UAV and Sensor Models

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I. Introduction

The intent of this document is to briefly describe the dynamic models and sensor models we are using in our Kalman filter. The coordinate frames of interest are defined and some general notation described in Section II. Section III provides a very brief description of the Kalman filter design that is used to produce state estimates of the simulation UAV. Section IV provides the state dynamics equations and Sections V through X describe the sensor models being used: IMU, digital compass, absolute presser, feature range, line of sight, and position and ground velocity from GPS.

II. Notation and Frames

There are three frames of reference that are of interest in this work. Figure 1 shows each of these frames of reference. The North-East-Down (NED) coordinate system is used as the inertial frame. The body frame is centered at the location of the UAV and is defined such that the x component points out the nose of the aircraft and the y component points along the right wing. The final frame is the feature camera frame. The origin of the camera frame is located at the camera. The z axis of the camera frame is defined to point out the lens, in the direction of viewing, and the x axis is defined to nominally point in the direction of travel of the UAV.

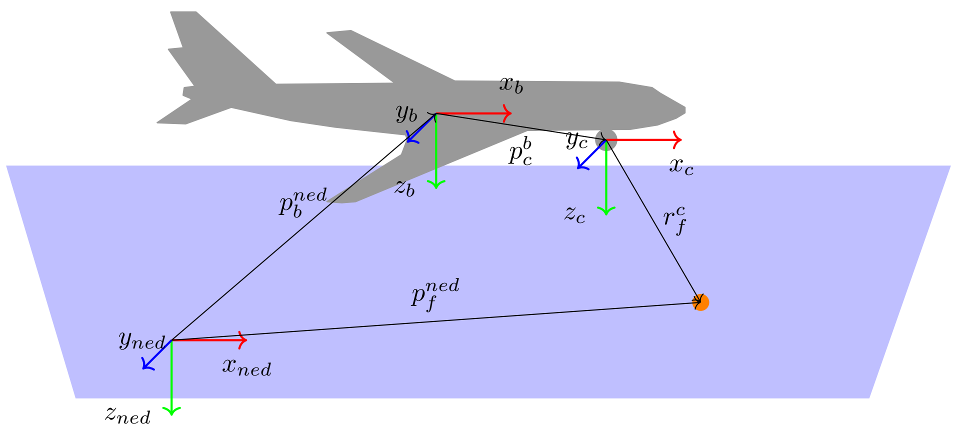

Fig. 1: Depiction of the coordinate frames used in this work.

The general notation used thought this work is listed in Table I. Additionally, when referencing a specific element of a vector, a second subscript is used. For example, p_b is used to denote the position of the body with $p_{b,n}^{ned}$ being the north component of p_b, expressed in the NED frame. Likewise, $p_{b,e}^{ned}$ and $p_{b,d}^{ned}$ are the east and down components, respectively. When applying discrete sensor measurements to an estimated value, $\hat{y}^-$ is known as the "a priori" estimate of y and $\hat{y}^+$ is known as the "posteriori" estimate of y.

Symbol	Description
y	The true value of the quantity/point/vector that y represents.
$\hat{y}$	The estimated value of the quantity/point/vector that y represents.
δy	The error in the estimated value of y.
$\tilde{y}$	The quantity/point/vector that y represents as measured by a noisy sensor.
$\bar{y}$	The nominal value of the quantity/point/vector that y represents.
y^{frame}	The point/vector that y represents expressed in the "frame" frame.
$y_{\text{specifier}}$	The "specifier" is part of the name of the variable. For example, p_b is the position, p, of the body of the UAV.

TABLE I: General notation patterns.

A few functions of importance to this work will now be defined. $[\cdot]_\times : \mathbb{R}^3 \rightarrow \mathbb{R}^{3\times 3}$ is the cross product matrix operator and is defined as

$$\left[x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right]_\times = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix},\ x \in \mathbb{R}^3$$

which results in the property that

$$a \times b = [a]_\times b, \ a, b \in \mathbb{R}^3.$$

There are two ways that orientation is represented in this work, quaternions and roll-pitch-yaw vectors. In both cases there is a need to make a rotation matrix that performs the described rotation. When converting a quaternion into a rotation matrix the function used is $R(q) : \mathbb{H} \to \mathbb{SO}(3)$ and is defined as

$$R\left(q = \begin{bmatrix} q_w \\ q_x \\ q_y \\ q_z \end{bmatrix}\right) = \begin{bmatrix} q_w^2 + q_x^2 - q_y^2 - q_z^2 & 2(q_x q_y - q_w q_z) & 2(q_x q_z + q_w q_y) \\ 2(q_x q_y + q_w q_z) & q_w^2 - q_x^2 + q_y^2 - q_z^2 & 2(q_y q_z - q_w q_x) \\ 2(q_x q_z - q_w q_y) & 2(q_y q_z + q_w q_x) & q_w^2 - q_x^2 - q_y^2 + q_z^2 \end{bmatrix}, \ q \in \mathbb{H}.$$

When converting a roll-pitch-yaw vector into a rotation matrix, we abuse notation and use the function $R(x) : \mathbb{R}^3 \to \mathbb{SO}(3)$, which is defined as

$$R\left(x = \begin{bmatrix} x_r \\ x_p \\ x_y \end{bmatrix}\right) = \begin{bmatrix} \cos(x_p)\cos(x_y) & \sin(x_r)\sin(x_p)\cos(x_y) - \cos(x_r)\sin(x_y) & \cos(x_y)\sin(x_p)\cos(x_r) + \sin(x_y)\sin(x_r) \\ \cos(x_p)\sin(x_y) & \sin(x_r)\sin(x_p)\sin(x_y) + \cos(x_r)\cos(x_y) & \cos(x_r)\sin(x_p)\sin(x_y) - \sin(x_r)\cos(x_y) \\ -\sin(x_p) & \sin(x_r)\cos(x_p) & \cos(x_r)\cos(x_p) \end{bmatrix}, \ x \in \mathbb{R}^3.$$

III. Kalman Filter

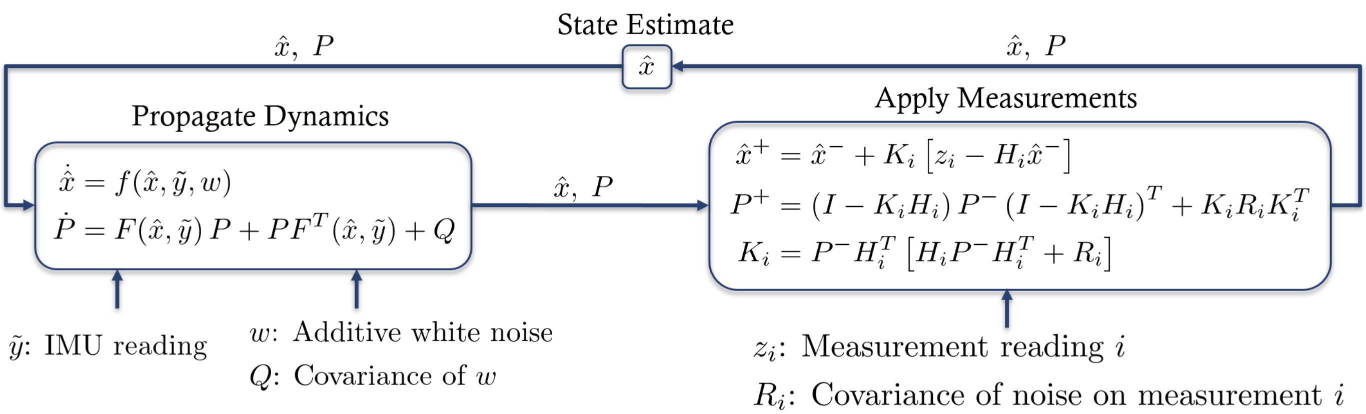

Fig. 2: Block diagram of a Kalman Filter.

Figure 2 shows a generic block diagram of a Kalman filter using model replacement. $F(\hat{x}, \tilde{y})$ is the linearization of the dynamics function $f(\hat{x}, \tilde{y}, w)$, i.e., $F(\hat{x}, \tilde{y}) = \frac{\partial f(\hat{x}, \tilde{y}, w)}{\partial \delta x}$. H_i is known as the measurement sensitivity matrix of measurement i, and is defined as $H_i = \frac{\partial \hat{z}_i(\hat{x})}{\partial \delta x}$ where $\hat{z}_i(\hat{x})$ is the navigation measurement model. As with any Kalman filter, the navigation state, $\hat{x}$, and error state covariance, P, are propagated using the dynamic equations. When discrete sensor measurements become available, they are used to update $\hat{x}$ and P. The notation $\hat{x}^+(t)$, or abbreviated $\hat{x}^+$, is the state estimate at time t after the discrete measurement update is applied. $\hat{x}^-$ is the estimate prior to applying the measurement update.

The difference between a baseline Kalman filter and one that uses model replacement is in how the navigation state and error state covariance is propagated. In a "standard" Kalman filter, the dynamics equations used are a function of the current state and the control inputs. This requires an in depth and accurate modeling of how forces and moments effect the states. The filter uses that information to calculate the dynamic update for the navigation state. When using model replacement, the IMU readings are used as inputs to the system. The linear accelerations and angular velocities that the IMU measures are used with kinematic relations to calculate the time derivative of $\hat{x}$. In other words, the IMU reading replaces the need to directly model how the control input will affect the navigation state.

IV. Dynamics Model

A. Nomenclature

$p_b^{ned} \in \mathbb{R}^3$	The position of the UAV, p_b, in the NED frame
$q_b^{ned} \in \mathbb{H}$	The quaternion that represents the rotation from the body frame to the NED frame
$\theta_b^{ned} \in \mathbb{R}^3$	The Euler angles that represent the rotation from the NED frame to the body frame
$v_b^{ned} \in \mathbb{R}^3$	The velocity vector of the UAV in the NED frame
$b_c \in \mathbb{R}$	The additive bias term for the compass sensor

$b_{ap} \in \mathbb{R}$	The additive bias term for the absolute presser sensor
$b_{fr} \in \mathbb{R}$	The additive bias term for the feature range sensor
$b_{los} \in \mathbb{R}^3$	The camera misalignment bias term for the feature line of sight sensor
$b_g \in \mathbb{R}^3$	The additive bias term for the gyroscope
$b_a \in \mathbb{R}^3$	The additive bias term for the accelerometer
$g^{ned} \in \mathbb{R}^3$	The acceleration vector felt from gravity in the NED frame
$\tilde{a}^b \in \mathbb{R}^3$	The linear accelerations of the UAV in the body frame as read from the accelerometer
$\tilde{\omega}^b \in \mathbb{R}^3$	The angular velocities of the UAV in the body frame as read from the gyroscope

B. State Spaces

The truth state vector is defined as

$$x = \begin{bmatrix} p_b^{ned} = \begin{bmatrix} p_{b,n}^{ned} \\ p_{b,e}^{ned} \\ p_{b,d}^{ned} \end{bmatrix} \\ q_b^{ned} \\ v_b^{ned} = \begin{bmatrix} v_{b,n}^{ned} \\ v_{b,e}^{ned} \\ v_{b,d}^{ned} \end{bmatrix} \\ b_c \\ b_{ap} \\ b_{fr} \\ b_{los} \\ b_g \\ b_a \end{bmatrix}.$$

The navigation state vector is defined as

$$\hat{x} = \begin{bmatrix} \hat{p}_b^{ned} = \begin{bmatrix} \hat{p}_{b,n}^{ned} \\ \hat{p}_{b,e}^{ned} \\ \hat{p}_{b,d}^{ned} \end{bmatrix} \\ \hat{q}_b^{ned} \\ \hat{v}_b^{ned} = \begin{bmatrix} \hat{v}_{b,n}^{ned} \\ \hat{v}_{b,e}^{ned} \\ \hat{v}_{b,d}^{ned} \end{bmatrix} \\ \hat{b}_c \\ \hat{b}_{ap} \\ \hat{b}_{fr} \\ \hat{b}_{los} \\ \hat{b}_g \\ \hat{b}_a \end{bmatrix}.$$

The error state vector is defined as

$$\delta x = \begin{bmatrix} \delta p_b^{ned} \\ \delta \theta_b^{ned} \\ \delta v_b^{ned} \\ \delta b_c \\ \delta b_{ap} \\ \delta b_{fr} \\ \delta b_{los} \\ \delta b_g \\ \delta b_a \end{bmatrix}$$

and is related to the truth and navigation states by

$$x = \begin{bmatrix} \hat{p}_b^{ned} + \delta p_b^{ned} \\ \begin{bmatrix} 1 \\ -\frac{1}{2}\delta\theta_b^{ned} \end{bmatrix} \otimes \hat{q}_b^{ned} \\ \hat{v}_b^{ned} + \delta v_b^{ned} \\ \hat{b}_c + \delta b_c \\ \hat{b}_{ap} + \delta b_{ap} \\ \hat{b}_{fr} + \delta b_{fr} \\ \hat{b}_{los} + \delta b_{los} \\ \hat{b}_g + \delta b_g \\ \hat{b}_a + \delta b_a \end{bmatrix}.$$

