Dark Matter and Dark Energy: Informal Problem Sheet 2

I Question 1

We start from the density profile for an NFW halo, using Equation (3.42) in the notes;

$$\rho(r) = \frac{\rho_s}{\left(\frac{r}{r_s}\right)\left(1+\frac{r}{r_s}\right)^2} \tag{I.1}$$

where r_s is some reference scale, and ρ_s is the density at this scale. We want the circular velocity $v_{\rm circ}$ at $r^\star = 1.5\,\text{kpc}$, which can find from the mass, $M(r^\star)$;

$$v_{\rm circ}(r^\star) = \sqrt{\frac{GM(r^\star)}{r^\star}} \tag{I.2}$$

So all that remains is to calculate the mass contained in a radius $r^\star$ given the NFW density profile, this is just an integration;

$$\begin{aligned}
M(r^\star) &= \int_0^r r^2 \mathrm{d}r \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta \mathrm{d}\theta\, \rho(r) \\
&= 4\pi \int_0^{r^\star} \mathrm{d}r\, \rho(r) \\
&= 4\pi \int_0^{r^\star} \mathrm{d}r\, \frac{\rho_s r^2}{\left(\frac{r}{r_s}\right)\left(1+\frac{r}{r_s}\right)^2} \\
&= 4\pi r_s \rho_s \int_0^{r^\star} \mathrm{d}r\, \frac{r}{\left(1+\frac{r}{r_s}\right)^2} \\
&= 4\pi r_s^3 \rho_s \int_0^{r^\star} \mathrm{d}r\, \frac{r}{(r_s+r)^2} \\
&= 4\pi r_s^3 \rho_s \int_0^{r^\star} \mathrm{d}r\, \left(\frac{r+r_s}{(r_s+r)^2} - \frac{r_s}{(r+r_s)^2}\right) \\
&= 4\pi r_s^3 \rho_s \left[\log(r+r_s) + \frac{r_s}{(r+r_s)}\right]_0^{r^\star} \\
&= 4\pi r_s^3 \rho_s \left(\log\left(\frac{r^\star + r_s}{r_s}\right) + \frac{r_s}{r_s + r^\star} - 1\right)
\end{aligned}$$

Simplifying this last expression we find that;

$$M(r^\star) = 4\pi r_s^3 \rho_s \left(\log\left(\frac{r^\star+r_s}{r_s}\right) - \frac{r^\star}{r^\star + r_s}\right) \tag{I.3}$$

Then we just need to put the numbers in;

$$\begin{aligned}
&G = 4.3\times 10^{-3}\,\text{km}^2\,\text{s}^{-2}\,\text{pc}\,M_\odot^{-1}, \\
&\qquad \rho_s = 0.4\,M_\odot\,\text{pc}^{-3},\; r_s = 1000\,\text{pc},\; r^\star = 1500\,\text{pc}
\end{aligned}$$

to find first the mass;

$$\begin{aligned}
M(r^\star) = 4\pi \cdot 0.4\,M_\odot\,\text{pc}^{-3}(1500\,\text{pc})^3 \\
\cdot \left(\log\left(\frac{2500\,\text{pc}}{1000\,\text{pc}}\right) - \frac{1500\,\text{pc}}{2500\,\text{pc}}\right)
\end{aligned}$$

which gives $M(r^\star) = 1.6\times 10^9\,M_\odot$. Finally we find the velocity;

$$\begin{aligned}
v_{circ}(r^\star) &= \sqrt{\frac{GM(r^\star)}{r^\star}} \\
&= \sqrt{\frac{4.3\times 10^{-3}\,\text{km}^2\,\text{s}^{-2}\,\text{pc}\,M_\odot^{-1}\cdot 1.6\times 10^9\,M_\odot}{1500\,\text{pc}}} \\
&= 68\,\text{km}\,\text{s}^{-1}
\end{aligned}$$

II Question 2

This is basically just an integration question, we are given;

$$\frac{\mathrm{d}n}{\mathrm{d}\omega} = \frac{1}{\pi^2 c^2}\frac{\omega^2}{e^{\frac{\hbar\omega}{kT}} - 1} \tag{II.1}$$

which implies that the total number density at some temperature T is given by;

$$n(T) = \int_0^\infty \mathrm{d}\omega\, \frac{1}{\pi^2 c^3}\frac{\omega^2}{e^{\frac{\hbar\omega}{kT}} - 1} \tag{II.2}$$

If we let;

$$x = \frac{\hbar\omega}{kT} \Rightarrow \mathrm{d}\omega = \frac{kT}{\hbar}\mathrm{d}x$$

So the integral becomes;

$$n(T) = \frac{(kT)^3}{\pi^2 c^3 \hbar^3}\int_0^\infty \mathrm{d}x\, \frac{x^2}{e^x - 1}$$

The integral is a bit involved, it turns out that the following general class of integrals is related to the Riemann zeta function via;

$$\int_0^\infty \mathrm{d}x\, \frac{x^n}{e^x - 1} = n!\zeta(n)$$

So we can read off;

$$n(T) = \frac{(kT)^3}{\pi^2 c^3 \hbar^3} 2!\zeta(3) = \frac{2(kT)^3\zeta(3)}{\pi^2 c^3 \hbar^3} \tag{II.3}$$

It remains to put the numbers in;

$$\begin{aligned}
&k = 1.4\times 10^{-23}\,\text{m}^2\text{kgs}^{-2}\text{K}^{-1},\; T = 2.7\,\text{K},\; \zeta(3)\sim 1.2, \\
&\qquad \hbar = 1.1\times 10^{-34}\,\text{m}^2\text{kgs}^{-1},\; c = 3\times 10^8\,\text{ms}^{-1}
\end{aligned}$$

which gives $n(2.7\,\text{K}) \sim 4\times 10^8\,\text{m}^{-3} \sim 400\,\text{cm}^{-3}$.

III Question 3

We are given the following;

- The dark matter forms as isothermal sphere, so;

$$\rho_{\rm DM}(r) = \rho_s\left(\frac{r_s}{r}\right)^2 \tag{III.1}$$

- The gas has a density profile that goes like r^{-3}, so;

$$\rho_{\mathrm{gas}}(r) = \rho_g \left(\frac{r_g}{r}\right)^3 \tag{III.2}$$

- The dark matter is the dominant gravitational component, so the mass $M(r)$ is dominated by the contribution $M_{\mathrm{DM}}(r)$

To find this mass we need to integrate the density;

$$\begin{aligned} M_{\mathrm{DM}}(r^\star) &= \int_0^{r^\star} r^2 \mathrm{d}r \int \mathrm{d}\Omega\, \rho_{\mathrm{DM}}(r) \\ &= 4\pi\rho_s r_s^2 \int_0^{r^\star} \mathrm{d}r \\ &= 4\pi\rho_s r_s^2 r^\star \end{aligned}$$

We can then apply the hydrostatic equilibrium equation;

$$\frac{\mathrm{d}P_{\mathrm{gas}}}{\mathrm{d}r} = -\frac{GM_{\mathrm{DM}}(r)\rho_{\mathrm{gas}}(r)}{r^2} \propto \frac{1}{r^4} \tag{III.3}$$

Integrating this we see that $P_{\mathrm{gas}}(r) \propto r^{-3}$, finally we need to use the ideal gas equation to deduce;

$$T(r) = \frac{PV}{Nk} \propto \frac{r^3}{r^3} \tag{III.4}$$

In other words, we find that the temperature is constant (isothermal).