Asymptotic Methods: Notation and Basic Definitions

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I Background Knowledge and Useful Formulae

Here is a brief summary of some of the formulae and review of the concepts which will be used frequently without comment in this course - these come mostly from Differential Equations IA and Methods, Complex Methods/Analysis IB. A few additional statements appear in small type, which means they are not examinable. There are a few exercises, mostly integrals, which you can do to help revise the material.

I.1 Real Gaussian Integrals and the Gamma Function

We will make regular use of the following formulae:

1. $\int_{-\infty}^{\infty} e^{-At^2}\,dt = \sqrt{\frac{\pi}{A}}$, for $A > 0$.
2. $\int_{0}^{\infty} t^{2n+1}\,e^{-At^2}\,dt = \frac{1}{2}\,n!\,A^{-(n+1)}$, for $A > 0$.
3. $\int_{-\infty}^{\infty} t^{2n}\,e^{-At^2}\,dt = \frac{(2n-1)!!}{2^n}\frac{\sqrt{\pi}}{A^{n+1/2}} = \frac{(2n)!}{n!2^{2n}}\frac{\sqrt{\pi}}{A^{n+1/2}}$, for $n \in \mathbb{N}$ and $A > 0$. This is derived either by differentiation with respect to A or integration by parts. (Non-examinable aside: in the latter case, you might like to notice that $(2n)!/(n!2^n)$ is the number of ways of pairing off $2n$ objects - this leads to a graphical representation for Gaussian integrals of polynomials like this which you will find in physics books as Feynman diagrams.)
4. $\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}\,dt$ is well-defined for $\Re z > 0$ and satisfies $\Gamma(n) = (n-1)!$ for positive integral n. (Proof: integration by parts exercise).
5. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, by change of variable $t = x^2$ to a Gaussian integral.
6. Integration by parts gives $\Gamma(z+1) = z\Gamma(z)$, so that
$$\Gamma(n+\frac{1}{2}) = (n-\frac{1}{2})(n-\frac{3}{2})\dots(\frac{1}{2})\Gamma(\frac{1}{2}) = (2n-1)!!\sqrt{\pi}/2^n\,.$$

It is important that you understand the relation between Gaussian integrals and the Gamma function:

Exercise I.1. Make the change of variables $t = x^2$ and relate items 2,1,3 and 4,5,6 respectively.

Exercise I.2. Calculate $\int_{-\infty}^{+\infty} \exp[-ax^2 + bx + c]\,dx$ for $a > 0$ and b, c arbitrary real numbers.

I.2 Fourier transform

The formulae you will mostly need are the *Fourier inversion* formula, the *Parseval identity* and the *Convolution theorem* given in items 1,4 and 5 below. Also recall the important principle, which is a consequence of the formulae in item 2, that the Fourier transform interchanges rapidity of decay at infinity and smoothness: a function is said to be *rapidly decreasing* if it decreases faster than any polynomial, i.e. $x^N f(x) \to 0$ as $|x| \to \infty$ for all positive integers N. It is equivalent to say that $\sup_{x\in\mathbb{R}} |x^n f(x)| < \infty$ for all positive integers n. Such functions play a natural role in the Fourier transform: for a precise formulation, define the Schwartz space of test functions:

$$\mathcal{S}(\mathbb{R}) = \{u \in C^\infty(\mathbb{R}) : |u|_{\alpha,\beta} = \sup_{x\in\mathbb{R}} |x^\alpha \partial^\beta u(x)| < \infty\,,\ \forall \alpha \in \mathbb{Z}_+, \beta \in \mathbb{Z}_+\,.\}$$

In words, this is the space of functions which are both smooth and all of whose derivatives are rapidly decreasing. This is a convenient space on which to define the Fourier transform because - on account of the interchange of rapidity of decrease with smoothness, as expressed in item 2 - $\mathcal{S}$ is invariant under the Fourier transform:

Theorem I.3. *The Fourier transform, i.e. the mapping*

$$\begin{aligned} \mathcal{F} : \mathcal{S}(\mathbb{R}) &\to \mathcal{S}(\mathbb{R})\,, \\ u &\mapsto \hat{u} \qquad \textit{where } \hat{u}(\xi) = \mathcal{F}_{x\to\xi}(u(x)) = \int_{\mathbb{R}} e^{-i\xi\cdot x} u(x)\, dx \end{aligned}$$

is a linear bijection whose inverse is the map $\mathcal{F}^{-1}$ which takes v to the function $\check{v} = \mathcal{F}^{-1}(v)$ given by

$$\check{v}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{+i\xi\cdot x} v(\xi)\, d\xi\,,$$

and the following hold:

