Stability Analysis

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Consider a dynamical system of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, y)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = g(x, y)$$

Since the dynamics depend only on the variables x, y themselves and nothing else, we call such a system <u>autonomous</u>. To understand this system, we would like to know the behavior of trajectories of $(x(t), y(t))$ over time. We can gain such an understanding qualitatively by plotting the <u>nullclines</u> of the system in the $x - y$ plane, given by

$$f(x, y) = 0$$
$$g(x, y) = 0$$

We can then construct the so-called <u>phase plane</u> by sketching trajectories in the $x - y$ plane. For simple systems, we can directly calculate trajectories by picking an initial condition and solving the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{f(x, y)}{g(x, y)}$$

However, this is usually impossible to do analytically, so we instead turn to the nullclines to guide us via the following rules:

- Trajectories can only cross the $x-$nullcline $f(x, y) = 0$ vertically (i.e. with $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$)
- Trajectories can only cross the $y-$nullcline $g(x, y) = 0$ horizontally (i.e. with $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$)
- Regions enclosed by the nullclines have $\frac{\mathrm{d}x}{\mathrm{d}y}$ with constant sign
- Crossings of the two nullclines are fixed points (stable/unstable) of the system

This last point is of great importance, as often what we are most interested in is the long-run behavior of the system. Thus, we would like to be able to know the behavior of the system near each of the fixed points. We can do so via standard stability analysis. Consider a fixed point (x^*, y^*) of the above system given, found by solving the equation

$$f(x^*, y^*) = g(x^*, y^*) = 0$$

To understand the system's behavior near this point, we analyze the dynamics at a nearby point

$$(\tilde{x}(t), \tilde{y}(t)) = (x^* + \delta x(t), y^* + \delta y(t))$$

to examine where it ends up in the limit of $t \to \infty$. If $(\tilde{x}(t), \tilde{y}(t)) \to (x^*, y^*)$, i.e. $(\delta x(t), \delta y(t)) \to (0, 0)$, as $t \to \infty$ then we know the fixed point (x^*, y^*) is stable.

Assuming $(\delta x(t), \delta y(t))$ to be very small, we can safely approximate the dynamics at $(\tilde{x}, \tilde{y})$ to first order:

$$\frac{\mathrm{d}\tilde{x}}{\mathrm{d}t} = \frac{\mathrm{d}\delta x}{\mathrm{d}t} = f(x^* + \delta x, y^* + \delta y) \approx f(x^*, y^*) + f_x(x^*, y^*)\delta x + f_y(x^*, y^*)\delta y$$
$$\frac{\mathrm{d}\tilde{y}}{\mathrm{d}t} = \frac{\mathrm{d}\delta y}{\mathrm{d}t} = g(x^* + \delta x, y^* + \delta y) \approx g(x^*, y^*) + g_x(x^*, y^*)\delta x + g_y(x^*, y^*)\delta y$$

where I have used the notation $f_z(a, b) = \frac{\partial f}{\partial z}\Big|_{x=a, y=b}$. Since $f(x^*, y^*) = g(x^*, y^*) = 0$, we can rewrite this approximation in matrix notation as follows:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{J}\mathbf{x}$$

where

$$\mathbf{x} = \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}$$

and

$$\mathbf{J} = \begin{bmatrix} f_x(x^*, y^*) & f_y(x^*, y^*) \\ g_x(x^*, y^*) & g_y(x^*, y^*) \end{bmatrix}$$

is the Jacobian of the vector-valued function $\mathbf{f}(x, y) = \begin{bmatrix} f(x, y) & g(x, y) \end{bmatrix}^{\mathrm{T}}$, evaluated at (x^*, y^*).

We now have a linear dynamical system that we can actually solve. Note what we have done: by picking a point very near to the fixed point and approximating its dynamics to first order, we have effectively linearized the dynamics around this fixed point, giving us a linear system that we can analyze and solve. Specifically, the solution to this linear system is given by

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$$

where λ_1, λ_2 and $\mathbf{v}_1, \mathbf{v}_2$ are the eigenvalues and eigenvectors of the 2×2 Jacobian matrix $\mathbf{J}$. Thus, if $\mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2) < 0$, we know that $e^{\lambda_1 t}, e^{\lambda_2 t} \rightarrow 0$ and therefore $(\delta x, \delta y) \rightarrow 0$ as $t \rightarrow \infty$, so we can conclude (x^*, y^*) is a stable fixed point. Otherwise, (x^*, y^*) could be either unstable, a saddle node, or a limit cycle (see table below). We therefore need only calculate the eigenvalues of the Jacobian matrix $\mathbf{J}$ to determine qualitative behavior around the fixed point (x^*, y^*):

