Complex Methods

1 Analytic Functions.

- Open set: An open set $D \subseteq \mathbb{C}$ is such that, $\forall z_0 \in D$, $\exists \delta > 0$: the disc $|z - z_0| < \delta \subseteq D$. Further, a neighbourhood of a point $z \in \mathbb{C}$ is an open set containing z.

- The extended complex plane is denoted $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$, and is conceptualised by use of the Riemann sphere with the point at infinity being the North pole. In particular, in order to investigate properties of $f(z)$ at ∞, we instead consider $f(\frac{1}{\zeta})$ at $\zeta = 0$. $(z = \frac{1}{\zeta})$

- Complex Differentiation: a complex differentiable function $f(z)$ with $f: \mathbb{C} \to \mathbb{C}$ is differentiable at z if:

$$f'(z) = \lim_{\delta z \to 0} \left\{ \frac{f(z+\delta z) - f(z)}{\delta z} \right\} \text{ exists.}$$

- $f(z)$ is analytic at z if $\exists$ a neighbourhood of z such that f' exists throughout. This is equivalent to f having a Taylor series. $f(z)$ is entire if it is analytic throughout $\mathbb{C}$.

- Cauchy-Riemann Equations.

We write $f(z) = u(x,y) + iv(x,y)$.

Now: i) Let $\delta z = \delta x$ (i.e along real axis).

Then: $$f'(z) = \lim_{\delta x \to 0} \left\{ \frac{u(x+\delta x, y) + iv(x+\delta x, y) - (u+iv)}{\delta x} \right\}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

ii) Let $\delta z = i\delta y$

Then: $$f'(z) = \lim_{\delta y \to 0} \left\{ \frac{u(x, y+\delta y) + iv(x, y+\delta y) - \{u + iv\}}{i\delta y} \right\}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

But since f is differentiable at z:

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$$\Longrightarrow \boxed{\left\{ \begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned} \right\}}$$

If u_x, u_y, v_x, v_y are continuous then the converse holds.

$\boxed{\text{N.B}}$ when dealing with sin/cos etc, can be useful to consider

$$\left\{ \begin{aligned} \sin(x+iy) &= \sin x \cos iy + \cos x \sin iy \\ &= \sin x \cosh y + i \cos x \sinh y. \end{aligned} \right\}$$

- <u>Harmonic Functions.</u> Two functions $\{u, v\}$ that satisfy the CR equations are called harmonic conjugates.

$\boxed{\text{N.B}}$ given either $u(x,y)$ or $v(x,y)$ we may find the other up to a constant. Normally express final result in terms of z.

Now, if $f(z) = u(x,y) + iv(x,y)$ is analytic, then:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2}$$

$$\Longrightarrow \boxed{\nabla^2 u = 0}.$$

and similarly $\nabla^2 v = 0$. So, the real and imaginary
parts of an analytic function satisfy Laplace's equation. (i.e they are harmonic).

• Multi-valued functions.

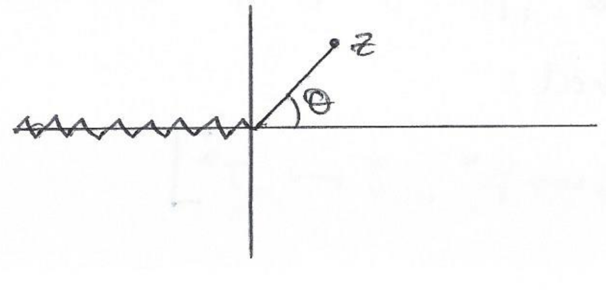
We first define a branch point; a branch point of a function is a point that is impossible to encircle with a curve on which the function is both continuous and single valued. (This is said to be a branch point singularity).

e.g • $\log(z-\alpha)$ has a branch point at $z=\alpha$.

• $z^{\alpha} = r^{\alpha} e^{i\alpha\theta}$ has a branch point at the origin for $\alpha \notin \mathbb{Z}$. (There will be a jump from $r_0^{\alpha} e^{2\pi i \alpha}$ to r_0^{α} to ensure continuity, so z^{α} not continuous unless $e^{2\pi i \alpha} = 1$).

• $\log z$ also has branch point at ∞ (consider $\log\left(\frac{1}{\xi}\right) = -\log \xi$ which has branch point at $\xi = 0$).

• We deal with branch points by the use of branch cuts. e.g for $\log z$ we exclude the negative real axis and do not allow any curve to cross the cut.



Taking $\theta \in (-\pi, \pi]$ defines the canonical branch of $\log z$ and defines its principal value.

Note that $\begin{cases} \text{at } z = x + i0^+, & \log z = \log|z| + i\pi \\ \text{at } z = x + i0^-, & \log z = \log|z| - i\pi \end{cases}$

So there is a discontinuity of $2\pi i$ across the cut.

We define a branch cut by either:

$\left\{\begin{array}{l} \text{1) defining function and } \theta \text{ parameter range.} \\ \text{2) Specifying location of branch cut and the} \\ \quad \text{value of the function at any point not} \\ \quad \text{on the cut.} \end{array}\right.$

- Multiple branch cuts. We may require more than one branch cut, which is achieved by considering each cut separately and investigating the discontinuities.

- Möbius Maps: consider $z \mapsto w = \dfrac{az+b}{cz+d}$

which we may consider as a map from $\mathbb{C}^* \mapsto \mathbb{C}^*$.

with $-\frac{d}{c} \mapsto \infty$, $\infty \mapsto \frac{a}{c}$.

