Note on Harmonic analysis

HaoYun Yao

November 9, 2024

Preface

This is originally my note when reading [DE14]. It gradually becomes a place for me to collect and my thought and I've learnt. The main focuses are number theory and representation theory of locally compact groups. Familiarity with undergraduate analysis, algebra, topology and measure theory are assumed. At certain point classical algebraic number theory is also assumed.

Contents

Chapter 1

Topological groups

We begin with some abstract nonsense. Fix a category $\mathcal{C}$ with finite products, i.e., with binary product and final object. Assume further there exists a faithful functor $\omega : \mathcal{C} \to \mathbf{Set}$ that preserves finite products.

Definition. A **group object** in $\mathcal{C}$ is an object G together with morphisms

$$\mathrm{mult} = \mathrm{mult}_G : G \times G \to G, \qquad \mathrm{inv} = \mathrm{inv}_G : G \to G, \qquad e = e_G : * \to G$$

in $\mathcal{C}$ such that the following hold

(i) Associativity:

$$\begin{tikzcd}
G \times (G \times G) \arrow[r, "\mathrm{id}_G \times \mathrm{mult}"] \arrow[dd, "\mathrm{canonical}"', "\wr"] & G \times G \arrow[dr, "\mathrm{mult}"] & \\
& & G \\
(G \times G) \times G \arrow[r, "\mathrm{mult} \times \mathrm{id}_G"'] & G \times G \arrow[ur, "\mathrm{mult}"'] &
\end{tikzcd}$$

(ii) Left and right identity:

$$\begin{tikzcd}
G \times * \arrow[r, "\mathrm{id}_G \times e"] & G \times G \arrow[dr, "\mathrm{mult}"] & \\
G \arrow[u, "\wr"'] \arrow[rr, "\mathrm{id}_G"] \arrow[d, "\wr"'] & & G \\
* \times G \arrow[r, "e \times \mathrm{id}_G"'] & G \times G \arrow[ur, "\mathrm{mult}"'] &
\end{tikzcd}$$

(iii) Left and right inverse:

$$\begin{tikzcd}
& G \times G \arrow[rr, "\mathrm{id}_G \times \mathrm{inv}"] & & G \times G \arrow[dr, "\mathrm{mult}"] & \\
G \arrow[ur, "\Delta"] \arrow[rr, "f"] \arrow[dr, "\Delta"'] & & * \arrow[rr, "e"] & & G \\
& G \times G \arrow[rr, "\mathrm{inv} \times \mathrm{id}_G"'] & & G \times G \arrow[ur, "\mathrm{mult}"'] &
\end{tikzcd}$$

Example 1.0.1.

(i) Group objects in **Set** are exactly all abstract groups.

(ii) Group objects in **Gp** are exactly all abelian groups.

Proof. Let (G, m, inv, e) be a group object in **Gp**. Since the initial object $*$ in **Gp** is the one-point group and $e : * \to G$ is a homomorphism, the image of e in G is 1_G. Let's identify e with 1_G. Since $m : G \times G \to G$ is a homomorphism, we have

$$m(g, h)m(x, y) = m(gx, hy)$$

Then

$$m(x, y) = m(xe, ey) = m(x, e)m(e, y) = xy$$

and

$$m(x, y) = m(ex, ye) = m(e, y)m(x, e) = yx$$

so $xy = yx$. We also prove that $m(x, y) = xy$, i.e., (G, m, inv, e) coincides with itself. □

Lemma 1.0.2. An object G in $\mathcal{C}$ is a group object if and only if the reprensentable functor $h_G : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ factors through the forgetful functor $\mathbf{Gp} \to \mathbf{Set}$.

Proof. Say G is a group object. The morphisms $\mathrm{mult}_G : G \times G \to G$, $\mathrm{inv}_G : G \to G$ and $e_G : * \to G$ give rise to natural transformations

$$h_G \times h_G \to h_G, \qquad h_G \to h_G, \qquad h_* \to h_G$$

Then $h_G(X) = \mathrm{Hom}_{\mathcal{C}}(X, G)$ is a group for each object X in $\mathcal{C}$, and $h_G(f) : h_G(Y) \to h_G(X)$ is a group homomorphism for each morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. Conversely, say $h_G : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ factors through $\mathbf{Gp} \to \mathbf{Set}$. Then each $h_G(X)$ is a group, and the group law $h_G(X) \times h_G(X) \to h_G(X)$ defines a natural transformation $h_G \times h_G \to h_G$. Indeed, this follows as $h_G(f)$ is a group homomorphism for each morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. Similarly the inversions and the identities glue to natural transformations $h_G \to h_G$, $h_* \to h_G$. By Yoneda lemma they correspond to morphisms $G \times G \to G$, $G \to G$, $* \to G$. The compatibility of among the transformations shows that these morphisms make G a group object. □

Hence a group object in $\mathcal{C}$ is exactly a pair

$$(G, F_G) \in \mathcal{C} \times [\mathcal{C}^{\mathrm{op}}, \mathbf{Gp}]$$

subject to the condition $h_G = \phi \circ F_G$, where $\phi : \mathbf{Gp} \to \mathbf{Set}$ is the forgetful functor. For such two pairs (G, F_G), (H, F_H), we say a morphism $f : G \to H$ in $\mathcal{C}$ defines a morphism $(G, F_G) \to (H, F_H)$ of the pair if

$${}_X h(f) : h_G(X) \to h_H(X)$$

is a group homomorphism. Here $h_G(X) = F_G(X)$ and $h_H(X) = F_H(X)$ as sets, so they are equipped with group structures. Alternatively (and more concretely), define the category $\mathbf{Gp}(\mathcal{C})$ of group objects in $\mathcal{C}$ as follows. Objects in $\mathbf{Gp}(\mathcal{C})$ are group objects in $\mathcal{C}$. For two group objects G, H, define

$$\mathrm{Hom}_{\mathbf{Gp}(\mathcal{C})}(G, H) = \{\phi \in \mathrm{Hom}_{\mathcal{C}}(G, H) \mid \mathrm{mult}_H \circ (\phi \times \phi) = \phi \circ \mathrm{mult}_G\}.$$

There is an obvious forgetful functor $\varphi : \mathbf{Gp}(\mathcal{C}) \to \mathcal{C}$. Recall $\mathcal{C}$ is equipped with a functor $\omega : \mathcal{C} \to \mathbf{Set}$ that preserves finite products. For each group object $(G, \mathrm{mult}, \mathrm{inv}, e)$ in $\mathcal{C}$, the pair

$$(\omega(G), \omega(\mathrm{mult}), \omega(\mathrm{inv}), \omega(e))$$

defines a group structure on $\omega(G)$. Similarly a morphism in $\mathbf{Gp}(\mathcal{C})$ maps to a group homomorphism under $\omega\varphi$. This defines a functor

$$\omega : \mathbf{Gp}(\mathcal{C}) \longrightarrow \mathbf{Gp}$$

fitting into the commutative diagram

$$\begin{tikzcd}
\mathbf{Gp}(\mathcal{C}) \arrow[r, "\omega"] \arrow[d, "\varphi"'] & \mathbf{Gp} \arrow[d, "\phi"] \\
\mathcal{C} \arrow[r, "\omega"] & \mathbf{Set}
\end{tikzcd}$$

Lemma 1.0.3. If $\mathcal{C}$ is complete, then so is $\mathbf{Gp}(\mathcal{C})$.

Proof. Let $F : J \to \mathbf{Gp}(\mathcal{C})$ be a diagram in $\mathcal{C}$. Since $\mathcal{C}$ is complete, the limit $\lim \varphi \circ F$ exists in $\mathcal{C}$. We must show there exists a natural group structure on $\lim \varphi \circ F$. As limits commute, we have

$$\lim(\varphi \circ F) \times \lim(\varphi \circ F) = \lim\left((\varphi \circ F) \times (\varphi \circ F)\right) \circ \Delta_J$$

where $\Delta_J : J \to J \times J$ is the diagonal embedding. So to have a morphism

$$\lim \varphi \circ F \times \lim \varphi \circ F \longrightarrow \lim \varphi \circ F$$

it suffices to construct natural transformation $((\varphi \circ F) \times (\varphi \circ F)) \circ \Delta_J \to \varphi \circ F$. The multiplication map on each $\varphi \circ F(i)$ gives such a map, and it is natural as F takes values in $\mathbf{Gp}(\mathcal{C})$, i.e., $F(i) \to F(j)$ commutes with multiplication. Hence we obtain a map

$$\mathrm{mult} : \lim \varphi \circ F \times \lim \varphi \circ F \longrightarrow \lim \varphi \circ F$$

Inversion inv and the identity map e on $\lim \varphi \circ F$ are defined in a similar way. It makes $\lim \varphi \circ F$ into a group object in $\mathcal{C}$, and the map $\lim \varphi \circ F \to \varphi \circ F(i)$ commutes with multiplication. Now define $\lim F$ to be this group object together with the limit cone $\lim \varphi \circ F \to \varphi \circ F(i)$. It is evident that $\lim F$ represents the limit of F in $\mathbf{Gp}(\mathcal{C})$. □

Lemma 1.0.4. If $\omega : \mathcal{C} \to \mathbf{Set}$ preserves limits, then so does the functor $\omega : \mathbf{Gp}(\mathcal{C}) \to \mathbf{Gp}$.

Proof. This follows from the construction of limits in $\mathbf{Gp}(\mathcal{C})$ and that of the functor $\omega : \mathbf{Gp}(\mathcal{C}) \to \mathbf{Gp}$. □

1.1 Topological groups

Definition. A **topological group** is a group object in **Top**. Put

$$\mathbf{TopGp} = \mathbf{Gp}(\mathbf{Top})$$

and call it the **category of topological groups**.

Lemma 1.1.1. A topological group G is T_2 if and only if it is T_1, if and only if $\{1_G\}$ is closed.

Proof. The last equivalence holds since translation is a homeomorphism. $T_2 \Rightarrow T_1$ is clear. Assume G is T_1. The diagonal $\Delta_G \subseteq G \times G$ is exactly the inverse image of 1_G under the continuous map $G \times G \to G$ given by $(x, y) \mapsto x^{-1}y$. Hence Δ_G is closed, proving G is T_2. □

1.1.1 Uniform structure

A topological group G has two uniform structures on hand. For each unit-neighborhood V, put

$$V_l := \{(x, y) \in G \times G \mid y \in xV\}$$
$$V_r := \{(x, y) \in G \times G \mid y \in Vx\}$$

We call $\{V_l \mid V \text{ is a unit-neighborhood of } G\}$ (resp. $\{V_r \mid V \text{ is a unit-neighborhood of } G\}$) the **left (resp. right) uniform structure on** G.

Lemma 1.1.2. Both the left and the right uniform structures recover the topology on G.

Proof. This is straightforward. For instance we verify the left uniform structure induces the topology on G. For each unit-neighborhood V of G, we have

$$V_l(e) := \{y \in G \mid (e, y) \in V_l\} = \{y \in G \mid y \in V\} = V.$$

Now the lemma follows from Proposition B.1.3. □

Corollary 1.1.2.1. A T_2 topological group is regular.

Proof. This is Corollary B.1.4.4. □

Definition. Let X be a uniform space and $f : G \to X$ be a function.

(i) f is **left** (resp. **right**) **uniformly continuous** if it is uniformly continuous with respect to the left (resp. right) uniform structure.

(ii) f is called **uniformly continuous** if it is both left and right uniform continuous.

Lemma 1.1.3. Let X be a metrizable TVS. Then any function $f \in C_c(G, X)$ is uniformly continuous.

Proof. Fix a metric d on X. Let $K = \mathrm{supp}(f)$. Fix $\varepsilon > 0$ and a compact unit-neighborhood V. Since f is continuous, for every $x \in G$ there exists a unit-neighborhood $V_x \subseteq V$ such that if $y \in xV_x$, then $d(f(y), f(x)) < \varepsilon/2$. Let U_x be the symmetric open unit-neighborhood with $U_x^2 \subseteq V_x$. Then the sets xU_x $(x \in KV)$ form an open cover of the compact set KV, so there exists $x_1, \dots, x_n \in KV$ such that $KV \subseteq x_1U_{x_1} \cup \dots \cup x_nU_{x_n}$. Put $U_i = U_{x_i}$ and $U = U_1 \cap \dots \cap U_n$; note that U is symmetric.

Let $x, y \in G$ with $x^{-1}y \in U$.

- If $x \notin KV$, then $y \notin K$, for $x \in yU^{-1} = yU \subseteq yV$. Then $f(x) = f(y) = 0$.
- If $x \in KV$, then there exists j with $x \in x_j U_j$, so that $y \in x_j U_j U \subseteq x_j V_{x_j}$. Hence

$$d(f(x), f(y)) \leqslant d(f(x), f(x_j)) + d(f(x_j), f(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So far we have found a unit-neighborhood U such that if $x^{-1}y \in U$, then $d(f(x), f(y)) < \varepsilon$, i.e., f is left uniformly continuous. It goes verbatim that f is right uniformly continuous, and hence the lemma follows. $\square$

1.2 Limits

For a Hausdorff topological group G, let $\mathcal{F}_G$ be a collection of closed normal subgroups such that

(a) every unit-neighborhood of G contains some element in $\mathcal{F}_G$,

(b) $\mathcal{F}_G$ is directed by inclusion, and

(c) either G is complete or some element in $\mathcal{F}_G$ is compact.

Theorem 1.2.1. Retain the notation above. Then the canonical map

$$G \longrightarrow \varprojlim_{H \in \mathcal{F}_G} G/H$$

is a continuous bijection.

Proof. Put $G' = \varprojlim_{H \in \mathcal{F}_G} G/H$. Injectivity follows from (a), and the continuity is clear. For surjectivity, let $(x_H)_H \in G'$. Then $\{x_H H \mid H \in \mathcal{F}_G\}$ is a family of closed subspace in G. If $H' \subseteq H \in \mathcal{F}_G$, then $x_{H'}H' \subseteq x_H H$ as x_H maps to $x_{H'}$ under $G/H \to G/H'$. Hence

$$\{x_H H \mid H \in \mathcal{F}_G\}$$

satisfies the finite intersection property. Any element in $\bigcap_{h \in \mathcal{F}_G} x_H H$ will map to $(x_H)_H$, so it remains to show they have nontrivial intersection. If some element in $\mathcal{F}_G$ is compact, then it is nontrivial by Corollary A.4.1.2. If G is complete, by (a) we see S is a Cauchy filter, so it has a nontrivial intersection. $\square$

Chapter 2

Haar measure

2.1 LCH Groups

Definition. A topological space X is **locally compact** if every point of X admits a compact neighborhood. An **LCH group** is a topological group whose underlying topology is locally compact Hausdorff.

1. A subset of a space X is **relatively compact** if its closure is compact in X.
2. A subset of a space X is **σ-compact** if it is a countable union of compact subsets.

Proposition 2.1.1. Let G be an LCH group.

1. For a closed subgroup $H \leqslant G$, the quotient space G/H is an LCH space.
2. G admits a σ-compact open subgroup.
3. The countable union of σ-compact open subgroups of G generates a σ-compact open subgroup.

Proof.

1. Let $x \in G$, and pick a compact neighborhood U of x in G. Then $UH \subseteq G/H$ is a compact neighborhood of xH in G/H, as $G \to G/H$ is continuous and open.

   To show G/H is Hausdorff, let $xH \neq yH \in G/H$. Let $U \subseteq G$ be a relatively compact open neighborhood of x with $\overline{U} \cap yH = \varnothing$ (U exists since $yH \subseteq G$ is closed). $\overline{U}$ being compact and H being closed, the set $\overline{U}H$ is also closed in G, so there exists a relatively compact open neighborhood V of y such that $V \cap UH = \varnothing$. Hence $VH \cap UH = \varnothing$.

2. Let V be a symmetric relatively compact open unit-neighborhood. For $n \in \mathbb{N}$, one has

   $$\overline{V}^n \subseteq \overline{V^n} \subseteq V.V^n = V^{n+1}$$

   so that $H := \bigcup_n \overline{V}^n = \bigcup_n V^{n+1}$, which is σ-compact, and is an open subgroup.

3. Let (L_n) be a sequence of σ-compact open subgroups of G. Put $L := \bigcup_n L_n$; then the subgroup generated by the L_n is $\bigcup_m L^m$, which is clearly open, and being a countable union of countable unions of σ-compact sets, it is also σ-compact.

$\square$

2.2 Haar Measure on LCH Groups

2.2.1 Haar measures

Definition. For a topological space X, put

$$C_c(X) = \{f : X \to \mathbb{C} \mid f \text{ is continuous with compact support}\}$$

where the **support** of a function f is defined as $\operatorname{supp} f := \overline{\{x \in X \mid f(x) \neq 0\}}$.

Let X be an LCH space, and let μ be an outer Radon measure. That μ is weakly inner regular can be paraphrased as follows. For each compact $K \subseteq U$, by Urysohn's Lemma we can find $f \in C_c^+(G)$ with $f|_K \equiv 1$, $0 \leqslant f \leqslant 1$ and $f|_{X \setminus U} \equiv 0$. Then

$$\mu(U) \geqslant \int_U f d\mu = \int_X f d\mu \geqslant \int_K f d\mu = \mu(K)$$

Hence

$$\mu(U) = \sup_{\substack{f \in C_c^+(G) \\ f \leqslant \mathbf{1}_U}} \int_X f d\mu$$

Proposition 2.2.1. Let μ be an outer Radon measure on an LCH space X. Then $C_c(X)$ with compact supports is dense in $L^p(X, \mu)$ for every $1 \leqslant p < \infty$.

Proof. Fix $1 \leqslant p < \infty$ and let V be the closure of $C_c(X)$ in $L^p(\mu)$. We must show $V = L^p := L^p(\mu)$. It suffices to show $\mathbf{1}_A \in V$ for all measurable sets A with finite measure. By outer regularity, we can find open sets $(U_n)_n$ containing A such that $\mathbf{1}_{U_n} \to \mathbf{1}_A$ in L^p, so we can assume A is open. Again by weakly inner regularity we reduce to the case A is compact.

For each $\varepsilon > 0$, pick open $U \supseteq A$ with $\mu(U \setminus A) < \varepsilon$. By Urysohn's Lemma, there exists $g \in C_c(X)$ with $g|_A \equiv 1$, $g|_{X \setminus U} \equiv 0$ and $0 \leqslant g \leqslant 1$. Then

$$\|\mathbf{1}_A - g\|_p^p = \int_{U \setminus A} |g(x)|^p d\mu(x) \leqslant \mu(U \setminus A) < \varepsilon$$

□

Definition. Let G be a LCH group.

1. A measure μ on G is called **left-invariant** if $\mu(xA) = \mu(A)$ for all $x \in G$ and measurable A.

2. A *non-zero* left-invariant outer Radon measure on G is called a **left Haar measure** on G.

A **right Haar measure** is similarly defined.

In what follows a **Haar measure** is usually referred to as a left Haar measure.

Theorem 2.2.2. Let G be a LCH group. Then there exists a Haar measure on G which is uniquely determined up to positive scalars.

The proof will occupy the following subsections. We first derive some properties of Haar measures.

Corollary 2.2.2.1. Let μ be a Haar measure on the LCH group G.

(i) Every nonempty open set U has strictly positive measure.

(ii) Every compact set has finite measure.

(iii) Every continuous positive function $f \geqslant 0$ with $\int_G f(x)d\mu(x) = 0$ vanishes identically.

(iv) The support of a measurable function f on G that is μ-integrable is contained is in a σ-compact open subgroup of G.

Proof.

(i) Suppose there exists a nonempty open set U with zero measure. Then for every compact set K in G, K can be covered by finitely many left-translations xU of U, so that K is also of measure zero. Being weakly inner regular, μ vanishes on every open set. Hence μ is identically zero, contradicting to the non-triviality assumption imposed in the definition of a Haar measure.

(ii) Recall that a Radon measure is locally finite. Hence every compact set can be covered by finitely many open sets with finite measure.

(iii) It implies that $\mu(f^{-1}(0,\infty)) = 0$, so by 1. it is empty.

(iv) It suffices to show the set $A = \{x \in X \mid f(x) \neq 0\}$ is contained in a σ-compact open subgroup L. We make the following reductions.

- $A = \bigcup_n A_n$, where $A_n = \{x \in X \mid |f(x)| \geqslant 1/n\}$ is of finite measure. Hence it suffices to show a set with finite measure is contained in a σ-compact open subgroup.
- By outer regularity, we can find open $U \supseteq A$ with $\mu(U) < \infty$, so it suffices to show an open set with finite measure is contained in a σ-compact open subgroup.

Let $H \leqslant G$ be any σ-compact open subgroup. Then G is a disjoint union of the xH with $x \in G$. Since $\mu(U) < \infty$, U can only meet countably many cosets xH, for either $xH \cap U = \varnothing$ or $\mu(xU \cap U) > 0$ by 1. Let L be the group generated by H and those cosets xH that meet U. Then $L \supseteq U \supseteq A$ and L is σ-compact and open.

□

The following lemma shows Haar measures turn out to determine the topology.

Lemma 2.2.3. Let G be an LCH group, μ a Haar measure and X a measurable set with $\infty > \mu(X) > 0$. Then the set $XX^{-1} = \{xy^{-1} \mid x, y \in X\}$ is a unit-neighborhood of G.

Proof. By weakly inner regularity we may assume X is compact. By outer regularity we can find an open neighborhood U of X with $\mu(X) \leqslant \mu(U) < 2\mu(X)$. By continuity and the compactness of X, we can find a unit-neighborhood W of G such that $WX \subseteq U$. Then for all $w \in W$, we must have $wX \cap X \neq \varnothing$, for otherwise $\mu(U) \geqslant 2\mu(X)$, a contradiction. But then $W \subseteq XX^{-1}$ as wanted. □

2.2.2 Existence

To prove Theorem, we invoke Riesz's representation theorem. It suffices to show that up to positive scalars there exists a nontrivial positive linear functional $I : C_c(G) \to \mathbb{C}$ that is invariant in the sense $I(L_x f) = I(f)$ for every $x \in G$ and $f \in C_c(F)$, where $L_x f(y) := f(x^{-1}y)$ is the left translation. Likewise define $R_x f(y) := f(xy)$. The "inverse" in the definition of L_x is made so that $L_{xy} = L_x L_y$.

Lemma 2.2.4. Let G be an LCH group and $\Lambda : C_c(G) \to \mathbb{C}$ be a positive linear functional. Then Λ is left-invariant if and only if its induced outer Radon measure on G is left-invariant.

Proof. The if part is obvious. For the only if part, let μ be an induced outer Radon measure on G. We need to show $\mu(xA) = \mu(A)$ for all measurable A and $x \in G$. By outer regularity, we may assume $A = U$ is open, and by weakly inner regularity we have

$$\mu(U) = \sup_{\substack{f \in C_c^+(G) \\ f \leqslant \mathbf{1}_U}} \int_X f d\mu = \sup_{\substack{f \in C_c^+(G) \\ f \leqslant \mathbf{1}_U}} \Lambda(f)$$

Thus for all $g \in G$,

$$\mu(gU) = \sup_{\substack{f \in C_c^+(G) \\ f \leqslant \mathbf{1}_{gU}}} \Lambda(f) = \sup_{\substack{f \in C_c^+(G) \\ L_{g^{-1}} f \leqslant \mathbf{1}_U}} \Lambda(f) = \sup_{\substack{f \in C_c^+(G) \\ f \leqslant \mathbf{1}_U}} \Lambda(L_g f) = \sup_{\substack{f \in C_c^+(G) \\ f \leqslant \mathbf{1}_U}} \Lambda(f) = \mu(U)$$

□

Definition. A function $f : G \to \mathbb{C}$ is **positive** if $f(G) \subseteq \mathbb{R}_{\geqslant 0}$; we write $f \geqslant 0$ in this case.

- Put $C_c^+(G)$ for the set of all positive continuous functions with compact support.
- For any two $f, g \in C_c^+(G)$ with $g \neq 0$, there exist finitely many $s_j \in G$ and positive numbers c_j such that for all $x \in G$, $f(x) \leqslant \sum_{j=1}^{n} c_j g(s_j^{-1} x)$, or simply $f \leqslant \sum_{j=1}^{n} c_j L_{s_j} g$.

  Indeed, let U be a compact neighborhood such that g does not vanishes on U and say $s_1 U, \ldots, s_n U$ cover the support of f. Put $c_j = \dfrac{\max\{f(x) \mid x \in s_j U\}}{\min\{g(x) \mid x \in U\}}$. Then for $x \in s_j U$, we have $f(x) \leqslant c_j g(s_j^{-1} x)$. Summing over $j = 1, \ldots, n$ gives the desired inequality.

For $f, g \in C_c^+(G)$ with $g \neq 0$, define the **index**

$$(f : g) = \inf \left\{ \sum_{j=1}^{n} c_j \mid \text{there exists } s_j \in G \text{ such that } f \leqslant \sum_{j=1}^{n} c_j L_{s_j} g \right\}$$

Lemma 2.2.5. For $f, f_1, f_2, g, h \in C_c^+(G)$ with $g, h \neq 0$ and $c > 0$, one has

1. $(L_y f : g) = (f : g)$ for every $y \in G$.
2. $(f_1 + f_2 : g) \leqslant (f_1 : g) + (f_2 : g)$.
3. $(cf : g) = c(f : g)$.
4. $f_1 \leqslant f_2 \Rightarrow (f_1 : g) \leqslant (f_2 : g)$.
5. $(f : h) \leqslant (f : g)(g : h)$.
6. $(f : g) \geqslant \dfrac{\max f}{\max g}$, where $\max f := \max\{f(x) \mid x \in G\}$.

In the following we fix a nonzero function $f_0 \in C_c^+(G)$. For $f, \phi \in C_c^+(G)$ with $\phi \neq 0$, define

$$J(f, \phi) = J_{f_0}(f, \phi) := \frac{(f : \phi)}{(f_0 : \phi)}$$

Lemma 2.2.6. For $f, g, \phi \in C_c^+(G)$ with $f, \phi \neq 0$ and $c > 0$, one has

1. $\dfrac{1}{(f_0 : f)} \leqslant J(f, \phi) \leqslant (f : f_0)$.
2. $J(L_s f, \phi) = J(f, \phi)$ for every $s \in G$.

3. $J(f+g,\phi) \leqslant J(f,\phi) + J(g,\phi)$.

4. $J(cf,\phi) = cJ(f,\phi)$.

Lemma 2.2.7. Let $f_1, f_2 \in C_c^+(G)$ and $\varepsilon > 0$. There exists a unit-neighborhood V in G such that

$$J(f_1,\phi) + J(f_2,\phi) \leqslant J(f_1+f_2,\phi) + \varepsilon$$

holds for every $\phi \in C_c^+(G)\backslash\{0\}$ with support in V.

Proof. Pick an $f' \in C_c^+(G)$ such that $f \equiv 1$ on the support of $f_1 + f_2$. Choose $\delta > 0$ such that

$$\delta(f_1 + f_2 : f_0) + (\delta + \delta^2)(f' : f_0) < \varepsilon$$

and set

$$f = f_1 + f_2 + \delta f', \qquad h_1 = \frac{f_1}{f}, \qquad h_2 = \frac{f_2}{f}$$

where we set $h_j(x) = 0$ if $f(x) = 0$. Then $h_j \in C_c^+(G)$ for $j = 1, 2$. For if $f(x) = 0$, then $f_j(x) = f'(x) = 0$ so that $x \notin \operatorname{supp}(f_1 + f_2)$. Hence there exists a unit-neighborhood U such that if $y \in xU$, then $f_1(y) + f_2(y) = 0$, implying $f_1(y) = f_2(y) = 0$. Hence $h_j \equiv 0$ in xU, demonstrating the continuity of h_j at x.

Being compactly supported, the h_j are uniformly continuous, so there exists a unit-neighborhood V such that for $x, y \in G$ with $x^{-1}y \in V$ and $j = 1, 2$, one has $|h_j(x) - h_j(y)| < \delta/2$. Let $\phi \in C_c^+(G)\backslash\{0\}$ with support in V, and choose finitely many $s_k \in G$, $c_k > 0$ with $f \leqslant \sum_k c_k L_{s_k}\phi$. Then $\phi(s_k^{-1}x) \neq 0$ (so that $s_k^{-1}x \in V$) implies $|h_j(x) - h_j(s_k)| < \delta/2$, and for all x, one has

$$\begin{aligned} f_j(x) = f(x)h_j(x) &\leqslant \sum_k c_k \phi(s_k^{-1}x) h_j(x) \\ &\leqslant \sum_k c_k \phi(s_k^{-1}x) \left(h_j(s_k) + \frac{\delta}{2} \right) \end{aligned}$$

so that $(f_j : \phi) \leqslant \sum_k c_k (h_j(s_k) + \delta/2)$, implying

$$(f_1 : \phi) + (f_2 : \phi) \leqslant \sum_k c_k (1 + \delta)$$

which yields

$$\begin{aligned} J(f_1,\phi) + J(f_2,\phi) &\leqslant J(f,\phi)(1+\delta) \\ &\leqslant (J(f_1+f_2,\phi) + \delta J(f',\phi))(1+\delta) \\ &= J(f_1+f_2,\phi) + \delta J(f_1+f_2,\phi) + (\delta+\delta^2)J(f',\phi) \\ &\overset{2.2.6.1}{\leqslant} J(f_1+f_2,\phi) + \delta(f_1+f_2 : f_0) + (\delta+\delta^2)(f' : f_0) \\ &< J(f_1+f_2,\phi) + \varepsilon \end{aligned}$$

□

Lemma 2.2.5.(5) together with $(f : f) = 1$ gives $\dfrac{1}{(f_0 : f)} \leqslant (f : f_0)$. For $f \in C_c^+(G)\backslash\{0\}$, let S_f be the compact interval

$$S_f := \left[\frac{1}{(f_0 : f)}, (f : f_0) \right]$$

The product $S := \prod_{f \in C_c^+(G)\backslash\{0\}} S_f$ is compact by Tychonov's theorem. For every $\phi \in C_c^+(G)\backslash\{0\}$, we get an element $J(f,\phi) \in S_f$ from Lemma 2.2.6.(1), hence an element $(J(f,\phi))_f \in S$.

For a unit-neighborhood V let

$$L_V := \overline{\left\{(J(f,\phi))_f \in S \mid \phi \in C_c^+(G)\backslash\{0\} \text{ with } \operatorname{supp}(\phi) \subseteq V\right\}} \underset{\text{closed}}{\subseteq} S$$

Since S is compact, the intersection

$$\bigcap \{L_V \mid V : \text{ unit neighborhood in } G\}$$

is nonempty by Corollary A.4.1.2. Choose an element $(I_{f_0}(f))_f$ in this intersection.

Lemma 2.2.8. $I := I_{f_0} : C_c^+(G) \to \mathbb{C}$ is a left-invariant positive homogeneous additive map.

Proof. By definition and Lemma 2.2.6.2. and 4., I is positive, left-invariant and homogeneous. Let $f, g \in C_c^+(G)\backslash\{0\}$. We need to show $I(f+g) = I(f) + I(g)$. For each $\varepsilon > 0$, by Lemma 2.2.7 we can find unit-neighborhood V such that

$$J(f,\phi) + J(g,\phi) \leqslant J(f+g,\phi) + \varepsilon$$

holds for every $\phi \in C_c^+(G)\backslash\{0\}$ with support in V. In particular, this forces $I(f)+I(g) \leqslant I(f+g)+\varepsilon$. Letting $\varepsilon \to 0^+$, and since $(I(f))_f \in \bigcap_V L_V$, we must have $I(f) + I(g) \leqslant I(f+g)$. The reversed inequality follows from Lemma 2.2.6.3. $\square$

Extending I by linearity we obtain a well-defined positive invariant linear functional $I : C_c(G) \to \mathbb{C}$. This proves the existence of a Haar measure on G.

2.2.3 Uniqueness

Lemma 2.2.9. Let ν be a Haar measure on G. Then for every $f \in C_c(G)$ the map

$$s \mapsto \int_G f(xs)d\nu(x)$$

is continuous on G.

Proof. We must show for each $s \in G$ there exists a neighborhood U of s such that for all $t \in U$ one has

$$\left|\int_G (f(xs) - f(xt))d\nu(x)\right| < \varepsilon$$

Replacing f by $R_s f$, we reduce to the case $s = e$. Let $K = \operatorname{supp}(f)$, and V a compact symmetric unit-neighborhood. For $t \in V$, one has $\operatorname{supp}(R_t f) \subseteq KV$. Since f is uniformly continuous, there exists a symmetric unit-neighborhood W such that for $t \in W$ one has $|f(xt) - f(x)| < \dfrac{\varepsilon}{\nu(KV)}$. Finally, for $t \in W \cap V$, we see

$$\begin{aligned}\left|\int_G (f(xt) - f(x))d\nu(x)\right| &\leqslant \int_{KV} |f(xt) - f(x)|d\nu(x) \\ &< \frac{\varepsilon}{\nu(KV)}\nu(KV) = \varepsilon\end{aligned}$$

$\square$

Suppose note μ, ν are two Haar measures. We aim to show that $\nu = c\mu$ for some $c > 0$. For $f \in C_c(G)$ with $\displaystyle\int_G f(t)d\mu(t) =: I_\mu(f) \neq 0$, set

$$D_f(s) := \frac{1}{I_\mu(f)} \int_G f(ts)d\nu(t)$$

The function D_f is continuous by Lemma 2.2.9. Let $g \in C_c(G)$. By Fubini's theorem and the invariance of the measures μ, ν, we obtain

$$\begin{aligned} I_\mu(f)I_\nu(g) = \int_G\int_G f(s)g(t)d\nu(t)d\mu(s) &\overset{\text{inv}}{=} \int_G\int_G f(s)g(s^{-1}t)d\nu(t)d\mu(s) \\ &\overset{\text{Fubini}}{=} \int_G\int_G f(s)g(s^{-1}t)d\mu(s)d\nu(t) \overset{\text{inv}}{=} \int_G\int_G f(ts)g(s^{-1})d\mu(s)d\nu(t) \\ &\overset{\text{Fubini}}{=} \int_G\int_G f(ts)g(s^{-1})d\nu(t)d\mu(s) \\ &= \int_G\left(\int_G f(ts)d\nu(t)\right)g(s^{-1})d\mu(s) \\ &= I_\mu(f)\int_G D_f(s)g(s^{-1})d\mu(s) \end{aligned}$$

Since $I_\mu(f) \neq 0$, we conclude

$$I_\nu(g) = \int_G D_f(s)g(s^{-1})d\mu(s)$$

Pick another $f' \in C_c(G)$ with $I_\mu(f') \neq 0$. It follows that

$$\int_G (D_f(s) - D_{f'}(s))g(s^{-1})d\mu(s) = 0$$

for every $g \in C_c(G)$. Replacing g with the function $\tilde{g}$ given by

$$\tilde{g}(s) = |g(s)|^2\overline{(D_f(s^{-1}) - D_{f'}(s^{-1}))}$$

one obtains

$$\int_G |(D_f(s) - D_{f'}(s))g(s^{-1})|^2 d\mu(s) = 0$$

It follows that $(D_f(s) - D_{f'}(s))g(s^{-1}) = 0$. Since g is arbitrary, $D_f = D_{f'}$; call this function D. Now for each $f \in C_c(G)$ with $I_\mu(f) \neq 0$, one has

$$\int_G f(t)d\mu(t)D(e) = \int_G f(t)d\nu(t)$$

By linearity, it follows that this equality holds everywhere in $C_c(G)$, and hence proving the uniqueness.

2.3 Modular Characters

Let G be an LCH group. For a Haar measure μ, we sometimes write $\operatorname{vol}(X, \mu)$ instead of $\mu(X)$ to stand for the volume of X measured by μ. In this section we fix a left Haar measure μ. For a topological group automorphism $\sigma : G \to G$, the assignment $X \mapsto \operatorname{vol}(\sigma(X), \mu)$ defines another left Haar measure on G, so by uniqueness there exists a unique positive scalar $\operatorname{mod}_G(\sigma) \in \mathbb{R}_{>0}$ such that $\operatorname{vol}(\sigma(X), \mu) = \operatorname{mod}_G(\sigma)\operatorname{vol}(X, \mu)$. The scalar $\operatorname{mod}_G(\sigma)$ is called the **modulus** of the automorphism σ. Again by uniqueness of Haar measures the modulus is independent of the choice of μ.

Similarly, for each $g \in G$ the assignment $X \mapsto \operatorname{vol}(Xg, \mu)$ defines a left Haar measure, so by uniqueness there exists $\Delta_G(g) > 0$ with $\operatorname{vol}(Xg, \mu) = \Delta_G(g)\operatorname{vol}(X, \mu)$. Again this is independent of the choice of μ. The map $\Delta_G : G \to \mathbb{R}_{>0}$ is called the **modular function/modular character** of G, a name we justify below. For now, notice that for each $g \in G$ if we denote $\operatorname{Inn}_g : G \to G$ the conjugation by g, i.e., $\operatorname{Inn}_g(x) := gxg^{-1}$, then $\operatorname{mod}_G(\operatorname{Inn}_g) = \Delta_G(g)^{-1}$.

Definition. An LCH group is called **unimodular** if the modular character is trivial.

Theorem 2.3.1. For convenience, write dx for the fixed left Haar measure μ.

1. The modular function $\Delta_G : G \to \mathbb{R}_{>0}$ is a continuous group homomorphism.

2. Let $[G, G]$ denote the closure of the commutator subgroup of G. Then $\Delta \equiv 1$ if $G/Z(G)[G, G]$ is compact. In particular, G is unimodular if G is abelian or compact.

3. If $\sigma : G \to G$ is a topological group automorphism, then
$$\int_G f(\sigma^{-1}(x))d\mu(x) = \mathrm{mod}_G(\sigma) \int_G f(x)d\mu(x)$$
is valid for all $f \in L^1(G)$. In a concise form, $d(\sigma(x)) = \mathrm{mod}_G(\sigma)dx$. In particular, for $y \in G$ one has
$$\int_G f(xy)dx = \Delta_G(y^{-1}) \int_G f(x)dx$$
or $d(xy) = \Delta_G(y)dx$.

4. For $f \in L^1(G)$,
$$\int_G f(x^{-1})\Delta(x^{-1})dx = \int_G f(x)dx$$
In a concise form, $d(x^{-1}) = \Delta_G(x^{-1})dx$.

