The Characteristic Problem in Ideal GRMHD

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ABSTRACT

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The full eigenvector decomposition for the flux Jacobian matrices of the equations of general relativistic magnetohydrodynamics (GRMHD) are found. The matter equations for GRMHD in the ideal limit can be written in a system of balance law equations. These equations can then be framed as a single 8x8 matrix equation written in terms of the fluid and magnetic field variables. Obtaining the full decomposition allows for the implementation of sophisticated numerical techniques. In this paper we provide the right and left eigenvectors as well as the wave speeds for use in such numerical modeling.

Keywords: [GRMHD, Numerical Relativity, Relativistic Plasmas, Relativistic Astrophysics]

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Chapter 1

Introduction

"...it is a foolish thing to make a long prologue, and to be short in the story itself."

—2 Maccabees 2:32

There is a myriad of astrophysical phenomena that consist of both strong gravitational and magnetic fields. This is the realm of general relativistic magnetohydrodynamics (GRMHD). Such systems need to be studied numerically due to the knotty nature of the evolution equations—they are nonlinear and in general have no simplifying symmetry. To this day many of the premier problems in GRMHD, including the binary neutron star merger, remain more or less unsolved. The urgency for numerical code that can handle such problems has intensified with the recent detection of gravitational waves and their corresponding electromagnetic counterpart (Abbott et al. 2017).

The primary difficulty in writing such a code lies in trying to accurately resolve the physics of multiple length scales without requiring unrealistic computational resources. Gravitational waveforms can be calculated via post-Newtonian methods coupled to black hole perturbation theory. On the other hand, there are many codes out there that claim to be high resolution GRMHD simulations i.e. they are meant to capture shock features and other instabilities that are thought to be key ingredients in important physics such as r-process nucleosynthesis and short gamma ray burst.

But to date, there appears to be no single code that can make quantitative predictions of both gravitational wave and other GRMHD physics. In order to study the cornucopia of interesting GRMHD phenomena inherent in processes like binary neutron star mergers or thick disk accretion onto a black hole, progress will have to be made in solving this resolution problem.

A powerful remedy to this problem was introduced in the early 1980s and is called adaptive mesh refinement (AMR) (Berger & Oliger 1984). This method was soon applied to non-relativistic fluids (Berger & Colella 1989), and is now used in full GRMHD simulations (see (Anderson et al. 2006), (Anderson et al. 2008), and (Dietrich et al. 2018) for examples). In particular, Anderson et al. show the computational advantage of AMR over more straightforward grid schemes (Anderson et al. 2006).

Perhaps just as important as grid choice is the chosen solution method. There are various groups who perform simulations in fairly standard ways e.g. flat spacetimes and basic approximate Riemann solvers. Duez et al. claim to solve the GRMHD equations without approximation, but only under specialized circumstances (Duez et al. 2005). Similar methods are used by Liu et al., who claim that strong magnetic fields will have a measurable effect on the gravitational waveform of a neutron star merger (Liu et al. 2008). They admit, however, that their code does not have the resolution necessary to resolve MHD instabilities. Recently, an impressive computation was done on the Japanese supercomputer "K" (Kiuchi et al. 2017). Merging a black hole and a neutron star, the group was able to observe several instabilities and their effects. They were not, however, able to make quantitiative predictions concerning r-process nucleosynthesis or gamma ray bursts, for example. Perhaps the lessen we learn from these models is that if we wish to understand more detailed physics, we're going to need more sophisticated numerical techniques.

One method that shows promise relies on the so-called characteristic decomposition, sometimes called the characteristic approach. The method is described in detail in a review by Font (Font 2003). The results of the present paper apply specifically to this method. We have called the process of finding the characteristic structure of the GRMHD equations the characteristic problem; hence the name of the paper.

The problem is this: Obtain the equations of motion for your system. In our case these result from the energy conservation equation. We then write these equations in a balance law form. Use a chain rule to exploit the hyperbolic nature of the equations. The flux Jacobian matrices can then be diagonalized i.e. we obtain their eigenvalues and eigenvectors, to reveal the wave speeds and modes of the system. These wave speeds and eigenvectors can be used in certain Godunov-type numerical schemes. This is the characteristic approach.

We are, by no means, the first to investigate this approach in numerical GRMHD. In the 1990s Banyuls et al. wrote a code that tackles non-relativistic hydrodynamics (no $\vec{B}$ field) using a characteristic approach (Banyuls et al. 1997). Antón et al. extended this work to GRMHD where, along with the usual slog of tests, were apparently able to model thick disk accretion onto a black hole. While no mention is made of r-process nucleosynthesis explicitly, it is thought that accretion onto a black hole following neutron star inspiral is a primary source for all r-process elements. A similar group calculated a renormalized set of eigenvectors for the GRMHD equations in 2009, with the goal of streamlining computation (Antón et al. 2009). About the same time, another group tested a code employing the full eigenvector decomposition to high resolution shock capturing techniques (Cerdá-Durán et al. 2008). Notably, they employ constraint damping techniques and allow for a variety of equations of state; a necessary parameter when studying neutron stars. While these efforts are encouraging, there appears to be no where in the literature where these codes or similar ones employing the characteristic approach are used in simulating the not-so-friendly-much-hairier-non-test-tube problems like a neutron star merger.

The remainder of the paper will go as follows: First we will discuss the equations of GRMHD; in particular the equations of motion. We will then show explicitly what we mean by the characteristic structure of the equations. From there we will proceed with the calculation of both the characteristic equation (from which we obtain the wave speeds) and a complete set of eigenvectors. Ultimately, our intent for the results of this paper is to find a home for them in a code of their own.

Now that our prologue is long, we shall make our story longer to avoid being foolish.

Chapter 2

The Equations of GRMHD

We are interested in coupling electromagnetism to a perfect fluid in strong gravitational fields. We will therefore be interested in Einstein's field equations

$$G_{ab} = 8\pi T_{ab}$$

and more particularly in the matter equations that govern GRMHD. To obtain the matter equations, we first define a stress energy tensor

$$T_{ab} = \Big[\rho_0(1+\varepsilon)+P\Big]u_a u_b + P g_{ab} + F_{ac}\,F_b{}^c - \frac{1}{4} g_{ab} F_{cd}\,F^{cd}$$

where the fluid variables are u_a, the four-velocity; ρ_0, the rest energy density; ε, the internal energy; and P, the pressure. In addition, g_{ab} is the metric and F_{ab} is the Maxwell tensor. Along with the stress energy tensor we require an Ohm's-type law coupling the fluid to the electric and magnetic fields; namely,

$$J_a + \rho_e u_a = \sigma F_{ab}\,u^b$$

However, because we're doing ideal MHD we make the approximation that $\sigma \to \infty$, so the above simplifies to

$$F_{ab}\,u^b = 0$$

A consequence of perfect conductivity is that the electric field vanishes in the frame of the fluid. Throughout our computation, then, we will only be concerned with the magnetic field, B_i.

The matter equations can be written as

$$\nabla_a T^{ab} = 0$$

$$\nabla_a * F^{ab} = 0$$

where $*F^{ab} \equiv \varepsilon^{abcd} F_{cd} /2$.

Before obtaining a more useful form of these equations, an onslaught of definitions must be made. First we define v_i to be the spatial fluid velocity so that we can define the Lorentz factor as $W = \frac{1}{1-v_i v^i}$. We define the fluid enthalpy as $h_e = \rho_0(1+\varepsilon)+P$. Further, we use the convention that $B_i B^i \equiv B^2$, $B^i v_i \equiv (Bv)$, etc. With these definitions, we can now define the so-called conserved variables, $\vec{q} = \left(D, S_i, \tau, B_i\right)$. They are, respectively, the rest mass energy density times the Lorentz factor, the momentum, the total energy density minus the rest mass energy density (essentially kinetic energy), and the magnetic field. We have already encountered the "primitive" variables, $\vec{w} = \left(\rho_0, v_i, P, B_i\right)$, (notice that the magnetic field is both a primitive and a conserved variable). We can write the conserved variables in terms of the primitive ones:

$$\begin{aligned}
D &= W\rho_0 \\
S_i &= \left[h_e W^2 + B^2\right] v_i - (Bv) B_i \\
\tau &= h_e W^2 + B^2 - P - \frac{1}{2}\left[(Bv)^2 + \frac{B^2}{W^2}\right] - D \\
B_i &= B_i
\end{aligned}$$

Finally, within the ADM formalism we define the lapse, α, and the shift vector field, β_i. We define the projector, $h_{ab} = g_{ab} + n_a n_b$, where, $n_a = (-\alpha, 0, 0, 0)$. For convenience we define a modified spatial velocity, $\bar{v}^k = v^k - \frac{\beta^k}{\alpha}$.

To the regular matter equations we must attach the baryon conservation equation:

$$\partial_t(\sqrt{h}D) + \partial_i\left[\sqrt{-g}D\left(v^i - \frac{\beta^i}{\alpha}\right)\right]$$

After all these definitions we can now write the matter equations in a more useful way:

$$\begin{aligned}
\partial_t(\sqrt{h}D) + \partial_i\left[\sqrt{-g}D\bar{v}^i\right] &= 0\\
\partial_t(\sqrt{h}S_b) + \partial_i\left[\sqrt{-g}\left((\perp T)^i{}_b - \frac{\beta^i}{\alpha}S_b\right)\right] &= \sqrt{-g}\left[{}^3\Gamma^i{}_{ab}(\perp T)^a{}_i + \frac{1}{\alpha}S_a\partial_b\beta^a - \frac{1}{\alpha}\partial_b\alpha E\right]\\
\partial_t(\sqrt{h}\tau) + \partial_i\left[\sqrt{-g}\left(S^i - Dv^i - \frac{\beta^i}{\alpha}\tau\right)\right] &= \sqrt{-g}\left[(\perp T)^{ab}K_{ab} - \frac{1}{\alpha}S^a\partial_a\alpha\right]\\
\partial_t(\sqrt{h}B^b) + \partial_i\left[\sqrt{-g}\left(B^b\bar{v}^i - B^i\bar{v}^b\right)\right] &= 0\\
-\frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}B^i\right) &= 0
\end{aligned}$$

where K_{ab} is the extrinsic curvature tensor.

2.1 The Matter Equations in Balance Law Form

In the previous section we wrote down the matter equations of GRMHD in a somewhat complicated looking form. When considering the characteristic problem, as we will be doing, one can write the matter equations in balance law form by ignoring all source terms. Furthermore, we can collect all terms with time derivatives and all terms with spatial derivatives into two 8×1 vectors we'll call $\vec{F}^0$ and $\vec{F}^k$, respectively. Thus, all the matter equations can be written succinctly as

$$\partial_t\vec{F}^0 + \partial_i\vec{F}^b = 0$$

where,

$$\vec{F}^0 = \sqrt{h}\begin{pmatrix} D \\ S_b \\ \tau \\ B^b \end{pmatrix} \qquad \vec{F}^b = \sqrt{-g}\begin{pmatrix} D\bar{v}^i \\ (\perp T)^i{}_b - \frac{\beta^i}{\alpha}S_b \\ S^i - Dv^i - \frac{\beta^i}{\alpha}\tau \\ B^b\bar{v}^i - B^i\bar{v}^b \end{pmatrix}$$

We have not previously defined the quantity $(\perp T)_{ij}$. Writing out one of its forms:

$$(\perp T)_{ij} = v_i S_j + P \cdot h_{ij} - \frac{1}{W^2}\left[B_i B_j - \frac{1}{2}h_{ij}B^2\right] - (Bv)\left[B_i v_j - \frac{1}{2}h_{ij}(Bv)\right]$$

and trading out all instances of the lapse and shift in favor of $\bar{v}^k$ we can write the forms of $\vec{F}^0$ and $\vec{F}^k$ out explicitly in terms of the primitive variables. (Note: $\sqrt{-g} = \alpha\sqrt{h}$ and we're going to change the indexes "b" and "i" to "i" and "k" respectively.)

$$\vec{F}^0 = \sqrt{h}\begin{pmatrix} W\rho_0 \\ \left(h_e W^2 + B^2\right)v_i - (Bv)B_i \\ h_e W^2 + B^2 - \frac{1}{2}\left[(Bv)^2 + B^2/W^2\right] - W\rho_0 - P \\ B^i \end{pmatrix}$$

and

$$\frac{\vec{F}^k}{\alpha\sqrt{h}} = \begin{pmatrix} W\rho_0\bar{v}^k \\ \left([h_e W^2 + B^2]v_i - (Bv)B_i\right)\bar{v}^k + h_i^k\left(P + \frac{1}{2}[(Bv)^2 + B^2/W^2]\right) - \left(B_i/W^2 + (Bv)v_i\right)B^k \\ \left(h_e W^2 + B^2 - \frac{1}{2}[(Bv)^2 + B^2/W^2] - W\rho_0 - P\right)\bar{v}^k + \left(P + \frac{1}{2}[(Bv)^2 + B^2/W^2]\right)v^k - (Bv)B^k \\ B^i\bar{v}^k - B^k\bar{v}^i \end{pmatrix}$$

To summarize, we have obtained the equations of motion for the matter of a general system in GRMHD. The equations comprise a system of eight equations and a constraint on there being no magnetic monopoles. We have collected the evolution equations into a single vector equation and made this equation a balance law by dropping the source terms. In defining our vectors, $\vec{F}^0$ and $\vec{F}^k$, we have written all quantities in terms of the primitive fluid variables. We are now primed to discuss the characteristic structure of the equations.

