Asymptotic Analysis of Oscillatory Integrals - the Stationary Phase method.

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I Introduction to Stationary Phase

The asymptotic analysis of integrals such as

$$I(x) = \int_a^b f(t)\, e^{ix\phi(t)}\, dt\,, \qquad (x \to +\infty)$$

hinges upon the following heuristic Localization Principle for Oscillatory Integrals:

> *The asymptotic expansion of $I(x)$ for large positive x is determined entirely by the points of stationary phase $\{t_\mu : \phi'(t_\mu) = 0\}$ (and possibly the endpoints a and b).*

To see why this is so under appropriate hypotheses, consider $[a, b] = [0, 2\pi]$ with periodic boundary conditions (so the endpoint contributions cancel out) when there is no point of stationary phase:

1. either $\phi'(t) \geq c > 0\ \forall t$ or $-\phi'(t) \geq c > 0\ \forall t$;
2. $f \in C^\infty(\mathbb{R})$ and $f(t + 2\pi) = f(t)\ \forall t$;
3. $\phi \in C^\infty(\mathbb{R})$ and $\phi(t + 2\pi) = \phi(t) + 2k\pi\ \forall t$ for some integer k.

The model case to have in mind is the Fourier coefficients c_n of a periodic function f

$$f(t) = \sum c_n e^{int}\,, \qquad c_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-int}\, f(t)\, dt$$

corresponding to the case $\phi(t) = -t$. In this case the identity

$$(in)^s c_n = \int_0^{2\pi} e^{-int}\, f^{(s)}(t)\, dt$$

(which follows by sucessive integration by parts) shows that if f is a 2π-periodic C^s function[1] then $c_n = O(n^{-s})$ as $n \to \infty$. The same is true in general:

[1] This means its derivatives up to order s exist and are continuous

Lemma 1. *Under conditions 1-3 above*

$$I(n) \;=\; \int_0^{2\pi} f(t)\,e^{in\phi(t)}\,dt \;=\; O(n^{-N})$$

as $n \to +\infty$ for all $N \in \mathbb{N}$.

Proof. Just as for the case of Fourier coefficients this is proved by repeated integration by parts.

$$\begin{aligned} I(n) \;&=\; \int_0^{2\pi} f(t)\,e^{in\phi(t)}\,dt \;=\; \Big[\frac{f(t)e^{in\phi(t)}}{in\phi'(t)}\Big]_{t=0}^{t=2\pi} - \frac{1}{in}\int_0^{2\pi}\Big(\frac{f(t)}{\phi'(t)}\Big)'\,e^{in\phi(t)}\,dt \\ &=\; \Big[\frac{f(t)e^{in\phi(t)}}{in\phi'(t)}\Big]_{t=0}^{t=2\pi} - \Big[\frac{e^{in\phi(t)}}{in\phi'(t)}\Big(\frac{f(t)}{in\phi'(t)}\Big)'(t)\Big]_{t=0}^{t=2\pi} \\ &\quad+\; \frac{1}{(in)^2}\int_0^{2\pi}\Big(\frac{1}{\phi'}\Big(\frac{f}{\phi'}\Big)'\Big)'(t)\,e^{in\phi(t)}\,dt\,. \end{aligned}$$

The boundary terms vanish by the periodicity assumptions, leaving

$$I(n) \;=\; \int_0^{2\pi} f(t)\,e^{in\phi(t)}\,dt \;=\; \frac{1}{(in)^2}\int_0^{2\pi}\Big(\frac{1}{\phi'}\Big(\frac{f}{\phi'}\Big)'\Big)'(t)\,e^{in\phi(t)}\,dt\,.$$

Now applying the quotient rule the integrand is continuous on account of condition 1 above, and therefore $I(n) = O(n^{-2})$. But this process can clearly be continued indefinitely, under the smooth periodicity assumptions in 2 and 3 above, since only $\phi'(t)$ ever appears in the denominator. Indeed the result of s such integration by parts can be written

$$I(n) \;=\; \int_0^{2\pi} f(t)\,e^{in\phi(t)}\,dt \;=\; \frac{1}{(in)^s}\int_0^{2\pi}(D^s f)(t)\,e^{in\phi(t)}\,dt\,,$$

where D is the operator whose action on an arbitrary function g is

$$Dg(t) \;=\; \frac{d}{dt}\Big(\frac{1}{\phi'(t)}g(t)\Big)$$

Applying the quotient rule will give an expression for $D^s f$ which is of the form $Q(t)/(\phi'(t)^{2s}$ where $Q(t)$ is some polynomial expression in $f(t)\ldots f^{(s)}(t)$ and $\phi'(t),\ldots\phi^{(s+1)}(t)$. It follows that under the assumptions 1-3 above $D^s f(t)$ is bounded, and hence the result follows. □

Remark 2. An alternative way to prove this would be to take advantage of the monotonicity of ϕ (which follows from assumption 1) to make a change of variables to $y = \phi(t)$.

