Gravitational Waves Derivation

Thursday 17 October 2024 22:13

Einstein equation: $\underline{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}}$

Consider $g_{\mu\nu} = h_{\mu\nu} + \eta_{\mu\nu}$ $|h_{\mu\nu}| \ll 1$ small perturbation

Raise and lower indices using $\eta_{\mu\nu}$, work to first order in $h_{\mu\nu}$ to get the following tensors

- $\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} \eta^{\mu\gamma} (h_{\alpha\gamma,\beta} + h_{\beta\gamma,\alpha} - h_{\alpha\beta,\gamma})$
- $R_{\mu\nu\alpha\beta} = \frac{1}{2} (h_{\mu\beta,\nu\alpha} + h_{\nu\alpha,\mu\beta} - h_{\mu\alpha,\beta\nu} - h_{\beta\nu,\mu\alpha})$
- $R_{\nu\beta} = \eta^{\mu\alpha} R_{\mu\nu\alpha\beta} = \frac{1}{2} (-\partial_\mu \partial^\mu h_{\nu\beta} + \partial_\mu \partial_\beta h_\nu{}^\mu + \partial_\mu \partial_\nu h_\beta{}^\mu - \partial_\beta \partial_\nu h_\mu{}^\mu)$
- $R = \eta^{\nu\beta} R_{\nu\beta} = -\partial_\mu \partial^\mu h + \partial_\mu \partial_\nu h^{\mu\nu}$ $\quad h = h_\alpha{}^\alpha$

$$\Rightarrow G_{\mu\nu} = \frac{1}{2} \{ -\partial_\alpha \partial^\alpha h_{\mu\nu} - \partial_\mu \partial_\nu h + \partial_\alpha \partial_\mu h_\nu{}^\alpha + \partial_\alpha \partial_\nu h_\mu{}^\alpha - \eta_{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta} + \eta_{\mu\nu} \partial_\alpha \partial^\alpha h \}$$

all to first order in $h_{\mu\nu}$

Now define $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu} \Rightarrow h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2} \bar{h} \eta_{\mu\nu}$ as $\bar{h} = \bar{h}_\mu{}^\mu = -h$. We can change coordinates as follows:

$x^\alpha \to \tilde{x}^\alpha = x^\alpha - \xi^\alpha(x)$, which induces the change:

$h_{\mu\nu} \to \tilde{h}_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ and therefore

$\bar{h}_{\mu\nu} \to \tilde{\bar{h}}_{\mu\nu} = \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial_\alpha \xi^\alpha$

In terms of $\bar{h}_{\mu\nu}$, the Einstein tensor becomes:

$$G_{\mu\nu} = \frac{1}{2} \{ -\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} \underline{- \eta_{\mu\nu} \partial_\alpha \partial_\beta \bar{h}^{\alpha\beta} + \partial_\mu \partial_\alpha \bar{h}_\nu{}^\alpha + \partial_\nu \partial_\alpha \bar{h}_\mu{}^\alpha} \}$$

The yellow terms all contain a term of the form $\partial_\alpha \bar{h}_\mu{}^\alpha$, so change coordinates as stated above to get

$\partial_\alpha \bar{h}_\mu{}^\alpha \to \partial_\alpha \bar{h}_\mu{}^\alpha + \partial_\alpha \partial^\alpha \xi_\alpha$

Choose ξ_α to set this equal to zero - solving the wave equation w/ source with given source, to get the Lorentz/DeDonder gauge: $\partial_\alpha \bar{h}_\mu{}^\alpha = 0$

This gives the linearised Einstein eqn: $\boxed{\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}}$

In a vacuum, $T_{\mu\nu} = 0$, giving:

$\boxed{\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = \Box \bar{h}_{\mu\nu} = 0}$, so we get plane wave solutions

$\underline{\bar{h}_{\mu\nu} = \mathrm{Re}(H_{\alpha\beta} e^{ik_\rho x^\rho})}$: k^μ is a real wavevector, with

$k_\mu k^\mu = 0 \Leftrightarrow$ wave eqn

$k^\alpha H_{\mu\alpha} = 0 \Leftrightarrow$ DeDonder gauge

Choose wave propogating in x^3 direction, so set

$k^\mu = k(1,0,0,1)$ $\quad k_\mu = k(-1,0,0,1)$ $\quad e^{ik_\rho x^\rho} = e^{-ik(t - x^3)}$

Gauge condition $\Rightarrow H_{\alpha 0} + H_{\alpha 3} = 0$. Gauge freedom:

$\bar{h}_{\alpha\beta} \to \bar{h}_{\alpha\beta} + \partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha - \eta_{\alpha\beta} \partial_\gamma \xi^\gamma$ which means

$\partial^\mu \bar{h}_{\alpha\mu} \to \partial^\mu \bar{h}_{\alpha\mu} + \partial_\alpha (\cancel{\partial^\mu \xi_\mu}) + \partial^\mu \partial_\mu \xi_\alpha - \partial_\alpha (\cancel{\partial_\gamma \xi^\gamma})$

gauge condition respected if $\Box \xi_\alpha = 0$ so get (real part supressed)

$\xi_\alpha = -i X_\alpha e^{ik_\rho x^\rho}$ const X_α so

$\partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha - \eta_{\alpha\beta} \partial_\gamma \xi^\gamma = (k_\alpha X_\beta + k_\beta X_\alpha - \eta_{\alpha\beta} k_\gamma X^\gamma) e^{ik_\rho x^\rho}$

which has the effect on soln. $H_{\alpha\beta} \to H_{\alpha\beta} + k_\alpha X_\beta + k_\beta X_\alpha - \eta_{\alpha\beta} k_\gamma X^\gamma$

Return to example above with $k = 1$, chose $X_\alpha = (A, 0, 0, B)$ $k_\gamma X^\gamma = A + B$

H_{01}, H_{02} unaffected but

$H_{00} \to H_{00} + k_0 X_0 + k_0 X_0 - \eta_{00}(A+B) = H_{00} - A + B$

$H_{03} \to H_{03} + k_0 X_3 + k_3 X_0 - \eta_{03}(A+B) = H_{03} - B + A$

$H_{00} + H_{03} \to H_{00} + H_{03}$ as required. Now repeat similarly for

$X_\alpha = (0, C, 0, 0)$ and $X'_\alpha = (0, 0, C', 0)$ to get
$H_{01} \to H_{01} - C \quad H_{02} \to H_{02} - C'$, and we can make
$H_{0\alpha} = H_{\alpha 0} = 0$ for all α. Then using $X_\alpha = (A, 0, 0, B)$
we can make $H_{ij} \to H_{ij} - \delta_{ij}(A+B)$ for $i,j = 1, 2$ so
choose $A+B$ s.t. $H_{11} = -H_{22}$. This gives:

$$H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

the <u>transverse traceless gauge</u> (TT)
with two independent polarisation states
in this gauge $h_{\mu\nu} = \bar{h}_{\mu\nu}$