Lattice Field Theories
BUSSTEPP 19 Lecture Notes

Biagio Lucini

Department of Mathematics, Computational Foundry, College of Science, Swansea University (Bay Campus), Fabian Way, Swansea SA1 8EN, UK

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Email address: b.lucini@swansea.ac.uk (Biagio Lucini)

1. Aims and scope

The purpose of this mini-course is to provide a taster of Lattice Field Theories and their applications to a broad audience of first year Ph.D. students in Theoretical Particle Physics having a different background and different research focus.
These notes have been extracted almost verbatim from Sect. 2 of

[1] B. Lucini and M. Panero, "SU(N) gauge theories at large N," Phys. Rept. **526**, 93 (2013) [arXiv:1210.4997 [hep-th]].

I am indebted to the coauthor of this work, Marco Panero, for co-designing and co-developing this presentation with myself. As the title of the above paper suggests, this Lattice presentation was part of a piece of work discussing SU(N) Gauge Theories in the large-N limit. While I do not assume the Reader to be familiar with the latter topic, I will keep the large-N perspective also in the notes, to provide an explicit example of how the Lattice as a tool can have broader applications than just Quantum Chromodynamics.
Of course, in a four-hour course, only so much ground can be covered. Henceforth, I encourage the Reader to supplement these notes with the avaible literature. Classic textbooks on Lattice Field Theories include:

[2] I. Montvay and G. Munster, "Quantum fields on a lattice," Cambridge University Press (1994);

[3] H. J. Rothe,"Lattice gauge theories: An Introduction," World Scientific (2012);

[4] J. Smit, "Introduction to quantum fields on a lattice: A robust mate," Cambridge University Press (2002).

Some relevant original works will be referenced throughout these notes.

As for the prerequisites, an intermediate course on Quantum Field Theory should enable the Reader to get acquainted with the general ideas underpinning Lattice Field Theories. A general introduction to the Quantum Field Theory concepts presented in the lectures can be found e.g. in

[5] M. E. Peskin and D. V. Schroeder,"An Introduction to quantum field theory," CRC Press (1995).

2. Introduction

For energies above 1 GeV, QCD is perturbative. Hence, a controlled expansion in $\alpha_s = g^2/(4\pi)$ can be used to compute physical quantities. This programme is successful at explaining experimental data obtained at particle collider facilities, and is indeed one of the key theoretical ingredients for the analysis of collision events observed at the LHC. However, as the energy is lowered below 1 GeV, α_s rapidly becomes of order one, and the perturbative expansion looses its predictive power. Hence, in order to derive from first principles phenomena that are typical of the non-perturbative domain of QCD, like chiral symmetry breaking and confinement, a different approach is needed. In the previous section, we have discussed the large-N expansion as a possible systematic expansion to study non-perturbative QCD and the gauge-gravity dual as a way to compute observables in the zeroth order in that expansion (corresponding to the $N \to \infty$ limit of the theory). In this

section, we shall discuss a complementary approach, based on numerical simulations of the theory discretized on a spacetime lattice using Monte Carlo techniques. Once the continuum limit is taken, numerical results for QCD obtained in the lattice framework can be compared to observations. Thanks to recent technical advances in the field of lattice gauge theory, not only do we have now a reasonable numerical proof that QCD is the correct theory of the strong interactions, but we can also trust the predictive power of lattice QCD calculations. The progress in the field has encouraged further development targeting a more theoretical understanding of gauge theories. Among possible directions, numerical studies of the 't Hooft large-N limit have generated a large volume of quantitative results that on the one hand significantly advances our understanding of the theory and on the other hand can be used to inform analytical approaches. In this section, we lay the foundations for understanding Monte Carlo results for SU(N) gauge theories in the 't Hooft large-N limit. While this part brings only limited benefit to the reader already familiar with numerical calculations on a spacetime lattice, it could provide a useful reference for those who are interested in using lattice results to inform or inspire analytical calculations. The exposition is pedagogical and aimed at underlining the main conceptual steps and the technicalities needed for a critical understanding of the numerical results and fixing the notation for later chapters rather than at providing detailed derivations of all the basic lattice results needed later on. The reader interested in this latter aspect is referred to the excellent textbooks on lattice gauge theories [1, 2, 3, 4, 5, 6]. The works reported in the early original literature on the subject are by now considered classic papers in the field. Although we have mentioned the main original references, there are many more to which we can not and we are not making justice. Once again, we refer the reader to the textbooks, which contain a better account of the early original contributions to the field.

This section is organized as follows. In subsec. 3 we discuss the discretization of the free scalar field and some general aspects of the recovery of the continuum limit. How to construct lattice variables for the gauge fields so that gauge invariance is respected in a discretized spacetime will be the subject of subsec. 4, which will also deal with subtleties connected with the discretization of fermionic fields. Numerical calculations will be discussed in subsec. 5, while in subsec. 6 we shall present how continuum large-N physics is extracted from numerical simulations. For the sake of definiteness, unless otherwise stated, in this section we only deal with four-dimensional theories.

3. The free scalar field

3.1. Path integral approach

The lattice approach is based on the path integral quantization. For the sake of definiteness, let us consider a free scalar theory, described by the Lagrangian

$$\mathcal{L}(\phi(x)) = \frac{1}{2}\partial^\mu\phi(x)\partial_\mu\phi(x) - \frac{1}{2}m^2\phi(x)^2, \tag{1}$$

where ϕ is a spin zero scalar field of mass m. The easiest way to quantize this theory is to use the canonical approach. With $x = (x_0, \vec{x})$, with x_0 being the temporal component and $\vec{x}$ the spatial component of the quadrivector x, we define the momentum $\Pi(x)$ as

$$\Pi(x) = \frac{\partial\mathcal{L}(\phi(x))}{\partial_0\phi(x)} \tag{2}$$

and we impose the usual equal-time ($x_0 = y_0$) commutation relations

$$\begin{aligned}[\phi(x), \Pi(y)] &= i\delta^3(\vec{x} - \vec{y}), \\ [\phi(x), \phi(y)] &= [\Pi(x), \Pi(y)] = 0.\end{aligned}$$

This approach naturally leads to the Fock space, in which the base states are multi-particle states labelled by the momenta of each single particle.

If we now consider an interaction that can be written as[1] $V(\phi) = \alpha f(\phi)$, with f e.g. a polynomial of degree k, for small α we can still start from the canonical quantization and treat the effect of the interaction perturbatively in α. It is then possible to compute systematically the n-point correlation function for arbitrary n

$$\mathcal{C}_n\left(\phi(x_i),\ \ldots,\ \phi(x_n)\right) = \langle \phi(x_1)\ \ldots \phi(x_n) \rangle \tag{3}$$

as a power series in α. Since all the observables can be expressed using the $\mathcal{C}_n$, all physical processes can be accessed in this way. The basic physical assumption underlying the perturbative expansion is that multiparticle states of well-defined momentum are still a good approximation of the eigenstates of the interacting theory, with corrections that can be accounted for by a systematic expansion in powers of α. However, in a quantum field theory α is not constant, but depends on the momentum. Hence, in order for the perturbative calculation to be valid, α needs to remain small for the relevant range of energies.

