Part-II Electrodynamics

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Lecture notes

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1 Introduction

We begin with a reminder of Maxwell's equations, which were discussed at length in the IB course *Electromagnetism*:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1.1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Faraday)} \tag{1.2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \,. \qquad \text{(Ampére)} \tag{1.4}$$

Here $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ are the electric and magnetic fields, respectively, $\rho(\mathbf{x}, t)$ is the electric charge density, and $\mathbf{J}(\mathbf{x}, t)$ is the electric current density (defined so that $\mathbf{J} \cdot d\mathbf{S}$ is the charge per time passing through a stationary area element $d\mathbf{S}$). The constants $\epsilon_0 = 8.854187 \cdots \times 10^{-12}\,\mathrm{kg}^{-1}\,\mathrm{m}^{-3}\,\mathrm{s}^4\,\mathrm{A}^2$ (defined) is the electric permittivity of free space, and $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{kg\,m\,s}^{-2}\,\mathrm{A}^{-2}$ (definition) is the permeability of free space. Via the force laws (see below), they serve to define the magnitudes of SI electrical units in terms of the units of mass, length and time. They are not independent since $\mu_0\epsilon_0$ has the units of $1/\text{speed}^2$ – indeed, as we review shortly, $\mu_0\epsilon_0 = 1/c^2$. By convention, we choose μ_0 to take the value given here, which then defines the unit of current (ampere) and hence charge (coulomb or ampere second). Since we define the metre in terms of the second (the latter defined with reference to the frequency of a particular hyperfine transition in caesium) such that $c = 2.99792458\,\mathrm{m\,s}^{-1}$, the value of ϵ_0 is then fixed as $1/(\mu_0 c^2)$.

With the inclusion of the *displacement current* $\mu_0\epsilon_0 \partial\mathbf{E}/\partial t$ in Eq. (1.4), the continuity equation is satisfied, which expresses the conservation of electric charge:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \,. \tag{1.5}$$

Maxwell's equations tell us how charges and currents generate electric and magnetic fields. We also need to know how charged particles move in electromagnetic fields, and this is governed by the *Lorentz force law*:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \,, \tag{1.6}$$

where q is the charge on the particle and $\mathbf{v}$ is its velocity. The continuum form of the force equation follows from summing over a set of charges; the force density (force per volume) is then

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B} \,, \tag{1.7}$$

in terms of the charge and current densities.

Maxwell's equations are a remarkable triumph of 19th century theoretical physics. In particular, they unify the phenomena of not only electricity and magnetism but also light. To see this, consider Maxwell's equations in free space ($\rho = 0$ and $\mathbf{J} = 0$). Taking the curl of Eq. (1.2), and using Eqs (1.1) and (1.4), gives[1]

$$\boldsymbol{\nabla}^2\mathbf{E} = \epsilon_0\mu_0 \frac{\partial^2\mathbf{E}}{\partial t^2} \,. \tag{1.8}$$

This is a wave equation for the electric field with wave speed $c = 1/\sqrt{\mu_0\epsilon_0}$. The equivalent equation for $\mathbf{B}$ follows from Eq. (1.4). We see that *electromagnetic waves* propagate in free space with speed that is the speed of light in vacuum.

Despite these successes, there is an apparent issue with Maxwell's equations in that they are not invariant under Galilean transformations. Recall that these are transformations of the form

$$t \to t \,, \qquad \mathbf{x} \to \mathbf{x} - \mathbf{u}t \,, \tag{1.9}$$

and so $\mathbf{v} \to \mathbf{v} - \mathbf{u}$, where $\mathbf{u}$ is the relative velocity of two reference frames. Newton's law $d\mathbf{p}/dt = \mathbf{F}$ (where $\mathbf{p} = m\mathbf{v}$ with m the mass of the particle) is invariant under such transformations since forces are frame-invariant in Newtonian dynamics. The Lorentz force law can be made Galilean invariant if it is assumed that the electric and magnetic fields transform under Galilean transformations as $\mathbf{E} \to \mathbf{E} + \mathbf{u} \times \mathbf{B}$ and $\mathbf{B} \to \mathbf{B}$. However, there is a problem in that Maxwell's equations predict waves that propagate at the fixed speed c, but under Galilean transformations the velocity of a wave must change in the same way as particle velocities under a change of frame. One possible way out of this is to assume that Maxwell's equations are not Galilean invariant in that they only hold in one particular reference frame. The problem with this is that, unlike the equations governing the propagation of sound waves in a gas say, it is not clear what defines the preferred frame. This interpretation began to look increasingly untenable with experiments (by Fizeau and Michelson and Morley in the second half of the 19th century) on the velocity of light in moving fluids, and this was a major motivation for Einstein introducing the framework of special relativity in 1905. Within special relativity, it is Galilean invariance and Newtonian mechanics that are

[1] Recall the identity $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}) - \boldsymbol{\nabla}^2\mathbf{E}$.

replaced by new relativity principles, while Maxwell's equations remain form-invariant under these new (Lorentz) transformations.

In this course we shall establish the form-invariance of Maxwell's equations under Lorentz transformations. In doing so, we shall learn how to rewrite electromagnetism in a more *covariant* formalism that makes manifest its invariance under Lorentz transformations. With this streamlined formalism, we shall discuss the generation of electromagnetic radiation by accelerating charges and develop action principles for the electromagnetic field, which is an important step towards developing a quantum version of the theory (see Part-III courses in Quantum Field Theory).

2 Review of electrostatics

We begin with a brief review of electrostatics. Here, we deal with *static* charge distributions, i.e., $\partial/\partial t = 0$ and $\mathbf{J} = 0$. Maxwell's equations for the electric and magnetic fields decouple in static situations, and those for the electric field become

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} , \quad \boldsymbol{\nabla} \times \mathbf{E} = 0 . \tag{2.1}$$

Since $\mathbf{E}$ has vanishing curl, we can write it in terms of the gradient of the *electrostatic potential* $\phi(\mathbf{x})$ as $\mathbf{E} = -\boldsymbol{\nabla}\phi$. Combining with $\boldsymbol{\nabla} \cdot \mathbf{E} = \rho/\epsilon_0$ gives the *Poisson equation*:

$$\nabla^2 \phi = -\rho/\epsilon_0 . \tag{2.2}$$

A typical problem in electrostatics is to determine the field within some region S due to specified charges within S, and subject to appropriate boundary conditions on the (closed) boundary ∂S. In such situations, Poisson's equation has a unique solution subject to *Dirichlet boundary conditions* (ϕ specified on ∂S), or a unique solution up to an irrelevant constant for *Neumann boundary conditions* (the normal derivative of ϕ specified on ∂S). Of course, these boundary conditions must result from other charges outside the region of interest.

Here, we are interested in the field of localised charged distributions with no boundary surfaces. In this case, we can assume that the potential decays to zero at spatial infinity. Since the Poisson equation is linear, we can solve it with the help of an appropriate *Green's function*, $G(\mathbf{x}; \mathbf{x}')$:

$$\phi(\mathbf{x}) = -\frac{1}{\epsilon_0} \int G(\mathbf{x}; \mathbf{x}') \rho(\mathbf{x}') \, d^3\mathbf{x}' . \tag{2.3}$$

The Green's function has to satisfy $\nabla^2 G(\mathbf{x}; \mathbf{x}') = \delta^{(3)}(\mathbf{x} - \mathbf{x}')$, where the Dirac delta function appears on the right, and tend to zero as $|\mathbf{x}| \to \infty$. Homogeneity and isotropy[2] demand that the Green's function only depends on $|\mathbf{x} - \mathbf{x}'|$. Taking $\mathbf{x}' = 0$, G is then a function of only $r \equiv |\mathbf{x}|$. Away from the origin, we have

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial G}{\partial r}\right) = 0\,, \tag{2.4}$$

for which the solution that decays at infinity is $G = -A/r$. We can fix the integration constant A by integrating $\nabla^2 G = \delta^{(3)}(\mathbf{x})$ over some region S that encloses the origin. Using the divergence theorem, we have

$$\int_{\partial S} \boldsymbol{\nabla} G \cdot d\mathbf{S} = 1\,. \tag{2.5}$$

Taking the region S to be a sphere of radius R, the integral on the left evaluates to $(A/R^2) \times 4\pi R^2$ and so $A = 1/(4\pi)$. The appropriate Green's function is thus

$$G(\mathbf{x};\mathbf{x}') = \frac{-1}{4\pi|\mathbf{x} - \mathbf{x}'|}\,, \tag{2.6}$$

and the solution of Poisson's equation is

$$\phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\,d^3\mathbf{x}'\,. \tag{2.7}$$

For a point charge at the origin, $\rho(\mathbf{x}) = q_1\delta^{(3)}(\mathbf{x})$, and the potential $\phi = q_1/(4\pi\epsilon_0 r)$. The associated electric field is

$$\mathbf{E} = \frac{q_1}{4\pi\epsilon_0 r^2}\hat{\mathbf{x}}\,, \tag{2.8}$$

where $\hat{\mathbf{x}} = \mathbf{x}/r$ is a unit vector in the direction of $\mathbf{x}$. The Lorentz force law (1.6) then gives the force on a charge q_2 in the field of q_1 as

$$\mathbf{F}_{21} = \frac{q_1 q_2}{4\pi\epsilon_0|\mathbf{x}_1 - \mathbf{x}_2|^3}(\mathbf{x}_2 - \mathbf{x}_1)\,, \tag{2.9}$$

where we have generalised to have q_1 at $\mathbf{x}_1$ and q_2 at $\mathbf{x}_2$. This is simply Coulomb's (inverse-square) law. Note that the force is a power-law and therefore has no preferred scale. Ultimately, this reflects the fact that, in a quantum treatment, the gauge boson that carries the electromagnetic force (the photon) is massless. If the photon did have mass, this would introduce a scale into the problem and the potential of a point charge would typically have the *Yukawa* form, $\phi \propto e^{-r/l}/r$ for some scale-length l inversely proportional to the mass. As an example of such force carriers, the gauge bosons that carry the weak and strong forces are massive with sub-atomic associated length-scales.

[2]Under rigid transformations of the charge density, i.e., translations and rotations, the potential should transform similarly. For example, if the charges are all translated by $\mathbf{a}$, the potential should translate to $\phi(\mathbf{x}) \to \phi(\mathbf{x} - \mathbf{a})$.

2.1 Multipole expansions

We now ask the question how does the potential and field due to a localised charge distribution vary at distances large compared to the size of the distribution? We might expect that if there is a non-zero total charge Q, asymptotically the field will look like that due to a point charge Q. This is indeed correct, but what happens if the total charge vanishes? To answer this, we consider the *multipole expansion* of Eq. (2.7).

Consider a charge distribution localised within a region of extent a. We aim to calculate the fields at distances from the charges that are large compared to a. As we shall see, these are generally given in terms of simple low-order multipoles of the charge distribution. Choose the origin close to the charges, so that all charges are within a of the origin. Observe at $\mathbf{x}$ where $r \equiv |\mathbf{x}| \gg a$. We now deal with the $1/|\mathbf{x}-\mathbf{x}'|$ factor in Eq. (2.7) by expanding in the small quantity $|\mathbf{x}'|/r$. Using

$$|\mathbf{x}-\mathbf{x}'|^2 = \mathbf{x}\cdot\mathbf{x} - 2\mathbf{x}\cdot\mathbf{x}' + \mathbf{x}'\cdot\mathbf{x}' \,, \tag{2.10}$$

we have

$$\begin{aligned}
|\mathbf{x}-\mathbf{x}'|^{-1} &= \frac{1}{r}\left(1 - 2\frac{\mathbf{x}\cdot\mathbf{x}'}{r^2} + \frac{|\mathbf{x}'|^2}{r^2}\right)^{-1/2} \\
&\approx \frac{1}{r}\left[1 + \frac{\mathbf{x}\cdot\mathbf{x}'}{r^2} - \frac{1}{2}\frac{|\mathbf{x}'|^2}{r^2} + \frac{3}{8}\left(\frac{-2\mathbf{x}\cdot\mathbf{x}'}{r^2}\right)^2 + \cdots\right] \\
&\approx \frac{1}{r} + \frac{\mathbf{x}\cdot\mathbf{x}'}{r^3} + \frac{3(\mathbf{x}\cdot\mathbf{x}')^2 - r^2|\mathbf{x}'|^2}{2r^5} + \cdots \,.
\end{aligned} \tag{2.11}$$

Note that the second term on the right is $O(a/r)$ times the first, and the second $O(a/r)^2$ times the first. Writing $3(\mathbf{x}\cdot\mathbf{x}')^2 - r^2|\mathbf{x}'|^2 = x_i x_j (3x_i' x_j' - \delta_{ij}|\mathbf{x}'|^2)$ (summation convention), we can write the potential in the form

$$\phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{\mathbf{x}\cdot\mathbf{P}}{r^3} + \frac{1}{2}\frac{Q_{ij}x_i x_j}{r^5} + \cdots\right) , \tag{2.12}$$

where the multipole moments of the charge distribution are

$$Q \equiv \int \rho(\mathbf{x}')\, d^3\mathbf{x}' \qquad \text{(Total charge or monopole)} \tag{2.13}$$

$$\mathbf{P} \equiv \int \rho(\mathbf{x}')\mathbf{x}'\, d^3\mathbf{x}' \qquad \text{(Dipole moment)} \tag{2.14}$$

$$Q_{ij} \equiv \int \rho(\mathbf{x}')(3x_i' x_j' - \delta_{ij}|\mathbf{x}'|^2)\, d^3\mathbf{x}' \qquad \text{(Quadrupole moment)}\,. \tag{2.15}$$

The quadrupole moment is a symmetric, trace-free tensor and therefore has five independent components. The potential and fields due to these multipole moments vary as

$$\begin{aligned} &\text{Monopole:} && \phi \sim r^{-1} && E \sim r^{-2} && (2.16)\\ &\text{Dipole:} && \phi \sim r^{-2} && E \sim r^{-3} && (2.17)\\ &\text{Quadrupole:} && \phi \sim r^{-3} && E \sim r^{-4}. && (2.18) \end{aligned}$$

As $r \to \infty$, the field is dominated by that of the lowest order non-vanishing multipole moment. For example, if the total charge vanishes, asymptotically the field will vary like a dipole (unless the dipole moment vanishes too).

A simple charge distribution with no total charge but a non-vanishing dipole moment results from taking a pair of oppositely charged particles, q and $-q$, and separating them by some vector $\mathbf{d}$. The dipole moment is then $\mathbf{P} = q\mathbf{d}$. The dipole moment is critical to inter-molecular bonding in chemistry. The strong hydrogen bonds between water molecules arise from interactions of the electric dipole moments of the polar molecules.

We can make a simple charge distribution with vanishing total charge and dipole moment by adding two dipoles end-to-end. This gives charges q at $\pm d$ along the x-axis and a charge of $-2q$ at the origin, say. The dipole vanishes by symmetry, but the quadrupole moment is

$$Q_{ij} = 2qd^2\,\mathrm{diag}(2,-1,-1)\,. \tag{2.19}$$

It is worth noting that, generally, the multipole moments of the charge distribution depend on the choice of origin. Specifically, if we shift the origin by $-\mathbf{a}$ (or equivalently move all the charges by $\mathbf{a}$), the total charge is invariant but the dipole changes to

$$\mathbf{P} \to \mathbf{P} + Q\mathbf{a}\,. \tag{2.20}$$

We see that only if the total charge vanishes is the dipole independent of the choice of origin. A little thought should convince you that the multipole moments have to change to preserve the value of the potential and field at a given physical location, since shifting the origin also changes $\mathbf{x}$. Note that if $Q \neq 0$, we can always choose the origin so that the dipole moment vanishes, i.e., take $\mathbf{a} = -\mathbf{P}/Q$.

Exercise: Show that the quadrupole moment transforms under a change of origin as

$$Q_{ij} \to Q_{ij} + [3(a_iP_j + a_jP_i) - 2\delta_{ij}\mathbf{a}\cdot\mathbf{P}] + Q\left(3a_ia_j - \delta_{ij}\mathbf{a}^2\right)\,. \tag{2.21}$$

It is therefore invariant if both the total charge and dipole vanish. More generally, the lowest order non-vanishing multipole moment is independent of the choice of origin.

2.2 Electrostatic energy

In building up a charge distribution, we generally have to do work to overcome repulsive forces. We can interpret this work as electrostatic potential energy of the charge distribution. If we bring up a test charge q from infinity to a point $\mathbf{y}$ in an electrostatic field, we have to do work $W = -q\int^{\mathbf{y}} \mathbf{E}\cdot d\mathbf{x}$. The integral is independent of the path taken since the field is *conservative* (i.e., curl free). In terms of the potential,

$$W = q\int^{\mathbf{y}} \boldsymbol{\nabla}\phi\cdot d\mathbf{x} = q\phi(\mathbf{y})\,. \tag{2.22}$$

Consider now a charge distribution $\rho(\mathbf{x})$ with associated potential $\phi(\mathbf{x})$. We want to figure out how much work it took to build up this distribution starting with infinitely separated charges. We can imagine doing this by incrementing the charge density from $\lambda\rho(\mathbf{x})$ to $(\lambda+d\lambda)\rho(\mathbf{x})$, i.e., by bringing in infinitesimal charges $\rho(\mathbf{x})d^3\mathbf{x}d\lambda$ to each volume $d^3\mathbf{x}$. Repeating this from $\lambda = 0$ to $\lambda = 1$ builds up the entire charge distribution. At each stage, the new charge that we bring in feels a potential $\lambda\phi(\mathbf{x})$ by linearity. The increment of work we have to do for each volume is thus $\lambda\phi(\mathbf{x})\rho(\mathbf{x})d^3\mathbf{x}d\lambda$. Integrating over λ from 0 to 1 gives the total work

$$W = \frac{1}{2}\int \rho(\mathbf{x})\phi(\mathbf{x})\,d^3\mathbf{x}\,. \tag{2.23}$$

This is the required electrostatic potential energy.

An alternative, and very useful, way of reinterpreting the electrostatic potential energy is to think of it being stored in the field itself. To see how this works, we use $\rho = \epsilon_0\boldsymbol{\nabla}\cdot\mathbf{E}$ in Eq. (2.23) to find

$$\begin{aligned} W &= \frac{1}{2}\epsilon_0\int \phi\boldsymbol{\nabla}\cdot\mathbf{E}\,d^3\mathbf{x} \\ &= \frac{1}{2}\epsilon_0\int \left[\boldsymbol{\nabla}\cdot(\phi\mathbf{E}) - \mathbf{E}\cdot\boldsymbol{\nabla}\phi\right]\,d^3\mathbf{x} \\ &= \frac{1}{2}\epsilon_0\int |\mathbf{E}|^2\,d^3\mathbf{x}\,, \end{aligned} \tag{2.24}$$

where we used the divergence theorem in the last step (noting that the product of the field and potential go to zero at least as fast as $1/r^3$ as $r\to 0$ for a localised charge distribution). We see that the *energy density* of the electrostatic field can be taken to be $\epsilon_0|\mathbf{E}|^2/2$.

3 Review of magnetostatics

In magnetostatics, we are interested in steady distributions of current hence $\partial/\partial t = 0$. Since current is just the flow of charge, the charges themselves are not static but the charge density is time independent. This means that we must have $\boldsymbol{\nabla}\cdot\mathbf{J}=0$. Maxwell's equation for the magnetic field now give

$$\boldsymbol{\nabla}\cdot\mathbf{B}=0\,,\quad \boldsymbol{\nabla}\times\mathbf{B}=\mu_0\mathbf{J}\,. \tag{3.1}$$

Since the magnetic field is divergence free, it can always be written as the curl of some *magnetic vector potential* $\mathbf{A}$, i.e., $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}$.

At this point, we have to face the issue of *gauge freedom.* The physical magnetic field does not uniquely determine the vector potential since both $\mathbf{A}$ and $\mathbf{A}+\boldsymbol{\nabla}\chi$ have the same curl.[3] We can use this gauge freedom to our advantage with a judicious choice of gauge. For magnetostatics it is convenient to adopt the *Coulomb gauge* in which $\boldsymbol{\nabla}\cdot\mathbf{A}=0$. (We shall generalise this to electrodynamics later.) We can always find such a vector potential since if we start with some $\mathbf{A}$ in a general gauge, a gauge transformation with $\nabla^2\chi=-\boldsymbol{\nabla}\cdot\mathbf{A}$ will yield a divergence-free vector potential. The Poisson equation for χ can always be solved (it is just the same problem as solving for ϕ given ρ in electrostatics).

Adopting the Coulomb gauge, and using $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}$ in the second of Eq. (3.1), we have

$$\begin{aligned}
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A})&=\mu_0\mathbf{J}\\
\Rightarrow\quad \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A})-\nabla^2\mathbf{A}&=\mu_0\mathbf{J}\\
\Rightarrow\quad -\nabla^2\mathbf{A}&=\mu_0\mathbf{J}\quad \text{Coulomb gauge)}\,.
\end{aligned} \tag{3.2}$$

This is just the vector form of Poisson's equation, and the solution with $\mathbf{A}$ tending to zero at infinity for a localised current distribution is

$$\mathbf{A}(\mathbf{x})=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,d^3\mathbf{x}'\,. \tag{3.3}$$

We can check that the gauge condition is satisfied as follows:

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{A}&=\frac{\mu_0}{4\pi}\int\mathbf{J}(\mathbf{x}')\cdot\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)d^3\mathbf{x}'\\
&=-\frac{\mu_0}{4\pi}\int\mathbf{J}(\mathbf{x}')\cdot\boldsymbol{\nabla}_{\mathbf{x}'}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)d^3\mathbf{x}'\\
&=-\frac{\mu_0}{4\pi}\int\left[\boldsymbol{\nabla}_{\mathbf{x}'}\cdot\left(\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right)-\frac{\boldsymbol{\nabla}_{\mathbf{x}'}\cdot\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right]d^3\mathbf{x}'\\
&=\frac{\mu_0}{4\pi}\int\frac{\boldsymbol{\nabla}_{\mathbf{x}'}\cdot\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,d^3\mathbf{x}'=0\,,
\end{aligned} \tag{3.4}$$

[3] The related problem in electrostatics is that the potential ϕ is only determined up to a constant by the electric field. We fixed this gauge freedom by taking $\phi=0$ at infinity.

where we used the divergence theorem in the step leading to the last line, noting that the current vanishes at infinity for a localised current distribution, and $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$ in the last equality. Note that $\boldsymbol{\nabla}_{\mathbf{x}'}$ is the gradient with respect to $\mathbf{x}'$. The magnetic field follows from taking the curl in Eq. (3.3):

$$\begin{aligned} \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} &= -\frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{x}') \times \boldsymbol{\nabla} \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) d^3\mathbf{x}' \\ &= \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{x}') \times \frac{(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \, d^3\mathbf{x}' \, . \end{aligned} \tag{3.5}$$

Finally, for a current I localised in a thin loop of wire we can replace $\mathbf{J}(\mathbf{x}')d^3\mathbf{x}' \to I d\mathbf{x}'$, and so

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \oint \frac{d\mathbf{x}' \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \, . \tag{3.6}$$

This is the Biot–Savart law.

3.1 Multipole expansion

As for electrostatics, we can ask what is the form of the magnetic field at a large distance from a localised current distribution. Using the first two terms of Eq. (2.11) in Eq. (3.3), we have

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi r} \int \left(1 + \frac{\mathbf{x} \cdot \mathbf{x}'}{r^2} + \cdots \right) \mathbf{J}(\mathbf{x}') \, d^3\mathbf{x}' \, . \tag{3.7}$$

To make further progress, we establish the following useful result. Since $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$ here, we have

$$\begin{aligned} 0 &= \int_S (x_j x_k \ldots x_l) \boldsymbol{\nabla} \cdot \mathbf{J} \, d^3\mathbf{x} \\ &= \int_S \frac{\partial}{\partial x_i} \left[J_i (x_j x_k \ldots x_l) \right] \, d^3\mathbf{x} - \int_S J_i \frac{\partial}{\partial x_i} (x_j x_k \ldots x_l) \, d^3\mathbf{x} \\ &= \int_{\partial S} (x_j x_k \ldots x_l) \mathbf{J} \cdot d\mathbf{S} - \int_S \left[(J_j x_k \ldots x_l) + \cdots + (x_j x_k \ldots J_l) \right] \, d^3\mathbf{x} \, , \end{aligned} \tag{3.8}$$

where we used the generalisation of the divergence theorem in going to the last line. Extending the region of integration S to be all space, the surface term then vanishes for a localised current distribution and so must the last term on the right of the final equality. Special cases of this result that are useful for our purposes here are

$$\int \mathbf{J} \, d^3\mathbf{x} = 0 \, , \qquad \int (J_i x_j + J_j x_i) \, d^3\mathbf{x} = 0 \, . \tag{3.9}$$

The first of these results means that the first (monopole) term in Eq. (3.7) vanishes: there is no magnetic analogue of the total charge Q of electrostatics. This is because there are no magnetic monopoles since $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$. For the second term in Eq. (3.7), we have

$$\begin{aligned} A_i(\mathbf{x}) &= \frac{\mu_0}{4\pi r^3} \int x_j x'_j J_i(\mathbf{x}')\, d^3\mathbf{x}' \\ &= \frac{\mu_0}{4\pi r^3} \frac{1}{2} \int x_j \left[x'_j J_i(\mathbf{x}') - x'_i J_j(\mathbf{x}') \right] d^3\mathbf{x}' \,, \end{aligned} \tag{3.10}$$

where we used the second of Eq. (3.9). It follows that

$$\begin{aligned} \mathbf{A}(\mathbf{x}) &= \frac{\mu_0}{4\pi r^3} \frac{1}{2} \int \left[\mathbf{x} \cdot \mathbf{x}' \mathbf{J}(\mathbf{x}') - \mathbf{x}' \mathbf{x} \cdot \mathbf{J}(\mathbf{x}') \right] d^3\mathbf{x}' \\ &= \frac{\mu_0}{4\pi r^3} \left(\frac{1}{2} \int \mathbf{x}' \times \mathbf{J}(\mathbf{x}')\, d^3\mathbf{x}' \right) \times \mathbf{x} \,. \end{aligned} \tag{3.11}$$

Defining the *magnetic dipole moment*,

$$\mathbf{m} \equiv \frac{1}{2} \int \mathbf{x}' \times \mathbf{J}(\mathbf{x}')\, d^3\mathbf{x}' \,, \tag{3.12}$$

we can write the vector potential at great distance from a localised current distribution as

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi r^3} \mathbf{m} \times \mathbf{x} + \cdots \,, \tag{3.13}$$

where the remaining terms are smaller than the dipole term by factors of a/r, where a is the spatial extent of the current distribution.

The magnetic moment is invariant under a change of origin since under $\mathbf{x}' \to \mathbf{x}' + \mathbf{a}$, the dipole transforms as

$$\mathbf{m} \to \mathbf{m} + \frac{1}{2} \int \mathbf{a} \times \mathbf{J}(\mathbf{x}')\, d^3\mathbf{x}' = \mathbf{m} \,, \tag{3.14}$$

since $\int \mathbf{J}\, d^3\mathbf{x} = 0$.

The magnetic field of a dipole follows from taking the curl of Eq. (3.13):

$$\begin{aligned} \mathbf{B}(\mathbf{x}) &= \frac{\mu_0}{4\pi} \boldsymbol{\nabla} \times \left(\frac{\mathbf{m} \times \mathbf{x}}{r^3} \right) \\ &= \frac{\mu_0}{4\pi} \left[\boldsymbol{\nabla} \left(\frac{1}{r^3} \right) \times (\mathbf{m} \times \mathbf{x}) + \frac{1}{r^3} \boldsymbol{\nabla} \times (\mathbf{m} \times \mathbf{x}) \right] \\ &= \frac{\mu_0}{4\pi} \left[-\frac{3}{r^5} \mathbf{x} \times (\mathbf{m} \times \mathbf{x}) + \frac{1}{r^3} (3\mathbf{m} - \mathbf{m}) \right] \\ &= \frac{\mu_0}{4\pi r^5} \left(3\mathbf{m} \cdot \mathbf{x}\mathbf{x} - r^2 \mathbf{m} \right) \,. \end{aligned} \tag{3.15}$$

This has exactly the same form as the electric field of a dipole. Indeed, noting that

$$\boldsymbol{\nabla}\left(\frac{\mathbf{m}\cdot\mathbf{x}}{r^3}\right) = \frac{\mathbf{m}}{r^3} - 3\frac{\mathbf{m}\cdot\mathbf{x}\mathbf{x}}{r^5}\,, \tag{3.16}$$

we can write the magnetic field of a dipole as a gradient of a dipole potential:

$$\mathbf{B}(\mathbf{x}) = -\frac{\mu_0}{4\pi}\boldsymbol{\nabla}\left(\frac{\mathbf{m}\cdot\mathbf{x}}{r^3}\right)\,. \tag{3.17}$$

For the case of current flowing in a thin wire confined to a plane, but otherwise of arbitrary shape, the magnetic moment can be written as the product of the current and the vector area (in the right-handed sense with respect to positive current) of the current loop, $\mathbf{S}$. This follows since, replacing $\mathbf{J}d^3\mathbf{x}$ with $Id\mathbf{x}$, we have

$$\begin{aligned}\mathbf{m} &= \frac{1}{2}\int \mathbf{x}\times\mathbf{J}\,d^3\mathbf{x}\\ &= \frac{I}{2}\oint \mathbf{x}\times d\mathbf{x}\\ &= I\mathbf{S}\,.\end{aligned} \tag{3.18}$$

For more complicated current distributions, for which the dipole may vanish, it may be necessary to consider higher multipoles of the current to obtain the field at great distances. A simple example is current flowing in a figure-of-eight loop. This is two opposite dipoles displaced from each other and so has no net magnetic dipole moment. Instead, the quadrupole will dominate the asymptotic field.