We define a circline to be a circle or a line with the key property that:

{Möbius maps take circlines to circlines}

Also, given $\alpha, \beta, \gamma, \alpha^*, \beta^*, \gamma^* \in \mathbb{C}^*$ we may find a Möbius map such that:

$[\alpha \mapsto \alpha^*, \ \beta \mapsto \beta^*, \ \gamma \mapsto \gamma^*]$

- Conformal Maps.

{Definition:} A conformal map $f: \mathcal{U} \to V$ where $\mathcal{U}$ and V are open sets $\subseteq \mathbb{C}$. is one which is analytic with non-zero derivative.

{Proposition:} A conformal map preserves the angle between curves.

Pf: Suppose $z_1(t)$ is a curve in $\mathbb{C}$ parametrised by $t \in \mathbb{R}$. and passes through z_0 when $t = t_1$. Suppose tangent is well-defined i.e. $z_1'(t_1) \neq 0$. and that the curve makes an angle $\phi = \arg\{z_1'(t_1)\}$ with the x-axis at z_0. Then consider the image curve $Z_1(t) = f(z_1(t))$. Then at t_1,

$$Z_1'(t_1) = z_1'(t_1) f'(z_1(t_1)) = z_1'(t_1) f'(z_0).$$

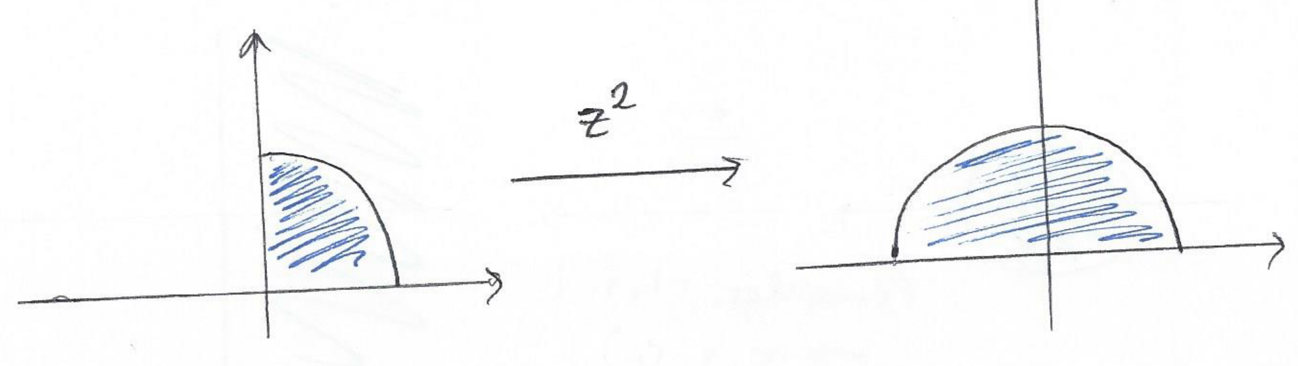
$$\rightarrow \arg Z_1'(t_1) = \arg\left(z_1'(t_1) f(z_0)\right) = \phi + \arg f'(z_0)$$

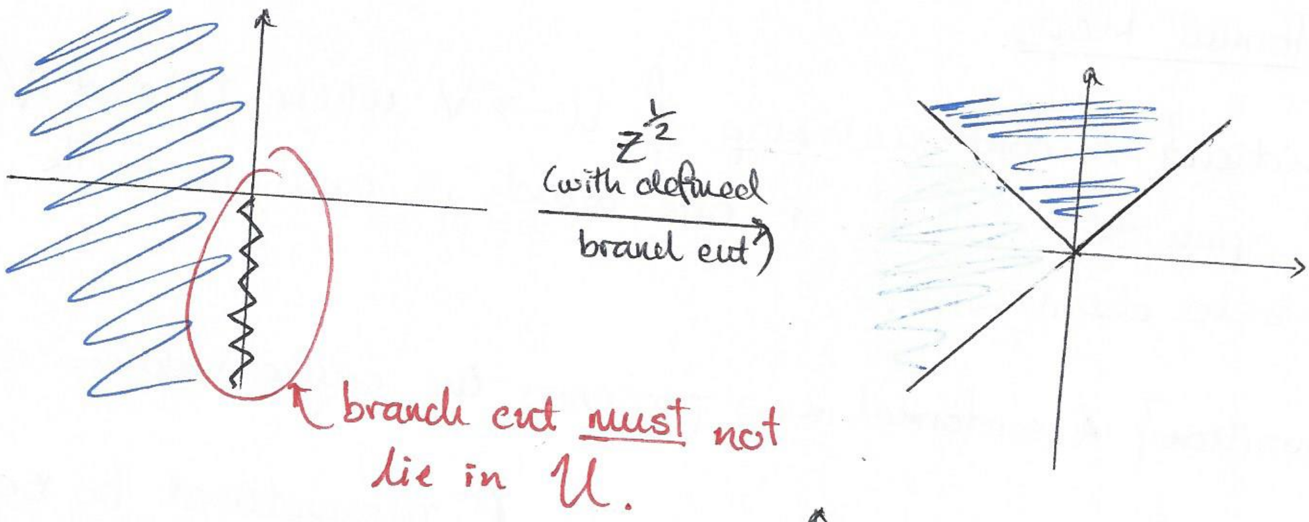
(exists since f conformal)

