Further Complex Methods

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1 Preliminaries

A complex number $z = x + iy$ is a pair (x, y) with complex conjugate $\bar{z} = x - iy$. If x, y are co-ordinates in $\mathbb{R}^2$ then any function can be written in terms of z and $\bar{z}$ using,

$$x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2i}$$

Functions of a complex variable

- A function of a complex variable is defined to be those functions that can be written entirely in terms of z, not $\bar{z}$. A function of a complex variable is **continuous** if,

$$\lim_{z \to z_0} f(z) = f(z_0)$$

- The **derivative** of a function of a complex variable is defined as,

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}$$

  For a function to be differentiable, the limit must be independent of the direction in which the limit is taken. If this is true, then this function is said to e differentiable at z. If $f'(z)$ exists, then $f(z)$ is continuous, but the converse does not necessarily hold.

The Cauchy-Riemann Equations

Suppose we write $f(z) = u(x, y) + iv(x, y)$, then consider,

$$\delta z f'(z) = \lim_{\delta z = \delta x + i\delta y} u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y) - u(x, y) - v(x, y)$$

Taking $\delta x, \delta y = 0$ in turn we find that,

$$\delta z f'(z) = u_x \delta x + iv_x \delta x = u_y \delta y + iv_y \delta y$$

These expressions must be the same, since they are the derivative taken in different directions. This gives, (using $\delta z = \delta x + i\delta y$),

$$f'(z) = u_x + iv_x = -iu_y + v_y$$

Which, comparing real and imaginary parts, gives us the **Cauchy-Riemann equations**:

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}} \tag{1.1}$$

If (1.1) holds (implicitly assuming the partial derivatives exist), then $f(z)$ is differentiable, and vice-versa.

Furthermore, if the Cauchy-Riemann equations hold, then both $u(x, y)$ and $v(x, y)$ are **harmonic**. Also, lines of constant u and lines of constant v are orthogonal by the following,

$$\nabla u = (u_x, u_y), \quad \nabla v = (v_x, v_y) \Rightarrow \nabla u \cdot \nabla v = u_x v_x + u_y v_y = 0$$

by (1.1)

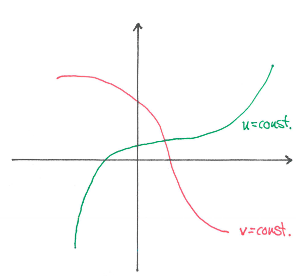


Figure 1.1
Lines of constant u and v

Analytic Functions

$f(z)$ is analytic at z_0 if $f(z)$ is differentiable in the neighbourhood of z_0. Similarly for some region $U \subset \mathbb{C}$.

1. e^z is analytic in the finite complex z-plane
2. $\bar{z}$ is analytic nowhere
3. $\frac{1}{z^3}$ is analytic everywhere except $z = 0$

An **entire** function is one that is analytic everywhere in the *finite* complex plane.

Singularities

1. **Isolated Singularity** A function is said to be an isolated singularity if it fails to be analytic at a point. For example, $\frac{1}{z^2}$ has an isolated singularity at $z = 0$. Suppose a function has an isolated singularity at $z = z_0$ then it can be expanded as a **Laurent series** around z_0,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

Suppose that $c_n = 0$ for all $n < -N$ where $N > 0$. Then the function has a **pole** of order N. For example, $\frac{1}{z^3}$ has a pole of order 3 at $z = 0$. The co-efficient c_{-1} is special, it is the **residue** of the pole at z_0.

2. **Removable Singularity** Here, we may redefine the function to be equal to the limit at the singularity, rendering $f(z)$ continuous. The c_n will be zero for all $n < 0$. For example, $\frac{\sin(z)}{z}$ has a removable singularity at $z = 0$.

3. **Essential Singularity** An essential singularity is where the order of the pole of an isolated singularity is infinite. For example, $e^{\frac{1}{z}}$.

Meromorphic Functions

These are functions of z that only have poles of finite order in the finite complex plane.

Cauchy's Theorem

$$\int_{\mathcal{C}} f(z)\,\mathrm{d}z = 2\pi i \sum_i \mathrm{res}(z_i) \tag{1.2}$$

The integral here is taken around $\mathcal{C}$ in the anticlockwise direction, and $f(z)$ is meromorphic.

Riemann Sphere

We now briefly remind ourselves of the construction of the complex plane as a sphere. We may perform a stereographic projection onto the plane such that the north pole corresponds with the point at ∞. Then, the mapping $z \to w = \frac{1}{z}$ maps the point at infinity to zero, and allows us to evaluate properties of the function such as singularities and residues on the extended complex plane, $\mathbb{C} \cup \{\infty\}$

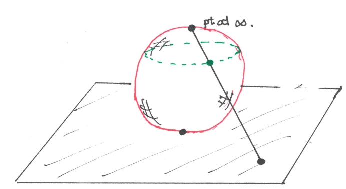


Figure 1.2
The Riemann Sphere

Multivalued Functions

Consider the square root of a complex variable $z = re^{i\theta}$, then,

$$z^{\frac{1}{2}} = r^{\frac{1}{2}} e^{\frac{i\theta}{2}}, \quad r \geq 0$$

As one goes around the circle, $\theta \to \theta + 2\pi$, and $z^{\frac{1}{2}} \to r^{\frac{1}{2}} e^{\frac{i\theta}{2}} e^{i\pi} = -r^{\frac{1}{2}} e^{\frac{i\theta}{2}}$. So we see that $f(z)$ changes sign as we go round the circle once. If we go round twice, it is invariant.

The effect of going round the circle is usually called **monodromy**. For the case $f(z) = z^{\frac{1}{2}}$ it is $(-1)^n$. The monodromy always forms a group, so in this case it is simply $\mathbb{Z}_2$. A point where the monodromy is not unity is called a **branch point**. For $z^{\frac{1}{2}}$, the origin is a branch point. By making the change of co-ordinates $z \to \frac{1}{z}$, we find that ∞ is also a branch point by the same method (consider a small circle at the north pole of the Riemann sphere). There is always more than one branch point, so remember to look at infinity.

Examples

1. For $f(z) = (z - z_0)^p$, if p is an integer, then f is single-valued. If $p = \frac{m}{n}$ where m and n are integers, then,

$$\begin{aligned} z &= z_0 + re^{i\theta} \\ (z - z_0)^p &= r^p e^{ip\theta} \\ &= r^{\frac{m}{n}} e^{\frac{im\theta}{n}} \end{aligned}$$

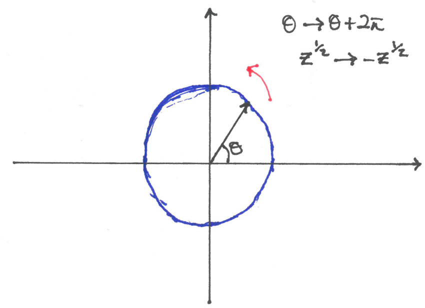


Figure 1.3
We consider z round a circle enclosing the origin

Now take $\theta \to \theta + 2\pi$, then $(z - z_0)^p \to r^{i\frac{m}{n}} e^{\frac{2\pi i m}{n}}$. So there is a change in the phase of the function by $e^{\frac{2\pi i m}{n}}$. Suppose one goes round $z = z_0$ s times, then the phase factor is $e^{\frac{2\pi i s m}{n}}$. So if $s = n$ then we get back to the original value, or indeed any multiple of n times. The monodromy is therefore $\{1, e^{\frac{2\pi i m}{n}}, e^{\frac{4\pi i m}{n}}, \ldots\}$. We see that the monodromy is the cyclic group of order n. Suppose that p is not rational, then one never gets back to the starting point. The monodromy for a single circle around z_0 is $e^{2i\pi p}$. The monodromy group is $\mathbb{Z}_\infty$.

2. Now consider $f(z) = \log(z)$, so $f(z) = \log(r) + i\theta$. So going around the circle at the origin once gives us an extra imaginary part, $2\pi i$. If one goes round the circle n times, $f(z) \to f(z) + 2\pi n i$. So there are an infinite number of possible values for $f(z)$. The monodromy is addition of $2\pi i n$.

   - $z = 0$ is a branch point, the monodromy group is the integers under addition.
   - $z = \infty$ is also a branch point since $\log(\frac{1}{z}) = -\log(z)$.

Back to branch cuts

A branch cut is a method of making $f(z)$ single-valued in the complex plane. Returning to $f(z) = z^{\frac{1}{2}}$, then f is single-valued if θ lies in a range of length 2π. Now restrict θ to run from 0 to 2π. The function is discontinuous across the positive real axis.

Failure of Analyticity

To get around this, exclude the positive real axis from the definition of the function. If z is real $z^{\frac{1}{2}}$ can still be defined either by,

1. Taking the limit from the top half-plane
2. Taking the limit from the bottom half-plane

There is, however, always a discontinuity across branch cuts. The branch cut extends all the way out to infinity since the discontinuity between $\theta = 0$ and $\theta = 2\pi$ is non-vanishing for all r.

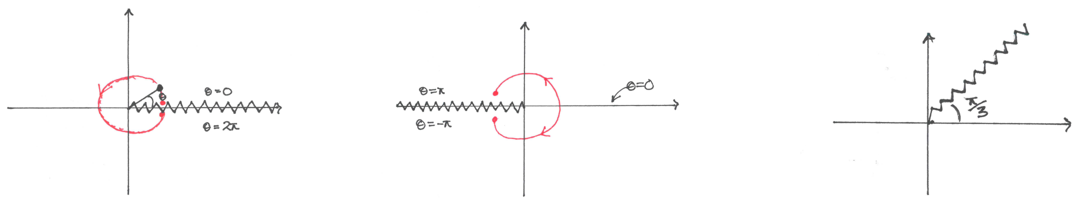
For a square root type branch cut, the discontinuity is always just a sign corresponding to the nature of the monodromy. This is not the only way to arrange a branch cut however. Another possibility is for θ to run from $-\pi$ to π. Other possibilities are also shown.

A more complicated example

Consider $f(z) = (z-1)^{\frac{1}{4}}$. There is a branch point at $z = 1$, then let $z = 1 + re^{i\theta}$, there are 4 possible values for $f(z)$. If we go around a circle enclosing $z = 1$ once, $(z-1)^{\frac{1}{4}} \to e^{\frac{i\pi}{2}}(z-1)^{\frac{1}{4}}$. Finally, letting $z \to \frac{1}{z}$ we are left with,

$$\left(\frac{1}{z} - 1\right)^{\frac{1}{4}} = w^{-\frac{1}{4}}(1-w)^{\frac{1}{4}}$$

So there is a branch point at $z = \infty$ also. We now look to make the function single valued by introducing a branch cut. We might,



(a) Along the positive real axis (b) Along the negative real axis (c) Another option

Figure 1.4: Various different options for the branch cut

- Let θ run from 0 to 2π
- Let θ run from $-\pi$ to π
- Any other way of diving up the complex z-plane to make $f(z)$ single-valued will do.

A contour integral

Consider,

$$I = \int_0^\infty \frac{x^{\frac{1}{2}}}{1+x^2}\,\mathrm{d}x$$

where x is a real variable in this case. We consider the more general complex problem,

$$\int_0^\infty \frac{z^{\frac{1}{2}}}{1+z^2}\,\mathrm{d}z$$

The integrand has branch points at 0 and ∞, with simple poles at $z = \pm i$. We restrict the argument to run between 0 and 2π.

1. Along $\mathcal{C}_1$, $z = x$ so one just gets,
$$\int_{\mathcal{C}_1} \frac{z^{\frac{1}{2}}}{1+z^2}\,\mathrm{d}z = I$$
2. Along $\mathcal{C}_2$, $z = Re^{i\theta}$ as $R \to \infty$. Then $\mathrm{d}z = iRe^{i\theta}\,\mathrm{d}\theta$, and,
$$\int_{\mathcal{C}_2} f(z)\,\mathrm{d}z = \int_0^{2\pi} iRe^{i\theta}\frac{R^{\frac{1}{2}}e^{\frac{i\theta}{2}}}{1+R^2e^{2i\theta}}\,\mathrm{d}\theta$$
3. Along $\mathcal{C}_3$, $z = 2i\pi$, then
$$\int_{\mathcal{C}_3} f(z)\,\mathrm{d}z = \int_\infty^0 \frac{x^{\frac{1}{2}}e^{i\pi}}{1+x^2e^{4i\pi}}\,\mathrm{d}x = I$$

(a) A branch cut at $z = 1$ (b) Another option

Figure 1.5: Another illustration of the options for the branch cuts

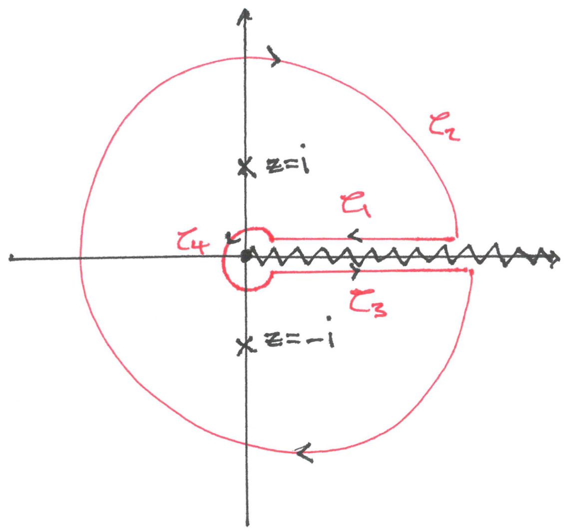


Figure 1.6
Contour, $\mathcal{C}$, with a branch cut along positive real axis

4. Finally on $\mathcal{C}_4$, $z = \epsilon e^{i\theta}$, $\mathrm{d}z = i\epsilon e^{i\theta}\,\mathrm{d}\theta$, then,

$$\int_{\mathcal{C}_4} f(z)\,\mathrm{d}z = \int_{2\pi}^{0} i\epsilon e^{i\theta} \frac{\epsilon^{\frac{1}{2}} e^{\frac{i\theta}{2}}}{1+\epsilon^2 e^{2i\theta}}\,\mathrm{d}\theta$$

But this is $\mathcal{O}(\epsilon^{\frac{3}{2}})$ so vanishes in the limit $\epsilon \to 0$.

Now what are the residues at $z = \pm i$? We first note we must use $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, i.e. not $\theta = -\frac{\pi}{2}$. Then we find that residues are $\frac{1}{2}e^{-\frac{i\pi}{4}}, \frac{1}{2}e^{\frac{3i\pi}{4}}$, which says that,

$$\int_0^\infty \frac{z^{\frac{1}{2}}}{1+z^2}\,\mathrm{d}z = 2\pi i(-1)\sin\left(\frac{\pi}{4}\right)$$

Which gives us finally that

$$I = \frac{\pi}{\sqrt{2}}$$

.

Another Branch Cut

Consider $f(z) = \sqrt{(z-a)(z-b)}$ where a is real and positive, b is real and negative. There are branch points at $z = a, b$. We ask whether there is a branch point at ∞? Let $w = \frac{1}{z}$, then,

(a) Branch cut for $0 \leq \theta_1 < 2\pi, \quad 0 \leq \theta_2 < 2\pi$

(b) Branch cut for $-\pi \leq \theta_1 < \pi, \quad 0 \leq \theta_2 < 2\pi$

Figure 1.7: Different choices of argument for the branch cut

$$\begin{aligned} f(w) &= \sqrt{\left(\frac{1}{w} - a\right)\left(\frac{1}{w} - b\right)} \\ &= \frac{1}{w}\sqrt{(1-aw)(1-bw)} \\ &= \frac{1}{w}\left(1 - \frac{aw}{2} - \frac{bw}{2} + \mathcal{O}(w^2)\right) \end{aligned}$$

Which is fine apart from a pole at ∞, so $w = 0$ is not a branch point, so $z = \infty$ is **not** a branch point. We now write $f(z) = \sqrt{r_1 r_2} e^{\frac{i(\theta_1+\theta_2)}{2}}$ where,

$$0 \leq \theta_1 < 2\pi, \quad 0 \leq \theta_2 < 2\pi$$

The effect is to introduce a branch cut along the real axis between a and b. Now consider instead,

$$-\pi \leq \theta_1 < \pi, \quad 0 \leq \theta_2 < 2\pi$$

This cut appears to end at $+\infty$ and $-\infty$ which is not a branch point. If we think in terms of the Riemann sphere however, we see that this is resolved by the branch cut simply going through the single north pole.

