Variational Principles

{Section 1}

Unless otherwise specified, f will be a function $f: \mathbb{R}^n \to \mathbb{R}$ and $\underline{x} = (x_1, \dots, x_n)$.

- **Stationary Points**

A stationary point is an $\underline{x_0}$ such that $\nabla f|_{\underline{x_0}} = 0$.

i.e $\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \dots = \frac{\partial f}{\partial x_n} = 0$.

Also, the Taylor expansion for f is:

$$\left\{ f(\underline{x}) = f(\underline{0}) + \underline{x} \cdot \underline{\nabla} f + \frac{1}{2} \sum_{i,j} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j} + O(x^3) \right\}$$

So, about a stationary point:

$$f(\underline{x}) = f(\underline{0}) + \frac{1}{2} \sum_{i,j} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j} + O(x^3)$$

Definition: (Hessian matrix) $\left\{ H_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j} \right\}$

Now, since $(H_{i,j})$ is symmetric, it is diagonal in some co-ordinate system (x'_i). In this co-ordinate system:

$$\left\{ f(\underline{x}) - f(\underline{0}) = \frac{1}{2} \sum_{i=1}^{n} \lambda_i (x'_i)^2 \right\}$$

where (λ_i) are the eigenvalues of $(H_{i,j})$.

Now, if $\begin{cases} \lambda_i > 0 \text{ for all } i \Rightarrow \underline{0} \text{ is a minimum} \\ \lambda_i < 0 \text{ for all } i \Rightarrow \underline{0} \text{ is a maximum} \\ \lambda_i > 0 \text{ for some } i, \lambda_i < 0 \text{ for all other } i \Rightarrow \underline{0} \text{ is a saddle point} \end{cases}$

If some $\lambda_i = 0$, then we have a degenerate stationary point.

(For $n = 2$, $\det H$, $\operatorname{tr} H$ is sufficient:

$$\begin{cases} \det H < 0 \Rightarrow \text{saddle} \\ \det H > 0, \begin{cases} \operatorname{tr} H > 0 \Rightarrow \text{maximum} \\ \operatorname{tr} H < 0 \Rightarrow \text{minimum} \end{cases} \end{cases}$$

)

• Convex Functions : Convex functions have nice properties. e.g all stationary points are minima, and can take at most one minimum value.

Definition: (Convex Set). A set $S \subseteq \mathbb{R}^n$ is convex if for any distinct $\underline{x}, \underline{y} \in S$, $t \in (0,1)$:

$$\left\{ (1-t)\underline{x} + t\underline{y} \in S \right\}$$

Definition: (Convex function) A function, $f: \mathbb{R}^n \to \mathbb{R}$ is convex if:

$$\left\{ \begin{array}{l} \text{i) Domain, } D(f)\text{, is convex} \\ \text{ii) } f\{(1-t)\underline{x} + t\underline{y}\} \leq (1-t)f(\underline{x}) + tf(\underline{y}) \end{array} \right\},$$

(if this inequality is strict, then the function is strictly convex.)

Now, suppose our function, f, is convex and once differentiable.

Define : $h(t) = (1-t)f(\underline{x}) + tf(\underline{y}) - f((1-t)\underline{x} + t\underline{y})$

By the definition of convexity, $h(t) \geq 0$ and $h(0) = 0$.

So: $$\frac{h(t) - h(0)}{t} \geq 0 \text{ for any } t \in (0,1)$$

$$\Rightarrow h'(0) \geq 0$$

On the other hand, $h'(0) = f(\underline{y}) - f(\underline{x}) - (\underline{y} - \underline{x}) \nabla f(\underline{x})$

So: $\left\{ f(\underline{y}) \geq f(\underline{x}) + (\underline{y} - \underline{x}) \cdot \nabla f(\underline{x}) \right\}$

This says that a convex function lies entirely above it's tangent plane.

{Corollary: A stationary point of a convex function is a global minimum. There can be more than one global minimum, but there is at most one if the function is strictly convex.}

Pf: Given x_0 : $\nabla f(x_0) = \underline{0}$.

$$\Rightarrow f(\underline{y}) \geq f(\underline{x_0}) + (\underline{y} - \underline{x}) \cdot \nabla f(\underline{x_0}) = f(\underline{x_0}).$$

We may rewrite the first-order convexity condition as:

$$(\underline{y}-\underline{x})\cdot\{\nabla f(\underline{y})-\nabla f(\underline{x})\} \geq f(\underline{x})-f(\underline{y})-(\underline{x}-\underline{y})\cdot\nabla f(\underline{y}) \geq 0$$

↑ from original condition.

So: $\left\{(\underline{y}-\underline{x})\cdot\{\nabla f(\underline{y})-\nabla f(\underline{x})\}\geq 0\right\}$.

which simply says that $\nabla f(\underline{x})$ is a non-decreasing function.

Consider a twice-differentiable function now. write $\underline{y}=\underline{x}+\underline{h}$ then, from the above condition:

$$\underline{h}\cdot(\nabla f(\underline{x}+\underline{h})-\nabla f(\underline{x}))\geq 0$$

$$\Rightarrow h_i\{h_j\nabla_i\nabla_j f + O(h^2)\}\geq 0$$

$$\Rightarrow h_i H_{ij} h_j + O(h^2)\geq 0$$

So, this is true for all $\underline{h}$ if H is positive semi-definite i.e the eigenvalues are non-negative. (if all positive, then H is definite).

Statement: Convexity $\Leftrightarrow$ Hessian is positive semi-definite

↑ definite if strictly convex.

· Legendre Transform: Important in classical mechanics + thermodynamics to transform between Lagrangian + Hamiltonian and Helmholtz free energy + enthalpy.

Given a function (differentiable), we essentially wish to transform to the conjugate variable, $p=\frac{df}{dx}$. e.g in classical mechanics, the conjugate momentum is $p=\frac{\partial L}{\partial \dot{x}}$.

Now, $f^*(p)=f(x(p))$ is not the right choice. In particular we wish to have the property:

$$\left\{\frac{df^*}{dp}=x\right\}$$ i.e x is the conjugate of p.

In terms of differentials:

$$df = \frac{df}{dx}dx = p\,dx$$

and $df^* = x\,dp$.

Consider product rule: $d(xp) = x\,dp + p\,dx$

So, defining $f^*(p) = x(p)\,p - f(x(p))$, then we satisfy the desired relation.

<u>Definition:</u> (Legendre Function) Given a function $f: \mathbb{R}^n \to \mathbb{R}$, the Legendre transform, f^* is defined by:

$$f^*(\underline{p}) = \sup_{\underline{x}} \{ \underline{p} \cdot \underline{x} - f(\underline{x}) \}$$

The domain of f^* is the set of $\underline{p} \in \mathbb{R}^n$ such that the supremum is finite.

Note that supremum is obtained when derivative of $\underline{p} \cdot \underline{x} - f(\underline{x})$ is zero i.e when $\underline{p} = \nabla f(\underline{x})$.

<u>Lemma:</u> f^* is always convex.

