Imprints of Inflation on the Cosmic Microwave Background

Abstract

Primordial fluctuations during inflation are realised as temperature and polarisation anisotropies in the Cosmic Microwave Background. The statistics of these fields places constraints on our models of primordial physics. This talk will explore the idea of the bispectrum as well as briefly looking forward.

1 Inflation

i The Horizon Problem

Why do we need inflation? It was originally proposed as a solution to the following problem, known as the Horizon Problem.

Horizon Problem

We observe a causally disconnected cosmic microwave background, and yet regions separated by causal boundaries have the same temperature to 1 part in 10^5.

Inflation looks to solve this by introducing a period of rapid expansion in the very early universe which drives initially causally connected, equilibrated regions out of contact. This then manifests itself as the picture we see today. This has been incredibly successful and provides the best effort at a solution that we currently have.

ii Models of Inflation

however, there is more than one way to do this, usually encoded in the dynamics of scalar fields governed by particular a particular Lagrangian. For example, we could consider simple scalar field inflation.

$$\mathcal{L} = \sqrt{-g}\left(\partial_\mu \phi \partial^\mu \phi - V(\phi)\right) \tag{1}$$

or more esoteric ideas such as DBI inflation.

$$\mathcal{L} = \frac{1}{g_s}\sqrt{-g}\left(\frac{\phi^4}{\lambda}\sqrt{1+\lambda\frac{\partial_\mu \phi \partial^\mu \phi}{\phi^4}} + m^2\phi^2\right) \tag{2}$$

Ultimately these produce different experimental signatures, for example, the B-modes currently being searched for POLARBEAR are governed by the form of the potential. We will investigate the statistics of the perturbations sourced by these primordial fields, in particular the divergences from a Gaussian distribution. As we've alluded to here, it's that tiny perturbations we brushed over in discussing the horizon problem In evolution evolution of the universe. They are the seeds of structure formation, and poetically, quantum fluctuations source galaxy clustering and all the large scale structure we see today.

2 The Cosmic Microwave Background

I don't think it is unreasonable to suggest that the CMB is the most precise and flexible data source Cosmologists currently have access to. Projects such as Euclid will look to extend the work done on the CMB to large scale structure surveys, however the most accurate parameter estimations reside firmly with the Planck team.

i What do we actually measure?

So what do we actually measure? With a little bit of thought we see we actually observe the CMB as a 2D sphere around us, which can describe by some field $M^X(\hat{\boldsymbol{n}})$. The X labels either the temperature or polarisation character of the data and $\hat{\boldsymbol{n}}$ is a unit vector. Since the data is on a sphere, we can decompose it into spherical harmonics;

$$M^X(\hat{\boldsymbol{n}}) = \sum_{lm} a^X_{lm} Y_{lm}(\hat{\boldsymbol{n}}) \tag{3}$$

then the a^X_{lm} are the CMB multipoles which we can extract from the data.

ii Transfer Functions and Primordial Fluctuations

Now we need to link the two regimes. How do we match up inflationary perturbations to the corresponding perturbations in the CMB? We'll denote the primordial fluctuations as $\zeta(\boldsymbol{k})$ from now on, which are model dependent importantly. Given these, we can use well-established so called transfer functions, $\Delta_l^X(\boldsymbol{k})$ to map these to the CMB field, $f_l^X(\boldsymbol{k})$ via;

$$f_l^X(\boldsymbol{k}) = \Delta_l^X(\boldsymbol{k})\zeta(\boldsymbol{k}) \tag{4}$$

iii The Bispectrum

We've now got enough formalism to introduce the key ideas of the bispectrum and the subsequent non-Gaussianity. Given out primordial fluctuations $\zeta(\boldsymbol{k})$, we define the bispectrum $B(k_1, k_2, k_3)$ as follows;

$$\langle \zeta(\boldsymbol{k}_1)\zeta(\boldsymbol{k}_2)\zeta(\boldsymbol{k}_3)\rangle = (2\pi)^3\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3)B(k_1, k_2, k_3) \tag{5}$$

