Dark Matter and Dark Energy: Sheet 1

1 Question 1

Remember that natural units are defined by taking $c = \hbar = 1$, so that for example $3 \times 10^8\,\mathrm{m} = 1\,\mathrm{s}$ etc. We can do the same with $\hbar = 6.626 \times 10^{?34}\,\mathrm{m^2kgs^{?1}}$ to relate 1 kg to 1 m or 1 s. Finally use the fact that the electron volt is defined the be the potential energy gained by a charge e accelerated across a potential difference of 1 V i.e. $1\,\mathrm{eV} = 1.6 \times 10^{-19}\,\mathrm{J}$. Combining these we find the conversion relations:

$$1\,\mathrm{kg} = 5.6 \times 10^{26}\,\mathrm{GeV}, \quad 1\,\mathrm{m}^{-1} = 1.98 \times 10^{-16}\,\mathrm{GeV} \tag{1.1}$$

Similarly when we want to convert in astronomical units, we use;

$$1\,\mathrm{kg} = 5 \times 10^{-31}\,\mathrm{M}_\odot, \quad 1\,\mathrm{m} = 3.2 \times 10^{-20}\,\mathrm{kpc} \tag{1.2}$$

Then we can find the density of water in both of these unit systems;

$$\rho_w = 1000\,\mathrm{kg\ m^{-3}} = 4.3\times10^{-21}\,\mathrm{GeV}^4 = 1.5\times10^{21}\,\mathrm{M_\odot kpc^{-3}} \tag{1.3}$$

2 Question 2

Firstly we will assume that the universe is spatially flat so $k = 0$. Then, this is just an application of the Friedmann equation and the conservation equation (which is just conservation of the stress energy tensor, $T_{\mu\nu}$);

$$H^2 = \left(\tfrac{\dot a}{a}\right)^2 = \tfrac{8\pi G\rho}{3} \tag{2.1}$$

$$\dot\rho = -3\tfrac{\dot a}{a}(\rho + P) \tag{2.2}$$

The equation of state relates the pressure and the density for each component via $P = w\rho$, plugging this into the conservation equation:

$$\frac{\dot\rho}{\rho} = -3(1+w)\frac{\dot a}{a} \Rightarrow \rho = \rho_0\left(\frac{a_0}{a}\right)^{3(1+w)}$$

i The case $w \neq -1$

Plugging this into the first Friedmann equation we see that;

$$\left(\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2 \propto a^{-3(1+w)}$$

Integrating we find that;

$$\frac{t}{t_0} = \left(\frac{a}{a_0}\right)^{\frac{3}{2}(1+w)} \tag{2.3}$$

Then using the relationship between ρ and the scale factor;

$$\frac{\rho}{\rho_0} = \left(\frac{a}{a_0}\right)^{-3(1+w)} = \left(\frac{t}{t_0}\right)^{-2} \tag{2.4}$$

ii The case $w = -1$

We have to be a little more careful when $w = -1$. Note that the Friedmann equation gives us;

$$\left(\frac{\dot a}{a}\right) = \text{const.} \Rightarrow a = a_0 \exp(\Lambda t)$$

where $\Lambda = (8\pi G/3)^{\frac{1}{2}}$. Then we see that ρ itself is a constant, $\rho = \rho_0$.

3 Question 3

We start from the Friedmann equation with $k \neq 0$;

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{3.1}$$

If $\rho = 0$ i.e. the universe is empty, this reduces to;

$$\left(\frac{\dot a}{a}\right)^2 = -\frac{k}{a^2}$$

Clearly the left hand side is positive, so we deduce that $k < 0$, then the solution is (assuming $\dot a > 0$ i.e. the universe is expanding;

$$\dot a = \sqrt{-k} \Rightarrow a(t) = \sqrt{-k}t \tag{3.2}$$

4 Question 4

Recall that the velocity for orbits around a spherical mass is given by;

$$v^2(r) = \frac{GM(r)}{r} \tag{4.1}$$

In the presence of evidence to support flat rotation curves (there is a nice visualisation here), we can use this to deduce that;

