CFT and gravity, problem set 1

This problem set corresponds to lectures 1 and 2. The exercises are likely too many for one session (or even two). Feel free to pick the few exercises you like the most. Also, feel free to use mathematica (or equivalent) as much as possible.

Exercise 1 ***Conformal and Weyl invariance of a free massless scalar field***

a) *Verify that the action*

$$S[\varphi] = \int d^d x \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi \tag{1}$$

describing the Gaussian fixed point is conformal invariant. In other words, show that

$$S[\varphi'] = S[\varphi] \ , \qquad \qquad \varphi'(x) = \left| \frac{\partial x'}{\partial x} \right|^{\Delta/d} \varphi(x') \tag{2}$$

where $x \to x'$ is a conformal transformation and $\Delta = \frac{d-2}{2}$. Hint: the only non-trivial transformation to check is the special conformal transformation.

b) *Now consider the action for a free scalar field in a curved background*

$$S[g_{\mu\nu}, \varphi] = \int d^d x \sqrt{g} \left(\frac{1}{2} g^{\mu\nu} \nabla_\mu \varphi \nabla_\mu \varphi + \frac{1}{2} \xi R \varphi^2 \right) \tag{3}$$

where R is the Ricci scalar of the metric $g^{\mu\nu}$ and ξ is a dimensionless coupling constant. Show that the action is Weyl invariant

$$S[\Omega^2 g_{\mu\nu}, \Omega^{-\Delta} \varphi] = S[g_{\mu\nu}, \varphi] \tag{4}$$

for $\Delta = \frac{d-2}{2}$ and $\xi = \frac{d-2}{4(d-1)}$. Hint: you will need to use the following formula

$$\tilde{R} = \Omega^{-2} \left[R - 2(d-1) g^{\mu\nu} \nabla_\mu \nabla_\nu \log \Omega - (d-2)(d-1) g^{\mu\nu} (\nabla_\mu \log \Omega)(\nabla_\nu \log \Omega) \right] \tag{5}$$

where $\tilde{R}$ is the Ricci scalar for the metric $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$. You can find the derivation of this formula in appendix D of Wald's book on General Relativity.

Obtain the stress-energy tensor for this theory and check that it is traceless and conserved on-shell (i.e., if φ solves the classical equations of motion).

Exercise 2 ***Conformal algebra and unitary bounds*** *The generators of the conformal algebra can be represented as follows*

$$\hat{P}_\mu = -i\partial_\mu \ , \qquad \hat{L}_{\mu\nu} = -i(x_\mu\partial_\nu - x_\nu\partial_\mu) \tag{6}$$
$$\hat{D} = -\, x\cdot\partial \ , \qquad \hat{K}_\mu = i(2x_\mu\, x\cdot\partial - x^2\partial_\mu)$$

a) *Show that the generators obey the following commutation relations*

$$[D, P_\mu] = P_\mu \ , \qquad [D, K_\mu] = -K_\mu \ , \qquad [K_\mu, P_\nu] = 2\delta_{\mu\nu}D - 2i\, M_{\mu\nu} \ ,$$
$$[M_{\mu\nu}, P_\alpha] = i\left(\delta_{\mu\alpha}P_\nu - \delta_{\nu\alpha}P_\mu\right) \ , \qquad [M_{\mu\nu}, K_\alpha] = i\left(\delta_{\mu\alpha}K_\nu - \delta_{\nu\alpha}K_\mu\right) \ ,$$
$$[M_{\alpha\beta}, M_{\mu\nu}] = i\left(\delta_{\alpha\mu}M_{\beta\nu} + \delta_{\beta\nu}M_{\alpha\mu} - \delta_{\beta\mu}M_{\alpha\nu} - \delta_{\alpha\nu}M_{\beta\mu}\right) \ . \tag{7}$$

b) *In a unitary representation, there is a positive definite inner product such that*

$$\hat{D}^\dagger = \hat{D} \ , \qquad \hat{K}_\mu^\dagger = \hat{P}_\mu \ , \qquad \hat{L}_{\mu\nu}^\dagger = \hat{L}_{\mu\nu} \ . \tag{8}$$

Show that unitarity implies that the dimension (or eigenvalue of $\hat{D}$) of a scalar primary state $|\mathcal{O}\rangle$ can not be lower than $\frac{d-2}{2}$, and that the bound is saturated by a state created by a free massless scalar field (obeying the equation of motion $\partial^2\mathcal{O}(x) = 0$).