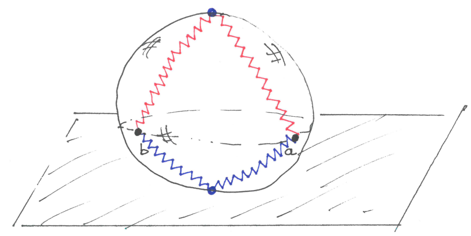


Figure 1.8
We understand the branch cut using the Riemann Sphere

2 Cauchy Principal Value of an Integral

Sometimes it is possible to construct the Cauchy Principal Value for an integral, defined to be,

$$\boxed{\mathcal{P}\int_a^b f(x)\,\mathrm{d}x = \lim_{\epsilon\to 0}\left\{\int_a^{x_0+\epsilon} f(x)\,\mathrm{d}x + \int_{x_0+\epsilon}^b f(x)\,\mathrm{d}x\right\}} \tag{2.1}$$

where $\mathcal{P}$ denotes the principal value. Now suppose that $f(x)$ has a single singularity at x_0. Then if the limit as $\epsilon \to 0$ exists, it defines the principal value. If the integral were convergent, *then the principal value coincides with the original value.*

Example.

$$\begin{aligned}\mathcal{P}\int_{-1}^2 \frac{1}{x}\,\mathrm{d}x &= \lim_{\epsilon\to 0}\left\{\int_{-1}^{-\epsilon}\frac{1}{x}\,\mathrm{d}x + \int_{\epsilon}^2 \frac{1}{x}\,\mathrm{d}x\right\}\\ &= \lim_{\epsilon\to 0}\left\{\log|x|\Big|_{-1}^{\epsilon} + \log|x|\Big|_{\epsilon}^{2}\right\}\\ &= \log 2\end{aligned}$$

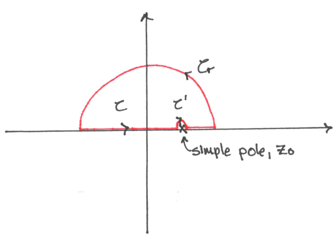


Figure 2.1
Principal Value contour

In the complex plane, this turn out to be rather convenient. Suppose one integrates a function that has a pole on the real axis. The principal value corresponds to the contour shown in Figure 2.1. For example, this contour might be closed in the top half plane. We apply a small modification to Cauchy's Theorem.

$$2\pi i\sum_i \operatorname{res} z_i = \int_{\mathcal{C}_r} f(z)\,\mathrm{d}z + \int_{\mathcal{C}} f(z)\,\mathrm{d}z + \int_{\mathcal{C}'} f(z)\,\mathrm{d}z \tag{2.2}$$

The second term represents the Principal Value. On $\mathcal{C}'$, $z = z_0 + \epsilon e^{i\theta}$ where θ runs from π to 0. Near $z = z_0$,

$$f(z) = \sum_{n=-1}^{\infty} c_n (z-z_0)^n$$

So f has a residue c_{-1} at $z = z_0$. So,

$$\begin{aligned}\int_{\mathcal{C}'} f(z)\,\mathrm{d}z &= \int_{\theta=\pi}^{\theta=0}\sum_{n=-1}^{\infty} c_n(z-z_0)^n\,\mathrm{d}z\\ &= \sum_{n=-1}^{\infty}\int_{\theta=\pi}^{\theta=0} c_n\epsilon^n e^{in\theta} i\epsilon e^{i\theta}\,\mathrm{d}\theta\\ &= \sum_{n=-1}^{\infty}\int_{\theta=\pi}^{\theta=0} ic_n\epsilon^{n+1}e^{i(n+1)\theta}\,\mathrm{d}\theta\end{aligned}$$

In the limit as $\epsilon \to 0$, only the $n = -1$ term contributes (which makes us note that higher order terms may cause issues). Then,

$$\int_{\mathcal{C}'} f(z)\,\mathrm{d}z = -i\pi c_{-1}$$

Example. *We calculate the principal value,*

$$\mathcal{P}\int_{-\infty}^{\infty}\frac{e^{iax}}{x}\,\mathrm{d}x$$

where a is real and positive, using a similar contour to the above, and applying our modified version of Cauchy's Theorem in (2.2). Then,

$$2\pi i\sum_i \operatorname{res} z_i = \int_{\mathcal{C}_r} f(z)\,\mathrm{d}z + \mathcal{P}\int_{-\infty}^{\infty}\frac{e^{iax}}{x}\,\mathrm{d}x - i\pi c_{-1}$$

This gives us,

$$0 = 0 + \mathcal{P}\int_{-\infty}^{\infty} \frac{e^{iax}}{x}\,\mathrm{d}x - i\pi$$

So we find that,

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{e^{iax}}{x}\,\mathrm{d}x = i\pi$$

We may take the imaginary part of the expression above to find that,

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{\sin(ax)}{x}\,\mathrm{d}x = \pi, \quad a > 0$$

But since the principal part of a convergent integral is the same as the integral, one finds that,

$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x}\,\mathrm{d}x = \pi$$

independent of the constant a.

Example. *Consider the integral,*

$$I = \int_{-\infty}^{\infty} \frac{e^{px} - e^{qx}}{1 - e^{x}}\,\mathrm{d}x, \quad 0 < p, q < 1$$

where the conditions of p, q ensure convergence. Now, for $z = 2i\pi n$ the denominator vanishes, however we show that these are removable singularities in the limit $x \to 0$,

$$\begin{aligned}\frac{e^{px} - e^{qx}}{1 - e^{x}} &= \frac{1 + px + \frac{(px)^2}{2} + \ldots - 1 - qx - \frac{(qx)^2}{2} - \ldots}{1 - 1 - x - \frac{x^2}{2} - \ldots} \\ &= \frac{p + \frac{p^2x}{2} + \ldots - q - \frac{q^2x}{2} - \ldots}{-(1 + \frac{x}{2} + \ldots)} \to -(p - q)\end{aligned}$$

Hence this integrand is the same as its principal value,

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{e^{px} - e^{qx}}{1 - e^{x}}\,\mathrm{d}x$$

We choose a rectangular contour, then applying the modified version of Cauchy's theorem, noting that the contour

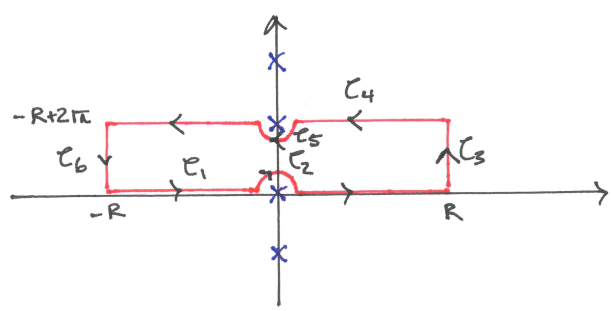


Figure 2.2
Rectangular contour

encloses no poles, we see that $\mathcal{C}_3$ and $\mathcal{C}_6$ gives no contribution as $R \to \infty$ since $0 < p, q < 1$. So,

$$0 = \mathcal{P}\int_{-R}^{R} \frac{e^{pz} - e^{qz}}{1 - e^{z}}\,\mathrm{d}z + \mathcal{P}\int_{-R+2i\pi}^{R+2i\pi} \frac{e^{pz} - e^{qz}}{1 - e^{z}}\,\mathrm{d}z - i\pi\,\mathrm{res}(0) - i\pi\,\mathrm{res}(2i\pi)$$

We now calculate the residues:

- $\text{res}_{z=0}\left(\frac{e^{pz}}{1-e^z}\right) = \lim_{z\to 0}\frac{ze^{pz}}{1-e^z} = -1$
- $\text{res}_{z=2i\pi}\left(\frac{e^{pz}}{1-e^z}\right) = \lim_{z\to 2i\pi}\frac{(z-2i\pi)e^{pz}}{1-e^z} = -e^{2i\pi(p-1)}$

Then, we consider simply the principal value:

$$\mathcal{P}\int_{-\infty}^{\infty}\frac{e^{pz}}{1-e^z}\,\mathrm{d}z$$

which satisfies,

$$0 = \mathcal{P}\int_{-\infty}^{\infty}\frac{e^{pz}}{1-e^z}\,\mathrm{d}z + \mathcal{P}\int_{-\infty}^{\infty}\frac{e^{p(z+2i\pi)}}{1-e^z}\,\mathrm{d}z - i\pi(-1) - i\pi(-e^{2i\pi(p-1)})$$

So we find that,

$$\begin{aligned}\mathcal{P}\int_{-\infty}^{\infty}\frac{e^{pz}}{1-e^z}\,\mathrm{d}z &= \frac{-i\pi(1+e^{2i\pi p})}{1-e^{2i\pi p}}\\ &= \frac{-i\pi(e^{-i\pi p}+e^{i\pi p})}{e^{-i\pi p}-e^{-i\pi p}}\\ &= \pi\cot(\pi p)\end{aligned}$$

So finally, we obtain,

$$I = \int_{-\infty}^{\infty}\frac{e^{px}-e^{qx}}{1-e^x}\,\mathrm{d}x = \pi(\cot(\pi p) - \cot(\pi q)) \tag{2.3}$$

2.1 The Hilbert Transform

The **Hilbert Transform** is a bit like the Fourier transform. We take a function of a real variable, s, denoted $f(s)$, which has no singularities on the real axis. Then the Hilbert transform is defined as,

$$\boxed{\mathcal{H}_f(t) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{f(s)}{t-s}\,\mathrm{d}s} \tag{2.4}$$

where t is also a real variable. Whilst this isn't a bad definition, it can be turned into something nicer,

$$\mathcal{H}_f(t) = \frac{1}{\pi}\lim_{\epsilon\to 0}\left[\int_{-\infty}^{t-\epsilon}\frac{f(s)}{t-s}\,\mathrm{d}s + \int_{t+\epsilon}^{\infty}\frac{f(s)}{t-s}\,\mathrm{d}s\right]$$

If we then shift the pole in s to the origin using $s' = s - t$ we see,

$$\begin{aligned}\mathcal{H}_f(t) &= \frac{1}{\pi}\lim_{\epsilon\to 0}\left[-\int_{-\infty}^{-\epsilon}\frac{f(s'+t)}{s'}\,\mathrm{d}s' - \int_{\epsilon}^{\infty}\frac{f(s'+t)}{s'}\,\mathrm{d}s'\right]\\ &= \frac{1}{\pi}\lim_{\epsilon\to 0}\left[-\int_{\infty}^{\epsilon}\frac{f(-s'+t)}{s'}\,\mathrm{d}s' - \int_{\epsilon}^{\infty}\frac{f(s'+t)}{s'}\,\mathrm{d}s'\right]\\ &= \frac{1}{\pi}\lim_{\epsilon\to 0}\left[\int_{\epsilon}^{\infty}\frac{f(t-s')-f(t+s')}{s'}\,\mathrm{d}s'\right]\end{aligned}$$

We now consider finding the inverse of the Hilbert transform. $\hat{\mathcal{H}}_f(\omega)$ is the Fourier transform of the Hilbert transform. Take a function $f(t)$, then the Fourier transform is,

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}\,\mathrm{d}t$$

So, we have,

$$\hat{\mathcal{H}}_f(\omega) = \frac{1}{\pi}\lim_{\epsilon\to 0}\left[\int_{-\infty}^{\infty}\mathrm{d}t\int_{\epsilon}^{\infty}\frac{1}{s}\big(f(t-s)-f(t+s)\big)e^{i\omega t}\,\mathrm{d}s\right]$$

We assume that we can interchange the order of integration without causing a problem, and take $t-s \to t$ in the first integral and $t+s \to t$ in the second integral. Then,

$$
\begin{aligned}
\hat{\mathcal{H}}_f(\omega) &= \frac{1}{\pi}\lim_{\epsilon\to 0}\left[\int_{-\infty}^{\infty}\mathrm{d}t\int_{\epsilon}^{\infty}\frac{1}{s}\left[f(t)e^{i\omega t}e^{i\omega s} - f(t)e^{i\omega t}e^{-i\omega s}\right]\mathrm{d}s\right] \\
&= \frac{1}{\pi}\lim_{\epsilon\to 0}\hat{f}(\omega)\int_{\epsilon}^{\infty}\left(e^{i\omega s} - e^{-i\omega s}\right)\mathrm{d}s \\
&= \frac{1}{\pi}\lim_{\epsilon\to 0}\hat{f}(\omega)\int_{\epsilon}^{\infty}\frac{2i\sin(\omega s)}{s}\,\mathrm{d}s \\
&= \frac{i}{\pi}\hat{f}(\omega)\int_{-\infty}^{\infty}\frac{\sin(\omega s)}{s}\,\mathrm{d}s
\end{aligned}
$$

Now, we consider $\mathcal{P}\int_{-\infty}^{\infty}\frac{e^{i\omega s}}{s}\,\mathrm{d}s$, this satisfies,

- $\mathcal{P}\int_{-\infty}^{\infty}\frac{e^{i\omega s}}{s}\,\mathrm{d}s = i\pi$ for $\mathcal{R}(\omega) > 0$
- $\mathcal{P}\int_{-\infty}^{\infty}\frac{e^{i\omega s}}{s}\,\mathrm{d}s = -i\pi$ for $\mathcal{R}(\omega) < 0$
- $\mathcal{P}\int_{-\infty}^{\infty}\frac{e^{i\omega s}}{s}\,\mathrm{d}s = 0$ for $\omega = 0$

So finally, we find that for real ω,

$$\hat{\mathcal{H}}_f(\omega) = i\hat{f}(\omega), \qquad \omega > 0 \tag{2.5}$$

$$\hat{\mathcal{H}}_f(\omega) = -i\hat{f}(\omega), \qquad \omega < 0 \tag{2.6}$$

$$\hat{\mathcal{H}}_f(\omega) = 0, \qquad \omega = 0 \tag{2.7}$$

We find from this that the Hilbert transform satisfies,

$$\hat{\mathcal{H}}_f\hat{\mathcal{H}}_f(\omega) = -f(\omega), \qquad \omega \neq 0 \tag{2.8}$$

and zero otherwise. Thus the Hilbert transform satisfies $\mathcal{H}^2 = -1$, and hence the inverse transform has $\mathcal{H}^{-1} = -\mathcal{H}$.

2.1.1 Kramers-Kronig Relations

We may use the Hilbert transform to discover the Kramers-Kronig relations. Consider a function $f(s)$ of a complex variable s, analytic with $\left|\frac{f(s)}{s}\right| \to 0$ as $|s| \to \infty$ in the upper half plane. Now we consider,

$$\int_{\mathcal{C}}\frac{f(s)}{t-s}\,\mathrm{d}s$$

where t is taken to be real. The contour $\mathcal{C}$ is shown in Figure 2.3. Integrating along the real axis gives,

$$\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{f(s)}{t-s}\,\mathrm{d}s + \frac{1}{\pi}\big(-i\pi\,\mathrm{res}(t)\big) = 0$$

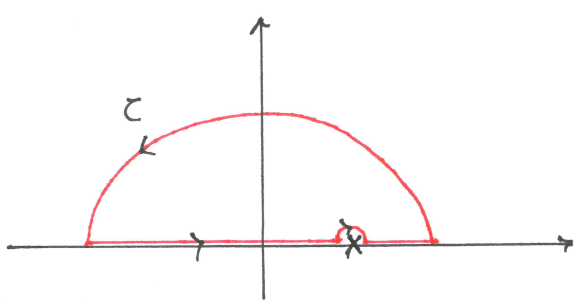


Figure 2.3
Kramers-Kronig contour

So we find that,

$$\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{f(s)}{t-s}\,\mathrm{d}s = -if(t)$$

Separating $f(s)$ into real and imaginary parts,

$$f(z) = u(x,y) + iv(x,y)\big|_{y=0}$$

Then we find that,

- Real part:
$$\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{u(x,0)}{t-x}\,\mathrm{d}x = -i\cdot iv(t,0) = v(t,0)$$
- Imaginary part:
$$\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{v(x,0)}{t-x}\,\mathrm{d}x = -u(t,0)$$

So we see that if u is known on the real axis, then v can also be found.

Example. *Consider the case* $u(x,0)=\cos(x)$*. Then,*

$$\begin{aligned}
v(t,0) &= \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\cos(s)}{t-s}\,\mathrm{d}s \\
&= \frac{-1}{\pi}\Re\left(\mathcal{P}\int_{-\infty}^{\infty}\frac{e^{is}}{s-t}\,\mathrm{d}s\right) \\
&= \frac{-1}{\pi}\Re\left(\mathcal{P}\int_{-\infty}^{\infty}e^{it}\frac{e^{i(s-t)}}{s-t}\,\mathrm{d}s\right) \\
&= \frac{-1}{\pi}\Re\left(e^{it}\mathcal{P}\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\,\mathrm{d}x\right) \\
&= \frac{-1}{\pi}\Re(i\pi e^{it}) \\
&= \sin(t)
\end{aligned}$$

This tells us what the real and imaginary parts are on the real axis,

$$u(x,0) + iv(x,0) = \cos x + i\sin x = e^{ix}$$

We can now construct $f(z)$ in the upper half plane since it is a function of only $z = x+iy$. Furthermore, e^{iz} decays fast enough, so it is analytic in the upper half plane.

3 Analytic Continuation and Special Functions

Suppose one takes an analytic function in the neighbourhood of a point z_0. Then it can be expanded as a Taylor series about that point.

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n$$

Now suppose that the radius of convergence is some ρ, so the series is defined on a region $\mathcal{D}$, and that the series is singular at some isolated point Q. Now form a power series about a point $R = z_1$ which does not lie on PQ. Then this power series defines a function $g(z)$ as a power series around R. Suppose this is defined on a region $\mathcal{D}'$. Then,

- $\mathcal{D}'$ may lie entirely in $\mathcal{D}$
- $\mathcal{D}'$ may have some overlap with $\mathcal{D}$, but also contain points outside of $\mathcal{D}$

If $f(z) = g(z)$ in the intersection $\mathcal{D} \cap \mathcal{D}'$ then $g(z)$ is the ***analytic continuation*** *of $f(z)$.*

The point of analytic continuation is that if $f(z)$ is defined in some domain, then one may be able to find a $g(z)$ that agrees with $f(z)$ somewhere but is defined in a different region.

