Gravitational Waves Notes 2

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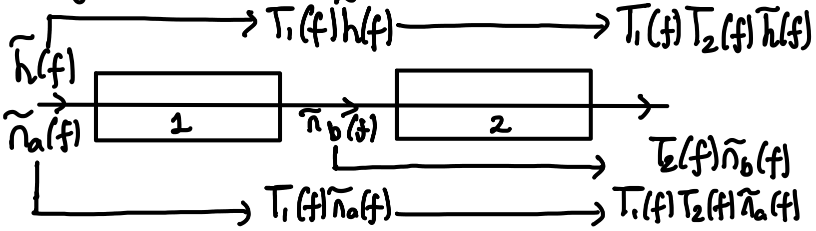
Maggiore Vol 1 - 7.1 - Noise Spectral Density

- Output of a GW detector is a time-series f.ex. phase shift of light recombining after travelling through interferometer
- Mixture of true GW signal and noise
- GW described by tensor h_{ij}, so input $\underline{h(t) = D^{ij} h_{ij}(t)}$
  D^{ij} = constant detector tensor
- Output of detector $\tilde{h}_{out}(f)$ in freq space is linear function of input via transfer function in linear system $\underline{\tilde{h}_{out}(f) = T(f)\tilde{h}(f)}$
- BUT output also has noise, so output in time space is
  $\underline{S_{out}(t) = h_{out}(t) + n_{out}(t)}$

- A detector can be modelled as a series of linear systems with their own transfer functions $T_i(f)$ and their own noises $n_i(f)$. The overall transfer function is given by $T(f) = \prod_{i=1}^{n} T_i(f)$ and the noises given by propagating the noise through the remaining transformers, shown here:



If $n_{out}(t)$ is noise output, then define total noise input as $\underline{\tilde{n}(f) = T^{-1}(f)\,\tilde{n}_{out}(f)}$ and $\underline{s(t) = h(t) + n(t)}$

- Detection problem: Separate $h(t)$ from $n(t)$
- Advantage of referring to everything as the inputs is $n(t)$ gives measure of min of $h(t)$, since $\mathcal{O}(D^{ij}) = 1$, whereas h_{out} depends on transfer function
- Assume noise is stationary: different Fourier components are uncorrelated, ensemble average of the components is
  $$\underline{\langle \tilde{n}^*(f)\tilde{n}(f')\rangle := \delta(f-f')\tfrac{1}{2}S_n(f)}$$
- $n(t)$ real $\Rightarrow \tilde{n}(-f) = \tilde{n}^*(f) \Rightarrow S_n(-f) = S_n(f)$
- $n(t)$ dimensionless assumed $\Rightarrow [S_n(f)] = Hz^{-1}$
- wlog $\langle n(t)\rangle = 0$
- If $f = f'$, restrict time interval to $-T/2 < t < T/2$ finite
  $\delta(f=0) \to \left[\int_{-T/2}^{T/2} dt\, e^{2\pi i f t}\right]\Big|_{f=0} = T$

$$\Rightarrow \underline{\langle |\tilde{n}(f)|^2\rangle = \tfrac{1}{2} S_n(f) T}$$

- On the interval $[-T/2, T/2]$, the Fourier modes have discrete frequencies $f_n = n/T$, so resolution in frequency is $\Delta f = 1/T$
  $\Rightarrow \tfrac{1}{2} S_n(f) = \langle |\tilde{n}(f)|^2\rangle \Delta f$
- $\langle n^2(t)\rangle = \langle n^2(t=0)\rangle = \int_{-\infty}^{\infty} df\, df' \langle \tilde{n}^*(f)\tilde{n}(f')\rangle$
  $= \tfrac{1}{2}\int_{-\infty}^{\infty} df\, S_n(f)$
  $= \int_0^{\infty} df\, S_n(f)$ $\rightarrow$ $S_n(f)$ is noise spectral density

or power spectral density

Specifically, single-sided as we only integrate over physical frequencies $f>0$.

- Note: $n(t) \to \lambda n(t) \Rightarrow S_n(f) \to \lambda^2 S_n(f)$, but strain sensitivity $\sqrt{S_n(f)}$ increases linearly.
- Technically, one should define the auto-correlation function $R(\tau) = \langle n(t+\tau) n(t) \rangle$ and perform the FT on this
$$\tfrac{1}{2} S_n(f) \equiv \int_{-\infty}^{\infty} d\tau\, R(\tau)\, e^{2\pi i f \tau}$$
$$\Rightarrow R(\tau) = \tfrac{1}{2}\int_{-\infty}^{\infty} df\, S_n(f)\, e^{-2\pi i f \tau} = \langle n(t+\tau) n(t) \rangle$$
$$\Rightarrow R(0) = \langle n^2(t) \rangle = \int_0^{\infty} df\, S_n(f)$$ which is equivalent to before if the FT of $n(t)$ exists.

