Chapter 14

Index Structures

It is not sufficient simply to scatter the records that represent tuples of a relation among various blocks. To see why, think how we would answer the simple query `SELECT * FROM R`. We would have to examine every block in the storage system to find the tuples of R. A better idea is to reserve some blocks, perhaps several whole cylinders, for R. Now, at least we can find the tuples of R without scanning the entire data store.

However, this organization offers little help for a query like

```
SELECT * FROM R WHERE a=10;
```

Section 8.4 introduced us to the importance of creating *indexes* to speed up queries that specify values for one or more attributes. As suggested in Fig. 14.1, an index is any data structure that takes the value of one or more fields and finds the records with that value "quickly." In particular, an index lets us find a record without having to look at more than a small fraction of all possible records. The field(s) on whose values the index is based is called the *search key*, or just "key" if the index is understood.

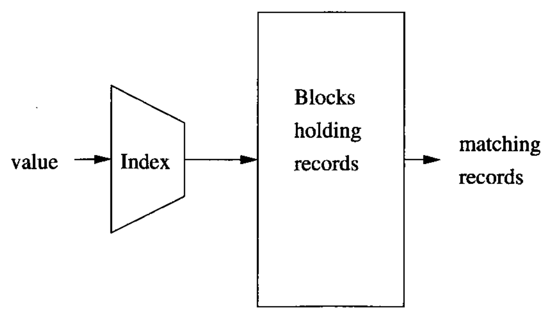

Figure 14.1: An index takes a value for some field(s) and finds records with the matching value

Different Kinds of "Keys"

There are many meanings of the term "key." We used it in Section 2.3.6 to mean the primary key of a relation. We shall also speak of "sort keys," the attribute(s) on which a file of records is sorted. We just introduced "search keys," the attribute(s) for which we are given values and asked to search, through an index, for tuples with matching values. We try to use the appropriate adjective — "primary," "sort," or "search" — when the meaning of "key" is unclear. However, in many cases, the three kinds of keys are one and the same.

In this chapter, we shall introduce the most common form of index in database systems: the B-tree. We shall also discuss hash tables in secondary storage, which is another important index structure. Finally, we consider other index structures that are designed to handle multidimensional data. These structures support queries that specify values or ranges for several attributes at once.

14.1 Index-Structure Basics

In this section, we introduce concepts that apply to all index structures. Storage structures consist of *files*, which are similar to the files used by operating systems. A *data file* may be used to store a relation, for example. The data file may have one or more *index files*. Each index file associates values of the search key with pointers to data-file records that have that value for the attribute(s) of the search key.

Indexes can be "dense," meaning there is an entry in the index file for every record of the data file. They can be "sparse," meaning that only some of the data records are represented in the index, often one index entry per block of the data file. Indexes can also be "primary" or "secondary." A primary index determines the location of the records of the data file, while a secondary index does not. For example, it is common to create a primary index on the primary key of a relation and to create secondary indexes on some of the other attributes.

We conclude the section with a study of information retrieval from documents. The ideas of the section are combined to yield "inverted indexes," which enable efficient retrieval of documents that contain one or more given keywords. This technique is essential for answering search queries on the Web, for instance.

14.1.1 Sequential Files

A *sequential file* is created by sorting the tuples of a relation by their primary key. The tuples are then distributed among blocks, in this order.

Example 14.1: Fig 14.2 shows a sequential file on the right. We imagine that keys are integers; we show only the key field, and we make the atypical assumption that there is room for only two records in one block. For instance, the first block of the file holds the records with keys 10 and 20. In this and several other examples, we use integers that are sequential multiples of 10 as keys, although there is surely no requirement that keys form an arithmetic sequence. □

Although in Example 14.1 we supposed that records were packed as tightly as possible into blocks, it is common to leave some space initially in each block to accomodate new tuples that may be added to a relation. Alternatively, we may accomodate new tuples with overflow blocks, as we suggested in Section 13.8.1.

14.1.2 Dense Indexes

If records are sorted, we can build on them a *dense index*, which is a sequence of blocks holding only the keys of the records and pointers to the records themselves; the pointers are addresses in the sense discussed in Section 13.6. The index blocks of the dense index maintain these keys in the same sorted order as in the file itself. Since keys and pointers presumably take much less space than complete records, we expect to use many fewer blocks for the index than for the file itself. The index is especially advantageous when it, but not the data file, can fit in main memory. Then, by using the index, we can find any record given its search key, with only one disk I/O per lookup.

Example 14.2: Figure 14.2 suggests a dense index on a sorted file. The first index block contains pointers to the first four records (an atypically small number of pointers for one block), the second block has pointers to the next four, and so on. □

The dense index supports queries that ask for records with a given search-key value. Given key value K, we search the index blocks for K, and when we find it, we follow the associated pointer to the record with key K. It might appear that we need to examine every block of the index, or half the blocks of the index, on average, before we find K. However, there are several factors that make the index-based search more efficient than it seems.

1. The number of index blocks is usually small compared with the number of data blocks.
2. Since keys are sorted, we can use binary search to find K. If there are n blocks of the index, we only look at $\log_2 n$ of them.

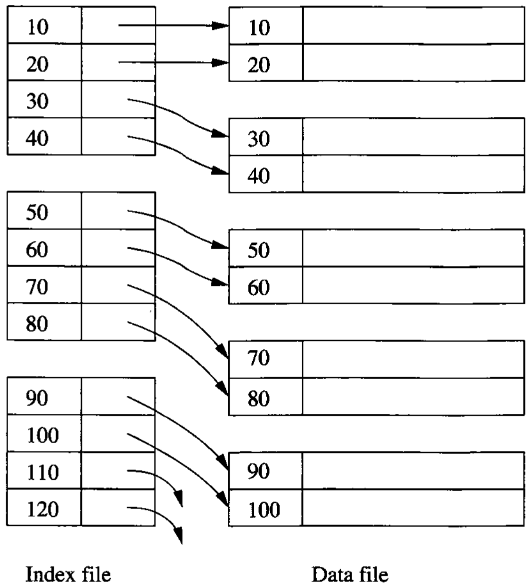

Figure 14.2: A dense index (left) on a sequential data file (right)

3. The index may be small enough to be kept permanently in main memory buffers. If so, the search for key K involves only main-memory accesses, and there are no expensive disk I/O's to be performed.

14.1.3 Sparse Indexes

A sparse index typically has only one key-pointer pair per block of the data file. It thus uses less space than a dense index, at the expense of somewhat more time to find a record given its key. You can only use a sparse index if the data file is sorted by the search key, while a dense index can be used for any search key. Figure 14.3 shows a sparse index with one key-pointer per data block. The keys are for the first records on each data block.

Example 14.3: As in Example 14.2, we assume that the data file is sorted, and keys are all the integers divisible by 10, up to some large number. We also continue to assume that four key-pointer pairs fit on an index block. Thus, the first sparse-index block has entries for the first keys on the first four blocks, which are 10, 30, 50, and 70. Continuing the assumed pattern of keys, the second index block has the first keys of the fifth through eighth blocks, which we assume are 90, 110, 130, and 150. We also show a third index block with first keys from the hypothetical ninth through twelfth data blocks. □

To find the record with search-key value K, we search the sparse index for the largest key less than or equal to K. Since the index file is sorted by key, a

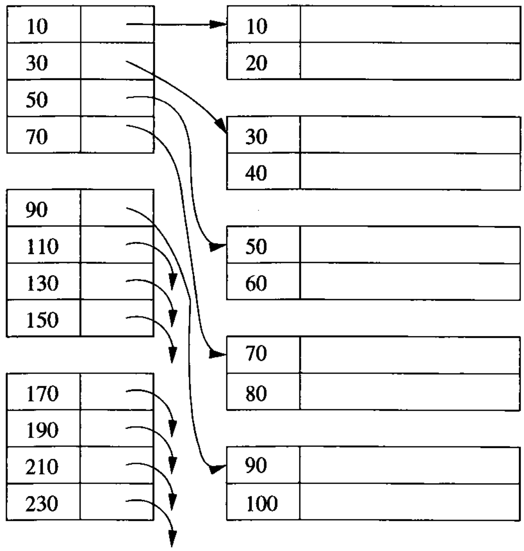

Figure 14.3: A sparse index on a sequential file

binary search can locate this entry. We follow the associated pointer to a data block. Now, we must search this block for the record with key K. Of course the block must have enough format information that the records and their contents can be identified. Any of the techniques from Sections 13.5 and 13.7 can be used.

14.1.4 Multiple Levels of Index

An index file can cover many blocks. Even if we use binary search to find the desired index entry, we still may need to do many disk I/O's to get to the record we want. By putting an index on the index, we can make the use of the first level of index more efficient.

Figure 14.4 extends Fig. 14.3 by adding a second index level (as before, we assume keys are every multiple of 10). The same idea would let us place a third-level index on the second level, and so on. However, this idea has its limits, and we prefer the B-tree structure described in Section 14.2 over building many levels of index.

In this example, the first-level index is sparse, although we could have chosen a dense index for the first level. However, the second and higher levels must be sparse. The reason is that a dense index on an index would have exactly as many key-pointer pairs as the first-level index, and therefore would take exactly as much space as the first-level index.

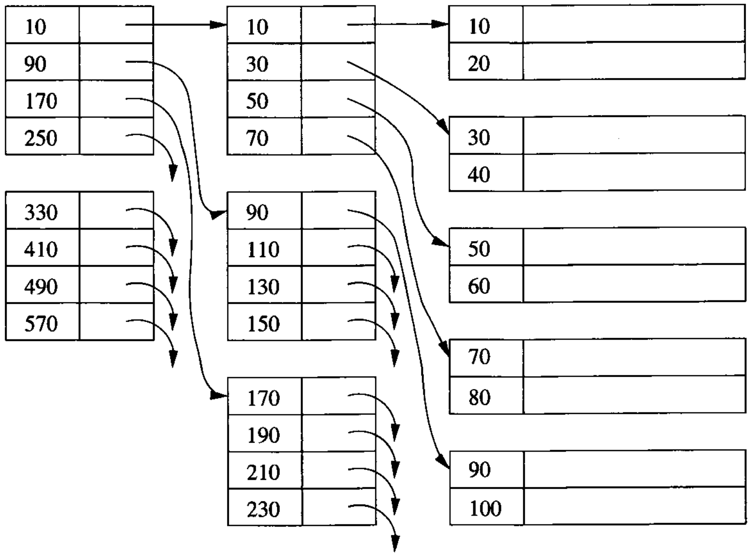

Figure 14.4: Adding a second level of sparse index

14.1.5 Secondary Indexes

A secondary index serves the purpose of any index: it is a data structure that facilitates finding records given a value for one or more fields. However, the secondary index is distinguished from the primary index in that a secondary index does not determine the placement of records in the data file. Rather, the secondary index tells us the current locations of records; that location may have been decided by a primary index on some other field. An important consequence of the distinction between primary and secondary indexes is that:

- Secondary indexes are always dense. It makes no sense to talk of a sparse, secondary index. Since the secondary index does not influence location, we could not use it to predict the location of any record whose key was not mentioned in the index file explicitly.

Example 14.4: Figure 14.5 shows a typical secondary index. The data file is shown with two records per block, as has been our standard for illustration. The records have only their search key shown; this attribute is integer valued, and as before we have taken the values to be multiples of 10. Notice that, unlike the data file in Fig. 14.2, here the data is not sorted by the search key.

