The Liouville-Green and WKB Approximations.

Send corrections to David Stuart dmas2@cam.ac.uk

I Liouville-Green Approximations

We now consider asymptotic analysis of solutions of equations of the form

$$\epsilon^2 y'' + q(x)y = 0\,.$$

We will consider the asymptotic description of solutions when $\epsilon \to 0$. We first note that ostensibly more general equations can be put into this form.

Exercise 1. Show that the equation $w'' + P(x)w' + Q(x)w = 0$ can be converted into the form $y'' + q(x)y = 0$ by a transformation $w(x) = y(x)e^{-\frac{1}{2}\int^x P(s)ds}$, and find q in terms of P and Q. Show that rescaling $x \to x/\epsilon$ converts $y'' + qy = 0$ into $\epsilon^2 y'' + qy = 0$.

To understand the nature of the asymptotic regime $\epsilon \to 0$, recall that if q is a constant then we can write down a solution of this equation; $w(x) = Ae^{iS}$ where $S = \epsilon^{-1}q^{\frac{1}{2}}x$. When $q > 0$ solutions are oscillatory with wavelength $2\pi\epsilon q^{-\frac{1}{2}}$. Thus when ϵ is very small with q a fixed function we are studying the regime in which the wavelength is very small compared to length scales over which q varies - it is in this situation that the Liouville-Green and WKBJ methods can be applied.

The idea roughly is to treat q as "locally approximately constant". To see what this means, search for a solution of the form $y(x) = a(x)e^{\frac{i}{\epsilon}S(x)}$. We think of this as describing waves which have local, near to the point x, wavelength $\approx 2\pi\epsilon/S'(x)$ so that both S and a are approximately constant over a wavelength (since they are functions of x not ϵx.) Substituting we get, by equating terms proportional to ϵ^0 and ϵ^1, the following two conditions:

$$\begin{aligned} -a(S')^2 + qa &= 0 \\ 2a'S' + aS'' &= 0. \end{aligned}$$

The $O(1)$ equation gives $S = \pm\int^z q^{\frac{1}{2}}\,dz$ and the $O(\epsilon)$ equation gives $a = q^{-\frac{1}{4}}$ (after integrating and using the $O(1)$ equation to substitute for S'). This leads to the Liouville-Green approximations to the differential equation:

$$y(x) \sim q^{-\frac{1}{4}}\left\{A_+ \exp\left(\frac{i}{\epsilon}\int^x q^{\frac{1}{2}}\,dx\right) + A_- \exp\left(-\frac{i}{\epsilon}\int^x q^{\frac{1}{2}}\,dx\right)\right\}\,, \tag{I.1}$$

when $q > 0$; if $q < 0$ a more convenient form is

$$y(x) \sim |q|^{-\frac{1}{4}}\left\{a_+ \exp\left(\frac{1}{\epsilon}\int^x |q|^{\frac{1}{2}}\,dx\right) + a_- \exp\left(-\frac{1}{\epsilon}\int^x |q|^{\frac{1}{2}}\,dx\right)\right\}. \tag{I.2}$$

Of course in the first (oscillatory) case cosines and sines can also be used.

Exercise 2. Check that you can relate this derivation to the one given previously in lectures, and extended in question 2 of the third sheet.

II The Airy Functions

The Airy functions are solutions of the equation $y''(x) = xy(x)$. Thus using the notation above,

$$q = -x\,,\ q^{\frac{1}{2}} = \pm i x^{\frac{1}{2}} \quad \text{and} \int \sqrt{q}\, dx = \pm i \frac{2}{3} x^{\frac{3}{2}}\,.$$

