Asymptotic Analysis of Laplace Integrals

Send corrections to David Stuart dmas2@cam.ac.uk

I Introduction

Laplace integrals are of the form

$$I(x) \;=\; \int_a^b f(t)\exp[x\phi(t)]\,dt\,. \tag{1}$$

The asymptotic analysis of these integrals in the limit $x \to +\infty$ is simplified immensely by the Principle of Localization, which be stated informally as follows.

> *The asymptotic expansion of $I(x)$ for large x is determined entirely by the contributions of arbitrarily small neighborhoods of the points $\{t_\mu\}$ at which ϕ attains its maximum value $M = \max_{[a,b]}\phi = \phi(t_\mu)$.*

A crucial example is $\phi(t) = -t$ and $a = 0, b = +\infty$, in which case Watson's Lemma gives the required asymptotic expansion.

Watson's Lemma *Consider*

$$I(x) = \int_0^\infty t^\alpha\, g(t) e^{-xt} dt, \quad x > 0.$$

Assume that g is a locally integrable function and that

(i) $\alpha > -1$

(ii) *g has an asymptotic expansion* $g(t) \sim \sum_{j=0}^\infty a_j t^{rj}, \quad (t \to 0^+)$ *, for some positive r ;*

(iii) *there exist positive constants K, β such that $|g(t)| \le Ke^{\beta t}$ for $t \ge 0$.*

Then $I(x)$ has an asymptotic expansion

$$I(x) \;\sim\; \sum_{j=0}^\infty a_j\,\frac{\Gamma(\alpha + rj + 1)}{x^{\alpha+rj+1}}, \qquad (x \to +\infty)\,.$$

The point $t = 0$ is here the unique point at which the maximum value of $\phi(t) = -t$ over $[0,\infty)$ is achieved. The contribution from any interval $[c,d]$ with $\infty > d > c > 0$ is sub-dominant because

$$\left| \int_c^d t^\alpha \, g(t) \, e^{-xt} \, dt \right| \leq K \exp[-xc] \times \int_c^d t^\alpha e^{\beta t} \, dt = o(x^{-N})$$

as $x \to +\infty$ for any N since $c > 0$. (Make sure you know why). A similar argument works for general integrals like (1), and also for the case $d = \infty$ - refer to the proof given in class of Watson's Lemma to see this if you are unsure.

To obtain the overall asymptotic expansion of an integral like (1) for large x carry out the following three steps.

- **Step One:** Find the points $\{t_\mu\}$ at which ϕ is maximized;
- **Step Two:** Expand in small neighbourhoods of the $\{t_\mu\}$ and obtain the expansions of the integral over the intervals $t_\mu - \epsilon \leq t \leq t_\mu + \epsilon$ (or over $t_\mu - \epsilon \leq t \leq t_\mu$ or $t_\mu \leq t \leq t_\mu + \epsilon$ if $t_\mu = b$ or $t_\mu = a$ respectively);
- **Step Three:** Combine all contributions to get an overall asymptotic expansion.

As well as Watson's Lemma and the definition of the Gamma function and formulae for moments of various Gaussian integrals, the following formulae are useful to have at hand:

$$\int_0^\infty t^z e^{-xt} dt = \frac{\Gamma(z+1)}{x^{z+1}}, \quad \Re z > -1$$

which is just a rescaling of the definition of $\Gamma(z+1)$, and the binomial expansion

$$\Big(1+u\Big)^\alpha = \sum_{j=0}^{\infty} \frac{\Gamma(\alpha+1)}{j!\Gamma(\alpha-j+1)} \, u^j \,,$$

which converges for $|u| < 1$.

Exercise I.1. In the case $\alpha \in \mathbb{N}$ verify that this formula agrees with the usual combinatorial formula for the binomial expansion.

II An example where the maximum is achieved at two points

Consider the integral

$$I(x) = \int_0^{\frac{3\pi}{2}} \exp[x\,(\sin t)^2] \, dt$$

for real $x > 0$. *Analyze the asymptotic behaviour of $I(x)$ as $x \to +\infty$ by the Laplace method, obtaining the first two terms in the asymptotic expansion, and giving an estimate for the size of the next term.*

Step One: determination of contributing points The function in the exponential $\phi(t) = (\sin t)^2$ has a maximum value of $+1$ which is attained at the internal point $\frac{\pi}{2}$ and at the end point $\frac{3\pi}{2}$. By the Laplace Localization Principle (see discussion below) it is to be expected that the asymptotics of $I(x)$ will be determined by contributions from arbitrarily small intervals

$$[\frac{\pi}{2} - \epsilon\,,\ \frac{\pi}{2} + \epsilon]\,,$$

and

$$[\frac{3\pi}{2} - \epsilon\,,\ \frac{3\pi}{2}]\,,$$

where ϵ is small and positive. Again, by the same principle, it is to be expected that these contributions will be independent of ϵ.

