# 3 Geometric Interpretation of OLS, Mean Variance of OLS, Partitioned Regression

## 3.1 Geometric Interpretation

Consider estimation of $\beta$ in the model:

$$y_i = x_i^{'}\beta + \varepsilon_i, \; i = 1, ..., n$$

This is equivalent in matrix form to: $Y = X\beta + \varepsilon$

The OLS estimator is: $\hat{\beta} = (X^{'}X)^{-1}X^{'}Y$

**Definition 3.1.1**

The Projection Matrix is defined as:

$$P_X = X(X^{'}X)^{-1}X^{'}$$

The Residual Maker Matrix is defined as:

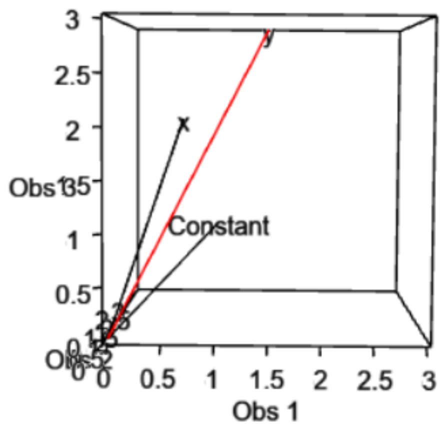
$$M_X = I - P_X$$

Then

$$\hat{Y} = X\hat{\beta} = P_X Y$$

$$\hat{\varepsilon} = Y - \hat{Y} = M_X Y$$

**Claim 3.1.1.** $P_X$ and $M_X$ are symmetric and idempotent.



Thus, $\hat{Y} = X\hat{\beta}$ is the orthogonal projection of the n-dimensional vector $Y$ onto the subspace spanned by the columns of $X$. Each column of X represents the n values that each regressor takes for every observation.

The "subspace" spanned by the columns of $X$ is the set of all linear combinations of the columns of $X$. The orthogonal projection of $Y$ onto this subspace is the closest point in the subspace to $Y$. This is because we solve:

$$\hat{\beta} = \underset{b}{\operatorname{argmin}} \sum (y_i - x_i'b)^2 = \underset{b}{\operatorname{argmin}} (Y - Xb)'(Y - Xb) = \underset{b}{\operatorname{argmin}} \|Y - Xb\|^2$$

**Example.** $k = n$

Clearly if we had k=n regressors, then the columns of $X$ would span the entire n-dimensional space and the projection would be the identity matrix. In this case, $\hat{Y} = Y$, and the residuals would be zero.

### 3.1.1 The Residual Vector

The difference between $Y$ and the projection of $Y$ onto the subspace is the residual vector $\hat{\varepsilon}$.

**Claim 3.1.2.** The residual vector is orthogonal to the subspace spanned by the columns of $X$ and so is orthogonal to each column of $X$ $X'\hat{\varepsilon} = 0$

**Proof.** Intuitively: This is because the projection of $Y$ onto the subspace is the closest point in the subspace to $Y$. If the residual vector were not orthogonal to the subspace, then we could move the projection of $Y$ onto the subspace along the residual vector and get a point that is closer to $Y$. This would contradict the fact that the projection of $Y$ onto the subspace is the closest point in the subspace to $Y$.

Algebraically:

$$X'\hat{\varepsilon} = X'(Y - \hat{Y}) = X'(Y - P_X Y) = X'(Y - X(X'X)^{-1}X'Y) = 0$$

□

## 3.2 Conditional Mean and Variance of OLS

### 3.2.1 Conditional Mean

**Claim 3.2.1.** $\hat{\beta}$ is a conditionally unbiased estimator of $\beta$

$$\mathbb{E}[\hat{\beta}|X] = \beta$$

**Proof.**

$$\hat{\beta} = (X'X)^{-1}X'Y = (X'X)^{-1}X'(X\beta + \varepsilon) = \beta + (X'X)^{-1}X'\varepsilon$$

$$\mathbb{E}[\hat{\beta}|X] = \beta + (X'X)^{-1}X'\mathbb{E}[\varepsilon|X] \overset{1}{=} \beta$$

1. via strict exogeneity $\mathbb{E}[\varepsilon|X] = 0$, do not need iid (e.g. can have a regressor $x_i = i$)

$\square$

Also only need strict exogeneity for a causal interpretation of $\beta$.