Substituting in to the Liouville-Green approximations above we are led to expect the asymptotic behaviour of the solutions of Airy's equation to be given by the following possibilities[1]:

$$\begin{aligned} y(x) &\sim x^{-\frac{1}{4}} \exp\left(\pm \frac{2}{3} x^{\frac{3}{2}}\right) & \qquad & x \to +\infty \\ &\sim |x|^{-\frac{1}{4}} \begin{matrix}\cos\\ \sin\end{matrix} \left(\frac{2}{3}\,|x|^{\frac{3}{2}}\right) & & x \to -\infty. \end{aligned}$$

This argument does not relate the asymptotic behaviour to the initial conditions imposed on the solution at (say) $x = 0$, and does not determine the constants in the asymptotics (and thus the phases in the oscillatory case are also left undetermined.) A more rigorous approach, which allows identification of the constants as well as proofs, is based on the integral representation for solutions of Airy's equation provided by the Laplace transform. It is first necessary to fix two particular solutions of the Airy equation which are convenient to work with. Recall that the origin is an ordinary point for the Airy equation, and there are two power series solutions, with infinite radius of convergence, linear combinations of which are used to define the Airy functions:

$$\begin{aligned} \mathrm{Ai}\,(x) &= 3^{-\frac{2}{3}} \sum_{n=0}^{\infty} \frac{x^{3n}}{3^{2n} n! \Gamma(n+\frac{2}{3})} - 3^{-\frac{4}{3}} \sum_{n=0}^{\infty} \frac{x^{3n+1}}{3^{2n} n! \Gamma(n+\frac{4}{3})}\,, \quad \text{and} \\ \mathrm{Bi}\,(x) &= 3^{-\frac{1}{6}} \sum_{n=0}^{\infty} \frac{x^{3n}}{3^{2n} n! \Gamma(n+\frac{2}{3})} + 3^{-\frac{5}{6}} \sum_{n=0}^{\infty} \frac{x^{3n+1}}{3^{2n} n! \Gamma(n+\frac{4}{3})}\,. \end{aligned}$$

These rather specific linear combinations are chosen on account of their behaviour at infinity. This can be analyzed via the following integral formulae:

$$\mathrm{Ai}\,(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(xs + \frac{1}{3}s^3)}\, ds = \frac{1}{\pi} \int_0^{\infty} \cos[xs + \frac{1}{3}s^3]\, ds\,, \quad x \in \mathbb{R}. \tag{I.3}$$

and

$$\mathrm{Bi}\,(x) = \frac{1}{\pi} \int_0^{\infty} \exp[xs - \frac{1}{3}s^3] + \sin[xs + \frac{1}{3}s^3]\, ds\,, \quad x \in \mathbb{R}. \tag{I.4}$$

To check these agree with the power series above it is sufficient to prove that they solve Airy's equation $y'' = xy$ and that the initial conditions at $x = 0$ for y and y' agree (since it is a second order linear equation.) For this it is best first to complexify the integrand to ensure rapid convergence. (As they stand, the integrals above are not absolutely convergent, and are only defined as improper integrals.) So consider first of all Ai (x), and replace s by $z = |z| \exp(i\theta)$ and consider the integral

$$\int_C e^{i\left(xz + \frac{z^3}{3}\right)}\, dz,$$

[1] Recall that in the oscillatory case this is strictly speaking not an asymptotic relation, as discussed in Stationary Phase notes.

where C is a contour in the complex $z-$plane. For large $|z|$ convergence requires $\sin(3\theta) > 0$, or

$$0 < \theta < \frac{\pi}{3} \text{ or } \frac{2\pi}{3} < \theta < \pi \text{ or } \frac{4\pi}{3} < \theta < \frac{5\pi}{3},$$

convergence being most rapid along the three rays $e^{i\pi/6}|z|$, $e^{i5\pi/6}|z|$ or $e^{-i\pi/2}|z|$ which bisect these three sectors. Applying the Jordan Lemma we can deform the contour along the real axis to a contour which for large $|z|$ lies in the domains shown.

