5

THERMODYNAMICS IN THE EXPANDING UNIVERSE

5.1 The Boltzmann Equation

For much of its early history, most of the constituents of the Universe were in thermal equilibrium, thereby making an equilibrium description a good approximation. However, there have been a number of very notable departures from thermal equilibrium—neutrino decoupling, decoupling of the background radiation, primordial nucleosynthesis, and on the more speculative side, inflation, baryogenesis, decoupling of relic WIMPs, etc. As we have previously emphasized, if not for such departures from thermal equilibrium, the present state of the Universe would be completely specified by the present temperature. The departures from equilibrium have led to important relics—the light elements, the neutrino backgrounds, a net baryon number, relic WIMPs (weakly-interacting massive particles), relic cosmologists, and so on.

As discussed in Chapters 2 and 3, once a species totally decouples from the plasma its evolution is very simple: particle number density decreasing as R^{-3} and particle momenta decreasing as R^{-1}. The evolution of the phase space distribution of a species which is in LTE or is completely decoupled is simple. It is the evolution of particle distributions around the epoch of decoupling that is challenging. Recall that the rough criterion for a particle species to be either coupled or decoupled involves the comparison of the interaction rate of the particle, Γ, with the expansion rate of the Universe, H:

$$\begin{aligned} \Gamma &\gtrsim H \quad \text{(coupled)} \\ \Gamma &\lesssim H \quad \text{(decoupled)} \end{aligned} \tag{5.1}$$

where Γ is the interaction rate (per particle) for the reaction(s) that keep the species in thermal equilibrium. The units of Γ, of course, are time^{-1}.

While this is a very useful rule of thumb which is usually surprisingly accurate, in order to properly treat decoupling one must follow the microscopic evolution of the particle's phase space distribution function $f(p^\mu, x^\mu)$. This of course is governed by the Boltzmann equation, which can be written as

$$\hat{\mathbf{L}}[f] = \mathbf{C}[f] \tag{5.2}$$

where $\mathbf{C}$ is the collision operator and $\hat{\mathbf{L}}$ is the Liouville operator. The familiar non-relativistic Liouville operator for the phase space density $f(\vec{\mathbf{v}}, \vec{\mathbf{x}})$ of a particle species of mass m subject to a force $\vec{\mathbf{F}} = d\vec{\mathbf{p}}/dt$ is

$$\hat{\mathbf{L}}_{\rm NR} = \frac{d}{dt} + \frac{d\vec{\mathbf{x}}}{dt}\cdot\vec{\nabla}_x + \frac{d\vec{\mathbf{v}}}{dt}\cdot\vec{\nabla}_v = \frac{d}{dt} + \vec{\mathbf{v}}\cdot\vec{\nabla}_x + \frac{\vec{\mathbf{F}}}{m}\cdot\vec{\nabla}_v. \tag{5.3}$$

The covariant, relativistic generalization of the Liouville operator is

$$\hat{\mathbf{L}} = p^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \frac{\partial}{\partial p^\alpha}. \tag{5.4}$$

Note, as expected, gravitational effects enter the equation only through the affine connection. For the FRW model the phase space density is spatially homogeneous and isotropic: $f = f(|\vec{\mathbf{p}}|, t)$ [or equivalently $f(E, t)$]. For the Robertson-Walker metric the Liouville operator is

$$\hat{\mathbf{L}}[f(E,t)] = E\frac{\partial f}{\partial t} - \frac{\dot{R}}{R}|\vec{\mathbf{p}}|^2 \frac{\partial f}{\partial E}. \tag{5.5}$$

Using the definition of the number density in terms of the phase space density

$$n(t) = \frac{g}{(2\pi)^3}\int d^3p f(E, t), \tag{5.6}$$

and upon integration by parts, the Boltzmann equation can be written in the form

$$\frac{dn}{dt} + 3\frac{\dot{R}}{R}n = \frac{g}{(2\pi)^3}\int \mathbf{C}[f]\frac{d^3p}{E}. \tag{5.7}$$

The collision term for the process $\psi + a + b + \cdots \longleftrightarrow i + j + \cdots$ is given by

$$\frac{g}{(2\pi)^3}\int \mathbf{C}[f]\frac{d^3p_\psi}{E_\psi} = -\int d\Pi_\psi d\Pi_a d\Pi_b \cdots d\Pi_i d\Pi_j \cdots$$

$$\times(2\pi)^4\delta^4(p_\psi + p_a + p_b \cdots - p_i - p_j \cdots)$$
$$\times \Big[|\mathcal{M}|^2_{\psi+a+b+\cdots\to i+j+\cdots} f_a f_b \cdots f_\psi (1\pm f_i)(1\pm f_j)\cdots$$
$$-|\mathcal{M}|^2_{i+j+\cdots\to\psi+a+b+\cdots} f_i f_j \cdots (1\pm f_a)(1\pm f_b)\cdots(1\pm f_\psi)\Big] \quad (5.8)$$

where f_i, f_j, , f_a, f_b, $\cdots$ are the phase space densities of species i, j, $\cdots$, a, b, $\cdots$; f_ψ is the phase space density of ψ (the species whose evolution we are focusing on); $(+)$ applies to bosons; $(-)$ applies to fermions; and

$$d\Pi \equiv g\frac{1}{(2\pi)^3}\frac{d^3p}{2E} \quad (5.9)$$

where g counts the internal degrees of freedom. The 4-dimensional delta function enforces energy and momentum conservation, and the matrix element squared, $|\mathcal{M}|^2_{i+j+\cdots\to\psi+a+b+\cdots}$, for the process $i+j+\cdots\to\psi+a+b+\cdots$, is averaged over initial and final spins, and includes the appropriate symmetry factors for identical particles in the initial or final states.[1] The topic of statistical mechanics in the expanding Universe is discussed by Wagoner [2], and in the monograph of Bernstein [2].

In the most general case, the Boltzmann equations are a coupled set of integral-partial differential equations for the phase space distributions of all the species present! Fortunately, in problems of interest to us, all but one (or two) species will have equilibrium phase space distribution functions because of their rapid interactions with other species, reducing the problem to a single integral-partial differential equation for the one species of interest, denoted by ψ.

There are two well motivated approximations that greatly simplify (5.8). The first is the assumption of T (or CP) invariance,[2] which implies

$$|\mathcal{M}|^2_{i+j+\cdots\to\psi+a+b+\cdots} = |\mathcal{M}|^2_{\psi+a+b+\cdots\to i+j+\cdots} \equiv |\mathcal{M}|^2. \quad (5.10)$$

The second simplification is the use of Maxwell-Boltzmann statistics for all species instead of Fermi-Dirac for fermions and Bose-Einstein for bosons.[3]

[1]For n identical particles of a given species in the initial or final state, a factor of $1/n!$. The rules for calculating $|\mathcal{M}|^2$ are given in many standard texts, including Quigg, and Bjorken and Drell [1].

[2]In Chapter 6 we will relax the assumption of CP invariance when we discuss baryogenesis. Since the only observation of CP violation is in the K^0–$\bar{K}^0$ system, this assumption is well justified.

[3]In the absence of a degenerate (i.e., $\mu_i \gtrsim T$) Fermi species or a Bose condensate,

In the absence of Bose condensation or Fermi degeneracy, the blocking and stimulated emission factors can be ignored, $1 \pm f \simeq 1$, and $f_i(E_i) = \exp[-(E_i - \mu_i)/T]$ for all species in kinetic equilibrium. With these two assumptions the Boltzmann equation may be cast in the familiar form

$$\dot{n}_\psi + 3Hn_\psi = -\int d\Pi_\psi d\Pi_a d\Pi_b \cdots d\Pi_i d\Pi_j \cdots (2\pi)^4 |\mathcal{M}|^2 \\ \times \delta^4(p_i + p_j \cdots - p_\psi - p_a - p_b \cdots)\left[f_a f_b \cdots f_\psi - f_i f_j \cdots\right] \quad (5.11)$$

where, as usual, $H \equiv \dot{R}/R$.

The significance of the individual terms is manifest: The $3Hn_\psi$ term accounts for the dilution effect of the expansion of the Universe, and the right hand side of (5.11) accounts for interactions that change the number of ψ's present. In the absence of interactions the solution to (5.11) is $n_\psi \propto R^{-3}$.

Finally, it is usually useful to scale out the effect of the expansion of the Universe by considering the evolution of the number of particles in a comoving volume. This is done by using the entropy density, s, as a fiducial quantity, and by defining as the dependent variable

$$Y \equiv \frac{n_\psi}{s}. \quad (5.12)$$

Using the conservation of entropy per comoving volume (sR^3 = constant), it follows that

$$\dot{n}_\psi + 3Hn_\psi = s\dot{Y}. \quad (5.13)$$

Furthermore, since the interaction term will usually depend explicitly upon temperature rather than time, it is useful to introduce as the independent variable

$$x \equiv m/T, \quad (5.14)$$

where m is any convenient mass scale (usually taken as the mass of the particle of interest). During the radiation-dominated epoch x and t are related by

$$t = 0.301 g_*^{-1/2} \frac{m_{Pl}}{T^2} = 0.301 g_*^{-1/2} \frac{m_{Pl}}{m^2} x^2 \quad (5.15)$$

the use of Maxwell-Boltzmann statistics introduces only a small quantitative change, as all three distribution functions are very similar (and much less than one) for momenta near the peak of the distribution. Moreover, for any non-relativistic species, Maxwell-Boltzmann statistics becomes exact in the limit $(m_i - \mu_i)/T \gg 1$.

so that the Boltzmann equation can be rewritten as

$$\frac{dY}{dx} = -\frac{x}{H(m)s}\int d\Pi_\psi d\Pi_a d\Pi_b \cdots d\Pi_i d\Pi_j \cdots (2\pi^4)|\mathcal{M}|^2$$
$$\times \delta^4(p_i + p_j \cdots - p_\psi - p_a - p_b \cdots)[f_a f_b \cdots f_\psi - f_i f_j \cdots] \quad (5.16)$$

where $H(m) = 1.67 g_*^{1/2} m^2/m_{Pl}$, and $H(x) = H(m)x^{-2}$.

We will now consider some specific applications of the formalism we have developed here to treat non-equilibrium thermodynamics.

5.2 Freeze Out: Origin of Species

If a massive particle species remained in thermal equilibrium until the present, its abundance, $n/s \sim (m/T)^{3/2}\exp(-m/T)$, would be absolutely negligible because of the exponential factor. If the interactions of the species freeze out (i.e., $\Gamma < H$) at a temperature such that m/T is not much greater than 1, the species can have a significant relic abundance today. We will now calculate that relic abundance.

First, suppose that the species is stable (or very long-lived compared to the age of the Universe when its interactions freeze out). In the next section we will consider the case where the species is unstable. Given that it is stable, only annihilation and inverse annihilation processes, e.g.,

$$\psi\bar{\psi} \longleftrightarrow X\bar{X}, \quad (5.17)$$

can change the number of ψ's and $\bar{\psi}$'s in a comoving volume.[4] Here X generically denotes all the species into which ψ's can annihilate. In addition, we assume that there is no asymmetry between ψ's and $\bar{\psi}$'s.[5]

We will also assume that all the species X, $\bar{X}$ into which ψ, $\bar{\psi}$ annihilate have thermal distributions with zero chemical potential. Because these particles will usually have additional interactions which are "stronger" than their interactions with ψ's, the assumption of equilibrium for the X's is almost always a good one. For example, let ψ, $\bar{\psi} = \nu$, $\bar{\nu}$ and X, $\bar{X} = e^-$, e^+; while the neutrinos only have weak interactions, the $e^\pm$'s have weak and electromagnetic interactions.

[4]For simplicity we will only consider $2 \leftrightarrow 2$ annihilation and creation processes; it is straightforward to generalize from here.

[5]Again, it is straightforward to generalize this to include the possibility of an excess (or deficit) of ψ's over $\bar{\psi}$'s.

