Part IB - Metric and Topological Spaces

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Metrics
Definition and examples. Limits and continuity. Open sets and neighbourhoods. Characterizing limits and continuity using neighbourhoods and open sets. [3]

Topology
Definition of a topology. Metric topologies. Further examples. Neighbourhoods, closed sets, convergence and continuity. Hausdorff spaces. Homeomorphisms. Topological and non-topological properties. Completeness. Subspace, quotient and product topologies. [3]

Connectedness
Definition using open sets and integer-valued functions. Examples, including intervals. Components. The continuous image of a connected space is connected. Path-connectedness. Path-connected spaces are connected but not conversely. Connected open sets in Euclidean space are path-connected. [3]

Compactness
Definition using open covers. Examples: finite sets and [0, 1]. Closed subsets of compact spaces are compact. Compact subsets of a Hausdorff space must be closed. The compact subsets of the real line. Continuous images of compact sets are compact. Quotient spaces. Continuous real-valued functions on a compact space are bounded and attain their bounds. The product of two compact spaces is compact. The compact subsets of Euclidean space. Sequential compactness. [3]

Contents

0 Introduction

In Analysis, we defined a lot of things for real numbers. For example, if (x_n) is a sequence in $\mathbb{R}$, $x_n \to x$ if

$$(\forall \varepsilon > 0)(\exists N)(\forall n > N)\, |x_n - x| < \varepsilon.$$

$f : \mathbb{R} \to \mathbb{R}$ is said to be continuous iff $f(x_n) \to f(x)$ whenever $x_n \to x$.

How can we make sense of these notions, when we are no longer talking about the reals? For example,

- Let $X = \{A \in M_{k\times k}(\mathbb{R}) : A^T = A\}$ be the set of real symmetric matrices. Define $F : X \to \mathbb{R}$ to be $F(a) =$ the largest eigenvalue of A. Is this continuous?
- Let $X = \{f : [0,1] \to \mathbb{R} : f \text{ is continuous}\}$. Let $F : X \to \mathbb{R}$ map $F(f) = f(\frac{1}{2})$. Is this continuous?

We also have a lot of theorems about continuous functions. For example, the intermediate value theorems says that if $f : [0,1] \to \mathbb{R}$ is continuous, $f(0) < 0 < f(1)$, then $\exists x \in [0,1]$ such that $f(x) = 0$. The maximum value theorem says that if $f : [0,1] \to \mathbb{R}$ is continuous, $\exists x \in [0,1]$ such that $f(x) \geq f(y)$ for all $y \in [0,1]$.

How can we extend these to non-$\mathbb{R}$ spaces?

We will answer these questions in this course.

1 Metric spaces

Let $(\mathbf{v}_n) = ((x_n, y_n))$ be a sequence in $\mathbb{R}^2$. What should it mean for $(\mathbf{v}_n) \to \mathbf{v} = (x, y)$.

Intuitively, it should mean $(x_n) \to x$ and $(y_n) \to y$. This is equivalent to saying that $|x_n - x| \to 0$ and $|y_n - y| \to 0$, or $(x_n - x)^2 + (y_n - y)^2 \to 0$. This in turn is equivalent to saying that $|\mathbf{v}_n - \mathbf{v}| \to 0$, where $|\mathbf{v}_n - \mathbf{v}|$ is the Euclidean distance from $\mathbf{v}_n$ to $\mathbf{v}$, defined as $|(x, y)| = \sqrt{x^2 + y^2}$.

So we can define sequence convergence just by considering the distances between points.

Definition (Metric space)**.** A *metric space* is a pair (X, d_X) where X is a set (the space) and d_X is a function $d_X : X \times X \to \mathbb{R}$ (the metric) such that (for all x, y, z.

- $d_X(x, y) \geq 0$ (non-negativity)
- $d_X(x, y) = 0$ iff $x = y$ (identity of indiscernibles)
- $d_X(x, y) = d_X(y, x)$ (symmetry)
- $d_X(x, z) \leq d_X(x, y) + d_X(y, z)$ (triangle inequality)

Example.

- Let $X = \mathbb{R}^n$. Let

$$d(\mathbf{v}, \mathbf{w}) = |\mathbf{v} - \mathbf{w}| = \sqrt{\sum_{i=1}^{n}(v_i - w_i)^2}.$$

  This is the Euclidean metric.

- Let X be a set, and

$$d_X(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

  The first three axioms are trivially satisfied. How about the fourth? The left hand side is either 0 or 1, and RHS is 0, 1, 2. It can only possibly fail if RHS $= 0$, but this means $x = y = z$. So LHS $= 0$ as well. So we are safe.

  This is the discrete metric.

Definition (Metric subspace)**.** Let (X, d_X) be a metric space, and $Y \subseteq X$. Then (Y, d_Y) is a metric space, where $d_Y(a, b) = d_X(a, b)$, and said to be a *subspace* of X.

Example. For example, $S^n = \{\mathbf{v} \in \mathbb{R}^{n+1} : \mathbf{v} = 1\}$, the n-dimensional sphere, is a subspace of $\mathbb{R}^{n+1}$.

Definition (Convergent sequences)**.** Let (x_n) be a sequence in a metric space (X, d_X). We say (x_n) *converges to* $x \in X$, written $x_n \to x$, if $(d(x_n, x)) \to 0$. Equivalently,

$$(\forall \varepsilon)(\exists N)(\forall n > N)\, d(x_n, x) < \varepsilon.$$

Example.

- If $(\mathbf{v}_n)$ is a sequence in $\mathbb{R}^k$ with the Euclidean metric, with $\mathbf{v}_n = (v_n^1, \cdots, v_n^k)$, $\mathbf{v} = (v^1, \cdots, v^k) \in \mathbb{R}^k$, then $\mathbf{v}_n \to \mathbf{v}$ iff $(v_n^i) \to v^i$ for all i.
- If X has the discrete metric, then $x_n \to x$ iff $x_n = x$ for all but finitely many n, ie. it is eventually always x (take $\varepsilon = \frac{1}{2}$).

Proposition. If (X, d) is a metric place, (x_n) is a sequence in X such that $x_n \to x$, $x_n \to x'$, then $x = x'$.

Proof. Given $\varepsilon > 0$. Then $\exists N$ such that $d(x_n, x) < \varepsilon/2$ if $n > N$, and $\exists N'$ such that $d(x_n, x') < \varepsilon/2$ if $n > N'$. Then if $n > \max(N, N')$, then

$$\begin{aligned} 0 \le d(x, x') &\le d(x, x_n) + d(x_n, x') \\ &= d(x_n, x) + d(x_n, x') \\ &\le \varepsilon. \end{aligned}$$

So $0 \le d(x, x') \le \varepsilon$ for all $\varepsilon > 0$. So $d(x, x') = 0$, and $x = x'$.

Note that we used all of the axioms. We used non-negativity when we said $0 \le d(x, x')$. We used the triangle inequality for the next inequality, and symmetry to swap $d(x, x_n)$ with $d(x_n, x)$. Now we use the identity of indiscernibles to say that since $d(x, x') = 0$, they must be equal. $\square$

Definition (Continuity)**.** If (X, d_X) and (Y, d_Y) are metric spaces, and $f : X \to Y$, we say f is *continuous* if $f(x_n) \to f(x)$ (in Y) whenever $x_n \to x$ (in X).

Example. Let $X = \mathbb{R}$ with the Euclidean metric. Let $Y = \mathbb{R}$ with the discrete metric. Then $f : X \to Y$ that maps $f(x) = x$ is not continuous. This is since $1/n \to 0$ in the Euclidean metric, but not in the discrete metric.

However, $g : Y \to X$ by $g(x) = x$ is continuous, since a sequence in Y that converges is eventually constant.

1.1 Examples of metric spaces

Example (Manhattan metric)**.** Let $X = \mathbb{R}^2$, and

$$d(\mathbf{x}, \mathbf{y}) = d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|.$$

The first three axioms are again trivial. To prove the triangle inequality, we have

$$\begin{aligned} d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) &= |x_1 - y_1| + |x_2 - y_2| + |y_1 - z_1| + |y_2 - z_2| \\ &\ge |x_1 - z_1| + |z_2 - z_2| \\ &= d(\mathbf{x}, \mathbf{z}), \end{aligned}$$

using the triangle inequality for $\mathbb{R}$. This is the distance from a point to another where you are only allowed to move horizontally or vertically, but not diagonally.

Example (British railway metric)**.** Let $x = \mathbb{R}^2$. We define

$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} |\mathbf{x} - \mathbf{y}| & \text{if } \mathbf{x} = k\mathbf{y} \\ |\mathbf{x}| + |\mathbf{y}| & \text{otherwise} \end{cases}$$

To explain the name of this metric, think of Britain with London as the origin. Since the railway system is ~~stupid~~ less than ideal, all trains go through London.

For example, if you want to go from Oxford to Cambridge (and obviously not the other way round), you go from Oxford to London, then London to Cambridge. So the distance traveled is the distance from London to Oxford plus the distance from London to Cambridge.

The exception is when the two destinations lie along the same line, in which case, you can directly take the train from one to the other without going through London, and hence the "if $\mathbf{x} = k\mathbf{y}$" clause.

Example (p-adic metric)**.** Let $p \in \mathbb{Z}$ be a prime number. Then define

$$|n|_p = \begin{cases} p^{-k} & n = p^k m, p \nmid m \\ 0 & n = 0 \end{cases}$$

Take $X = \mathbb{Z}$, $d_p(a, b) = |a - b|_p$. Observe that

$$|a - b|_p \leq \max\{|a|_p, |b|_p\}$$

by doing some funny stuff about divisibility. So the triangle inequality follows.

With respect to d_2, we have $1, 2, 4, 8, 16, 32, \cdots \to 0$.

Example (Uniform metric)**.** Let $X = C[0, 1]$, the set of all continuous functions on $[0, 1]$. Then define

$$d(f, g) = \max_{x \in [0,1]} |f(x) - g(x)|$$

which exists since continuous functions are bounded and attains their bounds.

For example, let $F : C[0, 1] \to \mathbb{R}$ be defined by $F(f) = f(\frac{1}{2})$. Then this is continuous with respect to the uniform metric on $C[0, 1]$ and the usual metric on $\mathbb{R}$.

If $f_n \to f$ in the uniform metric, then we have to show that $F(f_n) \to F(f)$, ie. $f_n(\frac{1}{2}) \to f(\frac{1}{2})$.

Let $a_n = |f_n(\frac{1}{2}) - f(\frac{1}{2})|$ and $b_n = \max |f_n(x) - f(x)|$. Then trivially

$$0 \leq a_n \leq b_n \to 0.$$

So $a_n \to 0$. So $f_n(\frac{1}{2}) \to f(\frac{1}{2})$.

1.2 Norms

Definition (Norm)**.** If V is a real vector space, a *norm* on V is a function $\|\cdot\| : V \to \mathbb{R}$ such that

- $\|\mathbf{v}\| \geq 0$ for all $\mathbf{v} \in V$
- $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.
- $\|\lambda\mathbf{v}\| = |\lambda|\|\mathbf{v}\|$
- $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$.

Example. Let $V = \mathbb{R}^n$. We can let

$$\|\mathbf{v}\|_1 = \sum_{i=1}^n |v_i|.$$

We can also let

$$\|\mathbf{v}\|_2 = \sqrt{\sum_{i=1}^n v_i^2},$$

the *Euclidean norm*.

Finally, we can have

$$\|\mathbf{v}\|_\infty = \max\{|v_i| : 1 \leq i \leq n\}.$$

In general, we have

$$\|\mathbf{v}\|_p = \left(\sum_{i=1}^n |v_i|^p\right)^{1/p}.$$

for any $1 \leq p \leq \infty$ is a norm, and $\|\mathbf{v}\|_\infty$ is the limit as $p \to \infty$.

Proof that these are norms are left as an exercise for the reader (in the example sheets).

