

# Structural Effects on Implicature Calculation

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## Abstract

This paper provides an investigation of Ignorance Inferences by looking at the superlative modifier *at least*. The formal properties of these inferences are characterized in terms of the epistemic conditions that they impose on the speaker, thereby establishing how much *can* and *must* be inferred about what the speaker is ignorant about. The paper makes two main contributions. First, it argues that the form of these inferences depends solely on the structural properties of the expression that *at least* is modifying, which do not necessarily coincide with semantic entailment. Rather, rank and order seems to matter: with totally ordered associates, *at least* triggers Ignorance Inferences that may be formally different than those obtained with partially ordered associates (Mencia 2016b). Second, it builds on neo-Gricean double alternative generation mechanisms (like Schwarz 2016) by showing that one of them must be provided by focus alternative semantics.

## 1 Introduction

Natural languages are furnished with a rich variety of expressions allowing speakers to convey that they are uncertain or ignorant about something, and thus that they cannot—or wish not to—commit to providing more information. These are expressions such as (a subset of) disjunctive statements, indefinites, modals, adverbs, evidentials, conditionals, mood markers, etc. Within this wide range of expressions, some seem to have the expression of uncertainty or ignorance more ingrained; in a sense, they seem to be “dedicated” markers of ignorance. Superlative modifiers like *at most* and *at least* constitute one such case. As illustration, consider (1):

- (1) a. #I have at most two daughters.  
b. #I have at least five fingers.

These examples are odd. The epistemic competence commonly assumed when we talk about progeny or our own body is at odds with the presence of superlative modifiers and their incompatibility with full knowledge. Contrast (1) with the felicitous (2), which differs only in that no speaker knowledge need be assumed.

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- (2) a. Bill has at most two daughters.
- b. That caterpillar has at least twenty legs.

This epistemic effect of superlative modifiers has been dubbed an Inference of Ignorance (II henceforth). The fact that superlative modifiers trigger IIs is uncontroversial. There also seems to be a growing consensus that IIs of superlative modifiers should be treated as the result of some or other pragmatic process. However, the controversy revolves around what exactly those pragmatic processes might be, and how we should formally characterize them. A number of different proposals, each introducing its own machinery, have been put forward. Semantically, superlative modifiers have been analyzed as modals (Geurts and Nouwen 2007), as *minima* and *maxima* operators (Nouwen 2010; Kennedy 2015), as inquisitive expressions (Coppock and Brochhagen 2013; Ciardelli et al. 2017), as operators of meta-speech acts (Cohen and Krifka 2014), and as epistemic indefinites (Nouwen, 2015). In deriving IIs, semantic accounts are often additionally augmented by pragmatic enrichment processes of various sorts, including neo-Gricean analyses (Büring 2007, Kennedy 2015, Schwarz 2016) and those relying on grammatical approaches to implicatures (Mayr 2013).

Despite this attention that superlative modifiers have attracted recently, investigation into IIs has asymmetrically focused on numerals or measure phrases—such as those in (1)/(2)—, leaving cases involving other categories (DPs, VPs, etc.) largely unexplored. But, as is common with other scalar modifiers, superlative modifiers may combine with a number of different types of complements or associates, consistently leading to IIs in all these environments:

- (3) a. At least *some students* came to the party. [HORN SCALES]  
        $\leadsto$  *the speaker is ignorant about whether all students came*
- b. At least *Bill and Sue* came to the party. [CARDINALITY SCALES]  
        $\leadsto$  *the speaker is ignorant about whether someone else came to the party*
- c. Sue won at least *the silver medal*. [LEXICAL SCALES]  
        $\leadsto$  *the speaker is ignorant about whether Sue won the gold medal*

This paper focuses on the IIs that arise with the modifier *at least* across these different types of associated scales. The first part of the paper is devoted to scrutinizing the exact form of IIs with *at least* with different types of scales and what they tell us about the speaker’s epistemic state in each case. Three key empirical points emerge from this investigation: (i) the nature of IIs is not uniform across associate types, since (ii) their exact form depends on the structural properties of the associate of *at least*—i.e. whether the domain of the associate is totally or partially ordered—and, in turn, (iii) these structural properties provide all the sufficient information to predict the correct IIs with *at least*, rendering notions such as semantic and contextual entailment between associates irrelevant.

These descriptive findings are used in the second part to provide a unified account for IIs. The specific account I endorse takes *at least* to be a scalar modifier interpreted relative to some focalized constituent. Building on earlier literature, IIs with *at least* arise as a kind of Quantity Implicature, derived in a neo-Gricean fashion. The calculation of implicatures with *at least* requires two sets of alternatives, as has already been proposed by Mayr (2013), Kennedy (2015) and Schwarz (2016) for numerals. The main innovation of the calculus presented here is that each set of alternatives relevant for the Gricean computation is provided by a different, independent, mechanism. The first method is the familiar substitution method within elements of a scale (Horn 1972, Sauerland 2004b, a.o.), where I take *at least* to form a “Horn set” with *only*, given the parallels between the two elements in terms of focus association (Schwarz 2016). In addition, a different set of alternatives is obtained by replacing

the focus-bearing constituent, i.e., *at least*'s associate, with contextually relevant alternatives (Rooth 1992, Fox and Katzir 2011). The resulting analysis (i) provides a homogeneous treatment of *at least* that (ii) applies to all contexts and (iii) captures all three empirical properties of *at least* statements described in the preceding paragraph.

## 2 Characterizing ignorance across contexts

### 2.1 Setting a baseline: *at least* and numerals

The first goal is to establish a reliable benchmark that will help determine the assertibility conditions of *at least*-statements across the board. Here we shall simply build on much previous research and look at cases where *at least* modifies a numeral associate. There is now a consensus that IIs of *at least* modifying a number  $n$  convey “partial ignorance”, whereby not every alternative value to  $n$  must necessarily constitute an epistemic possibility for the speaker (Kennedy 2015, Nouwen 2015, Schwarz 2016, a.o.). To see why, we can exploit the oddness that ensues when an II about proposition  $\phi$  clashes with a declaration of knowledge about  $\phi$ : cooperative speakers in ordinary contexts would not follow-up statement (4) with either (4a) or (4b), whereas a qualifying follow-up such as (4c) is felicitous and unproblematic.

- (4) Bill ate at least two apples...
- a. #but I know that he didn't eat exactly two.
  - b. #in fact, he did not eat more than two.
  - c. but I know that he didn't eat {four/three or four/between three or six/...}.

The felicity of the various cases in (4c) shows that when speakers chose an *at least*-statement, they are not committing themselves to be ignorant about all values above  $n$ ; they are, thus, only partially ignorant. Conversely, the infelicity of the follow-ups in (4a)/(4b) shows that, although speakers need not be totally ignorant about the exact value of  $n$ , they must still be ignorant about certain information. In this specific case, there must be at least two possibilities that speakers have to consider to be true: *exactly*  $n$  and *more than*  $n$  (Büring 2007). We shall thus informally summarize the assertibility conditions of *at least* with numeral associates as follows (Cohen and Krifka 2014, Spychalska 2015):

- (5) A proposition  $\phi$  of the form ‘*at least*  $n$   $P$ ’ is assertible by  $S$  if: (i) ‘*exactly*  $n$   $P$ ’ is compatible with all  $S$  knows, and (ii) ‘*more than*  $n$   $P$ ’ is compatible with all  $S$  knows.

These assertibility conditions correspond to what the epistemic state of a cooperative speaker has to be like so that a sentence like ‘*at least*  $n$   $P$ ’ can be uttered felicitously—assuming, of course, that the usual pragmatic principles are in place.

### 2.2 Ignorance beyond numbers

We now have a more precise—albeit informal—characterization of IIs that can be used as a benchmark to compare non-numeral cases. The non-numeral cases that are of special interest for us are those where the scales involved can be formally distinguished from the numeral scale. Taking the three examples in (3) above as a point of departure, we can identify two axes of variation: (i) semantic entailment ((3a)/(3b) vs. (3c)) and (ii) totally vs. partially ordered scales ((3a)/(3c) vs. (3b)).

Let us consider first *at least* with scales where the relation between its members is not driven by logical entailment. These usually fall into two categories: contextual-scales and so-called “ad hoc” or “lexical” scales (Hirschberg 1991). These scales are composed of elements that “outrank” each other but are either in a relation of contextual entailment or outright mutually exclusive. Take for instance the case of the lexical scale established by professorship ranks at US universities: *visiting professor, assistant professor, associate professor, full professor*. One cannot be an associate professor and a full professor at the same time, and yet one cannot be a full professor without having been an associate first. In this sense there is a common understanding that full professors outrank associate professors, but these ranks are not ordered by entailment.<sup>1</sup> As noted above, these pragmatic scales also trigger IIs, and so the sentence (6) below might convey ignorance as to the exact rank Bill holds. Furthermore, an inspection of this non-entailing scales with respect to numerals and determiners reveals that they all have parallel assertibility conditions.

- (6) Al is at least an assistant professor. . .
- a. #in fact, he has tenure.
  - b. #but I know that he does not have tenure.
  - c. but I know that he is not {an associate/a full} professor.

Suppose Al’s exact title was at stake and the speaker uttered (6). In doing so, she communicates ignorance about whether she is an assistant professor or holds some higher position, both of which are considered epistemic possibilities. She could be accused of being quite misleading if it was later revealed that she knew in fact, that Al has tenure (6a) or that Al does not have tenure (6b). Nevertheless, no conflict arises when additional knowledge about other ranks is conveyed (6c).

This clash is fully parallel to what we observed with numerals.<sup>2</sup> This resulting state of affairs reveals two facts, one for each of the two formal distinctions identified among the scales in (3). First, IIs remain the same irrespective of whether the members of the scales that *at least* acts on stand in a relation of entailment—semantic or contextual—or are mutually exclusive. Second, when *at least* associates with a scale whose members are totally ordered, there is a pair of IIs that are predictable and uniform across contexts:

- (7) Predictable IIs about proposition  $\phi$  with totally ordered associates for *at least*:
- a. The immediately higher ranked alternative to  $\phi$ , where for a pair  $x, y$ , if  $x > y$  and there is no  $z$  s.t.  $x > z > y$ ,  $x$  is immediately higher than  $y$ .
  - b. The exhaustive interpretation of  $\phi$ .

A second way in which scales associated with *at least* may vary formally from numeral associates has to do with their structural constitution. This is the case, for instance, of conjunctive plurals (e.g. as in (3b)), which provide domains that are only partially ordered, unlike the totally ordered

1 The difference between these ranks and those standing in a relation of contextual entailment amounts to the impact that additional, contextual premises may have in the entailment patterns. Take the scale representing the different types of school degrees available in the US educative system: *high-school degree, college degree, PhD degree*. Clearly *Sue has a college degree* does not semantically entail *Sue has a high-school degree*, but it does so contextually if we accept the additional premise that *All college graduates have college degrees*. In the case of lexical scales, there are no such premises—no additional information would ever make *Sue is an associate professor* entail that *Sue is an assistant professor*, and they remain always mutually exclusive.

2 Readers can easily check that this is the case also for scales other types of scales, such as entailing scales (e.g. quantificational-determiners), contextually entailing scales (e.g. school degrees in fn. 1) and even evaluative/preference scales (in the sense of Nakanishi and Rullmann 2009 and Biezma 2013, a.o.).

associates discussed above. The key difference between the two orderings is that partial orders admit elements that are not ordered with respect to each other; they are said to be “incomparable”.<sup>3</sup> For instance, assuming a domain with individuals *Bill*, *Sue* and *Al*, an expression of the form ‘*Al and Bill are Q*’ neither entails nor is entailed by ‘*Al and Sue are Q*’. Graphically:<sup>4</sup>

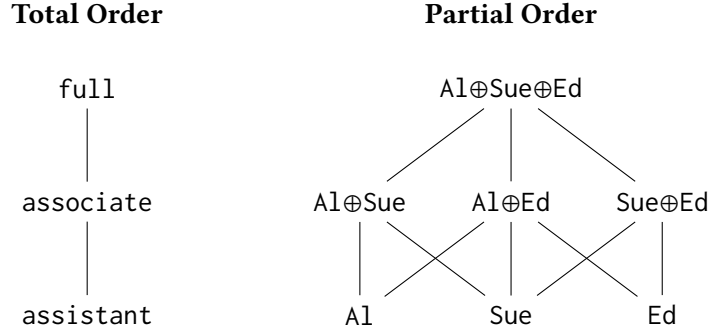


Figure 1: Structural differences between total vs. partial associates of *at least*

This structural difference is not innocuous: as it turns out, when *at least* acts on partially ordered associates none of the two conditions described in (7) hold mandatorily anymore. We begin examining condition (7a) first. Consider the following scenario.

