1 Basic Probability. Conditional expectation function.

1.1 Random Variables

Definition 1.1.1: Cumulative distribution function

The cumulative distribution function of X is defined as $F_X(x) \equiv P(X \leq x)$. A function F is a cdf iff:

1. $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$;
2. $F(\cdot)$ non-decreasing;
3. $F(\cdot)$ right-continuous; i.e., $\forall x_0$, $\lim_{x\downarrow x_0} F(x) = F(x_0)$.

Definition 1.1.2: Probability density function

For a continuous r.v., $f_X(x)$ defined as the function which satisfies $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$ for all x. A function f_X is a pdf iff:

1. $\forall x$, $f_X(x) \geq 0$;
2. $\int_{\mathbb{R}} f_X(x)\,dx = 1$.

f_X gives the probability of any event: $P(X \in B) = \int_{\mathbb{R}} 1_{(x\in B)} f_X(x)\,dx$.
A continuous (in all dimensions) random vector X has joint pdf $f_X(x_1,\ldots,x_n)$ iff $\forall A \subseteq \mathbb{R}^n$, $P(X \in A) = \int\cdots\int_A f_X(x_1,\ldots,x_n)\,dx_1\cdots dx_n$.

Exercise 1.1.1. Show that the standard normal density integrates to unity by showing (when $u > 0$):

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}uy^2}dy = \frac{1}{\sqrt{u}}.$$

Solution:-

$$\left[\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}dy\right]\left[\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\frac{1}{2}(x^2+y^2)}dxdy.$$

By changing to polar coordinates, $x^2+y^2 = r^2$ and $dxdy = rdrd\theta$. Thus, the desired integral becomes:

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} e^{-\frac{1}{2}ur^2}rdrd\theta = \frac{1}{u}$$

Setting $u = 1$ yields the desired result.

Definition 1.1.3: τ-th quantile

Let X be a random variable with distribution function F_X. The τ-th quantile of X is defined as the value x_τ such that

$$F_X^{-1}(\tau) = \inf\{x : F_X(x) \geq \tau\}$$

where $0 \leq \tau \leq 1$.

Why inf and not min?

Because F is right-continuous and non-decreasing, the superlevel sets of F are of the form $[a, \infty]$ where $a > -\infty$ or else the entire real line. When the superlevel set is the whole line, there is no min (among the reals), while the inf is -∞. For a = $+\infty$ the superlevel set is empty and so the inf = $+\infty$. These cases can potentially arise when $\tau = 0$ or $\tau = 1$ respectively. *If $\tau \in (0, 1)$ then we can replace inf with min.*

If X is discrete, then using minimum and infimum are equivalent, since the support is finite and attains a minimum at some point. However, a continuous X with infinite support will not achieve a minimum, hence the infimum is needed.

Example. The CDF of an Exponential distribution with parameter λ is given by

$$F(x; \lambda) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0, \\ 0 & x < 0. \end{cases}$$

The quantile function for Exponential(λ) is derived by finding the value of Q for which $1 - e^{-\lambda Q} = p$:

$$Q(p; \lambda) = \frac{-\ln(1-p)}{\lambda},$$

for $0 \leq p < 1$. The quartiles are therefore:

- First quartile ($p = 1/4$): $-\ln(3/4)/\lambda$
- Median ($p = 1/2$): $-\ln(1/2)/\lambda$
- Third quartile ($p = 3/4$): $-\ln(1/4)/\lambda$.

Definition 1.1.4: Expectation

For a function g, the expectation of $g(X)$ is defined as $\mathbb{E}g(X) = \int_{-\infty}^{\infty} g(x)f(x)\,dx$.

