Part IA - Dynamics and Relativity

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Basic concepts
Space and time, frames of reference, Galilean transformations. Newton's laws. Dimensional analysis. Examples of forces, including gravity, friction and Lorentz. [4]

Newtonian dynamics of a single particle
Equation of motion in Cartesian and plane polar coordinates. Work, conservative forces and potential energy, motion and the shape of the potential energy function; stable equilibria and small oscillations; effect of damping.

Angular velocity, angular momentum, torque.

Orbits: the $u(\theta)$ equation; escape velocity; Kepler's laws; stability of orbits; motion in a repulsive potential (Rutherford scattering). Rotating frames: centrifugal and coriolis forces. *Brief discussion of Foucault pendulum.* [8]

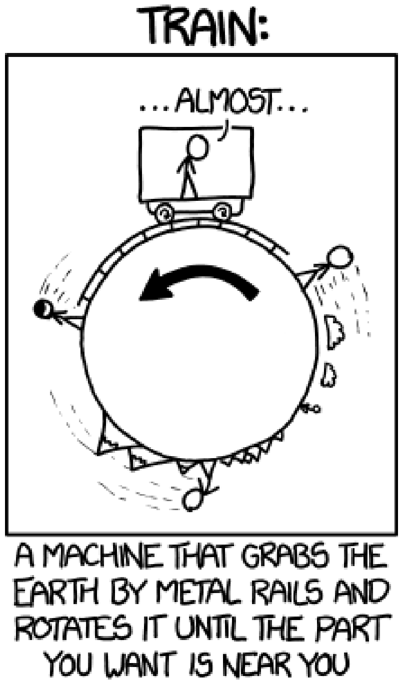
Newtonian dynamics of systems of particles
Momentum, angular momentum, energy. Motion relative to the centre of mass; the two body problem. Variable mass problems; the rocket equation. [2]

Rigid bodies
Moments of inertia, angular momentum and energy of a rigid body. Parallel axes theorem. Simple examples of motion involving both rotation and translation (eg. rolling). [3]

Special relativity
The principle of relativity. Relativity and simultaneity. The invariant interval. Lorentz transformations in $(1+1)$-dimensional spacetime. Time dilation and length contraction. The Minkowski metric for $(1+1)$-dimensional spacetime. Lorentz transformations in $(3+1)$ dimensions. 4-vectors and Lorentz invariants. Proper time. 4-velocity and 4-momentum. Conservation of 4-momentum in particle decay. Collisions. The Newtonian limit. [7]

Contents

1 Newtonian dynamics of particles



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We start with a few basic definitions:

Definition (Particle)**.** An *particle* is an object of insignificant size. It can be regarded as a point. It has a *mass* $m > 0$, and *electric charge* q.

Its position at time t is described by its *position vector*, $\mathbf{r}(t)$ or $\mathbf{x}(t)$ with respect to an origin O.

Definition (Frame of reference)**.** A *frame of reference* is choice of coordinate axes for $\mathbf{r}$. The axes may be fixed, moving, or accelerating relative to another frame.

With a frame of reference, we can write $\mathbf{r}$ in cartesian coordinates as (x, y, z)

Definition (Velocity)**.** The *velocity* of the particle is

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}.$$

and is tangent to the path or trajectory.

Definition (Acceleration)**.** The *acceleration* of the particle is

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{\mathrm{d}^2\mathbf{r}}{dt^2}.$$

Definition (Momentum)**.** The *momentum* of a particle is

$$\mathbf{p} = m\mathbf{v} = m\dot{\mathbf{r}}.$$

m is the *inertial mass* of the particle, and measures its reluctance to accelerate (cf. Newton's Second Law)

1.1 Newton's laws of motion

We begin by stating Newton's three laws of motion:

Law (Newton's First Law of Motion)**.** A body remains at rest, or moves uniformly in a straight line, unless acted on by a force. (This is in fact Galileo's Law of Inertia)

Law (Newton's Second Law of Motion)**.** The rate of change of momentum of a body is equal to the force acting on it (in both magnitude and direction).

Law (Newton's Third Law of Motion)**.** To every action there is an equal and opposite reaction: the forces of two bodies on each other are equal and in opposite directions.

The first law might seem redundant given the second if interpreted literally. Therefore, we should be interpreting it in the different way:

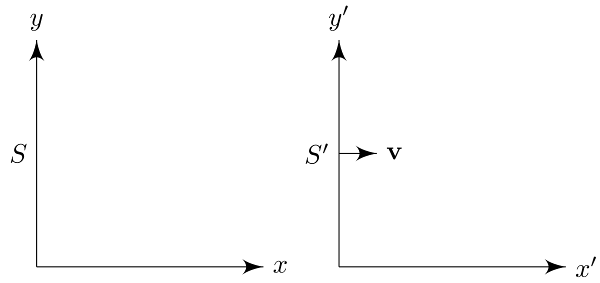
Note that the first law isn't always true. Take yourself as a frame of reference. When you move around your room, things will seem like they are moving around (relative to you). When you sit down, they stop moving. However, in reality, they've always been sitting there still. On second thought, this is because you, the frame of reference, is accelerating, not the objects. The first law only holds in frames that are themselves not accelerating. We call these *inertial frames*.

Definition (Inertial frames)**.** *Inertial frames* are frames of references in which the frames themselves are not accelerating. Newton's Laws only hold in inertial frames.

The we can take the first law to assert that inertial frames exists. Even though the Earth itself is rotating and orbiting the sun, for most purposes, any fixed place on the Earth counts as an inertial frame.

1.2 Galilean transformations

Inertial frames aren't unique. If S is an inertial frame, then any other frame S' in uniform motion relative to S is also inertial:



Assuming the frames coincide at $t = 0$, then

$$
\begin{aligned}
x' &= x - vt \\
y' &= y \\
z' &= z \\
t' &= t
\end{aligned}
$$

Generally, the position vector transforms as

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}t,$$

where $\mathbf{v}$ is the (constant) velocity of S' relative to S. The velocity and acceleration transform as follows:

$$\dot{\mathbf{r}}' = \dot{\mathbf{r}} - \mathbf{v}$$
$$\ddot{\mathbf{r}}' = \ddot{\mathbf{r}}$$

Definition (Galilean boost)**.** A Galilean boost is a change in frame of reference by

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}t$$
$$t' = t$$

for a fixed, constant $\mathbf{v}$.

In addition to Galilean boosts, we can construct new inertial frames by applying (any combination of) the following:

– Translations of space:
$$\mathbf{r}' = \mathbf{r} - \mathbf{r}_0$$

– Translations of time:
$$t' = t - t_0$$

– Rotation (and reflection):
$$\mathbf{r}' = R\mathbf{r}$$
with $R \in O(3)$.

These transformations together generate the Galilean group.

Law (Galilean relativity)**.** The *principle of relativity* asserts that the laws of physics are the same in inertial frames.

This implies that physical processes work the same

– at every point of space

– at all times

– in whichever direction we face

– at whatever constant velocity we travel.

In other words, the equations of Newtonian physics must have *Galilean invariance.*

Since the laws of physics are the same regardless of your velocity, velocity must be a *relative concept*, and there is no such thing as an “absolute velocity” that all inertial frames agree on.

However, all inertial frames must agree on whether you are accelerating or not (even though they need not agree on the direction of acceleration since you can rotate your frame). So acceleration is an *absolute* concept.

1.3 Newton's Second Law

Law. The *equation of motion* for a particle subject to a force F is

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F},$$

where $\mathbf{p} = m\mathbf{v} = m\dot{\mathbf{r}}$ is the (linear) momentum of the particle. We say m is the (inertial) mass of the particle, which is a measure of its reluctance to accelerate.

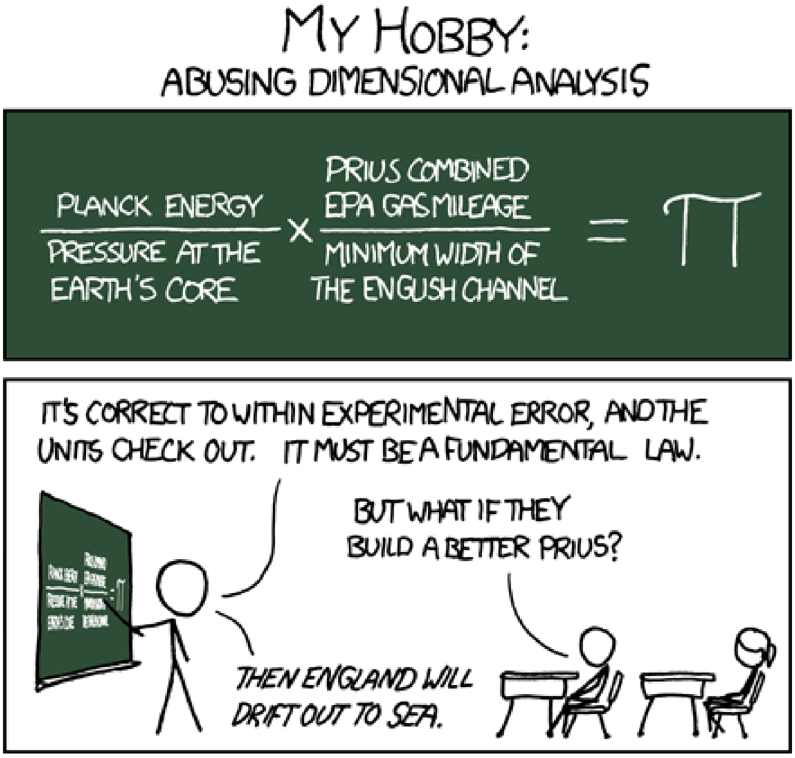
Usually, m is constant. Then

$$\mathbf{F} = m\mathbf{a} = m\ddot{\mathbf{r}}.$$

If $\mathbf{F}$ is specified as a function of $\mathbf{r}, \dot{\mathbf{r}}$ and t, then we have a second-order ordinary differential equation for $\mathbf{r}$.

To determine the solution, we need to specify the initial values of $\mathbf{r}$ and $\dot{\mathbf{r}}$, ie. the initial position and velocity. The trajectory of the particle is then uniquely determined for all future (and past) times.

2 Dimensional Analysis



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Physical quantities are not pure numbers, but have *dimensions*. In any equation, the dimensions have to be consistent.

For many problems in dynamics, three basic dimensions are sufficient:

– length, L

– mass, M

– time, T

The dimensions of a physical quantity X, denoted as $[X]$ are expressible uniquely in terms of L, M and T, eg.

– $[\text{area}] = L^2$

– $[\text{density}] = L^{-3}M$

– $[\text{velocity}] = LT^{-1}$

– $[\text{acceleration}] = LT^{-2}$

– $[\text{force}] = LMT^{-2}$ since the dimensions must satisfy the equation $F = ma$.

– $[\text{energy}] = L^2MT^{-2}$, eg. consider $E = mv^2/2$.

Physical constants also have dimensions, eg. $[G] = L^3M^{-1}T^{-2}$ by $F = GMm/r^2$.

We can only take sums and products of terms that have dimensions, and if we sum two terms, they must have the same dimension. More complicated functions of dimensional quantities are not allowed, eg. e^x makes no sense if x has a dimension, since

$$e^x = 1 + x + \frac{1}{2}x^2 + \cdots$$

and if x had a dimension, we would be summing up terms of different dimensions.

2.1 Units

A *unit* is a convenient standard physical quantity. In the SI system, there are base units corresponding to the basics dimensions. The three we need are

- meter (m) for length
- kilogram (kg) for mass
- second (s) for time

A physical quantity can be expressed as a pure number times a unit with the correct dimensions, eg.

$$G = 6.67 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}.$$

It is important to realize that SI units are chosen arbitrarily for historical reasons only. The equation of physics must work in any consistent system of units. This is captured by the fact that physical equations must be dimensionally consistent.

2.2 Scaling

Suppose we believe that a physical quantity Y depends on 3 other physical quantities X_1, X_2, X_3, ie. $Y = Y(X_1, X_2, X_3)$. Let their dimensions be as follows:

- $[Y] = L^aM^bT^c$
- $[X_i] = L^{a_i}M^{b_i}T^{c_i}$

Suppose further that we know that the relationship is a power law, ie.

$$Y = CX_1^{p_1}X_2^{p_2}X_3^{p_3},$$

where C is a dimensionless constant (ie. pure number). Since the dimensions must work out, we know that

$$\begin{aligned} a &= p_1a_1 + p_2a_2 + p_3a_3 \\ b &= p_1b_1 + p_2b_2 + p_3b_3 \\ c &= p_1c_1 + p_2c_2 + p_3c_3 \end{aligned}$$

for which there is a unique solution provided that the dimensions of X_1, X_2 and X_3 are independent.

Note that if X_i are not independent, eg. $X_1^2 X_2$ is dimensionless, then our law can involve more complicated terms such as $\exp(X_1^2 X_2)$ since the argument of exp is dimensionless.

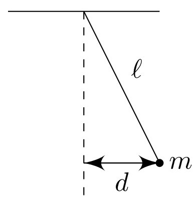
In general, if the dimensions of X_i are not independent, order them so that the independent terms $[X_1], [X_2], [X_3]$ are at the front. For each of the remaining variables, form a dimensionless quantity $\lambda_i = X_i X_1^{q_1} X_2^{q_2} X_3^{q_3}$. Then the relationship must be of the form

$$Y = f(\lambda_4, \lambda_5, \cdots) X_1^{p_1} X_2^{p_2} X_3^{p_3}.$$

where f is an arbitrary function of the dimensionless variables.

Formally, this results is described by the *Buckingham's Pi theorem*, but we will not go into details.

Example (Simple pendulum)**.**



We want to find the period P. We know that P could depend on

- mass m: $[m] = M$
- length ℓ: $[\ell] = L$
- gravity g: $[g] = LT^{-2}$
- initial displacement d: $[d] = L$

and of course $[P] = T$.

We observe that m, ℓ, g have independent dimensions, and with the fourth, we can form the dimensionless group d/ℓ. So the relationship must be of the form

$$P = f\left(\frac{d}{l}\right) m^{p_1} \ell^{p_2} g^{p_3},$$

where f is a dimensionless function. For the dimensions to balance,

$$T = M^{p_1} L^{p_2} L^{p_3} T^{-2p_3}.$$

So $p_1 = 0, p_2 = -p_3 = 1/2$. Then

$$P = f\left(\frac{d}{\ell}\right) \sqrt{\frac{\ell}{g}}.$$

While we cannot find the exact formula, we know that if both ℓ and d are quadrupled, then P will double.

3 Forces

3.1 Force and potential energy in one dimension

Consider a particle of mass m moving in a straight line with position $x(t)$. Suppose $F = F(x)$, ie. depends on position only. We define the potential energy as follows:

Definition (Potential energy)**.** Given a force field $F = F(x)$, we define the *potential energy* to be a function $V(x)$ such that

$$F = -\frac{\mathrm{d}V}{\mathrm{d}x}.$$

or

$$V = -\int F\,\mathrm{d}x.$$

V is defined up to a constant term.

The equation of motion is then

$$m\ddot{x} = -\frac{\mathrm{d}V}{\mathrm{d}x}.$$

Proposition. Suppose the equation of a particle satisfies

$$m\ddot{x} = -\frac{\mathrm{d}V}{\mathrm{d}x}.$$

Then the total energy

$$E = T + V = \frac{1}{2}m\dot{x}^2 + V(x)$$

is conserved, ie. $\dot{E} = 0$.

Proof.

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= m\dot{x}\ddot{x} + \frac{\mathrm{d}V}{\mathrm{d}x}\dot{x}\\ &= \dot{x}\left(m\ddot{x} + \frac{\mathrm{d}V}{\mathrm{d}x}\right)\\ &= 0\end{aligned}$$

□

Example. Consider the harmonic oscillator

$$V = \frac{1}{2}kx^2.$$

Then the equation of motion satisfy

$$m\ddot{x} = -kx.$$

This is the case of, say, Hooke's Law for a spring. But in general, small perturbations near potential wells are also harmonic oscillators.

The general solution of this is

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

with $\omega = \sqrt{k/m}$.

A and B are arbitrary constants, and are related to the initial position and velocity by $x(0) = A, \dot{x}(0) = \omega B$.

For a general potential energy $V(x)$, conservation of energy allows us to solve the problem formally:

$$E = \frac{1}{2}m\dot{x}^2 + V(x)$$

Since E is a constant, from this equation, we have

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \pm\sqrt{\frac{2}{m}(E - V(x))} \\ \pm\int\frac{\mathrm{d}x}{\sqrt{\frac{2}{m}(E - V(x))}} &= t - t_0 \end{aligned}$$

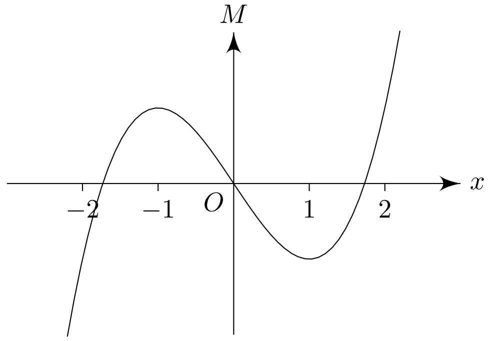
To find $x(t)$, we need to do the integral and then solve for x - not usually possible by analytical methods, but possible by numerical methods.

3.2 Motion in a potential

The graph of the potential energy $V(x)$ gives us a qualitative understanding of the dynamics, eg. is the particle trapped or can it escape to infinity?

Example. Consider $V(x) = m(x^3 - 3x)$. Note that this can be dimensionally consistent even though we add up x^3 and $-3x$, if we declare "3" to have dimension L^2.

We plot this as follows:



Suppose we release the particle from rest at $x = x_0$. Then $E = V(x_0)$. We can consider the different cases:

- $x_0 = \pm 1$: Particle stays there for all t.
- $-1 < x_0 < 2$: Particle oscillates back and forth in potential well
- $x_0 < -1$: Particle falls to $x = -\infty$.
- $x_0 > 2$: Particle overshoots well and continues to $x = -\infty$.
- $x_0 = 2$: Special case: The particle goes towards $x = -1$. How long does it take, and what happens next? Here $E = 2m$, we we noted previously

$$t - t_0 = -\int \frac{\mathrm{d}x}{\sqrt{\frac{2}{m}(E - V(x'))}}.$$

Let $x = -1 + \varepsilon(t)$. Then

$$\begin{aligned}\frac{2}{m}(E - V(x)) &= 4 - 2(-1+\varepsilon)^3 + 6(-1+\varepsilon)\\ &= 6\varepsilon^2 - 2\varepsilon^3.\end{aligned}$$

So

$$t - t_0 = -\int_3^{\varepsilon} \frac{\mathrm{d}\varepsilon'}{\sqrt{6\varepsilon'^2 - 2\varepsilon'^3}}$$

We reach $x = -1$ when $\varepsilon \to 0$. But for small ε', the integrand is approximately $\propto 1/\varepsilon'$, which integrates to $\ln \varepsilon' \to -\infty$ as $\varepsilon' \to 0$. So $\varepsilon = 0$ is achieved after infinite time, ie. it takes infinite time to reach $\varepsilon = 0$, or $x = -1$.

3.2.1 Equilibrium points

Definition (Equilibrium point)**.** A particle is in *equilibrium* if it has no tendency to move away. It will stay there for all time. Since $m\ddot{x} = -V'(x)$, the equilibrium points are the stationary points of the potential energy, ie.

$$V'(x_0) = 0;$$

Consider motion near an equilibrium point. Expand V in a Taylor series:

$$V(x) \approx V(x_0) + \frac{1}{2}V''(x_0)(x - x_0)^2.$$

Neglect the higher-order terms. Then the equation of motion is

$$m\ddot{x} = -V''(x_0)(x - x_0).$$

If $V''(x_0) > 0$, then V has a local minimum at x_0, and we have the potential of the harmonic oscillator. The equilibrium point is *stable* and the particle oscillates with angular frequency

$$\omega = \sqrt{\frac{V''(x_0)}{m}}.$$

This is only valid for small oscillations. This shows that small oscillations near stable equilibriums are stable.

If $V''(x_0) < 0$, then V has a local maximum at x_0. The equilibrium point is unstable. In this case,

$$x - x_0 \approx Ae^{\gamma t} + Be^{-\gamma t}$$

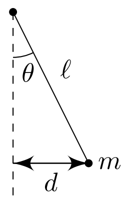
for

$$\gamma = \sqrt{\frac{-V''(x_0)}{m}}.$$

For almost all initial conditions, $A \neq 0$ and the particle will diverge from the equilibrium point, leading to a breakdown of the approximation.

