We look to show

$$Fib(n) = \frac{\phi^n - \psi^n}{\sqrt{5}}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$
Consider the base cases $Fib(0), Fib(1), Fib(2)$.
Clearly

$$Fib(0) = \frac{1-1}{\sqrt{5}} = 0$$

$$\begin{aligned} Fib(1) &= \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} \\ &= \frac{\sqrt{5}}{\sqrt{5}} \\ &= 1 \end{aligned}$$

$$\begin{aligned} Fib(2) &= \frac{(\frac{1+\sqrt{5}}{2})^2 - (\frac{1-\sqrt{5}}{2})^2}{\sqrt{5}} \\ &= \frac{\frac{2\sqrt{5}+2\sqrt{5}}{4}}{\sqrt{5}} \\ &= \frac{\sqrt{5}}{\sqrt{5}} \\ &= 1 \end{aligned}$$

We assume the following inductive hypotheses hold for $k > 2$:

$$Fib(k) = \frac{\phi^k - \psi^k}{\sqrt{5}}$$

$$Fib(k-1) = \frac{\phi^{k-1} - \psi^{k-1}}{\sqrt{5}}$$

We show

$$Fib(k+1) = Fib(k) + Fib(k-1)$$

$$
\begin{aligned}
&= Fib(k) + Fib(k-1) \\
&= \frac{\phi^k - \psi^k}{\sqrt{5}} + \frac{\phi^{k-1} - \psi^{k-1}}{\sqrt{5}} \\
&= \frac{\phi^k + \phi^{k-1} - (\psi^k + \psi^{k-1})}{\sqrt{5}} \\
&= \frac{\phi^{k+1}(\phi^{-1} + \phi^{-2}) - \psi^{k+1}(\psi^{-1} + \psi^{-2})}{\sqrt{5}} \\
&= \frac{\phi^{k+1}(\frac{1}{\phi}\phi) + \psi^{k+1}(\frac{1}{\psi}\psi)}{\sqrt{5}} \\
&= \frac{\phi^{k+1} + \psi^{k+1}}{\sqrt{5}}
\end{aligned}
$$

$\square$