Chapter 3

The Characteristic Structure

In the last section we were able to write the matter equations in the shorthand notation

$$\frac{\partial \vec{F}^0}{\partial t} + \frac{\partial \vec{F}^k}{\partial x^i} = 0$$

While this equation looks simple enough, there is yet more we can do to exploit the hyperbolic nature of the equations and ready them for numerical work. We ultimately want to solve for the primitive variables, $\vec{p}$, so we do what amounts to a chain rule on our vector equation:

$$\begin{aligned}
\frac{\partial \vec{F}^0}{\partial t} + \frac{\partial \vec{F}^k}{\partial x^i} &= 0 \\
\frac{\partial \vec{F}^0}{\partial \vec{p}}\frac{\partial \vec{p}}{\partial t} + \frac{\partial \vec{F}^k}{\partial \vec{p}}\frac{\partial \vec{p}}{\partial x^i} &= 0 \\
\mathscr{F}^0 \partial_t \vec{p} + \mathscr{F}^k \partial_i \vec{p} &= 0 \\
\partial_t \vec{p} + \left(\mathscr{F}^0\right)^{-1} \mathscr{F}^k \partial_i \vec{p} &= 0
\end{aligned}$$

where $\left(\mathscr{F}^0\right)^{-1}$ is the inverse of $\mathscr{F}^0$. Naively, it looks like we now have a system of advection equations with the wavespeed information hidden in the mysterious looking quantity $\left(\mathscr{F}^0\right)^{-1}\mathscr{F}^k$. If this quantity happened to be diagonal that is exactly what we would have. Not surprisingly, however, $\left(\mathscr{F}^0\right)^{-1}\mathscr{F}^k$ is a mess, as we'll see below. Nevertheless, we can try to diagonalize this matrix, i.e. find the eigenvalues and eigenvectors, and so obtain the characteristic wavespeeds and modes of the problem. This is what we mean by the characteristic structure of the equations, and this decomposition will be necessary for the computational methods we wish to eventually employ.

Before moving on to the bulk of the calculation, we'll employ a "trick" to make our path a little smoother. First we define $\vec{e} = (e^0, e^j, e^4, \hat{e}^j)^T$ as a stand in for the components of our right eigenvector (everything we say here has an analogy to the left eigenvectors as well). Let λ^k be a stand in for the eigenvalues. We want to solve

$$\begin{aligned}
\left(\mathscr{F}^0\right)^{-1} \mathscr{F}^k \vec{e} &= \lambda^k \vec{e} \\
\mathscr{F}^k \vec{e} &= \lambda^k \mathscr{F}^0 \vec{e} \\
(\mathscr{F}^k - \lambda^k \mathscr{F}^0)\vec{e} &= 0
\end{aligned}$$

Thus, instead of finding the inverse of $\mathscr{F}^0$ (a dizzying prospect) we need only do a linear combination between $\mathscr{F}^0$ and $\mathscr{F}^k$. Note that these two matrices are constructed from simple derivatives of $\vec{F}^0$ and $\vec{F}^k$. After writing the following identities

$$\begin{aligned}
\chi &= \frac{\partial P}{\partial \rho_0} \\
\kappa &= \frac{\partial P}{\partial \varepsilon} \\
\gamma &= \frac{\partial h_e}{\partial \rho_0} = 1 + \varepsilon + \chi \\
h_e {c_s}^2 &= \rho_0 \chi + \frac{P}{\rho_0} \kappa
\end{aligned}$$

where c_s is the sound speed, we write $\mathscr{F}^0$ and $\mathscr{F}^k$ in all their glory and leave the details to the dedicated reader.

$$\frac{\mathscr{F}^0}{\sqrt{h}} = \begin{pmatrix}
W & W^3 \rho_0 v_j & 0 & 0_j \\
W^2 \gamma v_i & h_{ij} Q + 2 h_e W^4 v_i v_j - B_i B_j & (\rho_0 + \kappa) W^2 v_i & 2 v_i B_j - B_i v_j - (Bv) h_{ij} \\
\gamma W^2 - W - \chi & (2 h_e W^4 + B^2 - W^3 \rho_0) v_j - (Bv) B_j & (\rho_0 + \kappa) W^2 - \kappa & (2 - W^{-2}) B_j - (Bv) B_j - (Bv) v_j \\
0^i & {0^i}_j & 0^i & {h^i}_j
\end{pmatrix}$$

where $Q = h_e W^2 + B^2$, and

$$\frac{\mathscr{F}^k}{\alpha\sqrt{h}} = \begin{pmatrix} W\bar{v}^k & W\rho_0(W^2 v_j \bar{v}^k + h_j{}^k) & 0^k & 0_j{}^k \\ W^2\gamma v_i \bar{v}^k + h_i^k \chi & A^k{}_{ij} & (\rho_0+\kappa)W^2 v_i \bar{v}^k + h_i^k \kappa & B^k{}_{ij} \\ (W^2\gamma - W - \chi)\bar{v}^k & C^k{}_j & (\rho_0+\kappa)W^2\bar{v}^k - \kappa(\bar{v}^k - v^k) & D^k{}_j \\ 0^{ik} & B^i h_j{}^k - B^k h_j{}^i & 0^{ik} & h^i{}_j \bar{v}^k - h^k{}_j \bar{v}^i \end{pmatrix}$$

where,

$$\begin{aligned}
A^k{}_{ij} &= (h_{ij}Q + 2h_e W^4 v_i v_j - B_i B_j)\bar{v}^k + (Qv_i - (Bv)B_i)h_j{}^k + h_i^k((Bv)B_j - B^2 v_j) \\
&\quad - (h_{ij}(Bv) + v_i B_j - 2B_i v_j)B^k \\
B^k{}_{ij} &= (2v_i B_j - B_i v_j - (Bv)h_{ij})\bar{v}^k - (h_{ij}/W^2 + v_i v_j)B^k + h_i^k((Bv)v_j + B_j/W^2) \\
&\quad - h_j{}^k(B_i/W^2 + (Bv)v_i) \\
C^k{}_j &= (2h_e W^4 v_j - (Bv)B_j + B^2 v_j - \rho_0 W^3 v_j)\bar{v}^k + (Q - W\rho_0)h_j{}^k + ((Bv)B_j - B^2 v_j)v^k - B_j B^k \\
D^k{}_j &= (2B_j - (Bv)v_j - B_j/W^2)\bar{v}^k + ((Bv)v_j + B_j/W^2)v^k - v_j B^k - (Bv)h_j{}^k
\end{aligned}$$

3.1 Characteristic Equation

Just like in simpler diagonalization problems we first find the eigenvalues, λ^k, by taking the determinant of our matrix, in our case $(\mathscr{F}^k - \lambda^k \mathscr{F}^0)$, and finding the roots of the resulting characteristic polynomial. To that end,

$$
det \mid \mathscr{F}^k - \lambda^k \mathscr{F}^0 \mid = (\alpha\sqrt{h})^8 \begin{vmatrix} W(\bar{v}^k - \lambda^k) & W^3\rho_0 v_j(\bar{v}^k - \lambda^k) + W\rho_0 h_j^{\ k} \\ W^2\gamma v_i(\bar{v}^k - \lambda^k) + h_i^{\ k}\chi & \tilde{A}^k{}_{ij} \\ (W^2\gamma - W - \chi)(\bar{v}^k - \lambda^k) + \chi v^k & \tilde{C}^k{}_j \\ 0^{ik} & B^i h_j^{\ k} - B^k h_j^{\ i} \end{matrix}
$$

$$
\begin{matrix} 0^k & 0_j^{\ k} \\ (\rho_0 + \kappa)W^2 v_i(\bar{v}^k - \lambda^k) + h_i^{\ k}\kappa & \tilde{B}^k{}_{ij} \\ \left[(\rho_0 + \kappa)W^2 - \kappa\right](\bar{v}^k - \lambda^k) + \kappa v^k & \tilde{D}^k{}_j \\ 0^{ik} & h^i{}_j(\bar{v}^k - \lambda^k) - h^k{}_j \bar{v}^i \end{vmatrix}
$$

where we have split the determinant into two lines and,

$$
\begin{aligned}
\tilde{A}^k{}_{ij} &= (h_{ij}Q + 2h_e W^4 v_i v_j - B_i B_j)(\bar{v}^k - \lambda^k) + (Qv_i - (Bv)B_i)h_j^{\ k} + h_i^{\ k}((Bv)B_j - B^2 v_j) \\
&\quad - (h_{ij}(Bv) + v_i B_j - 2B_i v_j)B^k \\
\tilde{B}^k{}_{ij} &= (2v_i B_j - B_i v_j - (Bv)h_{ij})(\bar{v}^k - \lambda^k) - (h_{ij}/W^2 + v_i v_j)B^k + h_i^{\ k}((Bv)v_j + B_j/W^2) \\
&\quad - h_j^{\ k}(B_i/W^2 + (Bv)v_i) \\
\tilde{C}^k{}_j &= (2h_e W^4 + B^2 - \rho_0 W^3)v_j(\bar{v}^k - \lambda^k) - (Bv)B_j(\bar{v}^k - \lambda^k) + (Q - W\rho_0)h_j^{\ k} \\
&\quad + ((Bv)B_j - B^2 v_j)v^k - B_j B^k \\
\tilde{D}^k{}_j &= (2B_j - (Bv)v_j - B_j/W^2)(\bar{v}^k - \lambda^k) + ((Bv)v_j + B_j/W^2)v^k - v_j B^k - (Bv)h_j^{\ k}
\end{aligned}
$$

The calculation of this determinant is so involved, that it deserves its own section.

3.1.1 Finding the Determinant

The present section will be boring to all except those who wish to make the calculation themselves and who want to check their work against ours. Before skipping ahead, however, the reader should be aware of two very important definitions in this section. For convenience, we define,

$$a^k = \bar{v}^k - \lambda^k$$
$$\Delta^{kk} = (a^k)^2 Q - 2(Bv)a^k B^k - \frac{1}{W^2} B^k B^k$$

where the first appears many times in the determinant as written at the end of the last section, and the second will be a most useful quantity later. The reader may now skip to the end of the section to find the characteristic equation without any loss of sanity. Those who wish to follow every indexed detail from here on out may wish consult Canto III, line 9 of Mr. Alighieri's *Inferno*.

At the end of last section we obtained,

$$\begin{vmatrix} Wa^k & W^3\rho_0 v_j a^k + W\rho_0 h_j^{\ k} & 0^k & 0_j^{\ k} \\ W^2\gamma v_i a^k + h_i^{\ k}\chi & \tilde{A}^k{}_{ij} & (\rho_0+\kappa)W^2 v_i a^k + h_i^{\ k}\kappa & \tilde{B}^k{}_{ij} \\ (W^2\gamma - W - \chi)a^k + \chi v^k & \tilde{C}^k{}_j & \left[(\rho_0+\kappa)W^2 - \kappa\right]a^k + \kappa v^k & \tilde{D}^k{}_j \\ 0^{ik} & B^i h_j^{\ k} - B^k h_j^{\ i} & 0^{ik} & h^i{}_j a^k - h^k{}_j \bar{v}^i \end{vmatrix}$$

dropping, for the moment, the $(\alpha\sqrt{h})^8$ outside the determinant. For simplicity we'll refer to rows (and columns) one through four instead of one through eight, ignoring the index structure unless explicitly stated. We will also use quotation, or "ditto" marks to signify that an entry has not changed since the last operation.

Add row 1 to row 3 to get:

$$\begin{vmatrix} " & " & " & " \\ " & " & " & " \\ (W^2\gamma - \chi)a^k + \chi v^k & \tilde{C}^k{}_j + W^3\rho_0 v_j a^k + W\rho_0 h_j^{\ k} & " & " \\ " & " & " & " \end{vmatrix}$$

Adding $-v_i$ times row 3 to row 2 gives:

$$\begin{vmatrix} " & " & " & " \\ \chi\left[h_i^{\,k}+(a^k-v^k)v_i\right] & \alpha^k{}_{ij} & \kappa\left[h_i^{\,k}+(a^k-v^k)v_i\right] & \tilde{B}^k{}_{ij}-v_i\tilde{D}^k{}_j \\ " & " & " & " \\ " & " & " & " \end{vmatrix}$$

where, $\alpha^k{}_{ij}=\tilde{A}^k{}_{ij}-\left(\tilde{C}^k{}_j+W^3\rho_0 v_j a^k+W\rho_0 h_j^{\,k}\right)v_i$.

Adding $-\frac{\chi}{\kappa}$ times column 3 to column 1 gives:

$$\begin{vmatrix} " & " & " & " \\ 0_i^k & " & " & " \\ W^2a^k\left[\gamma-\chi(1+\frac{\rho_0}{\kappa})\right] & " & " & " \\ 0^{ik} & " & " & " \end{vmatrix}$$

We arrive, now, at our first important simplification by adding $-W\left[\gamma-\chi(1+\frac{\rho_0}{\kappa})\right]$ times row 1 to row 3 to get:

$$\begin{vmatrix} Wa^k & " & " & " \\ 0_i^k & " & " & " \\ 0^k & \xi_j^{\,k} & " & " \\ 0^{ik} & " & " & " \end{vmatrix}$$

where, $\xi_j^{\,k}=\tilde{C}^k{}_j+\left(W^3\rho_0 v_j a^k+W\rho_0 h_j^{\,k}\right)\left[1-W\left(\gamma-\chi(1+\frac{\rho_0}{\kappa})\right)\right]$.

Having obtained a column of zeros, we may use the rules of computing determinants to write the above as Wa^k times the determinant of the $3x3$ block of the lower right corner of the array. We will eventually do this, but for the sake of not having to redefine columns and rows we leave it in. We do make a "quick" side-step here to simplify $\alpha^k{}_{ij}$ and $\xi_j^{\,k}$. Note that using the thermodynamic relations given near the beginning of this chapter we can find that $\gamma-\chi\left(1+\frac{\rho_0}{\kappa}\right)=\frac{h_e}{\rho_0\kappa}(\kappa-\rho_0 c_s{}^2)$.

