11 ML Asymptotics. Likelihood Ratio Test.

More rigour: Amemiya (1985)

11.0.1 Consistency of ML

Let z_i be iid with density $f(z;\theta_0)$ for $i = 1, \ldots, n$.

$$\hat{\theta}_{ML} = \underset{\theta}{\operatorname{argmax}} \sum_{i=1}^{n} \log f(z_i;\theta)$$

By Khinchine's LLN for any θ

$$\frac{1}{n}\sum_{i=1}^{n} \log f(z_i;\theta) \xrightarrow{p} \mathbb{E}_{\theta_0}[\log f(z;\theta)]$$

We can invoke KLLN as given z_i iid $\Rightarrow$ any function of z_i is also iid. We also need to assume the expectation exists. This is taken over the value of the true parameter, but the conditioned θ runs across the real line.

Proposition 11.0.1.

$$\hat{\theta}_{ML} \xrightarrow{p} \underset{\theta}{\operatorname{argmax}} \mathbb{E}_{\theta_0}[\log f(z;\theta)]$$

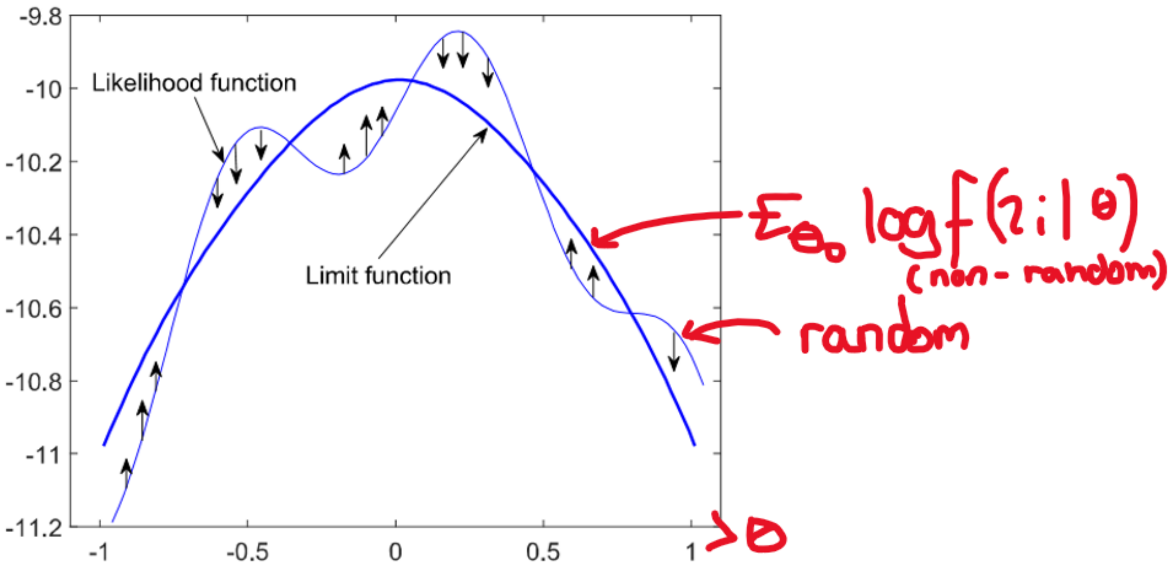
Proof.

$$\hat{\theta}_{ML} = \underset{\theta}{\operatorname{argmax}} \sum_{i=1}^{n} \log f(z_i;\theta) = \underset{\theta}{\operatorname{argmax}} \frac{1}{n}\sum_{i=1}^{n} \log f(z_i;\theta)$$

But:

$$\frac{1}{n}\sum_{i=1}^{n} \log f(z_i;\theta) \xrightarrow{p} \mathbb{E}_{\theta_0}[\log f(z;\theta)] \ \underline{\text{uniformly}} \Rightarrow \underset{\theta}{\operatorname{argmax}} \frac{1}{n}\sum_{i=1}^{n} \log f(z_i;\theta) \xrightarrow{p} \underset{\theta}{\operatorname{argmax}} \mathbb{E}_{\theta_0}[\log f(z;\theta)]$$

$\square$



Proposition 11.0.2. $E_{\theta_0} \log f(z_i; \theta)$ is maximised at the true value of parameter θ_0

