Part IA - Probability

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Basic concepts
Classical probability, equally likely outcomes. Combinatorial analysis, permutations and combinations. Stirling's formula (asymptotics for $\log n!$ proved). [3]

Axiomatic approach
Axioms (countable case). Probability spaces. Inclusion-exclusion formula. Continuity and subadditivity of probability measures. Independence. Binomial, Poisson and geometric distributions. Relation between Poisson and binomial distributions. Conditional probability, Bayes's formula. Examples, including Simpson's paradox. [5]

Discrete random variables
Expectation. Functions of a random variable, indicator function, variance, standard deviation. Covariance, independence of random variables. Generating functions: sums of independent random variables, random sum formula, moments.

Conditional expectation. Random walks: gambler's ruin, recurrence relations. Difference equations and their solution. Mean time to absorption. Branching processes: generating functions and extinction probability. Combinatorial applications of generating functions. [7]

Continuous random variables
Distributions and density functions. Expectations; expectation of a function of a random variable. Uniform, normal and exponential random variables. Memoryless property of exponential distribution. Joint distributions: transformation of random variables (including Jacobians), examples. Simulation: generating continuous random variables, independent normal random variables. Geometrical probability: Bertrand's paradox, Buffon's needle. Correlation coefficient, bivariate normal random variables. [6]

Inequalities and limits
Markov's inequality, Chebyshev's inequality. Weak law of large numbers. Convexity: Jensen's inequality for general random variables, AM/GM inequality.

Moment generating functions and statement (no proof) of continuity theorem. Statement of central limit theorem and sketch of proof. Examples, including sampling. [3]

Contents

0 Introduction

In every day life, we often encounter the use of the term probability, and they are used in many different ways. For example, we can hear people say:

(i) The probability that a fair coin will land heads is 1/2.

(ii) The probability that a selection of 6 members wins the National Lottery Lotto jackpot is 1 in $\binom{19}{6} = 13983816$ or 7.15112×10^{-8}.

(iii) The probability that a drawing pin will land 'point up' is 0.62.

(iv) The probability that a large earthquake will occur on the San Andreas Fault in the next 30 years is about 21%

(v) The probability that humanity will be extinct by 2100 is about 50%

The first two cases are things derived from logic. We know that the coin either lands heads or tails. By definition, a fair coin is equally likely to land heads or tail. So the probability of either must be 1/2.

The third is something probably derived from experiments. Perhaps we did 1000 experiments and 620 of the pins point up. The fourth and fifth examples belong to yet another category that talks about are beliefs and predictions.

We call the first kind "classical probability", the second kind "frequentist probability" and the last "subjective probability". In this course, we only consider classical probability.

Note that the material in this notes does not necessarily follow the order in which they are lectured.

1 Classical probability

1.1 Classical probability

Definition (Classical probability)**.** *Classical probability* applies in a situation when there are a finite number of equally likely outcome.

Consider the problem of points

A and B play a game in which they keep throwing coins. If a head lands, then A gets a point. Otherwise, B gets a point. The first person to get 10 points wins a prize.

Now suppose A has got 8 points and B has got 7. The game has to end because an earthquake struck. How should they divide the prize? We answer this by finding the probability of A winning. Someone must have won by the end of 19 rounds, ie. after 4 more rounds. If A wins at least 2 of them, then A wins.

The number of ways this can happen is $\binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 11$. So A should get 11/16 of the prize.

1.2 Sample space and events

Consider an experiment that has a random outcome.

Definition (Sample space)**.** The set of all possible outcomes is the *sample space*, Ω. We can lists the outcomes as $\omega_1, \omega_2, \cdots \in \Omega$. Each $\omega \in \Omega$ is an *outcome*.

Definition (Event)**.** A subset of Ω is called an *event*.

Definition (Set notations)**.** Given any two events $A, B \subseteq \Omega$,

- The *complement* of A is $A^C = A' = \bar{A} = \Omega \setminus A$.
- "A or B" is the set $A \cup B$.
- "A and B" is the set $A \cap B$.
- A and B are *mutually exclusive* or *disjoint* if $A \cap B = \emptyset$.
- If $A \subseteq B$, then A occurring implies B occurring.

Definition (Probability)**.** Suppose $\Omega = \{\omega_1, \omega_2, \cdots, \omega_N\}$. Let $A \subseteq \Omega$ be an event. Then the *probability* of A is

$$\mathbb{P}(A) = \frac{\text{Number of outcomes in } A}{\text{Number of outcomes in } \omega} = \frac{|A|}{N}.$$

Example. Suppose r digits are drawn at random from a table of random digits from 0 to 9. What is the probability that

(i) No digit exceeds k;

(ii) The largest digit drawn is k?

The sample space is $\Omega = \{(a_1, a_2, \cdots, a_r) : 0 \le a_i \le 9\}$. Then $|\Omega| = 10^r$.

Let $A_k = [\text{no digit exceeds } k] = \{(a_1, \cdots, a_r) : 0 \le a_i \le k\}$. Then $|A_k| = (k+1)^r$. So

$$P(A_k) = \frac{(k+1)^r}{10^r}.$$

Now let $B_k = [\text{largest digit drawn is } k]$. We can find this by finding all outcomes in which no digits exceed k, and subtract it by the number of outcomes in which no digit exceeds $k-1$. So $|B_k| = |A_k| - |A_{k-1}|$ and

$$P(B_k) = \frac{(k+1)^r - k^r}{10^r}.$$

1.3 Counting

Example. A menu has 6 starters, 7 mains and 6 desserts. How many possible meals combinations are there? Clearly $6 \times 7 \times 6 = 252$.

This is the fundamental rule of counting:

Theorem (Fundamental rule of counting)**.** Suppose we have to make r multiple choices in sequence. There are m_1 possibilities for the first choice, m_2 possibilities for the second etc. Then the total number of choices is $m_1 \times m_2 \times \cdots m_r$.

Example. How many ways can $1, 2, \cdots, n$ be ordered? The first choice has n possibilities, the second has $n-1$ possibilities etc. So there are $n \times (n-1) \times \cdots \times 1 = n!$.

Sampling with or without replacement

Suppose we have to pick n items from a total of x items. We can model this as follows: Let $N = \{1, 2, \cdots, n\}$ be the list. Let $X = \{1, 2, \cdots, x\}$ be the items. Then each way of picking the items is a function $f : N \to X$ with $f(i) =$ item at the ith position.

Definition (Sampling with replacement)**.** When we *sample with replacement*, after choosing at item, it is put back and can be chosen again. Then *any* sampling function f satisfies sampling with replacement.

Definition (Sampling without replacement)**.** When we *sample without replacement*, after choosing an item, we kill it with fire and cannot choose it again. Then f must be an injective function, and clearly we must have $X \geq n$.

We can also have sampling with replacement, but we require each item to be chosen at least once. In this case, f must be surjective.

Example. Suppose $N = \{a, b, c\}$ and $X = \{p, q, r, s\}$. How many injective functions are there $N \to X$?

When we choose $f(a)$, we have 4 options. When we choose $f(b)$, we have 3 left. When we choose $f(c)$, we have 2 choices left. So there are 24 possible choices.

Example. I have n keys in my pocket. We select one at random once and try to unlock. What is the possibility that I succeed at the rth trial?

Suppose we do it with replacement. We have to fail the first $r-1$ trials and succeed in the rth. So The probability is

$$\frac{(n-1)(n-1)\cdots(n-1)(1)}{n^r} = \frac{(n-1)^{r-1}}{n^r}.$$

Now suppose we are smarter and try without replacement. Then the probability is

$$\frac{(n-1)(n-2)\cdots(n-r+1)(1)}{n(n-1)\cdots(n-r+1)} = \frac{1}{n}.$$

Example (Birthday problem)**.** How many people are needed in a room for there to be a probability of at least a half that two people have the same birthday?

Suppose $f(r)$ is the probability that, in a room of r people, there is a birthday match.

We solve this by finding the probability of no match, $1 - f(r)$. The total number of possibilities of birthday combinations is 365^r. For nobody to have the same birthday, the first person can have any birthday. The second has 364 else to choose, etc.

$$\mathbb{P}(\text{no match}) = \frac{365 \cdot 364 \cdot 363 \cdots (366-r)}{365 \cdot 365 \cdot 365 \cdots 365}$$

If we calculate this with a computer, we find that $f(22) = 0.475695$ and $f(23) = 0.507297$.

While this might sound odd since 23 is small, we shouldn't be counting the number of people, but the number of pairs. With 23 people, we have $23 \times 22/2 = 253$ pairs, which is quite large compared to 365.

Sampling with or without regard to ordering

There are cases where we don't care about, say list positions. For example, if we pick two representatives from a class, the order of picking them doesn't matter.

Recall we have the function $f : N \to X$ and they map $f(1), f(2), \cdots, f(n)$. In general, after choosing the numbers, we can

- Leave the list alone
- Sort the list ascending. ie. we might get $(2, 5, 4)$ and $(4, 2, 5)$. If we don't care about list positions, these are just equivalent to $(2, 4, 5)$.
- Re-number each item by the number of the draw on which it was first seen. For example, we can rename $(2, 5, 2)$ and $(5, 4, 5)$ both as $(1, 2, 1)$. This happens if the labelling of items doesn't matter.
- Both of above. So we can rename $(2, 5, 2)$ and $(8, 5, 5)$ both as $(1, 1, 2)$.

Total number of cases

Combining these four possibilities with whether we have replacement, no replacement, or "everything has to be chosen at least once", we have 12 possible cases of counting. The most important ones are:

- Replacement + with ordering: the number of ways is x^n.
- Without replacement + with ordering: the number of ways is $x_{(n)} = x^{\underline{n}} = x(x-1)\cdots(x-n+1)$.
- Without replacement + without order: we only care which items get selected. The number of ways is $\binom{x}{n} = C_n^x = x_{(n)}/n!$.

– Replacement + without ordering: we only care how many times the item got chosen. We want to find ways to partition $n = n_1 + n_2 + \cdots n_k$. Say $n = 6$ and $k = 3$. We can write a particular partition as

$$**|*|***$$

So we have $n + k - 1$ symbols and $k - 1$ of them are bars. So the number of ways is $\binom{n+k-1}{k-1}$.

1.4 Multinomial coefficient

Suppose we fill successive positions in a list of length n, with replacement, from a set of k items. How many ways can this be done so that item i is used n_i times? (of course we must have $n_1 + n_2 + \cdots + n_k = n$)

In the case of $k = 2$, we have the binomial coefficient. In general, we have

Definition (Multinomial coefficient)**.** A *multinomial coefficient* is

$$\binom{n}{n_1, n_2, \cdots, n_k} = \binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1\cdots-n_{k-1}}{n_k} = \frac{n!}{n_1!n_2!\cdots n_k!}.$$

It is the number of ways to distribute k items into n positions with replacement, in which the ith position has n_i items.

Example. We know that

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \cdots + y^n.$$

If we have a trinomial, then

$$(x+y+z)^n = \sum_{n_1,n_2,n_3}\binom{n}{n_1, n_2, n_3}x^{n_1}y^{n_2}y^{n_3}.$$

Example. How many ways can we deal 52 cards to 4 player, each with a hand of 13? There are

$$\binom{52}{13, 13, 13, 13} = \frac{52!}{(13!)^4} = 53644737765488792839237440000 = 5.36 \times 10^{28}.$$

While computers are still capable of calculating that, what if we tried more ~~power~~ cards? Suppose each person has n cards. Then the number of ways is

$$\frac{(4n)!}{(n!)^4} = \frac{2^{8n}}{\sqrt{2}(\pi n)^{3/2}}.$$

We can use Stirling's Formula to approximate it:

1.5 Stirling's formula

Before we state and prove Stirling's formula, we prove a weaker (but examinable) version:

Proposition. $\log n! \sim n \log n$

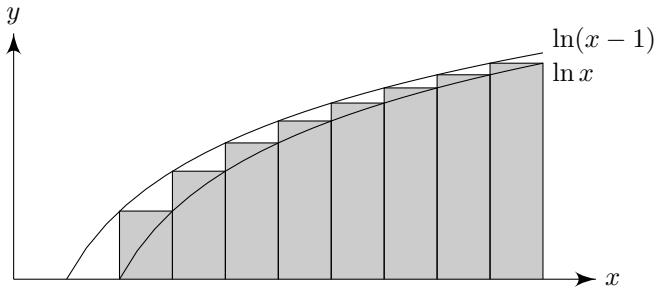
Proof. Note that

$$\log n! = \sum_{k=1}^{n} \log k.$$

Now we claim that

$$\int_1^n \log x \, \mathrm{d}x \leq \sum_1^n \log k \leq \int_1^{n+1} \log x \, \mathrm{d}x.$$

This is true by considering the diagram:



Perform the integral to obtain

$$n \log n - n + 1 \leq \log n! \leq (n+1)\log(n+1) - n;$$

Divide both sides by $n \log n$ and let $n \to \infty$. Both sides tend to 1. So

$$\frac{\log n!}{n \log n} \to 1.$$

□

Now we prove Stirling's Formula:

Theorem (Stirling's formula)**.** As $n \to \infty$,

$$\log\left(\frac{n! e^n}{n^{n+\frac{1}{2}}}\right) = \log\sqrt{2\pi} + O\left(\frac{1}{n}\right)$$

Corollary.

$$n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}$$

Proof. (non-examinable) Define

$$d_n = \log\left(\frac{n! e^n}{n^{n+1/2}}\right) = \log n! - (n + 1/2)\log n + n$$

Then

$$d_n - d_{n+1} = (n + 1/2)\log\left(\frac{n+1}{n}\right) - 1.$$

Write $t = 1/(2n+1)$. Then

$$d_n - d_{n+1} = \frac{1}{2t}\log\left(\frac{1+t}{1-t}\right) - 1.$$

We can simplifying by noting that

$$\log(1+t) - t = -\frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + \cdots$$
$$\log(1-t) + t = -\frac{1}{2}t^2 - \frac{1}{3}t^3 - \frac{1}{4}t^4 - \cdots$$

Then if we subtract the equations and divide by $2t$, we obtain

$$\begin{aligned} d_n - d_{n+1} &= \frac{1}{3}t^2 + \frac{1}{5}t^4 + \frac{1}{7}t^6 + \cdots \\ &< \frac{1}{3}t^2 + \frac{1}{3}t^4 + \frac{1}{3}t^6 + \cdots \\ &= \frac{1}{3}\frac{t^2}{1-t^2} \\ &= \frac{1}{3}\frac{1}{(2n+1)^2 - 1} \\ &= \frac{1}{12}\left(\frac{1}{n} - \frac{1}{n+1}\right) \end{aligned}$$

By summing these bounds, we know that

$$d_1 - d_n < \frac{1}{12}\left(1 - \frac{1}{n}\right)$$

Then we know that d_n is bounded below by d_1+ something, and is decreasing since $d_n - d_{n+1}$ is positive. So it converges to a limit A. We know A is a lower bound for d_n since (d_n) is decreasing.

Suppose $m > n$. Then $d_n - d_m < \left(\frac{1}{n} - \frac{1}{m}\right)\frac{1}{12}$. So taking the limit as $m \to \infty$, we obtain an upper bound for d_n: $d_n < A + 1/(12n)$.

However, all these results are useless if we don't know what A is. To find A, we have a small detour to prove a formula:

Take $I_n = \int_0^{\pi/2} \sin^n\theta \,\mathrm{d}\theta$. This is decreasing for increasing n as $\sin^n\theta$ gets smaller. We also know that

$$\begin{aligned} I_n &= \int_0^{\pi/2} \sin^n\theta \,\mathrm{d}\theta \\ &= \left[-\cos\theta\sin^{n-1}\theta\right]_0^{\pi/2} + \int_0^{\pi/2}(n-1)\cos^2\theta\sin^{n-2}\theta\,\mathrm{d}\theta \\ &= 0 + \int_0^{\pi/2}(n-1)(1+\sin^2\theta)\sin^{n-2}\theta\,\mathrm{d}\theta \\ &= (n-1)(I_{n-2} - I_n) \end{aligned}$$

So

$$I_n = \frac{n-1}{n}I_{n-2}.$$

We can directly evaluate the integral to obtain $I_0 = \pi/2$, $I_1 = 1$. Then

$$I_{2n} = \frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n}\pi/2 = \frac{(2n)!}{(2^n n!)^2}\frac{\pi}{2}$$
$$I_{2n+1} = \frac{2}{3}\cdot\frac{4}{5}\cdots\frac{2n}{2n+1} = \frac{(2^n n!)^2}{(2n+1)!}$$

So using the fact that I_n is decreasing, we know that

$$1 \leq \frac{I_{2n}}{I_{2n+1}} \leq \frac{I_{2n-1}}{I_{2n+1}} = 1 + \frac{1}{2n} \to 1.$$

Using the approximation $n! \sim n^{n+1/2}e^{-n+A}$, where A is the limit we want to find, we can approximate

$$\frac{I_{2n}}{I_{2n+1}} = \pi(2n+1)\left[\frac{((2n)!)^2}{2^{4n+1}(n!)^4}\right] \sim \pi(2n+1)\frac{1}{ne^{2A}} \to \frac{2\pi}{e^{2A}}.$$

Since the last expression is equal to 1, we know that $A = \log\sqrt{2\pi}$ □

Example. Suppose we toss a coin $2n$ times. What is the probability of equal number of heads and tails? The probability is

$$\frac{\binom{2n}{n}}{2^{2n}} = \frac{(2n)!}{(n!)^2 2^{2n}} \sim \frac{1}{\sqrt{n\pi}}$$

This approximation can be improved:

Proposition (non-examinable)**.** We use the $1/12n$ term from the proof above to get a better approximation:

$$\sqrt{2\pi}n^{n+1/2}e^{-n} + \frac{1}{12n+1} \leq n! \leq \sqrt{2\pi}n^{n+1/2}e^{-n} + \frac{1}{12n}.$$

Example. Suppose we draw 26 cards from 52. What is the probability of getting 13 reds and 13 blacks? The probability is

$$\frac{\binom{26}{13}\binom{26}{13}}{\binom{52}{26}} = 0.2181.$$

2 Axioms of probability

Definition (Probability space)**.** A *probability space* is a triple $(\Omega, \mathcal{F}, P)$. Ω is the *sample space*, $\mathcal{F}$ is a collection of subsets of Ω. $P : \mathcal{F} \to [0, 1]$ is the *probability measure*. $\mathcal{F}$ has to satisfy the following axioms:

(i) $\emptyset, \Omega \in \mathcal{F}$.

(ii) $A \in \mathcal{F} \Rightarrow A^C \in \mathcal{F}$.

(iii) $A_1, A_2, \cdots, \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} \in \mathcal{F}$.

And $\mathbb{P}$ has to satisfy the following *Kolmogorov axioms*:

(i) $0 \leq \mathbb{P}(A) \leq 1$ for all $A \in \mathcal{F}$

(ii) $\mathbb{P}(\Omega) = 1$

(iii) For any countable collection of events $A_1, A_2, \cdots$ which are disjoint, ie. $A_i \cap A_j = \emptyset$ for all i, j, we have

$$\mathbb{P}\left(\bigcup_i A_i\right) = \sum_i \mathbb{P}(A_i).$$

We say $\mathbb{P}(A)$ is the probability of the event A.

Definition (Probability distribution)**.** Let $\Omega = \{\omega_1, \omega_2, \cdots\}$. Choose $\{p_1, p_2, \cdots, \}$ such that $\sum_{i=1}^{\infty} p_i = 1$. Let $p(\omega_i) = p_i$. Then define

$$\mathbb{P}(A) = \sum_{\omega_i \in A} p(\omega_i).$$

This $\mathbb{P}(A)$ satisfies the above axioms, and $p_1, p_2, \cdots$ is the *probability distribution*

Theorem.

(i) $\mathbb{P}(\emptyset) = 0$

(ii) $\mathbb{P}(A^C) = 1 - \mathbb{P}(A)$

(iii) $A \subseteq B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$

(iv) $\mathbb{P}(A \subseteq B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

Proof.