After simplification we obtain,

$$
\begin{aligned}
\alpha^k{}_{ij} &= \big(h_{ij}Q - B_iB_j\big)a^k - (Bv)B_ih_j{}^k - \big(h_{ij}(Bv) - 2B_iv_j\big)B^k \\
&\quad + \big((Bv)B_j - B^2v_j\big)\big(h_i^k + v_i(a^k - v^k)\big) \\
\xi_j{}^k &= \big(2h_eW^4 + B^2\big)v_ja^k - (Bv)B_ja^k + Qh_j{}^k + \big((Bv)B_j - B^2v_j\big)v^k - B_jB^k \\
&\quad - W^2\big(W^2v_ja^k + h_j{}^k\big)h_e(1 - \frac{\rho_0}{\kappa}c_s{}^2)
\end{aligned}
$$

Now add $-\big((Bv)B_j - B^2v_j\big)/\kappa$ times column 3 to column 2.

$$
\begin{vmatrix}
Wa^k & W^3\rho_0v_ja^k + W\rho_0h_j{}^k & " & " \\
0_i^k & \tilde{\alpha}^k{}_{ij} & " & " \\
0^k & \tilde{\xi}_j{}^k & " & " \\
0^{ik} & B^ih_j{}^k - B^kh_j{}^i & " & "
\end{vmatrix}
$$

And,

$$
\begin{aligned}
\tilde{\alpha}^k{}_{ij} &= \big(h_{ij}Q - B_iB_j\big)a^k - (Bv)B_ih_j{}^k - \big(h_{ij}(Bv) - 2B_iv_j\big)B^k \\
\tilde{\xi}_j{}^k &= \big(2h_eW^4 + B^2\big)v_ja^k + Qh_j{}^k - B_jB^k - W^2\big(W^2v_ja^k + h_j{}^k\big)h_e(1 - \frac{\rho_0}{\kappa}c_s{}^2) \\
&\quad - W^2a^k\big(1 + \frac{\rho_0}{\kappa}\big)\big((Bv)B_j - B^2v_j\big)
\end{aligned}
$$

Add $-\big((Bv)v_j + B_j/W^2\big)/\kappa$ times column 3 to column 4 to get:

$$
\begin{vmatrix}
Wa^k & " & 0^k & 0_j{}^k \\
0_i^k & \tilde{\alpha}^k{}_{ij} & " & \beta^k{}_{ij} \\
0^k & \tilde{\xi}_j{}^k & " & \delta^k{}_j \\
0^{ik} & " & 0^{ik} & h_j{}^ia^k - h_j{}^k\bar{v}^i
\end{vmatrix}
$$

where, after much simplification, $\beta^k{}_{ij}$ and $\delta^k{}_j$ can be written,

$$
\begin{aligned}
\beta^k{}_{ij} &= -\big(B_iv_j + (Bv)h_{ij}\big)a^k - \big(h_{ij}B^k + h_j{}^kB_i\big)\frac{1}{W^2} + \big((Bv)B_j - B^2v_j\big)\big(h_i^k + v_i(a^k - v^k)\big) \\
\delta^k{}_j &= 2B_ja^k - v_jB^k - (Bv)h_j{}^k - W^2a^k\big((Bv)v_j + B_j\frac{1}{W^2}\big)\big(1 + \frac{\rho_0}{\kappa}\big)
\end{aligned}
$$

Next, we can do B^k times column 4 added to a^k times column 2. Remember, column 2 is actually three columns (the index j). So we can, by rules of the determinant, pull out three factors of $1/a^k$, which allows us to perform the operation at hand legally. Upon doing this we obtain,

$$(\frac{1}{a^k})^3 \begin{vmatrix} Wa^k & \left(W^3\rho_0 v_j a^k + W\rho_0 h_j{}^k\right)a^k & 0^k & 0_j{}^k \\ 0_i^k & a^k\tilde{\alpha}^k{}_{ij} + B^k\beta^k{}_{ij} & '' & \beta^k{}_{ij} \\ 0^k & a^k\tilde{\xi}_j{}^k + B^k\delta^k{}_j & '' & \delta^k{}_j \\ 0^{ik} & B^i h_j{}^k a^k - B^k h_j{}^k \bar{v}^i & 0^{ik} & h_j{}^i a^k - h_j{}^k \bar{v}^i \end{vmatrix}$$

At this stage the index notation really shows its power. Notice that if we contract column 4 with B^j, then row 4, column 4 is exactly the same as row 4, column 2 up to a factor of $h_j{}^k$. We have found, then, a serendipitous way to zero out the last entry in column 3 (actually the last entry in three columns). This operation amount to multiplying columns 6, 7, and 8 by B^1, B^2, and B^k respectively, adding them together and subtracting the result from columns 2, 3, and 4. How easy this would have been to see without the index notation is anybody's guess. Without further ado, then, we obtain,

why these

$$(\frac{1}{a^k})^3 \begin{vmatrix} Wa^k & \left(W^3\rho_0 v_j a^k + W\rho_0 h_j{}^k\right)a^k & 0^k & 0_j{}^k \\ 0_i^k & a^k\tilde{\alpha}^k{}_{ij} + B^k\beta^k{}_{ij} - B^l\beta^k{}_{il} h_j{}^k & '' & \beta^k{}_{ij} \\ 0^k & a^k\tilde{\xi}_j{}^k + B^k\delta^k{}_j - B^l\delta^k{}_l h_j{}^k & '' & \delta^k{}_j \\ 0^{ik} & 0_j{}^{ik} & 0^{ik} & h_j{}^i a^k - h_j{}^k \bar{v}^i \end{vmatrix}$$

where we have changed the dummy index in the contraction to "l" to avoid confusion with the "j" in $h_j{}^k$.

It is left to the reader to simplify the second and third entries in column 3, but rest assured that many simplifications await you. After grouping all terms into those multiplied by h_{ij} or those multiplied by B_j, the entry in row 2, column 2 becomes,

$$\begin{aligned}
a^k \tilde{\alpha}^k{}_{ij} + B^k \beta^k{}_{ij} - B^l \beta^k{}_{il} h_j{}^k &= h_{ij}\Big[(a^k)^2 Q - 2(Bv)a^k B^k - \frac{1}{W^2}B^k B^k\Big] \\
&\quad - B_i\Big[(a^k)^2 B_j - a^k\Big((Bv)h_j{}^k + v_j B^k\Big) - \frac{1}{W^2}h_j{}^k B^k\Big] \\
&= h_{ij}\Delta^{kk} - B_i E_j{}^{kk} \\
&= A_{ij}{}^{kk}
\end{aligned}$$

and the entry in row 3, column 2 becomes,

$$\begin{aligned}
a^k \tilde{\xi}_j{}^k + B^k \delta^k{}_j - B^l \delta^k{}_l h_j{}^k = a^k a^k &\Big[W^2\Big(W v_j - (Bv)B_j\Big) + \frac{\rho_0}{\kappa}W^2\Big(h_e W^2 c_s{}^2 v_j + B^2 v_j - (Bv)B_j\Big)\Big] \\
+ a^k &\Big[h_j{}^k W^2 (Bv)^2 - W^2 (Bv) v_j B^k \\
&\quad + \frac{\rho_0}{\kappa}\Big(\big(B^2 + W^2 (Bv)^2 + h_e W^2 c_s{}^2\big)h_j{}^k - B^k\big(W^2 (Bv) v_j + B_j\big)\Big)\Big] \\
- B^k &\Big[h_j{}^k (Bv) - v_j B^k\Big] \\
= C_j{}^{kk} &
\end{aligned}$$

At this point it behooves us to start gathering terms in their powers of a^k. Remember that $a^k = \bar{v}^k - \lambda^k$, and it is λ^k, the eigenvalues, that we are after. Said another way, we want to group our characteristic equation in powers of a^k as best we can, so that it is easier to solve for and identify the particular wave speeds.

We can now write:

$$det \mid \mathscr{F}^k - \lambda^k \mathscr{F}^0 \mid = (\alpha^2 h)^4 (\frac{1}{a^k})^3 \begin{vmatrix} Wa^k & \left(W^3\rho_0 v_j a^k + W\rho_0 h_j{}^k\right)a^k & 0^k & 0_j{}^k \\ 0_i^k & A_{ij}{}^{kk} & \kappa\left(h_i{}^k + v_i(a^k - v^k)\right) & \beta^k{}_{ij} \\ 0^k & C_j{}^{kk} & W^2 a^k(\rho_0 + \kappa) - \kappa(a^k - v^k) & \delta^k{}_j \\ 0^{ik} & 0_j{}^{ik} & 0^{ik} & h_j{}^i a^k - h_j{}^k \bar{v}^i \end{vmatrix}$$

which can immediately be simplified to

$$(\alpha^2 h)^4 (\frac{1}{a^k})^3 W a^k \begin{vmatrix} A_{ij}{}^{kk} & \kappa\left(h_i{}^k + v_i(a^k - v^k)\right) & \beta^k{}_{ij} \\ C_j{}^{kk} & W^2 a^k(\rho_0 + \kappa) - \kappa(a^k - v^k) & \delta^k{}_j \\ 0_j{}^{ik} & 0^{ik} & h_j{}^i a^k - h_j{}^k \bar{v}^i \end{vmatrix}$$

At this stage, we can once again use the index notation to our advantage, if, of course, we don't forget what it means. Notice that the bottom row actually represents a $3x4$ block of zeros with another $3x3$ block to the right, represented by the bottom right entry in our indexed matrix. In other words, we potentially have more rows of zeros we can use to simplify! The structure of the $3x3$ block looks something like,

$$\begin{matrix} a^k & 0 & -\bar{v}^1 \\ 0 & a^k & -\bar{v}^2 \\ 0 & 0 & -\lambda^k \end{matrix}$$

where all superscripts here denote a certain component, not a power. The first thing we notice is that the bottom row of our full $7x7$ determinant is zero except the entry in row 7, column 7; namely, $-\lambda^k$. After eliminating the 7th row and 7th column by the regular rules of the determinant, notice from the $3x3$ block written above, that we have two more rows whose entries are all zero save one entry. We can, in essence, eliminate the third row and third column in our full $3x3$, indexed determinant and simply pick up a factor of $-\lambda^k(a^k)^2$, i.e.

$$(\alpha^2 h)^4 (\frac{1}{a^k})^3 W a^k (-\lambda^k)(a^k)^2 \begin{vmatrix} A_{ij}{}^{kk} & \kappa\left(h_i^k v_i (a^k - v^k)\right) \\ C_j{}^{kk} & W^2 a^k(\rho_0 + \kappa) - \kappa(a^k - v^k) \end{vmatrix}$$

It looks like we're almost done, but don't be fooled: This is **NOT** a $2x2$ determinant to be calculated in the elementary way. This is in fact a $4x4$ determinant. We can, however, use a handy-dandy little formula for computing determinants in the specialized form we have obtained. In general,

$$\begin{vmatrix} a_{ij} & v_j \\ c_j & x \end{vmatrix} = det(a_{ij}) \cdot \left[x - \vec{c}^T (a_{ij})^{-1} \vec{v}\right] \leftarrow \text{Stay } \hat{j}$$

Switch to i

So, all that remains for us to do is to find the determinant and inverse of $A_{ij}{}^{kk}$ and the rest is history.

In calculating $det(A_{ij}{}^{kk})$ it might be helpful to remember that $A_{ij} = h_{ij}\Delta^{kk} - B_i E_j{}^{kk}$, and to write,

$$A_{ij}{}^{kk} = \begin{pmatrix} h_{11}\Delta - B_1E_1 & h_{12}\Delta - B_1E_2 & h_{13}\Delta - B_1E_3 \\ h_{21}\Delta - B_2E_1 & h_{22}\Delta - B_2E_2 & h_{23}\Delta - B_2E_3 \\ h_{31}\Delta - B_3E_1 & h_{32}\Delta - B_3E_2 & h_{33}\Delta - B_3E_3 \end{pmatrix}$$

where we have dropped the k indices for the moment. Taking this determinant by brute force, we find that all terms multiplied by either Δ or $(\Delta)^0$, cancel completely. Furthermore, we recognize that the terms multiplying Δ^3 are exactly the determinant of h_{ij}; namely, $h_{11}(h_{22}h_{33} - h_{32}h_{23}) - h_{12}(h_{21}h_{33} - h_{31}h_{23}) + h_{13}(h_{21}h_{32} - h_{31}h_{22})$, so that term can be simplified to $h\Delta^{kk}$.

The terms multiplying Δ^2 are not so obvious, unfortunately. But if you group things in terms of B_1E_1, B_1E_2, etc., you find that you have a nine term sum of the form, $C^{ij}B_iE_j$, where C^{ij} is the matrix of cofactors for h_{ij}. (Warning: up indices on C are used here to signify a sum with B_iE_j, not the inverse of the matrix of cofactors.) If we remember the Laplace or cofactor expansion for computing the determinant, we can replace C^{ij}, the cofactor matrix of h_{ij}, with a more useful quantity; namely,

$$h = h_{ij} C^{ij}$$
$$C^{ij} = h \cdot h^{ij}$$

where, in this case h^{ij} is, in fact the inverse of the projector, h_{ij}. Putting some of this together we find,

$$\begin{aligned} det({A_{ij}}^{kk}) &= h \cdot (\Delta^{kk})^3 - hB^j {E_j}^{kk} (\Delta^{kk})^2 \\ &= h \cdot (\Delta^{kk})^2 \big(\Delta^{kk} - B^j {E_j}^{kk}\big) \end{aligned}$$

By definition, $\Delta^{kk} - B^j {E_j}^{kk} = {A_j}^{jkk}$, which after some cancellation we find to equal $(a^k)^2 h_e W^2$. So,

$$det({A_{ij}}^{kk}) = h(\Delta^{kk})^2 h_e W^2 (a^k)^2$$

We now turn our attention to finding the inverse of ${A_{ij}}^{kk}$. We could do this directly with the formula,

$$A^{-1} = \frac{1}{det(A)} adj(A)$$

where, $adj(A)$, is the classical adjoint of A. Instead, we choose to use a "guess and check" method. We assume A^{-1} has the form,

$$(A^{-1})^{ij} = Xh^{ij} + YB^i E^j$$

and demand that $A_{ij}(A^{-1})^{ij}$ be the identity. (Note: we have, again, dropped the k indices for the moment.) We find,

$$\begin{aligned} A_{ij}(A^{-1})^{ij} &= \big(h_{ij}\Delta - B_i E_j\big)\big(Xh^{ij} + YB^i E^j\big) \\ &= X\big(\Delta - (BE)\big) + Y(BE)\big(\Delta - (BE)\big) \end{aligned}$$

where, $(BE) \equiv B^l E_l$, which we found previously to be equal to $\Delta - h_e W^2 (a^k)^2$. After a little staring, we realize that the above expression is equal to unity if $X = \frac{1}{\Delta}$, and $Y = \frac{1}{\Delta(\Delta - (BE))}$. Thus, we have found,

$$\left(A_{ij}{}^{kk}\right)^{-1} = \frac{1}{\Delta^{kk}}\left(h^{ij} + \frac{B^i E^j}{h_e W^2 (a^k)^2}\right) \tag{3.1}$$

Although we designed $(A_{ij}{}^{kk})^{-1} A_{ij}{}^{kk}$ to be equal to the identity, it is useful to check that this is true in the form we have just found.