c) *Show that a vector primary state $|\mathcal{O}^\alpha\rangle$ contained in a unitary representation must have dimension larger or equal to $d-1$. Show that when the bound is saturated, the state is created by a conserved current. Recall that for a spin 1 state,*

$$\hat{L}_{\mu\nu}|\mathcal{O}^\alpha\rangle = (M_{\mu\nu})^\alpha{}_\beta|\mathcal{O}^\beta\rangle \ , \qquad (M_{\mu\nu})^\alpha{}_\beta = i(\eta_{\nu\beta}\delta^\alpha_\mu - \eta_{\mu\beta}\delta^\alpha_\nu) \ . \tag{9}$$

Hint: Compute the norm of $P_\mu|\mathcal{O}^\mu\rangle$.

d) *Verify that the operator*

$$\hat{C} = \hat{D}^2 - \frac{1}{2}\left(\hat{K}_\mu\hat{P}^\mu + \hat{P}_\mu\hat{K}^\mu\right) + \frac{1}{2}\hat{L}_{\mu\nu}\hat{L}^{\mu\nu} \tag{10}$$

is a Casimir of the conformal algebra (i.e. it commutes with all its generators). Determine its value for a scalar and a vector primary state.

e) *Generalize questions **c)** and **d)** for symmetric traceless primary states $|\mathcal{O}^{\alpha_1\ldots\alpha_l}\rangle$. Recall that for spin l states,*

$$\hat{L}_{\mu\nu}|\mathcal{O}^{\alpha_1\ldots\alpha_l}\rangle = \sum_{i=1}^{l}(M_{\mu\nu})^{\alpha_i}{}_\beta|\mathcal{O}^{\alpha_1\ldots\alpha_{i-1}\beta\alpha_{i+1}\ldots\alpha_l}\rangle \ . \tag{11}$$

You should find that the dimension of such a state in a unitary theory must be greater or equal to $d-2+l$.

CFT and gravity, problem set 2

Here's a user guide, please pick your favorite: Exercises 1 and 2 should get you more acquainted with large N QFTs. Exercise 3 is very short, except for the bonus, and leads you to a nice coordinate system for AdS, which we'll use tomorrow (the bonus is for those among you who like changes of coordinates). Exercise 4 goes back to a basic feature of CFTs which we did not discuss in depth: the Operator product expansion. It's quite fun, I promise.

Exercise 1 ***The Euler characteristic.***

The Euler characteristic is defined as

$$\chi = F - E + V, \tag{1}$$

where F, E, V *are faces, edges and vertices of a polygonal surface.*

a) *Give an argument for the invariance of* χ *under continuous deformations of a surface. Such deformations can (i) distort the surface without changing the number of faces, edges and vertices, or (ii) shrink edges or faces to a point, turn a face into an edge etc. Find a set of elementary moves and show that those do not change* χ*.*

b) *Prove that if the polygonal surface is topologically equivalent to a sphere with* B *holes cut out and* H *handles stuck on it, then*

$$\chi = 2 - 2H - B. \tag{2}$$

Hint: *Start by a sphere and open holes or add handles to it.*

Exercise 2 ***The large*** N ***limit of a vector model***

Consider a massless field in the fundamental representation of $O(N)$, ϕ^i, $i = 1, \dots N$, *with an interaction term*

$$\mathcal{L}_{\text{int}} = \frac{\lambda}{24N}(\phi^i\phi^i)^2. \tag{3}$$

Consider the limit of large N *at fixed* λ*.*

a) *Which class of Feynman diagrams determines the two point function of* $\sigma = \phi^i\phi^i/\sqrt{N}$ *at leading order in* $1/N$*? Apply the normal ordering prescription: fields at the same point are never contracted.*

b) *Can you resum this class of diagrams and compute the two point function of* σ*? The computation is easier in momentum space. You will need to evaluate a loop integral. Perform the computation in a generic dimension* d*.*

c) *Consider now the IR limit of the correlator when* $2 < d < 4$, i.e. *the small momentum limit. The theory flows to a fixed point. What is the scaling dimension of* σ*? Is it the same as the dimension at the UV fixed point? (Notice that, in momentum space, constant terms or terms polynomial in the momentum correspond to contact terms and do not affect the correlation function at separated points).*

d) *Prove that the scaling dimensions of* ϕ^i *at leading order in* $1/N$ *equals the free field value.*