3.2 Magnetic energy

In establishing a current distribution, work must be done against induced electromotive forces. Incrementing the current density by $d\lambda\mathbf{J}(\mathbf{x})$, as we did in electrostatics, the flux linked by any current loop changes and the current-generating sources must do work. For a set of current loops, with final current I_i and each linking total flux Φ_i,[4] the work done in establishing the current (in addition to any Ohmic losses in the wires) is

$$W = \frac{1}{2}\sum_i I_i\Phi_i\,, \tag{3.19}$$

where the factor of 1/2 follows from linearity. The sum here is over all current loops. We can write this is in a more convenient form by noting that

$$\Phi = \int(\boldsymbol{\nabla}\times\mathbf{A})\cdot d\mathbf{S} = \oint \mathbf{A}\cdot d\mathbf{x} \tag{3.20}$$

[4] The magnetic flux linked by a current loop is $\Phi = \int \mathbf{B}\cdot d\mathbf{S}$, where the integral is over any surface bounded by the loop.

which follows from Stokes theorem. Making the usual replacement $Id\mathbf{x} \to \mathbf{J}d^3\mathbf{x}$, we have

$$\begin{aligned} W &= \frac{1}{2}\sum_i I_i \oint_i \mathbf{A}\cdot d\mathbf{x} \\ &= \frac{1}{2}\int \mathbf{A}\cdot\mathbf{J}\, d^3\mathbf{x}\,, \end{aligned} \tag{3.21}$$

where the volume integral can be taken over all space.

Exercise: Show that the expression on the right of Eq. (3.21) is gauge invariant as a consequence of $\boldsymbol{\nabla}\cdot\mathbf{J} = 0$.

As with electrostatics, we can fruitfully think of this work as being potential energy of the magnetic field. Eliminating $\mathbf{J}$ from Eq. (3.21) with $\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J}$, we have

$$\begin{aligned} W &= \frac{1}{2\mu_0}\int \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})\, d^3\mathbf{x} \\ &= \frac{1}{2\mu_0}\epsilon_{ijk}\int A_i\frac{\partial B_k}{\partial x_j}\, d^3\mathbf{x} \\ &= \frac{1}{2\mu_0}\epsilon_{ijk}\int \frac{\partial(A_iB_k)}{\partial x_j}\, d^3\mathbf{x} - \frac{1}{2\mu_0}\epsilon_{ijk}\int B_k\frac{\partial A_i}{\partial x_j}\, d^3\mathbf{x} \\ &= \frac{1}{2\mu_0}\int \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A})\, d^3\mathbf{x} \\ &= \frac{1}{2\mu_0}\int |\mathbf{B}|^2\, d^3\mathbf{x}\,, \end{aligned} \tag{3.22}$$

where we used the generalisation of the divergence theorem to show that the first term in the third line is vanishing for localised current distributions. It follows that we can take the magnetic energy density to be $|\mathbf{B}|^2/(2\mu_0)$.

4 Review of special relativity

You met the revolutionary ideas of special relativity in the IA course *Mechanics and Relativity*. Here we recap the main ideas and introduce a more streamlined notation that we shall use in the rest of the course.

Special relativity is based on the following two postulates:

- The laws of physics are the same in all inertial frames[5].
- Light signals in vacuum propagate rectilinearly and with the same speed c in all inertial frames.

The first postulate means that any equation of physics must take the same form when written in terms of spacetime coordinates in different inertial frames.

4.1 Lorentz transformations

The consequence of these two postulates is that for two inertial frames, S and S', with spatial axes aligned and with S' moving at speed $v = \beta c$ along the positive x-axis of S and with origins coincident at $t = t' = 0$, the spacetime coordinates of any event are related by the (standard) Lorentz transformation:

$$\begin{aligned} ct' &= \gamma(ct - \beta x) \\ x' &= \gamma(x - \beta ct) \\ y' &= y \\ z' &= z\,, \end{aligned} \tag{4.1}$$

where $\gamma \equiv (1-\beta^2)^{-1/2}$. The transformation for a general relative velocity $\mathbf{v}$, with axes still aligned, can be obtained by first co-rotating the spatial axes of S and S' so that the relative motion is along the x-axis, performing the above transformation, and then back-rotating. Such Lorentz transformations are called *Lorentz boosts* and they contain no additional relative (spatial) rotation between the two frames. Note that the Lorentz transformation of Eq. (4.1) reduces to the Galilean transformation for $\beta \ll 1$.

By construction, Lorentz transformations preserve the spacetime interval between two events, i.e.,

$$(\Delta\mathbf{x}')^2 - (c\Delta t')^2 = (\Delta\mathbf{x})^2 - (c\Delta t)^2\,. \tag{4.2}$$

If we write the spacetime coordinates as $x^\mu = (ct, x_i)$, with $\mu = 0$ labelling the time component, and $\mu = i = 1,2,3$ labelling the space components, we can write the invariant interval for infinitesimal separations in the form

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu\,. \tag{4.3}$$

[5]Inertial frames are frames of reference in which a freely-moving particle (i.e., one subject to no external forces) proceeds at constant velocity in accordance with Newton's first law. A frame of reference can be thought of as a set of test particles, all at rest with respect to each other, and each equipped with a synchronised clock. Laying the test particles out on a Cartesian grid and assigning spatial coordinates, spacetime coordinates can be assigned to any event in spacetime by the spatial coordinates of the test particle on whose wordline the event occurs, and by the time on the test particle's clock at the event.

Here, $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is the *Minkowski metric* and we have introduced the summation convention that *repeated spacetime (Greek) indices, with one upper and one lower, are summed over.*

More generally, we define *homogeneous Lorentz transformations* to be transformations from one system of spacetime coordinates x^μ to another x'^μ, with

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \,, \tag{4.4}$$

such that

$$\eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta} \,. \tag{4.5}$$

Equivalently, we can write this in matrix form as $\boldsymbol{\Lambda}^T\boldsymbol{\eta}\boldsymbol{\Lambda} = \boldsymbol{\eta}$. Such transformations leave $\eta_{\mu\nu}dx^\mu dx^\nu$ invariant since

$$\begin{aligned} \eta_{\mu\nu}dx'^\mu dx'^\nu &= \underbrace{\eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta}_{\eta_{\alpha\beta}} dx^\alpha dx^\beta \\ &= \eta_{\alpha\beta} dx^\alpha dx^\beta \,. \end{aligned} \tag{4.6}$$

Note that when we sum over repeated indices, the exact symbol we assign to the "dummy" index is irrelevant. For the standard transformation of Eq. (4.1), we have

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{4.7}$$

Homogeneous Lorentz transformations form a *group*. Closure in ensured since, for a composition of two transformations represented by the matrix product $\boldsymbol{\Lambda}_1\boldsymbol{\Lambda}_2$, we have

$$\begin{aligned} (\boldsymbol{\Lambda}_1\boldsymbol{\Lambda}_2)^T\boldsymbol{\eta}(\boldsymbol{\Lambda}_1\boldsymbol{\Lambda}_2) &= \boldsymbol{\Lambda}_2^T\boldsymbol{\Lambda}_1^T\boldsymbol{\eta}\boldsymbol{\Lambda}_1\boldsymbol{\Lambda}_2 \\ &= \boldsymbol{\Lambda}_2^T\boldsymbol{\eta}\boldsymbol{\Lambda}_2 \\ &= \boldsymbol{\eta} \,. \end{aligned} \tag{4.8}$$

Clearly, the set of homogeneous Lorentz transformations contains the identity, $\Lambda^\mu{}_\nu = \delta^\mu_\nu$, and every element is invertible[6] since $\boldsymbol{\Lambda}^T\boldsymbol{\eta}\boldsymbol{\Lambda} = \boldsymbol{\eta}$ implies that $\det(\boldsymbol{\Lambda})^2 = 1$. We shall be most interested in the subgroup of *restricted homogeneous Lorentz transformations*. These are transformations that are continuously connected to the identity, i.e., any element can be taken to the identity by smooth variation of its parameters. In

[6] The inverse of a Lorentz transformation is also a Lorentz transformation since multiplying $\boldsymbol{\Lambda}^T\boldsymbol{\eta}\boldsymbol{\Lambda} = \boldsymbol{\eta}$ on the left with $(\boldsymbol{\Lambda}^T)^{-1}$ and on the right with $\boldsymbol{\Lambda}^{-1}$ implies that $(\boldsymbol{\Lambda}^{-1})^T\boldsymbol{\eta}\boldsymbol{\Lambda}^{-1} = \boldsymbol{\eta}$. Physically, this makes sense since the inverse transforms back from x'^μ to x^μ.

particular, restricted Lorentz transformations have $\det(\mathbf{\Lambda}) = +1$ and[7] $\Lambda^0{}_0 \geq 1$. Physically, they correspond to transformations between inertial frames that exclude space inversion or time reversal.

Since $\eta_{\mu\nu}$ is symmetric, Eq. (4.5) puts 10 independent constraints on the 16 independent components of $\Lambda^\mu{}_\nu$. It follows that a general homogeneous Lorentz transformation is described by $16 - 10 = 6$ parameters. Three of these encode the relative velocity of the two frames (i.e., the boost part) and three any additional spatial rotation.

The invariant interval $\Delta s^2 = \eta_{\mu\nu}\Delta x^\mu x^\nu$ between two events allows us to classify all spacetime events in relation to some given event as being either *spacelike separated* ($\Delta s^2 > 0$), *timelike* ($\Delta s^2 < 0$) or null (lightlike, with $\Delta s^2 = 0$). At any event, the set of null-separated events defines the *light cone*. Light rays emitted from some event and propagating in vacuum have worldlines that lie on the lightcone of the emission event. Massive particles follow worldlines for which the tangent vector is timelike (i.e., has negative norm) and at any event a massive particle will be moving within the lightcone in spacetime. For timelike- and null-separated events, the temporal ordering of the events (i.e., the sign of Δt) is Lorentz invariant so we can speak of the future and past light cone. This is not the case for spacelike-separated events.

4.2 4-vectors

In Newtonian physics, it is very convenient to write our equations in terms of 3D vectors (or tensors) to make manifest their transformation properties under spatial rotations. We now generalise this idea to spacetime 4-vectors and tensors. By expressing our equations in terms of such objects, we can be sure that we are writing down equations that are form invariant under Lorentz transformations and hence satisfy the postulates of special relativity.

We define a 4-vector to be an object A^μ that transforms like the displacement dx^μ under Lorentz transformations, i.e.,

$$A^\mu \to \Lambda^\mu{}_\nu A^\nu \quad \text{for } x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \,. \tag{4.9}$$

It follows that $\eta_{\mu\nu}A^\mu A^\nu$ is invariant, defining a Lorentz-invariant norm. More generally, we can define a Lorentz-invariant scalar product between two 4-vectors, A^μ and B^μ as $\eta_{\mu\nu}A^\mu B^\nu$.

[7]Note that the 00 element of Eq. (4.5) gives

$$\begin{aligned} \eta_{00} &= \eta_{00}(\Lambda^0{}_0)^2 + \sum_i \eta_{ii}(\Lambda^i{}_0)^2 \\ \Rightarrow \quad (\Lambda^0{}_0)^2 &= 1 + \sum_i (\Lambda^i{}_0)^2 \geq 1 \,. \end{aligned}$$

We should properly call A^μ a *contravariant* 4-vector to distinguish it from a *covariant* 4-vector. To motivate the introduction of the latter, consider the transformation of the differential operator $\partial/\partial x^\mu$: the chain rule gives

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}\,. \tag{4.10}$$

Since

$$\frac{\partial x^\nu}{\partial x'^\mu}\underbrace{\frac{\partial x'^\mu}{\partial x^\rho}}_{\Lambda^\mu{}_\rho} = \delta^\nu_\rho\,, \tag{4.11}$$

it follows that $\partial x^\nu/\partial x'^\mu = (\Lambda^{-1})^\nu{}_\mu$, where the right-hand side is the matrix representing the inverse transformation. The Lorentz transformation law for partial derivatives is therefore

$$\frac{\partial}{\partial x'^\mu} = (\Lambda^{-1})^\nu{}_\mu\frac{\partial}{\partial x^\nu}\,. \tag{4.12}$$

Note the distinction to the transformation law for a contravariant 4-vector. It is convenient to *define* the quantities

$$\Lambda_\mu{}^\nu \equiv (\Lambda^{-1})^\nu{}_\mu\,; \tag{4.13}$$

we then define covariant 4-vectors, A_μ, to be objects that transform under Lorentz transformations as

$$A'_\mu = \Lambda_\mu{}^\nu A_\nu\,. \tag{4.14}$$

Note the consistent index placement on both sides of all equations.

We can express $\Lambda_\mu{}^\nu$ in terms of $\Lambda^\mu{}_\nu$ and the Minkowski metric directly by noting that

$$\boldsymbol{\Lambda}^T\boldsymbol{\eta}\boldsymbol{\Lambda} = \boldsymbol{\eta} \quad\Rightarrow\quad \boldsymbol{\Lambda}^{-1} = \boldsymbol{\eta}^{-1}\boldsymbol{\Lambda}^T\boldsymbol{\eta}\,, \tag{4.15}$$

or, in terms of components,

$$\begin{aligned}\Lambda_\mu{}^\nu = (\Lambda^{-1})^\nu{}_\mu &= \eta^{\nu\rho}\Lambda^\tau{}_\rho\eta_{\tau\mu}\\ &= \eta_{\mu\tau}\eta^{\nu\rho}\Lambda^\tau{}_\rho\,,\end{aligned} \tag{4.16}$$

where we have written the components of the inverse of the metric as $\eta^{\mu\nu}$. Numerically, the components $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ are the same, and

$$\eta^{\mu\rho}\eta_{\rho\nu} = \delta^\mu_\nu\,. \tag{4.17}$$

The *contraction* of a contravariant vector A_μ and a covariant vector B^μ is Lorentz invariant since

$$\begin{aligned}A'_\mu B'^\mu &= \Lambda_\mu{}^\nu A_\nu \Lambda^\mu{}_\tau B^\tau\\ &= (\Lambda^{-1})^\nu{}_\mu\Lambda^\mu{}_\tau A_\nu B^\tau\\ &= \delta^\nu_\tau A_\nu B^\tau\\ &= A_\nu B^\nu\,.\end{aligned} \tag{4.18}$$

Every contravariant vector can be mapped with the Minkowski metric into an associated covariant vector and vice versa. For example, we can write

$$A_\mu = \eta_{\mu\nu} A^\nu, \quad A^\mu = \eta^{\mu\nu} A_\nu \,. \tag{4.19}$$

Explicitly, this means that $A_0 = -A^0$ and $A_i = A^i$. We can easily check that $\eta_{\mu\nu}A^\nu$ is a covariant vector by considering its transformation law:

$$\begin{aligned} A'_\mu &= \eta_{\mu\nu} A'^\nu \\ &= \eta_{\mu\nu} \Lambda^\nu{}_\tau A^\tau \\ &= \underbrace{\eta_{\mu\nu} \Lambda^\nu{}_\tau \eta^{\tau\rho}}_{\Lambda_\mu{}^\rho} A_\rho \\ &= \Lambda_\mu{}^\rho A_\rho \,. \end{aligned} \tag{4.20}$$

The scalar product between two contravariant vectors, A^μ and B^μ, can then be written either as $\eta_{\mu\nu}A^\mu B^\nu$, or as a contraction $A^\mu B_\mu$ between the contravariant A^μ and the covariant B_μ (or the other way around).

For the covariant differential operator $\partial_\mu \equiv \partial/\partial x^\mu$, we can form the Lorentz-invariant (scalar) operator

$$\Box \equiv \eta^{\mu\nu}\partial_\mu \partial_\nu = -\frac{\partial^2}{\partial (ct)^2} + \nabla^2 \,. \tag{4.21}$$

This is the operator that appears in the wave equation for waves propagating at speed c.

Alternative take on covariant and contravariant components of 4-vectors: In 3D Euclidean space we are used to thinking of vectors as geometric objects, the archetypal example being displacement vectors that connect two points in space. Let us generalise this idea to spacetime so that the 4-vector $d\mathbf{x}$ connects two neighbouring events[8]. By taking unit displacements along the coordinate axes of some inertial frame, we can introduce a basis of four vectors $\mathbf{e}_\mu$. Here, the subscript μ labels the element of the basis. In terms of this basis, we can write

$$d\mathbf{x} = dx^\mu \mathbf{e}_\mu \,. \tag{4.22}$$

Under a Lorentz transformation (i.e., a passive reparameterisation of spacetime with a different set of inertial coordinates) the coordinate differentials dx^μ transform but the 4-vector itself is unchanged. This means that the basis vectors must also change to compensate the change in the dx^μ. Indeed, since $\mathbf{e}_\mu = \partial \mathbf{x}/\partial x^\mu$, we have

$$\mathbf{e}'_\mu = \frac{\partial \mathbf{x}}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial \mathbf{x}}{\partial x^\nu} = \frac{\partial x^\nu}{\partial x'^\nu}\mathbf{e}_\nu = \Lambda_\mu{}^\nu \mathbf{e}_\nu \,. \tag{4.23}$$

[8] We shall temporarily denote 4-vectors by boldface in this brief interlude.

We introduce an inner product on spacetime; this is a bilinear symmetric function of two 4-vectors, $\mathbf{A} = A^\mu \mathbf{e}_\mu$ and $\mathbf{B} = B^\mu \mathbf{e}_\mu$, defined by

$$\mathbf{A} \cdot \mathbf{B} \equiv \eta_{\mu\nu} A^\mu B^\nu \, . \tag{4.24}$$

It follows that we must have

$$\mathbf{e}_\mu \cdot \mathbf{e}_\nu = \eta_{\mu\nu} \, , \tag{4.25}$$

and it is easy to verify that this is preserved under Lorentz transformations.

The *dual basis* of 4-vectors, denoted $\mathbf{e}^\mu$, is defined by

$$\mathbf{e}^\mu \cdot \mathbf{e}_\nu = \delta^\mu_\nu \, , \tag{4.26}$$

so that $\mathbf{e}^\mu = \eta^{\mu\nu} \mathbf{e}_\nu$ and $\mathbf{e}_\mu = \eta_{\mu\nu} \mathbf{e}^\nu$. The dual basis transforms in the same way as the dx^μ under Lorentz transformations, $\mathbf{e}'^\mu = \Lambda^\mu{}_\nu \mathbf{e}^\nu$, to preserve the duality relation in Eq. (4.26). We can express a 4-vector $\mathbf{A}$ in terms of its components on either basis, i.e.,

$$\mathbf{A} = A^\mu \mathbf{e}_\mu = A_\mu \mathbf{e}^\mu \, . \tag{4.27}$$

As usual, we can extract the components by taking the inner product with the appropriate basis 4-vectors: $A^\mu = \mathbf{A} \cdot \mathbf{e}^\mu$ and $A_\mu = \mathbf{A} \cdot \mathbf{e}_\mu$. It follows that

$$A_\mu = (A^\nu \mathbf{e}_\nu) \cdot \mathbf{e}_\mu = \eta_{\mu\nu} A^\nu \, , \tag{4.28}$$

and, similarly, $A^\mu = \eta^{\mu\nu} A_\nu$. Under Lorentz transformations,

$$A'_\mu = \mathbf{A} \cdot \mathbf{e}'_\mu = \mathbf{A} \cdot (\Lambda_\mu{}^\nu \mathbf{e}_\nu) = \Lambda_\mu{}^\nu A_\nu \, , \tag{4.29}$$

so the A_μ (the *covariant* components) transform in the *same* way as the coordinate basis vectors. This is the origin of the covariant and contravariant terminology for the components.

We see that the covariant components A_μ and the contravariant components A^μ are an equivalent description of the same geometric object, the 4-vector $\mathbf{A}$. Some 4-vectors are more naturally represented by their contravariant components, an obvious example being the dx^μ of the spacetime displacement, while for others the contravariant components are more natural. An example of the latter are the $\partial/\partial x^\mu$ of the spacetime gradient operator.

4.2.1 Examples of 4-vectors in relativistic kinematics

Velocity 4-vector. Consider a point particle following a worldline $x^\mu(\tau)$, where τ is the particle's proper time (i.e., time as measured on a clock carried with the particle). The velocity 4-vector is defined by

$$u^\mu = dx^\mu / d\tau \, , \tag{4.30}$$

where we are dividing a 4-vector dx^μ by the scalar $d\tau$. We can express the proper time in terms of the time in any inertial frame S by noting that

$$\begin{aligned} c^2 d\tau^2 = -\eta_{\mu\nu} dx^\mu dx^\nu &= c^2 dt^2 - d\mathbf{x}^2 \\ &= c^2 dt^2 (1 - \mathbf{v}^2/c^2) \\ &= c^2 dt^2/\gamma^2 \,, \end{aligned} \tag{4.31}$$

where γ is the Lorentz factor associated with the particle's velocity $\mathbf{v}$ relative to S. It follows that $d\tau = dt/\gamma$, and so the components of the velocity 4-vector in S are

$$u^\mu = \gamma \frac{d}{dt}(ct, \mathbf{x}) = (\gamma c, \gamma \mathbf{v}) \,. \tag{4.32}$$

Note the norm of u^μ:

$$\eta_{\mu\nu} u^\mu u^\nu = (\eta_{\mu\nu} dx^\mu dx^\nu)/(d\tau)^2 = -c^2 \,. \tag{4.33}$$

Momentum 4-vector. For a massive particle of rest-mass m, the momentum 4-vector is obtained from the velocity 4-vector by multiplication by m:

$$p^\mu \equiv m u^\mu = (\gamma m c, \gamma m \mathbf{v}) \,. \tag{4.34}$$

The norm is $\eta_{\mu\nu} p^\mu p^\nu = -m^2 c^2$. In some inertial frame, the components of p^μ are E/c, the (total) energy measured in that frame, and $\mathbf{p}$, the relativistic 3-momentum, i.e.,

$$p^\mu = (E/c, \mathbf{p}) \quad \text{where } E = \gamma m c^2 \text{ and } \mathbf{p} = \gamma m \mathbf{v} \,. \tag{4.35}$$

For an isolated system, the 4-momentum is *conserved*, i.e., the relativistic 3-momentum and energy are conserved.

More generally, the 3-momentum of a particle is changed by a force $\mathbf{F}$ according to Newton's second law $d\mathbf{p}/dt = \mathbf{F}$. The interpretation of the time component of p^μ as the energy is then consistent since differentiating $\eta_{\mu\nu} p^\mu p^\nu = -m^2 c^2$ gives

$$\begin{aligned} E dE/dt &= c^2 \mathbf{p} \cdot d\mathbf{p}/dt \\ &= c^2 \mathbf{p} \cdot \mathbf{F} \\ \Rightarrow \quad dE/dt &= \mathbf{p} \cdot \mathbf{F}/(\gamma m) = \mathbf{v} \cdot \mathbf{F} \,, \end{aligned} \tag{4.36}$$

where the quantity on the right of the final equality is the usual rate of working of the force. We can introduce a 4-vector force F^μ as $F^\mu = dp^\mu/d\tau$; then

$$\begin{aligned} F^\mu &= \gamma \frac{d}{dt} \left(\frac{E}{c}, \mathbf{p} \right) \\ &= \left(\frac{\gamma}{c} \frac{dE}{dt}, \gamma \mathbf{F} \right) \\ &= \left(\frac{\gamma}{c} \mathbf{v} \cdot \mathbf{F}, \gamma \mathbf{F} \right) \,. \end{aligned} \tag{4.37}$$

The 4-force is necessarily orthogonal to p^μ (and so to u^μ) to preserve the norm of p^μ.

Acceleration 4-vector. The 4-acceleration a^μ is

$$a^\mu = du^\mu/d\tau\,. \tag{4.38}$$

Constancy of the norm $\eta_{\mu\nu}u^\mu u^\nu = -c^2$ implies that a^μ is orthogonal to u^μ. The components of a^μ are

$$\begin{aligned} a^\mu &= \gamma\frac{d}{dt}(\gamma c, \gamma\mathbf{v}) \\ &= \gamma\left(c\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\mathbf{v} + \gamma\mathbf{a}\right) \\ &= \gamma^2\left(\frac{\gamma^2}{c}\mathbf{v}\cdot\mathbf{a}, \frac{\gamma^2}{c^2}\mathbf{a}\cdot\mathbf{v}\mathbf{v} + \mathbf{a}\right)\,, \end{aligned} \tag{4.39}$$

where $\mathbf{a} = d\mathbf{v}/dt$ is the usual 3-acceleration and we used $d\gamma/dt = \gamma^3\mathbf{a}\cdot\mathbf{v}/c^2$, which follows from differentiating $\gamma^{-2} = 1 - \mathbf{v}\cdot\mathbf{v}/c^2$. In the *instantaneous rest frame* of the particle (i.e., the frame in which it is at rest at any instant), the components of the 4-acceleration are simply $a^\mu = (0, \mathbf{a}_{\text{rest}})$. It follows that the norm of a^μ is the square of the rest-frame acceleration: $\eta_{\mu\nu}a^\mu a^\nu = \mathbf{a}^2_{\text{rest}}$.

Example: A particle moves along the x-axis of some inertial frame S with constant rest-frame acceleration α. At proper time τ, suppose that the particle is at $x(\tau)$ and $t(\tau)$ in S and is moving with speed βc (along the x-axis). In its instantaneous rest frame, the components of the 4-acceleration are $a'^\mu = \alpha(0,1,0,0)$. Performing the inverse Lorentz transform back to S gives

$$a^\mu = du^\mu/d\tau = (\gamma\beta\alpha, \gamma\alpha, 0, 0)\,. \tag{4.40}$$

Since the velocity 4-vector is $u^\mu = (\gamma c, \gamma c\beta, 0, 0)$, we have

$$\frac{d(\gamma c)}{d\tau} = \gamma\beta\alpha \quad \text{and} \quad c\frac{d(\gamma\beta)}{d\tau} = \gamma\alpha\,. \tag{4.41}$$

To solve these, we note that $d\gamma/d\tau = \gamma^3\beta d\beta/d\tau$; combining with Eq. (4.41), we find

$$\frac{d\beta}{d\tau} = \frac{\alpha}{\gamma^2 c} = \frac{\alpha}{c}(1-\beta^2)\,. \tag{4.42}$$

The solution with $\beta = 0$ at $\tau = 0$ is

$$\beta(\tau) = \tanh(\alpha\tau/c) \quad \text{and} \quad \gamma(\tau) = \cosh(\alpha\tau/c)\,. \tag{4.43}$$

As expected, the speed asymptotes to c ($\beta = 1$). We can solve for the wordline of the particle using

$$\frac{dx}{d\tau} = \gamma\frac{dx}{dt} = \gamma c\beta = c\sinh(\alpha\tau/c)\,. \tag{4.44}$$

The solution with $x = c^2\alpha$ at $\tau = 0$ (for convenience) is

$$x(\tau) = \frac{c^2}{\alpha}\cosh(\alpha\tau/c)\,. \tag{4.45}$$

Similarly, $dt/d\tau = \gamma = \cosh(\alpha\tau/c)$ gives

$$t(\tau) = \frac{c}{\alpha}\sinh(\alpha\tau/c)\,, \tag{4.46}$$

taking $t = 0$ at $\tau = 0$. We see that the motion is hyperbolic in the x-t plane, and asymptotes to $x = c|t|$ as $|\tau| \to \infty$. The wordline is entirely within the "Rindler wedge", $x > c|t|$. Note that if we consider an observer at rest in S at the origin, sending out light signals to the accelerated particle, only signals emitted before $t = 0$ are ever received by the accelerated particle.

4.3 4-tensors

A tensor of type $\binom{p}{q}$ has p upper (contravariant) indices and q lower (covariant) indices. Such a tensor $T^{\mu\nu\dots}{}_{\rho\sigma\dots}$ transforms as

$$T'^{\mu\nu\dots}{}_{\rho\sigma\dots} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \dots \Lambda_\rho{}^\gamma \Lambda_\sigma{}^\delta \dots T^{\alpha\beta\dots}{}_{\gamma\delta\dots} \tag{4.47}$$

under $x'^\mu = \Lambda^\mu{}_\nu x^\nu$. As with 4-vectors, one can raise and lower indices with $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$, for example

$$T^{\mu\nu} = \eta^{\nu\alpha} T^\mu{}_\alpha \tag{4.48}$$

which takes a type-$\binom{1}{1}$ tensor to a type $\binom{2}{0}$.

