An analysis of car suspension system using a connected spring-mass damper model
–ME/CS/EE 75a–

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We develop a first-order model for analysis of the suspension of the vehicle. First we establish the spring-mass-damper system used and derive equations of motion. We then provide transfer function representations of our systems and perform frequency domain analysis and look at response characteristics of our system. Physical parameters are currently approximate and will looked to be improved and updated with more research and finalization of components.

I. The 1/4 model

We begin by examining the car suspension with each wheel and suspension component as a series of spring-connected masses with damping. Thus, we can first analyze a single component of the vehicle by simply looking at one such wheel system, and thereby accounting for 1/4 of the car. We will later extend the model to include moment effects of the car's center of mass, as well as interactions between the component.

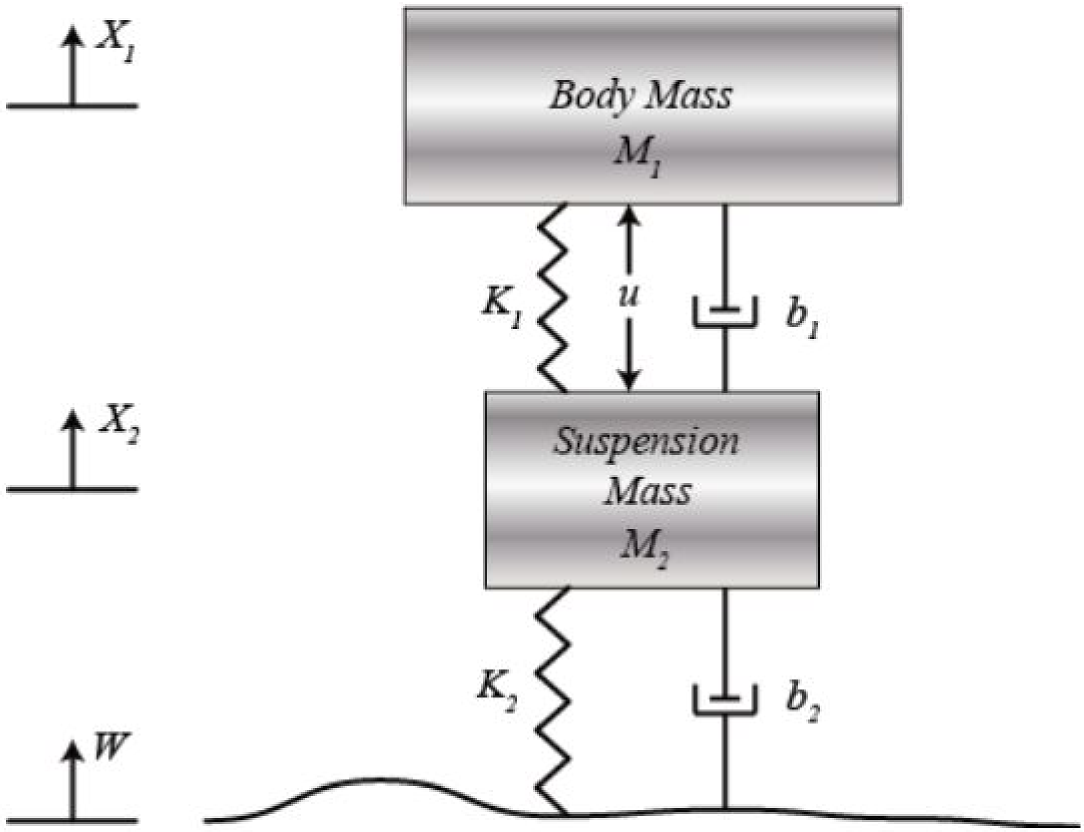

Figure 1: The connected spring-mass-damper used to model the car suspension.

We have the following parameters of interest:
X_1-Displacement of the vehicle
X_2- Displacement of the suspension system
W-Vertical displacement of driving surface
K_1-Spring constant for suspension system
K_2-Spring constant for wheel
b_1-Damping constant for suspension system
b_2-Damping constant for wheel
U-control input (if active suspension).

