Known unknowns*

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Abstract This paper provides an investigation of Ignorance Inferences by looking at the superlative modifier *at least* in non-embedding contexts. The formal properties of these IIs are characterized in terms of the epistemic conditions that it imposes on the speaker, thereby establishing how much *can* and *must* be inferred about what the speaker is ignorant about. The paper makes two main contributions. First, it builds on Schwarz's (2016a) neo-Gricean double alternative generation mechanism by arguing that one of them must be provided by focus alternative semantics (partially following Fox and Katzir 2011). Second, it argues that the form of these IIs depends on the structural properties of the expression that *at least* is modifying: with totally ordered associates, *at least* triggers IIs that are formally different than those obtained with partially ordered associates. The results improve upon previous analyses by deriving the right form of IIs in the rich variety of environments in which *at least* can appear.

1 Introduction

Superlative modifiers like *at most* and *at least* often convey that the speaker is uncertain or ignorant about something, and so she cannot commit to providing more information. This epistemic effect of superlative modifiers has been dubbed an Inference of Ignorance (II henceforth). As illustration, consider (1):

(1) a. #I have at most two daughters.
    b. #I have at least five fingers.

These examples are odd. The epistemic competence commonly assumed when we talk about progeny or our own body is at odds with the presence of superlative modifiers and their incompatibility with full knowledge. Contrast (1) with the felicitous (2), which differs only in that no speaker knowledge need be assumed.

(2) a. Bill has at most two daughters.
    b. That caterpillar has at least twenty legs.

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That superlative modifiers trigger IIs is uncontroversial. The controversy is about what exactly is the division of labor between semantics and pragmatics in deriving the various implications conveyed by sentences like (1). More specifically, it is debated whether IIs are encoded as part of the conventional meaning of superlative modifiers (Geurts and Nouwen 2007, Nouwen 2010) or whether they are the result of pragmatic processes (Büring 2007, Coppock and Brochhagen 2013b, Nouwen 2015, Schwarz 2016a, a.o.). To this end, a number of different proposals, each introducing its own machinery, have been put forward. On the semantic side, superlative modifiers have been analyzed as modals (Geurts and Nouwen 2007), as *minima* and *maxima* operators (Nouwen 2010), as inquisitive expressions (Coppock and Brochhagen 2013b), as operators of meta-speech acts (Cohen and Krifka 2014), and as epistemic indefinites (Nouwen 2015). On the pragmatic side, at least two main avenues of research have been pursued in deriving IIs: neo-Gricean analyses (Büring 2007, Schwarz 2016a) and those relying on grammatical approaches to implicatures (Mayr 2013).

Despite this attention that superlative modifiers have attracted recently, a careful investigation into the nature and form of their IIs has yet to be undertaken. That is, given a sentence containing a superlative modifiers, how much *can* be inferred about what the speaker is ignorant about? Is there anything that speakers *must* be obligatorily ignorant about in order to successfully use a superlative modifier? How does the context (grammatical and discursive) of superlative modifiers affect the form and presence of IIs?

This paper aims to contribute to these questions by offering a precise characterization of IIs with the superlative modifier *at least* and putting forward a theory that will capture those facts. I begin by sketching some desideratum that any satisfactory theory of *at least* must meet.

1.1 Two questions

The focus of the paper is on the following two related questions. The first question relates to the fact that *at least* may combine with a number of different types of complements or associates, consistently leading to IIs in all these environments. As shown in (3), IIs of *at least* might be present with a variety of scales (Hirschberg 1985).

(3) a. At least *some students* came to the party. [HORN SCALES]
$\rightsquigarrow$ *the speaker is ignorant about whether all students came*

b. At least *Bill and Sue* came to the party. [CARDINALITY SCALES]
$\rightsquigarrow$ *the speaker is ignorant about whether someone else came to the party*

c. Sue won at least *the silver medal*. [LEXICAL SCALES]
$\rightsquigarrow$ *the speaker is ignorant about whether Sue won the gold medal*

d. Bill ate at least *broccoli*. [EVALUATIVE SCALES; e.g for a preference scale ⟨ *broccoli, candy* ⟩]
$\rightsquigarrow$ *the speaker is ignorant about whether Bill ate candy*

Despite this flexibility of *at least*, investigation into IIs has asymmetrically focused on the numeral case, where *at least* modifies numerals or measure phrases, leaving the cases involving associates of other categories (DPs, VPs, etc.) largely unexplored. A tacit assumption in the literature is that *at least* behaves alike in IIs across-the-board. However, the fact that potential associates of *at least* show a rich variety of formal properties raises the possibility of non-uniformity in the nature of their IIs. Thus, a first empirical goal in this paper is to identify the precise nature of IIs across these various environments. Call this first question UNIFORMITY, as in (4).

(4) UNIFORMITY
Are the inferences that come with *at least* the same across the board, regardless of its type of associate?

The second question relates partly to the first one: Is there anything that the speaker *must* be ignorant about in order to felicitously use *at least*? Different formulations of IIs make different predictions as to what inferences can be made about the speaker's epistemic state. For instance, consider (5), and two possible characterizations of its IIs in terms of disjunctive statements:

(5) Bill is at least an assistant professor. [Coppock and Brochhagen 2013b: 10]

a. [*assistant professor*] or [*associate professor*] or [*full professor*]
$\rightsquigarrow$ *the speaker doesn't know whether Bill is an assistant, associate or full professor.*

b. [*assistant professor*] or [*more than an assistant professor*]
$\rightsquigarrow$ *the speaker doesn't know whether Bill is an assistant professor or some higher rank.*

At first glance, the difference between (5a) and (5b) may appear trivial. The two views in fact make diverging predictions about the kind of inferences one is licensed to make about the speaker's ignorance. Whereas formulations like (5a) predict that the speaker believes that every rank above assistant professor is a possible rank for

Bill, formulations like (5b) do not require that any particular rank above assistant professor is a mandatory epistemic possibility for the speaker, as long as one of them is. The issue bears on the distinction between two types of ignorance: total and partial Ignorance. By mode of illustration, consider (6) with the two relevant paraphrases.

(6) At least four people came to the party.

a. TOTAL IGNORANCE: For any number $n \geq 4$, the speaker is ignorant about whether or not exactly n-many friends came to the party.

b. PARTIAL IGNORANCE: The speaker is ignorant about whether or not exactly 4 or more than 4 friends came to the party.

For the numeral case, it has been shown that formulations akin to (5a), conveying total ignorance as in (6a), cannot be correct (Kennedy 2015, Mendia 2015, Nouwen 2015, Schwarz 2016a, a.o.). These accounts agree that when *at least* modifies a numeral *n*, the speaker intends to convey ignorance about two particular propositions: (i) that she is ignorant about whether *exactly n* is the case, and that she is ignorant as to whether *more than n* is the case. The question remains, however, as to whether the rest of the contexts in which *at least* may appear require the same type of ignorance. Thus, our second empirical goal is to identify exactly what kinds of information are compatible with an utterance containing *at least* across all the linguistic contexts in which it appears. Call this question PREDICTABILITY:

(7) PREDICTABILITY
Is there any proposition in particular about which the speaker must be ignorant about so that she can successfully use an *at least*-statement?

1.2 The plan

To summarize, a full characterization of IIs with *at least* across environments is lacking, and therefore it is presently unclear whether extant theories are adequate in their empirical coverage. By answering the aforementioned questions about the UNIFORMITY and PREDICTABILITY of *at least*'s IIs we provide further criteria for evaluating different theories. In the first part of this paper, I build on existing literature on the IIs of numerals to make precise the exact form of IIs with *at least* across contexts and what they tell us about the speaker's epistemic state, thereby providing an answer to these questions. Two key empirical points emerge from this investigation: (i) the nature of IIs is not uniform across associate types, and (ii) while ignorance with *at least* may always be partial (i.e. (5b)/(6b) above are better characterizations of the speaker's epistemic state), their exact

These descriptive findings will be used in the second part to provide a unified account for IIs. The specific account I endorse takes *at least* to be a scalar modifier interpreted relative to some focalized constituent. Building on earlier literature, IIs with *at least* arise as a kind of Quantity Implicature, derived in a neo-Gricean fashion. The calculation of implicatures with *at least* will require two sets of alternatives, as has already been proposed by Mayr (2013), Kennedy (2015) and Schwarz (2016a) or numerals. The main innovation of the calculus presented here is that each set of alternatives relevant for the Gricean computation is provided by a different, independent, mechanism. The first method is the familiar substitution method within elements of a Horn scale (Horn 1972, Sauerland 2004, a.o.), where I take *at least* to form a Horn scale with *only*, given the parallels between the two elements in terms of focus association (Schwarz 2016a). In addition, a different set of alternatives is obtained by replacing the focus-bearing constituent, i.e., *at least*'s associate, with contextually relevant alternatives (Rooth 1992, Fox and Katzir 2011).

The remainder of this paper is organized as follows. I first illustrate that *at least* is compatible with partial ignorance and outline what the speaker must be ignorant about across its different uses. I then briefly argue for two key assumptions that are crucial to the analysis of *at least* that is to follow–(i) that ignorance is pragmatic, and (ii) that *at least* is conventionally associated with focus–before turning to the analysis itself in §4. In §5 the results obtained are assessed–including open questions an unsolved problems.

2 Characterizing Ignorance across Contexts

2.1 Setting a baseline: *at least* and numerals

The main goal of this section is to provide a proper characterization of IIs that *at least* conveys across all contexts, by scrutinizing our intuitions about them. Following earlier literature (Büring 2007, Kennedy 2015 Mendia 2015, Nouwen 2015, Schwarz 2016a, Schwarz 2016b, a.o.), I show that the IIs of *at least* when its associate is a numeral simply convey partial ignorance. That this is so can be easily ascertained by looking into the following two examples:

(8) **Situation:** *Two commentators are talking on TV about a classic basketball game played in the 90's. They are commenting on the points that were scored in that game on triples. A commentator says:* <u>*Michael Jordan scored at least 30 points*</u>.

In (8) both commentators know that triples are three-point field goals in basketball, in contrast to the two points awarded for easier shots. They assume, too, that they are targeting an audience that is well versed in the rules of basketball, and so this information is shared by every agent in the conversation, active or passive. In this situation, the commentator's utterance is perfectly acceptable. This is an instance of partial ignorance: the addressee cannot draw an inference that the speaker is completely ignorant, since she knows that the speaker does know something–namely, that quantities of scores that are not tuples of three are not allowable options. Another manifestation of partial ignorance can be seen when *at least* is used in situations where the bounds denoted by the number expression are flexible. Consider the following example, from Nouwen 2015:

(9) **Situation:** *Bill forgot the password of his WIFI network. The only thing he remembers is that the password is between six and ten characters long. Bill says: The password is at least six characters long.*

In the situation above, Bill can utter (9) felicitously, even though his epistemic state excludes some of the values that in principle could be available. Moreover, the speaker can utter (9) without bringing attention to any misleading implicatures. In this regard, *at least* differs from multiple disjunctions, which demand that the speaker be ignorant with respect to every individual disjunct. It seems, then, that when using *at least* two alternative propositions–(10a) and (10b)–have a preferential status, as the possibility of them being true seems to be necessarily included in the epistemic state of the speaker.

(10) Bill ate at least two apples...

a. #but I know that he didn't eat only two.
b. #in fact, he did not eat more than two.
c. but I know that he didn't eat {four/three or four/between three or six/...}.

The conclusion is that for *at least* with numeral associates to be felicitous, the speaker need not be totally ignorant, but she must be ignorant about certain information. Specifically, we found that when uttering a sentence where *at least* modifies a numeral *n*, there must be at least two possibilities that the speaker has to consider to be true: *exactly n* and *more than n* (Büring 2007). We can informally summarize the assertibility conditions of *at least* as follows (cf. Cohen and Krifka 2014, Spychalska 2015):

(11) *Assertibility conditions of* at least-*numerals*: A proposition ϕ of the form *at least n P* is assertible by *S iff*:

a. *S* knows that ϕ is true **and** *S* knows that *less than n P* is false,
b. *exactly n P* is compatible with all *S* knows, and
c. *more than n P* is compatible with all *S* knows.

These assertibility conditions correspond to what the epistemic state of a cooperative speaker has to be like so that a sentence with *at least*-numeral can be uttered felicitously (assuming, for course, that all pragmatic principles are at work). To this respect, the assertibility conditions of *at least* are fully parallel to those of a disjunction, where each disjunct is required to be possibly true, and it is required not to be certainly true by the speaker. [1]

2.2 Ignorance beyond numerals

Our previous discussion offers us an answer to the question about PREDICTABILITY, posed in section §1, repeated below:

(7) PREDICTABILITY
Is there any proposition in particular about which the speaker must be ignorant about so that she can successfully use an *at least*-statement?

With *at least*, the exhaustive interpretation of the prejacent to the sentence uttered must necessarily constitute an epistemic possibility for the speaker; i.e., that *exactly n* is true. In addition, *more than n* must as well be a mandatory possibility. Thus, we may conclude that the answer to PREDICTABILITY in the case of numerals is positive: the speaker must be ignorant about whether *exactly n* or whether *more/less than n* is the case.

We can now aim to answer our second question, concerning UNIFORMITY, and assess whether our answer to this question affects in any way the results we obtained for PREDICTABILITY.

(4) UNIFORMITY
Are the inferences that come with *at least* the same across the board, regardless of its type of associate?

As mentioned before, a tacit assumption in the literature is that IIs of *at least* are uniform across associate types. As we saw in examples (3), *at least* seemed to convey the same kind of IIs irrespective of the differences in associate types. However, we now have a more precise characterization of IIs, and are equipped to ask whether the assertibility conditions stated in (11) are the same across different associate-types.

