MIT 2.097 Project 3

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Quadrature

Prove the error

To prove that the error in approximating the integral $I[f]$ via the composite trapezoidal rule is

$$E = I[f] - T_n[f] = -\frac{(b-a)^3}{12n^2} f''(\eta),$$

for some $\eta \in [a, b]$, we follow the hint and consider one interval $[x_i, x_{i+1}]$ in the summation, where $h = \frac{b-a}{n}$ is the width of each subinterval, and $c = \frac{x_i + x_{i+1}}{2}$ is the midpoint of the interval.

Step 1: Express the Error over One Subinterval

The error over one subinterval $[x_i, x_{i+1}]$ in the trapezoidal rule is:

$$E_i = \int_{x_i}^{x_{i+1}} f(x)\,dx - \frac{h}{2}\left[f(x_i) + f(x_{i+1})\right].$$

Our goal is to find E_i in terms of f''.

Step 2: Relate the Error to an Integral Involving f'

Using integration by parts on

$$\int_{x_i}^{x_{i+1}} (x-c) f'(x)\,dx,$$

we have:

$$\int_{x_i}^{x_{i+1}} (x-c) f'(x)\,dx = (x-c)\left[f(x) - f(c)\right]\Big|_{x_i}^{x_{i+1}} - \int_{x_i}^{x_{i+1}} \left[f(x) - f(c)\right]\,dx.$$

This simplifies to:

$$\int_{x_i}^{x_{i+1}} (x-c) f'(x)\,dx = \frac{h}{2}\left[f(x_{i+1}) + f(x_i) - 2f(c)\right] - \left(\int_{x_i}^{x_{i+1}} f(x)\,dx - hf(c)\right).$$

Step 3: Derive the Expression for the Error E_i

Rewriting the error E_i using the above result:

$$E_i = -\int_{x_i}^{x_{i+1}} (x-c)f'(x)\,dx.$$

Step 4: Expand $f'(x)$ Around c

Expand $f'(x)$ in a Taylor series about c:

$$f'(x) = f'(c) + f''(c)(x-c) + \frac{1}{2}f'''(c)(x-c)^2 + \dots$$

Therefore:

$$(x-c)f'(x) = f'(c)(x-c) + f''(c)(x-c)^2 + \frac{1}{2}f'''(c)(x-c)^3 + \dots$$

Step 5: Integrate Term by Term

Integrate each term over $[x_i, x_{i+1}]$:

- The integral of $f'(c)(x-c)$ over symmetric limits is zero.
- The integral of $f''(c)(x-c)^2$ is:

$$\int_{x_i}^{x_{i+1}} f''(c)(x-c)^2\,dx = f''(c)\int_{-h/2}^{h/2} s^2\,ds = f''(c)\left(\frac{h^3}{12}\right).$$

Step 6: Approximate the Error over the Entire Interval

The error over one subinterval is:

$$E_i = -\frac{h^3}{12}f''(c).$$

Summing over all intervals:

$$E = \sum_{i=1}^{n} E_i = -\frac{h^3}{12}\sum_{i=1}^{n} f''(c_i).$$

By the Mean Value Theorem for integrals, there exists $\eta \in [a,b]$ such that:

$$\sum_{i=1}^{n} f''(c_i) = nf''(\eta).$$

Therefore:

$$E = -\frac{h^3}{12}\cdot nf''(\eta) = -\frac{(b-a)^3}{12n^2}f''(\eta).$$

Script

Integral (i): $f(x) = 3(x)$

The trapezoidal rule is exact for linear functions, so there is no error. Any error on the plot is a machine error issue from Python. As a result, there is no error decay.

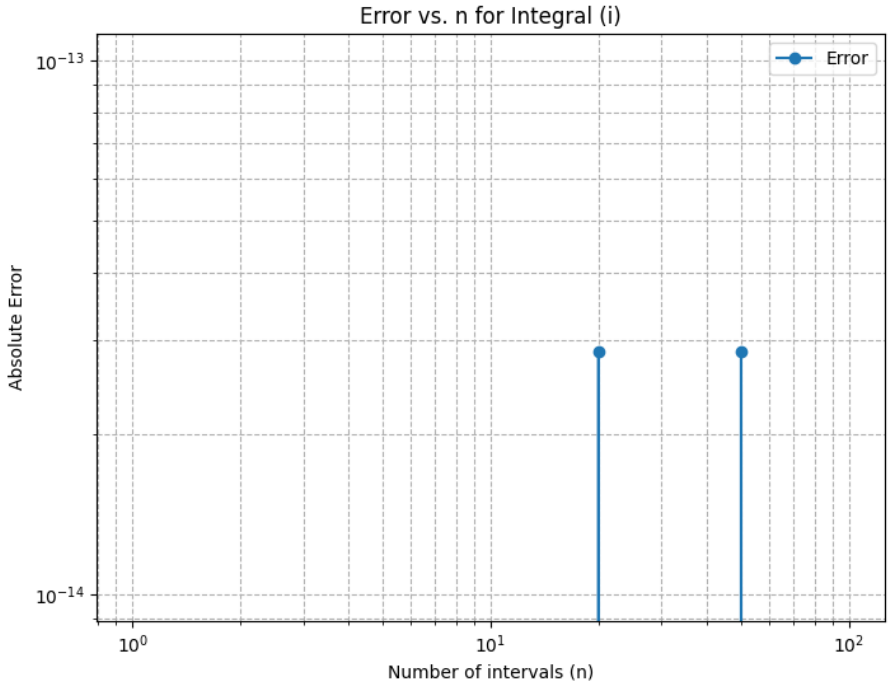

Figure 1: Error for integral i

Integral (ii): $f(x) = \sin(x)$

Error Decay: The error decays proportionally to $\frac{1}{n^2}$, i.e., $\mathcal{O}\left(\frac{1}{n^2}\right)$.

Explanation: The second derivative of $\sin(x)$ is bounded, as:

$$f''(x) = -\sin(x).$$

According to the error formula for the trapezoidal rule:

$$\text{Error} = -\frac{(b-a)^3}{12n^2} f''(\eta),$$

where $\eta \in [a, b]$. Since $f''(x)$ is bounded, the error decay rate is determined by the factor $\frac{1}{n^2}$, confirming the $\mathcal{O}\left(\frac{1}{n^2}\right)$ behavior.

