Dark Matter and Dark Energy

Malcolm Fairbairn

December 7, 2017

Contents

These notes are under constant revision. I give chocolate rewards to those who find errors.

Chapter 1

Units and Scales in the Universe

I want to start by discussing Units and Scales. These are very important, by using the right units in the right context, we can make our lives very much easier. We should all be able to change from one unit system to another one without problem. It will help you. Units are your friends, they are not to be feared.

1.1 S.I. Units

System Internationale Units as defined by the Bureau International des Poids et Mesures in Paris. Uses water as the relationship between length and weight. Not completely self consistent since a cubic metre of water weights 1000 kg not 1 kg. Everything can be reduced to kilograms, metres and seconds. e.g.

Quantity	Equation	Unit	in kg,m,s
Density	ρ =M/V	-	1 kg m^{-3}
Force	F=ma	Newton	1 kg m s^{-2}
Pressure	P=F/A	Pascal	1 kg m^{-1}s^{-2}
Energy	E= (1/2)mv^2	Joule	1 kg m^{2}s^{-2}
Frequency	f=E/h=$\omega/\hbar$	Hertz	1 s^{-1}

Table 1.1: SI Units

where Planck's constant $h = 6.626 \times 10^{-34}\text{m}^2\text{kg s}^{-1} = 6.626 \times 10^{-34}\text{Js}$.
So in these Units

Quantity	value	
Size of Hydrogen atom	5.2×10^{-11} m	dominated by electron cloud
Mass of Hydrogen atom	1.66×10^{-27} kg	dominated by proton mass
Height of human	1.7 m	roughly - I am not heightist.
Mass of Human	70 kg	roughly - I am not weightist.
Solar Mass	1.989×10^{30} kg	
Density at centre of neutron star	10^{18} kg m^{-3}	
Pressure at centre of neutron star	10^{32} Pa	
Average density in solar neighbourhood	9.5×10^{-21}kg m^{-3}	
Average density in Universe (matter only)	2.5×10^{-27}kg m^{-3}	

Table 1.2: Some typical Scales in S.I. Units

Newton's Gravitational Constant G= $6.67 \times 10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2}$.

1.2 cgs Units

Closely related to S.I. Units, although a cubic centimetre of water does actually weigh a gram. Used in USA and often used rather than S.I. units in Astrophysics.

Quantity	Equation	Unit	
Density	ρ =M/V	-	1 g cm^{-3}
Force	F=ma	dyne	1 g cm s^{-2}
Pressure	P=F/A	barye	1 g $cm^{-1}s^{-2}$
Energy	E= (1/2)mv^2	erg	1 g cm^2s^{-2}
Frequency	f=E/h=$\omega/\hbar$	Hertz	1 s^{-1}

Table 1.3: cgs Units

1 erg = 1 g cm^2s^{-2} = 1 kg ($\times10^{-3}$) m^2 ($\times10^{-4}$)s^{-2}=10^{-7}J
In cgs units, Planck's constant is h=6.626 $\times$ $10^{-27}cm^2g\ s^{-1}$=6.626$\times10^{-27}$erg s.
Newton's Gravitational Constant G= 6.67 $\times$ $10^{-8}cm^3g^{-1}s^{-2}$.

1.3 Taking speed of light $c = 1$

Minkowski metric in flat space time with Cartesian coordinates

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \tag{1.1}$$

In special and general relativity, we are tought to think of time as a coordinate much like space. Clearly there is a difference because it has a minus sign in front of it. Nevertheless, we can rotate from the x direction to the y direction using a rotation matrix, and we can rotate from the x direction to the t direction using a rotation matrix where the square root of minus one i is placed in front of the rotation angle. This is a Lorentz boost. So we can use the same units for space and time. Then velocities

$$v = \frac{\Delta space}{\Delta time} \tag{1.2}$$

are clearly dimensionless. Since we know the largest velocity is the speed of light c, we define the dimensions such that if you are moving through space at the same rate you are moving through time, then your speed $v = c = 1$. Both space and time have dimensions, but you can choose if that is metres or seconds or something else.

Since $E = mc^2$ if we set $c = 1$ then mass and energy have the same units, i.e. $[E] = [m]$ although the kinetic energy of a particle will be much less than its mass energy, unless it is relativistic.

Also, since for a photon $E = pc$ we know that if $c = 1$, momentum must have the same units as energy.

1.4 Taking $\hbar = 1$

The Quantum of action and of angular momentum is given by Planck's constant $\hbar$ which also tells us when quantum mechanical effects become important. For example, the angular momentum of a pencil which is spinning 3 times a second is about 3×10^{-7} J s which is a lot larger than $\hbar = 1.05 \times 10^{-34}$ J s. So we can see that essentially, $\hbar$ can also be seen as a conversion factor which reflects the fact that our normal units are very far from the quantum world.

We know from the Uncertainty Principle that

$$\Delta x \Delta p \geq \hbar/2 \tag{1.3}$$

and that

$$\Delta E \Delta t \geq \hbar/2 \tag{1.4}$$

So again, we can take units such that $\hbar = 1$ but then we have to look at what that means for units. Distance x must have opposite units to momentum p and energy E must have opposite units to t.

prefix		value	example	
-	-	eV	1 eV	chemical binding energies
k	kilo	keV	10^3 eV	x-ray
M	mega	MeV	10^6 eV	electron mass / nuclear binding energies
G	giga	GeV	10^9 eV	proton mass
T	tera	TeV	10^{12} eV	LHC centre of mass, about 10 times
				electroweak scale (i.e. 10 $\times$W,Z,Higgs mass)
P	peta	PeV	10^{15} eV	energy of neutrinos detected at icecube
E	exa	EeV	10^{18} eV	energy of Ultra high energy cosmic rays

Table 1.4: Some examples of Natural Units

1.5 Natural Units $\hbar = c = 1$

These are the units most often used in particle physics. First of all, we can see from the the fact that $[t] = [x]$ that $[E] = [p]$ but also we can see that $[t] = [x] = [E]^{-1} = [p]^{-1}$ so we only need one unit for all of them. We could use Joules, we could use metres, we could use kilograms or we could use seconds. Very often people use electron volt or eV which is the energy gained by a particle with the charge of the electron as it moves through one volt of potential difference in an electric field.

$$1\text{eV} = 1.602 \times 10^{-19}\text{J} \tag{1.5}$$

As particle accelerators developed, people use keV, MeV and now GeV. Maybe this will change in the future.

Quantity	Equation	Unit
Density	ρ =M/V	GeV^4
Force	F=ma	GeV^2
Pressure	P=F/A	GeV^4
Energy	E= (1/2)mv^2	GeV
Frequency	f=E/h=$\omega/\hbar$	GeV

Table 1.5: Some Quantities in Natural Units

1.6 The Planck Mass

There are many frontiers in physics, the word frontier is not very well defined! Sometimes it is convenient to think of three of them and come up with three axes, one axis represents

Quantity	value	
Size of Hydrogen atom	5.2×10^{-11} m	$2.63 \times 10^{5}\text{GeV}^{-1}$=0.26 keV^{-1}
Mass of Hydrogen atom	1.66×10^{-27} kg	0.938 GeV=938 MeV
Height of human	1.7 m	8.6×10^{15} GeV^{-1}
Mass of Human	70 kg	3.9×10^{28} GeV
Solar Mass	1.989×10^{30} kg	$=1.12 \times 10^{57}$ GeV
Density at centre of neutron star	10^{18} kg m^{-3}	0.004 GeV^4
Pressure at centre of neutron star	10^{32} Pa	5×10^{-6} GeV ^{4}
Average density in solar neighbourhood	9.5×10^{-21}kg m^{-3}	$4.5 \times 10^{-41}\text{GeV}^4$
Average density in Universe (matter only)	2.5×10^{-27}kg m^{-3}	$1.19\times 10^{-47}\text{GeV}^4$

Table 1.6: Some typical Scales in natural Units

relativistic speeds vs slow speeds, a second is strong gravity vs. weak gravity and the third is quantum vs. non-quantum regime. These three regimes are governed by Newton's constant G,

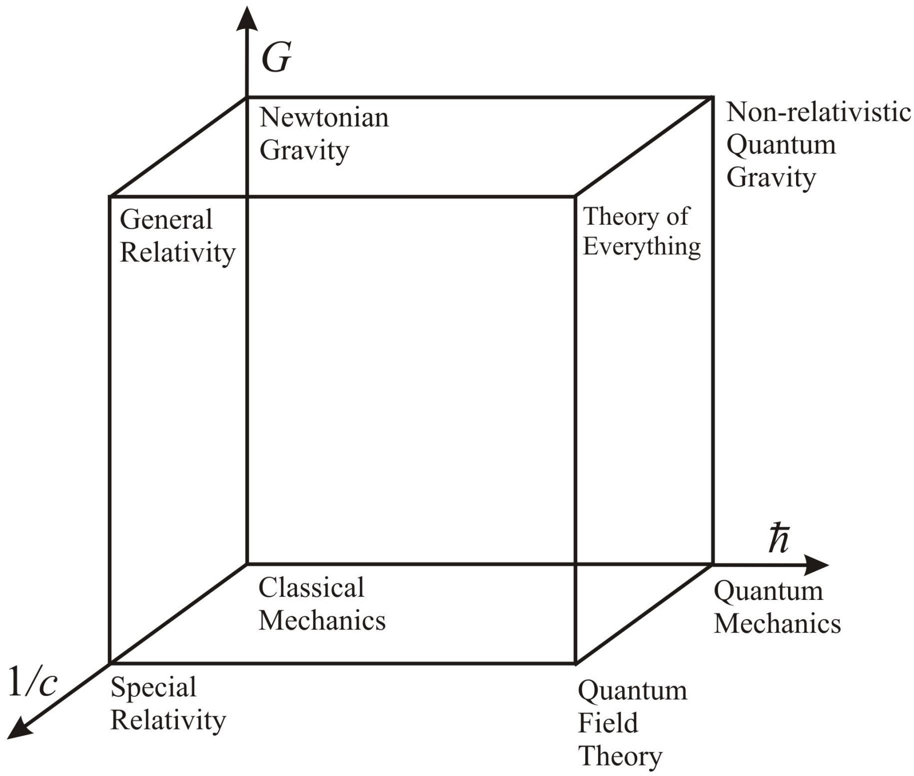

Figure 1.1: Cube of Physical Theories

Planck's constant $\hbar$ and the speed of light c. We can construct some physical quantities from a combination of these three quantities (in SI units). There is a mass, which we call the **Planck Mass**

$$M_{Pl} = \sqrt{\frac{\hbar c}{G}} = 2.176 \times 10^{-8}\text{kg} \tag{1.6}$$

which is quite heavy, about the size of a bacterium. The **Planck Length**

$$l_{Pl} = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-35}\text{m} \tag{1.7}$$

which is about 20 orders of magnitude shorter than a photon and the **Planck Time**

$$t_{Pl} = \sqrt{\frac{\hbar G}{c^5}} = 5.391 \times 10^{-44}\mathrm{s} \tag{1.8}$$

which is very short indeed Of course in natural units with $\hbar = c = 1$ then $M_{Pl} = l_{pl}^{-1} = t_{pl}^{-1} = 1.221 \times 10^{19}$GeV. One way to think about the Planck mass is where the quantum line hits the black hole line. Sometimes physicists using natural Units use $M_{Pl} = 1/\sqrt{8\pi G}$ rather than

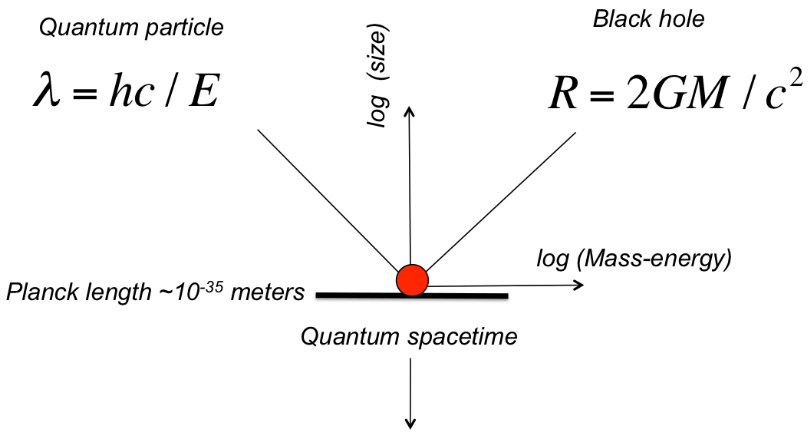

Figure 1.2: Physical meaning of the Planck Mass/Length

$M_{Pl} = 1/\sqrt{G}$ so you should always make sure which notation they have adopted.

1.7 Gravitational Natural Planck Units

Some people set $\hbar = c = G = 1$. If you are doing GR or trying to do quantum gravity, this makes a lot of sense. If you are combining particle physics with gravitational physics, it makes arguably less sense. If you are trying to make links between cosmology, high energy astrophysics and collider physics, it definitely makes things complicated. You should know it exists. We will not be using it in this course.

1.8 Astrophysical Units

In space, things are very large so metres are not a lot of use and GeV^{-1} are even worse. It helps to define the parsec One parsec is 3.26 light years which is about 3.086×10^{16}m. The nearest stellar system to the sun is alpha Centauri which is about 1.34 parsecs away. We are about 8.5 kpc from the centre of the Galaxy, the disk of which has a radius of about 15 kpc. Galaxies are typically seperated by about a Mpc.

We are moving around the Galaxy at about 230 km s^{-1} roughly in the direction of the constellation of Cygnus.

$$v_{rot}^2 = \frac{GM}{r} \qquad \rightarrow \qquad M = \frac{r v_{rot}^2}{G} \tag{1.9}$$

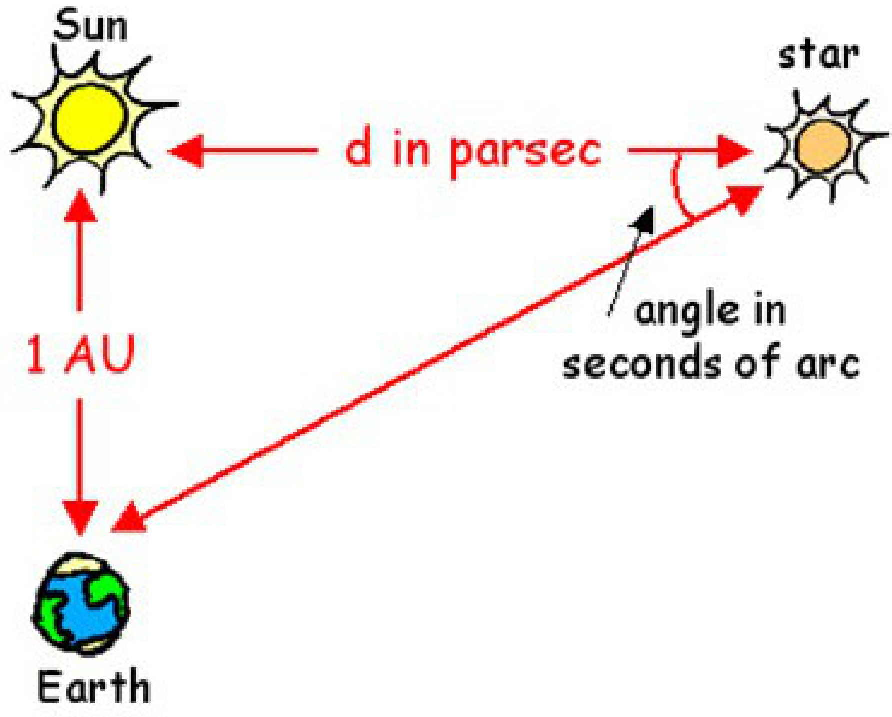

Figure 1.3: Definition of a Parsec

which is kind of a pain in the backside if we are using SI or cgs.

$$\begin{aligned} G &= 6.67 \times 10^{-11} \left(\frac{\mathrm{m}}{\mathrm{s}}\right)^2 \mathrm{m} \quad \mathrm{kg}^{-1} \\ &= 6.67 \times 10^{-11} \left(\frac{\mathrm{km}}{\mathrm{s}}\right)^2 \left(\frac{\mathrm{m}}{\mathrm{km}}\right)^2 \frac{\mathrm{kpc}}{M_\odot} \left(\frac{M_\odot}{\mathrm{kg}}\right) \left(\frac{\mathrm{m}}{\mathrm{kpc}}\right) \\ &= 4.299 \times 10^{-6} \left(\frac{\mathrm{km}}{\mathrm{s}}\right)^2 \frac{\mathrm{kpc}}{M_\odot} \end{aligned} \tag{1.10}$$

so now we can look again at equation (1.9)

$$M = \frac{r v_{rot}^2}{G} = \frac{8.5\mathrm{kpc}\left(200\mathrm{kms}^{-1}\right)^2}{4.288 \times 10^{-6} \left(\frac{\mathrm{km}}{\mathrm{s}}\right)^2 \frac{\mathrm{kpc}}{M_\odot}} = 8 \times 10^{10} M_\odot \tag{1.11}$$

which gives us roughly the enclosed mass of the Milky Way at the Solar Radius. We can ask how long it takes to rotate around the Galaxy?

$$t = \frac{2\pi r}{v_{rot}} = \frac{2\pi 8.5\mathrm{kpc}}{200\mathrm{kms}^{-1}} = 8.2 \times 10^{15}\mathrm{s} = 261\mathrm{Myr} \tag{1.12}$$

So you see it makes sense to use kpc (or perhaps pc for small galaxies like dwarf galaxies) and Myr and to measure masses in $M_\odot$. Velocities are better in kms^{-1} but remember 1 kpc Myr^{-1} is about 10^3 kms^{-1} so not a million miles off.

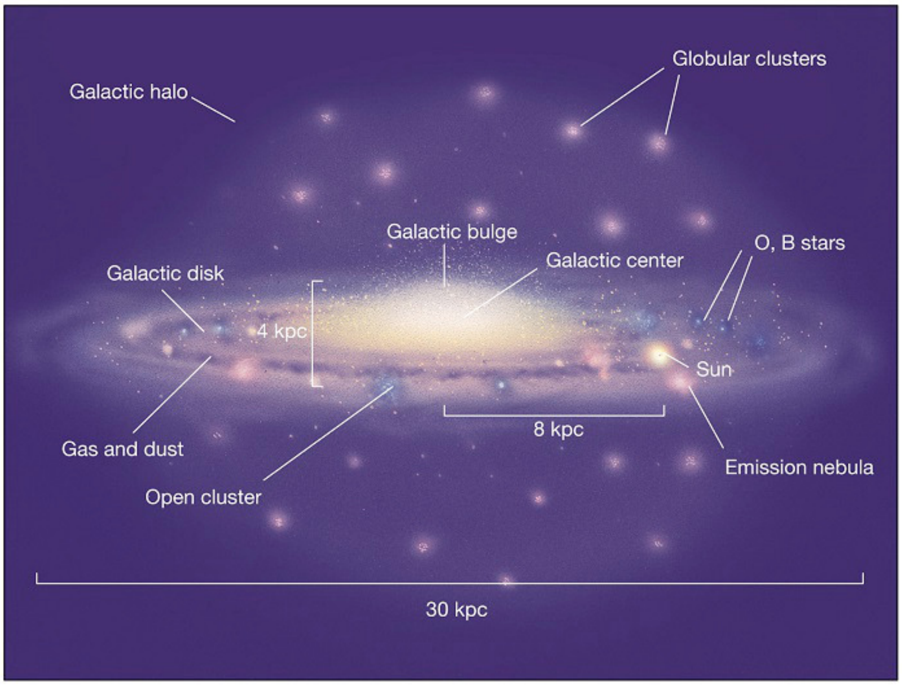

Figure 1.4: Schematic of the Milky Way

Chapter 2

Cosmic Dynamics

Accordingly, since nothing prevents the earth from moving, I suggest that we should now consider also whether several motions suit it, so that it can be regarded as one of the planets. For, it is not the center of all the revolutions. - **Nicolaus Copernicus**

This observation by Copernicus that the Earth was not the centre of the Universe has been extended to the Cosmological Principle that our location in the Universe is nowhere special, and also that there is no preferred direction in the Universe. To be slightly more precise, we assume that the Universe is isotropic and homogeneous, it looks the same in all directions and it doesn't change from place to place (i.e. it looks the same in all directions no matter where you stand).

We know that the Universe is dynamic however, both Newton's and Einstein's equations tell us this. Einstein's GR also tells us that the Universe can have spatial and space-time curvature. The cosmological principle is a statement about the spatial part of the Universe, so although we allow the Universe to change in the future and the past, at any given fixed time (of course, this is an ill-defined concept in any relativistic theory) everything is the same.

2.1 Robertson-Walker Metric and Friedman equations

The simplest space time metric of the Universe which satisfies the cosmological principle is the Robertson-Walker Metric

$$ds^2 = -c^2 dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \tag{2.1}$$

which is the metric with the greatest spatial symmetry (formally, the highest number of space-like Killing vectors.) The factor $a(t)$ is called the scale factor which the coordinate r, θ, ϕ are called the *co-moving coordinates* and they are co-moving because a particle at a fixed value of r will stay there even as $a(t)$ increases and the Universe expands. Things in the Universe are located and sit at a given r, θ, ϕ but get closer to each other or further apart as $a(t)$ changes over time. Of course in reality, galaxies don't sit still, they fly around, merge, collide etc but more about that later.

The spatial curvature is encoded in the parameter k which determines whether the Universe is spatially flat ($k = 0$), open ($k < 0$) or closed ($k > 0$).

Some people make the scale factor a dimensionless and r dimensionful, some people make a dimensionful and k dimensionless, some people like $k = -1, 0, 1$. You can make a carry the dimensions of length and let r be dimensionless. However, since many people these days assume $k = 0$ (MORE LATER) you can often see the other approach often being adopted.

Now if we differentiate the Robertson-Walker metric in the normal way we get the Christoffel symbols, and them and their derivatives allow us to formulate the ricci tensor $R_{\mu\nu}$ and

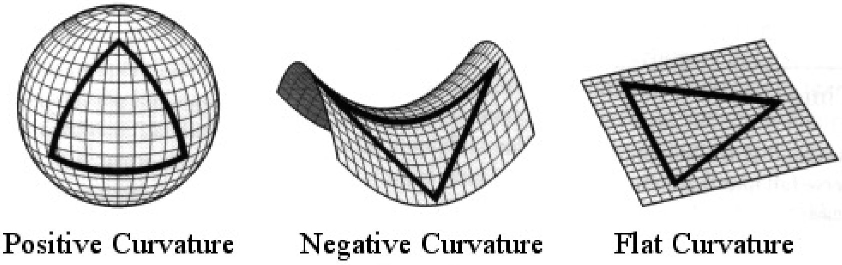

Figure 2.1: Different spatial curvatures of the Universe

consequently the Einstein Tensor $G_{\mu\nu}$.

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.2}$$

Where $T_{\mu\nu}$ is the stress energy tensor. The cosmological principle also applies to the stress energy in the Universe so that the net momentum is zero. also the net vorticity of the stuff in the Universe should be zero, or there will be a preferred direction corresponding to the axis around which that vorticith is occurring. So we are left with the time-time component of density $T_{00} = -\rho$ and the spatial components which are only pressure, i.e. $T_{11} = T_{22} = T_{33} = P$ and nothing else.

The coordinates r, θ, ϕ drop out which is a good thing if we want out Universe to obey the cosmological principle and we get

$$G_{00} \rightarrow 3\left(\frac{\dot{a}}{a}\right)^2 + 3\frac{k}{a^2} = 8\pi G\rho \tag{2.3}$$

$$G_{11} \rightarrow 2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = -8\pi G P \tag{2.4}$$

$$G_{00} - 3G_{11} \rightarrow \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \tag{2.5}$$

So the first thing we can see is that $\dot{a}$, i.e. the rate at which the Universe expands or contracts depends upon the density and pressure of stuff in the Universe, the spatial curvature of the Universe and the cosmological constant Λ. The quantity $\dot{a}/a = H$ is called the Hubble rate and its value today H_0 is called the Hubble constant. We will use H and $\dot{a}/a$ interchangeably so be careful.

The age of the Universe today is denoted t_0 and the scale factor today is denoted a_0.

2.1.1 Redshift

Photons which are emitted at some early time t_e and arrive today t_0 are stretched because the scale factor of the universe $a_0 > a_e$. This factor is called redshift z

$$1 + z = \frac{a_0}{a_e} \tag{2.6}$$

You can use z instead of time, $z = 0$ is today, $z = 1$ is when the scale factor is half the size it is today.

2.1.2 Energy density in the Universe

so we have seen that $3H^2 = 8\pi G\rho - ka^{-2}$ and we have measured H so we can start to makes some inferences about ρ and k. Of course there may be different kinds of stuff in the Universe so in general

$$\rho_{total} = \rho_x + \rho_y + \rho_z + ... \tag{2.7}$$

For example, the energy density of matter ρ_M and the energy density of radiation ρ_γ. Conservation of energy tells us that the covariant derivative (divergence) of the stress energy tensor should be zero. This actually comes also from the Bianchi identity which is a seperate concept in differential Geometry.

$$\frac{\nabla T_{\mu\nu}}{\partial x^\mu} = 0 \rightarrow \dot{\rho} = -3\frac{\dot{a}}{a}(\rho + P) \tag{2.8}$$

or you can also get this from the first law of thermodynamics, i.e, $dE = -PdV$. Note $TdS = 0$ since the expansion of the Universe is adiabatic normally, although in the Early Universe that can change.

In general an equation of state is a relationship between pressure, density and temperature. In cosmology, the equation of state is often simpler and is caled w and it is defined as the relationship between pressure and density. For cosmology usually we write simply

$$P = w\rho \tag{2.9}$$

and that is that. We will see later that the equation of state of dark energy *may* be a function of time, or at least people are testing that.

We multiply equation (2.8) by dt/da and then we get

$$\frac{d\rho}{da} = -\frac{3}{a}(\rho + P) = -\frac{3\rho}{a}(1 + w) \tag{2.10}$$

which has solution

$$\rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)} \tag{2.11}$$

Matter

The matter energy density ρ_M is due to "normal" dust particles which get diluted like the inverse of the volume as the Universe increases $\rho_M = n * m$ where n is the number density of particles and m is the mass of an individual dust particle.

$$\begin{aligned} n &= \frac{\text{number}}{volume} = \frac{\text{constant}_1}{a^3} \\ nm &= \frac{\text{number} \times \text{mass}}{volume} = \frac{\text{constant}_2}{a^3} \end{aligned}$$

so that $\rho_M \propto a^{-3}$ BUT for any density we know that $\rho \propto a^{-3(1+w)}$ So that for matter $w_M = 0$ in other words pressure $P = 0$ which is what we expect. Baryons are essentially matter (we ignore the electrons since their mass is so much smaller) at least on cosmological scales, as is dark matter (more later)

Radiation

This is the energy density in relativistic particles which includes all massless particles such as photons. Also maybe neutrinos, depending on their mass but we don't know for sure if they are non relativistic yet.

$$\rho_\gamma = n_\gamma E_\gamma \tag{2.12}$$

where n_γ is the number density of relativistic particles (such as photons) and E_γ is the energy of each particle. Now $n_\gamma \propto a^{-3}$ but also $E_\gamma \propto a^{-1}$ because light gets stretched as the Universe expands so that $\rho_\gamma \propto a^{-4} = a^{-3(1+w)}$ which means that $w_\gamma = 1/3$.

Radiation-Matter Equality

Since radiation redshifts more rapidly $\propto (1+z)^4$ than matter $\propto (1+z)^3$ At early times, the Universe is radiation dominated and at late times the Universe is matter dominated (if we forget other forms of radiation like Λ). The redshift of radiation and matter equality z_{eq} is the redshift at which their densities are equal to each other

$$\frac{\rho_M(z_{eq})}{\rho_\gamma(z_{eq})} = 1 = \frac{\rho_{M0}(1+z_{eq})^3}{\rho_{\gamma 0}(1+z_{eq})^4} \tag{2.13}$$

so that $1 + z_{eq} = \rho_{M0}/\rho_{\gamma 0}$.

2.1.3 Models with spatial curvature

Now recall that

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{2.14}$$

where $k > 0$ corresponds to a positively curved space, $k = 0$ corresponds to flat space and $k < 0$ corresponds to a negatively curved Universe. Remember some people set $k = -1, 0, 1$ in which case the radius of curvature of the Univers is set by $1/a$ while other people allow k to be a free number because they would like to set the scale factor today equal to one. Writing that in various equivalent ways $a_0 = a(\text{today}) = a(t_0) = a(z = 0) = 1$, in which case the radius of curvature is set by $\sqrt{|k|}$.