An equally simple case is that of a smooth compactly supported function $f : \mathbb{R} \to \mathbb{R}$, i.e. a function in the space

$$C_c^\infty(\mathbb{R}) \;=\; \{f \in C^\infty(\mathbb{R}) : \exists R > 0 \text{ such that } f(t) = 0 \text{ if } |t| \geq R\}\,.$$

Lemma 3. *Let $f \in C_c^\infty(\mathbb{R})$ and let $\phi \in C^\infty(\mathbb{R})$ satisfy either $\phi'(t) \geq c > 0\ \forall t$ or $-\phi'(t) \geq c > 0\ \forall t$. Then*

$$I(x) \;=\; \int_{-\infty}^{+\infty} f(t)\,e^{ix\phi(t)}\,dt \;=\; O(x^{-N})\ \forall N \in \mathbb{N}\,, \qquad (x \to +\infty)\,.$$

Proof. Exercise using the same integration by parts as in the proof of the previous Lemma, but making use of the hypothesis of compact support to see that the boundary terms vanish. $\square$

Remark 4. It is clear that the conclusion of the theorem holds for $I(x) = \int_a^b f(t)\, e^{ix\phi(t)}\, dt$ where $[a,b]$ is either a closed bounded interval and f is zero outside of a proper closed subinterval $[c,d] \subset (a,b)$ or if $[a,b]$ is a semi-infinite interval such as $[a,\infty)$ and f is zero outside of a proper bounded closed subinterval $[c,d] \subset (a,\infty)$. However, if f does not vanish in a neighbourhood of one of the endpoints, say the point $t = a$, it is clear that the integration by parts procedure will give *endpoint contributions* from such points even though they are not points of stationary phase.

Finally on this topic, there is a general result called the *Riemann-Lebesgue* Lemma which states that for any integrable function f

$$\boxed{\lim_{|x|\to\infty} \int_{\mathbb{R}} f(t)e^{ixt}\, dt = 0\,.}$$

This result however does not come with any quantitative information on the rate of decay. Two approaches to its formulation and proof are given below, after some simple examples of stationary phase integrals.

II Simple Examples

According to the Localization Principle just discussed the asymptotics of oscillatory integrals is determined by evaluating the contributions from each of the points of stationary phase, but with the additional twist that the endpoints of the interval may well contribute also - this becomes clear from some examples. Thus we follows the following four steps:

- **Step One:** Find the points $\{t_\mu\}$ at which $\phi'(t_\mu) = 0$;
- **Step Two:** Expand in small neighbourhoods of the $\{t_\mu\}$ and obtain the expansions of the integral over the intervals $t_\mu - \epsilon \le t \le t_\mu + \epsilon$ (or over $t_\mu - \epsilon \le t \le t_\mu$ or $t_\mu \le t \le t_\mu + \epsilon$ if $t_\mu = b$ or $t_\mu = a$ respectively);
- **Step Three:** ascertain whether or not there are endpoint contributions (in the sense of Remark 4) and if so use the integration by parts procedure to compute them;
- **Step Four:** Combine all contributions to get an overall asymptotic expansion.

In practice this is similar to the procedure for Laplace integrals although the mechanism behind the Localization Principle in the oscillatory case is completely different. Also observe that end points are more likely to be important as there is no real exponential damping out factor as in the case of Laplace integrals.

There are some specially simple cases in which the Cauchy theorem of complex analysis makes it possible to convert a stationary phase integral into a Laplace integral. Before attempting a real stationary phase problem we will consider a simple example of this type:

An exactly computable example. For the integral $\int_0^\infty t^\gamma e^{ixt^p} dt$ with $x > 0, p \ge 1, 0 > \gamma > -1$, the asymptotics is expected to be determined by the sole point of stationary phase $t = 0$. Actually the integral can be computed exactly: we assert that

$$\int_0^\infty t^\gamma e^{ixt^p} dt \;=\; \frac{1}{p}\left(\frac{1}{x}\right)^{\frac{\gamma+1}{p}} \Gamma\left(\frac{\gamma+1}{p}\right) \exp\left[\frac{i\pi}{2p}(\gamma+1)\right]\,. \tag{1}$$

To derive this apply the Cauchy theorem and Jordan's lemma in the sector of the upper half complex plane bounded by the real axis $t = \rho$, the ray $t = \rho e^{\frac{i\pi}{2p}}$ for $0 \le \rho \le R$ and the arc $t = Re^{i\theta}$, $0 \le \theta \le \frac{\pi}{2p}$. Apply the change of variables $z = t^p$ to convert the integral along the arc to the following integral on $\mathcal{A} = \{z = R^p e^{i\Theta}, 0 \le \Theta \le \frac{\pi}{2}\}$:

$$\int_{\mathcal{A}} e^{ixz}\, z^{\frac{\gamma+1}{p}} \frac{dz}{z}$$

which has limit zero as $R \to \infty$ by Jordan's Lemma as long as $\frac{\gamma+1}{p} < 1$, which holds by the assumptions. Therefore, taking the limit in the Cauchy theorem we find that the two integrals along the rays are equal, i.e.

