

Quantum Field Theory II

University of Cambridge Part III Mathematical Tripos

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ABSTRACT: These are the lecture notes for the Advanced Quantum Field Theory course given to students taking Part III Maths in Cambridge during Lent Term of 2017. The main aims are to discuss Path Integrals, the Renormalization Group, Wilsonian Effective Field Theory and non-Abelian Gauge Theories.

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Acknowledgments

Nothing in these lecture notes is original. In particular, my treatment is heavily influenced by several of the textbooks listed below, especially Vafa *et al.* and the excellent lecture notes of Neitzke in the early stages, then Schwartz and Weinberg's textbooks and Hollowood's lecture notes later in the course.

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Preliminaries

This course is the second course on Quantum Field Theory offered in Part III of the Maths Tripos, so I'll feel free to assume you've already taken the first course in Michaelmas Term (or else an equivalent course elsewhere). There may be some overlap between this course and certain other Part III courses this term. In particular, I'd expect the material here to complement the courses on *The Standard Model* and on *Applications of Differential Geometry to Physics* very well. In turn, I'd also hope this course is useful preparation for courses on *Supersymmetry* and *String Theory*.

Books & Other Resources

There are many (too many!) textbooks and reference books available on Quantum Field Theory. Different ones emphasize different aspects of the theory, or applications to different branches of physics or mathematics – indeed, QFT is such a huge subject nowadays that it is probably impossible for a single textbook to give an encyclopedic treatment (and absolutely impossible for a course of 24 lectures to do so). Here are some of the ones I've found useful while preparing these notes; you might prefer different ones to me.

- **Nair, V.P.**, *Quantum Field Theory: A Modern Perspective*, Springer (2005).
Although it isn't so well known, this is perhaps my favourite QFT book. It begins with a clear, concise discussion of all the standard perturbative material you'll find in any QFT course. However, unlike many books, it also makes clear that there's far more to QFT than just perturbation theory. Contains excellent discussions of the configuration space of field theories, ambiguities in quantization, approaches to strong coupling limits in QCD, and QFT at finite temperature.

This next list contains the stalwart QFT textbooks. You will certainly want to consult (at least) one of these repeatedly during the course. They'll also be very helpful for people taking the Standard Model course.

- **Peskin, M. and Schroeder, D.**, *An Introduction to Quantum Field Theory*, Addison-Wesley (1996).
An excellent QFT textbook, containing extensive discussions of both gauge theories and renormalization. Many examples worked through in detail, with a particular emphasis on applications to particle physics.
- **Schwartz, M.**, *Quantum Field Theory and the Standard Model*, CUP (2014).
The new kid on the block, honed during the author's lecture courses at Harvard. I really like this book – it strikes an excellent balance between formalism and applications (mostly to high energy physics), with fresh and clear explanations throughout.
- **Srednicki, M.**, *Quantum Field Theory*, CUP (2007).
This is also an excellent, very clearly written and very pedagogical textbook, with clearly compartmentalised chapters breaking the material up into digestible chunks. However, our route through QFT in this course will follow a slightly different path.

- **Zee, A.**, *Quantum Field Theory in a Nutshell*, 2nd edition, PUP (2010).

QFT is notorious for containing many technical details, and it's easy to get lost. This is a great book if you want to keep the big picture of what QFT is all about firmly in sight. It will put you joyfully back on track and remind you why you wanted to learn the subject in the first place. It's not the best place to work through detailed calculations, but that's not the point.

There are also a large number of books that are more specialized. Many of these are rather advanced, so I do not recommend you use them as a primary text. However, you may well wish to dip into them occasionally to get a deeper perspective on topics you particularly enjoy. This list is particularly biased towards my (often geometric) interests:

- **Banks, T.** *Modern Quantum Field Theory: A Concise Introduction*, CUP (2008).
I particularly enjoyed its discussion of the renormalization group and effective field theories. As it says, this book is probably too concise to be a main text.
- **Cardy, J.**, *Scaling and Renormalization in Statistical Physics*, CUP (1996).
A wonderful treatment of the Renormalization Group in the context in which it was first developed: calculating critical exponents for phase transitions in statistical systems. The presentation is extremely clear, and this book should help to balance the ‘high energy’ perspective of many of the other textbooks.
- **Coleman, S.**, *Aspects of Symmetry*, CUP (1988).
Legendary lectures from one of the most insightful masters of QFT. Contains much material that is beyond the scope of this course, but so engagingly written that I couldn't resist including it here!
- **Costello, K.**, *Renormalization and Effective Field Theory*, AMS (2011).
A pure mathematician's view of QFT. The main aim of this book is to give a rigorous definition of (perturbative) QFT via path integrals and Wilsonian effective field theory. Another major achievement is to implement this for gauge theories by combining BV quantization with the ERG. Repays the hard work you'll need to read it – for serious mathematicians only.
- **Deligne, P., et al.**, *Quantum Fields and Strings: A Course for Mathematicians*, vols. 1 & 2, AMS (1999).
Aimed at professional mathematicians wanting an introduction to QFT. They thus require considerable mathematical maturity to read, but most certainly repay the effort. Almost everything here is beyond the level of this course, but I can promise your appreciation of QFT will be deepened immeasurably by reading the lectures of Deligne & Freed on *Classical Field Theory* (vol. 1), Gross on the *Renormalization Group* (vol. 1), Gadwezki on CFTs (vol. 2), and especially Witten on *Dynamics of QFT* (vol. 2). (I recommend you read Witten on *anything*.)
- **Polyakov, A.**, *Gauge Fields and Strings*, Harwood Academic (1987).
A very original and very deep perspective on QFT, building a form of synthesis

of Polyakov's approach to strongly coupled QCD. Several of the most important developments in theoretical physics over the past couple of decades have been (directly or indirectly) inspired by ideas in this book.

- **Schweber, S.,** *QED and the Men Who Made It: Dyson, Feynman, Schwinger and Tomonaga*, Princeton (1994).

Not a textbook, but a tale of the times in which QFT was born, and the people who made it happen. It doesn't aim to dazzle you with how very great these heroes were¹, but rather shows you how puzzled they were, how human their misunderstandings, and how tenaciously they had to fight to make progress. Inspirational stuff.

- **Vafa, C., and Zaslow, E.,** (eds.), *Mirror Symmetry*, AMS (2003).

A huge book comprising chapters written by different mathematicians and physicists with the aim of understanding Mirror Symmetry in the context of string theory. Chapters 8 – 11 give an introduction to QFT in low dimensions from a perspective close to the one we will start with in this course. The following chapters could well be useful if you're taking the String Theory Part III course.

- **Weinberg, S.,** *The Quantum Theory of Fields*, vols. 1 & 2, CUP (1996).

Penetrating insight into everything it covers and packed with many detailed examples. The perspective is always deep, but it requires strong concentration to follow a story that sometimes plays out over several chapters. Weinberg's thesis is that QFT is the inevitable consequence of marrying Quantum Mechanics, Relativity and the Cluster Decomposition Principle (that distant experiments yield uncorrelated results). In this telling, particles play a primary role, with fields coming later; for me, that's backwards.

- **Zinn–Justin, J.** *Quantum Field Theory and Critical Phenomena*, 4th edition, OUP (2002).

Contains a very insightful discussion of the Renormalization Group and also a lot of information on Gauge Theories. Most of its examples are drawn from either Statistical or Condensed Matter Physics.

Textbooks are expensive. Fortunately, there are lots of excellent resources available freely online. I like these:

- **Dijkgraaf, R.,** *Les Houches Lectures on Fields, Strings and Duality*, <http://arXiv.org/pdf/hep-th/9703136.pdf>

An modern perspective on what QFT is all about, and its relation to string theory. For the most part, the emphasis is on more mathematical topics (*e.g.* TFT, dualities) than we will cover in the lectures, but the first few sections are good for orientation.

- **Hollowood, T.,** *Six Lectures on QFT, RG and SUSY*, <http://arxiv.org/pdf/0909.0859v1.pdf>

¹I should say 'are'; even now in 2017, Freeman Dyson still works at the IAS almost every day.

An excellent mini-series of lectures on QFT, given at a summer school aimed at end-of-first-year graduate students from around the UK. They put renormalization and Wilsonian Effective Theories centre stage. While the final two lectures on SUSY go beyond this course, I found the first three very helpful when preparing the current notes. We'll follow parts of these notes closely.

- **Neitzke, A.**, *Applications of Quantum Field Theory to Geometry*,
<https://www.ma.utexas.edu/users/neitzke/teaching/392C-applied-qft/>
Lectures aimed at introducing mathematicians to Quantum Field Theory techniques that are used in computing Seiberg–Witten invariants. I very much like the perspective of these lectures, and we'll Neitzke's notes closely for the first part of the course.
- **Osborn, H.**, *Advanced Quantum Field Theory*,
<http://www.damtp.cam.ac.uk/user/ho/Notes.pdf>
The lecture notes for a previous incarnation of this course, delivered by Prof. Hugh Osborn. They cover similar material to the current ones, but from a rather different perspective. If you don't like the way I'm doing things, or for extra practice, take a look here!
- **Polchinski, J.**, *Renormalization and Effective Lagrangians*,
<http://www.sciencedirect.com/science/article/pii/0550321384902876>
- **Polchinski, J.**, *Dualities of Fields and Strings*,
<http://arxiv.org/abs/1412.5704>
The first paper gives a very clear description of the ‘exact renormalization group’ and its application to scalar field theory. The second is a recent survey of the idea of ‘duality’ in QFT and beyond. We'll explore this if we get time.
- **Segal, G.**, *Quantum Field Theory* lectures,
[YouTube lectures](#)
Recorded lectures aiming at an axiomatization of QFT by one of the deepest thinkers around. I particularly recommend the lectures “*What is Quantum Field Theory?*” from Austin, TX, and “*Three Roles of Quantum Field Theory*” from Bonn (though the blackboards are atrocious!).
- **Tong, D.**, *Quantum Field Theory*,
<http://www.damtp.cam.ac.uk/user/tong/qft.html>
The lecture notes from the Michaelmas QFT course in Part III. If you feel you're missing some background from last term, this is an excellent place to look. There are also some [video lectures](#) from when the course was given at Perimeter Institute.
- **Weinberg, S.**, *What Is Quantum Field Theory, and What Did We Think It Is?*,
<http://arXiv.org/pdf/hep-th/9702027.pdf>

- Weinberg, S., *Effective Field Theory, Past and Future*,
<http://arXiv.org/pdf/0908.1964.pdf>

These two papers provide a fascinating account of the origins of effective field theories in current algebras for soft pion physics, and how the Wilsonian picture of Renormalization gradually changed our whole perspective of what QFT is about.

- Wilson, K., and Kogut, J. *The Renormalization Group and the ϵ -Expansion*,
Phys. Rep. **12** 2 (1974),
<http://www.sciencedirect.com/science/article/pii/0370157374900234>

One of the first, and still one of the best, introductions to the renormalization group as it is understood today. Written by someone who changed the way we think about QFT. Contains lots of examples from both statistical physics and field theory.

That's a huge list, and only a real expert in QFT would have mastered everything on it. I provide it here so you can pick and choose to go into more depth on the topics you find most interesting, and in the hope that you can fill in any background you find you are missing.

1 Introduction

Quantum Field Theory is, to begin with, exactly what it says it is: the quantum version of a field theory. But this simple statement hardly does justice to what is the most profound description of Nature we currently possess. As well as being the basic theoretical framework for describing elementary particles and their interactions (excluding gravity), QFT also plays a major role in areas of physics and mathematics as diverse as string theory, condensed matter physics, topology, geometry, combinatorics, astrophysics and cosmology. It's also extremely closely related to statistical field theory, probability and from there even to (quasi-)stochastic systems such as finance.

1.1 Choosing a QFT

To build a QFT, we start by picking the space on which it lives. Usually, this will be some smooth, Riemannian (or pseudo–Riemannian) manifold (M, g) of dimension $\dim(M) = d$. For example, for most applications to particle physics, we'd choose $(M, g) = (\mathbb{R}^4, \eta)$ where η is the Minkowski metric. However, this is far from being the only interesting choice. Many applications in condensed matter one sets either $(M, g) = (\mathbb{R}^3, \delta)$ with δ the flat Euclidean metric, or perhaps $M = U \subset \mathbb{R}^3$ to study field theory living in a sample of material that occupies some region U . The worldsheet description of string theory involves a QFT living on a Riemann surface $(\Sigma, [g])$ where only the conformal class $[g]$ of the metric needs to be specified, while applications of QFT to topological problems such as knot invariants make use of a certain gauge theory living on an arbitrary orientable three–manifold M with no metric at all. Whatever choice we make, in QFT the metric g is regarded as fixed – studying what happens when the metric itself has quantum fluctuations requires quantum gravity.

Having decided which universe we live in, our next choice is to pick which objects we wish to study. That is, we must choose the **fields**. The simplest choice is a scalar field, which is just a function on M . It'll often be useful to think of this a map

$$\phi : M \rightarrow \mathbb{R}, \mathbb{C}, \dots,$$

according to whether the scalar is real– or complex–valued. More generally, ϕ could describe a map

$$\phi : M \rightarrow N$$

from our space to some other (Riemannian) manifold N , known as the **target space**. For example, we'll see that we can think of ordinary non–relativistic Quantum Mechanics in terms of a $d = 1$ QFT living on an interval $I = [0, 1]$ known as the **worldline**, where the fields describe a map $\phi : I \rightarrow \mathbb{R}^3$. In particle physics, the pion field $\pi(x)$ describes a map $M \rightarrow G/H$ where M is our Universe and G and H are Lie groups. (In the specific case of pions in the Standard Model, it turns out that $G/H = (SU(2) \times SU(2))/SU(2)$.) In string theory, some of the worldsheet fields are scalars describing a map $\phi : \Sigma \rightarrow N$ embedding the worldsheet in a certain special type of Riemannian manifold N called a Calabi–Yau manifold.

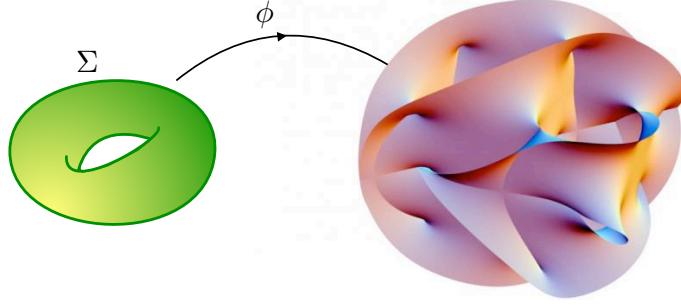


Figure 1: String Theory involves a QFT describing maps from a Riemann surface Σ to a Calabi–Yau manifold.

That's just scalars! We could also choose our fields to have non-trivial spin, or be gauge fields – connections on a principal G -bundle $P \rightarrow M$ as we'll see in chapter 8 – or carry charge under the gauge fields, meaning that they transform as a section of some bundle $E \rightarrow M$ associated to $P \rightarrow M$ by a choice of representation. For example, QED involves a photon A_μ which is a connection on a principal $U(1)$ -bundle and a charged electron, which is a section of the spin bundle on M tensored with a circle, representing the phase of the electron field. In some theories (particularly in $d > 4$), the fields include p -form fields, represented by tensors with p antisymmetric downstairs indices.

Whatever fields we pick, I'll let \mathcal{C} denote the **space of field configurations** on M . That is, \mathcal{C} every point $\phi \in \mathcal{C}$ corresponds to a configuration of the field – a picture of what the field looks like across the whole universe M . Since we allow our fields to have arbitrarily small bumps and ripples, \mathcal{C} is typically an infinite dimensional function space. Trying to understand the geometry and topology of this infinite dimensional space of fields, and then trying to do something useful with it is fundamentally what makes QFT difficult, but it's also what makes it interesting and powerful.

The next ingredient we need is to specify the **action** for our theory. This is a function

$$S : \mathcal{C} \rightarrow \mathbb{R} \tag{1.1}$$

on the space of fields. In other words, given a field configuration, the action produces a real number. We often write $S[\phi]$ for this number, as opposed to $S(\phi)$, just to remind us that the domain of S is infinite dimensional.

We often assume that this action is **local**, meaning that it can be written as

$$S[\phi] = \int_M d^d x \sqrt{g} \mathcal{L}(\phi(x), \partial\phi(x), \dots) \tag{1.2}$$

where the Lagrangian (density) \mathcal{L} depends on the value of ϕ and finitely many derivatives at just a *single* point in M . Note that this is actually a very strong restriction. A general polynomial function on \mathcal{C} would be more like

$$\int_{M^{\otimes n}} d^d x_1 d^d x_2 \cdots d^d x_n f(x_1, x_2, \dots, x_n) \phi(x_1) \phi(x_2) \cdots \phi(x_n)$$

involving the integral of the field at many different points, with some smearing function $f : M^{\otimes n} \rightarrow \mathbb{R}$. If we were to include such multi-local terms in our action, the resulting classical field equations would be integro-differential equations, so the behaviour of our field at one point $x \in M$ would depend what the field configuration looks like across all of M . This ‘action at a distance’ is usually thought to be unphysical, at least in classical physics. However, we’ll see later that QFT forces us to consider these non-local terms even if we try to rule them out in setting up the theory.

1.2 What do we want to compute?

In this course, the main tool we’ll use to

1.2.1 The Partition function

1.2.2 Correlation functions

Beyond the partition function, the most important object we wish to compute in any QFT are (normalized) **correlation functions**.

In the context of QFT, we’ll often choose the functions we insert to correspond to some quantity of physical interest that we wish to measure; perhaps the energy of the quantum field in some region, or the total angular momentum carried by some electrons, or perhaps temperature fluctuations in the CMB at different angles on the night sky. For reasons that will become apparent, we’ll often call these functions ‘operators’, though the terminology is somewhat inaccurate (particularly in zero dimensions).

1.2.3 Boundaries and Hilbert space

If our manifold M has boundaries, say $\partial M = \cup_i B_i$, then to specify the path integral we must choose some boundary conditions for the fields on each component of ∂M . We’ll see below that on each boundary component B_i , the possible configurations of the field naturally form a Hilbert space \mathcal{H}_i . Thus, on a manifold with boundary the path integral really defines a map

$$\otimes_i \mathcal{H}_i \rightarrow \mathbb{C}. \quad (1.3)$$

To compute this object, the idea is that once we decide what our fields look like on each B_i (in other words once we pick a state in each \mathcal{H}_i) we obtain a complex number by performing the path integral

$$\int_{\phi \in \mathcal{C} : \phi|_{B_i} = \varphi_i} D\phi e^{-S[\phi]/\hbar} \quad (1.4)$$

over those fields on M that agree with our chosen fields φ_i on each boundary component.

As a very important special case, suppose $M = N \times I$, where N is some $D - 1$ dimensional manifold and I is just an interval of length T with respect to the metric g on M . In this case the path integral gives us a map²

$$U(T) : \mathcal{H} \rightarrow \mathcal{H} \quad (1.5)$$

²In fact, in Minkowskian signature, this map is **unitary**. Unitarity is difficult to see from the path integral perspective and is why you spent time studying canonical quantization last term.

from the Hilbert space associated to the incoming boundary of M to that associated to the outgoing boundary. If we imagine that our interval I is made up of two smaller intervals I_1 and I_2 of lengths T_1 and T_2 , with $T_1 + T_2 = T$, then we can imagine breaking the path integral over $M \times I$ into ones over $M \times I_1$ and $M \times I_2$. The path integral over $M \times I_1$ involves picking some boundary values for the fields on the

$$U(T_1) \circ U(T_2) = U(T) . \quad (1.6)$$

By allowing ourselves to break up M into many thin slivers, this implies that

$$U(T) = \exp(-iTH) \quad (1.7)$$

where the Hermitian operator H is the **Hamiltonian** of the QFT and generates evolution of the fields along I . (In Euclidean signature we would instead write $U(T) = e^{-TH}$.)

2 QFT in zero dimensions

We'll embark on our journey from the simplest possible starting point: we'll study QFT in a space–time with zero dimensions. That's a very drastic simplification, and much of the richness of QFT will be absent here. Indeed, I expect many of the ideas in this chapter will be things you've met (long) before, although perhaps in a different context. Still, you shouldn't sneer. We'll see that even this simple case contains baby versions of ideas we'll study more generally later in the course, and it will provide us with a safe playground in which to check we understand what's going on. Furthermore, **it has been seriously conjectured** that full, non-perturbative string theory is itself a zero-dimensional QFT (though admittedly with infinitely many fields).

2.1 Partition functions and correlation functions in $d = 0$

If our space–time M is zero–dimensional and connected, then it must be just a single point. In zero dimensions, there are no lengths, so there is no notion of a metric. Similarly, the Lorentz group is trivial, hence all its representations are trivial. In other words, all fields must be scalars: there is no notion of the ‘spin’ of a field, simply because there is no notion of a Lorentz transformation.

In the simplest case, a ‘field’ on M is a map $\phi : \{\text{pt}\} \rightarrow \mathbb{R}$, or in other words just a real variable. The ‘space of all field configurations’ \mathcal{C} is also easy to describe: it's again just \mathbb{R} , because our entire universe M is just one point, so we completely specify what the field looks like by giving its value at this one point. The path integral measure $\mathcal{D}\phi$ becomes just the standard (Lebesgue) measure $d\phi$ on \mathbb{R} and the partition function

$$\mathcal{Z} = \int_{\mathbb{R}} d\phi e^{-S(\phi)/\hbar}, \quad (2.1)$$

is just a standard integral over the real line.

Now let's choose our action. In zero dimensions, there are no space–time directions along which we could differentiate our ‘field’, so there can be no kinetic terms. Thus, the action is just a function $S(\phi)$ of this one real variable. Typically, we'll take $S(\phi)$ to be a polynomial (with highest term of even degree), such as

$$S(\phi) = \frac{m^2}{2}\phi^2 \quad \text{or perhaps} \quad S(\phi) = \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4,$$

but all that really matters is that S is chosen so that the partition function (2.1) converges.

The coefficients of various powers of ϕ in the action are collectively known as **coupling constants**. The coefficients of ϕ^p with $p = 0, 1, 2$ have a slightly different status. Somewhat abstractly, we can think of the set of couplings as being coordinates on the infinite dimensional ‘space of theories’ in the sense that, at least for our single field ϕ with $S(\phi)$ analytic, the theory is specified once we choose values for all possible monomials ϕ^p with $p \geq 0$.

What we get for \mathcal{Z} depends on which action we picked, so the partition function then depends on the values of the coupling constants

$$\mathcal{Z} = \mathcal{Z}(m^2, g, \dots). \quad (2.2)$$

As a piece of notation, I'll often write \mathcal{Z}_0 for the partition function in the free theory, where the couplings of all but the term quadratic in the field(s) are set to zero.

Similarly, correlation functions are

$$\langle f \rangle := \frac{1}{\mathcal{Z}} \int_{\mathbb{R}} d\phi f(\phi) e^{-S(\phi)/\hbar} \quad (2.3)$$

where along with $e^{-S(\phi)/\hbar}$ we've inserted some other function $f(\phi)$ into the integral. We'll assume that f is sufficiently well-behaved that the integral (2.3) still exists. In particular, f should not grow so rapidly as $|\phi| \rightarrow \infty$ as to overcome the decay of $e^{-S(\phi)/\hbar}$. In practice, we'll restrict ourselves to the case that f is just a polynomial.

The usual way to think about correlation functions comes from probability. So long as the action $S(\phi)$ is \mathbb{R} -valued, $e^{-S} \geq 0$ so we can view $\frac{1}{\mathcal{Z}}e^{-S}$ as a *probability density* on the space of fields. The correlation function (2.3) is then just the *expectation value* $\langle f \rangle$ of $f(\phi)$ averaged over the space of fields with this measure, with the factor of $1/\mathcal{Z}$ ensuring that the probability measure is normalized. On a higher dimensional space-time M we'll be able to insert functions *at different points in space-time* into the path integral, and the correlation functions will probe whether there is any statistical relation between, say, two functions $f(\phi(x))$ and $g(\phi(y))$ at points $x, y \in M$.

Alternatively, recalling that the general partition function $\mathcal{Z}(m^2, g, \dots)$ depends on the values of all possible couplings, we see that correlation functions of polynomial operators describe the change in this general \mathcal{Z} as we infinitesimally vary some combination of the couplings, evaluated at the point in theory space corresponding to our original model. For example, in the simplest case that $f(\phi) = \phi^p$ is monomial, we have formally

$$\frac{1}{p!} \langle \phi^p \rangle = -\frac{\hbar}{\mathcal{Z}} \left. \frac{\partial}{\partial g_p} \mathcal{Z}(m^2, \lambda_i) \right|_* \quad (2.4)$$

where g_p is the coupling to $\phi^p/p!$ in the general action, and $*$ is the point in theory space where the couplings are set to their values in the specific action that appears in (2.3).

2.2 Free field theory

The simplest QFTs are free, meaning that the action is (at most) quadratic in the fields. As an example, suppose we have n fields ϕ^a with $a = 1, \dots, n$, thought of as a map $\phi : \{\text{pt}\} \rightarrow \mathbb{R}^n$. We choose the action to be the quadratic function

$$S(\phi) = \frac{1}{2} M(\phi, \phi) = \frac{1}{2} M_{ab} \phi^a \phi^b, \quad (2.5)$$

where $M : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is represented by a real, positive-definite, symmetric matrix. The partition function of this free, zero-dimensional QFT is the basic Gaussian integral

$$\mathcal{Z}_0 = \int_{\mathbb{R}^n} d^n \phi e^{-M(\phi, \phi)/2\hbar}$$

with the standard (Lebesgue) measure $d^n \phi$ on the space of fields, which is now just \mathbb{R}^n . To evaluate this, note that since M is a real symmetric matrix, its eigenvectors are orthogonal

so M can be diagonalized by some orthogonal transformation $O : \mathbb{R}^n \rightarrow \mathbb{R}^n$. The path integral measure is the standard measure $d^n\phi$ on \mathbb{R}^n , which is invariant under such an orthogonal transformation. In the basis of eigenvectors, the integral is just a product of n independent Gaussian integrals

$$\int_{\mathbb{R}} d\chi e^{-m\chi^2/2\hbar} = \sqrt{\frac{2\pi\hbar}{m}}, \quad (2.6)$$

where m is the eigenvalue of M . Multiplying all the contributions, we obtain

$$\mathcal{Z}_0 = \int_{\mathbb{R}^n} d^n\phi e^{-M(\phi,\phi)/2\hbar} = \frac{(2\pi\hbar)^{n/2}}{\sqrt{\det M}}, \quad (2.7)$$

where we have written the product of eigenvalues more invariantly as the determinant. Note that M being positive-definite ensures that $\det M > 0$ and the integral exists.

It will also be useful to consider a small generalization. We allow the action to include a linear source term $J_a\phi^a$:

$$S(\phi) = \frac{1}{2}M(\phi, \phi) + J \cdot \phi. \quad (2.8)$$

Completing the square, we have

$$\frac{1}{2}M(\phi, \phi) + J \cdot \phi = \frac{1}{2}M(\tilde{\phi}, \tilde{\phi}) - \frac{1}{2}M^{-1}(J, J) \quad (2.9)$$

where $\tilde{\phi} := \phi + M^{-1}(J, \cdot)$ are some translated coordinates on \mathbb{R}^n . (Our assumption that M was positive-definite also guarantees that M^{-1} exists.) Since $\tilde{\phi}$ differs from ϕ by a translation, the measure $d^n\tilde{\phi} = d^n\phi$. Therefore, in the presence of the source J the partition function is

$$\begin{aligned} \mathcal{Z}(J) &= \int_{\mathbb{R}^n} d^n\phi \exp\left(-\frac{1}{\hbar}\left(\frac{1}{2}M(\phi, \phi) + J \cdot \phi\right)\right) \\ &= \exp\left(\frac{1}{2\hbar}M^{-1}(J, J)\right) \int_{\mathbb{R}^n} d^n\tilde{\phi} e^{-M(\tilde{\phi}, \tilde{\phi})/2\hbar} = \exp\left(\frac{1}{2\hbar}M^{-1}(J, J)\right) \mathcal{Z}_0 \end{aligned} \quad (2.10)$$

where \mathcal{Z}_0 is the original partition function (2.7).

This generalization helps us to compute correlation functions. Suppose $\langle P(\phi) \rangle$ is a polynomial $P : \mathbb{R}^n \rightarrow \mathbb{R}$. By linearity of the integral, evaluation of the correlation function $\langle P(\phi) \rangle$ reduces to the case that P is a product of linear factors $\ell(\phi) = \ell \cdot \phi$, so we just need to compute

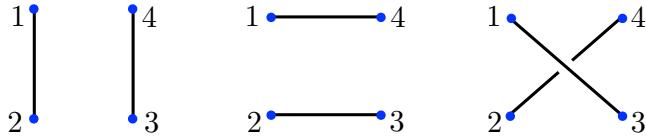
$$\langle \ell_1(\phi) \cdots \ell_p(\phi) \rangle = \frac{1}{\mathcal{Z}_0} \int_{\mathbb{R}^n} d^n\phi e^{-M(\phi, \phi)/2\hbar} \prod_{i=1}^p \ell_i(\phi). \quad (2.11)$$

If p is odd, then the integrand is an odd function of (at least one direction of) ϕ , so vanishes when integrated over \mathbb{R}^n . Let's evaluate the remaining case $p = 2k$. We have that

$$\begin{aligned}
\langle \ell_1(\phi) \cdots \ell_{2k}(\phi) \rangle &= \frac{1}{Z_0} \int_{\mathbb{R}^n} d^n \phi \left. \prod_{i=1}^{2k} \ell_i(\phi) e^{-M(\phi,\phi)/2\hbar - J(\phi)/\hbar} \right|_{J=0} \\
&= \frac{(-\hbar)^{2k}}{Z_0} \int_{\mathbb{R}^n} d^n \phi \left. \prod_{i=1}^{2k} \ell_i \cdot \frac{\partial}{\partial J} \left[e^{-M(\phi,\phi)/2 - J(\phi)/\hbar} \right] \right|_{J=0} \\
&= \left. \prod_{i=1}^{2k} \ell^{(i)} \cdot \frac{\partial}{\partial J} \left[\frac{\hbar^{2k}}{Z_0} \int_{\mathbb{R}^n} d^n \phi e^{-M(\phi,\phi)/2 - J(\phi)/\hbar} \right] \right|_{J=0} \\
&= \hbar^{2k} \prod_{i=1}^{2k} \ell^{(i)} \cdot \left. \frac{\partial}{\partial J} \left[e^{M^{-1}(J,J)/2\hbar} \right] \right|_{J=0}
\end{aligned} \tag{2.12}$$

The first line is a triviality: we want to know the correlation function in the original theory where $J = 0$. In going to the second line here we differentiated the action *wrt* J to bring down each factor of ϕ , in going to the third line we note that the integrand is absolutely convergent so the order of integration and differentiation may safely be exchanged, and the final line uses the result (2.10).

In the final expression of (2.12), for every derivative $\hbar \ell \cdot \partial/\partial J$ that acts on the exponential we get a factor of $M^{-1}(\ell, J)$. Because we'll set $J = 0$ at the end of the calculation, we can get a non-vanishing contribution to (2.12) only when exactly half the derivatives bring down such factors, while the other half then removes the J dependence in front of the exponential. Let σ be a way of joining the elements of the set $\{1, 2, \dots, 2k\}$ into pairs, and let Π_{2k} denote the set of all possible (complete) pairings. For example, for the set $\{1, 2, 3, 4\}$, Π_4 is



Then the correlation function is

$$\langle \ell_1(\phi) \cdots \ell_{2k}(\phi) \rangle = \hbar^k \sum_{\sigma \in \Pi_k} \prod_{i \in \{1, \dots, 2k\}/\sigma} M^{-1}(\ell_i, \ell_{\sigma(i)}) , \tag{2.13}$$

in other words, a sum over products of all inequivalent ways of connecting pairs of ℓ_i using M^{-1} . For example, we have

$$\langle \ell_1(\phi) \ell_2(\phi) \rangle = M^{-1}(\ell_1, \ell_2) \tag{2.14}$$

for the 2-point function, while the 4-point function is

$$\begin{aligned}
&\langle \ell_1(\phi) \ell_2(\phi) \ell_3(\phi) \ell_4(\phi) \rangle \\
&= M^{-1}(\ell_1, \ell_2) M^{-1}(\ell_3, \ell_4) + M^{-1}(\ell_1, \ell_3) M^{-1}(\ell_4, \ell_2) + M^{-1}(\ell_1, \ell_4) M^{-1}(\ell_2, \ell_3) .
\end{aligned} \tag{2.15}$$

In general, there are $|\Pi_{2k}| = (2k)!/(2^k k!)$ ways of joining $2k$ elements into pairs, so the $2k$ -point function receives $(2k)!/(2^k k!)$ contributions. In particular, we find

$$\langle (\ell \cdot \phi)^{2k} \rangle = \frac{(2k)!}{2^k k!} (\hbar M^{-1}(\ell, \ell))^k \quad (2.16)$$

when all of the ℓ_i 's are the same.

In dimensions $d > 0$, we will consider correlation functions where products of fields are inserted at different points in our space-time. In this case, M^{-1} is the **propagator** – the inverse of the quadratic term in the action. We can think of it as representing the response of one field insertion to the presence of another. The above picture of pairs of points connected by lines is then an example of **Feynman diagram** describing the correlation function. In $d = 0$, there's no sense in which the fields 'propagate' anywhere, and the diagrams are just a nifty way to keep track of the combinatorics. Since we're currently thinking just about free theory, our diagrams are rather trivial as they have no (internal) vertices at present.

Our result (2.13) for the correlation function is known as **Wick's theorem**, though in the $d = 0$ context of Gaussian distributions it's called **Isserlis' theorem** by probabalists. You met Wick's theorem last term from the point of view of canonical quantization, where it arose from decomposing the field operator $\hat{\phi}$ into creation and annihilation operators, and commuting these operators past one another. A closely related result for fermionic variables (that anticommute: $\theta^a \theta^b = -\theta^b \theta^a$) starts from the observation that, if θ^a are fermionic, then

$$\int d^n \theta d^n \bar{\theta} \exp(M(\bar{\theta}, \theta) + \bar{\eta} \cdot \theta + \bar{\theta} \cdot \eta) = e^{M^{-1}(\bar{\eta}, \eta)} \det(M) \quad (2.17)$$

in place of (2.10). You're asked to prove this identity in the first problem set.

2.3 Perturbation theory

Interesting theories involve interactions, so that the action $S(\phi)$ is not merely quadratic. In this case, integrals such as

$$\int_{\mathbb{R}^n} d^n \phi f(\phi) e^{-S(\phi)/\hbar} \quad (2.18)$$

become transcendental, even for simple actions $S(\phi)$ – including most of physical interest – and simple choices of $f(\phi)$. Typically, we do not know how to evaluate such integrals analytically.

We may hope to approximate (2.18) perturbatively by expanding around the classical limit $\hbar \rightarrow 0$. However, our integral cannot have a Taylor expansion around $\hbar = 0$, since any such Taylor expansion would have to converge for all \hbar in a disc $D \subset \mathbb{C}$ centered on the origin. But if the action is chosen so that the integral converges whenever $\hbar > 0$, then (2.18) surely *diverges* if we formally attempt continue into the region $\text{Re}(\hbar) < 0$.

Instead, we can obtain an **asymptotic expansions** for such path integrals. Suppose $S(\phi)$ is a smooth function that has a global minimum at a unique point $\phi = \phi_0 \in \mathbb{R}^n$, such

that the Hessian matrix $\partial_a \partial_b S(\phi_0)$ is positive-definite. Then (2.18) has an asymptotic expansion

$$\int_{\mathbb{R}^n} d^n \phi f(\phi) e^{-S(\phi)/\hbar} \sim (2\pi\hbar)^{n/2} \frac{f(\phi_0) e^{-S(\phi_0)/\hbar}}{\sqrt{\det(\partial_a \partial_b S(\phi_0))}} (1 + \hbar A_1 + \hbar^2 A_2 + \dots) \quad (2.19)$$

as $\hbar \rightarrow 0^+$. I won't prove this, but you should be able to prove it yourself if you've taken a course in asymptotic analysis. (The proof is an application of Laplace's method for asymptotic evaluation of integrals, essentially based on Watson's lemma. Don't worry if you don't know this stuff – I don't require that you can prove (2.19).) The leading term in this expansion is known as the **semi-classical** term, while higher order terms in \hbar are called **quantum corrections**.

Suppose in particular that the partition function has an asymptotic series

$$\begin{aligned} \mathcal{Z}(\hbar) &=: (2\pi\hbar)^{n/2} \frac{e^{-S(\phi_0)/\hbar}}{\sqrt{\det(\partial_a \partial_b S(\phi_0))}} I(\hbar) \\ &\sim (2\pi\hbar)^{n/2} \frac{e^{-S(\phi_0)/\hbar}}{\sqrt{\det(\partial_a \partial_b S(\phi_0))}} (a_0 + a_1 \hbar + a_2 \hbar^2 + \dots) \end{aligned} \quad (2.20)$$

so that $(a_0 + a_1 \hbar + a_2 \hbar^2 + \dots + a_N \hbar^N)$ are the first N terms in the asymptotic series for $I(\hbar)$. This statement means that, for all $N \in \mathbb{N}$

$$\lim_{\hbar \rightarrow 0^+} \frac{1}{\hbar^N} \left| I(\hbar) - \sum_{n=0}^N a_n \hbar^n \right| = 0. \quad (2.21)$$

In other words, with fixed N , for sufficiently small $\hbar \in \mathbb{R}_{\geq 0}$ the first N terms of the series differ from the exact answer by less than $\epsilon \lambda^N$ for *any* $\epsilon > 0$. (The difference is $o(N)$). However, since our series actually diverges, if we instead fix a value of $\hbar > 0$ and include more and more terms in the sum, we will eventually get worse and worse approximations to the answer. Perturbation theory thus tells us important, but not complete, information about our QFT.

Let's understand this in an example. Consider the $d = 0$ QFT with a single scalar field ϕ and action $S(\phi) = m^2 \phi^2/2 + \lambda \phi^4/4!$. We need to take $\lambda > 0$ for the partition function to converge, and we'll also assume $m^2 > 0$ so that the action has a unique minimum at $\phi_0 = 0$. Then the leading term in our asymptotic expansion is

$$(2\pi\hbar)^{1/2} \frac{e^{-S(\phi_0)}}{\sqrt{S''(\phi_0)}} = \frac{\sqrt{2\pi\hbar}}{m}, \quad (2.22)$$

since $S(\phi_0) = 0$ here. We recognize this as just the partition function $\mathcal{Z}(m, 0)$ of the free

theory at $\lambda = 0$. Going further, an asymptotic series for $\mathcal{Z}(m^2, \lambda)$ is given by

$$\begin{aligned}
\mathcal{Z}(m^2, \lambda) &= \int_{\mathbb{R}} d\phi e^{-\frac{1}{\hbar}(\frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4)} = \int_{\mathbb{R}} d\phi \left[e^{-m^2\phi^2/2\hbar} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-\lambda}{4!\hbar} \right)^n \phi^{4n} \right] \\
&\sim \sum_{n=0}^N \frac{1}{n!} \left(\frac{-\lambda}{4!\hbar} \right)^n \int_{\mathbb{R}} d\phi e^{-\frac{m^2}{2}\phi^2} \phi^{4n} = \frac{\sqrt{2\hbar}}{m} \sum_{n=0}^N \frac{1}{n!} \left(\frac{-\hbar\lambda}{3!m^4} \right)^n \int_0^\infty dx e^{-x} x^{2n+\frac{1}{2}-1} \\
&= \frac{\sqrt{2\hbar}}{m} \sum_{n=0}^N \frac{1}{n!} \left(\frac{-\hbar\lambda}{3!m^4} \right)^n \Gamma\left(2n + \frac{1}{2}\right).
\end{aligned} \tag{2.23}$$

In the first line we used the Taylor expansion of the entire function $e^{\lambda\phi^4/4!\hbar}$. In going to the second line we truncated this series to the first $N+1$ terms and then exchanged the order of the summation and integration. Note that it would not be legitimate to exchange the order of the *infinite* sum and the integral, because the original integral did not converge when \hbar (or λ) was negative. The remaining expressions follow upon substituting $x = m^2\phi^2/2\hbar$ and recognizing the integral representation of the gamma function. (Somewhat more laboriously, the ϕ integral can be obtained after repeated integration by parts.) Using the value of $\Gamma(z)$ at positive half-integers we have finally

$$\begin{aligned}
\mathcal{Z}(m^2, \lambda) &\sim \frac{\sqrt{2\pi\hbar}}{m} \sum_{n=0}^N (-)^n \frac{\hbar^n \lambda^n}{m^{4n}} \frac{1}{(4!)^n n!} \frac{(4n)!}{4^n (2n)!} \\
&= \frac{\sqrt{2\pi\hbar}}{m} \left[1 - \frac{\hbar\lambda}{8m^4} + \frac{35}{384} \frac{\hbar^2 \lambda^2}{m^8} + \dots \right]
\end{aligned} \tag{2.24}$$

as our asymptotic series for the partition function.

Let me make a few remarks. Firstly, upto the factor $\sqrt{2\pi\hbar}$, this series depends on \hbar and λ only through the combination $\hbar\lambda$. Recalling that the $\mathcal{Z}(m, \lambda)$ was only defined for $\lambda/\hbar > 0$, what we have obtained is *also* an asymptotic series for expansion in the coupling constant. This is typically the way perturbation theory is presented in QFT. Secondly, note we can see the divergence of the perturbation series directly here: From Stirling's approximation $n! \approx e^{n \ln n}$ as $n \rightarrow \infty$ we see that the coefficient of $(\hbar\lambda)^n/m^{4n}$ is approximately

$$\frac{1}{(4!)^n n!} \frac{(4n)!}{4^n (2n)!} \approx e^{n \ln n}.$$

Thus these coefficients asymptotically grow faster than exponentially with n , so the series (2.24) has zero radius of convergence. It is interesting to ask whether it is possible to recover the exact value of $\mathcal{Z}(m^2, \lambda)$ from its asymptotic series. Remarkably, a technique known as **Borel resummation** allows one to achieve this, at least in certain circumstances. You're invited to explore it for this example in the problem sheets.

Finally, I remark that $\mathcal{Z}(m^2, \lambda)$ itself should exist even if $m^2 < 0$, provided \hbar and λ are positive, because the exponential enhancement from the factor $e^{+|m^2|\phi^2/2\hbar}$ at small ϕ is eventually suppressed by the quartic term in the action. However, the asymptotic series (2.24) is not valid in this case, as we can see from the fact that the (Gaussian)

integrals in the second line of (2.23) require $m^2 > 0$ to converge. More fundamentally, the problem is that when $m^2 < 0$, the point $\phi = 0$ which we took to give the dominant contribution to the integral is now a (local) **maximum** of the action, the global minima being at $\phi_0 = \pm\sqrt{6m^2/\lambda}$. In physics terminology *we are expanding around the wrong vacuum*. Particles with $m^2 < 0$ are called **tachyons**, and they always signal an instability. Whether or not this instability is just due to a poor choice of perturbative expansion (as here), or whether the whole theory is unstable (meaning $\mathcal{Z}(m^2, \lambda)$ does not exist for $m^2 < 0$) is not always clear. The situation where the minimum of the action involves a non-zero value for some field is often associated with **spontaneous symmetry breaking**. You can learn more about this *e.g.* in the Part III courses on Statistical Field Theory, or the Standard Model.

2.3.1 Feynman diagrams

Consider again the terms in the series

$$\mathcal{Z}(m^2, \lambda) \sim \frac{\sqrt{2\pi\hbar}}{m} \sum_{n=0}^N (-)^n \frac{\hbar^n \lambda^n}{m^{4n}} \frac{1}{(4!)^n n!} \frac{(4n)!}{4^n (2n)!}. \quad (2.25)$$

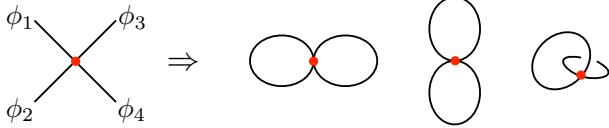
The powers of \hbar , λ and m here are essentially fixed by dimensional analysis, while the factor $\frac{1}{(4!)^n n!}$ comes straightforwardly from expanding the ϕ^4 term in the exponential. The remaining factor $(4n!)/4^n (2n)!$ is the number of ways of joining $4n$ elements into distinct pairs; indeed, we saw in the discussion of Wick's theorem that the integral $\int e^{-\phi^2/2} \phi^4 d\phi$ had a combinatoric interpretation in terms of pairings. Let's use this to interpret $\mathcal{Z}(m^2, \lambda)$ in terms of Feynman diagrams.

With the action $S(\phi) = m^2\phi^2/2 + \lambda\phi^4/4!$ the Feynman rules are simply²



where the propagator is constant since we are in zero dimensions. The minus sign in the vertex comes from the fact that we are expanding e^{-S} . To compute perturbation series in QFT, Feynman tells us to construct all possible graphs (not necessarily connected) using this propagator and vertex. In the case of the partition function $\mathcal{Z}(m^2, \lambda)$, we want vacuum graphs, *i.e.*, those with no external edges. In constructing all possible such graphs, we imagine the individual vertices carry their own unique ‘labels’, so that we can tell them apart, and that likewise each of the four ϕ fields present in a given vertex carries its own label. Thus, the term proportional to λ receives contributions from three individual graphs

²Henceforth, we set $\hbar = 1$.



corresponding to the three possible ways to join up the four ϕ fields into pairs.

The partition function itself is given by the sum of graphs

$$Z = \emptyset + \text{(disconnected graph)} + \text{(connected graph)} + \text{(connected graph)} + \text{(connected graph)} + \dots$$

$$= 1 + \frac{-\lambda}{8m^4} + \frac{\lambda^2}{48m^8} + \frac{\lambda^2}{16m^8} + \frac{\lambda^2}{128m^8} + \dots$$

where we include both connected and disconnected graphs, with the contribution of a disconnected graph being the product of the contributions of the two connected graphs. Notice that this requires that we assign a factor 1 to the trivial graph \emptyset (no vertices or edges), which is also included as the zeroth-order term in the sum.