Now consider the factor $[f_\psi f_{\bar\psi} - f_X f_{\bar X}]$ in the collision term in the Boltzmann equation. Since X, $\bar X$ are in thermal equilibrium (and for simplicity we assume they have zero chemical potential)[6]

$$\begin{aligned} f_X &= \exp(-E_X/T), \\ f_{\bar X} &= \exp(-E_{\bar X}/T). \end{aligned} \tag{5.18}$$

The energy part of the δ-function enforces $E_\psi + E_{\bar\psi} = E_X + E_{\bar X}$, so that

$$f_X f_{\bar X} = \exp[-(E_X + E_{\bar X})/T] = \exp[-(E_\psi + E_{\bar\psi})/T] = f_\psi^{\rm EQ} f_{\bar\psi}^{\rm EQ}, \tag{5.19}$$

since $f_\psi^{\rm EQ} \equiv \exp(-E_\psi/T)$ and $f_{\bar\psi}^{\rm EQ} \equiv \exp(-E_{\bar\psi}/T)$. Therefore, it follows that

$$[f_\psi f_{\bar\psi} - f_X f_{\bar X}] = [f_\psi f_{\bar\psi} - f_\psi^{\rm EQ} f_{\bar\psi}^{\rm EQ}]. \tag{5.20}$$

Now the interaction term can be written in terms of n_ψ, the actual number density of ψ's, and $n_\psi^{\rm EQ}$, the equilibrium number density of ψ's, as

$$\frac{dn_\psi}{dt} + 3Hn_\psi = -\langle\sigma_{\psi\bar\psi\to X\bar X}|v|\rangle[n_\psi^2 - (n_\psi^{\rm EQ})^2], \tag{5.21}$$

or,

$$\frac{dY}{dx} = \frac{-x\langle\sigma_{\psi\bar\psi\to X\bar X}|v|\rangle s}{H(m)}(Y^2 - Y_{\rm EQ}^2), \tag{5.22}$$

where $Y \equiv n_\psi/s = n_{\bar\psi}/s$ is the *actual* number of ψ, $\bar\psi$'s per comoving volume, and $Y_{\rm EQ} \equiv n_\psi^{\rm EQ}/s = n_{\bar\psi}^{\rm EQ}/s$ is the *equilibrium* number of ψ, $\bar\psi$'s per comoving volume. The thermally-averaged annihilation cross section times velocity is given by

$$\begin{aligned} \langle\sigma_{\psi\bar\psi\to X\bar X}|v|\rangle \equiv \left(n_\psi^{\rm EQ}\right)^{-2} \int d\Pi_\psi d\Pi_{\bar\psi} d\Pi_X d\Pi_{\bar X} (2\pi)^4 \\ \times\delta^4(p_\psi + p_{\bar\psi} - p_X - p_{\bar X})|\mathcal{M}|^2 \exp(-E_\psi/T)\exp(-E_{\bar\psi}/T). \end{aligned} \tag{5.23}$$

Now if we consider other annihilation channels for $\psi\bar\psi$, say $\psi\bar\psi$ to some final state F (not necessarily a two-body final state), there is an additional term in $\dot n_\psi$, which is simply (5.21) with $\langle\sigma_{\psi\bar\psi\to X\bar X}|v|\rangle$ replaced by

[6]While the Boltzmann equation is explicitly covariant, specifying the particle distribution functions singles out a frame—the comoving frame—and breaks the covariance. Once this is done, all quantities must be evaluated in this frame.

$\langle\sigma_{\psi\bar{\psi}\to F}|v|\rangle$. Summing over all annihilation channels yields the final result in terms of the *total* annihilation cross section $\langle\sigma_A|v|\rangle$

$$\begin{aligned}\frac{dn_\psi}{dt} + 3Hn_\psi &= -\langle\sigma_A|v|\rangle[n_\psi^2 - (n_\psi^{\rm EQ})^2] \\ \frac{dY}{dx} &= \frac{-x\langle\sigma_A|v|\rangle s}{H(m)}(Y^2 - Y_{\rm EQ}^2). \end{aligned} \tag{5.24}$$

In the non-relativistic ($x \gg 3$) and in the extreme relativistic ($x \ll 3$) regimes the equilibrium value of the number of ψ's per comoving volume has the following simple limiting forms,

$$\begin{aligned}Y_{EQ}(x) &= \frac{45}{2\pi^4}\left(\frac{\pi}{8}\right)^{1/2}\frac{g}{g_{*S}}x^{3/2}e^{-x} = 0.145\frac{g}{g_{*S}}x^{3/2}e^{-x} \quad (x \gg 3) \\ Y_{EQ}(x) &= \frac{45\zeta(3)}{2\pi^4}\frac{g_{\rm eff}}{g_{*S}} = 0.278\frac{g_{\rm eff}}{g_{*S}} \quad (x \ll 3) \end{aligned} \tag{5.25}$$

where $g_{\rm eff} = g$ (bosons) and $g_{\rm eff} = 3g/4$ (fermions).

This very convenient form of the Boltzmann equation that we have just derived has all the "features" that we would have expected: The destruction rate of ψ, $\bar{\psi}$'s per comoving volume is just proportional to the annihilation rate of ψ, $\bar{\psi}$'s, and the net rate of destruction is just balanced by inverse (creation) processes when $n_\psi = n_\psi^{\rm EQ}$. Moreover, creation processes are Boltzmann suppressed for $T \ll m$, because only a small fraction of $X\bar{X}$ pairs have sufficient KE to create a $\psi\bar{\psi}$ pair. In fact, we could have forgone the sleight-of-hand involved in rewriting f_X, $f_{\bar{X}}$ in terms of $f_\psi^{\rm EQ}$, $f_{\bar{\psi}}^{\rm EQ}$, and just derived (5.24) by invoking the *principle of detailed balance*—however, we would do so at the expense of pedagogy. Whenever all species except the one of interest are in thermal equilibrium, the creation processes can be obtained from the destruction processes by substituting $n \to n_{\rm EQ}$ for the species of interest.

Remembering that $H \propto x^{-2}$, so that $H(T) = x^{-2}H(m)$, we see that (5.24) can be cast in a very suggestive way,

$$\begin{aligned}\frac{x}{Y_{\rm EQ}}\frac{dY}{dx} &= -\frac{\Gamma_A}{H}\left[\left(\frac{Y}{Y_{\rm EQ}}\right)^2 - 1\right], \\ \Gamma_A &\equiv n_{\rm EQ}\langle\sigma_A|v|\rangle. \end{aligned} \tag{5.26}$$

In this form, we see that the change of ψ's per comoving volume is controlled by the *effectiveness of annihilations*, the usual Γ/H factor, times a measure of the deviation from equilibrium. It is then clear that when Γ/H is less than order unity, the relative change in the number of ψ's in a comoving volume becomes small, $-\Delta Y/Y \sim -(xdY/dx)/Y_{\rm EQ} \sim (\Gamma/H) \lesssim 1$, annihilations freeze out, and the number of ψ's in a comoving volume "freezes in."

The Boltzmann equation for the evolution of the abundance of a species is a particular form of the Riccati equation, for which there are no general, closed-form solutions. Before we solve the equation by approximate methods, let's consider the qualitative behavior of the solution. The annihilation rate Γ_A varies as n_{EQ} times the thermally-averaged annihilation cross section $\langle\sigma_A|v|\rangle$. In the relativistic regime, $n_{EQ} \sim T^3$, and like other rates, Γ_A will vary as some power of T. In the non-relativistic regime, $n_{EQ} \sim (mT)^{3/2}\exp(-m/T)$, cf. (3.52), so that Γ_A decreases exponentially. In either regime, Γ_A decreases as T decreases, and so eventually annihilations become impotent, roughly when $\Gamma_A \simeq H$, which for definiteness, say occurs for $x = x_f$ ("freeze out"). Thus, we expect that for $x \lesssim x_f$, $Y \simeq Y_{\rm EQ}$, while for $x \gtrsim x_f$ the abundance "freezes in:" $Y(x \gtrsim x_f) = Y_{\rm EQ}(x_f)$.

• *Hot Relics:* First consider the case of a particle species for which $x_f \lesssim 3$. In this case, freeze out occurs when the species is still relativistic and $Y_{\rm EQ}$ is not changing with time. Since $Y_{\rm EQ}$ is constant, the final value of Y is very insensitive to the details of freeze out (i.e., the precise value of x_f), and the asymptotic value of Y, $Y(x \to \infty) \equiv Y_\infty$, is just the equilibrium value at freeze out:

$$Y_\infty = Y_{\rm EQ}(x_f) = 0.278 g_{\rm eff}/g_{*S}(x_f) \quad (x_f \lesssim 3). \tag{5.27}$$

Thus the species freezes out with order unity abundance relative to s (or the number density of photons). Assuming the expansion remains isentropic thereafter (constant entropy per comoving volume), the abundance of ψ's today is (s_0 is the present entropy density)

$$n_{\psi 0} = s_0 Y_\infty = 2970 Y_\infty \ {\rm cm}^{-3} \tag{5.28}$$

$$= 825[g_{\rm eff}/g_*(x_f)] \ {\rm cm}^{-3}. \tag{5.29}$$

If, after freeze out, the entropy per comoving volume of the Universe should increase, say by a factor of γ, the present abundance of ψ's in a comoving volume would be diminished by γ: $Y_\infty = Y(x_f)/\gamma$. In the next Section we will discuss an example of entropy production.

A species that decouples when it is relativistic is often called a *hot relic*. The present relic mass density contributed by a hot relic of mass m is simple to compute:

$$\rho_{\psi 0} = s_0 Y_\infty m = 2.97 \times 10^3 Y_\infty (m/\text{eV})\ \text{eV cm}^{-3} \qquad (5.30)$$

$$\Omega_\psi h^2 = 7.83 \times 10^{-2} [g_{\text{eff}}/g_{*S}(x_f)](m/\text{eV}). \qquad (5.31)$$

Based upon the present age of the Universe we know that $\Omega_0 h^2 \lesssim 1$; applying this bound to the contribution of the species ψ to $\Omega_0 h^2$ we obtain a cosmological bound to the mass of the ψ:

$$m \lesssim 12.8\ \text{eV}[g_{*S}(x_f)/g_{\text{eff}}]. \qquad (5.32)$$

Light (mass $\lesssim$ MeV) neutrinos decouple when $T \sim$ few MeV, and $g_{*S} = g_* = 10.75$. For a single, 2-component neutrino species $g_{\text{eff}} = 2 \times (3/4) = 1.5$, so that $g_{\text{eff}}/g_{*S} = 0.140$. This implies that

$$\Omega_{\nu\bar{\nu}} h^2 = \frac{m_\nu}{91.5\ \text{eV}}, \qquad (5.33)$$

$$m_\nu \lesssim 91.5\ \text{eV}. \qquad (5.34)$$

This cosmological bound to the mass of a stable, light neutrino species is often referred to as the Cowsik-McClelland bound.[7]

Note that the present density of ψ's depends upon $g_{*S}(x_f)$. If a species decouples very early on, when g_{*S} is large, its present number density is proportionally smaller. As an example, consider a species with $g_{\text{eff}} = 1.5$ which decouples at a temperature $T \gtrsim 300$ GeV, when $g_{*S} \simeq g_* \gtrsim 106.75$. For such a species the present contribution to the energy density is

$$\Omega_\psi h^2 = \frac{m}{910\ \text{eV}}, \qquad (5.35)$$

about a factor of 10 less than that of a conventional neutrino species. We see here that a species that decouples when $g_{*S} \gg 1$ has a present abundance much less than that of the microwave photons, and if the species

[7]In their original paper, Coswik and McClelland [6] consider a 4-component neutrino species ($g = 4$), and took $\Omega < 3.8$, $h = 1/2$ and $T_\nu = T$, which resulted in the bound $m \lesssim 8$ eV.

is massless, a temperature much less than the photon temperature, $T_\psi \simeq (3.91/g_{*S})^{1/3}T$. For the latter reason, such a relic is often referred to as a *warm relic.* Examples of possible warm relics include a light gravitino, or a light photino (here, "light" means mass less than about a keV).

• *Cold Relics:* Now consider the more difficult case where freeze out occurs when the species is non-relativistic ($x_f \gtrsim 3$), and $Y_{\rm EQ}$ is decreasing exponentially with x. In this case the precise details of freeze out are important.