1. $u(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{u}(\xi) e^{i\xi\cdot x} d\xi$ *where* $\hat{u}(\xi) = \int_{\mathbb{R}} e^{-i\xi\cdot x} u(x)\, dx$ *(Fourier inversion),*
2. $\widehat{\partial^\alpha u}(\xi) = \mathcal{F}_{x\to\xi}(\partial^\alpha u(x)) = (i\xi)^\alpha \hat{u}(\xi)$ *and* $(\partial^\alpha \hat{u})(\xi) = \mathcal{F}_{x\to\xi}((-ix)^\alpha u(x))$ *for all* $x, \xi \in \mathbb{R}, \alpha \in \mathbb{Z}_+$,
3. $\int_{\mathbb{R}} \hat{v}(\xi) u(\xi)\, d\xi = \int_{\mathbb{R}} v(x) \hat{u}(x)\, dx\,,$
4. $\frac{1}{2\pi}\int_{\mathbb{R}} \overline{\hat{v}(\xi)} \hat{u}(\xi)\, d\xi = \int_{\mathbb{R}} \overline{v(x)} u(x)\, dx$ *(Parseval),*
5. $\widehat{u * v} = \hat{u}\hat{v}$ *where* $u * v = \int u(x-y)v(y)\, dy$ *(convolution).*

Exercise I.4. Calculate $K_t(x)$, the inverse Fourier transform of the function $\hat{K}_t(\xi) = e^{-t|\xi|^2}$, for fixed positive t and verify the Parseval identity $\frac{1}{2\pi}\int |\hat{K}_t|^2 d\xi = \int |K_t|^2 dx$. (You probably came across this Fourier transform computation in IB: make sure you know why it is justified to make a complex change of variables; reconsider after reviewing contour integrals in the next section if you are unsure.)

I.3 Holomorphic Functions

You will mostly need to have the *Cauchy theorem*, *integral formula* and the *residue formula* at your disposal and recall how to make use of these in calculating contour integrals, often in conjunction with *Jordan's lemma*. To start we recall that to integrate a complex valued function on a C^1 curve Γ we introduce a parametrization $t \mapsto z(t) = x(t) + iy(t)\,,\ t_0 \leq t \leq t_1\,$, of Γ, and then define

$$\int_\Gamma f(z)\,dz \;=\; \int_{t_0}^{t_1} f(z(t))\,\big(dx(t) + idy(t)\big) \;=\; \int_{t_0}^{t_1} f(z(t))\,\big(\dot{x}(t) + i\dot{y}(t)\big)\,dt$$

thus reducing to the case of a real integral with respect to the parameter t. This definition extends in the expected way to piece-wise C^1 curves by simply adding the contributions from each C^1 portion of the curve. It will always be assumed in what follows that curves are piece-wise C^1. A simple closed curve is a closed loop (i.e. it eventually comes back to its starting point) which has otherwise no self-intersections; circles, squares and triangles in the plane are all simple closed curves which admit piece-wise C^1 parametrizations. The Jordan curve theorem asserts that the complement of a simple closed curve Γ consists of two connected components - an interior Int(Γ) (which is bounded), and an exterior Ext(Γ) (which is unbounded).

A complex valued function $f : \Omega \to \mathbb{C}$ defined on an open set $\Omega \subset \mathbb{C}$ is said to be *holomorphic* if for each $z_0 \in \Omega$ there is a disc

$$D_r(z_0) \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z - z_0| < r\}\,,$$

of radius $r > 0$ on which f is given as a *convergent* power series

$$f(z) \;=\; \sum_{n=0}^{\infty} c_n (z - z_0)^n\,, \quad \text{for } |z - z_0| < r\,.$$

Theorem I.5. *For a complex valued function $f : \Omega \to \mathbb{C}$ the following are equivalent.*

1. *f is holomorphic in Ω;*
2. *There exists, for each $z_0 \in \Omega$, a complex number $f'(z_0)$ such that*[1]
$$\lim_{\substack{z \to z_0 \\ z \in \Omega \setminus \{z_0\}}} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)\,;$$
3. *f is continuous and*
$$\int_\Delta f(z)\,dz \;=\; 0$$
for all closed triangles $\Delta \subset \Omega$.

