Gravitational Waves 4

Tuesday 12 November 2024 11:57

Maggiore Vol 1 - 4.1 - Inspiral of Compact Binaries

- Binary system - two compact, point-like stars masses m_1, m_2 at positions $\underline{r}_1, \underline{r}_2$; reduced mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$
- Eqn of motion (Newtonian):

$$\ddot{\underline{r}} = -\frac{Gm}{r^3}\underline{r} \qquad \underline{r} = \underline{r}_2 - \underline{r}_1 \quad m = m_1 + m_2$$

- Circular orbits: frequency ω_s and orbital radius R related: $v = \omega_s R$ and acceleration $\frac{v^2}{R} = \frac{Gm}{R^2}$

$$\Rightarrow \underline{\omega_s^2 = \frac{Gm}{R^3}}$$

- Introduce the chirp mass: $M_c = \mu^{3/5} m^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$
- We have expressions for the GW amplitudes in this case:

$$h_+(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{gw}}{c}\right)^{2/3}\frac{1+\cos^2\theta}{2}\cos(2\pi f_{gw} t_{ret} + 2\phi)$$

$$h_\times(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{gw}}{c}\right)^{2/3}\cos\theta\sin(2\pi f_{gw} t_{ret} + 2\phi)$$

$$f_{gw} = \frac{\omega_{gw}}{2\pi} = \frac{\omega_s}{\pi}$$

- GW amplitudes depend on m_1, m_2 only through M_c (lowest order approx)
- In terms of Schwarzschild radius $R_c \equiv 2GM_c/c^2$ and reduced wavelength $\lambda\!\!\!^- = c/\omega_{gw}$

$$h_+(t) = \mathcal{A}\,\frac{1+\cos^2\theta}{2}\cos(\omega_{gw} t_{ret} + 2\phi)$$

$$h_\times(t) = \mathcal{A}\cos\theta\sin(\omega_{gw} t_{ret} + 2\phi)$$

$$\mathcal{A} = \frac{1}{2^{1/3}}\left(\frac{R_c}{r}\right)\left(R_c/\lambda\!\!\!^-\right)^{2/3}$$

- There is also an expression for the angular distribution of power radiated

$$\frac{dP}{d\Omega} = \frac{2}{\pi}\frac{c^5}{G}\left(\frac{GM_c\omega_{gw}}{2c^3}\right)^{10/3}\left[\left(\frac{1+\cos^2\theta}{2}\right)^2 + \cos^2\theta\right]$$

$$d\Omega = d(\cos\theta)\,d\phi$$

$$\Rightarrow P = \int \frac{dP}{d\Omega}d\Omega = \frac{2}{\pi}\frac{c^5}{G}\left(\frac{GM_c\omega_{gw}}{2c^3}\right)^{10/3}\int d\Omega\left[\left(\frac{1+\cos^2\theta}{2}\right)^2 + \cos^2\theta\right]$$

$$P = \frac{32}{5}\frac{c^5}{G}\left(\frac{GM_c\omega_{gw}}{2c^3}\right)^{10/3} \text{ total power radiated.}$$