If $V''(x_0) = 0$, then further work is required to determine the outcome.

Example. Consider the simple pendulum.



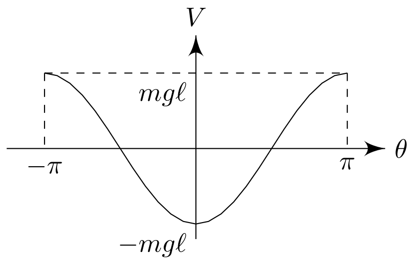
Suppose that the pendulum makes an angle θ with the vertical. The equation of motion is governed by

$$l\ddot{\theta} = -g\sin\theta.$$

The energy is

$$E = T + V = \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta.$$

Therefore $V \propto -\cos\theta$. We have a stable equilibrium at $\theta = 0$, and unstable equilibrium at $\theta = \pi$.



If $E > mg\ell$, then $\dot{\theta}$ never vanishes and the pendulum makes full circles.

If $-mg\ell < E < mg\ell$, then $\dot{\theta}$ vanishes at $\theta = \pm\theta_0$ for some $0 < \theta_0 < \pi$ ie. $E = -mg\ell\cos\theta_0$. The pendulum oscillates back and forth. It takes a quarter of a period to reach from $\theta = 0$ to $\theta = \theta_0$. Using the previous general solution, oscillation period P is given by

$$\frac{P}{4} = \int_0^{\theta_0} = \frac{\mathrm{d}\theta}{\sqrt{\frac{2E}{m\ell^2} + \frac{2g}{\ell}\cos\theta}}.$$

Since we know that $E = -mg\ell\cos\theta_0$, we know that

$$\frac{P}{4} = \sqrt{\frac{\ell}{g}}\int_0^{\theta_0}\frac{\mathrm{d}\delta}{\sqrt{2\cos\theta - 2\cos\theta_0}}.$$

The integral is difficult to evaluate in general, but for small θ_0, we can use $\cos\theta \approx 1 - \frac{1}{2}\theta^2$. So

$$P \approx \sqrt{\frac{\ell}{g}}\int_0^{\theta_0}\frac{\mathrm{d}\theta}{\sqrt{\theta_0^2 - \theta^2}} = 2\pi\sqrt{\frac{\ell}{g}}$$

and is independent of the amplitude θ_0. This is of course the result for the harmonic oscillator.

3.2.2 Force and potential energy in three dimensions

Consider a particle of mass m moving in 3D. The equation of motion is a vector equation

$$m\ddot{\mathbf{r}} = \mathbf{F}.$$

Definition (Kinetic energy)**.** We define the kinetic energy of the particle is

$$T = \frac{1}{2}m|\mathbf{v}|^2 = \frac{1}{2}m\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}.$$

If we want to know how it varies with time, we obtain

$$\frac{\mathrm{d}T}{\mathrm{d}t} = m\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}} = \mathbf{F}\cdot\dot{\mathbf{r}} = \mathbf{F}\cdot\mathbf{v}.$$

This is the power.

Definition (Power)**.** The *power* is the rate at which work is done on a particle by a force. It is given by

$$P = \mathbf{F}\cdot\mathbf{v}.$$

Definition (Work done)**.** The *work done* on a particle by a force is the change in kinetic energy caused by the force. The work done on a particle moving from $\mathbf{r}_1 = \mathbf{r}(t_1)$ to $\mathbf{r}_2 = \mathbf{r}(t_2)$ along a trajectory C is the line integral

$$W = \int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_{t_1}^{t_2}\mathbf{F}\cdot\dot{\mathbf{r}}\;\mathrm{d}t = \int_{t_1}^{t_2} P\;\mathrm{d}t.$$

Now we define a particular type of force known as *conservative force* that has many important properties.

Definition (Conservative force and potential energy)**.** A *conservative force* is a force field $\mathbf{F}(\mathbf{r})$ that can be written in the form

$$\mathbf{F} = -\nabla V.$$

V is the *potential energy function.*

Proposition. If $\mathbf{F}$ is conservative, then the energy

$$\begin{aligned} E &= T + V \\ &= \frac{1}{2}m|\mathbf{v}|^2 + V(\mathbf{r}) \end{aligned}$$

is conserved. Then the work done is equal to the change in potential energy, and is independent of the path taken between the end points.

In particular, if we travelled around a closed loop, no work is done.

Proof.

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + V\right) \\ &= m\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}} + \frac{\partial V}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t} \\ &= (m\ddot{\mathbf{r}} + \nabla V)\cdot\dot{\mathbf{r}} \\ &= (m\ddot{\mathbf{r}} - \mathbf{F})\cdot\dot{\mathbf{r}} \\ &= 0 \end{aligned}$$

In this case, the work done is

$$W = \int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = -\int_C (\nabla V)\cdot\mathrm{d}\mathbf{r} = V(\mathbf{r}_1) - V(\mathbf{r}_2).$$

□

3.3 Central forces

This is a special class of conservative force, where the potential depends only on the distance from the origin.

Definition (Central force)**.** A central force is a force with a potential $V(r)$ that depends only on the distance from the origin, $r = |\mathbf{r}|$. Note that a central force can be both attractive or repulsive.

Note the following useful formula

Proposition. $\nabla r = \hat{\mathbf{r}}$.

This is because the direction in which r increases most rapidly is $\mathbf{r}$, and the rate of increase is clearly 1. This can also be proved algebraically:

Proof. We know that

$$r^2 = x_1^2 + x_2^2 + x_3^2.$$

Then

$$2r\frac{\partial r}{\partial x_i} = 2x_i.$$

So

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} = (\hat{\mathbf{r}})_i.$$

□

Proposition. Let $\mathbf{F} = -\nabla V(r)$ be a central force. Then

$$\mathbf{F} = -\nabla V = -\frac{\mathrm{d}V}{\mathrm{d}r}\hat{\mathbf{r}},$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$ is the unit vector in the radial direction pointing away from the origin.

Proof. Using the proof above,

$$(\nabla V)_i = \frac{\partial V}{\partial x_i} = \frac{\mathrm{d}V}{\mathrm{d}r}\frac{\partial r}{\partial x_i} = \frac{\mathrm{d}V}{\mathrm{d}r}(\hat{\mathbf{r}})_i$$

□

Central forces give rise to an additional conserved quantity called *angular momentum.*

Definition (Angular momentum)**.** The *angular momentum* of a particle is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \dot{\mathbf{r}}.$$

Proposition. Angular momentum is conserved by a central force.

Proof.

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = m\dot{\mathbf{r}} \times \dot{\mathbf{r}} + m\mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{0} + \mathbf{r} \times \mathbf{F} = \mathbf{0}.$$

where the last equality comes from the fact that $\mathbf{F}$ is parallel to $\mathbf{r}$ for a central force. □

In general, for a non-central force, the rate of change of momentum is the *torque.*

Definition (Torque)**.** The *torque* G of a particle is the rate of change of momentum.

$$\mathbf{G} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{r} \times \mathbf{F}.$$

Note that $\mathbf{L}$ and $\mathbf{G}$ depends on the choice of origin. For a central force, only the angular momentum about the center of the force is conserved.

3.4 Gravity

Gravity is a conservative and central force.

Law. If a particle of mass M is fixed at a origin, then a second particle of mass m experiences a potential energy

$$V(r) = -\frac{GMm}{r},$$

where $G \approx 6.67 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$ is the *gravitational constant.*

The gravitational force experienced is then

$$F = -\nabla V = -\frac{GMm}{r^2}\hat{\mathbf{r}}.$$

Since the force is negative, particles are attracted to the origin.

Definition (Gravitaional potential and field)**.** The *gravitational potential* is the gravitational potential energy per unit mass. It is

$$\Phi_g(r) = -\frac{GM}{r}.$$

Note that *potential* is confusingly different from *potential energy*.

The *gravitational field* is the force per unit mass,

$$\mathbf{g} = -\nabla\Phi_g = -\frac{GM}{r^2}\hat{\mathbf{r}}.$$

These are properties of the mass M alone.

Then the potential energy of a second particle is $V = m\Phi_g$.

Proposition. The gravitational potential due to many fixed masses M_i at points $\mathbf{r}_i$ is is

$$\Phi_g(\mathbf{r}) = -\sum_i \frac{GM_i}{|\mathbf{r}-\mathbf{r}_i|}.$$

Again, $V = m\Phi_g$ for a particle of mass m.

Proposition. The external gravitational potential of a spherically symmetric object of mass M is the same as that of a point particle with the same mass at the center of the object, ie.

$$\Phi_g(r) = -\frac{GM}{r}.$$

Proof. cf. Vector Calculus □

Example. If you live on a spherical planet of mass M and radius R, and can move only a small distance $z \ll R$ above the surface, then

$$\begin{aligned} V(r) &= V(R+z)\\ &= -\frac{GMm}{r+z}\\ &= -\frac{GMm}{r}\left(1 - \frac{z}{R} + \cdots\right)\\ &\approx \text{const.} + \frac{GMm}{R^2}z\\ &= \text{const.} + mgz. \end{aligned}$$

where $g = GM/R^2 \approx 9.8\,\mathrm{m\,s^{-2}}$ for Earth. Usually we are lazy and just say that the potential is mgz.

Example. How fast do we need to jump to escape the gravitational pull of the Earth? If we jump upwards with speed v from the surface, then

$$E = T + V = \frac{1}{2}mv^2 - \frac{GMm}{R}.$$

Since after escape, we have $T \geq 0$ and $V = 0$, and energy is conserved, we must have $E \geq 0$ from the very beginning. ie.

$$v > v_{esc} = \sqrt{\frac{2GM}{R}}.$$

3.4.1 Inertial and gravitational mass

Mass appears in two equations:

$$m_i\ddot{\mathbf{r}} = \mathbf{F} \quad \text{(inertial mass)}$$

and

$$\mathbf{F} = -\frac{GM_g m_g}{r^2}\hat{\mathbf{r}} \quad \text{(gravitational mass).}$$

and they play totally different roles. The first is the resistance to motion, and the second is the response to gravitational forces. Conceptually, these are quite different, but experiment shows that they are indeed equivalent to each other, ie. $m_i = m_g$, with an accuracy of 10^{-12} or better.

This is only explained by Einstein's general theory of relativity.

We can further distinct the gravitational mass by "passive" and "active", ie. the amount of gravitational field generated by a particle (M), and the amount of gravitational force received by a particle (m), but they are still equal, and we end up calling all of them "mass".

3.5 Electromagnetism

Notation (Electric and magnetic field)**.**

(i) $\mathbf{E}(\mathbf{r}, t)$ is the electric field;

(ii) $\mathbf{B}(\mathbf{r}, t)$ is the magnetic field;

Law (Lorentz force law)**.** The *electromagnetic force* experienced by a particle with electric charge q is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

Law. The electron has charge

$$q_e = -1.6 \times 10^{-19}\,\mathrm{C}$$

and other particles' charges are integer multiples of this unit (unless you are a quark).

Law. For time-independent fields, $\mathbf{E}$ can be written as

$$\mathbf{E} = -\nabla\Phi_e,$$

where $\Phi_e(\mathbf{r})$ is the *electrostatic potential.* The electric force $q\mathbf{E}$ is then conserved.

Definition (Electrostatic potential)**.** The electrostatic potential is a function $\Phi_e(\mathbf{r})$ such that

$$\mathbf{E} = -\nabla\Phi_e.$$

Note that $\mathbf{E}$ is conservative, and $\mathbf{B}j$ acts perpendicular to the velocity, and so does no work. So

Proposition. For time independent $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$, the energy

$$E = T + V = \frac{1}{2}m|\mathbf{v}|^2 + q\Phi_e$$

is conserved.

Proof.

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= m\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}} + q(\nabla\Phi_e)\cdot\dot{\mathbf{r}}\\ &= (m\ddot{\mathbf{r}} - q\mathbf{E})\cdot\dot{\mathbf{r}}\\ &= (q\dot{\mathbf{r}}\times\mathbf{B})\cdot\dot{\mathbf{r}}\\ &= 0\end{aligned}$$

□

3.5.1 Point charges

Law (Columb's law)**.** A particle of charge Q, fixed at the origin, produces an electrostatic potential

$$\Phi_e = \frac{Q}{4\pi\varepsilon_0 r},$$

where $\varepsilon_0 \approx 8.85\times 10^{-12}\,\mathrm{m}^{-3}\,\mathrm{kg}^{-1}\,\mathrm{s}^2\,\mathrm{C}^2$.

The corresponding electric field is

$$\mathbf{E} = -\nabla\Phi_e = \frac{Q}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{r^2}.$$

The resulting force on a particle of charge q is

$$\mathbf{F} = q\mathbf{E} = \frac{Qq}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{r^2}.$$

Definition (Electric constant)**.** ε_0 is the *electric constant* or *vacuum permittivity* or *permittivity of free space.*

Electrostatic forces are closely analogous to gravitational forces, but there is an important difference: charges can be positive or negative. Thus electrostatic forces can be either attractive or repulsive, whereas gravity is always attractive.

3.5.2 Motion in a uniform magnetic field

Set $\mathbf{E} = \mathbf{0}$ and choose axes such that $\mathbf{B} = (0, 0, B)$ is constant.

According to the Lorentz force law, $\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$. The components of the equation of motion are

$$\begin{aligned} m\ddot{x} &= qB\dot{y} \quad (1)\\ m\ddot{y} &= -qB\dot{x} \quad (2)\\ m\ddot{z} &= 0\end{aligned}$$

So we have a uniform motion parallel to $\mathbf{B}$.

There are many ways to solve this system of equations. Here we solve it using complex numbers.

Let $\zeta = x + iy$. Then (1) + (2)i gives

$$m\ddot{\zeta} = -iqB\dot{\zeta}.$$

Then the solution is

$$\zeta = \alpha e^{-\omega t} + \beta,$$

where $\omega = qB/m$ is the *gyrofrequency*, and α and β are complex integration constants. We see that the particle goes in circles, with center β and radius α.

We can choose coordinates such that, at $t = 0$, $\mathbf{r} = 0$ and $\dot{\mathbf{r}} = (0, v, w)$, ie. $\zeta = 0$ and $\dot{\zeta} = iv$, and $z = 0$ and $\cdot z = w$.

The solution is then
$$\zeta = R(1 - e^{-i\omega t}).$$
with $R = v/\omega = (mv)/(qB)$ is the *gyroradius* or *Larmor radius.* Alternatively,
$$\begin{aligned}x &= R(1 - \cos\omega t)\\ y &= R\sin\omega t\\ z &= wt.\end{aligned}$$
This is circular motion in the plane perpendicular to $\mathbf{B}$:
$$(x - R)^2 + y^2 = R^2,$$
combined with uniform motion parallel to $\mathbf{B}$, ie. a helix.

We can also have a vectorial treatment: Start with
$$m\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \times \mathbf{B}$$
Let $\mathbf{B} = B\mathbf{n}$ with $|\mathbf{n}| = 1$. Then
$$\ddot{\mathbf{r}} = \omega\dot{\mathbf{r}} \times \mathbf{n},$$
with our gyrofrequency $\omega = qB/m$. We integrate once, assuming $\mathbf{r}(0) = \mathbf{0}$ and $\dot{\mathbf{r}}(0) = \mathbf{v}_0$.
$$\dot{\mathbf{r}} = \omega\mathbf{r} \times \mathbf{n} + \mathbf{v}_0. \tag{$*$}$$
Project $(*)$ parallel to and perpendicular to $\mathbf{B}$:

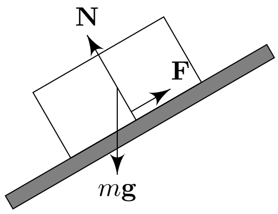
First dot $(*)$ with $\mathbf{n}$:
$$\dot{\mathbf{r}} \cdot \mathbf{n} = \mathbf{v}_0 \cdot \mathbf{n} = w = \text{const}.$$
We integrate again to obtain
$$\mathbf{r} \cdot \mathbf{n} = wt.$$
Then write $\mathbf{r} = (\mathbf{r} \cdot \mathbf{n})\mathbf{n} + \mathbf{r}_\perp$, with $\mathbf{r}_\perp \cdot \mathbf{n} = 0$.

The perpendicular component of $(*)$ gives
$$\dot{\mathbf{r}}_\perp = w\mathbf{r}_\perp \times \mathbf{n} + \mathbf{v}_0 - (\mathbf{v}_0 \cdot \mathbf{n})\mathbf{n}.$$
We solve this by differentiating again to obtain
$$\ddot{\mathbf{r}}_\perp = \omega\dot{\mathbf{r}}_\perp \times \mathbf{n} = -\omega^2\mathbf{r}_\perp + \omega\mathbf{v}_0 \times \mathbf{n},$$
which we can solve using particular integrals and complementary functions.

3.6 Friction

Energy is conserved at an atomic level, but does not appear to be conserved in many everyday processes. This is described by *friction.* We divide them into different categories:

Definition (Dry friction)**.** When solid objects are in contact, a *normal reaction force* $\mathbf{n}$ (perpendicular to the contact surface) prevents them from interpenetrating, while a *frictional force* $\mathbf{F}$ (tangential to the surface) resists relative tangential motion (sliding or slipping).



If no sliding occurs, then we have *static friction* of

$$|\mathbf{F}| \leq \mu_s |\mathbf{N}|,$$

where μ_s is the *coefficient of static friction.*

If sliding occurs, then we have *kinetic friction* of

$$|F| = \mu_k |\mathbf{N}|,$$

where μ_k is the *coefficient of kinetic friction.*

These coefficients are measures of roughness and depend on the two surfaces involved. For example, Teflon on Teflon has coefficient of around 0.04, what rubber on asphalt has about 0.8.

Usually, $\mu_s > \mu_k > 0$.

A hypothetical perfectly smooth surface has $\mu_s = \mu_k = 0$.

Definition (Fluid drag)**.** When a solid object moves through a fluid (ie. liquid or gas), it experiences a *drag force.*

There are two important regimes.

(i) Linear drag: for small things in viscous fluids moving slowly, eg. a single cell organism in water,
$$\mathbf{F} = -k_1 \mathbf{v}.$$
where $\mathbf{v}$ is the velocity of the object relative to the fluid, and $k_1 > 0$ is a constant. For a sphere of radius R, Stoke's Law gives
$$k_1 = 6\pi\mu R,$$
where μ is the viscosity of the fluid.

(ii) Quadratic drag: for large objects moving rapidly in less viscous fluid, eg. cars or tennis balls in air,
$$\mathbf{F} = -k_2 |\mathbf{v}|^2 \hat{\mathbf{v}}.$$

In either case, the object loses energy. The power exerted be the drag force is

$$\mathbf{F}\cdot\mathbf{v} = \begin{cases} -k_1|\mathbf{v}|^2 \\ -k_2|\mathbf{v}|^3 \end{cases}$$

Example. Consider a projectile moving in a uniform gravitational field and experiencing a linear drag force.

At $t = 0$, we throw the projectile with velocity $\mathbf{u}$ from $\mathbf{x} = \mathbf{0}$.

The equation of motion is

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = m\mathbf{g} - k\mathbf{v}.$$

We first solve for $\mathbf{v}$, and then deduce $\mathbf{x}$.

We use an integrating factor $\exp(\frac{k}{m}t)$ to obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{kt/m}\mathbf{v}\right) &= e^{kt/m}\mathbf{g} \\ e^{kt/m}\mathbf{v} &= \frac{m}{k}e^{kt/m}\mathbf{g} + \mathbf{c} \\ \mathbf{v} &= \frac{m}{k}\mathbf{g} + \mathbf{c}e^{-kt/m} \end{aligned}$$

Since $\mathbf{v} = \mathbf{u}$ at $t = 0$, $\mathbf{c} = \mathbf{u} - \frac{m}{k}\mathbf{g}$. So

$$\mathbf{v} = \dot{\mathbf{x}} = \frac{m}{k}\mathbf{g} + \left(\mathbf{u} - \frac{m}{k}\mathbf{g}\right)e^{-kt/m}.$$

Integrating once gives

$$\mathbf{x} = \frac{m}{k}\mathbf{g}t - \frac{m}{k}\left(\mathbf{u} - \frac{m}{k}\mathbf{g}\right)e^{-kt/m} + \mathbf{d}.$$

Since $\mathbf{x} = \mathbf{0}$ at $t = 0$. So

$$\mathbf{d} = \frac{m}{k}\left(\mathbf{u} - \frac{m}{k}\mathbf{g}\right).$$

So

$$\mathbf{x} = \frac{m}{k}\mathbf{g}t + \frac{m}{k}\left(\mathbf{u} - \frac{m}{k}\mathbf{g}\right)(1 - e^{-kt/m}).$$

In component form, let $\mathbf{x} = (x, y)$, $\mathbf{u} = (u\cos\theta, u\sin\theta)$, $\mathbf{g} = (0, -g)$. So

$$\begin{aligned} x &= \frac{mu}{k}\cos\theta(1 - e^{-kt/m}) \\ z &= -\frac{mgt}{k} + \frac{m}{k}\left(u\sin\theta + \frac{mg}{k}\right)(1 - e^{-kt/m}). \end{aligned}$$

We can characterize the strength of the drag force by the dimensionless constant $ku/(mg)$, with a larger constant corresponding to a larger drag force.