7.2 Pattern Functions & Angular Sensitivity

A GW with given propagation direction $\hat{n}$ is written
$$h_{ij}(t,\underline{x}) = \sum_{A=+,\times} e_{ij}^A(\hat{n}) \int_{-\infty}^{\infty} df\, \tilde{h}_A(f)\, e^{-2\pi i f(t - \hat{n}\cdot x/c)}$$

- For a ground based interferometer, $2\pi f\, \hat{n}\cdot\underline{x} \ll 1$, so neglect spatial dependence and we get
$$h_{ij}(t,\underline{x}) = \sum_{A=+,\times} e_{ij}^A(\hat{n}) \int_{-\infty}^{\infty} df\, \tilde{h}_A(f)\, e^{-2\pi i f t} = \sum_{A=+,\times} e_{ij}^A(\hat{n})\, h_A(t)$$

contracting with the tensor D^{ij} gives us
$$h(t,\underline{x}) = \sum_{A=+,\times} D^{ij} e_{ij}^A(\hat{n})\, h_A(t) \; ; \quad \boxed{F_A(\hat{n}) = D^{ij} e_{ij}(\hat{n})}$$

The $F_A(\hat{n})$ are detector pattern functions, so can write
$$\boxed{h(t) = h_+(t) F_+(\theta,\phi) + h_\times(t) F_\times(\theta,\phi)} \quad [\text{since } \hat{n} = \hat{n}(\theta,\phi)]$$

- This is all in terms of axes $(\hat{\underline{u}}, \hat{\underline{v}})$ in a plane orthogonal to $\hat{n}$; if we rotate by ψ in the transverse plane:
$$\hat{\underline{u}}' = \hat{\underline{u}}\cos\psi - \hat{\underline{v}}\sin\psi$$
$$\hat{\underline{v}}' = \hat{\underline{u}}\sin\psi + \hat{\underline{v}}\cos\psi$$
and the amplitudes of the polarisations change to
$$h_+' = h_+ \cos 2\psi - h_\times \sin 2\psi$$
$$h_\times' = h_+ \sin 2\psi + h_\times \cos 2\psi$$
$$(e_{ij}^+)'(\hat{n}) = \hat{u}_i'\hat{u}_j' - \hat{v}_i'\hat{v}_j' \qquad (e_{ij}^\times)'(\hat{n}) = \hat{u}_i'\hat{v}_j' + \hat{v}_i'\hat{u}_j'$$
$$\hookrightarrow = e_{ij}^+(\hat{n})\cos 2\psi - e_{ij}^\times(\hat{n})\sin 2\psi \qquad \hookrightarrow = e_{ij}^+(\hat{n})\sin 2\psi + e_{ij}^\times(\hat{n})\cos 2\psi$$

Together this implies
$$F_+'(\hat{n},\psi) = F_+(\hat{n})\cos 2\psi - F_\times(\hat{n})\sin 2\psi$$
$$F_\times'(\hat{n},\psi) = F_+(\hat{n})\sin 2\psi + F_\times(\hat{n})\cos 2\psi$$

- $\int \frac{d^2\hat{n}}{4\pi} F_+(\hat{n}) F_\times(\hat{n}) = 0$ - useful identity, independent of D^{ij}
- If we take the average over all ψ as well we get
$$\int_0^{2\pi} \frac{d\psi}{2\pi} F_+^2(\hat{n},\psi) = \int_0^{2\pi} \frac{d\psi}{2\pi} F_\times^2(\hat{n},\psi)$$

and $\langle F_+^2(\hat{n},\psi) \rangle = \langle F_\times^2(\hat{n},\psi) \rangle \qquad \langle \dots \rangle = \int_0^{2\pi} \frac{d\psi}{2\pi} \int \frac{d^2\hat{n}}{4\pi}$

with $F := \langle F_+^2 \rangle + \langle F_\times^2 \rangle = 2/5$ angular efficiency factor

- For interferometers: $F_+(\theta, \phi; \psi=0) = \frac{1}{2}(1+\cos^2\theta)\cos 2\phi$
$$F_\times(\theta,\phi;\psi=0) = \cos\theta \sin 2\phi$$
$$F = 2/5$$