C. State Propagation

The truth state dynamics equation is[1]

$$\dot{x} = \begin{bmatrix} \dot{p}_b^{ned} \\ \dot{q}_b^{ned} \\ \dot{v}_b^{ned} \\ \dot{b}_c \\ \dot{b}_{ap} \\ \dot{b}_{fr} \\ \dot{b}_{los} \\ \dot{b}_g \\ \dot{b}_a \end{bmatrix} = \begin{bmatrix} v_b^{ned} \\ \frac{1}{2} q_b^{ned} \otimes \begin{bmatrix} 0 \\ \tilde{\omega}^b - b_g \end{bmatrix} \\ R\left(q_b^{ned}\right)\left[\tilde{a}^b - b_a\right] + g^{ned} \\ \frac{-1}{\tau_c} b_c + w_{cb} \\ \frac{-1}{\tau_{ap}} b_{ap} + w_{apb} \\ \frac{-1}{\tau_{fr}} b_{fr} + w_{frb} \\ \frac{-1}{\tau_{los}} b_{los} + w_{losb} \\ \frac{-1}{\tau_g} b_g + w_{gb} \\ \frac{-1}{\tau_a} b_a + w_{ab} \end{bmatrix}.$$

The navigation state dynamics equation is

$$\dot{\hat{x}} = \begin{bmatrix} \dot{\hat{p}}_b^{ned} \\ \dot{\hat{q}}_b^{ned} \\ \dot{\hat{v}}_b^{ned} \\ \dot{\hat{b}}_c \\ \dot{\hat{b}}_{ap} \\ \dot{\hat{b}}_{fr} \\ \dot{\hat{b}}_{los} \\ \dot{\hat{b}}_g \\ \dot{\hat{b}}_a \end{bmatrix} = \begin{bmatrix} \hat{v}_b^{ned} \\ \frac{1}{2} \hat{q}_b^{ned} \otimes \begin{bmatrix} 0 \\ \tilde{\omega}^b - \hat{b}_g \end{bmatrix} \\ R\left(\hat{q}_b^{ned}\right)\left[\tilde{a}^b - \hat{b}_a\right] + g^{ned} \\ \frac{-1}{\tau_c} \hat{b}_c \\ \frac{-1}{\tau_{ap}} \hat{b}_{ap} \\ \frac{-1}{\tau_{fr}} \hat{b}_{fr} \\ \frac{-1}{\tau_{los}} \hat{b}_{los} \\ \frac{-1}{\tau_g} \hat{b}_g \\ \frac{-1}{\tau_a} \hat{b}_a \end{bmatrix}.$$

Note that $\dot{x}$ and $\dot{\hat{x}}$ are functions of x and $\hat{x}$ respectively, and the output of the IMU, $\tilde{a}^b$ and $\tilde{\omega}^b$.

V. IMU

The Inertial Measurement Unit (IMU) provides rapid measurements of the linear accelerations of the UAV in the body frame, a^b, and the angular velocities of the UAV in the body frame, ω^b. The IMU will typically provide these measurements as quickly, or quicker, then a Kalman filter can process. We are using the IMU model presented in [1] with the additional additive bias terms presented below.

A. Nomenclature

$a^b \in \mathbb{R}^3$	The true linear accelerations of the UAV in the body frame
$\omega^b \in \mathbb{R}^3$	The true angular velocities of the UAV in the body frame
$\eta_a, \eta_g \in \mathbb{R}^3$	The additive white process noise of the accelerometer and gyroscope, respectively
$Q_a, Q_g \in \mathbb{R}^{3\times 3}$	The covariance of η_a and η_g, respectively
$b_a, b_g \in \mathbb{R}^3$	The true additive sensor bias on the accelerometer and gyroscope, respectively

[1]The "true" model used in the simulation actually accounts for aerodynamics and wind considerations as described by [1]. The "truth" model used herein is solely for the sake of linear covariance analysis.

$\tau_{ab}, \tau_{gb} \in \mathbb{R}_+$	The FOGM time constants of b_a and b_g, respectively
$w_{ab}, w_{gb} \in \mathbb{R}^3$	The additive white process noise that drives b_a and b_g, respectively
$q_{ab}, q_{gb} \in \mathbb{R}_+$	The covariance of each element of w_a and w_g, respectively
$\sigma_{ab,ss}, \sigma_{gb,ss} \in \mathbb{R}_+$	The steady state standard deviation of b_a and b_g, respectively

B. Truth Model

The true measurement model is given by

$$\begin{bmatrix} \tilde{\omega}^b \\ \tilde{a}^b \end{bmatrix} = \begin{bmatrix} \omega^b + b_g \\ a^b + b_a \end{bmatrix} + \begin{bmatrix} \eta_g \\ \eta_a \end{bmatrix}$$

where a^b and ω^b come from the truth model, η_a and η_g are additive white process noises, and b_a and b_g are the true sensor biases. The Power Spectral Densities (PSD) of η_a and η_g are

$$\begin{aligned} E\left[\eta_a(t_0)\eta_a^T(t_1)\right] &= Q_a\delta(t_0 - t_1) & t_0, t_1 \in \mathbb{R} \\ E\left[\eta_g(t_0)\eta_g^T(t_1)\right] &= Q_g\delta(t_0 - t_1). \end{aligned}$$

The biases b_a and b_g are commonly modeled as First Order Gauss Markov (FOGM) processes with time constants τ_a and τ_g and driving white noise w_a and w_g. Expressed mathematically this means,

$$\dot{b}_a = \frac{-1}{\tau_{ab}} b_a + w_{ab} \qquad \dot{b}_g = \frac{-1}{\tau_{gb}} b_g + w_{gb}$$

where

$$\begin{aligned} E\left[w_{ab}(t_0)w_{ab}^T(t_1)\right] &= q_{ab}I_{3\times 3}\delta(t_0 - t_1) & t_0, t_1 \in \mathbb{R} \\ E\left[w_{gb}(t_0)w_{gb}^T(t_1)\right] &= q_{gb}I_{3\times 3}\delta(t_0 - t_1) \end{aligned}$$

and

$$q_{ab} = \frac{2\sigma_{ab,ss}^2}{\tau_{ab}} \qquad q_{gb} = \frac{2\sigma_{gb,ss}^2}{\tau_{gb}}.$$

C. Navigation Model

The expected measurement is extracted from $\tilde{\omega}^b$, $\tilde{a}^b$ as

$$\begin{bmatrix} \hat{\tilde{\omega}}^b \\ \hat{\tilde{a}}^b \end{bmatrix} = \begin{bmatrix} \tilde{\omega}^b - \hat{b}_g \\ \tilde{a}^b - \hat{b}_a \end{bmatrix}$$

where

$$\dot{\hat{b}}_a = \frac{-1}{\tau_{ab}} \hat{b}_a \qquad \dot{\hat{b}}_g = \frac{-1}{\tau_{gb}} \hat{b}_g.$$

VI. Digital Compass

Digital compasses are be used to get heading readings for the UAV, ψ. The digital compass model we are using is based on the model given in [1].