<u>Pf:</u>
$$\begin{aligned} f^*((1-t)\underline{p} + t\underline{q}) &= \sup_x \{ ((1-t)\underline{p}\cdot\underline{x} + t\underline{q}\cdot\underline{x}) - f(\underline{x}) \} \\ &= \sup_x \{ (1-t)(\underline{p}\cdot\underline{x} - f(\underline{x})) + t(\underline{q}\cdot\underline{x} - f(\underline{x})) \} \\ &\le (1-t)\sup_x \{ \underline{p}\cdot\underline{x} - f(\underline{x}) \} + t \sup_x \{ \underline{q}\cdot\underline{x} - f(\underline{x}) \} \\ &= (1-t) f^*(\underline{p}) + t f^*(\underline{q}) \end{aligned}$$

To show domain is convex, we note $f^*((1-t)\underline{p} + t\underline{q})$ is bounded so $(1-t)\underline{p} + t\underline{q}$ lies in domain of f^* $\therefore$ convex.

e.g.$_1$: $f(x) = \frac{1}{2}ax^2 \Rightarrow p = \frac{df}{dx} = ax$, so $f^*(p) = p \cdot \frac{p}{a} - \frac{1}{2}a\left(\frac{p}{a}\right)^2 = \boxed{\frac{p^2}{2a}}$

e.g.$_2$: $f(x) = cx$ $(c>0) \Rightarrow px - f(x) = (p-c)x$ which has no finite supremum unless $p = c$. So domain is simply $\{c\}$. So $f^*(p) = 0$.

Thm: If f is convex, differentiable with Legendre transform, f^*. Then:

$$\left\{ (f^*)^* = f \right\}$$

Pf: We have $f^*(\underline{p}) = (\underline{p}\cdot\underline{x}(\underline{p}) - f(\underline{x}(\underline{p})))$ where $\underline{x}(\underline{p})$ satisfies $\underline{p} = \nabla f(\underline{x}(\underline{p}))$. Differentiating w.r.t. $\underline{p}$ we have:

$$\begin{aligned}\nabla_i f^*(\underline{p}) &= x_i + p_j \nabla_i x_j(\underline{p}) - \nabla_i x_j(\underline{p}) \nabla_j f(\underline{x}) \\ &= x_i + p_j \nabla_i x_j(\underline{p}) - \nabla_i x_j(\underline{p})\, p_j \\ &= x_i\end{aligned}$$

So $\nabla f^*(\underline{p}) = \underline{x}$

Then: $\underline{p}$'s conjugate variable is $\underline{x}$

So: $$\begin{aligned} f^{**}(\underline{x}) &= (\underline{x}\cdot\underline{p} - f^*(\underline{p}))\big|_{\underline{p}=\underline{p}(\underline{x})} \\ &= \underline{x}\cdot\underline{p} - (\underline{p}\cdot\underline{x} - f(\underline{x})) \\ &= f(\underline{x}).\end{aligned}$$

Applying this to thermodynamics: Given a system, the energy is usually given in terms of entropy (S) and volume (V).

i.e $E = E(S,V)$.

Think in terms of gas inside a piston. Two things may affect energy:

i) We may push piston + modify volume, corresponding to work done by gas of $-pdV$

ii) Alternatively, we may heat it up inducing a heat change of TdS.

So: $$\left\{ dE = TdS - pdV \right\}$$

Comparing with chain rule, we have: $\frac{\partial E}{\partial S} = T$, $-\frac{\partial E}{\partial V} = p$.

To express the energy in terms of T (now the Helmholtz free energy) we obtain the conjugate function by taking the (negative) Legendre transform:

$$\left\{ F(T,V) = \inf_S \{E(S,V) - TS\} = E(S,V) - S\frac{\partial E}{\partial S} \right\}$$

If we transform with respect to V, we obtain the enthalpy.

· Lagrange Multipliers : We wish to solve problems of constrained maximisation.

e.g find maximum of $f(x,y)$ subject to $p(x,y)=0$ (i.e along a path.

We require $\underline{\nabla} f \cdot \underline{dl} = 0$ for $\underline{dl}$ parallel to path. Now normal to path is $\underline{\nabla} p$. So, our condition becomes:

$$\left.\begin{aligned} \underline{\nabla} f &= \lambda \underline{\nabla} p \\ p &= 0 \end{aligned}\right\} \text{ system to be solved for } x, y, \lambda.$$

e.g Find radius of smallest circle centred at origin that intersects $y = x^2 - 1$.

Three methods:

i) For intersection, $x^2+y^2=R^2$, $y=x^2-1$

So: $(x^2)^2 - x^2 + 1 - R^2 = 0$

$\Rightarrow \quad x^2 = \frac{1}{2} \pm \sqrt{R^2 - \frac{3}{4}}$

$\Rightarrow \quad \underline{R_{min} = \frac{\sqrt{3}}{2}}$.

ii) Want to minimise $f(x,y) = x^2+y^2$ subject to constraint $p(x,y) = y - x^2 + 1 = 0$.

Solve constraint to obtain $y = x^2 - 1$.

Then $R^2(x) = f(x, y(x)) = (x^2)^2 - x^2 + 1$

Look for stationary pts of $R^2(x)$

$\Rightarrow (R^2(x))' = 0 \Rightarrow x\left(x^2 - \frac{1}{2}\right) = 0$

$\Rightarrow x = 0,\ R = 1$ or $x = \frac{\sqrt{2}}{2}$, $\boxed{R = \frac{\sqrt{3}}{2}}$

iii) Lagrange Multipliers.

Find stationary points of function:

$$\phi(x, y, \lambda) = f(x,y) - \lambda p(x,y) = x^2 + y^2 - \lambda(y - x^2 + 1)$$

$$\Rightarrow \frac{\partial \phi}{\partial x} = 2x(1+\lambda) = 0$$
$$\frac{\partial \phi}{\partial y} = 2y - \lambda = 0$$
$$\frac{\partial \phi}{\partial \lambda} = y - x^2 + 1 = 0$$

$\Rightarrow x = 0, y = -1, R = \sqrt{x^2+y^2} = 1$

or $\lambda = -1, y = -\frac{1}{2}, x = \pm\frac{1}{\sqrt{2}}$

so $\boxed{R = \frac{\sqrt{3}}{2}}$

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e.g.2 for $\underline{x} \in \mathbb{R}^n$, find minimum of quadratic form

$$f(\underline{x}) = x_i A_{ij} x_j$$

on the surface $|\underline{x}|^2 = 1$.

i) Constraint imposes normalisation condition on $\underline{x}$, but $f(\underline{x})$ scales with $\underline{x}$. So, if we define:

$$\Lambda(\underline{x}) = \frac{f(\underline{x})}{|\underline{x}|^2} := \frac{f(\underline{x})}{g(\underline{x})}$$

Then the problem is unconstrained minimization of $\Lambda(\underline{x})$

So: $\nabla_i \Lambda(\underline{x}) = \frac{2}{g}\left\{ A_{ij} x_j - \frac{f}{g} x_i \right\}$

$$\Rightarrow \quad A\underline{x} = \Lambda \underline{x}$$

So, extremal values of Λ are eigenvalues of A. Then, smallest value is simply smallest eigenvalue.

ii) With Lagrange multipliers. Find stationary values of:

$$\phi(\underline{x}, \lambda) = f(\underline{x}) - \lambda(|\underline{x}|^2 - 1)$$

So: $\quad 0 = \nabla\phi \Rightarrow A_{ij} x_j = \lambda x_i$

and $\quad \frac{\partial \phi}{\partial \lambda} = 0 \Rightarrow |\underline{x}|^2 = 1$.

which gives the same solution.