- (8) **Context:** Sherlock Holmes went on vacation and let some of his friends celebrate a dinner on 221B Baker Street: Dr. Watson, Mrs. Hudson, Mycroft and Irene Adler. After vacation, he returns to his room only to discover that somebody messed with his chemistry set. Inspector Lestrade from Scotland Yard is with him, and asks: Who touched the chemistry set?
- (9) It was at least Mycroft and Mrs. Hudson...
- a. but certainly not Irene Adler.
  - b. but certainly not Dr. Watson.

Sherlock’s statement in (9) expresses an II about who exactly besides Mycroft and Mrs. Hudson touched the chemistry set. This II is nevertheless perfectly compatible with the knowledge that some other particular individual in the domain did not touch it, as shown by the felicity of the follow-ups in (9a)/(9b). The two follow-ups, however, correspond to alternative propositions that are in fact immediately higher than the prejacent, and the two statements should be odd according to our earlier conclusion in (7a) (cf. (4b)/(6b)), contrary to fact. Figure 2 shows the relevant portion of the domain, with the prejacent of (9) as its bottom element.<sup>5</sup>

We conclude, then, that (9) does not necessarily convey an II about the immediately higher ranked alternatives. Moreover, since the presence or absence of entailment (of any kind) between the different alternatives does not seem to play a role in the IIs that are conveyed with *at least*, it must be the different orderings that are to blame for the contrast.

3 Total orders are stronger in that they are connected: for any two distinct elements  $x, y$  of a totally ordered set, either  $x \leq y$  or  $y \leq x$ . With partial orders, instead, it could be that neither  $x \leq y$  nor  $y \leq x$  was the case for two distinct  $x, y$ .

4 For the moment, I will loosely refer to the nodes or elements in these structures as placeholders for both *at least*’s associates (of different syntactic categories) as well as full alternative propositions to the prejacent.

5 Here M stands for *Mycroft*, H for *Mrs. Hudson*, W for *Dr. Watson*, A for *Irene Adler*, M⊕H for *Mycroft and Mrs. Hudson*, etc.

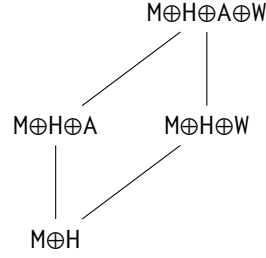


Figure 2: Structure of the domain in (8)/(9)

We turn now to (7b), the conclusion that *at least* conveys a mandatory  $\Pi$  about the exhaustive interpretation of the prejacent (see (4a)/(6a)). Here too we find a noticeable contrast. Suppose now that Sherlock provides the following answer, in the same context of (8).

(10) It was at least Mycroft and Mrs. Hudson, but not only them.

There does not seem to be anything wrong, odd or misleading about Sherlock’s answer in (10). Set in an ordinary detective dialog, we can take his contribution to be maximally informative: from his answer, Inspector Lestrade can learn that Sherlock knows that Mycroft and Mrs. Hudson did touch the chemistry set, but that they were not the only ones in doing so. That is, saying that at least them both touched the set is not at odds with an epistemic state where it is taken for granted that somebody else touched it as well. This is unlike the behavior of numerals, as a comparison to a numeral version of the same dialog in the same scenario reveals.

(11) IL. How many people do you think touched the chemistry set?  
SH. #It was at least two people, but not only two.

The contrast between (10) and (11) is clear, as (11) fails just like numerals did before—assuming, that is, that Sherlock is being genuine and maximally informative, given the evidence. This particular contrast between the answers in (10) and (11) led Mencia (2016b) to suggest the following generalization:<sup>6</sup>

- (12) Generalization on *at least*’s  $\Pi$ s (to be revised): [Mencia 2016b]
- a. When the associate of a *at least* is totally ordered, the exhaustive interpretation of the prejacent must necessarily constitute an epistemic possibility for the speaker;
  - b. When the associate of *at least* is partially ordered, the exhaustive interpretation of the prejacent can but need not constitute an epistemic possibility for the speaker.

As is, however, this generalization does not accurately represent the behavior of *at least*, and thus requires some qualification. In particular, the source of the asymmetry between (10) and (11) is not the lack of a connected domain, but more precisely the presence of immediately higher-ranked *incomparable* elements in the structure. This can be confirmed by looking at cases where *at least* is acting on associates that do have a unique immediately higher alternative but whose domain is nevertheless partially ordered. These are the “limiting cases” of partially ordered structures, cases

<sup>6</sup> Additionally, Mencia (2016b) presents a truth-value judgment study showing that speakers accept sentences of the form *at least a ⊕ b P but not only a ⊕ b P* far more often than the numeral counterparts *at least n P but not only n P*.



where the elements are dominated by unique higher ranked object, thus “flattening” the relevant portion of the domain and generating a sub-structure that is in fact totally ordered. In the structure depicted in Figure 2 we have two such cases:  $M \oplus H \oplus A$  and  $M \oplus H \oplus W$ . Now consider (13) in the same context of (8):

- (13) It was at least Mycroft, Mrs. Hudson and Irene Adler...
- a. #but certainly not just them.
  - b. #and clearly also Dr. Watson.

We have seen many such cases of oddness already: if Sherlock knew that Mycroft, Mrs. Hudson and Irene Adler *and someone else* did it as well, then he knew exactly who did it (all the individuals in the domain). Similarly, if Sherlock knew that the three of them *but nobody else* did it, then he would know for certain that the only culprits are the three of them. This is what the clashes in (13a)/(13b) reveal. This behavior is fully expected if, as we pointed out, the source of the asymmetry between totally vs. partially ordered associates is precisely whether the prejacent contains a unique immediately higher alternative or not. In (13) we picked a limiting case, effectively conflating the two cases by deeming most of the structure irrelevant, and the resulting statement behaved as though the domain was totally ordered. We can now amend the earlier generalization:

- (14) Generalization on *at least*’s IIs (final):
- a. When *at least* modifies an associate with a unique immediately higher alternative  $x$ , **both** the exhaustive interpretation of the prejacent **and**  $x$  must necessarily constitute an epistemic possibility for the speaker;
  - b. When *at least* modifies an associate with more than one immediately higher alternatives  $x_1, \dots, x_n$ , **neither** the exhaustive interpretation of the prejacent **nor** any of  $x_1, \dots, x_n$  must necessarily constitute epistemic possibilities for the speaker.

Notice that although clause (14b) states that *at least* conveys ignorance about no particular alternative in those contexts, the speaker must nevertheless be unable to settle which alternative is the case. Thus, denying all higher alternatives to the prejacent renders *at least* infelicitous:

- (15) #It was at least Mycroft and Mrs. Hudson, and nobody else.

To conclude, an examination of non-numeral associates reveals that the assertibility conditions of *at least* modifying a partially ordered scale differ significantly from those of totally ordered scales in certain contexts. More concretely, sentences of the form *at least n P* with some number  $n$  are only felicitous if the speaker takes the corresponding proposition *only/exactly n P* to be compatible with all she knows. In contrast, this is not always a requirement when *at least* modifies conjunctive plurals. The assertibility conditions of *at least* with conjunctions are determined by the number of available immediately higher alternatives to the prejacent, as described in (14).

### 2.3 Interim summary

There are minimal epistemic conditions that speakers must meet to successfully use *at least*, which in turn identify what is minimally predictable about the speaker’s epistemic state. With totally ordered associates *at least* triggers different IIs as compared to *at least* with partially ordered associates. The locus of the difference lies in what is required of the exhaustive interpretation of the prejacent and its immediately higher alternatives: the speaker **must** necessarily take these epistemic possibilities

into account when the prejacent has a unique immediately higher alternative, but she **need not** do so if these are non-unique. Finally, while IIs of *at least* are affected by the structural properties of its associates, it is largely irrelevant whether the different associates stand in a relation of entailment (either semantic or contextual) or are instead mutually exclusive. The IIs of *at least* remain invariant across varying entailment relations.

### 3 Laying the foundation

The discussion in section §2 reveals the properties that any adequate theory of *at least* must account for, properties that turned out to be more refined than previously noted. The remainder of the paper presents an account of *at least* that derives these properties. I begin by introducing some background assumptions and technical tools before turning to the formal analysis in §4.

#### 3.1 Background assumptions

I make two key background assumptions. The first one concerns the pragmatic nature of IIs. As indicated by our discussion of assertibility conditions, there is a tight connection between the felicity conditions of *at least* and the speaker’s communicative intentions. In fact, it could be argued that *at least*’s contribution to the discourse in non-embedded contexts is primarily to convey speaker’s ignorance (Coppock and Brochhagen 2013, Ciardelli et al. 2017). Yet, under certain circumstances, *at least*-statements seem felicitous even when their assertibility conditions are denied or blatantly unmet. This behavior bears the blueprint of a conversational implicature (Horn 1972, Grice 1975, Gazdar 1979, a.o.), and the consensus in the literature is that IIs of *at least* should be understood as such (see Büring 2007, Coppock and Brochhagen (2013), Mayr 2013, Kennedy 2015, Nouwen 2015, Schwarz 2016, a.o.). I take this much for granted and thus in the interest of space I do not provide here a justification of the implicature nature of *at least*.

A second key assumption regarding *at least* is the idea that focus serves a mediating role between its semantics and the IIs it gives rise to. This link, I argue, can shed light on when IIs are present/absent and also the precise nature of the IIs conveyed, but it requires the ancillary assumption that *at least* is in fact conventionally associated with focus, in the technical sense of Beaver and Clark (2008). Evidence for this claim is also extensive in the literature; the reader is referred to e.g. Krifka 1999, Beaver and Clark 2008, Coppock and Brochhagen 2013, Cohen and Krifka 2014, Mendia 2017, a.o.).

#### 3.2 Essential pragmatic calculus

To properly talk about IIs, we first need to know what it means to be ignorant about something. Assume that  $K$  and  $P$  stand for the familiar epistemic certainty and possibility operators, such that  $K_S\phi$  means *the speaker S knows that  $\phi$*  and  $P_S\phi$  means that  *$\phi$  is compatible with all S knows*.<sup>7</sup> Then, to be ignorant about a proposition  $\phi$  is expressed as follows:<sup>8</sup>

<sup>7</sup> The two operators  $K$  and  $P$  are interdefinable:  $K\phi \leftrightarrow \neg P\neg\phi$  and  $P\phi \leftrightarrow \neg K\neg\phi$  (Hintikka 1962). Hintikka’s system is an epistemic logic developed by enriching propositional calculus with the operator  $K$  and the three additional axioms  $K$  (distributivity;  $K(p \rightarrow q) \rightarrow (Kp \rightarrow Kq)$ ),  $T$  (reflexivity;  $Kp \rightarrow p$ ) and  $4$  (positive introspection;  $Kp \rightarrow KKp$ ). This is the  $KT4$  modal system, which I assume here. (See Hendricks and Symons 2014 on the adequacy of different logics to model knowledge and belief.)

<sup>8</sup> Square brackets ‘ $[\ ]$ ’ enclose propositions, so if  $\phi$  is a propositional variable,  $[\phi]$  stands for *the proposition that  $\phi$* .



$$(16) \quad \text{SIGNATURE OF IGNORANCE: } \neg K[\phi] \wedge \neg K\neg[\phi] \quad \leftrightarrow \quad P[\phi] \wedge P\neg[\phi]$$

(16) shows the technical notion of *ignorance* that I shall rely on, a notion stronger than mere lack of knowledge. By *being ignorant about  $\phi$*  I refer to a mental (epistemic) state of some agent in which she is unsure about the truth of  $\phi$ , i.e. it is necessary that the agent consider both  $\phi$  and  $\neg\phi$  live possibilities compatible with her knowledge (Hintikka 1962, 12-15). Thus, an inference of the form  $\neg K[\phi]$  is too weak to convey an II by itself. Sometimes I use the following notational convention, where  $I_S[\phi]$  means that *the speaker is ignorant about whether  $\phi$* :

$$(17) \quad I_S[\phi] \equiv \neg K_S[\phi] \wedge \neg K_S\neg[\phi] \quad \leftrightarrow \quad P_S[\phi] \wedge P_S\neg[\phi]$$

We turn now to the question of how to derive IIs of this form. I present here a rather condensed rendition of the neo-Gricean account of IIs as quantity implicatures, in the spirit of Horn (1972) and Gazdar (1979), putting together insights from both Hintikka's (1962) epistemic logic and Grice's (1975) theory of language in use (see also Gamut 1991, Sauerland 2004b, Fox 2007, Geurts 2010 a.m.o.). Assume that we are dealing with a cooperative speaker and that some version of the Maxims of Quality and Quantity are at work (Grice 1975).

