Methods

1 Vector Spaces.

- Some simple vector space preliminaries, leading to natural definition of an inner product:

$$(f, g) = \int_{\Sigma} f(x)^* g(x)\, d\mu \quad \left(\text{where } \mu \text{ is some measure}\right)$$

{Definition} Homogeneous Boundary Conditions. are such that if f and g satisfy the boundary conditions, so does $(\lambda f + \mu g)$.

2 Fourier Series.

{Definition} f is periodic if $\exists$ a fixed R: $f(x+R) = f(x)\ \forall x$, it is often convenient to consider this as a function $f: S^1 \to \mathbb{C}$ parametrised by θ.

- Now, we expand f in the basis $\{e^{in\theta} : n \in \mathbb{Z}\}$:

$$\Rightarrow \left[f(\theta) = \sum_{n \in \mathbb{Z}} \hat{f}_n e^{in\theta} \right]$$

$$\text{where } \left[\hat{f}_n = \frac{1}{2\pi}(e^{in\theta}, f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta} f(\theta)\, d\theta \right]$$

N.B if f is real, $(\hat{f}_n)^* = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta} f(\theta)\, d\theta\right)^*$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta} f(\theta)\, d\theta = \hat{f}_{-n}.$$

$$\Rightarrow f(\theta) = \hat{f}_0 + \sum_{n=1}^{\infty} (\hat{f}_n e^{in\theta} + \hat{f}_n^* e^{-in\theta})$$

$$\Longleftrightarrow f(\theta) = \hat{f}_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

$$\text{where } \left[a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(n\theta) f(\theta)\, d\theta,\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(n\theta) f(\theta)\, d\theta. \right]$$

- At an isolated discontinuity, the Fourier series returns the average of the limiting values from either side.

- Parseval's Theorem:

$$(f,f) = \int_{-\pi}^{\pi} |f(\theta)|^2 \, d\theta$$

$$= 2\pi \sum_{n\in\mathbb{Z}}' |\hat{f}_n|^2$$

Or equivalently: $$\boxed{\frac{1}{L}\int_{-L}^{L} \{f(x)\}^2 dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)}$$

- Miscellaneous formulas: For a function on $[-l, l]$ where

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left\{a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right\}$$

Then:
$$\begin{cases} a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx \\ a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx \\ b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx \end{cases}$$

and for half-range series:

$$\begin{cases} \text{cosine: } \quad a_n = \frac{2}{l}\int_{0}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx \quad (b_n = 0) \\ \text{sine: } \quad b_n = \frac{2}{l}\int_{0}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx \quad (a_n = 0). \end{cases}$$

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3 Sturm-Liouville Theory.

{Definition}. The adjoint A^{+} of a map $A: V \to V$ is such that:

$$(A^{+}\underline{u}, \underline{v}) = (\underline{u}, A\underline{v}) \quad \forall \underline{u}, \underline{v} \in V.$$

Then, A is self-adjoint if:

$$\boxed{(A\underline{u}, \underline{v}) = (\underline{u}, A\underline{v})}.$$

Now, suppose we have a self-adjoint map M, with eigenvector $\underline{v_i}$ and eigenvalue λ_i, so:

$$M\underline{v_i} = \lambda_i \underline{v_i}$$

Then: $\lambda_i(\underline{v_i}, v_i) = (v_i, Mv_i)$
$= (Mv_i, v_i)$
$= \lambda_i^*(v_i, v_i) \Rightarrow \lambda_i = \lambda_i^* \Rightarrow \lambda_i$ is $\boxed{\text{real!}}$

Further, we may show analogously that $\boxed{(v_i, v_j) = 0}$ for $i \neq j$.

- Now consider differential operators $\mathcal{L}$, first consider second order operators:

$$\mathcal{L}y = P\frac{d^2y}{dx^2} + R\frac{dy}{dx} - Qy.$$
$$= P\left\{\frac{d^2y}{dx^2} + \frac{R}{P}\frac{dy}{dx} - \frac{Q}{P}y\right\}$$
$$= P\left\{\exp\left(-\int\frac{R}{P}\right)\frac{d}{dx}\left(\exp\left(\int\frac{R}{P}dx\right)\frac{dy}{dx}\right) - \frac{Q}{P}y\right\}$$

Letting $p = \exp\left(\int\frac{R}{P}dx\right)$

$$\Rightarrow \mathcal{L}y = Pp^{-1}\left\{\frac{d}{dx}\left(p\frac{dy}{dx}\right) - \frac{Q}{P}py\right\}$$

Letting $q = \frac{Q}{P}p$ and dropping the factor of Pp^{-1} we are left with the Sturm-Liouville form:

$$\underline{\mathcal{L}y = \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y.} \quad (*)$$

We may show that, assuming p or the functions themselves are periodic, then:

$$(f, \mathcal{L}g) = (\mathcal{L}f, g) \Rightarrow \mathcal{L} \text{ is self-adjoint.}$$

- {Definition} $\boxed{(f,g)_w = \int_a^b f(x)^* g(x)\, w(x)\, dx}$

weight function.

{Definition}. An eigenfunction of weight $w(x)$ of $\mathcal{L}$ is a function $y: [a,b] \to \mathbb{C}$ satisfying:

$$\boxed{\mathcal{L}y = \lambda w y.} \quad \text{where } \lambda \text{ is the eigenvalue.}$$

Using the same inner product proof as before, we may show eigenvalues of Sturm-Liouville operator are real and eigenfunctions are orthogonal wrt the weighted inner product.

- On a compact domain, the eigenvalues $\lambda_1, \dots, \lambda_n, \dots$ form a countably infinite discrete sequence.

e.g Let $\mathcal{L} = \frac{d^2}{dx^2}$, $w(x) = 1$ on $[-L, L]$, then:

$$\frac{d^2 y_n}{dx^2} = \lambda_n y_n(x)$$

$$\Rightarrow \left\{ y_n(x) = \exp\left(\frac{in\pi x}{L}\right), \; \lambda_n = -\frac{n^2\pi^2}{L^2} \right\}.$$

which is just a Fourier series.

- Hermite Polynomials: satisfy:

$$\tfrac{1}{2}H'' - xH' = \lambda H$$

Now, $p(x) = \exp\left(-\int_0^x 2t \, dt\right) = e^{-x^2}$

$$\Rightarrow \frac{d}{dx}\left(e^{-x^2}\frac{dH}{dx}\right) = -2\lambda e^{-x^2} H(x)$$

So $w(x) = e^{-x^2}$.

To ensure boundary terms vanish, we want H to only grow polynomially.

It turns out:

$$\left(H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left\{ e^{-x^2} \right\}. \right)$$

• Now we look to solve forced equations:

$$\mathcal{L}g = f(x) = w(x)F(x).$$

Given a basis $\{y_n\}$ we may write:

$$g(x) = \sum_{n=1}^{\infty}{}' \hat{g}_n y_n(x) \quad \text{and} \quad F(x) = \sum_{n=1}^{\infty}{}' \hat{F}_n y_n(x)$$

Then: $$\mathcal{L}g = \sum_{n=1}^{\infty}{}' \hat{g}_n \lambda_n w(x) y_n(x)$$

$$\Rightarrow \sum_{n=1}^{\infty}{}' \hat{g}_n \lambda_n w(x) y_n(x) = \sum_{n=1}^{\infty}{}' \hat{F}_n w(x) y_n(x)$$

$$\Rightarrow \hat{g}_n = \frac{\hat{F}_n}{\lambda_n} \quad \text{provided } \lambda_n \neq 0.$$

In another form:

$$g(x) = \sum_n{}' \frac{1}{\lambda_n} (y_n, F)_w \, y_n(x)$$

$$= \int_a^b \left\{ \sum_n{}' \frac{1}{\lambda_n} y_n^*(t) y_n(x) w(t) F(t) \right\} dt$$

$$= \int_a^b G(x;t) F(t) w(t) \, dt$$

where $\left[G(x,t) = \sum_n{}' \frac{1}{\lambda_n} y_n^*(t) y_n(x) \right]$ is the Green's function.

• Parseval Theorem (II).

$$(f,f)_w = \sum_n{}' |f_n|^2$$

• Least-squares approximation: Suppose we approximate $f: \Omega \to \mathbb{C}$ by a finite set of eigenfunctions $y_n(x)$, which is:

$$g(x) = \sum_{k=1}^{n}{}' c_k y_k(x).$$

We wish to find "the best" c_k, we wish to minimise:

$$(f-g, f-g)_w = (f,f)_w + (g,g)_w - (f,g)_w - (g,f)_w.$$

$$\Rightarrow \quad 0 = \frac{\partial}{\partial c_j}(f-g, f-g) = \frac{\partial}{\partial c_j}\left\{ (f,f)_w + (g,g)_w - (f,g)_w - (g,f)_w \right\}$$

$$\Rightarrow \quad 0 = \frac{\partial}{\partial c_j}\left\{ \sum_{k=1}^{n} |c_k|^2 - \sum_{k=1}^{n} f_k^* c_k - \sum_{k=1}^{n} c_k^* f_k \right\}$$

$$\Rightarrow \quad 0 = c_j^* - f_j^* \Rightarrow c_j^* = f_j^*$$

Taking $\frac{\partial}{\partial c_j^*} \Rightarrow c_j = f_j$. The second-derivative confirms this is a minimum. So the coefficients of the infinite expansion are the best.

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4 Partial Differential Equations.

4.1 Laplace's Equation.

$$\boxed{\nabla^2 \phi = 0} \qquad \text{for } \phi : \mathbb{R}^n \to \mathbb{C}.$$

Functions that obey Laplace's equation are called harmonic.

- Let Ω be a compact domain with boundary $\partial\Omega$. Let $f : \partial\Omega \to \mathbb{R}$, then there is a unique solution to $\nabla^2\phi = 0$ on Ω with $\phi|_{\partial\Omega} = f$.

Pf: Suppose ϕ_1 and ϕ_2 are solutions : $\phi_1|_{\partial\Omega} = \phi_2|_{\partial\Omega} = f$

Now, let $\Phi = \phi_1 - \phi_2$. So $\Phi|_{\partial\Omega} = 0$. Then:

$$0 = \int_{\Omega} \Phi \nabla^2 \Phi \, d^n x = -\int_{\Omega} (\nabla\Phi)\cdot(\nabla\Phi) + \int_{\partial\Omega} \cancel{\Phi \nabla \Phi \cdot n} \, d^{n-1}x$$

$$\Rightarrow \quad 0 = -\int_{\Omega} \|\nabla\Phi\|^2 d^n x \Rightarrow \nabla\Phi = 0 \text{ throughout } \Omega$$

$$\Rightarrow \quad \Phi = 0 \Rightarrow \phi_1 = \phi_2.$$

- Dirichlet condition: $\phi|_{\partial\Omega} = f(x) \Rightarrow$ unique
- Neumann condition: $n \cdot \nabla\phi|_{\partial\Omega} = g(x) \Rightarrow$ unique up to constant.

- Laplace's equation on the unit disk. (special case).

Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$. Then the Laplace equation is:

$$0 = \frac{\partial^2 \phi}{\partial z \partial \bar{z}} \quad \text{where } z = x+iy.$$

$$\Rightarrow \phi(z,\bar{z}) = \psi(z) + \chi(\bar{z}) \quad \text{for some } \psi, \chi.$$

Suppose we have Dirichlet boundary conditions $\phi|_{\partial\Omega} = f(\theta)$, now f has a Fourier expansion since $\partial\Omega \cong S^1$:

$$\Rightarrow f(\theta) = \hat{f}_0 + \sum_{n=1}^{\infty} \hat{f}_n e^{in\theta} + \sum_{n=1}^{\infty} \hat{f}_{-n} e^{-in\theta}$$

But $z = e^{i\theta}$ on $\partial\Omega$

$$\Rightarrow f(\theta) = \hat{f}_0 + \sum_{n=1}^{\infty} \hat{f}_n z^n + \sum_{n=1}^{\infty} \hat{f}_{-n} \bar{z}^n$$

Then define:

$$\underline{\phi(z,\bar{z}) = \hat{f}_0 + \sum_{n=1}^{\infty} \hat{f}_n z^n + \sum_{n=1}^{\infty} \hat{f}_{-n} \bar{z}^n}$$

which satisfies the BCs and converges for $|z|<1$, and is of the form $\psi(z) + \chi(\bar{z}) \Rightarrow$ {unique solution.}.

- Separation of variables.

eg $\Omega = \{(x,y,z) \in \mathbb{R}^3 : 0 \le x \le a,\ 0 \le y \le b,\ z \ge 0\}$ with:

$$\begin{cases} \psi(0,y,z) = \psi(a,y,z) = 0 \\ \psi(x,0,z) = \psi(x,b,z) = 0 \\ \lim_{z\to\infty} \psi(x,y,z) = 0 \\ \psi(x,y,0) = f(x,y) \end{cases}$$

We try $\psi(x,y,z) = X(x)Y(y)Z(z)$.

$$\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

$$\Rightarrow \begin{cases} X'' = -\lambda X \\ Y'' = -\mu Y \\ Z'' = (\lambda+\mu) Z \end{cases} \Rightarrow \begin{cases} X = a\sin\sqrt{\lambda}x + b\cos\sqrt{\lambda}x \\ Y = c\sin\sqrt{\mu}y + d\cos\sqrt{\mu}y \\ Z = g\exp(\sqrt{\lambda+\mu}\,z) + h\exp(-\sqrt{\lambda+\mu}\,z) \end{cases}$$

Imposing BCs gives:

$$\begin{cases} b = 0, & \lambda = \left(\frac{n\pi}{a}\right)^2 \\ d = 0, & \mu = \left(\frac{m\pi}{b}\right)^2 \\ z \to 0 \Rightarrow g = 0. \end{cases}$$

So: $\psi_{nm}(x,y,z) = A_{n,m} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \exp(-S_{n,m} z)$

where $S_{n,m}^2 = \left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)\pi^2$

$$\Rightarrow \psi(x,y,z) = \sum_{n,m}{}' A_{n,m} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \exp(-S_{n,m} z)$$

Then, at $z=0$:

$$\sum_{n,m}{}' A_{n,m} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) = f(x,y)$$

Now: $$\left[\int_0^a \sin\left(\frac{k\pi x}{a}\right) \sin\left(\frac{n\pi x}{a}\right) dx = \delta_{k,n} \cdot \frac{a}{2},\right]$$

$$\Rightarrow \frac{ab}{4} A_{k,j} = \int_0^a dx \int_0^b dy \left\{ \sin\left(\frac{k\pi x}{a}\right) \sin\left(\frac{j\pi y}{b}\right) f(x,y) \right\}$$

- Laplace's equation in spherical polars

We use spherical co-ordinates defined by:

$$x_1 = r\sin\theta\cos\phi$$
$$x_2 = r\sin\theta\sin\phi$$
$$x_3 = r\cos\theta$$

Then: $$\left\{ \nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}. \right\}$$

We look for axisymmetric solutions:

$$\psi(r,\theta,\phi) = \boxed{\psi(r,\theta).}$$

We then separate variables:

$$\psi(r,\theta) = R(r)\,\Theta(\theta)$$

$$\Rightarrow \frac{\psi}{r^2}\left\{ \frac{1}{R}\frac{d}{dr}(r^2R') + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\right\} = 0$$

$$\Rightarrow \frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \lambda R \quad , \quad \frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -\lambda \sin\theta\,\Theta$$

$w(\theta) = \sin\theta$ in SL form.

① Legendre Polynomials:

We look at the angular part. It is usual to make substitution $x = \cos\theta \Rightarrow \frac{d}{d\theta} = -\sin\theta\frac{d}{dx}$ ($-1 \le x \le 1$ since $0 \le \theta \le \pi$).

$$\Rightarrow -\sin\theta\frac{d}{dx}\left(\sin\theta(-\sin\theta)\frac{d\Theta}{dx}\right) = -\lambda\sin\theta\,\Theta.$$

$$\Rightarrow \frac{d}{dx}\left\{(1-x^2)\frac{d\Theta}{dx}\right\} = -\lambda\Theta$$

It turns out that $\mathcal{L}$ is self-adjoint if $\Theta(x)$ is regular at $x = \pm 1$ (since $1-x^2$ vanishes). We try a power series:

$$\Theta(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow a_n(\lambda - n(n+1)) + a_{n+2}(n+2)(n+1) = 0$$

$$\Rightarrow a_{n+2} = \frac{n(n+1)-\lambda}{(n+2)(n+1)}a_n$$

It may be shown that the power series' (two indep. solns) will only converge if they terminate. So:

$$\lambda = l(l+1) \text{ for some } l.$$

This generates the Legendre polynomials, the first few are:

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \tfrac{1}{2}(3x^2-1) \ldots$$

where $\forall n$ we have $\boxed{P_n(1) = 1.}$

we also have:

$$\boxed{(P_l, P_k) = \frac{2\delta_{lk}}{2l+1}}$$

② Radial Part.

$$(r^2 R')' = l(l+1)R$$

we try $R(r) = r^\alpha \Rightarrow \alpha(\alpha+1) = l(l+1)$

$$\Rightarrow \alpha = l, \ \alpha = -(l+1)$$

So: $$\boxed{\phi(r,\theta) = \sum_{l=0}^{\infty}{}' \left\{ a_l r^l + \frac{b_l}{r^{l+1}} \right\} P_l(\cos\theta)}, \quad (*)$$

When we apply BC's, we normally require regularity at $r=0$
$\Rightarrow b_l = 0$.

Then on $\partial\Omega$, $\phi(a,\theta) = f(\theta)$

$$\Rightarrow f(\theta) = \sum{}' F_l P_l(\cos\theta)$$

weight function

$$\Rightarrow F_l = (P_l, f) = \int_0^\pi P_l(\cos\theta) f(\theta)\, d(\cos\theta)$$

Then $$\phi(r,\theta) = \sum_{l=0}^{\infty}{}' F_l \left(\frac{r}{a}\right)^l P_l(\cos\theta).$$

- Multipole Expansion for Laplace Equation.

We may show that

$$\phi(\underline{r}) = \frac{1}{|\underline{r} - \underline{r}'|} \text{ solves } \nabla^2\phi = 0$$

Further, $$\frac{1}{|\underline{r}-\underline{r}'|} = \frac{1}{r'} \sum_{l=0}^{\infty}{}' \left(\frac{r}{r'}\right)^l P_l(\underline{r}\cdot\underline{r}')$$

$$\Rightarrow \frac{1}{|\underline{r}-\underline{r}'|} = \frac{1}{r'} + \frac{r}{r'^2}\, \hat{\underline{r}}\cdot\underline{r}' + \cdots$$

dipole

monopole

• Laplace's equation in cylindrical co-ordinates.

In cylindrical polars we have:

$$\left[\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0\right]$$

We set the following BC's:

$$\begin{cases} \phi(a,\theta,z) = 0 \\ \phi(r,\theta,0) = f(r,\theta) \\ \lim_{z\to\infty}\phi(r,\theta,z) = 0 . \end{cases}$$

Separating variables $\phi(r,\theta,z) = R(r)\,\Theta(\theta)\,Z(z)$

$$\Rightarrow \left[\frac{1}{rR}(rR')' + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z} = 0\right]$$

$$\Rightarrow \left[Z'' = \mu Z\right]$$, then multiplying through by r^2:

$$\frac{r}{R}(rR')' + \frac{\Theta''}{\Theta} + \mu r^2 = 0$$

$$\Rightarrow \left[\Theta'' = -\lambda\Theta\right]$$

and $$\left[r^2R'' + rR' + (\mu r^2 - \lambda)R = 0\right]$$

Periodicity in θ gives:

$$\left[\Theta(\theta) = a_n\sin n\theta + b_n\cos n\theta\right] , \quad \lambda = n^2 \ (n\in\mathbb{N}).$$

BC that $\lim_{z\to\infty}\phi(r,\theta,z) = 0 \Rightarrow \left[Z(z) = c_\mu \exp(-\sqrt{\mu}z)\right]$

and then: $$\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{n^2}{r}R = -\mu r R$$

Letting $x = r\sqrt{\mu}$

$$\Rightarrow \left[x^2\frac{d^2R}{dx^2} + x\frac{dR}{dx} + (x^2 - n^2)R = 0\right] \quad (*)$$

BESSEL'S EQUATION.

We call the two independent solutions to this equation:

$$J_n(x) \quad \text{and} \quad Y_n(x).$$

We have that:

- J_n is regular at the origin and $J_n(x) \sim x^n$
- Conversely, the Y_n are not and:

$$\begin{cases} Y_0 \sim \ln x \\ Y_n \sim x^{-n} \end{cases}$$

Then: $\phi_{n\mu}(r,\theta,z) = (a_n \sin n\theta + b_n \cos n\theta) e^{\sqrt{\mu} z} \{ c_{\mu n} J_n(r\sqrt{\mu}) + d_{\mu n} Y_n(r\sqrt{\mu}) \}$

Regularity at $r=0$ $\Rightarrow$ $\boxed{d_{\mu,n} = 0}$

Now, imposing the condition $\phi(a,\theta,z) = 0$

$$\Rightarrow J_n(a\sqrt{\mu}) = 0$$

$$\Rightarrow \boxed{\sqrt{\mu} = \frac{k_{ni}}{a}} \quad \left(\mu = \frac{k_{ni}^2}{a^2} \right)$$

where k_{ni} is the i^{th} root of $J_n(x)$.

$$\Longrightarrow \boxed{\phi(r,\theta,z) = \sum_{n=0}^{\infty} \sum_i \{ A_{ni} \sin n\theta + B_{ni} \cos n\theta \} \exp\left(-\frac{k_{ni}}{a} z\right) J_n\left(\frac{k_{ni} r}{a}\right)}$$

We may show that:

$$\left[\int_0^a J_n\left(\frac{k_{nj} r}{a}\right) J_n\left(\frac{k_{ni} r}{a}\right) r \, dr = \frac{a^2}{2} \delta_{ij} \left[J_n'(k_{ni}) \right]^2 \right]$$

4.2 The Heat Equation

For a function $\phi : \mathbb{R}^n \times [0,\infty) \to \mathbb{C}$, the heat equation is:

$$\boxed{\frac{\partial \phi}{\partial t} = K \nabla^2 \phi}$$

As a start, we see that:

$$\frac{d}{dt} \int_{\mathbb{R}^n} \phi(\underline{x},t) \, d^n x = 0$$

Since $\frac{d}{dt}\int_{\mathbb{R}^n} \phi(\underline{x},t)\, d^n x = \int_{\mathbb{R}^n} \frac{\partial \phi}{\partial t}\, d^n x$

$$= \kappa \int_{\mathbb{R}^n} \nabla^2 \phi \, d^n x = 0 \quad \text{provided } \nabla\phi \to 0 \text{ as } |\underline{x}| \to \infty.$$

• Secondly, if $\phi(\underline{x},t)$ solves the heat equation then so do:

$$\left.\begin{aligned} &① \quad \phi_1(\underline{x},t) = \phi(\underline{x}-\underline{x}_0,\, t-t_0) \\ &② \quad \phi_2(\underline{x},t) = A\,\phi(\lambda \underline{x}, \lambda^2 t). \end{aligned}\right\}$$