We derive the equations of motions using standard force/acceleration considerations:

$$M_1\ddot{X}_1 = -b_1(\dot{X}_1 - \dot{X}_2) - K_1(X_1 - X_2) + U$$

$$M_2\ddot{X}_2 = b_1(\dot{X}_1 - \dot{X}_2) + K_1(X_1 - X_2) + b_2(\dot{W} - \dot{X}_2) + K_2(W - X_2) - U$$

We can develop transfer functions for the above processes to gain better insight into the frequency response characteristics of our suspension system and vehicle as a whole.
We primarily are interested in the transfer function between the road displacement and the displacements of X_1 and X_2. We take Laplace Transforms, assuming zero initial conditions.

$$(M_1s^2 + b_1s + K_1)X_1(s) - (b_1s + K_1)X_2(s) = U(s)$$

$$(b_1 s + K_1)X_1(s) + (M_2 s^2 + (b_1 + b_2)s + (K_1 + K_2))X_2(s) = (b_2 s + K_2)W(s) - U(s)$$

In matrix form:

$$\begin{pmatrix} (M_1 s^2 + b_1 s + K_1 & -(b_1 s + K_1) \\ -(b_1 s + K_1) & (M_2 s^2 + (b_1 + b_2)s + (K_1 + K_2)) \end{pmatrix} \begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} = \begin{pmatrix} U(s) \\ (b_2 s + K_2)W(s) - U(s) \end{pmatrix}$$

We thus have an equation of the form

$$\mathbf{A} \begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} = \begin{pmatrix} U(s) \\ (b_2 s + K_2)W(s) - U(s) \end{pmatrix}$$

where

$$\mathbf{A} = \begin{pmatrix} (M_1 s^2 + b_1 s + K_1 & -(b_1 s + K_1) \\ -(b_1 s + K_1) & (M_2 s^2 + (b_1 + b_2)s + (K_1 + K_2)) \end{pmatrix}$$

We can now look to isolate transfer functions between our inputs to the system to our displacements. We take the inverse of $\mathbf{A}$ and simplify to recover a form to read off the transfer functions from inputs to outputs.

$$\begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} = \frac{1}{det(\mathbf{A})} \begin{pmatrix} M_2 s^2 + b_2 s + K_2 & (b_1 b_2 s^2 + (b_1 K_1 + b_2 K_1)s + K_1 K_2 \\ -M_1 s^2 & (M_1 b_2 s^3 + (M_1 K_2 + b_1 b_{@})s^2 + (b_1 K_2 + b_2 K_1)s + K_1 K_2) \end{pmatrix} \begin{pmatrix} U(s) \\ W(s) \end{pmatrix}$$

We thus recover salient transfer functions for the input W to our respective outputs by setting $U = 0$

$$G_{X_1 W} = \frac{1}{det(\mathbf{A})}(b_1 b_2 s^2 + (b_1 K_1 + b_2 K_1)s + K_1 K_2)$$

$$G_{X_2 W} = \frac{1}{det(\mathbf{A})}(M_1 b_2 s^3 + (M_1 K_2 + b_1 b_2)s^2 + (b_1 K_2 + b_2 K_1)s + K_1 K_2)$$

$$G_{(X_1 - X_2)W} = \frac{1}{det(\mathbf{A})}(-M_1 b_2 s^3 - M1 K_2 s^2)$$

We can now perform analysis in the frequency domain via these transfer functions.

II. Analysis using specifications of suspension system

We can now look into some of the frequency response characteristics for our system. We take the following parameter sets based on the mass budget of the car and specifications of the requisite subsystems.
M_1- 110.35 kg
M_2- 65 kg
K_1- 400 lb/in
K_2- 1200 lb/in.
b_1-350; Damping constant for suspension system
b_2-4000; Damping constant for wheel
U=0, no control input.

We have relatively high uncertainty for the values of the damping constants; we will investigate the effect of varying these parameters later.