(i) Ω and $\emptyset$ are disjoint. So $\mathbb{P}(\Omega) + \mathbb{P}(\emptyset) = \mathbb{P}(\Omega \cup \emptyset) = \mathbb{P}(\Omega)$. So $\mathbb{P}(\emptyset) = 0$.

(ii) $\mathbb{P}(\Omega) = 1 = \mathbb{P}(A) + \mathbb{P}(A^C)$ since A and A^C are disjoint.

(iii) Write $B = A \cup (B \cap A^C)$. Then
$P(B) = \mathbb{P}(A) + \mathbb{P}(B \cap A^C) \geq \mathbb{P}(A)$.

(iv) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B \cap A^C)$. We also know that $\mathbb{P}(B) = \mathbb{P}(A \cap B) + \mathbb{P}(B \cap A^C)$. Then the result follows.

□

From above, we know that $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$. So we say that $\mathbb{P}$ is a *subadditive* function. We also know that $\mathbb{P}(A \cap B) + \mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$ (in fact they are equal!). We say $\mathbb{P}$ is *submodular*.

Definition (Limit of events)**.** A sequence of events $A_1, A_2, \cdots$ is *increasing* if $A_1 \subseteq A_2 \cdots$. Then we define the *limit* as

$$\lim_{n\to\infty} A_n = \bigcup_1^\infty A_n.$$

Similarly, if they are *decreasing*, ie. $A_1 \supseteq A_2 \cdots$, then

$$\lim_{n\to\infty} A_n = \bigcap_1^\infty A_n.$$

Theorem. If $A_1, A_2, \cdots$ is increasing or decreasing, then

$$\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}\left(\lim_{n\to\infty} A_n\right).$$

Proof. Take $B_1 = A_1$, $B_2 = A_2 \setminus A_1$. In general,

$$B_n = A_n \setminus \bigcup_1^{n-1} A_i.$$

Then

$$\bigcup_1^n B_i = \bigcup_1^n A_i, \quad \bigcup_1^\infty B_i = \bigcup_i^\infty A_n.$$

Then

$$\begin{aligned}
\mathbb{P}(\lim A_n) &= \mathbb{P}\left(\bigcup_1^\infty A_i\right) \\
&= \mathbb{P}\left(\bigcup_1^\infty B_i\right) \\
&= \sum_1^\infty \mathbb{P}(B_i) \text{ (Axiom III)} \\
&= \lim_{n\to\infty} \sum_{i=1}^n \mathbb{P}(B_i) \\
&= \lim_{n\to\infty} \mathbb{P}\left(\bigcup_1^n A_i\right) \\
&= \lim_{n\to\infty} \mathbb{P}(A_n).
\end{aligned}$$

and the decreasing case is proven similarly (or apply above to A_i^C). □

2.1 Boole's inequality

This is also known as the "union bound".

Theorem (Boole's inequality)**.** For any $A_1, A_2, \cdots$,

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

Proof. The axiom states a similar formula that only holds for disjoint sets. So we need a clever trick to make them disjoint. We define

$$\begin{aligned} B_1 &= A_1 \\ B_2 &= A_2 \setminus A_1 \\ B_i &= A_i \setminus \bigcup_{k=1}^{i-1} A_k. \end{aligned}$$

So we know that

$$\bigcup B_i = \bigcup A_i.$$

But the B_i are disjoint. So our Axiom (iii) gives

$$\mathbb{P}\left(\bigcup_i A_i\right) = \mathbb{P}\left(\bigcup_i B_i\right) = \sum_i \mathbb{P}\left(B_i\right) \leq \sum_i \mathbb{P}\left(A_i\right).$$

Where the last inequality follows from (iii) of the theorem above. □

Example. Suppose we have countably infinite number of biased coins. Let $A_k = [k\text{th toss head}]$ and $\mathbb{P}(A_k) = p_k$. Suppose $\sum_1^\infty p_k < \infty$. What is the probability that there are infinitely many heads?

The event "there is at least one more head after the ith coin toss" is $\bigcup_{k=i}^\infty A_k$. There are infinitely many heads if and only if there are unboundedly many coin tosses, ie. no matter how high i is, there is still at least more more head after the ith toss.

So the probability required is

$$\mathbb{P}\left(\bigcap_{i=1}^{\infty}\bigcup_{k=i}^{\infty} A_k\right) \leq \lim_{i\to\infty} \mathbb{P}\left(\bigcup_{k=i}^{\infty} A_k\right) = \lim_{i\to\infty} \sum_{k=i}^{\infty} p_k \to 0$$

Therefore $\mathbb{P}(\text{infinite number of heads}) = 0$.

Example (Erdos 1947)**.** Is it possible to colour a complete n-graph (ie. a graph of n vertices with edges between every pair of vertices) red and black such that no k-vertex complete subgraph with monochrome edges?

Erdös said this is possible if

$$\binom{n}{k} 2^{1-\binom{k}{2}} < 1.$$

We colour edges randomly, and let $A_i = [i\text{th subgraph has monochrome edges}]$. Then the probability that at least one subgraph has monochrome edges is

$$\mathbb{P}\left(\bigcup A_i\right) \leq \sum \mathbb{P}(A_i) = \binom{n}{k} 2 \cdot 2^{-\binom{k}{2}}.$$

The last expression is obtained since there are $\binom{n}{k}$ ways to choose a subgraph; a monochrome subgraph can be either red or black, thus the multiple of 2; and the probability of getting all red (or black) is $2^{-\binom{k}{2}}$.

If this probability is less than 1, then there must be a way to colour them in which it is impossible to find a monochrome subgraph, or else the probability is 1. So if $\binom{n}{k}2^{1-\binom{k}{2}} < 1$, the colouring is possible.

2.2 Inclusion-exclusion formula

Theorem (Inclusion-exclusion formula)**.**

$$\mathbb{P}\left(\bigcup_i^n A_i\right) = \sum_1^n \mathbb{P}(A_i) - \sum_{i_1<i_2} \mathbb{P}(A_{i_1} \cap A_{j_2}) + \sum_{i_1<i_2<i_3} \mathbb{P}(A_{i_1} \cap A_{i_2} \cap A_{i_3}) - \cdots + (-1)^{n-1}\mathbb{P}(A_1 \cap \cdots \cap A_n).$$

Proof. Perform induction on n. $n = 2$ is proven above.

Then

$$\mathbb{P}(A_1 \cup A_2 \cup \cdots A_n) = \mathbb{P}(A_1) + \mathbb{P}(A_2 \cup \cdots \cup A_n) - \mathbb{P}\left(\bigcup_{i=2}^n (A_1 \cap A_i)\right).$$

and we can apply the induction hypothesis for $n - 1$. □

Example. Let $1, 2, \cdots, n$ be randomly permuted to $\pi(1), \pi(2), \cdots, \pi(n)$. If $i \neq \pi(i)$ for all i, we say we have a *derangement*.

Let $A_i = [i \neq \pi(i)]$.

Then

$$\begin{aligned}\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) &= \sum_k \mathbb{P}(A_k) - \sum_{k_1<k_2} \mathbb{P}(A_{k_1} \cap A_{k_2}) + \cdots \\ &= n \cdot \frac{1}{n} - \binom{n}{2}\frac{1}{n}\frac{1}{n-1} + \binom{n}{3}\frac{1}{n}\frac{1}{n-1}\frac{1}{n-2} + \cdots \\ &= 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n-1}\frac{1}{n!} \\ &\to e^{-1}\end{aligned}$$

So the probability of derangement is $1 - \mathbb{P}(\bigcup A_k) \approx 1 - e^{-1} \approx 0.632$.

2.3 Bonferroni's inequalities

Recall that, from inclusion exclusion,

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(AB) - \mathbb{P}(BC) - \mathbb{P}(AC) + \mathbb{P}(ABC),$$

where $\mathbb{P}(AB)$ is a shorthand for $\mathbb{P}(A \cap B)$. If we only take the first three terms, then we get Boole's inequality

$$\mathbb{P}(A \cup B \cup C) \leq \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C).$$

In general

Theorem. Bonferroni's inequalities For any events $A_1, A_2, \cdots, A_n$ and $1 \leq r \leq n$,

$$\mathbb{P}\left(\bigcup_1^n A_i\right) \leq \sum_{i_1} \mathbb{P}(A_{i_1}) - \sum_{i_1<i_2} \mathbb{P}(A_{i_1}A_{i_2}) + \sum_{i_1<i_2<i_3} \mathbb{P}(A_{i_1}A_{i_2}A_{i_3}) + \cdots + \sum_{i_1<i_2<\cdots<i_r} \mathbb{P}(A_{i_1}A_{i_2}A_{i_3}\cdots A_{i_r})$$

If r is odd,

$$\mathbb{P}\left(\bigcup_1^n A_i\right) \geq \sum_{i_1} \mathbb{P}(A_{i_1}) - \sum_{i_1<i_2} \mathbb{P}(A_{i_1}A_{i_2}) + \sum_{i_1<i_2<i_3} \mathbb{P}(A_{i_1}A_{i_2}A_{i_3}) + \cdots - \sum_{i_1<i_2<\cdots<i_r} \mathbb{P}(A_{i_1}A_{i_2}A_{i_3}\cdots A_{i_r}).$$

If r is even. This can be proved by induction on n.

Example. Let $\Omega = \{1, 2, \cdots, m\}$ and $1 \leq j, k \leq m$. Write $A_k = \{1, 2, \cdots, k\}$. Then

$$A_k \cap A_j = \{1, 2, \cdots, \min(j, k)\} = A_{\min(j,k)}$$

and

$$A_k \cup A_j = \{1, 2, \cdots, \max(j, k)\} = A_{\max(j,k)}.$$

We also have $\mathbb{P}(k) = k/m$.

Now let $1 \leq x_1, \cdots, x_n \leq m$ be some numbers. Then Bonferroni's inequality says

$$\mathbb{P}\left(\bigcup A_{x_i}\right) \geq \sum \mathbb{P}(A_{x_i}) - \sum_{i<j} \mathbb{P}(A_{x_i} \cap A_{x_j}).$$

So

$$\max\{x_1, x_2, \cdots, x_n\} \geq \sum x_i - \sum_{i_1<i_2} \min\{x_1, x_2\}.$$

2.4 Independence

Definition (Independent events)**.** Two events A and B are *independent* if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

Otherwise, they are said to be *dependent*.

Proposition. If A and B are independent, then A and B^C are independent.

Proof.

$$\begin{aligned}\mathbb{P}(A \cap B^C) &= \mathbb{P}(A) - \mathbb{P}(A \cap B)\\ &= \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(B)\\ &= \mathbb{P}(A)(1 - \mathbb{P}(B))\\ &= \mathbb{P}(A)\mathbb{P}(B^C)\end{aligned}$$

□

Example. Roll two fair dice. Let A_1 and A_2 be the event that the first and second die is odd respectively. Let $A_3 = [\text{sum is odd}]$. The event probabilities are as follows:

Event	Probability
A_1	$1/2$
A_2	$1/2$
A_3	$1/2$
$A_1 \cap A_2$	$1/4$
$A_1 \cap A_3$	$1/4$
$A_1 \cap A_2 \cap A_3$	0

From the example above, we see that just because a set of events is pairwise independent does not mean they are independent all together. We define:

Definition (Independence of multiple events)**.** Events $A_1, A_2, \cdots$ are said to be *mutually independent* if

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}) = \mathbb{P}(A_{i_1})\mathbb{P}(A_{i_2}) \cdots \mathbb{P}(A_{i_r})$$

for any $i_1, i_2, \cdots i_r$ and $r \geq 2$.

Example. Let A_{ij} be the event that i and j roll the same. We roll 4 dice. Then

$$\mathbb{P}(A_{12} \cap A_{13}) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} = \mathbb{P}(A_{12})\mathbb{P}(A_{13}).$$

But

$$\mathbb{P}(A_{12} \cap A_{13} \cap A_{23}) = \frac{1}{36} \neq \mathbb{P}(A_{12})\mathbb{P}(A_{13})\mathbb{P}(A_{23}).$$

So they are not mutually independent.

We can also apply this concept to experiments. Suppose we model two independent experiments with $\Omega_1 = \{\alpha_1, \alpha_2, \cdots\}$ and $\Omega_2 = \{\beta_1, \beta_2, \cdots\}$ with probabilities $\mathbb{P}(\alpha_i) = p_i$ and $\mathbb{P}(\beta_i) = q_i$. Further suppose that these two experiments are independent, ie.

$$\mathbb{P}((\alpha_i, \beta_j)) = p_i q_j$$

for all i, j. Then we can have a new sample space $\Omega = \Omega_1 \times \Omega_2$.

Now suppose $A \subseteq \Omega_1$ and $B \subseteq \Omega_2$ are results of the two experiments. We can view them as subspaces of Ω by rewriting them as $A \times \Omega_2$ and $\Omega_1 \times B$. Then the probability

$$\mathbb{P}(A \cap B) = \sum_{\alpha_i \in A, \beta_i \in B} p_i q_i = \sum_{\alpha_i \in A} p_i \sum_{\beta_i \in B} q_i = \mathbb{P}(A)\mathbb{P}(B).$$

So we say the two experiments are "independent" even though the term usually refers to different events in the same experiment. We can generalize this to n independent experiments, or even countably many infinite experiments.

2.5 Important discrete distributions

By *discrete* we mean the sample space is countable. The sample space is $\Omega = \{\omega_1, \omega_2, \cdots\}$ and $p_i = \mathbb{P}(\{\omega_i\})$.

Definition (Bernoulli distribution)**.** Suppose we toss a coin. $\Omega = \{H, T\}$ and $p \in [0, 1]$. The *Bernoulli distribution*, denoted $B(1, p)$ has

$$\mathbb{P}(H) = p; \quad \mathbb{P}(T) = 1 - p.$$

Definition (Binomial distribution)**.** Suppose we toss a coin n times with probability p of getting heads. Then

$$\mathbb{P}(HHTT\cdots T) = pp(1-p)\cdots(1-p).$$

Then

$$\mathbb{P}(\text{two heads}) = \binom{n}{2} p^2 (1-p)^{n-2}.$$

In general,

$$\mathbb{P}(k \text{ heads}) = \binom{n}{k} p^k (1-p)^{n-k}.$$

We call this the *binomial distribution* and write it as $B(n, p)$.

Definition (Geometric distribution)**.** Suppose we toss a coin with probability p of getting heads. The probability of having a head after k consecutive tails is

$$p_k = (1-p)^k p$$

This is *geometric distribution*. We say it is *memoryless* because how many tails we've got in the past does not give us any information to how long I'll have to wait until I get a head.

Definition (Hypergeometric distribution)**.** Suppose we have an urn with n_1 red balls and n_2 black balls. We choose n balls. The probability that there are k red balls is

$$\mathbb{P}(k \text{ red}) = \frac{\binom{n_1}{k}\binom{n_2}{n-k}}{\binom{n_1+n_2}{n}}.$$

Definition (Poisson distribution)**.** The *Poisson distribution* denoted $P(\lambda)$ is

$$p_k = \frac{\lambda^k}{k!} e^{-\lambda}$$

for $k \in \mathbb{N}$.

What is this weird distribution? It is a distribution used to model rare events. Suppose that an event happens at a rate of λ. We can think of this as there being a lot of trials, say n of them, and each has a probability λ/n of succeeding. As we take the limit $n \to \infty$, we obtain the Poisson distribution.

Theorem (Poisson approximation to binomial)**.** Suppose $n \to \infty$ and $p \to 0$ such that $np = \lambda$. Then

$$q_k = \binom{n}{k} p^k (1-p)^{n-k} \to \frac{\lambda^k}{k!} e^{-\lambda}.$$

Proof.

$$\begin{aligned} q_k &= \binom{n}{k} p^k (1-p)^{n-k} \\ &= \frac{1}{k!} \frac{n(n-1)\cdots(n-k+1)}{n^k} (np)^k \left(1 - \frac{np}{n}\right)^{n-k} \\ &\to \frac{1}{k!} \lambda^k e^{-\lambda} \end{aligned}$$

since $(1 - a/n)^n \to e^{-a}$. □

2.6 Conditional probability

Definition (Conditional probability)**.** Suppose B is an event with $\mathbb{P}(B) > 0$. For any event $A \subseteq \Omega$, the *conditional probability of A given B* is

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Note that if A and B are independent, then

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(B)} = \mathbb{P}(A).$$

Example. In a game of poker, let A_i = [player i gets royal flush]. Then

$$\mathbb{P}(A_1) = 1.539 \times 10^{-6}.$$

and

$$\mathbb{P}(A_2|A_1) = 1.969 \times 10^{-6}.$$

It is significantly bigger, albeit still incredibly tiny. So we say "good hands attract".

If $\mathbb{P}(A|B) > \mathbb{P}(A)$, then we say that B attracts A. Since

$$\frac{\mathbb{P}(A \cap B}{\mathbb{P}(B)} > \mathbb{P}(A) \Leftrightarrow \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} > \mathbb{P}(B),$$

A attracts B if and only if B attracts A. We can also say A repels B if A attracts B^C.

Theorem.

(i) $\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B)$.

(ii) $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A|B \cap C)\mathbb{P}(B|C)\mathbb{P}(C)$.

(iii) $\mathbb{P}(A|B \cap C) = \frac{\mathbb{P}(A \cap B|C)}{\mathbb{P}(B|C)}$.

(iv) The function $\mathbb{P}(\cdot|B)$ restricted to subsets of B is a probability function (or measure).

Proof. Proofs of (i), (ii) and (iii) are trivial. Now prove (iv): by checking the axioms

(i) Let $A \subseteq B$. Then $\mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)} \leq 1$.

(ii) $\mathbb{P}(B|B) = \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1$.

(iii) Let A_i be disjoint events. Then

$$\begin{aligned}\mathbb{P}\left(\bigcup_i A_i \middle| B\right) &= \frac{\mathbb{P}(\bigcap A_i \cap B)}{\mathbb{P}(B)}\\ &= \frac{\mathbb{P}\left(\bigcap_i A_i\right)}{\mathbb{P}(B)}\\ &= \sum \frac{\mathbb{P}(A_i)}{\mathbb{P}(B)}\\ &= \sum \frac{\mathbb{P}(A_i \cap B)}{\mathbb{P}(B)}\\ &= \sum \mathbb{P}(A_i|B).\end{aligned}$$

□

2.7 Law of total probability

Definition (Partition)**.** A *partition of the sample space* is a collection of disjoint events $\{B_i\}_{i=0}^{\infty}$ such that $\bigcup_i B_i = \Omega$.

Proposition. If B_i is a partition of the sample space, and A is any event, then

$$\mathbb{P}(A) = \sum_{i=1}^{\infty} \mathbb{P}(A \cap B_i) = \sum_{i=1}^{\infty} \mathbb{P}(A|B_i)\mathbb{P}(B_i).$$

Example. A fair coin is tossed repeatedly. The gambler gets $+1$ for head, and -1 for tail. Continue until he is broke or achieves $\$a$. Let

$$p_x = \mathbb{P}(\text{goes broke}|\text{starts with } \$x),$$

and B_1 be the probability that he gets head on the first toss. Then

$$\begin{aligned}p_x &= \mathbb{P}(B_1)p_{x+1} + \mathbb{P}(B_1^C)p_{x-1}\\ p_x &= \frac{1}{2}p_{x+1} + \frac{1}{2}p_{x-1}\end{aligned}$$

We have two boundary conditions $p_0 = 1$, $p_a = 0$. Then solving the recurrence relation, we have

$$p_x = 1 - \frac{x}{a}.$$

Theorem (Bayes' formula)**.** Suppose B_i is a partition of the sample space, and A and B_i all have non-zero probability. Then for any B_i,

$$\mathbb{P}(B_i|A) = \frac{\mathbb{P}(A|B_i)\mathbb{P}(B_i)}{\sum_j \mathbb{P}(A|B_j)\mathbb{P}(B_j)}.$$

Note that the denominator is simply $\mathbb{P}(A)$ written in a fancy way.