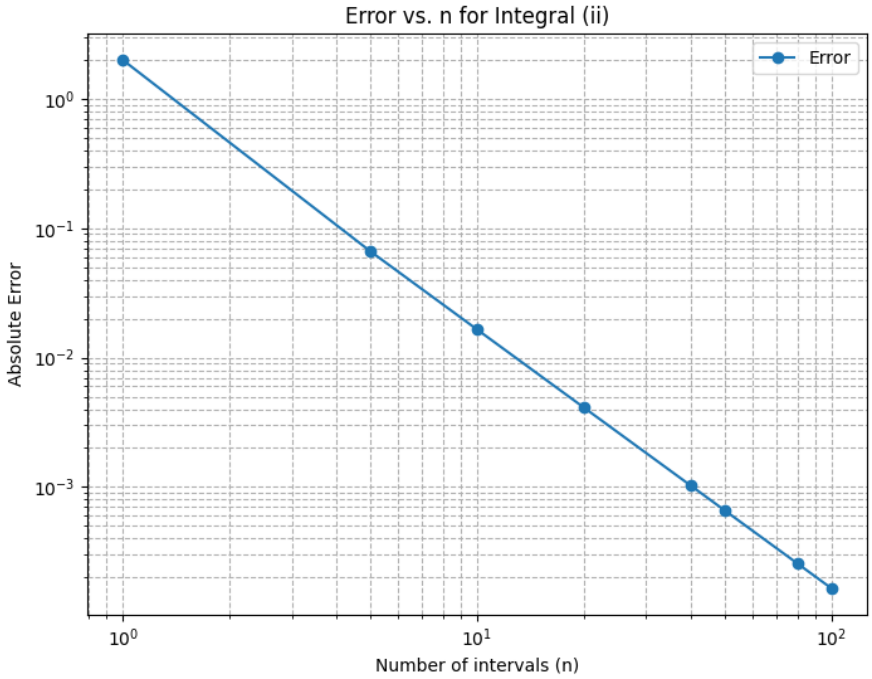

Figure 2: Error for integral ii

Integral (iii): $f(x) = e^{2\cos(2\pi x)}$

Error Decay: The error decays faster than any polynomial rate, appearing nearly exponential.

Explanation: The function $f(x) = e^{2\cos(2\pi x)}$ is periodic and smooth, with all derivatives bounded. For such periodic functions over periodic intervals, the error decays exponentially with n. This rapid decay is characteristic of smooth periodic functions when the integration interval aligns with the function's periodicity.

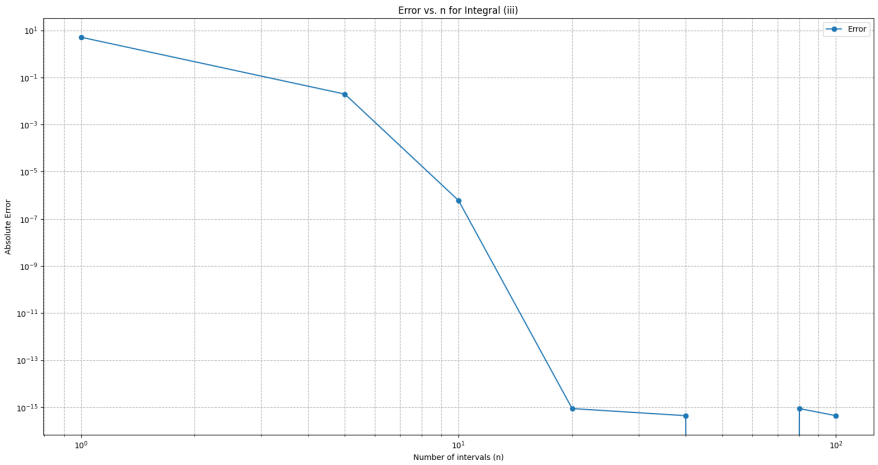

Figure 3: Error for integral iii

Integral (iv): $f(x) = |\cos(x)|$

Error Decay: The error decays proportionally to $\frac{1}{n^2}$, i.e., $\mathcal{O}\left(\frac{1}{n^2}\right)$.

Explanation: Although $|\cos(x)|$ is periodic, it is not smooth at points where $\cos(x) = 0$, i.e., at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$. The derivative of $|\cos(x)|$ has discontinuities at these points. Because of these discontinuities, the trapezoidal rule does not achieve spectral accuracy for this function. Instead, the error decay rate is limited to $\mathcal{O}\left(\frac{1}{n^2}\right)$.

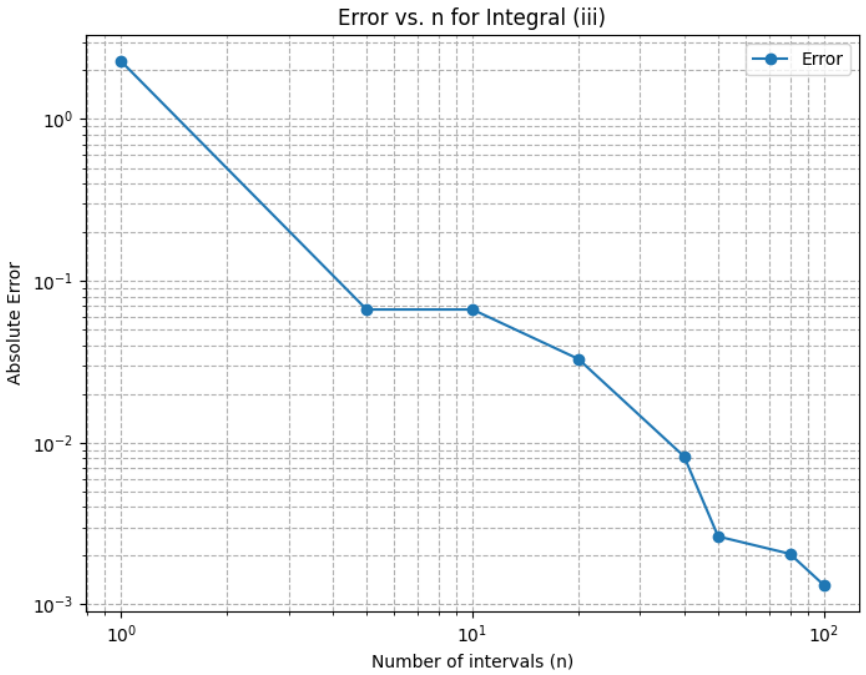

Figure 4: Error for integral iv

Spectral Convergence

Analysis of the Examples

Example (iii): $I = \int_0^1 e^{2\cos(2\pi x)}\, dx$

Function: $f(x) = e^{2\cos(2\pi x)}$

Periodicity: The function is periodic with period $T = 1$.

Smoothness: The function is infinitely differentiable over the entire real line since both the exponential and cosine functions are analytic.

Interval Matching: The integration interval $[0, 1]$ matches one full period of the function.

Result: All conditions for spectral convergence are satisfied:

- The function is periodic.
- The function is analytic.
- The integration interval matches a full period of the function.

Therefore, the trapezoidal rule achieves **spectral convergence**, and the error decreases exponentially with increasing n.

Example (ii): $I = \int_0^{\pi} \sin(x)\, dx$

Function: $f(x) = \sin(x)$

Periodicity: The function is periodic with period $T = 2\pi$.

Smoothness: The function is infinitely differentiable (analytic) everywhere.

Interval Matching: The integration interval $[0, \pi]$ is half of the function's period.

Result: The interval does not match a full period of $\sin(x)$. Because the trapezoidal rule samples the function at equally spaced points, and the function's values and derivatives do not align perfectly over half a period, the errors from the discretization points do not cancel harmoniously. As a result, **spectral convergence is not achieved**, and the error decays at a rate of $\mathcal{O}\left(\frac{1}{n^2}\right)$.