In other words, 3. says that $\Delta_G(x^{-1})dx$ is a right Haar measure, and 4. implies that $x \mapsto x^{-1}$ takes dx to $\Delta_G(x^{-1})dx$.

Proof.

3. It is clear for the case $f = \mathbf{1}_A$ with A measurable and $\mathrm{vol}(A) < \infty$. The general case follows as usual.

1.
   - Group homomorphism. For measurable A and $x, y \in G$, we have
   $$\Delta(xy)\mu(A) = \mu_{xy}(A) = \mu(Axy) = \mu_y(Ax) = \Delta(y)\mu(Ax) = \Delta(y)\mu_x(A) = \Delta(y)\Delta(x)\mu(A)$$
   Pick A such that $\mu(A) \neq 0$; then it follows that $\Delta(xy) = \Delta(x)\Delta(y)$.
   - Continuity. Let $f \in C_c(G)$ with $c = \int_G f(x)dx \neq 0$. By 3. one has
   $$\Delta(y) = \frac{1}{c} \int_G f(xy^{-1})dx = \frac{1}{c} \int_G R_{y^{-1}} f(x)dx$$

2. Since $\mathbb{R}_{>0}$ is abelian, we have $\Delta(aba^{-1}b^{-1}) = 1$ for all $a, b \in G$. Since Δ is continuous, we have $\Delta([G, G]) = 1$. If $g \in Z(G)$, then $\mathrm{vol}(Xg, \mu) = \mathrm{vol}(gX, \mu) = \mathrm{vol}(X, \mu)$, so $\Delta(g) = 1$. Hence Δ factors thorough $G/Z(G)[G, G]$. The only compact subgroup of $\mathbb{R}_{>0}$ is $\{1\}$, so $\Delta \equiv 1$.

4. Let $f \in C_c(G)$ and $I(f) = \int_G f(x^{-1})\Delta(x^{-1})dx$. By 3.
$$\begin{aligned} I(L_z f) &= \int_G f(z^{-1}x^{-1})\Delta(x^{-1})dx = \int_G f((xz)^{-1})\Delta(x^{-1})dx \\ &= \Delta(z^{-1}) \int_G f(x^{-1})\Delta((xz^{-1})^{-1})dx = \int_G f(x^{-1})\Delta(x^{-1})dx \\ &= I(f) \end{aligned}$$

Hence I is an invariant positive linear functional on $C_c(G)$, so by Lemma 2.2.4 and the uniqueness of Haar measures there exists $c > 0$ such that $I(f) = c \int_G f(x)dx$.

It remains to show $c = 1$. Let $\varepsilon > 0$ and choose a symmetric unit-neighborhood V with $|1 - \Delta(s)| < \varepsilon$ for every $s \in V$, and take a nonzero symmetric function $f \in C_c^+(V)$. Then

$$\begin{aligned}|1-c| \int_G f(x)dx &= \left| \int_G f(x)dx - I(f) \right| \leqslant \int_G |f(x) - f(x^{-1})\Delta(x^{-1})|dx \\ &= \int_V f(x)|1 - \Delta(x^{-1})|dx < \varepsilon \int_G f(x)dx\end{aligned}$$

so that $|1 - c| < \varepsilon$. Since ε is arbitrary, $c = 1$.

$\square$

Remark. Denote by $\mathrm{inv} : G \to G$ the inversion on G and consider the pushforward measure $\nu := \mathrm{inv}_* \mu$ of the left-invariant Haar measure μ on G. It is a Radon measure on G, and satisfies the integration formula, valid for all $f \in C_c(G)$

$$\int_G f(x)d\nu := \int_G f(x^{-1})d\mu.$$

See (2.4.25) and (2.4.26) for these statements. Now

$$\int_G f(xy)d\nu := \int_G f(y^{-1}x^{-1})d\mu = \int_G f(x^{-1})d\mu = \int_G f(x)d\nu$$

so ν is a right Haar measure. From 2. if we write μ as dx, then $\Delta(x^{-1})dx$ is also a right Haar measure, so the uniqueness says $dx^{-1} = c\Delta(x^{-1})dx$ for some unique scalar $c > 0$. This is exactly the same as the first part of the proof of 4.

Remark 2.3.2. In the literature people use right Haar measures to define modular character. To be precise, let μ_r be a right Haar measure on G. Since $X \mapsto \mathrm{vol}(hX, \mu_r)$ is also a right Haar measure, there exists a constant $\delta_G(h)$ such that

$$\mathrm{vol}(hX, \mu_r) = \delta_G(h) \,\mathrm{vol}(X, \mu_r).$$

Using the fact that $\Delta_G(g)^{-1}dg$ is a right Haar measure, one can show that

$$\delta_G = \Delta_G^{-1}.$$

Indeed, if we take $d\mu_r(g) = \Delta_G(g)^{-1}dg$, then

$$\begin{aligned}\mathrm{vol}(hX, \mu_r) &= \int_G \mathbf{1}_{hX}(g)d\mu_r(g) \\ &= \int_G \mathbf{1}_X(h^{-1}g)\Delta_G(g)^{-1}dg = \int_G \mathbf{1}_X(g)\Delta_G(hg)^{-1}dg = \Delta_G(h)^{-1}\,\mathrm{vol}(X, \mu_r)\end{aligned}$$

Proposition 2.3.3. Let G be a LCH group. TFAE:

(i) There exists $x \in G$ such that $\mathrm{vol}(\{x\}) > 0$.

(ii) $\mathrm{vol}(\{e\}) > 0$.

(iii) The counting measure is a Haar measure on G.

(iv) G is a discrete group.

In particular, a discrete group is unimodular.

Proof. That 1. $\Leftrightarrow$ 2. follows from the left-invariant of vol. Assume 2.. Then for any finite set $E \subseteq G$, $\mathrm{vol}(E) = \mathrm{vol}(\{e\})\#E$. The general case follows from monotonicity. For 3. $\Rightarrow$ 4., note that this implies every open set with finite measure is a finite set. By Hausdorff axiom one see every point in U is open. Hence G is discrete. Finally, if G is discrete, every singleton is open so that each of them has a positive measure. $\square$

Proposition 2.3.4. Let G be a LCH group. Then G has finite measure if and only if G is compact.

Proof. The if part is clear. Now assume $\mathrm{vol}(G) < \infty$. Let U be a compact unit-neighborhood. Then

$$\max\{n \in \mathbb{N} \mid \text{there exists } (x_i)_{1 \leqslant i \leqslant n} \subseteq G \text{ such that } x_i U \cap x_j U = \varnothing \text{ if } i \neq j\} < \infty$$

Let $z_1 U, \ldots, z_n U$ be such pairwise disjoint translation, and set K to be their disjoint union. Then K is compact, and for every $x \in G$, $xK \cap K \neq \varnothing$, so that $x \in KK^{-1}$. Hence $G = KK^{-1}$ is compact. $\square$

2.3.1 Examples

Example 2.3.5 (General linear groups)**.** Let k be a non-discrete locally compact Hausdorff topological field (c.f. §2.5; we will use some results therein) and V a finite dimensional vector space over k. Any Hausdorff vector space topology on V is the same. In particular, by taking a k-basis $V \cong k^n$, we easily see that the general linear group

$$\mathrm{GL}_V(k) := \{T \in \mathrm{End}_k V \mid T \text{ is invertible}\} = \{T \in \mathrm{End}_k V \mid \det T \neq 0\} \underset{\text{open}}{\subseteq} \mathrm{End}_k V$$

is an LCH group, and is isomorphic to $\mathrm{GL}_n(k)$ as topological groups, where $n = \dim_k V$.

Let T be a general element in $\mathrm{End}_k V$, and denote by dT a Haar measure on $\mathrm{End}_k V$ (as an additive group). Let $v_1, \ldots, v_n$ be a k-basis for V. Then there is an isomorphism

$$\begin{aligned} \mathrm{End}_k V &\longrightarrow V^n \\ T &\longmapsto (Tv_1, \ldots, Tv_n). \end{aligned}$$

By Fubini's theorem, for each $S \in \mathrm{End}_k V$ we then have $\mathrm{mod}_{\mathrm{End}_k V}(\ell_S) = \mathrm{mod}_V(S)^n$ (where $\ell_S : T \mapsto ST$), and by Corollary 2.5.3.1.(iii) the latter equals $\mathrm{mod}_k(\det S)^n$. In sum, we obtain the following integral formula

$$\int_{\mathrm{End}_k V} f(ST)dT = \mathrm{mod}_k(\det S)^n \int_{\mathrm{End}_k V} f(T)dT$$

valid for all $f \in C_c(\mathrm{End}_k V)$ and $S \in \mathrm{End}_k V$. It follows that the measure $\dfrac{dT}{\mathrm{mod}_k(\det T)^n}$ is a Haar measure of the general linear group $\mathrm{GL}_V(k)$.

Moreover, the group $\mathrm{GL}_V(k)$ is unimodular. It is a consequence that V is isomorphic to its linear dual $V^\vee$ (but in a non-canonical fashion). Precisely, if $v_1, \ldots, v_n$ is a k-basis for V, then we can form the dual basis $v_1^\vee, \ldots, v_n^\vee$. For $T \in \mathrm{End}_k V$, denote by $T^\vee \in \mathrm{End}_k V^\vee$ the corresponding element such that $T^\vee f = f \circ T$ for all $f \in V^\vee$. Define $\Phi : \mathrm{End}_k V \to (V^\vee)^n$ by

$$\Phi(T) = (v_1^\vee \circ T, \ldots, v_n^\vee \circ T) = (T^\vee v_1^\vee, \ldots, T^\vee v_n^\vee),$$

and write $\Psi : (V^\vee)^n \to \operatorname{End}_k V$ for its inverse. Then up to a scalar, for all $f \in C_c(\operatorname{End}_k V)$ the integral formula

$$\int_{\operatorname{End}_k V} f(T)dT = \int_{(V^\vee)^n} (f \circ \Psi)(\lambda_1, \ldots, \lambda_n) \otimes_{i=1}^n d\lambda_i$$

is valid. Here all $d\lambda_i$ are Haar measures on $V^\vee$ and $\otimes_{i=1}^n d\lambda_i$ is the product measure as constructed in Theorem D.4.7 (along with an induction). Then for $S \in \operatorname{End}_k V$, if we put $f_S : T \mapsto TS$, then

$$\begin{aligned}\int_{\operatorname{End}_k V} f(TS)dT = \int_{\operatorname{End}_k V} f_S(T)dT &= \int_{(V^\vee)^n} (f_S \circ \Psi)(\lambda_1, \ldots, \lambda_n) \otimes_{i=1}^n d\lambda_i \\ &= \int_{(V^\vee)^n} f(\Psi(\lambda_1, \ldots, \lambda_n)S) \otimes_{i=1}^n d\lambda_i\end{aligned}$$

Notice that $\Phi(TS) = (S^\vee \times \cdots \times S^\vee)\Phi(T)$, so

$$\Phi(\Psi(\lambda_1, \ldots, \lambda_n)S) = (S^\vee\lambda_1, \ldots, S^\vee\lambda_n).$$

Hence the last integral is equal to

$$\int_{(V^\vee)^n} f(\Psi(S^\vee\lambda_1, \ldots, S^\vee\lambda_n)) \otimes_{i=1}^n d\lambda_i$$

and by Corollary 2.5.3.1.(iii) it is

$$\operatorname{mod}_k(\det S^\vee)^n \int_{(V^\vee)^n} f(\Psi(\lambda_1, \ldots, \lambda_n)) \otimes_{i=1}^n d\lambda_i = \operatorname{mod}_k(\det S^\vee)^n \int_{\operatorname{End}_k V} f(TS)dT.$$

The matrix representation of $S^\vee$ with respect to the basis $v_1^\vee \ldots, v_n^\vee$ is the transpose of that of S with respect to $v_1, \ldots, v_n$, so $\det S^\vee = \det S$, whence proving

$$\int_{\operatorname{End}_k V} f(TS)dT = \operatorname{mod}_k(\det S)^n \int_{\operatorname{End}_k V} f(T)dT.$$

In particular, this shows $\dfrac{dT}{\operatorname{mod}_k(\det T)^n}$ is right-invariant. By linear algebra we see this measure is also invariant under transpose.

Example 2.3.6 (Special linear group)**.** We use the same notation as above. The special linear group $\mathrm{SL}_V(k)$ is defined by the short exact sequence

$$1 \longrightarrow \mathrm{SL}_V(k) \longrightarrow \mathrm{GL}_V(k) \xrightarrow{\det} k^\times \longrightarrow 1$$

In particular, it is a closed normal subgroup of $\mathrm{GL}_V(k)$, so it is unimodular by Corollary 2.4.8.2.

Example 2.3.7 (Matrices)**.** Let V, W be two finite dimensional Hausdorff k-vector spaces. The space $\operatorname{Hom}_k(V, W)$ also has a unique Hausdorff vector space topology, as it is finite dimensional. The general linear group $\mathrm{GL}_W(k)$ acts on $\operatorname{Hom}_k(V, W)$ from the left, so we can consider the modulus $\operatorname{mod}_{\operatorname{Hom}_k(V,W)}(g)$ for any $g \in \mathrm{GL}_V(k)$. To compute it, recall that $\operatorname{Hom}_k(V, W) \cong V^\vee \otimes_k W \cong W^{\oplus \dim_k V}$ and the isomorphism is $\mathrm{GL}_W(k)$-equivariant. Hence

$$\operatorname{mod}_{\operatorname{Hom}_k(V,W)}(g) = \operatorname{mod}_W(g)^{\dim_k V} = \operatorname{mod}_k(\det g)^{\dim_k V}.$$

Similarly, $\mathrm{GL}_V(k)$ acts on $\operatorname{Hom}_k(V, W)$ from the right, and we have $\operatorname{mod}_{\operatorname{Hom}_k(V,W)}(h) = \operatorname{mod}_k(\det h)^{\dim_k W}$ for all $h \in \mathrm{GL}_V(k)$.

Example 2.3.8 (Standard parabolic subgroups)**.** Again let V, W be two finite dimensional Hausdorff k-vector space. Consider

$$P_{V,W}(k) = \left\{ \begin{pmatrix} g & T \\ & h \end{pmatrix} \in \mathrm{GL}_{V\oplus W}(k) \mid g \in \mathrm{GL}_V(k), h \in \mathrm{GL}_K(k), T \in \mathrm{Hom}_k(W, V) \right\}.$$

This is a closed subgroup of $P_{n+m}(k)$, so it is also an LCH group. However, this is not unimodular as we will see soon. To compute its left Haar measure, from the multiplication law

$$\begin{pmatrix} g' & T' \\ & h' \end{pmatrix} \begin{pmatrix} g & T \\ & h \end{pmatrix} = \begin{pmatrix} g'g & g'T + T'h \\ & h'h \end{pmatrix}$$

we see if dg (resp. dh, dT) denotes a Haar measure on $\mathrm{GL}_V(k)$ (resp. $\mathrm{GL}_W(k)$, $\mathrm{Hom}_k(W, V)$), the product measure $dgdh\dfrac{dT}{\mathrm{mod}_k(\det g)^{\dim_k W}}$ is a left Haar measure on $P_{V,W}(k)$. Likewise, $dgdh\dfrac{dT}{\mathrm{mod}_k(\det h)^{\dim_k V}}$ is a right Haar measure on $P_{V,W}(k)$. In particular, this shows $P_{V,W}(k)$ is not unimodular, and the modular function is

$$\Delta_{\mathrm{GL}_{V,W}(k)}\left(\begin{pmatrix} g & T \\ & h \end{pmatrix}\right) = \frac{\mathrm{mod}_k(\det h)^{\dim_k V}}{\mathrm{mod}_k(\det g)^{\dim_k W}}.$$

More generally, if $V_1, \ldots, V_n$ are finite dimensional Hausdorff k-vector spaces, it is easy to proceed by induction to find a left and right Haar measure of the group

$$P_{V_1,\ldots,V_n}(k) = \left\{ \begin{pmatrix} g_1 & T_{12} & \cdots & T_{1n} \\ & g_2 & \cdots & T_{2n} \\ & & \ddots & \vdots \\ & & & g_n \end{pmatrix} \mid g_i \in \mathrm{GL}_{V_i}(k),\ T_{ij} \in \mathrm{Hom}_k(V_j, V_i) \right\}.$$

The modular function can be explicitly written down:

$$\Delta_{P_{V_1,\ldots,V_n}(k)}\left(\begin{pmatrix} g_1 & T_{12} & \cdots & T_{1n} \\ & g_2 & \cdots & T_{2n} \\ & & \ddots & \vdots \\ & & & g_n \end{pmatrix}\right) = \prod_{1\leqslant i<j\leqslant n} \frac{\mathrm{mod}_k(g_j)^{\dim_k V_i}}{\mathrm{mod}_k(g_i)^{\dim_k V_j}}$$

Example 2.3.9 (Unipotent radicals of standard parabolic subgroups)**.** Retain the notation in the previous example. Consider the subgroup

$$U_{V_1,\ldots,V_n}(k) := \left\{ \begin{pmatrix} \mathrm{id}_{V_1} & T_{12} & \cdots & T_{1n} \\ & \mathrm{id}_{V_2} & \cdots & T_{2n} \\ & & \ddots & \vdots \\ & & & \mathrm{id}_{V_n} \end{pmatrix} \mid T_{ij} \in \mathrm{Hom}_k(V_j, V_i) \right\} \underset{\text{closed}}{\leqslant} P_{V_1,\ldots,V_n}(k).$$

Moreover, it is a normal subgroup, so each $p \in P_{V_1,\ldots,V_n}(k)$ defines an automorphism Inn_p of $U_{V_1,\ldots,V_n}(k)$ by conjugation: $\mathrm{Inn}_p(u) := pup^{-1}$. It is direct to see $\otimes_{i<j} dT_{ij}$ is a left and right Haar measure on $U_{V_1,\ldots,V_n}(k)$, where each dT_{ij} is a Haar measure on $\mathrm{Hom}_k(V_j, V_i)$. In particular, $U_{V_1,\ldots,V_n}(k)$ is unimodular and for any $p \in P_{V_1,\ldots,V_n}(k)$ we have $\mathrm{mod}_{U_{V_1,\ldots,V_n}(k)}(\mathrm{Inn}_p) = \Delta_{P_{V_1,\ldots,V_n}(k)}(p^{-1})$.

Example 2.3.10 (Matrix rings over division rings)**.** Let k be a non-discrete locally compact Hausdorff topological field, and let D be a non-discrete locally compact Hausdorff topological division ring containing k. Let V be a finite dimensional Hausdorff left topological vector space over D. Let dT denote a Haar measure on $A := \operatorname{End}_D V$. We claim for $f \in C_c(A)$ and $U \in \operatorname{Aut}_D V$, we have

$$\int_A f(UT)dT = \operatorname{mod}_k(N_{A/k}(U)) \int_A f(T)dT$$

where $N_{A/k} : \operatorname{End}_D V \to k$ is the composition $\operatorname{End}_D V \xrightarrow{T \mapsto UT} \operatorname{End}_k A \xrightarrow{\det} k$. This follows from Corollary 2.5.3.1.(iii) once we regard A as a vector space over k. Now since

$$A^\times = \operatorname{Aut}_D V = \{T \in \operatorname{End}_D V \mid N_{A/k}(T) \neq 0\} \underset{\text{open}}{\subseteq} A$$

$A^\times$ is an LCH group. The equality above shows that $\dfrac{dT}{\operatorname{mod}_k(N_{A/k}(T))}$ is a left Haar measure on $A^\times$. In fact, the map $\operatorname{End}_D V \xrightarrow{T \mapsto UT} \operatorname{End}_k A \xrightarrow{\det} k$ coincides with $\operatorname{End}_D V \xrightarrow{T \mapsto TU} \operatorname{End}_k A \xrightarrow{\det} k$, so $A^\times$ is in fact unimodular.

2.4 Invariant Measures on Quotient Spaces

Let G be an LCH group and $H \leqslant G$ be a closed subgroup. A primary goal in this section is to find an G-invariant measure on the quotient space G/H. We start with making a precise definition.

Definition. Let X be a G-space. A measure ν on the Borel σ-algebra of X is called an **invariant measure** if $\nu(xA) = \nu(A)$ for every $x \in G$ and every measurable $A \subseteq X$.

We need a generalized version of Lemma 2.2.4, though the proof is exactly the same.

Lemma 2.4.1. Let X be an LCH G-space, $\Lambda : C_c(X) \to \mathbb{C}$ a positive linear functional and $\chi \in \operatorname{Hom}_{\mathbf{TopGp}}(G, \mathbb{R}_{>0})$. For each $g \in G$, define $g.\Lambda : f \mapsto \Lambda([x \mapsto f(gx)])$. Then $g.\Lambda = \chi^{-1}(g)\Lambda$ for all $g \in G$ if and only if the associated outer Radon measure μ satisfies $\mu(xA) = \chi(x)\mu(A)$ for all measurable $A \subseteq X$.

Proof. The if part is clear. For the only if part, let μ be an outer Radon measure on X induced by Λ. As in Lemma 2.2.4, it suffices to say $\mu(xU) = \chi(x)\mu(U)$ for open U and $x \in G$, and we have

$$\mu(U) = \sup_{\substack{f \in C_c^+(X) \\ f \leqslant \mathbf{1}_U}} \int_X f d\mu = \sup_{\substack{f \in C_c^+(X) \\ f \leqslant \mathbf{1}_U}} \Lambda(f)$$

Thus for all $x \in G$,

$$\mu(xU) = \sup_{\substack{f \in C_c^+(X) \\ f \leqslant \mathbf{1}_{xU}}} \Lambda(f) = \sup_{\substack{f \in C_c^+(X) \\ x^{-1}.f \leqslant \mathbf{1}_U}} \Lambda(f) = \sup_{\substack{f \in C_c^+(X) \\ f \leqslant \mathbf{1}_U}} \Lambda(x.f) = \sup_{\substack{f \in C_c^+(X) \\ f \leqslant \mathbf{1}_U}} \chi(x)\Lambda(f) = \chi(x)\mu(U)$$

□

With this lemma, we turn our eyes to invariant distributions. We begin our discussion with a rather general setting. Let $\chi \in \operatorname{Hom}_{\mathbf{TopGp}}(H, \mathbb{R}_{>0})$ and consider the space

$$C(G/H, \chi) := \{f \in C(G) \mid f(xh) = \chi(h)f(x) \text{ for all } x \in G,\, h \in H\}$$
$$C_c(G/H, \chi) := \{g \in C(G/H, \chi) \mid \operatorname{supp} g \text{ is compact modulo } H\}.$$

If we denote by $\pi : G \to G/H$ the quotient map, then for $g \in C_c(G/H, \chi)$ the imposed condition means $\pi(\operatorname{supp} g) \subseteq G/H$ is compact. For each $f \in C_c(G)$, define $f^{H,\chi} : G \to \mathbb{C}$ by

$$f^{H,\chi}(x) := \int_H f(xh)\chi^{-1}(h)dh$$

where dh is a fixed left Haar measure on H. The integral converges since f has compact support. When $\chi = \mathbf{1}_H$ is the trivial character, we write $f^{H,\mathbf{1}_H} = f^H$ for brevity.

Lemma 2.4.2.

(i) $f^{H,\chi} \in C_c(G/H, \chi)$ for all $f \in C_c(G)$.

(ii) The resulting map

$$\begin{aligned} C_c(G) &\longrightarrow C_c(G/H, \chi) \\ f &\longmapsto f^{H,\chi} \end{aligned}$$

is surjective. In addition, the fibre of $h \in C_c(G/H, \chi) \cap C^+(G)$ meets $C_c^+(G)$ nontrivially.

Proof.

(i) Let K be the support of f, let $x \in G$ and let U be a compact neighborhood of x. Then $\operatorname{supp} L_{y^{-1}} f|_H \subseteq U^{-1}K \cap H$ for all $y \in U$. Put $d = \operatorname{vol}(U^{-1}K \cap H, dh)$. Given $\varepsilon > 0$, by uniform continuity of f there is a neighborhood $V \subseteq U$ of x such that $|f(yh) - f(xh)| < \dfrac{\varepsilon}{d \cdot \sup\limits_{h \in U^{-1}K \cap H} \chi^{-1}(h)}$ for all $y \in V$. It follows that

$$|f^{H,\chi}(y) - f^{H,\chi}(x)| \leqslant \int_{U^{-1}K \cap H} |f(yh) - f(xh)|\chi^{-1}(h)dh < \varepsilon$$

for $y \in V$, so that $f^{H,\chi}$ is continuous at x.

Finally, $x \in \operatorname{supp} f^{H,\chi}$ implies $f(xh) \neq 0$ for some $h \in H$, so $x \in (\operatorname{supp} f)H$. Hence $\operatorname{supp} f^{H,\chi} \subseteq (\operatorname{supp} f)H$ so that $\operatorname{supp} f^{H,\chi}$ is compact modulo H.

(ii) We need a lemma.

Lemma 2.4.3. Let G be a LCH group and $H \leqslant G$ a closed subgroup. For every compact $C \subseteq G/H$, there exists compact $K \subseteq G$ such that $\pi(K) = C$.

Proof. Put $\pi : G \to G/H$ to be the canonical projection. For each $c \in C$, pick $y_c \in \pi^{-1}(C)$ and an open relatively compact neighborhood U_c of y_c. Since π is open, $\{\pi(U_c)\}_{c \in C}$ forms an open cover of C, and thus we can find $c_1, \ldots, c_n \in C$ such that $C \subseteq K'$ with $K' = \overline{U_{c_1}} \cup \cdots \cup \overline{U_{c_n}} \underset{\text{cpt}}{\subseteq} G$. It suffices to take $K = K' \cap \pi^{-1}(C)$. ☐

Let $g \in C_c(G/H, \chi)$ and use Lemma 2.4.3 to find $K \subseteq G$ a compact set such that $\pi(K) = \pi(\operatorname{supp} g)$, where $\pi : G \to G/H$ is the quotient map. By Urysohn's lemma we can find $\varphi \in C_c(G)$ with $\varphi \geqslant 0$ and $\varphi|_K \equiv 1$. Take $f = g\varphi/\varphi^{H,\chi}$ (note that this makes sense as $x \in \operatorname{supp} g$ implies $xh \in K$ for some $h \in H$ and $\varphi^{H,\chi}(x) > 0$); then

$$f^{H,\chi}(x) = \int_H \frac{g\varphi}{\varphi^{H,\chi}}(xh)\chi^{-1}(h)dh = \int_H \chi(h)g(x)\frac{\varphi(h)\chi^{-1}(h)}{\chi(h)\varphi^{H,\chi}(x)}dh = g(x)$$

proving the surjectivity. The last assertion is clear from the above construction.

□

Let $\Lambda : C_c(G/H, \chi) \to \mathbb{C}$ be a nontrivial G-invariant positive linear functional. Define $\widetilde{\Lambda} = \Lambda \circ [f \mapsto f^{H,\chi}] : C_c(G) \to \mathbb{C}$. Since Φ is G-equivariant, we see $\widetilde{\Lambda}$ is G-invariant, so by Lemma 2.2.4 it is given by a Haar measure dg on G. Then we have a integration formula

$$\int_G f(g)dg = \Lambda\left(\int_H f(gh)\chi^{-1}(h)dh\right)$$

valid for all $f \in C_c(G)$. Now for $h_0 \in H$, we have

$$\begin{aligned}\Delta_G(h_0)^{-1}\int_G f(g)dg = \int_G f(gh_0)dg &= \Lambda\left(\int_H f(ghh_0)\chi^{-1}(h)dh\right)\\ &= \Lambda\left(\Delta_H(h_0^{-1})\chi(h_0)\int_H f(gh)\chi^{-1}(h)dh\right).\end{aligned}$$

Hence we obtain $\Delta_H(h) = \Delta_G(h)\chi(h)$ for $h \in H$. This proves the necessary part of the following

Theorem 2.4.4. Let G be an LCH group, $H \leqslant G$ a closed subgroup and $\chi \in \mathrm{Hom}_{\mathbf{TopGp}}(H, \mathbb{R}_{>0})$. Then a necessary and sufficient condition for the existence of a nontrivial G-invariant positive linear functional on $C_c(G/H, \chi)$ is to have $\Delta_H = \chi\Delta_G|_H$. It is unique up to a positive scalar if it exists.

Proof. Now assume $\Delta_H = \chi\Delta_G|_H$. Define $\Lambda : C_c(G/H, \chi) \to \mathbb{C}$ by

$$\Lambda(g) = \int_G f(x)dx$$

where $f \in C_c(G)$ satisfies $f^{H,\chi} = g$. We must show $f^{H,\chi} = 0$ implies $\displaystyle\int_G f(x)dx = 0$. To this end, for each $\phi \in C_c(G)$, compute

$$\begin{aligned}0 &= \int_G\int_H f(xh)\chi^{-1}(h)\phi(x)dhdx = \int_H\left(\Delta_G(h)^{-1}\int_G f(x)\chi^{-1}(h)\phi(xh^{-1})dx\right)dh\\ &= \int_H\left(\Delta_H(h)^{-1}\chi(h)\int_G f(x)\chi^{-1}(h)\phi(xh^{-1})dx\right)dh)\\ &= \int_G\int_H f(x)\phi(xh)dhdx = \int_G \phi^H(x)f(x)dx.\end{aligned}$$

We can find ϕ with $\phi^H \equiv 1$ on the support of f, and thus $\displaystyle\int_G f(x)dx = 0$. Hence $\Lambda : C_c(G/H, \chi) \to \mathbb{C}$ is well-defined, and is clear that it is nonzero and positive. Λ is invariant for $L_x g = (L_x f)^{H,\chi}$ for each $x \in G$.

For the uniqueness, let $\Lambda : C_c(G/H, \chi) \to \mathbb{C}$ by any nontrivial G-invariant positive linear functional. The proof for the necessary part shows that we can choose a left Haar measure dg on G that makes the integration formula

$$\int_G f(g)dg = \Lambda\left(\int_H f(gh)\chi^{-1}(h)dh\right)$$

valid for all $f \in C_c(G)$. In other words, the following triangle

$$\begin{tikzcd} C_c(G) \arrow[r] \arrow[dr, "\int_G dx"'] & C_c(G/H, \chi) \arrow[d, "\Lambda"] \\ & \mathbb{C} \end{tikzcd}$$

is commutative. Since the upper-horizontal arrow is surjective by Lemma 2.4.2, it follows Λ is uniquely determined by the measure dg. This shows the uniqueness. □

Instead of the left coset space G/H, we sometimes encounter the right coset space $H\backslash G$. No significant difference occurs, as there is a canonical homeomorphism

$$\begin{aligned} G/H &\longrightarrow H\backslash G \\ gH &\longmapsto Hg^{-1} \end{aligned}$$

In particular, analogues of Lemma 2.4.3 and Lemma 2.4.2 hold for $G \to H\backslash G$ as well: we define

$$\begin{aligned} C(H\backslash G, \chi) &= \{f \in C(G) \mid f(hx) = \chi(h)f(x) \text{ for all } x \in G,\, h \in H\} \\ C_c(H\backslash G, \chi) &= \{g \in C(H\backslash G, \chi) \mid \operatorname{supp} g \text{ is compact modulo } H\} \end{aligned}$$

where for $f \in C_c(G)$ a function ${}^{H,\chi}f \in C_c(G/H, \chi)$ is similarly defined as

$${}^{H,\chi}f(x) = \int_H f(hx)\chi^{-1}(h)dh.$$

and dh is a right Haar measure on H. If $\Lambda : C_c(H\backslash G, \chi)$ is a G-invariant positive linear functional, then $\Lambda \circ [f \mapsto {}^{H,\chi}f] : C_c(G) \to \mathbb{C}$ is a right G-invariant positive linear functional, so it is induced by a right Haar measure dg on G satisfying an integration formula similar to the above one:

$$\int_G f(g)dg = \Lambda\left(\int_H f(hg)\chi^{-1}(h)dh\right) \tag{$\spadesuit$}$$

Note that $\Delta_G(g)dg$ (resp. $\Delta_H(h)dh$) is a left Haar measure on G (resp. H) (see Remark 2.3.2), so for $h_0 \in H$ we have

$$\int_G f(h_0g)dg = \int_G f(h_0g)\Delta_G^{-1}(g)\Delta_G(g)dg = \int_G f(g)\Delta_G^{-1}(h_0^{-1}g)\Delta_G(g)dg = \Delta_G(h_0)\int_G f(g)dg$$

and

$$\Lambda\left(\int_H f(h_0hg)\chi^{-1}(h)dh\right) = \Lambda\left(\Delta_H(h_0)\int_H f(hg)\chi^{-1}(h_0^{-1}h)dh\right) = \Delta_H(h_0)\chi(h_0)\Lambda\left(\int_H f(hg)\chi^{-1}(h)dh\right).$$

Equating them yields $\Delta_G|_H = \chi\Delta_H$. Reversing the process just like we did in Theorem 2.4.4 proves the

Theorem 2.4.5. Let G be an LCH group, $H \leqslant G$ a closed subgroup and $\chi \in \mathrm{Hom}_{\mathbf{TopGp}}(H, \mathbb{R}_{>0})$. Then a necessary and sufficient condition for the existence of a nontrivial G-invariant positive linear functional on $C_c(H\backslash G, \chi)$ is to have $\Delta_G|_H = \chi\Delta_H$. It is unique up to a positive scalar if it exists.

Similar to Corollary 2.2.2.1, we can prove

Lemma 2.4.6. Let G be an LCH group, $H \leqslant G$ a closed subgroup and $\chi \in \mathrm{Hom}_{\mathbf{TopGp}}(G, \mathbb{R}_{>0})$. If $\Lambda : C_c(H\backslash G, \chi) \to \mathbb{C}$ is a nontrivial G-invariant positive linear functional, then $\Lambda(f) > 0$ for $0 \neq f \in C_c(H\backslash G, \chi)^+$

Proof. Arrange Haar measures dg, dh on G and H so that ($\spadesuit$) holds. Use Lemma to choose $F \in C_c(G)^+$ with $F^{H,\chi} = f$. Then

$$\int_G F(g)dg = \Lambda(F^{H,\chi}) = \Lambda(f).$$

Hence if $\Lambda(f) = 0$, then $F \equiv 0$ by Corollary 2.2.2.1.(iii), which implies $f = F^{H,\chi} \equiv 0$. ☐

2.4.1 Quotient measure

Theorem 2.4.7. Let G be an LCH group and $H \leqslant G$ a closed subgroup such that $\Delta_H = \Delta_G|_H$. Given left Haar measures on G and H, there is a unique G-invariant Radon measure ν on G/H such that for every $f \in C_c(G)$ one has the quotient integral formula

$$\int_G f(x)dx = \int_{G/H}\int_H f(xh)dhd\nu(x).$$

We will always assume this normalization and call the ensuing measure on G/H the **quotient measure**.

Proof. By assumption and Theorem 2.4.4 (and its proof), there exists a unique G-invariant positive linear functional $\Lambda : C_c(G/H) \to \mathbb{C}$ such that

$$\int_G f(g)dg = \Lambda\left(\int_H f(gh)dh\right)$$

holds for all $f \in C_c(G)$. By Lemma 2.4.1 the functional Λ corresponds to an G-invariant Radon measure ν on G/H, so this finishes the proof. $\square$

Theorem 2.4.8. Retain the setting in Theorem 2.4.7. The quotient integral formula

$$\int_G f(x)dx = \int_{G/H}\int_H f(xh)dhd\nu(x)$$

remains valid for all $f \in L^1(G)$.

Proof. We may assume $f \geqslant 0$. By monotone convergence theorem we can assume f is a step function, and by linearity we can even reduce to the case $f = \mathbf{1}_A$ where A is measurable of finite Haar measure. We have to show

- $\mathbf{1}_A^H$ is measurable on G/H;
- $\displaystyle\int_{G/H} \mathbf{1}_A^H d\nu(x) = \int_G \mathbf{1}_A dx.$

We start with the case $A = U$ being open. Notice that $\mathbf{1}_U = \sup\limits_{\substack{\phi \in C_c(G)\\ 0 \leqslant \phi \leqslant \mathbf{1}_U}} \phi$. Let $x \in G$. Since $\mu(x^{-1}U) < \infty$, by Corollary 2.2.2.1.4. $x^{-1}U$ is contained in a σ-compact open subgroup of G; in particular, $x^{-1}U \cap H$ is σ-compact and open in H, so by applying Lemma D.4.6 we see

$$\mathbf{1}_U^H(x) = \int_H \sup_{\substack{\phi \in C_c(G)\\ 0 \leqslant \phi \leqslant \mathbf{1}_U}} \phi(xh)dh = \sup_{\substack{\phi \in C_c(G)\\ 0 \leqslant \phi \leqslant \mathbf{1}_U}} \int_H \phi(xh)dh$$

Since $G/H \ni x \mapsto \displaystyle\int_H \phi(xh)dh$ is continuous, it is ν-measurable[1]. A repeated use of Urysohn's Lemma and Lemma D.4.6 shows

$$\begin{aligned}\int_{G/H}\int_H \mathbf{1}_U(xh)dhd\nu(x) &= \int_{G/H}\int_H \sup_{0 \leqslant \phi \leqslant \mathbf{1}_U} \phi(xh)dhd\nu(x)\\ &= \sup_{0 \leqslant \phi \leqslant \mathbf{1}_U} \int_{G/H}\int_H \phi(xh)dhd\nu(x)\\ &= \sup_{0 \leqslant \phi \leqslant \mathbf{1}_U} \int_G \phi(x)dx = \int_G \sup_{0 \leqslant \phi \leqslant \mathbf{1}_U} \phi(x)dx = \int_G \mathbf{1}_U(x)dx\end{aligned}$$

[1] Recall for a continuous function with valued in $\mathbb{R}$ and $\mathbb{C}$, we always equip the codomain with the Borel σ-algebra.

so that the case $A = U$ is proved. If $A = K$ is a compact set, let V be a precompact open neighborhood of K. Then $\mathbf{1}_K = \mathbf{1}_V - \mathbf{1}_{V\backslash K}$ so that this case is done. For general A of finite measure and given $n \in \mathbb{N}$, by regularity and Lemma D.2.1, there are compact K_n and open U_n with $K_n \subseteq A \subseteq U_n$ and $\mu(U_n\backslash K_n) < \frac{1}{n}$. We can assume the K_n are increasing and the U_n are decreasing. Let $g = \lim_{n\to\infty} \mathbf{1}^H_{K_n}$ and $h = \lim_{n\to\infty} \mathbf{1}^H_{U_n}$; then

- g, h are integrable on G/H;
- $0 \leqslant g \leqslant \mathbf{1}^H_A \leqslant h$;
- $h - g \geqslant 0$ has integral zero.