(18) MAXIMS OF QUALITY: (i) Do not say what you believe to be false. (ii) Do not say what you do not have evidence for.

(19) MAXIM OF QUANTITY

If two propositions  $[\phi]$  and  $[\psi]$  are such that (i) the denotation of  $[\phi]$  asymmetrically entails  $[\psi]$ , (ii)  $[\phi]$  and  $[\psi]$  are relevant, and (iii) the speaker believes both  $[\phi]$  and  $[\psi]$  to be true, the speaker should choose  $[\phi]$  over  $[\psi]$ .<sup>9</sup>

The Maxims of Quality can be related to the operators  $K$  and  $P$  by Hintikka's (1962) principle of Epistemic Implication, whereby utterance of a sentence  $\phi$  by a speaker  $S$  commits  $S$  to the knowledge of  $\phi$ :  $\phi$  implicates  $\psi$  if  $K[\phi \wedge \neg\psi]$  is inconsistent. When a cooperative speaker  $S$  is following the Maxims of Quality, the addressee is allowed to infer that the utterance of  $\phi$  by  $S$  implicates that  $K_S\phi$ . The Maxim of Quantity provides a notion of strength: it ensures that given a number of true and relevant alternatives to the proposition that has been uttered, if a speaker is being cooperative, she should choose the semantically strongest, more informative alternative she has access to over the rest. In view of this definition of the Maxim of Quantity, it is useful to define the notion of Stronger Alternative (SA): An SA  $\psi$  of a proposition  $\phi$  is an alternative proposition that asymmetrically entails  $\phi$ :  $\psi$  is an SA of  $\phi$  iff  $\psi \rightarrow \phi$  and  $\phi \nrightarrow \psi$ . The set of SAs of a proposition  $\phi$  is expressed as  $SA(\phi)$  (as opposed to the set  $Alt(\phi)$  of all alternatives to  $\phi$ ). Thus, if we are to be cooperative, we have to provide the semantically strongest relevant and true proposition we can. We now define the weakest form of inference, a Primary Implicature, following Sauerland's (2004b) terminology. In addition, we also define the Implicature Base, the set of propositions resulting from conjoining the Quality Implicature with its Primary Implicatures.

(20) PRIMARY IMPLICATURE: The inference that  $\neg K\psi$ , for an SA  $\psi$ .

<sup>9</sup> Following common practice in model-theoretic semantics, we rely here on *semantic* entailment: a statement  $\phi$  is semantically entailed by a set of statements  $\Gamma$  iff every model in which all the statements in  $\Gamma$  are true is a model in which  $\phi$  is true. In possible worlds semantics, the more informative a statement, the further it reduces the worlds that are accessible from the evaluation world.

- (21) **IMPLICATURE BASE:** The set consisting of the Quality Implicature together with its Primary Implicatures.

As an illustration of how to derive an II in this framework, consider the following sentence (see also Spector 2006, Fox 2007, Schwarz 2016 a.m.o.):

- (22) Bill read Tintin or Asterix.

This sentence conveys that the speaker does not know which of the comics Bill read. The reasoning to derive such IIs proceeds as follows: First, assume that the speaker is being cooperative. This means that she is observing the Maxim of Quality. Assume moreover that there is no reason to believe that the speaker is not maximally informative, and so she observes the Maxim of Quantity as well. Upon hearing (22) (represented as  $[T \vee A]$ ), the addressee can conclude that the speaker thinks that this much is true. Thus, by the principle of Epistemic Implication, she concludes that  $K_S[T \vee A]$ . The proposition  $[T \vee A]$  has at least two stronger alternatives, the individual disjuncts  $[T]$  and  $[A]$ . This follows from the Maxim of Quantity:  $[T] \in SA([T \vee A])$ , since  $[T]$  is relevant and  $[T] \rightarrow [T \vee A]$ , but  $[T \vee A] \nrightarrow [T]$ . The same reasoning applies also to  $[A]$ . Following the Maxim of Quantity, the addressee concludes that if the speaker did not utter any one of the SAs, it must be because she did not have evidence enough, or maybe she did not know. Therefore, she infers the Primary Implicature that  $\neg K_S[T]$  and  $\neg K_S[A]$ . (23) below summarizes the process:

- (23) a. **ASSERTION:**  $[(22)] = [T \vee A]$   
 b. **EPISTEMIC IMPLICATION:**  $K_S[T \vee A]$   
 c.  $SA([(22)]) = \{[T], [A]\}$   
 d. **PRIMARY IMPLICATURES:**  $\neg K_S[T] \wedge \neg K_S[A]$   
 e. **IMPLICATURE BASE:**  $K_S[T \vee A] \wedge \neg K_S[T] \wedge \neg K_S[A]$

The Implicature Base contains all the information that the addressee may be able to deduce from the speaker's utterance without any further assumptions. These are not quite yet the IIs we want: in order to conform to the Signature of Ignorance, each disjunct must be an epistemic possibility for the speaker. Luckily, as many have shown, the task is trivial: given the properties of the operators  $K$  and  $P$  defined above,  $P_S[T]$  and  $P_S[A]$  are in fact entailed by the Implicature Base (see e.g. Spector 2006, Fox 2007, Schwarz 2016 a.o.).<sup>10</sup> In general, it is a feature of this type of neo-Gricean calculations that it derives IIs about pairs of Stronger Alternatives if they are (i) mutually exclusive and (ii) they jointly cover the logical space contributed by the meaning of the assertion. These Stronger Alternatives are often referred to as being *symmetric* (after Fox 2007), and the resulting possibility inferences  $\neg K_S\neg[T]$  and  $\neg K_S\neg[A]$  are said to be *pragmatically* entailed.

$$(24) \quad \overbrace{K_S[T \vee A] \wedge \neg K_S[T] \wedge \neg K_S[A]}^{\text{Implicature Base}} \wedge \underbrace{\neg K_S\neg[T]}_{\text{Pragmatic Entailment}} \wedge \underbrace{\neg K_S\neg[A]}_{\text{Pragmatic Entailment}}$$

Epistemic Implication    Primary Implicature    Primary Implicature    Pragmatic Entailment    Pragmatic Entailment

Thus, the Implicature Base alone provides all the necessary pieces to derive that the epistemic possi-

<sup>10</sup> To see that  $K[T \vee A] \wedge \neg K[T] \wedge \neg K[A] \rightarrow P[T] \wedge P[A]$  is the case, assume that  $\neg P[T]$ : since  $P[T]$  is equivalent to  $\neg K\neg[T]$ , then  $\neg P[T]$  is equivalent to  $\neg\neg K\neg[T]$ , which can be reduced to  $K\neg[T]$  by double negative. But  $K\neg[T]$  cannot be, since it contradicts the Primary Implicature in the premise. Thus, it must be the case that  $\neg K\neg[T]$ , which is equivalent to  $P[T]$ . (The same proof holds *mutatis mutandis* for  $P[A]$ .)

bility of every disjunct is a must.<sup>11</sup> It follows, too, that knowledge about the truth of any one of the particular disjuncts should not be allowed, in this case because both  $K_S[T]$  and  $K_S[\neg T]$  contradict the II that  $I_S[T]$ , and equivalently for  $K_S[A]$  and  $K_S[\neg A]$ .

The upshot of this discussion is that the choice of what counts as an SA is important: given the right choice of SAs, IIs may be entailed by the Implicature Base.<sup>12</sup> IIs of disjunctive statements can be derived by relying on independently needed formal principles, which provide the two necessary and sufficient ingredients to derive IIs about each particular disjunct: a suitable epistemic logic and the assumption that SAs are established by asymmetric entailment relations.

## 4 Calculating ignorance

This section provides a unified account of *at least* as a scalar modifier, following the idea that its IIs arise as Gricean conversational implicatures. Crucially, the analysis requires computing these implicatures by factoring in two sets of alternatives (Mayr 2013, Mendia 2016a, Schwarz 2016). The main innovation of the analysis presented here is that the set of alternatives relevant for the Gricean calculus is provided by two independent mechanisms. In addition to the neo-Gricean substitution method within elements of a Horn scale (Horn 1972, Sauerland 2004b, a.o.)—in this case between the focus particles *at least* and *only*—, a different set of alternatives is obtained by replacing the focus-bearing constituent with contextually relevant alternatives (Rooth 1992, cf. Fox and Katzir 2011).

The inclusion of each of the two sources of alternatives has its own motivation. Appealing to alternatives induced by focus-bearing constituents achieves three goals: to capture the focus sensitivity of *at least*; to exploit Roothian focus semantics and provide *at least* with the flexibility required to modify associates of various syntactic categories; and to access ordered structures that depend exclusively on *at least*'s associate, and thus need not rely on semantic entailment relationships. Appealing to a substitution method that generates *at least* and *only* alternatives fulfills two main tasks: it captures the fine sensitivity of *at least*'s IIs to the structural properties of its associate, and it allows ordinary pragmatic processes that rely on notions of informational strength to apply to ordered structures whose elements are logically (or contextually) independent—since, as we observed in §2, *at least* remains impervious to such considerations.

In what follows I show how, with these two modifications, we capture the three key properties of *at least*'s IIs mentioned above: (i) IIs of *at least* are partial, but with certain minimal conditions on speaker ignorance; (ii) these minimal conditions depend on the ordering properties of the associate and the location of the prejacent in the structure; and (iii) IIs are the same irrespective of the entailment properties of *at least*'s associates.

11 Above I ignored the scalar SA  $[T \wedge A]$ . Notice that after adding the corresponding Primary Implicature  $\neg K_S[T \wedge A]$ , the Implicature Base in (23e) does not entail that  $P_S[T \wedge A]$ , and so no II can be derived about  $[T \wedge A]$ . This may not be a bad thing, since implicatures associated with the conjunctive alternative to disjunctive statements can sometimes be strengthened to  $K_S[\neg(T \wedge A)]$  and so constitute a Secondary Implicature (or Scalar Implicature). In a classical neo-Gricean set-up, this strengthening requires an additional assumption often referred to as the “epistemic step” or “competence assumption” (see e.g., Geurts 2010 and the discussion in §5.1.3). This is not to say that (22) is incompatible with the speaker's ignorance as to whether Bill read both comics.

12 This is especially relevant when we consider disjunctions with multiple disjuncts. As Alonso-Ovalle (2006) showed, in order to calculate IIs with multiple disjuncts “sub-domain” alternatives—formed by smaller disjunctions each of whose individual disjuncts are part of the assertion—must be included (see also Spector 2006, Chierchia 2013 a.o.). Once the access to sub-domain alternatives is granted, the system presented in this paper can derive IIs of multiple disjuncts just the same.

## 4.1 Focus semantics

In order to calculate implicatures in a neo-Gricean framework, alternative propositions have to be ordered by the amount of information they convey. In the case of *at least*, such ordering is provided by focus alternatives. The semantics of focus delivers an ordinary semantic value and a focus semantic value that consists of a set of alternative propositions (Rooth 1985 *et seq.*). Then we can use this set of propositions to reason about plausible and more informative alternatives that the speaker could have uttered—just like we usually do in routine neo-Gricean pragmatics. I suggest that this constitutes the first set of relevant alternative propositions that is factored into the pragmatic calculus.

I take it that *at least* is a member of a limited class of focusing adverbs that bears a lexically determined dependency on focus, those which Beaver and Clark (2008) refer to as showing Conventional Association with Focus. (The theory of focus I am assuming corresponds then to an “intermediate” theory, in the sense of Rooth 1992.) Thus, *at least* always makes reference to focus-evoked alternatives compositionally derived throughout the semantic computation.

Informally, the meaning of a sentence  $S$  with some focalized constituent  $F$  is the set of propositions that obtains from  $S$  by making a substitution in the position corresponding to  $F$ . Alternatives to  $F$  are projected in a fully compositional fashion, hence it is a type-driven mechanism (Rooth 1985).<sup>13</sup> This is a two tier system delivering an ordinary semantic value  $\llbracket \cdot \rrbracket^o$  and a focus semantic value  $\llbracket \cdot \rrbracket^f$ .