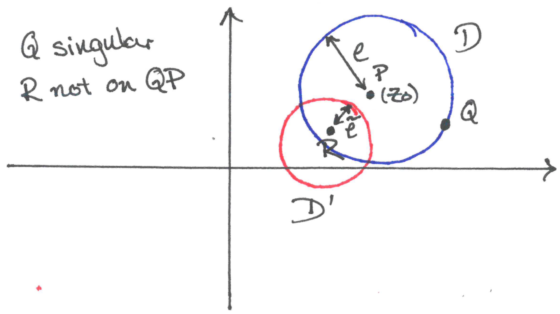


Figure 3.1
Analytic Continuation

Example. *Consider $f(z) = 1 + z + z^2 + \ldots$ which is convergent for $|z| < 1$. This series can be summed to give $g(z) = \frac{1}{1-z}$ such that inside the unit circle f agrees with g. $g(z)$ exists in $\mathbb{C}$ apart from the pole at $z = 1$, and therefore is an analytic continuation of $f(z)$ to the entire complex plane.*

Example. *Consider $f(z) = 1 + z^2 + z^4 + z^8 + z^{16} + \ldots$. This series is convergent for $|z| < 1$ but diverges at $z = 1$. Note that,*

$$f(z) = z^2 + (1 + z^4 + z^8 + z^{16} + \ldots = z^2 + f(z^2)$$

So, at $z = -1$, $f(-1) = 1 + f(1)$. $f(1)$ is divergent so there is a singularity at $z = -1$. Also,

$$f(z) = z^2 + z^4 + f(z^4)$$

which shows that if $z^4 = 1$, then $f(z)$ is singular, in other words, there must be singularities at $z = \pm i$. Extending this,

$$f(z) = z^2 + z^4 + \ldots + z^{2^n} + f(z^{2^n})$$

Thus, the 2^n roots of unity are always singular for all n. We see then that the whole of the unit circle in singular. Therefore, no series expansion will ever converge at $|z| = 1$, so no analytic continuation beyond the unit circle is possible.

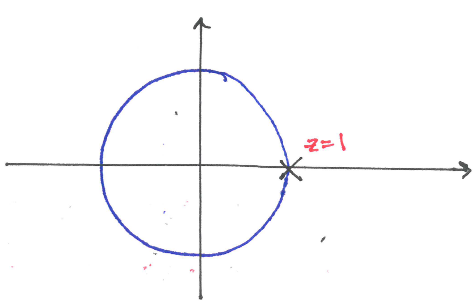


Figure 3.2
The pole at $z = 1$ for the first example

3.1 The Gamma Function

$$\boxed{\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t} \tag{3.1}$$

This converges provided that $\Re(z) > 0$. Suppose that z is a positive integer, then,

$$\int_0^\infty t^{n-1}e^{-t}\,\mathrm{d}t = (n-1)!$$

and so by (3.1),

$$\Gamma(n) = (n-1)!, \qquad n \in \mathbb{Z}^+ \tag{3.2}$$

Note that at $z = 0$, $\Gamma(z)$ diverges, but,

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t}\,\mathrm{d}t$$

exists for $\Re(z) > 0$. Integrating by parts,

$$\Gamma(z+1) = \Big[-t^z e^{-t}\Big]_0^\infty + \int_0^\infty z t^{z-1}e^{-t}\,\mathrm{d}t = z\Gamma(z)$$

Then, for $-1 < \Re(z) < 0$ we use this to define,

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}$$

since $\Gamma(z+1)$ exists for $z > -1$. Now, near $z = 0$, $\Gamma(1) = \int_0^\infty e^{-t}\,\mathrm{d}t = 1$. Thus, $\Gamma(z)$ has a simple pole at $z = 0$ which says that,

$$\Gamma(z) \sim \frac{1}{z} + \ldots$$

Iterating this procedure,

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{z(z+1)} = \ldots = \frac{\Gamma(z+n)}{z(z+1)\ldots(z+n-1)}$$

This gives us an analytic continuation of $\Gamma(z)$ for $\Re(z) > -n$. Since this is true for any n, then this defines $\Gamma(z)$ in the entire complex plane.

From this we see that the gamma function must have poles at all negative integers and 0. It is easy to calculate the residue of the pole at $z = -n$ by considering the coefficient $\frac{1}{z-n}$ in $\Gamma(-n)$,

$$\frac{\Gamma(1)}{(-n)(-n+1)\ldots(-1)}$$

So we see that the residue is $\frac{(-1)^n}{n!}$.

3.1.1 Euler's Limit Formula

This is a way of describing the exponential function we have seen before,

$$\lim_{n\to\infty}\left(1 - \frac{t}{n}\right)^n = \ldots = 1 - t + \frac{1}{2!}t^2 + \ldots + \frac{(-1)^p}{p!}t^p + \ldots = e^{-t}$$

Inspired by this result we define the *modified* Γ*-function*

$$\Gamma_n(z) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\,\mathrm{d}t, \qquad \Re(z) > 0 \tag{3.3}$$

such that as $n \to \infty$, we recover the Γ-function. Integrating by parts once:

$$\Gamma_n(z) = \left[\left(1 - \frac{t}{n}\right)^n \frac{t^z}{z}\right]_0^n - \int_0^n \left(1 - \frac{t}{n}\right)^{n-1}\left(\frac{-1}{n}\right) n\frac{t^z}{z}\,\mathrm{d}t$$

The boundary terms vanish since $\Re(z) > 0$, so,

$$\Gamma_n(z) = \int_0^n \left(1 - \frac{t}{n}\right)^{n-1} \frac{t^z}{z}\,\mathrm{d}t$$

Now, we keep integrating by parts, noting that the boundary terms will always vanish, to find that, after integrating $(n-1)$ times,

$$\Gamma_n(z) = \int_0^n t^{z+n-1} \frac{1}{z(z+1)(z+2)\dots(z+n-1)}\,\mathrm{d}t \frac{n(n-1)(n-2)\dots 1}{n \cdot n \dots n}$$

Computing the integral, we find that,

$$\begin{aligned}\Gamma_n(z) &= \frac{n^{z+n}}{z(z+1)\dots(z+n)} \cdot \frac{n!}{n^n} \\ &= \frac{n^z n!}{z(z+1)(z+2)\dots(z+n)}\end{aligned}$$

Now we take the limit as $n \to \infty$, to find Euler's limit formula for the Γ-function.

$$\boxed{\Gamma(z) = \lim_{n\to\infty} \frac{n^z n!}{z(z+1)\dots(z+n)}} \tag{3.4}$$

3.1.2 Weierstrass infinite product formula

Consider the following sequence,

$$u_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} - \log n$$

In the limit as $n \to \infty$ does this have a finite limit? We note,

$$u_n < \int_1^n \frac{1}{t}\,\mathrm{d}t < u_{n-1}$$

So $\frac{1}{n} < u_n < 1$. Now consider,

$$\begin{aligned}u_{n+1} - u_n &= \frac{1}{n+1} - \log(n+1) + \log n \\ &= \frac{1}{n+1} - \log\left(1 + \frac{1}{n}\right) \\ &= \frac{1}{n+1} - \frac{1}{n} - \dots < 0\end{aligned}$$

Thus the limit as $n \to \infty$ exists and happens to be give by the *Euler-Mascheroni constant.*

$$u_\infty \coloneqq \gamma = 0.5772156949\dots$$

Now look at $\frac{1}{\Gamma_n(z)}$ which is given by,

$$\begin{aligned}\frac{1}{\Gamma_n(z)} &= \frac{n^{-z} z(z+1)\dots(z+n)}{n!} \\ &= z e^{z\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\right)} \prod_{s=1}^{n} e^{-\frac{z}{s}}\left(1 + \frac{z}{s}\right) e^{-z\log n} \\ &= z e^{z u_n} \prod_{s=1}^{n} e^{-\frac{z}{s}}\left(1 + \frac{z}{s}\right)\end{aligned}$$

Taking the limit as $n \to \infty$, we recover,

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{s=1}^{\infty} e^{-\frac{z}{s}}\left(1 + \frac{z}{s}\right)$$

This is useful in the sense that is shows us that the *only* zeros of $\frac{1}{\Gamma(z)}$ are at $z = -n, n \in \mathbb{Z}$. Since these are single zeros, the *only* poles in $\Gamma(z)$ are at the negative integers and $z = 0$. Since $\frac{1}{\Gamma(z)}$ has no singularities in the finite complex plane, it is entire.

3.1.3 Reflection Formula

We see that,

$$\begin{aligned}
\frac{1}{\Gamma(z)\Gamma(1-z)} &= \lim_{n\to\infty} \frac{1}{n!}n^{-z}z(z+1)\dots(z+n)\frac{1}{n!}n^{-1+z}(1-z)(1-z+1)\dots(1-z+n) \\
&= \lim_{n\to\infty} \frac{1}{n}\frac{1}{(n!)^2}z(z+1)(1-z)(z+2)(2-z)\dots(z+n)(n-z)(n+1-z) \\
&= \lim_{n\to\infty} \frac{1}{n}z(1+z)(1-z)\left(1+\frac{z}{2}\right)\left(1-\frac{z}{2}\right)\dots\left(1+\frac{z}{n}\right)\left(1-\frac{z}{n}\right)(n+1-z) \\
&= z\prod_{s=1}^{\infty}\left(1-\frac{z^2}{s^2}\right)
\end{aligned}$$

Now $\prod_{s=1}^{\infty}\left(1-\frac{z^2}{s^2}\right)$ has zeros at $z=s$, where s is any integer and nowhere else aside from $s=0$. As $z\to 0$, $\prod_{s=1}^{\infty}\left(1-\frac{z^2}{s^2}\right)\to 1$, so we deduce,

$$\prod_{s=1}^{\infty}\left(1-\frac{z^2}{s^2}\right) = \frac{\sin\pi z}{\pi z}$$

and hence we find the reflection formula,

$$\boxed{\frac{1}{\Gamma(z)\Gamma(1-z)} = z\frac{\sin\pi z}{\pi z} = \frac{\sin\pi z}{\pi}} \tag{3.5}$$

3.1.4 Zeros of $\Gamma(z)$

We use the reflection formula derived in the last section,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}$$

Then, if $\Gamma(z)=0$, then $z\notin\mathbb{Z}$, implying that the right hand side is finite. In turn, this implies that $\Gamma(1-z)$ has a pole since $\Gamma(1-z)=\frac{\pi}{\Gamma(z)\sin\pi z}$. But this implies $z\in\mathbb{N}$ which is a contradiction. So we conclude $\Gamma(z)$ *has no zeros.* Now consider,

$$\begin{aligned}
\frac{2^{2z}\Gamma(z)\Gamma(z+\frac{1}{2})}{\Gamma(2z)} &= \lim_{n\to\infty}\left[2^{2z}\frac{n!n^z}{z(z+1)\dots(z+n)}\frac{n!n^zn^{\frac{1}{2}}}{\left(z+\frac{1}{2}\right)\left(z+\frac{3}{2}\right)\dots\left(z+n+\frac{1}{2}\right)}\frac{2z(2z+1)\dots(2z+2n)}{(2n)!(2n)^{2z}}\right] \\
&= \lim_{n\to\infty}\left[\frac{(n!)^2n^{\frac{1}{2}}2^{2n+1}}{(2n)!}\frac{z\left(z+\frac{1}{2}\right)\dots(z+n)}{z\left(z+\frac{1}{2}\right)\dots\left(z+n+\frac{1}{2}\right)}\right] \\
&= \lim_{n\to\infty}\left[\frac{(n!)^2 2^{2n+1}n^{-\frac{1}{2}}}{(2n)!\left(1+\frac{z+\frac{1}{2}}{n}\right)}\right]
\end{aligned}$$

Continuing from here we apply Stirling's approximation to the factorial function,

$$n! = \left(\frac{n}{e}\right)^n(2\pi n)^{\frac{1}{2}}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right) \tag{3.6}$$

Then,

$$\begin{aligned}
\frac{2^{2z}\Gamma(z)\Gamma(z+\frac{1}{2})}{\Gamma(2z)} &= \lim_{n\to\infty}\left[\frac{2^{2n+1}\left(\frac{n}{e}\right)^{2n}(2\pi n)}{\left(\frac{2n}{e}\right)^{2n}(4\pi n)^{\frac{1}{2}}n^{\frac{1}{2}}}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)\left(1-\frac{z+\frac{1}{2}}{n}\right)\right] \\
&= 2\sqrt{\pi}
\end{aligned}$$

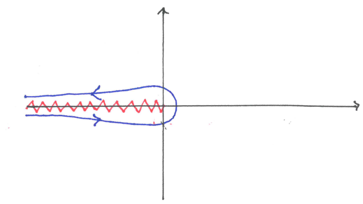
So we find that,

$$\Gamma(2z) = 2^{2z-1}\pi^{-\frac{1}{2}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \tag{3.7}$$

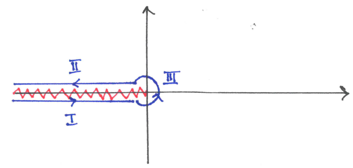
In particular, let $z=n$, then,

$$\Gamma\left(\frac{2n+1}{2}\right) = \frac{\sqrt{\pi}n!}{2^{2n-1}}\binom{2n-1}{n}$$

3.1.5 Integral Representation of $\Gamma(z)$



(a) The Hankel Contour



(b) Deformed Hankel Contour

We consider using Hankel's Contour when looking for an integral representation of the Γ-function. Now,

$$\int_{\mathcal{C}} e^t t^{-z}\,\mathrm{d}t$$

is defined $\forall z$. Now,

- On I: $t = \rho e^{-i\pi}$
- On II: $t = \rho e^{i\pi}$
- On III: $t = \epsilon e^{i\theta}$

Then, the integral becomes the following, where we note that the limit of the ϵ integral is zero if $\Re(z) < 1$,

$$\begin{aligned}
I &= -\int_{\infty}^{0} e^{-\rho}\rho^{-z}e^{i\pi z}\,\mathrm{d}\rho - \int_0^{\infty} e^{-\rho}\rho^{-z}e^{-i\pi z}\,\mathrm{d}\rho + \lim_{\epsilon\to 0}\int_{-\pi}^{\pi} ie^{\epsilon e^{i\theta}}\epsilon^{-z+1}e^{-i\theta z+i\theta}\,\mathrm{d}\theta \\
&= \left(e^{i\pi z} - e^{-i\pi z}\right)\int_0^{\infty} e^{-\rho}\rho^{-z}\,\mathrm{d}\rho, \qquad \Re(z) < 1 \\
&= \Gamma(1-z)\cdot 2i\sin\pi z \\
&= \frac{2\pi i}{\Gamma(z)}
\end{aligned}$$

And so, we find the integral representation of $\Gamma(z)$ to be,

$$\boxed{\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{\mathcal{C}} e^t t^{-z}\,\mathrm{d}t} \tag{3.8}$$

3.2 Beta Function

We define the Beta Function, $B(p,q)$ to be,

$$\boxed{B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,\mathrm{d}x} \tag{3.9}$$

This exists for $\Re(p), \Re(q) > 0$ and $x^{p-1}(1-x)^{q-1}$ are taken to be the principal values. We see from (3.9) that the Beta function has some simple properties,

1. Symmetry: Let $y = 1-x$, then we see that,

$$B(p,q) = B(q,p)$$

2. $B(p,q)$ is related to the Γ-function as follows,

$$\begin{aligned}
\Gamma(p)\Gamma(q) &= \int_0^{\infty} e^{-x}x^{p-1}\,\mathrm{d}x \cdot \int_0^{\infty} e^{-y}y^{q-1}\,\mathrm{d}y \\
&= 4\int_0^{\infty} e^{-u^2}u^{2p-1}\,\mathrm{d}u \int_0^{\infty} e^{-v^2}u^{2q-1}\,\mathrm{d}v
\end{aligned}$$

Letting $u = r\sin\theta$ and $v = r\cos\theta$, we find,

$$\Gamma(p)\Gamma(q) = 4\int_0^\infty e^{-r^2} r^{2p+2q-1}\,\mathrm{d}r \int_0^{\frac{\pi}{2}} (\cos\theta)^{2q-1}(\sin\theta)^{2p-1}\,\mathrm{d}\theta$$

Finally taking $r = s$ and $\sin\theta = t^{\frac{1}{2}}$ we see,

$$\begin{aligned}\Gamma(p)\Gamma(q) &= \int_0^\infty e^{-s}s^{p+q-1}\,\mathrm{d}s \int_0^1 t^{P-1}(1-t)^{q-1}\,\mathrm{d}t \\ &= \Gamma(p+q)B(p,q)\end{aligned}$$

So we conclude,

$$\boxed{B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}} \tag{3.10}$$

3.3 Riemann Zeta Function, $\zeta(s)$

We define the Riemann Zeta Function for $\Re(s) > 1$ to be,

$$\boxed{\zeta(s) = \sum_{n=1}^\infty n^{-s}} \tag{3.11}$$

It has important applications in analytic number theory, for example,

$$(1-2^{-s})\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \ldots - \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \ldots\right)$$

removes all even factors, whilst $(1-2^{-s})(1-3^{-s})$ removes all multiples of 3 from the series. Now, $\zeta(s)\prod_{p\text{ prime}}(1-p^{-s}) = 1$, so we find,

$$\boxed{\zeta(s) = \prod_{p\text{ prime}}\left(1-p^{-s}\right)^{-1}} \tag{3.12}$$

We already know that,

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,\mathrm{d}t$$

Now consider the following where we assume that z is such that the integral is defined,

$$\int_0^\infty e^{-zt}t^{s-1}\,\mathrm{d}t = z^{-s}\int_0^\infty e^{-u}u^{s-1}\,\mathrm{d}u = z^{-s}\Gamma(s)$$

This motivates writing z^{-s} as,

$$z^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty e^{-zt}t^{s-1}\,\mathrm{d}t$$

