Political Methodology III: Model Based Inference

Justin Grimmer

Professor
Department of Political Science
Stanford University

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Statistical Inference

- Model based inference:
- Assume: data generated via distributional process
- Defines a likelihood function: parameter values $\rightarrow$ likelihood of parameters, given data
- Derive estimators that identify values that maximize likelihood

General Likelihood Theory $\rightarrow$ Example $\rightarrow$ General Theory $\rightarrow$ Example

Likelihood Inference, the Basics

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We're going to be interested in making an inference about θ_0 using the observed data.

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- $L(\theta|\boldsymbol{y}) : \underbrace{\Theta}_{\text{parameter space}} \rightarrow \Re$
- Idea: values of $\theta \in \Theta$ will be more likely if they make the observed data a higher probability

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We also are only able to infer most likely value given modeling assumptions

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- $y_i = 1$ or $y_i = 0$

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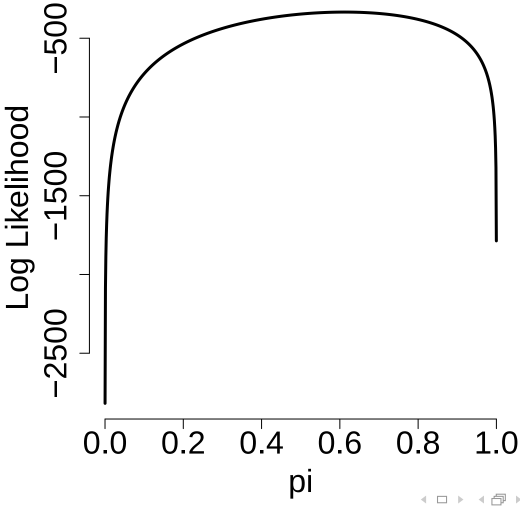
For a fixed set of observations, what does this look like?

Example: Bernoulli Trials: Simulated Example with $\pi = 0.6$



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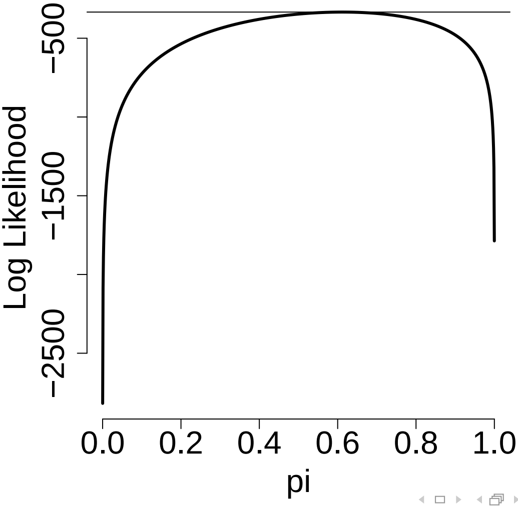

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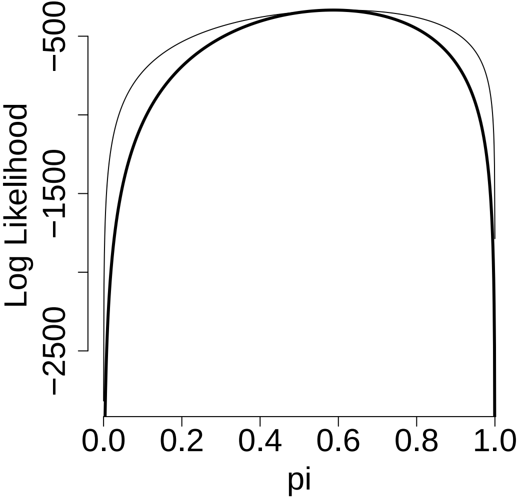