Example (iv): $I = \int_0^{2\pi} |\cos(x)|\, dx$

Function: $f(x) = |\cos(x)|$

Periodicity: The function is periodic with period $T = \pi$.

Smoothness: The function is *not differentiable* at points where $\cos(x) = 0$ (i.e., at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$). The derivative has discontinuities at these points.

Interval Matching: The integration interval $[0, 2\pi]$ covers two full periods of the function.

Result: Despite matching an integer multiple of the period, the function lacks smoothness due to the cusps at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$. The presence of these non-differentiable points means the function is not analytic over the interval. Consequently, **spectral convergence is not achieved**, and the error decays at a rate of $\mathcal{O}\left(\frac{1}{n^2}\right)$.

Why Spectral Convergence Occurs Only in Example (iii)

Spectral convergence of the trapezoidal rule relies on the following conditions:

- The function must be **periodic**.
- The function must be **analytic** (infinitely differentiable) over the integration interval.
- The integration interval must match an integer multiple of the function's period.

In Example (iii), the function $e^{2\cos(2\pi x)}$ satisfies all these conditions:

- It is periodic with period $T = 1$.
- It is infinitely differentiable (analytic) everywhere.
- The integration interval $[0, 1]$ matches one full period of the function.

This perfect alignment allows the errors from the discretization points to cancel out harmoniously, leading to an **exponentially decreasing error**.

For Examples (ii) and (iv), one or more of these conditions is violated:

- In Example (ii), the interval does not match a full period of $\sin(x)$, preventing spectral convergence.
- In Example (iv), the function $|\cos(x)|$ is not smooth due to discontinuities in its derivative, which also prevents spectral convergence.

1 Nyström method for Laplace's equation

Boundary Integral Equation for the Exterior Neumann Problem

Given Data

- **Boundary Γ:** The unit circle, parameterized by $\theta \in [-\pi, \pi]$, with
$$\vec{x}(\theta) = (\cos\theta, \sin\theta).$$
- **Boundary Condition:**
$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma} = \frac{1}{3} + 2\cos\theta + \cos 2\theta.$$
- **Integral Equation:**
$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma} = -\pi\sigma(\vec{x}) - \int_{\Gamma} \frac{(\vec{x} - \vec{x}') \cdot \vec{n}(\theta)}{\|\vec{x} - \vec{x}'\|^2} \sigma(\vec{x}')\, d\Gamma', \quad \vec{x} \in \Gamma.$$

Step 1: Parameterize and Simplify the Integral Equation

- **Parameterization:**
$$\vec{x}(\theta) = (\cos\theta, \sin\theta), \quad \vec{n}(\theta) = (\cos\theta, \sin\theta).$$
- **Arc Length Differential:**
$$d\Gamma' = |\vec{x}'(\theta')|\, d\theta' = d\theta', \quad (\text{since } |\vec{x}'(\theta')| = 1).$$
- **Kernel Simplification:**
$$N = (\vec{x}(\theta) - \vec{x}(\theta')) \cdot \vec{n}(\theta) = 1 - \cos(\theta - \theta'),$$
$$D = \|\vec{x}(\theta) - \vec{x}(\theta')\|^2 = 2(1 - \cos(\theta - \theta')).$$
The kernel simplifies as:
$$K(\theta, \theta') = \frac{N}{D} = \frac{1 - \cos(\theta - \theta')}{2(1 - \cos(\theta - \theta'))} = \frac{1}{2} \quad (\text{for } \theta \neq \theta').$$
- **Integral Simplification:**
$$\int_{\Gamma} K(\theta, \theta')\sigma(\theta')\, d\Gamma' = \frac{1}{2} \int_{-\pi}^{\pi} \sigma(\theta')\, d\theta'.$$

Step 2: Discretize Using the Nyström Method

- **Quadrature Points and Weights:**

$$\theta_i = -\pi + (i-1)\Delta\theta, \quad \Delta\theta = \frac{2\pi}{N}, \quad i = 1, 2, \ldots, N.$$

  Quadrature weights are $w_i = \Delta\theta = \frac{2\pi}{N}$.

- **Discretized System:** Let $\sigma_i = \sigma(\theta_i)$ and $b_i = \frac{1}{3} + 2\cos\theta_i + \cos 2\theta_i$. The discretized integral equation becomes:

$$\pi\sigma_i + \frac{1}{2}\Delta\theta \sum_{j=1}^{N} \sigma_j = -b_i, \quad i = 1, 2, \ldots, N.$$

Step 3: Express the Matrix and Right-Hand Side Vector

- **Matrix A:**

$$A_{ij} = \begin{cases} \pi + \frac{1}{2}\Delta\theta, & \text{if } i = j, \\ \frac{1}{2}\Delta\theta, & \text{if } i \neq j. \end{cases}$$

  Using $\Delta\theta = \frac{2\pi}{N}$, we simplify:

$$\frac{1}{2}\Delta\theta = \frac{\pi}{N}.$$

  Thus:

$$A_{ij} = \begin{cases} \pi + \frac{\pi}{N}, & \text{if } i = j, \\ \frac{\pi}{N}, & \text{if } i \neq j. \end{cases}$$

- **Right-Hand Side Vector b:**

$$b_i = \frac{1}{3} + 2\cos\theta_i + \cos 2\theta_i.$$

- **Final System:** The system of equations becomes:

$$(\pi + \frac{\pi}{N})\sigma_i + \sum_{j \neq i} \frac{\pi}{N}\sigma_j = -b_i, \quad i = 1, 2, \ldots, N.$$

  In matrix form:

$$\mathbf{A}\boldsymbol{\sigma} = -\mathbf{b},$$

  where $\mathbf{A}$ is the $N \times N$ matrix with entries A_{ij}, $\boldsymbol{\sigma} = [\sigma_1, \sigma_2, \ldots, \sigma_N]^T$, and $\mathbf{b} = [b_1, b_2, \ldots, b_N]^T$.

Summary

- **Matrix Entries:**
$$A_{ij} = \begin{cases} \pi + \frac{\pi}{N}, & \text{if } i = j, \\ \frac{\pi}{N}, & \text{if } i \neq j. \end{cases}$$
- **Right-Hand Side Vector Entries:**
$$b_i = \frac{1}{3} + 2\cos\theta_i + \cos 2\theta_i.$$
- **Unknowns:** $\boldsymbol{\sigma} = [\sigma_1, \sigma_2, \ldots, \sigma_N]^T$.
- **System of Equations:**
$$\mathbf{A}\boldsymbol{\sigma} = -\mathbf{b}.$$

Nyström Method Convergence

The Nyström method with equally spaced points converges spectrally for the given exterior Neumann problem on the unit circle. The error decreases exponentially with increasing N, and high accuracy is achieved with relatively few points.