Hence

$$\int_{G/H} \mathbf{1}^H_A dx = \int_{G/H} g(x)dx = \lim_{n\to\infty} \int_{G/H} \mathbf{1}^H_{K_n}(x)dx = \lim_{n\to\infty} \int_G \mathbf{1}_{K_n}(x)dx = \int_G \mathbf{1}_A(x)dx$$

□

Corollary 2.4.8.1. Suppose $H \leqslant G$ is a closed subgroup such that there exists a nonzero invariant Radon measure on G/H. Let $f : G \to \mathbb{C}$ be a measurable function such that $A = \{x \in G \mid f(x) \neq 0\}$ is σ-finite. If the iterated integral

$$\int_{G/H} \int_H |f(xh)|dhdx < \infty$$

exists, then $f \in L^1(G)$.

Proof. It suffices to show $|f| \in L^1(G)$. Choose an increasing sequence of measurable sets $(A_n)_{n\in\mathbb{N}}$ with finite measure whose union is A, and define $f_n : G \to \mathbb{C}$ by $f_n = \min\{|f| \cdot \mathbf{1}_{A_n}, n\}$. Then $(f_n)_{n\in\mathbb{N}}$ is an increasing sequence of integrable functions that converges pointwise to $|f|$. Then by Theorem

$$\int_G f_n(x)dx = \int_{G/H} \int_H f_n(xh)dhdx \leqslant \int_{G/H} \int_H |f_n(xh)|dhdx < \infty$$

The result follows from monotone convergence theorem. □

Corollary 2.4.8.2.

1. If $H \trianglelefteq G$ is a normal closed subgroup, then $\Delta_G|_H = \Delta_H$.
2. Let $H = \ker(\Delta_G : G \to \mathbb{R}_{>0})$ be the kernel. Then H is unimodular.

Proposition 2.4.9. Let $H \trianglelefteq G$ be a closed normal subgroup, and $\sigma : G \to G$ be an automorphism such that $\sigma(H) \subseteq H$ and $\sigma|_H : H \to H$ is an automorphism. Put $\overline{\sigma} : G/H \to G/H$ to be the induced automorphism. Then

$$\mathrm{mod}_G(\sigma) = \mathrm{mod}_H(\sigma|_H) \cdot \mathrm{mod}_{G/H}(\overline{\sigma})$$

Proof. Pick any Haar measure dg (resp. dh) on G (resp. on H), and denote by ν the resulting quotient measure on G/H. Then for $f \in C_c(G)$, we have

$$\int_G f(g)dg = \int_{G/H} \int_H f(hy)dhd\nu(y).$$

Replacing f with $f \circ \sigma^{-1}$, we obtain

$$\begin{aligned}\mathrm{mod}_G(\sigma) \int_G f(g)dg &= \int_{G/H} \int_H f(\sigma|_H^{-1}(h)\overline{\sigma}^{-1}(y))dhd\nu(y) \\ &= \mathrm{mod}_H(\sigma|_H) \cdot \mathrm{mod}_{G/H}(\overline{\sigma}) \int_{G/H} \int_H f(hy)dhd\nu(y) = \mathrm{mod}_H(\sigma|_H) \cdot \mathrm{mod}_{G/H}(\overline{\sigma}) \int_G f(g)dg.\end{aligned}$$

This finishes the proof. □

Proposition 2.4.10. Let G be an LCH group, $K \leqslant G$ a compact subgroup and $H \leqslant G$ a closed subgroup such that $G = HK$. Then one can arrange the Haar measure on $G,$, H, K in a way that for every $f \in L^1(G)$ one has

$$\int_G f(x)dx = \int_H \int_K f(hk)dkdh$$

Proof. The group $H \times K$ acts on G by $(h,k).g = hgk^{-1}$. This operation is transitive so that we have the identification $G = \dfrac{H \times K}{H \cap K}$, where we embed $H \cap K$ into $H \times K$ diagonally. Now

- the group $H \cap K$ is compact, so its modular character is trivial.
- The modular character of $H \times K$ is trivial on $H \cap K$.

By Theorem, there is a unique $H \times K$-invariant Radon measure on G up to scaling. We claim the Haar measure on G is also $H \times K$-invariant. Indeed, the Haar measure on G is H-invariant for H acts as left-translation. Since K is compact, we have $\Delta_G|_K \equiv 1$ so that

$$\int_G f(xk)dx = \Delta_G(k^{-1}) \int_G f(x)dx = \int_G f(x)dx$$

for all $k \in K$, $f \in C_c(G)$ by Theorem 2.3.1.3.

Now by uniqueness of the Haar measure, the Haar measure on G can be normalized in the way that the quotient integral formula

$$\int_H \int_K f(hk)dkdh = \int_{H \times K} f(hk)dh \otimes dk = \int_G \int_{H \cap K} f(x\ell)d\ell dx$$

holds for $f \in L^1(G)$. Finally, we see

$$\int_G \int_{H \cap K} f(x\ell)d\ell dx = \int_{H \cap K} \int_G f(x\ell)dxd\ell = \int_{H \cap K} \left(\Delta_G(\ell^{-1}) \int_G f(x)dx\right) d\ell = \int_G f(x)dx$$

□

2.4.2 Examples

Example 2.4.11. Consider the euclidean space $\mathbb{R}^n$ and equip it with the usual Lebesgue measure. The set of integer points $\mathbb{Z}^n$ is a discrete closed subgroup of $\mathbb{R}^n$, so we can choose the counting measure as its Haar measure and form the quotient measure μ on $\mathbb{R}^n/\mathbb{Z}^n$. By definition, for $f \in C_c(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} f(x)dx = \int_{\mathbb{R}^n/\mathbb{Z}^n} \sum_{v \in \mathbb{Z}^n} f(x+v)d\mu(x).$$

Take f to be the characteristic function of the closed unit cube C; then

$$1 = \int_{\mathbb{R}^n} \mathbf{1}_C(x)dx = \int_{\mathbb{R}^n/\mathbb{Z}^n} \#\{v \in \mathbb{Z}^n \mid v + x \in C\}d\mu(x)$$

The number $\#\{v \in \mathbb{Z}^n \mid v + x \in C\}$ is 1 if x represents a point of the open unit cube int C, and is $\geqslant 2$ if it represents a point of the boundary ∂C. Let us denote the image of ∂C in the quotient by S. Then

$$0 = \int_{\mathbb{R}^n} \mathbf{1}_{\partial C}(x)dx = \int_{\mathbb{R}^n/\mathbb{Z}^n} \#\{v \in \mathbb{Z}^n \mid v + x \in \partial C\}d\mu(x).$$

Note that $v + x \in \partial C$ for some $v \in \mathbb{Z}^n$ if and only if $x \in S$. Hence

$$0 = \int_S \#\{v \in \mathbb{Z}^n \mid v + x \in \partial C\}d\mu(x) \geqslant \mu(S) \geqslant 0$$

so that $\mu(S) = 0$. These altogether show that $\mathrm{vol}(\mathbb{R}^n/\mathbb{Z}^n, \mu) = 1$.

Example 2.4.12 (Quotient by open subgroups)**.** Let G be an LCH group and $U \leqslant G$ an open subgroup. In particular, U is closed and $\Delta_G|_U = \Delta_U$, so we can consider the coset space G/U, which is a discrete set. If dg is a Haar measure on G, then its restriction to U also defines a Haar measure on U. We show the quotient measure dx on G/U is exactly the counting measure.

Since G/U is discrete, $C_c(G/U) = \bigoplus_{x\in G/U} \mathbb{C}\mathbf{1}_{xU}$. Let $p \in G$ and pick any $\phi \in C_c(G)^+$ such that $\phi(p) = 1$. Then $f := \mathbf{1}_{pU}\phi/\phi^U$ satisfies $f^U = \mathbf{1}_{pU}$. Hence

$$\begin{aligned}\mathrm{vol}(pU, dx) = \int_{G/U} \mathbf{1}_{pU}(x)dx &= \int_G f(g)dg \\ &= \int_G \frac{\mathbf{1}_{pU}(g)\phi(g)}{\phi^U(g)}dg = \int_G \frac{\mathbf{1}_U(g)\phi(pg)}{\phi^U(pg)}dg = \frac{1}{\phi^U(p)}\int_U \phi(pg)dg = 1.\end{aligned}$$

Hence $\mathrm{vol}(pU, dx) = 1$ for all $p \in G$, so dx is the counting measure on G/U.

Example 2.4.13. Let G be an LCH group and $H \leqslant G$ an open subgroup. Let $K_G \leqslant G$ be another closed subgroup, and put $K_H := K_G \cap H \leqslant K$. Let dg and dk be Haar measures on G and K_G, and suppose $\Delta_G|_{K_G} \equiv \Delta_{K_G}$ so that the quotient measure ν exists on G/K_G. The measures dg and dk restrict to those on H and K_H, respectively, and there exists the quotient measure μ on H/K_H. The inclusion $H \subseteq G$ induces an open embedding

$$\iota : H/K_H \longrightarrow G/K_G,$$

Our goal is to show that for all measurables $X \subseteq H/K_H$, we have

$$\mu(X) = \nu(\iota(X)) := \iota^*\nu(X).$$

i.e., ν pulls back to μ. Equivalently, for $f \in C_c(H/K_H)$, we must show

$$\int_{K/K_H} f d\mu = \int_{G/K_G} \iota_! f d\nu$$

where $\iota_! f \in C_c(G/K_G)$ is the extension of f to G/K_G by zero, which is well-defined as ι is an open map. For this, we must write down explicitly the extension by zero map

$$\iota_! : C_c(H/K_H) \longrightarrow C_c(G/K_G)$$

This is easy: for $f \in C_c(H/K_H)$ and $g \in G$, we have $\iota_! f(g) = f(h)$ if $h \in gK_G$, and $\iota_! f(g) = 0$ if $H \cap gK_G = \varnothing$. Now, for $f \in C_c(H/K_H)$, we must prove

$$\int_{H/K_H} f d\mu = \int_{H/K_H} f d(\iota^*\nu) := \int_{G/K_G} \iota_! f d\nu$$

By Lemma 2.4.2, it suffices to take $f = F^{K_H}$ with $F \in C_c(H)$. We can also view $F \in C_c(G)$, so we can consider F^{K_G}. For $g \in G$, if $gk' = h$ for some $k' \in K_G$, $h \in H$, then $\iota_! F^{K_H}(g) = F^{K_H}(h)$, and

$$F^{K_G}(g) = F^{K_G}(h(k')^{-1}) = F^{K_G}(h) = \int_{K_G} F(hk)dk.$$

But $\operatorname{supp} F \subseteq H$, so we only need to consider $hk \in H$, or $k \in H \cap K_G = K_H$. Hence

$$F^{K_G}(g) = \int_{K_H} F(hk)dk = F^{K_H}(h).$$

In sum, this shows $\iota_! F^{K_H} = F^{K_G}\mathbf{1}_{HK_G} = (F\mathbf{1}_{HK_G})^{K_G}$. Then

$$\int_{G/K_G} \iota_! F^{K_H} d\nu = \int_{G/K_G} (F\mathbf{1}_{HK_G})^{K_G} d\nu = \int_G F(g)\mathbf{1}_{HK_G}(g)dg = \int_H F(g)\mathbf{1}_{HK_G}(g)dg = \int_H F(g)dg$$

The second last holds as $\operatorname{supp} F \subseteq H$. This shows what we want. It is worth noting that in the course of the discussion we've shown the diagram

$$\begin{tikzcd} C_c(H) \arrow[r, "\text{extend by zero}"] \arrow[d] & C_c(G) \arrow[d] \\ C_c(H/K_H) \arrow[r, "\iota_!"] & C_c(G/K_G) \end{tikzcd}$$

is not commutative, but nevertheless $\iota_! F^{K_H} = (F\mathbf{1}_{HK_G})^{K_G}$.

Example 2.4.14. Let G be an LCH group and $H \trianglelefteq G$ a closed normal subgroup. Suppose K_G be a closed subgroup of G with $\Delta_G|_{K_G} \equiv \Delta_{K_G}$ such that the map

$$\iota : H/K_H \longrightarrow G/K_G$$

induced by the inclusion $H \subseteq G$ is a homeomorphism, where $K_H := H \cap K_G$. Note that $\Delta_H|_{K_H} = \Delta_{K_H}$ as well, since $K_H \trianglelefteq K_G$. Pick any Haar measure dg, dh, dk_G, dk_H on G, H, K_G, K_H, and denote by μ, ν the resulting quotient measure on G/K_G and H/K_H.

We wish to compare $\iota_*\nu$ with μ, or compare $\iota^*\mu$ with ν. An easy computation is not enough to show the G-invariance of $\iota_*\mu$. However, it is easy to check the pullback measure $\iota^*\mu$ is H-invariant. Indeed, for $h \in H$ and $X \subseteq H/K_H$,

$$\iota^*\mu(hX) = \mu(\iota(hX)) = \mu(h\iota(X)) = \mu(\iota(X)) = \iota^*\mu(X).$$

Hence, by Theorem 2.4.7.(ii) $\alpha\nu = \iota^*\mu$ for some $\alpha > 0$. But then $\alpha\iota_*\nu = \iota_*\iota^*\mu = \mu$, so $\iota_*\nu$ is also G-invariant.

Example 2.4.15. Let G be an LCH group and $H \trianglelefteq$ be a closed subgroup such that $\Delta_G|_H = \Delta_H$. Pick any Haar measures dg, dh on G, H respectively. For any $c > 0$, let μ_c be the quotient measure on G/H of cdg by dh on H. Then $\mu_c = c\mu_1$ for any $c > 0$.

Indeed, let $f \in C_c(G/H)$ and pick any $F \in C_c(G)$ such that $F^H = f$. Then by quotient integral formula we have

$$\int_{G/H} f d\mu_c = \int F c dg = c\int_F dg = c\int_{G/H} f d\mu_1.$$

Since $f \in C_c(G/H)$ is arbitrary, this shows $\mu_c = c\mu_1$.

2.4.3 Eigenmeasure

Let G be an LCH group. Instead of considering left invariant Radon measures on G, for a continuous group homomorphism $\chi : G \to \mathbb{R}_{>0}$, let us consider the "$\chi$-eigenmeasure" μ, in the sense that $\mu(xA) = \chi(x)\mu(A)$ for all $x \in G$ and measurable $A \subseteq G$.

Lemma 2.4.16. For each a continuous group homomorphism $\chi : G \to \mathbb{R}_{>0}$, there exists a unique χ-eigen Radon measure, up to a positive scalar, on G.

Proof. For a function f on G, denote by $f\chi$ the function on G defined by $f\chi(x) := f(x)\chi(x)$. Then we have a linear isomorphism

$$\begin{aligned} \mathrm{Hom}_{\mathbb{C}}(C_c(G), \mathbb{C})^G &\longrightarrow \mathrm{Hom}_{\mathbb{C}}(C_c(G), \mathbb{C})^{\chi^{-1}} \\ \Lambda &\longmapsto \Lambda_\chi : f \mapsto \Lambda(f\chi) \end{aligned}$$

that sends positive to positive. The lemma now follows from Lemma 2.4.1 and the uniqueness of Haar measures. ☐

Theorem 2.4.17. Let $H \leqslant G$ be a closed subgroup and $\chi : G \to \mathbb{R}_{>0}$ a continuous group homomorphism. There exists a χ-eigen Radon measure on G/H if and only if $\chi(h)\Delta_G(h) = \Delta_H(h)$ for all $h \in H$.

Proof. Suppose μ is a χ-eigen Radon measure on G/H. For $f \in C_c(G)$, define $I : C_c(G) \to \mathbb{C}$ by

$$I(f) = \int_{G/H} (f\chi^{-1})^H d\mu.$$

This is a positive linear function on $C_c(G)$, and for $g \in G$,

$$g.I(f) = \int_{G/H}\int_H f(gxh)\chi^{-1}(xh)dhd\mu(x) = \chi(g)\int_{G/H} g^{-1}.(f\chi)^H d\mu = \chi(g)\chi^{-1}(g)I(f) = I(f).$$

Thus $I(f)$ is given by a Haar measure ν on G, i.e.,

$$\int_{G/H}\int_H f(xh)\chi^{-1}(xh)dhd\mu(x) = \int_G f(x)d\nu(x)$$

for all $f \in C_c(G)$. Now for $h_0 \in H$, one gets

$$\begin{aligned} \Delta_G(h_0)\int_G f(x)d\nu(x) &\overset{2.3.1.3}{=} \int_G f(xh_0^{-1})d\nu(x) = \int_G R_{h_0^{-1}} f(x)dx = \int_{G/H}\int_H R_{h_0^{-1}} f(xh)\chi^{-1}(xh)dhd\mu(x) \\ &\overset{2.3.1.3}{=} \chi^{-1}(h_0)\Delta_H(h_0)\int_{G/H}\int_H f(xh)dhd\mu(x) = \chi^{-1}(h_0)\Delta_H(h_0)\int_G f(x)d\nu(x) \end{aligned}$$

This shows $\chi(h_0)\Delta_G(h_0) = \Delta_H(h_0)$. The converse is proved in the same way as the proof of Theorem 2.4.4 with slight modification. ☐

2.4.4 Double coset spaces

Let G be an LCH group and $H, K \leqslant G$ be closed subgroups. Form the double coset space $H\backslash G/K$, denote by $\pi : G \to H\backslash G/K$ the canonical projection. Equip $H\backslash G/K$ with the final topology induced by π; then π is continuous and open.

Lemma 2.4.18.

(i) The final topology on $H\backslash G/K$ induced by $H\backslash G \to H\backslash G/K$ (resp. $G/K \to H\backslash G/K$) coincides with that of induced by π.

(ii) The topological space $H\backslash G/K$ is LCH.

Proof. Let $p : H\backslash G \to H\backslash G/K$ and $q : G \to H\backslash G$ be the canonical projections.

(i) For a set $U \subseteq H\backslash G/K$, if $p^{-1}(U)$ is open, then $\pi^{-1}(U) = q^{-1}p^{-1}(U)$ is open. Conversely, since q is surjective, we easily see that $q(\pi^{-1}(U)) = p^{-1}(U)$. In particular, if $\pi^{-1}(U)$ is open, so is $p^{-1}(U)$, as q is an open map. This proves the first statement of (i), and the other is proved similarly.

(ii) Put $X = H\backslash G$, which is LCH. Let $x \in X$ and U a compact neighborhood of x in X. Then $q(U)$ is a compact neighborhood of xK in X/K. To show X/K is Hausdorff, let $x, y \in X$ with $xK \neq yK$. Since K is closed, $yK \subseteq X$ is a closed set not containing x. Hence we can find opens $U, V \subseteq X$ such that $x \in U \subseteq X\backslash yK$ and $y \in V \subseteq X\backslash xK$ with $U \cap V = \varnothing$. Then $q(U) \cap q(V) = \varnothing$.

$\square$

Lemma 2.4.19. Suppose K is compact. For $\varepsilon > 0$ and $f \in C_c(G/K)$, there exists a unit-neighborhood U in G such that $|f(gx) - f(x)| < \varepsilon$ for all $x \in G/K$ and $g \in U$.

Proof. By Lemma 2.4.2, choose any $F \in C_c(G)$ with $F^K = f$. By uniform continuity of F there exists a compact unit-neighborhood U in G such that $|F(x) - F(y)| < \dfrac{\varepsilon}{\mathrm{vol}(K)}$ as long as $xy^{-1} \in U$. Then for $g \in U$, since $(gxk)(xk)^{-1} = g \in U$, we have

$$|F^K(gx) - F^K(x)| \leqslant \int_K |F(gxk) - F(xk)| dk < \mathrm{vol}(K) \cdot \frac{\varepsilon}{\mathrm{vol}(K)} = \varepsilon.$$

$\square$

Let $N = N_G(K)$ be the normalizer of K in G. Then N acts on $H\backslash G/K$ by

$$n.(HgK) := HgKn = HgnK,$$

and this makes $H\backslash G/K$ a right N-space. We ask the existence of right N-invariant Radon measure on $H\backslash G/K$. We begin by establishing an analogue of Lemma 2.4.2. For $f \in C_c(G)$ define ${}^H f^K : G \to \mathbb{C}$ by

$${}^H f^K(g) := \int_H \int_K f(hgk) dk dh = \int_H f^K(hg) dh = {}^H(f^K)(g)$$

In contrast to what we've done before, here respective measures dg, dh, dk on G, H, K are taken to be right Haar measures. We further assume K is *compact*, so dk is also left-invariant. The integral is well-defined as $g \mapsto f^K(g)$ has compact support. Moreover,

Lemma 2.4.20. ${}^H f^K$ defines a continuous function on $H\backslash G/K$ with compact support.

Proof. Put $C = \operatorname{supp} f$. For $(h, k) \in H \times K$, the support of $L_{h^{-1}} R_k f : g \mapsto f(hgk)$ is $h^{-1}Ck^{-1}$. Hence, if $g \notin HCK$, then $f(hgk) = 0$ for all $(h, k) \in H \times K$. The assumption of K being compact implies HCK is a closed set in G, so $\operatorname{supp} {}^H f^K \subseteq HCK$, and ${}^H f^K : H\backslash G/K \to \mathbb{C}$ has compact support. To show it is continuous, by Fubini we have

$${}^H f^K(g) = \int_K \int_H f(hgk) dh dk = \int_K {}^H f(gk) dk.$$

Since ${}^H f$ is continuous, it follows that ${}^H f^K$ is also continuous. $\square$

Lemma 2.4.21. Retain the above setting.

(i) For each compact set $C \subseteq H\backslash G/K$, there exists a compact set $C' \subseteq G$ such that $\pi(C') = C$..

(ii) The map defined by

$$\begin{aligned} C_c(G) &\longrightarrow C_c(H\backslash G/K) \\ f &\longmapsto {}^H f^G \end{aligned}$$

is surjective, and the fibre of $h \in C_c^+(H\backslash G/K)$ meets $C_c^+(G)$ nontrivially.

Proof.

(i) For each $c \in C$, pick $y \in \pi^{-1}(c)$ and a compact neighborhood U_c of y in G. Since π is open, $\{\pi(U_c) \mid c \in C\}$ forms an open cover of C, so there exists a finite set $F \subseteq C$ such that $C \subseteq \bigcup_{c\in F} \pi(U_c)$. Then $C' := \pi^{-1}(C) \cap \bigcup_{c\in F} U_c$ satisfies $\pi(C') = C$.

(ii) Let $F \in C_c(H\backslash G/K)\backslash\{0\}$ and set $C = \operatorname{supp} F$. By (i) we can find compact $C' \subseteq G$ such that $\pi(C') = C$. By Urysohn's Lemma we can find $\phi \in C_c(G)^+$ such that $\phi|_{C'} \equiv 1$. Then $f := \dfrac{(F\circ\pi)\phi}{{}^H\phi^K}$ satisfies ${}^H f^K = F$. The last assertion is clear.

□

For $n \in N$, the conjugation $\operatorname{Inn}_n : k \mapsto nkn^{-1}$ defines an automorphism on K, so we can consider its modulus $\operatorname{mod}_K(\operatorname{Inn}_n)$. Since K is compact, it satisfies $\operatorname{vol}(K, dk) = \operatorname{vol}(\operatorname{Inn}_n K, dk) = \operatorname{mod}_K(\operatorname{Inn}_n)\operatorname{vol}(K, dk)$. Hence $\operatorname{mod}_K(\operatorname{Inn}_n) = 1$, and

$$\int_K f(nkn^{-1})dk = \int_G f(k)dk$$

for all $k \in C(K)$.

Now suppose $H\backslash G/K$ admits a nonzero right N-invariant Radon measure μ. Define $I = I_\mu : C_c(G) \to \mathbb{C}$ by

$$I(f) = \int_{H\backslash G/K} {}^H f^K d\mu.$$

Since I is nontrivial and positive, so it is given by a positive Radon measure ν on G, i.e.,

$$\int_G f(x)d\mu(x) = \int_{H\backslash G/K}\int_H\int_K f(hxk)dkdhd\mu(x).$$

For $n \in N$, since μ is right N-invariant, one has

$$\begin{aligned} \int_G f(xn)d\nu(x) &= \int_{H\backslash G/K}\int_H\int_K f(hxkn)dkdhd\mu(x) = \int_{H\backslash G/K}\int_H\int_K f(hxnk)dkdhd\mu(x) \\ &= \int_{H\backslash G/K}\int_H\int_K f(hxk)dkdhd\mu(x) = \int_G f(x)d\nu(x). \end{aligned}$$

For $h' \in H$,

$$\begin{aligned} \int_G f(hx)d\nu(x) &= \int_{H\backslash G/K}\int_H\int_K f(h'hxk)dkdhd\mu(x) = \operatorname{mod}_H(\operatorname{Inn}(h')^{-1})\int_{H\backslash G/K}\int_H\int_K f(hh'xk)dkdhd\mu(x) \\ &= \operatorname{mod}_H(\operatorname{Inn}(h')^{-1})\int_{H\backslash G/K}\int_H\int_K f(hxk)dkdhd\mu(x) = \operatorname{mod}_H(\operatorname{Inn}(h')^{-1})\int_G f(x)d\nu(x). \end{aligned}$$

This proves the only if part of the following

Theorem 2.4.22. Let G be an LCH group and $H, K \leqslant G$ be closed subgroups with K compact. Put $N = N_G(K)$. The double coset space $H\backslash G/K$ admits a nonzero right N-invariant Radon measure if and only if G admits a Radon measure ν satisfying

(a) $\displaystyle\int_G f(xn)d\nu(x) = \int_G f(x)d\nu(x)$ for all $n \in N$, and

(b) $\displaystyle\int_G f(hx)d\nu(x) = \mathrm{mod}_H(\mathrm{Inn}(h)^{-1}) \int_G f(x)d\nu(x)$ for all $h' \in H$.

Proof. Let ν be a Radon measure on G satisfying (a) and (b). Define $I : C_c(H\backslash G/K) \to \mathbb{C}$ by

$$I(F) = \int_G f d\nu$$

where $f \in C_c(G)$ satisfies ${}^H f^K = F$. To show it is well-defined, let $f \in C_c(G)$ be such that ${}^H f^K = 0$. For any $\phi \in C_c(G)$, compute

$$\begin{aligned} 0 &= \int_G {}^H f^K \phi d\nu = \int_G \int_H \int_K f(hxk)\phi(x)dkdhd\mu(x) \overset{\text{Fubini}}{=} \int_H \int_K \int_G f(hxk)\phi(x)d\mu(x)dkdh \\ &\overset{\text{(a),(b)}}{=} \int_H \int_K \left(\mathrm{mod}_H(\mathrm{Inn}(h)^{-1}) \int_G f(x)\phi(h^{-1}xk^{-1})d\nu(x)\right) dkdh \\ &\overset{\text{Fubini}}{=} \int_G f(x) \left(\int_K \int_H \mathrm{mod}_H(\mathrm{Inn}(h)^{-1})\phi(h^{-1}xk^{-1})dhdk\right) d\nu(x) \\ &\overset{(2.3.1)}{=} \int_G f(x) \left(\int_K \int_H \phi(hxk)dhdk\right) d\nu(x) = \int_G f({}^H\phi^K)d\nu. \end{aligned}$$

Choosing $\phi \in C_c(G)$ such that ${}^H\phi^K \equiv 1$ shows $\displaystyle\int_G f\nu = 0$. By (a), I is right N-invariant, so by Riesz's representation theorem and Lemma 2.4.1, it defines a right N-invariant Radon measure on $H\backslash G/K$. $\square$

Example 2.4.23. Suppose further that G is unimodular and H is discrete. Then any Haar measure dg on G satisfies the conditions in Theorem 2.4.22. Hence there exists a unique right $N_G(K)$-invariant Radon measure ν on $H\backslash G/K$ satisfying

$$\int_G f(g)dg = \int_{H\backslash G/K} \sum_{\gamma \in H} \int_K f(\gamma xk)dkd\nu(x)$$

where dk is the Haar measure on K normalized so that $\mathrm{vol}(K, dk) = 1$. On the other hand, there exists a unique measure μ on G/K such that

$$\int_G f(g)dg = \int_{G/K} \int_K f(yk)dkd\mu(y).$$

Comparing two expressions, since $C_c(G) \to C_c(G/K)$ is surjective (2.4.2), we see the integral formula

$$\int_{G/K} f(y)d\mu(y) = \sum_{\gamma \in H} \int_{H\backslash G/K} f(\gamma x)d\nu(x)$$

holds for all $f \in C_c(G/K)$. Similarly, if we denote by ρ the quotient measure on $H\backslash G$, we have

$$\int_{H\backslash G} f(y)d\rho(y) = \int_{H\backslash G/K} \int_K f(xk)dkd\nu(x)$$

for all $f \in C_c(H\backslash G)$.

Example 2.4.24. Let G_i $(i = 1, 2)$ be unimodular and $K_i \leqslant G_i$ be a compact subgroup. Suppose further that $K_2 \leqslant G_2$ is open so that G_2/K_2 is discrete. Consider the inclusion $G_1 \times G_1 \times G_2$ given by $g \mapsto (g, e)$. Suppose Λ is an LCH group such that it can be viewed as subgroups of G_1, G_2; in particular, there is an inclusion $\Lambda \to G_1 \times G_2$. Assume its image in $G_1 \times G_2$ is discrete. Now consider the quotient

$$G_1/K_1 \xrightarrow{\quad\varphi\quad} G_1/K_1 \times G_2/K_2 \xrightarrow{\qquad} \Lambda\backslash G_1 \times G_2/K_1 \times K_2$$

The first arrow φ is open, and is measure preserving as long as G_1/K_1 on both sides use the same measure and G_2/K_2 uses counting measure.

If gK_1 and $g'K_1$ have the same image in $\Lambda\backslash G_1 \times G_2/K_1 \times K_2$, then $\gamma(gk_1, k_2) = (g', 1)$ for some appropriate elements, so $\gamma k_2 = 1$ and $\gamma g k_1 = g'$. The first relation says $\gamma \in \Lambda \cap K_2$, intersection taken place in G_2, and the second then implies that $g' \in (\Lambda \cap K_2)gK_1$, where we view $\Lambda \cap K_2$ as a subgroup of Λ and put it into G_1. Hence we obtain a well-defined map

$$\widetilde{\varphi} : \Lambda'\backslash G_1/K_1 \xrightarrow{\qquad} \Lambda\backslash G_1 \times G_2/K_1 \times K_2.$$

with $\Lambda' = \Lambda \cap K_2 \leqslant G_1$. To continue the discussion, we first show that $\Lambda' \leqslant G_1$ is discrete. Since $\Lambda \leqslant G_1 \times G_2$ is discrete, for $\gamma \in \Lambda \cap K_2$ and any compact neighborhood U_1 of γ in G_1, the set

$$\Lambda \cap (U_1 \times K_2) = \Lambda \cap ((\Lambda \cap U_1) \times (\Lambda \cap K_2))$$

is finite. In particular, $\Lambda' \cap U_1 \subseteq G_1$ is finite. Since G_1 is Hausdorff, by shrinking U_1, we then have $\Lambda' \cap U_1 = \{\gamma\}$.

Our goal is to show $\widetilde{\varphi}$ preserves measure. Note that this question makes sense since $\widetilde{\varphi}$ is an open embedding. Let dg_i be any Haar measures on G_i, and dk_i be Haar measures on K_i such that $dg_2|_{K_2} = dk_2$. On Λ and Λ' we take counting measures. What we want to show is the commutativity of the following diagram

$$\begin{tikzcd}
C_c(\Lambda'\backslash G_1/K_1) \arrow[r, "\widetilde{\varphi}_!"] \arrow[d, "\int dy"'] & C_c(\Lambda\backslash G_1 \times G_2/K_1 \times K_2) \arrow[dl, "\int dx"] \\
\mathbb{C} &
\end{tikzcd}$$

where we denote by dx (resp. dy) the double quotient measures. Consider the diagram

$$\begin{tikzcd}
C_c(G_1/K_1) \arrow[r, "\varphi_!"] \arrow[d] & C_c(G_1 \times G_2/K_1 \times K_2) \arrow[d] \\
C_c(\Lambda'\backslash G_1/K_1) \arrow[r, "\widetilde{\varphi}_!"] & C_c(\Lambda\backslash G_1 \times G_2/K_1 \times K_2)
\end{tikzcd}$$

as in Example 2.4.13, where the vertical arrows are "integration along fibre" as usual. Here for $f \in C_c(\Lambda'\backslash G_1/K_1)$, it is easy to see that $\widetilde{\varphi}_! f(g_1, g_2) = f(g_1)$ if $\Lambda(g_1, g_2)(K_1 \times K_2) \cap G_1 \times \{e\} \neq \varnothing$, and $\widetilde{\varphi}_! f(g_1, g_2) = 0$ otherwise. For $f \in C_c(G_1/K_1)$ and $(g_1, g_2) \in G_1 \times K_2$, one has ${}^{\Lambda}(\varphi_! f)(g_1, g_2) = \widetilde{\varphi}_!({}^{\Lambda'} f)(g_1, g_2)$

$${}^{\Lambda}(\varphi_! f)(g_1, g_2) = \sum_{\gamma \in \Lambda} f(\gamma g_1)\mathbf{1}_{eK_2}(\gamma g_2) = \sum_{\gamma \in \Lambda \cap K_2} f(\gamma g_1) = {}^{\Lambda'} f(g_1).$$

Hence $\widetilde{\varphi}_!({}^{\Lambda'}f) = {}^{\Lambda}(\varphi_! f)\mathbf{1}_{\Lambda(G_1\times\{e\})(K_1\times K_2)} = {}^{\Lambda}(\varphi_! f\mathbf{1}_{\Lambda(G_1\times\{e\})(K_1\times K_2)})$. Hence for $f \in C_c(G_1/K_1)$

$$\begin{aligned}\int_{\Lambda\backslash G_1\times G_2/K_1\times K_2} \widetilde{\varphi}_!({}^{\Lambda'}f)dx &= \int_{G_1\times G_2/K_1\times K_2} \varphi_! f\mathbf{1}_{\Lambda(G_1\times\{e\})(K_1\times K_2)} \\ &= \int_{G_1/K_1} f\mathbf{1}_{\Lambda(G_1\times\{e\})(K_1\times K_2)}|_{G_1\times\{e\}} = \int_{G_1/K_1} f = \int_{\Lambda'\backslash G_1/K_1} {}^{\Lambda'}f dy\end{aligned}$$

Here we repeatedly use formulas in (2.4.23). Since $C_c(G_1/K_1) \to C_c(\Lambda'\backslash G_1/K_1)$ is surjective (2.4.2), this shows the commutativity of the triangle depicted above.

2.4.5 Homomorphisms and measures

Let X, Y be two measurable spaces and μ a measure on X. For a measurable function $f : X \to Y$, we can consider the **pushforward measure** $f_*\mu$, which is a measure on Y defined as follows. If $A \subseteq X$ is any measurable set in X, then

$$f_*\mu(A) := \mu(f^{-1}(A)).$$

To see this is indeed a measure on Y, if $(A_i)_{i=1}^n$ is a sequence of measurable sets in X, then

$$f_*\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(f^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right)\right) = \mu\left(\bigcup_{i=1}^{\infty} f^{-1}(A_i)\right) \leqslant \sum_{n=1}^{\infty}\mu(f^{-1}(A_i)) = \sum_{n=1}^{\infty} f_*\mu(A_i).$$

If the A_i's are disjoint, then the equality holds. Also, we have $f_*\mu(\varnothing) = \mu(\varnothing) = 0$. This proves $f_*\mu$ is a measure on Y.

Lemma 2.4.25. Retain the above notation. For any measurable function $g : Y \to \mathbb{C}$, the integral formula holds:

$$\int_Y g d(f_*\mu) = \int_X (g \circ f) d\mu.$$

Proof. By MCT, it suffices to check the case when $g = \mathbf{1}_A$ with A a measurable set. This is clear, as

$$\int_Y \mathbf{1}_A d(f_*\mu) = \mathrm{vol}(A, f_*\mu) = \mathrm{vol}(f^{-1}(A), \mu) = \int_X \mathbf{1}_{f^{-1}(A)} d\mu = \int_X (\mathbf{1}_A \circ f) d\mu.$$

□

Lemma 2.4.26. Let X, Y be LCH spaces and $f : X \to Y$ a proper continuous map. If μ is a Radon measure on X, then the pushforward measure $f_*\mu$ is a Radon measure on Y.

Proof. Local finiteness is clear (the properness is used here). For U open in Y,

$$f_*\mu(U) = \mu(f^{-1}(U)) = \sup_{K \underset{\text{cpt}}{\subseteq} f^{-1}(U)} \mu(K) \leqslant \sup_{K \underset{\text{cpt}}{\subseteq} U} \mu(f^{-1}(K)) \leqslant \mu(f^{-1}(U)) = f_*\mu(U).$$

so $f_*\mu$ is weakly inner regular (the properness is not used here). Finally, for B Borel in Y, by Corollary A.7.2.1 we have

$$f_*\mu(B) = \mu(f^{-1}(B)) = \inf_{f^{-1}(B)\subseteq U \underset{\text{open}}{\subseteq} X} \mu(U) \geqslant \inf_{B\subseteq V \underset{\text{open}}{\subseteq} U} \mu(f^{-1}(V)) \geqslant \mu(f^{-1}(B)) = f_*\mu(B)$$

so $f_*\mu$ is outer regular. □

Lemma 2.4.27. Let G and H be LCH groups and $\varphi : G \to H$ a proper continuous homomorphism. If μ is a left Haar measure on G, then $\varphi_*\mu$ is a left Haar measure on H.

Proof. By the previous lemma, it remains to deduce the left invariance. Let $A \subseteq H$ be a measurable subset and $h \in H$. Let g be in the fibre of h. Then

$$\varphi_*\mu(hA) = \mu(\varphi^{-1}(hA)) = \mu(g\varphi^{-1}(A)) = \mu(\varphi^{-1}(A)) = \varphi_*\mu(A)$$

Thus $\varphi_*\mu$ is a Haar measure on H. □

Lemma 2.4.28. Let G and H be LCH groups and $\varphi : G \to H$ a surjective open continuous homomorphism with $\ker\varphi$ compact. Then φ is proper.