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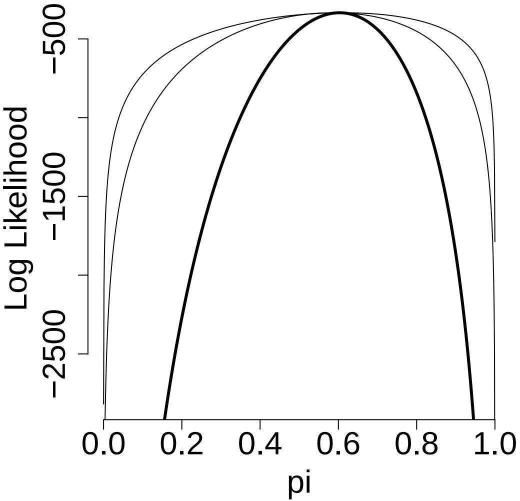

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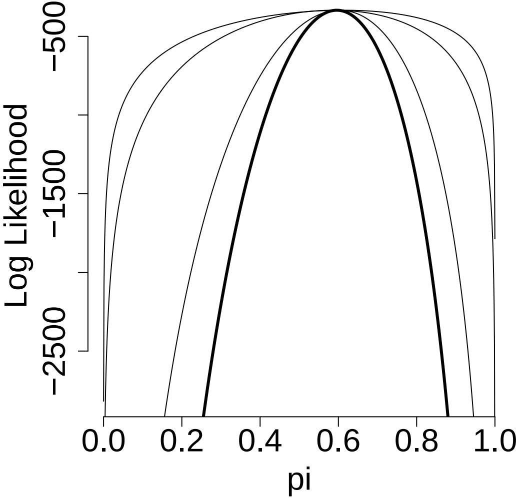

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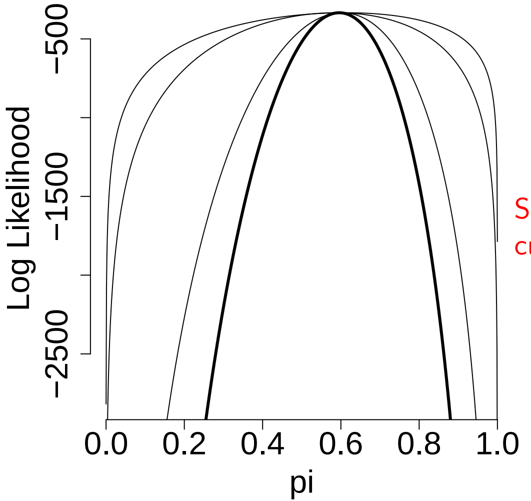

Second derivative captures this curvature

The Fisher Information measures the information that $\boldsymbol{y}$ conveys about the parameter $\boldsymbol{\theta}$. Define it using the two equivalent definitions:

Definition

The Fisher Information for a log-likelihood $l(\theta|\boldsymbol{Y})$ is

$$
\begin{aligned}
I(\theta) &= E\left[\left(\frac{\partial l(\theta|\boldsymbol{Y})}{\partial \theta}\right)^2 |\theta\right] \\
&= -E\left[\left(\frac{\partial^2 l(\theta|\boldsymbol{Y})}{\partial \theta \partial \theta}\right) |\theta\right]
\end{aligned}
$$

The observed Fisher information for a sample of n observations is given by

$$
I_n(\theta) = -\frac{\partial^2}{\partial \theta^2} l(\theta|\boldsymbol{y})
$$

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Inverting the information provides the asymptotic variance for the maximum likelihood estimator (under some regulatory conditions we will discuss later)

$$
\begin{aligned}
\text{Variance}(\theta^*) &= \frac{1}{I_n(\theta^*)} \\
\text{Standard Error}(\theta^*) &= \sqrt{\frac{1}{I_n(\theta^*)}}
\end{aligned}
$$

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- Curvature determines sampling distribution of maximum likelihood estimator

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1) The MLE gets as "close" as possible to the true answer

