Linear Algebra

based on the MIT 18.06 course.

AY25/26 Sem 2

This set of notes is based on the lectures as delivered in 2005 by Gilbert Strang and the MA1101R Linear Algebra I lecture notes by Ma Siu Lan and Victor Tan.

Contents

Matrices

A matrix is a rectangular array of numbers.

1.1 Linear Systems

Definition 1.1.1. A system of linear equations

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \\ &\vdots \\ a_nx + b_ny &= c_n \end{aligned} \tag{1}$$

can be represented by matrices in the form

$$AX = b$$

where A is the matrix of the coefficients, X is the matrix of the directional vectors, and b is a matrix of constants. Writing Eqs. (1) in this form we get

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}. \tag{2}$$

Definition 1.1.2. In the **column form** this is represented as

$$x \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + y \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \tag{3}$$

For AX, we say that AX is a combination of the columns of A.

Remarks. Any linear equation in the form $AX = b$ can be solved, given that the matrix A is a non-singular (or invertible) matrix.

Definition 1.1.3. The **linear combination** refers to the combination of x and y which solves Eqs. (3).

Definition 1.1.4. Gaussian elimination is an elimination algorithm. Multiples of the top row are subtracted from each lower row until the final matrix is that of an upper triangular matrix.

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 9 & 2 \\ 0 & 5 & 15 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 5 & 15 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix} \tag{4}$$

The equations are then solved through **back substitution**.

Definition 1.1.5. The **pivot** of the matrix above is the terms a_{11}, a_{22}, and a_{33} (with a_{11} being the first pivot and so on). Where the pivot $\neq 0$. The product of the pivots of a matrix gives its *determinant*.

Definition 1.1.6. **Elimination matrices** perform the elimination procedures in Gaussian Elimination. For example, for the processes in Eqs. 4

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 3 & 9 & 2 \\ 0 & 5 & 15 \end{bmatrix} &= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 5 & 15 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 5 & 15 \end{bmatrix} &= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix} \end{aligned} \tag{5}$$

Discussion. Eqs. (5) can be expressed as

$$E_{32}\left(E_{21}A\right) = U$$

which by the associative law is

$$\left(E_{32}E_{21}\right)A = U$$

where E_n is an elimination matrix.

1.2 Matrix Multiplication

It is given matrix A (of size $n \times m$) and matrix B (of size $m \times p$).

Definition 1.2.1. The size of the product matrix AB is $n \times m$.

Definition 1.2.2. For the matrix AB, the element at row j and column k is

$$\sum_{x=1}^{n} a_{jx}b_{xk}$$

1.3 Inverses

Definition 1.3.1. The **identity matrix** I of a matrix A is

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{6}$$

The size of I is the same as that of A.

Definition 1.3.2. If the **inverse** A^{-1} of the matrix A exists, then it is the matrix which fulfils

$$AA^{-1} = I \tag{7}$$

where I is the identity matrix.

Remarks. Such a matrix A is a invertible/non-singular matrix.

Definition 1.3.3. The inverse B^-1 **does not** exist when

$$\det B = 0, \text{ or}$$
$$B\mathbf{X} = 0 \text{ is solvable.}$$

Remarks. Such a matrix B is a singular matrix. Its inverse does not exist.

Exercise. Gauss-Jordan is an algorithm to solve for A^{-1} given A. Say we are given a matrix A, where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$$

To start, we combine A and I into a single augmented matrix

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]$$

Perform elimination methods such that the matrix to the left of the vertical division is that of I; A^{-1} would then be the matrix to the right of the vertical division.

$$\left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

Definition 1.3.4. The inverse of the product of two matrices AB is the matrix which fulfils

$$I = (AB)\left(B^{-1}A^{-1}\right) \tag{8}$$

Definition 1.3.5. The **transpose** A^T of a matrix A is the matrix where the rows and columns of A have been swapped.

$$A = \begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix} \rightarrow A^T = \begin{bmatrix} a & x \\ b & y \\ c & z \end{bmatrix}$$

We can write that for an element of the matrix A_{ij},

$$A_{ij} = A^T_{ji}$$

1.4 Elimination in the form A=LU

Discussion. Given that no row exchanges occured, Eqs. (5) was written in the form $EA = U$. It is preferable to write $EA = U$ in the form

$$A = LU$$

where $L = E^{-1}$.
In the matrix L, if no row exchanges occurred, the multipliers go directly into L.

1.4.1 With row exchanges

Definition 1.4.1. The permutation matrix P are identity matrices with reordered rows. P is used to execute row exchanges.

Remarks. For P, $P^{-1} = P^T$; for a $n \times n$ matrix, there are $n!$ permutations.

Discussion. Row exchanges become necessary when there is a 0 in the pivot spot. A row exchange with any row that has a non-zero digit in the pivot spot is necessary to perform elimination.

Definition 1.4.2. With row exchanges, $A = LU$ is written in the form

$$PA = LU$$

where P are permutation matrices.

Vector Spaces

2.1 Vector Spaces & Subspaces

Definition 2.1.1. The **vector space** is a space of vectors. In a *real* vector space, the basic operations

- $v + w$,
- cw, and
- any such linear combination (i.e. $cv + dw$).

remain closed (in the space).

Remarks. Real vector spaces must satisfy 8 conditions.

1. $x + y = y + x$,
2. $x + (y + z) = (x + y) + z$,
3. there is a "zero vector" such that $x + 0 = x$,
4. for each x there is a unique vector $-x$ such that $x + (-x) = 0$,
5. $1 \times x = x$,
6. $(c_1 c_2)\, x = c_1 (c_2 x)$,
7. $c\,(x + y) = cx + cy$, and
8. $(c_1 + c_2)\, x = c_1 x + c_2 x$.

These conditions are necessary to permit scalar multiplication and vector addition.

Exercise. Some examples of vector spaces are:

- $\mathbb{R}^2$ is the vector space of all 2-D real vectors.
- $\mathbb{R}^3$ is the vector space of all 3-D real vectors.
- In general, any space $\mathbb{R}^n$ is a vector space of all n-dimensional real vectors.

Definition 2.1.2. A **subspace** is a vector space within another vector space.

Exercise. Some examples of vector subspaces in the vector space $\mathbb{R}^{\nvDash}$

- all of $\mathbb{R}^{\nvDash}$,
- any line through $(0, 0)$ in $\mathbb{R}^{\nvDash}$, and
- the zero vector alone.

2.1.1 Columnspace & Nullspace

Definition 2.1.3. The **columnspace** $C(A)$ is the subspace of $\mathbb{R}^n$ which contains all the linear combinations of the columns of A.

Remarks. When the number of equations $m >$ the number of unknowns n, then $A\mathbf{x} = b$ does not always have a solution. $Ax = b$ only has a subspace when b is in the column space of A.

Definition 2.1.4. The **nullspace** $N(A)$ is the subspace of $\mathbb{R}^n$ which contains all the solutions to the $\mathbf{x}$ such that $A\mathbf{x} = 0$.