The Method of Steepest Descent

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In order to obtain asymptotics for integrals of the type

$$I(k) \;=\; \int_C f(z)\, e^{kh(z)}\, dz\,, \qquad (k \to +\infty)$$

where C is a contour in the complex plane and f, h are holomorphic functions, the idea is this.

> *The asymptotic behaviour of $I(k)$ for large positive k can be effectively analysed by using the Cauchy theorem to deform the contour C into another specially chosen contour Γ on which $\psi = \Im h = \psi_0$ is constant. This leads to a Laplace type integral*
>
> $$I(k) \;=\; \int_C f(z)\, e^{kh(z)}\, dz \;=\; e^{ik\psi_0} \int_\Gamma f(z)\, e^{k\phi(z)}\, dz\,, \qquad h(z) = \phi(z) + i\psi(z)\,,$$
>
> *the asymptotics of which is dominated by the saddle points $\{z_\mu : h'(z_\mu) = 0\}$, which correspond to local maxima of $e^{k\phi(z)}$ and possibly also end-points and singularities (if present).*

Remember the following facts about holomorphic functions $h = \phi + i\psi$:

- The level curves $\phi = \phi_0 = constant$ and $\psi = \psi_0 = constant$ are orthogonal families of curves (by the Cauchy-Riemann equations). The vector $\nabla\phi$ is directed along the curves $\psi = constant$, which are therefore either curves of steepest descent or of steepest ascent for ϕ (i.e., curves along which the rate of change of ϕ is either as large and negative as possible or as large and positive as possible.)

- By the maximum modulus principle, there are no strict local maxima for $|h|$ or $\phi = \Re h$ as functions on the plane - stationary points are saddle points. The reference to local maxima above means on the contour Γ, as a function of some parameter s used to parametrize Γ.

- The Cauchy-Riemann equations imply that if $\frac{d\phi}{ds} = 0$ at a point $z_0 = Z(s_0)$ on a curve $\Gamma = \{z = Z(s) : \psi(Z(s)) = \psi(z_0)\}$ parametrized by s, then $h'(z_0)$=0, i.e., z_0 is a saddle point.

Exercise 1. Make sure you understand how to prove the last item.

In general, the implementation of the steepest descent method involves the following steps:

- **Step One:** Find the saddle points $\{z_\mu\}$ at which $h'(z_\mu) = 0$ and work out the Taylor expansion[1] of h near each z_μ and hence find the curves of steepest descent emanating from z_μ.

[1] You will usually find it is quicker to work in complex coordinates for this.

- **Step Two:** Deform C to a contour consisting of paths of steepest descent (or contours), paying attention to
  - singularities (if any);
  - the behaviour at infinity - deforming contours at infinity is often justified by the Jordan Lemma in examples;
  - end-points.

  Then work out the contributions to the asymptotics as you have learned to do for Laplace type integrals, including contributions from end-points (see second example below), but also singularities (there may be residue contributions) as well as the usually dominant contributions from saddle points.

- **Step Three:** Combine all contributions to get an overall asymptotic expansion.

As a final point, you will find out from doing examples that in order to compute the leading behavior it is often sufficient to deform to asymptotically steepest descent contours, i.e., contours which coincide with the steepest descent contours only asymptotically in the neighborhood of the saddle points. However, to be sure you know what you are doing you should try to sketch and understand the entire curve of steepest descent which forms your contour.

Applying the Method of Steepest Descent as above can be a complicated procedure, but will hopefully become clear after working examples. It would be worthwhile and probably more effective to read the three steps above carefully again after working through the examples below. Also at the end there is a more general description of the paths of steepest descent/ascent associated to a saddle point of arbitrary order.