As a first case, let us consider *at least* with quantifiers like *some*, as in (12). The sentence conveys that the speaker is ignorant about how many students came, whether all of the students came, etc.

1 What the speaker knows for certain amounts to what *at least* asserts, namely the lower bound of the range that delimits the space of available possibilities. That this is the semantic contribution of numerals modified by *at least* is part of the standard analysis in terms of Generalized Quantifiers, as first presented in Barwise and Cooper (1981): $[\![\textit{at least } n]\!] = \lambda A.\lambda B \,.\, |A \cap B| \geq n$.

(12) At least some students came to the party.

We can examine these IIs using the same tests employed above to identify the assertibility conditions of *at least* with *some*. (Bear in mind that reproducing the judgments with *some* may require some more effort, most likely due to the inherent vagueness of the determiner).

(13) Bill ate at least some apples...

  a. #but I know that he didn't eat just some.

  b. #in fact, he didn't eat many.

  c. but I know that he didn't eat {most of them/all of them}.

Suppose that what is at stake is how many of the apples Bill ate. Suppose further that you knew that Bill ate almost all the apples, that is, that he ate more than just some apples. Then you would be quite misleading in saying that he ate *at least some*, since you could have been more informative. This clash between information known and information conveyed is what is brought by the follow-up in (13a). Similarly, if you knew that Bill barely ate a handful of apples, that is, that he did not eat more than just some, by saying *at least some* you would be conveying the false possibility that he ate more than just some apples, maybe many apples, maybe even all of them. Being in such an epistemic state, however, clashes with an *at least some* utterance in the same way that numerals do (13b). Finally, (13c) shows that no conflict arises when both knowledge about *at least some* and the possibility that *more than just some* are conveyed, suggesting that IIs conveyed by *at least* with *some* are also partial. (The reader is invited to check the facts with other quantifiers). This is exactly the same kind of behavior we observed with numerals: the exhaustive interpretation of the prejacent must constitute an epistemic possibility for the speaker, and so does the next relevant more informative alternative (i.e., in this case, many could be more than just some).

We cannot yet conclude in favor of UNIFORMITY. So far, the conclusion seems to be that associates like *some* share the relevant properties of numeral associates in what they demand the speaker's epistemic states to be with respect to II calculations. This is no coincidence: we must note that numerals and quantifiers like *some* share two important logical properties, relating to the fact that they are both ordered by logical strength, i.e. they constitute Horn scales (Horn 1972). First, for any term in $\{1,2,3,4,5,6...\}$ or {*some, many, most, all*}, instances of *x A are B* with one number/determiner term *x* semantically entail instances with any term to the left, but not to the right, as illustrated in (14).

(14) a. Bill ate five apples $\vDash$ Bill ate four apples

b. Bill ate many apples $\vDash$ Bill ate some apple

Second, Horn scales are totally ordered: for any member of a set that is totally ordered, it either entails or is entailed by every other member in the set. For instance, for any natural numbers n, i, an expression *n As are B* entails *n-i As are B* and is entailed by *n+i As are B*; moreover, for every number j, *j As are B* either entails or is entailed by *n As are B*.

However, *at least* can associate with elements that fail to have one or both of these properties. For instance, *at least* can associate with scales where, unlike with Horn scales, the relation between its members is not driven by logical entailment. These are often called "ad hoc", "pragmatic" or "lexical" scales (Hirschberg 1985), as they compose of elements that "outrank" each other but are nevertheless mutually exclusive; i.e. they are non-entailing scales. Take for instance the lexical scale established by professorship ranks at US universities: *visiting professor*, *assistant professor*, *associate professor*, *full professor*, etc. One cannot be an associate professor and a full professor at the same time, and yet one cannot be a full professor without having been an associate first. In this sense there is a common understanding that full professors outrank associate professors, but these ranks are not ordered by entailment.

(15) a. Bill is an associate professor $\nvDash$ Bill is an assistant professor

b. Bill won a silver medal $\nvDash$ Bill won a gold medal

c. Bill is a sergeant $\nvDash$ Bill is a private

These pragmatic scales also trigger IIs, and so the sentence (16) below might convey ignorance as to the exact rank Bill holds.

(16) Bill is at least an assistant professor.

An inspection of non-entailing scales with respect to the assertibility conditions obtained for numerals and determiners, reveals that they have parallel assertibility conditions.[2]

(17) Bill is at least an assistant professor. . .

a. #but I know that he does not have tenure.

b. #in fact, he has tenure.

c. but I know that he is not {an associate/a full} professor.

The (17a)/(17b) examples are infelicitous for the same reasons that the determiners were: it seems that asserting that Bill is at least an assistant professor commits the

2 In the US academic system only associate and higher ranked professors have been granted tenure.

speaker to the possibilities that Bill is in fact an assistant professor and that he might be of a higher rank. Thus, (17a) should be bad with any continuation conveying knowledge of tenure status. In turn, (17c) shows that ignorance can still be partial. As with the determiner case, no differences are found with non-entailing scales either, and so entailment cannot be a decisive factor in determining the implicative patterns of *at least*.[3]

It is also possible for *at least* to associate with elements that lack the second property, namely total ordering. As an illustration, let us consider conjunctive plurals. Assuming a domain with individuals $\{a, b, c, d\}$, an expression of the form $a \oplus b$ *are* B neither entails nor is entailed by $a \oplus c$ *are* B. [4] When dealing with partial orders it is useful to have names for those pairs of members that are not in an entailing relationship. For any two elements x, y of a partially ordered set P, if $x \leq y$ or $y \leq x$, then x and y are "comparable". Otherwise they are "incomparable". Thus, $a \oplus b$ and $a \oplus c$ in the previous example are incomparable, whereas $a \oplus b$ and $a \oplus b \oplus c$ are.

Let us take an example using *at least* with a conjunctive plural associate, as in (18) below, which conveys ignorance as to who exactly came to the party.

(18) At least Bill came to the party.

When we examine the assertibility conditions of *at least* modifying conjunctions, we find a difference compared to the other cases we have seen so far. Recall that with totally ordered scales, the exhaustive interpretation of the prejacent had to obligatorily constitute an epistemic possibility for the speaker. But now compare this with (20) in the context of (19) below.

3 By the same token, evaluative scales behave alike. Take, for instance, a preference scale formed by potential vacation destinations that Bill is considering, *Springfield* $\gg$ *NYC* $\gg$ *Rockport* $\gg$ *Jamaica*. Obviating the concessive reading (Nakanishi and Rullmann 2009, Biezma 2013), a speaker can express ignorance as to where exactly Bill went with a sentence like *Bill went at least to NYC*–for instance, in a situation where economic matters where decisive. The knowledge that is compatible with such utterance in an ordinary conversational situation parallels that of determiners and other lexical scales seen above.

(i) a. #but I know that he couldn't afford going to Rockport or Jamaica.
    b. #in fact, he went to Jamaica.
    c. but I know that he did not go to {Rockport/Jamaica}.

This kind of scales can be taken to be a variant of the pragmatic scales alluded to above, ones where the ordering is not provided by common knowledge but by idiosyncratic properties and attitudes of the agents participating in the conversation.

4 More explicitly, total orders are antisymmetric, transitive and total: for elements x, y, z of a totally ordered set T, if $x \leq y$ and $y \leq x$, then $x = y$; if $x \leq y$ and $y \leq z$, then $x \leq z$; and either $x \leq y$ or $y \leq x$. Partial orders are also antisymmetric and transitive, but they are reflexive instead of total, a weaker condition: all elements x in a partially ordered set P are such that they are related to themselves, $x \leq x$.

(19) **Context**: Sherlock Holmes went on vacation for a couple of days and let some of his friends celebrate a dinner on 221B Baker Street: Dr. Watson, Mrs. Hudson, Mycroft, Irene Adler and some of the Baker Street Irregulars. After vacation, he returns to his room only to discover that somebody has been messing with his chemistry set. Inspector Lestrade from Scotland Yard is with him, and asks:

(20) IL. Who do you think touched the chemistry set?
SH. It was at least Mycroft and Mrs. Hudson, but not only them.

There does not seem to be anything wrong with Sherlock's answer in (20). Set in an ordinary detective dialog, we can take his contribution to be maximally informative: from his answer, Inspector Lestrade can learn that Sherlock knows that Mycroft and Mrs. Hudson did touch the chemistry set, but that they were not the only ones in doing so. That is, saying that at least them both touched the set is not at odds with an epistemic state where it is taken for granted that somebody else touched it as well. This is unlike the behavior of numerals, as a comparison to a numeral version of the same dialog in the same scenario reveals.

(21) IL. How many people do you think that touched the chemistry set?
SH. #It was at least two people, but not only two.

The contrast between (20) and (21) is clear, and (21) with a numeral fails just like they did before–assuming, that is, that Sherlock is not feeling playful and he is being genuine and maximally informative, given the evidence. The critical point to be learned from this contrast is the following: when *at least* associates with a partially ordered associate, the exhaustive interpretation of the prejacent can but need not be among the epistemic possibilities considered by the speaker.[5]

Of course, Sherlock's answer in (20) is perfectly adequate also in the eventuality that he happened to know for a fact that there was somebody in particular who could have not touch the set; for instance, if he knew that it could have not been Irene Adler. This is the signature of partial ignorance again. Denying all more informative alternatives to the prejacent is the one situation that makes *at least* with partially ordered associates crash:

5 The intuitive contrast between the answers in (20) and (21) is supported by results from experimental research. Mendia (2016b) presents a study concerned precisely with the status of the exhaustive interpretation of the prejacent with superlative modifiers. Using a truth-value judgment task the author shows that speakers accepted sentences of the form *at least a⊕b P but not only a⊕b P* far more often than the numeral counterparts *at least n P but not only n P*.

(22) #It was at least Mycroft and Mrs. Hudson, and nobody else.

To conclude, an examination of non-numeral associates reveals that the assertibility conditions of *at least* modifying a partially ordered scale differ significantly from those of totally ordered scales. More concretely, sentences of the form *at least n P* with some number *n* are only felicitous if the speaker takes the corresponding proposition *only/exactly n P* to be compatible with all she knows. In turn, this is not a requirement when *at least* modifies conjunctive plurals. The assertibility conditions of *at least* with conjunctions can be informally stated, then, as follows:[6]

(23) *Assertibility conditions of* at least *with conjunctive plurals*: For a domain D $\{a, b, c, d\}$, a proposition ϕ of the form *at least* $a \oplus b$ P is assertible by S *iff*:

a. S knows that ϕ is true,

b. there is some x in D such that $a \oplus b \oplus x$ P is compatible with all S knows,

c. S knows that no x part of $a \oplus b$ is such that *only* x P.

2.3 Two answers

The first goal of the paper was to scrutinize the properties of the IIs that come with *at least* and, on the way, to answer two questions about the nature of these IIs; we dubbed these questions PREDICTABILITY and UNIFORMITY.

(7) PREDICTABILITY
Is there any proposition in particular about which the speaker must be ignorant about so that she can successfully use an *at least*-statement?

(4) UNIFORMITY
Are the inferences that come with *at least* the same across the board, regardless of its type of associate?

We found out that there are minimal epistemic conditions that speakers must meet to successfully use *at least*; in turn, these minimal conditions constitute what is minimally predictable about the speaker's epistemic state. The answer to the PREDICTABILITY question must be in the positive, then.

With respect to the UNIFORMITY question, we found out that the answer should be negative: *at least* with totally ordered associates trigger different IIs as compared to *at least* with partially ordered associates. The locus of the difference lies in what is required of the exhaustive interpretation of the prejacent: the speaker **must**

6 The expression "part of" can be understood in an intuitive way, akin to Link's (1983) two-place predicate $\prod$, the individual part relation, satisfying the biconditional $a \prod b \leftrightarrow a \oplus b = b$.

necessarily take this epistemic possibility into account with totally ordered scales, but she **need not** do so with partially ordered scales.

The lack of UNIFORMITY in *at least*'s inferences has an impact on the answer we provide for PREDICTABILITY as well, since both types of associates differ in the kind of ignorance that is predictable in each case. With totally ordered associates, listeners confronted with an *at least*-statement are able to immediately draw the inference that the speaker is ignorant as to whether the exhaustive interpretation of the prejacent is the case. With partially ordered associates, however, this is not the case, and all listeners are able to infer is that there is some alternative proposition that the speaker is ignorant about.

3 Laying the groundwork

The discussion in section §2 tells us that an adequate theory of *at least* must account for two key properties. First, it must account for the fact that ignorance with *at least* is partial, but with certain minimal conditions on speaker ignorance. Second, it must account for the fact that the nature of these minimal conditions differ depending on whether the associate constitutes a total or partial order. In what follows, I will present an account of *at least* that derives both properties. The analysis builds on a classical neo–Gricean implementation, with a twist. I argue that the calculation of implicatures with *at least* requires two sets of alternatives (Mayr 2013, Schwarz 2016a), provided by two independent mechanisms: the familiar substitution method within elements of a Horn scale (Horn 1972, Sauerland 2004), and Association with Focus, whereby a set of alternatives is obtained by replacing the focus-bearing constituent with contextually relevant alternatives (Rooth 1985, Fox and Katzir 2011).

Before turning to the formal implementation of this analysis, however, I will briefly mention two key background assumptions that need to be justified. The first has to do with the pragmatic treatment of IIs, the motivation for which is laid out in §3.1. The second concerns Association with Focus, which is necessary for deriving one of the two sets of alternatives needed for II-computation. This is discussed in §3.2 showing that *at least* is conventionally associated with focus.

