Political Methodology III: Model Based Inference

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Statistical Inference

- Model based inference:
- Assume: data generated via distributional process
- Thursday: Derived a likelihood for single/bivariate problems
- Today: Interested in conditional relationships
- Today: Build Linear Model Using Likelihood

Bivariate $\rightsquigarrow$ Multivariate $\rightsquigarrow$ Computational Model

Regression Model$\rightsquigarrow$ Time for Change

Model incumbent vote share, Y_i

1) X_{i1} =Incumbent Presidential Popularity (Net Approval/Disapproval)
2) X_{i2} =GDP Growth
3) X_{i3} =First-term Incumbent In Race

$\boldsymbol{X}_i = (X_{i1}, X_{i2}, X_{i3})$

Bivariate Regression Model via Maximum Likelihood

Linear regression model:

$$\begin{aligned} Y_i &= \beta_0 + \beta_1 X_{i1} + \epsilon_i \\ \epsilon_i &\sim \textsf{Normal}(0, \sigma^2) \end{aligned}$$

Bivariate Regression Model via Maximum Likelihood

Linear regression model:

$$\begin{aligned} Y_i &= \beta_0 + \beta_1 X_{i1} + \epsilon_i \\ \epsilon_i &\sim \text{Normal}(0, \sigma^2) \end{aligned}$$

Equivalently:

$$\begin{aligned} Y_i &\sim \text{Normal}(\mu_i, \sigma^2) \\ E[Y_i|X_{i1}] = \mu_i &= \beta_0 + \beta_1 X_{i1} \end{aligned}$$

Bivariate Regression Model via Maximum Likelihood

Linear regression model:

$$
\begin{aligned}
Y_i &= \beta_0 + \beta_1 X_{i1} + \epsilon_i \\
\epsilon_i &\sim \mathsf{Normal}(0, \sigma^2)
\end{aligned}
$$

Equivalently:

$$
\begin{aligned}
Y_i &\sim \mathsf{Normal}(\mu_i, \sigma^2) \\
E[Y_i|X_{i1}] = \mu_i &= \beta_0 + \beta_1 X_{i1}
\end{aligned}
$$

Parameters:

$$
\begin{aligned}
\boldsymbol{\beta} &= (\beta_0, \beta_1) \\
\sigma^2 &
\end{aligned}
$$

Bivariate Regression Model via Maximum Likelihood

$$L(\boldsymbol{\beta}, \sigma^2 | \boldsymbol{Y}, \boldsymbol{X}) = f(\boldsymbol{Y} | \boldsymbol{\beta}, \sigma^2, \boldsymbol{X})$$

Bivariate Regression Model via Maximum Likelihood

$$\begin{aligned} L(\boldsymbol{\beta}, \sigma^2 | \boldsymbol{Y}, \boldsymbol{X}) &= f(\boldsymbol{Y} | \boldsymbol{\beta}, \sigma^2, \boldsymbol{X}) \\ &= \prod_{i=1}^{N} f(Y_i | \boldsymbol{\beta}, \sigma^2, X_{i1}) \end{aligned}$$

Bivariate Regression Model via Maximum Likelihood

$$
\begin{aligned}
L(\boldsymbol{\beta}, \sigma^2 | \boldsymbol{Y}, \boldsymbol{X}) & = f(\boldsymbol{Y} | \boldsymbol{\beta}, \sigma^2, \boldsymbol{X}) \\
& = \prod_{i=1}^{N} f(Y_i | \boldsymbol{\beta}, \sigma^2, X_{i1}) \\
& = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(Y_i - \beta_0 - \beta_1 X_{i1})^2}{2\sigma^2} \right)
\end{aligned}
$$

Bivariate Regression Model via Maximum Likelihood

$$
\begin{aligned}
L(\boldsymbol{\beta}, \sigma^2 | \boldsymbol{Y}, \boldsymbol{X}) &= f(\boldsymbol{Y}|\boldsymbol{\beta}, \sigma^2, \boldsymbol{X}) \\
&= \prod_{i=1}^{N} f(Y_i|\boldsymbol{\beta}, \sigma^2, X_{i1}) \\
&= \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(Y_i - \beta_0 - \beta_1 X_{i1})^2}{2\sigma^2}\right) \\
&= \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\sum_{i=1}^{N} \frac{(Y_i - \beta_0 - \beta_1 X_{i1})^2}{2\sigma^2}\right)
\end{aligned}
$$

Bivariate Regression Model via Maximum Likelihood

$$L(\boldsymbol{\beta}, \sigma^2|\boldsymbol{Y}, \boldsymbol{X}) \quad = \quad \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\sum_{i=1}^{N} \frac{(Y_i - \beta_0 - \beta_1 X_{i1})^2}{2\sigma^2}\right)$$

Bivariate Regression Model via Maximum Likelihood

$$
\begin{aligned}
L(\boldsymbol{\beta}, \sigma^2 | \boldsymbol{Y}, \boldsymbol{X}) &= \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(- \sum_{i=1}^{N} \frac{(Y_i - \beta_0 - \beta_1 X_{i1})^2}{2\sigma^2} \right) \\
\log L(\boldsymbol{\beta}, \sigma^2 | \boldsymbol{Y}, \boldsymbol{X}) &= \underbrace{-\frac{N}{2} \log(2\pi)}_{\text{constant}} - \frac{N}{2} \log(\sigma^2) - \sum_{i=1}^{N} \frac{(Y_i - \beta_0 - \beta_1 X_{i1})^2}{2\sigma^2}
\end{aligned}
$$

Bivariate Regression Model via Maximum Likelihood

$$
\begin{aligned}
L(\boldsymbol{\beta}, \sigma^2|\boldsymbol{Y}, \boldsymbol{X}) &= \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\sum_{i=1}^{N} \frac{(Y_i - \beta_0 - \beta_1 X_{i1})^2}{2\sigma^2}\right) \\
\log L(\boldsymbol{\beta}, \sigma^2|\boldsymbol{Y}, \boldsymbol{X}) &= \underbrace{-\frac{N}{2}\log(2\pi)}_{\text{constant}} - \frac{N}{2}\log(\sigma^2) - \sum_{i=1}^{N} \frac{(Y_i - \beta_0 - \beta_1 X_{i1})^2}{2\sigma^2} \\
l(\boldsymbol{\beta}, \sigma^2|\boldsymbol{Y}, \boldsymbol{X}) &= -\frac{N}{2}\log(\sigma^2) - \sum_{i=1}^{N} \frac{(Y_i - \beta_0 - \beta_1 X_{i1})^2}{2\sigma^2} + c
\end{aligned}
$$

Bivariate Regression Model via Maximum Likelihood

$$
\begin{aligned}
L(\boldsymbol{\beta}, \sigma^2|\boldsymbol{Y}, \boldsymbol{X}) &= \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(- \sum_{i=1}^{N} \frac{(Y_i - \beta_0 - \beta_1 X_{i1})^2}{2\sigma^2}\right) \\
\log L(\boldsymbol{\beta}, \sigma^2|\boldsymbol{Y}, \boldsymbol{X}) &= \underbrace{-\frac{N}{2}\log(2\pi)}_{\text{constant}} - \frac{N}{2}\log(\sigma^2) - \sum_{i=1}^{N} \frac{(Y_i - \beta_0 - \beta_1 X_{i1})^2}{2\sigma^2} \\
l(\boldsymbol{\beta}, \sigma^2|\boldsymbol{Y}, \boldsymbol{X}) &= -\frac{N}{2}\log(\sigma^2) - \sum_{i=1}^{N} \frac{(Y_i - \beta_0 - \beta_1 X_{i1})^2}{2\sigma^2} + c
\end{aligned}
$$

Optimize with respect to:

- $\boldsymbol{\beta}$
- σ^2

Bivariate Regression Model via Maximum Likelihood

$$
\begin{aligned}
L(\boldsymbol{\beta}, \sigma^2|\boldsymbol{Y}, \boldsymbol{X}) &= \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\sum_{i=1}^{N}\frac{(Y_i - \beta_0 - \beta_1 X_{i1})^2}{2\sigma^2}\right) \\
\log L(\boldsymbol{\beta}, \sigma^2|\boldsymbol{Y}, \boldsymbol{X}) &= \underbrace{-\frac{N}{2}\log(2\pi)}_{\text{constant}} - \frac{N}{2}\log(\sigma^2) - \sum_{i=1}^{N}\frac{(Y_i - \beta_0 - \beta_1 X_{i1})^2}{2\sigma^2} \\
l(\boldsymbol{\beta}, \sigma^2|\boldsymbol{Y}, \boldsymbol{X}) &= -\frac{N}{2}\log(\sigma^2) - \sum_{i=1}^{N}\frac{(Y_i - \beta_0 - \beta_1 X_{i1})^2}{2\sigma^2} + c
\end{aligned}
$$

Optimize with respect to:

- $\boldsymbol{\beta}$
- σ^2

Bivariate Regression Model via Maximum Likelihood

$$\begin{aligned}
\frac{\partial l(\boldsymbol{\beta}, \sigma^2|\boldsymbol{Y}, \boldsymbol{X})}{\partial \beta_0} &= \sum_{i=1}^{N} \frac{Y_i - \beta_0 - \beta_1 X_{i1}}{\sigma^2} \\
\frac{\partial l(\boldsymbol{\beta}, \sigma^2|\boldsymbol{Y}, \boldsymbol{X})}{\partial \beta_1} &= \sum_{i=1}^{N} \frac{Y_i - \beta_0 - \beta_1 X_{i1}}{\sigma^2} X_{i1} \\
\frac{\partial l(\boldsymbol{\beta}, \sigma^2|\boldsymbol{Y}, \boldsymbol{X})}{\partial \sigma^2} &= -\frac{N}{2\sigma^2} + \sum_{i=1}^{N} \frac{(Y_i - \beta_0 - \beta_1 X_{i1})^2}{2(\sigma^2)^2}
\end{aligned}$$

Bivariate Regression Model via Maximum Likelihood

$$\frac{\partial l(\boldsymbol{\beta}, \sigma^2|\boldsymbol{Y}, \boldsymbol{X})}{\partial \beta_0} = \sum_{i=1}^{N} \frac{Y_i - \beta_0 - \beta_1 X_{i1}}{\sigma^2}$$

$$\frac{\partial l(\boldsymbol{\beta}, \sigma^2|\boldsymbol{Y}, \boldsymbol{X})}{\partial \beta_1} = \sum_{i=1}^{N} \frac{Y_i - \beta_0 - \beta_1 X_{i1}}{\sigma^2} X_{i1}$$

$$\frac{\partial l(\boldsymbol{\beta}, \sigma^2|\boldsymbol{Y}, \boldsymbol{X})}{\partial \sigma^2} = -\frac{N}{2\sigma^2} + \sum_{i=1}^{N} \frac{(Y_i - \beta_0 - \beta_1 X_{i1})^2}{2(\sigma^2)^2}$$

Score function: $s_i(\boldsymbol{\beta}, \sigma^2) = \nabla l(\boldsymbol{\beta}, \sigma^2|Y_i, X_i) =$

$$\left(\frac{Y_i - \beta_0 - \beta_1 X_{i1}}{\sigma^2}, \frac{Y_i - \beta_0 - \beta_1 X_{i1}}{\sigma^2} X_{i1}, -\frac{1}{2\sigma^2} + \frac{(Y_i - \beta_0 - \beta_1 X_{i1})^2}{2(\sigma^2)^2}\right)$$

Bivariate Regression Model via Maximum Likelihood

$$\frac{\partial l(\boldsymbol{\beta}, \sigma^2 | \boldsymbol{Y}, \boldsymbol{X})}{\partial \beta_0} = \sum_{i=1}^{N} \frac{Y_i - \beta_0 - \beta_1 X_{i1}}{\sigma^2}$$

$$\frac{\partial l(\boldsymbol{\beta}, \sigma^2 | \boldsymbol{Y}, \boldsymbol{X})}{\partial \beta_1} = \sum_{i=1}^{N} \frac{Y_i - \beta_0 - \beta_1 X_{i1}}{\sigma^2} X_{i1}$$

$$\frac{\partial l(\boldsymbol{\beta}, \sigma^2 | \boldsymbol{Y}, \boldsymbol{X})}{\partial \sigma^2} = -\frac{N}{2\sigma^2} + \sum_{i=1}^{N} \frac{(Y_i - \beta_0 - \beta_1 X_{i1})^2}{2(\sigma^2)^2}$$

Score function: $s_i(\boldsymbol{\beta}, \sigma^2) = \nabla l(\boldsymbol{\beta}, \sigma^2 | Y_i, X_i) =$

$$\left(\frac{Y_i - \beta_0 - \beta_1 X_{i1}}{\sigma^2}, \frac{Y_i - \beta_0 - \beta_1 X_{i1}}{\sigma^2} X_{i1}, -\frac{1}{2\sigma^2} + \frac{(Y_i - \beta_0 - \beta_1 X_{i1})^2}{2(\sigma^2)^2} \right)$$

$s_N(\boldsymbol{\beta}, \sigma^2) = \sum_{i=1}^{N} s_i(\boldsymbol{\beta}, \sigma^2)$.