Exercise 1.1.2. Suppose that Y is a continuous random variable with density $f(y)$ that is positive only if $y \geq 0$. If $F(y)$ is the distribution function, show that

$$\mathbb{E}(Y) = \int_0^\infty [1 - F(y)]dy$$

Solution:-

$$E(Y) = \int_0^\infty yf(y)dy = \int_0^\infty \left(\int_0^y dt\right) f(y)dy = \int_0^\infty \left(\int_t^\infty f(y)dy\right) dt$$

$$= \int_0^\infty P(Y > y)dy = \int_0^\infty [1 - F(y)]dy$$

Definition 1.1.5: Moment

For $n \in \mathbb{Z}$, the nth moment of X is $\mu'_n \equiv \mathbb{E}X^n$. Also denote $\mu'_1 = \mathbb{E}X$ as μ. The nth central moment is $\mu_n \equiv \mathbb{E}(X-\mu)^n$.

Two different distributions *can* have all the same moments, but only if the variables have unbounded support sets. Note that $\mathbb{E}X^n$ may not exist (the integral might be infinite), then we say the nth moment does not exist.

Notable moments and properties:

- The first raw moment is the mean, $\mu = \mathbb{E}[X]$
  - $\mathbb{E}[ag_1(X)+bg_2(X)+c] = a\mathbb{E}(g_1(X))+b\mathbb{E}(g_2(X))+c$ (i.e., expectation is a linear operator)
  - The mean is the MSE minimizing predictor for X; i.e., $\min_b \mathbb{E}(X-b)^2 = \mathbb{E}(X-\mathbb{E}X)^2$
  - If $X_1, \ldots, X_n$ mutually independent, then $\mathbb{E}[g_1(X_1) \cdot \cdots \cdot g_n(X_n)] = \mathbb{E}[g_1(X_1)] \cdot \cdots \cdot \mathbb{E}[g_n(X_n)]$.
- The second central moment is the variance, $\mathbb{E}[(x-\mu)^2]$
  - $Var(aX+bY) = a^2VarX + b^2VarY + 2abCov(X,Y)$
  - $\text{Var}(Y) = \mathbb{E}[\text{Var}(Y|X)] + \text{Var}(\mathbb{E}[Y|X])$ (i.e.: residual variance + regression variance)
  - $Var\mathbf{X} \equiv \mathbb{E}[\mathbf{X}\mathbf{X}'] - \mathbb{E}[\mathbf{X}]\mathbb{E}[\mathbf{X}]'$
  - $Var(\mathbf{X}+\mathbf{Y}) = Var(\mathbf{X}) + Cov(\mathbf{X},\mathbf{Y}) + Cov(\mathbf{X},\mathbf{Y})' + Var(\mathbf{Y})$;
  - $Var(\mathbf{AX}) = \mathbf{A}Var(\mathbf{X})\mathbf{A}'$.
  - $Cov(\mathbf{AX},\mathbf{BY}) = \mathbf{A}Cov(\mathbf{X},\mathbf{Y})\mathbf{B}'$;
  - $Cov(\mathbf{X},\mathbf{Y}) = Cov(\mathbf{Y},\mathbf{X})'$.
- The third central moment is the measure of lopsidedness of the distribution. When standardised by the standard deviation it is known as the skewness. Any symmetric distribution will have skewness of 0.
- The fourth central moment is a measure of the heaviness of the tail. When standardised by the standard deviation, it is known as the kurtosis:

$$\text{Kurt}[X] = \mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] = \frac{\mu_4}{\mu_2^2}.$$

Example. Find μ'_n for the uniform random variable with $\theta_1 = 0$ and $\theta_2 = \theta$.
By definition,

$$\mu'_n = E(Y^n) = \int_{-\infty}^{\infty} y^n f(y)\,dy = \int_0^{\theta} y^n \left(\frac{1}{\theta}\right) dy = \frac{y^{n+1}}{\theta(n+1)}\bigg|_0^{\theta} = \frac{\theta^n}{n+1}.$$

Thus,

$$\mu'_1 = \mu = \frac{\theta}{2}, \quad \mu'_2 = \frac{\theta^2}{3}, \quad \mu'_3 = \frac{\theta^3}{4},$$