An alternative approach for computing the correlation functions is the path integral. In this formulation, they are given by

$$\mathcal{C}_n\left(\phi(x_i),\ \ldots,\ \phi(x_n)\right) = \frac{\int \left(\mathcal{D}\phi(x)\right) \phi(x_1)\ \ldots \phi(x_n) e^{iS(\phi)}}{\int \left(\mathcal{D}\phi(x)\right) e^{iS(\phi)}}, \tag{4}$$

where ϕ has to be interpreted as a classical field and the integral has to be performed over all possible classical field configurations (this is indicated by the expression $(\mathcal{D}\phi(x))$). The integrand is weighted by the factor $e^{iS(\phi)}$, where

$$S(\phi) = \int \mathrm{d}^4x\ \mathcal{L}(\phi) \tag{5}$$

is the action evaluated over a field configuration $\phi(x)$. The denominator

$$Z = \int \left(\mathcal{D}\phi(x)\right) e^{iS(\phi)}, \tag{6}$$

needed to normalize the correlation functions, is what is referred to as the *path integral*. The advantage of the path integral formulation is that it can be used also for the interacting theory, irrespectively of the value of the coupling.

3.2. The scalar field on the lattice

At this stage, the path integral (6) is still a formal expression: in order to be able to use it, we need to give a prescription on how to perform the integration. To this purpose, we first perform a Wick rotation, which consists in the change of variable $\tau = ix_0$. In the new variables, up to an

[1] We assume that $V(\phi)$ is bounded from below.

overall minus sign, the metric is Euclidean. For this reason, the space of the vectors $x_E = (ix_0, \vec{x})$ is called *Euclidean space*. In Euclidean space, we write the Lagrangian as

$$\mathcal{L}_E(\phi) = \frac{1}{2}\partial_E^\mu \phi(x_E)\partial_{\mu,E}\phi(x_E) + \frac{1}{2}m^2\phi(x_E)^2, \tag{7}$$

where the metric is the identity (and in fact we could disregard the convention of lower and upper indices), and the action as

$$S_E(\phi) = \int \mathrm{d}^4 x_E \; \mathcal{L}_E(\phi). \tag{8}$$

Then, the path integral becomes

$$Z_E = \int (\mathcal{D}\phi(x_E))\, e^{-S_E(\phi)}. \tag{9}$$

Over eq. (6), Z_E has the advantage of having the damping factor $e^{-S_E(\phi)}$ replacing the oscillating factor $e^{iS(\phi)}$ in the integrand, which improves the convergence of the integral. Real-time correlation functions can be obtained from Euclidean-time correlation functions by analytic continuation. From now on, we will work in Euclidean space, and for convenience we will drop the subscript E from all expressions. For instance, we will write the Euclidean path integral as

$$Z = \int (\mathcal{D}\phi(x))\, e^{-S(\phi)}, \tag{10}$$

where it is understood that ϕ is defined in Euclidean space and S is the Euclidean action.

We still need to give an operational implementation of the measure $(\mathcal{D}\phi(x))$. To this purpose, we can consider a spacetime grid of spacing a (the *lattice*) and define ϕ only on sites of the grid, i.e. on points x such that $x = (n_0 a, n_1 a, n_2 a, n_3 a)$, with the n_i all integer. If we also impose that $0 \leq n_i \leq N_i - 1$, i.e. that the lattice size in the i-th direction is aN_i, then the path integral measure becomes a multidimensional integral. In the remainder of this section, we focus on the discretization of the theory, while in the next section we shall see how the continuum theory can be recovered from its discretized version.

For convenience, we rescale the field ϕ by a, obtaining the dimensionless combination $\varphi(i) = a\phi(ai)$, with $i = (n_0, n_1, n_2, n_3) = (n_0, \vec{n})$, in terms of which we formulate the free field scalar theory on the lattice. To this purpose, we replace the integral $\int \mathrm{d}^4 x$ with $a^4 \sum_i$, where i runs over all lattice points, and the derivative of ϕ with the finite difference

$$\partial_\mu \phi(i) \to \frac{\phi(i+\hat{\mu}) - \phi(i)}{a}, \tag{11}$$

where $\hat{\mu}$ is the unit vector in the direction μ. The lattice action becomes

$$S = \frac{1}{2}\sum_i \left[-\frac{1}{2}\left(\sum_{\mu=0}^{3} \varphi(i+\hat{\mu})\varphi(i) + \varphi(i)\varphi(i-\hat{\mu}) \right) + \left(\hat{m}^2 + 8\right) \right], \tag{12}$$

where $\hat{m} = am$, and the discretized path integral reads

$$Z = \int \left(\prod_i \mathrm{d}\varphi(i) \right) e^{-S(\varphi)}, \tag{13}$$

with S given in (12) and the product running over all lattice points. n-point Green functions are easily computed in momentum space, where it has to be considered that the momenta are cut-off at $p_{max} = \pi/a$ (in crystallography, this corresponds to the first Brillouin zone). For instance, for the two-point function (the lattice propagator), we have:

$$\langle \varphi(l)\varphi(m)\rangle = \int_{-\pi}^{\pi} \frac{d^4\hat{p}}{(2\pi)^4} \frac{e^{i\hat{p}\cdot(l-m)}}{4\sum_\mu \sin^2(\hat{p_\mu}/2) + \hat{m}^2}. \tag{14}$$

3.3. Continuum limit

The lattice path integral (13) is formally identical to the partition function of a statistical system with $N_1N_2N_3N_4$ degrees of freedom and Hamiltonian S. The problem of recovering the infinite volume limit of the original system is then mapped into the problem of performing the thermodynamic limit of the associated statistical system. This mapping allows typical concepts of statistical mechanics to be carried over to lattice field theory. Although we do not pursue the analogy further, important progress has been achieved exploiting this correspondence.

Once the infinite volume limit has been performed, the continuum limit can be recovered by taking the lattice spacing a to zero. At the classical level, this implies that the discretized action should reproduce the continuum action in the limit $a \to 0$. This request is easily fulfilled, since the lattice action has been constructed as a simple discretization of the continuum action. For the quantum theory, we need to systematically compute all lattice n-point functions and show that they converge to the corresponding continuum functions when $a \to 0$. Since a quantum field theory is uniquely specified by its n-point functions, this would suffice to prove that the lattice field theory reproduces the wanted field theory in the continuum limit. The above procedure can look tautological. However, there is no guarantee that the lattice theory obtained by naïve discretization of the path integral describe the wanted field theory in the continuum limit.