Example: Debye's Integral. Consider

$$I(k) = \int_C e^{k(\text{sech } v \sinh z - z)} dz, \quad v \neq 0, \quad k > 0,$$

where C is the so-called Sommerfeld contour depicted in Figure 3

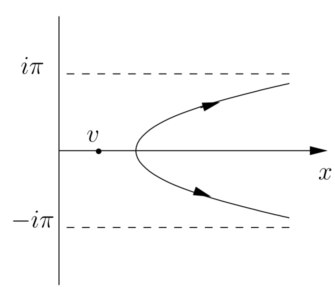

Figure 1: Debye's Integral

(On account of Cauchy's theorem it is not necessary to specify the contour exactly as the integral will not change if the contour is deformed without changing the asymptotes or passing through any singularities. In this case the integrand is an entire analytic function, so the integral will be the same for any contour of this shape with the same asymptotes - check that you understand that the integral is absolutely convergent with any contour asymptotic to the lines $z = \pm i\pi$ as it approaches infinity as shown.)

It turns out that $I(k)$ is proportional to $J_k(k\operatorname{sech} v)$, where J_k is the Bessel function of order k. Debye in 1954 rediscovered the method of steepest descent by studying the above integral (this method was introduced earlier by Riemann in connection with the hypergeometric function).

In the above case

$$h(z) = \operatorname{sech} v \sinh z - z, \quad h'(z) = \operatorname{sech} v \cosh z - 1, \quad z_0 = v.$$

Thus, we expect that the main contribution will come from the neighborhood of $z_0 = v$. Letting $z = v + \rho e^{i\theta}$, we find

$$h(z) = \operatorname{sech} v[\sinh v \cosh(\rho e^{i\theta}) + \cosh v \sinh(\rho e^{i\theta})] - \rho e^{i\theta} - v.$$

But

$$\cosh \varepsilon = \frac{e^{\varepsilon} + e^{-\varepsilon}}{2} = 1 + \frac{\varepsilon^2}{2} + O(\varepsilon^4), \quad \sinh \varepsilon = \varepsilon + O(\varepsilon^3), \quad \varepsilon \to 0.$$

Hence,

$$\begin{aligned} h(z) &= -v + \tanh v \left[1 + \frac{\rho^2}{2} e^{2i\theta} + O(\rho^4)\right] + \rho e^{i\theta} + O(\rho^3) - \rho e^{i\theta} \\ &= -v + \tanh v + \tanh v \frac{\rho^2}{2}(\cos 2\theta + i \sin 2\theta) + O(\rho^3), \quad \rho \to 0. \end{aligned}$$

The conditions $\sin 2\theta = 0$, $\cos 2\theta < 0$, imply $\theta = \pi/2$ and $\theta = 3\pi/2$. For $\theta = \pi/2$ we find the following contribution:

$$ie^{k(\tanh v - v)} \int_0^R e^{-k \tanh v \frac{\rho^2}{2}} d\rho \sim ie^{k(\tanh v - v)} \sqrt{\frac{2}{k \tanh v}} \int_0^\infty e^{-\tau^2} d\tau, \quad k \to \infty,$$

where we have used the substitution $k \tanh v \rho^2/2 = \tau^2$. Adding to the above the contribution from $\theta = 3\pi/2$ we find

$$I(k) \sim i\sqrt{\frac{2\pi}{k \tanh v}} e^{k(\tanh v - v)}, \quad k \to \infty.$$

Remark 2. *The choice $\theta = \pi/2$ or $3\pi/2$ corresponds to the tangent of the path of steepest descent through $z_0 = v$: more precisely this path is given by solving $\Im h(z) = 0$ which gives*

$$\textit{either} \quad \cosh x = \frac{1}{\operatorname{sech} v} \frac{y}{\sin y}, \quad \textit{or} \quad y = 0\,.$$

The first case is a curve looking very much like the original contour, but now chosen explicitly to be a path of steepest descent through v. It is easily seen to be tangent to the vertical at $z_0 = v$, corresponding to $\theta = \pi/2$ or $3\pi/2$ above. The alternative $y = 0$ corresponds to $\theta = 0$ or π, and is the path of steepest ascent emanating from v.

Exercise 3. Check that the curve of steepest descent described in the preceding remark has the asymptotes shown in the diagram.

