Calculus

based on the work of George B. Thomas Jr.

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This set of notes is adapted from *Thomas' Calculus* by George B. Thomas Jr.

Contents

Functions

1.1 Functions & Graphs

Definition 1.1.1. A **function** f form a domain set D to a range set Y is a rule that assigns a unique value $f(x)$ of D for each element in D.

Discussion. The domain D is the set of possible input values. The range Y is the set of possible output values. Each element in D has a **single**, **unique** value $f(x)$.The graph of f with domain D is the set

$$\{(x, f(x)|x \in D\}.$$

Axiom 1.1.2. A function f can only have one value $f(x)$ for each x in its domain. Any vertical line $x = a$ can only intersect f once.

Definition 1.1.3. A function f is a **piecewise function** if it has separate definitions for different parts of its domain.

$$f(x) = \begin{cases} f_1 & x \leq 0 \\ f_2 & x > 1 \end{cases}$$

Definition 1.1.4. A function $y = f(x)$ is an

$$\text{even function if } f(-x) = f(x),$$
$$\text{odd function if } f(-x) = -f(x),$$

for every x in the function's domain.

Definition 1.1.5. Two variables x, y are **proportional** if they are always a constant multiple of one another, i.e. $y = kx$. If $1 > k > -1$, x and y are said to be **inversely proportional**.

1.2 Composite Functions

Definition 1.2.1. The **composite function** is the function $f(g(x))$.

Remarks. x is the input for the function $g(x)$; $g(x)$ is then used as the input to f.

Definition 1.2.2. A graph is shifted vertically when we add k to f.

$$y = f(x) + k$$

Definition 1.2.3. A graph is shifted horizontally by h when we add h to x.

$$y = f(x + h)$$

1.3 Trigonometric Functions

Definition 1.3.1. The radian is defined as

$$s = r\theta, \tag{1}$$

where θ is in radians.

Definition 1.3.2. The degree is defined as

$$1^\circ = \frac{\pi}{180} rad.$$

Definition 1.3.3. By convention, the positive angle is measured counter-clockwise from the positive x-axis.

Definition 1.3.4. The basic trigonometric functions are

$$\begin{aligned} \sin\theta &= \frac{opp}{hyp} & \csc\theta &= \frac{hyp}{opp} = \frac{1}{\sin\theta} \\ \cos\theta &= \frac{adj}{hyp} & \sec\theta &= \frac{hyp}{adj} = \frac{1}{\cos\theta} \\ \tan\theta &= \frac{opp}{adj} = \frac{\sin\theta}{\cos\theta} & \cot\theta &= \frac{adj}{opp} = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} \end{aligned}$$

1.4 Trigonometric Identities

Definition 1.4.1. The **Pythagorean identity** is

$$\sin^2\theta + \cos^2\theta = 1 \tag{2}$$

Remarks. Dividing Eqs. (2) by $cos^2\theta$ and $\sin^2\theta$ gives

$$\begin{aligned} 1 + \tan^2\theta &= sec^2\theta \\ 1 + \cot^2\theta &= \csc^2\theta \end{aligned}$$

Definition 1.4.2. The **addition formulas** hold for al angles A and B.

$$\begin{aligned} \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\ \sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \end{aligned} \tag{3}$$

Remarks. All other (basic) trigonometric identities derive from Eqs. (2) and Eqs. (3).

Definition 1.4.3. The **double-angle formulas** derive from substituting θ for A and B in Eqs (3) and give

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ \sin 2\theta &= 2\sin\theta\cos\theta. \end{aligned} \tag{4}$$

Definition 1.4.4. The **half-angle formulas** derive by combining

$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1 \\ \cos^2\theta - \sin^2\theta &= cos2\theta \end{aligned}$$

which give

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \to \cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos 2\theta}{2}} \tag{5}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \to \sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos 2\theta}{2}} \tag{6}$$

Definition 1.4.5. The **Law of Cosines** (Cosine Law) is

$$c^2 = a^2 + b^2 - 2ab\cos\theta. \tag{7}$$

1.5 Exponential Functions

Definition 1.5.1. The exponential function of base a is

$$f(x) = a^x, a > 0.$$

Where $f(x) = a^x$, the following rules apply:

1. $a^x \times a^y = a^{x+y}$
2. $\frac{a^x}{a^y} = a^{x-y}$
3. $(a^x)^y = (a^y)^x = a^{xy}$
4. $a^x \times b^x = (ab)^x$
5. $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$

Definition 1.5.2. The **euler's number** e is defined as

$$\begin{aligned} e &= \lim_{x\to\infty}\left(\frac{n}{n+1}\right)^n \\ &= 1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!} + ... \end{aligned}$$

Definition 1.5.3. The **natural exponential function** refers to the function

$$f(x) = e^x$$

Definition 1.5.4. Given the function $y = y_0 e^{kx}$,

Exponential growth is when $k > 0$, and
Exponential decay is when $k < 0$.

1.6 Inverse Functions & Logarithms