For simplicity take $\rho = \rho_M \propto a^{-3}$ then equation (2.14) becomes

$$\dot{a}^2 = \frac{8\pi G}{3}\frac{\rho_0 a_0^3}{a} - k \tag{2.15}$$

- If $k = 0$ then as $a \to \infty$, $\dot{a} \to 0$ and the universe grinds to a halt at $t = \infty$.
- If $k < 0$, $\dot{a}$ never goes to zero and the Universe expands forever.
- If $k > 0$ then $\dot{a}$ becomes zero at a finite time. Now $3\ddot{a} = -4\pi G\rho a$ is always negative so if $k > 0$ the Universe collapses after a finite time $\to$ BIG CRUNCH

2.1.4 ρ_{crit} and Ω

This bit is super easy so long as you actually read it!

The Friedman equations look like

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \equiv \frac{8\pi G}{3}\rho_{crit} \tag{2.16}$$

So that if $k = 0$ then $\rho = \rho_M + \rho_\gamma + \rho_X + ... = \rho_{crit}$ or to put it another way, for each value of the Hubble rate H, there is is a density ρ_{crit} which corresponds to a spatially flat Universe $k = 0$

$$\rho_{crit} = \frac{3H_0^2}{8\pi G} \tag{2.17}$$

Now the Ω parameters are simply rescaled measurements of the density, measured in terms of the critical density, i.e. $\Omega_M = \rho_M/\rho_{crit}$, $\Omega_\Lambda = \rho_\Lambda/\rho_{crit}$ and more generally $\Omega_X = \rho_X/\rho_{crit}$.

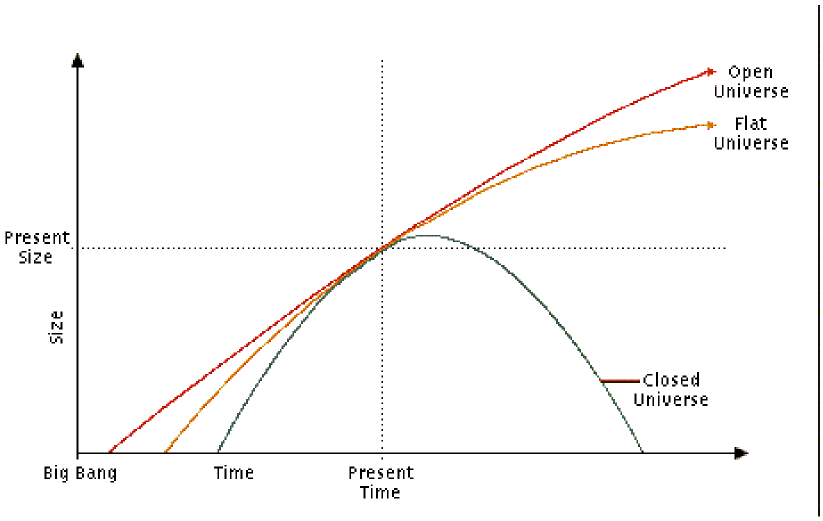

Figure 2.2: Different fates for different curvatures

Example

Now consider an open Universe $k < 0$ which means that $\rho < \rho_{crit}$. Lets say that ρ_M is 50% ρ_{crit} and ρ_γ is 10% ρ_{crit} then

$$\Omega = \Omega_M + \Omega_\gamma = 0.5 + 0.1 = 0.6 \tag{2.18}$$

Curvature written as Density

Sometimes it is convenient to treat curvature as a seperate source of density that we call Ω_k, such that

$$\Omega_M + \Omega_\gamma + \Omega_x + \Omega_y + ... + \Omega_k = 1 \tag{2.19}$$

then in this case all of the Ω_i (so long as you include Ω_k) will always sum to unity (i.e. 1) by definition and

$$\Omega_k = -\frac{k}{a_0^2 H_0^2} \tag{2.20}$$

2.2 Observations

2.2.1 Hubble Law and Hubble constant

As the Universe expands, galaxies move away from each other because $a(t)$ increases. The further they are from us the faster they appear to move.

If we consider a galaxy at co-moving distance r. At time t it is a distance $ra(t)$ away whereas at time t_0 it is a distance $ra(t_0)$ away so its velocity from us it

$$v_{hub} = r\frac{a(t_0) - a(t)}{t_0 - t} \tag{2.21}$$

Now for low redshift objects, $a(t) \sim a(t_0)$ and we can expand $a(t)$ in terms of $a(t_0)$

$$\begin{aligned} a(t) &= a(t_0) + \left.\frac{da}{dt}\right|_{t_0} (t - t_0) + \frac{1}{2}\left.\frac{d^2a}{dt^2}\right|_{t_0} (t - t_0)^2 + ... \\ &= a(t_0)\left[1 + H_0 (t - t_0) - \frac{1}{2} q_0 H_0^2 (t - t_0)^2 + ...\right] \end{aligned} \tag{2.22}$$

where q_0 is known as the deceleration parameter

$$q_0 = -\frac{\ddot{a}}{aH_0^2} = -\frac{4\pi G}{3H_0^2}(\rho + 3P) \tag{2.23}$$

The deceleration parameter used to be very important as the only observations we had of the Universe were at low redshift. With the development of better telescopes, nobody really talks about q_0 any more because now we can go to much higher redshifts and the pertubative expansion isn't really interesting any more, apart from to obtain the low redshift Hubble law. For small redshifts $H_0(t - t_0)$ is very small so only the first two terms are important.

$$v_{hub} = r\frac{a(t_0) - a(t)}{t_0 - t} = -r\frac{a(t_0)H_0\,(t - t_0)}{t_0 - t} = ra(t_0)H_0 = H_0 d \tag{2.24}$$

where d is the distance. Also

$$z \simeq \frac{H_0 d}{c} \tag{2.25}$$

2.2.2 Measuring the Hubble Constant at Low Redshift

Magnitudes

Astronomers measure brightness of objects in magnitudes. These are backward and logarithmic.

Object	m
Sun	-27
full moon	-12
venus	-5
Vega (star)	0
Andromeda Galaxy	+4.5
Neptune	+7

The values are apparent magnitudes m - related to brightness we see on earth. We can define the absolute magnitude M which is the magnitude the object would appear if it were 10 pc away.

$$m - M = 5\log d - 5 \tag{2.26}$$

where d is the distance in parsecs. If you have m and M you can get d. The Luminosity if the total outflow of energy of object (star etc) per second.

$$\log\left(\frac{L}{L_\odot}\right) = 0.4\,(M_\odot - M) \tag{2.27}$$

where $L_\odot$ is the luminosity of the Sun and $M_\odot$ is the absolute magnitude of the Sun.

Standard Candles

Standard Candles are astronomical objects we know the absolute magnitude (luminosity) of. E.g. Cepheid variables - stars which pulsate. The frequency of their pulsations is a direct function of their luminosity.

1. measure period of cepheid - calculate luminosity
2. measure apparent magnitude m with telescope $m - M \to d$
3. measure redshift of spectral lines in host galaxy (galaxy where Cepheid lives)

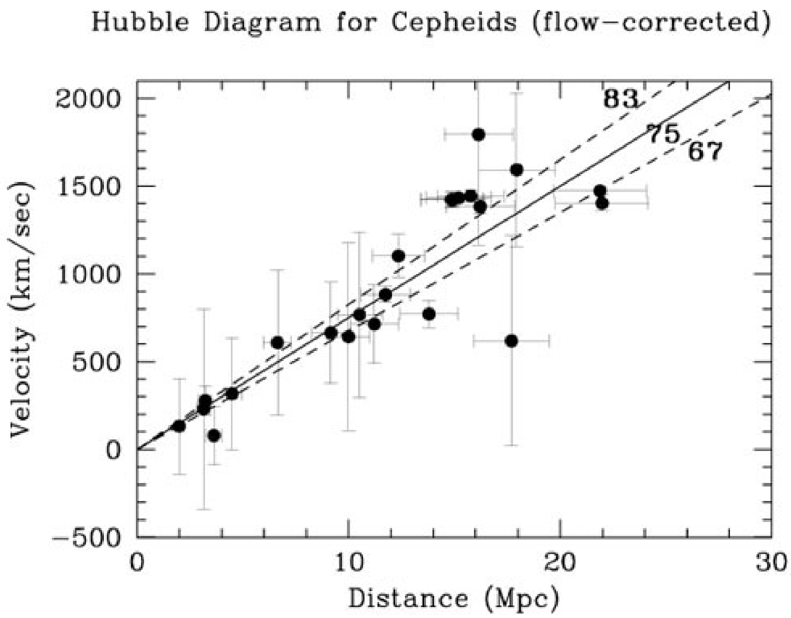

Figure 2.3: Measuring the Hubble Rate

4. draw Hubble diagram

Another type of standard candles are type 1a supernovae (Sn1a). Sn1a are thought to occur in binary systems when big stars pour matter onto small stars. degenerate white dwarf core develops inside the small star. When the core mass grows to $M_{Ch} \simeq 1.4 M_\odot$ i.e. the Chandrasekhar mass they explode and all have very similar magnitudes.

Type 1a supernovae are quite rare but can be observed at very large redshifts, $z > 1$.

Cepheid variables are more common but not observed at large distances (not bright enough). Cepheids are then used to obtain H_0.

Cepheid variable observations take place in galaxies which also harbour type 1a standard candles, which allow us to calibrate the distance ladder.

2.2.3 Measured value of the Hubble Constant.

The Hubble constant has a value of the order of 100 kms^{-1}Mpc^{-1}

$$H = h \times 100 \mathrm{kms}^{-1}\mathrm{Mpc}^{-1} \tag{2.28}$$

and the measured value of h is approximately $h = 0.65 - 0.75$ using equations such as (2.25), we simply compare m and M to get d and compare that to z. If we want to measure the evolution of the Hubble constant properly over the history of the Universe we need to take into account that d changes due to the evolution of the scale factor. More later.

2.3 The Hot Universe

The Universe is getting bigger, so that means that everything used to be closer together and hotter. We need to look at the physics of this early epoch. The Universe is full of microwave radiation which becomes less energetic as the Universe expands. The temperature of this Cosmic Microwave background radiation observed today is 2.73K$\sim 10^{-4}$ eV. At a redshift of $z \sim 1000$ the temperature of the radiation starts to become high enough to ionize hydrogen. The Universe becomes less transparent at this time and photons emitted before this point scatter off free electrons/ions. This is a boundary to what we can see with photons and is called the last scattering surface.

We can look in more detail at the thermal properties of particles and radiation in the early Universe.

For each particle species i, there is a distribution function

$$f_i(\vec{p}) = \frac{1}{\exp\left[\frac{E_i - \mu_i}{T}\right] \pm 1} \tag{2.29}$$

where $E_i = \sqrt{\vec{p_i}^2 + m_i^2}$ and μ_i is the chemical potential which we can safely reject under most situations in cosmology. The $+$ is for fermions and the $-$ is for bosons. Number density is then

$$n_i = \frac{g_i}{(2\pi)^3} \int f_i(\vec{p_i}) d^3p \tag{2.30}$$

where g_i is the number of internal degrees of freedom. For a photon, $g_\gamma = 2$ for the two spins. For electrons we have $g_e = 4$ (2 spins and antiparticles). Only left handed neutrinos exist in thermal equilibrium at low energies so we have $g_\nu = 2$.

The energy density is given by

$$\rho_i = \frac{g_i}{(2\pi)^3} \int E_i \vec{p} f_i(\vec{p_i}) d^3p \tag{2.31}$$

$$P_i = \frac{g_i}{(2\pi)^3} \int \frac{|\vec{p}|^2}{3E_i(\vec{p})} f_i(\vec{p_i}) d^3p \tag{2.32}$$

These integrals can be solved analytically in 2 regimes.

- $T \ll m$ i.e. non-relativistic matter

$$\begin{aligned} n_i &= g_i \left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T} \\ \rho_i &= m n_i \\ P_i &= T n_i \end{aligned} \tag{2.33}$$

  where I have set $k_B = 1$

- $T \gg m$ i.e. relativistic matter (always true for photons)

$$E = \left(p^2 + m^2\right)^{1/2} \rightarrow \frac{dE}{dp} = \frac{1}{2}\left(p^2 + m^2\right)^{-1/2} 2p = \frac{p}{E} \tag{2.34}$$

so $EdE = pdp$ and the phase space integral $d^3p \to 4\pi p^2 dp = 4\pi p E dE = 4\pi\sqrt{E^2 - m^2} E dE$

$$\begin{aligned}
\rho_i &= \frac{g_i}{2\pi^2}\int_0^\infty \frac{E^3 dE}{\exp[E/T] \pm 1} && (2.35)\\
&= \frac{\pi^2}{30} g_i T^4 (\mathrm{Bose-Einstein}) && (2.36)\\
&= \frac{7}{8}\frac{\pi^2}{30} g_i T^4 (\mathrm{Fermi-Dirac}) && (2.37)\\
& && (2.38)
\end{aligned}$$

and for the number densities we get

$$\begin{aligned}
n_i &= \frac{\zeta(3)}{\pi^2} g_i T^3 (\mathrm{Bose-Einstein})\\
&= \frac{3}{4}\frac{\zeta(3)}{\pi^2} g_i T^3 (\mathrm{Fermi-Dirac}) && (2.39)
\end{aligned}$$

where $\zeta(3) = 1.20206...$ Riemann Zeta Function.

Dividing ρ by n gives the average energy of a particle $\langle E \rangle_{BE} \sim 2.7T$ and $\langle E \rangle_{FD} \sim 3.15T$

So for the CMB radiation today, the density is $\rho_{\gamma 0} \sim T_\gamma^4 \sim 10^{-16}\mathrm{eV}^4$ whereas the density of matter $\rho_{M0} \sim 10^{-12}\mathrm{eV}^4$ so we were justified in neglecting the contribution of radiation to today's expansion rate. In fact $\Omega_{\gamma 0} \sim 10^{-4}$ and $z_{eq} \sim 5000$

Degrees of Freedom

In order to proceed we need to know more about the degrees of freedom g. This depends on the total number of relativistic degrees of freedom in thermal equilibrium with the plasma. Remember that

$$\rho_i = \frac{\pi^2}{30} g_i T^4 \tag{2.40}$$

and $\rho_{tot} = \sum_i \rho_i$. However any species will only contribute to the overall density of the radiation for as long as it is relativistic. As soon as T drops to the mass of the particle m it will become Boltzmann supressed and the correct expression for its density will be

$$\rho_i = m_i g_i \left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T} \tag{2.41}$$

which will be much less than equation (2.42) very quickly. Hence only the relativisic degrees of freedom are really significant in the overall density driving the expansion of the Universe at early times. We can therefore write

$$\rho_{tot} = \frac{\pi^2}{30} g_{eff} T^4 \tag{2.42}$$

where

$$g_{eff} = g_{eff}(T) = \sum_{i=bosons} g_i + \frac{7}{8}\sum_{i=fermions} g_i \tag{2.43}$$

and is only a sum over those particles with $m < T$. Remember that the factor $7/8$ comes from the integral over the Fermi-Dirac distribution.

Now for photons there are 2 degrees of freedom, one for each spin. For the 6 leptons $e, \nu_e, \mu, \nu_\mu, \tau, \nu_\tau$ there are for each species a particle and an anti-particle and 2 spins, so in total that is $6 \times 2 \times 2 = 24$ degrees of freedom. There are in total 6 quarks, which also each have an anti-particle and also two spins, however the quarks also have a colour charge under QCD, so they are either r,g or b, so the total number of degrees of freedom in quarks is $6 \times 2 \times 2 \times 3 = 72$. There are 8 gluons, each with two spins.

Note however that some of the leptons and some of the quarks are very heavy, for example, the top quark has a mass of about 173 GeV so at temperatures far below that it will not form a large part of the plasma and the effective degrees of freedom will not include it. The effective value of degrees of freedom g_{eff} is therefore a function of temperature.

The entropy density in the Universe should be conserved usually

$$s = \frac{2\pi^2}{45} g_{eff*} T^3 \tag{2.44}$$

in this quantity, g_{eff*} is not neccesarily the same as g_{eff} since the entropy is often assumed to be in one to one correspondence with the temperature of what is in thermal equilibrium (or later the photons, even though they are not in thermal equilibrium any more). For neutrinos for example, once they go out of thermal equilibrium, entropy is dumped into the photon field from the annihilation of electrons and positrons, so the neutrinos have a lower temperature than the photons today and they contribute less to the overall entropy. In this course we will not need to worry about the difference between g_{eff} and g_{eff*} too much.

2.3.1 Thermodynamics in the Expanding Universe

We set $k_B = 1$ so we measure T in eV. What does thermal equilibrium mean if the Universe is expanding? Formal answer very complicated but we can get a good handle on it. Assume some particle species at time t has a number density n, interaction cross section σ and velocity v. One can then define an interaction rate per particle $\Gamma = n\sigma v$ ($(n\sigma)^{-1}$ is the mean free path, time between collisions $\delta t = (n\sigma v)^{-1}$, rate $\Gamma = (\delta t)^{-1}$).

If $\Gamma(t) \gg H(t)$ the Universe doesn't expand much between collisions. We can then talk about thermal equilibrium.

2.3.2 Freeze out temperature

We know that $n \sim T^3$ and $\rho \sim T^4$. For an interaction involving the exchange of a massive gauge boson like the weak interaction $\sigma \propto s \sim T^2$. So $\Gamma = n\sigma v \simeq T^5$ ($v \sim c \sim 1$) and $H^2 \propto \rho \sim T^4$

So $\Gamma/H \propto T^3$ and as time moves on and T goes down, Γ will fall below H. This time/temperature is called the freeze out time/temperature. After that time very few interactions occur, the number density is frozen and only change due to dilution as the Universe expands.

2.3.3 Nucleosynthesis

Heavy elements, C, N, O, Fe, Si etc produced in stars. Very heavy elements Au, Ag, Pb, U etc produced in supernovae (hence rarety/value) but in the early universe only light elements are produced ^{4}He, ^{3}He, D, Li etc. Let us look at Hydrogen and Helium.

At early times $t \ll 1$sec $T \gg 1$ MeV the reactions

$$\begin{aligned} n &\leftrightarrow p + e^- + \bar{\nu}_e \\ \nu_e + n &\leftrightarrow p + e^- \\ e^+ + n &\leftrightarrow p + \bar{\nu}_e \end{aligned} \tag{2.45}$$

were not at all suppressed so the number of neutrons was equal to the number of protons. The Neutron-proton mass difference $\Delta m = m_n - m_p = 1.29\text{MeV}$ So as T cools towards 1 MeV the neutron and the proton number densities start to become different due to Boltzmann

$$\frac{n_n}{n_p} \simeq \frac{e^{-m_n/T}}{e^{-m_p/T}} = e^{-\Delta m/T} \tag{2.46}$$

The rate of the reactions (2.45) is very similar to neutrino reactions.

$$\Gamma(\nu_e + n \leftrightarrow p + e^-) \sim 2.1 \left(\frac{T}{\text{MeV}}\right)^5 \text{sec}^{-1} \tag{2.47}$$

which gives a freeze out temperature for those reactions of

$$T(\Gamma \sim H) = 0.8\text{MeV} \tag{2.48}$$

At $T = 0.8$ MeV, $n_n/n_p = e^{-\Delta m/T} = e^{-1.29\text{MeV}/0.8\text{MeV}} \sim 0.2$ so there are five times as many protons as neutrons. Weak interactions stop (apart from β decay of neutrons to protons). Main interaction is now

$$p + n \leftrightarrow d + \gamma \tag{2.49}$$

($d =^2 H =$ deuterium)

Deuterium Bottleneck and Helium formation

While $T > 0.1$ MeV $\Gamma(p + n \leftrightarrow d + \gamma) \gg H$ since deuterium has a very low binding energy and is photodissociated easily. Between $T = 0.8$ MeV and $T = 0.1$ MeV, some of the neutrons decay into protons. In fact

$$\left.\frac{n_n}{n_p}\right|_{T=0.1\text{MeV}} \sim 0.13 \tag{2.50}$$

at $T \sim 0.1$ MeV, deuterium is stable and 99.99% of the neutrons still around join ^{4}He nuclei via

$$\begin{aligned} d + d &\rightarrow {}^3He + n && (2.51)\\ {}^3He + d &\rightarrow {}^4He + p && (2.52)\\ & && (2.53) \end{aligned}$$

and

$$\begin{aligned} d + d &\rightarrow {}^3H + p && (2.54)\\ {}^3H + d &\rightarrow {}^4He + n && (2.55)\\ & && (2.56) \end{aligned}$$

we can now calculate the abundance of ^{4}He, Y

$$Y(^4\text{He}) = \frac{\text{number of nucleons in } ^4\text{He nuclei}}{\text{total number of nucleons}} \tag{2.57}$$

since nearly all neutrons ended up in ^{4}He we can write

$$Y(^4\text{He}) = \frac{4n_{\text{He}}}{n_{tot}} = \frac{4(n_n/2)}{n_n + n_p} = \left.\frac{2n_n}{n_n + n_p}\right|_{T=0.1\text{MeV}} \sim 0.24 \tag{2.58}$$

0.24 also comes from detailed nuclear simulations.

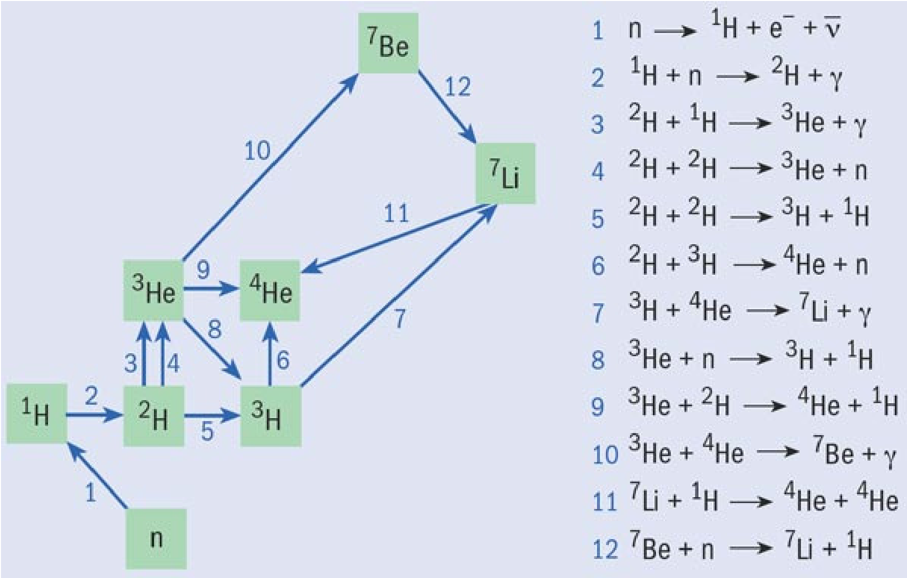

Figure 2.4: The 12 different nuclear reaction paths one should include if one wants to do primordial nucleosynthesis properly.

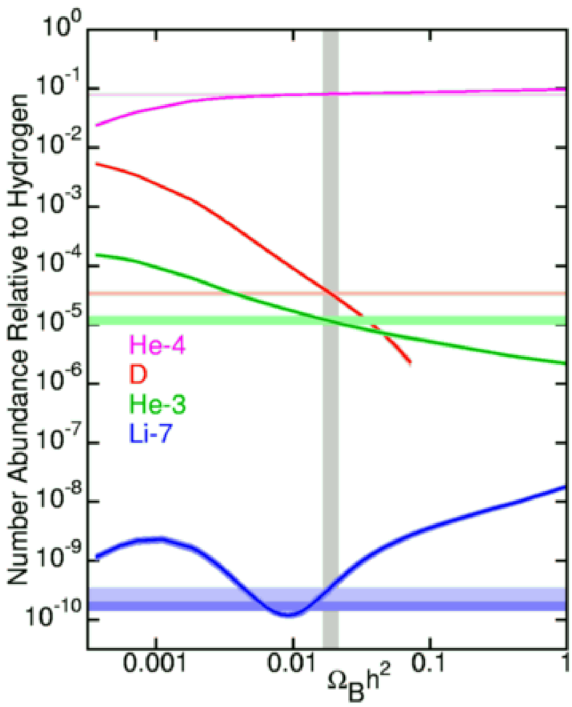

Figure 2.5: Abundances of different isotopes lead to different conclusions about the density of baryons

Constraint on number of neutrino species (and other light species)

Since these freeze out temperature 0.8 MeV, 0.1 MeV depend on $H = \dot{a}/a$ during nucleosynthesis and H depends on g_{eff}, nucleosynthesis tells us that there has to be only about 3 neutrinos. If there were 4 neutrinos, H would be bigger for a given T. Weak interactions would freeze out at a higher T, there would be more neutrons and $Y(\mathrm{He})$ would be bigger.

Baryon to Photon Ratio

If we look at unprocessed intergalactic dust, we see $Y(\mathrm{He}) \sim 0.24$. This is very good. Also can fit to D, Li, ^{3}He etc. Nucleosynthesis really makes us confident about the big bang model. BUT observations of d, Li, ^{3}He suggest a particular baryon to photon ratio and since we know 410 cm^{-3} photons we also have baryon density. $\Omega_B h^2 \simeq 0.02$ (B for baryons) so $\Omega_B \simeq 0.04$ and only around 4% of the energy density required to make the Universe flat given the observed value of the Hubble constants $h \simeq 0.7$ is in the form of baryons.

Chapter 3

Dark Matter Astrophysics

3.1 Ways to Measure Mass $M(r)$ in Spherical Objects

3.1.1 The Virial Theorem

We are going to look at particles bound inside a gravitational potential well, in fact the particles are bound together by their mutual self attraction. So we are going to assume that the positions and the velocities of the particles are bound for all time.

We define a quantity called the "virial" G

$$G = \sum_i p_i \cdot r_i \tag{3.1}$$

where p_i is the momentum of the ith particle and r_i is its position. Then we can see that

$$\frac{dG}{dt} = 2T + \sum_i F_i \cdot r_i \tag{3.2}$$

where we have defined T as being the total kinetic energy and also that force is the rate of change of momentum. Now this is the weird/dodgy bit, but it makes sense if you buy. We want to find the *time average* of both sides.

$$\left.\frac{(G(t) - G(0))}{t}\right|_{lim_{t\to\infty}} = 0 \tag{3.3}$$

because we have assumed that the function $G(t)$ is bounded and $t \to \infty$ this has to be the case. Therefore

$$0 = 2\langle T\rangle + \langle\sum_i F_i \cdot r_i\rangle \tag{3.4}$$

so now we have a relationship between the time average of kinetic energy $\langle T\rangle$ and some other quantity. What is that other quantity? Force is derivative of potential so

$$\sum_i F_i \cdot r_i = \sum_{i\neq j} -\nabla\left(V_{ij}\right) \cdot r_i \tag{3.5}$$

where V_{ij} is the potential energy between the ith and jth particles.

$$\begin{aligned}
\sum_i F_i \cdot r_i &= \sum_j \left\{ \sum_{i<j} -\nabla\left(V_{ij}\right) \cdot r_i + \sum_{j<i} -\nabla\left(V_{ij}\right) \cdot r_i \right\} \\
&= \sum_j \left\{ \sum_{i<j} -\nabla\left(V_{ij}\right) \cdot r_i + \sum_{i<j} -\nabla\left(V_{ji}\right) \cdot r_j \right\} \\
&= \sum_j \left\{ \sum_{i<j} -\nabla\left(V_{ij}\right) \cdot \left(r_i - r_j\right) \right\}
\end{aligned} \tag{3.6}$$

where we switched the dummy variables i and j in the second step. We also used the fact that the force of the ith particle on the jth particle is the minus of the other way around.

Then since the gravitational potential between two particles $V_{12} = Gm_1m_2/r$ we can write

$$\nabla\left(V_{ij}\right) \cdot \left(r_i - r_j\right) = -V_{ij} \tag{3.7}$$

so that

$$\langle \sum_i F_i \cdot r_i \rangle = \langle \sum_{i<j} V_{ij} \rangle = \langle V \rangle \tag{3.8}$$

so that now we have $0 = 2\langle T \rangle + \langle V \rangle$ or rather

$$2\langle T \rangle = -\langle V \rangle \tag{3.9}$$

So if we have a gravitationally bound object made up of particles moving around we can measure the velocity of those particles and directly relate them to the overall gravitational potential. Note that $\langle T \rangle$ contains the mass of the things moving around, not just their velocities.

This was used to calculate the mass of the Coma Cluster by Fritz Zwicky and he noticed that there was too much motion for the stars which were visible. His numbers are now said to be too uncertain, but nevertheless, if one does the exercise now with state of the art observations, there is certainly a need for dark matter.

$$U = -\int \frac{GM(r)}{r} dm = -\int_0^R \frac{G4\pi\rho r^3}{3r} 4\pi r^2 dr = -\frac{16}{15}\pi^2 G\rho^2 R^5 = -\frac{9}{15}\frac{GM^2}{R} \tag{3.10}$$

so then we can use the virial theorem to say

$$\langle v^2 \rangle = \frac{3}{5}\frac{GM}{R} \tag{3.11}$$

so now if we look at a dwarf spheroidal galaxy which has a half-light radius of about 100 pc and maybe 10^5 stars in it, we expect

$$\langle v^2 \rangle = \frac{3}{5} 4.299 \times 10^{-6} \left(\frac{\mathrm{km}}{\mathrm{s}}\right)^2 \frac{\mathrm{kpc}}{M_\odot} \frac{10^5 M_\odot}{0.1\mathrm{kpc}} \tag{3.12}$$

which gives $\sqrt{\langle v^2 \rangle} \sim 1\mathrm{km\ s^{-1}}$. However, what we actually observe is closer to $\sigma \sim 10\mathrm{km\ s^{-1}}$ which infers the presence of a much larger component of dark matter.

3.1.2 Hydrostatic Equilibrium of x-ray gas

We know from stellar structure courses that

$$\frac{dP}{dr} = -\frac{GM(r)\rho}{r^2} \tag{3.13}$$

but then using the fact that $PV = NkT$ we can show that

$$\frac{dP}{dr} = \frac{kT}{\mu}\frac{d\rho}{dr} + \frac{\rho k}{\mu}\frac{dT}{dr} = -\frac{GM(r)\rho}{r^2} \tag{3.14}$$

Where we havd used that μ is the mass of a single particle, so that $\rho V = N\mu$. Then, by re-arranging we can see that

$$M(r) = -\frac{kT}{\mu G}\left(\frac{d\ln\rho}{d\ln r} + \frac{d\ln T}{d\ln r}\right) r \tag{3.15}$$

so that by measuring the density and temperature of gas in a cluster as a function of radius, one can obtain $M(r)$ and compare it to the ammount of baryons that can be seen. So one needs ρ and T as a function of radius, but luckily the hot gas is emitting x-rays via Thermal Bremsstrahlung radiation which has emmisivity of the form

$$I(\omega, T) \propto \frac{Z^2 n_e n_n}{\sqrt{T}} \exp\left(-\frac{\hbar\omega}{kT}\right) \tag{3.16}$$

and so it depends on T but also on the density of electons n_e and nuclei n_n. So one can obtain $M(r)$. It is very clear from hot gas in clusters that there is additional mass required in addition to baryons to explain the temperature of the gas.

3.1.3 The Jeans Equation

If we want to model galaxies in more detail we use the Jeans equation:-

$$\frac{1}{\nu}\frac{\partial\left(\nu\sigma_r^2\right)}{\partial r} + \frac{2\beta\sigma_r^2}{r} = -\frac{GM(r)}{r^2} \tag{3.17}$$

where the anisotropy parameter

$$\beta = 1 - \frac{\sigma_t^2}{\sigma_r^2} \tag{3.18}$$

where σ_t and σ_r are the tangential and radial velocity dispersions and ν is the density of stars. For a dwarf spheroidal, σ_r, ν and β (and therefore σ_t^2) all refer to the stellar population, while $M(r)$ is all the mass, and is dominated by dark matter, the stars only making up a very small fraction of the total mass. We are interested in $M(r)$ but we are unable to measure β, so we have to marginalise.

3.1.4 Rotation curves

The rotational velocity for orbits around a spherical mass is

$$V^2 = \frac{GM(r)}{r} \tag{3.19}$$

In the outer parts of galaxies, the density of visible stuff is non-zero, but it is negligible compared to the stuff close to the centre. So while it is not exactly a perfect approximation, one expects that in the outer part of the galaxy $v_{circ} \propto r^{-0.5}$. What is observed is quite different, the rotational velocity around galaxies drops much less rapidly than expected at large radii, often flattening out.

It is not true that all Galaxy rotation curves are flat, but certainly some of them are. Some drop a lot more slowly than you would expect. Some actually rise within the radii that they can be observed, note that the rotation curves in the out parts of galaxies are often obtained by looking at gas beyond the extent of the obvious disk. The vast majority of rotation curves seem to be at odds with the baryonic content of the galaxy.

3.1.5 The Isothermal Sphere

How would we obtain flat rotation curves anyway? We need $v_{rot}^2(r) = GM(r)/r =$ constant so we need $M \propto r$

$$M(r') = \int_0^{r'} 4\pi\rho(r)r^2 dr \tag{3.20}$$

and if we say that

$$\rho(r) = \rho_s \left(\frac{r}{r_s}\right)^\gamma \qquad \rightarrow \qquad M(r) = \frac{4\pi}{(3+\gamma)}\rho_s \frac{r^{3+\gamma}}{r_s^\gamma} \tag{3.21}$$

where ρ_s is the value of the density at some radius r_s. We can see that $\gamma = -2$ will give us the correct rotation curve behaviour, so that

$$\rho(r) = \rho_s \frac{r_s^2}{r^2} \tag{3.22}$$

is what we need for $M(r) \propto r$ and flat rotation curves. This is called the isothermal sphere, fancy name, you should ask yourself 'why'? The answer is pretty simple if we use the Jeans equation. We assume that $\nu = \rho$ and that the thing we are interested in getting the σ_r^2 for is the same thing that is responsible for the $M(r)$ in the $GM(r)/r^2$ term.

$$\frac{1}{\rho}\frac{\partial\left(\rho\sigma_r^2\right)}{\partial r} + \frac{2\beta}{r}\sigma_r^2 = -\frac{GM(r)}{r^2} \tag{3.23}$$

We are going to show that for the density we obtained above, the sphere has the same temperature throughout, in other words, the average kinetic energy is constant. That also infers that $\sigma_t = \sigma_r$ and indeed we set $\beta = 0$ which simplifies the equation

$$\frac{1}{\rho}\frac{\partial\left(\rho\sigma^2\right)}{\partial r} = -2\sigma^2\frac{\rho_s r_s^2}{\rho(r)r^3} + \frac{\partial\sigma^2}{\partial r} = -\frac{GM(r)}{r^2} = -4\pi G\frac{\rho_s r_s^2}{r} \tag{3.24}$$

and we can now see straight away that

$$\sigma^2 = 2\pi G\rho_s r_s^2 \leftrightarrow \frac{\partial\sigma^2}{\partial r} = 0 \tag{3.25}$$

and so $\langle v^2\rangle = \textit{constant}$ hence 'isothermal'.

3.2 Spherical Collapse

Just as in Newton's theory, if we focus on a spherically symmetric region in GR we can ignore what is outside it and see how it evolves on its own. Imagine an overdense spherically symmetric region containing only matter.

$$\frac{d^2r}{dt^2} = \frac{-GM}{r^2} \tag{3.26}$$

which has solution

$$
\begin{aligned}
r &= A\,(1-\cos\theta) &(3.27)\\
t &= B\,(\theta-\sin\theta) &(3.28)\\
A^3 &= GMB^2 &(3.29)
\end{aligned}
$$

where θ is a parametric angle which goes from 0 to 2π. So we have an equation for t and a which map out a cycloid, the motion of the point on a wheel as it moves along a plane. This

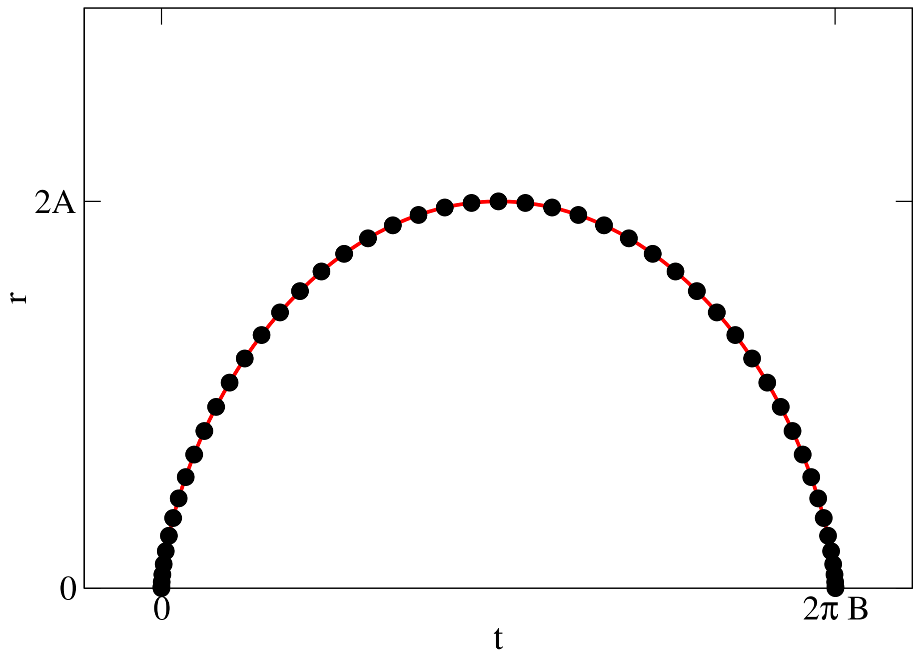

Figure 3.1: $r(t)$ for the overdense patch of Universe follows a cycloid pattern, schematically shown here. Dots are at evenly spaced values of θ.

corresponds to the size of the overdense region over time. You can think of the overdensity as being a seperate Universe which collapses in on itself. The maximum radius is at $r_{turn} = 2A$ which we call the turnaround radius. Now at that point the kinetic energy $T_{turn} = 0$ so we know all the energy is potential, in otherwords $E_{turn} = V_{turn} + T_{turn} = V_{turn} = -GM/r_{turn}$ whereas when virialised we know that $E_{vir} = T_{vir} + V_{vir}$ and $2T_{vir} = -V_{vir}$. We also know that energy is conserved so that $E_{vir} = E_{turn}$ which all taken together gives

$$E_{vir} = \frac{V_{vir}}{2} = -\frac{GM}{2r_{vir}} = E_{turn} = -\frac{GM}{r_{turn}} \tag{3.30}$$

which tells us the virialised radius $r_{vir} = r_{turn}/2 = A$ and that the density of the halo at that point

$$\rho_h = \frac{3M}{4\pi A^3} \tag{3.31}$$

Now complete collapse of the halo occurs at time $t_{vir} = 2\pi B$. We would like to ask ourselves, what is the density of the Universe when this happens? We assume that $\Omega_{tot} = \Omega_M = 1$ then

$$t = \frac{2}{3H} = \frac{2}{3}\sqrt{\frac{3}{8\pi G\rho}} = \sqrt{\frac{1}{6\pi G\rho}} \tag{3.32}$$

So that $\rho = (6\pi G t^2)^{-1} = (24\pi^3 G B^2)^{-1}$. The overdensity inside the halo is then

$$\frac{\rho_h}{\rho_{background}} = \frac{72\pi^3 G M B^2}{4\pi A^3} = 18\pi^2 = 178 \tag{3.33}$$

and you get slightly different answers depending upon the background cosmology you choose. However roughly speaking virialised halos have densities which are around 200 times the overall background density of the Universe at the point when they virialise. We would like to work out what redshift z_{vir} different halos virialise. If we say that the rough criterion is that

$$\rho_{vir} = 178 \times \rho_{M0} \left(1 + z_{vir}\right)^3 = 178 \times \frac{3\Omega_{M0} H_0^2}{8\pi G} \left(1 + z_{vir}\right)^3 \tag{3.34}$$

Then we have from the virial theorem

$$\langle \sigma^2 \rangle = \frac{GM}{R} \tag{3.35}$$

where now we have replaced v^2 with the velocity dispersion σ^2 which is basically the same thing.

$$\rho_{vir} = \frac{3M}{4\pi R^3} = \frac{3\sigma^6}{4\pi G^3 M^2} = 178 \times \frac{3\Omega_{M0} H_0^2}{8\pi G} \left(1 + z_{vir}\right)^3 \tag{3.36}$$

this gives us

$$(1 + z_{vir}) < X \left(\frac{\sigma}{100\text{kms}^{-1}}\right)^2 \left(\frac{10^{12} M_\odot}{M}\right)^{2/3} \left(\Omega_{m0} h^2\right)^{-1/3} \tag{3.37}$$

now calculate X.

So for the Milky way with $\sigma \sim 250\text{kms}^{-1}$ and $M \sim 10^{12} M_\odot$ we get $z_{vir} =$ and for Clusters with $\sigma \sim 1000\text{kms}^{-1}$ and $M \sim 10^{15} M_\odot$ we get $z_{vir} =$. So smaller halos form first.

3.2.1 Local Group Timing argument

We can apply these equations to the local group of the Milky Way and Andromeda (and triangulum etc.) Andromeda is about 0.75 Mpc away which is 2.3×10^{19} km while the age of the Universe is about 13.7 Gyr which is 4.3×10^{17}s. We observe the fact that Andromeda is moving towards us at 110 km/s.

$$r = A\left(1 - \cos\theta\right) \;\rightarrow\; \frac{dr}{d\theta} = A \sin\theta \tag{3.38}$$

$$t = B\left(\theta - \sin\theta\right) \;\rightarrow\; \frac{dt}{d\theta} = B\left(1 - \cos\theta\right) \tag{3.39}$$

Then we can write that

$$\frac{dr}{dt} = \frac{A \sin\theta}{B\left(1 - \cos\theta\right)} = \frac{2.3 \times 10^{19}\text{km}}{4.3 \times 10^{17}\text{s}} \frac{\sin\theta\left(\theta - \sin\theta\right)}{\left(1 - \cos\theta\right)^2} = 53 km/s \left[\frac{\sin\theta\left(\theta - \sin\theta\right)}{\left(1 - \cos\theta\right)^2}\right] \tag{3.40}$$

So we need the trigonometric factor in square brackets to be equal to about -2. This happens at about $\theta = 4.42$ which instantly gives us that $A = 1.8 \times 10^{19}$ km and that $B = 8.0 \times 10^{16}$ seconds. Now we remember that

$$M = \frac{A^3}{GB^2} = \frac{5.7 \times 10^{66} m^3}{6.67 \times 10^{-11} m^3 kg^{-1} s^{-2} 6.4 \times 10^{33} s^2} = 6.7 \times 10^{12} M_\odot \tag{3.41}$$

which is much more than we can see in the stars and gas of Andromeda and the Milky Way. Hence this is another argument for dark matter.

3.2.2 N-Body simulations

In order to simulate the distribution of dark matter, people generally use N-body simulations. If these simulations only contain dark matter and one assumes they are collisionless, this is simply a case of using Newton's inverse law of gravitation to update the positions and forces on all particles.

There are around 10^{11} stars in a galaxy like the Milky Way, and if the mass of the dark matter is around 100 GeV for example then there are about 10^{67} particles. Updating Newton's Law for every particle is a process which goes like $cputime \propto N^2$ so this would take a very long time! While it is possible to get this down to $cputime \propto N \ln N$, one still has to simulate a lot of mass with each particle, for example, the biggest simulations of the Milky Way have only around a billion particles in them, so each particle would represent a mass of 10^6 solar masses. That's a quarter of the mass of the central supermassive black hole.

If the dark matter starts without baryons then the smallest structures form first and the larger ones later. Typically in dark matter only simulations, the dark matter forms profiles with the functional form something like this

$$\rho(r) = \frac{\rho_s}{\left(\frac{r}{r_s}\right)\left(1 + \frac{r}{r_s}\right)^2} \tag{3.42}$$

which is called the Navarro-Frenk-White profile, normally known as the NFW profile. The parameter ρ_s is the density at the characteristic scale radius r_s where the density drops like r^{-2}. The density falls much less rapidly in the inner part of the halo compared to the outer part. Flat rotation curves, where $M(r) \sim \rho(r) r^3 \propto r$ must therefore occur close to the scale radius r_s. So we know that the scale radius must correspond to the very extent of the visible galaxy where the rotation curves are measured, showing that the dark matter halo must extend far beyond the baryonic part of the Galaxy.

You can define a virial radius, depending upon

$$\frac{3M(r_{vir})}{4\pi r_{vir}^3} = \rho_{av}(r < r_{vir}) = \Delta\rho_{crit} \tag{3.43}$$

where Δ is some overdensity that you have decided on, around 200 given the discussion above. It therefore marks the sphere within which the average density of the halo is bigger than the critical density by the factor Δ. You can also define a virial mass which is simply $M_{vir} = M(r_{vir})$.

The concentration parameter $c = r_{vir}/r_s$ is bigger for smaller haloes, reflecting the fact that the inner parts of halos retain some memory of the density of the Universe at the point they left the Hubble flow and collapsed.

3.2.3 The effect of Baryons on Dark Matter

The effect of baryons on the dark matter are mostly only important in the core. First one needs to talk about the effect of baryons. Baryons can lose energy, they get hot and emit photons. Stars form from gas and usually most of the mass of stars goes back into gas, so shocking and conservation of angular momentum can result in energy loss also. These factors together result in dark matter falling into the cores of galaxies and forming deep potential wells which draws in some dark matter, making it steeper.

3.3 Particle Properties of Dark Matter from Astrophysics

3.3.1 Free Streaming Length - Warm and Cold dark Matter

We have seen that small structures form first and then come together to conglomerate into larger structures. If dark matter is not cold then it will move a significant distance in the history of the Universe. If it moves in the early Universe, it will get further in today's Universe because it was expanding faster. The equation is

$$l_{FS} = \int_{t_{dec}}^{t_0} v(t) \frac{a_o}{a(t)} dt \tag{3.44}$$

where t_0 is today and t_{dec} is when the dark matter froze out of equilibrium with the standard model. Because of this, smaller structures will be wiped out if dark matter is produced very hot.

We can approximate this equation by saying that

$$l_{FS} \sim \int_0^{t(T=m)} c \frac{a_o}{a(t)} dt \tag{3.45}$$

where we assume that as long as the particle is relativistic it moves at speed $v = c$ but when it becomes non-relativistic, its motion is negligible and $v = 0$. Going to natural units with $c = 1$ we then have

$$l_{FS} \sim \int_0^{t(T=m)} \frac{a_o}{a(t)} dt = \int_{z=\infty}^{z(T=m)} \frac{dz}{H(t)} \tag{3.46}$$

We then use the fact that during the radiation dominated era, $H = 1.66\sqrt{g}T^2/M_{Pl}$. This comes from $H^2 = 8\pi G\rho/3$ and from $M_{Pl} = 1/\sqrt{G}$ and from equation (2.36). Also we know that (approximately) $T = T_0 a_0/a = T_0(1+z)$ then since $(1+z) \sim z$ at these high redshifts. In the following we will also use that $T_0 = 2.7K \sim 2 \times 10^{-4}$eV.

$$\begin{aligned} l_{FS} \sim \frac{M_{Pl}}{1.66\sqrt{g}T_0^2} \int_{z=\infty}^{z(T=m)} \frac{dz}{z^2} \sim 0.4 \frac{M_{Pl}}{T_0^2 z(T=m)} &= 0.4 \frac{10^{19}\text{GeV}}{(2\times 10^{-4}\text{eV})^2} \frac{2\times 10^{-4}\text{eV}}{m_\chi} \\ &\sim 10^{21}\text{m} \left(\frac{1\text{keV}}{m_\chi}\right) \end{aligned} \tag{3.47}$$

Now the density of matter in the Universe today is

$$\rho_{DM} = \Omega_M \frac{3H_0^2}{8\pi G} = 1.9 \times 10^{-26} \Omega_M h^2 \text{kgm}^{-3} \sim 3 \times 10^{-27} \text{kgm}^{-3} \tag{3.48}$$

So that the mass within 10^{21} m is about 10^{36} kg which is about $10^6 M_\odot$. Since the smallest objects in the Universe which we know are made of dark matter have about this mass, we know that the dark matter cannot have a mass less than about a keV if it was once in thermal equilibrium, because if it did then those objects would have been washed out.

More generally, this tells us that dark matter which is faster in the early Universe, or HOT dark matter is less favoured than dark matter which was colder in the early Universe, COLD dark matter. If the dark matter is somewhere in between then it is WARM dark matter. This may explain why we don't see as many dwarf spheroidal galaxies as we expect (the "missing satellite problem"), but it might be that they just don't have enough baryons in them because supernovae may have blown the baryons out.

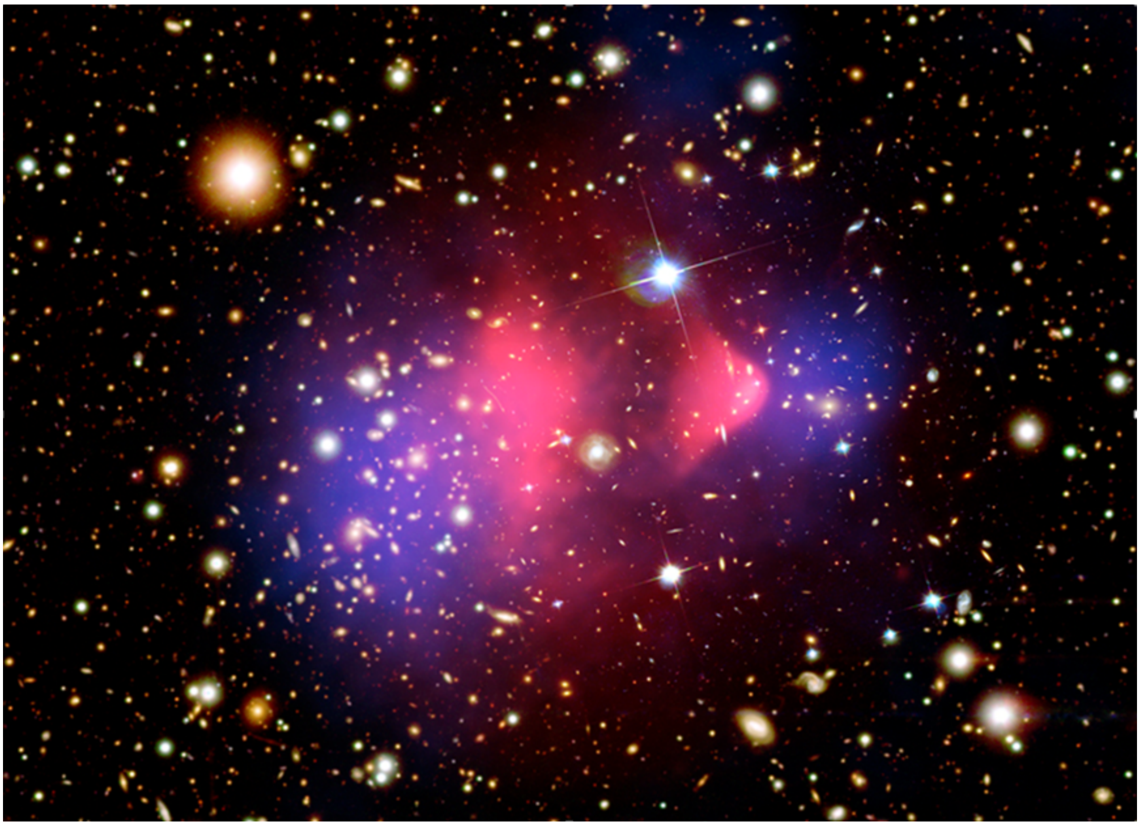

Figure 3.2: The Bullet Cluster. Optical images are overlaid with x-ray images (in red) and gravitational lensing images (in blue)

3.3.2 Self Interaction of Dark Matter - The Bullet Cluster

The Bullet Cluster is a very important piece of evidence which it is argued strongly supports the existance of dark matter. The Bullet Cluster is in fact thought to be two clusters which collided, the smaller cluster passing through the larger one. The name comes from x-ray images which show that the x-ray gas in the smaller cluster punched a hole in the larger cluster, leaving behind a characteirtic shockwave reminiscent of the damage done to materials when they are shot with a bullet.

It is clear that the gas in between the galaxies of these clusters is no longer in the same place as the galaxies associated with the clusters. The interpretation is that the galaxies have missed each other during the collision and passed straight through, while the intergalactic gas (the intracluster gas) from one cluster has collided with that from the other and slowed down in the process, so is lagging behind the cluster.

In clusters of galaxies, there are more baryons in the form of intergalactic gas rather than in the galaxies themselves, so most of the baryonic mass of the two colliding galaxies will be where the gas is. However, when we do gravitational lensing to see where the actual gas is, we see that it is where the galaxies are, not where the gas is. The interpretation of this observation is that while the gas has collided with itself, the galaxies haven't, and there is an additional component of mass which also hasn't collided with itself and is no longer in the same place as the gas. This is often cited as strong evidence against MOND, but also allows us to place a constraint on the self interaction of dark matter.

Lensing suggests that the core of the smaller cluster has a radius of 70 kpc and a density of around $\rho = 1.3 \times 10^{-25}$g cm^{-3}. It has an escape velocity of around 1900 km s^{-1} and observations of the gas dynamics suggest it crashed through the larger cluster at a speed of 4800 km s^{-1}.

There are various ways of using these observations to place a constraint on the self interaction

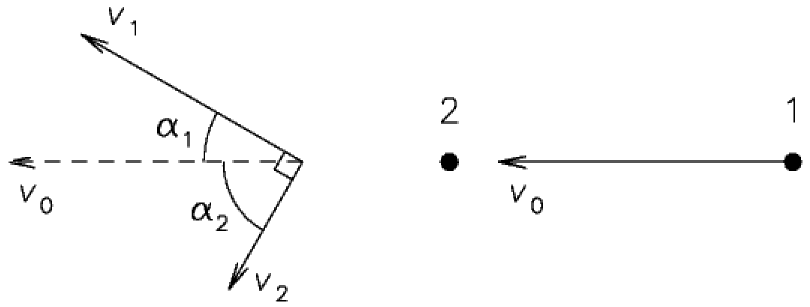

Figure 3.3: The Collision of two dark matter particles in the rest frame of the smaller cluster.

cross section of dark matter, all of which give answers which are similar to within an order of magnitude or so. I am going to present what I think is the simplest one, which involves the fact that the smaller cluster survives the collision and still exists on the other side, having punched through the larger cluster.

Let's look at the collision of two equal mass dark matter particles in the rest frame of the subcluster as shown in Fig. 3.3. In the smaller cluster's reference frame, particle 2 is at rest and particle 1, in the incoming flow, collides with it with a velocity

$$v_0 \approx 4800 \text{ km s}^{-1}, \tag{3.49}$$

where v_0 is the smaller cluster's velocity. Particle 1 scatters at an angle $\alpha_1 > 0$ with a velocity v_1, while particle 2 acquires a velocity v_2 at an angle $\alpha_2 > 0$. From energy and momentum conservation, $\alpha_1 + \alpha_2 = \pi/2$ and we can see that the velocities after scattering will be given by

$$v_1 = v_0 \cos\alpha_1, \quad v_2 = v_0 \sin\alpha_1. \tag{3.50}$$

Now if the collision results in one of the particles escaping and the other remaining in the smaller cluster then this interaction will not reduce the probability of the smaller cluster surviving. We would like to look at situations where both dark matter particles are given enough velocity to escape from the smaller cluster, consequently

$$\begin{aligned} \frac{v_1^2}{v_0^2} = \cos^2(\alpha_1) = 1 - \sin^2(\alpha_1) &> \frac{v_{esc}^2}{v_0^2} \\ \frac{v_2^2}{v_0^2} = \sin^2(\alpha_1) &> \frac{v_{esc}^2}{v_0^2} \end{aligned} \tag{3.51}$$

So that the criterion to have both particles obtaining a velocity greater than the escape velocity of the smaller cluster can be written

$$\frac{v_{esc}^2}{v_0^2} < \sin^2(\alpha_1) < 1 - \frac{v_{esc}^2}{v_0^2} \tag{3.52}$$

Now we go to the centre of mass frame of the particle collision. In this reference frame, particle 1 scatters by an angle $\theta = 2\alpha_1$ $(0 < \theta < \pi)$ and particle 2 scatters by $\pi - \theta$, so long as we maintain the definition and sense of α_1 and α_2 as set out in figure 3.3. The criterion set out in equation (3.52) is now

$$\frac{v_{esc}^2}{v_0^2} < \sin^2(\theta/2) < 1 - \frac{v_{esc}^2}{v_0^2} \tag{3.53}$$

and in this frame, we expect the scattering to be isotropic. Because of this, we need to take into account the fact that there is more solid angle around $\theta = \pi/2$ than around $\theta = 0$ or $\theta = \pi$

so we need to do a weighted integral in the normal way. Then the probability of ejection of both particles per collision is given by

$$\chi = \frac{\int_{\theta_{min}}^{\theta_{max}} 2\pi \sin\theta d\theta}{\int_0^{\pi} 2\pi \sin\theta d\theta} \tag{3.54}$$

where θ_{min} and θ_{max} will be determined by equation (3.53). We can solve this neatly

$$\begin{aligned}\int_{\theta_{min}}^{\theta_{max}} 2\pi \sin\theta d\theta &= 2\pi\left[-\cos\theta\right]_{\theta_{min}}^{\theta_{max}} = 2\pi\left[2\sin^2(\theta/2) - 1\right]_{\theta_{min}}^{\theta_{max}} \\ &= 2\pi\left(2 - 2\frac{v_{esc}^2}{v_0^2} - 1 - 2\frac{v_{esc}^2}{v_0^2} + 1\right) = 4\pi - 8\pi\frac{v_{esc}^2}{v_0^2}\end{aligned} \tag{3.55}$$

and since the bottom integral in χ is just 4π we have

$$\chi = 1 - 2\frac{v_{esc}^2}{v_0^2} \tag{3.56}$$

Now the optical depth τ of the oncoming dark matter wind for a single dark matter particle gives a measure of the probability of a dark matter particle in the smaller cluster colliding with one from the bigger cluster (the probability of not being hit will be $\exp(-\tau)$)

$$\tau = \frac{\sigma}{m}\Sigma \tag{3.57}$$

where m is the mass of the dark matter particle, σ is its self interaction cross section and Σ is the surface density of the larger cluster, around 0.3 g cm^{-2}.

Now we would like to ensure that the smaller cluster does not lose more than 30% of its mass, since the smaller cluster's overall mass to light ratio is about right for that kind of object. The probability of being hit is $1 - \exp(-\tau)$ and for $\tau \ll 1$, $1 - \exp(-\tau) \sim \tau$. We therefore require that $\chi\tau < 0.3$ which leads to

$$\frac{\sigma}{m} < 1\mathrm{cm}^2\mathrm{g}^{-1} \tag{3.58}$$

Which is actually quite a large cross section, for example, it corresponds to about 10^{-24}cm^2 for a GeV mass dark matter particle. This is very similar to the cross section corresponding to the nuclear force between atoms, so for example, neutrons would scatter off each other with this cross section (protons have electric charge so there would be a long range force.)

Note this assumes that the cross section is short range, in other words that the particles are a bit like little ping-pong balls. Technically, this corresponds to a Yukawa, short range interaction when the force carrying boson has a mass which sets the cross section. In the case where one considers long range forces, for example endowing the dark matter with a dark electric charge, the analysis is very much more complicated.

3.4 Measuring Matter Density - Galaxy Number Counts

We take a galaxy cluster and measure how many galaxies lie with an Abell radius $r_A \sim 1$ Mpc which is the typical scale of a galaxy cluster. Then we look at the quantity

$$\frac{N(< r_A)}{n r_A^3} \tag{3.59}$$

where n is the number density of galaxies throughout the Universe. This therefore is an indication of the overdensity of galaxies in clusters relative to the average throughout the Universe. We can the measure the velocity distribution of galaxies in clusters of size r_A which is measured to be

$$v^2 \simeq \left(950kms^{-1}\right)^2 (1 \pm 0.68) \tag{3.60}$$

then if we assume that the mass in the cluster is an isothermal sphere

$$M_d(< r_A) = \frac{2v^2 r_A}{G} = 4 \times 10^{14} h^{-1} M_\odot \tag{3.61}$$

Now if the galaxies trace the large scale mass distribution we can say that

$$\frac{N(< r_A)}{nr_A^3} \simeq \frac{M(< r_A)}{\rho_M r_A^3} \tag{3.62}$$

which gives us ρ_M, the density of matter. Typically we get values of the order of $\Omega_M \sim 0.3$ and we get similar results from other methods. This used to be a very important observation and for a long time in the 1980s and 1990s, before anyone thought $\Omega_{tot} = 1$ or $\Omega_\Lambda \sim 0.7$ they were pretty convinced that $\Omega_M \sim 0.3$ due to methods like this.

3.5 Modified Newtonian Dynamics (MOND)

The idea behind MOND is that rather than invoking the presence of dark matter to explain the behaviour of galaxies, we should instead use the fact that stars are moving in galaxies counter to our expectations because our theory of gravity is wrong for those regimes. We should use the observations to derive a modification to the normal Newtonian (Einsteinian) laws of gravity for very low accelerations. We write the normal acceleration due to gravity as

$$g_N = \frac{GM}{r^2} \tag{3.63}$$

then the idea behind MOND is that for relatively large gravitational accelerations we follow this, but for smaller accelerations, we follow a different law. In other words, if we write the acceleration as a then

$$\begin{aligned} a &\to g_N \qquad a \gg a_0 \\ a &\to \sqrt{g_N a_0} \qquad a \ll a_0 \end{aligned} \tag{3.64}$$

where a_0 is some reference acceleration that seperates the Newtonian regime from the MOND regime. This reference acceleration is very small, typically $a_0 \sim 10^{-10}$m s^{-2} The transition between the two regimes is set by some smooth function such that

$$\begin{aligned} \mu &\to 1 \qquad x \gg 1 \\ \mu &\to x \qquad x \ll 1 \end{aligned} \tag{3.65}$$

where $x = a/a_0$ and $\mu(x) = x/(1 + x)$ or $1 - exp(-x)$ or some appropriate function. Then you can write

$$a\mu(x) = \frac{GM}{r^2} \tag{3.66}$$

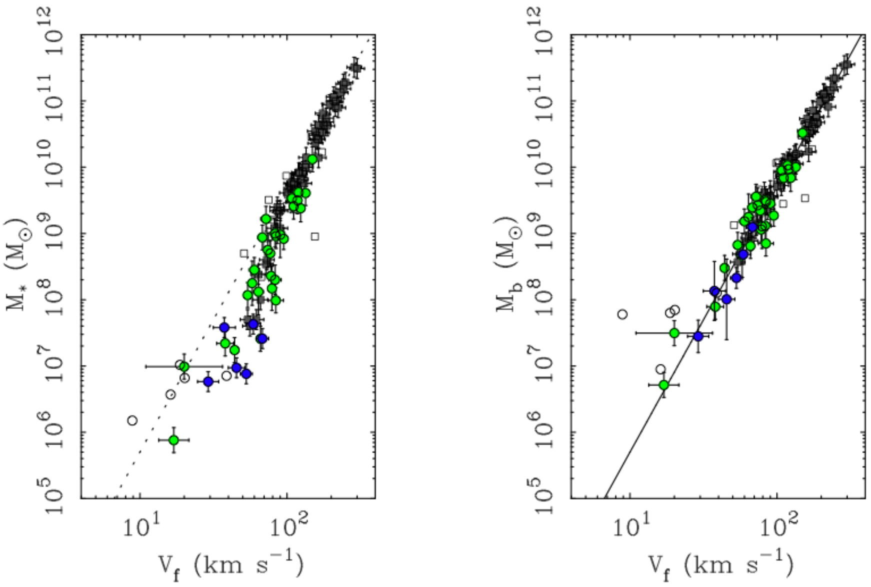

Figure 3.4: The stellar mass (left) and baryonic (right) Tully-Fisher relations. The baryonic mass is the sum of stellar and gas mass.

3.5.1 What does MOND do for you?

There are various features of Galaxies that it is claimed are fitted better by MOND, but the most obvious ones are flat rotation curves in the outer parts of galaxies. This can be seen very obviously by writing the equations for circular motion

$$\frac{v^2}{r} = a = \begin{cases} \frac{GM}{r^2} \to v^2 = \frac{GM}{r} & a \gg a_0 \\ \frac{\sqrt{a_0 GM}}{r} \to v^4 = a_0 GM & a \ll a_0 \end{cases} \tag{3.67}$$

which would explain flat rotation curves in the outer parts of galaxies quite well. MOND cannot explain the need for dark matter on cluster scales or on cosmological scales, although some people suggest that it does work better than dark matter on galactic scales. One thing which MOND does do well is that it explain an observed relationship between the rotational velocity of galaxies and their baryonic mass known as the Tully-Fisher relationship. This could in principle be solved by a combination of baryonic contraction, feedback etc with just dark matter and baryons, but this is complicated non-linear physics. The fact that MOND solves the problem without any complications is remarkable, in the sense that one should remark and be aware that it does.

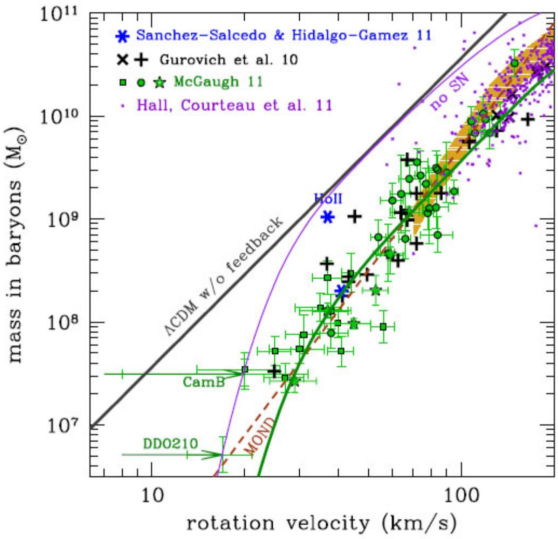

Figure 3.5: Baryonic Tully-Fisher relation (Mamon & Silk). Measurements are from HI measurements, where the velocity is the flat part of the rotation curve or from line-widths. The grey line is the naive dark matter prediction with no feedback, while the brown dashed line is the prediction from MOND. The green curved line comes from a model which includes baryonic feedback with three free parameters, vs. the one free parameter of MOND

3.6 Summary of Evidence for Dark Matter

- Galaxy Velocity Rotation Curves are higher than they should be in outer regions
- Velocity dispersion of stars indicates high mass:luminosity ratio in dwarf Galaxies
- Temperature of X-ray Gas in clusters > than what can be explained by baryonin matter
- Virial theorem applied to galaxies in clusters
- Galaxy Number counts
- The Bullet Cluster
- The fact that Ω_b measusred from Nucleosynthesis < what is required to explain H_0 or even Galaxy Number counts
- Gravitational Lensing of Galaxy Clusters compared to Baryonic content
- much more...

Chapter 4

WIMPS and Thermal relics

4.1 More on Freeze Out

Assume there is some species χ in the early Universe. n_χ is the number density of χ particles. If χ is in equilibrium

$$n_\chi = \frac{g_\chi}{(2\pi)^3} \int f_i(\vec{p_i}) d^3p \tag{4.1}$$

but χ is not necessarily in equilibrium. N.B. equilibrium is when the particle is created and annihilated at a rate Γ much greater than the expansion of the universe. This depends on the mass of the particle and the coupling (annihilation rate) of the particle with the rest of the plasma.

χ particles annihilate...

$$\chi + \bar{\chi} \to \sum_i x_i + \bar{x}_i \propto n_\chi^2 \sum_i \langle \sigma_{\chi\bar{\chi}\to\sum_i x_i\bar{x}_i} |\vec{v}| \rangle \tag{4.2}$$

where x_i are all the other particles in the plasma. The $\langle \sigma_{\chi\bar{\chi}\to\sum_i x_i\bar{x}_i} |\vec{v}| \rangle$ is referred to as the thermal averaging, so the cross section can depend upon energy, so for energetic particles hitting energetic particles, you may get a different cross section to slow particles hitting slow particles. Here we are integrating over the Maxwell Boltzmann distribution of both particles. The χ particles are also created from the plasma.

$$\sum_i x_i + \bar{x}_i \to \chi + \bar{\chi} \propto n_\chi^2 \sum_i \langle \sigma_{\sum_i x_i\bar{x}_i\to\chi\bar{\chi}} |\vec{v}| \rangle \tag{4.3}$$

At high T $\sigma_{\sum_i x_i\bar{x}_i\to\chi\bar{\chi}} = \sigma_{\chi\bar{\chi}\to\sum_i x_i\bar{x}_i} = \sigma_A$ so that if we consider the total number of particles n_χ in a single Unit of volume which is not expanding

$$\left.\frac{dn_\chi}{dt}\right|_{annihilation} = -\left(n_\chi^2 - n_{\chi EQ}^2\right) \langle \sigma_A |\vec{v}| \rangle \tag{4.4}$$

where the distinction $|_{annihilation}$ will be explained shortly. This shows us that the number density will follow the *thermal equilibrium number density* as set out in equation (2.33)

$$n_{\chi EQ} = g_\chi \left(\frac{m_\chi T}{2\pi}\right)^{3/2} e^{-m_\chi/T} \tag{4.5}$$

and if $n_\chi \neq n_{\chi EQ}$ then it will move towards it at a rate $\Gamma = n_\chi \langle \sigma_A |\vec{v}| \rangle$. Of course we have argued that this tracking of thermal equilibrium will only take place if $\Gamma \geq H$, and in order to see this explicitly in this equation we need to take into account the expansion of the Universe.

$$\left.\frac{dn}{dt}\right|_{total} = \frac{d}{dt}\left(\frac{N}{V}\right) = \frac{d}{dt}\left(\frac{N}{a^3}\right) = \frac{1}{a^3}\frac{dN}{dt} - 3\frac{N}{a^4}\frac{da}{dt} = \left.\frac{1}{a^3}\frac{dN}{dt}\right|_{annihilation} - 3Hn \tag{4.6}$$

One can see it this way, or one can simply use the covariant derivative, both of which which give the same answer

$$\frac{\nabla n_\chi}{dt} = \frac{dn_\chi}{dt} + 3Hn_\chi = -\langle\sigma_A|\vec{v}|\rangle\left(n_\chi^2 - n_{\chi EQ}^2\right) \tag{4.7}$$

So now we can see that the number density n_χ will only track $n_{\chi EQ}$ if it is not overwhelmed by the expansion term, which effectively damps the ability of the number density to keep up with the equilibrium number density

Now we define $Y = n_\chi/s$ where s is the entropy density

$$s = \frac{2\pi^2}{45} g_{eff} T^3 \tag{4.8}$$

so that

$$\frac{dY}{dt} = \frac{1}{s}\frac{dn_\chi}{dt} - \frac{n_\chi}{s^2}\frac{ds}{dt} \tag{4.9}$$

Now the entropy s =constant$\times T^3$ so that

$$\frac{1}{s}\frac{ds}{dt} = \frac{3}{T}\frac{dT}{dt} = -\frac{3}{a}\frac{da}{dt} = -3H \tag{4.10}$$

which means

$$\frac{dY}{dt} = \frac{1}{s}\left(\frac{dn_\chi}{dt} + 3Hn_\chi\right) = -\langle\sigma_A|\vec{v}|\rangle s\left(Y^2 - Y_{EQ}^2\right) \tag{4.11}$$

Next we introduce $x = m_\chi/T$ and note that

$$\frac{dx}{dt} = \frac{dx}{dT}\frac{dT}{dt} = -\frac{m_\chi}{T^2}\frac{dT}{dt} = \frac{m_\chi H}{T} = xH \tag{4.12}$$

which means we can write

$$\frac{dY}{dx} = -\frac{\langle\sigma_A|\vec{v}|\rangle s}{xH}\left(Y^2 - Y_{EQ}^2\right) \tag{4.13}$$

Which is the equation we will be solving. It is illustrative to divide by Y_{EQ} and multiply by x to obtain

$$\frac{x}{Y_{EQ}}\frac{dY}{dx} = -\frac{\Gamma}{H}\left[\left(\frac{Y}{Y_{EQ}}\right)^2 - 1\right] \tag{4.14}$$

where $\Gamma = n_{\chi EQ}\langle\sigma_A|\vec{v}|\rangle$ so when Γ drops below H we get freeze out, Y cannot keep up with Y_{EQ} and the co-moving (i.e. per unit r^3 not $a^3(t)r^3$) density is frozen. This is what we have been saying about when Γ drops below H particles freeze out. We can use this to find x_F the point of freezeout. We can do better than this though. In principle we may expect some energy dependance in our cross section so we can write $\langle\sigma_A|\vec{v}|\rangle = \sigma_0 x^{-n}$ then

$$\frac{dY}{dx} = -\frac{\lambda}{x^{n+2}}\left(Y^2 - Y_{EQ}^2\right) \tag{4.15}$$

where

$$\lambda = \left[\frac{\langle\sigma_A|\vec{v}|\rangle s}{xH}\right]_{x=1} = \frac{\sigma_0 s(T = m_\chi)}{H(T = m_\chi)} = \sqrt{\frac{\pi g_{eff}}{45}} M_{Pl} m_\chi \sigma_0 \tag{4.16}$$

Where the Planck mass is defined as as $M_{Pl} = 1/\sqrt{G} = 1.221 \times 10^{19}$ GeV when $\hbar = c = 1$. Then if we write $\Delta = Y - Y_{EQ}$ then while in equilibrium, $\Delta = 0$ and later $\Delta \sim Y >> 0$. One can write a differential equation for Δ

$$\frac{d\Delta}{dx} = -\frac{dY_{EQ}}{dx} - \frac{\lambda}{x^{n+2}}\Delta\left(2Y_{EQ} + \Delta\right) \tag{4.17}$$

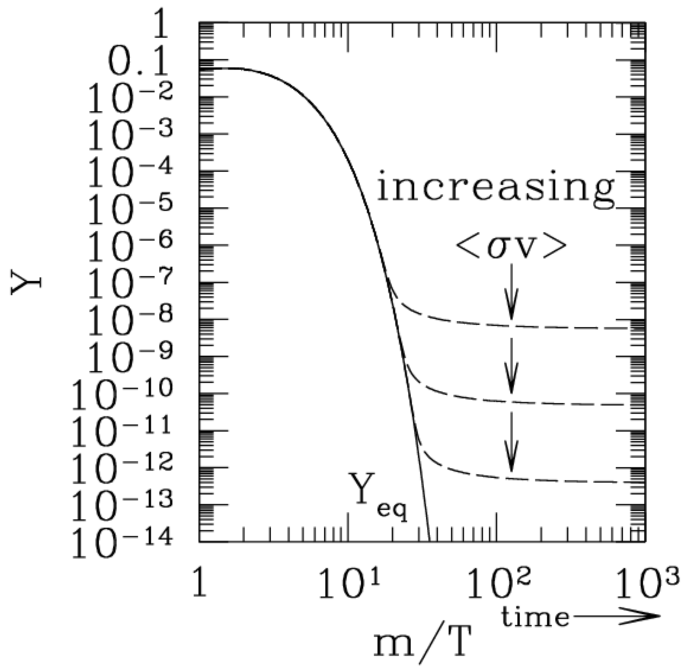

Figure 4.1: Freeze out, we can see the number density following, and then failing to follow the equilbrium number density.

At late times when $x >> x_F$ then $\Delta \simeq Y \gg Y_{EQ}$ and terms with Y_{EQ} and its derivative cane be dropped. Then

$$\frac{d\Delta}{dx} = -\frac{\lambda}{x^{n+2}}\Delta^2 \tag{4.18}$$

which we can integrate from $x = x_F$ to $x = \infty$ to get

$$Y_\infty = (n+1)x_F^{n+1}/\lambda. \tag{4.19}$$

To get x_F we need to set $H = \Gamma$

$$H = \sqrt{\frac{1}{5}\frac{2}{3}}\pi^{3/2}\sqrt{G}\sqrt{g}\frac{m_\chi^2}{x^2} = \Gamma = \frac{g_\chi}{(2\pi)^{3/2}}m_\chi^3 x^{-3/2}e^{-x}\sigma_0 x^{-n} \tag{4.20}$$

which gives us

$$x_F^{1/2-n}e^{-x_F} = \frac{8\pi^3}{3\sqrt{10}}\sqrt{Gg_{eff}}\frac{1}{g_\chi m_\chi \sigma_0} \tag{4.21}$$

Now it turns out the value of Y we are looking for

$$Y_\infty = \frac{n_\chi}{s} = \frac{\rho_\chi}{m_\chi s} = \frac{\rho_c \Omega_{DM}}{m_\chi s} = Y_\infty \sim 0.4\left(\frac{\text{eV}}{m_\chi}\right) \tag{4.22}$$

The entropy density of the Universe today is 2892 cm^{-3} and the density of dark matter is around 1251 eV cm^{-3} so this is why we need $Y_\infty \sim 0.4(\text{eV}/m_\chi)$.

Now, lets assume that $M_\chi = 100$ GeV and then $Y_\infty = 4 \times 10^{-12}$.

then that this will mean (as we will see) $T_F \sim 5$ GeV. At that temperature we have all the leptons, but only 5 of the quarks, so $g_{eff} = 91.5$. Let us also assume there are only two dark matter degrees of freedom so that $g_\chi = 2$. If we also assume that $n = 0$ we have

$$\lambda = 2.53 M_{Pl} m_\chi \sigma_0 \quad ; \quad x_F^{1/2}e^{-x_F} = \frac{125}{M_{Pl}m_\chi\sigma_0} = \frac{316}{\lambda} \quad ; \quad x_F = 4 \times 10^{-12}\lambda \tag{4.23}$$

where the first equation comes from (4.16), the second equation comes from (4.21) and the third equation comes from both (4.19) and (4.22). So we have two relations for λ as a function of x_F, we plot them both and see where they cross - around $x_F = 25.3$. (Sometimes they cross twice, in which case there are two solutions which give good relic density) Then we know that

$$\lambda = \frac{x_F}{Y_\infty} \to \sigma_0 = \frac{6.33 \times 10^{12}}{2.53 M_{Pl} m_\chi} = \frac{2 \times 10^{-9}}{\mathrm{GeV}^2} = 7.7 \times 10^{-37}\mathrm{cm}^2 \tag{4.24}$$

which we will come back to later. Note this infers that

$$\langle \sigma_A |\vec{v}| \rangle \sim 4.6 \times 10^{-27}\mathrm{cm}^3\mathrm{s}^{-1} \tag{4.25}$$

where we have use the fact that $v/c \sim \sqrt{x_F}$

We have seen that the amount of matter which is left over after freeze-out therefore depends upon its self annihilation cross section and its mass. We need to look at the kind of particles which could give such a cross section.

4.2 Cross Section estimates

4.2.1 Massless Bosons

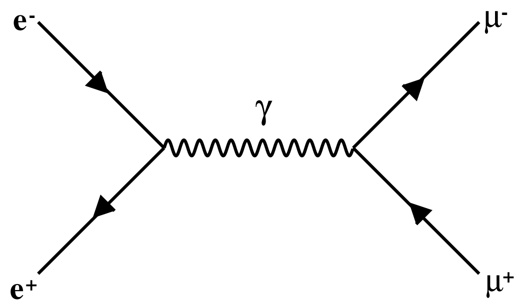

Figure 4.2: Electron-positron annihilation to form a muon and an anti-muon via a virtual photon.

If we look at figure 4.2 we can see that we have two incoming electrons, to be precise one electron and one positron (anti-electron), annihilating to form two muons. First thing to say is that since the mass of an electron is less than the mass of a muon, this would only take place if they have some high initial momentum. The probability of this process occuring depends upon the coupling between photons and electrons which is given by $\sqrt{\alpha_{em}} = e/\sqrt{4\pi}$ where e is the charge of the electron. It also depends upon the coupling between the photon and muons, which is also given by $\sqrt{\alpha_{em}}$.

There is another thing we need to worry about though, in the centre of momentum frame, once the electrons annihilate the total energy of the four vector of the system would have the form $p^\mu = (\sqrt{s}, 0, 0, 0)$. Here $\sqrt{s}$ is the centre of mass energy $s = 2|\vec{p}|^2 + 2m_e^2$ where $\vec{p}$ is the initial momentum of the electrons, which will be equal and opposite by definition in the centre of momentum frame.

Now this energy momentum at the moment the electrons annihilate $p^\mu = (\sqrt{s}, 0, 0, 0)$ is not a good momentum to be carried by photons. It is the momentum of a stationary particle with mass $\sqrt{s}$, stationary such that it is on a time-like trajectory through space-time. Photons have no mass which means they cannot stay still, you cannot get into a Lorentz frame where their trajectory is time-like, they have to travel through space as much as time. Because of this, we know the photon γ will be a *virtual* particle, it will have to "borrow" momentum-energy from the vacuum and then give it back within a short $\Delta t \sim \hbar/\Delta E$. This becomes more difficult as $\sqrt{s}$ increases. The overall difficulty needs to be reflected in the probability of the process occuring, so we have a suppression factor called the *propagator* since it is associated with the probability of the photon propagating from the left to the right. This propagator has the form for bosons $P(s) = 1/\left(s - m_{bos}^2\right)$ where the mass of the boson m_{bos} is zero for a massless photon. We then, going from left to right, obtain the amplitude $\sqrt{\alpha} \times (1/s) \times \sqrt{\alpha}$ or a probability / i.e. a cross section which is proportional to the amplitude squared

$$\sigma = const \times \phi \frac{\alpha^2}{s^2} \tag{4.26}$$

where ϕ is some factor related to the phase space available for the process to take place. If we were doing the calculation properly we would obtain the full expression for the phase space, but we can often do quite well just by looking at the dimension and knowing that we expect the cross section have dimension length squared so GeV^{-2} in natural units.

(These kind of naive estimations can go badly wrong if helicities of the particles create problems. However this is not what is happening here.) So we end up with

$$\sigma_{e^+e^- \to \mu^+\mu^-} \sim \frac{\alpha^2}{s} \tag{4.27}$$

while the true answer is $4\pi\alpha^2/(3s)$ (when the centre of mass energy is much larger than the muon mass, otherwise we need to take that into account of course) so our estimate is pretty good.

4.2.2 Massive Bosons

If we have a massive propagator then we need to include that too as it will change when the boson is happy to carry away the energy momentum of the collision, and when it isn't, so for example for the diagram in figure 4.3 we have a cross section

$$\sigma_{e^+\nu_e \to \mu^+\nu_\mu} \sim \frac{\alpha_w^2 s}{\left(s - M_W^2\right)^2} \tag{4.28}$$

so that this rises at low energies, becomes very large when $\sqrt{s} \sim M_W$ and then dies down again at high energies. Note here we have a slightly different value of α as it is related to α_{em} via the weak mixing angle but it is about the same. At $\sqrt{s} = M_W$ it looks like the cross section becomes singular, but actually what happens is that there is no virtuality about the particle at all - it is created 'on shell' - it has precisely the energy and momentum that it wants to

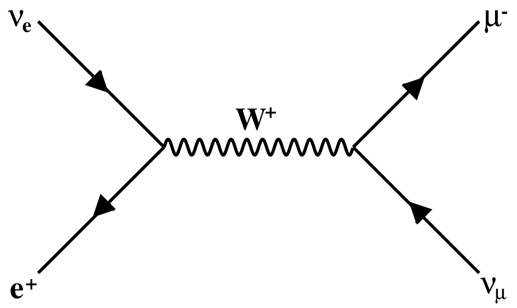

Figure 4.3: Positron-neutrino annihilation to form an anti-muon and via a W^+ boson which can be virtual or not depending upon $\sqrt{s}$.

have given its mass M_W and the centre of mass $\sqrt{s}$ so it exists happily as a classical particle. What then determines what happens is how long it takes for the particle to decay. If you think about the classical wave equation for a massive particle, adding an imaginary term to the mass will make the amplitude decrease exponentially over time, which when we quantize looks like a decay. In fact Γ is the decay rate of the particle, in other words it is the inverse of its lifetime.

So you add a width to the propagator $M_W \to M_W - i\Gamma_W/2$ and then

$$\left(s - M_W^2\right) \to \left[s - (M_W - i\Gamma_W/2)^2\right] = s - M_W^2 + iM_W\Gamma_W + \Gamma_W^2/4 \sim s - M_W^2 + iM_W\Gamma_W \quad (4.29)$$

where we have dropped the $(1/4)\Gamma_W^2$ term as we assume that $\Gamma \ll M_Z$ in order to have a well defined resonance. We can then write

$$\sigma_{e^+\nu_e \to \mu^+\nu_\mu} \sim \left|\frac{\alpha_w\sqrt{s}}{(s - M_W^2 + i\Gamma_W M_W)}\right|^2 = \frac{\alpha_w^2 s}{(s - M_W^2)^2 + M_W^2\Gamma^2} \quad (4.30)$$

where Γ_W is the decay rate of the W-boson into other particles. The width Γ is related to the uncertainty in the mass of unstable particles such as the W-boson, because it is shortlived, there is uncertainty in its energy and this makes it easier to produce from a variety of $\sqrt{s}$ around its actual mass.