where the delta function imposes momentum conservation. At the simplest level, if our initial fluctuations are Gaussian, then this 3-point function will vanish. Hence the bispectrum, which is highly model dependent fortunately, is a measure of the non-Gaussianity of the field. This is in contrast to a Gaussian field which is entirely determined by the mean and variance/power spectrum. Now, we need to match this up with the a_{lm}^X that we actually measure. A lengthy calculation tells us that,

$$\langle a_{l_1m_1}^{X_1} a_{l_2m_2}^{X_2} a_{l_3m_3}^{X_3}\rangle = \int \mathrm{d}\hat{\boldsymbol{n}}\, Y_{l_1m_1}(\hat{\boldsymbol{n}})Y_{l_2m_2}(\hat{\boldsymbol{n}})Y_{l_3m_3}(\hat{\boldsymbol{n}}) \times \left(\frac{2}{\pi}\right)^3 \int_{\mathcal{V}_k} \mathrm{d}\mathcal{V}\, (k_1k_2k_3)^2 B(k_1, k_2, k_3)\Delta_{l_1l_2l_3}^{X_1X_2X_3} \tag{6}$$

where $\Delta_{l_1l_2l_3}^{X_1X_2X_3} = \Delta_{l_1l_2l_3}^{X_1X_2X_3}(k_1, k_2, k_3)$ is a function of the transfer functions mentioned above. The point is this then; $B(k_1, k_2, k_3)$ is tightly constrained by the inflation model. This calculation shows us that we can directly predict the expected signal in the CMB given the bispectrum. This can then be compared to real measurements $\hat{a}_{lm}^X$ to test a given form of the bispectrum.

3 Non-Gaussianity

i Sources of Non-Gaussianity

So, having described what it is, and how we can measure it, where can non-Gaussianity come from? There are a number of sources that are important to understand if we want to extract the primordial contributions.

1. *Primordial sources* - non-Gaussianity produced in the early universe by inflation.
2. *Second-order sources* - generated by non-linearities in the transfer functions.
3. *Secondary non-Gaussianity* - generated by late time effects after recombination, for example lensing.
4. *Foreground non-Gaussianity* - galactic and extra-galactic sources.

ii Shape Functions and a simple example: Local Non-Gaussianity

With the fact that we will eventually have to decouple the different sources of non-Gaussianity, we will focus on a simple example of a model of primordial non-Gaussianity. First we make a definition. It is usual to write the bispectrum as;

$$B(k_1, k_2, k_3) = \frac{\mathcal{S}(k_1, k_2, k_3)}{(k_1 k_2 k_3)^2} \Delta_\zeta^2(k_*) \tag{7}$$

where $\Delta_\zeta^2(k_*) = k_*^3 \mathcal{P}_\zeta(k_*)$ is the curvature power spectrum evaluated at some intermediate scale (note inflation predicts an almost scale invariant power spectrum, so this makes sense). Now $\mathcal{S}(k_1, k_2, k_3)$ is known as the shape function. It is dimensionless so it only depends on (w.l.o.g) the ratios $x_2 := \frac{k_2}{k_1}$ and $x_3 := \frac{k_3}{k_1}$. Then the shape of the bispectrum is the dependence of $\mathcal{S}$ on these parameters subject to momentum conservation, i.e. $K := \frac{1}{3}(k_1 + k_2 + k_3)$ fixed. We can visualise this in Figure 3.1[1]; where for example, the squeezed limit represents the