$$\int_0^\infty t^\gamma e^{ixt^p} dt \;=\; \int_0^\infty \left(e^{\frac{i\pi}{2p}}\rho\right)^\gamma e^{-x\rho^p} e^{\frac{i\pi}{2p}} d\rho = e^{\frac{i\pi}{2p}(\gamma+1)} \int_0^\infty \rho^\gamma e^{-x\rho^p} d\rho. \tag{2}$$

Using the change of variables $x\rho^p = u$, we find

$$\int_0^\infty \rho^\gamma e^{-x\rho^p} d\rho = \frac{1}{x^{\frac{\gamma+1}{p}}} \frac{1}{p} \int_0^\infty u^{\frac{\gamma+1}{p}-1} e^{-u}\, du \;=\; \frac{1}{x^{\frac{\gamma+1}{p}}} \frac{1}{p} \Gamma\left(\frac{\gamma+1}{p}\right),$$

and then (2) gives the result (1) for $x > 0$.

The case $x < 0$ can be analyzed in a similar way, where the relevant ray is now $\rho e^{-i\pi/2p}$, (or just by complex conjugation). It is a consequence of this computation that the integral (1) exists as an improper integral, i.e. although not absolutely convergent the limit of the $\int_0^R$ does exist. (As remarked in the notes on complex Gaussian integrals it is possible to make sense of such oscillatory integrals more generally by using distributions, even when they do not exist as improper integrals.)

Stokes' Example: Contributions from Stationary Phase points. Consider

$$\int_0^\infty \cos\big(x(t^3 - t)\big)\, dt \;=\; \frac{1}{2}\Re \int_{-\infty}^\infty \exp[ix(t^3 - t)]\, dt\,.$$

Written as on the right, there are no endpoints so we only need to follows steps 1,2 and 4.The phase function is $\phi(t) = t^3 - t$. The stationary points are at $\pm\frac{1}{\sqrt{3}}$ and will make identical contributions by parity. Expanding $t = \frac{1}{\sqrt{3}} + s$ we find the phase function has (exact) Taylor expansion

$$\phi(\frac{1}{\sqrt{3}} + s) \;=\; \frac{-2}{3\sqrt{3}} + \sqrt{3}s^2 + s^3\,.$$

Thus this point is a local minimum. (The other is a local maximum). The fact that the second derivative is nonzero determines that the leading contribution is $O(x^{-\frac{1}{2}})$, as for Laplace type integrals. This completes **Step One.**

For **Step Two** the procedure is to make a local expansion in a neighbourhood of the points of stationary phase, leading to the following approximation for the contribution to the asymptotics from a neighbourhood of point $s = 0$:

$$\begin{aligned}\int_{\frac{1}{\sqrt{3}}-\epsilon}^{\frac{1}{\sqrt{3}}+\epsilon} \exp\Big[ix\phi(t)\Big]\,dt \;&=\; \int_{-\epsilon}^{+\epsilon} \exp\Big[\frac{-2ix}{3\sqrt{3}} + ix\sqrt{3}s^2\Big] \times \exp\Big[ixs^3\Big]\,ds \\ &= \int_{-\epsilon}^{+\epsilon} \exp\Big[\frac{-2ix}{3\sqrt{3}} + ix\sqrt{3}s^2\Big] \times \Big(1 + ixs^3 + \dots\Big)\,ds \\ &= \int_{-\infty}^{+\infty} \exp\Big[\frac{-2ix}{3\sqrt{3}} + ix\sqrt{3}s^2\Big]\,ds \;+ \dots \\ &= \exp\Big[\frac{-2ix}{3\sqrt{3}}\Big] \times \sqrt{\frac{\pi}{\sqrt{3}x}} \times e^{\frac{i\pi}{4}} \;+ \dots\,.\end{aligned}$$

The final line uses the formula for imaginary gaussian integrals (in fact a special case of the formula (1) in the preceding example, with $\gamma = 0, p = 2$). The penultimate relation holds on account of the Localization Principle.

As noted already, **Step Three** is null in this problem. For **Step Four** we appeal again to the Localization Principle to argue that we can combine the two contributions from arbitrarily small neighbourhoods of the points of stationary phase $\pm\frac{1}{\sqrt{3}}$ to obtain the leading term in the asymptotics for the integral:

$$\int_0^\infty \cos\big(x(t^3 - t)\big)\,dt \;=\; \sqrt{\frac{\pi}{\sqrt{3}x}}\,\cos\Big(\frac{\pi}{4} - \frac{2x}{3\sqrt{3}}\Big) \;+ \dots\,. \tag{3}$$

Note the presence of the rapidly oscillating factor - this is very typical of stationary phase problems - multiplying the $\sqrt{\frac{\pi}{\sqrt{3}x}}$ factor which is similar to the dominant terms arising in asymptotics of Laplace type integrals.

Remark 5. The oscillating factor is the reason it is not correct to use the notation $\sim$ in the asymptotic relation (3), since it would imply that the location of zeros was given exactly by the dominant term in the asymptotic expansion, which is generally not the case. See Bender and Orszag, Section 3.7 for a more detailed discussion of this issue. However many texts do not insist upon this point, and will write asymptotic relations of this type with the symbol $\sim$.