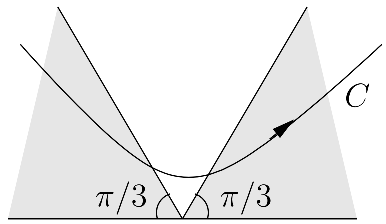

Figure 1: Path for Airy function

So let C be a path as shown asymptotic to the rays $e^{i\pi/6}|z|$ and $e^{i5\pi/6}|z|$. The integrand is now rapidly decreasing at infinity in the directions $e^{i\pi/6}$ and $e^{i5\pi/6}$, and differentiation through the integral is allowed and yields

$$\left(\frac{d^2}{dx^2} - x\right) Ai(x) = \frac{1}{2\pi}\int_C -(z^2+x)e^{i\left(xz+\frac{1}{3}z^3\right)}dz = \frac{i}{2\pi}\int_C \left(\frac{d}{dz}e^{i\left(xz+\frac{z^3}{3}\right)}\right) dz = 0 .$$

It follows that the integral definition of Ai (x) satisfies the Airy equation $-\frac{d^2y}{dx^2} + xy = 0$. It is now straightforward to check the initial conditions agree with those for the power series definition, and the identity of the two definitions is proved.

Exercise 3. Find a similar contour in the complex plane which is composed of paths which are asymptotic to one of the rays $e^{i\pi/6}|z|$, $e^{i5\pi/6}|z|$ or $e^{-i\pi/2}|z|$, such that the integral along this contour gives Bi , and hence show that the path integral above defines a solution of the Airy equation which can be identified with the power series definition of Bi (x) given above.

Here are the leading terms in the asymptotic expansions for these functions:

$$\text{Ai}\,(x) \;=\; \frac{e^{-\frac{2}{3}x^{\frac{3}{2}}}}{2\sqrt{\pi}x^{\frac{1}{4}}}\left(1 + O(|x|^{-\frac{3}{2}})\right), \qquad (x \to +\infty)\,, \tag{7.4}$$

and

$$\text{Ai}\,(x) \;=\; \frac{1}{\sqrt{\pi}|x|^{\frac{1}{4}}}\left(\sin\left(\frac{2}{3}|x|^{\frac{3}{2}} + \frac{\pi}{4}\right) + O(|x|^{-\frac{3}{2}})\right), \qquad (x \to -\infty)\,. \tag{7.5}$$

For Bi the asymptotics are

$$\text{Bi}\,(x) \;=\; \frac{e^{+\frac{2}{3}x^{\frac{3}{2}}}}{\sqrt{\pi}x^{\frac{1}{4}}}\left(1 + O(|x|^{-\frac{3}{2}})\right), \qquad (x \to +\infty)\,, \tag{7.6}$$

and

$$\mathrm{Bi}\,(x) \;=\; \frac{1}{\sqrt{\pi}|x|^{\frac{1}{4}}}\left(\cos\left(\frac{2}{3}|x|^{\frac{3}{2}}+\frac{\pi}{4}\right)+O(|x|^{-\frac{3}{2}})\right), \qquad (x\to-\infty)\,. \tag{7.8}$$

(The difference in factor of two in the dominant terms as $x \to +\infty$ is not a misprint. The complete series can be found in Bender and Orszag.

The leading terms given above can all be derived by techniques developed for asymptotic analysis of integrals. For example, to get the leading behaviour as $x \to +\infty$ for Ai (x) it is convenient to first rescale: introduce $\lambda = x^{\frac{3}{2}}$ as large parameter, and change the integration variable with $s = x^{\frac{1}{2}}t$, leading to the integral

$$\mathrm{Ai}\,(\lambda^{\frac{2}{3}}) \;=\; \int_C \exp[\lambda h(t)]\,dt\,, \qquad h(t) = i(t+\frac{t^3}{3})\,.$$