Proof. Under our assumption we have $G/\ker\varphi \cong H$ as topological groups, so it suffices to show $\pi : G \to G/\ker\varphi$ is proper. Let $C \subseteq G/\ker\varphi$ be a compact subset. By Lemma 2.4.2 we can find compact $K \subseteq G$ with $\pi(K) = C$. We claim $\pi^{-1}(C) = K\ker\varphi$.

- As $\pi(K\ker\varphi) = \pi(K) = C$, we have $K\ker\varphi \subseteq \pi^{-1}(C)$.
- Let $x \in \pi^{-1}(C)$. Then $\pi(x) \in C$, so we can find $k \in K$ such that $x\ker\varphi = k\ker\varphi$, or $k^{-1}x \in \ker\varphi$. Thus $x \in k\ker\varphi \subseteq K\ker\varphi$, so that $\pi^{-1}(C) \subseteq K\ker\varphi$.

Since $\ker\varphi$ is compact, being a product of two compact sets, $\pi^{-1}(C)$ is compact. Thus φ is a proper map. □

Remark 2.4.29. If G is σ-compact, then a continuous surjective homomorphism is automatically open. See Theorem 6.2.1.

Proposition 2.4.30. Let G and H be LCH groups and $\varphi : G \to H$ a surjective open continuous homomorphism with $\ker\varphi$ compact. Suppose A, B are two subset of H with finite measure, then

$$\frac{\mathrm{vol}(A)}{\mathrm{vol}(B)} = \frac{\mathrm{vol}(\varphi^{-1}(A)}{\mathrm{vol}(\varphi^{-1}(B))}.$$

Proof. This follows from previous lemmas. □

2.4.6 Quasi-invariant measures

Let G be an LCH group and $H \leqslant G$ a closed subgroup. Let μ be a Radon measure on the homogeneous space G/H. For $g \in G$ define the translate μ_g of μ by the formula $\mu_g(A) := \mu(gA)$.

Definition. Let the notations be as in above.

(i) μ is called **quasi-invariant** if all the measures μ_x have the σ-ideal of zero sets; equivalently, μ is quasi-invariant if the μ_x are mutually absolutely continuous.

(ii) μ is called **strongly quasi-invariant** if there exists a continuous map $\lambda : G \times G/H \to \mathbb{R}_{>0}$ such that $d\mu_x(p) = \lambda(x,p)d\mu(p)$ for all $x \in G$ and $p \in G/H$; equivalently, μ is strongly quasi-invariant if $\mu$is quasi-invariant and the Radon-Nikodym derivatives $(d\mu_x/d\mu)(p)$ is jointly continuous in (x,p).

Lemma 2.4.31. If μ is a nonzero quasi-invariant measure on G/H, then $\mu(U) > 0$ for every nonempty open set U.

Proof. Suppose otherwise. Say $\mu(U) = 0$ for some open set U. Then $\mu(gU) = 0$ for all $g \in G$ as μ_g and μ are mutually absolutely convergent. By weakly inner regularity it suffices to show μ vanishes on every compact set. But this is clear. $\square$

Lemma 2.4.32. Let V be a relatively compact symmetric open unit-neighborhood of G. Then there exists a set $A \subseteq G$ such that

(i) for every $g \in G$ there exists $a \in A$ with $gH \cap Va \neq \varnothing$, and

(ii) if $K \subseteq G$ is compact, $\#\{a \in A \mid KH \cap \overline{V}a \neq \varnothing\} < \infty$.

Proof. By Zorn's lemma there exists a maximal set $A \subseteq G$ such that if $a, b \in A$ then $a \neq VbH$ (note that this condition is symmetric in a and b). For any $g \in G$, gH intersects some Va, for otherwise $g \notin VaH$ for all $a \in A$, contradicting the maximality. Also, if $K \subseteq G$ is compact and $KH \cap \overline{V}a \neq \varnothing$ for infinitely many a, then we can find $(a_n)_n$ with each $a_n \in A$ distinct to each other and $(h_n)_n$ in H such that $a_n h_n \in \overline{V}K$ for all n. Since $\overline{V}K$ is compact, by passing to subsequence we can assume $a_n h_n$ converges to, say, $z \in \overline{V}K$. Pick a symmetric unit-neighborhood W in G such that $W^2 \subseteq V$. Then there are integers $n \neq m$ such that $a_n h_n$, $a_m h_m \in Wz$, and thus $a_n h_n \in V a_m h_m$. But then $a_n \in V a_m H$, a contradiction to the definition of A. $\square$

Lemma 2.4.33. There exists a continuous function $f : G \to [0, \infty)$ satisfying

(i) $\{y \in G \mid f(y) > 0\} \cap gH \neq \varnothing$ for all $g \in G$, and

(ii) $\operatorname{supp} f \cap KH$ is compact for every compact $K \subseteq G$.

Proof. Pick symmetric $h \in C_c^+(G)$ with $g(1) > 0$, let $V = h^{-1}(0, \infty)$, choose $A \subseteq G$ as in Lemma 2.4.32 for this V and set

$$f(g) = \sum_{a \in A} h(ga^{-1}).$$

By Lemma 2.4.32.(ii), this sum is finite when g is in a fixed compact subset of G, so f defines a continuous map on G. Since

$$\operatorname{supp} f = \overline{\bigcup_{a \in A} Va} \subseteq \bigcup_{a \in A} \overline{V}a$$

for any compact $K \subseteq G$, $\operatorname{supp} f \cap KH$ is contained in a finite union of the $\overline{V}a$ by Lemma 2.4.32.(ii) again, so $\operatorname{supp} f \cap KH$ is compact. Finally, by Lemma 2.4.32.(i), $\{y \in G \mid f(y) > 0\} = \bigcup_{a \in A} Va$ intersects every coset gH $(g \in G)$. $\square$

Definition. A **rho-function** for the pair (G, H) is a continuous map $\rho : G \to \mathbb{R}_{>0}$ such that

$$\rho(gh) = \frac{\Delta_H(h)}{\Delta_G(h)} \rho(g)$$

for all $g \in G$, $h \in H$.

Proposition 2.4.34. For any LCH group G and any closed subgroup $H \leqslant G$, (G, H) admits a rho-function.

Proof. Let f be as in Lemma 2.4.33 and set

$$\rho(g) := \int_H \frac{\Delta_G(h)}{\Delta_H(h)} f(gh)dh.$$

By Lemma 2.4.33.(ii), this integral converges for each $g \in G$ and defines a continuous function on G, and by (i) it is positive. Moreover,

$$\rho(gh_0) = \int_H \frac{\Delta_G(h)}{\Delta_H(h)} f(gh_0h)dh = \int_H \frac{\Delta_G(h_0^{-1}h)}{\Delta_H(h_0^{-1}h)} f(gh)dh = \frac{\Delta_H(h_0)}{\Delta_G(h_0)}\rho(g).$$

□

Lemma 2.4.35. If $f \in C_c(G)$ and $f^H = 0$, then the integral $\int_G f(g)\rho(g)dg = 0$ for every rho-function ρ.

Proof. By Lemma 2.4.2, we can find $\phi \in C_c^+(G)$ with $\phi^H = 1$ on $\pi(\operatorname{supp} f)$. Then

$$\begin{aligned} 0 &= \int_G \int_H \rho(g)\phi(g)f(gh^{-1})\Delta_H(h^{-1})dhdg \overset{2.3.1.3}{=} \int_H \int_G \rho(gh)\phi(gh)f(g)\Delta_H(h^{-1})\Delta_G(h^{-1})dgdh \\ &= \int_G \int_H \rho(g)\phi(gh)f(g)dhdg = \int_G f(g)\rho(g)\phi^H(g)dg = \int_G f(g)\rho(g)dg. \end{aligned}$$

Here ϕ is to make the integrals absolutely convergent. □

Theorem 2.4.36. Every strongly quasi-invariant measure on G/H arises from a rho-function constructed in Proposition 2.4.34, and all such measures are strongly equivalent.

Proof. Suppose μ is strongly equivalent, so that $(d\mu_x/d\mu)(p) = \lambda(x,p)$ is a positive continuous function on $G \times G/H$. For $x, y \in G$, since $\mu_{xy} = (\mu_x)_y$, the "chain rule" for the Radon Nikodym derivatives implies

$$\lambda(xy,p) = \lambda(x,yp)\lambda(y,p)$$

μ-a.e. in p. Since both sides are continuous functions, this identity must hold for every p by Lemma 2.4.31.

If $f \in C_c(G)$ and $y \in G$, we have

$$\begin{aligned} \int_{G/H}\int_H f(y^{-1}xh)\lambda(xh,H)^{-1}dhd\mu(xH) &= \int_{G/H}\int_H f(xh)\lambda(yxh,H)^{-1}\lambda(y,xH)dhd\mu(xH) \\ &= \int_{G/H}\int_H f(xh)\lambda(xh,H)^{-1}dhd\mu(xH) \end{aligned}$$

by the above identity. Hence

$$f \mapsto \int_{G/H}\int_H f(xh)\lambda(xh,H)^{-1}dhd\mu(xH)$$

defines a left-invariant positive linear function on $C_c(G)$, so there exists $c > 0$ such that

$$\int_{G/H}\int_H f(xh)\lambda(xh,H)^{-1}dhd\mu(xH) = c\int_G f(x)dx$$

Define $\rho(x) = c\lambda(x,H)$. Replacing f by $f\lambda(\cdot,H)$, we obtain

$$\int_{G/H}\int_H f(xh)dhd\mu(xH) = \int_G f(x)\rho(x)dx.$$

We show ρ is a rho-function. Indeed, for $h_0 \in H$,

$$\begin{aligned}\int_G f(x)\rho(xh_0)dx &= \Delta_G(h_0^{-1})\int_G R_{h_0^{-1}}f(x)\rho(x)dx \\ &= \Delta_G(h_0^{-1})\int_{G/H}\int_H f(xhh_0^{-1})dhd\mu(xH) \\ &= \Delta_G(h_0^{-1})\Delta_H(h_0)\int_{G/H}\int_H f(xh)dhd\mu(xH) = \Delta_G(h_0^{-1})\Delta_H(h_0)\int_G f(x)\rho(x)dx\end{aligned}$$

holds for every $f \in C_c(G)$, so

$$\rho(xh) = \frac{\Delta_H(h)}{\Delta_G(h)}\rho(x)$$

holds for every $x \in G$, $h \in H$. Since ρ is continuous and positive, this shows ρ is a rho-function.

It remains to show the last assertion. Suppose μ and μ' are strongly equivalent measures with associated rho-function ρ and ρ'. The functional equation for rho-function implies the ratio ρ'/ρ defines a positive continuous function ϕ on G/H. For $f \in C_c(G)$, we have $(f\rho'/\rho)^H = f^H\phi$, so

$$\int_{G/H} f^H d\mu' = \int_G f\rho' = \int_G f(\rho'/\rho)\rho = \int_{G/H} f^H\phi d\mu.$$

This proves $d\mu' = \phi d\mu$. □

2.5 Locally compact division rings

Definition.

(i) A **topological ring** R is a ring equipped with a topology such that the addition the multiplication are continuous.

(ii) A **topological division ring** is a topological ring D with identity such that every nonzero element is invertible and the inversion $D^\times \ni x \mapsto x^{-1} \in D^\times$ is continuous with respect to the subspace topology.

(iii) A **topological field** is a commutative topological division ring.

(iv) Let R be a topological ring. A **left topological R-module** M is a topological abelian group M together with a continuous map $R \times M \ni (r, m) \mapsto rm \to M$ making M an R-module.

(v) Let D be a topological division ring. A **left topological vector space over** D is a left topological D-module.

The goal of this subsection is to show all *non-discrete*[2] locally compact division rings are equipped with naturally defined valuations, in the sense of §8.

Let D be a non-discrete locally compact division ring and let μ be a Haar measure on D. For $a \in D^\times$, the left multiplication ℓ_a by a is an automorphism of the underlying abelian group, so we can consider the modulus $\mathrm{mod}_D(a) := \mathrm{mod}_D(\ell_a)$. This is a positive real number such that $\mu(aX) = \mathrm{mod}_D(a)\mu(X)$ for all measurable $X \subseteq D$. If we set $\mathrm{mod}_D(0) = 0$, then we obtain a function

$$\mathrm{mod}_D : D \longrightarrow \mathbb{R}_{\geqslant 0}$$

[2]This means the topology is not the discrete topology. We really need to exclude such a situation, since every division ring with the discrete topology is a locally compact topological division ring.

such that $\operatorname{mod}_D(a) = 0$ if and only if $a = 0$, and $\operatorname{mod}_D(xy) = \operatorname{mod}_D(x)\operatorname{mod}_D(y)$ for all $x, y \in D$.

Similarly, if V is a locally compact Hausdorff topological left vector space over D, then each $a \in D^\times$ acts on V by left multiplication ℓ_a, defining an automorphism of V. So we can consider its modulus $\operatorname{mod}_V(a) := \operatorname{mod}_V(\ell_a) \in \mathbb{R}_{>0}$. Again, we set $\operatorname{mod}_V(0) = 0$.

Lemma 2.5.1. Let V be an LCH topological left vector space over D. Then $\operatorname{mod}_V : D \to \mathbb{R}_{\geqslant 0}$ is continuous.

Proof. Let K be a compact unit-neighborhood of V, $a \in D$ and $\varepsilon > 0$. Let μ be a Haar measure on V; since μ is a Radon measure, there exists a neighborhood U of aK such that $\mu(U) \leqslant \mu(aK) + \varepsilon$. By continuity, there exists a neighborhood W_a of a in D such that $W_aK \subseteq U$; if $a \neq 0$, we assume $0 \notin W_a$. Then for $x \in W_a$,

$$\operatorname{mod}_D(x)\mu(K) = \mu(xK) \leqslant \mu(U) \leqslant \operatorname{mod}_D(a)\mu(K) + \varepsilon$$

so that $\operatorname{mod}_D(x) \leqslant \operatorname{mod}_D(a) + \mu(K)^{-1}\varepsilon$ for all $x \in W_a$. In particular, this shows mod_D is continuous at 0. Also, for $a \neq 0$, we have $\operatorname{mod}_D(x) \leqslant \operatorname{mod}_D(a^{-1}) + \mu(K)^{-1}\varepsilon$ for $x \in W_{a^{-1}}$, so that

$$\operatorname{mod}_D(x) \geqslant \operatorname{mod}_D(a) - \varepsilon \cdot \frac{\operatorname{mod}_D(a)\mu(K)^{-1}}{1 + \operatorname{mod}_D(a)^2\mu(K)^{-1}\varepsilon} \geqslant \operatorname{mod}_D(a) - \varepsilon \cdot \operatorname{mod}_D(a)^2\mu(K)^{-1}.$$

This shows mod_D is continuous at $a \in D^\times$. ☐

In particular, since D is non-discrete, for every $\varepsilon > 0$ we can find an $a \in D$ such that $0 < \operatorname{mod}_D(a) \leqslant \varepsilon$. By inversion, we see mod_D is an unbounded continuous function. As a consequence, we see D cannot be compact. Nevertheless, we have

Lemma 2.5.2. For $r \geqslant 0$, the set $B_r := \{a \in D \mid \operatorname{mod}_D(a) \leqslant r\}$ is compact.

Proof. Let K be a compact unit-neighborhood and let U be an unit-neighborhood such that $KU \subseteq K$. Since D is non-discrete, we can (and we do) choose $a \in K \cap U$ such that $0 < \operatorname{mod}_D(a) < 1$; then $a^n \in K$ for all $n \geqslant 1$. Since $\operatorname{mod}_D(a) < 1$, we see $\operatorname{mod}_D(a^n) \to 0$ as $n \to \infty$. In particular, the only limit point of the sequence $(a^n)_{n\geqslant 1}$ is 0, and since K is compact, we deduce that $a^n \to 0$ as $n \to \infty$.

Take any $r > 0$ and $x \in B_r$. Since $a^nx \to 0$ as $n \to \infty$, $m := \inf\{n \geqslant 0 \mid a^nx \in K\}$ exists. If $m \geqslant 1$, then $a^{m-1}x \notin K$ so that $a^mx \in K \backslash aK$. Set $d := \inf_{x \in \overline{K \backslash aK}} \operatorname{mid}_D(x)$; note that $d > 0$ since $\overline{K \backslash aK}$ is a compact set not containing 0. Let $N \geqslant 1$ be such that $\operatorname{mod}_D(a)^N \leqslant \dfrac{d}{r}$; then for $x \in B_r$ with $x \notin K$, we have

$$r\operatorname{mod}_D(a)^N \leqslant d \leqslant \operatorname{mod}_D(a^mx) \leqslant r\operatorname{mod}_D(a)^m$$

so that $N \geqslant m$. This shows $B_r \subseteq K \cup a^{-1}K \cup \cdots \cup a^{-N}K$. Since B_r is closed by Lemma 2.5.1, being contained in a finite union of compact sets, B_r is compact as well. ☐

Corollary 2.5.2.1.

(i) The sets B_r, $r > 0$ form a unit-neighborhood basis of D.

(ii) For $a \in D$, $\lim_{n\to\infty} a^n = 0$ if and only if $\operatorname{mod}_D(a) < 1$.

(iii) Any discrete division subring of D is finite.

Proof.

(i) Let K be any compact unit-neighborhood of D. Let $m > \sup_{x\in K} \mathrm{mod}_D(x)$ so that $K \subsetneq B_m$, and let $r' := \inf_{x\in\overline{B_m\backslash K}} \mathrm{mod}_D(x) > 0$; note that $\overline{B_m\backslash K}$ is compact by Lemma 2.5.2 and does not contain 0. Then $0 < r' \leqslant m$. If $0 < r < r'$, then $B_r \subseteq B_m$ and $B_r \cap \overline{B_m\backslash K} = \varnothing$, so that $B_r \subseteq K$.

(ii) This is proved during the proof of Lemma 2.5.2.

(iii) Let D' be any discrete division subring. Then $\mathrm{mod}_D(a) \leqslant 1$ for all $a \in (D')^\times$, for otherwise $(a^{-n})_n$ is a sequence in $(D')^\times$ tending to 0 by (ii), a contradiction to discreteness. Hence D' is a discrete subspace of a compact set B_1, so D' is finite.

□

Theorem 2.5.3. Let V be a Hausdorff topological left vector space over D. Let V' be a finite dimensional subspace and let $\{e_1, \ldots, e_n\}$ be a D-basis for V'. Then the map

$$\begin{aligned} D^n &\longrightarrow V' \\ (a_1, \ldots, a_n) &\longmapsto \sum_{i=1}^n a_i e_i \end{aligned}$$

is an isomorphism of topological left vector spaces over D. Also, V' is closed in V and V' is locally compact.

Proof. Denote by $T : D^n \to V'$ the map defined in the theorem. The space D^n is equipped with the product topology, and so by the definition T is continuous. It is also a D-isomorphism, so it remains to show T is an open map. In view of Corollary 2.5.2.1.(i) and the linearity of T, we only need to show $T((B_r)^n)$ is a unit-neighborhood of $0 \in V'$ for each $r > 0$. Set

$$S = \left\{ (a_1, \ldots, a_n) \in D^n \mid \sup_{1\leqslant i\leqslant n} \mathrm{mod}_D(a_i) = 1 \right\}.$$

We have $0 \notin S$, S is closed and is contained in $(B_1)^n$; by Lemma 2.5.2, S is compact. Hence, $0 \notin T(S)$ and $T(S)$ is a compact set. Since V is assumed to be Hausdorff, $T(S)$ is closed. By continuity we can find $\varepsilon > 0$ and a unit-neighborhood U of V such that $B_\varepsilon U \subseteq V\backslash T(S)$.

Let $r > 0$ and take $x \in D$ such that $0 < \mathrm{mod}_D(x) \leqslant r\varepsilon$. Let $0 \neq v = \sum_{i=1}^n a_i e_i \in V' \cap xU$, and let $j \in [n]$ be such that $\sup_{1\leqslant i\leqslant n} \mathrm{mod}_D(a_i) = \mathrm{mod}_D(a_j) > 0$. Then $a_j^{-1}(a_1, \ldots, a_n) \in S$, so $a_j^{-1}v \in T(S)$. Also, $a_j^{-1}v \in a_j^{-1}xU$, so by the choice of U and ε, it forces that $\mathrm{mod}_D(a_j^{-1}x) > \varepsilon$ and hence $\mathrm{mod}_D(a_j) < \varepsilon^{-1}\mathrm{mod}_D(x) \leqslant r$. This proves

$$V' \cap xU \subseteq T((B_r)^n),$$

which is what we want.

To show V' is closed in V, let W be the closure of V' in V. Let $w \in W$ and consider the subspace $Dw + V'$. By the first assertion, we see V' is closed in $Dw + V'$, so $w \in V'$. This shows $V' = W$ is closed in V. The last assertion is clear. □

Let V be a finite dimensional Hausdorff topological left vector space over D. By the previous theorem, V is locally compact, so the previous discussion applies. Generally, if $T : V \to V$ is a surjective D-linear map, then it is a D-isomorphism of V so $\mathrm{mod}_D(T)$ makes sense. If $T : V \to V$ is not surjective, we set $\mathrm{mod}_D(T) = 0$.

Corollary 2.5.3.1.

(i) Every finite dimensional left vector space over D has a unique Hausdorff topology making it a topological vector space over D.

(ii) If V is a locally compact Hausdorff topological left vector space over D, then $\dim_D V < \infty$ and $\mathrm{mod}_V(a) = \mathrm{mod}_D(a)^{\dim_D V}$ for all $a \in D$.

(iii) If V is a finite dimensional left vector space over D, D is commutative and $T : V \to V$ is a D-linear map, then $\mathrm{mod}_V(T) = \mathrm{mod}_D(\det T)$.

Proof.

(i) This follows from Theorem 2.5.3.

(ii) The last assertion follows from Fubini's Theorem. For the first assertion, let $a \in D$ with $0 < \mathrm{mod}_D(a) < 1$. By Corollary 2.5.2.1.(ii), we have $\lim_{n\to\infty} a^n = 0$, so by continuity of mod_V we have either $\mathrm{mod}_V(a) = 1$ or $\lim_{n\to\infty} \mathrm{mod}_V(a)^n = 0$. Hence $0 < \mathrm{mod}_V(a) \leqslant 1$.

Let W be a subspace of V of dimension $0 < d < \infty$. Since W is closed by Theorem 2.5.3, the quotient V/W is an LCH group, and D also acts on V/W continuously so that V/W is again a locally compact Hausdorff topological left vector space over D. By Proposition 2.4.9, we have

$$\mathrm{mod}_V(a) = \mathrm{mod}_W(a)\mathrm{mod}_{V/W}(a) \leqslant \mathrm{mod}_D(a)^d$$

Here $\mathrm{mod}_{V/W}(a) \leqslant 1$ by the first paragraph. Since $\mathrm{mod}_D(a) < 1$, this gives an upper bound for d. Since d is arbitrary, we deduce that V is finite dimensional over D.

(iii) This follows from the linear algebra that an automorphism can be written as a product of elementary operations and Fubini's Theorem.

□

Lemma 2.5.4. Let $\Gamma = \mathrm{mod}_D(D^\times) \leqslant \mathbb{R}_{>0}$. Then Γ is a closed subgroup of $\mathbb{R}_{>0}$, and $\mathrm{mod}_D : D^\times \to \Gamma$ is an open homomorphism.

Proof. Let $\Gamma' = \mathrm{mod}_D(D) = \Gamma \cup \{0\} \subseteq \mathbb{R}_{\geqslant 0}$. We show $\Gamma' \cap (0, r]$ is closed for each $r \geqslant 0$. We have $\Gamma' \cap [0, r] = \mathrm{mod}_D(B_r)$. Since mod_D is continuous and B_r is compact, $\Gamma' \cap [0, r]$ is compact. Since $\mathbb{R}_{\geqslant 0}$ is Hausdorff, it implies $\Gamma' \cap [0, r]$ is closed.

Let U be a unit-neighborhood of $D^\times$. Suppose otherwise $\mathrm{mod}_D(U)$ is not a neighborhood of 1 in Γ. Then there exists a sequence $(\gamma_n)_{n\geqslant 1} \subseteq \Gamma\backslash\mathrm{mod}_D(U)$ such that $\lim_{n\to\infty} \gamma_n = 1$. For each $n \geqslant 1$ pick $a_n \in D^\times$ with $\gamma_n = \mathrm{mod}_D(a_n)$. By Lemma 2.5.2, $(a_n)_n$ has a limit point, say $a \in D$, and by continuity $\mathrm{mod}_D(a) = 1$ so that $a \in \ker \mathrm{mod}_D$. Since $(\ker \mathrm{mod}_D)U$ is a neighborhood of $\ker \mathrm{mod}_D$, we see $a_n \in (\ker \mathrm{mod}_D)U$ for $n \gg 0$, implying $\gamma_n \in \mathrm{mod}_D(U)$, a contradiction. □

Theorem 2.5.5. There is a constant $A > 0$ such that

$$\mathrm{mod}_D(x + y) \leqslant A \max\{\mathrm{mod}_D(x), \mathrm{mod}_D(y)\}$$

for all $x, y \in D$. If we can choose $A = 1$, then $\Gamma := \mathrm{mod}_D(D^\times)$ is discrete in $\mathbb{R}_{>0}$. Moreover,

$$\sup_{x\in D,\, \mathrm{mod}_D(x)\leqslant 1} \mathrm{mod}_D(1 + x)$$

is the smallest A such that the inequality holds.

Proof. Let $A := \sup_{x \in D,\, \mathrm{mod}_D(x) \leqslant 1} \mathrm{mod}_D(1+x)$. Since B_1 is compact, $A < \infty$. By choosing $x = 0$, we also have $A \geqslant 1$. If $x = 0$ or $y = 0$, the equality is clear. Assume $x, y \neq 0$; further we can assume $\mathrm{mod}_D(y) \leqslant \mathrm{mod}_D(x)$. If $z := yx^{-1}$, then $\mathrm{mod}_D(z) \leqslant 1$ so that $\mathrm{mod}_D(1+z) \leqslant A$. Hence

$$\mathrm{mod}_D(x+y) = \mathrm{mod}_D(1+z)\mathrm{mod}_D(x) \leqslant A \max\{\mathrm{mod}_D(x), \mathrm{mod}_D(y)\}.$$

That A is minimal is clear.

Now assume $A = 1$. Then $\mathrm{mod}_D(1 + B_r) \subseteq [0,1]$. By Lemma 2.5.4, it is a unit-neighborhood of Γ. These together imply Γ is discrete. $\square$

Lemma 2.5.6. Let $F : \mathbb{Z}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ be a function such that $F(mn) = F(m)F(n)$ for all $m, n \in \mathbb{Z}_{\geqslant 0}$ and there exists $A > 0$ such that

$$F(m+n) \leqslant A \max\{F(m), F(n)\}$$

for all $m, n \in \mathbb{Z}_{\geqslant 0}$. Then $F \leqslant 1$ or there exists a $\lambda > 0$ such that $F(m) = m^\lambda$ for all $m \in \mathbb{Z}_{\geqslant 0}$.

Proof. If $F \equiv 0$ or $F \equiv 1$, then the lemma holds trivially. Assume F is not constant 0 and 1, so that by assumption we have $F(0) = 0$, $F(1) = 1$. Also, we have $F(m^k) = F(m)^k$ for all $m \in \mathbb{Z}_{\geqslant 0}$. Define $f : \mathbb{Z}_{\geqslant 0} \to \mathbb{R}$ by $f(m) = \max\{0, \log F(m)\}$ ($m \in \mathbb{Z}_{\geqslant 0}$); note that $F(m) = 0$ implies $f(m) = 0$. If we put $a = \max\{0, \log A\}$, we have

$$f(m^k) = kf(m), \qquad f(mn) \leqslant f(m) + f(n), \qquad f(m+n) \leqslant a + \max\{f(n), f(m)\}$$

for all $m, n \in \mathbb{Z}_{\geqslant 0}$. Now let $m, n \geqslant 2$ and write $m = \sum_{i=0}^{\ell} a_i n^i$ with $0 \leqslant a_i \leqslant n-1$ and $n^\ell \leqslant m < n^{\ell+1}$. Then

$$f(m) = f\left(\sum_{i=0}^{\ell} a_i n^i\right) \leqslant \ell a + \max_{0 \leqslant i \leqslant \ell} f(a_i n^i) \leqslant \ell a + \max_{0 \leqslant a \leqslant n-1} f(a) + \ell f(n).$$

Since $n^\ell \leqslant m$, we have $\ell \log n \leqslant m$ so that

$$\frac{f(m)}{\log m} \leqslant \frac{\max_{0 \leqslant a \leqslant n-1} f(a)}{\log m} + \frac{a + f(n)}{\log n}.$$

Replacing m by m^k, which does not change the LHS, and letting $k \to \infty$, we obtain

$$\frac{f(m)}{\log m} \leqslant \frac{a + f(n)}{\log n}.$$

Replacing n by n^k and letting $n \to \infty$ gives $\frac{f(m)}{\log m} \leqslant \frac{f(n)}{\log n}$. By symmetry this shows

$$\frac{f(m)}{\log m} = \frac{f(n)}{\log n} =: C \geqslant 0$$

for all $n, m \geqslant 2$ so that $f(m) = C \log m$. If $C = 0$, this shows $\log F(m) \leqslant 0$ for all $m \geqslant 0$ so that $F \leqslant 1$. If $C \neq 0$, then $f(m) \neq 0$ for $m \geqslant 2$ and hence $F(m) = m^C$, $m \geqslant 2$. This finishes the proof. $\square$

Return to our setting. Denote by 1_D the multiplicative identity of D. Define $F = F_D : \mathbb{Z} \to \mathbb{R}_{\geqslant 0}$ by $F(m) = \mathrm{mod}_D(m \cdot 1_D)$.

Lemma 2.5.7. If F is bounded, then $F \leqslant 1$ and $A = 1$.

Proof. The first follows from $F(mn) = F(m)F(n)$. For the second, by induction on n we have

$$\mathrm{mod}_D\left(\sum_{i=1}^{2^n} x_i\right) \leqslant A^n \max_{1\leqslant i\leqslant 2^n} \mathrm{mod}_D(x_i)$$

for $x_1, \ldots, x_{2^n} \in D$ and $n \geqslant 1$. By inserting some 0, we see $\mathrm{mod}_D\left(\sum_{i=1}^N x_i\right) \leqslant A^n \max_{1\leqslant i\leqslant N} \mathrm{mod}_D(x_i)$ for $N \leqslant 2^n$. Then for $x, y \in D$ and $n \geqslant$,

$$\begin{aligned}\mathrm{mod}_D(x+y)^{2^n} &\leqslant A^{n+1} \max_{0\leqslant i\leqslant 2^n}\left(\binom{2^n}{i}\mathrm{mod}_D(x)^i \mathrm{mod}_D(y)^{2^n-i}\right)\\ &\leqslant A^{n+1}\max\{\mathrm{mod}_D(x), \mathrm{mod}_D(y)\}^{2^n}\end{aligned}$$

Taking 2^n-th root and letting $n \to \infty$ prove the lemma. □

Theorem 2.5.8. Let D be a non-discrete locally compact division ring and put $F(m) := \mathrm{mod}_D(m \cdot 1_D)$ for $m \in \mathbb{Z}_{\geqslant 1}$.

(i) If $\mathrm{Char}\, D = p \geqslant 2$, then $F(m) = 1$ for $p \nmid m$, and $F(m) = 0$ for $p \mid m$.

(ii) If $\mathrm{Char}\, D = 0$, then D is a division algebra over $\mathbb{Q}_p$[3] of dimension $d < \infty$, and $F(m) = |m|_p^d$. Here p is a rational prime or ∞.

Proof. By Lemma 2.5.6, either there exists some $\lambda > 0$ such that $F(m) = m^\lambda$ for $m \geqslant 1$, or $F \leqslant 1$. Assume $F \leqslant 1$. Then $(m \cdot 1_D)_{m\geqslant 1} \subseteq B_1$, and by compactness of B_1 this sequence has a limit point $a \in B_1$. By Corollary 2.5.2.1.(i) there are infinitely many $m \in \mathbb{N}$ such that $\mathrm{mod}_D(m \cdot 1_D - a) \leqslant \varepsilon$ (for small $\varepsilon > 0$). In particular, we have $\mathrm{mod}_D(m \cdot 1_D - m' \cdot 1_D) \leqslant \varepsilon$ for $m \neq m'$ so that $F(n) < 1$ for some $n \in \mathbb{Z}_{\geqslant 1}$.

Let $p = \min\{n \in \mathbb{Z}_{\geqslant 1} \mid F(n) < 1\} > 1$. Since $F(nm) = F(n)F(m)$, p must be a prime. For $n \in \mathbb{Z}_{\geqslant 1}$, we have $F(np) < 1$, so that $F(1+np) = 1$ by Theorem 2.5.5 and Lemma 2.5.7. If $(m, p) = 1$, then $m^{p-1} \equiv 1 \pmod p$, so $F(m^{p-1}) = 1$, or $F(m) = 1$. If $\mathrm{Char}\, D = p' \geqslant 2$, then $F(p') = 0$ so that $p = p'$ by minimality. This proves (i). Assume $\mathrm{Char}\, D = 0$. Then $F(p) > 0$, and if we put $F(p) = p^{-c}$ for some $c > 0$, then for $m = p^n m' \geqslant 1$ with $(m', p) = 1$, we have

$$F(m) = F(p)^n F(m') = p^{-nc} = |p|_p^c = |m|_p^c.$$

In sum, we see if $\mathrm{Char}\, D = 0$, then there exists $p \leqslant \infty$ and $\lambda > 0$ such that $F = |\cdot|_p^\lambda$. Let us identity $\mathbb{Q}$ as a subfield of D via the map $r \mapsto r \cdot 1_D$. It follows that F is the function $|\cdot|_p^\lambda$ on $\mathbb{Q}$. By Corollary 2.5.2.1.(i), the subspace topology on $\mathbb{Q}$ is given by the metric $(x, y) \mapsto |x - y|_p$, so the closure of $\mathbb{Q}$ in D is isomorphic to $\mathbb{Q}_p$. Now regard D as a topological (left) vector space over $\mathbb{Q}_p$. By Corollary 2.5.3.1, $\dim_{\mathbb{Q}_p} D = d < \infty$ and $\mathrm{mod}_D(a) = \mathrm{mod}_{\mathbb{Q}_p}(a)^d$. The proof is complete once we notice that $\mathrm{mod}_{\mathbb{Q}_p} = |\cdot|_p$.

□

Definition. Let D be a non-discrete locally compact division ring.

(i) We say D is **of type** p if p is a rational prime and $\mathrm{mod}_D(p \cdot \mathbf{1}_D) < 1$.

(ii) We say D is **of real type** if D is a finite dimensional real division algebra.

If D is of type p, by the theorem we see $F \leqslant 1$, so the image $\mathrm{mod}_D(D^\times)$ is discrete by Theorem 2.5.5. This shows D cannot be connected.

[3] See the definition before Theorem 8.2.3

2.6 Convolution

We do not require an algebra to be unital or commutative.

Definition. Let G be an LCH group. For two measurable $f, g : G \to \mathbb{C}$ define the **convolution product** as

$$f * g(x) := \int_G f(y)g(y^{-1}x)dy$$

whenever the integral exists.

Theorem 2.6.1. Let $f, g \in L^1(G)$. Then

(i) The integral $f * g$ exists almost everywhere in x and defines a function in $L^1(G)$.

(ii) The L^1-norm satisfies $\|f * g\|_1 \leqslant \|f\|_1 \|g\|_1$.

The convolution product endows $L^1(G)$ with the structure of an algebra.

Proof. Note that f, g are measurable in the sense that the preimages of Borel sets under f, g are in the completed Borel σ-algebra. Define the function

$$\begin{aligned} \psi : G \times G &\longrightarrow \mathbb{C} \\ (y, x) &\longmapsto f(y)g(y^{-1}x) \end{aligned}$$

We can write ψ as

$$\begin{aligned} \psi : G \times G &\xrightarrow{\alpha} G \times G \xrightarrow{f \times g} \mathbb{C} \times \mathbb{C} \xrightarrow{\mu} \mathbb{C} \\ (y, x) &\longmapsto (y, y^{-1}x) \end{aligned}$$

where $\mu : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is multiplication. Since $f \times g$ and μ are measurable, to show ψ is measurable it suffices to show that α is. Since α is continuous, it is Borel-measurable, and we must show the preimage under α of a null set is again null (completeness). But this follows from the formula, valid for each $\phi \in C_c(G \times G)$.

$$\int_{G \times G} \phi(y, x) dx \otimes dy \overset{\text{Fubini}}{=} \int_G \int_G \phi(y, x) dx dy \overset{\text{inv}}{=} \int_G \int_G \phi(y, y^{-1}x) dx dy$$

Let $S(f)$ and $S(g)$ be the supports and f and g, respectively. By Corollary 2.2.2.1.4, $S(f)$ and $S(g)$ are σ-compact. The support of ψ is contained in $S(f) \times S(f)S(g)$, so it's also σ-compact. Hence by Fubini's Theorem,

$$\begin{aligned} \|f * g\|_1 &\leqslant \int_G \int_G |f(y)g(y^{-1}x)| dy dx = \int_G \int_G |f(y)g(y^{-1}x)| dx dy \\ &= \int_G \int_G |f(y)g(x)| dx dy \\ &= \|f\|_1 \|g\|_1 < \infty \end{aligned}$$

and Fubini's theorem again shows that $\psi(\cdot, x)$ is integrable a.e. in x and that $f * g \in L^1(G)$.