Example: Endpoint Contributions. Next we work out an example in which we have the opposite situation - no points of stationary phase and the interval endpoints determine the asymptotics. Consider

$$I(\omega) \;=\; \int_a^\infty f(t)\;\cos(\omega t)\;dt \tag{4}$$

as $\omega \longrightarrow \infty$, where $a\,,\,\omega$ are real, $f(t)$ is a smooth real function such that $f^{(s)}(t) \longrightarrow 0$ as $t \longrightarrow \infty$ and $f^{(s)} \in L^1(\mathbb{R})$ for all $s \geq 0$.

Consider to start with the integral $\int_a^\infty f(t)\,\exp[i\omega t]\,dt$ and perform an integration by parts to get

$$\int_a^\infty f(t)\,e^{i\omega t}\,dt \;=\; -\frac{f(a)e^{i\omega a}}{i\omega} \;-\; \frac{1}{i\omega}\int_a^\infty f'(t)\,e^{i\omega t}\,dt\,.$$

By the Riemann-Lebesgue lemma the remainder term can be bounded as

$$\Big|\frac{1}{i\omega}\int_a^\infty f'(t)\,e^{i\omega t}\,dt\Big| \;\leq\; \frac{1}{|\omega|}\Big|\int_a^\infty f'(t)\,e^{i\omega t}\,dt\Big| = o(\frac{1}{|\omega|})$$

since $f' \in L^1$ by assumption, justifying

$$\int_a^\infty f(t)\, e^{i\omega t}\, dt \;=\; -\frac{f(a)e^{i\omega a}}{i\omega} \;+\; o(\frac{1}{|\omega|}).$$

This structure repeats: after N integration by parts

$$\int_a^\infty f(t)\, e^{i\omega t}\, dt \;=\; \sum_{j=1}^{N} (-1)^j \frac{f^{(j-1)}(a)e^{i\omega a}}{(i\omega)^j} \;+\; (-1)^N \frac{1}{(i\omega)^N} \int_a^\infty f^{(N)}(t)\, e^{i\omega t}\, dt\,.$$

Again the fact that $f^{(N)} \in L^1$ implies that the final term is $o(\omega^{-N})$ as $\omega \to +\infty$, and hence

$$\int_a^\infty f(t)\, e^{i\omega t}\, dt \;=\; \sum_{j=1}^{N} (-1)^j \frac{f^{(j-1)}(a)e^{i\omega a}}{(i\omega)^j} \;+\; o(\frac{1}{|\omega|^N})$$

is valid as $\omega \to +\infty$. The expansion for I is just the real part of this.
Remark 6. The desired expansion is given by taking the real part - giving alternately $\sin \omega a$ and $\cos \omega a$ factors

$$\int_a^\infty f(t)\; \cos(\omega t)\; dt \;=\; \sum_{j=1}^{N} \frac{(-1)^j f^{(2j-1)}(a) \cos \omega a}{\omega^{2j}} + \sum_{j=1}^{N} \frac{(-1)^j f^{(2j-2)}(a) \sin \omega a}{\omega^{2j-1}} + o(\frac{1}{|\omega|^{2N}})\,.$$

- notice, however, that this is not strictly speaking an asymptotic expansion due to the presence of zeros in the oscillatory factors - this issue arises all the time in asymptotic analysis of oscillatory problems, because the approximation is unlikely to give the exact location of the zeros. The simplest thing to do which is accurate is to give an error bound for the remainder as above and remember that it is not actually an asymptotic expansion in the strict sense we have defined. See Section V below and Bender-Orszag, Section 3.7, for further discussion.

Observe that if $a = 0$ and $f^{(2s+1)}(a) = 0$ for all s then we see that the asymptotic expansion of $I(\omega)$ relative to the asymptotic sequence $\phi_j(\omega) = \omega^{-j}$ is the zero sequence[2]:

$$I(\omega) \;\sim\; \sum_{j=1}^{\infty} 0.\omega^{-j}\,, \qquad (\omega \to +\infty)\,.$$

As a test, this holds for example, if

$$I(\omega) \;=\; \int_0^\infty \frac{\cos(\omega t)}{1+t^2}\, dt \;=\; \frac{\pi}{2}\, e^{-\omega}\,, \qquad (\omega > 0)$$

which is indeed $o(\omega^{-N})$ as $\omega \to +\infty$ for all $N \in \mathbb{N}$.

