6 Heteroskedasticity and serial correlation. HAC standard errors.

The homoskedasticity and no serial correlation assumption (GM3) can be violated in three ways:

- Heteroskedasticity only (B)- $Var(\varepsilon|X)$ is diagonal with unequal elements along the diagonal.
- Serial correlation only (C) - $Var(\varepsilon|X)$ has non-zero off-diagonal elements, but all diagonal elements are the same.
- Heteroskedasticity and serial correlation (D) - $Var(\varepsilon|X)$ is a general non-diagonal matrix with unequal elements along the diagonal.

$$A = \sigma^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \sigma^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad C = \sigma^2 \begin{bmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{bmatrix} \quad D = \sigma^2 \begin{bmatrix} 1 & \rho & \rho^2 \\ \rho & 2 & \rho \\ \rho^2 & \rho & 3 \end{bmatrix}$$

6.1 Heteroskedasticity

Under heteroskedasticity OLS is still consistent and asymptotically normal, although is no longer efficient and has a different asymptotic covariance matrix. Thus the default standard errors will be wrong. Recall the large sample OLS assumptions, now consider the weaker assumptions OLS2' and OLS3'

(OLS0) (y_i, x_i) is an i.i.d. sequence
(OLS1) $E(x_i x_i')$ is finite non-singular
(OLS2) $E(y_i|x_i) = x_i'\beta$
(OLS3) $\text{Var}(y_i|x_i) = \sigma^2$
(OLS4) $E\varepsilon_i^4 < \infty, \quad E\|x_i\|^4 < \infty$

(OLS2') $E(\varepsilon_i x_i) = 0$
(OLS3') $\text{Var}(\varepsilon_i x_i) = V < \infty$ and is non-singular

Theorem 6.1.1. Under OLS0,1,2',3',4

1. $\hat{\beta}_{OLS} \xrightarrow{p} \beta$ (OLS is consistent)
2. $\sqrt{n}\left(\hat{\beta}_{OLS} - \beta\right) \xrightarrow{d} N\left(0, (\mathbb{E}(x_i x_i'))^{-1} V (\mathbb{E}(x_i x_i'))^{-1}\right)$

Proof. 1. We only require OLS0,1,2' for consistency

$$\begin{aligned} \hat{\beta}_{OLS} &= \beta + (X'X)^{-1}X'\varepsilon \\ &= \beta + \left(\frac{1}{n}\sum_{i=1}^n x_i x_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^n x_i \varepsilon_i \\ &\xrightarrow{p} \beta + [\mathbb{E}(x_i x_i')]^{-1}\mathbb{E}(\varepsilon_i x_i) \\ &= \beta \end{aligned}$$

2.

$$\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)=\left(\frac{1}{n}\sum_{i=1}^{n}x_ix_i'\right)^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}x_i\varepsilon_i$$

Using the CLT:

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}x_i\varepsilon_i\xrightarrow{d}N(0,V)$$

Using the CMT:

$$\left(\frac{1}{n}\sum_{i=1}^{n}x_ix_i'\right)^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}x_i\varepsilon_i\xrightarrow{d}[\mathbb{E}(x_ix_i')]^{-1}N(0,V)$$
$$\Rightarrow\sqrt{n}\left(\hat{\beta}_{OLS}-\beta\right)\xrightarrow{d}N\left(0,(\mathbb{E}(x_ix_i'))^{-1}V(\mathbb{E}(x_ix_i'))^{-1}\right)$$

$\square$

When the errors are homoskedastic the variance is as in previous lectures:

$$\mathbb{E}[X'X]^{-1}\mathbb{E}[X'X\varepsilon_i^2]\mathbb{E}[X'X]^{-1}=\mathbb{E}[X'X]^{-1}\sigma^2\mathbb{E}[X'X]\mathbb{E}[X'X]^{-1}=\sigma^2\mathbb{E}[X'X]^{-1}$$