Example (Screen test)**.** Suppose we have a screening test that tests whether a patient has a particular disease. We denote positive and negative results as $+$ and $-$ respectively, and D denotes the person having disease. Suppose that the test is not absolutely accurate, and

$$\begin{aligned}\mathbb{P}(+|D) &= 0.98\\ \mathbb{P}(+|D^C) &= 0.01\\ \mathbb{P}(D) &= 0.001.\end{aligned}$$

So what is the probability that a person has the disease given that he received a positive result?

$$\begin{aligned}\mathbb{P}(D|+) &= \frac{\mathbb{P}(+|D)\mathbb{P}(D)}{\mathbb{P}(+|D)\mathbb{P}(D) + \mathbb{P}(+|D^C)\mathbb{P}(D^C)}\\ &= \frac{0.98 \cdot 0.001}{0.098 \cdot 0.001 + 0.01 \cdot 0.999}\\ &= 0.09\end{aligned}$$

Since the disease is so rare, it is more likely that you don't have the disease and get a false positive.

Example. Consider the two following cases:

(i) I have 2 children, one of whom is a boy.

(ii) I have two children, one of whom is a son born on a Tuesday.

What is the probability that both of them are boys?

(i) $\mathbb{P}(BB|BB \cup BG) = \frac{1/4}{1/4+2/4} = \frac{1}{3}$.

(ii) Let B^* denote a boy born on a Tuesday, and B a boy not born on a Tuesday. Then

$$\begin{aligned}\mathbb{P}(B^*B^* \cup B^*B|BB^* \cup B^*B^* \cup B^*G) &= \frac{\frac{1}{14} \cdot \frac{1}{14} + 2 \cdot \frac{1}{14} \cdot \frac{6}{14}}{\frac{1}{14} \cdot \frac{1}{14} + 2 \cdot \frac{1}{14} \cdot \frac{6}{14} + 2 \cdot \frac{1}{14} \cdot \frac{1}{2}}\\ &= \frac{13}{27}.\end{aligned}$$

3 Discrete random variables

3.1 Discrete random variables

Definition (Random variable)**.** A *random variable* X taking values in a set Ω_X is a function $X : \Omega \to \Omega_X$. Ω_X is usually numbers, eg. $\mathbb{R}$ or $\mathbb{N}$.

A random variable basically assigns a "number" (or a thing in Ω_X) to each event (eg. assign 6 to the event "dice roll gives 6").

Definition (Discrete random variables)**.** A random variable is *discrete* if Ω_X is finite or countably infinite.

Notation. Let $T \subseteq \Omega_X$, define

$$\mathbb{P}(X \in T) = \sum_{\omega \in \Omega : X(\omega) \in T} p_\omega.$$

ie. the probability that the outcome is in T.

Example. X might be the value shown by rolling a fair die. Then $\Omega_X = \{1, 2, 3, 4, 5, 6\}$. We know that

$$\mathbb{P}(X = i) = \frac{1}{6}.$$

We call this the discrete uniform distribution.

Definition (Discrete uniform distribution)**.** A *discrete uniform distribution* is a discrete distribution with finitely many possible outcomes, in which each outcome is equally likely.

Example. If we roll two die, then $X + Y = i + j$, and

$$\Omega_{X+Y} = \{2, 3, \cdots, 12\}.$$

Notation. We write

$$\mathbb{P}_X(x) = \mathbb{P}(X = x).$$

We can also write $X \sim B(n, p)$ to mean

$$\mathbb{P}(X = r) = \binom{n}{r} p^r (1 - p)^{n-r}.$$

3.2 Expectation and variance

Definition (Expectation)**.** The *expectation* (or *mean*) of a real-valued X is equal to

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} p_\omega X(\omega).$$

provided this is absolutely convergent. Otherwise, we say the expectation doesn't exist. Alternatively,

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{x \in \Omega_X} \sum_{\omega : X(\omega) = x} p_\omega X(\omega) \\
&= \sum_{x \in \Omega_X} x \sum_{\omega : X(\omega) = x} p_\omega \\
&= \sum_{x \in \Omega_X} x P(X = x).
\end{aligned}$$

Example. Let X be the sum of the outcomes of two die. Then

$$\mathbb{E}[X] = 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + \cdots + 12 \cdot \frac{1}{36} = 7.$$

Note that $\mathbb{E}[X]$ can be non-existent if the sum is not absolutely convergent. However, it is possible for the expected value to be infinite:

Example (Petersburg paradox)**.** Suppose we play a game in which we keep tossing a coin until you get a tail. Suppose we get a tail on the ith round. Then I pay you $\$2^i$. The expected value is

$$\mathbb{E}[X] = \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 4 + \frac{1}{8} \cdot 8 + \cdots = \infty.$$

Example. We calculate the expected values of different distributions:

(i) Poisson $P(\lambda)$. Let $X \sim P(\lambda)$. Then

$$P_X(r) = \frac{\lambda^r e^{-\lambda}}{r!}.$$

So

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{r=0}^{\infty} rP(X = r) \\
&= \sum_{r=0}^{\infty} \frac{r\lambda^r e^{-\lambda}}{r!} \\
&= \sum_{r=1}^{\infty} \lambda \frac{\lambda^{r-1} e^{-\lambda}}{(r-1)!} \\
&= \lambda \sum_{r=0}^{\infty} \frac{\lambda^r e^{-\lambda}}{r!} \\
&= \lambda.
\end{aligned}$$

(ii) Let $X \sim B(n, p)$. Then

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{0}^{n} rP(x = r) \\
&= \sum_{0}^{n} r \binom{n}{r} p^r (1-p)^{n-r} \\
&= \sum_{0}^{n} r \frac{n!}{r!(n-r)!} p^r (1-p)^{n-r} \\
&= np \sum_{r=1}^{n} \frac{(n-1)!}{(r-1)![(n-1)-(r-1)]!} p^{r-1} (1-p)^{(n-1)-(r-1)} \\
&= np \sum_{0}^{n-1} \binom{n-1}{r} p^r (1-p)^{n-1-r)} \\
&= np.
\end{aligned}$$

Given a random variable X, we can create new random variables such as $X + 3$ or X^2. Formally, let $f : \mathbb{R} \to \mathbb{R}$ and X be a real-valued random variable. Then $f(X)$ is a new random variable that maps $\omega \mapsto f(X(\omega))$.

Example. if a, b, c are constants, then $a+bX$ and $(X-c)^2$ are random variables, defined as

$$(a + bX)(\omega) = a + bX(\omega)$$
$$(X - c)^2(\omega) = (X(\omega) - c)^2.$$

Theorem.

(i) If $X \geq 0$, then $\mathbb{E}[X] \geq 0$.

(ii) If $X \geq 0$ and $\mathbb{E}[X] = 0$, then $\mathbb{P}(X = 0) = 1$.

(iii) If a and b are constants, then $\mathbb{E}[a + bX] = a + b\mathbb{E}[X]$.

(iv) If X and Y are random variables, then $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$. This is true even if X and Y are not independent.

(v) $\mathbb{E}[X]$ is a constant that minimizes $\mathbb{E}[(X - c)^2]$ over c.

Proof.

(i) $X \geq 0$ means that $X(\omega) \geq 0$ for all ω. Then

$$\mathbb{E}[X] = \sum_\omega p_\omega X(\omega) \geq 0.$$

(ii) If there exists ω such that $X(\omega) > 0$ and $p_\omega > 0$, then $\mathbb{E}[X] > 0$. So $X(\omega) = 0$ for all ω.

(iii)
$$\mathbb{E}[a + bX] = \sum_\omega (a + bX(\omega))p_\omega = a = b\sum_\omega p_\omega = a + b\mathbb{E}[X].$$

(iv)
$$\mathbb{E}[X+Y] = \sum_\omega p_\omega[X(\omega)+Y(\omega)] = \sum_\omega p_\omega X(\omega)+\sum_\omega p_\omega Y(\omega) = \mathbb{E}[X]+\mathbb{E}[Y].$$

(v)
$$\begin{aligned}\mathbb{E}[(X - c)^2] &= \mathbb{E}[(X - \mathbb{E}[X] + \mathbb{E}[X] - c)^2] \\ &= \mathbb{E}[(X - \mathbb{E}[X])^2 + 2(\mathbb{E}[X] - c)(X - \mathbb{E}[X]) + (\mathbb{E}[X] - c)^2] \\ &= \mathbb{E}(X - \mathbb{E}[X])^2 + 0 + (\mathbb{E}[X] - c)^2.\end{aligned}$$

This is clearly minimized when $c = \mathbb{E}[X]$. Note that we obtained the zero in the middle because $\mathbb{E}[X - \mathbb{E}[X]] = \mathbb{E}[X] - \mathbb{E}[X] = 0$.

□

Theorem. For any random variables $X_1, X_2, \cdots X_n$, for which the following expectations exist,

$$\mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i].$$

Proof.

$$\sum_{\omega} p(\omega)[X_1(\omega) + \cdots + X_n(\omega)] = \sum_{\omega} p(\omega)X_1(\omega) + \cdots + \sum_{\omega} p(\omega)X_n(\omega).$$

□

Definition (Variance and standard deviation)**.** The *variance* of a random variable X is defined as

$$\mathrm{var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2].$$

The *standard deviation* is the square root of the variance, $\sqrt{\mathrm{var}(x)}$.

Theorem.

(i) $\mathrm{var}\, X \geq 0$. If $\mathrm{var}\, X = 0$, then $\mathbb{P}(X = \mathbb{E}[X]) = 1$.

(ii) $\mathrm{var}(a + bX) = b^2\, \mathrm{var}(X)$. This can be proved by expanding the definition and using the linearity of the expected value.

(iii) $\mathrm{var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$, also proven by expanding the definition.

Example (Binomial distribution)**.** Let $X \sim B(n, p)$ be a binomial distribution. Then $\mathbb{E}[X] = np$. We also have

$$\begin{aligned}
\mathbb{E}[X(X-1)] &= \sum_{r=0}^{n} r(r-1)\frac{n!}{r!(n-r)!}p^r(1-p)^{n-r} \\
&= n(n-1)p^2 \sum_{r=2}^{n} \binom{n-2}{r-2} p^{r-2}(1-p)^{(n-2)-(r-2)} \\
&= n(n-1)p^2.
\end{aligned}$$

The sum goes to 1 since it is the sum of all probabilities of a binomial $N(n-2, p)$ So $\mathbb{E}[X^2] = n(n-1)p^2 + \mathbb{E}[X] = n(n-1)p^2 + np$. So

$$\mathrm{var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = np(1-p) = npq.$$

Example (Poisson distribution)**.** If $X \sim P(\lambda)$, then $\mathbb{E}[X] = \lambda$, and $\mathrm{var}(X) = \lambda$, since $P(\lambda)$ is $B(n, p)$ with $n \to \infty, p \to 0, np \to \lambda$.

Example (Geometric distribution)**.** Suppose $\mathbb{P}(X = r) = q^r p$ for $r = 0, 1, 2, \cdots$.

Then

$$\begin{aligned}\mathbb{E}[X] &= \sum_0^\infty rpq^r \\ &= pq\sum_0^\infty rq^{r-1} \\ &= pq\sum_0^\infty \frac{\mathrm{d}}{\mathrm{d}q}q^r \\ &= pq\frac{\mathrm{d}}{\mathrm{d}q}\sum_0^\infty q^r \\ &= pq\frac{\mathrm{d}}{\mathrm{d}q}\frac{1}{1-q} \\ &= \frac{pq}{(1-q)^2} \\ &= \frac{q}{p}.\end{aligned}$$

Then

$$\begin{aligned}\mathbb{E}[X(X-1)] &= \sum_0^\infty r(r-1)pq^r \\ &= pq^2\sum_0^\infty r(r-1)q^{r-2} \\ &= pq^2\frac{\mathrm{d}^2}{\mathrm{d}q^2}\frac{1}{1-q} \\ &= \frac{2pq^2}{(1-q)^3}\end{aligned}$$

So the variance is

$$\mathrm{var}(X) = \frac{2pq^2}{(1-q)^3} + \frac{q}{p} - \frac{q^2}{p^2} = \frac{q}{p^2}.$$

3.3 Indicator random variables

Definition (Indicator function)**.** The *indicator function* or *indicator variable* $I[A]$ (or I_A) of an event $A \subseteq \Omega$ is

$$I[A](\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

We have the following properties (trivial to prove):

Proposition.

- $\mathbb{E}[I[A]] = \sum_\omega p(\omega)I[A](\omega) = \mathbb{P}(A)$.

- $I[A^C] = 1 - I[A]$.
- $I[A \cap B] = I[A]I[B]$.
- $I[A \cup B] = I[A] + I[B] - I[A]I[B]$.
- $I[A]^2 = I[A]$.

Example. Let $2n$ people (n husbands and n wives, with $n > 2$) sit alternate man-woman around the table randomly. Let N = number of couples sitting next to each other.

Let $A_i = [i\text{th couple sits together}]$. Then

$$N = \sum_{i=1}^{n} I[A_i].$$

Then

$$\mathbb{E}[N] = E\left[\sum I[A_i]\right] = \sum_{1}^{n} \mathbb{E}\big[I[A_i]\big] = n\mathbb{E}\big[I[A_1]\big] = n\mathbb{P}(A_i) = n \cdot \frac{2}{n} = 2.$$

We also have

$$\begin{aligned}\mathbb{E}[N^2] &= \mathbb{E}\left[\left(\sum I[A_i]\right)^2\right] \\ &= \mathbb{E}\left[\sum_i I[A_i]^2 + 2\sum_{i<j} I[A_i]I[A_j]\right] \\ &= n\mathbb{E}\big[I[A_i]\big] + n(n-1)\mathbb{E}\big[I[A_1]I[A_2]\big]\end{aligned}$$

We have $\mathbb{E}[I[A_1]I[A_2]] = \mathbb{P}(A_1 \cap A_2) = \frac{2}{n}\left(\frac{1}{n-1}\frac{1}{n-1} + \frac{n-2}{n-1}\frac{2}{n-1}\right)$. Plugging in, we ultimately obtain $\operatorname{var}(N) = \frac{2(n-2)}{n-1}$.

In fact, as $n \to \infty$, $N \sim P(2)$.

We can use these to prove the inclusion-exclusion formula:

Theorem (Inclusion-exclusion formula)**.**

$$\begin{aligned}\mathbb{P}\left(\bigcup_i^n A_i\right) = \sum_1^n \mathbb{P}(A_i) - \sum_{i_1<i_2} \mathbb{P}(A_{i_1} \cap A_{j_2}) + \sum_{i_1<i_2<i_3} \mathbb{P}(A_{i_1} \cap A_{i_2} \cap A_{i_3}) - \cdots \\ + (-1)^{n-1}\mathbb{P}(A_1 \cap \cdots \cap A_n).\end{aligned}$$

Proof. Let I_j be the indicator function for A_j. Write

$$S_r = \sum_{i_1<i_2<\cdots<i_r} I_{i_1} I_{i_2} \cdots I_{i_r},$$

and

$$s_r = \mathbb{E}[S_r] = \sum_{i_1<\cdots<i_r} \mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_r}).$$

Then

$$1 - \prod_{j=1}^{n}(1 - I_j) = S_1 - S_2 + S_3 \cdots + (-1)^{n-1} S_n.$$

And

$$\mathbb{P}\left(\bigcup_1^n A_j\right) = \mathbb{E}\left[1 - \prod_1^n (1 - I_j)\right] = s_1 - s_2 + s_3 - \cdots + (-1)^{n-1} s_n.$$

□

3.4 Independent random variables

Definition (Independent random variables)**.** Let $X_1, X_2, \cdots, X_n$ be discrete random variables. They are *independent* iff for any $x_1, x_2, \cdots, x_n$,

$$\mathbb{P}(X_1 = x_1, \cdots, X_n = x_n) = \mathbb{P}(X_1 = x_1) \cdots \mathbb{P}(X_n = x_n).$$

Theorem. If $X_1, \cdots, X_n$ are independent random variables, and $f_1, \cdots, f_n$ are functions $\mathbb{R} \to \mathbb{R}$, then $f_1(X_1), \cdots, f_n(X_n)$ are independent random variables.

Proof. Note that there can be many different x_i for which $f_i(x_i) = y_i$. When finding $\mathbb{P}(f_i(x_i) = y_i)$, we need to sum over all x_i such that $f_i(x_i) = f_i$. Then

$$\begin{aligned}
\mathbb{P}(f_1(X_1) = y_1, \cdots f_n(X_n) = y_n) &= \sum_{\substack{x_1 : f_1(x_1) = y_1 \\ \vdots \\ x_n : f_n(x_n) = y_n}} \mathbb{P}(X_1 = x_1, \cdots, X_n = x_n) \\
&= \sum_{\substack{x_1 : f_1(x_1) = y_1 \\ \vdots \\ x_n : f_n(x_n) = y_n}} \prod_{i=1}^{n} \mathbb{P}(X_i = x_i) \\
&= \prod_{i=1}^{n} \sum_{x_i : f_i(x_i) = y_i} \mathbb{P}(X_i = x_i) \\
&= \prod_{i=1}^{n} \mathbb{P}(f_i(x_i) = y_i).
\end{aligned}$$

Note that the switch from the second to third line is valid since they both expand to the same mess. □

Theorem. If $X_1, \cdots, X_n$ are independent random variables and all the following expectations exists, then

$$\mathbb{E}\left[\prod X_i\right] = \prod \mathbb{E}[X_i].$$

Proof. Write R_i for the range of X_i (or Ω_{X_i})

$$\begin{aligned}\mathbb{E}\left[\prod_1^n X_i\right] &= \sum_{x_1\in R_1}\cdots\sum_{x_n\in R_n} x_1x_2\cdots x_n \times \mathbb{P}(X_1=x_1,\cdots,X_n=x_n)\\ &= \prod_{i=1}^n \sum_{x_i\in R_i} x_i\mathbb{P}(X_i=x_i)\\ &= \prod_{i=1}^n \mathbb{E}[X_i].\end{aligned}$$

□

Corollary. Let $X_1,\cdots X_n$ be independent random variables, and $f_1, f_2,\cdots f_n$ are functions $\mathbb{R}\to\mathbb{R}$. Then

$$\mathbb{E}\left[\prod f_i(x_i)\right] = \prod \mathbb{E}[f_i(x_i)].$$

Theorem. If $X_1, X_2,\cdots X_1$ are independent random variables, then

$$\operatorname{var}\left(\sum X_i\right) = \sum \operatorname{var}(X_i).$$

Proof.

$$\begin{aligned}\operatorname{var}\left(\sum X_i\right) &= \mathbb{E}\left[\left(\sum X_i\right)^2\right] - \left(\mathbb{E}\left[\sum X_i\right]\right)^2\\ &= \mathbb{E}\left[\sum X_i^2 + \sum_{i\neq j} X_iX_j\right] - \left(\sum \mathbb{E}[X_i]\right)^2\\ &= \sum \mathbb{E}[X_i^2] + \sum_{i\neq j}\mathbb{E}[X_i]\mathbb{E}[X_j] - \sum(\mathbb{E}[X_i])^2 - \sum_{i\neq j}\mathbb{E}[X_i]\mathbb{E}[X_j]\\ &= \sum \mathbb{E}[X_i^2] - (\mathbb{E}[X_i])^2.\end{aligned}$$

□

Corollary. Let $X_1, X_2,\cdots X_n$ be independent identically distributed random variables (iid rvs). Then

$$\operatorname{var}\left(\frac{1}{n}\sum X_i\right) = \frac{1}{n}\operatorname{var}(X_1).$$

So we can reduce the variance of our average by increasing our sample size.

