Responsive optimal design with stimulus as design and state variable

Jamal Shabani

McMaster University

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Outline

Responsive optimal design

- Optimal design deals with finding the allocation of several materials in order to enhance the performance of a structure.

- Responsive materials are those materials whose properties can changed by external stimulus.
- Examples of responsive materials
  1. Shape Memory Alloy (SMA)
  2. Thermoelastic material
- Three design materials *i.e* **void/holes, non-responsive and responsive**
- **Goal:** Optimal allocation of (**void/holes, non-responsive and responsive**) materials and the **stimulus** or **stimulus control** that optimizes some general objective.

Outline

Problem statement

- Consider a ground domain $\Omega \in \mathbb{R}^d$, $d = 2, 3$ subject to boundary force f on a part $\Gamma_N \in \partial\Omega$ of its boundary and clamped on $\Gamma_D \in \partial\Omega$.

- Let χ_v, χ_s, and χ_r be a characteristic functions such that

$$\chi_{v,s,r}(x) = \begin{cases} 1 & x \in \text{Void, Non-responsive, Responsive,} \\ 0 & x \notin \text{Void, Non-responsive, Responsive.} \end{cases}$$

- We want the constraints

$$\chi_v(x) + \chi_s(x) + \chi_r(x) = 1, \forall x \in \Omega,$$

$$\int_\Omega \chi_{v,s,r} \, dx = \theta_{v,s,r}|\Omega| \text{ such that } \theta_v + \theta_s + \theta_r = 1.$$

- Design variable $\Phi = [\chi_v, \chi_s, \chi_r]^T$.

Problem statement

- Consider the constitutive law

$$\sigma(u_j) = \sum_{i=\{v,s,r\}} \chi_i \mathbb{C}_i \left(\mathrm{e}(u_j) - \beta_i s_j \mathrm{I}_d\right)$$

  where $\mathbb{C}_i$ denotes the Hooke's law for material i and $\mathrm{e}(u_j) = \frac{1}{2}(\nabla u_j + (\nabla u_j)^T)$ is the linearized strain associated to a displacement field u_j.

- β_i is a given parameter such that $\beta_v = \beta_s = 0, \beta_r = 1$, $s_j \in [-1, 1]$ is the stimulus associated with u_j and I_d is $d \times d$ identity matrix.

- Define the least square objective function

$$\mathcal{I}(u_1, \ldots, u_n) := \sum_{j=1}^{n} \frac{1}{2} \int_{\Omega_0} |u_j(x) - \bar{u}_j(x)|^2 \, dx, \qquad (1)$$

  where $\bar{u}_1, \ldots, \bar{u}_n$ are the target displacements.

Problem statement

- The displacement fields $u_j \in V$, $1 \leq j \leq n$, satisfy the weak form of the linearized elasticity system

$$\sum_{i=\{v,s,r\}} \int_\Omega \chi_i \mathbb{C}_i \left(\mathrm{e}(u_j) - \beta_i s_j \mathrm{I}_d\right) \cdot \mathrm{e}(v)\, dx = 0, \forall v \in V, \tag{2}$$

where

$$V := \left\{ v \in H^1(\Omega);\ v = 0 \text{ on } \Gamma_D \right\}. \tag{3}$$

- Mathematically, the responsive optimal design problem reads as

$$\begin{cases} \inf\limits_{(\Phi,\mathrm{s})} \ \mathcal{I}(u_1, u_2, \dots, u_n) \\ u_j \text{ satisfies the state equation (2)} \end{cases} \tag{4}$$

where $\mathrm{s} := (s_1, \dots, s_n)$.

