

Assume  $\mathbf{W}$  is a DFT matrix with transform length  $N = 3$ , and its Gram  $\mathbf{W}^* \mathbf{W}$ :

$$\mathbf{W} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \quad \mathbf{W}^* \mathbf{W} = \begin{pmatrix} a_{1,1}^2 + a_{2,1}^2 + a_{3,1}^2 & a_{1,1}a_{1,2} + a_{2,1}a_{2,2} + a_{3,1}a_{3,2} & a_{1,1}a_{1,3} + a_{2,1}a_{2,3} + a_{3,1}a_{3,3} \\ a_{1,1}a_{1,2} + a_{2,1}a_{2,2} + a_{3,1}a_{3,2} & a_{1,2}^2 + a_{2,2}^2 + a_{3,2}^2 & a_{1,2}a_{1,3} + a_{2,2}a_{2,3} + a_{3,2}a_{3,3} \\ a_{1,1}a_{1,3} + a_{2,1}a_{2,3} + a_{3,1}a_{3,3} & a_{1,2}a_{1,3} + a_{2,2}a_{2,3} + a_{3,2}a_{3,3} & a_{1,3}^2 + a_{2,3}^2 + a_{3,3}^2 \end{pmatrix}$$

Assume hop  $L = 2$ , construct frame operator  $\Phi$ :

$$\Phi = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{1,1} & a_{1,2} & a_{1,3} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2,1} & a_{2,2} & a_{2,3} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{3,1} & a_{3,2} & a_{3,3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{1,1} & a_{1,2} & a_{1,3} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{2,1} & a_{2,2} & a_{2,3} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{3,1} & a_{3,2} & a_{3,3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{1,1} & a_{1,2} & a_{1,3} & a_{1,3} \\ 0 & 0 & 0 & 0 & 0 & a_{2,1} & a_{2,2} & a_{2,3} & a_{2,3} \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$$

Main body of Gram  $\Phi^* \Phi$ :

$$\Phi^* \Phi = \begin{pmatrix} a_{1,2}^2 + a_{2,2}^2 + a_{3,2}^2 & a_{1,2}a_{1,3} + a_{2,2}a_{2,3} + a_{3,2}a_{3,3} & 0 & 0 & 0 & 0 & 0 \\ a_{1,2}a_{1,3} + a_{2,2}a_{2,3} + a_{3,2}a_{3,3} & a_{1,1}^2 + a_{1,3}^2 + a_{2,1}^2 + a_{2,3}^2 + a_{3,1}^2 + a_{3,3}^2 & a_{1,1}a_{1,2} + a_{2,1}a_{2,2} + a_{3,1}a_{3,2} & a_{1,1}a_{1,3} + a_{2,1}a_{2,3} + a_{3,1}a_{3,3} & 0 & 0 & 0 \\ 0 & a_{1,1}a_{1,2} + a_{2,1}a_{2,2} + a_{3,1}a_{3,2} & a_{1,2}^2 + a_{2,2}^2 + a_{3,2}^2 & a_{1,2}a_{1,3} + a_{2,2}a_{2,3} + a_{3,2}a_{3,3} & 0 & 0 & 0 \\ 0 & a_{1,1}a_{1,3} + a_{2,1}a_{2,3} + a_{3,1}a_{3,3} & a_{1,2}a_{1,3} + a_{2,2}a_{2,3} + a_{3,2}a_{3,3} & a_{1,1}^2 + a_{1,3}^2 + a_{2,1}^2 + a_{2,3}^2 + a_{3,1}^2 + a_{3,3}^2 & a_{1,1}a_{1,2} + a_{2,1}a_{2,2} + a_{3,1}a_{3,2} & a_{1,1}a_{1,3} + a_{2,1}a_{2,3} + a_{3,1}a_{3,3} & 0 \\ 0 & 0 & 0 & a_{1,1}a_{1,2} + a_{2,1}a_{2,2} + a_{3,1}a_{3,2} & a_{1,2}^2 + a_{2,2}^2 + a_{3,2}^2 & a_{1,2}a_{1,3} + a_{2,2}a_{2,3} + a_{3,2}a_{3,3} & 0 \\ 0 & 0 & 0 & a_{1,1}a_{1,3} + a_{2,1}a_{2,3} + a_{3,1}a_{3,3} & a_{1,2}a_{1,3} + a_{2,2}a_{2,3} + a_{3,2}a_{3,3} & a_{1,3}^2 + a_{2,3}^2 + a_{3,3}^2 & 0 \end{pmatrix}$$