It is useful to parameterize the temperature dependence of the annihilation cross section. On general theoretical grounds the annihilation cross section should have the velocity dependence $\sigma_A|v| \propto v^p$, where $p = 0$ corresponds to s-wave annihilation, $p = 2$ to p-wave annihilation, etc. Since $\langle v\rangle \sim T^{1/2}$, $\langle\sigma_A|v|\rangle \propto T^n$, $n = 0$ for s-wave annihilation, $n = 1$ for p-wave annihilation, etc. Therefore we parameterize $\langle\sigma_A|v|\rangle$ as

$$\langle\sigma_A|v|\rangle \equiv \sigma_0(T/m)^n = \sigma_0 x^{-n} \quad (for\ x \gtrsim 3). \tag{5.36}$$

With this parameterization, the Boltzmann equation for the abundance of ψ's becomes,

$$dY/dx = -\lambda x^{-n-2}(Y^2 - Y_{\rm EQ}^2), \tag{5.37}$$

where

$$\begin{aligned}\lambda &= \left[\frac{x\langle\sigma_A|v|\rangle s}{H(m)}\right]_{x=1} = 0.264(g_{*S}/g_*^{1/2})m_{Pl}\, m\, \sigma_0, \\ Y_{\rm EQ} &= 0.145(g/g_{*S})x^{3/2}e^{-x}. \end{aligned}\tag{5.38}$$

As we will now describe, this differential equation can be solved approximately to very good accuracy (better than 5%). To begin, consider the differential equation for $\Delta \equiv Y - Y_{\rm EQ}$, the departure from equilibrium,

$$\Delta' = -Y_{\rm EQ}' - \lambda x^{-n-2}\Delta(2Y_{\rm EQ} + \Delta), \tag{5.39}$$

where prime denotes d/dx. At early times ($1 < x \ll x_f$), Y tracks $Y_{\rm EQ}$ very closely, and both Δ and $|\Delta'|$ are small, so that an approximate solution is obtained by setting $\Delta' = 0$:

$$\begin{aligned}\Delta &\simeq -\lambda^{-1}x^{n+2}Y_{\rm EQ}'/(2Y_{\rm EQ} + \Delta) \\ &\simeq x^{n+2}/2\lambda. \end{aligned}\tag{5.40}$$

At late times ($x \gg x_f$), Y tracks $Y_{\rm EQ}$ very poorly: $\Delta \simeq Y \gg Y_{\rm EQ}$, and the terms involving $Y'_{\rm EQ}$ and $Y_{\rm EQ}$ can be safely neglected, so that

$$\Delta' = -\lambda x^{-n-2}\Delta^2. \tag{5.41}$$

Upon integration of (5.41) from $x = x_f$ to $x = \infty$, we obtain

$$Y_\infty = \Delta_\infty = \frac{n+1}{\lambda} x_f^{n+1}. \tag{5.42}$$

Now we must determine x_f. Recall $x = x_f$ is the time when Y ceases to track Y_{EQ}, or equivalently, when Δ becomes of order Y_{EQ}. Defining x_f by the criterion: $\Delta(x_f) = c\,Y_{\rm EQ}(x_f)$, c = numerical constant of order unity, the early time solution of (5.41) becomes $\Delta(x_f) \simeq x_f^{n+2}/\lambda(2+c)$, and the freeze-out criterion gives

$$x_f \cong \ln\left[(2+c)\lambda ac\right] - \left(n + \frac{1}{2}\right) \ln\left\{\ln\left[(2+c)\lambda ac\right]\right\} \tag{5.43}$$

where $a = 0.145(g/g_{*S})$. Note that x_f depends only logarithmically upon the numerical criterion for freeze out, i.e., the value of c, as does the final abundance. The results of a numerical integration of the Boltzmann equation are shown in Fig. 5.1.

Choosing $c(c+2) = n+1$ gives the best fit to the numerical results for the final abundance Y_∞ (to better than 5% for any $x_f \gtrsim 3$). With this choice

$$\begin{aligned} x_f &= \ln[0.038(n+1)(g/g_*^{1/2})m_{Pl}\, m\, \sigma_0] \\ &\quad - \left(n+\frac{1}{2}\right) \ln\left\{\ln\left[0.038(n+1)(g/g_*^{1/2})m_{Pl}\, m\, \sigma_0\right]\right\} \end{aligned} \tag{5.44}$$

$$Y_\infty = \frac{3.79(n+1)x_f^{n+1}}{(g_{*S}/g_*^{1/2})m_{Pl}\, m\, \sigma_0}. \tag{5.45}$$

We mention in passing that one could have obtained a very similar result to (5.45) by estimating x_f by the freeze-out criterion $\Gamma(x_f) \simeq H(x_f)$, and setting $Y_\infty = Y(x_f)$. The formulae for x_f and Y_∞ obtained this way differ very little; for x_f, the coefficient of the lnln term is $(-n+1/2)$ rather than $(-n-1/2)$, and for Y_∞, a factor of 5 instead of $3.79(n+1)$.

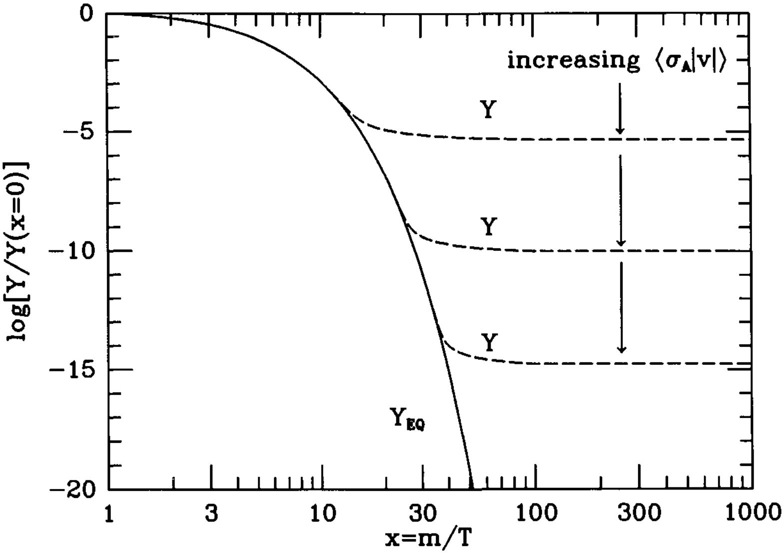

Fig. 5.1: The freeze out of a massive particle species. The dashed line is the actual abundance, and the solid line is the equilibrium abundance.

In some circumstances the annihilation cross section may be better approximated by $\langle\sigma_A|v|\rangle = \sigma_0 x^{-n}(1+bx^{-m})$, e.g., if both s-wave and p-wave annihilation processes are important. The modification to (5.45) is straightforward to compute: $Y_\infty \rightarrow Y_\infty/[1+(n+1)bx_f^{-m}/(m+n+1)]$, and $x_f \rightarrow x_f + \ln[1+b\{\ln(0.038(g/g_*^{1/2})m_{Pl}m\sigma_0)\}^{-m}]$.

As with a hot relic, the present number density and mass density of relic ψ's is easy to compute,

$$n_{\psi 0} = s_0 Y_\infty = 2970 Y_\infty \text{ cm}^{-3}$$

$$= 1.13\times 10^4 \frac{(n+1)x_f^{n+1}}{(g_{*S}/g_*^{1/2})m_{Pl}\, m\, \sigma_0} \text{ cm}^{-3} \qquad (5.46)$$

$$\Omega_\psi h^2 = 1.07\times 10^9 \frac{(n+1)x_f^{n+1} \text{ GeV}^{-1}}{(g_{*S}/g_*^{1/2})m_{Pl}\sigma_0}. \qquad (5.47)$$

It is very interesting to note that the relic density of ψ's is inversely proportional to the annihilation cross section and mass of the particle

$$Y_\infty = \frac{3.79(n+1)(g_*^{1/2}/g_{*S})x_f}{m\, m_{Pl}\langle\sigma_A|v|\rangle}. \tag{5.48}$$

The smaller the annihilation cross section, the greater the relic abundance—the weak prevail. Moreover, the present mass density ($\rho_{\psi 0} \propto mY_\infty$) only depends upon the annihilation cross section at freeze out, which for $n = 0$ (s-wave annihilation) is independent of temperature (and energy).

• *Two Examples of Cold Relics:* Now let's apply this formalism to two simple examples. First, consider a baryon symmetric Universe (not ours!), that is equal numbers of nucleons and antinucleons. Using the above machinery we can calculate the surviving relic abundance of nucleons and antinucleons. Taking the nucleon–antinucleon annihilation cross section to be $\langle\sigma_A|v|\rangle = c_1 m_\pi^{-2}$, where $m_\pi = 135$ MeV is the pion mass and c_1 is a numerical constant of order unity, we find that

$$\begin{aligned} x_f &\simeq 42 + \ln\, c_1, \\ T_f &\simeq 22\ \text{MeV}, \\ Y_\infty &\simeq 7\times 10^{-20} c_1^{-1}. \end{aligned} \tag{5.49}$$

Today, we know that the abundance of nucleons is $Y_\infty = n_B/s \simeq \eta/7 \sim$ (6 to 10) $\times 10^{-11}$, some 9 orders of magnitude larger than the relic abundance of nucleons in a baryon-symmetric Universe. As we will discuss later, this calculation provides strong evidence for the necessity of a baryon asymmetry—to prevent the so-called annihilation catastrophe in a baryon symmetric Universe. The presence of an excess of nucleons over antinucleons, of course, precludes this catastrophe, as all the excess nucleons necessarily survive annihilation. While the formalism developed here does not allow for a ψ–$\bar\psi$ asymmetry, it is straightforward to extend the formalism. In the case of the nucleon–antinucleon asymmetric Universe (like ours), the relic abundance of antinucleons can be computed (i.e., those that survive annihilation); it is

$$Y_\infty(\bar N) = 10^{18}\exp(-9\times 10^5). \tag{5.50}$$

This very, very small number illustrates an interesting feature of freeze out. In the symmetric case, as time goes on the annihilation rate falls exponentially:

$$\Gamma_A = n_{\rm EQ}\langle\sigma_A|v|\rangle \quad \propto \quad (m/T)^{3/2}\exp(-m/T)\langle\sigma_A|v|\rangle, \tag{5.51}$$

and annihilations necessarily quench as particles and antiparticles become very rare.[8] In the case of antinucleons in a baryon asymmetric Universe, the annihilation rate of antinucleons is: $\Gamma_A = n_N\langle\sigma_A|v|\rangle$, which does not decrease exponentially, as the nucleon abundance levels off at the value set by the asymmetry, $n_N \simeq n_B$. We will return to the origin of the all important baryon asymmetry in the next Chapter.

Next, consider the relic abundance of a hypothetical heavy, stable neutrino species of mass $m \gg$ MeV. Because of its large mass, such a neutrino species will decouple when it is non-relativistic (though not necessarily at the canonical $T \sim$ few MeV), and the formulae for a cold relic pertain.

Annihilation for such a species proceeds through Z^0 exchange to final states $i\bar{i}$; where $i = \nu_L, e, \mu, \tau, u, d, s, \cdots$ (ν_L denotes any lighter neutrino species). The annihilation cross section depends upon whether the heavy neutrino is of the Dirac or Majorana type; for $T \lesssim m \lesssim M_Z$, the annihilation cross section is

$$\begin{aligned}
\langle\sigma_A|v|\rangle_{\rm Dirac} &= \frac{G_F^2 m^2}{2\pi}\sum_i(1-z_i^2)^{1/2} \\
&\quad \times[(C_{V_i}^2 + C_{A_i}^2) + \frac{1}{2}z_i^2(C_{V_i}^2 + C_{A_i}^2)] \\
\langle\sigma_A|v|\rangle_{\rm Majorana} &= \frac{G_F^2 m^2}{2\pi}\sum_i(1-z_i^2)^{1/2} \\
&\quad \times[(C_{V_i}^2 + C_{A_i}^2)8\beta_i^2/3 + C_{A_i}^2 2z_i^2],
\end{aligned} \tag{5.52}$$

where $z_i = m_i/m$, β is the relative velocity, and C_V and C_A are given in terms of the weak isospin j_3, the electric charge q, and the Weinberg angle θ_W by $C_A = j_3$, $C_V = j_3 - 2q\sin^2\theta_W$.[9] The sum is over all quark and lepton species lighter than m.

[8] While annihilations cease to significantly affect the abundance of $\psi\bar{\psi}$'s after freeze out, annihilations do occur at a rate per comoving volume per Hubble time proportional to T^r ($r = 1+n$, RD; $r = 1.5+n$, MD)—and may have interesting consequences [7].

[9] We have assumed that the neutrino is less massive than M_Z. If the neutrino is more massive than the Z^0 the annihilation cross section will be $\sigma_A \sim \alpha^2/M^2$, and the calculation of freeze out will be modified.

In the Dirac case, annihilations proceed through the s-wave and $\langle\sigma_A|v|\rangle$ is velocity independent:

$$\sigma_0 \simeq c_2\, G_F^2 m^2/2\pi \tag{5.53}$$

where $c_2 \sim 5$. Taking $g = 2$ and $g_* \simeq 60$, from our formulae for x_f and Y_∞ we find

$$\begin{aligned} x_f &\simeq 15 + 3\ln(m/\mathrm{GeV}) + \ln(c_2/5) \\ Y_\infty &\simeq 6\times 10^{-9}\left(\frac{m}{\mathrm{GeV}}\right)^{-3}\left[1 + \frac{3\ln(m/\mathrm{GeV})}{15} + \frac{\ln(c_2/5)}{15}\right] \end{aligned} \tag{5.54}$$

from which we compute that

$$\Omega_{\nu\bar\nu}h^2 = 3(m/\mathrm{GeV})^{-2}\left[1 + \frac{3\ln(m/\mathrm{GeV})}{15}\right], \tag{5.55}$$

where we have included the identical relic abundance of the antineutrino species ($\Omega_{\nu\bar\nu} = 2\Omega_\nu$). Note that freeze out takes place at $T_F \simeq m/15 \simeq 70$ MeV(m/GeV)—before the interactions of light neutrinos freeze out. This is because as neutrinos annihilate and become rare, the annihilation process quenches. Requiring $\Omega_{\nu\bar\nu}h^2 \lesssim 1$ we obtain the so-called Lee-Weinberg bound:

$$m \gtrsim 2 \text{ GeV}. \tag{5.56}$$

Although it is often called the Lee-Weinberg bound [8], it was discovered independently by a number of people.