It remains to show the associativity and bilinearity. The bilinearity is clear. For associativity, let $f, g, h \in L^1(G)$. Then

$$\begin{aligned} (f * g) * h(x) &= \int_G (f * g)(y) h(y^{-1}x) dy = \int_G \int_G f(z) g(z^{-1}y) h(y^{-1}x) dz dy \\ &= \int_G \int_G f(z) g(z^{-1}y) h(y^{-1}x) dy dz = \int_G \int_G f(z) g(y) h(y^{-1}z^{-1}x) dy dz \\ &= \int_G f(z)(g * h)(z^{-1}x) dz = f * (g * h)(x) \end{aligned}$$

Using Fubini's Theorem to compute the integral is valid, which is justified in the preceding paragraphs. □

Proposition 2.6.2 (Young's convolution inequality)**.** Let G be unimodular. Let $1 \leqslant p, q, r \leqslant \infty$ with $\frac{1}{p}+\frac{1}{q} = \frac{1}{r}+1$. Then for $f \in L^p(G)$ and $g \in L^q(G)$, one has $f*g \in L^r(G)$ and $\|f * g\|_r \leqslant \|f\|_p \|g\|_q$

Proof.

$$\begin{aligned} |f * g(x)| &\leqslant \int_G |f(y)g(y^{-1}x)| dy \\ &= \int_G |f(y)|^{1-\frac{p}{r}} |g(y^{-1}x)|^{1-\frac{q}{r}} |f(y)^{\frac{p}{r}} g(y^{-1}x)^{\frac{q}{r}}| dy \\ &= \int_G (|f(y)|^p)^{\frac{r-p}{rp}} (|g(y^{-1}x)|^q)^{\frac{r-q}{rq}} |f(y)^p g(y^{-1}x)^q|^{\frac{1}{r}} dy \end{aligned}$$

We invoke Hölder's inequality. One checks that $r \geqslant p, q$ and

$$\frac{r-p}{rp} + \frac{r-q}{rq} + \frac{1}{r} = \frac{1}{r}\left(\frac{r}{p} - 1 + \frac{r}{q} - 1 + 1\right) = \frac{1}{r}(1 + r - 1) = 1$$

Hence

$$|f * g(x)| \leqslant \left(\int_G |f(y)|^p dy\right)^{\frac{r-p}{rp}} \left(\int_G |g(y^{-1}x)|^q dy\right)^{\frac{r-q}{rq}} \left(\int_G |f(y)^p g(y^{-1}x)^q| dy\right)^{\frac{1}{r}}$$

Taking r-power both sides, we see

$$|f * g(x)|^r \leqslant \|f\|_p^{r-p} \|g\|_q^{r-q} \int_G |f(y)^p g(y^{-1}x)^q| dy$$

Note that f^p, $g^q \in L^1(G)$, so by Theorem 2.6.1 the right hand side is integrable, so $f * g \in L^r(G)$ and

$$\|f * g\|_r^r \leqslant \|f\|_p^{r-p} \|g\|_q^{r-q} \|f\|_p^p \|g\|_p^q = \|f\|_p^r \|g\|_q^r$$

and thus $\|f * g\|_r \leqslant \|f\|_p \|g\|_q$. □

Lemma 2.6.3. For $f, g \in L^1(G)$ and $y \in G$, one has

$$R_y(f * g) = f * (R_y g) \qquad L_y(f * g) = (L_y f) * g$$

Proof.

$$R_y(f * g)(x) = (f * g)(xy) = \int_G f(z)g(z^{-1}xy)dz = \int_G f(z)R_y g(z^{-1}x)dx = f * (R_y g)(x)$$

Likewise for L. □

Theorem 2.6.4. The algebra $L^1(G)$ is commutative if and only if G is abelian.

Proof. Suppose G is abelian. Then

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)dy \overset{\text{inv}}{=} \int_G f(xy)g(y^{-1})dy \overset{2.3.1.4}{=} \int_G \Delta_G(y^{-1})f(xy^{-1})g(y)dy = (g * f)(x)$$

for $\Delta_G \equiv 1$. Suppose $L^1(G)$ is commutative and let $f, g \in L^1(G)$. For $x \in G$ we have

$$\begin{aligned} 0 = f * g(x) - g * f(x) &= \int_G \left(f(y)g(y^{-1}x) - g(y)f(y^{-1}x)\right) dy \\ &\overset{\text{inv}}{=} \int_G \left(f(xy)g(y^{-1}) - g(y)f(y^{-1}x)\right) dy \\ &\overset{2.3.1.4}{=} \int_G g(y) \left(\Delta_G(y^{-1})f(xy^{-1}) - f(y^{-1}x)\right) dy \end{aligned}$$

This holds for all $g \in L^1(G)$, so that $\Delta_G(y^{-1})f(xy^{-1}) - f(y^{-1}x) = 0$ for all $f \in C_c(G)$. Taking $x = 1$ gives $\Delta_G \equiv 1$, and thus G is unimodular and $f(xy^{-1}) = f(y^{-1}x)$ for every $f \in C_c(G)$, $x, y \in G$. This shows G is abelian. (If $xy^{-1} \neq y^{-1}x$ for some x, y, apply Urysohn's Lemma to find an f that make the equality fails.) □

Using convolution we may obtain an enhanced version of Lemma 2.2.3.

Lemma 2.6.5 (Steinhaus)**.** Let G be an LCH group and $X, Y \subseteq G$ be two positive finite measurable sets. Then $XY = \{xy \mid x \in X,\ y \in Y\}$ contains an open set in G.

Proof. By weakly inner regularity we can assume X, Y are compact. By Fubini,

$$\int_G \mathbf{1}_X * \mathbf{1}_Y(g)dg = \int_G \left(\int_G \mathbf{1}_X(h)\mathbf{1}_Y(h^{-1}g)dh \right) dg = \operatorname{vol}(X, dg)\operatorname{vol}(Y, dg) \neq 0.$$

Also,

$$\mathbf{1}_X * \mathbf{1}_Y(g) = \int_G \mathbf{1}_X(h)\mathbf{1}_Y(h^{-1}g)dh = \operatorname{vol}(X \cap gY^{-1}),$$

so $0 < \operatorname{vol}(X \cap gY^{-1})$ for some $g \in G$. Since

$$(X \cap gY^{-1})(X \cap gY^{-1})^{-1} \subseteq XYg$$

and the former set is a unit-neighborhood by Lemma 2.2.3, it follows that XY contains a neighborhood of g. □

2.6.1 Dirac net

Definition. A **Dirac function** is a function $\phi \in C_c(G)$ such that

(i) $\phi \geqslant 0$,

(ii) $\displaystyle\int_G \phi(x)dx = 1$ and

(iii) $\phi(x^{-1}) = \phi(x)$ for all $x \in G$.

A **Dirac family** is a family $(\phi_U)_U$ of Dirac functions indexed by the set $\mathcal{U}$ of all unit-neighborhoods U such that $\operatorname{supp}\phi_U \subseteq U$.

Lemma 2.6.6.

1. The convolution product of two Dirac functions is a Dirac function.
2. To every unit-neighborhood U there exists a Dirac function ϕ_U such that ϕ_U and $\phi_U * \phi_U$ have support inside U.

Proof.

1. Let ϕ, ψ be Dirac functions. That $\phi * \psi \geqslant 0$ is clear.

$$\int_G \phi*\psi(x)dx = \int_G\int_G \phi(y)\psi(y^{-1}x)dydx \overset{\text{Fubini}}{=} \int_G\int_G \phi(y)\psi(y^{-1}x)dxdy \overset{\text{inv}}{=} \left(\int_G \phi(y)dy\right)\left(\int_G \psi(x)dx\right) = 1$$

$$\phi * \psi(x^{-1}) = \int_G \phi(y)\psi(y^{-1}x^{-1})dy \overset{\text{inv}}{=} \int_G \phi(xy)\psi(y^{-1})dy = \int_G \psi(y^{-1}x^{-1})\psi(y)dy = \phi * \psi(x)$$

2. Let U be a given unit-neighborhood and let $W \subseteq U$ be any symmetric unit-neighborhood such that $W^2 \subseteq U$. By Urysohn's Lemma, there is an $h \in C_c(G)$ with $0 \neq h \geqslant 0$ and $\operatorname{supp}(h) \subseteq W$. Set $\phi_U(x) = h(x) + h(x^{-1})$ and normalize this function so that ϕ_U has integral 1. Then $\operatorname{supp}(\phi_U * \phi_U) \subseteq \operatorname{supp}(\phi_U)^2 \subseteq W^2 \subseteq U$.

□

2.6.2 Regular representations

Lemma 2.6.7. For given $1 \leqslant p < \infty$ and $g \in L^p(G)$, the maps $y \mapsto L_y g$ and $y \mapsto R_y g$ are continuous maps from G to $L^p(G)$. Moreover, $y \mapsto L_y g$ is uniformly continuous, and if G is unimodular, so is $y \mapsto R_y g$.

Proof. By the invariance of Haar integrals,

$$\|L_y g - L_x g\|_p = \left\|L_{x^{-1}y} g - g\right\|_p$$

so the uniform continuity will follow once the continuity at e is established. Likewise,

$$\|R_y g - R_x g\|_p = \Delta(x^{-1})^{1/p} \left\|R_{x^{-1}y} g - g\right\|_p$$

so the last assertion will follows for the same reason.

Now we prove the continuity at the unit element e. First consider the case $g \in C_c(G)$. Choose $\varepsilon > 0$ and put $K = \operatorname{supp}(g)$; then $\operatorname{supp}(L_y g) = yK$. Let U_0 be a compact symmetric unit-neighborhood. Then for $y \in U_0$, one has $\operatorname{supp}(L_y g) \subseteq U_0 K$. By uniform continuity there exists a unit-neighborhood $U \subseteq U_0$ such that for $y \in U$, $\|L_y g - g\|_G < \dfrac{\varepsilon}{\operatorname{vol}(U_0 K)^{\frac{1}{p}}}$. Then for $y \in U$,

$$\|L_y g - g\|_p = \left(\int_G |g(y^{-1}x) - g(x)|^p dx\right)^{\frac{1}{p}} < \varepsilon$$

For general g, pick $f \in C_c(G)$ with $\|f - g\|_p < \varepsilon/3$. Choose a unit-neighborhood U with $\|f - L_y f\|_p < \varepsilon/3$ for every $y \in U$. Then for $y \in U$,

$$\|g - L_y g\|_p \leqslant \|g - f\|_p + \|f - L_y f\|_p + \|L_y f - L_y g\|_p < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Note we use the fact $\|L_y f - L_y g\|_p = \|f - g\|_p$. For case for the right translation is similar, except in the last step we use $\|R_y f - R_y g\|_p = \Delta(y^{-1})^{1/p} \|f - g\|_p$ instead. □

Corollary 2.6.7.1. Let $1 \leqslant p, q \leqslant \infty$ satisfy $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then for $f \in L^p(G)$ and $g \in L^q(G)$, then convolution $f * g \in L^1(G)$ is right uniformly continuous.

Proof. Say $1 \leqslant q < \infty$, and let $x, y \in G$. By Hölder's inequality

$$|f * g(yx) - f * g(y)| = \left|\int_G f(t)(g((yx)^{-1}t) - g(y^{-1}t))dt\right| \leqslant \|f\|_p \|L_{yx} g - L_y g\|_q = \|f\|_p \|L_x g - g\|_q,$$

Then $f * g$ is right uniformly continuous by Lemma 2.6.7. □

Lemma 2.6.8. Let $H \leqslant G$ be a closed subgroup of an LCH group G such that G/H admits a nonzero G-invariant Radon measure; fix such a measure. For given $1 \leqslant p < \infty$ and $g \in L^p(G/H)$, the map $y \mapsto L_y g$ is a uniformly continuous map from G to $L^p(G/H)$.

Proof. This is a generalization of Lemma 2.6.7, and their proofs are the same. □

Lemma 2.6.9. Let $\varepsilon > 0$ and $1 \leqslant p < \infty$. For every $f \in L^p(G)$ there exists a unit-neighborhood U such that for every Dirac function ϕ_U with support in U one has

$$\|f * \phi_U - f\|_p < \varepsilon, \qquad \|\phi_U * f - f\|_p < \varepsilon$$

For every $f \in C(G)$ and every compact $K \subseteq G$ there exists a unit-neighborhood U such that for every Dirac function ϕ_U with support in U one has

$$\|f * \phi_U - f\|_K < \varepsilon, \qquad \|\phi_U * f - f\|_K < \varepsilon$$

where $\|g\|_K = \sup_{x \in K} |g(x)|$.

In other words this means that the net $(\phi_U * f)_U$ indexed by the set of all unit-neighborhoods, converges to f in the L^p sense if $f \in L^p(G)$, and compactly if $f \in C(G)$.

Proof. Since $p \geqslant 1$, the function $x \mapsto x^p$ is convex. Now by Jensen's inequality (applied to the measure $\phi_U(y)dy$), one has

$$\begin{aligned}
\|f * \phi_U - f\|_p^p &= \int_G \left| \int_G f(y)\phi_U(y^{-1}x)dy - f(x) \right|^p dx = \int_G \left| \int_G (f(xy) - f(x))\phi_U(y)dy \right|^p dx \\
&\leqslant \int_G \left(\int_G |(f(xy) - f(x))| \phi_U(y)dy \right)^p dx \\
&\leqslant \int_G \int_G |f(xy) - f(x)|^p \phi_U(y)dydx = \int_G \|R_y f - f\|_p^p \phi_U(y)dy
\end{aligned}$$

Then by Lemma 2.6.7, it suffices to pick U small. The other side is addressed similarly.

For the second, let $f \in C(G)$ and $K \subseteq G$ compact. Since $f|_K$ is uniformly continuous, for every $\varepsilon > 0$ there exists a unit-neighborhood U such that for all $x, y \in K$ with $y^{-1}x \in U$ one has $|f(y) - f(x)| < \varepsilon$. Now let ϕ_U be a Dirac function with support in U. Then

$$|f * \phi_U(x) - f(x)| \leqslant \int_G |f(xy) - f(x)|\phi_U(y)dy = \int_G |f(y) - f(x)|\phi_U(x^{-1}y)dy < \varepsilon$$

□

Part I

Structure of locally compact abelian groups

Chapter 3

Banach algebras

3.1 Banach Algebras

Definition.

1. A **Banach algebra** is an algebra $\mathcal{A}$ over $\mathbb{C}$ together with a norm $\|\cdot\|$ such that $(\mathcal{A}, \|\cdot\|)$ is a Banach space and that the norm is submultiplicative, i.e.

$$\|a \cdot b\| \leqslant \|a\| \|b\|$$

   holds for all $a, b \in \mathcal{A}$.

2. An algebra $\mathcal{A}$ is **unital** if the multiplication has an identity element, which we denote by $1_{\mathcal{A}}$.

3. If $\mathcal{A}$ is a unital Banach algebra, denote by $\mathcal{A}^\times$ the group of invertible elements in $\mathcal{A}$.

- In particular, the inequality implies the multiplication on a Banach algebra $\mathcal{A}$ is continuous, so that $\mathcal{A}$ is a topological ring.

- As usual, for $a \in \mathcal{A}$, $r > 0$, we use $B_r(a) = \{b \in \mathcal{A} \mid \|b - a\| < r\}$ to denote the open ball of radius r centered at a.

Definition. Let $\mathcal{A}$, $\mathcal{B}$ be Banach algebras.

1. A **homomorphism** of Banach algebras $\phi : \mathcal{A} \to \mathcal{B}$ is a continuous algebra homomorphism.

2. A **topological isomorphism** of Banach algebras is a homomorphism with continuous inverse.

3. An **(isometric) isomorphism** of Banach algebras is an topological isomorphism which is also an isometry.

Example 3.1.1.

1. For a compact space X, the $\mathbb{C}$-vector space $C(X) := C(X, \mathbb{R})$ of complex-valued continuous functions is a commutative Banach algebra with the sup-norm $\|f\|_X := \sup_{x \in X} |f(x)|$.

2. If G is an LCH group, then $(L^1(G), *)$ with $\|\cdot\|_1$ is a Banach algebra, and $L^1(G)$ is commutative iff G is abelian.

3. Let V be a Banach space. For $T \in \operatorname{End}_{\mathbb{C}}(V)$, define the **operator norm** by

$$\|T\|_{\mathrm{op}} := \sup_{v \neq 0} \frac{\|Tv\|}{\|v\|}$$

The operator T is **bounded** if $\|T\|_{\mathrm{op}} < \infty$, which is equivalent to saying that T is continuous. The set $\mathcal{B}(V)$ of all bounded operators on V is a Banach algebra with $\|\cdot\|_{\mathrm{op}}$.

Lemma 3.1.2. Let $\mathcal{A}$ be a unital Banach algebra and write $1 = 1_{\mathcal{A}}$. Then $\|1\| \geqslant 1$, and there exists an equivalent norm $\|\cdot\|'$ such that $(\mathcal{A}, \|\cdot\|')$ is again a Banach algebra with $\|1\|' = 1$.

Proof. Since $\|1\| \neq 0$ and $\|1\| \|1\| \geqslant \|1\|$, $\|1\| \geqslant 1$. For the second statement, define $\|\cdot\|'$ by

$$\|a\|' := \sup_{v \neq 0} \frac{\|av\|}{\|v\|}$$

Since $\|av\| \leqslant \|a\| \|v\|$, we have $\|a\|' \leqslant \|a\|$. Conversely,

$$\|a\|' = \sup_{v \neq 0} \frac{\|av\|}{\|v\|} \geqslant \frac{\|a\|}{\|1\|}$$

This shows $\|\cdot\|$ and $\|\cdot\|'$ are equivalent. Trivially, $\|1\|' = 1$. □

With Lemma in mind, we always assume the identity of a unital Banach algebra has norm 1.

Proposition 3.1.3. Let G be an LCH group. The algebra $L^1(G)$ is unital if and only if G is discrete.

Proof. If G is discrete, then every singleton has positive measure. The identity element of $L^1(G)$ is $\mathbf{1}_{\{e\}}$, where e is the identity element of G. Conversely, assume $L^1(G)$ is unital; say $\phi \in L^1(G)$ is the identity. Suppose G is not discrete. Then any unit-neighborhood U contains at least two points. By Urysohn's lemma there exists two Dirac functions ϕ_U and ψ_U with support in U such that $\operatorname{supp} \phi_U \cap \operatorname{supp} \psi_U = \varnothing$.

- Indeed, let $x \neq y \in U$ and neighborhood $x \in U_x$, $y \in U_y$ such that $U_x \cap U_y = \varnothing$ and $U_x^{-1} \cap U_y = \varnothing = U_x \cap U_y^{-1}$. Use Urysohn's lemma to find f_x, $f_y \in C_c(G)$ with $f_x, f_y \geqslant 0$ and $\operatorname{supp} f_x \subseteq U_x$, $\operatorname{supp} f_y \subseteq U_y$. Define $F_x(z) = f_x(z) + f_x(z^{-1})$ and $F_y(z)$ similarly. Then F_x and F_y are positive symmetric. One can normalize them so that they have integral 1.

In particular, $\|\phi_U - \psi_U\|_1 = 2$ for every U. Now by Lemma 2.6.9 (or its proof) we can find U such that $\|\phi_U * \phi - \phi\|_1 < 1$ and $\|\psi_U * \phi - \phi\|_1 < 1$. Hence

$$2 = \|\phi_U - \psi_U\|_1 \leqslant \|\phi_U * \phi - \phi\|_1 + \|\psi_U * \phi - \phi\|_1 < 2$$

a contradiction. Hence G is discrete. □

Lemma 3.1.4. Let $\mathcal{A}$ be a unital Banach algebra and let $a \in \mathcal{A}$ with $\|a\| < 1$. Then $1 - a \in \mathcal{A}^\times$ with inverse

$$(1-a)^{-1} = \sum_{n=0}^{\infty} a^n$$

The unit group $\mathcal{A}^\times$ is open in $\mathcal{A}$, and under the subspace topology, $\mathcal{A}^\times$ is a topological group.

Proof. Since $\|a\| < 1$, the series $\sum\limits_{n=0}^{\infty} a^n$ converges in $\mathcal{A}$, and

$$(1-a)\sum_{n=0}^{N} a^n = 1 - a^N \to 1 \text{ as } N \to \infty$$

Hence $\sum\limits_{n=0}^{\infty} a^n = (1-a)^{-1}$. For $x \in \mathcal{A}^\times$ and $y \in \mathcal{A}$ with $\|y-x\| < \left\|x^{-1}\right\|^{-1}$, we have

$$\left\|yx^{-1} - 1\right\| \leqslant \|y-x\| \left\|x^{-1}\right\| < 1$$

so $yx^{-1} \in \mathcal{A}^\times$, giving $y \in \mathcal{A}^\times$. This shows $\mathcal{A}^\times$ is open.

It remains to show that inversion is continuous in $\mathcal{A}^\times$. But for $x, y \in \mathcal{A}^\times$,

$$x^{-1} - y^{-1} = x^{-1}(y-x)y^{-1} = x^{-1}(y-x)(y^{-1} - x^{-1}) + x^{-1}(y-x)x^{-1}$$

Hence

$$(1 - x^{-1}(y-x))(x^{-1} - y^{-1}) = x^{-1}(y-x)x^{-1}$$

Fix x and let $\left\|x^{-1}(y-x)\right\| < \dfrac{1}{2}$ so that $1 - x^{-1}(y-x) \in \mathcal{A}^\times$. Then $x^{-1} - y^{-1} = \dfrac{x^{-1}(y-x)x^{-1}}{1 - x^{-1}(y-x)}$ and thus

$$\left\|x^{-1} - y^{-1}\right\| \frac{\left\|x^{-1}\right\|^2 \|y-x\|}{\|1 - x^{-1}(y-x)\|} \leqslant 2\left\|x^{-1}\right\|^2 \|y-x\|$$

Alternatively, for $\|a\| < 1$, we compute

$$\left\|(1-a)^{-1} - 1\right\| = \left\|\sum_{n=1}^{\infty} a^n\right\| \leqslant \sum_{n=1}^{\infty} \|a\|^n = \frac{1}{1 - \|a\|} - 1$$

so that $a \mapsto (1-a)^{-1}$ at $a = 0$. This implies the inversion $a \mapsto a^{-1} = (1-(1-a))^{-1}$ is continuous on $B_1(1)$, and hence on $xB_1(1)$ for all $x \in \mathcal{A}^\times$. $\square$

3.1.1 Spectrum

Let $\mathcal{A}$ be a unital Banach algebra. For $a \in \mathcal{A}$ we denote by

$$\operatorname{Res}(a) := \{\lambda \in \mathbb{C} \mid \lambda 1 - a \in \mathcal{A}^\times\}$$

the **resolvent set** of $a \in \mathcal{A}$. This is an open set in $\mathbb{C}$ for $\lambda \mapsto \lambda 1 - a$ is a continuous map and $\mathcal{A}^\times \subseteq \mathcal{A}$ is open. Its complement

$$\sigma_{\mathcal{A}}(a) := \mathbb{C} \backslash \operatorname{Res}(a)$$

is called the **spectrum** of a, and it is a closed set.

Example 3.1.5.

(a) $\mathcal{A} = M_n(\mathbb{C}) = \operatorname{End}(\mathbb{C}^n)$. Then the spectrum of $M \in \mathcal{A}$ consists of the eigenvalues of M in $\mathbb{C}$.

(b) Let X be a compact space and $\mathcal{A} = C(X)$. Then $\sigma(f) = \operatorname{Im} f = f(X)$ for any $f \in \mathcal{A}$.

(c) For a unital Banach algebra $\mathcal{A}$, we have $\sigma(ab)\backslash\{0\} = \sigma(ba)\backslash\{0\}$ for every $a, b \in \mathcal{A}$. For if $1 - ab$ is invertible with inverse c, then $1 - ba$ is invertible with inverse $1 + bca$:

$$(1-ba)(1+bca) = 1 + bca - ba - babca = 1 + b(c - 1 - abc)a = 1 + b(c(1-ab) - 1)a = 1$$

and $(1+bca)(1-ba) = 1 - ba + bca - bcaba = 1 - b(1 - c(1-ab))a = 1$.

Lemma 3.1.6. Let $\mathcal{A}$ be a unital Banach algebra. Then for every $a \in \mathcal{A}$ we have $\sigma_{\mathcal{A}}(a) \subseteq \{z \in \mathbb{C} \mid |z| \leqslant \|a\|\}$. In particular, $\sigma_{\mathcal{A}}(a)$ is compact.

Proof. If $|\lambda| > \|a\|$, then $\left\|\frac{a}{\lambda}\right\| < 1$ so that $\lambda 1 - a$ is invertible. ☐

Definition. Let $D \underset{\text{open}}{\subseteq} V$ and V be a Banach space. A map $f : D \to V$ is **holomorphic** if for every $z \in D$ the limit

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists in V.

- If f is holomorphic and $\alpha : V \to \mathbb{C}$ is continuous and $\mathbb{C}$-linear, then $\alpha \circ f : D \to \mathbb{C}$ is holomorphic in the usual sense.
- A holomorphic function is continuous.

Lemma 3.1.7. Let $a \in \mathcal{A}$. Then the map

$$\begin{aligned} f : \operatorname{Res}(a) &\longrightarrow \mathcal{A}^\times \\ \lambda &\longmapsto (\lambda 1 - a)^{-1} \end{aligned}$$

is holomorphic and vanishes at infinity.

Proof. For $\lambda, \rho \in \operatorname{res}(a)$, we have

$$f(\lambda) - f(\rho) = (\lambda 1 - a)^{-1}((\rho 1 - a) - (\lambda 1 - a))(\rho 1 - a)^{-1} = (\rho - \lambda) f(\lambda) f(\rho)$$

Put $\lambda = \rho + h$ with $h \in \mathbb{C}^\times$; then $\frac{1}{h}(f(\rho + h) - f(\rho)) = -f(\rho + h) f(\rho)$. Since f is continuous, letting $h \to 0$ gives $f'(\rho) = -f(\rho)^2$. This shows f is holomorphic.

For the last assertion, if $|\lambda| > 2\|a\|$, then

$$\left\|(\lambda 1 - a)^{-1}\right\| = |\lambda^{-1}| \left\|(1 - \lambda^{-1} a)^{-1}\right\| \leqslant |\lambda^{-1}| \sum_{n=0}^{\infty} \left\|\lambda^{-1} a\right\|^n \leqslant 2|\lambda|^{-1}.$$

This implies $\left\|(\lambda 1 - a)^{-1}\right\| \to 0$ as $|\lambda| \to \infty$, so f vanishes at infinity. ☐

Theorem 3.1.8. Let $\mathcal{A}$ be a unital Banach algebra, and let $a \in \mathcal{A}$. Then $\sigma_{\mathcal{A}}(a) \neq \varnothing$.

Proof. Suppose otherwise; then $a \neq 0$, $\operatorname{Res}(a) = \mathbb{C}$ and the map $f : \lambda \mapsto (\lambda 1 - a)^{-1}$ in the above lemma is bounded and entire. Hence for all $\alpha \in \mathcal{A}^\vee$, the composition $\alpha \circ f$ is bounded and entire, so it follows by Liouville's theorem that $\alpha \circ f$ is constant. Since f vanishes at infinity, we deduce that $\alpha \circ f \equiv 0$. By Hahn-Banach theorem, we see $f \equiv 0$, a contradiction. ☐

Corollary 3.1.8.1 (Gelfand-Mazur)**.** Let $\mathcal{A}$ be a unital Banach algebra that is also a division ring. Then $\mathcal{A} = \mathbb{C}1$.

Proof. For $a \in \mathcal{A}$, if $a \notin \mathbb{C}1$, then $\sigma_{\mathcal{A}}(a)$ is empty, which is absurd by Theorem. Hence $a \in \mathbb{C}1$. ☐

Definition. For an element a of a unital Banach algebra $\mathcal{A}$ we define the spectral radius $r(a)$ of a by

$$r(a) := \sup\{|\lambda| \mid \lambda \in \sigma_{\mathcal{A}}(a)\} \leqslant \|a\|$$

Theorem 3.1.9 (Spectral radius formula)**.** Let $\mathcal{A}$ be a unital Banach algebra. Then

$$r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}$$

Proof. If $\lambda 1 - a$ is not invertible, then neither is $\lambda^n 1 - a^n$ for each $n \in \mathbb{N}$. Hence $\lambda^n \in \sigma(a^n)$ and $|\lambda| \leqslant \|a^n\|^{\frac{1}{n}}$ so that $r(a) \leqslant \|a^n\|^{\frac{1}{n}}$. For the converse, we show $\limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}} \leqslant r(a)$. Recall

$$(\lambda 1 - a)^{-1} = \lambda^{-1}(1 - \lambda^{-1}a)^{-1} = \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}} \tag{$\diamond$}$$

holds for all $|\lambda| > \|a\|$. From the Cauchy estimate in complex analysis (c.f. Corollary D.7.5.1) we see expansion is in fact valid when $|\lambda| > r(a)$ (if $|\lambda| > r(a)$, then $(\lambda 1 - a)^{-1}$ is defined). It follows that $\limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}} < |\lambda|$ (view ($\diamond$) as a power series in λ^{-1}) for all $|\lambda| > r(a)$, and hence $\limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}} \leqslant r(a)$. ☐

Lemma 3.1.10. Suppose $\mathcal{B}$ is a unital Banach algebra and $\mathcal{A}$ is a closed subalgebra with $1 \in \mathcal{A}$. Then

$$\partial\sigma_{\mathcal{A}}(a) \subseteq \partial\sigma_{\mathcal{B}}(a) \subseteq \sigma_{\mathcal{B}}(a) \subseteq \sigma_{\mathcal{A}}(a)$$

for all $a \in \mathcal{A}$.

Proof. The only nontrivial part is the first containment. We prove it by contradiction: suppose $\lambda \in \partial\sigma_{\mathcal{A}}(a)\backslash\partial\sigma_{\mathcal{B}}(a)$. Then $\lambda \in \mathbb{C}\backslash\sigma_{\mathcal{B}}(a)$, for int $\sigma_{\mathcal{B}}(a) \subseteq$ int $\sigma_{\mathcal{A}}(a)$ is disjoint from $\partial\sigma_{\mathcal{A}}(a)$ by definition. In particular, $\lambda 1 - a$ is invertible in $\mathcal{B}$. Now let $(\lambda_n)_n \subseteq \mathbb{C}\backslash\sigma_{\mathcal{A}}(a)$ be a sequence that converges to λ. Then $\lambda_n 1 - a$ converges to $\lambda 1 - a$, and by continuity of inverse we see $(\lambda_n 1 - a)^{-1} \to (\lambda 1 - a)^{-1}$ in $\mathcal{B}$; since $\mathcal{A}$ is closed, in fact $(\lambda 1 - a)^{-1} \in \mathcal{A}$, or $\lambda \notin \sigma_{\mathcal{A}}(a)$ a contradiction. ☐

Example 3.1.11 (Disc-algebra)**.** Let $\mathbb{D} \subseteq \mathbb{C}$ be the open unit disc in $\mathbb{C}$. The **disc algebra** $\mathcal{A}$ is by definition the subalgebra of $C(\overline{\mathbb{D}})$ consisting of all functions that are holomorphic on $\mathbb{D}$. $\mathcal{A}$ is closed since a uniform limit of holomorphic functions is again holomorphic.

Let $\mathbb{T} = \partial\mathbb{D}$ be the circle group. By maximum principle, the restriction $\mathcal{A} \to C(\mathbb{T})$ is an isometry so that $\mathcal{A}$ can be viewed as a Banach subalgebra of $C(\mathbb{T})$. We know $\sigma_{\mathcal{A}}(f) = f(\overline{\mathbb{D}})$ and $\sigma_{C(\mathbb{T})}(f) = f(\mathbb{T})$ for all $f \in \mathbb{A}$.

Proposition 3.1.12 (Spectral mapping theorem for polynomials)**.** Let $\mathcal{A}$ is a unital algebra and $a \in \mathcal{A}$ with $\sigma_{\mathcal{A}}(a) \neq \varnothing$. If $p \in \mathbb{C}[z]$, then $\sigma_{\mathcal{A}}(p(a)) = p(\sigma_{\mathbb{A}}(a))$.

Proof. If p is a constant, this is clear. Suppose otherwise. For $\mu \in \mathbb{C}$, we write

$$p(z) - \mu = \lambda_0(\lambda_1 - z)\cdots(\lambda_n - z)$$

for some $n \in \mathbb{N}$, $\lambda_i \in \mathbb{C}$ with $\lambda_0 \neq 0$.

If $\mu \notin \sigma_{\mathcal{A}}(p(a))$, then $p(a) - \mu 1_{\mathcal{A}}$ is invertible, and hence so is each $\lambda_i 1_{\mathcal{A}} - a$. The converse is true obviously. Hence we have $\mu \in \sigma_{\mathcal{A}}(p(a))$ if and only if $\lambda_{\in}\sigma_{\mathbb{A}}(a)$ for some $1 \leqslant i \leqslant n$, and thus $\sigma_{\mathcal{A}}(p(a)) \subseteq p(\sigma_{\mathcal{A}}(a)$. For the reversed inclusion, if $\lambda \in \sigma_{\mathcal{A}}(a)$, then $p(a) - p(\lambda) = (\lambda 1_{\mathcal{A}} - a)b$ for some $b \in \mathcal{A}$, and hence $p(\lambda) \in \sigma_{\mathcal{A}}(p(a))$. ☐

3.1.2 Unitization

Definition. For any Banach algebra $\mathcal{A}$, the **unitization** of $\mathcal{A}$ is the Banach algebra

$$\mathcal{A}^e := \mathcal{A} \times \mathbb{C}$$

with obvious vector space structure, multiplication defined by $(a,\lambda)(b,\mu) = (ab + \mu a + \lambda b, \lambda\mu)$ and norm $\|(a,\lambda)\| := \|a\| + |\lambda|$.

- We view $\mathcal{A}$ as $\mathcal{A} \times \{0\} \subseteq \mathcal{A}^e$, and this is an embedding of Banach algebras.
- If $\mathcal{A}$ has unit 1, then we have an isomorphism of algebras

  $$\begin{aligned} \mathcal{A}^e &\longrightarrow \mathcal{A} \oplus \mathbb{C} \\ (a,\lambda) &\longmapsto (a + \lambda 1, \lambda) \end{aligned}$$

  where the multiplication on $\mathcal{A} \oplus \mathbb{C}$ is defined componentwise.
- The unitization defines a functor from the category of Banach algebras to the category of unital Banach algebras: for a homomorphism $\phi : \mathcal{A} \to \mathcal{B}$, define $\phi^e : \mathcal{A}^e \to \mathcal{B}^e$ by $\phi^e(a,\lambda) = (\phi(a),\lambda)$.

If $\mathcal{A}$ is a Banach space without unit, for each $a \in \mathcal{A}$ we define the **spectrum** of a to be

$$\sigma_{\mathcal{A}}(a) := \sigma_{\mathcal{A}^e}(a)$$

where we use the identification of $\mathcal{A}$ mentioned above.

Definition. Let X be an LCH space.

1. A function $f : X \to \mathbb{C}$ is said to **vanish at infinity** if for every $\varepsilon > 0$ there exists a compact set $K \subseteq X$ such that $|f(x)| < \varepsilon$ for all $x \in X \backslash K$.
2. Denote by $C_0(X)$ the subalgebra of $C(X)$ consisting of all continuous functions on X that vanish at infinity.

Then $C_0(X)$ is a Banach algebra with the sup-norm $\|f\|_X = \sup\limits_{x \in X} |f(x)|$. Note that $C_0(X)$ is unital if and only if X is compact, and in this case $C_0(X) = C(X)$.

Proof. We must show $C_0(X)$ is complete with respect to $\|\cdot\|_X$. Let $(f_n)_n$ be a Cauchy sequence in $C_0(X)$. About each $x \in X$ pick a compact neighborhood K_x. Then $(f_n|_{K_x})_n$ is uniformly Cauchy, so that we can define $f(x) := \lim\limits_{n\to\infty} f_n(x)$; it is clear f is well-defined and is continuous. Now fix $\varepsilon > 0$ and take $N \in \mathbb{N}$ such that $\|f_n - f_N\|_X < \varepsilon/3$ whenever $n \geqslant N$. Let $K \underset{\text{cpt}}{\subseteq} X$ such that $|f_N(x)| < \varepsilon/3$ for all $x \in X\backslash K$. For $x \in X - K$ there exists $m \geqslant N$ such that $|f(x) - f_m(x)| < \varepsilon/3$, and thus $|f(x)| \leqslant |f(x) - f_m(x)| + |f_m(x) - f_N(x)| + |f_N(x)| < \varepsilon$. This proves f vanishes at infinity. $\square$

Example 3.1.13. Let X be an LCH space and put $X^\infty = X \cup \{\infty\}$ to be the one point compactification of X. Every open neighborhood of ∞ in X^∞ is of the form $X\backslash K$ with $K \underset{\text{cpt}}{\subseteq} X$. With this in mind, we can identify $C_0(X)$ as a subspace of $C(X^\infty)$ consisting of all continuous functions f such that $f(\infty) = 0$. This justifies the notion "vanishing at infinity".

Lemma 3.1.14. There is a canonical topological isomorphism of Banach algebras $C(X^\infty) \cong C_0(X)^e$.

Proof. Define

$$\begin{aligned} \Phi : C(X^{\infty}) &\longrightarrow C_0(X)^e \\ f &\longmapsto (f - f(\infty), f(\infty)) \end{aligned}$$

It is clear that Φ is bijective.

- Homomorphism:

$$\begin{aligned} \Phi(f)\Phi(g) &= ((f - f(\infty))(g - g(\infty)) + g(\infty)(f - f(\infty)) + f(\infty)(g - g(\infty)), f(\infty)g(\infty)) \\ &= (fg - f(\infty)g(\infty), f(\infty)g(\infty)) = \Phi(fg) \end{aligned}$$

- Continuity:

$$\|\Phi(f) - \Phi(g)\| = \|f - f(\infty) - g + g(\infty)\|_X + |f(\infty) - g(\infty)| \leqslant 3\,\|f - g\|_{X^{\infty}}$$

- Continuous inverse: Let (f, λ), $(g, \rho) \in C_0(X)^e$.

$$\|(f + \lambda) - (g + \rho)\|_{X^{\infty}} \leqslant \|f - g\|_{X^{\infty}} + |\lambda - \rho| = \|f - g\|_X + |\lambda - \rho| = \|(f, \lambda) - (g - \rho)\|$$

$\square$

3.1.3 Gelfand transform

Definition. Let $\mathcal{A}$ be a commutative Banach algebra. The **structure space** $\Delta_{\mathcal{A}}$ is the set of all nonzero continuous algebra homomorphisms $\mathcal{A} \to \mathbb{C}$.