To work out the numerical factors, let D_n be the set of such graphs that contain precisely n vertices. Since each vertex comes with a power of the coupling λ , these diagrams will each contribute to the coefficient of λ^n in the expansion of $\mathcal{Z}(m^2, \lambda)$. Suppose there are $|D_n|$ graphs in this set. If, as above, we join up our labelled vertices in *every* possible way, then D_n will contain several elements that are identical as topological graphs, but differ just in the labelling of their vertices. We need to remove this overcounting. D_n is naturally acted on by the group $G_n = (S_4)^n \rtimes S_n$ that permutes each of the four fields in a given vertex (n copies of the permutation group S_4 on 4 elements) and also permutes the labels of each of the n vertices. This group has order $|G_n| = (4!)^n n!$, which is the same factor we saw before from expanding e^{-S} in powers of λ . Thus the asymptotic series may be rewritten as

$$\frac{\mathcal{Z}}{\mathcal{Z}_0} \sim \sum_{n=0}^{\infty} \left(\frac{-\lambda}{m^4} \right)^n \frac{|D_n|}{|G_n|}, \quad (2.26)$$

where $\mathcal{Z}_0 = \mathcal{Z}(m^2, 0)$. In detail, the power $(-\lambda)^n$ is the contribution of the coupling constants in each graph, the power of $(1/m^2)^{2n}$ comes from the fact that any vacuum diagram with exactly n 4-valent vertices must have precisely $2n$ edges, each of which contributes a factor of $1/m^2$. Finally, the factor $|D_n|$ is the number of diagrams that contribute at this order and the factor $1/|G_n|$ is the coefficient of this graph in expanding the exponential of the action perturbatively in the interactions.

There's another way to think of $|D_n|/|G_n|$ that is sometimes³ convenient. An **orbit** Γ of G_n in D_n is a set of labeled graphs in D_n that are identical up to a relabeling of their

³In practice, at least for the simple graphs we'll meet in this course, it's often just as quick to think through the possible ways a given topological graph Γ may be obtained by expanding out the vertices in e^{-S} and joining pairs of fields by propagators, as to work out the symmetry factor $|\text{Aut } \Gamma|$. I'll leave it to your taste.

fields and vertices, so that we can move from one labelled graph to another in the orbit using an element of G_n . Thus an orbit Γ is a topologically distinct graph in D_n . Let O_n be the set of such orbits. The **orbit stabilizer theorem**⁴ says that

$$\frac{|D_n|}{|G_n|} = \sum_{\Gamma \in O_n} \frac{1}{|\text{Aut } \Gamma|} \quad (2.27)$$

where $\text{Aut } \Gamma$ is the stabilizer of any element in Γ in G_n , *i.e.*, the elements of the permutation group G_n that don't alter the labeled graph. For example, if a graph in D_n involves a propagator joining two fields at the same vertex, then exchanging the labeling of those fields won't change the labeled graph. Similarly, if a pair of vertices are connected by two (or more) propagators, then exchanging the labels of the two (or more) legs on each vertex that are joined to these propagators does not change the labeled graph. Finally then, we can rewrite our asymptotic series (2.25) as

$$\begin{aligned} \frac{\mathcal{Z}}{\mathcal{Z}_0} &\sim \sum_{n=0}^{\infty} \left[\left(\frac{-\lambda}{m^4} \right)^n \sum_{\Gamma \in O_n} \frac{1}{|\text{Aut } \Gamma|} \right] \\ &= \sum_{\Gamma} \frac{1}{|\text{Aut } \Gamma|} \frac{(-\lambda)^{|v(\Gamma)|}}{(m^2)^{|e(\Gamma)|}}, \end{aligned} \quad (2.28)$$

in terms of a sum over Feynman graphs Γ , where $|v(\Gamma)|$ and $|e(\Gamma)|$ are respectively the number of vertices and edges of the graph Γ .

In zero dimensions, we've rederived the Feynman rule that we should weight each topologically distinct graph by $|v(\Gamma)|$ powers of (minus) the coupling constant $-\lambda$ and $|e(\Gamma)|$ powers of the propagator $1/m^2$, then divide by the symmetry factor $|\text{Aut } \Gamma|$ of the graph. More generally, if we have i different types of field, each with propagators $1/P_i$ and interacting via a set of vertices with couplings λ_α , then a graph Γ containing $|e_i(\Gamma)|$ edges of the field of type i and $|v_\alpha(\Gamma)|$ vertices of type α contributes an amount

$$\frac{1}{|\text{Aut } \Gamma|} \prod_i \frac{1}{P_i^{|e_i(\Gamma)|}} \prod_\alpha (-\lambda_\alpha)^{|v_\alpha(\Gamma)|}$$

to the perturbative series. To obtain the perturbative series for the partition function we sum this expression over both connected and disconnected vacuum graphs, including the trivial graph with no vertices.

It's often convenient to include just the connected graphs, as there are vastly fewer of them. We have

$$\frac{\mathcal{Z}}{\mathcal{Z}_0} = \exp \left[\sum_{\text{conn}} \frac{1}{|\text{Aut } \Gamma|} \prod_{i,\alpha} \frac{(-\lambda_\alpha)^{|v_\alpha(\Gamma)|}}{P_i^{|e_i(\Gamma)|}} \right] =: e^{-W+W_0} \quad (2.29)$$

where the sum in the exponential is only over connected, non-trivial graphs. The identity (2.29) easily visualized by writing the power series expansion of the *rhs*, defining the

⁴If you don't know this already, you can find a nicely explained proof on [Gowers's Weblog](#).

product of two connected graphs to be the disconnected graph whose two connected components are the original graphs. $W = \ln \mathcal{Z}$ is known as the **free energy** (so $W_0 = \ln \mathcal{Z}_0$ is the free energy of the Gaussian theory) and is one of the most important quantities in any QFT.

2.4 Supersymmetry and localization

This section is, I promise, really interesting – please read it! However, it’s not quite central to our story, so in 2017 I won’t discuss it in lectures. You can regard it as non-examinable.

For a generic QFT, the asymptotic series is as good a representation of the partition function (or correlation functions) as we can hope for, barring numerics. However, if the action is of a very special type, it may sometimes possible to evaluate the partition function and even certain correlation functions *exactly*. There are many mechanisms by which this might happen; this section gives a toy model of one of them, known as **localization** in supersymmetric theories.

Let’s take a theory where that in addition to our bosonic field ϕ , we have two fermionic fields ψ_1 and ψ_2 . With a zero-dimensional space–time, the space of fields is just $\mathbb{R}^{1|2}$. Given an action $S(\phi, \psi_i)$ the partition function is, as usual,

$$\mathcal{Z} = \int \frac{d\phi d\psi_1 d\psi_2}{\sqrt{2\pi}} e^{-S(\phi, \psi_i)} \quad (2.30)$$

where I’ve thrown a factor of $1/\sqrt{2\pi}$ into the measure for later convenience. Generically, we’d have to be content with a perturbative evaluation of \mathcal{Z} , using Feynman diagrams formed from edges for the ϕ and ψ_i fields, together with vertices from all the different vertices that appear in our action. For a complicated action, even low orders of the perturbative expansion might be difficult to compute in general.

However, let’s suppose the action takes the special form

$$S(\phi, \psi_1, \psi_2) = \frac{1}{2}(\partial h)^2 - \psi_1 \psi_2 \partial^2 h \quad (2.31)$$

where $h(\phi)$ is some (\mathbb{R} -valued) polynomial in ϕ and ∂h is its derivative wrt ϕ . Note that there can’t be any terms in S involving only one of the fermion fields since this term would itself be fermionic. There also can’t be higher order terms in the fermion fields since $\psi_i^2 = 0$ for a Grassmann variable, so the only thing special about this action is the relation between the purely bosonic piece and the second term involving $\psi_1 \psi_2$.

Now consider the transformations

$$\delta\phi = \epsilon_1 \psi_1 + \epsilon_2 \psi_2, \quad \delta\psi_1 = \epsilon_2 \partial h, \quad \delta\psi_2 = -\epsilon_1 \partial h \quad (2.32)$$

where ϵ_i are fermionic parameters. These are supersymmetry transformations in this zero-dimensional context; take the Part III Supersymmetry course to meet supersymmetry in higher dimensions. The most important property of these transformations is that they are *nilpotent*⁵. Under (2.32) the action (2.31) transforms as

$$\delta S = \partial h \partial^2 h (\epsilon_1 \psi_1 + \epsilon_2 \psi_2) - (\epsilon_2 \partial h) \psi_2 \partial^2 h - \psi_1 (-\epsilon_1 \partial h) \partial^2 h = 0 \quad (2.33)$$

⁵That is, $\delta_1^2 = 0$, $\delta_2^2 = 0$ and $[\delta_1, \delta_2] = 0$, where δ_1 is the transformation with parameter $\epsilon_2 = 0$, etc.. You should check this from (2.32) as an exercise!

and is thus invariant — this is what the special relation between the bosonic and fermionic terms in S buys us. (To obtain this result we used the fact that Grassmann variables anticommute.) It's also true that the integral measure $d\phi d^2\psi$ is likewise invariant; I'll leave this too as an exercise.

Supersymmetric QFTs are drastically simpler than generic ones, especially in zero dimensions. Let $\delta\mathcal{O}$ be the supersymmetry variation of some operator $\mathcal{O}(\phi, \psi_i)$ and consider the correlation function $\langle \delta\mathcal{O} \rangle$. Since $\delta S = 0$ we have

$$\langle \delta\mathcal{O} \rangle = \frac{1}{Z_0} \int d\phi d^2\psi e^{-S} \delta\mathcal{O} = \frac{1}{Z_0} \int d\phi d^2\psi \delta(e^{-S}\mathcal{O}) . \quad (2.34)$$

The supersymmetry variation here acts on both ϕ and the fermions ψ_i in $e^{-S}\mathcal{O}$. But if it acts on a fermion ψ_i then the resulting term does not contain that ψ_i and hence cannot contribute to the integral because $\int d\psi 1 = 0$ for Grassmann variables. On the other hand, if it acts on ϕ then while the resulting term may survive the Grassmann integral, it is a total derivative in the ϕ field space. Thus, provided \mathcal{O} does not disturb the decay of e^{-S} as $|\phi| \rightarrow \infty$, any such correlation function must vanish, $\langle \delta\mathcal{O} \rangle = 0$.

In particular, if we choose $\mathcal{O}_g = \partial g \psi_1$ for some $g(\phi)$, then setting the parameters $\epsilon_1 = -\epsilon_2 = \epsilon$ we have

$$0 = \langle \delta\mathcal{O}_g \rangle = \epsilon \langle \partial g \partial h - \partial^2 g \psi_1 \psi_2 \rangle . \quad (2.35)$$

The significance of this is that the quantity $\partial g \partial h - \partial^2 g \psi_1 \psi_2$ is the first-order change in the action under the *deformation* $h \rightarrow h + g$, again so long as g does not alter the behaviour of h as $|\phi| \rightarrow \infty$. The fact that $\langle \delta\mathcal{O}_g \rangle = 0$ tells that the partition function $Z[h]$, which we might think depends on all the couplings in the vertices in the polynomial h , is in fact largely insensitive to the detailed form of h because we can deform it by any other polynomial of the same degree or lower. The most important case is if we choose g to be proportional to h , then our deformation just rescales $h \rightarrow (1 + \lambda)h$ and so we see that $Z[h]$ is independent of the overall scale of h . By iterating this procedure, we can imagine rescaling h by a large factor so that the bosonic part of the action $(\partial h)^2/2 \rightarrow \Lambda^2 (\partial h)^2/2$. As $\Lambda \rightarrow \infty$, the factor e^{-S} exponentially suppresses any contribution to Z except from an infinitesimal neighbourhood of the critical points of h where $\partial h = 0$. This phenomenon is known as **localization** of the path integral.

It's now straightforward to work out the partition function. Near any such critical point ϕ_* we have

$$h(\phi) = h(\phi_*) + \frac{c_*}{2}(\phi - \phi_*)^2 + \dots \quad (2.36)$$

where $c_* = \partial^2 h(\phi_*)$, so the action (2.31) becomes

$$S(\phi, \psi_i) = \frac{c_*^2}{2}(\phi - \phi_*)^2 + c_* \psi_1 \psi_2 + \dots . \quad (2.37)$$

The higher order terms will be negligible as we focus on an infinitesimal neighbourhood of ϕ_* . Expanding the exponential in Grassmann variables the contribution of this critical

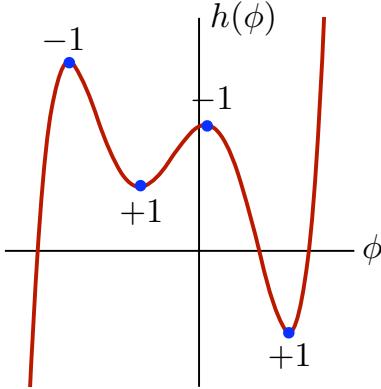


Figure 2: The supersymmetric path integral receives contributions just from infinitesimal neighbourhoods of the critical points of $h(\phi)$. These alternately contribute ± 1 according to whether they are minima or maxima.

point to the partition function is

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int d\phi d^2\psi e^{-c_*(\phi-\phi_*)^2/2} [1 - c_*\psi_1\psi_2] &= \frac{c_*}{\sqrt{2\pi}} \int d\phi e^{-c_*(\phi-\phi_*)^2} \\ &= \frac{c_*}{\sqrt{c_*^2}} = \text{sgn}(\partial^2 h|_{\phi_*}). \end{aligned} \quad (2.38)$$

Summing over all the critical points, the full partition function thus becomes

$$\mathcal{Z}[h] = \sum_{\phi_* : \partial h|_{\phi_*} = 0} \text{sgn}(\partial^2 h|_{\phi_*}) \quad (2.39)$$

and, as expected, is largely independent of the detailed form of h . In fact, if h is a polynomial of odd degree, then $\partial h = 0$ must have an even number of roots with $\partial^2 h$ being alternately > 0 and < 0 at each. Thus their contributions to (2.39) cancel pairwise and $\mathcal{Z}[h_{\text{odd}}] = 0$ identically. On the other hand, if h has even degree then it has an odd number of critical points and we obtain $\mathcal{Z}[h_{\text{ev}}] = \pm 1$, with the sign depending on whether $h \rightarrow \pm\infty$ as $|\phi| \rightarrow \infty$. (See figure 2.)

The fact that the partition function is so simple in this class of theories is a really remarkable result! To reiterate, we've found that for any form of polynomial $h(\phi)$, the partition function $\mathcal{Z}[h]$ is always either 0 or ± 1 . If we imagined trying to compute $\mathcal{Z}[h]$ perturbatively, then for a non-quadratic h we'd still have to sum infinitely diagrams using the vertices in the action. In particular, we could certainly draw Feynman graphs Γ with arbitrarily high numbers of loops involving both ϕ and ψ_i fields, and these graphs would each contribute to the coefficient of some power of the coupling constants in the perturbative expansion. However, by an apparent miracle, we'd find that these graphs always cancel themselves out; the net coefficient of each such loop graph would be zero with the contributions from graphs where either ϕ or $\psi_1\psi_2$ run around the loop contributing with opposite sign. The reason for this apparent miracle is the localization property of the supersymmetric integral.

In supersymmetric theories in higher dimensions, complications such as spin mean the cancellation can be less powerful, but it is nonetheless still present and is responsible for making supersymmetric quantum theories ‘tamer’ than non-supersymmetric ones. As an important example, diagrams where the Higgs particle of the Standard Model runs around a loop can have the effect of destabilizing the mass of the Higgs, sending it up to a very high scale. (We’ll understand this later on.) The Large Hadron Collider at CERN is currently searching for a hypothesized supersymmetric partner to the Higgs that many people postulated should be present so as to cancel these dangerous loop diagrams. The ultimate mechanism for this cancellation is just what we’ve seen above, though it’s power is filtered through the lens of a much more complicated theory.

I also want to point out that localization is useful for calculating much more than just the partition function. For each $a \in \{1, 2, 3, \dots\}$ suppose that $\mathcal{O}_a(\phi, \psi_i)$ is an operator that obeys $\delta\mathcal{O}_a = 0$, *i.e.* each operator is invariant under supersymmetry transformations (2.32). Then the (unnormalized) correlation function

$$\left\langle \prod_a \mathcal{O}_a \right\rangle = \int \frac{d\phi d^2\psi}{\sqrt{2\pi}} e^{-S} \prod_a \mathcal{O}_a \quad (2.40)$$

again localizes to the critical points of h . Once again, this is because deforming $h \rightarrow h + g$ leaves the correlator invariant since the deformation affects the correlation function as

$$\left\langle \prod_a \mathcal{O}_a \right\rangle \xrightarrow{h \rightarrow h+g} \left\langle \delta\mathcal{O}_g \prod_a \mathcal{O}_a \right\rangle = \left\langle \delta \left(\mathcal{O}_g \prod_a \mathcal{O}_a \right) \right\rangle = 0 \quad (2.41)$$

which vanishes by the same arguments as before. Here, we used the fact that $\delta\mathcal{O}_a = 0$ to write the operator on the *rhs* as a total derivative.

Of course, if any of the \mathcal{O}_a are already of the form $\delta\mathcal{O}'$, so that this \mathcal{O}_a is itself the supersymmetry transformation of some \mathcal{O}' , then $\langle \prod \mathcal{O}_a \rangle = 0$ which is not very interesting. The interesting operators are those which are δ -closed ($\delta\mathcal{O} = 0$) but not δ -exact ($\mathcal{O} \neq \delta\mathcal{O}'$). These operators describe the **cohomology** of the nilpotent operator δ . This is the starting-point for much of the mathematical interest in QFT: we can build supersymmetric QFTs that compute the cohomology of interesting spaces. For example, Donaldson’s theory of invariants of 4-manifolds that are homeomorphic but not diffeomorphic, and the Gromov–Witten generalization of intersection theory can both be understood as examples of (higher-dimensional) supersymmetric QFTs where the localization / cancellation is precise.

We’ll also meet essentially the same idea again in a slightly different context later in this course when we study BRST quantization of gauge theories.

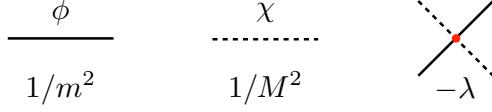
2.5 Effective theories: a toy model

Now I want to introduce a very important idea which will be central to our understanding of QFT in higher dimensions.

Suppose we have two real-valued fields ϕ and χ , so that the space of fields is \mathbb{R}^2 , and let the action be

$$S(\phi, \chi) = \frac{m^2}{2}\phi^2 + \frac{M^2}{2}\chi^2 + \frac{\lambda}{4}\phi^2\chi^2 \quad (2.42)$$

so that λ provides a coupling between the fields. We have the Feynman rules



which may be used to compute perturbative expressions for correlation functions such as

$$\langle f \rangle = \frac{1}{Z} \int_{\mathbb{R}^2} d\phi d\chi e^{-S(\phi,\chi)} f(\phi, \chi)$$

in the usual way. For example, we have

$$\begin{aligned} \ln \left[\frac{Z}{Z_0} \right] &= \text{Diagram: two circles connected by a horizontal line} + \text{Diagram: three circles connected by two horizontal lines} + \text{Diagram: four circles connected by three horizontal lines} + \text{Diagram: one circle with a dashed loop} \\ &= -\frac{\lambda}{4m^2 M^2} + \frac{\lambda^2}{16m^4 M^4} + \frac{\lambda^2}{16m^4 M^4} + \frac{\lambda^2}{8m^4 M^4} \end{aligned}$$

as the sum of connected vacuum diagrams, and also

$$\begin{aligned} \frac{1}{2} \langle \phi^2 \rangle &= \text{Diagram: single vertical line} + \text{Diagram: vertical line with a small loop} + \text{Diagram: vertical line with a large loop} + \text{Diagram: vertical line with a loop attached to the middle} + \text{Diagram: vertical line with a loop attached to the top} \\ &= \frac{1}{m^2} - \frac{\lambda}{2m^4 M^2} + \frac{\lambda^2}{4m^6 M^4} + \frac{\lambda^2}{2m^6 M^4} + \frac{\lambda^2}{4m^6 M^4} \end{aligned}$$

where the blue dots represent the insertions of the two powers of ϕ .

I want to arrive at this result in a different way. Suppose we're interested in correlation functions of operators that depend only on ϕ ; for example, we might imagine that the field χ has a very high mass such that our experiment isn't powerful enough to observe real χ production — we can only measure properties of the ϕ field. Having no idea what χ is doing suggests that we should perform its path integral first, *i.e.*, we average over χ configurations at each fixed ϕ . We define the **effective action** $S_{\text{eff}}(\phi)$ for the ϕ field to be the result of carrying out this χ integral. Thus

$$S_{\text{eff}}(\phi) := -\hbar \log \left[\int_{\mathbb{R}} d\chi e^{-S(\phi,\chi)/\hbar} \right] \quad (2.43)$$

where I've restored the powers of \hbar . Once we have found this effective action, we can use it to compute $\langle f \rangle$ for any observable that depends only on ϕ . Of course there's nothing mysterious here, we're simply choosing in which order to do our integrals.

In general computing S_{eff} can be difficult, but in our toy example it's straightforward because χ appears only quadratically in $S(\phi, \chi)$. We have

$$\int_{\mathbb{R}} d\chi e^{-S(\phi,\chi)/\hbar} = e^{-m^2 \phi^2 / 2\hbar} \sqrt{\frac{2\pi\hbar}{M^2 + \lambda\phi^2 / 2}} \quad (2.44)$$

where the first factor is the χ -independent part of the original action and the square root comes from the Gaussian integral over χ . Hence the effective action (2.45) is

$$\begin{aligned} S_{\text{eff}}(\phi) &= \frac{m^2}{2}\phi^2 + \frac{\hbar}{2}\ln\left[1 + \frac{\lambda}{2M^2}\phi^2\right] + \frac{\hbar}{2}\ln\frac{M^2}{2\pi\hbar} \\ &= \left(\frac{m^2}{2} + \frac{\hbar\lambda}{4M^2}\right)\phi^2 - \frac{\hbar\lambda^2}{16M^4}\phi^4 + \frac{\hbar\lambda^3}{48M^6}\phi^6 + \dots \\ &=: \frac{m_{\text{eff}}^2}{2}\phi^2 + \frac{\lambda_4}{4!}\phi^4 + \frac{\lambda_6}{6!}\phi^6 + \dots. \end{aligned} \quad (2.45)$$

The important point is that the effect of integrating out the ‘high energy field’ χ has *changed* the structure of the action. In particular, the mass term of the ϕ field has been shifted

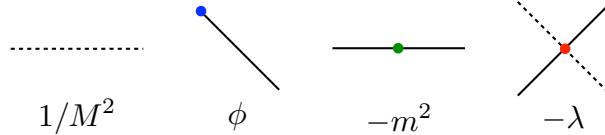
$$m^2 \rightarrow m_{\text{eff}}^2 = m^2 + \frac{\hbar\lambda}{2M^2}. \quad (2.46)$$

Even more strikingly, we’ve generated an infinite series of new coupling terms

$$\lambda_4 = -\frac{3\hbar}{2}\frac{\lambda^2}{M^4}, \quad \lambda_6 = 15\hbar\frac{\lambda^3}{M^6}, \quad \lambda_{2k} = (-1)^{k+1}\hbar\frac{(2k)!}{2^{k+1}k}\frac{\lambda^k}{M^{2k}} \quad (2.47)$$

describing self-interactions of ϕ . It’s important to observe that the ϕ mass shift and new ϕ self-interactions all vanish as $\hbar \rightarrow 0$; they are *quantum* effects. Notice also that they’re each suppressed by powers of the (high) mass M .

It’s useful to think in a little more detail about how these new couplings arise. We can perform the χ path integral using Feynman graphs, provided we remember that the ϕ field is not propagating, but may appear on an external leg. The χ propagator and vertices are



where the blue 1-valent vertex represents an insertion of the ϕ field, coming from expanding the action in powers of the vertices. These ingredients lead to the following perturbative construction of S_{eff} :

$$\begin{aligned} -S_{\text{eff}} &= \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} + \dots \\ &= -\frac{m^2}{2}\phi^2 - \frac{\lambda}{4M^2}\phi^2 + \frac{\lambda^2}{16M^4}\phi^4 - \frac{\lambda^3}{48M^6}\phi^6 \end{aligned}$$

where we note that since $-S_{\text{eff}}$ is the *logarithm* of the χ integral, only connected diagrams appear.

The diagrammatic expansion shows that the new interactions of ϕ have actually been generated by loops of the χ field. In our effective description that knows only about the

behaviour of the ϕ field, we can no longer ‘see’ the χ field ‘circulating’ around the loop. Instead, we perceive this just as a new interaction vertex for ϕ . (The fact that the new terms in S_{eff} come just from *single* loops of χ , and hence come with just a single power of \hbar , is special to the fact that χ appears in the original action (2.42) only quadratically. Generically there would be higher-order corrections.)

Using this effective action, we now find

$$\begin{aligned} \frac{1}{2}\langle\phi^2\rangle &= \text{[Feynman diagram: vertical line with blue dots at both ends]} + \text{[Feynman diagram: vertical line with blue dots at both ends, with a red dot on the line and a small loop attached to it]} + \\ &= \frac{1}{m_{\text{eff}}^2} - \frac{\lambda_4}{2m_{\text{eff}}^6} + \end{aligned}$$

where the propagator and vertices here are the ones appropriate for the effective action S_{eff} . Using the definition of the new couplings in terms of the original λ and M , this unsurprisingly agrees with our answer before, correct to order λ^2 . However, once we had the effective action, we arrived at this answer using just two diagrams, whereas above it required five. If we only care about a single correlation function then the work involved in first computing S_{eff} and then using the new set of Feynman rules to compute the low-energy correlator is roughly the same as just using the original action to compute this correlator directly. On the other hand, if we wish to compute many low-energy correlators then we’re clearly better off investing a little time to work out S_{eff} first.

The real point I wish to make is this: the way we experience the world is *always* through S_{eff} . Naively at least, we have no idea what new physics may be lurking just out of reach of our most powerful accelerators; there may be any number of new, hitherto undiscovered species of particle, or new dimensions of space-time, or even wilder new phenomena. However, when describing low-energy physics, we should only seek to describe the behaviour of the degrees of freedom (fields) that are relevant and accessible at the energy scale at which we’re conducting our experiments, *even if we happen to know what the more fundamental description is*. For example, a glass of water certainly consists of very many H₂O molecules, these molecules are bound states of atoms, each of which consist of many electrons orbiting around a central nucleus. In turn, this nucleus comprises of protons and neutrons stuck together by a strong force mediated by pions, and all these hadrons are themselves seething masses of quarks and gluons. But it would be very foolish to imagine we should describe the properties of water that are relevant in everyday life by starting from the Lagrangian for QCD.

Let me make one final comment. In the example above, we started from a very simple action in equation (2.42) and obtained a more complicated effective action (2.45) after integrating out the unobserved degree of freedom χ . A more generic case would start from a general action (invariant under $\phi \rightarrow -\phi$ and $\chi \rightarrow -\chi$ for simplicity)

$$S'(\phi, \chi) = \sum_{i,j} \frac{\lambda_{i,j}}{(2i)!(2j)!} \phi^{2i} \chi^{2j} \quad (2.48)$$

in which all possible even monomials in ϕ and χ are allowed. For example, we may have arrived at this action by integrating out some other field that was unknown in our above considerations. In this generic case, the effect of integrating out χ will not generate *new* interactions for ϕ — all possible even self-interactions are included anyway — but rather the values of the coupling constants $\lambda_{i,0}$ will get shifted, just as for the mass shift we saw above. In addition, because the χ path integral would now be very complicated, we can only reasonably expect to describe the shifted couplings as asymptotic series. Either way, the lesson to remember is that the effect of integrating out degrees of freedom is to change the values of the coupling constants in the effective action.

3 QFT in one dimension (= QM)

In one dimension there are two possible compact (connected) manifolds M : the circle S^1 and the interval I . We will parametrize the interval by $t \in [0, T]$ so that $t = 0$ and $t = T$ are the two point-like boundaries, while we will parametrize the circle by $t \in [0, T)$ with the identification $t \cong t + T$.

The most important example of a field on M is a map $x : M \rightarrow N$ to a Riemannian manifold (N, g) which we will take to have dimension n . That is, for each point t on our ‘space–time’ M , $x(t)$ is a point in N . It’s often convenient to describe N using coordinates. If an open patch $U \subset N$ has local co-ordinates x^a for $a = 1, \dots, n$, then we let $x^a(t)$ denote the coordinates of the image point $x(t)$. More precisely, $x^a(t)$ are the pullbacks to M of coordinates on U by the map x .

With these fields, the standard choice of action is

$$S[\phi] = \int_M \left[\frac{1}{2} g_{ab}(x) \dot{x}^a \dot{x}^b + V(x) \right] dt, \quad (3.1)$$

where $g_{ab}(x)$ is the pullback to M of the Riemannian metric on N and $\dot{x}^a = dx^a/dt$. We have also included in the action a choice of function $V : N \rightarrow \mathbb{R}$, or more precisely the pullback of this function to M , which is independent of worldline derivatives of x . In writing this action we have chosen one-dimensional metric on M to be just the flat Euclidean metric $\delta_{tt} = 1$. Under a small variation δx of x we have

$$\begin{aligned} \delta S &= \int_M \left[g_{ab}(x) \dot{x}^a \delta \dot{x}^b + \frac{1}{2} \frac{\partial g_{ab}(x)}{\partial x^c} \delta x^c \dot{x}^a \dot{x}^b + \frac{\partial V(x)}{\partial x^c} \delta x^c \right] dt \\ &= \int_M \left[-\frac{d}{dt} (g_{ac}(x) \dot{x}^a) + \frac{1}{2} \frac{\partial g_{ab}(x)}{\partial x^c} \dot{x}^a \dot{x}^b + \frac{\partial V(x)}{\partial x^c} \right] \delta x^c dt \end{aligned} \quad (3.2)$$

and requiring that this vanishes for arbitrary $\delta\phi(t)$ gives the Euler–Lagrange equations

$$\frac{d^2 x^a}{dt^2} + \Gamma_{bc}^a \dot{x}^b \dot{x}^c = g^{ab}(x) \frac{dV}{dx^b} \quad (3.3)$$

where $\Gamma_{bc}^a = \frac{1}{2} g^{ad} (\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc})$ is the Levi–Civita connection on N , again pulled back to the worldline.

The standard interpretation of all this is to image an arbitrary map $x(t)$ describes a possible *trajectory* a particle might in principle take as it travels through the space N . (See figure 2.) In this context, N is called the **target space** of the theory, while M (or its image $x(M) \subset N$) is known as the **worldline** of the particle. The field equation (3.3) says that when $V = 0$, classically the particle travels along a geodesic in (N, g) . V itself is then interpreted as a (non-gravitational) potential⁴ through which this particle moves.

⁴The absence of a minus sign on the *rhs* of (3.3) is probably surprising, but follows from the action (3.1). This is actually the *correct* sign with a Euclidean worldsheet, because under the Wick rotation $t \rightarrow it$ back to a Minkowski signature worldline, the *lhs* of (3.3) acquires a minus sign. In other words, in Euclidean time $\mathbf{F} = -m\mathbf{a}$!

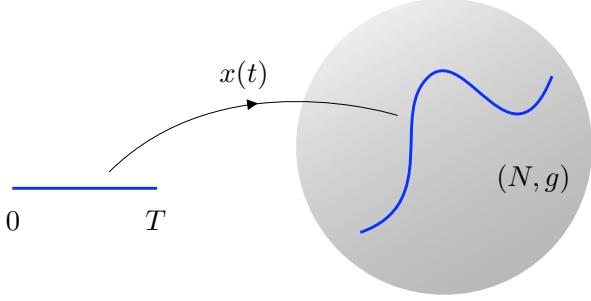


Figure 2: The theory (3.1) describes a map from an abstract worldline into the Riemannian target space (N, g) . The corresponding one-dimensional QFT can be interpreted as single particle Quantum Mechanics on N .

From this perspective, it's natural to think of the *target* space N as being the world in which we live, and computing the path integral for this action will lead us to single particle Quantum Mechanics, as we'll see below. However, we're really using this theory as a further warm-up towards QFT in higher dimensions, so I want you to also keep in mind the idea that the *worldline* M is actually ‘our space–time’ in a one-dimensional context, and the target space N can be some abstract Riemannian manifold unrelated to the space we see around us. For example, at physics of low-energy pions is described by a theory of this general kind, where M is our Universe and N is the group manifold $SU(3)$.

3.1 Quantum Mechanics

The usual way to do Quantum Mechanics is to pick a Hilbert space \mathcal{H} and a Hamiltonian H , which is a Hermitian operator $H : \mathcal{H} \rightarrow \mathcal{H}$. In the case relevant above, the Hilbert space would be $L^2(N)$, the space of square-integrable functions on N , and the Hamiltonian would usually be

$$H = \frac{1}{2}\Delta + V, \quad \text{where} \quad \Delta := \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^a} \left(\sqrt{g} g^{ab} \frac{\partial}{\partial x^b} \right) \quad (3.4)$$

is the Laplacian acting on functions in $L^2(N)$. The amplitude for the particle to travel from an initial point $y_0 \in N$ to a final point $y_1 \in N$ in Euclidean time T is given by

$$K_T(y_0, y_1) = \langle y_1 | e^{-HT} | y_0 \rangle, \quad (3.5)$$

which is known as the **heat kernel**. (Here I've written the *rhs* in the Heisenberg picture, which I'll use below. In the Schrödinger picture where states depend on time we would instead write $K_T(y_0, y_1) = \langle y_1, T | y_0, 0 \rangle$.)

The heat kernel is a function on $I \times N$ which may be defined to be the solution of the pde

$$\frac{\partial}{\partial t} K_t(x, y) + H K_t(x, y) = 0 \quad (3.6)$$

subject to the initial condition that $K_0(x, y) = \delta(x - y)$. I remind you that we're in Euclidean worldline time here, and in units where $\hbar = 1$ here. Rotating to Minkowski

signature by sending $t \rightarrow it$ and restoring the \hbar gives instead

$$i\hbar \frac{\partial}{\partial t} K_{it}(x, y) = H K_{it}(x, y) \quad (3.7)$$

that we recognize as Schrödinger's equation. In the simplest example where $N \cong \mathbb{R}^n$ with flat metric $g_{ab} = \delta_{ab}$ and vanishing potential $V = 0$, the heat kernel

$$K_T(x, y) = \frac{1}{(2\pi T)^{n/2}} \exp\left(-\frac{|x - y|^2}{2T}\right) \quad (3.8)$$

where $|x - y|$ is the Euclidean distance between x and y .

As you learned last term, Feynman showed that this heat kernel could also be represented as a path integral. The usual idea is to break the time interval T into N chunks, each of duration $\Delta t = T/N$. We can then write

$$\begin{aligned} \langle y_1 | e^{-HT} | y_0 \rangle &= \langle y_1 | e^{-H\Delta t} e^{-H\Delta t} \cdots e^{-H\Delta t} | y_0 \rangle \\ &= \int d^n x_1 \cdots d^n x_{N-1} \langle y_1 | e^{-H\Delta t} | x_{N-1} \rangle \cdots \langle x_2 | e^{-HT} | x_1 \rangle \langle x_1 | e^{-H\Delta t} | y_0 \rangle \\ &= \int \prod_{i=1}^{N-1} d^n x_i K_{\Delta t}(y_1, x_{N-1}) \cdots K_{\Delta t}(x_2, x_1) K_{\Delta t}(x_1, y_0). \end{aligned} \quad (3.9)$$

In the second line here we have inserted the identity operator $\int d^n x_i |x_i\rangle \langle x_i|$ on \mathcal{H} in between each evolution operator; in the present context this can be understood as the concatenation identity

$$K_{t_1+t_2}(x_3, x_1) = \int d^n x_2 K_{t_2}(x_3, x_2) K_{t_1}(x_2, x_1) \quad (3.10)$$

obeyed by convolutions of the heat kernel.

Now, while the flat space expression (3.8) for the heat kernel does not hold when g_{ab} is a more general Riemannian metric on N , in fact it is (almost) correct in the limit of small times. More precisely, it can be shown that the heat kernel always has the asymptotic form

$$\lim_{\Delta t \rightarrow 0} K_{\Delta t}(x, y) \sim \frac{1}{(2\pi\Delta t)^{n/2}} a(x) \exp\left(-\frac{d(x, y)^2}{2\Delta t}\right) \quad (3.11)$$

for small t , where $d(x, y)$ is the geodesic distance between x and y measured using the metric g , and where $a(x)$ is some polynomial in the Riemann curvature tensor that we won't need to be specific about. Therefore, splitting our original time interval $[0, T]$ into very many pieces of very short duration $\Delta t = T/N$ gives

$$\langle y_1 | e^{-HT} | y_0 \rangle = \lim_{N \rightarrow \infty} \left(\frac{1}{2\pi\Delta t}\right)^{\frac{nN}{2}} \int \prod_{i=1}^{N-1} d^n x_i a(x_i) \exp\left[-\frac{\Delta t}{2} \left(\frac{d(x_{i+1}, x_i)}{\Delta t}\right)^2\right] \quad (3.12)$$

as an expression for the heat kernel.

This more or less takes us to the path integral. If it is sensible to take the limits, then we can take

$$\mathcal{D}x \stackrel{?}{=} \lim_{N \rightarrow \infty} \left(\frac{1}{2\pi\Delta t} \right)^{\frac{nN}{2}} \prod_{i=1}^{N-1} d^n x_i a(x_i) \quad (3.13)$$

to be the path integral measure. Similarly, if the trajectory is at least once differentiable then $(d(x_{i+1}, x_i)/\Delta t)^2$ converges to $g_{ab}\dot{x}^a\dot{x}^b$ and we can write

$$\lim_{N \rightarrow \infty} \prod_{i=1}^{N-1} \exp \left[-\frac{\Delta t}{2} \left(\frac{d(x_{i+1}, x_i)}{\delta t} \right)^2 \right] = \exp \left[-\frac{1}{2} \int_0^T g_{ab} \dot{x}^a \dot{x}^b dt \right] \quad (3.14)$$

which recovers the action (3.1), with $V = 0$. (A more general heat kernel can be used to incorporate a non-zero potential.)

We'll investigate these limits further below. Accepting them for now, combining (3.13) & (3.14) we obtain the path integral expression

$$\langle y_1 | e^{-HT} | y_0 \rangle = \int_{C_T[y_0, y_1]} \mathcal{D}x \exp \left[-\frac{1}{2} \int_0^T g_{ab} \dot{x}^a \dot{x}^b dt \right], \quad (3.15a)$$

or in other words, the heat kernel can formally be written as

$$K_T(y_0, y_1) = \int_{C_T[y_0, y_1]} \mathcal{D}x e^{-S}. \quad (3.15b)$$

The integrals in these expressions are to be taken over the space $C_T[y_0, y_1]$ of all continuous maps $x : I \rightarrow N$ that are constrained to obey the boundary conditions $x(0) = y_0$ and $x(T) = y_1$.

3.1.1 The partition function

The partition function on the circle can likewise be given and interpretation in the operator approach to Quantum Mechanics. Tracing over the Hilbert space gives

$$\text{Tr}_{\mathcal{H}}(e^{-TH}) = \int d^n y \langle y | e^{-HT} | y \rangle = \int_N d^n y \int_{C_T[y, y]} \mathcal{D}x e^{-S} \quad (3.16)$$

using the path integral expression (3.15b) for the heat kernel. The path integral here is (formally) taken over all continuous maps $x : [0, T] \rightarrow N$ such that the endpoints are both mapped to the same point $y \in N$. We then integrate y everywhere over N^5 , erasing the memory of the particular point y . This is just the same thing as considering all continuous maps $x : S^1 \rightarrow N$ where the worldline has become a circle of circumference T . This shows that

$$\text{Tr}_{\mathcal{H}}(e^{-TH}) = \int_{C_{S^1}} \mathcal{D}x e^{-S} = \mathcal{Z}_{S^1}[N, g, V], \quad (3.17)$$

which is nothing but the partition function on S^1 . In higher dimensions this formula will be the basis of the relation between QFT and Statistical Field Theory, and is really the origin of the name ‘partition function’ for \mathcal{Z} in physics.

⁵In flat space, the heat kernel (3.8) obeys $K_T(y, y) = K_T(0, 0)$ so is independent of y . Thus if $N \cong \mathbb{R}^n$ with a flat metric, this final y integral does not converge. It will converge if N is compact, say by imposing that we live in a large box, or on a torus etc..

3.1.2 Operators and correlation functions

As in zero dimensions, we can also use the path integral to compute correlation functions of operators.

A **local operator** is one which depends on the field only at one point of the worldline. The simplest types of local operators come from functions on the target space. If $\mathcal{O} : N \rightarrow \mathbb{R}$ is a real-valued function on N , let $\hat{\mathcal{O}}$ denote the corresponding operator on \mathcal{H} . Then for any fixed time $t \in (0, T)$ we have

$$\langle y_1 | \hat{\mathcal{O}}(t) | y_0 \rangle = \langle y_1 | e^{-H(T-t)} \hat{\mathcal{O}} e^{-Ht} | y_0 \rangle \quad (3.18)$$

in the Heisenberg picture. Inserting a complete set of $\hat{\mathcal{O}}(x)$ eigenstates $\{|x\rangle\}$, this is

$$\begin{aligned} \int d^n x \langle y_1 | e^{-H(T-t)} \hat{\mathcal{O}}(x) | x \rangle \langle x | e^{-Ht} | y_0 \rangle &= \int d^n x \mathcal{O}(x) \langle y_1 | e^{-H(T-t)} | x \rangle \langle x | e^{-Ht} | y_0 \rangle \\ &= \int d^n x \mathcal{O}(x) K_{T-t}(y_1, x) K_t(x, y_0), \end{aligned} \quad (3.19)$$

where we note that in the final two expressions $\mathcal{O}(x)$ is just a number; the eigenvalue of $\hat{\mathcal{O}}$ in the state $|x\rangle$.

Using (3.15b), everything on the *rhs* of this equation can now be written in terms of path integrals. We have

$$\begin{aligned} \langle y_1 | e^{-H(T-t)} \hat{\mathcal{O}} e^{-Ht} | y_0 \rangle &= \int d^n x_t \left[\int_{C_{T-t}[y_1, x_t]} e^{-S} \times \mathcal{O}(x_t) \times \int_{C_t[x, y_0]} e^{-S} \right] \\ &= \int_{C_T[y_1, y_0]} \mathcal{D}x e^{-S} \mathcal{O}(x(t)), \end{aligned} \quad (3.20)$$

where we again note that integrating over all maps $x : [0, t] \rightarrow N$ with endpoint $x(t) = x_t$, then over all maps $x : [t, T] \rightarrow N$ with initial point $x(t)$ again fixed to x_t and finally integrating over all points $x_t \in N$ is the same thing as integrating over all maps $x : [0, T] \rightarrow N$ with endpoints y_0 and y_1 .

More generally, we can insert several such operators. If $0 < t_1 < t_2 < \dots < t_n < T$ then exactly the same arguments give

$$\begin{aligned} \langle y_1 | \hat{\mathcal{O}}_n(t_n) \cdots \hat{\mathcal{O}}_1(t_2) \hat{\mathcal{O}}_1(t_1) | y_0 \rangle &= \langle y_1 | e^{-H(T-t_n)} \hat{\mathcal{O}}_n(x) \cdots \hat{\mathcal{O}}_2(x) e^{-H(t_2-t_1)} \hat{\mathcal{O}}_1(x) e^{-Ht_1} | y_0 \rangle \\ &= \int_{C_T[y_0, y_1]} \mathcal{D}x e^{-S} \prod_{i=1}^n \mathcal{O}_i(x(t_i)) \end{aligned} \quad (3.21)$$

for the n -point correlation function. The hats on the $\hat{\mathcal{O}}_i$ remind us that the *lhs* involves *operators* acting on the Hilbert space \mathcal{H} . The objects \mathcal{O}_i inside the path integral are just ordinary functions, evaluated at the point $x(t_i) \in N$ ⁶.

Notice that in order to run our argument, it was very important that the insertion times t_i obeyed $t_i < t_{i+1}$: we would not have been able to interpret the *lhs* in the Heisenberg

⁶A more precise statement would be that they are functions on the space of fields $C_T[y_0, y_1]$ obtained by pullback from a function on N by the evaluation map at time t_i .

picture had this not been the case⁷. On the other hand, the insertions $\mathcal{O}_i(x(t_i))$ in the path integral are just functions and have no notion of ordering. Thus the expression on the right doesn't have any way to know which insertion times was earliest. For this to be consistent, for a general set of times $\{t_i\} \in (0, T)$ we must actually have

$$\int_{C_T[y_0, y_1]} \mathcal{D}x \left(e^{-S} \prod_{i=1}^n \mathcal{O}_i(x(t_i)) \right) = \langle y_1 | \mathcal{T}\{\prod_i \hat{\mathcal{O}}_i\} | y_0 \rangle \quad (3.22)$$

where the symbol \mathcal{T} on the *rhs* is defined by

$$\begin{aligned} \mathcal{T}\hat{\mathcal{O}}_1(t_1) &:= \mathcal{O}_1(t_1), \\ \mathcal{T}\{\hat{\mathcal{O}}_1(t_1) \hat{\mathcal{O}}_2(t_2)\} &:= \Theta(t_2 - t_1) \hat{\mathcal{O}}_2(t_2) \hat{\mathcal{O}}_1(t_1) + \Theta(t_1 - t_2) \hat{\mathcal{O}}_1(t_1) \hat{\mathcal{O}}_2(t_2), \\ &\vdots & &\vdots \end{aligned} \quad (3.23)$$

and so on, where $\Theta(t)$ is the Heaviside step function. By construction, these step functions mean that the *rhs* is now completely symmetric with respect to a permutation of the orderings. However, for any given choice of times t_i , only one term on the *rhs* can be non-zero. In other words, insertions in the path integral correspond to the **time-ordered product** of the corresponding operators in the Heisenberg picture.