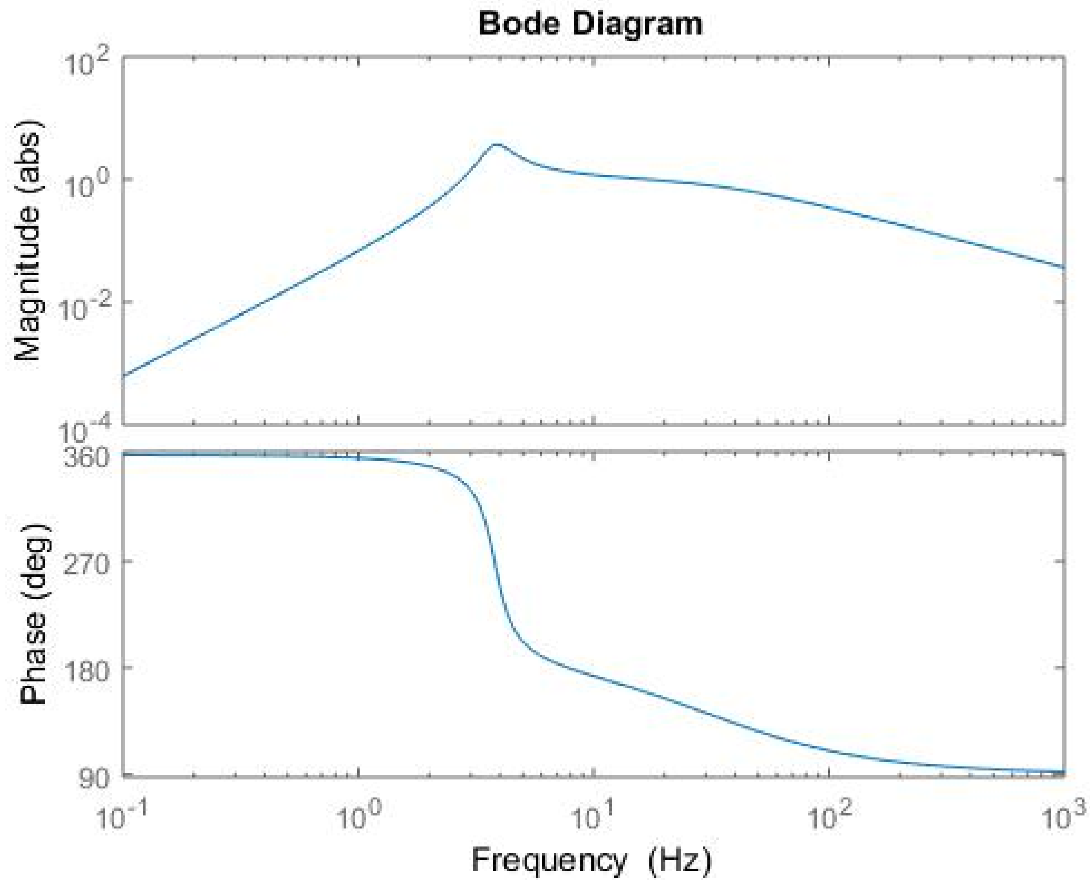

Figure 2: Bode Diagram for system with non-active suspension. We note resonance at frequency components of 3.5 Hz.