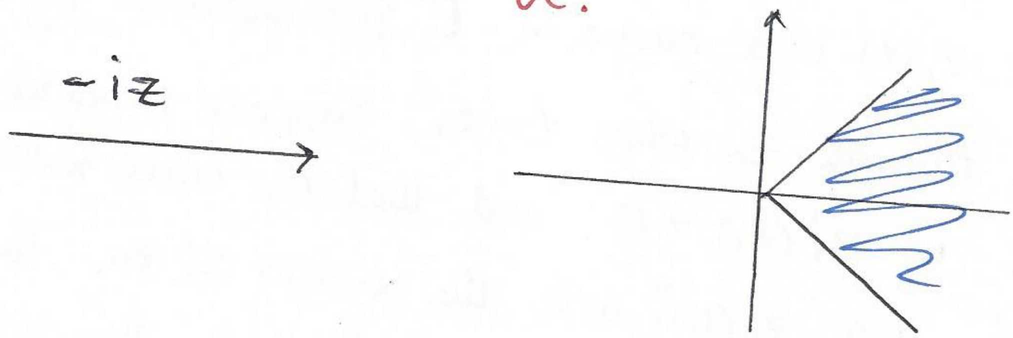
So: tangent direction rotated by $\arg f'(z_0)$ independent of z_1. The result follows.

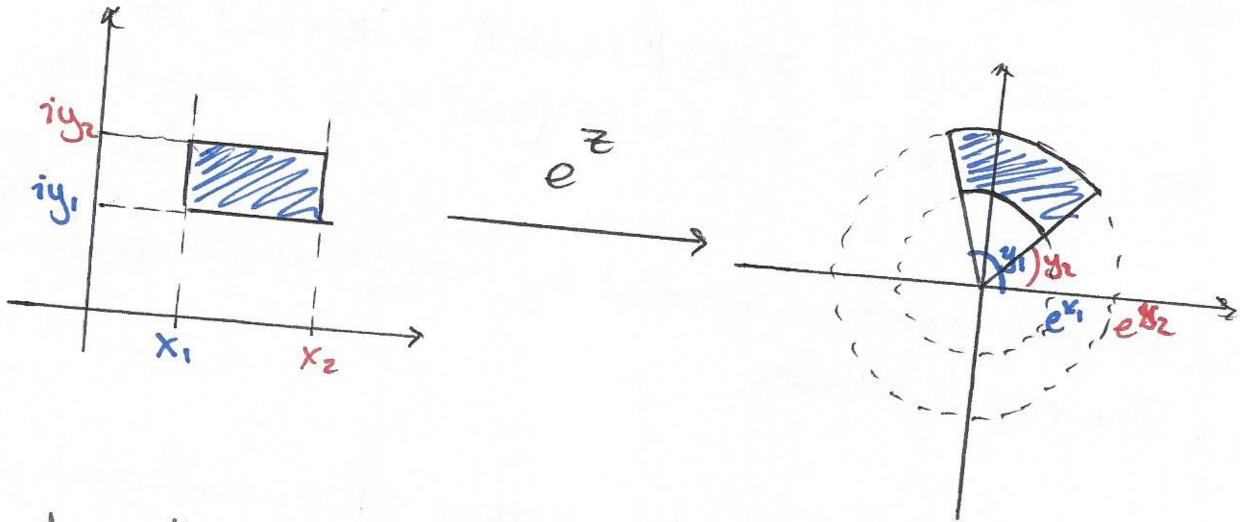
The easiest way to find the action of a conformal map on a set $\mathcal{U}$ is to consider its effect on $\partial\mathcal{U}$, since $\partial\mathcal{U} \mapsto \partial V$. Then take a point to see which region V is.

eg



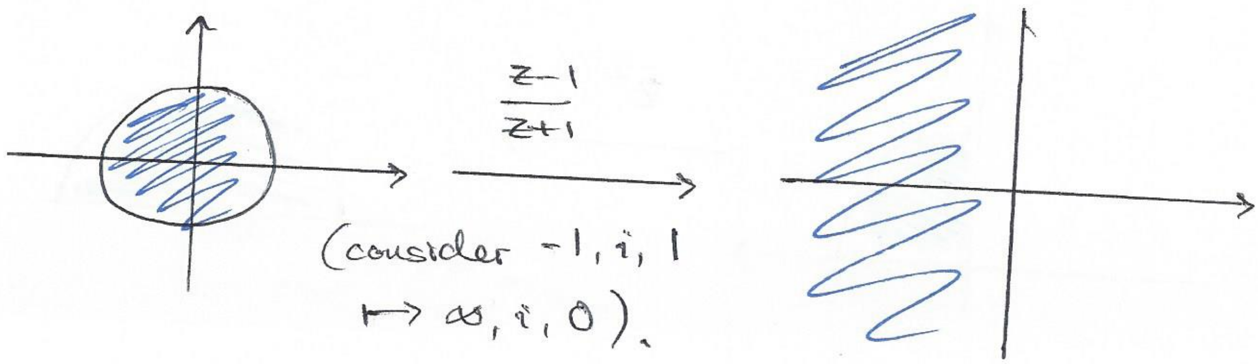






N.B logz does the reverse with suitable choice of branch cut.

Mobius maps take circlines → circlines.



- $\frac{1}{z}$ is particularly useful for acting on horizontal and vertical lines.