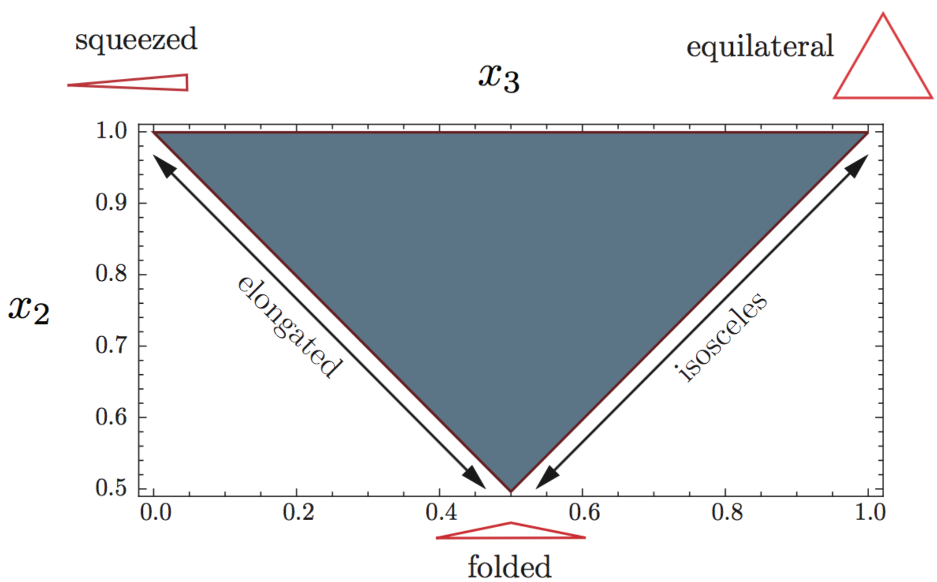

Figure 3.1
Momentum configurations of the bispectrum

case where $k_1 \ll k_2 \sim k_3$. Then finally, we can define the parameter $f_{\rm NL}$ as a measure of the amplitude of the non-Gaussianity;

$$f_{\rm NL}(K) = \frac{5}{18} \mathcal{S}(K, K, K) \tag{8}$$

for a scale invariant bispectra we can extract this and write,

$$B(k_1, k_2, k_3) = \frac{18}{5} f_{\rm NL} \frac{\mathcal{S}(k_1, k_2, k_3)}{(k_1 k_2 k_3)^2} \Delta_\zeta^2 \tag{9}$$

[1]Credit to Daniel Baumann's notes on Non-Gaussianity for this image

We're now in a position to look at an example: local non-Gaussianity. Suppose we have a Gaussian perturbation $\zeta_g(\boldsymbol{x})$, then we can model a non-Gaussian fluctuation by:

$$\zeta(\boldsymbol{x}) = \zeta_g(\boldsymbol{x}) + \frac{3}{5} f_{\mathrm{NL}} \zeta_g(\boldsymbol{x})^2 \tag{10}$$

Moving to Fourier space,

$$\zeta(\boldsymbol{k}) = \zeta_g(\boldsymbol{k}) + \zeta_{\mathrm{NL}}(\boldsymbol{k}) = \zeta_g(\boldsymbol{k}) + \frac{3}{5} f_{\mathrm{NL}} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \, \zeta_g(\boldsymbol{k}+\boldsymbol{p})\zeta_g(\boldsymbol{p}) \tag{11}$$

We can then use $\langle \zeta_g(\boldsymbol{k}_1)\zeta_g(\boldsymbol{k}_2)\rangle = (2\pi)^3 \mathcal{P}(k_1)\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2)$, to see that there will be contributions to $\langle \zeta(\boldsymbol{k}_1)\zeta(\boldsymbol{k}_2)\zeta(\boldsymbol{k}_3)\rangle$ of the form (see Appendix);

$$\langle \zeta_g(\boldsymbol{k}_1)\zeta_g(\boldsymbol{k}_2)\zeta_{\mathrm{NL}}(\boldsymbol{k}_3)\rangle = \frac{6}{5}(2\pi)^3 \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3) f_{\mathrm{NL}} \mathcal{P}(k_1)\mathcal{P}(k_2) \tag{12}$$

along with other cyclic permutations. So the full, bispectrum takes the formn;

$$B(k_1, k_2, k_3) = \frac{6}{5} f_{\mathrm{NL}} \left(\mathcal{P}(k_1)\mathcal{P}(k_2) + \mathcal{P}(k_2)\mathcal{P}(k_3) + \mathcal{P}(k_3)\mathcal{P}(k_1) \right)$$

if we then assume Δ_ζ is scale invariant i.e. $\mathcal{P}(k) = \Delta_\zeta k^{-3}$, then,

$$B(k_1, k_2, k_3) = \frac{6}{5} f_{\mathrm{NL}} \frac{\Delta^2}{(k_1 k_2 k_3)^2} \left(\frac{k_1^2}{k_2 k_3} + \frac{k_2^2}{k_3 k_1} + \frac{k_3^2}{k_1 k_2} \right) \tag{13}$$

and we can read off the shape function,

$$\mathcal{S}(k_1, k_2, k_3) = \frac{1}{3} \left(\frac{k_1^2}{k_2 k_3} + \frac{k_2^2}{k_3 k_1} + \frac{k_3^2}{k_1 k_2} \right) \tag{14}$$