However, the keys in the index file *are* sorted. The result is that the pointers in one index block can go to many different data blocks, instead of one or a few consecutive blocks. For example, to retrieve all the records with search key 20, we not only have to look at two index blocks, but we are sent by their pointers to three different data blocks. Thus, using a secondary index may result in

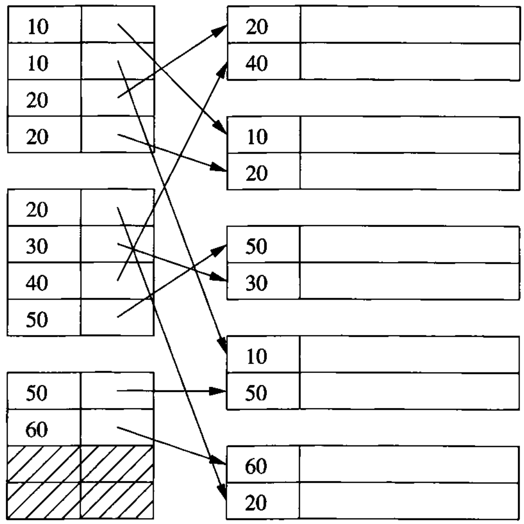

Figure 14.5: A secondary index

many more disk I/O's than if we get the same number of records via a primary index. However, there is no help for this problem; we cannot control the order of tuples in the data block, because they are presumably ordered according to some other attribute(s). □

14.1.6 Applications of Secondary Indexes

Besides supporting additional indexes on relations that are organized as sequential files, there are some data structures where secondary indexes are needed for even the primary key. One of these is the "heap" structure, where the records of the relation are kept in no particular order.

A second common structure needing secondary indexes is the *clustered file*. Suppose there are relations R and S, with a many-one relationship from the tuples of R to tuples of S. It may make sense to store each tuple of R with the tuple of S to which it is related, rather than according to the primary key of R. An example will illustrate why this organization makes good sense in special situations.

Example 14.5: Consider our standard movie and studio relations:

```
Movie(title, year, length, genre, studioName, producerC#)
Studio(name, address, presC#)
```

Suppose further that the most common form of query is:

```
SELECT title, year
FROM Movie, Studio
WHERE presC# = zzz AND Movie.studioName = Studio.name;
```

Here, *zzz* represents any possible certificate number for a studio president. That is, given the president of a studio, we need to find all the movies made by that studio.

If we are convinced that the above query is typical, then instead of ordering `Movie` tuples by the primary key `title` and `year`, we can create a *clustered file structure* for both relations `Studio` and `Movie`, as suggested by Fig. 14.6. Following each `Studio` tuple are all the `Movie` tuples for all the movies owned by that studio.

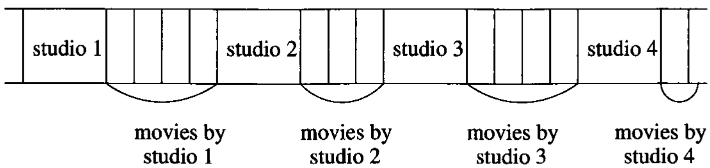

Figure 14.6: A clustered file with each studio clustered with the movies made by that studio

If we create an index for `Studio` with search key `presC#`, then whatever the value of *zzz* is, we can quickly find the tuple for the proper studio. Moreover, all the `Movie` tuples whose value of attribute `studioName` matches the value of `name` for that studio will follow the studio's tuple in the clustered file. As a result, we can find the movies for this studio by making almost as few disk I/O's as possible. The reason is that the desired `Movie` tuples are packed almost as densely as possible onto the following blocks. However, an index on any attribute(s) of `Movie` would have to be a secondary index. □

14.1.7 Indirection in Secondary Indexes

There is some wasted space, perhaps a significant amount of wastage, in the structure suggested by Fig. 14.5. If a search-key value appears n times in the data file, then the value is written n times in the index file. It would be better if we could write the key value once for all the pointers to data records with that value.

A convenient way to avoid repeating values is to use a level of indirection, called *buckets*, between the secondary index file and the data file. As shown in Fig. 14.7, there is one pair for each search key K. The pointer of this pair goes to a position in a "bucket file," which holds the "bucket" for K. Following this position, until the next position pointed to by the index, are pointers to all the records with search-key value K.

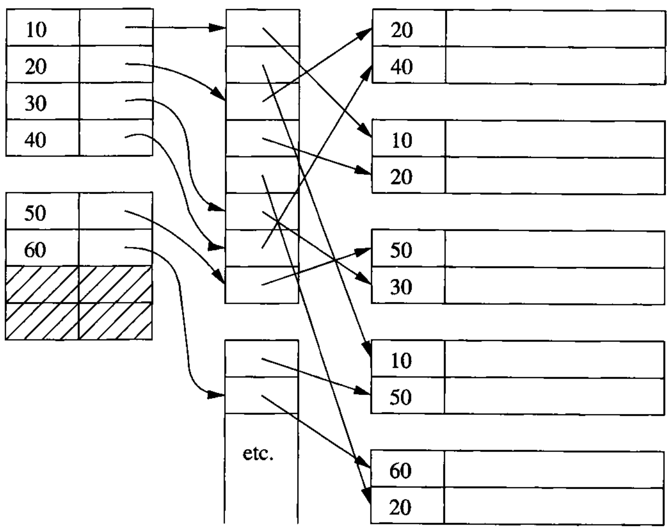

Figure 14.7: Saving space by using indirection in a secondary index

Example 14.6: For instance, let us follow the pointer from search key 50 in the index file of Fig. 14.7 to the intermediate "bucket" file. This pointer happens to take us to the last pointer of one block of the bucket file. We search forward, to the first pointer of the next block. We stop at that point, because the next pointer of the index file, associated with search key 60, points to the next record in the bucket file. □

The scheme of Fig. 14.7 saves space as long as search-key values are larger than pointers, and the average key appears at least twice. However, even if not, there is an important advantage to using indirection with secondary indexes: often, we can use the pointers in the buckets to help answer queries without ever looking at most of the records in the data file. Specifically, when there are several conditions to a query, and each condition has a secondary index to help it, we can find the bucket pointers that satisfy all the conditions by intersecting sets of pointers in memory, and retrieving only the records pointed to by the surviving pointers. We thus save the I/O cost of retrieving records that satisfy some, but not all, of the conditions.[1]

Example 14.7: Consider the usual `Movie` relation:

```
Movie(title, year, length, genre, studioName, producerC#)
```

[1] We also could use this pointer-intersection trick if we got the pointers directly from the index, rather than from buckets.

Suppose we have secondary indexes with indirect buckets on both `studioName` and `year`, and we are asked the query

```
SELECT title
FROM Movie
WHERE studioName = 'Disney' AND year = 2005;
```

that is, find all the Disney movies made in 2005.

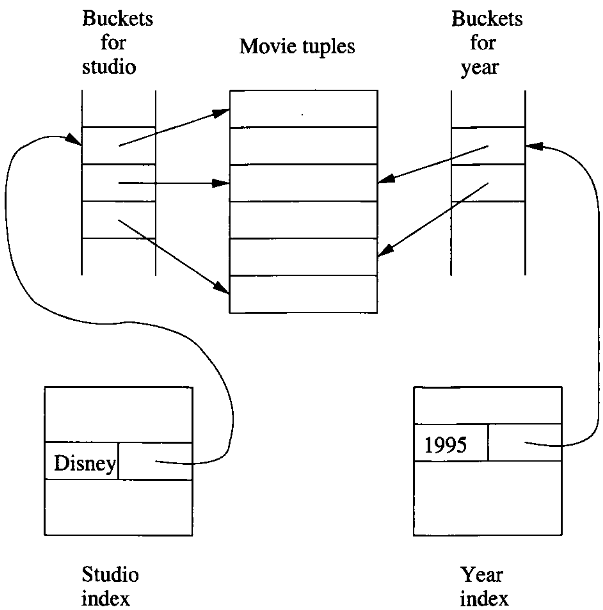

Figure 14.8: Intersecting buckets in main memory

Figure 14.8 shows how we can answer this query using the indexes. Using the index on `studioName`, we find the pointers to all records for Disney movies, but we do not yet bring any of those records from disk to memory. Instead, using the index on `year`, we find the pointers to all the movies of 2005. We then intersect the two sets of pointers, getting exactly the movies that were made by Disney in 2005. Finally, we retrieve from disk all data blocks holding one or more of these movies, thus retrieving the minimum possible number of blocks. □

14.1.8 Document Retrieval and Inverted Indexes

For many years, the information-retrieval community has dealt with the storage of documents and the efficient retrieval of documents with a given set of keywords. With the advent of the World-Wide Web and the feasibility of keeping all documents on-line, the retrieval of documents given keywords has become one of the largest database problems. While there are many kinds of queries that one can use to find relevant documents, the simplest and most common form can be seen in relational terms as follows:

- A document may be thought of as a tuple in a relation `Doc`. This relation has very many attributes, one corresponding to each possible word in a document. Each attribute is boolean — either the word is present in the document, or it is not. Thus, the relation schema may be thought of as

  ```
  Doc(hasCat, hasDog, ... )
  ```

  where `hasCat` is true if and only if the document has the word "cat" at least once.

- There is a secondary index on each of the attributes of `Doc`. However, we save the trouble of indexing those tuples for which the value of the attribute is `FALSE`; instead, the index leads us to only the documents for which the word is present. That is, the index has entries only for the search-key value `TRUE`.

- Instead of creating a separate index for each attribute (i.e., for each word), the indexes are combined into one, called an *inverted index.* This index uses indirect buckets for space efficiency, as was discussed in Section 14.1.7.

Example 14.8: An inverted index is illustrated in Fig. 14.9. In place of a data file of records is a collection of documents, each of which may be stored on one or more disk blocks. The inverted index itself consists of a set of word-pointer pairs; the words are in effect the search key for the index. The inverted index is kept in a sequence of blocks, just like any of the indexes discussed so far.

The pointers refer to positions in a "bucket" file. For instance, we have shown in Fig. 14.9 the word "cat" with a pointer to the bucket file. That pointer leads us to the beginning of a list of pointers to all the documents that contain the word "cat." We have shown some of these in the figure. Similarly, the word "dog" is shown leading to a list of pointers to all the documents with "dog." □

Pointers in the bucket file can be:

1. Pointers to the document itself.

2. Pointers to an occurrence of the word. In this case, the pointer might be a pair consisting of the first block for the document and an integer indicating the number of the word in the document.

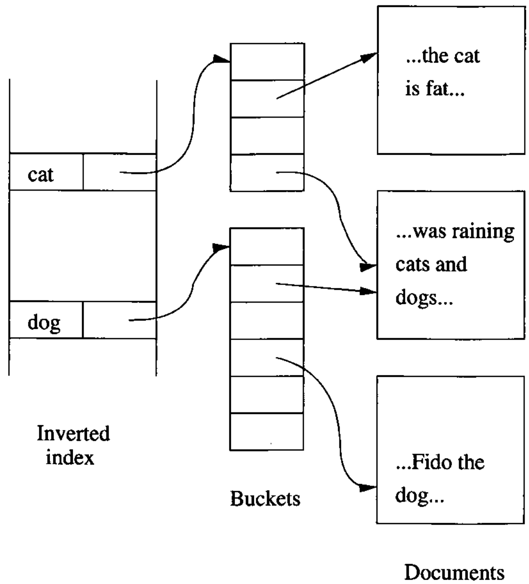

Figure 14.9: An inverted index on documents

When we use "buckets" of pointers to occurrences of each word, we may extend the idea to include in the bucket array some information about each occurrence. Now, the bucket file itself becomes a collection of records with important structure. Early uses of the idea distinguished occurrences of a word in the title of a document, the abstract, and the body of text. With the growth of documents on the Web, especially documents using HTML, XML, or another markup language, we can also indicate the markings associated with words. For instance, we can distinguish words appearing in titles, headers, tables, or anchors, as well as words appearing in different fonts or sizes.