The derivative terms in the action play an important role in evaluating these correlation functions. For suppose we had chosen our action to be just a potential term $\int V(x(t)) dt$, independent of derivatives $\dot{x}(t)$. Then, regularizing the path integral by dividing M into many small intervals as before, we would find that neighbouring points on the worldline completely decouple: unlike in (3.12) where the geodesic distance $d(x_{i+1}, x_i)^2$ in the heat kernel provides cross-terms linking neighbouring points together, we would obtain simply a product of independent integrals at each time step. Inserting functions $\mathcal{O}_i(x(t_i))$ that are likewise independent of derivatives of x into such a path integral would not change this conclusion. Thus, without the derivative terms in the action, we would have

$$\langle \mathcal{O}_1(t_1) \mathcal{O}_2(t_2) \rangle = \langle \mathcal{O}_1(t_1) \rangle \langle \mathcal{O}_2(t_2) \rangle \quad (3.24)$$

for all such insertions. In other words, there would be no possible non-trivial correlations between objects at different points of our (one-dimensional) Universe. This would be a very boring world: without derivatives, the number of people sitting in the lecture theatre would have nothing at all to do with whether or not a lecture was actually going on, and what you're thinking about right now would have nothing to do with what's written on this page.

This conclusion is a familiar result in perturbation theory. The kinetic terms in the action allow us to construct a **propagator**, and using this in Feynman diagrams enables us to join together interaction vertices at different points in space-time. As the name suggests, we interpret this propagator as a particle *traveling* between these two space-time interactions and the ability for particles to move is what allows for non-trivial correlation functions. Here we've obtained the same result directly from the path integral.

⁷Exercise: explain what goes wrong if we try to compute $\langle y_1 | e^{+TH} | y_1 \rangle$ with $T > 0$.

A wider class of local path integral insertions depend not just on x but also on its worldline derivatives \dot{x}, \ddot{x} etc.. In the canonical framework, with Lagrangian L we have

$$p_a = \frac{\delta L}{\delta \dot{x}^a} = g_{ab} \dot{x}^b \quad (3.25)$$

where the last equality is for our action (3.1). Thus we might imagine replacing the *function* $\mathcal{O}(x^a, \dot{x}^a)$ of x and its derivative in the path integral by the *operator* $\mathcal{O}(\hat{x}^a, g^{ab}(\hat{x})\hat{p}_b)$ in the canonical framework.

Now, probably the first thing you learned in Quantum Mechanics was that $[\hat{x}^a, \hat{p}_b] \neq 0$, so at least for generic functions the replacement

$$\mathcal{O}(x^a, \dot{x}^a) \rightarrow \mathcal{O}(\hat{x}^a, \hat{g}^{ab}\hat{p}_b)$$

is plagued by ordering ambiguities. For example, if we represent p_a by⁸ $-\partial/\partial x^a$, then should we replace

$$g_{ab} x^a \dot{x}^b \rightarrow -x^a \frac{\partial}{\partial x^a}$$

or should we take

$$g_{ab} x^a \dot{x}^b \rightarrow -\frac{\partial}{\partial x^a} x^a = -n - x^a \frac{\partial}{\partial x^a}$$

or perhaps something else? Even in free theory, we need to make a **normal ordering prescription** among the x 's and p 's to define what a composite operator means⁹.

From the path integral perspective, however, something smells fishy here. I've been emphasizing that path integral insertions $\mathcal{O}(x, \dot{x})$ are just ordinary *functions*, not operators. How can two ordinary functions fail to commute? To understand what's going on, we'll need to look into the definition of our path integral in more detail.

3.2 The continuum limit

In writing down the basic path integral (3.15b), we assumed it made sense to take the limit

$$\mathcal{D}x \stackrel{?}{=} \lim_{N \rightarrow \infty} \prod_{i=1}^N \left[\frac{1}{(2\pi\Delta t)^{n/2}} d^n x_i a(x_i) \right] \quad (3.26a)$$

to construct a measure on the space of fields. We also assumed it made sense to write

$$S[x] \stackrel{?}{=} \lim_{N \rightarrow \infty} \sum_{n=1}^{N-1} \Delta t \frac{1}{2} \left(\frac{x_{n+1} - x_n}{\Delta t} \right)^2 \quad (3.26b)$$

as the continuum action (here for a free particle).

Alternatively, instead of splitting the interval $[0, T]$ into increasingly many pieces, another possible way to define a regularized path integral starts by expanding each component of the field $x(t)$ as a Fourier series

$$x^a(t) = \sum_{k \in \mathbb{Z}} \tilde{x}_k^a e^{2\pi i t/T}.$$

⁸The absence of a factor of i on the *rhs* here is again a consequence of having a Euclidean worldline.

⁹And even there we may not be able to make a consistent choice. Read about the [Groenewald–Van Hove theorem](#) if you want sleepless nights.

We now regularize by truncating this to a finite sum with $|k| \leq N$. The (free) action for the truncated field is

$$S = \frac{2\pi}{T} \sum_{|k| \leq N} k^2 \delta_{ab} \tilde{x}_k^a \tilde{x}_k^b \quad (3.27)$$

and depends only on the Fourier coefficients. We might now try to define the path integral measure as the limit

$$\mathcal{D}x \stackrel{?}{=} \lim_{N \rightarrow \infty} \prod_{k=-N}^N \frac{d^n \tilde{x}_k}{(2\pi)^{n/2}} \quad (3.28a)$$

as an integral over more and more of these Fourier modes with higher and higher frequencies. The continuum action would then be taken to be the infinite series

$$S[x] \stackrel{?}{=} \lim_{N \rightarrow \infty} \frac{2\pi}{T} \sum_{|k| \leq N} k^2 \delta_{ab} \tilde{x}_k^a \tilde{x}_k^b \quad (3.28b)$$

which we hope converges.

The obvious question to ask is whether the limits in (3.26a) & (3.26b) or in (3.28a) & (3.28b) actually exist. Perhaps the single most important fact in QFT is that the answer to this question is “No!”.

3.2.1 The path integral measure

To prove this, let’s keep things simple and work just with the case that $N \cong \mathbb{R}^n$ with a flat metric, so that the space of fields is naturally an infinite dimensional *vector* space, where addition is given by pointwise addition of the fields at each t on the worldline.

We’ll start with the measure. In fact, it’s easy to prove that *there is no non-trivial Lebesgue measure on an infinite dimensional vector space*. To see this, first recall that for finite dimension D , $d\mu$ is a **Lebesgue measure** on \mathbb{R}^D if it assigns a strictly positive volume $\text{vol}(U) = \int_U d\mu > 0$ to every non-empty open set $U \subset \mathbb{R}^D$, if $\text{vol}(U') = \text{vol}(U)$ whenever U' may be obtained from U by translation, and finally if for every $x \in \mathbb{R}^D$ there exists at least one open neighbourhood U_x containing x for which $\text{vol}(U_x) < \infty$. The standard example is of course $d\mu = d^D x$. Now let $C_x(L)$ denote the open (hyper)cube centered on x and of side length L . This cube contains 2^D smaller cubes $C_{x_n}(L/2)$ of side length $L/2$, all of which are disjoint. Then

$$\text{vol}(C_x(L)) \geq \sum_{n=1}^{2^D} \text{vol}(C_{x_n}(L/2)) = 2^D \text{vol}(C_x(L/2)) \quad (3.29)$$

where the first inequality uses the fact that the measure is positive-definite, and the final equality uses translational invariance. We see that as $D \rightarrow \infty$, the only way the *rhs* can remain finite is if $\text{vol}(C_x(L/2)) \rightarrow 0$ for any finite L . So the measure must assign zero volume to any infinite dimensional hypercube. Finally, provided our vector space V is *countably* infinite (which both the Fourier series and discretized path integral make plain), we can cover *any* open $U \subset V$ using at most countably many such cubes $C(L/2)$, so $\text{vol}(U) = 0$ for any U and the measure must be identically zero.

3.2.2 Discretization and non-commutativity

The question of whether the discretized action itself converges will shed light on the puzzle of how x and p might not commute in the path integral. It suffices to consider the simplest case of a free particle in one dimension, so choose $N = \mathbb{R}$ and $V = 0$. Then if $0 < t_- < t < t_+ < T$ we have

$$\int_{C_T[y_0, y_1]} \mathcal{D}x e^{-S} x(t) \dot{x}(t_-) = \langle y_1 | e^{-H(T-t_+)} \hat{x} e^{-H(t_+-t)} \hat{p} e^{-Ht} | y_0 \rangle, \quad (3.30a)$$

when the insertion of x is later than that of \dot{x} , and

$$\int_{C_T[y_0, y_1]} \mathcal{D}x e^{-S} x(t) \dot{x}(t_+) = \langle y_1 | e^{-H(T-t)} \hat{p} e^{-H(t-t_+)} \hat{x} e^{-Ht_+} | y_0 \rangle \quad (3.30b)$$

when x is inserted at an earlier time than \dot{x} . Taking the limits $t_+ \rightarrow t$ from above and $t_- \rightarrow t$ from below, the difference between the *rhs* of (3.30a) & (3.30b) is

$$\langle y_1 | e^{-H(T-t)} [\hat{x}, \hat{p}] e^{-Ht} | y_0 \rangle = \langle y_1 | e^{-HT} | y_0 \rangle \quad (3.31)$$

which does not vanish. By contrast, the difference of the *lhs* seems to be automatically zero. What have we missed?

In handling the *lhs* of (3.30a)-(3.30b) we need to be careful. If we regularize the path integral by discretizing $[0, T]$, chopping it into chunks of width Δt , then we cannot pretend we are bringing the x and \dot{x} insertions any closer to each other than Δt without also taking account of the discretization of the whole path integral. Thus we replace

$$\lim_{t_- \uparrow t} [x(t) \dot{x}(t_-)] - \lim_{t_+ \downarrow t} [x(t) \dot{x}(t_+)]$$

by the discretized version

$$x_t \frac{x_t - x_{t-\Delta t}}{\Delta t} - x_t \frac{x_{t+\Delta t} - x_t}{\Delta t} \quad (3.32)$$

where we stop the limiting procedure as soon as x coincides with any part of the discretized derivative. The order of the factors of x_t and $x_{t \pm \Delta t}$ here doesn't matter; they're just ordinary integration variables.

Now consider the integral over x_t . Apart from the insertion of (3.32), the only dependence of the discretized path integral on this variable is in the heat kernels $K_{\Delta t}(x_{t+\Delta t}, x_t)$ and $K_{\Delta t}(x_t, x_{t-\Delta t})$. Using the explicit form of these kernels in flat space we have

$$\begin{aligned} & \int dx_t K_{\Delta t}(x_{t+\Delta t}, x_t) \left(x_t \frac{x_t - x_{t-\Delta t}}{\Delta t} - x_t \frac{x_{t+\Delta t} - x_t}{\Delta t} \right) K_{\Delta t}(x_t, x_{t-\Delta t}) \\ &= - \int dx_t x_t \frac{\partial}{\partial x_t} \left(K_{\Delta t}(x_{t+\Delta t}, x_t) K_{\Delta t}(x_t, x_{t-\Delta t}) \right) \\ &= \int dx_t K_{\Delta t}(x_{t+\Delta t}, x_t) K_{\Delta t}(x_t, x_{t-\Delta t}) = K_{2\Delta t}(x_{t+\Delta t}, x_{t-\Delta t}) \end{aligned} \quad (3.33)$$

where the second step is a simple integration by parts and the final step uses the concatenation property (3.10). The integration over x_t thus removes all the insertions from the

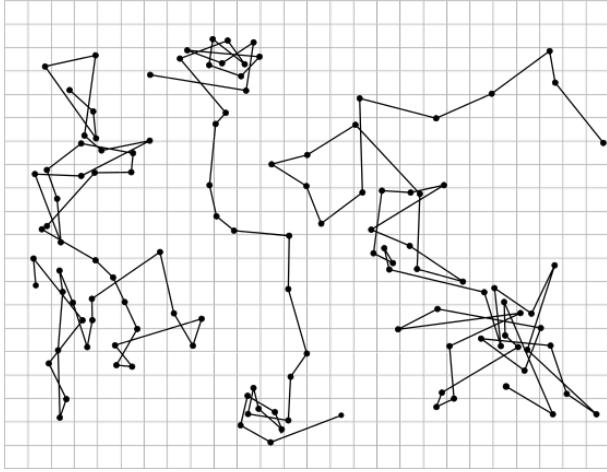


Figure 3: Stimulated by work of Einstein and Smoluchowski, Jean-Baptiste Perron made many careful plots of the locations of hundreds of tiny particles as they underwent Brownian motion. Understanding their behaviour played a key role in confirming the existence of atoms. A particle undergoing Brownian motion moves an average (rms) distance of \sqrt{t} in time t , a fact that is responsible for non-trivial commutation relations in the (Euclidean) path integral approach to Quantum Mechanics.

path integral, and the remaining integrals can be done using concatenation as before. We are thus left with $K_T(y_1, y_0) = \langle y_1 | e^{-HT} | y_0 \rangle$ in agreement with the operator approach.

There's an important point to notice about this calculation. Had we assumed the path integral included only maps $x : [0, T] \rightarrow N$ that are everywhere *differentiable*, rather than merely continuous, then the limiting value of (3.32) would necessarily vanish when $\Delta t \rightarrow 0$, contradicting the operator calculation. Non-commutativity arises in the path integral approach to Quantum Mechanics precisely because we're forced to include *non-differentiable* paths, *i.e.* our map $x \in C^0(M, N)$ but $x \notin C^1(M, N)$. But because our path integral includes non-differentiable maps we cannot assign any sensible meaning to $\lim_{\Delta t \rightarrow 0} (x_{t+\Delta t} - x_t)/\Delta t$ and the continuum action also fails to exist.

This non-differentiability is the familiar stochastic ('jittering') behaviour of a particle undergoing Brownian motion. It's closely related to a very famous property of random walks: that after a times interval t , one has moved through a net distance proportional to \sqrt{t} rather than $\propto t$ itself. More specifically, averaging with respect to the one-dimensional heat kernel

$$K_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t},$$

in time t , the mean squared displacement is

$$\langle (x - y)^2 \rangle = \int_{-\infty}^{\infty} K_t(x, y) (x - y)^2 dx = \int_{-\infty}^{\infty} K_t(u, 0) u^2 du = t \quad (3.34)$$

so that the rms average displacement from the starting point after time t is \sqrt{t} . Similarly,

our regularized path integrals yield a finite result because the average value of

$$x_{t+\Delta t} \frac{x_{t+\Delta t} - x_t}{\Delta t} - x_t \frac{x_{t+\Delta t} - x_t}{\Delta t} = \Delta t \left(\frac{x_{t+\Delta t} - x_t}{\Delta t} \right)^2,$$

which for a differentiable path would vanish as $\Delta t \rightarrow 0$, here remains finite.

3.2.3 Non-trivial measures?

The requirement that the measure be translationally invariant played an important role in the proof that the naive path integral measure $\mathcal{D}x$ doesn't exist. Do we really need this requirement? In fact, in one dimension, while neither $\mathcal{D}x$ nor $S[x]$ themselves have any continuum meaning, the limit

$$d\mu_W := \lim_{N \rightarrow \infty} \left[\prod_{i=1}^N \frac{dx_{t_i}}{(2\pi\Delta t)^{n/2}} \exp \left[-\frac{\Delta t}{2} \left(\frac{x_{t_{i+1}} - x_{t_i}}{\Delta t} \right)^2 \right] \right] \quad (3.35)$$

of the standard measures dx_{t_i} on \mathbb{R}^n at each time-step *together with* the factor e^{-S_i} does exist. The limit $d\mu_{|rmW}$ is known as the **Wiener measure** and, as you might imagine from our discussion above, it plays a central role in the mathematical theory of Brownian motion. Notice that the presence of the factor e^{-S_i} means that this measure is certainly not translationally invariant in the fields, avoiding the no-go theorem. For Bryce de Wit, the competition between the efforts of e^{-S} to damp out the contribution of wild field configurations and $\mathcal{D}x$ to concentrate on such fields was poetically “The eternal struggle between energy and entropy.”. Wiener's result means that in one dimension the contest is beautifully balanced.

In higher dimensions the situation is less clear. Certainly, the naive path integral measure does *not* exist. It is *believed* that Quantum Field Theories that are **asymptotically free** do have a sensible continuum limit, for reasons we'll see later in the course. The most important example of such a QFT is Yang–Mills theory in four dimensions: every physicist believes this exists, but you can still pick up \$1,000,000 from the Clay Institute for actually proving¹⁰ it.

Perhaps more surprisingly, there are plenty of very important field theories for which a continuum path integral measure, of any sort, almost certainly does *not* exist. The most famous example is General Relativity, but it is also true of both Quantum Electrodynamics (QED) and very likely even the Standard Model. Yet planets orbit around the Sun and satellites orbit around the Earth in exquisite agreement with the predictions of General Relativity, QED is the arena for the most accurate scientific measurements ever carried out, and the Standard Model is the Crown Jewel in our understanding of Nature at the subatomic level. Clearly, not having a well-defined continuum limit does *not* mean these theories are so hopelessly ill-defined as to be useless. On the contrary, we can define *effective* quantum theories in all these cases that make perfect sense: we just restrict ourselves to taking the path integral only over low-energy modes, or over some discretized

¹⁰Terms and conditions apply; see [here](#) for details.

version (such as putting the theory on a lattice). So long as we probe these theories within their domain of validity, they make powerful, accurate predictions. What lies beyond may not even be a QFT at all, but something else, perhaps String Theory.

3.3 Locality and Effective Quantum Mechanics

To appreciate the notion of an effective field theory in a simple setting, let's consider what happens in one dimension.

We imagine we have two different fields x and y on the same worldline, which we'll take to be a circle to avoid complications with end-points. I'll choose the action to be

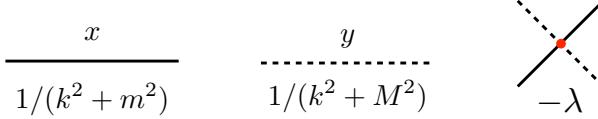
$$S[x, y] = \int_{S^1} \left[\frac{1}{2} \dot{x}^2 + \frac{1}{2} \dot{y}^2 + V(x, y) \right] dt \quad (3.36)$$

where the potential

$$V(x, y) = \frac{1}{2} (m^2 x^2 + M^2 y^2) + \frac{\lambda}{4} x^2 y^2 \quad (3.37)$$

allows the two fields to interact. In terms of the one-dimensional QFT, x and y look like interacting fields with masses m and M , while from the point of view of the target space \mathbb{R}^2 you should think of them as two harmonic oscillators with frequencies m and M , coupled together in a particular way. Of course, this coupling has been chosen to mimic what we did in section 2.6 in zero dimensions.

If we are interested in perturbatively computing correlation functions of (local) operators that are independent of $y(t)$, for example $\langle x(t_2)x(t_1) \rangle$, then we could proceed by directly using (3.36) to construct Feynman diagrams. We'd find ingredients



where k is the one-dimensional worldline momentum (which would be quantized in units of the inverse circumference of the circle). On the other hand, we learned in section 2.6 that for such a class of observables, it is expedient to first construct an effective action by integrating out the y field directly. We expect this effective action to contain infinitely many new self-interactions of x which together take into account the effect of the unobserved y field.

Let's repeat that calculation here. As far as the path integral over $y(t)$ is concerned, x is just a fixed background field so we have formally

$$\int \mathcal{D}y \exp \left[-\frac{1}{2} \int_0^T y \left(-\frac{d^2}{dt^2} + M^2 + \frac{\lambda}{2} x^2 \right) y^2 \right] = \det \left(-\frac{d^2}{dt^2} + M^2 + \frac{\lambda}{2} x^2 \right) \quad (3.38)$$

where I've imposed the boundary conditions $y\dot{y}|_{t=0,T} = 0$ on $y(t)$. Accordingly, the effective action for x is

$$S_{\text{eff}}[x] = \int_0^T \left[\frac{1}{2} \dot{x}^2 + \frac{m^2}{2} x^2 \right] dt - \text{tr} \ln \left(-\frac{d^2}{dt^2} + M^2 + \frac{\lambda}{2} x^2 \right) \quad (3.39)$$

where we've used the identity $\ln \det A = \text{tr} \ln A$ (which holds provided A is a trace-class operator; don't worry if you don't know what this means).

Remarkably, *the effective action for x is non-local!* To see this, suppose $G(t, t')$ is the worldline Green's function that obeys

$$\left(\frac{d^2}{dt^2} - M^2 \right) G(t, t') = \delta(t - t') \quad (3.40)$$

and so is the inverse of the operator $d^2/dt^2 - M^2$ on the circle. Explicitly one has

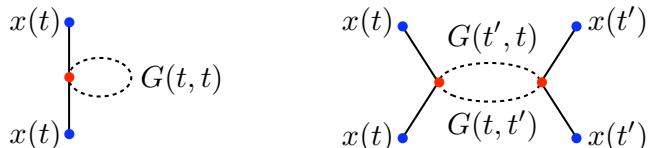
$$G(t, t') = \frac{1}{2M} \sum_{k \in \mathbb{Z}} e^{-M|t-t'+\beta k|} \quad (3.41)$$

where $\beta = 1/T$ is the inverse circumference and k represents the momentum modes. Now, using $\text{tr} \ln(AB) = \text{tr}(\ln A + \ln B) = \text{tr} \ln A + \text{tr} \ln B$ we have

$$\begin{aligned} \text{tr} \ln \left(-\frac{d^2}{dt^2} + M^2 + \frac{x^2}{2} \right) - \text{tr} \ln \left(-\frac{d^2}{dt^2} + M^2 \right) &= \text{tr} \ln \left(1 - \lambda \left(\frac{d^2}{dt^2} - M^2 \right)^{-1} \frac{x^2}{2} \right) \\ &= -\frac{\lambda}{2} \int_{S^1} dt G(t, t) x^2(t) - \frac{\lambda^2}{8} \int_{S^1 \times S^1} dt dt' G(t', t) x^2(t) G(t, t') x^2(t') + \dots \\ &= -\sum_{n=1}^{\infty} \frac{\lambda^n}{2^n n} \int_{(S^1)^n} dt_1 \cdots dt_n G(t_n, t_1) x^2(t_1) G(t_1, t_2) x^2(t_2) \cdots G(t_{n-1}, t_n) x^2(t_n) \end{aligned} \quad (3.42)$$

where the second term on the *lhs* is a divergent, but $x(t)$ independent constant. Integrating out y has indeed generated an infinite series of new interactions for $x(t)$, but except for the $\mathcal{O}(\lambda)$ term, these interactions are now *non-local*!

Again, it's instructive to see why this non-locality has arisen. The first two terms in the series (3.42) represent the Feynman diagrams



that arise in the perturbative evaluation of the y path integral. Unlike the trivial case of zero dimensions, here the y field is dynamical; in particular it has its own worldline propagator $G(t, t')$ that allows it to move around on the worldline. (Note that in these diagrams, I've drawn the external blue vertices at different places on the page just for clarity. Each external x field in the diagram on the left resides at the point $t \in S^1$, while the four x s on the right live pairwise at points t and t' .)

Non-locality is generally bad news in physics: the equations of motion we'd obtain from $S_{\text{eff}}[x]$ would be integro-differential equations stating that in order to work out the behaviour of the field x here, we first have to add up what it's doing everywhere else in the (one-dimensional) Universe. But we don't want the results of our experiment in CERN to

depend on what Ming the Merciless may or may not be having for breakfast over on the far side of the Galaxy. So how bad is it here?

From the explicit form (3.41) of the Green's function we see that $G(t, t')$ decays exponentially quickly when $t \neq t'$, with a scale set by the inverse mass M^{-1} of y . This suggests that the effects of non-locality will be small *provided we restrict attention to fields whose derivatives vary slowly on timescales $\sim M^{-1}$* . More specifically, expanding $x(t)$ we have

$$\begin{aligned} & \int dt dt' G(t, t')^2 x^2(t) x^2(t') \\ &= \int dt dt' G(t, t')^2 x^2(t) \left[x^2(t) + 2x(t)\dot{x}(t)(t - t') + \left(\dot{x}^2(t) + \frac{1}{2}x(t)\ddot{x}(t) \right) (t - t')^2 + \dots \right] \\ &= \int dt \left[\frac{\alpha}{M} x^4(t) + \frac{\beta}{M^3} \left(x^2 \dot{x}^2 + \frac{1}{2} x^2 \ddot{x} \right) + \frac{\gamma}{M^5} (\text{four derivative terms}) + \dots \right]. \end{aligned} \quad (3.43)$$

In going to the last line we have performed the t' integral. To do so, note that the Green's function $G(t, t')$ depends on t' only through the dimensionless combination $u = M(t - t')$. Thus we replace the factor $(t - t')^p$ in the p^{th} order term in the Taylor expansion by $(u/M)^p$ and change variables $dt' = du/M$ to integrate over the dimensionless quantity u . In particular, the infinite series of dimensionless constants $\alpha, \beta, \gamma, \dots$ are just some dimensionless numbers – their precise values don't matter for the present discussion.

The important point is that every new derivative of x in these vertices is *suppressed* by a further power of the mass M of the y field. Thus, so long as $\dot{x}, \ddot{x}, \dddot{x}, \dots$ are all small in units of M^{-1} , we should have a controllable expansion. In particular, if we truncate the infinite Taylor expansion and the infinite expansion (3.42) at any finite order, we will regain an apparently local effective action. This truncation is justified provided we restrict to processes where the momentum of the x field is $\ll M$.

However, once we start to probe energies $\sim M$ something will go badly wrong with our truncated theory. Assuming the original action (3.36) defined a unitary theory (at least in Minkowski signature), simply performing the exact path integral over y must preserve unitary. This is because we haven't yet made any approximations, just taken the first step to performing the full $\mathcal{D}x \mathcal{D}y$ path integral. All the possible states of the y field are still secretly there, encoded in the infinite series of non-local interactions for x . However, the approximation to keep just the first few terms in S_{eff} *can't* be unitary, because we're rejecting by hand various pieces of Feynman diagrams: we're throwing away some of the things y might have been doing.

The weak interactions are responsible for many important things, from the formation of light elements such as deuterium in the early Universe, to powering stars such as our Sun, to the radioactive β -decay of ^{14}C used in radiocarbon dating. Since the 1960s physicists have known that these weak interactions are mediated by a field called the W-boson and in 1983, the UA1 experiment at CERN discovered this field and measured its mass to be $M_W \simeq 80 \text{ GeV}$. Typically, β -decay takes place at much lower energies, so to describe them it makes sense to integrate out the dynamics of the W boson leaving us with an effective action for the proton, neutron, electron and neutrino that participate in the interaction.

This effective action contains an infinite series of terms, suppressed by higher and higher powers of the large mass M_W . Truncating this infinite effective action to its first few terms leads to Fermi's theory of β -decay which gives excellent results at low energies. However, if ones extrapolates the results obtained using this truncated action to high energies, one finds a violation of unitarity. The non-unitarity in Fermi's theory is what lead physicists to suspect the existence of the W-boson in the first place.

3.4 The worldline approach to perturbative QFT

In this chapter, we've been studying the case of QFT in 1 space-time dimension as a warm-up for the higher-dimensional QFTs we'll meet later on. Before proceeding, I'd like to point out an alternative approach to perturbative QFT that was invented by Feynman.

Let's start by considering maps $x : [0, T] \rightarrow \mathbb{R}^n$ with the free action

$$S[x] = \int_0^T dt \left[\frac{1}{2} \delta_{ab} \dot{x}^a \dot{x}^b + \frac{m^2}{2} \right] = \frac{m^2}{2} T + \int_0^T dt \frac{1}{2} \dot{x}^2, \quad (3.44)$$

where I've included a constant term $m^2/2$ in the Lagrangian. This may seem like a strange step; the constant term does not affect the dynamics of the field x in any way. Indeed, the path integral over x becomes

$$\int_{C_I[x,y]} \mathcal{D}x e^{-S} = e^{-Tm^2/2} \langle y | e^{-HT} | x \rangle \quad (3.45)$$

with the constant term in the action providing just an overall factor. Its true purpose will be revealed below.

With this action, the momentum conjugate to the field x^a is $p_a = \partial L / \partial \dot{x}^a = \dot{x}_a$, so the Hamiltonian is $H = p_a \dot{x}^a - L = p_a p^a / 2$, as expected for a free particle on \mathbb{R}^n . Therefore, by inserting complete sets of momentum eigenstates, we have

$$\begin{aligned} \langle y | e^{-HT} | x \rangle &= \int d^n p d^n q \langle y | p \rangle \langle p | e^{-HT} | q \rangle \langle q | x \rangle \\ &= \int \frac{d^n p}{(2\pi)^n} e^{ip \cdot (x-y)} e^{-Tp^2/2} \end{aligned} \quad (3.46)$$

and so the path integral becomes

$$\int_{C_I[x,y]} \mathcal{D}x e^{-S} = \int \frac{d^n p}{(2\pi)^n} e^{ip \cdot (x-y)} e^{-T(p^2+m^2)/2}. \quad (3.47)$$

(An alternative way to obtain the same result is to write the flat space heat kernel (3.8) as its inverse Fourier transform.) Feynman noticed that if we integrate this expression over all possible lengths $T \in (0, \infty)$ of our worldline, then we obtain

$$\int_0^\infty dT \int_{C_I[x,y]} \mathcal{D}x e^{-S} = 2 \int \frac{d^n p}{(2\pi)^n} \frac{e^{ip \cdot (x-y)}}{p^2 + m^2} \quad (3.48)$$

which is the Fourier transform of the momentum space propagator $1/(p^2 + m^2)$ for a scalar field $\Phi(x)$ of mass m on the *target* space \mathbb{R}^n .

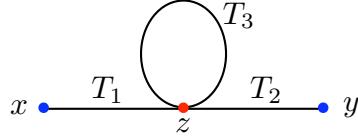
Feynman now realized that one could describe several such particles interacting with one another if one replaced the *worldline* I by a *worldgraph* Γ . For example, to obtain a perturbative evaluation of the r -point correlation function

$$\langle \Phi(x_1)\Phi(x_2)\dots\Phi(x_r) \rangle$$

of a massive scalar field $\Phi(x)$ in $\lambda\Phi^4$ theory on \mathbb{R}^n , one could start by considering a 1-dimensional QFT living on a 4-valent graph Γ with r end-points. This QFT is described by the action (3.44), where x is constrained to map each end-point of the graph to a different one of the Φ insertion points $x_i \in \mathbb{R}^n$. We assign a length T_e to each edge e of the graph, which in this context is often known as a **Schwinger parameter** of the graph. We now take the path integral over all maps $x : \Gamma \rightarrow \mathbb{R}^n$ and integrate over the Schwinger parameters of each edge.

Part of what is meant by an ‘integral over all maps $x : \Gamma \rightarrow \mathbb{R}^n$ ’ includes an integral over the location in \mathbb{R}^n to which each vertex of Γ is mapped. When we perform this integral, the factors of $e^{ip \cdot (x-y)}$ in the path integral (3.47) for each edge lead to target space momentum conserving δ -function at each vertex. As in (3.48), integrating over the Schwinger parameters generates a propagator $1/(p^2 + m^2)$ for each edge of the graph. Thus, after including a factor of $(-\lambda)^{|v(\Gamma)|}$ and dividing by the symmetry factor of the graph, our 1-dimensional QFT has generated the same expression as we would have obtained from Feynman rules for $\lambda\Phi^4$ on \mathbb{R}^n .

For example, the 4-valent graph with two end-points shown here:



corresponds to the path integral expression

$$\begin{aligned} & \frac{-\lambda}{4} \int_0^\infty dT_1 \int_{C_{T_1}[x,z]} \mathcal{D}x e^{-S} \times \int_0^\infty dT_2 \int_{C_{T_2}[y,z]} \mathcal{D}x e^{-S} \times \int_0^\infty dT_3 \int_{C_{T_3}[z,z]} \mathcal{D}x e^{-S} \\ &= \frac{-\lambda}{4} \int d^n z \frac{d^n p}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \frac{d^n \ell}{(2\pi)^n} \frac{e^{ip \cdot (x-z)}}{p^2 + m^2} \frac{e^{iq \cdot (y-z)}}{q^2 + m^2} \frac{e^{i\ell \cdot (z-z)}}{\ell^2 + m^2} \quad (3.49) \\ &= \frac{-\lambda}{4} \int \frac{d^n p}{(2\pi)^n} \frac{d^n \ell}{(2\pi)^n} \frac{e^{ip \cdot (x-y)}}{(p^2 + m^2)^2 (\ell^2 + m^2)}. \end{aligned}$$

This is the same order λ contribution to the 2-point function $\langle \Phi(x)\Phi(y) \rangle$ that we’d obtain from (Fourier transforming) the momentum space Feynman rules for $\lambda\Phi^4$ theory, with the graph treated as a Feynman graph in \mathbb{R}^n rather than a one-dimensional Universe.

To obtain the full perturbative expansion of $\langle \Phi(x_1)\Phi(x_2)\dots\Phi(x_n) \rangle$ we now sum over all graph topologies appropriate to our 4-valent interaction. Thus

$$\langle \Phi(x_1)\Phi(x_2)\dots\Phi(x_n) \rangle = \sum_{\Gamma} \frac{(-\lambda)^{|v(\Gamma)|}}{|\text{Aut } \Gamma|} \int_0^\infty d^{|e(\Gamma)|} T \int_{C_\Gamma[x_1, x_2, \dots, x_n]} \mathcal{D}\phi e^{-S_\Gamma[\phi]}, \quad (3.50)$$

where $|e(\Gamma)|$ and $|v(\Gamma)|$ are respectively the number of edges and vertices of Γ .

This worldline approach to perturbative QFT is close to the way Feynman originally thought about the subject, presenting his diagrams at the Pocono Conference of 1948. The relation of this approach to higher (four) dimensional QFT as we usually think about it was worked out by Dyson a year later, long before people used path integrals to compute anything in higher dimensions. Above, we've described just the simplest version of this picture, relevant for a scalar theory on the target space. There are more elaborate $D = 1$ QFTs that would allow us to obtain target space Quantum Mechanics for particles with spin, and we could also allow for more interesting things to happen at the interaction vertices of our worldgraphs. In this way, one can build up worldline approaches to many perturbative QFTs. This way of thinking can still be useful in practical calculations today, and still occasionally throws up conceptual surprises, but we won't pursue it further in this course.

Finally, I can't resist mentioning that what we've really been studying in this section is not merely one-dimensional QFT, but one-dimensional Quantum Gravity. In one dimension a Riemannian metric is just a 1×1 matrix $g_{tt}(t)$ with positive eigenvalues; in other words, a positive number at each point t of the worldline. General coordinate transformations act on this 1×1 matrix as

$$g_{tt} \rightarrow g_{t't'} = \frac{dt}{dt'} \frac{dt}{dt'} g_{tt} = \left(\frac{dt}{dt'} \right)^2 g_{tt}, \quad (3.51)$$

and so can be used to rescale the value of the metric to anything we like. Throughout this chapter, we've implicitly been using a coordinate system t on the worldline in which the worldline metric had been fixed to 1, which we're always free to do. The proper length T of the worldline interval I can be written

$$T = \int_I dt \sqrt{g_{tt}} = \int_I dt' \sqrt{g_{t't'}} \quad (3.52)$$

and is invariant under the general coordinate transform (3.51). In fact, in one dimension the length T is the *only* diffeomorphism invariant of a Riemannian metric, essentially because there is no 'room' for any sort of curvature. Thus, the integral over the lengths of all the edges of our graph in (3.50) is best thought of as an integral over the space of all possible Riemannian metrics on Γ , up to diffeomorphism invariance. Rather grandly, this is known as the **moduli space** of Riemannian metrics on Γ . Furthermore, in summing over graphs Γ we were really summing over the topological type of our one dimensional Universe. Notice that the vertices of our graphs are singularities of the one-dimensional Riemannian manifold, so we're allowing our Universe to have such wild (even non-Hausdorff) behaviour. So for fixed lengths T_e the path integral over $x(t)$ is the 'matter' QFT on a fixed background space Γ , while the integral over the lengths of edges in Γ together with the sum over graph topologies is Quantum Gravity.

This picture is also very close to perturbative String Theory. There, as you'll learn if you're taking the Part III String Theory course, the worldgraph Γ is replaced by a two dimensional worldsheet (Riemann surface) Σ , the $D = 1$ worldline QFT replaced by a

$D = 2$ worldsheet CFT¹¹. Likewise, the integral over the moduli space of Riemannian metrics on Γ becomes an integral over the moduli space of Riemann surfaces, and finally the sum over graphs is replaced by a sum over the topology of the Riemann surface. We know that the worldgraph approach to QFT only captures some aspects of perturbation theory, and in the following chapters we'll see that deeper insight is provided by QFT proper. Asking whether there's a similarly deeper approach to String Theory will take you to the mystic shores of String Field Theory, about which very little is known.

¹¹CFT = Conformal Field Theory.

4 Exact Properties of the Path Integral

In this chapter we begin our study of Quantum Field Theory proper: we take the dimension d of our space to be generic (certainly $d > 1$, and typically $d = 4$). Now, as you're no doubt aware, these theories are typically rather complicated. Except in very special circumstances, we can usually only hope to find a perturbative approximation to the path integral (or else we must seek to evaluate it numerically). As in the $d = 0$ example earlier, such perturbative expansions provide us with an asymptotic series that approximates the true path integral arbitrarily accurately as $\hbar \rightarrow 0$.

Given this situation, it'll be valuable to know what can be said *without* making any perturbative approximations. We'll start by taking a look at the role of symmetry in QFT and understanding how we can use it to constrain correlation functions — symmetries constrain the possible forms correlation functions may take *exactly*.

4.1 Symmetries of the quantum theory

One of the most important results in classical mechanics and classical field theory is *Noether's theorem*, stating that local symmetries of the action corresponds to a conserved charge. Let's recall how to derive this.

Consider the transformation

$$\delta_\epsilon \phi(x) = \epsilon f(\phi, \partial_\mu \phi) \quad (4.1)$$

where ϵ is an infinitesimal parameter, and $f(\phi, \partial_\mu \phi)$ is some function of the fields and their derivatives. The transformation is local if the function f depends on the values of the field and its derivatives only at the point $x \in M$, in which case you can think of it as being generated by the vector

$$V_f := \int_M d^d x \sqrt{g} f(\phi, \partial_\mu \phi) \frac{\delta}{\delta \phi(x)} \quad (4.2)$$

acting on the infinite dimensional space of fields. The transformation (4.1) is a *symmetry* if the action is invariant, $\delta S[\phi] = 0$, whenever the parameter ϵ is constant. Because it is invariant when ϵ is constant, if ϵ is now allowed to depend on position the change in the action must be proportional to the derivative of ϵ . In other words¹³

$$\delta_\epsilon S[\phi] = - \int_M * j \wedge d\epsilon = - \int_M g^{\mu\nu} j_\mu(x) \partial_\nu \epsilon(x) \sqrt{g} d^d x \quad (4.3)$$

for some function $j_\mu(x)$ known as the *current*.

However, when the equations of motion hold, the action is invariant under *any* small change in the fields. In particular, on the support of the equations of motion,

$$\delta_\epsilon S[\phi] = 0 \quad (4.4)$$

¹³The minus sign is a convention, included for later convenience. The first expression here is written in the language of differential forms, where we treat $j \in \Omega_M^1$ is a 1-form and where $* : \Omega_M^p \rightarrow \Omega_M^{d-p}$ is the Hodge star on the d -dimensional Riemannian manifold (M, g) . The second expression is just the same thing written in a local coordinate patch. Below, I'll often work in the sleeker language of forms, but I'll be sure to give the main results both ways. If you're uncomfortable with differential geometry, I recommend you repeat the derivations for yourself in the case $(M, g) = (\mathbb{R}^d, \delta)$.

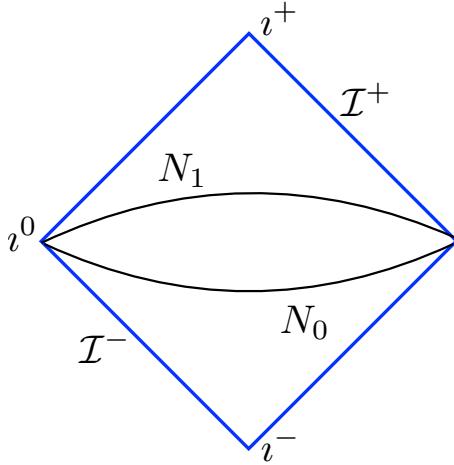


Figure 5: Classically, the charge (4.6) associated to a local symmetry is independent of the hypersurface over which it is integrated.

under the change (4.1) even if the parameter ϵ depends on position. Integrating by parts and choosing our function $\epsilon(x)$ to have compact support, we find that

$$d * j = 0 \quad \text{or equivalently} \quad \partial_\nu(\sqrt{g} g^{\mu\nu} j_\nu) = 0 \quad (4.5)$$

whenever the equations of motion are satisfied.

We define the *charge* Q corresponding to this transformation by

$$Q[N] := \int_N *j = \int_N j_\mu n^\mu \sqrt{g_N} d^{d-1}x \quad (4.6)$$

where N is any codimension-one hypersurface in the space-time M , n^μ is a unit normal vector to N (so $g(n, n) = 1$ and $g(n, v) = 0$ for any $v \in TN$), g_N denotes the restriction of the metric on M to N , and thus $\sqrt{g_N} d^{d-1}x$ is the $(d-1)$ -dimensional volume element on N . If $N_{0,1}$ are two such hypersurfaces bounding a region $M' \subset M$ of our space, then

$$Q[N_1] - Q[N_0] = \int_{N_1} *j - \int_{N_0} *j = \int_{\partial M'} *j = \int_{M'} d * j = 0 \quad (4.7)$$

where the third equality is Stokes' theorem and the final equality follows by the conservation equation (4.5). Thus $Q[N]$ depends on the choice of N only through its homology class. The most common application of this result is to take the surfaces $N_{0,1}$ to be constant time slices of Minkowski space-time¹⁴, as in figure 5. The statement that $Q[N_1] = Q[N_0]$ then becomes the statement that the charge Q is conserved under time evolution.

As a simple example, consider the action

$$S[\phi] = \frac{1}{2} \int_M d\bar{\phi} \wedge *d\phi + *V(|\phi|^2) \quad (4.8)$$

¹⁴Since the constant time slices $N_{0,1}$ are non-compact in this case, for our derivation to hold we should ensure that the current j decays sufficiently rapidly as we head towards spatial infinity so that the integrals defining $Q[N_{0,1}]$ converge.

for a complex scalar field. This action is invariant under the $U(1)$ transformation $\phi \mapsto e^{i\alpha}\phi$ that rotates the phase of ϕ by a constant amount α . Taking α to be infinitesimally small, we have

$$\delta\phi = i\alpha\phi, \quad \delta\bar{\phi} = -i\alpha\bar{\phi} \quad (4.9)$$

The corresponding current is $j_\mu = i(\partial_\mu\bar{\phi}\phi - \bar{\phi}\partial_\mu\phi)$ and so the charge associated with a hypersurface N is

$$Q[N] = i \int_N * (d\bar{\phi}\phi - \bar{\phi}d\phi). \quad (4.10)$$

In particular, if we are in flat space-time $(M, g) = (\mathbb{R}^{1,3}, \eta)$ and the fields decay rapidly as we approach spatial infinity, then the charge on a constant time hypersurface is

$$Q = i \int (\partial_0\bar{\phi}\phi - \bar{\phi}\partial_0\phi) d^3x \quad (4.11)$$

From the original action we have that the momentum conjugate to the field is

$$\pi = \frac{\delta\mathcal{L}}{\delta\partial_0\phi} = \partial_0\bar{\phi} \quad \text{and} \quad \bar{\pi} = \frac{\delta\mathcal{L}}{\delta\partial_0\bar{\phi}} = \partial_0\phi \quad (4.12)$$

and hence the charge can be written as

$$Q = i \int (\phi\pi - \bar{\phi}\bar{\pi}) d^3x \quad (4.13)$$

which indeed generates the transformations (4.9) via Poisson brackets.

4.1.1 Ward identities

The derivation of Noether's theorem used the classical equations of motion, so we must re-examine this in quantum theory. Suppose that some local field transformation $\phi \rightarrow \phi'(\phi)$ leaves the product of the action and path integral measure invariant, *i.e.*,

$$\mathcal{D}\phi e^{-S[\phi]} = \mathcal{D}\phi' e^{-S[\phi']} \quad (4.14)$$

In most cases, the symmetry transformation will actually leave *both* the action and measure invariant separately, but the weaker condition (4.14) is sufficient (and necessary).

In practice, we have much more familiarity with classical actions than with path integral measures, so one tends to look for symmetries of the action first and then hope these can be extended to symmetries of the measure. For example, if some action $S[\phi]$ depends on a field only through its derivatives $\partial^r\phi$ with $r \geq 1$, then this action is invariant under shifts of the field

$$\phi(x) \rightarrow \phi(x) + \phi_0 \quad (4.15)$$

where ϕ_0 is constant over M . Naïvely, the path integral measure $\mathcal{D}\phi$ instructs us to integrate over *all* continuous maps $\phi \in C^0(M, \mathbb{R})$ and we'd expect such an instruction not to care about a constant translation. However, to actually *define* what we mean by $\mathcal{D}\phi$ we must first pick a regularization and if we wish (4.14) to be valid under (4.15) then this regularization procedure must also be compatible with constant field translations. For

example, if $M = T^d$ we might expand $\phi(x)$ in terms of a Fourier series, and declare that we are only going to integrate over finitely many Fourier coefficients. The constant translation (4.15) only affects the lowest Fourier mode, and our regularized measure integrates over all values of this coefficient, so indeed (4.14) will hold.

As a further example, a theory living on $(M, g) = (T^d, \delta)$ will be invariant under $\text{SO}(d)$ rotations

$$x \rightarrow Lx, \quad \phi(x) \rightarrow \phi'(x) = \phi(Lx) \quad (4.16)$$

if the action is built from $\text{SO}(d)$ invariant combinations of the fields and their derivatives. We can regularize the path integral by integrating over all Fourier modes with the $\text{SO}(d)$ -invariant quantity $p^\mu p_\mu$ is less than some chosen cut-off¹⁵. However, if we chose instead to regularize our theory by replacing space by a simple cubic lattice $\Lambda \subset T^d$ and integrating over the values of the fields at each lattice site, then $\text{SO}(d)$ invariance would be broken to the discrete group of lattice symmetries.