- (25) a.  $\llbracket [Sue \text{ ate } [BROCCOLI]_F] \rrbracket^o = \text{ate}(Sue, broccoli)$   
 b.  $\llbracket [Sue \text{ ate } [BROCCOLI]_F] \rrbracket^f = \{x \in D_e \mid \text{ate}(Sue, x)\}$

As for the lexical entry of *at least*, assume a propositional version whereby it can directly take sets of propositions as arguments (Büring 2007).<sup>14</sup>

- (26)  $\llbracket at \text{ least } \rrbracket = \lambda C_{\langle st, t \rangle} . \lambda p_{\langle st \rangle} . \lambda w . \exists q [q \in C \wedge p \leq q \wedge q(w)]$

The association of *at least* with focus is no longer optional, as the set  $C$  is always constrained by the focus value of the associate. In addition to an ordinary semantic value, this definition produces a set of propositions determined by the focus semantic value. The lexical entry in (26) renders true a proposition  $p$  containing *at least* if there is some proposition  $q$  in the relevant set of alternative propositions which is at least as strong as  $p$  and  $q$  is true in the evaluation world.

In order to address the question of what focus sensitive expressions have in common, Rooth (1992, 1996) factors in the role of context on the semantic computation of sentences with focused constituents. In order for focus to be felicitous, the set of alternatives generated must be related to a contextually available set of alternatives  $C$ , where  $C$  is determined by contextually available or pragmatic information.

<sup>13</sup> A purely type-driven semantics of focus is known to overgenerate; see Katzir (2007) and Fox and Katzir (2011) for discussion and a proposal about how to fix it. For the purposes of this paper, I simply assume a semantic mode of composition whereby complex expressions like  $[\alpha, \beta]$  are compositionally derived by applying a function  $h$  pointwise, that is, to the meaning of  $[\alpha]$  and  $[\beta]$  separately (cf. Rooth 1996, 281):  $\llbracket [\alpha, \beta] \rrbracket^f \equiv \{h(X, Y) \mid X \in \llbracket \alpha \rrbracket^f, Y \in \llbracket \beta \rrbracket^f\}$ .

<sup>14</sup> This lexical entry leaves a number of questions open about how to better connect the syntactic properties of *at least* with its ability to semantically associate at a distance. In particular, it requires displacement of *at least* to a sentence initial position, and so it is insensitive to a number of limitations that *at least* shows (like, for instance, the inability to associate at a distance across subjects: *at least Bill ate an apple*, cannot mean the same as *Bill ate at least an apple*). Since the focus is on the IIs alone, I will not address these issues here.

- (27) Where  $\varphi$  is a syntactic phrase and  $C$  is a syntactically covert semantic variable,  $\varphi \sim C$  introduces the presupposition that  $C$  is a subset of  $\llbracket \varphi \rrbracket^f$  containing  $\llbracket \varphi \rrbracket^o$  and at least one other element. [Rooth 1996, 285]

In Roothian semantics, if  $XP$  is focused its focus-semantic value is the set of all the entities of its semantic type. The squiggle focus operator effectively limits a focus semantic value  $\llbracket XP \rrbracket^f$  to a contextually relevant set of alternatives  $C$  containing, minimally,  $\llbracket XP \rrbracket^o$  and one other element. Consider for instance a sentence like (28a) and its corresponding LF:

- (28) a. Sue is at least [an assistant professor]<sub>F</sub>  
 b. LF: [<sub>S1</sub> at least( $C$ ) [<sub>S2</sub> [<sub>S3</sub> Sue is [an assistant professor]<sub>F</sub> ]  $\sim C$ ]]

Uttered in an academic context,  $C$  is determined by  $\llbracket [\text{associate professor}]_F \rrbracket^f$  and thus it amounts to e.g.  $\{\text{assistant professor, associate professor, full professor, ...}\}$ . (The LF in (28b) follows Rooth in writing *at least*( $C$ ) to indicate that *at least* takes  $C$  as its domain argument.) The derivation of the ordinary and focus semantic values proceed routinely. (I use  $\llbracket XP \rrbracket^{f_c}$  to refer to the subset  $C$  of  $\llbracket XP \rrbracket^f$ .)

- (29) a.  $\llbracket 28a \rrbracket^f = \{P(\text{Sue}) \mid P \in D_{et}\}$   
 b.  $\llbracket 28a \rrbracket^o = \text{Sue is an assistant professor}$   
 c.  $\llbracket 28a \rrbracket^{f_c} = \left\{ \begin{array}{l} \dots \\ \text{Sue is a visiting professor,} \\ \text{Sue is an assistant professor,} \\ \text{Sue is an associate professor,} \\ \dots \end{array} \right\}$

Now *at least* can apply to the meanings in (29).

- (30)  $\llbracket (28a) \rrbracket = \lambda w. \exists q [q \in \llbracket (28a) \rrbracket^{f_c} \wedge \llbracket (28a) \rrbracket^o \leq q \wedge q(w)]$

As a consequence of this particular rendition of focus semantics, the strength of a proposition containing *at least* can only be assessed with respect to the focus semantic value of that proposition. The orderings induced by focus can be established conventionally, contextually or by the lexical properties of the focused constituents themselves. In the case of (28), the way in which professorships are ranked is set purely by conventions and world knowledge. In the case of numerals, *some*, conjunctive plurals, etc., the ordering is set by the entailment properties of the lexical items themselves. In this sense, an analysis of this type matches the predictions of those accounts that make use of a two alternative set strategy for numerals (notably Mayr 2013, Kennedy 2015 and Schwarz 2016)—modulo focus association—and extends it to virtually any constituent that *at least* can associate with.

## 4.2 Deriving alternatives

The set of alternatives obtained via focus-association can be further enriched by reasoning about plausible and more informative alternatives in the classical neo-Gricean sense. For (28) above, the focus value contributes a set of contextually restricted alternative expressions, which then expand accordingly to provide a set of propositions, as in (29c). Utilizing the same focus alternatives we can factor in about potential propositions that the speaker chose not to utter. Thus, for a proposition of the form [*at least*  $p$ ], we obtain the relevant set of propositions by replacing the prejacent  $p$  by every proposition in  $\llbracket p \rrbracket^{f_c}$ .

$$(31) \quad Alt_{\text{FOC}}(\llbracket 28a \rrbracket) = \{ \llbracket \text{at least } p \rrbracket \mid p \in \llbracket 28a \rrbracket^{f_c} \} = \left\{ \begin{array}{l} \dots \\ \text{Sue is at least a visiting professor,} \\ \text{Sue is at least an assistant professor,} \\ \text{Sue is at least an associate professor,} \\ \dots \end{array} \right\}$$

Notice that factoring in the set  $Alt_{\text{FOC}}$  in (31) serves an additional albeit crucial duty. No pragmatic enrichment process based on quantity/informativity considerations may apply to the propositions in (29c), since all the alternatives are mutually exclusive. This issue disappears if we consider alternative propositions that are themselves modified by *at least*: from the set (31) we can now identify the most informative alternatives as those in (32). In the case of (28) this ordering is obtained from *at least*'s associate conventionally, by world knowledge alone.

$$(32) \quad SA_{\text{FOC}}(\llbracket 28a \rrbracket^{f_c}) = \left\{ \begin{array}{l} \text{Sue is at least an associate professor,} \\ \text{Sue is at least a full professor,} \\ \text{Sue is at least a distinguished professor} \end{array} \right\}$$

The more informative a statement, the further it reduces the worlds that are epistemically accessible for the speaker from the evaluation world. For the alternatives in (32) this has the effect of ordering alternative propositions in terms of the size of compatible worlds—all else being equal: for a speaker truthfully uttering (28), the set of accessible worlds include worlds where Sue is either assistant, associate or full professor; e.g.  $W_{\text{ASSIST}}$ ,  $W_{\text{ASSOC}}$ ,  $W_{\text{FULL}}$ . Had speaker uttered *Sue is at least associate professor*, only  $W_{\text{ASSOC}}$  and  $W_{\text{FULL}}$  would be accessible. A cooperative speaker would then be urged to choose the latter over (28), in turn facilitating the usual competition-based pragmatic reasoning process.

Generally speaking, alternatives can be ordered by virtue of the lexical properties of its scales. This is no different in the case of *at least*. As advanced before, I suggest that *at least*-statements pragmatically compete with their corresponding *only*-statements. Traditional Horn scales like  $\{\text{some}, \text{all}\}$  or  $\{\text{or}, \text{and}\}$  are formed by sets of lexical items that stand in a relation of asymmetric entailment. Since *only* stands in an asymmetric entailment relation with *at least*, this seems a plausible option. Moreover, the fact that *only*, like *at least*, bears a conventionalized dependency on focus brings the connection between the two expressions closer. Thus, following the usual substitution method in neo-Gricean pragmatics, alternative propositions can be generated from the set of focus alternatives by swapping *at least* with *only*.<sup>15</sup> In this case, we generate the new set of alternative propositions in (33) by trading *at least* for *only*, and then we pick those alternatives that asymmetrically entail the prejacent to generate a second set of stronger alternatives, as in (34).

$$(33) \quad Alt_{\text{HS}}(\llbracket 28a \rrbracket) = \{ \llbracket \text{only } p \rrbracket \mid p \in \llbracket 28a \rrbracket^{f_c} \} = \left\{ \begin{array}{l} \dots \\ \text{Sue is only a visiting professor,} \\ \text{Sue is only an assistant professor,} \\ \text{Sue is only an associate professor,} \\ \dots \end{array} \right\}$$

<sup>15</sup> One may wonder whether the presuppositional properties of *only* first discussed in the classic analysis of Horn (1969) could interfere with the implicature calculation mechanism. While I do not have space to address this worry here, other formulations of this idea are also possible. For instance, one could think of replacing *only* with the silent exhaustivity operator *Exh* (Fox (2007), a.o.), as Schwarz (2016, fn.10) has pointed out. As in the present version, alternatives would be first generated by focus, optionally undergoing exhaustification through *Exh* thereafter. For now, assume Rooth's (1992) lexical entry for the exclusive *only*:  $\llbracket \text{only} \rrbracket = \lambda C_{(st,t)} . \lambda p_{(st)} . \lambda w . \forall q [q \in C \wedge q(w) \leftrightarrow p = q]$ .



$$(34) \quad SA_{HS}(\llbracket 28a \rrbracket) = \left\{ \begin{array}{l} \text{Sue is only an assistant professor,} \\ \text{Sue is only an associate professor,} \\ \text{Sue is only a full professor,} \\ \dots \end{array} \right\}$$

Thus, substitution of *at least* by *only* provides the second relevant set of alternatives that feeds the pragmatic calculus. Putting together both sets of Stronger Alternatives in (32) and (34), we get the final set of Stronger Alternatives over which we calculate implicatures.

$$(35) \quad SA(\llbracket 28a \rrbracket) = \left\{ \begin{array}{l} \text{Sue is at least an associate professor,} \\ \text{Sue is at least a full professor,} \\ \text{Sue is at least a distinguished professor} \\ \text{Sue is only an assistant professor,} \\ \text{Sue is only an associate professor,} \\ \text{Sue is only a full professor,} \\ \text{Sue is only a distinguished professor} \end{array} \right\}$$

As stated at the beginning of this section, each of the two modifications introduced in the process of generating the current set of alternatives serves its own purpose. Focus semantics grants us access to syntactically heterogeneous domains that, although ordered, may not be suitable for pragmatic competition based on informational strength. The inclusion of *at least* and *only* alternatives solves the issue by producing a set of alternatives where we do find semantic entailment relations. Thus, association with focus and *at least* ~ *only* competition manage jointly to create a set of alternative propositions amenable to pragmatic competition that is nevertheless congruent with the ordering established by *at least*'s associate.

### 4.3 Computing inferences

#### 4.3.1 Totally ordered associates

The set of Stronger Alternatives calculated above in (35) is sufficient to derive the right kind of IIs simply by following the Gricean style reasoning about conversational cooperation laid out in section §3.2. We continue to use the same example as above in (28), now expressed differently, for simplicity.<sup>16</sup>

$$(36) \quad SA(\llbracket 28a \rrbracket) = \left\{ \begin{array}{l} [\geq \text{Assoc}], [\geq \text{Full}], [\geq \text{Dist}], \\ [O \text{ Assis}], [O \text{ Assoc}], [O \text{ Full}], [O \text{ Dist}] \end{array} \right\}$$

Figure 3 below graphically represents the current entailment relations, including the utterance. Note that while all the  $[O \phi]$  alternatives are logically (and contextually) independent of each other, all *at least* alternatives either entail or are entailed by any other *at least* alternative. As a consequence, every *at least* statement, including the utterance itself, has two immediately stronger alternatives which are nevertheless logically independent of each other.

