

Part IB - Electromagnetism

Lectured by D. Tong
Notes taken by Dexter Chua

Lent 2015

Electromagnetism and Relativity

Review of Special Relativity; tensors and index notation. Lorentz force law. Electromagnetic tensor. Lorentz transformations of electric and magnetic fields. Currents and the conservation of charge. Maxwell equations in relativistic and non-relativistic forms. [5]

Electrostatics

Gauss's law. Application to spherically symmetric and cylindrically symmetric charge distributions. Point, line and surface charges. Electrostatic potentials; general charge distributions, dipoles. Electrostatic energy. Conductors. [3]

Magnetostatics

Magnetic fields due to steady currents. Ampere's law. Simple examples. Vector potentials and the Biot-Savart law for general current distributions. Magnetic dipoles. Lorentz force on current distributions and force between current-carrying wires. Ohm's law. [3]

Electrodynamics

Faraday's law of induction for fixed and moving circuits. Electromagnetic energy and Poynting vector. 4-vector potential, gauge transformations. Plane electromagnetic waves in vacuum, polarization. [5]

Contents

1	Introduction	3
1.1	Charge and Current	3
1.2	Forces and Fields	4
2	Electrostatics	5
2.1	Gauss' Law	5
2.2	Electrostatic potential	8
2.2.1	Point charge	9
2.2.2	Dipole	9
2.2.3	General charge distribution	10
2.2.4	Field lines and equipotentials	11
2.3	Electrostatic energy	11
2.4	Conductors	13
3	Magnetostatics	15
3.1	Ampere's Law	15
3.2	Vector potential	17
3.3	Magnetic dipoles	18
3.4	Magnetic forces	20
4	Electrodynamics	22
4.1	Induction	22
4.2	Magnetostatic energy	24
4.3	Resistance	26
4.4	Electromagnetic waves	27
4.5	Poynting vector	32
5	Electromagnetism and relativity	33
5.1	A review of special relativity	33
5.2	Vectors, co-vectors and tensors	34
5.3	Conserved currents	36
5.4	Gauge potentials and electromagnetic fields	36
5.4.1	Lorentz invariants	40
5.5	Maxwell Equations	41
6	The Lorentz force law	43

1 Introduction

Electromagnetism is important.

1.1 Charge and Current

The strength of the electromagnetic force experienced by a particle is determined by its (*electric*) *charge*. The SI unit of charge is the *Coulomb*. In this course, we assume that the charge can be any real number. However, at the fundamental level, charge is quantised. All particles carry charge $q = ne$ with $n \in \mathbb{Z}$,¹ and the basic unit $e \approx 1.6 \times 10^{-19} \text{C}$. For example, the electron has $n = -1$, proton has $n = +1$, neutron = $n = 0$.

In this course, it will be more useful to talk about *charge density* $\rho(\mathbf{x}, t)$.

Definition (Charge density). The *charge density* is the charge per unit volume. The total charge in a region V is

$$Q = \int_V \rho(\mathbf{x}, t) dV$$

The motion of charge is described by the *current density* $\mathbf{J}(\mathbf{x}, t)$.

Definition (Current and current density). For any surface S , the integral

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

counts the charge per unit time passing through S . I is the *current*, and \mathbf{J} is the *charge density*, “current per unit area”.

Intuitively, if the charge distribution $\rho(\mathbf{x}, t)$ has velocity $\mathbf{v}(\mathbf{x}, t)$, then (neglecting relativistic effects), we have

$$\mathbf{J} = \rho \mathbf{v}$$

Example. A wire is a cylinder of cross-sectional area A . Suppose there are n electrons per unit volume. Then

$$\rho = nq = -ne$$

$$\mathbf{J} = nq\mathbf{v}$$

$$I = nqvA$$

It is well known that charge is conserved. However, we can have a stronger statement- charge is conserved locally: it is not possible that a charge in a box disappears and instantaneously appears on the moon. If it disappears in the box, it must have moved to somewhere nearby.

This is captured by the *continuity equation*.

Law (Continuity equation).

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

¹Actually quarks have $n = \pm 1/3$ or $\pm 2/3$, but when we study quarks, electromagnetism becomes insignificant compared to the strong force. So for all practical purposes we can have $n \in \mathbb{Z}$.

The charge Q in some region V is

$$Q = \int_V \rho \, dV$$

So

$$\frac{dQ}{dt} = \int_V \frac{\partial \rho}{\partial t} \, dV = - \int_V \nabla \cdot \mathbf{J} \, dV = - \int_S \mathbf{J} \cdot d\mathbf{S}$$

where the last equality is by the divergence theorem. ie. the rate of change in charge is the rate of charge flowing out of the point.

We can take $V = \mathbb{R}^3$, the whole of space. If there are no currents at infinity, then

$$\frac{dQ}{dt} = 0$$

So the continuity equation implies the conservation of charge.

1.2 Forces and Fields

All forces are mediated by *fields*. In physics a *field* is a dynamical quantity which takes values at every point in space and time. The electromagnetic force is mediated by two fields:

- electric field $\mathbf{E}(\mathbf{x}, t)$
- magnetic field $\mathbf{B}(\mathbf{x}, t)$

Each of these fields is itself a 3-vector. There are two aspects to the force:

- Particles create fields
- Fields move particles

The second aspect is governed by the Lorentz force law:

Law (Lorentz force law).

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

The first aspect is governed by *Maxwell's equations*.

Law (Maxwell's Equations).

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 \\ \nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} &= \mu_0 \mathbf{j}\end{aligned}$$

where

- $\varepsilon_0 = 8.85 \times 10^{-12} \, \text{m}^{-3} \, \text{kg}^{-1} \, \text{s}^2 \, \text{C}^2$ is the electric constant
- $\mu_0 = 4\pi \times 10^{-6} \, \text{m} \, \text{kg} \, \text{C}^{-2}$ is the magnetic constant

These equations are special in a mathematical way. In fact, using quantum mechanics and relativity, it can be shown that these are the only equations that can possibly describe electromagnetism.

2 Electrostatics

Take a stationary charge density $\rho = \rho(\mathbf{x})$ with $\mathbf{J} = 0$. We look for time-independent solutions with $\mathbf{B} = 0$. We are left with two equations

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \\ \nabla \times \mathbf{E} &= 0\end{aligned}$$

and the other two equations just give $0 = 0$. Our goal is: Given ρ , find \mathbf{E} .

2.1 Gauss' Law

Consider a region $V \subseteq \mathbb{R}^3$ with boundary $S = \partial V$. We integrate the first equation over V :

$$\int \nabla \cdot \mathbf{E} \, dV = \int_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int_V \rho \, dV.$$

So

Law (Gauss' law).

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\varepsilon_0},$$

where Q is the total charge inside V .

Definition (Flux through surface). The *flux* of \mathbf{E} through the surface S is defined to be

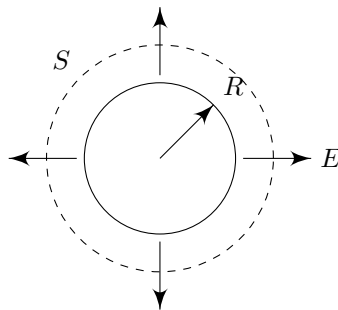
$$\int_S \mathbf{E} \cdot d\mathbf{S}.$$

Note: All surfaces containing charge Q have the same flux. Only charge inside S contributes to the flux. If there is a charge outside, then the field lines go in through one side of the surface and leave through the other. Then it gives no total flux.

From this, we can prove Coulomb's law:

Example. Gauss' law \Rightarrow Coulomb's force law.

We consider a spherically symmetric charge density $\rho(r)$. with $\rho(r) = 0$ for $r > R$, ie. all the charge is contained in a ball of radius R .



By symmetry, the force is the same in all directions and point outward radially. So

$$\mathbf{E} = E(r)\hat{\mathbf{r}}.$$

This immediately ensures that $\nabla \times \mathbf{E} = 0$.

Put S to be a sphere of radius $r > R$. Then

$$\begin{aligned} \int_S \mathbf{E} \cdot d\mathbf{S} &= \int_S E(r)\hat{\mathbf{r}} \cdot d\mathbf{S} \\ &= E(r) \int_S \hat{\mathbf{r}} \cdot d\mathbf{S} \\ &= E(r) \cdot 4\pi r^2 \\ &= \frac{Q}{\varepsilon_0} \end{aligned}$$

We were able to pull the $E(r)$ out since it is constant over the sphere.

Therefore

$$\mathbf{E}(r) = \frac{Q}{4\pi\varepsilon_0 r^2} \hat{\mathbf{r}}.$$

The Lorentz force law tells us that the force experienced by a second charge is

$$\mathbf{F}(\mathbf{r}) = \frac{Qq}{4\pi\varepsilon_0 r^2} \hat{\mathbf{r}},$$

which is Coulomb's law. Note that strictly speaking, this only holds when the charges are not moving. However, we can still use this because the corrections when they move are tiny.

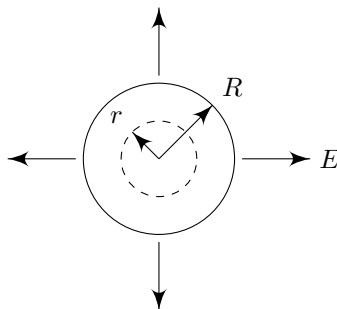
Example. Consider a uniform sphere with

$$\rho(r) = \begin{cases} \rho & r < R \\ 0 & r > R \end{cases}.$$

Outside, we know that

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r^2} \hat{\mathbf{r}}$$

Now suppose we are inside the sphere.



Then

$$\begin{aligned} \int_S \mathbf{E} \cdot d\mathbf{S} &= E(r)4\pi r^2 \\ &= \frac{Q}{\varepsilon_0} \left(\frac{r^3}{R^3} \right) \end{aligned}$$

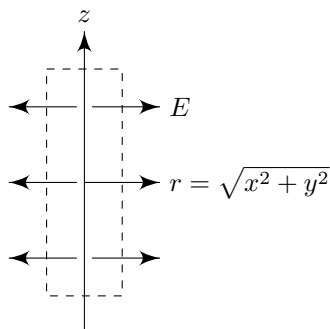
So

$$\mathbf{E}(r) = \frac{Qr}{4\pi\epsilon_0 R^3} \hat{\mathbf{r}},$$

and the field increases with radius.

Example (Line charge). Consider an infinite line with uniform charge density *per unit length* η .

We use cylindrical polar coordinates:



By symmetry, the field is radial, ie.

$$\mathbf{E}(r) = E(r) \hat{\mathbf{r}}.$$

Pick S to be a cylinder of length L and radius r . We know that the end caps do not contribute to the flux since the field lines are perpendicular to the normal. Also, the curved surface has area $2\pi rL$. Then

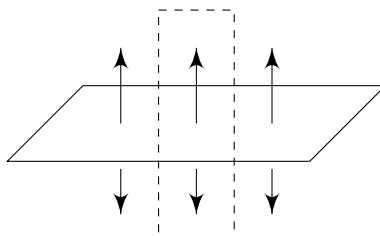
$$\int_S \mathbf{E} \cdot d\mathbf{S} = E(r) 2\pi rL = \frac{\eta L}{\epsilon_0}.$$

So

$$\mathbf{E}(r) = \frac{\eta}{2\pi\epsilon_0 r} \hat{\mathbf{r}}.$$

Note that the field varies as $1/r$, not $1/r^2$. Intuitively, this is because we have one more dimension of “stuff” compared to the point charge, so the field does not drop as fast.

Example (Surface charge). Consider an infinite plane $z = 0$, with uniform charge per unit area σ .



By symmetry, the field points vertically, and the field bottom is the opposite of that on top. we must have

$$\mathbf{E} = E(z) \hat{\mathbf{z}}$$

with

$$E(z) = -E(-z).$$

Consider a vertical cylinder of height $2z$ and cross-sectional area A . Now only the end caps contribute.

$$\begin{aligned} \int_S \mathbf{E} \cdot d\mathbf{S} &= E(z)A - E(-z)A \\ &= \frac{\sigma A}{\varepsilon_0}. \end{aligned}$$

So

$$E(z) = \frac{\sigma}{2\varepsilon_0}$$

and is constant.

Note that the electric field is discontinuous across the surface. We have

$$E(z \rightarrow 0+) - E(z \rightarrow 0-) = \frac{\sigma}{\varepsilon_0}.$$

Note that this is a general result that is true for any arbitrary surfaces and σ . We can prove this by considering a cylinder across the surface and then shrink it indefinitely. Then we find that

$$\hat{\mathbf{n}} \cdot \mathbf{E}_+ - \hat{\mathbf{n}} \cdot \mathbf{E}_- = \frac{\sigma}{\varepsilon_0}.$$

However, the components of \mathbf{E} tangential to the surface are continuous.

2.2 Electrostatic potential

In general, we will have to solve both $\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$ and $\nabla \times \mathbf{E} = \mathbf{0}$. However, we already know that the general form of the solution to the second equation is $\mathbf{E} = -\nabla\phi$ for some scalar field ϕ .

Definition (Electrostatic potential). If $\mathbf{E} = -\nabla\phi$, then ϕ is the *electrostatic potential*.

Then

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \nabla^2 \phi = \frac{\rho}{\varepsilon_0},$$

which is the *Poisson equation*. If we are in the middle of nowhere and $\rho = 0$, then we just get the *Laplace equation*.

Note the following:

- ϕ is only defined up to a constant. We usually fix this by insisting $\phi(\mathbf{r}) \rightarrow 0$ as $r \rightarrow \infty$. Note that this statement seems trivial, but this property of ϕ is actually very important and gives rise to a lot of interesting properties, as we will see later.
- The Poisson equation is linear. So if we have two charges $\rho = \rho_1 + \rho_2$, then $\phi = \phi_1 + \phi_2$ and $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$. This is the *principle of superposition*. Among the four fundamental forces of nature, electromagnetism is the only force with this property.

2.2.1 Point charge

Consider a point particle with charge Q at the origin. Then

$$\rho(\mathbf{r}) = Q\delta^3(\mathbf{r}).$$

The power 3 of the delta function indicates that it is for a 3-dimensional vector.

We have to solve

$$\nabla^2\phi = -\frac{Q}{\varepsilon_0}\delta^3(\mathbf{r}).$$

Away from the origin $\mathbf{r} = \mathbf{0}$, the solution is just

$$\phi = \frac{\alpha}{r} \quad \text{for some constant } \alpha$$

We check this by noting $\nabla r = \mathbf{r}/|r| = \hat{\mathbf{r}}$. So

$$\nabla\left(\frac{\alpha}{r}\right)^3 = \frac{-\alpha\mathbf{r}}{r^3}.$$

and thus

$$\nabla^2\phi = -\alpha\left(\frac{\nabla\cdot\mathbf{r}}{r^2} - 3\frac{\mathbf{r}\cdot\mathbf{r}}{r^5}\right) = 0.$$

The constant α is determined by the delta function. We integrate over a sphere centered at the origin to obtain

$$\int_V \nabla^2\phi \, dV = \int_S \nabla\phi \cdot d\mathbf{S} = -\frac{Q}{\varepsilon_0} \int_V \delta^3(r) \, dV = -\frac{Q}{\varepsilon_0}.$$

Substitute $\phi = \alpha/r$. $\nabla\phi = -\alpha/r^2$ points in the same direction as $d\mathbf{S}$, and is integrated over a surface area of $4\pi r^2$. So the surface integral is $-4\pi\alpha$ and $\alpha = Q/(4\pi\varepsilon_0)$. So

$$\mathbf{E} = -\nabla\phi = \frac{Q}{4\pi\varepsilon_0 r^2} \hat{\mathbf{r}}.$$

2.2.2 Dipole

Definition (Dipole). A *dipole* consists of two point charges, $+Q$ and $-Q$ at $\mathbf{r} = 0$ and $\mathbf{r} = -\mathbf{d}$ respectively. By the principle of superposition,

$$\phi = \frac{1}{4\pi\varepsilon_0} \left(\frac{Q}{r} - \frac{Q}{|\mathbf{r} + \mathbf{d}|} \right).$$

We can ask what the scalar field looks like at a distance $r \gg d$. We work with the vector version of Taylor expansion:

$$f(\mathbf{r} + \mathbf{d}) = f(\mathbf{r}) + \mathbf{d} \cdot \nabla f(\mathbf{r}) + \frac{1}{2}(\mathbf{d} \cdot \nabla)^2 f(\mathbf{r}) + \dots$$

So

$$\begin{aligned} \frac{1}{|\mathbf{r} + \mathbf{d}|} &= \frac{1}{r} - \mathbf{d} \cdot \nabla \left(\frac{1}{r} \right) + \frac{1}{2}(\mathbf{d} \cdot \nabla)^2 \left(\frac{1}{r} \right) \\ &= \frac{1}{r} - \frac{\mathbf{d} \cdot \mathbf{r}}{r^3} - \frac{1}{2} \left(\frac{\mathbf{d} \cdot \mathbf{d}}{r^3} - \frac{3(\mathbf{d} \cdot \mathbf{r})^2}{r^5} + \dots \right) \end{aligned}$$

For the dipole we have

$$\phi = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{r} - \frac{1}{r} + \frac{\mathbf{d} \cdot \mathbf{r}}{r^3} + \dots \right) \sim \frac{Q}{4\pi\epsilon_0} \frac{\mathbf{d} \cdot \mathbf{r}}{r^3}.$$

Definition (Electric dipole moment). We define the *electric dipole moment* is

$$\mathbf{p} = Q\mathbf{d}.$$

By convention, it points from -ve to +ve.

Then

$$\phi = \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{4\pi\epsilon_0 r^2},$$

and

$$\mathbf{E} = -\nabla\phi = \frac{1}{4\pi\epsilon_0} \left(\frac{3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{r^3} \right).$$

Note that this is *weird*.

2.2.3 General charge distribution

To find ϕ for a general charge distribution ρ , we use the Green's function for the Laplacian

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta^3(\mathbf{r} - \mathbf{r}'),$$

where

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi} \frac{1}{|\mathbf{r} - \mathbf{r}'|}.$$

We'll take $\rho(\mathbf{r}) \neq 0$ only on some compact region V . Then

$$\begin{aligned} \phi(\mathbf{r}) &= -\frac{1}{\epsilon_0} \int_V \rho(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') dV \\ &= \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV \end{aligned}$$

Note that inside the integral, we are integrating against \mathbf{r}' .

This works because if we apply ∇^2 to $\phi(\mathbf{r})$, then \mathbf{r}' terms are unaffected because we are differentiating wrt \mathbf{r} , and only $G(\mathbf{r}, \mathbf{r}')$ is differentiated to the delta function $\delta^3(\mathbf{r} - \mathbf{r}')$. Then the integral becomes $\rho(\mathbf{r}')$, which gives $\nabla^2\phi = \rho/\epsilon_0$, in accordance with Maxwell's equation.

Then

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= -\nabla\phi(\mathbf{r}) \\ &= -\frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{r}') \nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) dV \\ &= \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{r}') \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV \end{aligned}$$

So if we plug in a very complicated ρ , we get a very complicated \mathbf{E} !

However, we can ask what ϕ and \mathbf{E} look like very far from V , ie. $|\mathbf{r}| \gg |\mathbf{r}'|$.

We again use the Taylor expansion.

$$\begin{aligned} |\mathbf{r} - \mathbf{r}'| &= \frac{1}{r} + \mathbf{r}' \cdot \nabla \left(\frac{1}{r} \right) + \dots \\ &= \frac{1}{r} + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^3} + \dots \end{aligned}$$

Then

$$\begin{aligned} \phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{r}') \left(\frac{1}{r} + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^3} + \dots \right) dV \\ &= \frac{1}{4\pi\epsilon_0} \left(\frac{Q}{r} + \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2} + \dots \right) \end{aligned}$$

Where

$$\begin{aligned} Q &= \int_V \rho(\mathbf{r}') dV \\ \mathbf{p} &= \int_V \mathbf{r}' \rho(\mathbf{r}') dV \end{aligned}$$

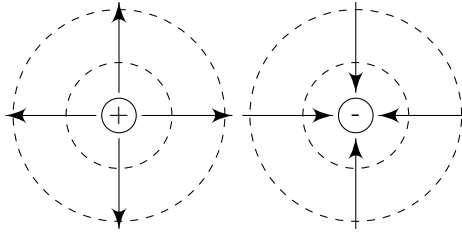
So if we have a huge lump of charge, we can consider it to be a point charge Q , plus some dipole correction terms.

2.2.4 Field lines and equipotentials

Definition (Field line). A *field line* is a continuous line tangent to the electric field \mathbf{E} . The density of lines is proportional to $|\mathbf{E}|$. They begin and end only at charges (and infinity). They never cross.

Definition (Equipotentials). *Equipotentials* are surfaces of constant ϕ . Because $\mathbf{E} = -\nabla\phi$, they are always perpendicular to field lines.

Example.



Use your imagination to join the two diagrams together to get the field lines for dipoles.

2.3 Electrostatic energy

We want to calculate how much energy is stored in the electric field. Recall from Dynamics and Relativity that a particle of charge q in a field $\mathbf{E} = -\nabla\phi$ has potential energy $U(\mathbf{r}) = q\phi(\mathbf{r})$.

$U(\mathbf{r})$ can be thought of as the work done in bringing the particle from infinity, as illustrated below:

$$\begin{aligned}
 \text{work done} &= - \int_{\infty}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} \\
 &= - \int_{\infty}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{r} \\
 &= q \int_{\infty}^{\mathbf{r}} \nabla \phi \cdot d\mathbf{r} \\
 &= q[\phi(\mathbf{r}) - \phi(\infty)] \\
 &= U(\mathbf{r})
 \end{aligned}$$

where we set $\phi(\infty) = 0$.

Now consider N charges q_i at positions \mathbf{r}_i . The total potential energy stored is the work done to assemble these particles. Let's put them in one by one.

(i) The first charge is free. The work done is $W_1 = 0$.

(ii) To place the second charge at position \mathbf{r}_2 takes work. The work is

$$W_2 = \frac{q_1 q_2}{4\pi\epsilon_0 |\mathbf{r}_1 - \mathbf{r}_2|}.$$

(iii) To place the third charge at position \mathbf{r}_3 , we do

$$W_3 = \frac{q_3}{4\pi\epsilon_0} \left(\frac{q_1}{|\mathbf{r}_1 - \mathbf{r}_3|} - \frac{q_2}{|\mathbf{r}_2 - \mathbf{r}_3|} \right)$$

(iv) etc.

The total work done is

Proposition.

$$U = \sum_{i=1}^N W_i = \frac{1}{4\pi\epsilon_0} \sum_{i < j} \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|}.$$

Equivalently,

$$U = \frac{1}{4\pi\epsilon_0} \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|}.$$

Now the potential at point \mathbf{r}_i due to all q_j with $j \neq i$ is

$$\phi(\mathbf{r}_i) = \frac{1}{4\pi\epsilon_0} \sum_{j \neq i} \frac{q_j}{|\mathbf{r}_i - \mathbf{r}_j|}.$$

So

$$U = \frac{1}{2} \sum_{i=1}^N q_i \phi(\mathbf{r}_i).$$

There is an obvious generalization to continuous charge distributions:

Proposition.

$$U = \frac{1}{2} \int \rho(\mathbf{r})\phi(\mathbf{r}) \, dV.$$

Note: There is a subtlety here, and there is an extra contribution to the work done in this integral. cf. official notes.

Now we can write

Proposition.

$$\begin{aligned} U &= \frac{\varepsilon_0}{2} \int (\nabla \cdot \mathbf{E})\phi \, dV \\ &= \frac{\varepsilon_0}{2} \int [\nabla \cdot (\mathbf{E}\phi) - \mathbf{E} \cdot \nabla \phi] \, dV \\ &= \frac{\varepsilon_0}{2} \int \mathbf{E} \cdot \mathbf{E} \, dV. \end{aligned}$$

2.4 Conductors

Definition (Conductor). A *conductor* is a region of space which contains lots of charges that are free to move.

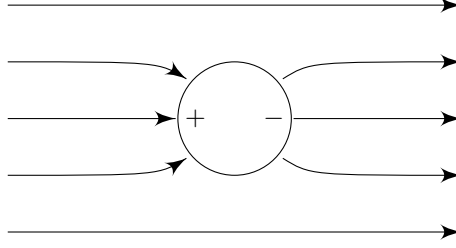
Note:

- In electrostatic situations, we must have $\mathbf{E} = 0$ inside a conductor. Otherwise, the charges inside the conductor would move till equilibrium. This almost describes the whole of the section: if you apply an electric field onto a conductor, the charges inside the conductor move until the external field is canceled out.
- Since $\mathbf{E} = 0$, ϕ is constant inside the conductor.
- Since $\nabla \cdot \mathbf{E} = \varepsilon/\varepsilon_0$, $\rho = 0$ inside a conductor, ie. the charges must cancel out inside. So any net charge has to live on the surface.
- $\mathbf{E} = -\nabla\phi$ is true, and ϕ constant in conductor means \mathbf{E} is perpendicular to the surface.
- If there is some surface charge σ , then the discontinuity of \mathbf{E} gives us $\mathbf{E}_{\text{outside}} = \frac{\sigma}{\varepsilon_0} \hat{\mathbf{n}}$.

There are two typical kinds of problems.

- (i) The total charge Q on a conductor is fixed.

Example. Consider a spherical conductor with $Q = 0$ in a constant electric field \mathbf{E} . Since the electric field lines must be perpendicular to the surface, they have to bend towards the conductor. Since field lines end and start at charges, there must be negative charges at the left and positive charges at the right.

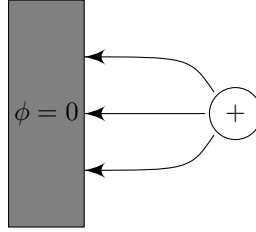


We get an induced surface charge and we want to figure out both \mathbf{E} and σ .

- (ii) The conductor is of fixed potential ϕ , usually taken to be $\phi = 0$. The conductor is said to be *grounded*, i.e. attached to the Earth.

Note: In the first case, the potential is constant in any particular environment, but can change when the environment changes. In this case, the potential is always 0 regardless of what the world looks like.

Example.



Suppose we have a conductor that fills all space $x < 0$. We ground it so that $\phi = 0$ through the plane. Then we place a charge q at $x = d > 0$.

We are looking for a solution with a source at $x = d$ and $\phi = 0$ at $x = 0$. We know that there is a unique solution.

We use the method of images, and pretend that instead of a conductor, we have a charge $-q$ at $x = -d$. The potential for this pair is just

$$\phi = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{\sqrt{(x-d)^2 + y^2 + z^2}} - \frac{q}{\sqrt{(x+d)^2 + y^2 + z^2}} \right].$$

This has $\phi = 0$ at $x = 0$, with a point source at $x = d$. Since this solves our original equation, it must be THE solution.

We can compute

$$\mathbf{E}_x = -\frac{\partial\phi}{\partial x} = \frac{q}{4\pi\epsilon_0} \left(\frac{x-d}{|\mathbf{r}-\mathbf{d}|^{3/2}} - \frac{x+d}{|\mathbf{r}+\mathbf{d}|^{3/2}} \right)$$

for $x > 0$. The induced surface charge is

$$\sigma = E_x \epsilon_0|_{x=0^+} = -\frac{q}{2\pi} \frac{d}{(d^2 + y^2 + z^2)^{3/2}}.$$

The surface charge is

$$q = \int \sigma \, dy \, dz = -q.$$

3 Magnetostatics

Charges always give rise to electric fields. Currents give rise to magnetic fields. We will look for time independent solutions to the Maxwell's equation with $\mathbf{J} \neq 0$ and $\rho = 0$. We need to solve

$$\begin{aligned}\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} \\ \nabla \cdot \mathbf{B} &= 0\end{aligned}$$

The goal is, given \mathbf{J} , find \mathbf{B} .

Note: $\rho = 0$ and $\mathbf{J} \neq 0$ means that we have equal positive and negative charges that can move.

Recall the conservation of charge:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

In this case, since $\rho = 0$, we must have

$$\nabla \cdot \mathbf{J} = 0.$$

These are *steady currents*.

3.1 Ampere's Law

Consider a surface S with boundary C . Current \mathbf{J} flows through S . We now integrate the first equation over the surface S to obtain

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S} = \oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{S}.$$

So

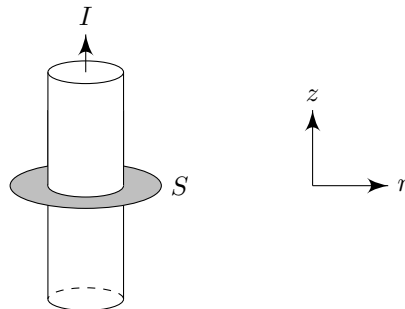
Law (Ampere's law).

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I,$$

where I is the current through the surface.

Example (A long straight wire). A wire is a cylinder with current I flowing through it.

We use cylindrical polar coordinates (r, φ, z) , where z is along the direction of the current, and r points in the radial direction.



We integrate over a disc that cuts through the wire horizontally.

We need $\oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$. By symmetry, we have

$$\mathbf{B}(\mathbf{r}) = B(r)\hat{\phi}.$$

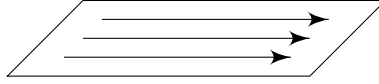
This automatically satisfies $\nabla \cdot \mathbf{B} = 0$. Then

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = B(r) \int_0^{2\pi} r d\varphi = 2\pi r B(r) = \mu_0 I.$$

So

$$\mathbf{B}(r) = \frac{\mu_0 I}{2\pi r} \hat{\phi}.$$

Example (Surface current). Consider the plane $z = 0$ with *surface current density* \mathbf{k} (i.e. current per unit length).



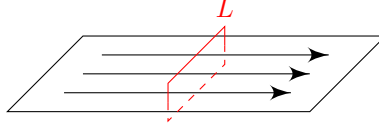
Take the x -direction to be the direction of the current, and the z direction to be the normal to the plane.

We imagine this situation by considering to be infinitely many copies of the above wire situation. Then the magnetic fields must point in the y -direction. By symmetry, we must have

$$\mathbf{B} = -B(z)\hat{y},$$

with $B(z) = -B(-z)$.

Consider a vertical rectangular loop of length L through the surface



Then

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = LB(z) - LB(-z) = \mu_0 kL$$

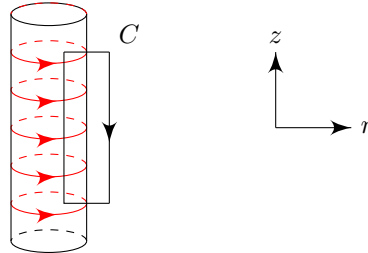
So

$$B(z) = \frac{\mu_0 k}{2} \quad \text{for } z > 0.$$

Similar to the electrostatic case, the magnetic field is constant, and the part parallel to the surface is discontinuous across the plane. This is a general result, i.e. across any surface,

$$\hat{\mathbf{n}} \times \mathbf{B}_+ - \hat{\mathbf{n}} \times \mathbf{B}_- = \mu_0 \mathbf{k}.$$

Example (Solenoid). A *solenoid* is a cylindrical surface current, usually made by wrapping a wire around a cylinder.



We use cylindrical polar coordinates with z in the direction of the extension of the cylinder. By symmetry, $\mathbf{B} = B(r)\mathbf{z}$.

Away from the cylinder, $\nabla \times \mathbf{B} = 0$. So $\frac{\partial B}{\partial r} = 0$, which means that $B(r)$ is constant outside. Since we know that $\mathbf{B} = \mathbf{0}$ at infinity, $\mathbf{B} = \mathbf{0}$ everywhere outside the cylinder.

To compute \mathbf{B} inside, use Ampere's law with a curve C . Note that only the vertical part (say of length L) inside the cylinder contributes to the integral. Then

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = BL = \mu_0 INL.$$

where N is the number of wires per unit length and I is the current in each wire (so INL is the total amount of current through the wires).

So $B = \mu_0 IN$.

3.2 Vector potential

For general current distributions \mathbf{J} , we also need to solve $\nabla \cdot \mathbf{B} = 0$.

This is telling us that there are no magnetic monopoles, ie. magnetic charges (cf. electric version, with $\rho = 0$)

We can solve this by writing $\mathbf{B} = \nabla \times \mathbf{A}$.

Definition (Vector potential). If $\mathbf{B} = \nabla \times \mathbf{A}$, then \mathbf{A} is the *vector potential*.

The other Maxwell equation then says

$$\nabla \cdot \mathbf{B} = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) = \mu_0 \mathbf{J}. \quad (*)$$

This is rather difficult to solve, but it can be made easier by noting that:

Proposition (Gauge invariance). \mathbf{A} is not unique. If \mathbf{A} is a vector potential, then if

$$\mathbf{A}' = \mathbf{A} + \nabla\chi$$

for any $\chi(\mathbf{x})$, then $\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times \mathbf{A}'$.

The transformation $\mathbf{A}' \mapsto \mathbf{A} + \nabla\chi$ is called a *gauge transformation*.

Definition ((Coulomb) gauge). Each \mathbf{A} is called a *gauge*. An \mathbf{A} such that $\nabla \cdot \mathbf{A} = 0$ is called a *Coulomb gauge*.

Proposition. We can always pick χ such that $\nabla \cdot \mathbf{A}' = 0$.

Proof. Suppose that $\mathbf{B} = \nabla \times \mathbf{A}$ with $\nabla \cdot \mathbf{A} = \psi(\mathbf{x})$. Then pick $\mathbf{A}' = \mathbf{A} + \nabla\chi$. Then we have

$$\nabla \cdot \mathbf{A}' = \nabla \cdot \mathbf{A} + \nabla^2 \chi = \psi + \nabla^2 \chi.$$

Pick χ such that $\nabla^2 \chi = -\psi$. This is the Poisson equation and there is always a solution. \square

If $\mathbf{B} = \nabla \times \mathbf{A}$ and $\nabla \cdot \mathbf{A} = 0$, then the Maxwell equation (*) is

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}.$$

Or, in Cartesian components,

$$\nabla^2 A_i = -\mu_0 J_i.$$

This is 3 copies of the Poisson equation, which we know how to solve using Green's functions. The solution is

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{J_i(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV,$$

or

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV,$$

both integrating over \mathbf{r}' .

We have to check *Coulomb gauge*:

$$\begin{aligned} \nabla \cdot \mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \cdot \nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) dV \\ &= -\frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \cdot \nabla' \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) dV \end{aligned}$$

We integrate by parts to obtain

$$= -\frac{\mu_0}{4\pi} \int \left[\nabla' \cdot \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) - \frac{\nabla' \cdot \mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right] dV.$$

where ∇' acts on \mathbf{r}' . On the last line, the first term vanishes because we have localized current, while the second vanishes because $\nabla \cdot \mathbf{J} = 0$. So $\nabla \cdot \mathbf{A} = 0$.

Law (Biot-Savart law). The magnetic field is

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A} = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} dV.$$

If the current is localized on a curve, this becomes

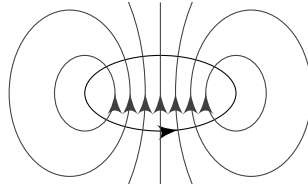
$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint_C \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \times d\mathbf{r}',$$

since $\mathbf{J}(\mathbf{r}')$ is non-zero near the curve.

3.3 Magnetic dipoles

We want to ask what \mathbf{B} looks like far from a localized current distribution.

Example (Current loop). Take a current loop of wire C , radius R and current I .



We can guess that \mathbf{B} looks like this, but we want to calculate it.

Since $\mathbf{B} = \nabla \times \mathbf{A}$, we know that

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV = \frac{\mu_0 I}{4\pi} \oint_C \frac{d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}.$$

Far from the loop, $|\mathbf{r} - \mathbf{r}'|$ is small and we can use the Taylor expansion.

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^3} + \dots$$

Then

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{r\pi} \oint_C \left(\frac{1}{r} + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^3} + \dots \right) d\mathbf{r}'.$$

Note that r is a constant of the integral, and we can take it out. The first $\frac{1}{r}$ term vanishes because it is a constant, and when we integrate along a closed loop, we get 0. So we only consider the second term.

We claim that for any constant vector \mathbf{g} ,

$$\oint_C \mathbf{g} \cdot \mathbf{r}' d\mathbf{r}' = \mathbf{S} \times \mathbf{g},$$

where $\mathbf{S} = \int d\mathbf{S}$ is the *vector area* of the surface bounded by C . It follows from Green's theorem for the function $f(r')$:

$$\oint f(r') d\mathbf{r}' = \int_S \nabla f \times d\mathbf{S},$$

taking $f(r') = \mathbf{g} \cdot \mathbf{r}'$. Then

$$\oint \mathbf{g} \cdot \mathbf{r}' d\mathbf{r}' = \int_S \mathbf{g} \times d\mathbf{S} = \mathbf{S} \times \mathbf{g}.$$

Using this, we have

$$\mathbf{A}(\mathbf{r}) \approx \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3},$$

where

Definition (Magnetic dipole moment). $\mathbf{m} = I\mathbf{S}$ is the *magnetic dipole moment*.

Then

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu_0}{4\pi} \left(\frac{3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3} \right).$$

This is the same form as \mathbf{E} for an electric dipole, except that there is no $1/r^2$ leading term here, because we have no magnetic monopoles.

After doing it for a loop, we can do it for a general current distribution:

Example. We have

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{J_i(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV$$

Instead of changing it into a line integral, we perform the Taylor expansion directly inside

$$= \frac{\mu_0}{4\pi} \int \left(\underbrace{\frac{J_i(\mathbf{r}')}{r}}_{\text{vanishes}} - \frac{J_i(\mathbf{r})(\mathbf{r} \cdot \mathbf{r}')}{r^3} + \dots \right) dV.$$

To prove that the term actually vanishes, we consider $\partial'_j(J_j r'_i)$, where $\partial_i = \frac{\partial}{\partial x_i}$. We have

$$\partial'_j(J_j r'_i) = \underbrace{(\partial'_j J_j)}_{=\nabla \cdot \mathbf{J}=0} r'_i + \underbrace{(\partial'_j r'_i)}_{=\delta_{ij}} J_i = J_i.$$

So the 1st term J_i is a total derivative and vanishes as we take the integral over \mathbb{R}^3 .

For the second term, we look at

$$\partial_j(J_j r_i r_k) = (\partial_j J_j) r_i r_k + J_i r_k + J_k r_i = J_k r_i + J_i r_k.$$

Apply this trick to

$$\int J_i r_j r'_j \, dV = \int \frac{r_j}{2} (J_i r'_j - J_j r'_i) \, dV,$$

where we discarded a total derivative $\partial'_j(J_j r_i; r'_u)$. Putting it back in vector notation,

$$\begin{aligned} \int J_i \mathbf{r} \cdot \mathbf{r}' \, dV &= \int \frac{1}{2} (J_i (\mathbf{r} \cdot \mathbf{r}') - r_i (\mathbf{J} \cdot \mathbf{r}')) \, dV \\ &= \int \frac{1}{2} [\mathbf{r} \times (\mathbf{J} \times \mathbf{r}')]_i \, dV. \end{aligned}$$

So the long-distance vector potential is again

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3},$$

with

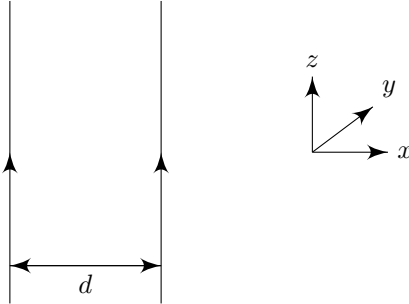
Definition (Magnetic dipole moment).

$$\mathbf{m} = \frac{1}{2} \int \mathbf{r}' \times \mathbf{J}(\mathbf{r}') \, dV.$$

3.4 Magnetic forces

We've seen that moving charge produces currents which generates a magnetic field. But we also know that a charge moving in a magnetic field experiences a force $\mathbf{F} = q\mathbf{r} \times \mathbf{B}$. So the currents will exert a force on each other.

Example (Two parallel wires).



We know that the force produced by each current is

$$\mathbf{B}_1 = \frac{\mu_0 I}{2\pi r} \hat{\phi}.$$

The particles on the second wire will feel as force

$$\mathbf{F} = q\mathbf{r} \times \mathbf{B}_1 = q\mathbf{r} \times \left(\frac{\mu_0 I_1}{2\pi d} \right) \hat{\mathbf{y}}.$$

But $\mathbf{J}_2 = nq\mathbf{v}$ and $I_2 = J_2 A$, where n is the density of particles and A is the cross-sectional area of the wire. So the number of particles per unit length is nA , and the force per unit length is

$$\mathbf{f} = na\mathbf{F} = \frac{\mu_0 I_1 I_2}{2\pi d} \hat{\mathbf{z}} \times \hat{\mathbf{y}} = -\mu_0 \frac{I_1 I_2}{2\pi d} \hat{\mathbf{x}}.$$

So if $I_1 I_2 > 0$, ie. the currents are in the same direction, the force is attractive. Otherwise the force is repulsive.

Example (General force). A current \mathbf{J}_1 , localized on some closed curve C_1 , sets up a magnetic field

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I_1}{4\pi} \oint_{C_1} \frac{\mathbf{r} - \mathbf{r}_1}{|\mathbf{r} - \mathbf{r}_1|^3} dV,$$

integrating over \mathbf{r}_1 .

A second current \mathbf{J}_2 on C_2 experiences the Lorentz force

$$\mathbf{F} = \int \mathbf{J}_2(\mathbf{r}) \times \mathbf{B}(\mathbf{r}) dV.$$

While we are integrating over all of space, the current is localized at a curve $C - 2$. So

$$\mathbf{F} = I_2 \oint_{C_2} d\mathbf{r}_2 \times \mathbf{B}(\mathbf{r}_2).$$

Hence

$$\mathbf{F} = \frac{I_1 I_2}{4\pi} \oint_{C_1} \oint_{C_2} d\mathbf{r}_2 \times \left(d\mathbf{r}_1 \times \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \right).$$

For well-separated currents, approximated by \mathbf{m}_1 and \mathbf{m}_2 , we claim that the force can be written as

$$\mathbf{F} = \frac{\mu_0}{4\pi} \nabla \left(\frac{3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})(\mathbf{m}_2 \cdot \hat{\mathbf{r}}) - (\mathbf{m}_1 \cdot \mathbf{m}_2)}{r^3} \right),$$

whose proof is too complicated to be included.

Note: A “magnet” we find on fridges consists of many aligned microscopic dipoles

4 Electrodynamics

We'll now look at time-dependent \mathbf{E} and \mathbf{B} fields.

4.1 Induction

We'll explore the Maxwell equation

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0.$$

In short, $\frac{\partial \mathbf{B}}{\partial t} \neq 0$ creates an \mathbf{E} that accelerates charges, which creates a current in a wire. This process is called induction. Consider a wire, which is a closed curve C , with a surface S .

We integrate over the surface S to obtain

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}.$$

By Stokes and commutativity of integration and differentiation (assuming S and C do not change in time), we have

$$\int_C \mathbf{E} \cdot d\mathbf{r} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S}.$$

Definition (Electromotive force (emf)). The *electromotive force* (emf) is

$$\mathcal{E} = \int_C \mathbf{E} \cdot d\mathbf{r}.$$

Note that this is not a force! It's the work done

Definition (Magnetic flux). The *magnetic flux* is

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S}.$$

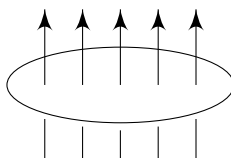
Then we have

Law (Faraday's law of induction).

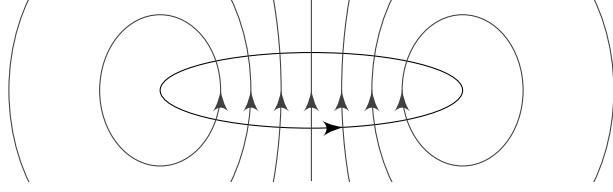
$$\mathcal{E} = -\frac{d\Phi}{dt}.$$

The minus sign is so important that it has its own name, called *Lenz's law*.

Example. Consider a circular wire with a magnetic field perpendicular to it.



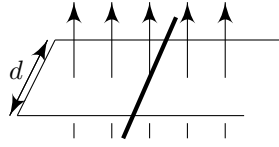
If we decrease \mathbf{B} such that $\dot{\Phi} < 0$, then $\mathcal{E} > 0$. So the current flows anticlockwise (viewed from above). The current generates its own \mathbf{B} . This acts to *increase* \mathbf{B} inside, which counteracts the initial decrease.



This means you don't get runaway behaviour.

There is a related way to induce a current: keep \mathbf{B} fixed and move wire.

Example.



Slide bar to the left with speed v . Each charge q will experience a Lorentz force

$$F = qvB,$$

in the counterclockwise direction.

The emf, defined as the work done per unit charge, is

$$\mathcal{E} = vBd,$$

because work is only done for particles on the bar.

Meanwhile, the change of flux is

$$\frac{d\Phi}{dt} = -vBd,$$

since the area decreases at a rate of $-vd$.

We again have

$$\mathcal{E} = -\frac{d\Phi}{dt}.$$

Note that we obtain the same formula but different physics - we used Lorentz force law, not Maxwell's equation.

Now we consider the general case: a moving loop $C(t)$ bounding a surface $S(t)$. As the curve moves, the curve sweeps out a cylinder S_c .