4.1.1 - Circular Orbits - The Chirp Amplitude

- Emission of GWs costs energy: source = sum of kinetic + potential
  $E_{orbit} = E_{kin} + E_{pot} = -\frac{Gm_1m_2}{2R}$
  $\Rightarrow$ if E_{orbit} becomes more and more negative, R must decrease.
- We have from above R decreases $\Rightarrow \omega_s$ increases $\Rightarrow$ more energy radiated $\Rightarrow R$ decreases $\therefore$ runaway process - leads to coalescence of the system
- Radial velocity: $\dot{R} = -\frac{2}{3}(\omega_s R)\frac{\dot{\omega}_s}{\omega_s^2}$; if in regime of $\dot{\omega}_s << \omega_s^2$, then $|\dot{R}| << \omega_s R$ = tangential velocity, so use approximation of circular orbit with slowly varying radius
- Writing in terms of chirp mass we get:
  $$E_{orbit} = -\left(\frac{G^2 M_c^5 \omega_{gw}^2}{32}\right)^{1/3}$$
- We can equate $P = -\frac{dE_{orbit}}{dt}$ to get, setting $f_{gw} = 2\pi\omega_{gw}$
  $$\dot{f}_{gw} = \frac{96}{5}\pi^{8/3}\left(\frac{GM_c}{c^3}\right)^{5/3} f_{gw}^{11/3}$$
- Integrating this $\Rightarrow f_{gw}$ diverges at time, say t_{coal}. If $\tau = t_{coal} - t$
  $$f_{gw} = \frac{1}{\pi}\left(\frac{5}{256}\frac{1}{\tau}\right)^{3/8}\left(\frac{GM_c}{c^3}\right)^{-5/8}$$
- We can put some reference values in to get:
  $$\tau \simeq 2.18\left(\frac{1.21 M_\odot}{M_c}\right)^{5/3}\left(\frac{100\,\text{Hz}}{f_{gw}}\right)^{8/3} s$$

  So if $M_c = 1.21 M_\odot$ and $f_{gw} = 10\,\text{Hz}$, radiation emitted $\approx$ 17 min to coalescence. If we have two bodies with $m_i = 1.4 M_\odot$ and $f_{gw} = 1\,\text{kHz}$, separation is 33 km; only possible for v. compact bodies with $r \approx 10$ km, so point-like assumption inaccurate.

- Another useful quantity is $\mathcal{N}_{cyc}$, number of cycles spent in detectors frequency bandwidth, $f \in [f_{min}, f_{max}]$. Period $T(t)$ of GW is slowly varying function of time, we get
  $\frac{dN_{cyc}}{dt} = \frac{1}{T} = f_{gw}$

$$\Rightarrow \mathcal{N}_{cyc} = \int_{t_{min}}^{t_{max}} f_{gw}\, dt = \int_{f_{min}}^{f_{max}} \frac{f_{gw}}{\dot{f}_{gw}}\, df_{gw}$$

we have an expression for $\dot{f}_{gw}$ in terms of f_{gw} so

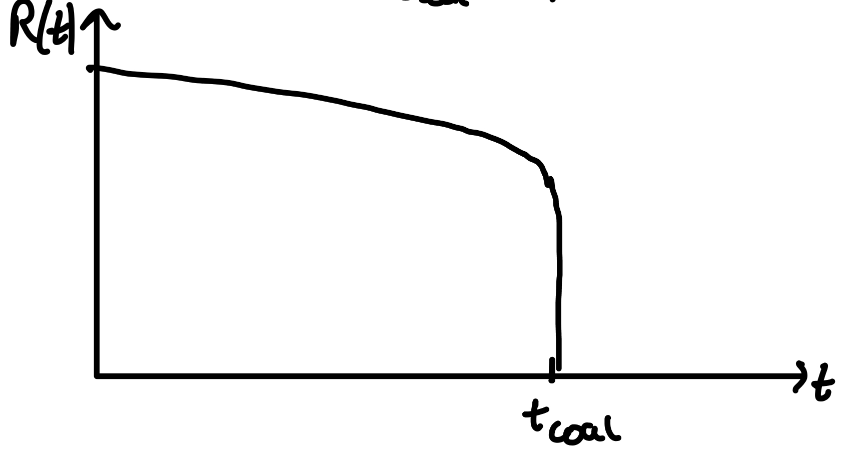
$$\mathcal{N}_{cyc} = \frac{1}{32\pi^{8/3}}\left(\frac{GM_c}{c^3}\right)^{-5/3}\left(f_{min}^{-5/3} - f_{max}^{-5/3}\right)$$

$$\simeq 1.6\times10^4\left(\frac{10\,\text{Hz}}{f_{min}}\right)^{5/3}\left(\frac{1.2M_\odot}{M_c}\right)^{5/3}$$

