5 Finite sample tests of linear hypotheses

5.1 Linear hypotheses

The t-test is appropriate when the null hypothesis is a real valued restriction. However, more generally there may be multiple restrictions on the coefficient vector $\boldsymbol{\beta}$. Suppose we have $p > 1$ restrictions, we can express a linear hypothesis about $\boldsymbol{\beta}$ in the form $\boldsymbol{R}_{p \times k} \boldsymbol{\beta}_{k \times 1} = \boldsymbol{q}_{p \times 1}$.

Example (Nerlove's returns to scale). Nerlove studied the regression of the total cost of electricity production on demand (Q_i) and factor prices (capital, labour and fuel):

$$\log TC_i = \beta_1 + \beta_2 \log Q_i + \beta_3 \log p_{C_i} + \beta_4 \log p_{L_i} + \beta_5 \log p_{F_i} + \varepsilon_i$$

Economic theory suggests that $\beta_2 = \frac{1}{r}$ where r is the degree of returns to scale. To test constant returns we can use $H_0 : \beta_2 = 1$, which is trivially linear in components of $\boldsymbol{\beta}$. Alternatively we can write

$$R\beta = q$$

with $R = (0, 1, 0, 0, 0)$ and $q = 1$.

Further the total cost must be homogenous of degree 1 with respect to factor prices (doubling cost of all inputs doubles total cost). To test this we can consider $H_0 : \beta_3 + \beta_4 + \beta_5 = 1$. If we were to reject this it would suggest model misspecification.

To test these hypotheses simulatenously consider:

$$R\beta = q \quad \text{with} \quad R = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

To test H_0: $\boldsymbol{R\beta} = \boldsymbol{q}$ vs. H_1: $\boldsymbol{R\beta} \neq \boldsymbol{q}$ we compute the vector $\boldsymbol{R\hat{\beta}} = \boldsymbol{q}$ and reject the null if this vector is "too large" depending on the distribution of $\hat{\boldsymbol{\beta}}$ under H_0.

Definition 5.1.1: Wald statistic

When restrictions are a linear function of coefficients $\boldsymbol{\beta}$, we can write the Wald statistic as

$$W = (R\hat{\beta} - q)'(R\hat{V}_{\hat{\beta}} R')^{-1}(R\hat{\beta} - q)$$

i.e. a weighted Euclidean measure of the length of the vector $R\hat{\beta} - q$.

Note:-

As the Wald statistic is symmetric in the argument $R\hat{\beta} - q$ it treats positive and negative alternatives symmetrically. Thus the inherent alternative is always two-sided.

The Wald statistic is not-invariant to a non-linear transformation/reparametrisation of the hypothesis. For example, asking whether $\beta_1 = 1$ is the same as asking whether $\log \beta_1 = 0$; but the Wald statistic for $\beta_1 = 1$ is not the same as the Wald statistic for $\log \beta_1 = 0$. This is because there is in general no neat relationship between the standard errors of β_1 and $\log \beta_1$, so it needs to be approximated.

Assuming normal regression:

$$\hat{\beta}|X \sim N(\beta, \sigma^2(X'X)^{-1})$$
$$R\hat{\beta}|X \sim N(R\beta, \sigma^2 R(X'X)^{-1}R')$$
$$R\hat{\beta} - q|X \sim N(R\beta - q, \sigma^2 R(X'X)^{-1}R')$$
$$\overset{H_0}{\sim} N(0, \sigma^2 R(X'X)^{-1}R')$$

We can thus standardise:

$$(\sigma^2 R(X'X)^{-1}R')^{-\frac{1}{2}}(R\hat{\beta} - q)|X \overset{H_0}{\sim} N(0, I_P)$$

$$(R\hat{\beta} - q)'(\sigma^2 R(X'X)^{-1}R')^{-1}(R\hat{\beta} - q)|X \overset{H_0}{\sim} \chi^2(p) \tag{5.1}$$

However, the true variance σ^2 is unknown, we thus replace it with the estimated $\hat{\sigma}^2$ to obtain the Wald statistic:

$$W = (R\hat{\beta} - q)'(\hat{\sigma}^2 R(X'X)^{-1}R')^{-1}(R\hat{\beta} - q)$$
$$= \frac{(R\hat{\beta} - q)'(\sigma^2 R(X'X)^{-1}R')^{-1}(R\hat{\beta} - q)}{\hat{\sigma}^2/\sigma^2}$$

Note that this distribution is not $\chi^2(p)$ since $\hat{\sigma}^2$ is itself a random variable. We must consider the joint distribution of $\hat{\sigma}^2$ and $\hat{\beta}$ to make progress.