Now we are equipped to use our handy-dandy formula we stated previously:

$$\begin{vmatrix} A_{ij}{}^{kk} & \kappa\left(h_i^k v_i(a^k - v^k)\right) \\ C_j{}^{kk} & W^2 a^k(\rho_0 + \kappa) - \kappa(a^k - v^k) \end{vmatrix}$$

$$= det(A_{ij}{}^{kk}) \cdot \left[W^2 a^k(\rho_0 + \kappa) - \kappa(a^k - v^k) - C_j{}^{kk} \cdot \left(A_{ij}{}^{kk}\right)^{-1} \cdot \kappa\left(h_i^k v_i(a^k - v^k)\right)\right]$$

So,

$$\begin{aligned} det(\mathscr{F}^k - \lambda^k \mathscr{F}^0) = {} & -\alpha^8 h^5 \lambda^k h_e w^3 (a^k)^2 (\Delta^{kk}) \\ & \cdot \Bigg\{ \left[(a^k)^2(h_e W^2 + B^2) - 2a^k (Bv) B^k - \frac{1}{W^2} B^k B^k\right] \cdot \left[\left(W^2(\rho_0 + \kappa) - \kappa\right) a^k + \kappa v^k\right] \\ & - C_j{}^{kk}\left[h^{ij} + \frac{B^i E^j}{h_e W^2 (a^k)^2}\right] \kappa\left(h_j{}^k + v_j(a^k - v^k)\right)\Bigg\} \end{aligned}$$

While the tale has been gripping, I must disappoint the reader by forgoing the algebra necessary to simplify the characteristic equation. (Writing neatly, it took me no less than five and one-half pages to do the calculation.) But to give a synopsis, it involves performing the necessary contractions, grouping in powers of a^k, and then factoring (as if by this point I needed to tell the reader how to do algebra). The dedicated reader who wishes to perform the calculation (or is being compelled to) may find it useful to remember $v^2 = 1 - 1/W^2$.

The characteristic equation as we obtained it:

$$det\left(\mathscr{F}^k - \lambda^k \mathscr{F}^0\right) = -\alpha^8 h^5 \rho_0 h_e W^3 \lambda^k a^k \Delta^{kk} \cdot \Bigg\{ h_e W^4 \left(1 - c_s^2\right)\left(a^k\right)^4$$

$$+ \left[\left(a^k\right)^2\left(h_e W^2 + B^2 + W^2 (Bv)^2\right) - c_s^2\left(a^k W (Bv) + \frac{B^k}{W^2}\right)^2\right] \cdot \left[\left(a^k - v^k\right)^2 - h^{kk}\right]\Bigg\}$$

3.2 The Wavespeeds

Before we ask our computer to find the roots of the above eighth order polynomial, let's look at it piece by piece and try to understand some of the physics behind what we've just done.

Upon setting the characteristic equation equal to zero we find that one of the solutions to the characteristic equation is $\lambda^k = 0$. In other words, one of the waves has zero wave speed or is trivial. It turns out that this could have been predicted earlier. We wrote the induction equation as

$$0 = \partial_t(\sqrt{h}B^b) + \partial_k\left[\sqrt{-g}\left(B^b\bar{v}^k - B^k\bar{v}^b\right)\right]$$

which, when $b = k$ has no flux term. In other words, B^k is static and its corresponding wave is trivial. Said one more way, the fact that we require our system to have no magnetic monopoles fixes the magnetic field in one direction and effectively reduces the size of our system.

There are seven nontrivial eigenvalues and therefore wave speeds left to calculate. Another simple case is the solution $a^k \to 0$ or $\lambda^k = \bar{v}^k$. In other words, one of the wave speeds corresponds to the (spatial) relativistic velocity of the fluid. We recognize that this wave speed describes the entropy wave. There will be one eigenvector associated with this wavespeed.

Next, is the case when $\Delta^{kk} = 0$, which corresponds to the so-called Alfvén waves. Δ^{kk} is a quadratic term in a^k, so there will be two wavespeeds each corresponding to its own eigenvector. For satisfaction, we can write out the explicit forms for these two wave speeds by remembering the definition of both Δ^{kk} and a^k.

$$\begin{aligned}
0 &= (a^k)^2 Q - 2(Bv)a^k B^k - \frac{1}{W^2}B^k B^k \\
a^k &= \frac{B^k}{Q}\left[(Bv) \pm \left((Bv)^2 + \frac{Q}{W^2}\right)^{1/2}\right] \\
\lambda^k &= \bar{v}^k + \frac{B^k}{Q}\left[(Bv) \pm \left((Bv)^2 + \frac{Q}{W^2}\right)^{1/2}\right]
\end{aligned}$$

where $Q = h_e W^2 + B^2$. These two solutions represent the speeds of the two Alfvén waves.

Finally, there is the quartic term in curly braces. The roots of this expression will describe the wavespeeds of the magnetosonic or magnetoacoustic waves. While there is technically an analytic solution to finding the roots of a general quartic expression, it is very messy and we gain no insight by writing it down. But, having the explicit expression in the characteristic equation, we can always solve this term numerically to obtain the four magnetosonic wave speeds.

Some numerical schemes require only the knowledge of some of the wave speeds; say, the fastest and the slowest. Still others require all wave speeds as we've just calculated. The schemes we're interested in, however, will also require the corresponding right and left eigenvectors.

Chapter 4

The Eigenvectors

In general, diagonalizing a matrix involves finding two sets of eigenvectors. We're not used to doing this in many of the toy physics problems we're handed, because almost all of the matricies in toy problems are symmetric. For symmetric matricies the left and right eigenvectors are identical. Our matrix of interest, $\mathscr{F}^k - \lambda^k \mathscr{F}^0$, however, is certainly not symmetric. As a result, the left and right eigenvectors will be different i.e. we'll have to find two sets instead of one.

Additionally, the eigenvalues of a square matrix need not be real. In general, the eigenvalues corresponding to the left eigenvectors are the complex conjugates of those corresponding to the right eigenvectors and *visa versa*. Because our system is physical, however, we expect (in fact demand) that our eigenvalues be real—they represent measurable quantities. We know that the wavespeeds for the Entropy and Alfvén, waves are positive for all possible values of the primitive variables. The conditions, if any, in which the magnetosonic wave speeds become complex have not been explored. We will, for lack of any signs of trouble, assume the are always real.

The rest of this chapter, indeed, the rest of this paper will be dedicated to finding the right and left eigenvectors and thereby completing the diagonalization process.

4.1 Right Eigenvectors

By calculating the right eigenvectors we mean finding the componants of the vector $\vec{e} = (e^0, e^j, e^4, \hat{e}^j)^T$ by solving the equation

$$\mathscr{F}^k \vec{e} = \lambda^k \mathscr{F}^0 \vec{e}$$

$$(\mathscr{F}^k - \lambda^k \mathscr{F}^0)\vec{e} = 0$$

Having previously calculated $\mathscr{F}^k - \lambda^k \mathscr{F}^0$, it is fairly straightforward to obtain:

$$0 = e^0 \cdot W a^k + (ve) W^3 \rho_0 a^k + e^k \cdot W \rho_0 \tag{4.1}$$

$$\begin{aligned}
0 = e^0 \Big[W^2 \gamma v_i a^k + \chi h_i{}^k \Big] & \qquad (4.2) \\
+ \Big\{ e_i \Big[Q \cdot a^k - (Bv) B^k \Big] + (ve) \Big[2 h_e W^4 v_i a^k - h_i{}^k B^2 + 2 B_i B^k \Big] & \\
+ (Be) \Big[- B_i a^k + h_i{}^k (Bv) - v_i B^k \Big] + e^k \Big[Q \cdot v_i - (Bv) B_i \Big] \Big\} & \\
+ e^4 \Big[(\rho_0 + \kappa) W^2 v_i a^k + h_i{}^k \kappa \Big] & \\
+ \Big\{ \hat{e}_i \Big[- (Bv) a^k - \frac{1}{W^2} B^k \Big] + (v\hat{e}) \Big[- B_i a^k - v_i B^k + h_i{}^k (Bv) \Big] & \\
+ (B\hat{e}) \Big[2 v_i a^k + \frac{1}{W^2} h_i{}^k \Big] - \hat{e}^k \Big[(Bv) v_i + \frac{1}{W^2} B_i \Big] \Big\} &
\end{aligned}$$

$$\begin{aligned}
0 = e^0 \Big[(W^2 \gamma - W - \chi) a^k + \chi v^k \Big] & \qquad (4.3) \\
+ \Big\{ (ve) \Big[(2 h_e W^4 + B^2 - W^3 \rho_0) a^k - B^2 v^k \Big] - (Be) \Big[(Bv)(a^k - v^k) + B^k \Big] + e^k (Q - W \rho_0) \Big\} & \\
+ e^4 \Big[(\rho_0 + \kappa) W^2 a^k + \kappa (v^k - a^k) \Big] & \\
+ \Big\{ - (v\hat{e}) \Big[(Bv)(a^k - v^k) + B^k \Big] + (B\hat{e}) \Big[(2 - \frac{1}{W^2}) a^k + \frac{1}{W^2} v^k \Big] - \hat{e}^k (Bv) \Big\} &
\end{aligned}$$

$$0 = B^i e^k - B^k e^i + \hat{e}^i a^k - \hat{e}^k \bar{v}^i \tag{4.4}$$

Before we get ambitious and start trying to manipulate the above equations let's remember that we've already found one solution: $\lambda^k = 0$, and therefore, $\vec{e} = (0, 0^j, 0, 0^j)^T$. But don't pat yourself on the back too soon; the other eigenvectors will not be so trivial to calculate.

There is one observation, however, that will make the road ahead a little less thorny, if even just ever so slightly. We wish to take advantage of a symmetry in equation (4.4). Remember, that this is actually three equations for values of i running from $(i = 1, 2, k)$. When we set $i = k$ the equation simplifies to

$$0 = -\lambda^k \hat{e}^k$$

If we are not considering the trivial case described above then we have found that $\hat{e}^k = 0$ for all eigenvectors. Essentially what we have done is reduce the size of our system to $7x7$.

4.1.1 The Entropy Wave

The entropy wave can be found by solving our system of equations in the special case when $a^k = 0$. Equation (4.1) becomes

$$0 = W\rho_0 \cdot e^k$$

With both e^k and $\hat{e}^k$ equal to zero, equation (4.4) simplifies to

$$0 = B^k e^i$$

which, as long as the magnetic field is nonzero in the k direction, implies $e^i = 0$.

With the preceeding simplifications the remaining equations become:

$$0 = e^0 \cdot \chi h_i^{\ k} + e^4 \cdot h_i^{\ k} \kappa + \left\{ -\hat{e}_i \frac{B^k}{W^2} + (v\hat{e}) \Big(-v_i B^k + (Bv) h_i^{\ k} \Big) + (B\hat{e}) \frac{h_i^{\ k}}{W^2} \right\} \tag{4.5}$$

$$0 = e^0 \cdot \chi v^k + e^4 \cdot v^k \kappa + \left\{ (v\hat{e}) \Big((Bv) v^k - B^k \Big) + (B\hat{e}) \frac{v^k}{W^2} \right\} \tag{4.6}$$

Contracting (4.5) with B^i we obtain the useful relation,

$$e^4 = -e^0 \frac{\chi}{\kappa}$$

again, assuming $B^k \neq 0$.

Finally, if we contract equation (4.5) with $\hat{e}^i$ we get;

$$0 = -\left[\frac{1}{W^2}\hat{e}_i\hat{e}^i + (v\hat{e})^2\right] \cdot B^k$$

because $\hat{e}^k = 0$. Since the metric is positive definite, $\hat{e}_i\hat{e}^i$ and $(v\hat{e})^2$ must both be positive. If the sum of two positive quantities equals zero, then both quantities must be zero. More importantly, since $\hat{e}_i\hat{e}^i = 0$, then $\hat{e}^i = 0$.

Reflecting, we have found six out of our eight components to be zero; namely, e^i and $\hat{e}^i$. Furthermore, we have a relationship between the remaining two components: $e^4 = -e^0\frac{\chi}{\kappa}$. Sticking to the form of $\vec{e}$ defined earlier, we can write the right eigenvector for the entropy wave as

$$\vec{e} = e^0\left(1, 0^j, -\frac{\chi}{\kappa}, 0^j\right)^T$$

where e^0 is our chosen normalization, and $(j = 1, 2, k)$.