Example: End-point contributions. We now consider an integral whose asymptotics are dominated by end-point contributions, and we will see how to use paths of steepest descent to determine these contributions. The integral is

$$I(k) = \int_0^1 \ln t e^{ikt} dt$$

Complexification of the above integral suggests consideration of

$$\tilde{I}(k) = \int_C \ln z e^{ikz} dz\,,$$

where the contour C can be chosen as any contour in the complex plane joining 0 to 1 into which the line segment $\{(x, 0) : 0 \le x \le 1\}$ can be deformed without encountering any singularities. (We take the branch cut for $\ln$ to be the negative real axis.) The integral $\tilde{I}(k)$ is one of the type under consideration with $h(z) = iz$, $h'(z) = i \neq 0$, and hence there do not exist any saddle points. Therefore we expect the main contribution will come from the end-points. To work these contributions out we find the curves of steepest descent emanating from the two end-points $z = 0$ and $z = 1$. Using $\rho \in \mathbb{R}$ as a parameter these curves are just vertical straight lines, paramterized respectively as

$$z = i\rho\,, \qquad \text{and} \qquad z = 1 + i\rho\,.$$

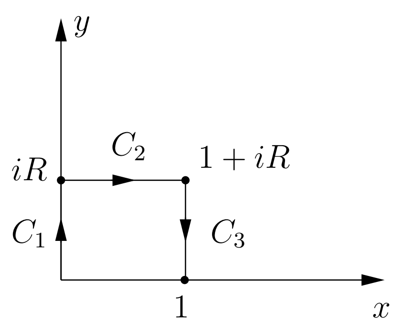

Figure 2: An End-point dominated Steepest Descent Integral

Substituting these paramterizations into the original integral leads to Laplace type integrals. Aiming to apply the Cauchy deformation theorem, we are led to take C to be the union of the contours shown in Figure 2. Hence, for any finite $R > 0$ we have

$$I(k) = i \int_0^R \ln(i\rho) e^{-k\rho} d\rho + \int_0^1 \ln(x + iR) e^{ik(x+iR)} dx - ie^{ik} \int_0^R \ln(1 + i\rho)^{-k\rho} d\rho.$$

Letting $R \to \infty$ we find

$$I(k) = i \int_0^\infty \ln(i\rho) e^{-k\rho} d\rho - ie^{ik} \int_0^\infty \ln(1 + i\rho) e^{-k\rho} d\rho = I_1(k) + I_2(k).$$

In I_1 we use the substitution $k\rho = \tau$, thus

$$I_1(k) = \frac{i}{k} \int_0^\infty \ln\left(\frac{i\tau}{k}\right) e^{-\tau} d\tau = \frac{i}{k}\left[\ln\left(\frac{i}{k}\right) \int_0^\infty e^{-\tau} d\tau + \int_0^\infty \ln \tau e^{-\tau} d\tau\right].$$

Thus,

$$I_1(k) = \frac{i}{k}\left[\frac{i\pi}{2} - \ln k - \gamma\right] = -\frac{i\ln k}{k} - \frac{\left(i\gamma + \frac{\pi}{2}\right)}{k}.$$

The integral I_2 can be computed via Watson's lemma:

$$\ln(1 + i\rho) = -\sum_{m=1}^{\infty} \frac{(-i\rho)^m}{m}.$$

Thus

$$I_2(k) \sim ie^{ik} \sum_{m=1}^{\infty} (-i)^m \frac{(m-1)!}{k^{m+1}}.$$

Hence,

$$I(k) \sim -\frac{i\ln k}{k} - \frac{i\gamma + \pi/2}{k} + ie^{ik} \sum_{m=1}^{\infty} (-i)^m \frac{(m-1)!}{k^{m+1}}, \quad k \to \infty.$$

Remark 4. *Notice that in computing end-point contributions in complex integrals it is often necessary to integrate on the steepest descent paths emanating from the end-points, and these may go far away from the original integral. Another example of this arises in the next example. Also, make sure you understand why the complex contour was chosen as it was rather than its reflection in the real axis.*

An Example solved by stationary phase and steepest descent (ii). The asymptotics for

$$\int_0^1 \exp[ixt^3]\, dt$$

which we derived previously by stationary phase can also be derived by the method of steepest descent. Thus let $t = \sigma + i\tau$ be a complex variable and define $h(t) = it^3$. We see that $t = 0$ is the only saddle point, but we must also consider the end-point $t = 1$. First of all, we find the paths of steepest descent through the points $t = 0$ and $t = 1$. By joining together two of these paths at the point at infinity, we will obtain the asymptotic expansion for $f(x)$ for large positive x.

