Cosmology

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Useful Information

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Text books: Textbooks that I have used in putting together these notes:

- *The Primordial Density Perturbation*, by Lyth and Liddle.
- *Primordial Cosmology*, by Peter and Uzan.
- *The Early Universe*, by Kolb and Turner.
- *Cosmology*, by Weinberg.

I would also like to thank Anthony Lewis, and Ed Copeland, Adam Moss and Tasos Avgoustidis for sharing their lecture notes.

Notation

- I use the $(-,+,+,+)$ metric convention.
- I use units where the speed of light $c = 1$, $\hbar = 1$ and the Boltzmann constant $k_B = 1$.

Contents

1 Introduction

Cosmology is the study of the universe as a whole system. It brings together General Relativity, particle physics, statistical physics and more. Our understanding of the universe is built on large, precision data sets. Like astronomy, cosmology is a purely observational science - we will only ever be able to see one universe.

The standard model of cosmology is ΛCDM, this says that the dominant components of our universe today are Cold Dark Matter, and a cosmological constant Λ, with sub-dominant amounts of baryonic matter and radiation. The evolution of these components in space-time is governed by general relativity, at early times the universe was very hot and dense, and it has been expanding and cooling ever since. As a result of this we can infer that at early times the universe contained a soup of relativistic fundamental particles. As the universe expands we pass through the electroweak phase transition, at around 1 TeV, quarks come together to form hadrons at around 1 GeV, these hadrons come together to form (light) atomic nuclei in a process called Big Bang Nucleosynthesis at around 1 MeV, eventually the universe has cooled so much that it becomes dominated by non-relativistic matter, known as the onset of matter domination, at around 10 eV, at this time structures can start to form. At around 1 eV electrons are captured by nuclei to form atoms, and photons 'decouple' from matter, a process known as recombination (this is a terrible choice of name). The universe continues to cool and structures continue to grow, until we arrive when energy scale is around 1 meV, and the universe is beginning to be dominated by the cosmological constant.

Today the energy budget of the universe is 68% dark energy / cosmological constant, 27% cold dark matter, and 5% 'ordinary matter' (this includes all of the baryonic matter, photons and neutrinos).

The observations that support this picture of the universe are too vast in number and type to list here. However key observations include:

- Detailed measurements of the Cosmic Microwave Background Radiation, which is the light that has been free streaming to us since recombination. This gives us both information about the state of the universe at recombination, from which we can infer the properties of the universe at earlier times, and also how it has evolved since.

- Observations of type 1a supernovae which are standard (standardise-able) candles, meaning they always have the same brightness, can be used to directly reconstruct the expansion history of the universe. These observations were key in determining that we do not live in a universe that is dominated by matter today.

- Observations of the distribution and growth of structure in the universe. This tells us about the initial conditions for matter fluctuations, the evolution of the universe, and whether we are correct to assume general relativity as our theory of gravity.

- Observations of galaxies, in particular their rotation and collision, tell us that there is a matter like component that is 'dark' (does not interact with photons) which clusters on galactic scales.

There are many fundamental questions still to be answered about our universe:

- What is the nature of dark matter?
- Why is the value of the cosmological constant so much smaller than particle physics energy scales?
- Why is there more matter than anti-matter?
- What set the initial conditions for the universe? Is the theory of inflation correct?

These notes are not a comprehensive cosmology course. Instead I will try to give you a flavour of the way that cosmology is done, as well as an overview of the history of the universe, and an idea of what's exciting right now.

2 Background Cosmology

We will assume that the evolution of cosmological space-time is well described by general relativity, and that it is, to a good approximation, homogeneous (the same at all points in space) and isotropic (the same in all directions). These assumptions can be checked against current data and are found to hold.[1] This implies that the spatial curvature is the same at all points in space, and only a function of time. It is easiest to parameterise this in terms of the **scale factor** $a(t)$ which determines the 'size' of the universe relative to today.

The result of these assumptions is that the metric describing a universe of this type has the form

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - Kr^2} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right) , \tag{1}$$

where K is the spatial curvature. $K > 0$ means a spherical geometry, $K = 0$ a flat geometry and $K < 0$ a hyperbolic geometry. This is the **Friedmann-Lemaitre-Robertson-Walker metric**, abbreviated as FLRW, or sometimes FRW, metric. We will assume $K = 0$ for these lectures, again this is an assumption that is well supported by current data. We normalise the scale factor so that $a(t_0) = 1$ where t_0 is today.