The following rules apply to tensor manipulations.

- Contractions may only be made between upper and lower indices. Contracting a $\binom{p}{q}$ tensor returns a $\binom{p-1}{q-1}$ tensor. For example $T^\alpha{}_\alpha$ is a Lorentz scalar since

$$\begin{aligned} T'^\alpha{}_\alpha &= \underbrace{\Lambda^\alpha{}_\rho \Lambda_\alpha{}^\delta}_{\delta^\delta_\rho} T^\rho{}_\delta \\ &= T^\rho{}_\rho\,. \end{aligned} \tag{4.49}$$

  Note that the specific symbol assigned to a contracted (dummy) index is irrelevant.

- Symmetrisation/antisymmetrisation may be performed only over upper or lower indices (but not mixed). Symmetrisation is denoted with round brackets and antisymmetrisation with square brackets:

$$T^{(\alpha\beta)} \equiv \frac{1}{2}\left(T^{\alpha\beta} + T^{\beta\alpha}\right) \quad \text{and} \quad T^{[\alpha\beta]} \equiv \frac{1}{2}\left(T^{\alpha\beta} - T^{\beta\alpha}\right)\,. \tag{4.50}$$

- Contraction, symmetrisation and index raising/lowering commute with Lorentz transformations. For example,

$$\begin{aligned}2T'^{(\alpha\beta)} &= T'^{\alpha\beta} + T'^{\beta\alpha} \\ &= \Lambda^\alpha{}_\sigma \Lambda^\beta{}_\rho T^{\sigma\rho} + \Lambda^\beta{}_\sigma \Lambda^\alpha{}_\rho T^{\sigma\rho} \\ &= \Lambda^\alpha{}_\rho \Lambda^\beta{}_\sigma \left(T^{\rho\sigma} + T^{\sigma\rho}\right) \\ &= 2\Lambda^\alpha{}_\rho \Lambda^\beta{}_\sigma T^{(\rho\sigma)} \,. \end{aligned} \tag{4.51}$$

- Partial differentiation takes a $\binom{p}{q}$ tensor to a $\binom{p}{q+1}$ tensor. For example, under a Lorentz transformation

$$\frac{\partial T^\alpha{}_\beta}{\partial x^\mu} \to \frac{\partial T'^\alpha{}_\beta}{\partial x'^\mu} = \Lambda_\mu{}^\rho \Lambda^\alpha{}_\sigma \Lambda_\beta{}^\tau \frac{\partial T^\sigma{}_\tau}{\partial x^\rho} \,, \tag{4.52}$$

  and $\partial_\mu T^\alpha{}_\beta$ is a $\binom{1}{2}$ tensor.

Finally, we return to the Minkowski metric. We now see that this is a $\binom{0}{2}$ tensor that takes the same form in all Lorentz frames. This follows since $(\mathbf{\Lambda}^{-1})^T \boldsymbol{\eta} \mathbf{\Lambda}^{-1} = \boldsymbol{\eta}$ implies

$$\eta_{\mu\nu} = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma \eta_{\rho\sigma} \,. \tag{4.53}$$

Similarly, $\eta^{\mu\nu}$ is a $\binom{2}{0}$ tensor and

$$\eta^\mu{}_\nu = \eta^{\mu\rho}\eta_{\rho\nu} = \delta^\mu_\nu \tag{4.54}$$

is of type $\binom{1}{1}$.

Having introduced all the machinery we need, we shall now see how to rewrite Maxwell's equations as 4-tensor equations, thus making manifest their Lorentz invariance.

5 Maxwell's equations and special relativity

5.1 Current 4-vector

We begin by considering the source terms ρ and $\mathbf{J}$ in Maxwell's equations from a relativistic perspective. We shall establish that

$$J^\mu = (\rho c, \mathbf{J}) \tag{5.1}$$

is a 4-vector (the current 4-vector) by considering the transformation properties of charge density and current density under Lorentz transformations.

Consider charges at rest in some frame S' giving a charge density ρ_0. In this frame $J'^\mu = (\rho_0 c, 0, 0, 0)$. How do these charges appear in a Lorentz boosted frame S in which the charges have velocity v along the x-axis? By length contraction, the number density is increased by the Lorentz factor γ, and hence the charge density is $\rho = \gamma\rho_0$. Moreover, since this density of charge has velocity $(v, 0, 0)$, the current density is $\mathbf{J} = (\gamma\rho_0 v, 0, 0)$. Hence in the frame S, $(\rho c, \mathbf{J}) = (\gamma\rho_0 c, \gamma\rho_0 v, 0, 0)$. But this is exactly what we get from applying an inverse Lorentz transformation to J'^μ since

$$\begin{aligned} J^0 &= \gamma(J'^0 + \beta J'^1) = \gamma\rho_0 c \\ J^1 &= \gamma(J'^1 + \beta J'^0) = \gamma\rho_0\beta c = \gamma\rho_0 v \,. \end{aligned} \tag{5.2}$$

This argument generalises to charges with a range of velocities by summing over sets of charges that are individually comoving.

The continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J} = 0\,, \tag{5.3}$$

can now be written in terms of the 4-current as

$$\partial_\mu J^\mu = 0\,, \tag{5.4}$$

which is manifestly form-invariant under Lorentz transformations.

5.2 Maxwell's equations and the 4-potential

In Sec. 3 we introduced the vector potential $\mathbf{A}$ for the magnetic field with $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$. Since $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ holds generally even for time-dependent fields, we can always write $\mathbf{B}$ in terms of $\mathbf{A}$. We cannot generally write $\mathbf{E} = -\boldsymbol{\nabla}\phi$ though, since

$$\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\,. \tag{5.5}$$

However, substituting $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$, we see that $\mathbf{E} + \partial\mathbf{A}/\partial t$ is curl-free, and so we can always write

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla}\phi\,, \tag{5.6}$$

where ϕ is the electric potential.

We met the idea of gauge transformations in Sec. 3. There, we argued that the physical magnetic field $\mathbf{B}$ is unchanged by a transformation of the vector potential $\mathbf{A}$ of the form $\mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\chi$. We now see that if the electric field is also to remain unchanged, we must simultaneously change the electric potential as $\phi \to \phi - \partial\chi/\partial t$ since then

$$\begin{aligned} -\frac{\partial}{\partial t}(\mathbf{A} + \boldsymbol{\nabla}\chi) - \boldsymbol{\nabla}\left(\phi - \frac{\partial \chi}{\partial t}\right) &= \mathbf{E} - \frac{\partial}{\partial t}\boldsymbol{\nabla}\chi + \boldsymbol{\nabla}\frac{\partial \chi}{\partial t} \\ &= \mathbf{E}\,. \end{aligned} \tag{5.7}$$

We now write the remaining two of Maxwell's equations,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{5.8}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\,, \tag{5.9}$$

in terms of ϕ and $\mathbf{A}$. (Note that in Eq. (5.9) we have used $\epsilon_0\mu_0 = 1/c^2$.) Equation (5.9) gives

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{A}) &= \mu_0 \mathbf{J} - \frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{A}}{\partial t} + \nabla\phi\right) \\ \Rightarrow \quad \nabla\nabla\cdot\mathbf{A} - \nabla^2\mathbf{A} &= \mu_0\mathbf{J} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \frac{1}{c^2}\nabla\frac{\partial\phi}{\partial t}\,. \end{aligned} \tag{5.10}$$

Combining terms, and recognising the wave operator $\Box = \nabla^2 - \partial^2/\partial(ct)^2$ acting on $\mathbf{A}$, we have

$$\Box\mathbf{A} - \nabla\left(\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) = -\mu_0\mathbf{J}\,. \tag{5.11}$$

We can write Eq. (5.8) in similar looking form:

$$\Box\left(\frac{\phi}{c}\right) + \frac{\partial}{\partial(ct)}\left(\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) = -\mu_0\rho c\,, \tag{5.12}$$

where we added and subtracted $\partial^2\phi/\partial(ct)^2$ and replaced ϵ_0 in favour of μ_0 and c. Let us introduce a candidate 4-vector potential with components

$$A^\mu \equiv (\phi/c, \mathbf{A})\,. \tag{5.13}$$

We can then combine Eqs. (5.11) and (5.12) into a single equation

$$\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = -\mu_0 J^\mu\,, \tag{5.14}$$

where $\partial^\mu = \eta^{\mu\nu}\partial_\nu = (-\partial/\partial(ct), \nabla)$, and J^μ is the 4-current. Clearly, this equation is manifestly covariant if A^μ is indeed a 4-vector, i.e., we have established that Maxwell's theory is Lorentz invariant provided that the potentials ϕ and $\mathbf{A}$ transform under Lorentz transformations like the time and space components of a 4-vector.

As well as being covariant under Lorentz transformations, Eq. (5.14) is gauge invariant as it must be. To see this, note that the gauge transformation $\phi \to \phi - \partial\chi/\partial t$ and $\mathbf{A} \to \mathbf{A} + \nabla\chi$ implies

$$A^\mu \to A^\mu + \partial^\mu\chi \tag{5.15}$$

for the 4-potential. The left-hand side of Eq. (5.14) is invariant under this transformation since

$$\begin{aligned} \Box A^\mu - \partial^\mu(\partial_\nu A^\nu) &\to \Box(A^\mu + \partial^\mu\chi) - \partial^\mu(\partial_\nu A^\nu + \Box\chi) \\ &= \Box A^\mu - \partial^\mu(\partial_\nu A^\nu)\,. \end{aligned} \tag{5.16}$$

Note also how the charge continuity equation (5.4) is built into Eq. (5.14); since

$$\partial_\mu \left[\Box A^\mu - \partial^\mu(\partial_\nu A^\nu)\right] = \Box(\partial_\mu A^\mu) - \Box(\partial_\nu A^\nu) = 0\,, \tag{5.17}$$

it follows that $\partial_\mu J^\mu = 0$ as required.

We shall return to Eq. (5.14) in Sec. 7 when we consider the emission of electromagnetic radiation. Being a sourced wave equation, it can be solved straightforwardly in terms of a suitable Green's function once the gauge is fixed. Before doing so, we shall consider an alternative relativistic formulation of Maxwell's equations that is more obviously related to their 3D first-order (in space and time derivatives) form, and for which gauge-invariance is manifest.

5.3 Maxwell field-strength tensor

We define the *Maxwell field-strength tensor*

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = 2\partial^{[\mu}A^{\nu]}\,, \tag{5.18}$$

which is a type-$\binom{2}{0}$ antisymmetric tensor. To motivate introducing $F^{\mu\nu}$ consider its components. For example,

$$\begin{aligned} F^{01} &= \partial^0 A^1 - \partial^1 A^0 \\ &= -\frac{\partial A_x}{\partial(ct)} - \frac{\partial(\phi/c)}{\partial x} \\ &= E_x/c\,, \end{aligned} \tag{5.19}$$

and

$$\begin{aligned} F^{12} &= \partial^1 A^2 - \partial^2 A^1 \\ &= \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \\ &= B_z\,. \end{aligned} \tag{5.20}$$

Evaluating the other components similarly gives

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}\,. \tag{5.21}$$

We see that the six linearly-independent components of $F^{\mu\nu}$ encode the $3+3$ components of the electric and magnetic fields. Note that under gauge transformations, $A^\mu \to A^\mu + \partial^\mu\chi$ and $F^{\mu\nu}$ is invariant since $\partial^{[\mu}\partial^{\nu]}\chi = 0$.

We now look to write Maxwell's equations directly in terms of the field-strength tensor. Since Maxwell's equations are first-order in space and time derivatives acting on the electric and magnetic fields, consider

$$\begin{aligned}\partial_\mu F^{\mu\nu} &= \partial_\mu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\\ &= \Box A^\nu - \partial^\nu(\partial_\mu A^\mu)\,.\end{aligned} \tag{5.22}$$

This is exactly the 4-vector on the left-hand side of Eq. (5.14) so we can write

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu\,. \tag{5.23}$$

This 4-vector equation is equivalent to the sourced Maxwell's equations (5.8) and (5.9). We can verify this explicitly; for example with $\nu = 0$ we have

$$\begin{aligned}\partial_\mu F^{\mu 0} &= -\mu_0 J^0\\ \Rightarrow\quad \frac{\partial F^{i0}}{\partial x_i} &= -\mu_0\rho c\\ \Rightarrow -\frac{1}{c}\boldsymbol{\nabla}\cdot\mathbf{E} &= -\mu_0\rho c\,,\end{aligned} \tag{5.24}$$

which gives Eq. (5.8). For $\nu = 1$, we have

$$\begin{aligned}\partial_\mu F^{\mu 1} &= -\mu_0 J^1\\ \Rightarrow\quad \frac{\partial F^{01}}{\partial(ct)} + \frac{\partial F^{21}}{\partial y} + \frac{\partial F^{31}}{\partial z} &= -\mu_0 J_x\\ \Rightarrow\quad \frac{\partial(E_x/c)}{\partial(ct)} - \frac{\partial B_z}{\partial y} + \frac{\partial B_y}{\partial z} &= -\mu_0 J_x\,,\end{aligned} \tag{5.25}$$

which is the x-component of Eq. (5.9).

Charge continuity is implicit in Eq. (5.23) due to the antisymmetry of $F^{\mu\nu}$; applying ∂_ν we have

$$\begin{aligned}\partial_\nu\partial_\mu F^{\mu\nu} &= -\mu_0\partial_\nu J^\nu\\ \Rightarrow\quad \partial_{[\nu}\partial_{\mu]} F^{\mu\nu} &= -\mu_0\partial_\nu J^\nu \qquad \text{(antisymmetry of } F^{\mu\nu}\text{)}\\ \Rightarrow\quad 0 &= -\mu_0\partial_\nu J^\nu\,.\end{aligned} \tag{5.26}$$

The other of Maxwell's equations, $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ and $\boldsymbol{\nabla}\times\mathbf{E} = -\partial\mathbf{B}/\partial t$, are implicit in the definition of the field-strength tensor as the antisymmetric derivative of the 4-potential. If we lower indices to obtain[9]

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{5.28}$$

[9]The components of $F_{\mu\nu}$ are

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c\\ E_x/c & 0 & B_z & -B_y\\ E_y/c & -B_z & 0 & B_x\\ E_z/c & B_y & -B_x & 0\end{pmatrix}\,. \tag{5.27}$$

taking a further derivative and antisymmetrising gives

$$\begin{aligned}\partial_{[\rho}F_{\mu\nu]} &= \frac{1}{3}\left(\partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} + \partial_\mu F_{\nu\rho}\right)\\ &= \frac{1}{3}\left[\partial_\rho\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) + \partial_\nu\left(\partial_\rho A_\mu - \partial_\mu A_\rho\right) + \partial_\mu\left(\partial_\nu A_\rho - \partial_\rho A_\nu\right)\right]\\ &= 0\,,\end{aligned} \tag{5.29}$$

where we used the antisymmetry of $F_{\mu\nu}$ in the first line.

The tensor equation

$$\partial_{[\rho}F_{\mu\nu]} = 0 \quad \text{or equivalently} \quad \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} + \partial_\mu F_{\nu\rho} = 0\,, \tag{5.30}$$

is $4\times 3\times 2/3! = 4$ independent equations corresponding to the scalar and vector-valued source-free Maxwell equations. To see this explicitly, consider taking $\rho = 0$, $\mu = 1$ and $\nu = 2$; then

$$\begin{aligned}\frac{\partial F_{12}}{\partial(ct)} + \frac{\partial F_{01}}{\partial y} + \frac{\partial F_{20}}{\partial x} &= 0\\ \Rightarrow \quad \frac{\partial B_z}{\partial(ct)} - \frac{\partial(E_x/c)}{\partial y} + \frac{\partial(E_y/c)}{\partial x} &= 0\\ \Rightarrow \quad -\frac{\partial B_z}{\partial t} &= \left(\boldsymbol{\nabla}\times\mathbf{E}\right)_z\,.\end{aligned} \tag{5.31}$$

Taking $0,1,3$ and $0,2,3$ give the y- and x-components of the Faraday equation (1.2). Finally, taking $1,2,3$ gives

$$\begin{aligned}\frac{\partial F_{23}}{\partial x} + \frac{\partial F_{12}}{\partial z} + \frac{\partial F_{31}}{\partial y} &= 0\\ \Rightarrow \quad \frac{\partial B_x}{\partial x} + \frac{\partial B_z}{\partial z} + \frac{\partial B_y}{\partial y} &= 0\\ \Rightarrow \quad \boldsymbol{\nabla}\cdot\mathbf{B} &= 0\,.\end{aligned} \tag{5.32}$$

We see that Maxwell's equations can be written in manifestly covariant form in terms of the field-strength tensor as

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu \quad \text{and} \quad \partial_{[\rho}F_{\mu\nu]} = 0\,. \tag{5.33}$$

The second of these equations is sometimes rewritten in terms of the *dual field-strength tensor*[10] ${}^*F^{\mu\nu}$ defined by

$${}^*F^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\,, \tag{5.34}$$

[10] This is analogous to what we do in 3D when we dualise the antisymmetric second-rank tensor $\partial_{[i}A_{j]}$ by contracting with ϵ_{ijk} to get the dual (pseudo-)vector $\boldsymbol{\nabla}\times\mathbf{A}$. The dual is properly called a pseudo-vector since ϵ_{ijk} is only invariant under those orthogonal transformations of coordinates that have unit determinant, i.e., avoid space inversion.

where the alternating (pseudo-)tensor is the fully-antisymmetric object defined by

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} 1 & \text{if } \mu\nu\rho\sigma \text{ is an even permutation of } 0,1,2,3 \\ -1 & \text{if } \mu\nu\rho\sigma \text{ is an odd permutation of } 0,1,2,3 \\ 0 & \text{otherwise}\,. \end{cases} \tag{5.35}$$

Under a Lorentz transformation, applying the usual tensor transformation law, we have

$$\epsilon^{\mu\nu\rho\sigma} \to \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\Lambda^{\rho}{}_{\gamma}\Lambda^{\sigma}{}_{\delta}\epsilon^{\alpha\beta\gamma\delta}\,. \tag{5.36}$$

But $\epsilon^{\mu\nu\rho\sigma} = \epsilon^{[\mu\nu\rho\sigma]}$ hence

$$\begin{aligned} \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\Lambda^{\rho}{}_{\gamma}\Lambda^{\sigma}{}_{\delta}\epsilon^{\alpha\beta\gamma\delta} &= \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\Lambda^{\rho}{}_{\gamma}\Lambda^{\sigma}{}_{\delta}\epsilon^{[\alpha\beta\gamma\delta]} \\ &= \Lambda^{\mu}{}_{[\alpha}\Lambda^{\nu}{}_{\beta}\Lambda^{\rho}{}_{\gamma}\Lambda^{\sigma}{}_{\delta]}\epsilon^{\alpha\beta\gamma\delta} \\ &= \Lambda^{[\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\Lambda^{\rho}{}_{\gamma}\Lambda^{\sigma]}{}_{\delta}\epsilon^{\alpha\beta\gamma\delta}\,, \end{aligned} \tag{5.37}$$

and so the transformed object is fully antisymmetric and so must be proportional to $\epsilon^{\mu\nu\rho\sigma}$. Considering the one linearly-independent component $0,1,2,3$, we have

$$\Lambda^{0}{}_{\alpha}\Lambda^{1}{}_{\beta}\Lambda^{2}{}_{\gamma}\Lambda^{3}{}_{\delta}\epsilon^{\alpha\beta\gamma\delta} = \det(\Lambda)\,, \tag{5.38}$$

where we have used the definition of the determinant. Since $\epsilon^{0123} = 1$, it follows that

$$\epsilon^{\mu\nu\rho\sigma} \to \det(\Lambda)\epsilon^{\mu\nu\rho\sigma}\,. \tag{5.39}$$

If we consider only restricted Lorentz transformations (recall these are continuously connected to the identity; see Sec. 4.1) we have $\det(\Lambda) = 1$ and $\epsilon^{\mu\nu\rho\sigma}$ is an invariant tensor under such transformations. Finally, note that on lowering indices, $\epsilon_{0123} = -1$. Returning to ${}^{*}F^{\mu\nu}$, we now see that it is a tensor under restricted Lorentz transformations.

Straightforward calculation, for example,

$${}^{*}F^{01} = \frac{1}{2}\left(\epsilon^{0123}F_{23} + \epsilon^{0132}F_{32}\right) = F_{23} = B_x\,, \tag{5.40}$$

establishes the components of ${}^{*}F^{\mu\nu}$ as

$${}^{*}F^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}\,. \tag{5.41}$$

These components are related to those of $F^{\mu\nu}$ by the duality transformation $\mathbf{E} \to c\mathbf{B}$ and $\mathbf{B} \to -\mathbf{E}/c$.

The covariant Maxwell equation $\partial_{[\rho}F_{\mu\nu]} = 0$ can be written in terms of ${}^*F_{\mu\nu}$ as follows:

$$\partial_{[\rho}F_{\mu\nu]} = 0 \quad \Rightarrow \quad \frac{1}{2}\epsilon^{\alpha\rho\mu\nu}\partial_\rho F_{\mu\nu} = 0\,, \tag{5.42}$$

so that

$$\partial_\mu {}^*F^{\mu\nu} = 0\,. \tag{5.43}$$

Finally, we can write Maxwell's covariant equations as the pair

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu \quad \text{and} \quad \partial_\mu {}^*F^{\mu\nu} = 0\,. \tag{5.44}$$

The lack of duality symmetry (i.e., there is no source term for the dual field strength) reflects the fact that there are no known magnetic charges (monopoles) in nature.

5.4 Lorentz transformations of E and B

Knowing that **E** and **B** form the components of the field-strength tensor, we can now calculate their transformation laws under Lorentz transformations. Generally, we have

$$F^{\mu\nu} \to F'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta}\,. \tag{5.45}$$

Specialising to the standard Lorentz boost of Eq. (4.7), we find, for example,

$$\begin{aligned} F'^{01} &= \Lambda^0{}_\alpha \Lambda^1{}_\beta F^{\alpha\beta} \\ &= \Lambda^0{}_1 \Lambda^1{}_0 F^{10} + \Lambda^0{}_0 \Lambda^1{}_1 F^{01} \\ \Rightarrow \quad E'_x/c &= (-\gamma\beta)(-\gamma\beta)(-E_x/c) + \gamma\gamma E_x/c \\ \Rightarrow \quad E'_x &= \gamma^2(1-\beta^2)E_x = E_x\,. \end{aligned} \tag{5.46}$$

Similarly,

$$\begin{aligned} F'^{02} &= \Lambda^0{}_\alpha \Lambda^2{}_\beta F^{\alpha\beta} \\ &= \Lambda^0{}_0 \Lambda^2{}_2 F^{02} + \Lambda^0{}_1 \Lambda^2{}_2 F^{12} \\ \Rightarrow \quad E'_y/c &= \gamma E_y/c - \gamma\beta B_z \\ \Rightarrow \quad E'_y &= \gamma(E_y - vB_z)\,, \end{aligned} \tag{5.47}$$

where $v = \beta c$. Repeating for the other components gives

$$\mathbf{E}' = \begin{pmatrix} E_x \\ \gamma(E_y - vB_z) \\ \gamma(E_z + vB_y) \end{pmatrix} \quad \text{and} \quad \mathbf{B}' = \begin{pmatrix} B_x \\ \gamma(B_y + vE_z/c^2) \\ \gamma(B_z - vE_y/c^2) \end{pmatrix}\,. \tag{5.48}$$

We see that the components of the fields along the direction of the relative velocity do not change, but the perpendicular components mix **E** and **B**. In the non-relativistic limit $v \ll c$, we recover the Galilean transformation $\mathbf{E} \to \mathbf{E} + \mathbf{v} \times \mathbf{B}$ and $\mathbf{B} \to \mathbf{B}$.

Example: In the inertial frame S, charges lie along the x-axis with charge per length λ. These generate an electric field with points out radially from the axis with magnitude $E = \lambda/(2\pi\epsilon_0 r)$ at distance r from the axis (from Gauss's law). In Cartesian coordinates, we have

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0(y^2+z^2)}\begin{pmatrix} 0 \\ y \\ z \end{pmatrix} . \tag{5.49}$$

The magnetic field vanishes since there is no current in S. We now transform to the frame S' moving at speed v along the x-axis to find

$$\mathbf{E}' = \frac{\lambda\gamma}{2\pi\epsilon_0(y^2+z^2)}\begin{pmatrix} 0 \\ y \\ z \end{pmatrix} = \frac{\lambda\gamma}{2\pi\epsilon_0(y'^2+z'^2)}\begin{pmatrix} 0 \\ y' \\ z' \end{pmatrix} , \tag{5.50}$$

$$\mathbf{B}' = \frac{\gamma v}{c^2}\frac{\lambda}{2\pi\epsilon_0(y^2+z^2)}\begin{pmatrix} 0 \\ z \\ -y \end{pmatrix} = \frac{\lambda\gamma v\mu_0}{2\pi(y'^2+z'^2)}\begin{pmatrix} 0 \\ z' \\ -y' \end{pmatrix} . \tag{5.51}$$

These fields should be consistent with the charges and currents in S'. By length contraction, the charge per length is $\gamma\lambda$ in S' and this does indeed generate the radial field in Eq. (5.50) by Gauss's law. The charges are all moving with velocity $(-v, 0, 0)$ in S', so the current in the wire is $-\gamma\lambda v$ along the x'-axis. This current generates an azimuthal field that circles the negative x'-axis in a right-handed sense. The magnitude follows from integrating $\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J}$ around a circular contour in the y'-z' plane centred on the x'-axis: $|\mathbf{B}'| = \mu_0 I'/(2\pi r')$, where $I' = \gamma\lambda v$ is the current in S'. Reassuringly, we see that direct calculation of the magnetic field from Maxwell's equations in S' gives results consistent with the Lorentz transformation of the fields calculated directly in S.

Example: The fields of a uniformly moving charge can be easily derived by transforming the electric field from the rest frame. Let the charge q be at rest in the inertial frame S at the origin. The electric field is simply the Coulomb field,

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0(x^2+y^2+z^2)^{3/2}}\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \tag{5.52}$$

The magnetic field vanishes in S. Transforming to S', which moves at speed v along the

x-axis, the electric field becomes

$$\mathbf{E}' = \frac{q}{4\pi\epsilon_0(x^2+y^2+z^2)^{3/2}}\begin{pmatrix} x \\ \gamma y \\ \gamma z \end{pmatrix}$$
$$= \frac{q\gamma}{4\pi\epsilon_0[\gamma^2(x'+vt')^2+y'^2+z'^2]^{3/2}}\begin{pmatrix} x'+vt' \\ y' \\ z' \end{pmatrix}, \tag{5.53}$$

where we further expressed the coordinates in S' in terms of those in S in the second equality. In S', the charge is at $\mathbf{x}' = (-vt', 0, 0)$ at time t' so we see that the electric field is radially out from the *current* position of the charge. However, the electric field is anisotropic. If we denote by $\mathbf{r}'$ the displacement vector connecting the current position of the charge to the observation point, so that $\mathbf{r}' = (x'+vt', y', z')$, and let ψ be the angle between that $\mathbf{r}'$ makes with the x'-axis (i.e.,, the direction of motion), we can write

$$\begin{aligned}\gamma^2(x'+vt')^2+y'^2+z'^2 &= (\gamma^2-1)(x'+vt')^2+\mathbf{r}'^2 \\ &= \mathbf{r}'^2\left(\gamma^2\beta^2\cos^2\psi+1\right) \\ &= \mathbf{r}'^2\gamma^2\left(1-\beta^2\sin^2\psi\right),\end{aligned} \tag{5.54}$$

where $\beta = v/c$ as usual. The electric field is therefore

$$\mathbf{E}' = \frac{q\hat{\mathbf{r}}'}{4\pi\epsilon_0\mathbf{r}'^2\gamma^2(1-\beta^2\sin^2\psi)^{3/2}}. \tag{5.55}$$

Along the x'-axis ($\psi = 0$), the field is smaller by a factor of $1/\gamma^2$ than for a charge at rest at the current position, while perpendicular to this ($\psi = \pi/2$) the field is enhanced by a factor of γ over that for a charge at rest. The field lines are thus concentrated close to the plane perpendicular to the motion for a rapidly moving particle.

There is also an azimuthal magnetic field with $cB'_y = \beta E'_z$ and $cB'_z = -\beta E'_y$.

Although the electric and magnetic fields transform in a non-trivial way under Lorentz transformations, there are two (scalar) quadratic invariants[11]. The first of these is $F_{\mu\nu}F^{\mu\nu}/2$, which is invariant by construction. In terms of the components,

$$\begin{aligned}\frac{1}{2}F_{\mu\nu}F^{\mu\nu} &= \frac{1}{2}\mathrm{Tr}\left[\begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}\begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}\right] \\ &= \mathbf{B}^2 - \mathbf{E}^2/c^2.\end{aligned} \tag{5.56}$$

[11]There is no invariant that is linear in the fields since $\eta_{\mu\nu}F^{\mu\nu} = 0$ (recall that the field-strength tensor is antisymmetric).