An alternate method to construct Gram  $\Phi^* \Phi$  without performing costly matrix multiplication:

Step 1 of overlap-add  $\mathbf{W}^* \mathbf{W}$  along diagonal, main body of Gram  $\Phi^* \Phi$ :

$$\begin{pmatrix} a_{1,2}^2 + a_{2,2}^2 + a_{3,2}^2 & a_{1,2}a_{1,3} + a_{2,2}a_{2,3} + a_{3,2}a_{3,3} & 0 & 0 & 0 & 0 \\ a_{1,2}a_{1,3} + a_{2,2}a_{2,3} + a_{3,2}a_{3,3} & a_{1,3}^2 + a_{2,3}^2 + a_{3,3}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 2 of overlap-add  $\mathbf{W}^* \mathbf{W}$  along diagonal, main body of Gram  $\Phi^* \Phi$ :

$$\begin{pmatrix} a_{1,2}^2 + a_{2,2}^2 + a_{3,2}^2 & a_{1,2}a_{1,3} + a_{2,2}a_{2,3} + a_{3,2}a_{3,3} & 0 & 0 & 0 & 0 \\ a_{1,2}a_{1,3} + a_{2,2}a_{2,3} + a_{3,2}a_{3,3} & a_{1,1}^2 + a_{1,3}^2 + a_{2,1}^2 + a_{2,3}^2 + a_{3,1}^2 + a_{3,3}^2 & a_{1,1}a_{1,2} + a_{2,1}a_{2,2} + a_{3,1}a_{3,2} & a_{1,1}a_{1,3} + a_{2,1}a_{2,3} + a_{3,1}a_{3,3} & 0 & 0 \\ 0 & a_{1,1}a_{1,2} + a_{2,1}a_{2,2} + a_{3,1}a_{3,2} & a_{1,2}^2 + a_{2,2}^2 + a_{3,2}^2 & a_{1,2}a_{1,3} + a_{2,2}a_{2,3} + a_{3,2}a_{3,3} & 0 & 0 \\ 0 & a_{1,1}a_{1,3} + a_{2,1}a_{2,3} + a_{3,1}a_{3,3} & a_{1,2}a_{1,3} + a_{2,2}a_{2,3} + a_{3,2}a_{3,3} & a_{1,3}^2 + a_{2,3}^2 + a_{3,3}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 3 of overlap-add  $\mathbf{W}^* \mathbf{W}$  along diagonal, main body of Gram  $\Phi^* \Phi$ :

$$\begin{pmatrix} a_{1,2}^2 + a_{2,2}^2 + a_{3,2}^2 & a_{1,2}a_{1,3} + a_{2,2}a_{2,3} + a_{3,2}a_{3,3} & 0 & 0 & 0 & 0 \\ a_{1,2}a_{1,3} + a_{2,2}a_{2,3} + a_{3,2}a_{3,3} & a_{1,1}^2 + a_{1,3}^2 + a_{2,1}^2 + a_{2,3}^2 + a_{3,1}^2 + a_{3,3}^2 & a_{1,1}a_{1,2} + a_{2,1}a_{2,2} + a_{3,1}a_{3,2} & a_{1,1}a_{1,3} + a_{2,1}a_{2,3} + a_{3,1}a_{3,3} & 0 & 0 \\ 0 & a_{1,1}a_{1,2} + a_{2,1}a_{2,2} + a_{3,1}a_{3,2} & a_{1,2}^2 + a_{2,2}^2 + a_{3,2}^2 & a_{1,2}a_{1,3} + a_{2,2}a_{2,3} + a_{3,2}a_{3,3} & 0 & 0 \\ 0 & a_{1,1}a_{1,3} + a_{2,1}a_{2,3} + a_{3,1}a_{3,3} & a_{1,2}a_{1,3} + a_{2,2}a_{2,3} + a_{3,2}a_{3,3} & a_{1,3}^2 + a_{2,3}^2 + a_{3,3}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Method that performs overlap-add  $\mathbf{W}^* \mathbf{W}$  along diagonal is in fact, equivalent to compute Gramian  $\Phi^* \Phi$  directly