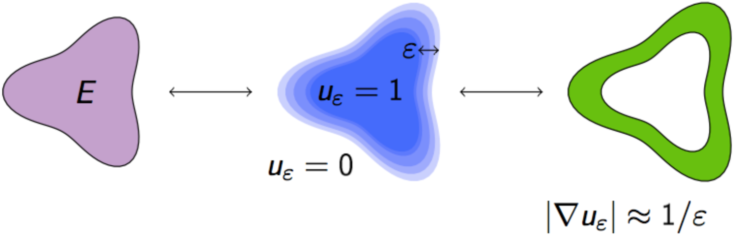
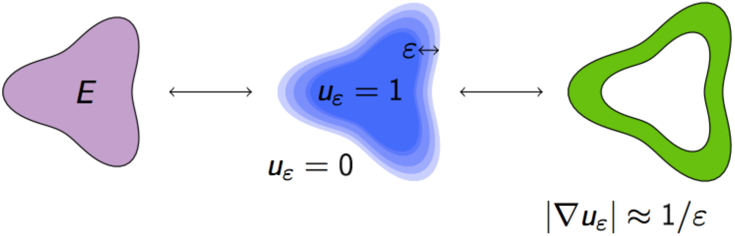
Phase field approximation

A phase field $u_\varepsilon : \mathbb{R}^d \to [0,1]$ is a smooth function which approximates the characteristic function $\mathbb{1}_E$ of a set E.

The area of ∂E is the perimeter of E.



Perimeter approximation

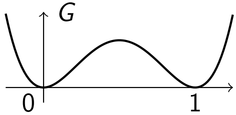


Thus, $\int \varepsilon |\nabla u_\varepsilon|^2 \mathrm{d}x \approx \frac{1}{\varepsilon} \mathit{Area} \approx \frac{1}{\varepsilon} \varepsilon P(E) = P(E) \quad$ as $\varepsilon \to 0$.

However, any constant function has zero energy! How to force u_ε to be close to a characteristic function, i.e. a binary function?

Perimeter approximation

Use a double-well potential, for instance $G(s) = \frac{1}{2}s^2(1-s)^2$.



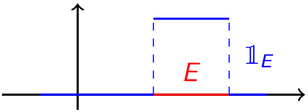
If $\sup_{\varepsilon} \left(\int \frac{1}{\varepsilon} G(u_\varepsilon) \mathrm{d}x \right) < +\infty$ then $u_\varepsilon \to 0$ or 1 a.e. as $\varepsilon \to 0$, i.e. u_ε approximates a characteristic function.

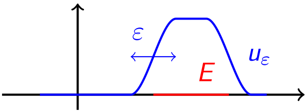
The Cahn-Hilliard functional

(Van der Waals)-Cahn-Hilliard energy

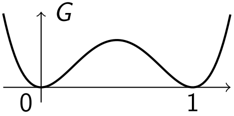
The phase-field approximation of perimeter is

$$P_\varepsilon(u) = \int \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} G(u) \right) \mathrm{d}x$$





where G is a double-well potential.



e.g.,

$$G(s) = \frac{1}{2} s^2 (1-s)^2$$

Phase-field approximation of perimeter

Convergence of P_ε (Modica, Mortola - 1977)

P_ε Γ-converges to

$$P(u) = \begin{cases} \lambda P(E) & \text{si } u = \mathbb{1}_E \in BV \\ +\infty & \text{otherwise} \end{cases}$$

where $\lambda = cst$ depends only on potential G.

Γ-convergence and minimizers

Let (F_n) equicoercive, Γ-converging to F. If, $\forall n$, u_n is a minimizer of F_n, then every cluster point u of (u_n) is a minimizer of F and $F(u) = \lim F_n(u_n)$.