When a symmetry of the classical action is broken by the regularized path integral measure, there are two possible outcomes. In the one hand, it may be that *some* regularization procedure that would have preserved the symmetry does exist — it's just that we didn't use it, whether through choice or ignorance. This was the case above with, where $\text{SO}(d)$ transformations were broken by the lattice, but not by the cut-off regularization. In this case, it turns out that if one computes any physical object such as a correlation function or scattering amplitude and then removes the regularization at the end of the calculation¹⁶, then the symmetry will be restored. Exactly this situation occurs for lattice treatments of gauge theories such as QCD. The only difficulty is in the intermediate steps of the calculations, where the symmetries are absent.

On the other hand, it may turn out that *no regularization procedure which preserves the symmetry exists*. In this case, the symmetry of the classical field theory is genuinely absent at the quantum level. Such symmetries are said to be *anomalous*; we'll consider them in more detail at the end of the course¹⁷. For now, we'll assume we've found a regulator that preserves our classical symmetry, so that (4.14) holds.

One of the main uses of symmetry is to deduce important restrictions on correlation functions. Consider a class of operators whose only variation under the transformation $\phi \rightarrow \phi'$ comes from their dependence on ϕ itself (such as scalar operators under rotations). Such operators transform as $\mathcal{O}(\phi) \rightarrow \mathcal{O}(\phi')$. At least on a compact manifold M we have

$$\begin{aligned} \int \mathcal{D}\phi e^{-S[\phi]} \mathcal{O}_1(\phi(x_1)) \cdots \mathcal{O}_n(\phi(x_n)) &= \int \mathcal{D}\phi' e^{-S[\phi']} \mathcal{O}_1(\phi'(x_1)) \cdots \mathcal{O}_n(\phi'(x_n)) \\ &= \int \mathcal{D}\phi e^{-S[\phi]} \mathcal{O}_1(\phi'(x_1)) \cdots \mathcal{O}_n(\phi'(x_n)) \end{aligned} \quad (4.17)$$

The first equality here is a triviality: we've simply relabeled ϕ by ϕ' as a dummy variable in the path integral. The second equality is non-trivial and uses the assumed symmetry (4.14)

¹⁵In Minkowski signature, the corresponding statements for $\text{SO}(1, d - 1)$ go through in essentially the same way for massive particles, but are more subtle when massless states are present.

¹⁶We'll understand later how, and in which circumstances, this can be done.

¹⁷Time permitting!

under the transformation $\phi \rightarrow \phi'$. We see that the correlation function obeys

$$\langle \mathcal{O}_1(\phi(x_1)) \cdots \mathcal{O}_n(\phi(x_n)) \rangle = \langle \mathcal{O}_1(\phi'(x_1)) \cdots \mathcal{O}_n(\phi'(x_n)) \rangle \quad (4.18)$$

so that it is *invariant* under the transformation.

For example, consider again the phase transformation

$$\phi \rightarrow \phi' = e^{i\alpha} \phi, \quad \bar{\phi} \rightarrow \bar{\phi}' = e^{-i\alpha} \bar{\phi} \quad (4.19)$$

that we examined above. The path integral measure will be invariant under this symmetry provided we integrate over as many modes of $\bar{\phi}$ as we do of ϕ . The Ward identity then implies that correlation functions built from local operators of the form $\mathcal{O}_i = \phi^{r_i} \bar{\phi}^{s_i}$ must obey

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = e^{i\alpha \sum_{i=1}^n (r_i - s_i)} \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle.$$

Allowing α to take different (constant) values shows that this correlator vanishes unless $\sum_i r_i = \sum_i s_i$. The symmetry thus imposes a **selection rule** on the operators we can insert if we wish to obtain a non-zero correlator.

For a second example, suppose $(M, g) = (\mathbb{R}^d, \delta)$ and consider a space-time translation $x \mapsto x' := x - a$ where a is a constant vector. Under this translation, we have

$$\phi(x) \mapsto \phi'(x) := \phi(x - a). \quad (4.20)$$

If the action and path integral measure are translation invariant and the operators \mathcal{O}_i depend on x only via their dependence on $\phi(x)$, then the Ward identity gives

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \langle \mathcal{O}_1(x - a) \cdots \mathcal{O}_n(x_n - a) \rangle$$

for any such vector a . Thus, having carried out the path integral, we'll be left with a function $f(x_1, x_2, \dots, x_n)$ that depends only on the *relative* positions $(x_i - x_j)$. Similarly, if the action & measure are invariant under rotations (or Lorentz transformations) $x \rightarrow Lx$ then a correlation function of scalar operators will obey

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \langle \mathcal{O}_1(Lx_1) \cdots \mathcal{O}_n(Lx_n) \rangle.$$

Combining this with the previous result shows that the correlator can depend only on the rotational (or Lorentz) invariant distances $(x_i - x_j)^2$ between the insertion points.

4.1.2 Currents and charges

As in the classical theory, any continuous symmetry comes with an associated current. Suppose that $\phi \rightarrow \phi' = \phi + \delta_\epsilon \phi$ is a symmetry of the path integral when ϵ is an infinitesimal constant parameter. Then, as in Noether's theorem, the variation of the action *and path integral measure* must be proportional to $\partial_\mu \epsilon$ when ϵ depends on position, so that

$$\mathcal{Z} = \int \mathcal{D}\phi' e^{-S[\phi']} = \int \mathcal{D}\phi e^{-S[\phi]} \left[1 - \int_M *j \wedge d\epsilon \right] \quad (4.21)$$

to lowest order. Notice that the current here may include possible changes in the path integral measure as well as in the action. The zeroth order term agrees with the partition function on the left, so the first order term must vanish and we have

$$0 = - \int_M * \langle j(x) \rangle \wedge d\epsilon = \int_M \epsilon(x) d * \langle j(x) \rangle, \quad (4.22)$$

if either $\partial M = 0$ or the fields decay sufficiently rapidly that there is no boundary contribution. For this to hold for arbitrary ϵ we must have $\partial^\mu \langle j_\mu(x) \rangle = 0$ so that the *expectation value* of the current obeys a conservation law, just as in classical physics.

Now let's see how the current insertions affect more general correlation functions. Consider a class of local operators that transform under $\phi \mapsto \phi' := \phi + \epsilon \delta\phi$ as

$$\mathcal{O}(\phi) \mapsto \mathcal{O}(\phi + \epsilon \delta\phi) = \mathcal{O}(\phi) + \epsilon \delta\mathcal{O} \quad (4.23)$$

to lowest order in ϵ , where we've defined $\delta\mathcal{O} := \delta\phi \partial\mathcal{O}/\partial\phi$. Then, accounting for both the change in the action and measure as well as in the operators,

$$\begin{aligned} \int \mathcal{D}\phi e^{-S[\phi]} \prod_{i=1}^n \mathcal{O}_i(\phi(x_i)) &= \int \mathcal{D}\phi' e^{-S[\phi']} \prod_{i=1}^n \mathcal{O}_i(\phi'(x_i)) \\ &= \int \mathcal{D}\phi e^{-S[\phi]} \left[1 - \int_M *j \wedge d\epsilon \right] \left[\prod_{i=1}^n \mathcal{O}_i(x_i) + \sum_{i=1}^n \epsilon(x_i) \delta\mathcal{O}_i(x_i) \prod_{j \neq i} \mathcal{O}_j(x_j) \right]. \end{aligned} \quad (4.24)$$

Again, the first equality is a triviality and the second follows upon expanding both $\mathcal{D}\phi' e^{-S[\phi']}$ and the operators to first order in the variable parameter $\epsilon(x)$. The ϵ -independent term on the *rhs* exactly matches the *lhs*, so the remaining terms must cancel. To first order in ϵ this gives

$$\int_M \epsilon(x) \wedge d * \left\langle j(x) \prod_{i=1}^n \mathcal{O}_i(x_i) \right\rangle = - \sum_{i=1}^n \left\langle \epsilon(x_i) \delta\mathcal{O}_i(x_i) \prod_{j \neq i} \mathcal{O}_j(x_j) \right\rangle, \quad (4.25)$$

after an integration by parts with $\epsilon(x)$ of compact support. Note that the derivative on the *lhs* hits the whole correlation function.

We'd like to strip off the parameter $\epsilon(x)$ and obtain a relation purely among the correlation functions themselves. In order to do this, we must handle the fact that the operator variations on the *rhs* are located only at the points $\{x_1, \dots, x_n\} \in M$. We thus write¹⁸

$$\epsilon(x_i) \delta\mathcal{O}_i(x_i) = \int_M * \delta^d(x - x_i) \epsilon(x) \delta\mathcal{O}_i(x_i) = \int_M \delta^d(x - x_i) \epsilon(x) \delta\mathcal{O}_i(x_i) \sqrt{g} d^d x$$

as an integral, so that all terms in (4.25) are proportional to $\epsilon(x)$. Since this may be chosen arbitrarily, we obtain finally

$$d * \left\langle j(x) \prod_{i=1}^n \mathcal{O}_i(x_i) \right\rangle = - * \sum_{i=1}^n \delta^d(x - x_i) \left\langle \delta\mathcal{O}_i(x_i) \prod_{j \neq i} \mathcal{O}_j(x_j) \right\rangle, \quad (4.26)$$

¹⁸Here, for an open set $U \subset M$ in curved space, my δ -function is defined to obey $\int_U \delta^d(x - x_i) \sqrt{g} d^d x = 1$ if $x_i \in U$ and 0 else, with an integration measure including the factor of \sqrt{g} .

or equivalently

$$\partial_\mu \left(g^{\mu\nu} \sqrt{g} \left\langle j^\mu(x) \prod_{i=1}^n \mathcal{O}_i(x_i) \right\rangle \right) = - \sum_{i=1}^n \delta^d(x - x_i) \left\langle \delta \mathcal{O}_i(x_i) \prod_{j \neq i} \mathcal{O}_j(x_j) \right\rangle \quad (4.27)$$

in terms of local co-ordinates on (M, g) . This is known as the *Ward identity* for the symmetry represented by $\phi \rightarrow \phi + \delta\phi$. It states that the divergence of a correlation function involving a current j^μ vanishes everywhere except at the locations of other operator insertions, and is the modification of the conservation law $d* \langle j(x) \rangle = 0$ for the expectation value of the current itself. Again, note that the divergence is taken *after* computing the path integral.

Let's integrate the Ward identity over some region $M' \subseteq M$ with boundary $\partial M' = N_1 - N_0$, just as we studied classically. We'll first choose M' to contain all the points $\{x_1, \dots, x_n\}$ so that the integral receives contributions from all of the terms on the *rhs* of (4.26). Then

$$\langle Q[N_1] \prod_i \mathcal{O}_i(x_i) \rangle - \langle Q[N_0] \prod_i \mathcal{O}_i(x_i) \rangle = - \sum_{i=1}^n \langle \delta \mathcal{O}_i(x_i) \prod_{j \neq i} \mathcal{O}_j(x_j) \rangle \quad (4.28)$$

where the charge $Q[N] = \int_N * j$ just as in the classical case. In particular, if $M' = M$ and M is closed (*i.e.*, compact without boundary) then we obtain

$$0 = \sum_{i=1}^n \langle \delta \mathcal{O}_i(x_i) \prod_{j \neq i} \mathcal{O}_j(x_j) \rangle \quad (4.29)$$

telling us that if we perform the symmetry transform throughout space-time then the correlation function is simply invariant, $\delta \langle \prod_i \mathcal{O}_i \rangle = 0$. This is just the infinitesimal version of the result we had before in (4.18).

On the other hand, if only one some of the x_i lie inside M' , then only some of the δ -functions will contribute. In particular, if $I \subseteq \{1, 2, \dots, n\}$ then we obtain

$$\langle Q[N_1] \prod_{i=1}^n \mathcal{O}_i(x_i) \rangle - \langle Q[N_0] \prod_{i=1}^n \mathcal{O}_i(x_i) \rangle = \sum_{i \in I} \langle \delta \mathcal{O}_i(x_i) \prod_{j \neq i} \mathcal{O}_j(x_j) \rangle. \quad (4.30)$$

whenever $x_i \in M'$ for $i \in I$. Only those operators enclosed in M' contribute to the changes on the *rhs*.

Note that the condition that M be closed cannot be relaxed lightly. On a manifold with boundary, to define the path integral we must specify some boundary conditions for the fields. The transformation $\phi \mapsto \phi'$ may now affect the boundary conditions, which lead to further contributions to the *rhs* of the Ward identity. For a relatively trivial example, the condition that the net charges of the operators we insert must be zero becomes modified to the condition that the *difference* between the charges of the incoming and outgoing states (boundary conditions on the fields) must be balanced by the charges of the operator insertions.

A much more subtle example arises when the space–time is non–compact and has infinite volume. In this case, the required boundary conditions as $|x| \rightarrow \infty$ are that our fields take some constant value ϕ_0 which lies at the minimum of the effective potential. Because of the suppression factor $e^{-S[\phi]}$, such field configurations will dominate the path integral on an infinite volume space–time. However, it may be that the (global) minimum of the potential is not unique; if $V(\phi)$ is minimized for any $\phi \in \mathcal{M}$ and our symmetry transformations move ϕ around in \mathcal{M} the symmetry will be **spontaneously broken**. You’ll learn much more about this story if you’re taking the Part III Standard Model course.

4.1.3 The Ward–Takahashi identity in QED

Ward identities can be derived for any symmetry transformation, but the name is often associated to the transformations

$$\psi \mapsto \psi' := e^{i\alpha} \psi, \quad \bar{\psi} \mapsto \bar{\psi}' := e^{-i\alpha} \bar{\psi}, \quad A_\mu \mapsto A'_\mu := A_\mu \quad (4.31)$$

which for constant α are symmetries of the QED action

$$S_{\text{QED}}[A, \psi] = \int d^4x \left[\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(iD^\mu - m)\psi \right]. \quad (4.32)$$

This was the case originally considered by Ward and Takahashi. The regularized path integral measure is also invariant under these transformations, *i.e.*,

$$\mathcal{D}\psi \mathcal{D}\bar{\psi} \mapsto \mathcal{D}\psi' \mathcal{D}\bar{\psi}' = \mathcal{D}\psi \mathcal{D}\bar{\psi} \quad (4.33)$$

provided our regularized measure involves integrating over an equal number of ψ and $\bar{\psi}$ modes. Thus these transformations are indeed symmetries of the path integral.

As above, we now promote α to a position–dependent parameter $\alpha(x)$, — this is *not* a gauge transformation because the gauge field A_μ itself remains unaffected. The path integral measure depends only on the fields ψ and $\bar{\psi}$, not their derivatives, so *if* our regularized measure also preserves the local symmetry¹⁹, the only contribution to the current will come from the action. One finds $j^\mu = \bar{\psi}\gamma^\mu\psi$, which of course is just the charged current to which the photon couples in QED.

Now consider the correlation function $\langle \psi(x_1)\bar{\psi}(x_2) \rangle$. Since $\delta\psi \propto \psi$ the Ward identity becomes

$$\partial_\mu \langle j^\mu(x)\psi(x_1)\psi(x_2) \rangle = -\delta^d(x - x_1)\langle \psi(x_1)\bar{\psi}(x_2) \rangle + \delta^d(x - x_2)\langle \psi(x_1)\bar{\psi}(x_2) \rangle \quad (4.34)$$

so that the vector $f_\mu(x, x_1, x_2) = \langle j_\mu(x)\psi(x_1)\bar{\psi}(x_2) \rangle$ is divergence free everywhere except at the insertions of the electron field.

¹⁹Note that in any case, the change in the measure here will be field independent.

The identity (4.40) is traditionally viewed in momentum space. We Fourier transform the two-point function of the electron field:

$$\begin{aligned} & \int d^4x_1 d^4x_2 e^{ik_1 \cdot x_1} e^{-ik_2 \cdot x_2} \langle \psi(x_1) \bar{\psi}(x_2) \rangle \\ &= \int d^4x_1 d^4x_2 e^{ik_1 \cdot (x_1 - x_2)} e^{i(k_1 - k_2) \cdot x_2} \langle \psi(x_1 - x_2) \bar{\psi}(0) \rangle \\ &= \delta^4(k_1 - k_2) S(k_1) \end{aligned} \quad (4.35)$$

where the first equality follows from translational invariance of the correlation function. Note that (unlike a Feynman diagram for scattering amplitudes) there's no requirement that the momenta here are on-shell; they're just the Fourier transforms of the insertion points. The previous equation defines the **exact electron propagator**,

$$S(k) := \int d^4x e^{ik \cdot x} \langle \psi(x) \bar{\psi}(0) \rangle \quad (4.36)$$

in momentum space. In perturbation theory, it represents the sum of all possible connected²⁰ Feynman diagrams that can be drawn in connecting the $\psi\bar{\psi}$ insertions together. (Thus, like the 2-point function $\langle \psi\bar{\psi} \rangle$, $S(k)$ carries a pair of Dirac spinor indices, which we've suppressed in the notation.) Specifically,



where the first line contains all possible connected contributions to the two-point function, and the second line writes these in terms of **one particle irreducible** (1PI) graphs: those connected graphs which cannot be turned into a disconnected graph by cutting any single internal line. The sum of such 1PI contributions is usually called the **electron self-energy** and denoted $\Sigma(\not{k})$. (This is also a matrix in spin space.) The exact electron propagator is related to $\Sigma(\not{k})$ by

$$S(k) = \frac{1}{i\not{k} + m - \Sigma(\not{k})} \quad (4.37)$$

by summing the geometric series above.

²⁰Recall that our correlation functions are normalized by the partition function, which is the sum of all *vacuum* diagrams.

In a similar way, we introduce the **exact electromagnetic vertex** $\Gamma_\mu(k_1, k_2)$ by the Fourier transform

$$\begin{aligned} & \int d^4x d^4x_1 d^4x_2 e^{ip \cdot x} e^{ik_1 \cdot x_1} e^{-ik_2 \cdot x_2} \langle j_\mu(x) \psi(x_1) \bar{\psi}(x_2) \rangle \\ &= \int d^4x d^4x_1 d^4x_2 e^{ip \cdot (x-x_2)} e^{ik_1 \cdot (x_1-x_2)} e^{i(p+k_1-k_2) \cdot x_2} \langle j_\mu(x-x_2) \psi(x_1-x_2) \bar{\psi}(0) \rangle \\ &=: \delta^4(p+k_1-k_2) S(k_1) \Gamma_\mu(k_1, k_2) S(k_2). \end{aligned} \quad (4.38)$$

Let's understand this definition. $\langle \psi(x_1) j_\mu(x) \bar{\psi}(x_2) \rangle$ will be given by the sum of all Feynman graphs connecting the electron field insertions at $x_{1,2}$ to the current at x . Recalling that $j_\mu = \bar{\psi} \gamma_\mu \psi$, we see that the leading contribution will simply come from a pair of propagators connecting $\psi(x_1)$ to $\bar{\psi}(x)$, and $\psi(x)$ to $\bar{\psi}(x_2)$ respectively. Further contributions will come from diagrams that correct each of these free propagators, turning them into the exact electron propagators on each side; *i.e.*

$$\langle \psi(x_1) j_\mu(x) \bar{\psi}(x_2) \rangle \supset \langle \psi(x_1) \bar{\psi}(x) \rangle \gamma_\mu \langle \psi(x) \bar{\psi}(x_2) \rangle. \quad (4.39)$$

These diagrams tell us nothing new about the *vertex*; they're already part of the exact electron propagator. We thus include factors of $S(k_1)$ and $S(k_2)$ in our definition.

The remaining contributions are the ones we care about. They involve graphs that connect the two exact electron propagators together in some way. For example, at leading order, we have the diagram



This class of diagrams is what contributes to $\Gamma_\mu(k_1, k_2)$, so

$$\Gamma_\mu(k_1, k_2) = \gamma_\mu + \text{quantum corrections}, \quad (4.40)$$

where we note that all the corrections to γ_μ come from loop diagrams.

Now let's return to our Ward identity (4.40). Taking the Fourier transform of the complete equation, we obtain

$$(k_1 - k_2)_\mu S(k_1) \Gamma^\mu(k_1, k_2) S(k_2) = iS(k_1) - iS(k_2) \quad (4.41)$$

or equivalently

$$(k_1 - k_2)_\mu \Gamma^\mu(k_1, k_2) = iS(k_2)^{-1} - iS(k_1)^{-1} \quad (4.42)$$

by acting with $S^{-1}(k_1)$ on the left and $S^{-1}(k_2)$ on the right. This identity was obtained by Takahashi; taking the limit $k_2 \rightarrow k_1$ gives

$$\Gamma_\mu(k, k) = -i \frac{\partial}{\partial k^\mu} S^{-1}(k) \quad (4.43)$$

which was the form originally obtained by Ward.

In previous chapters, we've seen that integrating out (high-energy) fields generically shifts the values of couplings in the (low-energy) effective action. Anticipating our story slightly, in QED, we'd expect that we can generate new contributions to both the electron kinetic term $\int i\bar{\psi}\not{D}\psi d^4x$ and the electron-photon vertex $\int \bar{\psi}\not{A}\psi d^4x$ (as well as the electron mass term). The significance of the Ward identity is that, provided the regulated path integral measure is compatible with the symmetry (4.37), *the quantum corrections to these two terms must be related*. In particular, using (4.43) in (4.48) we have

$$\begin{aligned} (k_1 - k_2)_\mu \Gamma^\mu(k_1, k_2) &= i(i\cancel{k}_2 + m - \Sigma(\cancel{k}_2) - i\cancel{k}_1 - m + \Sigma(\cancel{k}_1)) \\ &= (k_1 - k_2)_\mu \gamma^\mu + i(\Sigma(\cancel{k}_1) - \Sigma(\cancel{k}_2)) \end{aligned} \quad (4.44)$$

where we note that the ‘inverse electron propagator’ $S^{-1}(k)$ is nothing but the electron kinetic term in the action, written in momentum space. The fact that quantum corrections treat the whole covariant derivative $\int i\bar{\psi}\not{D}\psi d^4x$ together is important in ensuring that the quantum theory respects gauge transformations, as you'll explore further in the problem sets. In the early days of QED, before renormalization was systematically understood, the Ward identity provided a good check that the regularized theory was compatible with gauge invariance.

5 The Renormalization Group

Even a humble glass of pure water consists of countless H₂O molecules, which are made from atoms that involve many electrons perpetually executing complicated orbits around a dense nucleus, the nucleus itself is a seething mass of protons and neutrons glued together by pion exchange, these hadrons are made from the complicated and still poorly understood quarks and gluons which themselves maybe all we can make out of tiny vibrations of some string, or modes of a theory yet undreamed of. How then is it possible to understand anything about water without first solving all the deep mysteries of Quantum Gravity?

In classical physics the explanation is really an aspect of the Principle of Least Action: if it costs a great deal of energy to excite a degree of freedom of some system, either by raising it up its potential or by allowing it to whizz around rapidly in space–time, then the least action configuration will be when that degree of freedom is in its ground state. The corresponding field will be constant and at a minimum of the potential. This constant is the zero mode of the field, and plays the role of a Lagrange multiplier for the remaining low–energy degrees of freedom. You used Lagrange multipliers in mechanics to confine wooden beads to steel hoops. This is a good description at low energies, but my sledgehammer can excite degrees of freedom in the hoop that your Lagrange multiplier doesn’t reach.

We must re-examine this question in QFT because we’re no longer constrained to sit at an extremum of the action. The danger is already apparent in perturbation theory, for even in a process where all external momenta are small, momentum conservation at each vertex still allows for very high momenta to circulate around the loop and the value of these loop integrals would seem to depend on all the details of the high–energy theory. The Renormalization Group (RG), via the concept of *universality*, will emerge as our quantum understanding of why it is possible to understand physics at all.

5.1 Integrating out degrees of freedom

Suppose our QFT is governed by the action

$$S_{\Lambda_0}[\varphi] = \int d^d x \left[\frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi + \sum_i \Lambda_0^{d-d_i} g_{i0} \mathcal{O}_i(x) \right]. \quad (5.1)$$

Here we’ve allowed arbitrary local operators $\mathcal{O}_i(x)$ of dimension $d_i > 0$ to appear in the action; each \mathcal{O}_i can be a Lorentz–invariant monomial involving some number n_i powers of fields and their derivatives, *e.g.* $\mathcal{O}_i \sim (\partial\varphi)^{r_i} \varphi^{s_i}$ with $r_i + s_i = n_i$. For later convenience, I’ve included explicit factors of some energy scale Λ_0 in the couplings, chosen so as to ensure that the coupling constants g_{i0} themselves are dimensionless, but of course the action is at this point totally general. We’ve simply allowed all possible terms we can include to appear.

Given this action, we can define a regularized partition function by

$$\mathcal{Z}_{\Lambda_0}(g_{i0}) = \int_{C^\infty(M)_{\leq \Lambda_0}} \mathcal{D}\varphi e^{-S_{\Lambda_0}[\varphi]/\hbar} \quad (5.2)$$

where the integral is taken over the space $C^\infty(M)_{\leq \Lambda_0}$ of smooth functions on M whose energy is at most Λ_0 . The first thing to note about this integral is that it makes sense:

we've explicitly regularized the theory by declaring that we are only allowing momentum modes up to the cut-off¹⁹ Λ_0 . For example, there can be no UV divergences²⁰ in any perturbative loop integral following from (5.2), because the UV region is simply absent.

Now let's think what happens as we try to perform the path integral by first integrating those modes with energy between Λ_0 and $\Lambda < \Lambda_0$. The space $C^\infty(M)_{\leq \Lambda_0}$ is naturally a vector space with addition just being pointwise addition on M . Therefore we can split a general field $\varphi(x)$ as

$$\begin{aligned}\varphi(x) &= \int_{|p| \leq \Lambda_0} \frac{d^d p}{(2\pi)^d} e^{ip \cdot x} \tilde{\varphi}(p) \\ &= \int_{|p| \leq \Lambda} \frac{d^d p}{(2\pi)^d} e^{ip \cdot x} \tilde{\varphi}(p) + \int_{\Lambda < |p| \leq \Lambda_0} \frac{d^d p}{(2\pi)^d} e^{ip \cdot x} \tilde{\varphi}(p) \\ &=: \phi(x) + \chi(x),\end{aligned}\tag{5.3}$$

where $\phi \in C^\infty(M)_{\leq \Lambda}$ is the low-energy part of the field, while $\chi \in C^\infty(M)_{(\Lambda, \Lambda_0]}$ has high energy. The path integral measure on $C^\infty(M)_{\leq \Lambda_0}$ likewise factorizes as

$$\mathcal{D}\varphi = \mathcal{D}\phi \mathcal{D}\chi$$

into a product of measures over the low- and high-energy modes. Performing the integral over the high-energy modes χ provides us with an *effective action at scale Λ*

$$S_\Lambda^{\text{eff}}[\phi] := -\hbar \log \left[\int_{C^\infty(M)_{(\Lambda, \Lambda_0)}} \mathcal{D}\chi \exp(-S_{\Lambda_0}[\phi + \chi]/\hbar) \right] \tag{5.4}$$

involving the low-energy modes only. We call the process of integrating out modes *changing the scale* of the theory. We can iterate this process, integrating out further modes and obtaining a new effective action

$$S_{\Lambda'}^{\text{eff}}[\phi] := -\hbar \log \left[\int_{C^\infty(M)_{(\Lambda', \Lambda]}} \mathcal{D}\chi \exp(-S_\Lambda^{\text{eff}}[\phi + \chi]/\hbar) \right] \tag{5.5}$$

at a still lower scale $\Lambda' < \Lambda$. For this reason, equation (5.4) is known as the *renormalization group equation* for the effective action.

Separating out the kinetic part, we write the original action as

$$S_{\Lambda_0}[\phi + \chi] = S^0[\phi] + S^0[\chi] + S_{\Lambda_0}^{\text{int}}[\phi, \chi] \tag{5.6}$$

where $S^0[\chi]$ is the kinetic term

$$S^0[\chi] = \int d^d x \left[\frac{1}{2} (\partial \chi)^2 + \frac{1}{2} m^2 \chi^2 \right] \tag{5.7}$$

¹⁹In writing S_{Λ_0} in terms of dimensionless couplings, we used the same energy scale Λ_0 as we chose for the cut-off. This was purely for convenience.

²⁰On a non-compact space-time manifold M there can be IR divergences. This is a separate issue, unrelated to renormalization, that we'll handle later if I get time. If you're worried, think of the theory as living in a large box of side L with either periodic or reflecting boundary conditions on all fields. Momentum is then quantized in units of $2\pi/L$, so the space $C^\infty(M)_{\leq \Lambda_0}$ is finite-dimensional.

for χ and $S^0[\phi]$ is similar. Notice that the quadratic terms can contain no cross-terms $\sim \phi\chi$, because these modes have different support in momentum space. For the same reason, the terms in the effective interaction $S_{\Lambda_0}^{\text{int}}[\phi, \chi]$ must be at least cubic in the fields. Since ϕ is non-dynamical as far as the χ path integral goes, we can bring $S^0[\phi]$ out of the *rhs* of (5.4). Observing that the same ϕ kinetic action already appears on the *lhs*, we obtain ($\hbar = 1$)

$$S_{\Lambda}^{\text{int}}[\phi] = -\log \left[\int_{C^{\infty}(M)_{(\Lambda, \Lambda_0)}} \mathcal{D}\chi \exp(-S^0[\chi] - S_{\Lambda_0}^{\text{int}}[\phi, \chi]) \right] \quad (5.8)$$

which is the renormalization group equation for the effective *interactions*.

5.1.1 Running couplings and their β -functions

It should be clear that the partition function

$$\mathcal{Z}_{\Lambda}(g_i(\Lambda)) = \int_{C^{\infty}(M)_{\leq \Lambda}} \mathcal{D}\varphi e^{-S_{\Lambda}^{\text{eff}}[\varphi]/\hbar} \quad (5.9)$$

obtained from the effective action scale Λ (or at any lower scale) is exactly the same as the partition function we started with:

$$\mathcal{Z}_{\Lambda}(g_i(\Lambda)) = \mathcal{Z}_{\Lambda_0}(g_{i0}; \Lambda_0) \quad (5.10)$$

because we're just performing the remaining integrals over the low-energy modes. In particular, as the scale is lowered infinitesimally (5.10) becomes the differential equation

$$\Lambda \frac{d\mathcal{Z}_{\Lambda}(g)}{d\Lambda} = \left(\Lambda \frac{\partial}{\partial \Lambda} \Big|_{g_i} + \Lambda \frac{\partial g_i(\Lambda)}{\partial \Lambda} \frac{\partial}{\partial g_i} \Big|_{\Lambda} \right) \mathcal{Z}_{\Lambda}(g) = 0. \quad (5.11)$$

Equation (5.11) is known as the renormalization group equation for the partition function, and is our first example of a *Callan–Symanzik equation*. It just says that as we change the scale by integrating out modes, the couplings in the effective action S_{Λ}^{eff} vary to account for the change in the degrees of freedom over which we take the path integral, so that the partition function is in fact independent of the scale at which we define our theory, provided this scale is below our initial cut-off Λ_0 .

The fact that the couplings themselves vary, or ‘run’, as we change the scale is an important notion. As we saw in zero and one dimensions, it’s quite natural to expect the couplings to change as we integrate out modes, changing the degrees of freedom we can access at low scales. However, it seems strange: you’ve learned that the electromagnetic coupling

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}.$$

What can it mean for the fine structure constant to depend on the scale? We’ll understand the answer to such questions later.

With a generic initial action, the effective action will also take the general form

$$S_{\Lambda}^{\text{eff}}[\phi] = \int d^d x \left[\frac{Z_{\Lambda}}{2} \partial^{\mu} \phi \partial_{\mu} \phi + \sum_i \Lambda^{d-d_i} Z_{\Lambda}^{n_i/2} g_i(\Lambda) \mathcal{O}_i(x) \right], \quad (5.12)$$

where the *wavefunction renormalization factor* Z_{Λ} accounts for the fact that it's perfectly possible for the coefficient of the kinetic term itself to receive quantum corrections as we integrate out modes. (Z_{Λ} is not to be confused with the partition function \mathcal{Z}_{Λ} !) At any given scale, we can of course define a *renormalized field*

$$\varphi := Z_{\Lambda}^{1/2} \phi \quad (5.13)$$

in terms of which the kinetic term will be canonically normalized. We've also included a power of $Z_{\Lambda}^{1/2}$ in the definition of our scale Λ couplings so that these powers are removed once one writes the action in terms of the renormalized field.

Since the running of couplings is so important, we give it a special name and define the *beta-function* β_i of the coupling g_i to be its derivative with respect to the logarithm of the scale:

$$\beta_i := \Lambda \frac{\partial g_i}{\partial \Lambda}. \quad (5.14)$$

The β -functions for dimensionless couplings take the form

$$\beta_i(g_j(\Lambda)) = (d_i - d) g_i(\Lambda) + \beta_i^{\text{quant}}(g_j) \quad (5.15)$$

where the first term just compensates the variation of the explicit power of Λ in front of the coupling in (5.12). The second term β_i^{quant} represents the quantum effect of integrating out the high-energy modes. To actually compute this term requires us to perform the path integral and so will generically introduce dependence on all the other couplings in the original action (5.1), so that the β -function for g_i is a function of all the couplings $\beta_i(g_j)$.

Similarly, although at any given scale we can remove the wavefunction renormalization factor, moving to a different scale will generically cause it to re-emerge. We define the *anomalous dimension* γ_{ϕ} of the field ϕ by

$$\gamma_{\phi} := -\frac{1}{2} \Lambda \frac{\partial \ln Z_{\Lambda}}{\partial \Lambda} \quad (5.16)$$

Except for the fact that we've taken the derivative of the logarithm of $Z_{\Lambda}^{1/2}$, this is just the β -function for the coupling in front of the kinetic term. Like any β -function, γ_{ϕ} depends on the values of all the couplings in the theory. It gets its name for reasons that will be apparent momentarily. If our theory contained more than one type of field, then we'd have a wavefunction renormalization factor and anomalous dimension for each field²¹.

²¹In fact, in general we'd have a *matrix* of wavefunction renormalization factors, allowing different fields (of the same quantum numbers such as spin, charge *etc.*) to mix their identities as modes are integrated out.

5.1.2 Anomalous dimensions

Wavefunction renormalization plays an important role in correlation functions. Suppose we wish to compute the n -point correlator

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle := \frac{1}{Z} \int_{C^\infty(M)_{\leq \Lambda}} \mathcal{D}\phi e^{-S_\Lambda^{\text{eff}}[Z_\Lambda^{1/2}\phi; g_i(\Lambda_0)]} \phi(x_1) \cdots \phi_n(x_n) \quad (5.17)$$

of fields inserted at points $x_1, \dots, x_n \in M$ using the scale Λ theory, allowing for the possibility that we hadn't canonically normalized the field in the action. In terms of the canonically normalized field $\varphi := Z_\Lambda^{1/2}\phi$ this is

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = Z_\Lambda^{-n/2} \langle \varphi(x_1) \cdots \varphi(x_n) \rangle \quad (5.18)$$

since the change in the measure $\mathcal{D}\phi \rightarrow \mathcal{D}\varphi$ cancels as we've normalized by the partition function. Upon performing the φ path integral we will (in principle!) evaluate the remaining φ correlator as some function $\Gamma_\Lambda^{(n)}(x_1, \dots, x_n; g_i(\Lambda))$ that depends on the scale Λ couplings and on the fixed points $\{x_i\}$.

Now, if the field insertions just involve modes with energies $\ll \Lambda$ then we should be able to compute the same correlator using just a lower scale theory — the operator insertions will be unaffected as we integrate out modes in the range $(s\Lambda, \Lambda]$ for some fraction $s < 1$. Accounting for wavefunction renormalization gives

$$Z_{s\Lambda}^{-n/2} \Gamma_{s\Lambda}^{(n)}(x_1, \dots, x_n; g_i(s\Lambda)) = Z_\Lambda^{-n/2} \Gamma_\Lambda^{(n)}(x_1, \dots, x_n; g_i(\Lambda)), \quad (5.19)$$

or equivalently

$$\Lambda \frac{d}{d\Lambda} \Gamma_\Lambda^{(n)}(x_1, \dots, x_n; g_i(\Lambda)) = \left(\Lambda \frac{\partial}{\partial \Lambda} + \beta_i \frac{\partial}{\partial g_i} + n\gamma_\phi \right) \Gamma_\Lambda^{(n)}(x_1, \dots, x_n; g_i(\Lambda)) = 0 \quad (5.20)$$

infinitesimally. Equation (5.20) is the generalized Callan–Symanzik equation appropriate for correlation functions. Once again, it simply says that the couplings and wavefunction renormalization factors change as we lower the scale in such a way that correlation functions remain unaltered.

In a Poincaré invariant theory, correlation functions depend the distances between pairs of insertion points, as we saw in section (4.1.1). The typical size of these separations defines a new scale, quite apart from any choice of Λ , and we can use this to obtain an alternative interpretation of renormalization that is often useful. Integrate out modes in the range $(s\Lambda, \Lambda]$ as above, but having done so, let's now change coordinates on our space by $x^\mu \mapsto x'^\mu := sx$. The kinetic term $\int d^d x (\partial\phi)^2$ is invariant under this scaling provided we take the field to transform as

$$\phi(sx) = s^{(2-d)/2} \phi(x). \quad (5.21)$$

The remaining terms in the action are likewise unchanged by the rescaling provided we also rescale $\Lambda \rightarrow \Lambda/s$ in the opposite direction to x (as expected for an energy, rather than length, scale). Thus the energy scale $s\Lambda$ is restored to its original value Λ . It's important

to realize that these scalings have nothing to do with integrating out degrees of freedom in the path integral; they're just scalings.

Under the combined operations we find

$$\begin{aligned}\Gamma_{\Lambda}^{(n)}(x_1, \dots, x_n; g_i(\Lambda)) &= \left[\frac{Z_{\Lambda}}{Z_{s\Lambda}} \right]^{n/2} \Gamma_{s\Lambda}^{(n)}(x_1, \dots, x_n; g_i(s\Lambda)) \\ &= \left[s^{2-d} \frac{Z_{\Lambda}}{Z_{s\Lambda}} \right]^{n/2} \Gamma_{\Lambda}^{(n)}(sx_1, \dots, sx_n; g_i(s\Lambda)),\end{aligned}\tag{5.22}$$

where the first line uses the result (5.19) of integrating out modes, while the second line shows how correlation functions are related under the rescaling. Notice that the couplings g_i and wavefunction renormalization in the final expression are evaluated at the point $s\Lambda$ appropriate for the low-energy theory: the numerical values of these couplings are not affected by our subsequent rescaling.

Equation (5.22) has an important interpretation. First, notice that if we'd *started* with insertions at points x_i/s then we could equivalently write

$$\Gamma_{\Lambda}^{(n)}(x_1/s, \dots, x_n/s; g_i(\Lambda)) = \left[s^{2-d} \frac{Z_{\Lambda}}{Z_{s\Lambda}} \right]^{n/2} \Gamma_{\Lambda}^{(n)}(x_1, \dots, x_n; g_i(s\Lambda)).\tag{5.23}$$

On the left stands a correlation function computed in the theory with couplings $g_i(\Lambda)$ where the separations between operators are $|x_i - x_j|/s$. Thus, as $s \rightarrow 0$ this correlator probes the long distance, or *infra-red* properties of the theory. We see from the *rhs* that such IR correlators may equivalently be obtained by studying a correlation function where all separations are held constant, but we compute using a theory with different values $g_i(s\Lambda)$ for the couplings. This makes perfect sense: the IR properties of the theory are governed by the low-energy modes that survive as we integrate out more and more high-energy degrees of freedom.

This equation also allows us to gain insight into the meaning of the anomalous dimension γ_{ϕ} . The power of $s^{n(2-d)/2}$ on the *rhs* of (5.22) is the classical scaling behaviour we'd expect for an object of mass dimension $n(d-2)/2$. Equation (5.22) shows that the net effect of integrating out high-energy modes is to modify the expected classical scaling by a simple factor depending on the wavefunction renormalization. To quantify this, set $s = 1 - \delta s$ with $0 < \delta s \ll 1$. For each insertion of the field, (5.22) gives a factor

$$\left[s^{2-d} \frac{Z_{\Lambda}}{Z_{s\Lambda}} \right]^{1/2} = 1 + \left[\frac{d-2}{2} + \gamma_{\phi} \right] \delta s + \dots\tag{5.24}$$

with γ_{ϕ} as in (5.16). We see that the correlation function behaves *as if* the field scaled with mass dimension

$$\Delta_{\phi} = (d-2)/2 + \gamma_{\phi}\tag{5.25}$$

rather than the classical value $(d-2)/2$. Δ_{ϕ} is known as the *scaling dimension* of the field ϕ , and the anomalous dimension γ_{ϕ} is the difference between this scaling dimension and the naive classical dimension.

5.2 Renormalization group flow

In this section we'll build up a general understanding of how theories change as we probe them in the infra-red. This conceptual understanding, first developed by Kadanoff and Wilson in the context of condensed matter field theory, will stand us in good stead when we come to renormalize theories perturbatively in later sections. Such calculations are often rather technical — the general picture of the present section will prevent us from getting bogged down in the details.

5.2.1 Renormalization group trajectories

To start to understand what happens under renormalization, let's suppose we start with a theory where all the β -functions vanish. That is, we consider a special action where the initial couplings are tuned to particular values $g_{i0} = g_i^*$ such that $\beta_j|_{g_j=g_i^*} = 0$, so that the couplings for this particular theory in fact do *not* depend on scale. Such a theory is known as a *critical point* of the RG flow. A simple example, called the *Gaussian critical point*, is just free theory where $g_i^* = 0$ for all terms in the action except for the (massless) kinetic term. Clearly, the β -functions all vanish at this Gaussian critical point, since the free theory has no interactions which could be responsible for generating vertices as the cut-off is lowered. However, by tuning the initial couplings very carefully, we may be able to cause non-trivial quantum corrections to cancel precisely the classical rescaling term in (5.15) so that the beta functions vanish. Thus it may be possible, though difficult, to find other critical points beyond the Gaussian one.

The couplings g_i^* being independent of scale has important implications for correlation functions. Firstly, note that since it is a dimensionless function of the other couplings, the anomalous dimension $\gamma_\phi(g_i^*) := \gamma_\phi^*$ is likewise scale independent at a critical point. Then for the two-point correlation function (5.20) becomes

$$\Lambda \frac{\partial \Gamma_\Lambda^{(2)}(x, y)}{\partial \Lambda} = -2\gamma_\phi^* \Gamma_\Lambda^{(2)}(x, y) \quad (5.26)$$

showing that $\Gamma^{(2)}$ is a homogeneous function of the scale. By Lorentz invariance it can depend on the insertion points only through $|x - y|$, and dimensional analysis shows that $\langle \phi(x)\phi(y) \rangle = \Lambda^{d-2} G(\Lambda|x - y|, g_i^*)$ for some function of the dimensionless combination $\Lambda|x - y|$ and the dimensionless couplings g_i^* . Thus, at a critical point the two-point function must take the form

$$\Gamma_\Lambda^{(2)}(x, y; g_i^*) = \frac{\Lambda^{d-2}}{\Lambda^{2\Delta_\phi}} \frac{c(g_i^*)}{|x - y|^{2\Delta_\phi}} \sim \frac{c(g_i^*)}{|x - y|^{2\Delta_\phi}} \quad (5.27)$$

in terms of the scaling dimension

$$\Delta_\phi = (d - 2)/2 + \gamma_\phi \quad (5.28)$$

of ϕ , and where the constant $c(g_i^*)$ is independent of the insertion points. This power-law behaviour of correlation functions is characteristic of scale-invariant theories. In a theory where the interactions between ϕ insertions was due to some massive state traveling

from x to y , we'd expect the potential to decay as $e^{-m|x-y|}/|x-y|$ where m is the mass of the intermediate state. As in electromagnetism, the pure power-law we have found for this correlator is a sign that our states are massless, so that their effects are long-range.

Critical theories are certainly very special. The metric appears in the action, so changing the metric leads to a change in the partition function given by

$$\delta g^{\mu\nu}(x) \frac{\delta}{\delta g^{\mu\nu}(x)} \ln \mathcal{Z} = - \left\langle \frac{\delta S}{\delta g^{\mu\nu}(x)} \right\rangle = -\delta g^{\mu\nu}(x) \langle T_{\mu\nu}(x) \rangle, \quad (5.29)$$

the expectation value of the stress tensor $T_{\mu\nu}$. If the metric transformation is just a scale transformation then $\delta g^{\mu\nu} \propto g^{\mu\nu}$, so scale invariance of a theory at a critical point g_i^* implies that $\langle T_{\mu}^{\mu} \rangle = 0$. In fact, all known examples of Lorentz-invariant, unitary QFTs that are scale invariant are actually invariant under the larger group of conformal transformations and it's believed that all critical points of RG flows are CFTs²².

Now let's consider the behaviour of theories near to, but not at, a critical point. Since by definition the β -functions vanish when $g_i = g_i^*$, nearby we must have

$$\Lambda \frac{\partial g_i}{\partial \Lambda} \Big|_{g_j^* + \delta g_j} = B_{ij} \delta g_j + \mathcal{O}(\delta g^2) \quad (5.30)$$

where $\delta g_i = g_i - g_i^*$, and where B_{ij} is a constant (infinite dimensional!) matrix. Let σ_i be an eigenvector of B_{ij} , and let its eigenvalue be $\Delta_i - d$. Classically, we expected a dimensionless coupling to scale with a power of Λ determined by the explicit powers of Λ included in the action in (5.1), so that we'd have $\Delta_i = d_i$ classically. Just as for the correlation function in (5.20), the net effect of integrating out degrees of freedom is to modify this scaling so that near a critical point, the couplings really scale with a power of Λ determined by the eigenvalues of the linearized β -function matrix B_{ij} . The difference

$$\gamma_i := \Delta_i - d_i \quad (5.31)$$

is called the *anomalous dimension of the operator*, mimicking the anomalous dimension γ_ϕ of the field itself, while the quantity Δ_i itself is called the *scaling dimension* of the operator. If the quantum corrections vanished then the scaling dimension would coincide with the naive mass dimension of an operator obtained by counting the powers of fields and derivatives it contains.

