Syntax of Propositional Logic

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Semantics was developed as a means to interpret formal languages, and although "interpreting" does not suffice to account for the plethora of phenomena that we identify as "meaning", it does constitute a fundamental part of it. Thus, before we start doing semantics, we will first look into a couple of formal languages and their interpretations, so that we develop all the tools that we will need later to tackle heftier tasks, like interpreting natural languages. These materials are based on Partee et al. (1990, §5–§6.1, §6.5), Gamut (1991, §2–§2.3, 4.3–§4.35) and Cable (2014).

1 Warming up

Our general goal is to gain some insight into why the area of research known as Formal Semantics, as understood mainly in linguistics, converged–at least to a great extent–into "model theoretic" and "truth-conditional" frameworks. The justification of these choices is a topic that is usually absent from introductory semantics texts, so our goal is to re-do the path that some of pioneers walk, and discuss the reasons why these frameworks are useful for linguistic research, as well as some of their limitations.

Although Formal Semantics is often portrayed as being the linguists' affair, philosophers and computer scientists also contribute their fare share; it's just not "advertised" as such (at least in linguistic circles), I guess because the general goals and underlying assumptions are quite different in the different areas.

Studying the methods that logicians developed to interpret logical languages, such as Propositonal Logic, First Order Logic, etc., provides an excellent means to familiarize ourselves with the notion of interpretation *relative to a model*, a key mathematical device that has had ample impact in the way that we think nowadays about the notion of "interpretation".

And even about what we think of truth itself! We will discuss these issues when we compare mode-theoretic accounts of truth to proof-theoretic accounts.

The plan now is the following. We will begin by looking into a purely *syntactic* characterization Propositional Logic and First Order Logic, so that we can then precisely characterize what it is to be an interpretation of these formal languages.

The elephant in the room Before we continue any further, however, it is worth clarifying what we will **not** attempt to do. Anyone who has taken a semantics course at some point has been confronted with the question *What is meaning?* Formal Semantics, as practiced these days, does **not** provide an answer to that question. Formal Semantics is concerned mainly with one of the many aspects that contribute to the study meaning, namely **compositionality**: the study of how the interpretation of a complex expression arises from the interpretation of its composing parts, and the role that syntactic structure plays on the structure-to-meaning mapping.

For good measure, consider the following list of questions below, all taken from Katz (1972). There is no question that they all fall squarely under the umbrella of what counts as "meaning", and thus we should worry about finding an appropriate answer if we care about accounting for meaning in natural language. But, in truth, most of Formal Semantics has focused on topics ranging between points 11 and 14 below.

Things seem to move along, albeit slowly. Potts (2005) provided a first systematic (i.e. logical) study of conventional meaning, moving beyond points 11–14 in Katz's list. The work by our own Sarah Murray on evidentials also expands Formal Semantics in further directions.

1. What are synonymy and paraphrase?
2. What are semantic similarity and semantic difference?
3. What is antinomy?
4. What is superordination?
5. What are meaningfulness and semantic anomaly?
6. What is semantic ambiguity?
7. What is semantic redundancy?
8. What is semantic truth (analyticity, metalinguistic truth, etc.)?
9. What is semantic falsehood (contradiction, metalinguistic falsehood, etc.)?
10. What is semantically undetermined truth or falsehood (e.g., syntheticity)?
11. What is inconsistency?
12. What is entailment?
13. What is presupposition?
14. What is a possible answer to a question?
15. What is a self-answered question?

Here by "superordination" Katz means the relation of a universal proposition (or an absolute generalization) to a particular proposition (or an instance) in the same terms.

Here Katz is asking about those truths that are contingent on facts in the world, i.e. he is echoing the Kantian analytic/synthetic distinction.

2 Historical background

2.1 First steps

The first stab at building an argument, understood as reaching a conclusion from some set of premises, is due to the ancient Hindu and Greek philosophers. They focused on the notion of a **valid argument**. A valid argument is one such that it is impossible for the premises to be true and the conclusion to be false. The following are some such examples.

(1) a. *Valid argument*
All men are mortal. Socrates is a man. Therefore, Socrates is mortal.

b. *Invalid argument*
All beer is booze. All wine is booze. Therefore, all beer is wine.

A valid argument does **not** need true premises; it is required that, if the premises were true, the truth of the conclusion would be guaranteed. A valid argument whose premises are also true is said to be **sound**.

The key insight from the ancient philosophers, passed onto us mainly through Aristotle's work, is that many valid arguments seem to share the *same general* *syntactic form.* Some examples: "All Ps are Q; All S are a P; Therefore, all S are Q" (type AAA-1, "Barbara"), or "No Ps are Qs. All Ss are Ps. Therefore, no Ss are Qs" (type EAE-1, "Celarent").