$\Rightarrow$ ground based interferometers can follow evolution for $\sim10^8$ cycles

- As frequency increases, orbital radius shrinks; $\frac{\dot{R}}{R} = -\frac{1}{4\tau}$ $\left(\dot{R} = \frac{dR}{dt}\right)$

$$\Rightarrow \quad R(t) = R_0\left(\frac{t_{coal}-t}{t_{coal}-t_0}\right)^{1/4} = R_0\left(\frac{\tau}{\tau_0}\right)^{1/4}$$



- Combining this eqn with $f_{gw}(\tau)$ and $\omega_s(R)$ we get

$$\tau_0 = \frac{5}{256}\frac{c^5R_0^4}{G^3M^2\mu}$$

- Using Kepler's laws for initial orbital period and some numerical values

$$\tau_0 \simeq 9.829\times10^6\left(\frac{T_0}{1\text{hr}}\right)^{8/3}\left(\frac{M_\odot}{M}\right)^{2/3}\left(\frac{M_\odot}{\mu}\right)\text{yr}$$

$\Rightarrow$ under these assumptions, only binaries which at formation had orbital period $\lesssim 1$ day can have coalesced by emission of GWs.

- Particle that moves on a quasi-circular orbit has coords

$$x(t) = R(t)\cos\left(\tfrac{1}{2}\Phi(t)\right) \qquad y(t) = R(t)\sin\left(\tfrac{1}{2}\Phi(t)\right)$$

$$\Phi(t) = \int_{t_0}^{t} dt'\,\omega_{gw}(t')$$

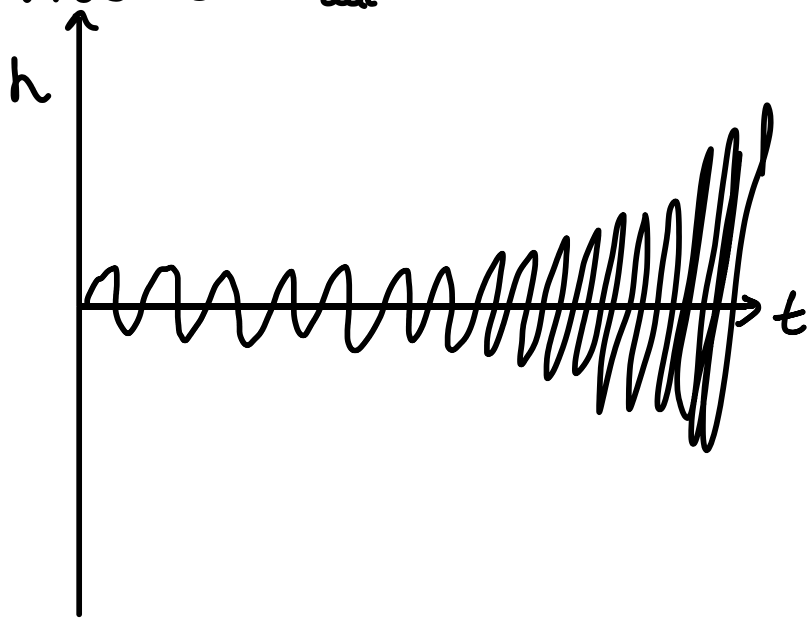
- For the quadrupole calculations, there are 3 differences in waveforms:

* $\omega_{gw} t \to \Phi(t)$ in the arg. of trig functions
* $\omega_{gw} \to \omega_{gw}(t)$ in factors multiplying trig functions
* contributions due to derivatives of R, ω_{gw}
  ↳ however $\dot{R}$, $\dot{\omega}_{gw}$ negligible as long as $\dot{\omega}_s \ll \omega_s^2$, which is a very good approx in the inspiral phase

$$\therefore \; h_+(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{gw}(t_{ret})}{c}\right)^{2/3}\left(\frac{1+\cos^2\iota}{2}\right)\cos\left[\Phi(t_{ret})\right]$$
$$h_\times(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{gw}(t_{ret})}{c}\right)^{2/3}\cos\iota\,\sin\left[\Phi(t_{ret})\right]$$