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Consistent Let $\widehat{\theta}_n$ be an estimator for θ, with sample size n. Then $\widehat{\theta}_n$ converges in probability to θ if, for all $\epsilon > 0$,

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Proposition

MLE is consistent: Assume $y_1, y_2, \ldots, y_n$ are simple random samples from $p(y|\theta_0)$. Define θ_n^* as the mle estimator with sample size n.
Then, as $n \rightarrow \infty$, $\theta_n^* \rightarrow \theta_0$ (in probability)

Properties of Maximum Likelihood Estimators

Simulated example with $\pi = 0.6$ and increasing n

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Distribution of $|\pi^* - 0.6|$

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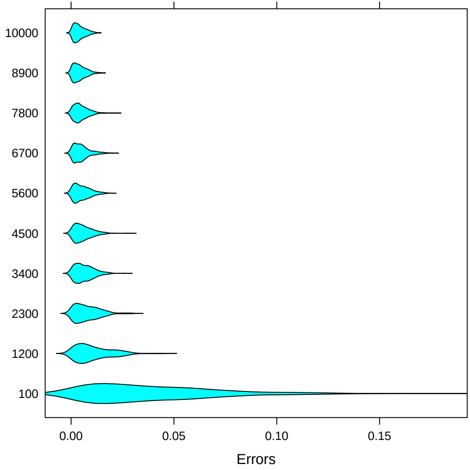

Definition

$\widehat{\theta}_n$, *with cdf* $F_n(x)$, *converges in distribution to random variable* Y *with cdf* $F(x)$ *if*

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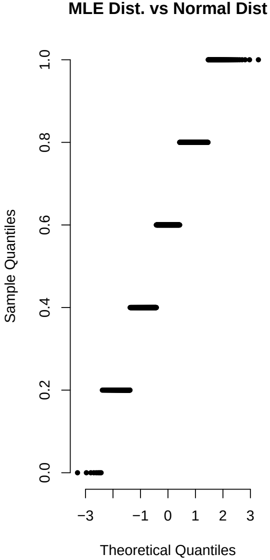
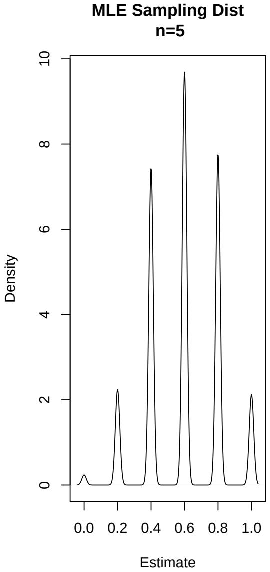
- MLE central limit theorem
- As we have more observations, the MLE converges, in distribution to a normal distribution

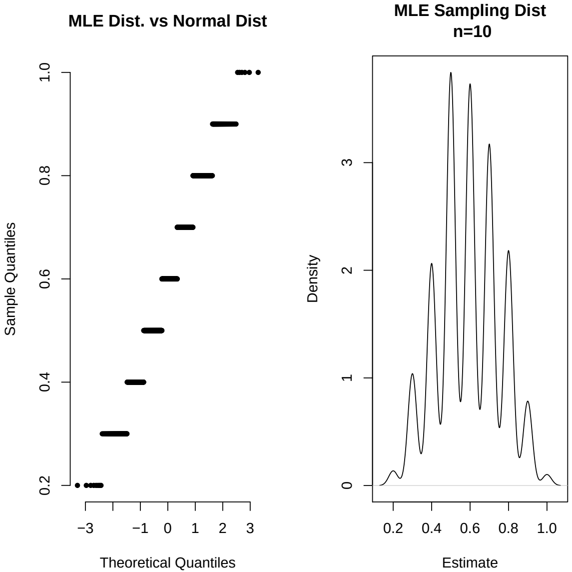
Central Limit Theorem for Maximum Likelihood Estimators