<sup>16</sup> The notational conventions are as follows: propositions are enclosed in square brackets, such that  $[\phi]$  stands for some proposition containing a relevant expression  $\phi$ , where  $\phi$  informally represents the associate of *at least* as a mnemonic of the relevant expression for the purpose of calculating implicatures. For instance, a sentence like *4 students came* is represented as  $[4]$  and *Al and Mary came* as  $[A \oplus M]$ . With modifiers,  $[\geq \phi]$  stands for *at least*  $\phi$  and  $[O \phi]$  for *only*  $\phi$ .

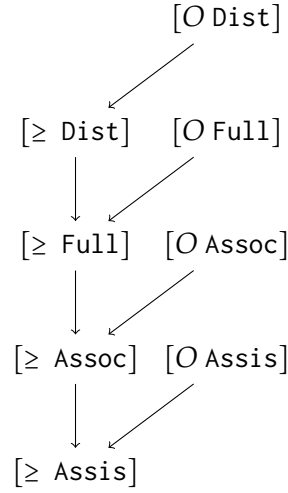


Figure 3: Entailment relations of (36) and its prejacent

Following standard neo-Gricean practice, when first confronted with an utterance by her interlocutor, a listener usually assumes that the speaker is being cooperative, and so she deduces that the proposition must be true given the speaker’s epistemic state; for the utterance (28), represented as  $[\geq \text{Assis}]$ , she deduces that  $K_S[\geq \text{Assis}]$  by Epistemic Implication. If there is no common understanding of the contrary, the listener may assume as well that the speaker is maximally informative, modulo relevance. If there is a Stronger Alternative that is known to be true and relevant, the speaker should have chosen it; since the speaker did not choose one of the Stronger Alternatives, it must be because she did not have sufficient grounds to claim so. Thus the listener is allowed to infer that the speaker does not possess such knowledge, thereby deriving a set of Primary Implicatures, (37d).

- (37) a. ASSERTION:  $[\geq \text{Assis}]$   
 b. EPISTEMIC IMPLICATION:  $K_S[\geq \text{Assis}]$   
 c.  $SA([\geq \text{Assis}]) = (36)$   
 d. PRIMARY IMPLICATURES:  
 $\neg K_S[\geq \text{Assoc}] \wedge \neg K_S[\text{O Assis}] \wedge \neg K_S[\text{O Assoc}] \wedge \neg K_S[\text{O Full}] \wedge \neg K_S[\text{O Dist}]$

Since the negation of the weakest *at least* alternative from the Stronger Alternative set (36) entails the rest of the stronger *at least* alternatives, only the weakest one is factored into the computation. Together with Epistemic Implication, these implicatures constitute the Implicature Base.

- (38) IMPLICATURE BASE:  
 $K_S[\geq \text{Assis}] \wedge \neg K_S[\geq \text{Assoc}] \wedge$   
 $\neg K_S[\text{O Assis}] \wedge \neg K_S[\text{O Assoc}] \wedge \neg K_S[\text{O Full}] \wedge \neg K_S[\text{O Dist}]$

As in the case of disjunction we illustrated in section §3.2, nothing else is required from the listener to draw ignorance about the speaker’s utterance, and at this point the task of deriving IIs from (38) is trivial: the Implicature Base entails that two and only two of the Stronger Alternatives in (36) must constitute epistemic possibilities for the speaker:  $\neg K_S \neg[\geq \text{Assoc}]$  and  $\neg K_S \neg[\text{O Assis}]$ . This follows simply from the properties of K and P and the fact that the associate of *at least* is totally ordered.

To see why this is the case, consider first  $\neg K_S \neg [\geq \text{Assoc}]$ , and assume, by *reductio*, that  $\neg K_S \neg [\geq \text{Assoc}]$  is not the case, i.e.  $K_S \neg [\geq \text{Assoc}]$ . What kind of epistemic state does such a speaker hold? Given the utterance, we know that  $K_S [\geq \text{Assis}]$  by Epistemic Implication. The space of possibilities denoted by  $[\geq \text{Assis}]$  jointly covered by the mutually exclusive stronger alternatives  $[\geq \text{Assoc}]$  and  $[O \text{ Assis}]$ , and thus one or the other must be true—otherwise the speaker would have uttered a false statement. Thus, if the speaker knew one of these two stronger statements is false, as with our initial assumption that  $K_S \neg [\geq \text{Assoc}]$ , then the other one would have to be the case; in this case,  $K_S [\geq \text{Assis}]$  and  $K_S \neg [\geq \text{Assoc}]$  together entail that  $K_S [O \text{ Assis}]$  is true. The problem is that  $K_S [O \text{ Assis}]$  directly contradicts the Primary Implicature that  $\neg K_S [O \text{ Assis}]$ , thereby rendering the Implicature Base inconsistent. Thus, it must be the case that  $\neg K_S \neg [\geq \text{Assoc}]$ , contrary to our initial assumption. A parallel reasoning shows that the second entailment  $\neg K_S \neg [O \text{ Assis}]$  also goes through. If  $\neg K_S \neg [O \text{ Assis}]$  were not true,  $K_S \neg [\geq \text{Assoc}]$ , it would entail that  $K_S [O \text{ Assis}]$  is true, which contradicts the Primary Implicature that  $\neg K_S [O \text{ Assis}]$ . Thus the final step is simply to acknowledge that the Epistemic Entailment and the Primary Implicatures gang up together to generate a set of pragmatic entailments that are formally identical to the Signature of Ignorance we defined in (16) above.

$$(39) \quad I_S [O \text{ Assis}] \wedge I_S [\geq \text{Assoc}]$$

As a consequence, a speaker uttering a statement that contains a *at least* is providing quite precise information about her epistemic state. No other epistemic possibilities are entailed. This last bit is paramount: it is important to recognize that the calculation presented above only goes through for cases where exactly two mutually exclusive Stronger Alternatives jointly exhaust the space of possibilities denoted by the utterance—i.e. they are symmetric. For the utterance  $[\geq \text{Assis}]$  those are the Stronger Alternatives  $[\geq \text{Assoc}]$  and  $[O \text{ Assis}]$  (see Figure 3). The issue is that if we generated IIs about every Stronger Alternative in (37c), we would predict that every SA must be an epistemic possibility for the speaker, contrary to fact. For further examination that the calculation fails to generate such unwanted IIs, take for instance the alternative proposition that *Sue is at least a full professor*,  $[\geq \text{Full}]$  and its corresponding Primary Implicature  $\neg K_S [\geq \text{Full}]$ . We have to show that no possibility inference  $\neg K_S \neg [\geq \text{Full}]$  is derived. As before, let us assume that  $K_S \neg [\geq \text{Full}]$  and try to derive a contradiction. The difference with the earlier case is that  $K_S [\geq \text{Assis}]$  and  $K_S \neg [\geq \text{Full}]$  together do not entail the truth of any other Stronger Alternative. In particular, it is still contingent whether  $[O \text{ Assoc}]$  is the case (and thus so is  $[\geq \text{Assoc}]$ ): knowing that Sue is no more than an associate professor does not commit the speaker to knowing that she is in fact an associate professor; given her utterance, she could be an assistant professor as well. A similar state of affairs holds of other stronger alternatives:  $K_S \neg [O \text{ Assoc}]$ , together with  $K_S [\geq \text{Assis}]$  does not commit the speaker to anything other than ruling out the possibility that Sue is an associate professor, for she could still be an assistant professor, a full professor, etc. (Section §5.1.3 discusses cases where all these stronger alternatives are further strengthened to Secondary Implicatures of the form  $K_S \neg [\phi]$ .) The inferred epistemic state of a collaborative speaker who uttered (28) while being certain she is not an associate professor would look as follows:

$$(40) \quad K_S [\geq \text{Assis}] \wedge K_S \neg [O \text{ Assoc}] \wedge I_S [O \text{ Assis}] \wedge I_S [\geq \text{Assoc}]$$

This accounts for the fact that IIs of *at least* only show partial ignorance: there are two and only two epistemic possibilities that are pragmatically entailed; the rest are simply contingent. This is, however, provided that the associate of *at least* is totally ordered. We can now see why: these

entailments are facilitated by a configuration where there are two (so-called symmetric) Stronger Alternatives that jointly exhaust the space of possibilities denoted by the assertion. As a consequence, one or the other corresponding Stronger Alternative must be true, and so negating any one of them entails the truth of the other. Moreover, the analysis predicts that these results should obtain as well for any *at least*-statement where the associate of *at least* is strictly ordered, as with other Horn sets, evaluative scales, etc. Finally, notice that these results track the perceived assertibility conditions of *at least* discussed in section §2, repeated here:

- (7) Predictable IIs about proposition  $\phi$  with totally ordered associates for *at least*:
- a. The immediately higher ranked alternative to  $\phi$ .
  - b. The exhaustive interpretation of  $\phi$ .

For *at least* to be assertible, a collaborative speaker must meet certain “epistemic criteria”. The two conditions middle clauses, (7b)/(7a) correspond to the epistemic possibilities entailed by the Implicature Base, and so they are to be observed. This accounts for the minimal pragmatic conditions that speakers must meet to successfully use *at least* without implying unwarranted additional IIs, i.e., while still conveying partial ignorance.

#### 4.3.2 Partially ordered associates

Let us now turn look at *at least* associating with expressions denoting partially ordered domains. Consider the following sentence with *at least* associating with a plural domain.

- (41) a. Liz saw at least  $[Bill]_F$   
b. LF:  $[_{S^1} \text{at least}(C) [_{S^2} [_{S^3} \text{Liz saw } [Bill]_F ] \sim C]]$

Assume a context with a reduced domain  $\{Bill, Sue, Ed\}$  of people that Liz could have seen. The derivation of the ordinary and focus semantic values proceeds as usual.

- (42) a.  $[[41a]]^f = \{saw(Liz, x) \mid x \in D_e\}$   
b.  $[[41a]]^o = \text{Liz saw Bill}$   
c.  $[[41a]]^{f_c} = \left\{ \begin{array}{l} \text{Liz saw Bill, Liz saw Sue, Liz saw Ed,} \\ \text{Liz saw Bill and Sue, Liz saw Bill and Ed, Liz saw Sue and Ed,} \\ \text{Liz saw Bill and Sue and Ed} \end{array} \right\}$

The truth-conditions are computed as in section §4.1, and as a result we obtain a lower bound on the range of allowable options. Then the derivation of alternatives and the calculation of implicatures proceeds exactly as in section §4.2 and section §4.3.

The crucial difference between partially vs. totally ordered associates lies in the entailments of the Implicature Base. In (41) *at least* associates with *Bill*, which is an element of an ordering of salient individuals. This domain, which takes the form of a join-semilattice (Link 1983), contains at least two pluralities that (i) are not comparable (i.e., not ordered with respect to each other), and that (ii) are minimally more informative than *Bill*: *Bill and Sue* and *Bill and Ed* (see Figure 1 above). As a consequence, the number of Stronger Alternatives that are minimally required to exhaust the space of possibilities denoted by the assertion is no longer two, but three—i.e. we no longer have a pair of symmetric alternatives: either Liz saw only Bill, or she saw at least Bill and Sue, or she saw at least Bill and Ed. This is illustrated in (43).

- (43) a.  $[\geq \text{Assist}] \leftrightarrow [O \text{ Assist}] \vee [\geq \text{Assoc}]$   
 b.  $[\geq B] \leftrightarrow [O B] \vee [\geq B \oplus S] \vee [\geq B \oplus E]$

Note that the logical signatures of the two cases above are isomorphic to the set of Primary Implicatures that Spector (2006) and Schwarz (2016) calculate for disjunctive statements with two and three disjuncts.

