Electromagnetism

1 Preliminaries

{Definition} Charge Density is charge / unit volume. Total charge in a region, V is:

$$\left\{ Q = \int_V \rho(\underline{x},t)\, dV \right\}$$

- For any surface S, given a current density $\underline{J}$:

$$\left\{ I = \int_S \underline{J} \cdot d\underline{S}. \right\}$$

e.g If the charge distribution $\rho(x,t)$ has velocity $\vec{v}(x,t)$, then:

$$\left\{ \vec{\underline{J}} = \rho \underline{v} \right\}$$

- Continuity Equation: Locally, we have the following:

$$\boxed{\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{J} = 0}$$

This implies that, if $Q = \int_V \rho(x,t) dV$

$$\Rightarrow \frac{dQ}{dt} = \int_V \frac{\partial \rho}{\partial t} dV = -\int_V \underline{\nabla} \cdot \underline{J}\, dV$$

$$\Rightarrow \frac{dQ}{dt} = -\int_{\partial V} \underline{J} \cdot d\underline{S}$$

In particular, if there are no currents at infinity, then:
$\frac{dQ}{dt} = 0. \Rightarrow$ conservation of charge.

- Forces + Fields: We have two fields, $\{\underline{E}(\underline{x},t)$ and $\underline{B}(\underline{x},t)\}$

① Lorentz Force Law:

$$\boxed{\underline{F} = q\{\underline{E} + \underline{v} \times \underline{B}\}}$$

③ Maxwell's Equations:

$$\boxed{\begin{aligned} \underline{\nabla}\cdot\underline{E} &= \frac{\rho}{\varepsilon_0} & \underline{\nabla}\times\underline{E} + \frac{\partial B}{\partial t} &= 0 \\ \underline{\nabla}\cdot\underline{B} &= 0 & \underline{\nabla}\times\underline{B} - \mu_0\varepsilon_0\frac{\partial E}{\partial t} &= \mu_0\underline{J} \end{aligned}}$$

2 Electrostatics

- We take:

$$\left\{\begin{aligned} \rho(\underline{x}) &= \rho \\ \underline{J} &= \underline{0} \\ \underline{B} &= \underline{0} \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \underline{\nabla}\cdot\underline{E} &= \frac{\rho(\underline{x})}{\varepsilon_0} \\ \underline{\nabla}\times\underline{E} &= 0 \end{aligned}\right.$$

- Gauss' Law: Consider a volume $V \subseteq \mathbb{R}^3$, then:

$$\int_V \underline{\nabla}\cdot E\, dV = \overbrace{\int_{\partial V} E\cdot d\underline{S}}^{\text{flux through surface.}} = \int_V \frac{\rho(\underline{x})}{\varepsilon_0}\, dV = \frac{Q}{\varepsilon_0}$$

$$\Rightarrow \boxed{\int_{\partial V} \underline{E}\cdot d\underline{S} = \frac{Q}{\varepsilon_0}}, \quad Q \text{ is charge enclosed in } V.$$

e.g Coulomb's Law: Take $\rho = \rho(r)$ (i.e spherically symmetric) and ensure that $\rho(r) = 0$ for $r > R$. Then $\underline{E} = E(r)\hat{\underline{r}}$ by symmetry, and let S_r be a sphere of radius $r > R$.

$$\int_{S_r} \underline{E}\cdot d\underline{S} = E(r)\int_{S_r} \hat{\underline{r}}\cdot d\underline{S} = 4\pi r^2 E(r)$$

$$\Rightarrow \underline{E(r) = \frac{Q}{4\pi\varepsilon_0 r^2}}, \quad \frac{Q}{\varepsilon_0} = \int_{V_a} \frac{\rho(\underline{x})}{\varepsilon_0}\, dV$$

e.g Line charge of charge density η / unit length $\{\overset{\text{symmetry}}{\underline{E} = E(r)\hat{\underline{r}}}\}$

$$\Rightarrow \underline{\underline{E}(r) = \frac{\eta}{2\pi\varepsilon_0 r}\hat{\underline{r}}} \quad \text{(in cylindrical co-ordinates).}$$

e.g Surface Charge: infinite plane $z=0$ with charge / unit area σ.

$$\Big\{\underbrace{\underline{E} = E(z)\hat{\underline{z}} \text{ and } |E(z)| = |E(-z)|}_{\text{symmetry.}\quad E(z) = -E(-z)}\Big\} \Rightarrow \underline{E(z) = \frac{\sigma}{2\varepsilon_0}}$$

- Surface discontinuity:

$$\boxed{\hat{\underline{n}}\cdot \underline{E}_+ - \hat{\underline{n}}\cdot \underline{E}_- = \frac{\sigma}{\varepsilon_0}}$$

- Electrostatic Potential: if $\nabla \times \underline{E} = \underline{0} \Rightarrow \underline{E} = -\nabla\phi$ for some ϕ which, combined with $\nabla\cdot\underline{E} = \frac{\rho}{\varepsilon_0}$

$$\Rightarrow \boxed{\nabla^2\phi = -\frac{\rho(\underline{x})}{\varepsilon_0}}$$

(i) Point charge. $\rho(\underline{r}) = Q\delta(\underline{r})$

$$\Rightarrow \nabla^2\phi = -\frac{Q}{\varepsilon_0}\delta(\underline{r})$$

For $\underline{r} \neq \underline{0}$, we have $\phi = \frac{\alpha}{r}$, then integrating over a sphere of radius r centred at the origin:

$$\Rightarrow -4\pi\alpha = -\frac{Q}{\varepsilon_0} \Rightarrow \alpha = \frac{Q}{4\pi\varepsilon_0}$$

So: $$\left\{\boxed{\phi = \phi(r) = \frac{Q}{4\pi\varepsilon_0 r}}\right\} \Rightarrow \left\{\underline{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\underline{r}}\right\}.$$

(ii) Dipole: Two charges, $+Q$ and $-Q$ at $\underline{r} = -\underline{d}$: by linearity:

$$\boxed{\phi = \frac{1}{4\pi\varepsilon_0}\left\{\frac{Q}{r} - \frac{Q}{|\underline{r}+\underline{d}|}\right\}}.$$

Now; for a general function $f(\underline{r})$:

$$f(\underline{r}+\underline{d}) = f(\underline{r}) + \underline{d}\cdot\nabla f(\underline{r}) + \frac{1}{2}(\underline{d}\cdot\nabla)^2 f(\underline{r}) + \cdots$$

$$\Rightarrow \frac{1}{|\underline{r}+\underline{d}|} \simeq \frac{1}{r} - \frac{\underline{d}\cdot\underline{r}}{r^3} - \frac{1}{2}\left\{\frac{\underline{d}\cdot\underline{d}}{r^3} - \frac{3(\underline{d}\cdot\underline{r})^2}{r^5}\right\} + \cdots$$

$$\Rightarrow \phi = \frac{Q}{4\pi\varepsilon_0}\left\{\frac{1}{r} - \frac{1}{r} + \frac{\underline{d}\cdot\underline{r}}{r^3} + \cdots\right\} \sim \frac{Q}{4\pi\varepsilon_0}\frac{\underline{d}\cdot\underline{r}}{r^3}$$

If we define $\underline{p} = Q\underline{d}$, the electric dipole moment. Then:

$$\underline{\phi = \frac{\underline{p}\cdot\hat{\underline{r}}}{4\pi\varepsilon_0 r^2}}$$

and $$\underline{\underline{E} = -\nabla\phi = \frac{1}{4\pi\varepsilon_0}\left\{\frac{3(\underline{p}\cdot\hat{\underline{r}})\hat{\underline{r}} - \underline{p}}{r^3}\right\}}$$

(iii) General Charge distribution: Use the Green's function:

$$\nabla^2 G(\underline{r}, \underline{r}') = \delta^3(\underline{r} - \underline{r}')$$

$$\Rightarrow \quad G(\underline{r}, \underline{r}') = -\frac{1}{4\pi|\underline{r} - \underline{r}'|}$$

$$\Rightarrow \quad \phi(\underline{r}) = -\frac{1}{\varepsilon_0} \int_V \rho(\underline{r}') G(\underline{r}; \underline{r}') d^3\underline{r}'$$