Step Two: Evaluation of various contributions Here you will generally choose between two possible methods: *either* change variables to put the integral into the form in Watson's Lemma and go from there, *or* expand directly the integrand. We will illustrate both techniques.

(i) Contribution from the point $\frac{3\pi}{2}$ by means of Watson's Lemma. Changing variables to $u = 1 - (\sin t)^2$ we see that the interval

$$t \in [\frac{3\pi}{2} - \epsilon\,,\ \frac{3\pi}{2}]$$

is mapped monotonically to the interval

$$u \in [0\,,\ \delta(\epsilon)]\,,$$

where $\delta(\epsilon) = 1 - \left(\sin(\frac{3\pi}{2} - \epsilon)\right)^2$, and $\lim_{\epsilon \to 0} \delta(\epsilon) = 0$ by continuity. Therefore

$$\int_{\frac{3\pi}{2}-\epsilon}^{\frac{3\pi}{2}} \exp[x\,(\sin t)^2]\,dt \;=\; e^x \int_0^{\delta(\epsilon)} \frac{e^{-xu}du}{2\sqrt{u}\sqrt{(1-u)}}\,.$$

Since we may assume $\delta(\epsilon)$ is arbitrarily small we may now expand the integrand using the binomial expansion:

$$(1-u)^{-\frac{1}{2}} \;=\; 1 + (-\frac{1}{2})(-u) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{1 \times 2}(-u)^2 + \cdots + (-1)^m \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} - m)\,m!}\,u^m + \ldots \qquad (2)$$

and then apply again the Laplace principle to replace the upper limit by $+\infty$, finally leading to the integral

$$\frac{e^x}{2} \int_0^\infty u^{-\frac{1}{2}} e^{-xu} f(u)\,du\,,$$

where $f(u)$ has the asymptotic expansion (2) as $u \to 0^+$. Applying Watson's Lemma, we find

$$\int_{\frac{3\pi}{2}-\epsilon}^{\frac{3\pi}{2}} \exp[x\,(\sin t)^2]\,dt \;\sim\; \frac{e^x}{2} \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} - m)\,m!}\,\frac{\Gamma(\frac{1}{2} + m)}{x^{\frac{1}{2}+m}}\,, \quad (x \to +\infty)$$

$$\sim\; \frac{e^x}{2}\sqrt{\frac{\pi}{x}}\Big(1 + \frac{1}{4x} + \ldots\Big)\,.$$

(ii) Contribution from the point $\frac{\pi}{2}$ by means of local expansion. For the integral around $\frac{\pi}{2}$ we can write $t = \frac{\pi}{2} + v$ so that

$$\begin{aligned}\int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}+\epsilon} \exp[x\,(\sin t)^2]\,dt \;&=\; e^x \int_{-\epsilon}^{+\epsilon} \exp[-x(\sin v)^2]\,dv \\ &=\; e^x \int_{-\epsilon}^{+\epsilon} e^{-xv^2}\,\exp\Big[-x\big((\sin v)^2 - v^2\big)\Big]\,dv \\ &=\; e^x \int_{-\epsilon}^{+\epsilon} e^{-xv^2}\,\sum_{n=0}^{\infty}(-x)^n \frac{\big((\sin v)^2 - v^2\big)^n}{n!}\,dv\,.\end{aligned}$$

This is now a sum of integrals of $v^{2r}e^{-xv^2}$. Let the coefficients C_{nj} be defined by the expansion of the integrand

$$v^{2n}\,\big((\frac{\sin v}{v})^2 - 1\big)^n \;=\; \sum_{j=2n}^{\infty} C_{nj}v^{2j}\,.$$

Applying the Localization Principle again we can convert the integral into one over the whole real line without affecting the asymptotic expansion. This leads to the integral

$$e^x \int_{-\infty}^{+\infty} e^{-xv^2}\,\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\sum C_{nj}v^{2j}\,dv\,.$$

Referring to the formula for Gaussian integrals

$$\int_{-\infty}^{\infty} v^{2j}\,e^{-xv^2}\,dv \;=\; \frac{(2j)!}{j!2^{2j}}\,\frac{\sqrt{\pi}}{x^{j+1/2}}$$

we end up with (using $C_{00} = 1$ and $C_{0j} = 0$ for $j \geq 1$)

$$\begin{aligned}\int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}+\epsilon} \exp[x\,(\sin t)^2]\,dt \;&\sim\; e^x\sqrt{\frac{\pi}{x}}\,\sum_{n=0}^{\infty}\sum_{j=2n}^{\infty}\frac{(-1)^n C_{nj}(2j)!}{n!j!2^{2j}x^{j-n}} \\ &\sim\; e^x\sqrt{\frac{\pi}{x}}\,\Big(C_{00} - \frac{3C_{12}}{4x} + \dots\Big)\,.\end{aligned}$$

