Methods of Applied Mathematics II: Asymptotic Methods

Peter D. Miller

July 28, 2002

Chapter 0

Themes of Asymptotic Analysis

0.1 Theme: Asymptotics, Convergence, and Divergence

We will spend some time in the course developing approximations for functions defined by integrals. For example, consider the famous "gamma" function:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt \tag{1}$$

defined here for $z > 0$. Here is a plot of this function. There is no elementary formula for the gamma

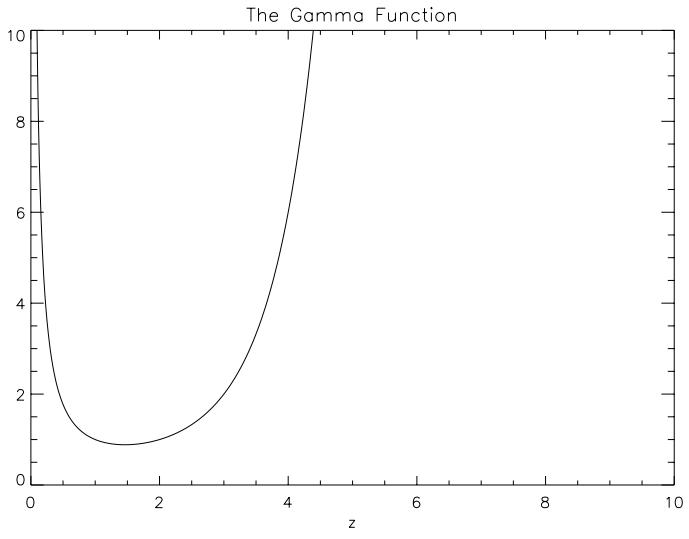

Figure 1: *A plot of the gamma function.*

function in terms of other functions. However we can seek accurate approximations in terms of more familiar functions. One approximation of the gamma function can be obtained by considering small positive values

of z. First use integration by parts:

$$\begin{aligned}\Gamma(z) &= \int_0^\infty t^{z-1}e^{-t}\,dt \\ &= \frac{1}{z}\int_0^\infty \frac{d}{dt}t^z\cdot e^{-t}\,dt \\ &= \frac{1}{z}\left[t^z e^{-t}\Big|_0^\infty + \int_0^\infty t^z e^{-t}\,dt\right] \\ &= \frac{1}{z}\int_0^\infty t^z e^{-t}\,dt\,.\end{aligned} \tag{2}$$

The last integral is in fact $\Gamma(z+1)$ so we have rediscovered the factorial recursion relation $\Gamma(z+1)=z\Gamma(z)$. In any case, when z is small and positive,

$$t^z = e^{z\log t} = 1 + z\log t + \frac{z^2}{2}(\log t)^2 + \ldots \tag{3}$$

So substituting and integrating term-by-term, we get

$$\begin{aligned}\Gamma(z+1) &= \int_0^\infty \left(1 + z\log t + \frac{z^2}{2}(\log t)^2 + \ldots\right)e^{-t}\,dt \\ &= 1 + z\int_0^\infty \log t\cdot e^{-t}\,dt + \frac{z^2}{2}\int_0^\infty (\log t)^2 e^{-t}\,dt + \ldots \\ &= 1 - \gamma z + \left(\frac{\gamma^2}{2} + \frac{\pi^2}{12}\right)z^2 + \ldots\,,\end{aligned} \tag{4}$$

where Euler's constant is $\gamma \approx 0.577216$. At last we get the *asymptotic approximation*

$$\Gamma(z) = \frac{1}{z} - \gamma + \left(\frac{\gamma^2}{2} + \frac{\pi^2}{12}\right)z + \ldots\,, \qquad \text{as} \qquad z\to 0\,. \tag{5}$$

By truncating this series at different numbers of terms, we get a number of approximations to $\Gamma(z)$ which are shown in Figure 2. Notice that the crudest approximation is in fact *exact* for $z=1$. But it gets worse than the others the closer z is to zero. All the approximations get better as z goes to zero. But the higher-order approximations get better faster as z goes to zero.

We can also find an approximation in terms of known functions valid for large positive z. The procedure for obtaining these approximations must wait until later in the course, but the result is the following *asymptotic expansion*:

$$\Gamma(z) \sim e^{-z}z^{z-1/2}\sqrt{2\pi}\left(1 + \frac{1}{12z} + \frac{1}{288z^2} + \ldots\right), \qquad \text{as} \qquad z\to\infty\,. \tag{6}$$

The first term of this asymptotic expansion is known as Stirling's formula. It is a remarkable approximation to the gamma function. Once again, by truncating this series at different places one gets a hierarchy of approximations to the gamma function, which are shown in Figure 3. It is clear that the approximations again form a series, as before. That is, each approximation in the hierarchy has the form of an additive correction to the previous approximation. But it is a very important point that *this series does not converge for any* z, which is the reason for using the notation $\sim$ instead of $=$. This is in contrast to the case of the expansion near $z=0$, which turns out to be a convergent Laurent series that converges uniformly for all complex z satisfying $0<|z|<1$.

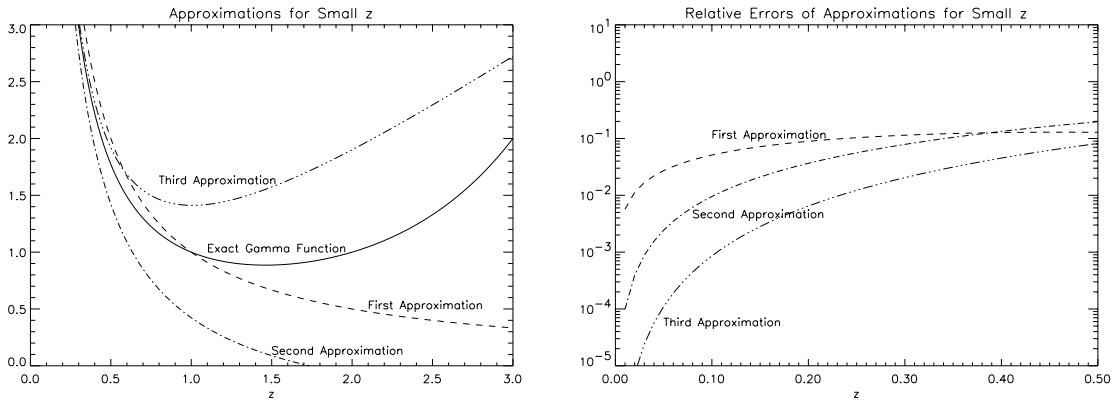

Figure 2: *Left: several increasingly accurate approximations of the gamma function valid for z near zero. Right: the relative errors of these approximations.*

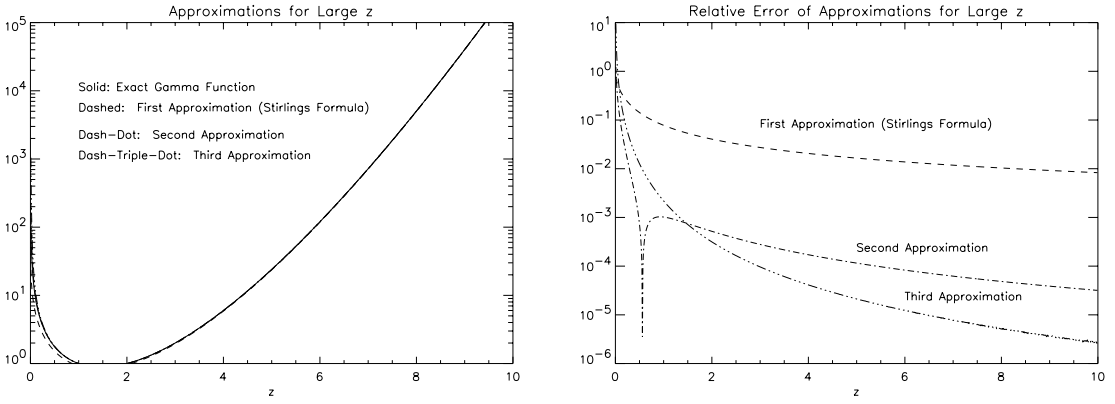

Figure 3: *Left: several increasingly accurate approximations of the gamma function valid for large positive z. Right: the relative errors of these approximations.*

The fact that most asymptotic series do not converge was a stumbling block for the theory of asymptotic expansions for many years. But since the work of Poincaré in the late nineteenth century, we understand how to interpret these divergent series. The main idea is this: the error R_n in keeping the first n terms goes to zero as $z \to \infty$ with n held fixed, but it does not go to zero as $n \to \infty$ with z held fixed.

Some main points to remember are:

1. Convergent methods give arbitrarily accurate approximations to the solution of a single fixed problem, *i.e.* finding the value of $\Gamma(1/3)$.

2. Asymptotic methods give arbitrarily accurate approximations to the solution of a family of problems indexed by a parameter, *i.e.* finding the value of $\Gamma(z)$ as z tends to infinity. As such, asymptotic approximations are always accompanied by a qualifier like $z \to 0$ or $z \to \infty$.

0.2 Theme: Other Parameters and Nonuniformity

Often we seek approximations that depend on more than one parameter, and then it is not always clear that an asymptotic approximation with respect to one parameter is valid for all values of the other parameters.

0.2.1 First example. Oscillations.

For example, consider the family of functions

$$x(t,\epsilon) = \cos(\sqrt{1+\epsilon}\cdot t)\,. \tag{7}$$

For each value of the parameter ϵ we have a periodic function of t. We can consider how these functions behave when ϵ is small, that is we can consider asymptotic approximations valid as $\epsilon \to 0$. It is perhaps natural to proceed as follows. For $|\epsilon| < 1$, we have the convergent Taylor series

$$\sqrt{1+\epsilon} = 1 + \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + \dots\,. \tag{8}$$

Also, for all δ we have

$$\cos(t+\delta) = \cos(t) - \sin(t)\delta - \cos(t)\frac{\delta^2}{2} + \dots\,. \tag{9}$$

Substituting the first series into the second, we get the asymptotic expansion

$$x(t,\epsilon) \sim \cos(t) - \frac{\epsilon}{2}t\sin(t) + \frac{\epsilon^2}{8}\left(t\sin(t) - t^2\cos(t)\right) + \dots\,. \tag{10}$$

Because it is an expansion in powers of ϵ, it is called an *asymptotic power series*. These approximations are compared with the function $x(t,\epsilon)$ for several values of ϵ in Figure 4 below. There are two things going on

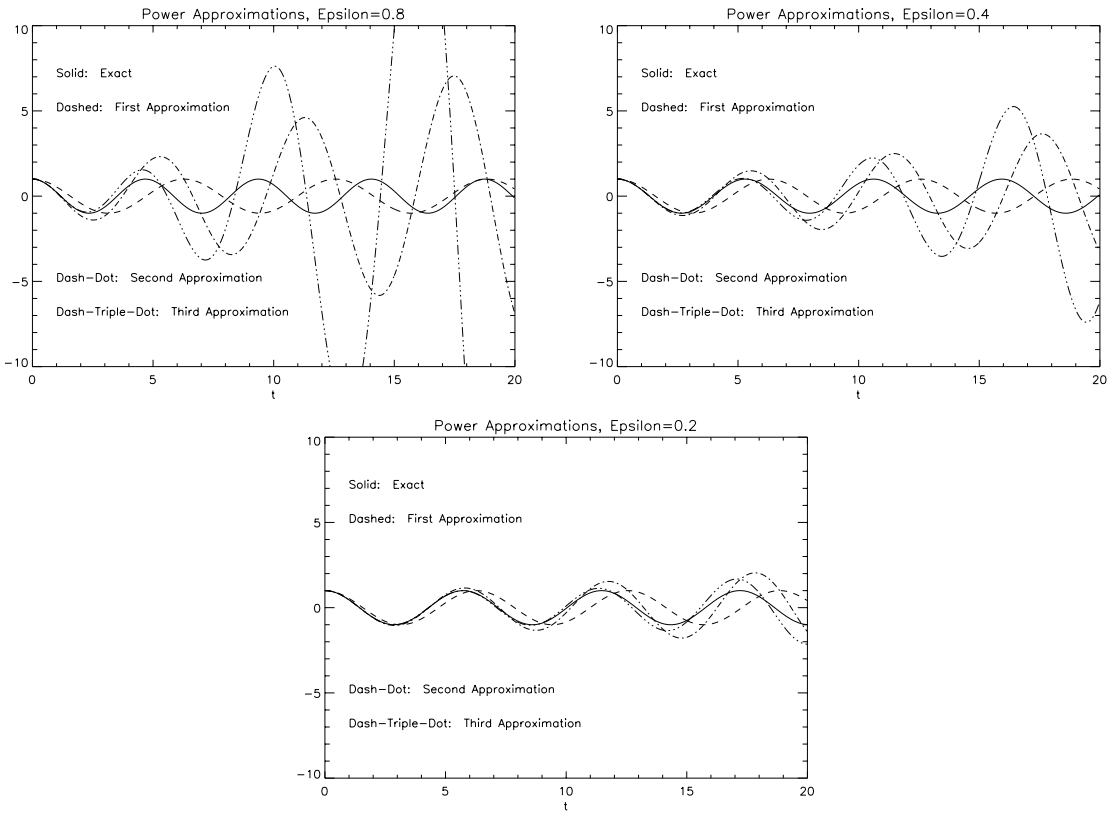

Figure 4: *Secular errors in power approximations. By "first approximation" we mean the leading term of (10), by "second approximation" we mean the first two terms of (10), and so on.*

here:

1. If you look at a fixed value of t, each approximation gets better and better as $\epsilon \to 0$.

2. But for any approximation, and for any small ϵ, the approximation ultimately gets very bad when t is big enough.

We say that the power series approximations are *not uniformly valid* with respect to t. The terms with coefficients depending on powers of t are causing the distortion of the waveform over long time scales. Such terms are called *secular terms* since they were first recognized in problems of celestial mechanics where they caused the distortion of approximations to the orbits of planets that appeared on the order of centuries (in French, century = siècle). The situation can be repaired somewhat by not expanding the cosine. Thus, one gets a sequence of approximations:

$$\begin{aligned} x(t,\epsilon) &\sim \cos(t) && \text{(first approximation)} \\ x(t,\epsilon) &\sim \cos\left(t + \frac{\epsilon}{2}t\right) && \text{(second approximation)} \\ x(t,\epsilon) &\sim \cos\left(t + \frac{\epsilon}{2}t - \frac{\epsilon^2}{8}t\right) && \text{(third approximation)} \\ &\vdots \end{aligned} \tag{11}$$

Notice that this hierarchy of approximants does not form a series. These approximations are compared with the original family of functions $x(t,\epsilon)$ in Figure 5. Each approximant still fails eventually (in t). But

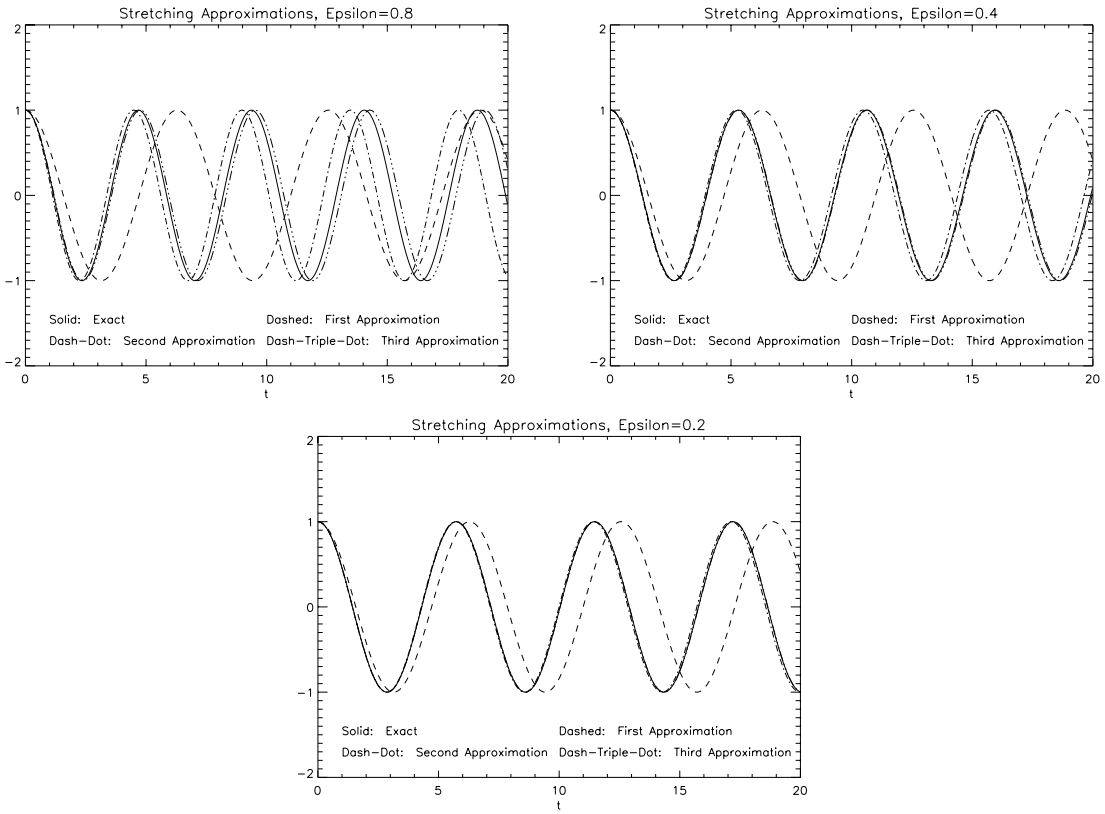

Figure 5: *Approximations valid on increasingly long time scales.*

now the higher-order approximations remain valid longer and longer. In fact, the interval of t in which the approximations converge as $\epsilon \to 0$ now expand as $\epsilon \to 0$ too!

0.2.2 Second Example. Boundary Layers.

As another example, let

$$m_+(\epsilon) = \frac{1}{2\epsilon}\left(\epsilon - 1 + \sqrt{1 + 2\epsilon - 3\epsilon^2}\right), \qquad m_-(\epsilon) = \frac{1}{2\epsilon}\left(\epsilon - 1 - \sqrt{1 + 2\epsilon - 3\epsilon^2}\right), \tag{12}$$

and consider the family of functions of x given by

$$u(x,\epsilon) = \frac{(e^{m_-(\epsilon)} - 1)e^{m_+(\epsilon)x} + (1 - e^{m_+(\epsilon)})e^{m_-(\epsilon)x}}{e^{m_-(\epsilon)} - e^{m_+(\epsilon)}}. \tag{13}$$

Again, we can ask how these functions behave as $\epsilon \to 0$. Using a technique you all will all be very good at shortly, one can seek an expansion in powers of ϵ:

$$u(x,\epsilon) \sim u_{\text{out}}(x,\epsilon) = e^{x-1} - \epsilon(x-1)e^{x-1} + \dots . \tag{14}$$

The approximations obtained by keeping only the explicit terms shown above are plotted along with the actual functions $u(x,\epsilon)$ for several values of ϵ in Figure 6. The approximations are clearly better and better

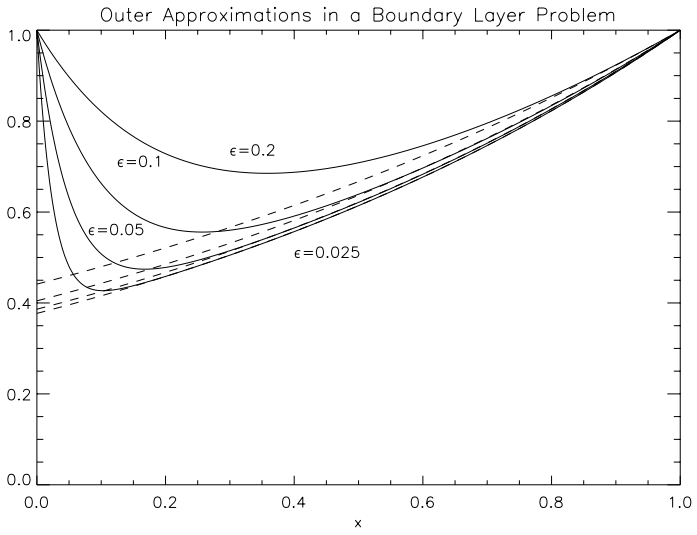

Figure 6: *Outer approximations in a boundary layer problem.*

as $\epsilon \to 0$ for each fixed x (compare with the secular approximations of $x(t,\epsilon)$). What goes wrong here is different, however. There is another part of the picture near $x = 0$ that is totally missed by these *outer approximations*, which naively assume that x is held fixed as $\epsilon \to 0$.

The situation can be repaired near $x = 0$ with a different sort of asymptotic expansion that captures the boundary layer behavior, an *inner expansion*. The inner expansion takes the form

$$u(x,\epsilon) \sim u_{\text{in}}(x,\epsilon) = \left(1 - \frac{1}{e}\right)e^{-\xi} + \frac{1}{e} + \epsilon\frac{\xi + 1 - e^{-\xi}}{e} + \dots , \tag{15}$$

where $\xi = x/\epsilon$. These approximations are in terms of a *rescaled variable* $\xi = x/\epsilon$ that acts like a magnifying glass blowing up the boundary layer behavior near $x = 0$. They are compared with $u(x,\epsilon)$ in Figure 7.

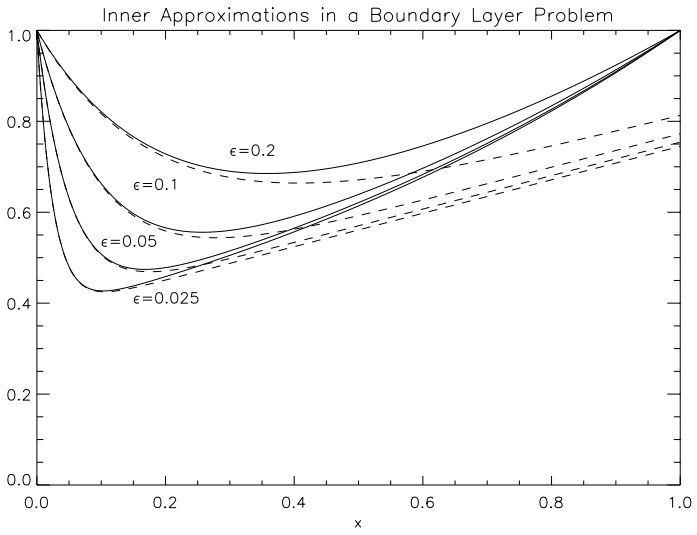

Figure 7: *Inner approximations in a boundary layer problem.*

0.3 Theme: Differential Equations

Where did these strange examples of nonuniform expansions come from? From the everyday world of differential equations. For example, the function $x(t, \epsilon)$ is the unique solution of the initial-value problem

$$\ddot{x} + (1 + \epsilon)x = 0\,, \qquad x(0) = 1\,, \dot{x}(0) = 0\,, \tag{16}$$

which represents a particle attached to a spring with a slightly perturbed spring constant when ϵ is a small number.

Similarly, the function $u(x, \epsilon)$ is the unique solution of the boundary-value problem

$$\epsilon u''(x) + (1 - \epsilon)u'(x) - (1 - \epsilon)u(x) = 0\,, \qquad u(0) = u(1) = 1\,, \tag{17}$$

which could represent, for example, the equilibrium deflection of a certain beam under certain loading stresses.

In both of these problems, the required expansions were obtained directly, by first solving the differential equations with side conditions explicitly, and then finding the appropriate asymptotic expansion. Usually, however, exact solutions to differential equations cannot be obtained in closed form, and we need to be obtain the expansions *directly from the differential equation, bypassing the solution formula.* We will concentrate on these techniques in the latter part of the course.

0.4 Theme: Partial Differential Equations and Canonical Physical Models

In the final part of the course, we will turn to wave mechanics problems described by partial differential equations. Because more independent variables are involved, when we seek asymptotic approximations we will find that conditions for validity of the expansion arise as *universal model equations.* For example, we will see that almost any situation in which long, small amplitude waves are propagating in one direction, the universal model is the *Korteweg-de Vries equation*

$$u_t + uu_x + u_{xxx} = 0\,. \tag{18}$$

On the other hand, whenever one has packets of small amplitude waves propagating in one direction, the universal model is a complex equation called the *nonlinear Schrödinger equation*

$$iu_t + u_{xx} \pm |u|^2 u = 0\,. \tag{19}$$

We will see how these equations may be derived from diverse physical situations via asymptotic analysis.

Chapter 1

The Nature of Asymptotic Approximations

1.1 Asymptotic Approximations and Errors

1.1.1 Order relations among functions.

Because we will need to estimate the error of the asymptotic approximations we will study in the course, and because the form of the estimates required is similar in each case, it is useful to have at hand some standard notation. This is the so-called "order notation".

Big-oh.

First, we concern ourselves with statements claiming that one function serves as a bound for another.

Definition 1 (Big-oh) *Let $f(z)$ and $g(z)$ be two complex-valued functions defined in some domain D of the complex plane whose closure contains a point z_0. Then we write*

$$f(z) = O(g(z)) \quad \text{in } D \text{ as } \quad z \to z_0 \tag{1.1}$$

if we can find constants $K > 0$ and $\delta > 0$ so that

$$|f(z)| \leq K|g(z)| \quad \text{whenever} \quad z \in D \quad \text{and} \quad 0 < |z - z_0| < \delta\,. \tag{1.2}$$

That is, f is bounded in magnitude by a multiple of g for z close enough to z_0 in D.

For asymptotic behavior for large z, the domain D must be unbounded and then the definition is modified as follows:

Definition 2 (Big-oh at infinity) *Let $f(z)$ and $g(z)$ be two complex-valued functions defined in an unbounded domain D of the complex plane. Then we say*

$$f(z) = O(g(z)) \quad \text{in } D \text{ as } \quad z \to \infty \tag{1.3}$$

if we can find constants $K > 0$ and $M > 0$ so that

$$|f(z)| \leq K|g(z)| \quad \text{whenever} \quad z \in D \quad \text{and} \quad |z| > M\,. \tag{1.4}$$

That is, f is bounded in magnitude by a multiple of g for z large enough in D.

In both cases, if D can be taken to be an open neighborhood of the point $z = z_0$ or $z = \infty$, then we don't refer to it explicitly. Also, if the inequality holds for all $z \in D$, without the added constraint of $0 < |z - z_0| < \delta$ or $|z| > M$, then we simply say $f(z) = O(g(z))$ in D. The role of the particular point z_0 is thus not terribly important in any given "big-oh" statement; it is any point in a neighborhood where the required bound holds.

Little-oh and asymptotic equivalence.

On the other hand, a more precise kind of statement can be given by considering limits as $z \to z_0$ or $z \to \infty$.

Definition 3 (Little-oh) *Let $f(z)$ and $g(z)$ be two complex-valued functions defined in some domain D of the complex plane whose closure contains a point z_0. Then we write*

$$f(z) = o(g(z)) \quad as \quad z \to z_0 \tag{1.5}$$

if for any given $\epsilon > 0$ we can find a corresponding $\delta(\epsilon) > 0$ so that

$$|f(z)| \leq \epsilon |g(z)| \quad whenever \quad z \in D \quad and \quad 0 < |z - z_0| < \delta(\epsilon)\,. \tag{1.6}$$

That is, f is smaller in magnitude than any multiple of g for $z \in D$ close enough to z_0 (how close depends on which multiple of g is being considered as the bound).

Similarly for asymptotics near infinity, we have the following.

Definition 4 (Little-oh at infinity) *Let $f(z)$ and $g(z)$ be complex-valued functions defined in an unbounded domain D of the complex plane. Then we write*

$$f(z) = o(g(z)) \quad as \quad z \to \infty \tag{1.7}$$

if for any given $\epsilon > 0$ we can find a corresponding $M(\epsilon) > 0$ such that

$$|f(z)| \leq \epsilon |g(z)| \quad whenever \quad z \in D \quad and \quad |z| > M(\epsilon)\,. \tag{1.8}$$

That is, f is smaller in magnitude than any multiple of g for $z \in D$ sufficiently large (how large depends on which multiple of g is being considered as the bound).

If there is a small enough number $\mu > 0$ so that $g(z) \neq 0$ for $0 < |z - z_0| < \mu$ and $z \in D$, then $f(z) = o(g(z))$ as $z \to z_0$ is the same thing as

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = 0\,. \tag{1.9}$$

The function g can vanish at z_0, but just not at other nearby points for this equivalent defintion to hold.

An even more precise kind of statement is the following.

Definition 5 (Asymptotic equivalence) *Two complex-valued functions $f(z)$ and $g(z)$ defined in some domain D of the complex plane whose closure contains a point z_0 are said to be asymptotically equivalent as $z \to z_0$ in D, written as*

$$f(z) \sim g(z) \quad as \quad z \to z_0 \quad in \quad D \tag{1.10}$$

precisely if

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = 1\,. \tag{1.11}$$

A similar definition holds for limits at $z = \infty$.

It is worth taking a moment at this point to discuss the role these various statements will play in characterizing an approximation. The idea is the following. For some possibly complicated function $f(z)$ and some point of interest z_0 we will try to cook up a simpler function $g(z)$ that is designed to approximate $f(z)$ in exactly the sense that $f(z) \sim g(z)$ as $z \to z_0$. As there could be many such functions $g(z)$ all approximating the same function $f(z)$ in this manner, the next goal is to characterize the error of the approximation. This is done in practice by looking at the error defined by the difference $e(z) = f(z) - g(z)$, and then seeking a simple bound given by a function $b(z) \geq 0$ such that we can prove either that $e(z) = O(b(z))$ or $e(z) = o(b(z))$ as $z \to z_0$. The bound $b(z)$ measures the size of the error of the asymptotic approximation $g(z)$ to the function $f(z)$.

1.1.2 Statements following from the order relations.

If all we know about two functions is that they satisfy one of the order relations, we can sometimes deduce other order relations. For example,

Proposition 1 (Little-oh implies big-oh) *If $f(z) = o(g(z))$ as $z \to z_0$ (or $z \to \infty$), then $f(z) = O(g(z))$ as $z \to z_0$ (or $z \to \infty$).*

Also,

Proposition 2 (Linear combinations) *Suppose that $f_n(z) = O(g_n(z))$ for $n = 1, \ldots, N$. Then, for any complex numbers $a_1, \ldots, a_N$, we have*

$$\sum_{n=1}^{N} a_n f_n(z) = O\left(\sum_{n=1}^{N} |a_n||g_n(z)|\right). \tag{1.12}$$

Proposition 3 (Integrals) *Suppose that $f(z,\mu)$ and $g(z,\mu)$ are such that for each fixed μ in some domain G of the complex μ-plane we have $f(z,\mu) = O(g(z,\mu))$ as $z \to z_0$. Then, for any path C lying in G,*

$$\int_C f(z,\mu)\, d\mu = O\left(\int_C |g(z,\mu)|\, |d\mu|\right), \qquad \text{as} \qquad z \to z_0\,. \tag{1.13}$$

On the other hand, it is usually very dangerous to differentiate order relations with respect to parameters, since the absolute values ignore oscillations that can make derivatives larger than you might think. For example, $\sin(\mu z) = O(1)$ for z real as $z \to \infty$ for all μ. But

$$\frac{d}{d\mu}\sin(\mu z) = z\cos(\mu z) \tag{1.14}$$

which grows as real $z \to \infty$. Also the only function satisfying $f(z) = O(0)$ is $f(z) \equiv 0$. So it is not true on two counts that $z\cos(\mu z) = O(0)$, which is what differentiating the order relation might imply.

Similarly, order relations are not generally preserved under differentiation with respect to z. For example, $\sin(z^2) = O(1)$ for z real as $z \to \infty$, but $2z\cos(z^2) \neq O(0)$ as differentiating the order relation might suggest. There are some circumstances under which order relations between functions are preserved after integration of both functions with respect to z. However, this is not true in general as the following example demonstrates. If $f(z) = 1$ and $g(z) = e^{iz}$, then $f(z) = O(g(z))$ as $z \to \infty$ with z real. But in this case

$$\int_0^z f(x)\, dx \neq O\left(\int_0^z g(x)\, dx\right) \tag{1.15}$$

since the integral of f grows with z while that of g does not.

1.1.3 Examples.

1. Let m and n be integers. If $n \geq m$, then $z^m = O(z^n)$ as $z \to \infty$ or simply for $|z| > 1$. Also, $z^n = O(z^m)$ as $z \to 0$ or simply for $|z| < 1$. If $n > m$ strictly, then $z^m = o(z^n)$ as $z \to \infty$ and $z^n = o(z^m)$ as $z \to 0$.

2. Above statements extend to arbitrary real numbers m and n if we select branch cuts for the power functions. For example, take the domain D to be the sector $\arg(z) \in (-\pi, \pi)$.

3. Comparing powers and logarithms. For any positive power $p > 0$ however small, we have

$$\log(z) = O(z^p) \qquad \text{as} \qquad z \to \infty \qquad \text{for } z \text{ real and positive}. \tag{1.16}$$

   Also, for any negative power $p < 0$ however small, we have

$$\log(z) = O(z^p) \qquad \text{as} \qquad z \to 0 \qquad \text{for } z \text{ real and positive}. \tag{1.17}$$

4. If $P(z)$ is any polynomial then $P(z) = o(e^z)$ as $z \to \infty$ with the real part of z positive, and $e^z = o(1/P(z))$ as $z \to \infty$ with the real part of z negative.

5. $2\cosh(z) \sim e^z$ as $z \to \infty$ in the right half-plane, but $2\cosh(z) \sim e^{-z}$ as $z \to \infty$ in the left half-plane. Stokes phenomenon.

1.2 Convergent and Asymptotic Series

1.2.1 Convergent power series.

Recall the Taylor series expansion of an *analytic* function $f(z)$ about a point $z = z_0$:

Definition 6 (Taylor series) *If $f(z)$ is analytic at z_0, then there exists an $R > 0$, the radius of convergence, such that the Taylor series*

$$S(z) = \sum_{n=0}^{\infty} S_n(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z)}{n!}(z - z_0)^n \tag{1.18}$$

converges absolutely whenever $|z - z_0| < R$. Moreover, for all z within the radius of convergence of z_0, $f(z) = S(z)$.

A function analytic at $z = \infty$ can be represented by a series in negative integer powers of z that is uniformly convergent for all z satisfying $|z| > 1/R$ for some $R > 0$. These series may be integrated and differentiated term-by-term within their radii of convergence.

Of particular interest is the remainder in the Taylor series, defined by

$$R_N(z) = S(z) - \sum_{n=0}^{N} S_n(z)\,. \tag{1.19}$$

We can get a formula for $R_N(z)$ by repeatedly viewing each derivative of $f(z)$ as a contour integral of its own derivative:

$$\begin{aligned}
f(z) &= f(z_0) + \int_{z_0}^{z} f'(s_1)\,ds_1 \\
&= f(z_0) + f'(z_0)(z - z_0) + \int_{z_0}^{z}\int_{z_0}^{s_1} f''(s_2)\,ds_2\,ds_1 \\
&= f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2}(z - z_0)^2 + \int_{z_0}^{z}\int_{z_0}^{s_1}\int_{z_0}^{s_2} f'''(s_3)\,ds_3\,ds_2\,ds_1 \qquad (1.20)\\
&\;\;\vdots \\
&= \sum_{n=0}^{N} S_n(z) + \int_{z_0}^{z}\int_{z_0}^{s_1}\cdots\int_{z_0}^{s_N} f^{(N+1)}(s_{N+1})\,ds_{N+1}\,ds_N\ldots ds_1\,.
\end{aligned}$$

Thus

$$R_N(z) = \int_{z_0}^{z}\int_{z_0}^{s_1}\cdots\int_{z_0}^{s_N} f^{(N+1)}(s_{N+1})\,ds_{N+1}\,ds_N\ldots ds_1\,. \tag{1.21}$$

Taylor's theorem asserts that, if z is held fixed with $|z - z_0| < R$, then

$$\lim_{N\to\infty} R_N(z) = 0\,, \tag{1.22}$$

which is the very meaning of the convergence of the infinite series S.

The remainder also has another property. If we take $|z - z_0| \le R_1 < R$, then it is easy to get the estimate

$$|R_n(z)| \le \frac{|z - z_0|^{N+1}}{(N+1)!} \sup_{|s-z_0|<R_1} |f^{(N+1)}(s)|\,, \tag{1.23}$$

by taking the paths of integration in (1.21) to all be radial. Thus, $R_n(z) = O((z - z_0)^{N+1})$ for $|z - z_0| < R_1$. Thus, not only does the reminder go to zero as N goes to infinity with z fixed, but also as $z \to z_0$ with N fixed.

1.2.2 Asymptotic series.

The latter property of Taylor series is also useful in approximation theory. Crucially, and perhaps surprisingly, it can be present in the absence of convergence of the formal series. This allows us to approximate nonanalytic functions.

Example: the exponential integral.

For example, the real exponential integral function $Ei(x)$ is defined for $x > 0$ by the integral

$$Ei(x) = \int_x^\infty e^{-t} t^{-1}\, dt\,. \tag{1.24}$$

This function is not analytic at $x = \infty$, so there is no convergent power series expansion there. But nonetheless we can integrate by parts in the formula an arbitrary number of times to get, exactly,

$$Ei(x) = e^{-x}\left[\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \ldots + \frac{(-1)^{N+1}(N-1)!}{x^N}\right] + (-1)^N N! \int_x^\infty e^{-t} t^{-(N+1)}\, dt\,. \tag{1.25}$$

This looks like the Nth partial sum of a series plus an explicit remainder $R_N(x)$.

The "infinite series" of terms in this expansion does not converge for any x. Even worse, the nth term in the series goes to infinity as n goes to infinity no matter what x is! However, for $x > 0$ we do have that

$$|R_N(x)| = \left|(-1)^N N! \int_x^\infty e^{-t} t^{-(N+1)}\, dt\right| \le N! x^{-(N+1)} e^{-x}\,, \tag{1.26}$$

so that $R_N(x) = O(e^{-x} x^{-(N+1)})$ as $x \to \infty$ with $x > 0$. Therefore, the error $R_N(x)$ goes to zero with N fixed as $x \to \infty$. And it eventually goes to zero at a faster rate with x the bigger N is. Because $R_N(x) = o(e^{-x} x^{-N})$, we get

$$Ei(x) = e^{-x}\left[\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \ldots + \frac{(-1)^{N+1}(N-1)!}{x^N}\right] + o(e^{-x} x^{-N}) \quad \text{as} \quad x \to \infty\,. \tag{1.27}$$

Since these approximations are valid in this sense for all N, we write them all compactly in the generic form

$$Ei(x) \sim e^{-x}\left[\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \ldots\right] \quad \text{as} \quad x \to \infty\,. \tag{1.28}$$

This is the *asymptotic expansion* of $Ei(x)$ as $x \to \infty$ with $x > 0$. Note that we cannot write $=$ because the infinite series of terms on the right-hand side is divergent.

Optimal approximation.

This is obtained by minimizing $|R_N(x)|$ with respect to N for a given fixed x. Typically, the error begins to decrease with N and then increase again. If we ask the error bounds to be roughly the same as N increases through some value N_0, then we are asking that (in the particular case of the expansion of $Ei(x)$),

$$N_0! x^{-(N_0+1)} e^{-x} \approx (N_0 - 1)! x^{-N_0} e^{-x}\,, \tag{1.29}$$

or simply $N_0 \approx x$. The particulars of this approach should be carefully modified for each specific problem. Obviously an explicit estimate for the remainder is most useful.

1.3 Asymptotic Sequences and Limit-Process Expansions

Here we begin a discussion of the general structure of hierarchies of asymptotic approximations.

Definition 7 (Asymptotic sequence) *A family of functions $\{\phi_n(z)\}$ indexed by $n = 1, 2, 3, \ldots$ is called an asymptotic sequence if whenever $n > m$, we have $\phi_n(z) = o(\phi_m(z))$ as $z \to z_0$.*

The functions in an asymptotic sequence get smaller and smaller down the line. Examples:

1. The sequence $\{(z-z_0)^{n-m}\}$ for any real m is asymptotic as $z \to 0$ in $-\pi < \arg(z-z_0) < \pi$.
2. The sequence $\{(\log(z))^{-n}\}$ is asymptotic as $z \to \infty$ with $z > 0$.
3. The sequence $\{e^z z^{-a_n}\}$ with $a_{k+1} > a_k$ is asymptotic as $z \to \infty$ (in any direction).
4. The sequence $1, z, z^2\log(z), z^2, z^3(\log(z))^2, z^3\log(z), z^3, z^4(\log(z))^3, \ldots$ is asymptotic for $z > 0$ as $z \to 0$.

Note that it is not necessary for all of the elements of an asymptotic sequence to be small as $z \to z_0$. Only for each element to be smaller than the preceding one.

The point of asymptotic sequences is that they are used to generate asymptotic expansions.

Definition 8 *Let $\{\phi_n(z)\}$ be an asymptotic sequence. Then the sum*

$$\sum_{n=0}^{N} a_n \phi_n(z) \tag{1.30}$$

is an asymptotic approximation of a function $f(z)$ if

$$f(z) - \sum_{n=0}^{N} a_n \phi_n(z) = o(\phi_N(z)) \quad \text{as} \quad z \to z_0\,. \tag{1.31}$$

If the above is true for each N in the infinite asymptotic sequence, then the formal infinite series

$$\sum_{n=0}^{\infty} a_n \phi_n(z) \tag{1.32}$$

is said to be an asymptotic expansion of $f(z)$ as $z \to z_0$.

If $f(z)$ has an asymptotic expansion as $z \to z_0$ with respect to the asymptotic sequence $\{\phi_n(z)\}$, then for each $N = 1, 2, 3, \ldots$ we have

$$f(z) = \sum_{n=0}^{N} a_n \phi_n(z) + o(\phi_N(z)) \quad \text{as} \quad z \to z_0\,. \tag{1.33}$$

We compactly represent all of these asymptotic approximations for each N by writing

$$f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z) \quad \text{as} \quad z \to z_0\,, \tag{1.34}$$

even though the infinite series on the right-hand side is formal, that is, it usually does not converge.

Some important points:

1. A given function $f(z)$ can have several different asymptotic expansions, with respect to different asymptotic sequences. For example if $f(z) = z^{-1} + e^{-z}(1-z^{-1})^{-1}$, then with respect to the sequence of negative powers $\{1, z^{-1}, z^{-2}, \ldots\}$ asymptotic as $z \to \infty$ on the positive real axis, we have the (finite) asymptotic expansion

   $$f(z) \sim z^{-1} \quad \text{as} \quad z \to \infty\,. \tag{1.35}$$

   Thus the exponential term is invisible to this asymptotic sequence. Such a term is said to be "transcendentally small" or "beyond all orders". However it may be captured by using a different asymptotic sequence, for example the sequence $\{1, z^{-1}, e^{-z}, e^{-z}z^{-1}, e^{-z}z^{-2}, \ldots\}$, asymptotic as $z \to \infty$ on the positive real axis. Now with respect to this sequence, the function has the expansion

   $$f(z) \sim z^{-1} + \sum_{n=1}^{\infty} e^{-z} z^{1-n} \quad \text{as} \quad z \to \infty\,. \tag{1.36}$$

2. A given asymptotic expansion, with the sequence $\{\phi_n(z)\}$ fixed and the coefficients $\{a_n\}$ fixed can correspond to many different functions $f(z)$. If

$$f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z) \quad \text{as} \quad z \to z_0 \tag{1.37}$$

then for any function $g(z)$ satisfying $g(z) = o(\phi_n(z))$ for all n,

$$f(z) + g(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z) \quad \text{as} \quad z \to z_0\,. \tag{1.38}$$

3. If a function $f(z)$ has an asymptotic expansion with respect to a given asymptotic sequence $\{\phi_n(z)\}$ then that expansion is unique, *i.e.* the coefficients a_n are determined systematically by a *limit process*:

$$\begin{aligned}
a_0 &= \lim_{z\to z_0} \frac{f(z)}{\phi_0(z)} \\
a_1 &= \lim_{z\to z_0} \frac{f(z) - a_0\phi_0(z)}{\phi_1(z)} \\
a_2 &= \lim_{z\to z_0} \frac{f(z) - a_0\phi_0(z) - a_1\phi_1(z)}{\phi_2(z)} \\
&\vdots \\
a_n &= \lim_{z\to z_0} \frac{f(z) - a_0\phi_0(z) - \ldots - a_{n-1}\phi_{n-1}(z)}{\phi_n(z)}\,.
\end{aligned} \tag{1.39}$$

4. Often the first term in an asymptotic expansion is all that is required in practice. It is called the *dominant term.*

1.4 The "Sum" of an Asymptotic Series

These asymptotic series expansions resemble convergent expansions in several ways. They both provide families of approximations for a given function $f(z)$. And they both can be systematically obtained by starting from the given function.

But with each convergent series, we can associate a unique function $f(z)$ by summing, while on the other hand we know that it is possible for a given asymptotic series to represent more than one function.

Definition 9 *Given a domain D whose closure contains the point z_0, the asymptotic sum of an asymptotic series corresponding to the sequence $\{\phi_n(z)\}$ asymptotic as $z \to z_0$ and the coefficients $\{a_n\}$ is the equivalence class $[f]$ of functions $f(z)$ defined in D that satisfy*

$$f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z) \quad \textit{as} \quad z \to z_0\,, \tag{1.40}$$

which means that

$$f(z) = \sum_{n=0}^{N} a_n \phi_n(z) + o(\phi_N(z)) \quad \textit{as} \quad z \to z_0 \tag{1.41}$$

for all N.

We can show the following.

Proposition 4 (Asymptotic summability) *Let $\{\phi_n\}$ be an asymptotic sequence as $z \to z_0$ in D, and let $\{a_n\}$ be an arbitrary sequence of complex constants. Then the asymptotic series*

$$\sum_{n=0}^{\infty} a_n \phi_n(z) \tag{1.42}$$

has an asymptotic sum. That is, there exists at least one function $f(z)$ for which

$$f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z) \qquad as \qquad z \to z_0 \,. \tag{1.43}$$

Proof: Without loss of generality, take all $a_n \neq 0$. Since $\phi_{n+1}(z) = o(\phi_n(z))$ as $z \to z_0$ it is also true that $a_{n+1}\phi_{n+1}(z) = o(a_n\phi_n(z))$ as $z \to z_0$. So we can find a decreasing sequence of radii $r_n > 0$ so that

$$|a_{n+1}\phi_{n+1}(z)| \leq \frac{1}{2}|a_n\phi_n(z)| \qquad \text{as long as} \qquad |z - z_0| \leq r_n \,. \tag{1.44}$$

Let the sequence of functions $\{\mu_n(z)\}$ be defined as follows:

$$\mu_n(z) = \begin{cases} 0\,, & |z - z_0| > r_n\,, \\ \dfrac{r_n - |z - z_0|}{r_n - r_{n+1}}\,, & r_{n+1} \leq |z - z_0| \leq r_n\,, \\ 1\,, & |z - z_0| < r_{n+1}\,, \end{cases} \tag{1.45}$$

and set

$$f(z) = \sum_{n=0}^{\infty} a_n \mu_n(z) \phi_n(z) \,. \tag{1.46}$$

First we need to establish that this infinite sum defines a function by actually being convergent. If the sequence $\{r_n\}$ converges to zero, then for any fixed z the sum essentially contains a finite number of terms and convergence is therefore not an issue. Otherwise, there are values of z for which all $\mu_n(z) \equiv 1$. But for these z values, $|z - z_0| < r_n$ for all n, and thus

$$|f(z)| = \left|\sum_{n=0}^{\infty} a_n \mu_n(z)\phi_n(z)\right| = \left|\sum_{n=0}^{\infty} a_n \phi_n(z)\right| \leq \sum_{n=0}^{\infty} |a_n\phi_n(z)| \leq |a_0\phi_0(z)| \sum_{n=0}^{\infty} \frac{1}{2^n} = 2|a_0\phi_0(z)| \,. \tag{1.47}$$

So the series converges for all z in D with the possible exception of z_0. It remains to show that the function $f(z)$ defined by the sum (1.46) has the given asymptotic expansion (1.42). Suppose N is fixed and that $|z - z_0| < r_N$. Then,

$$f(z) - \sum_{n=0}^{N} a_n \phi_n(z) = \sum_{n=N+1}^{\infty} a_n \mu_n(z) \phi_n(z) \tag{1.48}$$

so

$$\begin{aligned}
\left| f(z) - \sum_{n=0}^{N} a_n \phi_n(z) \right| &= \left| \sum_{n=N+1}^{\infty} a_n \mu_n(z) \phi_n(z) \right| \\
&\leq \sum_{n=N+1}^{\infty} |\mu_n(z)| \cdot |a_n \phi_n(z)| \\
&\leq \sum_{n=N+1}^{\infty} |a_n \phi_n(z)| \qquad (1.49) \\
&\leq |a_{N+1}\phi_{N+1}(z)| \sum_{n=N+1}^{\infty} \frac{1}{2^{n-N-1}} \\
&= 2|a_{N+1}\phi_{N+1}(z)| \\
&= o(\phi_N(z)) \,.
\end{aligned}$$

Therefore the formal (and generally nonconvergent) series (1.42) is indeed asymptotic to $f(z)$ given by the convergent series (1.46). □

1.5 Dominant Balances and Asymptotic Root-Finding

1.5.1 Example of regular expansions.

Consider finding the roots of the polynomial $P(x) = x^3 - x + 2\epsilon$ when ϵ is small. Because ϵ appears in the coefficients, it is natural to seek the roots in the form of asymptotic power series expansions. So, try an asymptotic expansion of the form

$$x \sim x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots \qquad (1.50)$$

for some unknown coefficients x_0, x_1, x_2 and so on, and substitute into the polynomial equation $P(x) = 0$:

$$(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots)^3 - (x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots) + 2\epsilon = 0 \,. \qquad (1.51)$$

Multiply out and collect like powers of ϵ. At "leading order", we have the terms proportional to ϵ^0:

$$x_0^3 - x_0 = 0 \qquad (1.52)$$

which has three solutions

$$x_0 = \begin{cases} 0 \\ 1 \\ -1 \end{cases} \qquad (1.53)$$

At the next order, we have the terms proportional to ϵ^1:

$$3x_0^2 x_1 - x_1 + 2 = 0 \,. \qquad (1.54)$$

This equation is linear in the correction x_1, given a value of x_0. It is *always* true that the equation you get for each correction beyond leading order will be linear. It is also usually true that the equations for the corrections depend on the solutions of the lower order corrections. So in this case,

$$x_1 = \frac{2}{1 - 3x_0^2} \qquad (1.55)$$

which is different for different leading order roots x_0. So, through order ϵ, the roots of $P(x) = 0$ are

$$\begin{aligned}
x &= 2\epsilon + o(\epsilon) \,, \\
x &= 1 - \epsilon + o(\epsilon) \,, \qquad (1.56) \\
x &= -1 - \epsilon + o(\epsilon) \,.
\end{aligned}$$

This elementary procedure has thus produced asymptotic approximations for all three roots of the cubic equation $P(x) = 0$. Because the problem for $\epsilon \neq 0$ is of the same type as the special case of the problem for $\epsilon = 0$, this is called a *regular perturbation problem.*

1.5.2 Example of singular expansions. Dominant balance.

Consider now finding the roots of $Q(x) = \epsilon x^3 + \epsilon x^2 - x - 1$ when ϵ is small. Try the same method as in the previous example, seeking roots as asymptotic power series $x \sim x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots$. We find at leading order the equation

$$-x_0 - 1 = 0 \tag{1.57}$$

which gives the unique solution $x_0 = -1$. At order ϵ we get

$$x_0^3 + x_0^2 - x_1 = 0 \tag{1.58}$$

or $x_1 = x_0^3 + x_0^2 = 0$. So through order ϵ we have the asymptotic approximation

$$x = -1 + o(\epsilon)\,. \tag{1.59}$$

But $Q(x)$ is a cubic polynomial and has three roots. So where are the other two roots? They are "at infinity". We can find them by rescaling the problem. For some unknown exponent P set

$$x = \epsilon^P y \tag{1.60}$$

where we think of keeping y bounded and also bounded away from zero as ϵ tends to zero. Then, the equation $Q(x) = 0$ becomes

$$\underset{\text{term I}}{\epsilon^{3P+1} y^3} + \underset{\text{term II}}{\epsilon^{2P+1} y^2} - \underset{\text{term III}}{\epsilon^P y} - \underset{\text{term IV}}{1} = 0\,. \tag{1.61}$$

We can try to find the correct value of P by using the *principle of dominant balance.* In this context, the principle of dominant balance says that the rescaled equation is consistent as ϵ tends to zero only if at least two terms correspond to the same power of ϵ (this is called a *balance*) and moreover the balance is *dominant* in the sense that every term not involved in the balance corresponds to a higher power of ϵ and is therefore asymptotically smaller than the balancing terms. Here is an exhaustive list of all possibilities for this problem:

1. **Balancing term III with term IV.** This requires choosing $P = 0$. This balance is dominant since terms I and II are then both $O(\epsilon)$ which is smaller than $O(1)$ as $\epsilon \to 0$. But it gives us nothing new in the sense that we already have an expansion for $P = 0$.

2. **Balancing term II with term IV.** This requires choosing $2P + 1 = 0$ or $P = -1/2$. This balance is rejected because it is not dominant; for example term III is then $O(\epsilon^{-1/2})$ which is bigger than either term involved in the balance.

3. **Balancing term I with term IV.** This requires choosing $3P + 1 = 0$ or $P = -1/3$. Not dominant since term III is then $O(\epsilon^{-1/3})$.

4. **Balancing term II with term III.** This requires choosing $2P + 1 = P$ or $P = -1$. Not dominant since term I is then $O(\epsilon^{-2})$ which is bigger than the size of the balancing terms $O(\epsilon^{-1})$.

5. **Balancing term I with term III.** This requires choosing $3P + 1 = P$ or $P = -1/2$. This balance is dominant, since the balancing terms are then $O(\epsilon^{-1/2})$ while terms II and IV are only $O(1)$.

6. **Balancing term I with term II.** This requires choosing $3P + 1 = 2P + 1$ or $P = 0$. Not dominant since term III is then only $O(1)$ while the balancing terms are smaller, being $O(\epsilon)$.

So the only new dominant balance we found is to take $P = -1/2$ which balances terms I and III to leading order. The equation becomes

$$y^3 + \epsilon^{1/2} y^2 - y - \epsilon^{1/2} = 0\,, \tag{1.62}$$

after dividing out the lowest power of ϵ. Suppose that $\epsilon > 0$ to clarify the square root as a real number that could be positive or negative. So we could set $\mu = \epsilon^{1/2}$ which goes to zero as ϵ does, and then we get

$$y^3 + \mu y^2 - y - \mu = 0\,. \tag{1.63}$$

As in the previous example, it now appears reasonable to try to expand y in an asymptotic power series in μ: $y \sim y_0 + \mu y_1 + \mu^2 y_2 + \ldots$. Substituting into the equation we find at leading order

$$y_0^3 - y_0 = 0\,, \tag{1.64}$$

which has solutions $y_0 = 0, 1, -1$. The solution $y_0 = 0$ violates our assumption that y should be bounded away from zero as μ tends to zero; moreover if we followed up this solution we would in fact obtain an expansion equivalent to the one we already found. So we consider only $y_0 = 1$ and $y_0 = -1$. At order μ we find

$$3y_0^2 y_1 + y_0^2 - y_1 - 1 = 0 \tag{1.65}$$

which for $y_0 = \pm 1$ gives $y_1 = 0$ in both cases. So, through order μ we have

$$y = \pm 1 + o(\mu)\,. \tag{1.66}$$

In terms of the original coordinate x, these expansions become

$$x = \pm \epsilon^{-1/2} + o(1)\,. \tag{1.67}$$

The problem of finding the roots of $Q(x)$ asymptotically as $\epsilon \to 0$ is a *singular* problem because the problem obtained by setting $\epsilon = 0$ is of a different nature than the problem for $\epsilon \neq 0$. Singular problems can often be made regular by insightful rescaling and the judicious use of the principle of dominant balance.

Chapter 2

Laplace's Method for Asymptotic Expansions of Integrals

2.1 Elementary Techniques for Integrals

There are two very basic techniques for obtaining asymptotic expansions of functions defined by integrals:

1. Repeated integration by parts.
2. Expansion of the integrand.

Both of these methods usually admit the explicit estimation of a remainder in the form of an integral, so it is possible to directly prove the validity of the expansion in the limit of interest.

As a first example of the first type of method, recall the expansion we obtained previously for the exponential integral:

$$Ei(x) = \int_x^\infty \frac{e^{-t}}{t} dt \sim e^{-x}\left[\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \dots\right] \quad \text{as} \quad x \to \infty \tag{2.1}$$

on the positive real axis. The second type of method is even more straightforward: find a convergent expansion of the integrand whose domain of convergence includes the whole path of integration, and just integrate term-by-term.

These two methods can be illustrated and contrasted in the example of the incomplete gamma function defined for $x > 0$ and $a > 0$ by the formula

$$\gamma(a, x) := \int_0^x e^{-t} t^{a-1}\, dt\,. \tag{2.2}$$

The relation with the usual gamma function is $\Gamma(a) = \gamma(a, +\infty)$. First imagine finding an asymptotic expansion of $\gamma(a, x)$ valid for a fixed in the limit $x \to 0$. Since the integral only involves values of t very close to zero, and since we know that the Taylor expansion of a analytic function of t at $t = 0$ is also asymptotic as $t \to 0$, we can expand

$$e^{-t} = \sum_{n=0}^{N} (-1)^n \frac{t^n}{n!} + r_N(t) \tag{2.3}$$

where the remainder satisfies

$$|r_N(t)| \le \frac{K}{(N+1)!}|t|^{N+1} \tag{2.4}$$

where we can choose the same constant $K > 0$ for all N and t satisfying $|t| < 1$, say. Then,

$$\gamma(a, x) = \sum_{n=0}^{N} \frac{(-1)^n}{n!} \int_0^x t^{n+a-1}\, dt + \int_0^x r_N(t) t^{a-1}\, dt = \sum_{n=0}^{N} \frac{(-1)^n x^{n+a}}{n!(n+a)} + \int_0^x r_N(t) t^{a-1}\, dt\,. \tag{2.5}$$

Using the estimate on the remainder of the Taylor series, we get

$$\left| \int_0^x r_N(t) t^{a-1}\, dt \right| \le \int_0^x |r_N(t)| t^{a-1}\, dt \le \frac{K}{(N+1)!} \int_0^x t^{N+a}\, dt = \frac{K x^{N+a+1}}{(N+1)!(N+a+1)}\,, \tag{2.6}$$

so the error in the N-term expansion of the integral is $O(x^{N+a+1}) = o(x^{N+a})$ as $x \to 0+$ which proves that the expansion is asymptotic as $x \to 0+$, that is,

$$\gamma(a,x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+a}}{n!(n+a)} \qquad \text{as} \qquad x \to 0+\,. \tag{2.7}$$

Now this expansion can be easily seen to be convergent for all $x > 0$, even though it is only asymptotic for $x \to 0+$. Thus it could in principle be used practically in two different ways:

1. Fix N and consequently get a smaller and smaller error by considering values of x closer and closer to zero. That is, use the asymptotic property of the expansion.

2. Fix $x > 0$ and consequently get a smaller and smaller error by taking N larger and larger. That is, use the convergent property of the expansion.

An important point, however, is that exploiting the convergent property to get more and more accurate approximations for fixed x requires more and more work to get the error smaller. Here is a concrete exploration of this property: suppose we want to calculate $\gamma(3/2, 100)$ using this convergent expansion. By a careful numerical integration, it is known that the exact value of the integral to six decimal places is $\gamma(3/2, 100) \approx 0.886227$. Denote by $S_N(a,x)$ the Nth partial sum:

$$S_N(a,x) := \sum_{n=0}^{N} \frac{(-1)^n x^{n+a}}{n!(n+a)}\,. \tag{2.8}$$

Then, we get the following summary of convergence:

$$\begin{aligned}
S_{10}(3/2,100) &\approx 2.16 \times 10^{15} \\
S_{100}(3/2,100) &\approx 5.24 \times 10^{42} \\
S_{200}(3/2,100) &\approx 2.09 \times 10^{25} \\
S_{250}(3/2,100) &\approx 3.50 \times 10^{7} \\
S_{270}(3/2,100) &\approx 1.03495 \\
S_{271}(3/2,100) &\approx 0.832 \\
S_{272}(3/2,100) &\approx 0.906 \\
S_{273}(3/2,100) &\approx 0.879 \\
S_{274}(3/2,100) &\approx 0.889 \\
S_{275}(3/2,100) &\approx 0.885 \\
S_{276}(3/2,100) &\approx 0.887 \\
S_{277}(3/2,100) &\approx 0.886
\end{aligned} \tag{2.9}$$

So to get only three digits of accuracy in the result requires 277 terms in the series! Worse yet, if you added up significantly fewer terms out of ignorance, the answer supplied by the truncated series would be wildly wrong!

This shows that the asymptotic property of the series as $x \to 0+$ is much more useful than the convergent property for all $x > 0$. To describe $\gamma(a,x)$ efficiently for x values that are not small, it is much better to develop a divergent asymptotic expansion valid as $x \to +\infty$. This can be done by first separating the integral as follows:

$$\gamma(a,x) = \int_0^{\infty} e^{-t} t^{a-1}\, dt - \int_x^{\infty} e^{-t} t^{a-1}\, dt = \Gamma(a) - \int_x^{\infty} e^{-t} t^{a-1}\, dt\,. \tag{2.10}$$

Because the latter integral is convergent it is necessary that it goes to zero as $x \to +\infty$. We can approximate it asymptotically by repeatedly integrating by parts, which is the first elementary technique we learned. So,

$$\begin{aligned}
\gamma(a,x) &= \Gamma(a) - e^{-x}x^{a-1} - (a-1)\int_x^\infty e^{-t}t^{a-2}\,dt \\
&= \Gamma(a) - e^{-x}x^{a-1} - (a-1)e^{-x}x^{a-2} - (a-1)(a-2)\int_x^\infty e^{-t}t^{a-3}\,dt \\
&\vdots \\
&= \Gamma(a) - e^{-x}x^{a-1}\left[1 + \frac{a-1}{x} + \frac{(a-1)(a-2)}{x^2} + \ldots + x^{-N}\prod_{n=1}^{N}(a-n)\right] + R_N(a,x)
\end{aligned} \tag{2.11}$$

where the error is

$$R_N(a,x) := -\prod_{n=1}^{N+1}(a-n)\cdot\int_x^\infty e^{-t}t^{a-N-2}\,dt\,. \tag{2.12}$$

It is easy to estimate (bound) the error:

$$\begin{aligned}
|R_N(a,x)| &= \prod_{n=1}^{N+1}|a-n|\cdot\int_x^\infty e^{-t}t^{a-N-2}\,dt \\
&\le x^{a-N-2}\prod_{n=1}^{N+1}|a-n|\cdot\int_x^\infty e^{-t}\,dt \\
&= e^{-x}x^{a-N-2}\prod_{n=1}^{N+1}|a-n|\,.
\end{aligned} \tag{2.13}$$

which proves that $R_N(a,x) = o(e^{-x}x^{a-N-1})$ as $x \to \infty$ with $a > 1$ fixed. Therefore although divergent, the following series

$$\gamma(a,x) \sim \Gamma(a) - e^{-x}x^{a-1}\left[1 + \frac{a-1}{x} + \frac{(a-1)(a-2)}{x^2} + \ldots\right] \tag{2.14}$$

is asymptotic as $x \to \infty$. By means of illustration, we can look at the accuracy of keeping just the first correction to the constant term $\Gamma(a)$; namely let us compare

$$\gamma(a,x) \qquad \text{with} \qquad \Gamma(a) - e^{-x}x^{a-1}\,. \tag{2.15}$$

Taking $a = 3/2$ and $x = 100$ as before, this simple approximation gives a value of ≈ 0.886227 which agrees with the exact result to (at least) six digits. Also, if a is a positive integer, the series truncates and gives the exact result.

2.2 Watson's Lemma

There is a whole collection of general techniques for obtaining asymptotic expansions of certain integrals that we will soon discuss: Laplace's method, the method of steepest descents, and the method of stationary phase. These methods allow one to figure out what a good candidate for an asymptotic expansion is, however the *proof* that the candidate expansion is asymptotic is usually different in different cases. However, many proofs involve reducing the problem to the study of a certain type of integral of the form

$$f(x) = \int_0^T e^{-xt}\phi(t)\,dt \tag{2.16}$$

for which we can *rigorously* compute asymptotics as $x \to +\infty$. Establishing the asymptotic expansion of such integrals is the aim of *Watson's Lemma*.

Proposition 5 (Watson's Lemma) *Suppose $\phi(t) = t^\lambda g(t)$ where $\lambda > -1$ and $g(t)$ is bounded in $[0,T]$ and analytic at $t = 0$. Then the integral (2.16) exists for all $x > 0$, and*

$$f(x) \sim \sum_{n=0}^{\infty} \frac{g^{(n)}(0)\Gamma(\lambda+n+1)}{n!x^{\lambda+n+1}} \qquad as \qquad x \to +\infty\,. \tag{2.17}$$

Perhaps the most interesting thing to observe is that the expansion does not involve the upper limit of integration T at all. All contributions to the integral come, in the limit $x \to +\infty$, from the neighborhood of $t = 0$. This does not mean that the value of $f(x)$ for any given x is independent of T. But the factor e^{-xt} becomes very concentrated near $t = 0$ for large positive x.

Here is a sketch of a proof of Watson's Lemma. First notice that without any loss of generality we can take $g(0) \neq 0$. If $g(0)$ turns out to be zero, then we can just factor out a t from the Taylor series for $g(t)$ about $t = 0$ and encorporate it into the t^λ factor by incrementing λ.

Next, we want to put the Taylor series for $g(t)$ in the integrand. But this series has a radius of convergence r which might be less than T. If $T > r$, then we can pick some positive $s < r$ and write

$$f(x) = \int_0^s e^{-xt}\phi(t)\,dt + \int_s^T e^{-xt}\phi(t)\,dt \tag{2.18}$$

and we can get the following bound:

$$\left|\int_s^T e^{-xt}\phi(t)\,dt\right| \leq \int_s^T e^{-xt}|t^\lambda g(t)|\,dt \leq \sup_{s<t<T}|t^\lambda g(t)|\int_s^T e^{-xt}\,dt = \sup_{s<t<T}|t^\lambda g(t)|\frac{e^{-xs}-e^{-xT}}{x}\,, \tag{2.19}$$

which is $o(x^{-p})$ for any positive p. So this is a really small contribution. It shows also incidentally why the upper limit of integration doesn't really matter in the final expansion.