Proof. Consider the KL divergence between $f(z;\theta)$ and $f(z;\theta_0)$:

$$E_{\theta_0} \log \frac{f(z;\theta_0)}{f(z;\theta)}$$

By construction the minimiser of the KL divergence must be the maximiser of $E_{\theta_0} \log f(z;\theta)$. By Jensen's inequality:

$$= -E_{\theta_0} \frac{\log f(z;\theta)}{\log f(z;\theta_0)} \geq -\log E_{\theta_0} \frac{f(z;\theta)}{f(z;\theta_0)} = -\log \int \frac{f(z;\theta)}{f(z;\theta_0)} f(z;\theta_0) dz = -\log 1 = 0$$

But we can achieve this bound by setting $\theta = \theta_0$ is a maximiser of $E_{\theta_0} \log f(z;\theta)$. □

Note:-

If there exists another maximiser θ_1, we must have $f(z;\theta_0) = f(z;\theta_1)$ for all z. In such a case, we say that a case, we say that the parameter is non-identified.
In the linear regression example , $\theta = (\beta', \sigma^2)$, would not be identified if $X'X$ has rank lower than k (perfect multicollinearity).
Pointwise convergence is not enough for consistency of the θ_{ML} estimator. Sufficient conditions are given by uniform convergence and "enough" curvature of $E_{\theta_0} \log f(z;\theta)$ around θ_0.

11.0.2 Asymptotic Normality of ML

Proposition 11.0.3.

$$\sqrt{n}(\hat{\theta}_{ML} - \theta_0) \xrightarrow{d} N(0, I^{-1}(\theta_0))$$

where $I(\theta_0)$ is the Fisher information matrix:

$$I_1(\theta_0) = Var\left[\frac{\partial}{\partial\theta} \log f(z;\theta_0)\right] = -\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2} \log f(z;\theta_0)\right] = E_{\theta_0}(H_1) = \mathbb{E}_{\theta_0}(\frac{H}{n})$$

Note $I_1(\theta_0)$ is the Fisher information for a single observation.
Define $I(\theta_0)$ as the Fisher information matrix for the sample.

This is the sum of the Fisher information for each observation $I(\theta_0) = nI_1(\theta_0)$, since $\log(z_i;\theta)$ is a function of iid z_i, and so is iid.

$$Var\left[\frac{\partial}{\partial\theta}L(\theta_0)\right] = Var\left[\frac{\partial}{\partial\theta}\sum_{i=1}^{n}\log f(z_i;\theta_0)\right] = nVar\left[\frac{\partial}{\partial\theta}\log f(z;\theta_0)\right] \text{ since iid}$$

Proof. Let $\Psi(\theta) = \frac{\partial}{\partial\theta}\frac{1}{n}L(\theta;Z)$, where

$$L(\theta;Z) = \sum_{i=1}^{n}\log f(z_i;\theta)$$

$\hat{\theta}_{ML}$ can be obtained as a solution to the likelihood equation: $\Psi(\hat{\theta}_{ML}) = \frac{\partial}{\partial\theta}\frac{1}{n}L(\hat{\theta}_{ML};Z) = 0$
Assuming consistency, $\hat{\theta}_{ML} \xrightarrow{p} \theta_0$, it makes sense to expand $\Psi(\hat{\theta}_{ML})$ around θ_0:

$$\Psi(\hat{\theta}_{ML}) = 0 = \Psi(\theta_0) + (\hat{\theta}_{ML} - \theta_0)\Psi'(\theta_0) + \frac{1}{2}(\hat{\theta}_{ML} - \theta_0)^2\Psi''(\tilde{\theta})$$

where $\tilde{\theta}$ is between $\hat{\theta}_{ML}$ and θ_0, such that the Taylor expansion is exact by the MVT.
Therefore when θ is scalar,

$$\sqrt{n}(\hat{\theta}_{ML} - \theta_0) = \frac{-\sqrt{n}\Psi(\theta_0)}{\Psi'(\theta_0) + (\hat{\theta}_{ML} - \theta_0)\Psi''(\tilde{\theta})/2}$$

But under the random sampling assumption:

$$\Psi(\theta_0) = \frac{1}{n}\sum_{i=1}^{n}\frac{\partial}{\partial\theta}\log f(z_i;\theta_0)\bigg|_{\theta=\theta_0} \xrightarrow{p} \frac{\partial}{\partial\theta}\mathbb{E}_{\theta_0}\log f(z;\theta_0)\bigg|_{\theta=\theta_0}$$

And with the Lindeberg-Levy CLT:

$$-\sqrt{n}\Psi(\theta_0) \xrightarrow{d} N\left(0, Var\left(\frac{\partial}{\partial\theta}\log f(z_i,\theta_0)\right)\right) = N\left(0, \frac{1}{n}I(\theta_0)\right)$$

Next, by Khinchine's LLN:

$$\Psi'(\theta_0) \xrightarrow{p} \frac{\partial^2}{\partial\theta^2}\mathbb{E}_{\theta_0}\log f(z;\theta_0)\bigg|_{\theta=\theta_0}$$

Finally, $(\hat{\theta}_M L - \theta_0)\Psi''(\tilde{\theta}) \xrightarrow{p} 0$ i.e. is $o_p(1)$, since $\hat{\theta}_{ML} - \theta_0 = o_p(1)$ and $\Psi''(\tilde{\theta})$ converges to a finite constant (Amemiya 1985, p. 67,ch 4).
Therefore by Slutsky's theorem:

$$\sqrt{n}(\hat{\theta}_{ML} - \theta_0) = \frac{-\sqrt{n}\Psi(\theta_0)}{\Psi'(\theta_0) + (\hat{\theta}_{ML} - \theta_0)\Psi''(\tilde{\theta})/2} \xrightarrow{d} \frac{N(0, \frac{1}{n}I(\theta_0))}{\mathbb{E}_{\theta_0}\frac{\partial^2}{\partial\theta^2}\log f(z;\theta_0)}$$

$$\therefore \sqrt{n}(\hat{\theta}_{ML} - \theta_0) \xrightarrow{d} \frac{N(0, \frac{1}{n}I(\theta_0))}{-\frac{1}{n}I(\theta_0)} = N(0, nI^{-1}(\theta_0))$$

In other words in large samples, $\hat{\theta}_{ML}$ is approximately normally distributed with mean θ_0 and variance $I^{-1}(\theta_0)$. □

This generalises straightforwardly to the case of a vector θ.

NOTE: $I(\theta_0)$ refers to the sample Fisher information matrix, which is $n \times I_1(\theta)$ - the finite information matrix of one observation. Thus saying $\hat{\theta}_{ML}$ is approximately normally distributed with mean θ_0 and variance $I^{-1}(\theta_0)$, means its variance is in fact $(1/n)I_1^{-1}(\theta_0)$, which goes to zero for large n and thus we have $\hat{\theta}_{ML} \xrightarrow{p} \theta_0$ as we found earlier.

11.1 Asymptotic efficiency of the maximum likelihood estimator

Proposition 11.1.1. θ_{ML} **is asymptotically efficient:**
Lowest asymptotic variance among all estimators that are

- asymptotically normal
- asymptotically unbiased
- regular

Recall the Cramér-Rao result:
Any unbiased estimator of θ_0 has variance no smaller than the inverse of the Fisher information. While suggestive of asymptotic efficiency here, it is a *finite* sample result and thus does not imply this.

11.1.1 Irregular Estimators

Hodges' Estimator

$$\theta_H = \begin{cases} \hat{\theta}_{ML} & \text{if } |\hat{\theta}_{ML}| \geq n^{-1/4} \\ 0 & \text{if } |\hat{\theta}_{ML}| < n^{-1/4} \end{cases}$$

Case 1: $\theta_0 \neq 0$
$\hat{\theta}_H$ is asymptotically equivalent to $\hat{\theta}_{ML}$. This is because $\hat{\theta}_{ML} \xrightarrow{p} \theta_0 \neq 0$, and $n^{-1/4} \to 0$, thus $|\hat{\theta}_{ML}| \geq n^{-1/4}$ will be true asymptotically, so $\hat{\theta}_H = \hat{\theta}_{ML}$ asymptotically.

Note:-

Big O, Little O Notation
$f(x) \in O(g(x))$ if $\exists\ K > 0$ and x_0 such that $|f(x)| \leq Kg(x)$ for all $x > x_0$.
$f(x) \in o(g(x))$ if $\forall\ K > 0\ \exists\ x_0$ such that $|f(x)| < Kg(x)$ for all $x > x_0$.

Product Rule: $f(x) = O(g(x))$ and $h(x) = O(k(x)) \Rightarrow f(x)h(x) = O(g(x)k(x))$
Little O $\Rightarrow$ Big O: $f(x) = o(g(x)) \Rightarrow f(x) = O(g(x))$

In probability:
$X_n \in O_P(\alpha_n)$ if $\forall \varepsilon > 0\ \exists\ K > 0$ and x_0 such that $\Pr(|f(x)| \leq Kg(x)) > 1 - \varepsilon$ for all $x > x_0$.
i.e.X_n/α_n is bounded up to an exceptional event of arbitrarily small (but fixed) positive probability, i.e. the ratio is 'bounded in probability.
$f(x) \in o_p(\alpha_n)$ if $\forall \varepsilon > 0\ \forall\ K > 0\ \exists\ x_0$ such that $\Pr(|f(x)| < Kg(x)) > 1 - \varepsilon$ for all $x > x_0$.

Case 2: $\theta_0 = 0$

Proposition 11.1.2. $|\hat{\theta}_{ML}| = O_p(n^{-1/2})$

Since $\sqrt{n}(\hat{\theta}_{ML} - \theta_0) \xrightarrow{d} N(0, I_1^{-1}(\theta_0))$, we know $\sqrt{n}(\hat{\theta}_{ML} - \theta_0) \in O_p(1)$, since its variance (and expectation) is finite and constant wrt n and so must be bounded in probability.
$\sqrt{n}(\hat{\theta}_{ML} - \theta_0) = \frac{\hat{\theta}_{ML} - \theta_0}{1\sqrt{n}} = O_p(1)$

$\Rightarrow \hat{\theta}_{ML} - \theta_0 = O_p(n^{-1/2})*$
$\therefore |\hat{\theta}_{ML}| = O_p(n^{-1/2})$

*(also loose intution from the product rule of normal big O, $\sqrt{n} = O_p(\sqrt{n})$)
Where let $\hat{\theta}_{ML} - \theta_0 \in O_p(\alpha_n)$
$\sqrt{n}(\hat{\theta}_{ML} - \theta_0) \in O_p(1) \Rightarrow O_P(\sqrt{n})O_P(\alpha_n) = O_P(\sqrt{n}\alpha_n) = O_P(1)$
$\Rightarrow \alpha_n = 1/\sqrt{n}$

Proposition 11.1.3. $|\hat{\theta}_{ML}| = o_p(n^{-1/4})$
$\sqrt{n}(\hat{\theta}_{ML} - \theta_0) \xrightarrow{d} N(0, I_1^{-1}(\theta_0))$ and $n^{-1/4} \xrightarrow{p} 0$ Thus by Slutsky's theorem: $n^{1/4}(\hat{\theta}_{ML} - \theta_0) \xrightarrow{d} 0 \Rightarrow n^{1/4}(\hat{\theta}_{ML} - \theta_0) \xrightarrow{p} 0$
$\Rightarrow \frac{(\hat{\theta}_{ML} - \theta_0)}{1/n^{1/4}} \xrightarrow{p} 0$
$\Rightarrow \lim_{n\to\infty} \mathbb{P}(|\frac{(\hat{\theta}_{ML} - \theta_0)}{1/n^{1/4}} - 0| > \varepsilon) = 0 \, \forall \varepsilon > 0$
$\Rightarrow \hat{\theta}_{ML} - \theta_0 = o_p(n^{-1/4})$ with the defintion of o_p
Intuitively as $n^{-1/4} > n^{-1/2}$, it makes sense that dividing by $n^{-1/4}$ binds more strictly (sends to zero) than dividing by $n^{-1/2}$, which already binds in probability (sends to a constant variance distribution).

When $\theta_0 = 0$ Hodges' estimator clearly imporves over $\hat{\theta}_{ML}$ because $|\hat{\theta}_{ML}| = o_p(n^{-1/4})$, which implies $\hat{\theta}_H = 0$ exactly asymptotically (with zero variance) for sufficiently large n.