We choose A such that:

$$\int_{\mathbb{R}^n} \phi_2(\underline{x},t)\, d^n x = \int_{\mathbb{R}^n} \phi(\underline{x},t)\, d^n x$$

$$\Longrightarrow \int_{\mathbb{R}^n} \phi_2(\underline{x},t)\, d^n x = A \int_{\mathbb{R}^n} \phi(\lambda \underline{x}, \lambda^2 t)\, d^n x$$

$$= A\lambda^{-n} \int_{\mathbb{R}^n} \phi(y, \lambda^2 t)\, d^n y \qquad (\text{where } y = \lambda \underline{x})$$

So, set $\boxed{A = \lambda^n}$

We may look for similarity solutions of the form:

$$\boxed{\phi(x,t) = \frac{1}{(\kappa t)^{n/2}} F(\eta) \quad \text{where} \quad \eta = \frac{x}{\sqrt{\kappa t}}}$$

e.g in 1+1 dimensions, $\Rightarrow F(\eta) = a\exp\left(-\frac{\eta^2}{4}\right)$ and normalisation gives $a = \frac{1}{\sqrt{4\pi}}$.

$$\Longrightarrow \quad \phi(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{x^2}{4\kappa t}\right)$$

• Now we look for uniqueness:

If $\phi : \Omega^n \times [0,\infty) \to \mathbb{R}$ satisfies $\frac{\partial \phi}{\partial t} = \kappa \nabla^2 \phi$ and obeys

① Initial conditions $\phi(\underline{x},0) = f(\underline{x}) \ \forall \underline{x} \in \Omega$

② Boundary condition $\phi(\underline{x},t)|_{\partial\Omega} = g(\underline{x},t) \ \forall t \in [0,\infty)$

Then ϕ is unique.

Pf: Suppose ϕ_1, ϕ_2 are solutions, and let $\Phi = \phi_1 - \phi_2$.

Now define $E(t) = \frac{1}{2}\int_\Omega \Phi^2 \, dV \Rightarrow E(t) \geq 0$. Now Φ satisfies the heat equation by linearity. Also, on $\partial\Omega$ and at $t=0$, $\Phi = 0$, and:

$$\frac{dE}{dt} = \int_\Omega \Phi \frac{d\Phi}{dt} \, dV$$

$$= \kappa \int_\Omega \Phi \nabla^2 \Phi \, dV$$

$$= \kappa \int_{\partial\Omega} \Phi \nabla \Phi \cdot d\underline{S} - \kappa \int_\Omega |\nabla \Phi|^2 dV$$

$$= -\kappa \int_\Omega (\nabla \Phi)^2 dV \leq 0$$

So E decreases with time but is always non-negative, and also $E = \Phi = 0$ at $t = 0 \Rightarrow E = 0$ always $\Rightarrow \Phi = 0 \ \forall t$

$\Rightarrow \boxed{\phi_1 = \phi_2}$.

e.g. Heat conduction in a finite rod: (of length $2L$)

① {Initial conditions} $\phi(x,0) = \Theta(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}$

② {Boundary conditions} $\phi(-L,t) = 0$, $\phi(L,t) = 1$.

we wish to apply separation of variables, however our initial + boundary conditions are inhomogenous. So we first look for any solution satisfying the boundary conditions (e.g time independent one).

So, we try $\phi_s(x,t) = \phi_s(x)$

$$\Rightarrow \frac{d^2\phi_s}{dx^2} = 0 \Rightarrow \left\{ \phi_s(x) = \frac{x+L}{2L} \right\}$$

Then, by linearity $\psi(x,t) = \phi(x,t) - \phi_s(x)$ obeys the heat equation with $\psi(-L,t) = \psi(L,t) = 0$,

and the initial condition:

$$\psi(x,0) = \Theta(x) - \frac{x+L}{2L}$$

Now, separate variables, $\psi(x,t) = X(x)T(t)$

$$\Rightarrow \quad T' = -\kappa\lambda T \ , \quad X'' = -\lambda X$$

$$\Rightarrow \quad \psi(x,t) = \left\{ a\sin(\sqrt{\lambda}x) + b\cos(\sqrt{\lambda}x) \right\} e^{-\kappa\lambda t}$$

Since IC's are odd $\Rightarrow b = 0$. and BC gives $\lambda = \frac{n^2\pi^2}{L^2}$

So:
$$\underline{\phi(x,t) = \phi_s(x) + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{\kappa n^2\pi^2 t}{L^2}\right)}$$

where $\quad a_n = \frac{1}{L}\int_{-L}^{L} \left\{ \Theta(x) - \frac{x+L}{2L} \right\} \sin\left(\frac{n\pi x}{L}\right) dx = \frac{1}{n\pi}$.

$\boxed{\text{N.B}}$ On a spherical domain; spherically symmetric solutions satisfy:

$$\frac{\partial\phi}{\partial t} = \frac{\kappa}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial\phi}{\partial r}\right)$$

Then separating variables $\phi(\cancel{x},t) = R(r)T(t)$ and letting

$$S(r) = rR(r)$$

$$\Rightarrow \left\{ T' = -\lambda^2 \kappa T \quad \text{and} \quad S'' = -\lambda^2 S. \right\}$$

4.3 The Wave Equation

In 1D:
$$\boxed{\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2}}$$

Separating variables and applying $\phi(0,t) = \phi(L,t) = 0$

$$\Rightarrow \quad \underline{\phi(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left\{ A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) \right\}}$$

• Energy:

The KE of a small element δx is:

$$\frac{1}{2}\mu\,\delta x \left(\frac{\partial\phi}{\partial t}\right)^2$$

$$\Rightarrow \quad \underline{K(t) = \frac{\mu}{2}\int_0^L \left(\frac{\partial\phi}{\partial t}\right)^2 dx}$$

The PE of a small element is:

$$\frac{T}{2}\,\delta x \left(\frac{\partial\phi}{\partial x}\right)^2$$

$$\left.\begin{aligned} &T(\delta s - \delta x) = T\left(\sqrt{\delta x^2 + \delta\phi^2} - \delta x\right) \\ &\simeq T\delta x\left\{1 + \tfrac{1}{2}\left(\tfrac{\partial\phi}{\partial x}\right)^2 + \dots\right\} - T\delta x \\ &\simeq \tfrac{T}{2}\,\delta x\left(\tfrac{\partial\phi}{\partial x}\right)^2 \end{aligned}\right.$$

$$\Rightarrow \quad \underline{V(t) = \frac{\mu}{2}\int_0^L c^2 \left(\frac{\partial\phi}{\partial x}\right)^2 dx} \qquad \left(c^2 = \frac{T}{\mu}\right).$$

Using the infinite series form for $\phi(x,t)$:

$$K(t) = \frac{\mu\pi^2 c^2}{4L}\sum_{n=1}^{\infty} n^2 \left\{A_n \sin\left(\frac{n\pi ct}{L}\right) - B_n \cos\left(\frac{n\pi ct}{L}\right)\right\}^2$$

$$V(t) = \frac{\mu\pi^2 c^2}{4L}\sum_{n=1}^{\infty} n^2 \left\{A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right\}^2$$

$$\Rightarrow \quad \underline{E(t) = \frac{\mu\pi^2 c^2}{4L}\sum_{n=1}^{\infty} n^2 (A_n^2 + B_n^2)}$$

{Proposition}. Suppose $\phi: \Omega \times [0,\infty) \to \mathbb{R}$ obeys $\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$ inside $\Omega \times (0,\infty)$ and is fixed at the boundary. Then E is constant.

$\boxed{\text{Pf}}$:

$$\frac{dE}{dt} = \int_\Omega \left\{\frac{\partial^2\phi}{\partial t^2}\frac{\partial\phi}{\partial t} + c^2 \nabla\left(\frac{\partial\phi}{\partial t}\right)\cdot\nabla\phi\right\} dV$$

integrating by parts on 2nd term.

$$\Rightarrow \quad \frac{dE}{dt} = \int_\Omega \frac{d\phi}{dt}\cancelto{0}{\left(\frac{\partial^2\phi}{\partial t^2} - c^2\nabla^2\phi\right)} dV + c^2\int_{\partial\Omega} \cancelto{0}{\frac{\partial\phi}{\partial t}}\nabla\phi\cdot d\underline{S}$$

$$\Rightarrow \quad \frac{dE}{dt} = 0.$$

{Proposition}. Suppose $\phi: \Omega \times [0,\infty) \to \mathbb{R}$ obeys $\frac{\partial^2\phi}{\partial t^2} = c^2\nabla^2\phi$ inside $\Omega \times (0,\infty)$ and, for some f, g, h:

$$\left.\begin{aligned} &\text{i) } \phi(\underline{x}, 0) = f(\underline{x}) \\ &\text{ii) } \frac{\partial\phi}{\partial t}(\underline{x}, 0) = g(\underline{x}) \\ &\text{iii) } \phi\big|_{\partial\Omega \times [0,\infty)} = h(\underline{x}) \end{aligned}\right\} \Rightarrow \underline{\phi \text{ is unique}}.$$

Pf: Let $\psi = \phi_1 - \phi_2$, then: $\frac{\partial^2 \psi}{\partial t^2} = c^2 \nabla^2 \psi$

and $\psi|_{\partial\Omega \times [0,\infty)} = \psi_{\Omega \times \{0\}} = \frac{\partial \psi}{\partial t}\Big|_{\Omega \times \{0\}} = 0$

Let $E\psi(t) = \frac{1}{2}\int_\Omega \left\{ \left(\frac{\partial \psi}{\partial t}\right)^2 + c^2 \underline{\nabla}\psi \cdot \underline{\nabla}\psi \right\} dV$

Then $E\psi$ is constant. Initially $\psi = \frac{\partial \psi}{\partial t} = 0 \Rightarrow E\psi(0) = 0$

So: $\frac{1}{2}\int_\Omega \left\{ \left(\frac{\partial \psi}{\partial t}\right)^2 + c^2 \nabla\psi \cdot \nabla\psi \right\} dV = 0$

Hence $\left(\frac{\partial \psi}{\partial t}\right) = 0 \implies \psi = 0$ since it is 0 at the beginning.

$\implies \boxed{\phi_1 = \phi_2}$

e.g Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, let $\phi(r,\theta,t)$ solve:

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = \nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

with $\phi|_{\partial\Omega} = 0$ (modelling at drum / membrane).