Since for the free theory the issue is trivial, let us consider the case of the interacting theory. Once the theory is discretized on a lattice, observables depend on the dimensionless couplings $(\hat{m}, \alpha, \dots)$, which in turn are functions of the lattice spacing. If we now take the lowest mass of the physical spectrum, M, if a continuum limit exist, we must have

$$\lim_{a\to 0} \hat{M}/a = M, \tag{15}$$

where $\hat{M}$ is the dimensionless mass determined on the lattice, and a reinstates the dimensions. The previous equation implies

$$\lim_{a\to 0} 1/\hat{M} = \infty, \tag{16}$$

where $1/\hat{M} = \xi$ can be interpreted as the correlation length (in dimensionless units) of the statistical system associated to the regularized quantum field theory. Hence, in the language of Statistical Mechanics, reaching the continuum limit means finding the values of the couplings for which the system is critical. Physically, this means that the correlation length of the system is much larger than the lattice spacing a, which implies that the system looses memory of the discretization. In general, whether values of the coupling exist such that the system is critical is a dynamical problem. If it happens, it might not be immediate to identify the corresponding quantum field theory in the infrared, since this is determined by the value of the n-point functions at the critical point. For SU(N) gauge theories, thanks to asymptotic freedom, the system is critical at the ultraviolet fixed point, where the theory is perturbative. Hence, the lattice provides

a way to regularize the theory in the ultraviolet, and removing the ultraviolet cut-off a corresponds to a well-defined renormalization procedure, in which an infrared scale (e.g. the string tension in the pure gauge case or the mass of the ρ meson in the presence of dynamical quarks) is fixed, and all spectral quantities are determined in terms of this scale.[2] It is important to emphasize that different ways to set the lattice scale (in terms of different physical observables) may lead to slightly different results at finite values of the lattice spacing, because different observables can be affected by different lattice artifacts. However, these differences disappear when results are extrapolated to the continuum limit $a \to 0$. In fact, once one has proved that the lattice discretized version of QCD has the right continuum limit, one could invert the logic and define QCD starting from the construction on the lattice. This would give a more rigorous way to specify the quantum field theory than canonical quantization, which has the problem of defining the theory starting from the perturbative Fock vacuum, which is very far from the real QCD vacuum. By the same token, using the lattice we can rigorously define the large-N limit *à la* 't Hooft of SU(N) gauge theories as the theory defined by all the $n-$point correlators when this limit is taken. In this approach, the lattice could be used to prove the existence of this limit, even in the non-perturbative regime.

In passing by, we notice that different prescriptions could have been used to construct the lattice action from the continuum one. For instance, we could request that the discrete derivative be defined as

$$\partial_\mu \phi(i) \to \frac{\phi(i+\hat{\mu}) - \phi(i-\hat{\mu})}{2a}. \tag{17}$$

The corresponding action would have differed from (12) by corrections that go to zero in the continuum limit. This ambiguity in defining the lattice action can be exploited to improve at the quantum level the convergence of the lattice theory to the continuum one as $a \to 0$.

4. Discretization of SU(N) gauge theories

4.1. Gauge fields

A good regularization of a quantum theory respects the crucial properties of the original theory, with the others recovered when the ultraviolet cut-off is removed. On the lattice, the property that we absolutely need to preserve is gauge invariance. This can be accomplished through parallel transports. Consider for instance continuum scalar electrodynamics (in Euclidean space). If the (complex) scalar field is coupled to an Abelian gauge field $A_\mu(x)$, the continuum derivative is replaced by the covariant derivative:

$$\partial_\mu \to D_\mu = \partial_\mu + i g_0 A_\mu(x), \tag{18}$$

where g_0 is the gauge coupling. Under gauge transformations defined by the function $\Lambda(x)$

$$A_\mu(x) \to A_\mu(x) - \partial_\mu \Lambda(x), \qquad \phi(x) \to e^{-i g_0 \Lambda(x)} \phi(x), \tag{19}$$

[2] An alternative way to set the lattice scale is based on Sommer's scale r_0 [7], which is defined as the distance at which the force F between a pair of external, infinitely heavy, fundamental color sources (i.e., a static quark-antiquark pair) satisfies: $r_0^2 F(r_0) = 1.65$. Comparison with phenomenological potential models shows that this length scale corresponds to approximately $r_0 = 0.5$ fm. For the lattice regularization of SU(3) Yang-Mills with the Wilson gauge action, a high-precision determination of the lattice spacing a in units of r_0, as a function of the lattice parameter β, was reported in ref. [8].

so that the Lagrangian

$$\mathcal{L}(\phi, A_\mu) = D_\mu\phi(x)D_\mu\phi^*(x) + m^2\phi(x)\phi^*(x) + \frac{1}{4}F_{\mu\nu}(x)F_{\mu\nu}(x)\ ,\ F_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{20}$$

is invariant. It is immediate to see that a naïve discretization of this Lagrangian will not preserve gauge invariance, the problem being that the finite difference mixes fields defined on different lattice points. This can be remedied as follows. We introduce the parallel transport along the link joining the sites i and $i+\hat{\mu}$ as

$$U_\mu(i) = e^{ig_0\int_x^{x+a\hat{\mu}} A_\mu(s)\mathrm{d}s}, \tag{21}$$

where $x = ai$. On the lattice, we make the replacement

$$D_\mu\phi(i) \to \frac{U_\mu(x)\phi(i+\hat{\mu}) - \phi(i)}{a}. \tag{22}$$

Gauge transformations act as usual on ϕ:

$$\phi(j) \to e^{-i\lambda(j)}\phi(j); \qquad \phi^*(j) \to e^{i\lambda(j)}\phi^*(j). \tag{23}$$

For the link variables, we have

$$U_\mu(j) \to e^{-i\lambda(j)}U_\mu(j)e^{i\lambda(j+\hat{\mu})}. \tag{24}$$

Using the fact that $U_{-\mu}(i) = U_\mu^*(i-\hat{\mu})$, it is easy to see that the terms involving ϕ in eq. (20) are invariant under discretized gauge transformations when using the prescription (22) for the covariant derivative. The part of the Lagrangian involving the field tensor $F_{\mu\nu}$ can also be expressed in terms of the link variables. The simplest possibility is given by the Wilson action [9]

$$S = \beta\sum_{j,\mu>\nu}\left(1 - \frac{1}{2}\left(U_{\mu\nu}(i) + U_{\mu\nu}^*(i)\right)\right) \tag{25}$$

where $\beta = 1/g_0^2$ and

$$U_{\mu\nu}(j) = U_\mu(j)U_\nu(j+\hat{\mu})U_\mu^*(j+\hat{\nu})U_\nu^*(j) \tag{26}$$

is the parallel transport of the gauge field around an elementary lattice square (*plaquette*). It is worth noticing that S defined in eq. (25) is manifestly gauge invariant. Expanding the exponentials defining the links at the leading order in a, we get

$$S \simeq \frac{a^4}{4}\sum_{j,\mu,\nu} F_{\mu\nu}(j)F_{\mu\nu}(j), \tag{27}$$

which is the naïve discretization of the gauge field action.