The saddle points for h are $\pm i$, so we wish to deform C into a path of steepest descent through $+i$. Notice that $h(i) = -\frac{2}{3}$. Writing $t = \sigma + i\tau$ we find the paths on which $\Im h = \Im h(i) = 0$ are determined by

$$\sigma(\sigma^2 - 3\tau^2 + 3) = 0\,.$$

The vertical line $\sigma = 0$, is a path of steepest ascent while the two branches of the hyperbola $\tau = 3^{-\frac{1}{2}}\sqrt{\sigma^2+3}$ are paths of steepest descent. Notice that these branches have slope $\frac{d\tau}{d\sigma} = \frac{\sigma}{\sqrt{3\sigma^2+9}}$ and so are asymptotic to the rays $e^{i\pi/6}|z|$, $e^{i5\pi/6}|z|$ discussed above, and Jordan's Lemma allows us to choose the contour formed from these two paths as C. Using τ to parametrize the integral, i.e., writing $t = \pm\sqrt{3}(\tau^2-1)^{\frac{1}{2}} + i\tau$ on C, and noting that the imaginary contributions cancel by symmetry, we are led to study the Laplace integral

$$\mathrm{Ai}\,(\lambda^{\frac{2}{3}}) \;=\; \frac{\lambda^{\frac{1}{3}}}{2\pi}\times 2\int_1^\infty \exp[\lambda(2\tau-\frac{8}{3}\tau^3)]\frac{\sqrt{3}\tau}{\sqrt{\tau^2-1}}\,d\tau\,.$$

With the changes of variable $\tau = 1 + u = 1 + v^2$ we find the dominant contribution is

$$\mathrm{Ai}\,(\lambda^{\frac{2}{3}}) \;\sim\; \frac{\lambda^{\frac{1}{3}}}{\pi}\sqrt{\frac{3}{2}}e^{-\frac{2\lambda}{3}}\int_0^\infty u^{-\frac{1}{2}}e^{-6\lambda u}\,du \;=\; \frac{\lambda^{\frac{1}{3}}}{\pi}\sqrt{\frac{3}{2}}e^{-\frac{2\lambda}{3}}\int_0^\infty 2\,e^{-6\lambda v^2}\,dv \;=\; \frac{\lambda^{-\frac{1}{6}}e^{\frac{-2\lambda}{3}}}{2\sqrt{\pi}}$$

which gives the leading asymptotic term displayed above for Ai (x) after substituting $\lambda = x^{\frac{3}{2}}$.

Exercise 4. Use the method of stationary phase to obtain the leading term in the asymptotic expansion as $x \to -\infty$ of Ai $(x$.

III The Connection Problem in the WKB Approximation

Consider now the problem of obtaining approximate solutions in the asymptotic regime $\epsilon << 1$ to an equation of the form $\epsilon^2 y''(x) + q(x)y(x)$, in the case that q changes sign. For simplicity assume that $q > 0$ when $x < 0$ and $q < 0$ when $x > 0$. In the regions $\mathrm{I} = \{x : x > 0\}$ and $\mathrm{III} = \{x : x < 0\}$ there are approximations given by the Liouville-Green method:

$$y_{\mathrm{I}}^{LG}(x) = |q|^{-\frac{1}{4}}\left(a_+\exp\left(\frac{1}{\epsilon}\int^x |q|^{\frac{1}{2}}\;dx\right) + a_-\exp\left(-\frac{1}{\epsilon}\int^x |q|^{\frac{1}{2}}\;dx\right)\right). \tag{I.5}$$

and, for $x < 0$,

$$y_{\rm III}^{LG}(x) \;=\; |q|^{-\frac{1}{4}}\left(A_+\exp\left(\frac{i}{\epsilon}\int^x |q|^{\frac{1}{2}}\,dx\right) + A_-\exp\left(-\frac{i}{\epsilon}\int^x |q|^{\frac{1}{2}}\,dx\right)\right)\,, \tag{I.6}$$

or (more conveniently as will soon become clear)

$$y_{\rm III}^{LG}(x) \;=\; |q|^{-\frac{1}{4}}\left(D\sin\left(\frac{1}{\epsilon}\int_x^0 |q|^{\frac{1}{2}}\,dx + \frac{\pi}{4}\right) + E\cos\left(\frac{1}{\epsilon}\int_x^0 |q|^{\frac{1}{2}}\,dx + \frac{\pi}{4}\right)\right)\,. \tag{I.7}$$