For the Majorana case, annihilation proceeds through both the s-wave and p-wave; however the formulae for x_f, Y_∞ and $\Omega_{\nu\bar\nu}h^2$ are similar. In Fig. 5.2 we show the contribution to $\Omega_0 h^2$ for a stable, massive neutrino species. For $m \lesssim$ MeV, $\Omega_{\nu\bar\nu}h^2 \propto m$ as the relic abundance is constant. For $m \gtrsim$ MeV, $\Omega_{\nu\bar\nu}h^2 \propto m^{-2}$ as the relic abundance decreases as m^{-3}. The relic mass density achieves its maximum for $m \sim$ MeV; neutrino masses less than about $92h^2$ eV, or more than about 2 GeV (Dirac) or about 5 GeV (Majorana) are cosmologically acceptable.

The calculation of the relic abundance of some hypothetical, massive stable particle species that was once in thermal equilibrium in the early Universe (*Origin of Species*, if you will) is one of the routine chores of an early-Universe cosmologist. Among the numerous potential relics venerated by cosmologists are hot or warm relics such as a light neutrino,

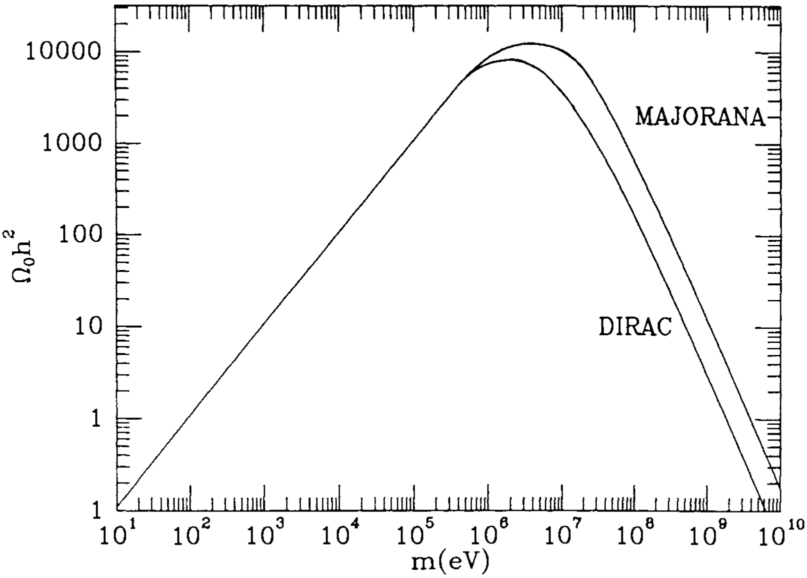

Fig. 5.2: The contribution to $\Omega_0 h^2$ for a stable neutrino species of mass m.

gravitino, photino, right-handed neutrino, etc., or cold relics such as a heavy neutrino, photino, sneutrino, higgsino, pyrgon, etc. In Chapters 7 and 10 we will discuss examples of non-thermal relics, monopoles and axions, respectively. The production of non-thermal relics differs from that of their thermal counterparts in that they were never in thermal equilibrium. In Chapter 9 we will stress the importance of early-Universe relics as candidates for the dark matter.

5.3 Out–of–Equilibrium Decay

We will now consider another kind of non-equilibrium process, the decay of a massive particle species which occurs out of equilibrium (i.e., after the particle has decoupled and has an abundance $Y \gg Y_{\rm EQ}$). As we shall see, such a process can produce considerable entropy.

Consider a non-relativistic and relatively long-lived particle species ψ that is decoupled and has a pre-decay abundance $Y_i = n_\psi/s$, (e.g., a 100 GeV gravitino whose lifetime is about 10^6 sec). Since the ψ is non-relativistic, its contribution to the energy density decreases as R^{-3}, but grows relative to the radiation energy density as R. If it is sufficiently long lived, it decays while dominating the energy density of the Universe, and thereby releases considerable entropy. Let's make a simple estimate of the entropy it releases. Suppose all the decays occur when $t \sim \tau$ (τ is the mean lifetime of the ψ). To make things interesting let us assume that its energy density dominates that of the radiation present before it decays.

The ψ decays at a time $t \sim H^{-1} \sim \tau$, when the temperature of the Universe is $T = T_D$, and the energy density of the Universe is $\rho \sim \rho_\psi = sY_i m$. Just before the ψ's decay T_D and τ are related by:

$$H^2(T_D) \equiv H_D^2 \sim G\rho \sim Y_i T_D^3 m/m_{Pl}^2 \sim \tau^{-2}. \tag{5.57}$$

Suppose that the ψ's decay into relativistic particles that rapidly (compared to the expansion timescale) thermalize, yielding a post-decay radiation density ρ_R. Then after the ψ's decay,

$$\rho_R \sim g_* T_{RH}^4, \tag{5.58}$$

which, by energy conservation, must also equal the energy density in ψ's just before their demise, $H_D^2 m_{Pl}^2$. Comparing (5.57) and (5.58) we find that the ratio of the entropy per comoving volume after decay to that before decay is

$$\frac{S_{\rm after}}{S_{\rm before}} \equiv \frac{g_* R^3 T_{RH}^3}{g_* R^3 T_D^3} \sim g_*^{1/4} \frac{Y_i m \tau^{1/2}}{m_{Pl}^{1/2}}. \tag{5.59}$$

In addition, it appears that the Universe has been heated up by the ψ decays:

$$T_{\rm after}/T_{\rm before} = T_{RH}/T_D = (S_{\rm after}/S_{\rm before})^{1/3}. \tag{5.60}$$

As we will now see, the estimate for the entropy increase is quite accurate; however, the temperature of the Universe never increases—rather, it just decreases more slowly than it would in the absence of ψ decays.

Now let's treat the problem more carefully. First consider the energy density in ψ particles: Due to ψ decays, the number of ψ's in a comoving volume ($\equiv R^3 n_\psi$) decreases according to the usual exponential decay law, $d(R^3 n_\psi)/dt = -\tau^{-1}(R^3 n_\psi)$, which gives

$$\dot{n}_\psi + 3Hn_\psi = -\tau^{-1} n_\psi. \tag{5.61}$$

Since the ψ's are non-relativistic, their energy density is given by $\rho_\psi = m\,n_\psi$, and

$$\dot{\rho}_\psi + 3H\rho_\psi = -\tau^{-1}\rho_\psi. \tag{5.62}$$

The solution is easily found:

$$\rho_\psi(R) = \rho_\psi(R_i)\left(\frac{R}{R_i}\right)^{-3}\exp(-t/\tau). \tag{5.63}$$

Let us assume that the energy released from ψ decays is rapidly (i.e., on a timescale less than the expansion timescale) converted into relativistic particles which are then thermalized. In this case the energy density of the Universe resides in two components: non-relativistic ψ particles and radiation. The decay of ψ's transfers energy from the former to the latter. The second law of thermodynamics applied to a comoving volume element implies that

$$dS = \frac{dQ}{T} = -d(R^3\rho_\psi)/T = \frac{R^3\rho_\psi}{T}(dt/\tau). \tag{5.64}$$

Using the fact that $S = (2\pi^2/45)g_* T^3 R^3$, (5.64) can be rewritten as

$$S^{1/3}\dot{S} = \left(\frac{2\pi^2}{45}g_*\right)^{1/3} R^4\rho_\psi/\tau. \tag{5.65}$$

A formal solution to this equation is easily obtained:

$$S^{4/3} = S_i^{4/3} + \frac{4}{3}\rho_\psi(R_i)R_i^4\tau^{-1}\int_{t_i}^{t}\left(\frac{2\pi^2 g_*}{45}\right)^{1/3}\frac{R(t')}{R_i}\exp(-t'/\tau)dt'. \tag{5.66}$$

Note that in the limit that g_* is constant, the first law of thermodynamics provides a simple means for obtaining the evolution equation for the radiation energy density ρ_R:

$$\begin{aligned} d(R^3\rho_R) &= -p_R d(R^3) - d(R^3\rho_\psi) = -\frac{\rho_R}{3}d(R^3) + (R^3\rho_\psi)dt/\tau \\ \dot{\rho}_R + 4H\rho_R &= \tau^{-1}\rho_\psi. \end{aligned} \tag{5.67}$$

This equation is equivalent to (5.65) if g_* is constant. If g_* is not constant, then p_R is not simply $\rho_R/3$, (5.67) is not valid, and (5.65) must be used. The physics of (5.67) is manifest: The $4H\rho_R$ term represents the usual red shift of the radiation energy density, and the ρ_ψ/τ term accounts for the energy input from ψ decays. In the absence of the ρ_ψ/τ term, the solution to (5.67) is just $\rho_R \propto R^{-4}$.

The expansion rate of the Universe is governed as usual by the Friedmann equation,

$$H^2 = (\dot{R}/R)^2 = \frac{8\pi}{3m_{Pl}^2}(\rho_\psi + \rho_R). \tag{5.68}$$

The energy density in radiation is related to the entropy per comoving volume by

$$\rho_R = \frac{3}{4}\Big(\frac{45}{2\pi^2 g_*}\Big)^{1/3} S^{4/3} R^{-4}. \tag{5.69}$$

Equations (5.68), (5.65) [or (5.67)], and (5.62) form a closed set of differential equations governing the evolution of R, S [or ρ_R], and ρ_ψ.

We will consider their solution in the interesting limit of significant entropy generation: Physically, this means that the ψ's come to dominate the energy density of Universe at some time before $t = \tau$. Suppose that this occurs at time $t = t_\psi$. Roughly speaking then, until the time $t = \tau$ when the energy density in ψ's begins to exponentially decrease, the Universe is matter dominated, and $R \propto t^{2/3}$. During this time $\rho_\psi \simeq \rho_\psi(R_i)(R_i^3/R^3) \propto t^{-2}$, and it is simple to solve (5.67) for the energy density in radiation

$$\rho_R = \rho_R(R_i)\left(\frac{R_i}{R}\right)^4 + \frac{5}{3}\rho_\psi(R_i)\frac{t_i^2}{t\tau}, \tag{5.70}$$

where $\rho(R_i)$ is the energy density at some initial epoch $t = t_i$, and for convenience we have taken t_i to be some time shortly after the ψ's begin to dominate the energy density. The physical significance of this solution is manifest: The first term represents the "primeval radiation" and the second that produced by ψ decays. From this we see that the ψ-produced radiation starts to be the dominant component when $t \simeq t_\psi(\tau/t_\psi)^{3/5}$. This means that until this time $\rho_R \propto R^{-4}$, and thereafter $\rho_R \propto t^{-1} \propto R^{-3/2}$. For $t \gtrsim \tau$, the "source term" ρ_ψ/τ dies away exponentially and ρ_R once again decreases as R^{-4}. Thus we see that ρ_R always *decreases*, albeit at a much slower rate when ψ decays are producing significant energy density. This fact is due to two things: (i) the exponential decay law—ψ's don't suddenly and simultaneously decay at $t = \tau$; (ii) the expansion of the Universe which red shifts the decay-produced radiation energy density ($\propto R^{-4}$) after it is produced. During the time interval from $t \simeq t_\psi(\tau/t_\psi)^{3/5}$ to $t \simeq \tau$, when the ψ produced radiation is the dominant radiation component, the entropy per comoving volume is also growing: $S \propto R^3\rho_R^{3/4} \propto R^{15/8} \propto t^{5/4}$.