A. Nomenclature

$\psi \in \mathbb{R}$	The true heading of the UAV
$\eta_c \in \mathbb{R}$	The additive white process noise of the compass
$q_c \in \mathbb{R}_+$	The covariance of η_c
$b_c \in \mathbb{R}$	The additive bias term for the compass
$\tau_{cb} \in \mathbb{R}_+$	The FOGM time constant of b_c
$w_{cb} \in \mathbb{R}$	The additive white process noise that drives b_c
$q_{cb} \in \mathbb{R}_+$	The covariance of w_{cb}
$\sigma_{cb,ss} \in \mathbb{R}_+$	The steady state standard deviation of b_c

B. Truth Model

The true measurement model is given by

$$\tilde{\psi} = \psi + b_c + \eta_c$$

where ψ comes from the truth model, η_c is additive white process noise, and b_c is the true sensor bias. The PSD of η_c is

$$E[\eta_c(t_0)\eta_c(t_1)] = q_c\delta(t_0 - t_1),\ t_0, t_1 \in \mathbb{R}.$$

In [1] b_c appears to be held constant, however we modeled it as a FOGM process with time constant τ_c and driving white noise w_c. Expressed mathematically,

$$\dot{b}_c = \frac{-1}{\tau_{cb}} b_c + w_{cb} \qquad \text{where} \qquad E[w_{cb}(t_0)w_{cb}(t_1)] = q_{cb}\delta(t_0 - t_1),\ t_0, t_1 \in \mathbb{R} \qquad \text{and} \qquad q_{cb} = \frac{2\sigma^2_{cb,ss}}{\tau_{cb}}.$$

C. Navigation Model

The expected measurement model is

$$\hat{\tilde{\psi}} = \hat{\psi} + \hat{b}_c \qquad \text{where} \qquad \dot{\hat{b}}_c = \frac{-1}{\tau_{cb}} \hat{b}_c.$$

D. Measurement Sensitivity Matrix

$$\hat{H}_c = \frac{\partial \hat{\tilde{\psi}}}{\partial \delta x} = \begin{bmatrix} 0_{1\times3} & \begin{bmatrix} 0 & 0 & -1 \end{bmatrix} & 0_{1\times3} & 1 & 0 & 0 & 0_{1\times3} & 0_{1\times3} & 0_{1\times3} \end{bmatrix}$$

VII. Absolute Pressure

Measurements of altitude can be inferred from the difference in the atmospheric presser at ground/sea level the measured atmospheric measurements on the vehicle. The absolute presser sensor model used in this work is based on [1].

A. Nomenclature

$y_{ap} \in \mathbb{R}_+$	The absolute atmospheric pressure difference between sea level and the altitude of the UAV
$\rho \in \mathbb{R}_+$	Air density at the flight location
$g \in \mathbb{R}_+$	The gravitational acceleration
$b_{ap} \in \mathbb{R}$	The additive bias term for absolute pressure sensor
$\eta_{ap} \in \mathbb{R}$	The additive white process noise of the absolute pressure sensor
$q_{ap} \in \mathbb{R}_+$	The covariance of η_{ap}
$\tau_{apb} \in \mathbb{R}_+$	The FOGM time constant of b_{ap}
$w_{apb} \in \mathbb{R}$	The additive white process noise that drives b_{ap}
$q_{apb} \in \mathbb{R}_+$	The covariance of w_{apb}
$\sigma_{apb,ss} \in \mathbb{R}_+$	The steady state standard deviation of b_{ap}

B. Truth Model

The true measurement model is given by

$$\tilde{y}_{ap} = -\rho g p^{ned}_{b,d} + b_{ap} + \eta_{ap}$$

where y_{ap} is the absolute presser difference between sea level and the vehicle's location, ρ is the air density at the vehicle location, g is the gravitational acceleration felt by the UAV in the body frame, $-p^{ned}_{b,d}$ is the altitude of the vehicle, b_{ap} is a temperature-related bias drift, and η_{ap} is zero-mean white noise. The PSD of η_{ap} is

$$E[\eta_{ap}(t_0)\eta_{ap}(t_1)] = q_{ap}\delta(t_0 - t_1),\ t_0, t_1 \in \mathbb{R}.$$

[1] does not give details about b_{ap}, but we again model it as a FOGM process. Expressed mathematically,

$$\dot{b}_{ap} = \frac{-1}{\tau_{apb}} b_{ap} + w_{apb} \qquad \text{where} \qquad E[w_{apb}(t_0)w_{apb}(t_1)] = q_{apb}\delta(t_0 - t_1),\ t_0, t_1 \in \mathbb{R} \qquad \text{and} \qquad q_{apb} = \frac{2\sigma^2_{apb,ss}}{\tau_{apb}}.$$

In this model ρ is assumed to be constant throughout the flight duration. For details on calculating ρ, see [1] page 133.

C. Navigation Model

The expected measurement model is

$$\hat{\tilde{y}}_{ap} = -\rho g \hat{p}^{ned}_{b,d} + \hat{b}_{ap} \qquad \text{where} \qquad \dot{\hat{b}}_{ap} = \frac{-1}{\tau_{apb}} \hat{b}_{ap}.$$

D. Measurement Sensitivity Matrix

$$\hat{H}_{ap} = \frac{\partial \hat{\tilde{y}}_{ap}}{\partial \delta x} = \begin{bmatrix} \begin{bmatrix} 0 & 0 & -g\rho \end{bmatrix} & 0_{1\times 3} & 0_{1\times 3} & 0 & 1 & 0 & 0_{1\times 3} & 0_{1\times 3} & 0_{1\times 3} \end{bmatrix}$$

VIII. Feature Range

Given a known feature, this measurement finds the range between the UAV and the feature. This sensor is only available when feature is visible to the UAV. We model whether or not the feature is visible based on the north-east distance between the feature and the UAV. Beyond a set radius, the feature is not visible.

A. Nomenclature

$p^{ned}_f \in \mathbb{R}^3$	The position of the feature in the NED frame
$b_{fr} \in \mathbb{R}$	The additive bias term for the feature range sensor
$\eta_{fr} \in \mathbb{R}$	The additive white process noise for the feature range sensor
$q_{fr} \in \mathbb{R}_+$	The covariance of η_{fr}
$\tau_{frb} \in \mathbb{R}_+$	The FOGM time constant of b_{fr}
$w_{frb} \in \mathbb{R}$	The additive white process noise that drives b_{fr}
$q_{frb} \in \mathbb{R}_+$	The covariance of w_{frb}
$\sigma_{frb,ss} \in \mathbb{R}_+$	The steady state standard deviation of b_{fr}

B. Truth Model

The true measurement model is given by

$$\tilde{r} = \left\| p^{ned}_f - p^{ned}_b \right\| + b_{fr} + \eta_{fr}$$

where p^{ned}_f is the position of the feature in the NED frame, b_{fr} is the true sensor bias, and η_{fr} is zero-mean white noise. The PSD of η_{fr} is

$$E[\eta_{fr}(t_0)\eta_{fr}(t_1)] = q_{fr}\delta(t_0 - t_1),\ t_0, t_1 \in \mathbb{R}.$$

b_{fr} is a FOGM process. Expressed mathematically,

$$\dot{b}_{fr} = \frac{-1}{\tau_{frb}} b_{fr} + w_{frb} \qquad \text{where} \qquad E[w_{frb}(t_0)w_{frb}(t_1)] = q_{frb}\delta(t_0 - t_1),\ t_0, t_1 \in \mathbb{R} \qquad \text{and} \qquad q_{frb} = \frac{2\sigma^2_{frb,ss}}{\tau_{frb}}.$$

C. Navigation Model

The navigation measurement model is given by

$$\hat{\tilde{r}} = \left\| p^{ned}_f - \hat{p}^{ned}_b \right\| + \hat{b}_{fr} \qquad \text{where} \qquad \dot{\hat{b}}_{fr} = \frac{-1}{\tau_{frb}} \hat{b}_{fr}.$$

D. Measurement Sensitivity Matrix

$$\hat{H}_{fr} = \frac{\partial \hat{\tilde{r}}}{\partial \delta x} = \begin{bmatrix} -\frac{p^{ned}_f - \hat{p}^{ned}_b}{\left\| p^{ned}_f - \hat{p}^{ned}_b \right\|} & 0_{1\times 3} & 0_{1\times 3} & 0 & 0 & 1 & 0_{1\times 3} & 0_{1\times 3} & 0_{1\times 3} \end{bmatrix}$$

IX. Bearing / Line Of Sight

The feature bearing model presented here is from [2] and [3]. It includes a misalignment bias and additive white noise. The Line Of Sight (LOS) measurement gives the relative position vector from the camera to the target, projected to the focal plane of the camera.

A. Nomenclature

$r_f^c \in \mathbb{R}^3$	The relative line-of-sight vector between the camera and feature in the camera frame
$p_f^{ned} \in \mathbb{R}^3$	The position of the feature in the NED frame
$p_c^b \in \mathbb{R}^3$	The position of the camera in the body frame
$b_{los} \in \mathbb{R}^3$	The camera misalignment bias term for the feature line of sight sensor
$R^T\left(q_b^{ned}\right) \in \mathbb{SO}(3)$	Rotation matrix from NED to body frame
$R_b^{\bar{c}} \in \mathbb{SO}(3)$	Rotation matrix from the body frame to the expected camera frame
$R_{\bar{c}}^c \in \mathbb{SO}(3)$	Rotation matrix from the expected camera frame to the true camera frame
$\eta_{los} \in \mathbb{R}^2$	The additive white process noise of the LOS sensor
$Q_{los} \in \mathbb{R}_+^{2\times 2}$	The covariance of η_{los}
$\tau_{los} \in \mathbb{R}_+$	The FOGM time constant of b_{los}
$w_{losb} \in \mathbb{R}^2$	The additive white process noise that drives b_{los}
$q_{losb} \in \mathbb{R}_+$	The covariance of w_{losb}
$\sigma_{losb,ss} \in \mathbb{R}_+$	The steady state standard deviation of b_{los}

B. Truth Model

The true measurement model is given by

$$\tilde{L} = \begin{bmatrix} r_x/r_z \\ r_y/r_z \end{bmatrix} + \eta_{los}$$

with

$$r_t^c = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = R_{\bar{c}}^c R_b^{\bar{c}} \left[R^T\left(q_b^{ned}\right)\left[p_f^{ned} - p_b^{ned}\right] - p_c^b \right]$$

where p_f^{ned} is the position of the feature in the NED frame, p_c^b is the position of the camera in the UAV body frame, $R_b^{\bar{c}}$ is the rotation from the body frame to the nominal camera frame, η_{los} is zero-mean white noise, and $R_{\bar{c}}^c$ is the rotation matrix from the nominal camera frame to the true camera frame. $R_{\bar{c}}^c$ is defined as

$$R_{\bar{c}}^c = R(b_{los}).$$

Note that b_{los} represents camera misalignment and is modeled as a FOGM process. Expressed mathematically,

$$\dot{b}_{los} = \frac{-1}{\tau_{losb}} b_{los} + w_{losb} \quad \text{where} \quad E\left[w_{losb}(t_0) w_{losb}^T(t_1)\right] = q_{losb} I_{2\times 2}\delta(t_0 - t_1),\ t_0, t_1 \in \mathbb{R} \quad \text{and} \quad q_{losb} = \frac{2\sigma_{losb,ss}^2}{\tau_{losb}}.$$

The PSD of η_{los} is

$$E\left[\eta_{los}(t_0)\eta_{los}^T(t_1)\right] = Q_{los}\delta(t_0 - t_1),\ t_0, t_1 \in \mathbb{R}.$$

C. Navigation Model

The navigation measurement model is given by

$$\hat{\tilde{L}} = \begin{bmatrix} \hat{r}_x/\hat{r}_z \\ \hat{r}_y/\hat{r}_z \end{bmatrix}$$

where

$$\hat{r}_t^c = \begin{bmatrix} \hat{r}_x \\ \hat{r}_y \\ \hat{r}_z \end{bmatrix} = \hat{R}_{\bar{c}}^c R_b^{\bar{c}} \left[R^T\left(\hat{q}_b^{ned}\right)\left[p_f^{ned} - \hat{p}_b^{ned}\right] - p_c^b \right] \qquad \text{with} \qquad \hat{R}_{\bar{c}}^c = R\left(\hat{b}_{los}\right) \qquad \text{and} \qquad \dot{\hat{b}}_{los} = \frac{-1}{\tau_{losb}} \hat{b}_{los}$$

D. Measurement Sensitivity Matrix

Using the chain rule is can be said that

$$\hat{H}_{los} = \frac{\partial \hat{\tilde{L}}}{\partial \hat{x}} = \frac{\partial \hat{\tilde{L}}}{\partial \hat{r}_t^c} \frac{\partial \hat{r}_c^t}{\partial \hat{x}}.$$

The two derivatives above are given as

$$\frac{\partial \hat{\tilde{L}}}{\partial \hat{r}_t^c} = \frac{\partial}{\partial \hat{r}_t^c}\left(\begin{bmatrix} \hat{r}_x/\hat{r}_z \\ \hat{r}_y/\hat{r}_z \end{bmatrix} \right) = \begin{bmatrix} 1/\hat{r}_z & 0 & -\hat{r}_x/\hat{r}_z^2 \\ 0 & 1/\hat{r}_z & -\hat{r}_y/\hat{r}_z^2 \end{bmatrix}$$

and

$$\frac{\partial \hat{r}_t^c}{\partial \hat{x}} = \begin{bmatrix} -R_{ned}^c & -R_{ned}^c \left[x_{diff}^{ned}\right]_\times & 0_{3\times3} & 0_{3\times1} & 0_{3\times1} & 0_{3\times1} & \left[R\left(\hat{b}_{los}\right) R_b^{\bar{c}} \left[R^T\left(\hat{q}_b^{ned}\right) x_{diff}^{ned} - p_c^b\right]\right]_\times & 0_{3\times3} & 0_{3\times3} \end{bmatrix}$$

where

$$x_{diff}^{ned} = p_f^{ned} - \hat{p}_b^{ned} \qquad \text{and} \qquad R_{ned}^c = R\left(\hat{b}_{los}\right) R_b^{\bar{c}} R^T\left(\hat{q}_b^{ned}\right).$$

For the derivation of $\frac{\partial \hat{\tilde{L}}}{\partial \hat{x}}$ see the appendix.

X. GNSS

A Global Navigation Satellite System (GNSS) or Global Positioning System (GPS) can be used to collect position information as well as velocities relative to the ground. The GPS model used in this work is based on the GPS model presented in [1].

A. Position

1) Nomenclature:

$b_{gps,p} \in \mathbb{R}^3$	The additive bias term of the GPS position sensor
$\eta_{gps,p} \in \mathbb{R}^3$	The additive white process noise of the GPS position sensor
$Q_{gps,p} \in \mathbb{R}_+^{3\times3}$	The covariance of $\eta_{gps,p}$
$\tau_{gps,pb} \in \mathbb{R}_+$	The FOGM time constant of $b_{gps,p}$
$w_{gps,pb} \in \mathbb{R}^3$	The additive white process noise that drives $b_{gps,p}$
$q_{gps,pb} \in \mathbb{R}_+$	The covariance of $w_{gps,pb}$
$\sigma_{gps,pb,ss} \in \mathbb{R}_+$	The steady state standard deviation of $b_{gps,p}$

2) Truth Model: The true measurement model is given by

$$\tilde{p}_b^{ned} = p_b^{ned} + b_{gps,p} + \eta_{gps,p}$$

where $b_{gps,p}$ is a FOGM bias, and $\eta_{gps,p}$ is additive white noise. The PSD of $\eta_{gps,p}$ is

$$E\left[\eta_{gps,p}(t_0)\eta_{gps,p}^T(t_1)\right] = Q_{gps,p}\delta(t_0 - t_1),\ t_0, t_1 \in \mathbb{R}.$$

$b_{gps,p}$ is a FOGM process and is defined by

$$\dot{b}_{gps,p} = \frac{-1}{\tau_{gps,pb}} b_{gps,p} + w_{gps,pb}$$

where

$$E\left[w_{gps,pb}(t_0)w_{gps,pb}^T(t_1)\right] = q_{gps,pb} I_{3\times3}\delta(t_0 - t_1),\ t_0, t_1 \in \mathbb{R} \qquad \text{and} \qquad q_{gps,pb} = \frac{2\sigma_{gps,pb,ss}^2}{\tau_{gps,pb}}.$$

3) Navigation Model: The expected measurement model is

$$\hat{\tilde{p}}_b^{ned} = \hat{p}_b^{ned} + \hat{b}_{gps,p} \qquad \text{where} \qquad \dot{\hat{b}}_{gps,p} = \frac{-1}{\tau_{gps,pb}} \hat{b}_{gps,p}.$$

4) Measurement Sensitivity Matrix:

$$\hat{H}_{gps,p} = \frac{\partial \hat{\tilde{p}}_b^{ned}}{\partial \delta x} = \begin{bmatrix} I_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times1} & 0_{3\times1} & 0_{3\times1} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \end{bmatrix}$$

B. Ground Velocities

$v_b \in \mathbb{R}$	The ground velocity of the aircraft
$\eta_{gps,v} \in \mathbb{R}$	The additive white process noise of the GPS velocity sensor
$q_{gps,v} \in \mathbb{R}_+$	The covariance of $\eta_{gps,v}$

1) Truth Model: The true measurement model is given by

$$\tilde{v}_b = \left\| \begin{bmatrix} v_{b,n}^{ned} \\ v_{b,e}^{ned} \end{bmatrix} \right\| + \eta_{gps,v}$$

where v_b is the ground velocity of the vehicle, and $\eta_{gps,v}$ is additive white noise. The PSD of $\eta_{gps,v}$ is

$$E[\eta_{gps,v}(t_0)\eta_{gps,v}(t_1)] = q_{gps,v}\delta(t_0 - t_1),\ t_0, t_1 \in \mathbb{R}.$$

2) Navigation Model: The expected measurement model is

$$\hat{\tilde{v}}_b = \left\| \begin{bmatrix} \hat{v}_{b,n}^{ned} \\ \hat{v}_{b,e}^{ned} \end{bmatrix} \right\|.$$

3) Measurement Sensitivity Matrix:

$$\hat{H}_{gps,v} = \frac{\partial \hat{\tilde{v}}_b}{\partial \delta x} = \begin{bmatrix} 0_{1\times3} & 0_{1\times3} & \begin{bmatrix} \dfrac{\begin{bmatrix} \hat{v}_{b,n}^{ned} \\ \hat{v}_{b,e}^{ned} \end{bmatrix}}{\left\| \begin{bmatrix} \hat{v}_{b,n}^{ned} \\ \hat{v}_{b,e}^{ned} \end{bmatrix} \right\|} & 0 \end{bmatrix} & 0 & 0 & 0 & 0_{1\times3} & 0_{1\times3} & 0_{1\times3} \end{bmatrix}$$

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[2] D. Woffinden and D. Geller, "Relative angles-only navigation and pose estimation for autonomous orbital rendezvous," *Journal of Guidance Control and Dynamics - J GUID CONTROL DYNAM*, vol. 30, pp. 1455–1469, 09 2007.

[3] R. Christensen, D. Geller, and M. Hansen, "Linear covariance navigation analysis of range and image measurement processing for autonomous lunar lander missions," in *2020 IEEE/ION Position, Location and Navigation Symposium (PLANS)*, 2020, pp. 1524–1535.

APPENDIX
LINE OF SIGHT LINEARIZATION

The feature bearing / line of sight sensor will now be linearized. First using the chain rule,

$$\hat{H}_{los} = \frac{\partial \hat{\tilde{L}}}{\partial \hat{x}} = \frac{\partial \hat{\tilde{L}}}{\partial \hat{r}_t^c} \frac{\partial \hat{r}_c^t}{\partial \hat{x}}.$$

$\frac{\partial \hat{\tilde{L}}}{\partial \hat{r}_t^c}$ can be calculated fairly easily as

$$\frac{\partial \hat{\tilde{L}}}{\partial \hat{r}_t^c} = \frac{\partial}{\partial \hat{r}_t^c}\left(\begin{bmatrix} \hat{r}_x/\hat{r}_z \\ \hat{r}_y/\hat{r}_z \end{bmatrix} \right) = \begin{bmatrix} 1/\hat{r}_z & 0 & -\hat{r}_x/\hat{r}_z^2 \\ 0 & 1/\hat{r}_z & -\hat{r}_y/\hat{r}_z^2 \end{bmatrix}.$$

However $\frac{\partial \hat{r}_c^t}{\partial \hat{x}}$ will be calculated using the perturbation method. Defining the true r_c^t as

$$r_t^c = R(b_{los}) R_b^{\bar{c}} \left[R^T\left(q_b^{ned}\right) \left[p_f^{ned} - p_b^{ned} \right] - p_c^b \right]$$

and the nominal as

$$\hat{r}_t^c = R\left(\hat{b}_{los}\right) R_b^{\bar{c}} \left[R^T\left(\hat{q}_b^{ned}\right) \left[p_f^{ned} - \hat{p}_b^{ned} \right] - p_c^b \right].$$

The perturbations can now be defined.

$$\begin{aligned} r_t^c &= \hat{r}_t^c + \delta r_t^c \\ R(b_{los}) &= \left[I - [\delta b_{los}]_\times \right] R\left(\hat{b}_{los}\right) \\ R\left(q_b^{ned}\right) &= \left[I - \left[\delta \theta_b^{ned}\right]_\times \right] R\left(\hat{q}_b^{ned}\right) \Rightarrow R^T\left(q_b^{ned}\right) = R^T\left(\hat{q}_b^{ned}\right) \left[I + \left[\delta \theta_b^{ned}\right]_\times \right] \\ p_b^{ned} &= \hat{p}_b^{ned} + \delta p_b^{ned} \end{aligned}$$

Substituting the perturbations into true equation yields

$$\hat{r}_t^c + \delta r_t^c = \left[I - [\delta b_{los}]_\times \right] R\left(\hat{b}_{los}\right) R_b^{\bar{c}} \left[R^T\left(\hat{q}_b^{ned}\right) \left[I + \left[\delta \theta_b^{ned}\right]_\times \right] \left[p_f^{ned} - \left[\hat{p}_b^{ned} + \delta p_b^{ned} \right] \right] - p_c^b \right].$$

Expanding

$$\begin{aligned} \hat{r}_t^c + \delta r_t^c =& \left[I - [\delta b_{los}]_\times \right] R\left(\hat{b}_{los}\right) R_b^{\bar{c}} \\ & \left[R^T\left(\hat{q}_b^{ned}\right) \left[I + \left[\delta \theta_b^{ned}\right]_\times \right] p_f^{ned} - R^T\left(\hat{q}_b^{ned}\right) \left[I + \left[\delta \theta_b^{ned}\right]_\times \right] \hat{p}_b^{ned} - R^T\left(\hat{q}_b^{ned}\right) \left[I + \left[\delta \theta_b^{ned}\right]_\times \right] \delta p_b^{ned} - p_c^b \right] \\ =& \quad \left[I - [\delta b_{los}]_\times \right] R\left(\hat{b}_{los}\right) R_b^{\bar{c}} R^T\left(\hat{q}_b^{ned}\right) \left[I + \left[\delta \theta_b^{ned}\right]_\times \right] p_f^{ned} \\ & - \left[I - [\delta b_{los}]_\times \right] R\left(\hat{b}_{los}\right) R_b^{\bar{c}} R^T\left(\hat{q}_b^{ned}\right) \left[I + \left[\delta \theta_b^{ned}\right]_\times \right] \hat{p}_b^{ned} \\ & - \left[I - [\delta b_{los}]_\times \right] R\left(\hat{b}_{los}\right) R_b^{\bar{c}} R^T\left(\hat{q}_b^{ned}\right) \left[I + \left[\delta \theta_b^{ned}\right]_\times \right] \delta p_b^{ned} \\ & - \left[I - [\delta b_{los}]_\times \right] R\left(\hat{b}_{los}\right) R_b^{\bar{c}} p_c^b \end{aligned}$$

$$
\begin{aligned}
= & \left[R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right) - [\delta b_{los}]_\times R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\right]\left[I + \left[\delta\theta_b^{ned}\right]_\times\right]p_f^{ned} \\
& - \left[R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right) - [\delta b_{los}]_\times R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\right]\left[I + \left[\delta\theta_b^{ned}\right]_\times\right]\hat{p}_b^{ned} \\
& - \left[R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right) - [\delta b_{los}]_\times R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\right]\left[I + \left[\delta\theta_b^{ned}\right]_\times\right]\delta p_b^{ned} \\
& - \left[R\left(\hat{b}_{los}\right)R_b^{\bar{c}} - [\delta b_{los}]_\times R\left(\hat{b}_{los}\right)R_b^{\bar{c}}\right]p_c^b \\
= & \ R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)p_f^{ned} - [\delta b_{los}]_\times R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)p_f^{ned} \\
& + R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\left[\delta\theta_b^{ned}\right]_\times p_f^{ned} - [\delta b_{los}]_\times R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\left[\delta\theta_b^{ned}\right]_\times p_f^{ned} \\
& - R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\hat{p}_b^{ned} + [\delta b_{los}]_\times R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\hat{p}_b^{ned} \\
& - R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\left[\delta\theta_b^{ned}\right]_\times \hat{p}_b^{ned} + [\delta b_{los}]_\times R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\left[\delta\theta_b^{ned}\right]_\times \hat{p}_b^{ned} \\
& - R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\delta p_b^{ned} + [\delta b_{los}]_\times R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\delta p_b^{ned} \\
& - R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\left[\delta\theta_b^{ned}\right]_\times \delta p_b^{ned} + [\delta b_{los}]_\times R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\left[\delta\theta_b^{ned}\right]_\times \delta p_b^{ned} \\
& - R\left(\hat{b}_{los}\right)R_b^{\bar{c}}p_c^b + [\delta b_{los}]_\times R\left(\hat{b}_{los}\right)R_b^{\bar{c}}p_c^b
\end{aligned}
$$

Removing higher order terms, shown below in red,

$$
\begin{aligned}
\hat{r}_t^c + \delta r_t^c = & \ R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)p_f^{ned} - [\delta b_{los}]_\times R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)p_f^{ned} \\
& + R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\left[\delta\theta_b^{ned}\right]_\times p_f^{ned} \color{red}{- [\delta b_{los}]_\times R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\left[\delta\theta_b^{ned}\right]_\times p_f^{ned}} \\
& - R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\hat{p}_b^{ned} + [\delta b_{los}]_\times R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\hat{p}_b^{ned} \\
& - R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\left[\delta\theta_b^{ned}\right]_\times \hat{p}_b^{ned} \color{red}{+ [\delta b_{los}]_\times R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\left[\delta\theta_b^{ned}\right]_\times \hat{p}_b^{ned}} \\
& - R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\delta p_b^{ned} \color{red}{+ [\delta b_{los}]_\times R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\delta p_b^{ned}} \\
& \color{red}{- R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\left[\delta\theta_b^{ned}\right]_\times \delta p_b^{ned} + [\delta b_{los}]_\times R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\left[\delta\theta_b^{ned}\right]_\times \delta p_b^{ned}} \\
& - R\left(\hat{b}_{los}\right)R_b^{\bar{c}}p_c^b + [\delta b_{los}]_\times R\left(\hat{b}_{los}\right)R_b^{\bar{c}}p_c^b.
\end{aligned}
$$

Grouping similar terms

$$
\begin{aligned}
\hat{r}_t^c + \delta r_t^c = & \ R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\left[p_f^{ned} - \hat{p}_b^{ned}\right] - R\left(\hat{b}_{los}\right)R_b^{\bar{c}}p_c^b \\
& + R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\left[\delta\theta_b^{ned}\right]_\times\left[p_f^{ned} - \hat{p}_b^{ned}\right] \\
& - R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\delta p_b^{ned} \\
& + [\delta b_{los}]_\times R\left(\hat{b}_{los}\right)R_b^{\bar{c}}\left[p_c^b - R^T\left(\hat{q}_b^{ned}\right)p_f^{ned} + R^T\left(\hat{q}_b^{ned}\right)\hat{p}_b^{ned}\right] \\
= & \ R\left(\hat{b}_{los}\right)R_b^{\bar{c}}\left[R^T\left(\hat{q}_b^{ned}\right)\left[p_f^{ned} - \hat{p}_b^{ned}\right] - p_c^b\right] \\
& + R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\left[\delta\theta_b^{ned}\right]_\times\left[p_f^{ned} - \hat{p}_b^{ned}\right] \\
& - R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\delta p_b^{ned} \\
& + [\delta b_{los}]_\times R\left(\hat{b}_{los}\right)R_b^{\bar{c}}\left[p_c^b + R^T\left(\hat{q}_b^{ned}\right)\left[\hat{p}_b^{ned} - p_f^{ned}\right]\right]
\end{aligned}
$$

Canceling the nominal

$$
\begin{aligned}
\delta r_t^c = & \ R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\left[\delta\theta_b^{ned}\right]_\times\left[p_f^{ned} - \hat{p}_b^{ned}\right] \\
& - R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\delta p_b^{ned} \\
& + [\delta b_{los}]_\times R\left(\hat{b}_{los}\right)R_b^{\bar{c}}\left[p_c^b + R^T\left(\hat{q}_b^{ned}\right)\left[\hat{p}_b^{ned} - p_f^{ned}\right]\right]
\end{aligned}
$$

Using the fact that $a \times b = -b \times a$

$$
\delta r_t^c = - R\left(\hat{b}_{los}\right)R_b^{\bar{c}}R^T\left(\hat{q}_b^{ned}\right)\left[p_f^{ned} - \hat{p}_b^{ned}\right]_\times \delta\theta_b^{ned}
$$

$$- R\Big(\hat{b}_{los}\Big) R_b^{\bar{c}} R^T\left(\hat{q}_b^{ned}\right) \delta p_b^{ned}$$
$$+ \Big[R\Big(\hat{b}_{los}\Big) R_b^{\bar{c}} \big[R^T\left(\hat{q}_b^{ned}\right)\big[p_f^{ned} - \hat{p}_b^{ned}\big] - p_c^b\big]\Big]_\times \delta b_{los}$$

Letting the δx coefficients be grouped into a matrix

$$\delta r_t^c = \begin{bmatrix} -R\Big(\hat{b}_{los}\Big) R_b^{\bar{c}} R^T\left(\hat{q}_b^{ned}\right) \\ -R\Big(\hat{b}_{los}\Big) R_b^{\bar{c}} R^T\left(\hat{q}_b^{ned}\right)\Big[p_f^{ned} - \hat{p}_b^{ned}\Big]_\times \\ 0_{3\times3} \\ 0_{1\times3} \\ 0_{1\times3} \\ 0_{1\times3} \\ \Big[R\Big(\hat{b}_{los}\Big) R_b^{\bar{c}} \Big[R^T\left(\hat{q}_b^{ned}\right)\Big[p_f^{ned} - \hat{p}_b^{ned}\Big] - p_c^b\Big]\Big]_\times \\ 0_{3\times3} \\ 0_{3\times3} \end{bmatrix}^T \delta x$$