Proof.

$$\begin{aligned}\operatorname{var}\left(\frac{1}{n}\sum X_i\right) &= \frac{1}{n^2}\operatorname{var}\left(\sum X_i\right)\\ &= \frac{1}{n^2}\sum \operatorname{var}(X_i)\\ &= \frac{1}{n^2} n\operatorname{var}(X_1)\\ &= \frac{1}{n}\operatorname{var}(X_1)\end{aligned}$$

□

Example. Let X_i be iid $B(1,p)$, ie. $\mathbb{P}(1) = p$ and $\mathbb{P}(0) = 1-p$. Then $Y = X_1 + X_2 + \cdots + X_n \sim B(n,p)$.

Since $\text{var}(X_i) = \mathbb{E}[X_i]^2 - (\mathbb{E}[X_i])^2 = p - p^2 = p(1-p)$, we have $\text{var}(Y) = np(1-p)$.

Example. Suppose we have two rods of unknown lengths a, b. We can measure the lengths, but is not accurate. Let A and B be the measured value. Suppose

$$\mathbb{E}[A] = a, \quad \text{var}(A) = \sigma^2$$

$$\mathbb{E}[B] = b, \quad \text{var}(B) = \sigma^2.$$

We can measure it more accurately by measuring $X = A+B$ and $Y = A-B$. Then we estimate a and b by

$$\hat{a} = \frac{X+Y}{2},\ \hat{b} = \frac{X-Y}{2}.$$

Then $\mathbb{E}[\hat{a}] = a$ and $\mathbb{E}[\hat{b}] = b$, ie. they are unbiased. Also

$$\text{var}(\hat{a}) = \frac{1}{4}\text{var}(X+Y) = \frac{1}{4}2\sigma^2 = \frac{1}{2}\sigma^2,$$

and similarly for b. So we can measure it more accurately by measuring the sticks together instead of separately.

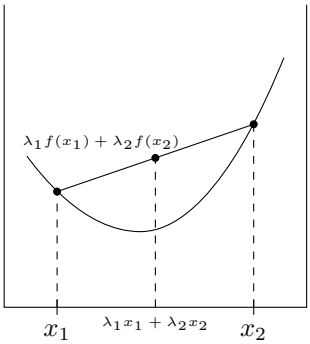
3.5 Inequalities

Here we prove a lot of different inequalities which may be useful for certain calculations. In particular, Chebyshev's inequality will allow us to prove the weak law of large numbers.

Definition (Convex function)**.** A function $f : (a,b) \to \mathbb{R}$ is *convex* if for all $x_1, x_2 \in (a,b)$ and $\lambda_1, \lambda_2 \geq 0$ such that $\lambda_1 + \lambda_2 = 1$,

$$\lambda_1 f(x_1) + \lambda_2 f(x_2) \geq f(\lambda_1 x_1 + \lambda_2 x_2).$$

It is *strictly convex* if the inequality above is strict (except when $x_1 = x_2$ or λ_1 or $\lambda_2 = 0$.



A function is *concave* if $-f$ is convex.

Proposition. If f is differentiable and $f''(x) \geq 0$ for all $x \in (a, b)$, then it is convex. It is strictly convex if $f''(x) > 0$.

Theorem (Jensen's inequality)**.** If $f : (a, b) \to \mathbb{R}$ is convex, then

$$\sum_{i=1}^{n} p_i f(x_i) \geq f\left(\sum_{i=1}^{n} p_i x_i\right)$$

for all $p_1, p_2, \cdots, p_n$ such that $p_i \geq 0$ and $\sum p_i = 1$, and $x_i \in (a, b)$.

This says that $\mathbb{E}[f(X)] \geq f(\mathbb{E}[X])$ (where $\mathbb{P}(X = x_i) = p_i$).

If f is strictly convex, then equalities hold only if all x_i are equal, ie. X takes only one possible value.

Proof. Induct on n. It is true for $n = 2$ by the definition of convexity. Then

$$\begin{aligned}
f(p_1x_1 + \cdots + p_nx_n) &= f\left(p_1x_1 + (p_2 + \cdots + p_n)\frac{p_2x_2 + \cdots + l_nx_n}{p_2 + \cdots + p_n}\right) \\
&\leq p_1 f(x_1) + (p_2 + \cdots p_n) f\left(\frac{p_2x_2 + \cdots + p_nx_n}{p_2 + \cdots + p_n}\right). \\
&\leq p_1 f_(x_1) + (p_2 + \cdots + p_n)\left[\frac{p_2}{(\,)} f(x_2) + \cdots + \frac{p_n}{(\,)} f(x_n)\right] \\
&= p_1 f(x_1) + \cdots + p_n(x_n).
\end{aligned}$$

where the $(\,)$ is $p_2 + \cdots + p_n$.

Strictly convex case is proved with $\leq$ replaced by $<$ by definition of strict convexity. □

Corollary (AM-GM inequality)**.** Given $x_1, \cdots, x_n$ positive reals, then

$$\left(\prod x_i\right)^{1/n} \leq \frac{1}{n} \sum x_i.$$

Proof. Take $f(x) = -\log x$. This is concave since its second derivative is $x^{-2} > 0$.

Take $\mathbb{P}(x = x_i) = 1/n$. Then

$$\mathbb{E}[f(x)] = \frac{1}{n} \sum -\log x_i = -\log \mathrm{GM}$$

and

$$f(\mathbb{E}[X]) = -\log \frac{1}{n} \sum x_i = -\log \mathrm{AM}$$

Since $f(\mathbb{E}[x]) \leq \mathbb{E}[f(x)]$, AM $\geq$ GM. Since $-\log x$ is strictly convex, AM = GM only if all x_i are equal. □

Theorem (Cauch-Schwarz inequality)**.** For any two random variables X, Y,

$$(\mathbb{E}[XY])^2 \leq \mathbb{E}[X^2]\mathbb{E}[Y^2].$$

Proof. If $Y = 0$, then both sides are 0. Otherwise, $\mathbb{E}[Y^2] > 0$. Let

$$w = X - Y \cdot \frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]}.$$

Then

$$\begin{aligned}\mathbb{E}[w^2] &= \mathbb{E}\left[X^2 - 2XY\frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]} + Y^2\frac{(\mathbb{E}[XY])^2}{(\mathbb{E}[Y^2])^2}\right] \\ &= \mathbb{E}[X^2] - 2\frac{(\mathbb{E}[XY])^2}{\mathbb{E}[Y^2]} + \frac{(\mathbb{E}[XY])^2}{\mathbb{E}[Y^2]} \\ &= \mathbb{E}[X^2] - \frac{(\mathbb{E}[XY])^2}{\mathbb{E}[Y^2]}\end{aligned}$$

Since $\mathbb{E}[w^2] \geq 0$, the Cauchy-Schwarz inequality follows. □

Theorem (Markov inequality)**.** If X is a random variable with $\mathbb{E}|X| < \infty$ and $\varepsilon > 0$, then

$$\mathbb{P}(|X| \geq \varepsilon) \leq \frac{\mathbb{E}|X|}{\varepsilon}.$$

Proof. We have

$$I[\{|X| \geq \varepsilon\}] \leq \frac{|X|}{\varepsilon}.$$

Since if $|X| \geq \varepsilon$, then LHS $= 1$ and RHS > 1; If $|X| \leq \varepsilon$, then LHS $= 0$ and RHS is non-zero.

Take the expected value to obtain

$$\mathbb{P}(|X| \geq \varepsilon) \leq \frac{\mathbb{E}|X|}{\varepsilon}.$$

□

Theorem (Chebyshev inequality)**.** If X is a random variable with $\mathbb{E}[X^2] < \infty$ and $\varepsilon > 0$, then

$$\mathbb{P}(|X| \geq \epsilon) \leq \frac{\mathbb{E}[X^2]}{\varepsilon^2}.$$

Proof. We have

$$I[\{|X| \geq \varepsilon\}] \leq \frac{x^2}{\varepsilon^2}.$$

Then take the expected value and the result follows. □

Note: that these not do not make any assumption about distributions of X.

Note: If $\mu = \mathbb{E}[X]$, then

$$\mathbb{P}(|X - \mu| \geq \varepsilon) \leq \frac{\mathbb{E}[(X-\mu)^2]}{\varepsilon^2} = \frac{\operatorname{var} X}{\varepsilon^2}$$

3.5.1 Weak law of large numbers

Theorem (Weak law of large numbers)**.** Let $X_1, X_2, \cdots$ be iid random variables, with mean μ and var σ^2.

Let $S_n = \sum_{i=1}^n X_i$.

Then for all $\varepsilon > 0$,

$$\mathbb{P}\left(\left|\frac{S_n}{n} - \mu\right| \geq \epsilon\right) \to 0$$

as $n \to \infty$.

We say, $\frac{S_n}{n}$ tends to μ (in probability), or

$$\frac{S_n}{n} \to_p \mu.$$

Proof. By Chebyshev,

$$\begin{aligned}
\mathbb{P}\left(\left|\frac{S_n}{n} - \mu\right| \geq \epsilon\right) &\leq \frac{\mathbb{E}\left(\frac{S_n}{n} - \mu\right)^2}{\varepsilon^2} \\
&= \frac{1}{n^2}\frac{\mathbb{E}(S_n - n\mu)^2}{\varepsilon^2} \\
&= \frac{1}{n^2\varepsilon^2}\operatorname{var}(S_n) \\
&= \frac{n}{n^2\varepsilon^2}\operatorname{var}(X_1) \\
&= \frac{\sigma^2}{n\varepsilon^2} \to 0
\end{aligned}$$

□

Note that we cannot relax the "independent" condition. For example, if $X_1 = X_2 = X_3 = \cdots = 1$ or 0, each with probability 1/2. Then $S_n/n \not\to 1/2$ since it is either 1 or 0.

Example. Suppose we toss a coin with probability p of heads. Then

$$\frac{S_n}{n} = \frac{\text{number of heads}}{\text{number of tosses}}.$$

Since $\mathbb{E}[X_i] = p$, then the weak law of large number tells us that

$$\frac{S_n}{n} \to p.$$

Theorem (Strong law of large numbers)**.**

$$\mathbb{P}\left(\frac{S_n}{n} \to \mu \text{ as } n \to \infty\right) = 1.$$

We say

$$\frac{S_n}{n} \to_{\text{as}} \mu,$$

where "as" means "almost surely".

Proof is left for Part II because it is too hard.

3.6 Covariance and correlation

Definition (Covariance)**.** Given two random variables X, Y, the *covariance* is

$$\operatorname{cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])].$$

Proposition.

(i) $\text{cov}(X, c) = 0$ for constant c.

(ii) $\text{cov}(X + c, Y) = \text{cov}(X, Y)$.

(iii) $\text{cov}(X, Y) = \text{cov}(Y, X)$.

(iv) $\text{cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$.

(v) $\text{cov}(X, X) = \text{var}(X)$.

(vi) $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y) + 2\,\text{cov}(X, Y)$.

(vii) If X, Y are independent, $\text{cov}(X, Y) = 0$.

These are all trivial to prove and proof is omitted.
Note that $\text{cov}(X, Y) = 0$ does not imply they are independent.

Example. Let $(X, Y) = (2, 0), (-1, -1)$ or $(-1, 1)$ with equal probabilities of $1/3$. These are not independent since $Y = 0 \Rightarrow X = 2$.

However, $\text{cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0 - 0 \cdot 0 = 0$.

Also if we randomly pick a point on the unit circle, and let the coordinates be (X, Y), then $\mathbb{E}[X] = \mathbb{E}[Y] = \mathbb{E}[XY] = 0$ by symmetry. So $\text{cov}(X, Y) = 0$ but X and Y are clearly not independent (they have to satisfy $x^2 + y^2 = 1$).

Definition (Correlation coefficient)**.** The *correlation coefficient* of X and Y is

$$\text{corr}(X, Y) = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)\,\text{var}(Y)}}.$$

Corollary. $|\,\text{corr}(X, Y)| \leq 1$.

Proof. Apply Cauchy-Schwarz to $X - \mathbb{E}[X]$ and $Y - \mathbb{E}[Y]$. □

3.7 Conditional distribution and expectation

Definition (Conditional distribution)**.** Let X and Y be random variables (in general not independent) with joint distribution $\mathbb{P}(X = x, Y = y)$. Then the *marginal distribution* (or simply *distribution*) of X is

$$\mathbb{P}(X = x) = \sum_{y \in \Omega_u} \mathbb{P}(X = x, Y = y).$$

The *conditional distribution* of X given Y is

$$\mathbb{P}(X = x | Y = y) = \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(Y = y)}.$$

The *conditional expectation* of X given Y is

$$\mathbb{E}[X | Y = y] = \sum_{x \int \Omega_X} x\mathbb{P}(X = x | Y = y).$$

We can view $\mathbb{E}[X|Y]$ as a random variable in Y: given a value of Y, we return the expectation of X.

Example. Consider a dice roll. Let $Y = 1$ denote an even roll and $Y = 0$ denote an odd roll. Let X be the value of the roll. Then $\mathbb{E}[X|Y] = 3 + Y$, ie 4 if even, 3 if odd.

Example. Let $X_1, \cdots, X_n$ be iid $B(1, p)$. Let $Y = X_1 + \cdots + X_n$. Then

$$\begin{aligned}\mathbb{P}(X_1 = 1|Y = r) &= \frac{\mathbb{P}(X_1 = 1, \sum_2^n X_i = r - 1)}{\mathbb{P}(Y = r)}\\ &= \frac{p\binom{n-1}{r-1}p^{r-1}(1-p)^{(n-1)-(r-1)}}{\binom{n}{r}p^r(1-p)^{n-1}} = \frac{r}{n}.\end{aligned}$$

So

$$\mathbb{E}[X_1|Y] = 1 \cdot \frac{r}{n} + 0\left(1 - \frac{r}{n}\right) = \frac{r}{n} = \frac{Y}{n}.$$

Note that this is a random variable!

Theorem. If X and Y are independent, then

$$\mathbb{E}[X|Y] = \mathbb{E}[X]$$

Proof.

$$\begin{aligned}\mathbb{E}[X|Y = y] &= \sum_x x\mathbb{P}(X = x|Y = y)\\ &= \sum_x x\mathbb{P}(X = x)\\ &= \mathbb{E}[X]\end{aligned}$$

□

We know that the expected value of a dice roll given it is even is 4, and the expected value given it is odd is 3. Since it is equally likely to be even or odd, the expected value of the dice roll is 3.5. This is formally captured by

Theorem (Tower property of conditional expectation)**.**

$$\mathbb{E}_Y[\mathbb{E}_X[X|Y]] = \mathbb{E}_X[X],$$

where the subscripts indicate what variable the expectation is taken over.

Proof.

$$\begin{aligned}\mathbb{E}_Y[\mathbb{E}_X[X|Y]] &= \sum_y \mathbb{P}(Y = y)\mathbb{E}[X|Y = y]\\ &= \sum_y \mathbb{P}(Y = y)\sum_x x\mathbb{P}(X = x|Y = y)\\ &= \sum_x \sum_y x\mathbb{P}(X = x, Y = y)\\ &= \sum_x x\sum_y \mathbb{P}(X = x, Y = y)\\ &= \sum_x x\mathbb{P}(X = x)\\ &= \mathbb{E}[X].\end{aligned}$$

□

This is also called the law of total expectation. We can state it as: suppose $A_1, A_2, \cdots, A_n$ is a partition of Ω. Then

$$\mathbb{E}[X] = \sum_{i:\mathbb{P}(A_i)>0} \mathbb{E}[X|A_i]\mathbb{P}(A_i).$$

3.8 Probability generating functions

Consider a random variable X, taking values $0, 1, 2, \cdots$. Let $p_r = \mathbb{P}(X = r)$.

Definition (Probability generating function (pgf))**.** The *probability generating function (pgf)* of X is

$$p(z) = \mathbb{E}[z^X] = \sum_{r=0}^{\infty} \mathbb{P}(X = r)z^r = p_0 + p_1 z + p_2 z^2 \cdots = \sum_{0}^{\infty} p_r z^r.$$

This is a power series (or polynomial), and converges if $|z| \leq 1$, since

$$|p(z)| \leq \sum_r p_r |z^r| \leq \sum_r p_r = 1.$$

We sometimes write as $p_X(z)$ to indicate the random variable.

Example. Consider a fair die. Then $p_r = 1/6$ for $r = 1, \cdots, 6$. Then

$$p(z) = \mathbb{E}[z^X] = \frac{1}{6}(z + z^2 + .., +z^6) = \frac{1}{6} z \left(\frac{1 - z^6}{1 - z}\right).$$

Theorem. The distribution of X is uniquely determined by its probability generating function.

Proof. $p_0 = p(0)$, $p_1 = p'(0)$ etc. (where p' is the derivative of p) In general,

$$\frac{\mathrm{d}^i}{\mathrm{d}z^i} p(z)\bigg|_{z=0} = \frac{r!}{(r-i)!} p_i.$$

So we can recover $(p_0, p_1, \cdots)$ from $p(z)$. □

Theorem (Abel's lemma)**.**

$$\mathbb{E}[X] = \lim_{z \to 1} p'(z).$$

If $p'(z)$ is continuous, then simply $\mathbb{E}[X] = p'(1)$.

Note that this theorem is trivial if $p'(1)$ exists, as long as we know that we can differentiate power series term by term. What is important here is that even if $p'(1)$ doesn't exist, we can still take the limit and obtain the expected value, eg. when $\mathbb{E}[x] = \infty$.

Proof.

$$p'(z) = \sum_{1}^{\infty} r p_r z^{r-1} \leq \sum_{1}^{\infty} r p_r = \mathbb{E}[X].$$

for $z \le 1$. So

$$\lim_{z\to 1} p'(z) \le \mathbb{E}[X].$$

For any ε, if we pick N large, then

$$\sum_1^N rp_r \ge \mathbb{E}[X] - \varepsilon.$$

So

$$\mathbb{E}[X] - \varepsilon \le \sum_1^N rp_r = \lim_{z\to 1} \sum_1^N rp_r z^r \le \lim_{z\to 1} \sum_1^\infty rp_r z^r = \lim_{z\to 1} p'(z).$$

So $\mathbb{E}[X] \le \lim_{z\to 1} p'(z)$. □

Theorem.

$$\mathbb{E}[X(X-1)] = \lim_{z\to 1} p''(z).$$

Proof. Same as above. □

Example. Consider the Poisson distribution. Then

$$p_r = \mathbb{P}(X = r) = \frac{1}{r!}\lambda^r e^{-\lambda}.$$

Then

$$p(z) = \mathbb{E}[z^X] = \sum_0^\infty z^r \frac{1}{r!}\lambda^r e^{-\lambda} = e^{\lambda z}e^{-\lambda} = e^{\lambda(z-1)}.$$

We can have a sanity check: $p(1) = 1$, which makes sense, since $p(1)$ is the sum of probabilities.

We have

$$\mathbb{E}[X] = \left.\frac{\mathrm{d}}{\mathrm{d}z} e^{\lambda(z-1)}\right|_{z=1} = \lambda,$$

and

$$\mathbb{E}[X(X-1)] = \left.\frac{\mathrm{d}^2}{\mathrm{d}x^2} e^{\lambda(z-1)}\right|_{z=1} = \lambda^2$$

So

$$\mathrm{var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

Theorem. Suppose $X_1, X_2, \cdots, X_n$ are independent random variables with pgfs $p_1, p_2, \cdots, p_n$. Then the pgf of $X_1 + X_2 + \cdots + X_n$ is $p_1(z)p_2(z)\cdots p_n(z)$.

Proof.

$$\mathbb{E}[z^{X_1+\cdots+X_n}] = \mathbb{E}[z^{X_1}\cdots z^{X_n}] = \mathbb{E}[z^{X_1}]\cdots\mathbb{E}[z^{X_n}] = p_1(z)\cdots p_n(z).$$

□

Example. Let $X \sim B(n, p)$. Then

$$p(z) = \sum_{r=0}^n \mathbb{P}(X = r)z^r = \sum \binom{n}{r} p^r(1-p)^{n-r}z^r = (pz + (1-p))^n = (pz+q)^n.$$

So $p(z)$ is the product of n copies of $pz + q$. But $pz + q$ is the pgf of $Y \sim B(1, p)$.