- (44) a.  $[A \vee B] \leftrightarrow [A] \vee [B]$   
 b.  $[A \vee B \vee C] \leftrightarrow [A] \vee [B] \vee [C]$

Like in (44), the disjunctions in the right-hand side of (43) constitute Stronger Alternatives individually entailing the left-hand side. We saw in §3.1 how this state of affairs works well for (44a), and how those results translate well to (43a) in the double-source approach that this paper explores. As both Spector (2006) and Schwarz (2016) note, the situation is somewhat troublesome for (44b). Statements with three disjuncts, such as *Liz read Tintin or Asterix or Spirou* convey IIs about each particular disjunct, but the basic neo-Gricean calculation presented in §3.1 fails to deliver them. In particular, for a statement  $[A \vee B \vee C]$  and Stronger Alternatives  $[A]$ ,  $[B]$  and  $[C]$ , the account predicts that the speaker's epistemic state is compatible with each *pair* of disjuncts  $[A \vee B]$ ,  $[A \vee C]$  and  $[B \vee C]$ , since the corresponding possibility inferences ( $\neg K_S \neg [A \vee B]$  and so on) are all entailed. Nevertheless, the account fails to produce parallel possibility inferences for each individual disjunct, of the form  $\neg K_S \neg [A]$ ,  $\neg K_S \neg [B]$  and  $\neg K_S \neg [C]$ , and thus we are stuck with the Primary Implicatures that  $\neg K_S [A]$ ,  $\neg K_S [B]$  and  $\neg K_S [C]$ , too weak to express IIs about each individual disjunct.<sup>17</sup>

What might be bad news for disjunctions with multiple disjuncts is nevertheless good news for (43b): as discussed in §2.2, statements like (41) do not necessarily require IIs about each one of the Stronger Alternatives, even if they are mutually exclusive and jointly cover the logical space of the asserted meaning. Thus, the difference between the logical signatures in (43) has major albeit welcome consequences when we compute the entailments of the Implicature Base. Again, consider *Liz saw at least Bill*. The calculation of the Implicature Base is summarized below.

- (45) a. ASSERTION:  $[\geq B]$   
 b. EPISTEMIC IMPLICATION:  $K_S[\geq B]$   
 c.  $SA([\geq B]) =$   

$$\left\{ \begin{array}{l} [O B], \\ [O B \oplus S], [O B \oplus E], [O B \oplus S \oplus E], \\ [\geq B \oplus S], [\geq B \oplus E], [\geq B \oplus S \oplus E] \end{array} \right\}$$
  
 d. PRIMARY IMPLICATURES:  

$$\left\{ \begin{array}{l} \neg K_S [O B] \wedge \\ \neg K_S [O B \oplus S] \wedge \neg K_S [O B \oplus E] \wedge \neg K_S [O B \oplus S \oplus E] \wedge \\ \neg K_S [\geq B \oplus S] \wedge \neg K_S [\geq B \oplus E] \wedge \neg K_S [\geq B \oplus S \oplus E] \end{array} \right\}$$
  
 e. IMPLICATURE BASE:  
 $K_S[\geq B] \wedge \neg K_S [O B] \wedge \neg K_S [\geq B \oplus S] \wedge \neg K_S [\geq B \oplus E]$

Unlike with totally ordered associates, the resulting Implicature Base does not entail that any one of the Stronger Alternatives constitutes an epistemic possibility for the speaker. Indeed, now we could negate any one Stronger Alternative without contradicting nor entailing the truth of any other. This

<sup>17</sup> Without further stipulations the accounts also delivers an inconsistent set when strengthening Primary Implicatures  $\neg K_S[\phi]$  to Secondary Implicatures  $K_S \neg[\phi]$  for each individual disjunct; see Sauerland 2004b, Alonso-Ovalle 2006, Spector 2006, Fox 2007 a.o., and §5.1.3 for further discussion.

is contrast with the behavior of disjuncts with multiple disjunctions: although the two constructions share the same logical signature, negating any one individual disjunct results in infelicity:

(46) Liz read Tintin, or Asterix, or Spirou, #but she didn't read Tintin.

Suppose, then, that the speaker knew that Liz did not see only Bill,  $K_S \neg[O B]$ . Conjoining this assumption with the Implicature Base results in a contingent set of propositions: all it says is that the speaker knows that Liz saw somebody else besides Bill, but she does not know who.

(47)  $K_S[\geq B] \wedge K_S \neg[O B] \wedge \neg K_S[O B] \wedge \neg K_S[\geq B \oplus S] \wedge \neg K_S[\geq B \oplus E]$

Knowing that  $K_S \neg[O B]$  does not settle the question as to which one of  $[\geq B \oplus S]$  or  $[\geq B \oplus E]$  is true, and so the speaker is predicted to be ignorant precisely about these two Stronger Alternatives: since together they carve out the remaining space of possibilities, negating one of them, e.g.,  $K_S \neg[B \oplus S]$  would in turn entail the truth of the second,  $K_S[B \oplus E]$ , contradicting once again the corresponding Primary Implicature that  $\neg K_S[B \oplus E]$  and resulting in an inconsistent set of beliefs. In these situations, all the speaker is allowed to infer is an epistemic state of the following form:

(48)  $K_S[\geq B] \wedge K_S \neg[O B] \wedge I_S[\geq B \oplus S] \wedge I_S[\geq B \oplus E]$

Of course, as noted in (15) and predicted by (45), although *at least* conveys ignorance about no particular alternative, the set of Stronger Alternatives, the speaker must nevertheless be ignorant about which alternative is the case. Denying all higher alternatives entails reveals that the speaker was in possession of further knowledge than expressed, rendering such statements infelicitous. In these case, (49) entails  $K_S[O B]$ , which although consistent, constitutes a violation of Quantity.

(49)  $K_S[\geq B] \wedge K_S \neg[\geq B \oplus S] \wedge K_S \neg[\geq B \oplus E]$

Note further that the limiting cases discussed in §2.2 behave as expected as well, since for all practical purposes they share the same logical signature as totally ordered associates (see (13)). Take for instance a statement of the form *Liz saw at least Bill and Sue*,  $[\geq B \oplus S]$ . Given the current domain, there are now again two mutually exclusive stronger alternatives that jointly cover the meaning of the utterance, and thus we derive IIs about each one of them.<sup>18</sup>

- (50) a. ASSERTION:  $[\geq B \oplus S]$   
 b. EPISTEMIC IMPLICATION:  $K_S[\geq B \oplus S]$   
 c.  $SA([\geq B \oplus S]) = \{[O B \oplus S], [\geq B \oplus S \oplus E]\}$   
 d. PRIMARY IMPLICATURES:  $\neg K_S[O B \oplus S] \wedge \neg K_S[\geq B \oplus S \oplus E]$   
 e. IMPLICATURE BASE:  $K_S[\geq B \oplus S] \wedge \neg K_S[O B \oplus S] \wedge \neg K_S[\geq B \oplus S \oplus E]$

The Implicature Base now entails  $\neg K_S \neg[\geq B \oplus S \oplus E]$  and  $\neg K_S \neg[\geq B \oplus S]$  and thus two IIs are predicted, capturing the “totally-ordered-like” behavior of the limiting case with partial orders. The inferences calculated in this fashion correspond again to the assertibility conditions of *at least* in the various cases discussed in §2, and conform to the observed felicity conditions on *at least*'s IIs, repeated below.

<sup>18</sup> I am glossing here over the issue as to whether alternatives that would be pragmatically implausible are ever generated, such as  $[\geq B \oplus S \oplus E]$  in this case. Since  $[O B \oplus S \oplus E]$  entails  $[\geq B \oplus S \oplus E]$ , the reasoning above goes through either way.



- (14) Generalization on *at least*'s IIs (final):
- a. When *at least* modifies an associate with a unique immediately higher alternative  $x$ , **both** the exhaustive interpretation of the prejacent **and**  $x$  must necessarily constitute an epistemic possibility for the speaker;
  - b. When *at least* modifies an associate with more than one immediately higher alternatives  $x_1, \dots, x_n$ , **neither** the exhaustive interpretation of the prejacent **nor** any of  $x_1, \dots, x_n$  must necessarily constitute epistemic possibilities for the speaker.

The first case corresponds to the epistemic possibilities entailed by the Implicature Base when *at least* modifies totally ordered associates or, as in (50) above, in the limiting case of partially ordered associates. The second case corresponds to the epistemic conditions found with the rest of partially ordered associates, which amount solely to the existence of some possibly unidentifiable Stronger Alternative that is compatible with the speaker's epistemic state. In these cases, the "epistemic criteria" of *at least* are weaker than with totally ordered associates and the limiting case. Thus, these constitute the minimal conditions that speakers must meet to successfully use *at least* without implying unwarranted additional IIs across contexts.

## 5 Conclusions and discussion

This paper accomplishes two things. From a descriptive standpoint, the paper identifies variation in the formal properties of the IIs conveyed by *at least* in different contexts. This variation reveals the need for a more fine grained characterization of IIs than it was previously known. Concretely, the paper shows that the exact form of *at least*'s IIs depends on the ordering properties of its associate, giving rise to two different profiles of conveying ignorance (see §5.2.2 for further discussion).

From a theoretical standpoint, the paper shows that these newly observed IIs can be derived as ordinary neo-Gricean Quantity implicatures by factoring in alternatives generated from two different sources, in the spirit of e.g. Schwarz and Shimoyama (2011), Mayr (2013), Kennedy (2015) and Schwarz (2016). Unlike earlier approaches, however, each set of alternatives is provided by a different, independent, mechanisms: focus alternatives and substitution of the scalar modifiers *at least* and *only* through pragmatic competition. Each method is justified on its own. Focus alternatives provide structured domains that depend exclusively on *at least*'s associate, and thus need not rely on semantic entailment relationships. Further generating alternatives by scalar substitution of *at least* and *only* permits to establish entailing relationships congruent with the ordered domains provided by *at least*'s associate. These alternatives are then fed into a vanilla neo-Gricean pragmatic enrichment process that captures the varying formal properties of *at least*'s IIs across different contexts.

One of the most obvious advantages of the present analysis is that it provides a uniform treatment to all cases where *at least* conveys IIs. The reasoning process that leads to these type of inferences is a general pragmatic mechanism triggered by external factors like conversational efficiency, speaker-hearer cooperation and rational behavior, and so the underlying mechanisms for calculating IIs across associate types are kept constant; no extra assumptions are required.

If the present analysis is on the right track it forces us to rethink the role of different factors at play in deriving IIs with *at least*. More generally, these results beget the question of what types of structures support conversational implicatures. This question has been thoroughly studied in the context of Secondary (scalar) Implicatures, where we can distinguish two prominent positions: *contextualism* and *logicism*. The following quotes illustrate the two opposing views:

- (51) a. CONTEXTUALISM [Hirschberg 1991, 93]  
 "...the orderings (that support) scalar implicatures) are partially (contextually) ordered sets [...] and any poset can support scalar implicatures."  
 b. LOGICISM [Magri 2017, 10]  
 "...the algorithm for the computation of scalar implicatures must be purely logical, namely blind to common knowledge."

The crucial difference between the two is that Contextualism allows pragmatic enrichment processes of the kind observed on Secondary Implicatures to operate on scales whose members may be logically independent, but contextually entailed (see fn. 1). Logicism denies this possibility. To my knowledge, this question has not been addressed from the point of view of Primary and Ignorance Implicatures, but a first look into it from the perspective of *at least* reveals an interesting state of affairs. We know from the earlier discussion that IIs of *at least* are not affected by the entailing properties of its associate. What the associate does contribute, however, is an ordering. So, at first instance, *at least*, through focus association, selects an associate that induces some type of ordering, either by relaying on entailment, context, conventions, world knowledge, etc. Here we must follow Contextualism. Then, the alternative generation algorithm derives *only*- and *at least*-alternatives that do establish semantic entailment relations. This permits ordinary pragmatic enrichment processes that rely on informative strength to operate on such sets of alternatives, and so at this point we must follow Logicism. We thus rely on context—not logic—to *establish* order, but we rely on logic—not context—to *reason* about order. Contextualism vs. Logicism seems to be a false dichotomy when it comes to deriving IIs with *at least*, since at some level or other we must adhere to *both*.

There is initial empirical evidence supporting this state of affairs. If Logicism was right, we could envision situations where logic undoes what context achieved. Degree expressions constitute good testing candidates. Properties of degrees such as *weight d-many kg.* become more informative the greater *d* is. But properties such as *d-many kg. are sufficient* become more informative the smaller *d* is. The former is upward monotone, while the latter is downward monotone, and the entailment patterns flip accordingly:

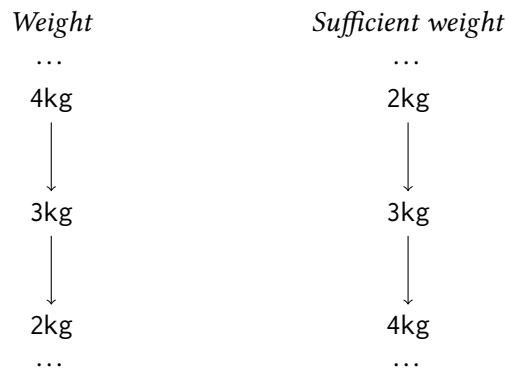


Figure 4: Flipping entailment pattern: *weight* vs *sufficient weight*

Given the entailment patterns in Figure 4, pure Logicism would predict a corresponding flip in the IIs obtained in each case. This is not what we find, however.