Lemma. If $\|\cdot\|$ is a norm on V, then

$$d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|$$

defines a metric on V.

Proof.

(i) $d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\| \geq 0$ by the definition of the norm.

(ii) $d(\mathbf{v}, \mathbf{w}) = 0 \Leftrightarrow \|\mathbf{v} - \mathbf{w}\} = 0 \Leftrightarrow \mathbf{v} - \mathbf{w} = \mathbf{0} \Leftrightarrow \mathbf{v} = \mathbf{w}$.

(iii) $d(\mathbf{w}, \mathbf{v}) = \|\mathbf{w} - \mathbf{v}\| = \|(-1)(\mathbf{v} - \mathbf{w})\| = |-1|\|\mathbf{v} - \mathbf{w}\| = d(\mathbf{v}, \mathbf{w})$.

(iv) $d(\mathbf{u}, \mathbf{v}) + d(\mathbf{v}, \mathbf{w}) = \|\mathbf{u} - \mathbf{v}\| + \|\mathbf{v} - \mathbf{w}\| \geq \|\mathbf{u} - \mathbf{w}\| = d(\mathbf{u}, \mathbf{w})$.

□

Definition (Inner product)**.** If V is a real vector space, an *inner product* on V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ such that

(i) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ for all $\mathbf{v} \in V$

(ii) $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

(iii) $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$.

(iv) $\langle \mathbf{v}_1 + \lambda \mathbf{v}_2, \mathbf{w} \rangle = \langle \mathbf{v}_1, \mathbf{w} \rangle + \lambda \langle \mathbf{v}_2, \mathbf{w} \rangle$.

Example.

(i) Let $V = \mathbb{R}^n$. Then
$$\langle \mathbf{v}, \mathbf{w}\rangle = \sum_{i=1}^n v_i w_i$$
is an inner product.

(ii) Let $V = C[0, 1]$. Then
$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x$$
is an inner product.

To show that inner products give norms, we need the Cauchy-Schwarz inequality:

Theorem (Cauchy-Schwarz inequality)**.** If $\langle \cdot, \cdot\rangle$ is an inner product, then
$$\langle \mathbf{v}, \mathbf{w}\rangle^2 \leq \langle \mathbf{v}, \mathbf{v}\rangle\langle \mathbf{w}, \mathbf{w}\rangle.$$

Proof. For any x, we have
$$\langle \mathbf{v} + x\mathbf{w}, \mathbf{v} + x\mathbf{w}\rangle = \langle \mathbf{v}, \mathbf{v}\rangle + 2x\langle \mathbf{v}, \mathbf{w}\rangle + x^2\langle \mathbf{w}, \mathbf{w}\rangle \geq 0.$$
Seen as a quadratic in x, since it is always non-negative, it can have at most one real root. So
$$(2\langle \mathbf{v}, \mathbf{w}\rangle)^2 - 4\langle \mathbf{v}, \mathbf{v}\rangle\langle \mathbf{w}, \mathbf{w}\rangle \leq 0.$$
So the result follows. $\square$

Lemma. If $\langle \cdot, \cdot\rangle$ is an inner product on V, then
$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}$$
is a norm.

Proof.

(i) $\|\mathbf{v}\| = \sqrt{\|\mathbf{v}, \mathbf{v}\|} \geq 0$.

(ii) $\|\mathbf{v}\| = 0 \Leftrightarrow \langle \mathbf{v}, \mathbf{v}\rangle \Leftrightarrow \mathbf{v} = 0$

(iii) $\|\lambda\mathbf{v}\| = \sqrt{\langle \lambda\mathbf{v}, \lambda\mathbf{v}\rangle} = \sqrt{\lambda^2\langle \mathbf{v}, \mathbf{v}\rangle} = |\lambda|\|\mathbf{v}\|$.

(iv)
$$\begin{aligned}(\|\mathbf{v}\| + \|\mathbf{w}\|)^2 &= \|\mathbf{v}\|^2 + 2\|\mathbf{v}\|\|\mathbf{w}\| + \|\mathbf{w}\|^2\\ &\geq \langle \mathbf{v}, \mathbf{v}\rangle + 2\langle \mathbf{v}, \mathbf{w}\rangle + \langle \mathbf{w}, \mathbf{w}\rangle\\ &= \|\mathbf{v} + \mathbf{w}\|^2\end{aligned}$$

$\square$

Example. We have the following norms on $C[0,1]$:

$$\|f\|_1 = \int_0^1 |f(x)|\,\mathrm{d}x$$
$$\|f\|_2 = \sqrt{\int_0^1 f(x)^2\,\mathrm{d}x}$$
$$\|f\|_\infty = \max_{x\in[0,1]} |f(x)|$$

The first two are the L^1 and L^2 norms, and the last is the uniform norm, since it induces the uniform metric.

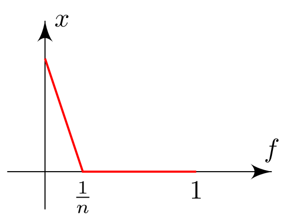
To show that these are norms, we have to show that they are 0 iff the function is constantly 0.

Lemma. Let $f \in C[0,1]$ satisfy $f(x) \geq 0$ for all $x \in [0,1]$. Then if $f(x)$ is not constantly 0, we have $\int_0^1 f(x)\,\mathrm{d}x > 0$.

Proof. Pick $x_0 \in [0,1]$ with $f(x_0) = a > 0$. Then since f is continuous, there is a δ such that $|f(x) - f(x_0)| < a/2$ if $|x - x_0| < \delta$. So $|f(x)| > a/2$ in this region.

Take
$$G(x) = \begin{cases} a/2 & |x - x_0| < \delta \\ 0 & \text{otherwise} \end{cases}$$
Then $f(x) \geq G(x)$ for all $x \in [0,1]$. So
$$\int_0^1 f(x)\,\mathrm{d}x \geq \int_0^1 G(x)\,\mathrm{d}x = \frac{a}{2}\cdot(2\delta) > 0.$$
$\square$

Example. Let $X = C[0,1]$, and let
$$d_1(f,g) = \|f - g\|_1 = \int_0^1 |f(x) - g(x)|\,\mathrm{d}x.$$
Let
$$f_n = \begin{cases} 1 - nx & x \in [0, \frac{1}{n}] \\ 0 & x \geq \frac{1}{n}. \end{cases}$$



Then
$$\|f\|_1 = \frac{1}{2}\cdot\frac{1}{n}\cdot 1 = \frac{1}{2n} \to 0$$
as $n \to \infty$. So $f_n \to 0$ in (X, d_1) where $0(x) = 0$.

On the other hand,

$$\|f_n\|_\infty = \max_{x\in[0,1]} \|f(x)\| = 1.$$

So $f_n \not\to 0$ in the uniform metric.

So the function $(C[0,1], d_1) \to (C[0,1], d_\infty)$ that maps $f \mapsto f$ is *not* continuous. Note that this is similar to the case that the constant function from the usual metric of $\mathbb{R}$ to the discrete metric of $\mathbb{R}$ is not continuous. However, the discrete metric is a *silly* metric, but d_1 is a genuine useful metric here.

Also the function $G : (C[0,1], d_1) \to (\mathbb{R}, \text{usual})$ with $G(f) = f(0)$ is not continuous, by the example above.

Note that G is a linear function, but is not continuous with respect to d_1. However, this does *not* mean that d_1 is a stupid metric to use. It turns out that no matter what norm we pick, we can always produce a linear function that is not continuous.

1.3 Open and closed subsets

Let (X, d) be a metric space.

Definition (Open and closed balls)**.** For any $x \in X$, $r \in \mathbb{R}$,

$$B_r(x) = \{y \in X : d(y, x) < r\}$$

is the *open ball* centered at x.

$$\bar{B}_r(x) = \{y \in X : d(y, x) \leq r\}$$

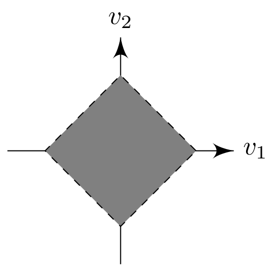
is the *closed ball* centered at X.

Example.

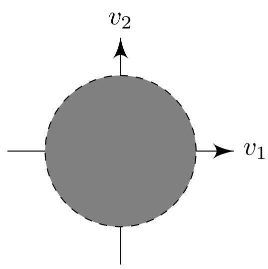
(i) When $X = \mathbb{R}$, $B_r(x) = (x - r, x + r)$. $\bar{B}_r(x) = [x - r, x + r]$.

(ii) When $X = \mathbb{R}^2$,

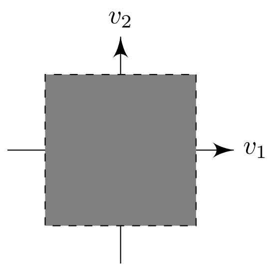
  (a) If d is the metric induced by the $\|\mathbf{v}\|_1 = \|v_1\| + \|v_2\|$, then an open ball is a rotated square.



  (b) If d is the metric induced by the $\|\mathbf{v}\|_2 = \sqrt{v_1^2 + v_2^2}$, then an open ball is an actual circle.



(c) If d is the metric induced by the $\|\mathbf{v}\|_\infty = \max\{|v_1|, |v_2|\}$, then an open ball is a square.



Definition (Open subset)**.** $U \subseteq X$ is an *open subset* if for every $x \in U$, $\exists \delta > 0$ such that $B_\delta(x) \subseteq U$.

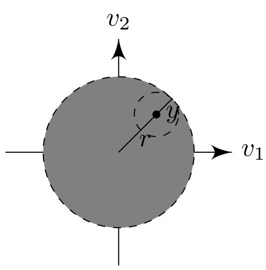
$C \subseteq X$ is a *closed subset* if $X \setminus C \subseteq X$ is open.

This is a very *very* important definition.

We first prove that this makes sense:

Lemma. The open ball $B_r(x) \subseteq X$ is an open subset, and the closed ball $\bar{B}_r(x) \subseteq X$ is a closed subset.

Proof. Given $y \in B_r(x)$, we must find $\delta > 0$ with $B_\delta(y) \subseteq B_r(x)$.



Since $y \in B_r(x)$, we must have $a = d(y, x) < r$. Let $\delta = r - a > 0$. Then if $z \in B_\delta(y)$, then

$$d(z, x) \leq d(z, y) + d(y, x) < (r - a) + a = r.$$

So $z \in B_r(x)$. So $B_r(y) \subseteq B_r(x)$ as desired.

The second statement is equivalent to $X \setminus \bar{B}_r(x) = \{y \in X : d(y, x) > r\}$ is open. The proof is very similar. $\square$

Note: $A \subseteq X$ being open depends on both A and X, not just A. For example, $[0, \frac{1}{2})$ is not an open subset of $\mathbb{R}$, but is an open subset of $[0, 1]$ (since it is $B_{\frac{1}{2}}(0)$), both with the Euclidean metric.

Note also that $A \subseteq X$ can be neither open nor closed. eg. $[0, 1)$ in $\mathbb{R}$ with the Euclidean metric. Also, $\mathbb{Q} \subseteq \mathbb{R}$ is neither open nor closed, since any open interval contains both rational and irrational numbers, and cannot be a subset of $\mathbb{Q}$ or $\mathbb{R} \setminus \mathbb{Q}$.

A subset can also be open *and* close. For example, let $X = [-1, 1] \setminus \{0\}$ with the Euclidean metric. Let $A = [-1, 0) \subseteq X$.. Then $A = B_1(-1)$ and $A = \bar{B}_{\frac{1}{2}}(-\frac{1}{2})$.

Definition (Open neighborhood)**.** If $x \in X$, an *open neighborhood* of x is an open $U \subseteq X$ with $x \in U$.

Lemma. If U is an open neighbourhood of x and $x_n \to x$, then $\exists N$ such that $x_n \in U$ for all $n > N$.