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e.g.3 Find the probability distribution $\{p_1, \dots, p_n\}$ satisfying $\sum_{i=1}^{n} p_i = 1$, that maximises the information entropy

$$S = -\sum_{i=1}^{n} p_i \log p_i$$

Look for stationary points of:

$$\phi(\underline{p}, \lambda) = -\sum_{i=1}^{n} p_i \log p_i - \lambda\left\{\sum_{i=1}^{n} p_i\right\} + \lambda$$

$$\frac{\partial \phi}{\partial p_i} = -\log p_i - (1+\lambda) = 0$$

$$\Rightarrow p_i = e^{-(1+\lambda)}$$

Now, this expression is independent of i $\therefore$ $p_i = k \ \forall i$

$$\Rightarrow \boxed{p_i = \frac{1}{n}}$$

{Section 2}

- **Functional Derivatives:**

Definition: (Functional) function that takes another real-valued function as an argument. Usually written $F[x]$., $x = x(t): \mathbb{R} \to \mathbb{R}$.

e.g Given medium of refractive index $n(\underline{x})$, time taken along a path $\underline{x}(t)$ is given by functional:

$$T[\underline{x}] = \int_{\underline{x}_0}^{\underline{x}_1} n(\underline{x})\, dt.$$

The particular functionals we will study are as follows: given $x(t): [\alpha, \beta] \to \mathbb{R}$, define:

$$\left\{ F[x] = \int_\alpha^\beta f(x, \dot{x}, t)\, dt \right\}$$

We wish to find stationary points of the functional $F[x]$. Suppose we vary $x(t)$ by an amount $\delta x(t)$. Then the corresponding change in $F[x]$, $\delta F[x]$ is given by:

$$\begin{aligned} \delta F[x] &= F[x + \delta x] - F[x] \\ &= \int_\alpha^\beta \{ f(x + \delta x, \dot{x} + \delta \dot{x}, t) - f(x, \dot{x}, t) \}\, dt \end{aligned}$$

Now, $f(x + \delta x, \dot{x} + \delta\dot{x}, t) \simeq f(x, \dot{x}, t) + (\delta x, \delta \dot{x}) \cdot \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial \dot{x}} \end{pmatrix} + o(\delta^2)$

$$= f(x, \dot{x}, t) + \delta x \frac{\partial f}{\partial x} + \delta \dot{x} \frac{\partial f}{\partial \dot{x}} + o(\delta^2)$$

So: $$\delta F[x] = \int_\alpha^\beta \left\{ \delta x \frac{\partial f}{\partial x} + \delta \dot{x} \frac{\partial f}{\partial \dot{x}} \right\} dt + o(\delta^2)$$

We integrate the second term by parts:

$$\int_\alpha^\beta \delta \dot{x} \frac{\partial f}{\partial \dot{x}}\, dt = \left[\delta x \frac{\partial f}{\partial \dot{x}} \right]_\alpha^\beta - \int_\alpha^\beta \delta x \frac{d}{dt} \left\{ \frac{\partial f}{\partial \dot{x}} \right\} dt$$

To get: $$\delta F[x] = \int_\alpha^\beta \delta x \left[\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}} \right) \right] dt + \left[\delta x \frac{\partial f}{\partial \dot{x}} \right]_\alpha^\beta$$

We require that the boundary term vanishes, by choice of suitable boundary conditions. e.g x fixed at $t = \alpha, \beta \Rightarrow \delta x(\alpha) = \delta x(\beta) = 0$.

Then: $$\delta F[x] = \int_\alpha^\beta \left(\delta x \frac{\delta F[x]}{\delta x(t)} \right) dt$$

where $\left\{ \frac{\delta F[x]}{\delta x(t)} = \frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}} \right) \text{ is the functional derivative.} \right\}$

For stationary points of $F[x]$, we require $\frac{\delta F[x]}{\delta x(t)} = 0$. So:

<u>Definition:</u> The Euler-Lagrange equations for a function $\underline{x}(t): \mathbb{R}^n \to \mathbb{R}^n$ are:

$$\left\{ \frac{\partial f}{\partial x_i} - \frac{d}{dt} \left\{ \frac{\partial f}{\partial \dot{x}_i} \right\} = 0 \right\} \text{ for all } i, \quad \alpha \leq t \leq \beta.$$

e.g {Geodesic in the plane, between A, B. The length is $\int_C dl = L[y]$ }

Choose an arbitrary parametrisation, $\underline{r}(t) = (x(t), y(t))$ for $t \in [0,1]$ such that $\underline{r}(0) = A$, $\underline{r}(1) = B$.

Now $dl = \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$

So: $$L[x,y] = \int_0^1 \sqrt{\dot{x}^2 + \dot{y}^2}\, dt = \int_0^1 f(x, y, \dot{x}, \dot{y}, t)\, dt$$

We have $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ since there is no explicit dependance.

Then: $$\frac{d}{dt} \left\{ \frac{\partial f}{\partial \dot{x}} \right\} = \frac{d}{dt} \left\{ \frac{\partial f}{\partial \dot{y}} \right\} = 0$$

$$\Rightarrow \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = c, \quad \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = s$$, c and s are constants

Now, $c^2+s^2=1$ so let $c=\cos\theta$, $s=\sin\theta$.

$$\Rightarrow \quad (\dot{x}\sin\theta)^2 = (\dot{y}\cos\theta)^2$$

$$\Rightarrow \quad \dot{x}\sin\theta = \pm\dot{y}\cos\theta$$

We choose θ such that we have a positive sign

$\Rightarrow \quad y\cos\theta = x\sin\theta + A$ which is straight line of slope $\tan\theta$.

• <u>First Integrals</u>: If $f(\underline{x}, \dot{\underline{x}}, t)$ does not depend on $\underline{x}$, then the Euler-Lagrange equations simplify to the <u>first integral</u>:

$$\frac{d}{dt}\left\{\frac{\partial f}{\partial \dot{x}}\right\} = 0 \quad \Rightarrow \left\{\frac{\partial f}{\partial \dot{x}} = \text{const.}\right\} \leftarrow \text{first integral.}$$

In physical problems, this corresponds to a conserved quantity. eg energy/momentum.

Now consider situation where f does not depend explicitly on t.

By the chain rule: $\displaystyle \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{dx}{dt}\frac{\partial f}{\partial x} + \frac{d\dot{x}}{dt}\frac{\partial f}{\partial \dot{x}}$

$$\Rightarrow \quad \frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{x}\frac{\partial f}{\partial x} + \ddot{x}\frac{\partial f}{\partial \dot{x}}$$

Substituting Euler-Lagrange equations $\left\{\frac{\partial f}{\partial x} = \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right)\right\}$ in gives:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{x}\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) + \ddot{x}\frac{\partial f}{\partial \dot{x}}$$

$$\Rightarrow \quad \frac{d}{dt}\left\{f - \dot{x}\frac{\partial f}{\partial \dot{x}}\right\} = \frac{\partial f}{\partial t}$$

So, if $\frac{\partial f}{\partial t} = 0$ then we have the first integral:

$$\left\{f - \dot{x}\frac{\partial f}{\partial \dot{x}} = \text{const.}\right\}$$

We may now define <u>action</u>. For a particle mass m,

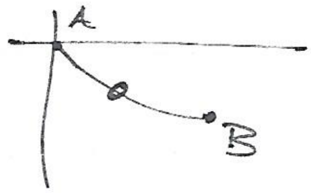
$$E = \tfrac{1}{2}mv^2 + U(\underline{x}), \text{ where } v = |\dot{\underline{x}}|^2$$

The action is defined as:

$$A = \int_A^B \sqrt{2m(E - U(\underline{x}))}\, dl$$

The principle of least action states that this quantity is minimised in the trajectory.

e.g. (Brachistochrone) What trajectory minimises time from A to B sliding along a wire?