Finally, let us compute the total entropy increase due to ψ decays, in the case that ψ's come to dominate the energy density of the Universe well before they decay. In general it follows from (5.67) that the ratio of the final ($t \gg \tau$) to initial ($t \ll \tau$) entropy per comoving volume can be written as

$$\begin{aligned}\frac{S_f}{S_i} &= \left[1+\frac{4}{3}\left(\frac{45}{2\pi^2 g_*(T_i)}\right)^{1/3}\frac{mY_i}{T_i}I\right]^{3/4},\\ I &= \tau^{-1}\int_0^\infty\left(\frac{2\pi^2 g_*}{45}\right)^{1/3}\frac{R(t)}{R_i}\exp(-t/\tau)dt.\end{aligned} \qquad (5.71)$$

The integral I depends upon the functional form of the time dependence of the scale factor $R(t)$. In the case where ψ's dominate the energy density of the Universe before they decay, I can be evaluated numerically:

$$I = 1.09\left(\frac{8\pi\rho_{\psi i}}{3m_{Pl}^2}\right)^{1/3}\tau^{2/3}\left(\frac{2\pi^2}{45}\right)^{1/3}\langle g_*^{1/3}\rangle, \qquad (5.72)$$

where the brackets indicate the appropriately-averaged value of $g_*^{1/3}$ over the decay interval. Bringing together all the numerical factors we find

$$\frac{S_f}{S_i} \simeq 1.83\langle g_*^{1/3}\rangle^{3/4}\frac{m\,Y_i\,\tau^{1/2}}{m_{Pl}^{1/2}}, \qquad (5.73)$$

which differs from our simple-minded estimate by only a numerical factor of order unity. Note that the temperature at the end of the decay epoch follows directly from the Friedmann equation:

$$H^2(t=\tau) \simeq \frac{1}{4}\tau^{-2} \simeq \frac{8\pi}{3m_{Pl}^2}\frac{\pi^2 g_*}{30}T_{RH}^4 \qquad (5.74)$$

$$T_{RH} = T(\tau) \simeq 0.55 g_*^{-1/4}(m_{Pl}/\tau)^{1/2}. \qquad (5.75)$$

And as it should, this estimate for T_{RH} agrees with that obtained from (5.60).

The numerical solution of the system of equations (5.68), (5.65) [or (5.67)], and (5.62) through the epoch of out of equilibrium decay of ψ is shown in Fig. 5.3. The dashed lines indicate the evolution of the energy densities in the absence of ψ decay. Also indicated is the increase in the entropy S due to ψ decay. For this example, $Y_i = 3.2 \times 10^{-5}$,

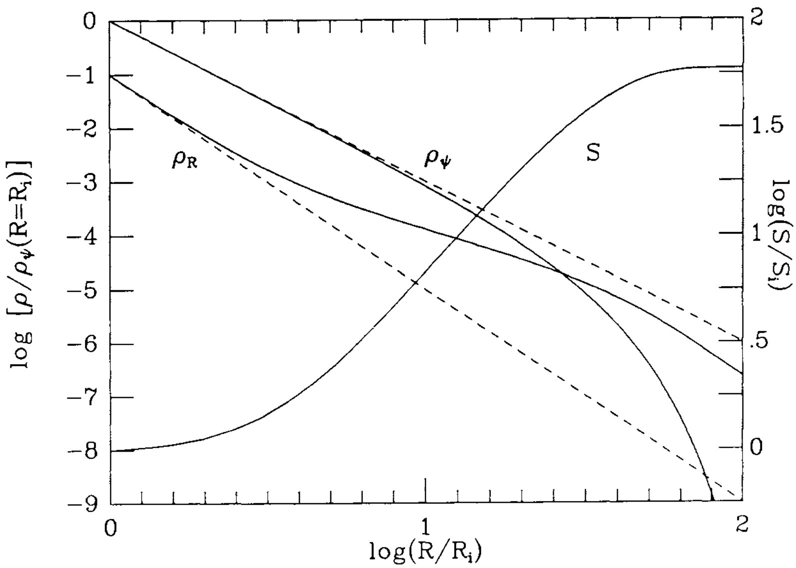

Fig. 5.3: The evolution of the radiation and ψ energy densities and entropy S through the epoch of out-of-equilibrium ψ decay (solid lines). Broken lines indicate the evolution of ρ_R and ρ_ψ in the absence of ψ decays.

$\rho_\psi(R_i)/\rho_R(R_i) = 10$, $H_i\tau = 100$, and $\tau/m_{Pl} = 10^{12}/m^2 g_*^{1/2}$. Equation (5.73) predicts $S_f/S_i = 59$, in excellent agreement with the numerical result shown in Fig. 5.3. Likewise, the analytic expression for the epoch when ψ-produced radiation starts to dominate the primeval radiation, $R/R_i \sim 3$, is also in agreement with the numerical results.

We will return to these equations when discussing the reheating of an inflationary Universe, as they also describe that process; in that case ψ is the scalar field that drives inflation. Before going on, let's consider an example of a hypothetical, long-lived particle that decays out of equilibrium, and produces significant entropy—the gravitino. The gravitino is the supersymmetric partner of the graviton. Its mass and lifetime,

$$m \simeq \mu^2/m_{Pl}, \tag{5.76}$$

$$\tau \simeq m_{Pl}^2/m^3 \simeq 9.8 \times 10^{13} m_{\rm GeV}^{-3}\ \sec \tag{5.77}$$

are determined by the scale of supersymmetry breaking μ. Like the graviton its interactions are so weak that it decouples very early on, while it is still relativistic. Thus we expect its initial abundance to be

$$Y_i = 0.278(g_{\rm eff}/g_{*S}) \sim 10^{-3}. \tag{5.78}$$

Using the formulae just derived, we find that

$$T_{RH} \sim m^{3/2}/m_{Pl}^{1/2} \sim 3\times 10^{-10} m_{\rm GeV}^{3/2}\ {\rm GeV} \tag{5.79}$$

$$S_f/S_i \sim 10^{-3}(m_{Pl}/m)^{1/2} \sim 10^7 m_{\rm GeV}^{-1/2}. \tag{5.80}$$

For the interesting value of the gravitino mass $m \sim 10^2$ to 10^3 GeV, gravitino decays produce an enormous amount of entropy after nucleosynthesis, seemingly excluding such a gravitino mass. This problem can be circumvented if the Universe inflated after gravitinos decoupled; then $Y_i \ll 10^{-3}$, as the only gravitinos present are those produced during the reheating process, a number which is much, much less than the equilibrium value. We will return to gravitinos and inflation again.

5.4 Recombination Revisited

Armed with our Boltzmann technology we will now calculate more precisely the residual ionization of the Universe. Recall from our earlier discussion in Chapter 3, that $X_e = n_p/(n_H + n_p)$ is the ionization fraction, where n_e, n_p and n_H are the number densities of free electrons, free protons and hydrogen atoms, and that for simplicity we have ignored the small abundances of the other light elements. The equilibrium ionization fraction satisfies

$$1 - X_e^{EQ} = (X_e^{EQ})^2 \frac{4\sqrt{2}\zeta(3)}{\sqrt{\pi}} \eta \left(\frac{T}{m_e}\right)^{3/2} \exp(B/T), \tag{5.81}$$

where η is the baryon-to-photon ratio and $B = 13.6$ eV is the binding energy of hydrogen. In the "post-recombination" era, where $X_e^{EQ} \ll 1$, the expression for X_e^{EQ} reduces to

$$X_e^{EQ} \simeq 0.51\eta^{-1/2}\left(\frac{m_e}{T}\right)^{3/4} \exp(-B/2T). \tag{5.82}$$

Following the evolution of the ionization fraction X_e is analogous to following the evolution of the abundance of a stable, massive particle species.

The key reaction there is annihilation ($\psi\bar{\psi} \longleftrightarrow X\bar{X}$), while here the key process is recombination ($e + p \longleftrightarrow H + \gamma$). By inspection we can write the Boltzmann equation for n_e:

$$\dot{n}_e + 3Hn_e = -\langle\sigma_{\rm rec}|v|\rangle \left(n_e^2 - (n_e^{\rm EQ})^2\right), \tag{5.83}$$

where $\langle\sigma_{\rm rec}|v|\rangle$ is the thermally-averaged recombination cross section. This cross section is given by

$$\langle\sigma_n|v|\rangle = \frac{4\pi^2\alpha}{m_e^2}\frac{B/n}{(3m_eT)^{1/2}}, \tag{5.84}$$

where $\langle\sigma_n|v|\rangle$ is the thermally-averaged cross section for $p + e \rightarrow H^*$ (nth excited level), and $E_n = 13.6\ {\rm eV}/n^2$ is the binding energy of the nth level. For simplicity, we will consider only the ground state ($n = 1$), so that

$$\begin{aligned} \langle\sigma_{\rm rec}|v|\rangle &= \frac{4\pi^2\alpha}{m_e^2}\frac{B}{(3m_eT)^{1/2}} \\ &= 4.7 \times 10^{-24}\ {\rm cm}^2 T_{\rm eV}^{-1/2} \end{aligned} \tag{5.85}$$

where $T_{\rm eV} = T/{\rm eV}$.

Taking the Universe to be matter-dominated, and defining $x = T_{\rm eV}^{-1}$, (5.83) can be rewritten as

$$X_e' = -\lambda x^{-2}(X_e^2 - (X_e^{EQ})^2), \tag{5.86}$$

$$\lambda = \left[\frac{n_B\langle\sigma_{\rm rec}|v|\rangle}{xH}\right]_{x=1} = 1.4 \times 10^5(\Omega_B h/\Omega_0^{1/2}), \tag{5.87}$$

$$X_e^{EQ} = 5.95 \times 10^7(\Omega_B h^2)^{-1/2} x^{3/4} \exp(-6.8x). \tag{5.88}$$

In the same way we did for the relic abundance of a massive particle species we can obtain an approximate solution for X_e^∞, the residual ionization fraction. Again, consider $\Delta = X_e - X_e^{EQ}$, the deviation from ionization equilibrium. The equation governing the evolution of Δ is

$$\Delta' = -X_e^{EQ\prime} - \lambda x^{-2}(\Delta + 2X_e^{EQ})\Delta. \tag{5.89}$$

For $x \lesssim x_f$ an approximate solution is obtained by setting $\Delta' = 0$,

$$\Delta \simeq -\frac{X_e^{EQ\prime} x^2}{\lambda(\Delta + 2X_e^{EQ})} \simeq \frac{3.4x^2}{\lambda} \tag{5.90}$$

On the other hand, for $x \gtrsim x_f$, $\Delta \simeq X_e \gg X_e^{EQ}$, $X_e^{EQ\prime}$, so that X_e^{EQ}, $X_e^{EQ\prime}$ can be neglected, and

$$\Delta' \simeq -\lambda x^{-2} \Delta^2. \tag{5.91}$$

Integrating from $x = x_f$ to $x = \infty$ we obtain

$$X_e^{\infty} = \frac{x_f}{\lambda}. \tag{5.92}$$

As before, "freeze out" ($x = x_f$) is found by using the approximate solution for $\Delta(x)$ to determine when the deviation from equilibrium ionization becomes of order unity, $\Delta \simeq X_e^{EQ}$:

$$\frac{3.4}{\lambda} x_f^2 \simeq 5.95 \times 10^7 (\Omega_B h^2)^{-1/2} x_f^{3/4} \exp(-6.8 x_f). \tag{5.93}$$

The value of x at freeze out is

$$\begin{aligned} x_f \;\; \simeq \;\; & \frac{1}{6.8} \ln\left[2.5 \times 10^{12} (\Omega_B/\Omega_0)^{1/2}\right] \\ & - \frac{5/4}{6.8} \ln\left\{\ln\left[2.5 \times 10^{12} (\Omega_B/\Omega_0)^{1/2}\right]\right\} \\ & \simeq 3.6 - 0.074 \ln(\Omega_B/\Omega_0). \end{aligned} \tag{5.94}$$

Taking $\Omega_B/\Omega_0 \sim 0.1$, the freeze in of the residual ionization occurs for $x_f \simeq 3.8$, or

$$T_F = \frac{1}{3.8}\ \mathrm{eV} \simeq 0.26\ \mathrm{eV}, \tag{5.95}$$

and the residual ionization fraction is

$$X_e^{\infty} \simeq \frac{x_f}{\lambda} \simeq 2.7 \times 10^{-5} (\Omega_0^{1/2}/\Omega_B h), \tag{5.96}$$

about comparable to the abundance of D or ^{3}He in the Universe. We note that using the poor man's criterion for freeze out, $(\Gamma_{rec}/H)|_{x_f} \simeq 1$, and setting $X_e^\infty = X_e^{EQ}(x_f)$, we would obtain a numerical factor of 2.9×10^{-5} rather than 2.7×10^{-5}.

5.5 Neutrino Cosmology

The fact that neutrinos decouple relatively early in the evolution of the Universe ($T \gtrsim$ few MeV) guarantees that neutrinos should have a substantial relic abundance today. In turn, this means that they may have important cosmological consequences. Moreover, their cosmological consequences can be used to place very significant constraints on their properties, constraints which go far beyond the reach of conventional terrestrial laboratories.