3.6.1 Effect of damping on small oscillations

Friction or drag causes oscillations about a potential minimum to be damped out. Energy is continually lost until the system comes to rest at the stable equilibrium.

Example. If a linear drag force is added to a harmonic oscillator, then the equation of motion becomes

$$m\ddot{\mathbf{x}} = -m\omega^2\mathbf{x} - k\dot{\mathbf{x}},$$

where ω is the angular frequency of the oscillator in the absence of damping. Rewrite as

$$\ddot{\mathbf{x}} + 2\gamma\dot{\mathbf{x}} + \omega^2 x = 0,$$

where $\gamma = k/2m > 0$. Solutions are $x = e^{\lambda t}$, where

$$\lambda^2 + 2\gamma\lambda + \omega^2 = 0,$$

or

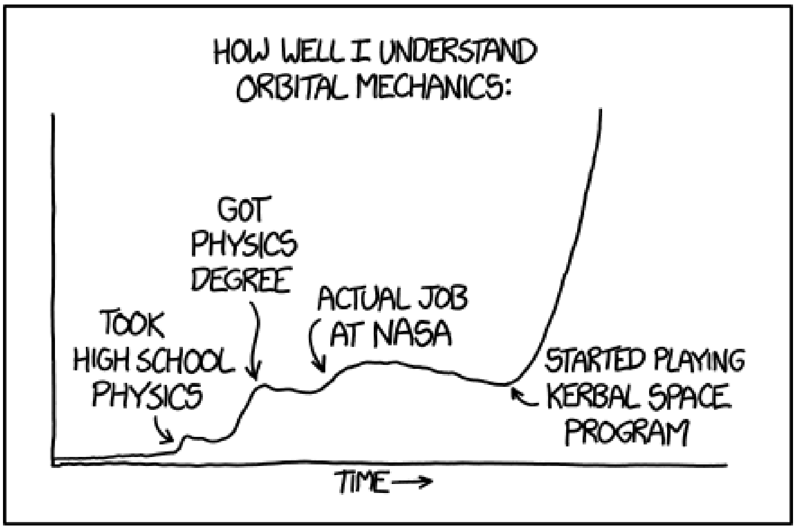
$$\lambda = -\gamma \pm \sqrt{\gamma^2 - \omega^2}.$$

If $\gamma > \omega$ (overdamped oscillator), roots are real and negative. So we have exponential decay.

If $0 < \gamma < \omega$ (underdamped oscillator), roots are complex with $\mathrm{Re}(\lambda) = -\gamma$. So we have decaying oscillations.

For details, refer to Differential Equations.

4 Orbits



We will study the motion in 3D of a particle in a central force,

$$m\ddot{\mathbf{r}} = -\nabla V(r).$$

The angular momentum $\mathbf{L} = m\mathbf{r} \times \dot{\mathbf{r}}$ is a constant vector, as previously shown. Furthermore $\mathbf{L} \cdot \mathbf{r} = 0$. Therefore, the motion takes place in a plane passing through the origin, and perpendicular to $\mathbf{L}$.

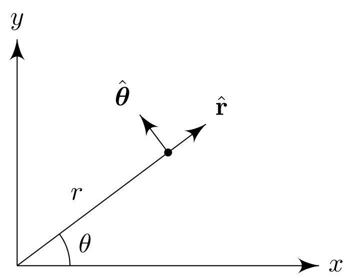
4.1 Polar coordinates in the plane

We choose our axes such that the orbital plane is $z = 0$. We introduce polar coordinates (r, θ):

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Notation. We define unit vectors in the directions of increasing r and increasing θ:

$$\hat{\mathbf{r}} = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}, \quad \hat{\boldsymbol{\theta}} = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}.$$



They form an orthonormal basis at each point, but depends on position:

Proposition.

$$\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}\theta} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \hat{\boldsymbol{\theta}}$$
$$\frac{\mathrm{d}\hat{\boldsymbol{\theta}}}{\mathrm{d}\theta} = \begin{pmatrix} -\cos\theta \\ -\sin\theta \end{pmatrix} = -\hat{\mathbf{r}}.$$

Proposition. For each particle with trajectory $\mathbf{r}(t)$, the polar coordinates r and θ depend on t, and so do the polar unit vectors. By the chain rule,

$$\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t} = \dot{\theta}\hat{\boldsymbol{\theta}}, \quad \frac{\mathrm{d}\hat{\boldsymbol{\theta}}}{\mathrm{d}t} = -\dot{\theta}\hat{\mathbf{r}}.$$

In terms of the polar unit vectors,

$$\mathbf{r} = r\hat{\mathbf{r}}.$$

The velocity is

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}.$$

The acceleration is

$$\begin{aligned}\ddot{\mathbf{r}} &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} - r\dot{\theta}^2\hat{\mathbf{r}} \\ &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}.\end{aligned}$$

Definition (Radial and angular velocity)**.** $\dot{r}$ is the *radial velocity*, and $\dot{\theta}$ is the *angular velocity.*

Example (Uniform motion in a circle)**.** $\dot{r} = 0$, $\dot{\theta} = \omega = $ const. So $\dot{\mathbf{r}} = r\omega\hat{\boldsymbol{\theta}}$.

The speed is

$$v = |\dot{\mathbf{r}}| = r|w| = \text{ const.}$$

The acceleration is

$$\ddot{\mathbf{r}} = -r\omega^2\hat{\mathbf{r}}.$$

In order to make a particle of mass m move uniformly in a circle, we must supply a *centripetal force* mv^2/r towards the center.

4.2 Motion in a central force field

Since $V = V(r)$, $\mathbf{F} = -\nabla V = \frac{\mathrm{d}V}{\mathrm{d}r}\hat{\mathbf{r}}$.

So Newton's 2nd law in polar coordinates is

$$m(\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} = -\frac{\mathrm{d}V}{\mathrm{d}r}\hat{\mathbf{r}}.$$

The θ component of this equation is

$$m(r\ddot{\theta} - 2\dot{r}\dot{\theta}) = 0.$$

So

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\theta}) = 0.$$

Write $L = mr^2\dot{\theta}$. This is the z component (ie. the only component) of the conserved angular momentum $\mathbf{L}$:

$$\begin{aligned}\mathbf{L} &= m\mathbf{r} \times \dot{\mathbf{r}}\\ &= mr\hat{\mathbf{r}} \times (\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}})\\ &= mr^2\dot{\theta}\ \hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}}\\ &= mr^2\dot{\theta}\ \hat{\mathbf{z}}.\end{aligned}$$

So are above equation states that L is constant, which is consistent with the conservation of angular momentum.

Introduce the angular momentum per unit mass:

Notation (Angular momentum per unit mass)**.** The *angular momentum per unit mass* is

$$h = \frac{L}{m} = r^2\dot{\theta} = \text{const.}$$

The radial (r) component of the equation of motion is

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{\mathrm{d}V}{\mathrm{d}r}.$$

We eliminate $\dot{\theta}$ using $r^2\dot{\theta} = h$ to obtain

$$m\ddot{r} = -\frac{\mathrm{d}V}{\mathrm{d}r} + \frac{mh^2}{r^3} = -\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}r},$$

where

$$V_{\text{eff}}(r) = V(r) + \frac{mh^2}{2r^2}.$$

We have reduced the problem to 1D motion in an (effective) potential - as studied previously.

The total energy of the particle is

$$\begin{aligned}E &= \frac{1}{2}m|\dot{\mathbf{r}}|^2 + V(r)\\ &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)\end{aligned}$$

(since $\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$, and $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are orthogonal)

$$\begin{aligned}&= \frac{1}{2}m\dot{r}^2 + \frac{mh^2}{2r^2} + V(r)\\ &= \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r).\end{aligned}$$

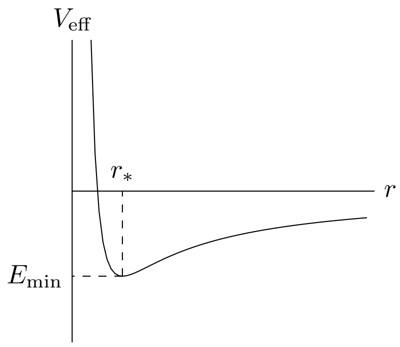
Example. Consider an attractive force following the inverse-square law (eg. gravity). Here

$$V = -\frac{mk}{r},$$

for some constant k. So

$$V_{\text{eff}} = -\frac{mk}{r} + \frac{mh^2}{2r^2}.$$

We have two terms of opposite signs and different dependencies on r. For small r, the second term dominates and V_{eff} is large. For large r, the first term dominates. Then V_{eff} asymptotically approaches 0 from below.



The minimum of V_{eff} is at

$$r_* = \frac{h^2}{k}, \quad E_{\min} = -\frac{mk^2}{2h^2}.$$

We have a few possible types of motion:

– If $E = E_{\min}$, then r remains at r_* and $\dot{\theta}h/r^2$ is constant. So we have a uniform motion in a circle.

– If $E_{\min} < E < 0$, then r oscillates and $\dot{r} = h/r^2$ does also. This is a non-circular, bounded orbit.

  Definition (Periapsis, apoapsis and apsides)**.** The points of minimum and maximum r in such an orbit are called the *periapsis* and *apoapsis*. They are collectively known as the *apsides*.

  Definition (Perihelion and aphelion)**.** For an orbit around the Sun, the periapsis and apoapsis are known as the *perihelion* and *aphelion*.

  In particular

  Definition (Perigee and apogee)**.** The perihelion and aphelion of the Earth are known as the *perigee* and *apogee*.

– If $E \geq 0$, then r comes in from ∞, reaches a minimum, and returns to infinity. This is an unbounded orbit.

In the case of motion in an inverse square force, the trajectories are conic sections (circles, ellipses, parabolae and hyperbolae).

4.2.1 Stability of circular orbits

Here we can deduce the stability of circular orbits. It is particularly important that the orbit in an inverse square force is stable!

Now consider a general potential energy $V(r)$. We have to answer two questions:

– Do circular orbits exist?

– If they do, are they stable?

For a circular orbit, $r = r_* = \text{const}$ for some value of $h \neq 0$ (if $h = 0$, then the object is just standing still!).

Since $\ddot{r} = 0$ constant r, we require

$$V'_{\text{eff}}(r_*) = 0.$$

The orbit is stable if r_* is a minimum of V_{eff}, ie.

$$V''_{\text{eff}}(r_*) > 0.$$

In terms of $V(r)$, we require

$$V'(r_*) = \frac{mh^2}{r_*^3}$$

and is stable for

$$V''(r_*) + \frac{3mh^2}{r_*^4} = V''(r_*) + \frac{3}{r_*}V'(r_*) > 0.$$

In terms of the radial force $F(r) = -V'(r)$, the orbit is stable if

$$F'(r_*) + \frac{3}{r}F(r_*) < 0.$$

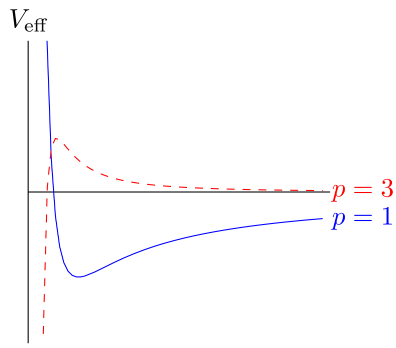
Example. Consider a central force with

$$V(r) = -\frac{mk}{r^p}$$

for some $k, p > 0$. Then

$$V''(r) + \frac{3}{r}V'(r) = \big(-p(p+1) + 3p\big)\frac{mk}{r^{p+2}} = p(2-p)\frac{mk}{r^{p+2}}.$$

So circular orbits are stable for $p < 2$. This is illustrated by the graphs of $V_{\text{eff}}(r)$ for $p = 1$ and $p = 3$.



4.3 Equation of the shape of the orbit

In general, we could determine $r(t)$ by integrating the energy equation

$$E = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r)$$

$$t = \pm\sqrt{\frac{m}{2}}\int\frac{\mathrm{d}r}{\sqrt{E - V_{\text{eff}}(r)}}$$

However, this is usually not practical, because we can't do the integral.

It is usually much easier to find the shape $r(\theta)$ of the orbit.

We can usually simplify the equation by introducing the new variable

Notation.
$$u = \frac{1}{r}.$$

Then
$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\dot{\theta} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\frac{h}{r^2} = -h\frac{\mathrm{d}u}{\mathrm{d}\theta}.$$

Then
$$\ddot{r} = \frac{\mathrm{d}}{\mathrm{d}t}\left(-h\frac{\mathrm{d}u}{\mathrm{d}\theta}\right) = -h\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}\dot{\theta} = -h\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}\frac{h}{r^2} = -h^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}.$$

This doesn't look very linear with u^2, but it will help linearizing the equation when we put in other factors.

The radial equation of motion
$$m\ddot{r} - \frac{mh^2}{r^3} = F(r)$$
then becomes

Proposition (Binet's equation)**.**
$$-mh^2u^2\left(\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u\right) = F\left(\frac{1}{u}\right).$$

For an inverse square force, $F(1/u)$ is proportional to u^2, and the equation is linear!

Given $F(r)$, we aim to solve this second order ODE for $u(\theta)$. If needed, we can then work out the time-dependence via
$$\dot{\theta} = hu^2.$$

4.4 The Kepler problem

4.4.1 Shapes of orbits

For a planet orbiting the sun,
$$V(r) = \frac{mk}{r},\quad F(r) = -\frac{mk}{r^2}$$
with $k = GM$. (For the Coulomb attraction of opposite charges, we have the same equation with $k = -\frac{Qq}{4\pi\epsilon_0 m}$.)

Binet's equation then becomes linear, and
$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = \frac{k}{h^2}.$$

We write the general solution as
$$u = \frac{k}{h^2} + A\cos(\theta - \theta_0),$$

where $A \geq 0$ and θ_0 are arbitrary constants.

If $A = 0$, then u is constant, and the orbit is circular. Otherwise, u reaches a maximum at $\theta = \theta_0$. This is the periapsis. We now re-define polar coordinates such that the periapsis is at $\theta = 0$. Then

Proposition. The orbit of a planet around the sun is given by

$$r = \frac{\ell}{1 + e\cos\theta}, \tag{$*$}$$

with $\ell = h^2/k$ and $e = Ah^2/k$. This is the polar equation of a conic, with a focus (the sun) at the origin.

Definition (Eccentricity)**.** The dimensionless parameter $e \geq 0$ in the equation of orbit is the *eccentricity* and determines the shape of the orbit.

($*$) can be re-written in Cartesian coordinates with $x = r\cos\theta$ and $y = r\sin\theta$ to obtain

$$(1 - e^2)x^2 + 2e\ell x + y^2 = \ell^2. \tag{$\dagger$}$$

There are three different possibilities:

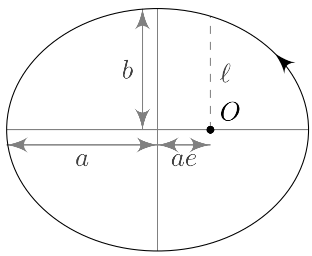
- Ellipse: $(0 \leq e < 1)$. r is bounded by

  $$\frac{\ell}{1+e} \leq r \leq \frac{\ell}{1-e}.$$

  ($\dagger$) can be put into the equation of an ellipse centered on $(-ea, 0)$,

  $$\frac{(x+ea)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

  where $a = \dfrac{\ell}{1-e^2}$ and $b = \dfrac{\ell}{\sqrt{1-e^2}} \leq a$.

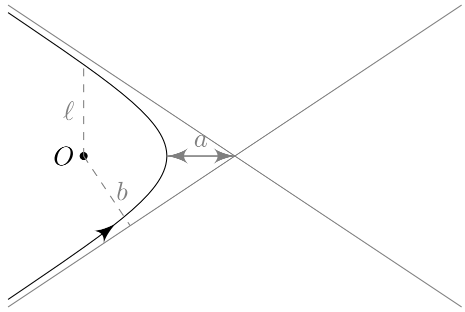
  

  a and b are the semi-major and semi-minor axis. ℓ is the *semi-latus rectum*. One focus of the ellipse is at the origin. If $e = 0$, then $a = b = \ell$ and the ellipse is a circle.

- Hyperbola: $(e > 1)$. For $e > 1$, $r \to \infty$ as $\theta \to \pm\alpha$, where $\alpha = \cos^{-1}(1/e) \in (\pi/2, \pi)$. Then ($\dagger$) can be put into the equation of a hyperbola centered on $(ea, 0)$,

  $$\frac{(x-ea)^2}{a^2} - \frac{y^2}{b^2} = 1,$$

  with $a = \dfrac{\ell}{e^2 - 1}$, $b = \dfrac{\ell}{\sqrt{e^2-1}}$.



This corresponds to an unbound orbit that is deflected (scattered) by an attractive force.

b is both the semi-minor axis and the *impact parameter*. It is the distance by which the planet would miss the object if there were no attractive force.

The asymptote is $y = \frac{b}{a}(x - ea)$, or

$$x\sqrt{e^2 - 1} - y = eb.$$

Alternatively, we have

$$(x, y) \cdot \left(\frac{\sqrt{e^2 - 1}}{e}, -\frac{1}{e}\right) = b$$

or $\mathbf{r} \cdot \mathbf{n} = b$, the equation of a line at a distance b from the origin.

- Parabola: $(e = 1)$. Then $(*)$ becomes

$$r = \frac{\ell}{1 + \cos\theta}.$$

We see that $r \to \infty$ as $\theta \to \pm\pi$. $(\dagger)$ becomes the equation of a parabola, $y^2 = \ell(\ell - 2x)$. The trajectory is similar to that of a hyperbola.

4.4.2 Energy and eccentricity

We can figure out which path a planet follows by considering its energy.

$$\begin{aligned} E &= \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}) - \frac{mk}{r} \\ &= \frac{1}{2}mh^2\left(\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right) + u^2\right) - mku \end{aligned}$$

Substitute $u = \dfrac{1}{\ell}(1 + e\cos\theta)$ and $\ell = \dfrac{h^2}{k}$, and it simplifies to

$$E = \frac{mk}{2\ell}(e^2 - 1),$$

which is independent of θ, as it must be.

So bounded orbits have $e < 1$, and thus $E < 0$. Unbounded orbits have $e > 1$ and $E > 0$. A parabolic orbit has $e = 1$, $E = 0$, and is "marginally bound".

The condition $E > 0$ is equivalent to $|\dot{\mathbf{r}}| > \sqrt{\frac{2GM}{r}} = v_{\text{esc}}$, which means you have enough kinetic energy to escape orbit.

4.4.3 Kepler's laws of planetary motion

Kepler came up with three laws of planetary motion through empirical observation:

Law (Kepler's first law)**.** The orbit of each planet is an ellipse with the Sun at one focus.

Note: In the solar system, planets generally have very low eccentricity (ie very close to circular motion), but asteroids and comets can have very eccentric orbits. In other solar systems, even planets have have highly eccentric orbits.

Law (Kepler's second law)**.** The line between the planet and the sun sweeps out equal areas in equal times.

Law (Kepler's third law)**.** The square of the orbital period is proportional to the cube of the semi-major axis, or

$$P^2 \propto a^3.$$

As we have seen, Law 1 follows from Newtonian dynamics and the inverse-square law of gravity.

Law 2 follows simply from the conservation of angular momentum: The area swept out by moving $\mathrm{d}\theta$ is $\mathrm{d}A = \frac{1}{2}r^2\,\mathrm{d}\theta$ (area of sector of circle). So

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\dot{\theta} = \frac{h}{2} = \text{const.}$$

and is true for *any* central force.

