The Characteristic Problem in Ideal General Relativistic Magnetohydrodynamics with Constraint Damping

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ABSTRACT

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In order to model general relativistic ideal magnetohydrodynamics, we couple the equations of general relativity, electromagnetism, and fluid dynamics. The right and left eigenvectors of the resulting characteristic equation, along with the corresponding eigenvalues, have previously been found in order to determine the wavespeeds and possible modes of the magnetohydrodynamic waves. Attempting to transfer these results into computational models can result in uncontrolled errors that would suggest the presence of magnetic monopoles and the failure of the no-monopole constraint to be satisfied correctly. In order to properly model our results, we include a constraint damping term with the coupled equations in order to maintain a magnetic divergence of zero over time. The eigenvectors and eigenvalues must be resolved with this constraint damping term. We solve for eigenvectors and eigenvalues that will allow for the correction of constraint violations. The resulting wavespeeds and modes will lead to possibilities for correct computational modeling through numerical methods.

Keywords: GRMHD, Relativistic MHD Simulation, Numerical Relativity, Relativistic Plasmas, Relativistic Astrophysics

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Chapter 1

Introduction

1.1 What is Ideal GRMHD?

Some of the most fascinating systems in our universe are areas of high gravitational and magnetic influences, such as regions near black holes and neutron stars. These types of systems are prevalent in the cosmos and deserve extensive study. In an effort to understand and describe these sytems, we must turn to general relativistic magnetohydrodynamics (GRMHD), a mathematical framework that is meant to describe electrically charged fluids under high gravitational influences.

The first component of GRMHD is general relativity. General relativity describes systems that are at high, relativistic speeds and under extreme gravitational influences. Most gravitation that we experience or observe can be characterized by classical or Newtonian gravity, which regards gravity as a force between masses. However, in areas that are under extreme gravitational influences, the familiar methods of describing motion no longer apply, and we must turn to general relativity or Einsteinian gravity. Einstein's theory of gravitation asserts that gravity is not a force between masses, but rather a curvature of spacetime that affects how mass moves. This theory is especially useful to describe some of the unusual behavior of matter near or within high gravitational influences. Thus general relativity is very important to describe material in the vicinity of black holes and neutron stars.

Magnetohydrodynamics (MHD) can be explained simply as the movement of fluids under electromagnetic influences, such as highly charged plasmas within accretion disks surrounding black holes or neutron stars. MHD combines the equations of fluid dynamics and electromagnetism to describe how these fluids will move when influenced by the currents and magnetic fields present.

Now it may be helpful to explain what we mean by "ideal." We are going to assume a conductivity of infinity in order to simplify some of our later work (don't worry, it still won't be simple). Assuming an infinite conductivity implies that within the comoving reference frame, any electric fields that form are instantaneously dispersed, thus when dealing with motion of the fluid we won't have to worry about electric fields. Taking the electric fields out of the picture is what makes these systems ideal. Although assuming an infinite conductivity may seem unrealistic, the plasma fields we are considering are very highly ionized and do have very high conductivity and magnetic fields dominate the dynamics of the material.

Now the motivation for our work is clear, whatever "our work" is, as we study some of the most strange and fascinating substances in the universe.

1.2 Implementation

So, how exactly are we going to study these fascinating objects? Sadly, we can't go poke our heads inside of a neutron star or take a sample of a black hole's accretion disk, so we resort to theoretical methods. As it turns out, our work is math. Without going into too much detail here, we will need to couple Einstein's field equations (which describe systems with Einsteinian gravity), the conservation equations of MHD, Maxwell's equations of electromagnetism, and the equation of state into one system that encapsulates all the forces at play in the realm of GRMHD.

Ultimately the work we have done is in preparation to numerically evolve physical simulations in GRMHD. In order to do this preparation, we'll not only need the equations described above, but we will also introduce a constraint damping term that will slightly complicate our system. This term is in place to damp any numerical errors that would build over time and suggest the presence of magnetic monopoles, which as far as we know don't exist, and thus we don't want to model in our system. After including this term we will end up with a coupled system in the flavor of a 9-component partial differential equation that describes motion in ideal GRMHD with added constraint damping. Finding the characteristic equation for this system will allow us to solve the eigenvalues and eigenvectors of the fluid, which correspond to wavespeeds and waveforms, respectively. These results build the framework for the early stages of numerical simulation.

Numerical simulation of magnetohydrodynamics has been achieved in the past, with different methods and in slightly different scenarios. Numerical simulations have been an active area of research for many years, with researchers exploring different methods to accurately and efficiently model the behavior of plasmas in the presence of magnetic fields. In particular, several papers have proposed different approaches for solving the equations of relativistic MHD, which is important for modeling phenomena such as relativistic jets and accretion disks as beforementioned.

Anderson et al. [1] and Neilsen et al. [2] both propose adaptive mesh refinement methods for solving the equations of relativistic MHD, which can allow for more efficient and accurate simulations of complex systems. Brio and Wu [3] present an upwind differencing scheme for solving the ideal MHD equations, which provides a stable and accurate numerical method for modeling shock waves and other discontinuities in the system. Meanwhile, Ibanez et al. [4] and Antón et al. [5] both propose Riemann solvers based on spectral decomposition and renormalized eigenvectors, respectively, which allow for accurate and stable simulations of relativistic MHD systems with strong magnetic fields.

Overall, these papers suggest that continued research into numerical methods for solving MHD equations is important for advancing our understanding of plasma behavior and predicting the behavior of complex systems. Future research may focus on improving the efficiency and accuracy of existing methods, as well as exploring new approaches for modeling relativistic MHD and other plasma phenomena. We aim to set up the framework for an accurate numerical simulation of such phenomena by solving the eigenvalues and eigenvectors of a characteristic structure built from the coupled physical equations at play. The analytical work for a system without constraint damping has previously been done by a former BYU student, Ryan Hatch [6], whose framework greatly guided the research reported here. However, now the work must be done with constraint damping incorporated in order to take care of small errors and prime this work for further numerical simulation.

Now we are in a position to dive into the math of ideal GRMHD.

Chapter 2

Methods

The focus of our research is finding the eigenvectors and eigenvalues of the characteristic equation associated with our coupled system, with constraint damping included. First we will demonstrate the coupling of our needed equations, then we will add constraint damping to the system, and finally we will solve for the desired quantities. As we will see, solving for the eigenvalues and eigenvectors will prove to be exceptionally in-depth.

2.1 Setup

We need to set up the equations of GRMHD, which requires coupling the equations that govern electromagnetism, fluids, and strong gravitational fields. We start with Einstein's field equations

$$G_{ab} = 8\pi T_{ab}$$

where G_{ab} is the gravity tensor and T_{ab} is the stress energy tensor. More specifically we want to obtain the matter equations which govern GRMHD. We define the stress energy tensor

$$T_{ab} = [\rho_0(1+\varepsilon)+P]u_a u_b + P g_{ab} + F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}$$

where u_a is the four-velocity, ρ_0 is the rest energy density, ε is the internal energy, P is pressure, g_{ab} is the metric and F_{ab} is the Maxwell tensor. Remembering our conditions for ideal MHD, we will take an Ohm's type law coupling the fluid to the electric and magnetic fields:

$$J_a + \rho_e u_a = \sigma F_{ab} u^b$$

which then, according to $\sigma \to \infty$, simplifies to

$$F_{ab} u^b = 0.$$

Remember that this means within the comoving reference frame, electric fields instantaneously disperse as they are formed.