**Claim 3.2.2.** $\hat{\beta}$ is an unconditionally unbiased estimator of $\beta$, provided expectations exist

$$\mathbb{E}[\hat{\beta}|X] = \beta$$

**Proof.** via law of iterated expectations

$$\mathbb{E}[\hat{\beta}] = \mathbb{E}[\mathbb{E}[\hat{\beta}|X]] = \mathbb{E}[\beta] = \beta$$

$\square$

### 3.2.2 Conditional Variance

**Theorem 3.2.1.**

$$Var(\hat{\beta}|X) = \sigma^2(X'X)^{-1}$$

**Lemma 3.2.1.** Unconditional Variance of a vector:

$$Var(z) = \mathbb{E}[(z - \mathbb{E}[z])(z - \mathbb{E}[z])'] = \mathbb{E}[zz'] - \mathbb{E}[z]\mathbb{E}[z']$$

**Corollary 3.2.1.** Conditional Variance of a vector:

$$Var(z|X) = \mathbb{E}[zz'|X] - \mathbb{E}[z|X]\mathbb{E}[z'|X]$$

Thus for $z = A(X)w$ where A is a matrix that depends on X we have:

$$Var(z|X) = \mathbb{E}[A(X)ww'A(X)'|X] - \mathbb{E}[A(X)w|X]\mathbb{E}[w'A(X)'|X]$$

$$= A(X)\mathbb{E}[ww'|X]A(X)' - A(X)\mathbb{E}[w|X]\mathbb{E}[w'|X]A(X)'$$
$$= A(X)Var(w|X)A(X)'$$

Therefore:

$$Var(\hat{\beta}|X) = Var(\beta + (X'X)^{-1}X'\varepsilon|X) = (X'X)^{-1}X'Var(\varepsilon|X)X(X'X)^{-1}$$

Then assuming homoskedasticity and no serial correlation: $Var(\varepsilon|X) = \sigma^2 I_n$

$$= (X'X)^{-1}X'\sigma^2 I_n X(X'X)^{-1} = \sigma^2(X'X)^{-1}$$

## 3.3 Partitioned Regression

To find formulae for conditional variances of component of $\hat{\beta}$ we can partition $X$ and $\beta$ into two parts:

$$X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$$

, $X_1$ is $n \times k_1$, $X_2$ is $n \times k_2$, $k_1 + k_2 = k$

$$\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

Then: $Y = X\beta + \varepsilon = X_1\beta_1 + X_2\beta_2 + \varepsilon$

**Theorem 3.3.1.**

$$Var(\hat{\beta}_1|X) = \sigma^2(X_1'M_2X_1)^{-1}$$

**Proof.** Recall that

$$\hat{\beta} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = (X'X)^{-1}X'Y$$

$$\begin{bmatrix} X_1 & X_2 \end{bmatrix}' \begin{bmatrix} X_1 & X_2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \end{bmatrix}' Y$$

thus

$$\begin{bmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} X_1'Y \\ X_2'Y \end{bmatrix}$$

this yields two equations in two unknowns:

$$X_1'X_1\hat{\beta}_1 + X_1'X_2\hat{\beta}_2 = X_1'Y$$

$$X_2'X_1\hat{\beta}_1 + X_2'X_2\hat{\beta}_2 = X_2'Y$$

Expressing $\hat{\beta}_1$ in terms of $\hat{\beta}_2$ and substituting into the second equation yields:

$$(X_2'X_1)(X_1'X_1)^{-1}(X_1'Y - (X_1'X_2)\hat{\beta}_2) + (X_2'X_2)\hat{\beta}_2 = X_2'Y$$
$$((X_2'X_2) - (X_2'X_1)(X_1'X_1)^{-1}(X_1'X_2))\hat{\beta}_2 = (X_2' - (X_2'X_1)(X_1'X_1)^{-1}X_1')Y$$
$$X_2'(I - X_1(X_1'X_1)^{-1}X_1')X_2\hat{\beta}_2 = X_2'(I - X_1(X_1'X_1)^{-1}X_1')Y$$

