# Fluid Dynamics.

## 1 Parallel Viscous Flow.

{Definition}. A fluid is a material that flows. We restrict our attention to Newtonian fluids.

{Definition}: A Newtonian fluid is a fluid with a linear relationship between stress and rate of strain, with the constant of proportionality being the viscosity. (Although we will usually take the inviscid approximation).

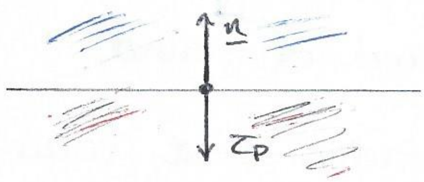
{Definition}. The stress is the force per unit area.

{Definition} The strain is the extension per unit length $\Rightarrow$ rate of strain concerns gradients of velocity.

- Stress: These are forces within the fluid.
  - First suppose we have a fluid of pressure $p$ acting on a surface with unit normal $\underline{n}$ poiting into the fluid.

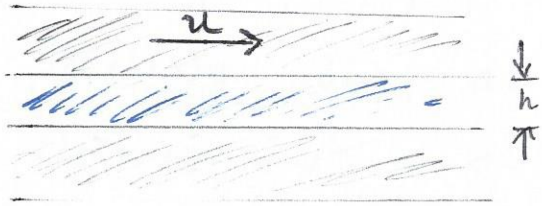
    This causes a normal stress, $\tau_p$:

    $$\boxed{\tau_p = -p\underline{n}} \quad \circledast$$

    If we have a pressure gradient, then this acts as a body force and may drive the flow.

  - Now consider two infinite plates with a fluid in between them. We keep the bottom plate and move the top one with speed $u$.

    Then, the tangential stress is the force per unit area required to move the top plate with velocity $u$.

    For a newtonian fluid: $\boxed{\tau_s \propto \frac{u}{h}}$, the velocity gradient.

{Definition} The dynamic viscosity, $\mu$, is the constant of proportionality.

i.e $$\boxed{\tau_s = \mu \frac{U}{h}}.$$

For a general flow, $u_T(\underline{x})$ is the velocity at $\underline{x}$, then the velocity gradient is $\frac{\partial u_T(\underline{x})}{\partial \underline{n}} \left(= \underline{n} \cdot \underline{\nabla} u_T(\underline{x})\right)$.

$$\Longrightarrow \boxed{\tau_s = \mu \frac{\partial u_T(\underline{x})}{\partial \underline{n}}} \quad (*)$$

• Steady parallel viscous flow:

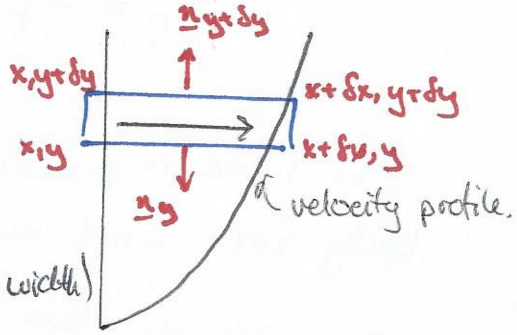
{Definition}. A steady flow is one that does not change with time.

For parallel flow, we assume $\underline{u}$ is of the form:

$$\underline{u} = (u(y), 0, 0).$$

To see that $u(y)$ does not depend on $x$, we appeal to [incompressibility]. This approximation is appropriate for speeds << speed of sound.

Now, consider test volume of fluid:
The box does not accelerate, so the forces should vanish:




i) {In the x-direction} (per unit transverse width)

$$p(x)\delta y - p(x+\delta x)\delta y + \tau_s(y+\delta y)\delta x + \tau_s(y)\delta x = 0$$

$$\Longrightarrow p(x)\delta y - p(x+\delta x)\delta y + \mu \frac{\partial u}{\partial y}(y+\delta y)\delta x + \mu \frac{\partial u}{\partial (-y)}(y)\delta x = 0$$

$$\Longrightarrow \frac{1}{\delta x}\left(p(x) - p(x+\delta x)\right) + \mu \cdot \frac{1}{\delta y}\left(\frac{\partial u}{\partial y}(y+\delta y) - \frac{\partial u}{\partial y}(y)\right) = 0$$

$$\Longrightarrow \boxed{-\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} = 0}.$$

ii) {In the y-direction} we get:

$$\boxed{-\frac{\partial p}{\partial y} = 0}.$$

If we allow an unsteady flow $\underline{u} = (u(y,t), 0, 0)$ and body force per unit volume $\underline{f} = (f_x, f_y, 0)$, density $\rho$, then:

$$\boxed{\begin{aligned} \rho \frac{\partial u}{\partial t} &= -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} + f_x \\ 0 &= -\frac{\partial p}{\partial y} + f_y \end{aligned}}$$

We now consider boundary conditions:

(1) <u>No-slip condition</u>: If boundary is stable:

$$\boxed{u_T = 0.}$$

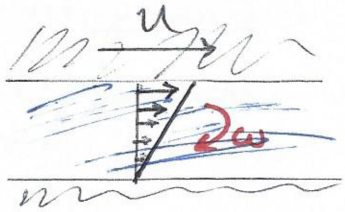
common for fluid-solid boundary.

(2) <u>Stress condition</u>: If a tangential stress $\tau$ is applied to fluid at boundary:

$$\boxed{-\mu \frac{\partial u_T}{\partial n} = \tau.}$$

- <u>Examples</u>:

(1) <u>Couette Flow</u>: driven by motion of boundary:




{<u>Assume</u>} • no pressure gradient
• steady flow:

$$\Longrightarrow \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{with} \quad \{u=0 \text{ on } y=0,\ u=U \text{ on } y=h\}$$

$$\Longrightarrow \boxed{u(y) = \frac{Uy}{h}}$$

(2) <u>Poiseuille Flow</u>: driven by a pressure gradient. We also include gravity:


$$\Longrightarrow \begin{cases} -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} = 0 & (\mathrm{I}) \\ -\frac{\partial p}{\partial y} - g\rho = 0 & (\mathrm{II}) \end{cases} \quad \text{with } \{u=0 \text{ on } y=0 \text{ and } y=h\}.$$

$(\mathrm{II}) \Rightarrow \quad p = -g\rho y + f(x)$ for some $f$.

$(\mathrm{I}) \Rightarrow \quad \mu \frac{\partial^2 u}{\partial y^2} = f'(x)$. Now $\mu \frac{\partial^2 u}{\partial y^2} = \lambda(y)$ and $f'(x) = K(x)$

$$\Longrightarrow \mu \frac{\partial^2 u}{\partial y^2} = f'(x) = G = \frac{P_1 - P_0}{L}$$

$$\Rightarrow \boxed{u = \frac{G}{2\mu} y(h-y)}.$$

Now, if we have a moving boundary and a pressure gradient, we may simply superpose the solutions.

- Derived properties of a flow:

{Definition}. The volume flux is the volume of fluid traversing a cross-section per unit time:

$$\boxed{q = \int_0^h u(y)\,dy} \quad \text{(per unit transverse width).}$$

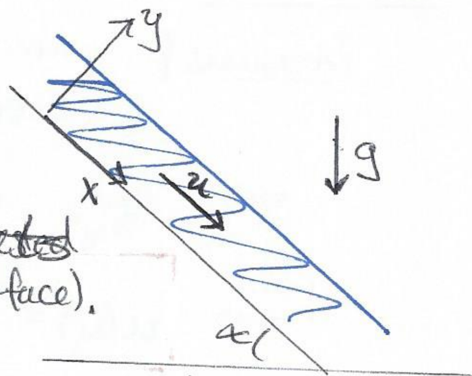
{Definition}. The vorticity of the flow is:

$$\boxed{\underline{\omega} = \underline{\nabla} \times \underline{u}}.$$

- More examples:

① Gravity-driven flow down a slope:

{Assume}:
- Flow is steady
- Atmospheric pressure does not vary over the vertical extend of the flow.
- $\mu_{air} << \mu_{liq}$, so air ~~exerted~~ no tangential stress (free surface).




$\Rightarrow$ (I) $\frac{\partial p}{\partial y} = -g\rho\cos\alpha$ (with $p = p_0$ on $y = h$)

$\Rightarrow \quad p = p_0 - g\rho(\cos\alpha)(y-h)$.

and (II) $\mu \frac{\partial^2 u}{\partial y^2} = -g\rho\sin\alpha$. (with $u = 0$ when $y = 0$).
and $\frac{\partial u}{\partial y} = 0$, $y = h$.

$$\Rightarrow \boxed{u = \frac{g\rho\sin\alpha}{2\mu} y(2h-y)},$$

## ⑤ Fluid initially at rest on flat surface.

At time $t=0$, boundary $y=0$ begins to move with speed $U$.


x-momentum equation gives:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \qquad \begin{cases} \bullet \text{ no forces} \\ \bullet \text{ no pressure gradient,} \end{cases}$$

where $\boxed{\nu = \frac{\mu}{\rho}}$ is the kinematic viscosity.