We now write the matter equation in the form

$$\nabla_a T^{ab} = 0$$
$$\nabla_a * F^{ab} = 0$$

with $*F^{ab} \equiv \varepsilon^{abcd} F_{cd}/2$. We aim to put these equations in a different, more useful form. First off, We write the primitive variables of our system:

$$\vec{w} = \begin{pmatrix} \rho_0 \\ v_i \\ P \\ B^i \end{pmatrix}$$

where "primitive" means the quantities have simple, intuitive meaning that can be observed in nature (for the most part). We will also write a similar form of the conserved variables, which, naturally, are not physically intuitive or measurable. However first we must define a number of quantities: v_i is the spatial fluid velocity and thus the Lorentz factor is $W = \frac{1}{1-v_i v^i}$. Fluid enthalpy is defined as $h_e = \rho_0(1+\varepsilon)+P$. We will also be using the tensor convention that $B_iB^i \equiv B^2$, $B^iv_i \equiv Bv$, etc. Now we define the conserved variables to be

$$\vec{q} = \begin{pmatrix} D \\ S_i \\ \tau \\ B^i \end{pmatrix},$$

where D is the rest mass energy density times the Lorentz factor, S_i is the momentum, τ is the difference of total energy density and rest mass energy density (similar to kinetic energy), and magnetic field. Notice that magnetic field is in both the primitive and conserved variable list. Now we write the conserved variables in terms of the primitive variables:

$$\begin{aligned} D &= \rho_0 \\ S_i &= [h_eW^2+B^2]v_i-(Bv)B_i \\ \tau &= h_eW^2+B^2-P-\frac{1}{2}\left[(Bv)^2+\frac{B^2}{W^2}\right]-D \\ B_i &= B_i \end{aligned}$$

.

Finally we use the Arnowitt-Deser-Misner formalism, or ADM formalism, a Hamiltoniam formulation of general relativity which is important for numerical evolution of relativity. In this formalism we have α, the lapse, β_i, the shift vector field, the projector $h_{ab} = g_{ab}+n_an_b$ where $n_a = (-\alpha,0,0,0)$, and finally we define a convenient modified spacial velocity $\bar{v}^k = v^k - \frac{\beta^k}{\alpha}$. We now attach the baryon conservation equation to the regular matter equations from before:

$$\partial_t(\sqrt{h}D)+\partial_i\left[\sqrt{-g}D\left(v^i-\frac{\beta^i}{\alpha}\right)\right].$$

We define K_{ab} as the extrinsic energy tensor. Finally we may write the matter equations in a more useful way:

$$\begin{aligned}
\partial_t(\sqrt{h}D)+\partial_i\left[\sqrt{-g}D\bar{v}^i\right] &= 0\\
\partial_t(\sqrt{h}S_b)+\partial_i\left[\sqrt{-g}\left((\perp T)^i{}_b-\frac{\beta^i}{\alpha}S_b\right)\right] &= \sqrt{-g}\left[{}^3\Gamma^i{}_{ab}(\perp T)^a{}_i+\frac{1}{\alpha}S_a\partial_b\beta^a-\frac{1}{\alpha}\partial_b\alpha E\right]\\
\partial_t(\sqrt{h}\tau)+\partial_i\left[\sqrt{-g}\left(S^i-Dv^i-\frac{\beta^i}{\alpha}\tau\right)\right] &= \sqrt{-g}\left[(\perp T)^{ab}K_{ab}-\frac{1}{\alpha}S^a\partial_a\alpha\right]\\
\partial_t(\sqrt{h}B^b)+\partial_i\left[\sqrt{-g}\left(B^b\bar{v}^i-B^i\bar{v}^b\right)\right] &= 0\\
-\frac{1}{\sqrt{h}}\partial_i(\sqrt{h}B^i) &= 0
\end{aligned}$$

Now we have a complicated looking mess of equations. To obtain an even more useful arrangement, we first ignore all source terms (this is allowed because we are going to consider the characteristic problem) and write the matter equations in balance law form. We will also define two 8×1 vectors $\vec{F}^0$ and $\vec{F}^k$, where $\vec{F}^0$ contains the terms of our matter equations connected to time derivatives, and $\vec{F}^k$ contains the terms connected to spatial derivatives,

$$\vec{F}^0=\sqrt{h}\begin{pmatrix} D\\ S_b\\ \tau\\ B^b\end{pmatrix}\qquad \vec{F}^b=\sqrt{-g}\begin{pmatrix} D\bar{v}^i\\ (\perp T)^i{}_b-\frac{\beta^i}{\alpha}S_b\\ S^i-Dv^i-\frac{\beta^i}{\alpha}\tau\\ B^b\bar{v}^i-B^i\bar{v}^b\end{pmatrix}$$

and now we can write a much simpler form of the matter equations:

$$\partial_t\vec{F}^0+\partial_i\vec{F}^b=0.$$

Let us take a brief pause to write out one form of the quantity $(\perp T)_{ij}$:

$$(\perp T)_{ij}=v_iS_j+P\cdot h_{ij}-\frac{1}{W^2}\left[B_iB_j-\frac{1}{2}h_{ij}B^2\right]-(Bv)\left[B_iv_j\frac{1}{2}h_{ij}(Bv)\right]$$

Noting that $\sqrt{-g}=\alpha\sqrt{h}$, we trade out α, the lapse, for $\vec{v}^k$ in order to write $\vec{F}^0$ and $\vec{F}^k$ only in terms of the primitive variables. Changing the indices b and i to i and k respectively, we have

$$\vec{F}^0 = \sqrt{h}\begin{pmatrix} W\rho_0 \\ (h_e W^2 + B^2)v_i - (Bv)B_i \\ h_e W^2 + B^2 - \frac{1}{2}[(Bv)^2 + B^2/W^2] - W\rho_0 - P \\ B^i \end{pmatrix}$$

and

$$\vec{F}^k = \alpha\sqrt{h}\begin{pmatrix} W\rho_0 \bar{v}^k \\ ([h_e W^2 + B^2]v_i - (Bv)B_i)\bar{v}^k + h_i^k(P + \frac{1}{2}[(Bv)^2 + B^2/W^2]) - (B_i/W^2 + (Bv)v_i)B^k \\ (h_e W^2 + B^2 - \frac{1}{2}[(Bv)^2 + B^2/W^2] - W\rho_0 - P)\bar{v}^k + (P + \frac{1}{2}[(Bv)^2 + B^2/W^2])v^k - (Bv)B^k \\ B^i \bar{v}^k - B^k \bar{v}^i \end{pmatrix}.$$

At this point we are almost ready to examine the characteristic structure of this system. However, we must first introduce our constraint damping term and make the proper adjustments.

2.2 Constraint Damping

As discussed before, when similar GRMHD numerical systems have been evolved, small errors build over time and suggest the presence of magnetic monopoles. In analytical terms, it is easy to satisfy $\nabla \cdot \vec{B} \to 0$ and to never contradict it. However, in numerical simulations this equation becomes more of an approximation such as $\nabla \cdot \vec{B} \to$ (something very small). This would generally be okay in terms of satisfying Maxwell's equations, but in numerical evolution the very small element can build on itself over time and eventually become something not so small. This error not only is at variance with Maxwell's equations, but physically this would lead to modeling magnetic monopoles. It is therefore crucial to introduce a constraint damping term to damp the divergence of magnetic field back down to 0, or at least very close to 0.

We introduce a new field ϕ, which will satisfy the equation $\partial_t \phi + \nabla \cdot F(\phi, B) = -s\phi$. This equation, when solved, will result in something similar to an exponential decay of ϕ over time and also will damp $\nabla \cdot \vec{B}$ to 0 over time. This will take care of any errors that arise in numerical simulation. We now need to contribute this equation to our system by splitting the time derivative and the spatail derivative into F^0 and F^k, respectively. We end up with

$$\vec{F}^0 = \sqrt{h}\begin{pmatrix} W\rho_0 \\ (h_e W^2 + B^2)v_i - (Bv)B_i \\ h_e W^2 + B^2 - \frac{1}{2}[(Bv)^2 + B^2/W^2] - W\rho_0 - P \\ B^i \\ \phi \end{pmatrix}$$

and

$$\vec{F}^k = \alpha\sqrt{h}\begin{pmatrix} W\rho_0\bar{v}^k \\ ([h_e W^2 + B^2]v_i - (Bv)B_i)\bar{v}^k + h_i^{\,k}(P + \frac{1}{2}[(Bv)^2 + B^2/W^2]) - (B_i/W^2 + (Bv)v_i)B^k \\ (h_e W^2 + B^2 - \frac{1}{2}[(Bv)^2 + B^2/W^2] - W\rho_0 - P)\bar{v}^k + (P + \frac{1}{2}[(Bv)^2 + B^2/W^2])v^k - (Bv)B^k \\ B^i\bar{v}^k - B^k\bar{v}^i \\ F(\phi, B) \end{pmatrix}.$$