Note that as before, we are assuming that the centre of mass energy is high enough that we don't need to worry about being able to create the muon, we are assuming that we have plenty of energy (the muon mass is 105 MeV). If we did need to worry about the muon mass then we would replace the factor of s on the top with something which reflected the phase space more accurately which would look like this:-

$$\sigma_{e^+\nu_e \to \mu^+\nu_\mu} \sim \frac{\alpha_w^2}{(s - M_W^2)^2 + M_W^2\Gamma^2}\frac{\left(s - m_\mu^2\right)^2}{s} \quad (4.31)$$

We can imagine the dark matter is a heavy neutrino, then the annihilation rate would be given by

$$\sigma_{\nu\bar{\nu}\to Z\to ALL} \sim \left|\frac{\alpha\sqrt{s}}{(s-M_Z^2+i\Gamma_Z M_Z)}\right|^2 = \frac{\alpha_{w'}^2 4m_{\nu DM}^2}{M_Z^4} \sim 10^{-37}\mathrm{cm}^2\left(\frac{m_{\nu DM}}{1GeV}\right)^2 \tag{4.32}$$

So we can see that $m_{\nu DM} \sim 3$ GeV gives us about the right number density of relic neutrinos. However, there is a problem here since the Z-boson also decays into pairs of quarks and leptons.

$$\Gamma_{Ztotal} = \Gamma_{Z\to e^+e^-} + \Gamma_{Z\to\mu^+\mu^-} + \Gamma_{Z\to u\bar{u}} + ... \tag{4.33}$$

so the total lifetime is much shorter depending upon how many species Z can decay into. Z particles are produced at the LHC but because the LHC is smashing protons together things are a bit messy. The Z particle has also been produced at LEP and its cross section measured carefully as a function of energy. Because we know that the width of the resonance for Z-boson

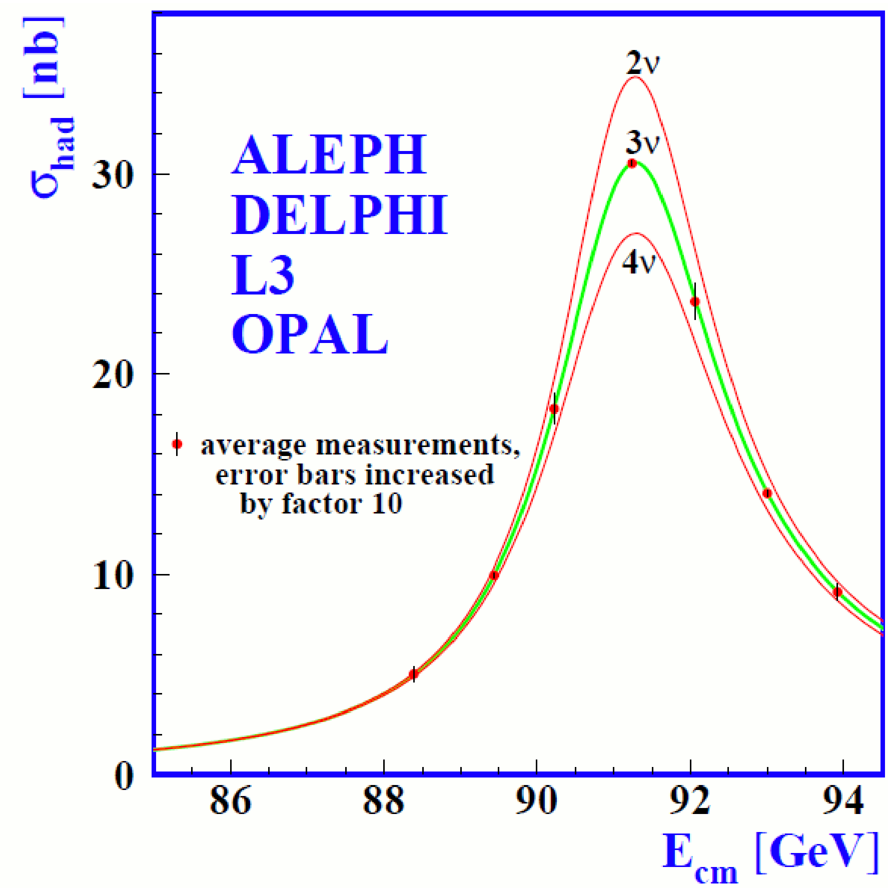

Figure 4.4: The cross section for the production of the Z-boson was measured in great detail at LEP. It does not allow for a 4th generation of neutrinos which are kinematically accesible when the Z decays.

production is set by the decay channels of the Z-boson, we can tell that there cannot be another Neutrino with a mass less than half the Z-boson mass, i.e. less than 45 GeV. So it does not appear that heavy neutrinos cannot be our thermal relics.

4.3 A model of dark matter

So we would like to consider a thermal relic particle which does give the correct relic abundance without messing up the precision electroweak physics observed at LEP, Tevatron, LHC and other colliders. We have seen that a heavy neutrino does not fit the bill. Let us consider another process in figure 4.5. Here we introduce another dark matter particle χ which couples

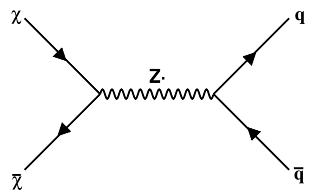

Figure 4.5: The coupling of dark matter χ with a new hypothetical Z-boson Z' which also couples to quarks .

to the standard model particles via a new Z-boson which we call Z'. Let us assume that it only couples to quarks. The annihilation cross section is then given by

$$\sigma_{\chi\bar{\chi}\to Z'\to q\bar{q}} \sim \frac{4g_q^2 g_\chi^2 s}{3\pi\left(s - M_{Z'}^2\right)^2 + M_{Z'}^2\Gamma_{Z'}^2} \tag{4.34}$$

where g_q is the coupling of the quarks to the Z' particle and g_χ is the coupling of the Z' to the dark matter particle. Then by setting the correct values of g_q, g_χ and $M_{Z'}$ we will be able to get the correct relic abundance.

4.4 Direct Detection

There is also the possibility of the dark matter scattering off quarks according to the diagram in figure 4.6 we see that this results in a scattering off Nuclei. In the Milky Way, the dark matter is moving at around 200 km/s which corresponds to v$\sim 10^{-3}$c. If we look at the scaterring of a dark matter particle off a nucleus at rest we get

$$m_\chi v_{\chi before} = m_\chi v_{\chi after} + m_A v_{A after} \qquad ; \qquad \frac{1}{2}m_\chi v_{\chi before}^2 = \frac{1}{2}m_\chi v_{\chi after}^2 + \frac{1}{2}m_A v_{A after}^2 \tag{4.35}$$

Where $before$ and $after$ mean before and after the collision. We can see that the typical recoil energy

$$E_{rec} = \frac{2m_A m_\chi^2}{\left(m_\chi + m_A\right)^2} v_\chi^2 = 2\frac{\mu_{A\chi}^2}{m_A} v_{\chi before}^2 \tag{4.36}$$

where m_A is the mass of the nucleus which is struck by the dark matter and m_χ is the mass of the dark matter particle. The reduced mass $\mu_{A\chi} = m_\chi m_A/(m_\chi + m_A)$.

If we approximate the Milky Way as being an isothermal sphere then we calculate the circular velocity $v^2 = GM/r = 2\sigma^2$ where here σ is the velocity dispersion of the isothermal sphere rather than a cross section. The value of the circular velocity of the Sun is around 220-230 kms^{-1}. In this way we see that $\sigma \sim 160$ kms^{-1} however since we are moing through the dark matter halo, we might expect the typical velocity with which we hit dark matter to be around 200 kms^{-1}. In other words, in dimensionless units, $v \sim 10^{-3}$ so that if the dark matter has the same mass as the target nucleus, e.g. 100 GeV dark matter and Xenon (roughly 130 nucleons) then $E_{rec} \sim m_\chi v^2 \sim 25 - 50$ keV. For lighter Dark matter, for example, for 10 GeV dark matter particle, $E_{rec} \sim 1$ keV.

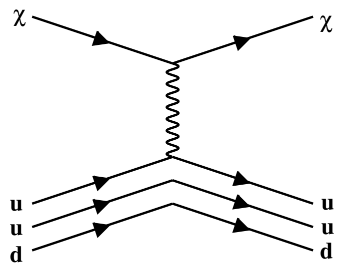

Figure 4.6: The scattering of dark matter χ of a nucleon via a new hypothetical Z-boson Z'.

The cross section for this happening is given by

$$\sigma^{SI}_{\chi N} = \frac{\mu^2_{\chi N}}{M^4_{Z'}} f^2_N \tag{4.37}$$

where the supescript SI means spin independent, so this is when the interaction of the boson with the nucleus does not depend upon the overall spin of the nucleus. The parameter $\mu_{\chi N}$ is the reduced mass of the nucleon (note it is nothing to do with muons, apologies, not my notation!) and

$$f_p = g_\chi \left(2g_u + g_d\right), \qquad f_n = g_\chi \left(g_u + 2g_d\right) \tag{4.38}$$

The deBroglie Wavelength of the dark matter particles will be $\lambda \sim h/(mv)$ and for a 10 GeV dark matter particle, this will correspond to around $\lambda \sim 10^{-3}$MeV^{-1} which is 2×10^{-14}m. And since the De Broglie Wavelength of the dark matter $\lambda \sim 1/(mv)$ is larger than the size of the nucleus, the matrix element contains a scatter off all A nucleons *coherently*, which means that the cross section is enhanced by A^2.

This is only the case for spin indepdendent cross sections, i.e. cross sections which do not depend upon the orientation of the dark matter spin and the nucleon spin. Those cross sections which do depend on this will not recieve this A^2 enhancement, since the spin of most of the nuclei are paired, and only nuclei with unpaired nucleons will be able to detect such interactions.

For more massive or more quickly moving dark matter, this A^2 starts to be supressed because the de Broglie wavelength starts to become comparable or smaller than the size of the nucleus, hence the interaction contains a *form factor* which takes this into account.

So now we imagine that $g_u = g_d = g_\chi = 0.1$ and that $M_{Z'} = 500$GeV and $m_\chi = 50$ GeV. Let us also imagine we are scattering off a particle with 100 nuclei in (lets approximate the mass of a nucleon to 1 GeV) then we have

$$\sigma^{SI}_{\chi A} = A^2 \sigma_{\chi N} = 100^2 \frac{\mu^2_{\chi N}}{M^4_{Z'}} f^2_N = 100^2 \frac{\left(100GeV \times 1GeV\right)^2}{\left(101GeV\right)^2 \times \left(500GeV\right)^4} 0.1^4 = 10^{-39} cm^2 = 10^{-3} pb \tag{4.39}$$

where here pb means pico-barns or 10^{-12} barns. A "barn" is a unit of cross section used by particle physicists corresponding to 10^{-24}cm^2 and it a relatively large cross section by particle physics standards (roughly the cross section for two neutrons to scatter off each other). It comes from the expression "couldn't hit the side of a barn", I think.

Now how big a detector do we need to see such an object? The local dark matter density is $\rho_{DM} = 0.3\text{GeVcm}^{-3}$ so for a particle moving relative to dark matter at rest at 200 kms^{-1}, the number of times it will be hit by a dark matter particle every second is

$$\frac{\rho_{dm}}{m_{dm}}\sigma_{\chi A} v = \frac{0.3\text{GeVcm}^{-3}}{50\text{GeV}} 10^{-39}\text{cm}^2 \times 2 \times 10^7\text{cms}^{-1} \sim 10^{-34}\text{s}^{-1} \tag{4.40}$$

so that if we would like one event per year we need around 10^{27} seperate nuclei. If the particular target we have in mind has 100 nucleons in each nuclei then the target mass we require is 10^{29} nuclei which weighs about 100 kg.

Such detectors are built with different detections mechanisms.

- **Scintillation** when a dark matter particle hits a nucleus it makes it moves relative to the other nuclei which creates some scintillation light in certain materials which is then picked up by sensitive detectors.

- **Ionisation** dark matter can ionise nuclei, leading to the freeing of electrons which can then be detected by being drifted in an electric field towards a cathode.

- **Phonons** a dark matter particle can heat a cryogenic system taking it from a superconducting to a non-superconducting state which can then be detected.

In practise we need to take into account the Maxwell-Bolztmann distribution of dark matter in the Milky Way, including the cut-off at the escape velocity. We also need to take into account the motion of the Earth around the Sun which changes the velocity distribution in the rest frame of the earth around the year - annual modulation.

4.5 Collider Constraints

There are two ways we could detect such a dark matter particle at the LHC, the first is dijets - if the boson couples to dark matter and standard model matter then it can also couple to standard model matter and standard model matter.

4.6 Self-Annihilating Dark Matter

Straight away we can see that if dark matter in N-body simulations concentrates inside the centre of galactic haloes then this is the place where we expect to see the most annhiliation. The equation for the number of annihilations will be given by

$$\frac{dn}{dt} = n^2 \langle \sigma_A |\vec{v}| \rangle \tag{4.41}$$

which will give us the annihilation in units of $s^{-1}m^{-3}$. More generally, if we assume the inner region of some dark matter halo from $r = 0$ to $r = r_{max}$ then we expect the annihilation to be

$$\frac{dE}{dt} = 2m_{dm} \int_{r=0}^{r=r_{max}} \left(\frac{\rho}{m_{dm}}\right)^2 \langle \sigma_A |\vec{v}| \rangle 4\pi r^2 dr \tag{4.42}$$

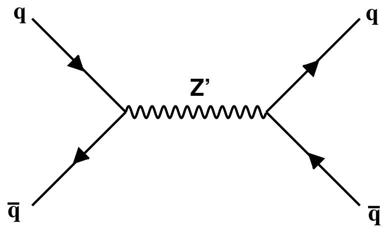

Figure 4.7: Extra bosons connecting the visible sector to the dark sector can lead to dijet processes in addition to those which exist in the standard model.

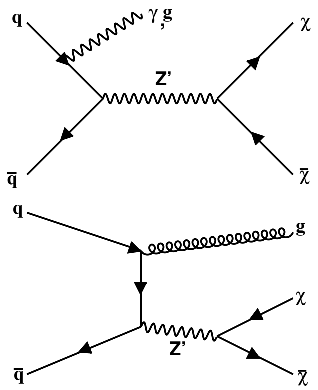

Figure 4.8: Monojet or Monophoton events are distinctive indicators of dark matter events.

and the precise channel that this manifests itself in depends upon the nature of the dark matter in question. This could be (very rarely) directly into gamma rays. More often it is into quarks which then hadronise producing gamma rays andprotons/anti-protons. There is currently a signal apparently coming from the galactic centre observed by the Fermi gamma rays telescope which could be consistent with gamma rays from dark matter annihilation. However it could also be consistent with unresolved astrophysical sources.

4.6.1 Diffusion-Loss Equation

For dark matter which annihilates into matter and anti-matter these particles are injected into the astrophysical environment. We then need to see how they evolve. In order to do this, we need to derive the diffusion loss equation

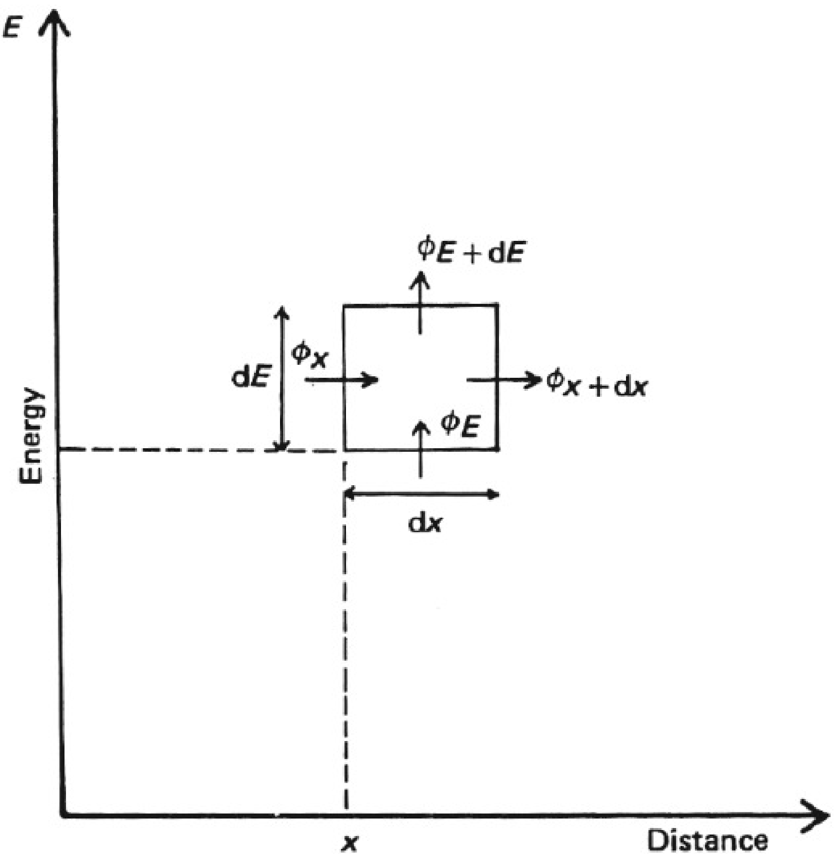

Figure 4.9: To derive the Diffusion Loss equation we split up space and Energy into cells and consider the flow of particles into and out of the cells in both dimensions E and x.

$$\begin{aligned}\frac{d}{dt}N(E,x,t)\,dEdx &= [\phi_x(E,x,t)-\phi_{x+dx}(E,x+dx,t)]\,dE \\ &+[\phi_E(E,x,t)-\phi_{E+dE}(E+dE,x,t)]\,dx \\ &+Q(E,x,t)\,dEdx \end{aligned} \tag{4.43}$$

where the number of particles between E and $E{+}dE$ and between x and $x{+}dx$ is $N(E,x,t)dEdx$ (see figure 4.9). Q is the rate of particles injected, for example from the annihilation of dark matter. The ϕ parameters are the flux out of the top and the side of the box. It is easy to see that this can be written

$$\frac{dN}{dt} = -\frac{\partial\phi_x}{\partial x} - \frac{\partial\phi_E}{\partial E} + Q \tag{4.44}$$

By definition, the flux

$$\phi_x = -D\frac{\partial N}{\partial x} \tag{4.45}$$

where D is the diffusion coefficient. So because of that

$$\frac{\partial\phi_x}{\partial x} = -D\frac{\partial^2 N}{\partial x^2} \tag{4.46}$$

or in three spatial dimensions we can generalise to $D\nabla^2 N$.

To calculate the difference in the energy flux ϕ_E between the top and the bottom of the box we need to consider how many particles are at the top of the box and how quickly they are losing energy and falling into the box vs. how many particles there are at the bottom of the box and how quickly they are losing energy and dropping out of the bottom. Mathematically:-

$$\frac{\partial \phi_E}{\partial E} = \frac{\partial}{\partial E}\left[N(E)\frac{dE}{dt}\right] = -\frac{\partial}{\partial E}\left[b(E)N(E)\right] \tag{4.47}$$

where $b(E) = -dE/dt$ is the energy loss.

We can now write the diffusion-loss equation as

$$\frac{dN}{dt} = D\nabla^2 N + \frac{\partial}{\partial E}\left[b(E)N(E)\right] + Q(E). \tag{4.48}$$

For high energy particles the main energy losses experienced are synchrotron radiation and "inverse" Compton Scattering (Compton scattering). Synchrotron radiation corresponds to the energy lost when charged particles spiral around magnetic fields and consequently emit radiation. Compton Scattering is when charged particles lose energy by scattering off radiation fields, see figure 4.10. To estimate this cross section, we need to know that the propagator for

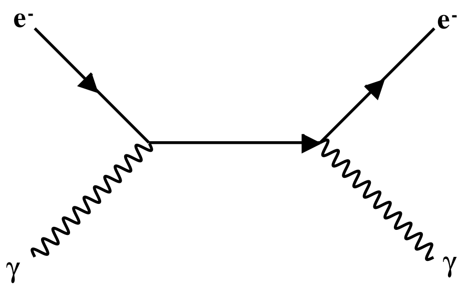

Figure 4.10: Compton Scaterring.

a fermion goes like $m_{fermion}^{-1}$ so that we can readily estimate that

$$\sigma(\gamma e \to \gamma e) \sim \frac{\alpha^2}{m_e^2} \tag{4.49}$$

The energy loss in the frame of the charged particle due to Compton scattering is simply $b(E) = \sigma_T c u_{rad}$ where u_{rad} is the energy density of radiation and $\sigma_T = 6.65 \times 10^{-25}\mathrm{cm}^2$ is the Thomson cross section.

$$\sigma_T = \frac{8\pi\alpha^2}{3m_e^2} \tag{4.50}$$

which supports our estimate for the Compton scatterig cross section. After Lorentz Boosting to the "rest frame" of the ambient radiation we get

$$b_{ICS}(E) = \frac{4}{3}\sigma_T c u_{rad}\beta^2\gamma^2 \sim \frac{4}{3}\sigma_T c u_{rad}\left(\frac{E}{m}\right)^2 \tag{4.51}$$

where the last approximate equality is for a relativistic particle. For the CMB, $u_{rad} = 0.26$eV cm^{-3} which is the same throughout the Universe. The energy density in starlight is comparable but is of course much bigger in the inner regions of the Milky Way.

We detect anti-matter in orbit (using AMS on the ISS at the moment) and try to work out if it could have come from the annihilation of dark matter which has lost energy and diffused throughout the galaxy.

Chapter 5

Alternatives to Thermal Relics

5.1 Axions

When one writes down a theory, in general one should first specify the field content (how many particles of what type charged under which fields) and then write down every Lorentz invariant and gauge invariant renormalizable term you can think of. We haven't done a lot of particle physics but there is a term like this which we should be able to write down in the case of the theory of quarks and gluons - QCD.

$$\mathcal{L}_{\mathrm{CP-viol.}} = \frac{\alpha_s}{4\pi}\,\theta\,\mathrm{tr}\,G_{\mu\nu}\tilde{G}^{\mu\nu} \equiv \frac{\alpha_s}{4\pi}\,\theta\,\frac{1}{2}\,\epsilon^{\mu\nu\alpha\beta}\,\mathrm{tr}\,G_{\mu\nu}G_{\alpha\beta}, \tag{5.1}$$

where G is the gluonic field strength, similar to the electromagnetic field tensor $F_{\mu\nu}$ and $\epsilon^{\mu\nu\alpha\beta}$ is the completely antisymmetric tensor of +1s and -1s. This is the same as writing the dot product between E and B. Similar to the strong coupling constant α_s, the fundamental parameter θ has to be determined experimentally. One of the most sensitive probes for it is the electric dipole moment of the neutron, arising from the CP-violating term given in Eq. (5.1). It should be of order

$$|d_n| \sim 10^{-16}\ \ |\theta|\ \ e\,\mathrm{cm}, \tag{5.2}$$

whereas the experimentally measured value of the electric diploe moment of the neutron is consistent with zero to a very high accuracy, requiring that $|\theta| < 10^{-10}$. There is no particular reason why this parameter should be so close to zero.

One of the ways people solve this is to promote the dimensionless parameter θ to a field a which we call the axion. We cannot replace θ with a directly as they have different dimensions, but we can replace θ with a/f_a where f_a is some mass scale which we talk about briefly in the slides.

The Lagrangian when we include an axion then looks like

$$\mathcal{L}_a = \frac{1}{2}\partial_\mu a \partial^\mu a + \frac{\alpha_s}{4\pi f_a}\,a\,\mathrm{tr}\,G^{\mu\nu}\tilde{G}_{\mu\nu} + \frac{s\alpha}{8\pi f_a}\,a\,F^{\mu\nu}\tilde{F}_{\mu\nu} \tag{5.3}$$

and what is more when a moves away from zero, it gives rise to complicated vacuum effects which give it an effective potential of the form

$$V = V_0\,[1 - \cos{(a/f_a)}] \tag{5.4}$$

because of this, the field prefers to move towards the minimum of the potential, which occurs when $a = 0$. So by promoting θ to a field a you have a dynamic solution as to why this term does not exist - it has a zero in front of it set by $\langle a\rangle = 0$ which arises dynamically.

Axions which Solve the Strong (i.e. QCD) CP problem generally have a relationship between $m_a f_a \sim m_\pi f_\pi$ where m_π is the pion mass and f_π is the pion decay constant. Both of these parameters have masses around 100 MeV.

In theories such as string theory there are sometimes a great number of axions which come from the way that the extra dimensions are compactified. For these axions, the same relationship between m_a and m_π does not exist and one can treat them to some degree as free parameters (depending upon which string theorist you talk to). In this situation, or any situation where one is agnostic about the precise value of the couplings, these particles are called "axion like particles" or ALPs.

5.1.1 Lagrangian for General Scalar fields

Here we do some background on scalar fields. The equation of motion for a scalar field is given by

$$m^2\phi - \partial_\mu \partial^\mu \phi = 0 \tag{5.5}$$

for flat space $\eta^{\mu\nu} = -1, 1, 1, 1$. Or, more generally

$$\frac{dV(\phi)}{d\phi} - \partial_\mu \partial^\mu \phi = 0 \tag{5.6}$$

for when $V(\phi) \neq (1/2)m^2\phi^2$. If we then assume that there are no spatial gradients for the field, in other words that it is homogeneous throughout space but can oscillate over time then we get

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi = \eta^{00}\partial_0\partial_0\phi = -\frac{\partial^2\phi}{\partial t^2} = m^2\phi \tag{5.7}$$

which has a very simple solution

$$\phi(t) = A\sin(mt) \tag{5.8}$$

however the pressure P and density ρ of the scalar field are given by

$$\begin{aligned} \rho &= \frac{1}{2}\dot{\phi}^2 + V(\phi) \\ P &= \frac{1}{2}\dot{\phi}^2 - V(\phi) \end{aligned} \tag{5.9}$$

Then the equation of state

$$w = \frac{P}{\rho} = \frac{\dot{\phi}^2 - 2V}{\dot{\phi}^2 + 2V} \tag{5.10}$$

will also vary over time. For the potential $V(\phi) = (1/2)m^2\phi^2$ we would like to find out the average equation of state as the field oscillates, in other words $\langle w \rangle$.

$$\begin{aligned} w &= \frac{\dot{\phi}^2 - 2V}{\dot{\phi}^2 + 2V} = \frac{A^2m^2\cos^2(mt) - A^2m^2\sin^2(mt)}{A^2m^2\cos^2(mt) + A^2m^2\sin^2(mt)} \\ &= \frac{\cos^2(mt) - \sin^2(mt)}{\cos^2(mt) + \sin^2(mt)} = \cos^2(mt) - \sin^2(mt) \end{aligned} \tag{5.11}$$

and the time average of this quantity

$$\langle w \rangle = \frac{\int_0^{t=2\pi/m} \left\{\cos^2(mt') - \sin^2(mt')\right\} dt'}{\int_0^{t=2\pi/m} dt''} = \left[\frac{\sin(2mt)}{4\pi}\right]_0^{t=2\pi/m} = 0 \tag{5.12}$$

So we can see that this oscillating scalar field has the same equation of state as matter.

Since this field behaves like matter, it can in fact play the role of dark matter, although strictly speaking it is more like a homogeneous bose-Einstein condensate throughout space. It does cluster like dark matter though. The differences are not obvious and even experienced physicists argue over interpretations.

Since we are in an expanding space-time and we understand General relativity, we should use covariant derivatives

$$\rightarrow g^{\mu\nu}\frac{\nabla}{\partial x^{\mu}}\frac{\partial}{\partial x^{\nu}}\phi - \frac{dV}{d\phi} = \frac{\partial^2\phi}{\partial x^{\mu}\partial x^{\nu}} - \Gamma^{\alpha}_{\mu\nu}\frac{\partial\phi}{\partial x^{\alpha}} - \frac{dV}{d\phi} \tag{5.13}$$

And assuming spatial isotropy ($\partial\phi/\partial x^i = 0$ where $i = 1,2,3$) this gives us a slightly different euqation of motion

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 \tag{5.14}$$

which takes into account the redshifting of the energy. This is exactly like a damped harmonic oscillator, but the damping comes from the expansion of the Universe. So as the universe expands, the energy density redshifts and the amplitude decreases ($A \rightarrow A(t)$).

5.1.2 The axion as dark matter

So if we assume that the axion is our scalar field then

$$\ddot{a} + 3H\dot{a} + \frac{dV}{da} = 0 \tag{5.15}$$

and it is easy to see that as long as H is large, then the field will not be able to move in its potential, since any positive $\dot{a}$ will lead to a negative $\ddot{a}$, preventing any acceleration of the field away from its value. In fact only when $H \ll m$ where m^2 is the second derivative of the potential with respect to a will the field be able to move (this is a damped harmonic oscillator and whether it is critically damped or not depends upon $H^2 - 4m^2$). Remember than

$$\cos\theta = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \tag{5.16}$$

so that close to the origin, the potential for the axion $\propto (1 - \cos\theta)$ is exactly like a quadratic mass potential.

So we think that if the axion is dark matter it obtains some value in the Early Universe, perhaps during inflation, and then only starts to move once the density of the Universe drops to the point where $H \sim m$ at which point it will start to move. Then it will oscillate and form the dark matter in the Universe.

5.1.3 Detecting axions

One way to detect axions is to use the coupling to the electromagnetic field strength which appears in equation (5.3). For simplicity we re-write the relevant part of the Lagrangian as

$$\mathcal{L} = \frac{1}{2}(\partial^{\mu}a\partial_{\mu}a - m^2a^2) - \frac{1}{4}\frac{a}{M}F_{\mu\nu}\widetilde{F}^{\mu\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

where $F_{\mu\nu}$ is the electromagnetic stress tensor and $\widetilde{F}_{\mu\nu} = \epsilon_{\mu\nu\rho\lambda}F_{\rho\lambda}$ is its dual, a denotes the axion, m is the axion mass and M is the inverse axion-photon coupling. Because of the $F_{\mu\nu}\tilde{F}^{\mu\nu}$ term, there is a finite probability for the photon to mix with the axion in the presence of a magnetic field. Mixing also occurs between photon components with different polarizations.

Technically, the mixing may be described as follows. We represent the photon field $A(t,x)$ as a superposition of fixed-energy components $A(x)e^{-i\omega t}$. If the magnetic field does not change significantly on the photon wavelength scale and the index of refraction of the medium $|n-1| \ll 1$, one can decompose the operators in the field equations as (for a photon moving in the z direction) $\omega^2 + \partial_z^2 \to 2\omega(\omega - i\partial_z)$, so that the field equations become Schrodinger-like,

$$i\partial_z \Psi = -\left(\omega + \mathcal{M}\right)\Psi \qquad ; \qquad \Psi = \begin{pmatrix} A_x \\ A_y \\ a \end{pmatrix}, \tag{5.17}$$

where

$$\mathcal{M} \equiv \begin{pmatrix} \Delta_p & 0 & \Delta_{Mx} \\ 0 & \Delta_p & \Delta_{My} \\ \Delta_{Mx} & \Delta_{My} & \Delta_m \end{pmatrix}.$$

The mixing is determined by the refraction parameter Δ_p, the axion-mass parameter Δ_m and the mixing parameter Δ_M. These three parameters are equal to

$$\begin{aligned} \Delta_{Mi} &= \frac{B_i}{2M} = 540 \left(\frac{B_i}{1\ \mathrm{G}}\right)\left(\frac{10^{10}\ \mathrm{GeV}}{M}\right)\mathrm{pc}^{-1}, \\ \Delta_m &= \frac{m^2}{2\omega} = 7.8\times 10^{-11}\left(\frac{m}{10^{-7}\ \mathrm{eV}}\right)^2\left(\frac{10^{19}\ \mathrm{eV}}{\omega}\right)\mathrm{pc}^{-1}, \\ \Delta_p &= \frac{\omega_p^2}{2\omega} = 1.1\times 10^{-6}\left(\frac{n_e}{10^{11}\ \mathrm{cm}^{-3}}\right)\left(\frac{10^{19}\ \mathrm{eV}}{\omega}\right)\mathrm{pc}^{-1}, \end{aligned}$$

respectively. Here $\omega_p^2 = 4\pi\alpha n_e/m_e$ is the plasma frequency squared (effective photon mass squared), n_e is the electron density, B_i, $i = x, y$ are the the components of the magnetic field B, m_e is the electron mass, α is the fine-structure constant and ω is the photon (axion) energy.

For constant magnetic field and electron density, the conversion probability is

$$P = \frac{4\Delta_M^2}{\left(\Delta_p - \Delta_m\right)^2 + 4\Delta_M^2}\sin^2\left(\frac{1}{2}L\Delta_{\mathrm{osc}}\right),$$

where

$$\Delta_{\mathrm{osc}}^2 = \left(\Delta_p - \Delta_m\right)^2 + 4\Delta_M^2$$

and we assumed that imaginary parts of all Δ's can be neglected. If B and n_e change spatially, the probability can be found by a numerical solution of Eqns. (5.17). The condition for strong mixing is

$$4\Delta_M^2 \gg \left(\Delta_p - \Delta_m\right)^2. \tag{5.18}$$

Other maximal-mixing conditions, which also must be met, are

$$\Delta_m \ll 2\Delta_M,$$

and

$$\Delta_p \ll 2\Delta_M,$$

which are equivalent to

$$\omega \gg 70\ \mathrm{eV}\left(\frac{m}{10^{-9}\ \mathrm{eV}}\right)^2\left(\frac{B}{\mathrm{G}}\right)^{-1}\left(\frac{M}{10^{10}\ \mathrm{GeV}}\right), \tag{5.19}$$

$$n_e \ll 10^{20}\ \mathrm{cm}^{-3}\left(\frac{\omega}{10^{19}\ \mathrm{eV}}\right)\left(\frac{B}{\mathrm{G}}\right). \tag{5.20}$$

In addition, to have large mixing one should require that the size L of the region in which conditions (5.19) and (5.20) are fulfilled should exceed the oscillation length,

$$L \gtrsim \frac{\pi}{\Delta_{\rm osc}},$$

that is

$$L \gtrsim 5.8 \times 10^{-3}\ \text{pc} \left(\frac{B}{\text{G}}\right)^{-1} \left(\frac{M}{10^{10}\ \text{GeV}}\right). \tag{5.21}$$

So that axions in a magnetic field can actually mix together with photons. This means that in principle, one can convert photons into axions in a magnetic field, leading to the concept of shining light through walls experiments. This is where one (tries) to convert photons into axions in a magnetic field, then send the beam of photons and axions into a wall. Only the axions will penetrate the wall, then you put a magnetic field on the other side of the wall and try to convert the axions back into photons on the other side.

Axions can also affect many astrophysical systems, they can carry away too much energy during supernova explosions etc. We expect axions to be produced in the Sun, in which case, one can detect them on Earth using experiments like CAST - the CERN axion Solar Telescope.

One can also look for dark matter axions converting back into microwaves in resonant cavities but this has to be extremely resonant, so that one can only look for one dark matter mass at a time. So far no such particles have been discovered.

5.2 Primordial Black Holes

In the early Universe, if there is a region of size R with a positive overdensity large enough that it will collapse before the Universe becomes matter dominated, it will form a black hole. The criterion for this to occur is simply that the gravitational energy

$$E \sim \rho R^3 \left[\frac{1}{2}\left(\frac{dR}{dt}\right)^2 - \frac{GM}{R}\right] \tag{5.22}$$

is negative and greater than the internal energy $U \sim PV \sim \rho V$ for radiation. So pertubations such that $G\rho R^2 \sim 1$ give rise to collapsing regions which will lead to black holes. Such pertubations require overdensities of $\delta \sim 1$ which are not very regularly seen.

The probability of finding such a large perturbation over a sphere of radius R is given by

$$p_R(\delta) = \frac{1}{\sqrt{2\pi}\sigma(R)} e^{-\frac{\delta^2}{2\sigma^2(R)}} \tag{5.23}$$

where

$$\sigma^2(R) = \left\langle \left(\frac{\delta M}{M}\right)^2_R \right\rangle = \frac{1}{2\pi^2}\int_0^\infty dk k^2 W^2_{TH}(kR)P(k) \tag{5.24}$$

where $P(k) = \langle|\delta_k|^2\rangle$ is the power spectrum of fluctuations and $W_{TH}(kR)$ is the fourier transform of the top-hat window function which probes the volume $V_W = \frac{4}{3}\pi R^3$. **This is beyond the scope of this course**, however, the probability of finding a pertubation on this scale is given by

$$\beta(M_H) = \frac{1}{\sqrt{2\pi}\sigma_H(t_k)}\int_{\delta_{min}}^{\delta_{max}} e^{-\frac{\delta^2}{2\sigma_H^2(t_k)}} d\delta \simeq \frac{\sigma_H(t_k)}{\sqrt{2\pi}\delta_{min}} e^{-\frac{\delta_{min}^2}{2\sigma_H^2(t_k)}} \tag{5.25}$$

where $\sigma_H^2(t_k) := \sigma^2(R)\big|_{t_k}$ where R is the horizon size ot time t_k and the last approximation is valid for $\delta_{min} \gg \sigma_H(t_k)$ and $(\delta_{max} - \delta_{min}) \gg \sigma_H(t_k)$. Since the fluctuations in the CMB tell

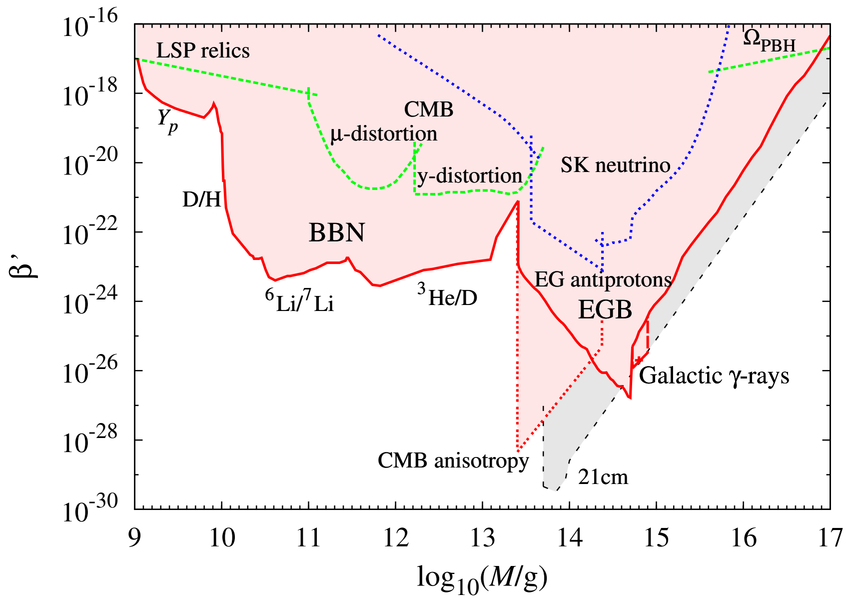

Figure 5.1: Constraints on different masses of primordial black holes due to different constraints

us that $\sigma \sim 10^{-4}$ so the probability of forming primordial black holes is very small. However, the CMB only gives us information about the power spectrum on large scale and if the power spectrum us not scale invariant, we can see that there is a bigger probability of forming black holes.

The density in the form of primoridal black holes is then given by

$$\Omega_{PBH} \simeq \Omega_r(1+z) \sim 10^6 \beta \left(\frac{t}{1\text{s}}\right)^{-1/2} \tag{5.26}$$

where so long as $M > 10^{15}$g. The reason for this is that Black holes have a Hawking temperature $T_H \sim 1/r_{Sch}$ where $r_{Sch} = 2GM/c^2$ and therefore evaporate as black bodies on a time scale

$$\tau(M) \simeq \frac{G^2 M^3}{\hbar c^4} \sim 10^{64} \left(\frac{M}{M_\odot}\right)^3 \text{yr} \tag{5.27}$$

simply according to the Stefan-Boltzmann law, so black holes smaller than 10^{15}g will have evaporate before now.

The constraints on primoridial black holes decaying are shown in figure 5.1, the evaporation products can mess up nucleosynthesis and can also great problems by being responsible for high energy gamma ray background radiation more than what is observed.

5.3 Sterile neutrinos

The neutrinos of the standard model are all left handed particles, but there could be right handed neutrinos which do not couple at all to the rest of the standard model at low energy. These neutrinos typically go out of equilibrium at high energies, long before nucleosynthesis. If they are light when they go out of equialibrium, then they have a number density today approximately the same as the photons in the CMB, the only different being the number of species which will suppres them slightly. In other words, $n \sim 10$ cm^{-3}. Then to obtain the correct density for dark matter, they will require masses of approximately 10^{-30}gcm$^{-3}/n =$

10^{-31}g which is around 100 eV- 10 keV. One starts to see that free streaming becomes very important for these species and the precise number of sterile neutrinos and the other species which froze out after sterile neutrinos determine whether these models are viable.

Sterile neutrinos will decay on a very large timescale. There is some limited evidence for a line around 3.5 keV which maybe sterile neutrinos.

Chapter 6

Cosmic Dynamics II

6.1 What is the age of the Universe?

OK so now we are in a situation to look at the implications of the expansion of the Universe. If everything is moving apart now, it means it was all closer to together at some point in the past, and a beginning to the Universe (depends on $\ddot{a}$) we have seen that

$$H^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} \tag{6.1}$$

and as we saw earlier, sometimes it is also convenient to write the curvature as a source of energy density

$$\rho_k = -\frac{3}{8\pi G}\frac{k}{a^2} \propto a^{-2} \tag{6.2}$$

Then $\Omega_M + \Omega_\Lambda + \Omega_k + \Omega_\gamma + \Omega_x = 1$ always. Also, since $a_0/a(t) = 1 + z$ we can write

$$H^2(z) = H_0^2\left[\Omega_\gamma (1+z)^4 + \Omega_M (1+z)^3 + \Omega_k (1+z)^2 + \Omega_\Lambda\right] \tag{6.3}$$

now we can rearrange the Hubble constant

$$H = \frac{1}{a}\frac{da}{dt} = \frac{a_0}{a}\frac{d(a/a_0)}{dt} = (1+z)\frac{d}{dt}\left[\frac{1}{1+z}\right] = \frac{-1}{1+z}\frac{dz}{dt} \tag{6.4}$$

So that we can write

$$\begin{aligned} dt &= \frac{-1}{(1+z)}\frac{dz}{H} \\ t_0 - t_1 &= \int_0^{z_1} \frac{dz}{(1+z)H(z)} \end{aligned} \tag{6.5}$$

And we can integrate this thing for any $H(z)$ so long as we have an idea what the density of stuff in the Universe is. Now a simple, perhaps the simplest and most generic situation is the case of a flat Universe containing only matter where the radiation and cosmological constant are either zero or negligible. In other words, $\Omega_\gamma = \Omega_k = \Omega_\Lambda = 0$ and $\Omega_M = 1$ then we can integrate the above equation

$$t_0 - t_1 = \int_0^{z_1} \frac{dz}{(1+z)H_0\sqrt{\Omega_M(1+z)^3}} = \frac{1}{H_0}\int_0^{z_1} \frac{dz}{(1+z)^{5/2}} \tag{6.6}$$

now we want to consider $t_1 \to 0$ which is the same as sending $z_1 \to \infty$ then we get

$$t_0 = \frac{2}{3H_0} \tag{6.7}$$

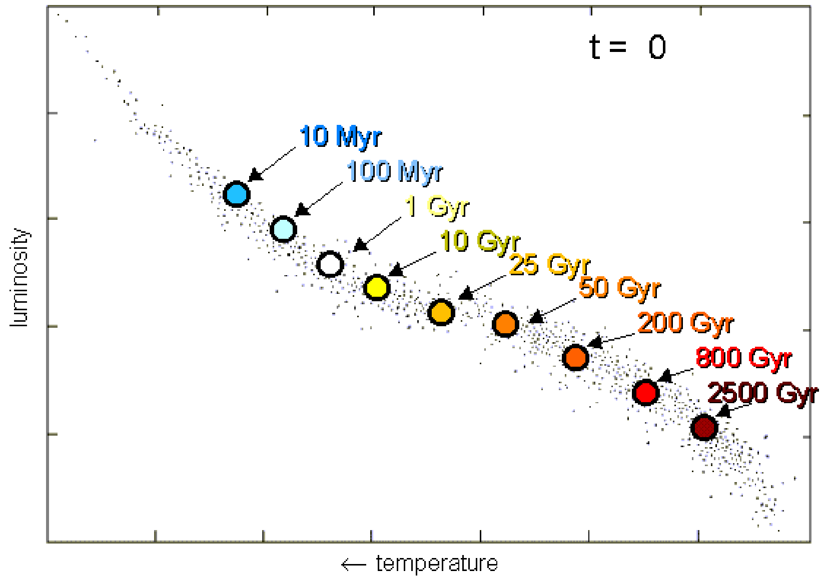

Figure 6.1: Different lifetimes of stars on the main sequence

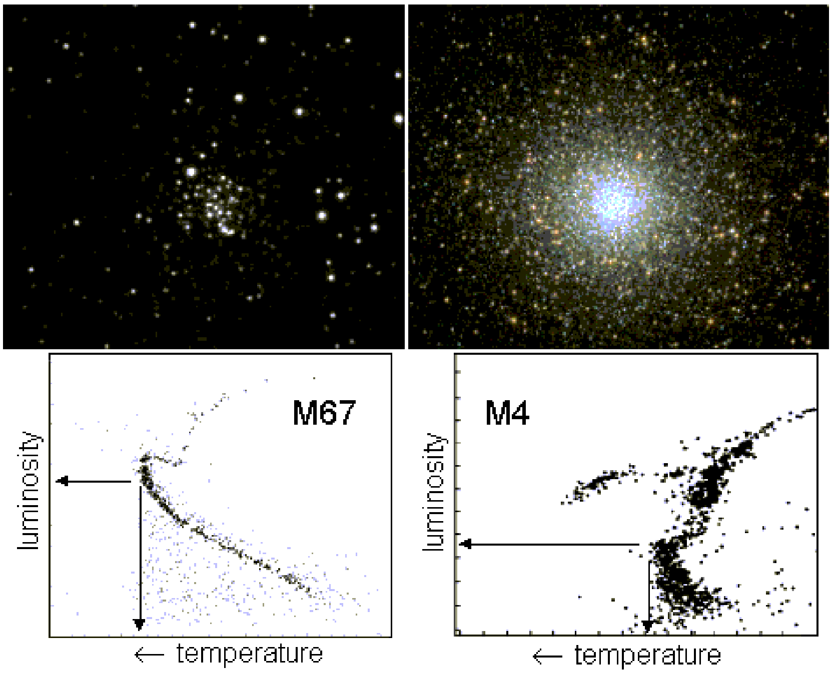

Figure 6.2: HR diagram of young (M67) and old (M4) Globular Clusters

Now a good modernish (2015ish) value is $H_0 = 70\mathrm{kms}^{-1}\mathrm{Mpc}^{-1}$. a megaparsec $Mpc = 3.0856 \times 10^{19}$km so $H_0 = 2.27 \times 10^{-18}\mathrm{s}^{-1}$ so that $t_0 = 2.94 \times 10^{17}\mathrm{s} = 9.3 \times 10^9$yr.

This is consistent with the age of the Sun and the Earth which are both about 4.5 billion years old. However it is not consistent with the stars we observe in globular clusters.

The Herzprung Russell (HR) diagram is the temperature-luminosity diagram for stars. When stars are burning hydrogen into helium they are said to be on the main sequence - a well defined line on HR diagram. Larger stars are more luminous and appear higher on the vertical luminosity scale. They also burn out of hydrogen faster and therefore leave the main sequence. You can therefore measure the age of star clusters by looking at which stars are still burning on the main sequence.

Some globular clusters are older than galaxies - they are the building blocks of galaxies according to some theories. By examining the main sequence cut-off we measure these globular clusters - they are around 12 billion years old!

This means that a Universe which only contains matter at the critical density will not work.

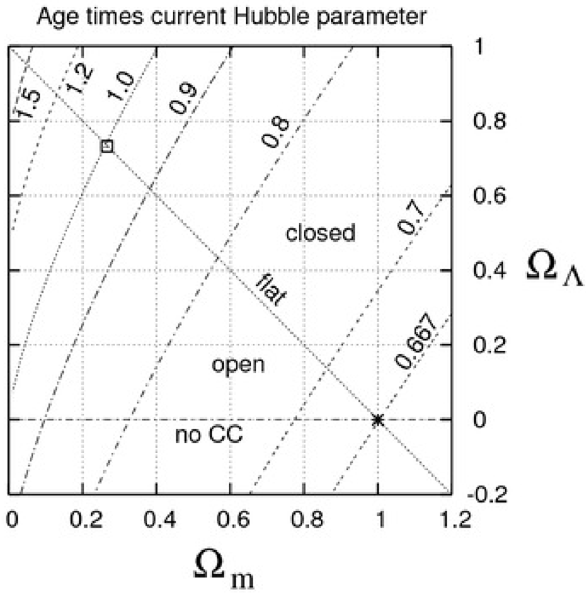

Figure 6.3: different ages, the numbers in the plot correspond to $H_0 t_0$

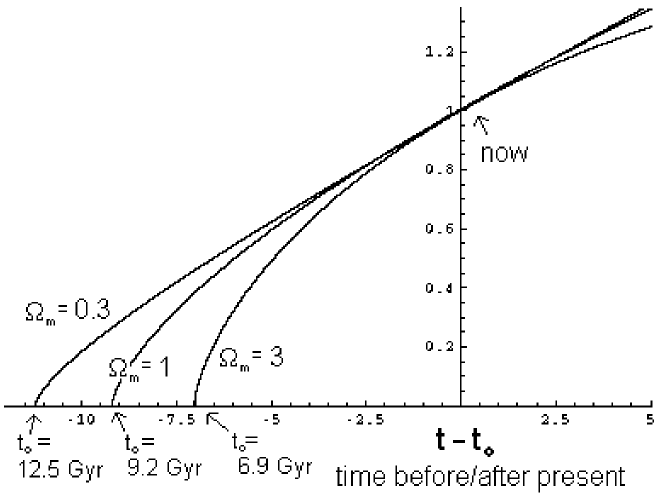

Figure 6.4: Scale factor evolution with respect to time for different comological parameters

6.2 Measuring the Hubble rate at High Redshift

If we want to see how the Hubble rate has changed over the history of the Universe, we need to look at larger distances and therefore larger redshifts. In those situations, the expansion used before is not enough and we need to look at the luminosity distance.

Luminosity Distance

If a supernova at comoving distance r sends out a photon at time t_{emit} the fraction we detect at time t_{obs} with a telescope of area A is

$$fraction = \frac{A}{4\pi\left(a(t_{obs}r)\right)^2} \tag{6.8}$$

BUT each photon is redshifted so that

$$E_\gamma(t_{obs}) = \frac{E_\gamma(t_{emit})}{1+z} \tag{6.9}$$

And the time between arrival of photons Δt is also stretched by a factor $(1+z)$. The flux F, which is the energy per unit area per second which arrives at the telescope is given by.

$$F = \frac{L}{4\pi a^2(t_0)r^2(1+z)^2} \tag{6.10}$$

where L is the luminosity, the total energy emitted by the source per second. In a flat space-time, $F = L/(4\pi d^2)$ where d is the distance so in an FRW universe with curved space and/or space-time we call $d_L = a(t_0)r(1+z)$ the Luminosity distance.

To fit the Hubble diagram and determine $\Omega_M, \Omega_\Lambda, \Omega_K$ etc., we need the exact expression for the luminosity distance $d_L(z)$. To get this, we need the relationship between r and z, in other words, photons we see now emitted at a redshift z come from what comoving distance r? To do this, first we establish the relationship between the comoving position of the source r and the time the photons were emitted t and then we find a relationship between t and z.

Photons take a light like trajectory, i.e. $ds^2 = 0$, and radial photons that arrive here do not change their value of θ or ϕ so we can write

$$ds^2 = 0 \to \frac{dr}{dt} = \frac{\sqrt{1-kr^2}}{a(t)} = \frac{\sqrt{1-kr^2}(1+z)}{a_0} \tag{6.11}$$

so that

$$\frac{a_0 dr}{\sqrt{1-kr^2}} = (1+z)dt = (1+z)dz\frac{dt}{dz} = -\frac{dz}{H(z)} \tag{6.12}$$

$$\int_0^r \frac{dr'}{\sqrt{1-kr'^2}} = \begin{cases} \frac{\arcsin(r\sqrt{k})}{\sqrt{k}} & k>0 \\ r & k=0 \\ \frac{\text{arcsinh}(r\sqrt{-k})}{\sqrt{-k}} & k<0 \end{cases} \tag{6.13}$$

which means we can then write the luminosity distance as

$$d_L = a(t_0)(1+z)r = \frac{c(1+z)}{H_0\sqrt{|\Omega_K|}}\mathcal{S}\left[\sqrt{|\Omega_K|}\int_0^z \frac{dz'}{\tilde{H}(z')}\right] \tag{6.14}$$

where $\tilde{H} = H/H_0$ and

$$\mathcal{S}[x] = \begin{cases} \sin(x) & k>0 \\ x & k=0 \\ \sinh(x) & k<0 \end{cases} \tag{6.15}$$

We then use this luminosity distance to work out the difference between the apparent and absolute magnitude. Using the luminosity distance (6.14) we are able to fit the type 1a supernovae luminosity-redshift relationship. A flat $\Omega_M = 1$ Universe without a cosmological constant is disfavoured, the data actually is difficult to reconcile with a Universe with no cosmological constant. This means that the Universe is accelerating. It also means that if the stuff really is a cosmological constant, it only became important recently - this is a problem.

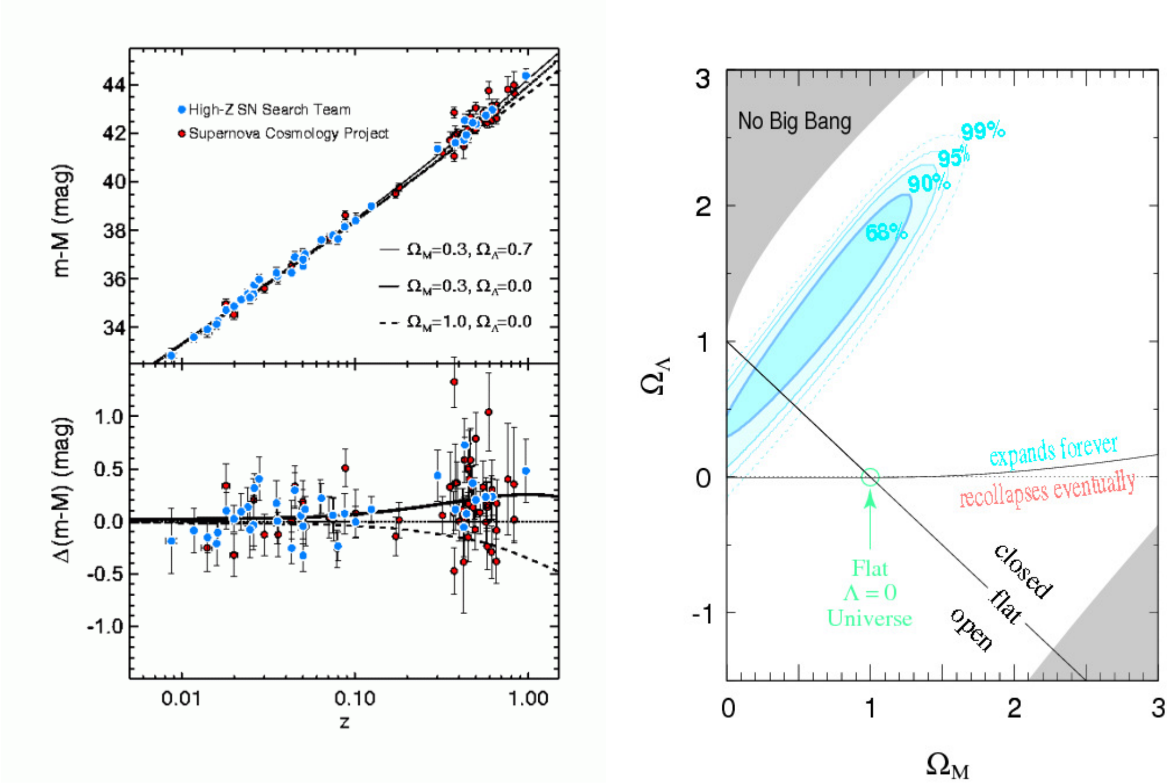

Figure 6.5: observations of type 1a supernovae disfavour a flat $\Omega_M = 1$ Universe

6.3 The CMB constraints

For the CMB radiation today, the temperature as measured by COBE is a very perfect black body corresponding to $T = 2.7$K. We will see that the density is $\rho_{\gamma 0} \sim T_\gamma^4 \sim 10^{-16}$eV^4 whereas the density of matter $\rho_{M0} \sim 10^{-12}$eV^4 so we are justified in neglecting the contribution of radiation to today's expansion rate. In fact $\Omega_{\gamma 0} \sim 10^{-4}$ and $z_{eq} \sim 5000$.

