1 Plausible reasoning

1.1 Deductive and plausible reasoning

- **Deductive reasoning** can be broken down into a **strong syllogism** (logical argument)

$$\frac{\begin{array}{c}\text{if } A \text{ is true, then } B \text{ is true}\\ A \text{ is true}\end{array}}{\text{therefore, } B \text{ is true}}$$

  and its inverse:

$$\frac{\begin{array}{c}\text{if } A \text{ is true, then } B \text{ is true}\\ B \text{ is false}\end{array}}{\text{therefore, } A \text{ is false.}}$$

- In almost all situations, we do not have the right information to allow this kind of reasoning, so we fall back on **weaker syllogisms**:

$$\frac{\begin{array}{c}\text{if } A \text{ is true, then } B \text{ is true}\\ B \text{ is true}\end{array}}{\text{therefore, } A \text{ becomes more plausible}}$$

- 'If A then B' expresses B only as a **logical** consequence of A, not necessarily a causal, physical consequence:
  - rain at 10 AM does not cause clouds at 9:45 AM
  - the logical connection is not in the uncertain causal direction (clouds $\Rightarrow$ rain), but the certain, though noncausal direction (rain $\Rightarrow$ clouds)
- another weak syllogism:

$$\frac{\begin{array}{c}\text{if } A \text{ is true, then } B \text{ is true}\\ A \text{ is false}\end{array}}{\text{therefore, } B \text{ becomes less plausible}}$$

- a still weaker syllogism:

$$\frac{\begin{array}{c}\text{if } A \text{ is true, then } B \text{ becomes more plausible}\\ B \text{ is true}\end{array}}{\text{therefore, } A \text{ becomes more plausible}}$$

- in doing plausible reasoning, the brain not only decides whether something becomes more or less plausible, but also the *degree* of plausibility
  - we depend very much on **prior information** (aka 'common sense') when evaluating the degree of plausibility of a new problem

1.2 Analogies with physical theories

1.3 The thinking computer

How could we build a machine which would carry out useful plausible reasoning, following clearly defined principles expressing an idealized common sense?

1.4 Introducting the robot

- Our robot will reason about **propositions** $\{A, B, C, \text{etc.}\}$
  - any proposition must have an unambiguous meaning and must be of the simple, definite logical type (i.e. Boolean, true/false)

1.5 Boolean algebra

- **logical product** or **conjunction**, denoted AB
  - both A and B are true (logical AND)
- **logical sum** or **disjunction**, denoted $A + B$
  - at least one of A, B is true (logical OR)
  - $A + B$ is equivalent to $B + A$
- if one of two propositions A, B is true iff the other is true, then A and B have the same **truth value**, denoted $A = B$
  - a primitive axiom of plausible reasoning: two propositions with the same truth value are equally plausible
- parentheses are used as in ordinary algebra, to indicate the order in which propositions are to be combined
  - in absence of parentheses, normal order of operations applies
- the **denial** of a proposition:
  - $\overline{A} \equiv A$ is false
  - $A = \overline{A}$ is false
  - Care is needed in unambiguous use of the bar:

$$\overline{AB} = AB \text{ is false;}$$
$$\bar{A}\bar{B} = \text{both } A \text{ and } B \text{ are false}$$

- Boolean algebra identities:

$$\text{idempotence:} \begin{cases} AA = A \\ A + A = A \end{cases}$$
$$\text{commutativity:} \begin{cases} AB = BA \\ A + B = B + A \end{cases}$$
$$\text{associativity:} \begin{cases} A(BC) = (AB)C = ABC \\ A + (B + C) = (A + B) + C = A + B + C \end{cases}$$
$$\text{distributivity:} \begin{cases} A(B + C) = AB + AC \\ A + (BC) = (A + B)(A + C) \end{cases}$$
$$\text{duality:} \begin{cases} \text{If } C = AB, \text{ then } \overline{C} = \overline{A} + \overline{B} \\ \text{If } D = A + B, \text{ then } \overline{D} = \bar{A}\bar{B} \end{cases}$$

– By applying the basic identities, further relations can be proven:

$$
\begin{aligned}
&\text{Let } \overline{B} = AD & \\
&A\overline{B} = AAD & \\
&A\overline{B} = AD & \text{idempotence} \\
&A\overline{B} = \overline{B} & \text{def. } \overline{B} \\
&A\overline{B} = \overline{B} + \overline{B} & \text{idempotence} \\
&A\overline{B} = \overline{B} + A\overline{B} & \text{def. } \overline{B} \\
&A\overline{B} = \bar{B}\bar{B} + A\overline{B} & \text{idempotence} \\
&A\overline{B} = \left(\overline{B} + A\right)\overline{B} & \text{distributivity} \\
&A = \overline{B} + A & \\
&B\overline{A} = \overline{A} & \text{duality}
\end{aligned}
$$

Therefore, if $\overline{B} = AD$, then $A\overline{B} = \overline{B}$, and $B\overline{A} = \overline{A}$.

- The proposition $A \Rightarrow B$ (A implies B) does not assert that either A or B is true
  - it means only that $A\overline{B}$ is false, or equivalently, $\left(\overline{A} + B\right)$ is true
  - this can also be written as the logical equation $A = AB$
    - i.e., given $A \Rightarrow B$, if A is true, then B must be true; or, if B is false then A must be false
    - This is what was stated in the strong syllogisms
    - However, if A is false, $A \Rightarrow B$ says nothing about B, and if B is true, $A \Rightarrow B$ says nothing about A
      - these are the cases in which the weak syllogisms *do* say something
      - plausible reasoning based on weak syllogisms is not a 'weakened' form of logic; it is an *extension* of logic with content not present in conventional deductive logic
- In formal logic, 'A implies B' means only that A and AB have the same truth value
  - in general, whether B is logically deducible from A depends not only on A and B, but the totality of propositions $(A, A', A'', ...)$ accepted as true and therefore available for use in the deduction

1.6 Adequate sets of operations

- Any number of propositions can be generated using the logical product (conjunction), logical sum (disjunction), implication, and negation operations
  - How large is the set of new propositions? Is it finite/infinite?
  - Are these four operations sufficient to generate every proposition?
  - Are any operations dispensable for generating every proposition?
- Logical NAND is defined as the negation of AND:

$$A \uparrow B \equiv \overline{AB} = \overline{A} + \overline{B}$$

Every logic function can be constructed with NAND alone:

$$
\begin{aligned}
&\overline{A} & \\
=&\overline{A} + \overline{A} & \text{idempotence} \\
=&A \uparrow A & \text{def. NAND}
\end{aligned}
$$

$$\begin{aligned} & AB \\ =& AB + AB && \text{idempotence} \\ =& \overline{AB} \uparrow \overline{AB} && \text{def. NAND} \\ =& (A \uparrow B) \uparrow (A \uparrow B) && \text{def. NAND} \end{aligned}$$

$$\begin{aligned} & A + B \\ =& AA + BB && \text{idempotence} \\ =& \overline{AA} \uparrow \overline{BB} && \text{def. NAND} \\ =& (A \uparrow A) \uparrow (B \uparrow B) && \text{def. NAND} \end{aligned}$$

- Logical NOR is defined as the negation of OR:

$$A \downarrow B \equiv \overline{A + B} = \bar{A}\bar{B}$$

Every logic function can also be constructed with NOR alone:

$$\begin{aligned} & \overline{A} \\ =& \bar{A}\bar{A} && \text{idempotence} \\ =& A \downarrow A && \text{def. NOR} \end{aligned}$$

$$\begin{aligned} & AB \\ =& (A + A)\,(B + B) && \text{idempotence} \\ =& \left(\overline{A + A}\right) \downarrow \left(\overline{B + B}\right) && \text{def. NOR} \\ =& (A \downarrow A) \downarrow (B \downarrow B) && \text{def. NOR} \end{aligned}$$

$$\begin{aligned} & A + B \\ =& (A + B)\,(A + B) && \text{idempotence} \\ =& \left(\overline{A + B}\right) \downarrow \left(\overline{A + B}\right) && \text{def. NOR} \\ =& (A \downarrow B) \downarrow (A \downarrow B) && \text{def. NOR} \end{aligned}$$

1.7 The basic desiderata

I Degrees of plausibility are represented by real numbers

- We adopt a nonessential, but natural, convention: greater plausibility shall correspond to a greater number
- It is also useful to assume continuity, i.e. an infinitesimally greater plausibility shall correspond to an infinitesimally greater number
- The plausibility assigned to a proposition A will, in general, depend on the truth value of proposition B, denoted $A|B$
  - this is ‘the conditional plausibility that A is true, given that B is true, or A given B
  - We avoid impossible problems, i.e. when $A|BC$ is specified, it is understood that B and C are compatible propositions

II Qualitative correspondence with common sense

- if old information C is updated to C' such that the plausibility for A is increased but the plausibility for B is unchanged, i.e, $(A|C') > (A|C)$ and $(B|AC') = (B|AC)$, then the plausibility that both A and B are true must increase, and the plausibility that A is false must decrease:

$$(AB|C') \geq (AB|C) \qquad\qquad \left(\overline{A}|C'\right) < \left(\overline{A}|C\right)$$

III Consistent reasoning

- **IIIa** If a conclusion can be reasoned out in more than one way, then every possible way must lead to the same result.
- **IIIb** All of the evidence relevant to a question is always taken into account. Conclusions are not based on an arbitrary subset of the information.
- **IIIc** Equivalent states of knowledge are represented by equivalent plausibility assignments.

There is only one set of mathematical operations for manipulating plausibilities that satisfies all of the desiderata listed above.

2 The quantitative rules

2.1 The product rule

- We seek a consistent rule relating the plausibility of the logical product AB to the plausibilities of A and B separately, in particular $AB|C$
  - The process of deciding that AB is true can be broken down into elementary decisions about A and B separately:

    1) decide that B is true; $(B|C)$

    2) having accepted B as true, decide that A is true. $(A|BC)$

    or equivalently,

    1') decide that A is true; $(A|C)$

    2') having accepted A as true, decide that B is true. $(B|AC)$