5.2 The joint distribution of $\hat{\sigma}^2$ and $\hat{\beta}$

Recall the definition of the variance estimator:

$$\hat{\sigma}^2 = \frac{\hat{\varepsilon}'\hat{\varepsilon}}{n - k}$$

To express this in terms of the population ε's examine the following, where we denote the residual maker matrix by $\mathbf{M_X} = \mathbf{I} - \mathbf{X}(\mathbf{X'X})^{-1}\mathbf{X'}$:

$$(n - k)\hat{\sigma}^2 = \hat{\varepsilon}'\hat{\varepsilon}$$
$$= (\mathbf{M_X y})'\mathbf{M_X y}$$
$$= (\mathbf{M_X}(\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}))'\mathbf{M_X}(\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon})$$
$$= \boldsymbol{\varepsilon}'\mathbf{M_X'}\mathbf{M_X}\boldsymbol{\varepsilon} \quad (\text{since } \mathbf{M_X X} = \mathbf{0})$$
$$= \boldsymbol{\varepsilon}'\mathbf{M_X}\boldsymbol{\varepsilon} \quad (\text{since } \mathbf{M_X'}\mathbf{M_X} = \mathbf{M_X}\mathbf{M_X} = \mathbf{M_X})$$

Since $\mathbf{M_X}$ is symmetric, it is positive definite when all eigenvalues are positive. Since it is also idempotent, $\mathbf{M_X^2} = \mathbf{M_X}$, all eigenvalues are either zero or one, meaning $\mathbf{M_X}$ is positive semi-definite.[1]

> **Lemma 5.2.1** (Spectral decomposition)**.** For every $n \times n$ real symmetric matrix, the eigenvalues are real and the eigenvectors can be chosen real and orthonormal. Thus a real symmetric matrix $\mathbf{A}$ can be decomposed as
> $$\mathbf{A} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q'}$$
> where $\mathbf{Q}$ is an orthogonal matrix whose columns are the real, orthonormal eigenvectors of $\mathbf{A}$, and $\boldsymbol{\Lambda}$ is a diagonal matrix whose entries are the eigenvalues of $\mathbf{A}$.

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[1] Alternatively since $\mathbf{M_X^2} = \mathbf{M_X}$ and $\mathbf{M_X'} = \mathbf{M_X}$, note that $\mathbf{v'M_X v} = \mathbf{v'M_X^2 v} = \mathbf{v'M_X'M_X v} = (\mathbf{v'M_X})'(\mathbf{M_X v}) = \|\mathbf{M_X v}\|^2$ for all $\mathbf{v} \in \mathbb{R}^n$.

The spectral decomposition of $\mathbf{M_X}$ is $\mathbf{M_X} = \mathbf{H\Lambda H'}$ where $\mathbf{HH'} = \mathbf{I_n}$ and $\mathbf{\Lambda}$ is diagonal with the eigenvalues of $\mathbf{M_X}$ along the diagonal. Since $\mathbf{M_X}$ is idempotent with rank $n - k$, it has $n - k$ eigenvalues equalling 1 and k eigenvalues equalling 0, so:

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{I}_{n-k} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_k \end{bmatrix}$$

In the normal regression $\boldsymbol{\varepsilon} \sim N(0, \mathbf{I_n}\sigma^2)$, we want to find the distribution of $\mathbf{H'\varepsilon}$. A linear combination of normals is also normal, meaning $\mathbf{H'\varepsilon}$ is normal with mean $\mathbb{E}[\mathbf{H'\varepsilon}] = \mathbf{H'}\mathbb{E}[\boldsymbol{\varepsilon}] = 0$ and variance $\mathrm{Var}(\mathbf{H'}e) = \mathbf{H'I_n}\sigma^2\mathbf{H} = \sigma^2\mathbf{H'H} = \mathbf{I_n}\sigma^2$. Thus $\mathbf{H'\varepsilon} \sim N(0, \mathbf{I_n}\sigma^2)$.