This shows that $X = Y_1 + Y_2 + \cdots + Y_n$ (which we already knew).

Example. If X and Y are independent Poisson random variables with parameters λ, μ respectively, then

$$\mathbb{E}[t^{X+Y}] = \mathbb{E}[t^X]\mathbb{E}[t^Y] = e^{\lambda(t-1)}e^{\mu(t-1)} = e^{(\lambda+\mu)(t-1)}$$

So $X + Y \sim \mathbb{P}(\lambda + \mu)$.

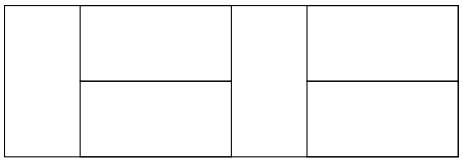
We can also do it directly:

$$\mathbb{P}(X + Y = r) = \sum_{i=0}^{r} \mathbb{P}(X = i, Y = r - i) = \sum_{i=0}^{r} \mathbb{P}(X = i)\mathbb{P}(X = r - i),$$

but is much more complicated.

We can use pgf-like functions to obtain some combinatorial results.

Example. Suppose we want to tile a $2 \times n$ bathroom by 2×1 tiles. One way to do it is



We can do it recursively: suppose there are f_n ways to tile a $2 \times n$ grid. Then if we start tiling, the first tile is either vertical, in which we have f_{n-1} ways to tile the remaining ones; or the first tile is horizontal, in which we have f_{n-2} ways to tile the remaining. So

$$f_n = f_{n-1} + f_{n-2},$$

which is simply the Fibonacci sequence, with $f_0 = f_1 = 1$.

Let

$$F(z) = \sum_{n=0}^{\infty} f_n z^n.$$

Then from our recurrence relation, we obtain

$$f_n z^n = f_{n-1} z^n + f_{n-2} z^n.$$

So

$$\sum_{n=2}^{\infty} f_n z^n = \sum_{n=2}^{\infty} f_{n-1} z^n + \sum_{n=2}^{\infty} f_{n-2} z^n.$$

Since $f_0 = f_1 = 1$, we have

$$F(z) - f_0 - z f_1 = z(F(z) - f_0) + z^2 F(z).$$

Thus $F(z) = (1 - z - z^2)^{-1}$. If we write

$$\alpha_1 = \frac{1}{2}(1 + \sqrt{5}), \quad \alpha_2 = \frac{1}{2}(1 - \sqrt{5}).$$

then we have

$$\begin{aligned} F(z) &= (1 - z - z^2)^{-1} \\ &= \frac{1}{(1 - \alpha_1 z)(1 - \alpha_2 z)} \\ &= \frac{1}{\alpha_1 - \alpha_2}\left(\frac{\alpha_1}{1 - \alpha_1 z} - \frac{\alpha_2}{1 - \alpha_2 z}\right) \\ &= \frac{1}{\alpha_1 - \alpha_2}\left(\alpha_1 \sum_{n=0}^{\infty} \alpha_1^n z^n - \alpha_2 \sum_{n=0}^{\infty} \alpha_2^n z^n\right). \end{aligned}$$

So

$$f_n = \frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{-\alpha_1 - \alpha_2}.$$

We can also consider *Dyck words*, such as (), or ((())()), ie any list of brackets that match (cf Lisp).

There is only one Dyck word of length 2, (). There are 2 of length 4, (()) and ()(). Similarly, there are 5 Dyck words of length 5.

Let C_n be the number of Dyck words of length $2n$. Each Dyck word can be split written as $(w_1)w_2$, where w_1 and w_2 are Dyck words. Since the lengths of w_1 and w_2 must sum up to $2(n-1)$,

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}. \qquad (*)$$

We again use pgf-like functions: let

$$c(x) = \sum_{n=0}^{\infty} C_n x^n.$$

From $(*)$, we can show that

$$c(x) = 1 + xc(x)^2.$$

We can solve to show that

$$c(x) = \frac{1 - \sqrt{1 - 4x}}{2x} = \sum_{0}^{\infty} \binom{2n}{n} \frac{x^n}{n+1},$$

noting that $C_0 = 1$. Then

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Sums with a random number of terms

We would like to consider a sum with a random number of terms. For example, an insurance company may receive a random number of claims, each demanding a random amount of money. Then we have a sum of a random number of terms. This can be answered using probability generating functions.

Example. Let $X_1, X_2, \cdots, X_n$ be iid with pgf $p(z) = \mathbb{E}z^X$. Let N be a random variable independent of X_i with pgf $h(z)$. What is the pgf of $S_n = X_1 + \cdots + X_N$?

$$\begin{aligned}
\mathbb{E}[z^{S_n}] &= \mathbb{E}[z^{X_1+\cdots+X_N}] \\
&= \mathbb{E}_N[\underbrace{\mathbb{E}_{X_i}[z^{X_1+\ldots+X_N}|N]}_{\text{assuming fixed } N}] \\
&= \sum_{n=0}^{\infty} \mathbb{P}(N=n)\mathbb{E}[z^{X_1+X_2+\cdots+X_n}] \\
&= \sum_{n=0}^{\infty} \mathbb{P}(N=n)\mathbb{E}[z^{X_1}]\mathbb{E}[z^{X_2}]\cdots\mathbb{E}[z^{X_n}] \\
&= \sum_{n=0}^{\infty} \mathbb{P}(N=n)(\mathbb{E}[z^{X_1}])^n \\
&= \sum_{n=0}^{\infty} \mathbb{P}(N=n)p(z)^n \\
&= h(p(z))
\end{aligned}$$

since $h(x) = \sum_{n=0}^{\infty} \mathbb{P}(N=n)x^n$.

So

$$\begin{aligned}
\mathbb{E}[S_N] &= \frac{\mathrm{d}}{\mathrm{d}z}h(p(z))\bigg|_{z=1} \\
&= h'(p(1))p'(1) \\
&= \mathbb{E}[N]\mathbb{E}[X_1]
\end{aligned}$$

To calculate the variance, use the fact that

$$\mathbb{E}[S_n(S_n-1)] = \frac{\mathrm{d}^2}{\mathrm{d}z^2}h(p(z))\bigg|_{z=1}.$$

Then we can find that

$$\operatorname{var}(S_N) = \mathbb{E}[N]\operatorname{var}(X_1) + \mathbb{E}[X_1^2]\operatorname{var}(N).$$

4 Interesting problems

4.1 Branching processes

Branching processes are used to model population growth by reproduction. Consider $X_0, X_1, \cdots$, where X_n is the number of individuals in the nth generation. We assume the following:

(i) $X_0 = 1$

(ii) Each individual lives for unit time and produces k offspring with probability p_k.

(iii) Suppose all offspring behave independently. Then

$$X_{n+1} = Y_1^n + Y_2^n + \cdots + Y_{X_n}^n,$$

where Y_i^n are iid random variables, which is the same as X_1 (the superscript denotes the generation).

It is useful to consider the pgf of a branching process. Let $F(z)$ be the pgf of Y_i^n. Then

$$F(z) = E[z^{Y_i^n}] = E[z^{X_1}] = \sum_{k=0}^{\infty} f_k z^k.$$

Define

$$F_n(z) = E[z^{X_n}].$$

Theorem.

$$F_{n+1}(z) = F_n(F(z)) = F(F(F(\cdots F(z)\cdots))) = F(F_n(z)).$$

Proof.

$$\begin{aligned}
F_{n+1}(z) &= \mathbb{E}[z^{X_{n+1}}] \\
&= \mathbb{E}[\mathbb{E}[z^{X_{n+1}}|X_n]] \\
&= \sum_{k=0}^{\infty} \mathbb{P}(X_n = k)\mathbb{E}[z^{X_{n+1}}|X_n = k] \\
&= \sum_{k=0}^{\infty} \mathbb{P}(X_n = k)\mathbb{E}[z^{Y_1^n+\cdots+Y_k^n}|X_n = k] \\
&= \sum_{k=0}^{\infty} \mathbb{P}(X_n = k)\mathbb{E}[z^{Y_1^n}]\mathbb{E}[z^{Y_2}]\cdots\mathbb{E}[z^{Y_n}] \\
&= \sum_{k=0}^{\infty} \mathbb{P}(X_n = k)(\mathbb{E}[z^{X_1}])^k \\
&= \sum_{k=0} \mathbb{P}(X_n = k)F(z)^k \\
&= F_n(F(z))
\end{aligned}$$

□

Theorem. Suppose $\mathbb{E}[X_1] = \sum kf_k = \mu$ and $\operatorname{var}(X_1) = \mathbb{E}[(X-\mu)^2] = \sum (k-\mu)^2 f_k < \infty$.

Then $\mathbb{E}[X_n] = \mu^n$, and

$$\operatorname{var} X_n = \frac{\sigma^2 \mu^{n-1}(\mu^n - 1)}{\mu - 1},$$

if $\mu \neq 1$, $n\sigma^2$ if $\mu = 1$.

Proof.

$$\begin{aligned}\mathbb{E}[X_n] &= \mathbb{E}[\mathbb{E}[X_n | X_{n-1}]] \\ &= \mathbb{E}[\mu X_{n-1}] \\ &= \mu \mathbb{E}[X_{n-1}]\end{aligned}$$

Then by induction, $\mathbb{E}[X_n] = \mu^n$ (since $X_0 = 1$).

To calculate the variance, note that

$$\operatorname{var}(X_n) = \mathbb{E}[X_n^2] - (\mathbb{E}[X_n])^2$$

and hence

$$\mathbb{E}[X_n^2] = \operatorname{var}(X_n) + (\mathbb{E}[X])^2$$

We then calculate

$$\begin{aligned}\mathbb{E}[X_n^2] &= \mathbb{E}[\mathbb{E}[X_n^2 | X_{n-1}]] \\ &= \mathbb{E}[\operatorname{var}(X_n) + (\mathbb{E}[X])^2 | X_{n-1}] \\ &= \mathbb{E}[X_{n-1} \operatorname{var}(X_1) + (\mu X_{n-1})^2] \\ &= \mathbb{E}[X_{n-1}\sigma^2 + (\mu X_{n-1})^2] \\ &= \sigma^2 \mu^{n-1} + \mu^2 \mathbb{E}[X_{n-1}^2]\end{aligned}$$

So

$$\begin{aligned}\operatorname{var} X_n &= \mathbb{E}[X_n^2] - (\mathbb{E}[X_n])^2 \\ &= \mu^2 \mathbb{E}[X_{n-1}^2] + \sigma^2 \mu^{n-1} - \mu^2 (\mathbb{E}[X_{n-1}])^2 \\ &= \mu^2 (\mathbb{E}[X_{n-1}^2] - \mathbb{E}[X_{n-1}]^2) + \sigma^2 \mu^{n-1} \\ &= \mu^2 \operatorname{var}(X_{n-1}) + \sigma^2 \mu^{n-1} \\ &= \mu^4 \operatorname{var}(X_{n-2}) + \sigma^2(\mu^{n-1} + \mu^n) \\ &= \cdots \\ &= \mu^{2(n-1)} \operatorname{var}(X_1) + \sigma^2(\mu^{n-1} + \mu^n + \cdots + \mu^{2n-3}) \\ &= \sigma^2 \mu^{n-1}(1 + \mu + \cdots + \mu^{n-1}).\end{aligned}$$

Of course, we can also obtain this using the probability generating function as well. □

Extinction probability

Let A_n be the event $X_n = 0$, ie extinction has occurred by the nth generation. Let q be the probability that extinction eventually occurs. Then

$$A = \bigcup_{n=1}^{\infty} A_n = [\text{extinction eventually occurs}].$$

Since $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$, we know that

$$q = \mathbb{P}(A) = \lim_{n\to\infty} \mathbb{P}(A_n) = \lim_{n\to\infty} \mathbb{P}(X_n = 0).$$

But

$$\mathbb{P}(X_n = 0) = F_n(0),$$

since $F_n(0) = \sum \mathbb{P}(X_n = k) z^k$. We know that

$$F(q) = F\left(\lim_{n\to\infty} F_n(0)\right) = \lim_{n\to\infty} F(F_n(0)) = \lim_{n\to\infty} F_{n+1}(0) = q.$$

So

$$F(q) = q.$$

Alternatively, using the law of total probability

$$q = \sum_k \mathbb{P}(X_1 = k)\mathbb{P}(\text{extinction}|X_1 = k) = \sum_k f_k q^k = F(q),$$

where the second equality comes from the fact that for the whole population to go extinct, each individual population must go extinct.

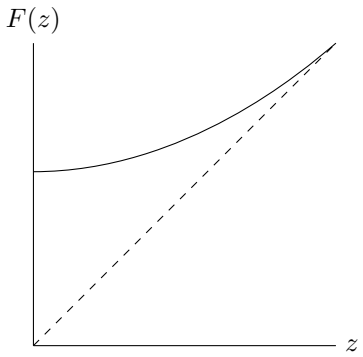
Theorem. The probability of extinction q is the smallest root to the equation $q = F(q)$. Suppose $\mu = \mathbb{E}[X_1]$, then if $\mu \leq 1$, then $q = 1$; if $\mu > 1$, then $q < 1$.

Proof. To show that it is the smallest root, let α be the smallest root. Then note that $0 \leq \alpha \Rightarrow F(0) \leq F(\alpha) = \alpha$ since F is increasing (proof: write the function out!). Hence $F(F(0)) \leq \alpha$. Continuing inductively, $F_n(0) \leq \alpha$ for all n. So

$$q = \lim_{n\to\infty} F_n(0) \leq \alpha.$$

So $q = \alpha$.

To show that $q = 1$ when $\mu \leq 1$, we show that $q = 1$ is the only root. We know that $F'(z), F''(z) \geq 0$ for $z \in (0, 1)$ (proof: write it out again!). So F is increasing and convex. Since $F'(1) = \mu \leq 1$, it must approach $(1, 1)$ from above the $F = z$ line. So it must look like this:



So $z = 1$ is the only root. □

4.2 Random walk and gambler's ruin

Here we'll study random walks, using gambler's ruin as an example.

Definition (Random walk)**.** Let $X_1, \cdots, X_n$ be iid random variables such that $X_n = +1$ with probability p, and -1 with probability $1-p$. Let $S_n = S_0 + X_1 + \cdots + X_n$. Then $(S_0, S_1, \cdots, S_n)$ is a *1-dimensional random walk*.

If $p = q = \frac{1}{2}$, we say it is a *symmetric random walk*.

Example. A gambler starts with $\$z$, with $z < a$, and plays a game in which he wins \$1 or loses \$1 at each turn with probabilities p and q respectively. What are

$$p_z = \mathbb{P}(\text{random walk hits } a \text{ before } 0 | \text{starts at } z),$$

and

$$q_z = \mathbb{P}(\text{random walk hits } 0 \text{ before } a | \text{starts at } z)?$$

He either wins his first game, with probability p, or loses with probability q. Then

$$p_z = qp_{z-1} + pp_{z+1},$$

for $0 < z < a$, and $p_0 = 0, p_a = 1$.

Try $p_z = t^z$. Then

$$pt^2 - t + q = (pt - q)(t - 1) = 0,$$

noting that $p = 1 - q$. If $p \neq q$, then

$$p_z = A1^z + B\left(\frac{q}{p}\right)^z.$$

Since $p_z = 0$, $A = -B$. Since $p_a = 1$, we obtain

$$p_z = \frac{1 - (p/q)^z}{1 - (p/q)^a}.$$

If $p = q$, then $p_z = A + Bz = z/a$.

Similarly, (or perform the substitutions $p \mapsto q$, $q \mapsto p$ and $z \mapsto a - z$)

$$q_z = \frac{(p/q)^z - (q/p)^a}{1 - (p/q)^a}$$

if $p \neq q$, and

$$q_z = \frac{a - z}{z}$$

if $p = q$. Since $p_z + q_z = 1$, we know that the game will eventually end.

What if $a \to \infty$? What is the probability of going bankrupt?

$$\begin{aligned}
\mathbb{P}(\text{path hits 0 ever}) &= \mathbb{P}\left(\bigcup_{a=z+1}^{\infty} [\text{path hits 0 before } a]\right) \\
&= \lim_{a\to\infty} \mathbb{P}(\text{path hits 0 before } a) \\
&= \lim_{a\to\infty} q_z \\
&= \begin{cases} (p/q)^z & p > q \\ 1 & p \leq q. \end{cases}
\end{aligned}$$

Duration of the game

Let D_z = expected time until the random walk hits 0 or a, starting from z. We first show that this is bounded. We know that there is one simple way to hit 0 or a: get $+1$ or -1 for a times in a row. This happens with probability $p^a + q^a$, and takes a steps. So even if this were the only way to hit 0 or a, the expected duration would be $\frac{a}{p^a+q^a}$. So we must have

$$D_z \leq \frac{a}{p^a + q^a}$$

This is a *very* crude bound, but it is sufficient to show that it is bounded, and we can meaningfully apply formulas to this finite quantity.

We have

$$\begin{aligned} D_z &= \mathbb{E}[\text{duration}] \\ &= \mathbb{E}[\mathbb{E}[\text{duration}|X_1]] \\ &= p\mathbb{E}[\text{duration}|X_1 = 1] + q\mathbb{E}[\text{duration}|X_1 = -1] \\ &= p(1 + D_{z+1}) + q(1 - D_{z-1}) \end{aligned}$$

So

$$D_z = 1 + pD_{z+1} + qD_{z-1},$$

subject to $D_0 = D_a = 0$.

We first find a particular solution by trying $D_z = Cz$. Then

$$Cz = 1 + pC(z + 1) = qC(z - 1).$$

So

$$C = \frac{1}{q - p},$$

for $p \neq q$. The we find the complementary solution. Try $D_z = t^z$.

$$pt^2 - t + q = 0,$$

which has roots 1 and q/p. So the general solution is

$$D_z = A + B\left(\frac{q}{p}\right)^z + \frac{z}{q - p}.$$

Putting in the boundary conditions,

$$D_z = \frac{z}{q - p} - \frac{a}{q - p} \cdot \frac{1 - (q/p)^z}{1 - (q/p)^a}.$$

If $p = q$, then to find the particular solution, we have to try

$$D_z = Cz^2.$$

Then we find $C = -1$. So

$$D_z = -z^2 + A + Bz.$$

Then the boundary conditions give

$$D_2 = z(a - z).$$

Using generating functions

We can use generating functions to solve the problem as well.

Let $U_{z,n} = \mathbb{P}(\text{random walk absorbed at 0 at } n|\text{start in } z)$.

We have the following conditions: $U_{0,0} = 1$; $U_{z,0=0}$ for $0 < z \leq a$; $U_{0,n} = U_{a,n} = 0$ for $n > 0$.

We define a pgf-like function.

$$U_z(s) = \sum_{n=0}^{\infty} U_{z,n} s^n.$$

We know that

$$U_{z,n+1} = pU_{z+1,n} + qU_{z-1}, n.$$

Multiply by s^{n+1} and sum on $n = 0, 1, \cdots$. Then

$$U_z(s) = psU_{z+1}(s) + qsU_{z-1}(s).$$

We try $U_z(s) = [\lambda(s)]^z$. Then

$$\lambda(s) = ps\lambda(s)^2 + s.$$

Then

$$\lambda_1(s), \lambda_2(s) = \frac{1 \pm \sqrt{1 - 4pqs^2}}{2ps}.$$

So

$$U_z(s) = A(s)\lambda_1(s)^z + B(s)\lambda_z(s)^2.$$

Since $U_0(s) = 1$ and $U_a(s) = 0$, we know that

$$A(s) + B(s) = 1$$

and

$$A(s)\lambda_1(s)^a + B(s)\lambda_2(s)^a = 0.$$

Then we find that

$$U_z(s) = \frac{\lambda_1(s)^a \lambda_2(s)^z - \lambda_2(s)^a \lambda_1(s)^z}{\lambda_1(s)^a - \lambda_2(s)^a}.$$

Since $\lambda_1(s)\lambda_2(s) = \frac{q}{p}$, we can "simplify" this to obtain

$$U_z(s) = \left(\frac{q}{p}\right)^z \cdot \frac{\lambda_1(s)^{a-z} - \lambda_2(s)^{a-z}}{\lambda_1(s)^a - \lambda_2(s)^a}.$$

We see that $U_z(1) = q_z$. We can apply the same method to find the generating function for absorption at a, say $V_z(s)$. Then the generating function for the duration is $U + Z$. Hence the expected duration is $D_z = U_z'(1) + V_z'(1)$.