[2]Recall that this is different to saying that $I(\omega) \sim 0$ the zero function, which implies that I vanishes for large ω

Example Consider the Bessel function of order n:

$$J_n(x) = \frac{1}{\pi} Re \int_0^{\pi} e^{i(x \sin t - nt)} dt.$$

Hence

$$J_n(n) = \frac{1}{\pi} Re \int_0^{\pi} e^{in(\sin t - t)} dt.$$

In order to evaluate the large n behavior of the above function we note that $\phi(t) = \sin t - t$, thus again the main contribution comes from the neighborhood of $t = 0$:

$$J_n(n) = \frac{1}{\pi} Re \int_0^{\pi} e^{in(-\frac{t^3}{6} + O(t^5))} dt.$$

Using the substitution

$$\frac{nt^3}{6} = \tau^3, \quad \text{or} \quad \left(\frac{n}{6}\right)^{\frac{1}{3}} t = \tau,$$

we find

$$J_n(n) \sim \frac{1}{\pi} \left(\frac{6}{n}\right)^{\frac{1}{3}} \int_0^{\infty} e^{-i\tau^3} d\tau, \quad n \to \infty.$$

Using (2) we find

$$\int_0^{\infty} e^{-i\tau^3} d\tau = \frac{1}{3} e^{-\frac{i\pi}{6}} \Gamma\left(\frac{1}{3}\right).$$

Thus,

$$J_n(n) \sim \frac{1}{3\pi} \cos\left(\frac{\pi}{6}\right) \Gamma\left(\frac{1}{3}\right) \left(\frac{6}{n}\right)^{\frac{1}{3}}, \quad n \to \infty.$$

III The Riemann Lebesgue Lemma

The first statement and proof is in the context of the Lebesgue integral, and so may only fully make sense if you have done P&M. The second formulation and proof is entirely within the framework of the Riemann integral familiar from IA Analysis.

Theorem 7 (Riemann-Lebesgue Lemma - 1st version)**.** *If $f \in L^1(\mathbb{R})$ then $\lim_{|x| \to +\infty} \int e^{ixt} f(t)\, dt = 0$.*

Proof. As we have seen (Lemma 3) the proof of the Riemann Lebesgue lemma is straightforward, using integration by parts, in the case that $f(t)$ is smooth and compactly supported, i.e. $f \in C_c^{\infty}(\mathbb{R})$. Using the density of $C_c^{\infty}(\mathbb{R})$ in the Lebesgue space $L^1(\mathbb{R})$ of absolutely integrable functions the result can be derived quickly from this. The precise density statement we need is this:

Lemma 8. *Given $f \in L^1(\mathbb{R})$ and $\epsilon > 0$ there exists a function $f_\epsilon \in C_c^{\infty}(\mathbb{R})$ such that $\|f_\epsilon - f\|_{L^1} < \frac{\epsilon}{2}$.*

This Lemma can be derived quickly from results in the P&M course, and may even have been stated explicitly - see in particular Lemma 7.4.2 and its proof in James Norris' course notes which are available on his website. But we know from Lemma 3 that since $f_\epsilon \in C_c^{\infty}(\mathbb{R})$ there holds $\lim_{|x| \to +\infty} \int e^{ixt} f_\epsilon(t)\, dt = 0$, so $\exists R_\epsilon > 0$ such that $\left|\int e^{ixt} f_\epsilon(t)\, dt\right| < \frac{\epsilon}{2}$ for $|x| \geq R_\epsilon$. But

$$\int e^{ixt} f(t)\, dt = \int e^{ixt} f_\epsilon(t)\, dt + \int e^{ixt} \left(f(t) - f_\epsilon(t)\right) dt$$

so that

$$\sup_{|x|\geq R_\epsilon} \Big|\int e^{ixt}\, f(t)\, dt\Big| \;\leq\; \sup_{|x|\geq R_\epsilon} \Big|\int e^{ixt}\, f_\epsilon(t)\, dt\Big| \;+\; \|f - f_\epsilon\|_{L^1} \leq \epsilon\,.$$

Since $\epsilon > 0$ is arbitrary this completes the proof. □

Theorem 9 (Riemann-Lebesgue Lemma - 2nd version)**.** *Let the function* $f\,:\,[a,b]\,\to\,\mathbb{R}$ *be* Riemann-integrable*, and* $x \in \mathbb{R}$*. Define the function* $I(x)$ *by*

$$I(x) \;=\; \int_a^b f(t)e^{ixt}\, dt\,. \tag{5}$$

Then $\lim_{|x|\to\infty} I(x) = 0\,.$

Remark 10. This statement is actually a consequence of the first since if f is extended by zero outside of $[a,b]$ the result is a function in $L^1(\mathbb{R})$. However we outline a method of proof which lies entirely within the framework of the Riemann integral.

Before giving the proof we recall the basic constructions of the Riemann definition of the integral. Consider the interval $[a,b]$, where $a < b$; a *partition* of $[a,b]$ is a finite set $P = \{a_0, ..., a_n\,:\, a = a_0 < a_1, ... < a_n = b\}$. Another partition $P*$ is said to be a *refinement* of P if $P \subset P*$ (that is every point of P is a point of $P*$). Given, two partitions, P_1 and P_2, we say that $P* = P_1 \cup P_2$ is their common refinement. Let $f\,:\,[a,b] \to \mathbb{R}$, be a bounded real function. Given the partition P the *upper* and *lower* Riemann sums are defined, respectively, by

$$U(P,f) := \sum_{i=1}^n M_i \Delta a_i, \qquad L(P,f) = \sum_{i=1}^n m_i \Delta a_i$$

where $\Delta a_i = a_i - a_{i-1}$, and for every $i \leq n$,

$$M_i := \sup\{f(x) : a_{i-1} \leq x \leq a_i\} \qquad m_i := \inf\{f(x) : a_{i-1} \leq x \leq a_i\}\,.$$