2.1 Redshift

Light traveling through the universe follows null geodesics of the metric in Equation (1). These are defined by

$$\int_{t_e}^{t_r} \frac{1}{a(t)} dt = \int_0^r dr , \tag{2}$$

where t_e and t_r are the times at which a wave was emitted and received respectively.

[1] There are a number of cosmological anomalies that *could* be explained by relaxing one of these assumptions, but the significance of these anomalies is weak and/or there are a number of other possible explanations for them.

Assume another wave is emitted at $t_e + 1/\nu_e$, where ν_e is the emission frequency, and received at $t_r + \delta t_r$. If we assume the time between the emission of the waves is short, then the distance they travel between being emitted and received is unchanged.

$$\int_{t_e}^{t_r} \frac{dt}{a(t)} = \int_{t_e+1/\nu_e}^{t_r+\delta t_r} \frac{dt}{a(t)} = r \; . \tag{3}$$

Rearranging this expression we find that

$$\frac{1}{\nu_e a(t_e)} = \frac{\delta t_r}{a(t_r)} \; , \tag{4}$$

writing $\delta t_r = 1/\nu_r$ we find

$$\frac{\nu_e}{\nu_r} = \frac{a(t_r)}{a(t_e)} \; . \tag{5}$$

If the signal is received on Earth 'today' then we define a new quantity, the **redshift**

$$1 + z = \frac{a(t_0)}{a(t_e)} = \frac{\nu_e}{\nu_0} \; . \tag{6}$$

2.2 The Friedmann Equation(s)

The Einstein equations with a cosmological constant are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu} \; . \tag{7}$$

If we assume that matter is a perfect fluid

$$T^{\mu}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \; , \tag{8}$$

where ρ and p are functions only of time, and that the metric has FLRW form then the Einstein equations become **the Friedmann equation**

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} \; , \tag{9}$$

and the **acceleration equation**

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \; . \tag{10}$$

We use an over-dot to denote differentiation with respect to time. The **Hubble 'constant'** is defined as $H = \dot{a}/a$.

To close this system of equations we also need the **conservation equation** $\nabla_\mu T^\mu_\nu = 0$, which becomes

$$\dot{\rho} + 3H(\rho + p) = 0 \ , \tag{11}$$

and an equation of state for the matter fluid, so that $p = w\rho$.

Example 1: A cosmological constant dominated universe. The Friedmann equation becomes

$$\frac{\dot{a}}{a} = \sqrt{\frac{\Lambda}{3}} \ , \tag{12}$$

and we find that

$$a(t) = \exp\left(\sqrt{\frac{\Lambda}{3}}(t - t_0)\right) \ , \tag{13}$$

and the universe expands exponentially.

Example 2: A matter dominated universe with constant equation of state, $p = w\rho$. The conservation equation becomes

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a}(1 + w) \ , \tag{14}$$

to give

$$\rho = \rho_0 a^{-3(1+w)} \ , \tag{15}$$

then, using the Friedmann equation, we find

$$a(t) \propto t^{\frac{2}{3(1+w)}} \ . \tag{16}$$

For non-relativistic matter $w = 0$ and

$$\rho(t) = \rho_0 a^{-3} \ , \tag{17}$$

so that the matter density dilutes with the volume of space, and we have

$$a \propto t^{2/3} \ . \tag{18}$$

For relativistic matter we have $w = 1/3$ and

$$\rho(t) = \rho_0 a^{-4} \ , \tag{19}$$

there is an additional dilution of the energy density because the frequency of the radiation is also redshifted, and we have

$$a \propto t^{1/2} \tag{20}$$

Note that we recover the same evolution as the cosmological constant case when $w = -1$.