Example 14.9: Figure 14.10 illustrates a bucket file that has been used to indicate occurrences of words in HTML documents. The first column indicates the type of occurrence, i.e., its marking, if any. The second and third columns are together the pointer to the occurrence. The third column indicates the document, and the second column gives the number of the word in the document.

We can use this data structure to answer various queries about documents without having to examine the documents in detail. For instance, suppose we want to find documents about dogs that compare them with cats. Without a deep understanding of the meaning of the text, we cannot answer this query precisely. However, we could get a good hint if we searched for documents that

a) Mention dogs in the title, and

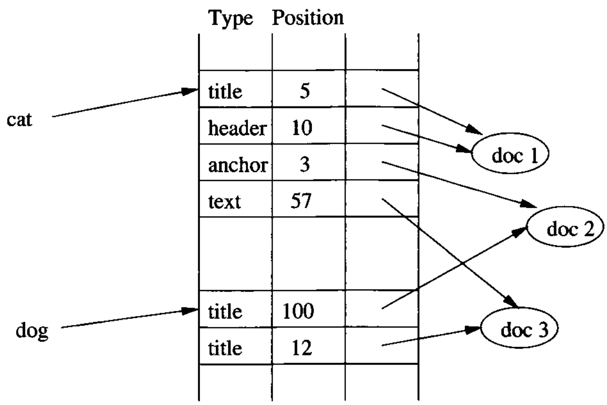

Figure 14.10: Storing more information in the inverted index

> **Insertion and Deletion From Buckets**
>
> We show buckets in figures such as Fig. 14.9 as compacted arrays of appropriate size. In practice, they are records with a single field (the pointer) and are stored in blocks like any other collection of records. Thus, when we insert or delete pointers, we may use any of the techniques seen so far, such as leaving extra space in blocks for expansion of the file, overflow blocks, and possibly moving records within or among blocks. In the latter case, we must be careful to change the pointer from the inverted index to the bucket file, as we move the records it points to.

b) Mention cats in an anchor — presumably a link to a document about cats.

We can answer this query by intersecting pointers. That is, we follow the pointer associated with "cat" to find the occurrences of this word. We select from the bucket file the pointers to documents associated with occurrences of "cat" where the type is "anchor." We then find the bucket entries for "dog" and select from them the document pointers associated with the type "title." If we intersect these two sets of pointers, we have the documents that meet the conditions: they mention "dog" in the title and "cat" in an anchor. □

14.1.9 Exercises for Section 14.1

Exercise 14.1.1: Suppose blocks hold either three records, or ten key-pointer pairs. As a function of n, the number of records, how many blocks do we need to hold a data file and: (a) A dense index (b) A sparse index?

More About Information Retrieval

There are a number of techniques for improving the effectiveness of retrieval of documents given keywords. While a complete treatment is beyond the scope of this book, here are two useful techniques:

1. *Stemming*. We remove suffixes to find the "stem" of each word, before entering its occurrence into the index. For example, plural nouns can be treated as their singular versions. Thus, in Example 14.8, the inverted index evidently uses stemming, since the search for word "dog" got us not only documents with "dog," but also a document with the word "dogs."

2. *Stop words*. The most common words, such as "the" or "and," are called *stop words* and often are excluded from the inverted index. The reason is that the several hundred most common words appear in too many documents to make them useful as a way to find documents about specific subjects. Eliminating stop words also reduces the size of the inverted index significantly.

Exercise 14.1.2: Repeat Exercise 14.1.1 if blocks can hold up to 30 records or 200 key-pointer pairs, but neither data- nor index-blocks are allowed to be more than 80% full.

! **Exercise 14.1.3:** Repeat Exercise 14.1.1 if we use as many levels of index as is appropriate, until the final level of index has only one block.

! **Exercise 14.1.4:** Consider a clustered file organization like Fig. 14.6, and suppose that ten records, either studio records or movie records, will fit on one block. Also assume that the number of movies per studio is uniformly distributed between 1 and m. As a function of m, what is the average number of disk I/O's needed to retrieve a studio and all its movies? What would the number be if movies were randomly distributed over a large number of blocks?

Exercise 14.1.5: Suppose that blocks can hold either three records, ten key-pointer pairs, or fifty pointers. Using the indirect-buckets scheme of Fig. 14.7:

a) If the average search-key value appears in 10 records, how many blocks do we need to hold 3000 records and its secondary index structure? How many blocks would be needed if we did *not* use buckets?

! b) If there are no constraints on the number of records that can have a given search-key value, what are the minimum and maximum number of blocks needed?

! Exercise 14.1.6: On the assumptions of Exercise 14.1.5(a), what is the average number of disk I/O's to find and retrieve the ten records with a given search-key value, both with and without the bucket structure? Assume nothing is in memory to begin, but it is possible to locate index or bucket blocks without incurring additional I/O's beyond what is needed to retrieve these blocks into memory.

Exercise 14.1.7: Suppose we have a repository of 1000 documents, and we wish to build an inverted index with 10,000 words. A block can hold ten word-pointer pairs or 50 pointers to either a document or a position within a document. The distribution of words is Zipfian (see the box on "The Zipfian Distribution" in Section 16.4.3); the number of occurrences of the ith most frequent word is $100000/\sqrt{i}$, for $i = 1, 2, \ldots, 10000$.

a) What is the averge number of words per document?

b) Suppose our inverted index only records for each word all the documents that have that word. What is the maximum number of blocks we could need to hold the inverted index?

c) Suppose our inverted index holds pointers to each occurrence of each word. How many blocks do we need to hold the inverted index?

d) Repeat (b) if the 400 most common words ("stop" words) are *not* included in the index.

e) Repeat (c) if the 400 most common words are not included in the index.

Exercise 14.1.8: If we use an augmented inverted index, such as in Fig. 14.10, we can perform a number of other kinds of searches. Suggest how this index could be used to find:

a) Documents in which "cat" and "dog" appeared within five positions of each other in the same type of element (e.g., title, text, or anchor).

b) Documents in which "dog" followed "cat" separated by exactly one position.

c) Documents in which "dog" and "cat" both appear in the title.

14.2 B-Trees

While one or two levels of index are often very helpful in speeding up queries, there is a more general structure that is commonly used in commercial systems. This family of data structures is called *B-trees*, and the particular variant that is most often used is known as a *B+ tree*. In essence:

- B-trees automatically maintain as many levels of index as is appropriate for the size of the file being indexed.

- B-trees manage the space on the blocks they use so that every block is between half used and completely full.

In the following discussion, we shall talk about "B-trees," but the details will all be for the B+ tree variant. Other types of B-tree are discussed in exercises.

14.2.1 The Structure of B-trees

A B-tree organizes its blocks into a tree that is *balanced*, meaning that all paths from the root to a leaf have the same length. Typically, there are three layers in a B-tree: the root, an intermediate layer, and leaves, but any number of layers is possible. To help visualize B-trees, you may wish to look ahead at Figs. 14.11 and 14.12, which show nodes of a B-tree, and Fig. 14.13, which shows an entire B-tree.

There is a parameter n associated with each B-tree index, and this parameter determines the layout of all blocks of the B-tree. Each block will have space for n search-key values and $n+1$ pointers. In a sense, a B-tree block is similar to the index blocks introduced in Section 14.1.2, except that the B-tree block has an extra pointer, along with n key-pointer pairs. We pick n to be as large as will allow $n+1$ pointers and n keys to fit in one block.

Example 14.10: Suppose our blocks are 4096 bytes. Also let keys be integers of 4 bytes and let pointers be 8 bytes. If there is no header information kept on the blocks, then we want to find the largest integer value of n such that $4n + 8(n+1) \leq 4096$. That value is $n = 340$. □

There are several important rules about what can appear in the blocks of a B-tree:

- The keys in leaf nodes are copies of keys from the data file. These keys are distributed among the leaves in sorted order, from left to right.

- At the root, there are at least two used pointers.[2] All pointers point to B-tree blocks at the level below.

- At a leaf, the last pointer points to the next leaf block to the right, i.e., to the block with the next higher keys. Among the other n pointers in a leaf block, at least $\lfloor (n+1)/2 \rfloor$ of these pointers are used and point to data records; unused pointers are null and do not point anywhere. The ith pointer, if it is used, points to a record with the ith key.

[2]Technically, there is a possibility that the entire B-tree has only one pointer because it is an index into a data file with only one record. In this case, the entire tree is a root block that is also a leaf, and this block has only one key and one pointer. We shall ignore this trivial case in the descriptions that follow.

- At an interior node, all $n + 1$ pointers can be used to point to B-tree blocks at the next lower level. At least $\lceil (n + 1)/2 \rceil$ of them are actually used (but if the node is the root, then we require only that at least 2 be used, regardless of how large n is). If j pointers are used, then there will be $j - 1$ keys, say $K_1, K_2, \ldots, K_{j-1}$. The first pointer points to a part of the B-tree where some of the records with keys less than K_1 will be found. The second pointer goes to that part of the tree where all records with keys that are at least K_1, but less than K_2 will be found, and so on. Finally, the jth pointer gets us to the part of the B-tree where some of the records with keys greater than or equal to K_{j-1} are found. Note that some records with keys far below K_1 or far above K_{j-1} may not be reachable from this block at all, but will be reached via another block at the same level.

- All used pointers and their keys appear at the beginning of the block, with the exception of the $(n + 1)$st pointer in a leaf, which points to the next leaf.

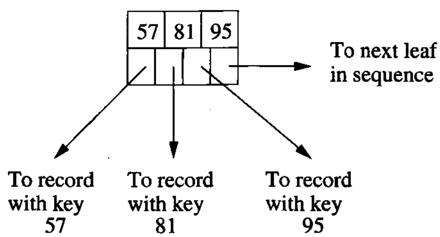

Figure 14.11: A typical leaf of a B-tree

Example 14.11: Our running example of B-trees will use $n = 3$. That is, blocks have room for three keys and four pointers, which are atypically small numbers. Keys are integers. Figure 14.11 shows a leaf that is completely used. There are three keys, 57, 81, and 95. The first three pointers go to records with these keys. The last pointer, as is always the case with leaves, points to the next leaf to the right in the order of keys; it would be null if this leaf were the last in sequence.

A leaf is not necessarily full, but in our example with $n = 3$, there must be at least two key-pointer pairs. That is, the key 95 in Fig. 14.11 might be missing, and if so, the third pointer would be null.