The generalization of the above discussion to SU(N) is immediate. Now, the link variables $U_\mu(j)$ are defined as the path-ordered exponential

$$U_\mu(j) = \mathcal{P}e^{ig_0\int_x^{x+a\hat{\mu}} A_\mu(s)\mathrm{d}s}, \tag{28}$$

where now $U_\mu(j)$ is a matrix in the SU(N) group, and the plaquette is the path-ordered product of links around the lattice plaquette:

$$U_{\mu\nu}(j) = U_\mu(j)U_\nu(j+\hat{\mu})U_\mu^\dagger(j+\hat{\nu})U_\nu^\dagger(j) \tag{29}$$

where as before $U_{-\mu}(j) = U_{j-\hat{\mu}}^\dagger(j)$. Under gauge transformation implemented by the SU(N)-valued function $\Omega(x)$, the $U_\mu(j)$ transforms as

$$U_\mu(j) \to \Omega(j)U_\mu(j)\Omega^\dagger(j+\hat{\mu}), \tag{30}$$

which is the lattice version of the continuum gauge transformation

$$A_\mu(x) \to \Omega(x)A_\mu(x)\Omega^\dagger(x) - \frac{i}{g_0}\Omega(x)\partial_\mu\Omega^\dagger(x). \tag{31}$$

In terms of $U_{\mu\nu}(j)$, the Wilson action reads

$$S = \beta \sum_{j,\mu>\nu} \left(1 - \frac{1}{N}\mathrm{ReTr}U_{\mu\nu}(j)\right). \tag{32}$$

where ReTr indicates the real part of the trace and $\beta = 2N/g_0^2$. Gauge invariance is guaranteed by the trace. The path integral reads

$$Z = \int \left(\prod_{j,\mu} \mathrm{d}U_\mu(j)\right) e^{-S}, \tag{33}$$

where each factor $\mathrm{d}U_\mu(j)$ in the measure is the Haar measure of SU(N) associated to the link $U_\mu(j)$. For our purposes, the most important property of the Haar measure is that it is uniform in the group. The action (32) yields the naïve discretization of the SU(N) Yang-Mills theory at the leading order in a as $a \to 0$, as it should. The lattice theory can also be shown to reproduce the correct continuum limit when the lattice spacing goes to zero. As we will see in more detail later, this is a non-trivial consequence of asymptotic freedom.

4.2. Fermions

Lattice discretization of fermion fields is not immediate. To see the origin of the problem and the possible solutions, we start by following closely the strategy we have used for the bosonic field. Given a continuum field $\psi(x)$, we define the (dimensionless) discretized fermion field $\hat{\psi}(i)$ as

$$\hat{\psi}(i) = a^{3/2}\psi(ia) \tag{34}$$

and its discretized derivative as

$$\hat{\partial}_\mu\hat{\psi}(i) = \frac{\hat{\psi}(i+\hat{\mu}) - \hat{\psi}(i-\hat{\mu})}{2}. \tag{35}$$

Note that, differently from the bosonic field case, we have used a symmetric definition for the derivative. By substituting these equations in the Euclidean Dirac action, we get the lattice fermionic action

$$S_f = \sum_{i,j,\alpha,\beta} \hat{\bar{\psi}}_\alpha(i) \left[\frac{1}{2}\sum_\mu \left(\gamma_\mu^E\right)_{\alpha\beta} \left(\delta_{i+\hat{\mu},j} - \delta_{i-\hat{\mu},j}\right) + \hat{m}\delta_{ij}\delta_{\alpha\beta}\right] \hat{\psi}_\beta(j), \tag{36}$$

where $\hat{m} = am$, with m the fermion mass. In this equation, Dirac indices are expressed with Greek letters, while Latin letters indicate lattice points. The γ_E are the Euclidean γ matrices, satisfying the anticommutation relations

$$\{\gamma_\mu^E, \gamma_\nu^E\} = 2\delta_{\mu\nu}. \tag{37}$$

In terms of the Minkowskian γ matrices, the Euclidean γ matrices are given by

$$\gamma_0^E = \gamma_0, \qquad \gamma_i^E = -i\gamma_i, \qquad i = 1, 2, 3. \tag{38}$$

It is convenient to define the Dirac operator D as

$$D_{\alpha\beta}(ij) = \left[\frac{1}{2}\sum_\mu \left(\gamma_\mu^E\right)_{\alpha\beta}\left(\delta_{i+\hat{\mu},j} - \delta_{i-\hat{\mu},j}\right) + \hat{m}\delta_{ij}\delta_{\alpha\beta}\right], \tag{39}$$

so that the fermion action can be written in a more compact form as

$$S_f = \sum_{i,j,\alpha,\beta} \hat{\overline{\psi}}_\alpha(i) D_{\alpha\beta}(ij) \hat{\psi}_\beta(j). \tag{40}$$

Fermionic correlation functions can be expressed in terms of the inverse Dirac operator. For instance, for the correlator, we have

$$\langle \hat{\psi}_\alpha(i) \hat{\overline{\psi}}_\beta(j) \rangle = D^{-1}_{\alpha\beta}(ij). \tag{41}$$

Using the momentum representation, we can rewrite this expression as

$$\langle \hat{\psi}_\alpha(i) \hat{\overline{\psi}}_\beta(j) \rangle = \int_{-\pi}^{\pi} \frac{d^4\hat{p}}{(2\pi)^4} \frac{-i\sum_\mu \gamma_\mu^E \sin(\hat{p}_\mu) + \hat{m}}{\sum_\mu \sin^2(\hat{p}_\mu) + \hat{m}^2} e^{i\hat{p}\cdot(i-j)}, \tag{42}$$

where $\hat{p} = ap$ and p is the continuum momentum. This expression reproduces the continuum Euclidean propagator of a free fermion when $a \to 0$, as it should. However, the same continuum form is obtained also when at least one of the components of the momentum $p_\mu = \pi/a$. Hence, each corner of the Brillouin zone contributes equally to the propagator, which means that the so-called naïve discretization, given by eq. (34), eq. (35) and eq. (36), gives rise to 16 fermion flavors in the continuum limit. This is the problem of fermion doubling. A well-known no-go theorem due to Nielsen and Ninomiya [10, 11] implies that doubling is an unavoidable consequence if one requires a discretized fermion theory that preserves chiral symmetry and is ultralocal (i.e. the action only involves couplings between fields in a localized region of space). Hence, in order to avoid doubling, one has to relax the request of chirality or the request of ultralocality. The Wilson solution to the problem of doubling was to relax the request of chirality [12]. In his approach, an irrelevant operator in the limit $a \to 0$ provides the doublers with an infinite mass in the continuum limit.[3] A different approach was proposed by Kogut and Susskind [16, 17, 18], who were able to reduce the number of doublers from 2^D (in D spacetime dimensions, assuming D to be even) to $2^{D/2}$ by spreading the four components of the Dirac spinor on the corners of the Brillouin zone (hence the name of "staggered fermions" for this discretization). More recently, a series of fermion discretizations, that avoid the doubling problem, by satisfying a *modified* form of