At a redshift of around $z = 1100$ the temperature of the CMB was about 3000 K which is when it starts to ionise the hydrogen in the Universe. This is the last time photons interact with hydrogen in the Early Universe, so it is called the last scattering surface.

Structure Formation

In cosmological structure formation theory, we take advantage of Birkhoff's theorem. Each sphere in the Universe evolves independently of what is outside that sphere(provided that the outside is spherically symmetric). This means that we can treat every little patch of the Universe like a mini-Universe with a density slightly different to the average density of the Universe. This is OK so long as the density contrast $(\rho - \rho_0)/\rho_0$ remains small (ρ_0 is the average background density of the Universe.) which from the CMB we know is true. A simple example of this is the fluctuations in the CMB temperature.

The relationship between temperature and density fluctuations at the last scattering surface is a bit subtle - denser regions of the Universe are hotter (and therefore brighter) but also the photons lose more energy as they climb out of those denser regions. The two effects work against each other. This is the Sachs Wolfe effect.

The result for a matter dominated Universe, which our Universe is at the Last Scattering Surface is

$$\left.\frac{\Delta T}{T}\right|_{observed} = \frac{1}{3}\Phi \tag{6.16}$$

where Φ is the gravitational potential, fluctuations in which are produced in the early Universe (perhaps during inflation) to look in more detail at how such perturbations evolve we need an equation for their evolution.

Perturbations

We are going to look at Newtonian Perturbations. This is not entirely appropriate for the CMB as we will see later but it will be useful for our purposes. In order to see how perturbations evolve, we need to combine three well known equations

1. **Continuity of mass**
$$\frac{\partial \rho}{\partial t} + \nabla (\rho \vec{v}) = 0 \tag{6.17}$$

2. **Euler Equation**
$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) = -\nabla \left(\frac{P}{\rho} \right) \tag{6.18}$$

3. **Poisson Equation**
$$\nabla^2 \Phi = 4\pi G \rho \tag{6.19}$$

We then expand all the quantities to first order e.g. expand to 1st order

$$\begin{aligned} \rho &= \rho_0 + \rho_1 \\ \Phi &= \Phi_0 + \Phi_1 \end{aligned} \tag{6.20}$$

so that for example $\rho(x,t) = \rho_0 + \rho_1$ where ρ_0 is the average density in the Universe. This perturbation theory breaks down when pertubations grow to become non-linear but it is OK around the Last Scattering Surface for the wavlengths we are interested in.

$$\delta = \frac{\rho(x,t) - \rho_0}{\rho_0} = \frac{\rho_1}{\rho_0} \tag{6.21}$$

Going to Fourier Space by breaking pertubations down into solutions of the form $\delta_k \propto \exp i \left(k_x \cdot x - \omega t \right)$ where $k_x = ak$ is the co-moving wavenumber leaves us with

$$\ddot{\delta_k} + 2\frac{\dot{a}}{a}\dot{\delta_k} + \left(\frac{k_x^2 v_s^2}{a^2(t)} - 4\pi G \rho_0 \right) \delta_k = 0 \tag{6.22}$$

where the speed of sound $v_s^2 = \partial p / \partial \rho$.

Jeans Criterion

for big k_x, there are only oscillatory solutions. The perturbations on these small scales cannot grow. Any physical length $\lambda = 2\pi a(t)/k_x$ and if

$$\lambda < \lambda_J = v_s \sqrt{\frac{\pi}{G\rho_0}} \tag{6.23}$$

there will be no growth. Physically, a sound wave can cross the volume and create a stabilising pressure before the stuff has time to gravitationally collapse. The length scale λ_J is called the Jean's length and the mass within such a sphere is called the Jeans mass

$$M_J = \frac{4\pi}{3}\left(\frac{\lambda_J}{2} \right)^3 = \left(\frac{\pi^{3/2}}{6} \right) \frac{v_s^3}{G^{3/2}\sqrt{\rho_0}} \simeq \frac{v_s^3 M_{Pl}^3}{\sqrt{\rho_0}} \tag{6.24}$$

Linear Growth of Structure above Jeans Length

For small k_x the pressure term is negligible and we can neglect it. Also there is evidence which suggests that dark matter is presureless. In this regime we look for solutions of the form $\delta = ct^n$. So for $k \gg 2\pi a\lambda_J$ or for dark matter

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\rho_0\delta = 0 \tag{6.25}$$

and we know that in the matter dominated epoch $a \propto t^{2/3}$ and $H = 2/(3t)$ so that

$$H^2 = \frac{8\pi G\rho}{3} \to 4\pi G\rho = \frac{2}{3t^2} \tag{6.26}$$

so that equation (6.25) gives us

$$n(n-1) + \frac{4n}{3} - \frac{2}{3} = 0 \tag{6.27}$$

which has solution

$$\delta = c_1 t^{2/3} + c_2 t^{-1} \tag{6.28}$$

i.e. a growing mode and a shrinking mode. The growing mode becomes dominant and the shrinking mode goes away. Note that since $a \propto t^{2/3}$ then $\delta \propto a$.

Oscillatory behaviour below Jeans Length and Qualitative picture of CMB oscillations

For large k_x the pressure term is dominant and we neglect instead the $-4\pi G\rho_0$ term.

$$\ddot{\delta}_k + 2\frac{\dot{a}}{a}\dot{\delta}_k + \left(\frac{k_x^2 v_s^2}{a^2(t)} - 4\pi G\rho_0\right)\delta_k = 0 \tag{6.29}$$

Note that as we wait the pressure term with the speed of sound in goes down like a^{-2} whereas the second term with the ambient density ρ_0 goes down like a^{-3} in the matter dominated era. So if we pick a particular comoving scale k_x its perturbations will grow until the density term drops below the pressure term.

In particular, since for baryons $P = \rho/3$ the speed of sound $v_s = c/\sqrt(3)$ and a scale $k = k_x/a$ will start oscillating when $k^{-1} = ct/\sqrt{3}$ or in other words, close to the time when it comes inside the horizon which is set by ct, the distance a photon can travel in the age of the Universe. Dark matter which has no pressure and $P = 0$ never oscillates, it just grows at different rates.

So small scales, large k enter the oscillatory, sub-horizon, sub-Jeans scale earlier and start to oscillate, whereas larger scales enter later. Different scales are therefore out of phase with each other, and the phase changes smoothly as you move through k. When the Universe cools to the Last Scattering surface, the baryons stop being a plasma, so their equation of state $P = 0$ and, like dark matter they stop oscillating immediately on all scales, all k so the fluctuations you see in the CMB today are a result of cutting off those perturbations and capturing the phase at different k which existed at that time.

WARNING these equations are only really valid inside the horizon and during the matter dominated epoch. Outside the horizon we need to use different equations which are obtained by doing perturbation theory in General Relativity which you can do in Andrew Pontzen's course next term. Since the growing modes correspond to scales which *are* outside the horizon, we should only really take this with a pinch of salt. Nevertheless, they don't oscillate outside the Jeans scale relativistic or not.

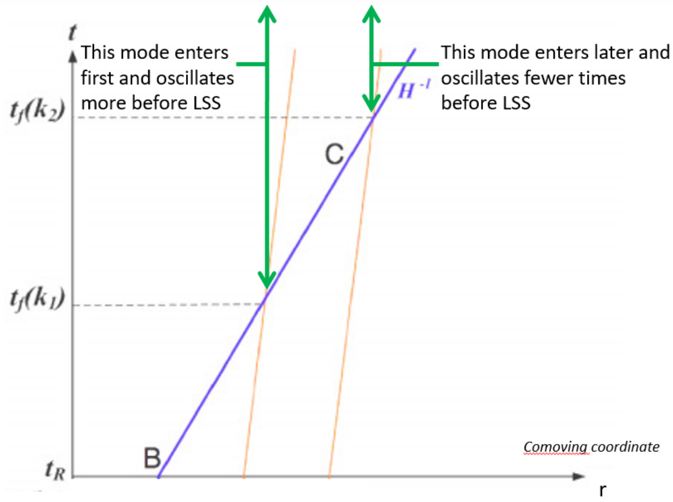

Figure 6.6: Smaller modes come inside the horizon and start to oscillate earlier, so that at t_{LSS} when the oscillations stop, they are at a different phase.

Perturbations in the CMB

We see the CMB through photons which were last coupled to baryons at the LSS ($z \sim 1000$). The Universe was already matter dominated at this time so the perturbations in density which are important are those in the form of matter. Also, we will see there is much more dark matter ($\Omega_M \sim 0.25$) than baryons $\Omega_B \sim 0.05$ and the dark matter does not couple to photons. Because it does not interact strongly with itself or anything else it is pressureless and hence has a very small Jeans radius. The perturbations therefore grow in the way we saw in the earlier section.

These perturbations act as potential wells for the baryons which fall into them. The pressure goes up and the baryons flow out again ($k \gg k_J \rightarrow$ oscillatory solutions) Therefore the initial density fluctuations act as the origins of waves which then spread outwards.

We introduce the parameter

$$R = \frac{3\rho_B}{4\rho_\gamma} \tag{6.30}$$

The speed of sound $v_s = c/\sqrt{3(1+R)}$. R determines the density of baryons vs. the density of radiation. Greater density of baryons makes them fall into over-densities more readily while the density of radiation acts to pull over densities apart. The two effects acting together give rise to waves. One can imagine qualitatively that if R is larger, then the dense regions of the waves will be denser, since the gravitational attraction of the baryons to each other will be more difficult for the radiation pressure to overcome. So this R parameter sets the overall zero point in the wave as it gets larger and smaller.

What do we observe in the CMB?

We have derived the oscillations in fourier space as a function of time η but we observe the CMB fluctuations in space. Instead of thinking of fixed k, we need to think of fixed η (η_{LSS}) and different k across the sky. We effectively then fourier transform the sky twice by using WMAP, Planck etc to get the correlation function $\langle \Delta T/T(x)\Delta T/T(x+\vec{r})\rangle$ which is usually expressed as a set of multipoles C_l with temperature anisotropy $\Delta T_l/T = \sqrt{C_l l(l+1)/2\pi}$.

Then, since the average $\langle \Delta T/T\rangle = 0$ we need to look at the variance, so we need to square what we did before.

So for a few baryons (R small) the squared peaks are roughly equal to each other whereas for many baryons (R big) the zero point shifts and the amplitude rises.

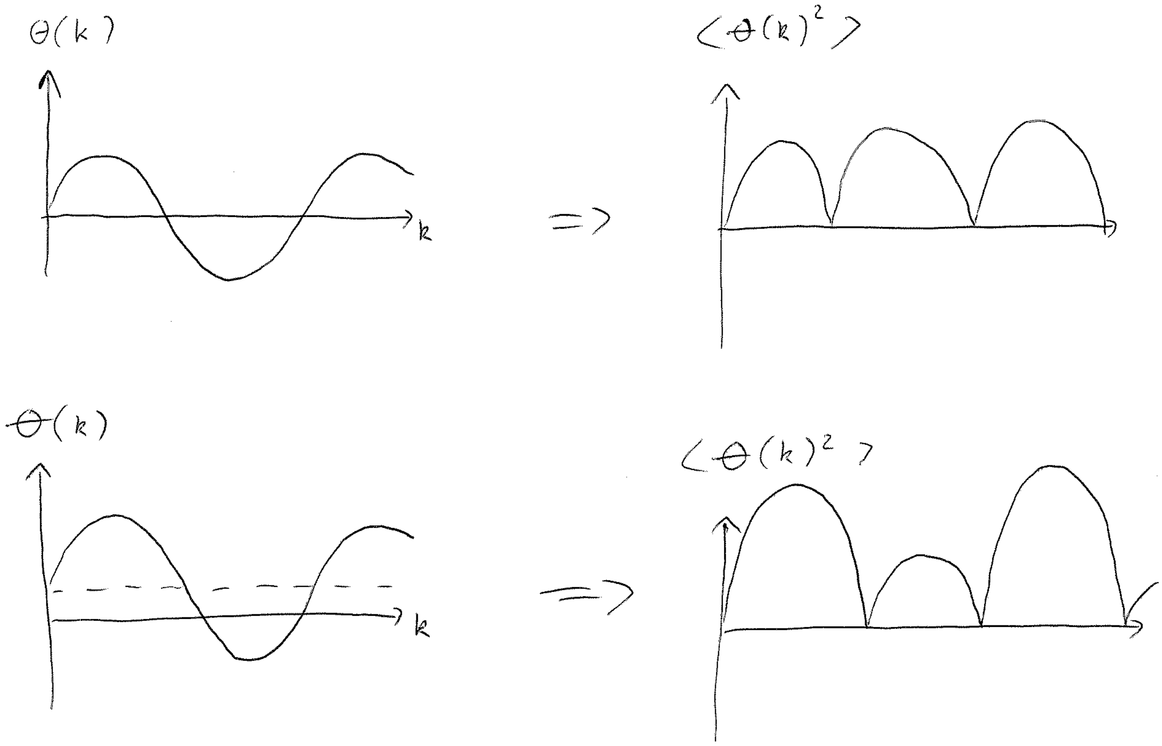

Figure 6.7: Demonstration of how baryons change amplitude between even and odd peaks.

Measuring Ω_{tot}- curvature of the Universe with the CMB

For long wavelengths $k < k_J$, oscill;ations cannot take place and the spectrum is flat. The position of the first peak therefore corresponds to the Jeans Length at the LSS or the "sound horizon"

$$r_s = \int_0^\eta v_s d\eta' = \int_0^t v_s \frac{1}{a(t)} dt \tag{6.31}$$

which is the distance that a sound wave can travel through the Universe between $t = 0$ and t_{LSS}. Since we know the physical size of this sound horizon, we can measure the angle, or multipole l, of the first peak. Then $\theta = d/d_A$ where d is the physical size and d_A is the angular distance $d_A = d_L/(1+z)^2$.

$$d_A = \frac{c}{|\Omega_K|^{1/2} H_0 (1+z_{LSS})} \mathcal{S}\left[|\Omega_K|^{1/2} \int_0^{z_{LSS}} \frac{dz'}{\sqrt{\Omega_\Lambda + \Omega_K(1+z')^2 + \Omega_M(1+z')^3}}\right] \tag{6.32}$$

where $\mathcal{S}$ is given by equation (6.15). The multipole of the first peak $l_1 \propto d_A$ and at the first approximation $l_1 \simeq 200\Omega_{tot}^{-1/2}$. And the observation? $l_1 = 197 \pm 6$ which suggests $\Omega_{tot} = 1$ and $k = 0$, i.e. a flat Universe.

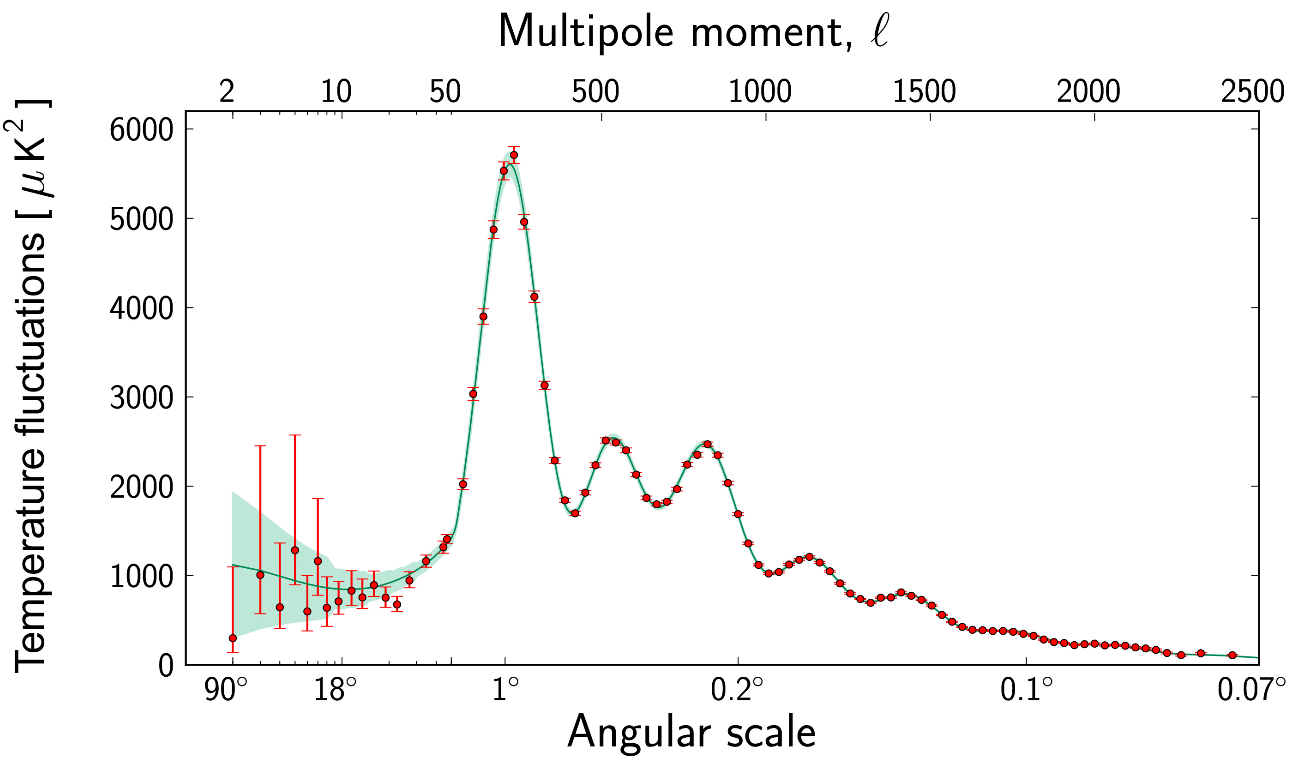

Figure 6.8: The actual data from the Planck Sattelite as a function of l.

6.4 Baryonic Acoustic Oscillations

Galaxies are clustered in the Universe. If you have one Galaxy, you are probably going to have another one near to it. They are clustered on and at the end of filaments and sheets surrounding large voids which are of typical size of about 100 Mpc. This is simply due to gravitational collapse making things cluster The Galaxy two point correlation function $\xi(r)$ is defined as the excess probability of finding another galaxy at a distance r from a particular galaxy relative to a uniform random distribution and averaged over the entire set:

$$dN(r) = \rho_{av} \left(1 + \xi(r)\right) dV_1 dV_2 \tag{6.33}$$

where ρ_{av} is the average density of galaxies in the Universe. The function usually follows a power law which drops off quickly with radius $\xi(r) \propto (r/r_0)^{-\gamma}$ where typical values of r_0 are about 8 Mpc and γ is around 1.8 but this varies for different types of galaxies. This arises simply because of collisionless gravitational collapse.

Now the sound waves in the Plasma left over from the oscillations before the last scattering surface mean that there is a slightly increased probability of finding an overdensity of baryons seperated by the characteristic scale corresponding to the peaks in the CMB. It turns out that

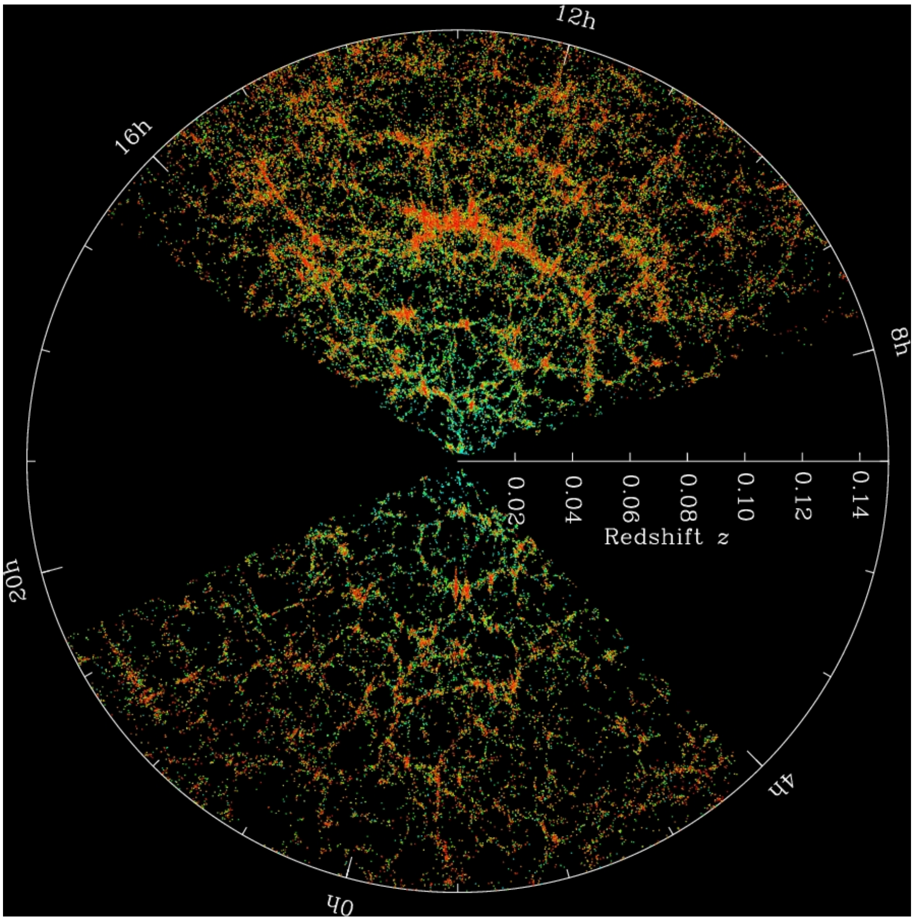

Figure 6.9: Distribution of Galaxies from the Sloan Digital Sky Survey.

this results in a slightly increased probability of forming a galaxies which are seperated by that scale. This has actually been detected in the two point function! It is a very small effect and very difficult to measure. However it does mean that we can observe the *same* comoving ruler at a totally different redshift ($z \sim 0.5$ rather than $z \sim 1100$). By performing the same analysis with te angular distance measurement at two different redshifts, we can break the degeneracy in the density content of the Universe.

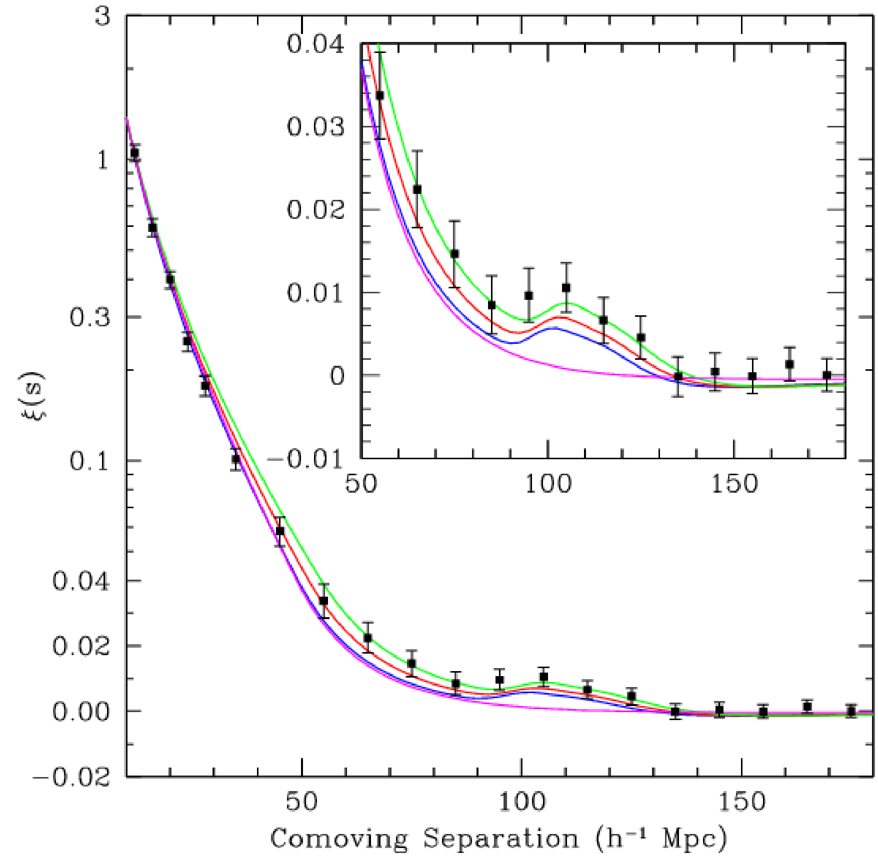

Figure 6.10: The Baryon Acoustic Oscillation bump is visible in the Sloan Digital Sky Survey data.

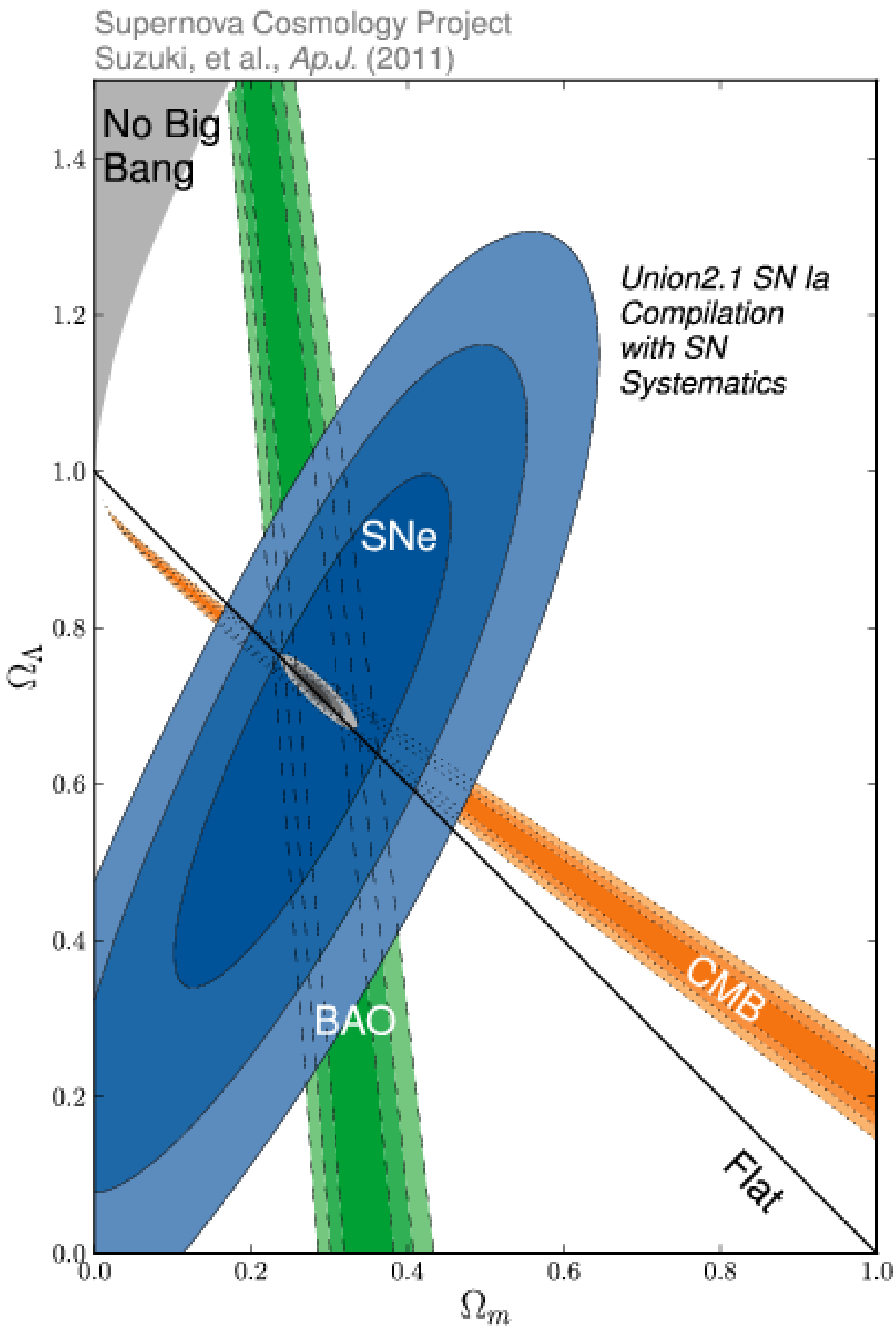

Figure 6.11: Combined constraints from Supernovae, CMB and BAO.