The change in flux

$$\begin{aligned} \Phi(t + \delta t) - \Phi(t) &= \int_{S(t+\delta t)} \mathbf{B}(t + \delta t) \cdot d\mathbf{S} - \int_{S(t)} \mathbf{B}(t) \cdot d\mathbf{S} \\ &= \int_{S(t+\delta t)} \left(\mathbf{B}(t) + \frac{\partial \mathbf{B}}{\partial t} \right) \cdot d\mathbf{S} - \int_{S(t)} \mathbf{B}(t) \cdot d\mathbf{S} + O(\delta t^2) \\ &= \int_{S(t)} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \left[\int_{S(t+\delta t)} - \int_{S(t)} \right] \mathbf{B}(t) \cdot d\mathbf{S} + O(\delta t^2) \end{aligned}$$

We know that

$$\left[\int_{S(t+\delta t)} - \int_{S(t)} \right] \mathbf{B}(t) \cdot d\mathbf{S} + \int_{S_c} \mathbf{B}(t) \cdot d\mathbf{S} = 0,$$

since the integral of \mathbf{B} over a closed surface is 0.

Write surface element on S_c as

$$d\mathbf{S} = (d\mathbf{r} \times \mathbf{v}) dt.$$

Then

$$\frac{d\Phi}{dt} = \lim_{\delta \rightarrow 0} \frac{\delta\Phi}{\delta t} = \int_{S(t)} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} - \int_{c(t)} (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{r}.$$

Using Maxwell's equation, we have

$$\frac{d\Phi}{dt} = - \int_C (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{r}.$$

which is the emf \mathcal{E} in general.

4.2 Magnetostatic energy

Suppose that a current I flows along a wire C . From magnetostatics, we know that this gives rise to a magnetic field \mathbf{B} , and hence a flux Φ given by

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S},$$

where S is the surface bounded by C .

Definition (Inductance). The *inductance* of a C

$$L = \frac{\Phi}{I},$$

the amount of flux it generates per unit current passing through C . It is a property only of the curve C .

While this concept is very important in engineering, the only place it will appear in this course is in the coming proof.

Example (The solenoid). Consider a solenoid of length ℓ and cross-sectional area A (with $\ell \gg \sqrt{A}$ so we can ignore end effects). We know that

$$B = \mu_0 IN,$$

where N is the number of turns of wire per unit length and I is the current. The flux through a single turn (pretending it is closed) is

$$\Phi_0 = \mu_0 INA.$$

So the total flux is

$$\Phi = \Phi_0 N \ell = \mu_0 I N^2 V,$$

where V is the volume, $A\ell$. So

$$L = \mu_0 N^2 V.$$

We can use the idea of inductance to compute the energy stored in magnetic fields. The idea is to compute the work done in building up a current.

Take a current I . Changing the current \Rightarrow magnetic field changes \Rightarrow induced emf,

$$\mathcal{E} = -\frac{d\Phi}{dt} = -L\frac{dI}{dt}.$$

This opposes the change in current by Lenz's law. In time δt , a charge $I\delta t$ flows around C . The work done is

$$\delta W = \mathcal{E}I\delta t = -LI\frac{dI}{dt}\delta t.$$

So

$$\frac{dW}{dt} = -LI\frac{dI}{dt} = -\frac{1}{2}L\frac{dI^2}{dt}.$$

So the work done to build up a current is

$$W = \frac{1}{2}LI^2 = \frac{1}{2}I\Phi.$$

Note that we dropped the minus sign because we switched from talking about the work done by the emf to the work done to oppose the emf.

This is identified with the energy stored in the system,

$$\begin{aligned} U &= \frac{1}{2}I \int_S \mathbf{B} \cdot d\mathbf{S} \\ &= \frac{1}{2}I \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} \\ &= \frac{1}{2}I \oint_C \mathbf{A} \cdot d\mathbf{r} \quad \text{using Stokes'} \\ &= \frac{1}{2} \int_{\mathbb{R}^3} \mathbf{J} \cdot \mathbf{A} \, dV \end{aligned}$$

Using Maxwell's equation $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$,

$$\begin{aligned} &= \frac{1}{2\mu_0} \int (\nabla \times \mathbf{B}) \cdot \mathbf{A} \, dV \\ &= \frac{1}{2\mu_0} \int [\underbrace{\nabla \cdot (\mathbf{B} \times \mathbf{A})}_{\text{vanishes if } C \text{ is localized}} + \mathbf{B} \cdot (\nabla \times \mathbf{A})] \, dV \\ &= \frac{1}{2\mu_0} \int \mathbf{B} \cdot \mathbf{B} \, dV. \end{aligned}$$

So

Proposition. The energy stored in a magnetic field is

$$U = \frac{1}{2\mu_0} \int \mathbf{B} \cdot \mathbf{B} \, dV.$$

In general, the energy stored in \mathbf{E} and \mathbf{B} is

$$U = \int \left(\frac{\epsilon_0}{2} \mathbf{E} \cdot \mathbf{E} + \frac{1}{2\mu_0} \mathbf{B} \cdot \mathbf{B} \right) \, dV.$$

Note that this does not follow from our results for pure magnetic fields and pure electric fields, since when both are present, they might interact in weird ways.

4.3 Resistance

The story so far is

$$\frac{d\Phi}{dt} = -\mathcal{E} \Rightarrow \text{accelerates charges}$$

In principle, we should be able to compute the current. But accelerating charges are complicated (emit light). Instead, we invoke a new effect, friction.

In a wire, this is called *resistance*. In most materials, the effect of resistance is that \mathcal{E} is proportional to the speed of the charged particles, rather than the acceleration.

We can think of the particles as accelerating for a very short period of time, and then reaching a terminal velocity. So

Law (Ohm's law).

$$\mathcal{E} = IR,$$

Definition (Resistance). The *resistance* is the R in Ohm's law.

Note: $\mathcal{E} = \int \mathbf{E} \cdot d\mathbf{r}$ and $\mathbf{E} = -\nabla\phi$. So $\mathcal{E} = V$, the potential difference. So Ohm's law says $V = IR$.

Definition. Resistivity and conductivity For the wire of length L and cross-sectional area A , we define the *resistivity* to be

$$\rho = \frac{AR}{L},$$

and the conductivity is

$$\sigma = \frac{1}{\rho}.$$

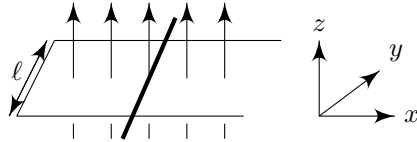
These are properties only of the substance and not the actual shape of the wire. Then Ohm's law reads

Law (Ohm's law).

$$\mathbf{J} = \sigma \mathbf{E}.$$

We can formally derive Ohm's law by considering the field and interactions between the electron and the atoms, but we're not going to do that.

Example.



Suppose the bar moves to the left with speed v . Suppose that the sliding bar has resistance R , and the remaining parts of the circuit are superconductors.

There are two dynamical variables, the position of the bar $x(t)$, and the current $I(t)$.

If a current I flows, the force on a small segment of the bar is

$$\mathbf{F} = IB\hat{\mathbf{y}} \times \hat{\mathbf{z}}$$

So the total force on a bar is

$$\mathbf{F} = IB\ell\hat{x}.$$

So

$$m\ddot{x} = IB\ell.$$

To determine the current

$$\mathcal{E} = -\frac{d\Phi}{dt} = -BD\dot{x}.$$

So Ohm's law gives

$$IR = -B\ell\dot{x}.$$

Hence

$$m\ddot{x} = -\frac{B^2\ell^2}{R}\dot{x}.$$

Integrating once gives

$$\dot{x}(t) = -ve^{-B^2\ell^2 t/mR}.$$

With resistance, we need to do work to keep a constant current. In time δt , the work needed is

$$\delta W = \mathcal{E}I\delta t = I^2 R\delta t$$

using Ohm's law. So

Definition (Joule heating). *Joule heating* is the energy lost in a circuit due to friction. It is given by

$$\frac{dW}{dt} = I^2 R.$$

4.4 Electromagnetic waves

Recall that the Maxwell equations are

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)\end{aligned}$$

So far we have studied all the equations apart from the $\mu_0\varepsilon_0\frac{\partial \mathbf{E}}{\partial t}$ term. Historically this term is called the *displacement current*.

The need of this term was discovered by purely mathematically, since people discovered that Maxwell's equations would be inconsistent with charge conservation without the term.

Without the term, the last equation is

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

Take the divergence of the equation to obtain

$$\mu_0 \nabla \cdot \mathbf{J} = \nabla \cdot (\nabla \times \mathbf{B}) = 0.$$

But charge conservation says that

$$\dot{\rho} + \nabla \cdot \mathbf{J} = 0.$$

These can both hold iff $\dot{\rho} = 0$. But we clearly can change the charge density - pick up a charge and move it elsewhere! Contradiction.

With the new term, taking the divergence yields

$$\mu_0 \left(\nabla \cdot \mathbf{J} + \epsilon_0 \nabla \cdot \frac{\partial \mathbf{E}}{\partial t} \right) = 0.$$

Since partial derivatives commute, we have

$$\epsilon_0 \nabla \cdot \frac{\partial \mathbf{E}}{\partial t} = \epsilon_0 \frac{\partial}{\partial t} (\nabla \cdot \mathbf{E}) = \dot{\rho}$$

by the first Maxwell's equation. So it gives

$$\nabla \cdot \mathbf{J} + \dot{\rho} = 0.$$

So with the new term, not only is Maxwell's equation consistent with charge conservation - it actually implies charge conservation.

We now look for solutions to Maxwell's equation in the case where $\rho = 0$ and $\mathbf{J} = 0$, ie. in nothing/vacuum.

Differentiating the fourth equation with respect to time,

$$\begin{aligned} \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} &= \frac{\partial}{\partial t} (\nabla \times \mathbf{B}) \\ &= \nabla \times \frac{\partial \mathbf{B}}{\partial t} \\ &= \nabla \left(\underbrace{\nabla \cdot \mathbf{E}}_{=\rho/\epsilon_0=0} \right) + \nabla^2 \mathbf{E} \quad \text{by vector identities} \\ &= \nabla^2 \mathbf{E}. \end{aligned}$$

So each component of \mathbf{E} obeys the wave equation

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = 0.$$

We can do exactly the same thing to show that \mathbf{B} obeys the same equation:

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = 0,$$

where the speed of the wave is

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

Recall from the first lecture that

- $\epsilon_0 = 8.85 \times 10^{-12} \text{ m}^{-3} \text{ kg}^{-1} \text{ s}^2 \text{ C}^2$
- $\mu_0 = 4\pi \times 10^{-6} \text{ m kg C}^{-2}$

So

$$c = 3 \times 10^8 \text{ m s}^{-1},$$

which is the speed of light!

We now look for plane wave solutions which propagate in the x direction, and are independent of y and z . So any derivatives wrt y and z are zero (ie doesn't change if we move in the y or z direction). So

$$\nabla \cdot \mathbf{E} = 0 \Rightarrow E_x \text{ is constant}$$

We take $E_x = 0$. wlog, assume $E_z = 0$, ie the wave propagate in the x direction and oscillates in the y direction. Then we look for solutions of the form

$$\mathbf{E} = (0, E(x, t), 0).$$

With

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} - \frac{\partial^2 \mathbf{E}}{\partial x^2} = 0.$$

The general solution is

$$E(x, t) = f(x - ct) + g(x + ct).$$

The most important solutions are the *monochromatic* waves

$$E = E_0 \sin(kx - \omega t).$$

Definition (Amplitude, wave number and frequency).