If (w.l.o.g) we take $k_1 \leq k_2 \leq k_3$, then the amplitude will be maximised for $k_1 \ll k_2 \sim k_3$ i.e. the squeezed limit we discussed above. This example and discussion was just to give a flavour of the sort of calculations that can be done. More importantly we can match this up with out earlier discussion. In this particular case, it can be shown that this squeezed limit *cannot* arise in single field inflation, supporting our claim earlier that we can use this to constrain our models. For completeness there are a number of other models of non-Gaussianity with the main ones being equilateral, $f_{\mathrm{NL}}^{\mathrm{equi}}$, or orthogonal, $f_{\mathrm{NL}}^{\mathrm{ortho}}$, which proceed in a similar way.

4 Conclusions and Future Directions

So what have we done? We've shown that given a model of inflation, i.e. a Lagrangian, $\mathcal{L}$ and gone through the following steps;

1. $\mathcal{L}$ has a characteristic spectrum of perturbations, $\zeta(\boldsymbol{k})$
2. This $\zeta(\boldsymbol{k})$ can be projected forward onto the CMB and be decoupled from the non-primordial sources.
3. The predicted shape functions, the amplitudes etc., can be compared to observation and potentially constrain the proposed model.

Looking forward, non-Gaussianity provides much more information on the very early universe than its Gaussian counterpart. Gaussian perturbations only have one free parameter (the power spectrum/variance) whereas non-Gaussian perturbations can have an unlimited number of free parameters. In reality, well motivated physical models constrain such parameters, and as we have seen, we can encode this in the 3-point function. Recently Planck has made the tightest constraints yet but projects such as POLARBEAR and the Simon's array will be able to search for much more subtle effects as seen in single field inflation for example. To sum up, I think the real excitement comes from the early universe element. The possibility to constrain very high energy theories is a desire shared across a number of fields. This is an attempt to realise that.

A Wick's Theorem and the 3-point function

We fill in the details for (12);

$$
\begin{aligned}
\langle \zeta_g(\boldsymbol{k}_1)\zeta_g(\boldsymbol{k}_2)\zeta_{\mathrm{NL}}(\boldsymbol{k}_3)\rangle &\propto \langle \zeta_g(\boldsymbol{k}_1)\zeta_g(\boldsymbol{k}_2)\int \frac{\mathrm{d}^3q}{(2\pi)^3}\zeta_g(\boldsymbol{q})\zeta_g(\boldsymbol{k}_3+\boldsymbol{q})\rangle \\
&= \int \frac{\mathrm{d}^3q}{(2\pi)^3}\langle \zeta_g(\boldsymbol{k}_1)\zeta_g(\boldsymbol{q})\rangle\langle \zeta_g(\boldsymbol{k}_2)\zeta_g(\boldsymbol{k}_3-\boldsymbol{q})\rangle \\
&= \int \frac{\mathrm{d}^3q}{(2\pi)^3}\mathcal{P}(k_1)\delta(\boldsymbol{k}_1+\boldsymbol{q})\langle \zeta_g(\boldsymbol{k}_2)\zeta_g(\boldsymbol{k}_3-\boldsymbol{q})\rangle \\
&= \mathcal{P}(k_1)\langle \zeta_g(\boldsymbol{k}_2)\zeta_g(\boldsymbol{k}_3+\boldsymbol{k}_1)\rangle \\
&= \mathcal{P}(k_1)\mathcal{P}(k_2)\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)
\end{aligned}
$$

where to go from the first line to the second line, we've used Wick's theorem to wrie the 4-point function as a product of 2-point functions.

References

[1] Daniel Baumann. Primordial non-gaussianity.

[2] Christian T. Byrnes. Lecture notes on non-gaussianity.

[3] J.F. Fergusson. Efficient optimal non-gaussian cmb estimators with polarisation. 2014.