- |   |  |
|---|--|
| (52) a. The apples weigh at least 3kg.<br>$\leadsto$ <i>S doesn't know whether exactly 3kg. or more</i>     | PREDICTIONS OF LOGICISM<br>✓ $I_S[O\ 3kg] \wedge I_S[\geq\ 4kg]$ |
| b. At least 3kg. of apples are sufficient.<br>$\leadsto$ <i>S doesn't know whether exactly 2kg. or less</i> | ✗ $I_S[O\ 3kg] \wedge I_S[\geq\ 2kg]$                            |

I take this as evidence that the entailment relations within the set of alternative propositions that give rise to IIs with *at least* must be congruent with the ordering determined by its associate. But then, if logic cannot undo this possibly contextual orderings, pure Logicism cannot be right for II derivation purposes with *at least*. It remains to be determined what the consequences are for other types of IIs (e.g. with disjuncts of different sizes), as well as whether and if so how this stance on the Contextualism vs. Logicism dichotomy bears on different theories of II generations (e.g. neo-Gricean vs grammatical).

In the remainder of the paper I discuss some loose ends and further predictions of the analysis defended here, and then conclude discussing some broader implications.

## 5.1 Loose ends

### 5.1.1 Further predictions

Semantically, the lexical entry for *at least* introduced in (26) quantifies existentially over some alternative that is ranked at least as high as the prejacent itself.

$$(26) \quad \llbracket at\ least \rrbracket = \lambda C_{\langle st, t \rangle} . \lambda p_{\langle st \rangle} . \lambda w . \exists q [q \in C \wedge p \leq q \wedge q(w)]$$

As a consequence, it is predicted that the prejacent of an *at least* statement is not entailed. This is motivated because in certain cases, when the alternatives are non-entailing, *at least* statements do not in fact entail their prejacent—e.g. *Sue is at least an assistant professor* does not entail *Sue is an assistant professor*. But, more generally, this lexical entry predicts that the prejacent is *never* entailed when the unmodified alternatives (the alternatives without *at least*) are not ordered by entailment. While the predictions work well for the scales discussed so far, there are cases where intuitions are not so clear. Consider the following two examples, due to an anonymous reviewer:

- (53) a. Smith is supervising at least [a first year student]<sub>F</sub>  
       b. At least [Amy and Bill]<sub>F</sub> form a team

As is, the analysis predicts that (53a) is compatible with Smith not supervising any first year student, and (53b) is compatible with Amy and Bill not forming a team. This prediction has, to my knowledge, not been investigated and I will leave the issue open here. But I would like to point out that in order to properly test such predictions, we must be careful to pick contexts that support the correct domains; for instance, in (53a) we should make sure that the relevant ordering is one of student seniority ranks (first year, second year, etc.) and not simply a plurality of students. Similarly, in (53b) we should avoid contexts where the domain includes pairs of individuals only, rather than ungrouped individuals.

When it comes to the varying form of IIs, the present account makes further predictions once we consider other types of partially ordered sets. An anonymous reviewer proposes the following variation on the Sherlock Holmes scenario in (8)

- (54) The situation is an in (8), with four people *A*, *B*, *C*, and *D* in the domain. As before, Sherlock is trying to figure out who touched his chemistry set, but now he additionally knows that *A*

would not do anything with  $C$  and  $D$  unless  $B$  is also involved. As before, Sherlock is asked: Who touched the chemistry set?

(55) ? At least  $A$  touched the chemistry set, but I'm sure it wasn't just  $A$ .

The resulting relevant sub-structure now looks as follows:

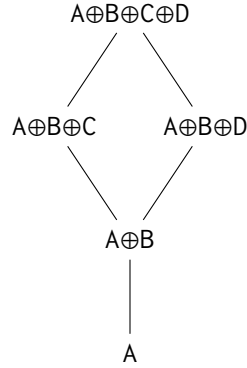


Figure 5: Structure of the domain in (54)

Because  $A$  has a unique immediately higher element, the relevant portion of the structure in (55) is predicted to behave as though *at least* operated on a totally ordered associate: *at least* cannot “see” the forking alternatives  $A \oplus B \oplus C$  and  $A \oplus B \oplus D$ . According to the present account, then, (55) is predicted to be infelicitous, since in that case Sherlock would have been in a position to assert that  $A \oplus B$  did commit the mischief (and perhaps somebody else). Judgements, however, are admittedly not so clear and the issue requires more investigation. Of particular interest is the putative contextual equivalence of alternatives  $A$  and  $A \oplus B$  in the present scenario, and whether that could lead to further flattening or pruning of the domain depicted in Figure 5.

Other cases that require further investigation are those pertaining conventional (non-entailing) scales that are partially ordered. French academic ranks would constitute one such case (due to an anonymous reviewer as well): in the French system there are parallel tracks of research-only positions and university positions:

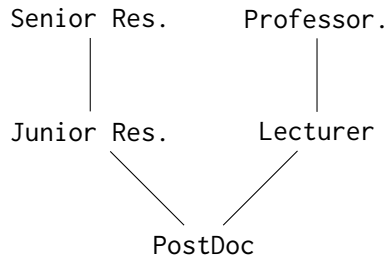


Figure 6: French academic system

In these cases, statements like *Mary is at least a postdoc* are predicted to be infelicitous if the speaker knew that Mary holds in fact some higher ranked position in the academic ladder. Once

again the facts are not as clear as the prediction itself.

### 5.1.2 Embedded *at least*

Büring (2007) first observed that the IIs of *at least* may disappear altogether under certain operators (for discussion see Schwarz and Shimoyama 2011 and Mayr 2013).<sup>19</sup> Consider (56) as illustration.

(56) Every student read at least two papers.

Sentence (56) is ambiguous, and under one of its interpretations it does not convey that the speaker ignorant as to how many papers every student read. While problematic for some approaches, e.g. Geurts and Nouwen (2007), Nouwen (2010), Penka (2010), double alternative set approaches to IIs easily account for the lack of IIs under universal quantifiers. Take  $\forall[\geq 2]$  to represent the meaning of (56), and  $K_S\forall[\geq 2]$  as the inference resulting from Epistemic Implication. In this analysis, IIs are derived if there is a pair of Stronger Alternatives that jointly exhausts the space of possibilities covered by the utterance. Note, however, that  $\forall[0\ 2]$  and  $\forall[\geq 3]$  fail to cover such space, since  $\forall[\geq 2]$  could be true by virtue of some students reading exactly two papers while other students read more than two, in which case neither  $\forall[0\ 2]$  nor  $\forall[\geq 3]$  would be true. As a consequence, none of the corresponding possibility implicatures are entailed,  $\neg K_S\neg\forall[0\ 2]$  and  $\neg K_S\neg\forall[\geq 3]$ , and no IIs are predicted. (The interpretation conveying ignorance is achieved by interpreting the universal quantifier under the scope of *at least*; see Büring 2007 and Kennedy 2015.)

### 5.1.3 Further strengthening and consistency

It is well known that *at least* does not give rise to scalar implicatures: (57) below does not support the strong inference that exactly two people came to the party.

(57) At least two people came to the party.

↗ *It is not the case that at least three people came to the party*

Any successful account of *at least* must derive the fact that such strengthening is blocked. In Gricean frameworks, scalar implicatures require the extra assumption that the speaker is maximally knowledgeable about the question that the proposition she is uttering is making a contribution to. That is, for a Stronger Alternative  $\phi$  by  $S$ , either  $K_S\phi$  or  $K_S\neg\phi$ , in so far as the result is consistent with the assertion and the Primary Implicatures (Sauerland 2004b, van Rooij and Schulz 2004). This assumption is usually referred to as the “epistemic step” (also the “competence” or “authority” assumption), and it is commonly assumed that the listener is free to consider the speaker an authority about Stronger Alternatives, unless these are preempted by IIs. Both Mayr (2013) and Schwarz (2016) show that, without further assumptions, a double-scale strategy in a neo-Gricean analysis of scalar implicatures would deliver the wrong results, leading to either the wrong implicatures or inconsistency. The problem is the following. Consider the case of (57). The analysis presented here only generates

<sup>19</sup> In the case of *at least*, IIs can be obviated in contexts that have been argued to involve some sort of universal quantification, like modal verbs, generics and imperatives. None of the examples below necessarily conveys ignorance.

- (i) a. Bill {must/has to/is required to} read at least two papers to get an A.
- b. Spiders have at least two eyes.
- c. Calculate at least one root of the equation  $8x^5 - 6x^4 - 83x^2 - 6x + 8 = 0$ .

IIs about two of the Stronger Alternatives:  $[O\ 2]$  and  $[\geq\ 3]$ . In principle, then, any other additional alternative  $\phi$  for which the system fails to generate the possibility implication  $\neg K_S \neg[\phi]$ ,  $\phi$  could be strengthened by the epistemic step to the stronger inference that  $K_S \neg[\phi]$ . Given the double alternative set approach to *at least*, there are many such Stronger Alternatives. Take for instance  $[O\ 3]$  and  $[\geq\ 4]$ . No IIs about these two alternatives are generated; in fact, taken separately, both  $K_S \neg[O\ 3]$  and  $K_S \neg[\geq\ 4]$  are contingent with the assertion and its IIs. The problem is that nothing in the neo-Gricean analysis exposed here preempts strengthening *both*. But together  $K_S \neg[O\ 3]$  and  $K_S \neg[\geq\ 4]$  entail  $K_S \neg[\geq\ 3]$ , contradicting the possibility inference  $\neg K_S \neg[\geq\ 3]$  that is part of the II. Thus, without further assumptions, factoring a secondary strengthening step in a neo-Gricean calculation predicts that no consistent inferences can be derived from (57).

In order to avoid such problems, Schwarz (2016) shows, we need to supplement traditional neo-Gricean analyses with a mechanism that preserves consistency during the derivation of Secondary Implicatures that is more advanced than that of Sauerland (2004b). The solution he proposes is to adapt Fox's (2007) idea that strengthening does not happen randomly, but rather it is only allowed if it does not lead to the necessary inclusion of any other Stronger Alternative, i.e. if the alternatives are "innocently excludable". By incorporating Innocent Exclusion into the neo-Gricean calculus, the full set of implicatures from (57): (i) is rendered consistent, (ii) entails the desired IIs, and (iii) avoids unattested implicatures. Consider a possible Epistemic Base for (57):

$$(58) \quad K_S[\geq\ 2] \wedge \neg K_S[\geq\ 3] \wedge \neg K_S[\geq\ 4] \wedge \neg K_S[O\ 2] \wedge \neg K_S[O\ 3] \wedge \neg K_S[O\ 4]$$

A proposition  $\phi$  is innocently excludable relative to a set  $A$  of Primary Implicatures if  $K_S \neg[\phi]$  is a member of *every* maximal subset of  $\{K_S \neg[\psi] : \neg K_S[\psi] \in A\}$  consistent with  $A$ . Schwarz (2016) shows that for cases like (57) none of the Stronger Alternatives in the set  $\{[O\ n], [\geq\ n+1]\}$  for all  $n \geq 3$  is innocently excludable. This is because for any number  $j \geq 3$ , there is some maximal subset of  $\{K_S \neg[O\ n], K_S \neg[\geq\ n+1]\}$  that is consistent with the set of Primary Implicatures and does not include either one of  $K_S \neg[O\ j]$  or  $K_S \neg[\geq\ j]$ . For instance, in the case of  $j = 3$ , one such set is  $\{K_S \neg[O\ 4], K_S \neg[\geq\ 4], K_S \neg[O\ 5], K_S \neg[\geq\ 5], K_S \neg[O\ 6], K_S \neg[\geq\ 6] \dots\}$ . Thus, while one can construct many sets consistent with the Primary Implicatures, not all maximal and consistent subsets contain  $K_S \neg[O\ j]$  (or  $K_S \neg[\geq\ j]$ , for the matter). The reasoning applies equally to the rest of alternatives in  $j \geq 3$ . As a consequence, any form of strengthening of  $\neg K_S[\phi]$  to  $K_S \neg[\phi]$  is blocked, either at first instance by existing Primary Implicatures, or at a second instance by Innocent Exclusion.