As an optional exercise you may deduce the following important fact: if f is the limit of a sequence of holomorphic functions which converge uniformly to f then f is also holomorphic. (We will not use this in this course.)

You should have at your fingertips the following:

[1]In statements like this it is to be understood that it is being asserted that the limit actually exists.

1. If f is holomorphic, except possibly for a set of isolated points $\{a_k\}$, in an open set which contains a simple closed curve Γ which does not meet the set $\{a_k\}$, then the *Cauchy Residue formula* holds:

$$\oint_\Gamma f(z)\,dz \;=\; 2\pi i \sum_{k:a_k \in \mathrm{Int}(\Gamma)} \mathrm{Res}(f;a_k)\,.$$

   The residue $\mathrm{Res}(f;a_k)$ is the coefficient of $(z-a_k)^{-1}$ in the Laurent expansion of $f(z)$ around the point $z=a_k$. As a special case, if the set $\{a_k\}$ is empty, or consists only of removable singularities, then $\oint_\Gamma f(z)\,dz \;=\; 0$ - this is the *Cauchy theorem*.

2. The *Cauchy Integral Formula*

$$f(z_0) \;=\; \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(w)}{w-z_0}\,dw\,. \tag{I.1}$$

   may be regarded as a special case of the Residue formula; it holds if f is holomorphic in Ω and $\overline{D}_r(z_0) \subset \Omega$.

3. If $\lim_{|z|\to\infty} |zf(z)| = 0$ then $\lim_{R\to\infty} \oint_{|z|=R} f(z)dz = 0$.

4. If $\lim_{|z|\to\infty} |f(z)| = 0$ and $k>0$ then if θ_1, θ_2 are in $[0,\pi]$ then

$$\lim_{R\to\infty} \int_{\substack{z=Re^{i\theta}\\ \theta_1\le\theta\le\theta_2}} f(z)\,e^{ikz}\,dz = 0\,.$$

   This result, known as *Jordan's Lemma*, is a just consequence of the fact that $\sin\theta \geq \frac{2\theta}{\pi}$ for $0 \leq \theta \leq \frac{\pi}{2}$ so that, for example,

$$\left|\int_{\substack{z=Re^{i\theta}\\ 0\le\theta\le\frac{\pi}{2}}} f(z)\,e^{ikz}\,dz\right| \;\leq\; \max_{|z|=R}|f(z)| \int_0^{\frac{\pi}{2}} e^{-2kR\theta/\pi}R d\theta \leq \frac{\pi}{2k}\max_{|z|=R}|f(z)|$$

   which $\to 0$ as $R\to\infty$.

From the Cauchy Integral formula differentiation gives immediately the Cauchy estimates:

Corollary I.6 (Cauchy Estimates)**.** *Let f be holomorphic and bounded in Ω, and $D_r(z_0) \subset \Omega$, with* $\sup_{D_r(z_0)} |f| = M < \infty$. *Then, for all* $n = 0, 1, 2\ldots$

$$|f^{(n)}(z_0)| \;\leq\; \frac{n!M}{r^n}\,.$$

From this you may prove that if f is holomorphic and bounded in Ω, then the power series representation $f(z) \;=\; \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z-z_0)^n$ actually converges in *all discs* $D_r(z_0) \subset \Omega$.. From this we can read off the very useful fact that, loosely speaking,

> *The radius of convergence of the Taylor series of a locally holomorphic function equals the distance to the nearest singularity.*

Exercise I.7. Evaluate $\int_0^\infty x^{-a}/(1+x)\,dx = \pi/\sin(\pi a)$ for $0<a<1$, by means of the keyhole contour.