Bivariate Regression Model via Maximum Likelihood

$$
\begin{aligned}
0 &= \sum_{i=1}^{N} \frac{Y_i - \beta_0^{*} - \beta_1^{*} X_{i1}}{\sigma^{2,*}} \\
0 &= \sum_{i=1}^{N} \frac{Y_i - \beta_0^{*} - \beta_1^{*} X_{i1}}{\sigma^{2,*}} X_{i1} \\
0 &= -\frac{N}{2\sigma^{2,*}} + \sum_{i=1}^{N} \frac{(Y_i - \beta_0^{*} - \beta_1^{*} X_{i1})^2}{2(\sigma^{2,*})^2}
\end{aligned}
$$

Algebra

Bivariate Regression Model via Maximum Likelihood

$$
\begin{aligned}
\beta_1^* &= \frac{\sum_{i=1}^N (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^N (X_i - \bar{X})^2} \\
\beta_0^* &= \bar{Y} - \beta_1^* \bar{X} \\
\sigma^{2,*} &= \frac{1}{N} \sum_{i=1}^N (Y_i - \beta_0^* - \beta_1^* X_i)^2
\end{aligned}
$$

Hessian Matrix

Define the Hessian for observation i as

$$H_i(\boldsymbol{\beta}, \sigma^2) \quad = \quad \begin{pmatrix} \frac{\partial l(\boldsymbol{\beta},\sigma^2|X_i,Y_i)}{\partial \beta_0 \beta_0} & \frac{\partial l(\boldsymbol{\beta},\sigma^2|X_i,Y_i)}{\partial \beta_0 \beta_1} & \cdots & \frac{\partial l(\boldsymbol{\beta},\sigma^2|X_i,Y_i)}{\partial \beta_0 \sigma^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial l(\boldsymbol{\beta},\sigma^2|X_i,Y_i)}{\partial \beta_0 \sigma^2} & \frac{\partial l(\boldsymbol{\beta},\sigma^2|X_i,Y_i)}{\partial \beta_1 \sigma^2} & \cdots & \frac{\partial l(\boldsymbol{\beta},\sigma^2|X_i,Y_i)}{\partial \sigma^2 \sigma^2} \end{pmatrix}$$

Bivariate Regression and Maximum Likelihood

Fisher Information matrix

$$\begin{aligned}
I(\boldsymbol{\beta}, \sigma^2) &= E[s_N(\boldsymbol{\beta}, \sigma^2)s_N(\boldsymbol{\beta}, \sigma^2)^{'}] \\
I(\boldsymbol{\beta}, \sigma^2) &= -E[H_i(\boldsymbol{\beta}, \sigma^2)] \text{ (under regularity conditions)} \\
I_{ij}(\boldsymbol{\beta}, \sigma^2) &= -E[\frac{\partial^2 l(\boldsymbol{\beta}, \sigma^2 | Y_i, \boldsymbol{X}_i)}{\partial \beta_i \beta_j} | \boldsymbol{\beta}, \sigma^2] \\
I_{ij}(\boldsymbol{\beta}, \sigma^2) &= - \int \frac{\partial^2 l(\boldsymbol{\beta}, \sigma^2 | Y_i, \boldsymbol{X}_i)}{\partial \beta_i \beta_j} f(Y_i | \boldsymbol{\beta}, \sigma^2, \boldsymbol{X}) d\boldsymbol{Y}
\end{aligned}$$

Observed Information Matrix

$$I_N(\boldsymbol{\beta}, \sigma^2) = -\sum_{i=1}^{N} H_i(\boldsymbol{\beta}, \sigma^2) = -\sum_{i=1}^{N} \nabla^2 \log l(\boldsymbol{\beta}, \sigma^2 | \boldsymbol{X}_i, Y_i)$$

Three Ways to Estimate Variance

$$
\begin{aligned}
\text{Var}(\widehat{\boldsymbol{\beta}, \sigma^2})_{OS} &= \left[\sum_{i=1}^{N} s_i(\boldsymbol{\beta}, \sigma^2) s_i(\boldsymbol{\beta}, \sigma^2)^{'}\right]^{-1} \quad \text{(BHHH estimator)} \\
\text{Var}(\widehat{\boldsymbol{\beta}, \sigma^2})_{OH} &= -\left[\sum_{i=1}^{N} H_i(\boldsymbol{\beta}, \sigma^2)\right]^{-1} \\
\text{Var}(\widehat{\boldsymbol{\beta}, \sigma^2})_{EH} &= -\left[\sum_{i=1}^{N} E[H_i(\boldsymbol{\beta}, \sigma^2)]\right]^{-1}
\end{aligned}
$$

Bivariate Regression Model via Maximum Likelihood

$$
\begin{aligned}
\frac{\partial^2 l(\boldsymbol{\beta}, \sigma^2 | \boldsymbol{Y}, \boldsymbol{X})}{\partial \beta_0 \partial \beta_0} &= -\sum_{i=1}^{N} \frac{1}{\sigma^2} \\
\frac{\partial^2 l(\boldsymbol{\beta}, \sigma^2 | \boldsymbol{Y}, \boldsymbol{X})}{\partial \beta_1 \partial \beta_1} &= -\sum_{i=1}^{N} \frac{X_i^2}{\sigma^2} \\
\frac{\partial^2 l(\boldsymbol{\beta}, \sigma^2 | \boldsymbol{Y}, \boldsymbol{X})}{\partial \sigma^2 \partial \sigma^2} &= \frac{N}{2(\sigma^2)^2} - \sum_{i=1}^{N} \frac{(Y_i - \beta_0 - \beta_1 X_{i1})^2}{(\sigma^2)^4} \\
\frac{\partial^2 l(\boldsymbol{\beta}, \sigma^2 | \boldsymbol{Y}, \boldsymbol{X})}{\partial \beta_0 \partial \beta_1} &= -\sum_{i=1}^{N} \frac{X_i}{\sigma^2} \\
\frac{\partial^2 l(\boldsymbol{\beta}, \sigma^2 | \boldsymbol{Y}, \boldsymbol{X})}{\partial \sigma^2 \partial \beta_1} &= -\sum_{i=1}^{N} \frac{(Y_i - \beta_0 - \beta_1 X_{i1}) X_i}{(\sigma^2)^2}
\end{aligned}
$$

Bivariate Regression Model via Maximum Likelihood

$$-E\left[\frac{\partial^2 l(\boldsymbol{\beta}, \sigma^2|\boldsymbol{Y}, \boldsymbol{X})}{\partial\beta_0\partial\beta_0}\right] = -E\left[-\frac{1}{\sigma^2}\right]$$

Bivariate Regression Model via Maximum Likelihood

$$
\begin{aligned}
-E\left[\frac{\partial^2 l(\boldsymbol{\beta}, \sigma^2 | \boldsymbol{Y}, \boldsymbol{X})}{\partial \beta_0 \partial \beta_0}\right] &= -E\left[-\frac{1}{\sigma^2}\right] \\
&= -\sum_{i=1}^{N} \int_{-\infty}^{\infty} -\frac{1}{\sigma^2} f(Y_i | \boldsymbol{\beta}, \sigma^2, X_i) dY_i
\end{aligned}
$$

Bivariate Regression Model via Maximum Likelihood

$$
\begin{aligned}
-E\left[\frac{\partial^2 l(\boldsymbol{\beta}, \sigma^2 | \boldsymbol{Y}, \boldsymbol{X})}{\partial \beta_0 \partial \beta_0}\right] &= -E\left[-\frac{1}{\sigma^2}\right] \\
&= -\sum_{i=1}^{N} \int_{-\infty}^{\infty} -\frac{1}{\sigma^2} f(Y_i | \boldsymbol{\beta}, \sigma^2, X_i) dY_i \\
&= \sum_{i=1}^{N} \frac{1}{\sigma^2} \int_{-\infty}^{\infty} f(Y_i | \boldsymbol{\beta}, \sigma^2, X_i) dY_i
\end{aligned}
$$

Bivariate Regression Model via Maximum Likelihood

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\begin{aligned}
-E\left[\frac{\partial^2 l(\boldsymbol{\beta}, \sigma^2|\boldsymbol{Y}, \boldsymbol{X})}{\partial \beta_0 \partial \beta_0}\right] &= -E\left[-\frac{1}{\sigma^2}\right] \\
&= -\sum_{i=1}^{N} \int_{-\infty}^{\infty} -\frac{1}{\sigma^2} f(Y_i|\boldsymbol{\beta}, \sigma^2, X_i) dY_i \\
&= \sum_{i=1}^{N} \frac{1}{\sigma^2} \int_{-\infty}^{\infty} f(Y_i|\boldsymbol{\beta}, \sigma^2, X_i) dY_i \\
&= \sum_{i=1}^{N} \frac{1}{\sigma^2} \times 1 = \frac{N}{\sigma^2}
\end{aligned}
$$

Bivariate Regression Model via Maximum Likelihood

$$
\begin{aligned}
-E[\frac{\partial^2 l(\boldsymbol{\beta}, \sigma^2|\boldsymbol{Y}, \boldsymbol{X})}{\partial \beta_1 \partial \beta_1}] &= -E[-\sum_{i=1}^{N} \frac{X_i^2}{\sigma^2}] \\
&= \sum_{i=1}^{N} \frac{X_i^2}{\sigma^2}
\end{aligned}
$$

Bivariate Regression Model via Maximum Likelihood

$$
\begin{aligned}
-E[\frac{\partial^2 l(\boldsymbol{\beta}, \sigma^2|\boldsymbol{Y}, \boldsymbol{X})}{\partial \beta_1 \partial \beta_1}] &= -E[-\sum_{i=1}^{N} \frac{X_i^2}{\sigma^2}] \\
&= \sum_{i=1}^{N} \frac{X_i^2}{\sigma^2} \\
-E[\frac{\partial^2 l(\boldsymbol{\beta}, \sigma^2|\boldsymbol{Y}, \boldsymbol{X})}{\partial \beta_0 \partial \beta_1}] &= -E[-\sum_{i=1}^{N} \frac{X_i}{\sigma^2}] \\
&= \sum_{i=1}^{N} \frac{X_i}{\sigma^2}
\end{aligned}
$$

$$-E[\frac{\partial^2 l(\boldsymbol{\beta}, \sigma^2|\boldsymbol{Y}, \boldsymbol{X})}{\partial \sigma^2 \partial \sigma^2}] = E[\frac{N}{2(\sigma^2)^2} - \sum_{i=1}^{N} \frac{(Y_i - \beta_0 - \beta_1 X_{i1})^2}{(\sigma^2)^4}]$$

$$
\begin{aligned}
-E[\frac{\partial^2 l(\boldsymbol{\beta}, \sigma^2|\boldsymbol{Y}, \boldsymbol{X})}{\partial\sigma^2\partial\sigma^2}] &= E[\frac{N}{2(\sigma^2)^2} - \sum_{i=1}^{N}\frac{(Y_i - \beta_0 - \beta_1 X_{i1})^2}{(\sigma^2)^4}] \\
&= -\frac{N}{2(\sigma^2)^2} + \sum_{i=1}^{N} E\left[\frac{(Y_i - \beta_0 - \beta_1 X_{i1})^2}{(\sigma^2)^4}\right]
\end{aligned}
$$

$$
\begin{aligned}
-E[\frac{\partial^2 l(\boldsymbol{\beta}, \sigma^2|\boldsymbol{Y}, \boldsymbol{X})}{\partial \sigma^2 \partial \sigma^2}] &= E[\frac{N}{2(\sigma^2)^2} - \sum_{i=1}^{N} \frac{(Y_i - \beta_0 - \beta_1 X_{i1})^2}{(\sigma^2)^4}] \\
&= -\frac{N}{2(\sigma^2)^2} + \sum_{i=1}^{N} E\left[\frac{(Y_i - \beta_0 - \beta_1 X_{i1})^2}{(\sigma^2)^4}\right] \\
&= -\frac{N}{2(\sigma^2)^2} + \sum_{i=1}^{N} \frac{\sigma^2}{(\sigma^2)^4}
\end{aligned}
$$

$$
\begin{aligned}
-E[\frac{\partial^2 l(\boldsymbol{\beta}, \sigma^2 | \boldsymbol{Y}, \boldsymbol{X})}{\partial \sigma^2 \partial \sigma^2}] &= E[\frac{N}{2(\sigma^2)^2} - \sum_{i=1}^{N} \frac{(Y_i - \beta_0 - \beta_1 X_{i1})^2}{(\sigma^2)^4}] \\
&= -\frac{N}{2(\sigma^2)^2} + \sum_{i=1}^{N} E\left[\frac{(Y_i - \beta_0 - \beta_1 X_{i1})^2}{(\sigma^2)^4}\right] \\
&= -\frac{N}{2(\sigma^2)^2} + \sum_{i=1}^{N} \frac{\sigma^2}{(\sigma^2)^4} \\
&= -\frac{N}{2(\sigma^2)^2} + \frac{2N}{2(\sigma^2)^2}
\end{aligned}
$$

$$
\begin{aligned}
-E[\frac{\partial^2 l(\boldsymbol{\beta}, \sigma^2 | \boldsymbol{Y}, \boldsymbol{X})}{\partial \sigma^2 \partial \sigma^2}] &= E[\frac{N}{2(\sigma^2)^2} - \sum_{i=1}^{N} \frac{(Y_i - \beta_0 - \beta_1 X_{i1})^2}{(\sigma^2)^4}] \\
&= -\frac{N}{2(\sigma^2)^2} + \sum_{i=1}^{N} E\left[\frac{(Y_i - \beta_0 - \beta_1 X_{i1})^2}{(\sigma^2)^4}\right] \\
&= -\frac{N}{2(\sigma^2)^2} + \sum_{i=1}^{N} \frac{\sigma^2}{(\sigma^2)^4} \\
&= -\frac{N}{2(\sigma^2)^2} + \frac{2N}{2(\sigma^2)^2} \\
&= \frac{N}{2(\sigma^2)^2}
\end{aligned}
$$

$$-E\left[\frac{\partial^2 l(\boldsymbol{\beta},\sigma^2|\boldsymbol{Y},\boldsymbol{X})}{\partial\sigma^2\partial\beta_1}\right] = -E\left[-\sum_{i=1}^{N}\frac{(Y_i-\beta_0-\beta_1X_{i1})X_i}{(\sigma^2)^2}\right]$$

$$
\begin{aligned}
-E\left[\frac{\partial^2 l(\boldsymbol{\beta}, \sigma^2 | \boldsymbol{Y}, \boldsymbol{X})}{\partial \sigma^2 \partial \beta_1}\right] &= -E\left[-\sum_{i=1}^{N} \frac{(Y_i - \beta_0 - \beta_1 X_{i1}) X_i}{(\sigma^2)^2}\right] \\
&= \frac{X_i}{(\sigma^2)^2}\left(\sum_{i=1}^{N} E[Y_i] - \beta_0 - \beta_1 X_{i1}\right)
\end{aligned}
$$

$$
\begin{aligned}
-E\left[\frac{\partial^2 l(\boldsymbol{\beta}, \sigma^2|\boldsymbol{Y}, \boldsymbol{X})}{\partial\sigma^2\partial\beta_1}\right] &= -E\left[-\sum_{i=1}^{N}\frac{(Y_i - \beta_0 - \beta_1 X_{i1})X_i}{(\sigma^2)^2}\right] \\
&= \frac{X_i}{(\sigma^2)^2}\left(\sum_{i=1}^{N} E[Y_i] - \beta_0 - \beta_1 X_{i1}\right) \\
&= \frac{X_i}{(\sigma^2)^2}\left(\sum_{i=1}^{N} \beta_0 + \beta_1 X_{i1} - \beta_0 - \beta_1 X_{i1}\right) = 0
\end{aligned}
$$