To work out terms explicitly, use the expansion for $\sin$ to get $\frac{\sin v}{v} = 1 - v^2/3! + v^4/5! - v^7/7! + \dots$ and hence

$$\begin{aligned}v^{2n}\,\big((\frac{\sin v}{v})^2 - 1\big)^n \;&=\; v^{2n}\Big(\frac{-2v^2}{3!} + \frac{v^4}{(3!)^2} + 2\frac{v^4}{5!} + \dots\Big)^n \\ &=\; \frac{(-1)^n}{3^n}v^{4n}\Big(1 - 3n \times (\frac{1}{36} + \frac{2}{5!})\,v^2 \;+\; \dots\Big)\,.\end{aligned}$$

Thus for $n \geq 1$ we have $C_{n,2n} = \frac{(-1)^n}{3^n}$ and $C_{n,2n+1} = \frac{2(-1)^{n-1}}{3^{n-1}\times 45}$, giving the first two terms as

$$\int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}+\epsilon} \exp[x\,(\sin t)^2]\,dt \;\sim\; e^x\sqrt{\frac{\pi}{x}}\,\Big(1 + \frac{1}{4x} + \dots\Big)\,.$$

Step Three: Derivation of Asymptotic Expansion to required order Combining the contributions from the two maximizing points computed in the previous section we obtain the first two terms in the asymptotic expansion as

$$I(x) \;=\; \int_0^{\frac{3\pi}{2}} \exp[x\,(\sin t)^2]\,dt \;\sim\; \frac{3e^x}{2}\sqrt{\frac{\pi}{x}}\left(1+\frac{1}{4x}+\dots\right).$$

It is clear from the computations above that the next term is $O(e^x/x^{5/2})$. Notice however that one has to be careful here - the presence of terms multiplying the exponential for example can change the conclusions easily, as should become clear after working through some more examples. For example, consider instead the integral

$$\tilde{I}(x) \;=\; \int_0^{\frac{3\pi}{2}} \exp[x\,(\sin t)^2]\,(1-\cos t)\,dt$$

In working out the contribution for example from the point $\frac{3\pi}{2}$ it is now necessary to include the factor $1-u^{\frac{1}{2}}$ in the integrand.

Exercise I.2. Obtain the first two terms in the expansion for $\tilde{I}(x)$ $(x \to +\infty)$.

III Laplace's Method: leading term for general order of vanishing

Consider the asymptotics of

$$I(x) = \int_a^b f(t)e^{-x\phi(t)}dt, \quad b > a \geq 0, \quad x \to +\infty\,,$$

for smooth f, ϕ, in the case that there is a *unique internal* maximum point c for $-\phi$, i.e., $\phi(c) = \max_{a\leq x\leq b}\,\phi(x)$ at which

$$\boxed{\phi'(c) = \phi''(c) = \cdots = \phi^{(s-1)}(c) = 0}$$

but $\boxed{\phi^{(s)}(c) > 0}$ for some *even* integer s. (It must be even or $-\phi$ wouldn't have a maximum at c.) Assume also $f(c) \neq 0$. Then for large positive x

$$I(x) \;\sim\; \frac{2e^{-x\phi(c)}f(c)\Gamma(\frac{1}{s})(s!)^{\frac{1}{s}}}{s\big(x\phi^{(s)}(c)\big)^{\frac{1}{s}}}$$

To derive this, expand in a neighbourhood of c, the integrand as:

$$f(t)e^{-x\phi(t)} \;=\; \big(f(c)+O(t-c)\big)\exp[-x\phi(c) - \frac{x}{s!}\phi^{(s)}(c)(t-c)^s + O((t-c)^{s+1})]\,.$$

Now apply again the Laplace Localization Principle and introduce $u = (t-c)^s$ as a new integration variable for $t > c$, so that $du = su^{\frac{s-1}{s}}\,dt$ and so

$$\begin{aligned}\int_c^{c+\epsilon} f(t)e^{-x\phi(t)}\,dt &\sim e^{-x\phi(c)} \int_c^{\infty} f(c)e^{-\frac{x}{s!}\phi^{(s)}(c)(t-c)^s}\big(1 + O(t-c) + O((t-c)^{s+1})\big)\,dt \\ &\sim f(c)e^{-x\phi(c)} \int_0^{\infty} e^{-\frac{x}{s!}\phi^{(s)}(c)u}\, s^{-1}\, u^{-\frac{s-1}{s}}\,(1 + O(u^{1/s}) + O(u^{1+1/s})\,du \\ &\sim \frac{f(c)e^{-x\phi(c)}\Gamma(\frac{1}{s})}{s} \times \left(\frac{s!}{x\phi^{(s)}(c)}\right)^{\frac{1}{s}} + O(x^{-\frac{2}{s}})\,.\end{aligned}$$

This is the leading contribution from the right side of the maximum point c. By parity the leading contribution from the left side is the same, leading to the formula stated.