Figure 14.12 shows a typical interior node. There are three keys, 14, 52, and 78. There are also four pointers in this node. The first points to a part of the B-tree from which we can reach only records with keys less than 14 — the first of the keys. The second pointer leads to all records with keys between the first and second keys of the B-tree block; the third pointer is for those records

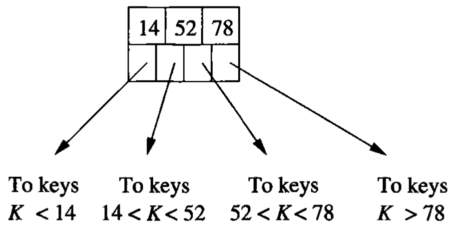

Figure 14.12: A typical interior node of a B-tree

between the second and third keys of the block, and the fourth pointer lets us reach some of the records with keys equal to or above the third key of the block.

As with our example leaf, it is not necessarily the case that all slots for keys and pointers are occupied. However, with $n = 3$, at least the first key and the first two pointers must be present in an interior node. □

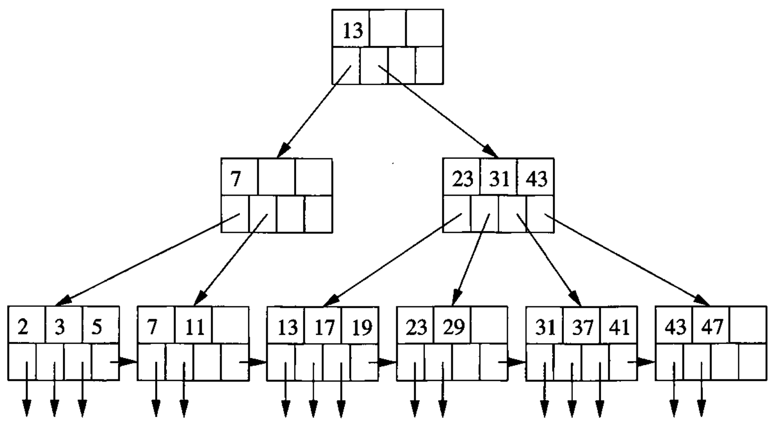

Figure 14.13: A B-tree

Example 14.12: Figure 14.13 shows an entire three-level B-tree, with $n = 3$, as in Example 14.11. We have assumed that the data file consists of records whose keys are all the primes from 2 to 47. Notice that at the leaves, each of these keys appears once, in order. All leaf blocks have two or three key-pointer pairs, plus a pointer to the next leaf in sequence. The keys are in sorted order as we look across the leaves from left to right.

The root has only two pointers, the minimum possible number, although it could have up to four. The one key at the root separates those keys reachable via the first pointer from those reachable via the second. That is, keys up to 12 could be found in the first subtree of the root, and keys 13 and up are in the second subtree.

If we look at the first child of the root, with key 7, we again find two pointers, one to keys less than 7 and the other to keys 7 and above. Note that the second pointer in this node gets us only to keys 7 and 11, not to *all* keys ≥ 7, such as 13.

Finally, the second child of the root has all four pointer slots in use. The first gets us to some of the keys less than 23, namely 13, 17, and 19. The second pointer gets us to all keys K such that $23 \leq K < 31$; the third pointer lets us reach all keys K such that $31 \leq K < 43$, and the fourth pointer gets us to some of the keys ≥ 43 (in this case, to all of them). □

14.2.2 Applications of B-trees

The B-tree is a powerful tool for building indexes. The sequence of pointers at the leaves of a B-tree can play the role of any of the pointer sequences coming out of an index file that we learned about in Section 14.1. Here are some examples:

1. The search key of the B-tree is the primary key for the data file, and the index is dense. That is, there is one key-pointer pair in a leaf for every record of the data file. The data file may or may not be sorted by primary key.

2. The data file is sorted by its primary key, and the B-tree is a sparse index with one key-pointer pair at a leaf for each block of the data file.

3. The data file is sorted by an attribute that is not a key, and this attribute is the search key for the B-tree. For each key value K that appears in the data file there is one key-pointer pair at a leaf. That pointer goes to the first of the records that have K as their sort-key value.

There are additional applications of B-tree variants that allow multiple occurrences of the search key[3] at the leaves. Figure 14.14 suggests what such a B-tree might look like.

If we do allow duplicate occurrences of a search key, then we need to change slightly the definition of what the keys at interior nodes mean, which we discussed in Section 14.2.1. Now, suppose there are keys $K_1, K_2, \ldots, K_n$ at an interior node. Then K_i will be the smallest new key that appears in the part of the subtree accessible from the $(i+1)$st pointer. By "new," we mean that there are no occurrences of K_i in the portion of the tree to the left of the $(i+1)$st subtree, but at least one occurrence of K_i in that subtree. Note that in some situations, there will be no such key, in which case K_i can be taken to be null. Its associated pointer is still necessary, as it points to a significant portion of the tree that happens to have only one key value within it.

[3]Remember that a "search key" is not necessarily a "key" in the sense of being unique.

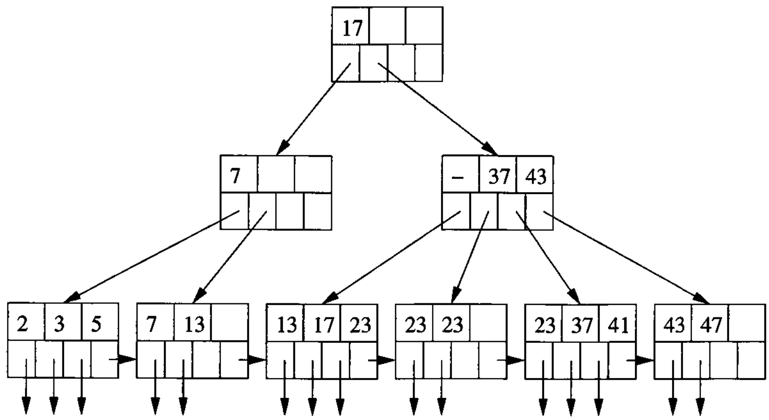

Figure 14.14: A B-tree with duplicate keys

Example 14.13: Figure 14.14 shows a B-tree similar to Fig. 14.13, but with duplicate values. In particular, key 11 has been replaced by 13, and keys 19, 29, and 31 have all been replaced by 23. As a result, the key at the root is 17, not 13. The reason is that, although 13 is the lowest key in the second subtree of the root, it is not a *new* key for that subtree, since it also appears in the first subtree.

We also had to make some changes to the second child of the root. The second key is changed to 37, since that is the first new key of the third child (fifth leaf from the left). Most interestingly, the first key is now null. The reason is that the second child (fourth leaf) has no new keys at all. Put another way, if we were searching for any key and reached the second child of the root, we would never want to start at its second child. If we are searching for 23 or anything lower, we want to start at its first child, where we will either find what we are looking for (if it is 17), or find the first of what we are looking for (if it is 23). Note that:

- We would not reach the second child of the root searching for 13; we would be directed at the root to its first child instead.

- If we are looking for any key between 24 and 36, we are directed to the third leaf, but when we don't find even one occurrence of what we are looking for, we know not to search further right. For example, if there were a key 24 among the leaves, it would either be on the 4th leaf, in which case the null key in the second child of the root would be 24 instead, or it would be in the 5th leaf, in which case the key 37 at the second child of the root would be 24.

□

14.2.3 Lookup in B-Trees

We now revert to our original assumption that there are no duplicate keys at the leaves. We also suppose that the B-tree is a dense index, so every search-key value that appears in the data file will also appear at a leaf. These assumptions make the discussion of B-tree operations simpler, but is not essential for these operations. In particular, modifications for sparse indexes are similar to the changes we introduced in Section 14.1.3 for indexes on sequential files.

Suppose we have a B-tree index and we want to find a record with search-key value K. We search for K recursively, starting at the root and ending at a leaf. The search procedure is:

BASIS: If we are at a leaf, look among the keys there. If the ith key is K, then the ith pointer will take us to the desired record.

INDUCTION: If we are at an interior node with keys $K_1, K_2, \ldots, K_n$, follow the rules given in Section 14.2.1 to decide which of the children of this node should next be examined. That is, there is only one child that could lead to a leaf with key K. If $K < K_1$, then it is the first child, if $K_1 \leq K < K_2$, it is the second child, and so on. Recursively apply the search procedure at this child.

Example 14.14: Suppose we have the B-tree of Fig. 14.13, and we want to find a record with search key 40. We start at the root, where there is one key, 13. Since $13 \leq 40$, we follow the second pointer, which leads us to the second-level node with keys 23, 31, and 43.

At that node, we find $31 \leq 40 < 43$, so we follow the third pointer. We are thus led to the leaf with keys 31, 37, and 41. If there had been a record in the data file with key 40, we would have found key 40 at this leaf. Since we do not find 40, we conclude that there is no record with key 40 in the underlying data.

Note that had we been looking for a record with key 37, we would have taken exactly the same decisions, but when we got to the leaf we would find key 37. Since it is the second key in the leaf, we follow the second pointer, which will lead us to the data record with key 37. □

14.2.4 Range Queries

B-trees are useful not only for queries in which a single value of the search key is sought, but for queries in which a range of values are asked for. Typically, *range queries* have a term in the WHERE-clause that compares the search key with a value or values, using one of the comparison operators other than = or <>. Examples of range queries using a search-key attribute k are:

```
SELECT * FROM R     SELECT * FROM R
WHERE R.k > 40;     WHERE R.k >= 10 AND R.k <= 25;
```

If we want to find all keys in the range $[a, b]$ at the leaves of a B-tree, we do a lookup to find the key a. Whether or not it exists, we are led to a leaf where

a could be, and we search the leaf for keys that are a or greater. Each such key we find has an associated pointer to one of the records whose key is in the desired range. As long as we do not find a key greater than b in the current block, we follow the pointer to the next leaf and repeat our search for keys in the range $[a, b]$.

The above search algorithm also works if b is infinite; i.e., there is only a lower bound and no upper bound. In that case, we search all the leaves from the one that would hold key a to the end of the chain of leaves. If a is $-\infty$ (that is, there is an upper bound on the range but no lower bound), then the search for "minus infinity" as a search key will always take us to the first leaf. The search then proceeds as above, stopping only when we pass the key b.

Example 14.15: Suppose we have the B-tree of Fig. 14.13, and we are given the range $(10, 25)$ to search for. We look for key 10, which leads us to the second leaf. The first key is less than 10, but the second, 11, is at least 10. We follow its associated pointer to get the record with key 11.

Since there are no more keys in the second leaf, we follow the chain to the third leaf, where we find keys 13, 17, and 19. All are less than or equal to 25, so we follow their associated pointers and retrieve the records with these keys. Finally, we move to the fourth leaf, where we find key 23. But the next key of that leaf, 29, exceeds 25, so we are done with our search. Thus, we have retrieved the five records with keys 11 through 23. □

14.2.5 Insertion Into B-Trees

We see some of the advantages of B-trees over simpler multilevel indexes when we consider how to insert a new key into a B-tree. The corresponding record will be inserted into the file being indexed by the B-tree, using any of the methods discussed in Section 14.1; here we consider how the B-tree changes. The insertion is, in principle, recursive:

- We try to find a place for the new key in the appropriate leaf, and we put it there if there is room.

- If there is no room in the proper leaf, we split the leaf into two and divide the keys between the two new nodes, so each is half full or just over half full.