Boundary conditions are: $\begin{cases} \bullet\ u=0,\ t=0 \\ \bullet\ u \to 0 \text{ as } y \to \infty,\ \forall t \\ \bullet\ u = U \text{ on } y=0 \text{ for } t>0. \end{cases}$

We approach this as a dimensional analysis problem: first:

Let: $T$ be our time scale

$\delta$ be our intrinsic length scale

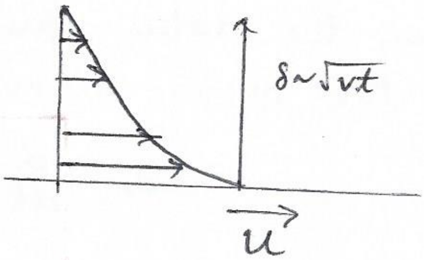
Then: $\frac{U}{T} \sim \nu \frac{U}{\delta^2} \Rightarrow \delta \sim \sqrt{\nu T}$

Now, let $u(y,t) = U f(\eta)$ where $\eta = \frac{y}{\delta} = \frac{y}{\sqrt{\nu t}}$

Then: $-\frac{1}{2}\eta f'(\eta) = f''(\eta)$ such that $f=1,\ \eta=0$.

$$\Rightarrow f(\eta) = \operatorname{erfc}\left(\frac{\eta}{2}\right)$$

$$\Rightarrow \boxed{u(y,t) = U \operatorname{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right)}$$




and the tangential stress is:

$$\tau_s = \mu \frac{\partial u}{\partial y} = \mu \frac{U}{\sqrt{\nu t}} \left(\frac{2}{\sqrt{\pi}}\right) e^{-y^2}\Big|_{y=0} = \frac{\mu U}{\sqrt{\pi \nu t}}$$

This may explain the speed of ocean currents due to wind for example as: $\left(\frac{\mu}{\sqrt{\nu}}\right)_{air} << \left(\frac{\mu}{\sqrt{\nu}}\right)_{water}$, which maintains surface tensions between the two fluids $\Rightarrow$ currents move slower than the air.

## 2 Kinematics

① <u>Eulerian Picture</u>: Equivalent to fixing measurement apparatus at a particular point and seeing how a quantity changes in time.

② <u>Lagrangian Picture</u>: We move along with the flow and measure our quantity along the trajectory/path. Consider a path $\underline{x}(t)$ and a time dependent field $f(\underline{x},t)$. Now, along $\underline{x}(t)$:

$$\frac{df}{dt}(\underline{x}(t),t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial t}$$

$$\Longrightarrow \frac{df}{dt}(\underline{x}(t),t) = \underline{\nabla} f \cdot \dot{\underline{x}} + \frac{\partial f}{\partial t}.$$

If $\underline{x}(t)$ is the path followed by a fluid particle then $\dot{\underline{x}}(t) = \underline{u}$. then we define the <u>material derivative</u>

$$\boxed{\frac{df}{dt} = \underbrace{\frac{Df}{Dt}}_{\text{Lagrangian derivative}} = \underbrace{\underline{u}\cdot\underline{\nabla} f}_{\text{advective derivative}} + \frac{\partial f}{\partial t}}$$

Eulerian time derivative.

• <u>Conservation of mass.</u>

Fix an arbitrary volume $D$ in space with boundary $\partial D$, outward normal $\underline{n}$. Then imagine there is a flow through the region: Now, the total fluid flow through the boundary is equivalent to the change in mass inside $D$.

$$\Longrightarrow \boxed{\frac{d}{dt}\int_D \rho \, dV = -\int_{\partial D} \rho \underline{u}\cdot\underline{n} \, dS}$$

By the divergence theorem:

$$\int_D \left(\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot(\rho\underline{u})\right) dV = 0$$

$$\Longrightarrow \frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot(\rho\underline{u}) = 0 \quad \text{everywhere.}$$ continuity eqn.

$$\Longrightarrow \frac{\partial \rho}{\partial t} + \underline{u}\cdot\underline{\nabla}\rho + \rho\underline{\nabla}\cdot\underline{u} = 0$$

But $\frac{\partial e}{\partial t} + \underline{u}\cdot\nabla e = \frac{De}{Dt}$

$$\Rightarrow \boxed{\frac{De}{Dt} + e\nabla\cdot\underline{u} = 0}$$

{Definition}. A fluid is incompressible if $\frac{De}{Dt} = 0$. and hence

$$\boxed{\underline{\nabla}\cdot\underline{u} = 0} \quad \text{(continuity equation)}.$$

• Kinematic Boundary Conditions.

Suppose the boundary has velocity $\underline{U}$, we define a local reference frame such that the boundary is stationary. Then in this frame, the fluid has relative velocity:

$$\underline{u}' = \underline{u} - \underline{U}$$

Fluids may not cross the boundary, so letting the normal at the boundary be $\underline{n}$, then:

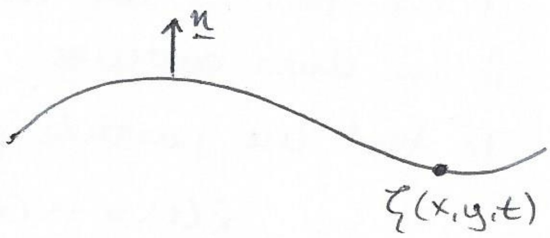
$$\underline{u}'\cdot\underline{n} = 0$$

$$\Rightarrow \boxed{\underline{u}\cdot\underline{n} = \underline{U}\cdot\underline{n}} \quad (\text{so if } \underline{U} = \underline{0} \Rightarrow \underline{u}\cdot\underline{n} = 0).$$

• Free surfaces:

We define the surface by:

$$z = \xi(x, y, t)$$




Or as a contour of:

$$F(x, y, z, t) = z - \xi(x, y, t) \quad (\text{ie } F = 0).$$

Then $\underline{n} \propto \underline{\nabla}F = (-\xi_x, -\xi_y, 1)$

Further, $\underline{U} = (0, 0, \xi_t)$

Let $\underline{u} = (u, v, w)$

Then we apply $\underline{u} \cdot \underline{n} = \underline{U} \cdot \underline{n}$

$$\Longrightarrow \quad -u\zeta_x - v\zeta_y + w = \zeta_t$$

$$\Longrightarrow \quad w = u\zeta_x + v\zeta_y + \zeta_t = \underline{u} \cdot \underline{\nabla}\zeta + \frac{\partial \zeta}{\partial t}$$

$$\Longrightarrow \quad \boxed{\frac{D\zeta}{Dt} = w}$$

- Streamfunction for incompressible flow: Suppose $\underline{\nabla} \cdot \underline{u} = 0$

This implies $\exists\, \underline{A}$ such that:

$$\underline{u} = \underline{\nabla} \times \underline{A}$$

If the flow is 2-dimensional, then $\underline{u} = (u(x,y,t), v(x,y,t), 0)$

Then $\underline{A}$ is of the form:

streamfunction.

$$\underline{A} = (0, 0, \psi(x,y,t))$$

$$\Longrightarrow \quad \underline{u} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0\right)$$

- Properties of the streamfunction:

i) Contours $\psi = c$ have normal $\underline{n} = \underline{\nabla}\psi = (\psi_x, \psi_y, 0)$.

Then $\underline{u} \cdot \underline{n} = \frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \psi}{\partial x} = 0$

$\Longrightarrow$ flow is tangent to contours of $\psi$. ← streamlines.

If the flow is time dependent, particles do not necessarily follow these contours

To find the particle paths we simply set:

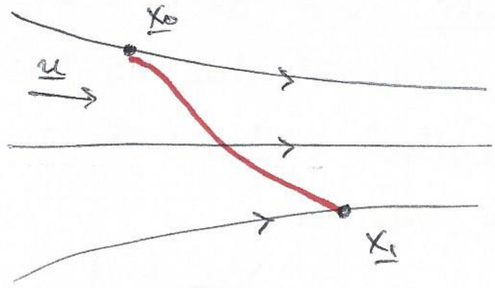
$$\dot{x}(t) = u(t), \quad \dot{y}(t) = v(t)$$

and eliminate $t$.

ii) The flow is faster when streamlines are closer together (so plot evenly spaced contours).

Consider volume flux crossing any curve from $\underline{x}_0$ to $\underline{x}_1$.

$$q = \int_{\underline{x}_0}^{\underline{x}_1} \underline{u} \cdot \underline{n} \, dl$$




Now, $\underline{n} \, dl = (-dy, dx)$.

$$\Rightarrow q = \int_{\underline{x}_0}^{\underline{x}_1} -\frac{\partial \psi}{\partial y} dy - \frac{\partial \psi}{\partial x} dx$$

$\Rightarrow$ $q = \psi(\underline{x}_0) - \psi(\underline{x}_1)$. so $q$ only depends on difference in value of $\psi$. So, for closer streamlines, we need higher speed to maintain same flux.

Also, $\psi$ is constant on a stationary rigid boundary since flow is tangential at boundary ($\underline{u} \cdot \underline{n} = 0$). i.e boundary is a streamline.

$\boxed{\text{N.B}}$ In plane polars: $\nabla \times (0, 0, \psi) = \left( r \frac{\partial \psi}{\partial \theta}, -\frac{\partial \psi}{\partial r}, 0 \right) = \underline{u}$
where $\underline{u} = (\dot{r}(r), \dot{\Theta}(\theta), \dot{z}(z))$

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## 3 Dynamics.