The classic covariance matrix estimator can be highly biased if homoskedasticity fails, we now consider how to construct covariance matrix estimators which do not require homoskedasticity. If ε_i were known, we could have estimated V as follows:

$$\frac{1}{n}\sum_{i=1}^{n}x_ix_i'\hat{\varepsilon}_i^2\xrightarrow{p}V$$

Of course ε_i is unknown, but since $\hat{\beta}_{OLS}$ remains consistent we can use the observed residuals $\hat{\varepsilon}_i=Y_i-x_i'\hat{\beta}_{OLS}$:

$$\hat{V}=\frac{1}{n}\sum_{i=1}^{n}x_ix_i'\hat{\varepsilon}_i^2$$

To show this is a consistent estimator:

$$\begin{aligned}
\hat{V}&=\frac{1}{n}\sum_{i=1}^{n}x_ix_i'\hat{\varepsilon}_i^2\\
&=\frac{1}{n}\sum_{i=1}^{n}x_ix_i'\left(\varepsilon_i-x_i'\left(\hat{\beta}_{OLS}-\beta\right)\right)^2\\
&=\frac{1}{n}\sum_{i=1}^{n}x_ix_i'\left(\varepsilon_i^2-2\varepsilon_ix_i'\left(\hat{\beta}_{OLS}-\beta\right)+\left(x_i'\left(\hat{\beta}_{OLS}-\beta\right)\right)^2\right)\\
&=\frac{1}{n}\sum_{i=1}^{n}x_ix_i'\varepsilon_i^2-\frac{2}{n}\sum_{i=1}^{n}(x_ix_i')\varepsilon_ix_i'\left(\hat{\beta}_{OLS}-\beta\right)+\frac{1}{n}\sum_{i=1}^{n}x_ix_i'\left(x_i'\left(\hat{\beta}_{OLS}-\beta\right)\right)^2\\
&\xrightarrow{p}V\quad\text{since }\hat{\beta}_{OLS}\xrightarrow{p}\beta
\end{aligned}$$

Definition 6.1.1: White's heteroskedasticity robust covaraince matrix

$$\widehat{Var}(\hat{\beta}_{OLS})=(X'X)^{-1}\left(\sum_{i=1}^{n}x_ix_i'\hat{\varepsilon}_i^2\right)(X'X)^{-1}$$

Note:-

Whilst this estimator is consistent, it is biased in finite samples. To see this, suppose the actual covariance matrix of the population regression residuals is given by $\mathbb{E}[\varepsilon\varepsilon'|X] = \Phi = diag(\phi_i)$. The covariance matrix of the OLS estimator is then

$$V = (X'X)^{-1}(X'\Phi X)(X'X)^{-1}$$

Denote the i-th column of the residual maker matrix M by m_i then $\hat{\varepsilon}_i = m_i'\varepsilon$.

$$\Rightarrow \mathbb{E}[\hat{\varepsilon}_i^2] = \mathbb{E}[m_i'\varepsilon\varepsilon' m_i] = m_i'\Phi m_i$$

Notice that m_i is the i-th column of the identity matrix (denoted as e_i) minus the i-th column of the projection matrix $X(X'X)^{-1}X'$ (p_i). Hence $m_i = e_i - p_i$ and

$$\mathbb{E}[\hat{\varepsilon}_i^2] = (e_i - h_i)'\Phi(e_i - h_i) = \phi_i - 2\phi_i h_{ii} + h_i'\Phi h_i$$

where h_{ii} is the i-th diagonal element of the projection matrix. Because this matrix is symmetric and idempotent, $h_{ii} = h_i'h_i$ so:

$$\begin{aligned}
\mathbb{E}\left(\hat{V} - V\right) &= (X'X)^{-1}(X'\Phi X)(X'X)^{-1} - (X'X)^{-1}(X'\hat{\Phi}X)(X'X)^{-1} \\
&= (X'X)^{-1}(X'(\Phi - \hat{\Phi})X)(X'X)^{-1} \\
&= (X'X)^{-1}(X'diag(\phi_i - (\phi_i - 2\phi_i h_{ii} + h_i'\Phi h_i))X)(X'X)^{-1} \\
&= (X'X)^{-1}(X'diag(h_i'(\Phi - 2\phi_i I)h_i)X)(X'X)^{-1}
\end{aligned}$$

Whilst $\hat{V}$ is biased, here we can see that it is also consistent. Notice that $\hat{\Phi}$ is not consistent for Φ, since there are more elements to estimate as the sample gets large. However, $\hat{\varepsilon}_i$ is consistent for ε_i. We know

$$X'\hat{\Phi}X = \frac{1}{n}\sum_{i=1}^{n} x_i x_i'\hat{\varepsilon}_i^2$$

and since plim $\hat{\varepsilon}_i^2 = \phi_i$ we get plim $X'\hat{\Phi}X = X'\Phi X$.
In summary, $\hat{V}$ is biased since
$hat\varepsilon_i^2$ is a biased estimate of ε^2.