- The splitting of nodes at one level appears to the level above as if a new key-pointer pair needs to be inserted at that higher level. We may thus recursively apply this strategy to insert at the next level: if there is room, insert it; if not, split the parent node and continue up the tree.

- As an exception, if we try to insert into the root, and there is no room, then we split the root into two nodes and create a new root at the next higher level; the new root has the two nodes resulting from the split as its children. Recall that no matter how large n (the number of slots for keys at a node) is, it is always permissible for the root to have only one key and two children.

When we split a node and insert it into its parent, we need to be careful how the keys are managed. First, suppose N is a leaf whose capacity is n keys. Also suppose we are trying to insert an $(n+1)$st key and its associated pointer. We create a new node M, which will be the sibling of N, immediately to its right. The first $\lceil (n+1)/2 \rceil$ key-pointer pairs, in sorted order of the keys, remain with N, while the other key-pointer pairs move to M. Note that both nodes N and M are left with a sufficient number of key-pointer pairs — at least $\lfloor (n+1)/2 \rfloor$ pairs.

Now, suppose N is an interior node whose capacity is n keys and $n+1$ pointers, and N has just been assigned $n+2$ pointers because of a node splitting below. We do the following:

1. Create a new node M, which will be the sibling of N, immediately to its right.

2. Leave at N the first $\lceil (n+2)/2 \rceil$ pointers, in sorted order, and move to M the remaining $\lfloor (n+2)/2 \rfloor$ pointers.

3. The first $\lceil n/2 \rceil$ keys stay with N, while the last $\lfloor n/2 \rfloor$ keys move to M. Note that there is always one key in the middle left over; it goes with neither N nor M. The leftover key K indicates the smallest key reachable via the first of M's children. Although this key doesn't appear in N or M, it is associated with M, in the sense that it represents the smallest key reachable via M. Therefore K will be inserted into the parent of N and M to divide searches between those two nodes.

Example 14.16: Let us insert key 40 into the B-tree of Fig. 14.13. We find the proper leaf for the insertion by the lookup procedure of Section 14.2.3. As found in Example 14.14, the insertion goes into the fifth leaf. Since this leaf now has four key-pointer pairs — 31, 37, 40, and 41 — we need to split the leaf. Our first step is to create a new node and move the highest two keys, 40 and 41, along with their pointers, to that node. Figure 14.15 shows this split.

Notice that although we now show the nodes on four ranks to save space, there are still only three levels to the tree. The seven leaves are linked by their last pointers, which still form a chain from left to right.

We must now insert a pointer to the new leaf (the one with keys 40 and 41) into the node above it (the node with keys 23, 31, and 43). We must also associate with this pointer the key 40, which is the least key reachable through the new leaf. Unfortunately, the parent of the split node is already full; it has no room for another key or pointer. Thus, it too must be split.

We start with pointers to the last five leaves and the list of keys representing the least keys of the last four of these leaves. That is, we have pointers P_1, P_2, P_3, P_4, P_5 to the leaves whose least keys are 13, 23, 31, 40, and 43, and

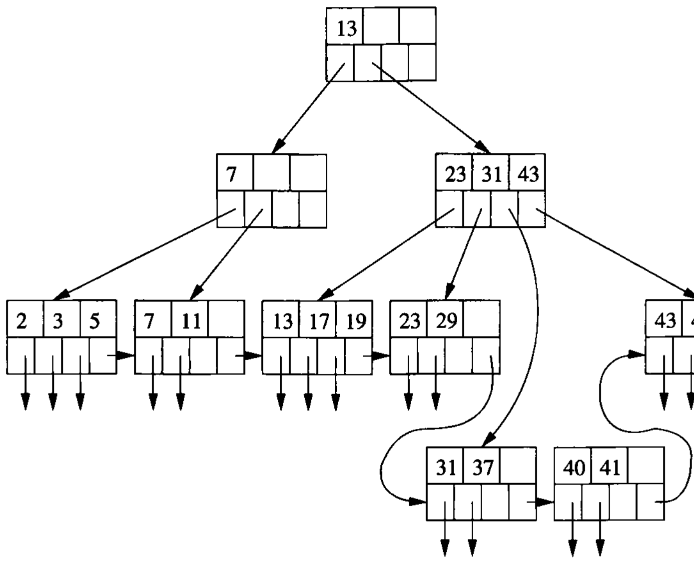

Figure 14.15: Beginning the insertion of key 40

we have the key sequence 23, 31, 40, 43 to separate these pointers. The first three pointers and first two keys remain with the split interior node, while the last two pointers and last key go to the new node. The remaining key, 40, represents the least key accessible via the new node.

Figure 14.16 shows the completion of the insert of key 40. The root now has three children; the last two are the split interior node. Notice that the key 40, which marks the lowest of the keys reachable via the second of the split nodes, has been installed in the root to separate the keys of the root's second and third children. □

14.2.6 Deletion From B-Trees

If we are to delete a record with a given key K, we must first locate that record and its key-pointer pair in a leaf of the B-tree. This part of the deletion process is essentially a lookup, as in Section 14.2.3. We then delete the record itself from the data file, and we delete the key-pointer pair from the B-tree.

If the B-tree node from which a deletion occurred still has at least the minimum number of keys and pointers, then there is nothing more to be done.[4] However, it is possible that the node was right at the minimum occupancy before the deletion, so after deletion the constraint on the number of keys is

[4]If the data record with the least key at a leaf is deleted, then we have the option of raising the appropriate key at one of the ancestors of that leaf, but there is no requirement that we do so; all searches will still go to the appropriate leaf.

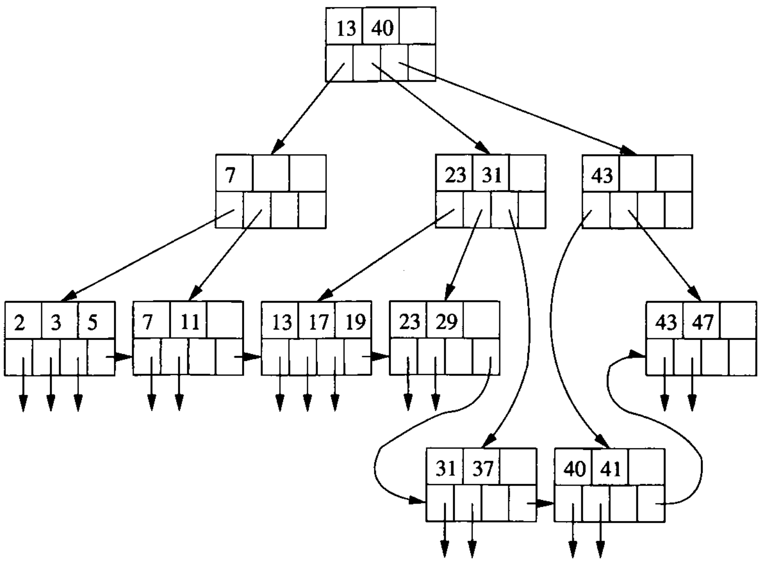

Figure 14.16: Completing the insertion of key 40

violated. We then need to do one of two things for a node N whose contents are subminimum; one case requires a recursive deletion up the tree:

1. If one of the adjacent siblings of node N has more than the minimum number of keys and pointers, then one key-pointer pair can be moved to N, keeping the order of keys intact. Possibly, the keys at the parent of N must be adjusted to reflect the new situation. For instance, if the right sibling of N, say node M, provides an extra key and pointer, then it must be the smallest key that is moved from M to N. At the parent of M and N, there is a key that represents the smallest key accessible via M; that key must be increased to reflect the new M.

2. The hard case is when neither adjacent sibling can be used to provide an extra key for N. However, in that case, we have two adjacent nodes, N and a sibling M; the latter has the minimum number of keys and the former has fewer than the minimum. Therefore, together they have no more keys and pointers than are allowed in a single node. We merge these two nodes, effectively deleting one of them. We need to adjust the keys at the parent, and then delete a key and pointer at the parent. If the parent is still full enough, then we are done. If not, then we recursively apply the deletion algorithm at the parent.

Example 14.17: Let us begin with the original B-tree of Fig. 14.13, before the insertion of key 40. Suppose we delete key 7. This key is found in the second leaf. We delete it, its associated pointer, and the record that pointer points to.

The second leaf now has only one key, and we need at least two in every leaf. But we are saved by the sibling to the left, the first leaf, because that leaf has an extra key-pointer pair. We may therefore move the highest key, 5, and its associated pointer to the second leaf. The resulting B-tree is shown in Fig. 14.17. Notice that because the lowest key in the second leaf is now 5, the key in the parent of the first two leaves has been changed from 7 to 5.

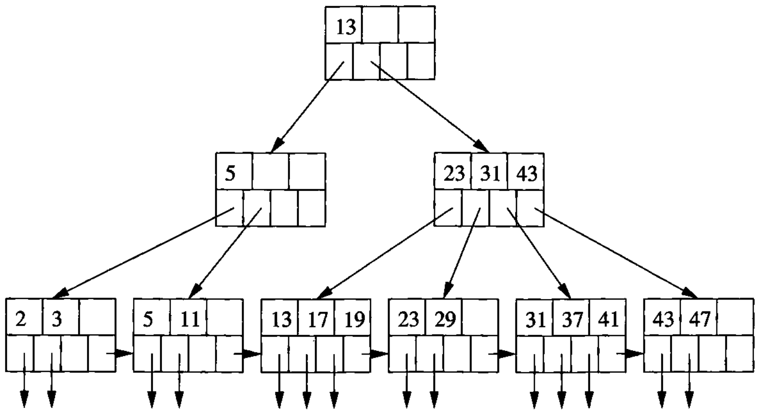

Figure 14.17: Deletion of key 7

Next, suppose we delete key 11. This deletion has the same effect on the second leaf; it again reduces the number of its keys below the minimum. This time, however, we cannot take a key from the first leaf, because the latter is down to the minimum number of keys. Additionally, there is no sibling to the right from which to take a key.[5] Thus, we need to merge the second leaf with a sibling, namely the first leaf.

The three remaining key-pointer pairs from the first two leaves fit in one leaf, so we move 5 to the first leaf and delete the second leaf. The pointers and keys in the parent are adjusted to reflect the new situation at its children; specifically, the two pointers are replaced by one (to the remaining leaf) and the key 5 is no longer relevant and is deleted. The situation is now as shown in Fig. 14.18.

The deletion of a leaf has adversely affected the parent, which is the left child of the root. That node, as we see in Fig. 14.18, now has no keys and only one pointer. Thus, we try to obtain an extra key and pointer from an adjacent sibling. This time we have the easy case, since the other child of the root can afford to give up its smallest key and a pointer.

The change is shown in Fig. 14.19. The pointer to the leaf with keys 13, 17,

[5]Notice that the leaf to the right, with keys 13, 17, and 19, is not a sibling, because it has a different parent. We could take a key from that node anyway, but then the algorithm for adjusting keys throughout the tree becomes more complex. We leave this enhancement as an exercise.