- The problem (4) is ill-posed
- How to make the problem (4) well-posed?
    - Perimeter penalization

Problem statement

- Let $\mathrm{D} = (D_v, D_s, D_r)$ be a partition of the domain Ω such that $\chi_{v,s,r} = \chi_{D_{v,s,r}}$ and define

$$\mathcal{P}(\chi) := \frac{1}{2} \sum_{i,j=\{v,s,r\}, i \neq j} \mathcal{H}^{d-1}\left(\partial D_i \cap \partial D_j \cap \Omega\right), \tag{5}$$

where $\mathcal{H}^{d-1}$ denotes the $d-1$ dimensional Hausdorff

- Given a small regularization parameter $\alpha > 0$, we study the problem

$$\inf_{(\Phi, \mathrm{s})} \quad \mathcal{I}(u_1, \ldots, u_n) + \alpha \mathcal{P}(\chi). \tag{6}$$

- **Problem:** It is not easy to implement (6).

The phase field approach to optimal design

- We introduce designs of the form $\rho = (\rho_v, \rho_s, \rho_r)$, where each $\rho_v, \rho_s, \rho_r \in H^1(\Omega; [0,1])$ is a smooth continuous density function.
- For any $\varepsilon > 0$, define the vector-valued Modica-Mortola functional

$$\mathcal{P}_\varepsilon(\rho) = \int_\Omega \frac{W(\rho)}{\varepsilon} + \varepsilon |D\rho|^2 \, dx. \tag{7}$$

- $W(\rho)$ is a non-negative multiple-well potential function vanishing only at three points $(1,0,0), (0,1,0)$ and $(0,0,1)$ and satisfying

$$d_{ij} := \inf \left\{ \int_0^1 W^{1/2}(\gamma(t)) |\gamma'(t)| \, dt; \gamma \in C^1((0,1); \mathbb{R}^m), \right.$$
$$\left. \gamma(0) = \rho_i, \gamma(1) = \rho_j \right\} = 1$$

The phase field approach to optimal design

- The space of admissible designs $\mathcal{D}_\rho$ and stimuli $\mathcal{S}$

$$\mathcal{D}_\rho := \left\{ \rho \in \left[H^1(\Omega; [0,1]) \right]^3, \sum_{i=\{v,s,r\}} \rho_i = 1, \int_\Omega \rho_{v,s,r}\, dx = \theta_{v,s,r} |\Omega| \right\},$$

$$\mathcal{S} := L^1 \left(\Omega, [-1,1]^n \right).$$

- The phase field regularization of (6) is then

$$\inf_{(\rho,\mathrm{s}) \in \mathcal{D}_\rho \times \mathcal{S}} \mathcal{I}(u_1, \ldots, u_n) + \alpha \mathcal{P}_\varepsilon(\rho). \tag{8}$$

where $u_j \in V$ satisfies the weak formulation

$$\sum_{i=\{v,s,r\}} \int_\Omega a(\rho_i) \mathbb{C}_i \left(\mathrm{e}(u_j) - \beta_i s_j \mathrm{I}_d \right) \cdot \mathrm{e}(v)\, dx = 0 \ \forall v \in V, \ 1 \le j \le n \tag{9}$$

and a is a continuous function such that $a(0) = 0$ and $a(1) = 1$.

Existence of solutions

- Define $\widetilde{\mathcal{I}}(\rho, \mathrm{s}) = \mathcal{I}\left(u_1(\rho, s_1), \ldots, u_n(\rho, s_n)\right)$.
- Consider the problem

$$\underset{(\rho,\mathrm{s})\in\mathcal{D}_\rho\times\mathcal{S}}{\arg\min}\ \widetilde{\mathcal{I}}(\rho, \mathrm{s}) + \widetilde{\mathcal{P}}_\varepsilon(\rho, \mathrm{s}) \xrightarrow{\Gamma} \underset{(\rho,\mathrm{s})\in\mathcal{D}_\rho\times\mathcal{S}}{\arg\min}\ \widetilde{\mathcal{I}}(\rho, \mathrm{s}) + \widetilde{\mathcal{P}}(\rho, \mathrm{s})$$

  where

$$\widetilde{\mathcal{P}}_\varepsilon(\rho, \mathrm{s}) := \begin{cases} \mathcal{P}_\varepsilon(\rho) & \text{if } (\rho, \mathrm{s}) \in \mathcal{D}_\rho \times \mathcal{S} \\ +\infty & \text{otherwise,} \end{cases}$$

- $\Gamma-$convergence is stable under continuous perturbation.
- It is enough to prove continuity of $\widetilde{\mathcal{I}}(\rho, \mathrm{s})$ w.r.t to ρ and s.