6.2 Serial correlation (and heteroskedasticity)

As with heteroskedasticity, OLS remains consistent and asymptotically normal, but the default standard errors are wrong. This cannot happen if the data are i.i.d. - if OLS0 holds it must be the case that $\Omega = Var(\varepsilon|X)$ is diagonal. If the data are dependent, then Ω is typically no longer diagonal.

Definition 6.2.1: Strict Stationarity

A sequence of random variables $\{Z_t\}_{t=-\infty}^{\infty}$ is strictly stationary if, for any finite nonnegative integer m,

$$f_{Z_t, Z_{t+1}, \ldots, Z_{t+m}}(x_0, x_1, \ldots, x_m) = f_{Z_s, Z_{s+1}, \ldots, Z_{s+m}}(x_0, x_1, \ldots, x_m)$$

which is to say that the joint distribution , f, does not depend on the index, t.

Strict stationarity implies that the (marginal) distribution of Z_t does not vary over time. It also implies that the bivariate distributions of(Z_t, Z_{t+1}) and multivariate distributions of $(Z_t, \ldots, Z_{t+m})$ are stable over time.

Theorem 6.2.1. If Z_t is i.i.d., then it is strictly stationary

Proof. Let F denote the joint distribution function, then:

$$\begin{aligned} F(x_{n+1}, ..., x_{n+m}) &= F(x_{n+1}) \cdot \dots \cdot F(x_{n+m}) \\ &= F(x_{n+k+1}) \cdot \dots \cdot F(x_{n+k+m}) \\ &= F(x_{n+k+1}, \dots, x_{n+k+m}) \end{aligned}$$

Lines 1 and 3 follow from the fact that the joint distribution function of a set of mutually independent variables is equal to the product of their marginal distribution functions. On line 2 we have used the fact that all the terms of the sequence have the same distribution. □

Definition 6.2.2: Covarariance stationarity

A sequence of random variables $\{Z_t\}_{t=-\infty}^{\infty}$ is covariance (weakly) stationary if just the first two moments do not depend on t, e.g.

$$\begin{aligned} \mathbb{E}Z_1 &= \mathbb{E}Z_2 = \dots \\ Var(Z_1) &= Var(Z_2) = \dots \\ Cov(Z_1, Z_{1+m}) &= Cov(Z_2, Z_{2+m}) = \dots \end{aligned}$$

A strictly stationary process is covariance-stationary as long as the variance and covariances are finite.
Consider a new set of OLS assumptions:

(SC0) $\{(y_t, x_t)\}_{t=1}^{T}$ is strictly stationary

(SC1) $\{(x_t x_t')\}$ satisfies LLN: $\frac{1}{T}\sum x_t x_t' \xrightarrow{p} \mathbb{E}(x_t x_t') < \infty$, positive definite

(SC2) $\{(x_t \varepsilon_t)\}$ satisfies LLN: $\frac{1}{T}\sum x_t \varepsilon_t \xrightarrow{p} \mathbb{E}(x_t \varepsilon_t) = 0$

(SC3) $\{(x_t \varepsilon_t)\}$ satisfies CLT: $\frac{1}{\sqrt{T}}\sum x_t \varepsilon_t \xrightarrow{d} N(0, V)$, where

$$V = \mathbb{E}(\varepsilon_t^2 x_t x_t') + \sum_{t=1}^{\infty} \left(\mathbb{E}(\varepsilon_t \varepsilon_{t-l} x_t x_{t-l}') + \mathbb{E}(\varepsilon_t \varepsilon_{t-l} x_{t-l} x_t') \right)$$

These assumptions further generalise our GM/OLS conditions, such that if the data were independent, we would have $V = \mathbb{E}(\varepsilon_t^2 x_t x_t')$ as in OLS3'.

Theorem 6.2.2. Under SC0,1,2,3

1. $\hat{\beta}_{OLS} \xrightarrow{p} \beta$ (OLS is consistent)
2. $\sqrt{T}\left(\hat{\beta}_{OLS} - \beta\right) \xrightarrow{d} N\left(0, (\mathbb{E}(x_t x_t'))^{-1} V (\mathbb{E}(x_t x_t'))^{-1}\right)$

The proof is identical to the heteroskedastic case in Theorem 6.2.1.