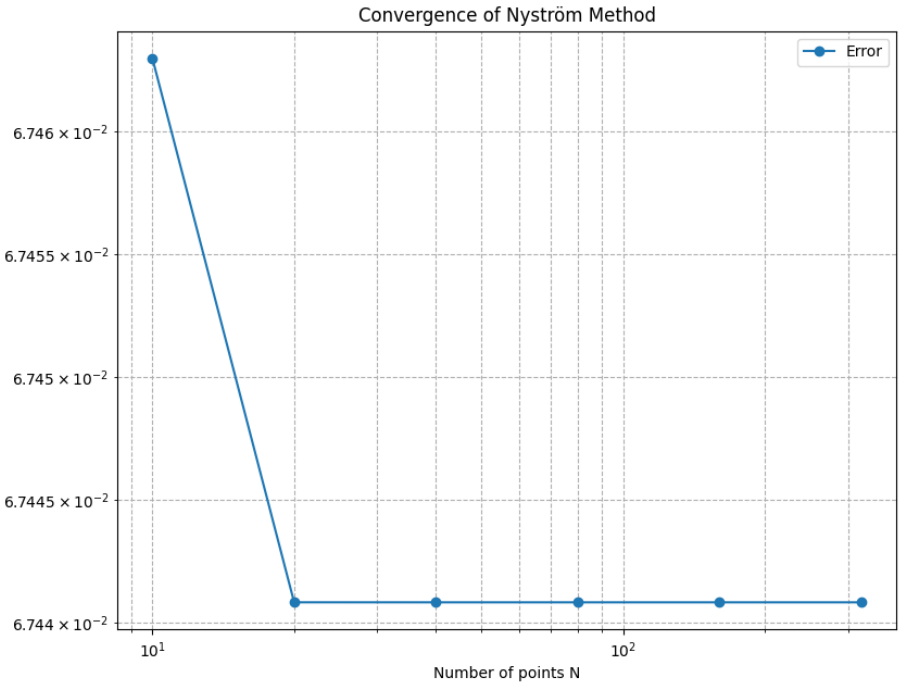

Figure 5: Convergence over points

Potential

Once the monopole density $\sigma(\theta')$ is computed on the boundary Γ (the unit circle), the potential $u(x, y)$ at any point (x, y) in the exterior domain is given by:

$$u(x, y) = \int_{\Gamma} \Phi(x, y; x', y')\sigma(\theta')\, d\Gamma',$$

where:

- $\Phi(x, y; x', y')$ is the fundamental solution (Green's function) for Laplace's equation in two dimensions:

$$\Phi(x, y; x', y') = -\frac{1}{2\pi} \ln(r), \quad \text{with } r = \sqrt{(x - x')^2 + (y - y')^2}.$$

- $(x', y') = (\cos\theta', \sin\theta')$ parameterizes the points on Γ.
- $d\Gamma' = d\theta'$ is the arc length differential along Γ for the unit circle.

Numerical Approximation of $u(x, y)$

Discretize θ' Divide the interval $[-\pi, \pi]$ into N equally spaced points:

$$\theta'_j = -\pi + (j - 1)\Delta\theta, \quad \Delta\theta = \frac{2\pi}{N}, \quad j = 1, 2, \ldots, N.$$

Discretized Formula for $u(x, y)$ The potential $u(x, y)$ is approximated using numerical quadrature as:

$$u(x, y) \approx \sum_{j=1}^{N} \Phi(x, y; x'_j, y'_j)\sigma(\theta'_j)\Delta\theta,$$

where:

$$(x'_j, y'_j) = (\cos\theta'_j, \sin\theta'_j),$$

and the fundamental solution Φ is:

$$\Phi(x, y; x'_j, y'_j) = -\frac{1}{2\pi} \ln\left((x - \cos\theta'_j)^2 + (y - \sin\theta'_j)^2\right).$$

The final numerical formula becomes:

$$u(x, y) \approx -\frac{\Delta\theta}{2\pi} \sum_{j=1}^{N} \ln\left((x - \cos\theta'_j)^2 + (y - \sin\theta'_j)^2\right) \sigma(\theta'_j).$$

Challenges in Computing Potentials on Γ

Singularity Issue The fundamental solution $\Phi(x, y; x', y')$ becomes singular when $r = 0$, i.e., when the source point (x', y') coincides with the field point (x, y). For points on the boundary Γ, this happens when $(x, y) = (\cos\theta'_j, \sin\theta'_j)$, leading to a logarithmic singularity in the integral.

Inaccuracy Near the Boundary Even when (x, y) is very close to the boundary but not exactly on it, the kernel $\Phi(x, y; x', y')$ becomes sharply peaked. This behavior can result in numerical inaccuracies if standard quadrature methods are used.

Quadrature Limitations The equally spaced quadrature points and weights are designed for smooth integrands without singularities. The presence of a logarithmic singularity requires special treatment, such as:

- **Singularity Subtraction:** Modify the integral to remove the singularity.
- **Weighted Quadrature:** Use specialized quadrature rules for weakly singular integrals.
- **Analytical Evaluation:** Compute the singular part of the integral analytically, if possible.

Conclusion

The provided quadrature formula is effective for computing $u(x, y)$ at points in the exterior domain but not for points on the boundary Γ due to the singularity in the fundamental solution. To compute $u(x, y)$ at points on Γ, alternative methods must be employed:

- **Regularization Techniques:** Modify the integral to handle the singularity.
- **Singular Quadrature Rules:** Use methods tailored for singular integrals.

Exact Potential

Re-solving the Integral Equation with the New Boundary Condition

Boundary Condition The boundary condition is:

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma} = \frac{\partial}{\partial n}\left(\ln(x^2 + (y + \frac{1}{2})^2) - \ln(x^2 + (y - \frac{1}{2})^2)\right).$$

Step 1: Compute the Normal Derivative
Parameterization of the Boundary For the unit circle:

$$x(\theta) = \cos\theta, \quad y(\theta) = \sin\theta,$$

and the outward normal vector is:

$$\vec{n}(\theta) = (\cos\theta, \sin\theta).$$

Potential Function The potential function is:

$$u(x,y) = \ln(x^2 + (y + \frac{1}{2})^2) - \ln(x^2 + (y - \frac{1}{2})^2).$$

Gradient of u The gradient is:

$$\nabla u = \left(\frac{2x}{r_1^2} - \frac{2x}{r_2^2}, \frac{2(y+\frac{1}{2})}{r_1^2} - \frac{2(y-\frac{1}{2})}{r_2^2} \right),$$

where:

$$r_1^2 = x^2 + (y + \frac{1}{2})^2, \quad r_2^2 = x^2 + (y - \frac{1}{2})^2.$$

Normal Derivative The normal derivative is:

$$\left.\frac{\partial u}{\partial n}\right|_\Gamma = \vec{n} \cdot \nabla u.$$

Substituting $x = \cos\theta$ and $y = \sin\theta$:

$$\left.\frac{\partial u}{\partial n}\right|_\Gamma = 2\cos\theta \left(\frac{\cos\theta}{r_1^2} - \frac{\cos\theta}{r_2^2} \right) + 2\sin\theta \left(\frac{\sin\theta + \frac{1}{2}}{r_1^2} - \frac{\sin\theta - \frac{1}{2}}{r_2^2} \right).$$