Proof. Since U is open, there exists some $\delta > 0$ such that $B_\delta(x) \subseteq U$. Since $x_n \to x$, $\exists N$ such that $d(x_n, x) < \delta$ for all $n > N$. This implies that $x_n \in B_\delta(x)$ for all $n > N$. So $x_n \in U$ for all $n > N$. $\square$

Definition (Limit point)**.** Let $A \subseteq X$. Then $x \in X$ is a *limit point* of A if there is a sequence $x_n \to x$ such that $x_n \in A$ for all n.

Example.

(i) If $a \in A$, then a is a limit point of A, by taking the sequence $a, a, a, a, \cdots$.

(ii) If $A = (0, 1) \subseteq \mathbb{R}$, then 0 is a limit point of A, eg. take $x_n = \frac{1}{n}$.

(iii) Every $x \in \mathbb{R}$ is a limit point of $\mathbb{Q}$.

Proposition. $C \subseteq X$ is a closed subset if and only if every limit point of C is an element of C.

Proof. ($\Rightarrow$) Suppose C is closed. Then $A = X \setminus C \subseteq X$ is open. Suppose X is a limit point of C, so $x_n \to x$, $x_n \in C$ for all n.

Suppose that $x \notin C$, then $x \in A$, which is open. So by the lemma, we know $\exists N$ such that $x_n \in A$ for all $n > N$. So $x_{N+1} \in A$ and $x_{N+1} \in C$, which is a contradiction. So we conclude that $x \in C$.

($\Leftarrow$) Suppose that C is not closed. Then A is not open. So $\exists x \in A$ such that $B_\delta(x) \not\subseteq A$ for all $\delta > 0$. This means that $B_\delta(x) \cap C \neq \emptyset$ for all $\delta > 0$.

So pick $x_n \in B_{\frac{1}{n}} \cap C$ for each $n > 0$. Then $x_n \in C$, $d(x_n, x) = \frac{1}{n} \to 0$. So $x_n \to x$. So x is a limit point of C which is not in C. $\square$

Now we come to the Really Important Result$^{\text{TM}}$:

Proposition (Characterization of continuity)**.** Let (X, d_x) and (Y, d_y) be metric spaces, and $f : X \to Y$. The following conditions are equivalent:

(i) f is continuous

(ii) If $x_n \to x$, then $f(x_n) \to f(x)$ (which is the definition of continuity)

(iii) For any closed subset $C \subseteq Y$, $f^{-1}(C)$ is closed in X.

(iv) For any open subset $U \subseteq Y$, $f^{-1}(U)$ is open in X.

(v) For any $x \in X$ and $\varepsilon > 0$, $\exists \delta > 0$ such that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$, or $d_x(x, z) < \delta \Rightarrow d_y(f(x), f(z)) < \varepsilon$.

Proof.

- $1 \Leftrightarrow 2$: by definition
- $2 \Rightarrow 3$: Suppose $C \subseteq Y$ is closed, and $x_n \to x$, where $x_n \in f^{-1}(C)$. We want to show that $x \in f^{-1}(C)$.

  We have $f(x_n) \to f(x)$ by (2) and $f(x_n) \in C$. So $f(x)$ is a limit point of C. Since C is closed, $f(x) \in C$. So $x \in f^{-1}(C)$. So every limit point of $f^{-1}(C)$ is in $f^{-1}(C)$. So $f^{-1}(C)$ is closed.
- $3 \Rightarrow 4$: If $U \subseteq Y$ is open, then $Y \setminus U$ is closed in X. So $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ is closed in X. So $f^{-1}(U) \subseteq X$ is open.
- $4 \Rightarrow 5$: Given $x \in X, \varepsilon > 0$, $B_\varepsilon(f(x))$ is open in Y. By (4), $f^{-1}(B_\varepsilon(f(x))) = A$ is open in X. Since $x \in A$, $\exists \delta > 0$ with $B_\delta(x) \subseteq A$. So
$$f(B_\delta(x)) \subseteq f(A) = f(f^{-1}(B_\varepsilon f(x))) \subseteq B_\varepsilon(f(x))$$
- $5 \Rightarrow 2$: Suppose $x_n \to x$. Given $\varepsilon > 0$, $\exists \delta > 0$ such that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$. Since $x_n \to x$, $\exists N$ such that $x_n \in B_\delta(x)$ for all $n > N$ Then $f(x_n) \in f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$ for all $n > N$. So $f(x_n) \to f(x)$.

□

Note that the third and fourth condition can be useful to decide if a subset is open or closed.

Example. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined as
$$f(x_1, x_2, x_3) = x_1^2 + x_2^4 x_3^6 + x_1^8 x_3^2.$$
Then this is continuous. So $\{\mathbf{x} \in \mathbb{R}^3 : f(x) \leq 1\} = f^{-1}((-\infty, 1])$ is closed in $\mathbb{R}^3$.

Lemma.

(i) Suppose $V_\alpha \subseteq X$ is open for all $\alpha \in A$. Then $U = \displaystyle\bigcup_{\alpha \in A} V_\alpha$ is open in X.

(ii) IF $V_1, \cdots, V_n \subseteq X$ are open, then so is $V = \displaystyle\bigcap_{i=1}^{n} V_i$.

Proof.

(i) If $x \in U$, then $x \in V_\alpha$ for some α. Since V_α is open, there exists $\delta > 0$ such that $B_\delta(x) \subseteq V_\alpha$. So $B_\delta(x) \subseteq \displaystyle\bigcup_{\alpha \in A} V_\alpha = U$. So U is open.

(ii) If $x \in V$, then $x \in V_i$ for all $i = 1, \cdots, n$. So $\exists \delta_i > 0$ with $B_{\delta_i}(x) \subseteq V_i$. Take $\delta = \min\{\delta_1, \cdots, \delta_n\}$. So $B_\delta(x) \subseteq V_i$ for all i. So $B_\delta(x) \subseteq V$. So V is open.

□

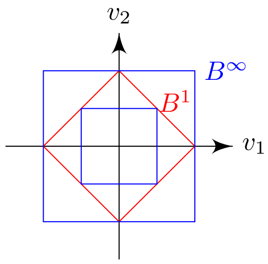
Note: An infinite intersection of open sets need not be open, eg. the intersection of all $(-\frac{1}{n}, \frac{1}{n})$ is $\{0\}$, which is not open.

2 Topological spaces

Definition (Inducing topology)**.** Two metrics d and d' on X *induce the same topology* if $U \subseteq X$ is open wrt to d iff $U \subseteq X$ is open wrt to d', ie. they produce the same open subsets.

If this is true, then $f : (X, d) \to (Y, d_Y)$ is continuous iff $f : (X, d') \to (Y, d_Y)$ is continuous, and $x_n \to x$ wrt d iff $x_n \to x$ wrt d'.

Example. Take $X = \mathbb{R}^n$ and $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_1$ and $d'(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_\infty$. These induce the same topology:
$\|\mathbf{v}\| = \sum_{i=1}^n |v_i|$ and $\|\mathbf{v}\|_\infty = \max_{1 \le i \le n} |v_i|$.
This implies that $\|\mathbf{v}\|_\infty \le \|\mathbf{v}\|_1 \le n\|\mathbf{v}\|\infty$.
So $B_r^\infty(x) \supseteq B_r^1(x) \supseteq B_{r/n}^\infty$.



If U is open with respect to d, and $x \in U$, then $\exists \delta > 0$ such that $B_\delta^1(x) \subseteq U$. So $B_{\delta/n}^\infty(x) \subseteq B_\delta^1(x) \subseteq U$.
Hence U is open with respect to d'. The other direction is similar.

Example. Let $X = C[0, 1]$. If $d(f, g) = \|f - g\|_1$ and $d'(f, g) = \|f - g\|_\infty$, they do not induce the same topology, since $(X, d) \to (X, d')$ by $f \mapsto f$ is not continuous.

If (X, d) is a metric space, to decide if $f : X \to Y$ or $g : Z \to X$ are continuous, it's enough to know what the open subsets of X are, and not the metric. And it turns out that in most cases, the metric isn't actually useful. So we can forget about the metrics, and keep the notion of open subsets:

2.1 Definitions

Recall that if X is a set, $\mathbb{P}(x) = \{A : A \subseteq X\}$ is the power set of X.

Definition (Topological space)**.** A *topological space* is a set X (the space) together with a set $\mathcal{U} \subseteq \mathbb{P}(X)$ (the topology) such that:

(i) $\phi, X \in \mathcal{U}$

(ii) If $V_\alpha \in \mathcal{U}$ for all $\alpha \in A$, then $\displaystyle\bigcup_{\alpha \in A} V_\alpha \in \mathcal{U}$.

(iii) If $V_1, \cdots, V_n \in \mathcal{U}$, then $\displaystyle\bigcap_{i=1}^n V_i \in \mathcal{U}$.

The elements of X are the *points*, and the elements of $\mathcal{U}$ are the open subsets of X.

Example. If (X, d) is a metric space, then

$$\mathcal{U} = \{U \subseteq X : U \text{ is open wrt to } d\}$$

is a topology.

Example.

(i) Let X be any set.

  (a) $\mathcal{U} = \{\phi, X\}$ is the *coarse topology* on X.

  (b) $\mathcal{U} = \mathbb{P}(X)$ is the *discrete topology* on X, since it is induced by the discrete metric.

  (c) $\mathcal{U} = \{A \subseteq X : X \setminus A \text{ is finite or } A = \emptyset\}$ is the *cofinite topology* on X.

(ii) Let $X = \mathbb{R}$, and $\mathcal{U} = \{(a, \infty) : a \in \mathbb{R}\}$ is the *order topology* on $\mathbb{R}$.

Definition (Continuous function)**.** If $f : X \to Y$ is a map of topological spaces, f is continuous if $f^{-1}(U)$ is open in X whenever U is open in Y.

Example.

(i) Any function $f : X \to Y$ is continuous if X has the discrete topology.

(ii) Any function $f : X \to Y$ is continuous if Y has the course topology.

(iii) If X and Y both have cofinite topology, then $f : X \to Y$ is continuous iff $f^{-1}(\{y\})$ is finite for every $y \in Y$.

Lemma. If $f : X \to Y$ and $g : Y \to Z$ is continuous, then so is $g \circ f : X \to Z$.

Proof. If $U \subseteq Z$ is open, g is continuous, then $g^{-1}(U)$ is open in Y. Since f is also continuous, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is open in X. □

Definition (Homeomorphism)**.** $f : X \to Y$ is a *homeomorphism* if

(i) f is a bijection

(ii) Both f and f^{-1} are continuous

Equivalently, f is a bijection and $U \subseteq X$ is open iff $f(U) \subseteq Y$ is open.

Two spaces are *homeomorphic* if there exists a homeomorphism between them, and we write $X \simeq Y$.

This is analogous to the concept of isomorphisms. Two spaces are homeomorphic if they are "the same".

Note: If $\phi : G_1 \to G_2$ is a bijective group homomorphism, then ϕ^{-1} is automatically a homomorphism. The analogous statement for continuous maps is false. f being continuous does not imply f^{-1} is continuous.

Example. Let $X = C[0, 1]$ with the topology induced by $\|\cdot\|_1$ and $Y = C[0, 1]$ with the topology induced by $\|\cdot\|_\infty$. Then $F : Y \to X$ by $f \mapsto f$ is continuous but F^{-1} is not.

Lemma. Homeomorphism is an equivalence relation.

(i) The identity map $I_X : X \to X$ is always a homeomorphism. So $X \simeq X$.

(ii) If $f : X \to Y$ is a homeomorphism, then so is $f^{-1} : Y \to X$. So $X \simeq Y \Rightarrow Y \simeq X$.

(iii) If $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms, then $g \circ f : X \to Z$ is an homeomorphism. So $X \simeq Y$ and $Y \simeq Z$ implies $X \simeq Z$.

Example.