So we are left looking at the integral from zero to s where s is some positive number inside the radius of convergence of the Taylor series for $g(t)$:

$$g(t) = \sum_{n=0}^{N} \frac{g^{(n)}(0)}{n!}t^n + r_N(t) \tag{2.20}$$

where there is some constant $K > 0$ so that

$$|r_N(t)| \leq \frac{Kt^{N+1}}{(N+1)!}\,. \tag{2.21}$$

Now we use this expansion in the integrand:

$$\int_0^s e^{-xt}t^\lambda g(t)\,dt = \sum_{n=0}^{N}\frac{g^{(n)}(0)}{n!}\int_0^s e^{-xt}t^{\lambda+n}\,dt + \int_0^s e^{-xt}t^\lambda r_N(t)\,dt\,, \tag{2.22}$$

and we have that

$$\left|\int_0^s e^{-xt}t^\lambda r_N(t)\,dt\right| \leq \int_0^s e^{-xt}t^\lambda|r_N(t)|\,dt \leq \frac{K}{(N+1)!}\int_0^s e^{-xt}t^{\lambda+N+1}\,dt\,. \tag{2.23}$$

To deal with the integrals in the sum from 0 to N in (2.22) and with the integral on the right-hand side of (2.23) amounts to the same problem. Namely, consider the integral

$$I_p(x) = \int_0^s e^{-xt}t^{\lambda+p}\,dt\,, \tag{2.24}$$

for $x \to +\infty$. Well, changing variables from t to $\tau = xt$, we get

$$\begin{aligned} I_p(x) &= x^{-(\lambda+p+1)} \int_0^{xs} e^{-\tau} \tau^{\lambda+p}\, d\tau \\ &= x^{-(\lambda+p+1)} \int_0^{\infty} e^{-\tau} \tau^{\lambda+p}\, d\tau - x^{-(\lambda+p+1)} \int_{xs}^{\infty} e^{-\tau} \tau^{\lambda+p}\, d\tau \qquad (2.25) \\ &= x^{-(\lambda+p+1)} \Gamma(\lambda+p+1) - x^{-(\lambda+p+1)} \int_{xs}^{\infty} e^{-\tau} \tau^{\lambda+p}\, d\tau\,. \end{aligned}$$

The second term here is extremely small. Since

$$\begin{aligned} \int_{xs}^{\infty} e^{-\tau} \tau^{\lambda+p}\, d\tau &= e^{-xs} \int_{xs}^{\infty} e^{xs-\tau} \tau^{\lambda+p}\, d\tau \\ &= e^{-xs} \int_0^{\infty} e^{-w} (w+xs)^{\lambda+p}\, dw \qquad (2.26) \\ &= e^{-xs} (xs)^{\lambda+p} \int_0^{\infty} e^{-w} \left(1 + \frac{w}{xs}\right)^{\lambda+p} dw \end{aligned}$$

and since $1 + \alpha \le e^{\alpha}$ for all positive α, we get

$$\left| \int_{xs}^{\infty} e^{-\tau} \tau^{\lambda+p}\, d\tau \right| \le e^{-xs} (xs)^{\lambda+p} \int_0^{\infty} e^{((\lambda+p)/(xs)-1)w}\, dw = e^{-xs} (xs)^{\lambda+p} \frac{xs}{xs-(\lambda+p)}\,, \qquad (2.27)$$

which is $o(x^q)$ for any real q. Therefore,

$$I_p(x) = x^{-(\lambda+p+1)} \Gamma(\lambda+p+1) + o(x^q) \quad \text{as} \quad x \to +\infty\,, \qquad (2.28)$$

for all q however negative. The $o(x^q)$ term is therefore beyond all orders with respect to the sequence of powers $\{x^{-(\lambda+n+1)}\}$ with respect to which we are constructing the asymptotic expansion and plays no further role. Using this result in (2.22) and (2.23) finally proves the asymptotic nature of the expansion given by Watson's Lemma.

2.3 Elementary Applications of Watson's Lemma

Watson's Lemma also holds under some more general circumstances. For example, x need not be strictly real and positive. It is sufficient if $|x| \to +\infty$ and $-\theta \le \arg(x) \le \theta$ for some angle θ satisfying $0 \le \theta < \pi/2$, *i.e.* if x is restricted to go to infinity in some sector in the right half-plane. Also, if the upper limit $T = +\infty$, then it is not necessary for $\phi(t)$ to be bounded as $t \to +\infty$ as long as the integral converges (exists) for sufficiently large x. So, for example, if

$$f(x) = \int_0^{\infty} e^{-xt} \log(1+t^2)\, dt \qquad (2.29)$$

then the integral exists for all $x > 0$ even though $\log(1+t^2)$ is unbounded for large t. The part of the argument in which the integral from s to T is shown to be negligible compared to any term in the asymptotic series must be modified. Here, we can get the bound we need (assuming x real, for simplicity) using the Cauchy-Schwarz inequality:

$$\begin{aligned} \left| \int_s^{\infty} e^{-xt} \log(1+t^2)\, dt \right| &= \int_s^{\infty} e^{-xt} \log(1+t^2)\, dt \\ &= \int_s^{\infty} t e^{-xt} \cdot \frac{\log(1+t^2)}{t}\, dt \qquad (2.30) \\ &\le \sqrt{\int_s^{\infty} t^2 e^{-2xt}\, dt} \sqrt{\int_s^{\infty} \frac{\log(1+t^2)^2}{t^2}\, dt}\,. \end{aligned}$$

The second integral now exists because of the t^{-2} decay. It is a number, independent of x. The first integral is elementary to evaluate using integration by parts (twice). We get

$$\int_s^\infty t^2 e^{-2xt}\,dt = \frac{1+2sx+2s^2x^2}{4x^3}e^{-2xs} \tag{2.31}$$

which is exponentially small (in particular $o(x^{-p})$ for any real p) as $x \to +\infty$. This shows that the rest of the proof of Watson's lemma goes through in this case as well. So since

$$\log(1+t^2) = \sum_{n=1}^\infty (-1)^{n+1}\frac{t^{2n}}{n} \tag{2.32}$$

for $0 \le t < 1$, we can take any positive $s < 1$ in the proof. We then get from Watson's Lemma:

$$\int_0^\infty e^{-xt}\log(1+t^2)\,dt \sim \sum_{n=1}^\infty \frac{(-1)^{n+1}(2n)!}{nx^{2n+1}} \quad \text{as} \quad x \to +\infty\,, \tag{2.33}$$

because $\Gamma(2n+1) = (2n)!$.

Lots of problems may involve finding approximations to integrals that are already in a form amenable to Watson's Lemma. For example, the formula

$$f(s) = \int_0^T e^{-st}\phi(t)\,dt \tag{2.34}$$

says that the function $f(s)$ is the *Laplace transform* of the signal $\phi(t)H(t)H(T-t)$ where $H(t)$ is the Heaviside step function:

$$H(t) = \begin{cases} 0\,, & \text{if} \quad t \le 0\,, \\ 1\,, & \text{if} \quad t > 0\,. \end{cases} \quad . \tag{2.35}$$

So in this case Watson's Lemma gives a way of finding accurate approximations for the Laplace transform of a signal when the transform variable s is large and confined to a sector in the right half-plane.

As another example, very important in its own right, let α and β be positive numbers, and consider the integral

$$f(x) = \int_{-\alpha}^\beta e^{-xt^2}\phi(t)\,dt\,, \tag{2.36}$$

where $\phi(t)$ is bounded, and analytic at $t = 0$. This integral doesn't appear at first glance to be of the type amenable to treatment by Watson's lemma, but it can be cast into such a form directly. First, if we let $m = \min(\alpha, \beta)$, we can show by a direct estimate that

$$f(x) - \int_{-m}^m e^{-xt^2}\phi(t)\,dt = o(x^p) \quad \text{as} \quad x \to +\infty \tag{2.37}$$

for any real power p, *i.e.* dropping the nonsymmetrical part of the domain of integration amounts to an exponentially small error. And then by symmetry,

$$\int_{-m}^m e^{-xt^2}\phi(t)\,dt = \int_{-m}^m e^{-xt^2}\phi_{\text{even}}(t)\,dt = 2\int_0^m e^{-xt^2}\phi_{\text{even}}(t)\,dt \tag{2.38}$$

where $\phi_{\text{even}}(t)$ is the *even part* of $\phi(t)$:

$$\phi_{\text{even}}(t) = \frac{1}{2}(\phi(t) + \phi(-t))\,. \tag{2.39}$$

The Taylor series for $\phi_{\text{even}}(t)$ consists of only the even power terms from the Taylor series for $\phi(t)$. So

$$\phi(t) = \sum_{n=0}^\infty a_n t^n \quad \text{implies} \quad \phi_{\text{even}}(t) = \sum_{n=0}^\infty a_{2n}(t^2)^n\,, \tag{2.40}$$

for t within the radius of convergence. Now changing variables to $\tau = t^2$, we get

$$2\int_0^m e^{-xt^2}\phi_{\text{even}}(t)\,dt = \int_0^{m^2} e^{-x\tau}\tau^{-1/2}\phi_{\text{even}}(\tau^{1/2})\,d\tau \tag{2.41}$$

and inside the radius of convergence ϕ_{even} is an analytic function of τ at $\tau = 0$ because

$$\phi_{\text{even}}(\tau^{1/2}) = \sum_{n=0}^{\infty} a_{2n}\tau^n\,. \tag{2.42}$$

So we may apply Watson's Lemma with $\lambda = -1/2$. We get

$$\int_{-\alpha}^{\beta} e^{-xt^2}\phi(t)\,dt \sim \sum_{n=0}^{\infty} \frac{a_{2n}\Gamma(n+1/2)}{x^{n+1/2}} \qquad \text{as} \qquad x \to +\infty\,. \tag{2.43}$$

Or, since $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(z+1) = z\Gamma(z)$, we find

$$\Gamma(n+1/2) = \frac{(2n)!}{2^{2n}n!}\sqrt{\pi}\,, \tag{2.44}$$

so

$$\int_{-\alpha}^{\beta} e^{-xt^2}\phi(t)\,dt \sim \sqrt{\frac{\pi}{x}}\sum_{n=0}^{\infty} \frac{a_{2n}(2n)!}{2^{2n}n!}x^{-n} \qquad \text{as} \qquad x \to +\infty\,. \tag{2.45}$$

2.4 Laplace's Method

In the statement of Watson's Lemma, one sees that all the asymptotic contributions to the integral

$$f(x) = \int_0^T e^{-xt}\phi(t)\,dt \tag{2.46}$$

as $x \to +\infty$ come from the neighborhood of $t = 0$, which is the maximum on $[0, t]$ of the function $h(t) = -t$ that is multiplied by x in the exponent. Likewise, all the asymptotic contributions to

$$f(x) = \int_{-\alpha}^{\beta} e^{-xt^2}\phi(t)\,dt \tag{2.47}$$

as $x \to +\infty$ come again from the neighborhood of $t = 0$, which is now the maximum on $[-\alpha, \beta]$ of the function $h(t) = -t^2$.

In 1820, Laplace himself considered real integrals of the general form

$$f(x) = \int_a^b e^{xh(t)}g(t)\,dt \tag{2.48}$$

in the limit $x \to +\infty$. His general strategy was to argue that the dominant contributions to $f(x)$ as $x \to +\infty$ will come from the immediate neighborhood of the point or points in $[a, b]$ where the function $h(t)$ achieves its maximum value.

Unlike in the elementary examples previously considered, it is usually not the case that

$$h_{\max} := \sup_{a \le t \le b} h(t) \tag{2.49}$$

is equal to zero. But by factoring out an exponential factor we can reduce the problem to one of this type:

$$f(x) = e^{xh_{\max}}\int_a^b e^{x\tilde{h}(t)}g(t)\,dt \tag{2.50}$$

where $\tilde{h}(t) := h(t) - h_{\max}$ which has a maximum value of 0.

2.4.1 Nonlocal contributions.

If one makes the reasonable assumption that $h(t)$ is a continuous function taking its maximum value at a finite number of values in $[a,b]$ at most, and also if one assumes that $g(t)$ is a bounded function, then we can prove that contributions to

$$f(x)e^{-xh_{\max}} = \int_a^b e^{x\bar{h}(t)} g(t)\, dt \tag{2.51}$$

are negligible compared to any power of x as $x \to +\infty$. For δ small, let $[a,b]_\delta$ be the interval $[a,b]$ takeaway open subintervals of length 2δ symmetrically placed around each point where $\tilde{h}(t) = 0$. Then, by continuity,

$$\sup_{t\in[a,b]_\delta} \tilde{h}(t) = -K_\delta < 0\,. \tag{2.52}$$

So, it follows that

$$\left| \int_{[a,b]_\delta} e^{x\bar{h}(t)} g(t)\, dt \right| \le \int_{[a,b]_\delta} e^{x\bar{h}(t)} |g(t)|\, dt \le e^{-xK_\delta} \int_{[a,b]_\delta} |g(t)|\, dt \tag{2.53}$$

which is exponentially small as $x \to +\infty$, for all $\delta > 0$ however small.

The parts of the integral $f(x)e^{-xh_{\max}}$ near each point in $[a,b]$ where $h(t)$ achieves its maximum value will be asymptotically more significant. Now we will see how to obtain these contributions.

2.4.2 Dominant contributions from the endpoints.

Suppose first that $h(t)$ achieves its maximum at the left endpoint, $t = a$. For some positive δ as small as we please, we want to find an asymptotic expansion for

$$I_{\text{left}}(x) := \int_a^{a+\delta} e^{x\bar{h}(t)} g(t)\, dt = \int_0^\delta e^{x\bar{h}(\tau+a)} g(\tau+a)\, d\tau\,. \tag{2.54}$$

We will suppose that $h(t)$ and $g(t)$ are analytic at $t = a$, and that δ is less than the radii of convergence of their series. Since the left endpoint is a maximum of $h(t)$, it must be the case that $\tilde{h}'(a) = h'(a) < 0$. In fact, we will take δ to be small enough so that $\tilde{h}'(t) < 0$ over the whole range of integration. This suggests a change of variables from τ to a new variable s so that

$$\tilde{h}(\tau + a) = -s\,. \tag{2.55}$$

Clearly, $\tau = 0$ corresponds to $s = 0$. Now because $\tilde{h}' \neq 0$ when $\tau = 0$, it follows from the implicit function theorem that this equation can be solved for $\tau = \tau(s)$ as an analytic function. Using this change of variables, we get

$$I_{\text{left}}(x) = \int_0^{-\bar{h}(a+\delta)} e^{-xs} g(\tau(s)+a)\tau'(s)\, ds\,. \tag{2.56}$$

This is of the form treatable by Watson's Lemma. We only need the Taylor series coefficients of $g(\tau(s))\tau'(s)$ to write down the asymptotic expansion as $x \to +\infty$. Since

$$\tilde{h}(\tau + a) = h'(a)\tau + \frac{h''(a)}{2}\tau^2 + \ldots \tag{2.57}$$

is the convergent Taylor series for $\tilde{h}$, the Taylor series for $\tau(s)$ must begin like

$$\tau(s) = -\frac{1}{h'(a)}s + O(s^2)\,. \tag{2.58}$$

Because it is a convergent series, it may be differentiated term-by-term, so

$$\tau'(s) = -\frac{1}{h'(a)} + O(s)\,. \tag{2.59}$$

Therefore, by Watson's Lemma, since $\Gamma(1) = 1$,

$$I_{\text{left}}(x) = -\frac{g(a)}{h'(a)}x^{-1} + O(x^{-2}) \quad \text{as} \quad x \to +\infty \tag{2.60}$$

where we have only kept the first term. This clearly dominates the exponentially small contributions obtained earlier.

Similarly, if $h(t)$ achieves its maximum at the right endpoint, then we may consider the integral

$$I_{\text{right}}(x) := \int_{b-\delta}^{b} e^{x\bar{h}(t)} g(t)\, dt = \int_0^{\delta} e^{x\bar{h}(b-\tau)} g(b-\tau)\, d\tau\,, \tag{2.61}$$

for $\delta > 0$ sufficiently small. This time using the analytic change of variables

$$\tilde{h}(b-\tau) = -s\,, \tag{2.62}$$

which exists because now we assume that $h'(b) > 0$, we apply Watson's Lemma and obtain

$$I_{\text{right}}(x) = \frac{g(b)}{h'(b)}x^{-1} + O(x^{-2}) \quad \text{as} \quad x \to +\infty\,. \tag{2.63}$$

2.4.3 Dominant contributions from interior maxima.

The case when $h(t)$ achieves its maximum at an interior point $t_{\max}$ of the interval $[a, b]$ is different. We want to consider the integral

$$I_{\text{mid}}(x) := \int_{t_{\max}-\delta}^{t_{\max}+\delta} e^{x\bar{h}(t)} g(t)\, dt = \int_{-\delta}^{\delta} e^{x\bar{h}(\tau+t_{\max})} g(\tau + t_{\max})\, d\tau\,. \tag{2.64}$$

Suppose again that h and g are analytic at $t_{\max}$. Then the interior maximum is a local maximum, and by Fermat's theorem $h'(t_{\max}) = 0$. Also, we must have $h''(t_{\max}) \le 0$. Let us now assume the generic condition $h''(t_{\max}) < 0$.

The idea in this case is to attempt the change of variables

$$\tilde{h}(\tau + t_{\max}) = -s^2\,. \tag{2.65}$$

For this to define an analytic transformation, we need to be able to solve for $\tau = \tau(s)$ as an analytic function in the neighborhood of the point $\tau = s = 0$. The implicit function theorem applied directly fails because the first derivatives vanish. But we notice that when τ is small,

$$\tilde{h}(\tau + t_{\max}) = \frac{h''(t_{\max})}{2}\tau^2 + O(\tau^3)\,, \tag{2.66}$$

so if the transformation exists it must be the case that near $\tau = s = 0$,

$$\tau(s) = s\sqrt{\frac{-2}{h''(t_{\max})}} + O(s^2) \quad \text{as} \quad s \to 0\,. \tag{2.67}$$

So, introduce a new unknown, say $v(s)$ and seek $\tau(s)$ in the form

$$\tau(s) = s\sqrt{\frac{-2}{h''(t_{\max})}} \cdot v(s)\,, \tag{2.68}$$

where $v(0) = 1$. Now with this substitution, divide the equation defining τ through by s^2:

$$F(v, s) = 1 + \sum_{n=2}^{\infty} \frac{h^{(n)}(t_{\max})}{n!} \left(\frac{-2}{h''(t_{\max})}\right)^{n/2} s^{n-2} v^n = 0\,. \tag{2.69}$$

Now, it is a direct matter to calculate the partial derivative:

$$\left.\frac{\partial F}{\partial v}\right|_{v=1,s=0} = -2 \neq 0 \tag{2.70}$$

and thus the implicit function theorem allows us to solve for $v = v(s)$ as an analytic function.

With the change of variables $\tau = \tau(s)$ the integral becomes

$$I_{\rm mid}(x) = \int_{-\sqrt{-\bar{h}(t_{\rm max}-\delta)}}^{\sqrt{-\bar{h}(t_{\rm max}+\delta)}} e^{-xs^2} g(\tau(s)+t_{\rm max})\tau'(s)\,ds\,. \tag{2.71}$$

This is exactly the sort of integral for which we developed an asymptotic representation as $x \to +\infty$ as an immediate consequence of Watson's Lemma. To write down this representation to just one term, we need

$$g(\tau(0)+t_{\rm max}) = g(t_{\rm max}) \quad \text{and} \quad \tau'(0) = \sqrt{\frac{-2}{h''(t_{\rm max})}}\,. \tag{2.72}$$

Therefore, we find that

$$I_{\rm mid}(x) = \sqrt{\frac{-2\pi}{xh''(t_{\rm max})}}\,g(t_{\rm max}) + O(x^{-3/2}) \quad \text{as} \quad x \to +\infty\,. \tag{2.73}$$

Again these local contributions dominate any contributions to the integral over $[a,b]$ that don't come from any maxima of $h(t)$. However, the most important observation here is that these contributions from interior maxima also dominate any contributions from the endpoints, which are $O(x^{-1})$ as compared with the much larger $O(x^{-1/2})$. Therefore, if it is known that $h(t)$ achieves its maximum at a point interior to $[a,b]$, then there is no need to include any contributions from the endpoints to leading order, even if the maximum of $h(t)$ is also achieved there.

2.4.4 Summary.

So if the function $h(t)$ achieves its maximum $h_{\rm max}$ at an interior point of the interval of integration $[a,b]$, then letting $t_{\rm max}^{(1)}, t_{\rm max}^{(2)}, \ldots, t_{\rm max}^{(M)}$ denote the interior points where the maximum is achieved, Laplace's method shows that

$$\int_a^b e^{xh(t)}g(t)\,dt = e^{xh_{\rm max}}\left(x^{-1/2}\sum_{i=1}^{M}\sqrt{\frac{-2\pi}{h''(t_{\rm max}^{(i)})}}\,g(t_{\rm max}^{(i)}) + O(x^{-3/2})\right) \quad \text{as} \quad x \to +\infty\,. \tag{2.74}$$

On the other hand, if $h(t)$ achieves its maximum *only* at $t = a$, then

$$\int_a^b e^{xh(t)}g(t)\,dt = e^{xh_{\rm max}}\left(-\frac{g(a)}{h'(a)}x^{-1} + O(x^{-2})\right) \quad \text{as} \quad x \to +\infty\,, \tag{2.75}$$

and if $h(t)$ achieves its maximum *only* at $t = b$, then

$$\int_a^b e^{xh(t)}g(t)\,dt = e^{xh_{\rm max}}\left(\frac{g(b)}{h'(b)}x^{-1} + O(x^{-2})\right) \quad \text{as} \quad x \to +\infty\,, \tag{2.76}$$

and if $h(t)$ achieves its maximum at both a and b but at no interior points, then the asymptotic description is the sum of (2.75) and (2.76).

These statements only hold if certain generic conditions are satisfied, like $h'(t)$ not vanishing for maxima at endpoints, and $h''(t)$ not vanishing for interior maxima. Other higher-order asymptotic formulas need to be developed in these special cases. Further generalizations of Laplace's method can be developed to handle integrals like

$$f(x) = \int_a^b e^{xh(x,t)}g(t)\,dt \tag{2.77}$$

where the locations of maxima change with x, and so on.

2.5 Application of Laplace's Method to Weakly Diffusive Regularization of Shock Waves

The simplest model for the development of shock waves is the initial-value problem for the partial differential equation for $u(x,t)$ given by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0\,, \qquad u(x,0) = u_0(x)\,. \tag{2.78}$$

If the initial condition $u_0(x)$ is a "bell-shaped" function with a single maximum, then under the evolution in time t, the top of the bell will move to the right faster than the bottom, and eventually will steepen until an infinite derivative develops. This is called the development of a shock wave. Laplace's method can help to give an answer to the question of what happens next.

2.5.1 The method of characteristics.

The shock wave equation can be solved as follows. Suppose that we consider a curve in the (x,t)-plane given by $x = x(t)$. Then, by the chain rule

$$\frac{d}{dt}u(x(t),t) = \frac{\partial u}{\partial t} + x'(t)\frac{\partial u}{\partial x}\,. \tag{2.79}$$

So, the shock wave equation can be interpreted as follows. If $x(t)$ is a curve in the plane that satisfies $x'(t) = u(x(t),t)$, then

$$\frac{d}{dt}u(x(t),t) = 0\,. \tag{2.80}$$

From this we see that $x''(t) = 0$, so the curve must in fact be a straight line. When $t = 0$, we can pass a straight line $x(t)$ through each point ξ of the x-axis; the slope of this line must be $x'(0) = u(\xi,0) = u_0(\xi)$. These lines are illustrated for the choice $u_0(x) = \mathrm{sech}(x)$ in the figure.

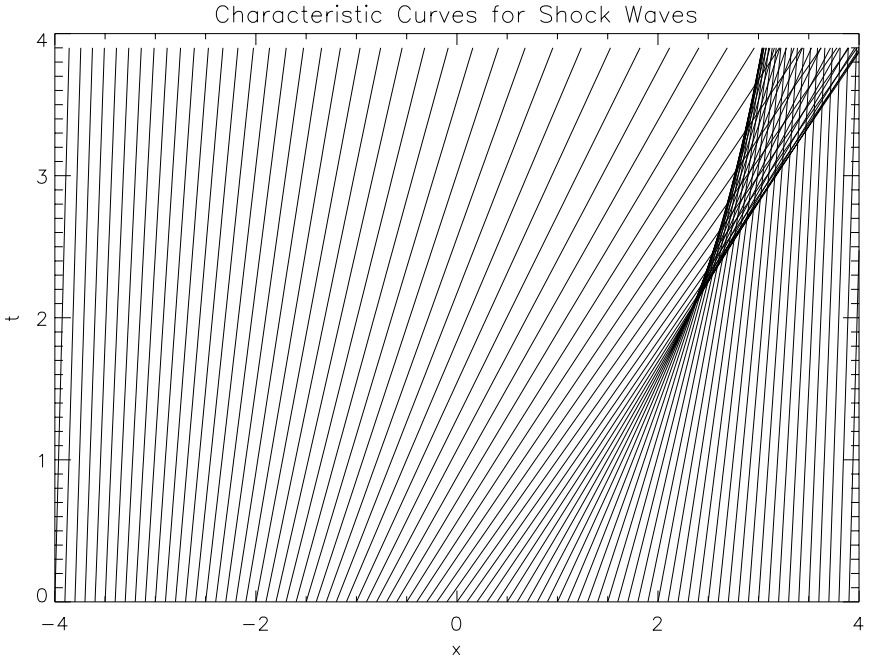

Figure 2.1: *The characteristic lines $x(t)$ for the shock wave equation with initial data $u_0(x) = \mathrm{sech}(x)$.*

Before the time of the shock wave, there is only one line passing through each (x,t). The solution $u(x,t)$ of the equation is then just the slope of this line. That is, on the line

$$x = \xi + u_0(\xi)t \tag{2.81}$$

The solution is simply $u = u_0(\xi)$. But after the shock forms, there is a zone in the (x,t)-plane where there are *three* characteristic lines passing through each point. The slopes of these lines give three possible candidates for a solution. The solution $u(x,t)$ predicted by the method of characteristics is a triple-valued function in this zone.

2.5.2 Regularization. Burgers' equation.

The point of view in physics is that whatever quantity (pressure, wave height, *etc.*) is being modeled must be single-valued. So when a model predicts multivaluedness, there is something wrong with the model. Usually one tries to add corrective terms to the model to fix the problem. One natural mechanism to mediate shock waves is diffusion. Consider $\nu > 0$ to be a small parameter (the viscosity), and look at the modified initial-value problem

$$\frac{\partial u^{(\nu)}}{\partial t} + u^{(\nu)}\frac{\partial u^{(\nu)}}{\partial x} = \nu\frac{\partial^2 u^{(\nu)}}{\partial x^2}\,, \qquad u^{(\nu)}(x,0) = u_0(x)\,. \tag{2.82}$$

This equation is called *Burgers' equation* and it is like the shock wave equation with a little bit of the heat (or diffusion) equation thrown in for good measure. When $\nu = 0$ we get back the shock wave equation. Also, it is known that when $\nu > 0$, the solution $u^{(\nu)}(x,t)$ exists for all time, which is *not* the case for $\nu = 0$. So what happens when ν is small but not zero? We can find out using Laplace's method.

2.5.3 Solving Burgers' equation. The Cole-Hopf transformation.

Burgers' equation can be solved transformed into the plain old diffusion equation by a remarkable change of variables attributed to Cole and Hopf, who independently rediscovered it in the early 1950's, but actually it had appeared much earlier in the textbook of Forsyth in the early 1900's. The idea is this: let

$$u^{(\nu)} = -\frac{2\nu}{\varphi^{(\nu)}}\frac{\partial\varphi^{(\nu)}}{\partial x} = -2\nu\frac{\partial}{\partial x}\log\varphi^{(\nu)}\,. \tag{2.83}$$

Then it is a direct matter to verify that $\varphi^{(\nu)}(x,t)$ satisfies

$$\frac{\partial\varphi^{(\nu)}}{\partial t} = \nu\frac{\partial^2\varphi^{(\nu)}}{\partial x^2}\,, \tag{2.84}$$

i.e. the diffusion equation with viscosity ν, with initial data

$$\varphi^{(\nu)}(x,0) = \varphi_0^{(\nu)}(x) = \exp\left(-\frac{1}{2\nu}\int_0^x u_0(\eta)\,d\eta\right)\,. \tag{2.85}$$

The solution of the linear diffusion equation can be expressed as an integral over "heat kernels":

$$\varphi^{(\nu)}(x,t) = \frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty}\varphi_0^{(\nu)}(\eta)e^{-(x-\eta)^2/4\nu t}\,d\eta\,. \tag{2.86}$$

Plugging this into formula (2.83), gives a formula for the solution of Burgers' equation with initial data $u_0(x)$:

$$u^{(\nu)}(x,t) = \frac{\displaystyle\int_{-\infty}^{\infty}\frac{x-\eta}{t}e^{-h/2\nu}\,d\eta}{\displaystyle\int_{-\infty}^{\infty}e^{-h/2\nu}\,d\eta} \tag{2.87}$$

where

$$h = h(\eta;x,t) = \int_0^\eta u_0(\xi)\,d\xi + \frac{(x-\eta)^2}{2t}\,. \tag{2.88}$$

2.5.4 Analysis of the solution in the zero viscosity limit.

The limit of zero viscosity can be analyzed by assuming x and t are fixed constants and studying the integrals in the formula (2.87) in the asymptotic limit $\nu \to 0+$. We use Laplace's method. The function $-h(\eta; x, t)$ behaves like $-\eta^2/2t$ as $\eta \to \pm\infty$ with x and t held fixed. Thus for $t > 0$ we are only dealing with interior maxima, which must occur at critical points. The condition $h'(\eta; x, t) = 0$ is:

$$u_0(\eta) - \frac{x-\eta}{t} = 0 \qquad \text{or} \qquad x = \eta + u_0(\eta)t \tag{2.89}$$

which determines η given x and t. It is the same as the equation for the characteristic lines.

If x and t are outside of the triple-valued zone for the shock wave equation, then there is exactly one solution $\eta = \eta(x, t)$ of this equation, and it necessarily corresponds to a maximum. Using the leading-order formula following from Laplace's method for the approximations of the two integrals, (2.87) becomes

$$u^{(\nu)} \sim \frac{x - \eta(x,t)}{t} = u_0(\eta(x,t)) \qquad \text{as} \qquad \nu \to 0+ \ . \tag{2.90}$$

This leading order formula is exactly the solution of the shock wave equation ($\nu = 0$) obtained previously by the method of characteristics.

On the other hand, if (x, t) is inside the triple-valued region of the shock-wave equation, then there are three characteristic lines through each point, and consequently three different critical numbers for $h(\nu; x, t)$. There are two local maxima with a local minimum inbetween. See the figure. However, the important thing

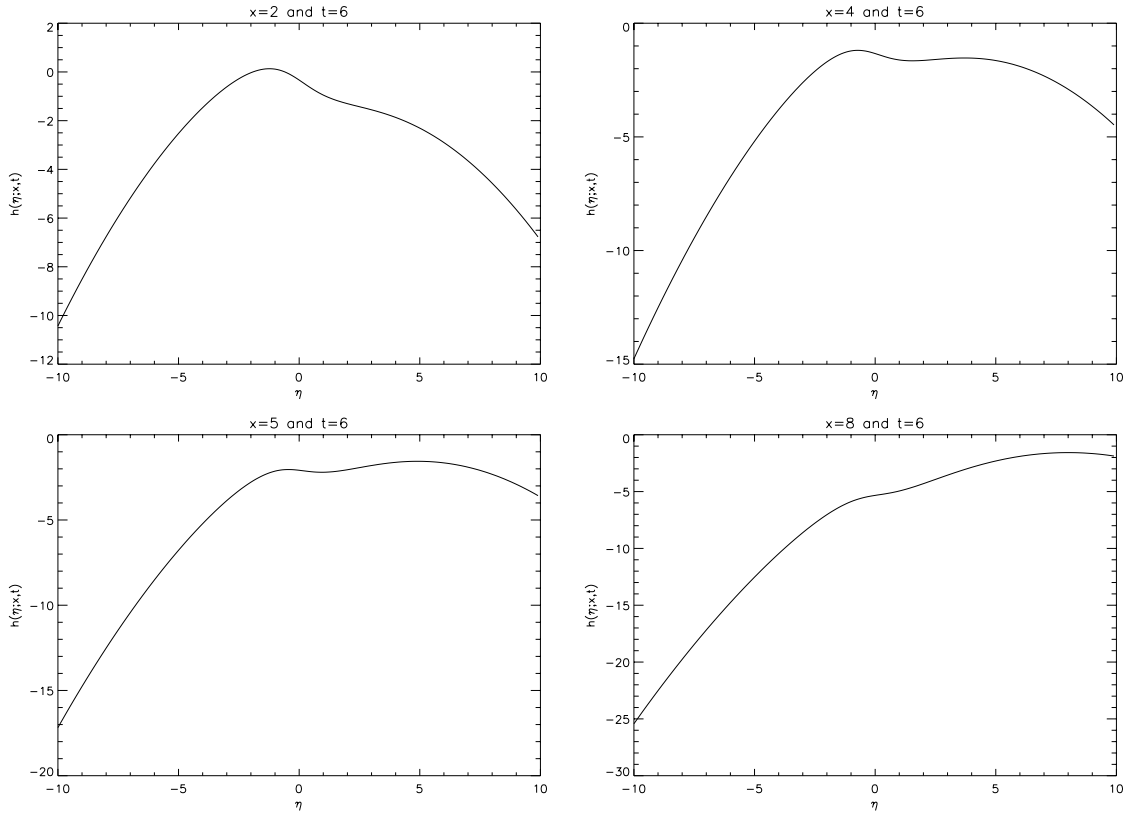

Figure 2.2: *Behavior of the function $h(\eta, x, t)$ as the point (x, t) crosses the shock zone. Here $u_0(x) = \mathrm{sech}(x)$. To the left and the right of the shock zone, there is only one maximum. Inside the shock zone there are two local maxima, only one of which is, however, the global maximum.*

to note here is that for each t there is a value of $x = x_{\mathrm{shock}}(t)$ where the location of the absolute maximum

of h switches over from the left local max to the right local max. If $x < x_{\text{shock}}(t)$, then Laplace's method applied to (2.87) gives

$$u^{(\nu)}(x,t) \sim \frac{x - \eta_{\text{left}}(x,t)}{t} = u_0(\eta_{\text{left}}(x,t)) \quad \text{as} \quad \nu \to 0+ \tag{2.91}$$

where $\eta_{\text{left}}(x,t)$ is the leftmost critical number for h. And if $x > x_{\text{shock}}(t)$, then

$$u^{(\nu)}(x,t) \sim \frac{x - \eta_{\text{right}}(x,t)}{t} = u_0(\eta_{\text{right}}(x,t)) \quad \text{as} \quad \nu \to 0+ \tag{2.92}$$

where $\eta_{\text{right}}(x,t)$ is the rightmost critical number for h. There is a discontinuity in the asymptotic solution exactly at $x = x_{\text{shock}}(t)$, which is the trajectory of the shock wave predicted by the zero-viscosity theory.

Chapter 3

The Method of Steepest Descents for Asymptotic Expansions of Integrals

3.1 General Considerations

3.1.1 The sort of integrals we consider.

The method of steepest descents is essentially a generalization of Laplace's method to contour integrals of complex-valued functions. Although many of the ideas carry over to integrals of more general form, the form that we will consider here is

$$I(\lambda) = \int_C e^{\lambda h(z)} g(z)\, dz\,. \tag{3.1}$$

The integral is a contour integral over a contour C in the complex z-plane. The functions $g(z)$ and $h(z)$ are complex-valued and defined for z on the contour C. Naturally, we assume that the functions g and h, the contour C, and the parameter λ are such that the integral gives a finite answer. In fact, the parameter λ will be assumed to be positive real without any loss of generality; if we want to consider an integral of the form

$$\int_C e^{\mu j(z)} g(z)\, dz \tag{3.2}$$

for μ complex, we just set $\lambda = |\mu|$ and $h(z) = e^{i \arg(\mu)} j(z)$ to write it in the form (3.1). We will be interested in the asymptotic behavior of $I(\lambda)$ as $\lambda \to +\infty$.

As the function $h(z)$ takes complex values, it is convenient to denote its real part by $\phi(z)$ and its imaginary part by $\psi(z)$. Thus $h(z) = \phi(z) + i\psi(z)$.

3.1.2 An application of Laplace's method and the phenomenon of catastrophic cancellation.

It is possible to get some elementary information about this integral directly using Laplace's method. Because the absolute value of the integral is always bounded by the integral of the absolute value,

$$\begin{aligned} |I(\lambda)| &\leq \int_C \left| e^{\lambda h(z)} g(z) \right| |dz| \\ &= \int_C e^{\lambda \phi(z)} |g(z)|\, |dz|\,. \end{aligned} \tag{3.3}$$

Let us denote this upper bound given by the latter integral by $B(\lambda)$. Upon parametrization by arc length s of C with $ds = |dz|$, $B(\lambda)$ is exactly of the form amenable to asymptotic treatment for large positive λ by Laplace's method. The main contributions to $B(\lambda)$ will come from the immediate neighborhoods of the

points on C where $\phi(z)$ takes its maximum value, say $\phi_{\max}$. Thus, if the maximum is taken at any interior point of C where the generic condition $d^2\phi/ds^2 < 0$ holds, then Laplace's method shows that

$$B(\lambda) = O(e^{\lambda\phi_{\max}}\lambda^{-1/2}) \text{ as } \lambda \to +\infty\,. \tag{3.4}$$

And if the maximum is taken only at an endpoint where the generic condition $d\phi/ds \neq 0$ holds, then Laplace's method implies that

$$B(\lambda) = O(e^{\lambda\phi_{\max}}\lambda^{-1}) \text{ as } \lambda \to +\infty\,. \tag{3.5}$$

This upper bound turns out to be much too crude most of the time. This is because the imaginary part of $h(z)$ introduces oscillations into the integrand that get ignored when we take absolute values because $|e^{i\psi(z)}| \equiv 1$. When we integrate (sum) over these oscillations, there is lots of cancellation, which makes the answer much smaller than if the phase factor were not there at all. In fact, it usually turns out that $|I(\lambda)|/B(\lambda)$ is exponentially small as $\lambda \to \infty$. To get accurate asymptotics for $I(\lambda)$ itself requires some additional techniques.

3.1.3 Analytic integrands and Cauchy's Theorem.

The main additional assumption that we need at the moment to make more progress is that the functions $g(z)$ and $h(z)$ are complex analytic functions at each point z of C. In this case, we know by analytic continuation that we may view g and h as analytic functions defined in some "ambient" neighborhood of the complex plane that contains the contour C. The advantage of this is that we may choose to study, in place of the integral over C defining $I(\lambda)$, another equivalent integral over a different contour.

To do this we need to recall some facts from complex variable theory.

Theorem 1 (Cauchy's Theorem) *Let C and C' be two smooth paths connecting the same endpoints with the same orientation, and suppose that $f(z)$ is analytic on both paths and also throughout the region of the complex z-plane in between them. Then*

$$\int_{C'} f(z)\,dz = \int_C f(z)\,dz\,. \tag{3.6}$$

Thus in a sense the integral is independent of the path.

It could perhaps be stressed that although the integrand has the same functional form on both paths of integration, the properties of $f(z)$ could be quite different on different paths. This fact can be used to our advantage.

Sometimes we would like to deform the path of integration even though the integrand $f(z)$ is not analytic throughout the region between the paths. A generalization of Cauchy's Theorem that is sometimes useful in this connection is the following.

Theorem 2 (Residue Theorem) *Let C and C' be two paths with the same orientation and the same endpoints, but otherwise not intersecting. Let C' be the path on the right of C. Suppose that $f(z)$ is analytic on C and C', and that in the region enclosed by the two curves the only singularities of $f(z)$ are poles of finite order at points $z_1, \ldots, z_n$. Let f_i denote the residue of $f(z)$ at z_i, that is, the coefficient of $(z - z_i)^{-1}$ in the Laurent series of $f(z)$ at z_i. Then*

$$\int_{C'} f(z)\,dz = \int_C f(z)\,dz + 2\pi i(f_1 + \ldots + f_n)\,. \tag{3.7}$$

Therefore it is still possible to move the path of integration in the presence of poles, but as my teacher used to say, "If you cross a pole, you have to pay the price." The price is $2\pi i$ times the residue.

For analytic integrands $f(z)$, like the case of $f(z) = e^{\lambda h(z)}g(z)$ with $g(z)$ and $h(z)$ analytic on C, the possibility of deforming the path of integration at will is an incredible freedom. It is also an awesome responsibility: we have to find the particular path that makes our analysis as easy as possible.

3.1.4 Saddle points and paths of steepest descent.

One idea that comes to mind given the freedom we have is to try to find the absolute maximum of the real part $\phi(z)$ of $h(z)$ and to deform the path of integration to pass through it. In fact, this is not possible for fundamental reasons. Let $z = x + iy$. In the neighborhood of each z where $h(z)$ is an analytic function, its real and imaginary parts are infinitely differentiable with respect to x and y and necessarily satisfy the *Cauchy-Riemann equations*:

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}\,. \tag{3.8}$$

It follows by cross-differentiation that

$$\Delta\phi := \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{and} \quad \Delta\psi := \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0\,, \tag{3.9}$$

that is, $\phi(x, y)$ and $\psi(x, y)$ are both *harmonic functions* that satisfy Laplace's equation. A fundamental fact about such functions is the following.

Theorem 3 (Maximum Principle) *Let ϕ be a harmonic function in a domain D with boundary ∂D, and suppose that ϕ is not a constant function. Then, the maximum value of ϕ may only be achieved on ∂D, if at all.*

Since $-\phi$ is harmonic in D whenever ϕ is, the same can be said about the minimum value. In particular it follows that there can be no local maxima of a harmonic function.

So we cannot proceed in this manner. But as ϕ can indeed have local maxima as a function of arc length on any given path C, we can still ask what we can hope to achieve by deforming the path. There are basically two desirable aims:

1. We can try to optimize the asymptotic approximation as $\lambda \to +\infty$ by getting a peak of $\phi(z)$ along C to be as "narrow" as possible. That is, we can try to get $d^2\phi/ds^2$ to be as negative as possible at the peak.

2. We can try to remove altogether the detrimental cancellation effects of increasingly wild oscillations as $\lambda \to +\infty$ by trying to select a path of integration along which the imaginary part ψ of h is not changing in the vicinity of the maximum (along C) of ϕ. Therefore we are trying to achieve $d\psi/ds = 0$ in a neighborhood of the peak.

It is a happy miracle that *both* of these aims can be achieved by making the contour of integration pass through a *critical point* $z_0 = x_0 + iy_0$ of $h(z)$ defined by the equation $h'(z_0) = 0$. Since

$$\frac{d}{dz} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \tag{3.10}$$

and since $h = \phi + i\psi$, we get

$$\begin{aligned} \frac{dh}{dz} &= \frac{1}{2}\left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} + i\left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right)\right) \\ &= \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} \\ &= \frac{\partial \psi}{\partial y} - i\frac{\partial \phi}{\partial y}\,. \end{aligned} \tag{3.11}$$

To get the last two lines we used the Cauchy-Riemann equations. This means that if $h'(z_0) = 0$, then we can separate real and imaginary parts and find

$$\frac{\partial \phi}{\partial x}(x_0, y_0) = 0 \quad \text{and} \quad \frac{\partial \psi}{\partial x}(x_0, y_0) = 0 \tag{3.12}$$

from the second line of (3.11) and also

$$\frac{\partial \psi}{\partial y}(x_0, y_0) = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial y}(x_0, y_0) = 0 \tag{3.13}$$

from the third line of (3.11). This means that $\nabla\phi(x_0, y_0) = \nabla\psi(x_0, y_0) = 0$. Although by the maximum principle, this simultaneous critical point of the two real functions ϕ and ψ cannot be an extremum of either, it can be (and usually is) a simple *saddle point*. By "simple", we mean that $h''(z_0) \neq 0$. Here you really need to see a picture of a saddle, and see what corresponds to a "mountain" and what corresponds to a "valley".

Let us suppose that we can deform the path of integration using Cauchy's Theorem or the residue theorem so that it passes through a critical point z_0 of h where ϕ and ψ both have simple saddle points. We ask the question of exactly how should we choose the path, given only that it is nailed down to z_0.

Consider trying to achieve the first of the two desirable aims, namely trying to get the peak of ϕ to be as narrow as possible. As a preliminary remark, note that we want the path to proceed over the saddle from a valley of ϕ to another valley. Otherwise, the dominant contributions to $I(\lambda)$ will not come from the neighborhood of z_0 at all. In most cases where the path of integration is an infinite contour, elementary convergence considerations will require that the tails of the path to eventually descend into valleys, so this is not much of a constraint. But we want the path that falls off of the saddle into both valleys as fast as possible. This will be the path of *steepest descent*. From multivariable calculus, we know that the path of steepest descent of ϕ is a curve tangent to $\nabla\phi$, since the gradient always points in the direction of steepest increase. Emanating from every simple saddle point there are four curves tangent to $\nabla\phi$. Two of them are paths of steepest ascent, which we ignore, and two of them are paths of steepest descent, which we would like to use as our contour of integration to get the peak to be as narrow as possible.

Consider now trying to satisfy the second desirable aim, namely to eliminate the oscillations leading to cancellation. Along a path passing through the critical point z_0, there will be no oscillations contributed by $e^{i\psi}$ if ψ is constant along the contour, and consequently if $\psi(x, y) = \psi(x_0, y_0)$ on the contour. Again from multivariable calculus, we know that if $\psi(x, y)$ is constant along a curve C (*i.e.* C is a *level set* of $\psi(x, y)$) then $\nabla\psi$ must be perpendicular to C at every point.

Now here comes the really amazing part. From the Cauchy-Riemann equations we get that

$$\nabla\phi = \begin{pmatrix} \dfrac{\partial \phi}{\partial x} \\ \\ \dfrac{\partial \phi}{\partial y} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \psi}{\partial y} \\ \\ -\dfrac{\partial \psi}{\partial x} \end{pmatrix} \tag{3.14}$$

which is a vector in the (x, y)-plane that is always perpendicular to $\nabla\psi$. So whenever C is parallel to $\nabla\phi$, it is also perpendicular to $\nabla\psi$, simply because ϕ and ψ are real and imaginary parts of the same analytic function h. Therefore, if we can arrange that our contour C is a path of steepest descent over the saddle point, then the same path is *automatically* a path along which the phase $\psi(x, y)$ is dead constant. So we manage to get both desirable conditions satisfied on the same path. Also, the path to use is uniquely selected in some neighborhood of z_0, which is mathematically satisfying.

3.1.5 Detailed local picture near a saddle point. Asymptotics for $I(\lambda)$ when λ is large.

When z is near the saddle point z_0, we can make the Taylor expansion:

$$h(z) = h(z_0) + \frac{h''(z_0)}{2}(z - z_0)^2 + O((z - z_0)^3)\,. \tag{3.15}$$

Now, let $\phi_0 := \phi(x_0, y_0)$ and $\psi_0 := \psi(x_0, y_0)$. Also, introduce the polar forms $h''(z_0) = ae^{i\alpha}$ with $a > 0$ and $z - z_0 = re^{i\theta}$ with $r > 0$. Separating the Taylor expansion of h into real and imaginary parts therefore gives

$$\begin{aligned} \phi &= \phi_0 + \frac{ar^2}{2}\Re(e^{i(\alpha+2\theta)}) + O(r^3) \\ &= \phi_0 + \frac{ar^2}{2}\cos(2\theta + \alpha) + O(r^3)\,, \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} \psi &= \psi_0 + \frac{ar^2}{2}\Im(e^{i(\alpha+2\theta)}) + O(r^3) \\ &= \psi_0 + \frac{ar^2}{2}\sin(2\theta+\alpha) + O(r^3)\,. \end{aligned} \tag{3.17}$$

Neglecting the effects of the $O(r^3)$ terms when r is small enough, we see that the real part ϕ will be less than its value ϕ_0 at the saddle for angles θ satisfying $\cos(2\theta+\alpha) < 0$ Therefore the directions coming off of the saddle with $2\theta+\alpha$ between $-3\pi/2$ and $-\pi/2$ or between $\pi/2$ and $3\pi/2$ correspond to descending into the valleys. Therefore

$$\text{For descent paths:} \quad \theta \in \left(-\frac{3\pi}{4} - \frac{\alpha}{2}, -\frac{\pi}{4} - \frac{\alpha}{2}\right) \quad \text{or} \quad \theta \in \left(\frac{\pi}{4} - \frac{\alpha}{2}, \frac{3\pi}{4} - \frac{\alpha}{2}\right). \tag{3.18}$$

Similarly, we can work out the angles θ for which we have ascent up to the mountains of ϕ:

$$\text{For ascent paths:} \quad \theta \in \left(-\frac{\pi}{4} - \frac{\alpha}{2}, \frac{\pi}{4} - \frac{\alpha}{2}\right) \quad \text{or} \quad \theta \in \left(\frac{3\pi}{4} - \frac{\alpha}{2}, \frac{5\pi}{4} - \frac{\alpha}{2}\right). \tag{3.19}$$

Also, we can approximate the curves on which $\psi = \psi_0$ by neglecting the order r^3 terms. Thus, the phase ψ remains constant (locally, near the saddle point) when $\sin(2\theta+\alpha) = 0$, which means that

$$\text{For constant phase:} \quad \theta = \frac{\pi}{2} - \frac{\alpha}{2} \quad \text{or} \quad \theta = -\frac{\pi}{2} - \frac{\alpha}{2} \quad \text{or} \quad \theta = -\frac{\alpha}{2} \quad \text{or} \quad \theta = \pi - \frac{\alpha}{2}\,. \tag{3.20}$$

You can see that the last two of these directions correspond to ascent paths, so we ignore them. But the first two paths are descent paths as well as constant phase paths. So we know that they are actually steepest descent paths too.

Because our analysis was only local, the true constant phase and steepest descent paths are only tangent to the lines with directions $\theta = (\pi-\alpha)/2$ and $\theta = -(\pi+\alpha)/2$. We imagine the selected contour of integration C^* to be placed, at least in a neighborhood of z_0, to agree *exactly* with this special path. Now on C^*,

$$h(z) - h(z_0) = \phi - \phi_0 + i(\psi - \psi_0) = \phi - \phi_0 \tag{3.21}$$

since $\psi \equiv \psi_0$ on C^*. Therefore $h(z) - h(z_0)$ is purely real on C^*. Also, since both parts of C^* leaving the saddle are descent paths for ϕ, we also find that $h(z) - h(z_0) \le 0$ and is only zero at the saddle point.

So we suppose that, possibly by taking into account some residue contributions, we are able to write the original integral $I(\lambda)$ in terms of another integral

$$I^*(\lambda) := \int_{C^*} e^{\lambda h(z)} g(z)\, dz\,. \tag{3.22}$$

In many applications we will have simply $I(\lambda) = I^*(\lambda)$ by Cauchy's Theorem. We now claim that $I^*(\lambda)$ may be immediately analyzed using Laplace's method for real integrals. Indeed, let $z(s)$ be an analytic real parametrization of C^* that is orientation-preserving, mapping the real interval $(-a, b)$ to C^* such that $z(0) = z_0$. Here a or b could be infinite, and often are. Then

$$I^*(\lambda) = e^{\lambda\phi_0} e^{i\lambda\psi_0} \int_{-a}^{b} e^{\lambda(h(z(s))-h(z_0))} \left[g(z(s))z'(s)\right] ds\,. \tag{3.23}$$

So there are some explicit factors multiplied by an integral. The integral is exactly of the sort for which Laplace's method can be used because the function multiplied by the large positive parameter λ in the exponent is purely real and takes its maximum value of zero at $s = 0$. If we let $H(s) = h(z(s)) - h(z_0)$, then Laplace's method says that

$$\int_{-a}^{b} e^{\lambda H(s)} \left[g(z(s))z'(s)\right] ds = \lambda^{-1/2}\sqrt{\frac{-2\pi}{H''(0)}}\, g(z_0)z'(0) + O(\lambda^{-3/2}) \quad \text{as} \quad \lambda \to +\infty\,. \tag{3.24}$$

This makes sense because we know that $H''(0) < 0$. Now, by the chain rule,

$$\frac{d^2H}{ds^2} = \frac{d^2z}{ds^2}\frac{dh}{dz} + \left(\frac{dz}{ds}\right)^2 \frac{d^2h}{dz^2}\,, \tag{3.25}$$

so when we set $s = 0$ which corresponds to the saddle point $z = z_0$ where by definition $h'(z_0) = 0$, we get

$$H''(0) = (z'(0))^2 h''(z_0)\,. \tag{3.26}$$

We can now see that the dependence on the parametrization $z(s)$ is about to disappear from the leading order asymptotic formula for the integral. The only catch is what is meant by the square root when we factor out the $z'(0)$ which is not necessarily real and positive. In fact we know for sure that

$$\left|\sqrt{\frac{-2\pi}{H''(0)}}z'(0)\right| = \sqrt{\frac{2\pi}{|h''(z_0)|}}\,, \tag{3.27}$$

and also the angle $\arg(z'(0))$ is (because the parametrization is orientation-preserving) the angle with which the selected contour C^* passes through the saddle point. Therefore,

$$\arg\left(\sqrt{\frac{-2\pi}{H''(0)}}z'(0)\right) = \frac{\pi}{2} - \frac{\alpha}{2} \quad \text{or} \quad -\frac{\pi}{2} - \frac{\alpha}{2} \tag{3.28}$$

depending on the valley from which the contour C^* rises toward the saddle. At last, we have a leading-order asymptotic formula for $I^*(\lambda)$:

$$I^*(\lambda) = e^{\lambda\phi_0}\lambda^{-1/2}\left(\sqrt{\frac{2\pi}{|h''(z_0)|}}g(z_0)e^{i(\lambda\psi_0+\beta)} + O(\lambda^{-1})\right) \quad \text{as} \quad \lambda \to +\infty\,, \tag{3.29}$$

where $\beta = (\pi - \alpha)/2$ or $\beta = -(\pi + \alpha)/2$ depending on the direction the contour C^* traverses the saddle.

As was the case with Laplace's method, it is possible to obtain higher-order terms (a whole asymptotic expansion) although it is tedious work and often the leading term is sufficiently accurate for applications. Also, the method can easily be generalized to handle higher-order saddle points (when $h''(z_0) = 0$) and their contributions, or situations in which the main contributions come from the endpoints rather than an interior maximum of $\phi(z)$ on the contour.

When it comes to *proving* the validity of the asymptotic approximation obtained from the method of steepest descents, it is necessary to argue that it is asymptotically correct to neglect the contributions to the integral $I(\lambda)$ coming from points on the contour C that are not close to the saddle point. Whereas this is easily taken care of under the umbrella of Laplace's method if the contour C can be taken to be the exact steepest descent contour C^* globally (not just near z_0), other arguments need to be improvised if C does not agree with C^* everywhere.

3.2 Examples

Here we present several examples of the application of the method of steepest descents.

3.2.1 The long-time asymptotics of diffusion processes.

Consider a distribution of some material in one dimension (say the material is confined to a long, thin pipe). The fundamental law governing this material is a *conservation law* stating that the material is neither created nor destroyed in the absence of any sources or sinks (again think of the material trapped in a pipe). If we isolate a piece of the pipe between x and $x + \Delta x$, and take $m(x, x + \Delta x, t)$ to denote the mass of the material in this portion of the pipe at time t, then the conservation of mass says that the only way this quantity can change with time is via transport of the material through the ends of the segment of pipe. Let $f(x, t)$ denote

the *flux* or flow rate of the material in the pipe to the right at coordinate x and time t. It has units of mass per unit time because it's a flow rate. Then, the conservation of mass can be written as

$$\frac{d}{dt}m(x, x+\Delta x, t) = f(x,t) - f(x+\Delta x, t)\,. \tag{3.30}$$

Dividing through by Δx and letting $\Delta x \to 0$ we get

$$\frac{\partial \rho}{\partial t} = -\frac{\partial f}{\partial x}\,, \tag{3.31}$$

where we suppose the limit

$$\rho(x,t) := \lim_{\Delta x \to 0} \frac{m(x, x+\Delta x, t)}{\Delta x} \tag{3.32}$$

exists and defines the *linear density* of the material in the pipe at position x and time t.

The physics of different materials and different modes of propagation through the pipe (convection, diffusion, *etc.*) gets reflected in different relationships between the flux $f(x,t)$ and the density $\rho(x,t)$. For a diffusion process, by definition there is a positive number ν (the diffusion coefficient) so that

$$f(x,t) := -\nu \frac{\partial \rho}{\partial x}\,. \tag{3.33}$$

This says: if there is a greater density of material immediately to the left of x than immediately to the right, then there will be a net flow of the material to the right proportional to the difference in densities. Inserting this relation into the conservation law gives the *diffusion equation* in one dimension:

$$\frac{\partial \rho}{\partial t} = \nu \frac{\partial^2 \rho}{\partial x^2}\,. \tag{3.34}$$

This is to be thought of as an evolution equation governing the development of the density function $\rho(x,t)$ with time t, given the initial value of the density everywhere in the pipe: $\rho(x,0) = \rho_0(x)$.

Let us develop the solution of the diffusion equation in the case of a doubly-infinite pipe: $x \in (-\infty, \infty)$. We can exploit *separation of variables* in this case to write down some particular solutions. Suppose that there is a solution $\rho(x,t)$ in the factorized form:

$$\rho(x,t) = X(x)T(t) \tag{3.35}$$

for two functions X and T of one variable. Inserting into the diffusion equation and dividing by $\nu X(x)T(t)$ gives

$$\frac{T'(t)}{\nu T(t)} = \frac{X''(x)}{X(x)}\,. \tag{3.36}$$

The only way a function of x only can equal a function of t only is if they are both equal to a constant, which we might write as $-k^2$. Thus, we have to find solutions of the following linear ordinary differential equations:

$$X''(x) + k^2 X(x) = 0 \qquad \text{and} \qquad T'(t) - \nu k^2 T(t) = 0\,. \tag{3.37}$$

These equations have constant coefficients, so they are easily solved in terms of exponentials. Up to multiplication by arbitrary constants,

$$X(x) = e^{ikx} \qquad \text{and} \qquad T(t) = e^{-\nu k^2 t}\,. \tag{3.38}$$

The equation for $X(x)$ also has solutions proportional to e^{-ikx} but we take this into account by just changing the separation constant from k to $-k$. This procedure gives us a family of factorizable solutions

$$\rho(x,t;k) := e^{ikx - \nu k^2 t} \tag{3.39}$$

of the diffusion equation, parametrized by a constant k. The only solutions of this form that are bounded for all x correspond to taking k to be real.

The diffusion equation is *linear*, so it has a *superposition principle*. This principle says that whenever $\rho_1(x,t),\ldots,\rho_n(x,t)$ are known solutions of the equation, then so is the linear combination $c_1\rho_1(x,t)+\ldots+c_n\rho_n(x,t)$ for any complex constants $c_1,\ldots,c_n$. Since we have a family of solutions $\rho(x,t;k)$ indexed by a continuous parameter k, it is natural to take a limit of Riemann sums of solutions and consider the integral

$$\rho(x,t)=\int_{-\infty}^{\infty}A(k)e^{ikx-\nu k^2t}\,dk \tag{3.40}$$

as a solution of the diffusion equation, where $A(k)$ is some complex-valued function of k. You can see that this integral satisfies the diffusion equation for $t>0$ just by differentiating under the integral sign.

Here is the question we want to address: what does this solution look like for large t? The answer to this question will describe the long-time asymptotics of a diffusion process. To be more precise, let us imagine moving through the material with some velocity c, so that $x=ct$. We will consider the asymptotic behavior of $\rho(x,t)$ with c fixed as $t\to+\infty$. So, consider the integral

$$I_c(t):=\rho(ct,t)=\int_{-\infty}^{\infty}A(k)e^{t(ikc-\nu k^2)}\,dk\,. \tag{3.41}$$

The exponent has the form $th(k)$ with $h(k):=ikc-\nu k^2$ which is of just the form for treatment by the method of steepest descents in the limit of large t.

For the moment, let us assume that $A(k)$ is an analytic function in a big enough strip surrounding the real axis in the complex k-plane. This turns out to be a condition on the decay properties of the function $\rho_0(x)$ describing the initial distribution of mass, but we will just assume it for now. Let's look for saddle points of the function $h(k)$. These are defined by the equation

$$h'(k_0)=ic-2\nu k_0=0\,. \tag{3.42}$$

This equation has a unique solution $k_0=ic/2\nu$. Moreover, at this point

$$h''(k_0)=-2\nu\neq 0 \tag{3.43}$$

so the saddle point is simple. Writing, as usual, $h=\phi+i\psi$, we have at the saddle $\phi_0=-c^2/4\nu$ and $\psi_0=0$. In this case, we can see the steepest descent path *explicitly*. Letting $k=k_{\rm r}+ik_{\rm i}$, we find that

$$\begin{aligned}\phi(k_{\rm r},k_{\rm i})-\phi_0 &= \Re\left(i(k_{\rm r}+ik_{\rm i})c-\nu(k_{\rm r}+ik_{\rm i})^2\right)+\frac{c^2}{4\nu}\\ &= -ck_{\rm i}-\nu k_{\rm r}^2+\nu k_{\rm i}^2+\frac{c^2}{4\nu}\\ &= \nu\left[\left(k_{\rm i}-\frac{c}{2\nu}\right)^2-k_{\rm r}^2\right]\,.\end{aligned} \tag{3.44}$$

Also, for the imaginary part of h (remember that $\psi_0=0$),

$$\begin{aligned}\psi(k_{\rm r},k_{\rm i}) &= \Im\left(i(k_{\rm r}+ik_{\rm i})c-\nu(k_{\rm r}+ik_{\rm i})^2\right)\\ &= ck_{\rm r}-2\nu k_{\rm r}k_{\rm i}\\ &= 2\nu\left(k_{\rm i}-\frac{c}{2\nu}\right)k_{\rm r}\,.\end{aligned} \tag{3.45}$$

The constant phase curves are curves in the $(k_{\rm r},k_{\rm i})$-plane made up of points that satisfy $\psi-\psi_0=\psi=0$, namely

$$\text{constant }\psi\text{:}\quad k_{\rm r}=0\qquad\text{and}\qquad k_{\rm i}=\frac{c}{2\nu}\,, \tag{3.46}$$

that is, straight vertical and horizontal lines through the saddle point. On general principles, two of these rays correspond to steepest ascent directions and two correspond to steepest descent directions. Similarly, the regions where $\phi - \phi_0$ changes sign are delimited by the straight lines obtained by solving $\phi - \phi_0 = 0$:

$$\text{mountain/valley boundaries:} \qquad k_{\rm i} = k_{\rm r} + \frac{c}{2\nu} \qquad \text{and} \qquad k_{\rm i} = -k_{\rm r} + \frac{c}{2\nu}\,. \tag{3.47}$$

These are diagonal lines with slopes ± 1 through the saddle point. It is easy to see from the exact formula for $\phi - \phi_0$ that the mountains of ϕ correspond to $\pi/4 < \arg(k - k_0) < 3\pi/4$ and $-3\pi/4 < \arg(k - k_0) < -\pi/4$, and that the valleys correspond to $-\pi/4 < \arg(k - k_0) < \pi/4$ and $3\pi/4 < \arg(k - k_0) < 5\pi/4$. This means that the original integration contour C, which is the real axis oriented from $-\infty$ to ∞, both starts and ends in valleys of ϕ.

We would like to consider deforming the path of integration to a new contour C^*, namely the horizontal contour from $-\infty + ic/2\nu$ to $\infty + ic/2\nu$ which we know to be the path of steepest descent through the saddle point. See the figure. The fact that we may move any finite part of C upwards to agree with C^* follows

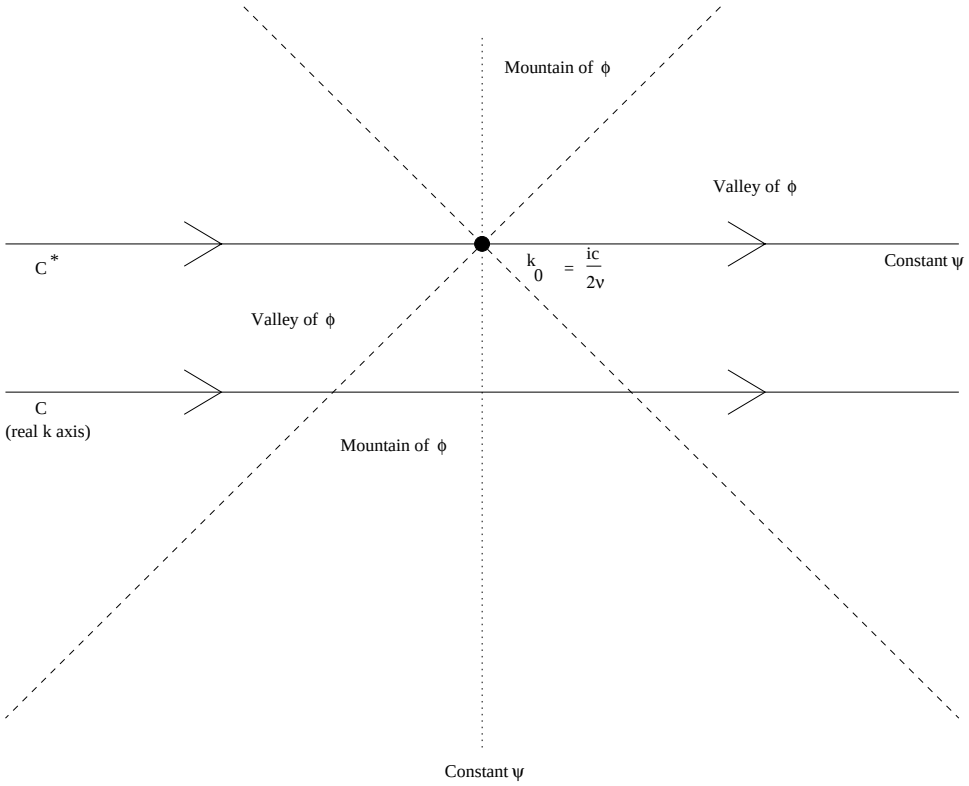

Figure 3.1: *The deformation of the contour C to the steepest descent contour C^* for the large t analysis of the integral $I_c(t)$.*

from Cauchy's theorem, since we are assuming $A(k)$ to be analytic in a sufficiently wide strip, and since $h(k)$ is entire. An additional little argument is required to justify moving the "tails" of C up to agree with C^*, but this is not hard to explain using just boundedness of $A(k)$ and the fact that C and C^* both start out in the same valley and then both end in the same valley. This means that, for functions $A(k)$ analytic and bounded in the strip including both C and C^*,

$$I_c(t) = \int_{C^*} A(k) e^{th(k)}\, dk = \int_{-\infty + ic/2\nu}^{\infty + ic/2\nu} A(k) e^{th(k)}\, dk\,. \tag{3.48}$$

Now on C^*, $h(k)$ is *real* with a single maximum $h_{\max} = \phi_0 = -c^2/4\nu$. Writing $k = ic/2\nu + z$, we get

$$I_c(t) = e^{-c^2 t/4\nu} \int_{-\infty}^{\infty} A(z + ic/2\nu) e^{-t\nu z^2}\, dz \tag{3.49}$$

and we can immediately apply Laplace's method (or Watson's Lemma) to the integral to find the leading term:

$$\rho(ct, t) = I_c(t) \sim e^{-c^2 t/4\nu} \sqrt{\frac{\pi}{\nu t}} A(ic/2\nu) \qquad \text{as} \qquad t \to +\infty\,. \tag{3.50}$$

So the solution decays exponentially to zero with time t in the frame of reference moving with constant velocity c. What is the meaning of the constant $A(ic/2\nu)$ in this asymptotic formula?