(Assuming charge contained in compact region)

$$\Rightarrow \quad \boxed{\phi(\underline{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\underline{r}')}{|\underline{r} - \underline{r}'|} d^3\underline{r}'}$$

If we apply the Taylor expansion again, we see that:

$$\phi(\underline{r}) \simeq \frac{1}{4\pi\varepsilon_0} \int_V \rho(\underline{r}') \left\{ \frac{1}{r} + \frac{\underline{r} \cdot \underline{r}'}{r^3} + \cdots \right\} d^3\underline{r}'$$

MULTIPOLE EXPANSION

$$\Rightarrow \left\{ \phi(\underline{r}) = \frac{1}{4\pi\varepsilon_0} \left(\underbrace{\frac{Q}{r}}_{\text{point charge.}} + \underbrace{\frac{\underline{p} \cdot \hat{\underline{r}}}{r^2}}_{\text{dipole.}} + \cdots \right) \right\}$$ e.g Quadrupole term:

where $Q = \int_V \rho(\underline{r}') d^3\underline{r}'$, $\underline{p} = \int_V \underline{r}' \rho(\underline{r}') dV'$

{Definition} Field lines are continuously tangent to the electric field. Density of lines $\propto |\underline{E}|$.

{Definition}. Equipotentials are surfaces of constant ϕ. Since $\underline{E} = -\underline{\nabla}\phi$, they are always perpendicular to field lines.

- Electrostatic Energy: Consider N charges, q_i at positions $\underline{r}_i$. Then total PE is energy required to assemble these particles (bringing in from infinity):

  1. First charge is free i.e $W_1 = 0$.
  2. Second charge requires work:

$$W_2 = \frac{q_1 q_2}{4\pi\varepsilon_0} \cdot \frac{1}{|\underline{r}_1 - \underline{r}_2|}$$

{ $U(\underline{r}) = q\phi(\underline{r})$ }.

  3. To place third charge:

$$W_3 = \frac{q_3}{4\pi\varepsilon_0} \left\{ \frac{q_1}{|\underline{r}_1 - \underline{r}_3|} + \frac{q_2}{|\underline{r}_2 - \underline{r}_3|} \right\}$$

Continuing this:

$$\mathcal{U} = \sum_{i=1}^{n} W_i = \frac{1}{4\pi\varepsilon_0} \sum_{i<j} \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|}$$

$$\iff \mathcal{U} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|}$$

If we write $\phi(\mathbf{r}_i) = \frac{1}{4\pi\varepsilon_0} \sum_{j \neq i} \frac{q_j}{|\mathbf{r}_i - \mathbf{r}_j|}$ as potential at $\mathbf{r}_i$ due to all other charges, then:

$$\mathcal{U} = \frac{1}{2} \sum_{i=1}^{n} q_i \phi(\mathbf{r}_i).$$

For a continuos distribution:

$$\mathcal{U} = \frac{1}{2} \int \rho(\mathbf{r}) \phi(\mathbf{r}) \, d^3\mathbf{r}.$$

$$\Rightarrow \mathcal{U} = \frac{\varepsilon_0}{2} \int (\nabla \cdot \mathbf{E}) \phi \, d^3\mathbf{r}$$

$$\Rightarrow \mathcal{U} = \frac{\varepsilon_0}{2} \int \left\{ \cancelto{0}{\nabla \cdot (\mathbf{E}\phi)} - \mathbf{E} \cdot \nabla\phi \right\} d^3\mathbf{r}.$$

$$\Rightarrow \boxed{\mathcal{U} = \frac{\varepsilon_0}{2} \int \mathbf{E} \cdot \mathbf{E} \, d^3\mathbf{r}}$$

• Conductors: In electrostatic situations we may derive some properties.

{Definition} Conductor is a region of space where charges are free to move.

① {Inside a conductor $\mathbf{E} = \mathbf{0}$} otherwise charges would move and would not be static. Also, if a field is applied to conductor, charges move to cancel external field.

② {$\rho = 0$ inside conductor (ie all charge on surface)}.

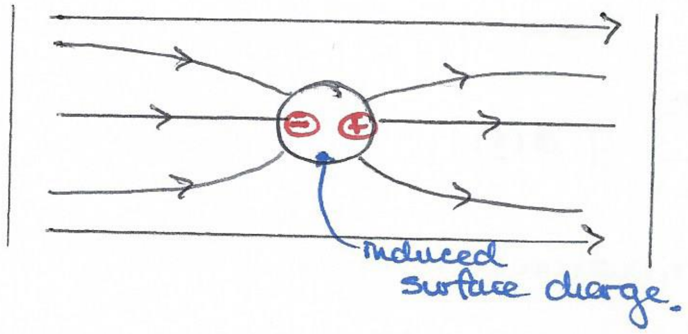
$\mathbf{E} = \mathbf{0}$ inside $\Rightarrow \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} = 0 \Rightarrow \rho = 0$ inside.

③ {surface is equipotential $\Rightarrow$ electric field normal to surface}.

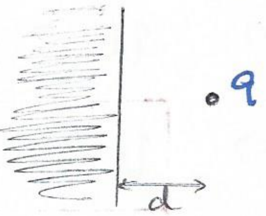
ϕ constant inside $\Rightarrow$ surface is equipotential,

Applying $\hat{n}\cdot(\underline{E}_+ - \underline{E}_-) = \frac{\sigma}{\varepsilon_0} \Rightarrow \underline{E}_{outside} = \frac{\sigma}{\varepsilon_0}\hat{n}$, from which we may calculate σ.

e.g {Spherical conductor with $Q=0$}



e.g {Ground conductor in presence of charge q}: Ground conductor so that $\phi=0$ throughout conductor in $x<0$. Place charge q at $x=d>0$: we apply the method of images with an image charge {$-q$ at $x=-d$}.



Then: $\phi_D = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q}{\sqrt{(x-d)^2+y^2+z^2}} - \frac{q}{\sqrt{(x+d)^2+y^2+z^2}}\right\}$

which satisfies $\phi_D(x=0) = 0$. Then, take:

$$\phi = \begin{cases} \phi_D & \text{for } x>0 \\ 0 & \text{for } x\leq 0 \end{cases}$$ and we are done.

Now, $\sigma = \varepsilon_0 E_x \cdot \hat{n}$.

Then: $E_x = -\frac{\partial\phi}{\partial x} = \frac{q}{4\pi\varepsilon_0}\left\{\frac{x-d}{|\underline{r}-\underline{d}|^{3/2}} - \frac{x+d}{|\underline{r}+\underline{d}|^{3/2}}\right\}$

$$\Rightarrow \sigma = \varepsilon_0 E_x|_{x=0} = -\frac{q}{2\pi}\frac{d}{(d^2+y^2+z^2)^{3/2}}$$

$\boxed{\text{N.B}}$ We may see that: $\int \sigma \, dy dz = -q$ is charge induced on surface $x=0$.

3 Magnetostatics

We study magnetic fields induced by steady currents i.e

$$\left\{\begin{array}{l} \underline{J} \neq \underline{0} \\ \rho = 0 \end{array}\right. \text{time independent solutions}\Bigg\} \Rightarrow \left\{\begin{array}{l} \nabla \times \underline{B} = \mu_0 \underline{J} \\ \nabla \cdot \underline{B} = 0 \end{array}\right.$$

no net charge. i.e +ve and −ve balance out.

$\iff \nabla \cdot \underline{J} = 0$ from continuity equation.

- **Ampère's Law:** Consider a surface S, with boundary $\mathcal{C}$:

$$\int_S (\nabla \times \underline{B}) \cdot d\underline{S} = \oint_{\mathcal{C}=\partial S} \underline{B} \cdot d\underline{r} = \mu_0 \int_S \underline{J} \cdot d\underline{S}$$

$$\Rightarrow \boxed{\oint_{\partial S = \mathcal{C}} \underline{B} \cdot d\underline{r} = \mu_0 \int_S \underline{J} \cdot d\underline{S} = \mu_0 I}$$

e.g {Long-straight wire} In cylindrical co-ordinates (r, φ, z):

$$\underbrace{\left\{ \underline{B}(r) = B(r)\hat{\underline{\varphi}} \right\}}_{\text{symmetry}} \Rightarrow \oint_{\mathcal{C}_r} \underline{B} \cdot d\underline{r} = 2\pi r B(r) = \mu_0 I$$

$$\Rightarrow \boxed{\underline{B}(r) = \frac{\mu_0 I}{2\pi r} \hat{\underline{\varphi}}}$$

e.g {Surface Current} surface current density $\underline{k}$. Take z-direction normal to plane: By superposition of above $\Rightarrow \overbrace{\left\{ \underline{B} = -B(z)\hat{\underline{y}} \right\}}^{\text{symmetry}}$ $B(z) = -B(-z)$.

($\hat{\underline{y}}$ normal to current in the plane).

Considering vertical rectangular loop, length L:

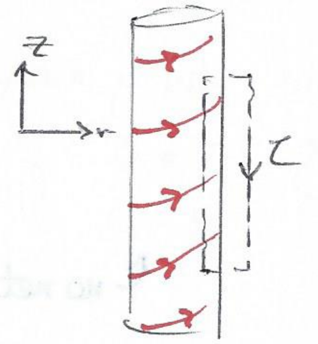
$$\oint_{\mathcal{C}} \underline{B} \cdot d\underline{l} = L\{B(z) - B(-z)\} = 2LB(z) = \mu_0 k L$$

$$\longrightarrow \boxed{\underline{B}(z) = \mathrm{sgn}(z)\frac{\mu_0 k}{2}\hat{\underline{y}}}$$

- **Surface Discontinuities:** Like in the electrostatic case we have:

$$\boxed{\underline{n} \times \underline{B}_+ - \underline{n} \times \underline{B}_- = \mu_0 \underline{k}.}$$

e.g <u>Solenoid</u>. By symmetry, $\underline{B} = B(r)\hat{\underline{z}}$



Now, $\nabla \times \underline{B} = 0$ away from cylinder

$$\Rightarrow \quad \frac{\partial B}{\partial r} = 0 \Rightarrow B \text{ constant.}$$

Now $\underline{B} = \underline{0}$ at infinity $\Rightarrow \boxed{\underline{B} = \underline{0}}$ outside cylinder everywhere.

Let N be number of turns / unit length then:

$$\oint_C \underline{B} \cdot d\underline{r} = BL = \mu_0 I N L$$

$$\Rightarrow \boxed{B = \mu_0 I N.} \text{ (inside cylinder)}.$$

• <u>Vector Potential</u>: Since we have $\nabla \cdot \underline{B} = 0 \Rightarrow \underline{B} = \nabla \times \underline{A}$ for some $\underline{A}$. Then:

$$\nabla \times \underline{B} = -\nabla^2 \underline{A} + \nabla(\nabla \cdot \underline{A}) = \mu_0 \underline{J}.$$

Now, if $\underline{A}$ is a vector potential, then for any function $\chi(\underline{x})$: $\underline{A}' = \underline{A} + \nabla \chi$ is also a vector potential of $\underline{B}$. The transformation $\underline{A} \mapsto \underline{A} + \nabla \chi$ is called a <u>guage transformation</u>.

{<u>Definition</u>}. A choice of $\underline{A}$ such that $\nabla \cdot \underline{A} = 0$ is called the <u>Colomb guage</u>.

We may always choose $\underline{A}'$ i.e χ such that $\nabla \cdot \underline{A}' = 0$.

<u>Pf</u>: Suppose $\underline{B} = \nabla \times \underline{A}$ and $\nabla \cdot \underline{A} = \psi(\underline{x})$. Then: for $\underline{A}' = \underline{A} + \nabla \chi$

$$\nabla \cdot \underline{A}' = \nabla \cdot \underline{A} + \nabla^2 \chi = \psi + \nabla^2 \chi$$

$$\Rightarrow \quad \nabla^2 \chi = -\psi \quad \text{which always has a solution}$$

If $\nabla \cdot \underline{A} = 0$ and $\underline{B} = \nabla \times \underline{A}$, then the Maxwell equation is:

$$\boxed{\begin{aligned} \nabla^2 \underline{A} &= -\mu_0 \underline{J} \\ \nabla^2 A_i &= -\mu_0 J_i \end{aligned}}$$

or in Cartesian co-ordinates:

The solution is then:

$$A_i(\underline{r}) = \frac{\mu_0}{4\pi}\int_{V'} \frac{J_i(\underline{r}')}{|\underline{r}-\underline{r}'|}\, dV'$$

$$\Rightarrow \quad \boxed{\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\int_{V'} \frac{\underline{J}(\underline{r}')}{|\underline{r}-\underline{r}'|}\, dV'}$$

We note that:

$$\underline{\nabla}\cdot\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\int \underline{J}(\underline{r}')\cdot\underline{\nabla}\left(\frac{1}{|\underline{r}-\underline{r}'|}\right) dV'$$

$$= -\frac{\mu_0}{4\pi}\int \underline{J}(\underline{r}')\cdot\underline{\nabla}'\left(\frac{1}{|\underline{r}-\underline{r}'|}\right) dV' \quad \leftarrow \text{differentiating wrt } \underline{r}'$$

$$= -\frac{\mu_0}{4\pi}\int \left\{ \underline{\nabla}'\cdot\left(\frac{\underline{J}(\underline{r}')}{|\underline{r}-\underline{r}'|}\right) - \frac{\underline{\nabla}'\cdot\underline{J}(\underline{r}')}{|\underline{r}-\underline{r}'|} \right\} dV'$$

We assume current is localised so $\underline{J}=\underline{0}$ on boundary $\Rightarrow$ first term zero. And $\underline{\nabla}'\cdot\underline{J}(\underline{r}')=0$ since currents are steady.

$$\Rightarrow \underline{\nabla}\cdot\underline{A}(\underline{r}) = 0 \quad \text{so is indeed a Coulomb guage.}$$

• <u>Biot-Savart Law:</u> $\left\{ \underline{B} = \underline{\nabla}\times\underline{A} = \frac{\mu_0}{4\pi}\int \underline{J}(\underline{r}')\times\frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^3}\, dV' \right\}$

If current is localised on a curve this becomes:

$$\boxed{\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi}\oint_C \frac{I\, d\underline{l}\times\underline{r}}{|\underline{r}|^3}}$$

• <u>Magnetic Dipoles:</u>

Consider a current loop: then:

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\oint_C \frac{I\, d\underline{r}'}{|\underline{r}-\underline{r}'|}$$

Far from the loop:

$$\frac{1}{|\underline{r}-\underline{r}'|} = \frac{1}{r} + \frac{\underline{r}\cdot\underline{r}'}{r^3} + \cdots$$

$$\Rightarrow \underline{A}(\underline{r}) = \frac{\mu_0 I}{4\pi} \oint_C \left\{ \frac{1}{r} + \frac{\underline{r}\cdot\underline{r}'}{r^3} + \cdots \right\} d\underline{r}'$$

Noting that r is a constant in the integration, we see first term $\to 0$. So we just consider second term:

We may show that, for any constant vector g:

$$\oint_C \underline{g}\cdot\underline{r}'\, d\underline{r}' = \underline{S}\times\underline{g}, \quad \underline{S} = \int d\underline{S} \text{ is vector area.}$$

[Pf: Follows from Green's Theorem:

$$\oint_C f(r')\, d\underline{r}' = \int_S \nabla f \times d\underline{S}$$

with $f(r') = \underline{g}\cdot\underline{r}'$ $\Rightarrow \oint_C \underline{g}\cdot\underline{r}'\, d\underline{r} = \int_S \underline{g}\times d\underline{S} = \underline{S}\times\underline{g}$]

Then: $\boxed{\underline{A}(\underline{r}) \simeq \frac{\mu_0}{4\pi}\frac{\underline{m}\times\underline{r}}{r^3}}$, $\underline{m} = I\underline{S}$ is the magnetic dipole moment.