We separate variables, $\phi(r,\theta,t) = R(r)\Theta(\theta)T(t)$

$$\implies \{T'' = -c^2 \lambda T\}, \quad \{\Theta'' = -\mu \Theta\}$$

$$\{r(rR')' + (r^2\lambda - \mu)R = 0\}$$

Then, since $\phi(t,r,\theta + 2\pi) = \phi(t,r,\theta) \Rightarrow \boxed{\mu = m^2}$

$$\implies r(rR')' + (r^2\lambda - m^2)R = 0$$

$$\implies \boxed{R(r) = a_m J_m(\sqrt{\lambda}\, r) + b_m Y_m(\sqrt{\lambda}\, r)}$$

Regularity ensures $b_m = 0$ and $\phi|_{\partial\Omega} = 0 \Rightarrow \boxed{\lambda = k_{mi}^2}$

So:

$$\begin{aligned}\phi(r,\theta,t) = &\sum_{i=0}^{\infty}{}' \{A_{0i}\sin(k_{0i}ct) + B_{0i}\cos(k_{0i}ct)\} J_0(k_{0i}r) \\ &+ \sum_{m=1}^{\infty}{}' \sum_{i=0}^{\infty}{}' \left(\{A_{mi}\cos(m\theta) + B_{mi}\sin m\theta\}\sin(k_{mi}ct) J_m(k_{mi}r)\right) \\ &+ \sum_{m=1}^{\infty}{}' \sum_{i=0}^{\infty}{}' \left(\{C_{mi}\cos(m\theta) + D_{mi}\sin(m\theta)\}\cos(k_{mi}ct) J_m(k_{mi}r)\right)\end{aligned}$$

$\boxed{N.B}$ If $\partial_t \phi(0,r,\theta) = g(r)$ and $\phi(0,r,\theta)$

$\Rightarrow A_{ni} = B_{ni} = C_{ni} = D_{ni} = 0$ for $n \neq 0$.
by symmetry.

and $\phi = 0$ at $t=0 \Rightarrow B_{0i} = 0$

$$\Rightarrow \boxed{\phi(r,\theta,t) = \sum_{i=0}^{\infty} A_{0i} \sin(k_{0i} c t) J_0(k_{0i} r)}$$

$$\Rightarrow \int_0^1 \sum_{i=0}^{\infty} k_{0i} c A_{0i} J_0(k_{0i} r) J_0(k_{0j} r)\, r\, dr$$

$$= \int_0^1 g(r) J_0(k_{0j} r)\, r\, dr$$

$$\Rightarrow \left\{ A_{0i} = \frac{2}{c k_{0i}} \cdot \frac{1}{[J_0'(k_{0i})]^2} \int_0^1 g(r) J_0(k_{0i} r)\, r\, dr \right\}$$

---

5 Distributions.

First consider the set of functions $D(\Omega)$ set of infinitely smooth functions on compact domain.

$$\text{e.g.} \quad \phi(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & |x|<1 \\ 0 & \text{otherwise} \end{cases} \in D(\Omega)$$

Then a distribution, T, is a map: $\boxed{T : D(\Omega) \to \mathbb{R}}$.

e.g. Define $T_f : D(\Omega) \to \mathbb{R}$ by:

$$T_f[\phi] = \int_\Omega f(x)\phi(x)\, dx.$$

{Definition} Dirac-delta function: is defined by: $\delta : D(\Omega) \to \mathbb{R}$.

$$\boxed{\delta[\phi] = \phi(0)}$$

We can differentiate distributions as follows:

$$\boxed{T'[\phi] = -T[\phi']}$$

and they may be generated from limits of ordinary functions:

e.g $G_n(x) = \frac{n}{\sqrt{\pi}} \exp(-n^2 x^2)$

$$\Rightarrow \quad G_n[\phi] \longrightarrow \delta[\phi] \text{ for any } \phi \text{ as } n \to \infty.$$

So: $\boxed{\delta'[\phi] = -\phi'(0).}$

· Properties of $\delta(x)$:

① Translation: $\int_{-\infty}^{\infty} \delta(x-a)\phi(x)\,dx = \int_{-\infty}^{\infty} \delta(y)\phi(y+a)\,dy = \boxed{\phi(a)}$

② Scaling: $\int_{-\infty}^{\infty} \delta(cx)\phi(x)\,dx = \int_{-\infty}^{\infty} \delta(y)\phi\left(\frac{y}{c}\right)\cdot\frac{dy}{|c|} = \boxed{\frac{1}{|c|}\phi(0)}$

③ Zeros: Suppose $f(x)$ is continuously differentiable with isolated simple zeros at x_i $\{i = 1, \dots, n\}$. Then in a neighbourhood of x_i: $f(x) \simeq (x - x_i)\frac{\partial f}{\partial x}\Big|_{x=x_i}$

$$\Rightarrow \int_{-\infty}^{\infty} \delta(f(x))\phi(x)\,dx = \sum_{i=1}^{n} \int_{-\infty}^{\infty} \delta\left((x-x_i)\frac{\partial f}{\partial x}\Big|_{x=x_i}\right)\phi(x)\,dx$$

$$\boxed{= \sum_{i=1}^{n} \frac{1}{|f'(x_i)|}\phi(x_i).}$$

$\boxed{\text{N.B}}$ On $[-L, L]$ we may expand $\delta(x)$ as a Fourier Series:

$$\delta(x) = \sum_{n\in\mathbb{Z}} \hat{\delta}_n e^{\frac{in\pi x}{L}}$$

with $\hat{\delta}_n = \frac{1}{2L}\int_{-L}^{L} e^{\frac{in\pi x}{L}}\delta(x)\,dx = \frac{1}{2L}$

$$\Rightarrow \boxed{\delta(x) = \frac{1}{2L}\sum_{n\in\mathbb{Z}} e^{\frac{in\pi x}{L}}}$$

- Green's Functions: $G(x;\xi)$ satisfies:

$$\boxed{\mathcal{L}G(x,\xi) = \delta(x-\xi).}$$

Then we may define:

$$y(x) = \int_a^b G(x;\xi)\, f(\xi)\, d\xi$$

where y solves $\mathcal{L}y = f$ on $[a,b]$.

Then: $\left\{ \mathcal{L}y = \int_a^b \mathcal{L}G(x;\xi) f(\xi) d\xi = \int_a^b \delta(x-\xi) f(\xi) d\xi = f \right\}$

We wish to find $G(x;\xi)$ subject to:

$$\boxed{G(a,\xi) = G(b,\xi) = 0.}$$

Then $\mathcal{L}G = 0$ on $[a,\xi) \cup (\xi, b]$ so it is expandable in a basis of solutions. Suppose $\{y_1, y_2\}$ are a basis of solutions with $\boxed{y_1(a) = y_2(b) = 0.}$ IMP

Then:

$$\boxed{G(x;\xi) = \begin{cases} A(\xi) y_1(x) & a \leq x < \xi \\ B(\xi) y_2(x) & \xi < x \leq b \end{cases}}$$

If G were discontinuous at $\xi \Rightarrow G'' \propto \delta'$ which cannot be accounted for. So G is continuous at ξ.

$$\Rightarrow \boxed{① \; A(\xi) y_1(\xi) = B(\xi) y_2(\xi)}$$

Now: $\displaystyle\int_{\xi-\varepsilon}^{\xi+\varepsilon} \left\{ \alpha(x) \frac{d^2G}{dx^2} + \beta(x) \frac{dG}{dx} + \gamma(x) G \right\} dx = \int_{\xi-\varepsilon}^{\xi+\varepsilon} \delta(x-\xi) dx = 1.$

Since G is continuous $\Rightarrow \displaystyle\int_{\xi-\varepsilon}^{\xi+\varepsilon} \gamma G\, dx = 0$ and G' is finite.

Since G' is finite $\Rightarrow \displaystyle\int_{\xi-\varepsilon}^{\xi+\varepsilon} \beta G'\, dx = 0.$

So: $$\lim_{\varepsilon \to 0} \int_{\xi-\varepsilon}^{\xi+\varepsilon} \alpha(x) \frac{d^2 G}{dx^2}\, dx = 1$$

integrating by parts.

$$\Rightarrow \lim_{\varepsilon \to 0} \left\{ \left[\alpha G'\right]_{\xi-\varepsilon}^{\xi+\varepsilon} + \int_{\xi-\varepsilon}^{\xi+\varepsilon} \alpha' G'\, dx \right\} = 1$$

$\to 0$ by finiteness of G'.

$$\Rightarrow \alpha(\xi)\left(\frac{\partial G}{\partial x}\bigg|_{\xi^+} - \frac{\partial G}{\partial x}\bigg|_{\xi^-} \right) = 1$$

$$\Rightarrow \boxed{(2)\ B(\xi)\, y_2'(\xi) - A(\xi)\, y_1'(\xi) = \frac{1}{\alpha(\xi)}}$$

Our two conditions at $x = \xi$ are:

(*) i) $A(\xi) y_1(\xi) = B(\xi) y_2(\xi)$ continuity

ii) $B(\xi) y_2'(\xi) - A(\xi) y_1'(\xi) = \frac{1}{\alpha(\xi)}$ jump.

Writing $W = y_1 y_2' - y_1' y_2$, we may show:

$$A(\xi) = \frac{y_2(\xi)}{\alpha(\xi) W(\xi)}, \quad B(\xi) = \frac{y_1(\xi)}{\alpha(\xi) W(\xi)}$$

Then, using the step function $\Theta(x)$:

$$\underline{G(x;\xi) = \frac{1}{\alpha(\xi) W(\xi)} \left\{ \Theta(\xi - x)\, y_2(\xi) y_1(x) + \Theta(x-\xi)\, y_1(\xi) y_2(x) \right\}}$$

$$\Rightarrow \left\{ y(x) = \int_a^b G(x;\xi) f(\xi)\, d\xi = \int_x^b \frac{f(\xi)}{\alpha(\xi) W(\xi)} y_2(\xi) y_1(x)\, d\xi + \int_a^x \frac{f(\xi)}{\alpha(\xi) W(\xi)} y_1(\xi) y_2(x)\, d\xi \right\}$$

note limits.

- <u>Green's Functions for initial value problems.</u>

We now try and solve $\mathcal{L}\phi = f(t)$: $y(t=t_0) = y'(t=t_0) = 0$.

Let $\{y_1(t), y_2(t)\}$ be any basis of solutions.

Then we may write: $\mathcal{L}G = \delta(t-\tau)$ and:

$$G(t;\tau) = \begin{cases} A(\tau)y_1(t) + B(\tau)y_2(t) & t_0 \le t < \tau \\ C(\tau)y_1(t) + D(\tau)y_2(t) & t > \tau \end{cases}$$

Initial conditions require that:

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

But $\det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = W \neq 0$ since $\{y_1, y_2\}$ is a basis

$\Rightarrow \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

So: $$G(t,\tau) = \begin{cases} 0 & t_0 \le t < \tau \\ C(\tau)y_1(t) + D(\tau)y_2(t) & t > \tau \end{cases}$$

Then, jump + continuity conditions give:

$$\left[\begin{array}{ll} \text{i)} & 0 = C(\tau)y_1(\tau) + D(\tau)y_2(\tau) \\ \text{ii)} & \frac{1}{\alpha(\tau)} = C(\tau)y_1'(\tau) + D(\tau)y_2'(\tau) \end{array}\right]$$

$$\Longrightarrow \left[y(t) = \int_{t_0}^{\infty} G(t,\tau)f(\tau)\,d\tau = \int_{t_0}^{t} G(t,\tau)f(\tau)\,d\tau \right]$$

---

6 Fourier Transform.

$$\boxed{\tilde{f}(k) = \mathcal{F}\{f\} = \int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx}$$

- Properties:

① Linearity: $\{\mathcal{F}\{c_1f_1 + c_2f_2\} = c_1\mathcal{F}(f_1) + c_2\mathcal{F}(f_2)\}$

② Translation

$$\mathcal{F}[f(x-a)] = \int_{-\infty}^{\infty} e^{-ikx} f(x-a)\,dx$$

$$= \int_{\mathbb{R}} e^{-ik(y+a)} f(y)dy$$

$$= e^{-ika} \int_{\mathbb{R}} e^{-iky} f(y)dy = e^{-ika} \mathcal{F}\{f(x)\}$$

$$\Rightarrow \left\{ \mathcal{F}\{f(x-a)\} = e^{-ika} \mathcal{F}\{f(x)\}. \right\}$$

③ Rephasing: $\left\{ \mathcal{F}\{e^{-ilx} f(x)\} = \tilde{f}(k+l). \right\}$

④ Scaling $\left\{ \mathcal{F}\{f(cx)\} = \frac{1}{|c|} \tilde{f}\left(\frac{k}{c}\right). \right\}$