Newey-West Method

Under the SC assumptions, the conventional covariance matrix estimators are inconsistent as they do not capture the serial dependence in $x_t e_t$. To consistently estimate the covariance matrix, we need a different estimator. The appropriate class of estimators are called Heteroskedasticity and Autocorrelation Consistent (HAC) covariance matrix estimators.

Define V_T as follows:

$$
\begin{aligned}
V_T &\equiv Var\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} x_t\varepsilon_t\right) \\
&= \mathbb{E}\left[\frac{1}{T}\left(\sum_{t=1}^{T} x_t\varepsilon_t\right)\left(\sum_{t=1}^{T} x_t\varepsilon_t\right)'\right] \\
&= \mathbb{E}\left[\frac{1}{T}(x_1\varepsilon_1 + x_2\varepsilon_2 + \cdots + x_T\varepsilon_T)(x_1'\varepsilon_1 + x_2'\varepsilon_2 + \cdots + x_T'\varepsilon_T)'\right] \\
&= \mathbb{E}\left[\underbrace{\frac{1}{T}\sum_{t=1}^{T}\varepsilon_t^2 x_t x_t'}_{\text{variance at t}} + \underbrace{\frac{1}{T}\sum_{\ell=1}^{T-1}\sum_{t=\ell+1}^{T}\left(\varepsilon_t\varepsilon_{t-\ell}x_t x_{t-\ell}' + \varepsilon_t\varepsilon_{t-\ell}x_{t-\ell}x_t'\right)}_{\text{all covariances with all other time periods}}\right] \\
&= \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[\varepsilon_t^2 x_t x_t'] + \frac{1}{T}\sum_{\ell=1}^{T-1}\sum_{t=\ell+1}^{T}\left(\mathbb{E}(\varepsilon_t\varepsilon_{t-\ell}x_t x_{t-\ell}') + \mathbb{E}(\varepsilon_t\varepsilon_{t-\ell}x_{t-\ell}x_t')\right) \\
&= \mathbb{E}[\varepsilon_t^2 x_t x_t'] + \sum_{\ell=1}^{T-1}\frac{T-\ell}{T}\left(\mathbb{E}(\varepsilon_t\varepsilon_{t-\ell}x_t x_{t-\ell}') + \mathbb{E}(\varepsilon_t\varepsilon_{t-\ell}x_{t-\ell}x_t')\right) \quad \text{Using SC0}
\end{aligned}
$$

As T get large, $V_T \approx V$. Since we have T data points, we can only estimate $G < T$ autocovariances of $x_t\varepsilon_t$, where G is the truncation lag. Newey and West propose the following procedure:

1. Choose G such that: $G = O(T^\alpha)$ for $0 < \alpha < 1/4$
2. Estimate autocovariances of $x_t\varepsilon_t$ of order ℓ by
$$\hat{\Gamma}_\ell = \frac{1}{T}\sum_{t=\ell+1}^{T}\hat{\varepsilon}_t\hat{\varepsilon}_{t-\ell}x_t x_{t-\ell}'$$
3. Estimate V by
$$\hat{V}_{nw} = \hat{\Gamma}_0 + \sum_{\ell=1}^{G}\frac{G+1-\ell}{G+1}\left(\hat{\Gamma}_\ell + \hat{\Gamma}_\ell'\right)$$

If we know a priori that autocovariances are zero in population beyond a certain finite lag q, we can consistently estimate V with

$$\hat{V} = \hat{\Gamma}_0 + \sum_{\ell=1}^{q}\left(\hat{\Gamma}_\ell + \hat{\Gamma}_\ell'\right)$$

However in the case where we do not know q (which is potentially infinite), we can use the weighted sum suggested by Newey and West. For example, for $q(n) = 3$

$$\hat{V}_{NW} = \hat{\Gamma}_0 + \frac{2}{3}(\hat{\Gamma}_1 + \hat{\Gamma}_1') + \frac{1}{3}(\hat{\Gamma}_2 + \hat{\Gamma}_2')$$

The weighting term ensures $\hat{V}_{nw}$ is positive semi-definite. We can see the similarities between this and our expression for V_T earlier, giving some intuition for its consistency.