Putting this into $\zeta(s) = \sum_{n=1}^\infty n^{-s}$ we find that,

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^\infty \frac{1}{\Gamma(s)}\int_0^\infty e^{-nt}t^{s-1}\,\mathrm{d}t \\ &= \frac{1}{\Gamma(s)}\int_0^\infty \frac{e^{-t}}{1-e^{-t}}t^{s-1}\,\mathrm{d}t\end{aligned}$$

which allows us to find an integral representation of the ζ-function. We compare this the the Bose-Einstein distribution in Statistical Physics.

$$\boxed{\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}}{e^t-1}\,\mathrm{d}t} \tag{3.13}$$

We now look to find an analytic continuation to a domain larger than $\Re(s) > 1$. Consider:

$$\int_{\mathcal{C}} \frac{t^{s-1}}{e^{-t}-1}\,\mathrm{d}t$$

where $\mathcal{C}$ is the Hankel contour. Now, $t = 0$ is a genuine s-branch point. The contour is defined as follows:

where $-\pi \leq \operatorname{Arg} z < \pi$. We have deformed the Hankel contour into the black curve, which results in:

- On I: $t = \rho e^{-i\pi}$
- On II: $t = \epsilon e^{i\theta}$
- On III: $t = \rho e^{i\pi}$

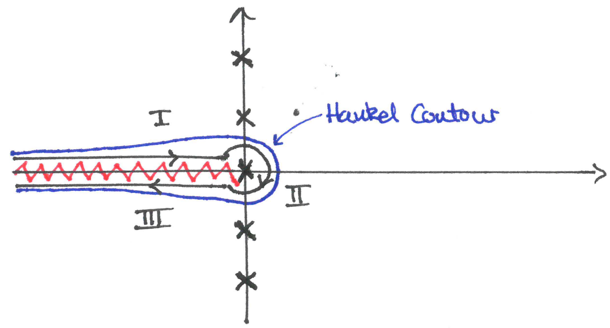


Figure 3.4
Hankel Contour for analytic continuation of $\zeta(s)$

On II the integrand isL

$$
\begin{aligned}
\lim_{\epsilon \to 0} \left[\frac{\epsilon e^{i\theta} i \epsilon^{s-1} e^{i(s-1)\theta} \, \mathrm{d}\theta}{e^{-\epsilon e^{i\theta}} - 1} \right] &= \lim_{\epsilon \to 0} \frac{i e^{is\theta} \epsilon^s}{-\epsilon e^{i\theta} + \frac{1}{2}\epsilon^2 e^{2i\theta} + \ldots - 1} \\
&= \lim_{\epsilon \to 0} \frac{i e^{is\theta} \epsilon^s}{-\epsilon e^{i\theta} \left(1 - \frac{1}{2}\epsilon e^{i\theta} + \ldots\right)} \\
&= \lim_{\epsilon \to 0} \epsilon^{s-1} (1 + \ldots)
\end{aligned}
$$

So the contribution round the small circle tends to zero for $\mathfrak{R}(s) > 1$. Then,

$$
\begin{aligned}
\int_{\mathcal{C}} \frac{t^{s-1}}{e^{-t} - 1} \, \mathrm{d}t &= \int_{\infty}^{0} \frac{\rho^{s-1} e^{-i\pi(s-1)} e^{-i\pi}}{e^{\rho} - 1} \, \mathrm{d}\rho + \int_0^{\infty} \frac{\rho^{s-1} e^{i\pi(s-1)} e^{i\pi}}{e^{\rho} - 1} \, \mathrm{d}\rho \\
&= \left(e^{i\pi s} - e^{-i\pi s}\right) \int_0^{\infty} \frac{\rho^{s-1}}{e^{\rho} - 1} \, \mathrm{d}\rho \\
&= \Gamma(s)\zeta(s) \cdot 2i \sin \pi s
\end{aligned}
$$

For $\mathfrak{R}(s) > 1$,

$$
\frac{1}{2i} \int_{\mathcal{C}} \frac{t^{s-1}}{e^{-t} - 1} \, \mathrm{d}t = \sin \pi s \zeta(s) \Gamma(s) \tag{3.14}
$$

Thus, this gives us an analytic continuation of the zeta function to the entire complex plane. Potential singularities can only come from the Γ-function, where $(1-s)$ must be zero or a negative integer, $s = 1, 2, 3, \ldots$.

Using $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, you would expect a singularity at $s = 1$ but not at $s = 2$. Suppose s is an integer, then there is no branch point. The only singularities are the poles at the origin and $2n\pi i$. We deform the Hankel contour into a small circle around the origin, then the integrand is,

$$
\frac{t^{s-1}}{e^{-t} - 1} = \frac{t^{s-1}}{1 - t + \frac{1}{2}t^2 + \ldots - 1} = \frac{t^{s-1}}{-t\left(1 - \frac{1}{2}t + \ldots\right)} = -t^{s-2}\left(1 + \frac{1}{2}t + \ldots\right)
$$

So we see that in the case $s = 2, 3, \ldots$, there is no singularity and the integral gives zero. At $s = 1$, the integrand has a pole of residue -1.

$$
\Rightarrow \int_{\mathcal{C}} \frac{t^{s-1}}{e^{-t} - 1} \, \mathrm{d}t = -2\pi i
$$

Near $s = 1$, we use $(1-s)\Gamma(1-s) = \Gamma(2-s)$,

$$
\Rightarrow \Gamma(1-s) = -\frac{1}{s-1} \Gamma(2-s)
$$

Now, $\Gamma(2-s)$ is finite and $\Gamma(1) = 1$ so $\Gamma(1-s) = -\frac{1}{s-1} + \ldots$. Using this, we have,

$$
\begin{aligned}
\zeta(s) &= \frac{1}{2\pi i} \left(\frac{-1}{s-1} + \ldots\right) \left(-2\pi i + \mathcal{O}(s-1)\right) \\
&= \frac{1}{s-1} + \text{ finite terms}
\end{aligned}
$$

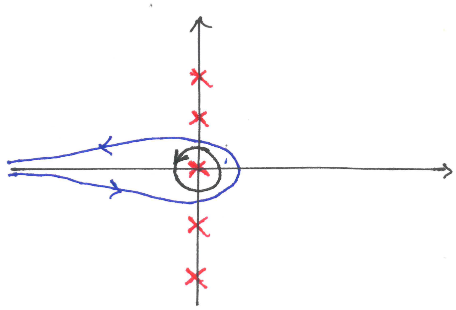


Figure 3.5
When s is an integer, there is no branch cut

Since the Hankel contour is well defined everywhere, the only singularity of $\zeta(s)$ in the finite complex plane is a simple pole of residue 1 at $s = 1$. Now choose $\Re(s) < 0$ and consider,

$$\int_{\mathcal{C}} \frac{t^{s-1}}{e^{-t}-1}\,\mathrm{d}t$$

where the contour, $\mathcal{C}$, is shown below. Now,

$$\int_{\text{Hankel}} \frac{t^{s-1}}{e^{-t}-1}\,\mathrm{d}t + \int_{\text{I,II,III,IV}} \frac{t^{s-1}}{e^{-t}-1}\,\mathrm{d}t = -2\pi i \sum_{n=-(N-1)}^{N-1} \operatorname*{res}_{z=2i\pi n}\left(\frac{t^{s-1}}{e^{-t}-1}\right)$$

It can be shown that the contribution from the large rectangle vanishes as $N \to \infty$. Then, in the limit,

$$\int_{\mathcal{C}} \frac{t^{s-11}}{e^{-t}-1}\,\mathrm{d}t = \lim_{n\to\infty}\left\{-2\pi i \sum_{n=-(N-1),n\neq 0}^{(N-1)} \operatorname*{res}_{z=2i\pi n}\left(\frac{t^{s-1}}{e^{-t}-1}\right)\right\}$$

The residue at $z = 2in\pi$ is most easily given by,

$$\lim_{z\to 2in\pi} \frac{(t-2in\pi)}{e^{-t}-1} t^{s-1}$$

So, using L'Hopital's rule, and noting that the residue depends on the half plane we are in (since $-\pi \leq \arg z < \pi$), we find that,

$$\operatorname*{res}_{z=2i\pi n}\left(\frac{t^{s-1}}{e^{-t}-1}\right) = \begin{cases} -(2\pi n)^{s-1} e^{i\frac{\pi}{2}(s-1)} & \text{in the upper half plane} \\ -(2\pi n)^{s-1} e^{i\frac{-\pi}{2}(s-1)} & \text{in the lower half plane} \end{cases}$$

So we deduce that,

$$\begin{aligned}\int_{\mathcal{C}} \frac{t^{s-11}}{e^{-t}-1}\,\mathrm{d}t &= 2\pi i \sum_{k=1}^{\infty} (2\pi n)^{s-1}\left\{e^{\frac{i\pi}{2}(s-1)} + e^{-\frac{i\pi}{2}(s-1)}\right\} \\ &= 2\pi i (2\pi)^{s-1} \sum_{k=1}^{\infty} n^{s-1}\left[-ie^{\frac{i\pi}{2}s} + ie^{-\frac{i\pi}{2}s}\right] \\ &= (2\pi)^s \zeta(1-s)\cdot 2i \sin\frac{\pi s}{2}\end{aligned}$$

Now recall that the original integral is related to the analytic continuation of $\zeta(s)$:

$$\int_{\mathcal{C}} \frac{t^{s-11}}{e^{-t}-1}\,\mathrm{d}t = \frac{2i\pi\zeta(s)}{\Gamma(1-s)}$$

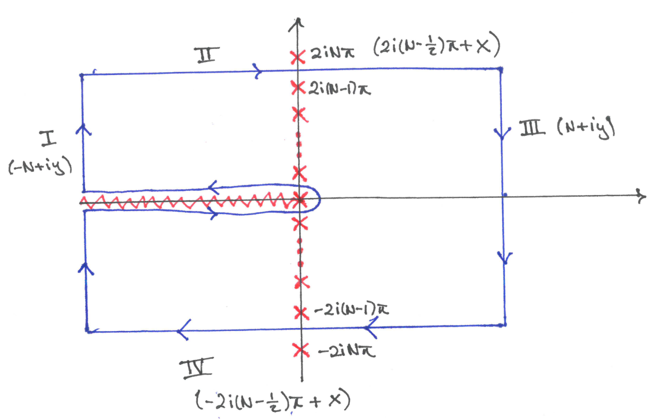


Figure 3.6
Closing the Hankel Contour

So we conclude that,

$$\zeta(1-s) = \frac{\zeta(s)(2\pi)^{1-s}}{\Gamma(1-s)\left(2\sin\frac{\pi s}{2}\right)}$$

Using the reflection formula for the Γ-function given in (3.5), we find,

$$\zeta(1-s) = \frac{\zeta(s)(2\pi)^{1-s}}{2\sin\frac{\pi s}{2}}\frac{\Gamma(s)\sin\pi s}{\pi}$$

We allows us to find the reflection formula for the ζ-function,

$$\boxed{\zeta(1-s) = 2\zeta(s)\Gamma(s)(2\pi)^{-s}\cos\frac{\pi s}{2}} \tag{3.15}$$

We'll use (3.15) to find some values for $\zeta(s)$, for example,

- Putting $s=1$,

$$\zeta(0) = \zeta(1)\cos\frac{\pi}{2}\cdot\Gamma(1)\frac{2}{2\pi}$$

  Now, $\zeta(s) \approx \frac{1}{s-1} +$ finite terms near $s=1$ and $\cos\frac{1}{2}\pi s = \cos\frac{1}{2}\pi + (s-1)\frac{\pi}{2}\left(-\sin\frac{1}{2}\pi\right) + o\left((s-1)^2\right) = -\frac{\pi}{2}(s-1) + o\left((s-1)^2\right)$. So we see that,

$$\zeta(s)\cos\frac{1}{2}\pi s = -\frac{\pi}{2} + o\left((s-1)\right)$$

  i.e. $\lim_{s\to 1}\zeta(s)\cos\frac{1}{2}\pi s = -\frac{\pi}{2}$ which gives us finally that,

$$\zeta(0) = \frac{-\pi}{2}\Gamma(1)\frac{2}{2\pi} = -\frac{1}{2} \qquad \Big(\neq (1+1+1+\cdots)\Big)$$

3.3.1 Zeros of the ζ-function

1. First we take $s = 2n+1$, so,

$$\zeta(-2n) = 2\zeta(2n+1)\Gamma(2n+1)\cdot\cos\frac{(2n+1)\pi}{2}\cdot(2\pi)^{-2n+1}$$

   Since all the terms remain finite, and $\cos\frac{(2n+1)\pi}{2} = 0$, we have found the triial zeros of the Riemann ζ-function,

$$\zeta(-2n) = 0$$

2. There is also a collection of non-trivial zeros:

$$\zeta(s) = 0, \qquad s = \frac{1}{2} + it, \quad t \in \mathbb{R}$$

   The Riemann Hypothesis says these are the only non-trivial zeros.

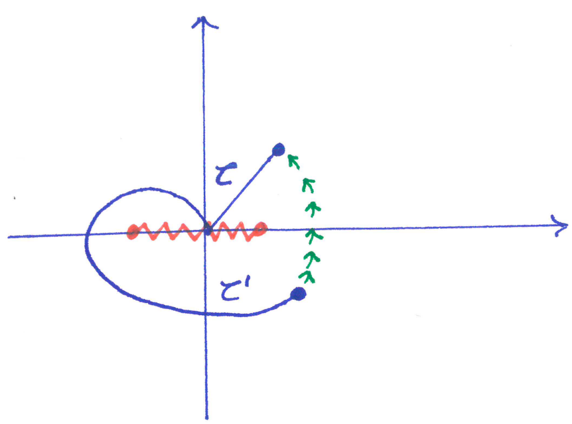


Figure 3.7
Illustration of winding around the branch cut

3.4 The arcsin Function

We might happily write:

$$\arcsin x = \int_0^x \frac{1}{\sqrt{1-\tilde{x}}}\,\mathrm{d}\tilde{x}$$

as a real function of x, but we know arcsin is a multi-valued function whereas the integral is single valued. Instead, we think about the integral in the complex plane, define:

$$\sin^{-1} z = \int_0^z \frac{1}{\sqrt{1-\tilde{z}}}\,\mathrm{d}\tilde{z}$$

We see there is a branch point at $\tilde{z} = \pm 1$, but not at $\tilde{z} = \infty$. Considering the whole integral near infinity however,

$$-\int_\infty^{\frac{1}{w}} \frac{w}{w^2\sqrt{1-w^2}}\,\mathrm{d}w = -\int_\infty^{\frac{1}{2}} \frac{1}{w}\frac{1}{\sqrt{1-w^2}}\,\mathrm{d}w$$

So we see that there is a pole at $w = 0$ in the integrand. From the diagram in Figure 3.7, the value along the real contour is the same as along the green contour. Suppose that the blue point is close to the black point, then the difference is 2π. Differences of $2n\pi$ come from winding around the singularity n times.

4 Elliptic Functions

Trigonometric functions are naturally defined on a circle,

$$\sin z = \sin(z + 2n\pi), \qquad \text{for all integers } n$$

i.e. they are periodic functions of period 2π. Elliptic functions are those that are doubly periodic. These are useful for analysis on a torus, and also in number theory. An *elliptic function* with periods ω_1 and ω_2 satisfies:

$$f(z) = f(z + 2m\omega_1 + 2n\omega_2), \qquad \text{for all integers } m, n$$

We construct a small cell where $f(z)$ is defined. Then an elliptic function is the same in all similar cells. This pattern repeats and covers the entire complex plane. An elliptic function is one that is doubly periodic and the only singularities are poles. We now consider some properties.

1. Take an elliptic function $f(z)$ and consider,

   $$\int_{\mathcal{C}} f(z)\,\mathrm{d}z$$

   where $\mathcal{C}$ is the trapezium which forms the boundary of a cell and does not intersect any singularity. You can always fix this by moving $\mathcal{C}$ to avoid any singularities. The reason one chooses no branch points in elliptic functions is to ensure that one can have double periodicity independently of where z is.

   Now, from the diagram, the periodicity in $f(z)$ implies that f on I is the same as on III, but with the orientation reversed, so they cancel, similarly II and IV cancel. This implies,

   $$\int_{\mathcal{C}} f(z)\,\mathrm{d}z = 2\pi i \sum \operatorname{res}(f(z)) = 0$$

   Thus the sum of the residues of the poles inside a cell is zero.

2. An elliptic function cannot have a single simple pole as its singularity inside a single cell, which follows from the statement above since the residue sum would not vanish.

3. Suppose one has an elliptic function with no poles in $\mathcal{C}$. Since the function is doubly periodic in the entire complex plane, including infinity, Liouville's Theorem implies the function is constant.

4. Suppose one tries to solve,

   $$f(z) = c, \qquad f(z) \text{ is elliptic}$$

   There is a nice formula that tells you the difference between the number of poles in a function and the number of zeros. Suppose $f(z) - c = 0$, then,

   $$\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f'(z)}{f(z) - c}\,\mathrm{d}z = \text{Number of poles} - \text{Number of zeros}$$

   Integrating around $\mathcal{C}$, where $\mathcal{C}$ is the boundary of a cell gives zero, since both $f'(z)$ and $f(z)$ are doubly periodic. In other words,

   $$\text{Number of poles} = \text{Number of zeros}$$

   Now, $f(z) - c$ has the same number of poles independently of c. Hence the number of solutions to $f(z) = c$ is independent of c.

Example. *Consider the singly periodic function,*

$$\frac{1}{\sin^2 x} = \sum_{n=-\infty}^{\infty} \frac{1}{(x - 2n\pi)^2}$$

which has a double pole at $x = 0, \pm 2\pi, \pm 4\pi, \ldots$.