For a stable neutrino species, we have seen that requiring their relic mass density to be not so great as to lead to a Universe that today is younger than 10 Gyr (i.e., $\Omega_0 h^2 \lesssim 1$) restricts any neutrino mass to be either less than about $92h^2$ eV, or greater than a few GeV. The limit of $92h^2$ eV is quite impressive when compared with the laboratory limits to the mass of the μ and τ neutrinos, 250 keV and 35 MeV respectively.

Now consider the possibility of an unstable neutrino species whose decay products are relativistic, even at the present epoch. It is clear that the mass density bound for such a species must be less stringent: from the epoch at which they decay (say, $z = z_D$) until the present, the mass density of the relativistic neutrino decay products decreases as R^{-4}, as opposed to the R^{-3} had the neutrinos not decayed. Roughly speaking then, the mass density today of the decay products is a factor of $(1+z_D)^{-1}$ less than that of a stable neutrino species.

The precise abundance of the neutrino decay products is very easy to compute. Denote the energy density of the relativistic decay products by ρ_D, and for simplicity assume that they do not thermalize. The equations governing the evolution of the daughter products are essentially the same equations we discussed earlier for a decaying particle species:

$$\begin{aligned} \dot\rho_D &+ 4H\rho_D = \rho_\nu/\tau, \\ \rho_\nu(R) &= \rho_\nu(R_i)\left(\frac{R}{R_i}\right)^{-3}\exp(-t/\tau), \end{aligned} \tag{5.97}$$

where R_i, t_i is some convenient epoch prior to decay, $t_i \ll \tau$. The relic density of the decay products is obtained by integrating (5.97):

$$\rho_D(t) = \rho_{\nu i}\tau^{-1}\left(\frac{R_i}{R}\right)^4 \int_i^t \frac{R(t')}{R_i}\exp(-t'/\tau)dt'. \tag{5.98}$$

Assuming that around the time the neutrinos decay ($t \sim \tau$) the scale factor $R \propto t^n$ ($n = 1/2$ radiation dominated; $n = 2/3$ matter dominated) we can evaluate this integral directly, and find that the present density of relic, relativistic particles from neutrino decays is

$$\rho_D(t_0) = n!\rho_\nu(t_0)\frac{R(\tau)}{R_0} \tag{5.99}$$

where $\rho_\nu(t_0)$ is the present density that neutrinos and antineutrinos would have had they not decayed, and $R(\tau)$ is the value of the scale factor at the time $t = \tau$. As expected, the present energy density of the decay products is less than that of a stable neutrino species, by a factor of $n!R(\tau)/R_0 \sim (1+z_D)^{-1}$.[10] During the matter-dominated epoch ($t \gtrsim 4.4\times 10^{10}(\Omega_0 h^2)^{-2}$ sec), $R(t)/R_0 = 2.9\times 10^{-12}(\Omega_0 h^2)^{1/3}t_{\rm sec}^{2/3}$, so that the reduction factor is

$$n!R(\tau)/R_0 = 2.6\times 10^{-12}(\Omega_0 h^2)^{1/3}\tau_{\rm sec}^{2/3}. \tag{5.100}$$

During the radiation-dominated epoch, $R(t)/R_0 = 2.4 g_*^{-1/12}\times 10^{-10}t_{\rm sec}^{1/2}$, so that the reduction factor is

$$n!R(\tau)/R_0 = 2.1\times 10^{-10}g_*^{-1/12}\tau_{\rm sec}^{1/2}. \tag{5.101}$$

Using the results of our earlier calculations for $\Omega_{\nu\bar\nu}h^2$, we obtain the following constraint to the epoch of decay (for neutrino masses which fall in the previously disallowed range)

$$\begin{aligned}
&m \lesssim 4\times 10^{11}{\rm eV}\,g_*^{1/12}\tau_{\rm sec}^{-1/2} && (\text{light}, \tau \lesssim t_{\rm EQ})\\
&m \lesssim 4\times 10^{13}{\rm eV}(\Omega_0 h^2)^{-1/3}\tau_{\rm sec}^{-2/3} && (\text{light}, \tau \gtrsim t_{\rm EQ})\\
\\
&m \gtrsim 3\times 10^{-5}{\rm GeV}\,g_*^{-1/24}\tau_{\rm sec}^{1/4} && (\text{heavy Dirac}, \tau \lesssim t_{\rm EQ})\\
&m \gtrsim 3\times 10^{-6}{\rm GeV}(\Omega_0 h^2)^{1/6}\tau_{\rm sec}^{1/3} && (\text{heavy Dirac}, \tau \gtrsim t_{\rm EQ})\\
\\
&m \gtrsim 7\times 10^{-5}{\rm GeV}\,g_*^{-1/24}\tau_{\rm sec}^{1/4} && (\text{heavy Majorana}, \tau \lesssim t_{\rm EQ})\\
&m \gtrsim 8\times 10^{-6}{\rm GeV}(\Omega_0 h^2)^{1/6}\tau_{\rm sec}^{1/3} && (\text{heavy Majorana}, \tau \gtrsim t_{\rm EQ})
\end{aligned} \tag{5.102}$$

[10]For reference, $(1/2)! = \sqrt{\pi}/2 \simeq 0.886$ and $(2/3)! \simeq 0.903$.

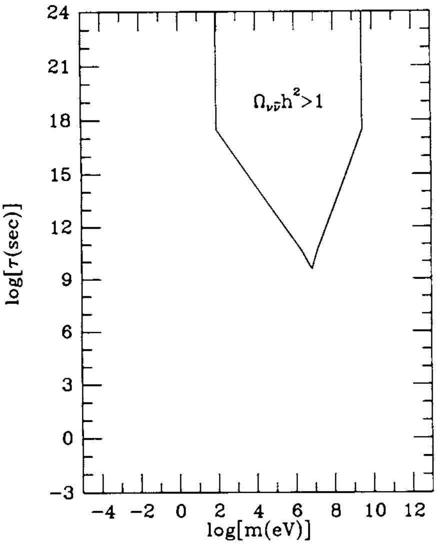

Fig. 5.4: The forbidden region of the neutrino mass–lifetime plane based upon the requirement that $\Omega_{\nu\bar\nu}h^2 \lesssim 1$.

This requirement excludes a region of the neutrino mass–lifetime plane as shown in Fig. 5.4.[11]

The limits just discussed apply irrespective of the nature of the decay products (so long as they are relativistic). If the decay products include "visible" particles, e.g., photons, $e^\pm$ pairs, pions, etc, much more stringent limits can be obtained. We will now consider the additional constraints which apply when the decay products include a photon.[12] The limits that follow depend both qualitatively and quantitatively upon the decay epoch, and we will consider five distinct epochs.

Before discussing these limits, it is useful to calculate the time at which the energy density of the massive neutrino species would dominate the energy density in photons. The energy density in photons is $\rho_\gamma = (\pi^2/15)T^4$, and assuming the neutrinos are NR, their energy density is $\rho_\nu = Y_\infty ms$.

[11]Consideration of the formation of structure in the Universe leads to a significantly more stringent constraint to the mass density of the relativistic decay products; as we shall discuss in Chapter 9, structure cannot grow in a radiation-dominated Universe. For a discussion of these constraints see [12].

[12]For the most part these same limits also apply if the decay products include $e^\pm$ pairs.

Taking $g_{*S} \simeq 4$, the energy densities are equal when $T \simeq 3Y_\infty m$. For heavy neutrinos Y_∞ is given by (5.54), and for light neutrinos, $Y_\infty \simeq 0.04$. Thus we find that the relic neutrino energy density will exceed the photon energy density at $T/m \lesssim 0.1$ for light neutrinos, and $T/m \lesssim 2\times 10^{-8} m_{\rm GeV}^{-3}$ for heavy neutrinos. Using $t \simeq 1$ sec$/T_{\rm MeV}^2$ for the age of the Universe, the epoch of matter domination (by massive neutrinos) is given by[13]

$$t(\mathrm{sec}) \simeq \begin{cases} 10^{14}(m/1\ \mathrm{eV})^{-2} & \text{light neutrinos} \\ 3\times 10^9 m_{\rm GeV}^4 & \text{heavy neutrinos.} \end{cases} \tag{5.103}$$

• $t_U \simeq 3\times 10^{17}\mathrm{sec} \leq \tau$: If the neutrino lifetime is greater than the age of the Universe, neutrinos will still be decaying at the present and decay-produced photons will contribute to the diffuse photon background. Assuming that the neutrinos are unclustered (the most conservative assumption), the differential number flux of decay-produced photons (per cm^2 sr sec erg) is

$$\frac{d\mathcal{F}_\gamma}{dEd\Omega} = \frac{n_\nu c}{4\pi\tau H_0} E^{-1}\left(\frac{E}{m/2}\right)^{3/2} \qquad (E \leq m/2) \tag{5.104}$$

where for simplicity we have assumed that each decay produces one photon of energy $m/2$ and that $\Omega_0 = 1$. Taking the number flux to be $d\mathcal{F}_\gamma/d\Omega \simeq E d\mathcal{F}_\gamma/dEd\Omega$ and $H_0 = 50$ km sec^{-1}Mpc^{-1}, we find

$$\begin{aligned} \frac{d\mathcal{F}_\gamma}{d\Omega} &\simeq 10^{29}\tau_{\rm sec}^{-1}\ \mathrm{cm^{-2}\,sr^{-1}\,sec^{-1}}\ \text{light neutrinos} \\ &\simeq 3\times 10^{22}\tau_{\rm sec}^{-1} m_{\rm GeV}^{-3}\ \mathrm{cm^{-2}\,sr^{-1}\,sec^{-1}}\ \text{heavy neutrinos} \end{aligned} \tag{5.105}$$

A summary of the observations of the diffuse photon background is shown in Fig. 5.5. The differential energy flux, $d\mathcal{F}/dEd\Omega$, is shown as a function of energy and wavelength. From this data, a very rough limit of

$$\frac{d\mathcal{F}_\gamma}{d\Omega} \lesssim \left(\frac{1\ \mathrm{MeV}}{E}\right) \mathrm{cm^{-2}sr^{-1}sec^{-1}}, \tag{5.106}$$

can be placed to the contribution of neutrino decay-produced photons to the photon background. Based upon this, the following lifetime limit

[13]Here, and throughout the following discussion, "light" will refer to neutrinos of mass less than an MeV, and "heavy" will refer to neutrinos of mass greater than an MeV (but less than M_Z).

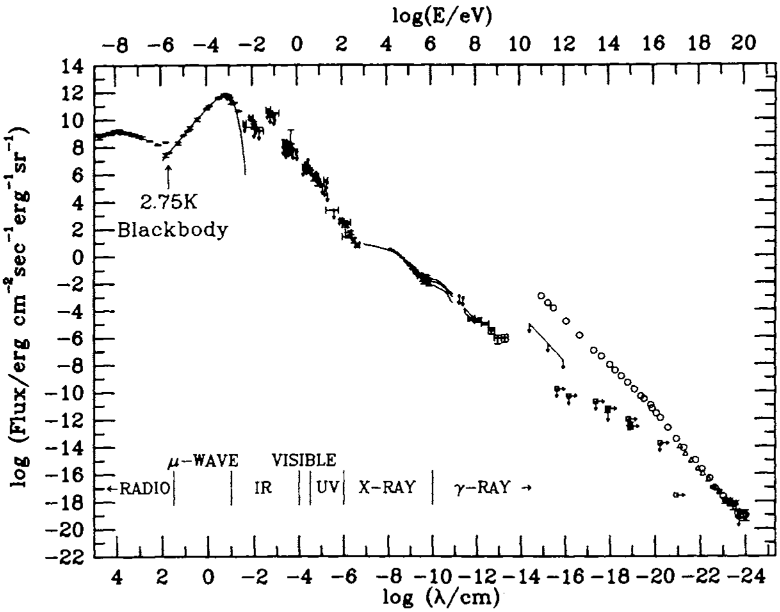

Fig. 5.5: The diffuse photon background for $10^5 \geq \lambda \geq 10^{-24}$ cm. Vertical arrows indicate upper limits, horizontal arrows indicate integrated flux ($> E$). Open circles and triangles indicate the total cosmic ray flux (photons plus hadrons) which places an upper limit to the photon flux (from [17]).

results:

$$\tau_{\rm sec} \geq \begin{cases} 10^{23} m_{\rm eV} & \text{light neutrinos} \\ 10^{25} m_{\rm GeV}^{-2} & \text{heavy neutrinos,} \end{cases} \tag{5.107}$$

applicable for neutrino lifetimes $\tau \gtrsim 3 \times 10^{17}$sec. The forbidden region of the mass–lifetime plane is shown in Fig. 5.6.