So: $$\underline{B}(\underline{r}) = \nabla\times\underline{A} = \frac{\mu_0}{4\pi}\left\{ \frac{3(\underline{m}\cdot\hat{\underline{r}})\hat{\underline{r}} - \underline{m}}{r^3} \right\}.$$

• {General Current Distribution:}

$$A_i(\underline{r}) = \frac{\mu_0}{4\pi}\int \frac{J_i(\underline{r}')}{|\underline{r}-\underline{r}'|}\, dV'$$

$$\Rightarrow A_i(\underline{r}) = \frac{\mu_0}{4\pi}\int \left\{ \frac{J_i(\underline{r}')}{r} - \frac{J_i(\underline{r})\,(\underline{r}\cdot\underline{r}')}{r^3} + \cdots \right\} dV'$$

Now, for first term:

$$\partial_j'(J_j r_i') = \overset{0\ (\nabla\cdot\underline{J}=0)}{\cancel{(\partial_j' J_j)}}\, r_i' + \underbrace{(\partial_j' r_i')}_{\delta_{ij}} J_j = J_i$$

So it a total derivative $\to 0$.

For second term:

$$\partial_j'(J_j r_i r_k) = (\partial_j J_j) r_i r_k + J_i r_k + J_k r_i = J_k r_i + J_i r_k$$

So: $$\int J_i r_j r_j'\, dV' = \int \frac{r_j}{2}(J_i r_j' - J_j r_i')\, dV'$$

$$\Rightarrow \int J_i \underline{r} \cdot \underline{r}' \, dV = \int \frac{1}{2} \{ J_i (\underline{r} \cdot \underline{r}') - r_i (\underline{J} \cdot \underline{r}') \} \, dV$$

$$= \int \frac{1}{2} \{ \underline{r} \times (\underline{J} \times \underline{r}') \}_i \, dV$$

$$\Rightarrow \boxed{\underline{A}(\underline{r}) \simeq \frac{\mu_0}{4\pi} \frac{\underline{m} \times \underline{r}}{r^3}}$$

where $\underline{m} = \frac{1}{2} \int \underline{r}' \times \underline{J}(\underline{r}') \, dV'$ is the magnetic dipole moment.

- Magnetic Forces. $\boxed{\underline{F} = q \dot{\underline{r}} \times \underline{B}}$.

For a general force, say a current $\underline{J}_1$ is localised to some closed curve $\mathcal{C}_1$, then this sets up a magnetic field:

$$\underline{B}_1(\underline{r}) = \frac{\mu_0 I_1}{4\pi} \oint_{\mathcal{C}_1} d\underline{r}_1 \times \frac{(\underline{r} - \underline{r}_1)}{|\underline{r} - \underline{r}_1|^3}$$

Then, the force experienced by a second current on $\mathcal{C}_2$ is:

$$\underline{F} = \int \underline{J}_2(\underline{r}) \times \underline{B}_1(\underline{r}) \, dV$$

$$\Rightarrow \underline{F} = I_2 \oint_{\mathcal{C}_2} d\underline{r}_2 \times \underline{B}_1(\underline{r})$$

$$\Rightarrow \boxed{\underline{F} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} d\underline{r}_2 \times \left(d\underline{r}_1 \times \frac{\underline{r}_2 - \underline{r}_1}{|\underline{r}_2 - \underline{r}_1|^3} \right)}$$

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4 Electrodynamics.

- Induction: $\boxed{\nabla \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = 0}$.

Integrating around a closed curve, $\mathcal{C}$, with surface S:

$$\boxed{\int_{\mathcal{C}} \underline{E} \cdot d\underline{r} = - \frac{d}{dt} \int_S \underline{B} \cdot d\underline{S}}$$

Assumes S does not change with time.

We define the electromotive force, $\mathcal{E}$, and the magnetic flux, Φ, as:

$$\mathcal{E} = \int_{C} \underline{E} \cdot d\underline{l} \ , \quad \Phi = \int_{S} \underline{B} \cdot d\underline{S} \, .$$

to give us Faraday's Law: $\left\{ \mathcal{E} = - \frac{d\Phi}{dt} \right\}$

Lenz's Law, current produced opposes change.

Now consider a changing surface $S(t)$

Then: $\Phi(t+\delta t) - \Phi(t)$

$$= \int_{S(t+\delta t)} \underline{B}(t+\delta t) \cdot d\underline{S} - \int_{S(t)} \underline{B}(t) \cdot d\underline{S}$$

$$= \int_{S(t+\delta t)} \left\{ \underline{B}(t) + \frac{\partial \underline{B}}{\partial t} \delta t \right\} \cdot d\underline{S} - \int_{S(t)} \underline{B}(t) \cdot d\underline{S} + O(\delta t^2)$$

$$= \delta t \int_{S(t)} \frac{\partial \underline{B}}{\partial t} \cdot d\underline{S} + \left\{ \int_{S(t+\delta t)} - \int_{S(t)} \right\} \underline{B}(t) \cdot d\underline{S} + O(\delta t)^2$$

Since $\nabla \cdot \underline{B} = 0$, $\int_{S(\text{closed})} \underline{B} \cdot d\underline{S} = 0 \Rightarrow \left\{ \int_{S(t+\delta t)} - \int_{S(t)} \right\} \underline{B} \cdot d\underline{S}$

$$+ \int_{S_c} \underline{B} \cdot d\underline{S} = 0$$

$$\Rightarrow \Phi(t+\delta t) - \Phi(t) = \delta t \int_{S(t)} \frac{\partial \underline{B}}{\partial t} \cdot d\underline{S} - \int_{S_c} \underline{B} \cdot d\underline{S}$$

Writing $d\underline{S} = (d\underline{r} \times \underline{v}) \delta t$ on S_c, and noting $\frac{\partial \underline{B}}{\partial t} = -\nabla \times \underline{E}$

$$\Rightarrow \frac{d\Phi}{dt} = \lim_{\delta t \to 0} \frac{\delta \Phi}{\delta t} = - \int_{C(t)} \underline{E} \cdot d\underline{r} - \int_{C(t)} (\underline{v} \times \underline{B}) \cdot d\underline{r}$$

$$\Rightarrow \boxed{\frac{d\Phi}{dt} = - \int_{C(t)} \left\{ \underline{E} + (\underline{v} \times \underline{B}) \right\} \cdot d\underline{r}}$$

- Magnetostatic Energy:

Define $\boxed{L = \dfrac{\Phi}{I}}$ as the inductance of a curve $\mathcal{C}$.

Now, $\mathcal{E} = -\dfrac{d\Phi}{dt} = -L\dfrac{dI}{dt}$.