Conservation of energy implies $\tfrac{1}{2}mv^2 = mgy$

$$\Longrightarrow v = \sqrt{2gy}$$

We wish to minimise $T = \int_A^B \frac{dl}{v}$

So: $$T = \frac{1}{\sqrt{2g}} \int \frac{\sqrt{dx^2+dy^2}}{\sqrt{y}} = \frac{1}{\sqrt{2g}} \int \sqrt{\frac{1+(y')^2}{y}}\, dx$$

There is no explicit x dependendance, so:

$$f - y' \frac{\partial f}{\partial y'} = \frac{1}{\sqrt{y(1+(y')^2)}} = \text{const.}$$

$$\Longrightarrow y(1+(y')^2) = c$$

This has the solution $$x = c(\theta - \sin\theta)$$ $$y = c(1-\cos\theta)$$

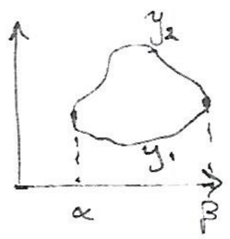
which is a <u>cycloid</u>.

• <u>Constrained variation of functionals</u>: Consider minimising $F[x]$ subject to $g(x(t)) = 0$. This is equivalent to the unconstrained stationary values of:

$$\phi_\lambda[x] = F[x] - \lambda(P[x] - c)$$

with respect to the function $x(t)$ and λ.

e.g. (Isoperimetric Problem). Consider string of fixed length, what is max area it can enclose? We may assume it encloses a convex region (else we can "push out"). Assuming this we split the curve into two parts:



We have $dA = \{y_2(x) - y_1(x)\}dx$

So: $A = \int_\alpha^\beta \{y_2(x) - y_1(x)\}dx$

Or alternatively $A[y] = \oint y(x)dx$

and the length is $L[y] = \oint dl = \oint \sqrt{1+(y')^2}\,dx$

So we look for stationary points of:

$$\phi_\lambda[y] = \oint \{y(x) - \lambda\sqrt{1+(y')^2}\}dx + \lambda L$$

Since there is no explicit dependance on x we use the first integral:

$$f - y'\frac{\partial f}{\partial y'} = \text{const.} = y_0$$

So: $y_0 = y - \lambda\sqrt{1+(y')^2} - \frac{\lambda(y')^2}{\sqrt{1+(y')^2}} = y - \frac{\lambda}{\sqrt{1+(y')^2}}$

$$\Rightarrow (y - y_0)^2 = \frac{\lambda^2}{1+(y')^2}$$

$$\Rightarrow (y')^2 = \frac{\lambda^2}{(y-y_0)^2} - 1$$

$$\Rightarrow \frac{(y-y_0)y'}{\sqrt{\lambda^2 - (y-y_0)^2}} = \pm 1$$

So: $d\left\{\sqrt{\lambda^2-(y-y_0)^2} \pm x\right\} = 0$

Then: $\lambda^2 - (y-y_0)^2 = (x-x_0)^2$

$$\Longrightarrow (x-x_0)^2 + (y-y_0)^2 = \lambda^2$$

which is a circle of radius λ.

So perimeter is $2\pi\lambda \Rightarrow \dfrac{L}{2\pi} = \lambda$

Then max. area is $\pi\lambda^2 = \boxed{\dfrac{L^2}{4\pi}}$

e.g.2 (Sturm-Liouville Problem) Let $\rho(x)$, $\sigma(x)$, $\omega(x)$ be real functions of x defined on $a \le x \le b$. Consider special case where ρ and ω are positive on $a \le x \le b$. We aim to find stationary points of the functional:

$$F[y] = \int_\alpha^\beta \left\{\rho(x)(y')^2 + \sigma(x)y^2\right\} dx$$

subject to $G[y] = \displaystyle\int_\alpha^\beta \omega(x) y^2\, dx = 1$.

Now, by the Euler-Lagrange equations, the functional derivatives of F, G are:

$$\frac{\delta F[y]}{\delta y} = 2\left\{-(\rho y')' + \sigma y\right\}$$

$$\frac{\delta G[y]}{\delta y} = 2(\omega y)$$

So, the Euler-Lagrange equation of $\Phi_\lambda[y] = F[y] - \lambda(G[y]-1)$

$$-(\rho y')' + \sigma y - \lambda \omega y = 0$$

which we write as an eigenvalue problem:

$$\mathcal{L}y = \lambda \omega y$$

where $\mathcal{L} = -\dfrac{d}{dx}\left(\rho \dfrac{d}{dx}\right) + \sigma$.

We note that $\mathcal{L}y = \lambda\omega y$ is linear, so if y is a soln. then Ay is.

But $G[Ay] = A^2$. So $G[y] = 1$ is a normalisation condition.

So, consider $\Lambda[y] = \frac{F[y]}{G[y]}$

$\Lambda[y]$ is not of the correct form to apply the Euler-Lagrange equations. However we vary it directly:

$$\delta\Lambda = \frac{1}{G}\delta F - \frac{F}{G^2}\delta G = \frac{1}{G}\{\delta F - \Lambda \delta G\}$$

when Λ is minimised, we have:

$$\delta\Lambda = 0 \Leftrightarrow \frac{\delta F}{\delta y} = \Lambda \frac{\delta G}{\delta y} \Leftrightarrow \mathcal{L}y = \Lambda w y$$

So at stationary values of $\Lambda[y]$, Λ is the associated Sturm-Liouville eigenvalue.

{<u>Section 3</u>}

Configuration space: in general, for N free particles, there will be 3N dimensions of configuration space:

$$\xi(t) = (x_1^1, x_1^2, x_1^3, \ldots, x_N^1, x_N^2, x_N^3).$$

These generalised co-ordinates need not be Cartesian co-ordinates and instead are better chosen with regard to the specific problem.

<u>Law</u>: (Hamilton's Principle). The actual path of a particle, $\xi(t)$, is the path that makes the action, $S[\xi]$, stationary, where:

$$\left\{ \begin{aligned} S[\xi] &= \int \mathcal{L}\, dt \\ \mathcal{L} &= T - V \end{aligned} \right\}$$

$\mathcal{L}$ ← Lagrangian, T ← KE, V ← PE.

e.g. for one particle in Euclidean 3-space: in Cartesian co-ordinates,
$T = \frac{1}{2}m|\dot{\underline{x}}|^2$, $V = V(\underline{x}, t)$

Then: $$S[\underline{x}] = \int_{t_0}^{t_1} \left\{\frac{1}{2}m|\dot{\underline{x}}|^2 - V(\underline{x}, t)\right\} dt.$$

and the Lagrangian is:

$$\mathcal{L}(\underline{x}, \dot{\underline{x}}, t) = \frac{1}{2}m|\dot{\underline{x}}|^2 - V(\underline{x}, t).$$

Apply the Euler-Lagrange equations to obtain,

$$\frac{\partial \mathcal{L}}{\partial \underline{x}} = \frac{d}{dt}\left\{\frac{\partial \mathcal{L}}{\partial \dot{\underline{x}}}\right\} \Rightarrow \boxed{m\ddot{\underline{x}} = -\nabla V}$$

which is Newton's Force Law.