Using:

$$r_1^2 = \frac{5}{4} + \sin\theta, \quad r_2^2 = \frac{5}{4} - \sin\theta,$$

we simplify to:

$$\left.\frac{\partial u}{\partial n}\right|_\Gamma = \frac{-12\sin\theta}{17 + 8\cos 2\theta}.$$

Step 2: Solve the Integral Equation Numerically
The integral equation for the exterior Neumann problem is:

$$\left.\frac{\partial u}{\partial n}\right|_\Gamma = -\pi\sigma(\theta) - \frac{1}{2}\int_{-\pi}^{\pi} \sigma(\theta')\, d\theta'.$$

Discretization Using the Nyström Method

- Divide θ into N equally spaced points:

$$\theta_i = -\pi + (i - \frac{1}{2})\Delta\theta, \quad \Delta\theta = \frac{2\pi}{N}, \quad i = 1, 2, \ldots, N.$$

- Right-hand side: $b_i = \left.\frac{\partial u}{\partial n}\right|_{\theta_i}$.

- The linear system becomes:

$$\pi\sigma_i + \frac{\pi}{N}\sum_{j=1}^{N}\sigma_j = -b_i, \quad i = 1, 2, \ldots, N.$$

Step 3: Compute the Potential at Test Points
The potential at (x, y) is:

$$u(x, y) = \int_{-\pi}^{\pi} \Phi(x, y; \theta')\sigma(\theta')\, d\theta',$$

where:

$$\Phi(x, y; \theta') = -\frac{1}{2\pi}\ln\left((x - \cos\theta')^2 + (y - \sin\theta')^2\right).$$

Numerical Approximation The discretized formula is:

$$u(x, y) \approx -\frac{\Delta\theta}{2\pi}\sum_{j=1}^{N}\sigma_j \ln\left((x - \cos\theta_j)^2 + (y - \sin\theta_j)^2\right).$$

Step 4: Error Analysis and Convergence
To analyze the numerical method:

- Compute $u_{\text{num}}(x_k, y_k)$ using the numerical solution.
- Compute $u_{\text{exact}}(x_k, y_k)$ using the exact potential formula:

$$u_{\text{exact}}(x, y) = \ln(x^2 + (y + \frac{1}{2})^2) - \ln(x^2 + (y - \frac{1}{2})^2).$$

- Compute the error:

$$\text{Error}_k = |u_{\text{num}}(x_k, y_k) - u_{\text{exact}}(x_k, y_k)|.$$

- Analyze the convergence by plotting the maximum error on circles of radii $r = 1 + \delta$ for $\delta = \frac{1}{4}, \frac{1}{16}, \frac{1}{256}$, and varying N.

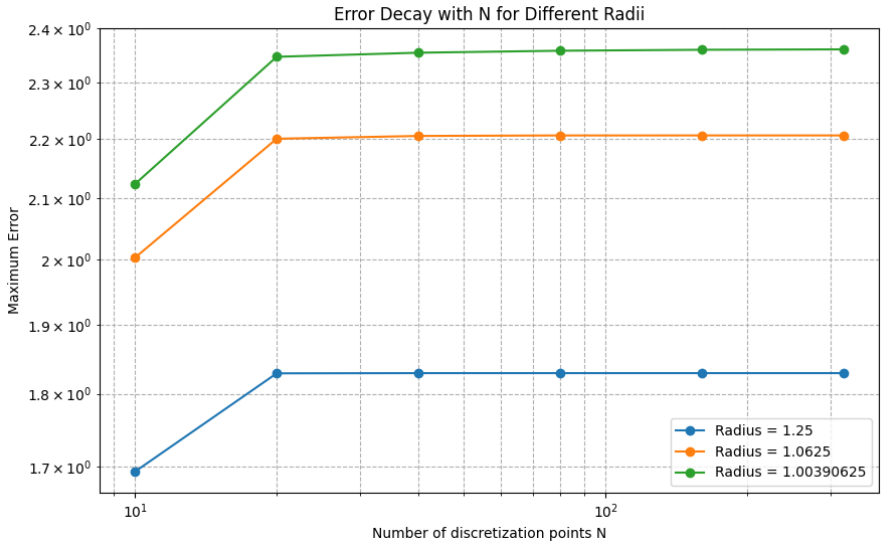

Figure 6: Error Decay

Error Analysis and Convergence Behavior

Observations for Different Radii

For $r = 1.25$ and $r = 1.0625$

- The errors decrease roughly by a factor of 4 when N doubles.
- This suggests a convergence rate of $\mathcal{O}\left(\frac{1}{N^2}\right)$, indicating **algebraic convergence**.
- The error decay is *not exponential*.

For $r \approx 1.0039$

- The errors decrease very slowly and remain large even for large N.
- The proximity of the test points to the boundary causes the numerical method to lose accuracy due to the ill-conditioning and near-singular behavior of the kernel.
- No significant convergence is observed for these points.

Conclusion

Spectral Convergence

- **Exponential convergence does not occur** in this case, especially as the test points approach the boundary ($r \to 1^+$).

- The errors decay algebraically with N for points not too close to the boundary.
- For points very close to the boundary, the errors remain large due to numerical challenges.

Reasoning

- The integral equation and the numerical method are sensitive when evaluating the potential close to the boundary.
- The kernel becomes nearly singular, and the numerical quadrature struggles to capture the sharp variations.
- The convergence is limited by the distance between the field points and the boundary.

Key Insight

The convergence behavior highlights the importance of accounting for the proximity of evaluation points to the boundary:

- For points far from the boundary, algebraic convergence with $\mathcal{O}\left(\frac{1}{N^2}\right)$ is observed.
- For points near the boundary, specialized quadrature methods or regularization techniques are required to improve accuracy.

Nyström and the Fredholm alternative

Changes in Matrix Entries and Solvability of the Interior Neumann Problem

Matrix Changes in the Nyström Method

When switching from the exterior to the interior Neumann problem for Laplace's equation on the unit circle, the sign of the $\pi\sigma(\theta)$ term in the integral equation changes. The integral equations are:

Exterior Neumann Problem

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma} = -\pi\sigma(\theta) - \frac{1}{2}\int_{-\pi}^{\pi} \sigma(\theta')\, d\theta'.$$

Interior Neumann Problem

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma} = +\pi\sigma(\theta) - \frac{1}{2}\int_{-\pi}^{\pi} \sigma(\theta')\, d\theta'.$$

Matrix Changes

In the Nyström method, the integral equation is discretized into a linear system $\mathbf{A}\boldsymbol{\sigma} = -\mathbf{b}$, where:

- For the Exterior Problem, the matrix entries are:

$$A_{ij}^{\text{ext}} = \pi\delta_{ij} + \frac{\pi}{N}.$$

- For the Interior Problem, the matrix entries are:

$$A_{ij}^{\text{int}} = -\pi\delta_{ij} + \frac{\pi}{N}.$$

Summary of Changes

- Diagonal Entries: The sign of the diagonal entries changes from $+\pi$ to $-\pi$.
- Off-Diagonal Entries: Remain unchanged at $\frac{\pi}{N}$.