⑤ Convolutions $\left\{ \mathcal{F}\{f * g\} = \mathcal{F}(f)\mathcal{F}(g) \right\}$ ✱

⑥ Derivatives : $\left\{ \mathcal{F}\{f'(x)\} = ik\,\mathcal{F}(f(x)) \right\}$ ✱

⑦ $\left\{ \mathcal{F}(xf(x)) = i\frac{d}{dk}\int_{-\infty}^{\infty} e^{-ikx} dx\, f(x) = i\tilde{f}'(k) \right\}$

Now consider a differential operator:

$$\mathcal{L}(\partial) = \sum_{r=0}^{p} c_r \frac{d^r}{dx^r} \quad \text{and} \quad \mathcal{L}(\partial)y = f$$

Then:

$$\mathcal{F}(\mathcal{L}(\partial)y) = \tilde{f}(f)$$

$$\Rightarrow c_0\tilde{y}(k) + c_1(ik)\tilde{y}(k) + \dots + c_p(ik)^p\tilde{y}(k) = \tilde{f}(k).$$

$$\Rightarrow \left[\tilde{y}(k) = \frac{\tilde{f}(k)}{\mathcal{L}(ik)} \right]$$

e.g Consider $\phi: \mathbb{R}^n \to \mathbb{C}$ and suppose

$$\nabla^2\phi - m^2\phi = \rho(\underline{x}). \quad \text{where } \nabla^2 = \sum_{i=1}^{n} \frac{d^2}{dx_i^2}$$

Define the n-dimensional Fourier transform:

$$\tilde{\phi}(\underline{k}) = \int_{\mathbb{R}^n} e^{-i\underline{k}\cdot\underline{x}} f(\underline{x})\, d\underline{x} = \mathcal{F}\{\phi(\underline{x})\}$$

Then: $\mathcal{F}\{\nabla^2\phi - m^2\phi\} = \mathcal{F}(e) = \tilde{e}(\underline{k})$

and $\displaystyle\int_{\mathbb{R}^n} e^{-i\underline{k}\cdot\underline{x}}\, \underline{\nabla}\cdot\underline{\nabla}\phi\, d\underline{x} = -\int_{\mathbb{R}^n} \underline{\nabla}(e^{-i\underline{k}\cdot\underline{x}})\cdot\underline{\nabla}\phi\, d\underline{x}$

$$= \int_{\mathbb{R}^n} \nabla^2(e^{-i\underline{k}\cdot\underline{x}})\,\phi\, d\underline{x} = -\underline{k}\cdot\underline{k}\int_{\mathbb{R}^n} e^{-i\underline{k}\cdot\underline{x}}\phi(\underline{x})\,d\underline{x}$$

$$= -|\underline{k}|^2\tilde{\phi}(\underline{k}).$$

So: $(-|\underline{k}|^2 - m^2)\tilde{\phi}(\underline{k}) = \tilde{e}(\underline{k})$

$$\Rightarrow \boxed{\tilde{\phi}(\underline{k}) = \frac{-\tilde{e}(\underline{k})}{|\underline{k}|^2 + m^2}}$$

- Fourier inversion theorem.

$$\boxed{f(x) = \mathcal{F}^{-1}\{\tilde{f}(k)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\tilde{f}(k)\,dk}$$

We have the duality: $\boxed{\mathcal{F}(f(x)) = 2\pi\mathcal{F}^{-1}(f(-x))}$.

- Parseval's Theorem for FT's.

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\tilde{f}(k)|^2\,dk = \frac{1}{2\pi}\|\tilde{f}\|^2$$

- FT's of distributions exist, for example:

$$\boxed{\mathcal{F}(\delta(x)) = 1}. \Rightarrow \mathcal{F}^{-1}(1) = \delta(x).$$

and $\boxed{\mathcal{F}(\delta(x-a)) = e^{ika}}$

and $\boxed{\mathcal{F}(e^{ilx}) = 2\pi\delta(k-l)}$.

$\Rightarrow$ for example $\left\{\mathcal{F}(\cos lx) = \mathcal{F}\left(\tfrac{1}{2}(e^{ilx} + e^{-ilx})\right)\right.$

$$\left. = \pi(\delta(k+l) + \delta(k-l))\right\}.$$

• Linear systems and response functions.

Consider system that modifies input $I(t)$ to produce output $O(t)$. We call FT of $I(t)$, $\tilde{I}(\omega)$ the resolution of I.

$$I(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\,\tilde{I}(\omega)\,d\omega$$

We specify a transfer function $\tilde{R}(\omega)$ such that:

$$O(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\,\tilde{R}(\omega)\tilde{I}(\omega)\,d\omega .$$

But then $O(t) = \mathcal{F}^{-1}\{\tilde{R}(\omega)\tilde{I}(\omega)\} = R * I$

$$\Rightarrow \boxed{O(t) = \int_{-\infty}^{\infty} R(t-u)\,I(u)\,du}$$

where $\quad R(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\,\tilde{R}(\omega)\,d\omega$

Causality ensures that $R(t) = 0$ for $t<0$

$$\Rightarrow \quad O(t) = \int_{0}^{t} R(t-u)\,I(u)\,du .$$

• General form of transfer function:

Suppose we have: $\quad I(t) = \mathcal{L}_m O(t)$

where $\quad \mathcal{L}_m = \sum_{i=0}^{m} a_i \frac{d^i}{dt^i}$ with $a_i \in \mathbb{C}$.

Taking FT's:

$$\tilde{I}(\omega) = \sum_{j=0}^{m} a_j\,(i\omega)^j\,\tilde{O}(\omega) .$$

$$\Rightarrow \quad \tilde{O}(\omega) = \frac{\tilde{I}(\omega)}{a_0 + i\omega a_1 + \cdots + (i\omega)^m a_m}$$

and hence: $\boxed{\tilde{R}(\omega) = \dfrac{1}{a_0 + i\omega a_1 + \cdots + (i\omega)^m a_m}}$ ✱

Say this polynomial has roots c_j with multiplicity k_j for $j=1,\dots,J$, then:

$$\tilde{R}(\omega) = \frac{1}{a_m}\prod_{j=1}^{J}\frac{1}{(i\omega-c_j)^{k_j}}$$

$$= \sum_{j=1}^{J}\sum_{r=1}^{k_j}\frac{\Gamma_{rj}}{(i\omega-c_j)^r} \quad \text{by partial fractions.}$$

for some $\Gamma_{rj}\in\mathbb{C}$.

We may show that:

$$\left[\, h_p(t) = \begin{cases} \frac{t^p}{p!}e^{\alpha t} & t>0 \\ 0 & \text{otherwise} \end{cases}\,\right]$$

has: $\left[\,\tilde{h}_p(\omega) = \frac{1}{(i\omega-\alpha)^{p+1}}\,\right]$ provided $Re(\alpha)<0$.

- May also be used to solve ODE's and PDE's. For PDE's check which variable has full domain $(-\infty < x_i < \infty)$ so FT exists, then take FT wrt 1/more variables e.g $\phi(x,y)\to\tilde{\phi}(k,y)$.

- Discrete Fourier Transform.

Suppose f is mostly concentrated in some region $[-R,S]$ i.e $|f(x)|<<1$. Then: $(R,S>0)$.

$$\tilde{f}(k)\simeq\int_{-R}^{S}e^{-ikx}f(x)\,dx.$$

Now, sample $f(x)$ at: $\left[\, x=x_j \doteq -R+j\frac{(R+S)}{N}\,\right]$

for N a large integer, $j=0,1,\dots,N-1$.

$$\Rightarrow \tilde{f}(k)\simeq\frac{R+S}{N}\sum_{j=0}^{N-1}f(x_j)e^{-ikx_j}$$

Suppose we may only store the result $\tilde{f}(k)$ for finitely many k, say:

$$\left[k = k_m = \frac{2\pi m}{R+S} \right]$$

Then: $\tilde{f}(k_m) \simeq \frac{R+S}{N} e^{\frac{2\pi i m R}{R+S}} \sum_{j=0}^{N-1}{}' f(x_j) e^{-\frac{2\pi i}{N} jm}$

$$= (R+S)\exp\left(2\pi i m \frac{R}{R+S}\right) \left\{ \frac{1}{N} \sum_{j=0}^{N-1}{}' f(x_j)\, \omega^{-jm} \right\}$$

where $\omega = e^{\frac{2\pi i}{N}}$ is an N^{th} root of unity.

- Let $F(m) = \frac{1}{N} \sum_{j=0}^{N-1}{}' f(x_j)\, \omega^{-jm}$.
- Now, in theory, we can reconstruct the values of $f(x_j)$ by solving linear equations, given $F(m)$. Now, let $G = \{1, \omega, \dots, \omega^{N-1}\}$ and consider a function $g : G \to \mathbb{C}$:

$$g(\omega^j) = f(\cancel{x}_j), \text{ then:}$$

$$\left\{ F(m) = \frac{1}{N} \sum_{j=0}^{N-1}{}' \omega^{-jm} g(\omega^j) \right\}$$

The space of all functions $g : G \to \mathbb{C}$ is isomorphic to $\mathbb{C}^N$ and has an inner product:

$$(f, g) = \frac{1}{N} \sum_{j=0}^{N-1}{}' f(\omega^j)^* g(\omega^j)$$

Let $e_m : G \to \mathbb{C}$ be the function $e_m(\omega^j) = \omega^{jm}$, then $\{e_m\}$ is an orthonormal basis wrt our inner product.

Then: $F(m) = \frac{1}{N} \sum_{j=0}^{N-1}{}' \omega^{-jm} f(x_j) = \frac{1}{N} \sum_{j=0}^{N-1}{}' e_m^*(\omega^j) g(\omega^j)$

$$= (e_m, g)$$

$$\Rightarrow \quad g = \sum_{m=0}^{N-1}{}' (e_m, g) e_m = \sum_{m=0}^{N-1}{}' F(m) e_m.$$

$$\Longrightarrow \left[f(x_j) = \sum_{m=0}^{N-1}{}' F(m)\, e_m(\omega^j) \right]$$

7 More PDEs.

- Characteristics: Given a PDE for some function $\phi(\underline{x})$, then given values of ϕ on a surface of dimension $(n-1)$ is called Cauchy data, and a PDE subject to these conditions is a Cauchy problem. It is well-posed if:

(1) A solution exists
(2) The solution is unique.
(3) The solution depends continuously on the auxiliary data.

requires a topology on the function space of solutions.