If V is independent of time, then so is $\mathcal{L}$, so we may obtain a first integral as follows:

$$\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial t} + \dot{\underline{x}}\frac{\partial \mathcal{L}}{\partial \underline{x}} + \ddot{\underline{x}}\frac{\partial \mathcal{L}}{\partial \dot{\underline{x}}}$$

$$\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial t} + \dot{\underline{x}}\frac{d}{dt}\left\{\frac{\partial \mathcal{L}}{\partial \dot{\underline{x}}}\right\} + \ddot{\underline{x}}\frac{\partial \mathcal{L}}{\partial \dot{\underline{x}}}$$

$$\Rightarrow \frac{d}{dt}\left\{\mathcal{L} - \dot{\underline{x}}\frac{\partial \mathcal{L}}{\partial \dot{\underline{x}}}\right\} = \frac{\partial \mathcal{L}}{\partial t}$$

So, if $\mathcal{L}$ is not explicitly dependent on time (ie $V = V(\underline{x})$), then:

$$\dot{\underline{x}}\frac{\partial \mathcal{L}}{\partial \dot{\underline{x}}} - \mathcal{L} = E \quad , E \text{ is a constant.}$$

e.g. for one particle:

$$E = m|\dot{\underline{x}}|^2 - \tfrac{1}{2}m|\dot{\underline{x}}|^2 + V(\underline{x}) = \tfrac{1}{2}m|\dot{\underline{x}}|^2 + V(\underline{x}) = \text{total energy.}$$

e.g.$_2$ Consider a central force field, $F = -\nabla V$ where $V = V(r)$.
Use spherical polar co-ordinates, (r, θ, ϕ) where:

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

Then, $T = \frac{1}{2}m|\dot{\underline{x}}|^2 = \frac{1}{2}m\left(\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)\right)$

So: $\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) - V(r)$

w.l.o.g, $\theta = \frac{\pi}{2}$ (motion is planar).

So: $\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - V(r)$

Then, Euler-Lagrange equations give:

$$\left\{\begin{array}{c} m\ddot{r} - mr\dot{\phi}^2 + V'(r) = 0 \\ \frac{d}{dt}\{mr^2\dot{\phi}\} = 0 \end{array}\right\}$$

So, $mr^2\dot{\phi} = h$ (angular momentum / unit mass).

Then $\dot{\phi} = \frac{h}{mr^2} \Rightarrow m\ddot{r} - \frac{mh^2}{r^3} + V'(r) = 0$

If we let $V_{eff}(r) = V(r) + \frac{mh^2}{2r^2}$

Then $m\ddot{r} = -V_{eff}'(r)$

So, in a gravitational field for example,

$$V_{eff}(r) = m\left\{-\frac{GM}{r} + \frac{h^2}{2r^2}\right\}$$

• <u>The Hamiltonian</u>: This formulation works with the conjugate momentum, $\left\{\underline{p} = \frac{\partial \mathcal{L}}{\partial \dot{\underline{x}}}\right\}$, which involves taking the Legendre transform of the Lagrangian.

The space of variables $(\underline{x}, \underline{p})$ is known as <u>phase space</u>.

Since the Legendre transform is self-inverse, we have:

$$\mathcal{L} = \underline{p}\cdot\dot{\underline{x}} - H(\underline{x},\underline{p}). \qquad \left\{\text{since } \dot{\underline{x}} \text{ is conjugate variable to } \underline{p}\right\}$$

where $\dot{\underline{x}} = \frac{\partial \mathcal{L}}{\partial \underline{p}} \Rightarrow \left\{H(\underline{x},\underline{p}) = \underline{p}\cdot\dot{\underline{x}} - \mathcal{L}\{\underline{x}, \dot{\underline{x}}(\underline{p})\}\right\}$

So, rewriting the action:

$$\left\{S[\underline{x},\underline{p}] = \int (\underline{p}\cdot\dot{\underline{x}} - H(\underline{x},\underline{p}))\, dt\right\}$$

(phase-space form.

The Euler-Lagrange equations are Hamilton's equations:

$$-\frac{\partial H}{\partial \underline{p}} + \dot{\underline{x}} = \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{\underline{p}}}\{\underline{p}\cdot\dot{\underline{x}} - H(\underline{x},\underline{p})\}\right\} = 0 \Rightarrow \dot{\underline{x}} = \frac{\partial H}{\partial \underline{p}}$$

$$-\frac{\partial H}{\partial \underline{x}} = \frac{d}{dt}\{\underline{p}\} = \dot{\underline{p}} \Rightarrow \dot{\underline{p}} = -\frac{\partial H}{\partial \underline{x}}$$

So: $\left\{\dot{\underline{x}} = \frac{\partial H}{\partial \underline{p}}, \quad \dot{\underline{p}} = -\frac{\partial H}{\partial \underline{x}}\right\}$

Solving these gives a trajectory in phase space parametrised by $\underline{x}$, $\underline{p}$, which are said to be canonically conjugate.

Symmetries + Noether's Theorem, Given:

$$F[x] = \int_\alpha^\beta f(x, \dot{x}, t)\, dt$$

suppose we transform variables, $t \to t^*(t)$, $x \to x^*(t^*)$. Then:

$$F[x] \to F^*[x^*] = \int_{\alpha^*}^{\beta^*} f(x^*; \dot{x}^*, t^*)\, dt^*$$

where $\alpha^* = t^*(\alpha)$, $\beta^* = t^*(\beta)$

Definition: (Symmetry), If $\{F^*[x^*] = F[x]\}$ for all x, α, β. Then $*$ is a symmetry.

example: 1) Consider $\{t \to t,\ x \to x + \varepsilon\}$ for some small ε. Then:

$$F^*[x^*] = \int_\alpha^\beta f(x + \varepsilon, \dot{x}, t)\, dt = \int_\alpha^\beta \left\{ f(x, \dot{x}, t) + \varepsilon \frac{\partial f}{\partial x} \right\} dt$$

So, the transformation is a symmetry if $\frac{\partial f}{\partial x} = 0$.

Also, we know if $\frac{\partial f}{\partial x} = 0$, then the first integral becomes:

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) = 0 \implies \frac{\partial f}{\partial \dot{x}} \text{ is conserved quantity.}$$

2) Consider $t \to t - \varepsilon$, $x \to x^*$: $x^*(t^*) = x(t)$. Then:

$$F^*[x^*] = \int_\alpha^\beta f(x, \dot{x}, t - \varepsilon)\, dt = \int_\alpha^\beta \left\{ f(x, \dot{x}, t) - \varepsilon \frac{\partial f}{\partial t} \right\} dt$$

So, symmetry if $\frac{\partial f}{\partial t} = 0$.

Again, the first integral becomes:

$$\frac{d}{dt}\left\{ f - \dot{x}\frac{\partial f}{\partial \dot{x}} \right\} = 0 \implies \left(f - \dot{x}\frac{\partial f}{\partial \dot{x}} \right) \text{ is conserved quantity.}$$

Thm: (Noether's Theorem) For every continuous symmetry of $F[x]$, the solutions (stationary points of $F[x]$) will have a corresponding conserved quantity.

Continuous means built up of infinitessimal transformations e.g reflection not continuous. If the continuity condition does not hold, then whilst there may be a symmetry, there is not necessarily a conserved quantity.

We consider symmetries that involve only the x-variable, which up to 1st order is: $\{x \to x(t) + \varepsilon h(t),\ t \to t\}$

By saying this is a symmetry is equivalent to saying for small ε,

$\delta F[x] = 0$.

On the other hand, since $x(t)$ is a stationary point of $F[x]$, we know if ε is non-constant but vanishes at the end-points, $\delta F = 0$.