4.1.2 Interlude

We now find ourselves considering the cases when a^k is different from zero i.e. we want to solve for the Alfvén and magnetosonic wave eigenvectors. For the moment we will simplify our equations generally with no assumptions yet to distinguish between different waves. Under this assumption equation (4.1) remains unchanged:

$$0 = e^0 \cdot a^k + (ve)W^2\rho_0 a^k + e^k \cdot \rho_0$$

Adding (4.1) to (4.3) we get:

$$\begin{aligned} 0 = e^0&\Big[(W^2\gamma - \chi)a^k + \chi v^k\Big] \\ &+\Big\{(ve)\Big[(2h_e W^4 + B^2)a^k - B^2 v^k\Big] - (Be)\Big[(Bv)(a^k - v^k) + B^k\Big] + e^k \cdot Q\Big\} \\ &+e^4\Big[(\rho_0 + \kappa)W^2 a^k + \kappa(v^k - a^k)\Big] \\ &+\Big\{-(v\hat{e})\Big[(Bv)(a^k - v^k) + B^k\Big] + (B\hat{e})\Big[(2 - \frac{1}{W^2})a^k + \frac{1}{W^2}v^k\Big] - \hat{e}^k(Bv)\Big\} \end{aligned} \tag{4.7}$$

Next we do $-v_i$ times (4.7) added to (4.2) to get:

$$
\begin{aligned}
0 = e^0\Big[-\chi\Big((v^k-a^k)v_i-h_i^{\ k}\Big)\Big] \qquad (4.8)\\
+\Big\{e_i\Big[Qa^k-(Bv)B^k\Big]+(ve)\Big[2B_iB^k+B^2\Big((v^k-a^k)v_i-h_i^{\ k}\Big)\Big]\\
+(Be)\Big[-B_ia^k+(Bv)\Big((a^k-v^k)v_i+h_i^{\ k}\Big)\Big]-e^k(Bv)B_i\Big\}\\
+e^4\Big[-\kappa\Big((v^k-a^k)v_i-h_i^{\ k}\Big)\Big]\\
+\Big\{\hat{e}_i\Big[-(Bv)a^k-\frac{B^k}{W^2}\Big]+(v\hat{e})\Big[-(Bv)\Big((v^k-a^k)v_i-h_i^{\ k}\Big)-B_ia^k\Big]\\
+(B\hat{e})\Big[-\frac{1}{W^2}\Big((v^k-a^k)v_i-h_i^{\ k}\Big)\Big]-\hat{e}^k\cdot\frac{B_i}{W^2}\Big\}
\end{aligned}
$$

If we contract equation (4.4) with B_i and v_i we get, respectively

$$0 = B^2\cdot e^k-B^k(Be)+(B\hat{e})a^k \qquad (4.9)$$

$$0 = (Bv)e^k-B^k(ve)+(v\hat{e})a^k \qquad (4.10)$$

Notice that we can use equation (4.10) to simplify (4.8) to

$$
\begin{aligned}
0 = (e^0\chi+e^4\kappa)\Big((a^K-v^k)v_i+h_i^{\ k}\Big) \qquad (4.11)\\
+\Big\{e_i\Big[(h_eW^2+B^2)a^k-(Bv)B^k\Big]+(ve)\Big[B_iB^k-B^2\Big((a^k-v^k)v_i+h_i^{\ k}\Big)\Big]\\
+(Be)\Big[-B_ia^k+(Bv)\Big((a^k-v^k)v_i+h_i^{\ k}\Big)\Big]\Big\}\\
+\Big\{\hat{e}_i\Big[-(Bv)a^k-\frac{B^k}{W^2}\Big]+(v\hat{e})\Big[(Bv)\Big((a^k-v^k)v_i+h_i^{\ k}\Big)\Big]\\
+(B\hat{e})\frac{1}{W^2}\Big[(a^k-v^k)v_i+h_i^{\ k}\Big]\Big\}
\end{aligned}
$$

Similarly, equation (4.7) contains the term $B^2\cdot e^k-B^k(Be)+a^k(B\hat{e})$, which is zero according to (4.9), so it simplifies to

$$
\begin{aligned}
0 = e^0&\left[(W^2\gamma-\chi)a^k+\chi v^k\right] \\
+(ve)\left[(2h_eW^4+B^2)a^k-B^2v^k\right]&+(Be)\left[(Bv)(v^k-a^k)\right]+e^k\cdot h_eW^2 \\
+e^4&\left[(\rho_0+\kappa)W^2a^k+\kappa(v^k-a^k)\right] \\
+(v\hat{e})\left[(Bv)(v^k-a^k)-B^k\right]&+(B\hat{e})\frac{1}{W^2}\left(v^k-a^k+W^2a^k\right)
\end{aligned}
\tag{4.12}
$$

If we contract the equation (4.11) with v^i, then multiply through by W^2, and in turn contract this same equation with B^i we get the following two equations:

$$
\begin{aligned}
0 = (e^0\chi+e^4\kappa)&(v^k-a^k+W^2a^k) \\
+(ve)\left[h_eW^4a^k-B^2(v^k-a^k)\right]&+(Be)\left[(Bv)(v^k-a^k)\right] \\
+(v\hat{e})\left[(Bv)(v^k-a^k)-B^k\right]&+(B\hat{e})\frac{1}{W^2}\left[v^k-a^k+W^2a^k\right]
\end{aligned}
\tag{4.13}
$$

$$
\begin{aligned}
0 = (e^0\chi+e^4\kappa)&\left[B^k-(Bv)(v^k-a^k)\right] \\
+(Be)\left[h_eW^2a^k-(Bv)^2(v^k-a^k)\right]&+(ve)\left[B^2(Bv)(v^k-a^k)\right] \\
-(B\hat{e})\frac{(Bv)}{W^2}\left[v^k-a^k+W^2a^k\right]&+(v\hat{e})\left[(Bv)\left(B^k-(Bv)(v^k-a^k)\right)\right]
\end{aligned}
\tag{4.14}
$$

Combining the two above equations by doing (4.14)+(Bv)(4.13) gives us (after much simplification)

$$
0 = (e^0\chi+e^4\kappa)\left(B^k+W^2a^k(Bv)\right)+(ve)h_eW^4a^k(Bv)+(Be)h_eW^2a^k \tag{4.15}
$$

It would now be helpful to establish a relationship between e^4 and e^0. To do this subtract (4.13) from (4.12) to get

$$
0 = e^0W^2a^k(\gamma-\chi)+e^4W^2a^k\rho_0+(ve)h_eW^4a^k+e^kh_eW^2
$$

Remembering the thermodynamic relations, $h_e\rho_0{c_s}^2={\rho_0}^2\chi+P\kappa$ and $\gamma=1+\varepsilon+\chi$, and the definition, $h_e = \rho_0(1+\varepsilon)+P$, we can rearrange the above into

$$0 = e^0 W^2 a^k \Big(\frac{h_e - P}{\rho_0}\Big) + e^4 W^2 a^k \rho_0 + (ve) h_e W^4 a^k + e^k h_e W^2$$

Finally, to this add $\left(-\frac{h_e W}{\rho_0}\right)$ times (4.1) and solve for e^4. This gives the useful relation

$$e^4 = e^0 \frac{P}{{\rho_0}^2} \tag{4.16}$$

Applying this last result to (4.15) and using one of the thermodynamic relations, we obtain

$$0 = e^0 \frac{h_e {c_s}^2}{\rho_0}\Big(B^k + W^2 a^k (Bv)\Big) + (ve) h_e W^4 a^k (Bv) + (Be) h_e W^2 a^k \tag{4.17}$$

Similarly, (4.13) becomes

$$\begin{aligned} 0 &= e^0 \frac{h_e {c_s}^2}{\rho_0}\big(v^k - a^k + W^2 a^k\big) \\ &+ (ve)\Big[h_e W^4 a^k - B^2 (v^k - a^k)\Big] + (Be)\Big[(Bv)(v^k - a^k)\Big] \\ &+ (v\hat{e})\Big[(Bv)(v^k - a^k) - B^k\Big] + (B\hat{e})\frac{1}{W^2}\Big[v^k - a^k + W^2 a^k\Big] \end{aligned}$$

The above can be simplified further if we utilize (4.13) and (4.14) to eliminate $(v\hat{e})$ and $(B\hat{e})$ leaving

$$0 = e^0 \frac{h_e {c_s}^2}{\rho_0}\big(v^k - a^k + W^2 a^k\big) + (ve)\big(h_e W^4 a^k - C^k\big) + (Be) D^k - e^k \frac{E^k}{a^k}$$

where,

$$\begin{aligned} C^k &= (v^k - a^k)\Big[B^2 - (Bv)\frac{B^k}{a^k}\Big] + \frac{B^k B^k}{a^k} \\ D^k &= (v^k - a^k)\Big[(Bv) + \frac{B^k}{W^2 a^k}\Big] + B^k \\ E^k &= (v^k - a^k)\Big[\frac{B^2}{W^2} + (Bv)^2\Big] + B^2 a^k - (Bv) B^k \end{aligned}$$

Lastly, we can eliminate the e^k term from the above by adding $\frac{E^k}{\rho_0 a^k}$ times (4.1) to it, producing

$$0 = \frac{e^0}{\rho_0}\Big(h_e {c_s}^2 (v^k - a^k + W^2 a^k) + E^k\Big) + (ve)\Big(h_e W^4 a^k - C^k + W^2 E^k\Big) + (Be) D^k \tag{4.18}$$

At this point we have effectively reduced our system to just two equations, (4.17) and (4.18), in two unknowns, (ve) and (Be), in terms of a normalization; e^0. We can write these two equations succinctly in a matrix equation:

$$\begin{pmatrix} h_e W^4 a^k - C^k + W^2 E^k & D^k \\ h_e W^4 (Bv) a^k & h_e W^2 a^k \end{pmatrix} \begin{pmatrix} ve \\ Be \end{pmatrix} = -e^0 \cdot \frac{1}{\rho_0} \begin{pmatrix} h_e c_s^2 \left(v^k - a^k + W^2 a^k\right) + E^k \\ h_e c_s^2 \left(B^k + W^2 a^k (Bv)\right) \end{pmatrix}$$

To solve, of course, we must invert the matrix on the left-hand side of the equation. It turns out that the determinant of this matrix is $h_e W^4 \Delta^{kk}$. Recall that setting the quantity Δ^{kk} equal to zero describes the Alfvén wave solutions. This means that for Alfvén waves the system degenerates, and we'll have to be a little cleaver in order to solve. But for magnetosonic waves $\Delta^{kk} \neq 0$ and $a^k \neq 0$. This will make solving for the magnetosonic wave eigenvectors more straightforward, even if tremendously complicated.

4.1.3 The Alfvén Waves

As previously stated $\Delta^{kk} = 0$ for Alfén waves and our system of equations becomes degenerate. In other words, our remaining two equations from the previous section must be the same up to an overall factor. This can only be true if $e^0 = 0$. (By (4.16) this also implies that $e^4 = 0$.) This result simplifies (4.17) to

$$(Be) = -(ve) W^2 (Bv) \tag{4.19}$$

and (4.1) becomes

$$e^k = -(ve) W^2 a^k. \tag{4.20}$$

Plugging this into (4.10) and (4.9), then rearranging we get

$$(v\hat{e}) = (ve) \left[W^2 (Bv) + \frac{B^k}{a^k} \right] \tag{4.21}$$

$$(B\hat{e}) = (ve) W^2 \left[B^2 + (Bv) \frac{B^k}{a^k} \right] \tag{4.22}$$

We have already shown that $\hat{e}^k = 0$, which leaves us with only e^i to calculate. From (4.4) and (4.20) we get

$$e^i = \frac{a^k}{B^k}\left[\hat{e}^i - (ve)W^2 B^i\right] \tag{4.23}$$

All that's left to do is to figure out e^i and $\hat{e}^i$ in terms of our chosen normalization, (ve). The good news is that we have sufficient information to do this. To see this more clearly, we write

$$e^i = \left(e_\perp{}^1, e_\perp{}^2, e^k\right) \qquad \hat{e}^i = \left(\hat{e}_\perp{}^1, \hat{e}_\perp{}^2, \hat{e}^k\right)$$

As such, we can write $(v\hat{e})$ and $(B\hat{e})$ as

$$(v\hat{e}) = v_1\hat{e}_\perp{}^1 + v_2\hat{e}_\perp{}^2 \qquad (B\hat{e}) = B_1\hat{e}_\perp{}^1 + B_2\hat{e}_\perp{}^2$$

Adding $(-\frac{B_2}{v_2})$ times the $(v\hat{e})$ equation to the $(B\hat{e})$ equation, and alternatively adding $(-\frac{B_1}{v_1})$ times the $(v\hat{e})$ equation to the $(B\hat{e})$ equation will lead to the following two results:

$$\begin{aligned}
\hat{e}_\perp{}^1 &= \frac{1}{v_1B_2 - v_2B_1}\left[B_2(v\hat{e}) - v_2(B\hat{e})\right] \\
&= \frac{(ve)}{v_1B_2 - v_2B_1}\left[B_2\left(W^2(Bv) + \frac{B^k}{a^k}\right) - v_2W^2\left(B^2 - (Bv)\frac{B^k}{a^k}\right)\right] \\
\hat{e}_\perp{}^2 &= \frac{1}{v_1B_2 - v_2B_1}\left[-B_1(v\hat{e}) + v_1(B\hat{e})\right] \\
&= \frac{(ve)}{v_1B_2 - v_2B_1}\left[-B_1\left(W^2(Bv) + \frac{B^k}{a^k}\right) + v_1W^2\left(B^2 - (Bv)\frac{B^k}{a^k}\right)\right]
\end{aligned}$$

upon applying (4.21) and (4.22).