Law 3 follows from this: the total area of the ellipse is $A = \pi ab = \frac{h}{2}P$ (by the second law). But $b^2 = a^2(1-e^2)$ and $h^2 = k\ell = ka(1-e^2)$. So

$$P^2 = \frac{(2\pi)^2 a^4(1-e^2)}{ka(1-e^2)} = \frac{(2\pi)^2 a^3}{k}.$$

4.5 Rutherford scattering

Consider motion in a *repulsive* inverse-square force,

$$V(r) = +\frac{mk}{r}, \quad F(r) = +\frac{mk}{r^2}.$$

For Coulomb repulsion of like charges,

$$k = \frac{Qq}{4\pi\varepsilon_0 m} > 0.$$

The solution is now

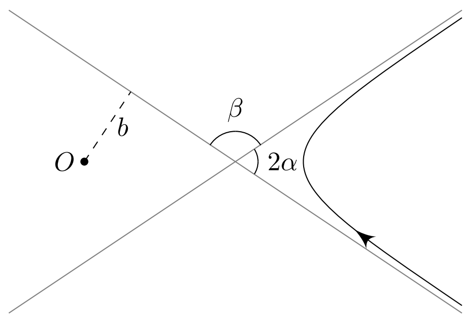
$$u = -\frac{k}{h^2} + A\cos(\theta - \theta_0).$$

We can take $A \geq 0, \theta_0 = 0$ wlog. Then

$$r = \frac{\ell}{e\cos\theta - 1},$$

with
$$\ell = \frac{h^2}{k}, \quad e = \frac{Ah^2}{k}.$$
We know that r and ℓ are positive. So we must have $e \geq 1$. Then $r \to \infty$ as $\theta \to \pm\alpha$, where $\alpha = \cos^{-1}(1/e)$.

The orbit is a hyperbola. Again,
$$\frac{(x - ea)^2}{a^2} - \frac{y^2}{b^2} = 1,$$
with $a = \frac{\ell}{e^2-1}$ and $b = \frac{\ell}{\sqrt{e^2-1}}$, but we consider the other branch of the hyperbola.



It seems as if the particle is deflected by O.

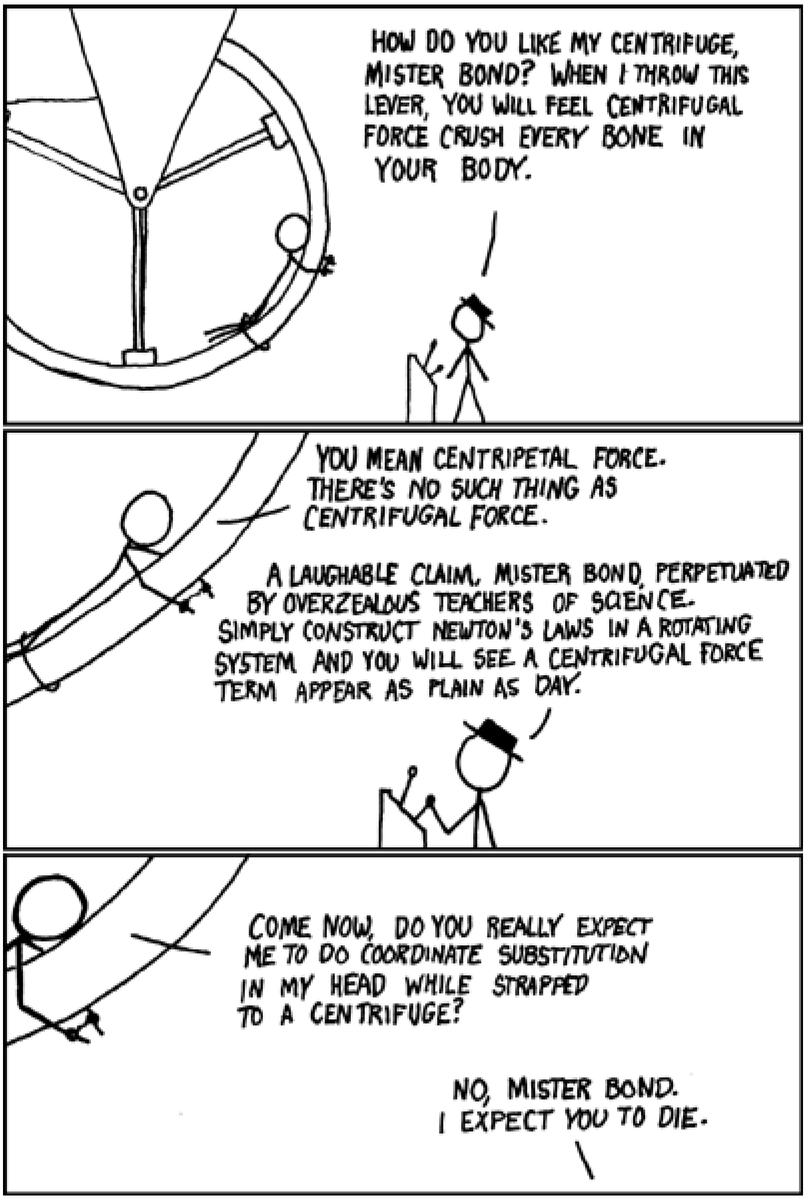
Far from the origin, the orbit tends towards uniform motion with speed v in a straight line. Then angular momentum per unit mass is $h = bv$ (velocity $\times$ perpendicular distance to O.

How does the scattering angle $\beta = \pi = 2\alpha$ depend on the impact parameter b and the incident speed v?

From the above,
$$\frac{1}{e} = \cos\alpha = \cos\left(\frac{\pi}{2} - \frac{\beta}{2}\right) = \sin\left(\frac{\beta}{2}\right),$$
So
$$b = \frac{\ell}{\sqrt{e^2 - 1}} = \frac{(bv)^2}{k}\tan\frac{\beta}{2}.$$
So
$$\beta = 2\tan^{-1}\left(\frac{k}{bv^2}\right).$$

So scattering angles approach π can be obtained for small impact parameters, $b \ll k/v^2$.

5 Rotating frames



Newton's laws hold only in inertial frames. A rotating frame is not inertial and the equations of motion are modified.

Let S be an inertial frame, and let S' be a non-inertial frame, rotating about the z axis with angular velocity $\omega = \dot{\theta}$ with respect to S.

Call the basis vectors $\mathbf{e}_i = \{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ and $\mathbf{e}'_i = \{\hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}'\}$.

Consider a particle at rest in $\mathbf{S}'$. From the perspective of S, its velocity is

$$\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_S = \boldsymbol{\omega} \times \mathbf{r},$$

where $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$ is the *angular velocity vector* (aligned with the rotation axis). This formula also applies to the basis vectors of S'.

$$\left(\frac{\mathrm{d}\mathbf{e}_i'}{\mathrm{d}t}\right)_S = \boldsymbol{\omega} \times \mathbf{e}_i'.$$

A general time-dependent vector $\mathbf{a}$ can be written as

$$\mathbf{a} = \sum a_i'(t)\mathbf{e}_i'.$$

From the perspective of S', with $\mathbf{e}_i'$ constant and its time derivative is

$$\left(\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}\right)_{S'} = \sum \frac{\mathrm{d}a_i'}{\mathrm{d}t}\mathbf{e}_i'.$$

In S, however, $\mathbf{e}_i'$ is not constant, and its time derivative is

$$\left(\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}\right)_S = \sum \frac{\mathrm{d}a_i}{\mathrm{d}t}\mathbf{e}_i' + \sum a_i'\boldsymbol{\omega} \times \mathbf{e}_i' = \left(\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega} \times \mathbf{a}.$$

This key identity applies to all vectors and can be written as an operator equation:

Proposition. If S is an inertial frame, and S' is rotating relative to S with angular velocity $\boldsymbol{\omega}$,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_S = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega} \times .$$

Applied to the position vector $\mathbf{r}(t)$ of a particle, it gives

$$\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_S = \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega} \times \mathbf{r}.$$

So the difference in velocity measured in the two frames is the relative velocity of the frames, which depends on position.

Applied a second time, and allowing for $\boldsymbol{\omega}$ to depend on time, it gives

$$\begin{aligned}\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_S &= \left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega}\times\right)\left(\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega} \times \mathbf{r}\right).\\ &= \left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{S'} + 2\boldsymbol{\omega} \times \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\end{aligned}$$

Since S is inertial, Newton's Second Law is

$$m\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_S = \mathbf{F}.$$

So

Proposition.

$$m\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right) = \mathbf{F} - 2m\boldsymbol{\omega} \times \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'} - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

Definition (Fictious forces)**.** The additional terms on the RHS of the equation of motion in rotating frames are *fictitious forces*, and are needed to explain the motion observed in S'. They are

- *Coriolis force*: $-2m\boldsymbol{\omega}\times\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'}$.
- *Euler force*: $-m\dot{\boldsymbol{\omega}}\times\mathbf{r}$
- *Centrifugal force*: $-m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$.

Usually we consider $\boldsymbol{\omega}$ to be constant and can neglect the Euler force.

A frame fixed to the Earth rotates with angular velocity

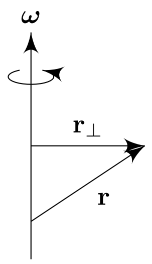
$$\omega=\frac{2\pi}{1\text{ day}}\approx 7.3\times 10^{-5}\,\mathrm{s}^{-1}.$$

5.1 The centrifugal force

Let $\boldsymbol{\omega}=\omega\hat{\boldsymbol{\omega}}$, where $|\hat{\boldsymbol{\omega}}|=1$. Then

$$-m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})=-m\big((\boldsymbol{\omega}\cdot\mathbf{r})\boldsymbol{\omega}-(\boldsymbol{\omega}\cdot\boldsymbol{\omega})\mathbf{r}\big)=m\omega^2\mathbf{r}_\perp,$$

where $\mathbf{r}_\perp=\mathbf{r}-(\mathbf{r}\cdot\hat{\boldsymbol{\omega}})\hat{\boldsymbol{\omega}}$, the projection of the position on the plane perpendicular to $\boldsymbol{\omega}$. So the centrifugal force is directed away from the axis of rotation and its magnitude is $m\omega^2$ times the distance form the axis.



Note that

$$\begin{aligned}\mathbf{r}_\perp\cdot\mathbf{r}_\perp&=\mathbf{r}\cdot\mathbf{r}-(\mathbf{r}\cdot\hat{\boldsymbol{\omega}})^2\\ \nabla(|\mathbf{r}_\perp|^2)&=2\mathbf{r}-2(\mathbf{r}\cdot\hat{\boldsymbol{\omega}})\hat{\boldsymbol{\omega}}=2\mathbf{r}_\perp.\end{aligned}$$

So

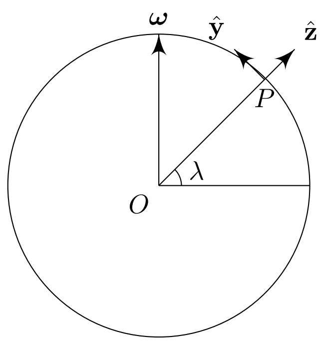
$$-m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})=-\nabla\left(-\frac{1}{2}m\omega^2|\mathbf{r}_\perp|^2\right)=-\nabla\left(-\frac{1}{2}m|\boldsymbol{\omega}\times\mathbf{r}|^2\right).$$

On a rotating planet, the gravitational and centrifugal forces per unit mass combine to make the *effective gravity*,

$$\mathbf{g}_{\text{eff}}=\mathbf{g}+\omega^2\mathbf{r}_\perp.$$

This gravity will not be vertically downwards. Consider a point P at latitude λ on the surface of a spherical planet of radius R.

We construct orthogonal axes:



with $\hat{\mathbf{x}}$ into the page. So $\hat{\mathbf{z}}$ is up, $\hat{\mathbf{y}}$ is North, and $\hat{\mathbf{x}}$ is East.

At P, we have

$$\begin{aligned}\mathbf{r} &= R\hat{\mathbf{z}}\\ \mathbf{g} &= -g\hat{\mathbf{z}}\\ \boldsymbol{\omega} &= \omega(\cos\lambda\hat{\mathbf{y}} + \sin\lambda\hat{\mathbf{z}})\end{aligned}$$

So

$$\begin{aligned}\mathbf{g}_{\text{eff}} &= \mathbf{g} + \omega^2\mathbf{r}_\perp\\ &= -g\hat{\mathbf{z}} + \omega^2 R\cos\lambda(\cos\lambda\hat{\mathbf{z}} - \sin\lambda\hat{\mathbf{y}})\\ &= -\omega^2 R\cos\lambda\sin\lambda\hat{\mathbf{y}} - (g - \omega^2 R\cos^2\lambda)\hat{\mathbf{z}}.\end{aligned}$$

So the angle α between $\mathbf{g}$ and $\mathbf{g}_{\text{eff}}$ is given by

$$\tan\alpha = \frac{\omega^2 R\cos\lambda\sin\lambda}{g - \omega^2 R\cos^2\lambda}.$$

This is 0 at the equator and the poles, and greatest when you are halfway between. However, this is still tiny on Earth, and does not affect our daily life.

5.2 The Coriolis force

This depends on the velocity in the rotating frame, $\mathbf{v} = \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'}$:

$$\mathbf{F} = -2m\boldsymbol{\omega}\times\mathbf{v},$$

like the Lorentz force on a charged particle in a magnetic field. However, this does not mean that particles will go around in circles, as in the case of magnetic fields, since we also have the centrifugal force in play.

Since this is always perpendicular to the velocity, it does no work.

Consider potion parallel to the Earth's surface. We only care about the effect of the Coriolis force on the horizontal trajectory, and ignore the vertical component that is tiny compared to gravity.

So we only take the vertical component of $\boldsymbol{\omega}$, $\omega\sin\lambda\hat{\mathbf{z}}$. The horizontal velocity $\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}}$ generates a horizontal Coriolis force:

$$-2m\omega\sin\lambda\hat{\mathbf{z}}\times\mathbf{v} = -2m\omega\sin\lambda(v_y\hat{\mathbf{x}} - v_x\hat{\mathbf{y}}).$$

In the Northern hemisphere ($0 < \lambda < \pi/2$), this causes a deflection towards the right. In the Southern Hemisphere, the deflection is to the left. The effect vanishes at the equator.

Note: Only the horizontal effect of horizontal motion vanishes at the equator. The vertical effects or those caused by vertical motion still exist.

Example. Suppose a ball is dropped from a tower of height h at the equator. Where does it land?

In the rotating frame,

$$\ddot{\mathbf{r}} = \mathbf{g} - 2\boldsymbol{\omega} \times \dot{\mathbf{r}} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

We work to first order in ω. Then

$$\ddot{\mathbf{r}} = \mathbf{g} - 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + O(\omega^2).$$

Integrate wrt t to obtain

$$\dot{\mathbf{r}} = \mathbf{g}t - 2\boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_0) + O(\omega^2),$$

where $\mathbf{r}_0$ is the initial position. We substitute into the original equation to obtain

$$\dot{\mathbf{r}} = \mathbf{g}\boldsymbol{\omega} \times \mathbf{g}t + O(\omega^2).$$

(where some new ω^2 terms are thrown into $O(\omega^2)$). We integrate twice to obtain

$$\mathbf{r} = \mathbf{r}_0 + \frac{1}{2}\mathbf{g}t^2 - \frac{1}{3}\boldsymbol{\omega} \times \mathbf{g}t^3 + O(\omega^2).$$

In components, we have $\mathbf{g} = (0, 0, -g)$, $\boldsymbol{\omega} = (0, \omega, 0)$ and $\mathbf{r}_0 = (0, 0, R + h)$. So

$$\mathbf{r} = \left(\frac{1}{3}\omega g t^3, 0, R + h - \frac{1}{2}gt^2\right) + O(\omega^2).$$

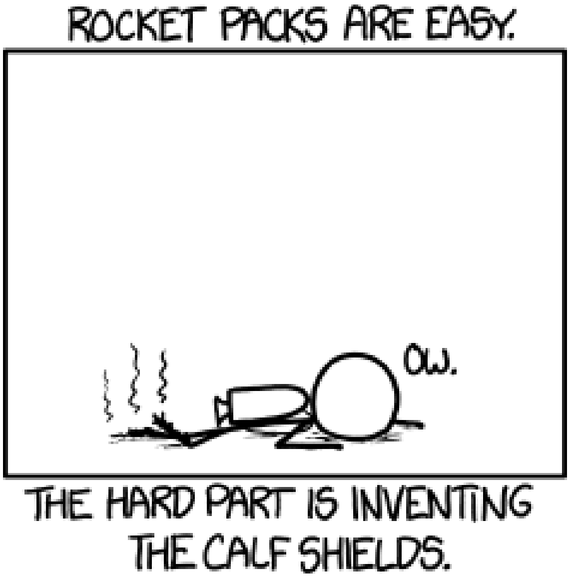
So the particle hits the ground at $t = \sqrt{2h/g}$, and its eastward displacement is $\frac{1}{3}wg\left(\frac{2h}{g}\right)^{3/2}$.

This can be understood in terms of angular momentum conservation in the non-rotating frame. At the beginning, the particle has the same angular velocity with the Earth. As it falls towards the Earth, to maintain the same angular momentum, the angular velocity has to increase to compensate for the decreased radius. So it spins faster than the Earth and drifts towards the East, relative to the Earth.

Example. Consider a pendulum that is free to swing in any plane, eg. a weight on a string. At the North pole, it will swing in a plane that is fixed in an inertial frame, while the Earth rotates beneath it. From the perspective of the rotating frame, the plane of the pendulum rotates backwards. This can be explained as a result of the Coriolis force.

At latitude λ, the plane rotates with period $\frac{1\text{ day}}{\sin\lambda}$ (the Coriolis force makes the particle deflect to the right. So the plane with keep turning rightwards as well).

6 Systems of particles



xkcd (Randall Munroe)

Consider a system of N interacting particles. Particle i has mass m_i, position $\mathbf{r}_i$, and momentum $\mathbf{p}_i = m_i\dot{\mathbf{r}}_i$. Note that the subscript denotes which particle it is referring to, not vector components.

Newton's Second Law for particle i is

$$m_i\ddot{\mathbf{r}}_i = \dot{\mathbf{p}}_i = \mathbf{F}_i,$$

where

$$\mathbf{F}_i = \mathbf{F}_i^{\text{ext}} + \sum_{j=1}^N \mathbf{F}_{ij},$$

where $\mathbf{F}_{ij}$ is the force on particle i by particle j, and $\mathbf{F}_i^{\text{ext}}$ is the external force on i, which comes from particles outside the system.

Since a particle cannot exert a force on itself, so $\mathbf{F}_{ii} = \mathbf{0}$. Also, Newton's third law requires that

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji}.$$

If we represent the forces as a matrix, then the matrix is antisymmetric.

For example, for gravitational forces,

$$\mathbf{F}_{ij} = -\frac{Gm_im_j(\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^3} = -\mathbf{F}_{ji}.$$

6.1 Motion of the center of mass

Definition (Total mass)**.** The *total mass* of the system is $M = \sum m_i$.

Definition (Center of mass)**.** The *center of mass* is located at

$$\mathbf{R} = \frac{1}{M}\sum_{i=1}^N m_i\mathbf{r}_i,$$

ie. the mass-weighted average position.

Definition (Total linear momentum)**.** The *total linear momentum* is

$$\mathbf{P} = \sum_{i=1}^{N} \mathbf{p}_i = \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i = M\dot{\mathbf{R}}.$$

This is equivalent to that of a single particle of mass M at the center of mass.

Definition (Total external force)**.** The *total external force* is

$$\mathbf{F} = \sum_{i=1}^{N} \mathbf{F}_i^{\mathrm{ext}}.$$

Then

Proposition.

$$M\ddot{\mathbf{R}} = \mathbf{F}.$$

Proof.

$$\begin{aligned} M\ddot{\mathbf{R}} &= \dot{\mathbf{P}} \\ &= \sum_{i=1}^{N} \dot{\mathbf{p}}_i \\ &= \sum_{i=1}^{N} \mathbf{F}_i^{\mathrm{ext}} + \sum_{i=1}^{N}\sum_{j=1}^{N} \mathbf{F}_{ij} \\ &= \mathbf{F} + \frac{1}{2}\sum_i \sum_j (\mathbf{F}_{ij} + \mathbf{F}_{ji}) \\ &= \mathbf{F} \end{aligned}$$

□

So the center of mass moves as a particle of mass M subject to a force $\mathbf{F}$. This is why Newton's Laws apply to macroscopic objects even though they are not individual particles.