4.1 Weierstrass Elliptic Function

Define the following function,

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{\substack{m,n \\ m,n \neq 0}} \frac{1}{(z - \Omega_{m,n})^2} - \frac{1}{\Omega_{m,n}^2}, \qquad \Omega_{m,n} = 2m\omega_1 + 2n\omega_2 \tag{4.1}$$

$\mathcal{P}(z)$ has a pole of order 2 at the origin, is even in z (which can be seen by sending $m \to -m$ and $n \to -n$, and also converges. Although it does not converge without the last term. We now look to find the periodicity, consider,

$$\mathcal{P}'(z) = \frac{-2}{z^3} + \sum_{\substack{m,n \\ m,n\neq 0}} \frac{-2}{(z-\Omega_{m,n})^3}$$
$$= -2\sum_{m,n} \frac{1}{(z-\Omega_{m,n})^3}$$

Now this expression is doubly periodic, $z \to z + 2p\omega_1 + 2q\omega_2$. Furthermore $\mathcal{P}'(z)$ is odd in z, either since $\mathcal{P}(z)$ is even or in the infinite sum taking $z \to -z$ and $m,n \to -m,-n$. Now consider,

$$\mathcal{P}'(z) = \mathcal{P}'(z+2\omega_1), \qquad \mathcal{P}' \text{ is periodic in } 2\omega_1$$

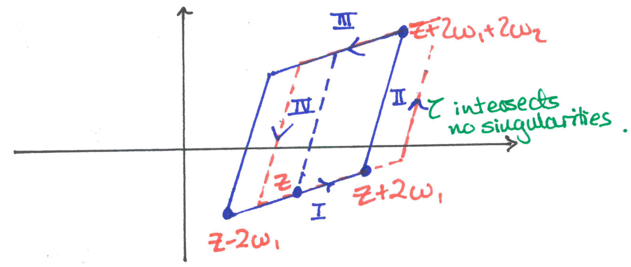


Figure 4.1
An example of a unit cell for an elliptic function

Integrating gives,

$$\mathcal{P}(z) = \mathcal{P}(z+2\omega_1) + k$$

where k is some integration constant. Now put $z = -\omega_1$ to find that,

$$\mathcal{P}(-\omega_1) = \mathcal{P}(\omega_1) + k$$

But $\mathcal{P}(z)$ is even so we find $k = 0$. Hence $\mathcal{P}(z)$ is periodic in ω_1 with period $2\omega_1$. We can perform an identical calculation for ω_2. Hence $\mathcal{P}(z)$ *is elliptic.*

Now we notice that the only singularities are double poles at the origin, and points conjugate to $z = 0$, i.e. $z = 2m\omega_1 + 2n\omega_2$. Also, $\mathcal{P}(z) - \frac{1}{z^2}$ is analytic at the origin and hence has a Taylor expansion in the neighbourhood.

$$\mathcal{P}(z) - \frac{1}{z^2} = z\mathcal{P}'(0) + \frac{1}{2}z^2\mathcal{P}''(0) + \frac{1}{6}z^3\mathcal{P}^{(3)}(0) + \frac{1}{24}z^4\mathcal{P}^{(4)}(0) + o(z^5)$$

Now, $\mathcal{P}'$ and $\mathcal{P}^{(3)}$ are both odd so $\mathcal{P}'(0) = 0$ and $\mathcal{P}^{(3)}(0) = 0$. Furthermore,

- $\mathcal{P}(z) - \frac{1}{z^2} = \sum_{\substack{m,n \\ m,n\neq 0}} \frac{1}{(z-\Omega_{m,n})^2} - \frac{1}{\Omega_{m,n}^2}$
- $\left(\mathcal{P}(z) - \frac{1}{z^2}\right)' = \sum_{\substack{m,n \\ m,n\neq 0}} \frac{-2}{(z-\Omega_{m,n})^3} = 0$
- $\left(\mathcal{P}(z) - \frac{1}{z^2}\right)'' = \sum_{\substack{m,n \\ m,n\neq 0}} \frac{6}{(z-\Omega_{m,n})^4}$
- $\left(\mathcal{P}(z) - \frac{1}{z^2}\right)^{(3)} = \sum_{\substack{m,n \\ m,n\neq 0}} \frac{-24}{(z-\Omega_{m,n})^5} = 0$
- $\left(\mathcal{P}(z) - \frac{1}{z^2}\right)^{(4)} = \sum_{\substack{m,n \\ m,n\neq 0}} \frac{120}{(z-\Omega_{m,n})^6}$

At $z = 0$, these are all "nice sums", we define $g_2 = 60\sum_{\substack{m,n \\ m,n\neq 0}} \Omega_{m,n}^{-4}$ and $g_3 = 140\sum_{\substack{m,n \\ m,n\neq 0}} \Omega_{m,n}^{-6}$ so that,

$$\mathcal{P}(z) = \frac{1}{z^2} + \frac{1}{20}g_2 z^2 + \frac{1}{28}g_3 z^4 + \mathcal{O}(z^6)$$

A knowledge of g_2, g_3 enables one in principle to go back and find the periods ω_1 and ω_2. Now,

$$\mathcal{P}'(z) = \frac{-2}{z^3} + \frac{1}{10}g_2 z + \frac{1}{7}g_3 z^3 + \mathcal{O}(z^5)$$
$$\mathcal{P}'(z)^2 = \frac{4}{z^6} - \frac{5}{2}g_2\frac{1}{z^2} - \frac{4}{7}g_3 + \mathcal{O}(z^2)$$
$$\mathcal{P}(z)^3 = \frac{1}{z^6} + \frac{3}{20}g_2\frac{1}{z^2} + \frac{3}{28}g_3 + \mathcal{O}(z^2)$$

So we find that,

$$\mathcal{P}'(z)^2 - 4\mathcal{P}(z)^3 = -g_2\frac{1}{z^2} - g_3 + \mathcal{O}(z^2)$$
$$\Rightarrow \mathcal{P}'(z)^2 - 4\mathcal{P}(z)^3 + g_2\mathcal{P}(z) + g_3 = \mathcal{O}(z^2)$$

But now all the terms in the left hand side are elliptic, so the right hand side must also be elliptic. Furthermore it has no singularities anywhere in the finite complex plane, therefore it is a constant by Liouville's Theorem. So the right hand side vanishes, and we are left with a non-linear differential equation for $\mathcal{P}(z)$.

$$\mathcal{P}'(z)^2 - 4\mathcal{P}(z)^3 + g_2\mathcal{P}(z) + g_3 = 0 \tag{4.2}$$

Separating variables, and taking the positive square root by convention, we find:

$$\int \frac{1}{\sqrt{4\mathcal{P}^3 - g_2\mathcal{P} - g_3}}\,\mathrm{d}\mathcal{P} = \int \mathrm{d}z = z + \text{const.}$$

We draw parallels with the integral form of arcsin, where now we see that the integration of cubic functions leads to a technology of elliptic functions.

5 Laplace's Method for solving differential equations

$$
\begin{aligned}
f(z) &= \int_{\mathcal{C}} e^{zt} g(t)\,\mathrm{d}t \\
f'(z) &= \int_{\mathcal{C}} e^{zt} \cdot t g(t)\,\mathrm{d}t \\
f''(z) &= \int_{\mathcal{C}} e^{zt} \cdot t^2 g(t)\,\mathrm{d}t
\end{aligned}
$$

Also, we can consider,

$$
\begin{aligned}
z f(z) &= \int_{\mathcal{C}} z e^{zt} g(t)\,\mathrm{d}t \\
&= \int_{\mathcal{C}} \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{zt}\right) g(t)\,\mathrm{d}t \\
&= \left[e^{zt} g(t)\right]_{\text{end points of } \mathcal{C}} - \int_{\mathcal{C}} e^{zt} g'(t)\,\mathrm{d}t
\end{aligned}
$$

Choosing a contour, $\mathcal{C}$, such that the boundary terms vanish (e.g. if $\mathcal{C}$ is closed) simply gives,

$$
z f(z) = -\int_{\mathcal{C}} e^{zt} g'(t)\,\mathrm{d}t
$$

Extending this further and again ensuring that the boundary terms vanish gives,

$$
z^2 f(z) = \int_{\mathcal{C}} e^{zt} g''(t)\,\mathrm{d}t
$$

Substituting expressions like this into differential equations for $f(z)$ can give nice differential equations for $g(t)$.

5.1 Airy's equation

$$
\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} - z f(z) = 0 \tag{5.1}
$$

Using Laplace's method, we find that (5.1) transforms into:

$$
\int_{\mathcal{C}} e^{zt}\left(t^2 g(t) + g'(t)\right)\mathrm{d}t = 0
$$

provided that the boundary term $[e^{zt} g(t)]$ vanishes. So we have turned the second order equation into a first order one,

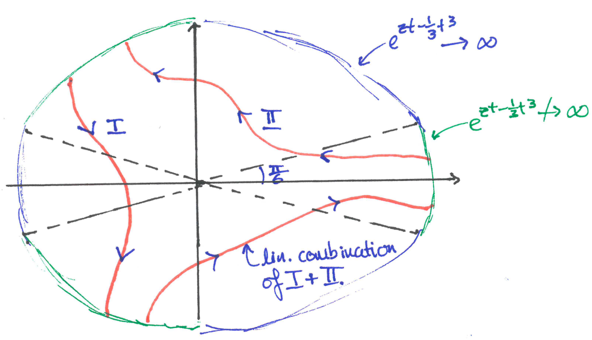


Figure 5.1
The linearly independent solutions of Airy's Equation

$$
\begin{aligned}
g'(t) + t^2 g(t) &= 0 \\
\Rightarrow \log g(t) &= -\frac{1}{3}t^3 + \text{const.} \\
\Rightarrow g(t) &= A e^{-\frac{1}{3}t^3}
\end{aligned}
$$

Thus, we find that,

$$
f(z) = \int_{\mathcal{C}} e^{zt-\frac{1}{3}t^3}\,\mathrm{d}t
$$

provided that $e^{zt-\frac{1}{3}t^3} = 0$ at the end points of $\mathcal{C}$. We can see that we can end the contour at ∞ in any black region, but starting and ending in the same region gives a contour which you can deform to zero. Hence $f(z = 0$.

- Contours that start in one black region and end in another cannot be deformed to zero.
- There are three different contours you can pick. However there should only be 2 linearly independent solutions. Actually, contour III is the same as I minus II. It is convenient to evaluate these integrals by picking a straight line from the origin to infinity in the middle of the region.

In the last case, $t = re^{i\theta}, \theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$, we are left with integrals of the form,

$$\int_0^\infty e^{zre^{i\theta}} e^{i\theta} e^{-\frac{1}{3}r^3 e^{3i\theta}} \, \mathrm{d}r$$

This integral can't be done in terms of elementary functions except when $z = 0$ where putting $\frac{1}{3}t^3 = x$ turns the integral into a Γ-function.

5.2 Confluent Hypergeometric Equation

$$\boxed{z\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} + (c-z)\frac{\mathrm{d}f}{\mathrm{d}z} - af = 0} \tag{5.2}$$

where a and c are in general complex numbers. We could try and find a series solution but Laplace's method is more interesting. We have,

$$\begin{aligned}
f(z) &= \int_{\mathcal{C}} e^{zt} g(t) \, \mathrm{d}t \\
\frac{\mathrm{d}f}{\mathrm{d}z} &= \int_{\mathcal{C}} e^{zt} t g(t) \, \mathrm{d}t \\
z\frac{\mathrm{d}f}{\mathrm{d}z} &= \int_{\mathcal{C}} z e^{zt} t g(t) \, \mathrm{d}t = \left[t g(t) e^{zt} \right]_{\mathcal{C}} - \int_{\mathcal{C}} e^{zt} \frac{\mathrm{d}}{\mathrm{d}t} \left(t g(t) \right) \mathrm{d}t \\
z\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} &= \left[t^2 g(t) \right]_{\mathcal{C}} - \int_{\mathcal{C}} e^{zt} \frac{\mathrm{d}}{\mathrm{d}t} \left(t^2 g(t) \right) \mathrm{d}t
\end{aligned}$$

Substituting this into (5.2) we are left we two conditions, the first is a first order ODE,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(-t^2 g(t) + t g(t) \right) + ((c-z)t - a) g(t) = 0 \tag{5.3}$$

As well as the boundary condition that the additional terms must vanish on the endpoints of $\mathcal{C}$, if there are any,

$$\left[t(t-1) g(t) e^{zt} \right]_{\mathcal{C}} \tag{5.4}$$

The first order transformed differential equation can be separated and then integrated to find,

$$\int \frac{\mathrm{d}g}{g} = \int \frac{(1-a) + (c-z)t}{t(t-1)} \, \mathrm{d}t$$

Using partial fractions we find,

$$\begin{aligned}
\int \frac{\mathrm{d}g}{g} &= \int \left(\frac{a-1}{t} - \frac{c-a-1}{1-t} \right) \mathrm{d}t \\
\log g &= (a-1)\log t + (c-a-1)\log(t-1) + \text{const.} \\
g &= t^{a-1}(1-t)^{c-a-1}
\end{aligned}$$

Thus, using our original ansatz,

$$f(z) = \int_{\mathcal{C}} e^{zt} t^{a-1} (1-t)^{c-a-1} \, \mathrm{d}t$$

where the choice of $\mathcal{C}$ in consistent with (5.4),

$$\left[t^a (1-t)^{c-a} e^{zt} \right]_{\mathcal{C}} = 0$$

The singularities in the complex t-plane are potentially at:

- $t = 0$: a branch point, zero or pole depending on a
- $t = 1$: a branch point, zero or pole depending on $(c - a)$
- $t = \infty$: e^{zt} may present a problem

Near ∞, put $t = \frac{1}{w}$, then $t^{a-1}(1-t)^{c-a-1}\mathrm{d}w$ becomes,

$$-w^{1-a}\left(1 - \frac{1}{w}\right)^{c-a-1}\frac{1}{w^2} = -w^{1-a-2-c+a+1}(w-1)^{c-a-1}\mathrm{d}w$$
$$= -w^{-c}(w-1)^{c-a-1}$$

So we see there is a branch point at ∞ depending on the value of c.

Example. *As an example, we take $c = \frac{5}{2}$ and $a = \frac{3}{2}$, $\Re(z) > 0$. Then the integrand is,*

$$e^{zt}t^{-\frac{5}{2}}$$

and is given by $w^{-\frac{5}{2}}$ at ∞. The boundary condition becomes,

$$\left[e^{zt}t^{\frac{3}{2}}(1-t)\right]_{\mathcal{C}}$$

which we note vanishes at $t = 0, 1$. We include some possible contours, and note that they form an over-complete basis for solutions with $\Re(z) > 0$. Note that we have included a branch cut along the negative real axis, and also that the sum of the two contours from zero is equivalent to the solution through 1, which explains our over-determination. If $\Re(z) < 0$ we may use the following contours,

(a) An overdetermined basis of solutions

(b) If $\Re(z) > 0$, we can use these contours

Example. *Suppose instead $\Re z > 0$, $a < 0$ and $c > 1$ where a, c are both integers. Hence there is a zero at $t = 1$ and a pole of order $(-a+1)$ at $t = 0$. There are no branch points or branch cuts, so possible contours are given by, Other contours will be linear combinations of the two solutions specified by these contours.*

(a) Suppose $z > 0$ and $a < 0$, $c > 1$ are both integers

(b) Suppose neither a nor c are integers

Let us now look at a more difficult example,

Example. *Suppose neither a nor c are integers and again $\Re(z) > 0$. Now $t = 0$ is a branch point. However, $(c - a)$ might or might not be an integer. If $(c - a)$ is not an integer, then,*

1. *$t = 1$ is a branch point*
2. *$t = \infty$ is also a branch point*

The condition that $\Re(z) > 0$ ensures that the ends of the contours lie in the left hand plane. The first choice is obvious, encircling a branch cut on the negative real axis. We have to be slightly cleverer when determining the other linearly independent solution however. Instead, we consider a cut starting at $t = 1$ which is not necessarily a straight line. This seems odd but is not really since you can run branch cuts anywhere you like so long as the function is single valued.