2.3 Useful Reference Quantities

It is helpful to define the **critical density**

$$\rho_c = \frac{3H^2}{8\pi G} , \tag{21}$$

which is the total density in a flat universe. We also define the **density parameter**

$$\Omega = \frac{\rho}{\rho_c} = \frac{8\pi G\rho}{3H^2} . \tag{22}$$

In a universe which contains matter, radiation and a cosmological constant the Friedmann equation can be rewritten as

$$\frac{H(z)^2}{H_0^2} = \Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 + \Omega_{\Lambda 0} . \tag{23}$$

2.4 Distances

The coordinate distance r is also known as the **comoving distance**, as we have factored out the expansion of the universe. The **proper distance** is $= a(t)r$. The comoving distance traveled by light in a given time is

$$r = \int \frac{dt}{a(t)} . \tag{24}$$

The **Luminosity Distance** is the distance to an object you would infer if you assumed that the inverse square law holds. If a source emits radiation with a luminosity L_s, and the flux received at a distance d_L, assuming the inverse square law, is F, then

$$d_L^2 = \frac{L_s}{4\pi F} . \tag{25}$$

Obviously, we know that the inverse square law doesn't hold in an expanding universe, but it is still useful to define the luminosity distance to an object in terms of its luminosity and the flux received.

We can relate luminosity distance to redshift in the following way: If a source emits an energy ΔE_S in a time Δt_S then its luminosity is

$$L_S = \frac{\Delta E_S}{\Delta t_S} . \tag{26}$$

The corresponding luminosity of radiation passing through a sphere of radius r is

$$L_r = \frac{\Delta E_r}{\Delta t_r} . \tag{27}$$

If we say that $\nu_e = 1/\Delta t_S$ and $\nu_r = 1/\Delta t_r$, then we know that the ratios of these frequencies can be related to the redshift by equation (6)

$$\frac{\Delta t_r}{\Delta t_S} = 1 + z = \frac{\Delta E_r L_S}{\Delta E_S L_r} . \tag{28}$$

But for radiation $\Delta E \propto \nu$ so that

$$\frac{\Delta E_r}{\Delta E_S} = \frac{\nu_r}{\nu_e} = \frac{1}{1+z} \,, \tag{29}$$

and we conclude that

$$\frac{L_S}{L_r} = (1+z)^2 \,, \tag{30}$$

which tells us that the number of photons per second decreases as the universe expands and that each photon looses energy as the universe expands.

The comoving distance traveled by the photons from the source is

$$r_S = \int_{t_1}^{t_0} \frac{dt}{a(t)} \,. \tag{31}$$

In the FRW metric the area of the surface of the sphere over which the photons are spread is $= 4\pi(a_0 r_S)^2$ at $t = t_0$. The observed flux of photons at a point is therefore

$$F = \frac{L_0}{4\pi(a_0 r_S)^2} \,, \tag{32}$$

and we find a relationship between redshift and luminosity distance

$$d_L = a_0 r_S (1+z) \,. \tag{33}$$

Note that, when $z \ll 1$, this reduces to the proper distance.

In general the luminosity distance to an object depends on the cosmological model. We can show that

$$r_S = \int_{t_1}^{t_0} \frac{1}{a(t)} dt = \frac{1}{a_0} \int_0^z \frac{1}{H(z')} dz' \,, \tag{34}$$

and as a result

$$d_L = (1+z) \int_0^z \frac{1}{H(z')} dz' \,. \tag{35}$$

The relationship between luminosity distance and redshift is a key part of determining the expansion history of the universe, eg using Type 1a supernovae as standard (or standardiseable) candles leads to the conclusion that the universe is currently cosmological constant dominated. It also forms part of the current H_0 tension.

3 The Thermal Universe

So far we have treated the matter in the universe only as a perfect fluid, of course it is really made up of the particles of the Standard Model. These particles can be bosons or fermions, they may be relativistic or non-relativistic, and this may change as the universe expands, and they may be in thermal equilibrium or out of equilibrium, and again this may change as the universe expands. In this section we still consider a homogeneous and isotropic universe, but we now allow for different particle species. We will determine how their different properties affects their abundance and temperature in the late universe.

We start by giving a brief overview of some statistical physics. Particles of type A, in thermal equilibrium are described by a distribution function of the form

$$f_A(p) = \frac{g_A}{(2\pi)^3} \frac{1}{e^{E_A/T_A} \pm 1} , \tag{36}$$

where g_A is the spin degeneracy factor, $E(p) = \sqrt{p^2 + m_A^2}$, m_A is the mass of A, and T_A is the temperature. We choose a plus sign for fermions and a minus sign for bosons. In this expression we have neglected the chemical potential, which will be a good approximation in the calculations that follow. The distribution function only depends on p, and there is no spatial dependence because of our assumptions of homogeneity and isotropy. The time dependence of the distribution is implicit through its dependence on temperature.