- Navier-Stokes Equation:

$$\boxed{\underbrace{\rho \frac{D\underline{u}}{Dt}}_{\text{mass. acceleration}} = \overbrace{-\nabla p}^{\text{pressure gradient}} + \underbrace{\mu \nabla^2 \underline{u}}_{\text{viscosity}} + \overbrace{\underline{f}}^{\text{body forces.}}}$$

i) Acceleration of a fluid particle is the material derivative of the velocity.

ii) $\nabla^2 \underline{u} = \underline{\nabla}(\underline{\nabla} \cdot \underline{u}) - \underline{\nabla} \times (\underline{\nabla} \times \underline{u})$ which, in an incompressible fluid reduces to $\nabla^2 \underline{u} = -\underline{\nabla} \times \underline{\omega}$.

In Cartesian co-ordinates:

$$\nabla^2 \underline{u} = (\nabla^2 u_x, \nabla^2 u_y, \nabla^2 u_z).$$

• Pressure: We first consider the pressure of a fluid at rest ($\underline{u} = \underline{0}$) due to the weight of fluid above it. This is called hydrostatic pressure. Then, the Navier-Stokes equation becomes:

$$\nabla p_H = \underline{f} = \rho \underline{g}$$

$$\Rightarrow \quad p_H = \rho \underline{g} \cdot \underline{x} + p_0 \quad (p_0 \text{ arbitrary constant}).$$

Suppose we have a body $D$ with boundary $\partial D$ and outward normal $\underline{n}$. Then the force is:

$$\underline{F} = -\int_{\partial D} p_H \, \underline{n} \cdot dS$$

$$= -\int_D \nabla p_H \, dV$$

$$= -\int_D \underline{g} \rho \, dV$$

$$= -\underline{g} \int_D \rho \, dV \boxed{= -M\underline{g}}$$

mass of fluid displaced

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So, we may split the total pressure in $p_H$, the hydrostatic pressure, and $p'$, the pressure caused/resulting from the motion. Then, $p = p_H + p'$ and the Navier-Stokes equation becomes:

$$\rho \frac{D\underline{u}}{Dt} = -\nabla p' + \mu \nabla^2 \underline{u}.$$

From which we will normally just drop the prime.

• Reynolds Number:

Suppose the flow has a characteristic speed $\mathcal{U}$, extrinsic length scale, $L$. We then define the time scale to be $T = \frac{L}{\mathcal{U}}$, and suppose that the pressure differences have magnitude $P$.

Then, dividing by $\rho$ in the Navier-Stokes equation:

$$\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\underline{u}$$

kinematic viscosity.

Then estimating the size of the terms:

$$\frac{U}{(L/U)} + \frac{U\cdot U}{L} = \frac{1}{\rho}\frac{P}{L} + \nu\frac{U}{L^2}$$

Dividing by $\frac{U^2}{L}$:

$$1 + 1 = \frac{P}{\rho U^2} + \frac{\nu}{UL}$$

inertia term. viscous term

{Definition} The Reynolds number is:

$$\boxed{Re = \frac{UL}{\nu}}$$

So, if Re is very large, the viscous term is small and we may neglect it. If Re is small, (ie small, slow moving object) then the viscous term becomes significant.

{Definition} Flows with the same geometry and Reynolds numbers are said to be dynamically similar.

Now, when [Re << 1,] the inertia terms are negligible and

$$P \sim \frac{\rho\nu U}{L} = \frac{\mu U}{L}$$

So, the pressure balances the shear stress.

$$\Rightarrow \quad 0 = -\nabla p + \mu\nabla^2\underline{u}. \quad \{(\text{with } \nabla\cdot\underline{u} = 0)\}$$

These are Stokes' Equations, and we find $\boxed{\underline{u} \propto \nabla p.}$

When $\boxed{Re >> 1}$ the viscous terms are negligible, then:

$P \sim \rho U^2$ and scales like the momentum flux.

Then we have the Euler equations:

$$\left\{ \begin{array}{l} \rho \frac{Du}{Dt} = -\nabla p \\ \nabla \cdot \underline{u} = 0 \end{array} \right\}$$

When we make this approximation, the order of the DE drops by 1 and we are forced to not satisfy all boundary conditions. We choose to give up the no-slip condition. This is okay so long as we are outside of the intrinsic length scale, $\delta$. In this region, the inertia and viscous terms are comparable, so:

$$U^2 \sim \frac{\nu U}{\delta^2}$$

$$\Longrightarrow \delta = \frac{\nu}{U} \quad \text{or} \quad \left\{ \frac{\delta}{L} = \frac{\nu}{UL} = \frac{1}{Re} \right\}$$

So, for large Re, $\delta$ is small.

- Stagnation Point Flow: We look for solution in $y \geq 0$ subject to $\underline{u} \to (Ex, -Ey, 0)$ as $y \to \infty$ where $E \geq 0$ is a constant and $\underline{u} = \underline{0}$ on $y = 0$.

The problem does not have an extrinsic length scale, so we look for a similarity solution in terms of:

$$\boxed{\eta = \frac{y}{\delta}} \text{ where } \quad \delta = \sqrt{\frac{\nu}{E}} \quad (\text{n.b } [\delta] = L)$$

so $\eta$ is dimensionless

We apply the incompressibility condition $\nabla \cdot \underline{u} = 0$, we see $\underline{u}$ must be of the form:

$$\underline{u} = (u, v, 0) = (Exg'(\eta), -E\delta g(\eta), 0).$$

We see that there is a streamfunction:

$$\psi = \sqrt{\nu E}\, x\, g(\eta)$$

$$\Rightarrow \begin{cases} \dfrac{\partial \psi}{\partial y} = \sqrt{\nu E}\, x\, g'(\eta) \cdot \dfrac{1}{\delta} = E x\, g'(\eta); \\ -\dfrac{\partial \psi}{\partial x} = -\sqrt{\nu E}\, g(\eta) = -E \delta\, g(\eta) \end{cases}.$$

Then, the Navier-Stokes equations have:

$$\begin{cases} u \dfrac{\partial u}{\partial x} + v \dfrac{\partial u}{\partial y} = -\dfrac{1}{\rho} \dfrac{\partial p}{\partial x} + \nu \left( \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} \right) \\ u \dfrac{\partial v}{\partial y} + v \dfrac{\partial v}{\partial y} = -\dfrac{1}{\rho} \dfrac{\partial p}{\partial y} + \nu \left( \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} \right) \end{cases}.$$

Substituting in $u$ and $v$ gives:

$$\begin{cases} (\mathrm{I}) \quad E^2 x \left( g'(\eta)^2 - g(\eta) g''(\eta) \right) = -\dfrac{1}{\rho} \dfrac{\partial p}{\partial x} - E^2 x\, g'''(\eta) \\ (\mathrm{II}) \quad E\sqrt{\nu E}\, g(\eta) g'(\eta) = -\dfrac{1}{\rho} \dfrac{\partial p}{\partial y} - E\sqrt{\nu E}\, g''(\eta) \end{cases}$$

we take $\frac{\partial}{\partial y}$ of (I) and $\frac{\partial}{\partial x}$ of (II) to elimate $p$ terms:

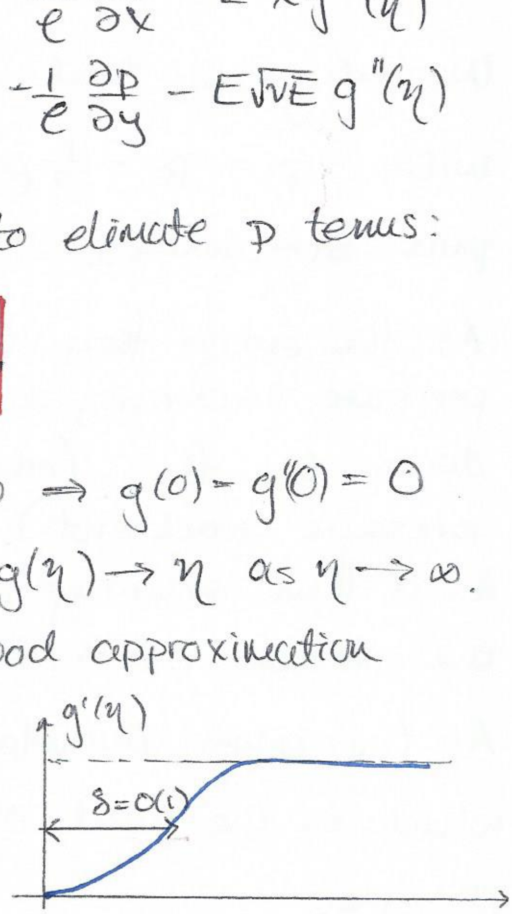
$$\Longrightarrow \boxed{g'(\eta) g'' - g(\eta) g'''(\eta) = g^{(4)}(\eta)}.$$

The no slip condition gives: $\underline{u} = \underline{0}$ on $y=0 \Rightarrow g(0) = g'(0) = 0$

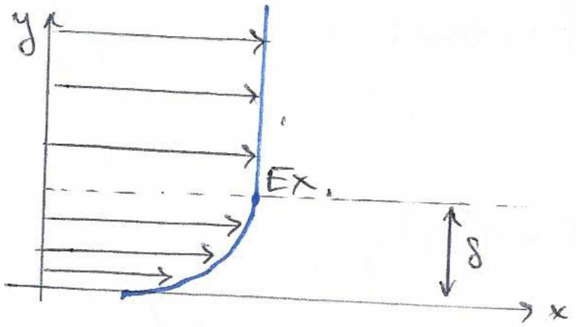
and as $y \to \infty$, we have $g'(\eta) \to 1$, $g(\eta) \to \eta$ as $\eta \to \infty$.