Using (4.23), we can now write all the componants of the right eigenvectors for the Alfvén waves:

$$
\begin{aligned}
e^0 &= 0 \\
{e_\perp}^1 &= \frac{(ve)}{v_1B_2 - v_2B_1}\left[B_2\Big(W^2(Bv) + \frac{B^k}{a^k}\Big) - v_2W^2\Big(B^2 - (Bv)\frac{B^k}{a^k}\Big)\right] - (ve)W^2B^1\frac{a^k}{B^k} \\
{e_\perp}^2 &= \frac{(ve)}{v_1B_2 - v_2B_1}\left[-B_1\Big(W^2(Bv) + \frac{B^k}{a^k}\Big) + v_1W^2\Big(B^2 - (Bv)\frac{B^k}{a^k}\Big)\right] - (ve)W^2B^2\frac{a^k}{B^k} \\
e^k &= -(ve)W^2a^k \\
e^4 &= 0 \\
{\hat{e}_\perp}^1 &= \frac{(ve)}{v_1B_2 - v_2B_1}\left[B_2\Big(W^2(Bv) + \frac{B^k}{a^k}\Big) - v_2W^2\Big(B^2 - (Bv)\frac{B^k}{a^k}\Big)\right] \\
{\hat{e}_\perp}^2 &= \frac{(ve)}{v_1B_2 - v_2B_1}\left[-B_1\Big(W^2(Bv) + \frac{B^k}{a^k}\Big) + v_1W^2\Big(B^2 - (Bv)\frac{B^k}{a^k}\Big)\right] \\
\hat{e}_k &= 0
\end{aligned}
$$

where a^k is either solution to $\Delta^{kk} = 0$.

4.1.4 The Magnetosonic Waves

In section 3.1.2 we left off last with

$$
\begin{pmatrix} h_eW^4a^k - C^k + W^2E^k & D^k \\ h_eW^4(Bv)a^k & h_eW^2a^k \end{pmatrix}
\begin{pmatrix} ve \\ Be \end{pmatrix}
= -e^0 \cdot \frac{1}{\rho_0}
\begin{pmatrix} h_ec_s^2\big(v^k - a^k + W^2a^k\big) + E^k \\ h_ec_s^2\big(B^k + W^2a^k(Bv)\big) \end{pmatrix}
$$

In the case of the magnetosonic waves $\Delta^{kk} \neq 0$ and we have the luxury of a non-singular system. Solving for (ve) and (Be) amounts to inverting the matrix on the left which gives us:

$$
\begin{pmatrix} ve \\ Be \end{pmatrix}
= -e^0 \cdot \frac{1}{\rho_0 h_eW^4\Delta^{kk}}
\begin{pmatrix} h_eW^2a^k & -D^k \\ -h_eW^4(Bv)a^k & h_eW^4a^k - C^k + W^2E^k \end{pmatrix}
\begin{pmatrix} h_ec_s^2\big(v^k - a^k + W^2a^k\big) + E^k \\ h_ec_s^2\big(B^k + W^2a^k(Bv)\big) \end{pmatrix}
$$

Explicitly this means that

$$(ve) = -e^0 \cdot \frac{1}{W^2\rho_0\Delta^{kk}}\left[a^k\left(h_e c_s^2(v^k - a^k + W^2 a^k) + (v^k - a^k)\left(\frac{B^2}{W^2} + (Bv)^2\right) + a^k B^2 - (Bv)B^k\right)\right.$$
$$\left. - \frac{c_s^2}{W^2}\left(B^k + W^2 a^k (Bv)\right)\left((v^k - a^k)\left((Bv) + \frac{B^k}{W^2 a^k}\right) + B^k\right)\right]$$

and

$$(Be) = -e^0 \cdot \frac{1}{W^2\rho_0\Delta^{kk}}\left\{W^2 a^k\left[(Bv)(1-c_s^2)\left((v^k - a^k)\left(\frac{B^2}{W^2} + (Bv)^2\right) + B^2 a^k - (Bv)B^k\right)\right.\right.$$
$$\left. + h_e c_s^2\left((Bv)(v^k - a^k) - B^k\right)\right]$$
$$\left. + \frac{c_s^2}{W^2}\left[\left(B^k + W^2 a^k (Bv)\right)\left((Bv)B^k(1-W^2) - v^k(Bv)\frac{B^k}{a^k} - W^2\left((v^k - a^k)(Bv)^2 + B^2 a^k\right) + \frac{B^k B^k}{a^k}\right)\right]\right\}$$

While the expressions for (ve) and (Be) are complicated, they are, nevertheless, explicit and in terms of e^0. We will find that our system is well determined, and our remaining task is simply to find each of the components of our eigenvector in terms of e^0, (ve), and (Be). From equation (4.1) we get:

$$e^k = -\frac{a^k}{\rho_0}\left(e^0 + W^2\rho_0(ve)\right)$$

With this, $(B\hat{e})$ and $(v\hat{e})$ are completely determined by the relations (4.10) and (4.9) obtained previously; namely,

$$(B\hat{e}) = \frac{1}{a^k}\left((Be)B^k - e^k B^2\right) \qquad (v\hat{e}) = \frac{1}{a^k}\left((ve)B^k - e^k(Bv)\right)$$

We already know that $\hat{e}^k = 0$ and that $e^4 = e^0 \cdot \frac{P}{\rho_0^2}$, so all that are left to calculate are e^i and $\hat{e}^i$ for $(i \neq k)$. From equation (4.4) we get

$$\hat{e}^i = \frac{1}{a^k}\left(e^i B^k - e^k B^i\right) \tag{4.24}$$

Alternatively,

$$e^i = \frac{1}{B^k}\left(\hat{e}^i a^k + e^k B^i\right) \tag{4.25}$$

As a check, notice that if $i = k$ in these last two expressions we recover $\hat{e}^k = 0$ and $e^k = e^k$.

Our last step is to solve the remaining indexed equation for either e^i or $\hat{e}^i$ and combine these in turn with either of the results above to get expressions for e^i and $\hat{e}^i$ in terms of e^0. A simiplified version of (4.2) reads

$$\begin{aligned}
0 = e^0\Big(\chi + \kappa\frac{P}{\rho_0^2}\Big)\Big((a^k - v^k)v_i + h_i^{\ k}\Big) + e_i\Big[Q\cdot a^k - (Bv)B^k\Big] \\
+(ve)\Big[B_iB^k - |B|^2\Big((a^k - v^k)v_i + h_i^{\ k}\Big)\Big] + (Be)\Big[-B_ia^k + (Bv)\Big((a^k - v^k)v_i + h_i^{\ k}\Big)\Big] \\
-\hat{e}_i\Big[(Bv)a^k + \frac{1}{W^2}B^k\Big] + (v\hat{e})\Big[(Bv)\Big((a^k - v^k)v_i + h_i^{\ k}\Big)\Big] \\
+(B\hat{e})\frac{1}{W^2}\Big((a^k - v^k)v_i + h_i^{\ k}\Big)
\end{aligned}$$

Solving for e^i and $\hat{e}^i$ becomes a matter of using (4.24) and (4.25) as well as the previous expression for (Be), (ve), $(B\hat{e})$, and $(v\hat{e})$ and moving terms around. We leave this step to the reader.

While it may be a bit unsatisfying to leave the components of the magnetosonic wave eigenvector unwritten, we have, nonetheless, obtained explicit expressions for all its components in terms of e^0, our chosen normalization.

4.2 Left Eigenvectors

Previously we solved $\mathscr{F}^k\vec{e} = \lambda^k\mathscr{F}^0\vec{e}$. Now we wish to solve $\vec{e}\mathscr{F}^k = \lambda^k\vec{e}\mathscr{F}^0$ for its eigenvalues and eigenvectors. Note that here $\vec{e}$ is a 1x8 row vector $\vec{e} = (e^0, e^i, e^4, \hat{e}_i)$ where, $i \in \{1, 2, k\}$.

In general the eigenvalues of the left eigenvector equation are simply the complex conjugate of the right eigenvalues. Since we found only real eigenvalues for the right-hand equation, the eigenvalues corresponding to the left and right eigenvectors are the same. Another way to see this is to note that for both left and right eigenvalue equations we must solve $det(\mathscr{F}^k - \lambda^k\mathscr{F}^0) = 0$, which will yield one unique characteristic equation.

Since $\mathscr{F}^k - \lambda^k\mathscr{F}^0$ is not symmetric, however, the left eigenvectors are not necessarily the same as the right. In fact, in general, they should not be the same. Multiplying $\mathscr{F}^k - \lambda^k \mathscr{F}^0$ on the left by our newly defined row vector yields a new set of equations:

$$0 = e^0 \cdot W a^k + (ve) W^2 \gamma a^k + e^k \cdot \chi + e^4 \Big[(W^2\gamma - W - \chi) a^k + \chi v^k \Big] \tag{4.26}$$

$$\begin{aligned} 0 = {} & e^0 \Big[W^3 \rho_0 v_j a^k + W \rho_0 h_j{}^k \Big] \\ & + \Big\{ e_j \Big[Q \cdot a^k - (Bv) B^k \Big] + (ve) \Big[2 h_e W^4 v_j a^k + Q \cdot h_j{}^k - B_j B^k \Big] \\ & + (Be) \Big[-B_j a^k - (Bv) h_j{}^k + 2 v_j B^k \Big] + e^k \Big[(Bv) B_j - B^2 v_j \Big] \Big\} \\ & + e^4 \Big[(2 h_e W^4 + B^2 - W^3 \rho_0) v_j a^k - (Bv) B_j a^k + (Q - W\rho_0) h_j{}^k + ((Bv) B_j - B^2 v_j) v^k - B_j B^k \Big] \\ & + (B\hat{e}) h_j{}^k - \hat{e}_j \cdot B^k \end{aligned} \tag{4.27}$$

$$0 = (ve) \Big[(\rho_0 + \kappa) W^2 a^k \Big] + e^k \cdot \kappa + e^4 \Big[(\rho_0 + \kappa) W^2 a^k + \kappa (v^k - a^k) \Big] \tag{4.28}$$

$$\begin{aligned} 0 = {} & \Big\{ - e_j \Big[(Bv) a^k + \frac{1}{W^2} B^k \Big] + (ve) \Big[2 B_j a^k - v_j B^k - (Bv) h_j{}^k \Big] \\ & - (Be) \Big[v_j a^k + \frac{1}{W^2} h_j{}^k \Big] + e^k \Big[(Bv) v_j + \frac{1}{W^2} B_j \Big] \Big\} \\ & + e^4 \Big[(2 - \frac{1}{W^2}) B_j a^k - (Bv) v_j a^k + ((Bv) v_j + \frac{1}{W^2} B_j) v^k - v_j B^k - (Bv) h_j{}^k \Big] \\ & + \hat{e}_j \cdot a^k - (\bar{v}\hat{e}) h_j{}^k \end{aligned} \tag{4.29}$$

To see if we can simplify the above set of equations we do something analogous to what we did for the right eigenvectors. First we investigate some of the column operations used to simplify the determinant in finding the characteristic equaiton. For instance, adding $-\chi/\kappa$ times (4.28) to (4.26) gets rid of the e^k term in (4.26). After grouping terms and upon using $\gamma - \chi(1 - \frac{\rho_0}{\kappa}) = \frac{h_e}{\rho_0 \kappa}(\kappa - \rho_0 c_s^2)$ the resultant equation looks like:

$$0 = e^0 - e^4 + \frac{h_e W}{\rho_0}(1 - \frac{\rho_0 c_s^2}{\kappa})(ve + e^4) \tag{4.30}$$

Staying completely general for the moment we make some contractions with our indexed equations. Contracting (4.27) with B^j yields (after many simplifications):

$$0 = W\rho_0(W^2(Bv)a^k + B^k)(e^0 - e^4) + h_e W^2(2W^2(Bv)a^k + B^k)(ve + e^4) \\ +(Be)h_e W^2 a^k \tag{4.31}$$

Now we do (4.27) contracted with v^j. After much less simplification:

$$\begin{gathered} 0 = W\rho_0(v^k - a^k + W^2 a^k)(e^0 - e^4) \\ +(ve)\Big[2h_e W^4 a^k(1-\frac{1}{W^2}) - 2(Bv)B^k + (h_e W^2 + B^2)(v^k + a^k)\Big] \\ +e^4\Big[(2h_e W^4 a^k(1-\frac{1}{W^2}) + \big((Bv)^2 - B^2(1-\frac{1}{W^2})\big)(v^k - a^k) + (h_e W^2 + B^2)v^k - (Bv)B^k\Big] \\ -(Be)(Bv)(v^k + a^k - 2B^k) + e^k\Big[(Bv)^2 - B^2(1-\frac{1}{W^2})\Big] \\ +(B\hat{e})v^k - (v\hat{e})B^k \end{gathered}$$

Before proceeding with other contractions, we perform an operation on the above equation to get it into a simpler form. Add to it $-\frac{1}{\kappa}\big((Bv)^2 - B^2(1-\frac{1}{W^2})\big)$ times (4.28) to get:

$$\begin{gathered} 0 = W\rho_0(v^k - a^k + W^2 a^k)(e^0 - e^4) \\ +\Big[2h_e W^4 a^k(1-\frac{1}{W^2}) - (Bv)B^k + (h_e W^2 + B^2)v^k - W^2 a^k(1+\frac{\rho_0}{\kappa})\big((Bv)^2 - B^2(1-\frac{1}{W^2})\big)\Big](ve + e^4) \\ +(ve)\Big[-(Bv)B^k + (h_e W^2 + B^2)a^k\Big] - (Be)(Bv)(v^k + a^k - 2B^k) \\ +(B\hat{e})v^k - (v\hat{e})B^k \end{gathered} \tag{4.32}$$

Now we contract (4.29) with B^j and v^j to get (respectively):

$$0 = 2\left[B^2 a^k - (Bv)B^k\right](ve + e^4) + \left[(Bv)^2 + \frac{1}{W^2}B^2\right](e^4 v^k + e^k) \tag{4.33}$$
$$-2(Be)\left[(Bv)a^k + \frac{1}{W^2}B^k\right] - e^4 \cdot (Bv)^2 a^k + (B\hat{e})a^k - (\bar{v}\hat{e})B^k$$
$$0 = \left[(Bv)a^k - B^k\right](ve + e^4) - (ve)(Bv)v^k + e^4 \cdot \frac{1}{W^2}B^k \tag{4.34}$$
$$-(Be)\frac{1}{W^2}(v^k - a^k + W^2 a^k) + e^k(Bv) + (v\hat{e})a^k - (\bar{v}\hat{e})v^k$$

It now might be helpful to start looking for specific solutions.