- Method of Characteristics:

Vector Fields and Integral Curves: Consider a smooth parametrised curve $C \subset \mathbb{R}^2$ which may be seen as a map: $X: \mathbb{R} \to \mathbb{R}^2$, $X: s \mapsto (x(s), y(s))$. Then the tangent vector, $\underline{v}$ is:

$$\underline{v} = \begin{pmatrix} \frac{dx(s)}{ds} \\ \frac{dy(s)}{ds} \end{pmatrix}$$

Now, consider $\phi: \mathbb{R}^2 \to \mathbb{C}$ restricted to C, so $\phi|_C: C \to \mathbb{C}$.
The directional derivative is:

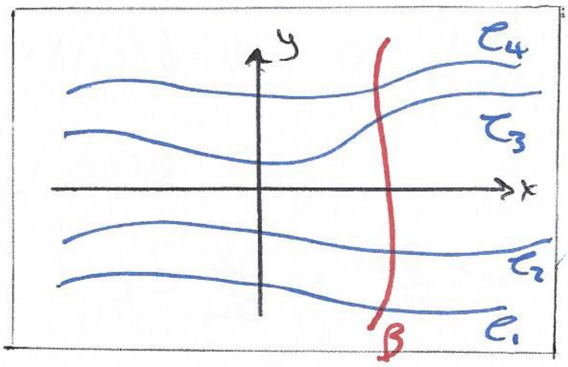
$$\frac{d\phi(x(s), y(s))}{ds} = \frac{dx(s)}{ds}\partial_x \phi|_C + \frac{dy(s)}{ds}\partial_y \phi|_C = \underline{v} \cdot \nabla\phi|_C$$

So, if $\underline{v} \cdot \nabla\phi|_C = 0$, then ϕ is constant along C. Now, consider $\underline{u} = \begin{pmatrix} \alpha(x,y) \\ \beta(x,y) \end{pmatrix}$, then an integral curve of $\underline{u}$ is a curve $C \subseteq \mathbb{R}^2$ such that:

$$\left\{ \frac{dx(s)}{ds} = \alpha(x,y), \quad \frac{dy(s)}{ds} = \beta(x,y). \right\}$$

Now, we define a transverse curve B that intersects every member of the family of integral curves, and is nowhere parallel. We fix s such that intersection occurs at $s=0$. If we parametrise B by t, then:

$$C_t = \{x = x(s,t),\ y = y(s,t)\}$$



where:

$$\left.\frac{\partial x(s,t)}{\partial s}\right|_t = \alpha\Big|_{C_t}, \quad \left.\frac{\partial y(s,t)}{\partial s}\right|_t = \beta\Big|_{C_t}.$$

Now, if the Jacobian $J = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}$, then we may find (t,s) in terms of (x,y). So, if $J \neq 0$, then the family of integral curves is space-filling + non-intersecting.

Suppose ϕ satisfies:

$$\alpha(x,y)\frac{\partial\phi}{\partial x} + \beta(x,y)\frac{\partial\phi}{\partial y} = 0$$

and define $\underline{V}(x,y) = \begin{pmatrix}\alpha(x,y)\\ \beta(x,y)\end{pmatrix}$, then $\underline{V}\cdot\underline{\nabla}\phi = 0$, then along any integral curve we have:

$$\frac{d\phi}{ds} = \frac{dx(s)}{ds}\frac{\partial\phi}{\partial x} + \frac{dy(s)}{ds}\frac{\partial\phi}{\partial y} = \underline{V}\cdot\underline{\nabla}\phi$$

$$\Rightarrow \left\{\left.\frac{\partial x(s,t)}{\partial s}\right|_t = \alpha(x,y),\quad \left.\frac{\partial y(s,t)}{\partial s}\right|_t = \beta(x,y)\right\}$$

So, the PDE becomes $\left.\frac{\partial\phi}{\partial s}\right|_t = 0$. For a well-posed problem we specify boundary data along B such that:

$$\phi(0,t) = h(t).$$

But $\left.\frac{\partial\phi}{\partial s}\right|_t = 0 \Rightarrow \underline{\phi(s,t) = h(t)}$, then invert with $t(x,y)$.

e.g. $\left.\frac{\partial\phi}{\partial x}\right|_y = 0$, $\phi(0,y) = f(y)$.

Here $\underline{V} = \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}dx/ds\\ dy/ds\end{pmatrix} \Rightarrow x = s + c,\ y = d.$

Initial curve is $y=t, x=0 \Rightarrow c=0,\ d=t.$

Now, $\frac{\partial \phi}{\partial s}\Big|_t = 0 \implies \phi(s,t) = h(t) = f(t)$

$$\implies \underline{\phi(x,y) = f(y).}$$

e.g.$_2$: $e^x \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} = 0, \quad \phi(x,0) = \cosh x$

$$\implies \frac{dx}{ds} = e^x, \quad \frac{dy}{ds} = 1 \implies e^{-x} = -s + c, \quad y = s + d$$

We want $x = t, y = 0$ $(s=0)$ $\implies e^{-x} = -s + e^{-t}, \quad y = s.$

Then $\frac{\partial \phi}{\partial s}\Big|_t = 0 \rightarrow \phi(s,t) = \phi(0,t) = \cosh t$

$$\implies \underline{\phi(x,y) = \cosh\{-\log(y + e^{-x})\}}.$$

e.g.$_3$ $\frac{\partial \phi}{\partial x} + 2\frac{\partial \phi}{\partial y} = ye^x, \quad \phi(x,x) = \sin x$

Here: $\frac{dx}{ds} = 1, \quad \frac{dy}{ds} = 2$ with $x = y = t$ at $s = 0$

$$\implies x(s,t) = s + t, \quad y(s,t) = 2s + t$$

$$\implies s = y - x, \quad t = 2x - y$$

Now, $\frac{\partial \phi}{\partial s}\Big|_t = ye^x = (2s+t)e^{s+t}, \quad \phi(0,t) = \sin t$

$$\implies \phi(s,t) = (2-t)e^t(1-e^s) + \sin t + 2se^{s+t}$$

$$\implies \underline{\phi(x,y) = (1 - 2x + y)e^{2x-y} + \sin(2x-y) + (y-2)e^x}.$$

- <u>Characteristics for 2^{nd} order PDEs</u>

Consider $\mathcal{L} = \sum_{i,j=1}^{n} \alpha^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x)\frac{\partial}{\partial x_i} + c(x)$

where $\alpha^{ij}(x) = \alpha^{ji}(x)$. Then the symbol, $\sigma(k,x)$ of $\mathcal{L}$ is defined to be:

$$\left\{\sigma(k,x) = \sum_{i,j=1}^{n} \alpha^{ij}(x)k_ik_j + \sum_{i=1}^{n} b^i(x)k_i + c(x).\right\}$$

Then principal part is:

$$\sigma_p(\underline{k}, x) = \sum_{i,j=1}^{n}{}' a^{ij}(x) k_i k_j.$$

Now, in general, $\sigma_p(\underline{k}, x) = \underline{k}^T A \underline{k}$ where $A = (\alpha^{ij})$. Since A is symmetric, all it's eigenvalues are real. Then, $\mathcal{L}$ is:

(1) Elliptic if all eigenvalues have same sign.
(2) Hyperbolic if all but one have same sign.
(3) Ultra-hyperbolic if there are more than one of each sign.
(4) Parabolic if A is degenerate (zero eigenvalue).

e.g Let $\mathcal{L} = a(x,y)\frac{\partial^2}{\partial x^2} + 2b(x,y)\frac{\partial^2}{\partial x \partial y} + c(x,y)\frac{\partial^2}{\partial y^2} + d(x,y)\frac{\partial}{\partial x}$

$$+ e(x,y)\frac{\partial}{\partial y} + f(x,y)$$

Then, $\mathcal{L}$ is: $\begin{cases} (1) \text{ elliptic} & \text{if } (b^2 - ac)(x,y) < 0 \\ (2) \text{ hyperbolic} & \text{if } (b^2 - ac)(x,y) > 0 \\ (3) \text{ parabolic} & \text{if } (b^2 - ac)(x,y) = 0 \end{cases}$

- Characteristic Surfaces: Given a differential operator, $\mathcal{L}$, let: $f(x_1, \dots, x_n) = 0$ define a surface $S \subseteq \mathbb{R}^n$. S is characteristic if:

$$\sum_{i,j}{}' a^{ij}(x) \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} = (\nabla f)^T A (\nabla f) = 0.$$

↑ we find $\underline{\nabla f}$ which is normal to surface and construct S.

Now,
- if $\mathcal{L}$ is elliptic, $(\nabla f)^T A (\nabla f) = 0$ has no non-zero real solutions.
- if $\mathcal{L}$ is parabolic, assume A has exactly one zero eigenvector, $\underline{n}$, and the other eigenvalues all have same sign.

Then: $\underline{n}^T A = A\underline{n} = 0$. $(\underline{n} \cdot \underline{n} = 1)$.

So, for any $\underline{\nabla f}$ we may write:

$$\nabla f = \underset{\text{along } \underline{n}}{\underline{n}(\underline{n}\cdot\nabla f)} + \underset{\text{orthogonal to } \underline{n}}{(\nabla f - \underline{n}(\underline{n}\cdot\nabla f))}$$

$$\Rightarrow \nabla f := \underline{n}(\underline{n}\cdot\nabla f) + \nabla_\perp f$$

Then: $(\nabla f)^T A (\nabla f) = \{\underline{n}(\underline{n}\cdot\nabla f) + \nabla_\perp f\}^T A \{\underline{n}(\underline{n}\cdot\nabla f) + \nabla_\perp f\}$

$$= (\nabla_\perp f)^T A (\nabla_\perp f).$$

Now, $\nabla_\perp f$ is in the positive/negative definite space of A $\Rightarrow$ no non-trivial solutions to $(\nabla_\perp f)^T A (\nabla_\perp f) = 0$.

So, characteristic surface has normal $\underline{n}$ $\therefore$ unique through any point $x \in \mathbb{R}^n$.

- If $\mathcal{L}$ is hyperbolic, all e-values but one have same sign. Suppose only one is negative, say $-\lambda$, with $\underline{m}$ corresponding eigenvector. Then:

$$(\nabla f)^T A (\nabla f) = \{\underline{m}(\underline{m}\cdot\nabla f) + \nabla_\perp f\} A \{\underline{m}(\underline{m}\cdot\nabla f) + \nabla_\perp f\}$$
$$= (\underline{m}\cdot\nabla f)^2\, \underline{m}^T A \underline{m} + (\nabla_\perp f)^T A (\nabla_\perp f)$$
$$= -\lambda(\underline{m}\cdot\nabla f)^2 + (\nabla_\perp f)^T A (\nabla_\perp f).$$

So, $(\nabla f)^T A (\nabla f) = 0$

$$\Rightarrow \underline{m}\cdot\nabla f = \pm\sqrt{\frac{(\nabla_\perp f)^T A (\nabla_\perp f)}{\lambda}}$$

Thus we have two possible characteristics at each $x \in \mathbb{R}^n$.

- <u>D'Alembert's Solution to Wave Equation.</u>

$$\frac{\partial^2\phi}{\partial t^2} - c^2\frac{\partial^2\phi}{\partial x^2} = 0$$

If we have: $\quad (\partial_x f, \partial_y f)\begin{pmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y)\end{pmatrix}\begin{pmatrix}\partial_x f \\ \partial_y f\end{pmatrix} = 0$

and $f(x,y) = 0$ is a characteristic, then we write $y = y(x)$

$$\Rightarrow 0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \text{ - since } \frac{df}{dx} = 0$$

$$\Rightarrow \frac{\partial_x f}{\partial_y f} = -\frac{dy}{dx}$$

$$\Rightarrow a(\partial_x f)^2 + 2b(\partial_x f)(\partial_y f) + c(\partial_y f)^2 = 0$$

$$\Rightarrow a\left(\frac{\partial_x f}{\partial_y f}\right)^2 + 2b\left(\frac{\partial_x f}{\partial_y f}\right) + c = 0$$

$$\Rightarrow a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right) + c = 0$$

$$\Rightarrow \frac{dy}{dx} = -\left\{\frac{-b \pm \sqrt{b^2 - ac}}{a}\right\}$$

So there is a solution if $b^2 - ac \geq 0$ (2 solns if hyperbolic and one if parabolic as predicted).