It is straightforward to verify the invariance of $\mathbf{B}^2 - \mathbf{E}^2/c^2$ directly from Eq. (5.48). The other invariant follows from contracting $F^{\mu\nu}$ with its dual:

$$
\begin{aligned}
-\frac{1}{4}F_{\mu\nu}{}^{*}F^{\mu\nu} &= -\frac{1}{4}\mathrm{Tr}\left[\begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}\right. \\
&\qquad\qquad \left.\times \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}\right] \\
&= \mathbf{E}\cdot\mathbf{B}/c\,.
\end{aligned}
\tag{5.57}
$$

The third contraction we might consider forming is ${}^{*}F_{\mu\nu}{}^{*}F^{\mu\nu}$. However, this does not yield an independent invariant since ${}^{*}F_{\mu\nu}{}^{*}F^{\mu\nu} = F_{\mu\nu}F^{\mu\nu}$. Higher-order invariants, such as $\det(F^{\mu}{}_{\nu})$ are also sometimes useful.

5.5 Lorentz force law

Recall that the relativistic force law

$$dp^{\mu}/d\tau = F^{\mu} \tag{5.58}$$

involves the 4-force F^{μ} with components

$$F^{\mu} = \left(\frac{\gamma}{c}\mathbf{v}\cdot\mathbf{F}, \gamma\mathbf{F}\right)\,, \tag{5.59}$$

where $\mathbf{F}$ is the 3-force on the particle and $\mathbf{v}$ is its velocity. Necessarily, $F^{\mu}u_{\mu} = 0$. For the Lorentz force law for a particle of charge q,

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})\,, \tag{5.60}$$

the rate of working $\mathbf{v}\cdot\mathbf{F} = q\mathbf{v}\cdot\mathbf{E}$ is from the electric field alone. Substituting into Eq. (5.59), we find the Lorentz 4-force

$$F^{\mu} = \gamma q\left(\frac{1}{c}\mathbf{v}\cdot\mathbf{E}, \mathbf{E} + \mathbf{v}\times\mathbf{B}\right)\,. \tag{5.61}$$

We now seek to express this 4-force in terms of the covariant field-strength tensor and 4-velocity.

Given the form of the components of F^{μ}, a sensible thing to try is $qF^{\mu\nu}u_{\nu}$. This is orthogonal to u^{μ} since the field-strength is antisymmetric. Using $u_{\mu} = \gamma(-c, \mathbf{v})$, we have

$$\begin{aligned} qF^{0\nu}u_\nu &= qF^{0i}u_i \\ &= q\gamma \mathbf{E}\cdot\mathbf{v}/c\,, \end{aligned} \tag{5.62}$$

which is indeed the required time component. For the space components, we have, for example,

$$\begin{aligned} qF^{1\nu}u_\nu &= qF^{10}u_0 + qF^{12}u_2 + qF^{13}u_3 \\ &= q\left(-\frac{E_x}{c}\right)(-\gamma c) + qB_z(\gamma v_y) - qB_y(\gamma v_z) \\ &= \gamma q(\mathbf{E}+\mathbf{v}\times\mathbf{B})_x\,, \end{aligned} \tag{5.63}$$

with the analogous result for the other two components. It follows that we can write the 4-force as $F^\mu = qF^{\mu\nu}u_\nu$, and the equation of motion of a point charge as

$$dp^\mu/d\tau = qF^{\mu\nu}u_\nu\,. \tag{5.64}$$

5.6 Motion in constant, uniform fields

We now consider the motion of a point particle of rest mass m and charge q in uniform electric and magnetic fields. The field-strength tensor is independent of spacetime position and the equation of motion (5.64) becomes

$$\frac{du^\mu}{d\tau} = \frac{q}{m}F^\mu{}_\nu u^\nu\,, \tag{5.65}$$

with $u^\mu = dx^\mu/d\tau$. The mixed components of the field-strength tensor are

$$F^\mu{}_\nu = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}. \tag{5.66}$$

The solution of Eq. (5.65) with initial condition $u^\mu = u^\mu(0)$ at $\tau = 0$ is

$$u^\mu(\tau) = \exp\left(\frac{q\tau}{m}F\right)^\mu{}_\nu u^\nu(0)\,, \tag{5.67}$$

where, as usual, the exponential is defined by its power series:

$$\exp\left(\frac{q}{m}F\tau\right)^\mu{}_\nu = \delta^\mu_\nu + \frac{q\tau}{m}F^\mu{}_\nu + \frac{1}{2}\left(\frac{q\tau}{m}\right)^2 F^\mu{}_\rho F^\rho{}_\nu + \cdots\,. \tag{5.68}$$

Integrating again gives the worldline

$$x^\mu(\tau) = x^\mu(0) + \left[\int_0^\tau \exp\left(\frac{q\tau'}{m}F\right)^\mu{}_\nu\, d\tau'\right] u^\nu(0)\,, \tag{5.69}$$

where the spacetime position at $\tau = 0$ is $x^\mu(0)$. Now, for an invertible matrix $\mathbf{M}$, we have

$$\begin{aligned}
\int_0^\tau \exp(\mathbf{M}\tau')\,d\tau' &= \int_0^\tau \left(\mathbf{I} + \mathbf{M}\tau' + \frac{1}{2}\mathbf{M}^2\tau'^2 + \frac{1}{3!}\mathbf{M}^3\tau'^3 + \cdots\right) d\tau' \\
&= \tau\mathbf{I} + \frac{1}{2}\mathbf{M}\tau^2 + \frac{1}{3!}\mathbf{M}^2\tau^3 + \frac{1}{4!}\mathbf{M}^3\tau^4 + \cdots \\
&= \mathbf{M}^{-1}\left(\mathbf{I} + \mathbf{M}\tau + \frac{1}{2}\mathbf{M}^2\tau^2 + \frac{1}{3!}\mathbf{M}^3\tau^3 + \cdots - \mathbf{I}\right) \\
&= \mathbf{M}^{-1}\left[\exp(\mathbf{M}\tau) - \mathbf{I}\right] .
\end{aligned} \tag{5.70}$$

With a suitable choice of spacetime origin, we can therefore write

$$x^\mu(\tau) = \frac{m}{q}(F^{-1})^\mu{}_\nu \exp\left(\frac{q\tau}{m}F\right)^\nu{}_\rho u^\rho(0) \quad \text{for } \det(F^\mu{}_\nu) \neq 0\,. \tag{5.71}$$

Exercise: Show that the quartic invariant

$$\det(F^\mu{}_\nu) = -(\mathbf{E}\cdot\mathbf{B})^2/c^2\,. \tag{5.72}$$

We first consider the case where $\mathbf{E}\cdot\mathbf{B} \neq 0$, i.e., both $\mathbf{E}$ and $\mathbf{B}$ are non-zero and are not perpendicular. (Note that this holds in all inertial frames, if true in one, due to the Lorentz invariance of $\mathbf{E}\cdot\mathbf{B}$.) We shall now show that a frame can always be found in which the $\mathbf{E}$ and $\mathbf{B}$ fields are parallel. The motion turns out to be rather simple in this frame, and the motion in a general frame can then by found by an appropriate Lorentz transformation.

For the case that $\mathbf{E}$ and $\mathbf{B}$ are not parallel in S, let us choose spatial axes such that both fields lie in the y-z plane. It follows that under a boost along the x-axis, the transformed fields are still in the y'-z' plane with components

$$\mathbf{E}' = \gamma\begin{pmatrix} 0 \\ E_y - vB_z \\ E_z + vB_y \end{pmatrix} \quad \text{and} \quad \mathbf{B}' = \gamma\begin{pmatrix} 0 \\ B_y + vE_z/c^2 \\ B_z - vE_y/c^2 \end{pmatrix} . \tag{5.73}$$

Since we want to demonstrate that for some $\beta = v/c$ with $|\beta| < 1$, we can set $\mathbf{E}'$ and $\mathbf{B}'$ parallel, consider $\mathbf{E}'\times\mathbf{B}'$ (which lies along the x'-axis):

$$\begin{aligned}
(\mathbf{E}'\times\mathbf{B}')_x &= \gamma^2\left[(E_y - vB_z)(B_z - vE_y/c^2) - (E_z + vB_y)(B_y + vE_z/c^2)\right] \\
&= \gamma^2\left[(\mathbf{E}\times\mathbf{B})_x(1+\beta^2) - v|\mathbf{B}|^2 - v|\mathbf{E}/c|^2\right] .
\end{aligned} \tag{5.74}$$

It follows that if we choose the velocity such that

$$\frac{\beta}{1+\beta^2} = \frac{(\mathbf{E}\times\mathbf{B})_x/c}{|\mathbf{B}|^2+|\mathbf{E}/c|^2}\,, \tag{5.75}$$

then the electric and magnetic fields will be parallel in S'. However, this is only possible if we can solve Eq. (5.75) for $|\beta|<1$. For given $|\mathbf{E}|$ and $|\mathbf{B}|$, the magnitude $|(\mathbf{E}\times\mathbf{B})_x| < |\mathbf{E}||\mathbf{B}|$ since $\mathbf{E}$ and $\mathbf{B}$ are not perpendicular. Writing $|\mathbf{B}| = \mu|\mathbf{E}|/c$ for some dimensionless $\mu > 0$, it follows that the magnitude of the right-hand side of Eq. (5.75) satisfies

$$\left|\frac{(\mathbf{E}\times\mathbf{B})_x/c}{|\mathbf{B}|^2+|\mathbf{E}/c|^2}\right| < \frac{\mu}{1+\mu^2}\,. \tag{5.76}$$

The maximum value of the function $\mu/(1+\mu^2)$ is $1/2$ (at $\mu=1$) and so the right-hand side of Eq. (5.75) is necessarily less than $1/2$. It follows that we can always find a β with $|\beta|<1$, such that Eq. (5.75) is satisfied and the transformed electric and magnetic fields are parallel.

Adopting a frame in which $\mathbf{E}$ and $\mathbf{B}$ are parallel, and now taking these to lie along the x-axis, we have $\mathbf{E}=(E,0,0)$ and $\mathbf{B}=(B,0,0)$ so that the field-strength tensor takes the block-diagonal form

$$F^\mu{}_\nu = \begin{pmatrix} 0 & E/c & 0 & 0 \\ E/c & 0 & 0 & 0 \\ 0 & 0 & 0 & B \\ 0 & 0 & -B & 0 \end{pmatrix}. \tag{5.77}$$

Noting that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \mathbf{I} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = -\mathbf{I}\,, \tag{5.78}$$

we have

$$\begin{aligned} \exp\begin{pmatrix} 0 & w \\ w & 0 \end{pmatrix} &= \mathbf{I} + w\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{w^2}{2}\mathbf{I} + \frac{w^3}{3!}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{w^4}{4!}\mathbf{I} + \cdots \\ &= \cosh w\mathbf{I} + \sinh w\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \end{aligned} \tag{5.79}$$

and

$$\begin{aligned} \exp\begin{pmatrix} 0 & w' \\ -w' & 0 \end{pmatrix} &= \mathbf{I} + w'\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \frac{w'^2}{2}\mathbf{I} - \frac{w'^3}{3!}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \frac{w'^4}{4!}\mathbf{I} + \cdots \\ &= \cos w'\mathbf{I} + \sin w'\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \tag{5.80}$$

It follows that

$$\exp\left(\frac{q\tau}{m}F\right)^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\left(\frac{q\tau}{mc}E\right) & \sinh\left(\frac{q\tau}{mc}E\right) & 0 & 0 \\ \sinh\left(\frac{q\tau}{mc}E\right) & \cosh\left(\frac{q\tau}{mc}E\right) & 0 & 0 \\ 0 & 0 & \cos\left(\frac{q\tau}{m}B\right) & \sin\left(\frac{q\tau}{m}B\right) \\ 0 & 0 & -\sin\left(\frac{q\tau}{m}B\right) & \cos\left(\frac{q\tau}{m}B\right) \end{pmatrix}, \tag{5.81}$$

which, from Eq. (5.71), gives the motion

$$\begin{aligned} ct(\tau) &= \frac{mc}{qE}\left[\sinh\left(\frac{q\tau}{mc}E\right)c\dot{t}(0) + \cosh\left(\frac{q\tau}{mc}E\right)\dot{x}(0)\right] \\ x(\tau) &= \frac{mc}{qE}\left[\cosh\left(\frac{q\tau}{mc}E\right)c\dot{t}(0) + \sinh\left(\frac{q\tau}{mc}E\right)\dot{x}(0)\right] \\ y(\tau) &= \frac{m}{qB}\left[\sin\left(\frac{q\tau}{m}B\right)\dot{y}(0) - \cos\left(\frac{q\tau}{m}B\right)\dot{z}(0)\right] \\ z(\tau) &= \frac{m}{qB}\left[\cos\left(\frac{q\tau}{m}B\right)\dot{y}(0) + \sin\left(\frac{q\tau}{m}B\right)\dot{z}(0)\right]. \end{aligned} \tag{5.82}$$

Notice how, with **E** and **B** parallel, the motion in the ct-x plane decouples from that in the y-z plane. The ct-x motion is a hyperbolic trajectory (like for the example of constant rest-frame acceleration given earlier) with

$$x^2 - (ct)^2 = \left(\frac{mc}{qE}\right)^2 \left([c\dot{t}(0)]^2 - \dot{x}^2(0)\right) > 0\,, \tag{5.83}$$

where the inequality follows from $[c\dot{t}(0)]^2 - \dot{\mathbf{x}}^2(0) = c^2$. Note that

$$[c\dot{t}(\tau)]^2 - \dot{x}^2(\tau) = [c\dot{t}(0)]^2 - \dot{x}^2(0)\,. \tag{5.84}$$

The motion in the y-z plane is circular with

$$y^2 + z^2 = \left(\frac{m}{qB}\right)^2 \left(\dot{y}^2(0) + \dot{z}^2(0)\right)\,. \tag{5.85}$$

The radius of the circle is controlled by the transverse proper-time speed $\left(\dot{y}^2 + \dot{z}^2\right)^{1/2}$ which is constant since

$$\dot{y}^2(\tau) + \dot{z}^2(\tau) = \dot{y}^2(0) + \dot{z}^2(0)\,. \tag{5.86}$$

Defining the *cyclotron frequency* $\omega_c \equiv qB/m$, we see that this is the angular frequency of the circular motion with respect to proper time. Combining the two motions gives, generally, a helical motion along the field direction. The radius of the circular projection is constant in time, but the spacing along the x-axis grows in time and asymptotically the spacing is constant in $\ln x$. As the speed along the x-axis approaches c, the period of the circular motion in t gets significantly dilated and the transverse speed $\left[(dy/dt)^2 + (dz/dt)^2\right]^{1/2}$ falls towards zero.

To complete our treatment of motion in constant, uniform fields we now consider the special cases for which $\mathbf{E} \cdot \mathbf{B} = 0$.

$\mathbf{B} = 0$. The motion can be obtained as the limit of Eq. (5.82) as $B \to 0$ (with an appropriate shift in the choice of origin which is infinite in the limit). The worldline is hyperbolic in the ct-x plane, but now linear in the y-z plane corresponding to circular motion in that plane with infinite radius. The velocity in the y-z plane (perpendicular to the electric field) asymptotes to zero.

$\mathbf{E} = 0$. The motion can be obtained as the limit of Eq. (5.82) as $E \to 0$. The motion in the ct-x plane is linear with $c\dot{t}$ and $\dot{x}$ constant, corresponding to a constant 3-velocity component along the x-axis (the direction of the magnetic field). Constancy of $c\dot{t}$ ensures that the speed of the particle is constant, as required since the magnetic field does no work. In the y-z plane, the motion is circular with proper angular frequency ω_c. If the 3-velocity along the field direction is v_x, and the perpendicular velocity is $\mathbf{v}_\perp$, the constant Lorentz factor is $\gamma^{-2} = 1 - v_x^2/c^2 - \mathbf{v}_\perp^2/c^2$ (and $\dot{t} = \gamma$) and the angular frequency of the circular motion with respect to coordinate time t is ω_c/γ. The spacing of the helical motion along the x-axis is constant. The radius r of the circular motion follows from

$$r^2 = \frac{1}{\omega_c^2}\left(\dot{y}^2(0) + \dot{z}^2(0)\right) = \frac{\gamma^2 \mathbf{v}_\perp^2}{\omega_c^2}\,. \tag{5.87}$$

This simple helical motion can be easily obtained directly from the equations of motion in 3D form. Since the speed is constant, $d\mathbf{p}/dt = \gamma m d\mathbf{v}/dt$ and the Lorentz force law gives

$$\frac{d\mathbf{v}}{dt} = \frac{q}{\gamma m}\mathbf{v} \times \mathbf{B}\,, \tag{5.88}$$

with $\gamma^{-2} = 1 - \mathbf{v}^2/c^2$. It follows that the component of $\mathbf{v}$ along the field is constant, while the perpendicular component precesses on a circle with angular frequency $\omega = qB/(\gamma m)$. The radius of the circle then follows from $|\mathbf{v}_\perp| = \omega r$.

$\mathbf{E} \perp \mathbf{B}$ *with non-zero* $\mathbf{E}$ *and* $\mathbf{B}$. We can always choose axes such that $\mathbf{E} = (0, E, 0)$ and $\mathbf{B} = (0, 0, B)$ (where E and B may be negative). If we now boost along the x-axis, we have

$$\mathbf{E}' = \begin{pmatrix} 0 \\ \gamma(E - vB) \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B}' = \begin{pmatrix} 0 \\ 0 \\ \gamma(B - vE/c^2) \end{pmatrix}\,. \tag{5.89}$$

If $|\mathbf{E}| > c|\mathbf{B}|$, we can set $\mathbf{B}' = 0$ by taking $\beta = cB/E$. We are then left with motion in a uniform electric field, which was discussed above. For $|\mathbf{E}| < c|\mathbf{B}|$, we can set $\mathbf{E}' = 0$ by taking $\beta = E/(cB)$ and we have helical motion in a constant magnetic field. Since a particle at rest in a magnetic field remains stationary, there exists a solution for $|\mathbf{E}| < c|\mathbf{B}|$ where in the original frame the particle moves at constant velocity in the direction perpendicular to the crossed fields. This is simply the velocity $\mathbf{v}$ such that electric and magnetic forces cancel, i.e., $q\mathbf{E} = -q\mathbf{v} \times \mathbf{B}$ requiring $\mathbf{v} = \mathbf{E} \times \mathbf{B}/\mathbf{B}^2$. The fact that charged particles with a specific velocity can move uniformly through crossed electric and magnetic fields can be used to select charged particles on their velocity. The special case $|\mathbf{E}| = c|\mathbf{B}|$ is left as an exercise.

Exercise: For perpendicular magnetic fields with $|\mathbf{E}| = c|\mathbf{B}|$ and $\mathbf{E} = (0, E, 0)$ and $\mathbf{B} = (0, 0, E/c)$, show that $ct(\tau)$ and $x(\tau)$ are, generally, cubic polynomials in τ, $y(\tau)$ is a quadratic polynomial, and $z(\tau)$ is linear.

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6 Energy and momentum of the electromagnetic field

In this section we shall discuss the energy and momentum densities carried by the electromagnetic field, and show how these are combined with the fluxes of energy and momentum into the spacetime *stress-energy tensor*.

6.1 Energy and momentum conservation

We begin by recalling Poynting's theorem, which expresses energy conservation in electromagnetism. In any inertial frame, the rate of work done per unit volume by the electromagnetic field on charged particles is $\mathbf{J} \cdot \mathbf{E}$. Note that magnetic field does no work since the magnetic force on any charge is perpendicular to its velocity. We now eliminate $\mathbf{J}$ in terms of the fields using Eq. (1.4) to find

$$\mathbf{J} \cdot \mathbf{E} = \frac{1}{\mu_0}\mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) - \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \,. \tag{6.1}$$

We can manipulate this further using

$$\boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{E}) - \mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) \,. \tag{6.2}$$

Replacing $\boldsymbol{\nabla} \times \mathbf{E}$ with $-\partial \mathbf{B}/\partial t$ from Eq. (1.2), and substituting in Eq. (6.1), we find

$$\mathbf{J} \cdot \mathbf{E} = -\frac{\partial}{\partial t}\left(\frac{\epsilon_0 \mathbf{E}^2}{2} + \frac{\mathbf{B}^2}{2\mu_0}\right) - \frac{1}{\mu_0}\boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{B}) \,. \tag{6.3}$$

Introducing the *Poynting vector*,

$$\mathbf{N} \equiv \mathbf{E} \times \mathbf{B}/\mu_0 \,, \tag{6.4}$$

and recalling the electromagnetic energy density

$$\epsilon = \frac{\epsilon_0 \mathbf{E}^2}{2} + \frac{\mathbf{B}^2}{2\mu_0}\,, \tag{6.5}$$

we have the (non-)conservation law

$$\frac{\partial \epsilon}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{N} = -\mathbf{J} \cdot \mathbf{E}\,. \tag{6.6}$$

The Poynting vector therefore gives the *energy flux* of the electromagnetic field, i.e., the rate at which electromagnetic energy flows through a surface element $d\mathbf{S}$ is $\mathbf{N} \cdot d\mathbf{S}$. Integrating Eq. (6.6) over a volume V gives

$$\frac{d}{dt}\int_V \epsilon\, d^3\mathbf{x} + \int_V \mathbf{E} \cdot \mathbf{J}\, d^3\mathbf{x} = -\int_{\partial V} \mathbf{N} \cdot d\mathbf{S}\,. \tag{6.7}$$

In words, this states that the rate of change of electromagnetic energy in the volume plus the rate at which the field does work on charges in the volume (i.e., the rate of change of mechanical energy in the volume) is equal to the rate at which electromagnetic energy flows *into* the volume.

We now perform a similar calculation for the force per unit volume on charged particles, $\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}$. Eliminating ρ with $\boldsymbol{\nabla} \cdot \mathbf{E} = \rho/\epsilon_0$ and $\mathbf{J}$ with Eq. (1.2) as above, we have the force density

$$\begin{aligned}
\rho\mathbf{E} + \mathbf{J} \times \mathbf{B} &= \epsilon_0 \mathbf{E} \boldsymbol{\nabla} \cdot \mathbf{E} + \frac{1}{\mu_0}(\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} \\
&= \epsilon_0 \mathbf{E} \boldsymbol{\nabla} \cdot \mathbf{E} + \frac{1}{\mu_0}(\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B} - \epsilon_0 \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) + \epsilon_0 \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t} \\
&= \epsilon_0 \mathbf{E} \boldsymbol{\nabla} \cdot \mathbf{E} + \frac{1}{\mu_0}(\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B} - \epsilon_0 \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) + \epsilon_0 (\boldsymbol{\nabla} \times \mathbf{E}) \times \mathbf{E}\,.
\end{aligned} \tag{6.8}$$

We can make the right-hand side look more symmetric between $\mathbf{E}$ and $\mathbf{B}$ by adding in $\mathbf{B}\boldsymbol{\nabla} \cdot \mathbf{B}/\mu_0 = 0$. Moreover, we can manipulate the curl terms with

$$(\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B} = \mathbf{B} \cdot \boldsymbol{\nabla}\mathbf{B} - \boldsymbol{\nabla}(\mathbf{B}^2)/2\,, \tag{6.9}$$

and similarly for $\mathbf{B} \to \mathbf{E}$, to give

$$\begin{aligned}
\rho\mathbf{E} + \mathbf{J} \times \mathbf{B} = {}& \epsilon_0 \left(\mathbf{E}\boldsymbol{\nabla} \cdot \mathbf{E} + \mathbf{E} \cdot \boldsymbol{\nabla}\mathbf{E} - \frac{1}{2}\boldsymbol{\nabla}\mathbf{E}^2 \right) + \frac{1}{\mu_0}\left(\mathbf{B}\boldsymbol{\nabla} \cdot \mathbf{B} + \mathbf{B} \cdot \boldsymbol{\nabla}\mathbf{B} - \frac{1}{2}\boldsymbol{\nabla}\mathbf{B}^2 \right) \\
& - \epsilon_0 \frac{\partial(\mathbf{E} \times \mathbf{B})}{\partial t}\,.
\end{aligned} \tag{6.10}$$

The first two terms on the right-hand side are total derivatives; they are (minus) the divergence of the symmetric *Maxwell stress tensor*

$$\sigma_{ij} \equiv -\epsilon_0 \left(E_i E_j - \frac{1}{2}\mathbf{E}^2 \delta_{ij} \right) - \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\mathbf{B}^2 \delta_{ij} \right)\,. \tag{6.11}$$

Further introducing the electromagnetic *momentum density*

$$\mathbf{g} \equiv \epsilon_0 \mathbf{E} \times \mathbf{B} = \mathbf{N}/c^2 \,, \tag{6.12}$$

we have a (non-)conservation law for momentum:

$$\frac{\partial g_i}{\partial t} + \frac{\partial \sigma_{ij}}{\partial x_j} = -\left(\rho \mathbf{E} + \mathbf{J} \times \mathbf{B}\right)_i \,. \tag{6.13}$$

The Maxwell stress tensor σ_{ij} encodes the flux of 3-momentum. In particular, the rate at which the ith component of 3-momentum flows through a surface element $d\mathbf{S}$ is given by $\sigma_{ij} dS_j$. Equation (6.13) can be written in integral form as

$$\frac{d}{dt} \int_V g_i \, d^3\mathbf{x} + \int_V (\rho \mathbf{E} + \mathbf{J} \times \mathbf{B})_i \, d^3\mathbf{x} = -\int_{\partial V} \sigma_{ij} \, dS_j \,, \tag{6.14}$$

which states that the rate of change of electromagnetic momentum in some volume V, plus the total force on charged particles (i.e., the rate of change of mechanical momentum) is equal to the rate at which electromagnetic momentum flows into V through the boundary.

Note that the Maxwell stress tensor is symmetric. This is true more generally (e.g. it holds for the stress tensor in continuum and fluid mechanics) as a consequence of angular momentum conservation; we shall briefly review this argument in Sec. 6.4. Note also that the momentum density and energy flux (Poynting vector) are related, $\mathbf{g} = \mathbf{N}/c^2$. We might have anticipated this result by thinking quantum mechanically of the electromagnetic field being composed of massless photons (all moving at the speed of light), with energy and momentum related by $E = pc$. A net energy flux will then necessarily be accompanied by a momentum density with $\mathbf{g} = \mathbf{N}/c^2$. In fact, the relation between momentum density and energy flux is universal and is necessary to ensure symmetry of σ_{ij} in every inertial frame (see also Sec. 6.4).

Given the integral form of the conservation law, Eq. (6.14), we note finally that for static situations, the total electromagnetic force on a charge and current distribution can be calculated by integrating the Maxwell stress tensor over any surface that fully encloses the sources.

6.2 Stress-energy tensor

We now develop a covariant formulation of the energy and momentum conservation laws derived above, seeking to unite them in a single spacetime conservation law. In doing so, we shall show that ϵ, $\mathbf{N}$ (and $\mathbf{g}$) and σ_{ij} are the components of a type-$\binom{2}{0}$ Lorentz tensor, the *stress-energy tensor*.

To motivate the introduction of the stress-energy tensor, first consider a simpler mechanical example. Massive particles, of rest mass m, are all at rest in some inertial frame S with (proper) number density n_0. In that frame, we clearly have

$$\epsilon = mc^2 n_0 \,, \quad \mathbf{N} = 0 = \mathbf{g} \,, \quad \sigma_{ij} = 0 \,. \tag{6.15}$$

Now consider performing a standard Lorentz boost to the frame S' in which the particles have 3-velocity $\mathbf{v}' = -(v, 0, 0)$. In S', the number density is γn_0 (by length contraction), the energy of each particle is γmc^2 and the momentum of each is $-\gamma m\mathbf{v}$. It follows that the energy density in S' is

$$\epsilon' = \gamma mc^2 \gamma n_0 = \gamma^2 \epsilon \,. \tag{6.16}$$

There is now a non-zero energy flux in S', with

$$N'_x = -\epsilon' v = -\gamma^2 v \epsilon \,, \tag{6.17}$$

and a momentum density with non-zero component

$$g'_x = (-\gamma mv)\gamma n_0 = -\gamma^2 v\epsilon/c^2 = N'_x/c^2 \,. \tag{6.18}$$

Finally, there is a flux of the x-component of 3-momentum along the x'-direction, so the stress tensor has a single non-zero component

$$\sigma'_{xx} = -v g'_x = \gamma^2 (v/c)^2 \epsilon \,. \tag{6.19}$$

Note how ϵ, $\mathbf{N}$ and σ_{ij} mix amongst themselves under Lorentz transformations. Let us tentatively write

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & cg_j \\ N_i/c & \sigma_{ij} \end{pmatrix} \,. \tag{6.20}$$

We shall now show that this object is indeed a tensor since under Lorentz transformations the components transform correctly. In the rest frame S, we have

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \,. \tag{6.21}$$

If $T^{\mu\nu}$ were a tensor, its components in S' would necessarily be given by $T'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta}$, where $\Lambda^\mu{}_\alpha$ for the standard boost is given in Eq. (4.7). Evaluating the transformation gives

$$\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta} = \begin{pmatrix} \gamma^2\epsilon & -\gamma^2\beta\epsilon & 0 & 0 \\ -\gamma^2\beta\epsilon & \gamma^2\beta^2\epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \epsilon' & cg'_x & 0 & 0 \\ N'_x/c & \sigma'_{xx} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \,, \tag{6.22}$$

where in the last equality we used the transformations derived above from physical arguments. Since the components on the right are exactly what we should have in S' from the identification in Eq. (6.20), we conclude that $T^{\mu\nu}$ is indeed a tensor.