Chapter 7

Dark Energy and the Cosmological Constant

We have seen that dark matter alone is not enough to explain the make up of the Universe, and we also need some large component (approximately 70%) which is not in teh form of radiation or matter but in the form of some energy component which does not redshift. The easiest thing to assume about this energy density is that it is in the form of a cosmological constant, but we do not know if this is the actual situation.

7.1 The Cosmological Constant

"The term is necessary only for the purpose of making possible a quasi-static distribution of matter, as required by the fact of the small velocities of the stars" (Einstein 1917).

Originally Einstein Suggested that his field equations should read

$$R_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{7.1}$$

However, when the covariant derivative is applied to the Ricci curvature tensor we find that $\nabla^\mu R_{\mu\nu} \neq 0$. We know that $\nabla^\mu T_{\mu\nu} = 0$ infers the conservation of energy and momentum so there is something wrong here. In fact Hilbert figured this out and started adding the $-(1/2)Rg_{\mu\nu}$. There was a mysterious and controversial change in Einstein's expression between a submitted and a refereed version of his paper on General Relativity. Some people accuse him of plagarism, the idea is that having seen Hilbert's correction, he realises and tries to cover his error. I personally have no idea but I think that since he did the hard conceptual work, I can forgive him cheating a little bit at the end in panic to make sure someone else didn't get all the credit, and anyway we will probably never know the full story. So the field equations are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{7.2}$$

which is the simplest combination of these quantities which has a zero covariant derivative. However, the constraint $\nabla^\mu(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) = 0$ of course has room for a constant of integration, so it can be written

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{7.3}$$

Einstein introduced this constant Λ and called it the "cosmological constant" in 1917. This was before the Friedman-Lemaitre-Robertson-Walker equations which came only later in 1922-24. Einstein realised very early on that his equations would lead to a dynamical Universe which would follow an expanding or contracting trajectory. He didn't like this conceptually and tried to stop the Universe from expanding using this cosmological constant.

It is easier to see how Λ can stop expansion using the FLRW equations.

$$H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} \tag{7.4}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3} \tag{7.5}$$

One can assume that $\rho = \rho_M$ i.e. a matter dominated Universe with $P = 0$ then one can set

$$\Lambda = 4\pi G\rho_M \qquad \rightarrow \qquad \ddot{a} = 0 \tag{7.6}$$

and then we see that the following choice for the curvature

$$\Lambda = 4\pi G\rho_M \quad \& \quad k = 4\pi G\rho_M a^2 \qquad \rightarrow \qquad \dot{a} = 0 \tag{7.7}$$

so that by setting the cosmological constant and the curvature to suitable values, there exists a solution where the expansion of the Universe is zero and the Universe remains static.

This solution, although valid, is unstable, since if ρ_M is just a little bit too big or too small, the Universe runs away from this fixed point. So this was an extremely fine tuned solution, not stable to perturbations. Einstein knew this, everyone knew this and no-one could come up with anything more sensible to stop the Universe from expanding. Einstein therefore rejected the cosmological constant and later told Gamow that it was his *"biggest blunder"*. It was not until the mid 1990s that people realised that getting rid of this term may have been premature.

Cosmological Constant as energy density

One can treat the cosmological constant as being just that - a cosmological constant, a *deus ex machina* term on the geometry side of Einstein's equations which comes from no-where with no obvious origin, or one can try and understand where the heck it may have come from.

We can write the cosmological constant as a form of energy density by moving it from the left, *geometry* side of the Einstein's equations to the right *energy/stuff* side.

$$\begin{aligned} \rho_{eff} &= \rho + \frac{\Lambda}{8\pi G} = \rho + \rho_\Lambda \\ P_{eff} &= P - \frac{\Lambda}{8\pi G} = P - P_\Lambda \end{aligned} \tag{7.8}$$

so that Λ acts like an energy density with negative pressure. We can write the relationship between pressure and density as

$$\begin{aligned} P &= w\rho \\ P_\Lambda &= w_\Lambda \rho_\Lambda = -\rho_\Lambda \\ w_\Lambda &= -1 \end{aligned} \tag{7.9}$$

where w_X is the equation of state of energy density X.

So lets now set $k = 0$ and set $\rho = 0$ but set $\rho_\Lambda \neq 0$ then

$$G_{00} \rightarrow 3\left(\frac{\dot{a}}{a}\right)^2 = 8\pi G\rho_\Lambda \tag{7.10}$$

which has solution

$$a(t) = a_0 \exp\left(\sqrt{\frac{8\pi G\rho_\Lambda}{3}}t\right) = a_0 e^{Ht} \tag{7.11}$$

so that Universe expands exponentially! *Inflationary* expansion. Best theory is that this happened in the early Universe (see Mairi's course) and possibly starting again now. MORE LATER. ρ_Λ therefore does not go down as the Universe expands, very unusual, although it does happen sometimes in quantum field theory. The energy density of normal matter gets diluted as the Universe expands.

If we *do* assume that we have a cosmological constant with an energy density ρ_Λ then we can see that as we go backwards in time to higher redshifts, the ratio between dark energy and all other forms of matter (i.e. dark matter and baryons) is given by

$$\frac{\rho_\Lambda}{\rho_{matter}} = \frac{\Omega_\Lambda}{\Omega_{matter}(1+z)^3} \tag{7.12}$$

such that we can define a redshift where the densities are equal to each other

$$z_{eq} = \left(\frac{\Omega_\Lambda}{\Omega_{matter}}\right)^{1/3} - 1 \sim 0.33 \tag{7.13}$$

which one can either think of as being very near to us or not. One can take the view that this was at a time $t =$ which is after all nearly half the age of the Universe ago, or one can think about the fact that this is at a ery low redshift and if we compare it to all the other events which occured in the Universe like the Last Scattering Surface ($z \sim 1000$), Matter-radiation equality ($z \sim 5000$) or nucleosynthesis ($z \sim 10^{10}$) it is extremely recent. This is the *why now?* conundrum.

It can be argued that this goes against the space-time generalisation of the Copernican principle, since the cosmological principle says that we are not in a special place in the Universe, the generalisation of that is that we are not in a special time in the Universe.

7.1.1 Phase Transitions

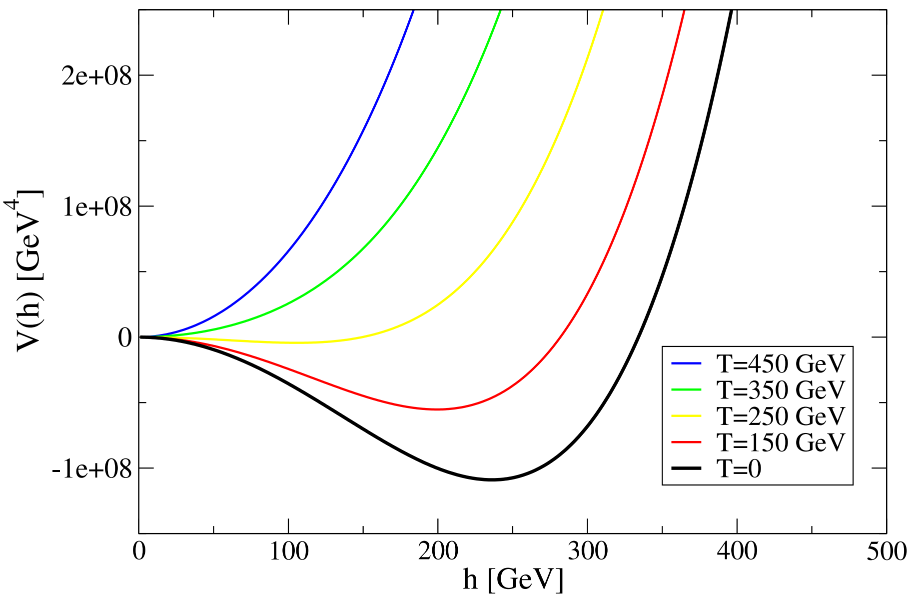

Figure 7.1: Higgs potential as the temperature decreases showing how the vacuum energy changes as we go from one vacuum to another at low temperatures.

If one considers for example the Higgs field, we find it has a potential of the form

$$V(h) = -\frac{1}{2}\mu^2 h^2 + \frac{1}{4}\lambda h^4 + \frac{1}{2}g^2 h^2 T^2 \tag{7.14}$$

where h is the Higgs field (a toy version where it is a once dimensional field), and T is the temperature. If we plot the potential we find that at different temperatures the minimum of the potential changes. At zero temperature we can see that

$$\langle h\rangle = v = \frac{m}{\sqrt{2\lambda}} = \frac{\mu}{\sqrt{\lambda}} \tag{7.15}$$

where $m^2 = \partial^2 V/\partial h^2$. So that at high temperatures, the Higgs field has expectation value $\langle h\rangle = 0$. We can see that it has a big difference in energy $\Delta V = \mu^4/4$ which is of order of the Higgs mass to the power 4, in other words $\Delta\rho = 10^8$ GeV^4. This change in the expectation value of the Higgs field as we go from big to low temperature is often referred to as the Electroweak Phase Transition.

Now how big is this compared to the observed expansion of the Universe today? We know that for a flat Universe

$$\rho_0 = \frac{3H_0^2}{8\pi G} \sim 10^{-47}\mathrm{GeV}^4 \sim \left(10^{-3}\mathrm{eV}\right)^4 \tag{7.16}$$

So when we look at this Higgs potential, we have to decide, where is zero on the vertical scale? The thing is that for particle physics we don't really care at all. The only thing which is important in particle physics is *differences* in energy, which is obviously quite different in cosmology where the *absolute* energy is what is important. So how does the vacuum decide that zero energy is after the Electroweak Phase transition? Also there are plenty of other phase transitions such as the QCD phase transition whch should give rise to an energy density change of about $\Delta\rho \sim$(100 MeV)^4 and presumably also a GUT phase transition which should give rise to $\Delta\rho \sim$(10^{16} GeV)^4!! All of these dwarf the actual energy density of the vacuum. We will see the situation actually is worse than this.

7.1.2 Quantum Fluctuations and the Lamb Shift

There is a slight mystery when one uses Dirac's equation to calculate the energy of level in the hydrogen atom and that is that the $^2S_{1/2}$ and the $^2P_{1/2}$ states have a slightly different energy when observed. The S state has a very slightly higher energy whereas they should have the same energy.

Consider a Hydrogen atom. We can think of the electron being confined within a box of size a so that the momentum $\Delta p \sim 1/a$ according to the Heisenberg Unicertainty principle ($\hbar = c = 1$) and you can write the total energy

$$E_{total} \sim \frac{1}{2ma^2} - \frac{\alpha}{a} \tag{7.17}$$

where $\alpha = e^2/\hbar c \sim 1/137$ is the fine structure constant. Differentiating and setting to zero, we find that $a \sim 1/(\alpha m)$ and $E = -\alpha^2 m/2$.

Inside the box there are many virtual photon fields with wavelength $k = 2\pi n/a$ with energy $E_k = k$ and electric field

$$\rho_{efield} = |\mathcal{E}_k|^2 = \frac{energy}{volume} = \frac{k}{a^3} \qquad \rightarrow \qquad |\mathcal{E}_k| \sim \frac{k^{1/2}}{a^{3/2}} \tag{7.18}$$

Where we use the fact that energy density $\rho \sim |\mathcal{E}|^2$ which you should know from the Poynting energy. That means in natural unites, electric field has mass 2, GeV^2. These photon fields accelerate the electron. We want to imagine the typical distance δr an electron gets buffeted by a field of wavelength k then we imagine that the field gives rise to an acceleration $\sqrt{\alpha}|\mathcal{E}_k|/m$ which we multiply by the timescale (time over one oscillation when E-field acts in same direction) to get the velocity, which we multiply again by the timescale to get the distance δr. The timescale $\tau \sim 1/k$ so we have

$$\delta r \sim \frac{\sqrt{\alpha}|\mathcal{E}_k|}{mk^2} = \frac{\sqrt{\alpha}}{mk^{3/2}a^{3/2}} \tag{7.19}$$

So then of course we need to estimate the total contribution from all k. Since these are random fluctuations, we need to add them in quadrature as they form a kind of random walk. Then we have

$$(\delta r)^2 = \Sigma_k(\delta r)^2_k \to a^3 \int d^3\mathbf{k}(\delta r)^2_k \sim a^3 \int k^2(\delta r)^2_k dk = \frac{\alpha}{m^2}\int \frac{dk}{k} \tag{7.20}$$

The limits on this integral are set by the size of the electron for small k and by the electron mass for large k since any such quantum fluctuation would be smaller than the electron and wouldn't have a well defined effect upon it. So we are left with

$$(\delta r)^2 = \frac{\alpha}{m^2}\ln(ma) = \frac{\alpha}{m^2}\ln(1/\alpha) \tag{7.21}$$

Now this quantum buffeting will have very little effect upon the overall energy as it will give rise to positive and negative displacements which largely cancel. You can see however this will not be the case at the origin, where the buffeting will reduce the amount of time the electron stays close to $r = 0$. This will be given approximately by δ_r^3/a^3 then

$$\Delta E = \frac{\alpha}{\delta r} \times \frac{(\delta r)^3}{a^3} = m\alpha^5 \ln(1/\alpha) \tag{7.22}$$

So that

$$\frac{\Delta E}{E} = \alpha^3 \ln\left(\frac{1}{\alpha}\right) \sim 10^{-6} \tag{7.23}$$

The $^2S_{1/2}$ state is spherically symmetric and peaked at the origin whereas the $^2P_{1/2}$ state goes to zero at the origin. So we expect that the electrons in the S state to be pushed away from the origin slightly increasing the energy, this will not happen in the P state.

The actual transition (the difference in energy between the S and P states in question) which is observed is 5.87μeV which is the correct order of magnitude, given that the ryderg constant (the energy of the electrons in hydrogen) is 13.6 eV.

These quantum fluctuations are therefore real and in principle we should worry about their energy because, again, while we do not care about anything other than the differences in energy when we are doing particle physics or quantum physics or QFT, in Cosmology, we care about all of the quatum fields.

If we estimate the energy density of these quantum fields in empty space, we get an integral

$$\langle\rho\rangle = \frac{1}{(2\pi)^3}\frac{1}{2}\int d^3\mathbf{k}\omega_k \tag{7.24}$$

which clearly for radiation will be divergent - it will just get bigger whatever we choose for the value of the cut-off.

$$\langle\rho\rangle = \frac{1}{(2\pi)^3}\frac{4\pi}{2}\int_0^M dk k^3 = \frac{1}{16\pi^2}M^4 \tag{7.25}$$

where M is some scale at which we expect the quantum fluctuations to stop. If we do this carefully (not shown here) we see that the expected pressure is such that $w = -1$ in other words that these vacuum fluctuations are exactly like a cosmological constant.

In the Lamb shift example, we saw that $M = m_e$ which is the electron mass. This would give us $\rho \sim 0.01(5 \times 10^5 eV)^4$ which is 10^{32} times too large to be the cosmological constant. However there is no reason to use the electron mass here, we are not talking about electrons, we are only talking about the vacuum fluctuations which don't care about the electrons. So what cut off should we choose?

The energy of a quantum fluctuation $E = \omega = k$ which is true for $k \gg m$ while the typical size of a quantum fluctuation is $r \sim k^{-1}$. The Schwarzschild radius is therefore $r_{Sch} = 2GM$ so that if

$$r_{Sch} = 2\frac{E}{M_{Pl}^2} \gg E^{-1} \qquad \to E > M_{Pl} \tag{7.26}$$

then a black hole forms, so this would be a natural cut-off, presumably something needs to stop quantum fluctuations of this magnitude. However that would lead to a vacuum energy of $\langle\rho\rangle \sim (10^{19}\mathrm{GeV})^4$ which is a factor 10^{120} too large.

We know that in theories like supersymmetry there are partner fermions for all bosons, and the vacuum contribution is negative for fermions. However we know that if supersymmetry exists it hasn't been discovered yet at the LHC and so it is at best a broken symmetry at low energies and the mass differences for particles and susy partners has to be of the order of a TeV which would mean that $\langle\rho\rangle \sim (10^{3}\mathrm{GeV})^4$ which is 10^{60} times too high.

7.1.3 Short scale deviations to gravity

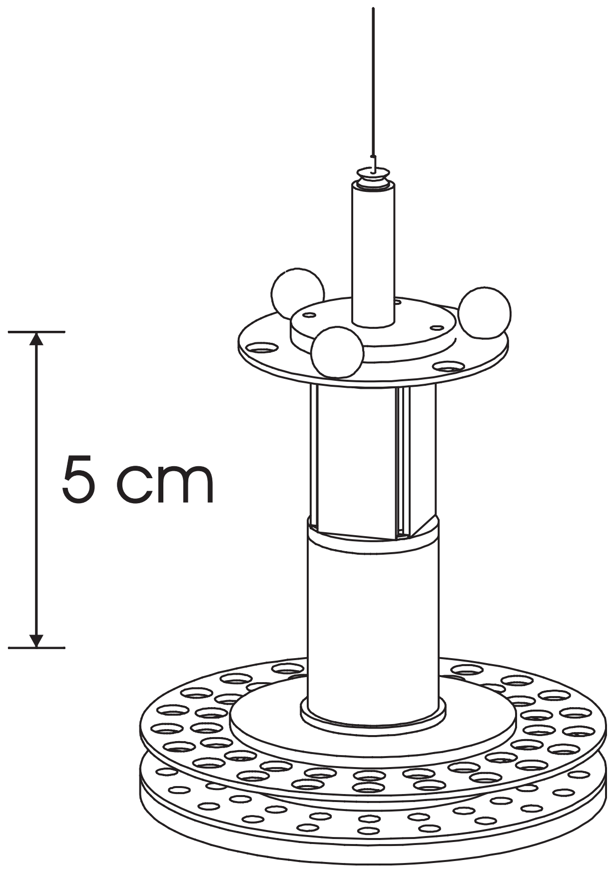

Figure 7.2: Example of Pendulum used by Adelberger group to measure inverse square law at short distances.

When we looked at the Lamb Shift, we said that the quantum fluctuations with wavelength shorter than the electron didn't really interact with it. This is not a bad approximation. If the *graviton* had a finite size then it might be insensitive to the quantum fluctuations with a smaller size. It would only respond to wavelengths larger than its size. This would explain the cosmological constant if the graviton had a finite size corresponding to about $10^3\mathrm{eV}^{-1}$ which is about 0.1 mm. If this was the case then we would expect deviations from the inverse square law on this scale taking the form.

$$V(r) = -G\frac{m_1 m_2}{r}[1 + \alpha \exp(-r/\lambda)] \ , \tag{7.27}$$

where λ will be around 0.1mm. This has been tested using special pendulums like the one shown in figure 7.2. Because of this, these models are somehwat disfavoured.

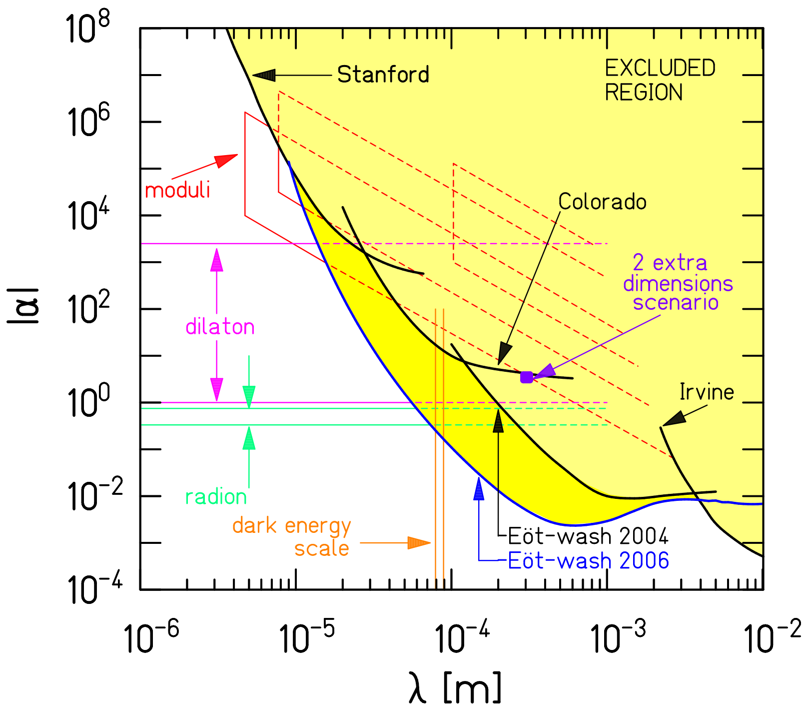

Figure 7.3: Results of the pendulum experiment showing no obvious deviation of inverse square law withing shaded region.

7.1.4 Cancelling the Cosmological Constant

This is based upon the work of Abbott in 1982. If we look again at the potential for the axion but then we add an extra couple of terms to it we get something like

$$V = V_a \left[1 - \cos\left(a/f_a\right)\right] + \epsilon\frac{a}{f_a} + V_0 \tag{7.28}$$

where V_0 is the background energy density of all the stuff in the Universe. V_a is the energy density height of the potential. The extra term proportional to a represents a slight offset in the potential which makes it increase slightly everytime one goes from a minimum at a to one at $a + 2\pi f_a$. The potential in fact has minima wherever $a = 2\pi N f_a$ which have energy density

$$V_N = 2\pi N\epsilon - V_a^4 + V_0 \tag{7.29}$$

and if we make $\epsilon < (10^{-3}eV)^4$ then we can always find a value of N which gives rise to a good value of ρ_{DE}, no matter what the value of V_0 is, positive or negative. The time taken to tunnel from one vacuum to another is exponentially surpressed depending upon the height of the Barrier. The tunelling rate per unit volume is given by

$$\frac{\Gamma}{V} = V_a \exp\left(-\frac{3M_{Pl}^4}{8V_N}\right) \tag{7.30}$$

which would result eventually in our getting down to a very small energy density but only after a very long period of time.

As one tunnels from one minimum to the next, the tunneling will take place in one place and move outwards, with a wall of energy moving outwards from the nucleaotion point which is referred to as a domain wall.

Bousso and Polchinski wanted to realise this in string theory as they realised that in that theory there are two dimensional objects which look a lot like the domain walls called 2-branes. A point charge (zero dimensional - 0D) like an electron has a vector potential A^μ and a 2-form tensor field strength $F^{\mu\nu}$. A one dimensional (1D) object like a string has a 2 form tensor potential $A^{\mu\nu}$ and a 3-form field strength $F^{\mu\nu\gamma}$. A 2D object like a membrane (sometimes called 2-brane) has a 3 form potential $A^{\alpha\beta\gamma}$ and a 4 form field strength $F^{\mu\nu\alpha\beta}$ which, in 3+1 dimensions, has no dynamics. So it looks exactly like a cosmological constant. The only thing it has is a charge, which represents how many branes have been nucleated and because each contributes a tension, this labels the contribution to the cosmological constant.

The tension of those 2-branes would then be related to the parameter ϵ, however the typical scale for that tension is given by the string scale, which is very often chose to be around 10^{16} GeV, too large.

Every time one nucleates a 2-brane, one changes the charge of the stack of branes which is masking the cosmological constant. The overall energy of the vacuum is then

$$V_{tot} = V_0 + V_{string}\Sigma_i n_i^2 q_i^2 \tag{7.31}$$

where V_{string} is very large, potentially $(10^{16}\text{GeV})^4$. Now the different i represent different charges, which could come from higher dimensions of these two dimensional objects being wrapped around different compact dimensions in the compact space (which could have very many different uncontratable loops). If there was only one such i then the jumps between adjacent vacuua would be very large. However, if i is very large, then as one looks at the number of grid points in the compact space, it is possible to get a spacing less than $(10^{-3}\text{eV})^4$. This means that String theory in some sense naturally has the possibility of very many different values of the cosmological constant being realised. However how the Universe picks the good one while also containing matter and radiation is still a mystery unsolved in this mechanism.

7.1.5 Quintessence

One can imagine that there is a field ϕ which has a very light mass, but a potential that it is not at the mimimum of. In the Early Universe, the field is frozen because $H \gg m$. It remains frozen. However in the late Universe, it starts to move. The equation of state

$$w = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)} \tag{7.32}$$

can be very close to -1 but if not exactly -1. These theories are mainly interesting because the equation of state can be different from -1 but apart from that they are not *particularly* interesting. They are a good thing to aim at though when one is making observations.

If $m \gg H$ then the field just rolls very quickly and is nothing like dark energy because the equation of state is wrong - $\dot{phi}$ is too large. To get anything interesting one needs $m \sim H_0$ and we know that $H^2 \sim \rho/M_{Pl}^2$ so that $m \sim \sqrt{\rho}/M_{Pl}$. We have seen that $\rho \sim (10^{-3}\text{eV})^4$ so that m needs to be $10^{-6}\text{eV}/10^{28}\text{GeV}$ which is about 10^{-33}eV. This is a very small mass.

7.1.6 Modified Gravity

It is possible that General Relativity is only valid on small scales, but on large scales there is a different theory. The normal Lagrangian which gives you General Relativity looks like

$$S = \frac{M_{Pl}^2}{2}\int d^4x\sqrt{-g}R + \int d^4x\sqrt{-g}\mathcal{L}_M \tag{7.33}$$

and we normally put dark energy in the matter Lagrangian $\mathcal{L}_M$. However it could be that the gravity part of the Lagrangian is changed, for example

$$S = \frac{M_{Pl}^2}{2}\int d^4x\sqrt{-g}\left(R - \frac{\mu^4}{R}\right) + \int d^4x\sqrt{-g}\mathcal{L}_M \tag{7.34}$$

where μ is a mass scale that must be fitted to the data. We can see straight away that only when R is very small will the extra term be significant. When R is large, like near a black hole or in the solar system then the extra term is not relevant. Variation of this action with respect to the metric gives new field equations

$$\left(1 + \frac{\mu^4}{R^2}\right)R_{\mu\nu} - \frac{1}{2}\left(1 - \frac{\mu^4}{R^2}\right)Rg_{\mu\nu} + \mu^4[g_{\mu\nu}\nabla_\alpha\nabla^\alpha - \nabla_\mu\nabla_\nu]R^{-2} = \frac{T^M_{\mu\nu}}{M_{Pl}^2} \tag{7.35}$$

from which it can be seen that for cosmological solutions there will exist a vacuum solution with $H \sim \mu$. Such $f(R)$ models can therefore give rise to late time acceleration which could be responsible for the apparent dark energy.

The solution of the field equations for a cosmological background lead to cosmological equations with higher derivative terms, for example for the field equations (7.35) the tt Friedman equation for a spatially flat universe becomes

$$3H^2 - \frac{\mu^4}{12(\dot{H} + 2H^2)^3}\left(2H\ddot{H} + 15H^2\dot{H} + 2\dot{H}^2 + 6H^4\right) = \frac{\rho_M}{M_{Pl}^2} \tag{7.36}$$

which means that there are more degrees of freedom in the space of solutions than for Einstein gravity with a cosmological constant or models of quintessence. This space of solutions needs to be compared with the data.

Many models of modified gravity lead to problems because they can violate solar system tests of GR.

7.1.7 Anthropic reasoning

One way to understand the smallness of the cosmological constant is to imagine that there are very many Universes which exist, all with different values of the cosmological constant ρ_{DE}.

If we calculate the number of observers in a Universe with a particular value of ρ_{DE} to be $\mathcal{A}(\rho_{DE})$ and the probability of having a particular value of ρ_{DE} is given by $\mathcal{P}(\rho_{DE})$ then the probability of being in a Universe where we observe such a value of ρ_{DE} is given by

$$\mathcal{P}_{obs}(\rho_{DE})\frac{\mathcal{A}(\rho_{DE})\mathcal{P}(\rho_{DE})}{\int_0^\infty \mathcal{A}(\rho_{DE})\mathcal{P}(\rho_{DE})d\rho_{DE}} \tag{7.37}$$

and if we do the maths, we find that the cosmological constant should not be more than around 100 times greater than what we observe. The value in our Universe is not that unlikely. However Anthropic methods are deeply controversial as we are turning our backs on the scientific method.