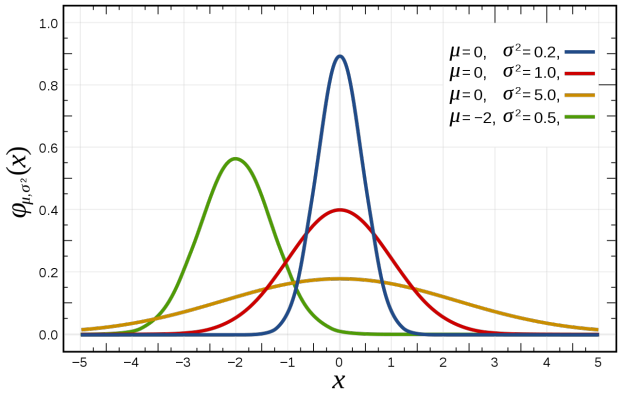
and so on.

1.2 Common Distributions

Normal (Gaussian)

PDF:

$$f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



- $\mathbb{E}[X] = \mu$
  $\mathbb{E}[(X-\mu)] = 0$
- $\mathbb{E}[X^2] = \mu^2 + \sigma^2$
  $\mathbb{E}[(X-\mu)^2] = \sigma^2$
- $\mathbb{E}[X^3] = \mu^3 + 3\mu\sigma^2$
  $\mathbb{E}[(X-\mu)^3] = 0$
- $\mathbb{E}[X^4] = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$
  $\mathbb{E}[(X-\mu)^4] = 3\sigma^4$

Properties

- The distribution is entirely characterised by the first two moments
- Square of standard normal is χ^2_1.
- If $X \sim N(\mu,\sigma^2)$, $Y \sim N(\gamma,\tau^2)$, and $X \perp\!\!\!\perp Y$, then $X+Y \sim N(\mu+\gamma, \sigma^2+\tau^2)$ (i.e., independent normals are additive in mean and variance).
- For a standard normal: $\mathbb{E}[Z^k] = 0$ if k odd, $\mathbb{E}[Z^k] = 1 \cdot 3 \cdot 5 \cdots (n-1)$ if k even.
- Ratio of independent standard normals is Cauchy ($\sigma = 1$, $\theta = 0$)

Lemma 1.2.1 (Stein's Lemma). If $g(\cdot)$ is differentiable with $\mathbb{E}|g'(X)| < \infty$, then $\mathbb{E}[g(X)(X-\mu)] = \sigma^2 \mathbb{E}g'(X)$.

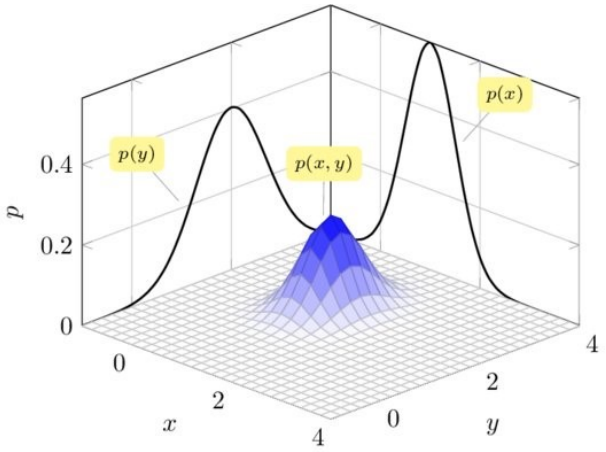
Proof. We shall prove in the case of a standard normal: $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$
Since $\int x \exp(-x^2/2)\,dx = -\exp(-x^2/2)$ we get from integration by parts:
$E[g(X)X] = \frac{1}{\sqrt{2\pi}} \int g(x) x \exp(-x^2/2)\,dx = \frac{1}{\sqrt{2\pi}} \int g'(x) \exp(-x^2/2)\,dx = E[g'(X)]$. □

Multivariate Normal

PDF:

$$\frac{1}{\sqrt{(2\pi)^k |\boldsymbol{\Sigma}|}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}$$

where $\mu = \mathbb{E}[\mathbf{X}]$ and $\boldsymbol{\Sigma}_{ij} = Cov(X_i, X_j)$



Bivariate Case

- $\boldsymbol{\mu} = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}$
- $\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}$