The connection problem is how to relate the constants in these approximate solutions on either side of the *turning point* at which $q = 0$ - this case $x = 0$ is the only turning point. (In general there may be several.) The idea is to assume that $q'(0) = -k < 0$ and that *in a region* $\rm II$ *very close to the turning point the solution can be approximated by solutions of the Airy equation*, appropriately rescaled:

$$y_{\rm II}^{Airy}(x) \;=\; \alpha\,\mathrm{Ai}\left(\left(\frac{k}{\epsilon^2}\right)^{\frac{1}{3}} x\right) \;+\; \beta\,\mathrm{Bi}\left(\left(\frac{k}{\epsilon^2}\right)^{\frac{1}{3}} x\right)\,. \tag{I.8}$$

The asymptotics of the Airy function allows us to match this "inner" solution, valid in a small interval around the turning point, with the outer Liouville-Green solutions. The point is that as $|x| \to 0$ we have the approximation $\int_0^{\pm|x|}\sqrt{|q|}dx \;\sim\; \pm\frac{2}{3}\sqrt{k}|x|^{\frac{3}{2}}$. This gives us the following approximations close to the turning point:

$$y_{\rm I}^{LG}(x) \;\sim\; |kx|^{-\frac{1}{4}}\left(a_+\exp\left(\frac{2}{3\epsilon}\sqrt{k}|x|^{\frac{3}{2}}\right) + a_-\exp\left(-\frac{2}{3\epsilon}\sqrt{k}|x|^{\frac{3}{2}}\right)\right)\,, \qquad (x \to 0^+)\,, \tag{I.9}$$

and

$$y_{\rm III}^{LG}(x) \sim |kx|^{-\frac{1}{4}}\left(D\sin\left(\frac{2}{3\epsilon}\sqrt{k}|x|^{\frac{3}{2}} + \frac{\pi}{4}\right) + E\cos\left(\frac{2}{3\epsilon}\sqrt{k}|x|^{\frac{3}{2}} + \frac{\pi}{4}\right)\right)\,, \qquad (x \to 0^-)\,. \tag{I.10}$$

However, for large values of $\left(\frac{k}{\epsilon^2}\right)^{\frac{1}{3}}$ it is possible to expand $y_{\rm II}^{Airy}$ using the Airy function asymptotics as

$$y_{\rm II}^{Airy}(x) \;=\; \alpha\,\frac{e^{-\frac{2}{3}\left(\frac{k}{\epsilon^2}\right)^{\frac{1}{2}}|x|^{\frac{3}{2}}}}{2\sqrt{\pi}\left(\frac{k}{\epsilon^2}\right)^{\frac{1}{12}}|x|^{\frac{1}{4}}}\left(1 + O(\epsilon|x|^{-\frac{3}{2}})\right) \;+\; \beta\,\frac{e^{+\frac{2}{3}\left(\frac{k}{\epsilon^2}\right)^{\frac{1}{2}}|x|^{\frac{3}{2}}}}{\sqrt{\pi}\left(\frac{k}{\epsilon^2}\right)^{\frac{1}{12}}|x|^{\frac{1}{4}}}\left(1 + O(\epsilon|x|^{-\frac{3}{2}})\right)$$

