Final Exam

FA-542

Due: December 17, 2023 at 11:59pm

Problem 1 (50pt)
Suppose that the daily log return of a pair of securities follows the following model:

$$\begin{cases} r_{1,t} & = & -0.05 & + & 0.1r_{1,t-1} & + & 0.05r_{2,t-1} & + & a_{1,t} \\ r_{2,t} & = & 0.1 & - & 0.1r_{1,t-1} & + & 0.3r_{2,t-1} & + & a_{2,t} \end{cases}$$

where a_t denotes a bivariate normal distribution with mean 0 and covariance

$$\Sigma := \begin{pmatrix} 0.4 & -0.1 \\ -0.1 & 0.2 \end{pmatrix}.$$

Any matrix operations can be computed in R. If any formulas require infinite series, you may approximate using the first 5 terms.

(a) Verify that the return series $\{r_t\}$ is a weakly stationary process.
Hint: The "polyroot" function can be used to find all roots of a polynomial in R and the "eigen" function can be used to find all eigenvalues of a matrix in R.

(b) (i) What is the mean vector of the return series r_t?
(ii) What is the covariance matrix of the return series r_t?
(iii) What are the lag-1, lag-2, and lag-5 cross-correlation matrices of the return series r_t?

(c) Assume that $r_0 = (-0.02, 0.08)^\top$ and $a_0 = (-0.08, 0.1)^\top$. Compute the 1-, 2-, and 3-step ahead forecasts of the return series at the forecast origin $t = 1$. What are the covariance matrices of the associated forecast errors?

(d) Create a report in pdf format and do the following:
(i) Simulate 1000 terms of this time series and plot the result.
(ii) Using the generated time series, find the sample mean and covariance. How does your sample mean vector compare with that computed analytically?
(iii) Using the generated time series, find the sample lag-1, lag-2, and lag-5 cross-correlation matrices.

(iv) Consider how you might use repeated simulations to forecast this time series. Use your method with 10,000 repeated simulations of the time series to forecast the 1-, 2-, and 3-step ahead returns with $r_0 = (-0.02, 0.08)^\top$ and $a_0 = (-0.08, 0.1)^\top$. What is the sample covariance of the errors? How do these values compare with those computed analytically?

(e) Create a report in pdf format and do the following:

(i) Simulate 1000 terms of this time series and plot the result. You may use the series constructed in (d)(i).

(ii) Using the generated time series, fit a univariate AR(1) model to each return series.

(iii) Compute the mean of both univariate models. How do these compare to those for the bivariate series?

(iv) Assume $r_{1,0} = -0.02$, $r_{2,0} = 0.08$, $a_{1,0} = -0.08$, and $a_{2,0} = 0.1$. Compute the 1-, 2-, and 3-step ahead forecasts of both of your univariate return series models at the forecast origin $t = 1$. What are the standard deviations of the associated forecast errors? How do these compare to those for the bivariate series? You may approach this problem either analytically or via simulations.

Problem 2 (30pt)
Create a report in pdf format and do the following:

(a) Download daily price data for January 1, 1990 through December 1, 2023 of Apple stock (AAPL) from Yahoo Finance. You may use the quantmod package in R for this purpose.

(b) Is there any evidence of serial correlations in the *weekly* log returns? Use the first 12 lagged autocorrelations and 5% significance level to answer this question. If yes, remove the serial correlations. If serial correlations exist, fit a *linear* model to account these serial correlations; justify your model to remove these serial correlations.
Note: If serial correlations do not exist, do not neglect the mean of the series.

(c) Obtain the residuals of your model from part (b) and test for ARCH effects. Use the first 6 lagged autocorrelations and 5% significance level to answer this question. If ARCH effects exist, fit a GARCH(1,1) model for the residual part.

(d) (i) Assuming $\epsilon_t \sim N(0, 1)$ i.i.d., what is the excess kurtosis of your GARCH(1,1) model?

(ii) Does this match the empirical excess kurtosis of your residuals? If not, what excess kurtosis of ϵ_t would be needed for the theoretical kurtosis to match the observed kurtosis? Provide a distribution for ϵ_t so that the theoretical excess kurtosis coincides with the observed kurtosis.

Problem 3 (20pt)
Create a report in pdf format and do the following:

(a) Download daily price data for January 1, 1980 through December 1, 2023 of Exxon Mobil stock (XOM) from Yahoo Finance. You may use the quantmod package in R for this purpose.

(b) Using any method discussed this semester, develop a time series model to predict *daily* log returns. Use data up to December 1, 2018 as the training data set and the remainder as the testing data. Briefly comment on the performance of your selected model. Justify the modeling choices made with, e.g., the appropriate statistical tests. Full credit will only be provided if rigorous justification for modeling choices are made.