In a time δt, a charge $I\delta t$ flows around $\mathcal{C}$, then the work done is:

$$\delta W = \mathcal{E} I \delta t = -LI\frac{dI}{dt}\delta t$$

$$\Rightarrow \frac{dW}{dt} = -LI\frac{dI}{dt} = -\frac{1}{2}L\frac{dI^2}{dt}$$

$$\Rightarrow W = \frac{1}{2}LI^2 = \frac{1}{2}\Phi I$$ is the work done to build up a current I

This work done is identified with the energy stored in the system; so:

$$U = \frac{1}{2}I\int_S \underline{B}\cdot d\underline{S} = \frac{1}{2}I\int_S (\underline{\nabla}\times\underline{A})\cdot d\underline{S}$$

$$\Rightarrow U = \frac{1}{2}I\int_{\mathcal{C}} \underline{A}\cdot d\underline{r} = \frac{1}{2}\int_{\mathbb{R}^3} \underline{J}\cdot\underline{A}\, dV$$

Now, $\underline{\nabla}\times\underline{B} = \mu_0\underline{J}$

$$\Rightarrow U = \frac{1}{2\mu_0}\int\left((\underline{\nabla}\times\underline{B})\cdot\underline{A}\right)dV = \frac{1}{2\mu_0}\int\left(\underline{\nabla}\cdot(\underline{B}\times\underline{A}) + \underline{B}\cdot(\underline{\nabla}\times\underline{A})\right)dV$$

Assuming $\underline{B}\times\underline{A}$ vanishes sufficiently fast at infinity:

$$\Rightarrow U = \frac{1}{2\mu_0}\int \underline{B}\cdot\underline{B}\, dV$$

So, the energy stored in $\underline{E}$ and $\underline{B}$ fields is:

$$\boxed{U = \frac{1}{2}\int\left\{\varepsilon_0\underline{E}\cdot\underline{E} + \frac{1}{\mu_0}\underline{B}\cdot\underline{B}\right\}dV}$$

• Resistance:

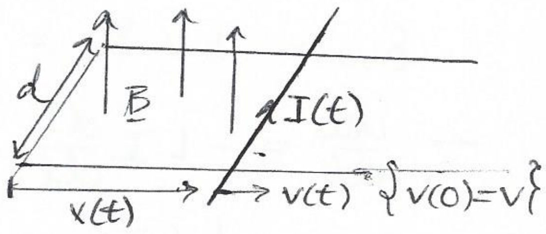
We have Ohm's Law: $\{\mathcal{E} = IR\}$ and also:

i) Resistivity: $\left\{\rho = \frac{AR}{l}\right\}$

ii) Conductivity: $\left\{\sigma = \frac{1}{\rho}\right\}$

More formally, Ohm's Law is: $\underline{J} = \sigma \underline{E}$.

e.g. {Bar with resistance R}



We have $d\underline{F} = IB\,\hat{y} \times \hat{z}$

$\Rightarrow \underline{F} = IBd\,\hat{x}$

$\Rightarrow m\ddot{x} = IBd$

Now, $\mathcal{E} = -\frac{d\Phi}{dt} = -Bd\dot{x} \Rightarrow IR = -Bl\dot{x}$

$$\Rightarrow m\ddot{x} = -\frac{B^2d^2}{R}\dot{x} \Rightarrow x(t) = -v\exp\left(-\frac{B^2d^2t}{mR}\right)$$

• Electromagnetic Waves: Solutions of Maxwell's Equations with $\underline{J} = \underline{0}$ and $\rho = 0$. So:

$$\left\{\begin{array}{ll} \nabla\cdot\underline{E} = 0 \quad , & \nabla\times\underline{E} = -\frac{\partial \underline{B}}{\partial t} \\ \nabla\cdot\underline{B} = 0 & \nabla\times\underline{B} = \mu_0\varepsilon_0\frac{\partial \underline{E}}{\partial t} \end{array}\right\}$$

$$\Rightarrow \mu_0\varepsilon_0\frac{\partial^2 \underline{E}}{\partial t^2} = \frac{\partial}{\partial t}(\nabla\times\underline{B})$$

$$\Rightarrow \mu_0\varepsilon_0\frac{\partial^2 \underline{E}}{\partial t^2} = \nabla\times\frac{\partial \underline{B}}{\partial t} = \nabla\times(\nabla\times\underline{E})$$

$$\Rightarrow \mu_0\varepsilon_0\frac{\partial^2 \underline{E}}{\partial t^2} = \nabla(\cancelto{0}{\nabla\cdot\underline{E}}) + \nabla^2\underline{E}$$

$$\Rightarrow \boxed{\mu_0\varepsilon_0\frac{\partial^2 \underline{E}}{\partial t^2} - \nabla^2\underline{E} = 0}$$

So each component of $\underline{E}$ satisfies the wave equation.

Similarly: $$\boxed{\mu_0 \varepsilon_0 \frac{\partial^2 \underline{B}}{\partial t^2} - \nabla^2 \underline{B} = 0}$$

with $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \, ms^{-1}$.

We look for plane wave solutions independent of y and z propagating in x-direction:

$$\Rightarrow \quad \underline{E}(\underline{x}) = (E_x(x,t), E_y(x,t), E_z(x,t))$$

Since $\nabla \cdot \underline{E} = 0 \Rightarrow E_x = \text{const}$, so take $E_x = 0$ wlog.

Assume $E_z = 0$, then:

$$\underline{E} = (0, E(x,t), 0).$$

$$\Rightarrow \quad E(x,t) = f(x-ct) + g(x+ct)$$

We have monochromatic solutions:

$$\underline{E = \underbrace{E_0}_{\text{amplitude}} \sin\{\underbrace{k}_{\text{wavenumber}}x - \overbrace{\omega}^{\text{frequency}} t\}}.$$

with $\boxed{\omega^2 = c^2 k^2}$ fixed by wave-equation. To solve for $\underline{B}$, use:

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

$$\Rightarrow \quad \underline{B} = (0, 0, B) \text{ for some } \underline{B}$$

$$\Rightarrow \quad \frac{\partial B}{\partial t} = -\frac{\partial E}{\partial x}$$

$$\Rightarrow \quad \underline{B(x,t) = \frac{E_0}{c} \sin\{kx - \omega t\}}.$$

Then, the most general monochromatic waves are:

$$\left\{ \underline{E} = \underline{E_0} \exp(i(\underline{k} \cdot \underline{x} - \omega t)), \quad \underline{B} = \underline{B_0} \exp(i(\underline{k} \cdot \underline{x} - \omega t)) \right\}$$

with $\boxed{\omega^2 = c^2 |\underline{k}|^2}$

Now: $\underline{\nabla}\cdot\underline{E}=0 \Rightarrow \boxed{\underline{k}\cdot\underline{E_0}=0}$

$\underline{\nabla}\cdot\underline{B}=0 \Rightarrow \boxed{\underline{k}\cdot\underline{B_0}=0}$

$\underline{\nabla}\times\underline{E}=-\frac{\partial B}{\partial t} \Rightarrow \boxed{\underline{k}\times\underline{E_0}=\omega\underline{B_0}}$

So, if E_0, B_0 are real, $\{k, E_0/c, B_0\}$ form an orthonormal set.

• {Definition}: A solution with real E_0, B_0 and $\underline{k}$ is said to be linearly polarized. If E_0 and B_0 are not real, then the polarization is not in a fixed direction:

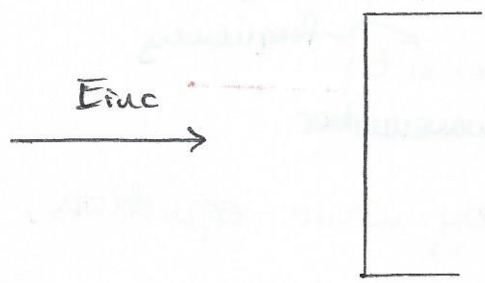
Say: $\underline{E_0}=\underline{\alpha}+i\underline{\beta}$, $\underline{\alpha},\underline{\beta}\in\mathbb{R}^3$, then:

$$Re(\underline{E}) = \underline{\alpha}\cos(\underline{k}\cdot\underline{x}-\omega t)-\underline{\beta}\sin(\underline{k}\cdot\underline{x}-\omega t).$$

Then $\underline{\nabla}\cdot\underline{E}=0 \Rightarrow \boxed{\underline{k}\cdot\underline{\alpha}=\underline{k}\cdot\underline{\beta}}$.

These are said to be elliptically polarized waves (if $(|\underline{\alpha}|=|\underline{\beta}|$, and $\underline{\alpha}\cdot\underline{\beta}=0)$, then we have circular polarisation)

e.g Conductor:



We have $\underline{E_{inc}}=E_0\hat{y}\exp(i(\cancel{\underline{k}\cdot\underline{x}}-\omega t))$, $\omega=ck$

We require $\underline{E}=\underline{0}$ inside conductor and $E_{\parallel}=0$ at surface. So $E_0\cdot\hat{y}|_{x=0}=0$. To achieve boundary conditions we add a reflected wave:

$$\underline{E}_{ref}=-E_0\hat{y}\exp(i(kx+\omega t))$$

$\Rightarrow$ $\underline{E}=\underline{E}_{inc}+\underline{E}_{ref}$ satisfying $\underline{E}\cdot\hat{y}|_{x=0}=0$.