Example:
$\pi = 0.6$
Increasing n

Central Limit Theorem for Maximum Likelihood Estimators



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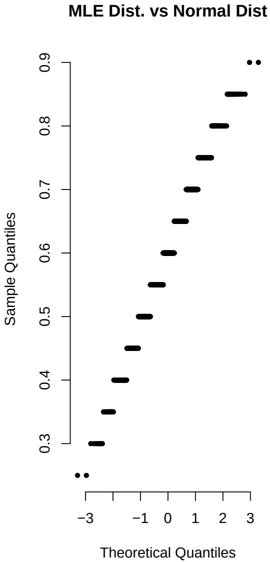

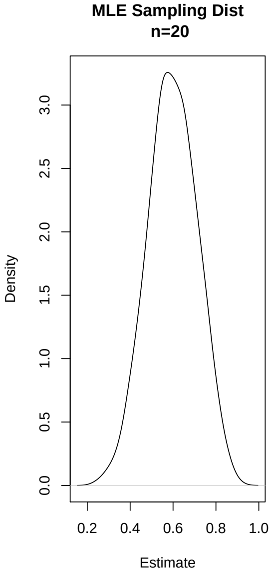

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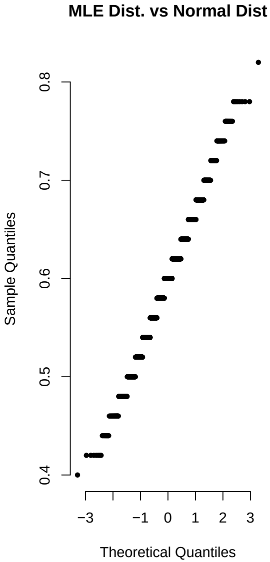

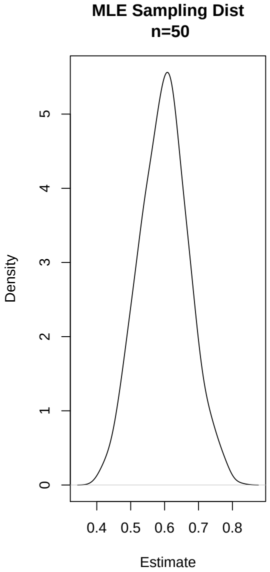

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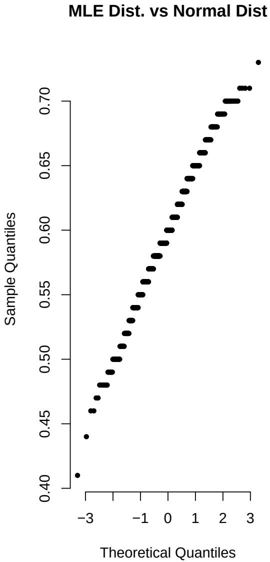

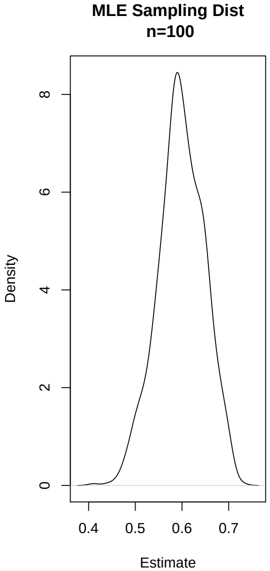

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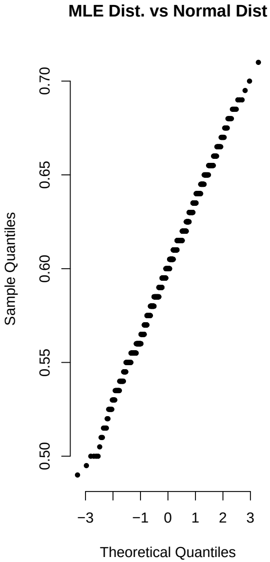