- If $\mathcal{A}$ is unital, then for each $m \in \Delta_{\mathcal{A}}$, $m(1) = m(1)^2$ and the assumption $m \neq 0$ imply $m(1) = 1$.
- For each $m \in \Delta_{\mathcal{A}}$, there exists precisely one extension $m^e : \mathcal{A}^e \to \mathbb{C}$ of m defined by

$$m^e(a, \lambda) = m(a) + \lambda$$

  Indeed, we must have $m^e(a, 0) = m(a)$ and $m^e(1_{\mathcal{A}^e}) = m^e(0, 1) = 1$.

- If $m' \in \Delta_{\mathcal{A}^e}$ is not extended from $\mathcal{A}$, then it must vanish on $\mathcal{A}$ by the uniqueness of extension. Hence we must have $m' = m_{\infty}$ is the **augmentation functional**, where $m_{\infty} \in \Delta_{\mathcal{A}^e}$ is defined by

$$m_{\infty}(a, \lambda) = \lambda$$

  In other words, $\Delta_{\mathcal{A}^e} = \{m^e \mid m \in \Delta_{\mathcal{A}}\} \cup \{m_{\infty}\}$.

Lemma 3.1.15. Let $\mathcal{A}$ be a commutative Banach algebra and $m \in \Delta_{\mathcal{A}} \subseteq \mathcal{A}^{\vee}$. Then $\|m\|_{\mathrm{op}} \leqslant 1$. If $\mathcal{A}$ is unital, $\|m\|_{\mathrm{op}} = 1$.

Proof. Suppose $\mathcal{A}$ is unital. Then

$$\|m\| = \sup_{\|a\|=1} |m(a)| \geqslant |m(1)| = 1$$

On the other hands, $m(a - m(a)1) = m(a) - m(a) = 0$ so $m(a) \in \sigma_{\mathcal{A}}(a)$; this gives $|m(a)| \leqslant \|a\|$, so that $\|m\| \leqslant 1$. If $\mathcal{A}$ is not unital, then we still have

$$1 = \|m^e\| = \sup_{\|(a,\lambda)\|=1} |m(a) + \lambda| \geqslant \sup_{\|(a,\lambda)\|=1, \lambda=0} |m(a) + \lambda| = \|m\|$$

or just $1 = \|m^e\| \geqslant \|m^e|_{\mathcal{A}}\|$. $\square$

Definition. For each normed space V, the **weak-*-topology** on $V^\vee$ is the initial topology induced by the set of functions $\{\delta_v \mid v \in V\}$, where $\delta_v : V^\vee \to \mathbb{C}$ is the evaluation at $v \in V$, namely

$$\delta_v(\alpha) := \alpha(v) \in \mathbb{C}$$

- The weak-*-topology is the same as the topology of pointwise convergence, i.e. the subspace topology given by the product space $\mathbb{C}^V = \prod_{v \in V} \mathbb{C}$.

  Proof. By definition a subbasis for the weak-*-topology consists of subsets of the form

  $$L(v, U) := \{\alpha \in V^\vee \mid \alpha(v) \in U\}$$

  where $U \underset{\text{open}}{\subseteq} \mathbb{C}$ and $v \in V$. Then by definition $\bigcap_{i=1}^{n} L(v_i, U_i)$ is the basis for the product space $\mathbb{C}^V$. ☐

For any commutative Banach algebra $\mathcal{A}$, we equip $\Delta_{\mathcal{A}}$ with the subspace topology induced from the weak-* topology on the dual space $\mathcal{A}^\vee$.

Theorem 3.1.16 (Banach-Alaoglu)**.** Let V be a complex normed space. Then the closed unit ball

$$\overline{B}' := \{f \in V^\vee \mid \|f\| \leqslant 1\} \subseteq V^\vee$$

is a compact Hausdorff space under the weak-* topology.

Proof. Since $\mathbb{C}^V$ is Hausdorff, so is $\overline{B}'$. For the compactness, consider the product $K := \prod_{v \in V} \overline{B_1(0)} \subseteq \mathbb{C}^V$ equipped with subspace topology; this is compact by Tychonov's theorem. Since $\overline{B}' \subseteq K$ and $\overline{B}' = V^\vee \cap K$, it suffices to show $V^\vee$ is closed in $\mathbb{C}^V$.

- Take $(x_v)_v \in \mathbb{C}^V$ such that $x_{v+w} \neq x_v + x_w$ for some $v, w \in V$. Then take respective neighborhoods U_v, U_w, U_{v+w} of x_v, x_w, x_{v+w} such that $(U_v + U_w) \cap U_{v+w} = \varnothing$. Then $U_v \times U_w \times U_{v+w} \times \prod \mathbb{C}$ is disjoint from $V^\vee$.

- If $(x_v)_v \in \mathbb{C}^V$ is such that $\lambda x_v \neq x_{\lambda v}$ for some $\lambda \in \mathbb{C}$ and $v \in V$, then take U, V such that $\lambda U \cap V = \varnothing$ with $x_v \in V$, $x_{\lambda v} \in U$. Then $U \times V \times \prod \mathbb{C}$ again disjoint from $V^\vee$.

☐

Lemma 3.1.17. Let $\mathcal{A}$ be a commutative Banach algebra. Then the inclusion

$$\begin{aligned} \Phi : \Delta_{\mathcal{A}} &\longrightarrow \Delta_{\mathcal{A}^e} \\ m &\longmapsto m^e \end{aligned}$$

is a homeomorphism onto its image.

Proof. Φ is clearly injective for $m^e|_{\mathcal{A}} = m$. To show it is a topological embedding, we have the following equivalence

$$\begin{aligned} m_j \to m \text{ in } \Delta_{\mathcal{A}} &\Leftrightarrow m_j(a) \to m(a) \text{ in } \mathbb{C} \text{ for each } a \in \mathcal{A} \\ &\Leftrightarrow m_j(a) + \lambda \to m(a) + \lambda \text{ in } \mathbb{C} \text{ for each } a \in \mathcal{A} \\ &\Leftrightarrow m_j^e(a) \to m^e(a) \text{ in } \mathbb{C} \text{ for each } a \in \mathcal{A}^e \\ &\Leftrightarrow m_j^e \to m^e \text{ in } \Delta_{\mathcal{A}}^e \end{aligned}$$

☐

Definition. For each $a \in \mathcal{A}$ define

$$\begin{aligned} \hat{a} : \Delta_{\mathcal{A}} &\longrightarrow \mathbb{C} \\ m &\longmapsto \hat{a}(m) := m(a) \end{aligned}$$

- $\hat{a}$ is continuous. Indeed, for each open U in $\mathbb{C}$,

$$\hat{a}^{-1}(U) = \{m \in \Delta_{\mathcal{A}} \mid m(a) \in U\} = \left(U \times \prod \mathbb{C}\right) \cap \mathbb{C}^{\mathcal{A}} \underset{\text{open}}{\subseteq} \mathbb{C}^{\mathcal{A}}$$

The association $\mathcal{A} \ni a \mapsto \hat{a} \in C(\Delta_{\mathcal{A}})$ is an algebra homomorphism, called the **Gelfand transform**.

Theorem 3.1.18. Let $\mathcal{A}$ be a commutative Banach algebra.

1. $\Delta_{\mathcal{A}}$ is an LCH space.
2. If $\mathcal{A}$ is unital, $\Delta_{\mathcal{A}}$ is compact.
3. For every $a \in \mathcal{A}$, $\hat{a} \in C_0(\Delta_{\mathcal{A}})$. The Gelfand transform

$$\begin{aligned} \mathcal{A} &\longrightarrow C_0(\Delta_{\mathcal{A}}) \\ a &\longmapsto \hat{a} \end{aligned}$$

   is an algebra homomorphism.
4. For every $a \in \mathcal{A}$ one has $\|\hat{a}\|_{\Delta_{\mathcal{A}}} \leqslant \|a\|$ so that the Gelfand transform is continuous.

Proof. Note that by a previous lemma, we have the containment $\Delta_{\mathcal{A}} \subseteq \overline{B}' \subseteq \mathcal{A}^{\vee}$ where $\overline{B}'$ is defined in the statement of Banach-Alaoglu. This means the closure $\overline{\Delta_{\mathcal{A}}}$ is thus compact in $\mathcal{A}^{\vee}$. We contend that

- $\overline{\Delta_{\mathcal{A}}} = \Delta_{\mathcal{A}}$ if $\mathcal{A}$ is unital. (This proves 2.)
- $\overline{\Delta_{\mathcal{A}}} = \Delta_{\mathcal{A}}$ or $\Delta_{\mathcal{A}} \cup \{0\}$ if $\mathcal{A}$ is not unital, where 0 is the zero map. (This proves 1.)

We first show that every element of $\overline{\Delta_{\mathcal{A}}}$ is an algebra homomorphism. The set of continuous algebra homomorphism is described by

$$S = \{(x_a)_a \in \mathcal{A}^{\vee} \mid x_{ab} = x_a x_b \text{ for all } a, b \in \mathcal{A}\}$$

Suppose $(x_a)_a \in \mathcal{A}^{\vee}$ is such that $x_{ab} \neq x_a x_b$ for some $a, b \in \mathcal{A}$. Then take respective neighborhoods U_a, U_b, U_{ab} of x_a, x_b, x_{ab} such that the product $U_a U_b$ is disjoint from U_{ab}. Then $(U_a \times U_b \times U_{ab} \times \prod \mathbb{C}) \cap \mathcal{A}^{\vee}$ is a neighborhood of $(x_a)_a$ that does not meet S. Hence S is closed, which implies that $\overline{\Delta_{\mathcal{A}}} \subseteq S$. Note that by definition $S \backslash \Delta_{\mathcal{A}} \subseteq \{0\}$. It shows our contention except the case $\mathcal{A}$ is unital. In the unital case, we have $\Delta_{\mathcal{A}} \subseteq (\{1\} \times \prod \mathbb{C}) \cap S$; the latter is a closed set in S so that $0 \notin \overline{\Delta_{\mathcal{A}}}$. Hence $\overline{\Delta_{\mathcal{A}}} = S = \Delta_{\mathcal{A}}$.

Now we prove 3. If $\Delta_{\mathcal{A}}$ is compact, the result ensues automatically. Otherwise, what we have proved above shows $\overline{\Delta_{\mathcal{A}}} = \Delta_{\mathcal{A}} \cup \{0\}$ coincides with the one point compactification of $\Delta_{\mathcal{A}}$. Taking a small neighborhood of 0 in $\overline{\Delta_{\mathcal{A}}}$ not touching 1 proves the vanishing at infinity of $\hat{a}$. Finally for 4., we have

$$\|\hat{a}\| = \sup_{m \in \Delta_{\mathcal{A}}} |\hat{a}(m)| = \sup_{m \in \Delta_{\mathcal{A}}} |m(a)| \leqslant \|m\| \, \|a\| = \|a\|$$

□

Example 3.1.19. Consider the case $\mathcal{A} = C_0(X)$ for an LCH space X. For each $x \in X$ let $m_x : \mathcal{A} \to \mathbb{C}$ be the evaluation map at x. Then $x \mapsto m_x$ is a homeomorphism from X to the structure space $\Delta_{\mathcal{A}}$. If X is not compact, consider the diagram

$$\begin{tikzcd} X^\infty \arrow[r] & \Delta_{C(X^\infty)} \\ X \arrow[r] \arrow[u, hook] & \Delta_{C_0(X)} \arrow[u, hook] \end{tikzcd}$$

where the vertical maps are canonical inclusions (Lemma 3.1.17 and 3.1.14) and the horizontal maps are $x \mapsto m_x$. If the case when the space is compact is shown, then the upper horizontal map is a homeomorphism, implying that the lower horizontal one is also a homeomorphism. Hence it suffices to deal with compact X.

Since both X and $\Delta_{\mathcal{A}}$ are compact Hausdorff, it suffices to show $x \mapsto m_x$ is a continuous bijection. The continuity is clear. The injectivity follows from Urysohn's lemma. For the surjectivity, let $m \in \Delta_{\mathcal{A}}$. Consider the vanishing set

$$V = \{x \in X \mid f(x) = 0 \text{ for all } f \in \ker m\}$$

If V is not empty, take $x \in V$. Then $\ker m = \ker m_x$. For each $f \in \Delta_{\mathcal{A}}$, $m_x(f - f(x)1) = 0$ so that $0 = m(f - f(x)1)$, namely, $m(f) = m(f(x)1) = m_x(f)$, showing the surjectivity. If V is empty, then for each $x \in X$ there exists $f_x \in \ker m$ such that $f_x(x) \neq 0$, and by continuity $f_x(y) \neq 0$ for $y \in U_x$ and some neighborhood U_x of x. By compactness we can choose $x_1, \ldots, x_n$ such that $U_{x_1} \cup \cdots \cup U_{x_n} = X$. Then $f_{x_1}^2 + \cdots + f_{x_n}^2$ vanishes nowhere on X, so it is unit in $C(X)$, contradicting to $f_{x_1}^2 + \cdots + f_{x_n}^2 \in \ker m$.

Lemma 3.1.20. Suppose that $\phi : \mathcal{A} \to \mathcal{B}$ is an algebra homomorphism between commutative Banach algebras such that $m \circ \phi \neq 0$ for every $m \in \Delta_{\mathcal{B}}$. Then the pullback map

$$\begin{aligned} \phi^* : \Delta_{\mathcal{B}} &\longrightarrow \Delta_{\mathcal{A}} \\ m &\longmapsto m \circ \phi \end{aligned}$$

is continuous. Moreover, it is a homeomorphism if it is bijective.

Proof. Let $U_1, \ldots, U_n$ be open sets in $\mathcal{C}$. We must show that for any $a_1, \ldots, a_n \in \mathcal{A}$, the set

$$S := \{m \in \Delta_{\mathcal{B}} \mid (m \circ \phi)(a_i) \in U_i \text{ for } i = 1, \ldots, n\}$$

is open in $\Delta_{\mathcal{B}}$. Indeed, if we put $b_i = \phi(a_i)$, then $\{m \in \Delta_{\mathcal{B}} \mid m(b_i) \in U_i \text{ for } i = 1, \ldots, n\}$ is contained in S, and the former set is open in $\Delta_{\mathcal{B}}$ by the very definition. Now consider the diagram

$$\begin{tikzcd} \Delta_{\mathcal{B}^e} \arrow[r, "\phi^{e*}"] & \Delta_{\mathcal{A}^e} \\ \Delta_{\mathcal{B}} \arrow[r, "\phi^*"] \arrow[u, hook] & \Delta_{\mathcal{A}} \arrow[u, hook] \end{tikzcd}$$

where $\phi^e : \mathcal{A}^e \to \mathcal{B}^e$ is the map defined by $\phi^e(a, \lambda) = (\phi(a), \lambda)$. ϕ^{e*} is well-defined for ϕ^* is. We have

$$(\phi^*(m))^e(a, \lambda) = \phi^*(m)(a) + \lambda = m(\phi(a)) + \lambda = m^e(\phi(a), \lambda) = m^e(\phi^e(a, \lambda)) = \phi^{e*}(m^e)(a, \lambda)$$

so the diagram commutes. If ϕ^* is bijective, so is ϕ^{e*}. Since Δ_e and $\Delta_{\mathcal{A}^e}$ are compact, ϕ^{e*} is a homeomorphism. By restricting to $\Delta_{\mathcal{B}}$ we see ϕ^* is also a homeomorphism if ϕ is bijective. $\square$

3.1.4 Maximal Ideals

Definition. Let $\mathcal{A}$ be an algebra. An linear subspace I of $\mathcal{A}$ is called an **ideal** if I is stable under the action of $\mathcal{A}$ on two sides.

1. I is **proper** if $I \subsetneq \mathcal{A}$.
2. I is **maximal** if I is proper and is maximal among all proper ideal with respect to the inclusion.

- If $\mathcal{A}$ is unital, then any proper ideal contains no unit in $\mathcal{A}$.
- In the case $\mathcal{A}$ is a unital Banach algebra, since $\mathcal{A}^\times$ is open, the closure of any proper ideal is still a proper ideal. In particular, a maximal ideal is closed.
- By a Zorn's lemma argument we see if $\mathcal{A}$ is a unital Banach algebra, then every proper ideal is contained in a maximal ideal.
- Let $\mathcal{A}$ be as above. If $\mathcal{A}$ is commutative, then the set of non unit elements forms a proper ideal.

Definition. Let $\mathcal{A}$ be a Banach algebra and $I \trianglelefteq \mathcal{A}$ a closed ideal. The quotient algebra $\mathcal{A}/I$ equipped with the quotient norm defined in Lemma E.1.9 is again a Banach algebra.

Proof. We still need to show $\|ab + I\| \leqslant \|a + I\| \, \|b + I\|$ for $a, b \in \mathcal{A}$.

$$\|a + I\| \, \|b + I\| \geqslant \inf_{x,y \in I} \|(a + x)(b + y)\| = \inf_{x,y \in I} \|ab + (ay + xb + xy)\| \geqslant \inf_{z \in I} \|ab + z\| = \|ab + I\|$$

□

Theorem 3.1.21. Let $\mathcal{A}$ be a commutative unital Banach algebra.

1. The map $m \mapsto \ker m$ is a bijection between $\Delta_{\mathcal{A}}$ and the maximal spectrum of $\mathcal{A}$.
2. $a \in \mathcal{A}^\times$ if and only if $m(a) \neq 0$ for every $m \in \Delta_{\mathcal{A}}$.
3. For $a \in \mathcal{A}$ one has $\sigma(a) = \operatorname{Im} \hat{a}$.

Proof. We first derive 2 and 3 from 1. The only if part of 2. is obvious. Conversely, if a is not invertible, then it is contained in some maximal ideal of $\mathcal{A}$, and by 1. this means $m(a) = 0$ for some $m \in \Delta_{\mathcal{A}}$. For 3. we have the following equivalences

$$\begin{aligned}
\lambda \in \sigma(a) &\Leftrightarrow a - \lambda 1 \notin \mathcal{A}^\times \\
&\Leftrightarrow m(a - \lambda 1) = 0 \text{ for some } m \in \Delta_{\mathcal{A}} \\
&\Leftrightarrow m(a) = \lambda \text{ for some } m \in \Delta_{\mathcal{A}} \\
&\Leftrightarrow \lambda \in \operatorname{Im} \hat{a}
\end{aligned}$$

It remains to show 1. Suppose $\ker m = \ker n$ for some $m, n \in \Delta_{\mathcal{A}}$. Then for each $a \in \mathcal{A}$, $m(a - m(a)) = 0 = n(a - m(a))$, i.e., $m(a) = n(a)$. Hence $m = n$. Now given an maximal ideal $\mathfrak{m}$ of $\mathcal{A}$. Then $\mathcal{A}/\mathfrak{m}$ is a unital Banach algebra that is also a field. By Gelfand-Mazur this implies $\mathcal{A}/\mathfrak{m} \cong \mathbb{C}$. This gives a nontrivial algebra homomorphism $m : \mathcal{A} \to \mathcal{A}/\mathfrak{m} \cong \mathbb{C}$ with kernel $\mathfrak{m}$. □

Corollary 3.1.21.1. Let $\mathcal{A}$ be a commutative non-unital Banach algebra. Then for all $a \in \mathcal{A}$, $\sigma(a) = \operatorname{Im} \hat{a} \cup \{0\}$.

Proof. By Theorem, $\sigma(a) = \operatorname{Im}\widehat{(a,0)}$, where $(a,0) \in \mathcal{A}^e$. Recall that $\Delta_{\mathcal{A}^e} = \{m^e \mid \in \mathcal{A}\} \cup \{m_\infty\}$. Hence

$$\operatorname{Im}\widehat{(a,0)} = \{m^e(a,0) \mid m \in \mathcal{A}\} \cup \{m_\infty(a,0)\} = \operatorname{Im}\hat{a} \cup \{0\}$$

□

3.2 The Gelfand-Naimark Theorem

3.2.1 C^*-algebra

Definition. Let $\mathcal{A}$ be an algebra over $\mathbb{C}$.

1. An **involution** $* : \mathcal{A} \to \mathcal{A}$ is an abelian group homomorphism such that for each $\lambda \in \mathbb{C}$ and $a, b \in \mathcal{A}$
   - $(\lambda a)^* = \overline{\lambda} a^*$.
   - $(ab)^* = b^* a^*$.
   - $(a^*)^* = a$.
2. $\mathcal{A}$ is called a **Banach *-algebra** if it is a Banach algebra equipped with an involution * such that $\|a^*\| = \|a\|$ for each $a \in \mathcal{A}$.
3. A Banach *-algebra $\mathcal{A}$ is called a C^***-algebra** if $\|a^*a\| = \|a\|^2$ for each $a \in \mathcal{A}$.
4. An element a of a Banach *-algebra is called **self-adjoint** if $a = a^*$.

Example 3.2.1.

1. Let H be a Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. Then the map $T \mapsto T^*$ is an involution, where T^* is the adjoint of T. $\mathcal{B}(H)$ is a C^*-algebra

   Proof. For each $v, w \in H$ one has $\langle Tv, w\rangle = \langle v, T^*w\rangle$. For each $v \in V$ by Cauchy-Schwarz

   $$\|Tv\|^2 = \langle T^*Tv, v\rangle \leqslant \|T^*Tv\|\,\|v\|$$

   so that $\dfrac{\|Tv\|^2}{\|v\|} \leqslant \dfrac{\|T^*Tv\|}{\|v\|}$. Hence $\|T\|^2 \leqslant \|T^*T\| \leqslant \|T^*\|\,\|T\|$, and thus $\|T\| \leqslant \|T^*\|$. By symmetry one has the equality, which forces $\|T\|^2 = \|T^*T\|$. □

2. Let X be an LCH space. Then $C_0(X)$ is a C^*-algebra where the involution is defined by the complex conjugation.

3. The disc algebra $\mathcal{A}$ (in Example 3.1.11) with involution defined by $f^*(z) := \overline{f(\overline{z})}$ is a Banach *-algebra but not a C^*-algebra.

   Proof. It is clear it is an Banach *-algebra. Let $f \in \mathcal{A}$.

   $$\|f^*f\| = \sup_{z\in\overline{\mathbb{D}}} |\overline{f(\overline{z})} f(z)| = \sup_{z\in\overline{\mathbb{D}}} |f(\overline{z}) f(z)|$$

   Consider the function $f(z) = e^{iz}$. Then

   $$\sup_{z\in\overline{\mathbb{D}}} |f(\overline{z}) f(z)| = \sup_{z\in\overline{\mathbb{D}}} |e^{i(\overline{z}+z)}| = 1$$

   On the other hand $\|f\|^2 = \sup |e^{iz}|^2 = \sup |e^{-\operatorname{Im}(z)}| = e^2$. □

Proposition 3.2.2. Let G be an LCH group. The Banach algebra $L^1(G)$ with the involution

$$f^*(x) := \Delta_G(x^{-1})\overline{f(x^{-1})}$$

is a Banach *-algebra but not a C^*-algebra unless G is trivial, in which case $L^1(G) = \mathbb{C}$.

Proof.

$$(f^*)^*(x) = \Delta_G(x^{-1})\overline{f^*(x^{-1})} = \Delta_G(x^{-1})\overline{\Delta_G(x)\overline{f(x)}} = f(x)$$

so it is an involution (the other conditions are obvious). Also

$$\|f^*\|_1 = \int_G |\Delta_G(x^{-1})\overline{f(x^{-1})}|dx \overset{2.3.1.4}{=} \int_G |f(x)|dx = \|f\|_1$$

For the last assertion, we need a lemma.

Lemma 3.2.3.

(a) Let X be an LCH space and let $x_1, \ldots, x_n \in X$ be distinct points. Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be any given numbers. Then there exists $f \in C_c(G)$ with $f(x_j) = \lambda_j$ for each j.

(b) Let G be an LCH group and $g \in C_c(G)$ be with the property $\left|\int_G g(y)dy\right| = \int_G |g(y)|dy$. Then there exists $\theta \in \mathbb{T}$ such that $g(x) \in \theta[0, \infty)$ for every $x \in G$.

Proof.

(a) Let $U_1, \ldots, U_n$ be pairwise disjoint open neighborhoods of $x_1, \ldots, x_n$ respective. By Urysohn's lemma we can find $f_i \in C_c(G)$ with $\operatorname{supp} f_i \subseteq U_i$ and $f_i(x_i) = 1$. Take $f(x) = \sum_{i=1}^{n} \lambda_i f_i(x)$.

(b) If $\int_G g(y)dy = 0$, then $\int_G |g(y)|dy = 0$ so that $g \equiv 0$. Let us suppose $\int_G g(y)dy \neq 0$. Replacing g by λg for some $\lambda \in \mathbb{T}$ we may assume $\int_G g(y)dy > 0$. Then

$$\begin{aligned} 0 &= \int_G |g(y)|dy - \int_G g(y)dy = \int_G |g(y)|dy - \operatorname{Re}\left(\int_G g(y)dy\right) \\ &= \int_G |g(y)|dy - \int_G \operatorname{Re}(g(y))dy \\ &= \int_G (|g(y)| - \operatorname{Re}(g(y)))dy \end{aligned}$$

so that $|g(y)| = \operatorname{Re}(g(y))$ for each $y \in G$. This means $g(y) \geqslant 0$.

□

Return to the proof of Proposition. We assume $L^1(G)$ is a C^*-algebra. Then for all $f \in C_c(G)$, one has $\|f * f^*\|_1 = \|f\|_1^2$, or

$$\int_G \left|\int_G \Delta_G(x^{-1}y)f(y)\overline{f(x^{-1}y)}dy\right| dx = \int_G \int_G |f(x)f(y)|dydx \overset{2.3.1.4}{=} \int_G \int_G \Delta_G(x^{-1}y)|f(y)\overline{f(x^{-1}y)}|dydx$$

It follows $\left|\int_G \Delta_G(x^{-1}y)f(y)\overline{f(x^{-1}y)}dy\right| = \int_G \Delta_G(x^{-1}y)|f(y)\overline{f(x^{-1}y)}|dy$, so by Lemma (b) for each $x \in G$ we can find θ_x such that $\Delta_G(x^{-1}y)f(y)\overline{f(x^{-1}y)} \in \theta_x[0, \infty)$ for all $y \in G$; since $\Delta_G \geqslant 0$, in fact $f(y)\overline{f(x^{-1}y)} \in \theta_x[0, \infty)$.

Suppose G is non-nontrivial. Pick $x_0 \neq 1$ in G. By Lemma (a) we can find $f \in C_G(x)$ with $f(x_0) = f(x_0^{-1}) = i$ and $f(1) = 1$. Now for $x = x_0$, we see

$$y = 1 \Rightarrow f(y)\overline{f(x^{-1}y)} = -i$$
$$y = x_0 \Rightarrow f(y)\overline{f(x^{-1}y)} = i$$

a contradiction. Hence G is trivial. □

Lemma 3.2.4. Suppose $\mathcal{A}$ is a C^*-algebra and that $a \in \mathcal{A}$ is self-adjoint, i.e., $a = a^*$. Then $r(a) = \|a\|$, where $r(a)$ is the spectral radius of a. Moreover, this holds for $a \in \mathcal{A}$ normal, i.e., $aa^* = a^*a$.

Proof. We already have $r(a) \leqslant \|a\|$. By C^* property have $\|a^2\| = \|a^*a\| = \|a\|^2$; it follows that $\|a^{2^n}\| = \|a\|^{2^n}$ for each n. By spectral radius formula, we obtain

$$r(a) = \lim_{n\to\infty} \left\|a^{2^n}\right\|^{\frac{1}{2^n}} = \|a\|$$

(If $\mathcal{A}$ is not unital, the spectrum of a is computed in the unitization of $\mathcal{A}$.)

Assume a normal. By C^*-property,

$$\|a^{2k}\| = \|(a^{2k})^*(a^{2k})\|^{\frac{1}{2}} = \|(a^*a)^{2k}\|^{\frac{1}{2}} = \|(a^*a)^k\|,$$

so by spectral radius formula we deduce $r(a) = r(a^*a)^{\frac{1}{2}}$. Since a^*a is self-adjoint,

$$r(a) = r(a^*a)^{\frac{1}{2}} = \|a^*a\|^{\frac{1}{2}} = \|a\|.$$

□

Corollary 3.2.4.1. If $\mathcal{A}$ is a C^*-algebra, then $\mathcal{A}$ has a unique C^*-norm.

Proof. Suppose that $\|\cdot\|$ is a norm on $\mathcal{A}$ such that $\mathcal{A}$ becomes a C^*-algebra. By Lemma for each $a \in \mathcal{A}$ we see $\|a\|^2 = \|a^*a\| = r(a^*a)$. The result now follows from the nature that the spectrum does nothing with the norm. □

Lemma 3.2.5. Let $\mathcal{A}$ be a C^*-algebra. Then $\mathcal{A}^e$ can be made into a C^*-algebra. Moreover, the embedding $\mathcal{A} \subseteq \mathcal{A}^e$ is isometric.

Proof. For $(a, \lambda) \in \mathcal{A}^e$, define $(a, \lambda)^* = (a^*, \overline{\lambda})$; it is clearly an involution on $\mathcal{A}^e$ extending that of on $\mathcal{A}$. For the norm, we first consider the case $\mathcal{A}$ being unital. Then there is an algebra homomorphism $\mathcal{A}^e \cong \mathcal{A} \oplus \mathbb{C}$, where the latter is equipped with componentwise multiplication. It is also a $*$-homomorphism if for each $(a, \lambda) \in \mathcal{A} \oplus \mathbb{C}$ we put $(a, \lambda)^* = (a^*, \overline{\lambda})$. Now defining $\|(a, \lambda)\| := \max\{\|a\|, |\lambda|\}$ clearly makes $\mathcal{A} \oplus \mathbb{C}$ a C^*-algebra and the embedding $a \mapsto (a, 0)$ is clearly isometric.

Now let us assume $\mathcal{A}$ is non-unital. Consider the homomorphism

$$L : \mathcal{A}^e \longrightarrow \mathcal{B}(\mathcal{A})$$
$$(a, \lambda) \longmapsto L_{(a,\lambda)} : b \mapsto ab + \lambda b$$

and define $\|(a, \lambda)\| := \|L_{(a,\lambda)}\|_{\text{op}}$.

1° $\|\cdot\|$ is a Banach algebra norm. It suffices to show L is injective. Suppose $ab + \lambda b = 0$ for all $b \in \mathcal{A}$.

- $\lambda = 0$. Then $ab = 0$ for all $b \in \mathcal{A}$. Letting $b = a^*$ and taking norm, by C^* property we have $0 = \|aa^*\| = \|a\|^2$ so $a = 0$.
- $\lambda \neq 0$. Then $b = \frac{a}{-\lambda}b$; we may simply assume $b = ab$. In particular, $a^* = aa^*$, implying a is self-adjoint. Hence $b^* = (ab)^* = b^*a$ for all $b \in \mathcal{A}$. It follows that a is the unit in $\mathcal{A}$, a contradiction.

It follows $(a, \lambda) = (0, 0)$ so that L is injective.

2° The inclusion $a \mapsto (a, 0)$ is isometric. On the one hand $\|ab\| \leqslant \|a\| \|b\|$ gives $\left\|L_{(a,0)}\right\| \leqslant \|a\|$. On the other hand $\|aa^*\| = \|a\| \|a^*\|$ tells $\left\|L_{(a,0)}\right\| \geqslant \|a\|$.

3° $\mathcal{A}^e$ is complete. Since $\mathcal{A}$ is a complete subspace of $\mathcal{A}^e$ of finite codimension, it follows from Lemma E.1.10 that $\mathcal{A}^e$ is also complete.

4° $\mathcal{A}^e$ is a C^*-algebra. Let $\varepsilon > 0$. By definition there exists $b \in \mathcal{A}$ with norm 1 such that $\|ab + \lambda b\| \geqslant \|(a, \lambda)\| (1 - \varepsilon)$. This implies

$$\begin{aligned}(1-\varepsilon)^2 \|(a,\lambda)\|^2 \leqslant \|ab + \lambda b\|^2 &= \|(ab+\lambda b)^* ab + \lambda b\| \\ &= \left\|(b^*, 0)(a^*, \overline{\lambda})(a, \lambda)(b, 0)\right\| \\ &\leqslant \|(b^*, 0)\| \left\|(a^*, \overline{\lambda})(a, \lambda)\right\| \|(b, 0)\| \\ &= \left\|(a^*, \overline{\lambda})(a, \lambda)\right\|\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows

$$\left\|(a, \lambda)^2\right\| \leqslant \left\|(a^*, \overline{\lambda})(a, \lambda)\right\| \leqslant \left\|(a^*, \overline{\lambda})\right\| \|(a, \lambda)\|$$

In particular we have $\|(a, \lambda)\| \leqslant \left\|(a^*, \overline{\lambda})\right\|$; by symmetry we have equality and hence we have equality everywhere.

□

Definition. If $\mathcal{A}$ is a Banach *-algebra and $a \in \mathcal{A}$, we define the **real part** and the **imaginary part** of a as

$$\operatorname{Re} a = \frac{1}{2}(a + a^*) \text{ and } \operatorname{Im} a = \frac{1}{2i}(a - a^*)$$

Then $\operatorname{Re} a$ and $\operatorname{Im} a$ are self-adjoint with $a = \operatorname{Re} a + i \operatorname{Im} a$.

3.2.2 Gelfand-Naimark

Lemma 3.2.6. If $\mathcal{A}$ is a commutative C^*-algebra, then $m(a^*) = \overline{m(a)}$ for every $a \in \mathcal{A}$ and every $m \in \Delta_{\mathcal{A}}$.

Proof. By passing to $\mathcal{A}^e$ if necessary, we may assume $\mathcal{A}$ is unital. If the statement holds for self-adjoint elements, then for general $a \in \mathcal{A}$,

$$m(a^*) = m(\operatorname{Re} a - i \operatorname{Im} a) = m(\operatorname{Re} a) - im(\operatorname{Im} a) = \overline{m(\operatorname{Re} a)} - \overline{im(\operatorname{Im} a)} = \overline{m(\operatorname{Re} a) + im(\operatorname{Im} a)} = \overline{m(a)}$$

Hence we may assume a is self-adjoint; in this case we must show $m(a) \in \mathbb{R}$. Write $m(a) = x + iy$ with $x, y \in \mathbb{R}$ and put $a_t = a + it$ for all $t \in \mathbb{R}$. Then

$$a_t^* a_t = (a - it)(a + it) = a^2 + t^2$$

and hence

$$x^2 + (y + t)^2 = |m(a_t)|^2 \leqslant \|a_t\|^2 = \|a_t^* a_t\| = \left\|a^2 + t^2\right\| \leqslant \|a\|^2 + t^2$$

so that $x^2 + y^2 + 2yt \leqslant \|a\|^2$, which forces $y = 0$. Thus $m(a) \in \mathbb{R}$ as we wished. □

Theorem 3.2.7 (Gelfand-Naimark)**.** If $\mathcal{A}$ is a commutative C^*-algebra, then the Gelfand transform

$$\begin{aligned} \mathcal{A} &\longrightarrow C_0(\Delta_{\mathcal{A}}) \\ a &\longmapsto \hat{a} \end{aligned}$$

is an isometric *-isomorphism. Furthermore, $\Delta_{\mathcal{A}}$ is compact if and only if $\mathcal{A}$ is unital.

Proof. We show $\mathcal{A} \to C_0(\Delta_{\mathcal{A}})$ is an isometric *-homomorphism with dense image $\hat{\mathcal{A}} = \{\hat{a} \mid a \in \mathcal{A}\}$. In particular, this shows $\hat{\mathcal{A}}$ is complete (thanks to isometry), and hence closed. Being dense, it follows that $\hat{\mathcal{A}} = C_0(\Delta_{\mathcal{A}})$. We first show it is an isometry. Indeed, by 3.1.21.3 and 3.2.4, for any $a \in \mathcal{A}$ one has

$$\|\hat{a}\|^2 = \|\bar{\hat{a}}\hat{a}\|^2 = \left\|\widehat{a^*a}\right\|^2 = r(a^*a) = \|a^*a\| = \|a\|^2$$

To show $\hat{\mathcal{A}}$ is dense in $C_0(\Delta_{\mathcal{A}})$, we use the Stone-Weierstrass, which is valid for $\Delta_{\mathcal{A}}$ is LCH by 3.1.18.

- $\hat{\mathcal{A}}$ separates points. Indeed, if $m, n \in \Delta_{\mathcal{A}}$ such that $\hat{a}(m) = \hat{a}(n)$ for each $a \in \mathcal{A}$, then $m = n$ since they agree everywhere.
- $\hat{\mathcal{A}}$ vanishes nowhere. This follows from our definition that $m \in \Delta_{\mathcal{A}}$ is nontrivial; hence we can find $a \in \mathcal{A}$ with $0 \neq m(a) = \hat{a}(m)$.
- $\hat{\mathcal{A}}$ is stable under complex conjugation. By 3.2.6, one has $\overline{\hat{a}(m)} = \overline{m(a)} = m(a^*) = \hat{a^*}(m)$ so that $\bar{\hat{a}} \in \hat{\mathcal{A}}$.

Hence $\hat{\mathcal{A}}$ is dense in $C_0(\Delta_{\mathcal{A}})$ with respect to the sup-norm. Finally, if $\Delta_{\mathcal{A}}$ is compact, then $\mathcal{A} \cong C_0(\Delta_{\mathcal{A}})$ is unital. The converse is 3.1.18. □

Let $\mathcal{A}$ and $\mathcal{B}$ be two commutative C^*-algebras. A morphism from $\mathcal{A}$ to $\mathcal{B}$ is a *-homomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ which defines a map $\Phi^* : \Delta_{\mathcal{B}} \to \Delta_{\mathcal{A}}$. This defines a category, and denote it by C^***-Alg**. Let **LCH** be a full subcategory of **Top** whose objects are locally compact Hausdorff spaces. In this section we met several (contravariant) functors:

$$\begin{aligned} F : \mathbf{LCH} &\longrightarrow C^*\textbf{-Alg} \\ X \xrightarrow{f} Y &\longmapsto C_0(X) \xleftarrow{f^*} C_0(Y) \end{aligned} \qquad \begin{aligned} G : C^*\textbf{-Alg} &\longrightarrow \mathbf{LCH} \\ \mathcal{A} \xrightarrow{\Phi} \mathcal{B} &\longmapsto \Delta_{\mathcal{A}} \xleftarrow{\Phi^*} \Delta_{\mathcal{B}} \end{aligned}$$

Gelfand-Naimark says that the Gelfand map defines a natural isomorphism from the identity functor to $F \circ G$: for $\Phi : \mathcal{A} \to \mathcal{B}$, one has a commutative diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\sim"] \arrow[d, "\Phi"'] & C_0(\Delta_{\mathcal{A}}) \arrow[d, "(\Phi^*)^*"] \\ \mathcal{B} \arrow[r, "\sim"] & C_0(\Delta_{\mathcal{B}}) \end{tikzcd}$$

On the other hand, the map in Example 3.1.19 defines a natural isomorphism from the identity functor of $G \circ F$: for $f : X \to Y$, one has a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[d, "f"'] & \Delta_{C_0(X)} \arrow[d, "(f^*)^*"] \\ Y \arrow[r] & \Delta_{C_0(Y)} \end{tikzcd}$$

(It is not quite clear from definition that F defines a functor, but the above commutative diagram shows that F is really a functor.) Hence, F and G define an equivalence of categories between **LCH** and C^***-Alg**.