Returning to electromagnetism, we have

$$T^{\mu\nu} = \frac{1}{\mu_0} \begin{pmatrix} \frac{1}{2}(\mathbf{E}^2/c^2 + \mathbf{B}^2) & (\mathbf{E}\times\mathbf{B})_j/c \\ (\mathbf{E}\times\mathbf{B})_i/c & \frac{1}{2}(\mathbf{E}/c)^2\delta_{ij} - E_iE_j/c^2 + \frac{1}{2}\mathbf{B}^2\delta_{ij} - B_iB_j \end{pmatrix} . \tag{6.23}$$

Since this is quadratic in the fields, we must be able to write it in terms of suitable contractions of the field-strength tensor $F^{\mu\nu}$ with itself. Some straightforward calculation will convince you that

$$T^{\mu\nu} = \frac{1}{\mu_0}\left(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right) . \tag{6.24}$$

This is symmetric as it must be. Noting the invariant $F^{\alpha\beta}F_{\alpha\beta} = 2(\mathbf{B}^2 - \mathbf{E}^2/c^2)$ [see Eq. (5.56)], we have, for example,

$$\begin{aligned} \mu_0 T^{00} &= F^{0\alpha}F^{0}{}_{\alpha} - \tfrac{1}{4}\eta^{00}2(\mathbf{B}^2 - \mathbf{E}^2/c^2) \\ &= F^{0i}F^{0}{}_{i} + \tfrac{1}{2}(\mathbf{B}^2 - \mathbf{E}^2/c^2) \\ &= \mathbf{E}^2/c^2 + \tfrac{1}{2}\mathbf{B}^2 - \tfrac{1}{2}\mathbf{E}^2/c^2 \\ &= \tfrac{1}{2}\left(\mathbf{B}^2 + \mathbf{E}^2/c^2\right) = \mu_0\epsilon\,. \end{aligned} \tag{6.25}$$

Exercise: Verify that Eq. (6.24) correctly reproduces the expressions for the energy flux and Maxwell stress tensor given in Eqs (6.4) and (6.11) respectively.

Note that $T^{\mu\nu}$ is trace-free since

$$T^{\mu}{}_{\mu} = \frac{1}{\mu_0}\left(F^{\mu\alpha}F_{\mu\alpha} - \frac{1}{4}\eta_{\mu\nu}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right) = 0\,, \tag{6.26}$$

where we used $\eta_{\mu\nu}\eta^{\mu\nu} = 4$.

6.3 Covariant conservation of the stress-energy tensor

Having introduced the stress-energy tensor $T^{\mu\nu}$, we expect to be able to write the conservation laws, Eqs (6.6) and (6.13), in the covariant form

$$\partial_\mu T^{\mu\nu} = \text{some 4-vector linear in the field strength tensor and current 4-vector}\,. \tag{6.27}$$

Taking the 0-component of $\partial_\mu T^{\mu\nu}$, we have

$$\begin{aligned}\partial_\mu T^{\mu 0} &= \frac{\partial T^{00}}{\partial(ct)} + \frac{\partial T^{0i}}{\partial x_i} \\ &= \frac{1}{c}\frac{\partial \epsilon}{\partial t} + \frac{1}{c}\boldsymbol{\nabla}\cdot\mathbf{N} \\ &= -\frac{1}{c}\mathbf{J}\cdot\mathbf{E}\,. \end{aligned} \tag{6.28}$$

Repeating for the i-component,

$$\begin{aligned}\partial_\mu T^{\mu i} &= \frac{\partial T^{0i}}{\partial(ct)} + \frac{\partial T^{ji}}{\partial x_j} \\ &= \frac{\partial g_i}{\partial t} + \frac{\partial \sigma_{ji}}{\partial x_j} \\ &= -(\rho\mathbf{E} + \mathbf{J}\times\mathbf{B})_i\,, \end{aligned} \tag{6.29}$$

so that

$$\partial_\mu T^{\mu\nu} = -\begin{pmatrix} \mathbf{J}\cdot\mathbf{E}/c \\ \rho\mathbf{E} + \mathbf{J}\times\mathbf{B} \end{pmatrix}. \tag{6.30}$$

The right-hand side (which must be a 4-vector) is $-F^{\nu}{}_{\mu}J^{\mu}$, since

$$F^{\nu}{}_{\mu}J^{\mu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}\begin{pmatrix} \rho c \\ J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} \mathbf{J}\cdot\mathbf{E}/c \\ \rho E_x + J_y B_z - J_z B_y \\ \rho E_y + J_z B_x - J_x B_z \\ \rho E_z + J_x B_y - J_y B_x \end{pmatrix}. \tag{6.31}$$

It follows that we can combine energy and momentum conservation into the covariant conservation law for the stress-energy tensor:

$$\partial_\mu T^{\mu\nu} = -F^{\nu}{}_{\mu}J^{\mu}\,. \tag{6.32}$$

The 4-vector on the right of this equation has a simple interpretation in the case of a simple convective flow of charges, each with charge q and number density n_0 in their rest frame. Then, $J^\mu = qn_0u^\mu$, where u^μ is the 4-velocity, and $F^{\nu}{}_{\mu}J^{\mu} = qn_0F^{\nu}{}_{\mu}u^{\mu}$ is the product of n_0 and the 4-force on each charge. It follows that, in this case, $F^{\nu}{}_{\mu}J^{\mu}$ is the 4-force per unit rest-frame volume.

We can establish the conservation law, Eq. (6.32), directly from the covariant form of Maxwell's equations. Lowering the ν index, we have

$$\begin{aligned}\mu_0\partial_\mu T^{\mu}{}_{\nu} &= \partial_\mu\left(F^{\mu\alpha}F_{\nu\alpha} - \tfrac{1}{4}\delta^{\mu}_{\nu}F_{\alpha\beta}F^{\alpha\beta}\right) \\ &= (\partial_\mu F^{\mu\alpha})F_{\nu\alpha} + F^{\mu\alpha}\partial_\mu F_{\nu\alpha} - \tfrac{1}{2}F^{\alpha\beta}\partial_\nu F_{\alpha\beta} \\ &= -\mu_0F_{\nu\alpha}J^\alpha + \tfrac{1}{2}\left(F^{\mu\alpha}\partial_\mu F_{\nu\alpha} + F^{\mu\alpha}\partial_\mu F_{\nu\alpha} - F^{\alpha\beta}\partial_\nu F_{\alpha\beta}\right)\,, \end{aligned} \tag{6.33}$$

where we used the first of Eq. (5.33) in the last equality. With some relabelling of the dummy indices, the sum of the final three terms can be shown to vanish by virtue of the second of Eq. (5.33),

$$\begin{aligned} F^{\mu\alpha}\partial_\mu F_{\nu\alpha} + F^{\mu\alpha}\partial_\mu F_{\nu\alpha} - F^{\alpha\beta}\partial_\nu F_{\alpha\beta} &= F^{\alpha\beta}\partial_\alpha F_{\nu\beta} + F^{\beta\alpha}\partial_\beta F_{\nu\alpha} - F^{\alpha\beta}\partial_\nu F_{\alpha\beta} \\ &= -F^{\alpha\beta}\left(\partial_\alpha F_{\beta\nu} + \partial_\beta F_{\nu\alpha} + \partial_\nu F_{\alpha\beta}\right) \\ &= 0\,, \end{aligned} \tag{6.34}$$

leaving $\partial_\mu T^\mu{}_\nu = -F_{\nu\mu}J^\mu$.

6.4 Symmetry of the stress-energy tensor

The stress-energy tensor is symmetric since $\mathbf{g} = \mathbf{N}/c^2$ and $\sigma_{ij} = \sigma_{ji}$. We demonstrated above that this was true for electromagnetism but it is true generally. In this section we shall briefly discuss why.

Consider the energy and momentum conservation laws for an isolated system:

$$\frac{\partial \epsilon}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{N} = 0 \tag{6.35}$$

$$\frac{\partial g_i}{\partial t} + \frac{\partial \sigma_{ij}}{\partial x_j} = 0\,. \tag{6.36}$$

The usual 3D angular momentum of the system is

$$\mathbf{L} \equiv \int \mathbf{x}\times\mathbf{g}\,d^3\mathbf{x}\,, \tag{6.37}$$

where the integral extends over all space. Taking the time derivative gives

$$\begin{aligned} \frac{dL_i}{dt} &= \epsilon_{ijk}\int x_j \frac{\partial g_k}{\partial t}\,d^3\mathbf{x} \\ &= -\epsilon_{ijk}\int x_j \frac{\partial \sigma_{kl}}{\partial x_l}\,d^3\mathbf{x} \\ &= -\epsilon_{ijk}\int \frac{\partial}{\partial x_l}\left(x_j\sigma_{kl}\right)\,d^3\mathbf{x} + \epsilon_{ijk}\int \sigma_{kj}\,d^3\mathbf{x} \\ &= -\epsilon_{ijk}\int x_j\sigma_{kl}\,dS_l + \epsilon_{ijk}\int \sigma_{kj}\,d^3\mathbf{x}\,. \end{aligned} \tag{6.38}$$

The first term on the right-hand side in the last line is the rate at which angular momentum is entering the (isolated) system from infinity and and must vanish for a localised field configuration. We then see that angular momentum of the isolated system is conserved *provided that* σ_{ij} *is symmetric* so that the last term on the right of Eq. (6.38) vanishes.

Now consider a Lorentz transformation. The stress-energy tensor transforms as

$$T'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}T^{\alpha\beta}\,. \tag{6.39}$$

Considering the 1-2 component, we have

$$\begin{aligned}\sigma'_{xy} &= \Lambda^{1}{}_{\alpha}\Lambda^{2}{}_{\beta}T^{\alpha\beta} \\ &= \Lambda^{1}{}_{0}\Lambda^{2}{}_{2}T^{02} + \Lambda^{1}{}_{1}\Lambda^{2}{}_{2}T^{12} \quad \text{(standard boost)} \\ &= -\gamma\beta T^{02} + \gamma T^{12}\,. \end{aligned} \tag{6.40}$$

Repeating for the 2-1 component, we have

$$\sigma'_{yx} = -\gamma\beta T^{20} + \gamma T^{21}\,. \tag{6.41}$$

Symmetry of σ_{ij} in all inertial frames therefore requires $T^{i0} = T^{0i}$ and so $\mathbf{g} = \mathbf{N}/c^2$.

A further consequence of the symmetry of the stress-energy tensor follows from considering the dynamics of the "centre of energy" of the system, $\int \epsilon\mathbf{x}\, d^3\mathbf{x}/\int \epsilon\, d^3\mathbf{x}$. Since the total energy is conserved, the dynamics of the centre of energy follow from those of $\int \epsilon\mathbf{x}\, d^3\mathbf{x}$; we have

$$\begin{aligned}\frac{d}{dt}\int \epsilon x_i\, d^3\mathbf{x} &= \int \frac{\partial\epsilon}{\partial t}x_i\, d^3\mathbf{x} \\ &= -\int \frac{\partial N_j}{\partial x_j}x_i\, d^3\mathbf{x} \\ &= -\int \frac{\partial}{\partial x_j}\left(x_i N_j\right)\, d^3\mathbf{x} + \int N_i\, d^3\mathbf{x} \\ &= -\int x_i N_j\, dS_j + \int N_i\, d^3\mathbf{x}\,. \end{aligned} \tag{6.42}$$

The first term on the right-hand side vanishes for a localised field distribution. If $\mathbf{N} = \mathbf{g}c^2$, then $\int \mathbf{N}\, d^3\mathbf{x}$ is constant by the conservation law for $\mathbf{g}$ [Eq. (6.36)], and we see from Eq. (6.42) that the centre of energy moves uniformly. In particular, in the zero momentum frame where $\int \mathbf{g}\, d^3\mathbf{x} = 0$, the centre of energy remains at rest.

The above argument shows that two theorems from mechanics for isolated systems – that the centre of energy moves uniformly and that angular momentum is conserved – are related in relativity. Indeed, one can introduce the object

$$S^{\mu\alpha\beta} \equiv x^{\alpha}T^{\mu\beta} - x^{\beta}T^{\mu\alpha} = -S^{\mu\beta\alpha}\,, \tag{6.43}$$

which transforms as a tensor under homogeneous Lorentz transformations. Then, for an isolated system, conservation of the stress-energy tensor implies that

$$\begin{aligned}\partial_{\mu}S^{\mu\alpha\beta} &= \delta^{\alpha}_{\mu}T^{\mu\beta} + x^{\alpha}\underbrace{\partial_{\mu}T^{\mu\beta}}_{=0} - \delta^{\beta}_{\mu}T^{\mu\alpha} - x^{\beta}\underbrace{\partial_{\mu}T^{\mu\alpha}}_{=0} \\ &= T^{\alpha\beta} - T^{\beta\alpha} = 0\,. \end{aligned} \tag{6.44}$$

Integrating this conservation law over space, the $0i$ component returns the centre-of-energy theorem and the ij component returns 3D angular momentum conservation.

6.5 Stress-energy tensor of a plane electromagnetic wave

As an example of computing the stress-energy tensor, we consider electromagnetic plane waves. Maxwell's equations admit wave-like solutions [see the wave equation in Eq. (1.8)] in free space:

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_0 f(\mathbf{k}\cdot\mathbf{x} - \omega t) \\ \mathbf{B} &= \mathbf{B}_0 f(\mathbf{k}\cdot\mathbf{x} - \omega t) \quad \text{with } \omega = c|\mathbf{k}| \,. \end{aligned} \tag{6.45}$$

If the function f is sinusoidal, with period 2π, these are harmonic waves propagating at speed c with wavelength $2\pi/k$ (with $k = |\mathbf{k}|$) and frequency $\omega/(2\pi)$. At any time, the phase of the wave, $\mathbf{k}\cdot\mathbf{x} - \omega t$, is constant on 3D planes perpendicular to the wavevector $\mathbf{k}$. Since $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$ in free space, and $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$, we must have

$$\mathbf{k}\cdot\mathbf{E}_0 = 0 \quad \text{and} \quad \mathbf{k}\cdot\mathbf{B}_0 = 0\,. \tag{6.46}$$

Furthermore, from $\boldsymbol{\nabla}\times\mathbf{E} = -\partial\mathbf{B}/\partial t$ we have

$$\mathbf{k}\times\mathbf{E}_0 = \omega\mathbf{B}_0 \quad \Rightarrow \quad \hat{\mathbf{k}}\times\mathbf{E}_0 = c\mathbf{B}_0\,, \tag{6.47}$$

where $\hat{\mathbf{k}}$ is a unit vector in the direction of $\mathbf{k}$, and so $c|\mathbf{B}_0| = |\mathbf{E}_0|$. It follows that $\mathbf{E}_0$, $\mathbf{B}_0$ and $\mathbf{k}$ form a right-handed triad of vectors.

We can write the fields in covariant form as follows. Phase differences must be Lorentz invariant so we can introduce the 4-wavevector

$$k^\mu = (\omega/c, \mathbf{k})\,. \tag{6.48}$$

This is a null 4-vector, $\eta_{\mu\nu}k^\mu k^\nu = 0$, since $\omega = ck$. The phase can then be written covariantly as $\mathbf{k}\cdot\mathbf{x} - \omega t = k_\mu x^\mu$. Specialising to $\mathbf{k}$ along the x-direction of some inertial frame, and $\mathbf{E}_0$ along y (so that $\mathbf{B}_0$ is along z), the components of the field-strength tensor in that frame are

$$F^{\mu\nu} = B_0 f(k_\mu x^\mu) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\,, \tag{6.49}$$

where $B_0 = |\mathbf{B}_0|$. Both invariants $F^{\mu\nu}F_{\mu\nu}$ and ${}^*F^{\mu\nu}F_{\mu\nu}$ vanish since $\mathbf{E}\cdot\mathbf{B} = 0$ and $|\mathbf{E}| = c|\mathbf{B}|$.

The stress-energy tensor, Eq. (6.23), reduces to $T^{\mu\nu} = F^{\mu\alpha}F^\nu{}_\alpha/\mu_0$ since $F^{\mu\nu}F_{\mu\nu} = 0$.

It follows that

$$T^{\mu\nu} = \frac{B_0^2 f^2(k_\mu x^\mu)}{\mu_0} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$= \frac{B_0^2 f^2(k_\mu x^\mu)}{\mu_0} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{6.50}$$

It follows that the energy density

$$\epsilon = B_0^2 f^2(k_\mu x^\mu)/\mu_0 \,, \tag{6.51}$$

which receives equal contributions from the electric and magnetic fields. The Poynting vector is along the x-direction, with

$$N_x = cB_0^2 f^2(k_\mu x^\mu)/\mu_0 \,. \tag{6.52}$$

Finally, the momentum flux has only one non-zero component:

$$\sigma_{xx} = B_0^2 f^2(k_\mu x^\mu)/\mu_0 \,. \tag{6.53}$$

Note that

$$\sigma_{xx} = N_x/c = cg_x = \epsilon \,. \tag{6.54}$$

These results are consistent with the photon picture in which photons with energy $E = pc$ (where p is the magnitude of their momentum) propagate at speed c along the x-direction. The energy density is then c times the momentum density, and the momentum flux is c times the momentum density[12].

Exercise: Consider reflection of a plane electromagnetic wave, propagating along the x-direction, from a perfect conductor whose plane surface is at $x = 0$. The conductivity is so high that the fields decay very rapidly inside the conductor and we can therefore approximate the fields as being zero inside. The **E** field is tangent to the surface and, since the tangential component is continuous, the **E** field must be zero at the surface of the conductor. The **B** field is also tangential but is now discontinuous at the surface with the discontinuity driven by a surface current flowing in a thin layer on the surface.

The total electric and magnetic fields (sum of incident and reflected waves) outside the conductor ($x < 0$) are therefore standing waves of the form

$$\mathbf{E}_{\rm tot} = B_0 c \,[f(kx - \omega t) - f(-kx - \omega t)]\,\hat{\mathbf{y}}$$
$$\mathbf{B}_{\rm tot} = B_0 \,[f(kx - \omega t) + f(-kx - \omega t)]\,\hat{\mathbf{z}} \,. \tag{6.55}$$

[12]The momentum carried through an area dS in the y-z plane in time dt is the total momentum of those photons within a volume $cdtdS$, i.e., $\sigma_{xx}dSdt = g_x cdtdS$.

Show that the energy density is

$$\epsilon = \frac{B_0^2}{\mu_0}\left[f^2(kx-\omega t) + f^2(-kx-\omega t)\right] , \tag{6.56}$$

which is the sum of the energy density in the incident and reflected waves. For harmonic waves, the time-averaged energy density is the same at all x. Show also that the Poynting vector has a single non-zero component

$$N_x = \frac{B_0^2 c}{\mu_0}\left[f^2(kx-\omega t) - f^2(-kx-\omega t)\right] , \tag{6.57}$$

which, again, is the sum of the incident and reflected energy fluxes. The Poynting vector vanishes at the surface of the conductor at all times, so no energy enters the conductor. This is still consistent with there being a surface current since the rate of energy dissipation by this current scales as the inverse of the conductivity and so vanishes in the limit of infinite conductivity assumed here. At other locations, the time-averaged Poynting vector vanishes for harmonic waves. Finally, show that the Maxwell stress-tensor is diagonal with

$$\begin{aligned} \sigma_{xx} &= \frac{B_0^2}{\mu_0}\left[f^2(kx-\omega t) + f^2(-kx-\omega t)\right] \\ \sigma_{yy} &= \frac{2B_0^2}{\mu_0} f(kx-\omega t) f(-kx-\omega t) , \end{aligned} \tag{6.58}$$

and $\sigma_{zz} = -\sigma_{yy}$. The $\sigma_{xx} = \epsilon$ is the sum of the incident and reflected momentum fluxes. However, the σ_{yy} and σ_{zz} components arise purely from interference of the incident and reflected waves. Since σ_{yy} and σ_{zz} only vary in the x-direction, the non-zero values are still consistent with the momentum density of the field and the conservation law (6.36).

There is a force exerted on the conductor ("radiation pressure") in the x-direction given per area by

$$\sigma_{xx}|_{x=0} = \frac{2B_0^2}{\mu_0} f^2(-\omega t) . \tag{6.59}$$

This follows from the integral form of the momentum conservation law, Eq. (6.14), noting the momentum density is zero inside the conductor. The *incident* momentum density at $x=0$ is $B_0^2 f^2(-\omega t)/(\mu_0 c)$, and so the radiation pressure is $2c$ times the incident momentum density. What is the physical origin of this force? It is the Lorentz force of the **B** field at the surface on the surface current distribution. To see that this is consistent, note that the surface current $\mathbf{J}_s$ (defined such that the current through a given line element dl lying in the surface and perpendicular to $\mathbf{J}_s$ is $|\mathbf{J}_s|dl$) needed to maintain the discontinuity in **B** is along the x-direction with $\mu_0 J_s = B_y|_{x=0}$. The force per area on this current is $J_s B_y|_{x=0}/2$ along the x-direction. The factor of $1/2$ here accounts for the average value of the magnetic field across the depth of the current distribution. Using $B_y|_{x=0} = 2B_0(-\omega t)$, we see that the Lorentz force per area is exactly σ_{xx} at the surface of the conductor, as required.

6.5.1 Radiation pressure of a photon "gas"

We can obtain an *isotropic* distribution of radiation by forming a random superposition of incoherent plane waves propagating in all directions with equal intensity. (Incoherent waves have no lasting phase relation amongst themselves, e.g. the radiation emitted by two separated thermal sources.) Such radiation is the classical analogue of a *photon gas*. Isotropy requires the (time-averaged) Poynting vector to vanish and that the Maxwell stress tensor $\sigma_{ij} = p\delta_{ij}$, where p is the pressure. The averaged stress-energy tensor is then diagonal with

$$\langle T^{\mu\nu} \rangle = \mathrm{diag}(\epsilon, p, p, p) \,. \tag{6.60}$$

The electromagnetic stress energy tensor is necessarily trace-free;

$$T^{\mu}{}_{\mu} = 0 \quad \Rightarrow \quad p = \epsilon/3 \,. \tag{6.61}$$

We see that for a photon gas the pressure is one-third of the energy density irrespective of the way that the energy is distributed over frequency (i.e., the radiation does not have to have a blackbody spectrum).

7 Radiation of electromagnetic waves

In this section we consider the *production* of electromagnetic radiation from time-dependent charge distributions. We first consider radiation from macroscopic charge and current densities involving non-relativistic motions, as, for example, in radiation from a radio antenna. We then move on to radiation from isolated charges in arbitrary motion. As we shall see, it is the acceleration of the charges that determines the radiation field. The theory we develop underlies extreme phenomena such as the *synchrotron radiation* emitted from charges circling relativistically in magnetic fields.

7.1 Retarded potentials

We begin by deriving the 4-potential generated causally by an arbitrary 4-current J^μ. Our starting point is Eq. (5.14) for the 4-vector potential A^μ:

$$\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = -\mu_0 J^\mu \,. \tag{7.1}$$

It is very convenient to make a gauge choice (the *Lorenz gauge*) such that $\partial_\mu A^\mu = 0$. This is clearly a Lorentz-invariant statement, which has the benefit of retaining covariance of the subsequent gauge-fixed equations under Lorentz transformations. It is always possible to make this gauge choice: starting in some general gauge we make a transformation $A^\mu \to A^\mu + \partial^\mu \chi$ with χ chosen to satisfy $\Box\chi = -\partial_\mu A^\mu$. As we show below, we can always solve this equation to determine a suitable χ. In the Lorenz gauge, Eq. (7.1) simplifies to

$$\Box A^\mu = -\mu_0 J^\mu \quad \text{or} \quad \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) A^\mu = -\mu_0 J^\mu \,, \tag{7.2}$$

which is a sourced wave equation.

We solve Eq. (7.2) by taking a Fourier transform in time, writing

$$A^\mu(t, \mathbf{x}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{A}^\mu(\omega, \mathbf{x}) e^{i\omega t}\, d\omega \,, \tag{7.3}$$

and similarly for J^μ. The sourced wave equation then becomes

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right) \tilde{A}^\mu = -\mu_0 \tilde{J}^\mu \,. \tag{7.4}$$

This is a sourced Helmholtz equation with $k^2 = \omega^2/c^2$. As for Poisson's equation in electrostatics, we can solve this with a Green's function $G(\mathbf{x}; \mathbf{x}')$ such that

$$\left(\nabla^2 + k^2\right) G(\mathbf{x}; \mathbf{x}') = \delta^{(3)}(\mathbf{x} - \mathbf{x}') \,, \tag{7.5}$$

so that

$$\tilde{A}^\mu(\omega, \mathbf{x}) = -\mu_0 \int G(\mathbf{x}; \mathbf{x}') \tilde{J}^\mu(\omega, \mathbf{x}')\, d^3\mathbf{x}' \,. \tag{7.6}$$

To describe the fields produced by localised sources, we require the Green's function to tend to zero as $|\mathbf{x}| \to 0$. Homogeneity and isotropy require that $G(\mathbf{x}; \mathbf{x}')$ is a function of $|\mathbf{x} - \mathbf{x}'|$ alone. Taking $\mathbf{x}' = 0$ and $r = |\mathbf{x}|$, away from $r = 0$ we have

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial G}{\partial r}\right) + k^2 G = 0 \,. \tag{7.7}$$

The solution that tends to zero as $r \to \infty$ is $G = -Ae^{-ikr}/r$. (The reason for choosing e^{-ikr} rather than e^{ikr} is to do with causality and will be explained shortly.) We fix the normalisation A by integrating Eq. (7.5) over a sphere of radius ϵ centred on $\mathbf{x}'$, and taking the limit as $\epsilon \to 0$. The integral of the $k^2 G$ term goes like ϵ^3/ϵ and so vanishes in the limit, leaving

$$\begin{aligned} &\lim_{r\to 0} 4\pi r^2 \partial G/\partial r = 1 \\ \Rightarrow \quad &\lim_{r\to 0} 4\pi r^2 A \left(\frac{e^{-ikr}}{r^2} + ik\frac{e^{-ikr}}{r}\right) = 1 \,, \end{aligned} \tag{7.8}$$

so that $A = 1/(4\pi)$ and

$$G(\mathbf{x}; \mathbf{x}') = -\frac{1}{4\pi}\frac{e^{-i\omega|\mathbf{x}-\mathbf{x}'|/c}}{|\mathbf{x} - \mathbf{x}'|} \,. \tag{7.9}$$

Inserting this Green's function into Eq. (7.6) and taking the inverse Fourier transform, gives

$$A^\mu(t,\mathbf{x}) = \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty}\frac{d\omega}{\sqrt{2\pi}}\int d^3\mathbf{x}'\,\frac{e^{i\omega(t-|\mathbf{x}-\mathbf{x}'|/c)}}{|\mathbf{x}-\mathbf{x}'|}\tilde{J}^\mu(\omega,\mathbf{x}')\,. \tag{7.10}$$

The integral over ω is the inverse Fourier transform of $\tilde{J}^\mu(\omega,\mathbf{x}')$ and returns $J^\mu(t_{\text{ret}},\mathbf{x}')$ where the *retarded time* $t_{\text{ret}} \equiv t-|\mathbf{x}-\mathbf{x}'|/c$. This gives our final result for the 4-potential from an isolated time-dependent charge distribution:

$$A^\mu(t,\mathbf{x}) = \frac{\mu_0}{4\pi}\int\frac{J^\mu(t_{\text{ret}},\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,d^3\mathbf{x}'\,. \tag{7.11}$$

This has a simple physical interpretation. Since the event at t_{ret} and $\mathbf{x}'$ is on the *past lightcone* of the event at t and $\mathbf{x}$, the field at any event x^μ is generated by the superposition of the fields due to currents at earlier times on the past lightcone of x^μ. This reflects causality in that electromagnetic disturbances propagate at the speed of light and that only earlier events can influence the field at any spacetime point[13]. The potential A^μ is called the *retarded potential.* We now see why we had to choose the e^{-ikr}/r form for the Green's function – had we chosen e^{ikr}/r we would have obtained the *advanced potential* for which the currents at $t+|\mathbf{x}-\mathbf{x}'|/c$ and $\mathbf{x}'$ influence the field at t and $\mathbf{x}$, i.e.,, acausal propagation.