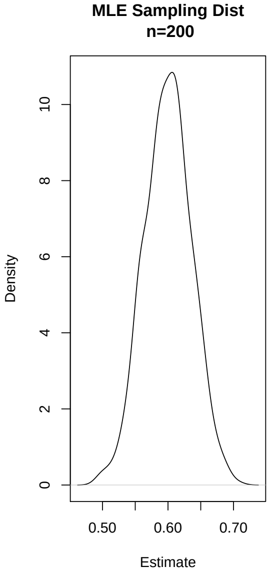

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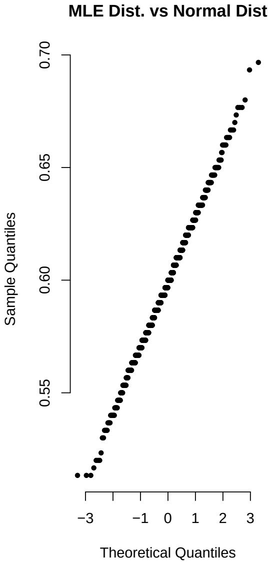

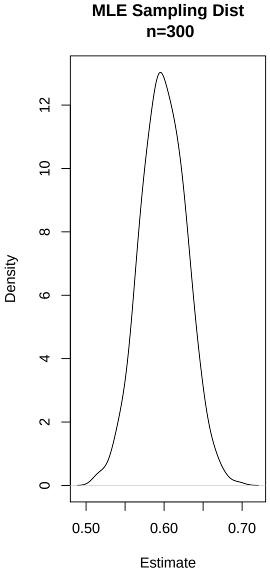

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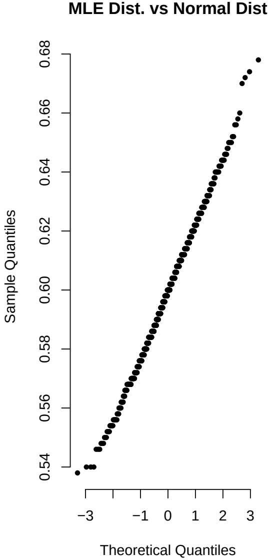

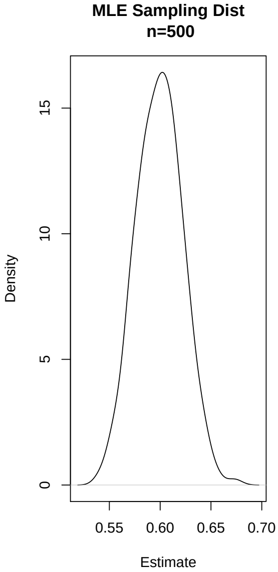

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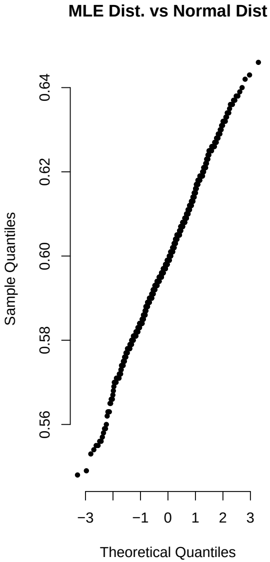