* We can show:
$$\Phi(\tau) = -2\left(\frac{5GM_c}{c^3}\right)^{-5/8}\tau^{5/8} + \Phi_0$$

so we can calculate:
$$h_+(\tau) = \frac{1}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{5}{c\tau}\right)^{1/4}\left(\frac{1+\cos^2\iota}{2}\right)\cos\left[\Phi(\tau)\right]$$
$$h_\times(\tau) = \frac{1}{r}\left(GM_c/c^2\right)^{5/3}\left(\frac{5}{c\tau}\right)^{1/4}\cos\iota\,\sin\left[\Phi(\tau)\right]$$
where $\tau = t_{coal} - t$



The FT of the signals are:

$$\tilde{h}_+(f) = Ae^{i\Psi_+(f)}\frac{c}{r}\left(\frac{GM_c}{c^3}\right)^{5/6}\frac{1}{f^{7/6}}\left(\frac{1+\cos^2\iota}{2}\right)$$
$$\tilde{h}_\times(f) = Ae^{i\Psi_\times(f)}\frac{c}{r}\left(\frac{GM_c}{c^3}\right)^{5/6}\frac{1}{f^{7/6}}\cos\iota$$
where $A = \frac{1}{\pi^{2/3}}\left(\frac{5}{24}\right)^{1/2}$ and the phases are:
$$\Psi_+(f) = 2\pi f(t_c + r/c) - \Phi_0 - \pi/4 + \tfrac{3}{4}\left(\frac{GM_c}{c^3}8\pi f\right)^{-5/3}$$
and $\Psi_\times(f) = \Psi_+ + \pi/2$

[need to go beyond Newtonian approx to distinguish noise]

* All calculations done in flat space: for BHs, corrections should be made. In Schwarzschild geometry, there is an inner most

Stable circular orbit, $r_{ISCO} = 6Gm/c^2$ $\quad m = m_1 + m_2$

↳ quasi-circular orbits valid only $r \gtrsim r_{ISCO}$, $\exists$ a max freq f_{max} where inspiral phase ends

- Kepler's Law $\Rightarrow (f_s)_{ISCO} = \frac{1}{6\sqrt{6}(2\pi)} \frac{c^3}{Gm}$
$\simeq 2.2\,\text{kHz}\left(\frac{M_\odot}{m}\right)$

For a typical NS-NS $(f_s)_{ISCO} \sim 800\,\text{Hz}$, BH-BH binary system typically have $f_s \sim$ mHz region

- Energy spectrum can be found as: $\frac{dE}{df} = \frac{\pi^{2/3}}{3G}(GM_c)^{5/3} f^{-1/3}$

So integrating up to the max frequency we are still in the inspiral phase we can estimate total energy radiated as

$$\Delta E_{rad} \sim \frac{\pi^{2/3}}{2G}(GM_c)^{5/3} f_{max}^{2/3}$$

$$\Delta E_{rad} \sim 4.2 \times 10^{-2} M_\odot c^2 \left(\frac{M_c}{1.21 M_\odot}\right)^{5/3} \left(\frac{f_{max}}{1\,\text{kHz}}\right)^{2/3}$$

and if we set $f_{max} = 2(f_s)_{ISCO}$ we get

$$\Delta E_{rad} \sim 8 \times 10^{-2} \mu c^2$$

- The binding energy in Schwarzschild metric of the ISCO

$$E_{binding} = \left(1 - \sqrt{8/9}\right)\mu c^2 \simeq 5.7 \times 10^{-2} \mu c^2$$

which is the total energy radiated in GWs when binary system is slowly inspiralling from an orbit with large separation down to ISCO.