Since σ_i is an eigenvector of B

$$\Lambda \frac{\partial \sigma_i}{\partial \Lambda} = (\Delta_i - d) \sigma_i + \mathcal{O}(\sigma^2) \quad (5.32)$$

and so the RG flow for σ_i is

$$\sigma_i(\Lambda) = \left(\frac{\Lambda}{\Lambda_0} \right)^{\Delta_i - d} \sigma_i(\Lambda_0) \quad (5.33)$$

at least to this order in the perturbation away from the critical point.

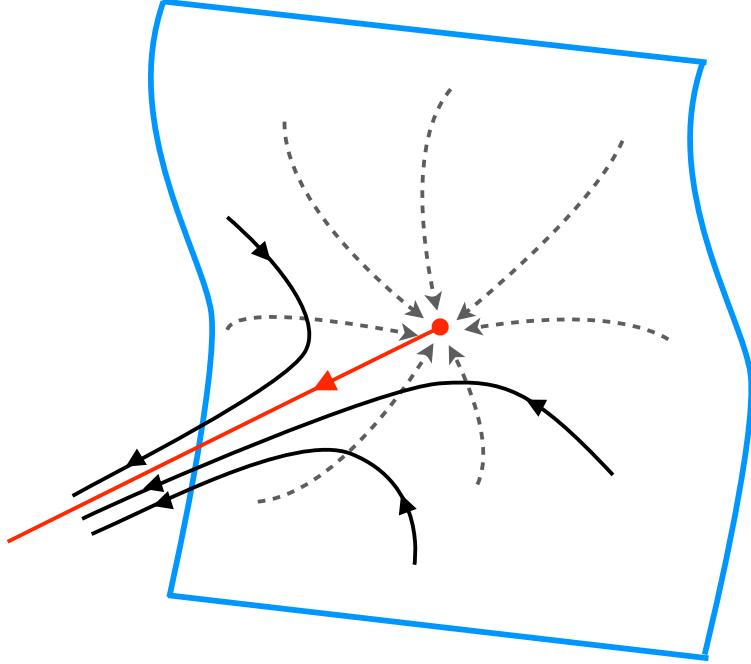


Figure 5: Theories on the critical surface flow (dashed lines) to a critical point in the IR. Turning on relevant operators drives the theory away from the critical surface (solid lines), with flow lines focussing on the (red) trajectory emanating from the critical point.

Now consider starting near a critical point and turning on the coupling to any operator with $\Delta_i > d$. According to (5.33) this coupling becomes smaller as the scale Λ is lowered, or as we probe the theory in the IR. We say that the corresponding operator is *irrelevant* since if we include it in the action then RG flow just makes us flow back to the critical point g_i^* . Classically, we can obtain operators with arbitrarily high mass dimension by including more and more fields and derivatives, so we expect that the critical point g_i^* sits on an infinite dimensional surface \mathcal{C} such that if we turn on any combination of operators that move us along \mathcal{C} , under RG flow we will end up back at the critical point. \mathcal{C} is known as the *critical surface* and we can think of the couplings of irrelevant operators as provided coordinates on \mathcal{C} , at least in the neighbourhood of g_i^* . (See figure 5.)

On the other hand, couplings with $\Delta_i < d$ grow as the scale is lowered and so are called *relevant*. If our action contains vertices with relevant couplings then RG flow will drive us away from the critical surface \mathcal{C} as we head into the IR. Starting precisely from a critical point and turning on a relevant operator generates what is known as a *renormalized trajectory*: the RG flow emanating from the critical point. As we probe the theory at lower and lower scales we evolve along the renormalized trajectory either forever or until we eventually meet another²² critical point g_i^{**} . Since each new field or derivative adds to the dimension of an operator, in fixed space-time dimension d there will be only finitely many

²²It's a theorem that this is always true in two dimensions. It is believed to be true also in higher dimensions, but the question is actually a current hot topic of research.

²³There are a few exotic examples where the theories flow to a limiting cycle rather than a fixed point.

(and typically only few) relevant operators, so the critical surface has finite codimension.

The remaining possibility is *marginal operators*, which have vanishing eigenvalues and so neither increase nor decrease under RG flow. At the Gaussian point, the scaling dimensions of operators are just given by their classical mass dimension, so we expect marginal operators to have scaling dimension $\Delta_i = d$. Near a critical point, quantum corrections can bring in a weak (typically logarithmic) dependence on scale to a classically marginal operator, making it either *marginally relevant* or *marginally irrelevant*. Provided the non-zero eigenvalues of these operators are sufficiently small, the size of such nearly marginal couplings can be unchanged for long periods of RG flow — although ultimately they will either be irrelevant or relevant. Such operators play an important role phenomenologically, as we will see.

A generic QFT will have an action that involves all types of operators and so lies somewhere off the critical surface. Under RG evolution, all the many irrelevant operators are quickly suppressed, while the relevant ones grow just as before. The flow lines of a generic theory thus strongly focus onto the renormalized trajectory, and so in the IR a generic QFT will closely resemble a theory emanating from the critical point, where only relevant operators have been turned on. The fact that many different high energy theories will flow to look the same in the IR is known as *universality*. It assures us that the properties of the theory in the IR are determined not by the infinite set of couplings $\{g_i\}$, but only by the couplings to a few relevant operators. We say that theories whose RG flows are all focussed onto the same trajectory emanating from a given critical point are in the same *universality class*. Theories in a given universality class could look very different microscopically, but will all end up looking the same at large distances. In particular, deep in the IR, these theories will all flow to the second critical point g_i^{**} . *This is the reason you can do physics!* To study a problem at a given energy scale you don't first need to worry about what the degrees of freedom at much higher energies are doing. They are, quite literally, irrelevant.

Let me emphasize that eigenvectors σ_i are generically *linear combinations* of the naive couplings in the action. Thus, turning on σ_i means we perturb away from the critical point by changing the couplings in front of the corresponding linear combination of our operators in the action. These RG ‘eigenoperators’ may be very different from any individual monomial in the fields you choose to include neatly in the effective interaction S^{eff} . A simple-looking individual operator \mathcal{O}_i that appears in (5.1) or is explicitly inserted into a correlation function could actually consist of many RG eigenfunctions. We say that operators *mix*, because a given operator transforms under RG flow into its dominant eigenfunction, which could look very much more complicated.

5.2.2 Counterterms and the continuum limit

So far, we've considered a *fixed* initial theory $S_{\Lambda_0}[\phi]$ with initial couplings g_{i0} , and examined how these couplings change as we probe the theory at long distances. Our definition of the

Dimension	Relevant operators	Marginal operators
$d = 2$	ϕ^{2k} for all $k \geq 0$	$(\partial\phi)^2, \phi^{2k}(\partial\phi)^2$ for all $k \geq 0$
$d = 3$	ϕ^{2k} for $k = 1, 2$	$(\partial\phi)^2, \phi^6$
$d = 4$	ϕ^2 for ≤ 3	$(\partial\phi)^2, \phi^4$
$d > 4$	ϕ^2 for $0 \leq k \leq 3$	$(\partial\phi)^2$

Table 1: *Relevant & marginal operators in a Lorentz invariant theory of a single scalar field in various dimensions, near the Gaussian critical point where the classical dimensions of operators are a good guide. Only the operators invariant under $\phi \rightarrow -\phi$ are shown. Note that the kinetic term $(\partial\phi)^2$ is always marginal, and the mass term ϕ^2 is always relevant.*

low-energy effective action as

$$S_{\Lambda}^{\text{eff}}[\phi] := -\hbar \log \left[\int_{C^{\infty}(M)_{(\Lambda, \Lambda_0)}} \mathcal{D}\chi \exp(-S_{\Lambda_0}[\phi + \chi]/\hbar) \right] \quad (5.34)$$

ensured that the partition function and correlation functions of low-energy observables were independent of the scale Λ . The question remains: what about dependence on the initial cut-off Λ_0 ? In this section we'll examine this by asking a sort of converse: Suppose we fix a particular low-energy theory (perhaps motivated by the results of some finite-scale experiments). How can we remove the high-energy cut-off, sending $\Lambda_0 \rightarrow \infty$, without affecting what the theory predicts for low-energy phenomena. We call this *taking the continuum limit* of our theory, since sending $\Lambda_0 \rightarrow \infty$ is allowing the field to fluctuate on arbitrarily small scales.

The key to achieving this lies with the universality of the renormalization group flow. First, suppose our initial couplings g_{i0} happen to lie on the critical surface \mathcal{C} , within the domain of attraction of g_i^* . Then as we raise the cut-off Λ_0 , the theory we obtain at any fixed scale Λ will be driven to the critical point g_i^* as all the irrelevant operators become arbitrarily suppressed by positive powers of Λ/Λ_0 . The critical point is a fixed point of the RG flow and is scale invariant, so we can happily send $\Lambda_0 \rightarrow \infty$. More precisely, whenever the theory S_{Λ_0} lives on the critical surface, the limit

$$\lim_{\Lambda_0 \rightarrow \infty} \left[\int_{C^{\infty}(M)_{(\Lambda, \Lambda_0)}} \mathcal{D}\chi \exp(-S_{\Lambda_0}[\phi + \chi]/\hbar) \right], \quad (5.35)$$

exists, provided we take this limit *after* computing the path integral. The resulting scale- Λ effective theory will be a CFT, independent of Λ . Since \mathcal{C} has only finite codimension, we only have to tune finitely many coefficients (those of all the relevant operators) in order to ensure that $g_{i0} \in \mathcal{C}$.

Theories such as Yang-Mills or QCD are *not* CFTs, but rather have relevant (and marginally relevant) operators turned on in their actions. How then can we understand the continuum limit of such theories? Consider a theory whose initial conditions are near, but not on \mathcal{C} . Universality of the RG flow shows that as we head into the IR, such a theory flows towards the critical point g_i^* for a while, but eventually diverges away, focussing on a

renormalized trajectory as in figure 5. Let μ denote the energy scale at which this theory passes closest to g_i^* . Since RG flow is determined by the initial conditions, μ depends only on the theory we started with. On dimensional grounds we must have

$$\mu = \Lambda_0 f(g_{i0}) \quad (5.36)$$

where $f(g_{i0})$ is some function of the dimensionless couplings $\{g_{i0}\}$ and $f = 0$ on \mathcal{C} , since all these theories flow to g_i^* exactly. To obtain a theory with relevant or marginal operators, we tune the initial couplings $\{g_{i0}\}$ so that μ remains *finite* as we take $\Lambda_0 \rightarrow \infty$. If $\text{codim}(\mathcal{C}) = r$ then this is one condition on r parameters — the coefficients of the relevant operators in the initial action. The theory we end up with thus depends on $(r - 1)$ parameters, together with the scale μ .

We achieve this tuning by introducing new *counterterms* $S_{\text{CT}}[\varphi, \Lambda_0]$ that depend on the fields ϕ as well as explicitly on the cut-off Λ_0 , modifying the initial action to

$$S_{\Lambda_0}[\varphi] \rightarrow S_{\Lambda_0}[\varphi] + S_{\text{CT}}[\varphi, \Lambda_0]. \quad (5.37)$$

The effective actions we considered before already contained all possible monomials in fields and their derivatives, so in this sense the counterterms add nothing new. However, the values of the counterterm couplings are to be chosen by hand — varying these couplings thus *changes* which initial high-energy theory we're considering, as opposed to running a set of couplings under RG flow, which just describes how the *same* theory appears at different scales. The counterterms are tuned so that the limit

$$e^{-S_{\Lambda}^{\text{eff}}[\phi]/\hbar} = \lim_{\Lambda_0 \rightarrow \infty} \left[\int_{C^\infty(M)_{(\Lambda, \Lambda_0)}} \mathcal{D}\chi \exp \left(-\frac{S[\phi + \chi]}{\hbar} - S_{\text{CT}}[\phi + \chi, \Lambda_0] \right) \right] \quad (5.38)$$

exists. Notice again that the limit is taken *after* performing the path integral. Sending $\Lambda_0 \rightarrow \infty$ defines a continuum QFT with finite (or *renormalized*) relevant couplings at scale Λ .

The reason for making S_{CT} explicit, rather than just treating the counterterms as a modification $\{g_{i0}\}$, is that in practice we work perturbatively. To evaluate the path integral in (5.38), we first compute quantum corrections to 1-loop order using the original action S . These 1-loop corrections will depend on the cut-off Λ_0 , and will be proportional to \hbar . In general, they will diverge as $\Lambda_0 \rightarrow \infty$ reflecting the fact that we lose control of the original theory if the cut-off is removed naïvely. However, vertices in S_{CT} provide further contributions to these quantum corrections. By tuning the values of the couplings in S_{CT} by hand, we can obtain a finite limit. Notice that S_{CT} comes with one extra power of \hbar in (5.38) compared to the original action. Thus, quantum corrections to the effective action arising from 1-loop diagrams of the original action should be matched by the *tree-level* contributions from S_{CT} . We'll get plenty of practice in doing this in the following sections.

There's one further possibility to consider. Suppose that to explain some experimental result, be it the scattering of pions and nucleons or the falling of apples, we need our

low-energy theory to contain irrelevant operators. If we really try to take the cut-off $\Lambda_0 \rightarrow \infty$, such operators will be arbitrarily suppressed at any finite energy scale. So their presence indicates that our theory cannot be valid up to arbitrarily high energies; there must be a finite energy scale at which new physics comes in to play. In the case of pion–nucleon scattering, this scale is ~ 217 MeV and indicates the presence of quarks, gluons and the whole structure of QCD. For radioactive β -decay, the scale is ~ 250 GeV and indicated the electroweak theory, while for gravity the scale is $\sim 10^{19}$ GeV, where probably the whole notion of QFT itself must give way. Perhaps most interesting of all are *marginally* irrelevant operators, like the quartic coupling (Φ^4) of the Higgs in the Standard Model. Strictly speaking, just like irrelevant operators, marginally irrelevant operators are arbitrarily suppressed as the cut-off is removed. However, they typically decay only logarithmically as Λ_0 is raised, rather than as a power law. Such operators thus afford us a tiny glimpse of new physics at exponentially high energy scales, far beyond the range of current accelerators.

5.3 Calculating RG evolution

It's time to think about how to calculate the quantum corrections to β -functions generated as we integrate out high energy modes. I'll this section I want to explain this in a way that I think is conceptually clear, and the natural generalization of what we have already seen in zero and one dimension. However, I'll warn you in advance that the techniques here are not the most convenient way to calculate β -functions.

5.3.1 Polchinski's equation

In perturbation theory, the *rhs* of (5.8) may be expanded as an infinite series of connected Feynman diagrams. If we wish to compute the low-energy effective interaction $S_\Lambda^{\text{int}}[\phi]$ as an integral over space–time in the usual way, then we should use the position space Feynman rules. As in section 3.4, the position space propagator $D^{(\chi)}(x, y)$ for the high-energy field χ is

$$D^{(\chi)}(x, y) = \int_{\Lambda < |p| \leq \Lambda_0} \frac{d^d p}{(2\pi)^d} \frac{e^{ip \cdot (x-y)}}{p^2 + m^2} \quad (5.39)$$

where we note the restriction to momenta in the range $\Lambda < |p| \leq \Lambda_0$. As usual, vertices from the high-energy action $S_{\Lambda_0}^{\text{int}}[\phi, \chi]$ come with an integration $\int d^d x$ over their location that imposes momentum conservation at the vertex. Now, diagrams that exclusively involve vertices which are independent of ϕ contribute just to a field-independent term on the *lhs* of (5.8). This term represents the shift in vacuum energy due to integrating out the χ field; we will henceforth ignore it²⁴. The remaining diagrams use vertices including at least one ϕ field, treated as external. Evaluating such a diagram leads to a contribution to the effective interaction $S_\Lambda^{\text{int}}[\phi]$ at scale Λ .

For general scales Λ and Λ_0 equation (5.8) is extremely difficult to handle; the integral on the right is a full path integral in an interacting theory. To make progress we consider

²⁴This is harmless in a non-gravitational theory, but is really the start of the **cosmological constant problem**.

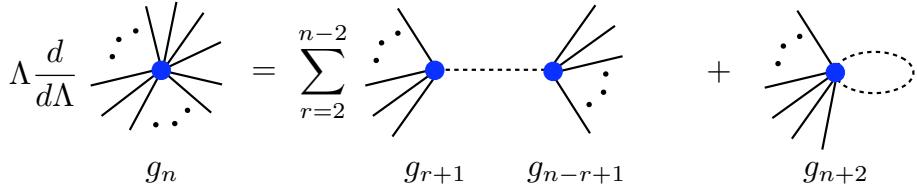


Figure 6: A schematic representation of the renormalization group equation for the effective interactions when the scale is lowered infinitesimally. Here the dashed line is a propagator of the mode with energy Λ that is being integrated out, while the external lines represent the number of low-energy fields at each vertex. All these external fields are evaluated at the same point x . The total number of fields attached to a vertex is indicated by the subscripts on the couplings g_i .

the case that we lower the scale only infinitesimally, setting $\Lambda = \Lambda_0 - \delta\Lambda$. To lowest order in $\delta\Lambda$, the χ propagator reduces to²⁵

$$D_\Lambda^{(\chi)}(x, y) = \frac{1}{(2\pi)^d} \frac{\Lambda^{d-1} \delta\Lambda}{\Lambda^2 + m^2} \int_{S^{d-1}} d\Omega e^{i\Lambda \hat{p} \cdot (x-y)} \quad (5.40)$$

as the range of momenta shrinks down, where $d\Omega$ denotes an integral over a unit $(d-1)$ -sphere in momentum space. This is a huge simplification! Since every χ propagator comes with a factor of $\delta\Lambda$, to lowest order in $\delta\Lambda$ we need only consider diagrams with a single χ propagator. Since ϕ is treated as an external field, we have only two possible classes of diagram: either the χ propagator links together two separate vertices in $S^{\text{int}}[\phi, \chi]$ or else it joins a single vertex to itself.

This diagrammatic representation of the process of integrating out degrees of freedom is shown in figure 6. It has a very clear intuitive meaning. The mode χ appearing in the propagator is the highest energy mode in the original scale Λ_0 theory. It thus probes the shortest distances we can reliably access using $S_{\Lambda_0}^{\text{eff}}$. When we integrate this mode out, we can no longer resolve distances $1/\Lambda_0$ and our view of the ‘local’ interaction vertices is correspondingly blurred. The graphs on the *rhs* of figure 6 represent new contributions to the n -point ϕ vertex in the lower scale theory coming respectively from two nearby vertices joined by a χ field, or a higher point vertex with a χ loop attached. Below scale Λ_0 we image that we are unable to resolve the short distance χ propagator.

We can write an equation for the change in the effective action that captures the information in the Feynman diagrams in figure 6. It was obtained by Polchinski²⁶, and is really just the infinitesimal version of Wilson’s renormalization group equation (5.4) for the effective action. Polchinski’s equation is

$$-\Lambda \frac{\partial S_\Lambda^{\text{int}}[\phi]}{\partial \Lambda} = \int d^d x d^d y \left[\frac{\delta S_\Lambda^{\text{int}}}{\delta \phi(x)} D_\Lambda(x, y) \frac{\delta S_\Lambda^{\text{int}}}{\delta \phi(y)} - D_\Lambda(x, y) \frac{\delta^2 S_\Lambda^{\text{int}}}{\delta \phi(x) \delta \phi(y)} \right], \quad (5.41)$$

²⁵To lowest order, it doesn’t matter whether we use Λ_0 or Λ in this expression.

²⁶Polchinski actually wrote a slightly more general version of the momentum space version of this equation, including source terms $\int J\phi$ in the effective action.

where $D_\Lambda(x, y)$ is the propagator (5.40) for the mode at energy Λ that is being integrated out. The variations of the effective interactions tell us how this propagator joins up the various vertices. Notice in the second term that since $S^{\text{int}}[\phi]$ is local, both the $\delta/\delta\phi$ variations must act at the same place if we are to get a non-zero result. On the other hand, the first term generates non-local contributions to the effective action since it links fields at x to fields at a different point y . In position space we expect a propagator at scale $\Lambda^2 + m^2$ to lead to a potential $\sim e^{-\sqrt{\Lambda^2+m^2}r}/r^{d-3}$ so this non-locality is mild and we can expand the fields in $\delta S^{\text{int}}/\delta\phi(y)$ as a series in $(x - y)$. This leads to new contributions to interactions involving derivatives of the fields, just as we saw in section 3.3 in one dimension. Finally, the minus signs in (5.41) comes from expanding $e^{-S^{\text{int}}[\phi]}$ to obtain the Feynman diagrams.

It's convenient to rewrite Polchinski's equation (5.41) as

$$\frac{\partial}{\partial t} e^{-S^{\text{int}}[\phi]} = - \int d^d x d^d y D_\Lambda(x, y) \frac{\delta^2}{\delta\phi(x) \delta\phi(y)} e^{-S^{\text{int}}[\phi]}, \quad (5.42)$$

in which form it reveals itself as a form of heat equation, with renormalization group ‘time’²⁷ $t \equiv \ln \Lambda$ and ‘Laplacian’

$$\Delta = \int d^d x d^d y D_\Lambda(x, y) \frac{\delta^2}{\delta\phi(x) \delta\phi(y)} \quad (5.43)$$

on the space of fields. Heat flow on a Riemannian manifold N is a strongly *smoothing* operation: if we expand a function $f : N \times \mathbb{R}_{>0} \rightarrow \mathbb{R}$ as

$$f(x, t) = \sum_k \tilde{f}_k(t) u_k(x)$$

in terms of a basis of eigenfunctions $u_k(x)$ of the Laplacian on N , then under heat flow the coefficients evolve as $\tilde{f}_k(t) = \tilde{f}_k(0) e^{-\lambda_k t}$. Consequently, all components $\tilde{f}_k(t)$ corresponding to positive eigenvalues λ_k are quickly damped away, with only the constant piece (with zero eigenvalue) surviving. This just corresponds to the well-known fact that a heat spreads out from areas of high concentration (such as a flame) until the whole room is at constant temperature. On a manifold with a pseudo-Riemannian (rather than Riemannian) manifold, some eigenvalues can be negative. These functions would then be enhanced under heat flow. Exactly the same thing happens under RG flow. Eigenfunctions of the Laplacian in (5.42) are combinations of operators in the effective interactions. Depending on the sign of their corresponding eigenvalues, these operators will be either enhanced or suppressed under the flow.

5.3.2 The local potential approximation

Polchinski's equation contains exact information about the behaviour of every possible operator under RG flow. Unfortunately, while it's structurally simple, actually solving this equation as it stands is prohibitively difficult, so we seek a more manageable approximation.

²⁷In the AdS/CFT correspondence, this RG time really does turn into an honest direction: into the bulk of anti-de Sitter space!

To obtain one, observe that except for the kinetic term, operators involving derivatives are irrelevant²⁸ whenever $d > 2$. This suggests that we can restrict attention to actions of the form

$$S_{\Lambda}^{\text{eff}}[\varphi] = \int d^d x \left[\frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi + V(\varphi) \right] \quad (5.44)$$

where the potential

$$V(\varphi) = \sum_k \Lambda^{d-k(d-2)} \frac{g_{2k}}{(2k)!} \varphi^{2k} \quad (5.45)$$

does not involve derivatives of ϕ . For simplicity, we've chosen $V(-\varphi) = V(\varphi)$, while the couplings g_{2k} are dimensionless as before. Neglecting the derivative interactions is known as the *local potential approximation*; it's important because it will tell us the shape of the effective potential experienced by a slowly varying field. Splitting the field $\varphi = \phi + \chi$ into its low- and high-energy modes as before, we expand the action as an infinite series

$$S_{\Lambda}^{\text{eff}}[\phi + \chi] = S_{\Lambda}^{\text{eff}}[\phi] + \int d^d x \left[\frac{1}{2} (\partial \chi)^2 + \frac{1}{2} \chi^2 V''(\phi) + \frac{1}{3!} \chi^3 V'''(\phi) + \dots \right]. \quad (5.46)$$

Notice that we have chosen a definition of ϕ so that it sits at a minimum of the potential, $V'(\phi) = 0$. This can always be arranged by adding a constant to ϕ , which is certainly a low-energy mode.

Now consider integrating out the high-energy modes χ . As before, we lower the scale infinitesimally, setting $\Lambda' = \Lambda - \delta\Lambda$ and working just to first order in $\delta\Lambda$. In any given Feynman graph, each χ loop comes with an integral of the form

$$\int_{\Lambda - \delta\Lambda < |p| \leq \Lambda} \frac{d^d p}{(2\pi)^d} (\dots) = \delta\Lambda \frac{\Lambda^{d-1}}{(2\pi)^d} \int_{S^{d-1}} d\Omega (\dots)$$

where $d\Omega$ denotes an integral over a unit $S^{d-1} \subset \mathbb{R}^d$ and (\dots) represents the propagators and vertex factors involved in this graph. As with Polchinski's equation, since each loop integral comes with a factor of $\delta\Lambda$, to lowest non-trivial order in $\delta\Lambda$ we need consider at most 1-loop diagrams for χ .

Suppose a particular graph involves a number v_i vertices containing i powers of χ and arbitrary powers of ϕ . Euler's identity tells us that a connected graph with e edges and ℓ loops obeys

$$e - \sum_i v_i = \ell - 1, \quad (5.47)$$

Since we're only integrating over the high scales modes, χ is the only propagating field. Furthermore, since we're integrating out χ completely, there are no external χ lines. Thus we also have the identity

$$2e = \sum_i i v_i \quad (5.48)$$

since every χ propagator is emitted and absorbed at some (not necessarily distinct) vertex. Eliminating e from (5.47) gives

$$\ell = 1 + \sum_i \frac{i-2}{2} v_i. \quad (5.49)$$

²⁸At least near the Gaussian critical point where classical scaling dimensions are a reasonable guide.

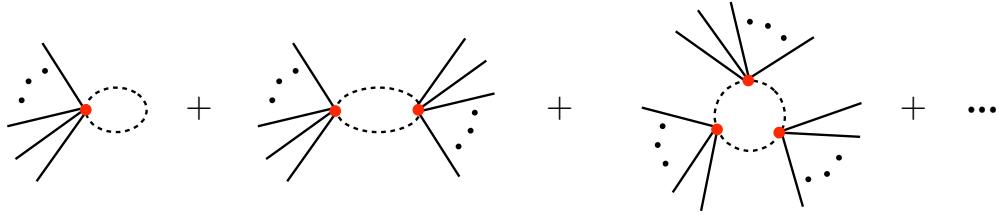


Figure 7: Diagrams contributing in the local potential approximation to RG flow. The dashed line represents a χ propagator with $|p| = \Lambda$, while the solid lines represent external ϕ fields. All vertices are quadratic in χ .

We only want to keep track of 1-loop diagrams, so we see that only the vertices with $i = 2$ χ lines (and arbitrary numbers of ϕ lines) are important. We can thus truncate the difference $S_{\Lambda}^{\text{eff}}[\phi + \chi] - S_{\Lambda}^{\text{eff}}[\phi]$ in (5.46) to

$$S^{(2)}[\chi] = \int d^d x \left[\frac{1}{2} \partial \chi^2 + \frac{1}{2} V''(\phi) \chi^2 \right] \quad (5.50)$$

so that χ appears only quadratically.

The diagrams that can be constructed from this action are shown in figure 7. If we make the temporary assumption that the low-energy field ϕ is actually constant, then in momentum space the quadratic action $S^{(2)}$ becomes

$$\begin{aligned} S^{(2)}[\chi] &= \int_{\Lambda - \delta \Lambda < |p| \leq \Lambda} \frac{d^d p}{(2\pi)^d} \tilde{\chi}(p) \left[\frac{1}{2} p^2 + \frac{1}{2} V''(\phi) \right] \tilde{\chi}(-p) \\ &= \frac{\Lambda^{d-1} \delta \Lambda}{2(2\pi)^d} (\Lambda^2 + V''(\phi)) \int_{S^{d-1}} d\Omega \tilde{\chi}(\Lambda \hat{p}) \tilde{\chi}(-\Lambda \hat{p}) \end{aligned} \quad (5.51)$$

using the fact that these modes have energies in a narrow shell of width $\delta \Lambda$.

Performing the path integral over χ is now straightforward. If the narrow shell contains N momentum modes, then from standard Gaussian integration

$$e^{-\delta \Lambda S_{\Lambda}^{\text{eff}}[\phi]} = \int \mathcal{D}\chi e^{-S_2[\chi, \phi]} = C \left(\frac{\pi}{\Lambda^2 + V''(\phi)} \right)^{N/2}. \quad (5.52)$$

On a non-compact manifold, N is actually infinite. To regularize it, we place our theory in a box of linear size L and impose periodic boundary conditions. The momentum is then quantized as $p_\mu = 2\pi n_\mu / L$ for $n_\mu \in \mathbb{Z}$ so that there is one mode per $(2\pi)^d$ volume in Euclidean space-time itself is L^d . Thus

$$N = \frac{\text{Vol}(S^{d-1})}{(2\pi)^d} \Lambda^{d-1} \delta \Lambda L^d \quad (5.53)$$

which diverges as the volume L^d of space-time becomes infinite. However, we can obtain a (correct) finite answer once we recognize that the cause of this divergence was ou simplifying

assumption that ϕ was constant. For spatially varying ϕ , we would instead obtain

$$\delta_\Lambda S^{\text{eff}}[\phi] = a \Lambda^{d-1} \delta\Lambda \int d^d x \ln [\Lambda^2 + V''(\phi)] \quad (5.54)$$

where the factor of $L^d \times \ln[\Lambda^2 + V''(\phi)]$ in (5.52) has been replaced by an integral over M . The constant

$$a := \frac{\text{Vol}(S^{d-1})}{2(2\pi)^d} = \frac{1}{(4\pi)^{d/2} \Gamma(d/2)} \quad (5.55)$$

is proportional to the surface area of a $(d-1)$ -dimensional unit sphere. Expanding the *rhs* of (5.54) in powers of ϕ leads to a further infinite series of ϕ vertices which combine with those present at the classical level in $V(\phi)$. Once again, integrating out the high-energy field χ has lead to a modification of the couplings in this potential.

We're now in position to write down the β -functions. Including the contribution from both the classical action and the quantum correction (5.54), the β -function for the ϕ^{2k} coupling is

$$\Lambda \frac{dg_{2k}}{d\Lambda} = [k(d-2) - d]g_{2k} - a\Lambda^{k(d-2)} \left. \frac{\partial^{2k}}{\partial \phi^{2k}} \ln [\Lambda^2 + V''(\phi)] \right|_{\phi=0}. \quad (5.56)$$

For instance, the first few terms in this expansion give

$$\begin{aligned} \Lambda \frac{dg_2}{d\Lambda} &= -2g_2 - \frac{ag_4}{1+g_2} \\ \Lambda \frac{dg_4}{d\Lambda} &= (d-4)g_4 - \frac{ag_6}{1+g_2} + \frac{3ag_4^2}{(1+g_2)^2} \\ \Lambda \frac{dg_8}{d\Lambda} &= (2d-6)g_6 - \frac{ag_8}{1+g_2} + \frac{15ag_4g_6}{(1+g_2)^2} - \frac{30ag_4^3}{(1+g_2)^3} \end{aligned} \quad (5.57)$$

as β -functions for the mass term, ϕ^4 and ϕ^6 vertices.

There are several things worth noticing about the expressions in (5.57). Firstly, each term on the right comes from a particular class of Feynman graph; the first term is the scaling behaviour of the classical ϕ^{2k} vertex, the second term involves a single χ propagator with both ends joined to the same valence $2k+2$ vertex, the third (when present) involves a pair of χ propagators joining two vertices of total valence $2k+4$, etc.. Secondly, we note that these Feynman diagrams are different to the ones that appeared in (5.41). By taking the local potential approximation, we have neglected any possible derivative terms that may have contributed to the running of the couplings in $V(\phi)$. The effect of this is seen in the higher-order terms that appear on the *rhs* of (5.57). From the point of view of the Wilson–Polchinski renormalization group equation, the local potential approximation effectively amounts to *solving* the β -function equations that follow from (5.41), writing the derivative couplings in terms of the non-derivative ones, and then substituting these back into the remaining β -functions for non-derivative couplings to obtain (5.57). The message is that the price to be paid for ignoring possible couplings in the effective action is more complicated β -functions. We will see this again in chapter ??, where β -functions will no longer be determined purely at one loop.

Finally, recall that $g_2 = m^2/\Lambda^2$ is the mass of the ϕ field in units of the cut-off. If this mass is very large, so $g_2 \gg 1$, then the quantum corrections to the β -functions in (5.57) are strongly suppressed. As for correlation functions near to, but not at, a critical point, this is as we would expect. A particle of mass m leads to a potential $V(r) \sim e^{-mr}/r^{d-3}$ in position space, so should not affect physics on scales $r \gg m^{-1}$.

5.3.3 The Gaussian critical point

From our discussion in section 5.2, we expect that the limiting values of the couplings in the deep IR will be a critical point of the RG evolution (5.56). The simplest type of critical point is the Gaussian fixed-point where $g_{2k} = 0 \forall k > 1$, corresponding to a free theory. Every one of the Feynman diagrams shown on the right of the Wilson renormalization group equation in figure 6 involves a vertex containing at least three fields (either χ or ϕ), so if we start from a theory where the couplings to each of these vertices are precisely set to zero, then no interactions can ever be generated. Indeed, (5.57) shows that the Gaussian point is indeed a fixed-point of the RG flow, with the mass term β -function $\beta_2 = -2g_2$ simply compensating for the scaling of the explicit power of Λ introduced to make the coupling dimensionless.

Last term you used perturbation theory to study ϕ^4 theory in four dimensions. Using perturbation theory means that you considered this theory in the neighbourhood of the Gaussian critical point so that the couplings could be treated as ‘small’. Let’s examine this again using our improved understanding of RG flow. Firstly, to find the behaviour of any coupling near to the free theory, as in equation (5.30) we should linearize the β -functions around the critical point. We’ll use our results (5.57) for a theory with an arbitrary polynomial potential $V(\phi)$. To linear order in the couplings, only the first two terms on the *rhs* of (5.57) contribute, giving

$$\beta_{2k} = \Lambda \frac{\partial g_{2k}}{\partial \Lambda} = (k(d-2) - d) g_{2k} - a g_{2k+2} \quad (5.58)$$

where $\delta g_{2k} = g_{2k} - g_{2k}^* = g_{2k}$ since $g_{2k}^* = 0$ for the Gaussian critical point. Writing the linearized β -functions in the form $\beta_i = B_{ij}g_j$ we see that the matrix B_{ij} is upper triangular, so its eigenvalues are simply its diagonal entries $k(d-2) - d$. In four dimensions, these eigenvalues are $2k - 4$, which is positive for $k \geq 3$. Thus, in four dimensions, deforming the free action by an operator of the form ϕ^{2k} with $k \geq 3$ is an *irrelevant* perturbation: turning on any such operator takes us away from the free theory in a direction along the critical surface, and we are pushed back to the free theory as the cut-off is lowered.

On the other hand, the mass term g_2 is a *relevant* deformation of the free action. Turning on even arbitrarily small masses will lead us away from the massless theory as the cut-off is lowered. Of course, once g_2 is large we cannot trust our linearized approximation (5.58), and the correct result (5.57) shows that the quantum corrections to g_2 are eventually suppressed as the mass becomes large in units of the cut-off.

The remaining coupling g_4 is particularly interesting. We’ve seen that for $k \geq 3$, the ϕ^{2k} interactions are irrelevant in $d = 4$ near the Gaussian fixed point, so at low energies

we may neglect them. β_4 then vanishes to linear order, so that the ϕ^4 coupling is marginal at this order. To study its behaviour, we need to go to higher order. From (5.57) we have

$$\beta_4 = \Lambda \frac{\partial g_4}{\partial \Lambda} = \frac{3}{16\pi^2} g_4^2 + \mathcal{O}(g_4^2 g_2) \quad (5.59)$$

to quadratic order, where we've again dropped the g_6 term. Equation (5.59) is solved by

$$\frac{1}{g_4(\Lambda)} = C - \frac{3}{16\pi^2} \ln \Lambda \quad (5.60a)$$

where C is an integration constant. Equivalently, we may write

$$g_4(\Lambda) = \frac{16\pi^2}{3 \ln(\mu/\Lambda)} \quad (5.60b)$$

in terms of some arbitrary scale μ . Since g_4 is the coefficient of the highest power of ϕ that appears in our potential, we must have $g_4 > 0$ if the action is to be bounded as $|\phi| \rightarrow \infty$, so we must choose $\mu > \Lambda$.

There are several important things to learn from this result. Firstly, we see that $g_4(\Lambda)$ decreases as $\Lambda \rightarrow 0$, ultimately being driven to zero. However, the scale dependence of g_4 is rather mild; instead of power-law behaviour we have only logarithmic dependence on the cut-off. Thus the ϕ^4 coupling, which was marginal at the classical level, because marginally irrelevant once quantum effects are taken into account. In the deep IR, we see only a free theory.

Secondly, away from the IR we notice that the integration constant μ determines a scale at which the coupling diverges. If we try to follow the RG trajectories back into the UV, perturbation theory will certainly break down before we reach $\Lambda \approx \mu$. The fact that the couplings in the action can be traded for energy scales μ at which perturbation theory breaks down is a ubiquitous phenomenon in QFT known as **dimensional transmutation**. We'll meet it many times in later chapters. The question of whether the ϕ^4 coupling *really* diverges as we head into the UV or just *appears to* in perturbation theory is rather subtle. More sophisticated treatments back up the belief that it does indeed diverge: in the UV we lose all control of the theory and in fact we do not believe that ϕ^4 theory really exists as a well-defined continuum QFT in four dimensions. This has important phenomenological implications for the Standard Model, through the quartic coupling of the scalar Higgs boson; take the Part III Standard Model course if you want to find out more.

The fact that the ϕ^4 coupling is not a free constant, but is determined by the scale and can even diverge at a finite scale $\Lambda = \mu$ should be worrying. How can we ever trust perturbation theory? The final lesson of (5.60b) is that if we want to use perturbation theory, we should always try to choose our cut-off scale so as to make the couplings as small as possible. In the case of ϕ^4 theory this means we should choose Λ as low as possible. In particular, if we want to study physics at a particular length scale ℓ , then our best chance for a weakly coupled description is to integrate out all degrees of freedom on length scales shorter than ℓ , so that $\Lambda \sim \ell^{-1}$.

5.3.4 The Wilson–Fisher critical point

The conclusion at the end of the previous section was that ϕ^4 theory does not have a continuum limit in $d = 4$. Since the only critical point is the Gaussian free theory we reach at low energies, four dimensional scalar theory is known as a **trivial** theory.

It's interesting to ask whether there are other, non-trivial critical points away from four dimensions. In general, finding non-trivial critical points is a difficult problem. Wilson and Fisher had the idea of introducing a parameter $\varepsilon := 4 - d$ which is treated as ‘small’ so that one is ‘near’ four dimensions. One then hopes that results obtained via the ε -**expansion** may remain valid in the physically interesting cases of $d = 3$ or even $d = 2$. From the local potential approximation (5.56) Wilson & Fisher showed that there is a critical point g_i^{WF} where

$$g_2^{\text{WF}} = -\frac{1}{6}\varepsilon + \mathcal{O}(\varepsilon^2), \quad g_4^{\text{WF}} = \frac{1}{3a}\varepsilon + \mathcal{O}(\varepsilon^2) \quad (5.61)$$

and $g_{2k}^{\text{WF}} \sim \varepsilon^k$ for all $k > 2$. We require $\varepsilon > 0$ to ensure that $V(\phi) \rightarrow 0$ as $|\phi| \rightarrow \infty$ so that the theory can be stable.

To find the behaviour of operators near to this critical point, once again we linearize the β -functions of (5.57) around g_{2k}^{WF} . Truncating to the subspace spanned by (g_2, g_4) we have

$$\Lambda \frac{\partial}{\partial \Lambda} \begin{pmatrix} \delta g_2 \\ \delta g_4 \end{pmatrix} = \begin{pmatrix} \varepsilon/3 - 2 & -a(1 + \varepsilon/6) \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} \delta g_2 \\ \delta g_4 \end{pmatrix}. \quad (5.62)$$

The matrix has eigenvalues $\varepsilon/3 - 2$ and ε , with corresponding eigenvectors

$$\sigma_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} -a(3 + \varepsilon/2) \\ 2(3 + \varepsilon) \end{pmatrix} \quad (5.63)$$

respectively. In $d = 4 - \varepsilon$ dimensions we have

$$a = \frac{1}{(4\pi)^{d/2}} \frac{1}{\Gamma(d/2)} \Big|_{d=4-\varepsilon} = \frac{1}{16\pi^2} + \frac{\varepsilon}{32\pi^2} (1 - \gamma + \ln 4\pi) + \mathcal{O}(\varepsilon^2) \quad (5.64)$$

where we have used the recurrence relation $\Gamma(z+1) = z\Gamma(z)$ and asymptotic formula

$$\Gamma(-\varepsilon/2) = -\frac{2}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon) \quad (5.65)$$

for the Gamma function as $\varepsilon \rightarrow 0$, where γ is the Euler–Mascheroni constant $\gamma \approx 0.5772$. Since ε is small the first eigenvalue is negative, so the mass term ϕ^2 is a *relevant* perturbation of the Wilson–Fisher fixed point. On the other hand, the operator $-a(3+\varepsilon/2)\phi^2 + 2(3+\varepsilon)\phi^4$ corresponding to σ_4 corresponds to an *irrelevant* perturbation. The projection of RG flows to the (g_2, g_4) subspace is shown in figure 8.

Although we've seen the existence of the Wilson–Fisher fixed point only for $0 < \varepsilon \ll 1$, more sophisticated techniques can be used to prove its existence in both $d = 3$ and $d = 2$ where it in fact corresponds to the Ising Model CFT. As shown in figure 8, both the Gaussian and Wilson–Fisher fixed-points lie on the critical surface, and a particular RG trajectory emanating from the Gaussian model corresponding to turning on the operator σ_4

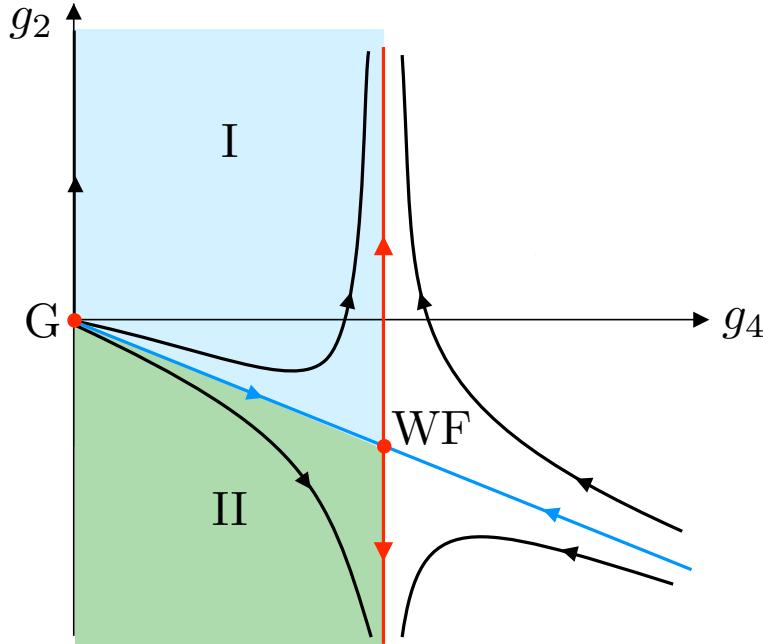


Figure 8: The RG flow for a scalar theory in three dimensions, projected to the (g_2, g_4) subspace. The Wilson–Fisher and Gaussian fixed points are shown. The blue line is the projection of the critical surface. The arrows point in the direction of RG flow towards the IR.

ends at the Wilson–Fisher fixed point in the IR. Theories on the line heading vertically out of the Gaussian fixed-point correspond to massive free theories, while theories in region I are massless and free in the deep UV, but become interacting and massive in the IR. These theories are parametrized by the scalar mass and by the strength of the interaction at any given energy scale. Theories in region II are likewise free and massless in the UV but interacting in the IR. However, these theories have $g_2 < 0$ so that the mass term is negative. This implies that the minimum of the potential $V(\phi)$ lies away from $\phi = 0$, so for theories in region II, ϕ will develop a vacuum expectation value, $\langle \phi \rangle \neq 0$. The RG trajectory obtained by deforming the Wilson–Fisher fixed point by a mass term is shown in red. All couplings in any theory to the right of this line diverge as we try to follow the RG back to the UV; these theories do not have well-defined continuum limits.

5.3.5 Zamolodchikov’s C –theorem

Polchinski’s equation showed that renormalization group flow could be understood as a form of heat flow. It’s natural to ask whether, as for usual heat flow, this can be thought of as a *gradient* flow so that there is some real positive function $C(g_i, \Lambda)$ that decreases monotonically along the flow. Notice that this implies $C = \text{const.}$ at a fixed point g_i^* , and that $C(g_i^*, \Lambda) > C(g_i^{**}, \Lambda')$ whenever a fixed point g_i^{**} may be reached by perturbing the theory a fixed point g_i^* by a relevant operator and flowing to the IR. In 1986, Alexander Zamolodchikov found such a function C for any unitary, Lorentz invariant QFT in two

dimensions.

Consider a two dimensional QFT whose (improved) energy momentum tensor is given by $T_{\mu\nu}(x)$. This is a symmetric 2×2 matrix, so has three independent components. Introducing complex coordinates $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$, we can group these components as

$$\begin{aligned} T_{zz} &:= \frac{\partial x^\mu}{\partial z} \frac{\partial x^\nu}{\partial z} T_{\mu\nu} = \frac{1}{2} (T_{11} - T_{22} - iT_{12}) \\ T_{\bar{z}\bar{z}} &:= \frac{\partial x^\mu}{\partial \bar{z}} \frac{\partial x^\nu}{\partial \bar{z}} T_{\mu\nu} = \frac{1}{2} (T_{11} - T_{22} + iT_{12}) \\ T_{z\bar{z}} &:= \frac{\partial x^\mu}{\partial z} \frac{\partial x^\nu}{\partial \bar{z}} T_{\mu\nu} = \frac{1}{2} (T_{11} + T_{22}) \end{aligned} \quad (5.66)$$

where $T_{\bar{z}z} = T_{z\bar{z}}$. This stress tensor is conserved, with the conservation equation being

$$0 = \partial^\mu T_{\mu\nu} = \partial_{\bar{z}} T_{zz} + \partial_z T_{\bar{z}\bar{z}} \quad (5.67)$$

in terms of the complex coordinates. Note that the stress tensor is a smooth function of z and \bar{z} .