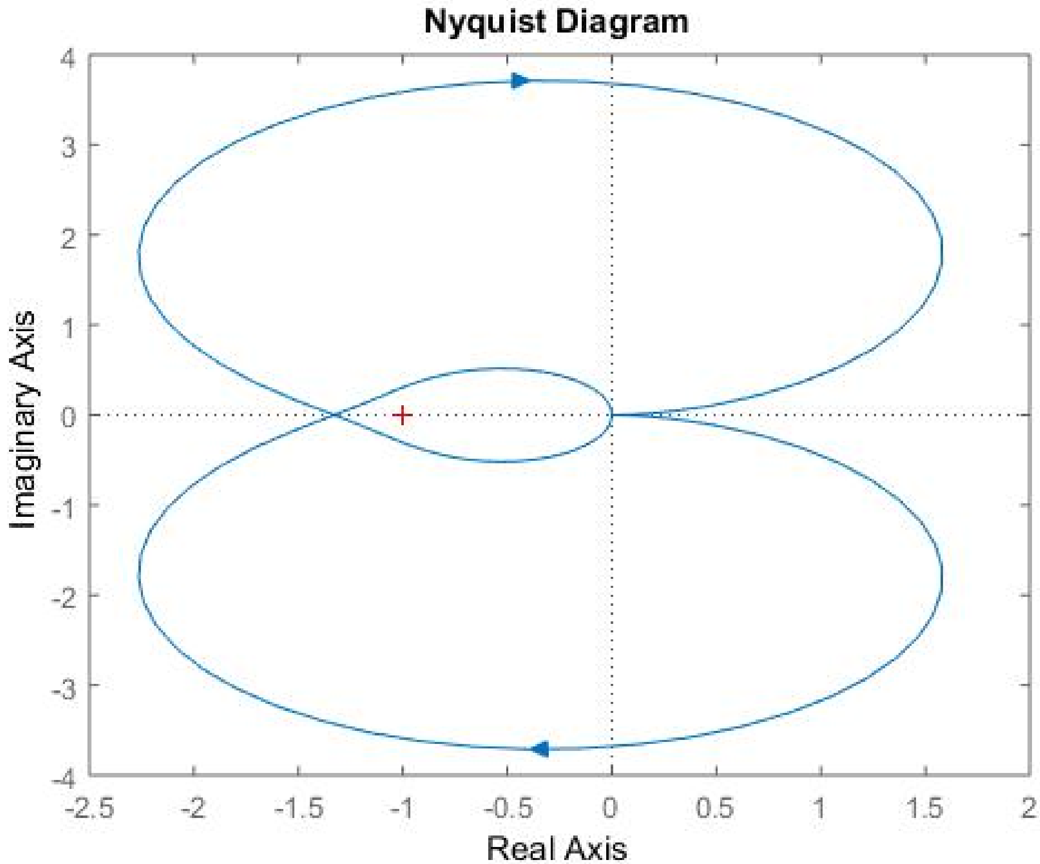

Figure 3: Nyquist Diagram for system. As corroborated by the step-response (system initially responds in the opposite direction), the system has right-half plane zeros, however our system is stable with no control input.