5 Continuous random variables

5.1 Continuous random variables

Consider Ω with a *continuum* of outcomes, like $\Omega = \{\omega : 0 \leq \omega \leq 2\pi\}$ (eg. a spinner). Then we have

$$\mathbb{P}(\omega \in [0, \theta]) = \frac{\theta}{2\pi}, \quad 0 \leq \theta \leq 2\pi.$$

Definition (Continuous random variable)**.** A *continuous random variable* X is a real-valued function $X : \Omega \to \mathbb{R}$ such that

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f(x)\, \mathrm{d}x,$$

where f is the *probability density function*, which satisfies

- $f \geq 0$
- $\int_{-\infty}^{\infty} f(x) = 1$.

Note that $\mathbb{P}(X = a) = 0$ since it is $\int_a^a f(x)\, \mathrm{d}x$. Then we also have

$$\mathbb{P}\left(\bigcup_{a \in \mathbb{Q}} [X = a]\right) = 0,$$

since it is a countable union of probability 0 events (and axiom 3 states that the probability of the countable union is the sum of probabilities, ie. 0).

Informally,

$$\mathbb{P}(X \in [x, x + \delta x]) = \int_x^{x+\delta x} f(z)\, \mathrm{d}z \approx f(x)\delta x.$$

Definition (Cumulative distribution function)**.** The *cumulative distribution function* (or simply *distribution function*) of a random variable X (discrete, continuous, or neither) is

$$F(x) = \mathbb{P}(X \leq x).$$

We can see that $F(x)$ is increasing and $F(x) \to 1$ as $x \to \infty$.

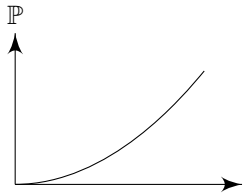
In the case of continuous random variables, we have

$$f(x) = \int_{\infty}^{x} f(z)\, \mathrm{d}z.$$

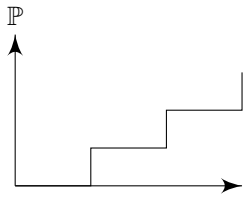
Then F is continuous and differentiable. In general, $F'(x) = f(x)$ whenever F is differentiable.

The name of *continuous* random variables comes from the fact that $F(x)$ is (absolutely) continuous.

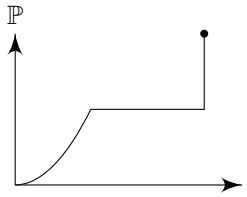
The cdf of a continuous random variable might look like this:



The cdf of a discrete random variable might look like this:



The cdf of a random variable that is neither discrete nor continuous might look like this:



Note that we always have

$$\mathbb{P}(a < x \leq b) = F(b) - F(a).$$

This will be equal to $\int_a^b f(x)\,\mathrm{d}x$ in the case of continuous random variables.

5.2 Certain important distributions

Definition (Uniform distribution)**.** The *uniform distribution* on $[a, b]$ has pdf

$$f(x) = \frac{1}{b-a}.$$

Then

$$F(x) = \int_a^x f(z)\,\mathrm{d}z = \frac{x-a}{b-a}$$

for $a \leq x \leq b$.

If X follows a uniform distribution on $[a, b]$, we write $X \sim U[a, b]$.

Definition (Exponential random variable)**.** The *exponential random variable with parameter* λ has pdf

$$f(x) = \lambda e^{-\lambda x}$$

and

$$F(x) = 1 - e^{-\lambda x}$$

for $x \geq 0$.

We write $X \sim \mathcal{E}(\lambda)$.

Then

$$\mathbb{P}(a \leq x \leq b) = \int_a^b f(z)\,\mathrm{d}z = e^{-\lambda a} - e^{-\lambda b}.$$

Proposition. The exponential random variable is *memoryless*, ie.

$$\mathbb{P}(X \geq x + z | X \geq x) = \mathbb{P}(X \geq z).$$

This means that, say if X measures the lifetime of a light bulb, knowing it has already lasted for 3 hours does not give any information about how much longer it will last.

Note: The geometric random variable is the discrete memoryless random variable.

Proof.

$$\begin{aligned}
\mathbb{P}(X \geq x + z | X \geq x) &= \frac{\mathbb{P}(X \geq x + z)}{\mathbb{P}(X \geq x)} \\
&= \frac{\int_{x+z}^{\infty} f(u)\,\mathrm{d}u}{\int_x^{\infty} f(u)\,\mathrm{d}u} \\
&= \frac{e^{-\lambda(x+z)}}{e^{-\lambda x}} \\
&= e^{-\lambda z} \\
&= \mathbb{P}(X \geq z).
\end{aligned}$$

□

Similarly, given that, you have lived for x days, what is the probability of dying within the next δx days?

$$\begin{aligned}
\mathbb{P}(X \leq x + \delta x | X \geq x) &= \frac{\mathbb{P}(x \leq X \leq x + \delta x)}{\mathbb{P}(X \geq x)} \\
&= \frac{\lambda e^{-\lambda x}\delta x}{e^{-\lambda x}} \\
&= \lambda \delta x.
\end{aligned}$$

So it is independent of how old you currently are, assuming your survival follows an exponential distribution.

In general, we can define the hazard rate to be

$$h(x) = \frac{f(x)}{1 - F(x)}.$$

Then

$$\mathbb{P}(x \leq X \leq x + \delta x | X \geq x) = \frac{\mathbb{P}(x \leq X \leq x + \delta x)}{\mathbb{P}(X \geq x)} = \frac{\delta x f(x)}{1 - F(x)} = \delta x \cdot h(x).$$

In the case of exponential distribution, $h(x)$ is constant and equal to λ.

5.3 Distribution of a function of a random variable

Theorem. If X is a continuous random variable with a pdf $f(x)$, and $h(x)$ is a continuous, strictly increasing function with $h^{-1}(x)$ differentiable, then $Y = h(X)$ is a random variable with pdf

$$f_Y(y) = f_X(h^{-1}(y))\frac{\mathrm{d}}{\mathrm{d}y}h^{-1}(y).$$

Proof.

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) \\ &= \mathbb{P}(h(X) \leq y) \\ &= \mathbb{P}(X \leq h^{-1}(y)) \\ &= F(h^{-1}(y)) \end{aligned}$$

Take the derivative with respect to y to obtain

$$f_Y(y) = F_Y'(y) = f(h^{-1}(y))\frac{\mathrm{d}}{\mathrm{d}y}h^{-1}(y).$$

□

It is often easier to redo the proof than to remember the result.

Example. Let $X \sim U[0,1]$. Let $Y = -\log X$. Then

$$\begin{aligned} \mathbb{P}(Y \leq y) &= \mathbb{P}(-\log X \leq y) \\ &= \mathbb{P}(X \geq e^{-y}) \\ &= (1 - e^{-y}). \end{aligned}$$

But this is the cumulative distribution function of $\mathcal{E}(1)$. So Y is exponentially distributed with $\lambda = 1$.

Theorem. Let $U \sim U[0,1]$. For any strictly increasing distribution function F, the random variable $X = F^{-1}U$ has distribution function F.

Proof.

$$\mathbb{P}(X \leq x) = \mathbb{P}(F^{-1}(U) \leq x) = \mathbb{P}(Y \leq F(x)) = F(x).$$

□

Note: This is true when F is not strictly increasing, provided we define $F^{-1}(u) = \inf\{x : F(x) \geq u, 0 < u < 1\}$.

Note: This can also be done for discrete random variables $\mathbb{P}(X = x_i) = p_i$ by letting

$$X = x_j \text{ if } \sum_{i=0}^{j-1} p_i \leq U < \sum_{i=0}^{j} p_i,$$

for $U \sim U[0,1]$.

5.4 Expectation and variance

Definition (Expectation)**.** The *expectation* (or *mean*) of a continuous random variable is

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xf(x)\,\mathrm{d}x,$$

provided not both $\int_0^\infty xf(x)\,\mathrm{d}x$ and $\int_{-\infty}^0 xf(x)\,\mathrm{d}x$ are infinite.

Theorem. If X is a continuous random variable, then

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X \geq x)\,\mathrm{d}x - \int_0^\infty \mathbb{P}(X \leq -x)\,\mathrm{d}x.$$

This result is more intuitive in the discrete case:

$$\sum_{x=0}^{\infty} x\mathbb{P}(X = x) = \sum_{x=0}^{\infty}\sum_{y=x+1}^{\infty} \mathbb{P}(X = y) = \sum_{x=0}^{\infty} \mathbb{P}(X > x),$$

where the first equality holds because on both sides, we have x copies of $\mathbb{P}(X = x)$ in the sum.

Proof.

$$\begin{aligned}
\int_0^\infty \mathbb{P}(X \geq x)\,\mathrm{d}x &= \int_0^\infty \int_x^\infty f(y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^\infty \int_0^\infty I[y \geq x] f(y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^\infty \left(\int_0^\infty I[x \leq y]\,\mathrm{d}x\right) f(y)\,\mathrm{d}y \\
&= \int_0^\infty yf(y)\,\mathrm{d}y.
\end{aligned}$$

We can similarly show that $\int_0^\infty \mathbb{P}(X \leq -x)\,\mathrm{d}x = -\int_{-\infty}^0 yf(y)\,\mathrm{d}y$. □

Example. Suppose $X \sim \mathcal{E}(\lambda)$. Then

$$\mathbb{P}(X \geq x) = \int_x^\infty \lambda e^{-\lambda t}\,\mathrm{d}t = e^{-\lambda x}.$$

So

$$\mathbb{E}[X] = \int_0^\infty e^{-\lambda x}\,\mathrm{d}x = \frac{1}{\lambda}.$$

Definition (Variance)**.** The *variance* of a continuous random variable is

$$\mathrm{var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (E[X])^2.$$

So we have

$$\mathrm{var}(X) = \int_{-\infty}^\infty x^2 f(x)\,\mathrm{d}x - \left(\int_{-\infty}^\infty xf(x)\,\mathrm{d}x\right)^2.$$

Example. Let $X \sim U[a, b]$. Then

$$\mathbb{E}[X] = \int_a^b \frac{1}{b-a}\, dx = \frac{a+b}{2}.$$

So

$$\begin{aligned}\mathrm{var}(X) &= \int_a^b b^2 \frac{1}{b-1}\, \mathrm{d}x - (\mathbb{E}[X])^2 \\ &= \frac{1}{12}(b-a)^2.\end{aligned}$$

5.5 Stochastic ordering and inspection paradox

Note: Stochastic is a fancy way of saying "random"

Recall our non-transitive dice. It is *weird* to have non-transitivity. So we want a way of linearly ordering them

Definition (Stochastic order)**.** We define the *stochastic order* as follows: $X \geq_{\mathrm{st}} Y$ if $\mathbb{P}(x > t) \geq \mathbb{P}(Y > t)$ for all t.

This is clearly transitive. Stochastic ordering implies expectation ordering, since

$$X \geq_{\mathrm{st}} Y \Rightarrow \int_0^\infty \mathbb{P}(X > x)\, \mathrm{d}x \geq \int_0^\infty \mathbb{P}(y > x) \Rightarrow \mathbb{E}[X] \geq \mathbb{E}[Y].$$

Note: We can also order things by hazard rate. Recall that

$$h(x) = \frac{f(x)}{1 - F(x)} = -\frac{\mathrm{d}}{\mathrm{d}x} \log(1 - F(x)).$$

Now suppose that n families have children attending a school. Family i has X_i children at the school, where $X_1, \cdots, X_n$ are iid random variables, with $P(X_i = k) = p_k$. Suppose that the average family size is μ.

Now pick a child at random. What is the probability distribution of his family size? Let J be the index of the family from which she comes. Then

$$\mathbb{P}(X_J = k | J = j) = \frac{\mathbb{P}(J = j, X_j = k)}{\mathbb{P}(J = j)}.$$

The denominator is $1/n$. The numerator is more complex. This would require the jth family to have k members, which happens with probability p_k, and that we picked a member from the jth family. This happens with probability $\mathbb{E}\left[\frac{k}{k+\sum_{i\neq j} X_i}\right]$. So

$$\mathbb{P}(X_J = k | J = j) = \mathbb{E}\left[\frac{nkp_k}{k + \sum_{i\neq j} X_i}\right].$$

Note that this is independent of j. So

$$\mathbb{P}(X_J = k) = \mathbb{E}\left[\frac{nkp_k}{k + \sum_{i\neq j} X_i}\right].$$

Also, $\mathbb{P}(X_1 = k) = p_k$. So

$$\frac{\mathbb{P}(X_J = k)}{\mathbb{P}(X_1 = k)} = \mathbb{E}\left[\frac{nk}{k + \sum_{i\neq j} X_i}\right].$$

This is increasing in k, and greater than 1 for $k > \mu$. So the average value of the family size of the child we picked is greater than the average family size. It can be shown that X_J is stochastically greater than X_1.

5.6 Jointly distributed random variables

Definition (Joint distribution)**.** Two random variables X, Y have *joint distribution* $F : \mathbb{R}^2 \mapsto [0, 1]$ with

$$F(x, y) = \mathbb{P}(X \leq x, Y \leq y)$$

The *marginal distribution* of X is

$$F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(X \leq x, Y < \infty) = F(x, \infty) = \lim_{y\to\infty} F(x, y)$$

Definition (Jointly distributed random variables)**.** We say $X_1, \cdots, X_n$ are *jointly distributed continuous random variables* and have *joint pdf* f if for any set $A \subseteq \mathbb{R}^n$

$$\mathbb{P}((X_1, \cdots, X_n) \in A) = \int_{(x_1,\cdots x_n)\in A} f(x_1, \cdots, x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

where

$$f(x_1, \cdots, x_n) \geq 0$$

and

$$\int_{\mathbb{R}^n} f(x_1, \cdots, x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n = 1.$$

Example. In the case where $n = 2$,

$$F(x, y) = \mathbb{P}(X \leq x, Y \leq y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x, y)\, \mathrm{d}x\, \mathrm{d}y.$$

If F is differentiable, then

$$f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y).$$

Theorem. If X and Y are jointly continuous random variables, then they are individually continuous random variables.

Proof. We prove this by showing that X has a density function.

We know that

$$\begin{aligned}\mathbb{P}(X \in A) &= \mathbb{P}(X \in A, Y \in (-\infty, +\infty)) \\ &= \int_{x\in A}\int_{-\infty}^{\infty} f(x, y)\, \mathrm{d}y\, \mathrm{d}x \\ &= \int_{x\in A} f_X(x)\, \mathrm{d}x\end{aligned}$$

So

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, \mathrm{d}y$$

is the (marginal) pdf of X. □

5.7 Independence of continuous random variables

Definition (Independent continuous random vairables)**.** Continuous random variables $X_1, \cdots, X_n$ are independent if

$$\mathbb{P}(X_1 \in A_1, X_2 \in A_2, \cdots, X_n \in A_n) = \mathbb{P}(X_1 \in A_1)\mathbb{P}(X_2 \in A_2) \cdots \mathbb{P}(X_n \in A_n)$$

for all $A_i \in \Omega_{X_i}$.

If we let F_{X_i} and f_{X_i} be the cdf, pdf of X, then

$$F(x_1, \cdots, x_n) = F_{X_1}(x_1) \cdots F_{X_n}(x_n)$$

and

$$f(x_1, \cdots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n)$$

are equivalent to the definition above.

Example. For example, if (X_1, X_2) takes a random value from $[0, 1] \times [0, 1]$, then $f(x_1, x_2) = 1$. Then we can see that $f(x_1, x_2) = 1 \cdot 1 = f(x_1) \cdot f(x_2)$. So X_1 and X_2 are independent.

On the other hand, if (Y_1, Y_2) takes a random value from $[0, 1] \times [0, 1]$ with the restriction that $Y_1 \leq Y_2$, then they are not independent, since $f(x_1, x_2) = 2I[Y_1 \leq Y_2]$, which cannot be split into two parts.

Proposition. For independent continuous random variables X_i,

(i) $\mathbb{E}[\prod X_i] = \prod \mathbb{E}[X_i]$

(ii) $\mathrm{var}(\sum X_i) = \sum \mathrm{var}(X_i)$

5.8 Geometric probability

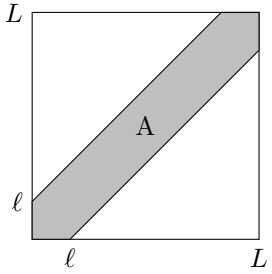
These outcomes can be visualized and probabilities can be found with the aid of a picture.

Example. Two points X and Y are chosen independently on a line segment of length L. What is the probability that $|X - Y| \leq \ell$? By "at random", we mean

$$f(x, y) = \frac{1}{L^2},$$

since each of X and Y have pdf $1/L$.

We can visualize this on a graph:



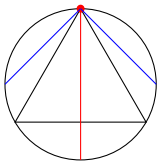
where A is the permitted region, and the two axes are the values of X and Y. The total area of the white part is simple the area of a square with length $L - \ell$. So the area of A is $L^2 - (L-\ell)^2 = 2L\ell - \ell^2$. So the desired probability is

$$\int_A f(x, y)\, \mathrm{d}x\, \mathrm{d}y = \frac{2\ell - \ell^2}{L^2}.$$

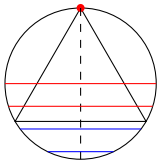
Example (Bertrand's paradox)**.** Suppose we draw a random chord in a circle. What is the probability that the length of the chord is greater than the length of the side of an inscribed equilateral triangle?

There are three ways of "drawing a random chord".

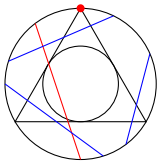
(i) We randomly pick two end points over the circumference independently. If we draw the inscribed triangle with the vertex at one end point, the length of the chord is longer than a side of the triangle if the other end point lies between the two other end points of the triangle. This happens with probability 1/3.



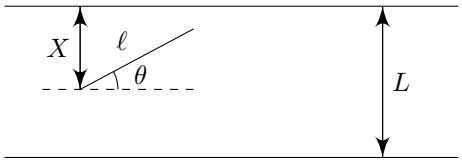
(ii) wlog the chord is horizontal, on the lower side of the circle. The mid-point is uniformly distributed along the middle (dashed) line. Then the probability of getting a long line is 1/2.



(iii) The mid point of the chord is distributed uniformly across the circle. Then you get a long line if and only if the mid-point lies in the smaller circle shown below. This occurs with probability 1/4.



Example (Buffon's needle)**.** A needle of length ℓ is tossed at random onto a floor marked with parallel lines a distance L apart, where $\ell \leq L$. What is $\mathbb{P}(A)$, where $A =$ [needle intersects a line]?