[3] A variant of Wilson fermions, including a "chirally twisted" mass term, was proposed in refs. [13, 14, 15].

chiral symmetry on the lattice [19, 20] (which goes over to the usual continuum chiral symmetry for $a \to 0$), have been proposed: these include the domain wall formulation [21, 22, 23, 24], the overlap formulation [25, 26, 27, 28, 29, 30], and the fixed-point formulation [31, 32, 33].[4] For a general discussion about lattice fermions and chiral symmetry, see refs. [35, 36, 37]. We stress that the physics in the continuum limit is independent of the lattice discretization used. However, a particular formulation could be more suited for a particular problem. In lattice calculations of SU(N) gauge theories with fermions, mostly the Wilson formulation has been used. The main motivations for this choice are the following:

- simulations using Wilson fermions are much faster than simulations using non-ultralocal fermions;
- unlike in the staggered fermion case, a generic number of flavors can be simulated in the Wilson approach;
- chiral symmetry can be recovered by tuning the bare quark mass to a critical value.[5]

The Wilson discretization starts from a modification of the Dirac operator with an additive term that goes like the Laplacian of $\hat{\psi}$ with strength controlled by a parameter r, conventionally taken to be equal to 1. Explicitly, this term reads

$$\Delta S_f = -\frac{r}{2} \sum_i \hat{\overline{\psi}}(i) \sum_\mu \left(\hat{\psi}(i+\hat{\mu}) + \hat{\psi}(i-\hat{\mu}) - 2\hat{\psi}(i) \right), \tag{43}$$

This yields the following Wilson Dirac operator

$$D^W_{\alpha\beta}(ij) = -\frac{1}{2} \sum_\mu \left[\left(r - \gamma^E_\mu\right)_{\alpha\beta} \delta_{i+\hat{\mu},j} + \left(r + \gamma^E_\mu\right)_{\alpha\beta} \delta_{i-\hat{\mu},j} \right] + (\hat{m} + 4r)\, \delta_{ij} \delta_{\alpha\beta}. \tag{44}$$

The resulting two-point function is

$$\langle \hat{\psi}_{\alpha(i)} \hat{\overline{\psi}}_\beta(j) \rangle = \int_{-\pi}^{\pi} \frac{d^4\hat{p}}{(2\pi)^4} \frac{-i \sum_\mu \gamma^E_\mu \sin(\hat{p}_\mu) + \hat{m}(p)}{\sum_\mu \sin^2(\hat{p_\mu}) + (\hat{m}(p)^2)} e^{i\hat{p}\cdot(i-j)}, \tag{45}$$

which is similar in form to the naïve propagator, the difference being that now the mass depends on the momentum:

$$m(p) = m + \frac{2r}{a} \sum_\mu \sin^2\left(\frac{a p_\mu}{2}\right) \tag{46}$$

(note that we have reinstated dimensionful units in this expression). When $a p_\mu \to 0$ for all μ, $m(p) \simeq m$, and the expected continuum result is recovered. Conversely, at the other corners of the Brillouin zone, $m(p)$ diverges. Hence, in the continuum limit the unphysical doublers get an infinite mass, decoupling from the action.

[4] A "chirally improved" variant of Wilson fermions was proposed in ref. [34].

[5] Although this value has to be found as a part of the simulation, this does not create particular problems in practical applications.

4.3. Gauge theories with fermionic matter

We conclude this brief introduction to lattice gauge theories with the case of fermionic matter coupled to gauge fields. For simplicity, we consider the case in which there are n_f fermion flavors having the same mass $\hat{m}$. In the presence of gauge interactions, for Wilson fermions the Dirac operator is given by

$$D^W_{\alpha\beta}(ij) = -\frac{1}{2}\sum_\mu \left[\left(r - \gamma^E_\mu\right)_{\alpha\beta} U_\mu(i)\delta_{i+\hat{\mu},j} + \left(r + \gamma^E_\mu\right)_{\alpha\beta} U^\dagger_\mu(j)\delta_{i-\hat{\mu},j} \right] + (\hat{m} + 4r)\,\delta_{ij}\delta_{\alpha\beta}. \quad (47)$$

For the sake of clarity, wherever the context is unambiguous, from now on we suppress the Dirac indices and the spacetime coordinates in D^W and in the fermion fields. In the most straightforward formulation, the path integral of the theory is given by

$$Z = \int \left(\prod_{j,\mu} \mathrm{d}U_\mu(j)\right) \left(\prod_{j,\alpha,l} \mathrm{d}\hat{\psi}^l_\alpha(j)\right) \left(\prod_{j,\alpha,l} \mathrm{d}\hat{\overline{\psi}}^l{}_\alpha(j)\right) e^{-S - \hat{\overline{\psi}}^l D_W \hat{\psi}^l}. \quad (48)$$

where l is the flavor index, running from 1 to n_f, and S is the gauge action (32). Performing explicitly the integration over the fermionic variables gives

$$Z = \int \left(\prod_{j,\mu} \mathrm{d}U_\mu(j)\right) (\det \mathrm{D_W})^{n_f}\, e^{-S}, \quad (49)$$

in which the determinant of D_W to the power of n_f appears.