Now, $\underline{u} = (Ex, -Ey, 0)$ is reached to a good approximation when:

$$\left\{ \underline{\eta \geq 1 \Rightarrow y \gtrsim \delta} \Rightarrow \right.$$




Then we see the following horizontal velocity profile:




At a scale $\delta$,

$$Re_\delta = \frac{U\delta}{\nu} \sim O(1).$$

acts as the boundary layer and for large scale:

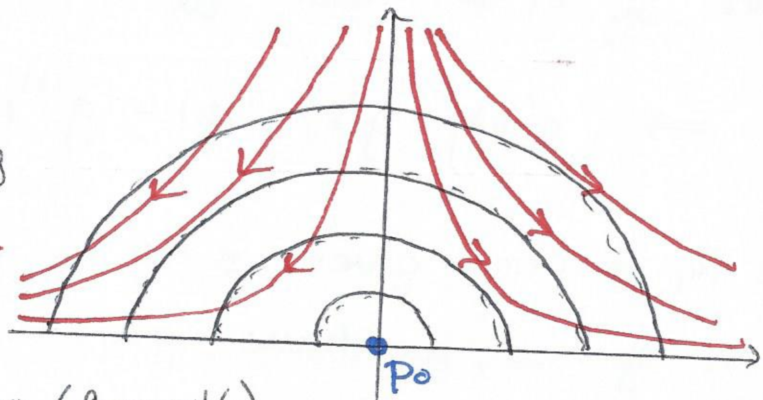
$$Re_L = \frac{UL}{\nu} >> 1.$$

So, when interested in scales larger than $\delta$ we ignore $y<\delta$ and imagine a rigid boundary at $y=\delta$ where no slip condition does not apply. When $Re_L >> 1$, we solve the Euler equations:

$$\left.\begin{aligned} \rho\frac{D\underline{u}}{Dt} &= -\nabla p + \underline{f} \\ \nabla\cdot\underline{u} &= 0 \end{aligned}\right\} \quad \underline{n}\cdot\underline{u} = 0 \text{ on stationary rigid boundary.}$$

We may see that $\underline{u} = (Ex, -Ey, 0)$ satisfies these equations with $\{p = p_0 - \frac{1}{2}\rho E^2(x^2+y^2)\}$. We plot contours of $p$ plus streamlines:

As flow enters from top, pressure increases, slowing down the flow, (adverse pressure gradient).
As it flows sideways, the pressure pushes the flow (favourable).




At the origin, the velocity is zero $\Rightarrow$ stagnation point, which is the point of highest pressure. This is a general principle.

- Inviscid approximation $(\nu=0)$ (incompressible also)

We will consider momentum to derive the Navier-Stokes equations in this simplification.

Consider an arbitrary volume $D$ with boundary $\partial D$, outward normal $\underline{n}$. Now, the total momentum of the fluid in $D$ may change due to 4 things:

i) Momentum flux across $\partial D$
ii) Surface pressure forces.
iii) Body forces.
(iv) viscous surface forces) ignore.

So:

$$\frac{d}{dt}\int_D \rho \underline{u}\, dV = -\int_{\partial D} \rho \underline{u}(\underline{u}\cdot\underline{n})\, dS - \int_{\partial D} p\underline{n}\, dS + \int_D \underline{f}\, dV$$

In suffix notation, this is:

$$\frac{d}{dt}\int_D \rho u_i\, dV = -\int_{\partial D} \rho u_i u_j n_j\, dS - \int_{\partial D} p n_i\, dS + \int_D f_i\, dV$$

$$\Rightarrow \int_D \left\{\rho \frac{\partial u_i}{\partial t} + \rho \frac{\partial}{\partial x_j}(u_i u_j)\right\} dV = \int_D \left\{-\frac{\partial p}{\partial x_i} + f_i\right\} dV$$

$$\Rightarrow \left\{\rho \frac{\partial u_i}{\partial t} + \rho \frac{\partial u_i}{\partial x_j} u_j + \rho u_i \frac{\partial u_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + f_i\right\}.$$

Now, $\frac{\partial u_j}{\partial x_j} = \nabla\cdot\underline{u} = 0$ by incompressibility, and the first two terms $\left\{\rho\frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho\left\{\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right\}_i = \left\{\frac{D\underline{u}}{Dt}\right\}_i\right\}$

$$\Rightarrow \quad \underline{\rho \frac{D\underline{u}}{Dt} = -\nabla p + \underline{f}} \quad \text{as before.}$$

Now, we consider a conservative force $\underline{f} = -\nabla\chi$, e.g. $\underline{f} = \rho\underline{g} = \nabla(\rho\underline{g}\cdot\underline{x})$ so $\chi = -\rho\underline{g}\cdot\underline{x} = g\rho z$ if $\underline{g} = (0,0,-g)$.

We first consider steady flow, i.e. $\frac{\partial \underline{u}}{\partial t}$ vanishes, then:

$$0 = -\int_{\partial D} \rho\underline{u}(\underline{u}\cdot\underline{n})\, dS - \int_{\partial D} p\underline{n}\, dS - \int_D \nabla\chi\, dV$$

$$\Rightarrow \int_{\partial D} \{ \rho \underline{u}(\underline{u}\cdot\underline{n}) + p\underline{n} + \chi \underline{n} \} dS = 0$$

momentum integral for steady flow.

We use the vector identity:

$$\left\{ \underline{u} \times (\nabla \times \underline{u}) = \nabla\left(\tfrac{1}{2}|\underline{u}|^2\right) - \underline{u}\cdot\nabla\underline{u} \right\}$$

Then the Euler momentum equation becomes.

$$\rho\frac{\partial \underline{u}}{\partial t} + \rho\nabla\left(\tfrac{1}{2}|\underline{u}|^2\right) - \rho\underline{u}\times(\nabla\times\underline{u}) = -\nabla p - \nabla \chi$$

Dotting with $\underline{u}$; we get Bernoulli's Equation:

$$\boxed{\frac{1}{2}\rho \frac{\partial |\underline{u}|^2}{\partial t} = -\underline{u}\cdot\nabla\left\{\tfrac{1}{2}\rho|\underline{u}|^2 + p + \chi\right\}}.$$

In the case of steady flow;

(*) $\boxed{H = \tfrac{1}{2}\rho|\underline{u}|^2 + p + \chi}$ is constant along streamlines.

If flow is not steady, we still have:

$$\frac{d}{dt}\int_D \tfrac{1}{2}\rho|\underline{u}|^2 dV = -\int_{\partial D} H\underline{u}\cdot\underline{n}\, dS.$$

H is transportable energy of the flow.

- It is often helpful to consider conservation of mass, where we have $q = UA = ua$ (for example). Also, we may use the momentum integral directly in conjuction with Euler equation to find forces etc.

- <u>Linear Flows.</u> Take a point $\underline{x}_0$, we may break the flow up into three parts:
$$\left\{ \begin{array}{l} \text{① Uniform flow} \\ \text{② Pure strain} \\ \text{③ Pure rotation.} \end{array} \right\}.$$

Now: $\quad u(\underline{x}) = u(\underline{x_0}) + (\underline{x} - \underline{x_0}) \cdot \nabla \underline{u}(\underline{x_0}) + \cdots$

$$\Rightarrow \quad u(\underline{x}) \simeq u(\underline{x_0}) + \underline{r} \cdot \underline{\nabla}\, \underline{u}\big|_{\underline{x_0}} \quad , \quad \underline{r} = \underline{x} - \underline{x_0}.$$

Now, $\underline{\nabla}\,\underline{u}$ is a rank 2 tensor, so we split it up into its symmetric and antisymmetric parts:

$$\underline{\nabla}\,\underline{u} = \frac{\partial u_i}{\partial x_j} = E_{ij} + \Omega_{ij}$$

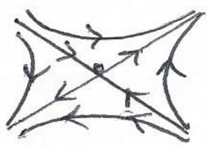
where: $\begin{cases} E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \\ \Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \end{cases}$

Writing $\underline{\omega} = \underline{\nabla} \times \underline{u}$, then $\underline{\omega} \times \underline{r} = (\underline{\nabla} \times \underline{u}) \times \underline{r}$

$$= r_j \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$= 2\Omega_{ij} r_j.$$

$$\Rightarrow \quad \boxed{\underline{u} = \underline{u_0} + E\underline{r} + \tfrac{1}{2}\underline{\omega} \times \underline{r}}$$



- Vorticity Equation:

Consider: $\quad \rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \underline{\nabla}\,\underline{u}\right) = -\nabla p - \nabla \chi + \mu \nabla^2 \underline{u}, \quad (*)$

Now, apply $\underline{u} \cdot \underline{\nabla}\,\underline{u} = \frac{1}{2}\nabla |\underline{u}|^2 - \underline{u} \times \underline{\omega}$., and take the curl of $(*)$

$$\Rightarrow \quad \frac{\partial \underline{\omega}}{\partial t} - \underline{\nabla} \times (\underline{u} \times \underline{\omega}) = \nu \nabla^2 \underline{\omega} \quad \text{since } \nabla \times \underline{\nabla} f = \underline{0}.$$

$\circ$ since $\underline{\nabla} \cdot (\underline{\nabla} \times \underline{f}) = 0$