Law (Conservation of momentum)**.** If $\mathbf{F} = \mathbf{0}$, then $\dot{\mathbf{P}} = \mathbf{0}$. So the total momentum is conserved.

Definition (Center of mass frame)**.** The *center of mass frame* is an inertial frame in which $\mathbf{R} = 0$ for all time.

Definition (Total angular momentum)**.** The *total angular momentum* of the system about the origin is

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i.$$

So

$$\begin{aligned}
\dot{\mathbf{L}} &= \sum_i \mathbf{r}_i \times \dot{\mathbf{p}}_i + \dot{\mathbf{r}}_i \times \mathbf{p}_i \\
&= \sum_i \mathbf{r}_i \times \left(\mathbf{F}_i^{\text{ext}} + \sum_j F_{ij}\right) + m(\dot{\mathbf{r}}_i \times \dot{\mathbf{r}}_i) \\
&= \sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} + \sum_i \sum_j \mathbf{r}_i \times \mathbf{F}_{ij} \\
&= \sum_i \mathbf{G}_i^{\text{ext}} + \frac{1}{2}\sum_i \sum_j (\mathbf{r}_i \times \mathbf{F}_{ij} + \mathbf{r}_j \times \mathbf{F}_{ji}) \\
&= \mathbf{G} + \frac{1}{2}\sum_i \sum_j (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij}
\end{aligned}$$

Usually the second sum is zero (see later), and we have

$$= \mathbf{G},$$

where

Definition (Total external torque)**.** The *total external torque* is

$$\mathbf{G} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}}.$$

The last line holds provided the "strong version" of Newton's Third Law holds:

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji} \text{ and is parallel to } (\mathbf{r}_i - \mathbf{r}_j).$$

This is true, at least, for gravitational and electrostatic forces. So the total angular momentum is conserved if $\mathbf{G} = \mathbf{0}$, ie the total external torque vanishes.

6.2 Motion relative to the center of mass

Let $\mathbf{r}_i = \mathbf{R} + \mathbf{r}_i^c$, where $\mathbf{r}_i^c$ is the position of particle i relative to the center of mass.

Then

$$\sum_i m_i \mathbf{r}_i^c = \sum m_i \mathbf{r}_i - \sum m_i \mathbf{R} = M\mathbf{R} - M\mathbf{R} = \mathbf{0}.$$

Also,

$$\sum_i m_i \dot{\mathbf{r}}_i^c = \mathbf{0}.$$

Using the equalities above, we can express the total linear momentum, angular momentum and kinetic energy in terms of $\mathbf{R}$ and $\mathbf{r}_i^c$ only:

$$\begin{aligned}
\mathbf{P} &= \sum_i m_i(\dot{\mathbf{R}} + \dot{\mathbf{r}}_i^c) = M\dot{\mathbf{R}}\\
\mathbf{L} &= \sum_i m_i(\mathbf{R} + \mathbf{r}_i^c) \times (\dot{\mathbf{R}} + \dot{\mathbf{r}}_i^c)\\
&= \sum_i m_i \mathbf{R} \times \dot{\mathbf{R}} + \mathbf{R} \times \sum_i m_i \dot{\mathbf{r}}_i^c + \sum_i m_i \mathbf{r}_i^c \times \dot{\mathbf{R}} + \sum_i m_i \mathbf{r}_i^c \times \dot{\mathbf{r}}_i^c\\
&= M\mathbf{R} \times \dot{\mathbf{R}} + \sum_i m_i \mathbf{r}_i^c \times \dot{\mathbf{r}}_i^c\\
T &= \frac{1}{2}\sum_i m_i |\dot{\mathbf{r}}_i|^2\\
&= \frac{1}{2}\sum_i m_I (\dot{\mathbf{R}} + \dot{\mathbf{r}}_i{}^c) \cdot (\dot{\mathbf{R}} + \dot{\mathbf{r}}_i^c)\\
&= \frac{1}{2}\sum_i m_i \dot{\mathbf{R}} \cdot \dot{\mathbf{R}} + \dot{\mathbf{R}} \cdot \sum_i m_i \dot{\mathbf{r}}_i^c + \frac{1}{2}\sum_i m_i \dot{\mathbf{r}}_i^c \cdot \dot{\mathbf{r}}_i^c \cdot \dot{\mathbf{r}}_i^c\\
&= \frac{1}{2}M|\dot{\mathbf{R}}|^2 + \frac{1}{2}\sum_i m_i |\dot{\mathbf{r}}_i^c|^2
\end{aligned}$$

In the case of the angular momentum and the kinetic energy, we see that they are composed of two parts - that of the center of mass and that of motion relative to center of mass.

If the forces are conservative in the sense that

$$\mathbf{F}_i^{\text{ext}} = -\nabla_i V_i(\mathbf{r}_i),$$

and

$$\mathbf{F}_{ij} = -\nabla_i V_{ij}(\mathbf{r}_i - \mathbf{r}_j),$$

where ∇_i is the gradient with respect to $\mathbf{r}_i$, then energy is conserved in the from

$$E = T + \sum_i V_i(\mathbf{r}_i) + \frac{1}{2}\sum_i\sum_j V_{ij}(\mathbf{r}_i - \mathbf{r}_j) = \text{const.}$$

6.3 The two-body problem

Consider two particles with no external forces. The center of mass is at

$$\mathbf{R} = \frac{1}{M}(m_i\mathbf{r}_i + m_2\mathbf{r}_2),$$

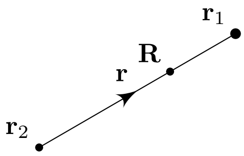
where $M = m_1 + m_2$.

Define the separation vector (or relative position vector)

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2.$$

Then

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_2}{M}\mathbf{r}, \quad \mathbf{r}_2 = \mathbf{R} - \frac{m_1}{M}\mathbf{r}.$$



Since $\mathbf{F} = \mathbf{0}$, $\ddot{\mathbf{R}} = \mathbf{0}$, ie. the center of mass moves uniformly.

Meanwhile,

$$\begin{aligned}
\ddot{\mathbf{r}} &= \ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2\\
&= \frac{1}{m_1}\mathbf{F}_{12} - \frac{1}{m_2}\mathbf{F}_{21}\\
&= \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\mathbf{F}_{12}
\end{aligned}$$

We can write this as

$$\mu\ddot{\mathbf{r}} = \mathbf{F}_{12}(\mathbf{r}),$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

is the *reduced mass.* This is the same as the equation of motion for *one particle* of mass μ with position vector $\mathbf{r}$ relative to a fixed origin - as studied previously.

For example, with gravity,

$$\mu\ddot{\mathbf{r}} = -\frac{Gm_1m_2\hat{\mathbf{r}}}{|\mathbf{r}|^2}.$$

So

$$\ddot{\mathbf{r}} = -\frac{GM\hat{\mathbf{r}}}{|\mathbf{r}|^2}.$$

For example, give a planet orbiting the Sun, both the planet and the sun moves in ellipses about their center of mass. The orbital period depends on the total mass.

It can be shown that

$$\begin{aligned}
\mathbf{L} &= M\mathbf{R}\times\dot{\mathbf{R}} + \mu\mathbf{r}\times\dot{\mathbf{r}}\\
T &= \frac{1}{2}M|\dot{\mathbf{R}}|^2 + \frac{1}{2}\mu|\dot{\mathbf{r}}|^2
\end{aligned}$$

by expressing $\mathbf{r}_1$ and $\mathbf{r}_2$ in terms of $\mathbf{r}$ and $\mathbf{R}$.

6.4 Variable-mass problem

Rockets, fireworks, falling raindrops, rolling snowballs, are all objects whose mass decreases or increases with time.

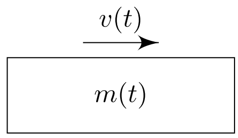
Newton's second law is

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F}, \quad \text{with } \mathbf{p} = m\dot{\mathbf{r}}.$$

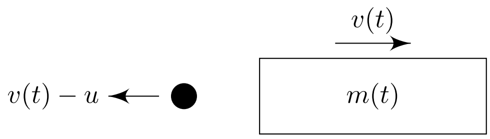
But we cannot simply apply this with $m = m(t)$, because these systems are not closed.

Consider a rocket moving in one dimension with mass $m(t)$ and velocity $v(t)$. The rocket propels itself forwards by burning fuel and ejecting the exhaust at velocity $-u$ relative to the rocket.

At time t:



At time $t + \delta t$, it ejects exhaust of mass $m(t) - m(t + \delta t)$ with velocity $v(t) - u + O(\delta t)$.



The change in total momentum of the system (rocket + exhaust) is

$$\begin{aligned}
\delta p &= m(t+\delta t)v(t+\delta t) + [m(t) - m(t+\delta t)][v(t) - u(t) + O(\delta t)] - m(t)v(t)\\
&= (m + \dot{m}\delta t + O(\delta t^2))(v + \dot{v}\delta t + O(\delta t^2)) - \dot{m}\delta t(v - u) + O(\delta t^2) - mv\\
&= (\dot{m}v + m\dot{v} - \dot{m}v + \dot{m}u)\delta t + O(\delta t^2)\\
&= (m\dot{v} + \dot{m}u)\delta t + O(\delta t^2).
\end{aligned}$$

Newton's second law gives

$$\lim_{\delta \to 0} \frac{\delta p}{\delta t} = F \text{ (the external force on rocket)}$$

So

Proposition (Rocket equation)**.**

$$m\frac{\mathrm{d}v}{\mathrm{d}t} + u\frac{\mathrm{d}m}{\mathrm{d}t} = F.$$

Example. When $F = 0$ (and u is constant), we have

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -u\frac{\mathrm{d}m}{dt}.$$

So

$$v = v_0 + u\log\left(\frac{m_0}{m(t)}\right),$$

Note that are expressing things in terms of the mass remaining m, not time t.

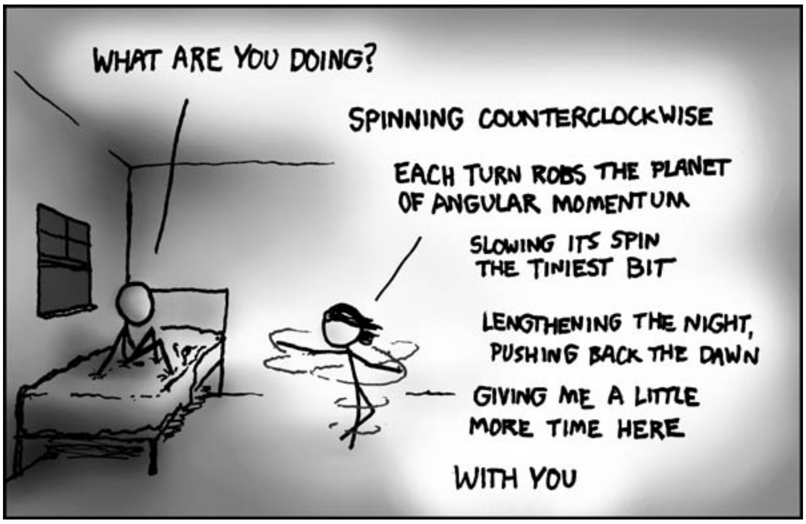
Note also that the velocity does not depend on the rate at which mass is ejected, only the velocity at which it is ejected.

Example. Consider a falling raindrop of mass $m(t)$, gathering mass from a stationary cloud. In this case, $u = v$. So

$$m\frac{\mathrm{d}v}{\mathrm{d}t} + v\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(mv) = mg,$$

with v measured downwards. To solve this, we need a model to determine the rate at which the raindrop gathers mass.

7 Rigid bodies



Definition (Rigid body). A *rigid body* is an extended object, consisting of N particles that are constrained such that the distance between any pair of particles, $|\mathbf{r}_i - \mathbf{r}_j|$, is fixed.

Intuitively, it is a solid object that cannot deform.

The possible motions of a rigid body are the continuous isometries of Euclidean space, ie. translations and rotations.

7.1 Angular velocity

Consider a single particle moving in a circle of radius s about the z axis. Its position and velocity vectors are

$$\begin{aligned}\mathbf{r} &= (s\cos\theta, s\sin\theta, z)\\ \dot{\mathbf{r}} &= (-s\dot\theta\sin\theta, s\dot\theta\cos\theta, 0).\end{aligned}$$

We can write

$$\dot{\mathbf{r}} = \boldsymbol{\omega}\times\mathbf{r},$$

where

$$\boldsymbol{\omega} = \dot\theta\hat{\mathbf{z}}$$

is the angular velocity vector.

In general,

$$\boldsymbol{\omega} = \dot\theta\hat{\mathbf{n}} = \omega\hat{\mathbf{n}},$$

where $\hat{\mathbf{n}}$ is a unit vector parallel to the rotation axis.

The kinetic energy of this particle is

$$\begin{aligned} T &= \frac{1}{2}m|\dot{\mathbf{r}}|^2 \\ &= \frac{1}{2}ms^2\dot{\theta}^2 \\ &= \frac{1}{2}I\omega^2 \end{aligned}$$

where $I = ms^2$ is the *moment of inertia*. This is the counterpart of "mass" in rotational motion (cf. $\frac{1}{2}mv^2$ vs $\frac{1}{2}I\omega^2$).

Definition (Moment of inertia)**.** The *moment of inertia* of a particle is

$$I = ms^2 = m|\hat{\mathbf{n}} \times \mathbf{r}|^2,$$

where s is the distance of the particle from the axis of rotation.

7.2 Moment of inertia

Consider a rigid body in which all N particles rotate about the same axis with the same angular velocity:

$$\dot{\mathbf{r}}_i = \boldsymbol{\omega} \times \mathbf{r}_i.$$

This ensures that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{r}_i - \mathbf{r}_j|^2 = 2(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j) \cdot (\mathbf{r}_i - \mathbf{r}_j) = 2\big(\boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{r}_j)\big) \cdot (\mathbf{r}_i - \mathbf{r}_j) = 0,$$

as required for a rigid body.

The rotational kinetic energy is

$$T = \frac{1}{2}\sum_{i=1}^N m_i|\dot{\mathbf{r}}_i|^2 = \frac{1}{2}I\omega^2,$$

where

Definition (Moment of inertia)**.** The *moment of inertia* of a rigid body about the rotation axis $\hat{\mathbf{n}}$ is

$$I = \sum_{i=1}^N m_i s_i^2 = \sum_{i=1}^N m_i|\hat{\mathbf{n}} \times \mathbf{r}_i|^2.$$

Definition. The angular momentum is

$$\mathbf{L} = \sum_i m_i\mathbf{r}_i \times \dot{\mathbf{r}}_i = \sum_i m_i\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i).$$

If we write

$$\boldsymbol{\omega} = \omega\hat{\mathbf{n}},$$

then the component of $\mathbf{L}$ in the direction of $\hat{\mathbf{n}}$ is

$$\begin{aligned} \mathbf{L} \cdot \hat{\mathbf{n}} &= \omega\sum_i m_i\hat{\mathbf{n}} \cdot (\mathbf{r}_i \times (\hat{\mathbf{n}} \times \mathbf{r}_i)) \\ &= \omega\sum_i m(\hat{\mathbf{n}} \times \mathbf{r}_i) \cdot (\hat{\mathbf{n}} \times \mathbf{r}_i) \\ &= I\omega \end{aligned}$$

However, $\mathbf{L}$ may not be parallel to to $\boldsymbol{\omega}$ in general. Using vector identities, we have

$$\mathbf{L} = \sum_i m_i\big((\mathbf{r}_i \cdot \mathbf{r}_i)\boldsymbol{\omega} - (\mathbf{r}_i \cdot \boldsymbol{\omega})\mathbf{r}_i\big)$$

Note that this is a linear function of $\boldsymbol{\omega}$. So we can write

$$\mathbf{L} = I\boldsymbol{\omega}$$

where I is the *inertia tensor* represented by a symmetric matrix with components

$$I_{jk} = \sum_i m_i(|\mathbf{r}_i|^2\delta_{jk} - (\mathbf{r}_i)_j(\mathbf{r}_i)_k),$$

where i refers to the index of the particle, and j, k are dummy suffixes summed over.

If the body rotates about a *principal axis*, ie. one of the three orthogonal eigenvectors of I, (eg. an axis of rotational symmetry if the body has one), then $\mathbf{L}$ is parallel to $\boldsymbol{\omega}$.

The relations $T = \frac{1}{2}I\omega^2$ and $L = I\omega$ for angular motion are analogous to the relations $T = \frac{1}{2}mv^2$ and $p = mv$ for linear motion.

7.3 Calculating the moment of inertia

For a solid body, we usually want to think of it as a continuous substance with a mass density, instead of individual point particles. So we replace the sum of particles by a volume integral weighted by the mass density $\rho(\mathbf{r})$.

Definition (Mass, center of mass and moment of inertia)**.** The *mass* is

$$M = \int \rho \;\mathrm{d}V.$$

The *center of mass* is

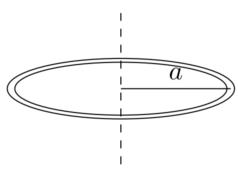
$$\mathbf{R} = \frac{1}{M}\int \rho\mathbf{r} \;\mathrm{d}V$$

The *moment of inertia* is

$$I = \int \rho s^2 \;\mathrm{d}V = \int \rho|\hat{\mathbf{n}} \times \mathbf{r}|^2 \;\mathrm{d}V.$$

In theory, we can study inhomogeneous bodies with varying ρ, but usually we mainly consider homogeneous ones with constant ρ throughout.

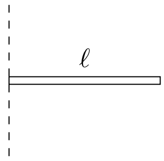
Example (Thin circular ring)**.** Suppose the ring has mass M and radius a, and a rotation axis through the center, perpendicular to the plane of the ring.



Then the moment of inertia is

$$I = Ma^2.$$

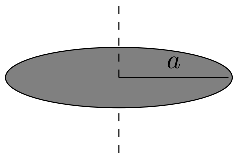
Example (Thin rod)**.** Suppose a rod has mass M and length ℓ. It rotates through one end, perpendicular to the rod.



The mass per unit length is M/ℓ. So the moment of inertia is

$$I = \int_0^\ell \frac{M}{\ell} x^2\,\mathrm{d}x = \frac{1}{3}M\ell^2.$$

Example (Thin disc)**.** Consider a disc of mass M and radius a, with a rotation axis through the center, perpendicular to the plane of the disc.



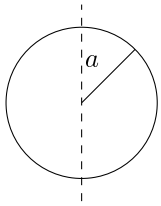
Then

$$\begin{aligned}
I &= \int_0^{2\pi}\int_0^a \underbrace{\frac{M}{\pi a^2}}_{\text{mass per unit length}} \underbrace{r^2}_{s^2} \underbrace{r\,\mathrm{d}r\,\mathrm{d}\theta}_{\text{area element}}\\
&= \frac{M}{\pi a^2}\int_0^a r^3\,\mathrm{d}r\int_0^{2\pi}\mathrm{d}\theta\\
&= \frac{M}{\pi a^2}\frac{1}{4}a^4(2\pi)\\
&= \frac{1}{2}Ma^2.
\end{aligned}$$

Now suppose that the rotation axis is in the plane of the disc instead (also rotating through the center). Then

$$\begin{aligned}
I &= \int_0^{2\pi}\int_0^a \underbrace{\frac{M}{\pi a^2}}_{\text{mass per unit length}} \underbrace{(r\sin\theta)^2}_{s^2} \underbrace{r\,\mathrm{d}r\,\mathrm{d}\theta}_{\text{area element}}\\
&= \frac{M}{\pi a^2}\int_0^a r^3\,\mathrm{d}r\int_0^{2\pi}\sin^2\theta\,\mathrm{d}\theta\\
&= \frac{M}{\pi a^2}\frac{1}{4}a^4\pi\\
&= \frac{1}{4}Ma^2.
\end{aligned}$$

Example. Consider a solid sphere with mass M, radius a, with a rotation axis though the center.