4.2.1 The Entropy Wave

As we did before, we must look at each specific eigenvalue or set of eigenvalues in order to find the corresponding eigenvector. We again ignore the eigenvalue $\lambda^k = 0$ as it still only produces a trivial eigenvector. And so we investigate the entropy wave were we take $a^k \to 0$. All relations produced in this section apply specifically to the entropy wave unless otherwise stated.

First we simplify equation (4.31) by setting $a^k = 0$, which produces

$$0 = \rho_0(e^0 - e^4) + h_e W(ve + e^4)$$

Combining this relation with (4.30) (which remains unchanged when $a^k = 0$) we get a very useful result:

$$0 = h_e W \frac{\rho_0 c_s^2}{\kappa}(ve + e^4)$$
$$0 = (ve + e^4) \tag{4.35}$$

Then, according to (4.30):

$$0 = (e^0 - e^4) \tag{4.36}$$

We also notice immediately that equations (4.26) and (4.28) both produce the relation

$$0 = e^k + e^4 v^k \tag{4.37}$$
$$e^k = -v^k e^4$$

Here we begin using e^0 as our normalization, in which case we have already discovered that $e^k = -v^k e^0$ and $e^4 = e^0$.

We recognize immediately that (4.37) appears in (4.33). Making this substitution, along with the others up to this point, (4.33) simplifies to

$$(Be) = -\frac{1}{2}W^2(\bar{v}\hat{e})$$

Equation (4.34) can now be solved with the preceeding information for either $(\bar{v}\hat{e})$ or (Be) yielding:

$$(\bar{v}\hat{e}) = \frac{2}{W^2}\left(\frac{B^k}{v^k}\right)\cdot e^0 \tag{4.38}$$
$$(Be) = -\frac{B^k}{v^k}\cdot e^0 \tag{4.39}$$

As we already have an expression for e^k, let's examine equations (4.27) and (4.29) for the condition $j \neq k$. Writing all possible quantities in terms of e^0 and of course setting $a^k = 0$ equation (4.27) becomes

$$e_j(Bv) + \hat{e}_j = -2v_j\frac{B^k}{v^k}\cdot e^0$$

Doing the same for (4.29) shows that e_j is identically zero for $j \neq k$. With this knowledge we can write

$$\hat{e}_j = -2v_j\frac{B^k}{v^k}\cdot e^0 \tag{4.40}$$

For $j \neq k$.

To obtain our last component we use the relation $(\bar{v}\hat{e}) = \frac{2}{W^2}\left(\frac{B^k}{v^k}\right)e^0$ noting that $(\bar{v}\hat{e}) = \bar{v}^j\hat{e}_j = \bar{v}^1\hat{e}_1 + \bar{v}^2\hat{e}_2 + \bar{v}^k\hat{e}_k$. Since we have expressions for $\hat{e}_j$ via (4.40), this expression can be easily solved for $\hat{e}_k$. Note: $\bar{v}^2$ means the second component of $\bar{v}$, not $\bar{v}$ squared. We find

$$\hat{e}_k = 2\frac{B^k}{\bar{v}^k v^k}\left[\frac{1}{W^2} + \bar{v}^1 v_1 + \bar{v}^2 v_2\right]e^0 \tag{4.41}$$

Thus the eigenvector for the entropy wave can be written:

$$\vec{e}_k = e^0\Big(1, 0^i, 1, -2v_1\frac{B^k}{v^k}, -2v_2\frac{B^k}{v^k}, 2\frac{B^k}{\bar{v}^k v^k}\big[\frac{1}{W^2} + \bar{v}^1 v_1 + \bar{v}^2 v_2\big]\Big)$$

4.2.2 The Alfvén Waves

Having obtained the entropy wave eigenvector, we turn our attention back to the full set of uncontracted equations. Only this time we rearrange the equations in some familiar terms before the contraction process. We write them here:

$$0 = e^0 - e^4 + \frac{h_e W}{\rho_0}(1 - \frac{\rho_0 c_s^2}{\kappa})(ve + e^4) \tag{4.42}$$

$$\begin{aligned} 0 = {} & (e^0 - e^4)W\rho_0(W^2 v_j a^k + h_j{}^k) + e_j(h_e W^2 + B^2)a^k - (Bv)(e_j B^k + (Be)h_j{}^k) \\ & +(ve + e^4)\Big[2h_e W^4 v_j a^k + (h_e W^2 + B^2)h_j{}^k - B_j B^k\Big] + (Be)(2v_j B^k - B_j a^k) \\ & +\big(e^k - e^4(a^k - v^k)\big)\big((Bv)B_j - B^2 v_j\big) + (B\hat{e})h_j{}^k - \hat{e}_j \cdot B^k \end{aligned} \tag{4.43}$$

$$0 = (ve + e^4)\Big[\big(1 + \frac{\rho_0}{\kappa}\big)W^2 a^k\Big] + e^k - e^4(a^k - v^k) \tag{4.44}$$

$$\begin{aligned} 0 = {} & (ve + e^4)\Big[2B_j a^k - v_j B^k - (Bv)h_j{}^k\Big] - a^k\Big[e_j(Bv) + (Be)v_j\Big] \\ & -\frac{1}{W^2}\big(e_j \cdot B^k + (Be)h_j{}^k\big) + \big(e^k - e^4(a^k - v^k)\big)\big((Bv)v_j + \frac{1}{W^2}B_j\big) \\ & +\hat{e}_j \cdot a^k - (\bar{v}\hat{e})h_j{}^k \end{aligned} \tag{4.45}$$

Equations (4.43) and (4.45) can each in turn be combined with equations (4.42) and (4.44) to produce a total of two vector equations in the quantities $(ve + e^4), e_j, (Be), (B\hat{e}), (\bar{v}\hat{e})$, and $\hat{e}_j$. After some cancellation we obtain:

$$0 = (ve+e^4)\Big[h_eW^2\big(W^2v_ja^k + \frac{\rho_0 c_s^2}{\kappa}(W^2v_ja^k + h_j^{\ k})\big) - W^2a^k\big(1+\frac{\rho_0}{\kappa}\big)\big((Bv)B_j - B^2v_j\big) \quad (4.46)$$
$$-B^2h_j^{\ k} - B_jB^k\Big]$$
$$+e_j(h_eW^2+B^2)a^k - (Bv)(e_j\cdot B^k + (Be)h_j^{\ k}) + (Be)(2v_jB^k - B_ja^k)$$
$$+(B\hat{e})h_j^{\ k} - \hat{e}_j\cdot B^k$$

and

$$0 = (ve+e^4)\Big[B_ja^k\big(1-\frac{\rho_0}{\kappa}\big) - W^2a^k(Bv)v_j\big(1+\frac{\rho_0}{\kappa}\big) - v_jB^k - (Bv)h_j^{\ k}\Big] \quad (4.47)$$
$$-a^k\Big[e_j(Bv)+(Be)v_j\Big] - \frac{1}{W^2}\big(e_j\cdot B^k + (Be)h_j^{\ k}\big) + \hat{e}_j\cdot a^k - (\bar{v}\hat{e})h_j^{\ k}$$

Now do $a^k(4.25)+B^k(4.47)$ to eliminate $\hat{e}_j$. After simplifications

$$0 = (ve+e^4)\Big[W^2(a^k)^2\big(Q\cdot v_j - (Bv)(B_j+v_j)\big) \quad (4.48)$$
$$+\frac{\rho_0}{\kappa}W^2a^k\Big(h_ec_s^2W^2v_ja^k + h_j^{\ k} + a^k\big((Bv)(B_j-v_j) - B^2v_j\big)\Big) - \big(v_jB^k + (Bv)h_j^{\ k}\big)a^k$$
$$-\big(B^2h_j^{\ k} + B_jB^k\frac{\rho_0}{\kappa}\big)a^k\Big] + e_j\cdot\Delta^{kk} + (Be)\Big[\big(v_jB^k - (Bv)h_j^{\ k}\big)a^k - B_j(a^k)^2 - \frac{1}{W^2}B^kh_j^{\ k}\Big]$$
$$+(B\hat{e})h_j^{\ k}a^k - (\bar{v}\hat{e})h_j^{\ k}B^k$$

remembering that $Q = h_eW^2+B^2$ and $\Delta^{kk} = Q(a^k)^2 - 2(Bv)B^ka^k - \frac{1}{W^2}B^kB^k$.

After this, do $(Bv)(3.26) - \frac{1}{W^2}(3.25)$ to get a second, independent vector equation:

$$0 = (ve+e^4)\Big[a^k\Big(2(Bv)B_j - \big(1+\frac{\rho_0}{\kappa}\big)\big(B^2v_j + W^2(Bv)^2v_j\big)\Big) \quad (4.49)$$
$$-h_eW^2\Big(v_ja^k + \frac{\rho_0c_s^2}{\kappa}\big(v_ja^k + \frac{1}{W^2}h_j^{\ k}\big)\Big) - (Bv)\big(v_jB^k + (Bv)h_j^{\ k}\big) + \frac{1}{W^2}(B^2h_j^{\ k} + B_jB^k)\Big]$$
$$-e_j\big(\frac{1}{W^2}Q\cdot^k + (Bv)^2a^k\big) + (Be)\Big(2v_jB^k - a^k\big((Bv)v_j + \frac{1}{w^2}B_j\big)\Big)$$
$$+\hat{e}_j\big((Bv)a^k + \frac{1}{W^2}B^k\big) - (B\hat{e})\frac{1}{W^2}h_j^{\ k} - (\bar{v}\hat{e})(Bv)h_j^{\ k}$$

One might be tempted at first to look at the specific cases when $j \neq k$ which would eliminate the terms containing $(B\hat{e})$ and $(\bar{v}\hat{e})$, for example. But before doing this, let's contract equations (4.48) and (4.49) with B^j. Doing so will lead to an important result. We obtain

$$\begin{aligned} 0 = (ve+e^4)\Big[&W^2(a^k)^2(Bv)\big(h_eW^2-(Bv)\big)+W^2a^k\frac{\rho_0}{\kappa}\Big(W^2a^k(Bv)h_ec_s^2+B^k+(Bv)^2a^k\Big) \\ &-\Big(2(Bv)+B^2\big(1+\frac{\rho_0}{\kappa}\big)\Big)B^ka^k\Big] \\ &+(Be)\Big[-B^2(a^k)^2-\frac{1}{W^2}B^kB^k+\Delta^{kk}\Big]+(B\hat{e})B^ka^k-(\bar{v}\hat{e})B^kB^k \end{aligned} \tag{4.50}$$

and

$$\begin{aligned} 0 = (ve+e^4)\Big[&a^k\Big(2B^2-\big(1+\frac{\rho_0}{\kappa}\big)\big(B^2+W^2(Bv)^2\big)\Big)-h_eW^2\Big(a^k+\frac{\rho_0c_s^2}{\kappa}\big(a^k+\frac{1}{W^2}\frac{B^k}{(Bv)}\big)\Big) \\ &-2(Bv)B^k\Big(1-\frac{1}{W^2}\frac{B^2}{(Bv)^2}\Big)\Big] \\ &+(Be)\Big[2B^k-a^k\frac{1}{W^2(Bv)}\big(h_eW^2+2B^2\big)\Big]+(B\hat{e})a^k-(\bar{v}\hat{e})B^k \end{aligned} \tag{4.51}$$

Fortuitously, there is a single operation between the preceeding two equations that eliminates both $(B\hat{e})$ and $(\bar{v}\hat{e})$ namely, $(4.50)-B^k(4.51)$, which gives us

$$0 = A^{kk}(ve+e^4)+B^{kk}(Be) \tag{4.52}$$

where,

$$\begin{aligned} A^{kk} \equiv\; &W^2(a^k)^2\Big[(Bv)\big(h_eW^2-(Bv)\big)+\frac{\rho_0}{\kappa}\big(W^2(Bv)h_ec_s^2+\frac{B^k}{a^k}-(Bv)^2\big)\Big] \\ &-a^kB^k\Big[2\big(B^2+(Bv)\big)-W^2(Bv)^2\big(1+\frac{\rho_0}{\kappa}\big)-h_eW^2\big(1+\frac{\rho_0c_s^2}{\kappa}\big)\Big] \\ &+B^kB^k\Big[h_e\frac{\rho_0c_s^2}{\kappa(Bv)}+2\Big((Bv)-\frac{B^2}{W^2(Bv)}\Big)\Big] \end{aligned}$$

and,

$$B^{kk} \equiv \Delta^{kk}-B^2(a^k)^2+\frac{a^kB^k}{W^2(Bv)}\big(h_eW^2+2B^2\big)-B^kB^k\big(\frac{1}{W^2}+2\big).$$

And so, although the expression is complicated, we have obtained (Be) in terms of $(ve+e^4)$. With the above relationship in (4.52), let's again consider equation (4.48). But let us consider the case when $j \neq k$. Equation (4.48) then simplifies to

$$0=(ve+e^4)\Big[W^2(a^k)^2\Big(Q\cdot v_j-(Bv)(B_j+v_j)+\frac{\rho_0}{\kappa}\big(h_e c_s^2 W^2 v_j+(Bv)(B_j-v_j)-B^2 v_j\big)\Big) \\ -a^k B^k\big(v_j+\frac{\rho_0}{\kappa}B_j\big)-\frac{A^{kk}}{B^{kk}}\cdot a^k\big(v_j B^k-B_j a^k\big)\Big]+e_j\cdot\Delta^{kk} \tag{4.53}$$

For $j \neq k$ the above equation is valid for both Alfvén waves and magnetosonic waves. We'll solve for the magnetosonic wave eigenvectors in the next section. But for Alfvén waves $\Delta^{kk}\to 0$ and $B^{kk}\neq 0$ thus,

$$0=(ve+e^4) \tag{4.54}$$

for Alfvén waves.