(Note to self: I don't think this is quite correct yet, need to get rid of the right term of phi's equation and determine how $\alpha\sqrt{h}$ or $\sqrt{h}$ are related)

2.3 The Characteristic Structure

We have written the matter equations in the form

$$\frac{\partial \vec{F}^0}{\partial t} + \frac{\partial \vec{F}^k}{\partial x^i} = 0.$$

In order to solve for the primitive variables, which will help us evolve our system numerically, we will perform a chain rule-type calculation on our system:

$$
\begin{aligned}
\frac{\partial \vec{F}^0}{\partial t} + \frac{\partial \vec{F}^k}{\partial x^i} &= 0 \\
\frac{\partial \vec{F}^0}{\partial \vec{p}} + \frac{\partial \vec{F}^k}{\partial \vec{p}}\frac{\partial \vec{p}}{\partial x^i} &= 0 \\
\mathscr{F}^0 \partial_t \vec{p} + \mathscr{F}^k \partial_i \vec{p} &= 0 \\
\partial_t \vec{p} + (\mathscr{F}^0)^{-1} \mathscr{F}^k \partial_i \vec{p} &= 0.
\end{aligned}
$$

We now have $\mathscr{F}^0$ and $\mathscr{F}^k$, which are matrix forms resulting from deriving our vectors F^0 and F^k by our vector $\vec{p}$, and $(\mathscr{F}^0)^{-1}$ is the inverse of $\mathscr{F}^0$. Writing them out fully:

$$
\frac{\mathscr{F}^0}{\sqrt{h}} = \begin{pmatrix}
W & W^3 \rho_0 v_j & 0 & 0_j & 0 \\
W^2 \gamma v_i & h_{ij} Q + 2 h_e W^4 v_i v_j - B_i B_j & (\rho_0 + \kappa) W^2 v_i & 2 v_i B_j - B_i v_j - (Bv) h_{ij} & 0^i \\
\gamma W^2 - W - \chi & (2 h_e W^4 + B^2 - W^3 \rho_0) v_j - (Bv) B_j & (\rho_0 + \kappa) W^2 - \kappa & (2 - W^{-2}) B_j - (Bv) B_j - (Bv) v_j & 0 \\
0^i & {0^i}_j & 0^i & {h^i}_j & 0^i \\
0 & 0_j & 0 & 0_j & 1
\end{pmatrix}
$$

$$
\frac{\mathscr{F}^k}{\alpha \sqrt{h}} = \begin{pmatrix}
W \bar{v}^k & W \rho_0 (W^2 v_j \bar{v}^k + h_j^{\ k}) & 0^k & 0_j^{\ k} & 0^k \\
W^2 \gamma v_i \bar{v}^k + h_i^{\ k} \chi & {\tilde{A}^k}_{\ ij} & (\rho_0 + \kappa) W^2 v_i \bar{v}^k + h_i^{\ k} \kappa & {\tilde{B}^k}_{\ ij} & 0^{ik} \\
(W^2 \gamma - W - \chi) \bar{v}^k & {\tilde{C}^k}_{\ j} & (\rho_0 + \kappa) W^2 \bar{v}^k - \kappa (\bar{v}^k - v^k) & {\tilde{D}^k}_{\ j} & 0^k \\
0^{ik} & B^i h_j^{\ k} - B^k h_j^{\ i} & 0^{ik} & {h^i}_j \bar{v}^k - {h^k}_j \bar{v}^i - h_j^{\ k} \eta \frac{\beta^i}{\alpha} & h^{ik} \\
0^k & {0^k}_j & 0^k & {h^k}_j & -\frac{\beta^k}{\alpha}
\end{pmatrix}
$$

where

$$
\begin{aligned}
\tilde{A}^{k}{}_{ij} &= (h_{ij}Q + 2h_e W^4 v_i v_j - B_i B_j)(\vec{v}^k - \lambda^k) + (Qv_i - (Bv)B_i)\, h_j{}^k + h_i^k \left((Bv)B_j - B^2 v_j\right) - \left(h_{ij}(Bv) + v_i B_j - 2B_i v_j\right) B^k \\
\tilde{B}^{k}{}_{ij} &= (2v_i B_j - B_i v_j - (Bv)h_{ij})(\vec{v}^k - \lambda^k) - \left(\frac{h_{ij}}{W^2} + v_i v_j\right) B^k + h_i^k \left((Bv)v_j + \frac{B_j}{W^2}\right) - h_j{}^k \left(\frac{B_i}{W^2} + (Bv)v_i\right) \\
\tilde{C}^{k}{}_{j} &= (2h_e W^4 + B^2 - \rho_0 W^3)v_j(\vec{v}^k - \lambda^k) - (Bv)B_j(\vec{v}^k - \lambda^k) + (Q - W\rho_0)h_j{}^k + \left((Bv)B_j - B^2 v_j\right) v^k - B_j B^k \\
\tilde{D}^{k}{}_{j} &= \left(2B_j - (Bv)v_j - \frac{B_j}{W^2}\right)(\vec{v}^k - \lambda^k) + \left((Bv)v_j + \frac{B_j}{W^2}\right) v^k - v_j B^k - (Bv)h_j{}^k
\end{aligned}
$$

and

$$
\begin{aligned}
\chi &= \frac{\partial P}{\partial \rho_0} \\
\kappa &= \frac{\partial P}{\partial \varepsilon} \\
\gamma &= \frac{\partial h_e}{\partial \rho_0} = 1 + \varepsilon + \chi \\
h_e {c_s}^2 &= \rho_0 \chi + \frac{P}{\rho_0}\kappa.
\end{aligned}
$$

As we can see, these matrices are quite complicated. Examining the quantity $(\mathscr{F}^0)^{-1}\mathscr{F}^k$ more closely, if we diagonalize this matrix by finding the eigenvalues and eigenvectors, we will have the characteristic wavespeeds and modes of the system. We will first define the components of the right eigenvectors as $\vec{e} = (e^0, e^j, e^4, \hat{e}^j, e^8)$, and the components of the eigenvalues as λ^k. We wish to solve

$$
\begin{aligned}
(\mathscr{F}^0)^{-1}\mathscr{F}^k\vec{e} &= \lambda^k\vec{e} \\
\mathscr{F}^k\vec{e} &= \lambda^k\mathscr{F}^0\vec{e} \\
(\mathscr{F}^k - \lambda^k\mathscr{F}^0)\vec{e} &= 0.
\end{aligned}
$$

So, rather than finding the inverse of $\mathscr{F}^0$, we can instead do a linear combination between $\mathscr{F}^0$ and $\mathscr{F}^k$ as shown above.

2.4 Finding the determinant

To solve for the eigenvectors and eigenvalues of $(\mathscr{F}^0)^{-1}\mathscr{F}^k$, we must obtain the characteristic equation. The first step to do this is a very long and complicated determinant, which will take up this section.