- <u>Solving Laplace's equation using conformal maps.</u>

Given Dirichlet BCs on ∂U on a non-standard domain $U \subseteq \mathbb{R}^2 \cong \mathbb{C}$, we apply the following scheme:

1) Find conformal map $f: U \to V$, V is "nice".
2) Map BCs on $\partial U \to \partial V$.
3) Solve $\nabla^2 \Phi = 0$ in V, with new BCs.
4) The required harmonic function $\phi(x,y)$ is given by:

$$\boxed{\phi(x,y) = \Phi\left(\mathrm{Re}\{f(x+iy)\}, \mathrm{Im}\{f(x+iy)\}\right)}.$$

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2. Contour Integration + Cauchy's Theorem.

- A curve is a continuous map $\gamma : [0,1] \to \mathbb{C}$, if it is closed, then $\gamma(0) = \gamma(1)$. Further, if it is simple, then it does not intersect itself. A contour is a piecewise smooth curve.

- <u>Contour Integral:</u>
$$\int_\gamma f(z)\,dz = \int_0^1 f(\gamma(t))\,\gamma'(t)\,dt.$$

e.g $I = \int_{\gamma_1} \frac{dz}{z}$, γ_1 is upper half unit circle. (clockwise)

Let $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta$

$$\Rightarrow I = \int_\pi^0 i\frac{e^{i\theta}}{e^{i\theta}}\,d\theta = -i\pi.$$

• An important property of the contour integral is:

If γ has length L and $|f(z)|$ is bounded by M on γ, then:

$$\left| \int_\gamma f(z)\,dz \right| \leq LM.$$

• Simply-connected domain: A domain D is simply-connected if it is connected and every closed curve in D encloses only points which are also in D.

• Cauchy's Theorem: If $f(z)$ is analytic in a simply-connected domain D, then for every simple closed contour, γ, in D

$$\boxed{\oint_\gamma f(z)\,dz = 0}$$

• Contour Deformation: Suppose γ_1 and γ_2 are contours from a to b, and f is analytic on γ_1 and γ_2 and between the contours, then:

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz.$$

• Cauchy's Integral Formula: Suppose $f(z)$ is analytic in a domain D and $z_0 \in D$. Then:

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}\,dz$$

where γ is a simple closed contour in D, encircling z_0 anti-clockwise.

If we think about this in terms of Laplace's equation, then knowing f on γ is essentially defining Dirichlet conditions on the boundary of a domain for the two harmonic functions

u, and v. We may differentiate Cauchy's formula to give:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

{N.B valid since integrand is continuous}

So f is infinitely differentiable.

3 Laurent Series + Singularities.

- If f is analytic at z_0, then it has a Taylor series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

in a neighbourhood of z_0.

- If instead, f is analytic in an annulus $R_1 < |z-z_0| < R_2$ then it has a Laurent series.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

Laurent series are unique, and we note that Taylor series are a special case of Laurent series.

e.g $f(z) = \frac{1}{z-a}$, $a \in \mathbb{C}$.

i) f is analytic in $|z| < |a|$, so it has a Taylor series about $z_0 = 0$.

$$\frac{1}{z-a} = -\frac{1}{a}\left(1-\frac{z}{a}\right)^{-1} = -\sum_{n=0}^{\infty} \frac{1}{a^{n+1}} z^n$$

ii) For $|z| > |a|$, f has a Laurent series:

$$\frac{1}{z-a} = \frac{1}{z}\left(1-\frac{a}{z}\right)^{-1} = \sum_{m=0}^{\infty} \frac{a^m}{z^{m+1}} = \sum_{n=-\infty}^{-1} a^{-n-1} z^n.$$

• The radius of convergence of a Laurent series about z_0 is the distance from z_0 to the closest singularity of $f(z)$.

• Zeros: The zeros of an analytic function $f(z)$ are of order N if in it's Taylor series, the first non-zero coefficient is a_N. Alternatively, order N if $f(z_0) = f'(z_0) = \cdots = f^{(N-1)}(z_0) = 0$ but $f^{(N)}(z_0) \neq 0$.

• When looking for Taylor series about z_0, often useful to consider writing the functions in terms of $\zeta = z - z_0$.

• Classification of singularities:

{Definition} Isolated singularity: Suppose f has a singularity at $z = z_0$, if there is a neighbourhood of z_0 such that $f(z)$ is analytic then z_0 is an isolated singularity. If there is no such neighbourhood, then it is an non-isolated singularity.

e.g $\operatorname{cosech} z$ and $\operatorname{cosech} \frac{1}{z}$.

i) $\operatorname{cosech} z$ has isolated singularities at $z = n\pi i$.

ii) $\operatorname{cosech} \frac{1}{z}$ has isolated singularities at $z = \frac{1}{n\pi i}$ and a non isolated one at $z = 0$.

$\boxed{\text{N.B}}$ $\operatorname{cosech} z$ also has essential, non-isolated singularity at $z = \infty$ because $\operatorname{cosech} \frac{1}{\zeta}$ has non-isolated singularity at $\zeta = 0$.

Classification:

① Check for branch point singularity.

② Check for non-isolated singularity

③ Otherwise consider Laurent series. $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$

a) If $a_n = 0$ for all $n < 0$, $\Rightarrow$ removable singularity

b) If $\exists N > 0$: $a_n = 0$ $\forall n < -N$ but $a_{-N} \neq 0$ $\Rightarrow$ order N pole at z_0.

c) if $\nexists N \Rightarrow$ essential isolated singularity.