The number density, density and pressure are defined in terms of the distribution function as

$$n_A = \int f_A(\vec{p}) d^3\vec{p} \, , \tag{37}$$

$$\rho_A = \int E(p) f_A(\vec{p}) d^3\vec{p} \, , \tag{38}$$

$$P_A = \int \frac{p^2}{3E(p)} f_A(\vec{p}) d^3\vec{p} \, . \tag{39}$$

3.1 Relativistic Species

If a species in thermal equilibrium is relativistic then $m_A \ll T$, and we can compute the number density, density and pressure which differ depending on whether the particle is a boson (B) or a fermion (F).

$$n_F = \frac{3}{4} n_B = \frac{3\zeta(3) g}{4\pi^2} T^3 , \tag{40}$$

where $\zeta(x)$ is the Riemann zeta function.

$$\rho_F = \frac{7}{8} \rho_B = \frac{7}{8} \frac{g\pi^2}{30} T^4 , \tag{41}$$

and $P = \rho/3$.

If there is more than one relativistic species then the total radiation density is

$$\rho_{\rm rad} = \sum_{\rm species} \rho_i \tag{42}$$

$$= \frac{g_* \pi^2}{30} T_\gamma^4 , \tag{43}$$

where the **number of (effectively massless) degrees of freedom** is

$$g_* = \sum_{\text{bosons}} g_i \left(\frac{T_i}{T_\gamma}\right)^4 + \frac{7}{8} \sum_{\text{bosons}} g_i \left(\frac{T_i}{T_\gamma}\right)^4 . \tag{44}$$

For example, when the temperature of the universe falls below 1 MeV, only the photon and neutrinos are relativistic. The photon has two degrees of freedom (two polarizations) and the neutrinos each have two degrees of freedom. So the resulting number of effective degrees of freedom is

$$\begin{aligned} g_* &= 2 + \frac{7}{8} \times 6 \times \left(\frac{T_n}{T_\gamma}\right)^4 & (45)\\ &= 3.36 , & (46) \end{aligned}$$

where the final equality relies on the computation of the neutrino temperature that we will compute in Section 3.5.

When the temperature is between 1 MeV and 100 MeV electrons and positrons are also relativistic. At this time there is no difference between the photon and neutrino temperatures as they are interacting through $e^+ \, e^-$ annihilation (this point will become clearer shortly) and so the effective number of degrees of freedom is

$$\begin{aligned} g_* &= 2 + \frac{7}{8} \times (3 \times 2 + 2 \times 2) & (47)\\ &= 10.75 . & (48) \end{aligned}$$

3.2 Entropy

We now briefly detour into some more thermodynamics in order to define another useful quantity. Starting from

$$dE = TdS - PdV , \tag{49}$$

we recall that, in a cosmological volume, $E = \rho V$ and that $V \sim a^3$. Then using the conservation equation we can show that

$$T\frac{dS}{dt} = 0 , \tag{50}$$

and we find that entropy is conserved.

We define the **entropy density** $s = S/V$. Again starting from equation (49) we can show that

$$\frac{d\rho}{dV} - T\frac{dS}{dV} = \frac{1}{V}(Ts - \rho - P) . \tag{51}$$

In equilibrium ρ and s depend on T and are independent of V, and so we find that $Ts - \rho - P = 0$. The entropy of the system in thermal equilibrium is dominated by that of the relativistic species, leading us to

$$s = \frac{2\pi^2 g_{*s}}{45} T_\gamma^3 , \tag{52}$$

where

$$g_{*s} = \sum_{\text{bosons}} g_i \left(\frac{T_i}{T_\gamma}\right)^3 + \frac{7}{8} \sum_{\text{bosons}} g_i \left(\frac{T_i}{T_\gamma}\right)^4 . \tag{53}$$