Suppose that $X \sim U[0,1]$ and $\theta \sim U[0,\pi]$. Then

$$f(x,\theta) = \frac{1}{L\pi}.$$

This touches a line if $X \leq \ell \sin\theta$. Hence

$$\mathbb{P}(A) = \int_{\theta=0}^{\pi} \underbrace{\frac{\ell \sin\theta}{\ell}}_{\mathbb{P}(X \leq \ell \sin\theta)} \frac{1}{\pi} \mathrm{d}\theta = \frac{2\ell}{\pi L}.$$

Since the answer involves π, we can estimate π by conducting repeated experiments! Suppose we have N hits out of n hits. Then an estimator for p is $\hat{p} = \frac{N}{n}$. Hence

$$\hat{\pi} = \frac{2\ell}{\hat{p}L}.$$

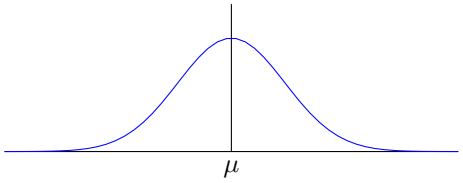
6 Normal distribution

Definition (Normal distribution)**.** The *normal distribution* with parameters μ, σ^2 has pdf

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),$$

for $-\infty < x < \infty$.

It looks approximately like this:



The *standard normal* is when $\mu = 0, \sigma^2 = 1$, ie. $X \sim N(0, 1)$. We usually write $\varphi(x)$ for the pdf and $\Phi(x)$ for the cdf of the standard normal.

We have to show that this makes sense, ie.

Proposition.

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}\, \mathrm{d}x = 1.$$

Proof. Substitute $z = \frac{(x-\mu)}{\sigma}$. Then

$$I = \int_{\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, \mathrm{d}z.$$

Then

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, \mathrm{d}x \int_{\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, \mathrm{d}y \\ &= \int_0^{\infty} \int_0^{2\pi} \frac{1}{2\pi} e^{-r^2/2} r\, \mathrm{d}r\, \mathrm{d}\theta \\ &= 1. \end{aligned}$$

□

We also have

Proposition. $\mathbb{E}[X] = \mu$.

Proof.

$$\begin{aligned} \mathbb{E}[X] &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} x e^{-(x-\mu)^2/2\sigma^2}\, \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (x-\mu) e^{-(x-\mu)^2/2\sigma^2}\, \mathrm{d}x + \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \mu e^{-(x-\mu)^2/2\sigma^2}\, \mathrm{d}x. \end{aligned}$$

The first term is antisymmetric about μ and gives 0. The second is just μ times the integral we did above. So we get μ. □

Proposition. $\text{var}(X) = \sigma^2$.

Proof. We have $\text{var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$. Substitute $Z = \frac{X-\mu}{\sigma}$. Then $\mathbb{E}[Z] = 0$, $\mathbb{E}[Z^2] = \frac{1}{\sigma^2}\mathbb{E}[X^2]$.

Then

$$\begin{aligned}\text{var}(Z) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z^2 e^{-z^2/2}\,\mathrm{d}z\\ &= \left[-\frac{1}{\sqrt{2\pi}} z e^{-z^2/2}\right]_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-z^2/2}\,\mathrm{d}z\\ &= 0 + 1\\ &= 1\end{aligned}$$

So $\text{var}\, X = \sigma^2$. □

Example. UK adult male heights are normally distributed with mean 70" and standard deviation 3". In the Netherlands, these figures are 71" and 3".

What is $\mathbb{P}(Y > X)$, where X and Y are the heights of randomly chosen UK and Netherlands males, respectively?

Now suppose that in both countries, the Olympic male basketball teams are selected from that portion of male whose hight is at least above 4" above the mean (which corresponds to the 9.1% tallest males of the country). What is the probability that a randomly chosen Netherlands player is taller than a randomly chosen UK player?

We have $X \sim N(70, 3^2)$ and $Y \sim N(71, 3^2)$. Then (cf. later lectures) $Y - X \sim N(1, 18)$.

$$\mathbb{P}(Y > X) = \mathbb{P}(Y - X > 0) = \mathbb{P}\left(\frac{Y - X - 1}{\sqrt{18}} > \frac{-1}{\sqrt{18}}\right) = 1 - \Phi(-1/\sqrt{18}),$$

since $\frac{(Y-X)-1}{\sqrt{18}} \sim (0, 1)$, and the answer is approximately 0.5931.

For the second part, we have

$$\mathbb{P}(Y > X | X \geq 74, Y \geq 75) = \frac{\int_{x=74}^{75} \phi_X(x)\,\mathrm{d}x + \int_{x=75}^{\infty}\int_{y=x}^{\infty} \phi_Y(y)\phi_X(x)\,\mathrm{d}y\,\mathrm{d}x}{\int_{x=74}^{\infty}\phi_X(x)\,\mathrm{d}x \int_{y=75}^{\infty}\phi_Y(y)\,\mathrm{d}y},$$

which is approximately 0.7558. So even though the Netherlands people are only slightly taller, if we consider the tallest bunch, the Netherlands people will be much taller on average.

6.1 Mode, median and sample mean

Definition (Mode and median)**.** Given a pdf $f(x)$, say $\hat{x}$ is a *mode* if

$$f(\hat{x}) \geq f(x)$$

for all x.

We say it is a median if

$$\int_{-\infty}^{\hat{x}} f(x)\,\mathrm{d}x = \frac{1}{2} = \int_{\hat{x}}^{\infty} f(x)\,\mathrm{d}x.$$

For a discrete random variable, the media is $\hat{x}$ such that

$$\mathbb{P}(X \leq \hat{x}) \geq \frac{1}{2}, \quad \mathbb{P}(X \geq \hat{x}) \geq \frac{1}{2}.$$

(we have a non-strict inequality since if the random variable always takes value 0, then both probabilities will be 1)

For $N(\mu, \sigma^2)$, the mean, mode median are all μ.

Definition (Sample mean)**.** If $X_1, \cdots, X_n$ is a random sample from some distribution, then the *sample mean* is

$$\bar{X} = \frac{1}{n}\sum_1^n X_i.$$

6.2 Distribution of order statistics.

Definition (Order statistics)**.** Let $Y_1, \cdots, Y_n$ be $X_1, \cdots, X_n$ arranged in increasing order, ie. $Y_1 \leq Y_2 \leq \cdots . \leq Y_n$. This is the *order statistics*.

We sometimes write $Y_i = X_{(i)}$.

Assume the X_i are iid with cdf F and pdf f. Then the pdf of Y_n is

$$\mathbb{P}(Y_n \leq y) = \mathbb{P}(X_1 \leq y \cdots, X_n \leq y) = \mathbb{P}(X_1 \leq y) \cdots \mathbb{P}(X_n \leq y) = F(y)^n.$$

So the pdf of Y_n is

$$\frac{\mathrm{d}}{\mathrm{d}y}F(y)^n = nf(y)F(y)^{n-1}.$$

So the pdf of Y is $nf(y)(1 - F(y))^{n-1}$.

Also,

$$\mathbb{P}(Y_1 \geq y) = \mathbb{P}(X_1 \geq y, \cdots, X_n \geq y) = (1 - F(y))^n.$$

What about the joint distribution of Y_1, Y_n?

$$\begin{aligned} G(y_1, y_n) &= \mathbb{P}(Y_1 \leq y_1, Y_n \leq y_n) \\ &= \mathbb{P}(Y_n \leq y_n) - \mathbb{P}(Y_1 \geq y_1, Y_n \leq y_n) \\ &= F(y_n)^n - (F(y_n) - F(y_1))^n. \end{aligned}$$

Then the pdf is

$$\frac{\partial^2}{\partial y_1 \partial y_n} G(y_1, y_n) = n(n-1)(F(y_n) - F(y_1))^{n-2} f(y_1) f(y_n).$$

We can think about this result in terms of the multinomial distribution. Suppose we want Y_1 to be within $[y_1, y_1 + \delta)$, and Y_n to be within $(y_n - \delta, Y - n + \delta]$, and all $n - 2$ others to be in between. Treat these regions as 3 bins, and we want 1 in each of the first and last ones, and $n - 2$ in the middle one. There are $\binom{n}{1, n-2, 1} = n(n-1)$ ways of doing so. The probability of each thing falling into the middle bin is $F(y_n) - F(y_1)$, and the probabilities of falling into the first and last bin are $f(y_1)\delta$ and $f(y_n)\delta$. Then the probability of $Y_1 \in [y_1, y_1 + \delta)$ and $Y_n \in (y_n - \delta, y_n]$ is

$$n(n-1)(F(y_n) - F(y_1))^{n-2} f(y_1) f(y_n) \delta^2,$$

and the result follows.

7 Transformation of random variables

Suppose $X_1, X_2, \cdots, X_n$ are random variables with joint pdf f. Let

$$\begin{aligned} Y_1 &= r_1(X_1, \cdots, X_n) \\ Y_2 &= r_2(X_1, \cdots, X_n) \\ &\vdots \\ Y_n &= r_n(X_1, \cdots, X_n). \end{aligned}$$

For example, we might have $Y_1 = \frac{X_1}{X_1+X_2}$ and $Y_2 = X_1 + X_2$.

Let $R \subseteq \mathbb{R}^n$ such that $\mathbb{P}((X_1, \cdots, X_n) \in R) = 1$, ie. R is the set of all values (X_i) can take.

Suppose S is the image of R under the above transformation, and the map $R \to S$ is bijective. Then there exists an inverse function

$$\begin{aligned} X_1 &= s_1(Y_1, \cdots, Y_n) \\ X_2 &= s_2(Y_1, \cdots, Y_n) \\ &\vdots \\ X_n &= s_n(Y_1, \cdots, Y_n). \end{aligned}$$

For example, if X_1, X_2 refers to the coordinates of a random point in Cartesian coordinates, Y_1, Y_2 might be the coordinates in polar coordinates.

Definition (Jacobian determinant)**.** Suppose $\frac{\partial S_i}{\partial y_j}$ exists and is continuous at every point $(y_1, \cdots, y_n) \in S$. Then the *Jacobian determinant* is

$$J = \frac{\partial(s_1, \cdots, s_n)}{\partial(y_1, \cdots, y_n)} = \det \begin{pmatrix} \frac{\partial s_1}{\partial y_1} & \cdots & \frac{\partial s_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial s_n}{\partial y_1} & \cdots & \frac{\partial s_n}{\partial y_n} \end{pmatrix}$$

Take $A \subseteq \mathbb{R}$ and $B = r(A)$. Then

$$\begin{aligned} \mathbb{P}((X_1, \cdots ., X_n) \in A) &= \int_A f(x_1, \cdots, X_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n \\ &= \int_B f(s_1(y_1, \cdots .y_n, s_2, \cdots, s_n)|J|\, \mathrm{d}y_1 \cdots\, \mathrm{d}y_n \\ &= \mathbb{P}((Y_1, \cdots Y_n) \in B). \end{aligned}$$

(cf. Vector Calculus) So

Proposition. $(Y_1, \cdots ., Y_n)$ has density

$$g(y_1, \cdots ., y_n) = f(s_1(y_1, \cdots, y_n), \cdots s_n(y_1, \cdots, y_n))|J|$$

if $(y_1, \cdots, y_n) \in S$, 0 otherwise.

Example. Suppose (X, Y) has density

$$f(x, y) = \begin{cases} 4xy & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

We see that X and Y are independent, with each having a density $f(x) = 2x$.

Define $U = X/Y$, $V = XY$. We will later show that they are not independent. Then we have $X = \sqrt{UV}$ and $Y = \sqrt{V/U}$.

The Jacobian is

$$\det \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix} = \det \begin{pmatrix} \frac{1}{2}\sqrt{v/u} & \frac{1}{2}\sqrt{u/v} \\ -\frac{1}{2}\sqrt{v/u^3} & \frac{1}{2}\sqrt{1/uv} \end{pmatrix} = \frac{1}{2u}$$

Note that this can be more easily found by considering

$$\det \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial u/\partial y \end{pmatrix} = 2u$$

and then inverting the matrix. So

$$g(u, v) = 2\sqrt{uv}\sqrt{\frac{v}{u}}\frac{1}{2u} = \frac{2v}{u},$$

if (u, v) is in the image S, 0 otherwise. So

$$g(u, v) = \frac{2v}{y} I[(u, v) \in S].$$

Since this is not separable, we know that U and V are not independent.

In the linear case, life is easy. Suppose

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = A \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = A\mathbf{X}$$

Then $\mathbf{X} = A^{-1}\mathbf{Y}$. Then $\frac{\partial x_i}{\partial y_j} = (A^{-1})_{ij}$. So $|J| = |\det(A^{-1})| = |\det A|^{-1}$. So

$$g(y_1, \cdots, y_n) = \frac{1}{|\det A|} f(A^{-1}\mathbf{y}).$$

Example. Suppose X_1, X_2 have joint pdf $f(x_1, x_2)$. Suppose we want to find the pdf of $Y = X_1 + X_2$. We let $Z = X_2$. Then $X_1 = Y - Z$ and $X_2 = Z$. Then

$$\begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = A\mathbf{X}$$

Then $|J| = 1/|\det A| = 1$. Then

$$g(y, z) = f(y - z, z)$$

So

$$g_Y(y) = \int_{-\infty}^{\infty} f(y - z, z)\, \mathrm{d}z = \int_{-\infty}^{\infty} f(z, y - z)\, \mathrm{d}z.$$

If X_1 and X_2 are independent, $f(x_1, x_2) = f_1(x_1)f_2(x_2)$. Then

$$g(y) = \int_{-\infty}^{\infty} f_1(z)f_2(y - z)\, \mathrm{d}z.$$

7.1 Non-injective transformations

We previously discussed transformation of random variables by injective maps. What if the mapping is not? There is no simple formula for that, and we have to work out each case individually.

Example. Suppose X has pdf f. What is the pdf of $Y = |X|$?

We use our definition. We have

$$\mathbb{P}(|X| \in x(a, b)) = \int_a^b f(x) + \int_{-b}^{-a} f(x)\,\mathrm{d}x = \int_a^b (f(x) + f(-x))\,\mathrm{d}x.$$

So

$$f_Y(x) = f(x) + f(-x),$$

which makes sense, since getting $|X| = x$ is equivalent to getting $X = x$ or $X = -x$.

Example. Suppose $X_1, \cdots, X_n$ are iid random variables. Then what is the joint pdf, say g, of the order statistics $Y_1, \cdots, Y_n$?

We have

$$g(y_1, \cdots y_n) = n! f(x_1) \cdots f(y_n),$$

if $y_1 \leq y_2 \leq \cdots \leq y_n$, 0 otherwise. We have this formula because there are $n!$ combinations of $x_1, \cdots, x_n$ that produces a given order statistics $y_1, \cdots, y_n$, and the pdf of each combination is $f(y_1) \cdots f(y_n)$.

Example. Suppose $X_1 \sim \mathcal{E}(\lambda), X_2 \sim \mathcal{E}(\mu)$ are independent random variables. Let $Y = \min(X_1, X_2)$. Then

$$\begin{aligned}\mathbb{P}(Y \geq t) &= \mathbb{P}(X_1 \geq t, X_2 \geq t)\\ &= \mathbb{P}(X_1 \geq t)\mathbb{P}(X_2 \geq t)\\ &= e^{-\lambda t} e^{-\mu t}\\ &= e^{-(\lambda+\mu)t}.\end{aligned}$$

So $Y \sim \mathcal{E}(\lambda + \mu)$.

Example. Let $X_1, \cdots, X_n$ be iid $\mathcal{E}(\lambda)$, and $Y_1, \cdots, Y_n$ be the order statistic. Let

$$\begin{aligned}Z_1 &= Y_1\\ Z_2 &= Y_2 - Y_1\\ &\vdots\\ Z_n &= Y_n - Y_{n-1}.\end{aligned}$$

These are the distances between the occurrences. We can write this as a $\mathbf{Z} = A\mathbf{Y}$, with

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0\\ -1 & 1 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\end{pmatrix}$$

Then $\det(A) = 1$ and hence $|J| = 1$. Suppose that the pdf of $Z_1, \cdots, Z_n$ is, say h. Then

$$\begin{aligned} h(z_1, \cdots, z_n) &= g(y_1, \cdots, y_n) \cdots 1 \\ &= n! f(y_1) \cdots f(y_n) \\ &= n! \lambda^n e^{-\lambda(y_1 + \cdots + y_n)} \\ &= n! \lambda^n e^{-\lambda(nz_1 + (n-1)z_2 + \cdots + z_n)} \\ &= \prod_{i=1}^{n} (\lambda i) e^{-(\lambda i) z_{n+1-i}} \end{aligned}$$

Since h is expressed as a product of n density functions, we have

$$Z_i \sim \mathcal{E}((n + 1 - i)\lambda).$$

with all Z_i independent.

8 Moment generating functions

If X is a continuous random variable, then the analogue of the pgf is the moment generating function:

Definition (Moment generating function)**.** The *moment generating function* of a random variable X is

$$m(\theta) = \mathbb{E}[e^{\theta X}].$$

For those θ in which $m(\theta)$ is finite, we have

$$m(\theta) = \int_{-\infty}^{\infty} e^{\theta x} f(x)\, \mathrm{d}x.$$

We will assume the following without proof:

Theorem. The mgf determines the distribution of X provided $m(\theta)$ is finite for all θ in some origin containing the origin.

Definition (Moment)**.** The rth *moment* of $X$is $\mathbb{E}[X^r]$.

Theorem. The r-th moment X is the coefficient of $\frac{\theta^r}{r!}$ in the power series expansion of $m(\theta)$, and is

$$\mathbb{E}[X^r] = \frac{\mathrm{d}^n}{\mathrm{d}\theta^n} m(\theta)\bigg|_{\theta=0} = m^{(n)}(0).$$

Proof. We have

$$e^{\theta X} = 1 + \theta X + \frac{\theta^2}{2!}X^2 + \cdots.$$

So

$$m(\theta) = \mathbb{E}[e^{\theta X}] = 1 + \theta\mathbb{E}[X] + \frac{\theta^2}{2!}\mathbb{E}[X^2] + \cdots$$

□

Example. Let $X \sim \mathcal{E}(\lambda)$. Then its mgf is

$$\mathbb{E}[e^{\theta X}] = \int_0^\infty e^{\theta x}\lambda e^{-\lambda x}\, \mathrm{d}x = \lambda \int_0^\infty e^{-(\lambda-\theta)x}\, \mathrm{d}x = \frac{\lambda}{\lambda - \theta},$$

where $0 < \theta < \lambda$. So

$$\mathbb{E}[X] = m'(0) = \frac{\lambda}{(\lambda-\theta)^2}\bigg|_{\theta=0} = \frac{1}{\lambda}.$$

Also,

$$\mathbb{E}[X^2] = m''(0) = \frac{2\lambda}{(\lambda-\theta)^3}\bigg|_{\theta=0} = \frac{2}{\lambda^2}.$$

So

$$\mathrm{var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Theorem. If X and Y are independent random variables with mgf $m_X(\theta), m_Y(\theta)$, then $X + Y$ has mgf $m_{X+Y}(\theta) = m_X(\theta)m_Y(\theta)$.

Proof.

$$\mathbb{E}[e^{\theta(X+Y)}] = \mathbb{E}[e^{\theta X}e^{\theta Y}] = \mathbb{E}[e^{\theta X}]\mathbb{E}[e^{\theta Y}] = m_X(\theta)m_Y(\theta).$$

□

9 More distributions

9.1 Cauchy distribution

Definition (Cauchy distribution)**.** The *Cauchy distribution* has pdf

$$f(x) = \frac{1}{\pi(1+x^2)}$$

for $-\infty < x < \infty$.

We check that this is a genuine distribution:

$$\int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)}\,\mathrm{d}x = \int_{-\pi/2}^{\pi/2} \frac{1}{\pi}\,\mathrm{d}\theta = 1$$

with the substitution $x = \tan\theta$. The distribution is a bell-shaped curve.

Proposition. The mean of the Cauchy distribution is undefined, while $\mathbb{E}[X^2] = \infty$.