Then: $\underline{B_{inc}}=\frac{E_0}{c}\hat{z}\exp(i(kx-\omega t))$

$\underline{B_{ref}}=\frac{E_0}{c}\hat{z}\exp(i(-kx-\omega t))$

which obeys $\underline{B}\cdot\hat{\underline{n}} = 0$ at the surface.

However, $\underline{B}\cdot\hat{\underline{z}} = \frac{2E_0}{c}e^{-i\omega t}$ at the surface, so we see a surface current $\underline{K}$, due to discontinuity, of:

$$\underline{K} = \pm\frac{2E_0}{\mu_0 c}\hat{\underline{y}}e^{-i\omega t}.$$ ie oscillating current can be seen as "cause" of reflected waves.

• <u>Poynting Vector</u>: The energy stored in a volume V is:

$$\mathcal{U} = \int_V\left(\frac{\varepsilon_0}{2}\underline{E}\cdot\underline{E} + \frac{1}{2\mu_0}\underline{B}\cdot\underline{B}\right)dV$$

Then: $$\frac{d\mathcal{U}}{dt} = \int_V\left(\frac{\varepsilon_0}{2}2\underline{E}\cdot\left\{(\nabla\times\underline{B})\frac{1}{\mu_0\varepsilon_0} - \frac{1}{\varepsilon_0}\underline{J}\right\} + \frac{1}{2\mu_0}\cdot 2\underline{B}\cdot(-\nabla\times\underline{E})\right)dV$$

$\uparrow \frac{\partial}{\partial t}(\underline{E}\cdot\underline{E}) = 2\underline{E}\cdot\frac{\partial\underline{E}}{\partial t}.$

$$\Rightarrow \frac{d\mathcal{U}}{dt} = \int_V\left(\frac{1}{\mu_0}\underline{E}\cdot(\nabla\times\underline{B}) - \underline{E}\cdot\underline{J} - \frac{1}{\mu_0}\underline{B}\cdot(\nabla\times\underline{E})\right)dV$$

But, $\left\{\underline{E}\cdot(\nabla\times\underline{B}) - \underline{B}\cdot(\nabla\times\underline{E}) = \nabla\cdot(\underline{E}\times\underline{B})\right\}$

$$\Rightarrow \frac{d\mathcal{U}}{dt} = -\int_V\underline{J}\cdot\underline{E}\,dV - \frac{1}{\mu_0}\int_S(\underline{E}\times\underline{B})\cdot d\underline{S}$$

Now, $\underline{J}\cdot\underline{E}$ term is work done on charged particles in V; giving us Poynting Theorem:

$$\underbrace{\frac{d\mathcal{U}}{dt} + \int_V\underline{J}\cdot\underline{E}\,dV}_{\text{energy of fields + particles.}} = \underbrace{-\frac{1}{\mu_0}\int_S\underline{S}\cdot d\underline{S}}_{\text{energy lost through surface.}}$$

where $\mu_0\underline{S} = (\underline{E}\times\underline{B})$ defines the Poynting vector.

So, e.g for linearly polarized waves:

$$\underline{E} = \underline{E_0}\sin(\underline{k}\cdot\underline{x}-\omega t), \quad \underline{B} = \frac{1}{c}(\hat{\underline{k}}\times\underline{E_0})\sin(\underline{k}\cdot\underline{x}-\omega t)$$

$$\Rightarrow \underline{S} = \frac{E_0^2}{c\mu_0}\hat{\underline{k}}\sin^2(\underline{k}\cdot\underline{x}-\omega t).$$

Time averaged over one period, $T = \frac{2\pi}{\omega}$:

$$\underline{\langle \underline{S} \rangle = \frac{E_0^2}{2c\mu_0} \hat{\underline{k}}}.$$

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5 Electromagnetism + Relativity

- <u>Vectors + Covectors:</u> we write the 4-vector:

$$\underline{X^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}}$$

and the Minkowski metric, $\eta_{\mu\nu}$ as: $\underline{\eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}$

Then: $\underline{X \cdot Y = X^T \eta Y}$. and $\underline{\|X\|^2 = X^T \eta X}$.

Now, define: $X_\mu = \begin{pmatrix} ct \\ -x \\ -y \\ -z \end{pmatrix}$

$$\Rightarrow X_\mu X^\mu = (ct)^2 - x^2 - y^2 - z^2 = \|X\|^2$$

So, in relativity, we must contract over one upper + one lower index. Now, given X_μ, we obtain X^μ by multiplying by $(\eta_{\mu\nu})^{-1} = \eta^{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$.

$$\Rightarrow \boxed{X^\mu = \eta^{\mu\nu} X_\nu}$$

- <u>Lorentz Transformations:</u> Suppose we have an inertial frame S' (ct', x', y', z') moving at a speed v in the x-direction relative to S (ct, x, y, z). Then:

$$\left\{ \begin{aligned} ct' &= \gamma \left(ct - \frac{v}{c} x \right) \\ x' &= \gamma \left(x - \frac{v}{c} ct \right) \\ y' &= y \\ z' &= z \end{aligned} \right\}$$

Lorentz Transformation

where $\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$.

This is represented by a matrix, $\Lambda^{\mu}{}_{\nu}$. Vectors transform according to:

$$\underline{X^{\mu} \longmapsto \Lambda^{\mu}{}_{\nu} X^{\nu}}$$

For $\Lambda^{\mu}{}_{\nu}$ to be a Lorentz transformation, it must satisfy:

$$\left\{ \Lambda^{\rho}{}_{\mu} \eta_{\rho\sigma} \Lambda^{\sigma}{}_{\nu} = \eta_{\mu\nu} \right\}$$

So, $\Lambda^{\mu}{}_{\nu}$ preserves the Minkowski metric. There are two classes of realisable Lorentz transformations. (non-time reversal/ reflections etc.).

i) {Rotations}

$$\underline{\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix}} \quad , \quad RR^T = I \text{ ie } R \text{ is orthogonal.}$$

ii) {Boosts}.

$$\underline{\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\frac{\gamma v}{c} & 0 & 0 \\ -\frac{\gamma v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

Now, we look at how X_{μ} transforms:

$$\begin{aligned} X_{\mu} \longmapsto X_{\mu}' &= \eta_{\mu\nu} (X^{\nu})' \\ &= \eta_{\mu\nu} \Lambda^{\nu}{}_{\sigma} X^{\sigma} \\ &= \eta_{\mu\nu} \Lambda^{\nu}{}_{\sigma} \eta^{\sigma\rho} X_{\rho} \\ &= \Lambda_{\mu}{}^{\rho} X_{\rho} \end{aligned}$$

So: $\underline{X_{\mu} \longmapsto \Lambda_{\mu}{}^{\rho} X_{\rho}}$

We may see that $\Lambda_\mu{}^\varrho$ is in fact the inverse of $\Lambda^\mu{}_\varrho$.