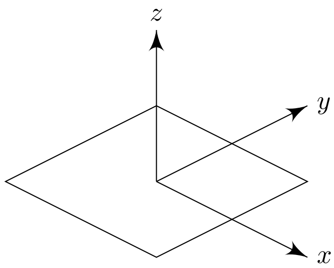
Using spherical polar coordinates (r, θ, ϕ) based on the rotation axis,

$$\begin{aligned}
I &= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \underbrace{\frac{M}{\frac{4}{3}\pi a^3}}_{\rho}\underbrace{(r\sin\theta)^2}_{s^2}\underbrace{r^2\sin\theta\;\mathrm{d}r\;\mathrm{d}\theta\;\mathrm{d}\phi}_{\text{volume element}}\\
&= \frac{M}{\frac{4}{3}\pi a^3}\int_0^a r^4\;\mathrm{d}r\int_0^{\pi}(1-\cos^2)\sin\theta\;\mathrm{d}\theta\int_0^{2\pi}\;\mathrm{d}\phi\\
&= \frac{M}{\frac{4}{3}\pi a^3}\cdot\frac{1}{5}a^5\cdot\frac{4}{3}\cdot 2\pi\\
&= \frac{2}{5}Ma^2.
\end{aligned}$$

Theorem (Perpendicular axis theorem)**.** For a two-dimensional object (a lamina), and three perpendicular axes x, y, z through the same spot, with z normal to the plane,

$$I_z = I_x + I_y,$$

where I_z is the moment of inertia about the z axis.



Note: this does NOT apply to 3D objects! For example, in a sphere, $I_x = I_y = I_z$.

Proof. Let ρ be the mass per unit volume. Then

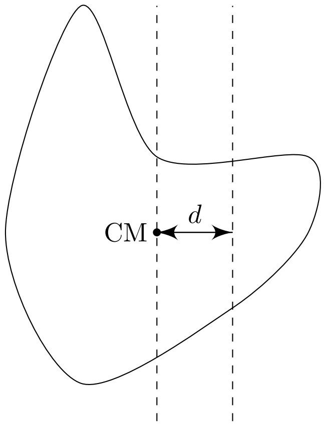
$$\begin{aligned}
I_x &= \int \rho y^2\;\mathrm{d}A\\
I_y &= \int \rho x^2\;\mathrm{d}A\\
I_z &= \int \rho(x^2+y^2)\;\mathrm{d}A = I_x + I_y.
\end{aligned}$$

□

Example. For a disc, $I_x = I_y$ by symmetry. So $I_z = 2I_x$.

Theorem (Parallel axis theorem)**.** If a rigid body of mass M has moment of inertia I^C about an axis passing through the center of mass, then its moment of inertia about a parallel axis a distance d away is

$$I = I^C + Md^2.$$



Proof. With a convenient choice of Cartesian coordinates such that the center of mass is at the origin and the two rotation axes are $x = y = 0$ and $x = d, y = 0$,

$$I^C = \int \rho(x^2 + y^2)\;\mathrm{d}V,$$

and

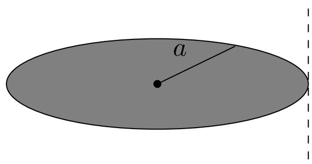
$$\int \rho\mathbf{r}\;\mathrm{d}V = \mathbf{0}.$$

So

$$\begin{aligned}
I &= \int \rho((x-d)^2 + y^2)\;\mathrm{d}V\\
&= \int \rho(x^2+y^2)\;\mathrm{d}V - 2d\int \rho x\;\mathrm{d}V + \int d^2\rho\;\mathrm{d}V\\
&= I^c + 0 + Md^2\\
&= I^c + Md^2.
\end{aligned}$$

□

Example. Take a disc of mass M and radius a, and rotation axis through a point on the circumference, perpendicular to the plane of the disc. Then



$$I = I^c + Ma^2 = \frac{1}{2}Ma^2 + Ma^2 = \frac{3}{2}Ma^2.$$

7.4 Motion of a rigid body

The general motion of a rigid body can be described as a translation of its centre of mass, following a trajectory $\mathbf{R}(t)$, together with a rotation about an axis through the center of mass. We write

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}_i^c.$$

Then

$$\dot{\mathbf{r}}_i = \dot{\mathbf{R}} + \dot{\mathbf{r}}_i^c.$$

Using this, we can break down the velocity and kinetic energy into translational and rotational parts.

If the body rotates with angular velocity $\boldsymbol{\omega}$ about the center of mass, then

$$\dot{\mathbf{r}}_i^c = \boldsymbol{\omega} \times \mathbf{r}_i^c.$$

Since $\mathbf{r}_i^c = \mathbf{r}_i - \mathbf{R}$, we have

$$\dot{\mathbf{r}}_i = \dot{\mathbf{R}} + \boldsymbol{\omega} \times \mathbf{r}_i^c = \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{R}).$$

On the other hand, the kinetic energy (calculated in previous lectures) is

$$\begin{aligned} T &= \frac{1}{2}M|\dot{\mathbf{R}}|^2 + \frac{1}{2}\sum_i m_i|\dot{\mathbf{r}}_i^c|^2 \\ &= \underbrace{\frac{1}{2}M|\dot{\mathbf{R}}|^2}_{\text{translational KE}} + \underbrace{\frac{1}{2}I^c\omega^2}_{\text{rotational}KE}. \end{aligned}$$

Sometimes we do not want to use the center of mass as the center. For example, if an item is held at the edge and spun around, we'd like to study the motion about the point at which the item is held, and not the center of mass.

So consider any point Q, with position vector $\mathbf{Q}(t)$ that is not the center of mass but moves with the rigid body, ie.

$$\dot{\mathbf{Q}} = \dot{\mathbf{R}} + \boldsymbol{\omega} \times (\mathbf{Q} - \mathbf{R}).$$

Usually this is a point inside the object itself, but we do not assume that in our calculation.

Then we can write

$$\begin{aligned} \dot{\mathbf{r}}_i &= \dot{\mathbf{R}} + \boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{R}) \\ &= \dot{\mathbf{Q}} - \boldsymbol{\omega} \times (\mathbf{Q} - \mathbf{R}) + \boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{R}) \\ &= \dot{\mathbf{Q}} + \boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{Q}). \end{aligned}$$

Therefore the motion can be considered as a translation of Q (with *different* velocity that the center of mass), together with rotation about Q (with the *same* angular velocity $\boldsymbol{\omega}$).

Equations of motion

As shown previously, the linear and angular momenta evolve according to

$$\dot{\mathbf{P}} = \mathbf{F} \quad \text{(total external force)}$$
$$\dot{\mathbf{L}} = \mathbf{G} \quad \text{(total external torque)}$$

These two equations determine the translational and rotational motion of a rigid body.

$\mathbf{L}$ and $\mathbf{G}$ depend on the choice of origin, which could be any point that is fixed in an inertial frame. More surprisingly, it can also be applied to the center of mass, even if this is accelerated: If

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i,$$

then

$$m_i \ddot{\mathbf{r}}_i^c = \mathbf{F}_i + m_i \ddot{\mathbf{R}}.$$

So there is a fictitious force $m_i \ddot{\mathbf{R}}$ in the center-of-mass frame. But the total torque of the fictitious forces about the center of mass is

$$\sum_i \mathbf{r}_i^c \times \left(-m_i \ddot{\mathbf{R}}\right) = -\left(\sum m_i \mathbf{r}_i^c\right) \times \ddot{\mathbf{R}} = 0 \times \dot{\mathbf{R}} = 0.$$

So we can still apply the above two equations.

In summary, the laws of motion apply in any inertial frame, or the center of mass (possibly non-inertial) frame.

Motion in a uniform gravitational field

In a uniform gravitational field $\mathbf{g}$, the total gravitational force and torque are the same as those that would act on a single particle of mass M located at the center of mass (which is also the *center of gravity*):

$$\mathbf{F} = \sum_i \mathbf{F}_i^{\text{ext}} = \sum_i m_i \mathbf{g} = M\mathbf{g},$$

and

$$\mathbf{G} = \sum_i \mathbf{G}_i^{\text{ext}} = \sum_i \mathbf{r} - i \times (m_i \mathbf{g}) = \sum m_i \mathbf{r}_i \times \mathbf{g} = M\mathbf{R} \times \mathbf{g}.$$

In particular, the gravitational torque about the center of mass vanishes: $\mathbf{G}^c = \mathbf{0}$.

We obtain a similar result for gravitational potential energy.

The gravitational potential in a uniform $\mathbf{g}$ is

$$\varphi_g = -\mathbf{r} \cdot \mathbf{g}.$$

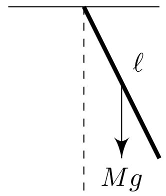
(since $\mathbf{g} = -\nabla \phi_g$ by definition)

So

$$\begin{aligned} V^{\text{ext}} &= \sum_i V_i^{\text{ext}} \\ &= \sum_i m_i (-\mathbf{r}_i \cdot \mathbf{g}) \\ &= M(-\mathbf{R} \cdot \mathbf{g}). \end{aligned}$$

Example (Thrown stick)**.** Suppose we throw a symmetrical stick. So the center of mass is the actual center. Then the center of the stick moves in a parabola. Meanwhile, the stick rotates with constant angular velocity about its center due to the absence of torque.

Example. Swinging bar.



This is an example of a *compound pendulum*.

Consider the bar to be rotating about the pivot (and not translating). Its angular momentum is $L = I\dot{\theta}$ with $I = \frac{1}{3}M\ell^2$. The gravitational torque about the pivot is

$$G = -Mg\frac{\ell}{2}\sin\theta.$$

The equation of motion is

$$\dot{L} = G.$$

So

$$I\ddot{\theta} = -Mg\frac{\ell}{2}\sin\theta,$$

or

$$\ddot{\theta} = -\frac{3g}{2\ell}\sin\theta.$$

which is exactly equivalent to a simple pendulum of length $2\ell/3$, with angular frequency $\sqrt{\frac{3g}{2\ell}}$.

This can also be obtained from an energy argument:

$$E = T + V = \frac{1}{2}I\dot{\theta}^2 - Mg\frac{\ell}{2}\cos\theta.$$

We differentiate to obtain

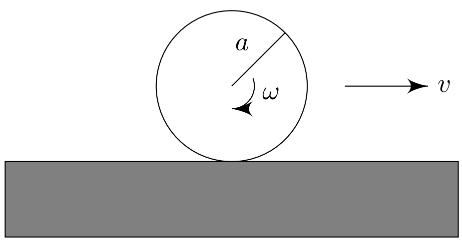
$$\frac{\mathrm{d}E}{\mathrm{d}t} = \dot{\theta}(I\ddot{\theta} + Mg\frac{\ell}{2}\sin\theta) = 0.$$

So

$$I\ddot{\theta} = -Mg\frac{\ell}{2}\sin\theta.$$

Sliding versus rolling

Consider a cylinder or sphere of radius a, moving along a stationary horizontal surface.



In general, the motion consists of a translation of the center of mass (with velocity v) plus a rotation about the center of mass (with angular velocity ω).

The horizontal velocity at the point of contact is

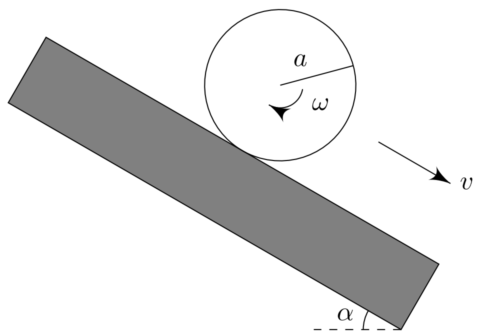
$$v_{\text{slip}} = v - a\omega.$$

For a pure sliding motion, $v \neq 0$ and $\omega = 0$, in which case $v - a\omega \neq 0$: the point of contact moves relative to the surface and kinetic friction may occur.

For a pure rolling motion, $v \neq 0$ and $\omega \neq 0$ such that $v - a\omega = 0$: the point of contact is stationary. This is the no-slip condition.

The rolling body can alternatively be considered to be rotating instantaneously about the point of contact (with angular velocity ω) and not translating.

Example (Rolling down hill)**.**



Consider a cylinder or sphere of mass M and radius a rolling down a rough plane inclined at angle α. The no-slip (rolling) condition is

$$v - a\omega = 0.$$

The kinetic energy is

$$T = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}\left(M + \frac{I}{a^2}\right)v^2.$$

The total energy is

$$E = \frac{1}{2}\left(M + \frac{I}{a^2}\right)\dot{x}^2 - Mgx\sin\alpha,$$

where x is the distance down slope. Energy is conserved (cf. later lectures). So

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \dot{x}\left(\left(M + \frac{I}{a^2}\right)\ddot{x} - Mg\sin\alpha\right) = 0.$$

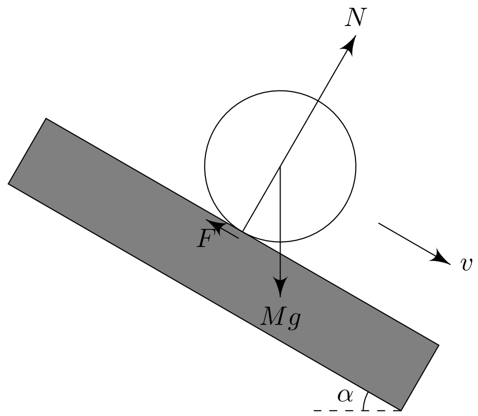
So

$$\left(M + \frac{I}{a^2}\right)\ddot{x} = Mg\sin\alpha.$$

For example, if we have a uniform solid cylinder,

$$I = \frac{1}{2}Ma^2 \quad \text{(as for a disc)}$$

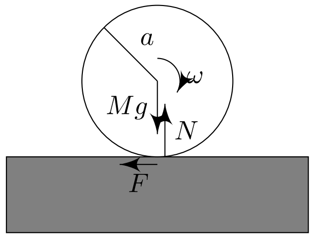
and so
$$\ddot{x} = \frac{2}{3} g \sin \alpha.$$
For a thin cylindrical shell,
$$I = Ma^2.$$
So
$$\ddot{x} = \frac{1}{2} g \sin \alpha.$$
Alternatively, we may do it in terms of forces and torques,



The equations of motion are
$$M\dot{v} = Mg \sin \alpha - F$$
and
$$I\dot{\omega} = aF.$$
While rolling,
$$\dot{v} - a\dot{\omega} = 0.$$
So
$$M\dot{v} = Mg \sin \alpha - \frac{I}{a^2}\dot{v},$$
leading to the same result.

Note that even though there is a frictional force, it does no work, since $v_{\text{slip}} = 0$. So energy is still conserved.

Example (Snooker ball)**.**



It is struck centrally so as to initiate translation, but not rotation. Sliding occurs initially. Intuitively, we think it will start to roll, and we'll see that's the case.

The constant frictional force is

$$F = \mu_k N = \mu_k M g,$$

which applies while $v - a\omega > 0$.

The moment of inertia about the center of mass is

$$I = \frac{2}{5} M a^2.$$

The equations of motion are

$$\begin{aligned} M\dot{v} &= -F \\ I\dot{\omega} &= aF \end{aligned}$$

Initially, $v = v_0$ and $\omega = 0$. Then the solution is

$$\begin{aligned} v &= v_0 - \mu_k g t \\ \omega &= \frac{5}{2}\frac{\mu_k g}{a} t \end{aligned}$$

as long as $v - a\omega > 0$. The slip velocity is

$$v_{\text{slip}} = v - a\omega = v_0 - \frac{7}{2}\mu_k g t = v_0\left(1 - \frac{t}{t_{\text{roll}}}\right),$$

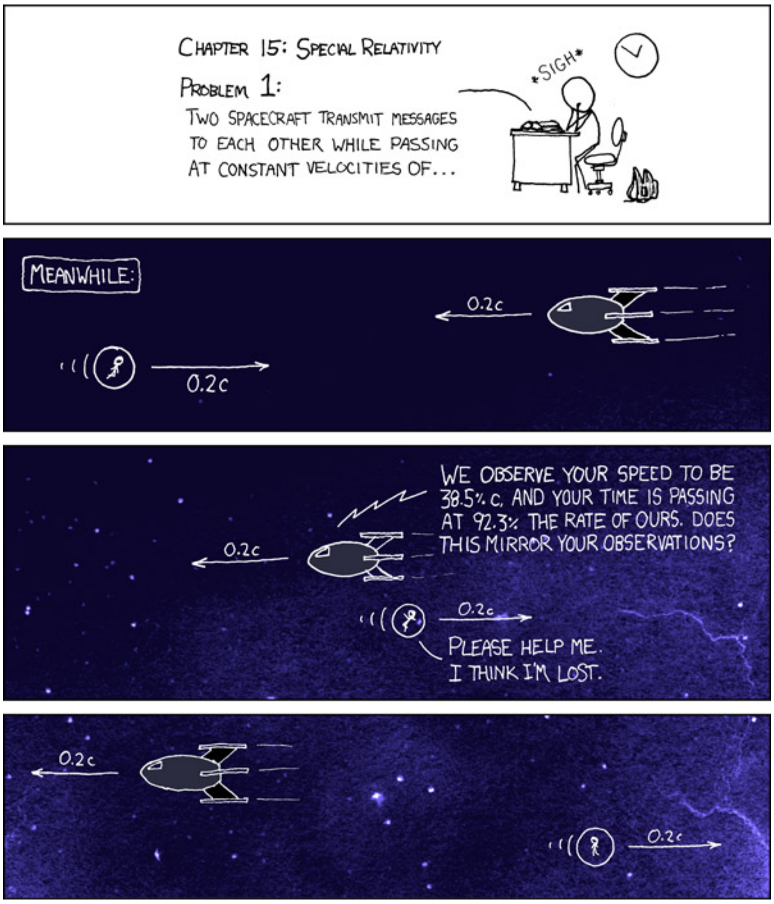
where

$$t_{\text{roll}} = \frac{2v_0}{7\mu_k g}.$$

This is valid up till $t = t_{\text{roll}}$. Then the slip velocity is 0, rolling begins and friction ceases.

At this point, $v = a\omega = \frac{6}{7}v_0$. The energy is then $\frac{5}{14}Mv_0^2 < \frac{1}{2}Mv_0^2$. So energy is lost to friction.

8 Special relativity



xkcd (Randall Munroe) CC-BY-NC 2.5

When particles move Extremely Fast$^{\text{TM}}$, Newtonian Dynamics becomes inaccurate and is replaced by Einstein's Special Theory of Relativity (1905).

Its effects are noticeable only when particles approach to the speed of light,

$$c = 299\,792\,458\,\mathrm{m\,s^{-1}} \approx 3 \times 10^8\,\mathrm{m\,s^{-1}}$$

This *really fast.*

The Special Theory of Relativity rests on the following postulate:

The laws of physics are the same in all inertial frames

This is the principle of relativity familiar to Galileo. Galilean relativity mentioned in the first chapter satisfies this postulate for dynamics. People then thought that Galilean relativity is what the world obeys. However, it turns out that there is a whole family of solutions that satisfy the postulate (for dynamics), and Galilean relativity is just one of them.

This is not a problem (yet), since Galilean relativity seems so intuitive, and we might as well take it to be the true one. However, it turns out that solving Maxwell's equations of electromagnetism gives an explicit value of the speed of light, c. This is independent of the frame of reference. So the speed of light must be the same in every inertial frame.

This is not compatible with Galilean relativity.

Consider the two inertial frames S and S', moving with relative velocity v. Then if light has velocity c in S, then Galilean relativity predicts it has velocity $c - v$ in S', which is wrong.

Therefore, we need to find a different solution to the principle of relativity that preserves the speed of light.

8.1 The Lorentz transformation

Consider again inertial frames S and S' whose origins coincide at $t = t' = 0$. For now, neglect the y and z directions, and consider the relationship between (x, t) and (x', t'). The general form is

$$x' = f(x, t), \quad t' = g(x, t),$$

for some functions f and g. This is not very helpful.

In any inertial frame, a free particle moves with constant velocity. So straight lines in (x, t) must map into straight liens in (x', t'). Therefore the relationship must be *linear*.

Given that the origins of S and S' coincide at $t = t' = 0$, and S' moves with relativity v relative to S, we know that the line $x = vt$ must map into $x' = 0$.

Combining these two information, the transformation must be of the form

$$x' = \gamma(x - vt), \tag{1}$$

for some factor γ that may depend on $|v|$ (*not* v itself. We can use symmetry arguments to show that γ should take the same value for velocities v and $-v$).

Note that Galilean transformation is compatible with this – just take γ to be always 1.