We first make two very important definitions:

$$
\begin{aligned}
a^k &= \bar{v}^k - \lambda^k \\
\Delta^{kk} &= (a^k)^2 Q - 2(Bv)a^k B^k - \frac{1}{W^2} B^k B^k
\end{aligned}
$$

where a^k is directly related to the eigenvalues and Δ^{kk} will become useful later on. We now wish to solve the determinant

$$\det|\mathscr{F}^k - \lambda^k \mathscr{F}^0| = (\alpha\sqrt{h})^9$$

which, after dropping the $(\alpha\sqrt{h})^9$ for now, is written out:

$$
\begin{vmatrix}
W(\bar{v}^k-\lambda^k) & W^3\rho_0 v_j(\bar{v}^k-\lambda^k)+W\rho_0 h_j{}^k & 0^k & 0_j{}^k & 0^k \\
W^2\gamma v_i(\bar{v}^k-\lambda^k)+h_i{}^k\chi & \tilde{A}^k{}_{ij} & (\rho_0+\kappa)W^2 v_i(\bar{v}^k-\lambda^k)+h_i{}^k\kappa & \tilde{B}^k{}_{ij} & 0^{ik} \\
(W^2\gamma-W-\chi)(\bar{v}^k-\lambda^k)+\chi v^k & \tilde{C}^k{}_j & [(\rho_0+\kappa)W^2-\kappa](\bar{v}^k-\lambda^k)+\kappa v^k & \tilde{D}^k{}_j & 0^k \\
0^{ik} & B^i h_j{}^k - B^k h_j{}^i & 0^{ik} & h^i{}_j(\bar{v}^k-\lambda^k)-h^k{}_j\bar{v}^i-h_j{}^k\eta\frac{\beta^i}{\alpha} & h^{ik} \\
0^k & 0^k{}_j & 0^k & h_j{}^k & -\frac{\beta^k}{\alpha}-\lambda^k
\end{vmatrix}
$$

What follows is a long and tedious but necessary process. To simplify our lives just a bit, we make the definition

$$\Psi^i = \eta\frac{\beta^i}{\alpha}$$

where Ψ^i can be thought of as a constraint damping term. We also recognize the relationship

$$
\begin{aligned}
\frac{\beta^k}{\alpha} &= v^k - \bar{v}^k \\
-\frac{\beta^k}{\alpha} &= -v^k + \bar{v}^k \\
-\frac{\beta^k}{\alpha} - \lambda^k &= -v^k + \bar{v}^k - \lambda^k \\
-\frac{\beta^k}{\alpha} - \lambda^k &= -v^k + a^k
\end{aligned}
$$

Replacing the correct terms with Ψ^i and a^k, we now have

$$\begin{vmatrix}
Wa^k & W^3\rho_0 v_j a^k + W\rho_0 h_j^{\ k} & 0^k & 0_j^{\ k} & 0^k \\
W^2\gamma v_i a^k + h_i^{\ k}\chi & \tilde{A}^k{}_{ij} & (\rho_0+\kappa)W^2 v_i a^k + h_i^{\ k}\kappa & \tilde{B}^k{}_{ij} & 0^{ik} \\
(W^2\gamma - W - \chi)a^k + \chi v^k & \tilde{C}^k{}_j & [(\rho_0+\kappa)W^2 - \kappa]a^k + \kappa v^k & \tilde{D}^k{}_j & 0^k \\
0^{ik} & B^i h_j^{\ k} - B^k h_j^{\ i} & 0^{ik} & h^i{}_j a^k - h^k{}_j \bar{v}^i - h_j^{\ k}\Psi^i & h^{ik} \\
0^k & 0^k{}_j & 0^k & h_j^{\ k} & a^k - v^k
\end{vmatrix}$$

Now, in order to simplify this determinant to a manageable level, we will perform a number of row and column operations where we refer to rows and columns one through five (which is represented) rather than one through nine (which we know is what we actually have). We will also use quotation marks to represent an entry that has been unchanged since last time.

Add row 1 to row 3:

$$\begin{vmatrix}
'' & '' & '' & '' & '' \\
'' & '' & '' & '' & '' \\
(W^2\gamma - \chi)a^k + \chi v^k & \tilde{C}^k{}_j + W^3\rho_0 v_j a^k + W\rho_0 h_j^{\ k} & '' & '' & '' \\
'' & '' & '' & '' & '' \\
'' & '' & '' & '' & ''
\end{vmatrix}$$

Add $-v_i$ times row 3 to row 2:

$$\begin{vmatrix}
'' & '' & '' & '' & '' \\
\chi[h_i^{\ k} + (a^k - v^k)v_i] & \alpha^k{}_{ij} & \kappa[h_i^{\ k} + (a^k - v^k)v_i] & \tilde{B}^k{}_{ij} - v_i\tilde{D}^k{}_j & '' \\
'' & '' & '' & '' & '' \\
'' & '' & '' & '' & '' \\
'' & '' & '' & '' & ''
\end{vmatrix}$$

where $\alpha^k{}_{ij} = \tilde{A}^k{}_{ij} - (\tilde{C}^k{}_j + W^3\rho_0 v_j a^k + W\rho_0 h_j^{\ k})v_i$.

Add $-\frac{\chi}{\kappa}$ times column 3 to column 1:

$$
\begin{vmatrix}
'' & '' & '' & '' & '' \\
0_i^k & '' & '' & '' & '' \\
W^2 a^k[\gamma-\chi(1+\frac{\rho_0}{\kappa})] & '' & '' & '' & '' \\
0^{ik} & '' & '' & '' & '' \\
'' & '' & '' & '' & ''
\end{vmatrix}
$$

Add $-W[\gamma-\chi(1+\frac{\rho_0}{\kappa})]$ times row 1 to row 3:

$$
\begin{vmatrix}
Wa^k & '' & '' & '' & '' \\
0_i^k & '' & '' & '' & '' \\
0^k & \xi_j{}^k & '' & '' & '' \\
0^{ik} & '' & '' & '' & '' \\
'' & '' & '' & '' & ''
\end{vmatrix}
$$

where $\xi_j{}^k = \tilde{C}^k{}_j + (W^3\rho_0 v_j a^k + W\rho_0 h_j{}^k)\left[1-W\left(\gamma-\chi(1+\frac{\rho_0}{\kappa})\right)\right]$.

At this point we will simplify $\alpha^k{}_{ij}$ and $\xi_j{}^k$. Using the thermodynamic relations, we can get $\gamma-\chi(1+\frac{\rho_0}{\kappa} = \frac{h_e}{\rho_0\kappa}(\kappa-\rho_0 c_s^2)$, and after simplifying we get

$$
\begin{aligned}
\alpha^k{}_{ij} &= (h_{ij}Q - B_iB_j)a^k - (Bv)B_ih_j{}^k - (h_{ij}(Bv) - 2B_iv_j)B^k + ((Bv)B_j - B^2v_j)(h_i{}^k + v_i(a^k - v^k)) \\
\xi_j{}^k &= (2h_eW^4 + B^2)v_ja^k - (Bv)B_ja^k + Qh_j{}^k + ((Bv)B_j - B^2v_j)v^k - B_jB^k - W^2(W^2v_ja^k + h_j{}^k)h_e(1-\frac{\rho_0}{\kappa}c_s^2)
\end{aligned}
$$

Moving on with matrix operations, add $-((Bv)B_j - B^2v_j)/\kappa$ times column 3 to column 2:

$$
\begin{vmatrix}
Wa^k & W^3\rho_0 v_j a^k + W\rho_0 h_j{}^k & '' & '' & '' \\
0_i^k & \tilde{\alpha}^k{}_{ij} & '' & '' & '' \\
0^k & \tilde{\xi}_j{}^k & '' & '' & '' \\
0^{ik} & B^ih_j{}^k - B^kh_j{}^i & '' & '' & '' \\
'' & '' & '' & '' & ''
\end{vmatrix}
$$

where

$$\tilde{\alpha}^k{}_{ij} = (h_{ij}q - B_iB_j)a^k - (Bv)B_ih_j{}^k - (h_{ij}(Bv) - 2B_iv_j)B^k$$
$$\tilde{\xi}_j{}^k = (2h_eW^4 + B^2)v_ja^k + Qh_j{}^k - B_jB^k - W^2(W^2v_ja^k + h_j{}^kh_e(1 - \frac{\rho_0}{\kappa}c_s^2) - W^2a^k(1 + \frac{\rho_0}{\kappa})((Bv)B_j - B^2v_j)$$

Now add $-((Bv)v_j + B_j/W^2)/\kappa$ times column 3 to column 4:

$$\begin{vmatrix} Wa^k & '' & 0^k & 0_j{}^k & '' \\ 0_i^k & \tilde{\alpha}^k{}_{ij} & '' & \beta^k{}_{ij} & '' \\ 0^k & \tilde{\xi}_j{}^k & '' & \delta^k{}_j & '' \\ 0^{ik} & '' & 0^{ik} & h_j{}^ia^k - h_j{}^k\bar{v}^k - h_j{}^k\Psi^i & '' \\ '' & '' & '' & '' & '' \end{vmatrix}$$

where

$$\beta^k{}_{ij} = -(B_iv_j + (Bv)h_{ij})a^k - (h_{ij}B^k + h_j{}^kB_i)\frac{1}{W^2}$$
$$\delta^k{}_j = 2B_ja^k - v_jB^k - (Bv)h_j{}^k - W^2a^k((Bv)v_j + B_j\frac{1}{W^2})(1 + \frac{\rho_0}{\kappa})$$

