Linear Algebra

for 93APM3 Applied Mathematics 3

AY25/26 Sem 2

This set of notes is based on *Linear Algebra with Applications* by Steven J. Leon.

Contents

Matrices

A matrix is a rectangular array of numbers.

1.1 Linear Systems

Definition 1.1.1. A system of linear equations (”the System”)

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \\ &\vdots \\ a_nx + b_ny &= c_n \end{aligned} \tag{1}$$

can be represented by matrices in the form

$$AX = b$$

where A is the matrix of the coefficients, X is the matrix of the directional vectors, and b is a matrix of constants. Writing Eqs. (1) in this form we get

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}. \tag{2}$$

Remarks. The matrix A is the *coefficient matrix* of the system.

Definition 1.1.2. In the **column form** this is represented as

$$x \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + y \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \tag{3}$$

For AX, we say that AX is a combination of the columns of A.

Remarks. Any linear equation in the form $AX = b$ can be solved, given that the matrix A is a non-singular (or invertible) matrix.

Definition 1.1.3. The **linear combination** refers to the combination of x and y which solves Eqs. (3).

Definition 1.1.4. The **augmented matrix** M_ag is the matrix in the form

$$\left[\begin{array}{c|c} A & b \end{array}\right]$$

The system of linear equations can be solved by performing *row operations* on the augmented matrix.

Definition 1.1.5. The **elementary row operations** are

1. interchange two rows,
2. multiply a row by a non-zero real number, and
3. replace a row by its sum with a multiple of another row.

Elementary row operations are performed on the matrix until the matrix is reduced to the *row echelon form*.

Definition 1.1.6. A matrix A is in **row echelon form** if

1. the first non-zero entry in each non-zero row is 1,
2. row k does not consist entirely of zeros, then the number of leading zeros in row $k+1$ is greater than that in row k, and
3. rows whose entries are all zero are below the rows having non-zero entries.

For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Definition 1.1.7. Gaussian elimination is the process of utilising the elementary row operations to bring a matrix into its row echelon form.

Definition 1.1.8. A is said to be in **reduced row echelon form** if

1. A is in row echelon form, and
2. the first non-zero entry in each row is the only non-zero entry in its column.

For example,

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Definition 1.1.9. Gauss-Jordan reduction is the process of utilising elementary row operations to bring a matrix to its reduced row echelon form.

Definition 1.1.10. Elementary matrices E perform the elimination procedures in Gaussian Elimination. For example,

$$\begin{gathered} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 3 & 9 & 2 \\ 0 & 5 & 15 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 5 & 15 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 5 & 15 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix} \end{gathered} \tag{4}$$

This can be expressed as

$$B = E_2 E_1 A$$

Definition 1.1.11. The matrix B is the **row equivalent** of A when there exists a finite sequence of elementary matrices $E_n, E_{n-1}, \cdots, E_1$ such that

$$B = E_n E_{n-1} \cdots E_1 A$$

1.2 Matrix Multiplication

It is given matrix A (of size $n \times m$) and matrix B (of size $m \times p$).

Definition 1.2.1. The size of the product matrix AB is $n \times m$.

Definition 1.2.2. For the matrix AB, the element at row j and column k is

$$\sum_{x=1}^{n} a_{jx} b_{xk}$$

1.3 Inverses

Definition 1.3.1. The **identity matrix** I of a matrix A is

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{5}$$

The size of I is the same as that of A.

Definition 1.3.2. If the **inverse** A^{-1} of the matrix A exists, then it is the matrix which fulfils

$$AA^{-1} = I \tag{6}$$

where I is the identity matrix.

Remarks. Such a matrix A is a invertible/non-singular matrix.

Definition 1.3.3. The inverse B^-1 **does not** exist when

$$\det B = 0, \text{ or}$$
$$B\mathbf{X} = 0 \text{ is solvable.}$$

Remarks. Such a matrix B is a singular matrix. Its inverse does not exist.