Exercise 3 *Global coordinates for Euclidean AdS are expressed as follows:*

$$\begin{aligned} X^0 &= R\cosh\tau\cosh\rho \\ X^\mu &= R\,\Omega^\mu\sinh\rho \\ X^{d+1} &= -R\sinh\tau\cosh\rho \end{aligned} \tag{4}$$

where Ω^μ $(\mu = 1, \ldots, d)$ *parametrizes a unit* $(d-1)-$*dimensional sphere. With the additional change of variable* $\tanh\rho = \sin r$*, the metric becomes*

$$ds^2 = \frac{R^2}{\cos^2 r}\left[d\tau^2 + dr^2 + \sin^2 r\, d\Omega_{d-1}^2\right] \ . \tag{5}$$

What is this space? What is its conformal boundary? Draw a picture of it!

Bonus*: compute the generators of the Euclidean conformal group (= isometries of AdS) in global coordinates:*

$$D = -iJ_{0,d+1}\ , \qquad P_\mu = J_{\mu 0} - J_{\mu,d+1}\ , \tag{6}$$

$$J_{\mu\nu} = J_{\mu\nu}\ , \qquad K_\mu = J_{\mu 0} + J_{\mu,d+1}\ , \tag{7}$$

with

$$J_{MN} = -i\left(X_M\frac{\partial}{\partial X^N} - X_N\frac{\partial}{\partial X^M}\right) . \tag{8}$$

Show that, in global coordinates, the conformal generators take the form

$$\begin{aligned} D &= -\frac{\partial}{\partial\tau}\ , \qquad J_{\mu\nu} = -i\left(\Omega_\mu\frac{\partial}{\partial\Omega^\nu} - \Omega_\nu\frac{\partial}{\partial\Omega^\mu}\right)\ , \\ P_\mu &= -ie^{-\tau}\left[\Omega_\mu\left(\partial_\rho + \tanh\rho\,\partial_\tau\right) + \frac{1}{\tanh\rho}\nabla_\mu\right]\ , \\ K_\mu &= ie^{\tau}\left[\Omega_\mu\left(-\partial_\rho + \tanh\rho\,\partial_\tau\right) - \frac{1}{\tanh\rho}\nabla_\mu\right]\ , \end{aligned}$$

where $\nabla_\mu = \frac{\partial}{\partial\Omega^\mu} - \Omega_\mu\Omega^\nu\frac{\partial}{\partial\Omega^\nu}$ *is the covariant derivative on the unit sphere* S^{d-1}.

Exercise 4 ***OPE in free theory and characters***

Since you learned about the operator product expansion (OPE) in a nice student talk, you can have fun solving this.

Consider a free scalar field in dimension d.

a) *Compute the following OPEs:*

$$\phi(x)\phi(y) \sim \, ? \tag{9}$$

$$\phi^2(x) \times \phi^2(y) \sim \, ? \tag{10}$$

The right hand side should be a sum of operators evaluated at one of the points, say:

$$\phi(x)\phi(y) \sim \frac{1}{(x-y)^{d-2}} + \phi(y)^2 + (x-y)^\mu \phi(x)\partial_\mu\phi(x) + \dots \tag{11}$$

Hint: use Wick theorem and notice that normal ordered products of coincident fields are non singular. Do not try to organize the OPE in terms of primaries, just obtain the expansion in terms of generic scaling operators of the schematic form

$$\phi^m \partial^n \phi^k \ . \tag{12}$$

b) *Consider the right hand side of the OPE $\phi \times \phi$. You should have gotten one operator per scaling dimension and spin. But this is a linear combination of primaries and descendants. How many primaries are there for each scaling dimension and spin? One way to count is to use characters. Consider the Hilbert space created by all the operators of the form*

$$\partial_{\mu_1} \dots \partial_{\mu_m} \phi \, \partial_{\nu_1} \dots \partial_{\nu_n} \phi \ . \tag{13}$$

In order to count them, start by showing that the following is true

$$\chi_1(q) \equiv \mathrm{Tr}_{H_1} q^D = \frac{q^{\frac{d-2}{2}}}{(1-q)^d} - \frac{q^{\frac{d-2}{2}+2}}{(1-q)^d} \ , \tag{14}$$

where H_1 is the space obtained by acting on the vacuum with the descendants of a single field ϕ. Hint: *recall that the equations of motion imply that some combinations of derivatives form null states and should be subtracted away.*

Now, argue that the character $\chi_2(q)$ for the Hilbert space H_2 produced by the operators (13) *is as follows:*

$$\chi_2(q) \equiv \mathrm{Tr}_{H_2} q^D = \frac{1}{2}\left(\chi_1(q)^2 + \chi_1(q^2)\right) \tag{15}$$