Theorem (Fundamental Theorem of $\Gamma-$Convergence)

Let X be a topological space. Let (F_n) be an equi-coercive sequence of functions and let F_n $\Gamma-$converges to F in X, then the minimizers of F_n converge to a minimizer of F.

Lemma (1)

Let $(\rho, \mathrm{s}) \in \mathcal{D}_\rho \times \mathcal{S}$ *and* $k_1, k_2 > 0$ *be such that for any* $1 \leq i \leq 3$ *and for any* $\Psi \in \mathrm{M}^{d\times d}_{\mathrm{sym}}$, *we have* $k_1 \Psi \cdot \Psi \leq \mathbb{C}_i \Psi \cdot \Psi \leq k_2 \Psi \cdot \Psi$. *There exists* $C > 0$ *such that if* $u_j \in V$ *satisfies* (2), *then*

$$\|u_j\|_{H^1(\Omega)} \leq C, \; 1 \leq j \leq n.$$

Proof.

By taking $u_j \in V$ as the test function, linearized weak form (9) becomes,

$$\sum_{i=\{v,s,r\}} \int_\Omega a(\rho_i)\mathbb{C}_i \mathrm{e}(u_j) \cdot \mathrm{e}(u_j)\, dx = \sum_{i=\{v,s,r\}} \int_\Omega a(\rho_i)\beta_i s_j \mathbb{C}_i \mathrm{I}_d \cdot \mathrm{e}(u_j)\, dx, \tag{10}$$

□

Existence of solutions

Proof (Cont.)

We get,

$$k_1\|e(u_j)\|_{L^2}^2 \leq \left\| \sum_{i=\{v,s,r\}} a(\rho_i)\beta_i s_j \mathbb{C}_i \mathrm{I}_d \right\|_{L^2} \|e(u_j)\|_{L^2},$$

and hence

$$\|e(u_j)\|_{L^2} \leq M.$$

By Korn's inequality, we have that the solution u_j is bounded

$$C\|u_j\|_{H^1} \leq \|e(u_j)\|_{L^2} \leq M \implies \|u_j\|_{H^1} \leq M.$$

□

Theorem (Continuity of displacements $\Longrightarrow$ continuity of $\widetilde{\mathcal{I}}(\rho, \mathrm{s})$)

Consider a sequence $(\rho_\varepsilon, \mathrm{s}_\varepsilon) \in \mathcal{D}_\rho \times S$ *of designs and stimuli such that* $\rho_\varepsilon \to \rho$, *in* $\left[L^1(\Omega)\right]^3$ *and* $\mathrm{s}_\varepsilon \rightharpoonup \mathrm{s}$ *in* $\left[L^2(\Omega)\right]^n$. *Let* $u_\varepsilon = (u_{1,\varepsilon}, \ldots, u_{n,\varepsilon})$ *(resp.* $u = (u_1, \ldots, u_n)$*) be the equilibrium displacements associated with* $(\rho_\varepsilon, \mathrm{s}_\varepsilon)$ *(resp.* (ρ, s)*). Then if the* $\mathbb{C}_i$ *satisfies the hypotheses of Lemma 2,* $u_\varepsilon \to u$ *in* $\left[L^2(\Omega)\right]^n$ *and* $\widetilde{\mathcal{I}}(\rho_\varepsilon, \mathrm{s}_\varepsilon) \longrightarrow \widetilde{\mathcal{I}}(\rho, \mathrm{s})$.