The steepest descent paths emanating from $t = 0$ are given by $\operatorname{Im} it^3 = 0$, i.e. they are the rays $\operatorname{Arg} t = \pm\pi/6, \pm\pi/2$ and $\operatorname{Arg} t = \pm 5\pi/6$. The cases $\pi/6, 5\pi/6, -\pi/2$ are lines of steepest descent moving away from $t = 0$ (while the others are lines of steepest ascent) - the ascent/descent characteristic alternates on going around the origin.

The steepest descent paths emanating from $t = 1$ are given by $\operatorname{Im} it^3 = 1$, i.e. writing $t = \sigma + i\tau$ we are interested in the cubics $\sigma^3 - 3\sigma\tau^2 = 1$. There is a branch passing through $t = 1$ which is vertical at that point, and reflection invariant in the σ-axis with asymptotes as they approach infinity $\tau = \pm\frac{1}{\sqrt{3}}\sigma$, i.e. the lines of steepest descent from $t = 0$. Call the upper branch of this C_2 .

Employing the Jordan lemma, we may joining the upper branch to the line $C_1 = \operatorname{Arg} t = \pi/6$ at infinity and applying the Cauchy theorem we get

$$f(x) \;=\; \int_{C_1} \exp[ixt^3]\, dt \;-\; \int_{C_2} \exp[ixt^3]\, dt\,.$$

On C_1 we make the change of variable $t = \sqrt[3]{u}\exp[i\pi/6]$ leading to

$$\begin{aligned}\int_{C_1} \exp[ixt^3]\, dt \;&=\; \frac{1}{3}\exp[i\pi/6]\int_0^{\infty} \exp[-xu]\frac{du}{u^{\frac{2}{3}}} \\ &=\; \frac{1}{3}\exp[i\pi/6]\Gamma(\frac{1}{3})x^{-\frac{1}{3}} \;=\; \Gamma(\frac{4}{3})\,\exp[i\pi/6]\, x^{-\frac{1}{3}}\,.\end{aligned}$$

On C_2 we make the change of variable $t = \sqrt[3]{1+iu}$ leading to

$$\begin{aligned}
\int_{C_2} \exp[ixt^3]\,dt \;&=\; \frac{1}{3}\,i \int_0^\infty \exp[ix - xu]\frac{du}{(1+iu)^{\frac{2}{3}}} \\
&=\; \frac{1}{3}\,i\,e^{ix} \sum\, i^j x^{-(j+1)} \sum\nolimits_{j=0}^{\infty} \frac{\Gamma(-\frac{2}{3}+1)}{\Gamma(-\frac{2}{3}+1-j)\Gamma(j+1)} \\
&\qquad \times \int_0^\infty e^{-t}t^j\,dt \\
&=\; \frac{1}{3}\,i\,e^{ix} \sum\, i^j x^{-(j+1)} \sum\nolimits_{j=0}^{\infty} \frac{\Gamma(-\frac{2}{3}+1)}{\Gamma(-\frac{2}{3}+1-j)}
\end{aligned}$$

where we used $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,dt$ and the Binomial expansion

$$(1+t)^\alpha \;=\; \sum\nolimits_{j=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-j)\Gamma(j+1)}\, t^j\,.$$

Recalling the identity $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, we have

$$\Gamma(\frac{1}{3})\Gamma(\frac{2}{3}) = \pi/\sin(\pi/3)\,, \quad \text{and } \;\Gamma(\frac{1}{3}-j)\Gamma(\frac{2}{3}+j) = (-1)^j\pi/\sin(\pi/3)\,.$$

From this we can rewrite the above expression as

$$\begin{aligned}
\int_{C_2} \exp[ixt^3]\,dt \;&=\; \frac{1}{3}\,e^{ix} \sum\nolimits_{j=0}^{\infty} (-1)^j i^{j+1} x^{-(j+1)} \frac{\Gamma(\frac{2}{3}+j)}{\Gamma(\frac{2}{3})} \\
&=\; e^{ix} \sum\nolimits_{j=0}^{\infty} (ix)^{-(j+1)} \frac{\Gamma(\frac{2}{3}+j)}{\Gamma(-\frac{1}{3})}\,.
\end{aligned}$$