Now, $\underline{\nabla} \times (\underline{u} \times \underline{\omega}) = \cancel{(\underline{\nabla} \cdot \underline{\omega})}\,\underline{u} + (\underline{\omega} \cdot \underline{\nabla})\underline{u} - \cancel{(\underline{\nabla} \cdot \underline{u})}\,\underline{\omega} - (\underline{u} \cdot \underline{\nabla})\underline{\omega}$

$$\Rightarrow \quad \boxed{\frac{\partial \underline{\omega}}{\partial t} + (\underline{u} \cdot \underline{\nabla})\underline{\omega} - (\underline{\omega} \cdot \underline{\nabla})\underline{u} = \nu \nabla^2 \underline{\omega}}.$$

Or, with the material derivative:

dissipation by vorticity.

$$\frac{D\underline{\omega}}{Dt} = \underline{\omega}\cdot\nabla\underline{u} + \nu\nabla^2\underline{\omega} \qquad \text{(vorticity equation)}.$$

amplification by stretching/twisting

In the case of an inviscid fluid, $\nu = 0$:

$$\frac{D\underline{\omega}}{Dt} = \underline{\omega}\cdot\nabla\underline{u}, \quad \text{taking the dot product with } \underline{\omega}:$$

$$\Rightarrow \frac{D}{Dt}\left\{\tfrac{1}{2}|\underline{\omega}|^2\right\} = (\underline{\omega}\cdot\nabla\underline{u})\cdot\underline{\omega}$$

$$= \underline{\omega}\,(E+\Omega)\cdot\underline{\omega}$$

$$= \omega_i\,(E_{ij}+\Omega_{ij})\,\omega_j$$

But $\Omega_{ij}$ is antisymmetric $\Rightarrow \omega_i\Omega_{ij}\omega_j = 0$. and there are principle axes such that $E$ is diagonal

$$\Rightarrow \frac{D}{Dt}\left\{\tfrac{1}{2}|\underline{\omega}|^2\right\} = E_1\omega_1^2 + E_2\omega_2^2 + E_3\omega_3^2$$

wlog assume $E_1 > 0$ ($E_1+E_2+E_3 = 0$ by incompressibility).
So suppose $E_2, E_3 < 0$, $\Rightarrow$ flow stretched in $\underline{e}_1$ direction and compressed radially. e.g for $\underline{\omega} = (\omega_1, 0, 0)$

$$\Rightarrow \frac{D}{Dt}\left(\tfrac{1}{2}\omega_1^2\right) = E_1\omega_1^2$$

So vortex grows exponentially. This is just conservation of angular momentum..

---

## 4 Inviscid, Irrotational Flow.

We now have an incompressible ($\nabla\cdot\underline{u} = 0$), inviscid ($\nu = 0$) and irrotational ($\nabla\times\underline{u} = \underline{0}$) flow. If $\nabla\times\underline{u} = \underline{\omega} = \underline{0}$ at $t=0$, then by the vorticity equation:

$$\left\{\frac{D\underline{\omega}}{Dt} = \underline{\omega}\cdot\nabla\underline{u} = \underline{0}\right\} \text{ so it remains zero } \forall t.$$

{Definition} Since $\nabla \times \underline{u} = \underline{0}$, we may write $\boxed{\underline{u} = \nabla \phi}$, for some $\phi$, the velocity potential.

- If we have $\nabla \cdot \underline{u} = 0 \Rightarrow \boxed{\nabla^2 \phi = 0}$, so the potential satisfies Laplace's equation. Flows whose potential satisfies Laplace's equation are called potential flows.

$\boxed{\text{N.B}}$ By linearity, solutions may be superposed (e.g uniform flow + point source.

- Three-dimensional Potential Flows:

In spherical co-ordinates $(r, \theta, \varphi)$:

$$\left\{ \nabla^2 \phi = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \phi}{\partial \varphi^2} \right\}$$

and
$$\left\{ \nabla \phi = \left( \frac{\partial \phi}{\partial r}, \frac{1}{r} \frac{\partial \phi}{\partial r}, \frac{1}{r\sin\theta} \frac{\partial \phi}{\partial \varphi} \right) \right\}$$

(I) $\{\phi = \phi(r)\}$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) = 0$$

$$\Rightarrow \frac{\partial \phi}{\partial r} = \frac{A}{r^2} \Rightarrow \phi(r) = -\frac{A}{r} + B$$

wlog, we set $B = 0$. So $\phi(r) = -\frac{A}{r}$.

Consider the volume flux $q$ across a sphere of radius $a$:

$$q = \int_S \underline{u} \cdot \underline{n}\, dS = \int_S u_r\, dS = \int_S \frac{\partial \phi}{\partial r}\, dS$$

$$= \int_S \frac{A}{a^2}\, dS = 4\pi A \Rightarrow \left\{ A = \frac{q}{4\pi} \right\}$$

$$\Rightarrow \phi(r) = -\frac{q}{4\pi r}$$

strength of source.

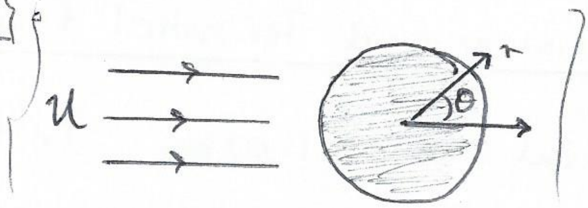
$\boxed{\text{N.B}}$ $\nabla^2\phi = q\,\delta(x) \implies \phi = -\frac{q}{4\pi r}$ is the Green's Function for $\nabla^2$.

(II) $\{\phi = \phi(r,\theta)\}$. From Methods, solution is:

$$\phi(r,\theta) = \sum_{n=0}^{\infty} \{A_n r^n + B_n r^{-n-1}\} P_n(\cos\theta).$$

and $\underline{u} = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, 0\right)$

eg {uniform flow past sphere:}




Upstream the flow is

$$\underline{u} = U\hat{\underline{x}}$$

$$\implies \phi = Ux = Ur\cos\theta$$

So, we have:

$$\begin{cases} \nabla^2\phi = 0 & r > a \\ \phi \to Ur\cos\theta & r \to \infty \\ \frac{\partial\phi}{\partial r} = 0 & r = a \end{cases}$$

($\underline{u}\cdot\underline{n} = 0$ on boundary) (no flow into sphere).

Now, since $P_1(\cos\theta) = \cos\theta$ and $P_n$ are orthogonal, our asymptotic condition requires:

$$\phi(r,\theta) = \left\{Ar + \frac{B}{r^2}\right\}\cos\theta$$

Then $\phi \to Ur\cos\theta \implies A = U$ and $\frac{\partial\phi}{\partial r} = 0 \implies B = \frac{Ua^3}{2}$

$$\implies \boxed{\phi(r,\theta) = U\left\{r + \frac{a^3}{2r^2}\right\}\cos\theta}$$

Then:

$$\begin{cases} u_r = \frac{\partial\phi}{\partial r} = U\left(1 - \frac{a^3}{r^3}\right)\cos\theta \\ u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\left(1 + \frac{a^3}{2r^3}\right)\sin\theta \end{cases}$$

We see at $r = a$, $\theta = 0, \pi$, $[u_r = u_\theta = 0]$.

and at N + S poles, $u_r = 0$, $u_\theta = \pm\frac{3U}{2}$, so velocity faster at top than uniform flow (windy at top of hill).

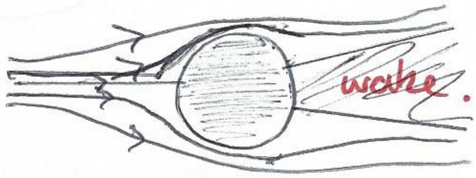
We now obtain the pressure on the surface. Apply Bernoulli's principle to the streamline $(a, \theta)$:

$$p_\infty + \frac{1}{2}\rho U^2 = p + \frac{1}{2}\rho U^2 \cdot \frac{9}{4}\sin^2\theta$$

$$\Rightarrow \quad p = p_\infty + \frac{1}{2}\rho U^2 \left\{1 - \frac{9}{4}\sin^2\theta\right\}.$$

So, $\left[\begin{array}{ll} p(\theta = 0, \pi) = p_\infty + \frac{1}{2}\rho U^2 & \text{high} \\ p(\theta = \frac{\pi}{2}, \frac{3\pi}{2}) = p_\infty - \frac{5}{8}\rho U^2. & \text{low} \end{array}\right]$

$\boxed{\text{N.B}}$ $p = f(\sin^2\theta)$ so front + back of sphere look the same, so fluid exerts no net force on the sphere. In practice, flow separates due to adverse pressure gradient at rear (caused by viscosity) to produce a wake.




e.g <u>Rising Bubble</u>: Get solution in bubble's stationary reference frame then translate back by $U$. We may show that KE of fluid is:

$$\int_{r>a} \frac{1}{2}\rho|u|^2 \, dV = \frac{\pi}{3}a^3\rho U^2 = \frac{1}{2}M_A U^2$$

where $M_A = \frac{1}{2}\left(\frac{4}{3}\pi a^3 \rho\right) = \frac{1}{2}M_D$, $M_D$ is mass displaced by the bubble, $M_A$ is called the added mass.

Now, since no energy is dissipated by viscosity:

$$\frac{d}{dt}\left\{\frac{1}{2}M_A U^2 - M_D g h\right\} = 0$$

$$\Rightarrow \quad M_A U \dot{U} - M_D g \dot{h} = 0$$

$$\Rightarrow \quad M_A U \dot{U} - M_D g U = 0$$

$$\Rightarrow \quad \boxed{\dot{U} = 2g.}$$

So bubble accelerates upwards at $2g$. (in an inviscid fluid).

