Machine Learning: Geometric aspects
Semester 2 2018

Overview

Smooth Manifolds

19/11/2018. Notes from Lee: Intro to smooth manifolds

A topological space M is a topological manifold of dimension n if:
• M is hausdorff
• M is second countable
• $\forall p \in M$ $\exists U$ a neighbourhood of p, $V \overset{\circ}{\subset} \mathbb{R}^n$ and a homeomorphism $\psi : U \to V$

(U, ψ) is the coordinate pair, with U the coordinate neighbourhood and ψ the (local) coordinate map. If $\psi(p) = 0$ the chart is centred at p. We can construct from a chart that contains p, a new chart centred at p by simply subtracting $\psi(p)$.

$\psi(p) = (x^1(p), \ldots, x^n(p))$ where $(x^1, \ldots, x^n)$ are the local coordinates on U.

If M is a topological manifold:

(a) M is locally path connected
(b) M is connected $\iff$ it is bath connected
(c) the components of M are the same as it's path components
(d) M has at most countably many components, each of which is a subset of M and a connected topological manifold

Smooth Manifolds:
We now add a smooth structure that enables calculus.
Note $f : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^m$ is smooth if the partial derivatives of all orders of every component exist.

Let M be a topological manifold. Let $(U, \phi), (V, \psi)$ are be two charts. If $U \cap V \neq \emptyset$, the composite map $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is the transition map from ϕ to ψ. The charts are said to be smoothly compatible if either $U \cap V = \emptyset$ or the transition map $\psi \circ \phi^{-1}$ is a diffeomorphism (Where smoothness is interpreted in the usual sense).

An *atlas* is a collection of charts whose domains cover M. An atlas $\mathcal{A}$ is a smooth atlas if every pair of charts is smoothly compatible. Equivalently if for every pair of charts ψ , ϕ, $\psi \circ \phi^{-1}$ is smooth then the atlas a smooth atlas. $\mathcal{A}$ is maximal if it is not contained in any other smooth atlas.

A *smooth structure* on a topological manifold M is a maximal smooth atlas. A smooth manifold is a pair $(M, \mathcal{A})$. Note it is possible to choose multiple smooth structures not compatible with each other for the same manifold.

Once we choose a chart (U, ϕ) on M, the coordinate map $\phi : U \to V$ can be thought of as an identification between U and V. We can then represent a point by it's coordinates $(x^1, \ldots, x^n) = \phi(p)$ and think of the tuple as being the point itself. $p = (x^1, \ldots, x^n)$ in local coordinates. We can also think of the map as being the identity map to simplify notation. e.g. $p = (x, y)$ or $p = (r, \theta)$

Einstein summation convention: We write $\sum_i x^i E_i = x^i E_i$ for example. Basis vectors are writien with lower indicies and components are written with upper indicies.

Smooth manifold examples:
• The zero manifold (countable discrete space).
• $\mathbb{R}^n$
• finite dimensional vector spaces
• $m \times n$ matricies, $M(m \times n, \mathbb{R})$ In the case of square matricies simply $M(n, \mathbb{R})$
• open sub-manifolds. Let $(M, \mathcal{A})$ be a smooth manifold.
If $U \overset{\circ}{\subset} M$ then take $\mathcal{A}_U = \{(V, \phi) \in \mathcal{A} : V \subset U\}$
• $GL(n, \mathbb{R})$ i.e. matricies with determinant non-zero. Open subset of set of matricies so clearly a smooth manifold.
• Matricies of maximal rank $M_m(m \times n, \mathbb{R})$
• $\mathbb{S}^n$
• $\mathbb{RP}^n$ Real projective plane
• if $M_1, \ldots, M_k$ are smooth manifolds of dimensions $n_1, \ldots, n_k$ Then $M_1 \times \ldots \times M_n$ is a smooth manifold with charts of the form $(U_1 \times \ldots \times U_k, \phi_1 \times \ldots \times \phi_k)$
• Grassman Manifolds (important). Let V be an n-dimensional real vector space. If $0 \leq k \leq n$, $G_k(V)$ is the set of all k-dimensional linear subspaces of V.

Need to generalise to manifolds with boundary. We use the closed n-dimensional upper half plane, $\mathbb{H}^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n : x^n \geq 0\}$. Int $\mathbb{H}^n$ and $\partial\mathbb{H}^n$ are the interior and boundaries respectively.

Smooth maps/functions:
If M is a smooth manifold, $f : M \to \mathbb{R}^k$ is said to be *smooth* if $\forall p \in M \exists (U, \phi)$ for M whose domain contains p such that $f \circ \phi^{-1}$ is smooth on $V = \phi(U) \subset \mathbb{R}^n$. $f \circ \phi^{-1}$ is the *coordinate represention* of f. Denote the set of smooth functions $f : M \to \mathbb{R}$ as $C^\infty(M)$. This is a vector space.