3.1 Ignorance is pragmatic

As indicated by their assertibility conditions in (11)/(23), there is a tight connection between the felicity conditions of *at least* and the speaker's communicative intentions. In fact, it could be argued that *at least*'s contribution to the discourse in non-embedded contexts is primarily to convey speaker's ignorance (Coppock and Brochhagen 2013b). Yet, under certain circumstances, *at least*-statements seem felicitous even when their assertibility conditions are denied or blatantly unmet. This behavior bears the blueprint of a conversational implicature (Grice 1975, Gazdar 1979, a.o.; particularly as formulated in Hirschberg 1985), and the consensus in the literature is that IIs of *at least* should be understood as such (see Büring 2007, Coppock and Brochhagen 2013b, Mayr 2013, Kennedy 2015, Mendia 2015, Nouwen 2015, Schwarz 2016a, a.o.). I take this much for granted and thus I do not provide here a full justification of the implicative nature of *at least*. Instead, I provide what to my knowledge is a new argument that IIs are pragmatic in nature, and refer the reader to the above mentioned works for completeness.

In neo-Gricean frameworks, IIs arise as a direct consequence of the mutual agreement that speakers are being maximally informative. In the absence of such agreement, there could be stronger relevant propositions that they could have remained silent about while still being cooperative. As it turns out, in situations where it is known that the speaker is not being maximally informative, (i.e. she is flouting the Maxim of Quantity), IIs do not arise.[7] As an illustration, consider the game show scenario (adapted from Fox 2014):

(24) **Situation:** In a TV game show, utterances by the host are not presupposed to be maximally informative. The contestant has won the biggest prize, which consists of one of two options: She either takes \$5000 in cash or she takes an envelope with an amount of cash unknown to her, but that the audience and the host already know. The contestant has to gamble. At some point, the host decides to give a hint that will help the contestant to assess her chances of picking the most profitable choice. Of course, the hint is such that it only provides part of the information available to the host, and this is common understanding for both the contestant and the audience. In this case, the host says: "<u>The envelope contains at least \$2500.</u>"

What is important about the host's utterance is that in no way is he trying to convey his ignorance about the situation, nor is he trying to be misleading. Dropping the assumption that he is conveying as much information as he has available makes the hint provided by the host appropriate even though he knows the exact quantity

7 Grice (1989: 44) discusses the case of disjunction:

> 'I can say to my children at some stage in a treasure hunt, "The prize is either in the garden or in the attic. I know that because I know where I put it, but I'm not going to tell you." Or I could just say (in the same situation) "The prize is either in the garden or in the attic", and the situation would be sufficient to apprise the children the fact that my reason for accepting the disjunction is that I know a particular disjunct to be true.'

The treasure hunt scenario illustrates that the cancellation of the II is contingent upon knowing whether the different agents in the conversation have agreed on being maximally informative or not.

that the envelope contains. And, more importantly, it precludes the contestant from drawing the inference that the host does not know how much money it contains. In fact, the contestant can be confident that the hint is true precisely because she takes the host to be an authority on the matter. Thus, IIs of *at least* depend on the assumption that the speaker is being maximally informative; when discourse participants drop this assumption, no IIs arise.

Knowing whether or not the speaker is being maximally informative is not part of one's linguistic competence, as one might gain such knowledge from several sources, ranging from world knowledge to tacit one-time assumptions between agents. To put differently, there is simply no way to tell from an *at least*-utterance alone whether the speaker is being maximally informative with respect to her epistemic state. The fact that IIs are crucially dependent on the goals of the conversation and on assumptions about the speakers intentions gives further support to the idea that these inferences are pragmatic in nature.

3.2 The role of focus

A second key ingredient in my analysis of *at least* is the idea that focus serves a mediating role between their semantics and the IIs they give rise to. This link, I argue, can shed light on when IIs are present/absent and also the precise nature of the IIs conveyed.

I start off with two independent observations. First, notice that the inferences that the addressee is allowed to infer are restricted by focus: IIs must covary with the associate of the *at least*, which in turn is determined by focus marking (Krifka 1999). This is easier to see when *at least* can associate at a distance with different focused elements in otherwise identical sentences (examples from Coppock and Brochhagen 2013a).

(25) a. The chair at least invited the $\underline{\text{postdoc}}_F$ to lunch.
$\rightsquigarrow$ *ignorance about whether someone else was invited*

b. The chair at least invited the postdoc $\underline{\text{to lunch}}_F$.
$\rightsquigarrow$ *ignorance about whether she got invited to something else*

This is a strong requirement: (25a) cannot carry the IIs of (25b) or vice versa.

The second observation is that IIs disappear–or at least the effect is diminished–when no focus is predicted under the scope of *at least*. For instance in the short Q&A pair in (26), the sentence (26b) directly answers the question in (26a): focus in (26b) as an answer to (26a) is predicted to be on [ten students] (see e.g., Rooth 1992, Roberts 2012), which in turn correlates with what the speaker is ignorant about, in this case the exact number of students that took Linguistics 101 (examples from Westera and Brasoveanu 2014).

(26) a. Exactly how many students took Linguistics 101?
b. At least ten students took Linguistics 101. [=(27b)]

In contrast, in (27b) the associate of *at least* does not directly answer (27a) and no constituent is predicted to bear focus. Correspondingly, the listener need not derive any II.

(27) a. Did at least ten students take Linguistics 101?
b. At least ten students took Linguistics 101. [=(26b)]

In sum, *at least* seems to associate with focused elements and their IIs covary with the constituent that bears focus. In addition, when the associate of *at least* does not bear focus, IIs need not arise. These two observations suggest that focus plays a pivotal role in determining what the IIs are about, and when they are present.[8]

4 Analysis

This section provides a unified account of *at least* a scalar modifier, following the idea that its IIs arise as neo-Gricean conversational implicatures. Crucially, the analysis requires computing these implicatures by factoring in two sets of alternatives (Mayr 2013, Schwarz 2016a). The main innovation of the analysis presented here is that the set of alternatives relevant for the Gricean calculus is provided by two independent mechanisms. In addition to the substitution method within elements of a Horn scale (Horn 1972, Sauerland 2004, a.o.), a different set of alternatives is obtained by replacing the focus-bearing constituent with contextually relevant alternatives (Rooth 1992, cf. Fox and Katzir 2011).

4.1 Background

To properly talk about IIs, we first need to know what it means to be ignorant about something. Assume that K and P stand for the familiar epistemic certainty and possibility operators, such that $\mathsf{K}_S\phi$ means *the speaker S knows that* ϕ and $\mathsf{P}_S\phi$ means that ϕ *is compatible with all S knows*. According to the properties that Hintikka (1962) ascribed to them, both operators K and P are interdefinable,

8 The relationship seems to be well-grounded by independent evidence as well. Beaver and Clark (2008) treat *at least* as a "minimizer", a particle that is conventionally Associated with Focus. Mendia (2016a) provides several pieces of empirical evidence supporting this a view, such as limitations in association with weak and extracted elements, backwards association, etc., concluding that a weaker relationship with focus (as in the case of *-est*, the bare form of the superlatives) is insufficient.

since $\mathsf{K}\phi \leftrightarrow \neg\mathsf{P}\neg\phi$ and $\mathsf{P}\phi \leftrightarrow \neg\mathsf{K}\neg\phi$.[9] In this system, the following equivalences follow: $\mathsf{K}\neg\phi \leftrightarrow \neg\mathsf{P}\phi$ and $\neg\mathsf{K}\phi \leftrightarrow \mathsf{P}\neg\phi$. Then, to be ignorant about a proposition ϕ is expressed as follows:[10]

(28) SIGNATURE OF IGNORANCE: $\neg\mathsf{K}[\phi] \wedge \neg\mathsf{K}\neg[\phi] \quad \leftrightarrow \quad \mathsf{P}[\phi] \wedge \mathsf{P}\neg[\phi]$

(28) shows the technical notion of *ignorance* that I shall rely on. To be ignorant about ϕ is a stronger notion than the mere lack of knowledge about ϕ. By *being ignorant about* ϕ I refer to a mental (epistemic) state of some agent in which she is unsure about the truth of ϕ. Crucially, in order to be ignorant about ϕ it is necessary that the agent consider both ϕ and $\neg\phi$ live possibilities compatible with her knowledge. It follows that not only does the agent not know the truth of ϕ, she also does not know the truth of $\neg\phi$. Hintikka (1962: 12-15) illustrates this difference by alluding to the contrast between *knowing that* ϕ and *knowing whether* ϕ:

(29) a. The speaker S does not know that ϕ: $\neg\mathsf{K}_S\phi$
b. The speaker S does not know whether ϕ: $\neg\mathsf{K}_S\neg\phi \wedge \neg\mathsf{K}_S\phi$

The distinction between (29a) and (29b) is in accordance with the intuition that when we are ignorant about whether ϕ, we consider both ϕ and $\neg\phi$ to be epistemic possibilities; I take this for granted here. Sometimes I use the following notational convention, where $\mathsf{I}_S[\phi]$ means that *the speaker is ignorant about whether* ϕ:

(30) $\mathsf{I}_S[\phi] \equiv \neg\mathsf{K}_S[\phi] \wedge \neg\mathsf{K}_S\neg[\phi] \quad \leftrightarrow \quad \mathsf{P}_S[\phi] \wedge \mathsf{P}_S\neg[\phi]$

We turn now to the question of how to derive IIs of this form. Gazdar (1979), putting together insights from both Hintikka's (1962) epistemic logic and Grice's (1975) theory of language use, argued that IIs can be derived as clausal quantity implicatures. Assume that we are dealing with a cooperative speaker and that some version of the Maxims of Quality are at work (Grice 1975).

9 For concreteness, I am assuming Hintikka's (1962) system, an epistemic logic developed by enriching propositional calculus with the operator K and the three additional axioms *K* (distributivity; $\mathsf{K}(p \to q) \to (\mathsf{K}p \to \mathsf{K}q)$), *T* (reflexivity; $\mathsf{K}p \to p$) and *4* (positive introspection; $\mathsf{K}p \to \mathsf{KK}p$). This is the *KT4* modal system. Whether *KT4* is the most adequate logic to model knowledge and belief is a matter subject to debate; see Hendricks and Symons (2014) for discussion.

10 I use square brackets '[]' to enclose propositions, so that if ϕ is a propositional variable, $[\phi]$ stands for *the proposition that* ϕ. In addition, I sometimes use $\mathsf{P}[\phi]$ and $\neg\mathsf{K}\neg[\phi]$ interchangeably, as well as $\neg\mathsf{K}[\phi]$ and $\mathsf{P}\neg[\phi]$, the choice depending on what expression is more intuitive on a case to case basis.

(31) MAXIMS OF QUALITY

a. Do not say what you believe to be false.

b. Do not say what you do not have evidence for.

The Maxims of Quality can be related to the operators K and P by Hintikka's (1962) principle of Epistemic Implication, whereby utterance of a sentence ϕ by a speaker S commits S to the knowledge of ϕ: ϕ implicates ψ if $\mathsf{K}[\phi \wedge \neg\psi]$ is inconsistent.[11] When a cooperative speaker S is following the Maxims of Quality, the addressee is allowed to infer that the utterance of ϕ by S implicates that $\mathsf{K}_S\phi$. This inference is sometimes also referred to as a Quality Implicature. In order to derive IIs, however, we need some notion of strength. Assume for now a characterization of the Maxim of Quantity defined in terms of asymmetric entailment.

(32) MAXIM OF QUANTITY
If two propositions $[\phi]$ and $[\psi]$ are such that (i) the denotation of $[\phi]$ asymmetrically entails $[\psi]$ (i.e., $[\phi] \rightarrow [\psi] \wedge \neg([\psi] \rightarrow [\phi])$), (ii) $[\phi]$ and $[\psi]$ are relevant, and (iii) the speaker believes both $[\phi]$ and $[\psi]$ to be true, the speaker should choose $[\phi]$ over $[\psi]$.

The Maxim of Quantity ensures that, given a number of true and relevant alternatives to the proposition that has been uttered, if a speaker is being cooperative, she should choose the semantically strongest alternative over the rest. In view of this definition of the Maxim of Quantity, it is useful to define the notion of Stronger Alternative (*SA*): An *SA* ψ of a proposition ϕ is an alternative proposition that asymmetrically entails ϕ: ψ is an *SA* of ϕ *iff* $\psi \rightarrow \phi$ and $\phi \nrightarrow \psi$. The set of *SA*s of a proposition ϕ is expressed as $SA(\phi)$ (as opposed to the set $Alt(\phi)$ of all alternatives to ϕ). According to the Maxim of Quantity, if we are to be cooperative, we have to provide the semantically strongest relevant and true proposition we can. Following the terminology in Sauerland (2004), we now define the weakest form of inference, a Primary Implicature. In addition, we also define the Implicature Base, the set of propositions resulting from conjoining the Quality Implicature with its Primary Implicatures.

(33) PRIMARY IMPLICATURE:
The inference that $\neg\mathsf{K}\psi$, for an *SA* ψ.

(34) IMPLICATURE BASE:
The set consisting of the Quality Implicature together with its Primary Implicatures.

11 This notion of "implication" is closer to that of "entailment" and it is not the one that I use when talking about implications in general.

As an illustration of how to derive an II in this framework, consider the following sentence

(35) Bill read Tintin or Asterix.