The coefficient of each power of q in χ_2 is the number of scaling operators of the kind (13). They are either primaries or descendants. Assume that the only primaries among the (13) are traceless symmetric with even spin ℓ and dimension $\Delta = d - 2 + \ell$ (why is this true?). Using the fact that such primaries are all conserved, compute their character:

$$\chi_\ell(q) = c(\ell)\frac{q^{d-2+\ell}}{(1-q)^d} - c(\ell-1)\frac{q^{d-1+\ell}}{(1-q)^d} \ , \qquad c(\ell) = \binom{\ell+d-1}{d-1} - \binom{\ell+d-3}{d-1} \ . \tag{16}$$

Finally, plug the whole thing in mathematica and check, for some choice of the dimension d and up to some order in q, that the following equality holds:

$$\chi_2(q) = \sum_{n=0}^{\infty} \chi_{2\ell}(q) \ . \tag{17}$$

So, how many primaries per spin are there in the $\phi \times \phi$ OPE?

CFT and gravity, problem set 3

Exercise 1 ***Free boson propagator in Euclidean AdS.*** *Consider a free scalar field with action*

$$S=\int_{AdS} dX\left[\frac{1}{2}(\nabla\phi)^2+\frac{1}{2}m^2\phi^2\right] . \tag{1}$$

The two-point function $\langle\phi(X)\phi(Y)\rangle$ *is given by the propagator* $\Pi(X,Y)$, *which obeys*

$$\left[\nabla_X^2-m^2\right]\Pi(X,Y)=-\delta(X,Y) . \tag{2}$$

From the symmetry of the problem it is clear that the propagator can only depend on the invariant $X\cdot Y$ *or equivalently on the chordal distance* $\zeta=(X-Y)^2$, *where we set* $R=1$. *Recalling that*

$$R^2\nabla^2_{AdS}\phi=\frac{1}{2}J_{AB}J^{BA}\phi=\left[-X^2\partial_X^2+X\cdot\partial_X\left(d+X\cdot\partial_X\right)\right]\phi , \tag{3}$$

show that

$$\Pi(X,Y)=\frac{\mathcal{C}_\Delta}{\zeta^\Delta}\,{}_2F_1\left(\Delta,\Delta-\frac{d}{2}+\frac{1}{2},2\Delta-d+1,-\frac{4}{\zeta}\right) , \tag{4}$$

where $2\Delta=d+\sqrt{d^2+(2m)^2}$ *and*

$$\mathcal{C}_\Delta=\frac{\Gamma(\Delta)}{2\pi^{\frac{d}{2}}\Gamma\left(\Delta-\frac{d}{2}+1\right)} . \tag{5}$$

Exercise 2 ***The CFT three-point function from AdS***

If we add a cubic coupling to the action of a free scalar in AdS (**??**)*:*

$$S=S_{\text{free}}+\frac{g}{3!}\int_{AdS}dX\,\phi^3(X) . \tag{6}$$

We obtain the following three-point function:

$$g\int_{AdS}dX\,\Pi(X,P_1)\Pi(X,P_2)\Pi(X,P_3)+O(g^3) , \tag{7}$$

where $\Pi(X,P_1)$ *is the bulk-to-boundary propagator, which you can take to be, up to a constant,*

$$\Pi(X,P_1)\propto\frac{1}{(-2P\cdot X)^\Delta} . \tag{8}$$

Compute the three-point function and check that it has the form dictated by conformal invariance of the boundary theory. It is helpful to use the integral representation

$$\frac{1}{(-2P\cdot X)^{\Delta}} = \frac{1}{\Gamma(\Delta)}\int_0^{\infty}\frac{ds}{s}s^{\Delta}e^{2sP\cdot X} \tag{9}$$

to bring the AdS integral to the form

$$\int_{AdS} dX e^{2X\cdot Q} \tag{10}$$

with Q a future directed timelike vector. Choosing the X^0 direction along Q and using Poincaré coordinates, it is easy to show that

$$\int_{AdS} dX e^{2X\cdot Q} = \pi^{\frac{d}{2}}\int_0^{\infty}\frac{dz}{z}\,z^{-\frac{d}{2}}e^{-z+Q^2/z}\ . \tag{11}$$