6 Integral Transforms for PDEs

6.1 The Fourier Transform

We use Fourier transform methods for solving the wave equation or the diffusion equation. For the sake of clarity, we recall the definitions, fixing the positions of the 2π factors.

$$\boxed{\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{ikx}\,\mathrm{d}x} \tag{6.1}$$

Whilst the inverse transform is given by,

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k) e^{-ikx}\,\mathrm{d}k} \tag{6.2}$$

We use these first to solve the wave equation in 1D, in exactly the same way as in IB. Consider,

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}, \qquad y = y(x,t) \tag{6.3}$$

Often we wish to solve the initial value problem, where $y(x,0) = f(x)$ and $\frac{\partial y}{\partial t}(x,0) = g(x)$ are specified. Taking the Fourier transform of (6.3) to get,

$$k^2 \tilde{y}(k,t) + \frac{1}{c^2}\frac{\partial^2 \tilde{y}}{\partial t^2} = 0 \tag{6.4}$$

Subject to $\tilde{y}(k,0) = \tilde{f}(k)$ and $\frac{\partial \tilde{y}}{\partial t}(k,0) = \tilde{g}(k)$, where $\tilde{f}$ and $\tilde{g}$ are simply the Fourier transforms of f and g. The general solution to (6.4) is of the form,

$$\tilde{y}(k,t) = A(k)e^{ikct} + B(k)e^{-ikct}$$

Introducing the boundary conditions we find that,

$$\tilde{y}(k,t) = \frac{1}{2}\left[\left(\tilde{f}(k) + \frac{\tilde{g}(k)}{ikc}\right)e^{ikct} + \left(\tilde{f}(k) - \frac{\tilde{g}(k)}{ikc}\right)e^{-ikct}\right]$$

Taking the inverse Fourier Transform, we see that $y(x,t)$ is given by,

$$y(x,t) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\left[\left(\tilde{f}(k) + \frac{\tilde{g}(k)}{ikc}\right)e^{-ik(x-ct)} + \left(\tilde{f}(k) - \frac{\tilde{g}(k)}{ikc}\right)e^{-ik(x+ct)}\right]\mathrm{d}k$$

Introducing the definition of $\tilde{f}$ and $\tilde{g}$ directly, and integrating, we find that,

$$y(x,t) = \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(w)\,\mathrm{d}w \tag{6.5}$$

6.2 Laplace Transforms

Suppose we have a function $f(t)$ defined for $t > 0$, then $\hat{f}(p)$ is the Laplace transform defined by,

$$\boxed{\hat{f}(p) = \int_{0}^{\infty} e^{-pt} f(t)\,\mathrm{d}t} \tag{6.6}$$

Whilst the inverse transform is defined by,

$$\boxed{f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt}\hat{f}(p)\,\mathrm{d}p} \tag{6.7}$$

Where in this case, the contour runs to the left of all singularities of $\hat{f}(p)$ as illustrated. We now turn our attention to the diffusion equation,

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} \tag{6.8}$$

in some suitable units. Now we can Laplace transform (6.8) to get useful results. A physical setting for the problem is the temperature, y, in a semi-infinite bar, which is initially at a temperature y_0, along with the condition that $y(x = 0) = 0 \forall t$. Taking this approach and referring back to IB Complex Methods for the precise form of the Laplace transforms we find that,

$$-y_0 + p\hat{y}(x,p) = \frac{\partial^2 \hat{y}}{\partial x^2}$$

Solving this, we obtain,

$$\hat{y}(x,p) = A(p)e^{x\sqrt{p}} + B(p)e^{-x\sqrt{p}} + \frac{y_0}{p}$$

We note that some care must be taken to define $\sqrt{p}$, in particular which branch to use. For p real and positive, we want $\sqrt{p}$ to be real and positive, then the boundary condition as $x \to \infty$ gives us that $A(p) = 0$. Thus,

$$\hat{y}(x,p) = B(p)e^{-x\sqrt{p}} + \frac{y_0}{p}$$

At $x = 0$, $y(0,t) = 0 \forall t > 0$ so $\hat{y}(0,p) = 0$. This gives us that $B(p) = -\frac{y_0}{p}$. Thus, $\hat{y}(x,p) = \frac{y_0}{p}\left(1 - e^{-x\sqrt{p}}\right)$. Now we look to find $y(x,t)$ by taking the inverse transform. We take $\pi < \operatorname{Arg} p < 3\pi$ so as to give the correct sign on top of the negative real axis. We see that the integrand has a singularity, and a branch point, at $p = 0$, as well as a branch point at infinity. Consider,

$$y(x,t) = y_0 \int_{\mathcal{C}} e^{pt}\frac{1}{p}\left(1 - e^{-\sqrt{p}x}\right) \mathrm{d}p$$

Where $\mathcal{C}$ is the contour illustrated. Now,

- On II: $p = \epsilon e^{i\theta}$, then as $\epsilon \to 0$, the integral becomes (up to some constant),

$$\int_{\pi}^{3\pi} \epsilon e^{i\theta}\epsilon\frac{1}{\epsilon e^{i\theta}}\left(1 - 1 + x\sqrt{\epsilon}e^{\frac{i\theta}{2}} + \dots\right)$$

  So II$\to 0$ as $\epsilon \to 0$.

- On III: $p = r^2 e^{3i\pi}$
- On I: $p = r^2 e^{i\pi}$

Putting this together we find,

$$y(x,t) = \frac{y_0}{2i\pi}\left[\int_{\infty}^{0} \frac{2r}{r^2}e^{-r^2 t}\left(1 - e^{irx}\right)\mathrm{d}r + \int_0^{\infty}\frac{2r}{r^2}e^{-r^2 t}\left(1 - e^{-irx}\right)\mathrm{d}r\right]$$

This, we find that,

$$y(x,t) = \frac{2y_0}{\pi}\int_0^{\infty}\frac{\sin xr}{r}e^{-r^2 t}\,\mathrm{d}r \tag{6.9}$$

Looking closer at (6.9), we see that the singularity at $r = 0$ is removable and the integral is finite and positive if $y_0 > 0$. This means that we picked the right branch.

6.3 Fourier transforms for forced oscillators

Take a response function $y(t)$ that satisfies the forced differential equation,

$$\boxed{\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + A\frac{\mathrm{d}y}{\mathrm{d}t} + By = f(t)} \tag{6.10}$$

In general we'll consider problems where $y(t) = \dot{y}(t) = f(t) = 0$ for $t < 0$. In this case, this is a linear response equation. TO gain more insight, we take the Fourier transform of (**??**) to find that,

$$-\omega^2\tilde{y}(\omega) - i\omega A\tilde{y}(\omega) + B\tilde{y}(\omega) = \tilde{f}(\omega) \tag{6.11}$$

Suppose that as $t \to -\infty$, $y(t) = \dot{y}(t) = 0$, then the idea of causality is that for $t < 0$, $y(t) = 0$. As it turns out, we can relate this causality to stability. Recall from IB Methods where we defined the *response function*,

$$\tilde{y}(\omega) = R(\omega)\tilde{f}(\omega)$$

Putting this into (6.11), we find that,

$$R(\omega) = \frac{1}{-\omega^2 - i\omega A + B} := \frac{-1}{(\omega-\omega_1)(\omega-\omega_2)}$$

In other words, R has two poles (we suppose they are distinct). Then, inverting the Fourier transform we find,

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\frac{-\tilde{f}(\omega)}{(\omega-\omega_1)(\omega-\omega_2)}\,\mathrm{d}\omega$$

Suppose $t < 0$, then we close in the top half plane, since then $e^{-i\omega t} \to 0$ as $\omega \to \infty$. Then,

$$y(t) = \frac{1}{2\pi}\int_{\mathcal{C}} e^{-i\omega t}\frac{-\tilde{f}(\omega)}{(\omega-\omega_1)(\omega-\omega_2)}\,\mathrm{d}\omega = i\sum_{\omega_i\in\mathcal{C}}\operatorname*{res}_{\omega=\omega_i}\left(e^{-i\omega t}\frac{-\tilde{f}(\omega)}{(\omega-\omega_1)(\omega-\omega_2)}\right)$$

A sufficient condition that $y(t) = 0$ for $t < 0$ is that there are no poles in the response function in the top half plane (we assume that $\tilde{f}(\omega)$ has no poles in the top half plane. So the response function can only have poles in the lower half plane as a consequence of causality. We now evaluate by closing in the lower half plane, noting that the residues are given by,

$$\left(\frac{1}{\omega_1-\omega_2}\tilde{f}(\omega_1)e^{-i\omega_1 t}\right), \qquad \left(\frac{1}{\omega_2-\omega_1}\tilde{f}(\omega_2)e^{-i\omega_2 t}\right)$$

We now see that if $e^{-i\omega_j t}$ blows up as a function of increasing t then there is an instability in the solution. But $\omega_j = \alpha_j + i\beta_j$ with $\beta_j < 0$. So, $e{-}i\omega_j t = e^{-i\alpha_j t}e^{\beta_j t}$, and there is no instability. This illustrates a deep relation between causality and stability.

7 Second Order Linear Differential Equations in $\mathbb{C}$

We consider the general second order linear differential equation of the form,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + p(z)\frac{\mathrm{d}y}{\mathrm{d}z} + q(z)y(z) = 0 \tag{7.1}$$

Let's suppose that $p(z)$ and $q(z)$ are analytic in the neighbourhood of a point z_0. To see that one can always construct solutions that are analytic in this neighbourhood, consider the following,

1. Eliminate the $\frac{\mathrm{d}y}{\mathrm{d}z}$ term first by letting,

   $$y(z) = u(z)\exp\left(-\frac{1}{2}\int_{z_0}^{z} p(\xi)\,\mathrm{d}\xi\right)$$

   Where the integral in the exponential exists since $p(\xi)$ is analytic in a neighbourhood of z_0. Differentiating twice and canceling off the non-zero exponential term we find that,

   $$u''(z) + \zeta(z)u(z) = 0, \qquad \zeta(z) = q(z) - \frac{1}{4}p^2(z) - \frac{1}{2}p'(z)$$

   Now, ζ is analytic in a neighbourhood of z_0.

2. We need to demonstrate the existence of two independent solutions to this ODE. Consider a series of functions of z, $v_n(z)$ starting with,

   $$v_0(z) = a_0 + a_1(z - z_0)$$

   where a_0 and a_1 are arbitrary complex constants. Now we define,

   $$v_n(z) = \int_{z_0}^{z} (\xi - z)\zeta(\xi)v_{n-1}(\xi)\,\mathrm{d}\xi \tag{7.2}$$

   In a neighbourhood of z_0, $|\zeta(z)| \leq M$ and $|v_0(z)| \leq \mu$ for some positive constants M and μ. Thus,

   $$|v_n(z)| \leq \frac{1}{n!}\mu M^n |z - z_0|^n$$

   Now we put $v(z) = \sum_{n=0}^{\infty} v_n(z)$ to find that,

   $$|v(z)| \leq \sum_{n=0}^{\infty} |v_n(z)| \leq \sum_{n=0}^{\infty} \frac{1}{n!}\mu M^n |z - z_0|^n = \mu e^{M|z - z_0|}$$

   Hence we see that $v(z)$ is convergent. Differentiating (7.2), we find that,

   $$\begin{aligned} v_n'(z) &= -\int_{z_0}^{z} \zeta(\xi)v_{n-1}(\xi)\,\mathrm{d}\xi \\ v_n''(z) &= -\zeta(z)v_{n-1}(z) \\ v''(z) &= \sum_{n=0}^{\infty} v_n''(z) = \sum_{n=1}^{\infty} -\zeta(z)v_{n-1}(z) \end{aligned}$$

   Furthermore, $\zeta(z)v(z) = \sum_{n=0}^{\infty} \zeta(z)v_n(z)$, so we find $v''(z) + \zeta(z)v(z) = 0$

Thus, we have found two linearly independent solutions of the differential equation in the neighbourhood of the point z_0 when $p(z)$ and $q(z)$ are analytic.

- If $p(z), q(z)$ are analytic at $z = z_0$, then this is termed a *regular point*
- Suppose that the only singularities of $p(z)$ and $q(z)$ are poles of order 1 and 2 respectively at $z = z_0$, then $z = z_0$ is a *regular singular point*
- If the singularities are worse then it is an *irregular singular point*

At a regular singular point, $p(z)$ has at most a simple pole, then $(z - z_0)p(z)$ is analytic. If $q(z)$ has at most a pole of order 2, then $(z - z_0)^2 q(z)$ is analytic. Then the aim is to transform all the coefficient functions in (7.1) into things which are analytic. We now take a closer look at regular singular points.

7.1 Regular Singular Points

Suppose that $p(z)$ has singularities that are at most simple poles and $q(z)$ has singularities that are at most poles of order 2. Then Frobenius' method works nicely in the neighbourhood of regular singular points. Suppose $p(z)$ and/or $q(z)$ has a pole(s) at $z = z_0$, then,

- $(z - z_0)p(z)$ is analytic
- $(z - z_0)^2 q(z)$ is analytic

Frobenius' method requires some kind of power series expansion around $z = z_0$,

$$y(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n+\sigma}, \qquad a_0 \neq 0$$

The series then starts at $(z - z_0)^\sigma$, where there is no requirement that σ is an integer. Differentiating we find that,

$$y'(z) = \sum_{n=0}^{\infty} a_n (n+\sigma)(z - z_0)^{n+\sigma-1}$$
$$y''(z) = \sum_{n=0}^{\infty} a_n (n+\sigma)(n+\sigma-1)(z - z_0)^{n+\sigma-2}$$

We put these relations in (7.1) to find that, after multiplying through by $(z - z_0)^2$,

$$\sum_{n=0}^{\infty} a_n (n+\sigma)(n+\sigma-1)(z - z_0)^{n+\sigma} + a_n(z - z_0)p(z)(n+\sigma)(z - z_0)^{n+\sigma} + a_n(z - z_0)^2 q(z)(z - z_0)^{n+\sigma} = 0 \quad (7.3)$$

Where we see that the second and third coefficients involve $(z - z_0)p(z)$ and $(z - z_0)^2 q(z)$ which are both analytic.

7.2 The Indicial Equation

At this point, we look at $n = 0$ to find the indicial equation. Taking out a factor of $(z - z_0)^\sigma$ and noting that $a_0 \neq 0$, we find that,

$$\sigma(\sigma - 1) + \sigma p_0 + q_0 = 0 \quad (7.4)$$

Where $(z - z_0)p(z) = p_0 + (z - z_0)p_1 + \ldots$ and $(z - z_0)^2 q(z) = q_0 + (z - z_0)q_1 + \ldots$. This is clearly a quadratic equation in σ, so there are two roots, we look at the possible scenarios.

(a) The two roots are distinct and do not differ by integers. Then equating equal powers will give two independent recursion relations for the a_n. Then we take (7.3), expand $p(z), q(z)$ around $z = z_0$ and equate powers to find the recurrence relation for each σ.

(b) The two roots are distinct and separated by integers. In this case we may or may not get two different recursion relations, or perhaps once works and the other is nonsensical. We are guaranteed to find one solution though, $y_1(z)$, say. Then the second solution may be found from the Wronskian,

$$\mathcal{W}(z) = y_1(z)y_2'(z) - y_2(z)y_1'(z)$$

Where $\mathcal{W}$ obeys the first order equation,

$$\mathcal{W}'(z) = -p(z)\mathcal{W}(z)$$

The resultant solution y_2 will likely involve a term $\log(z - z_0)$.

(c) If the roots coincide, one solution will be given as a power series, as above, however the other may only be found by using the Wronskian, and will involve $\log(z - z_0)$.

Now, if $(z-z_0)p(z)$ or $(z-z_0)^2q(z)$ are not analytic, then a semi-infinite power series won't exist. We now ask about the point at infinity. Letting $u = \frac{1}{z}$ as usual, we investigate the behaviour as $u \to 0$. Transforming (7.1) using $\frac{\mathrm{d}}{\mathrm{d}z} = -u^2\frac{\mathrm{d}}{\mathrm{d}u}$ and $\frac{\mathrm{d}^2}{\mathrm{d}z^2} = 2u^3\frac{\mathrm{d}}{\mathrm{d}u} + u^4\frac{\mathrm{d}^2}{\mathrm{d}u^2}$ gives us,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}u^2} + \left(\frac{2}{u} - \frac{p}{u^2}\right)\frac{\mathrm{d}y}{\mathrm{d}u} + \frac{q}{u^4}y = 0 \tag{7.5}$$

In order that $\left(\frac{2}{u} - \frac{p}{u^2}\right)$ has at most a simple pole at $u = 0$ and $\frac{q}{u^4}$ has at most a pole of order 2 at $u = 0$, we see that,

- q must vanish as $u \to 0$ as u^2 or faster, thus,

$$q \approx u^p, \quad p \geq 2$$

- Also, in order that $\frac{2}{u} - \frac{p}{u^2} = \frac{A}{u} + \ldots$, we see that we require,

$$p = ku + \mathcal{O}(u^2)$$

  as $u \to 0$. If p, q fail to meet these conditions then it is an irregular singular point.

Example. *Consider the following second-order equation,*

$$\frac{d^2y}{dz^2} + \frac{\gamma}{z}\frac{dy}{dz} + \frac{\delta}{z^2}y = 0 \tag{7.6}$$

We see that $p(z) := \frac{\gamma}{z}$ has a simple pole at $z = 0$, whilst $q(z) := \frac{\delta}{z^2}$ has a double pole there. Hence, $z = 0$ is a regular singular point. For $z \neq 0$, in the finite complex plane $p(z)$ and $q(z)$ are analytic. Furthermore, infinity is a regular singular point, in which case,

$$p(u) = \gamma u, \quad q(u) = \delta u^2$$

We expect therefore to find a power series expansion about $u = 0$. The solutions are $y = z^p$, where,

$$z^{p-2}\left(p(p-1) + \gamma p + \delta\right) = 0$$

Giving us,

$$p = \frac{1 - \gamma \pm \sqrt{\gamma^2 - 2\gamma + 1 - 4\delta}}{2}$$

If the roots are distinct then we have two independent solutions, however suppose the roots coincide, then,

$$\delta = \frac{1}{4}(\gamma - 1)^2$$

This corresponds to the solutions,

$$y_1(z) = z^{\frac{1-\gamma}{2}}, \quad y_2(z) = z^{\frac{1-\gamma}{2}}\log z$$

where we notice that the logarithmic singularity occurs.

We clarify the conditions for infinity to be an ordinary point, we require both of the following to be bounded at ∞,

$$z^4 q(z), \quad 2z - z^2 p(z)$$

Example. *Consider,*

$$\frac{d^2y}{dz^2} - y = 0$$

Here, we see the solutions are $y = e^z, e^{-z}$, and that $z = \infty$ is an irregular singular point (whilst all other points are ordinary). Examining the solutions, we see they both have essential singularities at ∞. This is a general principle, and indeed typically Frobenius' method will fail.

7.3 Classification of Regular Singular Points

We now look for second order linear ordinary differential equations with N regular singular points, with all other points being ordinary.