Properties

- A linear transformation of a normal is normal: if $\mathbf{X} \sim N_p(\mu, \boldsymbol{\Sigma})$, then for any $\mathbf{A} \in \mathbb{R}^{q\times p}$ with full row rank ($\Rightarrow q \leq p$), and any $\mathbf{b} \in \mathbb{R}^q$, we have $\mathbf{AX} + \mathbf{b} \sim N_q(\mathbf{A}\mu + \mathbf{b}, \mathbf{A\Sigma A'})$. In particular, $\boldsymbol{\Sigma}^{-1/2}(\mathbf{X} - \mu) \sim N(\mathbf{0}, \mathbf{I})$.
- The following transformations of $\mathbf{X} \sim N_p(\mu, \boldsymbol{\Sigma})$ are independent iff $\mathbf{A\Sigma B'} = Cov(\mathbf{AX}, \mathbf{BX}) = \mathbf{0}$:
  - $\mathbf{AX} \sim N(\mathbf{A}\mu, \mathbf{A\Sigma A'})$ and $\mathbf{BX} \sim N(\mathbf{B}\mu, \mathbf{B\Sigma B'})$,
  - $\mathbf{AX} \sim N(\mathbf{A}\mu, \mathbf{A\Sigma A'})$ and $\mathbf{X'BX} \sim \chi^2_{rk(\mathbf{B}\Sigma)}$ (where $\mathbf{B\Sigma}$ is an idempotent matrix),
  - $\mathbf{X'AX} \sim \chi^2_{rk(\mathbf{A}\Sigma)}$ and $\mathbf{X'BX} \sim \chi^2_{rk(\mathbf{B}\Sigma)}$ (where $\mathbf{A\Sigma}$ and $\mathbf{B\Sigma}$ are idempotent matrices).
- If X and Y are both normal and independent, this implies they are jointly normally distributed (i.e. (X, Y) is multivariate normal). However, a pair of jointly normal distributed variables need not be independent (would only be of if uncorrelated, $\rho = 0$).
- Independence and zero-covariance are equivalent for linear functions of normally distributed r.v.s.

Example (Individual normality $\not\Rightarrow$ joint normality). Consider $X \sim N(0, 1)$, and:

$$Y = \begin{cases} X, & \text{if } |X| \leq c \\ -X, & \text{if } |X| > c \end{cases} \quad \text{where } c > 0$$

When c is very small, $\text{corr}(X, Y) \approx -1$ and when c is very large, $\text{corr}(X, Y) \approx 1$. If the correlation is a continuous function of c, then there exists some c such that the correlation is 0. X and Y are uncorrelated, but clearly not independent since X completely determines Y. To show Y is normal:

$$\begin{aligned} P(Y \leq x) &= P(|X| < c \text{ and } X \leq x) + P(|X| > c \text{ and } -X \leq x) \\ &= P(|X| < c \text{ and } X \leq x) + P(|X| > c \text{ and } X \geq -x) \\ &= P(X \leq x) \end{aligned}$$

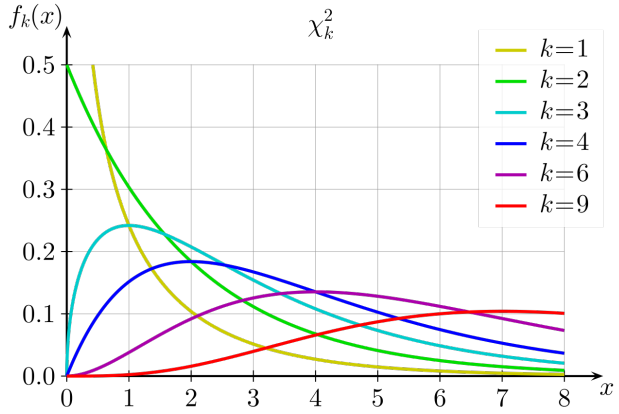
using the symmetry of $|X|$ and $|X| \leq c$. Note that $X - Y$ is not normally distributed due to the non-zero probability of $X - Y = 0$. However, a normal has no discrete part, i.e.: the probability of any point is 0. Thus, X and Y are not jointly normally distributed, even though they are individually normally distributed.