The two-point correlation functions of these stress tensor components are given by

$$\begin{aligned} \langle T_{zz}(z, \bar{z}) T_{zz}(0, 0) \rangle &= \frac{1}{z^4} F(|z|^2) \\ \langle T_{zz}(z, \bar{z}) T_{z\bar{z}}(0, 0) \rangle &= \frac{4}{z^3 \bar{z}} G(|z|^2) \\ \langle T_{z\bar{z}}(z, \bar{z}) T_{z\bar{z}}(0, 0) \rangle &= \frac{16}{|z|^4} H(|z|^2) \end{aligned} \quad (5.68)$$

where the explicit factors of z and \bar{z} on the *rhs* follow from Lorentz invariance, which also requires that the remaining functions F , G and H depend on position only through $|z|$. Like any correlation function, these functions will also depend on the couplings and scale Λ used to define the path integral.

The two-point function $\langle T_{z\bar{z}}(z, \bar{z}) T_{z\bar{z}}(0, 0) \rangle$ appearing here satisfies an important positivity condition. Using canonical quantization, we insert a complete set of QFT states to find

$$\begin{aligned} \langle T_{z\bar{z}}(z, \bar{z}) T_{z\bar{z}}(0, 0) \rangle &= \sum_n \langle 0 | \hat{T}_{z\bar{z}}(z, \bar{z}) e^{-H\tau} | n \rangle \langle n | \hat{T}_{z\bar{z}}(0, 0) | 0 \rangle \\ &= \sum_n e^{-E_n \tau} |\langle n | \hat{T}_{z\bar{z}}(0, 0) | 0 \rangle|^2 \end{aligned} \quad (5.69)$$

so that this two-point function is positive definite, and it follows that $H(|z|^2)$ is also positive definite.

Zamolodchikov now used a combination of this positivity condition and the current conservation equation to construct a certain quantity $C(g_i, \Lambda)$ that decreases monotonically along the RG flow. In terms of the two-point functions, current conservation (5.67) for the energy momentum tensor becomes

$$4F' + G' - 3G = 0 \quad \text{and} \quad 4G' - 4G + H' - 2H = 0 \quad (5.70)$$

where $F' = dF(|z|^2)/d|z|^2$ etc. We define the ***C*-function** to be

$$C(|z|^2) := 2F - G - \frac{3}{8}H \quad (5.71)$$

which obeys $dC/d|z|^2 = -3H/4$ by the current conservation equations. But by the positivity of the two-point function $\langle T_{z\bar{z}}(z, \bar{z}) T_{z\bar{z}}(0) \rangle$ this means that

$$r^2 \frac{dC}{dr^2} < 0 \quad (5.72)$$

so C decreases monotonically under two dimensional scaling transformations, or equivalently under two dimensional RG flow. The value of C at an RG fixed point can be shown to be the **central charge** of the CFT.

Ever since it was first proposed, physicists have searched for a generalization of Zamolodchikov's theorem to RG flows in higher dimensions. The two-dimensional quantity $T_{z\bar{z}}$ is just the trace T_μ^μ of the energy momentum tensor and in 1988 John Cardy proposed that a certain term — known as “ a ” — in the expansion of the two-point correlator of T_μ^μ plays the role of Zamolodchikov's C in any even number of dimensions. Cardy's conjecture was verified to all orders in perturbation theory the following year by DAMTP's own Hugh Osborn, while a complete, non-perturbative **proof** was finally given in 2011 by Zohar Komargodski & Adam Schwimmer.

6 Perturbative Renormalization

6.1 One-loop renormalization of $\lambda\phi^4$ theory

Consider the scalar theory

$$S_{\Lambda_0}[\phi] = \int \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \right] d^4x \quad (6.1)$$

with initial couplings m^2 and λ , defined for momentum modes $\leq \Lambda_0$. (The mass coupling is dimensionful here.) From the analysis of the previous chapter, we expect that near the Gaussian fixed point, the mass parameter is relevant, while the quartic coupling is marginally irrelevant. Let's see how these expectations are borne out in perturbative calculations.

Firstly, the mass term receives a correction from the Feynman diagram

If we're integrating out all momentum modes $|p| \leq \Lambda_0$ then this diagram is given by the Euclidean signature loop integral

$$\begin{aligned} -\frac{1}{2}\lambda \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2+m^2} &= -\lambda_0 \frac{\text{Vol}(S^3)}{2(2\pi)^4} \int_0^{\Lambda_0} \frac{p^3 dp}{p^2+m^2} \\ &= -\frac{\lambda m^2}{32\pi^2} \int_0^{\Lambda_0^2/m^2} \frac{x dx}{1+x} \\ &= \frac{\lambda}{32\pi^2} \left[\Lambda_0^2 - m^2 \ln \left(1 + \frac{\Lambda_0^2}{m^2} \right) \right], \end{aligned} \quad (6.2)$$

where the factor of $1/2$ is the symmetry factor of the diagram, and we have used the fact that $\text{Vol}(S^3) = 2\pi^2$. As expected, this result shows that the mass parameter is relevant: There's a quadratic divergence as we try to take the continuum limit $\Lambda_0 \rightarrow \infty$ (as well as a subleading logarithmic one).

If we wish to obtain finite results in the continuum limit, then we must tune the couplings (m^2, λ) in our scale- Λ_0 action so as to make (6.2) finite in the limit. Achieving such a tuning directly from (6.2) is complicated, but fortunately we don't need to do this. Recall from section 5.2.2 that we tune by modifying the action to include counterterms:

$$S_{\Lambda_0}[\phi] \rightarrow S_{\Lambda_0}[\phi] + \hbar S^{\text{CT}}[\phi, \Lambda_0] \quad (6.3)$$

where in this case the counterterm action is

$$S^{\text{CT}}[\phi, \Lambda_0] = \int \left[\frac{1}{2}\delta Z(\partial\phi)^2 + \frac{1}{2}\delta m^2\phi^2 + \frac{1}{4!}\delta\lambda\phi^4 \right] d^4x \quad (6.4)$$

with $(\delta Z, \delta m^2, \delta\lambda)$ representing our ability to adjust the couplings in the original action (including the coupling to the kinetic term — or wavefunction renormalization). These

counterterm couplings will depend explicitly on Λ_0 , as they represent the tuning that must be performed starting from the Λ_0 cut-off theory.

The fact that the counterterm action is proportional to \hbar means that classical contributions from S^{CT} contribute to the same order in \hbar as 1-loop diagrams from $S_{\Lambda_0}[\phi]$. Thus the mass term also receives a correction at order \hbar^0 from the tree diagram

where the cross represents the insertion of the counterterm δm^2 treated as a vertex. The full quantum contribution to the mass term is thus

$$\delta m^2 + \frac{\lambda}{32\pi^2} \left[\Lambda_0^2 - m^2 \ln \left(1 + \frac{\Lambda_0^2}{m^2} \right) \right] \quad (6.5)$$

to 1-loop accuracy.

6.1.1 The on-shell renormalization scheme

The *raison d'être* of counterterms is to ensure (6.5) has a finite continuum limit, so that they must cancel the part of (6.2) that diverges as $\Lambda_0 \rightarrow \infty$. This still leaves us a lot of freedom in choosing how much of the *finite* part of the loop integral can also be absorbed by the counterterms. There's no preferred way to do this, and any such choice is called a *renormalization scheme*. Ultimately, all physically measurable quantities (such as cross-sections, branching ratios, particle lifetimes *etc.*) should be independent of the choice of renormalization scheme.

One physically motivated choice is called the *on-shell scheme*. In this scheme, we fix the mass counterterm δm^2 by asking that, once we take the continuum limit, the pole in the exact propagator $\int d^4x e^{ip \cdot x} \langle \phi(x) \phi(0) \rangle$ in momentum space occurs at some experimentally measured value. (Recall that the cross-section σ has a peak at the location of poles (in the complex momentum plane) in the S-matrix, so this location is an experimentally measurable quantity.) For example, it would be natural to try to set up our original action so that the coupling m^2 is indeed this experimentally measured value. If we denote

$$M^2(p^2) = \text{sum of all 1PI contributions to the mass}^2 \text{ term} \quad (6.6)$$

then the exact propagator can be written as a geometric series

$$= \frac{1}{p^2 + m^2 + M^2(p^2)}.$$

Taking into account both the loop integral and the counterterm, to 1-loop accuracy we'd find the propagator

$$= p^2 + m^2 + \delta m^2 + \frac{\lambda}{32\pi^2} \left[\Lambda_0^2 - m^2 \ln \left(1 + \frac{\Lambda_0^2}{m^2} \right) \right] \quad (6.7)$$

Consequently, in this scheme we should choose

$$\delta m^2 = -\frac{\lambda}{32\pi^2} \left[\Lambda_0^2 - m^2 \left[\ln \left(1 + \frac{\Lambda_0^2}{m^2} \right) - 1 \right] \right] \quad (6.8)$$

so as to completely cancel the 1-loop contribution to the mass term.

Notice that we cannot sensibly take the limit $\Lambda_0 \rightarrow \infty$ in either the 1-loop correction or the counterterm separately, reflecting the fact that the path integral measure $D\phi$ over all modes in the continuum does not exist. However, we can take the continuum limit of the correlation function $\langle \phi(x)\phi(y) \rangle$ (or it's momentum space equivalent) *after* having computed the path integral. The combined contribution of the 1-loop diagram using the initial action and the tree-level diagram involving the counterterm (6.8) remains finite in the continuum. Of course, while our tuning (??) is good to 1-loop accuracy, if we computed the path integral to higher order in perturbation theory, including the contribution of higher-loop Feynman diagrams, we would have to make further tunings in δm^2 (proportional to higher powers of the coupling λ) so as to still retain a finite limiting result.

6.1.2 Dimensional regularization

While the idea of integrating out momenta only up to a cut-off Λ_0 is very intuitive, in more complicated examples it becomes very cumbersome to perform the loop integrals over $|p|$ with a finite upper limit. In studying the local potential approximation in different dimensions (such as for the Wilson–Fisher fixed point), we saw that different couplings change their behaviour, becoming either relevant, marginal or irrelevant, as the dimension of the space is changed. Since the irrelevant couplings always have vanishing continuum limits, whereas (untuned) relevant ones diverge in the continuum, this suggests that we can regularize our loop integrals by analytically continuing the dimension d of our space. I stress that this is purely a convenient device for regularizing loop integrals — there is no suggestion that Nature ‘really’ lives in non-integer dimensions. Furthermore, dimensional regularization is only a *perturbative* regularization scheme: whilst we shall see that it does allow us to regulate individual loop integrals over the full range $|p| \in [0, \infty)$, unlike imposing a cut-off or a lattice regularization, dimensional regularization does *not* provide any definition of a finite-dimensional path integral measure. Despite these conceptual shortcomings, its practical convenience makes it an essential tool in perturbative calculations, especially in gauge theories as we shall see later.

In d -dimensions, the quartic coupling λ has non-zero mass dimension $4 - d$, so we replace $\lambda \rightarrow \mu^{4-d}\lambda$ where the new λ is dimensionless, and μ is an arbitrary mass scale.

Thus, in dimensional regularization, we obtain the 1-loop correction to the mass coupling

$$\frac{1}{2}\lambda\mu^{4-d} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + m^2} = \lambda\mu^{4-d} \frac{\text{Vol}(S^{d-1})}{2(2\pi)^d} \int_0^\infty \frac{p^{d-1} dp}{p^2 + m^2}, \quad (6.9)$$

where $\text{Vol}(S^{d-1})$ is the surface volume of a unit sphere in d dimensions. To compute this, note that

$$\begin{aligned} \pi^{\frac{d}{2}} &= \int_{\mathbb{R}^d} e^{-(x,x)} d^d x = \text{Vol}(S^{d-1}) \int_0^\infty e^{-r^2} r^{d-1} dr \\ &= \frac{\text{Vol}(S^{d-1})}{2} \int_0^\infty e^{-u} u^{\frac{d}{2}-1} du \end{aligned} \quad (6.10)$$

and thus

$$\text{Vol}(S^{d-1}) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}. \quad (6.11)$$

The remaining integral is

$$\begin{aligned} \mu^{4-d} \int_0^\infty \frac{p^{d-1} dp}{p^2 + m^2} &= \frac{1}{2}\mu^{4-d} \int_0^\infty \frac{(p^2)^{d/2-1} d(p^2)}{p^2 + m^2} \\ &= \frac{m^2}{2} \left(\frac{\mu}{m}\right)^{4-d} \int_0^1 (1-u)^{\frac{d}{2}-1} u^{-\frac{d}{2}} du \\ &= \frac{m^2}{2} \left(\frac{\mu}{m}\right)^{4-d} \frac{\Gamma(\frac{d}{2})\Gamma(1-\frac{d}{2})}{\Gamma(1)}, \end{aligned} \quad (6.12)$$

where $u := m^2/(p^2 + m^2)$ and we have used the definition of the Euler beta-function

$$B(s, t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} = \int_0^1 u^{s-1} (1-u)^{t-1} du. \quad (6.13)$$

Combining the pieces the 1-loop contribution to the mass-shift is

$$\frac{m^2\lambda}{2} \frac{\text{Vol}(S^{d-1})}{2(2\pi)^d} \left(\frac{\mu}{m}\right)^{4-d} \frac{\Gamma(\frac{d}{2})\Gamma(1-\frac{d}{2})}{\Gamma(\frac{d}{2})} = \frac{m^2\lambda}{2(4\pi)^{d/2}} \left(\frac{\mu}{m}\right)^{4-d} \Gamma\left(1-\frac{d}{2}\right) \quad (6.14)$$

Expanding around $d = 4$ by setting $d = 4 - \epsilon$, this is

$$-\frac{m^2\lambda}{32\pi^2} \left[\frac{2}{\epsilon} - \gamma + \log\left(\frac{4\pi\mu^2}{m^2}\right) \right] + \mathcal{O}(\epsilon) \quad (6.15)$$

as $\epsilon \rightarrow 0$, where we've used the basic properties

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} \quad \text{and} \quad \Gamma(z) \sim \frac{1}{z} - \gamma + \mathcal{O}(\epsilon)$$

of the Gamma function, with $\gamma \approx 0.577$ being the Euler–Mascheroni constant. The divergence we saw as $\Lambda_0 \rightarrow \infty$ in the cut-off regularization has become a *pole* in $d = 4$ in dimensional regularization.

The simplest renormalization scheme is *minimal subtraction* (MS) — one simply chooses the counterterm

$$\delta m^2 = -\frac{m^2\lambda}{16\pi^2\epsilon} \quad \text{MS} \quad (6.16)$$

so as to remove the purely divergent parts of the loop diagrams. A more common scheme is *modified minimal subtraction* ($\overline{\text{MS}}$) in which one also removes the Euler–Mascheroni constant and the $\log 4\pi$ term, so that

$$\delta m^2 = -\frac{m^2 \lambda}{32\pi^2} \left(\frac{2}{\epsilon} - \gamma + \log 4\pi \right) \quad \overline{\text{MS}}. \quad (6.17)$$

We shall see later that these minimal subtraction schemes are rather different in character from the on–shell renormalization scheme. Nonetheless, it has become the most frequently used renormalization scheme in the literature.

6.1.3 Renormalization of the quartic coupling

The 1-loop correction to the quartic vertex is given by the three Feynman diagrams

which lead to the momentum space integrals

$$\frac{\lambda^2 \mu^{4-d}}{2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + m^2} \frac{1}{(p + k_1 + k_2)^2 + m^2} + \text{other channels} \quad (6.18)$$

in dimensional regularization. Because these Feynman diagrams involve two separate vertices, they will lead to non–local contributions; indeed, expanding (6.18) in powers of the momenta k_i generates new derivative interactions such as $\sim \phi^2 (\partial\phi)^2$. You should check that all such derivative terms are irrelevant as expected — they remain finite in the limit $d \rightarrow 4$ in dimensional regularization, and would in fact vanish in the continuum limit $\Lambda_0 \rightarrow \infty$ had we worked with a hard cut–off.

The k -independent part of each of the three loop diagrams involves the integral

$$\begin{aligned} \mu^{4-d} \int_0^\infty \frac{p^{d-1} dp}{(p^2 + m^2)^2} &= \frac{1}{2} \int_0^\infty \frac{(p^2)^{(d-2)/2}}{(p^2 + m^2)^2} d(p^2) \\ &= \frac{1}{2} \left(\frac{\mu}{m} \right)^{4-d} \int_0^1 u^{1-\frac{d}{2}} (1-u)^{\frac{d}{2}-1} du \\ &= \frac{1}{2} \left(\frac{\mu}{m} \right)^{4-d} \frac{\Gamma(2 - \frac{d}{2}) \Gamma(\frac{d}{2})}{\Gamma(2)} \end{aligned} \quad (6.19)$$

Combining all three diagrams, we have a total 1-loop contribution

$$3\lambda^2 \frac{\text{Vol}(S^{d-1})}{2(2\pi)^d} \frac{1}{2} \left(\frac{\mu}{m} \right)^{4-d} \frac{\Gamma(2 - \frac{d}{2}) \Gamma(\frac{d}{2})}{\Gamma(2)} = \frac{3\lambda^2}{2(4\pi)^{d/2}} \left(\frac{\mu}{m} \right)^{4-d} \Gamma\left(2 - \frac{d}{2}\right) \quad (6.20)$$

using our result (6.11) for $\text{Vol}(S^{d-1})$. Setting $d = 4 - \epsilon$, this becomes

$$\frac{3\lambda^2}{32\pi^2} \left(\frac{2}{\epsilon} - \gamma + \log \frac{4\pi\mu^2}{m^2} \right) + \mathcal{O}(\epsilon). \quad (6.21)$$

Consequently, in the $\overline{\text{MS}}$ scheme we choose our counterterm $\delta\lambda$ to be

$$\delta\lambda = \frac{3\lambda^2}{32\pi^2} \left(\frac{2}{\epsilon} - \gamma + \log 4\pi \right) \quad (6.22)$$

removing both the pole as $\epsilon \rightarrow 0$ and the $\gamma - \log 4\pi$ terms.

The remaining 1-loop correction to the quartic vertex leads to a β -function

$$\beta = \mu \frac{\partial \lambda}{\partial \mu} = \frac{3\lambda^2}{16\pi^2} \quad (6.23)$$

agreeing (to this order) with what we found in (??) for the local potential approximation. The fact that the β -function is positive shows that the quartic coupling is (marginally) irrelevant in $d = 4$. Thus, no matter how small we choose the interaction to be at some scale μ , if it is non-zero, then there is a scale $\mu' > \mu$ in the UV at which the coupling diverges. Of course, our perturbative treatment is not powerful enough to really say what occurs as this happens, but more sophisticated treatments indeed show that $\lambda\phi^4$ does not exist as a continuum QFT.

Notice that there are no 1-loop diagrams that can contribute to the wavefunction renormalization factor Z_Λ here — to obtain a correction to the kinetic term $(\partial\phi)^2$ we would need a momentum space diagram involving precisely two external ϕ fields. With a purely quartic interaction, the only such 1-loop diagram is the one relevant for the mass shift. However, the momentum running around the loop in this diagram does not involve any momentum being brought in by the external fields, and therefore cannot contribute the factor of k^2 necessary to be interpreted as a correction to the kinetic term. Put differently, this loop diagram is a purely *local* contribution, so does not affect any derivative terms. Thus $Z_\Lambda = 1$ to 1-loop accuracy. There are non-trivial wavefunction renormalization factors beginning at 2-loops.

6.2 One-loop renormalization of QED

The action for QED describes a massive charged Dirac spinor coupled to the electromagnetic field is

$$S_{\text{QED}}[A, \psi] = \int d^d x \left[\frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} + \bar{\psi} \not{D} \psi + m \bar{\psi} \psi \right] \quad (6.24)$$

where the covariant derivative in the fermion kinetic term is $\not{D}\psi = \gamma^\mu (\partial_\mu + iA_\mu)\psi$, and the Dirac matrices γ_μ obey $\{\gamma_\mu, \gamma_\nu\} = +2\delta_{\mu\nu}$ in Euclidean signature. (We'll understand more about covariant derivatives when we look at Yang–Mills theory.) In order for the covariant derivative $D_\mu = \partial_\mu + iA_\mu$ to make sense, the gauge field must have mass dimension 1 even in d dimensions. Thus, the electric charge e has dimensions $(4-d)/2$, so is relevant when $d < 4$, irrelevant in $d > 4$ and marginal in $d = 4$, at least to leading order. Introducing an arbitrary mass scale μ as before we introduce a dimensionless coupling $g(\mu)$ as

$$e^2 = \mu^{4-d} g^2(\mu). \quad (6.25)$$

To do perturbation theory, we'd like the kinetic terms to be canonically normalized, so we introduce a rescaled photon field $A_\mu^{\text{new}} = eA_\mu^{\text{old}}$. In terms of this rescaled field the action becomes

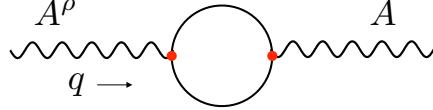
$$S_{\text{QED}}[A^{\text{new}}, \psi] = \int d^d x \left[\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(\not{\partial} + m)\psi + i\mu^{2-\frac{d}{2}} g \bar{\psi} \not{A} \psi \right] \quad (6.26)$$

with $g(\mu)$ appearing only in the electron–photon vertex, as befits a coupling. Notice that the new photon field has mass dimension $(d - 2)/2$, just like a scalar field.

6.2.1 Vacuum polarization: loop calculation

We'll now take a look at the simplest, and probably also the most important 1-loop graph in QED; the photon self-energy graph, also known as vacuum polarization.

Electromagnetic forces between charged particles are mediated by photon exchange. Quantum corrections modify the form of this propagator, for example by the 1-loop graph



where a virtual e^+e^- pair is formed and then reabsorbed. If we let $\Pi_{\text{loop}}^{\rho\sigma}(q)$ denote the 1PI contributions to the photon self-energy, then at one loop order the only contribution to $\Pi_{\text{loop}}^{\rho\sigma}$ is from the diagram above. The Feynman rules following from (6.26) give

$$\Pi_{\text{1 loop}}^{\rho\sigma}(q) = \mu^{4-d} g^2 \int \frac{d^d p}{(2\pi)^d} \frac{\text{tr}((-\text{i}\not{p} + m)\gamma^\rho(-\text{i}(\not{p} - \not{q}) + m)\gamma^\sigma)}{(p^2 + m^2)((p - q)^2 + m^2)}, \quad (6.27)$$

and we must now evaluate this integral.

To begin, note that

$$\frac{1}{AB} = \frac{1}{B - A} \left[\frac{1}{A} - \frac{1}{B} \right] = \int_0^1 \frac{dx}{[(1-x)A + xB]^2} \quad (6.28)$$

so that we can combine the two propagators in (6.27) as

$$\begin{aligned} \int_0^1 \frac{dx}{[(p^2 + m^2)(1-x) + ((p-q)^2 + m^2)x]^2} &= \int_0^1 \frac{dx}{[p^2 + m^2 - 2xp \cdot q + q^2 x]^2} \\ &= \int_0^1 \frac{dx}{[(p - qx)^2 + m^2 + q^2 x(1-x)]^2}. \end{aligned} \quad (6.29)$$

If we now change variables $p \rightarrow p' = p + qx$ then (6.27) becomes (dropping the prime)

$$\Pi_{\text{1 loop}}^{\rho\sigma}(q) = \mu^{4-d} g^2 \int \frac{d^d p}{(2\pi)^d} \int_0^1 dx \frac{\text{tr}((-\text{i}(\not{p} + \not{qx}) + m)\gamma^\rho(-\text{i}(\not{p} - \not{q}(1-x)) + m)\gamma^\sigma)}{[p^2 + \Delta]^2}, \quad (6.30)$$

where $\Delta := m^2 + q^2x(1-x)$.

The next step is to perform the traces over the Dirac matrices. We'll do this treating the Dirac spinors as having 4 components²⁹ as appropriate for our final goal of $d = 4$. Thus

$$\begin{aligned}\text{tr}(\gamma^\rho\gamma^\sigma) &= 4\delta^{\rho\sigma} \\ \text{tr}(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma) &= 4(\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\mu\nu}\delta^{\rho\sigma} + \delta^{\mu\sigma}\delta^{\nu\rho})\end{aligned}\quad (6.31)$$

in Euclidean signature, so that

$$\begin{aligned}\text{tr}((-i(p+qx) + m)\gamma^\rho(-i(p-q(1-x)) + m)\gamma^\sigma) \\ = 4[-(p+qx)^\rho(p-q(1-x))^\sigma + (p+qx)\cdot(p-q(1-x))\delta^{\rho\sigma} \\ -(p+qx)^\sigma(p-q(1-x))^\rho + m^2\delta^{\rho\sigma}]\ .\end{aligned}\quad (6.32)$$

Thus the loop integral becomes

$$\begin{aligned}\Pi_{1\text{loop}}^{\rho\sigma}(q) &= 4\mu^{4-d}g^2 \int \frac{d^dp}{(2\pi)^d} \int_0^1 dx \frac{1}{[p^2 + \Delta]^2} \\ &\times [-(p+qx)^\rho(p-q(1-x))^\sigma + (p+qx)\cdot(p-q(1-x))\delta^{\rho\sigma} \\ &\quad -(p+qx)^\sigma(p-q(1-x))^\rho + m^2\delta^{\rho\sigma}] ,\end{aligned}\quad (6.33)$$

which would be quadratically divergent in $d = 4$.

We're now ready to perform the loop integral. Observing that whenever $d \in \mathbb{N}$, any term involving an odd number of powers of momentum would vanish, we drop these terms. For the same reason, we replace

$$p^\mu p^\nu \rightarrow \frac{1}{d}\delta^{\mu\nu}p^2 \quad \text{and} \quad p^\mu p^\nu p^\rho p^\sigma \rightarrow \frac{(p^2)^2}{d(d+2)} [\delta^{\mu\nu}\delta^{\rho\sigma} + \delta^{\mu\rho}\delta^{\nu\sigma} + \delta^{\mu\sigma}\delta^{\nu\rho}]$$

where the tensor structure is fixed by Lorentz invariance and permutation symmetry, and the numerical factors are determined by contracting both sides with metrics. Finally, since the integrand now depends only on p^2 , the angular integrals may be performed trivially to obtain

$$\frac{d^dp}{(2\pi)^d} = \text{Vol}(S^{d-1}) \frac{p^{d-1} dp}{(2\pi)^d} = \frac{1}{(4\pi)^{d/2} \Gamma(d/2)} (p^2)^{\frac{d}{2}-1} d(p^2) \quad (6.34)$$

as in section 6.1.2. Thus (6.33) becomes

$$\begin{aligned}\Pi_{1\text{loop}}^{\rho\sigma}(q) &= 4\mu^{4-d} \frac{g^2}{(4\pi)^{\frac{d}{2}} \Gamma(\frac{d}{2})} \\ &\times \int_0^1 dx \int_0^\infty d(p^2) (p^2)^{\frac{d}{2}-1} \left[\frac{p^2(1-\frac{2}{d})\delta^{\rho\sigma} + (2q^\rho q^\sigma - q^2)x(1-x) + m^2\delta^{\rho\sigma}}{[p^2 + \Delta]^2} \right] .\end{aligned}\quad (6.35)$$

To go further, we use the integrals

$$\begin{aligned}\int_0^\infty d(p^2) \frac{(p^2)^{\frac{d}{2}-1}}{(p^2 + \Delta)^2} &= \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \frac{\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2})}{\Gamma(2)} \\ \int_0^\infty d(p^2) \frac{(p^2)^{\frac{d}{2}}}{(p^2 + \Delta)^2} &= \left(\frac{1}{\Delta}\right)^{1-\frac{d}{2}} \frac{\Gamma(1+\frac{d}{2})\Gamma(1-\frac{d}{2})}{\Gamma(2)}\end{aligned}\quad (6.36)$$

²⁹In certain supersymmetric theories, it is often convenient to work instead with d -dimensional spinors, which is known as *dimensional reduction*, rather than dimensional regularization.

that can be evaluated using the substitution $u = \Delta/(p^2 + \Delta)$ and the definition of the Euler B–function, just as we did in section 6.1.2.

Using these integrals to evaluate (6.35), altogether one finds that the 1-loop contribution to vacuum polarization is given by

$$\begin{aligned} \Pi_{\text{1 loop}}^{\rho\sigma}(q) &= -\frac{4g^2\mu^{4-d}}{(4\pi)^{d/2}} \Gamma\left(2-\frac{d}{2}\right) \\ &\quad \times \int_0^1 dx \left[\frac{\delta^{\rho\sigma}(-m^2 + x(1-x)q^2) + \delta^{\rho\sigma}(m^2 + x(1-x)q^2) - 2x(1-x)q^\rho q^\sigma}{\Delta^{2-\frac{d}{2}}} \right] \\ &:= (q^2 \delta^{\rho\sigma} - q^\rho q^\sigma) \pi_{\text{1 loop}}(q^2), \end{aligned} \quad (6.37)$$

where in the last line we have defined $\pi_{\text{1 loop}}(q^2)$ to be the dimensionless quantity

$$\pi_{\text{1 loop}}(q^2) = -\frac{8g^2(\mu)\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{d/2}} \int_0^1 dx x(1-x) \left(\frac{\mu^2}{\Delta}\right)^{2-d/2}. \quad (6.38)$$

and we recall that $\Delta = m^2 + q^2 x(1-x)$ (and that $d < 4$).

6.2.2 Counterterms in QED

The first thing to notice about our result (6.37) is that it, if all couplings remain constant, it will diverge in the physically interesting dimension because $\Gamma(2 - \frac{d}{2})$ has a pole when $d = 4$. To obtain a finite continuum result we must tune the initial couplings in the action, which as always we do by introducing counterterms. For QED the counterterms are

$$S^{\text{CT}}[A, \psi, \epsilon] = \int d^d x \left[\delta Z_3 \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \delta Z_2 \bar{\psi} \not{D} \psi + \delta m \bar{\psi} \psi \right]. \quad (6.39)$$

Adding these to the QED action (6.26) allows us to tune the initial values of the photon and electron wavefunction renormalizations, and the electron mass. The labels $(\delta Z_3, \delta Z_2)$ for the photon and electron wavefunction renormalization factors are conventional.

The fact that the entire kinetic term for the electron, including the gauge covariant derivative operator $\not{D} = \not{\partial} + ie\not{A}$, receives only one counterterm assumes that the regularized path integral preserves gauge invariance: Provided our regularized path integral is indeed gauge invariant, then $\not{\partial}\psi$ and $i\not{A}\psi$ cannot appear independently. Gauge invariance is maintained in lattice regularization, but would fail if one simply imposed a cut-off Λ_0 , because the requirement that fields only contain Fourier modes with $|p| \leq \Lambda_0$ is not preserved under the gauge transformation $\psi \rightarrow e^{i\chi(x)}\psi$, even if it is true of ψ and χ separately.³⁰ The desire to maintain manifest gauge invariance was one of the main motivation to use dimensional regularization in the first place, and our result (6.37) vindicates this decision: we see that the 1-loop correction $\Pi_{\text{1 loop}}^{\rho\sigma}(q^2)$ is proportional to $(q^2 \delta^{\rho\sigma} - q^\rho q^\sigma)$, so

$$q_\rho \Pi_{\text{1 loop}}^{\rho\sigma}(q) = 0. \quad (6.40)$$

³⁰In fact, the conceptually simple idea of integrating over modes only up to a cut-off *can* be done in gauge theory, but requires the introduction of a fair amount of technology beyond the scope of this course; see *e.g.* K. Costello's book cited in the introduction.

This signifies that the Ward identity for gauge transformations holds in the quantum theory (at least to one loop, but in fact it holds in general).

To fix the counterterm δZ_3 , note that the loop diagram (6.37) diverges in the physical dimension since $\Gamma(2 - d/2)$ has a pole as $d \rightarrow 4^-$. Indeed, setting $d = 4 - \epsilon$ we have

$$\pi_{\text{1 loop}}(q^2) \xrightarrow{d \rightarrow 4} -\frac{g^2(\mu)}{2\pi^2} \int_0^1 dx x(1-x) \left(\frac{2}{\epsilon} - \gamma + \ln \frac{4\pi\mu^2}{\Delta} \right) + \mathcal{O}(\epsilon) \quad (6.41)$$

where again γ is the Euler–Mascheroni constant. The contribution



from $-\frac{1}{4}\delta Z_3 F^{\rho\sigma}F_{\rho\sigma}$ must remove this pole, and in the $\overline{\text{MS}}$ scheme we'd set

$$\delta Z_3 = -\frac{g^2(\mu)}{12\pi^2} \left(\frac{2}{\epsilon} - \gamma + \ln 4\pi \right) \quad (6.42)$$

so as also to remove the contribution $\propto (-\gamma + \ln 4\pi)$. (To check that this counterterm does indeed cancel the pole, note that $\int_0^1 dx x(1-x) = \frac{1}{6}$.) Thus the total contribution to the effective photon self-energy at one loop is

$$\Pi^{\rho\sigma}(q) = (q^2\delta^{\rho\sigma} - q^\rho q^\sigma) \pi(q^2) \quad (6.43)$$

where

$$\pi(q^2) = +\frac{g^2(\mu)}{2\pi^2} \int_0^1 dx x(1-x) \ln \left[\frac{m^2 + x(1-x)q^2}{\mu^2} \right] \quad (6.44)$$

in the $\overline{\text{MS}}$ scheme.

Strikingly, the loop correction to the photon propagator has created the logarithm³¹

$$\ln [m^2 + x(1-x)q^2]$$

in momentum space. This is quite unlike anything you've seen at tree-level, where Feynman diagrams are always rational functions of momenta, but it is very similar to the logarithms we obtained from integrating out fields in lower dimensional examples. In the present case, the logarithm has a branch cut in the region $m^2 + x(1-x)q^2 < 0$, or in other words when

$$x(1-x)q_{\text{Lorentz}}^2 > m^2. \quad (6.45)$$

back in Lorentzian signature. Since $x(1-x) \leq 1/4$ for $x \in [0, 1]$, the smallest value of q^2 at which this branch cut is reached is

$$q_{\text{Lorentz}}^4 = 4m^2 \quad (6.46)$$

³¹Actually, it has produced a certain integral of a logarithm involving $x(1-x)$. This integral can be explicitly computed in terms of dilogarithms, but we won't need to know the result.

which is precisely the threshold energy for the creation of a *real* (as opposed to virtual) electron–positron pair.

At tree-level the photon propagator is

$$\Delta_{\mu\nu}^0(q) = \frac{1}{q^2} \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \quad (6.47)$$

in Lorenz (or Landau) gauge. In the quantum theory, the exact momentum space photon propagator $\Delta_{\mu\nu}(q)$ is then obtained by summing the geometric series

$$\begin{aligned} \Delta_{\mu\nu} &= \text{---} - \text{---}^{\text{1PI}} + \text{---}^{\text{1PI}} \text{---}^{\text{1PI}} \dots \\ &= \Delta_{\mu\nu}^0 - \Delta_{\mu\rho}^0 \Pi_\sigma^\rho \Delta_{\nu}^{0\sigma} + \Delta_{\mu\rho}^0 \Pi_\sigma^\rho \Delta_{\kappa}^{0\sigma} \Pi_\lambda^\kappa \Delta_{\nu}^{0\lambda} - \dots \end{aligned}$$

where $\Pi^{\mu\nu}$ denotes the sum of *all* one particle irreducible graphs with just two external photon lines, and the minus signs arise because we are in Euclidean signature with path integrals weighted by e^{-S} .

We've just found that

$$\Pi_\sigma^\rho = q^2 \left(\delta_\sigma^\rho - \frac{q^\rho q_\sigma}{q^2} \right) \pi(q^2) \quad (6.48)$$

where, to 1-loop accuracy, $\pi(q^2)$ is given by (6.44) in the $\overline{\text{MS}}$ scheme. In fact, one can show that (6.48) holds to all orders in the electromagnetic coupling as a consequence of the Ward identity: only the explicit form of $\pi(q^2)$ is modified. The factor in brackets projects onto the polarization states transverse to q^σ and obeys $P_\sigma^\rho P_\kappa^\sigma = P_\kappa^\rho$ as for any projection operator. Therefore the exact photon propagator may be written

$$\begin{aligned} \Delta_{\mu\nu}(q) &= \Delta_{\mu\nu}^0 - \Delta_{\mu\rho}^0 \Pi_\sigma^\rho \Delta_{\nu}^{0\sigma} + \Delta_{\mu\rho}^0 \Pi_\sigma^\rho \Delta_{\kappa}^{0\sigma} \Pi_\lambda^\kappa \Delta_{\nu}^{0\lambda} - \dots \\ &= \Delta_{\mu\nu}^0 (1 - \pi(q^2) + \pi^2(q^2) - \pi^3(q^2) \dots) \\ &= \frac{\Delta_{\mu\nu}^0}{1 + \pi(q^2)} \end{aligned} \quad (6.49)$$

by summing this geometric series.

Let's think about what this exact propagator tells us about the effective action for the photon that we'd obtain after performing the path integral over the electron field. The classical Maxwell action is

$$\begin{aligned} \frac{1}{4} \int F^{\mu\nu} F_{\mu\nu} d^d x &= \frac{1}{4} \int -i \left(q^\mu \tilde{A}^\nu(-q) - q^\nu \tilde{A}^\mu(-q) \right) i \left(q_\mu \tilde{A}_\nu(q) - q_\nu \tilde{A}_\mu(q) \right) d^d q \\ &= \frac{1}{2} \int q^2 \left(\delta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) \tilde{A}_\mu(-q) \tilde{A}_\nu(q) d^d q \end{aligned} \quad (6.50)$$

when written in momentum space. Note that $\Delta_{\mu\nu}^0$ is indeed the inverse of this momentum space kinetic term for polarizations transverse to q_μ . Thus, the exact photon propagator would follow from a momentum space quadratic term

$$S_{\text{eff}}^{(2)}[\tilde{A}] = \frac{1}{4} \int [1 + \pi(q^2)] q^2 \left(\delta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) \tilde{A}_\mu(-q) \tilde{A}_\nu(q) d^d q \quad (6.51)$$

where the electron loop effects are incorporated in the factor $[1 - \pi(q^2)]$. The part $\pi(0)$ of $\pi(q^2)$ that is independent of q^2 just provides an overall factor multiplying the classical action, and so corresponds straightforwardly to the position space term

$$S_{\text{eff}}^{(2)}[A] = \frac{1 + \pi(0)}{4} \int F^{\mu\nu}(z) F_{\mu\nu}(z) d^4 z = \frac{1}{4} \left[1 + \frac{g^2}{2\pi^2} \int_0^1 dx x(1-x) \ln \frac{m^2}{\mu^2} \right] \int F^{\mu\nu} F_{\mu\nu} d^4 z \quad (6.52)$$

where the second expression uses our result (6.44) for the 1-loop contribution to $\pi(0)$ in the $\overline{\text{MS}}$ scheme in $d = 4$. As expected, this is a contribution to photon wavefunction renormalization. Expanding $\pi(q^2)$ as a power series in q^2/m^2 shows that the remaining, q^2 -dependent terms correspond to an infinite series of higher derivative interactions of the schematic form $\partial^n F^{\mu\nu} \partial^n F_{\mu\nu}$. All these higher derivative couplings are irrelevant in $d = 4$ and may be expected to be small at energies much lower than the electron mass. (In particular, the original loop integrals for these terms were finite in $d = 4$.)

6.2.3 The β -function of QED

Knowing the effective action allows us to read off the β -function for the electric charge. To relate the photon kinetic term – involving wavefunction renormalization – to the β function for the electromagnetic coupling e , we first undo our rescaling $A_\mu^{\text{old}} = e A_\mu^{\text{new}}$ and work back in terms of the original gauge field A_μ^{old} . Then the quadratic term (6.52) in the effective action becomes

$$S_{\text{eff}}^{(2)}[A^{\text{old}}] = \frac{1}{4} \left[\frac{1}{g^2} + \frac{1}{2\pi^2} \int_0^1 dx x(1-x) \ln \frac{m^2}{\mu^2} \right] \int F^{\mu\nu} F_{\mu\nu} d^4 z. \quad (6.53)$$

In this way, we can view the vacuum polarization as a quantum correction to the value of $1/g^2$. Therefore

$$\mu \frac{\partial}{\partial \mu} \left(\frac{1}{g^2} \right) = -\frac{2}{g^3} \beta(g) = -\frac{1}{\pi^2} \int_0^1 dx x(1-x) \quad (6.54)$$

so that the β function for $g(\mu)$ is

$$\beta(g) := \mu \frac{\partial g}{\partial \mu} = \frac{g^3}{12\pi^2}. \quad (6.55)$$

Solving this $g(\mu)$ gives

$$\frac{1}{g^2(\mu)} = C - \frac{1}{6\pi^2} \ln \mu, \quad (6.56)$$

or equivalently

$$g^2(\mu) = \frac{g^2(\mu')}{1 - g^2(\mu')/6\pi^2 \ln(\mu/\mu')} \quad (6.57)$$

which fixes the coupling $g(\mu)$ at arbitrary scales in terms of its value at some arbitrary reference scale μ' . For example, we could choose μ' to be the scale of the (physical) electron mass, at which $g^2(m_e)/4\pi \approx 1/137$ is found experimentally.

As for the quartic coupling of $\lambda\phi^4$ theory, the fact that the β -function in QED is positive shows that the electromagnetic coupling is marginally irrelevant in $d = 4$, at least near the

Gaussian critical point $g = 0$. Consequently, it is believed that *pure QED does not exist as a continuum QFT* in four dimensions! This fact has no immediate phenomenological consequences, because taking $g^2(m_e)/4\pi \approx 1/137$ from experiment, the scale μ at which the coupling (6.57) diverges is fixed to be $\sim 10^{286}$ GeV, well beyond any point at which we claim to even vaguely trust QFT as a description of Nature. Nonetheless, the lesson from the β -function is that pure QED can only exist as a low-energy effective theory — in our world it unifies with the weak interactions at around ~ 100 GeV, where the physics of non-Abelian gauge theories comes into play.

Finally, I wish to point out a small peculiarity inherent in the $\overline{\text{MS}}$ renormalization scheme. In studying the local potential approximation in section 5.3.2 we discovered that the quantum contributions to the β -functions due to high energy states circulating in the loop became suppressed at scales much lower than the masses of these states. This made good sense: the heavy states decoupled from low-energy physics. However, in the $\overline{\text{MS}}$ scheme the β -function (6.54) shows no suppression at any scale and the coupling (6.57) still runs even at scales $\mu \ll m_e$. There is nothing wrong with this: as $\mu \rightarrow 0$ there is a balance in the effective action (6.51) between $e^2(\mu) \rightarrow 0$ and the growing effect of the loop contribution $\propto \ln(1/\mu^2) \rightarrow \infty$ so that the actual *observable physics* does remain constant. However, it's strange to have loop effects dominating the classical ones, so for some purposes it is better to proceed as follows. We consider *two* different theories. One includes the electron and is valid at scales $\mu \geq m_e$, while the second does not and is valid at scales $\mu \leq m_e$. Physical quantities in the two theories are matched at $\mu = m_e$. The effects of the electron (or other heavy particle) are then manually frozen out as we continue on to our other theory at lower scales. In particular, since pure Maxwell theory is free, for any $\mu \leq m_e$ the fine structure constant $\alpha(\mu) = g^2(\mu)/4\pi$ will remain frozen at its value $\approx 1/137$ at the electron mass.

We could avoid the need to decouple by hand had we used a renormalization scheme, such as on-shell renormalization, that fixes the value of the counterterm δZ_3 in terms of $\pi_{1\text{loop}}(q^2)$ at some definite scale $q^2 = \mu'^2$. That is, we set

$$\delta Z_3 = -\frac{g^2}{2\pi^2} \int_0^1 dx x(1-x) \left(\frac{2}{\epsilon} - \gamma + \ln \frac{4\pi\mu^2}{m^2 - x(1-x)\mu'^2} \right) \quad (6.58)$$

In this scheme, the β -function instead becomes

$$\beta(g) = \mu \frac{\partial g}{\partial \mu} = \frac{g^3}{2\pi^2} \int_0^1 dx x(1-x) \ln \frac{x(1-x)\mu'^2}{m^2 - x(1-x)\mu'^2} \quad (6.59)$$

in four dimensions. This does indeed approach zero when $\mu' \ll m_e$. However, for most purposes (particularly in more complicated theories such as Yang–Mills theory, or the full Standard Model) the $\overline{\text{MS}}$ scheme is so convenient that it's worth paying the price of having to decouple the electron by hand.

6.2.4 Physical interpretation of vacuum polarization

When light propagates through a region containing an insulating medium with no relevant degrees of freedom, on general grounds we expect the low-energy effective field theory to

be governed by an action that modifies the coefficients of the electric and magnetic fields in the usual Maxwell action by terms that respect the microscopic symmetries of the medium. In the present case, the medium is simply *the vacuum itself!* Since the vacuum is Lorentz invariant, these modifications must be proportional to the Lorentz invariant combination $E^2 - B^2 = F^{\mu\nu}F_{\mu\nu}$. In (6.52) we see explicitly that this is true. If we place a medium such as water in the presence of an electric field, it will become polarized due to the large dipole moment of the H₂O molecules. Likewise, at distances $\gtrsim 1/m$ the vacuum itself becomes a dielectric medium in which virtual electron–positron pairs form dipoles, polarizing the vacuum.