Proof.

$$\mathbb{E}[X] = \int_0^{\infty} \frac{x}{\pi(1+x^2)}\,\mathrm{d}x + \int_{-\infty}^{0} \frac{x}{\pi(1+x^2}\,\mathrm{d}x = \infty - \infty$$

which is undefined, but $\mathbb{E}[X^2] = \infty + \infty = \infty$. □

Suppose X, Y are independent Cauchy distributions. Let $Z = X + Y$. Then

$$\begin{aligned} f(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \frac{1}{\pi^2} \\ &= \frac{1}{(1+x^2)(1+(z-x)^2)} \\ &= \frac{1/2}{\pi(1+(z/2)^2)} \end{aligned}$$

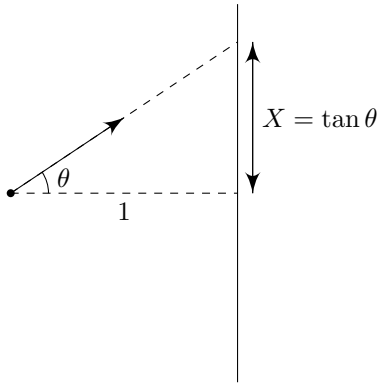
for all $-\infty < z < \infty$ (the integral can be evaluated using a tedious partial fraction expansion).

So $\frac{1}{2}Z$ has a Cauchy distribution. Alternatively the arithmetic mean of Cauchy random variables is a Cauchy random variable.

By induction, we can show that $\frac{1}{n}(X_1 + \cdots + X_n) \sim$ Cauchy distribution. This becomes a "counter-example" to things like the weak law of large numbers and the central limit theorem. The weirdness arises from the fact that it has no mean.

Proposition.

(i) If $\Theta \sim U[-\frac{\pi}{2}, \frac{\pi}{2}]$, then $X = \tan\theta$ has a Cauchy distribution. For example, if we fire a bullet at a wall 1 meter apart at a random random angle θ, the vertical displacement follows a Cauchy distribution.



(ii) If $X, Y \sim N(0, 1)$, then X/Y has a Cauchy distribution.

9.2 Gamma distribution

Example. Suppose $X_1, \cdots, X_n$ are iid $\mathcal{E}(\lambda)$. Let $S_n = X_1 + \cdots + X_n$. Then the mgf of S_n is

$$\mathbb{E}\left[e^{\theta(X_1+\ldots+X_n)}\right] = \mathbb{E}\left[e^{\theta X_1}\right]\cdots\mathbb{E}\left[e^{\theta X_n}\right] = \left(\mathbb{E}\left[e^{\theta X}\right]\right)^n = \left(\frac{\lambda}{\lambda-\theta}\right)^n.$$

Definition (Gamma distribution)**.** The *gamma distribution* $\Gamma(n, \lambda)$ has pdf

$$f(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!}.$$

We can show that this is a distribution by showing that it integrates to 1. We will show that this is the sum of n iid $\mathcal{E}(\lambda)$:

$$\begin{aligned}\mathbb{E}[e^{\theta X}] &= \int_0^\infty e^{\theta x}\frac{\lambda^n x^{n-1}e^{-\lambda x}}{(n-1)!}\,\mathrm{d}x\\ &= \left(\frac{\lambda}{\lambda-\theta}\right)^n\int_0^\infty \frac{(\lambda-\theta)^n x^{n-1}e^{-(\lambda-\theta)}}{(n-1)!}\,\mathrm{d}x\end{aligned}$$

The integral is just integrating over $\Gamma(n, \lambda - \theta)$, which gives one. So we have

$$\mathbb{E}[e^{\theta X}] = \left(\frac{\lambda}{\lambda-\theta}\right)^n$$

9.3 Beta distribution*

Suppose $X_1, \cdots, X_n$ be iid $U[0, 1]$. Let $Y_1 \le Y_2 \le \cdots \le Y_n$ be the order statistics. Then the pdf of Y_i is

$$f(y) = \frac{n!}{(i-1)!(n-i)!}y^{i-1}(y-y)^{n-i}.$$

Note that the leading term is the multinomial coefficient $\binom{n}{i-1,1,n-1}$. The formula is obtained using the same technique for finding the pdf of order statistics.

This is the *beta distribution*: $Y_i \sim \beta(i, n - i + 1)$. In general

Definition (Beta distribution)**.** The *beta distribution* $\beta(a, b)$ has pdf

$$f(x; a, b) = \frac{\Gamma(a, b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$$

for $0 \leq x \leq 1$.

This has mean $a/(a+b)$.

Its moment generating function is

$$m(\theta) = 1 + \sum_{k=1}^{\infty} \left(\prod_{r=0}^{k-1} \frac{a+r}{a+b+r} \right) \frac{\theta^k}{k!},$$

which is horrendous!

9.4 More on the normal distribution

9.4.1 Moment generating function

Suppose $X \sim N(\mu, \sigma^2)$. The mgf is found as follows:

$$E[e^{\theta X}] = \int_{-\infty}^{\infty} e^{\theta x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\sigma^2(x-\mu)^2} \, \mathrm{d}x.$$

Substitute $z = \frac{x-\mu}{\sigma}$. Then

$$\begin{aligned}
\mathbb{E}[e^{\theta X}] &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\theta(\mu+\sigma z)} e^{-\frac{1}{2}z^2} \, \mathrm{d}z \\
&= e^{\theta\mu + \frac{1}{2}\theta^2\sigma^2} \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\theta\sigma)^2}}_{\text{pdf of } N(\sigma\theta, 1)} \, \mathrm{d}z \\
&= e^{\theta\mu + \frac{1}{2}\theta^2\sigma^2}.
\end{aligned}$$

9.4.2 Functions of normal random variables

Theorem. Suppose X, Y are independent random variables with $X \sim N(\mu_1, \sigma_1^2)$, and $Y \sim (\mu_2, \sigma_2^2)$. Then

(i) $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

(ii) $aX \sim N(a\mu_1, a^2\sigma_1^2)$.

Proof.

(i)

$$\begin{aligned}
\mathbb{E}[e^{\theta(X+Y)}] &= \mathbb{E}[e^{\theta X}] \cdot \mathbb{E}[e^{\theta Y}] \\
&= e^{\mu_1\theta + \frac{1}{2}\sigma_1^2\theta^2} \cdot e^{\mu_2\theta + \frac{1}{2}\sigma_2^2\theta_2} \\
&= e^{(\mu_1+\mu_2)\theta + \frac{1}{2}(\sigma_1^2+\sigma_2^2)\theta^2}
\end{aligned}$$

which is the mgf of $N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

(ii)

$$\begin{aligned}\mathbb{E}[e^{\theta(aX)}] &= \mathbb{E}[e^{(\theta a)X}]\\ &= e^{\mu(a\theta)+\frac{1}{2}\sigma^2(a\theta)^2}\\ &= e^{(a\mu)\theta+\frac{1}{2}(a^2\sigma^2)\theta^2}\end{aligned}$$

□

9.4.3 Bounds on tail probabilities

Suppose $X \sim N(0,1)$. Write $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ for its pdf. It would be very difficult to find a closed form for its cumulative distribution function, but we can find an upper bound for it:

$$\begin{aligned}\mathbb{P}(X \geq x) &= \int_x^\infty \phi(t)\,\mathrm{d}t\\ &\leq \int_x^\infty \left(1+\frac{1}{t^2}\right)\phi(t)\,\mathrm{d}t\\ &= \frac{1}{x}\phi(x)\end{aligned}$$

To see the last step works, simply differentiate the result and see that you get $\left(1+\frac{1}{t^2}\right)\phi(t)\,\mathrm{d}t$. So

$$\mathbb{P}(X \geq x) \leq \frac{1}{x}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}.$$

Then

$$\log \mathbb{P}(X \geq x) \sim -\frac{1}{2}x^2.$$

9.5 Multivariate normal

Let $X_1, \cdots, X_n$ be iid $N(0,1)$. Then their joint density is

$$\begin{aligned}g(x_1, \cdots, x_n) &= \prod_{i=1}^n \phi(x_i)\\ &= \prod_1^n \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x_i^2}\\ &= \frac{1}{(2\pi)^{n/2}}e^{-\frac{1}{2}\sum_1^n x_i^2}\\ &= \frac{1}{(2\pi)^{n/2}}e^{-\frac{1}{2}\mathbf{x}^T\mathbf{x}},\end{aligned}$$

where $\mathbf{x} = (x_1, \cdots, x_n)^T$.

This result works if $X_1, \cdots, X_n$ are iid $N(0,1)$. Suppose we are interested in

$$\mathbf{Z} = \boldsymbol{\mu} + A\mathbf{X},$$

where A is an invertible $n \times n$ matrix. We can think of this as n measurements $\mathbf{Z}$ that are affected by underlying standard-normal factors $\mathbf{X}$. Then

$$\mathbf{X} = A^{-1}(\mathbf{Z} - \boldsymbol{\mu})$$

and

$$|J| = |\det(A^{-1})| = \frac{1}{\det A}$$

So

$$\begin{aligned} f(z_1, \cdots, z_n) &= \frac{1}{(2\pi)^{n/2}} \frac{1}{\det A} \exp\left[-\frac{1}{2}\big((A^{-1}(\mathbf{z}-\boldsymbol{\mu}))^T (A^{-1}(\mathbf{z}-\boldsymbol{\mu}))\big)\right] \\ &= \frac{1}{(2\pi)^{n/2} \det A} \exp\left[-\frac{1}{2}(\mathbf{z}-\mu)^T \Sigma^{-1} (\mathbf{z}-\boldsymbol{\mu})\right] \\ &= \frac{1}{(2\pi)^{n/2}\sqrt{\det \Sigma}} \exp\left[-\frac{1}{2}(\mathbf{z}-\boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{z}-\boldsymbol{\mu})\right]. \end{aligned}$$

where $\Sigma = AA^T$ and $\Sigma^{-1} = (A^{-1})^T A^{-1}$. We say

$$\mathbf{Z} = \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} \sim MUN(\boldsymbol{\mu}, \Sigma) \text{ or } N(\boldsymbol{\mu}, \Sigma).$$

This is the multivariate normal.

Then $\operatorname{cov}(Z_i, Z_j) = \mathbb{E}[(Z_i - \mu_i)(z_j - \mu_j)]$, which is the i, jth entry of Σ, since

$$\begin{aligned} \mathbb{E}[(Z-\mu)(Z-\mu)^T] &= \mathbb{E}[AX(AX)^T] \\ &= \mathbb{E}(AXX^TA^T) = A\mathbb{E}[XX^T]A^T \\ &= AIA^T \\ &= AA^T \\ &= \Sigma \end{aligned}$$

We have $\Sigma = \sigma^2$ when $n = 1$.

Now suppose $Z_1, \cdots, Z_n$ have covariances 0. Then $\Sigma = \operatorname{diag}(\sigma_1^2, \cdots, \sigma_n^2)$. Then

$$f(z_1, \cdots, z_n) = \prod_1^n \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{1}{2\sigma_i^2}(z_i - \mu_i)^2}.$$

Then $Z_1, \cdots, Z_n$ are independent, with $Z_i \sim N(\mu_i, \sigma_i^2)$. Note that it is only true for normal distributions that cov $= 0 \Rightarrow$ independent.

The moment generating function is

$$m(\boldsymbol{\theta}) = \mathbb{E}[e^{\boldsymbol{\theta}^T \mathbf{X}}] = \mathbb{E}[e^{\theta_1 X_1 + \cdots + \theta_n X_n}].$$

Bivariate normal

This is the special case of the multivariate normal when $n = 2$. Since there aren't too many terms, we can actually write them out.

The *bivariate normal* has

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \delta_2^2. \end{pmatrix}$$

Then

$$\operatorname{corr}(X_1, X_2) = \frac{\operatorname{cov}(X_1, X_2)}{\sqrt{\operatorname{var}(X_1)\operatorname{var}(X_2)}} = \frac{\rho\sigma_1\sigma_2}{\sigma_1\sigma_1} = \rho.$$

Then

$$\Sigma^{-1} = \frac{1}{1-\rho^2}\begin{pmatrix} \sigma_1^{-2} & -\rho\sigma_1^{-1}\sigma_2^{-1} \\ -\rho\sigma_1^{-1}\sigma_2^{-1} & \sigma_2^{-2} \end{pmatrix}$$

The joint mgf of the bivariate normal is

$$m(\theta_1, \theta_2) = e^{\theta_1\mu_1 + \theta_2\mu_2 + \frac{1}{2}(\theta_1\sigma_1^2 + 2\theta_1\theta_2\rho\sigma_1\sigma_2 + \sigma_1\sigma_2^2)}.$$

10 Central limit theorem

Suppose $X_1, \cdots, X_n$ are iid random variables with mean μ and variance σ^2. Let $S_n = X_1 + \cdots + X_n$. Then

$$\text{var}(S_n/\sqrt{n}) = \text{var}\left(\frac{S_n - n\mu}{\sqrt{n}}\right) = \sigma^2.$$

Theorem (Central limit theorem)**.** Let $X_1, X_2, \cdots$ be iid random variables with $\mathbb{E}[X_i] = \mu$, $\text{var}(X_i) = \sigma^2 < \infty$. Define

$$S_n = X_1 + \cdots + X_n.$$

Then for all finite intervals (a, b),

$$\lim_{n\to\infty} \mathbb{P}\left(a \le \frac{S_n - n\mu}{\sigma\sqrt{n}} \le b\right) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}\, \mathrm{d}t.$$

Note that the final term is the pdf of a standard normal. We say

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \to_D N(0, 1).$$

To show this, we will use the continuity theorem without proof:

Theorem (Continuity theorem)**.** If the random variables $X_1, X_2, \cdots$ have mgfs $m_1(\theta), m_2(\theta), \cdots$ and $m_n(\theta) \to m(\theta)$ as $n \to \infty$ for all θ, then $X_n \to_D$ the random variable with mgf $m(\theta)$.

We now provide a sketch-proof of the central limit theorem:

Proof. wlog, assume $\mu = 0, \sigma^2 = 1$ (otherwise replace X_i with $\frac{X_i - \mu}{\sigma}$).

Then

$$\begin{aligned} m_{X_i}(\theta) = \mathbb{E}[e^{\theta X_i}] &= 1 + \theta\mathbb{E}[X_i] + \frac{\theta^2}{2!}\mathbb{E}[X_i^2] + \cdots \\ &= 1 + \frac{1}{2}\theta^2 + \frac{1}{3!}\theta^3\mathbb{E}[X_i^3] + \cdots \end{aligned}$$

Now consider $S_n/\sqrt{n}$. Then

$$\begin{aligned} \mathbb{E}[e^{\theta S_n/\sqrt{n}}] &= \mathbb{E}[e^{\theta(X_1 + \ldots + X_n)/\sqrt{n}}] \\ &= \mathbb{E}[e^{\theta X_1/\sqrt{n}}] \cdots \mathbb{E}[e^{\theta X_n/\sqrt{n}}] \\ &= \left(\mathbb{E}[e^{\theta X_1/\sqrt{n}}]\right)^n \\ &= \left(1 + \frac{1}{2}\theta^2\frac{1}{n} + \frac{1}{3!}\mathbb{E}[X^3]\frac{1}{n^{3/2}} + \cdots\right)^n \\ &\to e^{\frac{1}{2}\theta^2} \end{aligned}$$

as $n \to \infty$ since $(1 + a/n)^n \to e^a$. And this is the mgf of the standard normal. Then the result follows from the continuity theorem. □

Note that this is not a very formal proof, since we have to require $\mathbb{E}[X^3]$ to be finite. Also, sometimes the moment generating function is not defined. But this will work for many "nice" distributions we will ever meet.

The proper proof uses the characteristic function

$$\chi_X(\theta) = E[e^{i\theta X}].$$

10.1 Normal approximation to binomial

Let $X_i \sim B(1,p)$. Then $S_n \sim B(n,p)$. So $\mathbb{E}[S_n] = np$ and $\text{var}(S_n) = p(1-p)$. So

$$\frac{S_n - np}{\sqrt{np(1-p)}} \to_D N(0,1).$$

Example. Suppose two planes fly a route. Each of n passengers chooses a plane at random. The number of people choosing plane 1 is $S \sim B(n, \frac{1}{2})$. Suppose the plane has s seats. What is

$$f(s) = \mathbb{P}(S > s),$$

ie. the plane is over-booked? We have

$$f(s) = \mathbb{P}(S > s) = \mathbb{P}\left(\frac{S - n/2}{\sqrt{n \cdot \frac{1}{2} \cdot \frac{1}{2}}} > \frac{s - n/2}{\sqrt{n}/2}\right).$$

Since

$$\frac{S - np}{\sqrt{n}/2} \sim N(0,1),$$

we have

$$f(s) \approx 1 - \Phi\left(\frac{s - n/2}{\sqrt{n}/2}\right).$$

eg, if $n = 1000$ and $s = 537$, then $\frac{S_n - n/2}{\sqrt{n}/2} \approx 2.34$, $\Phi(2.34) \approx 0.99$, and $f(s) \approx 0.01$. So the probability of overbooking is $1/100$ with 74 extra seats as buffer.

Example. An unknown proportion p of the electorate will vote Labour. It is desired to find p without an error not exceeding 0.005. How large should the sample be?

We estimate by

$$p' = \frac{S_n}{n},$$

where $X_i \sim B(1,p)$. Then

$$\begin{aligned}\mathbb{P}(|p' - p| \le 0.005) &= \mathbb{P}(|S_n - np| \le 0.005n) \\ &= \mathbb{P}\left(\underbrace{\frac{|S_n - np|}{\sqrt{np(1-p)}}}_{\approx N(0,1)} \le \frac{0.005n}{\sqrt{np(1-p)}}\right)\end{aligned}$$

We want $|p' - p| \le 0.005$ with probability ≥ 0.95. Then we want

$$\frac{0.005n}{\sqrt{np(1-p)}} \ge \Phi^{-1}(0.975) = 1.96.$$

(we use 0.975 instead of 0.95 since we are doing a two-tailed test) Since the maximum possible value of $p(1-p)$ is $1/4$, we have

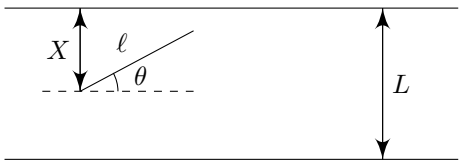
$$n \ge 38416.$$

In practice, we go by

$$\mathbb{P}(|p' < p| \le 0.03) \ge 0.95$$

This just requires $n \ge 1068$.

Example (Estimating π with Buffon's needle)**.** Recall that if we randomly toss a needle of length ℓ to a floor marked with parallel lines a distance L apart, the probability that the needle hits the line is $p = \frac{2\ell}{\pi L}$.



Suppose we toss the pin n times, and it hits the line N times. Then

$$N \approx N(np, np(1-p))$$

by the Central limit theorem. Write p' for the actual proportion observed. Then

$$\begin{aligned}\hat{\pi} &= \frac{2\ell}{(N/n)L}\\ &= \frac{\pi 2\ell/(\pi L)}{p'}\\ &= \frac{\pi p}{p + (p' - p)}\\ &= \pi\left(1 - \frac{p' - p}{p} + \cdots\right)\end{aligned}$$

Hence

$$\hat{\pi} - \pi \approx \frac{p - p'}{p}.$$

Since

$$p' \sim N\left(p, \frac{p(1-p)}{n}\right).$$

So

$$\hat{\pi} - \pi \sim N\left(0, \frac{\pi^2 p(1-p)}{np^2}\right) = N\left(0, \frac{\pi^2(1-p)}{np}\right)$$

We want a small variance, and that occurs when p is the largest. Since $p = 2\ell/\pi L$, this is maximized with $\ell = L$. In this case,

$$p = \frac{2}{\pi},$$

and

$$\hat{\pi} - \pi \approx N\left(0, \frac{(\pi - 2)\pi^2}{2n}\right).$$

If we want to estimate π to 3 dp,

$$\mathbb{P}(|\hat{\pi} - \pi| \leq 0.001) \geq 0.95$$

iff

$$0.001\sqrt{\frac{2n}{(\pi - 2)(\pi^2)}} \geq \Phi^{-1}(0.975) = 1.96$$

So $n \geq 2.15 \times 10^7$. So we can obtain π to 3 decimal places just by throwing a stick 20 million times! Isn't that exciting?