3.3 Non-relativistic Species

If a species in thermal equilibrium is non-relativistic then $m_A \gg T$, and we can approximate the number density, density and pressure

$$n \approx g \left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T} , \tag{54}$$

$$\rho = mn , \tag{55}$$

$$P \approx nT \ll \rho . \tag{56}$$

3.4 Decoupling

If a particle species interacts with photons at a high enough rate then they will share the same temperature. Here, 'high enough' means that there are many interactions in the time it takes the universe to expand significantly. The **interaction rate** is

$$\Gamma_A = n_T \langle \sigma v \rangle , \tag{57}$$

where n_T is the number density of the 'target particle' for the interaction, v is the particle's relative velocity and σ is the interaction cross section. We can see that, as a rule of thumb, particles are in thermal equilibrium when

$$\Gamma_A \gg H , \tag{58}$$

we will justify this condition more formally later. When this condition no-longer holds we say the particle has decoupled, and is no longer in thermal equilibrium.

We denote the time our particle species falls out of thermal equilibrium as t_D and approximate decoupling as an instantaneous process. Once a particle species is completely decoupled, and if it does not decay, the number of particles is conserved, but their momentum redshifts.

$$f_{t>t_D}(p,t) = f_{(\text{eq})}\left(\frac{a}{a_D} p, t_D\right) . \tag{59}$$

If decoupling happens when $T_D \gg m$ then the distribution function has the same form as that given above for a relativistic species in thermal equilibrium, but with temperature

$$T(t) = T_D \frac{a_D}{a(t)} . \tag{60}$$

3.5 Some Thermal History

When the temperature of the universe $T \ll 10^{12}$ K ≈ 100 MeV the only relativistic particles are the electron and positron, the neutrinos and the photon. We have shown that the effective number of degrees of freedom is $g_* = 10.75$. At this time the universe is dominated by radiation and

$$H(T) = \frac{\sqrt{8\pi}\rho_R^{1/2}}{\sqrt{3}m_P} \approx 5.44\frac{T^2}{m_P} \ . \tag{61}$$

Weak interaction process keep all of these particles in thermal equilibrium. The interaction cross section is

$$\sigma_F \approx G_F^2 E^2 \approx G_F^2 T^2 \ , \tag{62}$$

where G_F is the Fermi constant. If we assume that the neutrinos are approximately massless such that $v \sim 1$ and glossing over the details of which particles are the 'target' particles in this interaction we find that the interaction rate is

$$\Gamma \sim G_F^2 T^5 \ . \tag{63}$$

As a result we have

$$\frac{\Gamma_F}{H} \sim \frac{G_F^2 T^5 m_P}{T^2} \sim \left(\frac{T}{1 \text{ MeV}}\right)^3 \ , \tag{64}$$

and we find that neutrinos decouple at approximately 1 MeV.

Very shortly after this, when the temperature is around 0.5 MeV, (almost) all of the electrons and positrons will annihilate away. Before e^+e^- annihilation, but after neutrino decoupling

$$g_{*s} = 2 + \frac{7}{8}(2 \times 2) = \frac{11}{2} \ , \tag{65}$$

after e^+e^- annihilation

$$g_{*s} = 2 \ , \tag{66}$$

We know that entropy $S \sim sa^3 = (2\pi^2/45)g_{*S}(aT_\gamma)^3$ is conserved and so

$$g_{*s}(aT_\gamma)^3\big|_{\text{before}} = g_{*s}(aT_\gamma)^3\big|_{\text{after}} \ . \tag{67}$$

Before e^+e^- annihilation the neutrinos and photons have essentially the same temperature, and so we find that after e^+e^- annihilation

$$T_\nu^3 = \frac{4}{11}T_\gamma^3 \ . \tag{68}$$

So the relic neutrinos today are colder than the CMB photons. This result justifies our calculation of the effective number of degrees of freedom today.

3.6 Out of Equilibrium: The Boltzmann Equation

Out of equilibrium the dynamics of particles is governed by the **Boltzmann equation**:

$$\hat{L}[f_A] = \hat{C}_A[f_A] \ , \tag{69}$$

where $\hat{L} = d/d\lambda$ is the Liouville operator, and λ an affine parameter along a particle's world-line. $\hat{C}_A$ is the collision operator, which determines the change in the number of particles due to interactions or decays. The Boltzmann equation can be recast in terms of the particle number density, instead of the distribution function by integrating over three momentum.