So, for the wave equation: $a(x,t) = 1$, $b(x,t) = 0$, $c(x,t) = -c^2$

$$\Rightarrow \frac{dx}{dt} = \pm c \quad \Rightarrow \quad x \pm ct = \text{const.}$$

Let $u = x - ct$, $v = x + ct$

$$\Rightarrow \frac{\partial^2 \phi}{\partial u \partial v} = 0$$

$$\Rightarrow \phi = F(v) + G(u) = F(x+ct) + G(x-ct)$$

Now, $\phi(x,0) = G(x) + H(x) = f(x)$

$$\partial_t \phi(x,0) = -cG'(x) + cH'(x) = g(x)$$

$$\rightarrow 2H'(x) = f'(x) + \frac{1}{c} g(x)$$

$$\Rightarrow \begin{cases} H(x) = \frac{1}{2}\{f(x) - f(0)\} + \frac{1}{2c}\int_0^x g(y)\,dy \\ G(x) = \frac{1}{2}\{f(x) + f(0)\} - \frac{1}{2c}\int_0^x g(y)\,dy \end{cases}$$

$$\Rightarrow \boxed{\phi(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy.}$$

So, we see domain of dependence is $[x-ct, x+ct]$ or conversely given initial data at $(x_0, 0)$ it may influence solution in wedge defined by $x \pm ct = x_0$. with signals propagating at speed c. This is in contrast to parabolic equations. e.g Heat eqn.

Green's Functions for P.D.E's on $\mathbb{R}^n$.

FT's for diffusion equation.

$$\frac{\partial \theta}{\partial t} = D\frac{\partial^2 \theta}{\partial x^2}, \quad \theta(x,0) = f(x) \quad \text{and } \theta \to 0 \text{ as } |x| \to \infty$$

Take FT's wrt x $(x \to k)$

$$\Rightarrow \frac{\partial}{\partial t}\tilde{\theta}(k,t) = -Dk^2\tilde{\theta}(k,t), \quad \tilde{\theta}(k,0) = \tilde{f}(k)$$

$$\Rightarrow \tilde{\theta}(k,t) = \tilde{f}(k)\exp(-Dk^2 t)$$

We wish to apply convolution theorem, we associate,

$$\tilde{g}(k,t) = \exp(-Dk^2 t) \Leftrightarrow g(x,t) = \frac{1}{\sqrt{4\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right)$$

So, by convolution thm:

$$\boxed{\theta(x,t) = \frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} f(u)\overbrace{\exp\left(-\frac{(x-u)^2}{4Dt}\right)}^{S_d(x-u,t)\ \text{(source function)}}du}$$

e.g. Gaussian as IC: $f(x) = \sqrt{\frac{a}{\pi}}\,\theta_0\exp(-ax^2)$

$$\Rightarrow \theta(x,t) = \theta_0\sqrt{\frac{a}{\pi(1+4aDt)}}\exp\left(-\frac{ax^2}{(1+4aDt)}\right)$$

e.g.$_2$ $f(x) = \theta_0\,\delta(x)$

$$\Rightarrow \theta(x,t) = \theta_0\exp\left(-\frac{x^2}{4Dt}\right)\cdot\frac{1}{\sqrt{4\pi Dt}} = \theta_0\,\underbrace{S_d(x,t)}_{\text{source function.}}$$

Forced Heat Equation

$$\frac{\partial \theta_f(x,t)}{\partial t} - D\frac{\partial^2\theta_f(x,t)}{\partial x^2} = f(x,t), \quad \overset{\text{homogeneous}}{\boxed{\theta_f(x,0) = 0}}$$

N.B if we have inhomogeneous BC then we may solve unforced with inhom. then forced with hom. BCs and sum solutions.

We search for a Green's function, $G(x,t,y,\tau)$,

s.t:

$$\frac{\partial G(x,t,y,\tau)}{\partial t} - D\frac{\partial^2 G(x,t,y,\tau)}{\partial x^2} = \delta(x-y)\delta(t-\tau)$$

$$G(x,0,y,\tau) = 0$$

Then: $\theta_f(x,t) = \int_0^{\infty}\int_{-\infty}^{\infty} G(x,t,y,\tau) f(y,\tau)\,dy\,d\tau$,

Now, we note that $\mathcal{F}\{\delta(x-y)\} = e^{-iky}$ we get:

$$\frac{\partial}{\partial t}\tilde{G}(k,t;y,\tau) + Dk^2\tilde{G}(k,t;y,\tau) = e^{-iky}\delta(t-\tau)$$

$$\Rightarrow \frac{\partial}{\partial t}\left(e^{Dk^2t}\tilde{G}(k,t;y,\tau)\right) = e^{-iky+Dk^2t}\delta(t-\tau)$$

$$\Rightarrow e^{Dk^2t}\tilde{G}(k,t;y,\tau) = e^{-iky}\int_0^t e^{Dk^2u}\delta(u-\tau)\,du$$

So, if $t<\tau$, $\int_0^t e^{Dk^2u}\delta(u-\tau)\,du = 0$

$t>\tau$, $\int_0^t e^{Dk^2u}\delta(u-\tau)\,du = e^{Dk^2\tau}$

$$\Rightarrow \tilde{G}(k,t;y,\tau) = \Theta(t-\tau)e^{-iky}e^{-Dk^2(t-\tau)}$$

$$\Rightarrow G(x,t,y,\tau) = \frac{\Theta(t-\tau)}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-y)}e^{-Dk^2(t-\tau)}\,dk$$

$$= \frac{\Theta(t')}{2\pi}\int_{-\infty}^{\infty} e^{ikx'}e^{-Dk^2t'}\,dk$$

But this is FT of Gaussian from before:

$$\Rightarrow G(x,t;y,\tau) = \frac{\Theta(t-\tau)}{\sqrt{4\pi(D)(t-\tau)}}\exp\left(-\frac{(x-y)^2}{4D(t-\tau)}\right)$$

$$G(x,t;y,\tau) = \Theta(t-\tau)S_d(x-y,t-\tau)$$

Duhamel's Principle:

Given: $\begin{cases} \theta_t - D\theta_{xx} = 0, & \theta(x,\tau) = g(x) \\ \theta(x,t) = \int_{-\infty}^{\infty} g(u)S_d(x-u,t-\tau)\,du, & t>\tau \end{cases}$

and $\begin{cases} \theta_t - D\theta_{xx} = f(x,t), & \theta(x,0) = 0 \\ \theta_f(x,t) = \int_0^t \left[\int_{-\infty}^{\infty}[f(u,\tau)S_d(x-u,t-\tau)]\,du\right]d\tau \end{cases}$

Then if we think of $f(x,t)$ as being an IC at each fixed time τ. then $[\sim]$ represents effect to time t. Then $\int_0^t \ldots d\tau$ represents superposition. from all times τ onwards. The range of integration $\Rightarrow \tau > t$ which says data at time τ cannot affect past but only for later times $t \not> \tau$.

Forced wave equation

$$\frac{\partial^2 y_f(x,t)}{\partial t^2} - c^2 \frac{\partial^2 y_f(x,t)}{\partial x^2} = f(x,t)$$

$$\text{s.t } y_f(x,0) = 0, \quad \frac{\partial y_f}{\partial t}(x,0) = 0$$

We look for a Green's function $G(x,t,\xi,\tau)$ satisfying:

$$\frac{\partial^2 G(x,t,\xi,\tau)}{\partial t^2} - c^2 \frac{\partial^2 G(x,t,\xi,\tau)}{\partial x^2} = \delta(x-\xi)\delta(t-\tau)$$

Take FT's wrt x:

$$\frac{\partial^2 \tilde{G}(k,t,\xi,\tau)}{\partial t^2} + c^2k^2 \tilde{G}(k,t,\xi,\tau) = e^{-ik\xi}\delta(t-\tau)$$

$$\text{s.t } \tilde{G}, \ \frac{\partial \tilde{G}}{\partial t} = 0 \text{ at } t=0.$$

So, we are essentially finding a Green's function for $\frac{d^2y}{dt^2} + c^2k^2y = f(t)$, so Green's function satisfying

$$\frac{d^2G}{dt^2} + c^2k^2G = \underset{\uparrow \text{scale factor}}{\varepsilon}\,\delta(t-\tau)$$

This gives: $$\tilde{G}(k,t,\xi,\tau) = \frac{e^{-ik\xi}\sin(kc(t-\tau))\,\Theta(t-\tau)}{kc}$$

$$\Rightarrow G(x,t,\xi,\tau) = \frac{\Theta(t-\tau)}{2\pi c}\int_{-\infty}^{\infty} e^{ik(x-\xi)}\frac{\sin\{kc(t-\tau)\}}{k}\,dk$$

$$= \frac{\Theta(t-\tau)}{\pi c}\int_0^{\infty}\frac{\cos(k(x-\xi))\sin(kc(t-\tau))}{k}\,dk$$

$$= \frac{\Theta(t-\tau)}{2\pi c}\left\{\int_0^\infty \frac{\sin(k(x-\xi+c(t-\tau)))}{k}\,dk - \int_0^\infty \frac{\sin(k(x-\xi-c(t-\tau)))}{k}\,dk\right\}$$

$$= \frac{\Theta(t-\tau)}{\cancel{2\pi} 4c}\left\{\operatorname{sgn}(x-\xi+c(t-\tau)) - \operatorname{sgn}(x-\xi-c(t-\tau))\right\}$$

We see that $\Theta(t-\tau) \Rightarrow t > \tau$

Then, for $B > 0$, $\operatorname{sgn}(A+B) - \operatorname{sgn}(A-B) = 2\,\Theta(B-|A|)$

$$\Rightarrow G(x,t,\xi,\tau) = \frac{1}{2c}\,\Theta(c(t-\tau) - |x-\xi|)$$

We see that $G(x,t,\xi,\tau) \neq 0 \Rightarrow |x-\xi| < c(t-\tau)$, so a disturbance at $f(\xi,\tau)$ may only affect x for times $t > \tau + \frac{|x-\xi|}{c}$ i.e speed of propagation.

Now, $$y_f(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)\,d\xi\,d\tau$$

We may again apply Duhamel's principle with:

$$I(x,t) = \frac{1}{2c}\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)\,d\xi \qquad (t > \tau)$$

as the D'Alembert solution with initial data at time τ.

Poisson's Equation.

$$\nabla^2 u = -f(\underline{x})$$

on a domain D.

We define $\delta(\underline{r} - \underline{r_0}) = 0 \quad \forall\, \underline{r} \neq \underline{r_0}$

$$\int_D \delta(\underline{r}-\underline{r_0})\,dV = \begin{cases} 1 & \text{if } \underline{r_0} \in D \\ 0 & \text{otherwise} \end{cases}$$

and $$\int_D f(\underline{r})\,\delta(\underline{r}-\underline{r_0})\,dV = f(\underline{r_0}) \text{ if } \underline{r_0} \in D.$$

Free-space Green's function

The fundamental solution to Poisson's equation is the solution to:

$$\nabla^2 G(\underline{r};\underline{r_0}) = \delta(\underline{r}-\underline{r_0})$$

Since problem is rotationally symmetric about $\underline{r_0}$, solution can depend only on scalar distance from this point:

$$G(\underline{r};\underline{r_0}) = G(|\underline{r}-\underline{r_0}|) := G(r)$$

Define G_3 to be 3D Green's function and G_2 as 2D Green's function. Integrating over a sphere of radius r centred at $\underline{r_0}$ (or circle in 2D) we obtain.

$$\int_V \nabla^2 G_3\,dV = 1 = \int_{S_3} \nabla G_3(r)\cdot\underline{\hat{n}}\,dS$$

$$\int_S \nabla^2 G_2\,dV = 1 = \oint_{C} \nabla G_2(r)\cdot\underline{\hat{n}}\,dl$$

In both cases, $\nabla G\cdot\underline{\hat{n}} = \frac{dG}{dr}$

$$\Rightarrow \quad 4\pi r^2 \frac{dG_3}{dr} = 1, \quad 2\pi r\frac{dG_2}{dr} = 1$$

$$\Longrightarrow \quad G_3(r) = -\frac{1}{4\pi r} + C_3$$

$$G_2(r) = \frac{\log r}{2\pi} + C_2$$

In 3D we often apply far-field BC, $G_3(r)\to 0$ as $r\to\infty \Rightarrow C_3=0$ to get free-space Green's function:

$$G_3(r) = -\frac{1}{4\pi|\underline{r}-\underline{r_0}|}$$

N.B we must be careful using these singular functions throughout our domains.