In fact, although there are infinitely many different syllogisms allowed in syllogictic (or term) logic, only 256 are logically distinct, and of those only 24 are valid.

2.2 Further developments

Syllogistic logic was the first attempt at characterizing valid arguments purely in terms of their syntactic form. It was so successful that it was *the logic* from 300BC to 1800s. But it had its weaknesses. For one, it only applies to "syllogisms", a very restricted form of argument! For instance, syllogism cannot capture valid arguments based on relations: "All vegetables are plants. Therefore, every vegetable's part is a plant's part.

Think about how many inferences we can draw from natural language expressions (not even sentences!) without conforming to syllogistic forms.

For the greatest part of (the first half of) the 19^{th} century however logic was in the spotlight of mathematics and philosohy.

- In the early to mid 1800s there was a renewed interest in logic again, which fostered the first groundbreaking work in centuries (Boole, DeMorgan).
- There was a renewed interest in figuring out whether mathematics could derive from logic or "pure thought".
- Gottlob Frege's *Begriffsschrift* (1879). It is the basically the birth of modern logic: it's the first appearance of quantified variables, and encompasses a system that is basically First Order Logic that reconstructs much of the mathematics (known at the time) from first (basic) principles, and also a lots of natural language reasoning! (I.e. the kind of reaoning that syllogisms could capture and more.)
- After Frege, many other logicians developed and modified the key insights of Begriffsschrift: Peano introduced much of our modern logical notations (e.g. $\forall$, $\exists$, $\supset$), Principia Mathematica (1910-1913; but begun much earlier) attempted at reconstructing mathematics from pure logical principles,

Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens, Halle a. S.: Louis Nebert. Translated as Concept Script, a formal language of pure thought modelled upon that of arithmetic, by S. Bauer-Mengelberg in J. vanHeijenoort (ed.), From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931, Cambridge, MA: Harvard University Press, 1967.

The Principia saw also the birth of "types" (that we so much love in semantics) as an attempt to solve Russell's paradox.

The main idea about a "logic" is that we can follow a precisely defined formal notation for representing certain aspects of the "logical structure" of an assertion. So, the study of logic is really the study of *a* logical language. The main backbone of any logic is a set of *syntactically* defined rules for deriving formulas in the notation from other formulas in the notation.

If by "assertion" we mean "natural language assertion", then our formal notation will certainly not be able to capture *all* of its aspects.

Statements that can be derived (i.e. proven) from other statements following the reasoning "rules of the language" are called *theorems*.

A catch Everthing we have said about logic so far pertains to the *syntax* of logical languages: as with classical logic, one could set the goal of the enterprise to provide a purely syntactic characterization of what it is to be a valid argument. But, validity is only half the picture. An argument is valid if the truth of its premises entails the truth of its conclusion.. Thus, it would be self-contradictory to affirm the premises and deny the conclusion of a valid argument. In propositional logic, valid arguments are those that are true under *all* interpretations, i.e. tautologies. Thus, to the extent that a validity

Validity also requires that each sub-argument (or logical step) in your reasoning be valid.

depends on the form of the argument, it is more a *syntactic* than a *semantic* notion.

A fact that proof-theory capitalizes on. For instance the following is a valid argument.

What we want, instead, (at least in the long run) is a notion of validity that respects not just syntactic aspects of the language, but also whether the premises are true: we want a system where the syntactically well-formed conclusions that follow from true premises are also true. But this is tricky. The key questions here are: how can you *show* that your logical system derives only true inferences? And how do you show that it derives *all* the true inferences in the language (e.g. without making nonsensical mistakes)? To do this, we need a mathematically precise characterization of what it means for a formula to be true relative to an interpretation.

We are not so late. It wasn't until about the 1930's or so until this idea reached its modern form (e.g. Löwenheim, Gödel, Tarski).

3 Syntax (and informal semantics) of Propositional Logic

The system of Propositional Logic (PL) is intended to capture the inferences that depend upon the meaning of the so-called "sentential connectives": *and, or, if...then*, and *not.* We start firs by defining the **non-logical constants** of the language, its **logical constants** and its syntactic symbols.

(2) The non-logical constants of the language are an infinite set of proposition letters: $\{p, q, r, s, t, ... p_1, p_2, p_3, ...\}$.