when $\epsilon^{-\frac{2}{3}}x \to +\infty$, and

$$\begin{aligned} y_{\rm II}^{Airy}(x) \;=\; &\alpha\,\frac{1}{\sqrt{\pi}\left(\frac{k}{\epsilon^2}\right)^{\frac{1}{12}}|x|^{\frac{1}{4}}}\left(\sin\left(\frac{2}{3}\left(\frac{k}{\epsilon^2}\right)^{\frac{1}{2}}|x|^{\frac{3}{2}} + \frac{\pi}{4}\right) + O(\epsilon|x|^{-\frac{3}{2}})\right) \\ &+\;\beta\,\frac{1}{\sqrt{\pi}\left(\frac{k}{\epsilon^2}\right)^{\frac{1}{12}}|x|^{\frac{1}{4}}}\left(\cos\left(\frac{2}{3}\left(\frac{k}{\epsilon^2}\right)^{\frac{1}{2}}|x|^{\frac{3}{2}} + \frac{\pi}{4}\right) + O(\epsilon|x|^{-\frac{3}{2}})\right) \end{aligned}$$

when $\epsilon^{-\frac{2}{3}}x \to -\infty$.

The crucial point in this is the existence of an overlap region, defined by

$$\boxed{\epsilon^{\frac{2}{3}} \ll |x| \ll 1}$$

in which both types of asymptotic expansion are valid and can be compared. This leads to the *connection relations*:

$$a_+ = \beta\,\frac{(k\epsilon)^{\frac{1}{6}}}{\sqrt{\pi}} \qquad a_- = \alpha\,\frac{(k\epsilon)^{\frac{1}{6}}}{2\sqrt{\pi}}$$
$$D = \alpha\,\frac{(k\epsilon)^{\frac{1}{6}}}{\sqrt{\pi}} \qquad E = \beta\,\frac{(k\epsilon)^{\frac{1}{6}}}{\sqrt{\pi}}\,,$$

which imply that $a_+ = E$ and $2a_- = D$. This is the solution of the connection problem.

We note for use in the next example the following two consequences

(C1) The WKB approximate solution with a turning point at $x = 0$ which is given by

$$\frac{a_-}{|q(x)|^{\frac{1}{4}}}\exp\left[-\frac{1}{\epsilon}\int_0^x |q(t)|^{\frac{1}{2}}dt\right], \quad \text{for} \quad x > 0\,,$$

equals

$$\frac{2a_-}{|q(x)|^{\frac{1}{4}}}\sin\left[\frac{1}{\epsilon}\int_x^0 |q(t)|^{\frac{1}{2}}dt + \frac{\pi}{4}\right], \quad \text{when} \quad x < 0\,.$$

(C2) Reflecting in the origin (changing x to $-x$) we find similarly that if q is positive for $x > 0$ and negative for $x < 0$ the WKB approximate solution which is given by

$$\frac{b_-}{|q(x)|^{\frac{1}{4}}}\exp\left[-\frac{1}{\epsilon}\int_x^0 |q(t)|^{\frac{1}{2}}dt\right], \quad \text{for} \quad x < 0\,,$$

equals

$$\frac{2b_-}{|q(x)|^{\frac{1}{4}}}\sin\left[\frac{1}{\epsilon}\int_0^x |q(t)|^{\frac{1}{2}}dt + \frac{\pi}{4}\right], \quad \text{when} \quad x > 0\,.$$

Example: Quantum Mechanical Bound States - a two turning point problem. Consider the time–independent Schrödinger equation

$$\frac{d^2\psi}{dx^2} + \lambda^2\, q(x)\,\psi(x) = 0\,,$$

where $\lambda \gg 1$ denotes $\hbar^{-1}$ and $q(x)$ denotes $2m\big(E - V(x)\big)$. Suppose that

$$q(x) > 0 \quad \text{for} \quad a < x < b\,,$$

$$\text{and} \quad q(x) < 0 \quad \text{for} \quad -\infty < x < a \quad \text{and} \quad b < x < \infty\,,$$

and consider a bound state $\psi(x)$. Write down the possible Liouville–Green approximate solutions for $\psi(x)$ in each region, given that $\psi \longrightarrow 0$ as $|x| \longrightarrow \infty$.