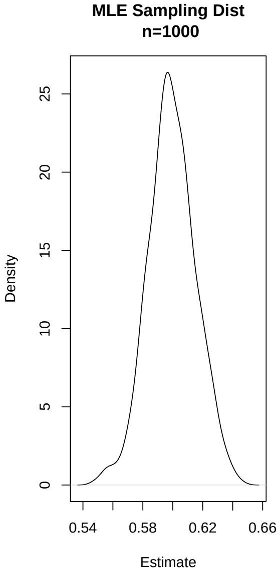

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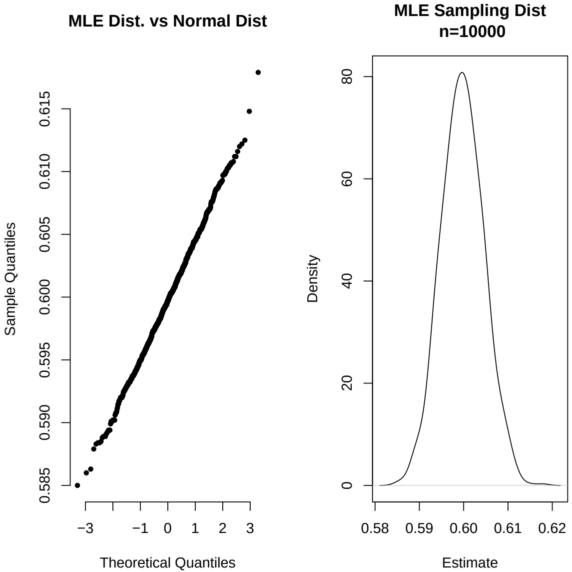

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Returning to Our Examples

Summary: θ^* is an MLE and θ_0 is the true value and we know the right distributional family

1) As $n \rightarrow \infty$, $\theta_n^* \rightarrow \theta_0$
2) As $n \rightarrow \infty$, $\sqrt{n}(\theta_n^* - \theta_0) \rightarrow \text{Normal}(0, \frac{1}{I(\theta_0)})$

where $I(\theta_0) = -\frac{\partial^2}{\partial \pi^2} l(\theta_0 | y)$ or curvature of log-likelihood at true value of θ_0

Two-parameter MLE

Multivariate Normal Distribution

Suppose that we have a vector of random variables,

$$\boldsymbol{X} = (X_1, X_2, \ldots, X_k)$$

Then we'll say that $X \sim$ Multivariate Normal$(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where,

$$\begin{aligned}
\boldsymbol{\mu} &= (\mu_1, \mu_2, \ldots, \mu_k) \\
\boldsymbol{\Sigma} &= \begin{pmatrix} \sigma_1^2 & Cov(X_1, X_2) & Cov(X_1, X_3) & \ldots & Cov(X_1, X_n) \\ Cov(X_1, X_2) & \sigma_2^2 & Cov(X_2, X_3) & \ldots & Cov(X_2, X_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Cov(X_1, X_k) & Cov(X_2, X_k) & Cov(X_3, X_k) & \ldots & \sigma_k^2 \end{pmatrix}
\end{aligned}$$

Multivariate Normal Distribution

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^{k/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})^{'}\boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu})\right)$$

Multivariate Version

Suppose that we simple random samples $\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_n$ from multivariate distribution $p(\boldsymbol{y}|\boldsymbol{\theta}_0)$.

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Inverting the Fisher-information matrix provides Variance-Covariance Matrix

Maximum Likelihood Estimation, Normal Distribution

Example 2:

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Our task:

- Obtain likelihood (summary estimator)
- Derive maximum likelihood estimators for μ and σ^2
- Characterize sampling distribution

Maximum Likelihood Estimation, Normal Distribution

Maximum Likelihood Estimation, Normal Distribution

$$L(\mu, \sigma^2 | \boldsymbol{y}) = \prod_{i=1}^{n} f(y_i | \mu, \sigma^2)$$

Maximum Likelihood Estimation, Normal Distribution

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L(\mu, \sigma^2 | \boldsymbol{y}) &= \prod_{i=1}^{n} f(y_i | \mu, \sigma^2) \\
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$$

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&= -\sum_{i=1}^{n} \frac{(y_i - \mu)^2}{2\sigma^2} - \frac{n}{2}\log(\sigma^2) + c
\end{aligned}
$$

Maximum Likelihood Estimation, Normal Distribution

Let's find μ^* and $(\sigma^2)^*$ that maximizes log-likelihood.