Now reverse the roles of the frames. From the perspective S', S moves with velocity $-v$. A similar argument leads to

$$x = \gamma(x' + vt'), \tag{2}$$

with the same factor γ, since γ only depends on $|v|$. Now consider a light ray (or photon) passing through the origin $x = x' = 0$ at $t = t' = 0$. Its trajectory in S is

$$x = ct.$$

We want a γ such that the trajectory in S' is

$$x' = ct'$$

as well, so that the speed of light is the same in each frame. Substitute these into (1) and (2)

$$\begin{aligned} ct' &= \gamma(c - v)t \\ ct &= \gamma(c + v)t' \end{aligned}$$

Multiply the two equations together and divide by tt' to obtain

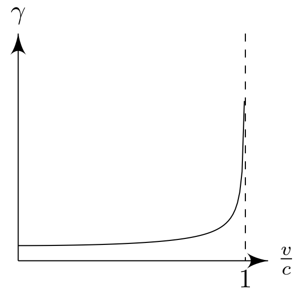
$$c^2 = \gamma^2(c^2 - v^2).$$

So
$$\gamma = \sqrt{\frac{c^2}{c^2 - v^2}} = \frac{1}{\sqrt{1 - (v/c)^2}}.$$

Definition (Lorentz factor)**.** The *Lorentz factor* is
$$\gamma = \frac{1}{\sqrt{1 - (v/c)^2}}.$$

Note that

- $\gamma \geq 1$ and is an increasing function of $|v|$.
- When $v \ll c$, then $\gamma \approx 1$, and we recover the Galilean transformation.
- When $|v| \to c$, then $\gamma \to \infty$.
- If $|v| \geq c$, then γ is imaginary, which is physically impossible (or at least *weird*).
- If we take $c \to \infty$, then $\gamma = 1$. So Galilean transformation is the transformation we will have if light is infinitely fast. Alternatively, in the world of Special Relativity, the speed of light is "infinitely fast".



For the sense of scale, we have the following values of γ at difference speeds:

- $\gamma = 2$ when $v = 0.866c$.
- $\gamma = 10$ when $v = 0.9965$.
- $\gamma = 20$ when $v = 0.999c$.

We still have to solve for the relation between t and t'. Eliminate x between (1) and (2) to obtain
$$x = \gamma(\gamma(x - vt) + vt').$$

So
$$t' = \gamma t - (1 - \gamma^{-2})\frac{\gamma x}{v} = \gamma\left(t - \frac{v}{c^2}x\right).$$

So we have

Law (Principle of Special Relativity)**.** Let S and S' be inertial frames, moving at the relative velocity of v. Then

$$\begin{aligned} x' &= \gamma(x - vt) \\ t' &= \gamma\left(t - \frac{v}{c^2}x\right), \end{aligned}$$

where

$$\gamma = \frac{1}{\sqrt{1 - (v/c)^2}}.$$

This is the *Lorentz transformations* in the standard configuration (in one spatial dimension).

We can invert this linear mapping to find (after some algebra)

$$\begin{aligned} x &= \gamma(x' + vt') \\ t &= \gamma\left(t' + \frac{v}{c^2}x'\right) \end{aligned}$$

Directions perpendicular to the relative motion of the frames are unaffected:

$$\begin{aligned} y' &= y \\ z' &= z \end{aligned}$$

Now we check that the speed of light is really invariant:

For a light ray travelling in the x direction in S:

$$x = ct, \quad y = 0, \quad z = 0.$$

In S', we have

$$\frac{x'}{t'} = \frac{\gamma(x - vt)}{\gamma(t - vx/c^2)} = \frac{(c - v)t}{(1 - v/c)t} = c,$$

as required.

For a light ray travelling in the Y direction in S,

$$x = 0, \quad y = ct, \quad z = 0.$$

In S',

$$\frac{x'}{t'} = \frac{\gamma(x - vt)}{\gamma(t - vx/c^2)} = -v,$$

and

$$\frac{y'}{t'} = \frac{y}{\gamma(t - vx/c^2} = \frac{c}{\gamma},$$

and

$$z' = 0.$$

So the speed of light is

$$\frac{\sqrt{x'^2 + y'^2}}{t'} = \sqrt{v^2 + \gamma^{-2}c^2} = c,$$

as required.

More generally, the Lorentz transformation implies

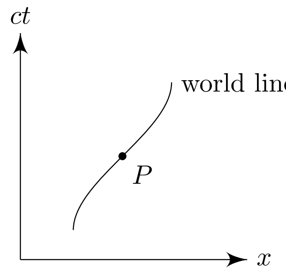
$$\begin{aligned}
c^2t'^2 - r'^2 &= c^2t'^2 - x'^2 - y'^2 - z'^2 \\
&= c^2\gamma^2\left(t - \frac{v}{c^2}x\right)^2 - \gamma^2(x - vt)^2 - y^2 - z^2 \\
&= \gamma^2\left(1 - \frac{v^2}{c^2}\right)(c^2t^2 - x^2) - y^2 - z^2 \\
&= c^2t^2 - x^2 - y^2 - z^2 \\
&= c^2t^2 - r^2.
\end{aligned}$$

We say that the quantity $c^2t^2 - x^2 - y^2 - z^2$ is *Lorentz-invariant.*

So if $\frac{r}{t} = c$, then $\frac{r'}{t'} = c$ also.

8.2 Spacetime diagrams

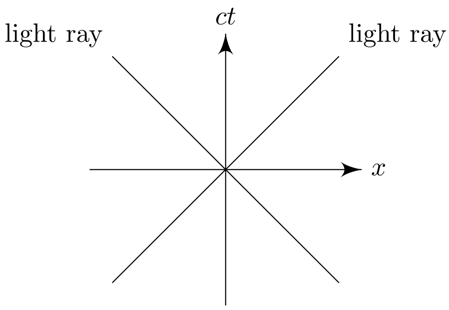
When considering one spatial dimension x and time t in an inertial frame S, we plot x on the horizontal axis and ct on the vertical axis. We use ct instead of t so that the dimensions make sense.



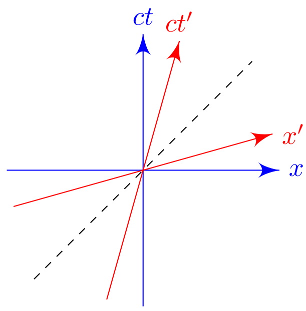
Definition (Spacetime)**.** The union of space and time in special relativity is called *Minkowski spacetime*. Each point P represents an *event*, labelled by coordinates (ct, x) (note the order!).

A particle traces out a *world line* in spacetime, which is straight if the particle moves uniformly.

Light rays moving in the x direction have world lines inclined at 45°.



We can also draw the axes of S', moving in the x direction at velocity v relative to S. The ct' axis corresponds to $x' = 0$, ie. $x = \frac{v}{c}(ct)$. The x' axis corresponds to $t' = 0$, ie. $ct = \frac{v}{c}x$.



Note that the x' and ct' axes are *not* orthogonal, but are symmetrical about the diagonal (dashed line). So they agree on where the world line of a light ray should lie on.

8.3 Relativistic physics

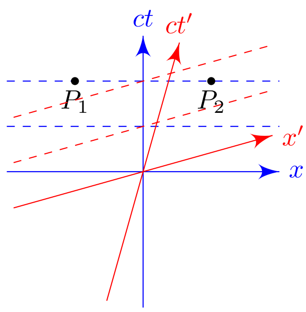
Simultaneity

Definition (Simultaneous events)**.** We say two events P_1 and P_2 are simultaneous in the frame S if $t_1 = t_2$.

They are represented in the following spacetime diagram by horizontal dashed lines.

However, events that are simultaneous in S' have equal values of t', and so lie on lines

$$ct - \frac{v}{c}x = \text{constant}.$$



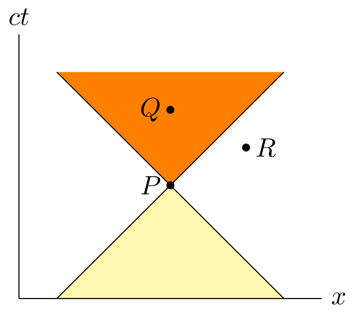
The lines of simultaneity of S' and those of S are different, and events simultaneous in S need not be simultaneous in S'. So simultaneity is relative. S thinks P_1 and P_2 happened at the same time, while S' thinks P_2 happens first.

Note that this is genuine disagreement. It is not due to effects like, it takes time for the light conveying the information to different observers. Our account above already takes that into account! (by not mentioning the existence of an observer :)

Causality

Although different people may disagree on the temporal order of events, the consistent ordering of cause and effect can be ensured.

Since things can only travel at at most the speed of light, P cannot affect R if R happens a millisecond after P but is at millions of galaxies away. We can draw a *light cone* that denotes the regions in which things can be influenced by P. These are the regions of space-time light (or any other particle) can possibly travel to. P can only influence events within its *future light cone*, and *be influenced* by events within its *past light cone*.



All observers agree that Q occurs after P. Different observers may disagree on the temporal ordering of P and R. However, since nothing can travel faster than light, P and R cannot influence each other. Since everyone agrees on how fast light travels, they also agree on the light cones, and hence causality. So philosophers are happy.

Time dilation

Suppose we have a clock that is stationary in S' (which travels at constant velocity v with respect to inertial frame S) ticks at constant intervals $\Delta t'$. What is the interval between ticks in S?

Lorentz transformation gives

$$t = \gamma\left(t' + \frac{v}{c^2}x'\right).$$

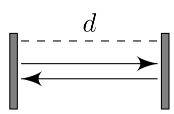
Since $x' =$ constant for the clock, we have

$$\Delta t = \gamma\Delta t' > \Delta t'.$$

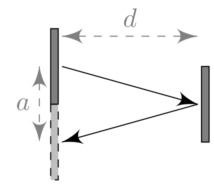
So the interval measured in S is greater! So moving clocks run slowly.

A non-mathematical explanation comes from Feynman (not lectured): Suppose we have a very simple clock: We send a light beam towards a mirror, and wait for it to reflect back. When the clock detects the reflected light, it ticks, and then sends the next light beam.

Then the interval between two ticks is the distance $2d$ divided by the speed of light.



From the point of view of an observer moving downwards, by the time light reaches the right mirror, it would have moved down a bit. So S sees



However, the distance travelled by the light beam is now $\sqrt{(2d)^2 + a^2} > 2d$. Since they agree on the speed of light, it must have taken longer for the clock to receive the reflected light in S. So the interval between ticks are longer.

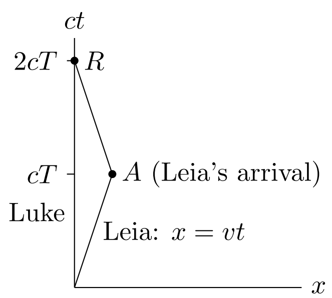
By the principle of relativity, all clocks must measure the same time dilation, or else we can compare the two clocks and know if we are "moving". So result follows.

This is famously evidenced by muons. Their half-life is around 2 microseconds (ie. on average they decay to something else after around 2 microseconds). They are created when cosmic rays bombard the atmosphere. However, even if they travel at the speed of light, 2 microseconds only allows it to travel 600 m, certainly not sufficient to reach the surface of Earth. However, we observe *lots* of muons on Earth. This is because muons are travelling so fast that their clocks run really slowly.

The twin paradox

Consider two twins: Luke and Leia. Luke stays at home. Leia travels at a constant speed v to a distant planet P, turns around, and returns at the same speed.

In Luke's frame of reference,



Leia's arrival (A) at P has

$$(ct, x) = (cT, vT).$$

The time experienced by Leia on her outward journey is

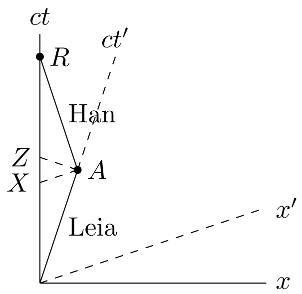
$$T' = \gamma\left(T - \frac{v}{c^2}T\right) = \frac{T}{\gamma}.$$

By Leia's return R, Luke has aged by $2T$, but Leia has aged by $\frac{2T}{\gamma} < 2T$. So she is younger than Luke, because of time dilation.

The paradox is: From Leia's perspective, Luke travelled away from her at speed and the returned, so he should be younger than her!

Why is the problem not symmetric?

We can draw Leia's initial frame of reference in dashed lines:



In Leia's frame, by the time she arrives at A, she has experienced a time $T' = \frac{T}{\gamma}$ as shown above. This event is simultaneous with event X in Leia's frame. Then in Luke's frame, the coordinates of X are

$$(ct, x) = \left(\frac{T'}{\gamma}, 0\right) = \left(\frac{T}{\gamma^2}, 0\right),$$

obtained through calculations similar to that above. So Leia thinks Luke has aged less by a factor of $1/\gamma^2$. At this stage, the problem *is* symmetric, and Luke also thinks Leia has aged less by a factor of $1/\gamma^2$.

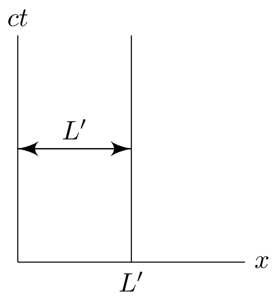
Things change when Leia turns around and changes frame of reference. To understand this better, suppose Leia meets a friend, Han, who is just leaving P at speed v. On his journey back, Han also thinks Luke ages T/γ^2. But in his frame of reference, his departure is simultaneous with Luke's event Z, not X, since he has different lines of simultaneity.

So the asymmetry between Luke and Leia occurs when Leia turns around. At this point, she sees Luke age rapidly from X to Z.

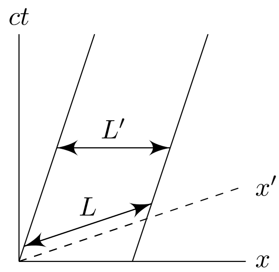
Length contraction

A rod of length L' is stationary in S'. What is its length in S?

In S', then length of the rod is the distance between the two ends at the same time. So we have



In S, we have



The lines $x' = 0$ and $x' = L'$ map into $x = vt$ and $x = vt + L'/\gamma$. So the length in S is $L = L'/\gamma < L'$. Therefore moving objects are contracted in the direction of motion.

Definition (Proper length)**.** The *proper length* is the length measured in an object's rest frame.

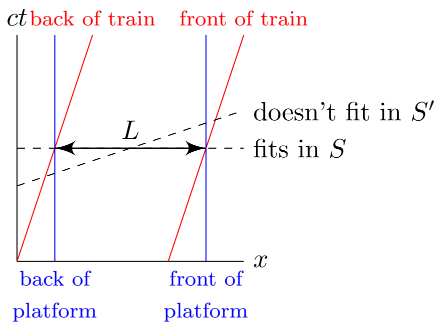
This is analogous to the fact that if you view a bar from an angle, it looks shorter than if you view it from the front. In relativity, what causes the contraction is not a spacial rotation, but a spacetime *hyperbolic* rotation.

Question: does a train of length L fit alongside a platform of length L if it travels through the station at a speed v such that $\gamma = 2$?

For the system of observers on the platform, the train contracts to a length $2L/\gamma = L$. So it fits.

But for the system of observers on the train, the platform contracts to length $L/\gamma = L/2$, which is much too short!

This can be explained by the difference of lines of simultaneity, since length is the distance between front and back *at the same time.*



Composition of velocities

A particle moves with constant velocity u' in frame S', which moves with velocity v relative to S. What is its velocity u in S?

The world line of the particle in S' is

$$x' = u't'.$$

In S, using the inverse Lorentz transformation,

$$u = \frac{x}{t} = \frac{\gamma(x' + vt')}{\gamma(t' + (v/c^2)x'} = \frac{u't' + vt'}{t' + (v/c^2)u't'} = \frac{u' + v}{1 + u'v/c^2}.$$

This is the formula for the relativistic composition of of velocities.

The inverse transformation is found by swapping u and u', and swapping the sign of v, ie.

$$u' = \frac{u - v}{1 - uv/c^2}.$$

Note:

- if $u'v \ll c^2$, then the transformation reduces to the standard Galilean addition of velocities $u \approx u' + v$.
- u is a monotonically increasing function of u for any constant v (with $|V| < c$.
- When $u' = \pm c$, $u = u'$ for any v, ie. the speed of light is constant in all frames of reference.
- Hence $|u'| < c$ iff $|u| < c$. This means that we cannot reach the speed of light by composition of velocities.

8.4 Geometry of spacetime

8.4.1 The invariant interval

Consider events P and Q with coordinates (ct_1, x_1) and (ct_2, x_2) separated by $\Delta t = t_2 - t_1$ and $\Delta x = x_2 - x_1$.

Definition (Invariant interval)**.** The *invariant interval* or *spacetime interval* between P and Q is defined as

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2.$$

Note that this quantity Δs^2 can be both positive or negative - so Δs might be imaginary!

Proposition. All inertial observers agree on the value of Δs^2.

Proof.

$$\begin{aligned} c^2 \Delta t'^2 - \Delta x'^2 &= c^2 \gamma^2 \left(\Delta t - \frac{v}{c^2}\Delta x\right)^2 - \gamma^2 (\Delta x - v\Delta t)^2 \\ &= \gamma^2 \left(1 - \frac{v^2}{c^2}\right)(c^2 \Delta t - \Delta x^2) \\ &= c^2 \Delta t - \Delta x^2. \end{aligned}$$

□

In three spatial dimensions,

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2.$$

For two infinitesimally separated events, we have

Definition (Line element)**.** The *line element* is

$$\mathrm{d}s^2 = c^2 \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2.$$

Our 4D spacetime looks like $\mathbb{R}^4$, if we just view each event as a string of 4 numbers. However, we have a special distance measure Δs^2, as opposed to the usual $\sqrt{x^2+y^2+z^2}$. So we give it a fancy name and call it a *Minkowski spacetime*. We say it has dimension $d = 1 + 3$.

Definition (Timelike, spacelike and lightlike separation)**.** Events with $\Delta s^2 > 0$ are *timelike separated*. It is possible to find inertial frames in which the two events occur in the same position, and are purely separated by time. Timelike-separated events lie within each other's light cones and can influence one another.

Events with $\Delta s^2 < 0$ are *spacelike separated*. It is possible to find inertial frame in which the two events occur in the same time, and are purely separated by space. Spacelike-separated events lie out of each other's light cones and cannot influence one another.

Events with $\Delta s^2 = 0$ are *lightlike* or *null separated*. In all inertial frames, the events lie on the boundary of each other's light cones. eg. different points in the trajectory of a photon are lightlike separated, hence the name.

Note: $\Delta s^2 = 0$ does not imply that P and Q are the same event.

8.4.2 The Lorentz group

The coordinates of an event P in frame S can be written as a *4-vector* (ie. 4-component vector) X. We write

$$X = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix},$$

with $\mu = 0, 1, 2, 3$.

The invariant interval between the origin and P can be written as an inner product

$$X \cdot X = X^T \eta X = c^2t^2 - x^2 - y^2 - z^2,$$

where

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

4-vectors with $X \cdot X > 0$ are called timelike, and those $X \cdot X < 0$ are spacelike. If $X \cdot X = 0$, it is lightlike or null.

A Lorentz transformation is a linear transformation transformation of the coordinates from one frame S to another S', represented by a 4×4 tensor ("matrix"):

$$X' = \Lambda X$$

Lorentz transformations can be defined as those that leave the inner product invariant:

$$(\forall X)(X' \cdot X' = X \cdot X),$$

which implies the matrix equation

$$\Lambda^T \eta \Lambda = \eta. \tag{$*$}$$

These also preserve $X \cdot Y$ if X and Y are both 4-vectors.

Two classes of solution to this equation are:

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix},$$

where R is a 3×3 orthogonal matrix. This rotates (or reflects) space and leaves time intact; and

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\beta = \frac{v}{c}$, and $\gamma = 1/\sqrt{1-\beta^2}$. Here we leave the y and z coordinates intact, and apply a Lorentz boost along the x direction.

The set of all matrices satisfying equation $(*)$ for the *Lorentz group* $O(1,3)$. It is generated by rotations and boosts, as defined above, which includes the absurd spacial reflections and time reversal.

The subgroup with $\det \Lambda = +1$ is the *proper Lorentz group* $SO(1,3)$.

The subgroup that preserves spatial orientation and the direction of time is the *restricted Lorentz group* $SO^+(1,3)$ (note that this is different from $SO(1,3)$, since if you do *both* spacial reflection and time reversal, the determinant of the matrix is still positive. We want to eliminate those as well!)