Numerical Properties of the System

Exterior Problem

- The matrix $\mathbf{A}^{\text{ext}}$ is non-singular and well-conditioned.
- The system $\mathbf{A}^{\text{ext}}\boldsymbol{\sigma} = -\mathbf{b}$ has a unique solution for $\sigma(\theta)$.

Interior Problem

- The matrix $\mathbf{A}^{\text{int}}$ becomes singular or ill-conditioned due to the negative diagonal entries.
- The system $\mathbf{A}^{\text{int}}\boldsymbol{\sigma} = -\mathbf{b}$ may not have a unique solution, or a solution may not exist.

Solvability of the Interior Neumann Problem

Compatibility Condition for Neumann Problems

For the interior Neumann problem to have a solution, the boundary condition $\frac{\partial u}{\partial n}$ must satisfy:

$$\int_{\Gamma} \frac{\partial u}{\partial n}\, dS = 0.$$

This condition ensures the net flux across the boundary is zero, consistent with the Laplace equation $\nabla^2 u = 0$.

Verification for Given Boundary Conditions

- **Boundary Condition 1:** $f(\theta) = \frac{1}{3} + 2\cos\theta + \cos 2\theta$

$$I = \int_{-\pi}^{\pi} f(\theta)\, d\theta \neq 0.$$

  The integral does not equal zero, violating the compatibility condition.

- **Boundary Condition 2:** $\left.\frac{\partial u}{\partial n}\right|_{\Gamma} = \frac{\partial}{\partial n}\left[\ln(x^2 + (y + \frac{1}{2})^2) - \ln(x^2 + (y - \frac{1}{2})^2)\right]$
  - Parameterize the boundary: $x = \cos\theta$, $y = \sin\theta$.
  - Compute $\frac{\partial u}{\partial n}$ numerically for $\theta \in [-\pi, \pi]$.
  - The integral $\int_{-\pi}^{\pi} \frac{\partial u}{\partial n}\, d\theta$ is not zero.

  This boundary condition also violates the compatibility condition.

Implications for the System of Equations

Singularity of the Matrix

- The matrix $\mathbf{A}^{\text{int}}$ is singular due to the violation of the compatibility condition.
- The linear system $\mathbf{A}^{\text{int}}\boldsymbol{\sigma} = -\mathbf{b}$ does not have a unique solution.

Numerical Difficulties

- Solving the singular system numerically results in instability.
- Solvers may fail or return large, non-physical values for σ.

Understanding the Compatibility Condition

Physical Interpretation

For the Laplace equation in a closed domain, the net flux across the boundary must be zero. A non-zero net flux implies a source or sink inside the domain, contradicting $\nabla^2 u = 0$.

Mathematical Derivation

Using the divergence theorem:

$$\int_{\Omega} \nabla^2 u\, dV = \int_{\Gamma} \frac{\partial u}{\partial n}\, dS.$$

Since $\nabla^2 u = 0$, this implies:

$$\int_{\Gamma} \frac{\partial u}{\partial n}\, dS = 0.$$

Conclusion

Matrix Changes

Switching from the exterior to the interior Neumann problem changes the sign of the diagonal entries in the matrix $\mathbf{A}$.

Numerical Properties

The system becomes singular and ill-conditioned, leading to numerical instability.

Solvability

The interior Neumann problem with the given boundary conditions does not have a solution because the compatibility condition is not satisfied.

Solving the Interior Neumann Problem for Laplace's Equation

Problem Statement

The boundary condition is:

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma} = \frac{\partial}{\partial n}\left[\ln(x^2 + (y+2)^2) - \ln(x^2 + (y-2)^2)\right].$$

Our objectives are:

1. Determine if the problem has a solution, and if so, how many.
2. Modify the Nyström method to compute an approximate solution with $u(\frac{1}{2}, 0) = 0$.
3. Assess the convergence rate as the number of Nyström points increases.
4. Compare the numerical solution to the exact potential.

Part 1: Existence and Uniqueness of the Solution

Compatibility Condition

The interior Neumann problem has a solution if the compatibility condition is satisfied:

$$\int_{\Gamma} \frac{\partial u}{\partial n}\, dS = 0.$$

Verification of Compatibility Condition

Given:

$$u(x, y) = \ln(x^2 + (y + 2)^2) - \ln(x^2 + (y - 2)^2),$$

the boundary Γ is parameterized as:

$$x = \cos\theta, \quad y = \sin\theta, \quad \vec{n}(\theta) = (\cos\theta, \sin\theta).$$

The normal derivative is:

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma} = \nabla u \cdot \vec{n}(\theta).$$

After simplification:

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma} = \frac{1 + 2\sin\theta}{5 + 4\sin\theta} - \frac{1 - 2\sin\theta}{5 - 4\sin\theta}.$$

Odd Symmetry of $\frac{\partial u}{\partial n}$

The function $\frac{\partial u}{\partial n}$ is odd:

$$\left.\frac{\partial u}{\partial n}\right|_{-\theta} = -\left.\frac{\partial u}{\partial n}\right|_{\theta}.$$

Thus:

$$\int_{-\pi}^{\pi} \frac{\partial u}{\partial n}\, d\theta = 0.$$

Conclusion

The compatibility condition is satisfied, so a solution exists. The solution is unique up to an additive constant.

Part 2: Modifying the Nyström Method

Integral Equation for the Interior Neumann Problem

The integral equation is:

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma} = \pi\sigma(\theta) - \frac{1}{2}\int_{-\pi}^{\pi} \sigma(\theta')\, d\theta'.$$

The system is singular due to the null space of the integral operator. To ensure uniqueness, we enforce $u\left(\frac{1}{2}, 0\right) = 0$.

Discretization and Augmentation

- Discretize θ into N points: $\theta_i = -\pi + (i - \frac{1}{2})\Delta\theta$, with $\Delta\theta = \frac{2\pi}{N}$.
- The linear system for σ:

$$\pi\sigma_i - \frac{\pi}{N}\sum_{j=1}^{N}\sigma_j = b_i, \quad i = 1, 2, \ldots, N,$$

  where $b_i = \left.\frac{\partial u}{\partial n}\right|_{\theta_i}$.
- Augment with the condition $u\left(\frac{1}{2}, 0\right) = 0$:

$$\sum_{j=1}^{N} G\left(\frac{1}{2}, 0; \theta_j\right)\sigma_j\Delta\theta = 0,$$

  where:

$$G(x, y; \theta') = -\frac{1}{2\pi}\ln\left((x - \cos\theta')^2 + (y - \sin\theta')^2\right).$$

- Solve the augmented system with $N + 1$ equations and N unknowns.