I.4 The Laplace Transform

This is an analytic continuation of the Fourier transform for functions $u(t)$ which

- vanish for $t \leq 0$, and
- have no more than exponential growth as $t \to +\infty$ in the sense that there exist real numbers C, β such that
$$|u(t)| \leq Ce^{\beta t}\,.$$

Assume in addition that u is continuous for $t > 0$, but its value is allowed to jump discontinuously from zero for negative time to some non-zero value $u(0) = \lim_{t\to 0+} u(t)$ at $t = 0$. Then the Laplace transform is given by

$$\tilde{u}(s) \;=\; \int_0^\infty e^{-st}\, u(t)\, dt\,.$$

This defines a function which is holomorphic for $\Re s > \beta$, and the Laplace inversion formula recovers the function for $t > 0$ by

$$u(t) \;=\; \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\,\tilde{u}(s)\, ds\,.$$

Any use made of the Laplace transform will reduce to these two formulae or simple deductions from them.

Exercise I.8 (*Convolution Theorem* for Laplace transform)**.** . Calculate the Laplace transform of $u * v(t) = \int_0^t u(t-\tau)v(\tau)d\tau$.

Exercise I.9. Show, by considering the convolution theorem for Laplace transforms or otherwise, that if $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt$ then $\Gamma(a)\Gamma(b) = B(a,b)\Gamma(a+b)$. Hence derive the identity

$$\Gamma(z)\,\Gamma(1-z) \;=\; \frac{\pi}{\sin(\pi z)}\,, \tag{I.2}$$

which can be used to define the Gamma function $\Gamma(z)$, and get information about it, in the region $\Re z \leq 0$ where the integral is not well defined. (You may refer to the integral $\int_0^\infty x^{-a}/(1+x)\,dx = \pi/\sin(\pi a)$ which you computed in the previous section.)

I.5 Complex Gaussian Integrals

The formulae for integrals of Gaussian functions can be extended to the complex plane via the Cauchy theorem, often used in conjunction with Jordan's lemma. For example, apply the Cauchy theorem to the holomorphic function $f(z) = \exp[-iAz^2]$, where $A > 0$, in the sector $\{z : \mathrm{Arg}z \in [-\pi/4, 0]\}$, and notice that when $z = r\exp[-i\pi/4]$ we have $f(z) = \exp[-Ar^2]$, which we know how to integrate from 0 to ∞, see above. Now the integral on $\mathrm{Arg}z = 0$ is an improper integral since $|\exp[-iAr^2]| = 1$, but by Cauchy's theorem we have the following identity for any $R > 0$:

$$\int_0^R \exp[-iAr^2]\,dr \;=\; \exp[-i\pi/4]\int_0^R \exp[-Ar^2]\,dr \;+\; \int_{\substack{-\pi/4\leq \mathrm{Arg}z\leq 0\\ |z|=R}} f(z)\,dz\,.$$

But by making the change of variable $w = z^2$ we have

$$\int_{\substack{-\pi/4 \leq \mathrm{Arg} z \leq 0 \\ |z|=R}} f(z)\, dz \;=\; \int_{\substack{-\pi/2 \leq \mathrm{Arg} w \leq 0 \\ |w|=R}} \exp[-iAw]\, \frac{dw}{2\sqrt{w}}\,,$$

which has limit zero as $R \to \infty$ by applying Jordan's lemma. It follows that

$$\lim_{R\to\infty} \int_0^R \exp[-iAr^2]\, dr \;=\; \lim_{R\to\infty} \exp[-i\pi/4] \int_0^R \exp[-Ar^2]\, dr \;=\; \frac{1}{2}\, \exp[-i\pi/4]\, \sqrt{\frac{\pi}{A}}\,.$$

Similarly, with the understanding $A > 0$ and that we are strictly speaking talking about improper integrals, we have the formulae

$$\int_{-\infty}^{\infty} \exp[\pm iAr^2]\, dr \;=\; \exp[\pm i\pi/4]\, \sqrt{\frac{\pi}{A}}\,.$$

Differentiation with respect to A gives formulae for the moments which are the expected extensions of those given above in the real case, such as:

$$\int_{-\infty}^{\infty} t^{2n} \exp[\pm iAt^2]\, dt \;=\; \frac{(2n-1)!!}{2^n} \frac{\sqrt{\pi}}{A^{n+1/2}} \exp[\pm i\pi/4]\,.$$

Remark I.10. A precise interpretation of these oscillatory integrals can be given using distributions, see Chapter 7 of Volume I of Hormander's "Analysis of Linear Partial Differential Operators".