Chi-Squared (χ^2)

PDF:

$$\chi^2_k = \sum_{i=1}^{k} Z_i^2$$

where $Z_i \overset{\text{iid}}{\sim} N(0,1)$



- $\mathbb{E}[X] = k$
  $\mathbb{E}[(X-k)] = 0$
- $\mathbb{E}[X^2] = k(k+2)$
  $\mathbb{E}[(X-k)^2] = 2k$
- $\mathbb{E}[X^3] = k(k+2)(k+4)$
  $\mathbb{E}[(X-k)^3] = 8k$
- $\mathbb{E}[X^4] = k(k+2)(k+4)(k+6)$
  $\mathbb{E}[(X-k)^4] = 12k^2 + 48k$

Properties

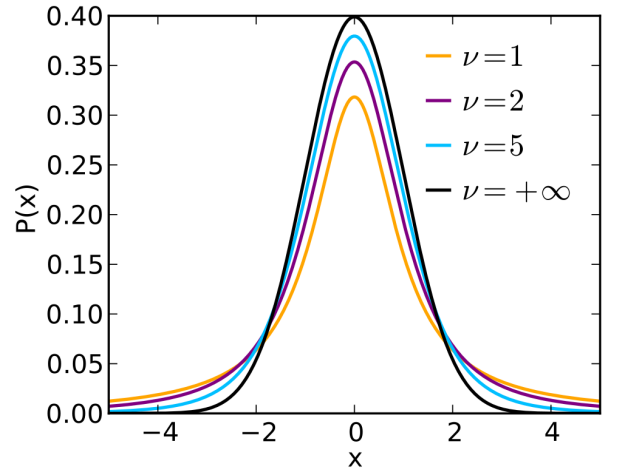
- If $X_1, \ldots, X_n$ are independent with $X_i \sim \chi^2_{p_i}$, then $\sum X_i \sim \chi^2_{\sum p_i}$ (i.e., independent chi squared variables add to a chi squared, and the degrees of freedom add).
- If $\mathbf{X} \sim N_n(\mu, \mathbf{\Sigma})$, then $(\mathbf{X}-\mu)'\mathbf{\Sigma}^{-1}(\mathbf{X}-\mu) \sim \chi^2_n$.
- If $\mathbf{X} \sim N_n(\mathbf{0}, \mathbf{I})$ and $\mathbf{P}_{n\times n}$ is an idempotent matrix, then $\mathbf{X}'\mathbf{P}\mathbf{X} \sim \chi^2_{rk(\mathbf{P})} = \chi^2_{\text{tr}(\mathbf{P})}$.
- If $\mathbf{X} \sim N_n(\mathbf{0}, \mathbf{I})$ then the sum of the squared deviations from the sample mean $\mathbf{X}'\mathbf{M}_\iota\mathbf{X} \sim \chi^2_{n-1}$.

Student's t

PDF:

$$t_\nu = \frac{Z}{\sqrt{X/\nu}} = c\left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

where $Z \sim N(0,1)$, $X \sim \chi^2_\nu$



- Mean: 0 for $\nu > 1$
- Variance: $\frac{\nu}{\nu-2}$ for $\nu > 2$, ∞ for $1 < \nu \leq 2$
- Skewness: 0 for $\nu > 3$
- Ex. kurtosis: $\frac{6}{\nu-4}$ for $\nu > 4$, ∞ for $2 < \nu \leq 4$

Why does the ν-th moment of t_ν not exist?

Consider the ν-th raw moment: $\int x^\nu c\left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} dx \approx \int c\nu^{\frac{\nu+1}{2}} x^{-1} dx$ when x is large. This

integral diverges, meaning the ν-th raw moment does not exist. A more rigorous proof requires the use of the Beta and Gamma functions.