Now,

$$\delta F = \int \left\{ f(x+\varepsilon h,\ \dot{x} + \varepsilon \dot{h} + \dot{\varepsilon} h,\ t) - f(x, \dot{x}, t) \right\} dt$$

$$= \int \left\{ \frac{\partial f}{\partial x} \varepsilon h + \frac{\partial f}{\partial \dot{x}} \varepsilon \dot{h} + \frac{\partial f}{\partial \dot{x}} \dot{\varepsilon} h \right\} dt$$

$$= \int \varepsilon \left\{ \frac{\partial f}{\partial x} h + \frac{\partial f}{\partial \dot{x}} \dot{h} \right\} dt + \int \left\{ \dot{\varepsilon} \frac{\partial f}{\partial \dot{x}} h \right\} dt$$

Consider the case where ε is constant. So:

$$\varepsilon \int \left\{ \frac{\partial f}{\partial x} h + \frac{\partial f}{\partial \dot{x}} \dot{h} \right\} dt = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} h + \frac{\partial f}{\partial \dot{x}} \dot{h} = 0$$

So:

$$\delta F = \int \left\{ \dot{\varepsilon} \frac{\partial f}{\partial \dot{x}} h \right\} dt$$

Now consider a variable ε that is non-constant but vanishes at the end-points, then, integrating by parts we obtain:

$$\int \varepsilon \frac{d}{dt} \left\{ \frac{\partial f}{\partial \dot{x}} h \right\} dt = 0$$

$\boxed{\text{COEFFICIENT OF } \dot{\varepsilon}}$

$$\Rightarrow \frac{d}{dt} \left\{ \frac{\partial f}{\partial \dot{x}} h \right\} = 0 \Rightarrow \left\{ \frac{\partial f}{\partial \dot{x}} h \text{ is the conserved quantity} \right\}$$

For example: Applying this to Hamiltonian mechanics: motion of particle is stationary point of:

$$S[\underline{x}, \underline{p}] = \int \left\{ \underline{p} \cdot \dot{\underline{x}} - \mathcal{H}(\underline{x}, \underline{p}) \right\} dt$$

$$\text{where } \mathcal{H}(\underline{x}, \underline{p}) = \frac{1}{2m} |\underline{p}|^2 + V(\underline{x})$$

1) Consider first case of free particle. ($V(\underline{x}) = 0$).

Since $S[\underline{x}, \underline{p}]$ depends only on $\dot{\underline{x}}$ (or $\underline{p}$), it is invariant under the transformation $\underline{x} \to \underline{x} + \underline{\varepsilon}$, $\underline{p} \to \underline{p}$

For a general $\underline{\varepsilon}$ that varies with time, we have:

$$\delta S = \int \left\{ \left[\underline{p} \cdot (\underline{\dot{x}} + \underline{\dot{\varepsilon}}) - \mathcal{H}(\underline{p}) \right] - \left[\underline{p} \cdot \underline{\dot{x}} - \mathcal{H}(\underline{p}) \right] \right\} dt$$

$$= \int \underline{p} \cdot \underline{\dot{\varepsilon}} \, dt$$

and hence, the momentum, $\underline{p}$ is conserved in the motion.

2) Time-translation invariance $\Rightarrow$ conservation of $\mathcal{H}(\underline{x}, \underline{p})$.

$\Leftrightarrow$ conservation of energy.

First integral is: $f - \dot{x} \dfrac{\partial f}{\partial \dot{x}}$ conserved.

$$\Rightarrow \left(\underline{p} \cdot \underline{\dot{x}} - \mathcal{H}(\underline{x}, \underline{p}) \right) - \underline{\dot{x}} \cdot \underline{p} = -\mathcal{H}(\underline{x}, \underline{p}) \text{ conserved}$$

$$\Rightarrow \mathcal{H}(\underline{x}, \underline{p}) \text{ conserved}$$

3) Suppose we have a potential which depends on radius, $V(|\underline{x}|)$. Then this has rotational symmetry. Choose an axis of rotation, $\underline{\omega}$.

Then:

$$\underline{x} \longrightarrow \underline{x} + \varepsilon \, \underline{\omega} \times \underline{x}$$

$$\underline{p} \longrightarrow \underline{p} + \varepsilon \, \underline{\omega} \times \underline{p}$$

Then the rotation does not affect $|\underline{x}|$, $|\underline{p}|$ and hence $\mathcal{H}(\underline{x}, \underline{p})$.

Now:

$$\delta S = \int \left(\underline{p} \cdot \frac{d}{dt} \left\{ \underline{x} + \varepsilon \, \underline{\omega} \times \underline{x} \right\} - \underline{p} \cdot \underline{\dot{x}} \right) dt$$

$$= \int \underline{p} \cdot \frac{d}{dt} \left\{ \varepsilon \, \underline{\omega} \times \underline{x} \right\} dt$$

$$= \int \underline{p} \cdot \left[\underline{\omega} \times \frac{d}{dt} (\varepsilon \underline{x}) \right] dt$$

$$= \int \underline{p} \cdot \left[\underline{\omega} \times (\dot{\varepsilon} \underline{x} + \varepsilon \underline{\dot{x}}) \right] dt$$

$$= \int \left\{ \dot{\varepsilon} \, \underline{p} \cdot (\underline{\omega} \times \underline{x}) + \varepsilon \, \underline{p} \cdot (\underline{\omega} \times \underline{\dot{x}}) \right\} dt$$

Since $\underline{p}$ is parallel to $\underline{\dot{x}}$, $\underline{p} \cdot (\underline{\omega} \times \underline{\dot{x}}) = 0$

$$= \int \dot{\varepsilon} \left(\underline{p} \cdot (\underline{\omega} \times \underline{x}) \right) dt$$

So: $\underline{\omega} \cdot (\underline{x} \times \underline{p})$ is a constant of the motion. Since the above holds for all $\underline{\omega}$ $\Rightarrow$ $\underline{L} = \underline{x} \times \underline{p}$ is conserved. This is the angular momentum.

{Section 4}

We now consider functions of the type: $y(x_1,\dots,x_m):\mathbb{R}^m \to \mathbb{R}^n$.
Then the functional is of the form:

$$\left\{ F[y] = \int \cdots \int f(y, \nabla y, x_1, \dots, x_m)\, dx_1 \cdots dx_m \right\}$$

where ∇y is the second-rank tensor defined by:

$$\nabla y = \left(\frac{\partial y}{\partial x_1}, \dots, \frac{\partial y}{\partial x_m} \right)$$

Instead of Euler-Lagrange equivalents, we consider instead directly variations in y, δy.

Example: {(Minimal surfaces in $\mathbb{E}^3$)} Consider a surface of least area subject to some boundary conditions. Suppose (x,y) are good co-ordinates to describe surface S, where $(x,y) \in D \subseteq \mathbb{R}^2$. Then S is defined by $z = h(x,y)$.
Then, the area is given by:

$$A[h] = \int_D \sqrt{1 + h_x^2 + h_y^2}\, dA \qquad \left(h_x = \frac{\partial h}{\partial x} \right)$$

Consider a variation, $h(x,y) \to h(x,y) + \delta h(x,y)$

Then: $$A[h+\delta h] = \int_D \sqrt{1 + (h_x + (\delta h)_x)^2 + (h_y + (\delta h)_y)^2}\, dA$$

$$= A[h] + \int_D \left\{ \frac{h_x (\delta h)_x + h_y (\delta h)_y}{\sqrt{1 + h_x^2 + h_y^2}} + O(\delta h^2) \right\} dA$$

Integrating by parts:

$$\delta A = -\int_D \delta h \left\{ \frac{\partial}{\partial x}\left(\frac{h_x}{\sqrt{1+h_x^2+h_y^2}} \right) + \frac{\partial}{\partial y}\left(\frac{h_y}{\sqrt{1+h_x^2+h_y^2}} \right) \right\} dA + O(\delta h^2)$$

plus boundary terms.