At this point it seems convenient to start using the quantity (ve) as our chosen normalization for Alfvén waves just as we did in the case of the right eigenvectors. Returning to equations (4.50) and (4.51) we do an independent operation to produce a relation different from (4.52). Do (4.50) added to B^k(4.51) to produce

$$0=2B^k\Big(a^k(B\hat{e})-B^k(\bar{v}\hat{e})\Big)+C^{kk}(ve+e^4) \tag{4.55}$$

where,

$$C^{kk}\equiv(a^k)^2\Big[W^2(Bv)\big(h_eW^2-(Bv)\big)+\frac{\rho_0W^2}{\kappa}\big(W^2(Bv)h_ec_s^2+\frac{B^k}{a^k}-(Bv)^2\big)+\frac{A^{kk}}{B^{kk}}\Big(B^2-\frac{\Delta^{kk}}{(a^k)^2}\Big)\Big] \\ +a^kB^k\Big[2\big(B^2-(Bv)\big)-\big(1+\frac{\rho_0}{\kappa}\big)\big(2B^2+W^2(Bv)^2\big)-h_eW^2\big(1+\frac{\rho_0c_s^2}{\kappa}\big)+\frac{A^{kk}}{B^{kk}}\frac{1}{W^2(Bv)}\big(h_eW^2+2B^2\big)\Big] \\ +B^kB^k\Big[\frac{A^{kk}}{B^{kk}}\big(\frac{1}{W^2}-2B^k\big)-h_e\frac{\rho_0c_s^2}{\kappa(Bv)}+2\big(\frac{B^2}{W^2(Bv)}-(Bv)\big)\Big].$$

As was the case with equations (4.52) and (4.53), equation (4.55) is valid for both magnetosonic and Alfvén waves. Considering the case when $\Delta^{kk} \to 0$, equations (4.52) and (4.55) reveal

$$(Be) = 0 \tag{4.56}$$

$$(\bar{v}\hat{e}) = \frac{a^k}{B^k}(B\hat{e}) \tag{4.57}$$

respectively.

From (4.42) and (4.54) we can write $e^0 = e^4 = -(ve)$. And from (4.44), $e^k = \left(v^k - a^k\right)(ve)$. With the preceeding simplifications equation (4.49) becomes

$$e_j\Big(\frac{1}{W^2}(h_e W^2 + B^2) + (Bv)^2\Big)a^k = \hat{e}_j\Big((Bv)a^k + \frac{1}{W^2}B^k\Big) - (B\hat{e})\Big(\frac{a^k}{B^k}(Bv) + \frac{1}{W^2}\Big)h_j^{\ k} \tag{4.58}$$

Setting $j \neq k$ in the above equation the coefficent attached to $(B\hat{e})$ vanishes and we are left with the relation

$$\begin{aligned} \hat{e}_j &= e_j \cdot a^k \left(W^2 a^k (Bv) + B^k\right)^{-1} \left(h_e W^2 + B^2 + W^2 (Bv)^2\right) \\ &= e_j \cdot \varphi \end{aligned} \tag{4.59}$$

for $j \neq k$.

Having an expression for $\hat{e}_j$ in terms of e_j, $(j \neq k)$, it would behoove us to find expressions for e_1 and e_2 (i.e. the first two components of e_j) in terms of our normalization, (ve). This can easily be done with the knowledge that $(Be) = 0$. Remember that (ve) and (Be) are just shorthand for the sums $v^1 e_1 + v^2 e_2 + v^k e_k$ and $B^1 e_1 + B^2 e_2 + B^k e_k$ respectively. We already know that $e_k = (v_k - a_k)(ve)$, so we have a system of two equations in two unknowns (the raised numbers here are indexes not exponents). They are

$$\begin{aligned} (ve) &= v^1 e_1 + v^2 e_2 + v^k (v_k - a_k)(ve) \\ 0 &= B^1 e_1 + B^2 e_2 + B^k (v_k - a_k)(ve) \end{aligned}$$

Ignoring the details this gives us

$$
\begin{aligned}
e_1 &= \frac{(ve)}{B^1v^2 - B^2v^1}\left[(B^kv^2 - B^2v^k)(a_k - v_k) - B^2\right] \\
e_2 &= \frac{(ve)}{B^2v^1 - B^1v^2}\left[(B^kv^1 - B^1v^k)(a_k - v_k) - B^1\right]
\end{aligned}
$$

Having found these two components we have also managed to find $\hat{e}_1$ and $\hat{e}_2$ from equation (4.59). Simply put, $\hat{e}_1 = e_1 \cdot \varphi$ and $\hat{e}_2 = e_2 \cdot \varphi$. All that remains now is to find $\hat{e}_k$. To do this we use relationship (4.57). Expanding the sums out we obtain

$$
\bar{v}^1\hat{e}_1 + \bar{v}^2\hat{e}_2 + \bar{v}^k\hat{e}_k = \frac{a^k}{B^k}(B^1\hat{e}_1 + B^2\hat{e}_2 + B^k\hat{e}_k)
$$

After solving for $\hat{e}_k$ and substituting in known values for $\hat{e}_1$ and $\hat{e}_2$ we can write all components of the Alfvén wave eigenvectors. They are:

$$
\begin{aligned}
e^0 &= -(ve) \\
e^1 &= \frac{(ve)}{B^1v^2 - B^2v^1}\left[(B^kv^2 - B^2v^k)(a_k - v_k) - B^2\right] \\
e^2 &= \frac{(ve)}{B^2v^1 - B^1v^2}\left[(B^kv^1 - B^1v^k)(a_k - v_k) - B^1\right] \\
e^k &= (v^k - a^k)(ve) \\
e^4 &= -(ve) \\
\hat{e}_1 &= e_1 \cdot \varphi \\
\hat{e}_2 &= e_2 \cdot \varphi \\
\hat{e}_k &= \frac{1}{B^k\lambda^k} \cdot \frac{(ve)\cdot\varphi}{B^1v^2 - B^2v^1}\left\{\left(B^1a^k - \bar{v}^1B^k\right)\left[\left(B^kv^2 - B^2v^k\right)\left(a_k - v_k\right) - B^2\right]\right. \\
&\quad \left. - \left(B^2a^k - \bar{v}^2B^k\right)\left[\left(B^kv^1 - B^1v^k\right)\left(a_k - v_k\right) - B^1\right]\right\}
\end{aligned}
$$

where, $\varphi = a^k\left(W^2a^k(Bv) + B^k\right)^{-1}\left(h_eW^2 + B^2 + W^2(Bv)^2\right)$ and a^k is either solution to $\Delta^{kk} = 0$. Notice that when $v^1 = v^2$ and $B^1 = B^2$ components $e^1, e^2, \hat{e}_1$, and $\hat{e}_2$ blow up. Under these same conditions $\hat{e}_k$ becomes the indeterminant form $0/0$.

4.2.3 The Magnetosonic Waves

To find the four magnetosonic wave eigenvectors, we must now consider $\Delta^{kk} \neq 0$ (in addition to $\lambda^k \neq 0$ and $a^k \neq 0$) and instead focus on finding the eight eigenvector components where a^k is one of the four solutions in the case where the quartic term of the characteristic equation goes to zero. We will not find the roots of the quartic expression here. Instead, let's find the components of the magnetosonic wave eigenvectors in terms of a^k and assume we can solve for a^k numerically.

To this end, recall equation (4.53)

$$\begin{aligned} 0 = (ve+e^4)\Big[W^2(a^k)^2\Big(Q\cdot v_j - (Bv)(B_j+v_j) + \frac{\rho_0}{\kappa}\Big(h_e c_c^2 W^2 v_j + (Bv)(B_j - v_j) - B^2 v_j\Big)\Big) \\ - a^k B^k\big(v_j + \frac{\rho_0}{\kappa}B_j\big) - \frac{A^{kk}}{B^{kk}}\cdot a^k\big(v_j B^k - B_j a^k\big)\Big] + e_j\cdot\Delta^{kk} \end{aligned} \tag{4.60}$$

Assuming now that $\Delta^{kk} \neq 0$ we can write

$$\begin{aligned} e_j = -(ve+e^4)\cdot\frac{1}{\Delta^{kk}} \quad &\Big[W^2(a^k)^2\Big(Q\cdot v_j - (Bv)(B_j+v_j) + \frac{\rho_0}{\kappa}\Big(h_e c_s^2 W^2 v_j + (Bv)(B_j - v_j) - B^2 v_j\Big) \\ &- a^k B^k\big(v_j + \frac{\rho_0}{\kappa}B_j\big) - \frac{A^{kk}}{B^{kk}}\cdot a^k\big(v_j B^k - B_j a^k\big)\Big] \\ = (ve+e^4)\cdot\Phi_j \end{aligned}$$

for $j \neq k$.

Similarly, if $j \neq k$, then we can use the previous result as well as the fact that $(Be) = -\frac{A^{kk}}{B^{kk}}\cdot(ve+e^4)$ to write equation (4.49) as

$$\begin{aligned} \hat{e}_j = -(ve+e^4)\Big((Bv)a^k + \frac{B^k}{W^2}\Big)^{-1}\Big\{a^k\Big(2(Bv)B_j - \big(1+\frac{\rho_0}{\kappa}\big)\big(B^2+W^2(Bv)^2\big)v_j - h_e W^2\Big(1+\frac{\rho_0 c_s^2}{\kappa}\Big)v_j\Big) \\ -(Bv)v_j B^k + B_j\frac{B^k}{W^2} - \frac{A^{kk}}{B^{kk}}\Big(2v_j B^k - a^k\big((Bv)v_j + \frac{B_j}{W^2}\big)\Big) + \frac{a^k}{\Delta^{kk}}\Big(Q\cdot\frac{1}{W^2} + (Bv)^2\Big) \\ \times\Big[W^2(a^k)^2\Big(Q\cdot v_j - (Bv)(B_j+v_j) + \frac{\rho_0}{\kappa}\Big(h_e c_s^2 W^2 v_j + (Bv)(B_j - v_j)\Big) + \frac{A^{kk}}{B^{kk}}\cdot\frac{B_j}{W^2}\Big) \\ - a^k B^k\Big(\frac{\rho_0}{\kappa}B_j + \Big(1+\frac{A^{kk}}{B^{kk}}\Big)v_j\Big)\Big]\Big\} \end{aligned}$$

or

$$\hat{e}_j = (ve + e^4) \cdot \Psi_j$$

for $j \neq k$.

Although the above expression is almost comically complicated, we have nevertheless succeeded in writing four components ($e^1, e^2, \hat{e}_1$, and $\hat{e}_2$) of the left magnetosonic wave eigenvector in terms of the normalization $(ve + e^4)$.

Recall equation (4.52):

$$(Be) = -\frac{A^{kk}}{B^{kk}} \cdot (ve + e^4)$$

Knowing e_1 and e_2 this equation allows us easily to solve for e_k. It is straightforward to obtain

$$e_k = -\frac{1}{B^k}\Big[B^1 e_1 + B^2 e_2 + \frac{A^{kk}}{B^{kk}}(ve + e^4)\Big]$$

where B^1 and B^2 are components of the magnetic field, not powers of B. We leave e_k in terms of e^1 and e^2, which are in turn given in terms of $(ve + e^4)$.

Similarly we rewrite equation (4.55)

$$0 = 2B^k\Big(a^k(B\hat{e}) - B^k(\bar{v}\hat{e})\Big) + C^{kk}(ve + e^4)$$

where C^{kk} was defined previously. This relation allows us to solve for $\hat{e}_k$. Again, foregoing the details we arrive at

$$\hat{e}_k = -\frac{1}{\lambda^k}\Big[\big(\bar{v}^1 - \frac{a^k}{B^k}B^1\big)\hat{e}_1 + \big(\bar{v}^2 - \frac{a^k}{B^k}B^2\big)\hat{e}_2 - \frac{C^{kk}}{2B^kB^k}(ve + e^4)\Big]$$

where $B^kB^k \neq B^{kk}$ and $\hat{e}_1$ and $\hat{e}_2$ are given in terms of $(ve + e^4)$.

We now have only e^0 and e^4 to solve for. Equation (4.44) is solved easily for e^4 giving

$$e^4 = \frac{1}{v^k - a^k}\Big[e^k + W^2 a^k\big(1 + \frac{\rho_0}{\kappa}\big)(ve + e^4)\Big]$$

Finally, from (4.42) we get

$$e^0 = e^4 - \frac{h_e W}{\rho_0}\big(1 - \frac{\rho_0 c_s^2}{\kappa}\big)(ve + e^4)$$

Thus, using our normalization $(ve + e^4)$, we have found expressions for all eight components of the magnetosonic wave eigenvectors where a^k in this case are the four solutions for when the quartic term in the characteristic equation is equal to zero.

Chapter 5

Conclusion

We have obtained the wave speeds and left and right eigenvectors for the equations of ideal GRMHD. These results can be used in numerical codes that require the full characteristic decomposition of the equations. Such codes have been written, but as far as we can tell have not been applied to real world simulations such as binary neutron star mergers. It is also noteworthy to mention that none of the papers which currently employ a characteristic approach also use adaptive mesh refinement.

One possible extensions to this work could be the inclusion of divergence cleaning. This would relax the $\nabla \cdot \vec{B} = 0$ constraint, and therefore change our results (there would no longer be a trivial wave speed and eigenvector, for example). Another possibility is to removed the ideal assumption made in Chapter 1, i.e. we could do the same calculation in resistive GRMHD. This could allow models using the characteristic approach to include potentially interesting physics such as magnetic reconnection.

Of course the most important step going forward is to build a code and to compare results with what has already been done. The hope is to eventually see a code which can model the complex, multi-scale physics of strong magnetic and gravitational fields.

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