- (i) E_0 is the *amplitude*
- (ii) k is the *wave number*.
- (iii) ω is the (*angular*) *frequency*.

The wave number is related to the wavelength by

$$\lambda = \frac{2\pi}{k}.$$

Since the wave has to travel at speed c , we must have

$$\omega^2 = c^2 k^2$$

So the value of k determines the value of ω , vice versa.

To solve for \mathbf{B} , we use

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

So $\mathbf{B} = (0, 0, B)$ for some B . Hence the equation gives.

$$\frac{\partial B}{\partial t} = -\frac{\partial E}{\partial x}.$$

So

$$B = \frac{E_0}{c} \sin(kx - \omega t).$$

Note that this is uniquely determined by \mathbf{E} , and we do not get to choose our favorite amplitude, frequency etc for the magnetic component.

We see that \mathbf{E} and \mathbf{B} oscillate in phase, orthogonal to each other, and orthogonal to the direction of travel. These waves are what we usually consider to be “light”.

Also note that Maxwell’s equations are linear, so we can add up two solutions to get a new one.

It is useful to sue complex notation. The most general monochromatic takes the form

$$\mathbf{E} = \mathbf{E}_0 \exp(i(\mathbf{k} \cdot \mathbf{x} - \omega t)),$$

and

$$\mathbf{B} = \mathbf{B}_0 \exp(i(\mathbf{k} \cdot \mathbf{x} - \omega t)),$$

with $\omega = c^2 |\mathbf{k}|^2$.

Definition (Wave vector). \mathbf{k} is the *wave vector*, which is real.

The “actual” solutions are just the real part of these expressions.

There are some restrictions to the values of \mathbf{E}_0 etc due to the maxwell’s equations:

$$\begin{aligned}\nabla \cdot \mathbf{E} = 0 &\Rightarrow \mathbf{k} \cdot \mathbf{E}_0 = 0 \\ \nabla \cdot \mathbf{B} = 0 &\Rightarrow \mathbf{k} \cdot \mathbf{B}_0 = 0 \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} &\Rightarrow \mathbf{k} \times \mathbf{E}_0 = \omega \mathbf{B}_0\end{aligned}$$

If \mathbf{E}_0 and \mathbf{B}_0 are real, then $\mathbf{k}, \mathbf{E}_0/c$ and \mathbf{B}_0 form a right-handed orthogonal triad of vectors.

Definition (Linearly polarized wave). A solution with real $\mathbf{E}_0, \mathbf{B}_0, \mathbf{k}$ is said to be *linearly polarized*.

This says that the waves oscillate up and down in a fixed plane.

If \mathbf{E}_0 and \mathbf{B}_0 are complex, then the polarization is not in a fixed direction. If we write

$$\mathbf{E}_0 = \boldsymbol{\alpha} + i\boldsymbol{\beta}$$

for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^3$, then the “real solution” is

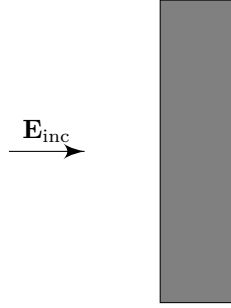
$$\text{Re}(\mathbf{E}) = \boldsymbol{\alpha} \cos(\mathbf{k} \cdot \mathbf{x} - \omega t) - \boldsymbol{\beta} \sin(\mathbf{k} \cdot \mathbf{x} - \omega t).$$

Note that $\nabla \cdot \mathbf{E} = 0$ requires that $\mathbf{k} \cdot \boldsymbol{\alpha} = \mathbf{k} \cdot \boldsymbol{\beta}$. It is not difficult to see that this traces out an ellipse.

Definition (Elliptically polarized wave). If \mathbf{E}_0 and \mathbf{B}_0 are complex, then it is said to be *elliptically polarized*. In the special case where $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$ and $\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = 0$, this is *circular polarization*.

We can have a simple application: why metals are shiny.

A metal is a conductor. Suppose the region $x > 0$ is filled with a conductor.



A light wave is incident on the conductor, ie

$$\mathbf{E}_{\text{inc}} = E_0 \hat{\mathbf{y}} \exp(i(kx + \omega t)),$$

with $\omega = ck$.

We know that inside a conductor, $\mathbf{E} = 0$, and at the surface, $\mathbf{E}_{\parallel} = 0$. So $\mathbf{E}_0 \cdot \hat{\mathbf{y}}|_{x=0} = 0$.

Then clearly our solution above does not satisfy the boundary conditions!
To achieve the boundary conditions, we add a reflected wave

$$\mathbf{E}_{\text{ref}} = -E_0 \hat{\mathbf{y}} \exp(i(-kx - \omega t)).$$

Then our total electric field is

$$\mathbf{E} = \mathbf{E}_{\text{inc}} + \mathbf{E}_{\text{ref}}.$$

Then this is a solution to Maxwell's equations since it is a sum of two solutions, and satisfies $\mathbf{E} \cdot \hat{\mathbf{y}}|_{x=0} = 0$ as required.

Maxwell's equations says $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$. So

$$\begin{aligned} \mathbf{B}_{\text{inc}} &= \frac{E_0}{c} \hat{\mathbf{z}} \exp(i(kx - \omega t)) \\ \mathbf{B}_{\text{ref}} &= \frac{E_0}{c} \hat{\mathbf{z}} \exp(i(-kx - \omega t)) \end{aligned}$$

This obeys $\mathbf{B} \cdot \hat{\mathbf{n}} = 0$, where $\hat{\mathbf{n}}$ is the normal to the surface. But we also have

$$\mathbf{B} \cdot \hat{\mathbf{z}}|_{x=0} = \frac{2E_0}{c} e^{-i\omega t},$$

and $\mathbf{B} = 0$ inside the conductor. So there *is* a magnetic field at the surface.

By our boundary conditions for conductors, this means that there's a surface current! This is given by

$$\mathbf{K} = \pm \frac{2E_0}{\mu_0 c} \hat{\mathbf{y}} e^{-i\omega t}.$$

So shining a light onto a metal will cause an oscillating current. We can imagine the process as the incident light hits the conductor, causes an oscillating current, which generates a reflected wave (since accelerating charges generate light - cf. next year)

We can do the same for light incident at an angle, and prove that the incident angle is equal to the reflected angle.

4.5 Poynting vector

Electromagnetic waves carry energy - that's how the Sun heats up the Earth! We will compute how much.

The energy stored in a field in a volume V is

$$U = \int_V \left(\frac{\varepsilon_0}{2} \mathbf{E} \cdot \mathbf{E} + \frac{1}{2\mu_0} \mathbf{B} \cdot \mathbf{B} \right) dV.$$

We have

$$\begin{aligned} \frac{dU}{dt} &= \int_V \left(\varepsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\mu_0} \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} \right) dV \\ &= \int_V \left(\frac{1}{\mu_0} \mathbf{E} \cdot (\nabla \times \mathbf{B}) - \mathbf{E} \cdot \mathbf{J} - \frac{1}{\mu_0} \mathbf{B} \cdot (\nabla \times \mathbf{E}) \right) dV. \end{aligned}$$

But

$$\mathbf{E} \cdot (\nabla \times \mathbf{B}) - \mathbf{B} \cdot (\nabla \times \mathbf{E}) = \nabla \cdot (\mathbf{E} \times \mathbf{B}),$$

by vector equations. So

$$\frac{dU}{dt} = - \int_V \mathbf{J} \cdot \mathbf{E} dV - \frac{1}{\mu_0} \int_S (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{S}.$$

Recall that the work done on a particle q moving with velocity v is $\delta W = q\mathbf{v} \cdot \mathbf{E} \delta t$. So the $\mathbf{J} \cdot \mathbf{E}$ term is the work done on a charged particles in V . We can thus write

Theorem (Poynting theorem).

$$\underbrace{\frac{dU}{dt} + \int_V \mathbf{J} \cdot \mathbf{E} dV}_{\text{Total change of energy in } V \text{ (fields + particles)}} = \underbrace{-\frac{1}{\mu_0} \int_S (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{S}}_{\text{Energy that escapes through the surface } S}.$$

Definition (Poynting vector). The *Poynting vector* is

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}.$$

This characterizes the energy transfer.

For a linearly polarized wave,

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_0 \sin(\mathbf{k} \cdot \mathbf{x} - \omega t) \\ \mathbf{B} &= \frac{1}{c} (\hat{\mathbf{k}} \times \mathbf{E}_0) \sin(\mathbf{k} \cdot \mathbf{x} - \omega t). \end{aligned}$$

So

$$\mathbf{S} = \frac{E_0^2}{c\mu_0} \hat{\mathbf{k}} \sin^2(\mathbf{k} \cdot \mathbf{x} - \omega t).$$

The average over $T = \frac{2\pi}{\omega}$ is thus

$$\langle \mathbf{S} \rangle = \frac{E_0^2}{2c\mu_0} \hat{\mathbf{k}}.$$

5 Electromagnetism and relativity

In this section, we try to “derive” Maxwell’s equations from special relativity. Of course, historically, Maxwell’s equations came first, and then special relativity was “derived” from it.

5.1 A review of special relativity

Here we do things in the reverse order of what we did in Dynamics and Relativity. We start with the equations, and then show that it has the properties we want (as opposed to what we physically observe, and find equations that describes it)

An observer on inertial frame S has coordinates (t, x, y, z) . An observer in inertial frame S' , moving with velocity $(v, 0, 0)$ with respect to S , has coordinates (t', x', y', z') . They are related by the *Lorentz transformation*

$$\begin{aligned} t' &= \gamma \left(t - \frac{v}{c^2} x \right) \\ x' &= \gamma (x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

with

$$\gamma = \frac{1}{\sqrt{1 - (v/c)^2}}.$$

We define the 4-vector

$$X^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix},$$

where μ is the coordinate ranging $\mu = 0, 1, 2, 3$.

Under a general Lorentz transformation

$$(X')^\mu = \Lambda^\mu_\nu X^\nu,$$

where Λ is a 4×4 matrix which obeys

$$\Lambda^\rho_\mu \eta_{\rho\sigma} \Lambda^\sigma_\nu = \eta_{\mu\nu},$$

where $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ is the *Minkowski metric*. There are two classes of Lorentz transformations:

(i) Rotations:

$$\Lambda^\mu_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix}$$

where $R^T R = 1$, ie. is an orthogonal matrix.

(ii) Boosts, eg. a boost in the x direction is

$$\Lambda^\mu_\nu = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

There are other extra *weird* transformations, but we are not going to discuss them.

All observers agree on the spacetime interval

$$X \cdot X = X^\mu \eta_{\mu\nu} X^\nu = c^2 t^2 - x^2 - y^2 - z^2.$$

Notation. We define 4-vectors to have indices up, eg. $X^\mu = \begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix}$.

We define the object with the index down to be $X_\mu = \begin{pmatrix} ct \\ -\mathbf{x} \end{pmatrix}$.

The advantage is that if we sum over repeated indices, one up and one down, we get

$$X^\mu X_\mu = x^0 x_0 + x^1 x_1 + x^2 x_2 + x^3 x_3 = c^2 t^2 - \mathbf{x}^2.,$$

which is a Lorentz invariant quantity.

In contrast, summing $X^\mu X^\mu = c^2 t^2 + \mathbf{x}^2$ is total nonsense.