Zero Regular Singular Points

Are there any second order ODEs with only regular points?, Consider,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + P(z)\frac{\mathrm{d}y}{\mathrm{d}z} + Q(z)y = 0$$

where $P(z) = \alpha_0 + \alpha_1 z + \ldots$ and $Q(z) = \beta_0 + \beta_1 z + \ldots$. But at ∞ for it to be a regular point, we need $z^4 Q(z)$ to be bounded, but this simply implies $Q(z) = 0$, so we are left with a first order equation.

One Regular Singular Point

Suppose we want a regular singular point at $z = z_1$, then define,

$$P(z) = \frac{\alpha(z)}{(z - z_1)}, \quad Q(z) = \frac{\beta(z)}{(z - z_1)^2}$$

where $\alpha(z) = \alpha_0 + \alpha_1(z - z_1) + \ldots$, and $\beta(z) = \beta_0 + \beta_1(z - z_1) + \ldots$. Again, in order that ∞ is an ordinary point, require that,

$$z^4 Q(z) = \frac{z^4 \beta(z)}{(z - z_1)^2}$$

is bounded, but again, this gives us that $\beta(z) = 0$, so once again we are left with a first order equation.

Two Regular Singular Points

Suppose we have two regular singular points at $z = z_1$ and $z = z_2$. Then,

$$P(z) = \frac{\alpha(z)}{z - z_1} + \frac{\beta(z)}{z - z_2}, \quad Q(z) = \frac{\gamma(z)}{(z - z_1)^2 (z - z_2)^2}$$

Where α, β, γ are analytic at $z = z_1, z_2$. Now as that $z = \infty$ is a regular point, requiring that $z^4 Q(z)$ is bounded. This condition tells us that $\gamma(z)$ is constant. Suppose $\gamma(z) = \gamma_0$, then ask that $2z - z^2 P(z)$ is bounded as $z \to \infty$. So,

$$2z - z^2 \left(\frac{\alpha(z)}{z - z_1} + \frac{\beta(z)}{z - z_2} \right)$$

must be bounded for large z. Since $\alpha(z) = \alpha_0 + \alpha_1(z - z_1) + \ldots$ and $\beta(z) = \beta_0 + \beta_1(z - z_2) + \ldots$, we require $\alpha_i = 0$ for all $i \geq 1$, as well as,

$$\alpha_0 + \beta_0 = 2$$

Thus, we find that the only class of differential equations with only two singular points is,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + \left(\frac{\alpha_0}{z - z_1} + \frac{\beta_0}{z - z_2} \right) \frac{\mathrm{d}y}{\mathrm{d}z} + \frac{\gamma_0}{(z - z_1)^2 (z - z_2)^2}, \quad \alpha_0 + \beta_0 = 1 \tag{7.7}$$

Consider the group $\mathrm{SL}_2(\mathbb{C})$, and its action as the Mobius Group which takes $z \mapsto w = \frac{az+b}{cz+d}$ with $ad - bc = 1$. This group can be used to map any two points z_1, z_2 to 0 and ∞. Such a transformation is,

$$w = \frac{z - z_1}{z - z_2}$$

This isn't quite a Mobius transformation but it will suffice. So,

$$z = \frac{w z_2 - z_1}{w - 1}$$

Now we transform the equation above using $\frac{\mathrm{d}y}{\mathrm{d}z} = \frac{z_1 - z_2}{(z - z_2)^2}$ etc. as well as the fact that $\alpha_0 + \beta_0 = 2$, to find that it is equivalent to,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}w^2} - \frac{\alpha_0}{w} \frac{\mathrm{d}y}{\mathrm{d}w} + \frac{\gamma_0}{w^2} y = 0 \tag{7.8}$$

This equation has a regular singular point at $w = 0$ and at $w = \infty$. The solutions are $y = w^p$, where p is found by substituting in $p(p-1) - \alpha p + \gamma = 0$. Or in terms of z,

$$y(z) = \left(\frac{z - z_1}{z - z^2} \right)^p, \qquad \left(\frac{z - z_1}{z - z^2} \right) \log \left(\frac{z - z_1}{z - z^2} \right)$$

1. Solutions to equations with two regular singular points are just powers (or powers with a logarithm)
2. If you use Frobenius' method, then $y(z)$ is of the form.

$$y(z) = \sum_{n=0}^{\infty} a_n (z - z + 0)^{n+\sigma}$$

3. Under the Mobius transformation, the roots of the indicial equation remain unchanged. This is an important observation.
4. Frobenius' method applied to $y(w)$ (as opposed to $y(z)$ is trivial in the sense that the recursion relation is uninteresting.

Three Regular Singular Points

We now suppose we have regular singular points at z_1, z_2, z_3 but not at ∞. So,

$$\begin{aligned} P(z) &= \frac{A}{z - z_1} + \frac{B}{z - z_2} + \frac{C}{z_3} \\ Q(z) &= \frac{1}{(z - z_1)(z - z_2)(z - z_3)} \left(\frac{D}{z - z_1} + \frac{E}{z - z_2} + \frac{F}{z - z_3} \right) \end{aligned}$$

The fact that there are no singularities at ∞ implies that $A, B, \ldots, F =$ const.. Now suppose the roots of the indicial equation are,

- At $z = z_1$: α_1, α_2
- At $z = z_2$: β_1, β_2
- At $z = z_3$: γ_1, γ_2

Then we find that,

$$\begin{aligned} A &= 1 - \alpha_1 - \alpha_2 \\ D &= \alpha_1 \alpha_2 (z_1 - z_2)(z_1 - z_3) \\ B &= 1 - \beta_1 - \beta_2 \\ E &= \beta_1 \beta_2 (z_2 - z_1)(z_2 - z_3) \\ C &= 1 - \gamma_1 - \gamma_2 \\ F &= \gamma_1 \gamma_2 (z_3 - z_1)(z_3 - z_2) \end{aligned}$$

We require $z^2 P(z) - 2z$ to be bounded at ∞ which implies that,

$$A + B + C = 2$$

So only 5 of the roots of the indicial equation are independent and,

$$\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1 + \gamma_2 = 1 \tag{7.9}$$

8 The Papperitz Symbol and the Hypergeometric Equation

We now define a structure called the Papperitz symbol, which contains all the information about the differential equation except the dependence of the roots of the indicial equation given in (7.9).

$$\mathcal{P}\left\{\begin{array}{cccc} z_1 & z_2 & z_3 & \\ \alpha_1 & \beta_1 & \gamma_1 & z \\ \alpha_2 & \beta_2 & \gamma_2 & \end{array}\right\}$$

Now, we make a Mobius transformation on z which take $z_1 \mapsto 0$,$z_2 \mapsto 1$ and $z_3 \mapsto \infty$. This is achieved by,

$$w = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \tag{8.1}$$

Note that this does *not* change the roots of the indicial equation. Then the Papperitz symbol, which represents a solution to the equation, is transformed into,

$$y = \mathcal{P}\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ \alpha_1 & \beta_1 & \gamma_1 & z \\ \alpha_2 & \beta_2 & \gamma_2 & \end{array}\right\}$$

Since Mobius transformations are invertible, we can always go back to $\{z_1, z_2, z_3\}$.

Now consider expanding $y(z)$ about $z = 0$. We find that,

$$y(z) = z^{\alpha_1}\sum_{n=0}^{\infty} a_n z^n \tag{8.2}$$

Similarly expanding around $z = 1$,

$$y(z) = (z-1)^{\beta_1}\sum_{n=0}^{\infty} b_n (z-1)^n \tag{8.3}$$

Where (8.2),(8.3) arise because α_1, β_1 are roots of the indicial equation at $z = 0, 1$. We make the followwing substitution,

$$y(z) = z^{\alpha_1}(z-1)^{\beta_1}\hat{y}(z)$$

This tells us that, near $z = 0, 1$, $\hat{y}$ must look like,

$$\sum_{n=0}^{\infty} a_n z^n, \qquad \sum_{n=0}^{\infty} b_n (z-1)^n$$

Thus, at $z = 0$, the roots of the indicial equation must be,

$$\alpha_1 - \alpha_1 (= 0), \quad (\alpha_2 - \alpha_1)$$

And at $z = 1$, the roots become,

$$0, \quad (\beta_2 - \beta_1)$$

As $z \to \infty$, $y(z) \sim z^{\alpha_1+\beta_1}\hat{y}(z)$. Thus this changes the roots at infinity by sending,

$$\gamma_1 \mapsto \gamma_1 - \alpha_1 - \beta_1, \quad \gamma_2 \mapsto \gamma_2 - \alpha_2 - \beta_2$$

So, without actually calculating anything, we conclude that $\hat{y}$ must obey,

$$\hat{y} = \mathcal{P}\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & \gamma_1 - \alpha_1 - \beta_1 & z \\ \alpha_2 - \alpha_1 & \beta_2 - \beta_1 & \gamma_2 - \alpha_2 - \beta_2 & \end{array}\right\} \tag{8.4}$$

Using the condition in (7.9), we can take the 6 parameters $\{\alpha_1, \ldots, \gamma_1, \gamma_2\}$ to 3 parameters $\{a, b, c\}$, and write this in a more convenient form.

$$\mathcal{P}\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{array}\right\} \tag{8.5}$$

So any 2^{nd} order linear DE with 3 regular singular points z_1, z_2, z_3 and roots of the indicial equation $\alpha_1, \ldots, \gamma_2$ can always be transformed into (8.5) by a Mobius transformation and taking out a factor of $z^{\alpha_1}(z-1)^{\beta_1}$.

As it turns out, pretty much every elementary function that you have encountered is in some sense solutions to this equation. For example $z^\alpha, \exp(z), \log z, J_\nu(z)$ etc. In a more classical form, (8.5) represents,

$$\boxed{z(1-z)\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + (c-(a+b+1)z)\frac{\mathrm{d}y}{\mathrm{d}z} - aby = 0} \tag{8.6}$$

This is the *Hypergeometric Equation* mentioned at the start of the chapter. We can use the Frobenius method and expand about $z=0$, where the roots of the indicial equation are $0, (1-c)$. We know there is a solution,

$$y = \sum_{n=0}^{\infty} a_n z^n$$

Then, looking at (8.6) we will always form a two-term recurrence relationship for the $\{a_n\}$. Solving this we find canonically the *hypergeometric series*,

$$\begin{aligned} y = 1 + \frac{ab}{c}z + \frac{a(a+1)b(b+1)}{2c(c+1)}z^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{6c(c+1)(c+2)}z^3 + \ldots \\ + \frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)}{n!c(c+1)\cdots(c+n-1)}z^n + \ldots \end{aligned} \tag{8.7}$$

We can rewrite this in terms of the Γ-function using,

$$a(a+1)\cdots(a+n-1) = \frac{(a+n-1)!}{(a-1)!} = \frac{\Gamma(a+n)}{\Gamma(a)} \quad \text{for } a \in \mathbb{Z}$$

Then, the hypergeometric series becomes the following, which converges in $|z|<1$

$$y(z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)} \cdot \frac{z^n}{n!}$$

But this is only one of the two possible solutions found by expanding about $z=0$. We usually define the hypergeometric series,

$$\boxed{F(a,b,c;z) := \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)} \cdot \frac{z^n}{n!}} \tag{8.8}$$

We can immediately note a few things about F, firstly it is symmetric in its first two arguments, $F(a,b,c;z) = F(b,a,c;z)$. Also, differentiating (8.8),

$$\begin{aligned} \frac{\mathrm{d}F(a,b,c;z)}{\mathrm{d}z} &= \sum_{n=1}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)} \cdot \frac{z^{n-1}}{(n-1)!} \\ &= \sum_{n=0}^{\infty} \frac{\Gamma((a+1)+n)\Gamma((b+1)+n) \cdot \frac{1}{c}\Gamma(c+1)}{\frac{1}{a}\Gamma(a+1) \cdot \frac{1}{b}\Gamma(b+1)\Gamma((c+1)+n)} \cdot \frac{z^n}{n!} \end{aligned}$$

So we find that,

$$\frac{\mathrm{d}F(a,b,c;z)}{\mathrm{d}z} = \frac{ab}{c}F(a+1,b+1,c+1;z) \tag{8.9}$$

If we look more closely at the hypergeometric series, we see that it is not defined for $c = 0, -1, -2, \ldots$ because of the poles in $\Gamma(c)$. Further is a or b is zero or a negative integer, the series will terminate after n terms.

8.1 Integral representation of the hypergeometric series

The first approach to this problem is due to Gauss and requires the properties of the Beta function. Recall that,

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \int_0^1 t^{p-1}(1-t)^{q-1}\,\mathrm{d}t, \quad \Re(p), \Re(q) > 0$$

Now consider,

$$\frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)} = \frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(b+n+c-b)}$$

But this is simply given by,

$$\frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)} = \int_0^1 t^{b+n-1}(1-t)^{c-b-1}\,\mathrm{d}t$$

where $\Re(c) > \Re(b) > 0$ for convergence. Then,

$$F(a,b,c;z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)\Gamma(b)} \cdot \frac{\Gamma(c)}{\Gamma(c-b)} \left[\int_0^1 t^{b+n-1}(1-t)^{c-b-1}\,\mathrm{d}t\right] \frac{z^n}{n!}$$

But, $\sum_{n=0}^{\infty} \frac{z^n}{n!}\frac{\Gamma(a+n)}{\Gamma(a)} = 1 + az + \frac{1}{2}a(a+1)\frac{z^2}{2!} + \ldots$, so,

$$\sum_{n=0}^{\infty} \frac{(zt)^n}{n!}\frac{\Gamma(a+n)}{\Gamma(a)} = 1 + atz + \frac{1}{2}a(a+1)\frac{t^2z^2}{2!} + \ldots = (1-tz)^{-a}$$

So, performing the sum over n inside the integral, we find the *Gauss Representation* of the hypergeometric series,

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,\mathrm{d}t \tag{8.10}$$

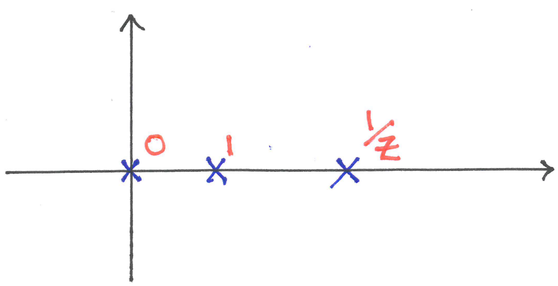


Figure 8.1
Arrange the cuts according to the problem in hand

In general there will be branch points at $t = 0, 1, \infty, \frac{1}{z}$, so we must arrange the cuts according to the problem in hand.

As long as the integral does not diverge we can always evaluate it by integrating along the real line between 0 and 1. There are potential singularities if z is real and greater than 1. This integral provides an analytic continuation of the hypergeometric series for all z except possibly on the real axis between 1 and ∞. Now we already know that,

$$F(a,b,c;0) = 1$$

But we can now compute $F(a,b,c;1)$ which is given by,

$$\begin{aligned} F(a,b,c;1) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-a-1}\,\mathrm{d}t \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \frac{\Gamma(b)\Gamma(c-b-a)}{\Gamma(c-a)} \end{aligned}$$

Where in the second line we have used the fact that the integral is in the form of a beta function. So,

$$F(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)} \tag{8.11}$$

This is possibly singular because of poles in the Γ-function, but we might expect this as the series is only guaranteed to converge for $|z| < 1$.

8.2 Barnes Representation of $F(a,b,c;z)$

If we recall the inverse Laplace transform, consider the following expression,

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} (-z)^t \frac{\Gamma(a+t)\Gamma(b+t)\Gamma(-t)}{\Gamma(c+t)}\,\mathrm{d}t \tag{8.12}$$

There are no branch points in the complex t-plane. However there are poles in $\Gamma(a+t), \ldots$

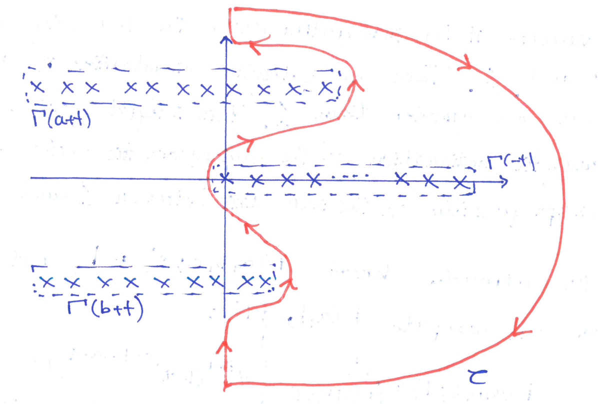


Figure 8.2
Closing the contour properly

The contour must run to the left of all the poles of $\Gamma(-t)$ and to the right of all the poles of $\Gamma(a+t)$ and $\Gamma(b+t)$. It is natural to close to the right as this gives a sum of the form $\sum_{n=0}^{\infty} \operatorname{res}(\cdot)$ and will induce terms such as $\Gamma(a+n)$ etc. Closing the contour to the right works for $|z|<1$. We now calculate the residues, recalling the reflection formula in (3.5). Then,

$$\lim_{z\to n}(z+n)\Gamma(-z) = \lim_{z\to n}\left\{\frac{(z-n)}{\sin\pi z}\frac{-\pi}{\Gamma(1+z)}\right\}$$
$$= \frac{1}{\pi\cos\pi n}\frac{-\pi}{\Gamma(1+n)} = -(-1)^n\frac{1}{n!}$$

So the Barnes integral becomes,

$$2\pi i\cdot\frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}z^n(-1)^n\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}$$
$$= F(a,b,c;z)\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} \tag{8.13}$$

Then, we have the following,

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}(-z)^t\frac{\Gamma(a+t)\Gamma(b+t)\Gamma(-t)}{\Gamma(c+t)}\,\mathrm{d}t = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}F(a,b,c;z) \tag{8.14}$$

Now suppose that c is a negative integer. Then the hypergeometric series must still solve the hypergeometric equation since it is linear. So consider,

$$\lim_{c\to -m}\frac{1}{\Gamma(c)}F(a,b,c;z) = \lim_{c\to -m}\left\{\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(a)\Gamma(b)}\right\}$$

The term $\Gamma(c+n)$ has poles for $n=0,1,2,\dots$ i.e. $c=-m,-m+1,\dots$. So the first term without a poles is at $n=m+1$. Start instead at $n=m+1$ since the first $(m+1)$ terms will give zero.

$$\begin{aligned}
\lim_{c\to -m}\frac{1}{\Gamma(c)}F(a,b,c;z) &= \sum_{n=m+1}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(a)\Gamma(b)}\cdot\frac{z^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{\Gamma(a+n+m+1)\Gamma(b+n+m+1)}{\Gamma(c+n+m+1)\Gamma(a)\Gamma(b)}\cdot\frac{z^{n+m+1}}{(n+m+1)!}\\
&= z^{m+1}\sum_{n=0}^{\infty}\frac{\Gamma(a+n+m+1)\Gamma(b+n+m+1)n!}{\Gamma(c+n+m+1)\Gamma(a)\Gamma(b)(m+n+1)!}\cdot\frac{z^n}{n!}\\
&= z^{m+1}\sum_{n=0}^{\infty}\frac{\Gamma(a+m+1+n)\Gamma(b+m+1+n)n!}{\Gamma(n+1)\Gamma(n+m+2)\Gamma(a)\Gamma(b)}\cdot\frac{z^n}{n!}\\
&= z^{m+1}\frac{\Gamma(a+m+1)\Gamma(b+m+1)}{(m+1)!\Gamma(a)\Gamma(b)}F(a+m+1,b+m+1,m+2;z)
\end{aligned}$$

Where in the second to last line we have used the fact that $c=-m$. Now this expression solves the hypergeometric equation even for $c=0,-1,-2,\dots$.