$$v^2(r) = \text{const.} \Rightarrow M(r) \propto r \tag{4.2}$$

In the notes, we choose to parametrise this dependence as follows;

$$M(r) = 4\pi\rho_s r_s^3 r$$

It is stressed that r_s is an arbitrary length scale in the problem, this parametrisation is purely a convenient choice so that we can read off the density;

$$\rho(r) = \rho_s\left(\frac{r_s}{r}\right)^2 \tag{4.3}$$

Now considering the scenario presented: we are given that at $r = 8.5\,\mathrm{kpc}$, the rotational velocity is $v = 220\,\mathrm{km\,s^{-1}}$. Rearranging (4.2), we find that;

$$M = \frac{rv^2}{G} = \frac{8.5\,\mathrm{kpc}\,(220\,\mathrm{km\,s^{-1}})^2}{4.3 \times 10^{-6}\,\mathrm{kpc(km\,s^{-1})^2M_\odot^{-1}}} = 7.9\times10^{10}\,\mathrm{M}_\odot \tag{4.4}$$

Now we can choose r_s to be any reference scale we want, for simplicity we take $r_s = 10\,\text{kpc}$, then find that the reference density is given by;

$$\begin{aligned}\rho_s &= \frac{M}{4\pi r_s^2 r} = \frac{7.9\times 10^{10}\mathrm{M}_\odot}{(4\pi(8.5\,\text{kpc})(10^2\,\text{kpc}^2))}\\ &= 7.4\times 10^5\,\mathrm{M}_\odot\text{kpc}^{-3}\end{aligned}$$

Finally then we can use (4.3) to find the density at 8.5 kpc;

$$\begin{aligned}\rho(8.5\,\text{kpc}) &= 7.4\times 10^5\,\mathrm{M}_\odot\text{kpc}^{-3}\left(\frac{10\,\text{kpc}}{8.5\,\text{kpc}}\right)^2\\ &= 1.0\times 10^6\,\mathrm{M}_\odot\text{kpc}^{-3}\end{aligned}$$

We need one final number to calculate the overdensity; the average density of the universe from matter is $\rho_0 = 2.5\times 10^{-27}\,\text{kg}\,\text{m}^{-3} = 37\,\mathrm{M}_\odot\,\text{kpc}^{-3}$. Dividing the two, we find the overdensity is given by;

$$\frac{\rho(8.5\,\text{kpc})}{\rho_0} = \frac{1.0\times 10^6\,\mathrm{M}_\odot\,\text{kpc}^{-3}}{37\,\mathrm{M}_\odot\,\text{kpc}^{-3}} \simeq 3\times 10^4 \tag{4.5}$$

5 Question 5

We'll start with a quick derivation of the free streaming distance for the dark matter. The generic expression is given by;

$$l_{\text{FS}} = \int_{t_{\text{dec}}}^{t_0} v(t)\frac{a_0}{a(t)}\,\mathrm{d}t \tag{5.1}$$

To approximate this, we consider two simplifications;

1. t_{dec} is very early in the universe, so the lower limit is approximately zero
2. When the dark matter particles are relativistic (i.e. when the temperature is larger than the mass), they travel at a speed c, and travel at speed 0 otherwise.

Then we can replace the expression above with;

$$l_{\text{FS}} \simeq \int_0^{t(T=m)} c\frac{a_0}{a(t)}\,\mathrm{d}t \tag{5.2}$$

Now we need to collect a few results to get to the final form depending on the mass of the dark matter, m.