The resulting state of affairs is that the neo-Gricean-style analyses that reason about speaker's beliefs need to be augmented with a fairly abstract and sophisticated mechanism of consistency preservation, Innocent Exclusion. This is a move that goes against the Gricean spirit, and it is at present unclear what the full ramifications might be. An alternative is to deny altogether that Secondary Implicatures should be explicated in a neo-Gricean framework, and derive them instead by means of grammatical exhaustivity operators like *Exh* (Fox 2007, Chierchia et al. 2012, a.o.), tailored to operate on alternatives at the grammatical level and not at the inferential level. One of the most compelling empirical arguments for this family of grammatical approaches to implicatures is the existence of embedded scalar implicatures. This fact alone seems to render the Gricean approach a non-starter, since in a Gricean framework the implicatures are always calculated at the propositional level (but see Geurts 2010 for a suggestion to derive embedded implicatures *à la* Grice). *Exh*-based theories however cannot derive epistemic inferences of any kind, neither Primary Implicatures nor IIs, providing some plausibility for approaches where general pragmatic considerations are factored into their derivation.



## 5.2 Broader implications

### 5.2.1 Ignorance in the grammar

The discussion on scalar implicatures above ended by pointing out that, if the strengthening of Primary to Secondary Implicatures is to be avoided for *at least*, some method of consistency preservation must be factored into the pragmatic calculation, e.g. Innocent Exclusion. Given the sophistication of such filtering procedures, this is more naturally achieved by resorting to exclusivity operators such as *Exh*, rather than domain general reasoning abilities. On the other hand, *Exh*-based theories are not well equipped to derive epistemic inferences, a limitation that has been addressed by proposing grammatical epistemic operators, like Meyer's (2013) *K*, that interact with *Exh* in such a way that IIs are in fact derived as semantic entailments. Notice, however, that IIs seem to be essentially a root-level phenomenon, and this resistance against embedding is more in line once again with neo-Gricean approaches (cf. Secondary Implicatures, which are famously embeddable). For instance, (59a) can only be contradiction free under an exclusive interpretation of the disjunction, which is usually attributed to a scalar implicature. As Sauerland (2004a) points out, if weaker implicatures were embeddable, (59b) should be able to convey something like this: when the speaker is sure that John or Mary worked and not sure that John worked, the result was good, but when the speaker is sure that John worked, the result was a mess. This is clearly not an available reading of (59b) (examples from Sauerland 2004a).

- (59) a. When John or Mary worked, the result was good, but when John and Mary worked, the result was a mess.  
b. #When John or Mary worked, the result was good, but when John worked, the result was a mess.

Recently, Fox (2016) has provided a conceptual reason to believe that, if relevance is closed under belief, IIs can only be derived by means of grammatical operations (see also Buccola and Haida 2019 for a full implementation). Unlike Meyer (2013), however, Fox (2016) advocates for a weaker position where the demand for *K* follows from Gricean pressures, such as the requirement to adhere to (some version of) the Maxims of Quality and Quantity, and thus *K* need not apply when e.g. the Maxim of Quantity is known to be deactivated. The choice of a parse with or without *K*, then, would depend on their plausibility in the context. In turn, the inability to embed weak implicatures may be indicative of an inability to embed *K*, suggesting that *K* should be grouped with those grammatical operations that are known to target the root level only (e.g. inversion in English questions, imperatives, certain type of speaker oriented adverbs, etc.).

All in all, there is a palpable tension here: what neo-Gricean analyses struggle to accomplish is precisely that what grammatical approaches are suited for, and vice-versa. Thus, whether it is better to keep a neo-Gricean approach with Innocent Exclusion, accept that IIs must be derived by means of grammatical operators, or some in-between hybrid approach is a question for the future.<sup>20</sup> Although

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<sup>20</sup> This is how one such hybrid approach would look like: Primary Implicatures, and thus IIs as well, are the product of reasoning about speaker's beliefs given their utterances, and thus non-embeddable. This is achieved by some non-specialized cognitive system based on general principles of rational behavior. The listener, however, is free to make the assumption, independently from how she reasons about the utterance itself, that her interlocutor is an expert on the subject matter. Since the empirical result of making such assumption is a family of implicatures that is embeddable (unlike Primary Implicatures and IIs), the strengthening of the speaker's utterance must happen by means that allow embedding, i.e. by some modular (linguistic) formal system that determines the truth-conditions of a sentence. This could be done for instance by invoking an exhaustivity operator *Exh*—after all, what is usually referred to as the epistemic step is nowhere to be

the account presented in this paper adheres to the view that IIs are to be derived by means of domain general reasoning abilities, I hope that its results may be reproduced by appealing to grammatical epistemic operators, if that turns out to be the right analysis of ignorance.

### 5.2.2 Towards a typology of ignorance

A corollary of this study is that there are a total of three formally distinguishable types of ignorance. Total ignorance is the strongest type of ignorance, the one conveyed by disjunction. In the general case, disjunctive statements convey ignorance about every individual disjunct—and every smaller disjunct in the case of multiple disjunctions—or, alternatively, about every alternative in its sub-domain (e.g. Alonso-Ovalle 2006, Chierchia 2013). This means that total ignorance is incompatible with any kind of positive knowledge about the relevant Stronger Alternatives. Schematically:  $K_S[\alpha_i \vee, \dots, \vee \alpha_j] \sim I_S[\alpha_i] \wedge, \dots, \wedge I_S[\alpha_j]$ .

This is not the case with partial ignorance, since partial ignorance is indeed compatible with some amount of knowledge. What that knowledge is, however, varies among different expressions. In the case of *at least* associating with totally ordered scales, such as numerals, ignorance is mandatory only about two alternative propositions: (i) the exhaustive interpretation of the pre-jacent and (ii) the immediately Stronger Alternative. (The same holds of partially ordered associates in the limiting case.) For convenience, say this is a case of “strong partial ignorance”. Schematically:  $K_S[\geq n] \sim I_S[O n] \wedge I_S[\geq n + 1]$ . Practically speaking, after a speaker utilizes an *at least n* expression, listeners can only be certain that she is ignorant about the truth of those two Stronger Alternative propositions, and ignorance about the rest of Stronger Alternatives is merely contingent.

Finally, when *at least* associates with partially ordered scales (to the exclusion of the limiting case), it does not entail ignorance about any one particular Stronger Alternative. Call this “weak partial ignorance”. This is formally equivalent to von Fintel’s (2000) “modal variation”, an epistemic inference conveyed by *ever*-free relatives, epistemic indefinites (Alonso-Ovalle and Menéndez-Benito 2010) and epistemic numbers (Anderson 2016, Mendia 2018). All weak partial ignorance amounts to is an impossibility to decide what is the witness of an existential claim; or, alternatively, an impossibility to determine what accessible world is epistemically the best:

(60) **Modal Variation:**

$$\exists w', w'' \in D_{S,w}[\{x : P(w')(x)\} \neq \{x : P(w'')(x)\}]$$

[for some property  $P$ , where  $D_{S,w}$  is the set of epistemically accessible worlds compatible with the speaker  $S$ ’s evidence in  $w$ ]

The thing to notice about weak partial ignorance is that it does not conform to the signature of ignorance defined above in (16), at least not immediately.

$$(16) \text{ SIGNATURE OF IGNORANCE: } \neg K[\phi] \wedge \neg K\neg[\phi] \quad \leftrightarrow \quad P[\phi] \wedge P\neg[\phi]$$

Even in a heavily reduced domain  $D$  containing just *Sue*, *Bill* and *Ed*, a sentence like *At least Bill came* expresses a proposition of the form  $K_S[\geq B]$  whose Stronger Alternatives cannot be strengthened to a full II. We saw why above: to carve out the space of available epistemic possibilities upon concluding that  $K_S[\geq B]$  requires at least three Stronger Alternatives:  $[O B]$ ,  $[\geq B \oplus S]$  and  $[\geq B \oplus E]$ , and thus

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found in Grice’s theory. In turn, *Exh* comes well equipped to avoid inconsistencies, and so one can remain Gricean, albeit only for the purposes of calculating weak implicatures. This would effectively remove the need to replicate the merits of grammatical approaches to scalar implicatures in the Gricean framework. To assess the plausibility of this line of thought, however, involves a much more in-depth examination that I have space for here.

we could negate any one of those three Stronger Alternatives without contradicting nor entailing the truth of any other. What  $K_S[\geq B]$  entails is the epistemic possibility that there is some individual  $x \in D$  other than Bill such that a proposition of the form  $[\geq B \oplus x]$  is compatible with all the speaker knows,  $P_S[\geq B \oplus x]$  or alternatively  $\neg K_S \neg[\geq B \oplus x]$ . Given  $D$ , denying that  $\neg K_S \neg[\geq B \oplus x]$  for all  $x \in D$  is equivalent to denying two possibilities,  $P_S[\geq B \oplus E]$  and  $P_S[\geq B \oplus S]$ , which would entail the truth of  $K_S[OB]$ , contradicting the corresponding Primary Implicature that  $\neg K_S[OB]$ . Crucially, however,  $P_S[\geq B \oplus x]$  should not be understood as  $P_S[\geq B \oplus S] \wedge P_S[\geq B \oplus E]$ , since the latter relates epistemic possibilities accessible to the speaker with particular individuals in the domain, and this is too strong an inference. Instead, weak partial ignorance is not ignorance about any particular such Stronger Alternative, but rather expresses that the speaker considers that one such Stronger Alternative is possibly the case, while she may not be able to determine which one. As a consequence, weak partial ignorance may (but need not) be compatible with a fair amount of knowledge, as was discussed earlier in §2.1 and §4.3. This is not to say, however, that there is no Stronger Alternative  $\phi$  such that the speaker's epistemic state contains  $I_S[\phi]$ ; after all, the speaker is surely ignorant about some such Stronger Alternatives. But given the ways in which the domain is structured, the listener is unable to retrieve the exact Stronger Alternative that the speaker is ignorant about; hence the “weakness” of this type of ignorance. Schematically, then:  $K_S[\geq B] \rightsquigarrow I_S[\geq B \oplus x]$ , for some  $x \in D$ .

Summarizing, we find that natural language expressions may convey various types of formally distinguishable Ignorance Inferences.

(61) For some set of Stronger Alternatives  $SA$  a speaker  $S$  may express that:

- a. **TOTAL IGNORANCE**  
 $S$  ignores the truth of all (sub-domain) Stronger Alternatives, they must all constitute epistemic possibilities for  $S$  (e.g. disjunction).
- b. **STRONG PARTIAL IGNORANCE**  
 $S$  ignores the truth of two Stronger Alternatives: (i) the exhaustive interpretation of the prejacent and (ii) its immediately Stronger Alternative; the two must constitute epistemic possibilities for  $S$  (e.g. *at least* with totally ordered associates: numerals, ranks, etc.).
- c. **WEAK PARTIAL IGNORANCE**  
 $S$  ignores the truth of at least one Stronger Alternative, but the listener has no means to retrieve which one (e.g. *at least* with partially ordered associates (excluding the limiting cases), *ever-free* relatives, epistemic indefinites/numbers, etc.).

It is worth noting that the three type of ignorance effects arise as the product of four possible sets of Stronger Alternatives that have distinct logical profiles:

(62) Logical signature of different Stronger Alternative sets giving rise to IIs:

- a. Sets where all  $SAs$  pairwise cover the logical space of the meaning of the assertion.
- b. Sets where only one pair of  $SAs$  exhausts the logical space of the assertion.
- c. Sets where no two  $SAs$  cover the logical space of the assertion, but all  $SAs$  jointly do.
- d. Sets of  $SAs$  that do not cover the entire logical space of the assertion.

The relations between these signatures and the types of ignorance described in (61) are straightforward in some cases: (62a) invariably conveys Total Ignorance (e.g. disjunction with only two disjuncts), (62b) conveys Strong Partial Ignorance (e.g. *at least* with totally ordered associates), and (62c) never conveys IIs (e.g. disjunctions and *at least* under universal operators of various sorts). The case of (62c) is nevertheless more interesting, since, depending on the set of Stronger Alternatives

it predicts IIs of varying strength. Take a disjunctive statement like  $[A \vee B \vee C]$ . If, as discussed in e.g. Spector (2006), we take the SAs of  $[A \vee B \vee C]$  to be *only* its individual disjuncts, then we predict Weak Partial Ignorance (for reasons discussed in §4.3.2). But if we take  $[A \vee B \vee C]$  to include also all its possible smaller disjuncts as SAs— $[A \vee B]$ ,  $[A \vee C]$  and  $[B \vee C]$  in this case—then the prediction is that it should convey Total Ignorance (given the entailments of its corresponding Implicature Base; see Alonso-Ovalle 2006). These new typology of ignorance can thus be deployed to determine what sets of SAs are at work in each case, and thus help refine the algorithms that generate them.

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