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Numerical Methods for Partial Differential Equations

Massachusetts Institute of Technology – Fall 2024

Project 4 – Time Dependent Finite Differences and Finite Volumes

version 1.1 *Due: Dec. 11, 2024*

Problem 1 - Convection equation (20pts)

We want to solve numerically the linear time dependent convection equation

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = 0, \tag{1}$$

using the method of lines. For simplicity, we will take $U = 1$. We will solve the equation on a domain $-10 < x < 10$ using periodic boundary conditions and an initial condition $u(x, 0) = 10e^{-x^2}$ until a time $T = 20$.

For the spatial discretization, we consider a uniform grid and two finite difference schemes:

i)

$$\frac{du_j}{dt} = -\frac{u_{j+1} - u_{j-1}}{2\Delta x}$$

ii)

$$\frac{1}{6}\left(\frac{du_{j+1}}{dt} + 4\frac{du_j}{dt} + \frac{du_{j-1}}{dt}\right) = -\frac{u_{j+1} - u_{j-1}}{2\Delta x}$$

Note that scheme ii) is implicit and requires the solution of a system of equations involving the mass matrix every time we need to evalaute $d\underline{u}/dt$.

For the time integration, we will use a fourth order Runge-Kutta scheme:

$$\alpha^1 = \Delta t\, f(u^i) \tag{2}$$

$$\alpha^2 = \Delta t\, f(u^i + \alpha^1/2) \tag{3}$$

$$\alpha^3 = \Delta t\, f(u^i + \alpha^2/2) \tag{4}$$

$$\alpha^4 = \Delta t\, f(u^i + \alpha^3) \tag{5}$$

$$u^{i+1} = u^i + \frac{1}{6}(\alpha^1 + 2\alpha^2 + 2\alpha^3 + \alpha^4). \tag{6}$$

The stability region for this scheme consists of all z such that $|1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}| \leq 1$. In particular, all points on the imaginary axis between $\pm i2\sqrt{2}$ are included.

Questions

1) (5 pts) Determine the order of spatial accuracy for each scheme.

2) (5 pts) Determine the allowable timestep for each scheme as function of Δx.

3) (5 pts) Set $\Delta x = 0.2$ and solve with scheme ii) and determine the error at $T = 20$ in the infitnity norm.

4) (5 pts) Solve the problem using scheme i) and choose an Δx so that the error is less than that determined in the previous question for scheme ii).

Problem 2 - Solitons (30 pts)

Problem Statement

J. Scott Russell wrote in 1844:

> *"I believe I shall best introduce this phenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel."*

In 1895, Korteweg and de Vries formulated the equation (the KdV Equation):

$$u_t + 6uu_x + u_{xxx} = 0, \tag{7}$$

which models Russell's observation. The term uu_x describes the sharpening of the wave and u_{xxx} the dispersion (i.e., waves with different wave lengths propagate with different velocities). When these two terms balance a propagating wave with unchanged form is the result. These waves with unchanged form are called *solitons*. The resulting waves have a special shape, and have some very distinct properties. For example, a soliton with a larger amplitude will be 'thinner' than one with a smaller relative amplitude. In addition, the amplitude and velocity are dependent.

Solitons are seen in several diverse areas of study. The primary application of solitons today is in optical fibers, where the linear dispersion of the fiber provides smoothing of the wave, and the non-linear properties give the sharpening. The result is a very stable and long-lasting pulse that is free from dispersion, which is a problem with traditional optical communication techniques.

The KdV equation and solitons also have relevance to Tsunamis The KdV equation is a governing PDE for shallow water waves (see Whitham, G.B, *Linear and Nonlinear Waves* Wiley, New York, 1974). Since a Tsunami, when it propagates in the open ocean, has a long wavelength in comparison to the depth of the water, it is considered a shallow water wave as it propagates through the deep ocean water. Hence, a simple model for Tsunami propagation in the open ocean would be a soliton, which is governed by the KdV equation.

Using direct substitution, we can show that the one-soliton solution

$$u_1(x,t) = \frac{v}{2\cosh^2\left(1/2\sqrt{v}(x - vt - x_0)\right)} \tag{8}$$

solves the KdV equation (7). Here, $v > 0$ and x_0 are arbitrary parameters.