It is not obvious that Eq. (7.11) is covariant under Lorentz transformations, since we are integrating over space alone. To see that it is covariant, we rewrite it as a spacetime integral as follows:

$$A^\mu(x) = \frac{\mu_0}{4\pi}\int J^\mu(y)\frac{\delta(y^0-x^0+|\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|}\,d^4y\,. \tag{7.12}$$

This looks more covariant since it involves the Lorentz-invariant measure[14] d^4y. However, we still have the issue of the delta-function. To see that this is Lorentz invariant, consider

$$\delta\left(\eta_{\mu\nu}(x^\mu-y^\mu)(x^\nu-y^\nu)\right)\Theta(x^0-y^0)\,, \tag{7.13}$$

where Θ is the Heaviside (unit-step) function, which equals unity if its argument is greater than zero and vanishes otherwise. The product of the two terms in Eq. (7.13) is therefore Lorentz invariant since the delta-function is non-zero only for null-separated x^μ and y^μ, and the temporal ordering of such events is Lorentz invariant. We now show that Eq. (7.13) is twice the term $\delta(y^0-x^0+|\mathbf{x}-\mathbf{y}|)/|\mathbf{x}-\mathbf{y}|$ in question in Eq. (7.12). This follows from a general result for the delta-function with a function $f(x)$ as its argument:

$$\delta\left(f(x)\right) = \sum_i \frac{\delta(x-x_{*,i})}{|f'(x_{*,i})|}\,, \tag{7.14}$$

[13]Recall that the temporal ordering of timelike and null-separated events is Lorentz invariant

[14]Under a Lorentz transformation, $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, the Jacobian $|\partial x'^\mu/\partial x^\nu|$ is unity since $\det(\Lambda^\mu{}_\nu)^2 = 1$.

where the sum is over the roots $x_{*,i}$ of $f(x)$, i.e.,, $f(x_{*,i}) = 0$, and $f'(x)$ is the derivative of $f(x)$. Writing out the components of $\eta_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu)$, we have

$$|\mathbf{x}-\mathbf{y}|^2 - (x^0 - y^0)^2 = (|\mathbf{x}-\mathbf{y}| + x^0 - y^0)(|\mathbf{x}-\mathbf{y}| - x^0 + y^0)\,, \tag{7.15}$$

so that

$$\begin{aligned}\delta\left(\eta_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu)\right) &= \frac{\delta(y^0 - x^0 + |\mathbf{x}-\mathbf{y}|)}{2|y^0 - x^0|} + \frac{\delta(y^0 - x^0 - |\mathbf{x}-\mathbf{y}|)}{2|y^0 - x^0|} \\ &= \frac{\delta(y^0 - x^0 + |\mathbf{x}-\mathbf{y}|)}{2|\mathbf{x}-\mathbf{y}|} + \frac{\delta(y^0 - x^0 - |\mathbf{x}-\mathbf{y}|)}{2|\mathbf{x}-\mathbf{y}|}\,. \end{aligned} \tag{7.16}$$

The Heaviside function in Eq. (7.13) selects only the first term on the right, showing that

$$\delta\left(\eta_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu)\right)\Theta(x^0 - y^0) = \frac{\delta(y^0 - x^0 + |\mathbf{x}-\mathbf{y}|)}{2|\mathbf{x}-\mathbf{y}|}\,, \tag{7.17}$$

as required. Equation (7.12) therefore involves a Lorentz-invariant integration of the 4-vector current over the past lightcone of the event x^μ, and so is properly covariant.

Exercise: Noting that the gauge condition $\partial_\mu A^\mu = 0$ takes the form $i\omega\tilde{A}^0/c + \boldsymbol{\nabla}\cdot\tilde{\mathbf{A}} = 0$ after Fourier transforming in time, verify that the right-hand side of Eq. (7.6) satisfies the gauge condition by virtue of charge conservation.

7.2 Dipole radiation

As a first application of the retarded potential derived above, we consider radiation from some macroscopic charge distribution in which all motions are non-relativistic (in some inertial frame). We shall be interested in the fields at large distances compared to the spatial extent of the charge distribution.

Specifically, consider the case where all charges are localised within some region of size a, and place the origin of spatial coordinates within this volume. To calculate the 4-potential at $\mathbf{x}$, where $r = |\mathbf{x}| \gg a$, we Taylor expand $|\mathbf{x}-\mathbf{x}'|$ in Eq. (7.11) as

$$\begin{aligned}|\mathbf{x}-\mathbf{x}'| &= r - \frac{\mathbf{x}\cdot\mathbf{x}'}{r} + \cdots \\ &= r - \hat{\mathbf{x}}\cdot\mathbf{x}' + \cdots\,,\end{aligned} \tag{7.18}$$

where $\hat{\mathbf{x}}$ is a unit vector in the direction of $\mathbf{x}$. Note that the second term on the right is smaller than the first by $O(a/r)$. Substituting into Eq. (7.11), we have

$$A^\mu(t,\mathbf{x}) = \frac{\mu_0}{4\pi r}\int J^\mu(t_{\mathrm{ret}},\mathbf{x}')\left(1 + \frac{\hat{\mathbf{x}}\cdot\mathbf{x}'}{r} + \cdots\right)d^3\mathbf{x}'\,, \tag{7.19}$$

where $t_{\rm ret} = t - r/c + \hat{\mathbf{x}}\cdot\mathbf{x}'/c + \cdots$. At $r \gg a$, we can safely neglect the $O(a/r)$ corrections in the round brackets in Eq. (7.19). However, the treatment of the dependence of $t_{\rm ret}$ on $\mathbf{x}'$ depends on the rate at which the source varies. For an oscillating charge distribution, with angular frequency ω, if $a/c \ll 1/\omega$ then the variation in $t_{\rm ret}$ over the source is short compared to the period and can be handled perturbatively via a Taylor expansion of J^μ in time. Note that the typical velocity of the charges is $a\omega$ and so we require this to be much less than c for an expansion to be valid. Equivalently, since the fields radiated by a source oscillating at ω oscillate at the same frequency, the associated wavelength λ must be large compared to a.

For sufficiently slowly-varying sources, we can neglect the variation of $t_{\rm ret}$ across the source. The leading-order contribution to the magnetic vector potential is then

$$\mathbf{A}(t,\mathbf{x}) = \frac{\mu_0}{4\pi r}\int \mathbf{J}(t - r/c, \mathbf{x}')\, d^3\mathbf{x}' \,. \tag{7.20}$$

We can express the right-hand side in terms of the time derivative of the electric dipole moment of the charge distribution as follows. Charge conservation, $\partial\rho/\partial t + \boldsymbol{\nabla}\cdot\mathbf{J} = 0$, implies that

$$\begin{aligned}
\dot{\mathbf{P}} &= \frac{d}{dt}\int \rho(\mathbf{x})\mathbf{x}\, d^3\mathbf{x} \\
&= \int \frac{\partial\rho}{\partial t}\mathbf{x}\, d^3\mathbf{x} \\
&= -\int (\boldsymbol{\nabla}\cdot\mathbf{J})\mathbf{x}\, d^3\mathbf{x}\,,
\end{aligned} \tag{7.21}$$

so that

$$\begin{aligned}
\dot{P}_i &= -\int \frac{\partial J_j}{\partial x_j} x_i\, d^3\mathbf{x} \\
&= \int J_j \frac{\partial x_i}{\partial x_j}\, d^3\mathbf{x} \\
&= \int J_i\, d^3\mathbf{x}\,,
\end{aligned} \tag{7.22}$$

where the second equality follows from the generalised divergence theorem and noting that the surface term vanishes for a localised current distribution. We therefore have

$$\mathbf{A}(t,\mathbf{x}) \approx \frac{\mu_0}{4\pi r}\dot{\mathbf{P}}(t - r/c)\,. \tag{7.23}$$

This is called the *electric dipole* approximation. For a source oscillating at frequency ω, we have $\mathbf{A} \sim e^{-i\omega(t-r/c)}/r$, i.e.,, outgoing spherical waves.

We can calculate the magnetic field from $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. We have to be careful to remember that $t - r/c$ depends on $\mathbf{x}$, so that for some function $f(t - r/c)$,

$$\begin{aligned}\frac{\partial f(t-r/c)}{\partial x_i} &= -\frac{1}{c} f'(t-r/c) \frac{\partial r}{\partial x_i} \\ &= -\frac{x_i}{rc} f'(t-r/c) \,. \end{aligned} \tag{7.24}$$

It follows that

$$\begin{aligned}\mathbf{B}(t, \mathbf{x}) &= -\frac{\mu_0}{4\pi r^2} (\boldsymbol{\nabla} r) \times \dot{\mathbf{P}}(t-r/c) - \frac{\mu_0}{4\pi rc} (\boldsymbol{\nabla} r) \times \ddot{\mathbf{P}}(t-r/c) \\ &= -\frac{\mu_0}{4\pi r^2} \hat{\mathbf{x}} \times \dot{\mathbf{P}}(t-r/c) - \frac{\mu_0}{4\pi rc} \hat{\mathbf{x}} \times \ddot{\mathbf{P}}(t-r/c) \,. \end{aligned} \tag{7.25}$$

Note that the first contribution to the field varies as $1/r^2$ while the second goes like $1/r$. For a source oscillating at ω, the relative size of the $1/r$ part compared to the $1/r^2$ part is $O(r/\lambda)$. We define the far-field (sometimes known as the radiation zone or the wave zone) as the asymptotic region $r \gg \lambda$. In the far-field, the $1/r$ term dominates the magnetic field, and describes outgoing spherical waves of radiation:

$$\mathbf{B}(t, \mathbf{x}) = -\frac{\mu_0}{4\pi rc} \hat{\mathbf{x}} \times \ddot{\mathbf{P}}(t-r/c) \qquad (r \gg \lambda) \,. \tag{7.26}$$

Note that the $\mathbf{B}$ field is perpendicular to $\hat{\mathbf{x}}$ and to $\ddot{\mathbf{P}}$, and that it is $\ddot{\mathbf{P}}$ that determines the radiation field.

For a set of charges $\{q_i\}$ located at $\{\mathbf{x}_i\}$, we have $\mathbf{P} = \sum_i q_i \mathbf{x}_i$ and so

$$\ddot{\mathbf{P}} = \sum_i q_i \ddot{\mathbf{x}}_i \,. \tag{7.27}$$

This shows that it is *acceleration* of the charges that generates the dipole radiation field.

We have to work a little harder for the electric potential since, with the same order of approximation as we used above for $\mathbf{A}$, the electric potential ϕ involves $\int \rho(t - r/c, \mathbf{x}')\, d^3\mathbf{x}'$, i.e., the total charge Q at time $t - r/c$. Since Q is independent of time for an isolated charge distribution, the potential is also time-independent to this level of approximation (it is just the usual asymptotic Coulomb field). We therefore have to work at higher order in a/λ to find the part of ϕ that describes radiation[15]. It is straightforward to do so, but here we shall follow the simpler approach of using the Lorenz gauge condition to find ϕ.

If we use Eq. (7.23) in $\boldsymbol{\nabla} \cdot \mathbf{A} + c^{-2} \partial\phi/\partial t = 0$, we find

$$\frac{\partial \phi}{\partial t} = \frac{\mu_0 c^2}{4\pi r^2} \hat{\mathbf{x}} \cdot \dot{\mathbf{P}}(t-r/c) + \frac{\mu_0}{4\pi rc} \hat{\mathbf{x}} \cdot \ddot{\mathbf{P}}(t-r/c) \,, \tag{7.28}$$

[15]The 3-current $\mathbf{J} \sim \rho \mathbf{v}$ is higher-order in $v/c \sim a/\lambda$ than ρ, which is why we can apparently work to lower order in a/λ when dealing with the vector potential.

which integrates to give

$$\phi(t,\mathbf{x}) = \frac{Q}{4\pi\epsilon_0 r} + \frac{1}{4\pi\epsilon_0 r^2}\hat{\mathbf{x}}\cdot\mathbf{P}(t-r/c) + \frac{1}{4\pi\epsilon_0 rc}\hat{\mathbf{x}}\cdot\dot{\mathbf{P}}(t-r/c)\,. \tag{7.29}$$

Here, we have included the Coulomb term from the constant charge Q, which appears as an integration constant. The second term is the usual dipole potential that would be generated by a static charge distribution, but here it depends on the electric dipole moment at the earlier time $t - r/c$. Typically, this term is smaller than the Coulomb term by $O(a/r)$. It is the third term that describes the radiation component of the potential. It varies as $1/r$ and is typically $O(r/\lambda)$ times the second term, and so dominates in the far-field.

We can form the electric field from $\mathbf{E} = -\partial\mathbf{A}/\partial t - \boldsymbol{\nabla}\phi$. Equations (7.23) and (7.29) give terms varying as $1/r$, $1/r^2$ and $1/r^3$. The part going as $1/r$ dominates in the far-field:

$$\begin{aligned}\mathbf{E}(t,\mathbf{x}) &= -\frac{\mu_0}{4\pi r}\left[\ddot{\mathbf{P}}(t-r/c) - \hat{\mathbf{x}}\hat{\mathbf{x}}\cdot\ddot{\mathbf{P}}(t-r/c)\right] \qquad (r\gg\lambda)\\ &= \frac{\mu_0}{4\pi r}\hat{\mathbf{x}}\times\left[\hat{\mathbf{x}}\times\ddot{\mathbf{P}}(t-r/c)\right].\end{aligned} \tag{7.30}$$

It originates from the time derivative of $\dot{\mathbf{P}}$ in $\mathbf{A}$ (Eq. 7.23) and the spatial derivative of $\dot{\mathbf{P}}(t-r/c)$ in ϕ (Eq. 7.29). This expression for the electric field in the far-field can also be obtained directly from the magnetic field with $\boldsymbol{\nabla}\times\mathbf{B} = c^{-2}\partial\mathbf{E}/\partial t$. The electric field is perpendicular to $\hat{\mathbf{x}}$ and lies in the plane formed from $\hat{\mathbf{x}}$ and $\ddot{\mathbf{P}}$.

We see that in the far-field,

$$\mathbf{E} = -c\hat{\mathbf{x}}\times\mathbf{B} \qquad (r\gg\lambda)\,, \tag{7.31}$$

so that $|\mathbf{E}| = c|\mathbf{B}|$, and $\mathbf{E}\cdot\mathbf{B} = 0$. This behaviour is just like for a plane wave with wavevector along $\hat{\mathbf{x}}$. This makes sense since in the far-field, the radius r is so large compared to the wavelength λ that on the scale of λ the spherical wavefronts are effectively planar.

7.2.1 Power radiated

Energy is carried off to spatial infinity from the source in the form of electromagnetic radiation. We can determine the rate at which energy is radiated by integrating the Poynting vector over a spherical surface at very large radius. Working in the far-field, we have

$$\begin{aligned}\mathbf{N} &= \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \\ &= \frac{c}{\mu_0}\mathbf{B}\times(\hat{\mathbf{x}}\times\mathbf{B}) \\ &= \frac{c}{\mu_0}|\mathbf{B}|^2\hat{\mathbf{x}} \\ &= \frac{\mu_0}{(4\pi r)^2 c}|\hat{\mathbf{x}}\times\ddot{\mathbf{P}}|^2\hat{\mathbf{x}}\,. \end{aligned} \tag{7.32}$$

This is radial, as expected. The magnitude of the Poynting vector varies as $1/r^2$, so the integral over a spherical surface of radius r is independent of r. If $\ddot{\mathbf{P}}$ is along the z-axis, $|\hat{\mathbf{x}}\times\ddot{\mathbf{P}}| = \sin\theta|\ddot{\mathbf{P}}|$, where θ is the angle that $\hat{\mathbf{x}}$ makes with the z-axis. In this case,

$$\mathbf{N} = \frac{\mu_0}{(4\pi r)^2 c}|\ddot{\mathbf{P}}|^2\sin^2\theta\,\hat{\mathbf{x}}\,. \tag{7.33}$$

The total radiated power then follows from

$$\begin{aligned}\int\mathbf{N}\cdot d\mathbf{S} &= \frac{\mu_0}{(4\pi)^2 c}|\ddot{\mathbf{P}}|^2\int_{-1}^{1}\sin^2\theta\,d\cos\theta\int_0^{2\pi}d\phi \\ &= \frac{\mu_0}{6\pi c}|\ddot{\mathbf{P}}|^2\,. \end{aligned} \tag{7.34}$$

The power radiated per solid angle is $|\mathbf{N}|r^2$ and varies as $\sin^2\theta$. The radiation is maximal in the plane perpendicular to $\ddot{\mathbf{P}}$ and there is no radiation along the direction of $\ddot{\mathbf{P}}$. The radiation is linearly polarized, with the electric field in the plane containing $\hat{\mathbf{x}}$ and $\ddot{\mathbf{P}}$. The magnetic field is perpendicular to $\mathbf{E}$.

7.3 Radiation from an arbitrarily moving point charge

We now consider radiation from a single point charge moving arbitrarily, i.e.,, we relax the assumption of non-relativistic motion. Specifically, we consider a point charge q following a wordline $y^\mu(\tau)$ in spacetime, where τ is proper time.

The 4-current is non-zero except on the worldline, and can be written covariantly as

$$J^\mu(x) = qc\int\delta^{(4)}\left(x-y(\tau)\right)\dot{y}^\mu(\tau)\,d\tau\,. \tag{7.35}$$

The 4D delta-function here is Lorentz-invariant (since the 4D measure d^4x is), and its appearance ensures that the integration over τ only selects that proper time (if any) for which $x^\mu = y^\mu(\tau)$. The 4D delta-function can be decomposed in any inertial frame as

$$\delta^{(4)}\left(x-y(\tau)\right) = \delta\left(ct-y^0(\tau)\right)\delta^{(3)}\left(\mathbf{x}-\mathbf{y}(\tau)\right)\,. \tag{7.36}$$

We can perform the integration over proper time in Eq. (7.35) using the result in Eq. (7.14) for the delta-function with a function as its argument, to give

$$J^\mu(t,\mathbf{x}) = qc\delta^{(3)}\left(\mathbf{x}-\mathbf{y}(t)\right)\dot{y}^\mu/\dot{y}^0\,, \tag{7.37}$$

where $\mathbf{y}(t)$ is the 3D location of the charge at coordinate time t. The $\mu = 0$ component gives $\rho(t,\mathbf{x}) = q\delta^{(3)}\left(\mathbf{x}-\mathbf{y}(\tau)\right)$, and the $\mu = i$ component gives the current density $\mathbf{J}(t,\mathbf{x}) = q\mathbf{v}(t)\delta^{(3)}\left(\mathbf{x}-\mathbf{y}(\tau)\right)$, where $\mathbf{v}(t)$ is the 3D velocity of the charge at time t. These are clearly the correct (non-covariant) expressions for the charge and current density due to a point charge, from which we establish the validity of the covariant result in Eq. (7.35).

7.3.1 Lienard–Weichert potentials and fields

Inserting Eq. (7.37) for the 4-current into Eq. (7.12) for the retarded potential, we find

$$\begin{aligned} A^\mu(x) &= \frac{\mu_0 qc}{4\pi}\int d\tau \int d^4z\,\delta^{(4)}\left(z-y(\tau)\right)\dot{y}^\mu(\tau)\frac{\delta(x^0-z^0-|\mathbf{x}-\mathbf{z}|)}{|\mathbf{x}-\mathbf{z}|} \\ &= \frac{\mu_0 qc}{4\pi}\int \dot{y}^\mu(\tau)\frac{\delta\left(x^0-y^0(\tau)-|\mathbf{x}-\mathbf{y}(\tau)|\right)}{|\mathbf{x}-\mathbf{y}(\tau)|}d\tau\,, \end{aligned} \tag{7.38}$$

where we performed the integration d^4z using the defining property of the delta-function. To make further progress, it is convenient to express the remaining 1D delta-function in covariant form using Eq. (7.17):

$$\frac{\delta\left(x^0-y^0(\tau)-|\mathbf{x}-\mathbf{y}(\tau)|\right)}{|\mathbf{x}-\mathbf{y}(\tau)|} = 2\Theta\left(R^0(\tau)\right)\delta\left(\eta_{\mu\nu}R^\mu(\tau)R^\nu(\tau)\right)\,, \tag{7.39}$$

where we defined $R^\mu(\tau) = x^\mu - y^\mu(\tau)$. The combination of the delta-function and Heaviside function selects the unique value of proper time when the charge's worldline intersects the past lightcone of x^μ. Denoting the proper time there by τ_* (which is a function of x^μ), and using

$$\frac{d}{d\tau}\left[\eta_{\mu\nu}R^\mu(\tau)R^\nu(\tau)\right] = -2\eta_{\mu\nu}R^\mu(\tau)\dot{y}^\nu(\tau)\,, \tag{7.40}$$

we have

$$\frac{\delta\left(x^0-y^0(\tau)-|\mathbf{x}-\mathbf{y}(\tau)|\right)}{|\mathbf{x}-\mathbf{y}(\tau)|} = \frac{\delta(\tau-\tau_*)}{|R^\mu(\tau_*)\dot{y}_\mu(\tau_*)|}\,. \tag{7.41}$$

Finally, using this in Eq. (7.38), and integrating over τ, we obtain the *Leinard–Wiechert potential* in covariant form:

$$A^\mu(x) = \frac{\mu_0 qc}{4\pi}\frac{\dot{y}^\mu(\tau_*)}{|R^\nu(\tau_*)\dot{y}_\nu(\tau_*)|}\,. \tag{7.42}$$

Equation (7.42) is a beautifully compact expression for the 4-potential due to an arbitrarily moving charge. However, to gain intuition for this result, it is useful to unpack it into time and space components in some inertial frame. Then, $\dot{y}^\mu(\tau_*) = \gamma(\tau_*)\,(c, \mathbf{v}(\tau_*))$, where $\mathbf{v}(\tau_*)$ is the 3-velocity at proper time τ_* and $\gamma(\tau_*)$ is the associated Lorentz factor. Furthermore, $R^\mu(\tau_*) = (c\Delta t(\tau_*), \mathbf{R}(\tau_*))$, where $\Delta t(\tau_*)$ is the time difference between the current event and the time that the charge intersected the past lightcone, and $\mathbf{R}(\tau_*) \equiv \mathbf{x} - \mathbf{y}(\tau_*)$. Note that $c\Delta t(\tau_*) = |\mathbf{R}(\tau_*)| = R(\tau_*)$. We now have

$$|R^\nu(\tau_*)\dot{y}_\nu(\tau_*)| = c\gamma(\tau_*)R(\tau_*)\left(1 - \hat{\mathbf{R}}(\tau_*)\cdot\mathbf{v}(\tau_*)/c\right)\,, \tag{7.43}$$

and the time and space components of Eq. (7.42) give

$$\phi(t, \mathbf{x}) = \frac{q}{4\pi\epsilon_0 R(\tau_*)\left(1 - \hat{\mathbf{R}}(\tau_*)\cdot\mathbf{v}(\tau_*)/c\right)} \tag{7.44}$$

$$\mathbf{A}(t, \mathbf{x}) = \frac{\mu_0 q\mathbf{v}(\tau_*)}{4\pi R(\tau_*)\left(1 - \hat{\mathbf{R}}(\tau_*)\cdot\mathbf{v}(\tau_*)/c\right)}\,. \tag{7.45}$$

Due to the presence of the $1 - \hat{\mathbf{R}}(\tau_*)\cdot\mathbf{v}(\tau_*)/c$ factors in the denominators of Eqs. (7.44) and (7.45), the potentials (and hence fields) are concentrated in the direction of motion, $\hat{\mathbf{R}}(\tau_*) \propto \mathbf{v}(\tau_*)$, for relativistic motion. This is a consequence of extreme aberration for a rapidly moving particle.

If we adopt an inertial frame such that the particle is at rest in that frame at τ_*, the Lienard–Wiechert potentials simplify to $\mathbf{A}(t, \mathbf{x}) = 0$ and

$$\phi(t, \mathbf{x}) = \frac{q}{4\pi\epsilon_0 R_{\text{rest}}(\tau_*)} \qquad \text{(rest-frame)}\,, \tag{7.46}$$

where $R_{\text{rest}}(\tau_*)$ is the distance between the observation event and $y^\mu(\tau_*)$ in this rest-frame. These *potentials* are exactly what one would expect for a stationary point charge. If instead we adopt a frame in which the charge is not at rest at τ_*, but is moving non-relativistically, the vector potential no longer vanishes but is approximately

$$\mathbf{A}(t, \mathbf{x}) \approx \frac{\mu_0 q\mathbf{v}(\tau_*)}{4\pi R(\tau_*)} \qquad \text{(non-relativistic)}\,. \tag{7.47}$$

This is consistent with the non-relativistic (dipole) approximation in Eq. (7.23) noting that $d\mathbf{P}/dt = q\mathbf{v}$ for a point charge.

To calculate the electric and magnetic fields of an arbitrarily moving charge, we return to the covariant form of the Lienard–Weichert potentials (Eq. 7.42) and form $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We have to remember that τ_* is a function of x^μ, so that, for example,

$\partial_\mu \dot{y}^\nu(\tau_*) = \ddot{y}^\nu(\tau_*)\partial_\mu\tau_*$. We can evaluate $\partial_\mu\tau_*$ by differentiating the null condition $\eta_{\alpha\beta}R^\alpha(\tau_*)R^\beta(\tau_*) = 0$ to find

$$\begin{aligned} 0 &= \eta_{\alpha\beta}\partial_\mu R^\alpha(\tau_*)R^\beta(\tau_*) \\ &= \eta_{\alpha\beta}\partial_\mu\left(x^\alpha - y^\alpha(\tau_*)\right)R^\beta(\tau_*) \\ &= \eta_{\alpha\beta}\left(\delta^\alpha_\mu - \dot{y}^\alpha(\tau_*)\partial_\mu\tau_*\right)R^\beta(\tau_*)\,. \end{aligned} \tag{7.48}$$

Solving for $\partial_\mu\tau_*$ gives

$$\frac{\partial\tau_*}{\partial x^\mu} = \frac{R_\mu(\tau_*)}{R^\nu(\tau_*)\dot{y}_\nu(\tau_*)}\,. \tag{7.49}$$

The other term we require in differentiating Eq. (7.42) is

$$\begin{aligned} \partial_\mu\left[R^\nu(\tau_*)\dot{y}_\nu(\tau_*)\right] &= \left(\delta^\nu_\mu - \dot{y}^\nu(\tau_*)\partial_\mu\tau_*\right)\dot{y}_\nu(\tau_*) + R^\nu(\tau_*)\ddot{y}_\nu(\tau_*)\partial_\mu\tau_* \\ &= \partial_\mu\tau_*\left[R^\nu(\tau_*)\ddot{y}_\nu(\tau_*) - \dot{y}^\nu(\tau_*)\dot{y}_\nu(\tau_*)\right] + \dot{y}_\mu(\tau_*) \\ &= \partial_\mu\tau_*\left[c^2 + R^\nu(\tau_*)\ddot{y}_\nu(\tau_*)\right] + \dot{y}_\mu(\tau_*)\,. \end{aligned} \tag{7.50}$$

Putting these pieces together, we find (exercise!), on antisymmetrising,

$$F_{\mu\nu}(x) = -\frac{\mu_0 qc}{4\pi(cR_{\text{rest}})}\left(\frac{R_\mu\ddot{y}_\nu - R_\nu\ddot{y}_\mu}{(cR_{\text{rest}})} + \frac{(R_\mu\dot{y}_\nu - R_\nu\dot{y}_\mu)}{(cR_{\text{rest}})^2}(c^2 + R^\alpha\ddot{y}_\alpha)\right)\,, \tag{7.51}$$

where all quantities on the right are evaluated at τ_*. Here, we have used

$$-R^\mu(\tau_*)\dot{y}_\mu(\tau_*) = cR_{\text{rest}}\,, \tag{7.52}$$

to express $F_{\mu\nu}$ in terms of R_{rest}, the radial distance between x^μ and $y^\mu(\tau_*)$ as measured in the frame in which the charge is at rest at τ_*.

Inspection of Eq. (7.52) reveals terms going like $1/R$, that depend on the 4-acceleration, and a term going like $1/R^2$ that depends only on the 4-velocity of the charge. The $1/R$ terms describe radiation, while the $1/R^2$ term turns out to be the field that would be generated from a charge moving uniformly with the same velocity at τ_* (see below).

Exercise: Extract the electric and magnetic fields from the field-strength tensor in Eq. (7.52) in a general inertial frame, recalling that the 4-acceleration has components

$$\ddot{y}^\mu = \gamma^2\left(\frac{\gamma^2}{c}\mathbf{v}\cdot\mathbf{a}, \frac{\gamma^2}{c^2}\mathbf{a}\cdot\mathbf{v}\mathbf{v} + \mathbf{a}\right)\,. \tag{7.53}$$

You should find (after a lengthy, but straightforward, calculation) that

$$\mathbf{E}(t,\mathbf{x}) = \frac{q}{4\pi\epsilon_0(1 - \mathbf{v}\cdot\hat{\mathbf{R}}/c)^3}\left(\frac{\hat{\mathbf{R}} - \mathbf{v}/c}{\gamma^2 R^2} + \frac{\hat{\mathbf{R}}\times[(\hat{\mathbf{R}} - \mathbf{v}/c)\times\mathbf{a}]}{Rc^2}\right)\,, \tag{7.54}$$

where all quantities on the right are evaluated at τ_*. For the magnetic field, you should find

$$\mathbf{B}(t,\mathbf{x}) = \frac{1}{c}\hat{\mathbf{R}}(\tau_*) \times \mathbf{E}(t,\mathbf{x})\,. \tag{7.55}$$

The exercise above derives the electric and magnetic fields from $F_{\mu\nu}$. Note the following points.