N.B. for removable singularities, if we redefine $f(z_0) = a_0$, then $f(z)$ will become analytic at $z = z_0$.

- Remember ∞, e.g. z^2 has double pole at ∞ since $\frac{1}{\zeta^2}$ has double pole at $\zeta = 0$.

- Residues: At an isolated singularity, z_0, $\operatorname{res}_{z=z_0} f(z)$ is the coefficient a_{-1} in it's Laurent expansion.

At a pole of order N:

$$\boxed{\operatorname{res}_{z=z_0} f(z) = \lim_{z \to z_0} \left\{ \frac{1}{(N-1)!} \frac{d^{N-1}}{dz^{N-1}} (z-z_0)^N f(z) \right\}}$$

N.B. L'Hôpitals rule is extremely useful here.

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4 The Calculus of Residues.

- The residue theorem: Suppose f is analytic in a simply-connected domain except at a finite number of isolated singularities $z_1, \ldots, z_n$, and γ encircles all singularities anticlockwise:

$$\boxed{\oint_\gamma f(z)\, dz = 2\pi i \sum_{k=1}^{n} \operatorname{res}_{z=z_k} f(z).}$$

- Aware how to approach integrals like $\int_0^\infty \frac{1}{1+x^4} dx$ etc.

Consider integrals of the form:

$$\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta$$

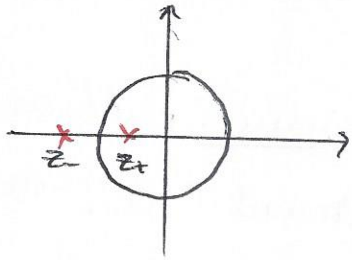
We substitute $\boxed{z = e^{i\theta}}$, then:

$$\cos\theta = \frac{1}{2}\left(z + z^{-1}\right)$$

$$\sin\theta = \frac{1}{2i}\left(z - z^{-1}\right)$$

and we end up with a closed contour integral around the unit circle.

e.g. $I = \displaystyle\int_0^{2\pi} \frac{d\theta}{a + \cos\theta}$, $a > 1$



Let $z = e^{i\theta} \Rightarrow dz = iz\,d\theta$.

$$\Rightarrow \quad I = \oint_\gamma \frac{(iz)^{-1}\,dz}{a + \frac{1}{2}(z + z^{-1})} = -2i \oint_\gamma \frac{dz}{z^2 + 2az + 1}$$

$$z^2 + 2az + 1 = 0 \Rightarrow z = z_\pm = -a \pm \sqrt{a^2 - 1}$$

Now, $\displaystyle\frac{1}{z^2 + 2az + 1} = \frac{1}{(z - z_-)(z - z_+)} \rightarrow \operatorname*{res}_{z = z_+} \left\{ \frac{1}{z^2 + 2az + 1} \right\} = \frac{1}{z_+ - z_-} = \frac{1}{2\sqrt{a^2 - 1}}$

$$\Rightarrow \quad I = -2i\left(\frac{2\pi i}{2\sqrt{a^2 - 1}}\right) = \frac{2\pi}{\sqrt{a^2 - 1}}$$

· <u>Keyhole contours</u>: (used for functions with a <u>branch cut</u>).

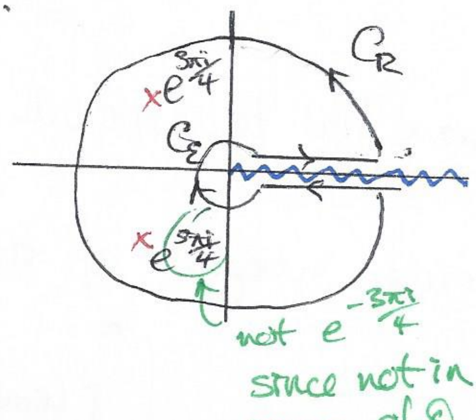
e.g. $I = \displaystyle\int_0^\infty \frac{x^\alpha}{1 + \sqrt{2}\,x + x^2}\,dx$ with $-1 < \alpha < 1$.

Take branch cut to be along positive real axis: and define $z^\alpha = r^\alpha e^{i\alpha\theta}$, $0 \leq \theta < 2\pi$.

Now: $\displaystyle\int_{\gamma_R} \frac{z^\alpha\,dz}{1 + \sqrt{2}z + z^2} = 2\pi R \cdot O(R^{\alpha - 2})$

$$= O(R^{\alpha - 1})$$

$$\longrightarrow 0$$



not $e^{-\frac{3\pi i}{4}}$ since not in range of θ defined in branch cut.

Along C_ε, let $z = \varepsilon e^{i\theta}$

$$\Rightarrow \int_{2\pi}^{0} \frac{\varepsilon^\alpha e^{i\alpha\theta}}{1+\sqrt{2}\,\varepsilon e^{i\theta} + \varepsilon^2 e^{2i\theta}}\, i\varepsilon e^{i\theta} d\theta$$

$$= O(\varepsilon^{\alpha+1}) \to 0$$

Now: just above branch cut:

$$\int_\varepsilon^R \frac{x^\alpha}{1+\sqrt{2}x + x^2}\, dx \to I$$

Just below the cut:

$$\int_R^\varepsilon \frac{x^\alpha e^{2\alpha\pi i} dx}{1+\sqrt{2}x + x^2} \to -e^{2\alpha\pi i} I$$