The first effect is that vacuum polarization leads to a measureable change in the Coulomb potential. Recall that in the non-relativistic limit (in Lorentzian signature), the Fourier transform of the (Feynman gauge) photon propagator $\delta^{\rho\sigma}/q^2$ is the Coulomb potential $V(r) = e^2/4\pi r$, as I hope is familiar from Rutherford scattering. Let's compute the 1-loop quantum corrections to this result. We consider a scattering process in which two spin $\frac{1}{2}$ charged particles interact electromagnetically. The Feynman diagrams

make contributions of the form

$$S(1, 2 \rightarrow 1', 2') = \frac{-e_1 e_2}{4\pi^2 q^2} \delta^4(p_1 + p_2 - p_{1'} - p_{2'}) [1 + \pi(q^2)] \bar{u}_{1'} \gamma^\mu u_1 \bar{u}_{2'} \gamma_\mu u_2 \quad (6.60)$$

where $u_{1,2}$ are the on-shell Dirac wavefunctions of the incoming particles and $\bar{u}_{1',2'}$ are the on-shell Dirac wavefunctions for the outgoing particles. The 1-loop diagram modifies the classical answer by the factor $[1 + \pi(q^2)]$. Non-relativistically, the energy transfer $q^0 \ll |\mathbf{q}|$ and

$$\bar{u}_{1'} \gamma^\mu u_1 \approx \begin{pmatrix} -i\delta_{m_1, m_{1'}} \\ \mathbf{0} \end{pmatrix},$$

where the factor of $\delta_{mm'}$ enforces that the spins of the two particles should be aligned. Thus, in the non-relativistic limit we have

$$S(1, 2 \rightarrow 1', 2') \approx \frac{-e_1 e_2}{4\pi^2 \mathbf{q}^2} \delta^4(p_1 + p_2 - p_{1'} - p_{2'}) [1 + \pi(\mathbf{q}^2)] \delta_{m_1 m_{1'}} \delta_{m_2 m_{2'}}. \quad (6.61)$$

We can compare this result to the calculation of scattering in non-relativistic quantum mechanics for a potential $V(\mathbf{r})$ in the Born approximation, where

$$S_{\text{Born}}(1, 2 \rightarrow 1', 2') = \frac{-e_1 e_2}{4\pi^2} \delta^4(p_1 + p_2 - p_{1'} - p_{2'}) \delta_{m_1 m_{1'}} \delta_{m_2 m_{2'}} \int d^3 \mathbf{r} V(\mathbf{r}) e^{-i\mathbf{q} \cdot \mathbf{r}} \quad (6.62)$$

This shows that the 1-loop corrected amplitude looks just like the amplitude we would find from Born level scattering off a modified classical potential $V_1(\mathbf{r})$ whose Fourier transform

is $e_1 e_2 [1 + \pi(\mathbf{q}^2)]/\mathbf{q}^2$, or in other words

$$V_1(\mathbf{r}) = \frac{e_1 e_2}{(2\pi)^3} \int d^3 \mathbf{q} e^{i\mathbf{q}\cdot\mathbf{r}} \frac{1 + \pi(\mathbf{q}^2)}{\mathbf{q}^2}. \quad (6.63)$$

To lowest order in $\pi(\mathbf{q}^2)$, this is just the potential energy

$$V(|\mathbf{r}|) = \int d^3 \mathbf{x} d^3 \mathbf{y} \frac{\rho_1(\mathbf{x}) \rho_2(\mathbf{y})}{|\mathbf{x} - \mathbf{y} + \mathbf{r}|} \quad (6.64)$$

produced at point \mathbf{r} from the electrostatic interaction of two charge distributions $\rho_1(\mathbf{x})$ and $\rho_2(\mathbf{y})$ defined by

$$\rho_{1,2}(\mathbf{r}) = e_{1,2} \delta^3(\mathbf{r}) + \frac{e_{1,2}}{2(2\pi)^3} \int d^3 \mathbf{q} e^{i\mathbf{q}\cdot\mathbf{r}} \pi(\mathbf{q}^2). \quad (6.65)$$

In particular, we see that

$$\begin{aligned} \int d^3 \mathbf{r} \rho_{1,2}(\mathbf{r}) &= e_{1,2} \left[1 + \frac{1}{2(2\pi)^3} \int d^3 \mathbf{r} d^3 \mathbf{q} e^{i\mathbf{q}\cdot\mathbf{r}} \pi(\mathbf{q}^2) \right] \\ &= e_{1,2} \left[1 + \frac{1}{2} \pi(0) \right] \\ &= e_{1,2}, \end{aligned} \quad (6.66)$$

where the last line uses the on-shell renormalization scheme result (??). Thus the total charge seen by the long-range part of the Coulomb potential is the same as the charge governing the interaction in Feynman diagrams. After a contour integration using the on-shell scheme result for $\pi(\mathbf{q}^2)$, one finds

$$\frac{\rho_{1,2}(\mathbf{r})}{e_{1,2}} = (1+L)\delta^3(\mathbf{r}) - \frac{e^2}{8\pi^3 r^3} \int_0^1 dx x(1-x) \left[1 + \frac{mr}{\sqrt{x(1-x)}} \right] \exp\left(\frac{-mr}{\sqrt{x(1-x)}}\right) \quad (6.67)$$

where L is the integral

$$L = \frac{e^3}{8\pi^3} \int d^3 \mathbf{r} \frac{1}{r^3} \left[\int_0^1 dx x(1-x) \left[1 + \frac{mr}{\sqrt{x(1-x)}} \right] \exp\left(\frac{-mr}{\sqrt{x(1-x)}}\right) \right]. \quad (6.68)$$

This integral diverges at short distances $r \rightarrow 0$. The interpretation is that the bare point charge of strength $e_1(1+L)$ sitting at $\mathbf{r} = 0$ polarizes the vacuum, attracting virtual particles of opposite charge towards it and repelling their antiparticles as they circulate around the loop. Thus the bare charge is partially *shielded* and we see only a finite charge e_1 .

7 Non-Abelian Gauge Theory

Perhaps the single most important class of quantum field theory to understand is non-Abelian Gauge Theory. The starting-point for any non-Abelian gauge theory is the statement that our world is not just a manifold M , but rather a *principal G -bundle*. In this section we'll begin by looking at these from a geometric and topological perspective, before going on to study particular

7.1 Principal bundles and vector bundles

$P \rightarrow M$. These words mean that P is a manifold that comes with a projection map $\pi : P \rightarrow M$, such that for any $x \in M$, $\pi^{-1}(x) \cong G$ for some Lie group G . The space $\pi^{-1}(x)$ is known as the *fibre* of P over x and is often denoted by P_x , while the space M is called the *base*. You should think of P as M with a copy of G attached at each point (see figure ??). In physics, the Lie group G is known as the *gauge group*, while in maths it's often called the *structure group*. For example, electromagnetism is the case $G = U(1)$, while for reasons nobody really understands³⁶ the Standard Model has $G = SU(3) \times SU(2) \times U(1)$.

Principal bundles come with a natural (right) group action $G : P \rightarrow P$ that preserves the fibres. In other words, if $p \in P$ is a point in the fibre over $x \in M$ then acting with a group element $g \in G$ gives another point $pg \in P$, with the property that $\pi(pg) = \pi(p) = x$ so that pg and p both lie in the *same* copy of the fibre. Thus the group action allows you to move around within each copy of G , but does not move you around in M .

To get more of a handle on these abstract ideas, it's useful to describe the situation just in a small region. Given an open set $U \subset M$, a *local trivialization* is a choice of isomorphism

$$\Phi : \pi^{-1}(U) \rightarrow U \times G \tag{7.1}$$

and so gives a way to identify $P|_U$ with $U \times G$. Explicitly, if we're given a point $p \in \pi^{-1}(U) \subset P$, then we can always write $\Phi(p) = (\pi(p), \phi(p))$ where $\pi(p) \in U \subset M$ is just whatever point p projects to, and $\phi(p)$ is some group element. Exactly which group element we get will of course depend on exactly how we choose Φ , but we require that this is compatible with the action of G on the bundle itself. In other words, if $G : p \rightarrow pg$, then

$$\Phi(pg) = (\pi(pg), \phi(pg)) = (\pi(p), \phi(p)g). \tag{7.2}$$

Notice that while it's true *locally* that any principal bundle looks like $U \times G$, this might not be true globally. The simplest example is to take $G = \mathbb{R}$ thought of as a one-dimensional Abelian group, and $M = S^1$. Then both the cylinder $S^1 \times \mathbb{R}$ and the Möbius strip are principal R -bundles — they both look locally like $U \times \mathbb{R}$ — but topologically they are different.

³⁶As you'll learn if you're taking the Part III Standard Model course, the running of the coupling constants for each of the three semi-simple factors, together with the particular representations of $SU(3) \times SU(2) \times U(1)$ in which quarks and leptons lie, suggests that this group may just be a low-energy remnant of a larger 'grand unified gauge group'. This grand unified group is often thought to be $SU(5)$. Or $SU(5) \times U(1)$. Or perhaps $SO(10)$. Or maybe E_6 . Like I said, no one really knows.

Now suppose that $\{U_\alpha\}$ are a collection of open sets in M . Given a local trivialization Φ_α of $\pi^{-1}(U_\alpha)$ on each region U_α , if the open sets overlap then we can ask how the trivializations are related on their common domain of definition $\pi^{-1}(U_\alpha \cap U_\beta)$. First, let's just consider what happens at one point $p \in U_\alpha \cap U_\beta$. By definition, both trivializations involve projecting p to the same point in M , but $\phi_\alpha(p)$ may be a different group element than $\phi_\beta(p)$. Nonetheless, since both $\phi_\alpha(p)$ and $\phi_\beta(p)$ are certainly both in G , we must be able to find a group element $t_{\alpha\beta} \in G$ such that

$$\phi_\beta(p) = \phi_\alpha(p)t_{\alpha\beta}. \quad (7.3)$$

This group element allows us to relate our two local trivializations at some point p . If we wish to compare our trivializations throughout $\pi^{-1}(U_\alpha \cap U_\beta)$ then we must allow $t_{\alpha\beta}$ to vary. Thus, given a pair of open sets U_α and U_β , we define a *transition function* to be a map

$$T_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow G. \quad (7.4)$$

which we can think of as a G -valued function $t_{\alpha\beta}(x)$, defined at each $x \in U_\alpha \cap U_\beta$. So that we can compare Φ_α with Φ_β as well as compare Φ_β with Φ_α , we ask that the transition functions are invertible: $t_{\beta\alpha}(x) = t_{\alpha\beta}^{-1}(x)$. We also impose the compatibility relation $T_\alpha \gamma = T_\alpha \beta \circ T_\beta \gamma$ on triple overlaps $U_\alpha \cap U_\beta \cap U_\gamma$, which says that the result of comparing Φ_α with Φ_γ is the same as first comparing Φ_α with Φ_β and then comparing Φ_β with Φ_γ in any region where all three are defined. Finally, it's natural to ask that all these transition functions vary smoothly over $U_\alpha \cap U_\beta$, so we will.

In physics, the most common case of all this is when $U_\alpha = U_\beta$ are actually the *same* set U , and we're just comparing two different ways of identifying $\pi^{-1}(U)$ with $U \times G$. In this case, the local trivializations are thought of as choices of *gauge*, while the transition function is usually called a *gauge transformation*. For example, a familiar case might be to choose $M \cong \mathbb{R}^{3,1}$ and $G \cong U(1)$, whereupon for each x we could write $t(x) = e^{i\lambda(x)} \in U(1)$ with $\lambda(x)$ a gauge parameter in electrodynamics. Another example that should be familiar from General Relativity is to take M to be (curved) space-time and G to be $GL(d, \mathbb{R})$. In this case, a local trivialization is a choice of *coordinate system* above an open patch U , whereas the transition functions $T_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(d, \mathbb{R})$ are general coordinate transformations. Of course, in any open region of M there could be many valid coordinate systems, and a large part of the Principle of Relativity is the statement that the actual *physics* doesn't depend on which coordinates (= local trivialization) we use. You probably also know that spaces with non-trivial topology (*e.g.* just a circle S^1 or sphere S^d) cannot be described by just a single set of coordinates, which is why all our constructions are defined only locally.

Let me also point out that the mathematical picture of principal bundles is also the inspiration for efforts to recover the Standard Model from higher dimensional theories, initiated in the 1930s by Kaluza and later by Klein. For example, the Lie group $SU(2)$ is isomorphic to S^2 , so a principal $SU(2)$ -bundle over 'our' space-time M can be thought of as a six dimensional space-time where every point of M comes with a copy of S^2 . If the sphere is very small, then present technology won't allow us to concentrate enough energy

in a small region so as to excite spherical harmonics on the S^2 , since these oscillations will involve some very high frequency set by the inverse radius. Thus, at low energies, this theory should involve just the constant modes on S^2 and so look like a theory on M . We might hope to find some remnant of the spherical extra dimensions by examining how low-energy fields transform under higher dimensional coordinate transformations. You can find out more about these ideas in the Extra Dimensions course in Part III.

7.1.1 Choosing a representation

As you learned in the Symmetries, Particles & Fields course, whenever we're given a Lie group it's often a good idea to look at the *representations* of this group. Recall that a representation ρ is a choice of vector space (usually either \mathbb{R}^r or \mathbb{C}^r) together with map

$$\rho : G \rightarrow \text{Mat}(r; \mathbb{R}) \quad \text{or} \quad \rho : G \rightarrow \text{Mat}(r; \mathbb{C}) \quad (7.5)$$

to the space of $r \times r$ matrices (with real or complex values) that tells us how elements of G act on the vector space. This map should be compatible with the group structure in the sense that

$$\rho(gh) = \rho(g) \circ \rho(h), \quad (7.6)$$

where on the left gh denotes multiplication in G , whereas the *rhs* denotes matrix multiplication.

For example, the fundamental representation of the rotation group $SO(3)$ represents elements $g \in SO(3)$ by 3×3 matrices that are orthogonal and have unit determinant; these matrices tell us how the components of a standard vector $\mathbf{v} \in \mathbb{R}^3$ change as we rotate. The same group also has a spinor representation where an element $g \in SO(3)$ is represented by a 2×2 unitary matrix U_g with unit determinant, often written in terms of the Pauli sigma matrices σ as $e^{i\alpha_g \cdot \sigma}$, where the parameters α_g depend on which group element we're considering. This representation tells us how the two complex components of a spinor (such as an electron wavefunction) change under rotations.

Now, if we have a principal bundle then we have not just a single copy of a Lie group G , but a whole family of copies, one at each point of M . If we pick a representation ρ , we thus get a whole family of vector spaces. This structure is known as a *vector bundle* $E \rightarrow M$. In a vector bundle the fibre $\pi^{-1}(x)$ at each point $x \in M$ is now a vector space — the one we got when we chose a G -representation. Just like above, a vector bundle has local trivializations $\Phi : \pi^{-1}(U) \cong U \times \mathbb{C}^r$ and transition functions (or gauge transformations) are maps $T_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{Mat}(r; \mathbb{C})$. If the principal bundle we started with is a subgroup of $GL(r; \mathbb{C})$ then these transition functions will preserve some structure on the vector space. For example, if $G = U(r) \subset GL(r; \mathbb{C})$ and we choose the fundamental r -dimensional (complex) representation, then the transition functions will be *unitary* matrices preserving the inner product $\sum_{a=1}^r |z^a|^2$ on each fibre, while if $G = SU(r)$ then the transition functions will additionally have *unit determinant*, and so will also preserve the top holomorphic form³⁷

$$\epsilon_{a_1 \dots a_r} dz^{a_1} \wedge dz^{a_2} \wedge \dots \wedge dz^{a_r}$$

³⁷Don't worry if you don't know what this means.

on each fibre.

Vector bundles are of relevance to physics because a charged matter field is a *section* of E . This is a map

$$s : M \rightarrow E \tag{7.7}$$

that obeys $\pi \circ s = \text{id}$. Given a choice of trivialization, we can think of this section as the assignment of a vector $s^a(x) \in \mathbb{C}^r$ (for a complex vector bundle) to each point x . If we change our local trivialization using a gauge transform, then the particular vector components we get will change according to

$$s_\beta(x) = s_\alpha(x) t_{\alpha\beta}(x) \tag{7.8}$$

where $t_{\alpha\beta}(x)$ are the transition functions. We'll sometimes write $\Omega_M^0(E)$ to denote the space of all smooth sections of $E \rightarrow M$.

As a simple example, consider a complex scalar field on M . Usually, we think of this as just a function $\phi : M \rightarrow \mathbb{C}$ where $\phi : x \mapsto \phi(x)$ is the value of the field at $x \in M$. However, if this scalar carries electric charge q , then it doesn't really have any preferred ‘value’ because, as you learnt in electrodynamics, we can change $\phi(x) \rightarrow e^{iq\lambda(x)}\phi(x)$ by a gauge transform. The correct interpretation of our charged scalar is that ϕ is really a section of a vector bundle $E \rightarrow M$ associated to the principal $U(1)$ bundle of electromagnetism. Once we pick a local trivialization — *i.e.* pick a gauge — then we can think of $\phi(x)$ as a (one component) vector, at least for some open region $U \subset M$. However, there's no preferred way to choose this gauge, and making different choices (changing gauge) will cause ϕ to transform as is familiar.

The electromagnetic example is special because the only irreducible representations of $U(1)$ are one-dimensional; they're just labelled by the *charge* q of the field, with $q = 0$ being the trivial representation. As you saw in Symmetries, Particles & Fields, non-Abelian groups typically have (infinitely) many different irreps. For example, **BLAH BLAH BLAH**

Notice that the statement that a section is a map $s : M \rightarrow E$ means that, once we've picked a gauge, the resulting field *depends only on* $x \in M$. This is where the notion of a gauge theory differs from Kaluza–Klein theory, where the fields are allowed to vary over the whole higher-dimensional space.

Above, we've constructed vector bundles from principal bundles by making a choice of representation. Finally, let me mention that we can also go the other way and construct a principal bundle starting from a vector bundle, at least in the case of a matrix Lie group (those that are subgroups of $GL(r)$). For given any rank r vector bundle $E \rightarrow M$, we define the *frame bundle* to be the principal $GL(r)$ -bundle whose fibre $\pi^{-1}(x)$ is the collection of all basis vectors (= frames) in the fibre E_x . There is a natural action of $GL(r)$ on this frame bundle, relating any pair of frames. If we wish to construct a principal G -bundle for a subgroup of $GL(r)$ then as above we just require that our basis vectors are compatible with some extra structure. For example, in the real case we obtain a principal $O(r)$ -bundle by asking that our basis vectors are orthogonal, and a principal $SO(r)$ bundle by asking that they also define a fixed volume element.

The most common Lie groups that arise in physics are indeed matrix Lie groups, so the two viewpoints are equivalent. However, in some exotic theories (especially string theory and some grand unified theories) exceptional Lie groups such as E_6 play an important role, so the fundamental picture is really that of principal bundles. With this caveat, I'll mostly use vector bundles from now on.

7.1.2 Connections and curvature

So far, we have described a vector bundle E as just a collection of vector spaces parametrized by a base space M . If we wish to write down the kinetic terms of any matter field, we will need derivatives and we'd usually write

$$v^\mu \partial_\mu \phi(x) \stackrel{?}{=} \lim_{\epsilon \rightarrow 0} \frac{\phi(x + \epsilon v) - \phi(x)}{\epsilon}, \quad (7.9)$$

where v^μ is a vector at x . However, for a charged field this expression is meaningless as it stands, because the two terms on the right live in completely different spaces: $\phi(x + \epsilon v)$ lives in the fibre $E_{x+\epsilon v}$ while $\phi(x)$ lives in a different copy E_x of the fibre.

To make sense of this, we need a way to compare vectors in different fibres, which is what a *connection*, or *covariant derivative* provides. This is a linear map

$$\nabla : \Omega_M^0(E) \rightarrow \Omega_M^1(E) \quad (7.10)$$

from the space of sections to the space of 1-forms (covectors) on M with values in E . The connection is defined by the properties of *linearity*:

$$\nabla(\alpha_1 s_1 + \alpha_2 s_2) = \alpha_1 \nabla(s_1) + \alpha_2 \nabla(s_2) \quad (7.11)$$

for any two sections s_1, s_2 and constants α_1, α_2 , and the *Leibniz rule*:

$$\nabla(fs) = df s + f \nabla(s), \quad (7.12)$$

where $f \in C^\infty(M)$ is a smooth function. More specifically, for every tangent vector v on M , the connection defines a derivative $v \cdot \nabla s = v^\mu \nabla_\mu(s)$, thought of as the derivative of our section $s(x)$ in the direction of v , and then the Leibniz rule says $v \cdot \nabla(fs) = f v^\mu \nabla_\mu s + (v^\mu \partial_\mu f)s$, where ∂_μ is the standard partial derivative of the function f . Notice that if ∇ and ∇' are any two connections, then the difference obeys

$$(\nabla - \nabla')(fs) = f(\nabla - \nabla')s. \quad (7.13)$$

Thus $(\nabla - \nabla')$ maps $\Omega_M^0(E) \rightarrow \Omega_M^1(E)$ in a way that is linear over *functions* $f \in C^\infty(M)$. Hence the difference between any two connections is an element of $\text{Hom}(E, E \otimes T^*M) \cong \text{End}(E) \otimes T^*M$.

To understand what this means, let's again look in a small region. Suppose we have a trivialization $\Phi : E|_U \rightarrow U \times \mathbb{C}^r$. Then in this region, any section $s : U \rightarrow E$ can be thought of as a vector-valued function on U , *i.e.* given $s : M \rightarrow E$, we can write $\Phi \circ s : U \rightarrow U \times \mathbb{C}^r$ with $\Phi(s(x)) = (x, s_U(x))$ for some vector $s_U(x)$. Then, within U , we have

$$\Phi(\nabla s)(x) = (x, \nabla s_U) \quad \text{where} \quad (\nabla s)_U = ds_U + A_U s_U \quad (7.14)$$

where A_U is a section of $U \otimes \text{End}(\mathbb{C}^r) \otimes T^*M|_U$ that is independent of the particular section s . The object A_U is thus a matrix-valued 1-form (or covector) known to mathematicians as a *connection 1-form*. In physics, A_U is the gauge field.

To make this more familiar, suppose U_α and U_β are each open sets in M with overlap $U_\alpha \cap U_\beta \neq \emptyset$. Suppose we're given trivializations $\Phi_\alpha : E|_{U_\alpha} \rightarrow U_\alpha \times \mathbb{C}^r$ and $\Phi_\beta : E|_{U_\beta} \rightarrow U_\beta \times \mathbb{C}^r$, and let $g_{\beta\alpha} : U_\beta \cap U_\alpha \rightarrow \text{End}(\mathbb{C}^r)$ denote the transition function. In physics, we'd say we're working in one gauge (given by Φ_α) in U_α , and in another gauge in U_β , and that $g_{\beta\alpha}$ is the gauge transformation that takes us between the two gauges. We have $s_\beta = g_{\beta\alpha}s_\alpha$ and similarly $(\nabla s)_\beta = g_{\beta\alpha}(\nabla s_\alpha)$, since these are sections of E and $E \otimes T^*M$, respectively. It now follows that the gauge fields A_α and A_β on the two open patches must be related by

$$A_\beta = -g_{\beta\alpha}dg_{\beta\alpha}^{-1} + g_{\beta\alpha}A_\alpha g_{\beta\alpha}^{-1}. \quad (7.15)$$

As a special case, U_β and U_α might both be the *same* region, in which case (7.15) tells us how A changes under a change of trivialization on a given region. Notice also that if $\text{rk}(E) = r = 1$, then $g(x) \in \text{End}(\mathbb{C}^r)$ is just a single function at each point of $x \in U_\beta \cap U_\alpha$ which we can write as $e^{i\lambda}$ whereupon (7.15) reduces to $A_\beta = A_\alpha - id\lambda$, which is the familiar behaviour under a gauge transformation of the vector potential in electromagnetism.

Given a connection $\nabla : \Omega_M^0(E) \rightarrow \Omega_M^1(E)$, we can extend its definition to sections of $E \otimes \wedge^p T^*M$. Such sections are to be thought of as p -forms³⁸ with values in sections of E . The space of such is written $\Omega_M^p(E)$. This extension is also called ∇ , and is again defined by

$$\begin{aligned} \text{linearity} \quad & \nabla(\mathfrak{s}_1 + \mathfrak{s}_2) = \nabla\mathfrak{s}_1 + \nabla\mathfrak{s}_2 \\ \text{Leibniz} \quad & \nabla(s\omega) = \nabla(s)\wedge\omega + s d\omega \end{aligned} \quad (7.16)$$

where $\mathfrak{s}_{1,2} \in \Omega_M^p(E)$ are p -form sections, while $s \in \Omega_M^0(E)$ is a regular section as above and $\omega \in \Omega_M^p$ is a p -form on M .

The construction above shows that the connection ∇ behaves as a sort of exterior derivative, generalized the usual de Rham exterior derivative d to the case of sections of vector bundles. However, there is a crucial difference. While $d^2 = 0$ automatically, it is not in general true that $\nabla^2 = 0$. To see this, note that

$$\begin{aligned} \nabla(\nabla(s\omega)) &= \nabla((\nabla s)\wedge\omega + s d\omega) \\ &= (\nabla^2 s)\wedge\omega - (\nabla s)\wedge d\omega + (\nabla s)\wedge d\omega + s d^2\omega \\ &= (\nabla^2 s)\wedge\omega \end{aligned} \quad (7.17)$$

where the second and third terms have cancelled and the last term is identically zero by the nilpotency of the de Rham exterior derivative. This calculation shows that $\nabla^2 : \Omega_M^p(E) \rightarrow \Omega_M^{p+2}(E)$ is linear over multiplication of the section by an arbitrary form,

$$\nabla^2(\mathfrak{s}\wedge\omega) = (\nabla^2\mathfrak{s})\wedge\omega.$$

³⁸A p -form on M can be thought of as a tensor with p contravariant indices, antisymmetrized on these indices. We write Ω_M^p for the space of such p -forms. Thus, if $\omega \in \Omega_M^p$ then in local coordinates on M we have $\omega = \omega_{\mu_1\mu_2\cdots\mu_p}(x) dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p}$.

It therefore must correspond to multiplication by some section $F_\nabla \in \Omega_M^2(\text{End}(E))$. In particular, if $s \in \Omega_M^0(E)$, we have

$$\nabla^2(s) = F_\nabla s. \quad (7.18)$$

The $\text{End}(E)$ -valued 2-form F_∇ is called the *curvature* of the connection.

To understand this more explicitly, let's again choose a local trivialization $\Phi : E|_U \rightarrow U \times \mathbb{C}^r$, with $(\nabla s)_U = ds_U + A_U s_U$. Then we have

$$\begin{aligned} (\nabla^2 s)_U &= \nabla(ds_U + A_U s_U) \\ &= d^2 s_U + d(A_U s_U) + A_U \wedge (ds_U + A_U s_U) \\ &= (dA_U + A_U \wedge A_U)s_U \end{aligned} \quad (7.19)$$

and indeed all the derivatives of s_U itself cancel out. Thus on $E|_U$ with the trivialization given by Φ , we can identify the curvature as

$$\begin{aligned} (F_\nabla)_U &= dA_U + A_U \wedge A_U = (\partial_\mu A_\nu + A_\mu A_\nu) dx^\mu \wedge dx^\nu \\ &= \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]) dx^\mu \wedge dx^\nu =: \frac{1}{2}F_{\mu\nu} dx^\mu \wedge dx^\nu \end{aligned} \quad (7.20)$$

where $F_{\mu\nu}$ are the curvature components. Recall that A_U is a matrix-valued 1-form, so that the commutator term $[A_\mu, A_\nu]$ does not in general vanish.

Given that $\nabla^2(\mathfrak{s}) = F_\nabla \mathfrak{s}$ for any section $\mathfrak{s} \in \Omega_M^p(E)$, so that ∇^2 acts as a purely multiplicative operator, it's interesting to compute what happens when we act with ∇ for a third time. On the one hand, we have

$$\nabla^3(\mathfrak{s}) = \nabla(F_\nabla \mathfrak{s}) = \nabla(F_\nabla)\mathfrak{s} + F_\nabla \nabla(\mathfrak{s}) \quad (7.21)$$

while on the other hand,

$$\nabla^3(\mathfrak{s}) = \nabla^2(\nabla \mathfrak{s}) = F_\nabla \nabla \mathfrak{s}. \quad (7.22)$$

The two equations (7.21) & (7.22) are compatible iff

$$\nabla(F_\nabla) = 0, \quad (7.23)$$

which is known as the *Bianchi identity* for the curvature F_∇ . In a local trivialization where $\nabla = d + A_U$ and $F_\nabla = dA_U + A_U \wedge A_U$ the Bianchi identity can be seen explicitly by calculating

$$\begin{aligned} \nabla(F_\nabla)|_U &= dF_\nabla + A_U \wedge F_\nabla - F_\nabla \wedge A_U \\ &= d(dA_U + A_U \wedge A_U) + A_U \wedge (dA_U + A_U \wedge A_U) - (dA_U + A_U \wedge A_U) \wedge A_U \\ &= dA_U \wedge A_U - A_U \wedge dA_U + A_U \wedge dA_U + A_U^3 - dA_U \wedge A_U - A_U^3 \\ &= 0, \end{aligned} \quad (7.24)$$

where we've used the fact that, in a local trivialization, the covariant derivative acts as $\nabla\phi = d\phi + A \wedge \phi - (-)^p \phi \wedge A$ on any section $\phi \in \Omega_M^p(\text{End}(E))$. This agrees with our global argument above.

7.1.3 Holonomy

Some other time.

7.2 Classical Yang–Mills theory

The first and most important example of a non–Abelian gauge theory was introduced to physics 1954 by Chen Ning Yang and Robert Mills, and then almost completely ignored for nearly a decade. From a phenomenological point of view, the importance of Yang–Mills theory arises because (as you’re surely aware) the Standard Model — the most fundamental description of Nature we currently possess — is at its heart a non–Abelian gauge theory based on $SU(3) \times SU(2) \times U(1)$, though it took much hard work and many further new ideas before this became apparent. From the perspective of a theoretical physicist, Yang–Mills is theory important also because it’s the *only* QFT in $d = 4$ that might a continuum limit, as realized by Coleman & Gross in 1973. In mathematics, Yang–Mills theory is at the heart of Simon Donaldson’s exploration of the wild world of four–manifolds. More recently, it’s even been related to the (geometric) Langlands Program. In a deep sense, Yang–Mills theory is the right four–dimensional analogue of geodesics in $d = 1$ and harmonic maps in $d = 2$.

7.2.1 The Yang–Mills action

To describe Yang–Mills theory, we pick a d –dimensional (pseudo-)Riemannian manifold (M, g) complete with a choice of metric g . The Yang–Mills action is then defined to be

$$S_{\text{YM}}[\nabla] = \frac{1}{2g_{\text{YM}}^2} \int_M \text{tr}(F_\nabla \wedge *F_\nabla) = \frac{1}{4g_{\text{YM}}^2} \int_M g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho}^a F_{\nu\sigma}^a \sqrt{g} d^d x \quad (7.25)$$

where g_{YM} is a coupling constant. The Yang–Mills action is thus just the (square of) the L^2 -norm of F_∇ with respect to the standard volume element on M supplied by the metric g . It’s the natural generalization of the Maxwell action

$$S_{\text{Max}}[\nabla] = \frac{1}{4e^2} \int F^{\mu\nu} F_{\mu\nu} d^4 x = \frac{1}{4e^2} \int (\mathbf{E} \cdot \mathbf{E} - \mathbf{B} \cdot \mathbf{B}) dt d^3 x \quad (7.26)$$

of electromagnetism, to which it reduces when $G = U(1)$ and $(M, g) = (\mathbb{R}^{3,1}, \delta)$. I’ve written the action as $S_{\text{YM}}[\nabla]$ to emphasize that we should treat the action as a function of the connection, not of the curvature. Again, this is familiar to you from deriving Maxwell’s (vacuum) equations as the Euler–Lagrange equations of (7.26).

From the point of view of physics, the most important difference between the Yang–Mills action for a non–Abelian group G and the Abelian (Maxwell) case is that, for non–Abelian G , *the Yang–Mills field interacts with itself*. We can see this by restricting to a local trivialization on $U \subset M$ where³⁹ $\nabla = d + A$ and $F = dA + A^2$. On this patch the

³⁹Henceforth, I’ll drop the subscript on A_U , as is common.

Yang–Mills action becomes

$$\begin{aligned} S_{\text{YM}}[A] &= \frac{1}{2g_{\text{YM}}^2} \int_U \text{tr}((dA + A^2) \wedge *(dA + A^2)) \\ &= \frac{1}{4g_{\text{YM}}^2} \int_U g^{\mu\nu} g^{\rho\sigma} \left(\partial_{[\mu} A_{\rho]}^a + \frac{1}{2} f_{bc}^a A_{[\mu}^b A_{\rho]}^c \right) \left(\partial_{[\nu} A_{\sigma]}^a + \frac{1}{2} f_{de}^a A_{[\nu}^d A_{\sigma]}^e \right) \sqrt{g} d^d x \end{aligned} \quad (7.27)$$

Thus we see that the action contains a cubic vertex of the schematic form $\sim AA dA$, and a quartic vertex $\sim A^4$. Thus, even in the absence of any charged matter, Yang–Mills theory is a non–trivial interacting theory.

At the classical level, these self–interactions make themselves felt via the Euler–Lagrange equations that follow from (7.25). Varying the connection $\nabla \rightarrow \nabla + \delta a$ where $\delta a \in \Omega_M^1(\text{End}(E))$, to first order in δa we have that

$$F_{\nabla+\delta a} - F_\nabla = \nabla(\delta a), \quad (7.28)$$

or in other words, $\delta F_{\mu\nu} = \nabla_{[\mu} \delta a_{\nu]}$. Therefore, varying the action gives⁴⁰

$$\delta S_{\text{YM}} = \frac{1}{g_{\text{YM}}^2} \int_M \text{tr}(\nabla \delta a \wedge *F_\nabla) = -\frac{1}{g_{\text{YM}}^2} \int_M \text{tr}(\delta a \wedge \nabla *F_\nabla) + \text{boundary terms} \quad (7.29)$$

so that the Euler–Lagrange equations are

$$\nabla *F_\nabla = 0, \quad (7.30)$$

or equivalently

$$0 = \nabla^\mu F_{\mu\nu}^a = \partial^\mu F_{\mu\nu}^a + \frac{1}{2} f_{bc}^a A^{b\mu} F_{\mu\nu}^c \quad (7.31)$$

in terms of components in a local trivialization. However you wish to write them, these are known as the *Yang–Mills equations*. We recall from (7.23) that the curvature of any connection automatically obeys the Bianchi identity $\nabla F_\nabla = 0$, or

$$\nabla_\mu F_{\nu\lambda} + \nabla_\nu F_{\lambda\mu} + \nabla_\lambda F_{\mu\nu} = 0 \quad (7.32)$$

or

$$\partial_\mu F_{\nu\lambda}^a + \partial_\nu F_{\lambda\mu}^a + \partial_\lambda F_{\mu\nu}^a + \frac{1}{2} f_{bc}^a \left(A_\mu^b F_{\nu\lambda}^c + A_\nu^b F_{\lambda\mu}^c + A_\lambda^b F_{\mu\nu}^c \right) = 0 \quad (7.33)$$

to be completely explicit. Mathematically, connections whose curvature obeys the Yang–Mills equation (7.30) are *critical points* of the function $S_{\text{YM}}[\nabla]$ defined on the space of *all* connections on $P \rightarrow M$.

Unlike the vacuum Maxwell equations, the Yang–Mills equations and the Bianchi identity (7.33) are *non–linear* p.d.e.s. The non–linearities arise because of the presence of A in the covariant derivative ∇ and the non–linear terms $\sim A^2$ in the curvature itself. The reason this happened is that the curvature F transforms in the adjoint representation,

⁴⁰As an exercise, you should go through this derivation for yourself, using the second line of (7.27) and checking you’re happy with where all the indices go. Once you’ve done this and checked you are happy, I hope you’ll begin to appreciate the usefulness of the form notation.

with $F \rightarrow gFg^{-1}$ under a gauge transformation $g(x)$. So any sort of differential equation obeyed by F will have to involve covariant derivatives, as these are the only derivatives that make geometric sense. Electrodynamics avoids this complication because the adjoint representation of an Abelian group is actually trivial, as we can see by the manipulation $gFg^{-1} = Fgg^{-1} = F$, which is allowed since all group elements commute. As a consequence of the non-linearity, unlike beams of light or radio waves in the Abelian case, we do not expect superposition of solutions, and propagating waves of Yang–Mills fields do not pass through one another freely.

Famously, General Relativity is also a geometric theory in which the field (Einstein) equations are a system of non-linear pdes. Indeed, there are many parallels between the two theories, most of which can be made apparent by treating General Relativity as a theory of connections on the tangent bundle $TM \rightarrow M$. Now I’m sure you can all write down several non-trivial solutions of the Einstein equations, probably including the Schwarzschild metric, various homogeneous cosmological models, the Kerr metric and perhaps a few others. Likewise, I certainly expect you’ve solved Maxwell’s equations in the presence of any number of weird charge configurations, including electrical circuits, solenoids and all manner of other things. However, I doubt that many of you know any non-trivial solutions of the Yang–Mills equations at all.

Why not? The answer turns out to be revealing⁴¹. Turning the question around, the reason you *do* know lots of solutions of the Maxwell or Einstein equations is simply that the role these equations play in Nature has been understood for over a century. Their weak field, Newtonian approximations have been known considerably longer, and the phenomena they describe are apparent in everyday life. By contrast, our technology has only just reached the point where we can perform any experiment in which the classical Yang–Mills equations are relevant.

The reason this is so is an effect known as the *mass gap*. Skipping ahead of our story, the path integral for Yang–Mills will roughly take the form $\int DA e^{-S_{\text{YM}}[\nabla]/\hbar}$. Because the coupling constant appears only as an overall factor in the Yang–Mills action (7.25), it plays the same role as \hbar ; the path integral depends on g_{YM} and \hbar only through the combination $g_{\text{YM}}^2 \hbar$ (at least in pure Yang–Mills). Thus we should expect that the quantum theory is well-approximated by the classical limit if $g_{\text{YM}} \rightarrow 0$ so that the theory is ‘weakly coupled’. However, we’ve seen already that coupling constants can run depending on the scale at which we examine the physics. Below, we’ll find that in a non-Abelian theory (with not too much matter), the gauge coupling actually *increases* as we view physics at lower and lower energy scales. In this region, Yang–Mills theory is an inherently *quantum* theory and can yield results that are very far from the classical story. For example, the energy scale at which QCD becomes strongly coupled is ~ 200 MeV. This is accessible to particle accelerators, but at everyday energy scales the role of Yang–Mills theory in Nature cannot be described without talking about the quantum theory.

In fact, when C.N. Yang first presented his work with Mills, he was strongly criticized

⁴¹No, it’s not that the Yang–Mills equations are ‘harder’ to solve than the Einstein equations — quite the contrary.

by Pauli. Pauli had spotted that there is no term $\sim A^2$ in the Yang–Mills action (it would not be compatible with gauge invariance), so quanta of the Yang–Mills field will be massless. He concluded that, as in both electromagnetism and gravity, these particles will be responsible for some long range force⁴². Needless to say, in Nature no other such forces exist: this is why the Yang–Mills equations were only contemplated in the middle of the 20th century. (Yes, strong and weak nuclear forces were known about earlier, but they’re certainly not long range.) Pauli’s conclusion was correct in the weak coupling approximation, but the classical Yang–Mills equations are a very poor guide to the low-energy physics.

7.2.2 Minimal coupling

We can also write down actions describing the coupling of Yang–Mills theory to charged matter. The simplest of these are the *minimally coupled* actions, which essentially says that you just take your favourite action for uncharged matter, and then replace all derivatives by gauge covariant derivatives.

For example, suppose we have a scalar field ϕ that lives in the adjoint representation of the gauge group so that $\phi \rightarrow h\phi h^{-1}$ under a gauge transformation $h(x)$. With minimal coupling, the kinetic terms for this scalar are

$$S_{\text{kin}}[\phi, \nabla] = \int_M \text{tr}(\nabla\phi \wedge * \nabla\phi) \quad (7.34)$$

and locally on $U \subset M$ this is

$$S_{\text{kin}}[\phi, A] = \frac{1}{2} \int_U g^{\mu\nu} \left(\partial_\mu \phi^a + \frac{1}{2} f_{bc}^a A_\mu^b \phi^c \right) \left(\partial_\nu \phi^a + \frac{1}{2} f_{de}^a A_\nu^d \phi^e \right) \sqrt{g} d^d x. \quad (7.35)$$

We can also construct potential terms for ϕ of the form

$$S_{\text{pot}}[\phi] = \int_U V(\phi) \sqrt{g} d^d x \quad (7.36)$$

where $V(\phi)$ is any gauge invariant polynomial in ϕ . For example, since ϕ is in the adjoint representation we can think of it as a matrix and then a simple choice would be

$$V(\phi) = \frac{m^2}{2} \text{tr}(\phi\phi) + \frac{\lambda}{4!} \text{tr}(\phi\phi\phi\phi) \quad (7.37)$$

where the traces ensure that $V(h\phi h^{-1}) = V(\phi)$.

Similarly, the minimally coupled action for a massive Dirac spinor ψ transforming the in fundamental representation of G is⁴³

$$S_{\text{Dir}}[\psi, \nabla] = \int_{\mathbb{R}^d} \bar{\psi} (\text{i}\not{\nabla} + m) \psi d^d x \quad (7.38)$$

⁴²Recall that in four dimensions, a particle of mass m gives rise to a potential $V(r) \sim e^{-mr}/r$. For any finite m the resulting force is negligible at distances $\gg 1/m$ from the centre of the potential, but when $m = 0$ the force can be felt right across the Universe.

⁴³I’ll write this just for $(M, g) = (\mathbb{R}^d, \delta)$, which will be sufficient for our purposes in this course. To do more we’d first need to discuss how to define spinors on a curved manifold.

where ∇ acts in the fundamental representation, while the conjugate spinor $\bar{\psi}$ transforms in the antifundamental as $\bar{\psi} \rightarrow \bar{\psi}h$. Explicitly, if $G = SU(N)$ and $i, j = 1, \dots, N$ label a basis of \mathbb{C}^N , then we think of the gauge field A_μ as an $N \times N$ matrix $(A_\mu)_i^j$ and the action is

$$S_{\text{Dir}}[\psi, \nabla] = \int_{\mathbb{R}^d} \bar{\psi}_i \delta_j^i (\mathrm{i}\gamma^\mu \partial_\mu + m) \psi^j + \mathrm{i}\bar{\psi}_i \gamma^\mu (A_\mu)_j^i \psi^i \, d^d x \quad (7.39)$$

where γ^μ are the Dirac γ -matrices. Note that S_{Dir} reduces to the usual electron action in the Abelian case $G = U(1)$ for QED.

For simple operators such as $(\partial\phi)^2$ the replacement $(\partial\phi)^2 \rightarrow (\nabla\phi)^2$ is unambiguous; once we've declared in which representation ϕ transforms, there is a unique notion of the covariant derivative acting on that representation. However, you may feel uneasy about treating a more complicated operator such as $\partial^\mu \partial^\nu \phi \partial_\mu \partial_\nu \phi$. Clearly our replacement prescription should involve $\mathrm{tr}(\nabla^\mu \nabla^\nu \phi \nabla_\mu \nabla_\nu \phi)$, but since $[\partial_\mu, \partial_\nu] = 0$ whereas $[\nabla_\mu, \nabla_\nu] \neq 0$, how can we tell whether or not the operator we end up with should include the antisymmetric part of μ and ν ? There is no unambiguous way to decide, but fortunately the issue is not very important: whether or not we include such terms in our initial action, if they are allowed by global symmetries then they will in any case be generated by quantum corrections to the effective action.

In the presence of charged matter, the Yang–Mills equations (7.30) are replaced by

$$\nabla^\mu F_{\mu\nu} = -g_{\text{YM}}^2 J_\nu \quad (7.40)$$

where $J^\nu(x) = \delta S_{\text{matter}}/\delta a_\nu(x)$ is the matter current. (The Bianchi identity still holds.) Notice that, since the matter action as a whole was invariant and a lives in the adjoint representation, the current J_μ also transforms in the adjoint. For example, in the case of our scalar above we have

$$J_\nu^a = \frac{1}{2} f_{bc}^a \phi^b (\nabla_\nu \phi)^c \quad (7.41)$$

whereas for the Dirac spinor

$$(J_\nu)_i^j = \mathrm{i}\bar{\psi}_i \gamma_\nu \psi^j. \quad (7.42)$$

In electromagnetism, the adjoint representation was trivial, so the electromagnetic current J^{em} satisfied a standard conservation law $\partial^\mu J_\mu^{\text{em}} = 0$. In the non–Abelian case, it only makes sense to differentiate J covariantly using ∇ acting in the adjoint representation. Using the equation of motion (7.40) we have

$$-g_{\text{YM}}^2 (\nabla^\nu J_\nu)^a = (\nabla^\nu (\nabla^\mu F_{\mu\nu}))^a = \frac{1}{2} ([\nabla^\nu, \nabla^\mu] F_{\mu\nu})^a = \frac{1}{4} f_{bc}^a F^{b\mu\nu} F_{\mu\nu}^c \quad (7.43)$$

where the second equality follows since $F_{\mu\nu}$ is antisymmetric in (μ, ν) , the third equality uses the facts that the commutator of covariant derivatives is F and that these derivatives act on the original F in the adjoint representation. Finally, this expression vanishes by antisymmetry of the structure constants $f_{bc}^a = -f_{cb}^a$. Thus we have a *covariant conservation law*

$$\nabla^\mu J_\mu = 0 \quad (7.44)$$

for our current in a non–Abelian theory. The differences between this conservation law and the naive conservation law $\partial^\mu J_\mu^{\text{Noether}}$ we found for Noether currents in section ?? have some profound consequences: you’ll explore these in Problem Set 3.

7.3 Quantum Yang–Mills theory

We’re now ready to consider the quantum theory of Yang–Mills. In the first few sections, we’ll treat the path integral formally as an integral over infinite dimensional spaces, without worrying about imposing cutoffs. We’ll turn to questions about using renormalization to make sense of these formal integrals in section ??.