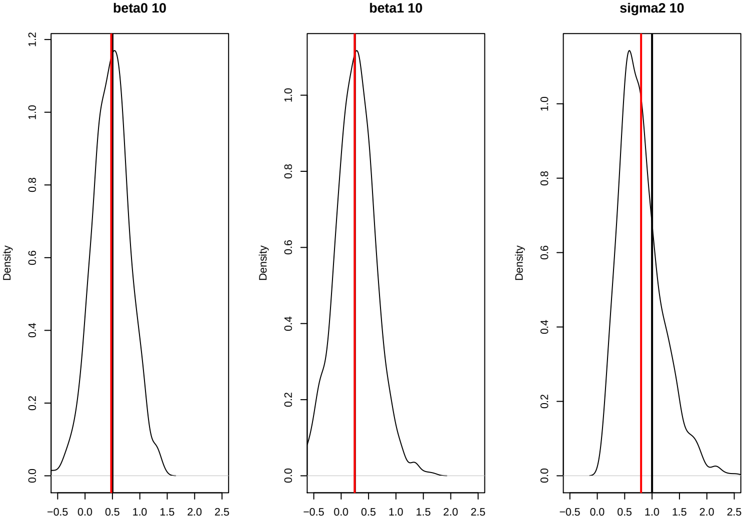
Bivariate Regression Model via Maximum Likelihood

$$
\begin{aligned}
I(\boldsymbol{\beta}, \sigma^2)_N &= \begin{pmatrix} \frac{N}{\sigma^2} & \sum_{i=1}^N \frac{X_i}{\sigma^2} & 0 \\ \sum_{i=1}^N \frac{X_i}{\sigma^2} & \sum_{i=1}^N \frac{X_i^2}{\sigma^2} & 0 \\ 0 & 0 & \frac{N}{2(\sigma^2)^2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\sigma^2}\boldsymbol{X}^{'}\boldsymbol{X} & \boldsymbol{0} \\ \boldsymbol{0} & \frac{N}{2(\sigma^2)^2} \end{pmatrix}
\end{aligned}
$$

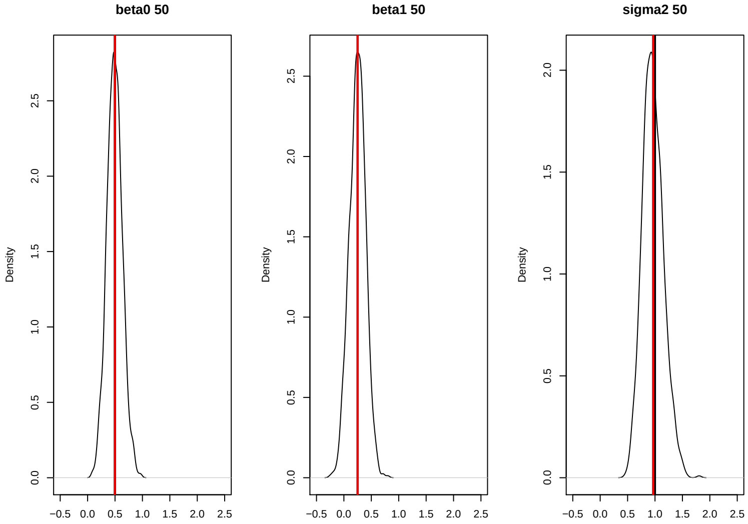
Inverting yields:

$$
\widehat{\text{Var}(\boldsymbol{\beta}, \sigma^2)} = \begin{pmatrix} \sigma^2 \left(\boldsymbol{X}^{'}\boldsymbol{X}\right)^{-1} & 0 \\ 0 & \frac{2(\sigma^2)^2}{N} \end{pmatrix}
$$

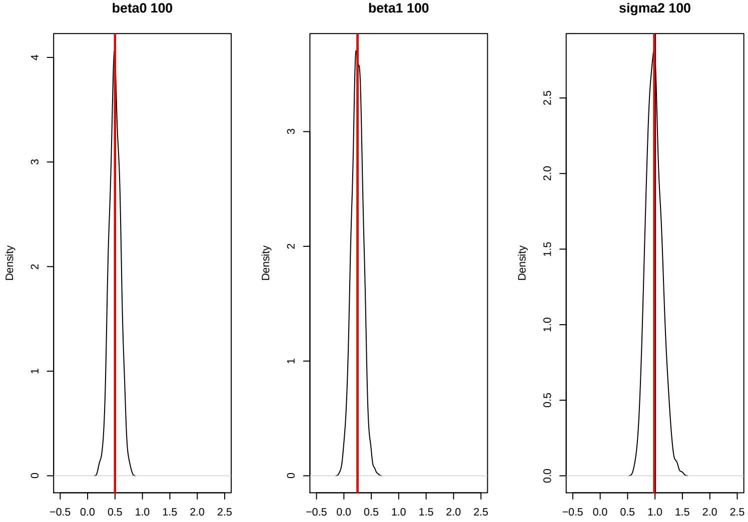
Bivariate Regression with Maximum Likelihood: Numerical Simulation



Bivariate Regression with Maximum Likelihood: Numerical Simulation



Bivariate Regression with Maximum Likelihood: Numerical Simulation



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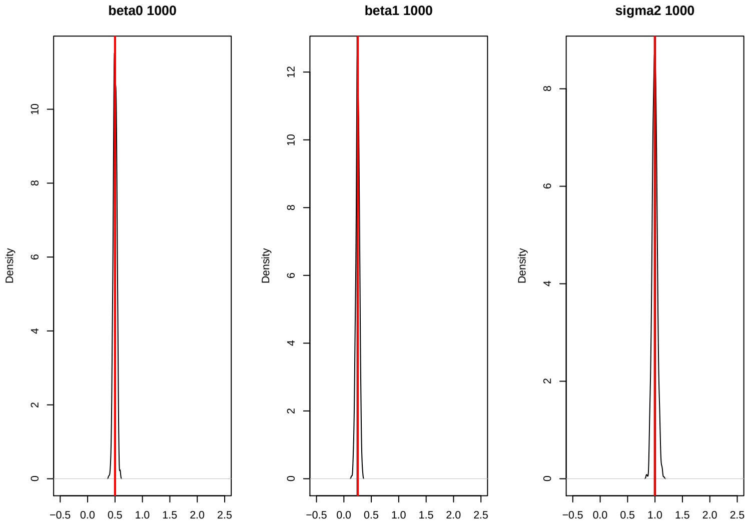

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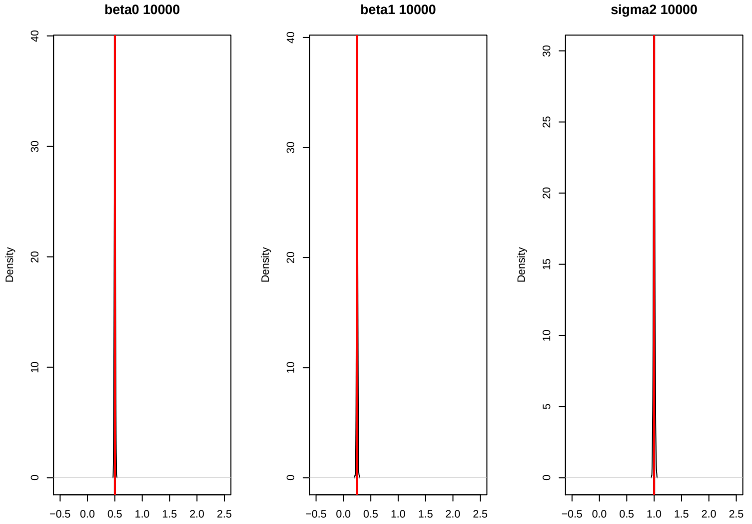
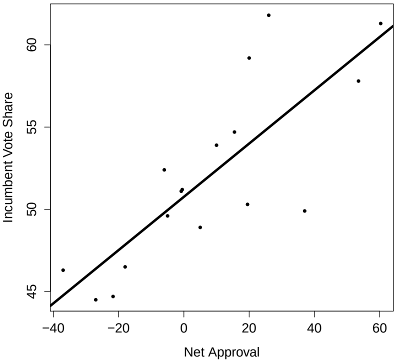
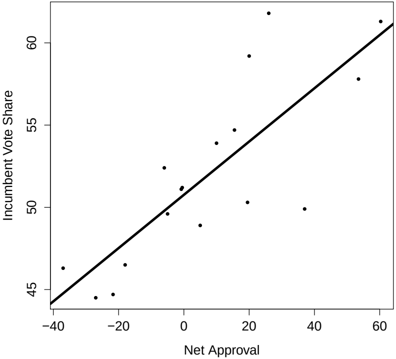


Bivariate Regression via Maximum Likelihood



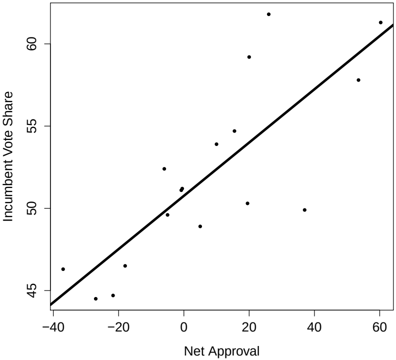
Bivariate Regression via Maximum Likelihood

$$\text{Incumbent}_i = \beta_0^* + \beta_1^* \text{Approval}_i + \epsilon_i$$



Bivariate Regression via Maximum Likelihood

$$\begin{aligned}\textsf{Incumbent}_i &= \beta_0^* + \beta_1^* \textsf{Approval}_i + \epsilon_i \\ \textsf{Incumbent}_i &= \underbrace{50.76 + 0.16 \times \textsf{Approval}_i}_{\widehat{\textsf{Incumbent}}_i} + \epsilon_i\end{aligned}$$



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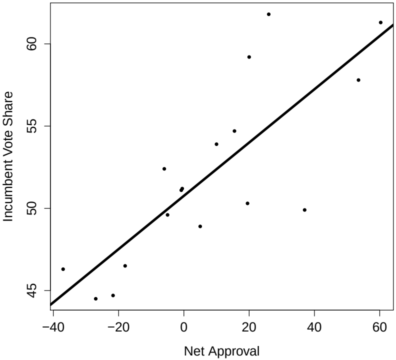

Multivariate Regression via Maximum Likelihood

Define:

$$\begin{aligned}
\boldsymbol{X}_i &= (1, X_{i1}, X_{i2}, X_{i3}) \\
\boldsymbol{\beta} &= (\beta_0, \beta_1, \beta_2)
\end{aligned}$$

Then our maximum likelihood estimators are:

$$\begin{aligned}
\boldsymbol{\beta}^* &= \left(\boldsymbol{X}^{'}\boldsymbol{X}\right)^{-1}\boldsymbol{X}^{'}\boldsymbol{Y} \\
\sigma^{2,*} &= \frac{1}{N}(Y - \boldsymbol{X}\boldsymbol{\beta})^{'}(Y - \boldsymbol{X}\boldsymbol{\beta}) \\
&= \frac{1}{N}\sum_{i=1}^{N}(Y_i - \beta_0 - \beta_1 X_{i1} - \beta_2 X_{i2} - \beta_3 X_{i3})^2
\end{aligned}$$

Then,

$$\boldsymbol{\beta}^{*}, \sigma^{2,*} \quad \rightarrow^{D} \quad \mathsf{MVN}((\boldsymbol{\beta}^{*}, \sigma^{2,*}), I(\boldsymbol{\beta}^{*}, \sigma^{2,*})^{-1})$$

Where

$$\mathsf{Var}(\widehat{\boldsymbol{\beta}^{*}, \sigma^{2,*}})) \quad = \quad \begin{pmatrix} \sigma^{2}(\boldsymbol{X}^{'}\boldsymbol{X})^{-1} & 0 \\ 0 & \frac{2(\sigma^{2})^{2}}{N} \end{pmatrix}$$

Analytic vs Computational

Analytic optimization:

- Calculus + pencil and paper
- Search for closed form solutions
- Big guarantees, but solutions can be hard (impossible) to obtain

Computational Optimization:

- Iterative procedure
- Search for approximate solution
- Fewer guarantees, but you'll obtain solutions

Overall strategy:
One-parameter Newton Raphson $\rightsquigarrow$ Multi-parameter Newton Raphson $\rightsquigarrow$ BFGS (Wednesday)

Newton-Raphson Method

Iterative procedure to find a root

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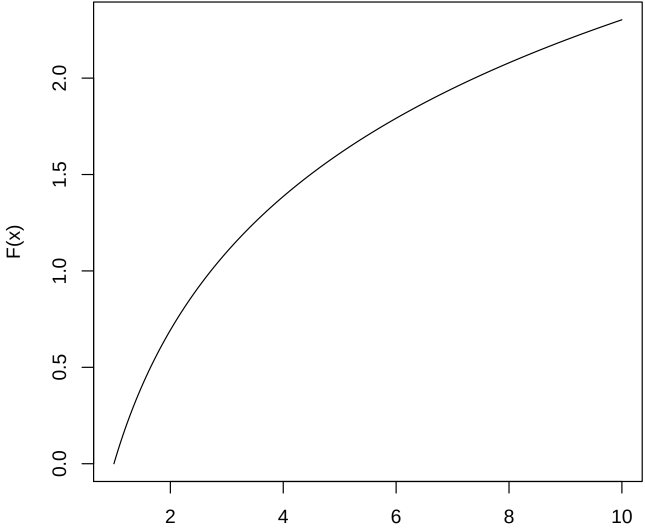
Approximate with tangent line, iteratively update

Tangent Line



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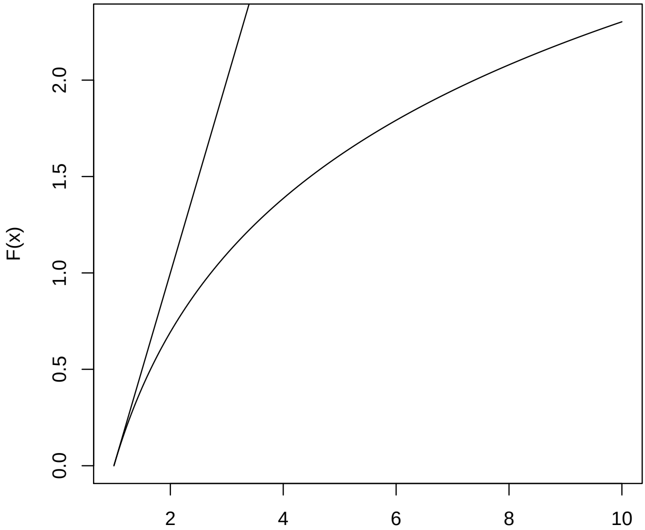

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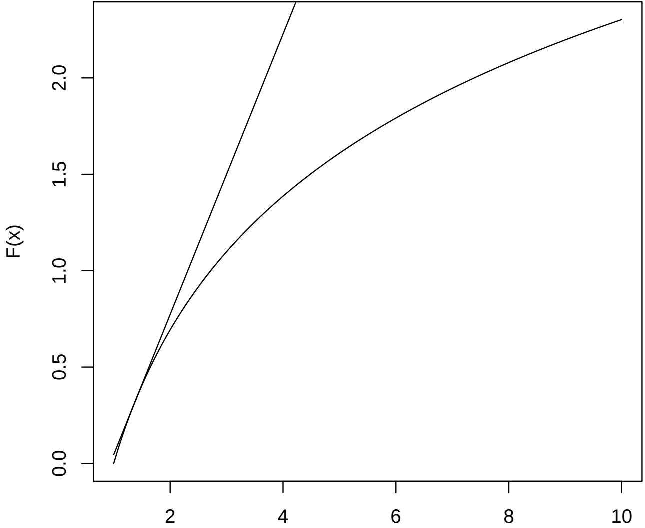