Since:
$$\Lambda^\varrho{}_\mu \eta_{\varrho\sigma} \Lambda^\sigma{}_\nu = \eta_{\mu\nu}$$
$$\Rightarrow \quad \Lambda^\varrho{}_\mu \eta_{\varrho\sigma} \Lambda^\sigma{}_\nu \eta^{\nu\tau} = \delta_\mu{}^\tau$$
$$\Rightarrow \quad \Lambda^\varrho{}_\mu \Lambda_\varrho{}^\tau = \delta_\mu{}^\tau.$$

- We may now define vectors and covectors:
  - i) Vectors have indices up and transform as:
  $$X^\mu \mapsto \Lambda^\mu{}_\nu X^\nu$$
  - ii) Covectors have indices down and transform as:
  $$X_\mu \mapsto \Lambda_\mu{}^\nu X_\nu$$

e.g {<u>4-derivative</u>}
$$\boxed{\partial_\mu = \frac{\partial}{\partial X^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)}.$$

We see that ∂_μ is a covectors as follows:
$$\partial_\mu \Rightarrow \frac{\partial}{\partial X^\mu} \mapsto \frac{\partial}{\partial X'^\mu} = \frac{\partial X^\nu}{\partial X'^\mu}\frac{\partial}{\partial X^\nu} = (\Lambda^{-1})^\nu{}_\mu \partial_\nu = \Lambda_\mu{}^\nu \partial_\nu .$$

- {<u>Definition</u>}. A tensor of type (m,n) is a quantity:
$$T^{\mu_1,\dots,\mu_m}{}_{\nu_1,\dots,\nu_n}$$

which transforms as:
$$\left\{ T'^{\mu_1,\dots,\mu_m}{}_{\nu_1,\dots,\nu_n} = \Lambda^{\mu_1}{}_{\varrho_1}\cdots\Lambda^{\mu_m}{}_{\varrho_m}\Lambda_{\nu_1}{}^{\sigma_1}\cdots\Lambda_{\nu_n}{}^{\sigma_n} T^{\varrho_1\dots\varrho_m}{}_{\sigma_1\dots\sigma_n} \right\}$$

- <u>Conserved Current:</u>

We define a 4-current, $\boxed{J^\mu = \begin{pmatrix} \varrho c \\ \underline{J} \end{pmatrix}}$

We show it transforms as a vector: e.g consider static charge density $\rho_0(\underline{x})$ with $\underline{J}=0$, then in a frame boosted by $\underline{v}$, new 4-current is:

$$J'^{\mu} = \Lambda^{\mu}{}_{\nu} J^{\nu} = \begin{pmatrix} \gamma\rho_0 c \\ -\gamma\rho_0 \underline{v} \end{pmatrix}$$ which makes sense.

Then the local continuity equation: $\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot\underline{J} = 0$

is simply:

$$\boxed{\partial_{\mu} J^{\mu} = 0}$$

- Gauge Potentials and Electromagnetic Fields:

Consider the scalar potential ϕ and vector potential $\underline{A}$.

We see that if we define:

$$\boxed{\begin{aligned} \underline{E} &= -\nabla\phi - \frac{\partial \underline{A}}{\partial t} \\ \underline{B} &= \underline{\nabla}\times\underline{A}. \end{aligned}}$$

then these satisfy the Maxwell Equations. We note that these choices of ϕ and $\underline{A}$ are not unique and we may shift them by any function $\chi(\underline{x},t)$ in a gauge transformation as follows:

$$\left\{ \begin{aligned} \phi &\mapsto \phi - \frac{\partial \chi}{\partial t} \\ \underline{A} &\mapsto \underline{A} + \nabla\chi \end{aligned} \right\}$$

and get the same E and $\underline{B}$. We may now define the 4-vector guage potential:

$$\boxed{A^{\mu} = \begin{pmatrix} \phi/c \\ \underline{A} \end{pmatrix}}$$

Then gauge transformations are simply:

$$\boxed{A_\mu \longmapsto A_\mu - \partial_\mu X}$$

and we define the anti-symmetric electromagnetic tensor:

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu}$$

This has 6 independent components (since diagonals all zero and $F_{\mu\nu} = -F_{\nu\mu}$). Finally we note that it is invariant under gauge transformation:

$$\begin{aligned} F_{\mu\nu} \longmapsto F'_{\mu\nu} &= \partial_\mu (A_\nu - \partial_\nu X) - \partial_\nu (A_\mu - \partial_\mu X) \\ &= \partial_\mu A_\nu - \partial_\mu \partial_\nu X - \partial_\nu A_\mu + \partial_\nu \partial_\mu X \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - \partial_\mu \partial_\nu X + \partial_\nu \partial_\mu X \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}. \end{aligned}$$

Now, we calculate some components, e.g:

$$F_{01} = \frac{1}{c}\frac{\partial}{\partial t}(-A_x) - \frac{\partial(\phi/c)}{\partial x} = \boxed{\frac{E_x}{c}}$$

indices down in A_μ

and $F_{02} = \frac{E_y}{c}$, $F_{03} = \frac{E_z}{c}$

and $F_{12} = \frac{\partial}{\partial x}(-A_y) - \frac{\partial}{\partial y}(-A_x) = -B_z$

$$\Rightarrow F_{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

$$\Rightarrow F^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}F_{\rho\sigma} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

Now, both $F_{\mu\nu}$ and $F^{\mu\nu}$ are tensors, so under a Lorentz transformation, we have: L17

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}F^{\rho\sigma}$$

e.g i) under rotation:

$$\underline{E}' = R\underline{E}$$
$$\underline{B}' = R\underline{B}.$$

ii) Under a boost in the x-direction:

$$\left\{\begin{array}{ll} E_x' = E_x & B_x' = B_x \\ E_y' = \gamma(E_y - vB_z) & B_y' = \gamma\left(B_y + \frac{v}{c^2}E_z\right) \\ E_z' = \gamma(E_z + vB_y) & B_z' = \gamma\left(B_z - \frac{v}{c^2}E_y\right) \end{array}\right\}$$

e.g {Boosted Line Charge} Infinite line along x-direction with uniform charge/unit length η has electric field:

$$\underline{E} = \frac{\eta}{2\pi\varepsilon_0(y^2+z^2)}\begin{pmatrix}0\\y\\z\end{pmatrix} \quad \text{and} \quad \underline{B} = \underline{0}$$

Now, with $\underline{v} = (v, 0, 0)$:

$$\underline{E}' = \frac{\eta\gamma}{2\pi\varepsilon_0(y^2+z^2)}\begin{pmatrix}0\\y\\z\end{pmatrix} = \frac{\eta\gamma}{2\pi\varepsilon_0(y'^2+z'^2)}\begin{pmatrix}0\\y'\\z'\end{pmatrix}$$

$$\underline{B}' = \frac{\eta\gamma v}{2\pi\varepsilon_0 c^2(y^2+z^2)}\begin{pmatrix}0\\z\\-y\end{pmatrix} = \frac{\mu_0\eta\gamma v}{2\pi\cdot(y'^2+z'^2)}\begin{pmatrix}0\\z'\\-y'\end{pmatrix}$$

$$\Longrightarrow \quad \underline{B}' = \frac{\mu_0 I'}{2\pi\sqrt{y'^2+z'^2}}\left(\hat{\varphi}'\right)$$

where $\hat{\varphi}' = \frac{1}{\sqrt{y'^2+z'^2}}\begin{pmatrix}0\\-z'\\y'\end{pmatrix}$ is the basis vector of cylindrical co-ordinates in S' and $I' = -\gamma\eta v$ is the current.

e.g {Boosted Point Charge}:

{In S}: $$\underline{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\underline{r}} = \frac{Q}{4\pi\varepsilon_0 (x^2+y^2+z^2)^{3/2}}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and $\underline{B} = \underline{0}$.

{In S'}: $$\underline{E}' = \frac{Q}{4\pi\varepsilon_0 (x^2+y^2+z^2)^{3/2}}\begin{pmatrix} x \\ \gamma y \\ \gamma z \end{pmatrix}$$

$$\Rightarrow \quad \underline{E}' = \frac{Q\gamma}{4\pi\varepsilon_0 \{\gamma^2(x'+vt')^2+y'^2+z'^2\}}\begin{pmatrix} x'+vt' \\ y' \\ z' \end{pmatrix}$$

Now, suppose the particle sits at $(-vt', 0, 0)$ in S', then at $t'=0$, $\underline{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$:

$$\begin{aligned} (\gamma^2(x'^2) + y'^2 + z'^2) &= (\gamma^2 - 1)x'^2 + r'^2 \\ &= \frac{v^2\gamma^2}{c^2}x'^2 + r'^2 \\ &= \left(\frac{v^2\gamma^2}{c^2}\cos^2\theta + 1\right)r'^2 \\ &= \gamma^2\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)r'^2 \end{aligned}$$

$$\Rightarrow \quad \underline{E}' = \frac{1}{\gamma^2\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{3/2}} \cdot \frac{1}{4\pi\varepsilon_0 r'^2}\,\underline{r}'$$

squashes the electric field.