Exercise I.3. Find the appropriate modification of this formula if c is an end point, i.e. $c = a$ or $c = b$.

Example: the L^p norm for large p Consider the L^p norm of g, i.e.

$$||g||_p = \left(\int_a^b |g(t)|^p dt\right)^{\frac{1}{p}}.$$

For simplicity assume that g is a smooth positive function which has a unique maximum value achieved at $t = c \in (a, b)$ at which $g''(c) < 0$. Then, by the preceding example with $s = 2$ and

$$\phi(t) = \ln|g(t)|, \quad f = 1, \quad \phi' = \frac{g'}{g}, \quad \phi'' = \frac{g''}{g} - \frac{g'^2}{g^2}, \quad \phi'(c) = 0,$$

we find

$$\begin{aligned}\int_a^b |g(t)|^p dt &\sim \sqrt{\frac{2\pi}{p}\frac{|g(c)|}{|g''(c)|}}\,e^{p\ln|g(c)|} \\ &= Ap^{-\frac{1}{2}}|g(c)|^p\,.\end{aligned}$$

Hence,

$$||g||_p \sim A^{\frac{1}{p}}|g(c)|p^{-\frac{1}{2p}}.$$

Thus, using

$$p^{-\frac{1}{2p}} = e^{-\frac{1}{2p}\ln p} \sim 1 + O\left(\frac{\ln p}{p}\right), \quad A^{\frac{1}{p}} = e^{\frac{1}{p}\ln A} = 1 + O\left(\frac{1}{p}\right),$$

we find

$$||g||_p \sim |g(c)|.$$

Remark I.4. If you've done P&M you may like to compare this with the argument you likely saw in that course: if $(\Omega, \mathcal{F}, \lambda)$ is a finite measure space (i.e. $\lambda(\Omega) < \infty$) and $g \in L^\infty(\Omega)$ then if $l < \|g\|_{L^\infty}$ there holds

$$l^p \lambda(\{x : |g(x)| > l\}) \leq \int_{\{x:|g(x)|>l\}} |g(x)|^p \, d\lambda(x) \leq \int_\Omega |g(x)|^p \, d\lambda(x) \leq \|g\|_{L^\infty}^p \lambda(\Omega) .$$

Taking the $\frac{1}{p}$th power gives

$$l \, \lambda(\{x : |g(x)| > l\})^{\frac{1}{p}} \leq \|g\|_{L^p} \leq \|g\|_{L^\infty} \lambda(\Omega)^{\frac{1}{p}} .$$

Since both $\lambda(\,\cdot\,)^{\frac{1}{p}}$ factors on left and right have limit one as $p \to \infty$, this gives

$$l \leq \liminf_{p\to\infty} \|g\|_{L^p} \leq \limsup_{p\to\infty} \|g\|_{L^p} \leq \|g\|_{L^\infty} .$$

Since $l < \|g\|_{L^\infty}$ was arbitrary this implies that $\lim_{p\to\infty} \|g\|_{L^p} = \|g\|_{L^\infty}$.

Example: Stirling's Formula Consider

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^x dt = \int_0^\infty e^{-t+x\ln t} \, dt ,$$

for $x \to +\infty$. The exponential is maximized when

$$-1 + \frac{x}{t} = 0 ,$$

i.e., when $t = x$ - a *moving maximum* (depends on x!). To "fix it" just substitute $t = xs$. Hence

$$\begin{aligned} \Gamma(x+1) &= \int_0^\infty e^{-xs} x^x s^x \, ds \\ &= x^{x+1} \int_0^\infty e^{-xs+x\ln s} \, ds. \end{aligned}$$

Using $s = 1 + \sigma$, we find

$$\begin{aligned} \Gamma(x+1) &= x^{x+1} \int_{-1}^\infty e^{-x(1+\sigma)+x(\sigma-\frac{\sigma^2}{2}+O(\sigma^3))} d\sigma \\ &\sim e^{-x} x^{x+1} \int_{-1}^\infty e^{-\frac{x\sigma^2}{2}} d\sigma \\ &\sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x , \quad x \to \infty, \end{aligned}$$

which is the Stirling formula.

Exercise I.5. Find the next term in the expansion of $\Gamma(x+1)$ as $x \to +\infty$.