Writing out the full form of the determinant:

$$\begin{vmatrix} Wa^k & W^3\rho_0v_ja^k + W\rho_0h_j{}^k & 0^k & 0_j{}^k & 0^k \\ 0_i^k & \tilde{\alpha}^k{}_{ij} & \kappa[h_i{}^k + v_i(a^k - v^k)] & \beta^k{}_{ij} & 0^{ik} \\ 0^k & \tilde{\xi}_j{}^k & [(\rho_0 + \kappa)W^2 - \kappa]a^k + \kappa v^k & \delta^k{}_j & 0^k \\ 0^{ik} & B^ih_j{}^k - B^kh_j{}^i & 0^{ik} & h^i{}_ja^k - h^k{}_j\bar{v}^i - h_j{}^k\Psi^i & h^{ik} \\ 0^k & 0^k{}_j & 0^k & h_j{}^k & a^k - v^k \end{vmatrix}$$

Notice that we have a column of zeros below the first term Wa^k, which will help us simplify the determinant greatly. We now wish to get rid of either the entry in row 4, column 5, or the entry in row 5, column 4 so that we can simplify our determinant even further. Subtracting row 5 times $h^{ik}/(a^k - v^k)$ from row 4 gives

$$\begin{vmatrix} Wa^k & W^3\rho_0 v_j a^k + W\rho_0 h_j^{\ k} & 0^k & 0_j^{\ k} & 0^k \\ 0_i^k & \tilde{\alpha}^k{}_{ij} & \kappa[h_i^{\ k} + v_i(a^k - v^k)] & \beta^k{}_{ij} & 0^{ik} \\ 0^k & \tilde{\xi}_j^{\ k} & [(\rho_0+\kappa)W^2 - \kappa]a^k + \kappa v^k & \delta^k{}_j & 0^k \\ 0^{ik} & B^i h_j^{\ k} - B^k h_j^{\ i} & 0^{ik} & h^i{}_j a^k - h^k{}_j \bar{v}^i - h_j^{\ k}\Psi^i - \frac{h_j^{\ k} h^{ik}}{a^k - v^k} & 0^{ik} \\ 0^k & 0^k{}_j & 0^k & h_j^{\ k} & a^k - v^k \end{vmatrix}$$

Using the fact that the last column is all zeros except the last value, and defining the value $f^i = \bar{v}^i + \Psi^i + \frac{h^{ik}}{a^k - v^k}$, we can simplify to

$$(a^k - v^k)\begin{vmatrix} Wa^k & W^3\rho_0 v_j a^k + W\rho_0 h_j^{\ k} & 0^k & 0_j^{\ k} \\ 0_i^k & \tilde{\alpha}^k{}_{ij} & \kappa[h_i^{\ k} + v_i(a^k - v^k)] & \beta^k{}_{ij} \\ 0^k & \tilde{\xi}_j^{\ k} & [(\rho_0+\kappa)W^2 - \kappa]a^k + \kappa v^k & \delta^k{}_j \\ 0^{ik} & B^i h_j^{\ k} - B^k h_j^{\ i} & 0^{ik} & h_j^{\ i} a^k - h_j^{\ k} f^i \end{vmatrix}$$

Now add B^k times column 4 to a^k times column 2, and remembering that column 2 is actually three columns, pull out three factors of $1/a^k$:

$$(a^k - v^k)(\frac{1}{a^k})^3\begin{vmatrix} Wa^k & (W^3\rho_0 v_j a^k + W\rho_0 h_j^{\ k})a^k & 0^k & 0_j^{\ k} \\ 0_i^k & a^k\tilde{\alpha}^k{}_{ij} + B^k\beta^k{}_{ij} & \kappa[h_i^{\ k} + v_i(a^k - v^k)] & \beta^k{}_{ij} \\ 0^k & a^k\tilde{\xi}_j^{\ k} + B^k\delta^k{}_j & [(\rho_0+\kappa)W^2 - \kappa]a^k + \kappa v^k & \delta^k{}_j \\ 0^{ik} & B^i h_j^{\ k} a^k - B^k h_j^{\ k}\bar{v}^i & 0^{ik} & h_j^{\ i} a^k - h_j^{\ k} f^i \end{vmatrix}$$

Now subtract column 4 contracted with B^j from column 2:

$$(a^k - v^k)(\frac{1}{a^k})^3\begin{vmatrix} Wa^k & (W^3\rho_0 v_j a^k + W\rho_0 h_j^{\ k})a^k & 0^k & 0_j^{\ k} \\ 0_i^k & a^k\tilde{\alpha}^k{}_{ij} + B^k\beta^k{}_{ij} - B^l\beta^k{}_{il} h_j^{\ k} & \kappa[h_i^{\ k} + v_i(a^k - v^k)] & \beta^k{}_{ij} \\ 0^k & a^k\tilde{\xi}_j^{\ k} + B^k\delta^k{}_j - B^l\delta^k{}_l h_j^{\ k} & [(\rho_0+\kappa)W^2 - \kappa]a^k + \kappa v^k & \delta^k{}_j \\ 0^{ik} & 0_j^{\ ik} & 0^{ik} & h_j^{\ i} a^k - h_j^{\ k} f^i \end{vmatrix}$$

The bottom righmost term $h_j^{\ i}a^k - h_j^{\ k} f^i$ is equivalent to the 3 x 3 block

$$\begin{array}{ccc} a^k & 0 & -f^1 \\ 0 & a^k & -f^2 \\ 0 & 0 & a^k - f^k \end{array}$$

and all terms to the left of this block are 0. So we can simplify further by taking out a factor of $(a^k)^2(a^k - f^k)$:

$$(a^k - v^k)(\frac{1}{a^k})^3(a^k - f^k)(a^k)^2 \begin{vmatrix} Wa^k & (W^3\rho_0 v_j a^k + W\rho_0 h_j^{\ k})a^k & 0^k \\ 0_i^{\ k} & a^k \tilde{\alpha}^k_{\ ij} + B^k \beta^k_{\ ij} - B^l \beta^k_{\ il} h_j^{\ k} & \kappa[h_i^{\ k} + v_i(a^k - v^k)] \\ 0^k & a^k \tilde{\xi}_j^{\ k} + B^k \delta^k_{\ j} - B^l \delta^k_{\ l} h_j^{\ k} & [(\rho_0 + \kappa)W^2 - \kappa]a^k + \kappa v^k \end{vmatrix}$$

Finally, we can take out a factor of Wa^k and our determinant simplifies to

$$(a^k - v^k)(\frac{1}{a^k})^3(a^k - f^k)(a^k)^2 Wa^k \begin{vmatrix} a^k \tilde{\alpha}^k_{\ ij} + B^k \beta^k_{\ ij} - B^l \beta^k_{\ il} h_j^{\ k} & \kappa[h_i^{\ k} + v_i(a^k - v^k)] \\ a^k \tilde{\xi}_j^{\ k} + B^k \delta^k_{\ j} - B^l \delta^k_{\ l} h_j^{\ k} & [(\rho_0 + \kappa)W^2 - \kappa]a^k + \kappa v^k \end{vmatrix}$$

We can now compute the determinant, which after much simplification becomes

$$-\alpha^8 h^5 \rho_0 h_e W^3 (a^k - v^k)(a^k - f^k)a^k \cdot$$
$$\Delta^{kk} \left\{ h_e W^4 (a^k)^4 (1 - c_s^2 + [(a^k)^2(h_e W^2 c_s^2 + B^2 + W^2(Bv)^2 - c_s^2(a^k W(Bv) + \frac{B^k}{W})^2] \cdot [(a^k - v^k)^2 - h^{kk}] \right\}$$

This, set equal to 0, is the characteristic equation we need in order to find the characteristic wavespeeds and waveforms.