(i) The open intervals $(0, 1) \simeq (a, b)$ for all $a, b \in \mathbb{R}$, where the topology is induced by the Euclidean metric, using the homeomorphism $x \mapsto a + (b - a)x$.

Similarly, $[0, 1] \simeq [a, b]$

(ii) $(-1, 1) \simeq \mathbb{R}$ by $x \mapsto \tan(\frac{\pi}{2}x)$.

(iii) $\mathbb{R} \simeq (0, \infty)$ by $x \mapsto e^x$.

(iv) $(a, \infty) \simeq (b, \infty)$ by $x \mapsto x + (b - a)$.

The fact that $\simeq$ is an equivalence relation implies that any 2 open intervals in $\mathbb{R}$ are homeomorphic.

Is $(0, 1)$ homeomorphic to $[0, 1]$? No, but that's hard to prove!

And is $\mathbb{R}$ homeomorphic to $\mathbb{R}^2$? We will be able to prove this a few lectures later. However, we will not be able to answer the question "is $\mathbb{R}^n \simeq \mathbb{R}^m$ for $n \neq m$" rigorously in this course.

To answer all these question, we need to introduce some tools.

2.2 Sequences

Definition (Open neighbourhood)**.** An *open neighbourhood* of $x \in X$ is an open set $U \subseteq X$ with $x \in U$.

Definition (Convergent sequence)**.** A sequence $x_n \to x \in X$ if for every open neighbourhood U of x, $\exists N$ such that $x_n \in U$ for all $n > N$.

Example.

(i) If X has the course topology, then any sequence x_n converges to every $x \in X$, since there is only one open neighbourhood of x.

(ii) If X has the cofinite topology, no two x_ns are the same, then $x_n \to x$ for every $x \in X$, since every open set can only have finitely many x_n not inside it.

That doesn't sound like what convergence should look like. Fortunately, there is a class of spaces where sequences behave nicely.

Definition (Hausdorff space)**.** A topological space X is *Hausdorff* if for all $x_1, x_2 \in X$ with $x_1 \neq x_2$, there exists open neighbourhoods U_1 of x_1, U_2 of x_2 such that $U_1 \cap U_2 = \emptyset$.

Lemma. If X is Hausdorff, x_n is a sequence in X with $x_n \to x$ and $x_n \to x'$, then $x = x'$, ie. limits are unique.

Proof. If $x \neq x'$, then there exists open neighbourhoods U, U' of x, x' with $U \cap U' = \emptyset$. Then $x_n \to x$ implies that $\exists N$ such that $x_n \in U$ for all $n > N$; $x_n \to x'$ implies that $\exists N'$ such that $x_n \in U'$ for all $n > N'$. Then for $n > \max(N, N')$, then $x_n \in U \cap U'$, which contradicts $U \cap U' = \emptyset$. So we must have $x = x'$. □

Example.

(i) If X has more than 1 element, then the course topology on X is not Hausdorff.

(ii) If X has infinitely many elements, the cofinite topology on X is not Hausdorff.

(iii) The discrete topology is always Hausdorff.

(iv) If (X, d) is a metric space, the topology induced by d is Hausdorff: for $x_1 \neq x_2$, let $r = d(x_1, x_2) > 0$. Then take $U_i = B_{r/2}(x_i)$. Then $U_1 \cap U_2 = 0$.

2.3 Closed sets

Definition (Closed sets)**.** $C \subseteq X$ is *closed* if $X \setminus C$ is an open subset of X.

Lemma.

(i) If C_α is a closed subset of X for all $\alpha \in A$, then $\bigcap_{\alpha \in A} C_\alpha$ is closed in X.

(ii) If $C_1, \cdots, C_n$ are closed in X, then so is $\bigcup_{i=1}^n C_i$.

Proof.

(i) Since C_α is closed in X, $X \setminus C_\alpha$ is open in X. So $\bigcup_{\alpha \in A}(X \setminus C_\alpha) = X \setminus \bigcap_{\alpha \in A} C_\alpha$ is open. So $\bigcap_{\alpha \in A} C_\alpha$ is closed.

(ii) If C_i is closed in X, then $X \setminus C_i$ is open. So $\bigcap_{i=1}^n (X \setminus C_i) = X \setminus \bigcup_{i=1}^n C_i$ is open. So $\bigcup_{i=1}^n C_i$ is closed.

□

Corollary. If X is Hausdorff and $x \in X$, then $\{x\}$ is closed in X.

Proof. For all $y \in x$, there exists open subsets U_y, V_y with $y \in U_y, x \in V_y$, $U_y \cap V_y = \emptyset$.

Let $C_y = X \setminus U_y$. Then C_y is closed, $y \notin C_y$, $x \in C_y$. So $\{x\} = \bigcap_{u \neq x} C_y$ is closed since it is an intersection of closed subsets. □

2.4 Closure and interior

2.4.1 Closure

Let X be a topological space and $A \subseteq X$. Define

$$\mathcal{C}_A = \{C \subseteq X : A \subseteq C \text{ and } C \text{ is closed in } X\}$$

Definition. The *closure* of A in X is

$$\bar{A} = \bigcap_{C \in \mathcal{C}_A} C.$$

Since X is closed in X (its complement $\emptyset$ is open in X), $\mathcal{C}_A \neq \emptyset$ and it makes sense to take the intersection.

Since $\bar{A}$ is an intersection of closed sets, it is closed in X. Also, if $C \in \mathcal{C}_A$, then $A \subseteq C$. So $A \subseteq \bigcap_{C \in \mathcal{C}_A} C = \bar{A}$.

Proposition. $\bar{A}$ is the smallest closed subset of X which contains A.

Proof. If $K \subseteq X$ is closed, $A \subseteq K$, then $K \in \mathcal{C}_A$. So $\bar{A} = \bigcap_{C \in \mathcal{C}_A} C \subseteq K$. □

We basically *defined* the closure such that it is the smallest closed subset of X which contains A.

However, while this "clever" definition makes it easy to prove the above property, it is rather difficult to directly use it to compute the closure.

We define

Definition (Limit point)**.** A *limit point* of A is an $x \in X$ such that there is a sequence $x_n \to x$ with $x_n \in A$ for all x_n.

Let

$$L(A) = \{x \in X : x \text{ is a limit point of } A\}$$

Lemma. If $C \subseteq X$ is closed, then $L(C) = C$.

Proof. Exactly the same as that for metric spaces. □

We proved the converse of this statement for metric spaces, but it is *false* for topological spaces in general!

Proposition. $L(A) \subseteq \bar{A}$.

Proof. If $A \subseteq C$, then $L(A) \subseteq L(C)$. If C is closed, then $L(C) = C$. So $C \in \mathcal{C}_A \Rightarrow L(A) \subseteq C$. So $L(A) \subseteq \bigcap_{C \in \mathcal{C}_A} C = \bar{A}$. □

Corollary. Suppose $C \subseteq X$ is closed and $A \subseteq C$ and $C \subseteq L(A)$, Then $C = \bar{A}$.

Proof. $C \subseteq L(A) \subseteq \bar{A} \subseteq C$, where the last step is since $\bar{A}$ is the smallest closed set containing A. So $C = L(A) = \bar{A}$. □

Example.

- Let $(a, b) \subseteq \mathbb{R}$. Then $\overline{(a, b)} = [a, b]$.
- Let $\mathbb{Q} \subseteq \mathbb{R}$. Then $\bar{\mathbb{Q}} = \mathbb{R}$.

– $\overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}$.

– In $\mathbb{R}^n$ with the Euclidean metric, $\overline{B_r(x)} = \bar{B}_r(X)$. In general, $\overline{B_r(x)} \subseteq \bar{B}_r(X)$, since $\bar{B}_r(x)$ is closed and $B_r(x) \subseteq \bar{B}_r(x)$, but these need not be equal.

  For example, if X has the discrete metric, then $B_1(x) = \{x\}$. Then $\overline{B_1(x)} = \{x\}$, but $\bar{B}_1(x) = X$.

Definition (Dense subset)**.** $A \subseteq X$ is *dense* in X if $\bar{A} = X$.

Example. $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are both dense in $\mathbb{R}$ with the usual topology.

2.4.2 Interior

Let $A \subseteq X$, and let $\mathcal{O}_A = \{U \subseteq X : U \subseteq A, U \text{ is open in } X\}$.

Definition (Interior)**.** The *interior* of A is

$$\operatorname{Int}(A) = \bigcup_{U \in \mathcal{O}} U.$$

Proposition. $\operatorname{Int}(A)$ is the largest open subset of X contained in X. Proof is similar to proof for closure.

What trick do we have to find the interior?

Proposition. $X \setminus \operatorname{Int}(A) = \overline{X \setminus A}$

Proof. $U \subseteq A \Leftrightarrow (X \setminus U) \supseteq (X \setminus A)$. Also, U open in $X \Leftrightarrow X \setminus U$ is closed in X.

So the complement of the largest open subset of X contained in A will be the smallest closed subset containing $X \setminus A$. ☐

Example. $\operatorname{Int}(\mathbb{Q}) = \operatorname{Int}(\mathbb{R} \setminus \mathbb{Q}) = \emptyset$.

2.5 New topologies from old

We start by revising some set theory:

If $Y \subseteq X$, then we have an inclusion map $\iota : Y \to X$ that sends $y \mapsto y$.

Definition (Subspace topology)**.** If X is a topological space and $Y \subseteq X$, the *subspace topology* on Y is given by: V is an open subset of Y if there is some U open in X such that $V = Y \cap U$.

If we simply write $Y \subseteq X$ and don't specify a topology, the subspace topology is assumed. For example, when we write $\mathbb{Q} \subseteq \mathbb{R}$, we are thinking of $\mathbb{Q}$ with the subspace topology inherited from $\mathbb{R}$.

Example. If (X, d) is a metric space and $Y \subseteq X$, then the metric topology on (Y, d) is the subspace topology, since $B_r^Y(y) = Y \cap B_r^X(y)$.

To show that this is indeed a topology, we need the following set theory facts:

$$Y \cap \left(\bigcup_{\alpha \in A} V_\alpha\right) = \bigcup_{\alpha \in A} (Y \cap V_\alpha)$$

$$Y \cap \left(\bigcap_{\alpha \in A} V_\alpha\right) = \bigcap_{\alpha \in A} (Y \cap V_\alpha)$$

Proposition. The subspace topology is a topology.

Proof.

(i) Since $\emptyset$ is open in X, $\emptyset = Y \cap \emptyset$ is open in Y.
Since X is open in X, $Y = Y \cap X$ is open in Y.

(ii) If V_α is open in Y, then $V_\alpha = Y \cap U_\alpha$ for some U_α open in X. Then

$$\bigcup_{\alpha \in A} V_\alpha = \bigcup_{\alpha \in A} (Y \cap U_\alpha) = Y \cap \left(\bigcup_{\alpha \in U} U_\alpha \right).$$

Since $\bigcup U_\alpha$ is open in X, so $\bigcap V_\alpha$ is open in Y.

(iii) If V_i is open in Y, then $V_i = Y \cap U_i$ for some open $U_i \subseteq X$. Then

$$\bigcap_{i=1}^{n} V_i = \bigcap_{i=a}^{n} (Y \cap U_i) = Y \cap \left(\bigcap_{i=a}^{n} U_i \right).$$

Since $\bigcap U_i$ is open, $\bigcap V_i$ is open.

□

Proposition. If Y has the subspace topology, $f : Z \to Y$ is continuous iff $\iota \circ f : Z \to X$ is continuous.

This is a defining property of the subspace topology.

Proof. ($\Rightarrow$) If $U \subseteq X$ is open, then $\iota^{-1}(U) = Y \cap U$ is open in Y. So ι is continuous. So if f is continuous, so is $\iota \circ f$.