Recalling the definition of the residual maker matrix, $M_x$, we define $M_1$ as the residual maker matrix for $X_1$:

$$M_1 = I - X_1(X_1'X_1)^{-1}X_1'$$

Therefore,

$$\hat{\beta}_2 = (X_2'M_1X_2)^{-1}X_2'M_1Y$$

and similarly

$$\hat{\beta}_1 = (X_1'M_2X_1)^{-1}X_1'M_2Y$$
$$Var(\hat{\beta}_1|X) = Var((X_1'M_2X_1)^{-1}X_1'M_2Y|X)$$

$$= (X_1'M_2X_1)^{-1}X_1'M_2Var(Y|X)M_2X_1(X_1'M_2X_1)^{-1}$$
$$= (X_1'M_2X_1)^{-1}X_1'M_2\sigma^2 I_n M_2X_1(X_1'M_2X_1)^{-1}$$
$$= \sigma^2(X_1'M_2X_1)^{-1}$$

Similarly,

$$Var(\hat{\beta}_2|X) = \sigma^2(X_2'M_1X_2)^{-1}$$

If $X_1$ and $X_2$ are 'almost' colinear, projection of $X_1$ onto spaces orthogonal to $X_2$ is almost zero. Thus $X_1'M_2X_1$ is almost zero and so $Var(\hat{\beta}_1|X)$ is very large. This is an example of multicollinearity.

□

### 3.3.1 FRISCH-WAUGH-LOVELL THEOREM

**Theorem 3.3.2.** The OLS estimator of $\beta_1$ in the regression of $Y$ on $X$ is the same as the OLS estimator of $\beta_1$ in the regression of $M_2Y$ on $M_2X_1$.

This is from a two step procedure:

1. Obtain $M_2Y$ by regressing $Y$ on $X_2$ and forming residuals. This is the portion of Y not correlated with $X_2$.

$$\hat{e} = Y - X_2(X_2'X_2)^{-1}X_2'Y = M_2Y$$

Obtain $M_2X_1$ by regressing $X_1$ on $X_2$. This is the portion of $X_1$ not correlated with $X_2$.

$$\hat{v} = X_1 - X_2(X_2'X_2)^{-1}X_2'X_1 = M_2X_1$$

2. Then regress $M_2Y$ on $M_2X_1$, equivalently $\hat{e}$ on $\hat{v}$. This measures the effect of $X_1$ on $Y$ after controlling for $X_2$.

**Proof.** Comparing the OLS estimators:

$$\hat{\beta}_1 = (X_1'M_2X_1)^{-1}X_1'M_2Y = (X_1'M_2'M_2X_1)^{-1}X_1'M_2'M_2Y$$
$$= [(M_2X_1)'(M_2X_1)]^{-1}(M_2X_1)'M_2Y$$

Thus the OLS estimator of $\beta_1$ in the regression of $Y$ on $X$ is the same as the OLS estimator of $\beta_1$ in the regression of $M_2Y$ on $M_2X_1$.

Then comparing regression residuals:

$$\hat{\varepsilon} = Y - X\hat{\beta} = Y - X_1\hat{\beta}_1 - X_2\hat{\beta}_2$$

Residual from step 2 of the partitioned regression is:

$$\tilde{\varepsilon} = M_2Y - M_2X_1\hat{\beta}_1 = M_2(Y - X_1\hat{\beta}_1) = M_2(Y - X_1\hat{\beta}_1 - X_2\hat{\beta}_2) = M_2\hat{\varepsilon} = \hat{\varepsilon}$$

This third equality holds because $M_2X_2 = 0$. Thus the residuals from the two regressions are the same and so the regression procedures are identical.