A useful example is particle annihilation, where the Boltzmann equation becomes

$$\dot{n}_A + 3Hn_A = -R_{A\bar{A}\to X\bar{X}} n_A n_{\bar{A}} + R_{X\bar{X}\to A\bar{A}} n_X n_{\bar{X}} \ , \tag{70}$$

where the R's are rate coefficients, $R = \langle \sigma v \rangle$. In equilibrium the rates $R_{A\bar{A}\to X\bar{X}}$ and $R_{X\bar{X}\to A\bar{A}}$ obey the **detailed balance** equation

$$R_{A\bar{A}\to X\bar{X}} n_A^{(eq)} n_{\bar{A}}^{(eq)} = R_{X\bar{X}\to A\bar{A}} n_X^{(eq)} n_{\bar{X}}^{(eq)} \ . \tag{71}$$

3.7 Freeze Out

In this section we will calculate the relic abundance for massive stable particles after they fall out of thermal equilibrium. If a particle of species A is stable, the number of particles can only be changed by annihilation, $A + \bar{A} \leftrightarrow X + \bar{X}$, where we will assume that species X is still in thermal equilibrium. The number densities are $n_A = n_{\bar{A}}$ and $n_X = n_{\bar{X}} = n_X^{(eq)}$. The Boltzmann equation becomes

$$\begin{aligned} \dot{n}_A + 3Hn_A &= -\langle \sigma_A v \rangle n_A n_{\bar{A}} + \langle \sigma_X v \rangle n_X^{(eq)} n_{\bar{X}}^{(eq)} && (72) \\ &= -\langle \sigma_A v \rangle (n_A^2 - n_A^{(eq)2}) \ . && (73) \end{aligned}$$

We define a comoving number density to factor out dilution $Y_A = n_A/s \propto a^3 n_A$. After some algebra we find that the Boltzmann equation can be rewritten as

$$\frac{d \ln Y_A}{d \ln a} = -\frac{\Gamma_A}{H} \left(1 - \left(\frac{Y_A^{(eq)}}{Y_A} \right)^2 \right) \ . \tag{74}$$

From this we can see why we have previously used the condition $\Gamma_A \gg H$ as the condition for thermal equilibrium. When $\Gamma_A \gg H$ the second factor on the right hand side of equation (74) must be small to compensate, and so $Y_A = Y_A^{(eq)}$. When $\Gamma_A \ll H$ the abundance 'freezes in' as the right hand side of equation (74) tends to zero, and after this time $Y_A = Y_A^{(eq)}\Big|_{T_f}$, where T_f is the freeze out temperature.

3.8 Thermal Relics and the WIMP 'Miracle'

A particle species decouples when $\Gamma_A \sim H$, and so

$$n|_f \sim \left.\frac{H}{\langle\sigma v\rangle}\right|_f \,, \tag{75}$$

and the relic abundance today is

$$\Omega_{A,0} = \frac{8\pi G m_A n_{A,0}}{3H_0^2} \,. \tag{76}$$

If we assume that the universe is dominated by radiation at the time of freeze out then $H_f^2 = (8\pi G/3)\rho_{\rm rad,f}$, and $\rho_{\rm rad} = (\pi^2/30)g_* T^4$. After freeze out we have seen that the comoving number density, Y_A, becomes constant, this means that the relic number density will fall at the same rate as the entropy density

$$\frac{n_{A,f}}{n_{A,0}} = \frac{g_{*f}T_f^3}{g_{*0}T_0^3} \,, \tag{77}$$

so that

$$\Omega_{A,0} = \left(\frac{8\pi G}{3}\right)^{3/2} \frac{\pi}{\sqrt{30}}\frac{m_A}{T_f}\frac{g_{*0}T_0^3}{\langle\sigma v\rangle H_0^2 g_{*f}^{1/2}} \tag{78}$$

$$= 1.5\times 10^{-10}\frac{m_A}{T_f}\frac{1}{\langle\sigma v\rangle g_{*f}^{1/2}}\ {\rm GeV}^{-2} \,, \tag{79}$$

where we have used $T_0 = 2.7$ K $= 2.3\times10^{-13}$ GeV, $H_0 \approx 70$ kms^{-1}Mpc^{-1} $= 1.5\times10^{-42}$ GeV and $g_{*0} = 3.36$. From this we can see that the relic abundance of a particle is proportional to its mass, and inversely proportional to its cross-section.