Exercise I.11. Calculate the inverse Fourier transform of $\exp[-it|\xi|^2]$. Let $\psi(x,t)$ be the solution of the Schrödinger equation

$$i\frac{\partial \psi}{\partial t} \;+\; \frac{\partial^2 \psi}{\partial x^2} \;=\; 0\,, \qquad \psi(x,0) = \psi_0(x)\,.$$

As you will recall from Methods IB, applying statement 2 from the Fourier transformation theorem I.3, and using integrating factors on the resulting ordinary differential equation for the Fourier transform <u>in x</u>, suggests the formula

$$\hat{\psi}(\xi,t) = \exp[-it|\xi|^2]\, \hat{\psi}_0(\xi)$$

Use the Convolution theorem and the Gaussian integrals given above to derive from this the formula for the inverse Fourier transform

$$\psi(x,t) \;=\; \frac{1}{(4\pi i t)^{\frac{1}{2}}} \int_{-\infty}^{+\infty} \exp[\frac{i|x-y|^2}{4t}] \psi_0(y)\, dy\,.$$

(If you do QM: compare this with the formulae in your notes, probably putting $\hbar = 1$ and $m = \frac{1}{2}$, evaluate the integral when $\psi_0(x) = \exp[-ax^2 + ikx]$, and interpret the answer.)

Exercise I.12 (Non-examinable)**.** There are multi-dimensional versions of the Gaussian integral formulae which you may like to try to prove as an exercise: if A is symmetric $n \times n$ matrix with Re A positive definite (this means it has positive eigenvalues) then

$$\int_{\mathbb{R}^n} \exp[-x^T A x]\, dx \;=\; \frac{\pi^{n/2}}{(\det A)^{1/2}}\,.$$

In the case that $A = iB$, with B an invertible real symmetric $n \times n$ matrix with n_+ positive eigenvalues and n_- negative eigenvalues, where $n_+ + n_- = n$, the formula is

$$\int_{\mathbb{R}^n} \exp[-ix^T B x]\, dx \;=\; \frac{\pi^{n/2}}{|\det B|^{1/2}} \exp[-i\frac{\pi}{4}(n_+ - n_-)]\,.$$

(Hint: to prove these, be wise diagonalize.)

I.6 Some Functions and Power Series

As well as the standard exponential and trigonometric functions, we will often use the Gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt\,, \qquad \Re z > 0\,,$$

which has been discussed above already - the above integral gives a function holomorphic in the domain $\Re z > 0$, and the functional relation (I.2) gives the unique continuation of this function to the complex plane, with poles at the negative integers. (Why is it unique?).

We will also use the Airy functions, which are solutions of the equation $y''(x) = xy(x)$. The origin is an ordinary point for this equation, and there are two power series solutions, linear combinations of which are used to define the Airy functions:

$$\mathrm{Ai}\,(x) \;=\; 3^{-\frac{2}{3}} \sum_{n=0}^{\infty} \frac{x^{3n}}{3^{2n} n! \Gamma(n+\frac{2}{3})} \;-\; 3^{-\frac{4}{3}} \sum_{n=0}^{\infty} \frac{x^{3n+1}}{3^{2n} n! \Gamma(n+\frac{4}{3})}\,, \quad \text{and}$$

$$\mathrm{Bi}\,(x) \;=\; 3^{-\frac{1}{6}} \sum_{n=0}^{\infty} \frac{x^{3n}}{3^{2n} n! \Gamma(n+\frac{2}{3})} \;+\; 3^{-\frac{5}{6}} \sum_{n=0}^{\infty} \frac{x^{3n+1}}{3^{2n} n! \Gamma(n+\frac{4}{3})}\,.$$

These rather specific linear combinations are chosen on account of their behaviour at infinity rather than at zero, as will become clear.

Exercise I.13. Apply the Frobenius method to obtain power series solutions to the Bessel equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \big(1 - \frac{\nu^2}{x^2}\big) y = 0$$

at the *regular* singular point $x = 0$, for the case that $\nu \notin \mathbb{Z}$, showing that

$$J_\nu(x) = \big(\frac{x}{2}\big)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n! \Gamma(n+\nu+1)}$$

and $Y_\nu(x) = \big(J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)\big)/\sin(\nu\pi)$ are two linearly independent solutions. In the case that ν is an integer it is necessary to obtain the second solution by introducing a logarithm also.

You are not expected to memorize these power series formulae.

II Basic Concepts and Definitions

We summarize the basic definitions of the subject- which arose in the work of Poincare - with which we will be working in the course.