Differentiate the one-soliton solution:

$$u(x,t) = 1/2\,\frac{v}{\left(\cosh\left(1/2\,\sqrt{v}\,(x - vt - x_0)\right)\right)^2} \tag{9}$$

$$u_x(x,t) = -1/2\,\frac{v^{3/2}\sinh\left(1/2\,\sqrt{v}\,(x - vt - x_0)\right)}{\left(\cosh\left(1/2\,\sqrt{v}\,(x - vt - x_0)\right)\right)^3} \tag{10}$$

$$u_{xx}(x,t) = 1/4\,\frac{v^2\left(2\,\left(\cosh\left(1/2\,\sqrt{v}\,(x - vt - x_0)\right)\right)^2 - 3\right)}{\left(\cosh\left(1/2\,\sqrt{v}\,(x - vt - x_0)\right)\right)^4} \tag{11}$$

$$u_{xxx}(x,t) = -1/2\,\frac{v^{5/2}\sinh\left(1/2\,\sqrt{v}\,(x - vt - x_0)\right)\left(\left(\cosh\left(1/2\,\sqrt{v}\,(x - vt - x_0)\right)\right)^2 - 3\right)}{\left(\cosh\left(1/2\,\sqrt{v}\,(x - vt - x_0)\right)\right)^5} \tag{12}$$

$$u_t(x,t) = 1/2\,\frac{v^{5/2}\sinh\left(1/2\,\sqrt{v}\,(x - vt - x_0)\right)}{\left(\cosh\left(1/2\,\sqrt{v}\,(x - vt - x_0)\right)\right)^3}. \tag{13}$$

Inserting into the KdV equation, and simplifying we get

$$u_t(x,t) + 6u(x,t)u_x(x,t) + u_{xxx}(x,t) = 0. \tag{14}$$

Questions

1) (5 pts) We will solve the KdV equation numerically using the method of lines and finite difference approximations for the space derivatives. Rewrite the equation as

$$\frac{\partial u}{\partial t} = -6uu_x - u_{xxx}, \tag{15}$$

and derive a second-order accurate finite difference approximation for the right hand side.

2) (10 pts) For the time integration, we will use a fourth order Runge-Kutta described in the previous problem.

Our equation (7) is non-linear, and to make a stability analysis we first have to linearize it. In this case, it turns out that the stability will be determined by the discretization of the third-derivative term u_{xxx}. Therefore, consider the simplified problem

$$\frac{\partial u}{\partial t} = -u_{xxx}, \tag{16}$$

and use von Neumann stability analysis to derive an expression for the maximum allowable time-step Δt in terms of Δx.

3) (15 pts) Write a program that solves the equation using your discretization. Solve it in the region $-10 \leq x \leq 10$ with a grid size $\Delta x = 0.05$, and use periodic boundary conditions:

$$u(-10) = u(10).$$

Integrate from $t = 0$ to $t = 10$, using an appropriate time-step that satisfies the stability condition you derived above. For each of the initial conditions below, plot the solution at $t = 10$ and comment on the results.

a. To begin with, use a single soliton (8) as initial condition, that is, $u(x,0) = u_1(x,0)$. Set $v = 20$ and $x_0 = 0$.

b. The one-soliton solution looks almost like a Gaussian. Try $u(x,0) = 10e^{-x^2}$.

c. Try the two-soliton solution $u(x,0) = \frac{6}{\cosh^2(x)}$.

d. Create "your own" two-soliton solution by superposing (adding) two one-soliton solutions with $v = 14$ and $v = 8$ (both with $x_0 = 0$).

e. Same as before, but with $v = 14, x_0 = -3$ and $v = 8, x_0 = 3$. Describe what happens when the two solitons cross (amplitudes, velocities), and after they have crossed.

Problem 3 - Traffic Flow (50 pts)

Problem Statement

Consider the traffic flow problem, described by the non-linear hyperbolic equation:

We shall first examine the formation of the shock wave using the non-linear hyperbolic equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0. \tag{17}$$

with $\rho = \rho(x,t)$ the density of cars (vehicles/km), and $u = u(x,t)$ the (average) velocity of the cars. Assume that the velocity u is given as a function of ρ:

$$u = u_{\text{max}} \left(1 - \frac{\rho}{\rho_{\text{max}}}\right). \tag{18}$$

With u_{max} the maximum speed and $0 \le \rho \le \rho_{\text{max}}$. The flux of cars is therefore given by:

$$f(\rho) = \rho u_{\text{max}} \left(1 - \frac{\rho}{\rho_{\text{max}}}\right). \tag{19}$$

We will solve this problem using a first order finite volume scheme:

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x}\left(F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}}\right). \tag{20}$$

For the numerical flux function we will consider two different options:

1) Roe's flux

The expression of the numerical flux is given by:

$$F^R_{i+\frac{1}{2}} = \frac{1}{2}[f(\rho_i) + f(\rho_{i+1})] - \frac{1}{2}\left|a_{i+\frac{1}{2}}\right|(\rho_{i+1} - \rho_i) \tag{21}$$

with

$$a_{i+\frac{1}{2}} = u_{\text{max}}\left(1 - \frac{\rho_i + \rho_{i+1}}{\rho_{\text{max}}}\right). \tag{22}$$

Note that $a_{i+\frac{1}{2}}$ satisfies

$$f(\rho_{i+1}) - f(\rho_i) = a_{i+\frac{1}{2}}(\rho_{i+1} - \rho_i) \tag{23}$$

2) Godunov's flux

In this case the numerical flux is given by:

$$F^G_{i+\frac{1}{2}} = f\left(\rho\left(x_{i+\frac{1}{2}}, t^{n+}\right)\right) = \begin{cases} \min\limits_{\rho \in [\rho_i, \rho_{i+1}]} f(\rho), & \rho_i < \rho_{i+1} \\ \max\limits_{\rho \in [\rho_i, \rho_{i+1}]} f(\rho), & \rho_i > \rho_{i+1}. \end{cases} \tag{24}$$

Note that for Godunov's scheme it is possible to come out with flux expressions that do not require a brute force search for the minimum/maximum of the flux between ρ_i and ρ_{i+1}.

Questions

1) (10 pts) For both Roe's scheme and Godunov's scheme, look at the problem of a traffic light turning green at time $t = 0$. We are interested in the solution at $t = 2$ using both schemes. What do you observe for each of the schemes? Explain briefly why the behavior you observe arises.

Use the following parameters:

$$\rho_{\max} = 1.0, \quad u_{\max} = 1.0, \quad \rho_L = \rho_{\max}, \quad \Delta x = \frac{1}{100}, \quad \Delta t = \frac{0.8\Delta x}{u_{\max}} \tag{25}$$

Consider a domain $x \in (-5, 5)$ with the following initial condition at the instant when the traffic light turns green

$$\rho(x,0) = \begin{cases} \rho_L, & x < 0 \\ 0, & x \geq 0. \end{cases} \tag{26}$$

For boundary conditons, set $\rho(-5,t) = \rho_{\max}$ and use extrapolation at $x = 5$ (i.e. set $\rho(5,t) = \rho(5 - \Delta x, t)$.)

2) (5 pts) Identify, as a function of ρ_i and ρ_{i+1}, when the Roe and Godunov's fluxes are equal and when they are different.

For the rest of this problem use only the scheme(s) which are valid models of the problem.

3) (20 pts) Simulate the effect of a traffic light at $x = -\frac{\Delta x}{2}$ which has a period of $T = T_1 + T_2 = 2$ units. Assume that the traffic light is $T_1 = 1$ units in red and $T_2 = 1$ units on green. Assume a sufficiently high flow density of cars (e.g. set $\rho = \frac{\rho_{\max}}{2}$ on the left boundary). Use an extrapolation boundary conditon on the right boundary and determine the average flow, or capacity of cars over a time period T.

The average flow can be approximated as

$$\dot{q} = \frac{1}{N_T}\sum_{n=1}^{N_T} f^n = \frac{1}{N_T}\sum_{n=1}^{N_T} \rho^n u^n \ , \tag{27}$$

where N_T is the number of timesteps for each period T. You should run your computation until $\dot{q}$ over a time period does not change. Note that by continuity $\dot{q}$ can be evaluated over any point in the interior of the domain (in order to avoid boundary condition effects, we consider only those points on the interior of the domain).

Note: A red traffic light can be modeled by simply setting $F_{i+\frac{1}{2}} = 0$ at the position where the traffic light is located.

4) (15 pts) Assume now that we simulate two traffic lights, one located at $x = -\frac{\Delta x}{2}$ and the other at $x = 0.1 - \frac{\Delta x}{2}$, both with period T. Calculate the road capacity (= average flow) for different delay factors. That is, if the first light turns green at time t, then the second light will turn green at time $t + \tau$. Solve for $\tau = k\frac{T}{20}, k = 0, \ldots, 19$. Plot your results of capacity vs. τ and determine the optimal delay τ.