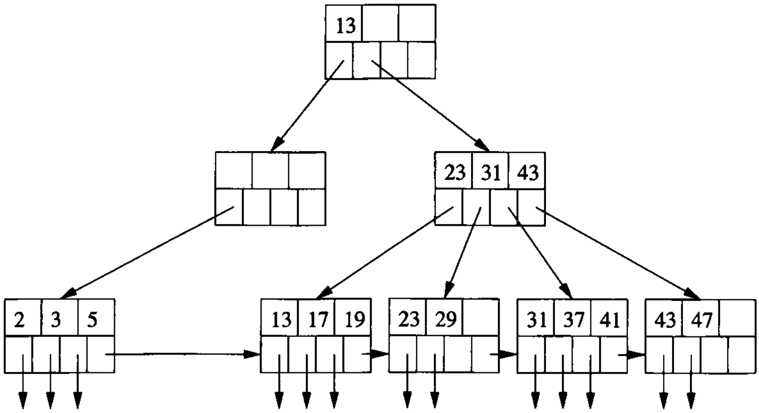

Figure 14.18: Beginning the deletion of key 11

and 19 has been moved from the second child of the root to the first child. We have also changed some keys at the interior nodes. The key 13, which used to reside at the root and represented the smallest key accessible via the pointer that was transferred, is now needed at the first child of the root. On the other hand, the key 23, which used to separate the first and second children of the second child of the root now represents the smallest key accessible from the second child of the root. It therefore is placed at the root itself. □

14.2.7 Efficiency of B-Trees

B-trees allow lookup, insertion, and deletion of records using very few disk I/O's per file operation. First, we should observe that if n, the number of keys per block, is reasonably large, then splitting and merging of blocks will be rare events. Further, when such an operation is needed, it almost always is limited to the leaves, so only two leaves and their parent are affected. Thus, we can essentially neglect the disk-I/O cost of B-tree reorganizations.

However, every search for the record(s) with a given search key requires us to go from the root down to a leaf, to find a pointer to the record. Since we are only reading B-tree blocks, the number of disk I/O's will be the number of levels the B-tree has, plus the one (for lookup) or two (for insert or delete) disk I/O's needed for manipulation of the record itself. We must thus ask: how many levels does a B-tree have? For the typical sizes of keys, pointers, and blocks, three levels are sufficient for all but the largest databases. Thus, we shall generally take 3 as the number of levels of a B-tree. The following example illustrates why.

Example 14.18: Recall our analysis in Example 14.10, where we determined that 340 key-pointer pairs could fit in one block for our example data. Suppose

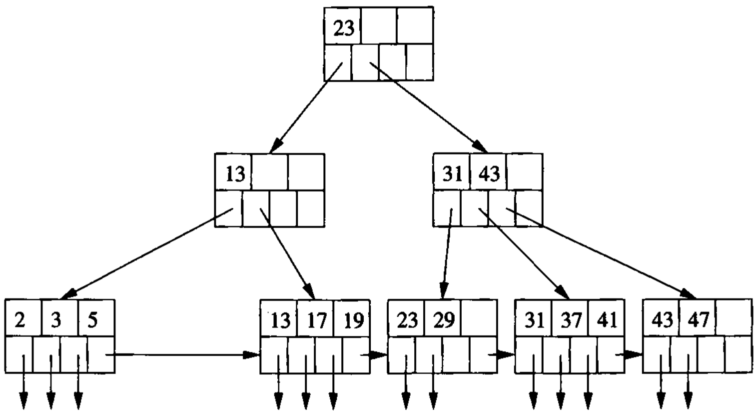

Figure 14.19: Completing the deletion of key 11

that the average block has an occupancy midway between the minimum and maximum, i.e., a typical block has 255 pointers. With a root, 255 children, and $255^2 = 65025$ leaves, we shall have among those leaves 255^3, or about 16.6 million pointers to records. That is, files with up to 16.6 million records can be accommodated by a 3-level B-tree. □

However, we can use even fewer than three disk I/O's per search through the B-tree. The root block of a B-tree is an excellent choice to keep permanently buffered in main memory. If so, then every search through a 3-level B-tree requires only two disk reads. In fact, under some circumstances it may make sense to keep second-level nodes of the B-tree buffered in main memory as well, reducing the B-tree search to a single disk I/O, plus whatever is necessary to manipulate the blocks of the data file itself.

14.2.8 Exercises for Section 14.2

Exercise 14.2.1: Suppose that blocks can hold either ten records or 99 keys and 100 pointers. Also assume that the average B-tree node is 70% full; i.e., it will have 69 keys and 70 pointers. We can use B-trees as part of several different structures. For each structure described below, determine (*i*) the total number of blocks needed for a 1,000,000-record file, and (*ii*) the average number of disk I/O's to retrieve a record given its search key. You may assume nothing is in memory initially, and the search key is the primary key for the records.

a) The data file is a sequential file, sorted on the search key, with 10 records per block. The B-tree is a dense index.

b) The same as (a), but the data file consists of records in no particular order, packed 10 to a block.

> **Should We Delete From B-Trees?**
>
> There are B-tree implementations that don't fix up deletions at all. If a leaf has too few keys and pointers, it is allowed to remain as it is. The rationale is that most files grow on balance, and while there might be an occasional deletion that makes a leaf become subminimum, the leaf will probably soon grow again and attain the minimum number of key-pointer pairs once again.
>
> Further, if records have pointers from outside the B-tree index, then we need to replace the record by a "tombstone," and we don't want to delete its pointer from the B-tree anyway. In certain circumstances, when it can be guaranteed that all accesses to the deleted record will go through the B-tree, we can even leave the tombstone in place of the pointer to the record at a leaf of the B-tree. Then, space for the record can be reused.

c) The same as (a), but the B-tree is a sparse index.

! d) Instead of the B-tree leaves having pointers to data records, the B-tree leaves hold the records themselves. A block can hold ten records, but on average, a leaf block is 70% full; i.e., there are seven records per leaf block.

e) The data file is a sequential file, and the B-tree is a sparse index, but each primary block of the data file has one overflow block. On average, the primary block is full, and the overflow block is half full. However, records are in no particular order within a primary block and its overflow block.

Exercise 14.2.2: Repeat Exercise 14.2.1 in the case that the query is a range query that is matched by 1000 records.

Exercise 14.2.3: Suppose pointers are 4 bytes long, and keys are 12 bytes long. How many keys and pointers will a block of 16,384 bytes have?

Exercise 14.2.4: What are the minimum numbers of keys and pointers in B-tree (*i*) interior nodes and (*ii*) leaves, when:

a) $n = 10$; i.e., a block holds 10 keys and 11 pointers.

b) $n = 11$; i.e., a block holds 11 keys and 12 pointers.

Exercise 14.2.5: Execute the following operations on Fig. 14.13. Describe the changes for operations that modify the tree.

a) Lookup the record with key 41.

b) Lookup the record with key 40.

c) Lookup all records in the range 20 to 30.

d) Lookup all records with keys less than 30.

e) Lookup all records with keys greater than 30.

f) Insert a record with key 1.

g) Insert records with keys 14 through 16.

h) Delete the record with key 23.

i) Delete all the records with keys 23 and higher.

Exercise 14.2.6: When duplicate keys are allowed in a B-tree, there are some necessary modifications to the algorithms for lookup, insertion, and deletion that we described in this section. Give the changes for: (a) lookup (b) insertion (c) deletion.

! **Exercise 14.2.7:** In Example 14.17 we suggested that it would be possible to borrow keys from a nonsibling to the right (or left) if we used a more complicated algorithm for maintaining keys at interior nodes. Describe a suitable algorithm that rebalances by borrowing from adjacent nodes at a level, regardless of whether they are siblings of the node that has too many or too few key-pointer pairs.

! **Exercise 14.2.8:** If we use the 3-key, 4-pointer nodes of our examples in this section, how many different B-trees are there when the data file has the following numbers of records: (a) 6 (b) 10 !! (c) 15.

! **Exercise 14.2.9:** Suppose we have B-tree nodes with room for three keys and four pointers, as in the examples of this section. Suppose also that when we split a leaf, we divide the pointers 2 and 2, while when we split an interior node, the first 3 pointers go with the first (left) node, and the last 2 pointers go with the second (right) node. We start with a leaf containing pointers to records with keys 1, 2, and 3. We then add in order, records with keys 4, 5, 6, and so on. At the insertion of what key will the B-tree first reach four levels?

14.3 Hash Tables

There are a number of data structures involving a hash table that are useful as indexes. We assume the reader has seen the hash table used as a main-memory data structure. In such a structure there is a ***hash function h*** that takes a search key (the ***hash key***) as an argument and computes from it an integer in the range 0 to $B-1$, where B is the number of *buckets*. A *bucket array*, which is an array indexed from 0 to $B-1$, holds the headers of B linked lists, one for each bucket of the array. If a record has search key K, then we store the record by linking it to the bucket list for the bucket numbered $h(K)$.

14.3.1 Secondary-Storage Hash Tables

A hash table that holds a very large number of records, so many that they must be kept mainly in secondary storage, differs from the main-memory version in small but important ways. First, the bucket array consists of blocks, rather than pointers to the headers of lists. Records that are hashed by the hash function h to a certain bucket are put in the block for that bucket. If a bucket has too many records, a chain of overflow blocks can be added to the bucket to hold more records.

We shall assume that the location of the first block for any bucket i can be found given i. For example, there might be a main-memory array of pointers to blocks, indexed by the bucket number. Another possibility is to put the first block for each bucket in fixed, consecutive disk locations, so we can compute the location of bucket i from the integer i.

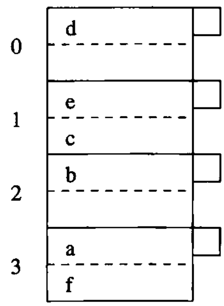

Figure 14.20: A hash table

Example 14.19: Figure 14.20 shows a hash table. To keep our illustrations manageable, we assume that a block can hold only two records, and that $B = 4$; i.e., the hash function h returns values from 0 to 3. We show certain records populating the hash table. Keys are letters a through f in Fig. 14.20. We assume that $h(d) = 0$, $h(c) = h(e) = 1$, $h(b) = 2$, and $h(a) = h(f) = 3$. Thus, the six records are distributed into blocks as shown. □

Note that we show each block in Fig. 14.20 with a "nub" at the right end. This nub represents additional information in the block's header. We shall use it to chain overflow blocks together, and starting in Section 14.3.5, we shall use it to keep other critical information about the block.

14.3.2 Insertion Into a Hash Table

When a new record with search key K must be inserted, we compute $h(K)$. If the bucket numbered $h(K)$ has space, then we insert the record into the block for this bucket, or into one of the overflow blocks on its chain if there is no room

Choice of Hash Function

The hash function should "hash" the key so the resulting integer is a seemingly random function of the key. Thus, buckets will tend to have equal numbers of records, which improves the average time to access a record, as we shall discuss in Section 14.3.4. Also, the hash function should be easy to compute, since we shall compute it many times.

A common choice of hash function when keys are integers is to compute the remainder of K/B, where K is the key value and B is the number of buckets. Often, B is chosen to be a prime, although there are reasons to make B a power of 2, as we discuss starting in Section 14.3.5. For character-string search keys, we may treat each character as an integer, sum these integers, and take the remainder when the sum is divided by B.

in the first block. If none of the blocks of the chain for bucket $h(K)$ has room, we add a new overflow block to the chain and store the new record there.