Properties

- If $X_1, \ldots, X_n$ are iid $N(\mu, \sigma^2)$, then $\sqrt{n}(\bar{X} - \mu)/\sigma \sim N(0, 1)$. However, we will generally not know σ. Using the sample variance rather than the true variance gives $\sqrt{n}(\bar{X} - \mu)/s \sim t_{n-1}$.
- If a t distribution has ν degrees of freedom, there are only $\nu - 1$ defined moments. ν has thicker tails than normal.
- t_1 is Cauchy distribution (the ratio of two independent standard normals). t_∞ is standard normal.

Example (Derive variance of Student's t). Consider $X \sim t_\nu$. When $\nu > 1$:

$$E(X) = 0$$

$$(t_\nu)^2 \sim F_{1,\nu} \Rightarrow E(X^2) = E(Y)$$

with $Y \sim F_{1,\nu}$, where $F_{1,\nu}$ is the F-distribution with $(1, \nu)$ degrees of freedom. $E(Y)$ exists if and only if $\nu > 2$:

$$E(Y) = E(X^2) = \frac{\nu}{\nu - 2}$$

We therefore have:

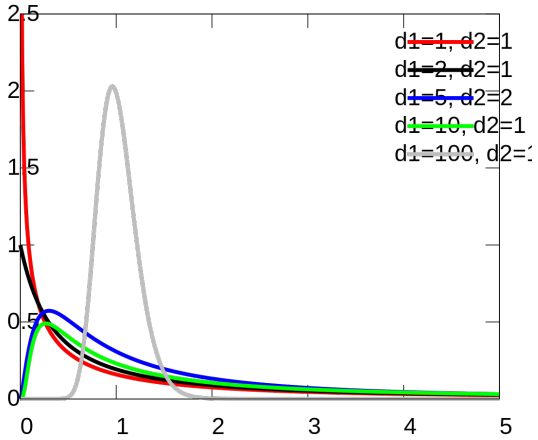
$$\text{var}(X) = E(X^2) - (E(X))^2 = \frac{\nu}{\nu - 2}$$

Snedecor's F

PDF:

$$F_{d_1,d_2} = \frac{X_1/d_1}{X_2/d_2}$$

where $X_1 \sim \chi^2_{d_1}$, $X_2 \sim \chi^2_{d_2}$



- Mean: $\frac{d_2}{d_2 - 1}$ for $d_2 > 2$
- Variance: $\frac{2d_2^2(d_1 + d_2 - 2)}{d_1(d_2 - 2)^2(d_2 - 4)}$, for $d_2 > 4$

Properties

- $1/F_{p,q} \sim F_{q,p}$ (i.e., the reciprocal of an F r.v. is another F with the degrees of freedom switched);
- $(t_q)^2 \sim F_{1,q}$;
- If $X \sim F_{p,q}$ then $Y = \lim_{q \to \infty} pX \sim \chi^2_p$

1.3 Conditional expectation function

Definition 1.3.1: Conditional distribution

Conditional distribution of Y given X is defined as

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} \quad \text{if } f_X(x) \neq 0$$

Conditional expectation $E(Y|X=x)$ is defined as

$$E(Y|X=x) = \int_y y f_{Y|X}(y|x)dy$$

Often, we will skip $X=x$ having in mind that $E(Y|X)$ is a function of random variable X. Hence, it is itself a random variable.
We can also condition for/on multiple coordinates: e.g., for (X_1, X_2, X_3, X_4) a continuous random vector, $f(x_3, x_4|x_1, x_2) \equiv f(x_1, x_2, x_3, x_4)/f_{X_1X_2}(x_1, x_2)$, where f is a joint pdf, and $f_{X_1X_2}$ is the marginal pdf in X_1 and X_2.

Note:-

Borel Paradox: Be careful when we condition on events of probability zero: two events of probability zero may be equivalent, but the probabilities conditional on the two events is different!