7.3 Matched Filtering

- $s(t) = h(t) + n(t)$; usually in the situation $|h(t)| \ll |n(t)|$
- Suppose we know the form of $h(t)$; then
$$\frac{1}{T}\int_0^T dt\, s(t) h(t) = \frac{1}{T}\int_0^T dt\, h^2(t) + \frac{1}{T}\int_0^T dt\, n(t) h(t)$$
- $h(t)$ and $n(t)$ are separately oscillating, the first integral on the RHS is definitely positive and a slowly varying function of time so grows as T for large T $\Rightarrow$ $\frac{1}{T}\int_0^T dt\, h^2(t) \sim h_0^2 \;\; \mathcal{O}(1)$
- $n(t), h(t)$ uncorrelated, so $\int_0^T dt\, n(t) h(t) \sim T^{1/2}$ for large T (c.f. random walk), so
$$\frac{1}{T}\int_0^T dt\, n(t) h(t) \sim \left(\frac{\tau_0}{T}\right)^{1/2} n_0 h_0$$
h_0 = characteristic amplitude of h $\quad n_0$ = characteristic amplitude of n
τ_0 = characteristic time
- As $T \to \infty$, second term goes to 0 and we have 'filtered out' noise - not possible to set $T = \infty$ in practise, but need only $h_0 > (\tau_0/T)^{1/2} n_0$
- Look for optimal signal to noise ratio. Let:
$$\hat{s} = \int_{-\infty}^{\infty} dt\, s(t) K(t) \qquad K(t) = \text{filter function}$$
- Signal-to-noise ratio is S/N where
$S = \mathbb{E}[\hat{s} \mid \text{signal present}]$ $\quad N = \text{rms}(\hat{s} \mid \text{signal absent})$
- $\langle n(t) \rangle = 0$ implies:
$$S = \int_{-\infty}^{\infty} dt\, \langle s(t) \rangle K(t) = \int_{-\infty}^{\infty} dt\, h(t) K(t)$$
$$= \int_{-\infty}^{\infty} df\, \tilde{h}(f) \tilde{K}^*(f)$$

- $N^2 = \left[\langle \hat{s}^2(t) \rangle - \langle \hat{s}(t) \rangle^2\right]_{h=0} = \langle \hat{s}^2(t) \rangle_{h=0}$
$$= \int_{-\infty}^{\infty} dt\, dt'\, K(t) K(t') \langle n(t) n(t') \rangle$$
$$= \int_{-\infty}^{\infty} dt\, dt'\, K(t) K(t') \int_{-\infty}^{\infty} df\, df'\, e^{2\pi i f t - 2\pi i f' t'} \langle \tilde{n}^*(f) \tilde{n}(f') \rangle$$
- From defn of PSD, this is $N^2 = \int_{-\infty}^{\infty} df\, \frac{1}{2} S_n(f) |\tilde{K}(f)|^2$
$$\Rightarrow \boxed{\frac{S}{N} = \frac{\int_{-\infty}^{\infty} df\, \tilde{h}(f) \tilde{K}^*(f)}{\left[\int_{-\infty}^{\infty} df\, \frac{1}{2} S_n(f) |\tilde{K}(f)|^2\right]^{1/2}}}$$

- Now want to know which K maximises this
- Define $(A|B) = \mathrm{Re}\int_{-\infty}^{\infty} df\, \frac{\tilde{A}^*(f) \tilde{B}(f)}{(1/2) S_n(f)}$
$$= 4\,\mathrm{Re}\int_0^{\infty} \frac{\tilde{A}^*(f) \tilde{B}(f)}{S_n(f)}$$
$$\Rightarrow \frac{S}{N} = \frac{(u|h)}{(u|u)^{1/2}} \qquad u(t) \text{ is func. s.t. } \tilde{u}(f) = \tfrac{1}{2} S_n(f) \tilde{K}(f)$$
- This is a scalar product, so the solution is simple; we

want to maximise U along h so choose $\tilde{U}(f)$ proportional to $\tilde{h}(f)$, $\boxed{\tilde{K}(f) = \text{const.}\ \frac{\tilde{h}(f)}{S_n(f)}}$

(const. arbitrary)

- This gives optimal $\frac{S}{N}$ as $\boxed{\left(\frac{S}{N}\right)^2 = 4\int_0^\infty df\ \frac{|\tilde{h}(f)|^2}{S_n(f)}}$