Among fermionic observables, we shall consider only zero-momentum correlation functions of isovector meson operators, which take the form

$$\mathcal{C}_{\Gamma\Gamma'}(t) = \sum_{\vec{x}} \left\langle \left(\hat{\overline{\psi}}^l(t,\vec{x})\Gamma\hat{\psi}^k(t,\vec{x})\right)^\dagger \left(\hat{\overline{\psi}}^l(0,\vec{0})\Gamma'\hat{\psi}^k(0,\vec{0})\right) \right\rangle, \quad (50)$$

where Γ and Γ' are two Euclidean Dirac matrices, $(0,\vec{0})$ is the conventional origin of the lattice and $l \neq k$. Integrating the previous expression over the fermion fields yields

$$\mathcal{C}_{\Gamma\Gamma'}(t) = -\sum_{\vec{x}} \left\langle \mathrm{tr}\left(\Gamma^\dagger (D^W)^{-1}(x,0)\Gamma'(D^W)^{-1}(0,x)\right)\right\rangle, \quad (51)$$

with $0 \equiv (0,\vec{0})$ and $x \equiv (t,\vec{x})$ and tr being the trace over Dirac indices. More explicitly, in the path integral formulation this expression reads

$$\mathcal{C}_{\Gamma\Gamma'}(t) = -\frac{1}{Z}\int \left(\prod_{j,\mu} \mathrm{d}U_\mu(j)\right) (\det \mathrm{D_W})^{n_f}\, \mathrm{tr}\left(\Gamma^\dagger (D^W)^{-1}(x,0)\Gamma'(D^W)^{-1}(0,x)\right) e^{-S}. \quad (52)$$

4.4. Quenched approximation

Fermionic observables like $\mathcal{C}_{\Gamma\Gamma'}$ require the evaluation of the fermionic determinant. This determinant can be expanded in fermionic loops. As an expansion parameter, one can use for instance the fermion mass. This gives rise to the so-called hopping parameter expansion. The leading order in this expansion simply consists in neglecting all fermionic loops, which means setting $\det D^W = 1$. This infinite fermion mass limit defines the *quenched approximation*. In

practical terms, working in the quenched approximation means neglecting the back-reaction of fermions on the gauge fields. Note in addition that the quenched theory is non-unitary.

In lattice QCD, in order to obtain an accurate numerical result it is crucial to evaluate the full fermionic determinant. However, in large-N QCD, at fixed quark mass fermion loops become less and less relevant. In fact, the quenched large-N limit coincides with the large-N limit in the theory with dynamical fermions. Hence, if we are interested only in the large-N result, the calculation can be performed in the quenched theory (note that, obviously, this does not hold for the evaluation of finite-N corrections).

A phenomenon that has been observed in lattice QCD is the delayed onset of unquenching effects: the quenched calculation proves to work even in a regime in which one would expect significant contributions from fermion loops [38]. This feature can be seen as another indication that the physical strong interaction is close to its large-N limit.

5. Monte Carlo calculations

In a SU(N) gauge theory, the evaluation of the path integral on a lattice of size $\mathcal{S}$ (in lattice units) involves the evaluation of $d = 4(N^2-1)\mathcal{S}$ integrals. Such a large number of integrals are impractical to be performed using grid methods. A stochastic approach, based on the observation that the path integral measure is reminiscent of the Boltzmann measure in statistical mechanics—and, hence, at fixed bare parameters only a subset of possible values of the variables will dominate the integral—is preferable. Moreover, grid methods are affected by a systematic error that is $O(1/\mathcal{S}^{s/d})$, where s is a number that depends on the approximation used by the given grid method (for instance, $s = 4$ for the popular Simpson method). Hence, when d is large, the error becomes unavoidably of order one. Monte Carlo methods provide the wanted stochastic approach: in a Monte Carlo calculation, field configurations are generated according to the path integral measure, which means that configurations recur according to the weight of their contribution to the path integral. Thanks to this property, the vacuum expectation value of an observable can be computed as a simple average over the configurations generated during the Monte Carlo simulation. Moreover, the error (which in this case is statistical and not systematic) can be kept under control, since it scales as $1/\sqrt{N_K}$, where N_K is the number of generated configurations.

The theory behind Monte Carlo calculations is based on Markovian processes. Consider a system that evolves through a sequence of states over discrete time. We indicate a generic state as C_m and the ensemble of all states (or configurations) as $\{C_m\}$. The evolution is determined by a probability P_{nm} to transition from C_n to C_m at any given time. The dynamics is said to be Markovian if the configuration realized at time t only depends on the configuration realized at time $t-1$, and not on the configurations realized at previous times. Under some technical assumptions that we do not specify here, one can prove that an asymptotic probability distribution characterizes a Markovian dynamics. This is called the equilibrium distribution. Monte Carlo algorithms are recipes to construct Markovian dynamics that have the path integral measure as the equilibrium distribution. In general, several different Markovian dynamics can generate the same equilibrium distribution. For some Markovian dynamics that satisfies the detailed balance relation

$$P_{nm}e^{-S(C_n)} = P_{mn}e^{-S(C_m)}, \tag{53}$$

where $S(C)$ is the action evaluated on the configuration C, the equilibrium distribution ρ_m is given by

$$\rho_m = \frac{e^{-S(C_m)}}{\sum_n e^{-S(C_n)}}, \tag{54}$$

in which it is easy to recognize the Boltzmann equilibrium distribution.

The problem of generating an ensemble of configurations dominating the path integral and approximating the observables with controlled precision becomes then the problem of defining an appropriate Markovian dynamics for our theory. Once we have done this, we can start from any arbitrary state and let the system evolve. After discarding a sufficient number of configurations at the beginning of the chain, the remaining ones would be distributed with the right statistics.[6]

An algorithm can be characterized by the efficiency with which it explores the configuration space. A good measure of the efficiency is the correlation time. In general, for a given observable $\mathcal{O}$,

$$\langle \mathcal{O}(t)\mathcal{O}(t+\tau)\rangle \propto e^{-t/\tau_{\mathcal{O}}}, \tag{55}$$

where the correlation time $\tau_{\mathcal{O}}$ depends on both $\mathcal{O}$ and the chosen Markovian dynamics. At fixed observable, $\tau_{\mathcal{O}}$ provides a measure of the efficiency of the algorithm: the smaller $\tau_{\mathcal{O}}$, the faster the configuration space is explored. Topological observables are notoriously more difficult to decorrelate than local observables. An algorithm that decorrelates fast topological observables can be considered efficient in general terms.

For SU(2) pure gauge theory, an efficient algorithm is the one proposed by Creutz [39] and later refined by Kennedy and Pendleton [40]. This algorithm belongs to the wider class of the heath-bath algorithms, i.e. of those algorithms in which detailed balance is obtained by generating the new configuration according to its Boltzmann weight. To further decrease the correlation time and increase ergodicity, it is possible to supplement the heath-bath with an overrelaxation step, in which the link variables are changed in such a way that the action remains constant [41, 42].

For SU(N), a Kennedy-Pendleton or overrelaxation update can be performed by a sequence of updates on different SU(2) subgroups. The idea of updating the SU(2) subgroups was proposed by Cabibbo and Marinari [43]. Although the Kennedy-Pendleton algorithm with a Cabibbo-Marinari cycle works well for SU(3), at very large N cycling over the SU(2) subgroups may become inefficient. To overcome this potential issue, it has been proposed to supplement the Cabibbo-Marinari update with an overrelaxation step over the whole SU(N) group [44, 45].