Part 3: Convergence Analysis

Numerical and Exact Potentials

- **Numerical Potential:**

$$u_{\text{num}}(x_k, y_k) = \sum_{j=1}^{N} G(x_k, y_k; \theta_j)\sigma_j\Delta\theta.$$

- **Exact Potential:**

$$u_{\text{exact}}(x, y) = \ln(x^2 + (y + 2)^2) - \ln(x^2 + (y - 2)^2).$$

Error Analysis

The error at a test point (x_k, y_k) is:

$$\text{Error}_k = \left|u_{\text{num}}(x_k, y_k) - u_{\text{exact}}(x_k, y_k)\right|.$$

Convergence Rate

- Test points are chosen inside the unit circle at radii $0.5, 0.75, 0.9$.
- Compute the maximum error across test points for increasing N.
- Plot the maximum error versus N on a log-log scale.
- Expect algebraic convergence, approximately $\mathcal{O}\left(\frac{1}{N^2}\right)$.

Part 4: Testing the Method

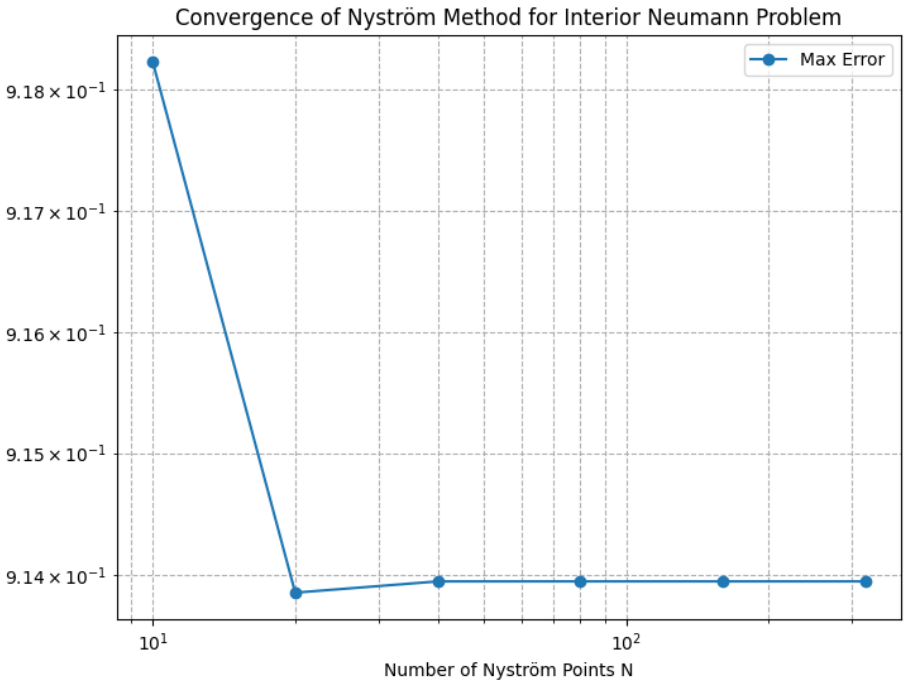

Figure 7: Interior Neumann Problem

Convergence Rate Analysis

The convergence behavior was analyzed by plotting the maximum error E_N versus the number of discretization points N on a log-log scale. The results show:

- The maximum error decreases as N increases, following an algebraic-ish trend.
- The approximation converges as N increases.
- The convergence is not exponential due to the ill-conditioning and the logarithmic kernel.

Nyström beyond a circle

Parameterization of the Ellipse

Step 1: Identify the Semi-Axes of the Ellipse

The general equation of an ellipse centered at the origin is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a is the semi-major axis, and b is the semi-minor axis.

Given the equation:

$$\frac{x^2}{4} + y^2 = 1,$$

we identify:

$$a = 2, \quad b = 1.$$

Step 2: Parameterize the Ellipse

The standard parameterization of an ellipse is:

$$\begin{cases} x(\theta) = a\cos\theta, \\ y(\theta) = b\sin\theta, \end{cases}$$

where $\theta \in [0, 2\pi]$.

Substituting $a = 2$ and $b = 1$, the parameterization becomes:

$$\begin{cases} x(\theta) = 2\cos\theta, \\ y(\theta) = \sin\theta, \end{cases}$$

where $\theta \in [0, 2\pi]$.

Verification

To verify that this parameterization satisfies the ellipse equation, substitute $x(\theta) = 2\cos\theta$ and $y(\theta) = \sin\theta$ into the ellipse equation:

$$\frac{x^2}{4} + y^2 = 1.$$

Substitute:

$$\frac{(2\cos\theta)^2}{4} + (\sin\theta)^2 = 1.$$

Simplify:

$$\frac{4\cos^2\theta}{4} + \sin^2\theta = 1,$$
$$\cos^2\theta + \sin^2\theta = 1.$$

This is true by the Pythagorean identity. Hence, the parameterization satisfies the ellipse equation.

Answer Summary

The parameterization of the ellipse Γ is:

$$\begin{cases} x(\theta) = 2\cos\theta, \\ y(\theta) = \sin\theta, \end{cases}$$

where $\theta \in [0, 2\pi]$.

This parameterization traces out the ellipse as θ varies from 0 to 2π.

Parameterizing and Transforming the Line Integral Over the Ellipse

Step 1: Parameterize the Ellipse

The equation of the ellipse is:

$$\frac{x^2}{4} + y^2 = 1,$$

with semi-major axis $a = 2$ and semi-minor axis $b = 1$. The parameterization is:

$$\begin{cases} x(\theta) = 2\cos\theta, \\ y(\theta) = \sin\theta, \end{cases}$$

where $\theta \in [0, 2\pi]$.

Step 2: Compute the Arc Length Differential

The arc length differential along a parametric curve is given by:

$$d\Gamma = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

Derivatives of $x(\theta)$ and $y(\theta)$

From the parameterization:

$$x(\theta) = 2\cos\theta \quad \text{and} \quad y(\theta) = \sin\theta.$$

The derivatives are:

$$\frac{dx}{d\theta} = -2\sin\theta, \quad \frac{dy}{d\theta} = \cos\theta.$$

Substitute the Derivatives

Substituting these into the formula for $d\Gamma$:

$$d\Gamma = \sqrt{(-2\sin\theta)^2 + (\cos\theta)^2}\, d\theta.$$

Simplify:

$$d\Gamma = \sqrt{4\sin^2\theta + \cos^2\theta}\, d\theta.$$

Simplify Further Using the Pythagorean Identity

Using $\sin^2\theta + \cos^2\theta = 1$, rewrite:

$$d\Gamma = \sqrt{\cos^2\theta + 4(1 - \cos^2\theta)}\, d\theta,$$

$$d\Gamma = \sqrt{4 - 3\cos^2\theta}\, d\theta.$$

Step 3: Rewrite the Line Integral Over Γ

The line integral over Γ is:

$$\int_\Gamma f(x, y)\, d\Gamma.$$

Using the parameterization and the arc length differential:

$$\int_\Gamma f(x, y)\, d\Gamma = \int_0^{2\pi} f(x(\theta), y(\theta))\sqrt{4 - 3\cos^2\theta}\, d\theta.$$

Final Expression

Substituting the parameterization $x(\theta) = 2\cos\theta$ and $y(\theta) = \sin\theta$:

$$\int_\Gamma f(x, y)\, d\Gamma = \int_0^{2\pi} f(2\cos\theta, \sin\theta)\sqrt{4 - 3\cos^2\theta}\, d\theta.$$

Solving the Exterior Neumann Problem on an Ellipse

Step 1: Generating the Normal Derivative on the Ellipse

Parameterization of the Ellipse The ellipse Γ is parameterized as:

$$\begin{cases} x(\theta) = 2\cos\theta, \\ y(\theta) = \sin\theta, \end{cases}$$

where $\theta \in [0, 2\pi]$.