II.1 Asymptotic Expansions on the Real Line

To start with we consider asymzptotic relations and expansions for functions of a real variable $x \in \mathbb{R}$.

1. The notation $f(x) = O(g(x))$ as $x \to x_0$ means that there exist numbers $M > 0, \rho > 0$ such that
$$|f(x)| \leq M|g(x)|\,, \quad \text{for } |x - x_0| \leq \rho\,.$$
In words: $f(x)$ is of order $g(x)$ as $x \to x_0$. For example, $2x + 4x^2$ is of order x as $x \to 0$, since $|2x + 4x^2| \leq 3|x|$ for $|x| \leq \frac{1}{2}$.

2. $f(x) = o(g(x))$ as $x \to x_0$ means that $\lim_{x\to x_0} \left|\frac{f(x)}{g(x)}\right| = 0$, i.e., that given $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - x_0| < \delta$ then $|f(x)| < \epsilon|g(x)|$. For example, $x^2 = o(x)$ as $x \to 0$, since $\left|\frac{x^2}{x}\right| = |x| < \epsilon$ if $|x| < \epsilon$.

3. Given two *functions* f and g, we write
$$f(x) \sim g(x), \qquad \Big(x \to x_0\Big)$$
means $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$, i.e., for all $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - x_0| < \delta$ then
$$|f(x) - g(x)| < \epsilon|g(x)|\,,$$
or more briefly $f - g = o(g)$ as $x \to x_0$. For example $\sin x \sim x\ (x \to 0)$. Notice that if $f(x) \sim 0\ (x \to 0)$ then f vanishes in a neighbourhood of $x = 0$.
It is sometimes useful to say $f(x)$ is an approximation of $g(x)$ correct to $O(\delta(x))$ (respectively $o(\delta(x))$) as $x \to x_0$ if $f(x) - g(x) = O(\delta(x))$ (respectively $o(\delta(x))$) as $x \to x_0$. For example x is an approximation to $\sin x$ to $O(x^3)$ as $x \to 0$. Similarly, $1 + x$ is an approximation of $1/(1 - x)$ correct to order $o(x)$ as $x \to 0$, since
$$\frac{1}{1-x} - (1 + x) = 1 + x + x^2 + \cdots - (1 + x) = O(x^2) = o(x) \ \text{ as } \ x \to 0.$$

4. A sequence of functions $\{\phi_j(x)\}$, $j = 1, 2, \cdots$, is called an *asymptotic sequence* as $x \to x_0$, if
$$\phi_{j+1}(x) = o(\phi_j(x))\,, \qquad \text{as } x \to x_0.$$

5. A function $f(x)$ is said to have an *asymptotic expansion* at x_0,
$$f(x) \sim \sum_{j=1}^{\infty} a_j \phi_j(x)\,, \qquad \Big(x \to x_0\Big)\,,$$

with respect to the asymptotic sequence $\{\phi_j\}_{j=1}^{\infty}$ if, for all $N \in \mathbb{N}$ there holds

$$f(x) - \sum_{j=1}^{N} a_j \phi_j(x) = o(\phi_N(x)), \qquad \text{as } x \to x_0 .$$

The first non-zero term in an asymptotic expansion is referred to as the *leading term*.

II.2 Asymptotic Expansions in the Complex Plane

The definitions in the preceding section extend in an obvious way to the case $x \in \mathbb{R}^d$, or when x is a complex variable $z \in \mathbb{C}$, or even to more general metric spaces. However, in interesting examples of asymptotics for complex functions we will find that the expansions only hold under some restriction on the way in which $z \to z_0$ - typically z will be restricted to some sector of the complex plane.

6. A complex function $f(z)$ is asymptotic to a function $g(z)$ as $z \to z_0$ in a sector $\{z : \text{Arg}\,(z - z_0) \in (\theta_0, \theta_1)\}$, i.e.,

$$f(z) \sim g(z) \qquad (z \to z_0; \text{Arg}\,(z - z_0) \in (\theta_0, \theta_1)),$$

if $f - g = o(g)$, or equivalently that $\lim_{z\to z_0}(f(z)/g(z)) = 1$, as $z \to z_0$ in this sector.