So, our minimal surface will satisfy:

$$\frac{\partial}{\partial x}\left(\frac{h_x}{\sqrt{1+h_x^2+h_y^2}} \right) + \frac{\partial}{\partial y}\left(\frac{h_y}{\sqrt{1+h_x^2+h_y^2}} \right) = 0$$

which simplifies to the minimal surface equation:

$$\left\{ (1+h_y^2) h_{xx} + (1+h_x^2) h_{yy} - 2 h_x h_y h_{xy} = 0 \right\}$$

We consider some special cases:

i) Obvious solution of $h(x,y) = Ax + By + C$ which represents a plane.

ii) If $|\nabla h|^2 << 1$, then $h_x^2; h_y^2$ are small so, approximately:

$$h_{xx} + h_{yy} = 0$$
$$\Rightarrow \nabla^2 h = 0$$

So harmonic functions are approximately minimal surfaces.

iii) (Cylindrically-symmetric solution) $h(x,y) = z(r)$ where $r = \sqrt{x^2+y^2}$. We have an ordinary differential equation:

$$rz'' + z' + (z')^3 = 0$$

which has the general solution:

$$z = A^{-1}\cosh(Ar) + B.$$

which is a catenoid.

Example 2: {Small Amplitude Oscillations of uniform string} Suppose we have string mass density, e, tension T.

Amplitude is given by $y(x;t)$, then KE is:

$$T = \frac{1}{2}\int_0^a e v^2\, dx = \frac{e}{2}\int_0^a \dot{y}^2\, dx$$

Potential energy is tension times length so:

$$V = T\int dl = T\int_0^a \sqrt{1+(y')^2}\, dx \simeq aT + \int_0^a \frac{1}{2}T(y')^2 dx$$

The aT term can be seen as ground-state energy, and does not affect position of stationary points. Action is:

$$S[y] = \int\int_0^a \left\{\frac{1}{2}e\dot{y}^2 - \frac{1}{2}T(y')^2\right\} dx\, dt$$

Applying Hamilton's principle, $\delta S[y] = 0$.

Now, $$\delta S[y] = \int\int_0^a \left\{ e\dot{y}\frac{\partial\{\delta y\}}{\partial t} - Ty'\frac{\partial\{\delta y\}}{\partial x}\right\} dx\, dt$$

Integrating by parts:

$$\delta S[y] = \iint_0^a \delta y \left(e\ddot{y} - Ty''\right) dx\,dt \; + \text{ boundary terms}$$

Assuming boundary terms vanish, we need:

$$\left\{ \ddot{y} - v^2 y'' = 0 \qquad \text{where } v^2 = T/e \right\}$$

Example 3: {Maxwell's Equations} We wish to derive the Lagrangian for the field itself (electromagnetic).

Definitions: e = electric charge density
$\underline{J}$ = electric current density.
ϕ = electric scalar potential
$\underline{A}$ = magnetic vector potential.
$\underline{E} = -\nabla\phi - \dot{\underline{A}}$ is electric field
$\underline{B} = \nabla \times \underline{A}$ is magnetic field.

We also pick convenient units where $c = \varepsilon_0 = \mu_0 = 1$.
The action is now given by:

$$S[\underline{A}, \phi] = \int \left(\tfrac{1}{2}\left\{ |\underline{E}|^2 - |\underline{B}|^2 \right\} + \underline{A}\cdot\underline{J} - \phi e \right) dV\,dt$$

$-\underline{E}\cdot(\nabla\phi + \dot{\underline{A}})$

Varying $\underline{A}$ by $\delta\underline{A}$ and ϕ by $\delta\phi$ we get:

$$\delta S = \int \left\{ -\underline{E}\cdot\Big(\underbrace{\nabla(\delta\phi)}_{\text{parts}} + \underbrace{\frac{\partial(\delta\underline{A})}{\partial t}}_{\text{parts}}\Big) - \underline{B}\cdot\nabla\times\delta\underline{A} + \delta\underline{A}\cdot\underline{J} - e\delta\phi \right\} dV\,dt$$

We integrate by parts: $\hookrightarrow \nabla\cdot\underline{E}$ $\quad \hookrightarrow \dot{\underline{E}}$

$$\delta S = \int \left\{ \delta\underline{A}\cdot\left(\dot{\underline{E}} - \nabla\times\underline{B} + \underline{J}\right) + \delta\phi\left(\nabla\cdot\underline{E} - e\right) \right\} dV\,dt$$

Since the coefficients have to be zero:

$$\left\{ \nabla\times\underline{B} = \underline{J} + \dot{\underline{E}} \;, \quad \nabla\cdot\underline{E} = e \right\}$$

and from the definitions of $\underline{E}, \underline{B}$,

$$\left\{ \nabla\cdot\underline{B} = 0 \;, \quad \nabla\times\underline{E} = -\dot{\underline{B}} \right\}$$

• **The Second Variation**: we need to look at second functional derivatives to establish maximum + minimum properties.

- Now, suppose $x(t) = x_0(t)$ is a solution of:

$$\frac{\delta F[x]}{\delta y(x)} = 0$$

To determine the type of stationary point we must expand $F[x+\delta x]$ to second order. Let $\delta x(t) = \varepsilon\,\xi(t)$ for $\varepsilon \ll 1$, and functionals of the form:

$$F[x] = \int_\alpha^\beta f(x,\dot{x},t)\,dt$$

with $\xi(\alpha) = \xi(\beta) = 0$.

Consider variation $x \to x + \delta x$.

$$f(x+\varepsilon\xi,\ \dot{x}+\varepsilon\dot{\xi},\ t) - f(x,\dot{x},t)$$
$$= \varepsilon\left(\xi\frac{\partial f}{\partial x} + \dot{\xi}\frac{\partial f}{\partial \dot{x}}\right) + \frac{\varepsilon^2}{2}\left(\xi^2\frac{\partial^2 f}{\partial x^2} + 2\xi\dot{\xi}\frac{\partial^2 f}{\partial x\partial\dot{x}} + \dot{\xi}^2\frac{\partial^2 f}{\partial \dot{x}^2}\right) + \mathcal{O}(\varepsilon^3)$$

Noting that $2\xi\dot{\xi} = (\xi^2)'$ and integrating parts we obtain:

$$= \varepsilon\xi\left\{\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right)\right\} + \frac{\varepsilon^2}{2}\left\{\xi^2\left\{\frac{\partial^2 f}{\partial x^2} - \frac{d}{dt}\left(\frac{\partial^2 f}{\partial x\partial\dot{x}}\right)\right\} + \dot{\xi}^2\frac{\partial^2 f}{\partial \dot{x}^2}\right\}$$

$+\ \mathcal{O}(\varepsilon^3)$
$+$ boundary terms which vanish

So: $$F[x+\varepsilon\xi] - F[x] = \int_\alpha^\beta\left\{\varepsilon\xi\left\{\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right)\right\} + \frac{\varepsilon^2}{2}\delta^2F[x,\xi] + \mathcal{O}(\varepsilon^3)\right\}dt$$

where $$\delta^2F[x;\xi] = \int_\alpha^\beta\left\{\xi^2\left[\frac{\partial^2 f}{\partial x^2} - \frac{d}{dt}\left(\frac{\partial^2 f}{\partial x\partial\dot{x}}\right)\right] + \dot{\xi}^2\frac{\partial^2 f}{\partial\dot{x}^2}\right\}dt$$

is a functional of both x and ξ. analogous to $\underline{\delta x}^T\underline{\underline{H}}\,\underline{\delta x}$

If $\delta^2F[x;\xi] > 0$ for all non-zero ξ. Then a solution $x_0(t)$ is an absolute minimum.

Example: (Geodesic's in the plane) We show that straight line is indeed shortest distance + not just stationary value.

Recall, $f = \sqrt{1+(y')^2}$

Then: $\frac{\partial f}{\partial y} = 0$, $\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+(y')^2}}$, $\frac{\partial^2 f}{\partial (y')^2} = \frac{1}{\sqrt{1+(y')^2}^3}$

So: $\delta^2 F[y, \xi] = \int_\alpha^\beta \frac{\dot{\xi}^2}{(1+(y')^2)^{3/2}} dx > 0$

So, any stationary function is indeed a minimum.