- Potential flow in 2D.

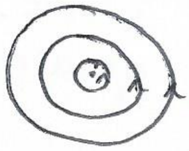
in polar co-ordinates: $\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$

and $\underline{u} = \underline{\nabla}\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial r}\right)$.

The general solution to Laplace's equation is:

$$\phi = A\log r + B\theta + \sum_{n=1}^{\infty}\{A_n r^n + B_n r^{-n}\}\begin{cases}\cos n\theta \\ \sin n\theta\end{cases}$$

e.g Point Source. $\nabla^2\phi = q\,\delta(x) \Rightarrow \boxed{\phi = \frac{q}{2\pi}\log r.}$

e.g Point Vortex.  no radial velocity $\Rightarrow \frac{\partial\phi}{\partial r} = 0$

So. $\phi = \phi(\theta) \Rightarrow \phi = B\theta.$

Now, consider the circulation around a loop:

$$K = \oint_{r=a} \underline{u}\cdot d\underline{l} = \int_0^{2\pi} \frac{B}{a}\cdot a\,d\theta = 2\pi B$$

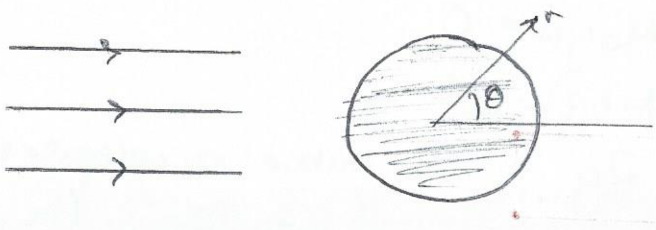
$$\Rightarrow \left\{\phi = \frac{K}{2\pi}\theta,\quad u_\theta = \frac{K}{2\pi r}.\right\}$$

Further, $\left[\underline{\nabla}\times\underline{u} = \underline{0} \text{ for } r \neq 0.\right]$

and $\oint_{\mathcal{C}} \underline{u}\cdot d\underline{l} = \begin{cases} K & \text{if } \mathcal{C} \text{ contains origin} \\ 0 & \text{otherwise.}\end{cases}$

$\Leftrightarrow$ infinite vorticity at origin, zero elsewhere.

e.g Uniform flow past cylinder.




$\nabla^2\phi = 0$ (on $r > a$)

$\phi \to Ur\cos\theta$ ($r \to \infty$)

$\frac{\partial\phi}{\partial r} = 0$ ($r = a$)

This has the solution:

$$\boxed{\phi = U\left\{r + \frac{a^2}{r}\right\}\cos\theta + \underbrace{\frac{K}{2\pi}\theta}}$$

allows for net circulation around cylinder, to account for vorticity in viscous boundary layer.

Then: $\left\{\begin{array}{l} u_r = U\left\{1 - \frac{a^2}{r^2}\right\}\cos\theta \\ u_\theta = -U\left\{1 + \frac{a^2}{r^2}\right\}\sin\theta + \frac{K}{2\pi r}\end{array}\right\} \Rightarrow \psi = Ur\sin\theta\left\{1 - \frac{a^2}{r^2}\right\} - \frac{K}{2\pi}\log r.$

streamfunction.

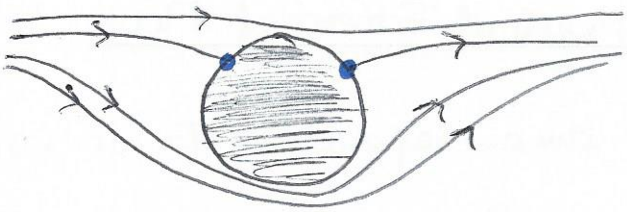
(i) $\{K = 0\}$ gives a flow similar to that around a sphere with no net force on cylinder and a symmetrical flow, with two stagnation points.

(ii) $\{K \neq 0\}$ First look at stagnation points:

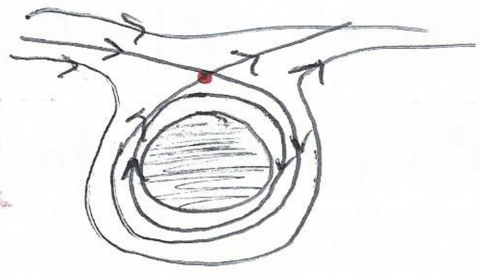
$$u_r = 0 \iff r = a \text{ or } \cos\theta = 0.$$

for $u_\theta|_{r=a} = 0 \Rightarrow \underline{K = 4\pi a U\sin\theta}.$

So: provided $|K| \leq 4\pi a U$, we have stagnation points on the boundary:



For $|K| > 4\pi a U$, there are no stagnation points on boundary. However, we still have a stagnation point when $\cos\theta = 0$ $\Rightarrow \theta = \pm\frac{\pi}{2}$. Then $u_\theta = 0 \Rightarrow$ solution for $r^*$, $\theta = \frac{\pi}{2}$.



Now, we consider the pressure, using Bernoulli's equation for steady potential flow:

$$p_\infty + \tfrac{1}{2}\rho U^2 = p + \tfrac{1}{2}\rho\left\{\frac{K}{2\pi a} - 2U\sin\theta\right\}^2$$

pressure on surface.

$$\Rightarrow p = p_\infty + \tfrac{1}{2}\rho U^2 - \frac{\rho K^2}{8\pi^2 a^2} + \frac{\rho K U\sin\theta}{\pi a} - 2\rho U^2\sin^2\theta$$

Since pressure is symmetrical before and after $\Rightarrow$ no force in x-direction. $\{p = f(\sin\theta)\}$.

However: $F_y = -\int_0^{2\pi} p\sin\theta\,\{a\,d\theta\} = -\int_0^{2\pi} \frac{\rho K U}{\pi a}\sin^2\theta\, a\,d\theta$

$$\Rightarrow \boxed{F_y = -\rho U K}.$$

- In general, the magnus force (lift force) that results due to interaction between flow $\underline{U}$ and vortex $\underline{K}$ is:

$$\underline{F} = \rho\underline{U} \times \underline{K}.$$

- Time-dependent Potential Flows (see Ex sheet 3).

Consider time-dependent Euler equation:

$$\rho\left\{\frac{\partial \underline{u}}{\partial t} + \nabla\{\tfrac{1}{2}|\underline{u}|^2\} - \underline{u}\times\underline{\omega}\right\} = -\nabla p - \nabla\chi$$

Assume $\underline{u} = \nabla\phi \Rightarrow \underline{\omega} = \underline{0}$, so:

$$\underline{\nabla}\left\{\rho\frac{\partial\phi}{\partial t} + \tfrac{1}{2}\rho|\underline{u}|^2 + p + \chi\right\} = 0$$

$$\Rightarrow \boxed{\rho\frac{\partial\phi}{\partial t} + \tfrac{1}{2}\rho|\underline{u}|^2 + p + \chi = f(t)}$$ independent of space.

e.g <u>Oscillations of a bubble</u>: Suppose we have bubble of radius $a(t)$, then spherically symmetric oscillations induce a flow that satisfies:

$$\begin{cases} \nabla^2\phi = 0 & (r>a) \\ \phi \to 0 & (r\to\infty) \\ \boxed{\frac{\partial\phi}{\partial r} = \dot{a}} & (r=a) \end{cases}$$

Now, for $\phi = \phi(r)$, $\nabla^2\phi = \frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)\right) = 0$

$$\longrightarrow \phi = \frac{A(t)}{r} \Rightarrow u_r = -\frac{A(t)}{r^2} \longrightarrow 0 \text{ as } r\to\infty$$

and $-\frac{A(t)}{a^2} = \dot{a} \rightarrow A(t) = -a^2\dot{a}$

$$\text{So: } \phi(r,t) = -\frac{a^2\dot{a}}{r}$$

Then: $\left.\frac{\partial\phi}{\partial t}\right|_{r=a} = \left\{-\frac{2a\dot{a}^2}{r} - \frac{a^2\ddot{a}}{r}\right\}\Big|_{r=a} = -\{a\ddot{a} + 2\dot{a}^2\}$

Applying Euler's equation on surface and at infinity:

$$-\rho\{a\ddot{a} + 2\dot{a}^2\} + \frac{1}{2}\rho\dot{a}^2 + p(a,t) = p_\infty$$

$$\Rightarrow \rho\left\{a\ddot{a} + \frac{3}{2}\dot{a}^2\right\} = p(a,t) - p_\infty.$$

Writing $a(t) = a_0 + \eta(t)$ where $|\eta(t)| << a_0$

$$\Rightarrow a\ddot{a} + \frac{3}{2}\dot{a}^2 = (a_0+\eta)\ddot{\eta} + \frac{3}{2}\dot{\eta}^2 = a_0\ddot{\eta} + O(\eta^2)$$

$$\Rightarrow \underline{\rho a_0\ddot{\eta} = p(a,t) - p_\infty.}$$

- Thermodynamics tells us that $\delta p$ is caused by a change in volume $\delta V$, and that if the expansion is rapid, the change is adiabatic and satisfies $PV^\gamma = $ const.

$$\Rightarrow \log p + \gamma\log V = \text{const.}$$

$$\Rightarrow \frac{\delta p}{p_\infty} = -\gamma\frac{\delta V}{V_0} = -3\gamma\frac{\eta}{a_0} \qquad \left(\frac{\delta V}{V} = 3\frac{\delta r}{r}\right)$$

$$\Rightarrow \rho a_0 \ddot{\eta} = -\frac{3\gamma p_\infty}{a_0}\eta$$

$$\Rightarrow \boxed{\ddot{\eta} = -\frac{3\gamma p_\infty}{\rho a_0^2}\eta}$$

which is SHM with $\omega = \left(\frac{3\gamma p_\infty}{\rho a_0^2}\right)^{1/2}$.

