Abstract Notes

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Lagrange's Theorem: If H is a subgroup of a finite group G, then $|H| \mid |G|$.

0.0.1 Order of an Element

The order of an element a, denoted $|a|$, is the smallest positive integer n such that $a^n = e$.

$\langle a \rangle =$ the group generated by a.

Prop. 1: If $a\ inG$, then $|a| \mid |G|$

Prop. 2: Take $\mathbb{Z}_p^*$ where p is prime. Then $|\mathbb{Z}_p^*| = p - 1$

Suppose G is a group and H and K are subgorups of G.
Then $HK = \{hk \mid h \in H \text{ and } k \in K\}$

Theorem: $|HK| = \dfrac{|H||K|}{|H \cap K|}$

0.0.2 Stabilizer

The stabilizer of i in G is the set of elements of G that fix i.
Denoted $\mathrm{stab}_G(i)d$.

Prop: $\mathrm{stab}_G(i)$ is a subgroup of G.

0.0.3 Orbit

The orbit of i in G is the set of all $g(i)$ where $g \in G$. Denoted $\mathrm{orb}_G(i)$

Theorem: $|G| = |\text{stab}_G(i)||\text{orb}_G(i)|$

0.0.4 Normal Subgroup

Let G be a group. A subgroup N is called a normal subgroup of G if $gN = Ng$ for all $g \in G$. This does not mean that $gn = ng$ for all $n \in N$. It does mean that $gn = n_1g$ for some $n_1 \in N$.

0.0.5 Index of a Subgroup

Let H be a subgroup of G. Then the index of H in G is the number of distinct left (or right) cosets of H in G.
index $= \frac{|G|}{|H|}$

0.0.6 External Direct Product

Let G_1 and G_2 be two groups. The external direct product is defined as
$G_1 \oplus G_2 = \{(x, y)|x \in G_1, y \in G_2\}$.
And the product is defined by $(x, y)(x_1, y_1) = (xx_1, yy_1)$.
Observe that if G_1 and G_2 are abelian, then $G_1 \oplus G_2$ is abelian.

0.0.7 Internal Direct Product

Suppose H and K are normal subgroups of G. G is called the internal direct product of H and K if $G = HK$ and $H \cap K = \{e\}$

Theorem: Let G be an abelian group of order n. Suppose p is a prime and $p|n$. Then there exists an element $g \in G$ that is of order p.

0.0.8 Center of G:

$Z(G) = \{g \in G|gh = hg \text{ for all } h \in G\}$

Prop: $Z(G)$ is a normal subgroup of G.

Theorem: $Z(G)$ is a normal subgroup of G.

0.0.9 Homeomorphism

A homeomorphism is a mapping

$$\begin{aligned} \phi : & G \mapsto G' \\ & e \mapsto e' \end{aligned}$$

if $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G$

0.0.10 Kernel

The kernel of ϕ is the set of all elements in G that are mapped to e'. Denoted $\ker\phi = \{g \mid g \in G, \phi(g) = e'\}$

Prop: $\ker\phi$ is a normal subgroup of G.

Theorem: Suppose $\phi : G \mapsto G'$ is an onto homeomorphism. Then $G/\ker\phi$