The remaining part of this argument, as I will present it here, is a very crude estimate. However it can be made much more precise by solving the Boltzmann equation for particles interacting at the weak scale. If a heavy particle with a weak cross section, freezes out when it is relativistic, or close to relativistic, so that $v \sim 1$ then, $\langle\sigma v\rangle \sim G_F^2 m_A^2$, where G_F is the Fermi constant. If freeze out happens in the very early universe then $g_{*f} \sim 100$. Then the relic abundance is of the right order of magnitude to be dark matter today $\Omega_{A,0} \sim 0.1$ if $m_A \sim$ GeV and $m_A/T_f \sim 10$. So no fine tunings of the model are required, and it is indeed quite straightforward to find dark matter candidates of this type in, for example, supersymmetry.

4 Perturbation Theory

The universe is, of course, not completely homogeneous. If it were, we wouldn't be here! To understand how structures evolve in the universe we use perturbation theory. In this section we will see the ideas behind cosmological perturbation theory, using Newtonian gravity. The same principles can be used to derive the full equations within general relativity, but the simplified approach avoids having to introduce a lot of new notation, and talking about gauge choices. These calculations will apply to our universe during matter domination, on scales well inside the cosmological horizon.

We take the metric to be

$$ds^2 = -(1+2\phi)dt^2 + (1-2\phi)\delta_{ij}dx^i dx^j \ , \tag{80}$$

and we assume that $|\phi| \ll 1$ and $|\partial_0 \phi| \ll |\nabla\phi|$. These assumptions will be consistent if $T^{00} = T_{00} = \rho$ dominates the energy momentum tensor. The Einstein equation (which becomes the Poisson equation) is

$$\nabla^2\phi = 4\pi G\rho \ , \tag{81}$$

the continuity equation and Euler equation become

$$\frac{d\rho}{dt} = -(\vec{\nabla}\cdot\vec{v})\rho \ , \tag{82}$$

$$\frac{d\vec{v}}{dt} = -\frac{1}{\rho}\vec{\nabla}P - \vec{\nabla}\phi \ , \tag{83}$$

where $\vec{v}$ is the matter fluid velocity. We have also assumed that $|v| \ll 1$, and $d/dt = \partial/\partial t + \vec{v}\cdot\vec{\nabla}$.

In a perturbed universe we write

$$\phi(\vec{x},t) = \phi^{(0)}(t) + \Phi(\vec{x},t) \tag{84}$$
$$\vec{u}(\vec{x},t) = H(t)\vec{r} + \vec{v}(\vec{x},t) \tag{85}$$
$$\rho(\vec{x},t) = \rho(t) + \delta\rho(\vec{x},t) = \rho(t)(1+\delta(\vec{x},t)) \tag{86}$$
$$P(\vec{x},t) = P(t) + \delta P(\vec{x},t) \ , \tag{87}$$

in Fourier space the first order perturbations of the continuity, Euler and Poisson equations are

$$a\dot{\delta}_k = -i\vec{k}\cdot\vec{v}_k \ , \tag{88}$$

$$a\dot{\vec{v}}_k + aH\vec{v}_k + i\vec{k}\Phi_k = -i\vec{k}\frac{\delta P_k}{\rho} \ , \tag{89}$$

$$-\frac{k^2}{a^2}\Phi_k = 4\pi G\rho\delta_k \ , \tag{90}$$

4.1 Scalar, Vector Decomposition

We now decompose the velocity vector $\vec{v}_k$ into vector and scalar parts

$$\vec{v}_k = \vec{v}_k^{(sc)} + \vec{v}_k^{(vec)} \ , \tag{91}$$

where $\vec{v}^{(sc)} \propto \vec{k}$ and $\vec{k} \cdot \vec{v}_k^{(vec)} = 0$. We will write $\vec{v}_k^{(sc)} = -i\vec{k}V_k/|k|$. The Euler equation can now be separated into two components, and the equation for the vector mode is