Chapter 3

Results

We will now begin solving the eigenvalues and eigenvectors of our characteristic equation, which correspond to the wavespeeds and wave modes, respectively. The wavespeeds correspond to certain waves, namely the Alfvén waves, fast magnetosonic waves, slow magnetosonic waves, and entropy waves.

In the context of magnetohydrodynamics, Alfven waves are waves that propagate along magnetic field lines in a plasma. Alfven waves can play a role in the dynamics of astrophysical systems, such as accretion disks around black holes and the interstellar medium. Fast magnetosonic waves, also known as fast magnetoacoustic waves or simply fast waves, are another type of wave that can propagate in a magnetized plasma. They are a combination of magnetic and acoustic waves, and can travel at speeds faster than the Alfven speed. Slow magnetosonic waves are another type of wave that can propagate in a magnetized plasma. They are also a combination of magnetic and acoustic waves, but travel at speeds slower than the Alfven speed. Entropy waves are waves that are associated with changes in the entropy of a plasma. They are typically found in plasmas that are not in thermal equilibrium and can play a role in the heating and cooling of plasmas.

We will now proceed to set up and solve the equations resulting from our characteristic structure.

3.1 Initial Equations

Remember that we mean to solve the eigenvectors $\vec{e} = (e^0, e^j, e^4, \hat{e}^j, e^8)^T$ through solving the equation

$$\mathscr{F}^k \vec{e} = \lambda^k \mathscr{F}^0 \vec{e}$$

which we will rearrange into

$$(\mathscr{F}^k - \lambda^k \mathscr{F}^0)\vec{e} = 0$$

Now, if we simply multiply our previously calculated $\mathscr{F}^k - \lambda^k \mathscr{F}^0$ by the eigenvalues $\vec{e} = (e^0, e^j, e^4, \hat{e}^j, e^8)^T$, we obtain the following equations.

$$0 = e^0 W a^k + (ve) W^3 \rho_0 a^k + e^k W \rho_0 \tag{3.1}$$

$$\begin{aligned} 0 = e^0 \big[W^2 \gamma v_i a^k + \chi h_i{}^k\big] &+ \Big\{ e^i \big[Q a^k - (Bv) B^k\big] + (ve)\big[2 h_e W^4 v_i a^k - h_i{}^k B^2 + 2 B_i B^k\big] \\ &+ (Be)\big[-B_i a^k + h_i{}^k (Bv) - v_i B^k\big] + e^k \big[Q v_i - (Bv) B_i\big] \Big\} \\ &+ e^4 \big[(\rho_0 + \kappa) W^2 v_i a^k + h_i{}^k \kappa\big] \\ &+ \Big\{ \hat{e}_i \big[-(Bv) a^k - \frac{1}{W^2} B^k\big] + (v\bar{e})\big[-B_i a^k - v_i B^k + h_i{}^k (Bv)\big] \\ &+ (B\hat{e})\big[2 v_i a^k + \frac{1}{W^2} h_i{}^k\big] - \hat{e}^k \big[(Bv) v_i + \frac{1}{W^2} B_i\big] \Big\} \end{aligned} \tag{3.2}$$

$$\begin{aligned} 0 = e^0 &\big[(W^2 \gamma - W - \chi) a^k + \chi v^k\big] \\ &+ \Big\{ (ve)\big[(2 h_e W^4 + B^2 - W^3 \rho_0) a^k - B^2 v^k\big] - (Be)\big[(Bv)(a^k - v^k) + B^k\big] + e^k (Q - W \rho_0) \Big\} \\ &+ e^4 \big[(\rho_0 + \kappa) W^2 a^k + \kappa (v^k - a^k)\big] \\ &+ \Big\{ -(v\hat{e})\big[(Bv)(a^k - v^k) + B^k\big] + (B\hat{e})\big[(2 - \frac{1}{W^2}) a^k + \frac{1}{W^2} v^k\big] - \hat{e}^k (Bv) \Big\} \end{aligned} \tag{3.3}$$

$$0 = B^i e^k - B^k e^i + \hat{e}^i a^k - \hat{e}^k \bar{v}^i - \hat{e}^k \psi^i + e^i h^{ik} \tag{3.4}$$

$$0 = \hat{e}^k + e^8 (a^k - v^k) \tag{3.5}$$

We can combine equations 3.4 and 3.5 to get the equation

$$0 = B^i e^k - B^k e^i + \hat{e}^i a^k - \hat{e}^k f^i \tag{3.6}$$

where f^i has been previously defined, and this relation will come in handy. If we simplify by letting $i = k$, then 3.6 becomes

$$\hat{e}^k \left[a^k - f^k \right] = 0 \tag{3.7}$$

This equation can only be true generally if $\hat{e}^k = 0$, and according to equation 3.5 this also implies that $e^8 = 0$. This means that the right eigenvectors simplify to the non-constrained eigenvectors found by Ryan Hatch [6] for the entropy, Alfvén, and magnetosonic waves. This reduces the extent of our agenda greatly, and now we can focus on finding the two new eigenvectors that result from adding constraint damping.

3.2 The New Eigenvectors

We will now focus on the final eigenvectors. To set this up, we recognize that the final two eigenvectors result from a new term in our characteristic equation, and letting this term be equal to 0 gives us

$$(a^k - v^k)(a^k - f^k) = 0,$$

which we can then expand into

$$(a^k)^2 - a^k \left(2v^k + \frac{\beta^k}{\alpha}(\eta - 1) \right) + \left((v^k)^2 + \frac{\beta^k}{\alpha}(\eta - 1)v^k - h^{kk} \right) = 0. \tag{3.8}$$

We will now use the quadratic equation to solve for a^k, which gives two values for the wave speeds:

$$a^k_{\pm} = \left(v^k + \frac{\beta^k}{2\alpha}(\eta - 1) \right) \pm \frac{\sqrt{\left(2v^k + \frac{\beta^k}{\alpha}(\eta - 1) \right)^2 - 4\left((v^k)^2 + \frac{\beta^k}{\alpha}(\eta - 1)v^k - h^{kk} \right)}}{2} \tag{3.9}$$

To solve for these two wavespeeds in their symbolic form would not be of much use to us, so we will rather solve for the eigenvectors generally with constraints and keep this form as is.

We are now ready to take a look at the five original equations and perform some helpful manipulations.

3.2.1 Preparations

We will first add Equation 3.1 to 3.3, resulting in

$$
\begin{aligned}
0 = {} & e^0\big[(W^2\gamma-\chi)a^k+\chi v^k\big] \\
& +\Big\{(ve)\big[(2h_eW^4+B^2)a^k-B^2v^k\big]-(Be)\big[(Bv)(a^k-v^k)+B^k\big]+e^kQ\Big\} \\
& +e^4\big[(\rho_0+\kappa)W^2a^k+\kappa(v^k-a^k)\big] \\
& +\Big\{-(v\hat{e})\big[(Bv)(a^k-v^k)+B^k\big]+(B\hat{e})\big[(2-\frac{1}{W^2})a^k+\frac{1}{W^2}v^k\big]-\hat{e}^k(Bv)\Big\}.
\end{aligned}
\tag{3.10}
$$

Next, we multiply $-v_i$ by 3.10 and add this to 3.2, which results in

$$
\begin{aligned}
0 = {} & e^0\big[-\chi((v^k-a^k)v_i-h_i^{\,k})\big] \\
& +\Big\{e^i\big[Qa^k-(Bv)B^k\big]+(ve)\big[2B_iB^k+B^2((v^k-a^k)v_i-h_i^{\,k})\big] \\
& +(Be)\big[-B_ia^k+(Bv)((a^k-v^k)v_i+h_i^{\,k}\big]-e^k(Bv)B_i\Big\} \\
& +e^4\big[-\kappa((v^k-a^k)v_i-h_i^{\,k})\big] \\
& +\Big\{\hat{e}_i\big[-(Bv)a^k-\frac{1}{W^2}B^k\big]+(v\hat{e})\big[-(Bv)((v^k-a^k)v_i-h_i^{\,k})-B_ia^k\big] \\
& +(B\hat{e})\big[-\frac{1}{W^2}((v^k-a^k)v_i-h_i^{\,k})\big]-\hat{e}^k\frac{B_i}{W^2}\Big\}.
\end{aligned}
\tag{3.11}
$$