To specify Yang–Mills theory, we had to pick a principal G bundle $P \rightarrow M$ together with a connection ∇ on P . So our first thought might be to try to define the Yang–Mills partition function as

$$Z_{\text{YM}}[(M, g), g_{\text{YM}}] \stackrel{?}{=} \int_{\mathcal{A}} DA e^{-S_{\text{YM}}[\nabla]} \quad (7.45)$$

where \mathcal{A} is the space of all connections on P . To understand what this integral might mean, first note that, given any two connections ∇ and ∇' , the 1-parameter family

$$\nabla^\tau := \tau \nabla + (1 - \tau) \nabla' \quad (7.46)$$

is also a connection for all $\tau \in [0, 1]$. For example, you can check that the *rhs* has the behaviour expected of a connection under any gauge transformation. Thus we can find a path in \mathcal{A} between any two connections. Since $\nabla' - \nabla \in \Omega_M^1(\mathfrak{g})$, we conclude that \mathcal{A} is an infinite dimensional affine space whose tangent space at any point is $\Omega_M^1(\mathfrak{g})$, the infinite dimensional space of all \mathfrak{g} –valued covectors on M . In fact, it’s easy to write down a flat (L^2 -)metric on \mathcal{A} using the metric on M :

$$ds_{\mathcal{A}}^2 = \int_M \text{tr}(\delta A \wedge * \delta A) = \frac{1}{2} \int_M g^{\mu\nu} \delta A_\mu^a \delta A_\nu^a \sqrt{g} d^d x . \quad (7.47)$$

In other words, given any two tangent vectors $(a_1, a_2) \in \Omega_M^1(\mathfrak{g})$ at the point $\nabla \in \mathcal{A}$,

$$ds_{\mathcal{A}}^2(a_1, a_2) = \int_M \text{tr}(a_1 \wedge * a_2) , \quad (7.48)$$

independent of where in \mathcal{A} we are. This is encouraging: \mathcal{A} just looks like an infinite dimensional version of \mathbb{R}^n , with no preferred origin since there is no preferred connection on P .

We might now hope that the path integral (7.45) means formally that we should pick an arbitrary base–point $\nabla_0 \in \mathcal{A}$, then write any other connection $\nabla = \nabla_0 + A$, with the measure DA indicating that we integrate over all $A \in \Omega_M^1(\mathfrak{g})$ using the translationally invariant measure on \mathcal{A} associated to the flat metric (7.47). (Such an infinite dimensional flat measure does not exist — we’re delaying this worry for now.) For a connection $\nabla = \nabla_0 + A$, the action becomes

$$\begin{aligned} S_{\text{YM}}[\nabla] &= \frac{1}{2g_{\text{YM}}^2} \int_M \text{tr}(F_\nabla \wedge * F_\nabla) \\ &= \frac{1}{2g_{\text{YM}}^2} \int_M \text{tr}(F_{\nabla_0} \wedge * F_{\nabla_0}) + \frac{1}{2g_{\text{YM}}^2} \int_M \text{tr}(\nabla_0 A + A^2) \wedge * (\nabla_0 A + A^2) . \end{aligned} \quad (7.49)$$

For example, on a topologically trivial bundle a standard choice would be to pick the trivial connection $\nabla_0 = \partial$ as base-point. Then $F_{\nabla_0} = 0$ and the action takes the familiar form

$$S_{\text{YM}}[\partial + A] = \frac{1}{2g_{\text{YM}}^2} \int_M \text{tr}(dA + A^2) \wedge *(dA + A^2). \quad (7.50)$$

The path integral (7.45) would be interpreted as an integral over all gauge fields A . However, in some circumstances we'll meet later (even when P is topologically trivial), it will be useful to choose a different base-point ∇_0 for which $F_{\nabla_0} \neq 0$, known as a *background field*. In this case, the first term on the *rhs* of (7.49) is the action for the background field and comes out of the path integral as an overall factor, while the remaining action for A involves the covariant derivative with respect to the background field.

Of course, there's a problem. By construction, the Yang–Mills action was invariant under gauge transformations, so the integrand in (7.45) is degenerate along gauge orbits. Consequently, the integral will inevitably diverge because we're vastly overcounting. You met this problem already in the case of QED during the Michaelmas QFT course. There, as here, the right thing to do is to integrate just over physically inequivalent connections — those that are *not* related by a gauge transform. In other words, the correct path integral for Yang–Mills should be of the form

$$Z_{\text{YM}}[(M, g), g_{\text{YM}}] = \int_{\mathcal{A}/\mathcal{G}} D\mu e^{-S_{\text{YM}}[\nabla]} \quad (7.51)$$

where \mathcal{G} is the space of all gauge transformations, so that \mathcal{A}/\mathcal{G} denotes the space of all gauge equivalence classes of connections: we do not count as different two connections that are related by a gauge transformation. Note that this definition means *gauge ‘symmetry’ does not exist* in Nature! We've *quotiented* by gauge transformations in constructing the path integral, so the resulting object has no knowledge of any sort of gauge transformations. They were simply a *redundancy* in our construction. The same conclusion holds if we compute correlation functions of any gauge invariant quantities, whether they be local operators built from gauge invariant combinations of matter fields, or Wilson loops running around some curves in space.

However, we're not out of the woods. Whilst \mathcal{A} itself was just an affine space, the space \mathcal{A}/\mathcal{G} is much more complicated. For example, it has highly non-trivial topology investigated by Atiyah & Jones, and by Singer. Certainly \mathcal{A}/\mathcal{G} is *not* affine, so we don't yet have any understanding of what the right measure $D\mu$ to use on this space is, even formally. In the case of electrodynamics, you were able to avoid this problem (at least in perturbation theory on \mathbb{R}^4) by picking a gauge, defining the photon propagator and just getting on with it. The non-linear structure of the non-Abelian theory means we'll have to consider this step in more detail.

7.3.1 A ghost story

The way to proceed was found by Feynman, de Witt, and by Faddeev & Popov. To understand what they did, let's warm up with a finite-dimensional example.

Figure 11: The gauge slice C should be chosen to be transverse to the gauge orbits.

Suppose we have a function $S : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined at any point on the (x, y) -plane, and suppose further that this function is invariant under rotations of the plane around the origin. We think of $S(x, y)$ as playing the role of our ‘action’ for ‘fields’ (x, y) , while rotations represent ‘gauge transformations’ leaving this action invariant. Of course, rotational invariance implies that $S(x, y) = h(r)$ in this example, where $h(r)$ is some function of the radius. We easily compute

$$\int_{\mathbb{R}^2} dx dy e^{-S(x,y)} = 2\pi \int_0^\infty dr r e^{-h(r)} \quad (7.52)$$

which will make sense for sufficiently well-behaved $f(r)$. The factor of 2π appears here because the original integral was rotationally symmetric: it represents the redundancy in the expression on the left of (7.52).

In the case of Yang–Mills, if we integrated over the space \mathcal{A} of all connections rather than over \mathcal{A}/\mathcal{G} , the redundancy would be infinite: while the volume $\text{vol}(G)$ is finite for compact structure groups G , the volume $\text{vol}(\mathcal{G})$ of the space of all gauge transformations is infinite — heuristically, you can think of this as a copy of $\text{vol}(G)$ at each point of M . What we’d like to do is understand how to keep the analogue of $\int_0^\infty dr r e^{-h(r)}$ in the gauge theory case, without the redundancy factor. However, neither the right set of gauge invariant variables (analogous to r) nor the right measure on \mathcal{A}/\mathcal{G} (generalizing $r dr$) are obvious in the infinite dimensional case.

Returning to (7.52), suppose C is any curve traveling out from the origin that intersects every circle of constant radius exactly once. More specifically, let $f(\mathbf{x})$ be some function with the properties

- For any point $\mathbf{x} \in \mathbb{R}^2$ there exist a rotation $R : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $f(R\mathbf{x}) = 0$
- f is non-degenerate on the orbits; i.e., $f(R\mathbf{x}) = f(\mathbf{x})$ iff R is the identity⁴⁴ in $\text{SO}(2)$.

The curve $C = \{\mathbf{x} \in \mathbb{R}^2 : f(\mathbf{x}) = 0\}$ then intersects every orbit of the rotation group exactly once and so is isomorphic to $\mathbb{R}^2/\text{SO}(2)$: we can think of the curve $C \subset \mathbb{R}^2$ as a way to embed the orbit space in the plane. Notice that the non-degeneracy property means that $f(\mathbf{x})$ itself is certainly not rotationally invariant. In anticipation of the application to Yang–Mills theory, we call $f(\mathbf{x})$ the *gauge fixing function* and the curve C it defines the *gauge slice*. (See figure 11.)

Now consider the integral

$$\int_{\mathbb{R}^2} dx dy \delta(f(\mathbf{x})) e^{-S(x,y)} \quad (7.53)$$

over all of \mathbb{R}^2 . Clearly, the δ -function restricts this integral to the gauge slice. However, the actual value we get depends on our choice of specific function $f(\mathbf{x})$; for example, even

⁴⁴Technically, we should restrict to $\mathbb{R}^2 - \{0\}$ to ensure this condition holds. For smooth functions $S(x, y)$ this subtlety won’t affect our results and I’ll ignore it henceforth.

replacing $f \rightarrow cf$ for some constant c (an operation which preserves the curve C) reduces the integral by a factor of $1/|c|$. Thus we cannot regard (7.53) as an integral over the moduli space $\mathbb{R}^2/\text{SO}(2)$ — it also depends on exactly how we embedded this moduli space inside \mathbb{R}^2 . The problem arose because the δ -function changes as we change $f(\mathbf{x})$. To account for this, define

$$\Delta_f(\mathbf{x}) := \left. \frac{\partial}{\partial \theta} f(R_\theta \mathbf{x}) \right|_0 \quad (7.54)$$

where the right hand side means we compute the rate of change of f with respect to a rotation R_θ through angle θ , evaluated at the identity $\theta = 0$. Notice that we only need to know how an *infinitesimal* rotation acts in order to compute this.

It's clear that the new integral

$$\int_{\mathbb{R}^2} dx dy |\Delta_f(\mathbf{x})| \delta(f(\mathbf{x})) e^{-S(x,y)} \quad (7.55)$$

involving the modulus of Δ_f doesn't change if we rescale f by a constant factor as above. Nor does it change if we rescale f by a non-zero, r -dependent factor $c(r)$, which means that (7.55) is completely independent of the choice of function used to define the gauge slice C . In fact, I claim that (7.55) is actually *independent of the particular gauge slice itself*. To see this, let f_1 and f_2 be any two different gauge-fixing functions. Since the curves $C_{1,2}$ they define each intersect every orbit of $\text{SO}(2)$ uniquely, we can always rotate C_1 into C_2 , provided we allow ourselves to rotate by different amounts at different values of the radius r . Thus we must have

$$f_2(\mathbf{x}) \propto f_1(R_{12}(r)\mathbf{x}) \quad (7.56)$$

for some r -dependent rotation $R_{12}(r)$ and where the proportionality factor depends at most on the radius. By rescaling invariance,

$$|\Delta_{f_2}(\mathbf{x})| \delta(f_2(\mathbf{x})) = |\Delta_{f_1}(\mathbf{x}')| \delta(f_1(\mathbf{x}')) \quad (7.57)$$

where we've defined $\mathbf{x}' := R_{12}\mathbf{x}$ for any point $\mathbf{x} \in \mathbb{R}^2$, whether it lies on our curves or not. Now, the statement that the action $S(\mathbf{x})$ is rotationally invariant means that it takes the same value all around every circle of constant radius, so $S(\mathbf{x}) = S(\mathbf{x}')$. Similarly,

$$dx' dy' = dx dy \quad (7.58)$$

because again this measure is rotationally invariant at every value of r .⁴⁵ Putting all this together, the integral in (7.55) is independent of the choice of gauge slice $\mathbb{R}^2/\text{SO}(2) \hookrightarrow \mathbb{R}^2$, as we wished to show.

⁴⁵Again, it's a good idea to check you're comfortable with this assertion by writing

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \alpha(r) & \sin \alpha(r) \\ -\sin \alpha(r) & \cos \alpha(r) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and explicitly working out the transformation of the measure, allowing for the fact that the angle $\alpha(r) = \alpha(\sqrt{x^2 + y^2})$ depends on the radius. You'll find the measure is nonetheless invariant.

As a concrete example, suppose we choose C to be the x -axis, defined by $f(\mathbf{x}) = y = 0$. With this choice, $f(R\mathbf{x}) = y \cos \theta - x \sin \theta$ where R represents anti-clockwise rotation through θ . Thus

$$\Delta_f(\mathbf{x}) = \frac{\partial}{\partial \theta} (y \cos \theta - x \sin \theta) \Big|_{\theta=0} = -x \quad (7.59)$$

and therefore our integral (7.55) becomes

$$\int_{\mathbb{R}^2} dx dy |\Delta_c(\mathbf{x})| \delta(f(\mathbf{x})) e^{-S(x,y)} = \int_{\mathbb{R}^2} dx dy |x| \delta(y) e^{-S(x,y)} = \int_{\mathbb{R}} dx |x| e^{-h(|x|)} \quad (7.60)$$

where in the last step we've used the fact that since $S(x, y)$ was rotationally invariant, along the line $y = 0$ it can only depend on $|x|$. Since $|x|$ is an even function of x , we have

$$\int_{\mathbb{R}} dx |x| e^{-h(|x|)} = 2 \int_0^\infty dr r e^{-h(r)}, \quad (7.61)$$

which disagrees with the radial part of our original integral by a factor of 2. What's gone wrong is that circles of constant r intersect the x -axis twice — when $x > 0$ and when $x < 0$ — and our gauge fixing condition $y = 0$ failed to account for this; in other words, it slightly failed the non-degeneracy property. We'll see below that this glitch is actually a model of something that also happens in the case of Yang–Mills theory.

To recap, what we've achieved with all this is that, for any non-degenerate gauge-fixing function f , we can write the desired integral over the space of orbits $\mathbb{R}^2/\text{SO}(2) \cong (0, \infty)$ as

$$\int_{\mathbb{R}^2/\text{SO}(2)} dr r e^{-h(r)} = \int_{\mathbb{R}^2} dx dy |\Delta_f(\mathbf{x})| \delta(f(\mathbf{x})) e^{-S(x,y)}. \quad (7.62)$$

The point is that the expression on the *rhs* refers only to functions and coordinates on the affine space \mathbb{R}^2 , and uses only the standard measure $dx dy$ on \mathbb{R}^2 . When the gauge orbits have dimension > 1 we must impose several gauge fixing conditions f^a , one for each transformation parameter θ^a . Then we take the integral to include a factor

$$|\Delta_f(\mathbf{x})| \prod_a \delta(f^a(\mathbf{x})) \quad (7.63)$$

where now Δ_f is the *Faddeev–Popov determinant*

$$\Delta_f(\mathbf{x}) := \det \left(\frac{\partial f^a(R\mathbf{x})}{\partial \theta^b} \right) \quad (7.64)$$

for a generic set of variables $\mathbf{x} \in \mathbb{R}^n$ where the action is invariant under some transformation $\mathbf{x} \rightarrow R\mathbf{x}$ (not necessarily a rotation). Again, this will allow us to write an integral over the space of orbits of these transformations in terms of an integral over the affine space \mathbb{R}^n . These are things we have access to in the gauge theory case⁴⁶ where the affine space in question is the space \mathcal{A} of all gauge fields, and the transformation group is the space \mathcal{G}

⁴⁶Modulo, as always, the problem that there is no Lebesgue measure on \mathcal{A} : this is what we'll treat with renormalization.

of all gauge transformations. Armed with these ideas, we now turn to the case of gauge theory.

In Yang–Mills theory, we can fix the gauge redundancy by picking a particular connection in each gauge equivalence class — in other words, by picking an embedding of $\mathcal{A}/\mathcal{G} \hookrightarrow \mathcal{A}$ specified by some gauge-fixing functional $f[A]$. The most common choices of gauge fixing functional are *local*, in the sense that $f[A]$ depends on the value of the gauge field just at a single point $x \in M$. Heuristically, we then restrict to $f[A(x)] = 0$ at every point $x \in M$ by inserting “ $\delta[f] = \prod_{x \in M} \delta(f[A(x)])$ ” in the path integral. We’ll consider how to interpret this infinite-dimensional δ -function below. The remaining ingredient we need is the Faddeev–Popov determinant

$$\Delta_f = \det \frac{\delta f^a[A^\lambda(x)]}{\delta \lambda^b(y)} \quad (7.65)$$

where $A^\lambda = A + \nabla \lambda$ denotes an infinitesimal gauge transformation of A with parameters $\lambda(x)$ valued in the adjoint. Like the δ -functional $\delta[f]$, this determinant is now of an infinite dimensional matrix; we’ll consider what this determinant means momentarily. With these ingredients, our Yang–Mills path integral can be written as

$$\int_{\mathcal{A}/\mathcal{G}} D\mu e^{-S_{\text{YM}}[\nabla]} = \int_{\mathcal{A}} DA |\Delta_f| \delta[f] e^{-S_{\text{YM}}[\nabla]}, \quad (7.66)$$

where the factor of $|\Delta_f| \delta[f]$ restricts us to an arbitrary gauge slice, but leaves no dependence on any particular choice of slice, as above. Again, the advantage of the *rhs* is that it refers only to the ‘naive’ integral measure over all connections.

Now let’s consider how to treat these $\delta[f]$ and Δ_f factors. Taking our lead from Fourier analysis, we introduce a new field h (sometimes called a *Nakanishi–Lautrup field*) and write

$$\delta[f] = \int Dh e^{-S_{\text{gf}}[h, \nabla]}, \quad (7.67)$$

where

$$S_{\text{gf}}[h, \nabla] := \int_M \text{tr}(h * f[A]) = \frac{1}{2} \int_M h^a f^a[A] \sqrt{g} d^d x \quad (7.68)$$

is the *gauge-fixing action*. The idea is that h is a Lagrange multiplier — performing the path integral over h imposes $f[A(x)] = 0$ throughout M . Notice that since we needed one gauge-fixing condition for every gauge parameter, we take h to lie in the adjoint representation, $h \in \Omega_M^0(\mathfrak{g})$. This does *not* imply that (7.67) is gauge invariant: indeed it *cannot* be if we wish to use it to fix a gauge! For the Faddeev–Popov determinant Δ_f , recall that if M is an $n \times n$ matrix and (c^i, \bar{c}_j) are n -component Grassmann variables, then $\det(M) = \int d^n c d^n \bar{c} \exp(\bar{c}_j M^j_i c^i)$. Applying the same idea here, we have⁴⁷

$$\det \frac{\delta f^a[A^\lambda(x)]}{\delta \lambda^b(y)} = \int Dc D\bar{c} e^{-S_{\text{gh}}[\bar{c}, c \nabla]} \quad (7.69)$$

⁴⁷One can show that the determinant is positive-definite, at least in a neighbourhood of the trivial connection. Thus, for the purposes of perturbation theory around the trivial background, we can drop the modulus sign. Non-perturbatively we must be more careful.

where

$$S_{\text{gh}}[\bar{c}, c, \nabla] := - \int_{M \times M} \bar{c}^a(x) \frac{\delta f^a[A^\lambda(x)]}{\delta \lambda^b(y)} c^b(y) \quad (7.70)$$

and the fields (c^a, \bar{c}^a) are *fermionic scalars*, again valued in the adjoint representation of G . They are known as *ghosts* (c) and *antighosts* (\bar{c}).

These ghosts certainly seem strange. In the Michaelmas QFT course you learned that in any unitary QFT, the spin–statistics theorem requires fermionic fields to have half–integral spin (at least for $d > 2$). How come we’re now allowed fermionic scalars? In section 7.3.2 we’ll understand that the space of states $\Psi[A, \bar{c}, c, h]$ including the ghosts is a complex vector space, but is not a Hilbert space: it’s inner product fails to be positive definite. Thus the theory including ghosts is indeed non–unitary. However, we’ll see that there *is* a positive–definite inner product on the space of *gauge invariant* states that are independent of the ghosts and Nakanishi–Lautrup fields. Thus, provided we only try to compute expectation values of gauge invariant operators that are independent of (\bar{c}, c, h) , we will have a unitary theory. The whole reason for introducing ghosts was just to remove unphysical gauge degrees of freedom in the naive path integral over the simple affine space \mathcal{A} — the ‘physically meaningful’ integral was always supposed to be taken over \mathcal{A}/\mathcal{G} , where no ghosts arise. Operators in \mathcal{A}/\mathcal{G} would certainly be written without ghosts.

Putting everything together, our Yang–Mills path integral can finally be written as

$$\int_{\mathcal{A}/\mathcal{G}} D\mu e^{-S_{\text{YM}}[\nabla]} = \int D[A, c, \bar{c}, h] \exp(-S_{\text{YM}}[\nabla] - S_{\text{gh}}[\bar{c}, c, \nabla] - S_{\text{gf}}[h, \nabla]) \quad (7.71)$$

where the integral on the *rhs* is formally to be taken over the space of all fields (∇, \bar{c}, c, h) . Everything on the right now looks like some form of action, so we can hope to compute it perturbatively using Feynman rules.

Let’s now make all this more concrete by seeing how it works in an example. An important, frequently occurring choice of gauge is *Lorenz*⁴⁸ *gauge* is to pick the trivial connection ∂ as a base-point by writing $\nabla = \partial + A$, and impose that A obeys

$$f^a[A] = \partial^\mu A_\mu^a(x) = 0 \quad (7.72)$$

for all $x \in M$ and for all $a = 1, \dots, \dim(G)$. An obvious reason to want to work in Lorenz gauge is that in the important case $(M, g) = (\mathbb{R}^d, \delta)$, it respects the $\text{SO}(d)$ invariance of the flat Euclidean metric. Since $A^\lambda = A + \nabla \lambda$ (where $\nabla = \partial + A$ is the connection associated to A), in the case of Lorenz gauge we have

$$\begin{aligned} f^a[A^\lambda] &= \partial^\mu (A_\mu + \nabla_\mu \lambda)^a = \partial^\mu A_\mu^a + \partial_\mu (\nabla_\mu \lambda)^a \\ &= \partial^\mu A_\mu^a + \partial^\mu (\partial_\mu \lambda^a + \frac{1}{2} f_{bc}^a A_\mu^b \lambda^c). \end{aligned} \quad (7.73)$$

Consequently, the matrix appearing in the Faddeev–Popov determinant is

$$\frac{\delta f^a[A^\lambda(x)]}{\delta \lambda^b(y)} = (\partial^\mu \nabla_\mu)_x^{ab} \delta^d(x - y) \quad (7.74)$$

⁴⁸Poor Ludvig Lorenz. Eternally outshone by Hendrik Lorentz to the point of having his work misattributed.

where the subscript x denotes the variable on which the differential operator $\partial^\mu \nabla_\mu$ acts. The factor of $\delta^d(x - y)$ arose because our gauge-fixing condition was local: the object $\partial^\mu(\nabla_\mu \lambda)$ lives at one point x , so we get nothing if we vary it wrt to changes in λ at some other point. While it may look scary to have a differential operator acting on these δ -functions, using this result in the ghost action yields

$$\begin{aligned} S_{\text{gh}}[\bar{c}, c, \nabla] &= - \int_{M \times M} \bar{c}^a(x) (\partial^\mu \nabla_\mu)_x^{ab} \delta^d(x - y) c^b(y) d^d y d^d x \\ &= - \int_M \bar{c}^a (\partial^\mu \nabla_\mu c)^a d^d x \\ &= \int_M (\partial^\mu \bar{c}^a) (\nabla_\mu c)^a d^d x \end{aligned} \quad (7.75)$$

where in the first step we integrated out y using the δ -function, recalling that the differential operators only care about x . Altogether, in Lorenz gauge we have the action

$$S[\nabla, c, \bar{c}, h] = \frac{1}{2g_{\text{YM}}^2} \int_M \text{tr}(dA + A^2) \wedge * (dA + A^2) + \int_M \text{tr}(d\bar{c} \wedge * \nabla c) + \int_M \text{tr}(h d*A) \quad (7.76)$$

Except for the strange spin/statistics of the ghost fields and the mixture of covariant and normal derivatives, this is now a perfectly respectable, local action for scalar fields coupled to the gauge field. Notice that in the Abelian case where the adjoint representation is trivial, the Lorenz gauge ghost action would be $\int_M d\bar{c} \wedge *dc$ and in particular would be independent of the gauge field A . Thus the ghosts would have completely decoupled, which is why you didn't meet them last term.

The remainder of this section is really just for enthusiasts: if your brain's already swimming, it's better to skip on to section 7.3.2.

If you're still reading, then recall that in the finite dimensional case, we needed our gauge fixing function to obey two conditions: that we can indeed always find a gauge transformation such that $f[A] = 0$ holds, and that once we've found it, this gauge is unique so that in particular starting on the gauge slice and performing any gauge transformation takes us off the slice.

Let's start by considering whether we *can* always solve (7.72). In other words, let $\nabla \in \mathcal{A}$ be some arbitrary connection and let $\mathcal{G}(\nabla)$ denote the orbit of ∇ under \mathcal{G} . Then we must show that there is always *some* $\nabla' \in \mathcal{G}(\nabla)$ whose connection 1-form obeys (7.72). You've seen how to do this in the Abelian case of electrodynamics: you noted that under a gauge transform, $A \rightarrow A' = A - d\lambda$ for some λ . The condition that A be in Lorenz gauge says

$$0 = \partial^\mu A'_\mu = \partial^\mu A_\mu - \Delta\lambda \quad (7.77)$$

where Δ is the Laplacian on M . Regarding this as a condition on λ , we must solve $\Delta\lambda = u$ where $u(x) = \partial^\mu A_\mu(x)$ is essentially arbitrary. This can always be done provided u is orthogonal to the kernel of the adjoint of the Laplacian in the L^2 norm on (M, g) . However, the Laplacian is self-adjoint ($\int_M \phi * \Delta\psi = \int_M (\Delta\phi) * \psi$) and $\ker \Delta$ consists of

constant functions, since if $u \in \ker \Delta$ then

$$0 = \int_M u * \Delta u = - \int_M du \wedge *du = -\|du\|^2 \quad (7.78)$$

whenever u has compact support. Hence $du = 0$ so u is constant. Thus, for any generic electromagnetic potential A_μ , we can find a gauge transform that puts it in Lorenz gauge. In the non-Abelian case, things are more complicated because the gauge transform of a connection is non-linear: $A \rightarrow A + \nabla_A \lambda$, whose value depends on which A we start with. It turns out that we can always solve (7.72), but the proof is considerably more complicated — one was found by Karen Uhlenbeck in 1982 (at least for some common choices of M), and an alternative proof was later found by Simon Donaldson.

We must still check that (7.72) singles out a *unique* representative, so that we count each gauge equivalence class only once. Encouragement comes from the fact that connections obeying (7.72) are orthogonal to connections that are pure gauge, with respect to the L^2 -metric (7.47) on \mathcal{A} . For if a_1 is a tangent vector at the point $\nabla \in \mathcal{A}$ that obeys $\nabla * a_1 = 0$, while $a_2 = \nabla \lambda$ is also a tangent vector at ∇ that points in the direction of an infinitesimal gauge transform, then

$$ds_{\mathcal{A}}^2(a_1, a_2) = \int_M \text{tr}(a_1 \wedge *a_2) = \int_M \text{tr}(a_1 \wedge *\nabla \lambda) = - \int_M \text{tr}((\nabla * a_1) \lambda) = 0 \quad (7.79)$$

using the Lorenz condition. Thus changing our connection in a way that obeys Lorenz gauge takes us in a direction that is *orthogonal* to the orbits of the gauge group. This certainly shows that starting from any base-point and integrating over all gauge fields along the slice incorporates only gauge inequivalent connections while we're *near* our base-point.

However, as in the finite dimensional example where the line $y = 0$ intersected each circle on constant radius twice, it doesn't guarantee that some other connection, far away along the gauge slice, isn't secretly gauge equivalent to one we've already accounted for. This troubling possibility is known as the *Gribov ambiguity*, after Vladimir Gribov who first pointed it out and showed it actually occurs in the case of Coulomb gauge $\partial^i A_i = 0$ (the indices just run over $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$). Somewhat later, Iz Singer showed that the Gribov ambiguity is in fact *inevitable*: no matter which gauge condition you pick, the gauge orbit *always* intersects the gauge slice more than once (at least for most reasonable M). To show this, Singer noted that \mathcal{A} is itself an infinite dimensional principle bundle over the space $\mathcal{B} := \mathcal{A}/\mathcal{G}$ where the group \mathcal{G} of all gauge transformations plays the role of the *structure* group. A gauge slice amounts to a global section of this bundle — *i.e.*, the choice of a unique point in \mathcal{A} for each point in \mathcal{B} . A result I won't prove states that a principal bundle only admits a global section when it's topologically trivial, so the existence of a global gauge slice would imply

$$\overset{?}{\mathcal{A}} \cong \mathcal{B} \times \mathcal{G}. \quad (7.80)$$

Since \mathcal{A} is an affine space, clearly $\pi_k(\mathcal{A}) = 0$ for all $k > 0$ (*i.e.* \mathcal{A} itself is topologically trivial and has no non-contractible cycles). However, Singer computed that $\pi_k(\mathcal{G}) \neq 0$ for at least some $k > 0$ which says that there *are* some non-contractible cycles in the space on the *rhs* of (7.80). Thus $\mathcal{A} \neq \mathcal{B} \times \mathcal{G}$, so \mathcal{A} is non-trivial as a principal bundle

over \mathcal{B} , and no global gauge choice exists. In practice, we'll work perturbatively, meaning we never venture far enough from our chosen base-point connection to meet any Gribov copies. Non-perturbatively, we'd have to cover \mathcal{A}/\mathcal{G} with different coordinate patches, pick different gauges in each one and then piece them together at the end. I'm not aware of anyone actually trying to do this.

7.3.2 BRST cohomology

We've seen that renormalization group flow generates an infinite series of interactions — every possible term that is not forbidden by symmetries of the original action and regularized measure — all but finitely many of which are strongly suppressed at low energies. The gauge-fixing and ghost terms in the previous section are *not* invariant under gauge transformations, so we cannot use gauge invariance as a criterion by which to restrict the possible terms that are generated by RG flow. But this is troubling: if gauge non-invariant terms are indeed allowed in the effective action, what is to stop (*e.g.*) a mass term⁴⁹ $\sim \text{tr} A_\mu A^\mu$ from being generated automatically in the quantum theory?

Becchi, Rouet & Stora and independently Tyurin showed that the full gauge-fixed action (??) in fact does possess a remarkable symmetry that remembers the gauge invariance of the original Yang–Mills action. Consider the transformations

$$\begin{aligned}\delta A_\mu &= \epsilon \nabla_\mu c & \delta \bar{c} &= i\epsilon h \\ \delta c &= -\frac{\epsilon}{2} [c, c] & \delta h &= 0\end{aligned}\tag{7.81}$$

where ϵ is a constant, anticommuting parameter. Note that $[c, c]^a = f_{bc}^a c^b c^c$ is not identically zero because the ghosts are Grassmann valued. Letting Ψ^i denote any of the fields $\{A_\mu^a, c^a, \bar{c}^a, h^a\}$, we will often write these transformations as $\delta\Psi = \epsilon \mathcal{Q}\Psi$ so that $\mathcal{Q}\Psi$ represents the *rhs* of (7.81) with the anticommuting parameter ϵ stripped away. Note that $\mathcal{Q}\Psi^i$ thus has opposite statistics to Ψ^i itself.

The expression for δA shows that, as far as the gauge field itself is concerned, these BRST transformations act just like a gauge transformation $A_\mu \rightarrow A_\mu + \nabla_\mu \lambda$ with gauge parameter $\lambda^a(x) = \epsilon c^a(x)$ given in terms of the ghost field. It follows that any gauge-invariant function of the connection alone, such as the original Yang–Mills action $S_{\text{YM}}[\nabla]$, is invariant under the transformations (7.81). To see that the rest of the action is also invariant under (7.81), we'll first show that $[\delta_1, \delta_2] = 0$, where $\delta_{1,2}$ are transformations with parameters $\epsilon_{1,2}$. Since

$$[\delta_1, \delta_2]\Psi^i = \delta_1(\epsilon_2 \mathcal{Q}\Psi^i) - \delta_2(\epsilon_1 \mathcal{Q}\Psi^i) = -(\epsilon_1 \epsilon_2 - \epsilon_2 \epsilon_1) \mathcal{Q}^2 \Psi^i = -2\epsilon_1 \epsilon_2 \mathcal{Q}^2 \Psi^i\tag{7.82}$$

using the fact that the parameters are fermionic and so anticommute with \mathcal{Q} . Therefore the statement $[\delta_1, \delta_2] = 0$ amounts to the statement that the transformation $\Psi \mapsto \mathcal{Q}\Psi$ is nilpotent.

⁴⁹Here I mean just a naive perturbative mass term for the gauge boson (or gluon), not the more sophisticated appearance of a mass gap in the non-perturbative theory, or phase transition to a massive theory by means of the Higgs mechanism.

For the Nakanishi–Lautrup field h , this assertion is trivial. Similarly, for the antighost \bar{c} we have

$$[\delta_1, \delta_2]\bar{c} = i\delta_1(\epsilon_2 h) - i\delta_2(\epsilon_1 h) = 0 \quad (7.83)$$

since h itself is invariant. For the gauge field,

$$\begin{aligned} [\delta_1, \delta_2]A_\mu &= \delta_1(\epsilon_2 \nabla_\mu c) - \delta_2(\epsilon_1 \nabla_\mu c) \\ &= \epsilon_2 \nabla_\mu(\delta_1 c) + \epsilon_2[\delta_1 A_\mu, c] - (1 \leftrightarrow 2) \\ &= (\epsilon_2 \epsilon_1 - \epsilon_1 \epsilon_2) \left[-\frac{1}{2} \nabla_\mu([c, c]) + [\nabla_\mu c, c] \right], \end{aligned} \quad (7.84)$$

which vanishes since $\nabla_\mu[c, c] = [\nabla_\mu c, c] + [c, \nabla_\mu c] = 2[\nabla_\mu c, c]$ using antisymmetry of the Lie bracket $[,]$ together with the fact that the ghosts are Grassmann valued. Finally, for the ghost itself we have

$$\begin{aligned} [\delta_1, \delta_2]c &= -\frac{\epsilon_2}{2}([\delta_1 c, c] + [c, \delta_1 c]) + \frac{\epsilon_1}{2}([\delta_2 c, c] + [c, \delta_2 c]) \\ &= \frac{\epsilon_2}{4}([\epsilon_1[c, c], c] + [c, \epsilon_1[c, c]]) - (1 \leftrightarrow 2) \\ &= \frac{1}{2}(\epsilon_2 \epsilon_1 - \epsilon_1 \epsilon_2)[[c, c], c]^a t_a. \end{aligned} \quad (7.85)$$

Because the ghosts anticommute, the expression $[[c, c], c]^a = f_{bc}^a f_{de}^b c^d c^e c^c$ must be totally antisymmetric on $\{d, e, c\}$ and hence it vanishes by the Jacobi identity. Thus $\mathcal{Q}^2 \Psi^i = 0$ for any single field $\Psi^i \in \{A, c, \bar{c}, h\}$.

Now let's show that the BRST transformation is nilpotent even when acting on an arbitrary functional $\mathcal{O}(A, c, \bar{c}, h)$ of the fields. We compute

$$\mathcal{Q}^2 \mathcal{O} = \mathcal{Q} \left((\mathcal{Q}\Psi^i) \frac{\delta \mathcal{O}}{\delta \Psi^i} \right) = \mathcal{Q}^2 \Psi^i \frac{\delta \mathcal{O}}{\delta \Psi^i} - \mathcal{Q}\Psi^i \mathcal{Q}\Psi^j \frac{\delta^2 \mathcal{O}}{\delta \Psi^j \delta \Psi^i}. \quad (7.86)$$

The first term vanishes by our calculations above. To see that the second term also vanishes, split the sums over all fields (labelled by i, j) into separate sums over bosonic fields $\Psi^i \in \{A_\mu, h\}$ and fermionic fields $\Psi^i \in \{c, \bar{c}\}$. In the case that i and j both refer either to bosonic or fermionic fields, the term cancels because $\mathcal{Q}\Psi$ has opposite statistics to Ψ^i itself, so that pre-factor is symmetric if the second derivatives are antisymmetric, and *vice-versa*. The mixed terms cancel among themselves.

We're now in position to see why the full, gauge-fixed action is BRST invariant. We have

$$\int \mathcal{Q} \text{tr}(\bar{c}f[A]) d^d x = \int \left[i \text{tr}(hf[A]) - \text{tr} \left(\bar{c} \frac{\delta f}{\delta \theta} \nabla_\mu c \right) \right] d^d x = S_{\text{gf}}[h, A] + S_{\text{gh}}[A, c, \bar{c}], \quad (7.87)$$

so the gauge-fixing and ghost terms in the action are the BRST transformation of $\text{tr}(\bar{c}f[A])$. Since BRST transformations are nilpotent, it follows that these terms are BRST invariant for any gauge-fixing functional $f[A]$. Combined with the gauge invariance of the original Yang–Mills action, this shows that BRST transformations preserve the full Yang–Mills

gauge-fixed action. Provided we regularize the path integral measure in a way that preserves this (as will be true perturbatively in dimensional regularization), BRST symmetry will be a symmetry of the quantum theory, and all new terms that are generated by RG flow will also be constrained to be BRST invariant. In particular, terms that depend only on the original gauge field will be constrained to be gauge invariant, preventing the appearance of a mass term $\sim A^2$ even at the quantum level.

Furthermore, if we restrict ourselves to computing correlation functions of operators $\{\mathcal{O}_i\}$ that are BRST invariant, the result will be completely independent of the particular choice of gauge-fixing functional $f[A]$, just as for the partition function. This is because $f[A]$ appears only in the BRST exact term $\mathcal{Q} \text{tr}(hf[A])$ in the action. Therefore, changing f leads to a change

$$\Delta \left\langle \prod_i \mathcal{O}_i \right\rangle = - \left\langle \int \mathcal{Q} \text{tr}(h \Delta f) d^d x \prod_i \mathcal{O}_i \right\rangle = - \int \left\langle \mathcal{Q} \left(\text{tr}(h \Delta f) \prod_i \mathcal{O}_i \right) \right\rangle d^d x \quad (7.88)$$

using the assumption $\mathcal{Q}\mathcal{O}_i = 0$ of BRST invariant operators. This vanishes because it is the integral of a BRST exact quantity, so is a total derivative on the space of fields.

It's revealing to view these results from a Hamiltonian / canonical quantization perspective. By Noether's theorem, the fact that the action is invariant under BRST transformations leads to a conserved charge, which you should check is given by⁵⁰

$$\begin{aligned} Q &= \int_N \text{tr} \left(\frac{1}{g_{\text{YM}}^2} \nabla c \wedge *F_\nabla + i h * \nabla c + \frac{1}{2} * d\bar{c} [c, c] \right) \\ &= \int \text{tr} \left(\frac{1}{g_{\text{YM}}^2} \nabla_i c F^{0i} + i h \nabla_0 c + \frac{1}{2} \partial_0 \bar{c} [c, c] \right) d^{d-1} x \end{aligned} \quad (7.89)$$

in the case of the Lorenz gauge action (??), where the second line is valid in the standard case that the co-dimension 1 surface N used to define the charge is the plane $x^0 = \text{const.}$

To be completed: BRST cohomology and relation to Hamiltonian approach.

7.4 Perturbative renormalization of Yang–Mills theory

Even though perturbation theory is of limited use at low energies, we'll begin our study of quantum Yang–Mills theory by trying to see what we can learn from it. As an incentive, we should expect that the perturbative description *will* be useful at high energies (where the renormalized coupling turns out to be small), so understanding perturbation theory will allow us to probe the UV behaviour of Yang–Mills theory.

7.4.1 Feynman rules in R_ξ gauges

If we write $\nabla = \partial + A$ to expand around the trivial connection, then $F = dA + A^2$, the Yang–Mills Lagrangian schematically contains terms

$$F^2 \sim (dA)^2 + A^2(dA) + A^4. \quad (7.90)$$

⁵⁰In a more sophisticated treatment, I'd point out that under canonical quantization this charge is really the Chevalley–Eilenberg differential of the infinite-dimensional Lie group \mathcal{G} of all gauge transformations, acting on the space of fields.

Thus, in addition to the purely kinetic term $(dA)^2$, we have a four-gluon interaction vertex A^4 and a three-gluon interaction vertex $A^2 dA$. Let's start by taking a closer look at the propagators and interactions that result from this gauge-fixed action. For simplicity, we'll consider just pure Yang-Mills theory on $(M, g) = (\mathbb{R}^d, \delta)$, with the gauge-fixing functional taken to be $f[A] = \partial^\mu A_\mu$.

As it stands, the gauge-fixing term $i \int \text{tr}(h \partial^\mu A_\mu)$ is a little awkward to work with; the field h is non-dynamical, but integrating it out introduces a δ -function into the path integral that we don't know how to handle. For this reason, it will be convenient to add the BRST exact term

$$-i\frac{\xi}{2} \int \mathcal{Q} \text{tr}(\bar{c}h) d^d x = \frac{\xi}{2} \int \text{tr}(hh) d^d x \quad (7.91)$$

to the action, where ξ is an arbitrary constant. Since this term is BRST exact, its presence doesn't affect the value of any correlation function of BRST invariant operators. However, integrating out h we now find the constraint $h = -i\partial^\mu A_\mu/\xi$. Inserting this back into the action gives

$$\int \left[i \text{tr}(h \partial^\mu A_\mu) + \frac{\xi}{2} \text{tr}(h^2) \right] d^d x = \frac{1}{2\xi} \int \text{tr}(\partial^\mu A_\mu \partial^\nu A_\nu) d^d x \quad (7.92)$$

which can be seen as a modification of the kinetic term of the gauge field. Combining this with the kinetic part of the original Yang-Mills action, one finds that the momentum space gluon propagator is

$$D_{\mu\nu}^{ab}(p) = -\frac{\delta^{ab}}{p^2} \left[\delta_{\mu\nu} - (1-\xi) \frac{p_\mu p_\nu}{p^2} \right] \quad (7.93)$$

in momentum space. This propagator is often said to be in R_ξ gauge. Since it originally appeared in front of a BRST exact term, the value of ξ can be chosen freely; common choices are $\xi = 0$ (Landau's choice – this recovers the original Lorenz gauge as for electromagnetism) and $\xi = 1$ (Feynman and 't Hooft's choice).

The gluons can interact among themselves via both the $A^2 dA$ and A^4 interaction vertices in the original action. In detail, the flat space three-gluon vertex is

$$\Gamma_{\mu\nu\lambda}^{abc}(k, p, q) = -g_{YM} f^{abc} [(k-p)_\lambda \delta_{\mu\nu} + (p-q)_\mu \delta_{\nu\lambda} + (q-k)_\nu \delta_{\lambda\mu}] \quad (7.94)$$

in momentum space, while the four-gluon vertex is

$$\begin{aligned} \Gamma_{\mu\nu\lambda\sigma}^{abcd} &= -g_{YM}^2 f^{abe} f^{cde} (\delta_{\mu\lambda} \delta_{\nu\sigma} - \delta_{\mu\sigma} \delta_{\nu\lambda}) - g_{YM}^2 f^{ace} f^{bde} (\delta_{\mu\nu} \delta_{\sigma\lambda} - \delta_{\mu\sigma} \delta_{\nu\lambda}) \\ &\quad - g_{YM}^2 f^{ade} f^{bce} (\delta_{\mu\nu} \delta_{\sigma\lambda} - \delta_{\mu\lambda} \delta_{\nu\sigma}), \end{aligned} \quad (7.95)$$

and is independent of the momenta since it is a local (non-derivative) interaction.

We must also consider ghost fields which can run around loops even if they do not appear externally. The gauge fixed action above yields a ghost propagator

$$C^{ab}(p) = \frac{\delta^{ab}}{p^2} \quad (7.96)$$

in momentum space. This is the standard form expected for massless scalars, except the fact that we get a + sign rather than a - sign reflects the fact that the ghosts are fermionic. Finally, in Lorenz gauge $\partial^\mu a_\mu = 0$ we have a $a_\mu bc$ vertex of the form

$$\Gamma^{abc}(p) = -g_{YM} f^{abc} p_\mu \quad (7.97)$$

where the component p_μ of the momentum of the antighost couples to the component a_μ of the gauge field. It's a really good exercise — and a standard exam question — to check you can derive all these terms from the action above.

The most likely feeling at this point is panic. It doesn't take much imagination to see that any attempt to use these vertices to construct Feynman diagrams will quickly run into a huge proliferation of terms. In fact, counting each term in the vertices (7.94) & (7.95) separately, even a fairly simple process like $2 \rightarrow 3$ gluon scattering receives contributions from $\sim 10,000$ terms, already just at *tree* level! In theories with charged matter, such as QCD and the Standard Model, there are further interactions coming from the gluons in the covariant derivatives, and at loop level there are further contributions from the ghosts.

On the one hand, perhaps this is just the way it is. After all, Yang–Mills theory is a complicated, non-linear theory. If you come along and prod it in a more or less arbitrary way (*i.e.* do perturbation theory), you should expect that the consequences will indeed be messy and complicated. But another possible response to the above is a slight feeling of nausea. The whole point of our treatment of Yang–Mills theory in terms of bundles, connections and curvature was to show how tremendously natural this theory is from a geometric perspective. Yet this naturality is badly violated by our splitting of the Yang–Mills action into $(da)^2$, $a^2 da$ and a^4 pieces, none of which separately have any geometric meaning. Surely there must be a different way to treat this theory — one that is less ugly, and treats the underlying geometric structure with more respect?

Many physicists sympathize with this view (me included). In fact, over the years various different ways to think about Yang–Mills theory have been proposed, ranging from viewing Yang–Mills theory as a type of string theory, to writing it in twistor space instead of space-time, to putting it on a computer. Some of these approaches have been highly successful, others only partially so. We'll take a brief look at a few in later sections. For now though, we must soldier on and do our best to understand the theory perturbatively in the neighbourhood of the trivial connection. To do otherwise would be somewhat akin to trying to understand differential geometry without first knowing what a vector is.

7.4.2 Yang–Mills is perturbatively renormalizable!

7.4.3 The β -function and asymptotic freedom

7.5 Further aspects

7.5.1 A string theory in disguise?

't Hooft's picture of YM = string theory. Mention ribbon diagrams (cf matrix models). Show expansion around $N = \infty$ can be rearranged as a genus expansion of the ribbon graph. Vague allusion to AdS/CFT.