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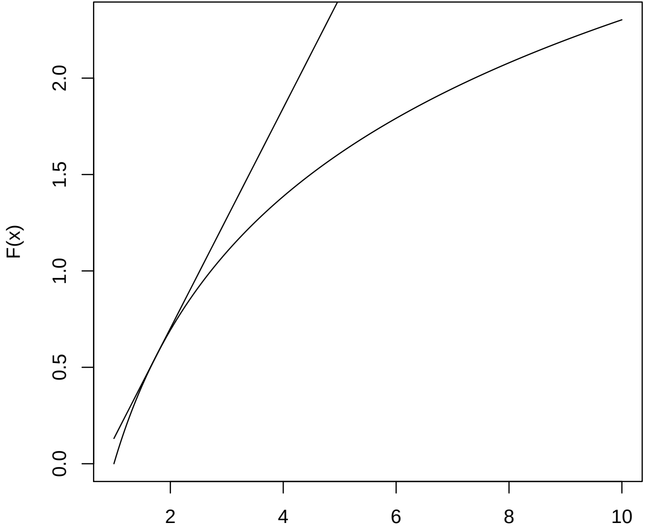

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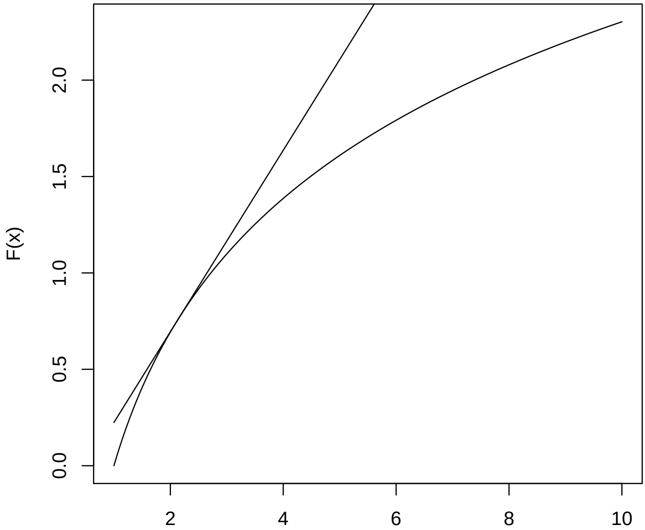

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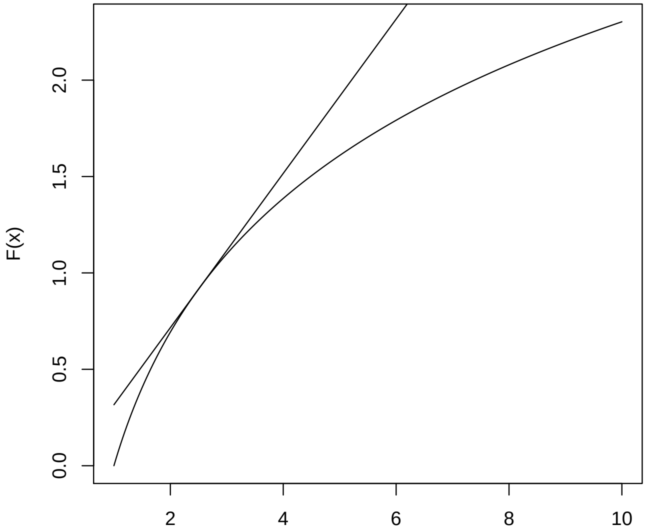

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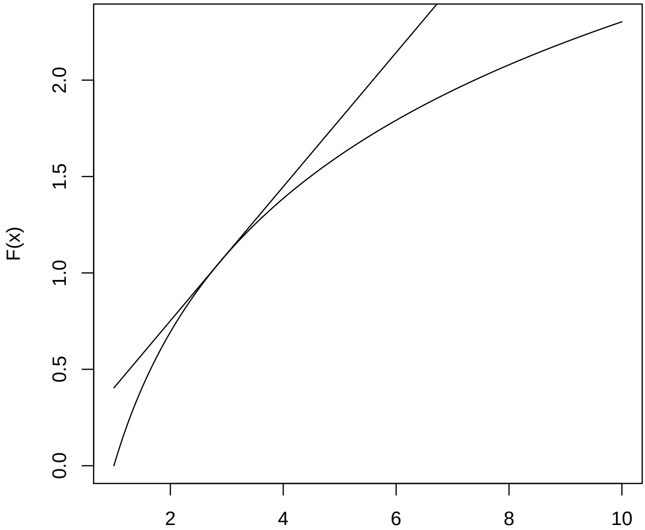

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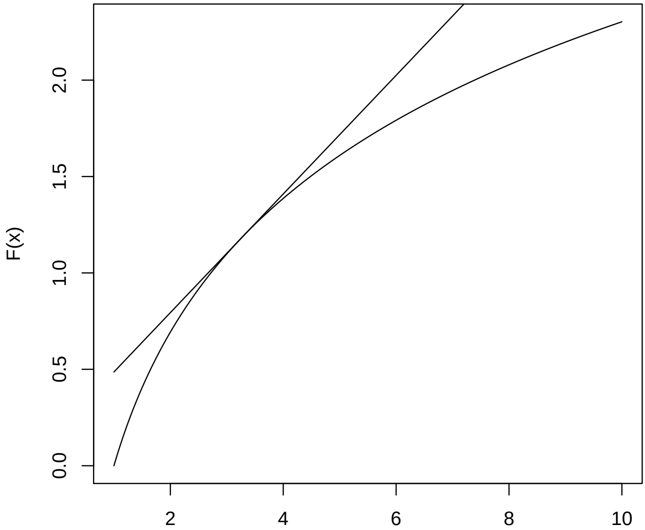

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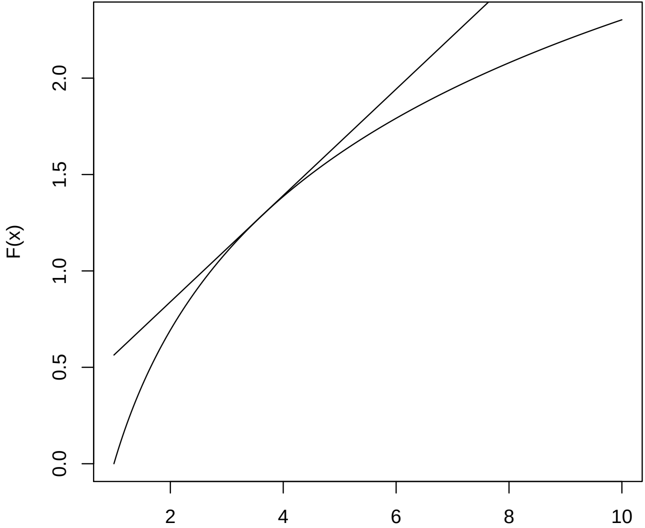

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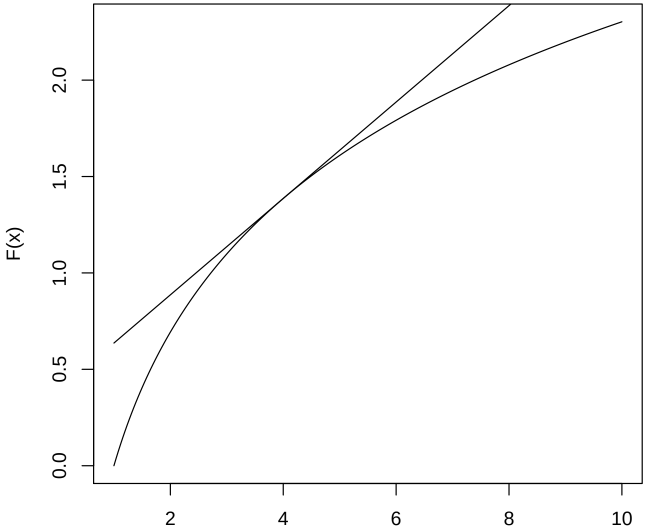

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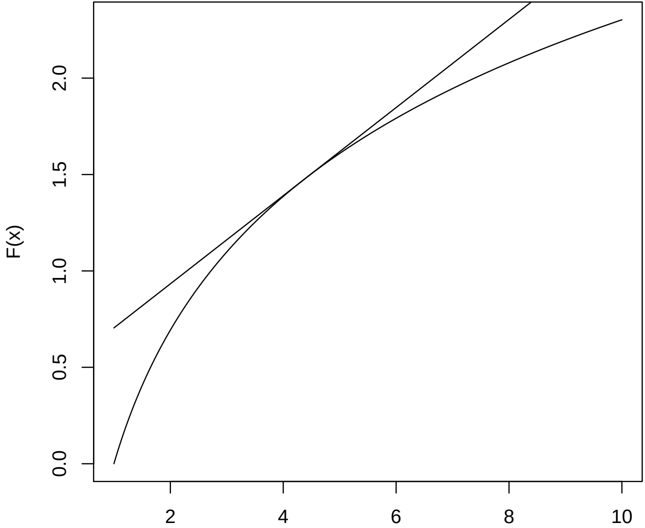

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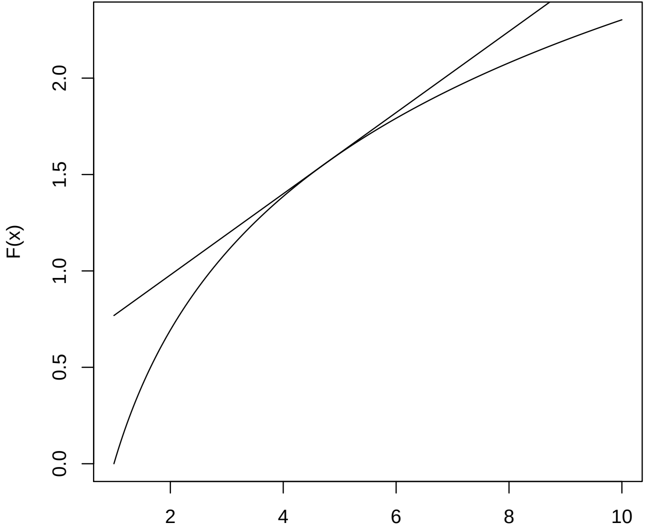

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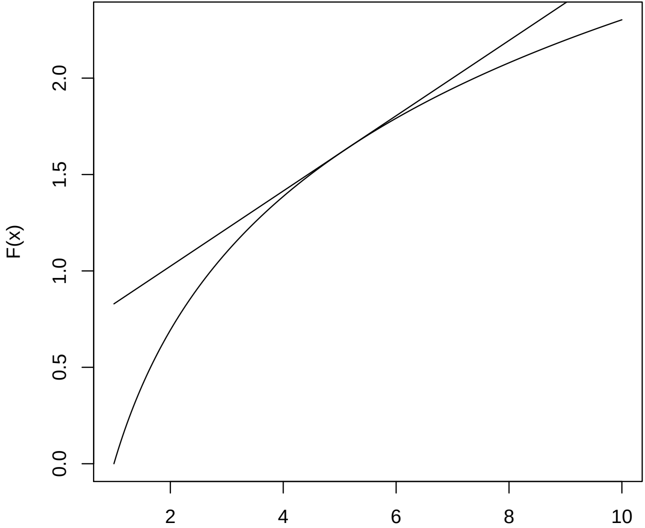

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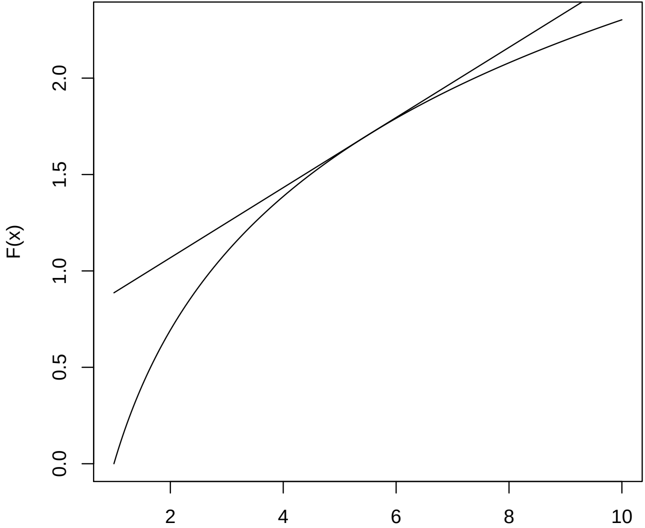

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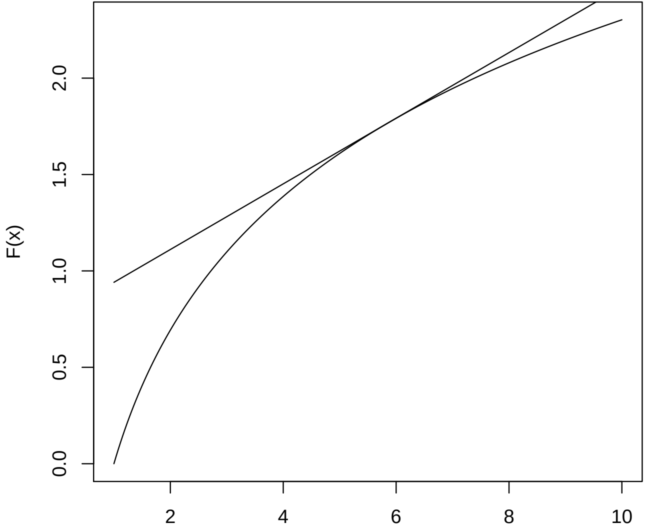

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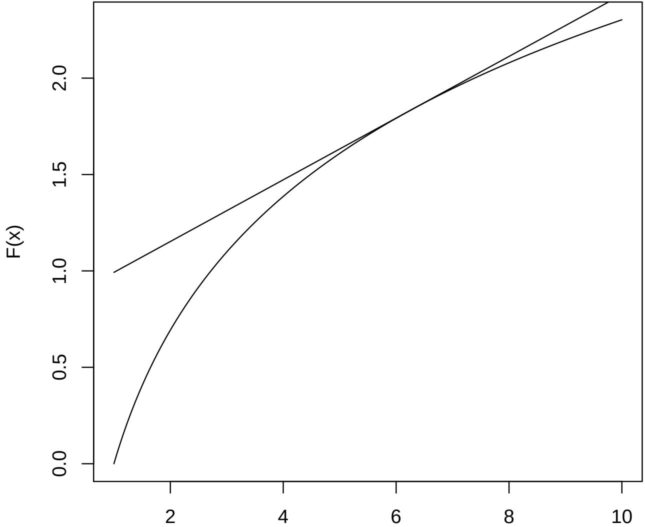