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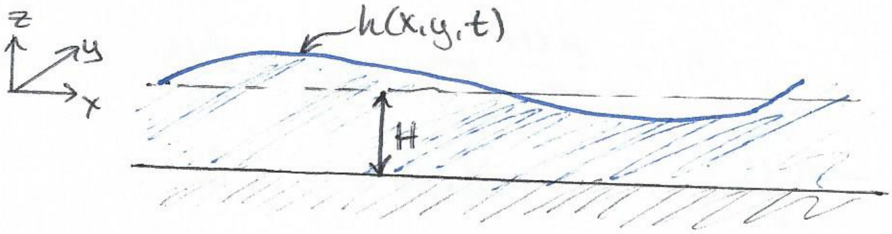
5 Water waves.




• Dimensional Analysis: Consider waves with $k = \frac{2\pi}{\lambda}$, (wavenumber) on a layer of water of depth H. Suppose fluid is inviscid. Then dimensionally, the wave speed c, depends on k, g and H:

$$c = \sqrt{gH}\, f(kH) \quad \rightarrow \text{dimensionless.}$$

(i) Deep water: $\{H >> \lambda\} \Rightarrow kH >> 1$, so we expect c not to depend on H as it is too big, thus the only way this may happen is if $f \propto \frac{1}{\sqrt{kH}}$

$$\Rightarrow \boxed{c = \alpha\sqrt{\frac{g}{k}}}$$

(ii) Shallow Water $\{H << \lambda\}$. Here we expect not to have a dependence on $\lambda$ (ie k), so f is constant:

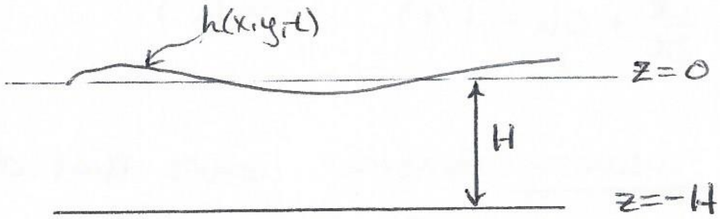
$$\Rightarrow \boxed{c = \beta\sqrt{gH}}$$

(N.B. These explain why waves come in parallel to shore (refraction due to speed differences))

Also explains why waves break as top of wave travels faster than trough.

• Equation and Boundary Conditions:

We assume the fluid is initially at rest (hence $\underline{\omega} = \nabla \times \underline{u}$ is initially zero and hence always zero) and is inviscid.




Along with incompressibility we have Laplace's equation:

$$\nabla^2 \phi = 0$$

with the following boundary conditions:

$$\begin{cases} ① \; u_z|_{z=-H} = \dfrac{\partial \phi}{\partial z}\Big|_{z=-H} = 0 & \text{(no flow through bottom).} \\ ② \; u_z|_{z=h(x,y,t)} = \dfrac{Dh}{Dt} = \dfrac{\partial h}{\partial t} + u_x \dfrac{\partial h}{\partial x} + u_y \dfrac{\partial h}{\partial y} & \text{(free surface).} \\ ③ \; p|_{z=h} = p_0 = \text{const.} & \end{cases}$$

We make a few simplifying assumptions:

$$\begin{cases} ① \; h \ll H & \text{(amplitude is small)} \\ ② \; \dfrac{\partial h}{\partial x}, \dfrac{\partial h}{\partial y} \ll 1 & \text{(waves are flat)} \end{cases}$$

So, in Bernoulli's equation at $z=h$:

$$\rho \frac{\partial \phi}{\partial t} + \frac{1}{2} \rho |\nabla \phi|^2 + g\rho h + p_0 = f(t)$$

We ignore $u_x \dfrac{\partial h}{\partial x}$, $u_y \dfrac{\partial h}{\partial y}$ and the whole of $|\nabla \phi|^2$ since it is small. Now, we write:

$$\frac{\partial \phi}{\partial z}\Big|_{z=h} = \frac{\partial \phi}{\partial z}\Big|_{z=0} + h \frac{\partial^2 \phi}{\partial z^2}\Big|_{z=0} + \cdots$$

and approximate $\dfrac{\partial \phi}{\partial z}\Big|_{z=h} = \dfrac{\partial \phi}{\partial z}\Big|_{z=0}$.

Then, we have:

$$\left.\begin{array}{ll} \nabla^2\phi = 0 & \{-H \le z \le 0\} \\ \dfrac{\partial\phi}{\partial z} = 0 & (z=-H) \\ \dfrac{\partial\phi}{\partial z} = \dfrac{\partial h}{\partial t} & (z=0) \\ \dfrac{\partial\phi}{\partial t} + gh = f(t) & (z=h) \end{array}\right\}$$

• Two-dimensional waves: Consider waves that do not depend on $y$: take:

$$h = h_0 \exp\{i(kx-\omega t)\}$$

Then, using the boundary condition at $z=0$: there is a solution of the form:

$$\phi = \hat{\phi}(z) e^{i(kx-\omega t)}$$

$$\Rightarrow \quad -k^2\hat{\phi} + \hat{\phi}'' = 0$$

$$\Rightarrow \quad \hat{\phi} = \phi_0 \cosh k(z+H). \qquad \left(\Rightarrow \frac{\partial\phi}{\partial z} = 0,\ z=-H\right)$$

Now, $\dfrac{\partial\phi}{\partial z} = \dfrac{\partial h}{\partial t}$ at $z=0$

$$\rightarrow \quad k\phi_0 \sinh kH = -i\omega h_0 \qquad (1)$$

Also, in Bernoulli's equation, we require:

$$-i\omega\hat{\phi}(z) e^{i(kx-\omega t)} + gh_0 e^{i(kx-\omega t)} = f(t)$$

not to depend on $x$

$$\Rightarrow \quad -i\omega\phi_0 \cosh kH + gh_0 = 0 \qquad (2)$$

These solve to give:

$$\boxed{\omega^2 = gk\tanh kH.}$$ dispersion relation.

Then, $c = \dfrac{\omega}{k} = \sqrt{\dfrac{g}{k}\tanh kH}$

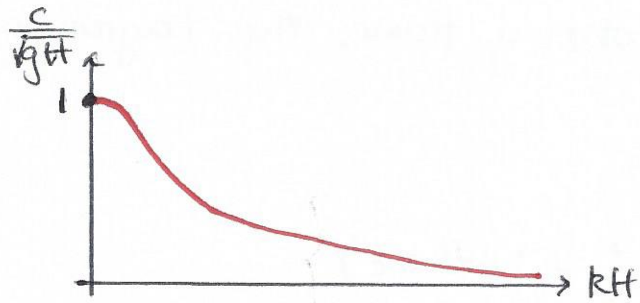
So, when:

(i) $\{kH \gg 1\}$ ie deep water, $\tanh kH \to 1$

$$\Rightarrow \quad c = \sqrt{\frac{g}{k}} \quad (\alpha = 1).$$

(ii) $\{kH \ll 1\}$ ie shallow water, $\tanh kH \to kH$

$$\Rightarrow \quad c = \sqrt{gH} \quad (\beta = 1)$$




We notice that this means that waves get disintegrated due to different wavelengths travelling faster than others.

- Group velocity: Suppose two waves with similar wave numbers $k_1, k_2$ travel close together, then:

$$\sin k_1 x + \sin k_2 x = 2 \sin\left(\frac{k_1+k_2}{2}x\right)\cos\left(\frac{k_1-k_2}{2}x\right)$$

Then, $\frac{k_1+k_2}{2} \simeq k_1$ and $\frac{k_1-k_2}{2}$ is small, so waves look like:

The group velocity is given by $\quad c_g = \frac{\partial \omega}{\partial k}$

So if e.g $\omega = \sqrt{gk} \Rightarrow c_g = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}c$.

- Rayleigh-Taylor Instability: We look at the case where we have water over air, so we replace $g \leftrightarrow -g$

$$\Rightarrow \quad \omega^2 = -gk \text{ in deep water.}$$

$$\Rightarrow \quad \omega = \pm i\sqrt{gk}$$

$$\Rightarrow \quad h \propto Ae^{\sqrt{gh}t} + Be^{-\sqrt{gh}t} \qquad \{h = h_0 \exp((ikx) - \sqrt{gh}t)\}$$

So we have an exponetially growing solution and the water falls + is unstable. Applicable to water through oil, mushroom clouds, supernovae etc.