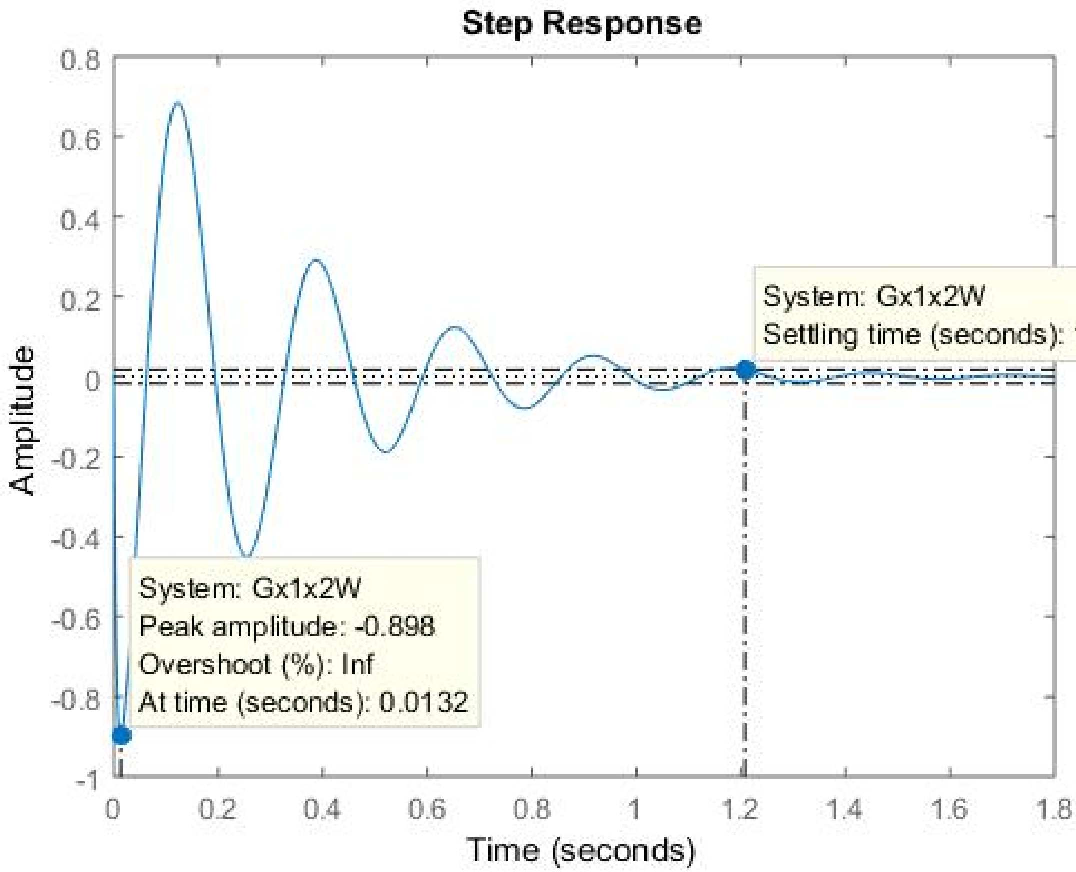

Figure 4: Step response. We note a that for a unit step in the road displacement, our suspension system is displaced by a factor of 0.856, then settles to 0 over a time of 1.77s.

```matlab
%% Car suspension modeling
%  James Blackwood
%  ME/EE/CS 75 a- Fall 2015
%
%  Develops and analyzes car suspension using connected spring-mass-damper
%  system.
%%
clear all;
close all;

% Definition of parameters
lbin2nm = 175.126835; % lb/in to N/m conversion

% Options to look at Bode plot magnitude in absolute units
% and frequency in Hz
opts = bodeoptions('cstprefs');
opts.MagUnits = 'abs';
opts.MagScale = 'log';
opts.FreqUnits = 'Hz';



M1 = 110.35; % quarter mass of vehicle, kg
M2 = 65; % suspension mass, kg
K1 = 400 * lbin2nm; % spring constant suspension system, N/m
K2 = 1200 * lbin2nm; % spring constant wheel system, N/m
b1 = 350; % damping constant suspension system, N*s/m
b2 = 4000; % damping constant wheel system, N*s/m

% set control input to 0 for non-active suspension
U = 0; % control input, m


%%
s = tf('s');

detA = ((M1*s^2+b1*s+K1)*(M2*s^2+(b1+b2)*s+(K1+K2))-(b1*s+K1)*(b1*s+K1));

% Explicit forms of transfer functions
Gx1W =  (b1*b2*s^2+(b1*K1+b2*K1)*s+K1*K2)/detA; % W to X1
Gx2W = (M1*b2*s^3+(M1*K2+b1*b2)*s^2+(b1*K2+b2*K1)*s+K1*K2)/detA; % W to X2
Gx1x2W = (-M1*b2*s^3-M1*K2*s^2)/detA; % W to X1-X2

Gx1x2U = ((M1+M2)*s^2+b2*s+K2)/detA; % U to X1-X2


figure;
bodeplot(Gx1x2W,opts);
figure;
nyquist(Gx1x2W);
figure;
step(Gx1x2W);
```

References

[1]Motivation for model, derivation of transfer functions
http://ctms.engin.umich.edu/CTMS/index.php?example=Suspensionsection=SystemModeling

[2]Hoosier racing tires information
www.hoosiertire.com/Fsaeinfo.htm

[3]Ohlins TTX25 MkII information
http://www.ohlinsusa.com/ohlins-ttx-25-fsae