There are typical subcategories of **LCH** and C^***-Alg**, namely the full subcategory of **LCH** whose objects are compact Hausdorff spaces, and the subcategory of C^***-Alg** whose objects are commutative unital C^*-algebras and morphisms are unital *-homomorphisms. Then F and G restrict to an equivalence of categories between them.

3.3 The Continuous Functional Calculus

Definition. Let $\mathcal{A}$ be a commutative unital C^*-algebra and $a \in \mathcal{A}$. The assignment

$$\begin{aligned} C(\sigma_{\mathcal{A}}(a)) &\longrightarrow C(\Delta_{\mathcal{A}}) \cong \mathcal{A} \\ f &\longmapsto f \circ \hat{a} =: f(a) \end{aligned}$$

is called the **(continuous) functional calculus** for a. (The last isomorphism is given by Gelfand-Naimark.)

- The assignment is well-defined for $\sigma_{\mathcal{A}}(a) = \operatorname{Im} \hat{a}$ by 3.1.21.3.

Definition. Let $\mathcal{A}$ be a C^*-algebra.

1. An element $a \in \mathcal{A}$ is **normal** if $aa^* = a^*a$.

2. For each $a \in \mathcal{A}$, denote by $C^*(a)$ the **smallest C^*-subalgebra of $\mathcal{A}$ containing a.**

3. If $\mathcal{A}$ is unital, $C^*(a, 1)$ is called the **unital C^*-algebra generated by** a.

- One sees that $C^*(a)$ is the closure of the subalgebra generated by a and a^*. Indeed, denote by X the latter space. X is a C^*-subalgebra of $\mathcal{A}$ for * is continuous. Since $C^*(a)$ is a closed *-algebra containing a and a^*, $X \subseteq C^*(a)$. By minimality $X = C^*(a)$.

- Similarly, $C^*(a, 1)$ is the closure of the subalgebra generated by 1, a and a^*.

- $a \in \mathcal{A}$ is normal if and only if $C^*(a)$ is commutative.

Lemma 3.3.1. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a *-homomorphism with $\mathcal{A}$ Banach * and $\mathcal{B}$ C^*. Then $\|\Phi(a)\| \leqslant \|a\|$ for every $a \in \mathcal{A}$. In particular, Φ is continuous.

Proof. To show Φ is continuous, we only need to show $\Phi^e : \mathcal{A}^e \to \mathcal{B}^e$ is continuous. If this is shown, then $\Phi = \Phi^e|_{\mathcal{A}}^{\mathcal{B}}$ is also continuous. Thus we can assume $\mathcal{A}$ and $\mathcal{B}$ are unital and Φ is unital. We can assume a is self-adjoint. Indeed, if the self-adjoint case is proved, then

$$\|\Phi(a)\|^2 = \|\Phi(a)\Phi(a)^*\| = \|\Phi(aa^*)\| \leqslant \|aa^*\| \leqslant \|a\| \, \|a^*\| = \|a\|^2$$

We claim $\sigma_{\mathcal{B}}(\Phi(a)) \subseteq \sigma_{\mathcal{A}}(a)$. To see this, suppose $\Phi(a) - \lambda = \Phi(a - \lambda)$ is not invertible in $\mathcal{B}$. Then $a - \lambda$ is not invertible, either. It follows that

$$\|\Phi(a)\| \overset{3.2.4}{=} r(\Phi(a)) \leqslant r(a) \leqslant \|a\|$$

$\square$

Lemma 3.3.2. Let $\mathcal{A} \subseteq \mathcal{B}$ be unital C^*-algebra. Then $\mathcal{B}^\times \cap \mathcal{A} = \mathcal{A}^\times$. In particular, for $a \in \mathcal{A}$, we have $\sigma_\mathcal{A}(a) = \sigma_\mathcal{B}(a)$.

Proof. The inclusion $\supseteq$ is clear, so we suppose $a \in \mathcal{B}^\times \cap \mathcal{A}$. First assume $\mathcal{A}$ and $\mathcal{B}$ are commutative. The restriction $\mathrm{res} : \Delta_\mathcal{B} \to \Delta_\mathcal{A}$ defines a continuous homomorphism, whence a map $\mathrm{res}^* : C(\Delta_\mathcal{A}) \to C(\Delta_\mathcal{B})$. Then the diagram

$$\begin{tikzcd} C(\Delta_\mathcal{A}) \arrow[r, "\sim"] \arrow[d, "\mathrm{res}^*"] & \mathcal{A} \arrow[d] \\ C(\Delta_\mathcal{B}) \arrow[r, "\sim"] & \mathcal{B} \end{tikzcd}$$

where the horizontal maps are Gelfand transforms, is commutative. Since a is invertible in $\mathcal{B}$, its image $\widehat{a} \in C(\Delta_\mathcal{B})$ has no zero, whence $\widehat{a} \in C(\Delta_\mathcal{A})$ has no zero as well. This proves $a \in \mathcal{A}^\times$.

Now assume $\mathcal{A}$ is commutative, while $\mathcal{B}$ possibly not. We can find $b \in \mathcal{B}$ such that $ab = ba = 1$. Then $a^*b = baa^*b = ba^*ab = ba^*$, i.e., b commutes with a^*. Similarly, $b = a^{-1}$ commutes with b^*. This means a is invertible in the commutative C^*-algebra $C^*(1, a, b)$ generated by $1, a, b$. Then $a \in \mathcal{A}^\times$ by the first paragraph.

Finally, drop commutative condition on $\mathcal{A}$. First consider the case a being normal. Then the C^*-algebra $C^*(1, a)$ is commutative, and the second paragraph shows $a \in C^*(1, a)^\times \subseteq \mathcal{A}^\times$. Last, suppose $a \in \mathcal{A}$ is arbitrary. Note that a is invertible if and only if aa^* and a^*a are invertible (purely algebraic). Since aa^* and a^*a are normal, the last paragraph implies aa^* and a^*a are invertible in $\mathcal{A}$, so a is invertible in $\mathcal{A}$ as well. □

Theorem 3.3.3. Let $\mathcal{A}$ be a unital C^*-algebra and let $a \in \mathcal{A}$ be a normal element. Then there exists a unique unital *-homomorphism

$$\Phi_a : C(\sigma(a)) \to \mathcal{A}$$

with the property that $\Phi_a(\mathrm{id}) = a$. Write $f(a) := \Phi_a(f)$. Then

(i) Φ_a is isometric with image $C^*(a, 1) \subseteq \mathcal{A}$.

(ii) If $\mathcal{A}$ is commutative, the $\widehat{\Phi_a(f)} = f \circ \hat{a}$.

(iii) If $\mathcal{A} = C^*(a, 1)$, then $\hat{a} : \Delta_\mathcal{A} \to \sigma(a)$ is a homeomorphism.

(iv) If $f : \sigma(a) \to \mathbb{C}$ has a power series expansion $f(z) = \sum_{n=0}^\infty a_n(z - z_0)^n$ which converges uniformly on $\sigma(a)$, then $f(a) = \sum_{n=0}^\infty a_n(a - z_0 1)^n$, and the series converges in norm.

Proof. The uniqueness follows from Stone-Weierstrass and Lemma 3.3.1. Let us prove the existence. Consider $a \in C^*(1, a) \subseteq \mathcal{A}$ and its Gelfand transform

$$\hat{a} : \Delta_{C^*(1,a)} \longrightarrow \mathrm{Im}\, \hat{a} = \sigma_{C^*(1,a)}(a) = \sigma_\mathcal{A}(a)$$

This is injective, for if $\hat{a}(m_1) = \hat{a}(m_2)$, then $m_1(a) = m_2(a)$; together with Lemma 3.2.6, we see m_1 and m_2 agree on the linear span of elements of the form $a^\ell(a^*)^k$ which is dense in $C^*(1, a)$, and thus $m_1 = m_2$. Since $C^*(1, a)$ is unital, $\Delta_{C^*(1,a)}$ is compact, and thus $\hat{a}$ is a homeomorphism. Thus we get a unital isometric *-isomorphism

$$\begin{aligned} \Psi : C(\sigma(a)) &\longrightarrow C(\Delta_{C^*(1,a)}) \\ f &\longmapsto f \circ \hat{a} \end{aligned}$$

Composing this with the inverse of the Gelfand transform $C^*(1,a) \to C(\Delta_{C^*(1,a)})$ gives us a unital isometric *-isomorphism

$$\Phi_a : C(\sigma(a)) \longrightarrow C^*(1,a)$$

Now compute $\Phi(1) = 1$ and $\Psi(\mathrm{id}) = \mathrm{id} \circ \hat{a} = \hat{a}$, so that $\Phi_a(\mathrm{id}) = a$. For the last assertion (iv), for each N put $f_N(z) = \sum\limits_{n=0}^{N} a_n(z - z_0)^n$. Then

$$\Psi_a(f_N) = \sum_{n=0}^{N} a_n(a - z_0 1)^n$$

and since Ψ_a is an isometry,

$$\|\Psi_a(f_N) - \Psi_a(f)\|_{\mathcal{A}} = \|\Psi_a(f_N - f)\|_{\mathcal{A}} = \|f_N - f\|_\infty \to 0 \text{ as } N \to \infty$$

$\square$

Example 3.3.4. Let X be a compact Hausdorff space. Then for each $f \in C(X)$, the functional calculus sends each function $g \in C(\sigma(f)) = C(f(X))$ to the function $g \circ f \in C(X)$. This follows quickly from the uniqueness part of Theorem and the fact that $g \mapsto g \circ f$ satisfies $\mathrm{id} \circ f = f$.

Corollary 3.3.31.1.

(a) If $a \in \mathcal{A}$ is normal, then a is self-adjoint if and only $\sigma_{\mathcal{A}}(a) \subseteq \mathbb{R}$.

(b) Let $\Psi : \mathcal{A} \to \mathcal{B}$ be a unital *-homomorphism between C^*-algebras and let $a \in \mathcal{A}$ be a normal element. Then $\Psi(a)$ is normal and $\sigma(\Psi(a)) \subseteq \sigma(a)$. Then diagram

$$\begin{tikzcd} C(\sigma(a)) \arrow[r, "\Phi_a"] \arrow[d, "\mathrm{res}"] & \mathcal{A} \arrow[d, "\Psi"] \\ C(\sigma(\Psi(a))) \arrow[r, "\Phi_{\Psi(a)}"] & \mathcal{B} \end{tikzcd}$$

commutes. In particular, one has $f(\Psi(a)) = \Psi(f(a))$ for every $f \in C(\sigma(a))$.

(c) Suppose $a \in \mathcal{A}$ is a normal element in the unital C^*-algebra $\mathcal{A}$ and let $f \in C(\sigma_{\mathcal{A}}(a))$. Then

(i) $f(a) \in \mathcal{A}$ is normal,

(ii) (Spectral mapping theorem) $\sigma_{\mathcal{A}}(f(a)) = f(\sigma_{\mathcal{A}}(a))$, and

(iii) $g(f(a)) = (g \circ f)(a)$ for all $g \in C(\sigma_{C(\sigma_{\mathcal{A}}(a))}(f)) = C(f(\sigma_{\mathcal{A}}(a))) \overset{\text{(ii)}}{=} C(\sigma_{\mathcal{A}}(f(a)))$.

Proof.

(a) $\Phi_a(\mathrm{id}) = a$, so $a^* = \Phi_a(\mathrm{id})^* = \Phi_a(\overline{\mathrm{id}})$. Since Φ_a is isometric, it is injective, so $a = a^*$ if and only if $\mathrm{id} = \overline{\mathrm{id}}$, if and only if $\sigma_{\mathcal{A}}(a) \subseteq \mathbb{R}$.

(b) The first two statements are clear. For the commutativity we have

$$\Psi\Phi_a(\mathrm{id}_{\sigma(a)}) = \Psi(a) = \Phi_{\Psi(a)}(\mathrm{id}_{\sigma(\Psi(a))}) = \Phi_{\Psi(a)}(\mathrm{res}\,\mathrm{id}_{\sigma(a)})$$

Since all maps are *-homomorphism, it follows from Stone-Weierstrass that the diagram commutes.

(c) (i) is clear since $f(a) \in C^*(1, a)$. For (iii), consider the commutative diagram

$$\begin{tikzcd} C(\sigma(f)) \arrow[r, "\Phi_f"] \arrow[d, "\mathrm{res}"] & C(\sigma_{\mathcal{A}}(a)) \arrow[d, "\Phi_a"] \\ C(\sigma(\Phi_a(f))) \arrow[r, "\Phi_{f(a)}"] & \mathcal{A} \end{tikzcd}$$

Note that by (ii) the left vertical arrow res is the identity map, so

$$g(f(a)) = \Phi_{f(a)}(g) = \Phi_{f(a)}(\operatorname{res} g) = \Phi_a \Phi_f(g) = \Phi_a(g \circ f) = (g \circ f)(a) = (g \circ f)(a)$$

It remains to show (ii). Trivially we have $\sigma_{\mathcal{A}}(f(a)) \subseteq \sigma_{C(\sigma_{\mathcal{A}}(a))}(f) = f(\sigma_{\mathcal{A}}(a))$. Note that the spectrum is preserved under $\mathbb{C}-$algebra isomorphisms; then

$$\sigma_{\mathcal{A}}(f(a)) = \sigma_{C^*(a,1)}(f(a)) = \sigma_{C(\sigma_{\mathcal{A}}(a))}(f) = f(\sigma_{\mathcal{A}}(a))$$

$\square$

Corollary 3.3.31.2. Suppose $\Psi : \mathcal{A} \to \mathcal{B}$ is an injective *-homomorphism between C^*-algebras. Then Ψ is already isometric.

Proof. By Lemma 3.3.1 we have $\|\Psi(a)\| \leqslant \|a\|$ for all $a \in \mathcal{A}$. As in the proof of Lemma 3.3.1, we may assume everything is unital. Assume there exists $a \in \mathcal{A}$ with $\|\Psi(a)\| < \|a\|$; we may assume $\|a\| = 1$. Then

$$\alpha := \|\Psi(a^* a)\| = \|\Psi(a)\|^2 < \|a\|^2 = \|a^* a\| = 1$$

Put $c = a^* a$; then c is self-adjoint with $\sigma(c) \subseteq [-1, 1]$ and $\sigma(\Psi(c)) \subseteq [-\alpha, \alpha]$. By Lemma 3.2.4 $\sigma(c)$ contains either 1 or -1 (note that $\sigma(c)$ is compact, so $r(a) = |\lambda|$ for some $\lambda \in \sigma(c)$), so we can find a function $0 \neq f \in C(\sigma(c))$ with $f \equiv 0$ on $\sigma(\Psi(c))$. Injectivity of Ψ shows that $0 \neq \Psi(f(c)) = f(\Psi(c)) = 0$, a contradiction. $\square$

Corollary 3.3.31.3 (Fuglede's)**.** Let $\mathcal{A}$ be a C^*-algebra and $a \in \mathcal{A}$ be normal. If $b \in \mathcal{A}$ is such that $ab = ba$, then $a^* b = ba^*$.

Proof. We may assume $\mathcal{A}$ is unital. Define $f : \mathbb{C} \to \mathcal{A}$ by

$$f(\lambda) = e^{i\lambda a^*} b e^{-i\lambda a^*}.$$

Clearly f is a holomorphic function. Write $\lambda a^* = c_1(\lambda) + ic_2(\lambda)$ with $c_i(\lambda)$ self-adjoint. Since a is normal, $c_1(\lambda)$ commutes with $c_2(\lambda)$ and

$$e^{i\lambda a^*} = e^{ic_1(\lambda)} e^{-c_2(\lambda)} = e^{2ic_1(\lambda)} e^{-(c_2(\lambda)+ic_1(\lambda))} = e^{2ic_1(\lambda)} e^{-i\bar{\lambda} a}.$$

Since $ab = ba$, we have $e^{-i\bar{\lambda} a} b = b e^{-i\bar{\lambda} a}$, so

$$f(\lambda) = e^{2ic_1(\lambda)} \left(e^{-i\bar{\lambda} a} b e^{i\bar{\lambda} a} \right) e^{-2ic_1(\lambda)} = e^{2ic_1(\lambda)} b e^{-2ic_1(\lambda)}.$$

As $c_1(\lambda)$ is self-adjoint, we have $\sigma(c_1(\lambda)) \subseteq \mathbb{R}$, so $e^{2ic_1(\lambda)}$ is bounded, which implies that $f(\lambda)$ is a bounded entire function. By Liouville's theorem, it follows that f is a constant. But

$$0 = f'(0) = i(a^* b - ba^*)$$

this shows $a^* b = ba^*$. $\square$

We now extend Theorem 3.3.3 to the non-unital case. Recall for a non-unital C^*-algebra $\mathcal{A}$, the spectrum of an element is computed in the unitization; this implies $0 \in \sigma(a)$ for all $a \in \mathcal{A}$. For each element $a \in \mathcal{A}$, let us defined

$$C_0(\sigma(a)) := \{f \in C(\sigma(a)) \mid f(0) = 0\}$$

This notation may cause confusion. However, observe that $\sigma(a)$ is the one-point compactification of $\sigma(a)\backslash\{0\}$, so $\{f \in C(\sigma(a)) \mid f(0) = 0\}$ is exactly the space $C_0(\sigma(a)\backslash\{0\}) \subseteq C(\sigma(a))$. This kind of justifies our confusing notation.

Corollary 3.3.31.4. Let $\mathcal{A}$ be a *non-unital* C^*-algebra and $a \in \mathcal{A}$ be a normal element. There exists a unique *-homomorphism

$$\Phi_a : C_0(\sigma(a)) \to \mathcal{A}$$

with the property that $\Phi_a(\mathrm{id}) = a$. Moreover, it is isometric with image $C^*(a) \subseteq \mathcal{A}$.

Proof. The uniqueness follows as in Theorem 3.3.3. For the existence, we apply Theorem 3.3.3 to the unitization $\mathcal{A}^e$ of $\mathcal{A}$. We then obtain a unique unital *-homomorphism $\Phi_a : C(\sigma(a)) \to \mathcal{A}^e$ with the following properties:

(i) $\Phi_a(\mathrm{id}) = a$, where $\mathrm{id} = \mathrm{id}_{\sigma(a)}$ and (by definition) $\sigma(a) = \sigma_{\mathcal{A}^e}((a, 0))$.

(ii) Φ_a is isometric with image $C^*(a, 1) \subseteq \mathcal{A}^e$.

Let us study the image of $C_0(\sigma(a))$ under Φ_a. Since Φ_a is isometric and $C_0(\sigma(a))$ is closed in $C(\sigma(a))$, $\Phi_a(C_0(\sigma(a)))$ is also closed. Clearly $C_0(\sigma(a))$ contains all polynomials in z and $\overline{z}$ without constant term, so $\Phi_a(C_0(\sigma(a)))$ contains a dense subset of $C^*(a)$; being closed, we have $C^*(a) \subseteq \Phi_a(C_0(\sigma(a))) \subsetneq C^*(a, 1)$.

Lemma 3.3.32. $\dim_{\mathbb{C}} C^*(a, 1)/C^*(a) = 1$.

Proof. Let $x \in C^*(a, 1)\backslash C^*(a)$. Since $C^*(a)$ is closed in $C^*(a, 1)$, we can find a sequence of polynomials with nonzero constant term $p_n(X, Y) \in \mathbb{C}[X, Y]$ such that $p_n(a, a^*) \to x$. Let $c_n = p_n(0, 0)$; then $c_n \to c$ for some $c \in \mathbb{C}$. Put $q_n(X, Y) := p_n(X, Y) - c_n$; then $q_n(a, a^*) \to x - c1$ (recall that $1 = 1_{\mathcal{A}^e} = (0, 1) \in \mathcal{A} \times \mathbb{C}$). But $C^*(a)$ is closed and $q_n(a, a^*) \in C^*(a)$, it follows that $x - c1 \in C^*(a)$. ☐

By the lemma, since $\Phi_a(C_0(\sigma(a)))$ is proper in $C^*(a, 1)$, it forces that $\Phi_a(C_0(\sigma(a))) = C^*(a)$, as desired. ☐

Chapter 4

More on C^*-algebras

4.1 Some decomposition results

We collect some terminology in C^*-algebra.

Definition. Let $\mathcal{A}$ be a C^*-algebra and $a \in \mathcal{A}$.

1. a is **normal** if $a^*a = aa^*$.
2. a is **self-adjoint** if $a = a^*$. Let $\mathcal{A}_{sa}$ denote the set of self-adjoint elements in $\mathcal{A}$.
3. a is **positive** if a is self-adjoint and $\sigma_{\mathcal{A}}(a) \subseteq [0, \infty)$. Let $\mathcal{A}_+$ denote the set of positive elements in $\mathcal{A}$. (Recall for the non-unital case, the spectrum is computed in the unitization.)
4. a is a **projection** if a is self-adjoint and $a^2 = a$.
5. $a, b \in \mathcal{A}$ are called **orthogonal** if $ab = ba = 0$.
6. If $\mathcal{A}$ is unital, a is called **unitary** if a is normal and $aa^* = 1$.

We can define an partial order $\leqslant$ on $\mathcal{A}_{sa}$ as follows. For $a, b \in \mathcal{A}_{sa}$, say $a \leqslant b$ if $b - a \in \mathcal{A}_+$. This is really a partial order, for if $0 \leqslant a \geqslant 0$, i.e., $\sigma(a) = \{0\}$, then by Lemma 3.2.4, $\|a\| = r(a) = 0$, so that $a = 0$.

Lemma 4.1.1. Let X be an LCH space. Consider the C^*-algebra $C_0(X)$; the norm is sup-norm, and the $*$ is complex conjugation.

(i) For a function $f \in C_0(X)$, $f \geqslant 0$ is positive if and only if $f(x) \geqslant 0$ for all $x \in X$.

(ii) If $f \in C_0(X)$ is positive, then for all $n \in \mathbb{N}$, there exists a unique positive function $g \in C_0(X)$ such that $f = g^n$.

(iii) For self-adjoint $f \in C_0(X)$, there exist unique positive functions f^+, $f^- \in C_0(X)$ such that $f = f^+ - f^-$ with $f^+ f^- = f^- f^+ = 0$.

Proof.

(i) Note that $\sigma(f) = f(X)$. Thus $f \geqslant 0 \Leftrightarrow \sigma(f) \subseteq [0, \infty) \Leftrightarrow f(X) \subseteq [0, \infty)$.

(ii) For the existence, just take $g \in C_0(X)$ defined by $g(x) := f(x)^{\frac{1}{n}}$. The uniqueness is clear for we are in $\mathbb{R}_{\geqslant 0}$.

(iii) Since f is self-adjoint, $f(X) \subseteq \mathbb{R}$. Take $f^+ := \max\{0, f\}$ and $f^- = \max\{0, -f\}$

$\square$

Proposition 4.1.2. Let $\mathcal{A}$ be a C^*-algebra and $a \in \mathcal{A}_+$ be positive. For each $n \in \mathbb{N}$, there exists a unique positive element $b \in \mathcal{A}_+$ with $b^n = a$.

Proof. If $\mathcal{A}$ is unital, we use Theorem 3.3.3; otherwise, we use Corollary 3.3.31.4 instead. Since $\sigma(a) \subseteq [0, \infty)$, we can find $f(x) \in C_0(\sigma(a))$ with $f \geqslant 0$ and $f(x)^n = x$ for $x \in \sigma(a)$. Define $b := f(a) \in \mathcal{A}$; b is positive by Spectral mapping theorem, and $b^n = a$. This shows the existence. For the uniqueness, let $b' \in \mathcal{A}_+$ be another such element. Then $a \in C^*(b')$, and then

$$b \in C^*(a) \subseteq C^*(b') \cong C_0(\sigma(b'))$$

The uniqueness now follows from Lemma 4.1.1.(ii). $\square$

Proposition 4.1.3. Let $\mathcal{A}$ be a C^*-algebra.

(i) Every element is a linear combination of two self-adjoint elements.

(ii) Every self-adjoint element can be written uniquely as $x - y$ with x, y positive and orthogonal.

(iii) Every element can be written as a linear combination of four positive elements.

(iv) If $\mathcal{A}$ is unital, every element is a linear combination of four unitary elements.

Proof. We already see that for $a \in \mathcal{A}$, we can write $a = \operatorname{Re} a + i \operatorname{Im} a$ with

$$\operatorname{Re} a = \frac{1}{2}(a + a^*) \text{ and } \operatorname{Im} a = \frac{1}{2i}(a - a^*)$$

The existence part of (ii) follows directly from Lemma 4.1.1(ii), and if $a = x - y$ with x, y positive and $xy = 0$, then $(a^2)^{\frac{1}{2}} = x + y$, so $x = \frac{1}{2}(a + (a^2)^{\frac{1}{2}})$ is uniquely determined by a. (iii) follows from (i) and (ii). For (iv), by (i) it suffices to show every self-adjoint element is a linear combination of two unitary elements. Let $a \in \mathcal{A}_{sa}$; scaling if necessarily, we may assume $\|a\| \leqslant 1$. Then $1 - a^2 \geqslant 0$; indeed, it is self-adjoint, and if we put $f(t) = 1 - t^2$, then by Spectral mapping theorem we see $\sigma(1 - a^2) = f(\sigma(a)) \subseteq [0, 1]$. Let $u_1 = a - i\sqrt{1 - a^2}$ and $u_2 = a + i\sqrt{1 - a^2}$; then the u_i are unitary, and $a = \frac{1}{2}(u_1 + u_2)$, as wanted. $\square$

4.2 Positive elements

Proposition 4.2.1. Let $\mathcal{A}$ be a C^*-algebra. Then $\mathcal{A}_+$ is a closed convex cone.

Proof. WLOG we assume $\mathcal{A}$ is unital. For $a \in \mathcal{A}_+$ and $\lambda \geqslant 0$, it is easy to check $\lambda a \in \mathcal{A}_+$, so we only need to show that the sum of two positive elements remains positive.

We first show that for $X \subseteq \mathbb{R}$ compact and $f \in C(X)$, f is positive if there exists $r \in \mathbb{R}_{\geqslant 0}$ such that $\|f - r\| \leqslant r$. If $r = 0$, then $f = 0$ so it is positive trivially. Suppose $r > 0$ but f is not positive; we then can find $t \in X$ with $f(t) < 0$. Then $|f(t) - r| > r$ so that $\|f - r\| = \sup_{t \in X} |f(t) - r| > r$, a contradiction. This proves our claim. Secondly, note that if $f \in \mathcal{A}$ is positive, then $\|f - r\| \leqslant \|f\|$ if $0 \leqslant r \leqslant \|f\|$. This can be seen easily by Lemma 3.2.4.

Now let $a, b \in \mathcal{A}_+$. Since $a+b$ is self-adjoint, we may identify $C^*(a+b, 1_\mathcal{A}) \cong C(\sigma_\mathcal{A}(a+b))$ and $a+b$ with $f(t) = t$. Thus it suffices to find $r \geqslant 0$ such that $\|a+b-r\| \leqslant r$. Take $r := \|a\| + \|b\|$; then

$$\|a+b-r\| = \|a+b-\|a\|-\|b\|\| \leqslant \|a-\|a\|\| + \|b-\|b\|\| \leqslant \|a\| + \|b\| = r$$

The last inequality follows from what mentioned in the end of the second paragraph. This shows $a+b$ is positive.

It remains to show $\mathcal{A}_+$ is closed. From the second paragraph we have

$$B := \{a \in \mathcal{A}_+ \mid \|a\| \leqslant 1\} = \{a \in \mathcal{A} \mid \|a - 1_\mathcal{A}\| \leqslant 1\} \cap \mathcal{A}_{sa}$$

The last two sets are closed, so B is closed as well. Thus $\mathcal{A}_+ = \mathbb{R}_{\geqslant 0} B$ is closed: if a_n is a sequence in $\mathbb{R}_{\geqslant 0} B$ with $a_n \to a \in \mathcal{A}$ and $a \neq 0$, then $\dfrac{a_n}{\|a_n\|} \to \dfrac{a}{\|a\|}$, and since the $\dfrac{a_n}{\|a_n\|} \in B$, it forces $\dfrac{a}{\|a\|} \in B$, so $a \in \mathbb{R}_{\geqslant 0} B$. ☐

Corollary 4.2.1.1. The poset $(\mathcal{A}_{sa}, \geqslant)$ is upward directed, i.e., any two element in $\mathcal{A}_{sa}$ has a common upper bound in $\mathcal{A}_{sa}$.

Proposition 4.2.2. For $a \in \mathcal{A}$, the element a^*a is positive. In particular, we have $\mathcal{A}_+ = \{a^*a \mid a \in \mathcal{A}\}$.

Proof. Every a^*a is self-adjoint. Suppose $-a^*a$ is positive. Since $\sigma_\mathcal{A}(-a^*a)\backslash\{0\} = \sigma_\mathcal{A}(-aa^*)\backslash\{0\}$, $-aa^*$ is positive as well. Write $a = b + ic$ for $b, c \in \mathcal{A}_{sa}$. Then

$$\begin{aligned} aa^* &= b^2 + ibc - icb + c^2 \\ a^*a &= b^2 - ibc + icd + c^2 \end{aligned}$$

so $a^*a = 2b^2 + 2c^2 + (-aa^*)$ is a sum of positive elements, implying a^*a is also positive. Hence $0 = \|a^*a\| = \|a\|^2$, and therefore $\|a\| = 0$.

Write $a^*a = b - c$ with $b, c \in \mathcal{A}_+$ and $bc = cb = 0$. We claim $c = 0$. We have

$$-(ac)^*(ac) = -ca^*ac = -c(b-c)c = c^3 \in \mathcal{A}_+$$

By the first paragraph, it follows $ac = 0$, and hence

$$0 = a^*ac = (b-c)c = -c^2 = -c^*c$$

so that $0 = \|c^*c\| = \|c\|^2$, i.e., $c = 0$. Hence $a^*a = b$ is positive.

For the last equality, note that if $a \geqslant 0$, then there exists a positive square root $a^{\frac{1}{2}}$ of a by Proposition 4.1.2, so $a = (a^{\frac{1}{2}})^* a^{\frac{1}{2}}$. ☐

Proposition 4.2.3. Let $\mathcal{A}$ be a C^*-algebra and $a, b \in \mathcal{A}_{sa}$ with $a \leqslant b$. Then

(i) for any $c \in \mathcal{A}$, we have $c^*ac \leqslant c^*bc$;

(ii) $\|a\| \leqslant \|b\|$;

(iii) if $\mathcal{A}$ is unital and $a, b \in \mathcal{A}^\times \cap \mathcal{A}_+$, then $b^{-1} \leqslant a^{-1}$

Proof.

(i) Since $b - a \geqslant 0$, it has a positive square root $(b-a)^{\frac{1}{2}}$ by Proposition 4.1.2. Thus

$$c^*bc - c^*ac = c^*(b-a)c = ((b-a)^{\frac{1}{2}}c)^*((b-a)^{\frac{1}{2}}c) \geqslant 0$$

by Proposition 4.2.2.

(ii) WLOG we may assume $\mathcal{A}$ is unital. By continuous functional calculus, we have $b \leqslant \|b\|$; to see this, we can prove it in $C^*(b, 1_{\mathcal{A}}) \cong C(\sigma_{\mathcal{A}}(b))$, and this becomes an obvious identity $t \leqslant \max_{t \in \sigma_{\mathcal{A}}(b)} |t|$. Hence $a \leqslant b \leqslant \|b\|$. Finally, the same reasoning in $C^*(a, 1_{\mathcal{A}}) \cong C(\sigma_{\mathcal{A}}(a))$ gives $\|a\| \leqslant \|b\|$.

(iii) Note that for $c \in \mathcal{A}^\times$, we have $\sigma_{\mathcal{A}}(c^{-1}) = (\sigma_{\mathcal{A}}(c))^{-1}$, so $b^{-1}, a^{-1} \in \mathcal{A}_+$ as well. By (i) we have

$$b^{-\frac{1}{2}}ab^{-\frac{1}{2}} \leqslant b^{-\frac{1}{2}}bb^{-\frac{1}{2}} = 1_{\mathcal{A}}$$

so (ii) implies $\left\|b^{-\frac{1}{2}}a^{\frac{1}{2}}\right\|^2 \leqslant 1$. (Here we use the convention that the identity in a unital Banach algebra has norm 1.) Thus

$$a^{\frac{1}{2}}b^{-1}a^{\frac{1}{2}} \leqslant \left\|a^{\frac{1}{2}}b^{-1}a^{\frac{1}{2}}\right\| \leqslant 1 = 1_{\mathcal{A}}$$

By (i) we have

$$b^{-1} = a^{-\frac{1}{2}}\left(a^{\frac{1}{2}}b^{-1}a^{\frac{1}{2}}\right)a^{-\frac{1}{2}} \leqslant a^{-\frac{1}{2}}1_{\mathcal{A}}a^{-\frac{1}{2}} = a^{-1}$$

□

Proposition 4.2.4. Let $\mathcal{A}$ be a C^*-algebra.

(i) Every projection is positive. More generally, if $p \in \mathcal{A}$ is idempotent, i.e. $p^2 = p$, then $\sigma_{\mathcal{A}}(p) \subseteq \{0, 1\}$.

(ii) Let $p, q \in \mathcal{A}$ be projections. Then $p \leqslant q$ if and only if $pq = qp = p$.

Proof. (i) follows from Spectral mapping theorem for polynomials: if we take $f(z) = z - z^2$, then

$$f(\sigma_{\mathcal{A}}(p)) = \sigma_{\mathcal{A}}(f(p)) = \sigma_{\mathcal{A}}(0) = \{0\}.$$

so we must have $\sigma_{\mathcal{A}}(p) \subseteq \{0, 1\}$. For (ii), first assume $p \leqslant q$. Then $0 \leqslant q - p \leqslant 1_{\mathcal{A}^e} - p$, so by Proposition 4.2.3.(i)

$$0 = p0p \leqslant p(q-p)p \leqslant p(1_{\mathcal{A}^e} - p)p = p - p = 0$$

and thus $0 = p(q-p)p = pqp - p$, whence

$$\|qp - p\|^2 = \|(qp-p)^*(qp-p)\| = \|pqp - pqp - pqp + p\| = \|p - pqp\| = 0$$

i.e., $qp = p$. Also, $pq - p = (qp - p)^* = 0$. Conversely, suppose $pq = qp = p$. Then

$$(q-p)(q-p) = q^2 - qp - pq + p^2 = q - p - p + p = q - p$$

and $(q-p)^* = q - p$, so $q - p$ is a projection. By (i), $q - p \geqslant 0$, or $q \geqslant p$. □

4.3 Approximate identity

Definition. Let $\mathcal{B}$ be a Banach algebra. A net $(e_\lambda)_{\lambda\in\Lambda}$ is called a **left approximate identity** if for all $a \in \mathcal{B}$ we have

$$a = \lim_{\lambda\in\Lambda} e_\lambda a, \qquad \text{or} \qquad \lim_{\lambda\in\Lambda} \|a - e_\lambda a\| = 0$$

Similarly we define **right approximate identity**. A net in $\mathcal{B}$ is called an **approximate identity** if it is simultaneously a left and right approximate identity.

Lemma 4.3.1. Let $\mathcal{A}$ be a C^*-algebra and let $\mathcal{A}_+^1 := \{a \in \mathcal{A}_+ \mid \|a\| < 1\}$. Then the poset $(\mathcal{A}_+^1, \leqslant)$ is upward directed.

Proof. Put $\mathcal{A}^e$ to be its unitization if $\mathcal{A}$ is not unital, and $\mathcal{A}^e = \mathcal{A}$ if it is unital. First consider $a, b \in \mathcal{A}_+$. Then $1_{\mathcal{A}^e} + a, 1_{\mathcal{A}^e} + b \in (\mathcal{A}^e)^\times \cap \mathcal{A}_+^e$. Suppose $a \leqslant b$. Then $1_{\mathcal{A}^e} + a \leqslant 1_{\mathcal{A}^e} + b$, so $(1_{\mathcal{A}^e} + b)^{-1} \leqslant (1_{\mathcal{A}^e} + a)^{-1}$ by Proposition 4.2.3.(iii) and

$$a(1_{\mathcal{A}^e} + a)^{-1} = 1_{\mathcal{A}^e} - (1_{\mathcal{A}^e} + a)^{-1} \leqslant 1_{\mathcal{A}^e} - (1_{\mathcal{A}^e} + b)^{-1} = b(1_{\mathcal{A}^e} + b)^{-1}$$

Note that $a(1_{\mathcal{A}^e} + a)^{-1}$ and $b(1_{\mathcal{A}^e} + b)^{-1}$ are both in $\mathcal{A}_+^1$. To see this, expanding in power series we can see $a(1_{\mathcal{A}^e} + a)^{-1} \in \mathcal{A}$. Since $\sigma_{\mathcal{A}^e}(1_{\mathcal{A}^e} + a) \subseteq [1, \|a\| + 1]$, we have

$$\left\|a(1_{\mathcal{A}^e} + a)^{-1}\right\| \leqslant \frac{\|a\|}{\|1_{\mathcal{A}^e} + a\|} \leqslant \frac{\|a\|}{1 + \|a\|} < 1$$

Now suppose $a, b \in \mathcal{A}_+^1$. Let $x = a(1_{\mathcal{A}^e} - a)^{-1}$ and $y = b(1_{\mathcal{A}^e} - b)^{-1}$ and $c = (x+y)(1_{\mathcal{A}^e} + x + y)^{-1}$. Since $x \leqslant x + y$, by the first paragraph we have

$$a = x(x + 1_{\mathcal{A}^e})^{-1} \leqslant (x + y)(1_{\mathcal{A}^e} + x + y)^{-1} = c$$

Similarly, $b \leqslant c$. ∎

Theorem 4.3.2. Every C^*-algebra $\mathcal{A}$ has an increasing approximate identity consisting of positive elements with norm less than 1. If $\mathcal{A}$ is separable, the constructed approximate identity can be countable.