Example 14.20: Suppose we add to the hash table of Fig. 14.20 a record with key g, and $h(g) = 1$. Then we must add the new record to the bucket numbered 1. However, the block for that bucket already has two records. Thus, we add a new block and chain it to the original block for bucket 1. The record with key g goes in that block, as shown in Fig. 14.21. □

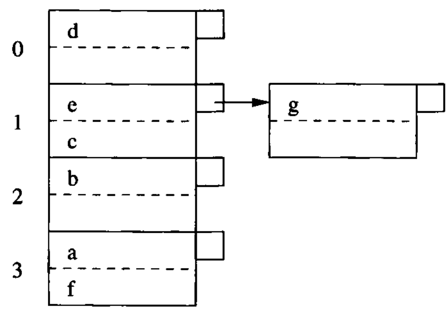

Figure 14.21: Adding an additional block to a hash-table bucket

14.3.3 Hash-Table Deletion

Deletion of the record (or records) with search key K follows the same pattern as insertion. We go to the bucket numbered $h(K)$ and search for records with that search key. Any that we find are deleted. If we are able to move records around among blocks, then after deletion we may optionally consolidate the blocks of a bucket into one fewer block.[6]

Example 14.21 : Figure 14.22 shows the result of deleting the record with key c from the hash table of Fig. 14.21. Recall $h(c) = 1$, so we go to the bucket numbered 1 (i.e., the second bucket) and search all its blocks to find a record (or records if the search key were not the primary key) with key c. We find it in the first block of the chain for bucket 1. Since there is now room to move the record with key g from the second block of the chain to the first, we can do so and remove the second block.

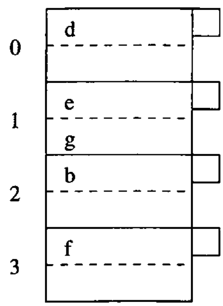

Figure 14.22: Result of deletions from a hash table

We also show the deletion of the record with key a. For this key, we found our way to bucket 3, deleted it, and "consolidated" the remaining record at the beginning of the block. □

14.3.4 Efficiency of Hash Table Indexes

Ideally, there are enough buckets that most of them fit on one block. If so, then the typical lookup takes only one disk I/O, and insertion or deletion from the file takes only two disk I/O's. That number is significantly better than straightforward sparse or dense indexes, or B-tree indexes (although hash tables do not support range queries as B-trees do; see Section 14.2.4).

However, if the file grows, then we shall eventually reach a situation where there are many blocks in the chain for a typical bucket. If so, then we need to search long lists of blocks, taking at least one disk I/O per block. Thus, there is a good reason to try to keep the number of blocks per bucket low.

The hash tables we have examined so far are called *static hash tables*, because B, the number of buckets, never changes. However, there are several kinds of *dynamic hash tables*, where B is allowed to vary so it approximates the number

[6]A risk of consolidating blocks of a chain whenever possible is that an oscillation, where we alternately insert and delete records from a bucket, will cause a block to be created or destroyed at each step.

of records divided by the number of records that can fit on a block; i.e., there is about one block per bucket. We shall discuss two such methods:

1. Extensible hashing in Section 14.3.5, and

2. Linear hashing in Section 14.3.7.

The first grows B by doubling it whenever it is deemed too small, and the second grows B by 1 each time statistics of the file suggest some growth is needed.

14.3.5 Extensible Hash Tables

Our first approach to dynamic hashing is called *extensible hash tables.* The major additions to the simpler static hash table structure are:

1. There is a level of indirection for the buckets. That is, an array of pointers to blocks represents the buckets, instead of the array holding the data blocks themselves.

2. The array of pointers can grow. Its length is always a power of 2, so in a growing step the number of buckets doubles.

3. However, there does not have to be a data block for each bucket; certain buckets can share a block if the total number of records in those buckets can fit in the block.

4. The hash function h computes for each key a sequence of k bits for some large k, say 32. However, the bucket numbers will at all times use some smaller number of bits, say i bits, from the beginning or end of this sequence. The bucket array will have 2^i entries when i is the number of bits used.

Example 14.22: Figure 14.23 shows a small extensible hash table. We suppose, for simplicity of the example, that $k = 4$; i.e., the hash function produces a sequence of only four bits. At the moment, only one of these bits is used, as indicated by $i = 1$ in the box above the bucket array. The bucket array therefore has only two entries, one for 0 and one for 1.

The bucket array entries point to two blocks. The first holds all the current records whose search keys hash to a bit sequence that begins with 0, and the second holds all those whose search keys hash to a sequence beginning with 1. For convenience, we show the keys of records as if they were the entire bit sequence to which the hash function converts them. Thus, the first block holds a record whose key hashes to 0001, and the second holds records whose keys hash to 1001 and 1100. □

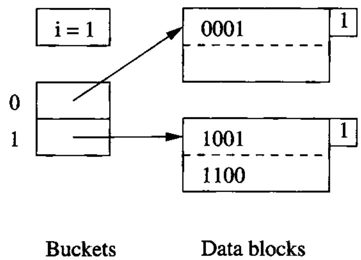

Figure 14.23: An extensible hash table

We should notice the number 1 appearing in the "nub" of each of the blocks in Fig. 14.23. This number, which would actually appear in the block header, indicates how many bits of the hash function's sequence is used to determine membership of records in this block. In the situation of Example 14.22, there is only one bit considered for all blocks and records, but as we shall see, the number of bits considered for various blocks can differ as the hash table grows. That is, the bucket array size is determined by the maximum number of bits we are now using, but some blocks may use fewer.

14.3.6 Insertion Into Extensible Hash Tables

Insertion into an extensible hash table begins like insertion into a static hash table. To insert a record with search key K, we compute $h(K)$, take the first i bits of this bit sequence, and go to the entry of the bucket array indexed by these i bits. Note that we can determine i because it is kept as part of the data structure.

We follow the pointer in this entry of the bucket array and arrive at a block B. If there is room to put the new record in block B, we do so and we are done. If there is no room, then there are two possibilities, depending on the number j, which indicates how many bits of the hash value are used to determine membership in block B (recall the value of j is found in the "nub" of each block in figures).

1. If $j < i$, then nothing needs to be done to the bucket array. We:

   (a) Split block B into two.

   (b) Distribute records in B to the two blocks, based on the value of their $(j + 1)$st bit — records whose key has 0 in that bit stay in B and those with 1 there go to the new block.

   (c) Put $j + 1$ in each block's "nub" (header) to indicate the number of bits used to determine membership.

   (d) Adjust the pointers in the bucket array so entries that formerly pointed to B now point either to B or the new block, depending on their $(j + 1)$st bit.

Note that splitting block B may not solve the problem, since by chance all the records of B may go into one of the two blocks into which it was split. If so, we need to repeat the process on the overfull block, using the next higher value of j and the block that is still overfull.

2. If $j = i$, then we must first increment i by 1. We double the length of the bucket array, so it now has 2^{i+1} entries. Suppose w is a sequence of i bits indexing one of the entries in the previous bucket array. In the new bucket array, the entries indexed by both $w0$ and $w1$ (i.e., the two numbers derived from w by extending it with 0 or 1) each point to the same block that the w entry used to point to. That is, the two new entries share the block, and the block itself does not change. Membership in the block is still determined by whatever number of bits was previously used. Finally, we proceed to split block B as in case 1. Since i is now greater than j, that case applies.

Example 14.23: Suppose we insert into the table of Fig. 14.23 a record whose key hashes to the sequence 1010. Since the first bit is 1, this record belongs in the second block. However, that block is already full, so it needs to be split. We find that $j = i = 1$ in this case, so we first need to double the bucket array, as shown in Fig. 14.24. We have also set $i = 2$ in this figure.

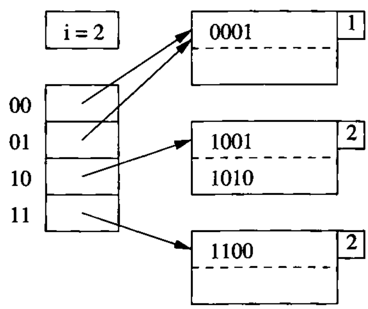

Figure 14.24: Now, two bits of the hash function are used

Notice that the two entries beginning with 0 each point to the block for records whose hashed keys begin with 0, and that block still has the integer 1 in its "nub" to indicate that only the first bit determines membership in the block. However, the block for records beginning with 1 needs to be split, so we partition its records into those beginning 10 and those beginning 11. A 2 in each of these blocks indicates that two bits are used to determine membership. Fortunately, the split is successful; since each of the two new blocks gets at least one record, we do not have to split recursively.

Now suppose we insert records whose keys hash to 0000 and 0111. These both go in the first block of Fig. 14.24, which then overflows. Since only one bit is used to determine membership in this block, while $i = 2$, we do not have to adjust the bucket array. We simply split the block, with 0000 and 0001 staying, and 0111 going to the new block. The entry for 01 in the bucket array is made to point to the new block. Again, we have been fortunate that the records did not all go in one of the new blocks, so we have no need to split recursively.

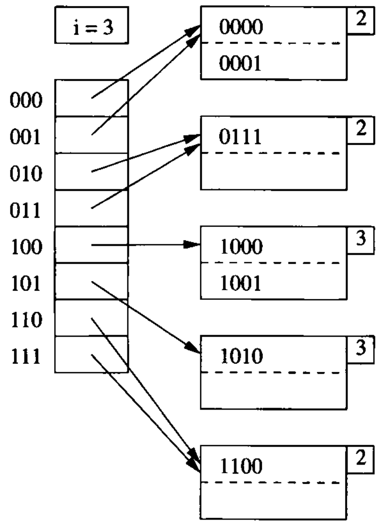

Figure 14.25: The hash table now uses three bits of the hash function

Now suppose a record whose key hashes to 1000 is inserted. The block for 10 overflows. Since it already uses two bits to determine membership, it is time to split the bucket array again and set $i = 3$. Figure 14.25 shows the data structure at this point. Notice that the block for 10 has been split into blocks for 100 and 101, while the other blocks continue to use only two bits to determine membership. □

14.3.7 Linear Hash Tables

Extensible hash tables have some important advantages. Most significant is the fact that when looking for a record, we never need to search more than one data block. We also have to examine an entry of the bucket array, but if the bucket array is small enough to be kept in main memory, then there is no disk I/O needed to access the bucket array. However, extensible hash tables also suffer from some defects:

1. When the bucket array needs to be doubled in size, there is a substantial amount of work to be done (when i is large). This work interrupts access to the data file, or makes certain insertions appear to take a long time.

2. When the bucket array is doubled in size, it may no longer fit in main memory, or may crowd out other data that we would like to hold in main memory. As a result, a system that was performing well might suddenly start using many more disk I/O's per operation.

3. If the number of records per block is small, then there is likely to be one block that needs to be split well in advance of the logical time to do so. For instance, if there are two records per block as in our running example, there might be one sequence of 20 bits that begins the keys of three records, even though the total number of records is much less than 2^{20}. In that case, we would have to use $i = 20$ and a million-bucket array, even though the number of blocks holding records was much smaller than a million.

Another strategy, called *linear hashing*, grows the number of buckets more slowly. The principal new elements we find in linear hashing are:

- The number of buckets n is always chosen so the average number of records per bucket is a fixed fraction, say 80%, of the number of records that fill one block.

- Since blocks cannot always be split, overflow blocks are permitted, although the average number of overflow blocks per bucket will be much less than 1.

- The number of bits used to number the entries of the bucket array is $\lceil \log_2 n \rceil$, where n is the current number of buckets. These bits are always taken from the *right* (low-order) end of the bit sequence that is produced by the hash function.