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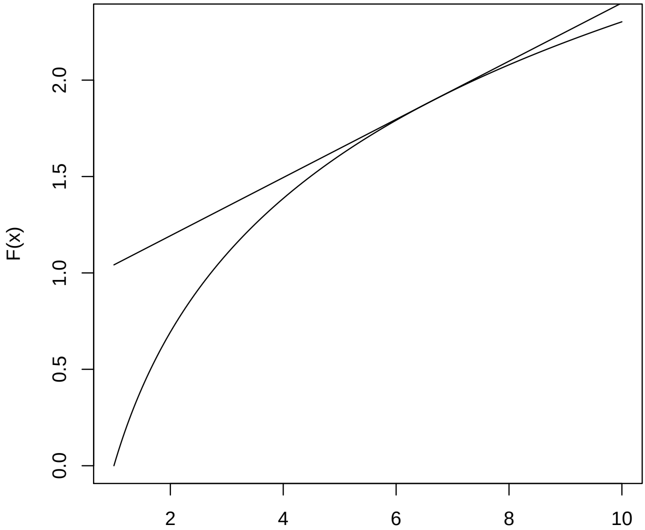

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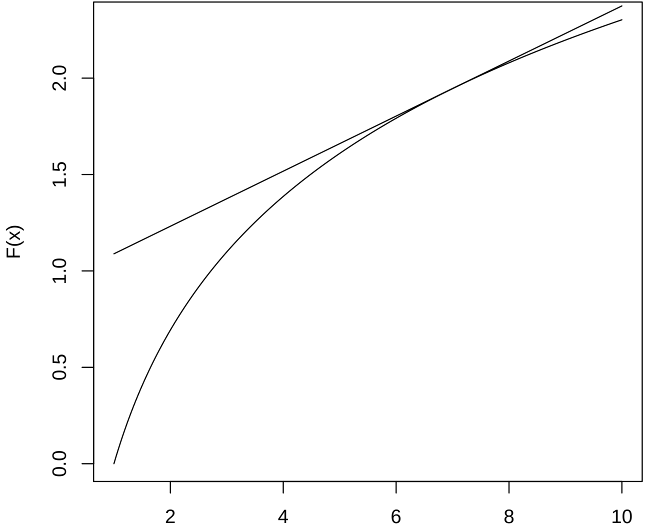

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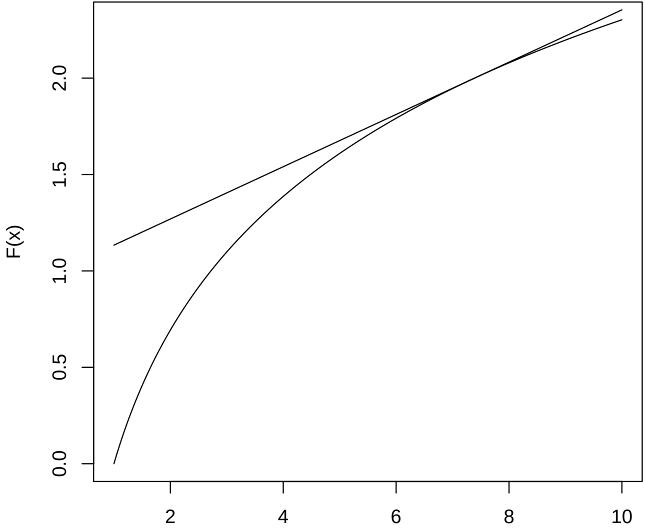

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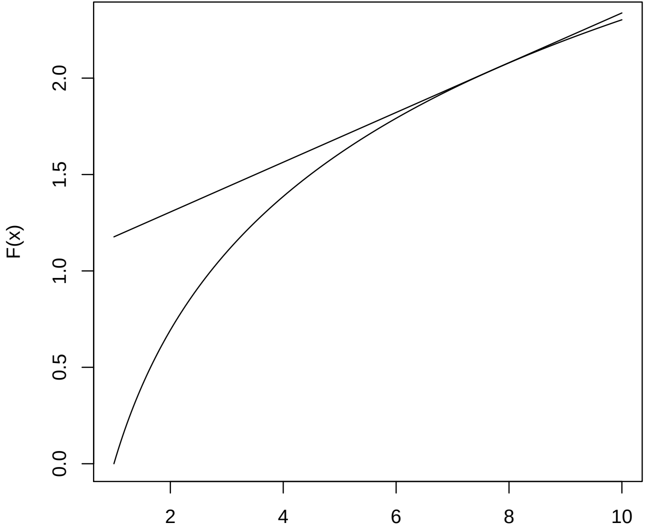

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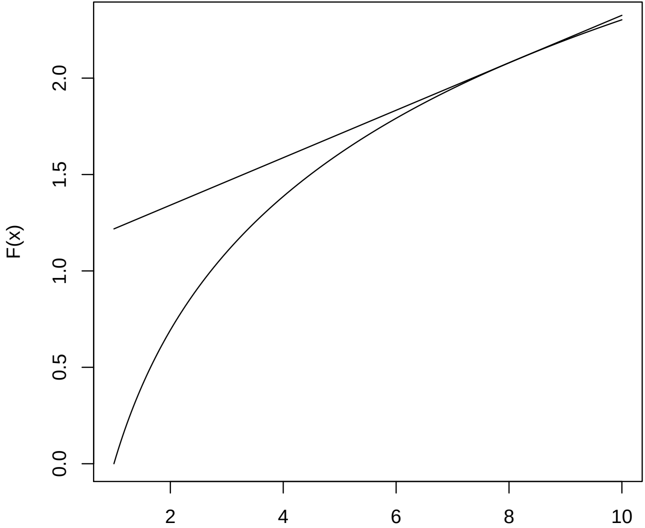

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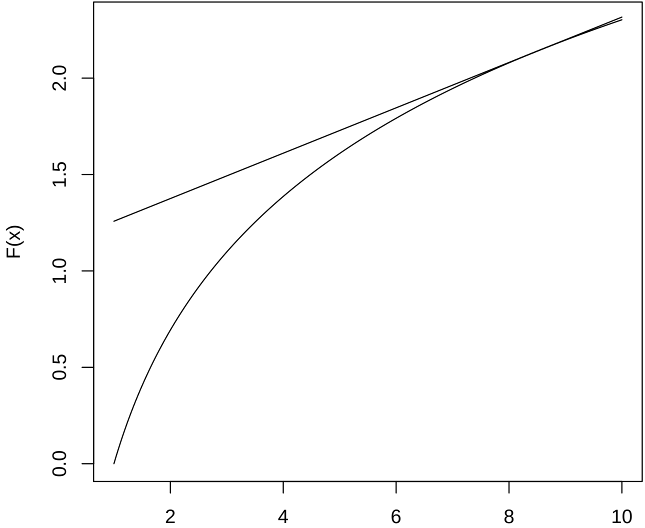

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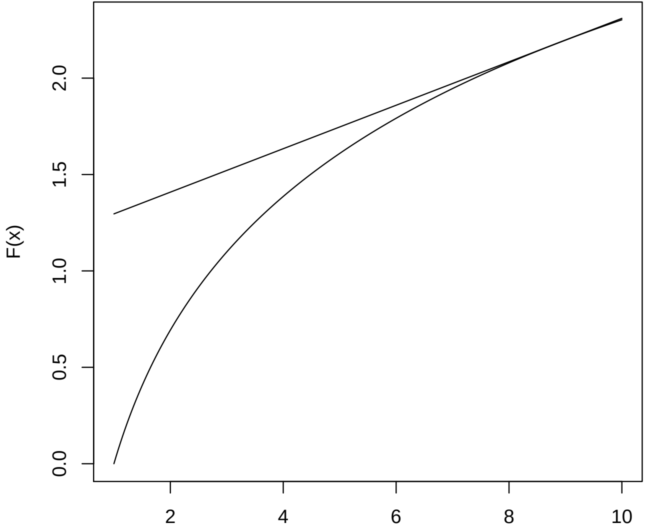

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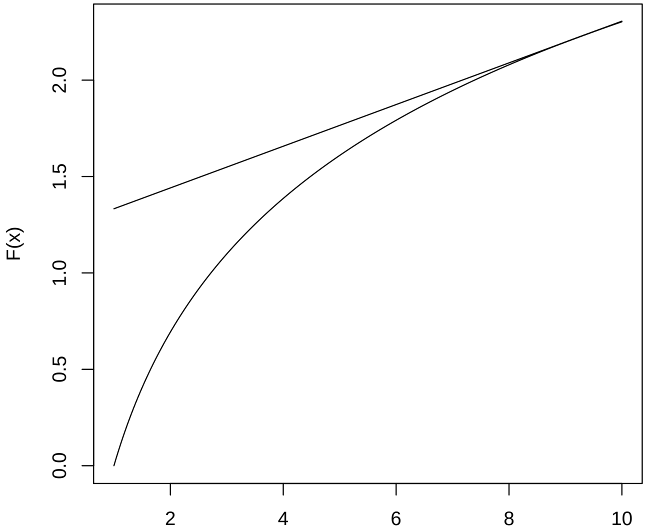

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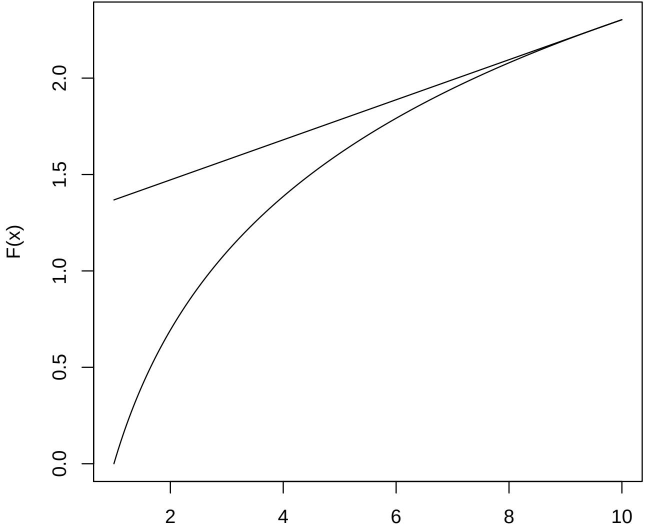

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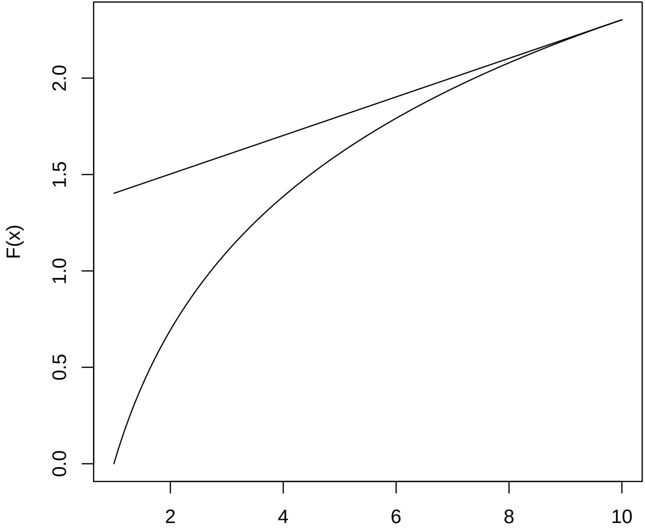

Tangent Line

Formula for Tangent line at x_0:

Tangent Line

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$$g(x) = f^{'}(x_0)(x - x_0) + f(x_0)$$

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Suppose we have some initial guess x_0. We're going to approximate $f^{'}(x)$ with the tangent line to generate a new guess

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$$g(x) \quad = \quad f^{''}(x_0)(x - x_0) + f^{'}(x_0)$$

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$$\begin{aligned} g(x) &= f^{''}(x_0)(x - x_0) + f^{'}(x_0) \\ 0 &= f^{''}(x_0)(x_1 - x_0) + f^{'}(x_0) \end{aligned}$$

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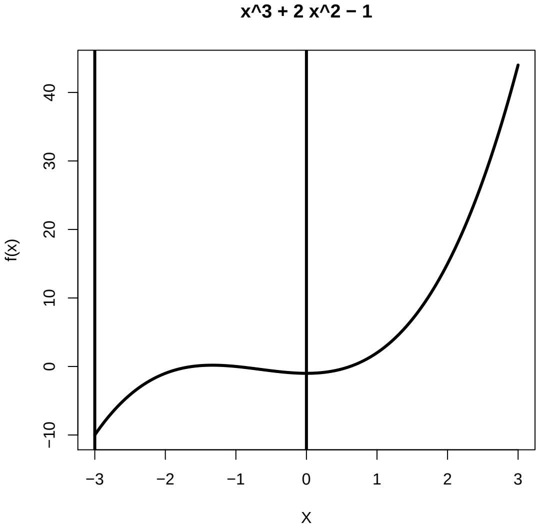
$$\begin{aligned}
g(x) &= f^{''}(x_0)(x - x_0) + f^{'}(x_0) \\
0 &= f^{''}(x_0)(x_1 - x_0) + f^{'}(x_0) \\
x_1 &= x_0 - \frac{f^{'}(x_0)}{f^{''}(x_0)}
\end{aligned}$$

Example Function

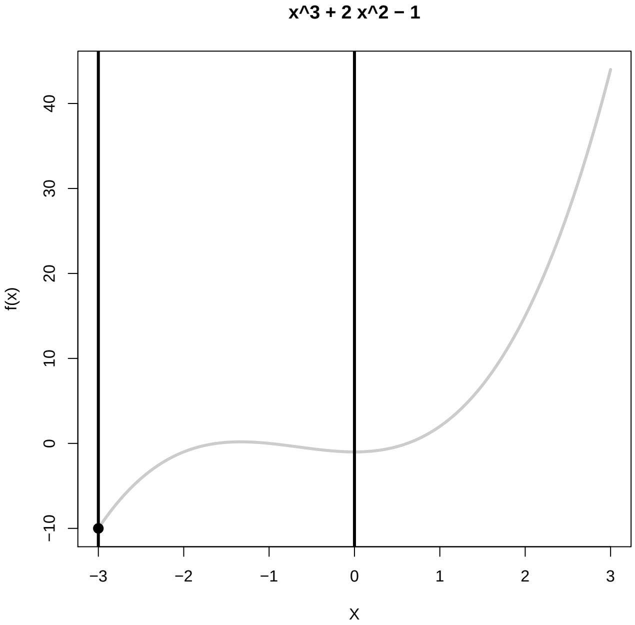
$f(x) = x^3 + 2x^2 - 1$ find x that maximizes $f(x)$ with $x \in [-3, 0]$