The *upper* and *lower* Riemann integrals of f over $[a,b]$, written respectively as $\int_a^{b-} f(x)dx$and $\int_{a-}^b f(x)dx$, are defined by

$$\int_a^{b-} f(x)dx := \inf\{U(P,f) : P \text{ is a partition of } [a,b]\} \tag{6}$$

$$\int_{a-}^{b} f(x)dx := \sup\{L(P,f) : P \text{ is a partition of } [a,b]\}. \tag{7}$$

Here, $\inf$ and $\sup$ are taken over all partitions of $[a,b]$. If

$$\int_a^{b-} f(x)dx = \int_{a-}^{b} f(x)dx \tag{8}$$

f is said to be *Riemann integrable on* [a,b] and we denote this common value by

$$\int_a^{b} f(x)dx = \int_a^{b-} f(x)dx = \int_{a-}^{b} f(x)dx \tag{9}$$

The Riemann Criterion for Integrability: *the function* $f : [a,b] \to \mathbb{R}$ *is Riemann-integrable* if and only if *for every* $\epsilon > 0$ *there exists a partition* P_ϵ *of* $[a,b]$ *such that* $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$.

In words this criterion says that a Riemann integrable function can be approximated by a piecewise constant function (and the converse). The idea of the following proof of the Riemann Lebesgue Lemma is to use this to reduce the problem to the piecewise constant case. In addition to linearity, recall the following properties of the Riemann integral[3] on a bounded interval $[a,b]$:

1. if f is piecewise constant, continuous, or monotone it is Riemann integrable ;
2. the product of two Riemann integrable functions is itself Riemann integrable;
3. if f is Riemann integrable then so is $|f|$ and $|\int_a^b f(t)\,dt| \le \int_a^b |f(t)|\,dt$.

Proof of Theorem 9. Since $f(x)$ is Riemann Integrable, for every $\epsilon > 0$ $\exists$ a partition $P = \{a_i\}_0^n$ as above such that $U(P,f) - L(P,f) < \frac{\epsilon}{2}$. Moreover the definition of the Riemann integral above implies that

$$L(P,f) \le \int_a^b f(t)dt \le U(P,f)\,.$$

It follows that if l is the piecewise constant function equal to m_i on $[a_{i-1}, a_i)$ then it is Riemann integrable and

$$\Big|\int_a^b f(t)\,dt - \int_a^b l(t)\,dt\Big| \le \frac{\epsilon}{2}\,.$$

But $f \ge l$ everywhere so this implies that actually

$$\int_a^b \big|f(t) - l(t)\big|\,dt \le \frac{\epsilon}{2}\,.$$

Next, e^{ikt} is continuous and so, by the properties above, $l(t)e^{ikt}$ is Riemann Integrable with

$$\int_a^b l(t)e^{ixt}\,dt = \sum_{i=1}^n m_i \frac{e^{ixa_i} - e^{ixa_{i-1}}}{ix}\,,$$

which is a finite sum of terms converging to zero as $x \to +\infty$. It follows that there exists R_ϵ such that if $|x| > R_\epsilon$ then $|\int_a^b l(t)e^{ixt}\,dt| < \frac{\epsilon}{2}$.

Finally, by linearity of the integral, the triangle inequality and the properties above

$$\begin{aligned} \Big|\int_a^b f(t)e^{ixt}\,dt\Big| &\le \Big|\int_a^b l(t)e^{ixt}\,dt\Big| + \Big|\int_a^b \big(f(t) - l(t)\big)e^{ikt}\,dt\Big| \\ &\le \frac{\epsilon}{2} + \int_a^b \big|f(t) - l(t)\big|\,dt \le \epsilon\,. \end{aligned}$$

for $|x| > R_\epsilon$, completing the proof since $\epsilon > 0$ is arbitrary. $\square$

[3]See your IA notes or the appendix to the book *Fourier Analysis* by Stein and Shakarchi.

IV An Example solved by stationary phase and steepest descent

Consider the integral

$$f(x) \;=\; \int_0^1 \exp[ixt^3]\,dt$$

for real $x > 0$. We can analyze the asymptotic behaviour of $f(x)$ for large x either by stationary phase or by steepest descent - here we will carry out the analysis by the former method; the latter method is in the next handout. The phase function $\psi(t) = t^3$ has its only stationary point $t = 0$, and so according to the method of stationary phase the leading contribution to $f(x)$ should be

$$\begin{aligned}\int_0^\infty \exp[ixt^3]\,dt \;&=\; \frac{1}{3}\int_0^\infty s^{-\frac{2}{3}}\exp[ixs]\,ds \;=\; \frac{1}{3}\Gamma\big(1/3\big)\;\exp[i\pi/6]\,x^{-\frac{1}{3}}\\ &=\; \Gamma\big(4/3\big)\;\exp[i\pi/6]\,x^{-\frac{1}{3}}\,.\end{aligned}$$