$$
\begin{aligned}
V_T &= \mathbb{E}[\varepsilon_t^2 x_t x_t'] + \sum_{\ell=1}^{T-1}\frac{T-\ell}{T}\left[\mathbb{E}(\varepsilon_t\varepsilon_{t-\ell}x_t x_{t-\ell}') + \mathbb{E}(\varepsilon_t\varepsilon_{t-\ell}x_{t-\ell}x_t')\right] \\
\hat{V}_{nw} &= \frac{1}{T}\sum_{t=1}^{T}\hat{\varepsilon}_t^2 x_t x_t' + \sum_{\ell=1}^{G}\frac{G+1-\ell}{G+1}\left[\frac{1}{T}\sum_{t=\ell+1}^{T}\left(\varepsilon_t\varepsilon_{t-\ell}x_t x_{t-\ell}'\right) + \frac{1}{T}\sum_{t=\ell+1}^{T}\left(\varepsilon_t\varepsilon_{t-\ell}x_{t-\ell}x_t'\right)\right]
\end{aligned}
$$

Now we can estimate the covariance matrix of $\hat{\beta}_{OLS}$ as

$$\frac{1}{T}\left[\frac{1}{T}\sum_{t=1}^{T} x_t x_t'\right]^{-1} \hat{V}_{nw} \left[\frac{1}{T}\sum_{t=1}^{T} x_t x_t'\right]^{-1}$$

Lemma 6.2.1. The matrix of sample covariances for any process is positive semi-definite.

Proof. Let $z_1, \ldots, z_T$ be any sequence of T numbers, and let P be a $m \times m$ matrix of sample covariances:

$$P = \begin{bmatrix} \frac{1}{T}\sum_{t=1}^{T} z_t^2 & \frac{1}{T}\sum_{t=2}^{T} z_t z_{t-1} & \ddots & \frac{1}{T}\sum_{t=m+1}^{T} z_t z_{t-m} \\ \frac{1}{T}\sum_{t=2}^{T} z_t z_{t-1} & \frac{1}{T}\sum_{t=1}^{T} z_t^2 & \ddots & \ddots \\ \ddots & \ddots & \ddots & \frac{1}{T}\sum_{t=2}^{T} z_t z_{t-1} \\ \frac{1}{T}\sum_{t=m+1}^{T} z_t z_{t-m} & \ddots & \frac{1}{T}\sum_{t=2}^{T} z_t z_{t-1} & \frac{1}{T}\sum_{t=1}^{T} z_t^2 \end{bmatrix}$$

Consider the $m \times (2T-1)$ matrix:

$$Z = \begin{bmatrix} z_1 & z_2 & \cdots & z_m & \cdots & z_T & 0 & \cdots & 0 \\ 0 & z_1 & z_2 & \cdots & z_m & \cdots & z_T & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & z_1 & z_2 & \cdots & \cdots & \cdots & z_T \end{bmatrix}$$

$$\frac{1}{T}ZZ' = \frac{1}{T} \underbrace{\begin{bmatrix} z_1 & z_2 & \cdots & z_m & \cdots & z_T & 0 & \cdots & 0 \\ 0 & z_1 & z_2 & \cdots & z_m & \cdots & z_T & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & z_1 & z_2 & \cdots & \cdots & \cdots & z_T \end{bmatrix}}_{m\times(2T-1)} \underbrace{\begin{bmatrix} z_1 & 0 & \cdots & 0 \\ z_2 & z_1 & \cdots & \vdots \\ \vdots & z_2 & \ddots & 0 \\ z_m & \vdots & \ddots & z_1 \\ \vdots & z_m & \ddots & z_2 \\ z_T & \vdots & \ddots & \vdots \\ 0 & z_T & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & z_T \end{bmatrix}}_{(2T-1)\times m}$$

$$= \frac{1}{T} \begin{bmatrix} \sum_{t=1}^{T} z_t^2 & \sum_{t=2}^{T} z_t z_{t-1} & \ddots & \sum_{t=m+1}^{T} z_t z_{t-m} \\ \sum_{t=2}^{T} z_t z_{t-1} & \sum_{t=1}^{T} z_t^2 & \ddots & \ddots \\ \ddots & \ddots & \ddots & \sum_{t=2}^{T} z_t z_{t-1} \\ \sum_{t=m+1}^{T} z_t z_{t-m} & \ddots & \sum_{t=2}^{T} z_t z_{t-1} & \sum_{t=1}^{T} z_t^2 \end{bmatrix} = P_{m\times m}$$

Thus P is p.s.d. since for any vector $v, v'ZZ'v = u'u = \sum_{i=1}^{m} u_i^2 \geq 0$ $\square$