7. A complex function f has an asymptotic expansion $\sum_{j=0}^{\infty} c_j(z - z_0)^j$ as $z \to z_0$ in a sector $\{z : \text{Arg}\,(z - z_0) \in (\theta_0, \theta_1)\}$, written

$$f(z) \sim \sum_{j=0}^{\infty} c_j(z - z_0)^j \qquad (z \to z_0; \text{Arg}\,(z - z_0) \in (\theta_0, \theta_1)),$$

if $f(z) - \sum_{j=0}^{N} c_j(z-z_0)^j = o((z-z_0)^N$ as $z \to z_0$ in the given sector. [Consideration of simple examples indicates that an asymptotic expansion or relation may fail on the rays $\text{Arg}\,(z-z_0) = \theta_j$ which bound the sector in which it does hold. Thus a more precise statement of what is meant is that if $z \to z_0$ with $\text{Arg}(z-z_0) \in [\theta_0+\epsilon, \theta_1-\epsilon]$ for small positive ϵ then $f(z) - \sum_{j=0}^{N} c_j(z - z_0)^j = o((z - z_0)^N$.]

As an example, consider the asymptotic behavior of $\sinh(z^{-1})$, as $z \to 0$, in the complex plane. Letting $z = \rho e^{i\theta}$, we find

$$\sinh\left(\frac{e^{-i\theta}}{\rho}\right) = \frac{1}{2}\exp[\frac{1}{\rho}(\cos\theta - i\sin\theta)] - \frac{1}{2}\exp[-\frac{1}{\rho}(\cos\theta - i\sin\theta)].$$

Thus,

$$\sinh\left(z^{-1}\right) \sim \left[\begin{array}{ll} \frac{1}{2}e^{z^{-1}}, & \cos\theta > 0 \\ & \\ -\frac{1}{2}e^{-z^{-1}}, & \cos\theta < 0 \end{array}\right., \qquad z \to 0.$$

Hence, the asymptotic expansion changes discontinuously across the rays $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$. In the region $\cos\theta > 0$ the term $\frac{1}{2}e^{z^{-1}}$ is called the dominant term, while the term $-\frac{1}{2}e^{-z^{-1}}$ is said to be sub-dominant. Across the Stokes lines there is evidently an interchange of dominance/sub-dominance.

This phenomenon, called Stoke's phenomenon, occurs often in applications. In particular, it occurs in connection to the asymptotic expansion of linear ODEs in the vicinity of an *irregular* singular point.

Exercise II.1. To clarify this point further, develop asymptotic expansions for $\tanh z$ as $z \to \infty$ in different sectors, and identify the *Stokes lines* which separate the regions in which different expansions hold.

Before leaving the introductory first section of the course, make sure that in addition to the basic definitions above you understand:

8. that if a function f has an asymptotic expansion, then f determines the coefficients uniquely, but that many functions can have the same asymptotic expansion;
9. the notions of dominant, sub-dominant and the Stokes phenomenon and lines (Section 3.7 in Bender-Orszag) and be aware that some books interchange use of the words Stokes line and anti-Stokes line, so always check which convention is intended;
10. that you can perform most basic operations (including addition, multiplication and integration) on asymptotic expansions *but not differentiation - why not?*, see the discussion in Section 3.8 of Bender-Orszag and the problems on the first exercise sheet.

Subtleties: (a) The definition in item 3. that a function f is asymptotic to another function should be really distinguished from the definition of asymptotic expansion in item 5, even though most books use the same symbol $\sim$ for the two concepts. For example, the function $\exp[-\frac{1}{x^2}]$ has an asymptotic expansion as $x \to 0$ with all zero coefficients, i.e.,

$$\exp[-\frac{1}{x^2}] \sim \sum 0 \,.\, x^j \,,$$

since $\exp[-\frac{1}{x^2}] = o(x^N)$ for all $N \in \mathbb{Z}_+$, but it is not asymptotic to the zero function in the sense of item 3. (Indeed all functions asymptotic to zero in the sense of item 3. actually vanish in some neighborhood of zero.) If $f(x) \sim \sum_{j=0}^{\infty} a_j x^j$, while $g(x) \sim \sum_{j=0}^{\infty} - a_j x^j$, as $x \to 0$, then $f + g$ is asymptotic to the zero sequence since it is straight forward to check that $f + g = o(x^N)$ for all N, but $f + g$ need not be asymptotic to zero in the sense of item 3.