Now contract 3.6 with B_i and then v_i separately to get two equations:

$$0 = B^2e^k-B^k(Be)+(B\hat{e})a^k-\hat{e}^k(Bf) \tag{3.12}$$

$$0 = (Bv)e^k-B^k(ve)+(v\hat{e})a^k-\hat{e}^k(vf) \tag{3.13}$$

We can now use equation 3.13 to simplify equation 3.11, which results in

$$
\begin{aligned}
0 = {} & (e^0\chi+e^4\kappa)((a^k-v^k)v_i+h_i^{\,k}) \\
& +\Big\{e_i\big[(h_eW^2+B^2)a^k-(Bv)B^k\big]+(ve)\big[B_iB^k-B^2((a^k-v^k)v_i+h_i^{\,k})\big] \\
& +(Be)\big[-B_ia^k+(Bv)((a^k-v^k)v_i+h_i^{\,k})\big]\Big\} \\
& +\Big\{\hat{e}_i\big[-(Bv)a^k-\frac{B^k}{W^2}\big]+(v\hat{e})\big[(Bv)((a^k-v^k)v_i+h_i^{\,k})\big]+(B\hat{e})\frac{1}{W^2}\big[(a^k-v^k)v_i+h_i^{\,k}\big]\Big\} \\
& -B_i\hat{e}^k(vf+\frac{1}{W^2}).
\end{aligned}
\tag{3.14}
$$

Similarly, we can use equation 3.12 to simplify equation 3.10, resulting in

$$
\begin{aligned}
0 = e^0\big[(W^2\gamma-\chi)a^k+\chi v^k\big]+(ve)\big[(2h_eW^4+B^2)a^k-B^2v^k\big]+(Be)\big[(Bv)(v^k-a^k)\big] \\
+e^kh_eW^2+e^4\big[(\rho_0+\kappa)W^2a^k+\kappa(v^k-a^k)\big]+(v\hat{e})\big[(Bv)(v^k-a^k)-B^k\big] \\
+(B\hat{e})\frac{1}{W^2}(v^k-a^k+W^2a^k)+\hat{e}^k\big[(Bf)-(Bv)\big].
\end{aligned}
\tag{3.15}
$$

Our next step is to obtain two separate equations by contracting equation 3.14 with v^i and multiplying it by W^2, and by contracting the same equation 3.14 with B^i. This results in the following equations:

$$
\begin{aligned}
0 = (e^0\chi+e^4\kappa)(v^k-a^k+W^2a^k)+(ve)\big[h_eW^4a^k-B^2(v^k-a^k)\big]+(Be)\big[(Bv)(v^k-a^k)\big] \\
+(v\hat{e})\big[(Bv)(v^k-a^k)-B^k\big]+(B\hat{e})\frac{1}{W^2}\big[v^k-a^k+W^2a^k\big]+\hat{e}^k(Bv)\big[W^2(vf)+1\big]
\end{aligned}
\tag{3.16}
$$

$$
\begin{aligned}
0 = (e^0\chi+e^4\kappa)\big[B^k-(Bv)(v^k-a^k)\big]-(B\hat{e})\frac{(Bv)}{W^2}\big[v^k-a^k+W^2a^k\big] \\
+(ve)\big[B^2(Bv)(B^k-a^k)\big]+(v\hat{e})\big[(Bv)(B^k-(Bv)(v^k-a^k))\big] \\
+(Be)\big[h_eW^2a^k-(Bv)^2(v^k-a^k)\big]-B^2\hat{e}^k\big[(vf)+\frac{1}{W^2}\big].
\end{aligned}
\tag{3.17}
$$

We simplify these expressions by adding (Bv) times 3.16 to 3.15, which gives us

$$
\begin{aligned}
0 = (e^0\chi+e^4\kappa)\big(B^k+W^2a^k(Bv)\big)+(ve)h_eW^4a^k(Bv) \\
+(Be)h_eW^2a^k+\hat{e}^k\big[B^2[(Bf)-(Bv)]+(Bv)^2[W^2(vf)+1]\big].
\end{aligned}
\tag{3.18}
$$

We now remember the thermodynamic relations $h_e\rho_0c_s^2=\rho_0^2\chi+P\kappa$ and $\gamma=1+\varepsilon+\chi$, which we can use to manipulate our equations 3.15 and 3.16. We subtract equations 3.16 from 3.15 with these definitions plugged in to get the useful relation

$$
e^4 = e^0\frac{P}{\rho_0^2}-\frac{\hat{e}^k}{\rho_0W^2a^k}\big[(Bf)-(Bv)(2+(vf)W^2)\big].
\tag{3.19}
$$

Now we will use the thermodynamic relation $h_e=\rho_0(1+\varepsilon)+P$ and use equation 3.19 to simplify equation 3.18 to get

$$
0 = e^0\frac{h_ec_s^2}{\rho_0}\left(B^k+W^2a^k(Bv)\right)+(ve)h_eW^4a^k(Bv)+(Be)h_eW^2a^k+\hat{e}^k\big[Z'\big],
\tag{3.20}
$$

where

$$
Z=(Bf)-(Bv)\big(2+(vf)W^2\big)
$$

and

$$Z' = B^2[(Bf) - (Bv)] + (Bv)^2\left[W^2(vf) + 1\right] - \frac{\kappa Z}{\rho_0 W^2 a^k}.$$

We do a similar thing to equation 3.16, and then use equations 3.12 and 3.13 to eliminate $(v\hat{e})$ and $(B\hat{e})$, which gives us

$$\begin{aligned} 0 = e^0 \frac{h_e c_s^2}{\rho_0}(v^k - a^k + W^2 a^k) + (ve)(h_e W^4 a^k - C^k) + (Be)D^k - e^k \frac{E^k}{a^k} \\ + \hat{e}^k \Bigg[(Bv)W^2(vf) + (Bv) + \frac{1}{W^2 a^k}\left(\frac{-Z\kappa}{\rho_0} + (Bf)\right)\left[v^k - a^k + W^2 a^k\right] \\ + \frac{(vf)}{a^k}\left[(Bv)(v^k - a^k) - B^k\right]\Bigg] \end{aligned}$$

where

$$\begin{aligned} C^k &= (v^k - a^k)\left[B^2 - (Bv)\frac{B^k}{a^k}\right] + \frac{(B^k)^2}{a^k} \\ D^k &= (v^k - a^k)\left[(Bv) + \frac{B^k}{W^2 a^k}\right] + B^k \\ E^k &= (v^k - a^k)\left[\frac{B^2}{W^2} + (Bv)^2\right] + B^2 a^k - (Bv)B^k \\ \tilde{Z} &= (Bv)W^2(vf) + (Bv) \\ &\quad + \frac{1}{W^2 a^k}\left(\frac{-Z\kappa}{\rho_0} + (Bf)\right)\left[v^k - a^k + W^2 a^k\right] + \frac{(vf)}{a^k}\left[(Bv)(v^k - a^k) - B^k\right]. \end{aligned}$$

Now we can get rid of the term e^k by adding equation 3.1 multiplied by $\frac{E^k}{\rho_0 a^k}$, which gives us

$$\begin{aligned} 0 = e^0 \frac{h_e c_s^2}{\rho_0}(v^k - a^k + W^2 a^k) + (ve)(h_e W^4 a^k - C^k) + (Be)D^k - e^k \frac{E^k}{a^k} \\ + \hat{e}^k \Bigg[(Bv)W^2(vf) + (Bv) + \frac{1}{W^2 a^k}\left(\frac{-Z\kappa}{\rho_0} + (Bf)\right)\left[v^k - a^k W^2 a^k\right] \\ + \frac{(vf)}{a^k}\left[(Bv)(v^k - a^k) - B^k\right]\Bigg]. \end{aligned} \tag{3.21}$$