This sentence conveys the II that the speaker does not know which of the comics Bill read. The reasoning proceeds as follows: First, assume that the speaker is being cooperative. This means that she is observing the Maxim of Quality. Assume moreover that there is no reason to believe that the speaker is not maximally informative, and so she observes the Maxim of Quantity as well. Upon hearing (35) (represented as $[\mathtt{T} \vee \mathtt{A}]$), the addressee can conclude that the speaker thinks that this much is true. Thus, by the principle of Epistemic Implication, she concludes that $\mathsf{K}_S[\mathtt{T} \vee \mathtt{A}]$. The proposition $[\mathtt{T} \vee \mathtt{A}]$ has at least two stronger alternatives, the individual disjuncts $[\mathtt{T}]$ and $[\mathtt{A}]$. This follows from the Maxim of Quantity: $[\mathtt{T}] \in SA([\mathtt{T} \vee \mathtt{A}])$, since $[\mathtt{T}]$ is relevant and $[\mathtt{T}] \rightarrow [\mathtt{T} \vee \mathtt{A}]$, but $[\mathtt{T} \vee \mathtt{A}] \nrightarrow [\mathtt{T}]$. The same reasoning applies also to $[\mathtt{A}]$. Following the Maxim of Quantity, the addressee concludes that if the speaker did not utter any one of the *SA*s, it must be because she did not have evidence enough, or maybe she did not know. Therefore, she infers the Primary Implicatures that $\neg\mathsf{K}_S[\mathtt{T}]$ and $\neg\mathsf{K}_S[\mathtt{A}]$. (36) below summarizes the process:

(36) a. ASSERTION: $[(35)] = [\mathtt{T} \vee \mathtt{A}]$
b. EPISTEMIC IMPLICATION: $\mathsf{K}_S[\mathtt{T} \vee \mathtt{A}]$
c. $SA([\mathtt{T} \vee \mathtt{A}]) = \{[\mathtt{T}], [\mathtt{A}]\}$
d. PRIMARY IMPLICATURES: $\neg\mathsf{K}_S[\mathtt{T}] \wedge \neg\mathsf{K}_S[\mathtt{A}]$
e. IMPLICATURE BASE: $\mathsf{K}_S[\mathtt{T} \vee \mathtt{A}] \wedge \neg\mathsf{K}_S[\mathtt{T}] \wedge \neg\mathsf{K}_S[\mathtt{A}]$

The Implicature Base contains all the information that the addressee may be able to deduce from the speakers utterance without any further assumptions. In particular, according to (36e), the addressee can conclude that the speaker knows that Bill read either Tintin or Asterix, that it is not the case that she knows that Bill read Tintin and it is not the case that she knows that Bill read Asterix. These are not quite yet the IIs we want. The last step to derive the right IIs from (36e) involves deriving that each disjunct is an epistemic possibility by the speaker, i.e., $\mathsf{P}_S[\mathtt{T}]$ and $\mathsf{P}_S[\mathtt{A}]$. Luckily, the task is trivial: given the properties of the operators K and P defined above, $\mathsf{P}_S[\mathtt{T}]$ and $\mathsf{P}_S[\mathtt{A}]$ are in fact entailed by the Implicature Base.[12]

12 To see that $\mathsf{K}[\mathtt{T} \vee \mathtt{A}] \wedge \neg\mathsf{K}[\mathtt{T}] \wedge \neg\mathsf{K}[\mathtt{A}] \rightarrow \mathsf{P}[\mathtt{T}] \wedge \mathsf{P}[\mathtt{A}]$ is the case, assume that $\neg\mathsf{P}[\mathtt{T}]$: since $\mathsf{P}[\mathtt{T}]$ is equivalent to $\neg\mathsf{K}\neg[\mathtt{T}]$, then $\neg\mathsf{P}[\mathtt{T}]$ is equivalent to $\neg\neg\mathsf{K}\neg[\mathtt{T}]$, which can be reduced to $\mathsf{K}\neg[\mathtt{T}]$ by double negative. But $\mathsf{K}\neg[\mathtt{T}]$ cannot be, since it contradicts the Primary Implicature in the premise. Thus, it must be the case that $\neg\mathsf{K}\neg[\mathtt{T}]$, which is equivalent to $\mathsf{P}[\mathtt{T}]$. (The same proof holds *mutatis mutandis* for $\mathsf{P}[\mathtt{A}]$.)

(37) ENTAILMENTS OF (36e): $\mathsf{K}_S[\mathtt{T} \vee \mathtt{A}] \wedge \neg\mathsf{K}_S[\mathtt{T}] \wedge \neg\mathsf{K}_S[\mathtt{A}] \wedge \neg\mathsf{K}_S\neg[\mathtt{T}] \wedge \neg\mathsf{K}_S\neg[\mathtt{A}]$

Thus, the Implicature Base alone provides all the necessary pieces to derive that the epistemic possibility of every disjunct is a must.[13] It follows, too, that knowledge about the truth of any one of the particular disjuncts should not be allowed, in this case because both $\mathsf{K}_S[\mathtt{T}]$ and $\mathsf{K}_S\neg[\mathtt{T}]$ contradict the II that $\mathsf{I}_S[\mathtt{T}]$–similarly for $\mathsf{K}_S[\mathtt{A}]$ and $\mathsf{K}_S\neg[\mathtt{A}]$.

The upshot of this discussion is that the choice of what counts as an *SA* is important: given the right choice of *SA*s, IIs may be entailed by the Implicature Base.[14] IIs of disjunctive statements can be derived by relying on independently needed formal principles, which provide the two necessary–and sufficient–ingredients to derive IIs about each particular disjunct: a suitable epistemic logic and the assumption that *SA*s are established by asymmetric entailment relations. In the following sections I extend the same pragmatic calculus to *at least*.

4.2 Calculating Ignorance

The previous section has set what the formal principles are that are responsible for calculating IIs. In what follows, I show how considering both Horn scale alternatives and focus alternatives provides the right kind of input to the implicature calculating mechanism so that it derives IIs of the right form in *at least*'s case.

4.2.1 Step 1: Focus alternatives and truth-conditions

In order to calculate implicatures in a neo-Gricean framework, alternative propositions have to be ordered. In the case of *at least*, one such ordering can be provided by asymmetric entailment relations among propositional focus alternatives. The semantics of focus delivers an ordinary semantic value and a focus semantic value that consists of a set of alternative propositions (Rooth 1985 *et seq.*). Then we can use this

13 Above I ignored the scalar *SA* $[\mathtt{T} \wedge \mathtt{A}]$. Notice that after adding the corresponding Primary Implicature $\neg\mathsf{K}_S[\mathtt{T} \wedge \mathtt{A}]$, the Implicature Base in (36e) does not entail that $\mathsf{P}_S[\mathtt{T} \wedge \mathtt{A}]$, and so no II can be derived about $[\mathtt{T} \wedge \mathtt{A}]$. This may not be a bad thing, since implicatures associated with the conjunctive alternative to disjunctive statements can sometimes be strengthened to $\mathsf{K}_S\neg[\mathtt{T} \wedge \mathtt{A}]$ and so constitute a Secondary Implicature (or Scalar Implicature). In a classical neo-Gricean set-up, this strengthening cannot happen without further ado, usually in the form of an additional assumption often referred to as the "epistemic step" or "competence assumption" (see e.g., Geurts 2010). This is not to say that (35) is incompatible with the speaker's ignorance as to whether Bill read both comics.

14 This is especially relevant when we consider disjunctions with multiple disjuncts. As Alonso-Ovalle (2006) showed, in order to calculate IIs with multiple disjuncts "sub-domain" alternatives–formed by smaller disjunctions each of whose individual disjuncts are part of the assertion–must be included (see also Chierchia 2013). Once the access to sub-domain alternatives is granted, the system presented in this paper can derive IIs of multiple disjuncts just the same.

set of propositions to reason about plausible and more informative alternatives that the speaker could have uttered–just like we usually do in routine neo-Gricean pragmatics. I suggest that this constitutes the first set of relevant alternative propositions that is factored into the pragmatic calculus.

I take it that the conclusions obtained from section §3.2 support the claim that *at least* is a member of a limited class of focusing adverbs that bears a lexically determined dependency on focus, those which Beaver and Clark (2008) refer to as showing Conventional Association with Focus. (The theory of focus I am assuming corresponds then to "intermediate" theory, in the sense of Rooth 1992.) Thus, *at least* always makes reference to focus-evoked alternatives compositionally derived throughout the semantic computation.

Informally, the meaning of a sentence S with some focalized constituent F is the set of propositions that obtains from S by making a substitution in the position corresponding to F. Alternatives to F are projected in a fully compositional fashion, hence it is a type-driven mechanism (Rooth 1985).[15] This is a two tier semantic system delivering an ordinary semantic value $[\![\cdot]\!]^o$ and a focus semantic value $[\![\cdot]\!]^f$.

(38) a. $[\![[Sue\ ate\ [broccoli]_F]\!]^o = ate(Sue, broccoli)$
b. $[\![[Sue\ ate\ [broccoli]_F]\!]^f = \{x \in D_e \mid ate(Sue, x)\}$

As for the lexical entry of *at least*, assume a propositional version whereby it can directly take sets of propositions as arguments (Büring 2007).[16]

(39) $[\![at\ least]\!] = \lambda C_{\langle st,t\rangle}.\lambda p_{\langle st\rangle}.\lambda w.\exists q[q \in C \wedge p \leq q \wedge q(w)]$

The association of *at least* with focus is no longer optional, as the set C is always constrained by the focus value of the associate. In addition to an ordinary semantic value, this definition produces a set of propositions determined by the focus semantic value. The lexical entry in (39) renders true a proposition p containing *at least* if there is some proposition q in the relevant set of alternative propositions which is at least as strong as p and q is true in the evaluation world.

15 A purely type-driven semantics of focus is known to overgenerate; see Katzir (2007) and Fox and Katzir (2011) for discussion and a proposal about how to fix it. For the purposes of this paper, I simply assume a semantic mode of composition whereby complex expressions like $[\alpha, \beta]$ are compositionally derived by applying a function h pointwise, that is, to the meaning of $[\alpha]$ and $[\beta]$ separately (cf. Rooth 1996: 281): $[\![\alpha, \beta]\!]^f \equiv \{h(X,Y) | X \in [\![\alpha]\!]^f, Y \in [\![\beta]\!]^f\}$.

16 This lexical entry leaves a number of questions open about how to better connect the syntactic properties of *at least* with its ability to semantically associate at a distance. In particular, it requires displacement of *at least* to a sentence initial position, and so it is insensitive to a number of limitations that *at least* shows (like, for instance, the inability to associate at a distance across subjects: *at least Bill ate an apple*, cannot mean the same as *Bill ate at least an apple*). Since the focus is on the IIs alone, I will not address these issues here.

In order to address the question of what focus sensitive expressions have in common, Rooth (1992, 1996) factors in the role of context on the semantic computation of sentences with focused constituents. In order for focus to be felicitous, the set of alternatives generated must be related to a contextually available set of alternatives C, where C is determined by contextually available or pragmatic information.

(40) Where φ is a syntactic phrase and C is a syntactically covert semantic variable, $\varphi \sim C$ introduces the presupposition that C is a subset of $[\![\varphi]\!]^f$ containing $[\![\varphi]\!]^o$ and at least one other element. [Rooth 1996: 285]

In Roothian semantics, if XP is focused its focus-semantic value is the set of all the entities of its semantic type. The squiggle focus operator effectively limits a focus semantic value $[\![\text{XP}]\!]^f$ to a contextually relevant set of alternatives C containing, minimally, $[\![\text{XP}]\!]^o$ and one other element. Consider for instance a sentence like (41a) and its corresponding LF.

(41) a. Sue is at least [an assistant professor]$_F$
b. LF: [$_{S^1}$ at least C [$_{S^2}$ [$_{S^3}$ Sue is [an assistant professor]$_F$] $\sim C$]]

Uttered in an academic context, we can assume that C as determined by $[\![$[associate professor]$_F]\!]^f$ amounts to {*assistant professor, associate professor, full professor,*. . . }. The derivation of the ordinary and focus semantic values proceed routinely. (I use $[\![\text{XP}]\!]^{f_c}$ to refer to the subset C of $[\![\text{XP}]\!]^f$.)

(42) a. $[\![\text{41a}]\!]^f = \{P(Sue) \mid P \in D_{et}\}$
b. $[\![\text{41a}]\!]^o =$ *Sue is an assistant professor*
c. $[\![\text{41a}]\!]^{f_c} = \left\{\begin{array}{l} \ldots \\ \textit{Sue is a visiting professor}, \\ \textit{Sue is an assistant professor}, \\ \textit{Sue is an associate professor}, \\ \ldots \end{array}\right\}$

Now *at least* can apply to the meanings in (42). (I include explicit references to the ordinary and focus semantic values, for clarity.)

(43) *Asserted content of propositions with* at least
$[\![(\text{41a})]\!] = \lambda w.\exists q[q \in [\![(\text{41a})]\!]^{f_c} \wedge [\![(\text{41a})]\!]^o \leq q \wedge q(w)]$

Minimally, *at least* conveys a semantic bound on the range of values that are allowable: in a context where Bill has three dogs, the definition correctly captures that the following sentence should be false.

(44) Bill has at least four dogs.

These truth-conditions account already for one of the clauses in the assertibility conditions listed in section §2, as the oddness of the follow-ups below signal.

(45) a. Bill has at least four dogs, #in fact he has three.
b. Bill invited at least Sue and Liz, #in fact he only invited Liz.