- Suppose i bits of the hash function are being used to number array entries, and a record with key K is intended for bucket $a_1 a_2 \cdots a_i$; that is, $a_1 a_2 \cdots a_i$ are the last i bits of $h(K)$. Then let $a_1 a_2 \cdots a_i$ be m, treated as an i-bit binary integer. If $m < n$, then the bucket numbered m exists, and we place the record in that bucket. If $n \leq m < 2^i$, then the bucket m does not yet exist, so we place the record in bucket $m - 2^{i-1}$, that is, the bucket we would get if we changed a_1 (which must be 1) to 0.

Example 14.24: Figure 14.26 shows a linear hash table with $n = 2$. We currently are using only one bit of the hash value to determine the buckets of records. Following the pattern established in Example 14.22, we assume the hash function h produces 4 bits, and we represent records by the value produced by h when applied to the search key of the record.

We see in Fig. 14.26 the two buckets, each consisting of one block. The buckets are numbered 0 and 1. All records whose hash value ends in 0 go in the first bucket, and those whose hash value ends in 1 go in the second.

Also part of the structure are the parameters i (the number of bits of the hash function that currently are used), n (the current number of buckets), and r

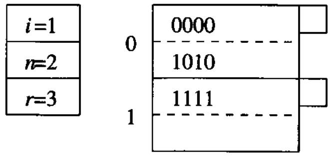

Figure 14.26: A linear hash table

(the current number of records in the hash table). The ratio r/n will be limited so that the typical bucket will need about one disk block. We shall adopt the policy of choosing n, the number of buckets, so that there are no more than $1.7n$ records in the file; i.e., $r \leq 1.7n$. That is, since blocks hold two records, the average occupancy of a bucket does not exceed 85% of the capacity of a block. □

14.3.8 Insertion Into Linear Hash Tables

When we insert a new record, we determine its bucket by the algorithm outlined in Section 14.3.7. We compute $h(K)$, where K is the key of the record, and we use the i bits at the end of bit sequence $h(K)$ as the bucket number, m. If $m < n$, we put the record in bucket m, and if $m \geq n$, we put the record in bucket $m - 2^{i-1}$. If there is no room in the designated bucket, then we create an overflow block, add it to the chain for that bucket, and put the record there.

Each time we insert, we compare the current number of records r with the threshold ratio of r/n, and if the ratio is too high, we add the next bucket to the table. Note that the bucket we add bears no relationship to the bucket into which the insertion occurs! If the binary representation of the number of the bucket we add is $1a_2 \cdots a_i$, then we split the bucket numbered $0a_2 \cdots a_i$, putting records into one or the other bucket, depending on their last i bits. Note that all these records will have hash values that end in $a_2 \cdots a_i$, and only the ith bit from the right end will vary.

The last important detail is what happens when n exceeds 2^i. Then, i is incremented by 1. Technically, all the bucket numbers get an additional 0 in front of their bit sequences, but there is no need to make any physical change, since these bit sequences, interpreted as integers, remain the same.

Example 14.25: We shall continue with Example 14.24 and consider what happens when a record whose key hashes to 0101 is inserted. Since this bit sequence ends in 1, the record goes into the second bucket of Fig. 14.26. There is room for the record, so no overflow block is created.

However, since there are now 4 records in 2 buckets, we exceed the ratio 1.7, and we must therefore raise n to 3. Since $\lceil \log_2 3 \rceil = 2$, we should begin to think of buckets 0 and 1 as 00 and 01, but no change to the data structure is necessary. We add to the table the next bucket, which would have number 10. Then, we split the bucket 00, that bucket whose number differs from the added bucket only in the first bit. When we do the split, the record whose key hashes to 0000 stays in 00, since it ends with 00, while the record whose key hashes to 1010 goes to 10 because it ends that way. The resulting hash table is shown in Fig. 14.27.

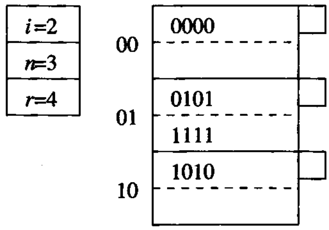

Figure 14.27: Adding a third bucket

Next, let us suppose we add a record whose search key hashes to 0001. The last two bits are 01, so we put it in this bucket, which currently exists. Unfortunately, the bucket's block is full, so we add an overflow block. The three records are distributed among the two blocks of the bucket; we chose to keep them in numerical order of their hashed keys, but order is not important. Since the ratio of records to buckets for the table as a whole is 5/3, and this ratio is less than 1.7, we do not create a new bucket. The result is seen in Fig. 14.28.

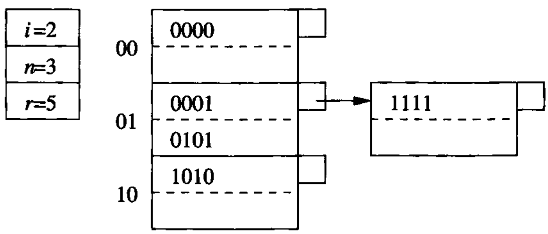

Figure 14.28: Overflow blocks are used if necessary

Finally, consider the insertion of a record whose search key hashes to 0111. The last two bits are 11, but bucket 11 does not yet exist. We therefore redirect this record to bucket 01, whose number differs by having a 0 in the first bit. The new record fits in the overflow block of this bucket.

However, the ratio of the number of records to buckets has exceeded 1.7, so we must create a new bucket, numbered 11. Coincidentally, this bucket is the one we wanted for the new record. We split the four records in bucket 01, with 0001 and 0101 remaining, and 0111 and 1111 going to the new bucket. Since bucket 01 now has only two records, we can delete the overflow block. The hash table is now as shown in Fig. 14.29.

Notice that the next time we insert a record into Fig. 14.29, we shall exceed

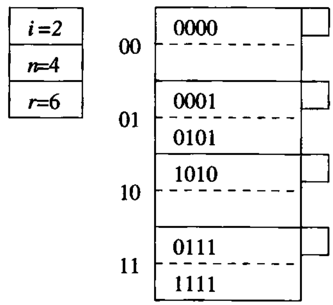

Figure 14.29: Adding a fourth bucket

the 1.7 ratio of records to buckets. Then, we shall raise n to 5 and i becomes 3. □

Lookup in a linear hash table follows the procedure we described for selecting the bucket in which an inserted record belongs. If the record we wish to look up is not in that bucket, it cannot be anywhere.

14.3.9 Exercises for Section 14.3

Exercise 14.3.1: Show what happens to the buckets in Fig. 14.20 if the following insertions and deletions occur:

i. Records g through j are inserted into buckets 0 through 3, respectively.

ii. Records a and b are deleted.

iii. Records k through n are inserted into buckets 0 through 3, respectively.

iv. Records c and d are deleted.

Exercise 14.3.2: We did not discuss how deletions can be carried out in a linear or extensible hash table. The mechanics of locating the record(s) to be deleted should be obvious. What method would you suggest for executing the deletion? In particular, what are the advantages and disadvantages of restructuring the table if its smaller size after deletion allows for compression of certain blocks?

! **Exercise 14.3.3:** The material of this section assumes that search keys are unique. However, only small modifications are needed to allow the techniques to work for search keys with duplicates. Describe the necessary changes to insertion, deletion, and lookup algorithms, and suggest the major problems that arise when there are duplicates in each of the following kinds of hash tables: (a) simple (b) linear (c) extensible.

! **Exercise 14.3.4:** Some hash functions do not work as well as theoretically possible. Suppose that we use the hash function on integer keys i defined by $h(i) = i^2 \bmod B$, where B is the number of buckets.

a) What is wrong with this hash function if $B = 10$?

b) How good is this hash function if $B = 16$?

c) Are there values of B for which this hash function is useful?

Exercise 14.3.5: In an extensible hash table with n records per block, what is the probability that an overflowing block will have to be handled recursively; i.e., all members of the block will go into the same one of the two blocks created in the split?

Exercise 14.3.6: Suppose keys are hashed to four-bit sequences, as in our examples of extensible and linear hashing in this section. However, also suppose that blocks can hold three records, rather than the two-record blocks of our examples. If we start with a hash table with two empty blocks (corresponding to 0 and 1), show the organization after we insert records with hashed keys:

a) $0000, 0001, \ldots, 1111$, and the method of hashing is extensible hashing.

b) $0000, 0001, \ldots, 1111$, and the method of hashing is linear hashing with a capacity threshold of 100%.

c) $1111, 1110, \ldots, 0000$, and the method of hashing is extensible hashing.

d) $1111, 1110, \ldots, 0000$, and the method of hashing is linear hashing with a capacity threshold of 75%.

Exercise 14.3.7: Suppose we use a linear or extensible hashing scheme, but there are pointers to records from outside. These pointers prevent us from moving records between blocks, as is sometimes required by these hashing methods. Suggest several ways that we could modify the structure to allow pointers from outside.

!! **Exercise 14.3.8:** A linear-hashing scheme with blocks that hold k records uses a threshold constant c, such that the current number of buckets n and the current number of records r are related by $r = ckn$. For instance, in Example 14.24 we used $k = 2$ and $c = 0.85$, so there were 1.7 records per bucket; i.e., $r = 1.7n$.

a) Suppose for convenience that each key occurs exactly its expected number of times.[7] As a function of c, k, and n, how many blocks, including overflow blocks, are needed for the structure?

[7] This assumption does not mean all buckets have the same number of records, because some buckets represent twice as many keys as others.

b) Keys will not generally distribute equally, but rather the number of records with a given key (or suffix of a key) will be *Poisson distributed.* That is, if λ is the expected number of records with a given key suffix, then the actual number of such records will be i with probability $e^{-\lambda}\lambda^i/i!$. Under this assumption, calculate the expected number of blocks used, as a function of c, k, and n.

! **Exercise 14.3.9:** Suppose we have a file of 1,000,000 records that we want to hash into a table with 1000 buckets. 100 records will fit in a block, and we wish to keep blocks as full as possible, but not allow two buckets to share a block. What are the minimum and maximum number of blocks that we could need to store this hash table?

14.4 Multidimensional Indexes

All the index structures discussed so far are *one dimensional*; that is, they assume a single search key, and they retrieve records that match a given search-key value. Although the search key may involve several attributes, the one-dimensional nature of indexes such as B-trees comes from the fact that values must be provided for all attributes of the search key, or the index is useless. So far in this chapter, we took advantage of a one-dimensional search-key space in several ways:

- Indexes on sequential files and B-trees both take advantage of having a single linear order for the keys.
- Hash tables require that the search key be completely known for any lookup. If a key consists of several fields, and even one is unknown, we cannot apply the hash function, but must instead search all the buckets.

In the balance of this chapter, we shall look at index structures that are suitable for multidimensional data. In these structures, any nonempty subset of the fields that form the dimensions can be given values, and some speedup will result.

14.4.1 Applications of Multidimensional Indexes

There are a number of applications that require us to view data as existing in a 2-dimensional space, or sometimes in higher dimensions. Some of these applications can be supported by conventional DBMS's, but there are also some specialized systems designed for multidimensional applications. One way in which these specialized systems distinguish themselves is by using data structures that support certain kinds of queries that are not common in SQL applications.

One important application of multidimensional indexes involves geographic data. A *geographic information system* stores objects in a (typically) two-dimensional space. The objects may be points or shapes. Often, these databases