8.4.3 Rapidity

Focus on the upper left 2×2 matrix of Lorentz boosts in the x direction. Write

$$\Lambda[\beta] = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}.$$

Combining two boosts in the x direction, we have

$$\Lambda[\beta_1]\Lambda[\beta_2] = \begin{pmatrix} \gamma_1 & -\gamma_1\beta_1 \\ -\gamma_1\beta_1 & \gamma_1 \end{pmatrix} \begin{pmatrix} \gamma_2 & -\gamma_2\beta_2 \\ -\gamma_2\beta_2 & \gamma_2 \end{pmatrix} = \Lambda\left[\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}\right]$$

after some messy algebra. This is just the velocity composition formula as before.

We can compare this with spatial rotation. Recall that

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

with

$$R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2).$$

For Lorentz boosts, we can define

Definition (Rapidity)**.** The *rapidity* of a Lorentz boot is ϕ such that

$$\beta = \tanh\phi, \quad \gamma = \cosh\phi, \quad \gamma\beta = \sinh\phi.$$

Then

$$\Lambda[\beta] = \begin{pmatrix} \cosh\phi & \sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} = \Lambda(\phi).$$

The rapidities add like rotation angles:

$$\Lambda(\phi_1)\Lambda(\phi_2) = \Lambda(\phi_1 + \phi_2).$$

This shows the close relation betweens spatial rotations and Lorentz boosts. Lorentz boots are simply *hyperbolic* rotations in spacetime!

8.5 Relativistic kinematics

A particle moves along a trajectory $\mathbf{x}(t)$ in S. Its velocity is $\mathbf{u}(t) = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$.

However, there is a better description of the trajectory.

Recall that we defined "proper length" as the length in the item in its rest frame. Similarly, we can define the *proper time*.

First consider a particle at rest in S', with $\mathbf{x}' = \mathbf{0}$. The invariant interval between events on its world line is $\Delta s^2 = c^2\Delta t'^2$.

Then the particle's world line is just a vertical line.

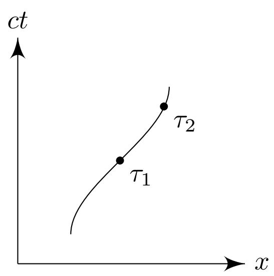
Definition (Proper time)**.** The *proper time* τ is defined such that

$$\Delta\tau = \frac{\Delta s}{c}$$

τ is the time experienced by the particle.

Note that the equation $\Delta\tau = \frac{\Delta s}{c}$ holds in all frames, since Δs is a Lorentz invariant. We can (and should) use τ even if we are not in the rest frame of the particle.

The world line of a particle can be parametrized using the proper time: $\mathbf{x}(\tau, t(\tau))$.



Infinitesimal changes are related by

$$\mathrm{d}\tau = \frac{\mathrm{d}s}{c} = \frac{1}{c}\sqrt{c^2\,\mathrm{d}t^2 - |\mathrm{d}\mathbf{x}|^2} = \sqrt{1 - \frac{|\mathbf{u}|^2}{c^2}}\,\mathrm{d}t.$$

Thus

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma_u$$

with

$$\gamma_u = \frac{1}{\sqrt{1 - \frac{|\mathbf{u}|^2}{c^2}}}.$$

The total time experienced by the particle along a segment of its world line is

$$T = \int \mathrm{d}\tau = \int \frac{1}{\gamma_u}\,\mathrm{d}t.$$

We can then define the *position 4-vector* and *4-velocity*.

Definition (Position 4-vector and 4-velocty)**.** The *position 4-vector* is

$$X(\tau) = \begin{pmatrix} ct(\tau) \\ \mathbf{x}(\tau) \end{pmatrix}.$$

Its *4-velocity* is defined as

$$U = \frac{\mathrm{d}X}{\mathrm{d}\tau} = \begin{pmatrix} c\frac{\mathrm{d}t}{\mathrm{d}\tau} \\ \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} \end{pmatrix} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\begin{pmatrix} c \\ \mathbf{u} \end{pmatrix} = \gamma_u \begin{pmatrix} c \\ \mathbf{u} \end{pmatrix},$$

where $\mathbf{u} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$.

Another common notation is

$$X = (ct, \mathbf{x}),\quad U = \gamma_u(c, \mathbf{u}).$$

If frames S and S' are related by $X' = \Lambda X$, then the 4-velocity also transforms as $U' = \Lambda U$.

Definition (4-vector)**.** A *4-vector* is a 4-component vectors that transforms in this way under a Lorentz transformation, ie. $X' = \Lambda X$.

When using suffix notation, the indices are written above (superscript) instead of below (subscript). The indices are written with Greek letters which range from 0 to 3. So we have X^μ instead of X_i, for $\mu = 0, 1, 2, 3$. If we write X_μ instead, it means a different thing. This will be explained more in-depth in the electromagnetism course (and you'll get more confused).

U is a 4-vector because X is a 4-vector and τ is a Lorentz invariant. Note that $\mathrm{d}X/\mathrm{d}t$ is *not* a 4-vector.

Note that this definition of 4-vector is analogous to that of a tensor - things that transform nicely according to our rules. Then τ would be a scalar, ie. rank-1 tensor, while t is just a number, not a scalar.

The inner product $U\cdot U = U'\cdot U'$ is Lorentz invariant, ie. the same in all inertial frames. In the rest frame of the particle, $U = (c, 0)$. So $U\cdot U = c^2$.

In any other frame, $Y = \gamma_u(c, \mathbf{u})$. So

$$Y\cdot Y = \gamma_u^2(c^2 - |\mathbf{u}|^2) = c^2$$

as expected.

Transformation of velocities revisited

We have seen that velocities cannot be simply added in relativity. However, the 4-velocity does transform linearly, according to the Lorentz transform:

$$U' = \Lambda U.$$

In frame S, consider a particle moving with speed u at an angle θ to the x axis in the xy plane. This is the most general case for motion not parallel to the Lorentz boost.

Its 4-velocity is

$$U = \begin{pmatrix} \gamma_u c \\ \gamma_u u \cos\theta \\ \gamma_u u \sin\theta \\ 0 \end{pmatrix}, \quad \gamma_u = \frac{1}{\sqrt{1 - u^2/c^2}}.$$

With frames S and S' in standard configuration (ie. origin coincide at $t = 0$, S' moving in x direction with velocity v relative to S),

$$U' = \begin{pmatrix} \gamma_{u'} c \\ \gamma_{u'} u' \cos\theta' \\ \gamma_{u'} u' \sin\theta' \\ 0 \end{pmatrix} \begin{pmatrix} \gamma_v & -\gamma_v v/c & 0 & 0 \\ -\gamma_v v/c & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_u c \\ \gamma_u u \cos\theta \\ \gamma_u u \sin\theta \\ 0 \end{pmatrix}$$

Instead of evaluating the whole matrix, we can divide different rows to get useful results.

The ratio of the first lines xc gives

$$u' \cos\theta' = \frac{u\cos\theta - v}{1 - \frac{uv}{c^2}\cos\theta},$$

just like the composition of parallel velocities.

The ratio of the third to second line gives

$$\tan\theta' = \frac{u \sin\theta}{\gamma_v(u\cos\theta - v)},$$

which describes *aberration*, a change in the direction of motion of a particle due to the motion of the observer. Note that this isn't just a relativistic effect! If you walk in the rain, you have to hold your umbrella obliquely since the rain seems to you that they are coming from an angle. The relativistic part is the γ_v factor in the denominator.

This is also seen in the aberration of starlight ($u = c$) due to the Earth's orbital motion. This causes small annual changes in the apparent positions of stars.

4-momentum

Definition (4-momentum)**.** The *4-momentum* of a particle of mass m is

$$P = mU = m\gamma_u \begin{pmatrix} c \\ \mathbf{u} \end{pmatrix}$$

The 4-momentum of a system of particles is the sum of the 4-momentum of the particles, and is conserved in the absence of external forces.

The spatial components of P are the *relativistic 3-momentum*,

$$\mathbf{p} = m\gamma_u \mathbf{u},$$

which differs from the Newtonian expression by a factor of γ_u. Note that $|\mathbf{p}| \to \infty$ as $|\mathbf{u}| \to c$.

What is the interpretation of the time component P^0 (ie. the first time component of the P vector)? We expand for $|\mathbf{u}| \ll c$:

$$P^0 = m\gamma c = \frac{mc}{\sqrt{1 + |\mathbf{u}|^2/c^2}} = \frac{1}{c}\left(mc^2 + \frac{1}{2}m|\mathbf{u}|^2 + \cdots\right).$$

We have a constant term mc^2 plus a kinetic energy term $\frac{1}{2}m|\mathbf{u}|^2$, plus more tiny terms, all divided by c. So this suggests that P^0 is indeed the energy for a particle, and the remaining $\cdots$ terms are relativistic corrections for our old formula $\frac{1}{2}m|\mathbf{u}|^2$ (the mc^2 term will be explained later). So we interpret P as

$$P = \begin{pmatrix} E/c \\ \mathbf{p} \end{pmatrix}$$

Definition (Relativistic energy)**.** The *relativistic energy* of a particle is $E = P^0$. So

$$E = m\gamma c^2 = mc^2 + \frac{1}{2}m|\mathbf{u}|^2 + \cdots$$

Note that $E \to \infty$ as $|\mathbf{u}| \to c$.

For a stationary particle,

$$E = mc^2.$$

This implies that mass is a form of energy. m is sometimes called the *rest mass*.

The energy of a moving particle, $m\gamma_u c^2$, is the sum of the rest energy mc^2 and kinetic energy $m(\gamma_u - 1)c^2$.

Since $P \cdot P = \frac{E^2}{c^2} - |\mathbf{p}|^2$ is a Lorentz invariant (lengths of 4-vectors are always Lorentz invariant) and equals m^2c^2 in the particle's rest frame, we have the general relation between energy and momentum

$$E^2 = |\mathbf{p}|^2c^2 + m^2c^4$$

In Newtonian physics, mass and energy are separately conserved. In relativity, mass is not conserved. Instead, it is just another form of energy, and the total energy, including mass energy, is conserved.

Mass can be converged into kinetic energy and vice versa (eg. atomic bombs!)

Massless particles

Particles with zero mass ($m = 0$), eg. photons, can have non-zero momentum and energy because they travel at the speed of light ($\gamma = \infty$).

In this case, $P \cdot P = 0$. So massless particles have light-like (or null) trajectories, and no proper time can be defined for such particles.

Other massless particles in the Standard Model of particle physics include the gluon.

For these particles, energy and momentum are related by

$$E^2 = |\mathbf{p}|^2c^2.$$

So

$$E = |\mathbf{p}|c.$$

Thus

$$P = \frac{E}{c}\begin{pmatrix} 1 \\ \mathbf{n} \end{pmatrix},$$

where $\mathbf{n}$ is a unit (3-)vector in the direction of propagation.

According to quantum mechanics, fundamental "particles" aren't really particles but have both particle-like and wave-like properties (if that sounds confusing, yes it is!). Hence we can assign it a *de Broglie wavelength*, according to the *de Broglie relation*:

$$|\mathbf{p}| = \frac{h}{\lambda}$$

where $h \approx 6.63 \times 10^{-34}\,\mathrm{m^2\,kg\,s^{-1}}$ is *Plank's constant.*

For massless particles, this is consistent with *Planck's relation*:

$$E = \frac{hc}{\lambda} = h\nu,$$

where $\nu = \frac{c}{\lambda}$ is the *wave frequency.*

Newton's second law in special relativity

Definition (4-force)**.** The *4-force* is

$$F = \frac{\mathrm{d}P}{\mathrm{d}\tau}$$

This equation is the relativistic counterpart to Newton's second law.

It is related to the 3-force $\mathbf{F}$ by

$$F = \gamma_u \begin{pmatrix} \mathbf{F}\cdot\mathbf{u}/c \\ \mathbf{F} \end{pmatrix}$$

Expanding the definition of the 4-force componentwise, we obtain

$$\frac{\mathrm{d}E}{\mathrm{d}\tau} = \gamma_u \mathbf{F}\cdot\mathbf{u} \Rightarrow \frac{\mathrm{d}E}{\mathrm{d}t} = \mathbf{F}\cdot\mathbf{u}$$

and

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau} = \gamma_u \mathbf{F} \Rightarrow \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F}$$

Equivalently, for a particle of mass m,

$$F = mA,$$

where

$$A = \frac{\mathrm{d}U}{\mathrm{d}\tau}$$

is the 4-acceleration.

We have

$$U = \gamma_u \begin{pmatrix} c \\ \mathbf{u} \end{pmatrix}$$

So

$$A = \gamma_u \frac{\mathrm{d}U}{\mathrm{d}t} = \gamma_u \begin{pmatrix} \dot\gamma_u c \\ \gamma_u \mathbf{a} + \dot\gamma_u \mathbf{u}. \end{pmatrix}$$

where $\mathbf{a} = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}$ and $\dot\gamma_u = \gamma_u^3 \frac{\mathbf{a}\cdot\mathbf{u}}{c^3}$.

In the instantaneous rest frame of a particle, $\mathbf{u} = \mathbf{0}$ and $\gamma_u = 1$. So

$$U = \begin{pmatrix} c \\ \mathbf{0} \end{pmatrix}, \quad A = \begin{pmatrix} 0 \\ \mathbf{a} \end{pmatrix}$$

Then $\mathbf{U}\cdot\mathbf{A} = 0$. Since this is a Lorentz invariant, we have $\mathbf{U}\cdot\mathbf{A} = 0$ in all frames.

8.6 Particle physics

Many problems can be solved using the conservation of 4-momentum,

$$P = \begin{pmatrix} E/c \\ \mathbf{p} \end{pmatrix},$$

for a system of particles.

Definition (Center of momentum frame)**.** The *center of momentum (CM) frame*, or *zero momentum frame*, is an inertial frame in which the total 3-momentum is $\sum \mathbf{p} = 0$.

This exists unless the system consists of one or more massless particle moving in a single direction.

Particle decay

A particle of mass m_1 decays into two particles of masses m_2 and m_2.

We have

$$P_1 = P_2 + P_3.$$

ie.

$$\begin{aligned} E_1 &= E_2 + E_3 \\ \mathbf{p}_1 &= \mathbf{p}_2 + \mathbf{p}_3. \end{aligned}$$

In the CM frame (ie. the rest frame of the original particle),

$$\begin{aligned} E_1 = m_1 c^2 &= \sqrt{|\mathbf{p}_2|^2 c^2 + m_2^2 c^4} + \sqrt{|\mathbf{p}_3|^2 c^2 + m_2^2 c^4} \\ &\geq m_2 c^2 + m_3 c^2. \end{aligned}$$

So decay is possible only if

$$m_1 \geq m_2 + m_3.$$

(Recall that mass is not conserved in relativity!)

Example. A possible decay path of the Higgs' particle can be written as

$$\begin{gathered} \mathrm{h} \to \gamma\gamma \\ \text{Higgs particle} \to 2 \text{ photons} \end{gathered}$$

This is possible by the above criterion, because $m_\mathrm{h} \geq 0$, while $m_\gamma = 0$.

The full conservation equation is

$$P_\mathrm{h} = \begin{pmatrix} m_\mathrm{h} c \\ \mathbf{0} \end{pmatrix} = P_{\gamma_1} + P_{\gamma_2}$$

So

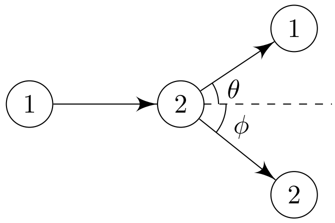
$$\begin{aligned} \mathbf{p}_{\gamma_1} &= p_{\gamma_2} \\ E_{\gamma_1} = E_{\gamma_2} &= \frac{1}{2} m_\mathrm{h} c^2. \end{aligned}$$

Particle scattering

When two particles collide and retain heir identities, the total 4-momentum is conserved:

$$P_1 + P_2 = P_3 + P_4$$

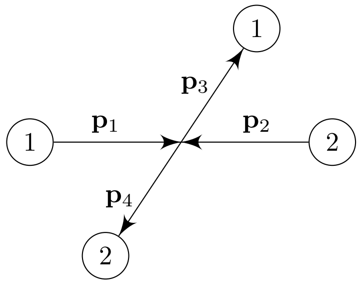
In the laboratory frame S, suppose that particle 1 travels with speed u and collides with particle 2 (at rest).



In the CM frame S',

$$\mathbf{p}_1' + \mathbf{p}_2' = 0 = \mathbf{p}_3' + \mathbf{p}_4'.$$

Both before and after the collision, the two particles have equal and opposite 3-momentum.



The scattering angle θ' is undetermined and can be thought of as being random. However, we can derive some conclusions about the angles θ and ϕ in the laboratory frame.

(staying in S' for the moment) Suppose the particles have equal mass m. They then have the same speed v in S'.

Choose axes such that

$$P_1' = \begin{pmatrix} m\gamma_v c \\ m\gamma_v v \\ 0 \\ 0 \end{pmatrix}, \quad P_2' = \begin{pmatrix} m\gamma_v c \\ -m\gamma_v v \\ 0 \\ 0 \end{pmatrix}$$

and after the collision,

$$P_3' = \begin{pmatrix} m\gamma_v c \\ m\gamma_v v\cos\theta' \\ m\gamma_v v\sin\theta' \\ 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} m\gamma_v c \\ -m\gamma_v v\cos\theta' \\ -m\gamma_v v\sin\theta' \\ 0 \end{pmatrix}.$$

We then use the Lorentz transformation to return to the laboratory frame S. The relative velocity of the frames is v. So the Lorentz transform is

$$\Lambda = \begin{pmatrix} \gamma_v & \gamma_v v/c & 0 & 0 \\ \gamma_v v/c & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and we find

$$P_1 = \begin{pmatrix} m\gamma_u c \\ m\gamma_u u \\ 0 \\ 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} mc \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

where

$$u = \frac{2v}{1 + v^2/c^2},$$

(cf. velocity composition formula)

Considering the transformations of P_3' and P_4', we obtain

$$\tan\theta = \frac{\sin\theta'}{\gamma_v(1 + \cos\theta')} = \frac{1}{\gamma_v}\tan\frac{\theta'}{2},$$

and

$$\tan\phi = \frac{\sin\theta'}{\gamma_v(1 - \cos\theta')} = \frac{1}{\gamma_v}\cot\frac{\theta'}{2}.$$

Multiplying these expressions together, we obtain

$$\tan\theta\tan\phi = \frac{1}{\gamma_v^2}.$$

So even though we do not know what θ and ϕ might be, they *must* be related by this equation.

In the Newtonian limit, where $|\mathbf{v}| \ll c$, we have $\gamma_v \approx 1$. So

$$\tan\theta\tan\phi = 1,$$

ie. the outgoing trajectories are perpendicular in S.

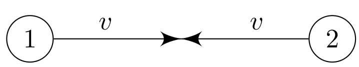
Particle creation

Collide two particles of mass m fast enough, and you create an extra particle of mass M.

$$P_1 + P_2 = P_3 + P_4 + P_5,$$

where P_5 is the momentum of the new particle.

In the CM frame,



$$P_1 + P_2 = \begin{pmatrix} 2m\gamma_v c \\ \mathbf{0} \end{pmatrix}$$

We have

$$P_3 + P_4 + P_5 = \begin{pmatrix} (E_3 + E_4 + E_5)/c \\ \mathbf{0} \end{pmatrix}$$

So

$$2m\gamma_v c^2 = E_3 + E_4 + E_5 \geq 2mc^2 + Mc^2.$$

So in order to create this new particle, we must have

$$\gamma_v \geq 1 + \frac{M}{2m}.$$

Alternatively, it occurs only if the initial kinetic energy in the CM frame satisfies

$$2(\gamma_v - 1)mc^2 \geq Mc^2.$$

If we transform to a frame in which the initial speeds are u and 0 (ie. stationary target), then

$$u = \frac{2v}{1 + v^2/c^2}$$

Then

$$\gamma_u = 2\gamma_v^2 - 1.$$

So we require

$$\gamma_u \geq 2\left(1 + \frac{M}{2m}\right)^2 - 1 = 1 + \frac{2M}{m} + \frac{M^2}{2m}.$$

This means that the initial kinetic energy in this frame must be

$$m(\gamma_u - 1)c^2 = \left(2 + \frac{M}{2m}\right) Mc^2,$$

which could be much larger than Mc^2, especially if $M \gg m$, which usually the case. For example, the mass of the Higgs boson is 130 times the mass of the proton. So it would be much advantageous to collide two beams of protons head on, as opposed to hitting a fixed target.