To clarify the relation between $A(k)$ and the initial data $\rho_0(x)$, set $t = 0$ in the formula for $\rho(x, t)$ to find

$$\rho_0(x) = \int_{-\infty}^{\infty} A(k) e^{ikx}\, dk\,, \tag{3.51}$$

which says that $\rho_0(x)$ is the *inverse Fourier transform* of the function $A(k)$. The other half of the transform pair is the *Fourier transform*

$$A(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho_0(x) e^{-ikx}\, dx\,. \tag{3.52}$$

Therefore, it follows that

$$A(ic/2\nu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho_0(x) e^{cx/2\nu}\, dx\,. \tag{3.53}$$

This formula gives a kind of constraint on the "cone" of possible constant velocity trajectories for which the asymptotic formula we just obtained is valid. Clearly, $A(ic/2\nu)$ can only exist if the initial condition $\rho_0(x)$ decays faster than $e^{-|c||x|/2\nu}$. So suppose it is known, for example that $\rho_0(x)$ decays like a constant times $e^{-\sigma|x|}$ as $x \to \pm\infty$. Then $A(ic/2\nu)$ will exist only for $|c| < c_{\max} := 2\nu\sigma$. So for initial data that only decays exponentially (or less) there is a maximum velocity for which the asymptotic behavior for long time is given by our formula. For "supersonic" velocities larger in magnitude than $c_{\max}$, the larger tails of $\rho_0(x)$ have some influence that must be taken into account by other methods.

3.2.2 Asymptotics of special functions. Airy functions. Stokes' phenomenon.

One of the most important differential equations in applied mathematics is *Airy's equation*:

$$w''(x) - xw(x) = 0\,. \tag{3.54}$$

It comes up in asymptotic analysis as a canonical model for transition phenomena, as we will see later in the course. This equation cannot be solved in terms of simple combinations of elementary functions, as easy as it looks. However, we can seek solutions in terms of integrals of exponentials. Namely, suppose C is a contour with endpoints (possibly infinite) a and b on which $q(z)$ is a given function, and consider seeking a solution of (3.54) in the form

$$w(x) = \frac{1}{2\pi i} \int_C q(z) e^{zx}\, dz\,. \tag{3.55}$$

The idea is to try to pick the function $q(z)$ and the contour C so that this function satisfies Airy's equation. Substituting into (3.54), we get

$$\frac{1}{2\pi i} \int_C z^2 q(z) e^{zx}\, dz - \frac{1}{2\pi i} \int_C x q(z) e^{zx}\, dz = 0\,. \tag{3.56}$$

Now, integrate by parts in the second integral to get:

$$\frac{1}{2\pi i} \int_C \left[z^2 q(z) + q'(z)\right] e^{zx}\, dz - \frac{1}{2\pi i} q(z) e^{zx} \Big|_{z=a}^{z=b} = 0\,. \tag{3.57}$$

So if it is possible to select $q(z)$ satisfying $q'(z) + z^2 q(z) = 0$ and also to choose the contour C so that the boundary terms vanish, then we will have in (3.55) a solution of Airy's equation. This method is generally applicable when the equation for $q(z)$ turns out to be simpler than the original equation for $w(x)$.

In this case, we can solve for $q(z)$ explicitly up to a constant, since

$$-z^2 = \frac{q'(z)}{q(z)} \equiv \frac{d}{dz} \log(q(z)) \tag{3.58}$$

and we may integrate both sides with respect to z to find, up to a constant factor,

$$q(z) = e^{-z^3/3}. \tag{3.59}$$

Now, the function $q(z)e^{zx} = e^{zx - x^3/3}$ is an exponential, so it never vanishes, except when $z \to \infty$ the right way. So we must use an infinite contour if we want to make the boundary terms vanish. When z is large, no matter what x is, the factor $q(z)$ determines the behavior of the product. This factor will go to zero as $z \to \infty$ in any sector of the complex plane where $-z^3/3$ has a negative real part. This gives three sectors of the z-plane, each with opening angle of $\pi/3$, in which the contour C may begin and end. Moreover, it must begin and end in different sectors, or else the integral will be identically zero by Cauchy's theorem. So there are, up to signs and analytic deformations, three possibilities for the contour C illustrated in the figure. We

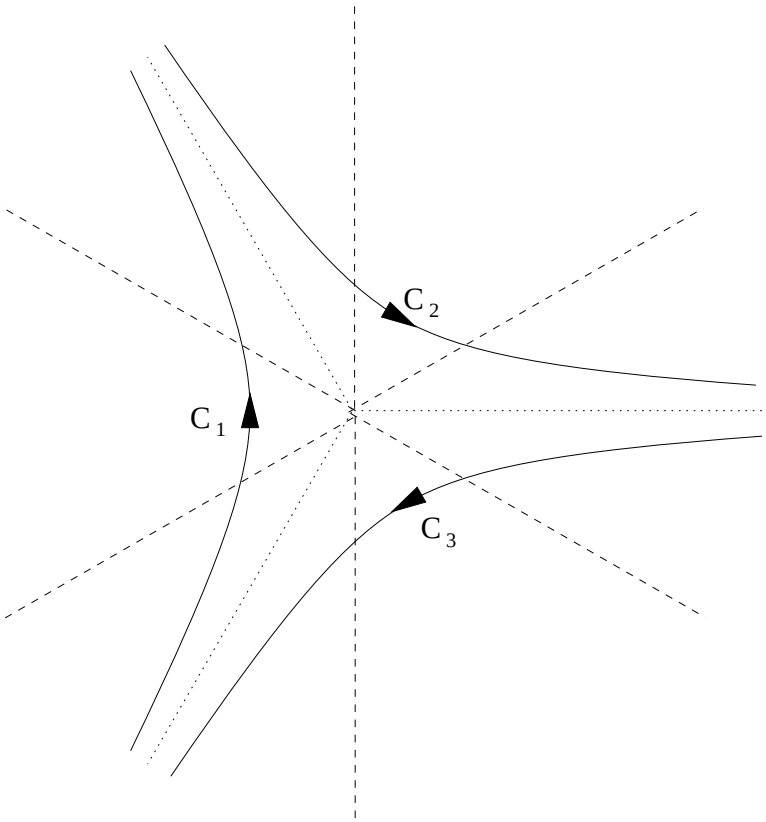

Figure 3.2: *The three admissible types of contours for integral solutions of Airy's equation.*

therefore have three different solutions represented by integrals,

$$w_i(x) := \frac{1}{2\pi i} \int_{C_i} e^{zx - z^3/3}\, dz \qquad \text{for} \qquad i = 1, 2, 3\,. \tag{3.60}$$

But Airy's equation (3.54) is second-order and can have only two linearly independent solutions. What about our three formulas? Well, they are not independent, since Cauchy's formula shows that

$$w_1(x) + w_2(x) + w_3(x) \equiv 0\,. \tag{3.61}$$

The particular function $w_1(x)$ is called the *Airy function* and is usually denoted by $Ai(x)$. It is an entire analytic function of x. Using the integral formula, we can find its asymptotic behavior as $x \to \infty$ in various sectors of the complex x-plane. Pick a phase angle κ for x and set $x = re^{i\kappa}$. Then

$$Ai(re^{i\kappa}) = \frac{1}{2\pi i}\int_{C_1} e^{re^{i\kappa}z}e^{-z^3/3}\,dz \tag{3.62}$$

and we want to consider $r \to +\infty$. The trouble here is that if we try to use the method of steepest descents, we would want to take $h(z) = e^{i\kappa}z$ which has no saddle points. So (like in the example of using Laplace's method to study the behavior $\Gamma(z)$) we need to change variables in the integral first. Set $z = u\sqrt{r}$ and then, since the "stretched" contour can be deformed back to C_1, we have

$$Ai(re^{i\kappa}) = \frac{r^{1/2}}{2\pi i}\int_{C_1} e^{r^{3/2}(e^{i\kappa}u - u^3/3)}\,du\,. \tag{3.63}$$

Let us denote the integral by $I_\kappa(r^{3/2})$, where

$$I_\kappa(\lambda) = \int_{C_1} e^{\lambda h_\kappa(u)}\,du \qquad \text{with} \qquad h_\kappa(u) := e^{i\kappa}u - \frac{u^3}{3}\,. \tag{3.64}$$

We are going to find leading-order asymptotics for $I_\kappa(\lambda)$ as $\lambda \to +\infty$ (which is the same as $r \to +\infty$) and explore how the procedure we use depends on the angle κ. Let us for the time being restrict our attention to angles κ satisfying the strict inequality $-\pi < \kappa < \pi$.

To find the saddle points, set $h'_\kappa(u_0) = 0$ to find

$$u_0 = u_0^{\mathrm{R}} := e^{i\kappa/2} \qquad \text{and} \qquad u_0 = u_0^{\mathrm{L}} := -e^{i\kappa/2}\,, \tag{3.65}$$

so there are *two* saddle points, antipodal points on the unit circle in the u-plane. Here the notation of the superscripts reminds us that over the range of angles $-\pi < \kappa < \pi$, u_0^{R} is in the right half-plane, and u_0^{L} is in the left half-plane. Note that

$$h_\kappa(u_0^{\mathrm{R}}) = \frac{2}{3}\cos\left(\frac{3}{2}\kappa\right) + i\frac{2}{3}\sin\left(\frac{3}{2}\kappa\right) \qquad \text{and} \qquad h_\kappa(u_0^{\mathrm{L}}) = -\frac{2}{3}\cos\left(\frac{3}{2}\kappa\right) - i\frac{2}{3}\sin\left(\frac{3}{2}\kappa\right)\,. \tag{3.66}$$

Writing $\phi_\kappa(u) = \Re(h_\kappa(u))$ and $\psi_\kappa(u) = \Im(h_\kappa(u))$, we see that for the real part of h_κ at the two critical points,

$$\phi_\kappa(u_0^{\mathrm{R}}) > \phi_\kappa(u_0^{\mathrm{L}}) \qquad \text{for} \qquad -\frac{\pi}{3} < \kappa < \frac{\pi}{3} \tag{3.67}$$

and

$$\phi_\kappa(u_0^{\mathrm{R}}) < \phi_\kappa(u_0^{\mathrm{L}}) \qquad \text{for} \qquad -\pi < \kappa < -\frac{\pi}{3} \qquad \text{and} \qquad \frac{\pi}{3} < \kappa < \pi\,. \tag{3.68}$$

Also, for the imaginary part of h_κ at the two critical points,

$$\psi_\kappa(u_0^{\mathrm{R}}) = \psi_\kappa(u_0^{\mathrm{L}}) \qquad \text{for} \qquad \kappa = -\frac{2\pi}{3}\,, \qquad \kappa = 0\,, \qquad \text{and} \qquad \kappa = \frac{2\pi}{3}\,. \tag{3.69}$$

For these three angles, there is a path of constant phase $\psi_\kappa(u)$ connecting the two saddle points; this path is simultaneously a steepest descent path from the saddle point for which $\phi_\kappa(u)$ is greater and a steepest ascent path from the saddle point for which $\phi_\kappa(u)$ is less.

Let us determine an appropriate deformation of the path C_1 for the asymptotic calculation of $Ai(re^{i\kappa})$ in the limit of $r \to \infty$ by the method of steepest descents. First we will consider increasing the phase angle from $\kappa = 0$ to $\kappa = \pi$. In the following figures we trace the corresponding behavior of the level curves of

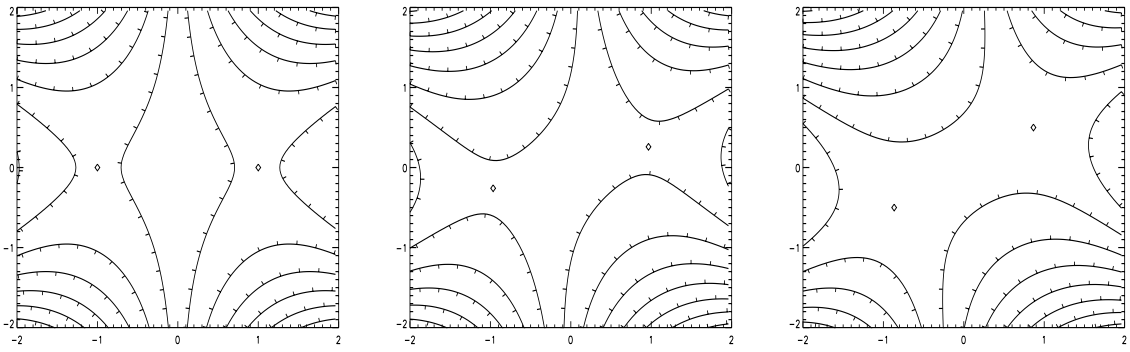

Figure 3.3: *Contour plots of* $\phi_\kappa(u) = \Re(h_\kappa(u))$. *Left:* $\kappa = 0$. *Middle:* $\kappa = \pi/6$. *Right:* $\kappa = \pi/3$.

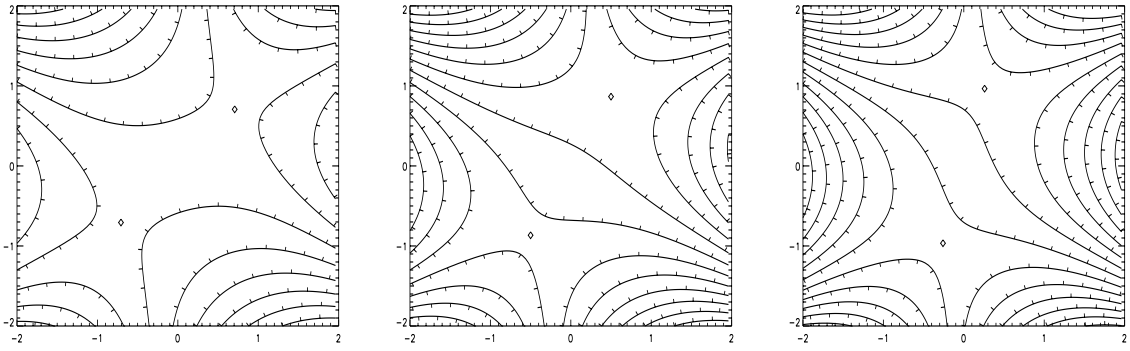

Figure 3.4: *Contour plots of* $\phi_\kappa(u) = \Re(h_\kappa(u))$. *Left:* $\kappa = \pi/2$. *Middle:* $\kappa = 2\pi/3$. *Right:* $\kappa = 5\pi/6$.

$\phi_\kappa(u)$. In each plot, the small lines connected to each contour are "hanging" downhill. For $\kappa = 0$, the path C_1 can clearly be deformed to agree with the global path of steepest descent from the left-most saddle point u_0^{L}. With this steepest descent path chosen for the path of integration, the integral $I_\kappa(\lambda)$ has an asymptotic expansion given essentially by Laplace's method; we only need to take into account that the angle of passage over the saddle is upwards, so $\beta = \pi/2$. It is interesting to note that in this case the dominant contribution to the integral comes from the neighborhood of the saddle point for which ϕ_κ is *less*. This shows that it is not at all necessary for the path to traverse the "highest" saddle point, as long as the path is the global steepest descent path for *some* saddle point. Note also that it is not possible to deform C_1 into the steepest descent path for u_0^{R} without changing the value of the integral. The situation is similar as κ increases up to the value $\pi/3$. The path of integration may be deformed in each case to agree with the global steepest descent contour of the saddle point u_0^{L} which therefore provides the dominant contribution to the integral even though it is "lower" than u_0^{R}. The angle β of passage over the saddle varies continuously with κ.

When κ increases through the value $\pi/3$, the two saddle points momentarily have the same real part, and then the saddle point u_0^{L} becomes "higher" than u_0^{R}. It remains possible to deform C_1 to the global steepest descent contour from the saddle u_0^{L}, although as κ increases toward $2\pi/3$ the steepest descent contour begins to pass very close to the other saddle point, near which it makes a sharp turn of nearly 90°.

At $\kappa = 2\pi/3$, something very strange happens. The steepest descent contour from the saddle point at u_0^{L} now runs right into the other saddle point, since $\psi_{2\pi/3}(u_0^{\mathrm{R}}) = \psi_{2\pi/3}(u_0^{\mathrm{L}})$ which places the two saddle points on the same level of $\psi_{2\pi/3}(u)$. Upon descending to u_0^{R} from u_0^{L}, we must make a left turn of exactly 90° to switch onto the steepest descent contour from u_0^{R} that descends into the correct valley of $\phi_{2\pi/3}(u)$. The global steepest descent curve in this case passes through two saddle points, and has a sharp corner in it, but

the dominant contribution still comes from the neighborhood of the point $u_0^{\rm L}$ because this saddle point is "higher" than the other one.

In the range $2\pi/3 < \kappa < \pi$, the global contour of steepest descent from $u_0^{\rm L}$ is no longer suitable for deformation from C_1, because only one end of this contour descends into a valley of $\phi_\kappa(u)$ together with C_1. One might think that this indicates the need to make use of the other saddle point; but one can easily see that the global contour of steepest descent from $u_0^{\rm R}$ is also unsuitable for deformation from C_1 for the same reason. In fact, the dominant contribution to $I_\kappa(\lambda)$ continues to come from the neighborhood of $u_0^{\rm L}$. There are two ways to see this:

1. Use a path C^* that agrees exactly with the steepest descent path from $u_0^{\rm L}$ locally, but not globally. Suppose we follow the exact steepest descent path for $u_0^{\rm L}$ from infinity in the lower half-plane where C_1 originates up to and just beyond the saddle, for example we might continue beyond the saddle for a small distance of δ in arc length. Then, if δ is small enough, one can see that it is possible to continue in a path along which $\phi_\kappa(u)$ is strictly decreasing (although not in the steepest possible manner) and arrive at the other saddle point $u_0^{\rm R}$. From here, it is possible to make a left turn and follow the steepest descent path from $u_0^{\rm R}$ into the appropriate valley of $\phi_\kappa(u)$. Because all along the piece of C^* starting from the distance of δ away from $u_0^{\rm L}$ and ending at $u_0^{\rm R}$, $\phi_\kappa(u)$ is dominated by its largest value at the initial endpoint, it is possible to show that this contribution is exponentially small compared to the local contribution from $u_0^{\rm L}$. Our estimate of this part of the integral will be very crude and bigger than optimal since there are oscillations present in the integrand as well ($\psi_\kappa(u)$ is not constant) leading to cancellation. But the crude upper bound is enough for our purposes, since there is in this case a bigger contribution around.

2. Recall the identity (3.61). This is the more elegant solution to our difficulties. It shows us that we may swap the integral over the path C_1 for the sum of integrals over $-C_2$ and $-C_3$ (the minus sign indicating reversed orientation). The path $-C_3$ may be deformed to the global steepest descent contour for the saddle point at $u_0^{\rm L}$, and the path $-C_2$ may be deformed to the global steepest descent contour for the saddle point at $u_0^{\rm R}$. Each integral may therefore be calculated by Laplace's method. Since in this range of angles $\phi_\kappa(u_0^{\rm L}) > \phi_\kappa(u_0^{\rm R})$, the contribution of the saddle point $u_0^{\rm L}$ will exponentially dominate that of the saddle point $u_0^{\rm R}$, so the former gives the complete asymptotic series of $I_\kappa(\lambda)$.

So over the whole range of angles $0 \le \kappa < \pi$, $I_\kappa(\lambda)$ is dominated by a contribution from the saddle point at $u_0^{\rm L}$. That is, the asymptotic expansion of $I_\kappa(\lambda)$ is the same as the expansion of the integral of the same integrand taken over a small contour segment of fixed length, containing $u_0^{\rm L}$, and contained in the global steepest descent contour from $u_0^{\rm L}$. Very similar reasoning shows that the same holds for the range of angles $-\pi < \kappa \le 0$. For the saddle point of interest, we have

$$h_\kappa''(u_0^{\rm L}) = -2u_0^{\rm L} = 2e^{i\kappa/2}\,, \tag{3.70}$$

so $\alpha = \kappa/2$. At this saddle point, we can tell from the phase angle α that the steepest descent directions locally have directions $\beta = \pi/2 - \kappa/4$ and $\beta = -\pi/2 - \kappa/4$. Now, when $\kappa = 0$, we are clearly passing vertically over the saddle with $\beta = \pi/2$, and the angle of passage varies continuously for $\kappa \in (-\pi, \pi)$. So for the angle β of passage appearing in the leading-order steepest descents formula, we use $\beta = \pi/2 - \kappa/4$ over the whole range $\kappa \in (-\pi, \pi)$. Since at the distinguished saddle point

$$h_\kappa(u_0^{\rm L}) = -\frac{2}{3}e^{3i\kappa/2} \quad \text{so} \quad \phi_0 = -\frac{2}{3}\cos\left(\frac{3\kappa}{2}\right) \quad \text{and} \quad \psi_0 = -\frac{2}{3}\sin\left(\frac{3\kappa}{2}\right)\,, \tag{3.71}$$

and since $g(u) \equiv 1$ for this problem, we obtain the asymptotic formula

$$I_\kappa(\lambda) = e^{-\frac{2\lambda}{3}\cos(3\kappa/2)}\lambda^{-1/2}\left(i\sqrt{\pi}e^{-i(\frac{2\lambda}{3}\sin(3\kappa/2)+\kappa/4)} + O(\lambda^{-1})\right) \quad \text{as} \quad \lambda \to \infty \tag{3.72}$$

as long as κ is restricted to the open interval $(-\pi, \pi)$. Since $Ai(x) = |x|^{1/2} I_{\arg(x)}(|x|^{3/2})/2\pi i$, we finally find the formula

$$Ai(x) = \frac{1}{2x^{1/4}\sqrt{\pi}}e^{-2x^{3/2}/3}\left[1 + O\left(|x|^{-3/2}\right)\right] \quad \text{as} \quad x \to \infty \quad \text{with} \quad -\pi < \arg(x) < \pi\,. \tag{3.73}$$

It is interesting to observe that the leading-order asymptotic formula for $Ai(x)$ is not an entire analytic function of x (it has a branch cut along the negative real axis in the complex x-plane), even though $Ai(x)$ is. In fact, a completely different asymptotic formula holds on the negative real x-axis. Here, the effects of *both* saddle points must be taken into account with weights of equal magnitude, which leads to a kind of interference effect. The fact that completely different asymptotic formulas can hold for representations of entire functions as $x \to \infty$ in different sectors is called *Stokes' phenomenon*. Here we see exactly why this happens. Even though $Ai(x)$ is an entire analytic function of x, when x tends to infinity, different saddle points play a role at different angles. When the angle goes through π, there is an abrupt change in the asymptotic behavior that reflects the change in the topology of the level curves.

Chapter 4

The Method of Stationary Phase for Asymptotic Expansions of Integrals

4.1 General Considerations

4.1.1 Oscillatory integrals and intuition.

The method of stationary phase was developed in the late 1800's by G. G. Stokes and Lord Kelvin to treat integrals of the form

$$I(\lambda) = \int_a^b g(t) e^{i\lambda h(t)}\, dt\,, \tag{4.1}$$

which is an integral over the interval $[a, b]$ of the real t-axis, where $g(t)$ and $h(t)$ are real-valued functions. We are considering λ to be real and taking $\lambda \to +\infty$. We could also consider $\lambda \to -\infty$ just by swapping $h(t)$ for $-h(t)$. This kind of integral is called an *oscillatory integral* because when λ is large, so is the slope of $\lambda h(t)$ as long as the slope of $h(t)$ is not zero. The slope of $\lambda h(t)$ is the frequency of oscillation of $e^{i\lambda h(t)}$ as t varies from a to b. So when λ is very large, the frequency of these oscillations of the integrand along the path of integration is also very large.

If the functions $g(t)$ and $h(t)$ are analytic in $[a, b]$, then they may be extended into the complex t-plane in a neighborhood of this real interval. Then, the path of integration from a to b can be deformed into the complex t-plane by Cauchy's Theorem, so that $I(\lambda)$ may be written as a contour integral along a contour C with endpoints a and b on the real axis. So if it is possible to find saddle points of $ih(t)$ in the complex t-plane, and if it is possible to take C to agree (at least in part) with the steepest descents path C^* passing over the saddle, then we may calculate the asymptotic expansion of $I(\lambda)$ using the method of steepest descents.

But it is also possible to obtain an asymptotic expansion for $I(\lambda)$ without going into the complex plane, which is advantageous if g and h are not analytic in $[a, b]$. This is the *method of stationary phase*. The latter method is even useful if g and h are analytic since its motivation is completely different from the ideas behind the method of steepest descents, which it historically pre-dates.

The idea of the method of stationary phase is that the oscillations present in $e^{i\lambda h(t)}$ when λ is large will lead to catastrophic cancellation as long as $h(t)$ is changing with t. Near points where $h(t)$ is not changing, there will be less cancellation and the contribution to $I(\lambda)$ is expected to be larger. See Figure 4.1.

4.1.2 Nonlocal contributions. The Riemann-Lebesgue Lemma.

The idea of this cancellation can be placed on rigorous footing. The tool for this we borrow from Fourier transform theory.

Proposition 6 (Riemann-Lebesgue lemma) *Let $f(x)$ be absolutely integrable on $[a, b]$ where a or b might be infinite:*

$$\int_a^b |f(x)|\, dx < \infty\,. \tag{4.2}$$

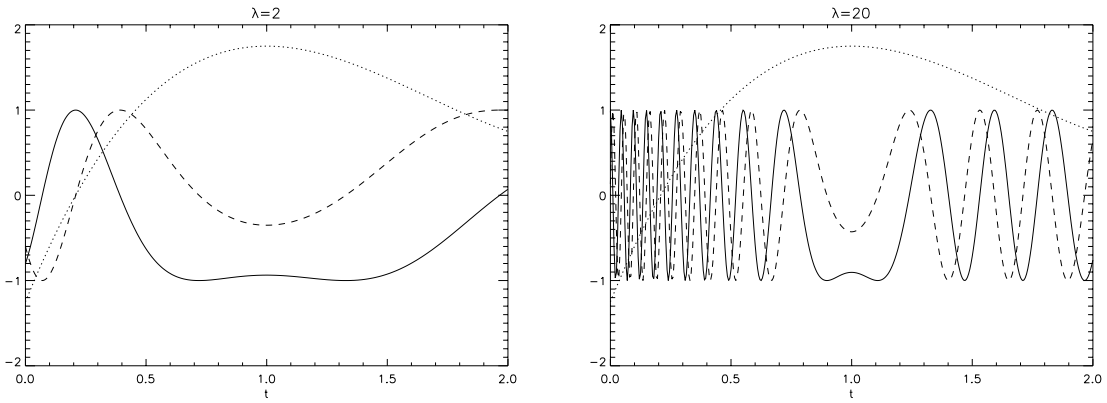

Figure 4.1: *Catastrophic cancellation for large λ except near critical points of $h(t)$. Dotted curve: $h(t)$. Solid curve:* $\cos(\lambda h(t)) = \Re(e^{i\lambda h(t)})$. *Dashed curve:* $\sin(\lambda h(t)) = \Im(e^{i\lambda h(t)})$.

Then, the integral

$$J(\lambda) := \int_a^b f(x) e^{i\lambda x}\, dx \tag{4.3}$$

is a continuous function for all real λ and $J(\lambda) \to 0$ as $\lambda \to \infty$ or $\lambda \to -\infty$.

The function $J(\lambda)$ is almost the Fourier transform of f. A proof of the Riemann-Lebesgue lemma can be based upon other more fundamental results, namely the Lebesque Dominated Convergence Theorem, and the fact that smooth functions are dense in the space $L^1[a,b]$ of absolutely integrable functions on $[a,b]$.

Proof of the Riemann-Lebesgue Lemma: To establish the continuity of $J(\lambda)$, write

$$J(\lambda+\epsilon) - J(\lambda) = \int_a^b f(x) e^{i\lambda x} \left[e^{i\epsilon x} - 1\right]\, dx\,. \tag{4.4}$$

Now since

$$\left| f(x) e^{i\lambda x} \left[e^{i\epsilon x} - 1\right] \right| \le 2|f(x)| \tag{4.5}$$

which gives an upper bound that is integrable on $[a,b]$ and independent of ϵ, and since for almost every $x \in [a,b]$,

$$f(x) e^{i\lambda x} \left[e^{i\epsilon x} - 1\right] \to 0 \quad \text{as} \quad \epsilon \to 0\,, \tag{4.6}$$

it follows from the Lebesgue Dominated Convergence Theorem that

$$J(\lambda+\epsilon) - J(\lambda) \to 0 \quad \text{as} \quad \epsilon \to 0\,. \tag{4.7}$$

That is, $J(\lambda)$ is a continuous function of λ.

To establish the decay of $J(\lambda)$, we first note that it is easy to prove decay if $f(x)$ is continuously differentiable, and $f'(x)$ is absolutely integrable. For then, we can integrate by parts:

$$J(\lambda) = \frac{1}{i\lambda}\int_a^b f(x) \frac{d}{dx}(e^{i\lambda x})\, dx = \frac{f(b)e^{i\lambda b}}{i\lambda} - \frac{f(a)e^{i\lambda a}}{i\lambda} - \frac{1}{i\lambda}\int_a^b f'(x) e^{i\lambda x}\, dx\,. \tag{4.8}$$

In particular, $J(\lambda) = O(\lambda^{-1})$ as $\lambda \to \pm\infty$ for such functions. But we will only need the fact that for each $\delta > 0$ there is an $M^f(\delta) > 0$ (generally depending on the function f, which explains the superscript) so that the condition $|\lambda| > M^f(\delta)$ is sufficient to guarantee that $|J(\lambda)| < \delta$.

For more general absolutely integrable functions $f(x)$, we approximate by nice functions of the type described above. Such functions are dense in $L^1[a,b]$, which means that when $f \in L^1[a,b]$, there is for each

$\delta > 0$ a continuously differentiable function $f_\delta \in L^1[a,b]$ with $f'_\delta(x) \in L^1[a,b]$ also, so that

$$\int_a^b |f(x) - f_\delta(x)|\, dx < \delta\,. \tag{4.9}$$

We need to show that for any $\epsilon > 0$, there exists some $N(\epsilon) > 0$ such that $|\lambda| > N(\epsilon)$ implies $|J(\lambda)| < \epsilon$. Write

$$J(\lambda) = \int_a^b f_{\epsilon/2}(x) e^{i\lambda x}\, dx + \int_a^b \left[f(x) - f_{\epsilon/2}(x)\right] e^{i\lambda x}\, dx \tag{4.10}$$

so that, by the triangle inequality and the approximating property of the functions $\{f_\delta(x)\}$,

$$|J(\lambda)| \le \left| \int_a^b f_{\epsilon/2}(x) e^{i\lambda x}\, dx \right| + \int_a^b |f(x) - f_{\epsilon/2}(x)|\, dx < \left| \int_a^b f_{\epsilon/2}(x) e^{i\lambda x}\, dx \right| + \frac{\epsilon}{2}\,. \tag{4.11}$$

Now, we restrict λ. Take $|\lambda| > N(\epsilon) := M^{f_{\epsilon/2}}(\epsilon/2)$. Then we have

$$|J(\lambda)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{4.12}$$

as desired. That is, $J(\lambda)$ decays to zero as $\lambda \to \pm\infty$. This completes our proof of the Riemann-Lebesgue Lemma. □

We will from now on be more concerned with how to exploit this result to help us develop asymptotic expansions. For example, suppose that $h(t)$ is continuously differentiable and that $[t_1, t_2]$ is a subinterval of $[a,b]$ where it is known that $h'(t) \neq 0$. Then, h is either strictly increasing or strictly decreasing in $[t_1, t_2]$ so we can change variables by setting $\tau = h(t)$ and then

$$\int_{t_1}^{t_2} g(t) e^{ih(t)}\, dt = \int_{h(t_1)}^{h(t_2)} \frac{g(t(\tau))}{h'(t(\tau))} e^{i\lambda\tau}\, d\tau \tag{4.13}$$

which goes to zero as $\lambda \to \infty$ by the Riemann-Lebesgue lemma as long as $g(t)$ is absolutely integrable. We can make a more precise statement about the *rate* of decay to zero if we assume some more smoothness of g and h. Let $G(\tau) := g(t(\tau))/h'(t(\tau))$. If G is differentiable then we can integrate by parts to find

$$\begin{aligned}
\int_{t_1}^{t_2} g(t) e^{ih(t)}\, dt &= \int_{h(t_1)}^{h(t_2)} G(\tau) e^{i\lambda\tau}\, d\tau \\
&= \frac{1}{i\lambda} \int_{h(t_1)}^{h(t_2)} G(\tau) \frac{d}{d\tau} e^{i\lambda\tau}\, d\tau \\
&= \frac{1}{i\lambda} \left[\frac{g(t_2)}{h'(t_2)} e^{i\lambda h(t_2)} - \frac{g(t_1)}{h'(t_1)} e^{i\lambda h(t_1)} - \int_{h(t_1)}^{h(t_2)} G'(\tau) e^{i\lambda\tau}\, d\tau \right].
\end{aligned} \tag{4.14}$$

Now if we apply the Riemann-Lebesgue lemma to the final integral we see that it goes to zero as $\lambda \to \infty$. The boundary terms do not go to zero as $\lambda \to \infty$; however they are oscillatory and are certainly bounded functions of λ. This means that the whole expression in square brackets is bounded as $\lambda \to \infty$, and we get the result that

$$\int_{t_1}^{t_2} g(t) e^{ih(t)}\, dt = O(\lambda^{-1}) \qquad \text{as} \qquad \lambda \to \infty\,. \tag{4.15}$$

Also, if we can take $t_1 = -\infty$ and $t_2 = +\infty$ (*i.e.* if h is a differentiable strictly monotonic function for all real t), then it has to be the case that the boundary terms vanish since g must go to zero at $t = \pm\infty$ for the integral to exist at all. In this case, we can say slightly more:

$$\int_{t_1}^{t_2} g(t) e^{ih(t)}\, dt = o(\lambda^{-1}) \qquad \text{as} \qquad \lambda \to \infty\,. \tag{4.16}$$

The estimate (4.15) plays a similar role in the method of stationary phase as the estimate (2.53) played in Laplace's method. It allows us to throw away most parts of the interval $[a, b]$ as long as we end up showing that the remaining contributions to $I(\lambda)$ are asymptotically dominant, bigger than λ^{-1}. Note however that whereas the estimate (2.53) was *exponentially small*, the estimate we get here, (4.15), is only *algebraically small*. In Laplace's method, the much stronger control of the nonlocal contributions allows us to write down a whole asymptotic expansion of the integral using only local information from the neighborhood of the maximum. In the method of stationary phase, the nonlocal contributions can be much larger, and consequently we may be forced to take them into account in computing corrections to the leading order asymptotic formula we will obtain.

4.1.3 Contributions from the neighborhood of a stationary phase point.

If $h(t)$ is differentiable and $h'(t) \neq 0$ on the whole finite or half-infinite interval $[a, b]$, then the estimate (4.15) immediately gives the leading-order asymptotic approximation

$$I(\lambda) = \frac{1}{i\lambda}\left[\frac{g(b)}{h'(b)}e^{i\lambda h(b)} - \frac{g(a)}{h'(a)}e^{i\lambda h(a)}\right] + o(\lambda^{-1}) \qquad \text{as} \qquad \lambda \to \infty\,. \tag{4.17}$$

If $a = -\infty$ and $b = \infty$ in the same circumstances, then it is more challenging to obtain the leading term, which is necessarily smaller. You can visualize it this way: the cancellation near a finite endpoint due to wild oscillations of the integrand is less catastrophic than the cancellation near an interior point because near an interior point contributions from the left and right halves of the neighborhood tend to cancel each other out, and this particular kind of cancellation cannot happen at an endpoint.

So we expect that the dominant contributions to $I(\lambda)$ will come from arbitrarily small neighborhoods of those points $t_k \in [a, b]$ for $k = 1, \ldots, n$ where $h'(t_k) = 0$. At such points, the phase angle of the complex exponential $e^{i\lambda h(t)}$ momentarily stops turning (reversing direction if $h''(t_k) \neq 0$). For this reason, the points t_k are called *stationary phase points*.

The generic situation is that $h''(t_k) \neq 0$ at the stationary phase point t_k. Let's find the leading contribution to $I(\lambda)$ from the neighborhood of this point. Pick any $\delta > 0$ (think of it as being small) and consider

$$I_k^\delta(\lambda) := \int_{t_k-\delta}^{t_k+\delta} g(t)e^{i\lambda h(t)}\,dt\,. \tag{4.18}$$

Since δ is small, and h is twice differentiable, we have

$$h(t) - h_k = \frac{h''(t_k)}{2}(t - t_k)^2 + o((t - t_k)^2)\,, \qquad \text{as} \qquad t \to t_k\,, \tag{4.19}$$

where h_k means $h(t_k)$. Using the same kind of reasoning we used in discussing Laplace's method, we can invoke the implicit function theorem to introduce a change of variables from t to s that exactly satisfies the equation

$$h(t) - h_k = \frac{h''(t_k)}{2}s^2\,. \tag{4.20}$$

Thus, if we let $\mu = h''(t_k)/2$, we get

$$I_k^\delta(\lambda) = e^{i\lambda h_k}\int_{s_-}^{s_+} k(s)e^{i\lambda\mu s^2}\,ds \qquad \text{where} \qquad s_\pm = s(t_k \pm \delta) = \pm\sqrt{\frac{2(h(t_k \pm \delta) - h(t_k))}{h''(t_k)}}\,, \tag{4.21}$$

and we are writing $k(s) := g(t(s))t'(s)$. We decompose this integral as follows:

$$\begin{aligned} I_k^\delta(\lambda) &= e^{i\lambda h_k} k(0) \int_{s_-}^{s_+} e^{i\lambda\mu s^2}\, ds + e^{i\lambda h_k} \int_{s_-}^{s_+} (k(s)-k(0)) e^{i\lambda\mu s^2}\, ds \\ &= e^{i\lambda h_k} k(0) \int_{-\infty}^{\infty} e^{i\lambda\mu s^2}\, ds - e^{i\lambda h_k} k(0) \int_{-\infty}^{s_-} e^{i\lambda\mu s^2}\, ds - e^{i\lambda h_k} k(0) \int_{s_+}^{\infty} e^{i\lambda\mu s^2}\, ds \\ &\quad + e^{i\lambda h_k} \int_{s_-}^{s_+} (k(s)-k(0)) e^{i\lambda\mu s^2}\, ds \\ &= J_1 + J_2 + J_3 + J_4\,. \end{aligned} \tag{4.22}$$

The dominant contribution will come from J_1 as we will now see. We study the four terms one by one.

The term J_1 is a tale by itself. Begin by rescaling to extract the $\lambda\mu$ dependence:

$$J_1 = \frac{k(0) e^{i\lambda h_k}}{\sqrt{\lambda|\mu|}} \int_{-\infty}^{\infty} e^{i\,\mathrm{sgn}(\mu)\tau^2}\, d\tau\,. \tag{4.23}$$

To evaluate this integral carefully, begin by recalling the meaning of the improper integral, namely that

$$\int_{-\infty}^{\infty} e^{i\,\mathrm{sgn}(\mu)\tau^2}\, dt := \lim_{R\to\infty} \int_{-R}^{R} e^{i\,\mathrm{sgn}(\mu)\tau^2}\, d\tau\,. \tag{4.24}$$

Letting $\tau = e^{i\pi\,\mathrm{sgn}(\mu)/4} z$ so that $\tau^2 = i\,\mathrm{sgn}(\mu) z^2$, we get a contour integral

$$\int_{-R}^{R} e^{i\,\mathrm{sgn}(\mu)\tau^2}\, d\tau = e^{i\pi\,\mathrm{sgn}(\mu)/4} \int_L e^{-z^2}\, dz\,, \tag{4.25}$$

where the contour L is the straight-line segment beginning at $z = -Re^{i\pi\,\mathrm{sgn}(\mu)/4}$ and ending at $z = Re^{i\pi\,\mathrm{sgn}(\mu)/4}$. Since the integrand is entire, we can use Cauchy's Theorem to deform the contour partly back to the real axis, and partly lying along the circle of radius R. In the latter integrals, we parametrize the contour by the angle θ and then combine them. Thus

$$\int_L e^{-z^2}\, dz = \int_{-R}^{R} e^{-x^2}\, dx + 2R \int_0^{\pi\,\mathrm{sgn}(\mu)/4} e^{-R^2(\cos(2\theta)+i\sin(2\theta))}\, d\theta\,. \tag{4.26}$$

Now we estimate the integral over θ as follows. Suppose first that $\mu > 0$. Then we have

$$\left| \int_0^{\pi/4} e^{-R^2(\cos(2\theta)+i\sin(2\theta))}\, d\theta \right| \le \int_0^{\pi/4} e^{-R^2\cos(2\theta)}\, d\theta\,. \tag{4.27}$$

The integral on the right-hand side of this inequality is exactly the kind we know how to approximate for large R using Laplace's method! Since the absolute maximum of $-\cos(2\theta)$ over the interval $\theta \in [0, \pi/4]$ occurs at the endpoint $\theta = \pi/4$ where the cosine vanishes and has a nonzero derivative, we know immediately from Laplace's method that

$$\int_0^{\pi/4} e^{-R^2\cos(2\theta)}\, d\theta = O(R^{-2}) \quad \text{as} \quad R\to\infty\,. \tag{4.28}$$

Similarly, if $\mu < 0$, then we have

$$\left| \int_0^{-\pi/4} e^{-R^2(\cos(2\theta)+i\sin(2\theta))}\, d\theta \right| \le \int_{-\pi/4}^{0} e^{-R^2\cos(2\theta)}\, d\theta = O(R^{-2}) \quad \text{as} \quad R\to\infty\,. \tag{4.29}$$

So after multiplying by R, the last term still goes to zero as $R\to\infty$. We have shown that

$$J_1 = \frac{k(0) e^{i(\lambda h_k + \pi\,\mathrm{sgn}(\mu)/4)}}{\sqrt{\lambda\mu}} \int_{-\infty}^{\infty} e^{-x^2}\, dx\,. \tag{4.30}$$

Now to finish the job we need to integrate the Gaussian, for which we recall the following very interesting trick. Let

$$I := \int_{-\infty}^{\infty} e^{-x^2}\, dx\,. \tag{4.31}$$

Since x is a dummy variable, we get that

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy\,. \tag{4.32}$$

The double integral can be reparametrized with polar coordinates:

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\, r\, dr\, d\theta = 2\pi\int_0^{\infty} e^{-r^2}\, r\, dr\,. \tag{4.33}$$

The extra r comes from the Jacobian of the change of variables $(x, y) \to (r, \theta)$, and it is just what we needed to change variables and evaluate the integral exactly. So at last we get

$$I^2 = \pi \quad \text{and therefore} \quad I = \sqrt{\pi}\,. \tag{4.34}$$

Putting these results together and using the fact that $k(0) = g(t_k)$ which follows from the exact change of variables, we finally get

$$J_1 = g(t_k)e^{i(\lambda h_k + \pi\,\mathrm{sgn}(\mu)/4)}\sqrt{\frac{\pi}{\lambda\mu}} = g(t_k)e^{i(\lambda h(t_k) + \pi\,\mathrm{sgn}(h''(t_k))/4)}\sqrt{\frac{2\pi}{\lambda|h''(t_k)|}}\,. \tag{4.35}$$

The terms J_2 and J_3 are handled as follows. Consider J_2 for concreteness and change variables to scale out the $\lambda\mu$:

$$J_2 = -\frac{e^{i\lambda h_k}k(0)}{\sqrt{\lambda|\mu|}}\int_{-\infty}^{\sqrt{\lambda|\mu|}\cdot s_-} e^{i\,\mathrm{sgn}(\mu)\tau^2}\, d\tau\,. \tag{4.36}$$

The upper limit of the integral is going to $-\infty$ as $\lambda \to +\infty$ because $s_- < 0$. Since we know from our previous analysis that the integral from $-\infty$ to $+\infty$ exists, it must be the case that the integral on the right-hand side of (4.36) goes to zero as $\lambda \to \infty$. That is,

$$J_2 = o(\lambda^{-1/2}) \quad \text{as} \quad \lambda \to \infty\,, \tag{4.37}$$

so it is asymptotically *smaller* than J_1 and negligible by comparison. The same argument shows that $J_3 = o(\lambda^{-1/2})$ as $\lambda \to \infty$ and is therefore also negligible. In fact, both of these estimates can be beefed up to $O(\lambda^{-1})$.

Now we move on to J_4, which we write as follows:

$$J_4 = \frac{e^{i\lambda h_k}}{2}\int_{s_-}^{0} \frac{k(s)-k(0)}{s}e^{i\lambda\mu s^2}\, 2s\, ds + \frac{e^{i\lambda h_k}}{2}\int_0^{s_+} \frac{k(s)-k(0)}{s}e^{i\lambda\mu s^2}\, 2s\, ds\,. \tag{4.38}$$

The fraction in the integrand is integrable at $s = 0$ because of cancellation. In each term we make a change of variables:

$$J_4 = \frac{e^{i\lambda h_k}}{2}\int_0^{s_-^2} \frac{k(-\sqrt{\tau})-k(0)}{\sqrt{\tau}}e^{i\lambda\mu\tau}\, d\tau + \frac{e^{i\lambda h_k}}{2}\int_0^{s_+^2} \frac{k(\sqrt{\tau})-k(0)}{\sqrt{\tau}}e^{i\lambda\mu\tau}\, d\tau\,. \tag{4.39}$$

These integrals both go to zero as $\lambda \to \infty$ by the Riemann-Lebesgue Lemma. The question is whether they go to zero faster than J_1. If we assume that the integrand is differentiable (which means that $g(t)$ is differentiable at $t = t_k$) then we may integrate by parts and *then* apply the Riemann-Lebesgue Lemma to find that

$$J_4 = O(\lambda^{-1}) \quad \text{as} \quad \lambda \to \infty\,. \tag{4.40}$$

We have therefore found that the leading contribution to $I(\lambda)$ from the neighborhood of a stationary phase point t_k at which $g(t)$ is differentiable and $h(t)$ is twice differentiable with $h''(t_k) \neq 0$ is given by

$$I_k^\delta(\lambda) \sim g(t_k) e^{i(\lambda h(t_k) + \pi \operatorname{sgn}(h''(t_k))/4)} \sqrt{\frac{2\pi}{\lambda |h''(t_k)|}} \quad \text{as} \quad \lambda \to \pm\infty \,. \tag{4.41}$$

Note that, as usual, the asymptotic result is independent of the size δ of the neighborhood under consideration as long as it only contains the single stationary phase point t_k. The local error near each t_k is generally $O(\lambda^{-1})$, which is the same asymptotic size as the contributions to $I(\lambda)$ coming from outside the δ-neighborhoods of each t_k. Therefore, taking all n stationary phase points in the interval $[a, b]$ into consideration, and assuming that they are all nondegenerate $h''(t_k) \neq 0$, we get the asymptotic formula

$$I(\lambda) = \sum_{k=1}^{n} g(t_k) e^{i(\lambda h(t_k) + \pi \operatorname{sgn}(h''(t_k))/4)} \sqrt{\frac{2\pi}{\lambda |h''(t_k)|}} + O(\lambda^{-1}) \quad \text{as} \quad \lambda \to \pm\infty \,. \tag{4.42}$$

Note that, unlike the case in Laplace's method where only the critical points corresponding to the absolute maximum were important asymptotically, here *all stationary phase points contribute equally to the integral.*

By following the same basic steps, generalized formulas may be found to cover cases where there is a degenerate stationary phase point where $h''(t_k) = 0$. Such points result in larger contributions to $I(\lambda)$ than in the nondegenerate case because the phase remains stationary "longer" near t_k as $h(t)$ is a "flatter" function there.

4.2 Examples

4.2.1 The long-time asymptotics of linear dispersive waves.

In virtually every physical situation in which waves are propagating

- in one space dimension
- with small amplitude
- in the absence of dissipation or gain
- in a fixed homogeneous medium

the appropriate model for the wave field $u(x, t)$, where x is space and t is time, is a linear, constant-coefficient, partial differential equation of the general form

$$P\left(-i\frac{\partial}{\partial x}, i\frac{\partial}{\partial t}\right) u(x, t) = 0 \tag{4.43}$$

where $P(k, \omega)$ is a polynomial in k and ω with real coefficients. For example,

1. The *wave equation*

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{4.44}$$

is of this form with $P(k, \omega) = c^2 k^2 - \omega^2$.

2. The *Schrödinger equation*

$$i\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2} \frac{\partial^2 \psi}{\partial x^2} = 0 \tag{4.45}$$

which describes the evolution of the wave function of free quantum particles of mass 1, is of this form with $P(k, \omega) = \hbar\omega - \hbar^2 k/2$.

3. The *linear Korteweg-de Vries equation*
$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{4.46}$$
describing the amplitude of long water waves, is of this form with $P(k,\omega) = \omega + k^3$.

4. The *Klein-Gordon equation*
$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + m^2 u = 0 \tag{4.47}$$
which comes from relativistic quantum field theory for particles with mass m, is of this form with $P(k,\omega) = c^2k^2 - \omega^2 + m^2$.

All of these problems are solved in the same way, by separation of variables. Equivalently, seek particular solutions in the form of exponentials:
$$\text{Try:} \qquad u(x,t) = e^{i(kx-\omega t)}\,. \tag{4.48}$$
This is a traveling wave of *wavenumber* k and *frequency* ω. Alternatively, the waveform has a *wavelength* given by $\lambda = 2\pi/k$ and a *period* given by $T = 2\pi/\omega$. Since we can write $e^{i(kx-\omega t)}$ in the form $e^{ik(x-v_p t)}$ with $v_p = \omega/k$, we see that the peaks and troughs of this waveform are moving to the right with speed $v_p = \omega/k$, which is called the *phase velocity*.

Question: for which values of ω and k will this be a solution of the equation? We notice that:
$$-i\frac{\partial}{\partial x}e^{i(kx-\omega t)} = ke^{i(kx-\omega t)} \qquad \text{and} \qquad i\frac{\partial}{\partial t}e^{i(kx-\omega t)} = \omega e^{i(kx-\omega t)}\,. \tag{4.49}$$
Consequently,
$$P\left(-i\frac{\partial}{\partial x}, i\frac{\partial}{\partial t}\right)e^{i(kx-\omega t)} = P(k,\omega)e^{i(kx-\omega t)}\,. \tag{4.50}$$
So the exponential will satisfy (4.43) if the parameters k and ω are linked by the relation $P(k,\omega) = 0$. The function $P(k,\omega)$ is called the *symbol* of the partial differential equation and the equation $P(k,\omega) = 0$ is referred to as the *dispersion relation* for (4.43). It tells us what kinds of waves are allowed as solutions.

The general solution of (4.43) should be sought as a superposition of these particular solutions. Usually, we try to use an integral with respect to k:
$$u(x,t) = \int_{-\infty}^{\infty} A(k)e^{i(kx-\omega t)}\,dk \tag{4.51}$$
where ω and k are linked by the dispersion relation $P(k,\omega) = 0$. To make this effective, we have to solve the dispersion relation for ω as a function of k. There could be several branches of the solution. For example, in the case of the wave equation, $P(k,\omega) = \omega^2 - c^2k^2 = 0$ implies either $\omega = \omega_+(k) := ck$ or $\omega = \omega_-(k) := -ck$. Suppose that the branches are denoted by $\omega_1(k), \ldots, \omega_n(k)$. Then we seek the general solution in the form
$$u(x,t) = \sum_{j=1}^{n} \int_{-\infty}^{\infty} A_j(k)e^{i(kx-\omega_j(k)t)}\,dk \tag{4.52}$$
where the $A_j(k)$ are arbitrary functions. They are connected with the initial data for (4.43), namely the values of u and possibly its derivatives with respect to t at $t = 0$, via Fourier transforms.

As was the case in our study of the diffusion equation, it is important to ask what this formula implies about the solution, since usually the integrals cannot be evaluated in closed form. Let us again consider the question of how the solution behaves in the limit $t \to \infty$ in a frame of reference moving with velocity v, a fixed parameter. In this moving frame, the field is given by
$$u(x = vt, t) = \sum_{j=1}^{n} \int_{-\infty}^{\infty} A_j(k)e^{it(kv-\omega_j(k))}\,dk\,. \tag{4.53}$$

Now, the frequencies $\omega_j(k)$ are all real, so the function $h_j(k) := kv - \omega_j(k)$ is a real-valued function of k over the range of integration. So the asymptotic behavior of the integral can be determined using the method of stationary phase. The stationary phase points k_{jm} satisfy the relation:

$$h_j'(k_{jm}) = v - \omega_j'(k_{jm}) = 0\,. \tag{4.54}$$

So, the index m is just ranging over all solutions of this equation for v fixed. Whether these points are simple stationary points is determined by looking at

$$h_j''(k_{jm}) = -\omega_j''(k_{jm})\,. \tag{4.55}$$

Thus, a simple stationary point k_{jm} is characterized by the equation $\omega_j''(k_{jm}) \neq 0$. If all stationary points turn out to be simple, then the formula implied by the method of stationary phase is:

$$u(vt,t) = \sum_{j=1}^{n}\sum_{m} A_j(k_{jm}) e^{it(vk_{jm}-\omega_j(k_{jm}))} e^{-i\pi\,\mathrm{sgn}(\omega_j''(k_{jm}))/4}\sqrt{\frac{2\pi}{t|\omega_j''(k_{jm})|}} + o(t^{-1/2}) \qquad \text{as} \qquad t \to +\infty\,. \tag{4.56}$$

It is instructive to eliminate v and put back x by setting $v = x/t$. Then the formula becomes

$$u(x,t) = \sum_{j=1}^{n}\sum_{m} A_j(k_{jm}) e^{-i\pi\,\mathrm{sgn}(\omega_j''(k_{jm}))/4}\sqrt{\frac{2\pi}{t|\omega_j''(k_{jm})|}} e^{i(k_{jm}x-\omega_j(k_{jm})t)} + o(t^{-1/2}) \tag{4.57}$$

as t and x both go to infinity with their ratio $v = x/t$ held fixed.

Here, we really see the structure of the solution for long times. Firstly, the asymptotic solution is a linear superposition of the elementary complex exponential exact solutions $e^{i(kx-\omega t)}$. We may ask the question of exactly which exponential waves appear near a given x and t (both large enough). To answer this question, we need to remember that since $v = x/t$, the stationary phase points are really functions of x and t, since they depend on the parameter v. Since both x and t are large, v does not change very much if x and t are varied over a few wavelengths or periods. So v and therefore the stationary phase points k_{jm} are *slowly varying* functions of x and t. Thus, near a given x and t, one sees slowly modulated waves with wavenumber k_{jm} satisfying the equation

$$\frac{x}{t} - \omega_j'(k_{jm}) = 0 \qquad \text{or} \qquad x = \omega_j'(k_{jm})t\,. \tag{4.58}$$

The quantity $\omega'(k)$ therefore looks like a velocity. For general dispersion relations and for general k, this velocity is *not* the same as the phase velocity $v_p = \omega(k)/k$. By contrast, we call the velocity $v_g = \omega'(k)$ the *group velocity*. So our asymptotic formula says that what you see in the neighborhood of position x and time t (here neighborhood means within a few wavelengths or periods) is a slowly modulated superposition of precisely those waves whose group velocity is equal to x/t.

Secondly, the amplitude of each of these waves is decaying in time like $t^{-1/2}$. The rate of this decay process is controlled by the size of $\omega_j''(k_{jm})$. The smaller this number is, the more slowly the amplitude decays. This number is called the *dispersion* of the waves.

4.2.2 The semiclassical dynamics of free particles in quantum mechanics.

Quantum mechanics provides us with a great historical example of a case when the equation satisfied by waves was obtained after the fact, from knowledge of the dispersion relation. The fundamental starting point is the fact that for free particles, the energy E (which is usually made up of a kinetic part T and a potential part V) is purely kinetic. If a free particle has momentum p, then the formula for kinetic energy from classical mechanics is

$$E = T := \frac{p^2}{2m}\,. \tag{4.59}$$

Now into this classical relation we insert two quantum-mechanical hypotheses. First, experiments carried out on beams of particles of momentum p indicated that these beams diffract just like ordinary light beams, where the light has wavelength $\lambda = 2\pi/k$ and k, the wavenumber, is related to the momentum by

$$p = \hbar k \tag{4.60}$$

which is the famous de Broglie relation. Here $\hbar$ is Planck's constant. We think of k as the wavenumber of the "matter wave" associated with a free particle of momentum p. Although it was proposed on a theoretical basis by de Broglie in the early 1920's this relation was experimentally verified in an experiment by Davisson and Germer in 1927 involving electron diffraction in nickel crystals. We take this as one hypothetical relation for quantum-mechanical waves. Second, in an effort to explain the spectrum of blackbody radiation, Planck himself argued in 1900 that electromagnetic radiation of angular frequency Ω could only be emitted or absorbed in discrete amounts of energy called "quanta". He specified that the quantum energy of a field with frequency Ω was

$$E = \hbar\Omega\,. \tag{4.61}$$

This fact was later verified in the sense that it explains the photoelectric effect, as Einstein showed in his Nobel prize winning work.

Let us suppose that a particle that we want to represent by a "matter wave" is set into motion by absorbing one quantum of radiation from an electromagnetic field of frequency Ω. The totally kinetic energy of the particle will then be $E = \hbar\Omega$, which must also be equal to $p^2/2m = \hbar^2k^2/2m$. Now the matter wave must have a frequency itself, say ω. Since it is interacting with the electromagnetic field in order to absorb some of its energy, it is natural to suppose that the two fields must be in *resonance*, which means that they have the same frequency: $\omega = \Omega$. Therefore the energy of the matter wave can be expressed in terms of its own frequency as $E = \hbar\omega$. Equating this to the kinetic energy expression in terms of momentum and thus wavenumber, we find

$$\hbar\omega = \frac{\hbar^2k^2}{2m} \tag{4.62}$$

which is a dispersion relation $P(k,\omega) = 0$ corresponding to $P(k,\omega) = \omega - \hbar k^2/2m$. Waves of the form $\psi = e^{i(kx-\omega t)}$ for which k and ω are linked by this dispersion relation necessarily satisfy the partial differential equation

$$i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = 0\,. \tag{4.63}$$

This is the Schrödinger equation for a free particle of mass m.

The solution $\psi(x,t)$ of Schrödinger's equation is called a *wave function* and is necessarily a complex-valued function of x and t because of the $i = \sqrt{-1}$ in the equation. This latter fact confounded early researchers because they felt the need to interpret $\psi(x,t)$ as the "density" of the electron at position x and time t. So it was interesting to determine what the limiting behavior of wave functions would be as $\hbar \to 0$, which is the limit in which quantum phenomena disappear. In this limit electron beams cease to diffract and radiation fields are not quantized in multiples of $\hbar\Omega$.

We now undertake an analysis of the limit $\hbar \to 0$ in the solutions of Schrödinger's equation, the so-called *semiclassical limit*. We expect the classical dynamics of free particles to appear somehow in the limit. Note that we really don't have the freedom to let $\hbar$ go to zero, since it is a fixed given number, namely $\hbar = 1.05 \times 10^{-34}\mathrm{kg}\cdot\mathrm{m}^2/\mathrm{s}$ (units of *action*). However, if other characteristic action scales in the problem are much larger than $\hbar$, then an asymptotic description in the limit $\hbar \to 0$ will be useful. From now on, we suppose that the units are chosen so that $m = 1$.

Before we attack this problem using the method of stationary phase, let us think about what should happen to ψ as $\hbar$ goes to zero. The fact is, it is not clear from Schrödinger's equation because if we set $\hbar = 0$ we don't have anything left over. In analogy with our experience in finding roots of equations, we might say that the assumption that ψ goes to some limit as $\hbar \to 0$ is not a dominant balance. Some sort of rescaling is necessary to elicit the correct asymptotic behavior. The idea here is the following: separate $\psi(x,t)$ into its phase and amplitude by writing

$$\psi(x,t) = A(x,t)e^{iS(x,t)/\hbar} \tag{4.64}$$

where A and S are both real-valued functions. Plug this into Schrödinger's equation and separate real and imaginary parts. We get

$$A\frac{\partial S}{\partial t} + \frac{1}{2}A\left(\frac{\partial S}{\partial x}\right)^2 = \frac{\hbar^2}{2}\frac{\partial^2 A}{\partial x^2} \quad \text{and} \quad \frac{\partial A}{\partial t} + \frac{\partial S}{\partial x}\frac{\partial A}{\partial x} + \frac{1}{2}A\frac{\partial^2 S}{\partial x^2} = 0\,. \tag{4.65}$$

Now it appears to be a dominant balance to take A and S to be functions tending to nice limits as $\hbar \to 0$. If we follow this idea, then to leading order we could neglect the terms with positive powers of $\hbar$. In particular, we expect that in the limit $\hbar \to 0$, we would have $S \to S_0$ which would satisfy the equation

$$\frac{\partial S_0}{\partial t} + \frac{1}{2}\left(\frac{\partial S_0}{\partial x}\right)^2 = 0\,. \tag{4.66}$$

Or, differentiating with respect to x and setting $u = \partial S_0/\partial x$ we get

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0\,. \tag{4.67}$$

Of course, this is precisely the nonlinear shock wave equation we studied as an application of Laplace's method. It is solved by the method of characteristics. Suppose the initial condition is $u(x,0) = u_0(x)$. Information from the point $x = \xi$ at $t = 0$ propagates along the characteristic straight line $x = u_0(\xi)t + \xi$, along which u remains constant. This linear propagation with constant velocity $u_0(\xi)$ is the way in which the classical limit of free-streaming particles is recovered from the solution of the Schrödinger equation. To see this better, remember that velocity is related to momemtum and mass by $v = p/m = \hbar k/m$ and recall the correspondence

$$ke^{i(kx-\omega t)} = -i\frac{\partial}{\partial x}e^{i(kx-\omega t)}\,. \tag{4.68}$$

If we formally extend this correspondence to the wave function ψ we get $k\psi \approx -i\partial\psi/\partial x$ and inserting the form $\psi = Ae^{iS/\hbar}$ we find $\hbar k \approx \partial S/\partial x \approx u$. So the field u can be interpreted as particle momentum, and then we see that the shock wave equation is the classical statement of conservation of momentum or Newton's first law: in the absence of applied forces, a particle with mass m and momentum p travels in a straight line with constant velocity $v = p/m$ (and hence constant momentum).

Suppose the initial condition $u(x,0)$ is a positive, bump-shaped function of x. This means that a collection of rapidly moving particles (corresponding to the momentum values near the peak) are trapped between two regions of more slowly moving particles (corresponding to the momentum values in the tails). Sooner or later, the fast moving particles are going to collide with the more slowly moving particles. This occurs when the characteristic lines first intersect. Now we know that this phenomenon leads to a region of the (x,t)-plane in which the method of characteristics leads to a triple-valued solution for the momentum field $u(x,t)$. Now, as was the case with Burgers' equation, it turns out that the neglected terms in the equation for S become important exactly when the shock wave develops and the characteristics first begin to intersect. To figure out what actually happens after the shock first forms, we have to take the neglected terms into account and analyze more carefully the limit $\hbar \to 0$. Whereas in the case of Burgers' equation the effect of the weakly-dissipative correction served to pick out a precise location for a propagating shock wave, the effect of the correction terms here will lead to new phenomena. From the quantum mechanical point of view, once the classical particles are occupying the same space, they will not collide, but rather *interfere* as would waves. We can see all of this explicitly using the method of stationary phase.

Suppose we study the initial-value problem

$$i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2}\frac{\partial^2\psi}{\partial x^2} = 0 \quad \text{subject to} \quad \psi(x,0) = a(x)\exp\left(i\frac{s(x)}{\hbar}\right)\,, \tag{4.69}$$

where a and s are given real-valued functions of x with $s'(x)$ having a single maximum. We know that the general solution is given by the superposition formula

$$\psi(x,t) = \int_{-\infty}^{\infty} B(k)\exp\left(i\left(kx - \frac{\hbar k^2}{2}t\right)\right)\,dk\,. \tag{4.70}$$

Before we study this formula, we should first figure out what $B(k)$ is. Setting $t = 0$ and referring back to our discussion of the solution of the diffusion equation, we see that the formula for $B(k)$ is given by the Fourier transform of $\psi(x,0)$

$$B(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(x)\exp\left(i\left(\frac{s(x)}{\hbar} - kx\right)\right)\,dx\,. \tag{4.71}$$

We can approximate this integral easily using the method of stationary phase. To do this, set $k = \kappa/\hbar$ so that the integral becomes

$$B(\kappa/\hbar) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(x)e^{ih(x)/\hbar}\,dx \qquad \text{where} \qquad h(x) = s(x) - \kappa x\,. \tag{4.72}$$

Considering the limit $\hbar \to 0$ with $\hbar > 0$, the stationary phase points satisfy

$$h'(x) = s'(x) - \kappa = 0\,. \tag{4.73}$$

Since we are assuming that $s'(x)$ is positive and has a single maximum with value $u_{\max}$, there are

- Two simple stationary phase points, say $x_-(\kappa)$ and $x_+(\kappa)$, as long as $0 < \kappa < u_{\max}$.
- No stationary phase points if $\kappa < 0$ or $\kappa > u_{\max}$.