Proof.

For each $(\rho_\varepsilon, \mathrm{s}_\varepsilon)$ we can find sequence u_ε such that u_ε is the solution of the linearized elasticity PDE associated with $(\rho_\varepsilon, \mathrm{s}_\varepsilon)$. Since u_ε is uniformly bounded in H^1 we can extract a convergent subsequence u_{ε_n} converging weakly to some $u \in H^1$. Recalling

$$\widetilde{\mathcal{I}}(\rho_\varepsilon, \mathrm{s}_\varepsilon) = \mathcal{I}\left(u_{1,\varepsilon}(\rho_\varepsilon, s_{1,\varepsilon}), \ldots, u_{n,\varepsilon}(\rho_\varepsilon, s_{n,\varepsilon})\right),$$

□

Proof (Cont.)

and since $u_{\varepsilon_n} \to u$ strongly in L^2 we can pass the limit inside to get

$$\begin{aligned}\widetilde{\mathcal{I}}(\rho_\varepsilon, \mathrm{s}_\varepsilon) &= \mathcal{I}\left(u_{1,\varepsilon}(\rho_\varepsilon, s_{1,\varepsilon}), \ldots, u_{n,\varepsilon}(\rho_\varepsilon, s_{n,\varepsilon})\right) \\ &\to \mathcal{I}\left(u_1(\rho, s_1), \ldots, u_n(\rho, s_n)\right) \\ &= \widetilde{\mathcal{I}}(\rho, \mathrm{s}).\end{aligned}$$

□

Note

The proof above assumes the minimizing sequence $(\rho_\varepsilon, \mathrm{s}_\varepsilon)$ is compact.

Numerical implementation

- The constraint $\rho_v + \rho_s + \rho_r = 1 \implies \rho_v = 1 - \rho_s - \rho_r$ and introduce $\tilde{\rho} = (\rho_s, \rho_r)$.
- We add pernalties $Q(\tilde{\rho}, \mathrm{s})$ and $V_C(\tilde{\rho})$ where

$$Q(\tilde{\rho}, \mathrm{s}) = \int_\Omega (\rho_v^2 + \rho_s^2) s^2 \, dx = \int_\Omega ((1 - \rho_s - \rho_r)^2 + \rho_s^2) s^2 \, dx$$

and

$$V_C(\tilde{\rho}) = \nu_s \int_\Omega \rho_s \, dx + \nu_r \int_\Omega \rho_r \, dx.$$

- We consider the problem:

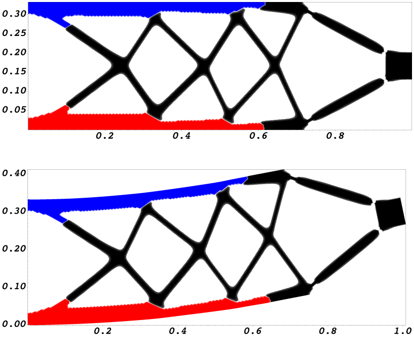
$$\begin{cases} \min\limits_{(\rho, \mathrm{s}) \in \mathcal{D}_\rho \times \mathcal{S}} \displaystyle\sum_{j=1}^{n} \frac{1}{2} \int_{\Omega_0} |u_j - \bar{u}_j|^2 \, dx + \alpha \mathcal{P}_\varepsilon(\tilde{\rho}, \mathrm{s}) + Q(\tilde{\rho}, \mathrm{s}) + V_C(\tilde{\rho}) \\ u_j \text{ satisfies the weak form (9)} \end{cases} \tag{11}$$