Theorem 1.3.1 (Law of Iterated Expectations). $\mathbb{E}X = \mathbb{E}[\mathbb{E}(X|Y)]$, provided the expectations exist. More generally, when $\mathcal{L} \subseteq \mathcal{M}$ (i.e., $\mathcal{L}$ contains less information, $\mathcal{M}$ contains more),

$$\mathbb{E}[X|\mathcal{L}] = \mathbb{E}[\mathbb{E}(X|\mathcal{M})|\mathcal{L}] = \mathbb{E}[\mathbb{E}(X|\mathcal{L})|\mathcal{M}].$$

Proof.

$$E(Y) = \int_y y f_Y(y)dy = \int_x \int_y y f_{XY}(x,y)dxdy = \int_x \int_y y f_{YX}(x,y)dydx$$
$$= \int_x \int_y y f_{Y|X}(y|x) f_X(x)dydx = \int_x E(Y|X=x) f_X(x)dx = E(E(Y|X)).$$

□

Theorem 1.3.2. $\mathbb{E}(Y|X)$ is the MSE $= E(Y - g(X))^2$ minimising predictor of Y based on knowledge of X.

Proof.

$$E(Y-g(X))^2 = E[Y - E(Y|X) + E(Y|X) - g(X)]^2$$
$$= E[Y - E(Y|X)]^2 + 2E[(Y - E(Y|X))(E(Y|X) - g(X))] + E[E(Y|X) - g(X)]^2$$

Using the law of iterated expectations: $E(Z) = E(E(Z|X))$

$$E[(Y - E(Y|X))(E(Y|X) - g(X))] = E(E[(Y - E(Y|X))(E(Y|X) - g(X))]|X)$$

Bring terms explained fully by X outside expectation

$$= E([E(Y|X) - g(X)]E\{[Y - E(Y|X)]|X\})$$

Expand conditional expectation

$$\begin{aligned} &= E([E(Y|X) - g(X)]\{E(Y|X) - E(Y|X)\}) \\ &= 0 \end{aligned}$$

Therefore,

$$\begin{aligned} 2E[(Y - E(Y|X))(E(Y|X) - g(X))] = 0 \Rightarrow& \\ E(Y - g(X))^2 &= E[Y - E(Y|X)]^2 + E[E(Y|X) - g(X)]^2 \\ &\geq E[Y - E(Y|X)]^2. \end{aligned}$$

and CEF is the best conditional predictor of Y □

Lemma 1.3.1 (Leibniz Rule). Let $f(x,t)$ be a continuously differentiable function then, for the function

$$F(t) = \int_{a(t)}^{b(t)} f(x,t)\,dx$$

the derivative of $F(t)$ with respect to t is given by

$$\frac{dF}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}\,dx + f(b(t),t) \cdot \frac{db}{dt} - f(a(t),t) \cdot \frac{da}{dt}$$

Theorem 1.3.3. The conditional median $med(Y|X)$ is the expected absolute error $= E(|Y - g(X)||X = x)$ minimizing predictor of Y based on knowledge of X.

The following proof is a complete version of the outline Alexei presents in the notes. A brief (similar) proof is given at the end.

Proof.

$$\begin{aligned} E(|Y - g(X)||X = x) &= \int_{-\infty}^{\infty} |y - g(x)| f_{Y|X}(y|x) dy \\ &= \int_{g(x)}^{\infty} (y - g(x)) f_{Y|X}(y|x) dy + \int_{-\infty}^{g(x)} (g(x) - y) f_{Y|X}(y|x) dy. \end{aligned}$$

Assume that $f_{Y|X}$ is zero to the left of some constant A, and is unity to the right of some constant B. The problem is:

$$\min_{g(x)} \left\{ \phi = \int_{g(x)}^{A} (y - g(x)) f_{Y|X}(y|x) dy + \int_{-B}^{g(x)} (g(x) - y) f_{Y|X}(y|x) dy \right\}$$

Applying Leibniz rule, we have:

$$\begin{aligned} \frac{d\phi}{dg(x)} &= \int_{A}^{g(x)} (1) f_{Y|X}(y|x) dy + (g(x) - g(x))(1) - (g(x) - A)(0) \\ &+ \int_{g(x)}^{B} (-1) f_{Y|X}(y|x) dy + (B - g(x))(0) - (g(x) - g(x))(1) \end{aligned}$$