Let $\mathbf{u} = \mathbf{H'\varepsilon}$, and partition $\underset{n \times 1}{\mathbf{u}} = \begin{bmatrix} \underset{(n-k) \times 1}{\mathbf{u_1}} \\ \underset{k \times 1}{\mathbf{u_2}} \end{bmatrix}$ where $\mathbf{u_1} \sim N(0, \mathbf{I_n}\sigma^2)$, then we have

$$\begin{aligned}
(n-k)\hat{\sigma}^2 &= \varepsilon'\mathbf{M_X}\varepsilon \\
&= \varepsilon'\mathbf{H\Lambda H'}\varepsilon \\
&= \mathbf{u'} \begin{bmatrix} \mathbf{I}_{n-k} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_k \end{bmatrix} \mathbf{u} \\
&= [\mathbf{u'_1} \ \mathbf{u'_2}] \begin{bmatrix} \mathbf{I}_{n-k} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_k \end{bmatrix} \begin{bmatrix} \mathbf{u_1} \\ \mathbf{u_2} \end{bmatrix} \\
&= \mathbf{u'_1 u_1}
\end{aligned}$$

where $\mathbf{u'_1 u_1}$ is the sum of $n - k$ squared standard normals , thus it is distributed χ^2_{n-k}. Since $\boldsymbol{\varepsilon}$ is independent of $\hat{\boldsymbol{\beta}}$ it follows that $\hat{\sigma}^2$ is independent of $\hat{\boldsymbol{\beta}}$ as well.

Theorem 5.2.1. In normal regression,

$$\frac{(n-k)\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-k}$$

and is independent of $\hat{\boldsymbol{\beta}}$.

Corollary 5.2.1. In normal regression satisfying GM1-3, the normalised Wald statistic $\frac{W}{p}$, is distributed as $F(p, n - k)$ under the null.

Proof.

$$\frac{W}{p} = \frac{(R\hat{\beta} - q)'(\sigma^2 R(X'X)^{-1}R')^{-1}(R\hat{\beta} - q)/p}{\hat{\sigma}^2/\sigma^2} \sim \frac{\chi^2(p)/p}{\chi^2(n-k)/(n-k)} \sim F(p, n-k).$$

Where we have used 5.1 in the numerator, and Theorem 5.2.1 in the denominator. $\square$

Consider a special case of testing a single restriction, that the j-th coefficient is zero. Then $R\hat{\beta}_j - q = \beta_j$:

$$\hat{\beta}_j | X \overset{H_0}{\sim} N(0, \sigma^2(X'X)^{-1}_{ij})$$

$$\frac{\hat{\beta}_j}{\sqrt{\sigma^2(X'X)^{-1}_{jj}}} | X \overset{H_0}{\sim} N(0, 1)$$

As before σ^2 is unknown, we can substitute in $\hat{\sigma}^2$, but the distribution will change:

$$t = \frac{\hat{\beta}_j}{\sqrt{\hat{\sigma}^2 (X'X)_{jj}^{-1}}}$$

$$= \frac{\hat{\beta}_j / \sqrt{\sigma^2 (X'X)_{jj}^{-1}}}{\sqrt{\frac{(n-k)\hat{\sigma}^2}{\sigma^2}/(n-k)}}$$

$$t|X \overset{H_0}{\sim} \frac{N(0,1)}{\sqrt{\chi^2(n-k)/(n-k)}}$$

$$\overset{H_0}{\sim} t(n-k)$$

Where we are using the fact that the numerator and denominator are independent conditional on X. Note that the square of the t-statistic equals the F-statistic for testing the single restriction.

$$t^2(n-k) = \left(\frac{N(0,1)}{\sqrt{\chi^2(n-k)/(n-k)}} \right)^2$$

$$= \frac{\chi^2(1)/1}{\chi^2(n-k)/(n-k)}$$

$$= F(1, n-k)$$

It is preferable to use the t-statistic since we can test one-sided alternatives, by squaring it we kill the sign of $\hat{\beta}_j$, making it impossible to differentiate between left and right sided alternatives.

5.3 The familiar form of the F-statistic

Consider the following test:
$$H_0 : R\beta = q \text{ vs. } H_1 : R\beta \neq q.$$