The residues at $z = e^{3\pi i/4}$ and $z = e^{5\pi i/4}$ are:

$$\frac{e^{3\alpha\pi i/4}}{\sqrt{2}i}, \quad \frac{e^{5\alpha\pi i/4}}{-\sqrt{2}i}$$

$$\Rightarrow (1-e^{2\alpha\pi i}) I = 2\pi i \left\{ \frac{e^{3\alpha\pi i/4}}{\sqrt{2}i} - \frac{e^{5\alpha\pi i/4}}{\sqrt{2}i} \right\}$$

$$\Rightarrow e^{\alpha\pi i}(e^{-\alpha\pi i} - e^{\alpha\pi i}) I = \sqrt{2}\pi e^{\alpha\pi i}(e^{-\alpha\pi i/4} - e^{\alpha\pi i/4})$$

$$\Rightarrow \qquad I = \frac{\sqrt{2}\pi \sin\left(\frac{\alpha\pi}{4}\right)}{\sin(\alpha\pi)}.$$

- <u>Rectangular contours</u>, often used for trigonometric and hyperbolic functions, Useful to utilise:

$$\cosh(x+iy) = \cosh R \cos y + i \sinh R \sin y.$$

to provide bounds.

Also used when considering $\cot z$, where we use a square contour to avoid the singularities (at $\boxed{z = n\pi}$) $n \neq 0$ ← simple poles
and a multiple pole at $\boxed{z=0}$.
To show the sides integrals tend to zero, we use:

$$|\cot \pi z| = \left|\cot\left(\left(N+\tfrac{1}{2}\right) + i\pi y\right)\right| = |-\tan i\pi y| \quad \text{along sides}$$
$$= |\tanh \pi y| \leq 1$$

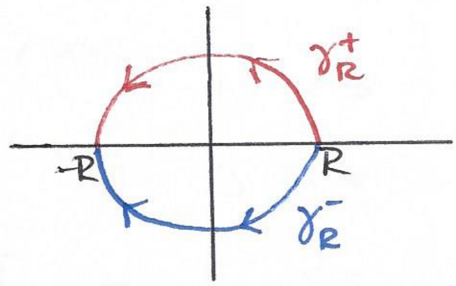
and
$$|\cot \pi z| = \frac{\sqrt{\cosh^2 (N+\frac{1}{2})\pi - \sin^2 \pi x}}{\sqrt{\sinh^2 (N+\frac{1}{2})\pi + \sin^2 \pi x}} \leq \coth\left(N+\tfrac{1}{2}\right)\pi \quad \text{on top}$$
$$\leq \coth \frac{\pi}{2}.$$

So $\cot \pi z$ is bounded $\Rightarrow \displaystyle\int \frac{\cot \pi z}{z^n}\, dz \to 0$ as $N \to \infty$.

- Jordan's Lemma: Suppose f is analytic except for a finite number of singularities and $f(z) \to 0$ as $|z| \to \infty$.

Then, for any $\lambda > 0$:

$$\boxed{\int_{\gamma_R^+} f(z) e^{i\lambda z}\, dz \to 0.}$$



{Similarly, if $\lambda < 0$, $\displaystyle\int_{\gamma_R^-} f(z) e^{i\lambda z}\, dz \to 0$ }

Proof: LEARN.

For $\theta \in [0, \frac{\pi}{2}]$, we have $\sin\theta \geq \frac{2\theta}{\pi}$

So: $$\left|\int_{\gamma_R} f(z) e^{i\lambda z}\, dz\right| = \left|\int_0^{\pi} f(Re^{i\theta}) e^{i\lambda Re^{i\theta}} iRe^{i\theta}\, d\theta\right|$$

$$\leq R\int_0^{\pi} |f(Re^{i\theta})|\left|e^{i\lambda Re^{i\theta}}\right| d\theta$$

$$\leq 2R \sup_{z\in\gamma_R} |f(z)| \int_0^{\frac{\pi}{2}} e^{-\cancel{2}\lambda R\sin\theta}\, d\theta$$

$$\leq 2R \sup_{z\in\gamma_R} |f(z)| \int_0^{\frac{\pi}{2}} e^{-\frac{2\lambda R\theta}{\pi}}\, d\theta$$

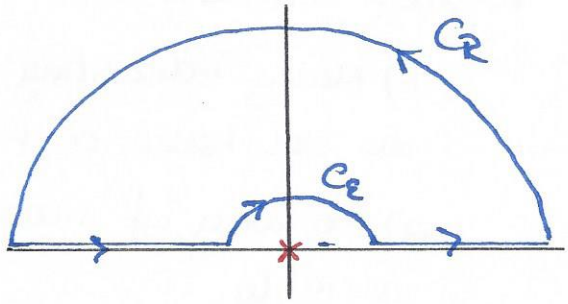
$$= \frac{\pi}{\lambda}\left(1-e^{-\lambda R}\right) \sup_{z\in\gamma_R} |f(z)| \longrightarrow 0$$

Similarly for $\lambda<0$. $\square$

eg $I=\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx$, to use Jordan's lemma, we integrate $\frac{e^{iz}}{z}$. Since $\frac{e^{iz}}{z}$ is singular at the origin, we write:

$$I=\operatorname{Im}\left\{\lim_{\substack{\varepsilon\to 0\\ R\to\infty}}\left\{\int_{-R}^{-\varepsilon}\frac{e^{ix}}{x}\,dx+\int_{\varepsilon}^{R}\frac{e^{ix}}{x}\,dx\right\}\right\}$$

and use the following contour:



So:

$$\int_{-R}^{-\varepsilon}\frac{e^{iz}}{z}\,dz+\int_{\varepsilon}^{R}\frac{e^{iz}}{z}\,dz=-\int_{C_\varepsilon}\frac{e^{iz}}{z}\,dz-\int_{C_R}\frac{e^{iz}\,dz}{z}$$

By Jordan's lemma, $\int_{C_R}\frac{e^{iz}}{z}\,dz\to 0$. On C_ε, let $z=\varepsilon e^{i\theta}$

$$\Rightarrow \int_{\pi}^{0}\frac{1+O(\varepsilon)}{\varepsilon e^{i\theta}}\, i\varepsilon e^{i\theta}\, d\theta = -i\pi + O(\varepsilon)$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx = \operatorname{Im}\{i\pi\} = \pi .$$

5 Transform Theory.

{Definition} Fourier Transform of $f(x)$ is:

$$\boxed{\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx}$$

and the inverse transform is:

$$\boxed{f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} \, dk}$$

Cauchy principal value.

(More precisely: $\frac{1}{2}\{f(x^+) + f(x^-)\} = \frac{1}{2\pi} \, PV \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} \, dk$)

- When using the calculus of residues to calculate inverse Fourier transforms, we usually use a semicircular contour (upper half plane if $x>0$, lower if $x<0$) and then apply Jordan's Lemma.

- Laplace Transform. We first look at the limitations of the Fourier transform:

  1) Many interesting functions grow exponentially, and do not have a FT.
  2) No way of incorporating initial or BC in transform variable.

The Laplace transform is a solution, however only defined for functions $f(t)$: $[f(t) = 0 \text{ for } t<0]$,

Normally we omit $H(t)$ when writing functions. So $f(t) = e^t \Leftrightarrow f(t) = e^t H(t)$.

The Laplace Transform of $f(t)$ is:

$$\boxed{\tilde{f}(p) = \int_0^\infty f(t) e^{-pt}\, dt.}$$

We write $\hat{f} = \mathcal{L}\{f\}$.

- <u>Standard properties / standard transforms.</u>

<u>Properties</u>:

i) $\mathcal{L}\{\alpha f + \beta g\} = \alpha \mathcal{L}\{f\} + \beta \mathcal{L}\{g\}$.

ii) $\mathcal{L}\{f(t-t_0) H(t-t_0)\} = e^{-pt_0} \hat{f}(p)$

iii) $\mathcal{L}\{f(\lambda t)\} = \frac{1}{\lambda} \hat{f}\left(\frac{p}{\lambda}\right)$

iv) $\mathcal{L}\{e^{p_0 t} f(t)\} = \hat{f}(p - p_0)$

(*) v) $\mathcal{L}\{f'(t)\} = p\hat{f}(p) - f(0)$

(*) vi) $\mathcal{L}\{f^{(n)}(t)\} = p^n \hat{f}(p) - \sum_{k=1}^{n} p^{n-k} f^{(k-1)}(0)$

(*) vii) $\mathcal{L}\{t^n f(t)\} = (-1)^n \hat{f}^{(n)}(p)$.

(*) viii) $\mathcal{L}\{f * g\} = \hat{f}(p)\hat{g}(p)$ useful for integral equations.

(*) ix) Asymptotic Limits: for integration constants.

$$p\hat{f}(p) \to \begin{cases} f(0) & \text{as } p \to \infty \\ f(\infty) & \text{as } p \to 0 \end{cases},$$

<u>Standard Transforms</u>:

i) $\mathcal{L}\{\delta(t)\} = 1$

ii) $\mathcal{L}\{1\} = \frac{1}{p}$

iii) $\mathcal{L}\{t^n\} = \frac{n!}{p^{n+1}}$

iv) $\mathcal{L}\{e^{-\alpha t}\} = \frac{1}{p+\alpha}$

v) $\mathcal{L}\{\sin \omega t\} = \frac{\omega}{p^2 + \omega^2}$

vi) $\mathcal{L}\{\cos\omega t\} = \dfrac{p}{p^2+\omega^2}$

vii) $\mathcal{L}\{\sinh\alpha t\} = \dfrac{\alpha}{p^2-\alpha^2}$

viii) $\mathcal{L}\{\cosh\alpha t\} = \dfrac{p}{p^2-\alpha^2}$

ix) $\mathcal{L}\{e^{-\alpha t}\sin\omega t\} = \dfrac{\omega}{(s+\alpha)^2+\omega^2}$ (from shifting property)

- Inverse Laplace Transform: We use the Bromwich inversion formula.

$$\boxed{f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{f}(p)e^{pt}\,dp}$$

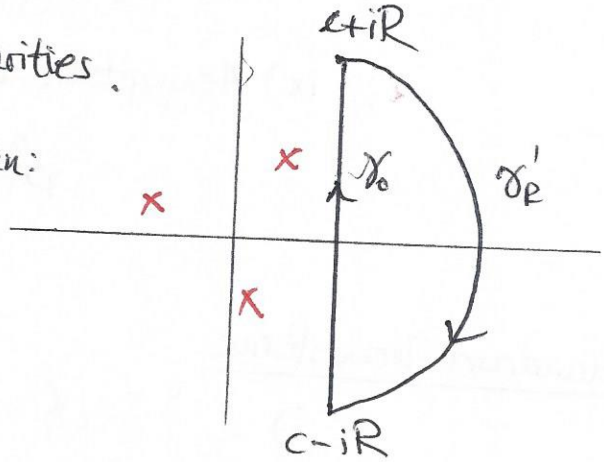
where c is real and chosen such that the Bromwich inversion contour, γ, lies to the right of all singularities.