Examples of the hypergeometric function

1. $F(1,1,2;z) = -\frac{1}{z}\log(1-z)$
2. $F(\frac{1}{2},1,\frac{3}{2};z) = \frac{1}{2z}\log\left(\frac{1+z}{1-z}\right)$
3. $F(\frac{1}{2},1,\frac{3}{2};-z^2) = \frac{1}{z}\arctan z$
4. $F(-a,a,\frac{1}{2},\sin^2 z) = \cos 2\pi z$

8.3 A second solution to the hypergeometric equation

We start this section by studying the Papperitz symbol,

$$\mathcal{P}\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{array}\right\}$$

Suppose c is not an integer then we see from the Papperitz symbol that there is a solution that starts with z^{1-c}, so consider,

$$y = z^{1-c}\hat{y}$$

and see what differential equation $\hat{y}$ satisfies. The effect is to move the roots of the indicial equation by $(1-c)$. Note however that this does not affect the roots at $z = 1$. Then the Papperitz symbol becomes,

$$\begin{aligned}\hat{y} = \mathcal{P}&\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ c-1 & 0 & a-c+1 & z \\ 0 & c-a-b & b-c+1 & \end{array}\right\}\\ = \mathcal{P}&\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & a-c+1 & z \\ c-1 & c-a-b & b-c+1 & \end{array}\right\}\\ = \mathcal{P}&\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & a' & z \\ 1-c' & c'-a'-b' & b' & \end{array}\right\}\end{aligned}$$

for some a', b', c'. Comparing to the line above, we see that,

$$c' = 2-c, \quad a' = a-c+1, \quad b' = b-c+1$$

So this new Papperitz symbol has a solution $F(a', b', c'; z)$. Thus if one multiplies this by z^{1-c}, it must satisfy the original hypergeometric equation. Thus the two linearly independent solutions are,

$$y_1 = F(a, b, c; z), \qquad y_2 = z^{1-c}F(a-c+1, b-c+1, 2-c; z) \tag{8.15}$$

If instead c is an integer, you will have to find the second solution using the Wronskian, which satisfies,

$$\mathcal{W} = y_1 y_2' - y_2 y_1', \quad \mathcal{W}' = -p\mathcal{W}$$

where $y'' + py' + qy = 0$. This leads to a logarithmic singularity at $z = 0$. Now we return our attention to the original form of the Papperitz symbol,

$$\mathcal{P}\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{array}\right\}$$

In a similar vein to before, there must be a solution that starts $(1-z)^{c-a-b}$. So performing the same trick, let,

$$y = (1-z)^{c-a-b}\tilde{y}$$

This changes the roots at $z = 1, \infty$,

$$\begin{aligned}\tilde{y} = \mathcal{P}&\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & a+b-c & c-b & z \\ 1-c & 0 & c-a & \end{array}\right\} = \mathcal{P}\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & a+b-c & c-a & z \\ 1-c & 0 & c-b & \end{array}\right\}\\ = \mathcal{P}&\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & a' & z \\ 1-c' & c'-a'-b' & b' & \end{array}\right\}\end{aligned}$$

So, identifying a', b', c',

$$a' = c-a, \quad b' = c-b, \quad c' = c$$

Then, applying the same transformation on y_2, we find two more solutions to the hypergeometric equation,

$$y_3 = (1-z)^{c-a-b}F(c-a, c-b, c; z), \qquad y_4 = (1-z)^{c-a-b}z^{1-c}F(1-a, 1-b, 2-c; z) \tag{8.16}$$

So y_3 and y_4 are two linearly independent solutions of the hypergeometric equation. Now y_1 must be a linear combination of y_3 and y_4,

$$y_1 = \alpha y_3 + \beta y_4 \tag{8.17}$$

Expanding around $z = 0$,

$$y_1 = 1 + \frac{ab}{c}z + \ldots = \alpha\{1 - (c-a-b)z + \ldots\}\left\{1 + \frac{(c-a)(c-b)}{c}z + \ldots\right\} + \beta z^{1-c}\{1 + \ldots\} \tag{8.18}$$

But c is not an integer, so we find that $\alpha = 1, \beta = 0$. So we obtain the following identity,

$$F(a, b, c; z) = (1-z)^{c-a-b}F(c-a, c-b, c; z) \tag{8.19}$$

Recall that the roots of the indicial equation are invariant under Möbius transformations. But there are Möbius transformations that interchange $\{0, 1, \infty\}$. For example, $z \mapsto 1-z$ interchanges $z = 0, 1$. The Papperitz symbol transforms as,

$$\mathcal{P}\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{array}\right\} \mapsto \mathcal{P}\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & a & 1-z \\ c-a-b & 1-c & b & \end{array}\right\}$$

Identifying this with,

$$\mathcal{P}\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & a' & w = 1-z \\ 1-c' & c'-a'-b' & b' & \end{array}\right\}$$

Gives us,

$$a' = a, \quad b' = b, \quad c' = 1 + a + b - c$$

This leads to two linearly independent solutions,

$$y_5 = F(a, b, 1+a+b-c; 1-z), \qquad (1-z)^{c-a-b}F(c-b, c-a, 1+c-a-b; 1-z) \tag{8.20}$$

Now, as before, y_1 must be a linear combination of y_5 and y_6,

$$y_1 = \alpha y_5 + \beta y_6$$

We know that $F(a, b, c; 0) = 1$ (so long as c is not a negative integer) and also from the integral representation,

$$F(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \Re(c-a-b) > 0$$

Substituting $z = 0, 1$ into the linear relation above,

$$1 = \alpha \cdot \frac{\Gamma(1+a+b-c)\Gamma(1-c)}{\Gamma(1+b-c)\Gamma(1+a-c)} + \beta \cdot \frac{\Gamma(1+c-a-b)\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)} \tag{8.21}$$

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} = \alpha \tag{8.22}$$

From this we deduce that,

$$\beta = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}$$

Where this second result either follows from solving the equations above, or using the symmetry,

$$a \mapsto c-b, \quad b \mapsto c-a, \quad c \mapsto c$$

which sends $y_5 \mapsto y_6$. Returning to the Papperitz symbol form y_1 and y_2,

$$\mathcal{P}\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{array}\right\}$$

We can apply a Möbius transformation that interchanges 0 and ∞, $z \mapsto \frac{1}{z}$, leaving $z = 1$ invariant. Then swapping 0 and ∞,

$$\mathcal{P}\left\{\begin{matrix} 0 & 1 & \infty & \\ a & 0 & 0 & \frac{1}{z} \\ b & c-a-b & 1-c & \end{matrix}\right\}$$

The solution starts with z^a, so taking out $\left(\frac{1}{z}\right)^a$, we see,

$$z^{-a}\mathcal{P}\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a & \frac{1}{z} \\ b-a & c-a-b & 1+a-c & \end{matrix}\right\} := z^{-a}\mathcal{P}\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a' & \frac{1}{z} \\ 1-c' & c'-a'-b' & b' & \end{matrix}\right\}$$

gives us,

$$a' = a, \quad b' = 1 + a - c, \quad c' = 1 + a - b$$

This, leaves us with one new solution,

$$y_7 = z^{-a} F\left(a, 1+a-c, 1+a-b; \frac{1}{z}\right) \tag{8.23}$$

Or if we had taken out z^{-b} instead,

$$y_8 = z^{-b} F\left(b, 1+b-c, 1+b-a, \frac{1}{z}\right) \tag{8.24}$$

The last example of this type involves swapping 1 and ∞ via $z \mapsto \frac{z}{z-1}$. Operating on the Pappertiz symbol,

$$\begin{aligned} &\mathcal{P}\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{matrix}\right\} \\ \mapsto\ &\mathcal{P}\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & a & 0 & \frac{z}{z-1} \\ 1-c & b & c-a-b & \end{matrix}\right\} \\ \mapsto\ &\left(\frac{z}{z-1} - 1\right)^a \mathcal{P}\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a & \frac{z}{z-1} \\ 1-c & b-a & c-b & \end{matrix}\right\} \\ &= (z-1)^{-a}\mathcal{P}\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a & \frac{z}{z-1} \\ 1-c & b-a & c-b & \end{matrix}\right\} \end{aligned}$$

Then, we obtain,

$$y_9 = (z-1)^{-a} F\left(a, c-b, c; \frac{z}{z-1}\right), \qquad y_{10} = z^{1-c}(z-1)^{a+c-1} F\left(a-c+1, 1-b, 2-c; \frac{z}{z-1}\right) \tag{8.25}$$

We could derive even more formulae in an equivalent way via any permutation of $\{0, 1, \infty\}$.

9 The Confluent Hypergeometric Function

The confluent hypergeometric function is a limiting process of the hypergeometric function, and includes examples such as the exponential functions. Start with the hypergeometric equation,

$$z(1-z)\frac{\mathrm{d}^2y}{\mathrm{d}z^2}+(c-(a+b+1)z)\,\frac{\mathrm{d}y}{\mathrm{d}z}-aby=0 \tag{9.1}$$

which has regular singular points at $z=0,1,\infty$. Replacing $z\mapsto\frac{z}{b}$, then the regular singular point at $z=1$ is mapped to $z=b$. Making this substitution and taking out an overall factor of b transforms the equation into,

$$z\left(1-\frac{z}{b}\right)\frac{\mathrm{d}^2y}{\mathrm{d}z^2}+\left(c-z-\frac{a+1}{b}z\right)\frac{\mathrm{d}y}{\mathrm{d}z}-ay=0 \tag{9.2}$$

Then, taking the limit as $b\to\infty$ leads to the confluent hypergeometric equation,

$$\boxed{z\frac{\mathrm{d}^2y}{\mathrm{d}z^2}+(c-z)\frac{\mathrm{d}y}{\mathrm{d}z}-ay=0} \tag{9.3}$$

Which has a regular singular point at $z=0$, but an irregular singular point at $z=\infty$. Many of its properties may be deduced from the hypergeometric equation/function by replacing $z\mapsto\frac{z}{b}$ and then taking the limit $b\to\infty$. We can do this to the hypergeometric series to find,

$$\boxed{M(a,c;z)=\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(c)}{\Gamma(c+n)\Gamma(a)}\cdot\frac{z^n}{n!}} \tag{9.4}$$

The second solution can be found from the hypergeometric series,

$$z^{1-c}M(1+a-c,2-c;z) \tag{9.5}$$

A few examples include,

1. $M(a,a;z)=\exp(z)$
2. The Bessel function, $J_\nu(z)=\frac{1}{\Gamma(\nu+1)\left(\frac{z}{2}\right)^{\nu}e^{-iz}M\left(\frac{1}{2}+\nu,1+2\nu;2iz\right)}$

9.1 Integral representation of confluent hypergeometric function

Start from the following,

$$\int_0^1 t^{a+n-1}(1-t)^{c-a-1}\,\mathrm{d}t=\frac{\Gamma(a+n)\Gamma(c-a)}{\Gamma(c+n)} \tag{9.6}$$

Substituting this into the series in (9.4) gives,

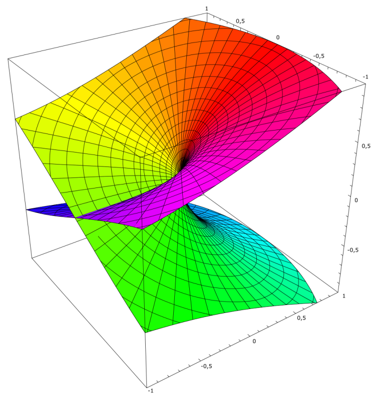
$$M(a,c;z)=\sum_{n=0}^{\infty}\left\{\frac{\Gamma(c)}{\Gamma(a)}\cdot\frac{z^n}{n!}\cdot\frac{1}{\Gamma(c-a)}\int_0^1 t^{a+n-1}(1-t)^{c-a-1}\,\mathrm{d}t\right\}$$

Assuming the integral exists, interchange the sum and the integral to find that,

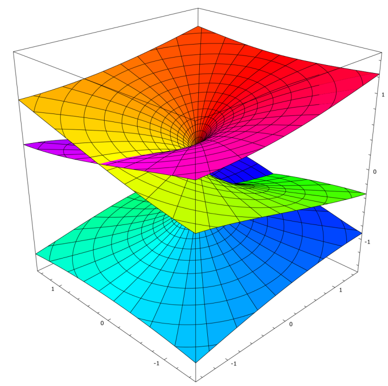
$$M(a,c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}e^{zt}\,\mathrm{d}t \tag{9.7}$$

9.2 Multivalued functions again

Consider the function $f(z)=z^{\frac{1}{2}}$, we could deal with the multivaluedness by saying that all values of the argument are allowed. This is done by considering different planes, known as *Riemann Sheets.* If we keep careful track of the phases, then there is no ambiguity. We can apply a similar logic to $f(z)=z^{\frac{1}{3}}$ where now there are three sheets, with each encircling of the origin inducing another factor of $\exp(\frac{2\pi i}{3})$. Now recall the definition of the beta function



(a) Riemann Surface for $z^{\frac{1}{2}}$



(b) Riemann Surface for $z^{\frac{1}{3}}$

again,

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \int_0^1 t^{p-1}(1-t)^{q-1}\,\mathrm{d}t, \quad \Re(p,q) > 0$$

Consider the following integral,

$$I = \int_{\mathcal{P}}^{(1^+,0^+,1^-,0^-)} t^{p-1}(1-t)^{q-1}\,\mathrm{d}t \tag{9.8}$$

Where the notation 1^+ means go round $t = 1$ in the positive direction etc. Then,

1. $\int_0^1 t^{p-1}(1-t)^{q-1}\,\mathrm{d}t$
2. $-e^{2\pi i(q-1)} \cdot \int_0^1 t^{p-1}(1-t)^{q-1}\,\mathrm{d}t$
3. $e^{2\pi i(q-1)} \cdot e^{2\pi i(p-1)} \cdot \int_0^1 t^{p-1}(1-t)^{q-1}\,\mathrm{d}t$
4. $-e^{2\pi i(p-1)} \cdot \int_0^1 t^{p-1}(1-t)^{q-1}\,\mathrm{d}t$

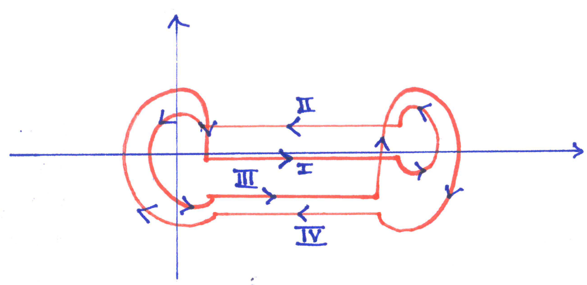


Figure 9.2
Contour encircling $t = 0$ and $t = 1$

Summing these contributions we find that,

$$\begin{aligned} I &= \int_0^1 t^{p-1}(1-t)^{q-1}\,\mathrm{d}t \left\{1 + e^{2\pi iq} + e^{2\pi ip}e^{2\pi iq} - e^{2\pi iq}\right\} \\ &= \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\left(1 - e^{2\pi ip}\right)\left(1 - e^{2\pi iq}\right) \end{aligned}$$

which gives us an analytic continuation of the integral representation of the Beta function. We can apply the same technique to the integral representation of $M(a,c;z)$, using the reflection formula for the Γ-function.

$$M(a,c;z) = -\frac{\Gamma(1-a)\Gamma(1+a-c)}{4\pi^2 e^{i\pi z}} \int_{\mathcal{P}}^{(1^+,0^+,1^-,0^-)} t^{a-1}(1-t)^{c-a-1}e^{zt}\,\mathrm{d}t \tag{9.9}$$