Proposition 5.3.1. The normalised Wald statistic is equivalent to the following formula for the F-statistic when testing linear restrictions:

$$F = \frac{W}{p} = \frac{(RSS_r - RSS_u)/p}{RSS_u/(n-k)}$$

Proof. Let us impose the null hypothesis $R\beta = q$ when minimising the sum of squared residuals, denote the solution as the restricted least squares estimator $\tilde{\beta}$:

$$\min_{\beta}(Y - X\beta)'(Y - X\beta) \quad \text{s.t.} \quad R\beta = q$$

$$\mathcal{L}(\beta) = (Y - X\beta)'(Y - X\beta) + \lambda'(R\beta - q)$$

$$\frac{\partial \mathcal{L}}{\partial \beta} = -2X'(Y - X\tilde{\beta}) + R'\lambda = 0$$

$$\Rightarrow X'Y - X'X\tilde{\beta} = R'\left(\frac{\lambda}{2}\right)$$

$$\Rightarrow (X'X)^{-1}X'Y - (X'X)^{-1}X'X\tilde{\beta} = (X'X)^{-1}R'\left(\frac{\lambda}{2}\right)$$

Define the usual (unrestricted) OLS estimate as $\hat{\beta} = \hat{\beta}_{OLS} = (X'X)X'Y$

$$\Rightarrow \hat{\beta} - \tilde{\beta} = (X'X)^{-1}R'\left(\frac{\lambda}{2}\right)$$

$$\Rightarrow \tilde{\beta} = \hat{\beta} - (X'X)^{-1}R'\left(\frac{\lambda}{2}\right)$$

$$\Rightarrow R\tilde{\beta} = R\hat{\beta} - R(X'X)^{-1}R'\left(\frac{\lambda}{2}\right)$$

Since $R\tilde{\beta} = q$:

$$q = R\hat{\beta} - R(X'X)^{-1}R'\left(\frac{\lambda}{2}\right)$$

$$R\hat{\beta} - q = R(X'X)^{-1}R'\left(\frac{\lambda}{2}\right)$$

$$\Rightarrow (R(X'X)^{-1}R')^{-1}(R\hat{\beta} - q) = \frac{\lambda}{2}$$

Thus,

$$\tilde{\beta} = \hat{\beta} - (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}(R\hat{\beta} - q)$$

Now from the corresponding restricted and unrestricted residuals,

$$\hat{\varepsilon} = Y - X\hat{\beta}$$

$$\tilde{\varepsilon} = Y - X\tilde{\beta} = X\hat{\beta} + \hat{\varepsilon} - X\tilde{\beta} = \hat{\varepsilon} + X(\hat{\beta} - \tilde{\beta})$$

Since $\hat{\varepsilon}'X = 0$ [a]

$$\tilde{\varepsilon}'\tilde{\varepsilon} = (\hat{\varepsilon} + X(\hat{\beta} - \tilde{\beta}))'(\hat{\varepsilon} + X(\hat{\beta} - \tilde{\beta}))$$
$$= \hat{\varepsilon}'\hat{\varepsilon} + \hat{\varepsilon}'X(\hat{\beta} - \tilde{\beta}) + (\hat{\beta} - \tilde{\beta})'X'\hat{\varepsilon} + (\hat{\beta} - \tilde{\beta})'X'X(\hat{\beta} - \tilde{\beta})$$
$$= \hat{\varepsilon}'\hat{\varepsilon} + (\hat{\beta} - \tilde{\beta})'X'X(\hat{\beta} - \tilde{\beta})$$

and substituting $\hat{\beta} - \tilde{\beta} = (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}(R\hat{\beta} - q)$,

$$\tilde{\varepsilon}'\tilde{\varepsilon} - \hat{\varepsilon}'\hat{\varepsilon} = ((X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}(R\hat{\beta} - q))'\cancel{X'X(X'X)^{-1}}R'\left(R(X'X)^{-1}R'\right)^{-1}(R\hat{\beta} - q)$$

$$= (R\hat{\beta} - q)'\left(R(X'X)^{-1}R'\right)^{-1}\cancel{R(X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}}(R\hat{\beta} - q)$$

$$= (R\hat{\beta} - q)'\left(R(X'X)^{-1}R'\right)^{-1}(R\hat{\beta} - q)$$

Finally,

$$\frac{W}{p} = \frac{(R\hat{\beta} - q)'\left(R(X'X)^{-1}R'\right)^{-1}(R\hat{\beta} - q)/p}{\sigma^2} = \frac{(\tilde{\varepsilon}'\tilde{\varepsilon} - \hat{\varepsilon}'\hat{\varepsilon})/p}{\frac{\hat{\varepsilon}'\hat{\varepsilon}}{n-k}} = \frac{(RSS_r - RSS_u)/p}{RSS_u/(n-k)}$$

$\square$

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[a]I.e.: Unrestricted OLS residuals uncorrelated with regressors, see lecture 2 for an explanation