If we take $x_-(\kappa) < x_+(\kappa)$, then since $s'(x)$ is increasing at $x_-(\kappa)$ and decreasing at $x_+(\kappa)$ we get

$$h''(x_-(\kappa)) > 0 \qquad \text{and} \qquad h''(x_+(\kappa)) < 0\,. \tag{4.74}$$

Applying the method of stationary phase, we find that as $\hbar \to 0$,

$$\begin{aligned} B(\kappa/\hbar) \quad = \quad & \frac{e^{i\pi/4}\sqrt{\hbar}}{\sqrt{2\pi}}\left[\frac{a(x_-(\kappa))}{\sqrt{|s''(x_-(\kappa))|}}\exp\left(i\frac{s(x_-(\kappa)) - \kappa x_-(\kappa)}{\hbar}\right)\right. \\ & \left. - \frac{a(x_+(\kappa))}{\sqrt{|s''(x_+(\kappa))|}}\exp\left(i\frac{s(x_+(\kappa)) - \kappa x_+(\kappa)}{\hbar}\right)\right] + O(\hbar) \end{aligned} \tag{4.75}$$

as long as κ is in the range of values in which there are two stationary phase points. For other values of κ, $B(\kappa/\hbar) = O(\hbar)$ because there are no stationary phase points.

To make use of this calculation, we change variables in the integral for $\psi(x,t)$ by setting $k = \kappa/\hbar$. We get

$$\psi(x,t) = \frac{1}{\hbar}\int_{-\infty}^{\infty} B(\kappa/\hbar)e^{iH(\kappa)/\hbar}\,d\kappa \qquad \text{where} \qquad H(\kappa) := \kappa x - \frac{1}{2}\kappa^2 t\,. \tag{4.76}$$

Inserting our stationary phase formula for $B(\kappa/\hbar)$ we get a sum of two terms, one for $x_-(\kappa)$ and one for $x_+(\kappa)$:

$$\psi(x,t) \sim \frac{e^{i\pi/4}}{\sqrt{2\pi\hbar}}\int_0^{u_{\max}} \frac{a(x_-(\kappa))}{\sqrt{|s''(x_-(\kappa))|}}e^{i\Phi_-(\kappa)/\hbar}\,d\kappa - \frac{e^{i\pi/4}}{\sqrt{2\pi\hbar}}\int_0^{u_{\max}} \frac{a(x_+(\kappa))}{\sqrt{|s''(x_+(\kappa))|}}e^{i\Phi_+(\kappa)/\hbar}\,d\kappa \tag{4.77}$$

as $\hbar \to 0$, where $\Phi_\pm(\kappa) := s(x_\pm(\kappa)) - \kappa x_\pm(\kappa) + \kappa x - \kappa^2 t/2$. These integrals can also be approximated for small $\hbar$ using the method of stationary phase. Let x and t be fixed. Then the stationary phase points satisfy

$$\Phi'_\pm(\kappa) = s'(x_\pm(\kappa))x'_\pm(\kappa) - \kappa x'_\pm(\kappa) - x_\pm(\kappa) + x - \kappa t = 0\,. \tag{4.78}$$

Now remember that the points $x_\pm(\kappa)$ are, by definition, solutions of the equation $s'(x_\pm(\kappa)) - \kappa = 0$ so the equation simplifies to

$$\Phi'_\pm(\kappa) = x - x_\pm(\kappa) - \kappa t = 0 \qquad \text{or} \qquad x - \kappa t = x_\pm(\kappa)\,. \tag{4.79}$$

Taking $s'(\cdot)$ of both sides of this latter equation and again using the definition of $x_\pm(\kappa)$ gives

$$s'(x - \kappa t) = \kappa\,. \tag{4.80}$$

This equation says that a stationary point κ is a reciprocal slope of a characteristic line passing through the point (x,t). We have to figure out which slopes correspond to solutions of the equation $x - \kappa t = x_+(\kappa)$ and which correspond to solutions of the equation $x - \kappa t = x_-(\kappa)$. But this is not hard. Suppose that $x = \kappa t + \xi$ is a characteristic line passing through (x,t). The value of ξ is the x-intercept. If ξ is greater than the value

of x where $s'(x)$ achieves its maximum it is equal to $x_+(\kappa)$. And if ξ is less than the value of x where $s'(x)$ achieves its maximum it is equal to $x_-(\kappa)$.

So suppose that x and t are such that there is only one characteristic line passing through (x,t), and its intercept $x=\xi$ at $t=0$ lies to the left of the maximum of $s'(x)$. Then the equation $\Phi'_+(\kappa)=0$ has no solutions and the equation $\Phi'_-(\kappa)=0$ has a unique solution κ_0 equal to the reciprocal slope of the characteristic line. In this case, the second integral will be negligible compared with the first since it has no stationary phase points. Using the leading-order formula to analyze the first integral with its unique stationary phase point κ_0 gives

$$\psi(x,t) = e^{i\pi(1+\mathrm{sgn}(\Phi''_-(\kappa_0)))/4}\frac{a(\xi)}{\sqrt{|s''(\xi)|\cdot|\Phi''_-(\kappa_0)|}}e^{i\Phi_-(\kappa_0)/\hbar}+o(1) \tag{4.81}$$

as $\hbar\to 0$. Similarly if the point (x,t) has just one characteristic line passing through it with reciprocal slope κ_0 whose x-intercept $x=\xi$ at $t=0$ lies to the right of the maximum of $s'(x)$, then the first integral is negligible and we find

$$\psi(x,t) = -e^{i\pi(1+\mathrm{sgn}(\Phi''_+(\kappa_0)))/4}\frac{a(\xi)}{\sqrt{|s''(\xi)|\cdot|\Phi''_+(\kappa_0)|}}e^{i\Phi_+(\kappa_0)/\hbar}+o(1) \tag{4.82}$$

as $\hbar\to 0$.

Note that in both of these cases, *i.e.* whenever there is only one characteristic line passing through (x,t), the solution has the following properties. It is neither small nor large asymptotically, but is highly oscillatory. The amplitude is proportional to the amplitude $a(\xi)$ of the initial condition carried along the characteristic trajectory. We say that the amplitude is passively transported by the phase (which determines the characteristics). Also, the solution is given by a single exponential function, so there is no cancellation. This is more clear when we look at the square modulus:

$$|\psi(x,t)|^2 = \frac{a(\xi)^2}{|s''(\xi)|\cdot|\Phi''_\pm(\kappa_0)|}+o(1) \tag{4.83}$$

as $\hbar\to 0$, which holds for both cases. This function has no zeros as we vary x and t.

Now, if we pick values of x and t inside the triple-valued region for the shock wave equation, then there are three characteristic lines passing through (x,t). It is not hard to see that exactly one of these has a reciprocal slope satisfying $x-\kappa t = x_-(\kappa)$ and the remaining two have reciprocal slopes satisfying $x-\kappa t = x_+(\kappa)$. That is, two of the characteristic lines intercept the x-axis at $t=0$ at points to the right of the maximum of $s'(x)$ and one hits at a point to the left of the maximum. Let's label the three reciprocal slopes κ_1, κ_2, and κ_3, in order according to their x-intercepts at $t=0$ arranged from left to right. The corresponding x-intercepts are $\xi_1<\xi_2<\xi_3$. Now when we apply the method of stationary phase to approximate $\psi(x,t)$, we have to take into account one stationary phase point $\kappa=\kappa_1$ in the first integral and two stationary phase points $\kappa=\kappa_2$ and $\kappa=\kappa_3$ in the second integral. The solution takes the asymptotic form of a sum of three terms

$$\begin{aligned}\psi(x,t) \quad = \quad & e^{i\pi(1+\mathrm{sgn}(\Phi''_-(\kappa_1)))/4}\frac{a(\xi_1)}{\sqrt{|s''(\xi_1)|\cdot|\Phi''_-(\kappa_1)|}}e^{i\Phi_-(\kappa_1)/\hbar} \\ & -e^{i\pi(1+\mathrm{sgn}(\Phi''_+(\kappa_2)))/4}\frac{a(\xi_2)}{\sqrt{|s''(\xi_2)|\cdot|\Phi''_+(\kappa_2)|}}e^{i\Phi_+(\kappa_2)/\hbar} \\ & -e^{i\pi(1+\mathrm{sgn}(\Phi''_+(\kappa_3)))/4}\frac{a(\xi_3)}{\sqrt{|s''(\xi_3)|\cdot|\Phi''_+(\kappa_3)|}}e^{i\Phi_+(\kappa_3)/\hbar}+o(1)\end{aligned} \tag{4.84}$$

as $\hbar\to 0$. Again, the field is neither large nor small in general. But now if we look at $|\psi(x,t)|^2$, we will get cross terms from the three exponential functions so that $|\psi(x,t)|^2$ will itself be a rapidly varying function of x and t. The pattern of maxima and minima present in the semiclassical solution in this regime can be interpreted as the quantum interference pattern of the colliding particles.

Let's compare the solution of the coupled equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\hbar^2}{2}\frac{\partial}{\partial x}\left(\frac{1}{A}\frac{\partial^2 A}{\partial x^2}\right) \quad \text{and} \quad \frac{\partial A}{\partial t} + u\frac{\partial A}{\partial x} + \frac{1}{2}A\frac{\partial u}{\partial x} = 0\,, \tag{4.85}$$

with that of Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}\,, \tag{4.86}$$

both of which reduce to the simple shock wave equation for $u(x,t)$ when $\hbar = 0$ or $\nu = 0$. The shock wave equation is difficult to interpret when the solution develops singularities. Both of these "extensions" of the equation can be solved for all x and t which gives us two different ways of regularizing the shock phenomenon. Yet these two systems have drastically different asymptotic behaviors. One of them (Burgers' equation) essentially cuts through the three-sheeted structure with a knife, putting a shock wave at exactly the trajectory where asymptotics switch over from one critical point to another. Away from this shock trajectory, the solution of Burgers' equation looks just like one of the sheets of the solution of the shock wave equation. On the other hand, the other system (Schrödinger's equation) fills in the whole triple-valued region with a wildly oscillating sea of maxima. It is as though the necessarily single-valued solution of Schrödinger's equation were trying to sample all three branches of the shock wave solution by wildly oscillating among them. This kind of oscillatory fix for shock waves is called a *dispersive regularization*.

An alternative way to obtain these asymptotic formulas is to express the solution of Schrödinger's equation more directly in terms of the initial data, essentially using a Green's function. So by Fourier theory, the solution formula is

$$\psi(x,t) = \int_{-\infty}^{\infty} B(k)e^{ikx - i\hbar tk^2/2}\,dk = \int_{-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(y,0)e^{-iky}\,dy\,e^{ikx-i\hbar tk^2/2}\,dk\,. \tag{4.87}$$

Exchanging order of integration (which is, technically speaking, only justified if $t \neq 0$), we get

$$\psi(x,t) = \int_{-\infty}^{\infty} K(x-y,t)\psi(y,0)\,dy \tag{4.88}$$

where

$$K(d,t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikd - i\hbar tk^2/2}\,dk\,. \tag{4.89}$$

This function is the *Green's function* for our initial-value problem. Completing the square in the exponent gives

$$K(d,t) = \frac{e^{id^2/2\hbar t}}{2\pi}\int_{-\infty}^{\infty} e^{-i\hbar t(k-d/\hbar t)^2/2}\,dk = \frac{e^{id^2/2\hbar t}}{2\pi}\int_{-\infty}^{\infty} e^{-i\hbar tu^2/2}\,du\,. \tag{4.90}$$

Treating this latter integral exactly as we did in working out the leading-order formula for the contribution of simple stationary phase points, we finally find

$$K(d,t) = \frac{e^{-i\pi\,\mathrm{sgn}(t)/4}}{\sqrt{2\pi\hbar|t|}}e^{id^2/2\hbar t}\,. \tag{4.91}$$

Note the similarity of the Green's function with the heat kernel for taken for imaginary t (the latter is the Green's function for the diffusion equation, which we used when we studied Burgers' equation).

Since the initial data has the form $\psi(y,0) = a(y)\exp(is(y)/\hbar)$, we get a solution formula for Schrödinger's equation in the form

$$\psi(x,t) = \frac{e^{-i\pi\,\mathrm{sgn}(t)/4}}{\sqrt{2\pi\hbar|t|}}\int_{-\infty}^{\infty} a(y)e^{is(y)/\hbar}e^{i(x-y)^2/2\hbar t}\,dy = \frac{e^{-i\pi\,\mathrm{sgn}(t)/4}}{\sqrt{2\pi\hbar|t|}}\int_{-\infty}^{\infty} a(y)e^{iL(y)/\hbar}\,dy\,, \tag{4.92}$$

where $L(y) := s(y) + (x-y)^2/2t$. The stationary phase points satisfy $L'(y) = s'(y) - (x-y)/t = 0$. The connection with our previous methodology comes from introducing a new variable κ defined for fixed x and t in terms of y by the equation $y = x - \kappa t$, or $\kappa = (x-y)/t$. Once again, we obtain exactly as many stationary phase points as there are characteristic rays through the point (x,t). But we avoided the extra step of asymptotically approximating the Fourier transform $B(k)$ along the way.

Chapter 5

Transform Integrals and their Asymptotic Analysis

5.1 Fourier and Laplace Transforms

In general terms, a *transform* is a "change of variables" at the level of functions. Just like changes of variables sometimes simplify algebraic problems, function transforms can simplify integral and/or differential equations. The price one pays for this simplification is that the transform and its inverse, even when given by explicit integral formulas, rarely admit exact analytical evaluation. This is why asymptotic analysis plays an important role in this field.

Different kinds of transforms are adapted to different kinds of problems, and you can even invent your own (we will do this shortly) but over the years several particular transforms have emerged as being universally useful. These include the Mellin transform, the Hankel transform, the Legendre transform, and the Hilbert transform, but the most famous by far are the *Fourier transform* and the *Laplace transform* along with their variants.

First we define the Fourier transform.

Definition 10 (Fourier Transform Pair) *Let $f(x)$ be a function that is square integrable on $(-\infty, \infty)$, that is,*

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty \,. \tag{5.1}$$

Then the Fourier Transform of f is a new square integrable function $\hat{f}(k)$ defined for almost all real k by

$$\hat{f}(k) := \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx \,. \tag{5.2}$$

The function $f(x)$ can be reciprocally expressed as the Inverse Fourier Transform of $\hat{f}$ according to the formula

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} \, dk \,. \tag{5.3}$$

Sometimes we use the notation $\mathcal{F}f := \hat{f}$, and $\mathcal{F}^{-1}\hat{f} := f$.

Next we define the Laplace transform.

Definition 11 (Laplace Transform Pair) *Let $f(t)$ be a function defined for real $t > 0$ so that for some real C we have $f(t) = O(e^{Ct})$ as $t \to +\infty$. Then the Laplace Transform of f is a new function $F(s)$ analytic for $\Re(s) > C$ and defined by*

$$F(s) := \int_{0}^{\infty} f(t) e^{-st} \, dt \,. \tag{5.4}$$

The function $f(t)$ can be reciprocally expressed as the Inverse Laplace Transform of F by the formula

$$f(t) = \frac{1}{2\pi i} \int_{D-i\infty}^{D+i\infty} F(s) e^{ts}\, ds \tag{5.5}$$

where D is any number greater than C. In other words, the path of integration must lie to the right of all singularities of $F(s)$. Sometimes we use the notation $\mathcal{L}f := F$ and $\mathcal{L}^{-1}F := f$.

The fact that both of these transforms are based on exponential functions means that they are well-adapted to problems involving differential operators. So, it is easy to see that if f is n times differentiable with all derivatives square integrable, then

$$\mathcal{F}\frac{d^n f}{dx^n} = (ik)^n \mathcal{F}f \tag{5.6}$$

so differentiation of the function living in the "direct space" is the same thing as multiplication by ik in the "transform space". Similarly for the Laplace transform of a function whose first n derivatives are transformable, we have

$$\mathcal{L}\frac{d^n f}{dt^n} = s^n \mathcal{L}f\,. \tag{5.7}$$

The utility of facts like these in solving problems can be illustrated by the solution of the diffusion equation in terms of Fourier transforms. Suppose we want to find the solution $\varphi(x,t)$ of

$$\frac{\partial \varphi}{\partial t} = \nu \frac{\partial^2 \varphi}{\partial x^2}\,, \tag{5.8}$$

subject to the initial condition $\varphi(x,0) = \varphi_0(x)$, the latter being a square integrable function of $x \in \mathbb{R}$. This is a problem posed in the "direct space". Let us apply the operator $\mathcal{F}$ to the equation, keeping in mind that t is a parameter as far as the transform is concerned, so that $\mathcal{F}$ maps $\varphi(x,t)$ to a new function $\hat{\varphi}(k,t)$. We find:

$$\frac{\partial \hat{\varphi}}{\partial t} = -\nu k^2 \hat{\varphi}\,. \tag{5.9}$$

The initial condition for this equation is $\hat{\varphi}(k,0) = \hat{\varphi}_0(k)$. This is the same problem viewed in the "transform space" or the "transform domain". It is solved independently for each k as a simple ordinary differential equation, and this is the great advantage of working in the transform domain. Thus,

$$\hat{\varphi}(k,t) = \hat{\varphi}_0(k) e^{-\nu k^2 t}\,. \tag{5.10}$$

To recover the solution $\varphi(x,t)$ at later times t, we must apply the operator $\mathcal{F}^{-1}$ to the result, which amounts to writing the solution as an integral over k. The method of solving the equation is expressed in the diagram shown in Figure 5.1.

5.2 Contour Deformations, Asymptotic Analysis, and Contributions of Branch Points

So we see that the solution formula for any problem treated by integral transforms like the Fourier and Laplace transform will ultimately be expressed as an integral that cannot typically be computed in terms of elementary functions. So it is of some interest to calculate asymptotic approximations since they give us a way of extracting concrete information from such integral formulas. We have already seen this several times in the course, in our studies of the diffusion equation, of general linear dispersive wave equations, and of the semiclassical Schrödinger equation. Here we make a few more comments about integrals of this sort, and show how to take into account some slightly more exotic singularities than we have considered before.

First let us look at inverse Fourier transform integrals. These are of the form

$$f(x) := \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx}\, dk\,, \tag{5.11}$$

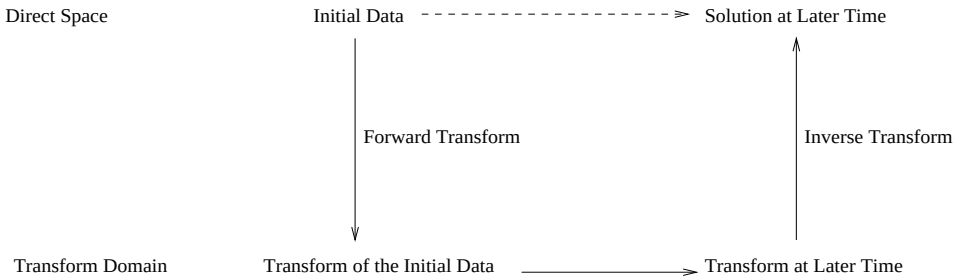

Figure 5.1: *The algorithm of solving a problem in the direct space (dashed arrow) and in the transform space (solid arrows). Although the horizontal part of the solution by transforms is trivial compared to the direct space solution, the extra steps of the forward and inverse transforms involving computing integrals that can rarely be found in closed form.*

where we will assume that $\hat{f}$ is in fact analytic on the real k axis. The singularities of $\hat{f}$ in the complex k-plane can in principle be very complicated, but at first pass it is reasonable to suppose that they are all isolated one from another, and even that there are only a finite number of them. The question is: in this kind of situation what can one learn about $f(x)$? The main tool we have in addressing this question is Cauchy's Theorem, which allows us to deform the path of integration away from the real axis in an attempt to simplify the integral. There are two obstructions to deforming the contour: the isolated singularities of $\hat{f}$ in the finite k-plane, and the behavior of the integrand $\hat{f}(k)e^{ikx}$ as k tends to infinity in different sectors. The latter determines our options in moving the infinite "tails" of the contour of integration.

Suppose that $x < 0$. Then the exponential factor e^{ikx} is decaying into the lower half k-plane. This means that if $\hat{f}(k)$ is analytic and, say, uniformly bounded in a strip containing the real axis, then the real axis may be replaced with a path of integration from left to right that is pushed down uniformly into the lower half-plane somewhat (if $x > 0$ we would not be able to push down the tails of the contour even though finite parts of the contour could be deformed). By Cauchy's Theorem the path may be deformed in this direction until the first singularity of $\hat{f}(k)$ is reached. In fact, as long as $\hat{f}(k)$ does not grow too fast at infinity (in particular if it is $O(k^p)$ for some real power p), then without changing the value of the integral, we can deform the path to coincide with a number of separate inverted U-shaped paths, one belonging to each singularity. See Figure 5.2.

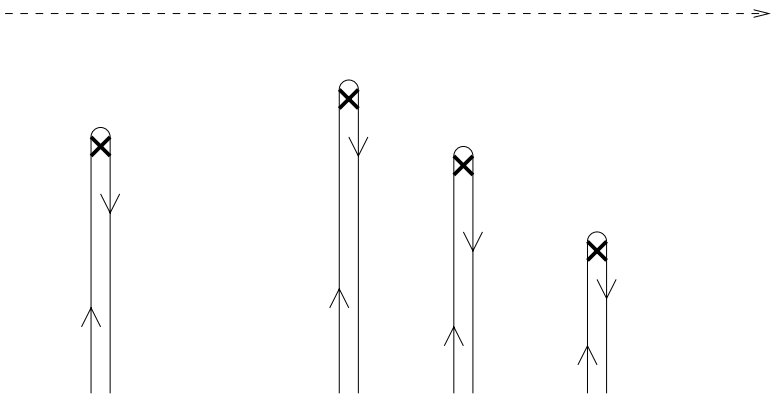

Figure 5.2: *The deformation of the contour of an inverse Fourier transform integral into the lower half k-plane for $x < 0$. The old contour is the real axis shown dashed. The new contour, shown in solid, consists of an isolated inverted U-shaped component for each singularity in the lower half plane.*

The behavior of the integrand in the infinite tails of the contour can prevent a deformation like that shown in Figure 5.2 from being possible. For example, recall the inverse Fourier transform integral for the

solution of the diffusion equation in which case we have $\hat{f}(k) = \hat{\varphi}_0(k)e^{-\nu k^2 t}$ and $t > 0$. In this case, it is typically only permissible to deform the contour in such a way that the ends go to infinity in symmetrical sectors of angle $\pi/2$ centered on the positive and negative real k axis. If the tails of the contour are moved outside these sectors, then the factor $e^{-\nu k^2 t}$ becomes large and typically dominates the integrand so that the integral no longer exists.

To continue our discussion, we thus suppose that the behavior of the function $\hat{f}(k)$ for large k in the upper or lower half-planes plays no role in the convergence of the inverse Fourier transform integral, and thus for $x < 0$ the contour may indeed be taken to be as illustrated in Figure 5.2. If a given singularity in the lower half-plane is a pole, then the Residue Theorem allows the corresponding U-shaped component of the contour to be neglected at the cost of $-2\pi i$ times the residue of the pole. The minus here comes from the orientation of the contour, which encircles the pole in a clockwise direction. The residue of the integrand at a pole $k = k_j$ in the lower half-plane is simply the residue of $\hat{f}(k)$ at k_j times $e^{ik_j x}$. These contributions to the integral coming from poles of $\hat{f}(k)$ are *exact*, since all we are using here is the fact that $x < 0$. That is, we are not (yet) considering asymptotics as $x \to -\infty$.

If, on the other hand, the singularity $k = k_j$ is a branching point, then there are several things to consider. First, $\hat{f}(k)$ is fundamentally a multivalued function, and it has several branches in the vicinity of the singularity. However, we know exactly which particular branch of the integrand $\hat{f}(k)$ we mean on the U-shaped contours precisely because its value is obtained there by analytic continuation from the real k-axis. In other words, we are implicitly taking the branch cuts, if any, all to lie inside the thin strip enclosed by each U-shaped contour component. In this case, it is not possible to deform the contour beyond the branch point, as there is no theorem like the Residue Theorem to help us do this. However, under certain conditions we can find the asymptotic behavior of this piece of the integral as $x \to -\infty$ using Watson's Lemma. Let U_j denote the U-shaped path corresponding to the branch point $k = k_j$. Then making the change of variables $\kappa = -i(k - k_j)$ we get

$$\int_{U_j} \hat{f}(k)e^{ikx}\,dk = ie^{ik_j x}\left[\int_{-\infty}^{0} \hat{f}(k_j + i(\kappa + i0))e^{-x\kappa}\,d\kappa - \int_{-\infty}^{0} \hat{f}(k_j + i(\kappa - i0))e^{-x\kappa}\,d\kappa\right]. \tag{5.12}$$

where by $\kappa \pm i0$ we mean the boundary value taken on the cut from the upper or lower half-plane for κ. This change of variables just rotates the U-shaped path about k_j by $-\pi/2$ and centers it at $\kappa = 0$. Now, it is often the case that k_j is a branch point of the sort for which there is a neighborhood of $\kappa = 0$ and a real number γ_j for which $\hat{f}(k_j + i\kappa)\kappa^{-\gamma_j}$ is analytic and nonzero, being equal to some function $g(\kappa)$ that has a Taylor series at $\kappa = 0$. We will suppose that this is the case; other types of branch points include branch points of the above type where γ_j is complex, logarithmic branch points, and complicated branch points where the local behavior is a sum of the elementary types described here. Let us suppose for the moment that $\gamma_j > -1$. By the way, by $\kappa^{-\gamma_j}$ we mean exactly the usual branch of this multivalued function that is cut on the negative real κ axis and that is positive for κ real and positive. Now for $\kappa < 0$ we have

$$\hat{f}(k_j + i(\kappa \pm i0)) = (\kappa \pm i0)^{\gamma_j} g(\kappa) = e^{\pm i\pi\gamma_j}(-\kappa)^{\gamma_j} g(\kappa)\,. \tag{5.13}$$

So changing variables to $\nu = -\kappa$ we find

$$\begin{aligned}\int_{U_j} \hat{f}(k)e^{ikx}\,dk &= ie^{ik_j x}\left[e^{i\pi\gamma_j}\int_0^\infty \nu^{\gamma_j} g(-\nu)e^{x\nu}\,d\nu - e^{-i\pi\gamma_j}\int_0^\infty \nu^{\gamma_j} g(-\nu)e^{x\nu}\,d\nu\right] \\ &= -2e^{ik_j x}\sin(\pi\gamma_j)\int_0^\infty \nu^{\gamma_j} g(-\nu)e^{x\nu}\,d\nu\,.\end{aligned} \tag{5.14}$$

Applying Watson's Lemma to the latter integral to find the asymptotic behavior as $x \to -\infty$ gives a whole asymptotic expansion in terms of the Taylor coefficients of $g(-\nu)$ at $\nu = 0$. As usual, the first term is the most interesting. Namely, we get the asymptotic result that

$$\int_{U_j} \hat{f}(k)e^{ikx}\,dk = -2e^{ik_j x}\sin(\pi\gamma_j)g(0)\Gamma(\gamma_j + 1)(-x)^{-(\gamma_j+1)}(1 + O(|x|^{-1})) \quad \text{as} \quad x \to -\infty\,. \tag{5.15}$$

If $\gamma_j = -1$ then k_j is a simple pole and its contribution is taken care of by the Residue Theorem. If $\gamma_j < -1$, then Watson's Lemma can still be applied, but first it is necessary integrate by parts in the integral over the inverted U-shaped contour, perhaps several times, with the aim of antidifferentiating the factor of $(k-k_j)^{\gamma_j}$ until its exponent is indeed greater than negative one. The price you pay for this comes from differentiating the other part of the integrand with respect to k, including the exponential e^{ikx}. This brings down one power of x for each time you have to integrate by parts, and consequently the contribution from such an integral at the end of the day will be e^{ik_jx} times something that grows like a power of $-x$. But this is still exponentially small as $x \to -\infty$. Finally, if the branching is more complicated, say requiring γ_j to be complex, or including logarithms, then the analysis required to obtain the leading contribution as $x \to -\infty$ will be different.

Notice that whether the contribution to $f(x)$ comes from a pole and is exact, or whether it comes from a simple branch point and we have to approximate the contribution for large negative x, we find that each isolated singularity k_j of $\hat{f}(k)$ results in an exponentially small contribution of order e^{ik_jx}. Comparing these exponents, we see that as $x \to -\infty$ only the singularities that are closest to the real axis are important. All other singularities are exponentially small by comparison. So as $x \to -\infty$, the asymptotic behavior of $f(x)$ is completely determined by the singularity or singularities of $\hat{f}(k)$ in the lower half-plane that have the largest imaginary part. Of course, if there are no singularities at all in the lower half-plane, and the contour may be deformed toward $k = -i\infty$ as explained above without changing the value of the integral, then $f(x)$ is identically zero for all $x < 0$.

When $x > 0$, it is not possible to deform the path of integration from the real axis into the lower half-plane. However, deformation into the upper half-plane is certainly possible, and again the contour can often be pushed all the way to $i\infty$ leaving only U-shaped components surrounding each singularity of $\hat{f}(k)$ in the upper half-plane. The procedure for finding the contribution of each pole or simple branch point is similar as was the case for $x < 0$. Now a singularity k_j in the upper half-plane contributes a term to $f(x)$ of the order of the exponential function e^{ik_jx}. Once again, these are all exponentially small, and all contributions are dominated by the singularities closest to the real axis. For $x > 0$, the dominant behavior of $f(x)$ comes from the singularities of $\hat{f}(k)$ in the upper half-plane with the smallest imaginary part. If there are no singularities of $\hat{f}(k)$ then $f(x)$ vanishes for all $x > 0$. Note that the dominant contribution fails to have an oscillatory character if and only if the corresponding singularity k_j lies on the imaginary axis.

Exactly the same kind of ideas apply to inverse Laplace transforms. Consider the integral

$$f(t) := \frac{1}{2\pi i} \int_{D-i\infty}^{D+i\infty} F(s)e^{st}\, ds \tag{5.16}$$

where D exceeds the real part of all singularities of $F(s)$. Here, we are implicitly taking $F(s)$ to be an analytic function on the contour of integration, and to permit relatively free deformation of the tails of the contour of integration, we will suppose that the analytic continuation of $F(s)$ from the contour behaves sub-exponentially as $s \to \infty$ in any direction; for example we might suppose that $F(s) = O(s^p)$ for some power p as $s \to \infty$. The first observation is that in these circumstances if $t < 0$, then the path of integration may be deformed by Cauchy's Theorem to the right, and since there are no singularities to stop it, the integral is identically equal to zero. So this is consistent with the Laplace transform itself, which only cares about the function values for $t > 0$. For $t > 0$, we can deform the contour of integration to the left, and now there will be a contribution from each singularity of $F(s)$, which we may take to be isolated points $s = s_j$. If the singularity at $s = s_j$ is a pole, the the contribution can be evaluated exactly, and will simply be $2\pi i e^{s_jt}$ times the residue of $F(s)$ at s_j. Here the sign is positive because the orientation of the contour is counterclockwise around the pole. This contribution can be exponentially large or small as $t \to +\infty$ depending on whether s_j lies to the right or the left of the imaginary axis. If, on the other hand, s_j is a simple branch point, then the contribution can be approximated as $t \to +\infty$ using Watson's Lemma. The calculation is very similar to what we worked out above for the Fourier transform so we will not repeat it here. But the upshot is that the contribution will again be of order e^{s_jt}. So now comparing the exponents, we see that the dominant contribution to $f(t)$ as $t \to +\infty$ comes from precisely those singularities of $F(s)$ to the left of the contour that have the largest real parts. Note that in the Laplace transform case, the function $f(t)$ will have a neutrally stable character (no exponential growth or decay) for large $t > 0$ if and only if the dominant singularity lies exactly on the imaginary s-axis.

5.3 Generalized Transform Methods and Using Asymptotic Analysis to Select Particular Solutions

With inverse Fourier and Laplace transforms, the integration contour is specified fairly precisely and the value of the transform in the integrand is known precisely on the contour of integration, even if there may be singularities away from the contour that place limits on contour deformation. However, there is another class of problems where neither the contour nor the integrand is specified so precisely, and part of the problem then becomes how to select these to achieve desired auxiliary properties of the solution.

This is best illustrated by example in order to see how such an indeterminate situation arises naturally. So suppose we would like to solve the third-order differential equation

$$2xw'''(x) + 9w''(x) - 2xw(x) = 0 \tag{5.17}$$

and we are only interested in those solutions that satisfy $w(x) \to 0$ as $x \to -\infty$. We can try to obtain solutions in the form of integrals, just like we did with Airy's equation.

So, for some function $q(z)$ and some contour C from $z = a$ to $z = b$, we try to seek a solution in the form

$$w(x) = \frac{1}{2\pi i} \int_C q(z) e^{zx}\, dz\,. \tag{5.18}$$

Substituting into the differential equation, differentiating under the integral sign, and integrating by parts to convert the multiplication by x into differentiation with respect to z, we get

$$\frac{1}{2\pi i} \int_C e^{zx} \left[3z^2 q(z) - 2(z^3-1)q'(z)\right]\, dz + \frac{1}{2\pi i} q(z) e^{zx} (z^3-1) \Big|_a^b = 0\,. \tag{5.19}$$

So if we can find a function $q(z)$ satisfying the first-order equation

$$2(z^3-1)q'(z) = 3z^2 q(z) \tag{5.20}$$

and then choose a contour C perhaps adapted to the specific function $q(z)$ so that the boundary terms vanish at the endpoints (which may be at infinity in some sectors) then our integral formula will solve the differential equation.

Now, unlike in the case of the Airy equation, here the auxiliary equation for $q(z)$ does not have any nontrivial solutions that are entire analytic functions of z. In fact, the solutions are not even single-valued in the complex z-plane. To see this, multiply through by $q(z)$ so that the equation becomes:

$$(z^3-1)\frac{d}{dz} q(z)^2 = q(z)^2 \frac{d}{dz}(z^3-1)\,. \tag{5.21}$$

Or, dividing through by $(z^3-1)^2$ and collecting terms on one side of the equals sign:

$$\frac{d}{dz}\left[\frac{q(z)^2}{(z^3-1)}\right] = 0\,. \tag{5.22}$$

Therefore, $q(z)^2 = K(z^3-1)$ for any constant K. Now since $q(z)^2$ has simple zeros at $z = 1$, $z = e^{2\pi i/3}$ and $z = e^{-2\pi i/3}$, $q(z)$ will vanish like a square root at these three points. To obtain a concrete function $q(z)$, we must select branch cuts in the complex z-plane emanating from each of these three points. This will determine $q(z)$ up to a constant multiple (and a sign in selecting the branch of the square root, which we absorb into the constant).

The job now is to choose the branch cuts for $q(z)$ and an appropriate contour C so that the boundary terms vanish, at least for all x sufficiently negative (which is our domain of interest) and so that as $x \to -\infty$, the integral $w(x)$ vanishes. First look at the boundary terms. The only place they vanish in the finite z-plane is when z agrees with any of the three branching points. So, for example, we can choose a contour C that joins a pair of branching points. But since we are considering $x < 0$, we see that the integration path may

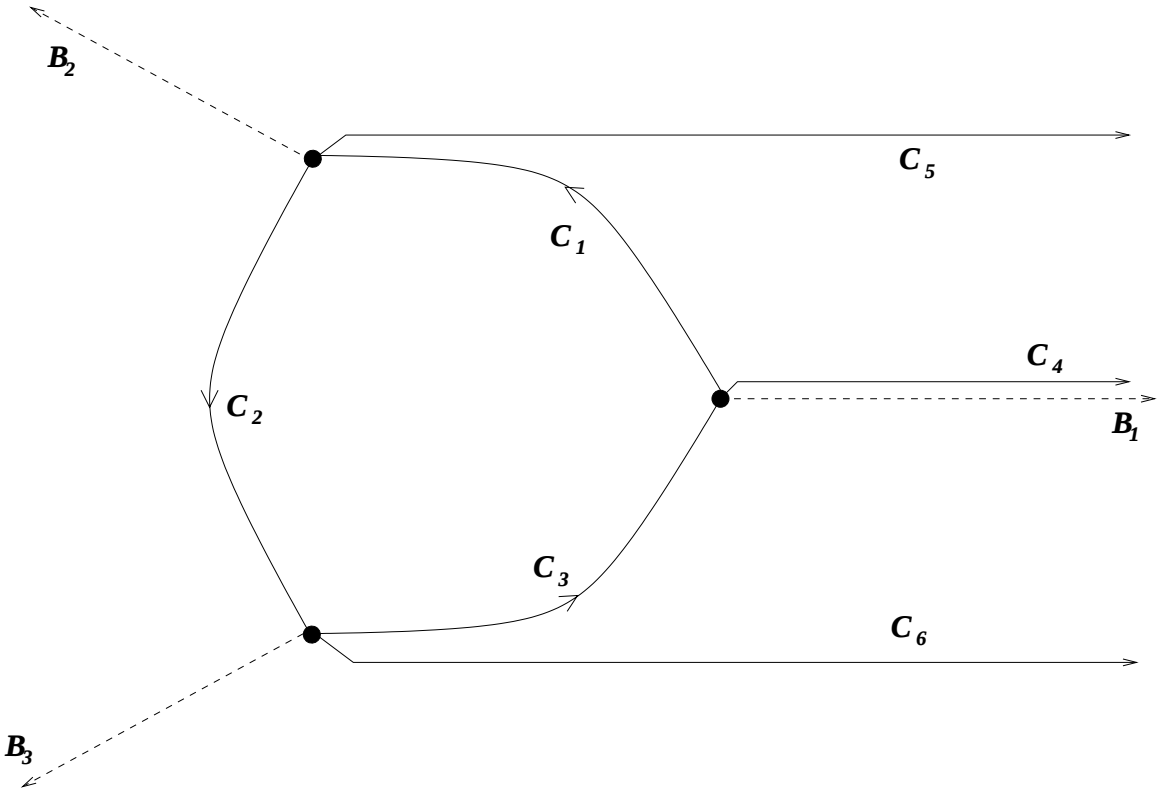

Figure 5.3: *A branch of $q(z)$ showing its cuts with dashed lines, and six admissible contours yielding solutions of the differential equation with $x < 0$.*

also go off to infinity as long as it does so in the right half-plane. Then as $z \to \infty$ the factor e^{zx} will kill all growth associated with $q(z)(z^3 - 1)$ and the corresponding boundary term will vanish. Choosing the branch cuts to radiate outward to infinity from each branch point, we get a picture like that shown in Figure 5.3. There are evidently three contours, C_1, C_2, and C_3 that are acceptable for all x, since their boundary terms vanish regardless of the sign of x. On the other hand, for $x < 0$, we can also use the contours C_4, C_5, and C_6. We are taking C_4 to lie just above the branch cut B_1 on the positive real axis. If we took it to lie just below instead, this would simply amount to a change of sign in the integral.

Now the equation is only third order, so these cannot all be independent. A third order linear equation has exactly three linearly independent solutions. So there must be linear relations among the integrals. Let us introduce the specific notation

$$w_k(x) := \frac{1}{2\pi i} \int_{C_k} q(z) e^{zx} \, dz \,. \tag{5.23}$$

Here k varies between one and six. The factor of $1/2\pi i$ is not really necessary here, since $q(z)$ is only specified up to a constant multiple. It is traditionally put here because it is useful when $q(z)$ has poles since then the residue contributions come in without the usual multiple of $2\pi i$. Given this notation, it is easy to see immediately from the figure that

$$w_1(x) + w_2(x) + w_3(x) = 0 \tag{5.24}$$

for all x as must be the case by Cauchy's theorem; the sum can be rewritten as a single integral over a single circular path inside of which the integrand is analytic. Also, since paths of integration may be deformed to infinity in the right half-plane when $x < 0$, we can also get the relations

$$w_1(x) = w_4(x) - w_5(x) \quad \text{and} \quad w_3(x) = w_6(x) + w_4(x) \quad \text{as long as} \quad x < 0 \,. \tag{5.25}$$

For $x < 0$ it looks to be convenient to take the three independent solutions to be $w_4(x)$, $w_5(x)$, and $w_6(x)$. The paths of integration for these three integrals are all straight horizontal half-lines.

Now the final question arises: how do we determine which of these three integrals (or which linear combinations of them) correspond to solutions of the original differential equation that vanish as $x \to -\infty$. We can treat directly each of these integrals as $x \to -\infty$ using Watson's Lemma.

For $w_5(x)$, we make the change of variable $t = z - e^{2\pi i/3}$. Then because $\cos(2\pi/3) = -1/2$ and $\sin(2\pi/3) = \sqrt{3}/2$,

$$w_5(x) = e^{-x/2} e^{ix\sqrt{3}/2} \frac{1}{2\pi i} \int_0^\infty q(e^{2\pi i/3} + t) e^{xt}\, dt\,. \tag{5.26}$$

Now in a neighborhood of $t = 0$, we can write

$$q(e^{2\pi i/3} + t) = t^{1/2} g_5(t) \tag{5.27}$$

where $g_5(t)$ is an analytic function with $g_5(0) \neq 0$ and $t^{1/2}$ means the usual square root function whose branch cut is on the negative real t axis and that is positive for positive t. This is just because $q(t)$ vanishes exactly like a square root at each branching point. This puts the integrand in exactly the form we need to use Watson's Lemma with $\lambda = 1/2$. Using the fact that $\Gamma(3/2) = \Gamma(1/2)/2 = \sqrt{\pi}/2$, we find that

$$w_5(x) \sim e^{-x/2} e^{ix\sqrt{3}/2} \frac{g_5(0)}{4i\sqrt{\pi}(-x)^{3/2}} \quad \text{as} \quad x \to -\infty\,. \tag{5.28}$$

Thus, $w_5(x)$ is exponentially large as $x \to -\infty$ and certainly does not fit the bill. A similar analysis of $w_6(x)$ using Watson's Lemma gives

$$w_6(x) \sim e^{-x/2} e^{-ix\sqrt{3}/2} \frac{g_6(0)}{4i\sqrt{\pi}(-x)^{3/2}} \quad \text{as} \quad x \to -\infty\,. \tag{5.29}$$

Here $g_6(t)$ is the analytic function $q(e^{-2\pi i/3} + t)t^{-1/2}$ in the vicinity of $t = 0$. The value $g_6(0)$ is related to $g_5(0)$ through details of the cut structure of the function $q(z)$. But it is just a number, and again we see that this solution of the differential equation is exponentially large as $x \to -\infty$.

To confirm our suspicion that it is in fact $w_4(x)$ that satisfies our auxiliary boundary condition at $x = -\infty$, we use Watson's Lemma to study this integral. Letting $t = z - 1$, we find

$$w_4(x) = \frac{e^x}{2\pi i} \int_0^\infty q(1 + t) e^{xt}\, dt\,. \tag{5.30}$$

Letting $g_4(t)$ be the analytic function $q(1 + t)t^{-1/2}$ near $t = 0$ on the contour of integration, and keeping in mind that what we mean by $q(1 + t)$ in defining $g_4(t)$ is its value infinitessimally above the cut B_1, we again apply Watson's Lemma with $\lambda = 1/2$ to find

$$w_4(x) \sim e^x \frac{g_4(0)}{4i\sqrt{\pi}(-x)^{3/2}} \quad \text{as} \quad x \to -\infty\,. \tag{5.31}$$

Thus, $w_4(x)$ and its constant multiples, are the only solutions of the differential equation that satisfy our boundary condition at $x = -\infty$. This illustrates how asymptotic analysis of integrals actually helps determine the appropriate "transform" to use (in the sense of which branch cut structure for multivalued $q(z)$ and which contour to use in the inverse transform integral) in many practical problems.

Note that in this problem the solutions $w_4(x)$, $w_5(x)$, and $w_6(x)$ fail to exist for $x > 0$ because the integrals defining them diverge. On the other hand, for $x > 0$ there are new possibilities for the path of integration that we could not consider for $x < 0$, namely three paths emerging from the branch points and going to infinity in the left half-plane. The fact that the nature of the set of solutions changes suddenly when x passes through zero is related to the fact that $x = 0$ is a *singular point* for the third order differential equation. In particular, when $x = 0$ the order of the differential equation differs from its value for $x \neq 0$. In the next chapter, we will discuss these singularities in more detail.

Chapter 6

Asymptotic Behavior of Solutions of Linear Second-Order Differential Equations

For the most part, up until this point in the course we have been studying various methods of obtaining asymptotic approximations of functions for which we had an explicit representation in terms of an integral. The only exception was our investigation of finding roots of polynomial equations in the form of asymptotic expansions. In that case the starting point was not an explicit formula for the roots in which ϵ appears as a parameter (no such formulae exist for polynomials of degree 5 or more anyway) but rather an equation that describes the roots implicitly, namely the polynomial equation $P(x) = 0$.

In fact, this latter situation represented by the root-finding problem is much more common than the situation in which an explicit solution formula exists, even in the form of an integral. Usually we start from an equation that the unknown satisfies and try to obtain asymptotic information about the solution of the equation *without first solving the equation*. In dealing with such problems, a rigorous approach must have two steps:

1. There must be a systematic method of determining the correct form of an asymptotic sequence and then obtaining the corresponding coefficients in an asymptotic series, making use only of the form of the equation satisfied by the unknown.

2. There must be some analytical machinery to justify an expansion produced by the formal methods of step 1. Namely, it must be shown after the fact that the equation actually has a solution that is asymptotically described by the formal series found in step 1.

The second step is the new feature for problems that cannot be solved directly from the outset. However, it often turns out to be the case that good intuition applied to the formal expansion procedure in step 1 can be recycled to assist in constructing the solid proofs that are needed in step 2. Most first courses in asymptotic methods concentrate on the various methods of systematically obtaining expansions and deemphasize the importance of the second step of proving the validity of the expansion obtained. This course will proceed similarly, however you should always keep in mind that without the second step the job is really only half finished.

6.1 Linear Second-Order Ordinary Differential Equations

In this chapter, we will be concerned with ordinary differential equations of the form

$$a_2(z)w''(z) + a_1(z)w'(z) + a_0(z)w(z) = 0\,. \tag{6.1}$$

Here w is a complex-valued function of a complex variable z. We will suppose that the coefficients $a_2(z)$, $a_1(z)$, and $a_0(z)$ are rational functions of z, which means that they are ratios of polynomials. Although we

cannot find representations for solutions of this equation in general in terms of integrals, we will be interested in finding the asymptotic behavior of solutions as $z \to z_0$. Sometimes we will take $z_0 = \infty$.

Before we discuss the general theory of second-order differential equations with rational coefficients, we will show how the problem can be reduced to a kind of canonical form which makes our work simpler. First note that if we divide by $a_2(z)$, the equation takes the form

$$w''(z) + p(z)w'(z) + q(z)w(z) = 0\,, \tag{6.2}$$

where $p(z) := w_1(z)/w_2(z)$ and $q(z) := w_0(z)/w_2(z)$ are again rational functions of z. Next, if we pick a point $z = a$ and a path from this point into a neighborhood of z-values of interest, and set

$$w(z) = y(z) \exp\left(-\frac{1}{2}\int_a^z p(\zeta)\,d\zeta\right) \tag{6.3}$$

where the integral is taken along the specified path (this is important to take into account if $p(z)$ has any residues in order that the integral be well-defined, but in the absence of residues any path will do) then the function $y(z)$ satisfies

$$y''(z) + f(z)y(z) = 0 \qquad \text{where} \qquad f(z) := q(z) - \frac{1}{2}p'(z) - \frac{1}{4}p(z)^2\,. \tag{6.4}$$

Here $f(z)$ is again a rational function of z if p and q are. So it is sufficient to consider equations of this canonical form with $f(z)$ being a rational function.

6.1.1 Analytic theory. Monodromy matrices. Regular singular and irregular singular points.

The fundamental existence theorem we need is the following.

Theorem 4 (Analytic Existence Theorem) *Let z_0 be a point in the complex plane where $f(z)$ is analytic. Then there is a domain D_0 containing z_0 and two analytic functions $y_1(z)$ and $y_2(z)$ defined in D_0 that satisfy $y_j''(z) + f(z)y_j(z) = 0$ for $j = 1$ and $j = 2$ and also the auxiliary conditions $y_1(z_0) = y_2'(z_0) = 1$ and $y_1'(z_0) = y_2(z_0) = 0$.*

A point z_0 at which Theorem 4 holds is called an *ordinary point* for the differential equation. Since the differential equation is linear, any linear combination $y(z) := \alpha y_1(z) + \beta y_2(z)$ also satisfies the equation where α and β are arbitrary complex constants. Therefore we can always find a unique solution of the differential equation satisfying $y(z_0) = \alpha$ and $y'(z_0) = \beta$ for given α and β.

Now the functions $y_1(z)$ and $y_2(z)$ can be analytically continued along a path γ starting at z_0 in the complex z-plane. In the course of this continuation we can arrive in a neighborhood D_1 of a point z_1 on γ that may not be in the original domain D_0. That is, by analytic continuation of $y_1(z)$ and $y_2(z)$ originally defined only in D_0 along γ we get two new analytic functions $\tilde{y}_1(z)$ and $\tilde{y}_2(z)$ defined in a neighborhood D_1 that may be disjoint from D_0. The question then arises: do the functions $\tilde{y}_1(z)$ and $\tilde{y}_2(z)$ analytic in D_1 have anything at all to do with the differential equation that $y_1(z)$ and $y_2(z)$ satisfy in the (possibly different) domain D_0? It turns out that $\tilde{y}_1(z)$ and $\tilde{y}_2(z)$ satisfy the differential equation in D_1. This follows from two facts:

Property 1: If $g(z)$ is defined in a domain D_1 as the analytic continuation along γ of a function $f(z)$ originally given in D_0, and if $h(z)$ is defined in D_1 as the analytic continuation along γ of $f'(z)$, then $g'(z) = h(z)$ for all $z \in D_1$. That is, derivative relationships are preserved under analytic continuation.

Property 2: Suppose that $f_1(z), \ldots, f_n(z)$ are analytic functions in D_0 that satisfy an analytic equation of the form $F(f_1(z), \ldots, f_n(z)) = 0$ for all $z \in D_0$. Let $g_1(z), \ldots, g_n(z)$, analytic in D_1, be the analytic continuations of the corresponding $f_j(z)$ along γ. Then $F(g_1(z), \ldots, g_n(z)) = 0$ for all $z \in D_1$. That is, analytic relations among functions that hold in some open set of z-values persist under analytic continuation.

To use these facts to obtain the desired result, just use the function $F(u,v,w) := u + vw$ and start with the functions $f_1(z) := y_j''(z)$, $f_2(z) := f(z)$, and $f_3(z) := y_j(z)$ in D_0, which continue along γ to the functions $g_1(z) := \tilde{y}_j''(z)$ (according to Property 1), $g_2(z) := f(z)$ (which is well-defined and single-valued in the whole z-plane), and $g_3(z) := \tilde{y}_j(z)$. Then from Property 2 we deduce that $y_j''(z) + f(z)y_j(z) = 0$ in D_0 implies that $\tilde{y}_j''(z) + f(z)\tilde{y}_j(z) = 0$ in D_1. So the differential equation is also preserved under analytic continuation of the solutions $y_1(z)$ and $y_2(z)$. Also, the two functions that start out linearly independent in D_0 remain linearly independent under analytic continuation. If this were not the case, then we could arrive at some neighborhood D_1 in which there were constants c_1 and c_2 not both zero, such that the continuations in D_1 satisfied $c_1\tilde{y}_1(z) + c_2\tilde{y}_2(z) = 0$ identically in D_1. But since analytic continuation is reversible, by Property 2 the equation $c_1y_1(z) + c_2y_2(z)$ would have to hold identically in the original neighborhood D_0. But this contradicts the presumed linear independence of $y_1(z)$ and $y_2(z)$ in D_0.

In fact, the analytic continuation of solutions of the differential equation from the original neighborhood D_0 can only fail at points in the complex z-plane where $f(z)$ has singularities. Since $f(z)$ is a rational function, we are assuming that these singularities of $f(z)$ are isolated poles. Suppose $f(z)$ has a pole at a point $z = z_*$. Although we cannot analytically continue the solutions $y_1(z)$ and $y_2(z)$ through z_*, we can pick a path γ from $z_0 \neq z_*$ that makes a closed loop surrounding the point z_*. We take γ to be oriented in the counterclockwise direction. Now, we know that analytic continuation of $y_1(z)$ and $y_2(z)$ around γ from D_0 to itself will result in some functions $\tilde{y}_1(z)$ and $\tilde{y}_2(z)$ defined in the same neighborhood D_0 that are also linearly independent and that also satisfy the same differential equation. Since there can only be two linearly independent solutions of any second-order differential equation, there must be a matrix of constants, a *monodromy matrix* such that

$$\begin{bmatrix} \tilde{y}_1(z) \\ \tilde{y}_2(z) \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} y_1(z) \\ y_2(z) \end{bmatrix}. \tag{6.5}$$

The monodromy matrix has to be invertible because both pairs of functions are linearly independent. Now this matrix depends on both the neighborhood D_0 along γ where we start and finish and also on the particular solutions $y_1(z)$ and $y_2(z)$ of the equation that we start with at z_0. Suppose, however, that we started with a set of functions $Y_1(z)$ and $Y_2(z)$ that are defined to be related to $y_1(z)$ and $y_2(z)$ as linear combinations:

$$\begin{bmatrix} y_1(z) \\ y_2(z) \end{bmatrix} =: \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} Y_1(z) \\ Y_2(z) \end{bmatrix}, \tag{6.6}$$

where the matrix of constants is invertible. This is a system of two equations that hold identically among analytic functions; therefore it is preserved under analytic continuation. So if $\tilde{Y}_1(z)$ and $\tilde{Y}_2(z)$ are the analytic continuations of $Y_1(z)$ and $Y_2(z)$ then we also have

$$\begin{bmatrix} \tilde{y}_1(z) \\ \tilde{y}_2(z) \end{bmatrix} =: \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} \tilde{Y}_1(z) \\ \tilde{Y}_2(z) \end{bmatrix} \tag{6.7}$$

with the same matrix of constants. Therefore the functions $Y_1(z)$ and $Y_2(z)$ and their analytic continuations $\tilde{Y}_1(z)$ and $\tilde{Y}_2(z)$ about the circular loop γ must satisfy

$$\begin{bmatrix} \tilde{Y}_1(z) \\ \tilde{Y}_2(z) \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}^{-1} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} Y_1(z) \\ Y_2(z) \end{bmatrix} := \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} Y_1(z) \\ Y_2(z) \end{bmatrix}. \tag{6.8}$$

The monodromy matrix $\{C_{ij}\}$ relating $Y_1(z)$ and $Y_2(z)$ with their analytic continuations around γ is thus *similar* to the "original" monodromy matrix $\{c_{ij}\}$. The same thing is true if we started with a neighborhood of a different point of the same curve γ. We therefore come to the conclusion that while the monodromy matrices are different for different choices of starting points, the two eigenvalues of the monodromy matrix are fundamental invariants of the differential equation and the curve γ that goes around the pole z_* of $f(z)$. In fact, they don't even depend on the curve γ, as long as z_* is the only singularity of $f(z)$ inside and the curve goes around in the counterclockwise direction.

Let ω_1 and ω_2 be the eigenvalues of a monodromy matrix $\{C_{ij}\}$ corresponding a counterclockwise circuit around the pole z_* of $f(z)$ beginning with two independent solutions $Y_1(z)$ and $Y_2(z)$. Neither of the

eigenvalues can be zero because the monodromy matrix is invertible. Suppose $\omega_1 \neq \omega_2$. Then corresponding to each eigenvalue there is a nonzero left (row) eigenvector,

$$[d_{j1}\ \ d_{j2}] \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \omega_j [d_{j1}\ \ d_{j2}] \tag{6.9}$$

and the row vectors $[d_{11}\ \ d_{12}]$ and $[d_{21}\ \ d_{22}]$ are linearly independent. It follows that the functions $y_1(z)$ and $y_2(z)$ defined as linear combinations

$$\begin{bmatrix} y_1(z) \\ y_2(z) \end{bmatrix} := \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} Y_1(z) \\ Y_2(z) \end{bmatrix} \tag{6.10}$$

have the property that under analytic continuation around any closed loop γ enclosing only the pole z_* in the counterclockwise sense, $y_j(z) \to \omega_j y_j(z)$. That is, these two linearly independent solutions of the differential equation go into simple multiples of themselves under analytic continuation about z_*. Pick now concrete numbers ρ_1 and ρ_2 (generally complex) so that $\omega_j = e^{2\pi i \rho_j}$. These two numbers are only determined up to adding on integers, but choose them concretely nonetheless. Now it is easy to check that the functions

$$V_1(z) := (z - z_*)^{-\rho_1} y_1(z) \qquad \text{and} \qquad V_2(z) := (z - z_*)^{-\rho_2} y_2(z) \tag{6.11}$$

go into themselves upon analytic continuation around any circuit γ because the multiplicative factor of ω_j picked up by $y_j(z)$ is cancelled by a factor of ω_j^{-1} picked up by the factor $(z - z_*)^{-\rho_j}$. So $V_1(z)$ and $V_2(z)$ are single-valued in any annular neighborhood of z_*. It follows from the general theory of analytic functions that $V_1(z)$ and $V_2(z)$ can be represented by convergent Laurent series. Going back to $y_1(z)$ and $y_2(z)$ we find that there are convergent series:

$$y_j(z) = (z - z_*)^{\rho_j} \sum_{n=-\infty}^{\infty} a_{jn} (z - z_*)^n . \tag{6.12}$$

The annulus of convergence is $0 < |z - z_*| < R$ for some $R > 0$ being the distance from z_* to the nearest distinct singularity of $f(z)$. Sometimes, the apparent singularity at $z = z_*$ is in fact removable in which case we actually have convergence for $0 \leq |z - z_*| < R$.

Continuing with the case of $\omega_1 \neq \omega_2$ there are now two possibilities. Either the Laurent series corresponding to the two eigenvalues both contain only a finite number of negative powers of $z - z_*$ or at least one series contains an infinite number of negative powers. In the former case, we say that z_* is a *regular singular point* of the differential equation, and in the latter case, we say that z_* is an *irregular singular point* of the equation.

Now if $\omega_1 = \omega_2 =: \omega$, then there is always at least one nonzero left (row) eigenvector of the matrix $\{C_{ij}\}$ corresponding to the degenerate eigenvalue ω. This row vector corresponds to a solution of the differential equation, say $u(z)$, that goes into $\omega u(z)$ upon analytic continuation around γ. This solution can be represented in terms of a Laurent series in the same way as described above. Now let $v(z)$ be any other linearly independent solution of the equation. Then whatever we get by continuing $v(z)$ around γ it must be a linear combination of $u(z)$ and $v(z)$; that is $v(z) \to au(z) + bv(z)$. This means that the monodromy matrix relative to the solutions $u(z)$ and $v(z)$ is exactly:

$$\begin{bmatrix} u(z) \\ v(z) \end{bmatrix} \to \begin{bmatrix} \omega & 0 \\ a & b \end{bmatrix} \begin{bmatrix} u(z) \\ v(z) \end{bmatrix} , \tag{6.13}$$

which must have degenerate eigenvalues both equal to ω. This forces $b = \omega$. Now this means that under analytic continuation around the circuit γ,

$$\frac{v(z)}{u(z)} \to \frac{v(z)}{u(z)} + 2\pi i \kappa \qquad \text{where} \qquad \kappa := \frac{a}{2\pi i \omega} . \tag{6.14}$$

Therefore there is some number κ (possibly zero) so that the fraction v/u goes into itself *plus* $2\pi i\kappa$. The function $\kappa \log(z - z_*)$ also has this property. This means that the combination

$$V_3(z) := \frac{v(z)}{u(z)} - \kappa \log(z - z_*) \tag{6.15}$$

is single-valued and analytic in an annular neighborhood of $z = z_*$, and consequently can be represented as a convergent Laurent series in positive and negative integer powers of $z - z_*$. Therefore in the case that $\omega_1 = \omega_2 = \omega$, there is one solution of the form

$$u(z) = (z - z_*)^\rho \sum_{n=-\infty}^{\infty} a_n (z - z_*)^n \tag{6.16}$$

and another solution of the form

$$v(z) = (z - z_*)^\rho \sum_{n=-\infty}^{\infty} b_n (z - z_*)^n + \kappa (z - z_*)^\rho \log(z - z_*) \sum_{n=-\infty}^{\infty} a_n (z - z_*)^n \,, \tag{6.17}$$

where ρ is a concrete number satisfying $e^{2\pi i \rho} = \omega$. Once again, the point z_* is a regular singular point if both series contain a finite number of terms corresponding to negative powers of $z - z_*$, and otherwise z_* is an irregular singular point.

6.1.2 Reduction of order.

Here we recall a useful method that essentially reduces the problem of finding the solutions of a second order linear differential equation to finding just any one nontrivial solution. Suppose we know one solution $y = y_0(z)$ of the differential equation $y''(z) + f(z)y(z) = 0$. Substitute $y(z) = y_0(z)v(z)$ and find

$$y_0''(z)v(z) + 2y_0'(z)v'(z) + y_0(z)v''(z) + f(z)y_0(z)v(z) = 0 \,, \tag{6.18}$$

which, since $y_0(z)$ is already a known solution reduces to

$$u'(z) + \frac{2y_0'(z)}{y_0(z)} u(z) = 0 \qquad \text{where} \qquad u(z) := v'(z) \,. \tag{6.19}$$

It is easily verified that the solution of this first-order differential equation for $u(z)$ is simply $u(z) = Cy_0(z)^{-2}$ where C is an arbitrary constant. Upon integrating to obtain $v(z)$ and multiplying by $y_0(z)$ we obtain the general solution of the second order problem $y''(z) + f(z)y(z) = 0$ in terms of two arbitrary constants C and w and any given nonzero solution $y_0(z)$ of the problem in the form

$$y(z) = Cy_0(z) \int_w^z \frac{d\zeta}{y_0(\zeta)^2} \,. \tag{6.20}$$

Therefore, sometimes we can use asymptotic information about one solution to obtain an asymptotic description of another from this direct formula.

6.2 Convergent Series Solutions

At points z_0 where the equation $y''(z) + f(z)y(z) = 0$ has an ordinary point or a regular singular point, it is possible to obtain expansions of solutions that are convergent and also asymptotic as $z \to z_0$.

6.2.1 Series solutions at ordinary points.

If z_0 is an ordinary point for the equation $y''(z) + f(z)y(z) = 0$, which is the same thing as saying that $f(z)$ is analytic at $z = z_0$, then we know from Theorem 4 that all solutions near z_0 are themselves analytic functions. Therefore they may be expressed as series in nonnegative integer powers of $z - z_0$, and we may directly seek them in this form. Suppose that the Taylor expansion of $f(z)$ at z_0 is

$$f(z) = \sum_{n=0}^{\infty} f_n (z - z_0)^n \,. \tag{6.21}$$

We seek a solution $y(z)$ in the same form with coefficients $\{y_n\}$. Since the series for $y(z)$ is a convergent power series, it is absolutely convergent and may be differentiated term-by-term to find the series for $y''(z)$. Similarly, the series for $f(z)$ and $y(z)$ may be multiplied with like powers of $z-z_0$ collected together to obtain the Taylor series for the product $f(z)y(z)$. This gives us a power series representation of $y''(z)+f(z)y(z)$. This quantity vanishes if and only if all of its Taylor coefficients do, so equating to zero all coefficients of powers of $z-z_0$ we find a hierarchy of equations:

$$\begin{aligned} 2y_2 + f_0 y_0 &= 0 \\ 6y_3 + f_0 y_1 + f_1 y_0 &= 0 \\ 12y_4 + f_0 y_2 + f_1 y_1 + f_2 y_0 &= 0 \\ &\vdots \\ n(n-1)y_n + \sum_{k=0}^{n-2} f_k y_{n-k-2} &= 0 \\ &\vdots \end{aligned} \tag{6.22}$$

The first two coefficients y_0 and y_1 are arbitrary, but after that all further coefficients may be solved for successively. It can be shown directly that the series so obtained converges regardless of the choice of y_0 and y_1 and represents a solution of the differential equation. Since for this solution $y(z_0)=y_0$ and $y'(z_0)=y_1$ specifying these first two coefficients is the same thing as solving the initial value problem for the differential equation.

Now being Taylor series, these expansions are convergent for each z in the radius of convergence. But we also know that such series are automatically asymptotic as well in the limit $z\to z_0$. That is, any finite truncation of the Taylor series obtained systematically from the above prescription at a point z_0 where $f(z)$ is known to be analytic will provide an asymptotic approximation of the true solution $y(z)$ as $z\to z_0$.

The point at infinity is an ordinary point of the differential equation if the point $w=0$ is an ordinary point of the equation obtained by changing the independent variable from to $w=1/z$. If we set $Y(w):=wy(1/w)$ and let $F(w):=f(1/w)$, then by the chain rule we find that the differential equation goes over into

$$Y''(w)+\frac{F(w)}{w^4}Y(w)=0\,. \tag{6.23}$$

This shows that infinity is an ordinary point for the original differential equation if $f(z)=O(z^{-4})$ as $z\to\infty$. Under this condition, solutions of $y''(z)+f(z)y(z)=0$ that are analytic at $z=\infty$ may be obtained by applying the method described above to the modified equation for $Y(w)$ in the vicinity of $w=0$. Again, any finite truncation of the convergent series obtained in this manner provides an accurate asymptotic approximation of the true solution given by the sum of the series, in the limit $z\to\infty$.

6.2.2 Series solutions at regular singular points. The method of Frobenius.

A similar systematic method due to Frobenius can be used to obtain convergent expansions of solutions near z_* when this point is a regular singular point of the equation. At every regular singular point z_* we know on general principles that there exists at least one solution that can be written as a convergent expansion in the form

$$y(z)=(z-z_*)^\rho\sum_{n=0}^{\infty}a_n(z-z_*)^n\,, \tag{6.24}$$

where $a_0\neq 0$. Here the particular value of ρ may differ from the value we picked when we discussed the general theory by an integer. In fact here ρ is at this point unknown; part of the problem is to determine

it. In any case, the integer difference can be thought of as having been selected precisely to make sure that the lowest order term in the series corresponds to a power of ρ (in other words whatever power is the most negative in the Laurent series of the single-valued factor is encorporated into ρ).