($\Leftarrow$) Suppose we know that $\iota \circ f$ is continuous. Given $V \subseteq Y$ is open, we know that $V = Y \cap U = \iota^{-1}(U)$. So $f^{-1}(V) = f^{-1}(\iota^{-1}(U))) = (\iota \circ f)^{-1}(U)$ is open, since $\iota \circ f$ is continuous. So f is continuous. □

Example. $D^n = \{\mathbf{v} \in \mathbb{R}^n : |\mathbf{v}| \leq 1\}$ is the n-dimensional unit disk. $S^{n-1} = \{\mathbf{v} \in \mathbb{R}^n : |\mathbf{v}| = 1\}$ is the $n-1$-dimensional sphere.

We have

$$\mathrm{Int}(D^n) = \{\mathbf{v} \in \mathbb{R}^n : |\mathbf{v}| < 1\} = B_1(\mathbf{0}).$$

We want to show that $\mathrm{Int}(D^n) \simeq \mathbb{R}^n$. We choose a homeomorphism $f : [0, 1) \mapsto [1, \infty)$. Then $\mathbf{v} \mapsto f(|\mathbf{v}|)\mathbf{v}$ is a homeomorphism $\mathrm{Int}(D^n) \to \mathbb{R}^n$.

2.5.1 Products

If X and Y are sets, the product is

$$X \times Y = \{(x, y) : x \in X, y \in Y\}$$

We have the projection functions $\pi_1 : X \times Y \to X$, $\pi_2 : X \times Y \to Y$ by

$$\pi_1(x, y) = x, \quad \pi_2(x, y) = y.$$

If $A \subseteq X, B \subseteq Y$, then we have $A \times B \subseteq X \times Y$.

Definition (Product topology)**.** If X and Y are topological spaces, the *product topology* on $X \times Y$ is given by:

$U \subseteq X \times Y$ is open if for every $(x, y) \in U$, there exists $V_x \subseteq X, W_Y \subseteq Y$ open neighbourhoods of x and y such that $V_X \times W_Y \subseteq U$.

Example.

- If $V \subseteq X$ and $W \subseteq Y$ are open, then $V \times W \subseteq X \times Y$ is open (take $V_X = V$, $W_Y = W$).
- The product topology on $\mathbb{R} \times \mathbb{R}$ is same as the topology induced by the $\|\cdot\|_\infty$, hence is also the same as the topology induced by $\|\cdot\|_2$ or $\|\cdot\|_1$. Similarly, the product topology on $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ is also the same as that induced by $\|\cdot\|_\infty$

The defining property is that $f : Z \mapsto X \times Y$ is continuous iff $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.

Note that if $U \subseteq X \times Y$, then

$$U = \bigcup_{(x,y) \in U} V_x \times W_y.$$

So $U \subseteq X \times Y$ is open if and only if it is a union of sets of the form $V \times W$ where $V \subseteq X$ and $W \subseteq Y$ are open.

Definition (Basis)**.** Let $\mathcal{U}$ be a topology on X. A subset $\mathcal{B} \subseteq \mathcal{U}$ is a *basis* if "$U \in \mathcal{U}$ iff U is a union of sets in $\mathcal{B}$".

Example.

- $\{V \times W : V \subseteq X, W \subseteq Y \text{ are open}\}$ is a basis for the product topology for $X \times Y$.
- If (X, d) is a metric space, then

  $$\{B_{1/n}(x) : n \in \mathbb{N}^+, x \in X\}$$

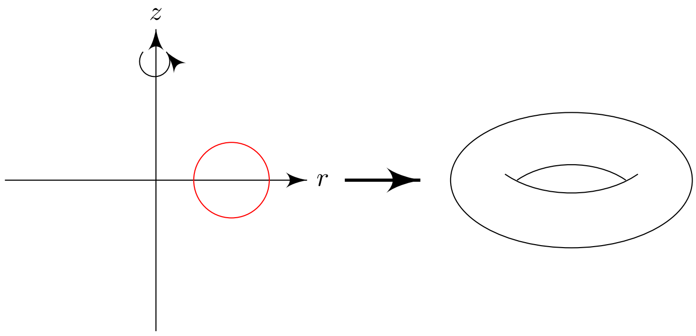
  is a basis for the topology induced by d.

Example.

- $(0, 1) \times (0, 1) \times \cdots \times (0, 1)^n \subseteq \mathbb{R}^n$ is the open $n-$dimensional cube in $\mathbb{R}^n$. Since $(0, 1) \simeq \mathbb{R}$, we have $(0, 1)^N \simeq \mathbb{R}^N \simeq \operatorname{Int}(D^n)$.
- $[0, 1] \times S^n \simeq [1, 2] \times S^n \simeq \{\mathbf{v} \in \mathbb{R}^{n+1} : 1 \leq |\mathbf{v}| \leq 2\}$, where the last homeomorphism is given by $(t, \mathbf{w}) \mapsto t\mathbf{w}$ with inverse $\mathbf{v} \mapsto (|\mathbf{v}|, \hat{\mathbf{v}})$.
- Let $A \subseteq \{(r, z) : r > 0\} \subseteq \mathbb{R}^2$, $R(A)$ be the set obtained by rotating A around the z axis. Then $R(A) \simeq S \times A$ by

  $$(x, y, z) = (\mathbf{v}, z) \mapsto (\hat{\mathbf{v}}, (|\mathbf{v}|, z)).$$

  In particular, if A is a circle, then $R(A) \simeq S^1 \times S^1 = T^2$ is the two-dimensional torus.



2.5.2 Quotients

If X is a set and $\sim$ is an equivalence relation on X, then the quotient $X/\sim$ is the set of equivalence classes. The project $\pi : X \to X/\sim$ is defined as $\pi(x) = [x]$, the equivalence class containing X.

Definition (Quotient topology)**.** If X is a topological space, the quotient topology on $X/\sim$ is given by: U is open in $X/\sim$ if $\pi^{-1}(U)$ is open in X.

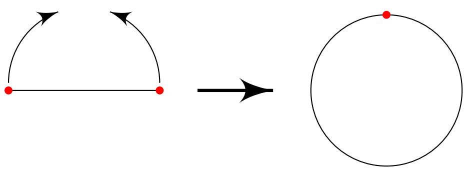
The defining property is $f : X/\sim \to Y$ is continuous iff $f \circ \pi : X \to Y$ is continuous.

Example.

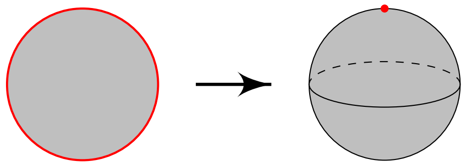
- Let $X = \mathbb{R}$, $x \sim y$ if $x - y \in \mathbb{Z}$. Then $X/\sim = \mathbb{R}/\mathbb{Z} \simeq S^1$, given by $[x] \mapsto (\cos 2\pi x, \sin 2\pi x)$.
- Let $X = \mathbb{R}^2$. Then $\mathbf{v} \sim \mathbf{w}$ iff $\mathbf{v} - \mathbf{w} \in \mathbb{Z}^2$. Then $X/\sim = \mathbb{R}^2/\mathbb{Z}^2 = (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z}) \sim S^1 \times S^1 = T^2$. Similarly, $\mathbb{R}^n/\mathbb{Z}^n = T^n = S^1 \times S^1 \times \cdots \times S^1$.
- If $A \subseteq X$, define $\sim$ by $x \sim y$ iff $x = y$ or $x, y \in A$. This groups everything in A together and leaves everything else alone.

  We often write this as X/A. Note that this is not consistent with the notation we just used above!

  - Let $X = [0, 1]$ and $A = \{0, 1\}$, then $X/A \sim S^1$ by, say, $t \mapsto (\cos 2\pi t, \sin 2\pi t)$. Intuitively, the equivalence relation says that the two end points of $[0, 1]$ are "the same". So we join the ends together to get a circle.

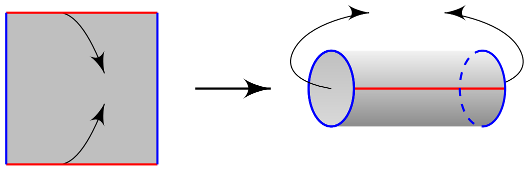
    

  - Let $X = D^n$ and $A = S^{n-1}$. Then $X/A \sim S^n$. This can be pictured as pulling the boundary of the disk together to a point to create a closed surface



– Let $X = [0,1] \times [0,1]$ with $\sim$ given by $(0, y) \sim (1, y)$ and $(x, 0) \sim (x, 1)$, then $X/{\sim} \simeq S^1 \times S^1 = T^2$, by, say

$$(x, y) \mapsto \big((\cos 2\pi x, \sin 2\pi x), (\cos 2\pi y, \sin 2\pi y)\big)$$



Similarly, $T^3 = [0,1]^3/{\sim}$, where the equivalence is analogous to above.

Note: Even if X is Hausdorff, $X/{\sim}$ may not be! For example, $\mathbb{R}/\mathbb{Q}$ is not Hausdorff.

3 Connectivity

Definition (Connected space)**.** A topological space X is *disconnected* if X can be written as $A \cup B$, where A and B are disjoint, non-empty open subsets of X. We say A and B *disconnect* X.

A space is *connected* if it is not disconnected.

Note: Being connected is a property of a *space*, not a subset. When we say "A is a connected subset of X", it means A is connected with the subspace topology inherited from X.

Being (dis)connected is a *topological* property, ie. if X is (dis)connected, and $f: X \to Y$ is a homeomorphism, then Y is (dis)connected: A is open in X iff $f(A)$ is open in Y. So A and B disconnect X iff $f(A)$ and $f(B)$ disconnect Y.

Example.

- If X has the coarse topology, it is connected.
- If X has the discrete topology and at least 2 elements, it is disconnected.
- If $X \subseteq \mathbb{R}$ such that there is an $\alpha \in \mathbb{R}$, with some $a, b \in X$ such that $a < \alpha < b$, then X is disconnected, by $A = X \cap (-\infty, \alpha)$, $B = X \cap (\alpha, \infty)$. For example, $(0, 1) \cup (1, 2)$ is disconnected.

Proposition. X is disconnected iff there exists a continuous surjective $f : X \to \{0, 1\}$ with the discrete topology.

Proof. ($\Rightarrow$) If A and B disconnect X, define

$$f(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B \end{cases}$$

Then $f^{-1}(\emptyset)$, $f^{-1}(\{0, 1\}) = X$, $f^{-1}(\{0\}) = A$ and $f^{-1}(\{1\}) = B$ are all open. So f is continuous. Also, since A, B are non-empty, f is surjective.

($\Leftarrow$) Given $f : X \mapsto \{0, 1\}$ surjective and continuous, define $A = f^{-1}(\{0\})$, $B = f^{-1}(\{1\})$. Then A and B disconnect X. □

Theorem. $[0, 1]$ is connected.

Note that $\mathbb{Q} \cap [0, 1]$ is *dis*connected, since we can pick our favorite irrational number a, and then $\{x : x < a\}$ and $\{x : x > a\}$ disconnect the interval. So we better use something special about $[0, 1]$.

The key property is that every non-empty $A \subseteq [0, 1]$ has a supremum.

Proof. Suppose A and B disconnect $[0, 1]$. wlog, assume $1 \in B$. A is non-empty. So $\alpha = \sup A$ exists. Then either

- $\alpha \in A$. Then $\alpha < 1$, since $1 \in B$. Since A is open, $\exists \varepsilon > 0$ with $B_\varepsilon(\alpha) \subseteq A$. So $\alpha + \frac{\varepsilon}{2} \in A$, contradicting supremality of α; or
- $\alpha \notin A$. Then $\alpha \in B$. Since B is open, $\exists \varepsilon > 0$ such that $B_\varepsilon(\alpha) \subseteq B$. Then $a \leq \alpha - \varepsilon$ for all $a \in A$. This contradicts α being the *least* upper bound of A.

Either option gives a contradiction. So A and B cannot exist and $[0, 1]$ is connected. $\square$

If $f : X \to Y$, $\operatorname{Im} f = \{f(x) : x \in X\} \subseteq Y$ has the subspace topology inherited from Y.