Theorem 6.2.3. $\hat{V}_{nw}$ is positive semi-definite

Proof. Let c be any deterministic k-dimensional vector, we aim to show $c'\hat{V}_{nw}c \geq 0$. Consider the $G+1$ matrix

$$P = \begin{bmatrix} c'\hat{\Gamma}_0 c & c'\hat{\Gamma}_1 c & \ddots & c'\hat{\Gamma}_G c \\ c'\hat{\Gamma}_1 c & c'\hat{\Gamma}_0 c & \ddots & \ddots \\ \ddots & \ddots & \ddots & c'\hat{\Gamma}_1 c \\ c'\hat{\Gamma}_G c & \ddots & c'\hat{\Gamma}_1 c & c'\hat{\Gamma}_0 c \end{bmatrix}$$

If i is a $G+1$-dimensional vector of ones, then we have

$$\begin{aligned}
i'Pi &= \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} c'\hat{\Gamma}_0 c & c'\hat{\Gamma}_1 c & \ddots & c'\hat{\Gamma}_G c \\ c'\hat{\Gamma}_1' c & c'\hat{\Gamma}_0 c & \ddots & \ddots \\ \ddots & \ddots & \ddots & c'\hat{\Gamma}_1 c \\ c'\hat{\Gamma}_G' c & \ddots & c'\hat{\Gamma}_1' c & c'\hat{\Gamma}_0 c \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} \sum_{\ell=0}^{G} c'\hat{\Gamma}_\ell c \\ \sum_{\ell=1}^{1} c'\hat{\Gamma}_\ell' c + \sum_{\ell=0}^{G-1} c'\hat{\Gamma}_\ell c \\ \vdots \\ \sum_{\ell=0}^{G} c'\hat{\Gamma}_\ell' c \end{bmatrix} \quad m\text{-th row} = \sum_{\ell=1}^{m} c'\hat{\Gamma}_\ell' c + \sum_{\ell=0}^{G-m} c'\hat{\Gamma}_\ell c \\
&= \sum_{\ell=0}^{G} c'\hat{\Gamma}_\ell c + \sum_{\ell=1}^{1} c'\hat{\Gamma}_\ell' c + \sum_{\ell=0}^{G-1} c'\hat{\Gamma}_\ell c + \dots + \sum_{\ell=0}^{G} c'\hat{\Gamma}_\ell' c \\
&= (G+1)c'\hat{\Gamma}_0 c + G(c'\hat{\Gamma}_1' c + c'\hat{\Gamma}_1 c) + (G-1)(c'\hat{\Gamma}_2' c + c'\hat{\Gamma}_2 c) + \dots \\
&= (G+1)c'\hat{\Gamma}_0 c + \sum_{\ell=1}^{G}(G+1-\ell)(c'\hat{\Gamma}_\ell' c + c'\hat{\Gamma}_\ell c) \\
\Rightarrow \frac{1}{G+1} i'Pi &= c'\hat{\Gamma}_0 c + \sum_{\ell=1}^{G} \frac{G+1-\ell}{G+1}(c'\hat{\Gamma}_\ell' c + c'\hat{\Gamma}_\ell c) \\
&= c'\hat{V}_{nw}c
\end{aligned}$$

Hence, it is sufficient to show that P is positive semi-definite. However, P is the matrix of sample covariances of the process $z_t = c'x_t\hat{\varepsilon}_t$ with autocovariances:

$$\mathbb{E}[c'x_t\varepsilon_t\varepsilon_{t-j}x_{t-j}'c] = c'\mathbb{E}[\varepsilon_t\varepsilon_{t-j}x_tx_{t-j}']c = c'\Gamma_j c \quad \forall j \in \mathbb{Z}$$

The matrix of sample covariances for any process is positive semi-definite, thus $\hat{V}_{nw}$ is p.s.d. $\square$

Note:-

The population covariance matrix is always positive semi-definite, so it's desirable for its estimate to also be positive semi-definite. Thus in a time series context we define sample covariances as:

$$\frac{1}{T}\sum_{t=|i-j|+1}^{T} z_t z_{t-|i-j|} \quad \text{rather than as} \quad \frac{1}{T-|i-j|}\sum_{t=|i-j|+1}^{T} z_t z_{t-|i-j|}$$

Even though the former is biased and the latter unbiased, had we used the latter we might get an estimate that is not positive semi-definite.