A consequence of this particular rendition of focus semantics for *at least* is that the strength of a proposition containing it can only be assessed with respect to the focus semantic value of that proposition. The orderings induced by focus can be established both pragmatically or by the lexical properties of the focused constituents themselves. In the case of (41), the way in which professorships are ranked is set purely by pragmatics and world knowledge. In the case of numerals, *some*, conjunctive plurals, etc., the ordering is set by the entailment properties of the lexical items themselves. An analysis of this type matches the predictions of those accounts that make use of a two alternative set strategy (notably Mayr 2013, Kennedy 2015 and Schwarz 2016a)–modulo focus association–and extends it to virtually any constituent that *at least* can associate with.

4.2.2 Step 2: Deriving alternatives

The main component of the analysis defended here is that the alternatives pertinent to the pragmatic calculus are generated by two independent mechanisms: focus alternatives and substitution by Horn scale-mates. The previous section illustrated one way to obtain a set of alternative proposition from the semantics of focus. We now can use this set of propositions to reason about plausible and more informative alternative propositions that the speaker could have uttered, just like we usually do in ordinary neo-Gricean pragmatics. I suggest that this constitutes the one set of relevant alternative propositions that is factored into the pragmatic calculus.

For instance, for the sentence *Sue is at least an assistant professor* in (41) above, the focus value provides the set of contextually restricted propositional alternatives in (42c). Utilizing the same focus alternatives we can reason about potential propositions that the speaker chose not to utter. For a proposition ϕ of the form [*at least p*], we calculate the relevant set of propositions by replacing the prejacent p by every proposition in $[\![\mathrm{p}]\!]^{f_c}$.

(46) $Alt_{\text{FOC}}(\llbracket 41a \rrbracket) = \{\llbracket \text{at least } p \rrbracket \mid p \in \llbracket 41a \rrbracket^{f_c}\} =$

$$\left\{\begin{array}{l} \ldots \\ \textit{Sue is at least a visiting professor}, \\ \textit{Sue is at least an assistant professor}, \\ \textit{Sue is at least an associate professor,} \\ \ldots \end{array}\right\}$$

From the set (46) we identify the most informative alternatives, as in 47, since only stronger alternatives are relevant for Gricean reasoning. In the case of (41) the ordering is provided by world knowledge alone.[17]

(47) $SA_{\text{FOC}}(\llbracket 41a \rrbracket^{f_c}) = \left\{\begin{array}{l} \textit{Sue is at least an associate professor,} \\ \textit{Sue is at least a full professor,} \\ \textit{Sue is at least a distiguished professor} \end{array}\right\}$

All the propositions in (47) are such that they outrank the uttered proposition by virtue of instantiating a better alternative, i.e., a higher ranked professorship. This already looks like a set amenable to draw inferences from.

Focus alternatives are not the only way alternative propositions can be ordered, however. In the case of Horn scales, alternatives are ordered by virtue of the lexical properties of its scale-mates. This is no different in the case of *at least*. As advanced before, I suggest that *at least* participates in a Horn scale together with *only*. Traditional Horn scales like {*some, all*} or {*or, and*} are formed by sets of lexical items that stand in a relation of asymmetric entailment. Since *at least* stands in an asymmetric entailment relation with *only*, this seems a plausible option. Moreover, the fact that *only*, like *at least*, bears a conventionalized dependency on focus brings the connection between the two expressions closer.

Thus, following the usual substitution method in neo-Gricean pragmatics (e.g., Sauerland 2004) alternative propositions can be generated from the set of focus alternatives by swapping *at least* with *only*.[18] In this case, we generate the new set

17 There are notions of informativity that could be applied here; in a Roberts (2012) style model of discourse, one could think of measuring informativity by keeping track of relevant propositions that address a particular question under discussion. Thus, *at least n+1* is more informative than *at least n* because the set of potential answers to the question under the discussion contains one less member. In this conception of informativity, the role of entailment is diminished.

18 One may wonder whether the presuppositional properties of *only* first discussed in the classic analysis of Horn (1969) could interfere with the implicature calculation mechanism. While I do not have space to address this worry here, other formulations of this idea are also possible. For instance, one could think of replacing *only* with the silent exhaustivity operator *Exh* (Fox (2007), a.o.), as Schwarz (2016a: fn.10) has pointed out. As in the present version, alternatives would be first generated by focus, optionally undergoing exhaustification through *Exh* thereafter. For now, assume Rooth's (1992) lexical entry for the exclusive *only*: $\llbracket \text{only} \rrbracket = \lambda C_{\langle st,t \rangle}.\lambda p_{\langle st \rangle}.\lambda w.\forall q[q \in C \wedge q(w) \leftrightarrow p = q]$.

of alternative propositions in (48) by trading *at least* for *only*, and then we pick those alternatives that asymmetrically entail the prejacent to generate a second set of stronger alternatives, as in (49).

(48) $Alt_{\mathrm{HS}}(\llbracket 41\mathrm{a} \rrbracket) = \{\llbracket \text{only } p \rrbracket \mid p \in \llbracket 41\mathrm{a} \rrbracket^{f_c}\} =$
$$\left\{\begin{array}{l} \ldots \\ \textit{Sue is only a visiting professor}, \\ \textit{Sue is only an assistant professor}, \\ \textit{Sue is only an associate professor}, \\ \ldots \end{array}\right\}$$

(49) $$SA_{\mathrm{HS}}(\llbracket 41\mathrm{a} \rrbracket) = \left\{\begin{array}{l} \textit{Sue is only an assistant professor}, \\ \textit{Sue is only an associate professor}, \\ \textit{Sue is only a full professor}, \\ \ldots \end{array}\right\}$$

Thus, Horn scales provide the second relevant set of alternatives that feeds the pragmatic calculus. Putting together both sets of Stronger Alternatives in (47) and (49), we get the final set of Stronger Alternatives over which we calculate implicatures.

(50) $$SA(\llbracket 41\mathrm{a} \rrbracket) = \left\{\begin{array}{l} \textit{Sue is at least an associate professor}, \\ \textit{Sue is at least a full professor}, \\ \textit{Sue is at least a distiguished professor} \\ \textit{Sue is only an assistant professor}, \\ \textit{Sue is only an associate professor}, \\ \textit{Sue is only a full professor}, \\ \textit{Sue is only a distinguished professor} \end{array}\right\}$$

4.2.3 Step 3: Calculating implicatures

Totally ordered associates The set of Stronger Alternatives calculated above in (50) is sufficient to derive the right kind of IIs simply by following the Gricean style reasoning about conversational cooperation laid out in section §4.1. We continue to use the same example as above in (41), now expressed differently, for simplicity.[19]

(51) $$SA(\llbracket 41\mathrm{a} \rrbracket) = \left\{\begin{array}{l} [\geq \texttt{Assoc}], [\geq \texttt{Full}], [\geq \texttt{Dist}], \\ [O\ \texttt{Assis}], [O\ \texttt{Assoc}], [O\ \texttt{Full}], [O\ \texttt{Dist}] \end{array}\right\}$$

19 The notational conventions are as follows: propositions are enclosed in square brackets, such that $[\phi]$ stands for some proposition containing a relevant expression ϕ, where ϕ informally represents the associate of *at least* as a mnemonic of the relevant expression for the purpose of calculating implicatures. For instance, a sentence like *4 students came* is represented as $[\texttt{4}]$ and *Al and Mary came* as $[\texttt{A} \oplus \texttt{M}]$. With modifiers, $[\geq\ \phi]$ stands for $[\textit{at least } \phi]$ and $[O\ \phi]$ for $[\textit{only } \phi]$.

Following the standard neo-Gricean practice, when first confronted with an utterance by her interlocutor, a listener usually assumes that the speaker is being cooperative, and so she deduces that the proposition must be true given the speaker's epistemic state; for the utterance (41), represented as $[\geq \texttt{Assis}]$, she deduces that $\mathsf{K}_S[\geq \texttt{Assis}]$ by Epistemic Implication. If there is no common understanding of the contrary, the listener may assume as well that the speaker is maximally informative–modulo relevance. Thus, if there are logically Stronger Alternatives that are both true and relevant, the speaker should have chosen one; since the speaker did not choose one of the Stronger Alternatives, it must be because she did not have sufficient grounds to claim so. Thus the listener is allowed to infer that the speaker does not possess such knowledge, thereby deriving a set of Primary Implicatures, (52d).

(52) a. ASSERTION: $[\geq \texttt{Assis}]$
b. EPISTEMIC IMPLICATION: $\mathsf{K}_S[\geq \texttt{Assis}]$
c. $SA([\geq \texttt{Assis}])$ = (51)
d. PRIMARY IMPLICATURES:
$\neg\mathsf{K}_S[\geq \texttt{Assoc}] \wedge \neg\mathsf{K}_S[O\,\texttt{Assis}] \wedge \neg\mathsf{K}_S[O\,\texttt{Assoc}] \wedge \neg\mathsf{K}_S[O\,\texttt{Full}] \wedge \neg\mathsf{K}_S[O\,\texttt{Dist}]$

Since the negation of the weakest *at least* alternative from the Stronger Alternative set (51) entails the rest of the stronger *at least* alternatives, only the weakest one is factored into the computation (see . Together with the Epistemic Implication, these implicatures constitute the Implicature Base.

(53) IMPLICATURE BASE:
$\mathsf{K}_S[\geq \texttt{Assis}] \bigwedge \neg\mathsf{K}_S[\geq \texttt{Assoc}] \bigwedge$
$\neg\mathsf{K}_S[O\,\texttt{Assis}] \wedge \neg\mathsf{K}_S[O\,\texttt{Assoc}] \wedge \neg\mathsf{K}_S[O\,\texttt{Full}] \wedge \neg\mathsf{K}_S[O\,\texttt{Dist}]$

As in the case of disjunction we illustrated in section §4.1, nothing else is required from the listener to draw ignorance about the speaker's utterance, and at this point the task of deriving IIs from (53) is trivial: the Implicature Base entails that two and only two of the Stronger Alternatives in (51) must constitute epistemic possibilities for the speaker: $\neg\mathsf{K}_S\neg[\geq \texttt{Assoc}]$ and $\neg\mathsf{K}_S\neg[O\ \texttt{Assis}]$. This follows simply from the properties of the operators K and P and the fact that the associate of the *at least* is totally ordered.

To see why this is the case, consider first $\neg\mathsf{K}_S\neg[\geq \texttt{Assoc}]$. If $\neg\mathsf{K}_S\neg[\geq \texttt{Assoc}]$ were not true, $\mathsf{K}_S\neg[\geq \texttt{Assoc}]$, it would entail that $\mathsf{K}_S[O\ \texttt{Assis}]$ is true, given our assumption that $\mathsf{K}_S[\geq \texttt{Assis}]$ holds. But $\mathsf{K}_S[O\ \texttt{Assis}]$ directly contradicts the Primary Implicature that $\neg K_S[O\ \texttt{Assis}]$, rendering the Implicature Base inconsistent. Thus, it must be the case that $\neg\mathsf{K}_S\neg[\geq \texttt{Assoc}]$. A similar reasoning shows that the second entailment $\neg\mathsf{K}_S\neg[O\ \texttt{Assis}]$ also goes through. If $\neg\mathsf{K}_S\neg[O\ \texttt{Assis}]$ were not true, $\mathsf{K}_S\neg[\geq \texttt{Assoc}]$, it would entail that $\mathsf{K}_S[O\ \texttt{Assis}]$ is true, which contradicts the

Primary Implicature that $\neg\mathsf{K}_S[O\ \texttt{Assis}]$. Thus the final step is simply to acknowledge that the Epistemic Entailment and the Primary Implicatures gang up together to generate a set of pragmatic entailments that are formally identical to the Signature of Ignorance we defined in (28) above.

(54) $\mathsf{I}_S[O\ \texttt{Assis}] \wedge \mathsf{I}_S[\geq\ \texttt{Assoc}]$

As a consequence, a speaker uttering a statement that contains a *at least* is providing quite precise information about her epistemic state. No other epistemic possibilities are entailed. For instance, take the alternative proposition that *Sue is only an associate professor*, $[O\ \texttt{Assoc}]$. The listener will deduce a Primary Implicature of the form $\neg\mathsf{K}_S[O\ \texttt{Assoc}]$. In this case, the corresponding epistemic possibility, $\neg\mathsf{K}_S\neg[O\ \texttt{Assoc}]$, is a contingent statement, not entailed by the Implicature Base. In fact, one could negate it, $\mathsf{K}_S\neg[O\ \texttt{Assoc}]$, without fear of contradicting any Primary Implicature or entailing any other relevant Stronger Alternative proposition. The epistemic state of a collaborative speaker who uttered (41) while being certain she is not an associate professor would look as follows:

(55) $\mathsf{K}_S[\geq\ \texttt{Assis}]\ \wedge\ \mathsf{K}_S\neg[O\ \texttt{Assoc}]\ \wedge\ \mathsf{I}_S[O\ \texttt{Assis}]\ \wedge\ \mathsf{I}_S[\geq\ \texttt{Assoc}]$

This accounts for the fact IIs of *at least* only show partial ignorance: there are two and only two epistemic possibilities that are pragmatically entailed; the rest may but need not. This is, however, provided that the associate of *at least* is totally ordered. We can now see why: these entailments are facilitated by a configuration where there are two Stronger Alternatives that jointly exhaust the space of possibilities denoted by the assertion. As a consequence, one or the other corresponding Stronger Alternative must be true, and so negating any one of them entails the truth of the other. Moreover, the analysis predicts that these results should obtain as well for any *at least*-statement where the associate of *at least* is strictly ordered, like other Horn scales, evaluative scales, etc.

Finally, notice that these results track the assertibility conditions of *at least* as defined in section §2, and so we can now generalize them to totally ordered associates.