(3) The logical constants (and their informal semantics) of the language are:

a. $\neg$ (negation) "it's not the case that"

b. $\wedge$ (conjunction) "and"

c. $\vee$ (disjunction) "or (inclusive)"

d. $\rightarrow$ (material implication) "if...then"

(4) The syntactic symbols are "(" and ")".

With these definitions we can now recursively define the set of Well Formed Formulae (WFF) in the language. By a **formula** we simply mean a syntactic object that has been constructed according to the syntactic rules of the language (and that can be given a semantic meaning by means of an interpretation; more on that later).

"WFF" is pronounced often as "woof".

(5) The set of Well Formed Formulae of PL, WFF, is the smallest set s.t.:

a. If ϕ a proposition letter, then $\phi \in WFF$.

b. If $\phi \in WFF$, then $\neg\phi \in WFF$.

c. If $\phi, \psi \in WFF$, then $(\phi \wedge \psi) \in WFF$.

d. If $\phi, \psi \in WFF$, then $(\phi \vee \psi) \in WFF$.

e. If $\phi, \psi \in WFF$, then $(\phi \rightarrow \psi) \in WFF$.

The fact that the set WFF is given a recursive definition means that we can use mathematical induction to prove certain facts about it. A classical one is that for all $w \in WFF$, w has either 0 or an even number of parenthesis.

We can now use our recursive syntax of PL together with the our informal knowledge of the logical connectives in (3) to "translate" some sentences of English, with the help of a "key". For instance, if we had: p = "It is raining",

q = "Bill is singing", l = "Liz comes", s = "Sue comes", b= "Bob is happy" and r = "Rob is happy":

(6) a. $(\neg p \wedge (p \rightarrow q))$
It is not raining, and if it is raining, then Bill is singing.

b. $((l \vee s) \rightarrow (\neg b \wedge \neg r))$
If Liz or Sue come, then Bob and Rob are not happy.

4 Natural deduction

We have defined a formal language in such a way that we can determine, for any given sentence, whether the sentence belongs to the the language or not. I.e. we have a set of "words" (the propositional constants) which may combine through a set of specific rules to form well-formed formulae. The question, now, is: how can be sure–i.e prove–that a given formula is in WFF?

In this sense, we are using a notion of language much closer to that used in automata and formal language theory than in our every-day linguistic practice.

We can provide an answer to this question stated purely in syntactic terms. We will develop a proof system that will provide a full characterization of **valid inference** in PL. The plan is to lay out some rules–again, stated purely in syntactic terms–for deriving formulae in PL *from other formulae*.

As an added bonus, we will see how these syntactic rules intuitively capture certain key aspects of the ordinary meaning of the English words *and*, *or*, *not* and *if,...then*.

The key notion of the proof system is, thus, that of a **derivation** of a $w \in WFF$ from another $v \in WFF$. A derivation is a finite, numbered list of formulae, where each formula is accompanied by a coded statement indicating how it entered the derivation (a **justification**). Such justifications can be of one the following two forms: (i) an **Assumption**, indicating that the formula is an assumption (or premise), and (ii) a **Rule application**, written *Rule n,...m*, where "Rule" is the name of the name of one of the derivation rules (to be introduced below) and "n,..., m" are the numbers of the formulae in the derivation that provide the input to the rule.

Some terminology: the **assumptions** of the derivation are the formulae in the derivation that are accompanied by the justification "Assumption". The **conclusion** of the derivation is the final line in the derivation, our goal.

Once a conclusion is reached, we may express it by means of a **sequent**: $(\Gamma \vdash \phi)$, or simply $\Gamma \vdash \phi$ when there is no ambiguity, states that there is a derivation of ϕ (the conclusion of the sequent) from Γ, a set of statements that constitute the assumptions of the sequent. The sequent $\Gamma \vdash \phi$ means that "there is a proof whose conclusion is ϕ and whose undischarged assumptions are all in Γ". Notice that Γ may be empty, in which case we simply write $\vdash \phi$. It could also be that Γ is infinite.

Often $\Gamma \vdash \phi$ is read as "Γ 'entails' ϕ", but it is important to keep in mind that this is a notion of *syntactic* entailment, determined entirely in terms of provability: if we can assume that Γ is the case, then we can show that ϕ is the case too.

4.1 Conjunction Introduction and Elimination: $\wedge$I and $\wedge$E

The first rule is that of **conjunction introduction**, $\wedge$I. This rule allows us to derive a complex proof of the form $(\phi \wedge \psi)$ from two proofs (or assumptions): one of ϕ and one proof of ψ. The general form of the rule is given below:

Written also as "I$\wedge$", "'and' introduction, "&I" or "$\wedge$ intro", among others.