$$a\dot{\vec{v}}_k^{(vec)} + aH\vec{v}_k^{(vec)} = 0 \ . \tag{92}$$

The solution to this equation decays as $1/a$. In cosmological perturbation theory we neglect modes that decays with the expansion of the universe. The remaining equations for the system are now

$$\dot{\delta}_k + \frac{|k|}{a}V_k = 0 \ , \tag{93}$$

$$\dot{V}_k + HV_k - \frac{|k|}{a}\Phi_k = \frac{|k|}{a}\frac{\delta P_k}{\rho} \ , \tag{94}$$

$$-\frac{k^2}{a^2}\Phi_k = 4\pi G\rho\delta_k = \frac{3}{2}H^2\Omega_m\delta_k \ , \tag{95}$$

where $\Omega_m = 8\pi G\rho/(3H^2)$. By substitution we find the following equation for the scalar mode

$$\ddot{\delta}_k + 2H\dot{\delta}_k - \frac{3}{2}H^2\Omega_m\delta_k = -\frac{k^2}{a^2}\frac{\delta P_k}{\rho} \ . \tag{96}$$

It's possible to show that δP_k is negligible on cosmological scales, and as there is no remaining k dependence in the equation for δ, this equation has growing and decaying solutions D_1 and D_2. During matter domination these behave as $D_1 \propto t^{2/3}$ and $D_2 \propto t^{-1}$. Again we neglect the decaying mode. We can then straightforwardly show that the Newtonian potential Φ is independent of time. These equations can, similarly, be solved for late time dark energy domination.

4.2 Cold Dark Matter and Baryons

There is not only one type of matter in the universe, and the perturbations of different types of matter may evolve slightly differently. An important distinction is between cold dark matter and baryonic matter. The evolution equations for mater perturbations become

$$\dot{\delta}_c + \frac{k}{a}V_c = 0 \tag{97}$$

$$\dot{V}_c + HV_c - \frac{k}{a}\Phi = 0 \tag{98}$$

$$\dot{\delta}_B + \frac{k}{a}V_B = 0 \tag{99}$$

$$\dot{V}_B + HV_B - \frac{k}{a}\Phi = \frac{k}{a}\frac{\delta P_B}{\rho_B} \ , \tag{100}$$

and as a result we find

$$\ddot{\delta}_B + 2H\dot{\delta}_B - \frac{3}{2}H^2\delta + \left(\frac{k}{a}\right)^2 c_s^2\delta_B = 0 \tag{101}$$

$$\ddot{\delta}_c + 2H\dot{\delta}_c - \frac{3}{2}H^2\delta = 0 \ , \tag{102}$$

where $c_s^2 = \delta P_B/\delta\rho_B$ is the speed of sound of baryons.

If $(k/a)^2 c_s^2 \delta_B$ is negligible in these equations, then baryonic and CDM perturbations grow in the same way. If $(k/a)^2 c_s^2 \delta_B \gg 3H^2/2$ on some scales, then we can define the **Jeans scale** $k_J(t)$ as the threshold scale for this condition to be satisfied. We can also define corresponding **Jeans wavelength**, $\lambda_J = 2\pi a/k_J$, and **Jeans mass**, $M_J = (4/3)\pi\rho(\lambda_J/2)^3$, which is the amount of matter in a sphere of radius λ_J. If $\delta_B = \delta_c \propto t^{2/3}$ then we find

$$k_J = \frac{a}{c_s}\sqrt{8\pi G\rho}\ , \tag{103}$$

and

$$M_J = \frac{\pi^{5/2}}{2^{5/2}\times 3}\frac{c_s^3}{G^{3/2}\rho^{1/2}}\ . \tag{104}$$

We can show, using thermodynamical considerations to obtain c_s, that $M_J \sim 10^4 M_\odot$ today, and scales as $(1+z)^{3/2}$ at earlier times. On scales below the Jeans length, the baryon fluid oscillates, and this places a lower limit on the mass of the first baryonic objects that can form.

A fully relativistic analysis of cosmological perturbations has tensor, vector and scalar modes (and one needs to be careful about the choice of gauge). There is much more physics to be found working in the full formalism, including oscillations in the photon baryon fluid, which get imprinted on the photons at the point of decoupling, and give rise to the peak structure in the CMB.