Proposition. If $f : X \to Y$ is continuous, and X is connected, then $\operatorname{Im} f$ is also connected.

Proof. Suppose A and B disconnect $\operatorname{Im} f$. Then $A, B \subseteq \operatorname{Im} f$ are open. So $A = \operatorname{Im} f \cap A'$ and $B = \operatorname{Im} F \cap B'$ for some A', B' open in Y. Then $f^{-1}(A) = f^{-1}(A')$ and $f^{-1}(B) = f^{-1}(B)$ are open in X, since f is continuous.

Since A, B are non-empty, $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty. Also, $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\emptyset) = \emptyset$. Finally, $A \cup B = \operatorname{Im} f$. So $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = X$.

So $f^{-1}(A)$ and $f^{-1}(B)$ disconnect X, contradicting our hypothesis. So $\operatorname{Im} f$ is connected. $\square$

Theorem (Intermediate value theorem)**.** Suppose $f : X \to \mathbb{R}$ is continuous and X is connected. If $\exists x_0, x_1$ such that $f(x_0) < 0 < f(x_1)$, then $\exists x \in X$ with $f(x) = 0$.

Proof. Suppose no such x exists. Then $0 \not\in \operatorname{Im} f$ while $0 > f(x_0) \in \operatorname{Im} f$, $0 < f(x_1) \in \operatorname{Im} f$. Then $\operatorname{Im} f$ is disconnected (from our previous example), contradicting X being connected. $\square$

Corollary. If $f : [0, 1] \to \mathbb{R}$ is continuous with $f(0) < 0 < f(1)$, then $\exists x \in [0, 1]$ with $f(x) = 0$.

Note: if X is disconnected, then there exists a continuous surjective $f : X \mapsto \{0, 1\}$. Then let $g(x) = f(x) - \frac{1}{2}$. Then g does not satisfy the intermediate value theorem. So the intermediate value theorem holds for X if and only if X is connected.

3.1 Path connectivity

Definition (Path connectivity)**.** A topological space X is *path connected* if for all points $x_0, x_1 \in X$, there is a continuous $\gamma : [0, 1] \to X$ with $\gamma(0) = x_0$, $\gamma(1) = x_1$. We say γ is a *path* from x_0 to x_1.

Example.

(i) $(a, b), [a, b), (a, b], \mathbb{R}$ are all path connected (paths given by linear functions).

(ii) $\mathbb{R}^n$ is path connected (eg. $\gamma(t) = t\mathbf{x}_1 + (1 - t)\mathbf{x}_0$).

(iii) $\mathbb{R}^n \setminus \{0\}$ is path-connected for $n > 1$ (paths either line segments or bent line segments).

Proposition. If X is path connected, then X is connected.

Proof. Suppose X is path connected but not connected. Then there is a continuous surjective $f : X \to \{0, 1\}$. Choose x_0, x_1 with $f(x_0) = 0$, $f(x_1) = 1$. Since X is path connected, there is a path $\gamma : [0, 1] \to X$ with $\gamma(0) = x_0, \gamma(1) = x_1$. Then $f \circ \gamma : [0, 1] \to \{0, 1\}$ is continuous and surjective. So $[0, 1]$ is not connected, which is a contradiction. $\square$

We can use connectivity to distinguish spaces.

Lemma. Suppose $f : X \to Y$ is a homeomorphism and $A \subseteq X$, then $f|_A : A \to f(A)$ is a homeomorphism.

Proof. Since f is a bijection, $f|_A$ is a bijection. If $U \subseteq f(A)$ is open, then $U = f(A) \cap U'$ for some $U' \subseteq Y$ open in Y. So $f|_A^{-1}(U) = f^{-1}(U) \cap A$ is open in A. So $f|_A$ is continuous. Similarly, for $(f|_A)^{-1}$. $\square$

Example. $[0,1] \not\simeq \mathbb{R}$. So $[0,1] \not\simeq (0,1) \simeq \mathbb{R}$.

If $f : [0,1] \to \mathbb{R}$ is a homeomorphism, let $A = (0,1]$. Then $f|_A : (0,1] \to \mathbb{R} \setminus f(0)$ is a homeomorphism. But $(0,1]$ is connected and $\mathbb{R} \setminus f(0)$ is disconnected. Contradiction.

Similarly, $[0,1) \not\simeq [0,1]$ and $[0,1) \not\simeq (0,1)$. Also, $\mathbb{R}^n \not\simeq \mathbb{R}$ for $n > 1$, and S' is not homeomorphic to any subset of $\mathbb{R}$.

3.1.1 Higher connectivity*

Recall that $S^0 = \{-1, 1\} \simeq \{0, 1\} \subseteq \mathbb{R}$ and $D^1 = [-1, 1] \simeq [0, 1] \subseteq \mathbb{R}$.

Then we have: X is path-connected iff any continuous $f : S^0 \to X$ extends to a continuous $\gamma : D^1 \to X$ with $\gamma|_{S^0} = f$.

We know that $\mathbb{R} - p$ is not path connected while $\mathbb{R}^2 - q$ is path connected. So for any p, q, $\mathbb{R} - p \not\simeq \mathbb{R}^2 - q$. So $\mathbb{R} \not\simeq \mathbb{R}^2$.

Definition (n-connectedness)**.** X is *n-connected* if any continuous $f : S^k \to X$ (with $k \leq n$) extends to a continuous $F : D^{k+1} \to X$ such that $F|_{S^k} = f$.

Then $\mathbb{R}^n - p$ is m-connected iff $m \leq n - 2$. So $\mathbb{R}^n - p \not\simeq \mathbb{R}^m - q$ unless $n = m$. So $\mathbb{R}^n \not\simeq \mathbb{R}^m$.

But our first statement about the connectedness of $\mathbb{R}^n - p$ is *very* hard to prove! We'll need algebraic topology for that.

3.2 Components

3.2.1 Path components

If $\gamma_1 : [0,1] \to X$ is a path from x to y and $\gamma_2 : [0,1] \to X$ is a path from y to z, Then $\gamma_1 * \gamma_2 : [0,1] \to X$ by

$$t \mapsto \begin{cases} \gamma(2t) & t \in [0, 1/2] \\ \gamma(2t-1) & t \in [1/2, 1] \end{cases}$$

is a path from x to z (formed by joining the two paths together). (Exercise: prove γ is continuous!)

Definition. Define $x \sim y$ if there is a path from x to y in X.

Lemma. $\sim$ is an equivalence relation.

Proof.

(i) For any $x \in X$, let $\gamma_x : [0,1] \to X$ be $\gamma(t) = x$, the constant path. Then this is a path from x to x. So $x \sim x$.

(ii) If $\gamma : [0, 1] \to X$ is a path from x to y, then $\bar{\gamma} : [0, 1] \to X$ by $t \mapsto \gamma(1 - t)$ is a path from y to x. So $x \sim y \Rightarrow y \sim x$.

(iii) If γ_1 is a path form x to y and γ_2 is a path from y yo z, then $\gamma_2 * \gamma_1$ is a path from x to z. So $x \sim y, y \sim z \Rightarrow x \sim z$.

$\square$

Definition (Path components)**.** Equivalence classes of the relation "$x \sim y$ if there is a path from x to y" are *path components* of X.

3.2.2 Connected components

Proposition. Suppose $Y_\alpha \subseteq X$ is connected for all $\alpha \in T$ and that $\bigcap_{\alpha \in T} Y_\alpha \neq \emptyset$. Then $Y = \bigcup_{\alpha \in T} Y_\alpha$ is connected.

Proof. Suppose A and B disconnect Y. Then A and B are open in Y. So $A = Y \cap A'$ and $B = Y \cap B'$, where A', B' are open in X. Let

$$A_\alpha = Y_\alpha \cap A = Y_\alpha \cap A', \quad B_\alpha = Y_\alpha \cap B = Y_\alpha \cap B'.$$

Then they are open. Since $Y = A \cup B$, we have

$$Y_\alpha = Y \cap Y_\alpha = (A \cup B) \cap Y_\alpha = A_\alpha \cup B_\alpha.$$

Since $A \cap B = \emptyset$, we have

$$A_\alpha \cap B_\alpha = Y_\alpha \cap (A \cap B) = \emptyset.$$

So A_α, B_α are disjoint. So Y_α is connected but is the disjoint union of open subsets A_α, B_α.

By definition of connectivity, this can only happen if $A_\alpha = \emptyset$ or $B_\alpha = \emptyset$.

However, $\displaystyle\bigcap_{\alpha \in T} Y_\alpha = \emptyset$. So pick $y \in \displaystyle\bigcap_{\alpha \in T} Y_\alpha$. So $y \in Y$. Then either $y \in A$ or $y \in B$. wlog, assume $y \in A$. Then $y \in Y_\alpha$ for all α implies that $y \in A_\alpha$ for all α. So A_α is non-empty for all α. So B_α is empty for all α. So $B = \emptyset$.

So A and B did not disconnect Y. Contradiction. $\square$

If $x \in X$, define $\mathcal{C}(x) = \{A \subseteq X : x \in A \text{ and } A \text{ is connected}\}$.

Definition (Connected component)**.** $C(x) = \displaystyle\bigcup_{A \in \mathcal{C}(x)} A$ is the connected component of x.

Since $\{x\} \in \mathcal{C}(x)$, we have $x \in C(x)$. Also, $x \in \bigcap_{A \in \mathcal{C}} A$ implies that $\bigcap_{A \in \mathcal{C}(x)} A \neq \emptyset$. So $C(x)$ is connected. So $C(x)$ is the largest connected subset of X which contains x.

Lemma. If $y \in C(x)$, then $C(y) = C(x)$.

Proof. Since $y \in C(x)$ and $C(x)$ is connected, $C(x) \subseteq C(y)$. So $x \in C(y)$. Then $C(y) \subseteq C(x)$. So $C(x) = C(y)$. $\square$

It follows that $x \sim Y$ if $x \in C(y)$ is an equivalence relation and the connected components of X are the equivalence classes.

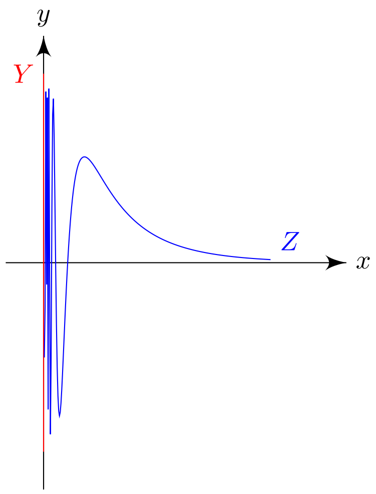
Example.

- Let $X = (-\infty, 0) \cup (0, \infty) \subseteq \mathbb{R}$. Then the connected components are $(-\infty, 0)$ and $(0, \infty)$, which are also the path components.
- Let $X = \mathbb{Q} \subseteq \mathbb{R}$. Then $C(x) = \{x\}$ for all $x \in X$. In this case, we say X is totally disconnected.

Note: $C(x)$ and $X \setminus C(x)$ need not disconnect X, even though it is the case in our first example. For example, in Example 2, $C(x) = \{x\}$ is not open.

Note: Path connected spaces are connected. So the path component containing X is a subset of $C(x)$, but they need not be equal.

Example. Let $Y = \{(0, y) : y \in R\} \subseteq \mathbb{R}^2$ be the y axis.
Let $Z = \{(x, \frac{1}{x}\sin\frac{1}{x}) : x \in (0, \infty)\}$.



Let $X = Y \cup Z \subseteq \mathbb{R}^2$. We claim that Y and Z are the path components of X. Since Y and Z are path connected, it suffices to show that there is no continuous $\gamma : [0, 1] \to X$ with $\gamma(0) = (0, 0)$, $\gamma(1) = \sin 1$.