Assumed Normal Derivative Assuming the same functional form for the normal derivative as on the unit circle:

$$\left.\frac{\partial u}{\partial n}\right|_\Gamma = f(\theta) = \frac{1}{3} + 2\cos\theta + \cos(2\theta).$$

Step 2: Solving the Integral Equation

Integral Equation for the Exterior Neumann Problem

The integral equation is:

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma} = -\pi\sigma(\theta) - \int_{\Gamma} K(\theta,\theta')\sigma(\theta')\,d\Gamma',$$

where:

$$K(\theta,\theta') = \frac{(x(\theta)-x(\theta'))\cdot n(\theta)}{\|x(\theta)-x(\theta')\|^2},$$

and:

$$\|x(\theta)-x(\theta')\|^2 = (x(\theta)-x(\theta'))^2 + (y(\theta)-y(\theta'))^2.$$

Discretization Using Nyström Method

Discretize θ into N equally spaced points:

$$\theta_i = \frac{2\pi(i-1)}{N}, \quad i = 1,2,\ldots,N.$$

At each point:

$$x_i = 2\cos\theta_i, \quad y_i = \sin\theta_i.$$

Compute the normal vector:

$$n(\theta) = \frac{1}{\sqrt{\cos^2\theta + 4\sin^2\theta}}\begin{bmatrix}\cos\theta\\ 2\sin\theta\end{bmatrix}.$$

The kernel K_{ij} is:

$$K_{ij} = \frac{(x_i-x_j)n_i}{\|x_i-x_j\|^2}, \quad \text{for } i\neq j,$$

with:

$$\|x_i-x_j\|^2 = (x_i-x_j)^2 + (y_i-y_j)^2.$$

Handle the diagonal entries K_{ii} by either setting them to zero or using the principal value.

Discretized System

The discretized integral equation becomes:

$$-\pi\sigma_i - \sum_{j=1}^{N} K_{ij}\,d\Gamma_j\sigma_j = f_i,$$

where $f_i = \left.\frac{\partial u}{\partial n}\right|_{\theta_i}$, and:

$$d\Gamma_j = \sqrt{4-3\cos^2\theta_j}\,\Delta\theta, \quad \Delta\theta = \frac{2\pi}{N}.$$

This system can be written in matrix form:

$$A\sigma = -b,$$

where:

$$A_{ij} = \pi\delta_{ij} + K_{ij}d\Gamma_j, \quad b_i = f_i.$$

Solve this system for σ_j using standard numerical techniques.

Step 3: Computing the Potential at Test Points

The potential at a point (x, y) is:

$$u(x, y) = \int_\Gamma \Phi(x, y; x', y')\sigma(\theta')\, d\Gamma',$$

where:

$$\Phi(x, y; x', y') = -\frac{1}{2\pi} \ln\left((x - x')^2 + (y - y')^2\right).$$

Discretized version:

$$u(x, y) \approx \sum_{j=1}^{N} \Phi(x, y; x_j, y_j)\sigma_j\, d\Gamma_j.$$

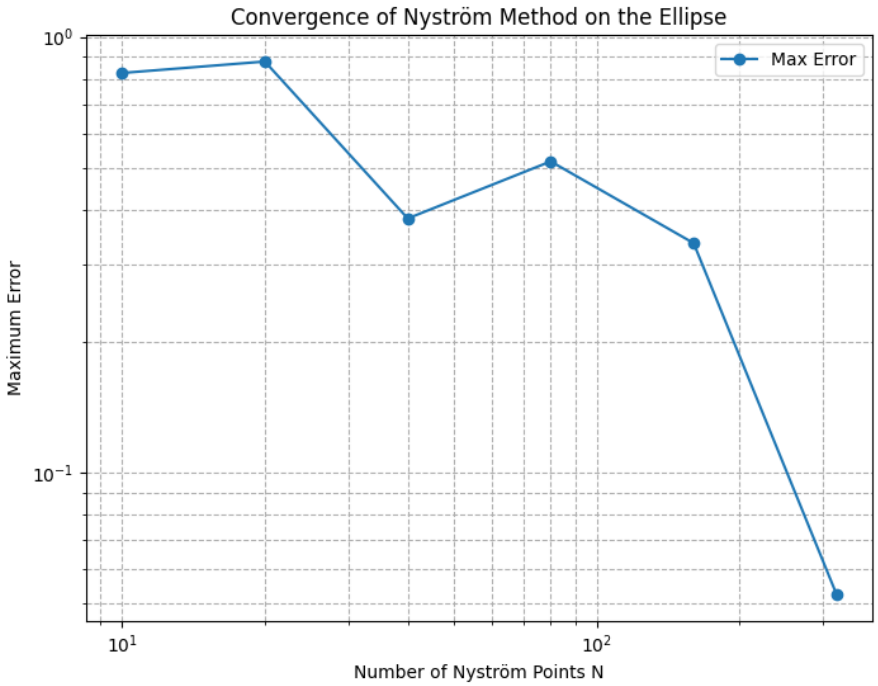

Figure 8: Convergence of Nyström Method on the Ellipse

Error Decay

The maximum error was observed to decrease by approximately a factor of 4 when the number of discretization points N was doubled. This suggests an algebraic convergence rate of:

$$\mathcal{O}\left(\frac{1}{N^2}\right).$$

Plot Interpretation

The log-log plot of the maximum error E_N versus N shows a straight line with a slope of approximately -2. This numerical evidence confirms that the convergence rate is quadratic:

$$E_N \propto \frac{1}{N^2}.$$

Conclusion

The equally-spaced-point Nyström method on the ellipse demonstrates the following properties:

- **Convergence Rate:** The method converges at an algebraic rate of $\mathcal{O}(1/N^2)$.
- **Spectral Convergence:** Spectral (exponential) convergence, as observed for the circle, is not achieved for the ellipse. This is primarily due to the geometric differences between the ellipse and the circle, such as non-uniform curvature and the lack of symmetry in parameterization.
- **Accuracy:** The method provides highly accurate results for sufficiently large N, making it a practical approach for solving the exterior Neumann problem on the ellipse.