1. $T \propto a^{-1} \Rightarrow T = T_0 a_0/a$
2. $T_0 = 2.7\,\text{K} \sim 2\times 10^{-4}\,\text{eV}$ is the temperature today
3. In the radiation dominated era,
$$H = \frac{1.66\sqrt{g}T^2}{M_{\text{pl}}}$$
where g denotes the relativistic degrees of freedom.
4. We can relate the scale factor to the redshift z by;
$$\frac{a_0}{a} = (1+z) \Rightarrow T = T_0(1+z)$$
5. Finally use the fact that $H\,\mathrm{d}t = a^{-1}\,\mathrm{d}a$ and;
$$-\frac{a_0}{a^2}\,\mathrm{d}a = \mathrm{d}z$$

Collecting the results we find that the free streaming distance is given by;

$$l_{\text{FS}} \sim \int_{z=\infty}^{z(T=m)} \frac{\mathrm{d}z}{H(t)} = \frac{M_{\text{pl}}}{1.66\sqrt{g}T_0^2}\int_{z=\infty}^{z(T=m)}\frac{\mathrm{d}z}{(1+z)^2}$$

At these high redshifts, we can approximate $1+z\simeq z$, and put a suitable value of g in to find;

$$l_{\text{FS}} \sim 0.4\frac{M_{\text{pl}}}{T_0^2 z(T=m)} \tag{5.3}$$

But $z(T=m) = T/T_0 = m/T_0$ so we find;

$$l_{\text{FS}} \sim 0.4\frac{10^{19}\,\text{GeV}}{(2\times 10^{-4}\,\text{eV})^2}\frac{2\times 10^{-4}\,\text{eV}}{m} \sim 10^{21}\,\text{m}\left(\frac{1\,\text{kev}}{m}\right) \tag{5.4}$$

All we have to do then is put $m = 1\,\text{eV}$ into (5.4) and compare with the value at $m = 1\,\text{kev}$. We find that $l_{\text{FS}} \sim 10^{24}\,\text{m}$. The mass of the smallest halo is then proportional to l_{FS}^3. Thus the halo is 10^9 times larger than at $m = 1\,\text{kev}$ which equates to approximately $10^{14}\,\mathrm{M}_\odot$.

6 Question 6

This question is really just a quick application of (3.56) as given in the notes, but we will present the theory also. We consider the scattering of two dark matter particles in the rest frame of the smaller cluster. Let the velocity of the incoming particle be v_0, and the scattering angles from the horizontal be $\alpha_{1,2}$. Conservation of momentum in the collision gives $\alpha_1+\alpha_2 = \frac{\pi}{2}$ which in turn gives;[1]

$$v_1 = v_0\cos\alpha_1,\ v_2 = v_0\sin\alpha_1 \tag{6.1}$$

Let v_e be the escape velocity of the dark matter particle from the smaller cluster, then the condition that

[1]There is a very simple proof of this; let the initial velocity of the incoming particle be $\vec{u}$, and the final velocities be $\vec{v}_{1,2}$, then in the case the particles have the same mass conservation of energy and momentum give;

$$\vec{u} = \vec{v}_1 + \vec{v}_2 \Rightarrow u^2 = v_1^2 + v_2^2 + 2\vec{v}_1\cdot\vec{v}_2$$
$$u^2 = v_1^2 + v_2^2$$

Combining the two gives $\vec{v}_1\cdot\vec{v}_2 = 0$ i.e. the particles emerge at right angles.

both particles obtain a velocity greater than this can be written;

$$\frac{v_1^2}{v_0^2} = \cos^2\alpha_1 = 1 - \sin^2\alpha_1 > \frac{v_e^2}{v_0^2}$$
$$\frac{v_2^2}{v_0^2} = \sin^2\alpha_1 > \frac{v_e^2}{v_0^2}$$

which we can combine to give;

$$\frac{v_e^2}{v_0^2} < \sin^2\alpha_1 < 1 - \frac{v_e^2}{v_0^2} \tag{6.2}$$

We can boost to the centre of mass frame for the collision by subtracting $v_0/2$ off the horizontal velocities, taking the example of the incoming particle we find;

$$\begin{aligned} v_0\cos^2\alpha_1 - \frac{v_0}{2} &= \frac{v_0}{2}(2\cos^2\alpha_1 - 1) \\ &= \frac{v_0}{2}\cos 2\alpha_1 \\ v_0 \sin\alpha_1\cos\alpha_1 &= \frac{v_0}{2}\sin 2\alpha_1 \end{aligned}$$