We can write

$$X_\mu = \eta_{\mu\nu} X^\nu.$$

In general, we can use $\eta_{\mu\nu}$ to lower indices. To raise indices, we'll use

$$\eta^{\mu\nu} = \text{diag}(+1, -1, -1, -1).$$

Note that $\eta^{\mu\nu} \eta_{\nu\rho} = \delta^\mu_\rho$. We can write

$$X^\mu = \eta^{\mu\nu} X_\nu.$$

Note that we must not use $\eta_{\mu\nu}$ instead of $\eta^{\mu\nu}$ or else we will be summing over two down indices, which is nonsense.

Rule. Whenever we sum over repeated indices, we *must* have one up, one down.

This ensures that we get a Lorentz-invariant quantity.

5.2 Vectors, co-vectors and tensors

A 4-vector transforms as $X^\mu \mapsto X'^\mu = \Lambda^\mu_\nu X^\nu$.

Definition (Co-vectors). We call X_ν a *co-vector*.

These transform as

$$\begin{aligned} X_\mu \mapsto X'_\mu &= \eta_{\mu\nu} X'^\nu \\ &= \eta_{\mu\nu} \Lambda^\nu_\sigma X^\sigma \\ &= \eta_{\mu\nu} \Lambda^\nu_\sigma \eta^{\sigma\rho} X_\rho. \end{aligned}$$

Using our rules for lowering and raising indices, we write this as

$$X_\mu \mapsto X'_\mu = \Lambda_\mu^\rho X_\rho.$$

Note that Λ_μ^ρ is different from Λ^ρ_μ !

We can think of Λ as the components of the inverse matrix of Λ^{-1} . This is because

$$\Lambda_\mu^\rho \eta_{\rho\sigma} \Lambda^\sigma_\nu = \eta_{\mu\nu}.$$

So

$$\Lambda^\rho_\mu \Lambda_{\rho\sigma} = \eta_{\mu\nu}.$$

Therefore

$$\Lambda^\rho_\mu \Lambda_\rho^\sigma = \delta_\mu^\sigma.$$

Since the LHS is the product of two components of a matrix, not a matrix itself. So we can invert the order to obtain

$$\Lambda_\rho^\sigma \Lambda^\rho_\mu = \delta_\mu^\sigma$$

Compare this to

$$(\Lambda^{-1})^\sigma_\rho \Lambda^\rho_\mu = \delta_\mu^\sigma.$$

So

$$(\Lambda^{-1})^\sigma_\rho = \Lambda_\rho^\sigma.$$

Definition (4-derivative). The relativistic generalization of ∇ is the *4-derivative* is define to be

$$\frac{\partial}{\partial X^\mu} = \left(\frac{1}{c} \frac{\partial}{\partial t}, \nabla \right).$$

Note that ∂_μ is a co-vector, not a 4-vector. We show this by consider a test function $\Phi(x)$ over Minkowski space. Then we want to show that

$$\partial_\mu \Phi = \left(\frac{1}{c} \frac{\partial \Phi}{\partial t}, \nabla \Phi \right)$$

transforms as a co-vector.

This follows from the chain rule*

$$\partial_\mu = \frac{\partial}{\partial X^\mu} \mapsto \frac{\partial}{\partial X'^\mu} = \frac{\partial X^\nu}{\partial X'^\mu} \frac{\partial}{\partial X^\nu} = (\Lambda^{-1})^\nu_\mu \partial_\nu = \Lambda_\mu^\nu \partial_\nu.$$

Definition (Tensor). A *tensor* of type (m, n) is a quantity

$$T^{\mu_1 \dots \mu_m}_{\nu_1 \dots \nu_n}$$

which transforms as

$$T'^{\mu_1 \dots \mu_m}_{\nu_1 \dots \nu_n} = \Lambda^{\mu_1}_{\rho_1} \dots \Lambda^{\mu_m}_{\rho_m} \Lambda_{\nu_1}^{\sigma_1} \dots \Lambda_{\nu_n}^{\sigma_n} \times T^{\rho_1, \dots, \rho_m}_{\sigma_1, \dots, \sigma_n}.$$

To build theories consistent with special relativity, we

- (i) Work only with tensors
- (ii) Make sure indices match in equations.

From now on, this is what special relativity means

5.3 Conserved currents

Recall that *local* charge conservation states that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

We said that this says that we can have a charge disappear on Earth and appear on Moon. This is because it would be travelling faster than light! So we should be able to write this relativistically. Write

$$J^\mu = \begin{pmatrix} \rho c \\ \mathbf{J} \end{pmatrix}$$

We want to show that this makes sense, ie. it transforms via left multiplication by Λ^μ_ν .

Consider a static charge density $\rho_0(\mathbf{x})$ with $\mathbf{J} = 0$. In a frame boosted by \mathbf{v} , the current should become

$$J'^\mu = \Lambda^\mu_\nu J^\nu = \begin{pmatrix} \gamma \rho_0 c \\ \gamma \rho_0 \mathbf{v} \end{pmatrix}.$$

So does it make sense that $\rho' = \gamma \rho_0$? Yes, since we have Lorentz contraction. So if we have the same amount of stuff (ie. charge), there will be more charge per unit volume. Also, $\mathbf{J}' = -\gamma \rho_0 \mathbf{v}$, as expected, since this is just the charge density times the velocity.

So this is a sensible definition.

With this definition of J^μ , Local charge conservation is just

$$\partial_\mu J^\mu = 0.$$

This is invariant under Lorentz transformation, since the indices work out (ie. the Λ^μ_ν and λ_σ^ρ cancel out).

We see that once we have the right notation, the laws of physics are so short and simple!

5.4 Gauge potentials and electromagnetic fields

Recall that we introduced ϕ and \mathbf{A} to help solve two of Maxwell's equations:

$$\begin{aligned} \nabla \times \mathbf{E} &= 0 \Leftrightarrow \mathbf{E} = -\nabla \phi \\ \nabla \cdot \mathbf{B} &= 0 \Leftrightarrow \mathbf{B} = \nabla \times \mathbf{A}. \end{aligned}$$

In general, these equations are

$$\nabla \times + \frac{\partial \mathbf{B}}{\partial t} = 0, \quad \nabla \cdot \mathbf{B} = 0.$$

So we cannot use $\mathbf{E} = -\nabla \phi$ directly if \mathbf{B} changes.

We solve these by introducing $\phi(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x}, t)$, and writing

$$\begin{aligned} \mathbf{E} &= -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} &= \nabla \times \mathbf{A}. \end{aligned}$$

So we can still use the vector and scalar potential.

Note: ϕ and \mathbf{A} are not unique. We can shift them by

$$\begin{aligned}\phi &\mapsto \phi - \frac{\partial \chi}{\partial t} \\ \mathbf{A} &\mapsto \mathbf{A} + \nabla \chi\end{aligned}$$

for any function $\chi(\mathbf{x}, t)$. These are *gauge transformations*. Plug them into the equations and you will see that you get the same \mathbf{E} and \mathbf{B} .

Now we have ended up with four objects, 1 from ϕ and 3 from A , so we can put them into a 4-vector *gauge potential*:

$$A^\mu = \begin{pmatrix} \phi \\ \mathbf{A} \end{pmatrix}, \quad A_\mu = \begin{pmatrix} \phi/c \\ \mathbf{A} \end{pmatrix}$$

We will assume that this makes sense (ie. is a genuine tensor), and show that this assumption will lead to Maxwell's equations.

Gauge transformations are now

$$A_\mu \mapsto A_\mu - \partial_\mu \chi.$$

We define the anti-symmetric electromagnetic tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Since this is antisymmetric, the diagonals are all 0, and $A_{\mu\nu} = -A_{\nu\mu}$. So this thing has $(4 \times 4 - 4)/2 = 6$ independent components. So it could encapsulate information about the electric and magnetic fields (and nothing else).

We note that F is invariant under gauge transformations:

$$F_{\mu\nu} \mapsto F_{\mu\nu} - \partial_\mu \partial_\nu \chi + \partial_\nu \partial_\mu \chi = F_{\mu\nu}.$$

We can compute components of $F_{\mu\nu}$.

$$F_{01} = \frac{1}{c} \frac{\partial}{\partial t} A_x - \frac{\partial \phi/c}{\partial x} = \frac{E_x}{c}.$$

Note that we have $-A_x$ instead of A_x since we have indices down. We can similarly obtain E_y/c and E_z/c .

We also have

$$F_{12} = \frac{\partial}{\partial x} (-A_y) - \frac{\partial}{\partial y} (-A_x) = -B_z.$$

So

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

Raising both indices, we have

$$F^{\mu\nu} = \eta^{\mu\rho} \eta^{\nu\sigma} F_{\rho\sigma} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

So the electric fields are inverted and the magnetic field is intact (which makes sense, since moving indices up and down will flip the sign of the spacial components. The electric field has one, while the magnetic field has two. So the magnetic field is swapped twice and is intact).

Both $F_{\mu\nu}$ and $F^{\mu\nu}$ are tensors. Under a Lorentz transformation, we have

$$F'^{\mu\nu} = \Lambda^\mu_\rho \Lambda^\nu_\sigma F^{\rho\sigma}.$$

For example under rotation

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix},$$

we find that

$$\begin{aligned} \mathbf{E}' &= R\mathbf{E} \\ \mathbf{B}' &= R\mathbf{B} \end{aligned}$$

Under a boost by v in the x -direction, we have

$$\begin{aligned} E'_x &= E_x \\ E'_y &= \gamma(E_y - vB_z) \\ E'_z &= \gamma(E_z + vB_y) \\ B'_x &= B_x \\ B'_y &= \gamma\left(B_y + \frac{v}{c^2}E_z\right) \\ B'_z &= \gamma\left(B_z - \frac{v}{c^2}E_y\right) \end{aligned}$$

Example (Boosted line charge). An infinite line along the x direction with uniform charge per unit length, η , has electric field

$$\mathbf{E} = \frac{\eta}{2\pi\epsilon_0(y^2 + z^2)} \begin{pmatrix} 0 \\ y \\ z \end{pmatrix}$$

The magnetic field is $\mathbf{B} = 0$. Plugging this into the equation above, an observer in frame S' boosted with $\mathbf{v} = (v, 0, 0)$, ie. parallel to the wire, sees

$$\mathbf{E} = \frac{\eta\gamma}{2\pi\epsilon_0(y^2 + z^2)} \begin{pmatrix} 0 \\ y \\ z \end{pmatrix} = \frac{\eta\gamma}{2\pi\epsilon_0(y'^2 + z'^2)} \begin{pmatrix} 0 \\ y' \\ z' \end{pmatrix}$$

Also,

$$\mathbf{B} = \frac{\eta\gamma v}{2\pi\epsilon_0\sigma^2(y^2 + z^2)} \begin{pmatrix} 0 \\ z \\ -y \end{pmatrix} = \frac{\eta\gamma v}{2\pi\epsilon_0\sigma^2(y'^2 + z'^2)} \begin{pmatrix} 0 \\ z' \\ -y' \end{pmatrix}$$

In frame S' , the charge density is Lorentz contracted to $\gamma\eta$. The magnetic field can be written as

$$\mathbf{B} = \frac{\mu_0 I'}{2\pi\sqrt{y'^2 + z'^2}} \hat{\varphi}',$$

where $\hat{\phi}' = \frac{1}{\sqrt{y'^2 + z'^2}} \begin{pmatrix} 0 \\ -z' \\ y' \end{pmatrix}$ is the basis vector of cylindrical coordinates, and $I' = -\gamma\eta v$ is the current.

This is what we calculated from Ampere's law previously. But we didn't *use* Ampere's law here. We used Gauss' law, and then applied a Lorentz boost.

We see that magnetic fields are relativistic effects of electric fields. They are what we get when we apply a Lorentz boost to an electric field. So relativity is not only about *very fast objects*TM. It is there when you stick a magnet onto your fridge!

Example (Boosted point charge). A boosted point charge generates a current, but is not the steady current we studied in magnetostatics. As the point charge moves, the current density moves.

A point charge Q , at rest in frame S has

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}} = \frac{Q}{4\pi\epsilon_0^2 (x^2 + y^2 + z^2)^{3/2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

and $\mathbf{B} = 0$.

In frame S' , boosted with $\mathbf{v} = (v, 0, 0)$, we have

$$\mathbf{E}' = \frac{Q}{4\pi\epsilon_0 (x^2 + y^2 + z^2)^{3/2}} \begin{pmatrix} x \\ \gamma y \\ \gamma z \end{pmatrix}$$

We need to express this in terms of x', y', z' , instead of x, y, z . Then we get

$$\mathbf{E}' = \frac{Q\gamma}{4\pi\epsilon_0 (\gamma^2 (x' + vt')^2 + y'^2 + z'^2)} \begin{pmatrix} x' + vt' \\ y' \\ z' \end{pmatrix}$$

Suppose that the particle sits at $(-vt', 0, 0)$ in S' . Let's look at the electric at $t' = 0$. Then the radial vector is $\mathbf{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$. In the denominator, we write

$$\begin{aligned} \gamma^2 (x'^2 + y'^2 + z'^2) &= (\gamma^2 - 1)x'^2 + r'^2 \\ &= \frac{v^2 \gamma^2}{c^2} x'^2 + r'^2 \\ &= \left(\frac{v^2 \gamma^2}{c^2} \cos^2 \theta + 1 \right) r'^2 \\ &= \gamma^2 \left(1 - \frac{v^2}{c^2} \sin^2 \theta \right) r'^2 \end{aligned}$$

where θ is the angle between the x' axis and \mathbf{r}' .

So

$$\mathbf{E}' = \frac{1}{\gamma^2 \left(1 - \frac{v^2}{c^2} \sin^2 \theta \right)^{3/2}} \frac{Q}{4\pi\epsilon_0 r'^2} \hat{\mathbf{r}}'.$$

The factor $1/\gamma^2 \left(1 - \frac{v^2}{c^2} \sin^2 \theta\right)^{3/2}$ squashes the electric field in the direction of motion.

This result was first discovered by Lorentz from solving Maxwell's equations directly, which lead to him discovering the Lorentz transformations.

There is also a magnetic field

$$\mathbf{B} = \frac{\mu_0 Q \gamma}{4\pi(\gamma^2(x' + vt')^2 + y'^2 + z'^2)^{3/2}} \begin{pmatrix} 0 \\ z' \\ -y' \end{pmatrix}.$$

5.4.1 Lorentz invariants

There are two combinations of \mathbf{E} and \mathbf{B} that all observers agree on. To find these, we just need to contract indices on tensors.

- (i) $\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = \frac{\mathbf{E}^2}{c^2} - \mathbf{B}^2$. This is similar to the Lagrangian, where \mathbf{E}^2/c^2 is like the kinetic energy and \mathbf{B}^2 is like the potential energy.
- (ii) To describe the second invariant, we need to introduce a new object in Minkowski space. This is the fully anti-symmetric tensor

$$\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \mu\nu\rho\sigma \text{ is even permutation of } 0123 \\ -1 & \mu\nu\rho\sigma \text{ is odd permutation of } 0123 \\ 0 & \text{otherwise} \end{cases}$$

This is analogous to the ε_{ijk} in \mathbb{R}^3 , only in 4-dimensional space only. Under a Lorentz transformation,

$$\varepsilon'^{\mu\nu\rho\sigma} = \Lambda^\mu{}_\kappa \Lambda^\nu{}_\lambda \Lambda^\rho{}_\beta \Lambda^\sigma{}_\gamma \varepsilon^{\kappa\lambda\alpha\beta}$$

Since $\varepsilon^{\mu\nu\rho\sigma}$ is fully-antisymmetric, we know that

$$\varepsilon'^{\mu\nu\rho\sigma} = a \varepsilon^{\mu\nu\rho\sigma}$$

for some constant a . To figure out what a is, test a single component:

$$\varepsilon'^{0123} = \Lambda^0{}_\kappa \Lambda^1{}_\lambda \Lambda^2{}_\beta \Lambda^3{}_\gamma \varepsilon^{\kappa\lambda\alpha\beta} = \det \Lambda.$$

Lorentz transformations have $\det \Lambda = +1$ (rotations, boosts), or $\det \Lambda = -1$ (reflections, time reversal).

We will restrict to Λ such that $\det \Lambda = +1$. Then $a = 1$, and $\varepsilon^{\mu\nu\rho\sigma}$ is invariant. In particular, it is a tensor.

We can finally apply this to electromagnetic fields. The *dual electromagnetic tensor*

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix}$$

The factor of a half is because, say \tilde{F}^{12} get contributions from both F_{03} and F_{30} , so we need to average the sum to avoid double-counting. This is yet another antisymmetric matrix.

This is obtained from $F^{\mu\nu}$ through the substitution $\mathbf{E} \mapsto c\mathbf{B}$ and $\mathbf{B} \mapsto -\mathbf{E}/c$. Note the minus sign!

Then $\tilde{F}^{\mu\nu}$ is a tensor. We can construct the other Lorentz invariant using $\tilde{F}^{\mu\nu}$. We don't get anything new if we contract this with itself, since $\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} = -F_{\mu\nu}F^{\mu\nu}$. Instead, the second Lorentz invariant is

$$\frac{1}{4}\tilde{F}^{\mu\nu}F_{\mu\nu} = \mathbf{E} \cdot \mathbf{B}/c.$$

5.5 Maxwell Equations

To write out Maxwell's Equations, we have to write them out using tensors. There aren't many of those, and it turns out there is only one way we can do it:

$$\begin{aligned}\partial_\mu F^{\mu\nu} &= \mu_0 J^\nu \\ \partial_\mu \tilde{F}^{\mu\nu} &= 0.\end{aligned}$$

As we said before, if we find the right way of writing equations, they look really simple! We don't have to worry ourselves with where the c and μ_0, ε_0 go!

We can show that if we insist that our equation is linear, then this is the only form of equations we can write down. We might be able to move the μ_0 and J^ν around, but those can be eliminated by physical (as opposed to mathematical) arguments.

We now check that these agree with the Maxwell's equations

First work with the first equation: when $\nu = 0$, we are left with

$$\partial_i F^{i0} = \mu_0 J^0,$$

where i ranges over 1, 2, 3. Then We have

$$\nabla \cdot \left(\frac{\mathbf{E}}{c} \right) = \mu_0 \rho c.$$

So we have

$$\nabla \cdot \mathbf{E} = c^2 \mu_0 \rho = \frac{\rho}{\varepsilon_0}$$

When $\nu = i$ for some $i = 1, 2, 3$, we have

$$\partial_\mu F^{\mu i} = \mu_0 J^i.$$

So after some tedious checking, we have

$$\frac{1}{c} \frac{\partial}{\partial t} \left(-\frac{\mathbf{E}}{c} \right) + \nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

With the second equation, when $\nu = 0$, we have

$$\partial_i \tilde{F}^{i0} = 0 \Rightarrow \nabla \cdot \mathbf{B} = 0.$$

$\nu = i$ gives

$$\partial_\mu \tilde{F}^{\mu i} = 0 \Rightarrow \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E}.$$

So we recover Maxwell's equations. Then we now see why the J^ν term appears in the first equation and not the second - it tells us that there is only electric charge, not magnetic charge.

Note: Each law is actually a 4-vector equation with $\nu = 0, 1, 2, 3$ (cf. $\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0$ is a vector equation, with one equation for each component $i = 1, 2, 3$). We say that the equations are *covariant*.

Note: We have $\partial_\nu \partial_\mu F^{\mu\nu} = 0$ due to anti-symmetry. So $\partial_\nu J^\nu = 0$, which is charge conservation. Note that previously derived charge conservation from Maxwell's Equation without using SR, but this derivation is much simpler!

Note: We can solve $\partial_\mu F^{\mu\nu} = 0$ by writing

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Then

$$\partial_\mu \tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \partial_\mu F_{\rho\sigma} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \partial_\mu (\partial_\rho A_\sigma - \partial_\sigma A_\rho) = 0$$

by symmetry.

This means that we can also write the Maxwell equations as

$$\partial_\mu F^{\mu\nu} \mu_0 J^\nu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

6 The Lorentz force law

The final aspect of electromagnetism is the Lorentz force law for a particle with charge q moving with velocity \mathbf{u} :

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

To write this in relativistic form, we use the proper time τ (time experienced by particle), which obeys

$$\frac{dt}{d\tau} = \gamma(\mathbf{u}) = \frac{1}{\sqrt{1 - u^2/c^2}}.$$

We define the 4-velocity $U = \frac{dX}{d\tau} = \gamma \begin{pmatrix} c \\ \mathbf{u} \end{pmatrix}$, and 4-momentum $P = \begin{pmatrix} E/c \\ \mathbf{p} \end{pmatrix}$, where E is the energy.

The Lorentz force law can be written as

$$\frac{dP^\mu}{d\tau} = qF^{\mu\nu}U_\nu.$$

We show that this does give our original Lorentz force law:

When $\mu = 1, 2, 3$, we obtain

$$\frac{d\mathbf{p}}{d\tau} = q\gamma(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

By the chain rule, since $\frac{dt}{d\tau} = \gamma$, we have

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

Note that here $\mathbf{p} = m\gamma\mathbf{v}$, the relativistic momentum, not the ordinary momentum.

But how about the $\mu = 0$ component? We get

$$\frac{dP^0}{d\tau} = \frac{1}{c} \frac{d\text{Energy}}{d\tau} = \frac{q}{c} \gamma \mathbf{E} \cdot \mathbf{v}.$$

This says that

$$\frac{d\text{Energy}}{dt} = q\mathbf{E} \cdot \mathbf{v},$$

which is our good old formula for the work done by an electric field.

Example (Motion in a constant field). Suppose that $\mathbf{E} = (E, 0, 0)$ and $\mathbf{u} = (v, 0, 0)$. Then

$$m \frac{d(\gamma u)}{dt} = qE.$$

So

$$m\gamma u = qEt.$$

So

$$u = \frac{dx}{dt} = \frac{qEt}{\sqrt{m^2 q^2 E^2 t^2 / c^2}}.$$

Note that $u \rightarrow c$ as $t \rightarrow \infty$. Then we can solve to find

$$x = \frac{mc^2}{q} \left(\sqrt{1 + \frac{q^2 E^2 t^2}{mc^2}} - 1 \right)$$

For small t , $x \approx \frac{1}{2}qEt^2$, as expected.

Example (Motion in constant magnetic field). Suppose $\mathbf{B} = (0, 0, B)$. Then we start with

$$\frac{dP^0}{d\tau} = 0 \Rightarrow \text{Energy} = m\gamma c^2 = \text{constant}.$$

So $|\mathbf{u}|$ is constant. Then

$$m \frac{\partial \gamma \mathbf{u}}{\partial t} = q \mathbf{u} \times \mathbf{B}$$

So

$$m\gamma \frac{d\mathbf{u}}{dt} = q \mathbf{u} \times \mathbf{B}$$

since $|\mathbf{u}|$ is constant. This is the same equation we saw in Dynamics and Relativity, for a particle in a magnetic field, except for the extra γ term. It goes in circles with frequency

$$\omega = \frac{qB}{m\gamma}.$$