FOC:

$$0 = \int_A^{g(x)} f_{Y|X}(y|x)dy - \int_{g(x)}^B f_{Y|X}(y|x)dy \Rightarrow \int_A^{g(x)} f_{Y|X}(y|x)dy = \int_{g(x)}^B f_{Y|X}(y|x)dy$$

Hence, $g(x)$ must be the value of Y such that $P(Y \leq g(x)|X = x) = P(Y > g(x)|X = x)$. That is, $g(x)$ must be the median of the conditional distribution $F_{Y|X}$.
To verify that we have minimized $E(|Y - g(x)||X = x)$:

$$\frac{d^2\phi}{dg(x)^2} = \frac{\partial}{\partial g(x)}\left(\int_A^{g(x)} f_{Y|X}(y|x)dy - \int_{g(x)}^B f_{Y|X}(y|x)dy\right)$$

$$= \int_A^{g(x)} 0 f_{Y|X}(y|x)dy + 1\left(\frac{dg(x)}{dg(x)}\right) - 1\left(\frac{dA}{dg(x)}\right) - \int_{g(x)}^B 0 f_{Y|X}(y|x)dy + 1\left(\frac{dB}{dg(x)}\right) - 1\left(\frac{dg(x)}{dg(x)}\right)$$

$= [0 + 1 - 0] - [0 + 0 - 1] = 2(> 0)$ so we are characterising a minimum. □

Also, note that if we let $A \to -\infty$ and $B \to \infty$, the support of F can be taken to be the whole real line, so there is no loss of generality in establishing the above result with a support of $[A, B]$.

Alternative Proof

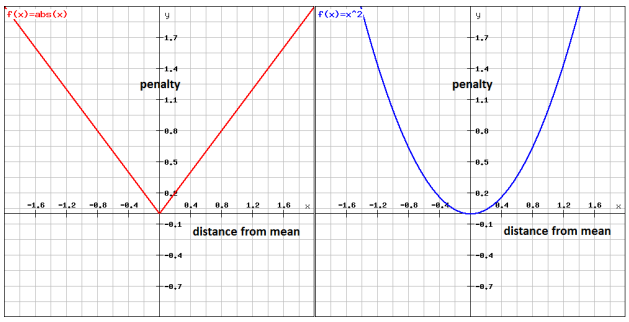
$$\frac{d}{dc}E(|X - c|) = E\left(\frac{d}{dc}|X - c|\right) = E\left(\frac{-(X - c)}{|X - c|}\right)$$

$$= E\left[1_{\{X<c\}} - 1_{\{X>c\}}\right] = P(X < c) - P(X \geq c)$$

$$\frac{d}{dc}E(|X - c|) = 0 \Rightarrow P(X < c) = P(X > c) = \frac{1}{2}$$

By definition of the median, $c = med(X)$ □

MAE vs MSE



- MAE $= E|Y - g(X)|$
- MSE $= E(Y - g(X))^2$

- MAE imposes a linear penalty on errors, i.e.: each deviation from the mean is given a proportional corresponding error.
- MSE is a squared proportional relationship between deviation and penalty. This will make sure that the further you are away from the mean, the proportionally more you will be penalized. Using this penalty function, outliers are deemed proportionally more informative than observations near the mean.

Because the MAE is a more robust estimator of scale than the sample variance or standard deviation, it works better with distributions without a mean or variance, such as the Cauchy distribution.

Weighted MAE

If underprediction is marginally less or more costly as overprediction, it makes sense to minimize the expectation of

$$\tau 1(Y > g(X))(Y - g(X)) + (1 - \tau)1(Y \leq g(X))(g(X) - Y)$$

with $\tau \in (0, 1)$. For example, parameter $\tau < 1/2$ would correspond to situations where the underprediction is less costly than overprediction. Following the same logic as above, we can show that *the corresponding best predictor would be τ-th quantile $\tau(X)$ of the conditional distribution of Y given X.*
Below we have (from left to right): $\tau = 1$ (no cost to overprediction), $\tau = 0$ (no cost to underprediction) and $\tau = 0.3$ (cost to both, but relatively more to overprediction.)