(7) Rule of $\wedge$I:

1	$\vdots$	
	$\vdots$	
m_i	ϕ	
	$\vdots$	
m_2	ψ	
	$\vdots$	
n	$(\phi \wedge \psi)$	$\wedge$I, m_i, m_2

There is also a corresponding **Sequent Rule** $\wedge$I: If $\Gamma \vdash \phi$ and $\Delta \vdash \psi$ are correct sequents, then $(\Gamma \cup \Delta \vdash (\phi \wedge \psi)$ is a correct sequent. (Notice that, as usual, parenthesis may be ommitted when there is no risk of ambiguity.)

With this rule it does not matter what step m_1 or m_2 comes first. For instance:

(8)

1	p	assumption
2	q	assumption
3	r	assumption
4	$p \wedge r$	$\wedge$I, 1, 3
5	$(p \wedge r) \wedge q$	$\wedge$I, 2, 4

Because the derivation in (8) is a **correct derivation**, we are allowed to assert that $\{p, q, r\} \vdash (p \wedge r) \wedge q$. The intuitive motivation (or connection) to English is that if we can truthfully assert some sentence "S_1", and some sentence "S_2", then we can also assert the sentence "S_1 and S_2".

The rule of **conjunction elimination**, or $\wedge$E, is in some sense its opposite: it allows us two ways of drawing conclusions, one for each of the conjuncts that a complex formula with the main sign $\wedge$ is connecting.

The corresponding Sequent Rule of $\wedge$E reads: If sequent $\Gamma \vdash (\phi \wedge \psi)$ is correct, then so are both sequents $\Delta \vdash \phi$ and $\Delta \vdash \psi$.

(9) Rule of $\wedge$E, (i):

1	$\vdots$	
	$\vdots$	
m	$(\phi \wedge \psi)$	
	$\vdots$	
n	ϕ	$\wedge$E, m

(10) Rule of $\wedge$E, and (ii):

1	$\vdots$	
	$\vdots$	
m	$(\phi \wedge \psi)$	
	$\vdots$	
n	ψ	$\wedge$E, m

As before, the connection to English is quite apparent as well: if we can (in English) assert some sentence "S_1 and S_2", then we can also assert "S_1" and "S_2" separately. Bellow is an illustration that uses both I$\wedge$ and E$\wedge$ to show that $p \wedge (q \wedge r) \vdash (p \wedge q) \wedge r$:

(11)

1	$p \wedge (q \wedge r)$	assumption
2	p	$\wedge$E, 1
3	$q \wedge r$	$\wedge$E, 1
4	q	$\wedge$E, 3
5	r	$\wedge$E, 3
6	$p \wedge q$	$\wedge$I, 2, 4
7	$(p \wedge q) \wedge r$	$\wedge$I, 5, 7

4.2 Implication Introduction and Elimination: $\rightarrow$I and $\rightarrow$E

Let us first consider the rule of *conditional elimination.* The intuitive idea here is that if we can (in English) assert the sentence "If S_1 then S_2", and we can assert the sentence "S_1", then we can also assert "S_2" (i.e. modus ponens).

(12) Rule of $\Rightarrow$E:

1	$\vdots$	
	$\vdots$	
m_i	$\phi \rightarrow \psi$	
	$\vdots$	
m_2	ϕ	
	$\vdots$	
n	ψ	$\Rightarrow$E, m_i, m_2

Sequent Rule $\Rightarrow$E: If $\Gamma \vdash \psi$ and $\Delta \vdash (\psi \rightarrow \phi)$ are both correct sequents, then the sequent $\Gamma \cup \Delta \vdash \phi$ is also correct.

The following example uses all the rules we have seen so far to prove that $\{p \wedge r, r \rightarrow q\} \vdash p \wedge q$:

(13)

1	$p \wedge r$	assumption
2	$r \rightarrow q$	assumption
3	p	$\wedge$E, 1
4	r	$\wedge$E, 1
5	q	$\Rightarrow$E, 2, 4
6	$p \wedge q$	$\wedge$I, 3, 5

The rule of **conditional introduction**, $\Rightarrow$I complicates things a bit, although the intuitive motivation is also quite clear: if whenever we assume "S_1" it follows that "S_2", then we can assert "If S_1 then S_2". That is, S_2 depends on the condition that S_1 holds.

(14) Rule of $\Rightarrow$I:

1	$\vdots$	
	$\vdots$	
m	$\quad\phi$	assumption
	$\quad\vdots$	
$n-1$	$\quad\psi$	
n	$\phi \to \psi$	$\Rightarrow$I, m, $n-1$

Sequent Rule $\to$I: If the sequent $\Gamma \cup \{\phi\} \vdash \psi$ is correct, the so is the sequent $\Gamma \vdash (\psi \to \phi)$.