After a sequence of thermalized updates, the expectation value of an observable can be computed as the average of the values of the observable evaluated over the various configurations of the Markov chain. For instance, if $\mathcal{O}(i)$ is the value of the observable $\mathcal{O}$ at the i-th configuration in a Markov chain of length N_K, we define

$$\overline{\mathcal{O}}_{N_K} = \frac{1}{N_K}\sum_{i=1}^{N_K}\mathcal{O}(i). \tag{56}$$

$\langle\mathcal{O}\rangle$, the vacuum expectation value of $\mathcal{O}$, is found as

$$\langle\mathcal{O}\rangle = \lim_{N_K\to\infty}\overline{\mathcal{O}}_{N_K}. \tag{57}$$

$\overline{\mathcal{O}}_{N_K}$ is a controlled estimate for $\langle\mathcal{O}\rangle$, in the sense that

$$\left(\frac{\Delta\langle\mathcal{O}\rangle}{\langle\mathcal{O}\rangle}\right)^2 = \frac{\langle\mathcal{O}\rangle^2 - \overline{\mathcal{O}}^2_{N_K}}{\langle\mathcal{O}\rangle^2} \propto N_K^{-1}. \tag{58}$$

[6] A correct identification of the transient requires an analysis of the numerical stability of the observables against the length of this cut.

Therefore, in a numerical simulation, $\langle\mathcal{O}\rangle$ can in principle be obtained with the desired accuracy, by tuning the length of the Markov chain. The statistical uncertainty $\Delta\mathcal{O}$ can also be quantified. However, due to the correlation of the configurations, simple Gaussian statistics is not applicable in a straightforward way. Nevertheless, Gaussian statistics can be applied after the values of the observable have been averaged over bins of size equal to or larger than $\tau_{\mathcal{O}}$. Practically, this means that the number of independent estimates of $\mathcal{O}$ is not N_K, as one would naïvely think, but rather $N_K/\tau_{\mathcal{O}}$.

Another issue that needs to be taken into account in order to provide a reliable estimator for $\langle\mathcal{O}\rangle$ is the bias. Technically, one says that an estimate is biased if it does not agree with the analytical value when the latter is computable. While any bias will disappear for $N_K \to \infty$, a bias could appear for naïve estimates at finite N_K. There are standard methods to remove biases, the most popular ones being jack-knife and bootstrap. Although the details of those methods will not be discussed any further (we refer to ref. [46] for a pedagogical introduction), it is important to be aware that extracting a numerical value and the corresponding error for an observable measured in numerical simulations using Monte Carlo methods requires a careful analysis, the reason being that the data are correlated. This also applies to fits of Monte Carlo data [47]. Modern lattice calculations, including those discussed in this review, use robust procedures to estimate errors on observables and fitting parameters. The particular procedure used in each calculation is generally discussed in the corresponding original publication, to which we refer for details.

6. Lattice simulations of SU(N) gauge theories

Monte Carlo calculations in lattice gauge theory aim at computing numerically values of physical observables. Typical quantities that are computed are connected correlation functions of two operators at zero momentum:

$$\mathcal{C}(\tau) = \frac{1}{N_t}\sum_{t=0}^{N_t-1}\Big(\langle\mathcal{O}_1^\dagger(t)\mathcal{O}_2(t+\tau)\rangle - \langle\mathcal{O}_1^\dagger(t)\rangle\langle\mathcal{O}_2(t+\tau)\rangle\Big)\,, \tag{59}$$

where N_t is the number of sites in the Euclidean-time direction of the lattice, and the operator is averaged over all the other three spatial coordinates, to project onto zero-momentum states. We have assumed that the Euclidean-time direction is compact, and that periodic (antiperiodic) boundary conditions along it are imposed for bosonic (respectively: fermionic) fields. For asymptotically large N_t,

$$\mathcal{C}(\tau) \propto e^{-m\tau}, \tag{60}$$

where m is the mass of the lowest-lying state that connects $\mathcal{O}_1|0\rangle$ to $\mathcal{O}_2|0\rangle$. If $\mathcal{O}_i$ are traces of closed loop operators carrying well-defined J^{PC} quantum numbers (note that in this context J refers to the dihedral group, to which the group of spatial rotations is reduced by the lattice structure), $\mathcal{C}(\tau)$ will identify glueball states. If the $\mathcal{O}_i$ operator is a fermion bilinear with well-defined J^{PC}, then $\mathcal{C}$ will be saturated by mesonic states. Finally, if the $\mathcal{O}_i$ are Polyakov loops wrapping around a compact spatial direction (with periodic boundary conditions), then the propagating states will be torelons, which will have a mass that is proportional to the string tension. Note that, in practice, extracting a signal with sufficient accuracy for a meaningful determination of the mass can be challenging. This problem can be overcome, by building various operators with the same quantum numbers that can be used in a variational approach (see e.g. ref. [48]). The variational calculation also allows one to extract masses of excitations in the given channels.

Another important remark is that the quality of the numerical data does not decrease with N: simple large-N counting arguments show that the ratio noise over signal is constant in N for pure gluonic correlators, while improves as $1/\sqrt{N}$ for fermionic correlators.

Thermodynamic properties can be studied in a finite temperature setup, in which the lattice has $N_s^3 \times N_t$ sites (N_s is the number of sites in each spatial direction, while N_t is the number of sites in the temporal direction), with $N_s \gg N_t$, and, again, periodic (antiperiodic) boundary conditions for bosonic (fermionic) fields along the compact direction, whose size is related to the temperature T by $T = 1/(aN_t)$. The deconfinement temperature is identified by looking at the peak of the susceptibility of the order parameter, which for a deconfining transition in the Yang-Mills theory is the Polyakov loop.[7] The scaling of the position (in β) of this susceptibility as $N_s \to \infty$ allows to identify the critical value of β (which is a function of N_t and of the order of the phase transition). The framework that enables us to perform those studies is the theory of finite size scaling.

Numerical studies of a lattice gauge theory with Monte Carlo techniques involve the following steps:

1. on a lattice of fixed size $\mathcal{S}$ and at a fixed value of the lattice coupling β (and of the hopping parameter κ, in the presence of dynamical fermions), evaluate numerically the vacuum expectation values of operators corresponding to physical observables;
2. at the same lattice couplings, perform numerical simulations on lattices of larger volume, in order to extrapolate to the thermodynamic limit;
3. repeat these two steps at various couplings in order to determine the value of the observables in the continuum limit.

In the programme, there are two extrapolation processes. Both of them involve fitting the Monte Carlo data to analytical behaviors whose leading order in the correcting parameters is known. For instance, on a lattice of linear size L, in the chiral limit the mass of the pseudoscalar meson m_π receives finite size corrections that, at the leading order, are proportional to $e^{-m_\pi L}$ [49]. Fitting the lattice data on various sizes with this *Ansatz* in the region in which the asymptotic behavior is reached (which has to be determined as a part of the simulation) allows one to extract m_π in the infinite volume limit.