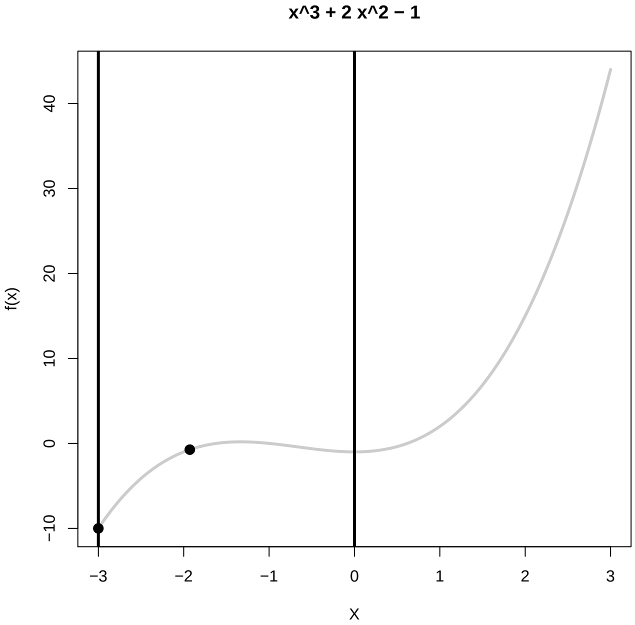


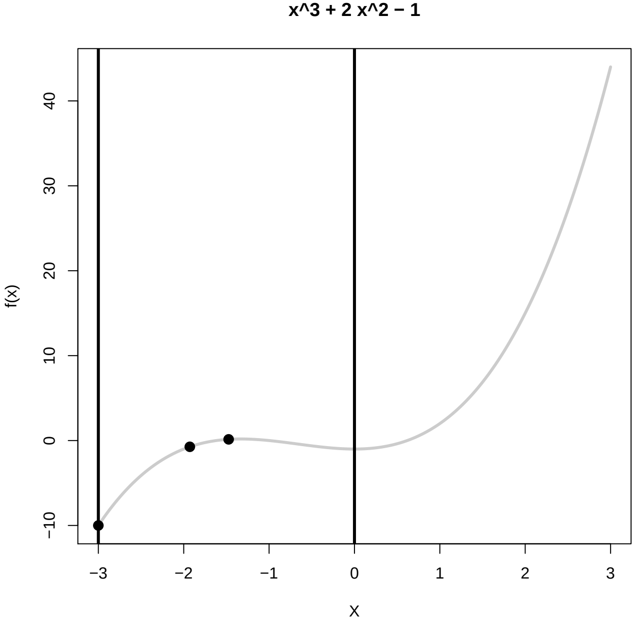
$$
\begin{aligned}
f^{'}(x) &= 3x^2 + 4x \\
f^{''}(x) &= 6x + 4
\end{aligned}
$$

Suppose we have guess x_t then the next step is:

$$
x_{t+1} = x_t - \frac{3x_t^2 + 4x_t}{6x_t + 4}
$$


x^3 + 2 x^2 − 1
f(x)
−10
0
10
20
30
40
−3
−2
−1
0
1
2
3
X


x^3 + 2 x^2 − 1
f(x)
40
30
20
10
0
−10
−3
−2
−1
0
1
2
3
X


x^3 + 2 x^2 − 1
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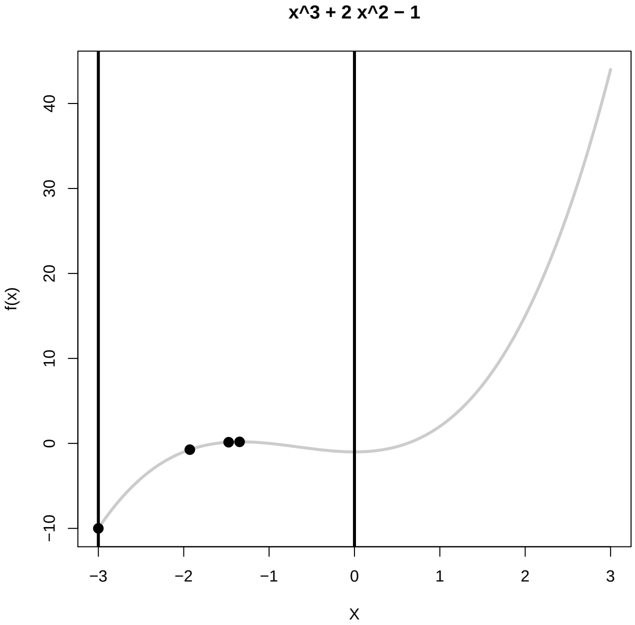
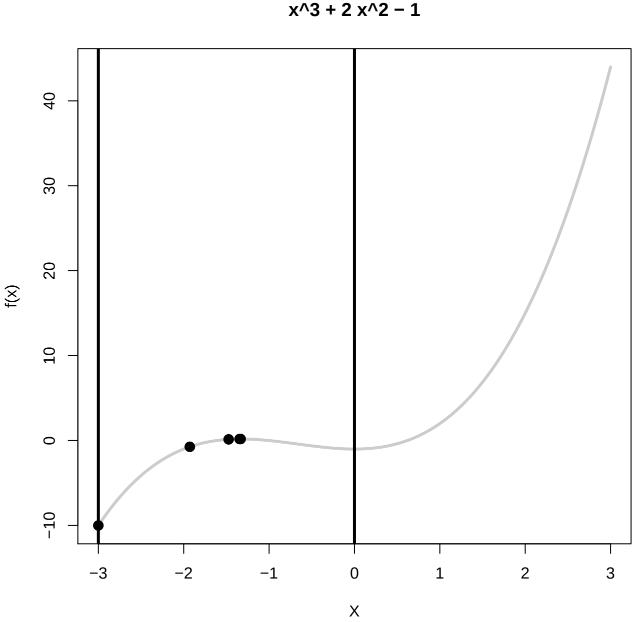
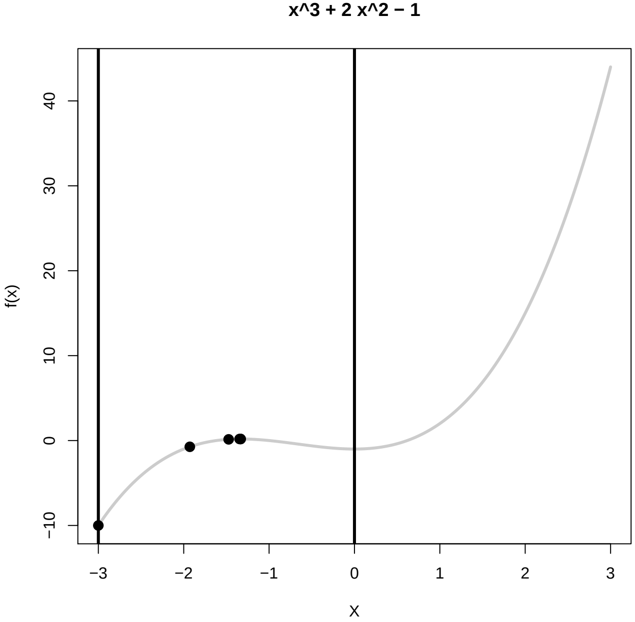
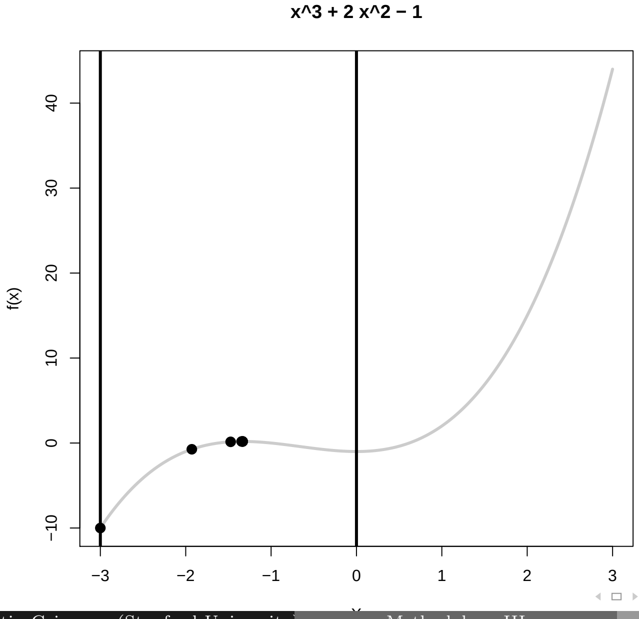

x^3 + 2 x^2 − 1
f(x)
−10
0
10
20
30
40
−3
−2
−1
0
1
2
3
X

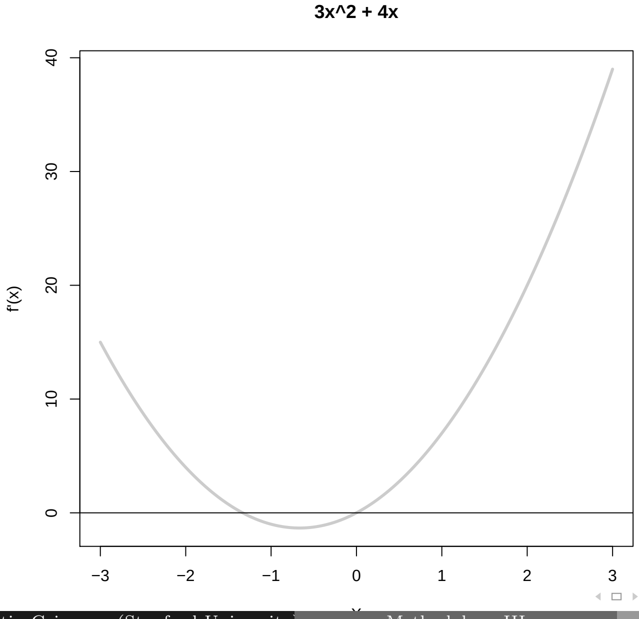

x^3 + 2 x^2 − 1
f(x)
40
30
20
10
0
−10
−3
−2
−1
0
1
2
3
X


x^3 + 2 x^2 − 1
f(x)
X
−10
0
10
20
30
40
−3
−2
−1
0
1
2
3

$x^* = -1.3333$



What is Happening with the Roots



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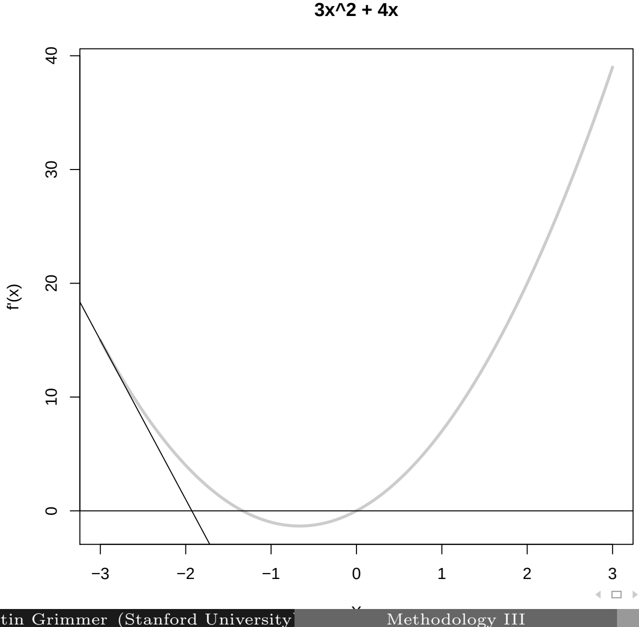

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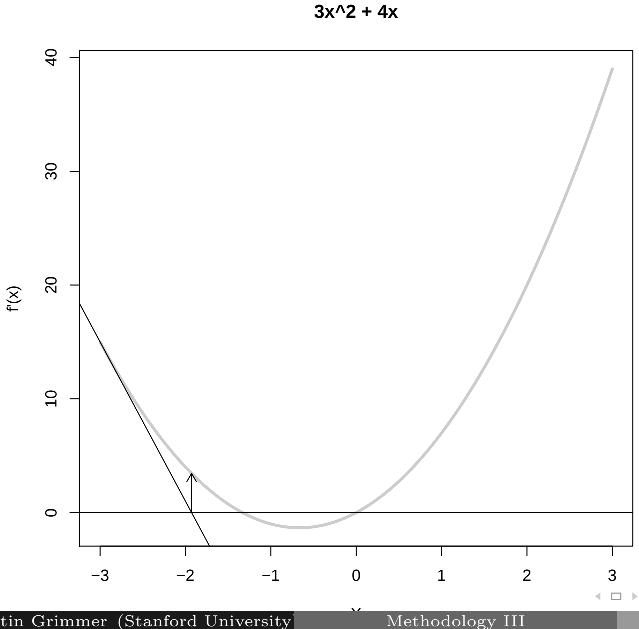

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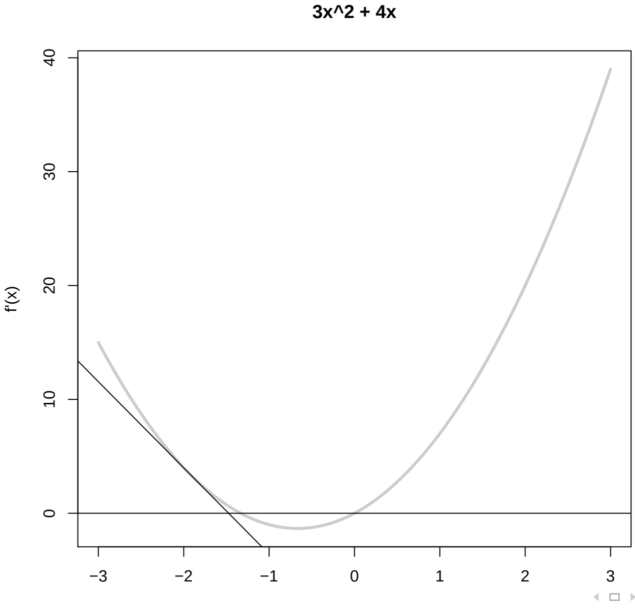