Exercise. We use **Gauss-Jordan elimination** to solve for A^{-1} given A. Say we are given a matrix A, where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$$

To start, we combine A and I into a single augmented matrix

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]$$

Perform elimination methods such that the matrix to the left of the vertical division is that of I; A^{-1} would then be the matrix to the right of the vertical division.

$$\left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

Definition 1.3.4. The inverse of the product of two matrices AB is the matrix which fulfils

$$I = (AB)\left(B^{-1}A^{-1}\right) \tag{7}$$

Definition 1.3.5. The **transpose** A^T of a matrix A is the matrix where the rows and columns of A have been swapped.

$$A = \begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix} \rightarrow A^T = \begin{bmatrix} a & x \\ b & y \\ c & z \end{bmatrix}$$

We can write that for an element of the matrix A_{ij},

$$A_{ij} = A^T_{ji}$$

1.4 Elimination in the form A=LU

Discussion. Given that no row exchanges occured, Eqs. (4) was written in the form $EA = U$. It is preferable to write $EA = U$ in the form

$$A = LU$$

where $L = E^{-1}$.
In the matrix L, if no row exchanges occurred, the multipliers go directly into L.

1.4.1 With row exchanges

Definition 1.4.1. The permutation matrix P are identity matrices with reordered rows. P is used to execute row exchanges.

Remarks. For P, $P^{-1} = P^T$; for a $n \times n$ matrix, there are $n!$ permutations.

Discussion. Row exchanges become necessary when there is a 0 in the pivot spot. A row exchange with any row that has a non-zero digit in the pivot spot is necessary to perform elimination.

Definition 1.4.2. With row exchanges, $A = LU$ is written in the form

$$PA = LU$$

where P are permutation matrices.

Vector Spaces

2.1 Vector Spaces

Definition 2.1.1. The **vector space** is a space of vectors. In a *real* vector space, the basic operations

- $v + w$,
- cw, and
- any such linear combination (i.e. $cv + dw$) of these operations.

remain closed (in the space).

Remarks. Real vector spaces must satisfy 8 conditions.

1. $x + y = y + x$,
2. $x + (y + z) = (x + y) + z$,
3. there is a "zero vector" such that $x + 0 = x$,
4. for each x there is a unique vector $-x$ such that $x + (-x) = 0$,
5. $1 \times x = x$,
6. $(c_1 c_2)\, x = c_1\, (c_2 x)$,
7. $c\,(x + y) = cx + cy$, and
8. $(c_1 + c_2)\, x = c_1 x + c_2 x$.

These conditions are necessary to permit scalar multiplication and vector addition.

Exercise. Some examples of vector spaces are:

- $\mathbb{R}^2$ is the vector space of all 2-D real vectors.
- $\mathbb{R}^3$ is the vector space of all 3-D real vectors.
- In general, any space $\mathbb{R}^n$ is a vector space of all n-dimensional real vectors.

2.2 Subspaces

Definition 2.2.1. A **subspace** S is a vector space within another vector space. S must be a non-empty subset of V, and must satisfy:

1. $\alpha\mathbf{x} \in S$ for $\mathbf{x} \in S$ for any scalar α, and
2. $\mathbf{x} + \mathbf{y} \in S$ whenever $\mathbf{x}, \mathbf{y} \in S$.

Exercise. Some examples of vector subspaces in the vector space $\mathbb{R}^{\nvDash}$

- all of $\mathbb{R}^{\nvDash}$,
- any line through $(0, 0)$ in $\mathbb{R}^{\nvDash}$, and
- the zero vector alone.

Definition 2.2.2. The **Nullspace** $N(A)$ of the space A is the set of all solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

$$N(A) = \{x \in A | A\mathbf{x} = \mathbf{0}\}$$

Definition 2.2.3. The **linear combination** of the vectors $v_1, v_2, \cdots, v_n$, which are vectors in a vector space V is the sum of these vectors in the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_n v_n \tag{8}$$

Definition 2.2.4. The **span** is the set of all linear combinations of $v_1, v_2, \cdots, v_n$ and is denoted by $\text{Span}\{v_1, \cdots, v_n\}$.

The set $\{v_1, \cdots v_n\}$ is a **spanning set** for V is and only if every vector in V can be written as a linear combination of $v_1, v_2, \cdots v_n$.

2.3 Linear Independence

Definition 2.3.1. The vectors $v_1, v_2, \cdots, v_n$ in a vector space V are **linearly independent** if

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

implies that $c_1, c_2, \cdots, c_n$ must be 0.

Definition 2.3.2. The vectors $v_1, v_2, \cdots, v_n$ in a vector space V are **linearly dependent** if there exists scalars such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

2.4 Basis & Dimension

Definition 2.4.1. The **basis** of a vector is formed by the set of vectors $\{v_1 \cdots v_n\}$ if and only if

1. $v_1, \cdots, v_n$ are linearly independent, and
2. $v_1, \cdots, v_n$ set spans V.

A vector space can have one or more vector bases.