But in finite samples, Hodge's estimator behaves poorly for $\theta \approx 0$. Asymptotically, this is reflected in its erratic behaviour when true value of parameter is drifting towards zero so that $\theta = h/\sqrt{n}$ for some $h \in \mathbb{R}$. For such sequences of θ, $\hat{\theta}_H$ is inconsistent. we have:

$$\sqrt{n}(\hat{\theta}_H - \theta_0) = \sqrt{n}(\hat{\theta}_H - h/\sqrt{n}) \to -h$$

Regular estimators would have the same asymptotic distribution for any value of $h/\sqrt{n}$ (a small change in parameter should not change the distribution of the estimator too much)

11.2 Likelihood Ratio Test

Suppose that the likelihood function is in general given by $L(\theta; Z) \equiv f(Z, \theta)$, where Z is a vector of data and θ is a vector of parameters. Consider testing the null hypothesis $H_0 : \theta \in \Theta_0$ against the alternative $H_1 : \theta \in \Theta_1$, where $\Theta_0 \cap \Theta_1 = \emptyset$.

The likelihood ratio test is defined by the following procedure:
Reject H_0 if

$$LR(Z) = \frac{\sup_{\theta \in \Theta_0} L(\theta; Z)}{\sup_{\theta \in \Theta_0 \cup \Theta_1} L(\theta; Z)} > c$$

. where c is chosen as a critical value so as to satisfy $max_{\theta\in\Theta_0} \Pr(LR(Z) > c) = \alpha$, where α is the significance level of the test (probability of Type 1 error).

Theorem 11.2.1. Neyman-Pearson Lemma:
When $\Theta_0 = \theta_0$ and $\Theta_1 = \theta_1$ (i.e. single values of the parameter vector), the likelihood ratio test is the most powerful test of size α.

11.2.1 Likelihood Ratio Test of linear restrictions in normal regression

Proposition 11.2.1. We show the LR test to be equivalent to the F test, as the LR statistic is a monotone transformation of the F statistic.

Consider a hypothesis $R\beta = r$ about coefficients of linear regression with normal errors:

$$Y = X\beta + \varepsilon, \varepsilon|X \sim N(0, \sigma^2 I)$$

The uncoonstrained ML estimates of β and σ^2 are in such a model $\hat{\beta}_{OLS}$ and $\hat{\sigma^2}_{Ml} = RSS_u/n$.

We have $\log(max_\theta L(Y, \theta|X))$ (unrestricted)

$$= \log \left[\left(\frac{1}{\sqrt{2\pi}|\sigma^2 I|^{-1/2}} \right)^n \exp(-\frac{1}{2\sigma^2}(Y - X\beta)'(Y - X\beta)) \right] \Bigg|_{\theta=\hat{\theta}_{ML}}$$

$$= -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\hat{\sigma}^2_{ML}) - \frac{1}{2\hat{\sigma}^2_{ML}}(Y - X\hat{\beta}_{ML})'(Y - X\hat{\beta}_{ML})$$

$$= -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\frac{RSS_u}{n}) - \frac{1}{2}\frac{RSS_u}{RSS_u/n}$$

$$= -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(RSS_u) - \frac{n}{2}$$

Similarly under the restrictions we can show that:

$$\log(max_{\theta \in \Theta_0} L(Y, \theta|X)) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(RSS_r) - \frac{n}{2}$$

where RSS_r is the restricted residual sum of squares.

Therefore the log likelihood ratio statistic for the test of $R\beta = r$ against $R\beta \neq r$ is:

$$LR = -2\left[-\frac{n}{2}\log(\frac{RSS_r}{n}) + \frac{n}{2}\log(\frac{RSS_u}{n}) \right] = n\log(\frac{RSS_r}{RSS_u})$$

$$= n\left[\log\left(\frac{p}{n-k}\frac{(RSS - r - RSS_u)/p}{RSS_u/(n-k)} + 1 \right) \right]$$

$$= n\left[\log\left(\frac{p}{n-k}\frac{W}{p} + 1 \right) \right]$$

Thus LR statistic is a monotone transformation of the F statistic $= W/p$ so that LR test and F test must be equivalnet in the context of testing the linear restrictions in normal regression model. But unlike F test, LR test provides a formidable tool for testing hyportheses in much broader contexts.

Finding c:

$$P(LR > c) = P(n\log(1 + \frac{p}{n-k}F) > c)$$

$$= P(F > \frac{n-k}{p}(e^{c/n} - 1)) = \alpha$$

Thus as we know the F distribution:

$$\frac{n-k}{p}(e^{c/n} - 1) = F_{1-\alpha}(p, n-k)$$

$$\Rightarrow c = n\log(F_{1-\alpha}(p, n-k)\frac{p}{n-k} + 1)$$