$$
\begin{aligned}
l(\mu, \sigma^2 | \boldsymbol{y}) &= -\sum_{i=1}^{n} \frac{(y_i - \mu)^2}{2\sigma^2} - \frac{n}{2}\log(\sigma^2) + c \\
\frac{\partial l(\mu, \sigma^2)|\boldsymbol{y})}{\partial \mu} &= -\sum_{i=1}^{n} \frac{2(y_i - \mu)}{2\sigma^2} \\
\frac{\partial l(\mu, \sigma^2)|\boldsymbol{y})}{\partial \sigma^2} &= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(Y_i - \mu)^2
\end{aligned}
$$

Maximum Likelihood Estimation, Normal Distribution

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\begin{aligned}
0 &= -\sum_{i=1}^{n} \frac{2(y_i - \mu^*)}{2\sigma^2} \\
0 &= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^{n} (y_i - \mu^*)^2
\end{aligned}
$$

Solving for μ and σ^2 yields,

$$
\begin{aligned}
\mu^* &= \frac{\sum_{i=1}^{n} y_i}{n} \\
\sigma^2 &= \frac{1}{n} \sum_{i=1}^{n} (y_i - \overline{y})^2
\end{aligned}
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Maximum Likelihood Estimation, Normal Distribution

Multivariate analogy: observed Fisher information matrix (Negative Hessian) (Negative matrix of second derivatives)

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$$I_n(\mu^*, \widehat{\sigma}^2) = -\begin{pmatrix} \frac{\partial^2 l(\mu,\sigma^2|\boldsymbol{y})}{\partial \mu^2} & \frac{\partial^2 l(\mu,\sigma^2|\boldsymbol{y})}{\partial \sigma^2 \partial \mu} \\ \frac{\partial^2 l(\mu,\sigma^2|\boldsymbol{y})}{\partial \sigma^2 \partial \mu} & \frac{\partial^2 l(\mu,\sigma^2|\boldsymbol{y})}{\partial^2 \sigma^2} \end{pmatrix}$$

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Taking derivatives and evaluating at MLE's yields

$$I_n(\mu^*, \widehat{\sigma}^2) = \begin{pmatrix} \frac{n}{\widehat{\sigma}^2} & 0 \\ 0 & \frac{n}{(\widehat{\sigma}^2)^2} \end{pmatrix}$$

Maximum Likelihood Estimation, Normal Distribution

Multivariate analogy: observed Fisher information matrix (Negative Hessian) (Negative matrix of second derivatives)

$$I_n(\mu^*, \widehat{\sigma}^2) = -\begin{pmatrix} \frac{\partial^2 l(\mu, \sigma^2|\boldsymbol{y})}{\partial \mu^2} & \frac{\partial^2 l(\mu, \sigma^2|\boldsymbol{y})}{\partial \sigma^2 \partial \mu} \\ \frac{\partial^2 l(\mu, \sigma^2|\boldsymbol{y})}{\partial \sigma^2 \partial \mu} & \frac{\partial^2 l(\mu, \sigma^2|\boldsymbol{y})}{\partial^2 \sigma^2} \end{pmatrix}$$

Taking derivatives and evaluating at MLE's yields

$$I_n(\mu^*, \widehat{\sigma}^2) = \begin{pmatrix} \frac{n}{\widehat{\sigma}^2} & 0 \\ 0 & \frac{n}{(\widehat{\sigma}^2)^2} \end{pmatrix}$$

Therefore, as $n \rightarrow \infty$, we have that

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$$p(\mu, \sigma^2) \quad \rightarrow^d \quad \text{Multivariate Normal}\left(\left(\overline{y}, \frac{1}{n}\sum_{i=1}^{n}(y_i - \overline{y})^2 \right), \begin{pmatrix} \frac{\widehat{\sigma}^2}{n} & 0 \\ 0 & \frac{(\widehat{\sigma}^2)^2}{n} \end{pmatrix} \right)$$

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This is an asymptotic result: results will vary with small sample sizes

Up next:

1) Linear regression in maximum likelihood
2) Logit/Probit
3) Numerical optimization