Definition 2.4.2. If the basis of V has n elements, then V is said to be of n-dimensions. If there exists a finite set of vectors that span V, then V is said to be finite dimensional; V is otherwise infinitely dimensional.

2.5 Row and Column Space

Definition 2.5.1. The **row space** $R(A)$ is the subspace of $\mathbb{R}^n$ which contains all the linear combinations of the rows of the matrix A.

$$\text{Row}(A) = R(A) = \text{Lin}\{r_1, r_2, \cdots, r_n\}$$

where $r_1, r_2, \cdots, r_n$ are the rows of A.

Definition 2.5.2. The **rank** of A

$$\text{rank}(A)$$

is the dimension of the row space of A.

Definition 2.5.3. The **column space** $C(A)$ is the subspace of $\mathbb{R}^n$ which contains all the linear combinations of the columns of the matrix A.

$$\text{Col}(A) = C(A) = \text{Lin}\{c_1, c_2, \cdots, c_n\}$$

where $c_1, c_2, \cdots, c_n$ are the columns of A.

2.6 Eigenvalues and Eigenvectors

Given that A is a square matrix of size $n \times n$. If there exists a **non-zero vector** x such that

$$Ax = \lambda x,$$

then

Definition 2.6.1. the scalar λ is a **eigenvalue** or **characteristic value** of A, and

Definition 2.6.2. the vector x is an **eigenvector** or **characteristic vector** of λ.

Discussion. The eigenvalues λ and eigenvalues x of A can be determined by

1. solving the characteristic equation $\det(A - \lambda I) = 0$ which determines the eigenvalues λ, and then
2. for each of the eigenvalues λ, solve the homogenous system $(A - \lambda I)v = 0$ which determines the corresponding eigenvector(s).

Theorem 2.6.3. The product of the eigenvalues λ_n of A gives the determinant of A.

$$\Pi_{i=1}^{n}(\lambda_n) = \det(A)$$

Theorem 2.6.4. For the eigenvectors λ_n and λ_m to be linearly independent, they must correspond to different eigenvalues x_n and x_m.

2.6.1 Similar Matrices

Definition 2.6.5. The matrices A and B are said to be **similar matrices** if there exists a non-singular matrix S such that

$$B = S^{-1}AS$$

Theorem 2.6.6. If two matrices A and B are similar, then A and B have the same characteristic equation and hence the same eigenvalues.

2.7 Diagonalisation

Definition 2.7.1. The $n \times n$ matrix A is said to be **diagonalisable** if there exists a non-singular matrix X and diagonal matrix D such that

$$X^{-}1AX = D$$

then X diagonalises A.

Remarks. For A to be diagonalisable, A must have n linearly independent eigenvectors, which must correspond to distinct eigenvalues. If A is diagonalisable, then

1. the column vectors of X correspond to the eigenvectors x of A,

$$X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

2. the diagonal elements of D correspond to the eigenvalues of A.

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

The square matrix is diagonalisable if it has n linearly independent eigenvalues. It does not need to have n distinct eigenvectors.

Exercise. Powers of matrices
If A is diagonalizable, then A can be factored into the form XDX^{-1}. Therefore,

$$A^k = XD^kX^{-1} = X \begin{bmatrix} (\lambda_1)^k & & & \\ & (\lambda_2)^k & & \\ & & \ddots & \\ & & & (\lambda_n)^k \end{bmatrix} X^{-1}$$

2.8 Systems of Linear Differential Equations

Definition 2.8.1. A system of differential equations (of the first degree)

$$\begin{aligned} {y'}_1 &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\ {y'}_2 &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \\ &\vdots \\ {y'}_n &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n \end{aligned} \tag{9}$$

and can be represented in the matrix form

$$\mathbf{y}' = A\mathbf{y} \tag{10}$$

Remarks. Any equation in the form

$$\frac{dy}{dx} = kx$$

can be solved with the general solution

$$x = Ae^{kt}, A \in \mathbb{R}$$

Thus, for the case $n > 1$,

$$\mathbf{Y} = \begin{bmatrix} x_1 e^{\lambda t} \\ x_2 e^{\lambda t} \\ \vdots \\ x_n e^{\lambda t} \end{bmatrix} = e^{\lambda t}\mathbf{x}$$

Definition 2.8.2. The **initial value problem** is any problem of the form

$$\begin{aligned} \mathbf{Y}' &= A\mathbf{Y} \\ \mathbf{Y}(0) &= \mathbf{Y}_0 \end{aligned} .$$

Any such problem in the form will have a unique solution.