(See (1) and surrounding discussion for how compute this final integral.) This determination of the leading term of course ignores the expected endpoint contribution from the point $t = 1$ - this contribution was "deleted" when we replaced $\int_0^1$ by $\int_0^\infty$. In view of the example (4) it might be expected to give a contribution of $O(\frac{1}{x})$, and this is indeed the case. The easiest way to derive the next order term would be to write

$$f(x) \;=\; \int_0^\infty \exp[ixt^3]\,dt \;-\; \int_1^\infty \exp[ixt^3]\,dt$$

and then change variables in the second term to $s = t^3$ and integrating by parts:

$$\begin{aligned}\int_1^\infty \exp[ixt^3]\,dt \;&=\; \frac{1}{3}\int_1^\infty s^{-\frac{2}{3}}\,e^{ixs}\,ds\\ &=\; \Big[\frac{s^{-\frac{2}{3}}e^{ixs}}{3ix}\Big]_{s=1}^\infty \;-\; \Big(-\frac{2}{3}\Big)\frac{1}{3ix}\int_1^\infty s^{-\frac{5}{3}}\,e^{ixs}\,ds\\ &=\; -\frac{e^{ix}}{3ix} \;+\; o(\frac{1}{x})\end{aligned}$$

by the Riemann-Lebesgue Lemma. Returning to the original integral, this implies

$$f(x) = \Gamma\big(4/3\big)\;\exp[i\pi/6]\,x^{-\frac{1}{3}} + \frac{e^{ix}}{3ix} \;+\; o(\frac{1}{x})\,.$$

Clearly this process continues to generate an asymptotic expansion for the contribution from the endpoint $t = 1$. In the next (non-examinable) section we illustrate in this case[4] an analytical device ("partitions of unity") which allows one to carry out the computation of asymptotic expansions arising from localized contributions when elementary procedures such as that just given are not available - in particular in higher dimensions.

In order to compute the next terms in the asymptotic expansion, one should first recall that the contribution from regions away from any points of stationary phase is rapidly decreasing, i.e. smaller than x^{-N} for any $n \in \mathbb{N}$, *but only as long as there are no endpoint contributions from the integration by parts.* In order to evaluate the contributions of the endpoint at $t = 1$ to the asymptotic expansion of $f(x)$ we introduce a localization (or bump) function $b(t)$ with the properties:

[4]Although it is really overkill in this example, as the previous paragraph shows, it illustrates a technique which is essential and usefl in more general situations.

1. b is smooth and $0 \leq b(t) \leq 1$ for all $t \in \mathbb{R}$;

2. $b(t) = 1$ for $t \leq 2\delta$;

3. $b(t) = 0$ for $t \geq 3\delta$;

for some positive number $\delta < 1/4$. Now, using these properties, write

$$
\begin{aligned}
f(x) &= \int_0^1 \exp[ixt^3]\,dt = \int_0^1 b(t)\,\exp[ixt^3]\,dt + \int_0^1 (1-b(t))\exp[ixt^3]\,dt \\
&= \int_0^\infty b(t)\,\exp[ixt^3]\,dt + \int_0^1 (1-b(t))\exp[ixt^3]\,dt \\
&= \int_0^\infty \exp[ixt^3]\,dt - \int_0^\infty (1-b(t))\,\exp[ixt^3]\,dt + \int_\delta^1 (1-b(t))\exp[ixt^3]\,dt \\
&= \int_0^\infty \exp[ixt^3]\,dt - \int_\delta^\infty (1-b(t))\,\exp[ixt^3]\,dt + \int_\delta^1 (1-b(t))\exp[ixt^3]\,dt\,.
\end{aligned}
$$

The first integral is evaluated explicitly to give the leading contribution as above. The second integral can be safely integrated by parts indefinitely since the point of stationary phase is outside of the interval of integration $[\delta, \infty]$; indeed, using the boundedness of b and the fact that its derivatives all vanish for t near to the endpoints $t = \delta$ and $t = \infty$, we can write

$$
\begin{aligned}
\int_\delta^\infty (1-b(t))\,\exp[ixt^3]\,dt &= \frac{1}{ix}\int_\delta^\infty \frac{(1-b(t))}{3t^2}\,d(\exp[ixt^3]) \\
&= -\frac{1}{(ix)^2}\int_\delta^\infty \frac{1}{3t^2}\frac{d}{dt}\left(\frac{(1-b(t))}{3t^2}\right) d(\exp[ixt^3])\,.
\end{aligned}
$$

This leads to the conclusion that the second integral is $O(x^{-2})$. In fact the process can be continued indefinitely to establish that the integral is in fact rapidly decreasing (exercise).