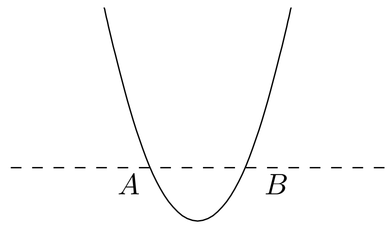

Figure 2:

Assume that $q(x)$ may be approximated by $q'(A)\,(x-A)$ near $x = A$, where $q'(A) > 0$, and by $q'(B)\,(x-B)$ near $x = B$, where $q'(B) < 0$.

Show that the energies E of bound states should be given approximately by

$$\lambda \int_A^B q^{\frac{1}{2}}(x) \;=\; \left(n + \frac{1}{2}\right)\pi \qquad\qquad (n \;=\; 0, 1, 2, \;\ldots\;).$$

This is known as the the WKB quantization condition, and can be thought of as a proper quantum mechanical justification of the Bohr-Sommerfeld rule of the "old quantum theory".

To solve this problem we make use of the two consequences of the connection problem just given. Since we are interested in normalizable solutions, we aim to find a WKB approximate solution which decays both as $x \to +\infty$ and as $x \to -\infty$. To achieve this we just translate the origin to obtain what we need from (C1) and (C2):

- The WKB approximate solution with a turning point at $x = B$ which decays like

  $$\frac{C_1}{|q(x)|^{\frac{1}{4}}} \exp\left[-\lambda \int_B^x |q(t)|^{\frac{1}{2}} dt\right], \quad (x \to +\infty),$$

  is given by

  $$\frac{2C_1}{|q(x)|^{\frac{1}{4}}} \sin\left[\lambda \int_x^B |q(t)|^{\frac{1}{2}} dt + \frac{\pi}{4}\right], \quad \text{when} \quad A < x < B.$$

- Similarly the WKB approximate solution with a turning point at $x = A$ which decays like

  $$\frac{C_2}{|q(x)|^{\frac{1}{4}}} \exp\left[-\lambda \int_x^A |q(t)|^{\frac{1}{2}} dt\right], \quad (x \to -\infty),$$

  is given by

  $$\frac{2C_2}{|q(x)|^{\frac{1}{4}}} \sin\left[\lambda \int_A^x |q(t)|^{\frac{1}{2}} dt + \frac{\pi}{4}\right], \quad \text{when} \quad A < x < B.$$

In order to match the above solutions in $A < x < B$ these two possible forms for the solution in the interval $A < x < B$ must agree up to multiplication by a constant. Noting that

$$\begin{aligned}\sin\left[\lambda \int_x^B |q(t)|^{\frac{1}{2}} dt + \frac{\pi}{4}\right] &= -\sin\left[\lambda \int_B^x |q(t)|^{\frac{1}{2}} dt - \frac{\pi}{4}\right] \\ &= -\sin\left\{\lambda \int_A^x |q(t)|^{\frac{1}{2}} dt + \frac{\pi}{4} - \left[\lambda \int_A^B |q(t)|^{\frac{1}{2}} dt + \frac{\pi}{2}\right]\right\},\end{aligned}$$

we see that the condition

$$\lambda \int_A^B |q(x)|^{\frac{1}{2}} dx + \frac{\pi}{2} \;=\; n\pi \quad n \in \mathbb{Z}$$

must hold in order to have a normalizable WKB approximate solution. If $q(x) = 2m(E - V(x))$, then

$$\sqrt{2m}\lambda \int_A^B \sqrt{E - V(x)} dx = (n + \frac{1}{2})\pi, \quad n = 0, 1, 2, \cdots .$$

Exercise 5. Work the approximate eigenvalues out in the case of the harmonic oscillator

$$V(x) \;=\; \frac{1}{2} m\omega^2 x^2$$

and, remembering $\lambda \;=\; \hbar^{-1}$, compare with the exact eigenvalues from your Quantum Mechanics IB notes.