Further, if M, N are two smooth manifolds. $F : M \to N$ is a *smooth map* if $\forall p \in M \exists (U, \phi)$ containing p and (V, ψ) containing $F(p)$ such that $F(U) \subset V$ and $\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$ is smooth. $\hat{F} = \psi \circ F \circ \phi^{-1}$ is the *coordinate representation* of F with respect to the the given coordinates.

Diffeomorphisms: $F : M \to N$ is a *diffeomorphism* if it is a smooth, bijective map with a smooth inverse. M is diffeomorphic to N sometimes written $M \approx N$.

$F : M \to N$ is a *local diffeomorphism* if $\forall p \in M \exists$ a neighbourhood U such that $F(U)$ is open in N and $F|_U : U \to F(U)$ is a diffeomorphism. Clearly every local diffeomorphism is an open map. Analogous to topological spaces being identified by homeomorphisms.

Interesting: If $\mathcal{A}_1$ and $\mathcal{A}_2$ are two smooth structures on $\mathbb{R}$, then $\exists$ a diffeomorphism $F : (\mathbb{R}, \mathcal{A}_1) \to (\mathbb{R}, \mathcal{A}_2)$. More generally, every topological manifold of dimension less than or equal to 3 has a smooth structure unique up to diffeomorphism. Higher dimensions the question remains unanswered. For $\mathbb{R}^n$, as long as $n \neq 4$, $\mathbb{R}^n$ has a unique smooth structure up to diffeomorphism. However $\mathbb{R}^4$ has uncountably many smooth structures none of which are diffeomorphic to each other(lol wtf).

Lie groups:
This is a smooth manifold G that is also a group under the multiplication map $m : G \times G \to G$ such that $m(g, h) = gh$ and $i(g) = g^{-1}$ (inversion) are smooth.
Skipped the rest of this chapter, may come back

Tangent Vectors:
If p is a point of M, a linear map $X : C^\infty(M) \to \mathbb{R}$ is called a *derivation* at p if:

$$X(fg) = f(p)Xg + g(p)Xf$$

$\forall f, g \in C^\infty(M)$. The set of all derivations at p is a vector space called the *tangent space* to M at p denoted by T_pM.If $M = \mathbb{R}^n$ this space is isomorphic to the geometric tangent space $\mathbb{R}^n_a = \{(a, v) : v \in \mathbb{R}^n\}$. Can thus visualise tangent vectors as arrows tangent to M whose base points are attatched to M.

For any $a \in \mathbb{R}^n$ the n-derivations

$$\left.\frac{\partial}{\partial x^i}\right|_a$$

given by

$$\left.\frac{\partial}{\partial x^i}\right|_a f = \frac{\partial f}{\partial x^i}(a)$$

form a basis for $T_a(\mathbb{R}^n)$
If M, N are smooth manifolds and $F : M \to N$ a smooth map, for each $p \in M$ define a map $F_* : T_pM \to T_{F(p)}N$ called the *pushforward* by:

$$F_*X(f) = X(f \circ F)$$

if $f \in C^\infty(N)$ then $f \circ F \in C^\infty(M)$
Some properties $F : M \to N, G : N \to P$ are smooth maps:
(a) $F_* : T_pM \to T_{F(P)}N$ is linear.
(b) $(G \circ F)_* = G_* \circ F_* : T_pM \to T_{G \circ F(p)}P$
(c) $(\mathrm{Id}_M)_* = \mathrm{Id}_{T_pM} : T_pM \to T_pM$
(d) If F is a diffeomorphism, then F_* is an isomorphism.

If $p \in M$ and $X \in T_pM$, and f, g are smooth functions on M that agree on some neighborhood of p then $Xf = Xg$.

If (U, ϕ) is a smooth coordinate chart on M, ϕ is a diffeomorphism from U to $V \overset{\circ}{\subset} \mathbb{R}^n$. $\phi_* : T_pM \to T_{\phi(p)}\mathbb{R}^n$ is an isomorphism. Since this has a basis as above, the pushforward of this basis under ϕ_*^{-1} form a basis for T_pM. We collapse the notation thusly:

$$\left.\frac{\partial}{\partial x^i}\right|_p = (\phi^{-1})_* \left.\frac{\partial}{\partial x^i}\right|_{\phi(p)}$$

This acts on a smooth function $f : U \to \mathbb{R}$ by:

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \frac{\partial \hat{f}}{\partial x^i}(\hat{p})$$

Which is the derivative of the coordinate representation of g at the coordinate representation of p. Such vectors are called the coordinate vectors at p asssociated with the given coordiante system. In $\mathbb{R}^n$ these correspond exactly to the vetors e_i under the isomorphism $T_a\mathbb{R}^n \leftrightarrow \mathbb{R}^n_a$.

The coordinate vectors form a basis for T_pM
$\forall X \in T_pM, X = X^i \frac{\partial}{\partial x^i}\big|_p$.
$X^j = (X^i \frac{\partial}{\partial x^i})(x^j) = X^j$

F_* is represented in terms of the coordinate bases by the jacobian matrix of the coordinated representative of F.

21/11/2018:
Adding a metric:

Main paper notes / questions:

Defintitions/Glossary: