2.097J / 6.7330J / 16.920J
Numerical Methods for Partial Differential Equations
Massachusetts Institute of Technology - Fall 2024

Project 3 - Boundary Element Methods

Version 1.2 *Due: November 27, 2024*

The first problem in this project is a warm-up on quadrature, followed by several questions about the Nyström boundary element method. For the Nyström method, you should have plots showing the method's exponential convergence (error versus number of points), and plots of σ or other variables where appropriate.

Problem 1 - Quadrature (15 pts)

This question concerns the trapezoidal rule. The aim is to prove the error associated with this rule, then implement the rule to approximate some integrals, and hopefully observe this error in practice. Finally we shall approximate the integral of a periodic function for which we observe a curious phenomenon, namely that the trapezoidal rule is spectrally accurate for periodic functions.

Consider the definite integral

$$I[f] = \int_a^b f(x)\mathrm{d}x.$$

In class, we proved the error in approximating I via the composite midpoint rule on n intervals is

$$I[f] - \frac{(b-a)}{n}\sum_{i=1}^{n} f\left(a + \frac{b-a}{n}\left(i - \frac{1}{2}\right)\right) = \frac{(b-a)^3}{24n^2}f''(\eta) \tag{1}$$

for η some point in $[a, b]$. In order to prove this, we made use of the Taylor series about the midpoint of the interval $c = (a + b)/2$,

$$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}, \tag{2}$$

for some ξ between x and c. The final term is called "Lagrange's remainder term" - a derivation of this is provided in the Lecture notes.

We noted from (1) that the error depends on the second derivative of f, hence the midpoint rule is exact when integrating polynomials of degree less than or equal to 1. Another rule which is exact for polynomials of degree less than or equal to 1 is the trapezoidal rule.

The trapezoidal rule on a single interval is

$$\int_a^b f(x)\mathrm{d}x \approx \frac{b-a}{2}(f(a)+f(b)).$$

This is equivalent to approximating the area under $f(x)$ by a trapezoid. It can also be seen as the average of the left and right Riemann sums. To improve the accuracy, one can subdivide $[a,b]$ into n intervals and use the rule on each interval. This leads to the composite trapezoidal rule:

$$\int_a^b f(x)\mathrm{d}x \approx \frac{b-a}{n}\left[\frac{f(a)+f(b)}{2}+\sum_{i=1}^{n-1} f\left(a+i\frac{b-a}{n}\right)\right] \tag{3}$$

(a) (4 pts)Prove that the error in approximating I via the composite trapezoidal rule is

$$-\frac{(b-a)^3}{12n^2}f''(\eta)$$

for some $\eta \in [a,b]$.

[Hint: Consider one interval in the summation in (3), $[x_i, x_{i+1}]$ and let $c=(x_i+x_{i+1})/2$ be the midpoint of this interval. Perform integration by parts on the following

$$\int_{x_i}^{x_{i+1}} (x-c)f'(x)\mathrm{d}x$$

and you should observe an integral form of the error. Then Taylor expand f about c and carry out the integration.]

(b) (8 pts) Write a script to implement the composite trapezoidal rule (3). Then use it to approximate the following integrals for number of intervals $n = 1, 5, 10, 20, \ldots, 100$. For each calculate the absolute error between the exact integral (which can be calculated by hand) and the approximated value, and plot the error as a function of n on a loglog plot. State how fast the error decays for each ($\mathcal{O}(1/n)$, $\mathcal{O}(1/n^2)$, $\mathcal{O}(1/n^3), \ldots$), and explain why this is the case if you can.

(i)

$$I = \int_2^{10} 3x\mathrm{d}x.$$

(ii)

$$I = \int_0^{\pi} \sin(x)\mathrm{d}x.$$

(iii)

$$I = \int_0^1 \mathrm{e}^{2\cos(2\pi x)}\mathrm{d}x$$

Exact: `2.279585302336067` (or besseli(0,2) in Matlab)

(iv)

$$I = \int_0^{2\pi} |\cos(x)|\mathrm{d}x$$

(c) (3 pts)You might have noticed that the error for example (iii) above converges spectrally. That is, the error decreases exponentially with increasing numbers of points, which happens when the trapezoidal rule is applied to some periodic functions. There are other periodic functions in the above set of examples, why doesn't the error converge spectrally for those other examples? (For more details on this, see the article below - also posted on Canvas).

The Exponentially Convergent Trapezoidal Rule, Trefethen and Weiderman
`http://epubs.siam.org/doi/pdf/10.1137/130932132`

Problem 2 - Nyström method for Laplace's equation (35 pts)

For this problem, the boundary is the unit circle (all points x, y such that $x^2 + y^2 = 1$). The boundary condition on the circle is of Neumann type, specifically, the normal derivative of the potential (with the normal pointing out of the circle) is

$$\left.\frac{\partial u}{\partial n}\right|_\Gamma = \frac{1}{3 + 2\cos\theta + \cos 2\theta}$$

where $\theta \in [-\pi, \pi]$, $x = \cos(\theta)$, and $y = \sin(\theta)$

For the 2-D Laplace's equation, the monopole integral formulation of the interior Neumann problem is given by

$$\frac{\partial u_\Gamma(\vec{x})}{\partial n_{\vec{x}}} = +\pi\sigma(\vec{x}) - \int_\Gamma^{PV} \frac{(\vec{x} - \vec{x}')^T n_{\vec{x}}}{\|\vec{x} - \vec{x}'\|^2}\sigma(\vec{x}')d\Gamma' \qquad \vec{x} \in \Gamma$$

and for the exterior Neumann problem, the monopole integral formulation is given by

$$\frac{\partial u_\Gamma(\vec{x})}{\partial n_{\vec{x}}} = -\pi\sigma(\vec{x}) - \int_\Gamma^{PV} \frac{(\vec{x} - \vec{x}')^T n_{\vec{x}}}{\|\vec{x} - \vec{x}'\|^2}\sigma(\vec{x}')d\Gamma' \qquad \vec{x} \in \Gamma.$$

a) (10 pts) Use a Nyström method to solve the exterior Neumann problem, using equally spaced quadrature points on the circle, and equal quadrature weights. Note that you should be able to get an analitic expression for all the integrals. Give an expression for the entries of the matrix and right hand side vector.

b) (10 pts) How quickly does the Nyström method (with equally spaced points) converge? That is, how quickly does the monopole density converge as you increase the number of points? Create and submit a plot of the error versus number of points to demonstrate convergence (you may find it helpful to check against the exact solution in the appendix below).

c) (5 pts) Once you have computed the monopole density, you can compute the potential at x, y point in the exterior domain using the equally-spaced-point quadrature approach. Give a formula for that potential, and comment about its usefulness for computing potentials at pointson the surface Γ.

d) (10 pts) Suppose the boundary condition for the exterior Neumann problem on the unit circle is specified as the normal derivative (pointing from interior to exterior) of given potential, as in

$$\frac{\partial u}{\partial n}\bigg|_{\Gamma} = \frac{\partial}{\partial n}\left(\log\sqrt{x^2 + (y + \frac{1}{2})^2} - \log\sqrt{x^2 + (y - \frac{1}{2})^2}\right).$$

Re-solve your integral equation for this boundary condition, and use your formula from part (c) to compare the difference between the exact potential (using the analytic formula, the sum of logs, above) and the numerical approximation from integrating your computed surface density. Examine the error at n test points on three test circles of radii $1 + \frac{1}{4}$, $1 + \frac{1}{16}$, and $1 + \frac{1}{256}$. How do the errors on each circle decay with n? Does exponential (or spectral convergence) occur for large enough n?

Problem 3 - Nyström and the Fredholm alternative(20 pts)

For this problem, we will examine solving the *interior* Neumann problem on the unit circle with the above equally-spaced-point Nyström method.

a) (5 pts) How do the matrix entries change when switching from the exterior to the interior Neumann problem. And how do the numerical properties of the system of equations change? Does the interior Neumann problem with either of the boundary conditions given in Problem 2 have a solution?

[Hint: Determine the condition that the normal derivative $\partial u/\partial n$ must satisfy for a solution to exist, and check if this condition holds. You can do this verification analitically or numerically.]

b) (15 pts) Suppose the boundary condition for the interior Neumann problem on the unit circle is specified as the normal derivative (pointing from interior to exterior) of given potential, as in

$$\frac{\partial u}{\partial n}\bigg|_{\Gamma} = \frac{\partial}{\partial n}\left(\log\sqrt{x^2 + (y + 2)^2} - \log\sqrt{x^2 + (y - 2)^2}\right).$$

Does the interior Neumann problem, with this boundary condition, have a solution? And if so, how many? Can you modify your Nyström method to compute an approximate solution given the potential at one point, in particular suppose that is $u(1/2, 0) = 0$? How fast is your approximation converging as you increase the number of Nyström points? Follow the strategy in part (d) of problem 2 to test your method.

Problem 4 - Nyström beyond a circle (30 pts)

In this problem you will develop an approach to solving the *exterior* Neumann problem using the equally-spaced-point Nyström method, but for the case where the boundary, Γ, is an ellipse. That is, $x, y \in \Gamma$ if

$$\frac{x^2}{4} + y^2 = 1.$$

a) (5pts) Determine the function that maps from $\theta \in [0, 2\pi]$ to $x, y \in \Gamma$.

b) (10 pts) Rewrite the line integral over the ellipse as an integral with respect to θ over the interval $[0, 2\pi]$. Take note to preserve the relationship between $d\Gamma$ and $d\theta$ as you change variables.

c) (15 pts) Use the potential function in question one, part d, to generate $\left.\frac{\partial u}{\partial n}\right|_{\Gamma}$, solve the exterior Neumann problem on the ellipse using the above transformation and the equally-spaced-point Nyström method. Then use the approach in problem 2, part d, to determine if your answer is accurate. And how fast does your approach converge as you increase the number of points? Show convergence plots.

Appendix

Exact solution for verification of numerical solution in 2(a), 2(b):
Let

$$f(\theta) := \frac{1}{3 + 2\cos\theta + \cos 2\theta}.$$

Then the exact solution to the exterior Neumann problem stated above is given by

$$\sigma(\theta) = -\frac{1}{\pi}\left(\frac{I}{2} + f(\theta)\right), \quad \theta \in [-\pi, \pi],$$

where

$$I := -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\mathrm{d}\theta.$$