We can proceed along similar lines as before by inserting this expansion into the equation, differentiating term-by-term to find the series for $y''(z)$, and multiplying by the Laurent series for $f(z)$ (which necessarily has poles in this singular case) to find the series for the product $f(z)y(z)$. Again, we separate like powers of $z - z_*$ and set the coefficients to zero. Here different things happen depending on the order of the pole that $f(z)$ has at z_*. First suppose that the pole is simple, so that

$$f(z) = \frac{f_{-1}}{z - z_*} + \sum_{n=0}^{\infty} f_n (z - z_*)^n \,, \tag{6.25}$$

with $f_{-1} \neq 0$. Then the sequence of equations that results is

$$\begin{aligned}
\rho(\rho-1)a_0 &= 0 \\
(\rho+1)\rho a_1 + f_{-1}a_0 &= 0 \\
(\rho+2)(\rho+1)a_2 + f_{-1}a_1 + f_0 a_0 &= 0 \\
&\vdots \\
(\rho+n)(\rho+n-1)a_n + \sum_{k=0}^{n-1} f_{k-1} a_{n-k-1} &= 0 \\
&\vdots
\end{aligned} \tag{6.26}$$

Now since $a_0 \neq 0$, from the first equation in the hierarchy (6.26) we must have either $\rho = 0$ or $\rho = 1$. But the choice $\rho = 0$ is not consistent with the second equation in the hierarchy (6.26), since $f_{-1}a_0 \neq 0$. So in fact we must choose $\rho = 1$. Then, a_0 is arbitrary as long as it is not zero, but then all further coefficients are determined systematically from successively solving the equations of the hierarchy with $\rho = 1$. In particular it always turns out that $a_1 \neq 0$. Once again it may be shown directly that the series so obtained actually converges to a function that satisfies the differential equation. This is an *analytic* solution since it is really a series in positive integer powers of $z - z_*$ whose constant term is zero. No other solution of the equation can be obtained by the method of Frobenius in this case. To find the other solution we can appeal to reduction of order. The reduction of order formula shows that since $a_1 \neq 0$, all other solutions besides this analytic one contain terms proportional to $\log(z - z_*)$. So we see that in the case when $f(z)$ has a simple pole at $z = z_*$, the point z_* is a regular singular point for which the monodromy matrix always has degenerate eigenvalues with $\omega_1 = \omega_2 = 1$ and only one left eigenvector.

Next suppose that $f(z)$ has a double pole at $z = z_*$ so its Laurent series is

$$f(z) = \frac{f_{-2}}{(z - z_*)^2} + \frac{f_{-1}}{z - z_*} + \sum_{n=0}^{\infty} f_n (z - z_*)^n \,. \tag{6.27}$$

Now the hierarchy of equations we obtain is the following:

$$\begin{aligned} \rho(\rho-1)a_0 + f_{-2}a_0 &= 0 \\ (\rho+1)\rho a_1 + f_{-2}a_1 + f_{-1}a_0 &= 0 \\ &\vdots \\ (\rho+n)(\rho+n-1)a_n + f_{-2}a_n + \sum_{k=0}^{n-1} f_{n-k-2}a_k &= 0 \\ &\vdots \end{aligned} \tag{6.28}$$

Since $a_0 \neq 0$, the first equation becomes a quadratic equation for ρ called the *indicial equation* for the point $z = z_*$:

$$I(\rho) := \rho^2 - \rho + f_{-2} = 0\,. \tag{6.29}$$

Let the two (generally complex) roots of this equation be ρ_1 and ρ_2, with the labeling convention that whenever possible $\Re(\rho_1) > \Re(\rho_2)$.

If we set $\rho = \rho_1$, then we see that the nonzero coefficient a_0 is arbitrary, but all further coefficients may always be solved for in succession, and as before it can be directly shown that this procedure produces a convergent series that sums to a solution of $y''(z) + f(z)y(z) = 0$. The procedure cannot fail for $\rho = \rho_1$ because the coefficient of a_n is just $I(\rho_1 + n)$ which cannot be zero for any positive n if indeed ρ_1 is a solution of the indicial equation with the largest real part. The solution obtained is proportional to the arbitrary constant a_0.

If the indicial equation has a double root, so that $\rho_2 = \rho_1 = 1/2$, then clearly this procedure can give nothing new by taking $\rho = \rho_2$. Since the two values of ρ agree, this is a case in which the two eigenvalues of the monodromy matrix are also equal: $\omega_1 = \omega_2 = \omega$. In this case the reduction of order formula can be used to find another linearly independent solution. From that formula, we see that since $y_0(z)^{-2} \sim a_0^{-2}(z - z_*)^{-1}$, we will necessarily obtain a solution with logarithms. So this is again a case when the monodromy eigenvalues are degenerate and there is only one eigenvector.

If the indicial equation has distinct roots, then the only possible way that the same procedure could fail to provide a new solution of similar form upon taking $\rho = \rho_2$ would be if it were not possible to solve for one of the coefficients a_m with $m > 0$. This might be the case if $\rho_1 - \rho_2 = m$ which means that the two roots of the indicial equation differ by an integer. Note that the corresponding monodromy matrix eigenvalues would then satisfy $\omega_2 = \omega_1 = \omega$ so again this is a degenerate situation in which we may not have enough eigenvectors. Since with $\rho = \rho_2$ the coefficient of a_m is $I(\rho_2 + m) = I(\rho_1) = 0$, the equation for a_m is only consistent if by some miracle it turned out that

$$Q_m := \sum_{k=0}^{m-1} f_{n-k-2}a_k = 0\,. \tag{6.30}$$

If this miraculous situation holds, then the coefficient a_m can be assigned any value at all, and the procedure continues with solving for the remaining coefficients, an algorithm that cannot fail and will always produce a convergent series that represents a new solution of the differential equation. This is a case in which there are indeed two linearly independent eigenvectors belonging to the degenerate monodromy eigenvalue ω. On the other hand, if we do not witness such a miracle since $Q_m \neq 0$, then we cannot solve for a_m and the process halts and fails to produce a new solution of the differential equation. In this case we again turn to the reduction of order formula. It is an exercise to verify that the coefficient of the logarithmic term contributed by the integral is proportional to Q_m, and so again a logarithmic term will always be present in this situation, meaning that there is only one eigenvector for the degenerate monodromy eigenvalue ω.

If $f(z)$ has a triple pole at $z = z_*$, or higher order, then the Frobenius method produces a hierarchy of equations the very first of which cannot be satisfied since $a_0 \neq 0$. So in these cases there are no solutions

of this type, which would correspond to Laurent series with a finite number of negative power terms. That is, the point $z = z_*$ is not a regular singular point for the differential equation. Therefore regular singular points correspond to only to points where $f(z)$ has a simple or double pole. In regard to the point at infinity, the same method we used previously to bring this point to $w = 0$ shows that infinity is a regular singular point if $f(z)$ vanishes at infinity like z^{-3} or z^{-2}.

6.3 Asymptotic Series Solutions at Irregular Singular Points

At ordinary points and regular singular points, the Laurent series always terminate after a finite number of negative power terms, so these series are not just convergent but also asymptotic as $z \to z_0$. On the other hand, at an irregular singular point $z = z_*$, the Laurent series is necessarily doubly infinite, and although certainly convergent in an annulus surrounding z_*, it is of no use whatsoever as an asymptotic series since there is no dominant term.

So to obtain asymptotic approximations of solutions of differential equations near irregular singular points we need to adopt a new method totally different from the convergent methods that the existence theory relies on so heavily. It is traditional to study irregular singularities at infinity. If there is an irregular singularity at a finite point $z = z_*$, one should first use a transformation similar to the one we used earlier to move this point to infinity. So we are studying the differential equation $y''(z) + f(z)y(z) = 0$ and its solutions for large complex z, in the case when $f(z)$ is asymptotically the same size as z^{-1} or possibly larger as $z \to \infty$.

One special case that falls into this category is the case when $f(z)$ is just a constant, say $f(z) \equiv k^2$. Then we can solve the equation exactly and two linearly independent solutions are $y_1(z) = e^{ikz}$ and $y_2(z) = e^{-ikz}$. These functions obviously have essential singularities at $z = \infty$. Their Laurent series around $z = \infty$ are just the usual Taylor series of these functions, which contain an infinite number of positive powers of z, and therefore provide no asymptotic information for large z. In a sense the exponentials themselves are their own best asymptotic approximations, and any attempt to take them apart fails to capture their essentially singular behavior near $z = \infty$. On the other hand, the quantities in the exponents, $\pm ikz$, are very simple functions as $z \to \infty$. Therefore the exponential function $\exp(\cdot)$ does some useful service in that it is a familiar and simple function that maps pole singularities into essential singularities.

The point of this example is to motivate introducing an exponential transformation for studying the asymptotics for large z for more general functions $f(z)$. Suppose we seek a solution of the equation $y''(z) + f(z)y(z)$ in the form $y(z) = e^{\phi(z)}$. This is a nonlinear transformation, so it converts the linear equation for $y(z)$ into a nonlinear equation for $\phi(z)$, namely upon cancelling the exponential factor,

$$\phi''(z) + [\phi'(z)]^2 + f(z) = 0\,. \tag{6.31}$$

Although nonlinear, this equation is better suited to asymptotic analysis because the term $f(z)$ appears on its own and we are assuming that we know everything we need to about $f(z)$. In particular, since $f(z)$ is a rational function it has a singly-infinite Laurent series at $z = \infty$ that is not only convergent for z sufficiently large but is also an asymptotic power series as $z \to \infty$. That is, $f(z)$ has an asymptotic expansion of the form

$$f(z) \sim z^p \sum_{n=0}^{\infty} \frac{f_n}{z^n} \qquad \text{as} \qquad z \to \infty\,, \tag{6.32}$$

where $f_0 \neq 0$ and p is an integer. We can obtain this expansion directly from any given formula for $f(z)$ by computing the coefficients f_n via a limit process. Since we are assuming infinity to be irregular singular, the integer power p here is at least -1.

Even though $f(z)$ is expanded in integer powers of z it is not clear that this must be the case for $\phi(z)$ given the nonlinear nature of the problem. We have to determine the correct asymptotic sequence with respect to which we can develop an expansion for $\phi(z)$. We can do this systematically using the principle of dominant balance. That is, we try to choose each term in the sequence in order so that at each stage there are at least two terms in the equation that are of the same order of magnitude as $z \to \infty$ and such that all other terms not involved in the balance are negligible by comparison.

Since the dominant contribution to $f(z)$ is the term $f_0 z^p$ with $p \geq -1$, we only need to take this contribution into account to find the dominant contribution to $\phi(z)$. There are three possible balances to consider:

1. Balancing $\phi''(z)$ with $[\phi'(z)]^2$. This suggests that the dominant contribution to $\phi(z)$ comes from a function $\phi_0(z)$ that satisfies the equation $\phi_0''(z) + [\phi_0'(z)]^2 = 0$. This is really a first-order equation for $\phi_0'(z)$ in disguise. Its general solution is $\phi_0'(z) = (z-a)^{-1}$ where a is an integration constant. Consequently, we get $\phi_0(z) = \log(z-a) + b$ where b is another integration constant. So both terms involved in the balance are of magnitude $(z-a)^{-2} = O(z^{-2})$ at infinity. This is strictly smaller than the leading term for $f(z)$ in the case of an irregular singularity, so the balance is not dominant.

2. Balancing $\phi''(z)$ with $f(z)$. This suggests that the dominant contribution to $\phi(z)$ comes from a function $\phi_0(z)$ that satisfies the equation $\phi_0''(z) + f_0 z^p = 0$. If $p = -1$ the general solution of this equation can be written in the form
$$\phi_0(z) = -f_0 \int_1^z \log(\zeta)\, d\zeta + az + b\,, \tag{6.33}$$
where a and b are integration constants. To see whether this balance is dominant, we have to compare the size of the balancing terms, in this case z^{-1}, with the size of the omitted term. Since in this case $[\phi_0'(z)]^2 = [-f_0 \log(z) + a]^2 \sim f_0^2 \log(z)^2$, we see that this omitted term is always asymptotically bigger than z^{-1}, so this balance is not dominant for $p = -1$. If $p > -1$, then the general solution of the equation for $\phi_0(z)$ is
$$\phi_0(z) = -\frac{f_0}{(p+1)(p+2)} z^{p+2} + az + b\,. \tag{6.34}$$
In this case the omitted term is $[\phi_0'(z)]^2 = [-f_0 z^{p+1}/(p+1) + a]^2$ which again since $f_0 \neq 0$ always dominates the terms involved in the balance which are only of order z^p. So this is not a dominant balance.

3. Balancing $[\phi'(z)]^2$ with $f(z)$. This suggests that the dominant contribution to $\phi(z)$ comes from a function $\phi_0(z)$ that satisfies $[\phi_0'(z)]^2 + f_0 z^p = 0$. Taking square roots, we get $\phi_0'(z) = \pm i f_0^{1/2} z^{p/2}$. For all $p \geq -1$ we therefore get
$$\phi_0(z) = a \pm \frac{2 i f_0^{1/2}}{p+2} z^{1+p/2}\,, \tag{6.35}$$
where a is an integration constant. In this case, the omitted term is $\phi''(z) \sim \phi_0''(z) = O(z^{p/2-1})$. The balance will therefore be dominant if $p/2 - 1 < p$ or $p > -2$ which holds for all irregular singularities at infinity.

So we accept the final balance as the dominant one. That is, we are approximating $\phi(z)$ by a function $\phi_0(z)$ that satisfies the equation
$$[\phi_0'(z)]^2 + f_0 z^p = 0 \quad \text{implying} \quad \phi_0(z) = a \pm \frac{2 i f_0^{1/2}}{p+2} z^{1+p/2}\,, \tag{6.36}$$
for some integration constant a. Now the difference between a true solution $\phi(z)$ of this equation and our guess $\phi_0(z)$ is the error defined by $r_0(z) := \phi(z) - \phi_0(z)$. Substituting this into the equation for $\phi(z)$ and using the equation that $\phi_0(z)$ satisfies, we get a differential equation for $r_0(z)$:
$$\phi_0''(z) + r_0''(z) + 2\phi_0'(z) r_0'(z) + [r_0'(z)]^2 + (f(z) - f_0 z^p) = 0\,. \tag{6.37}$$
Since now we know that the leading term in $f(z) - f_0 z^p$ is $f_1 z^{p-1}$ (at least if $f_1 \neq 0$, otherwise we have to keep looking for the first nonzero contribution), we can try to use the principle of dominant balance to find an asymptotically dominant contribution $\phi_1(z)$ to the error $r_0(z)$.

If the approximation of $\phi(z)$ by $\phi_0(z)$ is indeed to be asymptotic, it must be the case that $r_0(z) = o(\phi_0(z))$ as $z \to \infty$. We make the assumption (to be justified after the fact) that this relation carries over to derivatives as well: $r_0'(z) = o(\phi_0'(z))$ and $r_0''(z) = o(\phi_0''(z))$ as $z \to \infty$. This would be the case if all corrections turned

out to be powers of z, for example. Our assumption means that the only candidates for dominant terms in the equation for the error $r_0(z)$ are: $\phi_0''(z)$, $2\phi_0'(z)r_0'(z)$, and $f_1 z^{p-1}$. Moreover, if $p > 0$ then $f_1 z^{p-1}$ asymptotically dominates $\phi_0''(z)$ (and if $p = 0$, $\phi_0''(z) \equiv 0$), whereas if $p = -1$ then $\phi_0''(z)$ asymptotically dominates $f_1 z^{p-1}$. These considerations show that we can expect the dominant contribution $\phi_1(z)$ to the error $r_0(z)$ to satisfy

$$\begin{aligned} 2\phi_0'(z)\phi_1'(z) + f_1 z^{p-1} &= 0 \qquad \text{if} \quad p \geq 0 \\ \phi_0''(z) + 2\phi_0'(z)\phi_1'(z) &= 0 \qquad \text{if} \quad p = -1\,. \end{aligned} \tag{6.38}$$

These are directly integrated as follows:

$$\phi_1(z) = \begin{cases} \pm\dfrac{if_1}{pf_0^{1/2}} z^{p/2} + c & \text{if} \quad p > 0 \\ \pm\dfrac{if_1}{2f_0^{1/2}} \log(z) + c & \text{if} \quad p = 0 \\ \log\left((\pm i f_0^{1/2})^{-1/2} z^{1/4}\right) + c & \text{if} \quad p = -1\,, \end{cases} \tag{6.39}$$

where c is an integration constant. In fact, we can take $c = 0$ since we have already taken an additive arbitrary constant into account in $\phi_0(z)$. In each case, we see that $\phi_1(z) = o(\phi_0(z))$ as $z \to \infty$, and that this relation carries over to derivatives as well. This justifies our previous approximations.

So at this point, we are writing $\phi(z)$ in the form

$$\phi(z) = \phi_0(z) + \phi_1(z) + r_1(z)\,, \tag{6.40}$$

where $r_1(z)$ is the error at this order. Again we can write a differential equation for $r_1(z)$ using the equations satisfied by $\phi_0(z)$ and $\phi_1(z)$, and find a leading approximation $\phi_2(z)$ to $r_1(z)$. The idea is to continue this process, introducing a sequence of approximations $\phi_k(z)$ and higher-order errors $r_k(z)$ until we get to the point that dominant balance considerations indicate that $r_k(z) = o(1)$ as $z \to \infty$. Then we will have an asymptotic representation of a solution of the differential equation $y''(z) + f(z)y(z) = 0$ in the form

$$y(z) = e^{\phi_0(z)+\ldots+\phi_k(z)} e^{r_k(z)} = e^{\phi_0(z)+\ldots+\phi_k(z)} e^{o(1)} = e^{\phi_0(z)+\ldots+\phi_k(z)}(1 + o(1)) \qquad \text{as} \quad z \to \infty\,. \tag{6.41}$$

Therefore we will have continued the process far enough that the *relative error* of the approximation is small when z is large.

Let us illustrate this concretely for the equation $y''(z) + f(z)y(z) = 0$ with $f(z) = 1 + 1/z + 1/z^2$. So in this case we have $p = 0$. The leading term in the logarithm $\phi(z)$ of the solution $y(z)$ is in this case

$$\phi_0(z) = a \pm iz\,. \tag{6.42}$$

The second term, according to the general formula found above is

$$\phi_1(z) = \pm\frac{i}{2}\log(z) \tag{6.43}$$

where we are taking the constant c to be zero. The signs are meant to correspond in these two formulas. The exact equation for the error $r_1(z) := \phi(z) - \phi_0(z) - \phi_1(z)$ is therefore in this case

$$\left[\frac{3}{4} \mp \frac{i}{2}\right]\frac{1}{z^2} + r_1''(z) + [r_1'(z)]^2 \pm i\left[2 + \frac{1}{z}\right] r_1'(z) = 0\,. \tag{6.44}$$

It is an exercise to check that the dominant balance here is achieved by taking

$$\mp 2ir_1'(z) \sim \left[\frac{3}{4} \mp \frac{i}{2}\right]\frac{1}{z^2} \qquad \text{as} \quad z \to \infty\,, \tag{6.45}$$

which gives $r_1(z) = O(1/z)$. This shows that $r_1(z)$ goes to zero as $z \to \infty$, and consequently we have obtained approximations to the solution of the differential equation with asymptotically small relative error:

$$y(z) = Ce^{\pm iz} z^{\pm i/2}(1 + o(1)) \quad \text{as} \quad z \to \infty\,. \tag{6.46}$$

Here the constant C is just e^a.

If $f(z)$ had been more singular at infinity, then we would have had to go to higher order in the expansion of $\phi(z)$ before we would have found that the error was asymptotically small. Typically this procedure generates a sequence of successive corrections $\phi_k(z)$ that are a finite number of descending positive powers of z followed by a logarithm and then a chain of descending negative powers of z. When we truncate the chain anywhere after the logarithm first appears we then have a small relative error. The positive powers stay in the exponent, and the logarithm contributes a factor of a power of z.

6.4 Stokes' Phenomenon

Stokes' phenomenon arises when we consider the way that the asymptotic expansions of solutions of differential equations near irregular singular points, obtained by the formal procedure described above, relate to true solutions of the differential equation in question. When we consider the analogous question near ordinary points or regular singular points, the relation is trivial because the asymptotic expansions converge for all z in neighborhood of the point of interest (with the possible exception of the point itself). So the expansions do not only represent solutions asymptotically as $z \to z_0$, but they actually *are* solutions themselves.

But near an irregular singular point the situation is altogether different. First of all, the two formal asymptotic series we obtained have the property that one of them is totally dominant over the other as $z \to \infty$ along most rays in the complex plane. This means that one of the asymptotic expansions is so small compared to the other that if we attempt to represent it with respect to the asymptotic sequence used to develop the larger expansion, all coefficients will come out to be zero. This means that any component of a true solution of the differential equation that is proportional to the smaller expansion will be invisible to asymptotic expansion procedures if there is any component of the larger expansion present at all. This leads to the Stokes phenomenon: a given solution of the differential equation, when expanded asymptotically near an irregular singular point at infinity, will appear to have asymptotics in different sectors of the complex plane that are given by different formulas. There usually appear to be abrupt jumps in the asymptotic behavior along the rays that separate these sectors.

We can illustrate this with Airy's differential equation $y''(z) - zy(z) = 0$, whose only singular point is an irregular singular point at infinity. In Chapter 3, we already studied one exact solution of this equation that can be expressed as an integral, namely the Airy function $Ai(z)$. Using the method of steepest descents we found that this function was described asymptotically as $z \to \infty$ by the following formula:

$$Ai(z) = \frac{1}{2z^{1/4}\sqrt{\pi}} e^{-2z^{3/2}/3} \left[1 + O\left(|z|^{-3/2}\right)\right] \quad \text{as} \quad z \to \infty \quad \text{with} \quad -\pi < \arg(z) < \pi\,. \tag{6.47}$$

Now, the idea is that the Airy function $Ai(z)$ is a rather special solution of the differential equation in that a single asymptotic formula is valid for large z in the whole complex plane except for one ray, namely the negative real axis. To see how special this solution is, recall that a linearly independent solution of the equation is given by $Ai(ze^{2\pi i/3})$, which is the Airy function rotated by a one-third turn in the complex z-plane. This means that the general solution of Airy's equation is

$$y(z) = \alpha Ai(z) + \beta Ai(ze^{2\pi i/3})\,, \tag{6.48}$$

where α and β are arbitrary complex constants. To see what the asymptotic behavior of this solution is, first we should find the asymptotic behavior of $Ai(ze^{2\pi i/3})$. Using the correct definitions for the power functions relative to their branch cuts on the negative real axis, we find that

$$\left(ze^{2\pi i/3}\right)^{3/2} = \begin{cases} -z^{3/2} & \text{for} \quad -\pi < \arg(z) < \pi/3 \\ \\ z^{3/2} & \text{for} \quad \pi/3 < \arg(z) < \pi\,, \end{cases} \tag{6.49}$$

and

$$\left(ze^{2\pi i/3}\right)^{1/4} = \begin{cases} e^{i\pi/6}z^{1/4} & \text{for} \quad -\pi < \arg(z) < \pi/3 \\ e^{-i\pi/3}z^{1/4} & \text{for} \quad \pi/3 < \arg(z) < \pi\,. \end{cases} \tag{6.50}$$

Therefore, $Ai(ze^{2\pi i/3})$ has two different looking asymptotic expansions:

$$Ai(ze^{2\pi i/3}) = \frac{e^{-i\pi/6}}{2z^{1/4}\sqrt{\pi}}e^{2z^{3/2}/3}\left[1+O\left(|z|^{-3/2}\right)\right] \quad \text{as} \quad z\to\infty \quad \text{with} \quad -\pi < \arg(z) < \pi/3\,, \tag{6.51}$$

and

$$Ai(ze^{2\pi i/3}) = \frac{e^{i\pi/3}}{2z^{1/4}\sqrt{\pi}}e^{-2z^{3/2}/3}\left[1+O\left(|z|^{-3/2}\right)\right] \quad \text{as} \quad z\to\infty \quad \text{with} \quad \pi/3 < \arg(z) < \pi\,. \tag{6.52}$$

Now, the exponential function $e^{2z^{3/2}/3}$ is asymptotically dominant over $e^{-2z^{3/2}/3}$ as $z\to\infty$ for $-\pi/3 < \arg(z) < \pi/3$. The situation is reversed in the other two thirds of the complex plane. So $Ai(z)$ is exponentially small for $-\pi/3 < \arg(z) < \pi/3$, and exponentially large otherwise. And $Ai(ze^{2\pi i/3})$ is exponentially small for $-\pi < \arg(z) < -\pi/3$ and exponentially large otherwise. When both functions are large, they are of the same magnitude. So if we calculate the asymptotic behavior of the linear combination $y(z)$, we will find:

$$y(z) = \begin{cases} \dfrac{\alpha}{2z^{1/4}\sqrt{\pi}}e^{-2z^{3/2}/3}\left[1+O\left(|z|^{-3/2}\right)\right] & \text{for} \quad -\pi < \arg(z) < -\pi/3 \\ \dfrac{\beta e^{-i\pi/6}}{2z^{1/4}\sqrt{\pi}}e^{2z^{3/2}/3}\left[1+O\left(|z|^{-3/2}\right)\right] & \text{for} \quad -\pi/3 < \arg(z) < \pi/3 \\ \dfrac{\alpha+\beta e^{i\pi/3}}{2z^{1/4}\sqrt{\pi}}e^{-2z^{3/2}/3}\left[1+O\left(|z|^{-3/2}\right)\right] & \text{for} \quad \pi/3 < \arg(z) < \pi \end{cases} \tag{6.53}$$

as $z\to\infty$. This expansion presumes that none of the coefficients α, β, nor $\alpha+\beta e^{i\pi/3}$ is zero. But under this generic condition, the solution $y(z)$ is exponentially large as $z\to\infty$ in all three sectors. The rays separating the sectors are called *Stokes lines*. At these lines the asymptotic behavior of a typical solution undergoes a sudden change. Also, in each sector, the component of the true solution that is exponentially small by comparison never appears in the asymptotic expansion. If present it is "beyond all orders", or "transcendentally small".

So the lesson of Stokes' phenomenon is that an asymptotic expansion of a solution near an irregular singular point that is obtained from the differential equation rather than from a solution formula (if one exists) is not likely to represent any solution of the differential equation in a full neighborhood of the singular point. Different formulas will be necessary in different sectors. An asymptotically large exponential expansion in a given sector will necessarily represent many different solutions of the differential equation. On the other hand, an asymptotically small exponential expansion can represent only one solution, up to a constant multiple.

The α and β dependent constants in the expansion of the general solution are called the *monodromy data* of the solution relative to the irregular singular point at infinity. One problem that one can consider is whether there exists a true solution of the differential equation whose asymptotic expansion corresponds to a given set of monodromy data. We will not consider this problem here, but it is a good example of an *inverse problem* associated with a differential equation. The corresponding direct problem is simply to compute the monodromy data from a given solution, like we did for the general solution to Airy's equation.

6.5 Asymptotic Expansions for Solutions of Differential Equations Involving a Large or Small Parameter

Many problems from applied mathematics involve differential equations with an auxiliary parameter. For example, this parameter can be a dimensionless physical parameter like a Reynolds number or Nusselt

number appearing in the differential equations of fluid dynamics. It can also be an eigenvalue or separation constant in a boundary-value problem. This idea will become an important theme in the rest of the course. For now we will look at this idea from the point of view of second-order linear ordinary differential equations like we have studied recently. Namely, we will consider differential equations of the form

$$y''(x) + f(x;\lambda)y(x) = 0 \tag{6.54}$$

where λ is the parameter. We are writing x instead of z for the independent variable because for the time being we want to restrict attention to solutions on the real axis. For each different value of λ, the function f is different and the solutions of the equation consequently depend on the value of λ as well, although we usually suppress this when we write $y = y(x)$.

6.5.1 Regular perturbation problems.

The idea is to consider the asymptotic behavior of solutions of this differential equation in the limit $\lambda \to \lambda_0$, where λ_0 might be infinity. If $f(x;\lambda_0)$ is a sensible function of x, and if the convergence of $f(x;\lambda)$ to $f(x;\lambda_0)$ is uniform over the region of x-values of interest which we will assume to be bounded for the moment, then what we have on our hands is what is called a *regular perturbation problem*. In principle the problem can be solved for $\lambda = \lambda_0$ and we are interested in the way that the solutions are perturbed by adjusting slightly the value of λ.

An example of a problem of this type is:

$$y''(x) + (k^2 + \epsilon\mu(x))y(x) = 0 \tag{6.55}$$

where k is a fixed real constant while $\mu(x)$ is a bounded function of x, and the parameter of interest is ϵ which we shall take to be going to zero. This equation describes the propagation of waves in a weakly inhomogeneous medium, with the function $\epsilon\mu(x)$ describing the inhomogeneity. When $\epsilon = 0$, the problem can be solved exactly, and the general solution is $y_0(x) = a_+e^{ikx} + a_-e^{-ikx}$ for some arbitrary constants a_+ and a_-. The idea of a regular perturbation method is to take the weak inhomogeneity into account by computing successive corrections to this leading order solution of higher and higher order in ϵ. These corrections indicate the way that the weak inhomogeneity begins to influence the simple solutions for $\epsilon = 0$. In a regular situation like this, it is natural to seek an asymptotic solution in the form of a formal power series in ϵ:

$$y(x) \sim \sum_{n=0}^{\infty} \epsilon^n y_n(x)\,. \tag{6.56}$$

Here, x is a fixed parameter as far as the asymptotic expansion is concerned. Later we will concern ourselves with how to relax this restriction by letting x grow large or shrink as $\epsilon \to 0$. So, differentiating with respect to x just means doing so term-by-term in this expansion. That is,

$$y''(x) \sim \sum_{n=0}^{\infty} \epsilon^n y_n''(x)\,. \tag{6.57}$$

For example, inserting $y(x) = y_0(x) + \epsilon y_1(x) + O(\epsilon^2)$ into the differential equation, and applying the limit process, we get that $y_1(x)$ satisfies the equation

$$y_1''(x) + k^2 y_1(x) = -\mu(x)y_0(x) = -a_+\mu(x)e^{ikx} - a_-\mu(x)e^{-ikx}\,. \tag{6.58}$$

Since we know two linearly independent solutions of the corresponding homogeneous equation, namely $y_1 = e^{ikx}$ and $y_1 = e^{-ikx}$, we can seek the solution of this inhomogeneous problem using *variation of parameters*. Set $y_1(x) = b_+(x)e^{ikx} + b_-(x)e^{-ikx}$ and substitute into the equation for $y_1(x)$. First computing $y_1'(x)$, we get

$$y_1'(x) = b_+'(x)e^{ikx} + ikb_+(x)e^{ikx} + b_-'(x)e^{-ikx} - ikb_-(x)e^{-ikx}\,. \tag{6.59}$$

Since we are replacing one unknown function $y_1(x)$ by two unknown functions $b_+(x)$ and $b_-(x)$, we are allowed to impose one condition on these two functions. Let us impose the condition that

$$b_+'(x)e^{ikx} + b_-'(x)e^{-ikx} = 0\,. \tag{6.60}$$

Then taking one more derivative and inserting into the equation for $y_1(x)$ gives

$$ikb'_+(x)e^{ikx} - ikb'_-(x)e^{-ikx} = -a_+\mu(x)e^{ikx} - a_-\mu(x)e^{-ikx}\,. \tag{6.61}$$

Combining this equation with the restriction on $b_+(x)$ and $b_-(x)$ that we already imposed, we can solve for $b'_+(x)$ and $b'_-(x)$:

$$b'_+(x) = -\mu(x)\frac{a_+ + a_-e^{-2ikx}}{2ik} \quad \text{and} \quad b'_-(x) = \mu(x)\frac{a_+e^{2ikx} + a_-}{2ik}\,. \tag{6.62}$$

So, through order ϵ, the perturbation solution is

$$\begin{aligned} y(x) &= \left[a_+ + \epsilon b_+(0) - \frac{\epsilon}{2ik}\int_0^x \mu(u)(a_+ + a_-e^{-2iku})\,du\right]e^{ikx} \\ &+ \left[a_- + \epsilon b_-(0) + \frac{\epsilon}{2ik}\int_0^x \mu(u)(a_+e^{2iku} + a_-)\,du\right]e^{-ikx} + O(\epsilon^2)\,. \end{aligned} \tag{6.63}$$

Here, a_+, a_-, $b_+(0)$, and $b_-(0)$ are constants that can be chosen to satisfy some auxiliary conditions on the solution. Various things are clear from this formula. If the inhomogeneity $\mu(x)$ has either an average value that is nonzero, or Fourier components at wavenumbers $2k$ or $-2k$ (double the wavenumber of the leading order waveform), then the integrals will be growing functions of x which indicates that the inhomogeneity will have a long-range effect on the waves. Otherwise, the integrals will be bounded functions of x and the inhomogeneity will not have much of an effect at this order.

6.5.2 Singular perturbation problems without turning points.

There are significant issues in the above regular perturbation problem connected with the uniformity of the $O(\epsilon^2)$ error with respect to x. We will address these later in the course, but for now we should consider the class of *singular perturbation problems* which represents cases in which the function $f(x;\lambda)$ does not tend to any limit as $\lambda \to \lambda_0$. For ease of discussion in these cases we will assume that $\lambda_0 = \infty$, which is always possible by a redefinition of the parameter λ. Although there are many ways that $f(x;\lambda)$ can fail to have a limit as $\lambda \to \infty$, we will concentrate on problems where $f(x;\lambda)$ becomes infinitely large as $\lambda \to \infty$. Now if we are going to use the fact that $f(x;\lambda)$ is becoming large as $\lambda \to \infty$, we must in particular assume that $f(x;\lambda)$ is never zero for the x-values of interest. Later we will learn how to deal with values of x near zeros of $f(x;\lambda)$, the so-called *turning points*, but for now we will just assume that there aren't any.

The idea of f becoming large should be familiar to us. This is exactly what happens when there is no parameter but when we are close to a singular point of f. We saw that in that case it was useful to make an exponential transformation of the problem by setting $y = e^\phi$. In this case, such a transformation puts the equation in the (equivalent) form

$$\phi''(x) + [\phi'(x)]^2 + f(x;\lambda) = 0\,. \tag{6.64}$$

As before, we can try to find an asymptotic series for $\phi(x)$, this time holding x fixed and sending $\lambda \to \infty$. The precise form of this expansion will depend on the form of $f(x;\lambda)$. To begin with, suppose that $f(x;\lambda)$ has the asymptotic expansion

$$f(x;\lambda) \sim \lambda^2 \sum_{n=0}^{\infty} \lambda^{-n} f_n(x)\,, \tag{6.65}$$

as $\lambda \to \infty$ for each fixed x in the range of interest. Here the working assumption that f does not vanish for the x-values of interest means that $f_0(x) \neq 0$. The large size of $f(x;\lambda)$ is built into the factor λ^2 that multiplies the whole expansion. We can seek $\phi(x)$ in the form of an asymptotic expansion as well:

$$\phi(x) \sim \sum_{n=0}^{\infty} g_n(\lambda)\phi_n(x)\,. \tag{6.66}$$

Here the functions $g_n(\lambda)$ have to form an asymptotic sequence as $\lambda \to \infty$. Part of the problem is to figure out what this asymptotic sequence is for a given $f(x;\lambda)$.

To see what the leading-order contribution looks like, insert $\phi(x) = g_0(\lambda)\phi_0(x) + o(g_0(\lambda))$ and $f(x;\lambda) = \lambda^2 f_0(x) + o(\lambda^2)$ into the differential equation:

$$g_0(\lambda)\phi_0''(x) + g_0(\lambda)^2[\phi_0'(x)]^2 + o(g_0(\lambda)^2) + \lambda^2 f_0(x) + o(\lambda^2) = 0\,. \tag{6.67}$$

Here, the term $o(g_0(\lambda)^2)$ encorporates two terms from the square both of which are small compared to $g_0(\lambda)^2$ for fixed x. A dominant balance must therefore involve at least one of the two explicit unknown terms and the leading term $\lambda^2 f_0(x)$ of $f(x;\lambda)$. If we try to balance the second derivative term with $f(x;\lambda)$ by taking $g_0(\lambda) = \lambda^2$, then we see that this balance is not dominant as $\lambda \to \infty$ because the term proportional to $g_0(\lambda)^2$ will be larger. On the other hand, balancing $g_0(\lambda)^2[\phi_0'(x)]^2$ with $f(x;\lambda)$ will indeed be consistent. This means that we should select $g_0(\lambda)^2 = \lambda^2$ or $g_0(\lambda) = \lambda$. With this choice of the first function in the asymptotic sequence $\{g_n(\lambda)\}$, we find that $\phi_0(x)$ satisfies the equation

$$[\phi_0'(x)]^2 + f_0(x) = 0\,. \tag{6.68}$$

The general solution of this problem is obtained by taking a square root and integrating:

$$\phi_0(x) = \pm i \int_{x_0}^{x} [f_0(s)]^{1/2}\, ds\,. \tag{6.69}$$

Here x_0 is an arbitrary point in the range of x-values under consideration. The square root is well-defined by continuation from $s = x_0$ to $s = x$ given a choice of sign when $s = x_0$ since $f_0(x)$ does not vanish by assumption.

The process continues at the next order by using the fact that $g_0(\lambda) = \lambda$ and substituting $\phi(x) = \lambda\phi_0(x) + g_1(\lambda)\phi_1(x) + o(g_1(\lambda))$ and $f(x;\lambda) = \lambda^2 f_0(x) + \lambda f_1(x) + o(\lambda)$ into the differential equation. We get

$$\lambda\phi_0''(x) + g_1(\lambda)\phi_1''(x) + o(g_1(\lambda)) + 2\lambda g_1(\lambda)\phi_0'(x)\phi_1'(x) + g_1(\lambda)^2[\phi_1'(x)]^2 + o(g_1(\lambda)^2) + \lambda f_1(x) + o(\lambda) = 0\,, \tag{6.70}$$

where again the term $o(g_1(\lambda)^2)$ represents two terms of the same size, small compared with $g_1(\lambda)^2$. This looks messy, but since $g_1(\lambda) = o(\lambda)$ as $\lambda \to \infty$ in order for the sequence of gauge functions to be asymptotic, only a few terms are candidates for being involved in a dominant balance. So, we see that the term $g_1(\lambda)\phi_1''(x)$ is necessarily negligible compared with $\lambda\phi_0''(x)$. Likewise the term $g_1(\lambda)^2[\phi_1'(x)]^2$ is necessarily negligible compared with $2\lambda g_1(\lambda)\phi_0'(x)\phi_1'(x)$. So once again we are left with three terms to compare. Here we can see that all three terms will be locked into a three-way (and hence automatically dominant) balance if we take $g_1(\lambda) = 1$. Then the equation for $\phi_1(x)$ is

$$\phi_0''(x) + 2\phi_0'(x)\phi_1'(x) + f_1(x) = 0\,. \tag{6.71}$$

Therefore,

$$\phi_1'(x) = -\frac{1}{2}\left[\frac{\phi_0''(x)}{\phi_0'(x)} + \frac{f_1(x)}{\phi_0'(x)}\right] \qquad \text{or} \qquad \phi_1(x) = \log\left([\phi_0'(x)]^{-1/2}\right) - \frac{1}{2}\int_{x_1}^{x} \frac{f_1(s)}{\phi_0'(s)}\, ds\,. \tag{6.72}$$

Here x_1 is an arbitrary limit of integration, which absorbs the integration constant. The function $\phi_0(\cdot)$ may be eliminated from this formula using its defnition (6.69) in terms of $f_0(x)$ and a choice of sign.

The process can, of course, be continued. The next gauge function can be seen to be $g_2(\lambda) = \lambda^{-1}$, and we will then arrive at an equation for $\phi_2(x)$ that can be explicitly integrated. In this example, the whole asymptotic sequence therefore turns out to be just $\{\lambda^{1-n}\}$ for $n = 0, 1, 2, \ldots$. For more exotic ways that $f(x;\lambda)$ can depend on λ (beyond simple powers, say when $f(x;\lambda)$ involves exponential functions or logarithms of λ), the sequences that are needed can turn out to be different. In any case, since the next correction to calculate is $o(1)$, we have obtained enough terms in the expansion that the relative error in $y(x) = e^{\phi}(x)$ goes to zero as $\lambda \to \infty$. The form of the solution we have thus obtained is

$$y(x) = \frac{C}{[f_0(x)]^{1/4}} \exp\left(\pm i\lambda \int_{x_0}^{x} [f_0(s)]^{1/2}\, ds\right) \exp\left(\pm\frac{i}{2}\int_{x_1}^{x} \frac{f_1(s)}{[f_0(s)]^{1/2}}\, ds\right)(1 + o(1)) \tag{6.73}$$

as $\lambda \to \infty$. Shifts in the limits of integration x_0 and x_1 can be encorporated into the arbitrary multiplicative constant C. This single formula represents the asymptotic expansions of two linearly independent solutions of the differential equation, one for each choice of sign in the exponent.

In this formula, we clearly see that we should expect difficulties if there are isolated values of x where $f_0(x) = 0$, since the solution becomes infinitely large at these turning points. This is certainly inaccurate if the original function $f(x;\lambda)$ was analytic at the turning point for all sufficiently large λ, since in this case we know that the solutions themselves have to be analytic at the turning point. So something else has to be taken into account near such points. We will take this topic up shortly, however for now the message is that when a formal asymptotic expansion displays singularities, it usually means that the assumptions under which the expansion was derived are failing, and therefore new assumptions need to be supplied. On the other hand, if no such singularities are observed, one should have some confidence that the expansion indeed accurately represents solutions of the differential equation under investigation.

On the other hand, if $f_0(x)$ is identically zero, then what it really means is simply that the function $f(x;\lambda)$ is not as big as you first thought. That is, now $f(x;\lambda)$ has the asymptotic expansion

$$f(x;\lambda) \sim \lambda \sum_{n=0}^{\infty} \lambda^n f_{n+1}(x) \tag{6.74}$$

as $\lambda \to \infty$. If we just introduce a new parameter by setting $\lambda = \zeta^2$, so that ζ is also going to infinity, then the series is of exactly the same form as before, except that the odd powers of ζ are missing from the expansion. However, as long as now $f_1(x)$ does not have any zeros, the same procedure as before will provide the solution in this case. The function ϕ can be expanded relative to the asymptotic sequence $\{\zeta, 1, \zeta^{-1}, \zeta^{-2}, \ldots\}$ which is the same thing as $\{\lambda^{1/2}, 1, \lambda^{-1/2}, \lambda^{-1}, \ldots\}$.

As a brief historical remark, several special cases of the above procedure are associated with certain individuals. The case when $f(x;\lambda) = \lambda^2 f_0(x)$, that is, when only the first term is present, is called *Liouville's equation*. In the case when the first three terms are present the expansion procedure is often called the *WKB method*. WKB is an abbreviation for the names of three physicists: Wentzel, Kramers, and Brillouin. Sometimes in physics books you see the same method called WKBJ, and the J stands for Jeffreys.

6.5.3 Problems with turning points. Inner asymptotics.

As we observed above, the asymptotic procedure of substituting $y(x) = e^{\phi(x)}$ into the equation $y''(x) + f(x;\lambda)y(x) = 0$ and developing $\phi(x)$ in an asymptotic series as $\lambda \to \infty$ fails if $f(x;\lambda)$ vanishes at isolated values of x, which we called *turning points*. If $f(x;\lambda)$ is analytic at a turning point then the true solutions of the differential equation are also analytic there for each λ.

There are two problems connected with turning points. The first problem attempts to resolve the analyticity of the true solution at the turning point by finding an asymptotic representation of solutions in a little region of the x-axis near the turning point. This region shrinks in size as $\lambda \to \infty$, and the asymptotic procedure essentially "zooms in" on the solution in the vicinity of the turning point. An asymptotic expansion that is valid in such a shrinking neighborhood is typically called an *inner expansion*. So by contrast the expansions obtained when x stays away from all turning points are called *outer expansions*. We have already seen an example of an inner asymptotic expansion in the homework when we "blew up" the shock structure for Burgers' equation in the zero viscosity limit.

The second problem, sometimes called the *connection problem* concerns the way that a true solution of the differential equation is represented near a turning point by a three-layer sandwich structure consisting of one outer expansion on each side and an inner expansion in the middle. In particular, one wants to know how to determine the appropriate integration constants in one of the outer expansions given the precise form of the outer expansion on the other side of the turning point. This is accomplished by an asymptotic *matching* procedure that smoothly joins expansions together in a common regime of validity.

For simplicity, we will work with Liouville's equation

$$y''(x) + \lambda^2 f_0(x) y(x) = 0 \tag{6.75}$$

where $x \in (a, b)$ and on this interval $f_0(x)$ is a strictly increasing differentiable function with $f_0(x_0) = 0$ for a unique turning point $x_0 \in (a, b)$. The analogous situation with $f_0(x)$ decreasing through a turning

point may be treated in this way by first changing x to $-x$. We are interested in the asymptotic behavior of solutions of this problem when $\lambda \to +\infty$. For x trapped between a and any point strictly to the left of x_0, $f_0(x)$ is bounded above by a negative number, and in this region we have a two-parameter family of outer asymptotic solutions of the form

$$y(x) \sim y_{\text{out}}^{\text{left}}(x) := \frac{A_{\text{left}}}{|f_0(x)|^{1/4}} \exp\left(\lambda \int_{x_0}^{x} |f_0(s)|^{1/2}\, ds\right) + \frac{B_{\text{left}}}{|f_0(x)|^{1/4}} \exp\left(-\lambda \int_{x_0}^{x} |f_0(s)|^{1/2}\, ds\right). \tag{6.76}$$

On the other side of the turning point, say for x trapped between any point strictly to the right of x_0 and b, $f_0(x)$ is bounded below by a positive number, and so here we have a two-parameter family of outer asymptotic solutions of the form

$$y(x) \sim y_{\text{out}}^{\text{right}}(x) := \frac{A_{\text{right}}}{[f_0(x)]^{1/4}} \exp\left(i\lambda \int_{x_0}^{x} [f_0(s)]^{1/2}\, ds\right) + \frac{B_{\text{right}}}{[f_0(x)]^{1/4}} \exp\left(-i\lambda \int_{x_0}^{x} [f_0(s)]^{1/2}\, ds\right). \tag{6.77}$$

Note that to the right of x_0, the outer asymptotic formula consists of two terms that are of the same order of magnitude, both being oscillatory functions of x. On the other hand, to the left of x_0, the outer asymptotic formula consists of one exponentially growing term (the one proportional to B_{left}) and one exponentially decaying term. This situation should remind us of what we learned when we discussed Stokes' phenomenon. Namely, that if there is any component at all of the exponentially large solution, *i.e.* if $B_{\text{left}} \neq 0$, then the exponentially small solution will be transcendentally small by comparison and will therefore disappear as far as the asymptotic sequence of the large solution is concerned. On the other hand, if $B_{\text{left}} = 0$ exactly, then the leading order asymptotics will be given by what is left over, namely the exponentially small component proportional to A_{left}.

A typical well-posed connection problem is the following. Suppose we are given a true solution $y(x)$ of the differential equation whose asymptotic expansion for x-values to the left of the turning point is of the form (6.76) with $B_{\text{left}} = 0$. This means (i) that the solution is decaying exponentially away from the turning point in this region, and (ii) that the true solution is completely determined by the value of the constant A_{left} in (6.76). Now to the right of the turning point the same solution $y(x)$ is presumably represented asymptotically by an expression of the form (6.77) with some constants A_{right} and B_{right}. Since $y(x)$ is determined by A_{left} and determines the constants A_{right} and B_{right}, it should be possible to express the latter two directly in terms of A_{left}. Solving the connection problem in this context therefore describes exactly which oscillations appear to the right of the turning point given only that the solution is decaying to the left of the turning point.

The connection problem can be solved by working out what to do with Liouville's equation as $\lambda \to +\infty$ for x-values near x_0 and how the corresponding inner asymptotic solution relates to what is going on to the left and right. When x is close to x_0, we can write a Taylor approximation for $f_0(x)$ as follows:

$$f_0(x) = \nu^2(x - x_0) + O((x - x_0)^2) = \nu^2(x - x_0)\,[1 + O(x - x_0)] \qquad \text{where} \qquad \nu^2 = f_0'(x_0) > 0\,. \tag{6.78}$$

Now consider the change of variables $x \to z$ given by $x - x_0 = -\alpha z$ for some $\alpha > 0$ that we will determine. Let $y(x) = Y(z) = Y(-(x - x_0)/\alpha)$. Using the chain rule to transform the derivatives, Liouville's equation becomes

$$Y''(z) - \alpha^3\lambda^2\nu^2 z Y(z) + O(\alpha^4\lambda^2 z) z Y(z) = 0 \tag{6.79}$$

with the help of the Taylor approximation for $f_0(x)$. Now simplify matters by picking

$$\alpha = (\lambda\nu)^{-2/3}\,. \tag{6.80}$$

This puts the equation in the form

$$Y''(z) - zY(z) = O(\lambda^{-2/3} z) z Y(z)\,. \tag{6.81}$$

This will be a small perturbation of Airy's differential equation that we studied previously as long as the range of values of z satisfies $z = o(\lambda^{2/3})$. This makes the coefficient of the unwanted term on the right-hand side negligible compared with $zY(z)$. This is a range of z-values that is expanding as $\lambda \to +\infty$. Note that

the condition $z = o(\lambda^{2/3})$ is equivalent to the statement that $x = x_0 + o(1)$ as $\lambda \to \infty$, so we must simply be in some shrinking neighborhood of the turning point for the approximation of $Y(z)$ by a solution of Airy's equation to be valid.

As described in Chapter 3, the equation $Y_{\rm in}''(z) - zY_{\rm in}(z) = 0$ (we are thinking of $Y(z)$ as a perturbation of $Y_{\rm in}(z)$) has two linearly independent solutions that can be constructed as contour integrals. One of them, $Ai(z)$, we have studied in great detail. A common choice of a second linearly independent solution is the function $Bi(z)$ defined as $Bi(z) := -iw_1(z) - 2iw_3(z) = -iAi(z) - 2iw_3(z)$ where the functions $w_i(z)$ were specified in terms of contour integrals over paths C_i in the complex plane. In particular, we recall the following asymptotic formulae for $Ai(z)$:

$$Ai(z) = \begin{cases} \dfrac{1}{2z^{1/4}\sqrt{\pi}} e^{-2z^{3/2}/3} \left[1 + O\left(z^{-3/2}\right)\right] & z \to +\infty \\ \dfrac{1}{|z|^{1/4}\sqrt{\pi}} \left[\sin\left(\dfrac{2}{3}|z|^{3/2} + \dfrac{\pi}{4}\right) + O\left(|z|^{-3/2}\right)\right] & z \to -\infty\,. \end{cases} \tag{6.82}$$

Similar straightforward analysis of the integral $w_3(z)$ using the method of steepest descent, when combined with this information about $Ai(z)$ gives the following asymptotic formulae for the function $Bi(z)$:

$$Bi(z) = \begin{cases} \dfrac{1}{z^{1/4}\sqrt{\pi}} e^{2z^{3/2}/3} \left[1 + O\left(z^{-3/2}\right)\right] & z \to +\infty \\ \dfrac{1}{|z|^{1/4}\sqrt{\pi}} \left[\cos\left(\dfrac{2}{3}|z|^{3/2} + \dfrac{\pi}{4}\right) + O\left(|z|^{-3/2}\right)\right] & z \to -\infty\,. \end{cases} \tag{6.83}$$

It is important to keep in mind that despite the appearance of these formulae, both $Ai(z)$ and $Bi(z)$ are entire analytic functions of z and therefore provide a smooth transition between oscillatory behavior as $z \to -\infty$ and exponential decay (in the case of $Ai(z)$) or growth (in the case of $Bi(z)$) as $z \to +\infty$. Of course this is exactly what we need at a turning point, since the outer solutions are oscillatory on one side of the turning point and growing or decaying on the other.

So we now make use of these exact solutions of Airy's equation to write the solution of Liouville's equation near the turning point in the asymptotic form of a linear combination of $Ai(z)$ and $Bi(z)$:

$$y(x) = Y(z) \sim Y_{\rm in}(z) := A_{\rm in} Ai(z) + B_{\rm in} Bi(z) \quad \text{as} \quad \lambda \to +\infty\,, \tag{6.84}$$

with $z = o(\lambda^{2/3})$. To complete our "layer cake" picture of the true solution $y(x)$ near the turning point x_0, we have to somehow relate the three sets of coefficients $(A_{\rm left}, B_{\rm left})$, $(A_{\rm in}, B_{\rm in})$ and $(A_{\rm right}, B_{\rm right})$. This is the procedure known as matching of asymptotic expansions.

It works like this. Let x be a λ-dependent point near the turning point x_0 chosen so that:

1. $x \to x_0$ as $\lambda \to \infty$. The corresponding value of z is therefore $o(\lambda^{2/3})$.
2. $x \to x_0$ as $\lambda \to \infty$ slowly enough so that the corresponding value of z is going to either $+\infty$ or $-\infty$ (depending on whether $x - x_0$ is positive or negative) as λ grows.

The first condition makes sure that the inner asymptotic expansion of $y(x)$ is valid, and the second condition makes sure that the inner expansion is well-represented by the asymptotic formulae for $Ai(z)$ and $Bi(z)$ for large $|z|$. These conditions on x describe an intermediate range of values called an *overlap domain* in which both inner and outer expansions are expected to represent the true solution accurately. In such an overlap domain, the inner and outer expansions should have the same form, which allows their coefficients to be matched.

We already have expansions of $Y_{\rm in}(z)$ for large z, so we need to find expansions of $y_{\rm out}^{\rm left}(x)$ and $y_{\rm out}^{\rm right}(x)$ as $x \to x_0$ to finish the job. First suppose that $x \to x_0-$, so we are looking at $y_{\rm out}^{\rm left}(x)$. It all comes down to the Taylor approximation of $f_0(x)$. This approximation implies the following:

$$\begin{aligned} |f_0(x)|^{-1/4} &= \nu^{-1/2}|x - x_0|^{-1/4}\left[1 + O\left(|x - x_0|\right)\right] &= \lambda^{1/6}\nu^{-1/3}z^{-1/4}\left[1 + O(\lambda^{-2/3}z)\right] \\ \int_{x_0}^{x} |f_0(s)|^{1/2}\, ds &= -\frac{2}{3}\nu|x - x_0|^{3/2} + O\left(|x - x_0|^{5/2}\right) &= -\frac{2}{3}\lambda^{-1}z^{3/2} + \lambda^{-1}O\left(\lambda^{-2/3}z^{5/2}\right) \end{aligned} \tag{6.85}$$

as $x \to x_0-$, and because we are in the overlap domain, $z \to +\infty$. Note that we are indicating an absolute error for the integral and a relative error for $|f_0(x)|^{-1/4}$; this is because when the integral is put into the exponent, the absolute error will become a relative error since $\exp(A + O(\epsilon)) = \exp(A)\exp(O(\epsilon)) = \exp(A)(1 + O(\epsilon))$ if $\epsilon \to 0$. Consequently for z corresponding to an x value to the left of the turning point and yet in the overlap domain, we get

$$\begin{aligned} y(x) \sim y_{\text{out}}^{\text{left}}(x) &= \frac{\lambda^{1/6}\nu^{-1/3}A_{\text{left}}}{z^{1/4}} e^{-2z^{3/2}/3}\left[1 + O(\lambda^{-2/3}z)\right]\left[1 + O(\lambda^{-2/3}z^{5/2})\right] \\ &\quad + \frac{\lambda^{1/6}\nu^{-1/3}B_{\text{left}}}{z^{1/4}} e^{2z^{3/2}/3}\left[1 + O(\lambda^{-2/3}z)\right]\left[1 + O(\lambda^{-2/3}z^{5/2})\right], \end{aligned} \tag{6.86}$$

as $z \to +\infty$ in the overlap domain. Since z is large, the product of relative errors may be replaced simply by $1 + O(\lambda^{-2/3}z^{5/2})$, and these errors will therefore all be negligible if $z = o(\lambda^{4/15})$. This is consistent with $z = o(\lambda^{2/3})$ as $\lambda \to \infty$, but it is nonetheless a restriction on the size of the overlap domain. Now looking at these formulae, we see that these expansions match the inner expansions as $z \to +\infty$ if the coefficients are connected by the relations

$$\begin{aligned} A_{\text{in}} &= 2\sqrt{\pi}\lambda^{1/6}\nu^{-1/3}A_{\text{left}} \\ B_{\text{in}} &= \sqrt{\pi}\lambda^{1/6}\nu^{-1/3}B_{\text{left}}\,. \end{aligned} \tag{6.87}$$

Next suppose that $x \to x_0+$, so we are examining the outer expansion $y_{\text{right}}(x)$. In this case the Taylor expansion of $f_0(x)$ near x_0 gives us

$$\begin{aligned} [f_0(x)]^{-1/4} &= \nu^{-1/2}(x - x_0)^{-1/4}\Big[1 + O\,(x - x_0)\Big] = \lambda^{1/6}\nu^{-1/3}|z|^{-1/4}\left[1 + O(\lambda^{-2/3}|z|)\right] \\ \int_{x_0}^{x} [f_0(s)]^{1/2}\,ds &= \frac{2}{3}\nu(x - x_0)^{3/2} + O\left((x - x_0)^{5/2}\right) = \frac{2}{3}\lambda^{-1}|z|^{3/2} + \lambda^{-1}O\left(\lambda^{-2/3}|z|^{5/2}\right) \end{aligned} \tag{6.88}$$

as $x \to x_0+$ and because we are in the overlap domain, $z \to -\infty$. Assuming as before that we are in an appropriate overlap domain so that $z = o(\lambda^{4/15})$, we can therefore rewrite the outer expansion as

$$\begin{aligned} y(x) \sim y_{\text{out}}^{\text{right}}(x) &= \frac{\lambda^{1/6}\nu^{-1/3}}{|z|^{1/4}}\left(A_{\text{right}}e^{-i\pi/4} + B_{\text{right}}e^{i\pi/4}\right)\left[\cos\left(\frac{2}{3}|z|^{3/2} + \frac{\pi}{4}\right) + O\left(\lambda^{-2/3}|z|^{5/2}\right)\right] \\ &\quad + \frac{\lambda^{1/6}\nu^{-1/3}}{|z|^{1/4}}\left(A_{\text{right}}e^{i\pi/4} + B_{\text{right}}e^{-i\pi/4}\right)\left[\sin\left(\frac{2}{3}|z|^{3/2} + \frac{\pi}{4}\right) + O\left(\lambda^{-2/3}|z|^{5/2}\right)\right], \end{aligned} \tag{6.89}$$

as $z \to -\infty$ in the overlap domain. Here we see that this expansion matches the inner expansion for $z \to -\infty$ if the coefficients are related by

$$\begin{aligned} A_{\text{in}} &= \sqrt{\pi}\lambda^{1/6}\nu^{-1/3}(A_{\text{right}}e^{i\pi/4} + B_{\text{right}}e^{-i\pi/4}) \\ B_{\text{in}} &= \sqrt{\pi}\lambda^{1/6}\nu^{-1/3}(A_{\text{right}}e^{-i\pi/4} + B_{\text{right}}e^{i\pi/4})\,. \end{aligned} \tag{6.90}$$

Taken together, the relations (6.87) and (6.90) effectively connect the coefficients of the outer solution on the left of the turning point with those of the outer solution on the right of the turning point. One simply eliminates A_{in} and B_{in} between these two sets of equations. But there is more information in these relations as well, because they explain the nature of the inner solution of Liouville's equation near the turning point. In particular, we see that the inner coefficients scale like $\lambda^{1/6}$ compared to the outer coefficients. This means that the inner solution, besides being compressed into a shrinking region of size approximately $\lambda^{-2/3}$, has an amplitude that grows like $\lambda^{1/6}$. This explains how the blowing up of the outer solutions at $x = x_0$ is fixed up by the equation restoring analyticity of the solution. The outer solutions blow up as x moves in toward the turning point, and when their amplitude becomes about $\lambda^{1/6}$ (which occurs when $|x - x_0|$ is about $\lambda^{-2/3}$) the Airy functions cap off the developing singularity analytically.

This analysis is appropriate when the turning point is simple, in the sense that $f_0'(x_0) \neq 0$, and also that there are not other turning points that are too nearby. In more degenerate situations, the precise statements made above do not apply, but the spirit of the approach still survives. Namely, when the outer solutions fail, one should rescale the problem in the neighborhood of the turning point(s) until a canonical model problem (here it was the Airy differential equation) emerges to describe the inner asymptotics. In the best cases the solutions of this model problem are well-understood and their asymptotic behavior can be found. This then permits the matching of the outer expansions across the turning point.

6.5.4 Problems with more than one turning point. The Bohr-Sommerfeld quantization rule.

Often the function $f_0(x)$ has several zeros, in which case analysis of the sort described above can be applied individually for each turning point. As an example of this kind of problem, consider the equation

$$-\frac{\hbar^2}{2}\psi''(x) + V(x)\psi(x) = E\psi(x)\,. \tag{6.91}$$

This is the stationary Schrödinger equation describing the states of fixed energy E for a quantum particle moving in the potential energy field $V(x)$. We suppose that $V(x)$ has the form of a potential well, with a single minimum, so that $V(x)$ is strictly decreasing to the left of the minimum and strictly increasing to the right.

One of the central problems of quantum mechanics is to determine the values of the energy E for which the Schrödinger equation has a solution that is in $L^2(\mathbb{R})$ as a function of x. Thus, we are studying an eigenvalue problem. Generally this is a very difficult problem. However, in a nearly classical situation, the parameter $\hbar$ may be taken to be small and it is an old and famous idea try to exploit this fact to obtain approximations for the eigenvalues E.

Let us fix E to be a number above the minimum value of $V(x)$ so that the function $V(x) - E$ has exactly two zeros, $x_-(E)$ to the left of the minimum and $x_+(E)$ to the right of the minimum. See the figure. Upon

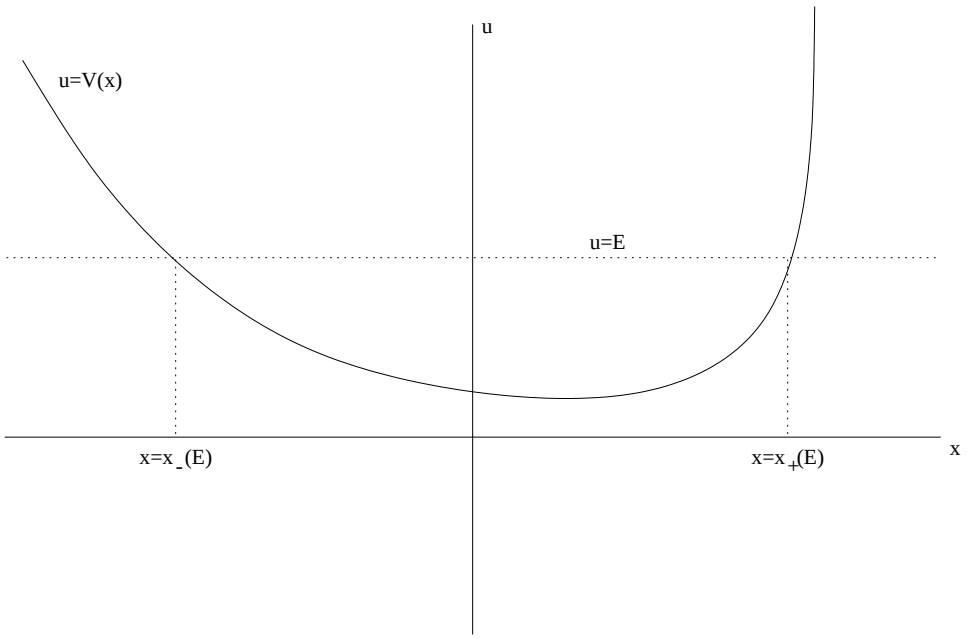

Figure 6.1: *The turning points $x_-(E)$ and $x_+(E)$ and the way they depend on the energy E.*

dividing the equation through by the coefficient of $\psi''(x)$, we see that we have exactly a Liouville equation for $\psi(x)$ with the large parameter $\lambda = 2/\hbar^2$ and with $f_0(x) := E - V(x)$. The two x-values $x_\pm(E)$ are turning

points for this problem. Since $V(x)$ is decreasing at $x_-(E)$, we see that $f_0(x)$ is increasing there so we have a turning point of exactly the type discussed previously. At the other turning point the function $f_0(x)$ is decreasing, and to solve the connection problem there we need to use a transformation of x to reverse left and right before we can apply directly the results we previously obtained.

Since we are only interested in solutions $\psi(x)$ that can be in $L^2(\mathbb{R})$, we must insist on exponential decay toward the left away from $x_-(E)$ and also toward the right away from $x_+(E)$. The number E will be an approximate eigenvalue of the problem if these two outer solutions match consistently through their respective turning points into the oscillatory region between the two turning points. So for some constant A_{left}, let us develop the unique solution $\psi_L(x)$ having the asymptotic representation as $\hbar \to 0$ for $x < x_-(E)$

$$\psi_L(x) = \frac{A_{\text{left}}}{(V(x)-E)^{1/4}} \exp\left(\frac{\sqrt{2}}{\hbar}\int_{x_-(E)}^{x} (V(s)-E)^{1/2}\,ds\right)[1+o(1)]\,. \tag{6.92}$$

Applying the connection formulae developed previously, we see that the solution $\psi_L(x)$ necessarily has an asymptotic description in the region $x_-(E) < x < x_+(E)$ between the two turning points of the form

$$\begin{aligned}\psi_L(x) &= \frac{A_{L,\text{mid}}}{(E-V(x))^{1/4}} \exp\left(\frac{i\sqrt{2}}{\hbar}\int_{x_-(E)}^{x} (E-V(s))^{1/2}\,ds\right) \\ &\quad + \frac{B_{L,\text{mid}}}{(E-V(x))^{1/4}} \exp\left(-\frac{i\sqrt{2}}{\hbar}\int_{x_-(E)}^{x} (E-V(s))^{1/2}\,ds\right) + o(1)\end{aligned} \tag{6.93}$$

as $\hbar \to 0$, where $A_{L,\text{mid}} = e^{-i\pi/4}A_{\text{left}}$ and $B_{L,\text{mid}} = e^{i\pi/4}A_{\text{left}}$.

Similarly, we can develop the unique solution $\psi_R(x)$ having an asymptotic representation as $\hbar \to 0$ for $x > x_+(E)$ of the form

$$\psi_R(x) = \frac{A_{\text{right}}}{(V(x)-E)^{1/4}} \exp\left(-\frac{\sqrt{2}}{\hbar}\int_{x_+(E)}^{x} (V(s)-E)^{1/2}\,ds\right)[1+o(1)] \tag{6.94}$$

for some constant A_{right}. A solution of the differential equation with this asymptotic behavior for $x > x_+(E)$ is necessarily exponentially decaying away from $x_+(E)$ to the right. This solution also has a representation for $x_-(E) < x < x_+(E)$ of the form

$$\begin{aligned}\psi_R(x) &= \frac{A_{R,\text{mid}}}{(E-V(x))^{1/4}} \exp\left(-\frac{i\sqrt{2}}{\hbar}\int_{x_+(E)}^{x} (E-V(s))^{1/2}\,ds\right) \\ &\quad + \frac{B_{R,\text{mid}}}{(E-V(x))^{1/4}} \exp\left(\frac{i\sqrt{2}}{\hbar}\int_{x_+(E)}^{x} (E-V(s))^{1/2}\,ds\right) + o(1)\,.\end{aligned} \tag{6.95}$$

The connection formulae giving $A_{R,\text{mid}}$ and $B_{R,\text{mid}}$ in terms of A_{right} are easily obtained from our prior results by a "flipping" change of variables. We find that $A_{R,\text{mid}} = e^{-i\pi/4}A_{\text{right}}$ and $B_{R,\text{mid}} = e^{i\pi/4}A_{\text{right}}$.

We can say that E represents an approximate eigenvalue of the problem if the two solutions $\psi_L(x)$ and $\psi_R(x)$, each of which is uniquely specified by a boundary condition at infinity outside the turning points, match each other to leading order in the oscillatory region between the turning points. To compare the two solutions in this region, we rewrite the integrals in the asymptotic description of $\psi_L(x)$ as follows:

$$\begin{aligned}\psi_L(x) &= \frac{A_{L,\text{mid}}e^{i\Phi(E)/\hbar}}{(E-V(x))^{1/4}} \exp\left(\frac{i\sqrt{2}}{\hbar}\int_{x_+(E)}^{x} (E-V(s))^{1/2}\,ds\right) \\ &\quad + \frac{B_{L,\text{mid}}e^{-i\Phi(E)/\hbar}}{(E-V(x))^{1/4}} \exp\left(-\frac{i\sqrt{2}}{\hbar}\int_{x_+(E)}^{x} (E-V(s))^{1/2}\,ds\right) + o(1)\end{aligned} \tag{6.96}$$

where the *phase integral* is given by

$$\Phi(E) := \sqrt{2} \int_{x_-(E)}^{x_+(E)} (E - V(s))^{1/2}\, ds\,. \tag{6.97}$$

By a direct comparison, we see that the asymptotic representations of $\psi_L(x)$ and $\psi_R(x)$ will match in the central region $x_-(E) < x < x_+(E)$, up to negligible terms of size $o(1)$ as $\hbar \to 0$, if

$$A_{\text{right}} = ie^{-i\Phi(E)/\hbar} A_{\text{left}} \quad \text{and} \quad A_{\text{right}} = -ie^{i\Phi(E)/\hbar} A_{\text{left}}\,. \tag{6.98}$$

These equations are overdetermined and will only be consistent if

$$\Phi(E) = \pi\hbar \left(n + \frac{1}{2}\right) \tag{6.99}$$

where $n = 0, 1, 2, 3, 4, \ldots$. This should be viewed as a condition that assigns to each nonnegative integer n an approximate eigenvalue $E = E_n$. The approximate eigenvalues E_n form an increasing sequence. In principle you find E_n by inverting the function Φ.

The relation (6.99) is one of the most famous formulae in physics and applied mathematics. It is called the *Bohr-Sommerfeld quantization rule*. It says that the approximate eigenvalues E correspond to the phase integral taking half-integer multiples of $\pi\hbar$. As $\hbar$ goes to zero, the eigenvalues become more and more densely packed, as the average distance between eigenvalues scales like $\hbar$. The phase integral has an interpretation in classical mechanics as the *action* associated with the periodic orbit in the potential well (imagine a ball rolling from one side of the well to the other) of energy E. The oldest form of quantum mechanics, long before the age of wave functions and the Schrödinger equation, was a modified version of classical mechanics invented by Niels Bohr. The only essential modification was to say that a periodic orbit in a classical mechanical system could only have values of action equal to a half-integer multiple of $\pi\hbar$. Here we see how this ad-hoc idea of Bohr appears naturally in the later version of quantum mechanics based on waves. Bohr's quantized action values correspond to approximate energy eigenvalues of a boundary value problem for "matter waves".

Chapter 7

Perturbation Methods for Boundary Value Problems

7.1 Singular Perturbation Theory for Second-Order Linear Equations

To develop the main ideas, we will consider a finite real interval $x \in [\alpha, \beta]$ and a two-point boundary value problem of the form

$$\begin{aligned} &\epsilon \frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = 0 \qquad \text{for} \qquad x \in (\alpha, \beta) \\ &\text{boundary conditions:} \qquad y(\alpha) = A\,, \qquad y(\beta) = B\,, \end{aligned} \tag{7.1}$$

where A and B are numbers. Here, $a(x)$ and $b(x)$ are real-valued continuous functions and we are seeking a "classical" solution that is twice differentiable in the interior and that extends continuously to the endpoints. In fact we will also take $a(x)$ to be differentiable in what follows below.

The parameter ϵ is taken to be positive, and we are interested in the possibility of the way that the solution (if it exists and is unique) depends on ϵ as it tends to zero through positive values. This is a singular perturbation problem because the *reduced equation* that we obtain by setting $\epsilon = 0$ is of the first order rather than second order. Usually this reduced equation does not have enough free integration constants in its solution to satisfy the given boundary conditions, even approximately. Therefore the reduced equation on its own gives a very poor model of the true nature of solutions to the boundary value problem.

7.1.1 Asymptotic existence of solutions.

The first question to ask is whether (7.1) has any solutions at all, and whether they are unique. The main result we need from the general theory of boundary value problems is the following.

Theorem 5 (Fredholm alternative) *The boundary value problem (7.1) has a unique solution if and only if the same problem with $A = 0$ and $B = 0$ has only the trivial solution of $y(x) \equiv 0$.*

The boundary value problem (7.1) with $A = B = 0$ is called the *homogeneous problem*, and any solution of the homogeneous problem is called a *homogeneous solution*. Clearly the zero function is always a homogeneous solution; the issue is whether there are any homogeneous solutions that are not identically zero.

The uniqueness part of the theorem is easy to understand. Suppose there were a nontrivial (not identically zero) homogeneous solution, say $y_0(x) \not\equiv 0$. If we were lucky enough to find any solution $y(x)$ of (7.1) for more general A and B, then the function $y(x)+y_0(x)$ would be another distinct solution of the same problem. The existence part of the theorem is harder and relies on the theory of integral equations of Fredholm type.

Now we want to determine whether our boundary value problem (7.1) has a unique solution at least for $\epsilon > 0$ but small enough. This means we are looking for homogeneous solutions $y_0(x)$. A convenient first step

is to put the problem in a more standard form like we did back when we discussed a similar looking equation in the case of rational coefficients (here the coefficients need only be continuous, and we are concerned only with real x). So, make the change of variable:

$$y_0(x) = w(x) \exp\left(-\frac{1}{2\epsilon}\int_\alpha^x a(s)\,ds\right). \tag{7.2}$$

Since $a(x)$ is continuous, it is integrable over the finite interval (α,β) and therefore the exponential factor is never zero and is always finite. Consequently, the homogeneous boundary conditions $y_0(\alpha) = y_0(\beta) = 0$ imply $w(\alpha) = w(\beta) = 0$. While leaving the boundary conditions invariant, the transformation changes the form of the differential equation:

$$\epsilon\frac{d^2w}{dx^2} - V(x;\epsilon)w = 0\,, \qquad \text{where} \qquad V(x;\epsilon) := \frac{a(x)^2}{4\epsilon} + \frac{a'(x)}{2} - b(x)\,. \tag{7.3}$$

Since $y_0(x) \equiv 0$ is the same thing as $w(x) \equiv 0$, if we can show that all homogeneous solutions of this modified boundary value problem for $w(x)$ are trivial, we will have shown the same thing for the boundary value problem of interest, namely (7.1).

Multiply (7.3) through by $w(x)$ and integrate from α to β:

$$\epsilon\int_\alpha^\beta w(s)\frac{d^2w}{ds^2}(s)\,ds - \int_\alpha^\beta V(s;\epsilon)w(s)^2\,ds = 0\,. \tag{7.4}$$

Now we can integrate by parts in the first integral:

$$\epsilon w(\beta)\frac{dw}{dx}(\beta) - \epsilon w(\alpha)\frac{dw}{dx}(\alpha) = \epsilon\int_\alpha^\beta\left[\frac{dw}{ds}(s)\right]^2 ds + \int_\alpha^\beta V(s;\epsilon)w(s)^2\,ds\,. \tag{7.5}$$

Due to the homogeneous boundary conditions on $w(x)$, the boundary terms vanish. We have a sum of two terms equal to zero. The first term is clearly nonnegative when $\epsilon > 0$. Suppose now we knew that

$$\text{Positivity condition:} \qquad V(x;\epsilon) \geq 0 \qquad \text{whenever} \qquad x \in [\alpha,\beta]\,. \tag{7.6}$$

Then, the second term would be nonnegative as well, and since the sum is zero, each term would have to be zero individually. Therefore, for $\epsilon > 0$ however small, the positivity condition implies

$$\int_\alpha^\beta\left[\frac{dw}{ds}(s)\right]^2 ds = 0 \tag{7.7}$$

This in turn implies that $dw/dx \equiv 0$ for all x in the interval (α,β). Taking into account the boundary conditions, we see that in fact $w(x) \equiv 0$. So, if the positivity condition holds, all homogeneous solutions $y_0(x)$ are trivial, and our boundary value problem (7.1) has a unique solution.

Whether the positivity condition is satsified by the function $V(x;\epsilon)$ is a delicate issue in general, depending on the detailed features of the functions $a(x)$ and $b(x)$. But if we fix the functions $a(x)$ and $b(x)$, and let ϵ tend toward zero through positive values, we see that for all $\epsilon > 0$ sufficiently small, the first term will dominate the latter two (which are missing the growing factor of ϵ^{-1}) for all x values where $a(x) \neq 0$. *Consequently, if the coefficient $a(x)$ never vanishes over the closed interval $[\alpha,\beta]$, then there exists some $\epsilon_0 > 0$ such that taking $0 < \epsilon < \epsilon_0$ guarantees positivity of $V(x;\epsilon)$ and therefore solvability of the boundary value problem (7.1).* Since the limit of interest is ϵ going to zero, this is often good enough for us.

The condition that $a(x)$ is never zero is sufficient for existence and uniqueness of solutions, but it is not always necessary. For example, another condition that is also sufficient for existence and uniqueness but that admits zeros of $a(x)$ is simply that

$$\frac{a'(x)}{2} - b(x) \geq 0 \qquad \text{whenever} \qquad x \in [\alpha,\beta]\,. \tag{7.8}$$

If this condition holds then the positivity condition holds for all $\epsilon > 0$, not just for small values. The boundary value problem (7.1) has a unique solution in this case as well. This condition is important because changes of sign in $a(x)$ can model transitional phenomena in nature.

7.1.2 Phenomenology of boundary layers.

Let us recall a problem we first looked at in Chapter 0. Consider the boundary value problem

$$\epsilon \frac{d^2y}{dx^2} + (1-\epsilon)\frac{dy}{dx} - (1-\epsilon)y = 0\,, \qquad y(0) = y(1) = 1\,. \tag{7.9}$$

This is a problem nearly of the type described above, with some added dependence in the functions $a(x)$ and $b(x)$ on ϵ. Since $a'(x) \equiv 0$ and $b(x)$ is negative as long as $\epsilon < 1$, this problem has a unique solution for all positive $\epsilon < 1$. This problem is unusual because the differential equation is linear and has constant coefficients, and consequently we can find its solution exactly by plugging in exponentials. Substituting $y = e^{mx}$ into the equation we find that we will have found a solution provided that m satisfies the quadratic equation

$$\epsilon m^2 + (1-\epsilon)m - (1-\epsilon) = 0\,. \tag{7.10}$$

Using the quadratic formula, we solve for m and find two solutions:

$$m_+(\epsilon) = \frac{1}{2\epsilon}\left(\epsilon - 1 + \sqrt{1+2\epsilon-3\epsilon^2}\right)\,, \qquad m_-(\epsilon) = \frac{1}{2\epsilon}\left(\epsilon - 1 - \sqrt{1+2\epsilon-3\epsilon^2}\right)\,. \tag{7.11}$$

The general solution of our equation is therefore a linear combination of these two exponentials with arbitrary coefficients. Picking the coefficients to satisfy the boundary conditions $y(0) = y(1) = 1$ results in the unique solution formula:

$$y(x;\epsilon) = \frac{(e^{m_-(\epsilon)} - 1)e^{m_+(\epsilon)x} + (1 - e^{m_+(\epsilon)})e^{m_-(\epsilon)x}}{e^{m_-(\epsilon)} - e^{m_+(\epsilon)}}\,. \tag{7.12}$$

Strictly speaking, we have found in this formula the unique solution of the boundary value problem for each $\epsilon > 0$. It is not obvious to the eye what these solutions look like, or how they behave, even though they are exact. However, using this formula we can apply a limit process to determine more usefully how $y(x,\epsilon)$ behaves as $\epsilon \to 0$. First, let us fix x and seek an expansion in powers of ϵ:

$$y(x;\epsilon) = e^{x-1} - \epsilon(x-1)e^{x-1} + O(\epsilon^2)\,. \tag{7.13}$$

The approximations $y_{\rm out}(x;\epsilon) := e^{x-1} - \epsilon(x-1)e^{x-1}$ obtained by keeping only the two explicit terms shown above are plotted along with the actual functions $y(x;\epsilon)$ for several values of ϵ in Figure 7.1. The approximations are clearly better and better as $\epsilon \to 0$ for each fixed x. They hit the boundary condition at $x = 1$ on the nose but get the boundary condition at $x = 0$ totally wrong. At the same time that these approximations are accurate for fixed x, there is another part of the picture near $x = 0$ that is totally missed by these *outer approximations*, which naively assume that x is held fixed as $\epsilon \to 0$. This "missing" part of the solution, where the true solution $y(x;\epsilon)$ changes rapidly near the left boundary at $x = 0$ is what we mean by a *boundary layer*.

A more accurate asymptotic description of the exact solution in the boundary layer can be obtained using a different sort of asymptotic expansion, an *inner expansion*. If we look at the exact solution formula (7.12) for $y(x;\epsilon)$, and take into account what the exponents $m_\pm(\epsilon)$ are, we see the ubiquitous appearance of a variable $\xi := x/\epsilon$. The inner expansion is obtained by substituting $x = \epsilon\xi$ into the exact solution formula (7.12) and finding the asymptotic expansion of $y(x;\epsilon)$ as $\epsilon \to 0$ with ξ held fixed this time. It takes the form

$$y(x;\epsilon) = \left(1 - \frac{1}{e}\right)e^{-\xi} + \frac{1}{e} + \epsilon\frac{\xi + 1 - e^{-\xi}}{e} + O(\epsilon^2)\,. \tag{7.14}$$

The inner approximations, here $y_{\rm in}(x;\epsilon) := (1-e^{-1})e^{-\xi} + e^{-1} + \epsilon e^{-1}(\xi + 1 - e^{-\xi})$, are in terms of a *rescaled variable* $\xi = x/\epsilon$ that acts like a magnifying glass blowing up the boundary layer behavior near $x = 0$. They are compared with the exact solution $y(x;\epsilon)$ in Figure 7.2. The inner approximations get the boundary condition at $x = 0$ right, but do not satisfy the boundary condition at $x = 1$.

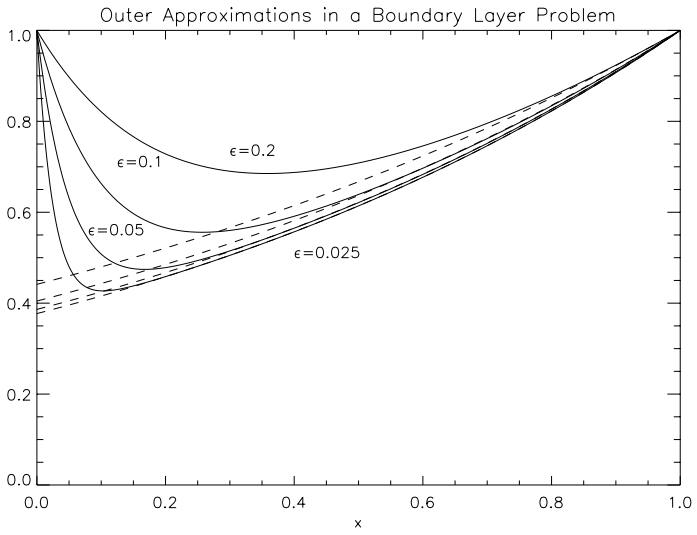

Figure 7.1: *Outer approximations in a boundary layer problem. The dashed curves correspond to* $y_{\rm out}(x;\epsilon)$ *for several values of* ϵ*. The solid curves correspond to the exact solution* $y(x;\epsilon)$ *for the same values of* ϵ*.*

7.1.3 Naive expansions and outer solutions.

We want to start considering how to take advantage of the smallness of the parameter ϵ in the boundary value problem (7.1) to find asymptotic information about the unique solution. Note that only for the most special cases of functions $a(x)$ and $b(x)$ is it possible to solve (7.1) exactly, either by elementary functions or in terms of contour integrals. So we have to have a technique for finding all the sorts of details described in the specific example analyzed above directly from the boundary value problem, without the luxury of an exact solution formula.

The obvious perturbation procedure is to consider the ordinary differential equation we are studying to be a small perturbation of the reduced problem obtained by setting $\epsilon = 0$. The reduced problem is thus:

$$a(x)\frac{dy^0}{dx} + b(x)y^0 = 0 \qquad \text{(Reduced Problem)}\,. \tag{7.15}$$

Its general solution is obtained by finding the explicit integrating factor. So we get

$$y^0(x) = C\exp\left(-\int_\alpha^x \frac{b(s)}{a(s)}\,ds\right) \tag{7.16}$$

where C is an arbitrary constant of integration. The approximation of dropping the term $\epsilon d^2y/dx^2$ will be a good one near x-values where $y(x)$ has a second derivative that is bounded uniformly as $\epsilon \to 0$. At the moment, we do not know whether this will be the case at a given x. But let us assume it to be true and continue with higher-order corrections. We can generate an asymptotic series for $y(x)$ whose first term is $y^0(x)$ by determining the functions in the corresponding sequence $\{g_n(\epsilon)\}$ systematically by the method of dominant balances. In this case, it is easy to see that the correct sequence is $g_n(\epsilon) := \epsilon^n$ for $n = 0, 1, 2, \ldots$. Seeking an asymptotic expansion

$$y(x) \sim \sum_{n=0}^{\infty} \epsilon^n y^n(x) \qquad \text{as} \qquad \epsilon \to 0\,, \tag{7.17}$$

additionally assuming that

$$\frac{dy}{dx} \sim \sum_{n=0}^{\infty} \epsilon^n \frac{dy^n}{dx} \qquad \text{and} \qquad \frac{d^2y}{dx^2} \sim \sum_{n=0}^{\infty} \epsilon^n \frac{d^2y^n}{dx^2} \qquad \text{as} \qquad \epsilon \to 0\,, \tag{7.18}$$

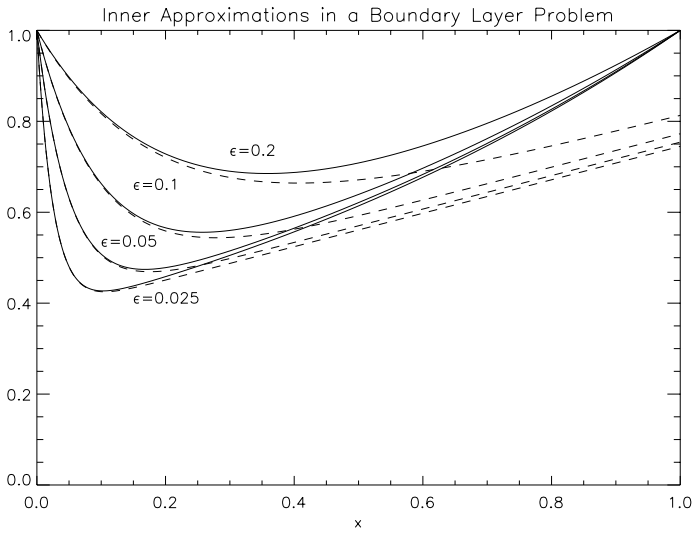

Figure 7.2: *Inner approximations in a boundary layer problem. The dashed curves correspond to* $y_{\rm in}(x;\epsilon)$ *for several values of* ϵ*. The solid curves correspond to the exact solution* $y(x;\epsilon)$ *for the same values of* ϵ*.*

and applying the limit process order-by-order, we get a sequence of equations for the coefficients $y^n(x)$. The leading coefficient we already found. The remaining coefficients satisfy the recursion relation

$$a(x)\frac{dy^n}{dx} + b(x)y^n = -\frac{d^2y^{n-1}}{dx^2} \qquad \text{for} \qquad n = 1, 2, 3, \ldots . \tag{7.19}$$

At each stage the right-hand side is known explicitly from the previous stage (and two differentiations). Each equation of the hierarchy is therefore easily solved for the unknown $y^n(x)$ as before, by using an integrating factor.

If the left-hand boundary $x = \alpha$ is included in the region of validity of this outer expansion (we still don't know how to tell whether this is the case) then it must satisfy the appropriate boundary condition. Since the boundary condition $y(\alpha) = A$ is independent of ϵ, this means that we must take

$$y^0(\alpha) = A \qquad \text{and} \qquad y^n(\alpha) = 0 \qquad \text{for} \qquad n = 1, 2, 3, \ldots . \tag{7.20}$$

Since each coefficient $y^n(x)$ is determined as the solution of a first-order linear equation, there is exactly one constant of integration at each order and therefore the boundary conditions (7.20) at $x = \alpha$ determine the asymptotic expansion completely and unambiguously.

Similarly, if the right-hand boundary $x = \beta$ is in the region of validity of the outer expansion, then each coefficient function must satisfy a boundary condition at $x = \beta$:

$$y^0(\beta) = B \qquad \text{and} \qquad y^n(\beta) = 0 \qquad \text{for} \qquad n = 1, 2, 3, \ldots . \tag{7.21}$$

These conditions would also determine all coefficients in the asymptotic expansion for the same reason.

Although it does happen occasionally, it is truly unlikely that an expansion matching the boundary conditions at $x = \alpha$ — which has no additional free parameters — would coincidentally also satisfy the boundary conditions at $x = \beta$, and vice-versa. In the generic situation, then, one of two things is happening:

1. Either the left endpoint $x = \alpha$ or the right endpoint $x = \beta$ fails to be in the region of validity of the outer asymptotic expansion due to the presence of a boundary layer. There may be boundary layers at both boundaries in some problems.

2. Both endpoints are in isolated regions of validity of distinct outer asymptotic expansions. In this case there is a region of sharp transition that is not attached to any boundary at all, but is in the interior of the interval (α, β). This transition is called an *internal layer*.

Now we move on to discuss how to find the boundary or internal layers and how to determine the asymptotic expansion for the unique solution of (7.1) in such layers.

7.1.4 Inner solutions. Boundary layers and internal layers.

Boundary and internal layers are in general places where the outer expansion technique of holding x fixed and sending ϵ to zero fails to capture the behavior of the true unique solution of the boundary value problem (7.1) because the latter has large derivatives, and the term $\epsilon d^2y/dx^2$ cannot be neglected. Now we can further argue on this basis that boundary and internal layers are "thin" in the sense that they describe the solution near certain fixed x-values lying in intervals that are shrinking in size as $\epsilon \to 0$. If somehow $\epsilon d^2y/dx^2$ were not negligible over some subinterval of (α, β) of fixed size as $\epsilon \to 0$, then both dy/dx and y would also have to become of size ϵ^{-1} somewhere over this interval (just integrate to see this). But then the terms retained in the reduced equation (which would be $O(\epsilon^{-1})$) would again dominate over the term $\epsilon d^2y/dx^2$ (which would be $O(1)$). So the second derivative term can only be important in thin layers, that shrink in size as ϵ tends to zero.

So since a layer always occurs near a distinguished point, say $x_0 \in [\alpha, \beta]$, and is always "thin", we can guess that its thickness is proportional to some function of ϵ, say $\delta(\epsilon)$ that is $o(1)$ as $\epsilon \to 0$. Without knowing what x_0 is or what the thickness $\delta(\epsilon)$ is, we can make a change of variables that zooms in on the layer:

$$z = \frac{x - x_0}{\delta(\epsilon)} \qquad \text{or} \qquad x = x_0 + \delta(\epsilon) z \,. \tag{7.22}$$

Thus, when x differs from x_0 by about $\delta(\epsilon)$, we get that z is about one in magnitude. And if x is held fixed, then z must go to infinity like $\delta(\epsilon)^{-1}$ as $\epsilon \to 0$ to "catch up". The variable z is known as an *inner variable* because it tunes over an order one range through the inside of the shrinking layer. By analogy, the original independent variable x is often called the *outer variable*. Making this change of variables in the differential equation of the boundary value problem (7.1), we get, for a function $Y(z) = y(x) = y(x_0 + \delta(\epsilon)z)$,

$$\frac{\epsilon}{\delta(\epsilon)^2} \frac{d^2Y}{dz^2} + \frac{a(x_0 + \delta(\epsilon)z)}{\delta(\epsilon)} \frac{dY}{dz} + b(x_0 + \delta(\epsilon)z)Y = 0 \,. \tag{7.23}$$

We suppose that the functions $a(x)$ and $b(x)$ have asymptotic expansions as $x \to x_0$. Since both of these functions are being assumed to be continuous here, the leading terms of these expansions are constants or smaller. So $a(x_0 + \delta(\epsilon)z) = a(x_0) + o(1)$ and $b(x_0 + \delta(\epsilon)z) = b(x_0) + o(1)$ where the terms $o(1)$ here are small as $x \to x_0$. If z is held fixed, they are also small as $\epsilon \to 0$, which implies $\delta(\epsilon) \to 0$ by assumption. The various sorts of expansions that appear depend on the nature of the functions at $x = x_0$. For example, if $a(x)$ is analytic at x_0, then it has a convergent (and therefore asymptotic) power series expansion there, so as $\epsilon \to 0$ with z fixed we will have

$$a(x_0 + \delta(\epsilon)z) \sim a(x_0) + a'(x_0)\delta(\epsilon)z + \frac{1}{2}a''(x_0)\delta(\epsilon)^2 z^2 + \ldots \,. \tag{7.24}$$

Now we study the rescaled differential equation (7.23) with z held fixed. We assume that $\delta(\epsilon) \to 0$ as $\epsilon \to 0$ and note that the hallmark of being in a layer of rapid transition is that the term with d^2Y/dz^2 must be included in any dominant balance. So this term can balance with one or the other of the remaining terms (or both). Suppose first that $a(x_0) \neq 0$ and that $b(x_0) \neq 0$. Then if we balance the first and third terms of (7.23) we find that $\delta(\epsilon) = \epsilon^{1/2}$. But this is not a dominant balance because the second term is then $O(\epsilon^{-1/2})$ while the balancing terms are only $O(1)$. On the other hand, if we balance the first and second terms of (7.23) we find that $\delta(\epsilon) = \epsilon$, which is a dominant balance. If there is a layer at such an x_0, it is necessarily of "thickness" ϵ. The same balance is dominant if $b(x_0) = 0$, as long as we still have $a(x_0) \neq 0$, as is easily seen.

So continuing with the case $a(x_0) \neq 0$, we should take $\delta(\epsilon) = \epsilon$ and seek an asymptotic expansion for Y with z held fixed (which corresponds to x inside the layer). The reduced equation ($\epsilon = 0$) for Y is

$$\frac{d^2Y^0}{dz^2} + a(x_0)\frac{dY^0}{dz} = 0 \,. \tag{7.25}$$

This is a constant coefficient problem whose general solution is easily obtained. Namely,

$$Y^0(z) = c_1 + c_2 e^{-a(x_0)z}, \tag{7.26}$$

where c_1 and c_2 are arbitrary integration constants. The general solution of the reduced inner equation is therefore exponentially decaying to the right if $a(x_0) > 0$, and to the left if $a(x_0) < 0$. On the other hand, it is exponentially growing in either one direction or the other. Now, it is not feasible to have an inner layer solution that grows exponentially out of a layer; the solution near the point x_0 would then become exponentially large over a short layer thickness, which cannot be matched onto any reasonable outer solution of the sort that we obtained previously. This means that

1. If $a(x_0) > 0$, then there can only be a layer at x_0 if $x_0 = \alpha$, the left boundary point. The solution decays to the right, out of the boundary layer.

2. If $a(x_0) < 0$, then there can only be a layer at x_0 if $x_0 = \beta$, the right boundary point. The solution decays to the left, out of the boundary layer.

3. The only way that there can be an internal layer at x_0 is if $a(x_0) = 0$, in which case a different scaling of the layer may be required to achieve a dominant balance in the equation (7.23) for $Y(z)$.

Once the appropriate scaling has been determined that resolves the layer, a complete asymptotic expansion of the inner solution $Y(z)$ can be developed, starting with the leading term solving the reduced inner equation. If the layer is a left boundary layer at $x_0 = \alpha$, then the inner asymptotic expansion of the solution in the boundary layer must satisfy the corresponding boundary conditions, namely since $z = 0$ corresponds to $x = \alpha$,

$$Y^0(0) = A \quad \text{and} \quad Y^n(0) = 0 \quad \text{for} \quad n = 1, 2, 3, \ldots . \tag{7.27}$$

Similarly, if the layer is a right boundary layer at $x_0 = \beta$, then the boundary conditions are

$$Y^0(0) = B \quad \text{and} \quad Y^n(0) = 0 \quad \text{for} \quad n = 1, 2, 3, \ldots . \tag{7.28}$$

Note that in the latter case we are using the inner solutions for $z \leq 0$ while in the former case we are using them for $z \geq 0$. If the layer is internal, then we get no information directly from the boundary conditions and we need the solution for both positive and negative z. Also, notice that whereas an outer expansion valid near a boundary is completely determined by the boundary conditions (no free integration constants), boundary layer expansions still involve one undetermined integration constant at each order, even after satisfying the boundary condition. This is because each differential equation in the hierarchy determining the expansion coefficients of $Y(z)$ in a layer is always of second order instead of first order.

7.1.5 Matching of asymptotic expansions. Intermediate variables. Uniformly valid asymptotics.

We have seen how to isolate the various boundary and/or internal layers, and how to obtain asymptotic expansions of the most general solutions of the governing differential equation for (7.1) valid in different regions and consistent with the boundary conditions at $x = \alpha$ and $x = \beta$. There now comes the question of how to determine the remaining integration constants so that in each region the corresponding expansion is uniquely determined and represents the true unique solution of the boundary value problem (7.1) as $\epsilon \to 0$.

Consider a point x_0, possibly a boundary point, where there is known to be a layer. In the layer, there is an inner variable z and an asymptotic sequence $\{1, \mu_1(\epsilon), \mu_2(\epsilon), \ldots\}$ such that the solution of the boundary value problem (7.1) is asymptotically represented in the form

$$y(x) \sim Y^0(z) + \sum_{n=1}^{\infty} \mu_n(\epsilon) Y^n(z) \quad \text{as} \quad \epsilon \to 0 \quad \text{with } z \text{ held fixed}. \tag{7.29}$$

As described above and as will be made clear in several examples, the particular functions $\mu_n(\epsilon)$ must be chosen in accordance with the principle of dominant balance using asymptotic expansions of $a(x)$ and $b(x)$

as $x \to x_0$. For left boundary layers or internal layers, the same solution of the boundary value problem (7.1) has an outer representation for $x > x_0$ fixed, which is always an asymptotic sequence in powers of ϵ of the form

$$y(x) \sim \sum_{n=0}^{\infty} \epsilon^n y^n(x) \quad \text{as} \quad \epsilon \to 0 \quad \text{with } x \text{ held fixed}. \tag{7.30}$$

We want to see how to make these expansions, which are supposed to represent the same function, match up with each other. The main idea is to introduce an *intermediate variable*, say w, that somehow can't decide whether it is in the layer or not. Concretely, let w be defined by

$$w = \frac{x - x_0}{\chi(\epsilon)} = \frac{\delta(\epsilon)}{\chi(\epsilon)} z \tag{7.31}$$

so it is concretely related to both the inner and outer variables as soon as we pick the scale $\chi(\epsilon)$. In order that w be intermediate, we want $x \to x_0$ when $\epsilon \to 0$ with w fixed, which forces us to select $\chi(\epsilon)$ to satisfy

$$\chi(\epsilon) \to 0 \quad \text{as} \quad \epsilon \to 0\,. \tag{7.32}$$

But at the same time, we want $z \to \infty$ when $\epsilon \to 0$ with w held fixed, which forces the condition

$$\frac{\delta(\epsilon)}{\chi(\epsilon)} \to 0 \quad \text{as} \quad \epsilon \to 0\,. \tag{7.33}$$

These two conditions limit the intermediate scale $\chi(\epsilon)$. It has to go to zero with ϵ, but not as fast as the inner scale, $\delta(\epsilon)$.

The procedure of matching the two expansions is to truncate both at some finite number of terms, and change the variable from z to w in the truncated inner expansion and from x to w in the truncated outer expansion. The resulting expressions are then each re-expanded in the limit $\epsilon \to 0$ with w fixed. To order the terms correctly, it becomes important to place further restrictions on the intermediate scale $\chi(\epsilon)$, and typically one finds more restrictions as one tries to match more terms. The two expansions, thus expressed in terms of the same variable, are said to *match* if they have the same form. When the expansions match, their common form is an asymptotic solution of the differential equation valid as $\epsilon \to 0$ with w held fixed. It is called $y_{\text{match}}(w)$.

The matching of expansions allows one to relate the constants of integration across the various layers, from the left boundary all the way to the right. When the expansions have been matched, each represents the true unique solution of the boundary value problem (7.1) asymptotically as $\epsilon \to 0$. Often one is only interested in the solution in one particular region of the interval $[\alpha, \beta]$, but nonetheless to determine the solution here it is necessary to do the matching everywhere. The boundary conditions influence the solution in a nonlocal way, which is why boundary value problems are harder than initial value problems.

Sometimes, however, one is interested in an approximation to the solution of the boundary value problem (7.1) that is uniformly valid over the whole interval $[\alpha, \beta]$. For this purpose, it does not suffice to give different formulae in different layers, so a single formula should be constructed. Consider a simple case, that of a single boundary layer at $x_0 = \alpha$. Then, there is one inner expansion, and one outer expansion that must be matched to determine the solution in the boundary layer. A uniformly valid approximation to leading order is given by

$$y(x) \sim y^0(x) + Y^0(z) - y^0_{\text{match}}(w)\,, \tag{7.34}$$

in which we substitute explicitly what w and z are in terms of x. This formula is uniformly close to $y(x)$ as ϵ tends to zero. To see this, notice that if we are in the boundary layer, so that z is fixed, then w is going to zero with ϵ so the combination $y^0(x) - y^0_{\text{match}}(w)$ vanishes and the inner asymptotic solution remains. Likewise, if we are outside the boundary layer then x is fixed and therefore w and z are both going to infinity such that the combination $Y^0(z) - y^0_{\text{match}}(w)$ goes to zero with ϵ, leaving behind the outer solution $y^0(x)$.

7.1.6 Examples.

First Example: As a first example, consider the boundary value problem (7.1) on the interval $[\alpha, \beta] = [-1, 1]$ with $a(x) = 1 + x^2$ and $b(x) = x$. The boundary conditions are $y(-1) = 0$ and $y(1) = 2$. This problem

has a unique solution for all sufficiently small ϵ because $a(x) \geq 1 > 0$ on $[-1,1]$. For the same reason, the only possible layer structure is a boundary layer near $x_0 = -1$. First let us find the outer solution, which should be valid as $\epsilon \to 0$ for x fixed and positive. Setting

$$y(x) \sim y^0(x) + \epsilon y^1(x) + \dots , \tag{7.35}$$

plugging into the differential equation (7.1) and collecting powers of ϵ, we find first that

$$y^0(x) = C \exp\left(- \int_1^x \frac{s}{1+s^2}\, ds \right) = C \sqrt{\frac{2}{1+x^2}}\,. \tag{7.36}$$

For the first correction $y^1(x)$, we obtain the following equation:

$$(1+x^2)\frac{dy^1}{dx} + xy^1 = -C \frac{d^2}{dx^2}\sqrt{\frac{2}{1+x^2}}\,. \tag{7.37}$$

The solution of this equation is

$$y^1(x) = \frac{C}{32}\sqrt{\frac{2}{1+x^2}}\left(\frac{24x}{(1+x^2)^2} + \frac{4x}{(1+x^2)} + 4\arctan(x) + 32D - 8 - \pi \right), \tag{7.38}$$

where D is another integration constant. The process can be continued to higher and higher order if so desired. Now, since there can only be a boundary layer at $x = -1$, the outer expansion must take care of the boundary condition at $x = 1$. This determines the constants C and D, and in fact all others that arise in the outer expansion at higher order. The condition $y^0(1) = 2$ requires that we choose $C = 2$, and the condition $y^1(1) = 0$ (similar homogeneous boundary conditions are satisfied by all higher order corrections) requires that we choose $D = 0$. So the outer expansion is completely determined by the differential equation and the boundary condition at $x = 1$.

At the left boundary, the boundary condition $y(-1) = 1$ is not satisfied by the outer expansion, since $y^0(-1) = 2$ and $y^1(-1) = (8-\pi)/16$. So we consider finding an inner expansion near the left boundary. The asymptotic expansions of $a(x)$ and $b(x)$ as $x \to -1$ are analytic and finite because both functions are polynomials.

$$a(x) = 2 - 2(x+1) + (x+1)^2 \quad \text{and} \quad b(x) = -1 + (x+1)\,. \tag{7.39}$$

In terms of the inner variable $z = (x+1)/\epsilon$, the differential equation becomes exactly

$$\frac{d^2Y}{dz^2} + \left[2 - 2\epsilon z + \epsilon^2 z^2\right]\frac{dY}{dz} + \left[-\epsilon + \epsilon^2 z\right] Y = 0\,. \tag{7.40}$$

Inserting the asymptotic expansion

$$Y(z) \sim Y^0(z) + \epsilon Y^1(z) + \dots \tag{7.41}$$

and collecting powers of ϵ gives first of all, for $Y^0(z)$, the reduced equation

$$\frac{d^2Y^0}{dz^2} + 2\frac{dY^0}{dz} = 0\,. \tag{7.42}$$

The general solution is simply

$$Y^0(z) = c_1 + c_2 e^{-2z}\,, \tag{7.43}$$

where c_1 and c_2 are the two integration constants. For $Y^1(z)$, we find the equation

$$\frac{d^2Y^1}{dz^2} + 2\frac{dY^1}{dz} = 2z\frac{dY^0}{dz} + Y^0\,. \tag{7.44}$$

The solution of this equation can be obtained easily using variation of parameters, given our explicit knowledge of $Y^0(z)$. We obtain

$$Y^1(z) = \left[\frac{c_1}{4}(2z-1) + d_1\right] + \left[\frac{c_2}{4}(4z^2+2z+1) + d_2\right] e^{-2z}\,, \tag{7.45}$$

where d_1 and d_2 are two new integration constants. This process can be continued indefinitely, generating higher and higher order corrections. Now the inner expansion is against the left boundary and so must satisfy the boundary condition at $x = -1$ which corresponds to $z = 0$. The boundary condition $Y^0(0) = 0$ requires that $c_1 + c_2 = 0$, which fails to completely determine $Y^0(z)$. Similarly, the boundary condition $Y^1(0) = 0$ requires that $d_1 + d_2 = c_1/2$, which fails to completely determine $Y^1(z)$. At each stage, the boundary condition at $x = -1$ will determine one of two free integration constants, leaving one constant undetermined at each order.

So to determine the remaining constants, we have to match the inner and outer expansions. Let $\chi(\epsilon)$ be a scale that lies asymptotically between ϵ and 1. Often we write this as $\epsilon \ll \chi(\epsilon) \ll 1$ which means exactly that as $\epsilon \to 0$ we have both $\chi(\epsilon) \to 0$ and $\epsilon/\chi(\epsilon) \to 0$. Introduce the intermediate variable $w = (x+1)/\chi(\epsilon) = \epsilon z/\chi(\epsilon)$.

First we reexpand the inner solution. Holding w fixed as ϵ tends to zero (along with $\chi(\epsilon)$ and $\epsilon/\chi(\epsilon)$), we have the explicit expansion:

$$Y^0(z) = c_1(1 - e^{-2z}) = c_1(1 - e^{-2w\chi(\epsilon)/\epsilon}) = c_1 + \text{exponentially small terms}\,. \tag{7.46}$$

In the same limit we have, from our explicit formula for $Y^1(z)$,

$$\begin{aligned} Y^1(z) &= \left[\frac{c_1}{4}(2z-1) + d_1\right] - \left[\frac{c_1}{4}(4z^2+2z+1) + d_1 - \frac{c_1}{2}\right] e^{-2z} \\ &= \left[\frac{c_1}{4}\left(2w\frac{\chi(\epsilon)}{\epsilon} - 1\right) + d_1\right] - \left[\frac{c_1}{4}\left(4w^2\frac{\chi(\epsilon)^2}{\epsilon^2} + 2w\frac{\chi(\epsilon)}{\epsilon} + 1\right) + d_1 - \frac{c_1}{2}\right] e^{-2w\chi(\epsilon)/\epsilon} \\ &= \frac{c_1 w}{2}\frac{\chi(\epsilon)}{\epsilon} - \frac{c_1}{4} + d_1 + \text{exponentially small terms}\,. \end{aligned} \tag{7.47}$$

Consequently,

$$Y^0(z) + \epsilon Y^1(z) = c_1 + \frac{c_1 w}{2}\chi(\epsilon) + \left(d_1 - \frac{c_1}{4}\right)\epsilon + \text{exponentially small terms}\,, \tag{7.48}$$

as $\epsilon \to 0$ with w held fixed.

Next we reexpand the outer solution. Substituting $x = -1 + \chi(\epsilon)w$ we obtain by a Taylor expansion

$$y^0(x) = 2\sqrt{\frac{2}{1 + (-1 + \chi(\epsilon)w)^2}} = 2 + \chi(\epsilon)w + O(\chi(\epsilon)^2)\,, \tag{7.49}$$

as $\epsilon \to 0$ with w held fixed. A similar direct expansion of $y^1(x)$ gives in this same limit

$$y^1(x) = -\frac{8+\pi}{8} + O(\chi(\epsilon))\,. \tag{7.50}$$

Consequently, we get

$$y^0(x) + \epsilon y^1(x) = 2 + w\chi(\epsilon) - \frac{8+\pi}{8}\epsilon + O(\epsilon\chi(\epsilon)) + O(\chi(\epsilon)^2)\,, \tag{7.51}$$

as $\epsilon \to 0$ with w held fixed.

Now we match, which means that we compare the two-term inner and outer expansions in the overlap domain where w is held fixed. The leading term of the inner expansion is just the constant c_1, since everything else vanishes as $\epsilon \to 0$. Similarly the leading term of the outer expansion is just the number 2. Matching the expansions to leading order then requires that we take

$$c_1 = 2\,. \tag{7.52}$$

With this choice, the terms proportional to $\chi(\epsilon)$ in both inner and outer expansions are also matched for free. Notice that we did not need any particular details of the intermediate scale $\chi(\epsilon)$ aside from the assumption

that $\epsilon \ll \chi(\epsilon) \ll 1$ in order to match at leading order. Now we would like to continue matching at the next order in hopes of determining the constant d_1. Here, we note that while the term $O(\epsilon\chi(\epsilon))$ is clearly negligible compared to both ϵ and $\chi(\epsilon)$, we do not know how the term $O(\chi(\epsilon)^2)$ compares in magnitude with ϵ. If we assume further that $\chi(\epsilon)^2 \ll \epsilon$, then the term $O(\chi(\epsilon)^2)$ will indeed be negligible and we can match the terms in the two expansions proportional to ϵ. This higher-order matching gives

$$d_1 = -\left(\frac{1}{2} + \frac{\pi}{8}\right) . \tag{7.53}$$

The additional assumption we made on the scale $\chi(\epsilon)$ is that $\epsilon \ll \chi(\epsilon) \ll \epsilon^{1/2}$. Thus in order to obtain the higher-order matching, we were forced to limit the size of the overlap domain beyond what was necessary to achieve the leading-order matching. This is a typical feature of matching problems. The overlap domain gets smaller and smaller as higher and higher order matching is required.

The terms that were matched and were common to both expansions form an approximation of the solution of the boundary value problem (7.1) that is valid as $\epsilon \to 0$ with w held fixed, that is, in the overlap domain. This approximation we denote as $y_{\text{match}}(w)$. It is naturally a function of w, but of course may be re-written later in terms of x. In this problem, $y_{\text{match}}(w)$ is given by

$$y_{\text{match}}(w) := 2 + \chi(\epsilon)w - \frac{8+\pi}{8}\epsilon\,. \tag{7.54}$$

Even though we don't know what $\chi(\epsilon)$ is precisely, we know that by definition $\chi(\epsilon)w = x + 1$. Therefore, the overlap domain solution can also be written as

$$y_{\text{match}}(w) = x + 3 - \frac{8+\pi}{8}\epsilon\,. \tag{7.55}$$

Although we can already determine many details about the behavior of the unique solution of (7.1) as $\epsilon \to 0$ through positive values from a study of our inner and outer expansions in which the integration constants through second order have now been determined completely by matching, it is convenient to build a single function that represents the true solution accurately over the whole interval $[-1, 1]$. This approximation is:

$$y(x) \sim \left(y^0(x) + \epsilon y^1(x)\right) + \left(Y^0(z) + \epsilon Y^1(z)\right) - y_{\text{match}}(w)\,. \tag{7.56}$$

In using this formula in practice, we restore the variable x by setting $w = (x+1)/\chi(\epsilon)$ and $z = (x+1)/\epsilon$.

We can easily see how these different types of expansions approximate the true solution of the boundary value problem (7.1) by simply plotting the explicit approximations. See Figure 7.3

Second Example: For the second example, we want to explore how the boundary layer matching procedure can be changed if $a(x)$ is not analytic at the boundary. Let $a(x) = 12x^{1/3}$, and $b(x) \equiv 1$ in the boundary value problem (7.1). The interval is $[\alpha, \beta] = [0, 1]$, and the boundary conditions are $y(0) = y(1) = 1$.

To verify the existence of a solution to the boundary value problem (7.1), we calculate

$$\frac{a'(x)}{2} - b(x) = 2x^{-2/3} - 1 \geq 1 \qquad \text{whenever} \qquad x \in [0, 1]\,. \tag{7.57}$$

Therefore the positivity condition is satisfied for all $\epsilon > 0$ guaranteeing existence and uniqueness of the solution.

Since $a(x) > 0$ for all $x > 0$, there can be no boundary layer at $x = 1$ nor can there be any internal layers. Therefore the only possibility is that of a boundary layer at $x = 0$.

First we study the outer asymptotics. With x fixed as $\epsilon \to 0$, the reduced equation is simply

$$12x^{1/3}\frac{dy^0}{dx} + y^0 = 0\,. \tag{7.58}$$

Its unique solution that satisfies the boundary condition $y^0(1) = 1$ is

$$y^0(x) = e^{(1-x^{2/3})/8}\,. \tag{7.59}$$

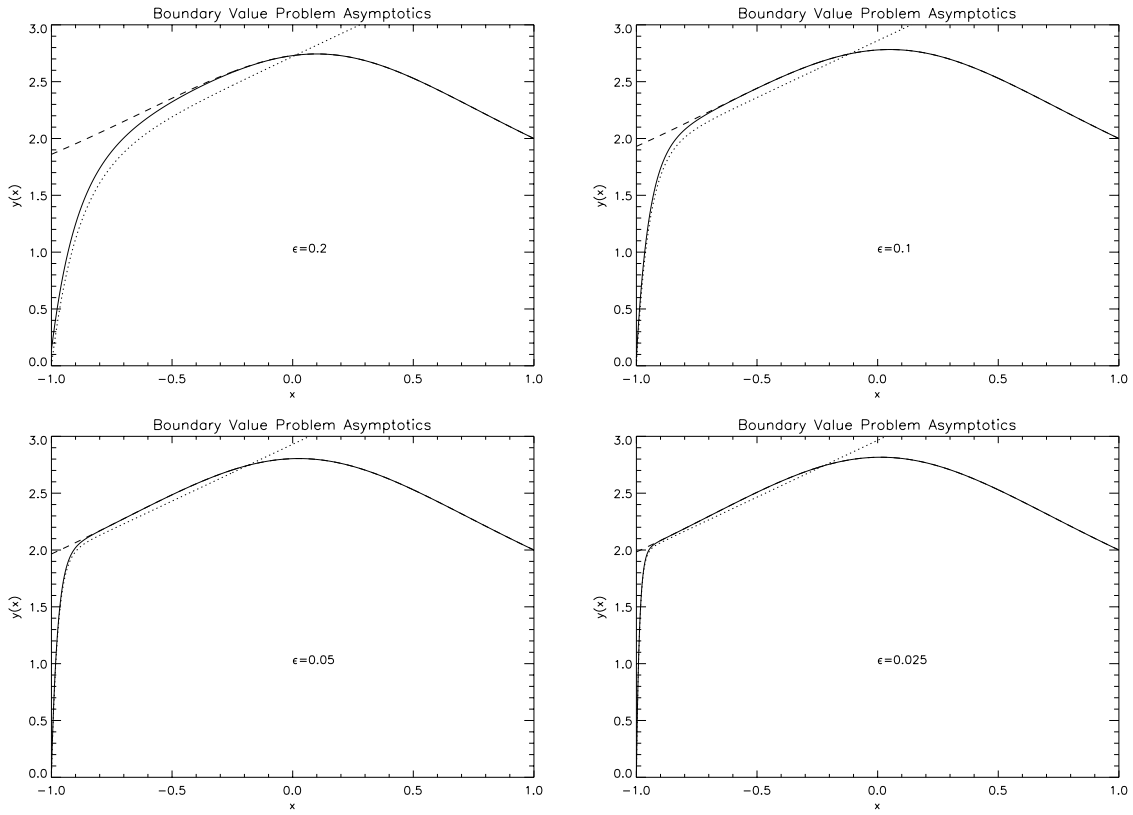

Figure 7.3: *The inner (dotted), outer (dashed), and uniformly valid (solid) two-term asymptotic approximations to the solution of the boundary value problem (7.1) with* $a(x) = 1 + x^2$ *and* $b(x) = x$*, on the interval* $[-1, 1]$ *with* $y(-1) = 0$ *and* $y(1) = 2$*. The results are displayed for four different values of* ϵ*. Upper left:* $\epsilon = 0.2$*. Upper right:* $\epsilon = 0.1$*. Lower left:* $\epsilon = 0.05$*. Lower right:* $\epsilon = 0.025$*.*

Higher-order corrections may be computed systematically by using an asymptotic series for $y(x)$ in nonnegative integer powers of ϵ. We will only keep the leading term at this time.

To study the inner asymptotics near in the anticipated boundary layer at $x = 0$, we need to introduce an inner variable by $z = x/\delta(\epsilon)$ where $\delta(\epsilon)$ is $o(1)$ because the layer must be thin, but otherwise is a gauge function that must be determined by dominant balance arguments with z held fixed. With the change of variables from x to z, and with $y(x) = Y(z)$, the differential equation becomes, exactly,

$$\frac{\epsilon}{\delta(\epsilon)^2}\frac{d^2Y}{dz^2} + 12\delta(\epsilon)^{-2/3}z^{1/3}\frac{dY}{dz} + Y = 0\,. \tag{7.60}$$

The dominant balance is between the second derivative term and the first derivative term, and is achieved by taking $\delta(\epsilon) = \epsilon^{3/4}$. Since $\delta(\epsilon)$ is the thickness of the boundary layer, here we see that the boundary layer is asymptotically thicker than it was in the last example. With this choice of $\delta(\epsilon)$, the equation can be written as

$$\frac{d^2Y}{dz^2} + 12z^{1/3}\frac{dY}{dz} + \epsilon^{1/2}Y = 0\,. \tag{7.61}$$

The reduced equation that determines the leading order approximation $Y^0(z)$ in the boundary layer with z fixed as $\epsilon \to 0$ is therefore

$$\frac{d^2Y^0}{dz^2} + 12z^{1/3}\frac{dY^0}{dz} = 0\,. \tag{7.62}$$

We can solve this equation first for dY^0/dz by finding an integrating factor and then we get Y^0 by simply integrating. So,

$$\frac{dY^0}{dz} = Ce^{-9z^{4/3}}, \tag{7.63}$$

where C is an integration constant. Integrating, we get

$$Y^0(z) = 1 + C\int_0^z e^{-9s^{4/3}}\,ds = D + \frac{C}{4\sqrt{3}}\int_0^{9z^{4/3}} e^{-t}t^{3/4-1}\,dt\,. \tag{7.64}$$

The integration constant has been chosen to satisfy the boundary condition $Y^0(0) = 1$. The integral is an incomplete gamma function. Again, this process can be continued, seeking an asymptotic series for $Y(z)$ in ascending powers of $\epsilon^{1/2}$. We will only keep this leading term, however.

Since we are only using the leading approximations, and since $Y^0(z)$ goes to a constant as $z \to +\infty$ (moving out of the boundary layer from inside) while $y^0(x)$ goes to a constant as $x \to 0+$ (moving into the boundary layer from outside), the matching takes a particularly simple form. Namely, we simply demand that

$$Y^0(+\infty) = y^0(0)\,. \tag{7.65}$$

In this case, the condition is

$$1 + \frac{C}{4\sqrt{3}}\Gamma(3/4) = e^{1/8} \quad \text{implying} \quad C = \frac{4\sqrt{3}(e^{1/8}-1)}{\Gamma(3/4)}\,. \tag{7.66}$$

The intermediate asymptotic description of the solution is just the common constant asymptote: $y_{\text{match}}(w) := Y^0(+\infty) = y^0(0)$.

With all integration constants determined, the leading order uniformly valid approximation to the solution of the boundary value problem (7.1) in this case is:

$$y(x) \sim y^0(x) + Y^0(z) - y_{\text{match}}(w) = e^{(1-x^{2/3})/8} + 1 + \frac{e^{1/8}-1}{\Gamma(3/4)}\int_0^{9x^{4/3}/\epsilon} e^{-t}t^{3/4-1}\,dt - e^{1/8}\,. \tag{7.67}$$

This leading-order uniformly valid approximation is compared with the outer and inner approximations in Figure 7.4.

Third Example: This example illustrates some of what can happen if $a(x)$ has zeros. Consider the boundary value problem (7.1) on the interval $x \in [-1,1]$ with $y(-1) = 3$ and $y(1) = 1$. The coefficient functions are $a(x) = x$ and $b(x) = -1 - x/4$. This problem has a unique solution for all $\epsilon > 0$ despite the fact that a has zeros because $a'(x)/2 - b(x) = 3/2 + x/4 \geq 5/4 > 0$ for $x \in [-1,1]$. Now, there cannot be a left boundary layer at $x = -1$ because $a(-1) < 0$, and there also cannot be a right boundary layer at $x = 1$ because $a(1) > 0$. So the only possibility is an internal layer at $x = 0$, where $a(x)$ vanishes.

The outer solutions at leading order are obtained from the reduced equation in which we have simply set $\epsilon = 0$. A solution of the reduced outer equation that satisfies the left boundary condition is

$$y_{\text{L}}^0(x) = -3xe^{(x+1)/4}\,. \tag{7.68}$$

Similarly, a leading-order outer solution satisfying the right boundary condition is

$$y_{\text{R}}^0(x) = xe^{(x-1)/4}\,. \tag{7.69}$$

As the outer approximation satisfying the boundary condition at $x = -1$ fails to satisfy the boundary condition at $x = +1$ and vice versa, the two outer portions of the solution must be matched through the internal layer. The two outer solutions cross each other at $x = 0$, but in a nonsmooth way that should be cleared up by a careful analysis in the internal layer.

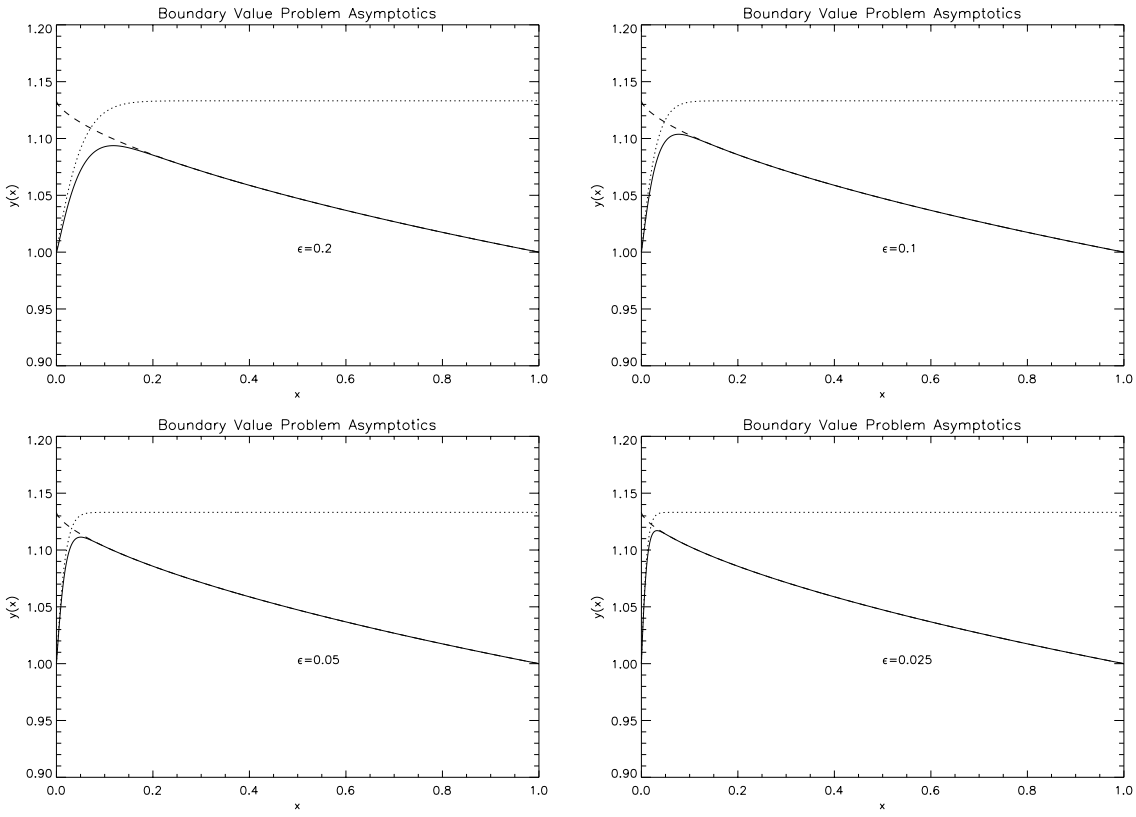

Figure 7.4: *The inner (dotted), outer (dashed), and uniformly valid (solid) one-term asymptotic approximations to the solution of the boundary value problem (7.1) with* $a(x) = 12x^{1/3}$ *and* $b(x) = 1$*, on the interval* $[0, 1]$ *with* $y(0) = y(1) = 1$*. The results are displayed for four different values of* ϵ*. Upper left:* $\epsilon = 0.2$*. Upper right:* $\epsilon = 0.1$*. Lower left:* $\epsilon = 0.05$*. Lower right:* $\epsilon = 0.025$*.*

We begin by investigating the inner solutions in the internal layer at $x = 0$. Let the inner variable be $z = x/\delta(\epsilon)$ with $\delta(\epsilon) << 1$ as $\epsilon \to 0$. We have to pick $\delta(\epsilon)$ in order to achieve a dominant balance in the limit $\epsilon \to 0$ with z fixed. With $y(x) = Y(z)$, the differential equation becomes exactly

$$\frac{\epsilon}{\delta(\epsilon)^2}\frac{d^2Y}{dz^2} + z\frac{dY}{dz} - (1 + \delta(\epsilon)z/4)Y = 0\,. \tag{7.70}$$

In this case, the dominant balance is for $\delta(\epsilon) = \epsilon^{1/2}$, which involves all three terms equally. With this choice, the reduced inner equation is

$$\frac{d^2Y^0}{dz^2} + z\frac{dY^0}{dz} - Y^0 = 0\,. \tag{7.71}$$

Now, the function $Y^0(z) = z$ obviously solves this equation. We can find another linearly independent solution by reduction of order. Set $Y^0(z) = zu(z)$ and get

$$\frac{d^2u}{dz^2} + \left(\frac{2}{z} + z\right)\frac{du}{dz} = 0\,. \tag{7.72}$$

Solving for du/dz, we get

$$\frac{du}{dz} = \frac{Ce^{-z^2/2}}{z^2}\,, \tag{7.73}$$

where C is an integration constant. Integrating to find $u(z)$ and multiplying by z to get back to $Y^0(z)$, we get

$$Y^0(z) = Dz + Cz \int_{-\infty}^{z} \frac{e^{-s^2/2}}{s^2}\, ds\,, \tag{7.74}$$

where D is another integration constant. It is best to rewrite the integral by integrating by parts once. Thus, the solution becomes

$$Y^0(z) = \tilde{D}z + \tilde{C}\left(e^{-z^2/2} + z\int_{-\infty}^{z} e^{-s^2/2}\, ds\right)\,, \tag{7.75}$$

where $\tilde{C}$ and $\tilde{D}$ are modified integration constants (still completely arbitrary). This form makes it clear that the inner leading order solution is analytic at $z = 0$.

Now we match the inner solution onto the two outer solutions, and therefore determine the integration constants $\tilde{C}$ and $\tilde{D}$. To illustrate the process of matching somewhat more concretely than we have done previously, let us fix an intermediate scale asymptotically in between the inner scale of $\epsilon^{1/2}$ and the outer scale of 1. So to be concrete, let us introduce the intermediate variable $w = \epsilon^{-1/4}x = \epsilon^{1/4}z$. Holding w fixed we have as $\epsilon \to 0$,

$$\begin{aligned} y_L^0(x) &= -3e^{1/4}\epsilon^{1/4}w + O(\epsilon^{1/2}) \\ y_R^0(x) &= e^{-1/4}\epsilon^{1/4}w + O(\epsilon^{1/2})\,. \end{aligned} \tag{7.76}$$

To use the same limit process to study $Y^0(z)$, we have to know whether w is positive or negative, since $Y^0(z)$ has different behavior depending on whether z is going to positive or negative infinity. So as $\epsilon \to 0$ we get

$$Y^0(z) = \begin{cases} \tilde{D}\epsilon^{-1/4}w + \text{exponentially small}\,, & w < 0 \\ \left(\tilde{D} + \tilde{C}\sqrt{2\pi}\right)\epsilon^{-1/4}w + \text{exponentially small}\,, & w > 0\,. \end{cases} \tag{7.77}$$

We have to match the expansion of $Y^0(z)$ for $w < 0$ to the expansion of $y_L^0(x)$ and match the expansion of $Y^0(z)$ for $w > 0$ to the expansion of $y_R^0(x)$. This gives:

$$\tilde{D} = \epsilon^{1/2}(-3e^{1/4}) \quad \text{and} \quad \tilde{C} = \epsilon^{1/2}\frac{e^{-1/4} + 3e^{1/4}}{\sqrt{2\pi}}\,. \tag{7.78}$$

With this choice of the constants, the inner approximation $Y^0(z)$ smoothly "fills in" the corner between the two outer approximations. An internal layer like this is sometimes called a *corner layer*. The fact that the integration constants are both order $\epsilon^{1/2}$ means that the actual offset of the corner from the x-axis is asymptotically proportional to $\epsilon^{1/2}$.

Our asymptotic description at leading order is therefore given by the following:

$$y(x) \sim \begin{cases} y_L^0(x)\,, & x < 0 \quad \text{fixed} \\ Y^0(z)\,, & z = \epsilon^{-1/2}x \quad \text{fixed} \\ y_R^0(x)\,, & x > 0 \quad \text{fixed}\,. \end{cases} \tag{7.79}$$

The terms we matched between $y_L^0(x)$ and $Y^0(z)$ were

$$y_{L,\text{match}}^0 := -3e^{1/4}\epsilon^{1/4}w = -3e^{1/4}x\,, \tag{7.80}$$

and the terms we matched between $y_R^0(x)$ and $Y^0(z)$ were

$$y_{R,\text{match}}^0 := e^{-1/4}\epsilon^{1/4}w = e^{-1/4}x\,. \tag{7.81}$$

We would like to use these to construct a single function that is a uniformly valid approximation to the solution of the boundary value problem over the whole interval $[-1, 1]$. Since we have matching into the

internal layer from both the right and the left, we might try a piecewise formula of the type we used in the boundary layer problems:

$$y_{\rm unif}(x) := \begin{cases} y_L^0(x) + Y^0(\epsilon^{-1/2}x) - y^0_{L,\rm match}\,, & x < 0 \\ \\ y_R^0(x) + Y^0(\epsilon^{-1/2}x) - y^0_{R,\rm match}\,, & x > 0\,. \end{cases} \tag{7.82}$$

Although it looks sloppy to break things up sharply at the origin, in fact our matching implies that $y_{\rm unif}(x)$ is actually a C^1 function of x throughout the interval $[-1, 1]$. Higher-order expansions and matching is required to get more smoothness of the approximation. The solution formulas we have obtained are compared for several values of ϵ in Figure 7.5.

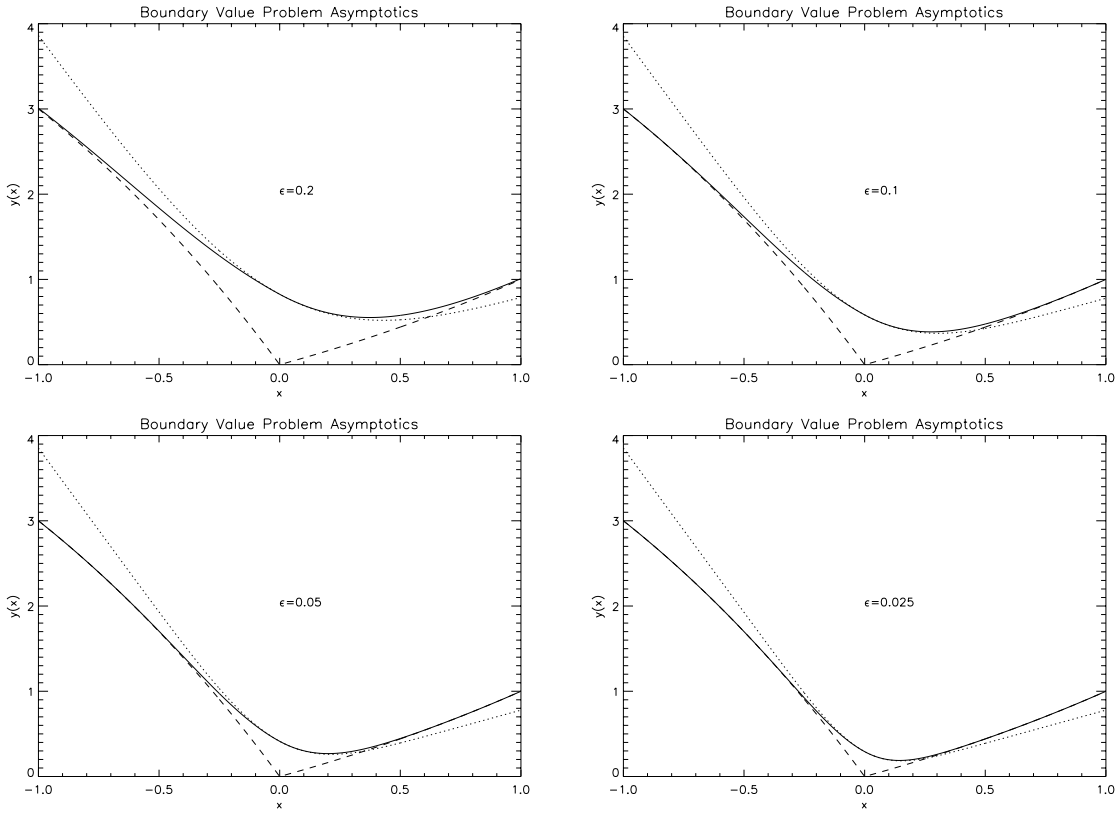

Figure 7.5: *The two leading order outer solutions (dashed), the leading order inner solution (dotted), and the uniformly valid approximator* $y_{\rm unif}(x)$ *(solid) for several values of* ϵ*. Upper left:* $\epsilon = 0.2$*. Upper right:* $\epsilon = 0.1$*. Lower left:* $\epsilon = 0.05$*. Lower right:* $\epsilon = 0.025$*.*

One final remark about this example. In our matching procedure, we adopted a concrete choice of the intermediate scale, namely $\chi(\epsilon) = \epsilon^{1/4}$. One might have some concern that since our uniformly valid approximation involves the common terms $y_{L,\rm match}$ and $y_{R,\rm match}$, our solution might involve this scale in some way. In fact, if one repeats the analysis with a general $\chi(\epsilon)$ satisfying only $\epsilon^{1/2} \ll \chi(\epsilon) \ll 1$ as $\epsilon \to 0$, exactly the same formulae will appear. The choice of a particular intermediate scale is an artifact of the matching process, and although it is important that a range of accessible intermediate scales exists for matching to be possible at all, the asymptotic results cannot depend on the choice of any particular scale in this range.

7.2 Nonlinear Problems

It is important to point out and emphasize that the general techniques of (i) looking for outer expansions, (ii) rescaling to find layers and looking for inner expansions, and (iii) matching of asymptotic expansions, have applicability far beyond the realm of the linear boundary value problem (7.1). For nonlinear problems, the rudimentary existence and uniqueness theory is not nearly as well-developed as in the linear case (where it is really as complete as it can be). However, if we are confident that there is a solution for one reason or another (say by physical arguments or evidence from numerical simulations), the relatively sad state of the existence/uniqueness theory is of little consequence regarding asymptotically computing solutions.

So here we consider in detail the solution of a problem described by J. Kevorkian and J. D. Cole. The boundary value problem is

$$\begin{aligned} &\epsilon\frac{d^2y}{dx^2} + y\frac{dy}{dx} - y = 0 \qquad \text{for} \qquad x \in (0,1) \\ &y(0) = A \qquad \text{and} \qquad y(1) = B\,. \end{aligned} \tag{7.83}$$

The numbers A and B are real and independent of ϵ. We want to develop the solution of this boundary value problem for small positive ϵ and assess the dependence of the asymptotic formulae on the fixed parameters A and B.

Actually, we do not need to consider all possible values of A and B, only about "half" of them. Suppose $y(x)$ is the solution of (7.83). Consider the function defined by the formula

$$u(x) := -y(1-x)\,. \tag{7.84}$$

So you just put $1-x$ into the formula for y everywhere there was previously just x, and flip the sign, to get u. Using the chain rule, we see that this function satisfies

$$\epsilon\frac{d^2u}{dx^2} + u\frac{du}{dx} - u = -\left(\epsilon y''(1-x) + y(1-x)y'(1-x) - y(1-x)\right) = 0 \tag{7.85}$$

where the prime means differentiation with respect to its argument, here $v = 1-x$, and we get zero at the end because y satisfies (7.83). Also $u(0) = -y(1) = -B$ and $u(1) = -y(0) = -A$. So if we know the solution of (7.83) with boundary conditions $y(0) = A$ and $y(1) = B$, we also know the solution of the same problem with $y(0) = -B$ and $y(1) = -A$. The mapping that sends (A, B) to $(-B, -A)$ is just a reflection through the line $B = -A$. So without any loss of generality, we can assume that $B \geq -A$.

The outer solutions of this problem come, as usual, from seeking a regular perturbation expansion (no rescaling):

$$y(x) \sim y^0(x) + \epsilon y^1(x) + \epsilon^2 y^2(x) + \ldots\,, \tag{7.86}$$

and making the usual assumption that the derivatives of $y(x)$ have corresponding asymptotic expansions in powers of ϵ. In regard to keeping track of the ordering of terms, the new wrinkle is the nonlinear term. But we just do the multiplication and collect together all the product terms of the same order. So

$$\begin{aligned} y\frac{dy}{dx} &= \left[y^0 + \epsilon y^1 + \epsilon^2 y^2 + O(\epsilon^3)\right]\left[\frac{dy^0}{dx} + \epsilon\frac{dy^1}{dx} + \epsilon^2\frac{dy^2}{dx} + O(\epsilon^3)\right] \\ &= y^0\frac{dy^0}{dx} + \epsilon\left[y^0\frac{dy^1}{dx} + y^1\frac{dy^0}{dx}\right] + \epsilon^2\left[y^0\frac{dy^2}{dx} + y^1\frac{dy^1}{dx} + y^2\frac{dy^0}{dx}\right] + O(\epsilon^3)\,. \end{aligned} \tag{7.87}$$

So the hierarchy of equations obtained for the coefficients $y^n(x)$ is:

$$y^0\frac{dy^0}{dx} - y^0 = 0\,, \tag{7.88}$$

$$y^0\frac{dy^1}{dx} + \frac{dy^0}{dx}y^1 - y^1 = -\frac{d^2y^0}{dx^2}\,, \tag{7.89}$$

$$y^0\frac{dy^2}{dx} + \frac{dy^0}{dx}y^2 - y^2 = -\frac{d^2y^1}{dx^2} - y^1\frac{dy^1}{dx}\,, \tag{7.90}$$

and so on. A key point here is to notice that after the first nonlinear equation that must be solved for $y^0(x)$, all further corrections come from solving linear first order equations, all of the same form, namely $Ly^n(x) = f^n(x)$ for some known function $f^n(x)$, where the differential operator L is in this case

$$Lu(x) := \left[y^0(x)\frac{d}{dx} + \frac{dy^0}{dx}(x) - 1 \right] u(x) = y^0\frac{du}{dx} + \frac{dy^0}{dx}u - u\,. \tag{7.91}$$

The nonlinear equation for $y^0(x)$ factors:

$$\text{Either} \quad y^0 = 0 \quad \text{or} \quad \frac{dy^0}{dx} - 1 = 0\,. \tag{7.92}$$

Therefore, either $y^0(x) \equiv 0$ or $y^0(x) = x - x_0$, where x_0 is a constant of integration. Now we don't know at this time where or if boundary or internal layers will occur, but certainly the solution $y^0 \equiv 0$ will not be of much use in satisfying either boundary condition in general. So we set it aside momentarily and consider $y^0(x) = x - x_0$. Notice that since the second derivative of this linear function vanishes, $y^0(x) = x - x_0$ is actually an *exact* solution of the full nonlinear equation for $y(x)$ including all three terms. This means that in this case all higher-order corrections that we calculate by solving the hierarchy of equations order-by-order will turn out to be zero. So we see that if $B = A + 1$, then the function $y(x) = x + A$ is the *exact* solution of the fully nonlinear boundary value problem (7.83) for *all* ϵ. Otherwise, we can certainly identify $y^0(x) = y^0_L(x) := x + A$ and $y^0(x) = y^0_R(x) := x + B - 1$ as two separate outer solutions that satisfy respectively the boundary conditions at $x = 0$ and $x = 1$. Some kind of layer will be required to connect the solutions together, and it is our job to find it.

To find the layer, introduce an inner variable $z = (x - x_0)/\delta(\epsilon)$ with x_0 the unknown position of the layer and $\delta(\epsilon) \ll 1$ the unknown thickness of the layer. Letting $Y(z) = y(x)$, the full differential equation goes over into

$$\frac{\epsilon}{\delta(\epsilon)^2}\frac{d^2Y}{dz^2} + \frac{1}{\delta(\epsilon)}Y\frac{dY}{dz} - Y = 0\,. \tag{7.93}$$

If we want the layer to include the previously neglected effect of the second derivative term, we have to find a dominant balance including this term. Assuming for the moment that $Y(0) \neq 0$, which means that the true solution $y(x)$ does not vanish at the location of the layer, we see that the only balance possible is to take $\delta(\epsilon) = \epsilon$ which makes the derivative terms dominant. Thus, the reduced equation for the leading order inner approximation $Y^0(z)$ will be

$$\frac{d^2Y^0}{dz^2} + Y^0\frac{dY^0}{dz} = 0\,. \tag{7.94}$$

To solve this equation, we notice that the second term is a perfect derivative, so the equation becomes

$$\frac{d}{dz}\left[\frac{dY^0}{dz} + \frac{1}{2}Y^0(z)^2\right] = 0 \qquad \text{implying} \qquad \frac{dY^0}{dz} + \frac{1}{2}Y^0(z)^2 = C \tag{7.95}$$

where C is a constant of integration. Let us write C in the form $C = \pm D^2/2$ where D is real. Although nonlinear the first order equation we are staring at is separable, so we can write it in the form

$$\frac{1}{2}dz = \frac{dY^0}{\pm D^2 - (Y^0)^2}\,. \tag{7.96}$$

If we choose $C < 0$ (minus sign), then upon integrating we find

$$-\frac{1}{D}\tan^{-1}\left(\frac{Y^0}{D}\right) = \frac{1}{2}(z - z_0) \qquad \text{or} \qquad Y^0(z) = -D\tan\left(\frac{D}{2}(z - z_0)\right)\,, \tag{7.97}$$

where z_0 is an integration constant. This solution blows up periodically for finite z, so it cannot represent any smooth solution of the boundary value problem (7.83). It cannot be matched onto anything. On the other hand, if we choose $C > 0$ (plus sign), then upon integrating we find

$$\frac{1}{2D}\log\left(\frac{D + Y^0}{D - Y^0}\right) = \frac{1}{2}(z - \tilde{z_0})\,, \tag{7.98}$$

where $\tilde{z_0}$ is an integration constant. If we take $\tilde{z_0} = z_0 \in \mathbb{R}$, then we get

$$Y^0(z) = D \tanh\left(\frac{D}{2}(z - z_0)\right), \tag{7.99}$$

which is bounded for all z and a good candidate for a layer solution. And if we take $\tilde{z_0} = \pi i/D + z_0$ with $z_0 \in \mathbb{R}$, then we get

$$Y^0(z) = D \coth\left(\frac{D}{2}(z - z_0)\right). \tag{7.100}$$

This latter solution blows up when $z = z_0$, but is finite otherwise. Therefore, it is possible to find a semi-infinite range of z values in which it is bounded, and can serve as a layer solution. But as the domain of z-values is only semi-infinite, it must be a boundary layer and not an internal layer. Both the tanh solution and the coth solution are invariant under the transformation $D \to -D$ so we may assume $D > 0$ without loss of generality. The tanh solution is increasing in z and satisfies $Y^0(-\infty) = -D$ and $Y^0(+\infty) = +D$. The coth solution has two branches, both decreasing functions of z: one that satisfies $Y^0(-\infty) = -D$ and decreases to $-\infty$ at $z = z_0-$, and one that decreases from $+\infty$ at $z = z_0+$ and satisfies $Y^0(+\infty) = D$. These solutions all decay exponentially fast onto their horizontal asymptotes. These solutions are shown in Figure 7.6. There is a singular solution of the inner equation (7.94) that comes from taking the integration

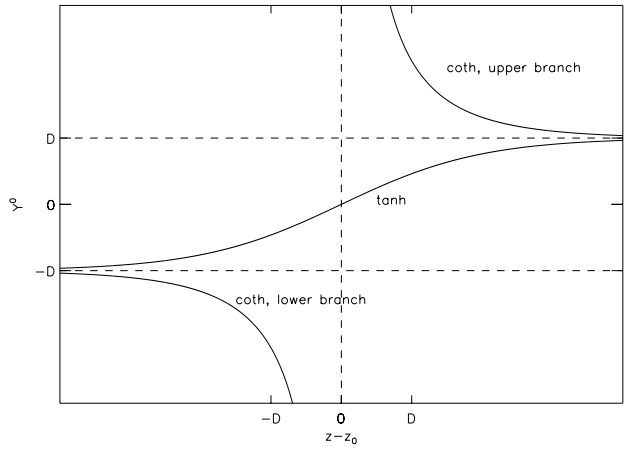

Figure 7.6: *The possible inner solutions $Y^0(z)$. Asymptotes are shown with dashed lines.*

constant D to be zero. Then $Y^0(z)$ has to satisfy

$$\frac{dY^0}{dz} + \frac{1}{2}(Y^0)^2 = 0 \quad \text{or} \quad -\frac{dY^0}{(Y^0)^2} = \frac{1}{2}dz \quad \text{giving} \quad Y^0(z) = \frac{2}{z - z_0}. \tag{7.101}$$

This solution decays to a horizontal asymptote of zero, but at an algebraic rate instead of exponentially fast.

Now we begin to see where the layers can be, and how the global solution is matched together for different values of A and B. We first try to find approximate solutions that make use of only the outer solutions $y_L^0(x)$ and/or $y_R^0(x)$ and any layers required to join them to boundaries or each other. Let us first look for boundary layers at $x = 0$. If $A > B - 1$, then the inner solution in the boundary layer will have to begin at A and then decay downward as $z \to +\infty$ onto the outer solution $y_R^0(x)$ matching the other boundary condition. Since $y_R^0(0) = B - 1$ the leading-order matching is of the simplest type, namely $Y^0(+\infty) = y_R^0(0) = B - 1$. The only solution that descends to a horizontal asymptote is the upper branch of the coth solution. The parameters z_0 and D are then chosen so that $Y^0(z)$ satisfies $Y^0(0) = A$ and $Y^0(+\infty) = B - 1$. In particular, we get $D = B - 1$ which must be positive. So this situation happens when $B > 1$ and $A > B - 1$. Matching the boundary condition at $z = 0$ then requires that

$$z_0 = \frac{1}{1 - B} \log\left(\frac{A + B - 1}{A - B + 1}\right). \tag{7.102}$$

The uniformly valid solution in this case is, to leading order, $y_{\rm unif}(x) = y_R^0(x) + Y^0(z) - (B-1)$ which is plotted in Figure 7.7 to indicate the structure of the solution. On the other hand, if $A < B-1$, then the

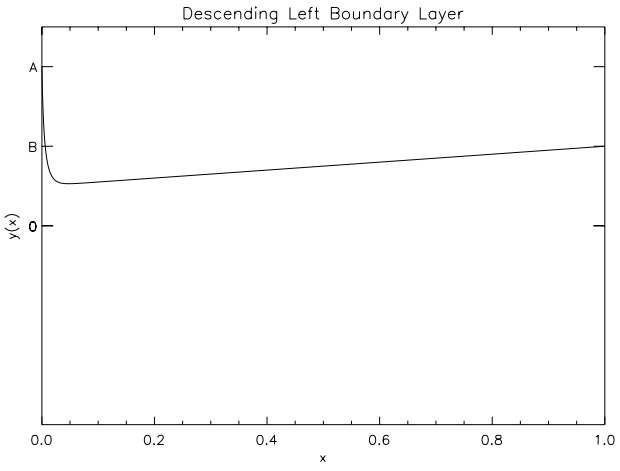

Figure 7.7: *The case $A > B-1$ and $B > 1$ in which there is a boundary layer at $x = 0$. This is a plot of the leading order uniformly valid approximation $y_{\rm unif}(x)$ for $\epsilon = 0.01$ and $A = 4$, $B = 2$.*

inner solution in the boundary layer at $x = 0$ would have to decay upward as $z \to +\infty$ to match onto the outer solution $y_R^0(x)$. The only inner solution that increases to a horizontal asymptote as $z \to +\infty$ is the tanh solution. For matching, the asymptote has to be $D = B-1$. And since the inner solution has to satisfy the boundary condition $Y^0(0) = A$, we must have $A > -D = 1 - B$ for this kind of boundary layer to be possible. Satisfying the boundary condition at $z = 0$ requires that z_0 be given by

$$z_0 = \frac{1}{1-B} \log\left(\frac{B-1+A}{B-1-A}\right) . \tag{7.103}$$

The leading-order uniformly valid approximation of the solution of (7.83) is of the same form as before, but with the tanh solution for $Y^0(z)$. It is plotted in Figure 7.8. This exhausts all the possibilities for boundary

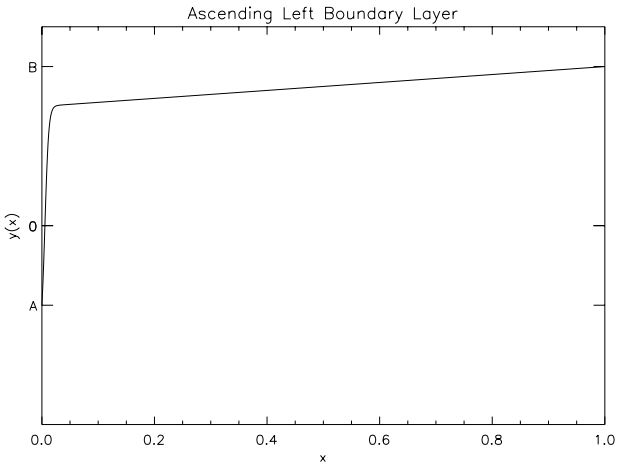

Figure 7.8: *The case $A < B-1$ and $A > 1-B$ in which there is a boundary layer at $x = 0$. This is a plot of the leading order uniformly valid approximation $y_{\rm unif}(x)$ for $\epsilon = 0.01$ and $A = -2$, $B = 4$.*

layers at $x = 0$ using our inner solutions $Y^0(z)$ to match onto $y_R^0(x)$. The situation in regard to boundary layers at $x = 1$ is similar and can be obtained using the symmetry $y \to -y$, $B \to -A$ and $A \to -B$ discussed earlier.

But we should notice that the tanh inner solution may also function perfectly well as an internal layer that connects between $y_L^0(x)$ on the left and $y_R^0(x)$ on the right. Since the tanh solution is increasing, from $-D$ at $z = -\infty$ to D at $z = +\infty$, such an internal layer can happen at some x_0 where $y_L(x_0) = -D$ and $y_R(x_0) = D$. Eliminating D between these two gives us the location of the layer:

$$x_0 = \frac{1 - A - B}{2} \,. \tag{7.104}$$

In order that $0 < x_0 < 1$ (the layer has to occur in the domain), we need A and B to satisfy $|A+B| < 1$, and in order to have $D > 0$ we need $y_R(x_0) > 0$ or $B - A > 1$. In this case, the constant z_0 is not determined by the leading-order matching process. But this constant just amounts to a shift of the location of the boundary layer of order ϵ, and is not terribly significant. We take it to be $z_0 = 0$ for concreteness. As in the example where we found an internal corner layer, a uniformly valid formula for the solution can be given in piecewise form:

$$y_{\text{unif}}(x) = \begin{cases} y_L^0(x) + Y^0(z) - y_L^0(x_0) \,, & x < x_0 \\ \\ y_R^0(x) + Y^0(z) - y_R^0(x_0) \,, & x > x_0 \,. \end{cases} \tag{7.105}$$

Given the shape of this internal layer, this is called a *shock layer*. Its functional form in terms of the tanh function is virtually identical with that of the shock wave for Burgers' equation in the zero-viscosity limit. A shock layer solution is illustrated in Figure 7.9.

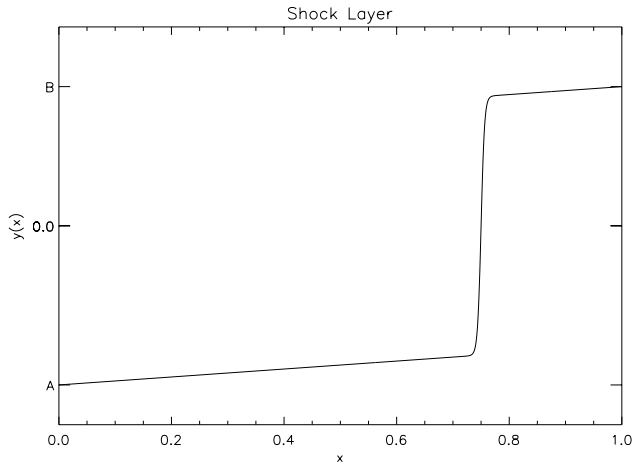

Figure 7.9: *The case* $|A + B| < 1$ *and* $B > 1 + A$ *in which there is a shock layer at* $x = x_0 = (1 - A - B)/2$. *This is a plot of the leading order uniformly valid approximation* $y_{\text{unif}}(x)$ *for* $\epsilon = 0.01$ *and* $A = -4$, $B = 3.5$.

We have now exhausted all possible structures that take into account the ingredients of the outer solutions $y_L^0(x)$ and $y_R^0(x)$, and the different types of inner solutions $Y^0(z)$. But there still remain many possible values of the boundary conditions A and B for which we have no results yet. To handle these cases, we recall that there is another solution of the outer equation, namely $y^0(x) \equiv 0$, that we put aside previously because it did not generally satisfy any boundary conditions. But this does not preclude its use in an intermediate part of the solution occupying a subinterval of fixed size between the two boundaries. It is this ingredient that we need to complete the picture for the remaining types of boundary conditions. We study these cases now.

If $A < 0$ and $B > 0$ but $A > B - 1$, then one can imagine a solution which starts out on the left like $y_L^0(x)$ and follows this until some point x_L at which $y_L^0(x_L) = 0$, and then switches to the zero solution until reaching the point x_R at which $y_R^0(x_R) = 0$, and then switches to $y_R^0(x)$. A solution of this sort is possible if

we can argue the existence of two corner layers that smoothly join the linear functions $y_L^0(x)$ and $y_R^0(x)$ onto the zero solution. To find such corner solutions, we note that near such a corner y must be small. So we can try to scale y as well as x. For linear differential equations, a change of scale of the dependent variable makes no difference, but in nonlinear problems scaling of the unknown is a powerful and essential tool. So we seek a corner-type solution by setting $y(x) = \mu(\epsilon) f(z)$ where the inner variable $z = (x - x_0)/\delta(\epsilon)$ and we are thinking of $x_0 = x_R$ or $x_0 = x_L$. Inserting this rescaling into the nonlinear equation in (7.83) and dividing through by $\mu(\epsilon)$ gives

$$\frac{\epsilon}{\delta(\epsilon)^2}\frac{d^2 f}{dz^2} + \frac{\mu(\epsilon)}{\delta(\epsilon)} f \frac{df}{dz} - f = 0\,. \tag{7.106}$$

Now with two scales to play with, several dominant balances are possible, and in such a situation the best idea is to try to find the richest balance, *i.e.* the one involving the most terms. In fact, with the right choice of $\delta(\epsilon)$ and $\mu(\epsilon)$ we can balance all three terms. This requires taking $\delta(\epsilon) = \mu(\epsilon) = \epsilon^{1/2}$. The resulting equation is then

$$\frac{d^2 f}{dz^2} + f\frac{df}{dz} - f = 0\,. \tag{7.107}$$

To see whether this equation supports corner layer type solutions, we study it using the phase plane. Let $v = df/dz$. Then, the system takes the form of a first-order system for f and v:

$$\begin{aligned} \frac{df}{dz} &= v \\ \frac{dv}{dz} &= f \cdot (1 - v)\,. \end{aligned} \tag{7.108}$$

This system has a unique equilibrium point at the origin in the (f, v)-plane. If f and v are close to the equilibrium and therefore small, we can linearize by dropping the quadratic term $-fv$ from the right-hand side of the second equation in (7.108). The linearized system can be solved by considering the sum $f + v$ and difference $f - v$ as new unknowns and adding and subtracting the two equations. One finds

$$\begin{aligned} \frac{d}{dz}(f + v) &\approx (f + v) \\ \frac{d}{dz}(f - v) &\approx -(f - v)\,. \end{aligned} \tag{7.109}$$

Therefore for some constants $C_\pm$ we get $f + v \approx 2C_+ e^z$ and $f - v \approx 2C_- e^{-z}$, and therefore

$$f(z) \approx C_+ e^z + C_- e^{-z} \quad \text{and} \quad v(z) \approx C_+ e^z - C_- e^{-z}\,, \tag{7.110}$$

when f and v are small and therefore near the equilibrium. This means that the equilibrium at the origin is a saddle point, or an unstable fixed point of the nonlinear coupled system for f and v. We can find the shape of the global orbits in the phase plane that correspond to the stable and unstable manifolds of this equilibrium by noting that

$$\frac{dv}{df} = f\frac{1 - v}{v} \tag{7.111}$$

which is a separable equation. Its solutions satisfy

$$-v - \log(1 - v) = \frac{1}{2}f^2 + C \tag{7.112}$$

where C is an integration constant. For the orbits emerging from the origin we must have $C = 0$. The two stable and two unstable manifolds obtained by solving for f as a function of v with $C = 0$ are shown in Figure 7.10 along with the asymptote $v = 1$, which is also an exact orbit of the nonlinear system. This latter orbit corresponds to the family of exact solutions $f(z) = z + K$ for any constant K. So now we can see that the stable manifold coming into the fixed point from the horizontal asymptote $v = 1$ at $f = -\infty$ is a corner layer solution that matches from $f \sim z$ as $z \to -\infty$ exponentially onto $f = 0$. And the unstable

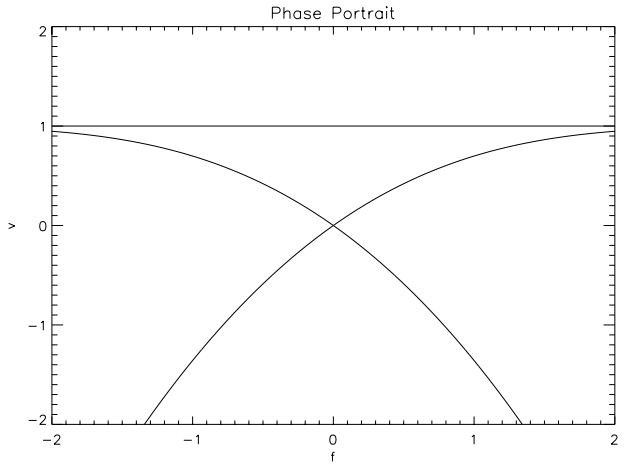

Figure 7.10: *The phase plane of the system (7.108) showing the stable and unstable manifolds of the equilibrium at the origin as well as the orbit* $v = 1$.

manifold leaving the fixed point exponentially fast and going to $f = +\infty$ along $v = 1$ is a corner layer solution matching the zero solution onto a solution $f \sim z$ as $z \to +\infty$. Although it is not convenient to write these solutions in closed form, it justifies the matching of the three types of outer solutions via two corner layers and produces a solution of the nonlinear boundary value problem (7.83) of the type shown qualitatively in Figure 7.11.

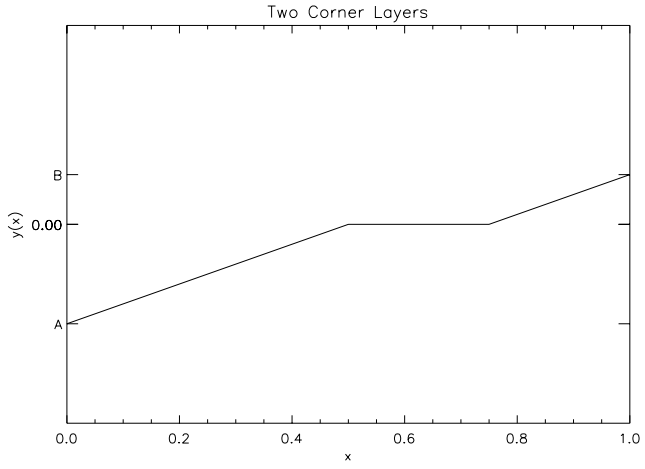

Figure 7.11: *The case* $A < 0$, $B > 0$, *and* $B < 1 + A$ *in which the two outer solutions* $y_L^0(x)$ *and* $y_R^0(x)$ *are joined by two corner layers at* $x = x_L$ *and* $x = x_R$ *onto the trivial solution* $y \equiv 0$. *This is a plot of the three outer solutions only for* $A = -1/2$, $B = 1/4$.

The last cases to investigate for $B > -A$ involve what happens to the above situation when one or both of the two corner points is pushed out past the boundaries of the domain. If $A > 0$ and $0 < B < 1$, then the corner point x_R survives, but x_L has been pushed through the left boundary. Somehow we have to get from the value of $y = A > 0$ down to match onto the zero solution, which continues until it smoothly merges onto $y_R^0(x)$ via the surviving corner point at $x = x_R$. To do this, we now call into service the singular solution of the inner equation (7.94), $Y^0(z) = 2/(z - z_0)$. If we pick $z_0 = -2/A$, then $Y^0(0) = A$, and as $z \to +\infty$ we

have $Y^0(z) \to 0$ algebraically. So this matches onto the zero solution to the right of $x = 0$, but not as fast as the exponential matching we have seen previously. A solution of this type is illustrated in Figure 7.12. Finally, if $A > 0$ and $B < 0$, then an algebraic boundary layer is also required at $x = 1$, in addition to the

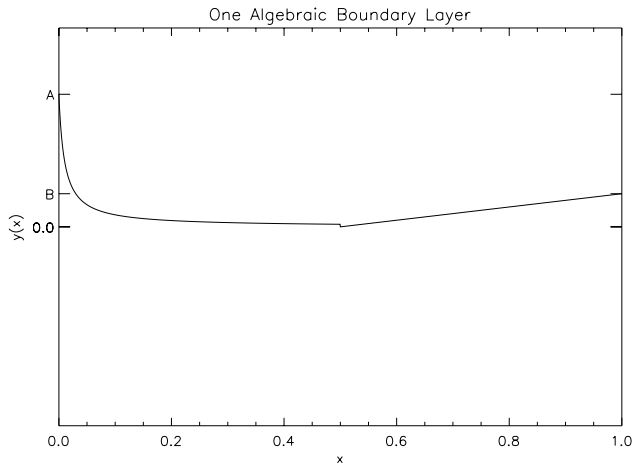

Figure 7.12: *The case $A > 0$ and $0 < B < 1$, in which a boundary layer at $x = 0$ decays slowly onto the zero solution which is in turn joined onto the outer solution $y_R^0(x)$ by a corner layer at $x = x_R$. This is a plot of the left boundary layer for $\epsilon = 0.01$ and the outer solution $y_R^0(x)$ for $A = 2$ and $B = 1/2$. The small jump in the graph near the corner point at $x = x_R$ is an indication of the fact that for this value of ϵ the algebraic decay from the left boundary layer is still quite slow.*

layer described above at $x = 0$. Both layers match onto the zero solution in the middle of the domain. The solution of (7.94) describing the right boundary layer is again a singular solution, this time with z_0 selected to satisfy the boundary condition $Y^0(0) = B$. The appropriate solution is thus $Y^0(z) = 2/(z - z_0)$ with $z_0 = -2/B$. The corresponding solution is illustrated in Figure 7.13.

The full (A, B)-plane has now been investigated, and it is clear that there is an enormous variety of behavior possible in this single problem. But the main message is that the same asymptotic techniques of singular perturbation theory that worked for linear problems also apply equally in many nonlinear cases as well. The various parts of the (A, B)-plane that we studied above in detail are summarized in Figure 7.14.

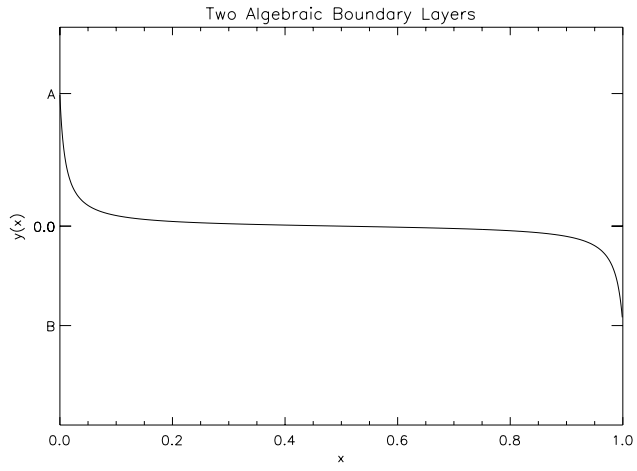

Figure 7.13: *The case $A > 0$ and $B < 0$, in which two boundary layers given by singular solutions of the inner equation are required to match onto the zero solution in the center of the domain. This is a plot for $\epsilon = 0.01$ and $A = 2$ while $B = -3/2$.*

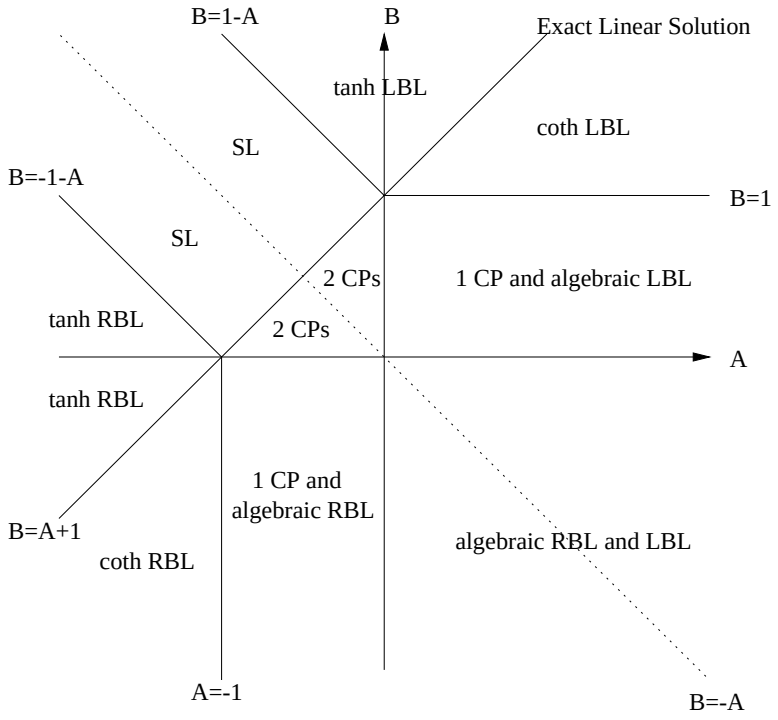

Figure 7.14: *The (A, B)-plane showing the different sorts of asymptotic solutions to the nonlinear boundary problem corresponding to the boundary conditions $y(0) = A$ and $y(1) = B$. The labels are: "LBL" means left boundary layer, "RBL" means right boundary layer, "SL" means shock layer, and "CP" means corner point.*

Chapter 8

Perturbation Theory of Oscillatory Phenomena

8.1 Perturbed Eigenvalue Problems

Before studying oscillations properly, we study a class of problems that have some important features in common with other problems we will examine shortly. And these problems are very important in applications themselves. Namely, we consider the eigenvalue problem

$$\mathbf{A}\mathbf{x} + \epsilon\mathbf{B}\mathbf{x} = \lambda\mathbf{x}\,, \tag{8.1}$$

where $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices, $\mathbf{x}$ is an n-component column vector (the eigenvector), and λ is a complex number (the eigenvalue). The eigenvalue problem consists of finding compatible pairs $(\mathbf{x}, \lambda)$ with $\mathbf{x} \neq \mathbf{0}$ that satisfy (8.1) given the matrices $\mathbf{A}$ and $\mathbf{B}$.

The idea of treating this as a perturbation problem is to suppose that the matrix $\mathbf{A}$ is one for which we know all eigenvectors and eigenvalues, and then we want to take ϵ to be small and work out the asymptotic behavior of eigenpairs $(\mathbf{x}, \lambda)$ by posing an appropriate asymptotic expansion whose leading term comes from the $\epsilon = 0$ reduced problem.

Actually, we could treat this problem in part using the methods we have already learned. The eigenvalues λ of (8.1) are the roots of the characteristic polynomial $\det(\mathbf{A} + \epsilon\mathbf{B} - \lambda\mathbb{I}) = 0$ where $\mathbb{I}$ is the $n \times n$ identity matrix. This polynomial is a perturbation of the reduced characteristic polynomial $\det(\mathbf{A} - \lambda\mathbb{I})$, whose roots we are assuming we know. So the problem of finding the eigenvalues λ for (8.1) could be viewed as a problem of asymptotic root finding exactly like we studied in Chapter 1. But this technique is not so efficient (in particular it requires writing down the whole characteristic polynomial, which is complicated if n is large) and does not give any information about the eigenvectors. It also obscures certain structure that this problem has that we can exploit in solving more complicated problems.

8.1.1 Nondegenerate theory.

Suppose first that λ_0 is a simple eigenvalue of $\mathbf{A}$ with eigenvector $\mathbf{x}_0$. This eigenvector is determined up to multiplication by any nonzero complex number. When we turn on the perturbation by looking at the problem (8.1) for small ϵ, we expect that both the eigenvalue and the eigenvector will change slightly. So we pose two asymptotic expansions in powers of ϵ:

$$\begin{aligned} \lambda &\sim \lambda_0 + \epsilon\lambda_1 + \epsilon^2\lambda_2 + \dots \\ \mathbf{x} &\sim \mathbf{x}_0 + \epsilon\mathbf{x}_1 + \epsilon^2\mathbf{x}_2 + \dots\,, \end{aligned} \tag{8.2}$$

which are expected to be asymptotic to true solutions of the eigenvalue problem in the limit $\epsilon \to 0$. The expansion for λ is of the same form as we used in studying the regular perturbation theory of simple roots

of polynomials, and the corresponding form of the expansion for the eigenvector is guessed by analogy. Substituting into (8.1) and equating powers of ϵ we find that the terms of order one are $\mathbf{A}\mathbf{x}_0 = \lambda_0\mathbf{x}_0$ which is satisfied automatically by assumption. The terms of order ϵ are then

$$\mathbf{A}\mathbf{x}_1 - \lambda_0\mathbf{x}_1 = \lambda_1\mathbf{x}_0 - \mathbf{B}\mathbf{x}_0\,. \tag{8.3}$$

Let's view this as an equation for the correction $\mathbf{x}_1$ to the eigenvector. It is an equation of the form $\mathbf{L}\mathbf{x}_1 = \mathbf{f}$ where $\mathbf{L}$ is the matrix $\mathbf{A} - \lambda_0\mathbb{I}$, and $\mathbf{f}$ is the right-hand side of (8.3). We might like to solve this problem by writing $\mathbf{x}_1 = \mathbf{L}^{-1}\mathbf{f}$. Trouble is, because λ_0 is a root of the characteristic polynomial of $\mathbf{A}$, $\det(\mathbf{L}) = \det(\mathbf{A} - \lambda_0\mathbb{I}) = 0$, so $\mathbf{L}^{-1}$ does not exist.

To see what to do, we need to recall from linear algebra that associated with the eigenvalue λ_0 of $\mathbf{A}$ there is also a nontrivial "left eigenvector" $\mathbf{y}_0$ that satisfies the equation

$$\mathbf{y}_0^\dagger\mathbf{A} = \lambda_0\mathbf{y}_0^\dagger \tag{8.4}$$

where the dagger means transpose and componentwise complex conjugation. So $\mathbf{y}_0$ is a column vector, but $\mathbf{y}_0^\dagger$ is the corresponding row vector with complex-conjugated elements. Alternatively, we can view $\mathbf{y}_0$ as an ordinary eigenvector of the conjugate-transpose matrix $\mathbf{A}^\dagger$ with complex-conjugate eigenvalue λ_0^*. If $\mathbf{A}$ is a Hermitian matrix, so $\mathbf{A}^\dagger = \mathbf{A}$, then its eigenvalues are all real and we may take $\mathbf{y}_0 = \mathbf{x}_0$. Since λ_0 is a simple eigenvalue of $\mathbf{A}$, there is only one left eigenvector up to multiplication by a nonzero complex number. If we multiply (8.3) on the left by $\mathbf{y}_0^\dagger$, then according to (8.4) the left-hand side becomes the zero vector $\mathbf{0}$. So, if the equation (8.3) is to be consistent, it is necessary that

$$\mathbf{y}_0^\dagger\mathbf{f} = 0 \quad \text{or} \quad \lambda_1\mathbf{y}_0^\dagger\mathbf{x}_0 - \mathbf{y}_0^\dagger\mathbf{B}\mathbf{x}_0 = 0\,. \tag{8.5}$$

This is a *solvability condition* for the singular equation $\mathbf{L}\mathbf{x}_1 = \mathbf{f}$. It says that the only way that there could possibly be a solution $\mathbf{x}_1$ of the equation (8.3) would be if the right-hand side were in the column space or range of the matrix $\mathbf{L} = \mathbf{A} - \lambda_0\mathbb{I}$. In fact, since λ_0 is not degenerate, there is only this one solvability condition, so (8.5) is sufficient as well as necessary in this case.

The solvability condition (8.5) determines the correction λ_1 to the eigenvalue. Rearranging (8.5) we find

$$\lambda_1 = \frac{\mathbf{y}_0^\dagger\mathbf{B}\mathbf{x}_0}{\mathbf{y}_0^\dagger\mathbf{x}_0}\,. \tag{8.6}$$

With this choice, we may find $\mathbf{x}_1$, although it will not be unique, being determined only up to the addition of any multiple of $\mathbf{x}_0$ (which is in the kernel of $\mathbf{L}$). If we know the full spectral decomposition of $\mathbf{A}$ we can see how to find $\mathbf{x}_1$ more explicitly. For example if we stack all of the eigenvectors into columns of a matrix $\mathbf{S}$ that is automatically invertible if all eigenvalues of $\mathbf{A}$ are distinct, then

$$\mathbf{A}\mathbf{S} = \mathbf{S}\mathbf{\Lambda} \tag{8.7}$$

where $\mathbf{\Lambda}$ is a diagonal matrix of eigenvalues. Then (8.3) becomes

$$(\mathbf{\Lambda} - \lambda_0\mathbb{I})\mathbf{S}^{-1}\mathbf{x}_1 = \mathbf{S}^{-1}\mathbf{f}\,. \tag{8.8}$$

The matrix $\mathbf{\Lambda} - \lambda_0\mathbb{I}$ has only one zero on the diagonal because λ_0 is a simple eigenvalue, and this means that the vector on the left-hand side has one entry that is zero. But the left eigenvectors $\mathbf{y}_k^\dagger$ may be taken to be the rows of $\mathbf{S}^{-1}$, since (8.7) implies that $\mathbf{S}^{-1}\mathbf{A} = \mathbf{\Lambda}\mathbf{S}^{-1}$. Therefore, by (8.5) the vector $\mathbf{S}^{-1}\mathbf{f}$ has a zero in the same slot. So we just drop this row and solve the remaining equations for $\mathbf{x}_1$. This is a system of $n-1$ equations in n unknowns, so $\mathbf{x}_1$ will not be unique. Since we already have an arbitrary multiple of $\mathbf{x}_0$ at leading order, there is really no need for any more of this direction, so it is traditional to choose $\mathbf{x}_1$ to be perpendicular to $\mathbf{x}_0$, fixing the remaining degree of freedom in $\mathbf{x}_1$ by adjoining the additional equation

$$\mathbf{y}_0^\dagger\mathbf{x}_1 = 0\,. \tag{8.9}$$

Higher order corrections may be computed similarly. At each stage one is faced with a solvability condition for $\mathbf{x}_k$ which must be satisfied by an appropriate choice of λ_k. In fact, it is always the same

solvability condition as (8.5), but with a different right-hand side. The general form of the equation for $\mathbf{x}_k$ follows directly from (8.1):

$$\mathbf{A}\mathbf{x}_k - \lambda_0 \mathbf{x}_k = \sum_{j=1}^{k} \lambda_j \mathbf{x}_{k-j} - \mathbf{B}\mathbf{x}_{k-1} \tag{8.10}$$

where the right-hand side is explicitly known from the previous orders except for the term proportional to λ_k, and where the left-hand side involves the same matrix $\mathbf{L} = \mathbf{A} - \lambda_0 \mathbb{I}$.

8.1.2 Degenerate theory.

If the known eigenvalue λ_0 of the matrix $\mathbf{A}$ is degenerate in the sense that it is a multiple root of the characteristic polynomial of multiplicity d with $1 < d \leq n$, then there are two distinct possibilities:

1. The matrix $\mathbf{A}$ has a full eigenspace spanned by d linearly independent eigenvectors, $\mathbf{x}_0^{(1)}, \ldots, \mathbf{x}_0^{(d)}$ all corresponding to the eigenvalue λ_0.

2. The eigenspace of $\mathbf{A}$ corresponding to the eigenvalue λ_0 has dimension somewhere between 1 and $d-1$ inclusive.

If $\mathbf{A}^\dagger = \mathbf{A}$, then only the first case is possible. If $\mathbf{A}$ is not Hermitian, and if the second case also holds, then $\mathbf{A}$ is not diagonalizable, and the best we can do is to put it into Jordan Normal Form.

Here we briefly show how the expansion procedure changes if the eigenspace of λ_0 is complete (of full dimension d). The first change occurs with the leading order equation $\mathbf{A}\mathbf{x}_0 = \lambda_0 \mathbf{x}_0$. This equation now has d linearly independent solutions, so rather than picking any one of them arbitrarily, we do the democratic thing and represent the eigenvector as a general element of the eigenspace, *i.e.* write it as a linear combination:

$$\mathbf{x}_0 = c_1 \mathbf{x}_0^{(1)} + c_2 \mathbf{x}_0^{(2)} + \ldots + c_d \mathbf{x}_0^{(d)} \,. \tag{8.11}$$

Moving on to the terms of order ϵ, we find the same equation (8.3), but now $\mathbf{x}_0$ is not a fixed vector, but rather the general linear combination (8.11). Associated with λ_0 there are in this diagonalizable degenerate case d linearly independent left eigenvectors $\mathbf{y}_0^{(1)}, \ldots, \mathbf{y}_0^{(d)}$, each of which satisfies

$$\mathbf{y}_0^{(k)\dagger} \mathbf{A} = \lambda_0 \mathbf{y}_0^{(k)\dagger} \tag{8.12}$$

for $k = 1, \ldots, d$. Each of these left eigenvectors gives a solvability condition for (8.3), via multiplication on the left, which kills the left-hand side by definition. Thus we have d independent solvability conditions

$$\mathbf{y}_0^{(k)\dagger} \mathbf{B}\mathbf{x}_0 = \lambda_1 \mathbf{y}_0^{(k)\dagger} \mathbf{x}_0 \,. \tag{8.13}$$

Taking into account the representation (8.11) of $\mathbf{x}_0$ as a linear combination, we can write these solvability conditions in a compelling form. If we build a column vector $\mathbf{c}$ of the numbers $c_1, \ldots c_d$, and if we make the eigenvectors $\mathbf{x}_0^{(1)}, \ldots \mathbf{x}_0^{(d)}$ the columns of a matrix $\mathbf{X}$ (n rows and d columns) and likewise make the left eigenvectors $\mathbf{y}_0^{(1)}, \ldots, \mathbf{y}_0^{(d)}$ the columns of a matrix $\mathbf{Y}$ of the same dimensions, then the totality of the solvability conditions (8.13) can be written in the form

$$\mathbf{Y}^\dagger \mathbf{B}\mathbf{X}\mathbf{c} = \lambda_1 \mathbf{Y}^\dagger \mathbf{X}\mathbf{c} = \lambda_1 \mathbf{c} \,, \tag{8.14}$$

with the second equality following if we have chosen the left eigenvectors $\mathbf{y}_0^{(k)\dagger}$ to be the rows of the inverse of the eigenvector matrix as we have discussed previously. The matrix $\mathbf{Y}^\dagger \mathbf{B}\mathbf{X}$ is a square matrix of dimension $d \times d$. This is therefore another eigenvalue problem: only for certain eigenvalues λ_1 will there be nonzero vectors $\mathbf{c}$ that satisfy the equation. If the eigenvalue problem (8.14) is nondegenerate, so there are d distinct eigenvalues $\lambda_1 = \lambda_1^{(1)}, \ldots, \lambda_1^{(d)}$, then we see that the perturbation of $\mathbf{A}$ by $\epsilon\mathbf{B}$ has broken the degeneracy by "selecting" d preferred directions within the eigenspace of λ_0, each of which is associated with a distinct eigenvalue $\lambda \sim \lambda_0 + \epsilon\lambda_1^{(k)}$. A generic perturbation thus fully "unfolds" the d-fold eigenvalue λ_0 of $\mathbf{A}$.

If the eigenspace of λ_0 is not complete (has dimension less than d) then the asymptotic expansions of eigenvalues and eigenvectors for (8.1) can involve negative or fractional powers of ϵ. The reason for this is that when the degenerate eigenvalue λ_0 of the matrix $\mathbf{A}$ is split into several distinct eigenvalues by the influence of the perturbation $\epsilon\mathbf{B}$, the perturbed problem suddenly has more eigenvectors than the unperturbed problem did. The extra eigenvectors must be "brought in from infinity" by rescaling in the same way that we accounted for asymptotically large roots in singularly perturbed root-finding problems. As we know from the corresponding discussion in Chapter 1, the exponent required to achieve this comes from dominant balance arguments and need not be an integer. See Hinch for some details about this possibility.

8.2 Periodic Boundary Conditions and Mathieu's Equation

Similar methods as we just developed for studying matrix eigenvalue problems carry over to certain problems involving differential equations. The matrix $\mathbf{A} + \epsilon\mathbf{B}$ in (8.1) could in principle be replaced with a linear operator in some Banach space, for example a differential operator acting on a space of functions satisfying some kind of side conditions. Whether the general formal methods apply in any particular case depends on whether the operators (or their inverses if they are invertible) are Fredholm, with index zero, so the Fredholm alternative applies. While this is the case for many differential operators acting on common spaces of functions, we will not need the general theory, instead using more common sense ideas in each particular case.

An example of this theme is the problem of looking for periodic solutions $y(t)$ of Mathieu's equation

$$y''(t) + [\delta + \epsilon\cos(t)]\, y(t) = 0\,. \tag{8.15}$$

Here, δ and ϵ are real parameters, and we will want to be thinking of ϵ as being small (compared to δ). The general theory of linear differential equations with periodic coefficients is known as *Floquet theory*. We will mention a few of the relevant points here.

Suppose we find two linearly independent solutions, say $y_1(t)$ and $y_2(t)$, of (8.15). Then, because the coefficient of $y(t)$ is periodic with period 2π, the two functions $\tilde{y}_1(t) := y_1(t+2\pi)$ and $\tilde{y}_2(t) := y_2(t+2\pi)$ must again be solutions of the same differential equation. But this equation only has two linearly independent solutions, so there must be some 2×2 matrix $\mathbf{T}$ of constants such that

$$\begin{bmatrix} \tilde{y}_1(t) \\ \tilde{y}_2(t) \end{bmatrix} = \mathbf{T} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}. \tag{8.16}$$

The matrix $\mathbf{T}$ is called the *monodromy matrix*.

Let us summarize the basic properties of the monodromy matrix $\mathbf{T}$. First, the entries of $\mathbf{T}$ are real constants if the solutions $y_1(t)$ and $y_2(t)$ are real. Second, we have the following result.

Proposition 7 *If $y_1(t)$ and $y_2(t)$ are linearly independent solutions of Mathieu's equation (8.15) then* $\det(\mathbf{T}) = 1$.

Proof: It can be directly checked using (8.15) that the Wronskian

$$W[y_1, y_2](t) := y_1(t)y_2'(t) - y_1'(t)y_2(t) \tag{8.17}$$

is independent of t. This is a special case of Abel's Theorem from the theory of ordinary differential equations. Now, if (8.16) holds, then so does

$$\begin{bmatrix} \tilde{y}_1(t) & \tilde{y}_1'(t) \\ \tilde{y}_2(t) & \tilde{y}_2'(t) \end{bmatrix} = \mathbf{T} \begin{bmatrix} y_1(t) & y_1'(t) \\ y_2(t) & y_2'(t) \end{bmatrix} \tag{8.18}$$

because the elements of $\mathbf{T}$ are constants. Using the definition of $\tilde{y}_1(t)$ and $\tilde{y}_2(t)$ this can be written as

$$\begin{bmatrix} y_1(t+2\pi) & y_1'(t+2\pi) \\ y_2(t+2\pi) & y_2'(t+2\pi) \end{bmatrix} = \mathbf{T} \begin{bmatrix} y_1(t) & y_1'(t) \\ y_2(t) & y_2'(t) \end{bmatrix}. \tag{8.19}$$

The proposition now follows upon taking the determinant of both sides and using Abel's Theorem. □

Thirdly, we have the following.

Proposition 8 *If* $\mathbf{T}$ *is the monodromy matrix corresponding to linearly independent solutions* $y_1(t)$ *and* $y_2(t)$ *of Mathieu's equation (8.15), and if* $\hat{\mathbf{T}}$ *is the monodromy matrix corresponding to two different linearly independent solutions* $\hat{y}_1(t)$ *and* $\hat{y}_2(t)$ *of the same equation, then there exists an invertible* 2×2 *matrix* $\mathbf{C}$ *of constants such that*

$$\hat{\mathbf{T}} = \mathbf{C}\mathbf{T}\mathbf{C}^{-1} . \tag{8.20}$$

In particular, this implies that the eigenvalues of the monodromy matrix are independent of the particular choice of starting functions $y_1(t)$ *and* $y_2(t)$.

Proof: This is a direct calculation; take $\mathbf{C}$ to be the matrix of constants relating the solutions $\hat{y}_1(t)$ and $\hat{y}_2(t)$ to the solutions $y_1(t)$ and $y_2(t)$. □

Suppose that μ is an eigenvalue of $\mathbf{T}$. There is then a particular linear combination $y_\mu(t)$ of $y_1(t)$ and $y_2(t)$ that satisfies

$$y_\mu(t + 2\pi) = \mu y_\mu(t) . \tag{8.21}$$

This means that if ρ is a number so that $\mu = e^{2\pi\rho}$, then $y_\mu(t)$ can be written in the form

$$y_\mu(t) = e^{\rho t} u_\rho(t) \qquad \text{where} \qquad u_\rho(t + 2\pi) = u_\rho(t) . \tag{8.22}$$

The solutions are therefore periodic solutions times exponentials, a fact that is known in solid state physics as the Bloch theorem. The eigenvalues μ of $\mathbf{T}$ are called the *Floquet multipliers* of (8.15). They do not depend on the particular choice of starting solutions $y_1(t)$ and $y_2(t)$.

Since the solutions $y_1(t)$ and $y_2(t)$ used to generate the monodromy matrix could be taken to be real, the eigenvalues of $\mathbf{T}$ must either be both real themselves or form a complex-conjugate pair. Also, since $\det(\mathbf{T}) = 1$ the product of the eigenvalues is one. This means that if the Floquet multipliers are not real, then they necessarily lie on the unit circle. If both Floquet multipliers of (8.15) are distinct and lie on the unit circle, so that $|\mu| = 1$, then the solutions of (8.15) cannot grow in t, just oscillate. This is a *stable* case of Mathieu's equation. On the other hand if the eigenvalues are real, then because $\det(\mathbf{T}) = 1$ there is one inside and one outside and consequently one solution is exponentially growing in t and one solution is exponentially decaying in t. This is an *unstable* case for Mathieu's equation. On the boundary between stability and instability there are two possibilities: either $\mu_1 = \mu_2 = 1$ (the *periodic* case) or $\mu_1 = \mu_2 = -1$ (the *antiperiodic* case).

Mathieu's equation (8.15) will be in one of the four cases, stable, unstable, periodic, or antiperiodic, depending on the values of the parameters δ and ϵ. If one restricts to a typical curve $\delta = \delta(\epsilon)$, then the equation (8.15) will be periodic or antiperiodic at only isolated points on the curve. These points separate ranges of stability and instability along the curve $\delta = \delta(\epsilon)$. This suggests that we should expect the periodic and antiperiodic points for Mathieu's equation in the (δ, ϵ)-plane to lie on curves themselves, curves which separate whole two-dimensional regions of stability and instability.

Let us seek the curves $\delta = \delta(\epsilon)$ along which Mathieu's equation has a 2π-periodic solution. The idea is that we can see exactly what is going on if $\epsilon = 0$ so we should treat ϵ as a perturbation to compute approximations to the curves $\delta(\epsilon)$ valid for small ϵ. This is an eigenvalue problem of the sort we considered previously. The eigenvalue is $\lambda = -\delta$ and the space in which we seek eigenvectors is the space of twice differentiable functions $y(t)$ that satisfy 2π-periodic boundary conditions: $y(t+2\pi) = y(t)$. We should try to apply the same methods as we did previously, namely seek solutions in the form of asymptotic expansions:

$$\begin{aligned} \delta &\sim \delta_0 + \epsilon\delta_1 + \epsilon^2\delta_2 + \dots \\ y(t) &\sim y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \dots \end{aligned} \tag{8.23}$$

presumed to be valid in the limit $\epsilon \downarrow 0$ and substitute into (8.15), collecting like powers of ϵ.

At leading order, we find

$$y_0''(t) + \delta_0 y_0(t) = 0 . \tag{8.24}$$

The general solution of this equation is $y_0(t) = A\cos(\sqrt{\delta_0}t) + B\sin(\sqrt{\delta_0}t)$ for arbitrary constants A and B. This solution is only periodic if $\delta_0 \geq 0$. Furthermore, $y_0(t)$ will satisfy $y_0(t + 2\pi) = y_0(t)$ only if

$$\delta_0 = 0, 1, 4, 9, \dots \qquad \text{that is} \qquad \delta_0 = n^2 , \quad n \in \mathbb{Z} . \tag{8.25}$$

We could pick any of these, but let us chase up what happens if $\delta_0 = 1$. At the next order, we find the equation

$$y_1''(t) + y_1(t) = -\delta_1 y_0(t) - \cos(t) y_0(t) \,. \tag{8.26}$$

Since we already know two homogeneous solutions of this equation, we can solve for $y_1(t)$ using variation of parameters, *i.e.* seeking a solution in the form $y_1(t) = a(t)\cos(t) + b(t)\sin(t)$. So

$$y_1'(t) = -a(t)\sin(t) + b(t)\cos(t) + a'(t)\cos(t) + b'(t)\sin(t) \,. \tag{8.27}$$

Since we have introduced two unknown functions $a(t)$ and $b(t)$, we can place one arbitrary constraint on them, so we take as our constraint

$$a'(t)\cos(t) + b'(t)\sin(t) = 0 \,. \tag{8.28}$$

This removes the last two terms from $y_1'(t)$ from the picture. Differentiating again and inserting into (8.26), we get

$$-a'(t)\sin(t) + b'(t)\cos(t) = -\delta_1 A\cos(t) - \delta_1 B\sin(t) - A\cos^2(t) - B\sin(t)\cos(t) \,. \tag{8.29}$$

Using both (8.28) and (8.29) we can solve for $a'(t)$ and $b'(t)$:

$$\begin{aligned} a'(t) &= \delta_1 A\sin(t)\cos(t) + \delta_1 B\sin^2(t) + A\sin(t)\cos^2(t) + B\sin^2(t)\cos(t) \\ b'(t) &= -\delta_1 A\cos^2(t) - \delta_1 B\sin(t)\cos(t) - A\cos^3(t) - B\sin(t)\cos^2(t) \,. \end{aligned} \tag{8.30}$$

If $y_1(t)$ is going to satisfy $y_1(t+2\pi) = y_1(t)$, then we need $a(t)$ and $b(t)$ to also be periodic with period 2π. Now, from (8.30) we see that both $a'(t)$ and $b'(t)$ are indeed 2π-periodic. But when we integrate to get $a(t)$ and $b(t)$, we will again have 2π-periodic functions only if the *average* or *mean value* of $a'(t)$ and $b'(t)$ over a period is equal to zero. Otherwise, the integral of the constant mean value will give rise to linear growth of $a(t)$ and $b(t)$.

This means that to find $y_1(t)$ in the class of 2π-periodic functions, we must satisfy the *solvability conditions*:

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} a'(t)\,dt &= \frac{\delta_1 B}{2} = 0 \\ \frac{1}{2\pi}\int_0^{2\pi} b'(t)\,dt &= -\frac{\delta_1 A}{2} = 0 \,. \end{aligned} \tag{8.31}$$

The only way we can solve these equations without taking $A = B = 0$ (which would not be useful because it would just kill the leading order solution) is to take $\delta_1 = 0$. Then, for $a(t)$ and $b(t)$ we get

$$\begin{aligned} a(t) &= -\frac{A}{3}\cos^3(t) + \frac{B}{3}\sin^3(t) + a_0 \\ b(t) &= -A\sin(t) + \frac{A}{3}\sin^3(t) + \frac{B}{3}\cos^3(t) + b_0 \,, \end{aligned} \tag{8.32}$$

where a_0 and b_0 are integration constants. As was the case in our study of eigenvalue problems, when one has to impose solvability conditions on the right-hand side of a linear equation, the solution one obtains is only determined up to homogeneous solutions. Since a_0 and b_0 simply augment the values of A and B which are already arbitrary, we can simply take $a_0 = b_0 = 0$ at this stage. Thus,

$$y_1(t) = -\frac{A}{3}\cos^4(t) + \frac{B}{3}\sin^3(t)\cos(t) - A\sin^2(t) + \frac{A}{3}\sin^4(t) + \frac{B}{3}\cos^3(t)\sin(t) \,. \tag{8.33}$$

To get a nontrivial correction to δ, we evidently must go to higher order. At the next order, the equation we find from (8.15) is

$$y_2''(t) + y_2(t) = -\delta_2 y_0(t) - \cos(t) y_1(t) \,. \tag{8.34}$$

We solve this equation also by variation of parameters, taking $y_2(t) = c(t)\cos(t) + d(t)\sin(t)$. Skipping the intermediate steps, we find that

$$\begin{aligned} c'(t) &= \delta_2 y_0(t)\sin(t) + y_1(t)\sin(t)\cos(t) \\ d'(t) &= -\delta_2 y_0(t)\cos(t) - y_1(t)\cos^2(t)\,. \end{aligned} \tag{8.35}$$

Again, to solve for $y_2(t)$ in the class of functions satisfying $y_2(t+2\pi) = y_2(t)$, we need $c(t)$ and $d(t)$ also to be 2π-periodic. This means we need to impose the solvability conditions

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} c'(t)\,dt &= \frac{\delta_2 B}{2} + \frac{B}{24} = 0 \\ \frac{1}{2\pi}\int_0^{2\pi} d'(t)\,dt &= -\frac{\delta_2 A}{2} + \frac{5A}{24} = 0\,. \end{aligned} \tag{8.36}$$

There are two ways to satisfy these solvability conditions (again, without taking both $A = B = 0$). One could take

$$A = 0 \quad \text{and} \quad \delta_2 = -\frac{1}{12}\,, \tag{8.37}$$

leaving B arbitrary, or one could take

$$B = 0 \quad \text{and} \quad \delta_2 = \frac{5}{12}\,, \tag{8.38}$$

leaving A arbitrary. Although having satisfied the solvability conditions one could now find $c(t)$ and $d(t)$ as 2π-periodic functions and consequently obtain $y_2(t)$, we will stop at having obtained the first interesting corrections to two distinct curves $\delta = \delta(\epsilon)$ passing through $\delta = 1$ at $\epsilon = 0$ along which Mathieu's equation has periodic solutions with period 2π. One solution, of the form

$$y(t) \sim B\left[\sin(t) + \frac{\epsilon}{3}\left(\sin^3(t)\cos(t) + \sin(t)\cos^3(t)\right)\right] \tag{8.39}$$

exists when

$$\delta(\epsilon) \sim 1 - \frac{\epsilon^2}{12} \quad \text{as} \quad \epsilon \to 0\,, \tag{8.40}$$

and another solution, of the form

$$y(t) \sim A\left[\cos(t) + \frac{\epsilon}{3}\left(\sin^4(t) - 3\sin^2(t) - \cos^4(t)\right)\right] \tag{8.41}$$

exists when

$$\delta(\epsilon) \sim 1 + \frac{5\epsilon^2}{12} \quad \text{as} \quad \epsilon \to 0\,. \tag{8.42}$$

Note that in this problem we started with a solution of fundamental period 2π when $\epsilon = 0$ and that this solution remained 2π-periodic under perturbation (as long as we detune δ accordingly). However, if we had started with value of $\delta_0 = n^2 \geq 4$, the solution at $\epsilon = 0$ would have had a shorter fundamental period than 2π, *i.e.* the fundamental period would have been $2\pi/n$. Now the general Floquet theory does not guarantee that shorter periods can exist; we must regard the $\epsilon = 0$ case as being special in this regard. What we should expect is that we can tune δ in such a way that under perturbation the solution continues to have period 2π, but not that it continues to have period $2\pi/n$. Thus all averages should be taken over intervals of length 2π, no matter what the fundamental period of the solution is when $\epsilon = 0$. Similar remarks apply to the antiperiodic solutions which generally have fundamental period 4π but which can have shorter fundamental periods when $\epsilon = 0$. Here one averages over intervals of length 4π.