• $t_{rec} \simeq 6 \times 10^{12} (\Omega_0 h^2)^{-1/2}$sec $\leq \tau \leq t_U$: If neutrinos decay after recombination, but before the present epoch, then the decay-produced photons will not interact and should appear today in the diffuse photon background. Again, for simplicity assume that each neutrino decay produces one photon of energy $m/2$. Then the present flux of such photons is

$$\begin{aligned} \frac{d\mathcal{F}_\gamma}{d\Omega} &= \frac{n_\nu c}{4\pi} \\ &\simeq 3 \times 10^{11}\ \text{cm}^{-2}\,\text{sr}^{-1}\,\text{sec}^{-1} \qquad \text{light neutrinos} \\ &\simeq 4 \times 10^4\ m_{\rm GeV}^{-3}\ \text{cm}^{-2}\,\text{sr}^{-1}\,\text{sec}^{-1} \qquad \text{heavy neutrinos} \end{aligned} \tag{5.108}$$

where we have assumed that when the neutrino species decays it is non relativistic, so that each decay-produced photon has energy $E \simeq m/2(1+z_D)$ today, where $(1+z_D) \simeq 3.5 \times 10^{11}\,(\Omega_0 h^2)^{-1/3}\tau_{\rm sec}^{-2/3}$. Comparing these flux estimates to our rough estimate of the diffuse background flux we obtain the constraints,

$$\begin{aligned} m &\lesssim 2\times 10^{6}(\Omega_0 h^2)^{-1/3}\tau_{\rm sec}^{-2/3}\,{\rm eV} \qquad \text{light neutrinos} \\ m &\gtrsim 8\times 10^{-3}(\Omega_0 h^2)^{1/6}\tau_{\rm sec}^{1/3}\,{\rm GeV} \qquad \text{heavy neutrinos,} \end{aligned} \tag{5.109}$$

applicable for neutrino lifetimes in the range $3.5\times 10^{11}(\Omega_0 h^2)^{-1/3}{\rm sec} \lesssim \tau \lesssim 3\times 10^{17}{\rm sec}$. For very light neutrino species the assumption that the species decays when it is non relativistic breaks down. If the species decays after $t = t_{therm} \simeq 10^6$ sec and before the present epoch, and is relativisic when it decays, the decay-prduced photons will be comparable in energy and number to the CMBR photons, and will cause significant distortions to the CMBR [17]. Thus a neutrino species that decays while relativistic in the time interval $10^6 \lesssim t \lesssim 3\times 10^{17}$ sec is forbidden. The excluded region is $200 \lesssim (\tau_{\rm sec}/m_{\rm eV}) \lesssim 4\times 10^{20}(\Omega_0 h^2)^{1/3}$, for

$$m_{\rm eV} \lesssim \begin{cases} 3.5\times 10^{8}(\Omega_0 h^2)^{-1/3}\tau_{\rm sec}^{-2/3} & \tau_{\rm sec} \gtrsim 4.4\times 10^{10}(\Omega_0 h^2)^{-2} \\ 4.6\times 10^{6}\tau_{\rm sec}^{-1/2} & \tau_{\rm sec} \lesssim 4.4\times 10^{10}(\Omega_0 h^2)^{-2}. \end{cases} \tag{5.110}$$

The forbidden regions of the mass–lifetime plane are shown in Fig. 5.6.

• $t_{therm} \simeq 10^6{\rm sec} \leq \tau \leq t_{rec}$: For neutrino decays that occur during this epoch, the decay-produced photons can scatter with electrons, which can in turn scatter with CMBR photons, thereby changing the spectral shape of the CMBR. However, during this epoch processes that alter the number of photons in the CMBR, e.g., the double Compton process, $\gamma + e \rightarrow \gamma+\gamma+e$, are not effective (i.e., $\Gamma < H$). Therefore, the result of dumping significant amounts of electromagnetic energy density from neutrino decays is a Bose-Einstein spectrum (with $\mu_\gamma \neq 0$) for the CMBR. As discussed in Chapter 1, the CMBR is to a very good precision a black body. Thus, any electromagnetic energy density resulting from neutrino decays during this epoch must be much less than that in the CMBR itself. Recalling that

$$\begin{aligned} \frac{\rho_\nu}{\rho_\gamma} &= \frac{mY_\infty s}{\rho_\gamma} \\ &\simeq 0.1m/T \qquad \text{light neutrinos} \end{aligned}$$

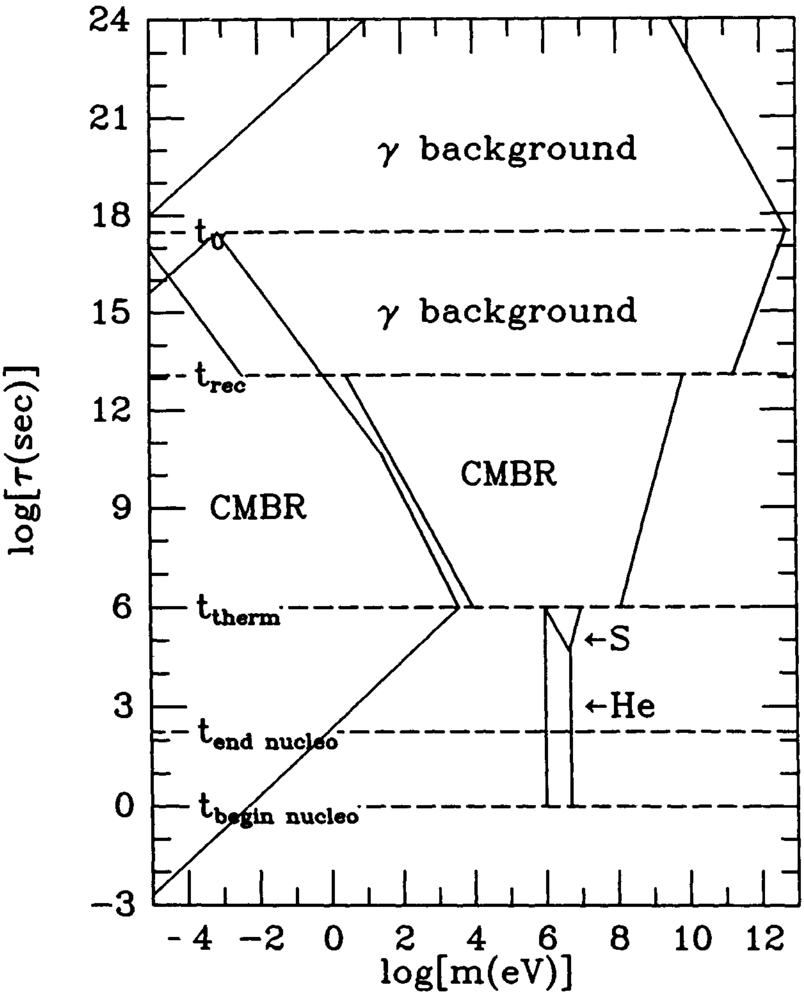

Fig. 5.6: Cosmological limits to the mass and lifetime of an unstable neutrino species that decays radiatively.

$$\frac{\rho_\nu}{\rho_\gamma} \simeq 2\times 10^{-8} m_{\rm GeV}^{-3} m/T \qquad \text{heavy neutrinos,} \tag{5.111}$$

and requiring that $\rho_\nu/\rho_\gamma \lesssim 1$, we obtain the following limits for a neutrino species that decays during this epoch:

$$\begin{aligned} m &\lesssim 10^7 \tau_{\rm sec}^{-1/2}\,\text{eV} \quad \text{light neutrinos} \\ m &\gtrsim 4\times 10^{-3}\,\tau_{\rm sec}^{1/4}\,\text{GeV} \quad \text{heavy neutrinos,} \end{aligned} \tag{5.112}$$

where we have taken $t_{\rm sec} \simeq T_{\rm MeV}^{-1/2}$. These limits are applicable for neutrino lifetimes in the range $10^6\text{sec} \lesssim \tau \lesssim 10^{13}\text{sec}$. The forbidden region of the mass–lifetime plane is shown in Fig. 5.6.[14]

• $t_{end\ nucleo} \simeq 3$ min $\leq \tau \leq t_{therm}$: For neutrino decays that occur during this epoch, the decay-produced photons can be thermalized into the CMBR because both Compton and double Compton scattering are effective ($\Gamma > H$). However, in so doing the entropy per comoving volume is increased (as we discussed in Section 5.3). This has the effect of decreasing further the present value of η relative its value during primordial nucleosynthesis: In the standard scenario, $\eta = 4\eta_{BBN}/11$. As discussed in Chapter 1, luminous matter (necessarily baryons) provides $\Omega_{LUM} \sim 0.01$, and thus provides direct evidence that today $\eta \gtrsim 4\times 10^{-11}$. On the other hand, as discussed in Chapter 4, primordial nucleosynthesis indicates that at the time of nucleosynthesis η corresponded to a present value of (3 to 10) $\times 10^{-10}$. Thus any entropy production after the epoch of nucleosynthesis must be less than a factor of $\sim 10^{-9}/4\times 10^{-11} \sim 30$. Recalling our formula for entropy production by a decaying species we obtain the bound

$$30 \gtrsim S_f/S_i \simeq 1.83 \langle g_*^{1/3}\rangle^{3/4} \frac{mY_\infty \tau^{1/2}}{m_{Pl}^{1/2}}, \tag{5.113}$$

which leads to the limits

$$\begin{aligned} 10^9 &\gtrsim m_{\rm eV}\,\tau_{\rm sec}^{1/2} \quad \text{light neutrinos} \\ 10^7 &\gtrsim m_{\rm GeV}^{-2}\,\tau_{\rm sec}^{1/2} \quad \text{heavy neutrinos,} \end{aligned} \tag{5.114}$$

[14] A neutrino species that decays after nucleosynthesis and produces photons of energy greater than 30 MeV can lead to photofission of the light elements produced during nucleosynthesis. Constraints that follow from this are discussed in the paper by Lindley which is preprinted in Chapter 4 of *Early Universe: Reprints*.

applicable for neutrino lifetimes in the range 200 sec $\lesssim \tau \lesssim 10^6$ sec. This bound too is shown in Fig. 5.6.

• $t_{begin\ nucleo} \simeq 1$ sec $\leq \tau \leq t_{end\ nucleo}$: If the neutrino lifetime is longer than about a sec, then massive neutrinos can contribute significantly to the mass density of the Universe during nucleosynthesis, potentially leading to an increase in ^{4}He production. Recall, only the equivalent of 1 additional neutrino species can be tolerated without overproducing ^{4}He. One additional neutrino species is about equivalent to the energy density contributed by photons. Since the crucial epoch is when the neutron-to-proton ratio freezes out ($t \sim 1$ sec, $T \sim 1$ MeV), the constraint that follows is $(\rho_\nu/\rho_\gamma)_{T\simeq \mathrm{MeV}} \lesssim 1$. This results in the mass limit

$$m \gtrsim 5 \times 10^{-3}\mathrm{GeV} \quad \text{heavy neutrinos.} \tag{5.115}$$

Note there is no corresponding limit for a light species because a light species is just one additional relativistic neutrino species. This limit, which is applicable to a heavy neutrino species with lifetime greater than about 1 sec, is shown in Fig. 5.6.

• $\tau \ll 1$ sec: A neutrino species that decays earlier than about 1 sec after the bang disappears without leaving much of a cosmological trace. Its decay products thermalize before primordial nucleosynthesis, and its only effect is to increase the entropy per comoving volume. If we understood the origin of the baryon-to-entropy ratio in great detail, and could predict its "pre-nucleosynthesis" value, then we could use entropy production by the decaying neutrino species to obtain constraints for very short lifetimes.

• *Astrophysical Implications:* Neutrino decay into visible modes can have "astrophysical" effects too. As the detection of neutrinos from SN 1987A dramatically demonstrated, type II supernovae are a copious source of neutrinos. The integrated flux of neutrino-decay-produced photons from type II supernovae that have occurred throughout the history of the Universe can be used to obtain a very stringent bound to acceptable neutrino masses and lifetimes.