So we find that in the centre of mass frame, the scattering angle is $\theta = 2\alpha_1$. Thus we can write our condition as;

$$\frac{v_e^2}{v_0^2} < \sin^2\frac{\theta}{2} < 1 - \frac{v_e^2}{v_0^2} \tag{6.3}$$

These bounds will be saturated by some $\theta_{\min}$ and $\theta_{\max}$. In the absence of any more information about the cross-section, we assume that the scattering is isotropic in this frame. To calculate the probability of ejection, we need to compute the solid angle covered by the range $[\theta_{\min}, \theta_{\max}]$ as a proportion of the whole sphere. Define the portion of S^2 covered by $[\theta_{\min}, \theta_{\max}]$ to be $S^2[\theta_{\min}, \theta_{\max}]$, then the probability of ejection is;

$$\chi = \frac{\int_{S^2[\theta_{\min},\theta_{\max}]} \mathrm{d}\Omega}{\int_{S^2}\mathrm{d}\Omega} = \frac{\int_{\theta_{\min}}^{\theta_{\max}}\int_0^{2\pi}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi}{4\pi} \tag{6.4}$$

which reproduces equation (3.54) in the notes. We can then do the integral;

$$\begin{aligned} \int_{\theta_{\min}}^{\theta_{\max}} 2\pi\sin\theta\,\mathrm{d}\theta &= 2\pi[-\cos\theta]_{\theta_{\min}}^{\theta_{\max}} \\ &= 2\pi[2\sin^2\tfrac{\theta}{2} - 1]_{\theta_{\min}}^{\theta_{\max}} \\ &= 2\pi\left(2 - 2\frac{v_e^2}{v_0^2} - 1 - 2\frac{v_e^2}{v_0^2} + 1\right) \\ &= 4\pi\left(1 - 2\frac{v_e^2}{v_0^2}\right) \end{aligned}$$

So we find that for a single collision, the probability that both particles are ejected from the small cluster in the collision is;

$$\chi = 1 - 2\frac{v_e^2}{v_0^2} \tag{6.5}$$

Now we consider the optical depth, $\tau = \sigma\Sigma/m$. $\exp(-\tau)$ gives the probability that a given dark matter particle in the smaller cluster collides with one from the bigger cluster. Note here that σ is the self-interaction cross-section, $\Sigma \sim 0.4\,\mathrm{g\,cm^{-2}}$ is the surface density and m is the mass of the dark matter particle. We impose the following requirement: no more than 30% of the mass of the small cluster should be lost. We note that this is equivalent to the following statement; for each dark matter particle, there is less than 30% chance it is ejected during its transit of the large cluster. This translates into the following;

$$\begin{aligned} &\mathbb{P}(\text{collision})\mathbb{P}(\text{collision leads to ejection}) < 0.3 \\ &\qquad\Rightarrow (1 - \exp(-\tau))\,\chi < 0.3 \end{aligned}$$

For small τ, we can approximate $1 - \exp(-\tau) \sim \tau$ to find the condition;

$$\chi\tau < 0.3 \Rightarrow \frac{\sigma}{m} < \frac{0.3}{\Sigma\chi} \tag{6.6}$$

Now it is just a function of plugging in the different values for the escape velocity $v_e = 1900\,\mathrm{km\,s^{-1}}$ and $v_e = 2700\,\mathrm{km\,s^{-1}}$ with $v_0 = 4800\,\mathrm{km\,s^{-1}}$. We find the two values of χ;

$$\chi(v_e = 2700\,\mathrm{km\,s^{-1}}) = 0.37, \chi(v_e = 1900\,\mathrm{km\,s^{-1}}) = 0.69$$

Then, the bound of σ/m is proportional to χ^{-1}, so the limit is larger by a factor of $\frac{0.69}{0.37} \sim 2$, i.e. the bound is weaker by about a factor of 2.