The function $\pi_2 \circ \gamma : [0, 1] \to \mathbb{R}$ projecting the path to the y direction is continuous. So it is bounded. Let M be such that $\pi_2 \circ \gamma(t) \leq M$ for all $t \in [0, 1]$. Let $W = X \cap \mathbb{R} \times (-\infty, M]$ be the part of X that lies below $y = M$. Then $\operatorname{Im} \gamma \subseteq W$.

Then W is disconnected: pick t_0 with $\frac{1}{t_0}\sin\frac{1}{t_0} > M$. Then $W \cap (-\infty, t)$ and $W \cap (t, \infty)$ disconnect W_0. This is a contradiction, since γ is continuous and $[0, 1]$ is connected.

We also claim that X is connected: suppose A and B disconnect X. Then since Y and Z are connected, either $Y \subseteq A$ or $Y \subseteq B$; $Z \subseteq A$ or $Z \subseteq B$. If both $Y \subseteq A, Z \subseteq A$, then $B = \emptyset$, which is not possible.

So wlog assume $A = Y$, $B = Z$. This is also impossible, since Y is not open in X as it is not a union of balls. Hence X must be connected.

Proposition. If $U \subseteq \mathbb{R}^n$ is open and connected, then it U is path-connected.

Proof. Let A be a path component of U. We first show that A is open.

Suppose $a \in A$. Since U is open, $\exists \varepsilon > 0$ such that $B_\varepsilon(a) \subseteq U$. We know that $B_\varepsilon(a) \simeq \mathrm{Int}(D^n)$ is is path-connected (eg. use line segments connecting the points). Since A is a path component and $a \in A$, we must have $B_\varepsilon(a) \subseteq A$. So A is an open subset of U.

Now suppose $b \in U \setminus A$. Then since U is open, $\exists \varepsilon > 0$ such that $B_\varepsilon(b) \subseteq U$. Since $B_\varepsilon(b)$ is path-connected, so if $B_\varepsilon(b) \cap A \neq \emptyset$, then $B_\varepsilon(b) \subseteq A$. But this implies $b \in A$, which is a contradiction. So $B_\varepsilon(b) \cap A = \emptyset$. So $B_\varepsilon(b) \subseteq U \setminus A$. Then $U \setminus A$ is open.

So $A, U \setminus A$ are disjoint open subsets of U. Since U is connected, we must have $U \setminus A$ empty (since A is not). So $U = A$ is path-connected. $\square$

4 Compactness

Suppose $\mathcal{U} \subseteq \mathbb{P}(X)$ is a topology on X.

Definition (Open cover)**.** An *open cover* of X is a subset $\mathcal{V} \subseteq \mathcal{U}$ such that

$$\bigcup_{V \in \mathcal{V}} V = X.$$

We say $\mathcal{V}$ *covers* X.

If $\mathcal{V}' \subseteq \mathcal{V}$, and $\mathcal{V}'$ covers X, then we say $\mathcal{V}'$ is a *subcover* of V.

Definition (Compact space)**.** A topological space X is *compact* if every open cover $\mathcal{V}$ of X has a finite subcover $\mathcal{V}' = \{V_1, \cdots, V_n\} \subseteq \mathcal{V}$.

This is a weird definition, but it is one of the most important concepts of topology.

Example.

(i) If X is finite, then $\mathbb{P}(X)$ is finite. So any open cover of X is finite. So X is compact. While this sounds dumb, it is actually profound - compactness is the next best thing to being finite.

(ii) Let $X = \mathbb{R}$ and $\mathcal{V} = \{(-R, R) : R \in \mathbb{R}, R > 0\}$. Then this is an open cover with no finite subcover. So $\mathbb{R}$ is not compact. Hence all open intervals are not compact since they are not homeomorphic to $\mathbb{R}$.

(iii) Let $X = [0, 1] \cap \mathbb{Q}$. Let

$$U_n = X \setminus (\alpha - 1/n, \alpha + 1/n).$$

for some irrational α in $(0, 1)$ (eg. $\alpha = \sqrt{2}^{-1}$).

Then $\bigcup_{n>0} U_n = X$ since α is irrational. Then $\mathcal{V} = \{U_n : n \in \mathbb{Z} > 0\}$ is an open cover of X. Since this has no finite subcover, X is not compact.

Theorem. $[0, 1]$ is compact.

Again, since this is not true for $[0, 1] \cap \mathbb{Q}$, we must use a special property of reals.

Proof. Suppose $\mathcal{V}$ is an open cover of $[0, 1]$. Let

$$A = \{a \in [0, 1] : [0, a] \text{ has a finite subcover of } \mathcal{V}\}.$$

ie. the set of all a such that $[0, a]$ has a finite subcover.

First show that A is non-empty. Since V covers $[0, 1]$, in particular, there is some V_0 that contains 0. So $\{0\}$ has a finite subcover V_0. So $0 \in A$.

Next we note that by definition, if $0 \leq b \leq a$ and $a \in A$, then $b \in A$.

Now let $\alpha = \sup A$. Suppose $\alpha < 1$. Then $\alpha \in [0, 1]$.

Since $\mathcal{V}$ covers X, let $\alpha \in V_\alpha$. Since V_α is open, there is some ε such that $B_\varepsilon(\alpha) \subseteq V_\alpha$. By definition of α, we must have $\alpha - \varepsilon \in A$. So $[0, \alpha - \varepsilon/2]$ has a finite subcover. Add V_α to that subcover to get a finite subcover of $[0, \alpha + \varepsilon/2]$. Contradiction (technically, it will be a finite subcover of $[0, \eta]$ for $\eta = \min(\varepsilon/2, 1 - \alpha)$, in case $\alpha + \varepsilon/2$ gets too large).

So we must have $\alpha = \sup A = 1$.

Now we argue as before: $\exists V_1 \in \mathcal{V}$ such that $1 \in V_1$ and $\exists \varepsilon > 0$ with $(1-\varepsilon, 1] \subseteq V_q$. Since $1 - \varepsilon \in A$, there exists a finite $\mathcal{V}' \subseteq V$ which covers $[0, 1-\varepsilon/2]$. Then $\mathcal{W} = \mathcal{V}' \cup \{V_1\}$ is a finite subcover of V. $\square$

Proposition. If X is compact and C is closed subset of X, then C is also compact.

The idea is given an open cover of C, add *one* open set U that covers everything else, get a finite subcover of this by compactness of X, and remove U.

Proof. Suppose $\mathcal{V}$ is an open cover of C. Say $\mathcal{V} = \{V_\alpha : \alpha \in T\}$. For each α, since V_α is open in C, $V_\alpha = C \cap V'_\alpha$ for some V'_α open in X. Also, since $\bigcup_{\alpha \in T} V_a = C$, we have $\bigcup_{\alpha \in T} V'_\alpha \supseteq C$.

Since C is closed, $U = X \setminus C$ is open in X. So $\mathcal{W} = \{V'_\alpha : \alpha \in T\} \cup \{U\}$ is an open cover of X. Since X is compact, $\mathcal{W}$ has a finite subcover $\mathcal{W}' = \{V'_{\alpha_1}, \cdots, V'_{\alpha_n}, U\}$ (U may or may not be in there, but it doesn't matter). Now $U \cap C = \emptyset$. So $\{V_{\alpha_1}, \cdots, V_{\alpha_n}\}$ is a finite subcover of C. $\square$

Proposition. Let X be a Hausdorff space. If $C \subseteq X$ is compact, then C is closed in X.

Proof. Let $U = X \setminus C$. We will show that U is open.

For any x, we will find a U_x such that $U_x \subseteq U$ and $x \in U_x$. Then $U = \bigcup_{x \in U} U_x$ will be open since it is as union of open sets.

To construct U_x, fix $x \in U$. Since X is Hausdorff, so for each $y \in C$, $\exists U_{xy}, W_{xy}$ open neighbourhoods of x and y respectively with $U_{xy} \cap W_{xy} = \emptyset$.

Then $\mathcal{W} = \{W_{xy} \cap C : y \in C\}$ is an open cover of C. Since C is compact, there exists a finite subcover $\mathcal{W}' = \{W_{xy_1} \cap C, \cdots, W_{xy_n} \cap C\}$.

Let $U_x = \bigcap_{i=1}^n U_{xy_i}$. Then U_x is open since it is a finite intersection of open sets. To show $U_x \subseteq U$, note that $W_x = \bigcup_{i=1}^n W_{xy_i} \subseteq C$ since $\{W_{xy_i} \cap C\}$ is an open cover. We also have $W_x \cap U_x = \emptyset$. So $U_x \subseteq U$. So done. $\square$

Definition (Bounded metric space)**.** A metric space (X, d) is *bounded* if there exists $M \in \mathbb{R}$ such that $d(x, y) \leq M$ for all $x, y \in X$.

Example. $A \subseteq \mathbb{R}$ is bounded iff $A \subseteq [-N, N]$ for some $N \in \mathbb{R}$.

Note: Being bounded is *not* a topological property. For example, $(0,1) \simeq \mathbb{R}$ but $(0, 1)$ is bounded while $\mathbb{R}$ is not. It depends on d, not just the topology it induces.

Proposition. A compact metric space (X, d) is bounded.

Proof. Pick $x \in X$. Then $V = \{B_r(x) : r \in \mathbb{R}^+\}$ is an open cover of X. Since X is compact, there is a finite subcover $\{B_{r_1}(x), \cdots, B_{r_n}(x)\}$.

Let $R = \max\{r_1, \cdots, r_n\}$. Then $d(x, y) < R$ for all $y \in X$. So

$$d(y, z) \leq d(y, x) + d(x, z) < 2R$$

For any $y, z \in X$. So X is bounded. $\square$

Theorem (Heine-Borel)**.** $C \subseteq \mathbb{R}$ is compact iff C is closed and bounded.

Since $\mathbb{R}$ is a metric space (hence Hausdorff), C is also a metric space.

So if C is compact, C is closed in $\mathbb{R}$, and C is bounded, by our previous two propositions.

Conversely, if C is closed and bounded, then $C \subseteq [-N, N]$ for some $N \in \mathbb{R}$. Since $[-N, N] \simeq [0, 1]$ is compact, and $C = C \cap [-N, N]$ is closed in $[-N, N]$, C is compact.

Corollary. If $A \subseteq \mathbb{R}$ is compact, $\exists \alpha \in A$ such that $\alpha \geq a$ for all $a \in A$.

Proof. Since A is compact, it is bounded. Let $\alpha = \sup A$. Then by definition, $\alpha \geq a$ for all $a \in A$. So it is enough to show that $\alpha \in A$.

Suppose $\alpha \notin A$. Then $\alpha \in \mathbb{R} \setminus A$. Since A is compact, it is closed in $\mathbb{R}$. So $\mathbb{R} \setminus A$ is open. So $\exists \varepsilon > 0$ such that $B_\varepsilon(\alpha) \subseteq \mathbb{R} \setminus A$, which implies that $a \leq \alpha - \varepsilon$ for all $a \in A$. This contradicts the assumption that $\alpha = \sup A$. So we can conclude $\alpha \in A$. □

We call $\alpha = \max A$ the maximum element of A.

Proposition. If $f : X \to Y$ is continuous and X is compact, then $\operatorname{Im} f \subseteq Y$ is also compact.

Note that we previously proved the same proposition with "compact" replaced by "connected".

Proof. Suppose $\mathcal{V} = \{V_\alpha : \alpha \in T\}$ is an open cover of $\operatorname{Im} f$. Since V_α is open in $\operatorname{Im} f$, we have $V_\alpha = \operatorname{Im} f \cap V'_\alpha$, where V'_α is open in Y. Then

$$W_\alpha = f^{-1}(V_\alpha) = f^{-1}(V'_\alpha)$$

is open in X. If $x \in X$ and $f(x)$ is in V_α for some α, then $x \in W_\alpha$. Then $\mathcal{W} = \{W_\alpha : \alpha \in T\}$ is an open cover of X.