Finally, the third integral can also be integrated by parts similarly, except that the upper limit of integration $t = 1$ does now contribute since $1 - b = 1$ for $t > 3/4 > 3\delta$. This leads to

$$
\begin{aligned}
\int_\delta^1 (1-b(t))\,\exp[ixt^3]\,dt &= \frac{1}{ix}\int_\delta^1 \frac{(1-b(t))}{3t^2}\,d(\exp[ixt^3]) \\
&= \frac{1}{3ix}e^{ix} - \frac{1}{(ix)^2}\int_\delta^1 \frac{1}{3t^2}\frac{d}{dt}\left(\frac{(1-b(t))}{3t^2}\right) d(\exp[ixt^3])\,.
\end{aligned}
$$

Again, as an exercise, check that this process can be continued indefinitely, with the upper limit of integration contributing to the asymptotics at each order in $\frac{1}{x}$, leading to the full asymptotic expansion

$$
f(x) \sim \Gamma\big(4/3\big)\,\exp[i\pi/6]\,x^{-\frac{1}{3}} - e^{ix}\sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{2}{3})}{(ix)^{n+1}\,\Gamma(-\frac{1}{3})}\,.
$$

Notice how there is no dependence on b because b is identically zero in a neighborhood of $t = 1$ and $1 - b$ is identically zero in a neighborhood of $t = \delta$. This reflects the fact that integrals with rapidly oscillating phases are asymptotically concentrated in arbitrarily small neighborhoods of the points of stationary phase. The bump function b is just introduced as a technical device to reveal this localization property. The pair of functions b and $1 - b$ obviously have the property $b(t) + 1 - b(t) = 1$ - this is the property of a partition of unity: in general situations one first covers the entire space by open sets $\mathcal{U}_\alpha$ chosen to have the property that the integrand is simple on each one - e.g. there is at most one point of stationary phase in each of the sets, and the integrand can be expanded locally on each set. One then uses a theorem which says that (under conditions on the space) there is a collection of smooth functions $\{b_\alpha\}$ such that $\sum_\alpha b_\alpha(t) = 1$, and chosen in such a way that the support[5] of each b_α is a subset of $\mathcal{U}_\alpha$. If all works out well there is a computable asymptotic expansion[6]. for each α. The overall asymptotics is then obtained by inserting the identity $\sum_\alpha b_\alpha(t) = 1$ inside the integral, computing the expansion for each α and then combining to give the complete expansion.

V Generalized Asymptotic Expansions

One way to incorporate the problem, which was raised in Remarks 5 and 6 and is discussed in more detail in Section 3.7 of Bender and Orszag, of describing and incorporating oscillatory functions in asymptotics is the notion of a *generalized asymptotic expansion.* Consider what we obtained for (4) above, and rewrite our conclusion in terms of the asymptotic sequence $\phi_N(\omega) = \omega^{-N}$ (as $\omega \to +\infty$). We can write the first equation in Remark 6 as

$$\int_a^\infty f(t)\ \cos(\omega t)\ dt \,=\, \sum_{j=1}^{N} a_j(\omega)\,+\,o(\phi_N(\omega))\,.$$

where

$$a_{2j}(\omega) = \frac{(-1)^j f^{(2j-1)}(a)\cos\omega a}{\omega^{2j}} \quad \text{and} \quad a_{2j-1}(\omega) = \frac{(-1)^j f^{(2j-2)}(a)\sin\omega a}{\omega^{2j-1}}\,.$$

This is an example of a generalized asymptotic expansion - the expansion consists of a set of functions whose partial sum gives an approximation to the given function controlled by *another* asymptotic sequence $\phi_j(\omega)$. Compared to the original definition of asymptotic expansion, we have to some extent decoupled the asymptotic sequence with respect to which the expansion is defined from the functions appearing in the actual expansion - these are no longer required to be multiples of the ϕ_j. This is a useful formalism and many analysis books use asymptotic expansions in this sense.

Finally, you may like to consider this question:

> *Is it true that if a function f admits a generalized asymptotic expansion, and this asymptotic expansion does actually converge to a limit in the usual sense (as a convergent series), then this limit must agree with f?*

You may like to consider this with reference to the sum[7] $\sum_{n=1}^\infty \delta_{nl}$ which is equal to 1 as a convergent sum for all $l \in \mathbb{N}$, but is also a generalized asymptotic expansion[8] as $l \to +\infty$ for the zero function, i.e., fix the asymptotic sequence $\phi_n(l) = l^{-n}$ as $l \to +\infty$, then

$$0 \sim \sum_{n=1}^\infty \delta_{nl}$$

in the sense that

$$\left|0 - \sum_{n=1}^{N} \delta_{nl}\right| = o(\phi_N(l))\,, \qquad (l \to +\infty)\,,$$

since the left hand side is actually zero for $l > N$. The crucial issue is the different order in which limits are taken in asymptotic as opposed to convergent series.

[5]The support of a function is the smallest closed set on whose complement the function vanishes.

[6]It may be zero to all orders if there are no points of stationary phase, endpoints etc

[7]Thanks to Clive Wells for this example.

[8]We here consider functions of the discrete variable $l \in \mathbb{N}$ as $l \to \infty$, but this should not cause confusion.