- The magnetic field is always perpendicular to the electric field and to $\hat{\mathbf{R}}(\tau_*)$.
- The radiation ($1/R$) part of the fields is sourced by the 3-acceleration. The electric field has an angular dependence given by $\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \mathbf{v}/c) \times \mathbf{a}]$, and is therefore perpendicular to $\hat{\mathbf{R}}(\tau_*)$, like the magnetic field. The angular dependence is more complicated than for dipole radiation [Eq. (7.30)], due to the presence of the $\mathbf{v}/c$ correction to $\hat{\mathbf{R}}$. However, in the non-relativistic limit ($|\mathbf{v}| \ll c$), the radiation fields reduce to those derived in the dipole approximation in Sec. 7.2 (noting that $d^2\mathbf{P}/dt^2 = q\mathbf{a}$ for a point charge). The magnitudes of the fields satisfy the usual relation for radiation, $|\mathbf{E}| = c|\mathbf{B}|$, even for non-relativistic motion of the charge.
- The $1/R^2$ contribution to the fields are the same as for a charge moving uniformly with the same velocity at τ_*. To see this, recall the following expression for the electric field from a uniformly moving charge (see the example in Sec. 5.4),

  $$\mathbf{E} = \frac{q\gamma\mathbf{r}}{4\pi\epsilon_0\left[\gamma^2(\mathbf{r}\cdot\mathbf{v})^2/c^2 + \mathbf{r}^2\right]^{3/2}}\,, \tag{7.56}$$

  where $\mathbf{r}$ points from the *current* position of the charge to the observation point. Let us express this in terms of $\mathbf{R}(\tau_*)$ in the case that the charge is moving uniformly. In time $R(\tau_*)/c$, a uniformly moving charge displaces by $\mathbf{v}R(\tau_*)/c$, and so

  $$\mathbf{r} = \mathbf{R}(\tau_*) - \mathbf{v}R(\tau_*)/c = R(\tau_*)[\hat{\mathbf{R}}(\tau_*) - \mathbf{v}/c]\,. \tag{7.57}$$

  Using this relation, it is straightforward to show that

  $$\gamma^2(\mathbf{r}\cdot\mathbf{v})^2/c^2 + \mathbf{r}^2 = \gamma^2 R^2(\tau_*)\left(1 - \hat{\mathbf{R}}(\tau_*)\cdot\mathbf{v}/c\right)^2\,, \tag{7.58}$$

  and so Eq. (7.57) reduces to the first term on the right of Eq. (7.54).

7.3.2 Power radiated

We now calculate the rate at which energy is radiated by an arbitrarily moving point charge. We shall first calculate this in the instantaneous rest frame of the charge and then use Lorentz transformations to deduce the rate in a general inertial frame.

In the frame in which the particle is at rest at proper time τ_*, the radiation part of electric field on the forward lightcone through $y^\mu(\tau_*)$ is, from Eq. (7.54),

$$E(t, \mathbf{x}) = \frac{q}{4\pi\epsilon_0 R_{\text{rest}} c^2} \hat{\mathbf{R}} \times (\hat{\mathbf{R}} \times \mathbf{a}_{\text{rest}}) \qquad \text{(Instantaneous rest frame)}\,, \tag{7.59}$$

where $\mathbf{a}_{\text{rest}}$ is the 3-acceleration in the instantaneous rest frame. The magnetic field is $\mathbf{B} = \hat{\mathbf{R}} \times \mathbf{E}/c$, so the Poynting vector is

$$\begin{aligned} \mathbf{N} &= \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \\ &= \frac{1}{c\mu_0} \mathbf{E} \times (\hat{\mathbf{R}} \times \mathbf{E}) \\ &= \frac{1}{c\mu_0} |\mathbf{E}|^2 \hat{\mathbf{R}}\,, \end{aligned} \tag{7.60}$$

which is radial. We can get the radiated power by integrating over a spherical surface of radius R_{rest} at time t (i.e., a section through the forward lightcone corresponding to a constant time surface in the instantaneous rest frame). Since the charge is at rest at τ_*, the radiation emitted in proper time $d\tau$ around τ_* crosses this spherical surface in (coordinate) time[16] $dt = d\tau$. It follows that the energy radiated in the instantaneous rest frame in proper time $d\tau$ is

$$\begin{aligned} dE_{\text{rest}} &= d\tau \int \mathbf{N} \cdot d\mathbf{S} \\ &= \frac{R_{\text{rest}}^2 d\tau}{c\mu_0} \int |\mathbf{E}|^2 \, d\Omega\,, \end{aligned} \tag{7.61}$$

[16] This would not be true if the charge were moving at τ_*. Generally, the time taken for the radiation emitted in $d\tau_*$ to cross a spherical surface of radius $R(\tau_*)$ can be got from Eq. (7.49). Rearranging gives

$$d\tau_* = \frac{R_\mu(\tau_*) dx^\mu}{R^\nu(\tau_*) \dot{y}_\nu(\tau_*)}\,.$$

Taking $dx^\mu = (cdt, \mathbf{0})$ gives

$$d\tau_* = \frac{dt}{\gamma(\tau_*) \left[1 - \hat{\mathbf{R}}(\tau_*) \cdot \mathbf{v}(\tau_*)/c\right]}\,.$$

This is just the usual Doppler effect. In the instantaneous rest frame, it reduces to $d\tau_* = dt$, as advertised.

where $d\Omega$ is the element of solid angle (i.e., $d\Omega = d^2\hat{\mathbf{R}}$). Taking the rest-frame acceleration to be along the z-axis, we have

$$|\hat{\mathbf{R}} \times (\hat{\mathbf{R}} \times \mathbf{a}_{\text{rest}})| = \sin\theta |\mathbf{a}_{\text{rest}}| \,, \tag{7.62}$$

so that on integrating over solid angle

$$\frac{dE_{\text{rest}}}{d\tau} = \frac{\mu_0 q^2 |\mathbf{a}_{\text{rest}}|^2}{6\pi c} \,. \tag{7.63}$$

This simple result is exactly what we would have got by applying the dipole approximation, Eq. (7.34), in the instantaneous rest frame, with $d^2\mathbf{P}/dt^2 \to q\mathbf{a}_{\text{rest}}$. This is not unexpected – the dipole approximation holds in situations where all velocities are non-relativistic and this is exactly true in the instantaneous rest frame. Indeed, the dipole approximation provides a useful short-cut to deriving Eq. (7.63).

Now consider the 4-momentum radiated in time $d\tau$. Since equal power is emitted along $\hat{\mathbf{R}}$ and $-\hat{\mathbf{R}}$ in the instantaneous rest frame, the total 3-momentum of the radiation is zero. We therefore have

$$dP^\mu = d\tau \left(\frac{1}{c} \frac{dE_{\text{rest}}}{d\tau}, \mathbf{0} \right) \tag{7.64}$$

in the rest frame. Lorentz transforming to an inertial frame in which the charge has velocity $\mathbf{v}(\tau_*)$ at τ_*, we have

$$\frac{dE'}{c} = dP'^0 = \gamma(\tau_*) dP^0 = \frac{\gamma(\tau_*) d\tau}{c} \frac{dE_{\text{rest}}}{d\tau} \,. \tag{7.65}$$

The time elapsed between the events $y^\mu(\tau_*)$ and $y^\mu(\tau_* + d\tau)$ in the general frame is $dt' = \gamma(\tau_*) d\tau$ by time dilation, hence

$$\frac{dE'}{dt'} = \frac{dE_{\text{rest}}}{d\tau} \,. \tag{7.66}$$

We see that the rate at which energy is radiated by the accelerated charge is Lorentz invariant and so, generally, the instantaneous power radiated is

$$\frac{dE}{dt} = \frac{\mu_0 q^2 a_\mu a^\mu}{6\pi c} = \frac{\mu_0 q^2 \gamma^4}{6\pi c} \left[\mathbf{a}^2 + \gamma^2 (\mathbf{a} \cdot \mathbf{v}/c)^2 \right] \,, \tag{7.67}$$

where a^μ is the acceleration 4-vector (so that $a^\mu a_\mu = |\mathbf{a}_{\text{rest}}|^2$). This result is known as the *relativistic Larmor formula*. (Its non-relativistic limit was first worked out by Larmor in 1897.)

Non-examinable aside: If the above argument is not to your taste, the same expression for the power radiated in a general frame can be obtained by direct calculation. The radiation part of the electric field is now given by

$$\mathbf{E}(t, \mathbf{x}) = \frac{q}{4\pi\epsilon_0} \frac{\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \mathbf{v}/c) \times \mathbf{a}]}{R(1 - \mathbf{v} \cdot \hat{\mathbf{R}}/c)^3 c^2} \,, \tag{7.68}$$

where all quantities on the right are evaluated at τ_*. The Poynting vector is still given by Eq. (7.60), but when integrating this over the surface of a sphere of radius $R(\tau_*)$ at time t, to determine the radiated power, we have to be careful to account for Doppler shifts that relate *dt at the source* to the increment in time for radiation to pass through the radius $R(\tau_*)$. Following the arguments in the footnote on Page 61, and noting that the coordinate time between the events at τ_* and $\tau_* + d\tau$ on the charge's worldline is $\gamma(\tau_*)d\tau$, the power emitted per solid angle (i.e.,, energy per solid angle per coordinate time *at source*) is

$$\begin{aligned} \frac{dE}{dtd\Omega} &= \frac{1}{c\mu_0}\left(1 - \frac{\mathbf{v}\cdot\hat{\mathbf{R}}}{c}\right)|\mathbf{E}|^2 R^2 \\ &= \frac{\mu_0 q^2}{(4\pi)^2 c}\frac{\left|\hat{\mathbf{R}}\times[(\hat{\mathbf{R}}-\mathbf{v}/c)\times\mathbf{a}]\right|^2}{\left(1-\mathbf{v}\cdot\hat{\mathbf{R}}/c\right)^5}. \end{aligned} \tag{7.69}$$

Note that the angular dependence of the radiated power is more complicated than the $\sin^2\theta$ dependence in the instantaneous rest frame. In particular, for a particle moving relativistically, the power is significantly concentrated in the direction of motion, $\hat{\mathbf{R}} \propto \mathbf{v}$. After some vector algebra, Eq. (7.69) can be reduced to

$$\frac{dE}{dtd\Omega} = \frac{\mu_0 q^2}{(4\pi)^2 c}\left(2\frac{(\hat{\mathbf{R}}\cdot\mathbf{a})(\mathbf{a}\cdot\mathbf{v}/c)}{\left(1-\mathbf{v}\cdot\hat{\mathbf{R}}/c\right)^4} + \frac{|\mathbf{a}|^2}{\left(1-\mathbf{v}\cdot\hat{\mathbf{R}}/c\right)^3} - \frac{(\hat{\mathbf{R}}\cdot\mathbf{a})^2}{\gamma^2\left(1-\mathbf{v}\cdot\hat{\mathbf{R}}/c\right)^5}\right). \tag{7.70}$$

Integrating over solid angle[17] directly gives the final result in Eq. (7.67).

7.4 Scattering

As a final application of the material developed in this section on radiation of electromagnetic waves, we consider the issue of scattering. An incident electromagnetic wave will cause a free charge, such as an electron, to oscillate and the acceleration that accompanies the oscillation will cause the charge to radiate. In this manner, the incident radiation is *scattered* into other directions. A similar thing happens for systems of bound positive and negative charges (e.g. an atom or molecule) for which the total charge vanishes. The incident field induces an oscillating electric dipole moment that radiates, scattering the incident radiation.

For the case of electron scattering, consider incident fields that are weak enough not to induce relativistic motions. In the frame in which the electron is at rest (on average),

[17]A straightforward, but lengthy, way to do this is to align $\mathbf{v}$ with the z-axis and $\mathbf{a}$ in the x-z plane, and express the components of $\hat{\mathbf{R}}$ in spherical polar coordinates.

the equation of motion of the electron in the incident electromagnetic wave is simply

$$m_e \ddot{\mathbf{x}} = -e\mathbf{E}\,, \tag{7.71}$$

where m_e is the mass of the electron, $-e$ is the charge, and $\mathbf{E}$ is the incident field evaluated at the location of the electron. For an incident wave at angular frequency ω, we can evaluate the right-hand side of Eq. (7.71) at the average position of the electron provided that the wavelength of the incident wave is large compared to the amplitude of the oscillation induced upon the electron. In this case, the electron oscillates at frequency ω and with amplitude $eE_0/(m_e\omega^2)$, where E_0 is the amplitude of the electric field in the incident wave. This oscillation amplitude will be small compared to the wavelength provided that

$$\frac{eE_0}{m_e c\omega} \ll 1\,. \tag{7.72}$$

However, since the maximum speed of the electron is the product of the frequency and amplitude, i.e., $eE_0/(m_e\omega)$, Eq. (7.72) is equivalent to the requirement that the electron motion is non-relativistic and so will hold by assumption.

The acceleration of the electron is therefore $-e\mathbf{E}/m_e$ and, using the Larmor formula (7.67) (and taking $\mathbf{v} = 0$), the radiated power is, after averaging over time,

$$\left\langle \frac{dE}{dt} \right\rangle = \frac{\mu_0}{6\pi c}\frac{e^4 E_0^2}{2m_e^2}\,. \tag{7.73}$$

The time-averaged magnitude of the Poynting vector of the incident radiation is

$$\langle |\mathbf{N}| \rangle = \frac{E_0^2}{2\mu_0 c}\,, \tag{7.74}$$

and the ratio of the radiated (scattered) power to this incident flux defines the *Thomson cross-section*,

$$\sigma_{\mathrm{T}} = \frac{\mu_0^2 e^4}{6\pi m_e^2} = \frac{e^4}{6\pi\epsilon_0^2 m_e^2 c^4}\,. \tag{7.75}$$

This is the effective geometric cross-section of the electron to scattering. Note that it is independent of frequency. The Thomson cross-section can be neatly written in the form

$$\sigma_{\mathrm{T}} = \frac{8\pi}{3} r_e^2\,, \tag{7.76}$$

where the *classical electron radius* r_e is defined by

$$\frac{e^2}{4\pi\epsilon_0 r_e} = m_e c^2\,. \tag{7.77}$$

(One can think of r_e as being proportional to the radius that a uniform, extended charge e would have if the rest-mass energy were all electrostatic.)

The other type of scattering we shall consider here is scattering of long wavelength radiation ($\lambda \gg$ physical size of scatterer) from bound charges. This is known as *Rayleigh scattering*. For bound charges, provided that the frequency of the incident wave is small compared to any resonant frequency (or transition frequency in a quantum description), the bound charges will collectively displace following the applied force from the incident wave. This induces an oscillating electric dipole moment of the form $\mathbf{P} = \alpha\mathbf{E}$, where $\mathbf{E}$ is the incident electric field at the position of the (small) scatterer, and α is the *polarizability*.

The wavelength of the scattered radiation is the same as that of the incident radiation and is large compared to the size of the scatterer. We are therefore in the dipole limit, and the radiated power is given by Eq. (7.34). Since $\ddot{\mathbf{P}} = -\alpha\omega^2\mathbf{E}$, the time-averaged radiated power is

$$\left\langle \frac{dE}{dt} \right\rangle = \frac{\mu_0}{6\pi c}\frac{\alpha^2\omega^4 E_0^2}{2} \,. \tag{7.78}$$

Dividing by the time-averaged incident energy flux gives the cross-section for Rayleigh scattering:

$$\sigma = \frac{\mu_0^2\alpha^2\omega^4}{6\pi} \propto \frac{1}{\lambda^4} \,. \tag{7.79}$$

Note the $1/\lambda^4$ wavelength dependence of the cross-section, so that short wavelengths (high frequencies) are scattered more than longer wavelengths. Rayleigh scattering underlies many natural phenomena. For example, the blue hue of the overhead sky is due to the (mostly) nitrogen and oxygen molecules in the atmosphere preferentially scattering the blue (short wavelength) component of sunlight. Furthermore, red sunsets follow from preferentially scattering blue light out of the line of sight towards the setting sun.

8 Variational principles

Recall that in non-relativistic classical mechanics, the equation of motion $m\ddot{\mathbf{x}} = -\boldsymbol{\nabla}V$ for a particle of mass m moving in a potential $V(\mathbf{x}, t)$ may be derived by extremising the *action*

$$S = \int_{t_1}^{t_2} \left(\frac{1}{2}m\dot{\mathbf{x}}^2 - V(\mathbf{x}) \right) dt \,. \tag{8.1}$$

In other words, we look for the path $\mathbf{x}(t)$, with fixed end-points at times t_1 and t_2, for which the action is stationary under changes of path. Note that the action has dimensions of energy $\times$ time.

The action is the integral of the *Lagrangian*

$$L = \frac{1}{2}m\dot{\mathbf{x}}^2 - V(\mathbf{x}) \,. \tag{8.2}$$

The Euler–Lagrange equations,

$$\frac{\partial L}{\partial \mathbf{x}} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{x}}}\right) , \tag{8.3}$$

determine the stationary path as that satisfying

$$-\boldsymbol{\nabla} V = \frac{d}{dt}(m\dot{\mathbf{x}}) \tag{8.4}$$

(subject to the boundary conditions at t_1 and t_2). This is just the expected Newtonian equation of motion.

In this short section, we shall generalise this action principle to relativistic motion of point particles moving in prescribed electromagnetic fields, and extend the principle to the field itself. The ultimate motivation for this is that the starting point for quantizing any classical theory is the action. We will not have much to say on quantization in this course (although we shall discuss the Schrödinger equation for particles in magnetic fields in Sec. 9), but the quantization of fields is fully developed in Part-III courses such as *Quantum Field Theory* and *Advanced Quantum Field Theory*.

8.1 Relativistic point particles in external fields

We start by seeking an action principle for a free relativistic point particle of rest mass m. We shall then include the effect of an external electromagnetic field.

The non-relativistic action for a free particle is just the integral of the kinetic energy $m\dot{\mathbf{x}}^2/2$ over time. A relativistic version of the action principle must involve a Lorentz-invariant action, and reduce to the non-relativistic version in the limit of speeds much less than c. A good candidate for the relativistic Lagrangian is $-mc^2/\gamma$, where γ is the Lorentz factor of the particle. To see this, note that $dt/\gamma = d\tau$ (the proper time increment) and so the action is Lorentz invariant – it is just the proper time along the path between the two fixed spacetime endpoints. Moreover,

$$\begin{aligned} -mc^2\left(1-\frac{\mathbf{v}^2}{c^2}\right)^{1/2} &= -mc^2\left(1-\frac{1}{2}\frac{\mathbf{v}^2}{c^2}+\cdots\right) \\ &= -mc^2 + \frac{1}{2}m\mathbf{v}^2 + \cdots , \end{aligned} \tag{8.5}$$

and so the action reduces to the usual Newtonian one in the non-relativistic limit up to an irrelevant constant ($-mc^2$). Writing the Lagrangian as

$$L = -mc^2\left(1-\mathbf{v}^2/c^2\right)^{1/2} , \tag{8.6}$$

the Euler–Lagrange equations give

$$\frac{d}{dt}\left(\frac{\mathbf{v}}{(1-\mathbf{v}^2/c^2)^{1/2}}\right) = 0\,, \tag{8.7}$$

since $\partial L/\partial \mathbf{x} = 0$. This is just the statement that the relativistic 3-momentum is conserved, as required for a free particle.

We now want to add in the effect of an external electromagnetic field. The non-relativistic form of the Lagrangian for a particle moving in a potential $V(\mathbf{x}, t)$ suggests adding $-q\phi$ to the relativistic Lagrangian, where q is the charge of the particle. However, this alone is insufficient since the magnetic field would not appear in the equation of motion. A further issue is related to gauge-invariance: the action principle must be invariant under gauge transformations of the form

$$\phi \to \phi - \frac{\partial \chi}{\partial t}, \qquad \mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\chi\,. \tag{8.8}$$

This means that the *change* in action between two paths with fixed endpoints $\mathbf{x}_1$ and $\mathbf{x}_2$ at t_1 and t_2 must be gauge-invariant. We can fix both of these issues by including a further term $q\mathbf{A}\cdot\mathbf{v}$ in the Lagrangian:

$$L = -mc^2\left(1-\mathbf{v}^2/c^2\right)^{1/2} - q\phi + q\mathbf{A}\cdot\mathbf{v}\,. \tag{8.9}$$

To see how this gives a gauge-invariant action principle, note that under a gauge transformation

$$\begin{aligned} S &\to S + q\int_{t_1}^{t_2}\left(\frac{\partial\chi}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\chi\right)dt \\ &= S + q\int_{t_1}^{t_2}\frac{d\chi}{dt}\,dt \\ &= S + q\left[\chi(t_2,\mathbf{x}_2) - \chi(t_1,\mathbf{x}_1)\right]\,. \end{aligned} \tag{8.10}$$

As the endpoints are fixed, the gauge transformation does not affect the change in the action (although it does change the absolute value of the action).

Finally, we need to check that we recover the correct equation of motion. The ith component of the Euler–Lagrange equations gives

$$\frac{d}{dt}\left(\frac{mv_i}{(1-\mathbf{v}^2/c^2)^{1/2}}\right) + q\frac{dA_i}{dt} = -q\frac{\partial\phi}{\partial x_i} + qv_j\frac{\partial A_j}{\partial x_i}\,. \tag{8.11}$$

Expanding the total (convective) derivative dA_i/dt gives the required equation of motion:

$$\begin{aligned} \frac{dp_i}{dt} &= -q\frac{\partial\phi}{\partial x_i} + qv_j\frac{\partial A_j}{\partial x_i} - q\frac{\partial A_i}{\partial t} - qv_j\frac{\partial A_i}{\partial x_j} \\ &= qE_i + q\left[\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A})\right]_i \\ &= q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)_i\,, \end{aligned} \tag{8.12}$$

where $\mathbf{p} = \gamma m \mathbf{v}$.

We can formulate the action principle more covariantly by recalling that the kinetic part of the action is just proportional to the proper time along the path, and can be written as

$$
\begin{aligned}
-mc^2 \int d\tau &= -mc \int (-\eta_{\mu\nu} dx^\mu dx^\nu)^{1/2} \\
&= -mc \int \left(-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right) d\lambda \, ,
\end{aligned}
\tag{8.13}
$$

for some arbitrary parameter λ along the path. This action is invariant to a reparameterisation of the path with a different λ[18]. We must now look for spacetime paths that extremise the action between fixed endpoint events. For the electromagnetic part of the action, note that

$$
A_\mu dx^\mu = (-\phi + \mathbf{v} \cdot \mathbf{A}) \, dt \, ,
\tag{8.14}
$$

so the full action can be written covariantly as

$$
S = -mc \int \left(-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{1/2} d\lambda + q \int A_\mu \frac{dx^\mu}{d\lambda} \, d\lambda \, .
\tag{8.15}
$$

It is straightforward to verify the gauge-invariance of changes in the action now since, under $A_\mu \to A_\mu + \partial_\mu \chi$, we have

$$
A_\mu \frac{dx^\mu}{d\lambda} \to A_\mu \frac{dx^\mu}{d\lambda} + \frac{d\chi}{d\lambda} \, .
\tag{8.16}
$$

The last term on the right is a total derivative and so does not contribute to changes in the action for fixed endpoints.

For the covariant action, the paths are parameterised by λ so the relevant Euler–Lagrange equations are

$$
\frac{\partial L}{\partial x^\mu} = \frac{d}{d\lambda} \left(\frac{\partial L}{\partial (dx^\mu / d\lambda)} \right) ,
\tag{8.17}
$$

where the Lagrangian

$$
L = -mc \left(-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{1/2} + q A_\mu \frac{dx^\mu}{d\lambda} \, .
\tag{8.18}
$$

The required derivatives of the Lagrangian are

$$
\frac{\partial L}{\partial x^\mu} = q (\partial_\mu A_\nu) \frac{dx^\nu}{d\lambda} \, ,
\tag{8.19}
$$

[18]It is more convenient to use a general parameter than τ itself since the latter would imply the *constraint* $\eta_{\mu\nu}(dx^\mu/d\tau)(dx^\nu/d\tau) = -c^2$, which would have to be accounted for when varying the path.

and

$$\frac{\partial L}{\partial(dx^\mu/d\lambda)} = \frac{mc}{\left[-\eta_{\mu\nu}(dx^\mu/d\lambda)(dx^\nu/d\lambda)\right]^{1/2}}\eta_{\mu\nu}\frac{dx^\nu}{d\lambda} + qA_\mu \,. \tag{8.20}$$

Substituting these derivatives into the Euler–Lagrange equations, and using $dA_\mu/d\lambda = (dx^\nu/d\lambda)\partial_\nu A_\mu$, we find

$$\begin{aligned} mc\frac{d}{d\lambda}\left(\frac{\eta_{\mu\nu}dx^\nu/d\lambda}{\left[-\eta_{\mu\nu}(dx^\mu/d\lambda)(dx^\nu/d\lambda)\right]^{1/2}}\right) &= q\frac{dx^\nu}{d\lambda}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \\ &= qF_{\mu\nu}\frac{dx^\nu}{d\lambda} \,. \end{aligned} \tag{8.21}$$

This equation is reparameterisation invariant, as expected since the action is. It simplifies if we take λ equal to the proper time,

$$\frac{dp_\mu}{d\tau} = qF_{\mu\nu}u^\nu \,, \tag{8.22}$$

where p^μ is the 4-momentum and $u^\mu = dx^\mu/d\tau$ is the 4-velocity. This is the expected covariant equation of motion (Eq. 5.64).

8.2 Action principle for the electromagnetic field

We now extend the action principle to the electromagnetic field itself. Rather than dealing with functions of a single variable, like $x^\mu(\lambda)$, we now have fields defined over spacetime. The action will therefore involve the integral over spacetime (with the Lorentz-invariant measure d^4x introduced in Sec. 7) of a Lorentz-invariant *Lagrangian density* $\mathcal{L}$. This Lagrangian density must also be gauge-invariant. The variational principle is then to find field configurations such that the action is stationary for arbitrary changes in the field that vanish on some closed surface in spacetime.

The covariant Maxwell equations involve first derivatives of $F_{\mu\nu}$ or, equivalently, second derivatives of A_μ. This suggests considering a Lagrangian density $\mathcal{L} \propto F_{\mu\nu}F^{\mu\nu}$, with A_μ the underlying field to vary, for the part of the action describing the free electromagnetic field. Adding this to the action in Eq. (8.15) for a particle in an external field, we are therefore led to consider a total action of the form

$$S = -\frac{1}{4\mu_0 c}\int F_{\mu\nu}F^{\mu\nu}\, d^4x - \sum mc\int\left(-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{1/2} d\lambda + \sum q\int A_\mu\frac{dx^\mu}{d\lambda}\, d\lambda \,, \tag{8.23}$$

for a system of N point particles interacting the electromagnetic field. Here, the summations are over the particles, each of which has its own worldline $x^\mu(\lambda)$. Note that the first term in the action, describing the free electromagnetic field, has the correct dimensions of energy $\times$ time. The numerical factor is chosen to get the correct field equation (see below).

Varying the $x^\mu(\lambda)$ for each particle gives an equation of motion like Eq. (8.22) for each particle. For the A_μ variation, we note that the interaction (i.e., third) term in the action can be rewritten in terms of the 4-current of the particles,

$$J^\mu(x) = \sum qc \int \delta^{(4)}\left(x - y(\lambda)\right) \frac{dy^\mu}{d\lambda}\, d\lambda \tag{8.24}$$

using

$$\begin{aligned} \int A_\mu J^\mu \, d^4x &= \sum qc \int d\lambda \int d^4x \, \delta^{(4)}\left(x - y(\lambda)\right) \frac{dy^\nu}{d\lambda} A_\mu(x) \\ &= \sum qc \int A_\mu\left(y(\lambda)\right) \frac{dy^\mu}{d\lambda}\, d\lambda\,. \end{aligned} \tag{8.25}$$

Varying A_μ by δA_μ in the action therefore gives

$$\begin{aligned} \delta S &= -\frac{2}{4\mu_0 c} \int (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) F^{\mu\nu}\, d^4x + \frac{1}{c}\int \delta A_\mu J^\mu \, d^4x \\ &= -\frac{1}{\mu_0 c}\int \left(F^{\mu\nu}\partial_\mu \delta A_\nu - \mu_0 J^\nu \delta A_\nu\right)\, d^4x \\ &= \frac{1}{\mu_0 c}\int \left(\partial_\mu F^{\mu\nu} + \mu_0 J^\nu\right)\delta A_\nu \, d^4x\,. \end{aligned} \tag{8.26}$$

Here, we have integrated by parts going to the last equality and dropped the surface term since the variation δA_ν vanishes on the boundary. If the action is at an extremum, the change δS must vanish *for all* δA_μ. This can only be the case if

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu\,, \tag{8.27}$$

which is the desired (sourced) Maxwell equation. The other covariant Maxwell equation, $\partial_{[\rho}F_{\mu\nu]} = 0$, is implied by our assumption that $F_{\mu\nu}$ is derived from the 4-potential A_ν.