(56) *Assertibility conditions of* at least *with total orders*:
A proposition ϕ of the form *at least n P* is assertible by a speaker *S iff*:

a. *S* knows that ϕ is true,
b. *exactly n P* is compatible with all *S* knows,
c. *more than n P* is compatible with all *S* knows, and
d. *S* knows that *less than n P* is not the case.

For *at least* to be assertible, a collaborative speaker must meet certain "epistemic criteria". The first clause, (56a) is given by Epistemic Implication, and the last clause, (56d), follows from the truth-conditions of *at least*. The two middle clauses, (56b)/(56c) correspond to the epistemic possibilities entailed by the Implicature Base, and so they are to be observed. This accounts for the minimal conditions that speakers must meet to successfully use *at least* without implying unwarranted additional IIs, i.e., while still conveying partial ignorance.

Partially ordered associates Let us now turn to the cases where *at least* associates with expressions that are only partially ordered. As we saw in section §2, the assertibility conditions of *at least* in these cases differ from total orders in not requiring that the exhaustive interpretation of the prejacent be an epistemic possibility. We see below how the present analysis derives this difference. Consider the following sentence with *at least* associating with a conjunctive plural.

(57) a. Liz saw at least $[\text{Bill}]_F$

b. LF: $[_{S^1}$ at least C $[_{S^2}$ $[_{S^3}$ Liz saw $[\text{Bill}]_F$ $]$ $\sim C]]$

Assume a context with a reduced domain {*Bill, Sue, Ed*} of people that Liz could have seen. The derivation of the ordinary and focus semantic values proceeds as usual.

(58) a. $[\![57\text{a}]\!]^f = \{saw(Liz,x) \mid x \in D_e\}$

b. $[\![57\text{a}]\!]^o =$ *Liz saw Bill*

c. $[\![57\text{a}]\!]^{f_C} =$
$$\left\{\begin{array}{l} \textit{Liz saw Bill, Liz saw Sue, Liz saw Ed,} \\ \textit{Liz saw Bill and Sue, Liz saw Bill and Ed, Liz saw Sue and Ed,} \\ \textit{Liz saw Bill and Sue and Ed} \end{array}\right\}$$

The truth-conditions are computed as in section §4.2.1, and as a result we obtain a lower bound on the range of allowable options. Then the derivation of alternatives and the calculation of implicatures proceeds exactly as in section §4.2.2 and section §4.2.3.

The crucial difference between partially *vs.* totally ordered associates lies in the entailments of the Implicature Base. In (57a) *at least* associates with *Bill*, which is an element of an ordering of salient individuals in the domain. This domain, which takes the form of a join-semilattice (Link 1983), contains at least two pluralities that (i) are not comparable (i.e., not ordered with respect to each other), and that (ii) are minimally more informative than *Bill*: *Bill and Sue* and *Bill and Ed* (see Figure 1). As a consequence, the number of Stronger Alternatives that are minimally required to exhaust the space of possibilities denoted by the prejacent is no longer two, but

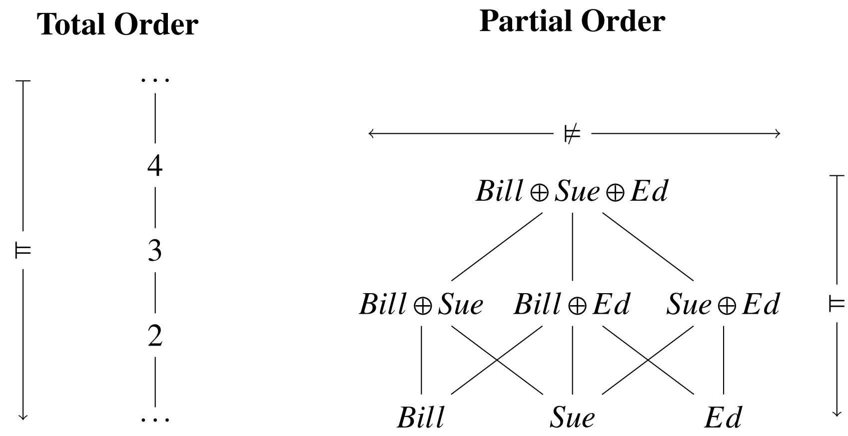

Figure 1 Entailment patterns of scales with different ordering properties

three. For an assertion like *Liz is at least an assistant professor*, we carve out all the possibilities with two Stronger Alternatives, she is either an assistant professor or she holds a higher rank. But in order to exhaust all possibilities for *Liz saw at least Bill* in the previous situation, we need three: she either saw only Bill, or she saw at least Bill and Sue, or she saw at least Bill and Ed. This is illustrated in (59).

(59) a. $[\geq \texttt{Assist}] \leftrightsquigarrow [O\,\texttt{Assist}] \vee [\geq \texttt{Assoc}]$
b. $[\geq \texttt{B}] \leftrightsquigarrow [O\,\texttt{B}] \vee [\geq \texttt{B}\oplus\texttt{S}] \vee [\geq \texttt{B}\oplus\texttt{E}]$

This difference has major consequences when we compute the entailments of the Implicature Base. Again, consider *Liz saw at least Bill*. The calculation of the Implicature Base is summarized below.

(60) a. ASSERTION: $[\geq \texttt{B}]$
b. EPISTEMIC IMPLICATION: $\mathsf{K}_S[\geq \texttt{B}]$
c. $SA([\geq \texttt{B}]) =$
$$\left\{\begin{array}{l}[O\,\texttt{B}],\\ [O\,\texttt{B}\oplus\texttt{S}],[O\,\texttt{B}\oplus\texttt{E}],[O\,\texttt{B}\oplus\texttt{S}\oplus\texttt{E}],\\ [\geq \texttt{B}\oplus\texttt{S}],[\geq \texttt{B}\oplus\texttt{E}],[\geq \texttt{B}\oplus\texttt{S}\oplus\texttt{E}]\end{array}\right\}$$
d. PRIMARY IMPLICATURES:
$$\left\{\begin{array}{l}\neg\mathsf{K}_S[O\,\texttt{B}]\wedge\\ \neg\mathsf{K}_S[O\,\texttt{B}\oplus\texttt{S}]\wedge\neg\mathsf{K}_S[O\,\texttt{B}\oplus\texttt{E}]\wedge\neg\mathsf{K}_S[O\,\texttt{B}\oplus\texttt{S}\oplus\texttt{E}]\wedge\\ \neg\mathsf{K}_S[\geq \texttt{B}\oplus\texttt{S}]\wedge\neg\mathsf{K}_S[\geq \texttt{B}\oplus\texttt{E}]\wedge\neg\mathsf{K}_S[\geq \texttt{B}\oplus\texttt{S}\oplus\texttt{E}]\end{array}\right\}$$
e. IMPLICATURE BASE:
$\mathsf{K}_S[\geq \texttt{B}] \wedge \neg\mathsf{K}_S[O\,\texttt{B}]\wedge\neg\mathsf{K}_S[\geq \texttt{B}\oplus\texttt{S}]\wedge\neg\mathsf{K}_S[\geq \texttt{B}\oplus\texttt{E}]$

Unlike with totally ordered associates, the resulting Implicature does not entail that any one of the Stronger Alternatives constitutes an epistemic possibility for the speaker. Indeed, now we could negate any Stronger Alternative without contradicting nor entailing the truth of any other. Thus, suppose that the speaker knew that it is not the case that Liz saw only Bill, $\mathsf{K}_S\neg[O\ \mathtt{B}]$. Conjoining this assumption with the Implicature Base results in a contingent set of propositions: all it says is that the speaker knows that Liz saw somebody else besides Bill, but she does not know who exactly.

(61) $\mathsf{K}_S[\geq \mathtt{B}] \wedge \mathsf{K}_S\neg[O\ \mathtt{B}] \wedge\neg\mathsf{K}_S[O\ \mathtt{B}]\wedge\neg\mathsf{K}_S[\geq \mathtt{B}\oplus\mathtt{S}]\wedge\neg K_S[\geq \mathtt{B}\oplus\mathtt{E}]$

Knowing that $\mathsf{K}_S\neg[O\ \mathtt{B}]$ does not settle the question as to which one of $[\geq\ \mathtt{B}\oplus\mathtt{S}]$ or $[\geq\ \mathtt{B}\oplus\mathtt{E}]$ might be true, and so the speaker is predicted to be ignorant precisely about these two Stronger Alternatives: since together they carve out the remaining space of possibilities, negating one of them, e.g., $\mathsf{K}_S\neg[\mathtt{B}\oplus\mathtt{S}]$ would in turn entail the truth of the second, $\mathsf{K}_S[\mathtt{B}\oplus\mathtt{E}]$, contradicting once again the corresponding Primary Implicature that $\neg\mathsf{K}_S[\mathtt{B}\oplus\mathtt{E}]$ and resulting in an inconsistent set of beliefs. In these situations, all the speaker is allowed to infer is an epistemic state of the following form.

(62) $\mathsf{K}_S[\geq \mathtt{B}] \wedge \mathsf{K}_S\neg[O\ \mathtt{B}] \wedge \mathsf{I}_S[\geq \mathtt{B}\oplus\mathtt{S}] \wedge \mathsf{I}_S[\geq \mathtt{B}\oplus\mathtt{E}]$

The allowed inferences correspond again to the assertibility conditions of *at least* associating with conjunctive plurals, repeated below from (23).

(23) *Assertibility conditions of* at least *with partial orders*:
For a domain D $\{a,\ b,\ c,\ d\}$, a proposition ϕ of the form *at least* $a\oplus b$ P is assertible by a speaker S *iff*:

a. S knows that ϕ is true,

b. there is some x in D such that $a\oplus b\oplus x$ P is compatible with all S knows,

c. S knows that no x part of $a\oplus b$ is such that *only* x P.

The first and last clauses follow from the principle of Epistemic Implication and the truth-conditions of *at least*, respectively. The middle clause corresponds to the epistemic possibilities entailed by the Implicature Base, which in this case amount solely to the existence of some possibly unidentifiable Stronger Alternative that is compatible with all the speaker knows. As before, these constitute the minimal conditions that speakers must meet to successfully use *at least* when modifying a partially ordered associate without implying unwarranted additional IIs.

The "epistemic criteria" in these cases are weaker than with totally ordered associates: the exhaustive interpretation of the prejacent, need no longer be an obligatory epistemic possibility for the speaker.

5 Conclusions and discussion

5.1 Assessment

The goal of the analysis defended above was to account for the properties of the IIs described in section §2. In order to do so, it is argued that it suffices with (i) a basic epistemic logic and (ii) the assumption that we need to factor in alternatives generated from two different sources, focus alternatives and Horn scales (cf. Mayr 2013, Kennedy 2015, Schwarz 2016a). As shown in section §3, there is independent evidence showing that IIs of *at least* are conversational implicatures (in the sense of Grice 1975 and Gazdar 1979) and that they show a lexical dependency on focus placement (in the sense of Beaver and Clark 2008). Taken together, these ingredients allowed us to analyze *at least* in a neo-Gricean fashion and account for the properties of the IIs described in section §2. In what follows I summarize and briefly discuss the merits of the analysis.

FLEXIBILITY One of the most obvious advantages of the present analysis is that it provides a uniform treatment to all cases where *at least* is allowed. The reasoning process that leads to IIs is a general pragmatic mechanism triggered by external factors like conversational efficiency and speaker-hearer cooperation, and so the underlying mechanisms for calculating IIs across associate types are kept constant, no extra assumptions are required. This is greatly facilitated by appealing to and reusing focus alternatives, making it possible for *at least* to refer to sets of expressions that would otherwise not be naturally ordered, thereby failing to constitute a scale.

STRENGTH OF IGNORANCE As many others already noted (see Schwarz (2016a) for a recent discussion), the epistemic inferences that can be drawn from *at least*-statements are somewhat weaker than those of disjunction. A disjunctive statement conveys IIs about each one of the individual disjuncts, a property referred to as total ignorance. Instead, *at least* does not convey ignorance about all the individual values in the relevant range that the speaker could be ignorant about. The analysis provided here is able to calculate IIs that are equivalent to the simplest disjunction possible (Büring 2007), thereby accounting for the fact that *at least* only conveys partial ignorance, while still being compatible with total ignorance. Moreover, given that totally *vs.* partially ordered associates correspond to disjuncts of different complexity, different types of partial ignorance are predicted for the two, as attested by the contrasts between e.g. Horn scales and conjunctive plural associates.

ASSERTIBILITY CONDITIONS As shown in the previous section, the assertibility conditions of *at least* are no longer surprising under the present analysis. The account defended here provides a precise definition of the "epistemic requirements" that speakers must meet in order to successfully use *at least*-expression–under the assumption that we are not dealing with a conversational situation where some pragmatic principle is being obviously disregarded. The assertibility conditions of *at least* are kept constant and derived via a general schema of implicature calculation. However, it was shown that different factors may affect the assertibility conditions of *at least* in certain cases. Three such main factors were mentioned above, namely focus placement, pragmatic assumptions about the discourse, and the ordering of the *at least*'s associate. Accordingly, the paper shows that the assertibility conditions of *at least* vary in predictable ways when there is a change in the status of any one of these factors.

Two of these three factors are accounted for by assuming (i) obligatory focus association and (ii) that IIs arise as conversational implicatures. As it was shown in section §2, focus placement determines what the IIs that *at least* is conveying are about. Put it otherwise, it is not possible for an *at least*-statement with a focused phrase XP_F to convey an II about some other phrase YP in the sentence. Conversely, in the absence of a focused phrase in the sentence, no IIs arise, or its effects are severely diminished. The analysis defended here handles this dependency by relying on focus alternatives to generate a set of alternative propositions that are later factored into the pragmatic calculus. This assumption was defended in section §3.2 above.