Since X is compact, so there's a finite subcover $\{W_{\alpha_1}, \cdots, W_{\alpha_n}\}$ of $\mathcal{W}$.

Since $V_\alpha \subseteq \operatorname{Im} f$, $f(W_\alpha) = f(f^{-1}(V_\alpha)) = V_\alpha$. So

$$\{V_{\alpha_1}, \cdots, V_{\alpha_n}\}$$

is a finite subcover of V. □

Theorem (Maximum value theorem)**.** If $f : X \to \mathbb{R}$ is continuous and X is compact, then $\exists x \in X$ such that $f(x) \geq f(y)$ for all $y \in X$.

Proof. Since X is compact, $\operatorname{Im} f$ is compact. Let $\alpha = \max\{\operatorname{Im} f\}$. Then $\alpha \in \operatorname{Im} f$. So $\exists x \in X$ with $f(x) = \alpha$. Then by definition $f(x) \geq f(y)$ for all $y \in X$. □

Corollary. If $f : [0, 1] \to \mathbb{R}$ is continuous, then $\exists x \in [0, 1]$ such that $f(x) \geq f(y)$ for all $y \in [0, 1]$

Proof. $[0, 1]$ is compact. □

4.1 Products and quotients

4.1.1 Products

Recall the product topology on $X \times Y$. $U \subseteq X \times Y$ is open if it is a union of sets of the form $V \times W$ such that $V \subseteq X, W \subseteq Y$ are open.

Theorem. If X and Y are compact, then so is $X \times Y$.

Proof. First consider the special type of open cover $\mathcal{V}$ of $X \times Y$ such that every $U \in \mathcal{V}$ has the form $U = V \times W$, where $V \subseteq X$ and $W \subseteq Y$ are open.

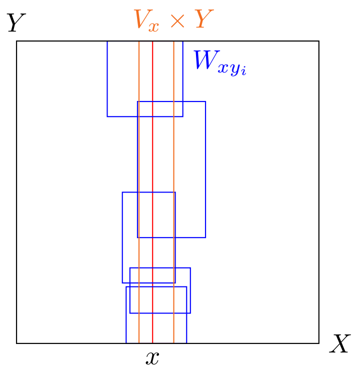
For every $(x, y) \in X \times Y$, there is $U_{xy} \in V$ with $(x, y) \in U_{xy}$. Then let

$$U_{xy} = V_{xy} \times W_{xy},$$

where $V_{xy} \subseteq X$, $W_{xy} \subseteq Y$ are open, $x \in V_{xy}, y \in W_{xy}$.

Fix $x \in X$. Then $\mathcal{W}_x = \{W_{xy} : y \in Y\}$ is an open cover of Y. Since Y is compact, there is a finite subcover $\{W_{xy_1}, \cdots, W_{xy_n}\}$.

Then $V_x = \bigcap_{i=1}^n V_{xy_i}$ is a finite intersection of open sets. So V_x is open in X. Moreover, $\mathcal{V}_x = \{U_{xy_1}, \cdots, U_{xy_n}\}$ covers $V_x \times Y$.



Now $\mathcal{O} = \{V_x : x \in X\}$ is an open cover of X. Since X is compact, there is a finite subcover $\{V_{x_1}, \cdots, V_{x_m}\}$. Then $\mathcal{V}' = \bigcup_{i=1}^m \mathcal{V}_{x_i}$ is a finite subset of V, which covers all of $X \times Y$.

In the general, case, suppose $\mathcal{V}$ is an open cover of $X \times Y$. For each $(x, y) \in X \times Y$, $\exists U_{xy} \in \mathcal{V}$ with $(x, y) \in U_{xy}$. Since U_{xy} is open, $\exists V_{xy} \in X, W_{xy} \subseteq Y$ open with $V_{xy} \times W_{xy} \subseteq U_{xy}$ and $x \in V_{xy}$, $y \in W_{xy}$.

Then $\mathcal{O} = \{V_{xy} \times W_{xy} : (x, y) \in (X, Y)\}$ is an open cover of $X \times Y$ of the type we already considered above. So it has a finite subcover $\{V_{x_1y_1} \times W_{x_1y_1}, \cdots, V_{x_ny_n} \times W_{x_ny_n}\}$. Now $V_{x_iy_i} \times W_{x_iy_i} \subseteq U_{x_iy_i}$. So $\{U_{x_1y_1}, U_{x_ny_n}\}$ is a finite subcover of $X \times Y$. $\square$

Example. The unit cube $[0, 1]^n = [0, 1] \times [0, 1] \times \cdots \times [0, 1]$ is compact.

Corollary (Heine-Borel in $\mathbb{R}^n$)**.** $C \subseteq \mathbb{R}^n$ is compact iff C is closed and bounded.

Proof. If C is bounded, $C \subseteq [-N, N]^n$ for some $N \in \mathbb{R}$, which is compact. The rest of the proof is exactly the same as for $n = 1$. $\square$

4.1.2 Quotients

We begin with a handy proposition.

Proposition. Suppose $f : X \to Y$ is a continuous bijection. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof. We show that f^{-1} is continuous. To do this, it suffices to show $(f^{-1})^{-1}(C)$ is closed in Y whenever C is closed in X. By hypothesis, f is a bijection . So $(f^{-1})^{-1}(C) = f(C)$.

Supposed C is closed in X. Since X is compact, C is compact. Since f is continuous. So $f(C) = (\operatorname{Im} f|_C)$ is compact. Since Y is Hausdorff and $f(C) \subseteq Y$ is compact, $f(C)$ is closed. □

We will apply this to quotients.

Recall that if $\sim$ is an equivalence relation on X, $\pi : X \to X/{\sim}$ is continuous iff $f \circ \pi : X \to Y$ is continuous.

Corollary. Suppose $f : X/{\sim} \to Y$ is a bijection, X is compact, Y is Hausdorff, and $f \circ \pi$ is continuous, then f is a homeomorphism.

Proof. Since X is compact and $\pi : X \mapsto X/{\sim}$ is continuous, $\operatorname{Im} \pi \subseteq X/{\sim}$ is compact. Since $f \circ \pi$ is continuous, f is continuous. So we can apply the proposition. □

Example. Let $X = D^2$ and $A = S^1 \subseteq X$. Then $f : X/A \mapsto S^2$ by $(r, \theta) \mapsto (1, \pi r, \theta)$ in spherical coordinates is a homeomorphism.

We can check that f is a continuous bijection and D^2 is compact. So $X/A \simeq S^2$.

4.2 Sequential compactness and completeness

Definition (Sequential compactness)**.** A topological space X is *sequentially compact* if every sequence (x_n) in X has a convergent subsequence (that converges to a point in X!).

Example. $(0, 1) \subseteq \mathbb{R}$ is not sequentially compact since no subsequence of $(1/n)$ converges to any $x \in (0, 1)$.

There is a reason why we called it sequential *compactness*. We first need a lemma.

Lemma. Let (x_n) be a sequence in a metric space (X, d) and $x \in X$. Then (x_n) has a subsequence converging to X iff for every $\epsilon > 0$, $x_n \in B_\varepsilon(x)$ for infinitely many n $(*)$.

Proof. If $(x_{n_i}) \to x$, then for every ϵ, we can find I such that $i > I$ implies $x_{n_j} \in B_\varepsilon(x)$ by definition of convergence. So $(*)$ holds.

Now suppose $(*)$ holds. We will construct a sequence $x_{n_i} \to x$ inductively. Take $n_0 = 0$. Suppose we have defined $x_{n_0}, x_{n_{i-1}}$.

By hypothesis, $x_n \in B_{1/i}(x)$ for infinitely many n. Take n_i to be smallest such n with $n_i > n_{i-1}$.

Then $d(x_{n_i} x) < \frac{1}{i}$ implies that $x_{n_i} \to x$. □

Theorem. If (X, d) is a compact *metric space*, then X is sequentially compact.

Note that this is a funny theorem, since both compactness and sequential compactness are topological properties, but this theorem requires the existence of a metric.

Proof. Suppose x_n is a sequence in X with no convergent subsequence. Then for any $y \in X$, there is no subsequence converging to y. By lemma, there exists $\epsilon > 0$ such that $x_n \in B_\varepsilon(y)$ for only finitely many n.

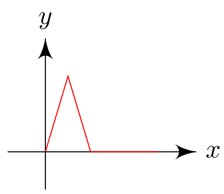
Let $U_y = B_\varepsilon(y)$. Now $\mathcal{V} = \{U_y : y \in X\}$ is an open cover of X. Since X is compact, there is a finite subcover $\{U_{y_1}, \cdots, U_{y_m}\}$. Then $x_n \in \bigcup_{i=1}^m U_{y_i} = X$ for only finitely many n. This is nonsense, since $x_n \in X$ for all n!

So x_n must have a convergent subsequence. $\square$

Note that it is also true that if X is a sequentially compact metric space, then X is compact, but we won't prove this. So for metric spaces, compactness and sequential compactness are the same.

Example. Let $X = C[0, 1]$ with the topology induced d_∞ (uniform norm). Let

$$f_n(x) = \begin{cases} nx & x \in [0, 1/n] \\ 2 - nx & x \in [1/n, 2/n] \\ 0 & x \in [2/n, 1] \end{cases}$$



Then $f_n(x) \to 0$ for all $x \in [0, 1]$. We now claim that f_n has no convergent subsequence.

Suppose $f_{n_i} \to f$. Then $f_{n_i}(x) \to f(x)$ for all $x \in [0, 1]$. So $f_{n_i}(x) \to 0$ for all $x \in [0, 1]$, then $f(x) = 0$. However, $d_\infty(f_{n_i}, 0) = 1$. So $f_{n_i} \not\to 0$.

It follows that $B_1(0) \subseteq X$ is not sequentially compact. So it is not compact.

4.3 Completeness

Definition (Cauchy sequence)**.** Let (X, d) be a metric space. A sequence (x_n) in X is *Cauchy* if for every $\varepsilon > 0$, $\exists N$ such that $d(x_n, x_m) \subseteq \varepsilon$ for all $n, m \geq N$.

Example.

(i) $x_n = \sum_{k=1}^n 1/k$ is *not* Cauchy.

(ii) Let $X = (0, 1) \subseteq \mathbb{R}$ with $x_n = \frac{1}{n}$. Then this is Cauchy but does not converge.

(iii) If $x_n \to x \in X$, then x_n is Cauchy (since $d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) = 2\varepsilon$ for sufficiently large n, m)

(iv) Let $X = \mathbb{Q} \subseteq \mathbb{R}$. Then the sequence $(2, 2.7, 2.71, 2.718, \cdots,)$ is Cauchy but does not converge in $\mathbb{Q}$.

Definition (Complete space)**.** A metric space (X, d) is *complete* if every Cauchy sequence in X converges converges to a limit in X.

Example. $(0, 1)$ and $\mathbb{Q}$ are not complete.

Proposition. If X is a compact metric space, then X is complete.

Proof. Let x_n be a Cauchy sequence in X. Since X is sequentially compact, there is a convergent subsequence $x_{n_i} \to x$. Given $\varepsilon > 0$, pick N such that $d(x_n, x_m) < \varepsilon/2$ for $n, m \geq N$.

Pick I such that $n_I \geq N$ and $d(x_{n_i}, x) < \varepsilon/2$ for all $i > I$. Then for $n \geq n_I$, $d(x_n, x) \leq d(x_n, x_{n_I}) + d(x_{n_I}, x) < \varepsilon$. So $x_n \to x$. □

Corollary. $\mathbb{R}^n$ is complete.

Proof. If $x_n \subseteq \mathbb{R}^n$ is Cauchy, then $x_n \subseteq \bar{B}_R(0)$ for some R, and $\bar{B}_R(0)$ is compact. So it converges. □

Note that completeness is not a topological property. $\mathbb{R} \simeq (0, 1)$ but one is complete and the other is not.