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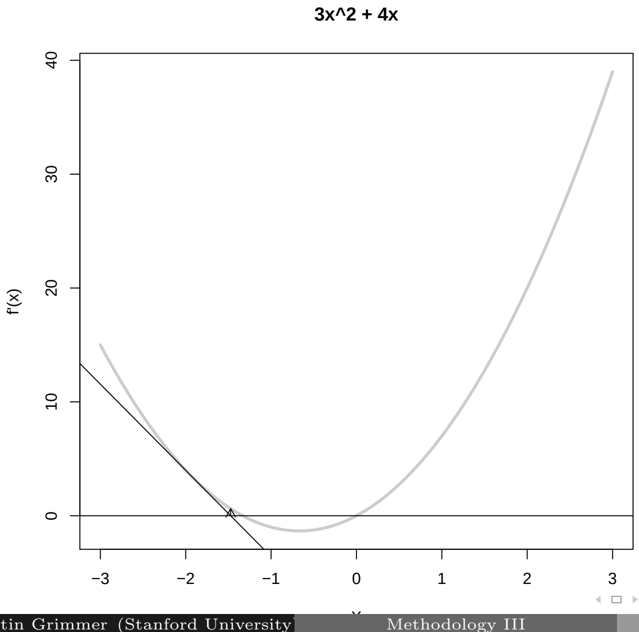

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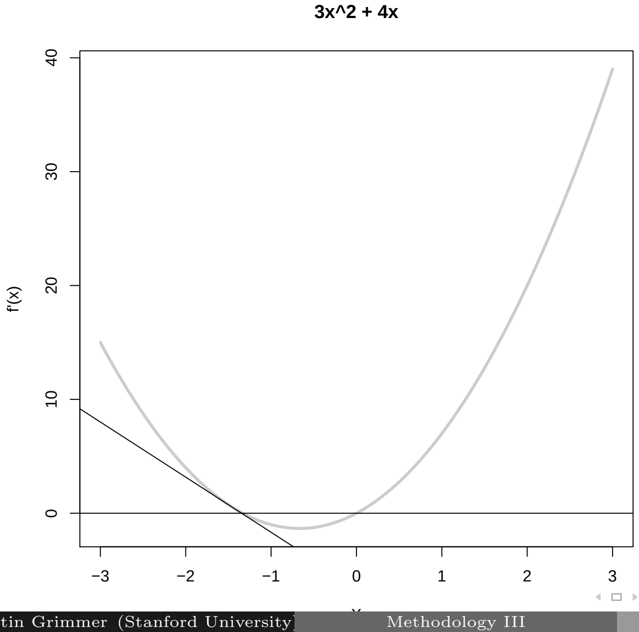

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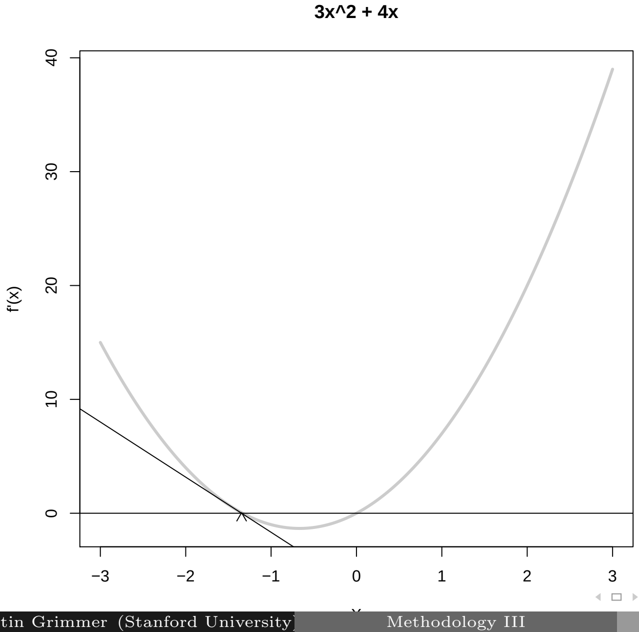

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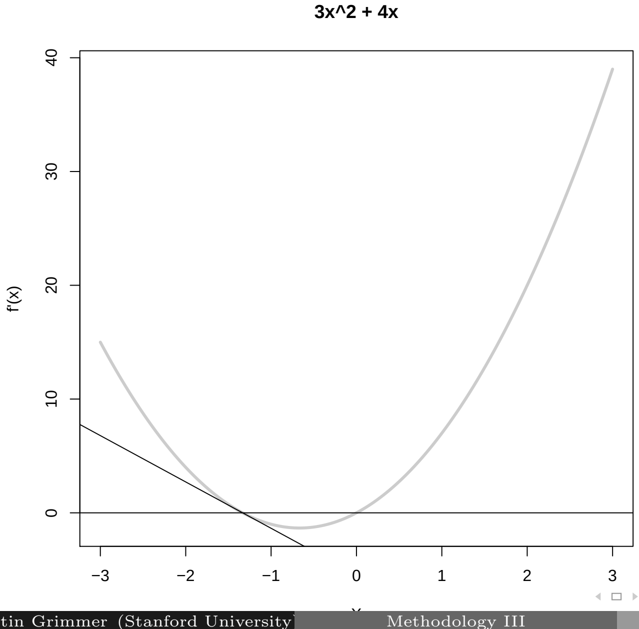

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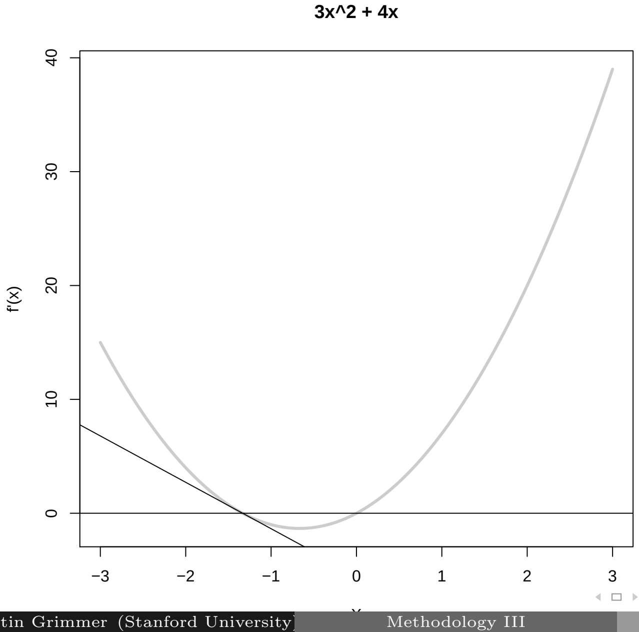

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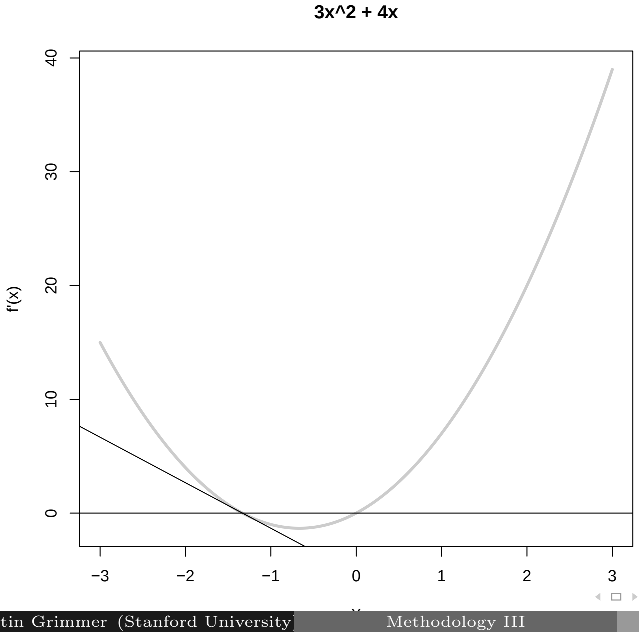

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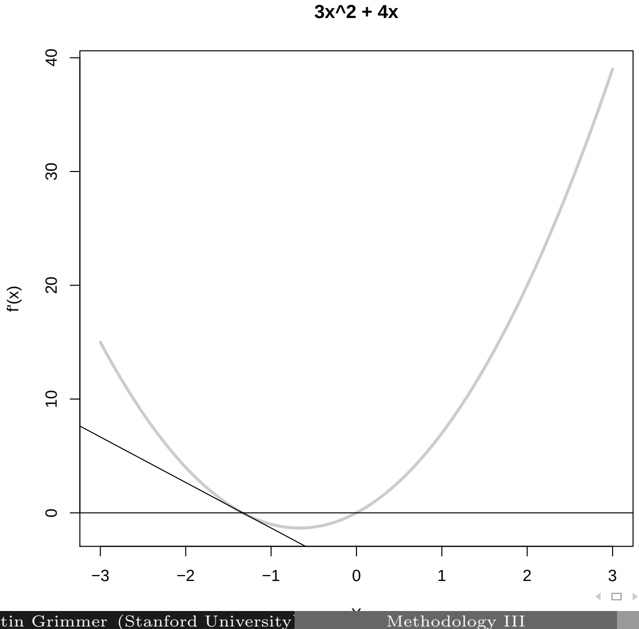

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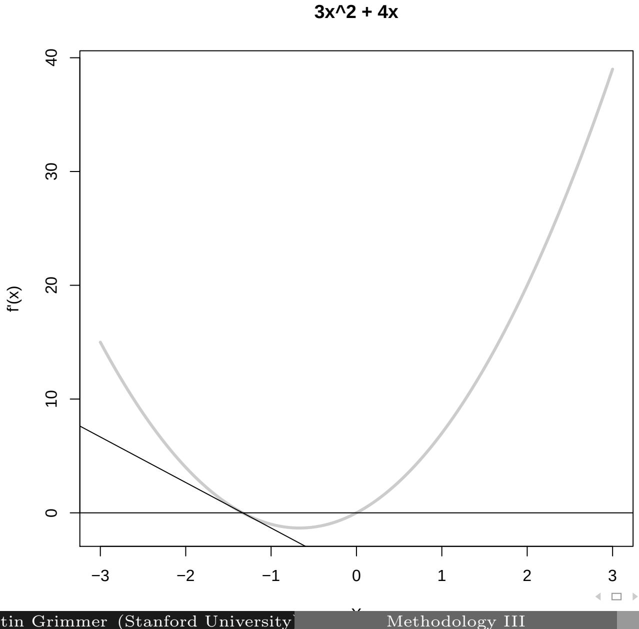

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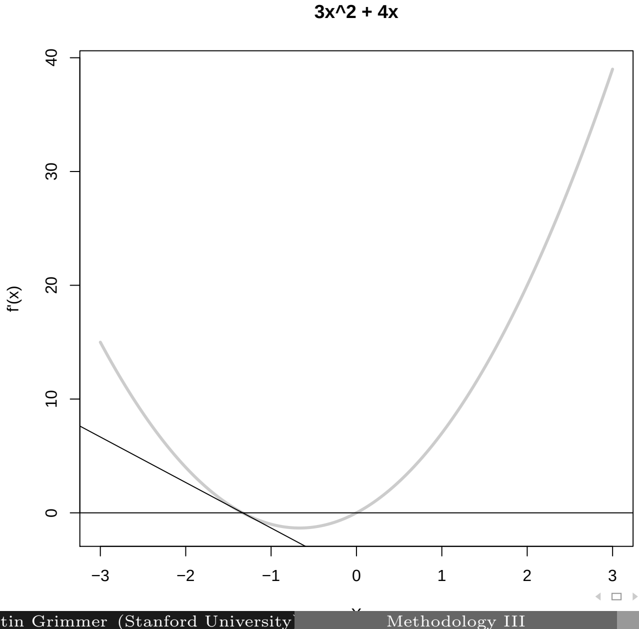