Numerical implementation

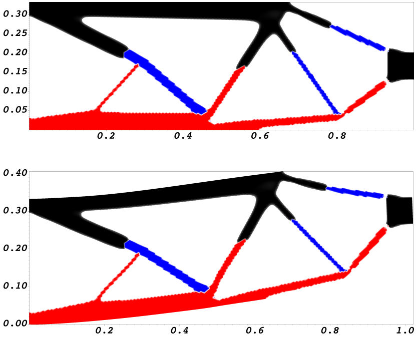
- We apply the adjoint method to compute the derivative of the objective function
- By collecting all terms explicitly depending on $s \implies$ closed form of an optimal stimulus.
- We tested two different numerical approaches.
  1. **Staggered** approach $\implies$ minimize w.r.t $\tilde{\rho}$ only and update s explicitly.
  2. **Monolithic** approach $\implies$ minimize w.r.t $\tilde{\rho}$ and s simultaneously.
- Our numerical implementation uses Firedrake and TAO

Numerical results

- Target displacement $\bar{u} = (0, 1)$.



Objective function $= 4.17 \times 10^{-3}$



Objective function $= 4.49 \times 10^{-1}$

Figure: Staggered (left) vs. Monolithic (right) approach. Composite plot of both material density and the stimulus in the reference (top) and deformed (bottom) configuration for stiffness ratio $E_3/E_2 = 1.0$.

Outline

Stimulus governed by poisson PDE

- Let the design variable $\tilde{\rho} = [\rho_s, \rho_r, g]^T$.
- $g \in [-1, 1]$ is the stimulus control function.
- The problem can be stated as:

$$\begin{cases} \min\limits_{(\rho, g)} \ \dfrac{1}{2} \displaystyle\int_{\Omega_0} |u - \bar{u}|^2 \, dx + \alpha \mathcal{P}_\varepsilon(\tilde{\rho}, s) + Q(\tilde{\rho}, g) + V_C(\tilde{\rho}) \\ u \text{ and } s \text{ satisfy the weak forms (12) and (13)} \end{cases}$$

$$\sum_{i=\{v,s,r\}} \int_\Omega a(\rho_i) \mathbb{C}_i \left(\mathrm{e}(u) - \beta_i s \mathrm{I}_d \right) \cdot \mathrm{e}(v) \, dx = 0 \ \forall v \in V, \tag{12}$$

$$\sum_{i=\{v,s,r\}} \int_\Omega k(\rho_i) \nabla s \cdot \nabla q \, dx - \int_\Omega g q \, dx = 0 \tag{13}$$

Stimulus governed by poisson PDE

- Target displacement $\bar{u} = (0, 1)$.

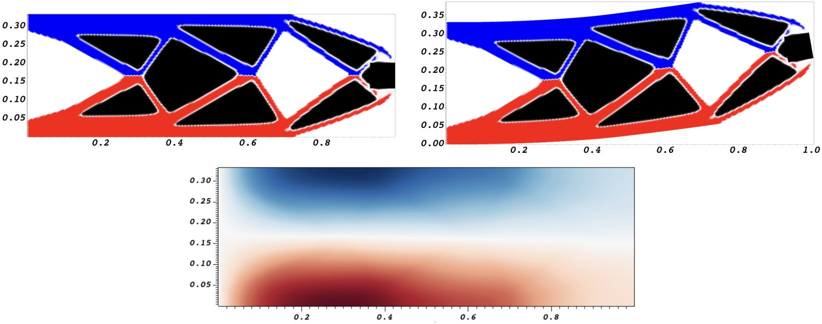


Figure: Composite plot of both material density and the heat source in the reference (top) and deformed configuration (bottom) for stiffness ratio $E_3/E_2 = 100$. The red and blue represent the area of the responsive material with heat source values 1 and -1 respectively.

Outline

Conclusion

- Responsive optimal design
- Responsive optimal design with stimulus as a design variable
  - Least square objective function with n target displacements
  - A paper was submitted titled "Systematic design of compliant morphing structures: a phase-field approach".
- Responsive optimal design with stimulus as a state variable
  - Stimulus governed by Poisson PDE
- Stimulus governed by Transient heat PDE (**Show some animations.)**

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Thank You!