8.3 Weakly Nonlinear Oscillations and Strained Coordinates

Let us continue our discussion of periodic phenomena by passing to a nonlinear context. Although what we are about to do can be generalized to many types of problems, we can illustrate the main points with a nonlinear differential equation of the form

$$y'' + F(y) = 0 \tag{8.43}$$

for some unknown function $y(t)$. This problem is autonomous, meaning that it does not contain t explicitly. Although we cannot solve this problem explicitly except for special cases of the function $F(y)$, we can make some headway by noticing that there is a constant of motion sometimes called an *energy integral*. To find this, multiply (8.43) by y' and notice that the left-hand side becomes a perfect derivative:

$$\frac{d}{dt}\left(\frac{1}{2}[y'(t)]^2 + V(y(t))\right) = 0 \tag{8.44}$$

where $V(y)$ is any function satisfying $V'(y) = F(y)$. This means that the energy function

$$E(y, y') := \frac{1}{2}[y']^2 + V(y) \tag{8.45}$$

is constant in t when $y(t)$ solves (8.43). It is generally a different constant for different solutions. You can find the appropriate value of E for any solution if you know the values of y and y' at any given time t, for example at the initial instant $t = 0$.

Usually $F(y)$ has zeros, so possibly by shifting y by a constant, we can assume that $F(0) = 0$. Let us further assume that $F'(0) > 0$. This means that near $y = 0$,

$$V(y) = \int_0^y F(s)\, ds = \frac{F'(0)}{2}y^2 + O(y^3) \qquad \text{as} \qquad y \to 0\,. \tag{8.46}$$

Therefore, sufficiently close to the origin in the (y, y') phase plane, the level sets of $E(y, y')$ will be approximate ellipses:

$$E(y, y') \sim \frac{1}{2}[y']^2 + \frac{F'(0)}{2}y^2 = E_0\,. \tag{8.47}$$

Note that if $F'(0) < 0$, then instead of ellipses, we have hyperbolas because the energy is locally indefinite. The true level curves are of course not ellipses. However, they will be closed curves at least close enough to the origin in the phase plane. The solutions $y(t)$ to (8.43) that live on these closed level curves of $E(y, y')$ are necessarily periodic functions of t.

Under the condition $F'(0) > 0$ we can try to find the corresponding small-amplitude periodic solutions of (8.43) using perturbation theory. By rescaling the time by a factor of $\sqrt{F'(0)}$ and by an appropriate rescaling of y, we can take the equation (8.43) in the form

$$y'' + y + \epsilon G(y) = 0 \tag{8.48}$$

where $G(0) = G'(0) = 0$. Let us study in detail some particular cases.

8.3.1 Weak cubic nonlinearity.

Suppose that $G(y) = ry^3$. We will seek the periodic solution of (8.48) satisfying the initial conditions $y(0) = \alpha$ and $y'(0) = \beta$ in the form of an asymptotic power series

$$y(t) \sim y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \dots\,, \tag{8.49}$$

and attempt to determine the coefficients systematically by substituting into (8.48) and collecting like powers of ϵ. At leading order we find

$$y_0'' + y_0 = 0 \tag{8.50}$$

which has the periodic solution $y_0(t) = A\cos(t) + B\sin(t)$. To satisfy the initial conditions (which are independent of ϵ) we have to take $A = \alpha$ and $B = \beta$. At order ϵ we obtain

$$y_1'' + y_1 = -ry_0^3 = -r\alpha^3\cos^3(t) - 3r\alpha^2\beta\sin(t)\cos^2(t) - 3r\alpha\beta^2\sin^2(t)\cos(t) - r\beta^3\sin^3(t)\,. \tag{8.51}$$

Solve for $y_1(t)$ using variation of parameters. Setting $y_1(t) = a(t)\cos(t) + b(t)\sin(t)$ subject to $a'(t)\cos(t) + b'(t)\sin(t) = 0$ and (8.51), we find

$$\begin{aligned} a'(t) &= r\alpha^3\sin(t)\cos^3(t) + 3r\alpha^2\beta\sin^2(t)\cos^2(t) + 3r\alpha\beta^2\sin^3(t)\cos(t) + r\beta^3\sin^4(t) \\ b'(t) &= -r\alpha^3\cos^4(t) - 3r\alpha^2\beta\sin(t)\cos^3(t) - 3r\alpha\beta^2\sin^2(t)\cos^2(t) - r\beta^3\sin^3(t)\cos(t)\,. \end{aligned} \tag{8.52}$$

To integrate nonlinear trigonometric expressions like these, a most useful method is to rewrite them as linear expressions in harmonics, that is, to find their Fourier series. I like to think of it systematically in the following way:

1. First, replace $\sin(t)$ and $\cos(t)$ by their equivalents in terms of exponentials e^{it} and e^{-it}.
2. Next, expand out all products until you have a sum of individual terms proportional to e^{ikt} for various integers k.
3. Finally replace e^{ikt} by $\cos(kt) + i\sin(kt)$ to get back to sines and cosines.

So, for example,

$$\begin{aligned} \sin^2(t)\cos^2(t) &= -\frac{1}{16}\left(e^{it} - e^{-it}\right)^2\left(e^{it} + e^{-it}\right)^2 \\ &= -\frac{1}{16}\left(e^{2it} - 2 + e^{-2it}\right)\left(e^{2it} + 2 + e^{-2it}\right) \\ &= -\frac{1}{16}\left(e^{4it} + e^{-4it} - 2\right) \\ &= \frac{1}{8} - \frac{1}{8}\cos(4t)\,. \end{aligned} \tag{8.53}$$

Using this method, we find

$$\begin{aligned} a'(t) &= \frac{r\alpha^3}{8}\left(2\sin(2t) + \sin(4t)\right) + \frac{3r\alpha^2\beta}{8}\left(1 - \cos(4t)\right) \\ &\quad + \frac{3r\alpha\beta^2}{8}\left(2\sin(2t) - \sin(4t)\right) + \frac{r\beta^3}{8}\left(3 - 4\cos(2t) + \cos(4t)\right) \\ b'(t) &= -\frac{r\alpha^3}{8}\left(3 + 4\cos(2t) + \cos(4t)\right) - \frac{3r\alpha^2\beta}{8}\left(2\sin(2t) + \sin(4t)\right) \\ &\quad - \frac{3r\alpha\beta^2}{8}\left(1 - \cos(4t)\right) - \frac{r\beta^3}{8}\left(2\sin(2t) - \sin(4t)\right)\,. \end{aligned} \tag{8.54}$$

Now, since we already satisfied the initial conditions $y(0) = \alpha$ and $y'(0) = \beta$ with the leading order solution, we need to find $a(t)$ and $b(t)$ subject to the initial conditions $a(0) = b(0) = 0$. This just means integrating the right-hand sides of (8.54) from zero to t, a straightforward task in the Fourier series representation.

Although everything works just as well for arbitrary α and β, let us simplify our lives in this example by taking $\beta = 0$. Then

$$\begin{aligned} a(t) &= \int_0^t a'(s)\,ds = -\frac{r\alpha^3}{8}\left(\cos(2t) - 1\right) - \frac{r\alpha^3}{32}\left(\cos(4t) - 1\right) \\ b(t) &= \int_0^t b'(s)\,ds = -\frac{3r\alpha^3}{8}t - \frac{r\alpha^3}{4}\sin(2t) - \frac{r\alpha^3}{32}\sin(4t)\,. \end{aligned} \tag{8.55}$$

Consequently our formula for the first order correction to the solution of (8.48) is

$$y_1(t) = -\frac{r\alpha^3}{32}\left[4\cos(2t)\cos(t) + \cos(4t)\cos(t) - 5\cos(t) + 8\sin(2t)\sin(t) + \sin(4t)\sin(t) + 12t\sin(t)\right]. \tag{8.56}$$

Let us look carefully at the final term in the formula (8.56) for $y_1(t)$. Unlike all the other terms, this term is not a periodic function. It grows linearly with t. This term is problematic for two reasons:

1. We know on general principles that the true solution we seek is a periodic function of t.

2. The asymptotic approximation $y(t) \sim y_0(t)+\epsilon y_1(t)$, which by assumption must satisfy $\epsilon y_1(t) = o(y_0(t))$, can only do so if $t \ll \epsilon^{-1}$. Otherwise the second term will not be small compared to the first as $\epsilon \to 0$. In other words, the approximation we have developed can only be valid on time scales that are not too long.

Such a problematic term in an asymptotic expansion is called a *secular term*. The etymology of this word is interesting: it comes from the French literature on celestial mechanics. In French, *siècle* means "century" and when researchers saw these terms appearing in their expansions, they noted that given the size of ϵ, these terms would appear to affect the orbits of the planets on time scales that were about 100 years long. In this case the final term in (8.56) is more properly called a *mixed secular term* because it includes both linear growth and trigonometric oscillation. Also, the terminology is sometimes overloaded in the sense that people often refer to the terms appearing on the right-hand side of (8.51) that lead to the unbounded terms $y_1(t)$ as "secular terms".

We need to think about how to remove the secular terms from the picture if we are going to obtain an accurate picture of the dynamics. The main idea here is that *for nonlinear oscillators the frequency of oscillation depends on the amplitude*. If we look at our method of asymptotic expansion, we find that at each order we will have to solve an equation with the same left-hand side as (8.51) and the solution will therefore involve functions of period 2π along with some unbounded terms. But our expansion procedure does not allow the fundamental period to be anything but 2π. In the true problem, the period may be close to 2π when ϵ is small, but will never be exactly 2π. The appearance of secular terms indicates that we got the frequency slightly wrong, and this means that the leading-order approximation $y_0(t)$ will get 180° out of phase with the true solution $y(t)$ after a long time, so the two will no longer be close as $\epsilon \to 0$.

We can try to adjust the frequency using a method called the *method of strained coordinates* or *Lindstedt's method*, although the main ideas were due to Poincaré. Going back to the leading-order problem for $y_0(t)$, suppose we took the solution in the form

$$y_0(t;\epsilon) = \alpha\cos(\omega t) + \beta\sin(\omega t), \tag{8.57}$$

where ω is a fundamental angular frequency to be determined. We suppose that it has an asymptotic expansion in powers of ϵ:

$$\omega \sim 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \dots, \tag{8.58}$$

as $\epsilon \to 0$. The coefficients ω_k are numbers that have to be determined. If we now re-expand $y_0(t;\epsilon)$ for t fixed as $\epsilon \to 0$, we can just use the Taylor series for sine and cosine to find

$$y_0(t;\epsilon) = \alpha\cos(t) + \beta\sin(t) + \epsilon\omega_1\left(-\alpha t\sin(t) + \beta t\cos(t)\right) + O(\epsilon^2). \tag{8.59}$$

Still seeking a solution of the form $y(t) \sim y_0(t;\epsilon) + \epsilon y_1(t) + \dots$, we substitute into (8.48) using the expansion (8.59) and collect like powers of ϵ. The leading order equation is satisfied automatically, and the equation that replaces (8.51) at order ϵ is

$$y_1'' + y_1 = -r\left(\alpha\cos(t) + \beta\sin(t)\right)^3 - \omega_1\left[-2\alpha\cos(t) - 2\beta\sin(t)\right]. \tag{8.60}$$

Now we have the parameter ω_1 at our disposal, and the idea is to choose it precisely so that this equation is solvable in the class of periodic functions. Let us see how this works in practice by assuming for simplicity of

exposition that $\beta = 0$. Solving for $y_1(t)$ by variation of parameters by setting $y_1(t) = a(t)\cos(t) + b(t)\sin(t)$, we find

$$\begin{aligned} a'(t) &= r\alpha^3 \sin(t)\cos^3(t) - 2\alpha\omega_1 \sin(t)\cos(t) \\ b'(t) &= -r\alpha^3 \cos^4(t) + 2\alpha\omega_1 \cos^2(t)\,. \end{aligned} \tag{8.61}$$

The solvability condition that $a(t)$ and $b(t)$ be bounded functions of t is that the average values of the right-hand sides in (8.61) over the fundamental period are both zero. By a direct calculation made easier by first finding the Fourier series representations of the nonlinear terms we find

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} a'(s)\,ds &= 0 \\ \frac{1}{2\pi}\int_0^{2\pi} b'(s)\,ds &= -\frac{3r\alpha^3}{8} + \omega_1\alpha\,. \end{aligned} \tag{8.62}$$

Therefore, $y_1(t)$ will be a bounded oscillatory function of t provided we take

$$\omega_1 = \frac{3r\alpha^2}{8}\,. \tag{8.63}$$

Now it is possible to solve for $y_1(t)$, although we already have important information in the leading-order approximation. Namely, the approximation

$$y(t) \sim \alpha\cos\left(\left[1 + \frac{3\epsilon r\alpha^2}{8}\right]t\right) \tag{8.64}$$

is now uniformly valid over time intervals that are much longer than was possible with our naive expansion procedure. In fact, this approximation first fails when $t \sim \epsilon^{-2}$, an order of magnitude improvement over $t \sim \epsilon^{-1}$. Higher-order corrections both to y_0 and to the frequency ω can be calculated systematically. One just uses (8.59) and picks the value of ω_k so that the solvability condition is satisfied in the differential equation for $y_k(t)$.

8.3.2 Weak nonlinearity and weak damping.

The same reasoning applies in certain nonconservative contexts as well. For example, consider the equation

$$y'' + y + \epsilon\left[ry^2 + y'\right] = 0\,, \tag{8.65}$$

where we are now considering $\epsilon > 0$. This equation does not have periodic solutions. If we multiply by y' we see that (8.65) can be written in the form

$$\frac{d}{dt}\left[\frac{1}{2}[y']^2 + \frac{1}{2}y^2 + \frac{\epsilon r}{3}y^3\right] = -\epsilon[y']^2\,. \tag{8.66}$$

So with $\epsilon > 0$, the energy integral is no longer constant along trajectories. Instead, it is decreasing. Since near the origin in the phase plane the energy is concave up, the trajectories near the origin all have to spiral slowly inwards toward the minimum. Even though the solutions are not periodic, they certainly do not grow with t, so we can still try to solve the problem by identifying secular terms and removing them using the method of strained coordinates.

For convenience let us take the initial conditions $y(0) = 0$, $y'(0) = \beta$. We substitute into (8.65) the asymptotic ansatz:

$$\begin{aligned} y(t) &\sim ue^{i\omega^+ t} + u^* e^{-i\omega^- t} + \epsilon y_1(t) + \epsilon^2 y_2(t) + \ldots \\ \omega^+ &\sim 1 + \epsilon\omega_1^+ + \epsilon^2\omega_2^+ + \ldots \\ \omega^- &\sim 1 + \epsilon\omega_1^- + \epsilon^2\omega_2^- + \ldots \end{aligned} \tag{8.67}$$

and collect like powers of ϵ. We have chosen to write the leading order solution in terms of exponentials this time for future convenience. Note that we are expanding both frequencies individually, that is, we are trying to use two "strained coordinates" in this problem. The initial conditions require that $u = -i\beta/2$. Taylor expanding the exponentials for small ϵ gives

$$y(t) = -\frac{i\beta}{2}e^{it} + \frac{i\beta}{2}e^{-it} + \frac{\epsilon\beta\omega_1^+ t}{2}e^{it} + \frac{\epsilon\beta\omega_1^- t}{2}e^{-it} + \epsilon y_1(t) + O(\epsilon^2)\,. \tag{8.68}$$

The equation we obtain at order ϵ for $y_1(t)$ is

$$y_1'' + y_1 = -i\beta\omega_1^+ e^{it} + i\beta\omega_1^- e^{-it} + \frac{r\beta^2}{4}\left(e^{2it} - 2 + e^{-2it}\right) - \frac{\beta}{2}\left(e^{it} + e^{-it}\right)\,. \tag{8.69}$$

For this equation to be solvable in the class of bounded functions of t, there can be no terms proportional to either e^{it} or e^{-it} on the right-hand side (this can be deduced directly by variation of parameters). Thus, there are two solvability conditions obtained by taking the coefficients of e^{it} and e^{-it} to both vanish:

$$\omega_1^+ = \frac{i}{2} \quad \text{and} \quad \omega_1^- = -\frac{i}{2}\,. \tag{8.70}$$

With the first corrections to the frequencies ω^+ and ω^- determined in this way, we can now solve for $y_1(t)$ subject to the initial conditions $y_1(0) = y_1'(0) = 0$, and we are guaranteed to obtain a uniformly bounded correction to y_0.

There are several features to point out here. The most obvious is the fact that the corrections ω_1^+ and ω_1^- to the frequencies of oscillation are both imaginary. This means that the leading order approximation takes the form

$$y(t) \sim \beta e^{-\epsilon t/2}\sin(t) \tag{8.71}$$

that is, the imaginary components of the frequency capture the decay of the periodic solution due to the damping. Furthermore, the leading term in the solution is not of the form $\beta\sin(\omega t)$ for any ω, even complex valued. In other words, it was essential that we used exponentials in our expansion rather than trigonometric functions since we want to capture gradual changes in amplitude. This is a clue that in general it is wise to work with exponentials if one does now know in advance whether the solutions will be exactly periodic or not. The next observation is that unlike in the previous example, the nonlinearity plays no role in adjusting the frequency at order ϵ, even though it is a "stronger" nonlinearity, being of second order instead of third. The reason it does not affect the frequency at this order is that it generates only harmonics that are not in *resonance* with the left-hand side of (8.69). In particular, the second harmonic will appear explicitly in the solution for $y_1(t)$ subject to the solvability conditions.

8.4 The Method of Multiple Scales

The method of strained coordinates is one of many methods that have been invented to aid in the systematic removal of secular terms from asymptotic expansions when it is known ahead of time that they do not belong for one reason or another. Another such method of note is the *method of averaging* which is particularly important in investigating perturbations of completely integrable Hamiltonian mechanical systems. This method is discussed in many textbooks on classical mechanics.

One disadvantage of the method of strained coordinates is that it requires knowing in advance some information about what kind of phenomena to expect from the solutions of the differential equation under investigation. You have to guess what "coordinates" should be "strained". There is, however, another very powerful formalism that removes some of the guesswork from the problem. Avoiding specifics as it does, the method can be applied systematically to a huge variety of singular perturbation problems, including both oscillatory initial-value problems and also singular boundary-value problems. The cost for all of this is that if applied haphazardly, the method can give nonsense results. The formalism we are speaking about is the *method of multiple scales.*

The idea of the method of multiple scales is that the hallmark of all singular perturbation problems for ordinary differential equations is the presence of many different scales in the exact solution of the problem.

For example, in the boundary-value problems we looked at, the total solution was built out of smooth parts mediated by thin layers in which the solution changes violently. Although we did not discuss it in the course, the same phenomena also occur when one studies so-called *relaxation oscillations* which are periodic solutions of differential equations (the most famous equation of this type is the van der Pol equation describing a nonlinear electric circuit) in which fundamentally different physical processes control the dynamics in different parts of the period of motion. Also, in the initial-value problems we have been looking at, the solutions involve a *microscale* (the unperturbed period of motion) and one or more *macroscales* (the characteristic timescales of amplitude modulation or decay). Somehow all of these different scales should be built into the leading-order approximation as $\epsilon \to 0$ in order to obtain uniform validity.

Although systematic, the method is best illustrated by example, so let us recall the problem of the oscillator with weak cubic nonlinearity:

$$y'' + y + \epsilon r y^3 = 0\,, \tag{8.72}$$

with $y(0) = \alpha$ and $y'(0) = 0$. We want to anticipate the possibility that y could depend explicitly on several different time scales:

$$T_0 := t\,, \quad T_1 := \epsilon t\,, \quad T_2 := \epsilon^2 t\,, \quad \ldots \quad T_k := \epsilon^k t\,, \tag{8.73}$$

where the number of scales to take varies from problem to problem. The variable T_0 is usually called the *fast time* and the other variables are *slow times*. We take the point of view that $y(t)$ is best represented as a function Y depending explicitly on all of these time scales:

$$y(t;\epsilon) = Y(T_0, T_1, T_2, \ldots, T_k; \epsilon)\,. \tag{8.74}$$

Now if this is the case, we can write down the form of all derivatives with respect to time t by the chain rule. So

$$y'(t;\epsilon) = \frac{\partial Y}{\partial T_0} + \epsilon \frac{\partial Y}{\partial T_1} + \epsilon^2 \frac{\partial Y}{\partial T_2} + \ldots + \epsilon^k \frac{\partial Y}{\partial T_k}\,. \tag{8.75}$$

And similarly

$$y''(t;\epsilon) = \frac{\partial^2 Y}{\partial T_0^2} + 2\epsilon \frac{\partial^2 Y}{\partial T_0 \partial T_1} + \epsilon^2 \left[\frac{\partial^2 Y}{\partial T_1^2} + 2\frac{\partial^2 Y}{\partial T_0 \partial T_2}\right] + \ldots + \epsilon^{2k} \frac{\partial^2 Y}{\partial T_k^2}\,. \tag{8.76}$$

Now when we substitute into (8.72) we will have swapped an ordinary differential equation for y for a partial differential equation in $k+1$ variables for Y. This seems to have made matters worse, except that we can take advantage of the presence of many independent variables to choose the dependence of Y on the slower times to remove secular terms from the asymptotic expansion of Y. So, into

$$\frac{\partial^2 Y}{\partial T_0^2} + 2\epsilon \frac{\partial^2 Y}{\partial T_0 \partial T_1} + \epsilon^2 \left[\frac{\partial^2 Y}{\partial T_1^2} + 2\frac{\partial^2 Y}{\partial T_0 \partial T_2}\right] + \ldots + \epsilon^{2k} \frac{\partial^2 Y}{\partial T_k^2} + Y + \epsilon r Y^3 = 0 \tag{8.77}$$

we substitute the asymptotic expansion

$$Y \sim Y_0 + \epsilon Y_1 + \epsilon^2 Y_2 + \ldots \tag{8.78}$$

and collect like powers of ϵ.

The equation we obtain at order 1 is

$$\frac{\partial^2 Y_0}{\partial T_0^2} + Y_0 = 0\,. \tag{8.79}$$

The solution of this partial differential equation is simple to obtain by analogy with the ordinary differential equation $y'' + y = 0$. Namely,

$$Y_0 = A e^{iT_0} + A^* e^{-iT_0} \tag{8.80}$$

where $A = A(T_1, T_2, \ldots, T_k)$ is an unknown complex-valued function. Now, the equation we get at order ϵ is

$$\begin{aligned} \frac{\partial^2 Y_1}{\partial T_0^2} + Y_1 &= -r Y_0^3 - 2\frac{\partial^2 Y_0}{\partial T_0 \partial T_1} \\ &= -r A^3 e^{3iT_0} - 3r A^2 A^* e^{iT_0} - 3r A A^{*2} e^{-iT_0} - r A^{*3} e^{-3iT_0} - 2i \frac{\partial A}{\partial T_1} e^{iT_0} + 2i \frac{\partial A^*}{\partial T_1} e^{-iT_0}\,. \end{aligned} \tag{8.81}$$

To ensure that Y_1 be a bounded function of T_0, we need to satisfy the solvability conditions that the coefficients of the resonant terms $e^{\pm iT_0}$ on the right-hand side vanish. So we must take

$$2i\frac{\partial A}{\partial T_1} + 3r|A|^2 A = 0\,. \tag{8.82}$$

Then, Y_1 will be a bounded function of T_0, and in particular will be composed of third harmonics $e^{\pm 3iT_0}$ upon solving

$$\frac{\partial^2 Y_1}{\partial T_0^2} + Y_1 = -rA^3 e^{3iT_0} - rA^{*3} e^{-3iT_0}\,. \tag{8.83}$$

When we solve this equation for Y_1, we should remember that the A on the right-hand side is a function only of the variables $T_1, \ldots, T_k$ and is therefore a constant as far as T_0 is concerned.

We have thus determined the dependence of the leading order solution on the fast scale T_0 and the first slow scale T_1. In fact, this is enough to recover all of our previous results as we will now see. The differential equation we obtained for A is nonlinear, but it is easily solved as follows. If we multiply (8.82) through by A^* and subtract its complex conjugate, we get

$$\frac{\partial}{\partial T_1}|A|^2 = 0\,, \tag{8.84}$$

so the amplitude of A is a constant. Therefore, inserting $A = Re^{i\theta}$ into (8.82) with R being a constant, we get

$$-2\frac{\partial \theta}{\partial T_1} + 3rR^2 = 0 \qquad \text{or} \qquad \theta = \frac{3r}{2}R^2 T_1 + \theta_0 \tag{8.85}$$

where θ_0 might depend on the slower times T_2, T_3, and so on. Therefore, our leading-order approximation to the solution $y(t;\epsilon)$ is

$$y(t;\epsilon) \sim Y_0 = Re^{i\theta_0} e^{3i\epsilon r R^2 t/2} e^{it} + Re^{-i\theta_0} e^{-3i\epsilon r R^2 t/2} e^{-it} \tag{8.86}$$

where we have put back t in favor of T_0 and T_1. Satisfying the initial conditions $y(0) = \alpha$ and $y'(0) = 0$ to leading order requires that we take $R = \alpha/2$ and $\theta_0 = 0$ which recovers our previous result, namely

$$y(t;\epsilon) \sim Y_0 = \alpha \cos\left(\left[1 + \frac{3\epsilon r \alpha^2}{8}\right] t\right) \tag{8.87}$$

as $\epsilon \to 0$.

Chapter 9

Weakly Nonlinear Waves

While the method of multiple scales can be usefully applied to singular perturbation problems for ordinary differential equations, it is most well-known as a method that applies equally well to partial differential equations. This is especially true for nonlinear equations describing the propagation of waves, when we can treat the nonlinearity as a perturbation. This is the subject of weakly nonlinear waves.

9.1 Deriving Canonical Partial Differential Equations Using the Method of Multiple Scales

Recall that when we applied the method of multiple scales to ordinary differential equations, we did not obtain the asymptotic solution in a single step; rather we obtained a new differential equation for the complex amplitude A as a function of the slow time T_1 from the solvability condition. Before we could work out what the weakly nonlinear oscillations looked like, we had to solve this new equation for $A(T_1)$.

The same idea carries over when we look at nonlinear waves. We will see that our perturbation procedure leads us to write down partial differential equations for certain functions of slow variables in order to eliminate secular terms. However, the emphasis now shifts, because we will be less interested in solving these new equations directly, and more interested in observing the way that the *same few model equations always arise as solvability conditions.* These model equations are consequently very important in their own right since each solution of the model implies facts about every individual nonlinear wave problem in which the same model arises from a perturbation procedure. For this reason, we view these model equations as being *canonical.*

This idea of the same model equation appearing in many different problems in an asymptotic limit should be familiar to us. For example, we know that in turning point problems, no matter what the exact coefficients in the equation are, Airy's differential equation describes the local transition as long as the turning point is nondegenerate (a generic condition).

We will now illustrate how to use the method of multiple scales to analyze weakly nonlinear waves by studying a particular example. The nonlinear partial differential equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sin(u) = 0 \tag{9.1}$$

is called the *sine-Gordon equation.* The name comes from the fact that the class of equations with the term $\sin(u)$ in (9.1) replaced by an arbitrary, generally nonlinear, function of u has been known for many years as the family of *Klein-Gordon equations.* Thus the sine-Gordon equation (9.1) is a special case of a Klein-Gordon equation involving the sine function.

The sine-Gordon equation (9.1) describes directly at least two distinct physical phenomena:

1. The mechanical oscillations of a chain of coupled pendula, or any system that can be approximately modeled in this way, for example the torsional vibrations of base-pairs on RNA molecules. We can think of it this way: each oscillator in the chain is a pendulum with angular displacement $u_n(t)$ and the

corresponding restoring force $-\sin(u_n(t))$. But each pendulum is coupled to its left and right nearest neighbors by linear springs, so there is also a contribution to the force on the nth pendulum from each neighbor:

$$\frac{d^2u_n}{dt^2} = -\sin(u_n) + k(u_{n+1} - u_n) + k(u_{n-1} - u_n)\,. \tag{9.2}$$

Here k is a normalized spring constant of the coupling. If the displacements of neighboring pendula are close, then a continuum limit approach makes sense. In such an approach, we suppose the existence of some smooth function $u(x,t)$ of which $u_n(t)$ is a sampled value for $x = nh$. Here h is the *lattice spacing*. If we set $k = 1/h^2$, then Taylor expansion of $u(x,t)$ in x shows that (9.2) goes over into the sine-Gordon equation (9.1) in the *continuum limit* of $h \to 0$. This mechanical analogy is a good way to visualize the solutions of the sine-Gordon equation.

2. The dynamics of the relative phase of quantum mechanical wavefunctions on either side of a long superconducting Josephson junction.

Also, the sine-Gordon equation (9.1) is important mathematically for at least two reasons:

1. Its solutions intrinsically describe surfaces with constant negative curvature. The sine-Gordon equation is therefore of great importance in differential geometry.

2. It is a *completely integrable system*. This means that the sine-Gordon equation (9.1) is one of a handful of nonlinear partial differential equations for which there is a mathematical framework for solving certain initial value problems *exactly*. There exists a tool called the *inverse-scattering transform* for studying integrable equations like (9.1), which should be viewed as a nonlinear analogue of the Fourier transform for solving linear partial differential equations.

Although there is the tool of the inverse-scattering transform available for studying the sine-Gordon equation (9.1), we can also obtain a great deal of information somewhat more directly (although approximately rather than exactly) using perturbation theory. What we want to do is to consider solutions of (9.1) for which $u(x,t)$ is small. We can do this by introducing a small perturbation parameter ϵ and setting $u = \epsilon U$. The sine-Gordon equation (9.1) thus becomes, exactly,

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} + \frac{1}{\epsilon}\sin(\epsilon U) = 0\,. \tag{9.3}$$

Now since the sine is not a linear function, the equation for $U(x,t)$ contains ϵ in an essential way, and consequently $U(x,t)$ will depend on ϵ too. But since as $\epsilon \to 0$ we have the Taylor expansion

$$\frac{1}{\epsilon}\sin(\epsilon U) = U - \frac{\epsilon^2}{6}U^3 + \ldots = U + O(\epsilon^2)\,, \tag{9.4}$$

we see that the equation (9.3) for $U(x,t)$ is a small perturbation of a linear equation (the reduced equation) when ϵ is small. Now on this basis we would like to posit a perturbation expansion of $U(x,t)$ in powers of ϵ. Since only even powers of ϵ appear to all orders in the Taylor expansion (9.4), it might seem that a series in positive integer powers of $\delta := \epsilon^2$ would be appropriate. However, it turns out that there is some advantage to using odd powers of ϵ too; in any case if these orders were not necessary we would find out later by deducing that the corresponding coefficients in the asymptotic series were zero. So let us proceed with the expansion

$$U = U_0 + \epsilon U_1 + \epsilon^2 U_2 + O(\epsilon^3)\,, \tag{9.5}$$

and insert into (9.3), collecting powers of ϵ.

At order ϵ^0 we find, as promised, a linear reduced equation

$$\frac{\partial^2 U_0}{\partial t^2} - \frac{\partial^2 U_0}{\partial x^2} + U_0 = 0\,. \tag{9.6}$$

The basic solutions of this linear equation are exponential traveling waves $e^{i(kx-\omega t)}$ where the wavenumber k and angular frequency ω are linked by the *dispersion relation*

$$\omega^2 = k^2 + 1\,. \tag{9.7}$$

The corresponding functions $\omega_+(k)$ and $\omega_-(k)$ are thus the upper and lower branches of a hyperbola. See Figure 9.1. We can pick concrete values of k and ω satisfying (9.7) and obtain a real-valued solution for

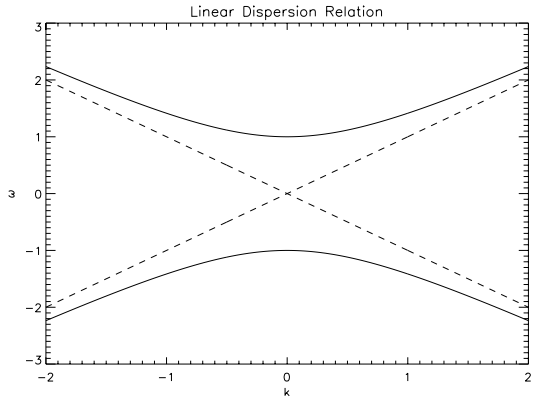

Figure 9.1: *The two branches $\omega_+(k)$ and $\omega_-(k)$ of the dispersion relation for the linearized sine-Gordon equation, as functions of k, with the hyperbolic asymptotes shown with dashed lines.*

$U_0(x,t)$ by writing

$$U_0(x,t) = Ae^{i(kx-\omega t)} + A^* e^{-i(kx-\omega t)}\,, \tag{9.8}$$

where A is a complex constant, whose complex conjugate is A^*. The "minus" expontial is also admitted because the dispersion relation is symmetric under changing $k \to -k$ and $\omega \to -\omega$.

Now, when we go to higher order in ϵ we know, roughly speaking, what is going to happen. The equation for U_1 is

$$\frac{\partial^2 U_1}{\partial t^2} - \frac{\partial^2 U_1}{\partial x^2} + U_1 = 0 \tag{9.9}$$

which we could solve by taking $U_1 \equiv 0$ (this is a consequence of only even powers of ϵ appearing in (9.3)), and then for U_2 we would have

$$\frac{\partial^2 U_2}{\partial t^2} - \frac{\partial^2 U_2}{\partial x^2} + U_2 = \frac{1}{6}U_0^3 = \frac{A^3}{6}e^{3i\theta} + \frac{A^2A^*}{2}e^{i\theta} + \frac{AA^{*2}}{2}e^{-i\theta} + \frac{A^{*3}}{6}e^{-3i\theta} \tag{9.10}$$

where we are writing $\theta = kx - \omega t$. The terms $e^{\pm i\theta}$ are certainly *resonant*, in that they will cause secular terms in the solution for U_2. It is also in principle possible for the terms $e^{\pm 3i\theta}$ to be resonant. This will be the case if it also turns out that the pairs $(\pm 3k, \pm 3\omega)$ satisfy the dispersion relation (9.7) given that k and ω do, admittedly an unusual situation but one to keep in mind. The reason that having exponentials on the right hand side whose wavenumbers and frequencies satisfy (9.7) leads to secular terms is easy to see in this case. Let k and ω satisfy (9.7) and let B be a complex constant. Consider the linear equation

$$\frac{\partial^2 U_2}{\partial t^2} - \frac{\partial^2 U_2}{\partial x^2} + U_2 = Be^{i\theta} \tag{9.11}$$

with $\theta = kx - \omega t$. Using (9.7), it is easy to check that a particular solution is given by

$$U_2(x,t) = \frac{Bit}{2\omega}e^{i\theta} \tag{9.12}$$

which grows linearly in t. An arbitrary solution for U_2 will differ from this growing solution by a homogeneous solution, which can be obtained by Fourier transforms. The general homogeneous solution that is bounded in x is also bounded in t. Consequently, all solutions that are bounded in x must be linearly growing in t. These statements hold whenever the right-hand side is a simple wave that satisfies the dispersion relation (9.7). So in principle we have to check for all possible resonances.

We anticipate that we can eliminate these linearly growing secular terms by introducing multiple time scales $T_0 = t$, $T_1 = \epsilon t$, and $T_2 = \epsilon^2 t$, and choosing the "constant" A to depend on these slow time scales in such a way that the offending terms are removed from the right-hand side. But we should think about the implications of these steps for a moment. If we introduce dependence on slow time scales, then according to our experience with the method of strained coordinates, this modulation amounts to altering the fundamental frequency ω somewhat. But according to the dispersion relation (9.7) the wavenumber k and the frequency ω are linked, and therefore we expect that it will be necessary to strain the wavenumber k whenever we strain the frequency ω, to maintain consistency. If straining frequencies is equivalent to introducing multiple time scales, then straining wavenumbers is equivalent to introducing multiple spatial scales. In fact, one additional scale will do, as it turns out. We thus introduce the variables $X_0 = x$ and $X_1 = \epsilon x$.

So we suppose that $U(x,t) = U(X_0, X_1, T_0, T_1, T_2)$ and insert into (9.3) using the chain rule to compute the derivatives:

$$\begin{aligned} \frac{\partial^2 U}{\partial t^2} &= \frac{\partial^2 U}{\partial T_0^2} + 2\epsilon \frac{\partial^2 U}{\partial T_0 \partial T_1} + \epsilon^2 \left[\frac{\partial^2 U}{\partial T_1^2} + 2 \frac{\partial^2 U}{\partial T_0 \partial T_2} \right] \\ \frac{\partial^2 U}{\partial x^2} &= \frac{\partial^2 U}{\partial X_0^2} + 2\epsilon \frac{\partial^2 U}{\partial X_0 \partial X_1} + \epsilon^2 \frac{\partial^2 U}{\partial X_1^2} \,. \end{aligned} \tag{9.13}$$

We use the same expansion (9.5) as before, in powers of ϵ. The terms at order $O(\epsilon^0)$ are essentially the linear equation (9.6) in new variables

$$\frac{\partial^2 U_0}{\partial T_0^2} - \frac{\partial^2 U_0}{\partial X_0^2} + U_0 = 0 \tag{9.14}$$

with the wavetrain solution

$$U_0 = A e^{i\theta} + \text{c. c.} \tag{9.15}$$

where $\theta = kX_0 - \omega T_0$ and k and ω satisfy (9.7). Here and in the following "c. c." denotes the complex conjugate of the terms preceding. We are treating $A = A(X_1, T_1, T_2)$ as a function of the slow variables, but constant in X_0 and T_0.

The terms appearing at order $O(\epsilon)$ are

$$\frac{\partial^2 U_1}{\partial T_0^2} - \frac{\partial^2 U_1}{\partial X_0^2} + U_1 = -2 \frac{\partial^2 U_0}{\partial T_0 \partial T_1} + 2 \frac{\partial^2 U_0}{\partial X_0 \partial X_1} = 2i \left[\omega \frac{\partial A}{\partial T_1} + k \frac{\partial A}{\partial X_1} \right] e^{i\theta} + \text{c. c.} \tag{9.16}$$

Because the exponentials $e^{\pm i\theta}$ are resonant, we will have secular terms in U_1 unless we choose the dependence of A on X_1 and T_1 according to

$$\frac{\partial A}{\partial T_1} + \frac{k}{\omega} \frac{\partial A}{\partial X_1} = 0 \,. \tag{9.17}$$

Since ω depends on k according to the dispersion relation (9.7), we can differentiate (9.7) implicitly with respect to k. Regardless of whether we are on the branch $\omega = \omega_+(k)$ or $\omega = \omega_-(k)$, we find that

$$\frac{k}{\omega_\pm(k)} = \omega_\pm'(k) \,. \tag{9.18}$$

Consequently, the general solution of (9.17) can be written in the form

$$A = f(X_1 - \omega_\pm'(k) T_1) \tag{9.19}$$

where $f(\xi)$ is an arbitrary function. Thus we learn that on the time scale T_1, the modulation of the wave amplitude amounts to a rigid translation to the right with a speed equal to the *group velocity* $v_g := \omega_\pm'(k)$. Having chosen A to satisfy (9.17), we have made the right-hand side of (9.16) identically zero. Although we could take any homogeneous solution as U_1, we keep in mind that we already began our expansion with a certain homogeneous solution, and since we would just be adding more of the same, we take U_1 to be the zero solution.

With $U_1 \equiv 0$, we can write down the equation appearing at order $O(\epsilon^2)$:

$$\begin{aligned} \frac{\partial^2 U_2}{\partial T_0^2} - \frac{\partial^2 U_2}{\partial X_0^2} + U_2 &= -\frac{\partial^2 U_0}{\partial T_1^2} - 2\frac{\partial^2 U_0}{\partial T_0 \partial T_2} + \frac{\partial^2 U_0}{\partial X_1^2} + \frac{1}{6}U_0^3 \\ &= \left[-\frac{\partial^2 A}{\partial T_1^2} + 2i\omega \frac{\partial A}{\partial T_2} + \frac{\partial^2 A}{\partial X_1^2} + \frac{1}{2}A^2 A^* \right] e^{i\theta} + \frac{1}{6}A^3 e^{3i\theta} + \text{c. c.} \end{aligned} \tag{9.20}$$

Assuming the exponential $e^{\pm 3i\theta}$ is not resonant, the solvability condition for a bounded solution U_2 is the equation

$$-\frac{\partial^2 A}{\partial T_1^2} + 2i\omega \frac{\partial A}{\partial T_2} + \frac{\partial^2 A}{\partial X_1^2} + \frac{1}{2}A^2 A^* = 0\,. \tag{9.21}$$

With this choice of dependence of A on T_2, we are guaranteed that the solution for U_2 will not contain secular terms. But the correction will now not be zero, since the equation is not homogeneous. There are nonresonant harmonics $e^{\pm 3i\theta}$ on the right-hand side that will give rise to contributions to U_2 proportional to these same harmonics.

The two solvability conditions (9.17) and (9.21) can be put into a more transparent form with a change of independent variables. Namely, based on the discussion following (9.17) we anticipate the utility of the change of variables $(X_1, T_1) \to (\xi, \tau)$ given by

$$\xi = X_1 - \omega'_{\pm}(k) T_1 \qquad \text{and} \qquad \tau = T_1\,. \tag{9.22}$$

Indeed from the Jacobian of this transformation, we immediately find that (9.17) becomes simply

$$\frac{\partial A}{\partial \tau} = 0 \tag{9.23}$$

or in words, in the frame of reference moving with the group velocity, the amplitude A is independent of $\tau = T_1$. With the same change of variables, and using (9.23), we find that (9.21) becomes

$$2i\omega_{\pm}(k)\frac{\partial A}{\partial T_2} + \left(1 - [\omega'_{\pm}(k)]^2\right)\frac{\partial^2 A}{\partial \xi^2} + \frac{1}{2}A^2 A^* = 0\,. \tag{9.24}$$

Once again, there is some information to be gleaned from implicitly differentiating the dispersion relation (9.7). Namely, we find that

$$[\omega'_{\pm}(k)]^2 + \omega_{\pm}(k)\omega''_{\pm}(k) = 1\,. \tag{9.25}$$

Therefore, we may finally write (9.21) in the form

$$i\frac{\partial A}{\partial T_2} + \frac{\omega''_{\pm}(k)}{2}\frac{\partial^2 A}{\partial \xi^2} + \frac{1}{4\omega_{\pm}(k)}|A|^2 A = 0\,. \tag{9.26}$$

Thus, while the shape of the wave envelope A is stationary in the group velocity frame over timescales of length $1/\epsilon$ (in the original t variable, since T_1 being order one means $t = O(1/\epsilon)$), on timescales of length $1/\epsilon^2$, the envelope is evolving in the group velocity frame according to the nonlinear partial differential equation (9.26). This equation is one of the most famous equations in all of applied mathematics, the *cubic nonlinear Schrödinger equation*.

There are so many remarkable things about the cubic nonlinear Schrödinger equation that it is hard to know where to begin. For one thing, it turns out that this equation is, like the sine-Gordon equation we began with, completely integrable and therefore exactly solvable by an inverse-scattering transform. It is very important to point out however that this integrability is not inherited from that of the sine-Gordon equation. If you look carefully, you will see that what we have done here (through order ϵ^2) would have been exactly the same had we begun with the Klein-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u - \frac{1}{6}u^3 = 0 \tag{9.27}$$

instead of the sine-Gordon equation. But the Klein-Gordon equation with the cubic approximation to $\sin(u)$ is *not* completely integrable. So in the course of our perturbation analysis it would seem that in general we get something for nothing! The next remarkable thing about the cubic nonlinear Schrödinger equation is the ubiquity with which it arises in perturbation theory. It has already been pointed out that our derivation was only sensitive to the first two terms in the Taylor expansion of $\sin(u)$. But the amazing thing is that a similar derivation leads to an equation of the form (9.26) under much more general circumstances. We will see another concrete example shortly, but for now it should be said clearly that whenever weakly nonlinear, nearly monochromatic, one-dimensional waves with wavenumber k and frequency ω are propagating in the presence of strong dispersion (meaning $\omega''(k) \neq 0$) and in the absence of dissipation or unusual resonances (*i.e.* if the pair $(3k, 3\omega)$ turns out to satisfy the dispersion relation as well as the pair (k, ω), a non-generic situation), the complex wave amplitude A will satisfy an equation of the general form

$$i\frac{\partial A}{\partial T_2} + \frac{\omega''(k)}{2}\frac{\partial^2 A}{\partial \xi^2} + \beta|A|^2 A = 0\,, \tag{9.28}$$

where $T_2 = \epsilon^2 t$, $\xi = \epsilon(x - \omega'(k)t)$, and where only the parameter β is not set-in-stone, and must be determined by a procedure similar to what we just carried out above for the sine-Gordon equation.

9.2 Example: Waves in Molecular Chains

To further illustrate the application of the method of multiple scales to the derivation of canonical model equations, we consider a problem of nonlinear vibrations in a monatomic chain. The setting is the classical mechanics of a chain of particles of equal mass, say m, each interacting with its two nearest neighbors by an interatomic force law that we can visualize as a nonlinear spring. See Figure 9.2. This sort of model

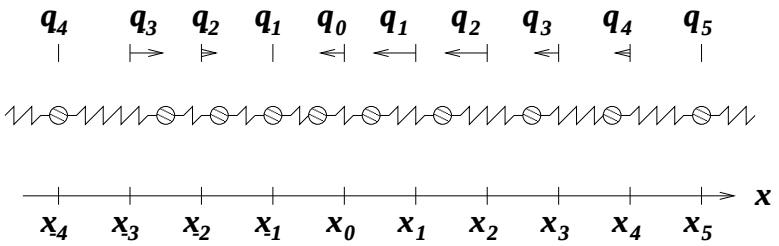

Figure 9.2: *The molecular chain showing displacements q_n of the groups from equally spaced equilibrium positions x_n.*

arises in several branches of applied mathematics. For example, the study of deformations of α-helix protein molecules can be reduced (under certain conditions) to a problem of this type. In this case, the individual masses are the peptide groups in the chain, which are connected one to the next by hydrogen bonds. It is also a basic model in solid-state physics. A very famous story concerns three physicists, Fermi, Pasta, and Ulam, who proposed a model of this type in an attempt to explain the process of *thermalization* in solid materials. Thermalization refers to the way that energy that is in the system at time $t = 0$ ultimately becomes equally distributed among all possible vibrational modes in accordance with the laws of statistical thermodynamics; it is the reason why hitting a piece of metal repeatedly for a long time with a hammer makes the metal start to feel hot to the touch. Fermi, Pasta, and Ulam knew that if the springs were all linear, then any energy you put into a particular vibrational mode will stay there forever, and consequently they thought that thermalization must be the result of weakly nonlinear effects in the springs, which serve to mix the energy among the linear vibrational modes. In fact their (very primitive by today's standards) numerical simulations seemed to show otherwise; namely that the energy dispersed among the various vibrational modes for a short time, but then spontaneously came back into just a few modes of vibration some time later! In other words, their simulations indicated that the energy does not thermalize at all in spite of the nonlinearities in the springs. This was sufficiently surprising that it warranted further investigation, and it

seems that the nonlinear molecular chain model, sometimes called the Fermi-Pasta-Ulam lattice, has not at this time been used to successfully explain thermalization. On the other hand, it has been a prototype for studying unexpectedly coherent phenomena ever since.

The total energy of this mechanical system is a sum of the total kinetic energy T and the total potential energy V. The kinetic energy is related to the motion of the particles. If we assign to the particle whose displacement from equilibrium at time t is $q_n(t)$ the momentum

$$p_n(t) := m\frac{dq_n}{dt}(t) \tag{9.29}$$

then the kinetic energy of this particle is $p_n^2/2m$. Consequently the total kinetic energy of the whole chain is a sum:

$$T = \sum_{n=-\infty}^{\infty} \frac{p_n^2}{2m}\,. \tag{9.30}$$

We assume that particles are only really moving very much in some finite part of the lattice, so that the infinite sum converges. On the other hand, the potential energy is stored in the compressed or extended springs. Let V_n be the potential energy of the spring connecting the particles with displacements q_n and q_{n-1}. We normalize V_n to be zero when $q_n - q_{n-1} = 0$, that is, when the spring is neither stretched nor compressed relative to its equilibrium configuration. Assuming the springs are all identical, then V_n is just a function $V(s_n)$ of the stretching, $s_n := q_n - q_{n-1}$, that vanishes when $s_n = 0$. For simplicity, we will also assume that the function $V(\cdot)$ is analytic. The total potential energy of the system is then the sum:

$$V = \sum_{n=-\infty}^{\infty} V(q_n - q_{n-1})\,. \tag{9.31}$$

Once again, we assume that $s_n \to 0$ for large n fast enough given the function $V(\cdot)$ so that the sum converges. This means that far enough away from the center of all the action, the masses are not moving much and are approximately equally spaced with spacing Δx.

The equations of motion for the masses are Hamilton's equations corresponding to the total energy $H = T + V$. These equations are:

$$\frac{dq_n}{dt} = \frac{\partial H}{\partial p_n} = \frac{p_n}{m} \tag{9.32}$$

and

$$\frac{dp_n}{dt} = -\frac{\partial H}{\partial q_n} = V'(q_{n+1} - q_n) - V'(q_n - q_{n-1})\,. \tag{9.33}$$

Eliminating the momenta p_n, we can combine these into a system of coupled second order equations, Newton's equations, for the dynamical variables $q_n(t)$:

$$m\frac{d^2q_n}{dt^2} = V'(q_{n+1} - q_n) - V'(q_n - q_{n-1})\,, \tag{9.34}$$

for $n \in \mathbb{Z}$. Note that if all the particles are equally spaced with spacing Δx, the force terms on the right-hand side become simply $V'(0) - V'(0) = 0$, so the system is indeed in equilibrium in such a configuration.

For weakly nonlinear oscillations, we should suppose that the displacements from equilibrium are very small, so that we may take into account nonlinear effects perturbatively. Thus we set $q_n = \epsilon Q_n$ where ϵ is a small parameter. Making this substitution and expanding V in Taylor series about $s_n = 0$ for ϵ small, the equations of motion become

$$\begin{aligned} m\frac{d^2Q_n}{dt^2} &= V''(0)\left[Q_{n+1} - 2Q_n + Q_{n-1}\right] + \epsilon\frac{V'''(0)}{2}\left[(Q_{n+1} - Q_n)^2 - (Q_n - Q_{n-1})^2\right] \\ &\quad + \epsilon^2\frac{V^{(IV)}(0)}{6}\left[(Q_{n+1} - Q_n)^3 - (Q_n - Q_{n-1})^3\right] + O(\epsilon^3)\,. \end{aligned} \tag{9.35}$$

This is evidently of the form of a formally small perturbation of a system of linear ordinary differential equations for the rescaled displacements $Q_n(t)$.

We anticipate that as we try to compute higher order terms in an asymptotic expansion in powers of ϵ we will encounter secular terms that will ruin the asymptotic nature of our subsequent approximations. Therefore we want to cut to the chase and introduce the slow spatial and temporal scales from the start. Let the time scales be $T_0 = t$, $T_1 = \epsilon t$, and $T_2 = \epsilon^2 t$. Introduce also the spatial scale $X_1 = \epsilon n$. This latter relation is a little strange, since n is a discrete variable, but we would like to consider X_1 as a continuous variable. Whatever philosophical discomfort it causes us at the moment will be justified soon, however. Thus, we are interested in seeking a solution of the form

$$Q_n(t) = Q_n^{(0)}(X_1, T_0, T_1, T_2) + \epsilon Q_n^{(1)}(X_1, T_0, T_1, T_2) + \epsilon^2 Q_n^{(2)}(X_1, T_0, T_1, T_2) + O(\epsilon^3)\,. \tag{9.36}$$

We are thinking of n and X_1 as independent spatial variables and of T_0, T_1, and T_2 as independent time scales. We know how to expand the time derivatives using the chain rule, but what about the differencing with respect to n? Well, using the definition of X_1, we see that

$$\begin{aligned} Q_{n\pm1}(t) &= Q_{n\pm1}^{(0)}(X_1 \pm \epsilon, T_0, T_1, T_2) + \epsilon Q_{n\pm1}^{(1)}(X_1 \pm \epsilon, T_0, T_1, T_2) + \epsilon^2 Q_{n\pm1}^{(2)}(X_1 \pm \epsilon, T_0, T_1, T_2) + O(\epsilon^3) \\ &= Q_{n\pm1}^{(0)}(X_1, T_0, T_1, T_2) + \epsilon \left[Q_{n\pm1}^{(1)}(X_1, T_0, T_1, T_2) \pm \frac{\partial Q_{n\pm1}^{(0)}}{\partial X_1}(X_1, T_0, T_1, T_2) \right] \\ &\quad + \epsilon^2 \left[Q_{n\pm1}^{(2)}(X_1, T_0, T_1, T_2) \pm \frac{\partial Q_{n\pm1}^{(1)}}{\partial X_1}(X_1, T_0, T_1, T_2) + \frac{1}{2}\frac{\partial^2 Q_{n\pm1}^{(0)}}{\partial X_1^2}(X_1, T_0, T_1, T_2) \right] + O(\epsilon^3)\,. \end{aligned} \tag{9.37}$$

So it is possible to collect the coefficients of powers of ϵ order by order. At order $O(\epsilon^0)$ we find the reduced equation, the advertised linear equation

$$m\frac{\partial^2 Q_n^{(0)}}{\partial T_0^2} = V''(0)\left[Q_{n+1}^{(0)} - 2Q_n^{(0)} + Q_{n-1}^{(0)} \right]. \tag{9.38}$$

Elementary traveling wave solutions can be sought in the form

$$Q_n^{(0)} = e^{i(\delta n - \omega T_0)}\,. \tag{9.39}$$

It is immediate to check that these will solve (9.38) if δ and ω satisfy the *dispersion relation*:

$$D(\delta, \omega) := m\omega^2 + V''(0)\left[e^{i\delta} - 2 + e^{-i\delta} \right] = 0 \quad \text{or} \quad \omega^2 = \frac{2V''(0)}{m}\left[1 - \cos(\delta) \right]. \tag{9.40}$$

Evidently, for bounded motions, we need to have $V''(0) > 0$. This ensures that the equilibrium configuration is *stable*. The appearance of trigonometric functions in the dispersion relation is a new feature for us, and is the hallmark of systems with discrete translational symmetry like crystals. The two branches of the dispersion relation are displayed in Figure 9.3. We look for a solution that is a modulated wave with discrete wavenumber $\delta \in (-\pi, \pi)$, so with a corresponding value of ω (upper or lower branch) we have a real solution in the form

$$Q_n^{(0)} = Ae^{i\theta} + \text{c. c.} \tag{9.41}$$

where $\theta = \delta n - \omega t$.

Now we are ready to move on to order $O(\epsilon)$. Here, we find the equation

$$\begin{aligned} L[Q^{(1)}] &= -2m\frac{\partial^2 Q_n^{(0)}}{\partial T_0 \partial T_1} + V''(0)\left[\frac{\partial Q_{n+1}^{(0)}}{\partial X_1} - \frac{\partial Q_{n-1}^{(0)}}{\partial X_1} \right] \\ &\quad + \frac{V'''(0)}{2}\left[(Q_{n+1}^{(0)} - Q_n^{(0)})^2 - (Q_n^{(0)} - Q_{n-1}^{(0)})^2 \right] \\ &= \left[2im\omega\frac{\partial A}{\partial T_1} + 2iV''(0)\sin(\delta)\frac{\partial A}{\partial X_1} \right] e^{i\theta} + iA^2 V'''(0)\left[\sin(2\delta) - 2\sin(\delta) \right] e^{2i\theta} + \text{c. c.} \end{aligned} \tag{9.42}$$

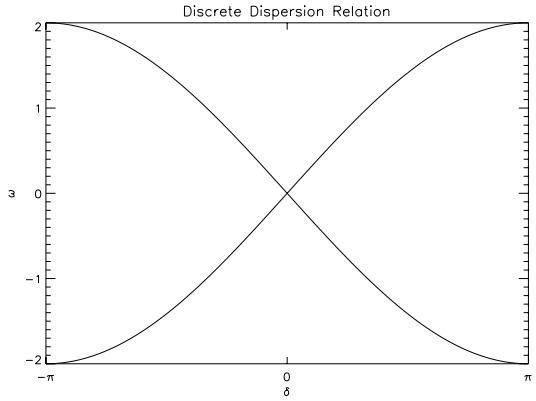

Figure 9.3: *The two branches of the discrete dispersion relation for linear motions of the lattice. The frequency is normalized to* $\sqrt{V''(0)/m}$.

where for general f, $L[f]$ denotes the linear expression

$$L[f] := m\frac{\partial^2 f_n}{\partial T_0^2} - V''(0)\left[f_{n+1} - 2f_n + f_{n-1}\right] , \tag{9.43}$$

and where we have used the explicit form (9.41) of the leading order solution to compute the right-hand side. Generally, the second harmonic terms on the right-hand side of (9.42) are nonresonant, since $D(2\delta, 2\omega)$ (*cf.* the dispersion relation (9.40)) is rarely zero given that $D(\delta,\omega) = 0$. On the other hand, the fundamental exponential $e^{i\theta}$ is resonant, and its presence will lead to secular terms unless we choose A so as to kill the corresponding coefficient. Therefore we take A to satisfy the solvability condition

$$2im\omega\frac{\partial A}{\partial T_1} + 2iV''(0)\sin(\delta)\frac{\partial A}{\partial X_1} = 0 . \tag{9.44}$$

Differentiating the dispersion relation (9.40) implicitly with respect to δ, we see that this equation can equivalently be written as

$$\frac{\partial A}{\partial T_1} + \omega'(\delta)\frac{\partial A}{\partial X_1} = 0 , \tag{9.45}$$

which indicates rigid motion of the wave envelope at the group velocity $v_g := \omega'(\delta)$. With A chosen to satisfy (9.45), the right-hand side of (9.42) will contain only nonresonant second harmonics. We can find a particular solution for (9.42) simply by assuming the form

$$Q_n^{(1)} = Be^{2i\theta} + B^* e^{-2i\theta} \tag{9.46}$$

for some complex constant B. Substituting into (9.42), we find that

$$B = -\frac{iA^2V'''(0)}{D(2\delta, 2\omega)}\left[\sin(2\delta) - 2\sin(\delta)\right] . \tag{9.47}$$

The nonresonance condition $D(2\delta, 2\omega) \neq 0$ is clearly crucial here. Although we can add to our particular solution for $Q_n^{(1)}$ an arbitrary homogeneous solution, which is a superposition of waves satisfying the dispersion relation (9.40), we take the point of view that we have already selected such a wave at leading order, and that adding something of the same type at this order would be contributing nothing interesting. Therefore we have the solution

$$Q_n^{(1)} = -\frac{iA^2V'''(0)}{D(2\delta, 2\omega)}\left[\sin(2\delta) - 2\sin(\delta)\right]e^{2i\theta} + \text{c. c.} \tag{9.48}$$

Now we can move onto order $O(\epsilon^2)$. We find for $Q_n^{(2)}$ the equation of motion

$$\begin{aligned} L[Q^{(2)}] &= \left[2im\omega\frac{\partial A}{\partial T_2} - m\frac{\partial^2 A}{\partial T_1^2} + V''(0)\cos(\delta)\frac{\partial^2 A}{\partial X_1} - K|A|^2 A\right] e^{i\theta} + \text{c. c.} \\ &\quad + \text{second and third harmonics}, \end{aligned} \tag{9.49}$$

where

$$K := 2V^{(IV)}(0)(\cos(\delta)-1)^2 + \frac{2[V'''(0)]^2(2\sin(\delta)-\sin(2\delta))^2}{D(2\delta,2\omega)}\,. \tag{9.50}$$

Generally the third harmonics are not resonant as was the case with the second harmonics, *i.e.* usually we have $D(3\delta,3\omega)\neq 0$ if $D(\delta,\omega)=0$. Thus the terms that must be eliminated to avoid secular terms in $Q^{(2)}$ are those proportional to the fundamental wave and its complex conjugate. The solvability condition is thus

$$2im\omega\frac{\partial A}{\partial T_2} - m\frac{\partial^2 A}{\partial T_1^2} + V''(0)\cos(\delta)\frac{\partial^2 A}{\partial X_1} - K|A|^2 A = 0\,. \tag{9.51}$$

With this condition satisfied, we may solve for $Q_n^{(2)}$ which we will find has components proportional to the second and third harmonics $e^{\pm 2i\theta}$ and $e^{\pm 3i\theta}$, but as desired is a bounded function.

Once again there is a great deal of utility to be had in the change of variables

$$\tau = T_1 \quad \text{and} \quad \xi = X_1 - \omega'(\delta)T_1\,. \tag{9.52}$$

With this change of variables, the solvability condition (9.45) takes the simple form

$$\frac{\partial A}{\partial \tau} = 0 \tag{9.53}$$

and (9.51) can be written as

$$i\frac{\partial A}{\partial T_2} + \frac{\omega''(\delta)}{2}\frac{\partial^2 A}{\partial \xi^2} + \beta|A|^2 A = 0\,, \tag{9.54}$$

where the real parameter β is given by

$$\beta := -\frac{m\omega^3}{4[V''(0)]^2}\left(V^{(IV)}(0) + 4[V'''(0)]^2\sin^2(\delta)\right)\,. \tag{9.55}$$

Thus we have once again arrived at the cubic nonlinear Schrödinger equation as a canonical envelope equation. The behavior of solutions of (9.54) is totally different depending on whether $\beta\omega''(\delta) > 0$ (the so-called "focusing" case) or $\beta\omega''(\delta) < 0$ (the so-called "defocusing" case). In the focusing case, broad packets (A nearly constant in ξ) are *modulationally unstable* and break up into little packets called *solitons*. On the other hand, broad packets are stable in the defocusing case. Now from looking at Figure 9.3, or by using the exact formula (9.40) we see that $\omega''(\delta)$ and $\omega(\delta)$ always have opposite signs. Thus, we see that whether we are in the stable or unstable case depends only on the sign of $V^{(IV)}(0)$ and its size relative to $4[V'''(0)]^2\sin^2(\delta)$. If $V^{(IV)}(0)$ is positive, meaning that the nonlinear springs become stiffer with larger deviations from equilibrium, then we are necessarily in the focusing case. On the other hand, if $V^{(IV)}(0)$ is sufficiently negative given δ, which corresponds to springs that become significantly "softer" when the amplitude increases, then we are in the defocusing case.

An important point to make is that the *dispersion* $\omega''(\delta)$ vanishes on either branch if $\delta = 0$. In this case, the nonlinear Schrödinger equation (9.54) is the wrong model, and we should go back to the drawing board. Here the idea is that $\delta = 0$ corresponds to *long waves*. When δ is close to zero, the displacement $Q_n^{(0)}$ is not changing much at all from particle to particle in the lattice. So suppose we go back to the original equations of motion (9.35) for $Q_n(t)$ and make the ansatz that

$$Q_n(t) = Q(X_1, T_1)\,, \tag{9.56}$$

that is, a smooth function of X_1 and T_1. Inserting into (9.35), we find that

$$m\frac{\partial^2 Q}{\partial T_1^2} = V''(0)\frac{\partial^2 Q}{\partial X_1^2} + \epsilon^2\left[\frac{V''(0)}{12}\frac{\partial^4 Q}{\partial X_1^4} + V'''(0)\frac{\partial Q}{\partial X_1}\frac{\partial^2 Q}{\partial X^2}\right] + O(\epsilon^3)\,. \tag{9.57}$$

If we make the change of variables

$$\tau = \epsilon^2 T_1 \quad \text{and} \quad \xi = X_1 - \sqrt{\frac{V''(0)}{m}}T_1 = X_1 - \omega'(0)T_1\,, \tag{9.58}$$

Then in terms of these new independent variables the equation becomes

$$-2\sqrt{mV''(0)}\frac{\partial^2 Q}{\partial\tau\partial\xi} = \frac{V''(0)}{12}\frac{\partial^4 Q}{\partial\xi^4} + V'''(0)\frac{\partial Q}{\partial\xi}\frac{\partial^2 Q}{\partial\xi^2} + O(\epsilon)\,. \tag{9.59}$$

So, we see that with the ansatz

$$q_n(t) = \epsilon Q(\xi,\tau) \quad \text{with} \quad \xi = \epsilon\left(n - \sqrt{\frac{V''(0)}{m}}t\right) \quad \text{and} \quad \tau = \epsilon^3 t\,, \tag{9.60}$$

the function $F := \partial Q/\partial\xi$ asymptotically satisfies the partial differential equation

$$2\sqrt{mV''(0)}\frac{\partial F}{\partial\tau} + V'''(0)F\frac{\partial F}{\partial\xi} + \frac{V''(0)}{12}\frac{\partial^3 F}{\partial\xi^3} = 0\,. \tag{9.61}$$

This model equation is called the *Korteweg-de Vries equation*. As it happens, it is also one of the completely integrable nonlinear partial differential equations, in fact the most famous one! The constant coefficients can be normalized out of the picture by rescaling F and τ by ϵ-independent factors. Thus,

$$\frac{\partial F}{\partial\tau} + F\frac{\partial F}{\partial\xi} + \frac{\partial^3 F}{\partial\xi^3} = 0\,, \tag{9.62}$$

holds for the rescaled F with the rescaled time τ. The Korteweg-de Vries equation was the first dispersive nonlinear wave equation ever recognized to have the remarkable mathematical property of complete integrability. The initial observation was essentially made in 1965 by Zabusky and Kruskal, who were conducting numerical experiments using the Korteweg-de Vries equation as a long-wave asymptotic model for the Fermi-Pasta-Ulam lattice. The amazing mathematical method for solving the Korteweg-de Vries equation, the inverse-scattering transform, appeared in 1967 in a paper of Gardner, Greene, Kruskal, and Miura.

While we should be amazed that looking at the same physical system of coupled nonlinear oscillators yields, via the method of multiple scales, two of the most famous integrable nonlinear partial differential equations (there are many others) in different weakly nonlinear limits, we should also be aware that this approach also often leads to models that are not necessarily integrable. On the other hand, these equations have a canonical form, and appear again and again in problem after problem. For example, if there had been some dissipation and driving in our nonlinear chain, we would have arrived at an envelope equation for A known as the complex Ginzburg-Landau equation, which looks like the cubic nonlinear Schrödinger equation, but has complex coefficients. While many people are interested in Ginzburg-Landau type equations because they are ubiquitous and therefore important, the nonconservative properties make the analysis of the model equation more difficult.