Not all functions are convex, so we may still ask whether a solution of the Euler-Lagrange equations is a local minimum.

We consider:

$$\delta^2 F[x_0, \xi] = \int_\alpha^\beta \{ e(t)\dot{\xi}^2 + \sigma(t)\xi^2 \} dt$$

where $e(t) = \frac{\partial^2 f}{\partial \dot{x}^2}\Big|_{x=x_0}$, $\sigma(t) = \left\{ \frac{\partial^2 f}{\partial x^2} - \frac{d}{dt}\left(\frac{\partial^2 f}{\partial x \partial \dot{x}}\right) \right\}\Big|_{x=x_0}$

A necessary condition for this is:

$$\{ e(t) \geq 0 \} \quad \text{(Legendre condition)}.$$

Example: (Brachistochrone Problem) we have:

$$T[x] \propto \int_\alpha^\beta \sqrt{\frac{1+\dot{x}^2}{x}}\, dt$$

So; $e(t) = \frac{\partial^2 f}{\partial \dot{x}^2}\Big|_{x_0} > 0$, $\sigma(t) = \frac{1}{2x^2\sqrt{x(1+\dot{x}^2)}} > 0$

So, the cycloid does minimise T.

Now, we wish to find a sufficient condition for $\delta^2 F[x, \xi] > 0$.

- <u>Jacobi Condition for local minima of $F[x]$</u>

We have the strong Legendre condition, $e(t) > 0$. Now, for a solution, x_0, to the Euler-Lagrange equations, we have:

$$\delta^2 F[x_0, \xi] = \int_\alpha^\beta (e(t)\dot{\xi}^2 + \sigma(t)\xi^2)\, dt$$

where $e(t)$, $\sigma(t)$ are as before. Assume $e(t)>0$ for $\alpha<t<\beta$, and that $\xi(\alpha)=\xi(\beta)=0$. When is this sufficient for $\delta^2F>0$.

Note that: $$0=\int_\alpha^\beta (w\xi^2)'\,dt \quad \text{since } \xi(\alpha)=\xi(\beta)=0.$$

and so: $$0=\int_\alpha^\beta \{2w\xi\dot{\xi}+\dot{w}\xi^2\}\,dt$$

Then: $$\delta^2F=\int_\alpha^\beta\left[e\dot{\xi}^2+2w\xi\dot{\xi}+(\sigma+\dot{w})\xi^2\right]dt$$

So. $$\delta^2F=\int_\alpha^\beta\left\{e\left(\dot{\xi}+\frac{w}{e}\xi\right)^2+\left(\sigma+\dot{w}-\frac{w^2}{e}\right)\xi^2\right\}dt$$

which is non-negative if $w^2=(\sigma+\dot{w})e$

So, as long as this equation has a solution, $\delta^2F>0$. We also note that $\delta^2F=0 \Rightarrow \dot{\xi}+\frac{w}{e}\xi=0 \Rightarrow \xi(x)=C\exp\left(-\int_\alpha^x\frac{w(s)}{e(s)}ds\right)$

But $\xi(\alpha)=0=C \Rightarrow \xi=0$ is only case where $\delta^2F=0$ may hold.

Now, let $w=-\frac{e\dot{u}}{u}$, then:
$$e\left(\frac{\dot{u}}{u}\right)^2=\sigma-\left(\frac{e\dot{u}}{u}\right)^{\cdot}=\sigma-\frac{(e\dot{u})'}{u}+e\left(\frac{\dot{u}}{u}\right)^2$$

So: $$-(e\dot{u})'+\sigma u=0 \quad \text{(Jacobi Accesory Equation)}$$

From the definition of w, $\left(\frac{\dot{u}}{u}\right)$, we require $u\neq 0$ on $[\alpha,\beta]$.

So, if we can find non-zero $u(x)$, then $\delta^2F>0$

<u>Example</u>: (Geodesics on unit sphere). For any curve C we have:
$$L=\int_C\sqrt{d\theta^2+\sin^2\theta\, d\phi^2}$$

If θ is a good parameter for the curve, then:
$$L[\phi]=\int_{\theta_1}^{\theta_2}\sqrt{1+\sin^2\theta\,(\phi')^2}\,d\theta$$

Or alternatively, if ϕ is a good parameter.

$$L[\Theta] = \int_{\phi_1}^{\phi_2} \sqrt{(\Theta')^2 + \sin^2\Theta}\, d\phi$$

Consider this second case,

$$f(\theta, \Theta') = \sqrt{(\Theta')^2 + \sin^2\Theta}$$

So:
$$\frac{\partial f}{\partial \Theta} = \frac{\sin\Theta\cos\Theta}{\sqrt{(\Theta')^2 + \sin^2\Theta}}, \quad \frac{\partial f}{\partial \Theta'} = \frac{\Theta'}{\sqrt{(\Theta')^2 + \sin^2\Theta}}$$

Since $\frac{\partial f}{\partial \phi} = 0$, we have the first integral,

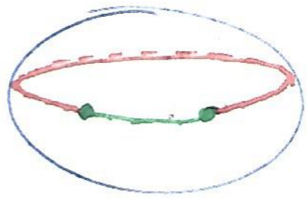
$$\text{const.} = f - \theta' \frac{\partial f}{\partial \theta'} = \frac{\sin^2\Theta}{\sqrt{(\Theta')^2 + \sin^2\Theta}}$$

$$\Rightarrow \quad c\sin^2\Theta = \sqrt{(\theta')^2 + \sin^2\Theta} \qquad (c \geq 1)$$

Consider case $c=1$ which occurs where $\theta' = 0$.

$$\Rightarrow \sin^2\theta = \sin\Theta \Rightarrow (\Theta = 0) \text{ or } \Theta = \frac{\pi}{2}$$

So, $\theta(\phi) = \frac{\pi}{2}$, which generates two arcs, we ask which one minimises $L[\phi]$.



Now,
$$\frac{\partial^2 f}{\partial (\theta')^2} = \frac{\sin^2\Theta}{(\sqrt{(\Theta')^2 + \sin^2\Theta})^3} = \frac{1}{c^3\sin^4\Theta} = 1$$

$$\frac{\partial^2 f}{\partial \Theta^2} = \frac{\cos 2\Theta}{\sqrt{(\Theta')^2 + \sin^2\Theta}} - \frac{\sin^2\Theta\cos^2\Theta}{(\sqrt{(\Theta')^2 + \sin^2\Theta})^3} = -1$$

$$\frac{\partial^2 f}{\partial\Theta\partial\Theta'} = 0$$

So, $e(\phi) = 1$, $\sigma(\phi) = -1$

And thus; $\delta^2 L[\Theta_0, \xi] = \int_{\phi_1}^{\phi_2} \{ (\xi')^2 - \xi^2 \} d\phi$

The Jacobi Accesory Equation is:

$$u'' + u = 0 \Rightarrow u \propto \sin\phi - \gamma\cos\phi \text{ for any } \gamma.$$

So, $u = 0$ when $\tan\phi = \gamma$.

We note the LHS is periodic with period π, so for any ϕ_γ which is the first zero of $\tan\phi = \gamma$, there is another zero π radians later. So:

1) $\exists$ nowhere zero solution to Jacobi Accesory equations assuming the equitorial arc is such that ϕ increases by no more than π. Hence, the shorter of the two solutions is indeed a local minimum for $L[\phi]$.

2) $\nexists$ a nowhere-zero solution if ϕ increases by more than π, so $\delta^2 \cancel{L} \leq 0 \Rightarrow$ Longer arc is not even local minimiser for $L[\phi]$.