Green's Identities

For two scalar fn's ϕ and ψ in some volume V with surface S,

$$\int_V \nabla\cdot(\phi\nabla\psi)\,dV = \int_V (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)\,dV$$
$$= \int_S \phi\nabla\psi\cdot\hat{\underline{n}}\,dS$$

Interchanging ϕ and ψ gives:

(Green's 2nd Identity).

$$\int_S\left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)dS = \int_V\{\phi\nabla^2\psi - \psi\nabla^2\phi\}\,dV$$

We wish to use $\psi = G_3$, but we must check that it still holds since div. theorem requires regularity of functions.

We see $\nabla^2 G_3 = \delta(\underline{r}-\underline{r_0}) \Rightarrow \int_V \phi\nabla^2 G_3\,dV = \phi(\underline{r_0})$

So: $\phi(\underline{r_0}) = \int_V G_3(\underline{r};\underline{r_0})\nabla^2\phi\,dV + \int_S\left(\phi(\underline{r})\frac{\partial G_3(\underline{r};\underline{r_0})}{\partial n} - G_3(\underline{r};\underline{r_0})\frac{\partial\phi(\underline{r})}{\partial n}\right)dS$

Now, define $\underline{r_0} = \underline{0}$ for simplicity, then define V_ε as V without ball, B_ε centred at origin, radius ε, with surface S_ε, cut out of V. Outside of V_ε, G_3 is non-singular and $\nabla^2 G_3 = 0$. So:

$$\int_{V_\varepsilon}(\phi\nabla^2 G_3 - G_3\nabla^2\phi)\,dV = -\int_{V_\varepsilon} G_3\nabla^2\phi\,dV$$
$$= \int_S\left(\phi\frac{\partial G_3}{\partial n} - G_3\frac{\partial\phi}{\partial n}\right)dS + \int_{S_\varepsilon}\left(\phi\frac{\partial G_3}{\partial n} - G_3\frac{\partial\phi}{\partial n}\right)dS$$

Now, $G_3 = -\frac{1}{4\pi\varepsilon}$ on S_ε, so;

$$-\int_{S_\varepsilon} G_3\frac{\partial\phi}{\partial n}\,dS = \frac{1}{4\pi\varepsilon}\int_{S_\varepsilon}\frac{\partial\phi}{\partial n}\,dS = \frac{4\pi\varepsilon^2}{4\pi\varepsilon}\bar{A}_\varepsilon$$

where $\bar{A}_\varepsilon$ is average value of $\frac{\partial\phi}{\partial n}$ on S_ε. Since ϕ is regular at origin, $\bar{A}_\varepsilon \to \left.\frac{\partial\phi}{\partial n}\right|_0$ as $\varepsilon\to 0$, so $-\int_{S_\varepsilon} G_3\frac{\partial\phi}{\partial n}\,dS \to 0$.

For first term of S_ε integral, derivative acts on $G_3 = -\frac{1}{4\pi r}$ and outward normal points in negative r direction

$$\Rightarrow \int_{S_\varepsilon} \phi \frac{\partial G_3}{\partial n} dS = -\frac{1}{4\pi\varepsilon^2}\int_{S_\varepsilon} \phi \, dS = -\bar{\phi}$$

where $\bar{\phi}$ is average value of ϕ on S_ε. $\bar{\phi} \to \phi(0)$ as $\varepsilon \to 0$

So: $$\int_{S_\varepsilon} \phi \frac{\partial G_3}{\partial n} dS \to -\phi(0) \quad \text{as } \varepsilon \to 0.$$

Then: $$\phi(\underline{r_0}) - \int_V G_3 \nabla^2\phi \, dV = \int_S \left(\phi \frac{\partial G_3}{\partial n} - G_3 \frac{\partial \phi}{\partial n}\right) dS \quad \text{in limit as } \varepsilon \to 0$$

as reqd.

Now, substitute $\nabla^2\phi = -f$, with $\phi = u$ into eqn to get:

$$u(\underline{r_0}) = \int_V (-f(\underline{r}))\, G_3(\underline{r};\underline{r_0})\, dV + \int_S \left\{ u \frac{\partial G_3(\underline{r};\underline{r_0})}{\partial n} - G_3(\underline{r};\underline{r_0}) \frac{\partial u}{\partial n} \right\} dS$$

Note: Setting $f=0$ gives soln. to Laplace's equation on interior of domain. in terms of values of u, $\frac{\partial u}{\partial n}$ on boundary, δD. If D is closed and bounded, then specification of u already uniquely determines u on the interior. and $\frac{\partial u}{\partial n}$ may not be freely specified. Similarly, $\frac{\partial u}{\partial n}$ on boundary determines u up to constant. (Dirichlet + Neumann)
Not a useful expression for solving for u in D but useful to prove further properties of harmonic functions.

We now look to solve Dirichlet boundary problem with Green's functions.

Comment on Neumann condition. Consistency condition must be satisfied.

Suppose $\frac{\partial u}{\partial n} = h(\underline{r})$ on ∂D, then by divergence theorem:

$$\int_{\partial D} h \, dS = \int_{\partial D} \nabla u \cdot \underline{n} \, dS = \int_D \nabla^2 u \, dV = -\int_D f \, dV$$

Dirichlet Green's Functions

Dirichlet Green's function for Laplacian operator on domain D is defined to be the function $G(\underline{r}, \underline{r_0})$ such that:

1) $G(\underline{r}, \underline{r_0}) = G_3(\underline{r}, \underline{r_0}) + H(\underline{r}, \underline{r_0})$

   where H is finite throughout D (including at $\underline{r} = \underline{r_0}$) and H satisfies Laplace's equation through D.

2) $G(\underline{r}, \underline{r_0}) = 0$ on ∂D, ie H modifies G_3 such that G is zero on ∂D.

We note that G also satisfies $\nabla^2 G = 0$ for $\underline{r} \neq \underline{r_0}$.

We may now, assuming existence of this G, find solution to Poisson's equation on domain D with dirichlet BC's.

Let $\nabla^2 u = -f$ in D with $u = h(\underline{r})$ on ∂D. Substituting $G_3 = G - H$, we have:

$$u(\underline{r_0}) = \int_{\partial D} \left\{ u \frac{\partial (G-H)}{\partial n} - (G-H) \frac{\partial u}{\partial n} \right\} dS + \int_D (-f)(G-H) \, dV$$

Now, H is Harmonic through D, so by Green's second identity:

$$\int_{\partial D} \left\{ u \frac{\partial H}{\partial n} - H \frac{\partial u}{\partial n} \right\} dS = -\int (-f) H \, dV$$

So, the H terms in the equation cancel and we get:

$$u(\underline{r_0}) = \int_{\partial D} \left(u(\underline{r}) \frac{\partial G(\underline{r};\underline{r_0})}{\partial n} - G(\underline{r};\underline{r_0}) \frac{\partial u(\underline{r})}{\partial n} \right) dS$$

$$+ \int_D (-f(\underline{r})) G(\underline{r};\underline{r_0})\, dV$$

Now, choosing H s.t $G = 0$ on ∂D, and $u(\underline{r}) = h(\underline{r})$ on ∂D we have:

$$u(\underline{r_0}) = \int_{\partial D} h(\underline{r}) \frac{\partial G(\underline{r};\underline{r_0})}{\partial n} dS + \int_D \{-f(\underline{r})\} G(\underline{r};\underline{r_0})\, dV$$

Method of Images:

For domains with sufficient symmetry, we may sometimes use method of images to construct G. It may also be used for forced heat + wave equations, as well as for homogeneous equations.

The key is to match BC's

e.g Laplace equation, Dirichlet Green's function for half-space.

Consider $D = \{(x,y,z) : z > 0\}$, we wish to find solution to:

$$\nabla^2 u = 0, \quad u(\underline{x}) \to 0 \text{ as } |\underline{x}| \to \infty, \quad u(x,y,0) = h(x,y).$$

We write $\underline{r} = \underline{x} = (x,y,z)$ and $\underline{r_0} = \underline{x_0}^+ = (x_0, y_0, z_0)$.

We know that G_3 satisfies all conditions except homogeneity.

Now, start from free-space Green's function, and try and add some other solutions to get required BC. Imagine there is image of the special point outside of D, same vertical distance away from boundary as special point is. To cancel values at boundary, consider image as point source, then define:

$$G(\underline{x};\underline{x}_0) = \frac{-1}{4\pi|\underline{x}-\underline{x}_0^+|} + \frac{1}{4\pi|\underline{x}-\underline{x}_0^-|}$$

opposite sign.

$$= -\frac{1}{4\pi}\left\{(x-x_0)^2+(y_0-y_0)^2+(z-z_0)^2\right\}^{\frac{1}{2}}$$

$$+\frac{1}{4\pi}\left\{(x-x_0)^2+(y-y_0)^2+(z+z_0)^2\right\}^{\frac{1}{2}}.$$

Since $x_0^- \notin D$, second term satisfies laplace's equation everywhere in D so we have condition (i). Also BC's now satisfied on $\frac{1}{4\pi|\underline{x}-\underline{x}_0^-|}$ remains finite for any $\underline{x}\in D$.

So we have Dirichlet Green's function.

Now, $\left.\frac{\partial G}{\partial n}\right|_{z=0} = -\left.\frac{\partial G}{\partial z}\right|_{z=0}$ (negative z-direction = outward normal)

$$= \frac{1}{4\pi}\left\{\frac{z+z_0}{|\underline{x}-\underline{x}_0^-|^3} - \frac{z-z_0}{|\underline{x}-\underline{x}_0^+|^3}\right\}$$

$$= \frac{z_0}{2\pi}\left\{(x-x_0)^2+(y-y_0)^2+z_0^2\right\}^{-\frac{3}{2}}$$

So: $$u(x_0,y_0,z_0) = \frac{z_0}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{(x-x_0)^2+(y-y_0)^2+z_0^2\right\}^{-\frac{3}{2}} h(x,y)\,dx\,dy.$$

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e.g Images for wave problems.

We now apply method of images to Green's functions for forced wave + heat eqns. e.g consider domain:

$$D=\{0\le x<\infty\} \quad \text{with Dirichlet condition } G(0,t,\xi,\tau)=0$$

Then for image, we add equal amplitude Green's function with opposite sign at reflected special point.

So: $$G(x,t,\xi,\tau) = \frac{\Theta(c(t-\tau)-|x-\xi|)}{2c} - \frac{\Theta(c(t-\tau)-|x+\xi|)}{2c}$$

Similarly, for homogeneous Neuman BC, we make derivative an odd function

$$\left.\frac{\partial G}{\partial x}\right|_{x=0} = 0$$

i.e $G(x,t,\xi,\tau) = \dfrac{\Theta(c(t-\tau)-|x-\xi|)}{2c} + \dfrac{\Theta(c(t-\tau)-|x+\xi|)}{2c}$

and the image has same sign.

For sufficiently small, +ve x; $|x-\xi| = \xi - x$, $|x+\xi| = x+\xi$

So; $\left.\dfrac{\partial G}{\partial x}\right|_{x=0} = \dfrac{1}{2c}\{\delta(c(t-\tau)-|0-\xi|)[1] + \delta(c(t-\tau)-|0+\xi|)(-1)$
$= 0.$