Using equations 3.20 and 3.21 with a third, currently unspecified equation would allow us to solve for the remainder of the eigenvectors through a system of equations. Finding this third equation proves to be tricky, but doable. We will first contract equation 3.2 with B^i. Then, without going into detail, we will use equations 3.12, 3.13, 3.19 and 3.1 substituted in to get rid of any terms not related to e^0, (ve), (Be) or $\hat{e}^k$. This process results in the equation

$$0 = e^0(L^k) + (ve)(M^k) + (Be)(N^k) + \hat{e}^k(R^k) \tag{3.22}$$

where

$$
\begin{aligned}
L^k &= \left[W^2\gamma(Bv)a^k + \chi(B^k)^2\right] + \left[\frac{-a^k}{\rho_0}h_e W^2(Bv)\right] + \left[\frac{P}{\rho_0}\left[(\rho_0+\kappa)W^2(Bv)a^k + B^k\kappa\right]\right] \\
M^k &= \left[2a^k h_e W^4(Bv)\right] + \left[-h_e W^4 a^k(Bv)\right] + \left[B^2 B^k\right] \\
N^k &= \left[h_e W^2 a^k - (Bv)B^k\right] + \left[-B^k(Bv)\right] \\
R &= \left[(Bv)^2 + \frac{B^2}{W^2}\right] + \left[(vf)B^2\right] + \left[-(Bf)(Bv)\right] - \left[\frac{[(Bf)-(Bv)(2+(vf)W^2)]}{\rho_0 W^2 a^k}\left[(\rho_0+\kappa)W^2(Bv)a^k + B^k\kappa\right]\right]
\end{aligned}
$$

This third equation allows us to set up a system of equations that will reveal the final eigenvector values. We will set up a matrix with our three equations, namely 3.20, 3.21, and 3.22:

$$
\begin{bmatrix} h_e W^4 a^k - C^k + W^2 E^k & D^k & \tilde{Z} \\ h_e W^4(Bv)a^k & h_e W^2 a^k & Z' \\ M^k & N^k & R \end{bmatrix}
\begin{bmatrix} ve \\ Be \\ \hat{e}^k \end{bmatrix}
= -e^0\frac{1}{\rho_0}
\begin{bmatrix} h_e c_s^2(v^k - a^k + W^2 a^k) + E^k \\ h_e c_s^2(B^k + W^2 a^k(Bv)) \\ L^k\rho_0 \end{bmatrix}.
$$

We will call the 3x3 matrix on the left Ω. WE find the inverse of Ω by finding the determinant,

$$
\begin{aligned}
|\Omega| = (a^k)^2 h_e^2 R W^6 &+ D^k M^k Z' + N^k(C^k - E^k W^2)Z' \\
&- a^k h_e W^2\left[C^k R + M^k\tilde{Z} + W^2((Bv)D^k R - E^k R + N^k Z' - (Bv)N^k\tilde{Z})\right].
\end{aligned}
\tag{3.23}
$$

Then the inverse is

$$
\Omega^{-1} = \frac{1}{|\Omega|}
\begin{bmatrix}
a^k h_e R W^2 - N^k Z' & D^k R - N^k\tilde{Z} & D^k Z' - a^k h_e W^2\tilde{Z} \\
-a^k(Bv)h_e R W^4 + M^k Z' & C^k R - RW^2(E^k + A^k h_e W^2) + M^k\tilde{Z} & (C^k - E^k W^2)Z' + a^k h_e W^4(-Z' + (Bv)\tilde{Z}) \\
a^k h_e W^2(-M^k + (Bv)N^k W^2) & D^k M^k + N^k(C^k - W^2(E^k + a^k h_e W^2)) & a^k h_e W^2(-C^k + W^2(-(Bv)D^k + E^k + a^k h_e W^2))
\end{bmatrix}
$$

We can now multiply both sides of our lovely matrix equation by Ω^{-1}, which will be quite a laborious step. We leave the grunt work to a symbolic computational program such as Mathematica, which gives us a very long and highly complicated looking result that we will leave out here. The tools to obtain this result have been specified for the few to whom this result would be of any use. For those who don't need the full symbolic result, rest assured that all eigenvectors are now fully accounted for.

3.2.2 Specifics

For those whom the specifics of our last few eigenvectors would greatly bless, we will walk through the steps of how to find them, foregoing vast detail. We first write the resulting equations for (ve), (Be) and $e\hat{k}$ in matrix form, which manifests as

$$\begin{bmatrix} ve \\ Be \\ \hat{e}^k \end{bmatrix} = -e^0 \frac{1}{\rho_0} \Omega^{-1} \begin{bmatrix} h_e c_s^2 (v^k - a^k + W^2 a^k) + E^k \\ h_e c_s^2 (B^k + W^2 a^k (Bv)) \\ L^k \rho_0 \end{bmatrix}.$$

An appropriate matrix multiplication gives us a full equation for $\hat{e}^k$, so we now focus on the two remaining equations:

$$ve = e^0 \Gamma$$

$$Be = e^0 \Pi$$

Where Γ and Π are the result of doing the matrix multiplication with Ω^{-1} and the column vector on the right hand side. We now expand (ve) and (Be):

$$v^1 e_1 + v^2 e_2 + v^k e_k = e^0 \Gamma \tag{3.24}$$

$$B^1 e_1 + B^2 e_2 + B^k e_k = e^0 \Pi \tag{3.25}$$

We again would need a third equation to solve for three unknowns. Instead, we will get rid of e^k by first rearranging 3.1 to get

$$e^k = \frac{-a^k h_{kk} \left(\frac{e^0}{\rho_0} + v^1 e_1 + v^2 e_2 \right)}{1 + a^k v_k}, \tag{3.26}$$

and then putting equations 3.24 and 3.25 in terms of e_1 and e_2. We now only have two unknowns, which is perfect for two equations. At this point the only step left is to algebraically solve for these two values, which again can more quickly, easily, and accurately be done on a symbolic computational program, such as Mathematica. The result is then primed for numerical computational work.

We have every eigenvector accounted for, so let us summarize: (Be) and (ve) have just been solved for, and the same equations can be used in a straightforward manner to find $(B\hat{e})$ and $(v\hat{e})$. We have e^k given above, and $\hat{e}^k$ is given at the end of the last section. Using these, e^4 can easily be found by substituting in equations 3.2 and 3.3 where needed. The relation between e^8 and $\hat{e}^k$ is given by equation 3.5. Finally, we have the complete set of right eigenvectors $\vec{e} = (e^0, e^j, e^4, \hat{e}^j, e^8)$. We now have the basis of numerical work which satisfies the magnetic monopole constraint of ideal general relativistic magnetohydrodynamics.

Chapter 4

Conclusion

Overall, GRMHD provides a mathematical framework for describing electrically charged fluids under high gravitational influences. By combining the equations of fluid dynamics, electromagnetism, and general relativity, we can accurately model the behavior of plasmas in areas of high gravitational and magnetic influences such as black holes and neutron stars. Although we cannot directly observe these phenomena, numerical simulations based on GRMHD equations have been successful in predicting their behavior in the past.

We have provided the framework for a numerical simulation of GRMHD principles based on the characteristic structure of coupled equations of general relativity, electromagnetism, and fluid dynamics. We have found the right eigenvectors of this system with constraint damping, which allows time evolution of the system without allowing small errors to build over time and represent something similar to a magnetic monopole.

Continued work will include finding the left eigenvectors, which is largely the same process as finding the right eigenvectors. We will also need to address renormalization, which takes care of numerical singularities resulting from eigenvalues being equal to one another. Physically, this is fine, but mathematically it must be taken care of. Finally, we may need to try other constraint damping methods and discover what different numerical simulations give us in order to gain greater insight into plasma behavior and what methods may or may not give proper results.

The continued research and development of numerical methods for solving MHD equations is crucial for advancing our understanding of plasma behavior and predicting the behavior of complex systems in the universe. With the right framework and numerical methods, we can unlock some of the secrets of the universe's most fascinating and enigmatic objects.

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