There are some key aspects of $\Rightarrow$I that we have to keep in mind: (i) The derivation above is acceptable only as long as no line $n < j$ makes reference to any lines from m to $n-1$. (ii) Once the conditonal assumption has been "used up", we say that the formulae occurring on lines m to $n-1$ have been "discharged". (iii) The key restriction is that once a formula has been discharged, no subsequent lines can make reference to it.

Or "dropped", "withdrawn", etc.

It is now useful to revisit our earlier statement that certain formulae follow from no assumptions. It follows from this that there are certain derivations that follow from no premises, which we express by not writing anything to the left of the turnstile symbol. For instance, the following is one such valid inference: $\vdash (p \wedge q) \to p$.

(15)

1	$\quad p \wedge q$	assumption
2	$\quad p$	$\wedge$E, 1
3	$(p \wedge q) \to p$	$\wedge$I, 3, 5

The following is a somewhat more convoluted derivation without premises showing that $\vdash ((p \to q) \to r) \to (p \to (q \to r))$.

(16)

1	$\quad (p \to q) \to r$	assumption
2	$\quad\quad p$	assumption
3	$\quad\quad\quad q$	assumption
4	$\quad\quad\quad p \wedge q$	$\wedge$I, 2, 3
5	$\quad\quad\quad r$	$\Rightarrow$E, 1, 4
6	$\quad\quad q \to r$	$\Rightarrow$I, 3, 5
7	$\quad p \to (q \to r)$	$\Rightarrow$I, 2, 6
8	$((p \to q) \to r) \to (p \to (q \to r))$	$\Rightarrow$I, 1, 7

4.3 Disjunction Introduction and Elimination: $\vee$I and $\vee$E

This rules deal with inclusive disjunction. Consider first **disjunction inclusion**, $\vee$I. From a natural language standpoint this is also a somewhat strange rule, since it allows us to conclude that $(\psi \vee \phi)$ follows from ϕ. If we know that "S_1" is the case, what is the point of stating that "S_1 or S_2"? But the fact of the matter is, that our $(\psi \vee \phi)$ statement remain true, regardless of its usefulness as an assertion. The rule is as follows:

Sequent rule: If at least one of $\Gamma \vdash \phi$ and $\Gamma \vdash \psi$ is a correct sequent, then the sequent $\Gamma \vdash \phi \vee \psi$ is correct.

(17) Rule of $\vee$I, (i):

1	$\vdots$	
	$\vdots$	
m	ϕ	
	$\vdots$	
n	$\phi \vee \psi$	$\vee$I, m

(18) Rule of $\vee$I, and (ii):

1	$\vdots$	
	$\vdots$	
m	ψ	
	$\vdots$	
n	$\phi \vee \psi$	$\vee$I, m

The rule of *disjunction elimination*, $\vee$I, is again a little more involved. To know that $\phi \vee \psi$ is to know that at least one of the two holds, and nothing more. Thus, χ is to follow from $\phi \vee \psi$ in either way, it will have to follow in both ways. In other words, χ follows from $\phi \vee \psi$ just in case it can be derived from ϕ and also from ψ. That is, a conclusion χ may be drawn from $\{\phi \vee \psi, \phi \to \chi, \psi \to \chi\}$. The rule:

Sequent Rule $\vee$E: If the sequents $\Gamma \cup \{\phi\} \vdash \chi$ and $\Delta \cup \{\psi\} \vdash \chi$ are correct, then the sequent $(\Gamma \cup \Delta \cup \{(\phi \vee \psi)\} \vdash \chi$ is also correct.

(19) Rule of $\vee$E:

1	$\vdots$	
	$\vdots$	
m_1	$\phi \vee \phi$	
	$\vdots$	
m_2	$\phi \to \chi$	
	$\vdots$	
m_3	$\psi \to \chi$	
	$\vdots$	
n	χ	$\vee$E, m_1, m_2, m_3

Let's derive $q \vee p$ from $p \vee q$:

(20)

1	$p \vee q$		assumption
2		p	assumption
3		$q \vee p$	$\vee$I, 2
4	$p \to (q \to p)$		$\Rightarrow$I, 2, 3
5		q	assumption
6		$q \vee p$	$\vee$I, 5
7	$q \to (q \to p)$		$\Rightarrow$I, 5, 6
8	$q \vee p$		$\vee$E, 1, 4, 7

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