• Lorentz Invariants: We contract indices; it turns out there are two such combinations:

(i) $$\boxed{\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = -\frac{\underline{E}^2}{c^2} + \underline{B}^2}$$

We now define the Levi-Cevita symbol in Minkowski spacetime.

$$\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \mu\nu\rho\sigma \text{ is an even permutation of } 0123 \\ -1 & \mu\nu\rho\sigma \text{ is an odd permutation of } 0123. \\ 0 & \text{otherwise} \end{cases}$$

We may show that the only fully anti-symmetric tensors in Minkowski space are multiples of $\varepsilon^{\mu\nu\rho\sigma}$. So under Lorentz transformation:

$$\varepsilon'^{\mu\nu\rho\sigma} = \Lambda^{\mu}{}_{\kappa}\Lambda^{\nu}{}_{\lambda}\Lambda^{\rho}{}_{\alpha}\Lambda^{\sigma}{}_{\beta}\,\varepsilon^{\kappa\lambda\alpha\beta}$$

So, since $\varepsilon^{\kappa\lambda\alpha\beta}$ is fully anti-symmetric, so is $\varepsilon'^{\mu\nu\rho\sigma}$

$$\Rightarrow \quad \varepsilon'^{\mu\nu\rho\sigma} = \gamma_{\varepsilon}\,\varepsilon^{\mu\nu\rho\sigma} \quad \text{for some } \gamma. \text{ (fixed).}$$

Now, $\varepsilon'^{0123} = \Lambda^{0}{}_{\kappa}\Lambda^{1}{}_{\lambda}\Lambda^{2}{}_{\alpha}\Lambda^{3}{}_{\beta}\,\varepsilon^{\kappa\lambda\alpha\beta} = \det \Lambda$

We have restricted to Λ with $\det \Lambda = 1$ (no reflections)

$$\Rightarrow \quad \varepsilon'^{\mu\nu\rho\sigma} = \varepsilon^{\mu\nu\rho\sigma} \quad \text{so it is } \{\text{invariant}\}.$$

Now, the <u>dual electromagnetic tensor</u> is defined to be:

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix}$$

$\boxed{\text{N.B}}$ $\tilde{F}_{\mu\nu}$ is obtained from $F_{\mu\nu}$ by $\underline{E} \mapsto c\underline{B}$, $\underline{B} \mapsto -\underline{E}/c$

This gives us the second Lorentz invariant:

$$\text{(ii)}. \quad \boxed{\frac{1}{4}\tilde{F}^{\mu\nu}F_{\mu\nu} = \frac{\underline{E}\cdot\underline{B}}{c}}$$

• <u>Maxwell's Equations</u>: It turns out these are:

$$\boxed{\begin{aligned} \partial_\mu F^{\mu\nu} &= \mu_0 J^\nu \\ \partial_\mu \tilde{F}^{\mu\nu} &= 0 \end{aligned}}$$

e.g (i) $\partial_i F^{i0} = \mu_0 J^0$

$$\Rightarrow \quad \underline{\nabla} \cdot \left(\frac{\underline{E}}{c} \right) = \mu_0 \rho c$$

$$\Rightarrow \quad \underline{\nabla} \cdot \underline{E} = \mu_0 c^2 \rho = \frac{\rho}{\varepsilon_0} . \Rightarrow \underline{\nabla} \cdot \underline{E} = \frac{\rho}{\varepsilon_0} .$$

and for $\nu = i$, $i = 1, 2, 3$, we get:

$$\frac{1}{c} \frac{\partial}{\partial t} \left(- \frac{\underline{E}}{c} \right) + \underline{\nabla} \times \underline{B} = \mu_0 \underline{J}$$

$$\Rightarrow \quad \underline{\nabla} \times \underline{B} - \frac{1}{\mu_0 \varepsilon_0} \frac{\partial \underline{E}}{\partial t} = \mu_0 \underline{J} .$$

(ii) $\partial_i \tilde{F}^{i0} = 0 \quad \Rightarrow \quad \underline{\nabla} \cdot \underline{B} = 0$

$$\partial_\mu \tilde{F}^{\mu i} = 0 \quad \Rightarrow \quad \frac{\partial \underline{B}}{\partial t} + \underline{\nabla} \times \underline{E} = 0 .$$

Also, note that since $F^{\mu\nu}$ is antisymmetric:

$$\partial_\nu \partial_\mu F^{\mu\nu} = 0 \quad \text{by contraction}$$

$$\Rightarrow \quad \{ \partial_\nu J^\nu = 0 \} \quad \text{(continuity equation)} .$$

• Lorentz Force Law: Consider a particle of charge q, and velocity $\underline{u}$. Then:

$$\frac{d\underline{p}}{dt} = q \left(\underline{E} + \underline{u} \times \underline{B} \right) .$$

In relativistic form, we use the proper time, τ, which satisfies:

$$\frac{dt}{d\tau} = \gamma(\underline{u}) = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

We define the 4-velocity: $U^\mu = \frac{dX^\mu}{d\tau} = \gamma \begin{pmatrix} c \\ \underline{u} \end{pmatrix}$

and the 4-momentum: $P^\mu = \begin{pmatrix} E/c \\ \underline{p} \end{pmatrix}$.

Then, the Lorentz force law may be written as:

$$\boxed{\frac{dP^{\mu}}{d\tau} = qF^{\mu\nu}U_{\nu}}$$

When $\mu = 1, 2, 3$:

$$\frac{d\underline{p}}{d\tau} = q\gamma\left(\underline{E} + \underline{u} \times \underline{B}\right) \qquad \text{where } \boxed{\underline{p} = m\gamma\underline{v}}$$

$$\Rightarrow \left\{ \frac{d\underline{p}}{dt} = q\left(\underline{E} + \underline{u} \times \underline{B}\right) \text{ as reqd.} \right\}$$

$\mu = 0$ gives:

$$\frac{1}{c}\frac{dE}{d\tau} = \frac{q}{c}\gamma\,\underline{E}\cdot\underline{u}$$

$$\Rightarrow \left\{ \frac{dE}{dt} = q\,\underline{E}\cdot\underline{u} \right\} \text{ which is work done by electric field.}$$

e.g. Suppose $\underline{E} = (E, 0, 0)$ and $\underline{u} = (v, 0, 0)$

$$\Rightarrow \quad m\frac{d(\gamma u)}{dt} = qE$$

$$\rightarrow \quad m\gamma u = qEt$$

$$\Rightarrow \quad u = \frac{dx}{dt} = \frac{qEt}{\sqrt{m^2 + \frac{q^2E^2t^2}{c^2}}}$$

Since $u \rightarrow c$ as $t \rightarrow \infty$, we solve to find:

$$x = \frac{mc^2}{q}\left(\sqrt{1 + \frac{q^2E^2t^2}{mc^2}} - 1\right) \simeq \frac{1}{2}qEt^2 \text{ for small } t \text{ as expected.}$$

e.g. $\underline{B} = (0, 0, B)$:

$$\frac{dP^0}{d\tau} = 0 \Rightarrow E = m\gamma c^2 = \text{const.} \quad \{\text{So } |\underline{u}| \text{ is const.}\}$$

$$\Rightarrow \quad m\frac{\partial(\gamma\underline{u})}{\partial t} = q\,\underline{u} \times \underline{B}$$

$$\Rightarrow \left\{ m\gamma \frac{d\underline{u}}{dt} = q\,\underline{u} \times \underline{B} \right\}$$

which is circular motion with frequency $\omega = \dfrac{qB}{m\gamma}$