To factorize the remaining integrals over s_1, s_2, s_3 and z it is helpful to change to the variables t_1, t_2, t_3 and z using

$$s_i = \frac{\sqrt{z\,t_1t_2t_3}}{t_i}\ . \tag{12}$$

Exercise 3 ***AdS black holes, the Bekenstein-Hawking formula and the entropy in CFT***

Consider a CFT on a sphere of radius R and at temperature T. The entropy is a non-trivial function of the dimensioless combination RT, which should scale in the high temperature regime as

$$S \sim (RT)^{d-1}\ . \tag{13}$$

Let us compute this function assuming the CFT is well described by Einstein gravity with asymptotically AdS boundary conditions. There are two possible bulk geometries that asymptote to the Euclidean boundary $S^1\times S^{d-1}$. The first is pure AdS

$$ds^2 = R^2\left[\frac{dr^2}{1+r^2} + \left(1+r^2\right)d\tau^2 + r^2 d\Omega_{d-1}^2\right] \tag{14}$$

with Euclidean time periodically identified and the second is Schwarzschild-AdS

$$ds^2 = R^2\left[\frac{dr^2}{f(r)} + f(r)d\tilde{\tau}^2 + r^2 d\Omega_{d-1}^2\right]\ , \tag{15}$$

where $f(r) = 1 + r^2 - \frac{m}{r^{d-2}}$. At the boundary $r = r_{max} \gg 1$, both solutions should be conformal to $S^1 \times S^{d-1}$ with the correct radii. Show that this fixes the periodicities

$$\Delta\tau = \frac{1}{TR}\frac{r_{max}}{\sqrt{1+r_{max}^2}} \ , \qquad \Delta\tilde{\tau} = \frac{1}{TR}\frac{r_{max}}{\sqrt{f(r_{max})}} \ . \tag{16}$$

Show also that regularity of the metric at the horizon (15) implies the periodicity

$$\Delta\tilde{\tau} = \frac{4\pi}{f'(r_H)} = \frac{4\pi}{r_H d + \frac{d-2}{r_H}} \ , \tag{17}$$

*where $r = r_H$ is the largest zero of $f(r)$. (*Hint: *perform a change of coordinates and compare the near horizon geometry to flat space in polar coordinates). Notice that this implies a minimal temperature for Schwarzschild black holes in AdS, $T > \frac{\sqrt{d(d-2)}}{2\pi R}$. For any given temperature larger than the minimum, there are two black holes. The solution with smaller r_h (small black hole) is unstable, while the one with larger r_h (big black hole) is stable, as it can be seen by computing their specific heat (notice that the energy of a black hole is proportional to its mass M).*

Both (14) and (15) are stationary points of the Euclidean action:

$$I[G] = \frac{1}{\ell_P^{d-1}} \int d^{d+1}w\sqrt{G}\left[\mathcal{R} - 2\Lambda\right] \ . \tag{18}$$

Therefore, we must compute the value of the on-shell action in order to decide which one dominates the path integral. Show that the difference of the on-shell actions is given by

$$I_{BH} - I_{AdS} = -2S_d\frac{R^{d-1}}{\ell_P^{d-1}}\left[r_{max}^d\Delta\tau - \left(r_{max}^d - r_H^d\right)\Delta\tilde{\tau}\right] \tag{19}$$

$$\longrightarrow S_d\frac{R^{d-1}}{\ell_P^{d-1}}\frac{1}{TR}r_H^{d-2}(1-r_H^2) \tag{20}$$

where S_d is the area of a unit $(d-1)$-dimensional sphere and in the last step we took the limit $r_{max} \to \infty$. Conclude that the black hole only dominates the bulk path integral when $r_H > 1$, which corresponds to $T > \frac{d-1}{2\pi R}$. This is the Hawking-Page phase transition – see S.W. Hawking and D.N. Page, Commun. Math. Phys. 87 (1983) 577 It is natural to set the free-energy of the AdS phase to zero because this phase corresponds to a gas of gravitons around the AdS background whose free energy does not scale with the large *parameter* $(R/\ell_P)^{d-1}$*. Therefore, the free energy of the black hole phase is given by*

$$F_{BH} = \frac{1}{R} S_d \frac{R^{d-1}}{\ell_P^{d-1}} r_H^{d-2}(1 - r_H^2) \ . \tag{21}$$

Verify that the thermodynamic relation $\frac{\partial F}{\partial T} = -S$ *agrees with the Bekenstein-Hawking formula for the black hole entropy. Now take the large temperature limit and recover the correct scaling* (13)*.*