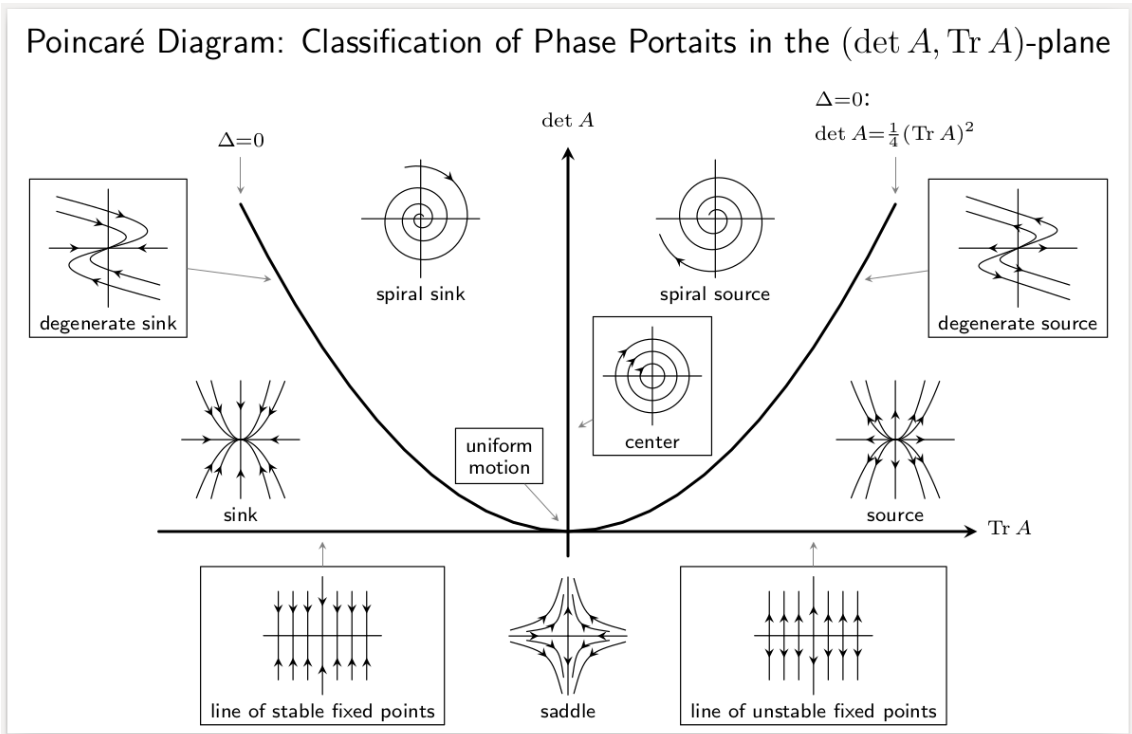
$$\begin{aligned} \mathbf{J}\mathbf{v} &= \lambda\mathbf{v} \\ \Leftrightarrow (\mathbf{J} - \lambda\mathbf{I})\mathbf{v} &= \mathbf{0} \\ \Rightarrow |\mathbf{J} - \lambda\mathbf{I}| &= \mathbf{0} \quad \text{for non-zero } \mathbf{v} \\ \Leftrightarrow \lambda^2 - \underbrace{\mathrm{Tr}[\mathbf{J}]}_{T} + \underbrace{|\mathbf{J}|}_{D} &= 0 \\ \Rightarrow \lambda_{\pm} &= \frac{T \pm \sqrt{T^2 - 4D}}{2} \end{aligned}$$

where the third line follows from the fact that, for there to be a non-zero vector $\mathbf{v}$ that satisfies the equation in the second line, the matrix $\mathbf{J} - \lambda\mathbf{I}$ must have a non-zero nullspace and therefore not be full-rank, which implies that its determinant $|\mathbf{J} - \lambda\mathbf{I}|$ must be 0. We can thus easily derive the following conditions for stability of the fixed point:

$$\begin{aligned} T &< 0 \\ D &> 0 \end{aligned}$$

The full picture is given by the table and figure below.

Fixed point	$\mathrm{Tr}[\mathbf{J}]$	$\mathrm{Det}[\mathbf{J}]$	Real part	Imaginary part
stable node	$T < 0$	$T^2 > 4D > 0$	$\mathrm{Re}(\lambda_\pm) < 0$	$\mathrm{Im}(\lambda_\pm) = 0$
stable spiral	$T < 0$	$4D > T^2 > 0$	$\mathrm{Re}(\lambda_\pm) < 0$	$\mathrm{Im}(\lambda_\pm) \neq 0$
unstable node	$T > 0$	$T^2 > 4D > 0$	$\mathrm{Re}(\lambda_\pm) > 0$	$\mathrm{Im}(\lambda_\pm) = 0$
unstable spiral	$T > 0$	$4D > T^2 > 0$	$\mathrm{Re}(\lambda_\pm) > 0$	$\mathrm{Im}(\lambda_\pm) \neq 0$
center (limit cycle??)	$T = 0$	$D > 0$	$\mathrm{Re}(\lambda_\pm) = 0$	$\mathrm{Im}(\lambda_\pm) \neq 0$
saddle	-	$D < 0$	$\mathrm{Re}(\lambda_+) > 0 > \mathrm{Re}(\lambda_-)$	$\mathrm{Im}(\lambda_\pm) = 0$
star/degenerate node	$T^2 = 4D$	$D \geq 0$	$\mathrm{Re}(\lambda_+) = \mathrm{Re}(\lambda_-)$	$\mathrm{Im}(\lambda_\pm) = 0$


Poincaré Diagram: Classification of Phase Portaits in the $(\det A, \mathrm{Tr}\, A)$-plane
$\det A$
$\Delta = 0$
$\Delta = 0$:
$\det A = \frac{1}{4}(\mathrm{Tr}\, A)^2$
spiral sink
spiral source
degenerate sink
degenerate source
center
uniform motion
sink
source
$\mathrm{Tr}\, A$
line of stable fixed points
saddle
line of unstable fixed points