Absence of focus is not the only way to waive the requirement of computing IIs with *at least*. IIs are pragmatic implicatures that show a strong context-dependency and do not arise across all types of conversational situations. This accounts for the second of three factors that can have an impact on the assertibility conditions of *at least*. In particular, IIs depend crucially on the common understanding among the participants in the conversation that the speaker is being maximally informative. Conversational situations where the Maxim of Quantity is predicted to be suspended are shown to correlate with a lack of IIs. The analysis can correctly capture such context-dependency by relying on the Maxim of Quantity, which can be easily suspended. The assumption that IIs arise a conversational implicatures is defended in section §3.1.

NON-UNIFORMITY The present analysis predicts that the status of the IIs differ across different types of associates, thus deriving the patterns of non-uniformity observed in section §2. The exhaustive interpretation of the prejacent necessarily constitutes an epistemic possibility for the speaker only if the associate of *at least* is totally ordered. If the associate is partially ordered, this possibility is merely that, a possibility, and it is not required to figure under the options that the speaker has to consider mandatorily. The analysis presented in this paper is able to capture this point of variation between the numeral case and some phrasal associates of *at least* without further stipulations. The ordering of the associate constitutes the third and final factor that might have an impact on the assertibility conditions of *at least*, and no extra assumptions are required to derive it.

Scalar Implicatures It is well known that *at least* does not give rise to scalar implicatures: (63) below does not support the strong inference that exactly two people came to the party.

(63) At least two people came to the party.
$\nrightarrow$ *It is not the case that at least three people came to the party*

Any successful account of *at least* must derive the fact that such strengthening is blocked. In Gricean frameworks, scalar implicatures require the extra assumption that the speaker is maximally knowledgeable about the question that the proposition she is uttering is making a contribution to. That is, for a Stronger Alternative ϕ by S, either $\mathsf{K}_S\phi$ or $\mathsf{K}_S\neg\phi$, in so far as the result is consistent with the assertion and the Primary Implicatures (Sauerland 2004, van Rooij and Schulz 2004). This assumption is commonly referred to as the "epistemic step" (also the "competence" or "authority" assumption), and it is commonly assumed that the listener by default takes the speaker to be competent about Stronger Alternatives, unless these are preempted by IIs. Both Mayr (2013) and Schwarz (2016a) show that, without further assumptions, a double Horn scale strategy in a neo-Gricean analysis of scalar implicatures would deliver the wrong results, leading to either the wrong implicatures or inconsistency.[20] In order to assume avoid such problems, Schwarz (2016a) shows, we need to supplement traditional neo-Gricean analyses with a mechanism that preserves consistency during the derivation of implicatures that is more advanced than that of Sauerland (2004). The solution he proposes is to adapt Fox's (2007) idea that strengthening does not happen randomly, but rather it is only allowed if it does not lead to the necessary inclusion of any other Stronger Alternative, i.e. if the alternatives are "innocently excludable". By incorporating

20 The problem is as follows. Consider the case of (63). The analysis presented above only generates IIs about two of the Stronger Alternatives: $[O\ 2]$ and $[\geq 3]$. In principle, then, any other additional alternative ϕ for which the system fails to generate the possibility implication $\neg\mathsf{K}_S\neg[\phi]$, ϕ could be strengthened by the epistemic step to the stronger inference that $\mathsf{K}_S\neg[\phi]$. Given the double alternative set approach to *at least*, there are many such Stronger Alternatives. Take for instance $[O\ 3]$ and $[\geq 4]$. No IIs about any of them are generated; in fact, taken separately, both $\mathsf{K}_S\neg[O\ 3]$ and $\mathsf{K}_S\neg[\geq 4]$ are contingent with the assertion and its IIs. The problem is that nothing in the neo-Gricean analysis exposed here preempts strengthening *both*. But together $\mathsf{K}_S\neg[O\ 3] \wedge \mathsf{K}_S\neg[\geq 4]$ entails $\mathsf{K}_S\neg[\geq 3]$ contradicting the possibility inference $\neg\mathsf{K}_S\neg[\geq 3]$ that is part of the II. The system, thus, predicts that no consistent inferences can be derived from (63).

Innocent Exclusion into the neo-Gricean calculus, the full set of implicatures from (63): (*i*) is rendered consistent, (*ii*) entails the desired IIs, and (*iii*) avoids unattested implicatures. Consider the Epistemic Base for (63):

(64) $K_S[\geq 2] \wedge \neg \mathsf{K}_S[\geq 3] \wedge \neg \mathsf{K}_S[\geq 4] \wedge \neg \mathsf{K}_S[O\, 2] \wedge \neg \mathsf{K}_S[O\, 3] \wedge \neg \mathsf{K}_S[O\, 4]$

For some proposition ϕ to be innocently excludable relative to a set of Primary Implicatures A means that $\mathsf{K}_S\neg[\phi]$ is a member of *every* maximal subset of $\{\mathsf{K}_S\neg[\psi] : \neg \mathsf{K}_S[\psi] \in A\}$ consistent with A. Schwarz (2016a) shows that none of the Stronger Alternatives in the set $\{[O\ n], [\geq n+1]\}$ for all $n \geq 3$ is innocently excludable. This is because for any $i \geq 3$, there is some maximal subset of $\{\mathsf{K}_S\neg[O\ n], \mathsf{K}_S\neg[\geq n+1]\}$ (where $n \geq 3$) that is consistent with the set of Primary Implicatures and does not include either one of $\mathsf{K}_S\neg[O\ i]$ or $\mathsf{K}_S\neg[\geq i]$. For instance, in the case of $i = 3$, $\{\mathsf{K}_S\neg[O\, 4], \mathsf{K}_S\neg[\geq 4], \mathsf{K}_S\neg[O\, 5], \mathsf{K}_S\neg[\geq 5], \mathsf{K}_S\neg[O\, 6], \mathsf{K}_S\neg[\geq 6] \ldots\}$ is one such set. Thus, while one can construct many such sets consistent with the Primary Implicatures, this not the case for *all* subsets taken together. As a consequence, any form of strengthening of $\neg\mathsf{K}_S[\phi]$ to $\mathsf{K}_S\neg[\phi]$ is blocked with *at least*, either at first instance by existing Primary Implicatures, or at a second instance by Innocent Exclusion.

EMBEDDED *at least* The analysis also makes the prediction that when *at least* appears under a universal operator, IIs disappear. This was first observed by Büring (2007) (for discussion see Schwarz and Shimoyama 2011 and Mayr 2013).[21] Consider (65) as illustration.

(65) Every student read at least two papers.

Sentence (65) is ambiguous, and under one of its interpretations it does not convey that the speaker ignorant as to how many papers every student read. Double alternative set approaches to IIs account for the lack of IIs under universal quantifiers. Take $\forall[\geq 2]$ to represent the meaning of (65), and $\mathsf{K}_S\forall[\geq 2]$ as the inference resulting from Epistemic Implication. In this analysis, IIs are derived by virtue of finding a pair of Stronger Alternatives that jointly exhaust the space of possibilities covered by the utterance. Note, however, that $\forall[O\, 2]$ and $\forall[\geq 3]$ fail to cover such space, since $\forall[\geq 2]$ could be true by virtue of some students reading exactly two papers while other students read more than two, in which case neither $\forall[O\, 2]$ nor $\forall[\geq 3]$ would

21 In the case of *at least*, IIs can be obviated in contexts that have been argued to involve some sort of universal quantification, like modal verbs, generics and imperatives. None of the examples below necessarily conveys ignorance.

(i) a. Bill {must/has to/is required to} read at least two papers to get an A.
b. Spiders have at least two eyes.
c. Calculate at least one root of the equation $8x^5 - 6x^4 - 83x^2 - 6x + 8 = 0$.

be true. As a consequence, none of the corresponding possibility implicatures are entailed, $\neg \mathsf{K}_S \neg \forall [O\, 2]$ and $\neg \mathsf{K}_S \neg \forall [\geq 3]$, and no IIs are predicted. (The interpretation conveying ignorance is achieved by interpreting the universal quantifier under the scope of *at least*; see Büring 2007 and Kennedy 2015.)

5.2 What is with *at most*?

The analysis presented here is intended to explain the properties of *at least*. One could have the expectation, however, that all things being equal, *at least* and *at most* behave alike except in their monotonic properties. For instance, the intuitive assertibility conditions of *at most* seem to replicate those of *at least*–modulo monotonicity.

(66) Bill ate at most six apples… [cf. (10)]

a. #but I know that he didn't eat exactly six.
b. #in fact, he did not eat less than six apples.
c. but I know that he didn't eat {four/three or four/between two or five/…}.

(67) *Assertibility conditions of* at most *with numerals*: A proposition ϕ of the form *at most n P* is assertible by a *S iff*: [cf. (11)]

a. *S* knows that ϕ is true **and** *S* knows that *more than n P* is not the case,
b. *exactly n P* is compatible with all *S* knows, and
c. *less than n P* is compatible with all *S* knows.

Moreover, in when *at most* modifies partially ordered associates, it seems that our intuitions line with those of *at least*. Below is the same scenario used for *at least* in section §2, minimally modified for superlative modifier *at most*.

(68) **Context**: Sherlock Holmes went on vacation for a couple of days and let some of his friends celebrate a dinner on 221B Baker Street: Dr. Watson, Mrs. Hudson, Mycroft, Irene Adler and some of the Baker Street Irregulars. After vacation, he returns to his room only to discover that somebody has been messing with his chemistry set. Inspector Lestrade of Scotland Yard is with him, and asks:

(69) IL. Who do you think touched the chemistry set?
SH. It was at most Mycroft, Mrs. Hudson, and two of the irregulars, but not all of them touched it.

Sherlock's answer in (69) is meant to support an epistemic state where he knows for a fact that each member of a selected group of suspects could have touched his chemistry set, and yet not all of them did touch it, although maybe more than one of them did. In my own and my informants' judgment, this is a fine sentence given the depicted scenario.

So, provided our intuitions about (66)/(68) are correct, what precludes us from importing wholesale the present account to *at most*? It would seem that simply making the corresponding adjustments to the lexical entry of *at most* regarding the asserted upper bound and its monotonic properties should suffice, all else being equal. However, all else is *not* equal when it comes to *at least* and *at most*. The following contrast seems to cast reasonable doubt on the possibility of an immediate extension

(70) A. Sue is at least an assistant professor.
     B. No, she is only an assistant professor.

(71) A. Sue is at most an assistant professor.
     B. #No she is only an assistant professor.

In (70) B's response in (70B) correct A's *at least*-statement with an *only*-statement, thereby addressing A's state of ignorance. This is not possible with *at most*, as illustrated by (71). I believe the contrast between (70) and (71) is indicative that, while a speaker uttering a sentence of the form [*at least* ϕ] could consider [*only* ϕ] as both a grammatical and plausible alternative to her utterance, this is not the case for *at most*. That is, in an ordinary *at most* sentence like *at most Bill and Al came*, the sentence *only Bill and Al came* would never figure as a plausible alternative to the former.

We could impute the source of the contrast to the joint action of the presuppositional content of *only* and the entailment properties of *at least* versus *at most*: in many theories of *only*, [*only* ϕ] requires a higher ranked alternative that does not hold, thereby bringing an existential commitment about the denied higher ranked alternative. This is not in conflict with *at least* (see the definition in 39), but it is problematic for *at most*, which is usually assumed to convey an upper bound.[22] Thus, *only* could never be a relevant option when reasoning about an *at most*-statement because it presuppositions clash with the assertive force of *at most*. In neo-Gricean terms, this means that a putative Horn scale {*at most, only*} is not a feasible option.

These findings may suggest that there is a greater divergence between *at least* and *at most* than previously thought (cf. Penka 2015). A fruitful avenue for future research is to carefully investigate whether this is in fact the case, a question that I leave open here.

22 For concreteness, assume: $[\![\textit{at most}]\!] = \lambda C_{\langle st,t\rangle}.\lambda p_{\langle st\rangle}.\lambda w.\forall q[q \in C \wedge q(w) \rightarrow q \leq p]$.

5.3 Types of ignorance

A corollary of this study is that there are a total of three formally distinguishable types of ignorance. Total ignorance is the strongest type of ignorance, the one conveyed by disjunction. In the general case, disjunctive statements convey ignorance about every individual disjunct–and every smaller disjunct in the case of multiple disjunctions–or, alternatively, about every alternative in its sub-domain (e.g. Alonso-Ovalle 2006). This means that total ignorance is incompatible with any kind of positive knowledge about the relevant alternatives. Schematically: $\mathsf{K}_S[\alpha_i \vee, ..., \vee\, \alpha_j] \rightsquigarrow \mathsf{I}_S[\alpha_i] \wedge, ..., \wedge\, \mathsf{I}_S[\alpha_j]$.

This is not the case with partial ignorance, since partial ignorance is indeed compatible with some amount of knowledge. What that knowledge is, however, varies among different expressions. In the case of *at least* associating with totally ordered scales, such as numerals, ignorance is mandatory only about two alternative propositions: (i) the exhaustive interpretation of the prejacent and (ii) the immediately Stronger Alternative. For convenience, say this is a case of "strong partial ignorance". Schematically: $\mathsf{K}_S[\geq\ n] \rightsquigarrow \mathsf{I}_S[O\ n] \wedge \mathsf{I}_S[\geq\ n+1]$. Practically speaking, after a speaker utilizes an *at least n* expression, listeners can only be certain that she is ignorant about the truth of those two Stronger Alternative propositions, and ignorance about the rest of Stronger Alternatives is a contingent matter.