Asymptotic freedom dictates that the continuum limit of the theory is reached for $\beta \to \infty$. In fact, at the lowest order, perturbation theory[8] predicts that

$$a = \frac{1}{\Lambda_{\rm lat}} e^{-\frac{12\pi^2}{11N^2}\beta} \tag{61}$$

(up to subleading corrections), where $\Lambda_{\rm lat}$ denotes the dynamically generated mass scale in the lattice regularization scheme. This formula implies that the lattice spacing goes (exponentially) to zero when $\beta \to \infty$. The variation of an observable with a is predicted by the Callan-Symanzik equation. The existence of a well-defined continuum limit implies that for two observables of the same mass dimension $\mathcal{O}_1$ and $\mathcal{O}_2$

$$\lim_{\beta\to\infty} \frac{\hat{\mathcal{O}}_1}{\hat{\mathcal{O}}_2} = \frac{\mathcal{O}_1}{\mathcal{O}_2}, \tag{62}$$

[7] Strictly speaking, the Polyakov loop is a well-defined order parameter only on an infinite lattice, because there can not be any phase transition in a system with finitely many degrees of freedom. For this reason, one normally considers the *modulus* of the average Polyakov loop in a configuration as a (pseudo-)order parameter, and identifies the location of the critical point by monitoring its susceptibility, when the system parameters are varied.

[8] For SU(N) gauge theories, in the lattice scheme the Symanzik β-function, which determines the variation of a as a function of the coupling g_0, is known to three loops [50, 51].

where $\hat{\mathcal{O}}_i = a^{-d_i}\mathcal{O}_i$ and d_i is the mass dimension of $\mathcal{O}_i$. At the leading non-trivial order in a, near the continuum limit[9]

$$\frac{\hat{\mathcal{O}}_1}{\hat{\mathcal{O}}_2} = \frac{\mathcal{O}_1}{\mathcal{O}_2} + O(a^2 M^2), \tag{63}$$

where M^2 denotes an observable with mass dimension 2 [52, 53]. This expression implies that, asymptotically, lattice corrections are quadratic in the lattice spacing. Values of β for which observables fulfill eq. (63) are said to be in the scaling region.

We stress again that, in order for the extrapolations described above to be meaningful, the system must be in the correct regime. For instance, in a small volume, deconfinement might arise and as a result the numerical data extracted in this phase are not simply related with their infinite volume limit. Similarly, the limit corresponding to strong (bare) lattice gauge coupling ($\beta \to 0$) is heavily dominated by discretization artifacts, whose properties (like, e.g., the existence of confinement [9] and the finiteness of the mass gap [54], which were proven analytically in the early days of lattice QCD) are not necessarily relevant for the continuum theory.[10] Nevertheless, strong coupling expansions are a useful theoretical tool in lattice gauge theory, and are characterized by a finite convergence radius [60, 58]. For the Wilson discretization of SU(N) Yang-Mills theories in four spacetime dimensions, it is known that, for $N \geq 5$, the range of (weak) couplings, which is analytically connected to the continuum limit, is separated from the strong-coupling regime by a strong, first-order bulk transition—which is signalled by a discontinuity in the expectation value of the plaquette, and which is *not* related to any symmetry breaking pattern—, while for $N < 5$ a crossover connects the strong- and weak-coupling regimes [61]. In order to probe the region of couplings analytically connected to the continuum limit, the simulations have to be performed at values of bare lattice 't Hooft coupling λ_0 smaller than the critical value corresponding to the bulk transition leading to the strong-coupling regime—which, for $N \to \infty$, has been numerically estimated to be at 0.3596(2) [62].[11]

SU(N) Yang-Mills theory dynamically generates a scale. The whole physical spectrum can be expressed in terms of this scale. Hence, in order to meaningfully compare theories at different N, a scale needs to be fixed. This could be for instance the string tension σ (i.e. the asymptotic slope of the confining potential at large distances) or the deconfinement temperature T_c. Once the choice has been made, large-N arguments predict the scaling with N of all other quantities relevant for the infrared dynamics. In particular, if a well-defined large-N theory exists, all spectral quantities should have a finite large-N limit. From perturbation theory, we expect that the leading finite-N corrections are of order $1/N^2$ for the gauge theory, and of order $1/N$ for the theory with dynamical fermions. Taking further the perturbative argument, one would expect that pure gauge observables can be expressed in a power series in $1/N^2$, while in the presence of dynamical fermions the power series is in $1/N$. In this language, the proximity of SU(3) to SU(∞) means that the series converges for $N = 3$. Moreover, the large-N approach is useful to describe QCD if for a comprehensive set of observables a reasonable approximation (to the order

[9] We are assuming that boundary conditions do not introduce corrections that are proportional to a, which is not always the case, but it is true in most simulations—including, in particular, those that use (anti-)periodic boundary conditions.

[10] Indeed, in the limit of strong bare lattice gauge coupling, even compact U(1) lattice gauge theory is confining [55]—while it is in a Coulomb phase at weak coupling [56, 57, 58, 59].

[11] It is interesting to note that this value is close to some estimates, worked out with (truncated) analytical expansions based on the large-N limit, which were already obtained in the early 1980's [63, 64].

of few percents) can be obtained by retaining only few leading corrections, with the quality of the approximation systematically improving when higher-order corrections are added. Note that, in the large-N limit, the quantity of reference, that one uses for comparing the results in theories with a different number of colors N, does not play any rôle: different quantities may be affected by finite-N corrections with different coefficients, but each of them tends to a well-defined value in the $N \to \infty$ limit.

We conclude this section with a brief discussion about the question, whether the continuum and the large-N limits commute. As pointed out in ref. [65], in general the interchange of these two limits may be non-trivial [66, 67]—especially if there exists a set of degrees of freedom, whose number does not grow with N, but which nevertheless have a strong effect on the dynamics at the cut-off scale at any finite N. However, as discussed in refs. [65, 68], for the gauge theories that we are presently interested in, one can safely assume that the continuum and the large-N limits commute. In other words, if we want to study the theory at infinite N, we can either take first the continuum limit at fixed N and then the large-N limit, or take first the large-N limit at fixed cut-off a and then the continuum limit. In the latter approach, the lattice spacing is kept fixed across the various N, by simulating the various theories for β such that the value of a physical quantity (e.g. the string tension or the critical temperature) has a predefined value in units of the lattice spacing a. While performing the large-N limit at fixed lattice spacing should be seen as an intermediate step towards getting continuum large-N physics, this approach can prove convenient in calculations that are particularly demanding from the computational point of view.

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