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## 6 Fluid Dynamics on a rotating frame

- Equations of Motion: In a rotating frame, the Lagrangian acceleration is given by:

$$\left\{ \frac{Du}{Dt} + 2\underline{\Omega} \times \underline{u} + \underline{\Omega} \times (\underline{\Omega} \times \underline{x}) \right\}$$

$$\Rightarrow \rho \left\{ \frac{\partial u}{\partial t} + \underline{u} \cdot \nabla \underline{u} + 2\underline{\Omega} \times \underline{u} \right\} = -\nabla p - \rho \underline{\Omega} \times (\underline{\Omega} \times \underline{x}) + g\rho$$

We may show that the {centrifugal force $\underline{\Omega} \times \underline{\Omega} \times \underline{x})$ is negligible} in comparison to gravity. $\left( \frac{|\underline{\Omega} \times (\underline{\Omega} \times \underline{x})|}{|g|} \simeq 4 \times 10^{-3} \right)$

Further, we consider motions for which:

$$|\underline{u} \cdot \nabla \underline{u}| << |2\underline{\Omega} \times \underline{u}|$$

$$\Rightarrow \quad \frac{U^2}{L} << \Omega U$$

$$\Rightarrow \quad \boxed{R_o = \frac{U}{\Omega L} << 1} \quad \text{(Rossby number)}$$

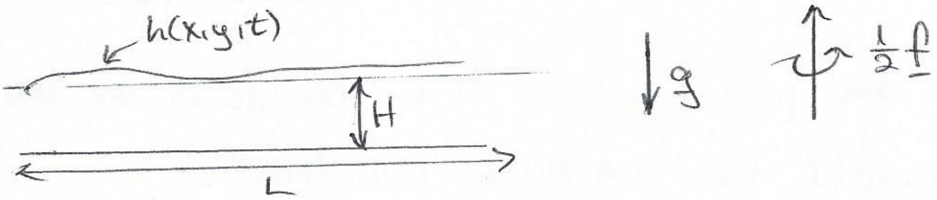
(We may show in our atmosphere both these approximations are valid).

$$\Rightarrow \quad \boxed{\frac{\partial u}{\partial t} + 2\underline{\Omega} \times \underline{u} = -\frac{1}{\rho} \nabla p + g}$$

We conventionally write $2\Omega = f$, the Coriolis parameter/planetary vorticity.

• Shallow-water Equations: (surprisingly a good model for the atmosphere + oceans).

with $p = p_0$ on $z = h$




Writing $\underline{u} = (u, v, w)$ and using $\underline{\nabla} \cdot \underline{u} = 0$

$$\Rightarrow \frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}$$

We consider motions where $L >> H$

$$\Rightarrow \frac{W}{H} = \frac{U}{L}, + \frac{V}{L} \Rightarrow W << U, V.$$

So we approximate $\underline{u} = (u, v, 0)$, and $\underline{f} = (0, 0, f)$.

$$\Rightarrow \boxed{\begin{aligned} \frac{\partial u}{\partial t} - fv &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ \frac{\partial v}{\partial t} + fu &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \\ 0 &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \end{aligned}}$$

Last equation gives: $\{p = p_0 - g\rho(h - z)\}$, then putting this into the first two gives:

$$\boxed{\begin{aligned} \frac{\partial u}{\partial t} - fv &= -g\frac{\partial h}{\partial x} \\ \frac{\partial v}{\partial t} + fu &= -g\frac{\partial h}{\partial y} \end{aligned}}$$

• RHS's are independent of $z$ and usually initial conditions have $u, v$ independent of $z$ $\Rightarrow$ velocities always independent of $z$.

• <u>Geostrophic Balance</u>. In a steady flow, the time derivative vanishes

$$\Rightarrow \quad \begin{cases} u = \frac{\partial}{\partial y}\left\{-\frac{gh}{f}\right\}\left(= \frac{\partial}{\partial y}\left(-\frac{P}{\rho f}\right)\right) \\ v = -\frac{\partial}{\partial x}\left\{-\frac{gh}{f}\right\}\left(= -\frac{\partial}{\partial x}\left(-\frac{P}{\rho f}\right)\right) \end{cases}$$

Streamlines are places where $h$ is constant i.e surfaces of constant height i.e pressure constant.

{<u>Definition</u>} (<u>Shallow-water streamfunction</u>).

$$\psi = -\frac{gh}{f}.$$

Near a low pressure zone, there is a pressure gradient pushing flow towards low pressure area. In the rotating frame, there is a Coriolis force that balances this force (geostrophic balance).

• Now consider horizontal surface $D$:

$$\frac{d}{dt}\int_D \rho h \, dV = -\int_{\partial D} h\rho \underline{u}_H \cdot \underline{n} \, dS \quad \text{by continuity:}$$

↑ horizontal velocity

$$\Rightarrow \int_D \frac{\partial(\rho h)}{\partial t} dV = -\int_D \nabla_H \cdot (\rho h \underline{u}_H) \, dV$$

where $\nabla_H = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0\right)$.

$$\Rightarrow \frac{\partial h}{\partial t} + \nabla_H \cdot (h \underline{u}_H) = 0.$$

Now, suppose we have small oscillations i.e $h = h_0 + \eta(x,y,t)$ $\eta \ll h_0$. and write $\underline{u} = (u(x,y), v(x,y), 0)$

$$\Rightarrow \quad \frac{\partial \underline{u}}{\partial t} + \underline{f} \times \underline{u} = -g \nabla \eta \; (\mathrm{I})$$ is eqn of motion

and the continuity equation becomes:

$$\frac{d\eta}{dt} + h_0 \underline{\nabla} \cdot \underline{u} = 0 \quad (\mathrm{II})$$ (ignoring higher order terms)

Taking the curl of (I)

$$\Rightarrow \quad \frac{\partial \underline{\zeta}}{\partial t} + \underline{f}\, \underline{\nabla} \cdot \underline{u} = 0 \quad , \quad \underline{\zeta} = \underline{\nabla} \times \underline{u}$$

$$\Rightarrow \quad \boxed{\frac{\partial}{\partial t}\left(\underline{\zeta} - \frac{\eta}{h_0}\underline{f}\right) = \frac{d\underline{Q}}{dt} = 0}$$

where $\underline{Q} = \underline{\zeta} - \frac{\eta}{h_0}\underline{f}$ is the potential vorticity.

Now, take divergence of (I):

$$\frac{\partial}{\partial t}(\underline{\nabla} \cdot \underline{u}) - \underline{f} \cdot \underline{\nabla} \times \underline{u} = -g \nabla^2 \eta$$

$$\Rightarrow \quad -\frac{1}{h_0}\frac{\partial^2 \eta}{\partial t^2} - \underline{f} \cdot \underline{\zeta} = -g \nabla^2 \eta$$

and we have $\underline{\zeta} = \underline{Q}_0 + \frac{\eta}{h_0}\underline{f}$ $\quad (\underline{Q}_0 = \underline{Q}(x, y, 0))$

$$\Rightarrow \quad \frac{\partial^2 \eta}{\partial t^2} - g h_0 \nabla^2 \eta + \underline{f} \cdot \underline{f}\, \eta = -h_0 \underline{f} \cdot \underline{Q}_0$$

eg


We have: $$\underline{Q}_0 = \begin{cases} -\frac{\eta_0}{h_0} \underline{f} & x > 0 \\ \frac{\eta_0}{h_0} \underline{f} & x < 0 \end{cases}$$ since there is no movement (ie $\underline{\zeta}(t=0) = \underline{0}$)

We look for a steady state such that $\left[\frac{\partial \eta}{\partial t} = 0\right]$
and one that is independent of $y$ i.e $\left[\frac{\partial \eta}{\partial y} = 0\right]$

$\Rightarrow \quad \eta = \eta(x)$

$$\Rightarrow \quad \frac{\partial^2 \eta}{\partial x^2} - \frac{f^2}{g h_0}\eta = \frac{f}{g} Q_0 = \mp \frac{f^2}{g h_0}\eta_0$$

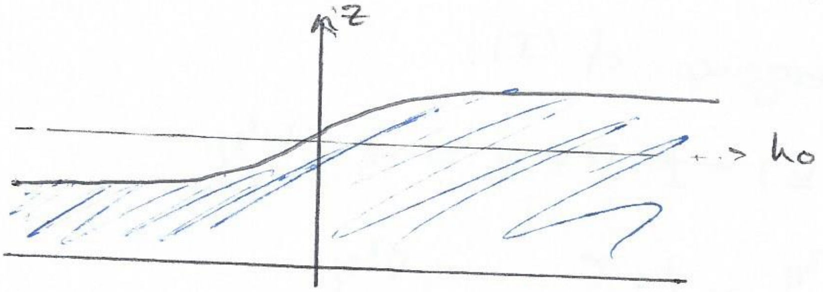
Now define $\boxed{R = \frac{\sqrt{g h_0}}{f}}$ (Rossby radius of deformation)

fundamental length scale for rotating problems.

$$\Rightarrow \quad \frac{d^2 \eta}{dx^2} - \frac{1}{R^2}\eta = \mp \frac{1}{R^2}\eta_0$$

We require: $\eta \to \pm \eta_0$ as $x \to \pm\infty$ and $\eta, \frac{d\eta}{dx}$ continuous at $x = 0$:

$$\Rightarrow \quad \eta = \begin{cases} \eta_0 \left(1 - \exp(-x/R)\right) & x > 0 \\ -\eta_0 \left(1 - \exp(+x/R)\right) & x < 0 \end{cases}$$




where the horizontal pressure gradient is in geostrophic balance with the Coriolis force due to flow in y-direction of:

$$v = \frac{g}{f}\frac{\partial \eta}{\partial x} = \eta_0 \sqrt{\frac{g}{h_0}}\, e^{-|x|/R}.$$