Finally, when *at least* associates with partially ordered scales, it does not entail ignorance about any one particular Stronger Alternative. Call this "weak partial ignorance". This is formally equivalent to von Fintel's (2000) "modal variation", an epistemic inference conveyed by *ever*-free relatives, epistemic indefinites (Alonso-Ovalle and Menéndez-Benito 2010) and epistemic numbers (Mendia 2018). All weak partial ignorance amounts to is an impossibility to decide what accessible world is epistemically the best:

(72) **Modal Variation**:
$\exists w', w'' \in D_{S,w}[\{x : P(w')(x)\} \neq \{x : P(w'')(x)\}]$
[where $D_{S,w}$ is the set of epistemically accessible worlds compatible with the speaker S's evidence in w]

The thing to notice about weak partial ignorance is that it does not conform to the signature of ignorance defined above in (28).

(28) SIGNATURE OF IGNORANCE: $\neg\mathsf{K}[\phi]\wedge\neg\mathsf{K}\neg[\phi] \quad \leftrightarrow \quad \mathsf{P}[\phi]\wedge\mathsf{P}\neg[\phi]$

Even in a heavily reduced domain D containing just *Sue*, *Bill* and *Ed*, a sentence like *At least Bill came* expresses a proposition of the form $\mathsf{K}_S[\geq \mathtt{B}]$ whose Stronger Alternatives cannot be strengthened to a full II. We saw why above: to carve out the space of available epistemic possibilities upon concluding that $\mathsf{K}_S[\geq \mathtt{B}]$ requires at least three Stronger Alternatives: $[O\ \mathtt{B}]$, $[\geq\ \mathtt{B}\oplus\mathtt{S}]$ and $[\geq\ \mathtt{B}\oplus\mathtt{E}]$, and thus we could negate any one of those three Stronger Alternatives without contradicting nor entailing the truth of any other. What $\mathsf{K}_S[\geq \mathtt{B}]$ entails is the epistemic possibility that there is some individual $x\in D$ other than Bill such that a proposition of the form $[\geq \mathtt{B}\oplus x]$ is compatible with all the speaker knows, $\mathsf{P}_S[\geq \mathtt{B}\oplus x]$ or alternatively $\neg\mathsf{K}_S\neg[\geq \mathtt{B}\oplus x]$. Given D, denying that $\neg\mathsf{K}_S\neg[\geq \mathtt{B}\oplus x]$ is equivalent to denying two possibilities, $\mathsf{P}_S[\geq \mathtt{B}\oplus\mathtt{E}]$ and $\mathsf{P}_S[\geq \mathtt{B}\oplus\mathtt{S}]$, which wound entail the truth of $\mathsf{K}_S[O\ \mathtt{B}]$, contradicting the corresponding Primary Implicature that $\neg\mathsf{K}_S[O\ \mathtt{B}]$. Crucially, however, $\mathsf{P}_S[\geq \mathtt{B}\oplus x]$ should not be understood as $\mathsf{P}_S[\geq \mathtt{B}\oplus\mathtt{S}]\ \wedge\ \mathsf{P}_S[\geq \mathtt{B}\oplus\mathtt{E}]$, since the latter relates epistemic possibilities accessible to the speaker with particular individuals in the domain, and this is too strong an inference. Instead, weak partial ignorance is not ignorance about any particular such Stronger Alternative, but rather expresses that the speaker considers that one such Stronger Alternative is possibly the case, while she may not able to determine which one. As a consequence, weak partial ignorance may (but need not) be compatible with a fair amount of knowledge, as was discussed earlier in §2.2 and §4.2.3. Schematically, then: $\mathsf{K}_S[\geq\ \mathtt{B}] \rightsquigarrow \mathsf{I}_S[\geq\ \mathtt{B}\oplus x]$, for some $x\in D$.

Summarizing, we find that natural expressions may convey three types of formally distinguishable Ignorance Inferences.

(73) For some set of Stronger Alternatives *SA* a speaker *S* may express:

a. *Total Ignorance*. *S* ignores the truth of all Stronger Alternatives in *SA*, they must all constitute epistemic possibilities *S* (e.g. disjunction).

b. *Strong Partial Ignorance* . *S* ignores the truth of two Stronger Alternatives: (*i*) the exahustive interpretation of the prejacent and (*ii*) the (possibly non-exhaustive) immediately Stronger Alternative; the two must constitute epistemic possibilities for *S* (e.g. *at least* with totally ordered associates).

c. *Weak Partial Ignorance*. *S* does not entail ignorance about any one Stronger Alternative, instead signals that the speaker is ignorant about at least one such (possibly unidentifiable) Stronger Alternative (e.g. *at least* with partially ordered associates, epistemic indefinites.)

6 Summary

This paper accomplishes two things. It provides a close examination of the Ignorance Inferences (IIs) conveyed by *at least* in all contexts in which it appears, and establishes that it conveys partial ignorance, but one that depends on the structural properties of its associate. We ask two questions about the nature of these inferences, which we dubbed PREDICTABILITY and UNIFORMITY.

(4) UNIFORMITY
Are the inferences that come with *at least* the same across the board, regardless of its type of associate?

(7) PREDICTABILITY
Is there any proposition in particular about which the speaker must be ignorant about so that she can successfully use an *at least*-statement?

The answer to the UNIFORMITY question should be negative: associates of *at least* that are totally ordered trigger IIs that are formally different than those associates that are partially ordered. The difference lies in what is required of the exhaustive interpretation of the prejacent: the speaker **must** necessarily take this epistemic possibility into account with totally ordered associates, but she **need not** do so with partially ordered associates. The answer to the question about PREDICTABILITY is also positive: in order to felicitously use *at least*, speakers must meet certain epistemic conditions. In turn, these conditions correlate with what is minimally predictable about the speaker's epistemic state after she utters an *at least*-statement. However, given UNIFORMITY, what can be inferred from an *at least*-statement varies depending on the type of the associate.

The second part of the paper develops an analysis of IIs that uniformly accounts for both numeral and phrasal cases. The account, based on a basic epistemic logic, defends the idea that we need to factor in alternatives generated from two different mechanisms and makes the novel claim that one such mechanism is focus alternatives. The analysis improves on previous approaches in a number of respects. It builds on Schwarz's (2016a) and extends a double alternative set neo-Gricean approach to a variety of scales other than the numeral, providing a general theory of IIs and their required alternatives.

References

Alonso-Ovalle, L. (2006). *Disjunction in Alternative Semantics*. PhD thesis, UMass Amherst.

Alonso-Ovalle, L. and Menéndez-Benito, P. (2010). Modal indefinites. *Natural Language Semantics*, 18:1–31.

Barwise, J. and Cooper, R. (1981). Generalized quantifiers and natural language. *Linguistics and Philosophy*, 4(2):159–219.

Beaver, D. I. and Clark, B. Z. (2008). *Sense and Sensitivity: how focus determines meaning*. Wiley-Blackwell, Malden, MA.

Biezma, M. (2013). Only one at least: refining the role of discourse in building alternatives. In Shwayder, K., editor, *Proceedings of Penn Linguistics Colloquium*, volume 36, pages 12–19.

Büring, D. (2007). The least *at least* can do. In *Proceedings of WCCFL*, volume 26, pages 114–130.

Chierchia, G. (2013). *Logic in grammar*. Oxford University Press.

Cohen, A. and Krifka, M. (2014). Superlative quantifiers as modifiers of meta-speech acts. *Linguistics and Philosophy*, 37(1):41–90.

Coppock, E. and Brochhagen, T. (2013a). Diagnosing truth, interactive sincerity, and depictive sincerity. In Snider, T., editor, *Proceedings of SALT*, pages 358–375.

Coppock, E. and Brochhagen, T. (2013b). Raising and resolving issues with scalar modifiers. *Semantics & Pragmatics*, 6:1–57.

Fox, D. (2007). Free choice and the theory of scalar implicatures. In Sauerland, U. and Stateva, P., editors, *Presupposition and Implicature in Compositional Semantics*, pages 71–120. Palgrave.

Fox, D. (2014). Cancelling the maxim of quantity: Another challenge for a gricean theory of scalar implicatures. *Semantics & Pragmatics*, 7(5):1–20.

Fox, D. and Katzir, R. (2011). On the characterization of alternatives. *Natural Language Semantics*, 19(1):87–107.

Gazdar, G. (1979). *Pragmatics: Implicature, Presupposition and Logical Form*. Academic Press.

Geurts, B. (2010). *Quantity implicatures*. Cambride University Press.

Geurts, B. and Nouwen, R. (2007). *At least* et al.: the semantics of scalar modifiers. *Language*, 83(3):533–559.

Grice, P. H. (1975). Logic and conversation. In Cole, P. and Morgan, J. L., editors, *Syntax and semantics*, volume 3, pages 41–58. Academic Press.

Grice, P. H. (1989). *Studies in the Way of Words*. Harvard University Press.

Hendricks, V. and Symons, J. (2014). Epistemic logic. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. http://plato.stanford.edu/archives/spr2014/entries/logic-epistemic/, spring 2014 edition.

Hintikka, J. (1962). *Knowledge and Belief*. Cornell University Press.

Hirschberg, J. (1985). *A Theory of Scalar Implicature*. PhD thesis, University of Pennsylvania.

Horn, L. (1969). A presuppositional approach to *only* and *even*. In Binnick, R., editor, *Meeting of the Chicago Linguistic Society*, volume 5, pages 98–107.

Horn, L. (1972). *On the Semantic Properties of Logical Operators in English*. PhD thesis, University of California, Los Angeles.

Katzir, R. (2007). Structurally-defined alternatives. *Linguistics and Philosophy*, 30(6):669–690.

Kennedy, C. (2015). A "de-fregean" semantics (and neo-gricean pragmatics) for modified and unmodified numerals. *Semantics & Pragmatics*, 8(10):1–44.

Krifka, M. (1999). At least some determiners aren't determiners. In Turner, K., editor, *The semantics/pragmatics interface from different points of view*. Elsevier Science B.V.

Link, G. (1983). The logical analysis of plurals and mass terms. In Bauerle, R., Schwarze, C., and von Stechow, A., editors, *Meaning, use and interpretation of language*, pages 302–323. De Gruyter.

Mayr, C. (2013). Implicatures of modified numerals. In Caponigro, Ivan & Cecchetto, C., editor, *From grammar to meaning: the spontaneous logicality of language*, pages 139–171. Cambridge University Press.

Mendia, J. A. (2015). Conveying ignorance: Ignorance inferences with superlative numeral modifiers. In Bellamy, K., Karvovskaya, E., Kohlberger, M., and Saad, G., editors, *Proceedings of ConSOLE XXIII*, pages 150–174.

Mendia, J. A. (2016a). Focusing on scales: Superlatives as scalar modifiers. In Hammerly, C. and Prickett, B., editors, *Proceedings of NELS 46*, volume 3, pages 11–24. Concordia University. Talk presented at NELS 46. Concordia University (Montreal).

Mendia, J. A. (2016b). Reasoning with Partial Orders. In Chereches, A., Ashton, N., Lutz, D., and Snider, T., editors, *Proceedings of SALT 26*, pages 489–505.

Mendia, J. A. (2018). Epistemic numbers. Proceedings of SALT 28.

Nakanishi, K. and Rullmann, H. (2009). Epistemic and concessive interpretations of at least.

Nouwen, R. (2010). Two kinds of modified numerals. *Semantics and pragmatics*, 3(3):1–41.

Nouwen, R. (2015). Modified numerals: the epistemic effect. In Alonso-Ovalle, L. and Menéndez-Benito, P., editors, *Epistemic Indefinites*. Oxford University Press.

Penka, D. (2015). *At most* at last. In Csipak, E. and Zeijlstra, H., editors, *Proceedings of Sinn und Bedeutung 19*, pages 463–480. Sinn und Bedeutung.

Roberts, C. (1996/2012). Information structure in discourse: Towards an integrated formal theory of pragmatics. *Semantics and Pragmatics*, 5(6):1–69.

Rooth, M. (1985). *Association with Focus*. PhD thesis, University of Massachusetts, Amherst.

Rooth, M. (1992). A theory of focus interpretation. *Natural Language Semantics*, 1(1):117–121.

Rooth, M. (1996). Focus. In Lappin, S., editor, *The Handbook of Contemporary Semantic Theory*, pages 271–297. Blackwell Publishers, Oxford.

Sauerland, U. (2004). Scalar implicatures in complex sentences. *Linguistics and Philosophy*, 27(3):367–391.

Schwarz, B. (2016a). Consistency preservation in quantity implicature: the case of *at least*. *Semantics & Pragmatics*, 9(1):1–47.

Schwarz, B. (2016b). *At least* and ignorance: a reply to Coppock & brochhagen (2013). *Semantics and Pragmatics*, 9(10).

Schwarz, B. and Shimoyama, J. (2011). Negative islands and obviation by *wa* in japanese degree questions. In Li, N. and Lutz, D., editors, *Proceedings of SALT 20*, pages 702–719.

Spychalska, M. (2015). At least not false, at most possible: between truth and assertibility of superlative quantifiers. *Synthese*, pages 1–32.

van Rooij, R. and Schulz, K. (2004). Exhaustive interpretation of complex sentences. *Journal of Logic, Language and Information*, 13:491–519.

von Fintel, K. (2000). Whatever. In Jackson, B. and Matthews, T., editors, *Proceedings of SALT X*, pages 27–39, Cornell University, Ithaca, NY. CLC Publications.

Westera, M. and Brasoveanu, A. (2014). Ignorance in context: The interaction of modified numerals and qud. In Snider, T., D'Antonio, S., and Weigand, M., editors, *Proceedings of SALT*, volume 24, pages 414–431.

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