(b) The condition $f - g = o(g)\,, (x \to x_0)$ implies that f has the same zeros as g near to x_0. This should always be kept in mind when using asymptotic relations for oscillatory functions, see the discussion in Section 3.8 of Bender and Orszag.

(c) The definition of an asymptotic expansion for a function f only refers to the sequence of partial sums $\sum_{j=1}^{N} a_j \phi_j$ in relation to the function f: although the formal notation $\sum_{j=1}^{\infty} a_j \phi_j$ frequently appears in discussions of asymptotic expansions, the infinite sum itself need not exist and is never made use of in the usual strict sense mathematical sense.

For the remainder of the course we will be working with these concepts in various specific problems involving integrals and differential equations. Here is a list of selected readings from [1] and [2, Chapter 6] which could usefully be read in conjunction with the lectures, organized according to the course syllabus as given in the schedules.

III Further Reading

III.0.1 Asymptotic expansions: definitions and simple examples - 4 lectures

Definition (Poincare) of $\phi(z) \sim \sum a_n z^{-n}$; examples; elementary properties; uniqueness; Stokes' phenomenon.

1. Review power series solutions of ODEs from Part IA by reading Sections 3.1-3.3 of [1], and then read Section 3.4 in [1].
2. Sections 3.5 and 3.6 in [1], and Section 6.1.1 in [2].
3. Section 3.7 in [1].
4. Section 3.8 in [1].

III.0.2 Asymptotic expansions of integrals - 7 lectures

*Integration by parts. Watson's lemma and Laplace's method. Riemann–Lebesgue lemma and method of stationary phase. The method of steepest descent (including derivation of higher order terms). Airy function, *and application to wave theory of a rainbow*.*

5. Sections 6.1 to 6.3 in [1] and/or Sections 6.1.2 and 6.2.1 in [1, Chapter 6].
6. Section 6.4 in [1] and/or Sections 6.2.2 and 6.2.3 in [2, Chapter 6].
7. Section 6.5 in [1].
8. Section 6.3 in [2, Chapter 6].
9. Section 6.6 in [1].
10. Sections 6.4 and 6.5 in [2, Chapter 6].
11. Section 6.7 in [1].

III.0.3 Asymptotic expansions of solutions of differential equations - 4 lectures

Asymptotic solution of second-order linear differential equations, including Liouville–Green functions (proof that they are asymptotic not required) and WKBJ with the quantum harmonic oscillator as an example.

12. Review Sturm-Liouville theory and Green functions from methods IB, then read Sections 10.1 and 10.3 in [1].
13. Section 10.4 in [1].
14. Section 10.5 in [1].
15. Sections 10.6 and 10.7 in [1].

III.0.4 Further developments - one lecture

*Further discussion of Stokes' phenomenon. *Asymptotics 'beyond all orders'*.*

16. Section 6.6 in [2, Chapter 6], the article [3] and by the same author "Asymptotics, superasymptotics, hyperasymptotics", in *Asymptotics Beyond All Orders*, H. Segur, S. Tanveer, and H. Levine, eds., Plenum, Ams- terdam, 1991, pp. 114. (in the Moore library, QC20.7.A85 .N386 1991), or publication 234 in the website in [3].

References

[1] C. Bender and S. Orszag, Advanced Mathematical methods for Scientists and Engineers. Available on reserve in the Moore library at QA371 .B46 1999, or in the UL at 348:01.c.10.52 or 349:1.c.95.372. Should also be in most college libraries, or buy from eg

`http://www.abebooks.co.uk/servlet/SearchResults?an=orszag+and+bender&sts=t`

[2] M. Ablowitz and A. Fokas, Complex variables: introduction and applications, QA331.7 .A25 2003, should be accessible in many libraries:

`http://idiscover.lib.cam.ac.uk/primo-explore/search?vid=44CAM_TEST&lang=en_US&search_scope=default_scope&sortby=rank&query=any,contains,ablowitz%20and%20fokas`

or possibly online at

`http://lib.myilibrary.com/Open.aspx?id=238730`

[3] Berry, M V, 1989, Stokes phenomenon; smoothing a Victorian discontinuity, Publ.Math.of the Institut des Hautes tudes scientifiques, 68 211-221. Available as publication number 190 at

`https://michaelberryphysics.wordpress.com/publications/`