Combining the two expressions, we end up with

$$f(x) \;=\; \Gamma(\frac{4}{3})\,\exp[i\pi/6]\,x^{-\frac{1}{3}} \;-\; e^{ix} \sum\nolimits_{j=0}^{\infty} (ix)^{-(j+1)} \frac{\Gamma(\frac{2}{3}+j)}{\Gamma(-\frac{1}{3})}\,.$$

This expansion actually holds throughout the right half plane $\{x \in \mathbb{C} : \mathrm{Re}\,x > 0\}$, since this is all that is needed to justify the treatment of the Laplace type integral on the contour C_2. We may obtain quickly from the above the asymptotic behaviour of $f(x)$ as x approaches infinity (i) along the imaginary axis, and (ii) in the left half plane $\{x \in \mathbb{C} : \mathrm{Re}\,x < 0\}$ as follows:

(i) If $x = ir$ then $f(ir) = \int_0^1 e^{-rt^3} dt$ can be analyzed as a Laplace integral. If $r > 0$ then $t = 0$ dominates, and if $r < 0$ then $t = 1$ dominates.

(ii) Using $\overline{f(x)} = f(-\overline{x})$ for complex conjugates gives the asymptotic expansion in the left half plane.

Concluding Remarks. After familiarizing yourself with the method through these examples the three steps to be taken in applying the method described above should now make more sense. An additional point worth making now is that these examples have hopefully indicated a general picture of the paths of steepest descent/ascent emanating from a saddle point - we will now state this explicitly. The point z_0 is called a saddle point of order N if the first N derivatives of $h(z)$ vanishes at z_0 and the $N+1$ derivative is different than zero:

$$\left.\frac{d^m h(z)}{dz^m}\right|_{z=z_0} = 0, \quad m = 1, \cdots, N,$$

$$\left.\frac{d^{N+1} h(z)}{dz^{N+1}}\right|_{z=z_0} = ae^{i\alpha}, \quad a > 0, \quad \alpha \in \mathbb{R}\,.$$

If $N = 1$, z_0 is a *simple saddle point*. The directions of the steepest descent and of steepest ascent are defined, with respect to polar coordinates (ρ, θ) centered at z_0, by

$$\theta = -\frac{\alpha}{N+1} + (2m+1)\frac{\pi}{N+1}, \quad m = 0, 1, ..., N \qquad \text{(descent)}$$

and

$$\theta = -\frac{\alpha}{N+1} + 2m\frac{\pi}{N+1}, \quad m = 0, 1, \cdots, N \qquad \text{(ascent)}\,.$$

Indeed,

$$h(z) - h(z_0) \sim \frac{(z-z_0)^{N+1}}{(N+1)!}\left.\frac{d^{N+1}h(z)}{dz^{N+1}}\right|_{z=z_0} = \frac{(\rho e^{i\theta})^{N+1}}{(N+1)!}ae^{i\alpha}, \qquad (z \to z_0)\,.$$

Thus,

$$h(z) - h(z_0) \sim \frac{\rho^{N+1}a}{(N+1)!}\left[\cos(\alpha + (N+1)\theta) + i\sin(\alpha + (N+1)\theta)\right], \qquad (z \to z_0)\,.$$

Hence, the paths of steepest descent are tangent to the rays starting at z_0 defined by

$$\sin(\alpha + (N+1)\theta) = 0, \quad \cos(\alpha + (N+1)\theta) < 0,$$

whereas for the paths of steepest ascent the $\cos$ above is positive. If $N = 1$, then the paths of steepest descent and ascent are given respectively by

$$\theta = -\frac{\alpha}{2} + \frac{\pi}{2}, \quad -\frac{\alpha}{2} + \frac{3\pi}{2}$$

and

$$\theta = -\frac{\alpha}{2}, \quad -\frac{\alpha}{2} + \pi.$$

Note that there is an alternation between paths of steepest descent and ascent as the saddle point z_0 is encircled.