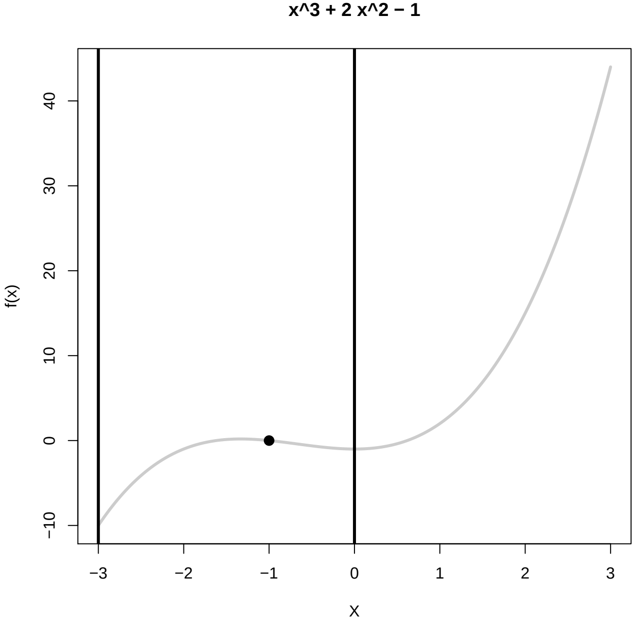
x^3 + 2 x^2 − 1
f(x)
−10
0
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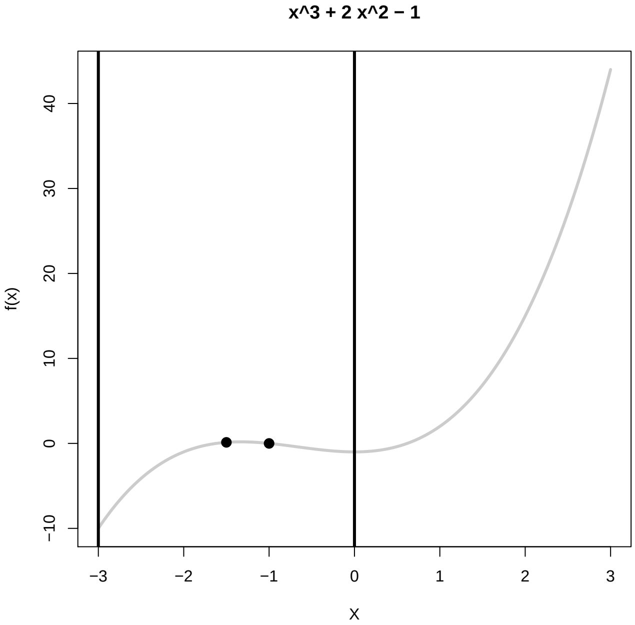
x^3 + 2 x^2 − 1
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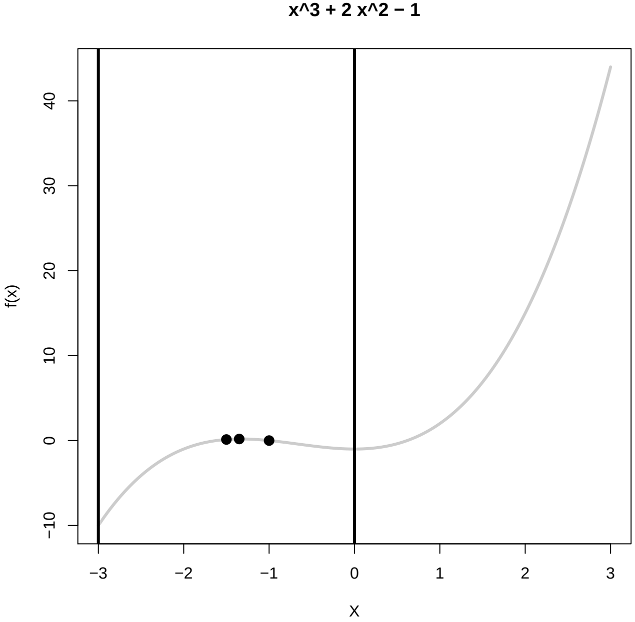
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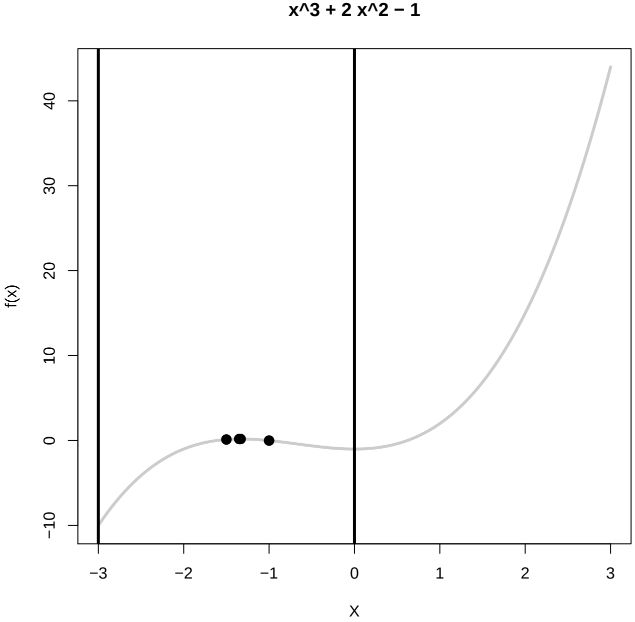
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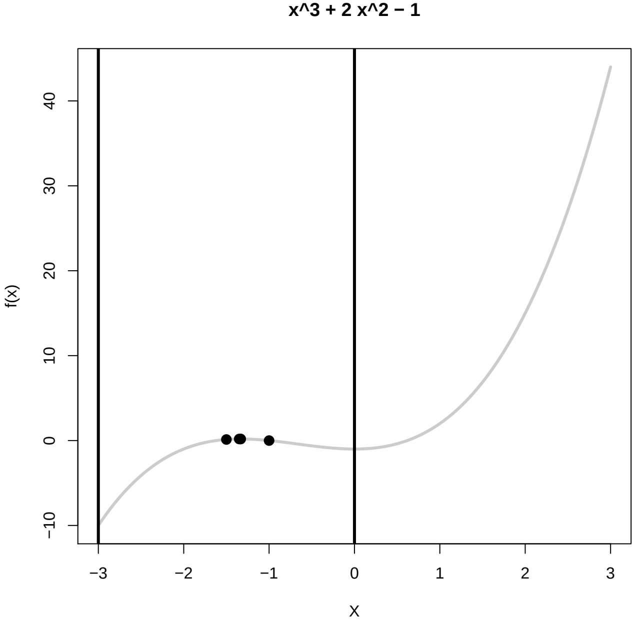
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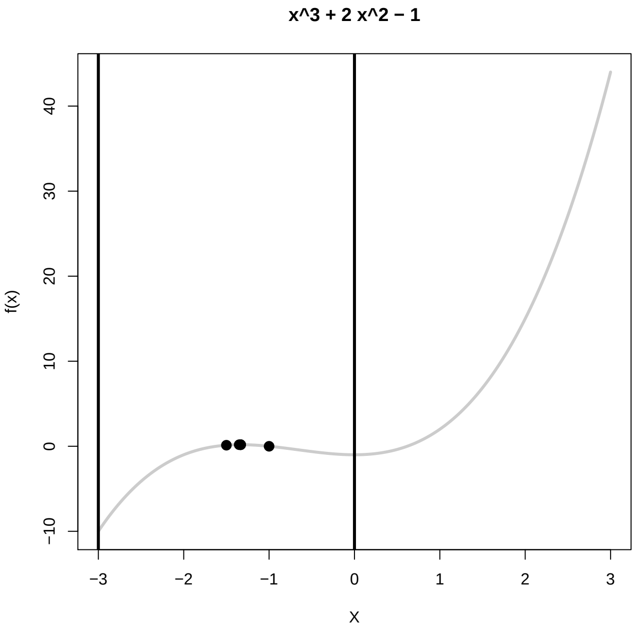
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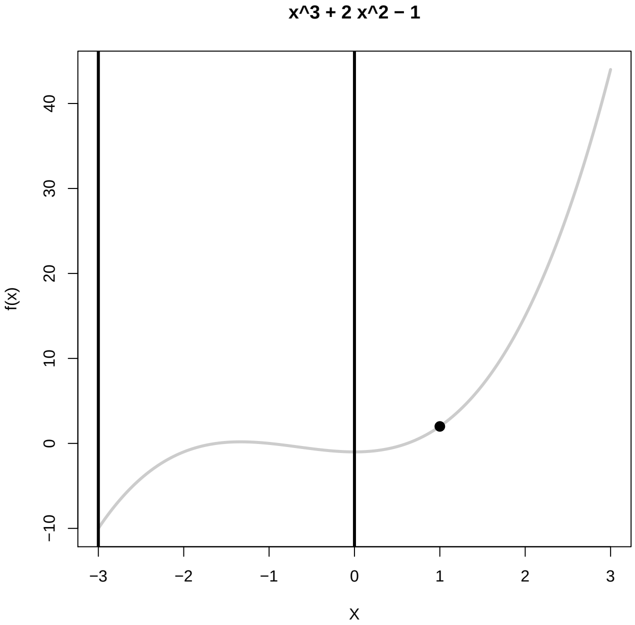
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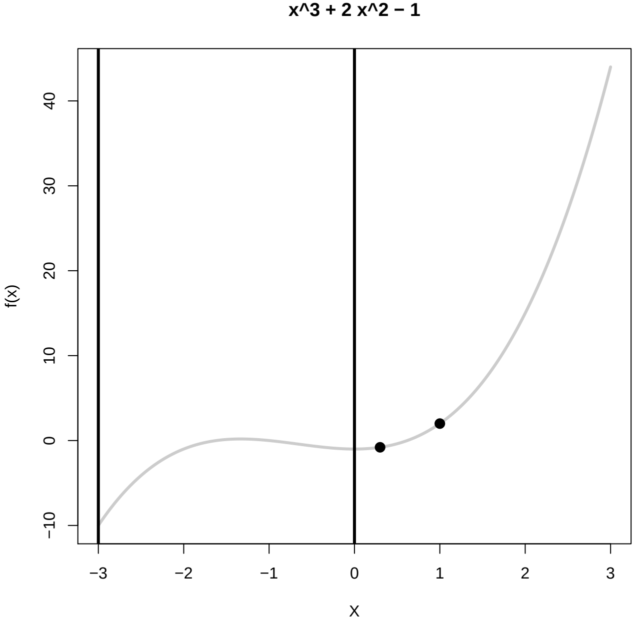
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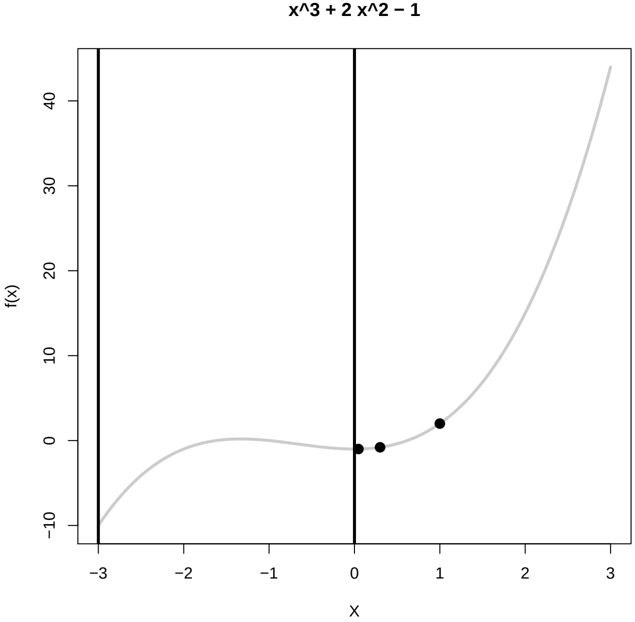
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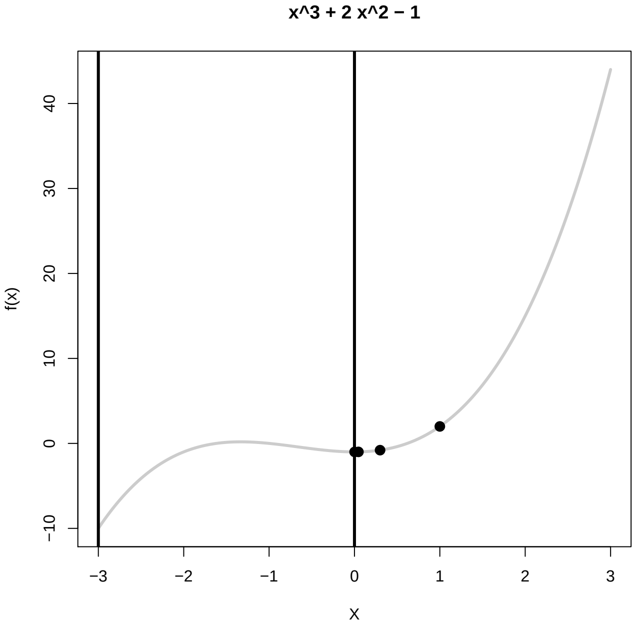
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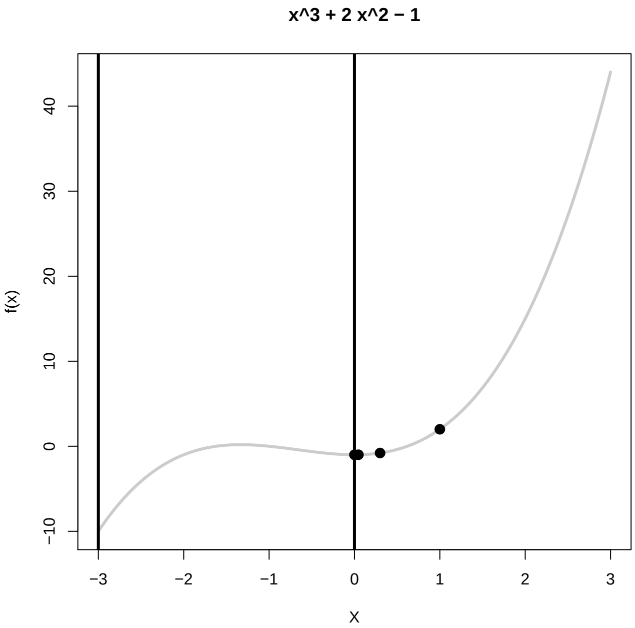
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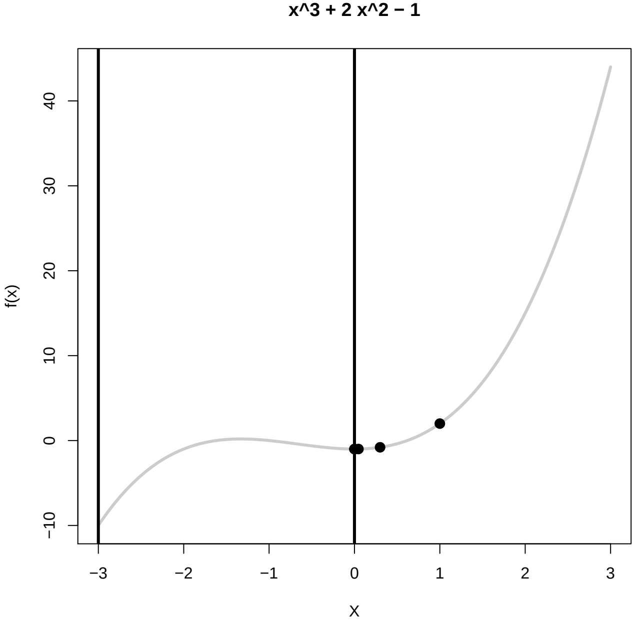
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Suppose $f : \Re^n \rightarrow \Re$. Suppose we have guess $\boldsymbol{x}_t$.

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Derivation (intuition): Approximate function with tangent plane. Find value of x_{t+1} that makes the plane equal to zero. Update again.

Multivariate Newton Raphson: Multivariate Linear Regression Model

Suppose that we have guess: $\left(\boldsymbol{\beta}_t, \sigma^2_t\right)$

$$
\begin{aligned}
(\boldsymbol{\beta}_{t+1}, \sigma^2_{t+1} &= \left(\boldsymbol{\beta}_t, \sigma^2_t\right) \underbrace{-\mathbf{H}(f)((\left(\boldsymbol{\beta}_t, \sigma^2_t\right))^{-1}}_{\text{Information}} \underbrace{\nabla f((\left(\boldsymbol{\beta}_t, \sigma^2_t\right)}_{\text{Score}} \\
\left(\boldsymbol{\beta}_{t+1}, \sigma^2_{t+1}\right) &= \left(\boldsymbol{\beta}_t, \sigma^2_t\right) + I_N(\boldsymbol{\beta}_t, \sigma^2_t)^{-1} \left(\sum_{i=1}^{N} s(Y_i | X_i, \boldsymbol{\beta}_t, \sigma^2_t) \right)
\end{aligned}
$$

So we have:

$$
\left(\boldsymbol{\beta}_{t+1}, \sigma^2_{t+1}\right) = \left(\boldsymbol{\beta}_t, \sigma^2_t\right)
$$

$$
+ \begin{pmatrix} \sigma^2 \left(\boldsymbol{X}^{'} \boldsymbol{X}\right)^{-1} & \mathbf{0} \\ \mathbf{0} & \frac{2(\sigma^2)^2}{N} \end{pmatrix} \left(\sum_{i=1}^{N} \left(\frac{\left(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta}_t\right)^{'} \boldsymbol{X}}{\sigma^2}, -\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4} (\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta})^{'} (\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta}) \right) \right)
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`R` code

1) Multivariate regression
2) Information $\rightsquigarrow$ learning about uncertainty from model
3) Numerical optimization

Probit/Logit + BFGS next!!