In most cases, this reduces to: (when $\left[\hat{f}(p) = o(|p|^{-1})\right.$ as $\left.|p|\to\infty\right]$).

$$\boxed{f(t) = \sum_{k=1}^{n} \operatorname*{res}_{p=p_k}\left\{\hat{f}(p)e^{pt}\right\}}.$$

Proof: For $t<0$, γ encloses no singularities.

Now, if $\hat{f}(p) = o(|p|^{-1})$ as $|p|\to\infty$, then:



$$\left|\int_{\gamma_R'} \hat{f}(p)e^{pt}\,dp\right| \le \pi R e^{ct}\sup_{p\in\gamma_R'}|\hat{f}(p)| \cdot$$

$$\to 0 \text{ as } R\to\infty.$$

So, (with a slight modification to Jordan's lemma if $\hat{f}(p)$ decays less rapidly),

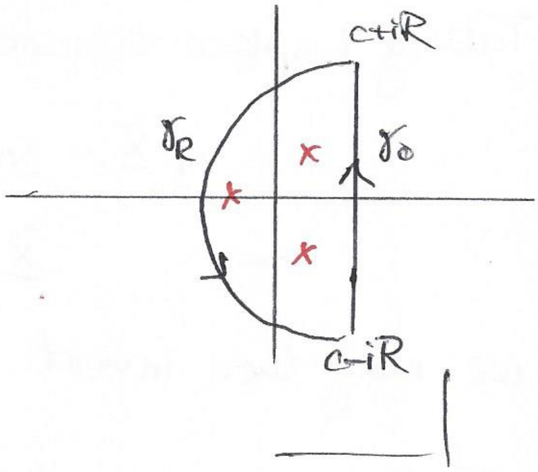
$$\int_{\gamma_0} \hat{f}(p)e^{pt}\,dp \to \int_{\gamma} \hat{f}(p)e^{pt}\,dp = 0 \text{ by Cauchy's Theorem}$$

$\Longrightarrow \quad f(t) = 0 \quad$ for $t < 0$.

For $t > 0$ we close contour to left.

Again $\displaystyle\int_{\gamma_R} \to 0$ as $R \to \infty$,

then the result follows from the residue theorem.



e.g. $\hat{f}(p) = p^{-n}$, which satisfies $\hat{f}(p) \to 0$ as $|p| \to \infty$.

Then, for $t > 0$, $\quad f(t) = \operatorname*{res}_{p=0} \left\{ \dfrac{e^{pt}}{p^n} \right\} = \lim_{p \to 0} \left\{ \dfrac{1}{(n-1)!} \dfrac{d^{n-1}}{dp^{n-1}} \left\{ e^{pt} \right\} \right\}$

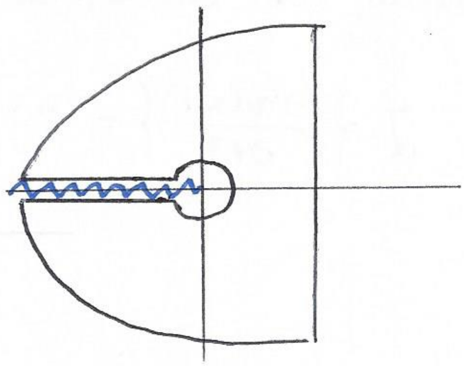
n^{th} order pole.

$$\Longrightarrow \quad f(t) = \frac{t^{n-1}}{(n-1)!}$$

- If $\hat{f}(p) \not\to 0$ as $|p| \to \infty$, we have to use the full inversion formula.

- Also, if $\hat{f}(p)$ has a branch point, we must use a Bromwich keyhole contour as shown:

Examples in the problem sheet 3.



- Obviously may be used to solve ODE's with initial conditions. Also can be used to solve systems of ODEs.

For example: $\quad \dot{\underline{x}} = M\underline{x} \quad , \quad \underline{x}(0) = \underline{x}_0$

Taking Laplace transforms yields:

$$p\tilde{\underline{x}} - \underline{x}_0 = M\tilde{\underline{x}}$$

$$\Rightarrow \quad \tilde{\underline{x}} = (pI - M)^{-1}\underline{x}_0$$

we may then invert to find $\underline{x}(t)$.

- Finally we may use Laplace transforms to solve PDE's where we transform the variable in which the initial data is given.

e.g $\quad \dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^2 u(x,t)}{\partial x^2} \quad , \quad u(0,t) = 0, \; u(1,t) = 0$

We transform $u(x,t) \mapsto U(x,s)$ $\qquad u(x,0) = 3\sin(2\pi x)$.

initial data.

$$\Rightarrow \quad \frac{d^2 U(x,s)}{dx^2} = sU(x,s) - u(x,0)$$

and solve the resulting ODE for fixed s, noting that

$u(0,t) = 0 \;, \; u(1,t) = 0 \;\Rightarrow\; U(0,s) = 0, \; U(1,s) = 0$.

Finally we transform back, $U(x,s) \mapsto u(x,t)$.

$\boxed{\text{N.B}}$ $\mathcal{L}\left\{\dfrac{\partial^2 u(x,t)}{\partial t^2}\right\} = s^2 U(x,s) - su(x,0) - u_t(x,0)$