Each type II supernova releases about 3×10^{53} ergs of energy in thermal neutrinos with average energy about 12 MeV—or about $N_{\nu\bar{\nu}} \simeq 5 \times 10^{57}$ neutrinos and antineutrinos of each species. The historical (last 1000 yr) type II rate in our own galaxy is about 1 per 30 yr (give or take a factor of 3), and the observed extragalactic rate is roughly $1.1h^2$per 100 yr per $10^{10} L_{B\odot}$ [15]. Using the measured mean blue luminosity density of the Universe, $\mathcal{L}_{B\odot} \sim 2.4h \times 10^8 L_{B\odot}\,\mathrm{Mpc}^{-3}$, this translates into a present type II rate (per volume) of $\Gamma_{SN} \simeq 2.5h^3 \times 10^{-85}\,\mathrm{cm}^{-3}\,\mathrm{sec}^{-1}$. Assuming that the type

II rate has been constant over the history of the Universe, the differential photon number flux from the decay of supernova-produced neutrinos is

$$\frac{d\mathcal{F}_\gamma}{d\Omega dE} = \frac{9}{5\sqrt{2}} \frac{\Gamma_{SN} t_U^2 N_{\nu\bar{\nu}}}{4\pi \langle E_\nu \rangle \tau / m} \frac{1}{\langle E_\nu \rangle^{1/2} E^{1/2}}. \tag{5.116}$$

For simplicity we have assumed that the supernovae neutrinos are mono-energetic: $E_\nu = \langle E_\nu \rangle \simeq 12$ MeV, that each decay-produced photon carries half the energy of the parent neutrino, and a flat Universe. Comparing the expected photon number flux at energy $\langle E_\gamma \rangle = \langle E_\nu \rangle / 6 \simeq 2$ MeV,

$$\langle E_\gamma \rangle \frac{d\mathcal{F}_\gamma}{d\Omega dE} \simeq \frac{1}{2} \frac{\Gamma_{SN} t_U^2 N_{\nu\bar{\nu}} m}{4\pi \langle E_\nu \rangle \tau}, \tag{5.117}$$

with the measured diffuse γ-ray flux at a few MeV, $3 \times 10^{-3} \mathrm{cm}^{-2}\,\mathrm{sr}^{-1}\,\mathrm{s}^{-1}$, we obtain the following constraint:

$$\tau_{\rm sec} \gtrsim 5 \times 10^{12} (\Gamma_{SN} / 3 \times 10^{-85} \mathrm{cm}^{-3} \mathrm{sec}^{-1}) m_{\rm eV}. \tag{5.118}$$

This bound applies to neutrino species light enough to be produced in supernovae ($m \lesssim 10$ MeV) and which decay outside the envelope of the exploding star ($\tau_{\rm sec} \gtrsim 10^{-5} m_{\rm eV}$).[15] This constraint is shown in Fig. 5.7.[16]

For a neutrino species that decays within the envelope of the exploding star, and thereby deposits energy in the envelope a different bound can be derived. Any energy deposited by neutrino decays in the envelope will be thermalized and radiated in the visible part of the spectrum. The energy radiated by SN 1987A in the visible was only about 10^{47} ergs, while each neutrino species carries off about 10^{53} ergs! The energy which is deposited in the envelope by a hypothetical unstable neutrino species is

$$E_{DEP} \simeq N_{\nu\bar{\nu}} \langle E_\nu \rangle \min[R_{BSG} / \tau_{LAB}, 1] \tag{5.119}$$

$$\simeq \min[10^{48} m_{\rm eV} / \tau_{\rm sec}\, \mathrm{ergs},\ 10^{53}\, \mathrm{ergs}] \tag{5.120}$$

where $R_{BSG} \sim 3 \times 10^{12}$ cm is the radius of the envelope of the progenitor blue super giant (Sanduleak −69 202, by name), and $\tau_{LAB} = \langle E_\nu \rangle \tau / m$ is

[15] In deriving (5.116) we have assumed the lab lifetime, $\tau_{LAB} \simeq \langle E_\nu \rangle \tau / m$, is larger than the age of the Universe ($\tau \gtrsim 10^{11} m_{\rm eV}$ sec). If not, $d\mathcal{F}_\gamma / d\Omega \simeq \Gamma_{SN} t_U N_{\nu\bar{\nu}} / 8\pi \simeq 1$ $\mathrm{cm}^{-2}\mathrm{sr}^{-1}\mathrm{s}^{-1}$—which is ruled out, independent of m and τ.

[16] Based upon γ-ray observations of SN 1987A made by the SMM spacecraft a similar, more direct, and slightly more restrictive bound obtains [15].

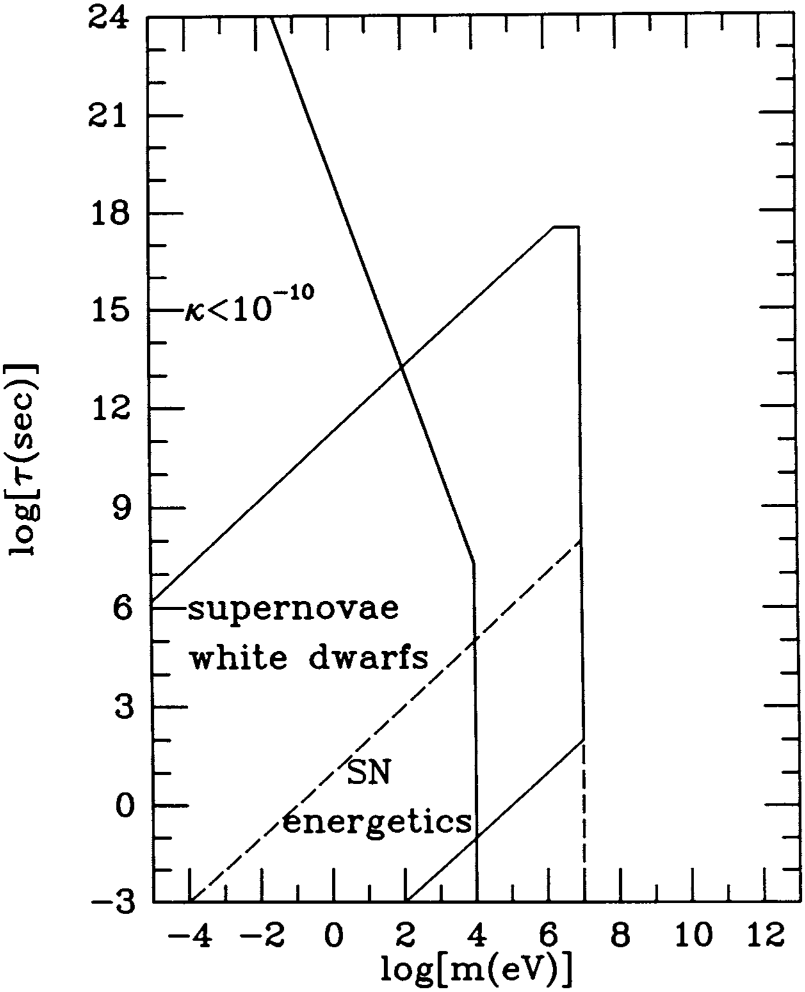

Fig. 5.7: Astrophysical limits to the mass and lifetime of an unstable neutrino that decays radiatively.

the neutrino lifetime in the rest frame of the supernova. Comparing this to the observed energy of 10^{47} ergs, we obtain the bounds

$$\begin{array}{ll} m_{\rm eV}/\tau_{\rm sec} \lesssim 0.1 & (\tau_{\rm sec} \gtrsim 10^{-5} m_{\rm eV}) \\ m_{\rm eV} \gtrsim 10^{7} & (\tau_{\rm sec} \lesssim 10^{-5} m_{\rm eV}) \end{array} \qquad (5.121)$$

This constraint too is shown in Fig. 5.7.

A neutrino species that can decay radiatively, $\nu_j \longrightarrow \nu_i + \gamma$, necessarily has an electromagnetic coupling that can be quantified as a transition magnetic moment, $\mu_{ij} = \kappa_{ij}(e/2m_e)$. The transition magnetic moment and neutrino mass and lifetime are related by

$$\begin{aligned} \tau^{-1} &= \alpha_{EM} \kappa^2 m^3 / 8 m_e^2, \\ \kappa &= 0.44 \tau_{\rm sec}^{-1/2} m_{\rm eV}^{-3/2}, \end{aligned} \qquad (5.122)$$

where we have assumed $m_j \gg m_i$. The transition moment leads to an electromagnetic correction to ν–e scattering. Laboratory limits to ν–e scattering through the transition moment leads to the bound $\kappa_{e\mu} \lesssim 10^{-8}$, or

$$\tau_{\rm sec} \gtrsim 2 \times 10^{15} m_{\rm eV}^{-3} \qquad (\nu_\mu \rightarrow \nu_e + \gamma). \qquad (5.123)$$

Further, such a transition moment leads to neutrino pair emission from white dwarfs and red giants through the process *plasmon* $\rightarrow \nu_i \nu_j$. For $\kappa \sim 10^{-10}$ to 10^{-11} plasmon $\nu\bar{\nu}$ emission can be a very significant cooling mechanism for these objects, and can effect their evolution. Based upon this, a limit of $\kappa_{ij} \lesssim 10^{-10}$ or so has been derived for $m \lesssim 10$ keV (see, e.g., the paper of Beg, Marciano, and Ruderman, in [15]). This translates to the limit

$$\tau_{\rm sec} \gtrsim 2 \times 10^{19} m_{\rm eV}^{-3} \qquad (m \lesssim 10 \text{ keV}). \qquad (5.124)$$

All of the astrophysical and cosmological constraints just discussed are summarized in Figs. 5.6 and 5.7. These constraints serve to illustrate how a large variety of cosmological and astrophysical observations can be used to probe particle properties in regimes beyond the reach of the terrestrial laboratory. We return in Chapter 10 to discuss another such very interesting example: the axion.

5.6 Concluding Remarks

In the first part of this Chapter we have focused on the treatment of non-equilibrium processes in the expanding Universe. As we have stressed, and will continue to stress, departures from equilibrium are responsible for most of the cosmological relics—without such departures the past history of the Universe would be irrelevant. The Boltzmann equation is the primary theoretical tool for dealing with non-equilibrium in the expanding Universe, and we have used it to consider recombination, the freeze out of the relic abundance of a massive particle species, and entropy production by the out-of-equilibrium decay of a massive relic.

Of particular interest was the relic abundance of a massive particle species. As we discussed in Chapters 1 and 4, most of the mass density in the Universe is dark, and if $\Omega_0 \gtrsim 0.15$, the dark matter is unlikely to be baryonic. The prime suspects in that case are relic particles from the early history of the Universe—e.g., a light neutrino species of mass $92h^2$ eV, or a heavy neutrino species of mass a few GeV would provide $\Omega_0 \simeq 1$. As we will discuss in Chapter 9, there are almost countless other candidates—axions, photinos, higgsinos, superheavy monopoles, etc., and most of the candidates arise as thermal relics. In Chapters 7 and 10, we will discuss the other class of relics, non-thermal relics; they include the axion, the monopole, and cosmic string.

"Cold relics" have the interesting property that their present abundance is inversely proportional to their mass and annihilation cross section. Consider a hypothethical species X, with mass m, 2 degrees of freedom, and an annihilation cross section, $\langle\sigma_A|v|\rangle = a \times 10^{-37}$ cm^2 (i.e., $n = 0$). It follows from our discussion of freeze out that

$$
\begin{aligned}
x_f &= 18 + \ln[a(m/\mathrm{GeV})] \\
Y_\infty &= \frac{2.7 \times 10^{-9}}{a(m/\mathrm{GeV})} \\
\Omega_X h^2 &= \frac{0.043 x_f}{a} \simeq \frac{0.77}{a}
\end{aligned}
\qquad (5.125)
$$

where we have taken $g_* = 60$. Note that up to logarithmic factors, i.e., $x_f \sim 18 + \ln\cdots$, $\Omega_X h^2$ is determined by the annihilation cross section, which is parameterized by a. It is also apparent that for $\Omega_X h^2$ to be of order unity, the annihilation cross section must be characteristic of a weak process. It is now clear why a heavy neutrino, and many of the supersymmetric partners, are prime dark matter candidates—all interact (with ordinary matter) with roughly weak strength.[16]

Because all interactions of relic dark matter particles with ordinary matter—annihilation into ordinary matter, pair production, elastic scattering, etc.—are set by the annihilation cross section (and its scaling with energy or temperature), the fact that $\Omega_X h^2 \sim 1$ fixes the annihilation cross section to be order 10^{-36} to 10^{-37} cm^2 has a multitude of implications. The production of dark matter particles at accelerators, or their direct detection necessarily involves "weak processes." There have been a variety of interesting proposals for the detection of relic dark matter particles—observing γ-rays from their annihilations in the galactic halo, observing high energy neutrinos from the annihilations of relic dark matter particles that are captured by and accumulate in the sun, and detecting the tiny (order keV) energy they deposit when they scatter with ordinary matter in a cryogenic, bolometric detector. All of the relevant cross sections are set by $\Omega_X h^2$—and fortunately for us, a weak cross section appears large enough to make the aforementioned detection schemes feasible.

5.7 References

Rules and conventions for calculating matrix elements, as well as matrix elements for some of the processes considered in this Chapter can be found in many references, e.g.,

1. C. Quigg, *Gauge Theories of the Strong, Weak, and Electromagnetic Interactions* (Benjamin/Cummings, Menlo Park, Calif. 1983); J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).

Statistical mechanics in the expanding Universe is also discussed in

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