9 Scalar electrodynamics and superconductivity

In this section we discuss how to include electromagnetism (and in particular magnetic fields) in the non-relativistic quantum mechanics of a massive, charged spin-0 particle. Our treatment of the electromagnetic field will remain classical; quantization of the electromagnetic field requires the more involved machinery of Quantum Field Theory (see Part-III courses). The non-relativistic scalar electrodynamics that we develop provides a phenomenological (but powerful) description of *superconductivity* that we briefly introduce in Sec. 9.2.

9.1 Electromagnetism and non-relativistic quantum mechanics

We begin by recalling the general procedure for quantizing the non-relativistic dynamics of a massive particle (see e.g. the Part-II course *Principles of Quantum Mechanics*) moving in a potential $V(\mathbf{x}, t)$. The non-relativistic Lagrangian is just the kinetic energy minus the potential energy:

$$L = \frac{1}{2}m\mathbf{v}^2 - V(\mathbf{x}) \,, \tag{9.1}$$

where m is the mass and $\mathbf{v} = \dot{\mathbf{x}}$ is the 3-velocity (overdots denoting derivatives with respect to coordinate time in this section). From the Lagrangian, we construct the *canonical momentum*

$$\boldsymbol{\pi} \equiv \frac{\partial L}{\partial \dot{\mathbf{x}}} = m\mathbf{v} \,, \tag{9.2}$$

which here is just the usual mechanical 3-momentum. The *Hamiltonian* is $H \equiv \boldsymbol{\pi}\cdot\dot{\mathbf{x}} - L$, and here is just the total energy

$$H = \frac{1}{2m}\boldsymbol{\pi}^2 + V(\mathbf{x}, t) \,, \tag{9.3}$$

and is conserved if the potential V is time independent. To quantize, we promote $\boldsymbol{\pi}$ and $\mathbf{x}$ to operators satisfying the canonical commutation relations

$$[\hat{x}_i, \hat{\pi}_j] = i\hbar\delta_{ij} \,. \tag{9.4}$$

In the coordinate representation, where the state of the system is (fully) described by a time-dependent *wavefunction* $\Psi(\mathbf{x}, t)$, we have $\hat{\boldsymbol{\pi}} \to -i\hbar\boldsymbol{\nabla}$ and $\hat{\mathbf{x}} \to \mathbf{x}$ (i.e.,, the action of $\hat{\mathbf{x}}$ on the state vector $|\Psi\rangle$ reduces to $\mathbf{x}\Psi$). The wavefunction evolves according to the *time-dependent Schrödinger equation*

$$\hat{H}|\Psi\rangle = i\hbar\frac{\partial|\Psi\rangle}{\partial t} \,. \tag{9.5}$$

In the coordinate representation, this reduces to

$$\frac{1}{2m}\left(\frac{\hbar}{i}\boldsymbol{\nabla}\right)^2\Psi + V(\mathbf{x}, t)\Psi = i\hbar\frac{\partial\Psi}{\partial t} \,. \tag{9.6}$$

We now consider the case of a particle coupled to electromagnetism through its charge q. The non-relativistic Lagrangian is, from Eq. (8.9),

$$L = \frac{1}{2}m\mathbf{v}^2 - q\phi + q\mathbf{A}\cdot\mathbf{v} \,, \tag{9.7}$$

The canonical momentum evaluates to

$$\boldsymbol{\pi} = m\mathbf{v} + q\mathbf{A} \,. \tag{9.8}$$

Note this is not just the mechanical momentum, but includes the magnetic vector potential too. Expressing the mechanical momentum as $m\mathbf{v} = \boldsymbol{\pi} - q\mathbf{A}$, the Hamiltonian $H = \boldsymbol{\pi} \cdot \dot{\mathbf{x}} - L$ simplifies to

$$H = \frac{1}{2m}(\boldsymbol{\pi} - q\mathbf{A})^2 + q\phi\,, \tag{9.9}$$

giving the time-dependent Schrödinger equation

$$\frac{1}{2m}\left(\frac{\hbar}{i}\boldsymbol{\nabla} - q\mathbf{A}\right)^2 \Psi + q\phi\Psi = i\hbar\frac{\partial\Psi}{\partial t}\,. \tag{9.10}$$

The observable properties of the system described by Eq. (9.10) must be gauge-invariant. In other words, if we work in a different gauge, with $\mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\chi$ and $\phi \to \phi - \partial\chi/\partial t$, we will generally get a different solution for Ψ, but the observables must be the same. Since global phase changes of Ψ are not observable, we consider[19]

$$\Psi \to e^{iq\chi/\hbar}\Psi\,. \tag{9.11}$$

The particular phase change is chosen so that the Schrödinger equation is still solved by the transformed Ψ, $\mathbf{A}$ and ϕ if it was solved in the original gauge. To check this is the case, note that

$$\begin{aligned}\left(\frac{\hbar}{i}\boldsymbol{\nabla} - q(\mathbf{A} + \boldsymbol{\nabla}\chi)\right)\left(e^{iq\chi/\hbar}\Psi\right) &= e^{iq\chi/\hbar}\left(\frac{\hbar}{i}\boldsymbol{\nabla} - q\mathbf{A}\right)\Psi \\ \Rightarrow \qquad \left(\frac{\hbar}{i}\boldsymbol{\nabla} - q\mathbf{A}\right)\Psi &\to e^{iq\chi/\hbar}\left(\frac{\hbar}{i}\boldsymbol{\nabla} - q\mathbf{A}\right)\Psi\,,\end{aligned} \tag{9.12}$$

and also

$$\begin{aligned}\left[i\hbar\frac{\partial}{\partial t} - q\left(\phi - \frac{\partial\chi}{\partial t}\right)\right]\left(e^{iq\chi/\hbar}\Psi\right) &= e^{iq\chi/\hbar}\left[i\hbar\frac{\partial}{\partial t} - q\phi\right]\Psi \\ \Rightarrow \qquad \left(i\hbar\frac{\partial}{\partial t} - q\phi\right)\Psi &\to e^{iq\chi/\hbar}\left(i\hbar\frac{\partial}{\partial t} - q\phi\right)\Psi\,.\end{aligned} \tag{9.13}$$

Hence, we see that under the gauge transformation $\mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\chi$ and $\phi \to \phi - \partial\chi/\partial t$, the wavefunction also must transform according to Eq. (9.11).[20] The phase change does not alter any observable properties, for example the probability density $|\Psi|^2$.

[19]Gradients in the phase can have observable consequences, for example they alter the probability current. We shall see below how to construct a gauge-invariant current.

[20]Note how the coupling to electromagnetism promotes the *global* $U(1)$ symmetry of the usual Schrödinger theory (i.e., the freedom to make global phase changes that are constant in spacetime) to a *local* $U(1)$ symmetry (spacetime-dependent phase changes). Indeed, the idea that introducing gauge fields to promote a global symmetry of a theory to a local one also introduces interactions to the theory, which are mediated by the gauge field, is a key one in particle physics and underlies the standard model.

The (probability) current density will play an important role in our discussion of superconductivity. This satisfies

$$\frac{\partial|\Psi|^2}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J}_\Psi = 0\,, \tag{9.14}$$

where, recall, we interpret $|\Psi|^2$ as the probability density for locating the particle. In the presence of the magnetic vector potential, the probability current density must differ from the usual expression,

$$\mathbf{J}_\Psi = -\frac{i\hbar}{2m}\left(\Psi^*\boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\Psi^*\right)\,, \tag{9.15}$$

since the latter is neither conserved nor gauge-invariant. The appropriate generalisation is to replace the derivative $\boldsymbol{\nabla}$ with the *covariant derivative*

$$\boldsymbol{\mathcal{D}} \equiv \boldsymbol{\nabla} - i\frac{q}{\hbar}\mathbf{A}\,, \tag{9.16}$$

that appears (squared) on the left side of the Schrödinger equation (9.10). We see, from Eq. (9.12), that it is this derivative whose action on a wavefunction is covariant under gauge transformations (i.e.,, the result transforms the same way as the wavefunction itself). To see that we properly obtain a conserved current, we write the Schrödinger equation as

$$-\frac{\hbar^2}{2m}\boldsymbol{\mathcal{D}}^2\Psi = \left(i\hbar\frac{\partial}{\partial t} - q\phi\right)\Psi\,, \tag{9.17}$$

so that

$$\begin{aligned} i\hbar\frac{\partial|\Psi|^2}{\partial t} &= \left(i\hbar\frac{\partial\Psi}{\partial t}\right)\Psi^* - \Psi\left(i\hbar\frac{\partial\Psi}{\partial t}\right)^* \\ &= \left(i\hbar\frac{\partial\Psi}{\partial t} - q\phi\Psi\right)\Psi^* - \Psi\left(i\hbar\frac{\partial\Psi}{\partial t} - q\phi\Psi\right)^* \\ &= -\frac{\hbar^2}{2m}\left[\Psi^*\boldsymbol{\mathcal{D}}^2\Psi - \Psi\left(\boldsymbol{\mathcal{D}}^2\Psi\right)^*\right]\,. \end{aligned} \tag{9.18}$$

Now, for any Ψ and Φ

$$\boldsymbol{\nabla}\cdot\left(\Phi^*\boldsymbol{\mathcal{D}}\Psi\right) = \left(\boldsymbol{\mathcal{D}}\Phi\right)^*\cdot\boldsymbol{\mathcal{D}}\Psi + \Phi^*\boldsymbol{\mathcal{D}}\cdot\left(\boldsymbol{\mathcal{D}}\Psi\right)\,, \tag{9.19}$$

so that Eq. (9.18) becomes

$$i\hbar\frac{\partial|\Psi|^2}{\partial t} = -\frac{\hbar^2}{2m}\left[\boldsymbol{\nabla}\cdot\left(\Psi^*\boldsymbol{\mathcal{D}}\Psi\right) - \boldsymbol{\nabla}\cdot\left(\Psi(\boldsymbol{\mathcal{D}}\Psi)^*\right)\right]\,. \tag{9.20}$$

This establishes that an appropriate probability current is

$$\begin{aligned} \mathbf{J}_\Psi &= -\frac{i\hbar}{2m}\left[\Psi^*\boldsymbol{\mathcal{D}}\Psi - \Psi(\boldsymbol{\mathcal{D}}\Psi)^*\right] \\ &= -\frac{i\hbar}{2m}\left(\Psi^*\boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\Psi^*\right) - \frac{q}{m}|\Psi|^2\mathbf{A}\,. \end{aligned} \tag{9.21}$$

Finally, in terms of the phase θ of the wavefunction, where $\Psi = |\Psi|e^{i\theta}$, the current can be written as

$$\mathbf{J}_\Psi = \frac{\hbar|\Psi|^2}{m}\left(\boldsymbol{\nabla}\theta - \frac{q}{\hbar}\mathbf{A}\right), \tag{9.22}$$

so it is the gauge-invariant gradient of the phase that determines the current density.

The Schrödinger equation (9.10) can be derived from an action $S = \int L\,dt$, where the Lagrangian

$$L = \int \left[\frac{1}{2}\Psi^*\left(i\hbar\frac{\partial}{\partial t} - q\phi\right)\Psi + \frac{1}{2}\Psi\left(\left(i\hbar\frac{\partial}{\partial t} - q\phi\right)\Psi\right)^* - \frac{\hbar^2}{2m}|\boldsymbol{\mathcal{D}}\Psi|^2\right] d^3\mathbf{x}\,. \tag{9.23}$$

This is manifestly gauge-invariant. The asymmetry between time and spatial derivatives reflects the lack of Lorentz invariance of the non-relativistic Schrödinger equation. Varying Ψ, we have

$$\begin{aligned}\delta S &= \int \left[\frac{1}{2}\delta\Psi^*\left(i\hbar\frac{\partial}{\partial t} - q\phi\right)\Psi + \frac{1}{2}\Psi^*\left(i\hbar\frac{\partial}{\partial t} - q\phi\right)\delta\Psi\right. \\ &\qquad\qquad \left. - \frac{\hbar^2}{2m}\left(\boldsymbol{\mathcal{D}}\delta\Psi\right)^*\cdot\boldsymbol{\mathcal{D}}\Psi + \text{c.c.}\right] dt\,d^3\mathbf{x} \\ &= \int \left[\delta\Psi^*\left(i\hbar\frac{\partial}{\partial t} - q\phi\right)\Psi - \frac{\hbar^2}{2m}\left(\boldsymbol{\nabla}\cdot\left(\delta\Psi^*\boldsymbol{\mathcal{D}}\Psi\right) - \delta\Psi^*\boldsymbol{\mathcal{D}}^2\Psi\right) + \text{c.c.}\right] dt\,d^3\mathbf{x}\,,\end{aligned} \tag{9.24}$$

where we integrated by parts (in time) and dropped the boundary term, and used Eq. (9.19). If we now use the divergence theorem, we find

$$\delta S = \int \left[\delta\Psi^*\left(i\hbar\frac{\partial}{\partial t} - q\phi\right)\Psi + \frac{\hbar^2}{2m}\delta\Psi^*\boldsymbol{\mathcal{D}}^2\Psi + \text{c.c.}\right] dt\,d^3\mathbf{x}\,. \tag{9.25}$$

Since this must vanish for all $\delta\Psi$, we recover the Schrödinger equation in the form of Eq. (9.17).

9.2 Application to superconductivity

In this section we apply the ideas developed above to give a phenomenological description of the equilibrium (hence static) properties of superconductors in magnetic fields.

Superconductivity is observed in many metals and alloys below a (material-dependent) transition temperature T_c. The superconducting state is defined by the following electrodynamic properties.

- The electrical resistivity disappears completely (at zero frequency) and so currents can flow persistently with no dissipation. This phenomenon was first observed in mercury ($T_c = 4.1\,\mathrm{K}$) by Onnes in 1911.

- Perfect diamagnetism, i.e.,, the magnetic field inside a superconductor vanishes. If a magnetic field is established in a sample above T_c by external currents, as the sample is cooled below T_c (super-)currents are induced on the surface of the superconductor that exactly cancel the magnetic field in the bulk of the sample. This effect is called the Meissner effect and was first observed by Meissner & Ochensfeld in 1933. Note the distinction from a perfect conductor, which resists any change in magnetic flux and so would not expel a pre-existing field from its interior on cooling.

The microscopic theory of (conventional) superconductivity is due to Bardeen, Cooper & Schrieffer (BCS; 1957). It is beyond the scope of this course to give a detailed description of this theory and we shall make do with a brief summary of the key ideas. The most important idea is that of a net attractive interaction between electrons near the Fermi surface mediated by phonons (quantized vibrations of the ionic lattice). In essence, the mutual Coulomb repulsion of the delocalised electrons is over-screened by the motion of the lattice as the ions are attracted to electron overdensities. A phonon-mediated interaction neatly explains the observed *isotope effect*, whereby the superconducting properties of a metallic element depend on the specific isotope (hence nuclear mass). The consequence of the net attraction between electrons is that they can form bound pairs, known as Cooper pairs, which therefore have integer spin and so behave as bosons. The Pauli exclusion principle does not apply to the bosonic Cooper pairs, so they can call occupy the same two-electron state. The extent of the pair wavefunction is very large (around $10^{-7}\,\mathrm{m}$) compared to the typical ionic separation (around $10^{-10}\,\mathrm{m}$), so that the *collective* behaviour of the pairs is critical for the superconducting state.

The transport properties of a superconductor, for example the electrical conductivity, arise from non-equilibrium motion of the Cooper pairs (plus any contribution from electrons in the normal phase). At any finite temperature, some pairs are dissociated by thermal excitation and their electrons occupy the usual quasi-single-electron levels of the normal (non-superconducting) phase. Above the critical temperature all pairs are dissociated and the electrons occupy the usual eigenstates of the independent electron approximation. The transition from normal to superconducting behaviour is an example of a *second-order phase transition*[21]. Roughly speaking, there is no latent heat associated with a second-order phase transition, but the heat capacity is discontinuous. A perhaps more familiar example is the ferromagnetic phase transition, whereby the

[21]Phase transitions are covered in more detail at the end of the Lent-term *Statistical Physics* Part-II course.

magnetization of a sample in zero applied magnetic field changes continuously from zero as the sample is cooled through the Curie temperature.

9.2.1 Ginzburg–Landau theory

Ginzburg–Landau theory provides a general *phenomenological* description of second-order phase transitions. Its application to superconductivity pre-dates the microscopic BCS theory. The basic idea is to characterise the superconducting state by a complex order parameter $\psi(\mathbf{x})$, which can crudely be thought of as a one-particle wavefunction describing the centre-of-mass motion of a Cooper pair of electrons[22]. The theory is constructed so that the equilibrium value of the order parameter vanishes above the critical temperature T_c, but it acquires a non-zero (temperature-dependent) value below T_c. The value of $|\psi|^2$ can be interpreted as the number density of electron pairs forming the superconducting state.

The key quantity in Ginzburg–Landau theory is the *free energy functional*, which is a functional of the order parameter ψ and the magnetic field in the sample. The idea is to minimise the free energy functional to determine the form of these fields in equilibrium, subject to the external constraints imposed on the system such as its temperature, volume and the magnetic field generated by external currents. The (equilibrium) order parameter is supposed to be continuous through the critical temperature, so close to T_c we can write the free energy functional as a Taylor expansion in $|\psi|$ about $\psi = 0$. Moreover, we expect gradients in the order parameter to cost energy, which we can penalise with terms proportional to $|\boldsymbol{\nabla}\psi|^2$ in the free energy functional. Finally, to include coupling to the magnetic field, we make the gradient terms gauge-invariant by replacing $\boldsymbol{\nabla} \to \boldsymbol{\nabla} - iq\mathbf{A}/\hbar$ and include the magnetic field energy. The microscopic theory tells us that we should take $q = -2e$ since we are dealing with pairs of electrons. Putting these ingredients together, we have the free energy functional

$$\mathcal{F}[\psi, \mathbf{A}] = \int \left(\frac{1}{2\mu_0}|\mathbf{B}|^2 + \frac{\hbar^2}{2m} \left| \left(\boldsymbol{\nabla} - i\frac{q\mathbf{A}}{\hbar} \right) \psi \right|^2 + \alpha(T)|\psi|^2 + \frac{1}{2}\beta(T)|\psi|^4 \right) d^3\mathbf{x} , \tag{9.26}$$

where $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. The normalisation of the gradient-squared term is conventional and is chosen by analogy with the action for the Schrödinger equation. Being a phenomenological description, the mass m has to be chosen by comparison with experiment or from the appropriate limit of a microscopic theory. On the basis of BCS theory, we expect $m \sim 2m_e^*$, where m_e^* is an appropriate effective mass of the electrons moving in the periodic potential of the unperturbed lattice of ions. Finally, at any temperature, we require $\beta > 0$ so that $\mathcal{F}$ is bounded from below. We shall see below how to choose $\alpha(T)$ such that we get the desired phase transition to the superconducting state at T_c.

[22] In ferromagnetism, the appropriate order parameter is the magnetic moment per volume.

The free energy functional in Eq. (9.26) is gauge-invariant under the usual transformations $\mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\chi$ and $\psi \to e^{iq\chi/\hbar}\psi$. Varying the action with respect to ψ (following the steps used for the Schrödinger equation in the previous section), we find

$$-\frac{\hbar^2}{2m}\left(\boldsymbol{\nabla} - i\frac{q\mathbf{A}}{\hbar}\right)^2 \psi + \alpha(T)\psi + \beta(T)|\psi|^2\psi = 0\,. \tag{9.27}$$

This is like the Schrödinger equation but with a non-linear term $\beta|\psi|^2\psi$. Varying with respect to $\mathbf{A}$, we have

$$\delta\mathcal{F} = \int \left\{ \frac{\hbar^2}{2m}\left[\left(-i\frac{q\delta\mathbf{A}}{\hbar}\psi\right)\cdot(\boldsymbol{\mathcal{D}}\psi)^* + (\boldsymbol{\mathcal{D}}\psi)\cdot\left(i\frac{q\delta\mathbf{A}}{\hbar}\psi^*\right)\right] + \frac{1}{\mu_0}(\boldsymbol{\nabla}\times\delta\mathbf{A})\cdot\mathbf{B}\right\} d^3\mathbf{x}\,, \tag{9.28}$$

where, recall, $\boldsymbol{\mathcal{D}} = \boldsymbol{\nabla} - iq\mathbf{A}/\hbar$. Noting that

$$(\boldsymbol{\nabla}\times\delta\mathbf{A})\cdot\mathbf{B} = \boldsymbol{\nabla}\cdot(\delta\mathbf{A}\times\mathbf{B}) + \delta\mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})\,, \tag{9.29}$$

on dropping the resulting surface term in $\delta\mathcal{F}$, we find the (static) Ampére equation,

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J}_s\,, \tag{9.30}$$

where the *supercurrent*

$$\begin{aligned}\mathbf{J}_s &= -\frac{i\hbar q}{2m}\left[\psi^*\boldsymbol{\mathcal{D}}\psi - \psi(\boldsymbol{\mathcal{D}}\psi)^*\right] \\ &= -\frac{i\hbar q}{2m}\left[\psi^*\left(\boldsymbol{\nabla} - i\frac{q\mathbf{A}}{\hbar}\right)\psi - \text{c.c.}\right]\,.\end{aligned} \tag{9.31}$$

Note that this is just the product of the charge q and the usual gauge-invariant probability current for a wavefunction ψ. Generally, we must solve the non-linear Schrödinger equation (9.27) and the Ampére equation (9.30) simultaneously, with appropriate boundary conditions, to find the magnetic field and supercurrent in the superconductor.

We begin by considering the case of a uniform (homogeneous) superconductor in zero applied magnetic field. Assuming $\mathbf{A} = 0$, there exist constant solutions to Eq. (9.27) with

$$\psi\left[\alpha(T) + \beta(T)|\psi|^2\right] = 0\,. \tag{9.32}$$

Now, we want the superconducting order parameter to vanish for temperatures higher than the critical temperature T_c, so that no electrons are in the superconducting phase. We can ensure this is the case by taking $\alpha(T) > 0$ for $T > T_c$, since, recall, $\beta(T) > 0$ for all T. Below T_c we want a non-zero (equilibrium) order parameter, which requires $\alpha(T) < 0$ for $T < T_c$; then

$$|\psi|^2 = -\alpha(T)/\beta(T) \qquad (T < T_c)\,. \tag{9.33}$$

In the vicinity of the critical temperature, generically we expect $\alpha(T) = a(T - T_c)$ for some constant a, and $\beta(T) \approx b$ for some other constant b.

For zero field and with ψ uniform, the free energy functional becomes

$$\mathcal{F} \propto \alpha(T)|\psi|^2 + \beta(T)|\psi|^4/2 \,. \tag{9.34}$$

For $T > T_c$, this has a single minimum at $|\psi| = 0$, but for $T < T_c$, the point $|\psi| = 0$ becomes a local maximum and the minimum free energy is realised at $|\psi| = -\alpha/\beta$. We see that the equilibrium value of the order parameter smoothly transitions from $|\psi| = 0$ above the critical temperature to $|\psi| = \sqrt{-\alpha/\beta}$ below T_c, as appropriate for a second-order phase transition.

Meissner effect

Ginzburg–Landau theory naturally accounts for the expulsion of magnetic flux from a superconductor. To see this, we write the supercurrent in terms of the gauge-invariant phase gradient, following Eq. (9.22):

$$\mathbf{J}_s = \frac{q\hbar|\psi|^2}{m}\left(\boldsymbol{\nabla}\theta - \frac{q}{\hbar}\mathbf{A}\right) , \tag{9.35}$$

where $\psi = |\psi|e^{i\theta}$. We expect the magnetic field to decay sharply inside the superconductor. If we assume that we can ignore spatial variations in $|\psi|$ over this transition region, we can approximate $|\psi|^2 = -\alpha(T)/\beta(T)$[23]. Taking the curl, we have

$$\boldsymbol{\nabla} \times \mathbf{J}_s \approx -\frac{q^2|\psi|^2}{m}\mathbf{B} \,, \tag{9.36}$$

which combines with the Ampére equation (9.30) to give

$$\begin{aligned} \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{B}) &= -\frac{\mu_0 q^2|\psi|^2}{m}\mathbf{B} \\ \Rightarrow \qquad \boldsymbol{\nabla}^2\mathbf{B} &= \frac{\mu_0 q^2|\psi|^2}{m}\mathbf{B} \,. \end{aligned} \tag{9.37}$$

[23] This approximation is not really justified. Equation (9.27) defines a characteristic *coherence length* ξ over which the magnitude of ψ will vary. By writing ψ in dimensionless form by dividing by $\sqrt{-\alpha/\beta}$ (for $T < T_c$), the remaining length scale in the zero-field limit is

$$\xi = \frac{\hbar}{\sqrt{2m|\alpha|}} \,.$$

It is over this scale that ψ recovers to the homogeneous value $\sqrt{-\alpha/\beta}$. Since $\alpha \to 0$ as $T \to T_c$, the coherence length is large near the critical temperature. However, as we shall see shortly, the magnetic field only penetrates a distance $\lambda \propto 1/\sqrt{|\alpha|}$, which may be comparable to ξ. In general, one must consider variations in the order parameter and the magnetic field, in which case both lengths ξ and λ enter the problem. Note that their ratio $\kappa \equiv \lambda/\xi$ is (almost) independent of temperature.

Consider the case of a plane surface at $x = 0$, with the superconductor filling the region $x > 0$. Take the magnetic field along the z-direction. Then the solution of Eq. (9.37) for $x > 0$ that is finite at $x \to \infty$ is

$$B_z(x) = B_0 e^{-x/\lambda}\,, \tag{9.38}$$

where B_0 is the magnetic field outside the superconductor and the *penetration depth*

$$\lambda^2 = \frac{m}{\mu_0 q^2 |\psi|^2}\,. \tag{9.39}$$

(We have assumed that $|\psi|^2$ is constant.) We see that a static magnetic field only penetrates a distance of order λ into the superconductor. Supercurrents flow within this surface region, screening the interior of the superconductor from the applied magnetic field. Taking $q = -2e$, $m \sim 2m_e$ (twice the free electron mass) and $|\psi|^2 = n_s$ roughly equal to half the number density of free electrons for the normal state (which is reasonable well below T_c), we find $\lambda \sim 10^{-8}$–$10^{-7}\,\mathrm{m}$. This is large compared to the ionic separation but small compared to typical macroscopic scales.

Flux quantization

The form of the supercurrent in Eq. (9.35) has a very striking implication: the magnetic flux linked by a superconductor is quantized. To see this, consider a superconducting ring with a magnetic field threading the hole in the centre. Since the magnetic field inside the superconductor is appreciable only near the surface, the same is true of the supercurrent (by Ampére's equation). In the bulk of the superconductor, where $\mathbf{J}_s = 0$, we have

$$\mathbf{A} = \frac{\hbar}{q}\boldsymbol{\nabla}\theta\,, \tag{9.40}$$

where θ is the phase of ψ. If we integrate this equation around a closed path deep inside the superconductor that encircles the hole, we have

$$\begin{aligned} \oint \mathbf{A}\cdot d\mathbf{x} &= \frac{\hbar}{q}\oint \boldsymbol{\nabla}\theta\cdot d\mathbf{x} \\ \Rightarrow \qquad \int \mathbf{B}\cdot d\mathbf{S} &= \frac{\hbar}{q}\Delta\theta\,, \end{aligned} \tag{9.41}$$

where we used Stokes theorem to relate $\oint \mathbf{A}\cdot d\mathbf{x}$ to the magnetic flux $\Phi = \int \mathbf{B}\cdot d\mathbf{S}$ linked by the circuit (hence ring). The change in phase $\Delta\theta$ on traversing the circuit must be an integer multiple n of 2π since ψ must be single-valued. It follows that

$$\Phi = \frac{2\pi\hbar n}{2e}\,, \tag{9.42}$$

where we have used the BCS result $q = -2e$, i.e., the flux linked by the ring is quantized in units of Φ_0 where

$$\Phi_0 = \frac{\pi\hbar}{e} = 2.07\times 10^{-15}\,\mathrm{T\,m^2}\,. \tag{9.43}$$

Flux quantization was first observed in 1961 (by Deaver & Fairbank and Doll & Näbauer), providing compelling evidence for the validity of a description of superconductivity by a complex order parameter ψ. Moreover, the observed value of the *fluxoid* Φ_0 validates the idea that the current is carried by pairs of electrons with $q = -2e$.