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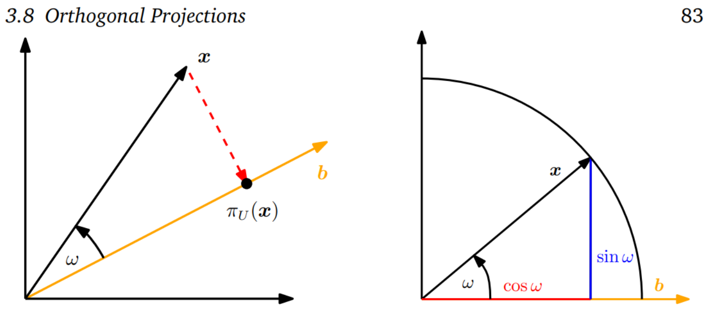
# 3.4 Appendix: Projection Onto a Line

Assume inner product is the dot proucts, defined as $x'y = \sum_{i=1}^{n} x_i y_i$




(a) Projection of $\boldsymbol{x} \in \mathbb{R}^2$ onto a subspace $U$ with basis vector $\boldsymbol{b}$.

(b) Projection of a two-dimensional vector $\boldsymbol{x}$ with $\|\boldsymbol{x}\| = 1$ onto a one-dimensional subspace spanned by $\boldsymbol{b}$.

where $x$ is projected onto a one-dimensional subspace $U \subseteq \mathbb{R}^n$ spanned by basis vector $b$. This goes through the origin.

When projecting $x \in \mathbb{R}^n$ onto $U$, we want to find the vector $\pi_U(X) \in U$ that is closest to x.

**Proposition 3.4.1.** As before we minimise $||x - \pi_U(x)||^2$. This implies that $x - \pi_U(x)$ is orthogonal to $U$ and thus also orthogonal to the basis vector $b$.

$$\langle x - \pi_U(x), b \rangle = 0$$

**Proposition 3.4.2.** Further, the projection $\pi_U(x)$ must be an element of $U$ and so is a scalar multiple of $b$, which spans U. Hence:

$$\pi_U(x) = \lambda b$$

for some $\lambda \in \mathbb{R}$

### 3.4.1 Finding $\lambda$

Substituting Prop 1.4.2 into 1.4.1 we get:

$$\langle x - \lambda b, b \rangle = 0$$

Exploiting the bilinearity of the inner product:

$$\langle x, b \rangle - \lambda \langle b, b \rangle = 0$$

$$\Rightarrow \lambda = \frac{\langle x, b \rangle}{\langle b, b \rangle} = \frac{\langle x, b \rangle}{||b||^2} = \frac{x'b}{b'b}$$

### 3.4.2 Finding $\pi_U(x)$

Since $\pi_U(x) = \lambda b$, we have:

$$\pi_U(x) = \frac{x'b}{b'b} b$$

The length of $\pi_U(x)$ is:

$$||\pi_U(x)|| = ||\lambda b|| = |\lambda|\,||b||$$

Thus the projection acts as a coordinate of $\pi_U(x)$ in the direction of $b$.

Using the dot product as the inner product we have:

$$= \frac{|x'b|}{||b||^2}||b|| = |cos(\theta)|\,||x||\,||b||\frac{||b||}{||b||^2} = |cos(\theta)|\,||x||$$

### 3.4.3 The Projection Matrix $P_\pi$

As projection is a linear mapping, there exists a matrix $P_\pi$ such that:

$$\pi_U(x) = P_\pi x$$

With the dot as the inner product and

$$\pi_U(x) = \lambda b = b\lambda = b\frac{b'x}{||b||^2} = \frac{bb'}{||b||^2}x$$

Thus

$$P_\pi = \frac{bb'}{||b||^2}$$

## 3.5 Projection Onto a General Subspace

We find a projection of $x \in \mathbb{R}^n$ onto a subspace $U \subseteq \mathbb{R}^n$ with $dim(U) = m \geq 1$. Assume that $b_1, ..., b_m$ is an ordered basis for $U$. Any projection $\pi_U(x)$ onto $U$ can be written as a linear combination of the basis vectors: such that $\pi_U(x) = \sum_{i=1}^m \lambda_i b_i$. We follow the same three step procedure as before:

### 3.5.1 Finding $\lambda_1, ..., \lambda_m$

We find coordinates $\lambda_1, ..., \lambda_m$ such that the linear combination

$$\pi_U(x) = \sum_{i=1}^m \lambda_i b_i = \mathbf{B}\vec{\lambda}$$

$$\mathbf{B} = \begin{bmatrix} \vec{b_1} & ... & \vec{b_m} \end{bmatrix}, \in \mathbb{R}^{n\times m}, \vec{\lambda} = \begin{bmatrix} \lambda_1 \\ ... \\ \lambda_m \end{bmatrix} \in \mathbb{R}^m$$

is such that $\pi_U(x)$ is the closest point in $U$ to $x$. This implies that $x - \pi_U(x)$ is orthogonal to $U$ and thus also orthogonal to each basis vector $b_i$. Thus we obtain simultaneous equations:

$$\langle x - \pi_U(x), b_1\rangle = b_1'(x - \pi_U(x)) = 0$$

$$\vdots$$

$$\langle x - \pi_U(x), b_m\rangle = b_m'(x - \pi_U(x)) = 0$$

as $\pi_U(x) = \mathbf{B}\vec{\lambda}$ we have:

$$b_1'(x - \mathbf{B}\vec{\lambda}) = 0$$

$$\vdots$$
$$b_m'(x - \mathbf{B}\vec{\lambda}) = 0$$

thus we obtain a homogeneous system of linear equations:

$$\begin{bmatrix} b_1' \\ \vdots \\ b_m' \end{bmatrix} (x - \mathbf{B}\vec{\lambda}) = 0$$

$$\Leftrightarrow \mathbf{B}'(x - \mathbf{B}\vec{\lambda}) = 0$$
$$\Leftrightarrow \mathbf{B}'\mathbf{B}\vec{\lambda} = \mathbf{B}'x$$
$$\Leftrightarrow \vec{\lambda} = (\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'x$$

where we require that $\mathbf{B}'\mathbf{B}$ is invertible, which is true if and only if $\mathbf{B}$ has full column rank, which is true if and only if the basis vectors $b_1, ..., b_m$ are linearly independent.

### 3.5.2 Finding $\pi_U(x)$

We have that $\pi_U(x) = \mathbf{B}\vec{\lambda}$ and so:

$$\pi_U(x) = \mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'x$$

### 3.5.3 The Projection Matrix $P_\pi$

As projection is a linear mapping, there exists a matrix $P_\pi$ such that:

$$\pi_U(x) = P_\pi x$$

Thus

$$P_\pi = \mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'$$

## 3.6 Appendix: OLS Estimator Equivalence

**Claim 3.6.1.**

$$\hat{\beta} = (X'X)^{-1}X'Y = \frac{\sum(X_i - \bar{X})(Y_i - \bar{Y})}{\sum(X_i - \bar{X})^2}$$
$$\Leftrightarrow X$$

includes a constant

Let us take the case for $k = 1$, i.e. $X$ is a vector of length $n$. Then: suppose $X$ includes a constant, i.e. $X = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$. Then let $\tilde{x_i} = (1, x_i)'$ Then $X = (\tilde{x_1}, ..., \tilde{x_n})'$ Thus:

$$(X'X)^{-1}X'Y = (\sum_{i=1}^{n} \tilde{x_i}\tilde{x_i}')^{-1} \sum_{i=1}^{n} \tilde{x_i}y_i$$

$$= \left[ \sum_{i=1}^{n} \begin{bmatrix} 1 & x_i \\ x_i & x_i^2 \end{bmatrix} \right]^{-1} \sum_{i=1}^{n} \begin{bmatrix} 1 \\ x_i \end{bmatrix} Y_i$$

$$= \left[ n \begin{bmatrix} 1 & \bar{x} \\ \bar{x} & \frac{1}{n}\sum_{i=1}^{n} x_i^2 \end{bmatrix} \right]^{-1} n \begin{bmatrix} \bar{y} \\ \frac{1}{n}\sum_{i=1}^{n} x_i y_i \end{bmatrix}$$

$$= \frac{1}{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2} \begin{bmatrix} \frac{1}{n}\sum_{i=1}^{n} x_i^2 & -\bar{x} \\ -\bar{x} & 1 \end{bmatrix} \begin{bmatrix} \bar{y} \\ \frac{1}{n}\sum_{i=1}^{n} x_i y_i \end{bmatrix}$$

The second component is the estimate fo the slope coefficient, and the first component is the estimate of the intercept coefficient. Thus we have:

$$\hat{\beta} = \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2}$$