Proof. Let $\Lambda = (\mathcal{A}_+^1, \leqslant)$ and $u_\lambda = \lambda$ for all $\lambda \in \Lambda$. For each $p \geqslant 1$, we claim (u_λ) is an approximate identity of $\mathcal{A}$. We must show that

$$a = \lim_{\lambda\in\Lambda} u_\lambda a = \lim_{\lambda\in\Lambda} a u_\lambda$$

for all $a \in \mathcal{A}$. By linearity it suffices to show this holds for $a \in \mathcal{A}_+$. Let $\Gamma : C^*(a) \to C_0(\sigma_{\mathcal{A}}(a)) = C_0(X)$ be the inverse of the continuous functional calculus, where $X = \sigma_{\mathcal{A}}(a)\backslash\{0\}$. Put $f = \Gamma(a)(= \mathrm{id})$ Let $0 < \varepsilon < 1$ and $K := \{x \in X \mid |f(x)| \geqslant \varepsilon\}$.

Let $\delta > 0$ such that $\delta < 1$ and $(1 - \delta)\|f\| < \varepsilon$. Define $g_\delta \in C_0(X)$ by setting $g_\delta(x) = \delta$ for $x \in K$ and $g \equiv 0$ outside a open neighborhood of K; this is possible since $K \subseteq X$ is compact. Then $g_\delta \in C_0(X)_+^1$ and $\|g_\delta f - f\| < \varepsilon$: for $x \in K$, $|\delta f(x) - f(x)| \leqslant (1 - \delta)\|f\| < \varepsilon$, and for $x \notin K$, $|g_\delta(x) f(x) - f(x)| \leqslant \varepsilon(1 - g_\delta(x)) < \varepsilon$. Since Γ is isometric, it preserves norm, so that $\Gamma^{-1}(g_\delta) = \mu$ for some $\mu \in \Lambda$, and

$$\|u_\mu a - a\| = \|a u_\mu - a\| = \|g_\delta f - f\| < \varepsilon$$

Suppose $\lambda \in \Lambda$ such that $\mu \leqslant \lambda$. Then $1_{\mathcal{A}^e} - u_\lambda \leqslant 1_{\mathcal{A}^e} - u_\mu$, so

$$a^{\frac{1}{2}}(1_{\mathcal{A}^e} - u_\lambda)a^{\frac{1}{2}} \leqslant a^{\frac{1}{2}}(1_{\mathcal{A}^e} - u_\mu)a^{\frac{1}{2}}$$

and hence

$$\|a - au_\lambda\| = \left\|a^{\frac{1}{2}}(1_{\mathcal{A}^e} - u_\lambda)a^{\frac{1}{2}}\right\| \leqslant \left\|a^{\frac{1}{2}}(1_{\mathcal{A}^e} - u_\mu)a^{\frac{1}{2}}\right\| = \|a - au_\mu\| < \varepsilon$$

Therefore, $a = \lim\limits_{\lambda\in\Lambda} u_\lambda a = \lim\limits_{\lambda\in\Lambda} au_\lambda$.

Now assume $\mathcal{A}$ is separable and let $\{a_n\}_{n\in\mathbb{N}}$ be a countable dense subset of $\mathcal{A}$. Let $\lambda_1 \in \Lambda$ be such that $\|a_1 u_\lambda - a_1\|, \|u_\lambda a_1 - a_1\| < \dfrac{1}{2}$ for $\lambda \geqslant \lambda_1$. Then choose $\lambda_2 \geqslant \lambda_1$ such that $\|a_j u_\lambda - a_j\|, \|u_\lambda a_j - a_j\| < \dfrac{1}{4}$ for $j = 1, 2$ and $\lambda \geqslant \lambda_2$. Continuing inductively, we obtain $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq \Lambda$ such that for $j \in \mathbb{N}$

$$\|a_j u_{\lambda_n} - a_j\|, \|u_{\lambda_n} a_j - a_j\| \to 0 \text{ as } n \to \infty$$

Now let $\varepsilon > 0$ and for $a \in \mathcal{A}$, let $j \in \mathbb{N}$ such that $\|a - a_j\| < \dfrac{\varepsilon}{3}$. Take $n \gg 0$ such that $\|a_j u_{\lambda_m} - a_j\| < \dfrac{\varepsilon}{3}$ for $m \geqslant n$. Then

$$\|au_{\lambda_m} - a\| \leqslant \|(a - a_j)u_{\lambda_m}\| + \|a_j u_{\lambda_m} - a_j\| + \|a_j - a\| < \varepsilon$$

for $\|u_{\lambda_m}\| \leqslant 1$. Similarly $\|u_{\lambda_m} a - a\| < \varepsilon$. This finishes the proof. ☐

Remark 4.3.3. In the following when we talk about an approximate identity of a C^*-algebra, we always assume it is an increasing net consisting of positive elements with norm less than 1.

Definition. A C^*-algebra is called **σ-unital** if it possesses a sequential approximate identity.

- The above theorem then shows that every separable C^*-algebra is σ-unital. Moreover, it admits an increasing sequential approximate identity consisting of positive element with norm less than 1.

4.4 Hereditary Subalgebras and Ideals

Definition. Let $\mathcal{A}$ be a C^*-algebra.

1. A C^*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ is called **hereditary** if for $a, b \in \mathcal{A}$ with $a \leqslant b$, $b \in \mathcal{B}$ implies $a \in \mathcal{B}$.
2. For a projection $p \in \mathcal{A}$, the C^*-subalgebra $p\mathcal{A}p$ is called a **corner**.

Lemma 4.4.1. Let $I \subseteq \mathcal{A}$ be a closed left ideal. Then I has a right approximate identity.

Proof. Observe that $I \cap I^*$ is a C^*-subalgebra of $\mathcal{A}$, so it admits an approximate identity $(u_\lambda)_{\lambda\in\Lambda}$. Let $a \in I$; then $a^*a \in I \cap I^*$, and

$$\begin{aligned}\|a - au_\lambda\|^2 &= \|(a - au_\lambda)^*(a - au_\lambda)\| \\ &\leqslant \|a^*a - u_\lambda a^*a\| + \|u_\lambda(a^*a - u_\lambda a^*a)\| \leqslant 2\|a^*a - a^*au_\lambda\| \to 0\end{aligned}$$

☐

Theorem 4.4.2. Let $\mathcal{A}$ be a C^*-algebra and $I \trianglelefteq \mathcal{A}$ a closed (two-sided) ideal.

(i) $I = I \cap I^*$; in particular, I is a *-ideal of $\mathcal{A}$.

(ii) $\mathcal{A}/I$ is a C^*-algebra with quotient norm (c.f. Lemma E.1.9) and the induced *-operation from $\mathcal{A}$.

Proof. Let (u_λ) be the right approximate identity constructed in the lemma. Then for $a \in I$,

$$a^* = (\lim_\lambda au_\lambda)^* = \lim_\lambda u_\lambda a^* \in I$$

since I is a closed ideal. Now for $a \in \mathcal{A}$, we have

$$\|a + I\| := \inf_{b \in I} \|a + b\| = \lim_\lambda \|a - u_\lambda a\| = \lim_\lambda \|a - au_\lambda\|$$

This is because $\|a + b\| \leqslant \|a - au_\lambda\| + \|au_\lambda + b\|$ and $au_\lambda \in I$. Hence

$$\|a^*a + I\| = \lim_\lambda \|a^*a(1 - u_\lambda)\| \geqslant \lim_\lambda \|(1 - u_\lambda)a^*a(1 - u_\lambda)\| = \lim_\lambda \|a(1 - u_\lambda)\|^2 = \|a + I\|^2$$

The second equality results from $\|1 - u_\lambda\| \leqslant 1$. For the reverse inequality, note that

$$\|a^* + I\| = \lim_\lambda \|(a - au_\lambda)^*\| = \lim_\lambda \|a - au_\lambda\| = \|a + I\|$$

so that $\|aa^* + I\| \leqslant \|a + I\| \, \|a^* + I\| = \|a + I\|^2$. ☐

Corollary 4.4.2.1. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a homomorphism of C^*-algebras. Then

(i) Φ is norm-decreasing;

(ii) $\Phi(\mathcal{A})$ is a C^*-subalgebra of $\mathcal{B}$;

(iii) Φ is injective if and only if Φ is isometric.

Proof. (i) is Lemma 3.3.1, and (iii) is Corollary 3.3.31.2. For (ii), since $\mathcal{A}/\ker\Phi \to \mathcal{B}$ is injective, it is isometric by (iii), implying the image $\Phi(\mathcal{A})$ is closed in $\mathcal{B}$, as wanted.

☐

Corollary 4.4.2.2. Let $\mathcal{A}$ be a C^*-algebra. If $I \trianglelefteq \mathcal{A}$ is a closed ideal, and $J \trianglelefteq I$ is a closed ideal of I, then J is also a closed ideal of $\mathcal{A}$.

Proof. Let (u_λ) be an approximate identity of I. Then for $a \in \mathcal{A}$ and $x \in J$, we have $ax = a\lim_\lambda u_\lambda x = \lim_\lambda (au_\lambda)x \in J$ since $au_\lambda \in I$. ☐

Chapter 5

Duality for Abelian Groups

5.1 The Dual Group

Definition.

1. A locally compact Hausdorff abelian group will be called an **LCA group** for short.
2. A **(unitary) character** of an LCA group A is a continuous group homomorphism $\chi : A \to S^1$.
3. For an LCA group A, denote by $\widehat{A}$ the set of all characters on A, which is a group under pointwise multiplication, called the **Pontryagin dual**, or the **dual group** of A.

Example 5.1.1.

1. Consider the additive group $\mathbb{R}$. For each $t \in \mathbb{R}$, define a character

$$\begin{aligned} \chi_t : \mathbb{R} &\longrightarrow S^1 \\ x &\longmapsto e^{2\pi i x t} \end{aligned}$$

The association $t \mapsto \chi_t$ is a group isomorphism $\mathbb{R} \cong \widehat{\mathbb{R}}$.

Proof. Let $f : \mathbb{R} \to S^1$ be a continuous group homomorphism. This means f is a continuous map with a functional equation $f(x+y) = f(x)f(y)$ for each $x, y \in \mathbb{R}$, and $f(0) = 1$. Pick a $\delta \in \mathbb{R}$ such that $\alpha := \int_0^\delta f(t)dt \neq 0$. Then the functional equation implies

$$\alpha f(x) = \int_0^\delta f(t+x)dt = \int_x^{x+\delta} f(t)dt$$

Since f is continuous, the right hand side is differentiable and thus f is itself differentiable. Differentiating the functional equation with respect to x, we obtain $f'(x+y) = f'(x)f(y)$; if we put $A = f'(0)$ and let $x = 0$, we derive a differential equation $f'(y) = Af(y)$ of f. It has a unique solution $f(y) = f(0)e^{Ay} = e^{Ay}$. Since f is bounded, $\operatorname{Re} A = 0$. Let us write $s = 2\pi i t$ for a unique $t \in \mathbb{R}$. Then

$$f(y) = e^{2\pi i t y} = \chi_t(y)$$

□

2. $\widehat{\mathbb{Z}} \cong \mathbb{R}/\mathbb{Z}$, and $\widehat{\mathbb{R}/\mathbb{Z}} \cong \mathbb{Z}$. The isomorphisms are the same as the case of $\mathbb{R}$.

Definition. Let X and Y be spaces. The **compact-open topology** on Y^X is the topology generated by the subbasis consisting of the sets

$$L(K, U) := \{f : X \to Y \mid f(K) \subseteq U\}$$

where $K \underset{\text{cpt}}{\subseteq} X$ and $U \underset{\text{open}}{\subseteq} Y$.

- The compact-open topology is finer than the product topology on Y^X.
- If Y is Hausdorff, then Y^X is Hausdorff. Indeed, for $f \neq g$ pick x such that $f(x) \neq g(x)$. Being Hausdorff, we can find disjoint open sets U and V with $f(x) \in U$ and $g(x) \in V$. Then $L(\{x\}, U)$ and $L(\{x\}, V)$ are disjoint open neighborhood of f and g, respectively.
- For each $x \in X$, the evaluation map $\delta_x : Y^X \to Y$ is continuous. Indeed, for each $U \underset{\text{open}}{\subseteq} Y$, $\delta_x^{-1}(U) = L(\{x\}, U)$ is open in Y^X.

Lemma 5.1.2. Let $\mathcal{K}$ be a collection of compact subsets of X containing a neighborhood basis at each point of X (such a collection exists when, for example, X is LCH), and let $\mathcal{B}$ be a subbasis for the open sets of Y. Then the sets $L(K, U)$ with $K \in \mathcal{K}$ and $U \in \mathcal{B}$ form a subbasis for the compact-open topology.

Proof. Since $L(K, U) \cap L(K, V) = L(K, U \cap V)$, we may assume $\mathcal{B}$ is a basis for the topology of Y. We must show for each $K \underset{\text{cpt}}{\subseteq} X$, $U \underset{\text{open}}{\subseteq} Y$ and $f \in L(K, U)$, we can find $K_1, \ldots, K_n \in \mathcal{K}$ and $U_1, \ldots, U_n \in \mathcal{B}$ such that $f \in \bigcap_{i=1}^{n} L(K_i, U_i) \subseteq L(K, U)$.

For each $x \in K$, take an open neighborhood $f(x) \in U_x \in \mathcal{B}$ in U and a $K_x \in \mathcal{K}$ such that $x \in f(K_x) \subseteq U_x$. Since K is compact, there exist $x_1, \ldots, x_n$ such that $K \subseteq K_{x_1} \cup \cdots \cup K_{x_n}$. Then $f \in \bigcap_{i=1}^{n} L(K_{x_i}, U_{x_i}) \subseteq L(K, U)$. $\square$

Lemma 5.1.3. Let Y be a metric space. Then for each $f \in C(X, Y)$, the sets of the form

$$B_{K,\varepsilon}(f) = \{g \in C(X, Y) \mid d_K(f, g) := \sup_{x \in K} d_Y(f(x), g(x)) < \varepsilon\}$$

with $\varepsilon > 0$ and $K \underset{\text{cpt}}{\subseteq} X$ forms a neighborhood basis of f.

Proof. We show that for each $L(K, U)$ and $f \in L(K, U)$ there exists a ball $B_{K,\varepsilon}(f) \subseteq L(K, U)$. Since $f(K)$ is compact, there is a distance $\varepsilon > 0$ from $f(K)$ to the complement of U; thus if $g \in B_{\varepsilon/2}(f)$, then $g(K) \subseteq U$ so that $g \in L(K, U)$.

It remains to show $B_{K,\varepsilon}(f)$ is open in compact-open topology. Let $g \in B_{K,\varepsilon}(f)$ and put $\delta = \varepsilon - d_K(f, g)$. Since $g(K)$ is compact, it is covered by $B_{\delta/3}(g(x_i))$ for some $x_1, \ldots, x_n \in K$. Put $K_i = K \cap \overline{g^{-1}(B_{\delta/3}(g(x_i)))}$; then

$$g(K_i) \subseteq g(\overline{g^{-1}(B_{\delta/3}(g(x_i)))}) \subseteq \overline{g(g^{-1}(B_{\delta/3}(g(x_i))))} \subseteq \overline{B_{\delta/3}(g(x_i))} \subseteq B_{\delta/2}(g(x_i)) =: U_i$$

Then for $h \in \bigcap_{i=1}^{n_g} L(K_i, U_i)$ and $x \in K_i$, one has

$$d_Y(f(x), h(x)) \leqslant d_Y(f(x), g(x)) + d_Y(g(x), g(x_i)) + d_Y(g(x_i), h(x)) \leqslant d_K(f, g) + \frac{\delta}{2} + \frac{\delta}{2} < \varepsilon$$

so that $h \in B_{K,\varepsilon}(f)$. Therefore, $g \in \bigcap_{i=1}^{n_g} L(K_i, U_i) \subseteq B_{K,\varepsilon}(f)$, i.e. $B_{K,\varepsilon}(f)$ is open. $\square$

Corollary 5.1.3.1. If X is compact and Y is a metric space, then the compact open topology on the space of continuous functions $C(X, Y)$ is the same as the topology induced by the sup-norm.

Lemma 5.1.4. Let X, Y, Z be topological spaces and $\varphi : X \to Y$ a continuous map. Then the map

$$\begin{aligned} \varphi^* : C(Y, Z) &\longrightarrow C(X, Z) \\ g &\longmapsto g \circ \varphi \end{aligned}$$

is continuous in compact-open topology.

Proof. Let $K \underset{\text{cpt}}{\subseteq} X$ and $U \underset{\text{open}}{\subseteq} Z$. Then

$$(\varphi^*)^{-1} L(K, U) = \{g \in C(Y, Z) \mid (g \circ \varphi)(K) \subseteq U\} = L(\varphi(K), U)$$

is open, as $\varphi(K) \subseteq Y$ is compact. ☐

Definition. Let A be an LCA group. We equip $\widehat{A}$ with the subspace topology of the compact-open topology on $\mathbb{C}^A$.

- $\widehat{A}$ is a closed subspace of the space of continuous functions $C(A)$. Indeed, write

  $$\widehat{A} = \bigcap_{x,y \in A} \{\chi : A \to S^1 \mid \chi(xy) = \chi(x)\chi(y)\}$$

  It suffices to show each set in the intersection is closed. Let $x, y \in A$. Suppose $f \in C(A)$ is such that $f(xy) \neq f(x)f(y)$. Take open neighborhoods U, V, W of $f(x)$, $f(y)$, $f(xy)$, respectively, such that W is disjoint from $UV = \{uv \mid (u, v) \in U \times V\}$. Since A is LCH, we can find compact $x \in K_U \subseteq f^{-1}(U), y \in K_V \subseteq f^{-1}(V)$, $xy \in K_W \subseteq f^{-1}(W)$. Then all functions g in $L(K_U, U) \cap L(K_V, V) \cap L(K_W, W)$ satisfy $g(xy) \neq g(x)g(y)$.

Example 5.1.5.

1. The compact open topology on $\widehat{\mathbb{R}} \cong \mathbb{R}$ is the usual topology on $\mathbb{R}$; see Theorem 7.1.3 for a proof.
2. $\widehat{S^1} \cong \mathbb{Z}$.

Proposition 5.1.6. With the compact-open topology $\widehat{A}$ is a Hausdorff topological group.

Proof. We have to show the map

$$\begin{aligned} \Phi : \widehat{A} \times \widehat{A} &\longrightarrow \widehat{A} \\ (\chi, \psi) &\longmapsto \chi\psi^{-1} \end{aligned}$$

is continuous. For (χ, ψ), (χ', ψ') and $x \in A$ one has

$$\begin{aligned} |\chi(x)\psi^{-1}(x) - \chi'(x)\psi'^{-1}(x)| &\leqslant |\chi(x)\psi^{-1}(x) - \chi(x)\psi'^{-1}(x)| + |\chi(x)\psi'^{-1}(x) - \chi'(x)\psi'^{-1}(x)| \\ &= |\psi^{-1}(x) - \psi'^{-1}(x)| + |\chi(x) - \chi'(x)| \end{aligned}$$

Let $K \subseteq A$ be compact and let $\varepsilon > 0$. Then

$$B_{K,\varepsilon}(\chi\psi^{-1}) = \{\gamma \in \widehat{A} \mid \left\|\gamma - \chi\psi^{-1}\right\|_K < \varepsilon\}$$

is an open neighborhood of $\chi\psi^{-1}$, and sets of this form are a neighborhood basis (Lemma 5.1.3). The estimate above shows that $B_{K,\varepsilon/2}(\chi) \times B_{K,\varepsilon/2}(\psi) \subseteq \Phi^{-1}(B_{K,\varepsilon}(\chi\psi^{-1}))$, so Φ is continuous. ☐

Proposition 5.1.7. Let A be an LCA group.

1. If A is compact, then $\widehat{A}$ is discrete.

2. If A is discrete, then $\widehat{A}$ is compact.

Proof.

1. Suppose A is compact. Put $U = \{z \in S^1 \mid \operatorname{Re} z > 0\}$. Since A is compact, the set $L(A, U)$ is open by definition. But U contains no nontrivial subgroup of S^1, so if $\chi \in L(A, U)$, then $\chi(A) = \{1\}$; hence $L(A, U)$ contains only trivial character.

2. Suppose A is discrete. Then the compact subspaces of A are precisely the finite subsets of A. Hence the compact open topology on $\widehat{A}$ coincides with the topology of pointwise convergence, and $\widehat{A} \subseteq (S^1)^A$ is a closed subspace; by Tychonov's theorem, the latter space is compact, and hence so is $\widehat{A}$.

□

Lemma 5.1.8. Let G, H be two LCA groups, and $\varphi \in \operatorname{Hom}_{\mathbf{TopGp}}(G, H)$. Then

$$\begin{aligned} \varphi^* : \widehat{H} &\longrightarrow \widehat{G} \\ \chi &\longmapsto \chi \circ \varphi \end{aligned}$$

is a continuous homomorphism.

Proof. Lemma 5.1.4 □

Lemma 5.1.9. Let G, H be two LCA groups. The canonical map

$$\begin{aligned} \widehat{G \times H} &\longrightarrow \widehat{G} \times \widehat{H} \\ \chi &\longmapsto (\chi|_{G \times \{1\}}, \chi|_{\{1\} \times H}) \end{aligned}$$

is an isomorphism of topological groups.

Proof. We must show this is a homeomorphism. By Lemma 5.1.4, this is a continuous map. To show the inverse is continuous, let $K \subseteq G \times H$ be compact and $U \subseteq S^1$ be open. Then the image of $L(K, U)$ is

$$L(\pi_G(K), U) \times L(\pi_H(K), U),$$

where $\pi_G : G \times H \to G$ and $\pi_H : G \times H \to H$ are canonical projections. Since this is open in the product, this proves the continuity of the inverse map. □

5.2 The Fourier Transform

Definition. Let A be an LCA group. For $f \in L^1(A)$ we define the **Fourier transform** of f to be the map $\widehat{f} : \widehat{A} \to \mathbb{C}$ defined by

$$\widehat{f}(\chi) := \int_A f(x)\overline{\chi(x)}dx$$

This integral exists since χ is bounded.

Lemma 5.2.1. For $f, g \in L^1(A)$ and $\chi \in \widehat{A}$, one has $|\hat{f}(\chi)| \leqslant \|f\|$ and $\widehat{f * g} = \hat{f}\hat{g}$.

Proof. The first assertion is clear. The second follows from a direct computation:

$$\begin{aligned}
\widehat{f * g}(\chi) &= \int_G (f * g)(x)\overline{\chi(x)}dx \\
&= \int_G \left(\int_G f(y)g(y^{-1}x)dy \right) \overline{\chi(x)}dx \\
&\overset{\text{Fubini}}{=} \int_G \int_G f(y)\overline{\chi(y)}g(y^{-1}x)\overline{\chi(y^{-1}x)}dxdy \\
&\overset{\text{inv}}{=} \int_G f(y)\overline{\chi(y)}dy \cdot \int_G g(x)\overline{\chi(x)}dx = \widehat{f}\widehat{g}
\end{aligned}$$

□

Lemma 5.2.2. For $f \in L^1(A)$ we have $\widehat{f^*} = \overline{\widehat{f}}$ so that the Fourier transform $\widehat{\cdot} : L^1(A) \to C(\widehat{A})$ is $*$-invariant.

Proof. Let $\chi \in \widehat{A}$. Then

$$\widehat{f^*}(\chi) = \int_A \Delta_A(y^{-1})\overline{f(y^{-1})}\chi(y)dy = \overline{\int_A f(y^{-1})\chi(y^{-1})dy} \overset{2.3.1.4}{=} \overline{\int_A f(y)\chi(y)dy} = \overline{\widehat{f}}(\chi)$$

□

Theorem 5.2.3. Let A be an LCA group. The map

$$\begin{aligned}
d : \widehat{A} &\longrightarrow \Delta_{L^1(A)} \\
\chi &\longmapsto d_\chi
\end{aligned}$$

where $d_\chi(f) := \displaystyle\int_A f(x)\overline{\chi(x)}dx$, is a homeomorphism. In particular,

1. $\widehat{A}$ is locally compact Hausdorff (Theorem 3.1.18) so that $\widehat{A}$ is an LCA group.
2. For every $f \in L^1(A)$, the Fourier transform $\widehat{f} \in C_0(\widehat{A})$. (Theorem 3.1.18) Precisely, the triangle commutes

$$\begin{tikzcd}
L^1(A) \arrow[rr, "\text{Fourier}"] \arrow[dr, "\text{Gelfand}"'] & & C_0(\widehat{A}) \\
& C_0(\Delta_{L^1(A)}) \arrow[ur, "d^*"'] &
\end{tikzcd}$$

Proof. It follows from the above lemma that $d_\chi \in \Delta_{L^1(A)}$.

- Injectivity. If $d_\chi = d_\psi$, then $\displaystyle\int_A f(x)\overline{(\chi(x) - \psi(x))}dx = 0$ for all $f \in L^1(G)$; in particular, letting $f = \chi - \psi$ we see $\displaystyle\int_A |\chi(x) - \psi(x)|^2 dx = 0$ so that $\chi = \psi$.
- Surjectivity. Let $m \in \Delta_{L^1(A)}$. Since $C_c(A)$ is dense in $L^1(A)$, we can find $g \in C_c(A)$ such that $m(g) \neq 0$. For each $x \in A$ define

$$\chi(x) = \frac{\overline{m(L_x g)}}{\overline{m(g)}}$$

That m is continuous implies χ is continuous, and for $x, y \in A$, one compute

$$m(L_x g)m(L_y g) = m(L_x g * L_y g) = m(L_{xy} g * g) = m(L_{xy} g)m(g)$$

so that $\chi(xy) = \chi(x)\chi(y)$, i.e. $\chi : A \to \mathbb{C}^\times$ is a group homomorphism.

Let $f \in C_c(A)$. Then by Lemma D.7.4 we can view $f * g$ as $\displaystyle\int_A f(x) L_x g dx$. Since m is a bounded linear functional, by Proposition D.7.1 one obtains

$$\begin{aligned}\int_A f(x)\overline{\chi(x)}dx &= \frac{1}{m(g)}\int_A f(x) m(L_x g) dx = \frac{1}{m(g)} m\left(\int_A f(x) L_x g dx\right) \\ &= \frac{1}{m(g)} m(f * g) = \frac{m(f)m(g)}{m(g)} = m(f)\end{aligned}$$

Let (ϕ_U) be a Dirac net in $C_c(A)$. Then $\phi_U * \overline{\chi}$ converges pointwise to $\overline{\chi}$, so for $x \in A$ and $\varepsilon > 0$ there exists a unit-neighborhood U such that

$$\begin{aligned}|\chi(x)| \leqslant |\phi_U * \overline{\chi}(x)| + \varepsilon &= \left|\int_A L_x \phi_U(y)\overline{\chi(y)}dy\right| + \varepsilon \\ &= |m(L_x\phi_U)| + \varepsilon \leqslant \lim_U \|L_x \phi_U\|_1 + \varepsilon = 1 + \varepsilon\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get $|\chi(x)| \leqslant 1$ for every $x \in A$; but $\chi(x^{-1}) = \chi(x)^{-1}$ so it is equality. In conclusion $\chi \in \widehat{A}$ and $m = d_\chi$.

- Continuity. Let $f \in L^1(A)$ and $U \subseteq \mathbb{C}$ be open. We must show the inverse image under d the set $M(f, U) := \{m \in \Delta_{L^1(A)} \mid m(f) \in U\}$ is open in $\widehat{A}$, i.e.

$$d^{-1}M(f, U) = \{\chi \in \widehat{A} \mid d_\chi(f) \in U\} = \{\chi \in \widehat{A} \mid \hat{f}(\chi) \in U\} \underset{\text{open}}{\subseteq} \widehat{A}$$

Let $\chi \in d^{-1}M(f, U)$ and $\varepsilon > 0$ such that $B_\varepsilon(\hat{f}(\chi)) \subseteq U$. Let $g \in C_c(A)$ with $\|f - g\|_1 < \varepsilon/3$. Put $K = \operatorname{supp}(g)$ and $V := B_{K,\varepsilon/3\|g\|_1}(\chi)$ (Lemma 5.1.3). Then for all $\psi \in V$ we have

$$\begin{aligned}|\hat{f}(\chi) - \hat{f}(\psi)| &\leqslant |\hat{f}(\chi) - \hat{g}(\chi)| + |\hat{g}(\chi) - \hat{g}(\psi)| + |\hat{g}(\psi) - \hat{f}(\psi)| \\ &< \|f - g\|_1 + \|g\|_1 \frac{\varepsilon}{3\|g\|_1} + \|f - g\|_1 \\ &< \varepsilon\end{aligned}$$

so that $\hat{f}(\psi) \in U$. Hence $\chi \in V \subseteq d^{-1}M(f, U)$, proving that $d^{-1}M(f, U)$ is open.

- Continuous inverse. We prove the following lemma.

Lemma 5.2.4. Let $\chi_0 \in \widehat{A}$, $K \subseteq A$ compact and $\varepsilon > 0$. Then there exist $\ell \in \mathbb{N}$, $f_0, \ldots, f_\ell \in L^1(A)$, and $\delta > 0$ such that for $\chi \in \widehat{A}$ the condition $|\hat{f}_j(\chi) - \hat{f}_j(\chi_0)| < \delta$ for all $j = 0, \ldots, \ell$ implies $|\chi(x) - \chi_0(x)| < \varepsilon$ for all $x \in K$; in other words,

$$d_{\chi_0} \in \bigcap_{j=0}^{\ell} M(f_j, B_\delta(\hat{f}_j(\chi_0))) \subseteq dB_{K,\varepsilon}(\chi_0)$$

Proof. We first make a reduction. For $f \in L^1(A)$, one has

$$\hat{f}(\chi) - \hat{f}(\chi_0) = \widehat{f\overline{\chi_0}}(\chi\overline{\chi_0}) - \widehat{f\overline{\chi_0}}(1)$$

so that we may assume $\chi_0 = 1$. Let $f \in L^1(A)$ with $\widehat{f}(1) = \displaystyle\int_A f(x)dx = 1$. Then there exists a unit neighborhood U of A such that $\|L_u f - f\|_1 < \dfrac{\varepsilon}{3}$ for all $u \in U$. Since K is compact, there are finitely many $x_1, \ldots, x_\ell \in A$ such that $K \subseteq x_1 U \cup \cdots \cup x_\ell U$. Set $f_j := L_{x_j} f$ and

$f_0 = f$, and let $\delta = \dfrac{\varepsilon}{3}$. Let $\chi \in \widehat{A}$ with $|\hat{f}_j(\chi) - 1| < \dfrac{\varepsilon}{3}$ for every $j = 0, \dots, \ell$. Now let $x \in K$. Then there exists $1 \leqslant j \leqslant \ell$ and $u \in U$ such that $x = x_j u \in x_j U$. One has

$$\begin{aligned}
|\chi(x) - 1| &= |\overline{\chi(x)} - 1| \\
&\leqslant |\overline{\chi(x)} - \overline{\chi(x)}\hat{f}(\chi)| + |\overline{\chi(x)}\hat{f}(\chi) - \hat{f}_j(\chi)| + |\hat{f}_j(\chi) - 1| \\
&= |1 - \hat{f}(\chi)| + |\widehat{L_x f}(\chi) - \widehat{L_{x_j} f}(\chi)| + |\hat{f}_j(\chi) - 1| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

where the last inequality uses

$$|\widehat{L_x f}(\chi) - \widehat{L_{x_j} f}(\chi)| \leqslant \left\|L_x f - L_{x_j} f\right\|_1 = \|L_u f - f\|_1 < \frac{\varepsilon}{3}$$

□

5.3 Group C^*-Algebra

Let $f \in L^1(A)$ and $\phi, \psi \in L^2(A)$. For every $y \in A$, one has

$$|\langle L_y \phi, \psi\rangle| \leqslant \|L_y \phi\|_2 \|\psi\|_2 = \|\phi\|_2 \|\psi\|_2$$

This implies that the integral $\displaystyle\int_A f(y)\langle L_y\phi, \psi\rangle dy$ exists, and one has the estimate

$$\left|\int_A f(y)\langle L_y \phi, \psi\rangle\right| \leqslant \|f\|_1 \|\phi\|_2 \|\psi\|_2$$

In other word, the anti-linear map

$$\begin{aligned}
L^2(A) &\longrightarrow \mathbb{C} \\
\psi &\longmapsto \int_A f(y)\langle L_y\phi, \psi\rangle dy
\end{aligned}$$

is bounded. By Riesz's representation theorem, there exists a unique vector, denoted by $L(f)\phi$, in $L^2(A)$ such that

$$\langle L(f)\phi, \psi\rangle = \int_A f(y)\langle L_y \phi, \psi\rangle dy$$

for all $\psi \in L^2(A)$. The above estimate gives $|\langle L(f)\phi, \psi\rangle| \leqslant \|f\|_1 \|\phi\|_2 \|\psi\|_2$; in particular, for $\psi = L(f)\phi$ one concludes

$$\|L(f)\phi\|_2^2 \leqslant \|f\|_1 \|\phi\|_2 \|L(f)\phi\|_2$$

so that $\|L(f)\phi\|_2 \leqslant \|f\|_1 \|\phi\|_2$, which implies the map

$$\begin{aligned}
L(f) : L^2(A) &\longrightarrow L^2(A) \\
\phi &\longmapsto L(f)\phi
\end{aligned}$$

is bounded, and hence continuous.

Lemma 5.3.1. If $f \in L^1(A)$ and $\phi \in L^1(A) \cap L^2(A)$, then $L(f)\phi = f * \phi = \phi * f$.

Proof. By Proposition 2.6.2, $f * \phi \in L^1(A) \cap L^2(A)$. Let $\psi \in C_c(A)$. Then

$$\begin{aligned}
\langle f * \phi, \psi \rangle &= \int_A \int_A f(y)\phi(y^{-1}x)\overline{\psi(x)}dydx \\
&\overset{\text{Fubini}}{=} \int_A f(y) \left(\int_A L_y\phi(x)\overline{\psi(x)}dx \right) dy \\
&= \int_A f(y)\langle L_y\phi, \psi \rangle dy \\
&= \langle L(f)\phi, \psi \rangle
\end{aligned}$$

Since $C_c(G)$ is dense in $L^2(A)$, this implies $L(f)\phi = f * \phi$. ☐

Lemma 5.3.2. The map

$$\begin{aligned}
L : L^1(A) &\longrightarrow \mathcal{B}(L^2(A)) \\
f &\longmapsto L(f)
\end{aligned}$$

is an injective continuous homomorphism of Banach *-algebras.

Proof.

- Continuity. Since L is clearly linear and $\|L(f)\|_{\text{op}} \leqslant \|f\|_1$, so L is bounded and thus continuous.
- Homomorphism. Let $f, g \in L^1(A)$ and $\phi \in C_c(A)$. One computes

  $$L(f * g)(\phi) \overset{5.3.1}{=} (f * g) * \phi = f * (g * \phi) \overset{5.3.1}{=} f * L(g)\phi \overset{5.3.1}{=} L(f)L(g)\phi$$

  Lemma 5.3.1 is valid for $L(g)\phi = g * \phi \in L^1(A) \cap L^2(G)$ by Proposition 2.6.2. Since $C_c(A)$ is dense in $L^2(G)$, this implies $L(f * g) = L(f)L(g)$.
- Injectivity. Suppose $f \in L^1(A)$ with $L(f)\phi = 0$ for all $\phi \in L^2(A)$. In particular, $f * \psi = 0$ for all $\psi \in C_c(G)$. Letting ψ run over a Dirac net of A, we see $f = 0$.
- *-equivariant. Recall that since A is abelian

  $$f^*(x) = \Delta_A(x^{-1})\overline{f(x^{-1})} = \overline{f(x^{-1})}$$

  For $\phi, \psi \in C_c(A)$, one has

  $$\begin{aligned}
  \langle L(f)\phi, \psi \rangle = \langle f * \phi, \psi \rangle &= \int_A \int_A f(y)\phi(y^{-1}x)\overline{\psi(x)}dxdy \\
  &\overset{2.3.1}{=} \int_A \int_A f(y^{-1})\phi(yx)\overline{\psi(x)}dxdy \\
  &\overset{\text{inv}}{=} \int_A \int_A \phi(x)f(y^{-1})\overline{\psi(y^{-1}x)}dxdy \\
  &= \int_A \int_A \phi(x)\overline{\overline{f(y^{-1})}\psi(y^{-1}x)}dxdy \\
  &= \langle \phi, f^* * \psi \rangle
  \end{aligned}$$

  so that $L(f^*) = L(f)^*$.

☐

Definition. For an LCA group A, the **group C^*-algebra** $C^*(A)$ is defined to be the closure of $L(L^1(A))$ in the C^*-algebra $\mathcal{B}(L^2(A))$.

- Since $L^1(A)$ is a commutative Banach algebra, $C^*(A)$ is a commutative C^*-algebra.

Theorem 5.3.3. The pullback map $L^* : \Delta_{C^*(A)} \to \Delta_{L^1(A)}$ is a homeomorphism.

Proof. Since the image of L is dense in $C^*(A)$, it follows that $m \circ L \neq 0$ for all $m \in \Delta_{C^*(A)}$ and that L^* is injective. By Lemma 3.1.20, it suffices to show L^* is surjective.

To show this, let $m \in \Delta_{L^1(A)}$ and $\chi \in \widehat{A}$ such that $m = d_\chi$, i.e., $m(f) = \hat{f}(\chi)$ for all $f \in L^1(A)$. We have to show m is continuous in C^*-norm, for then it has a continuous extension to $C^*(A)$. To this end, let $\mu_0 \in \Delta_{C^*(A)}$ be fixed. Then there exists $\chi_0 \in \hat{A}$ such that $\mu_0(L(f)) = \hat{f}(\chi_0)$ for all $f \in L^1(A)$. Then for $f \in L^1(A)$,

$$m(f) = \int_A f(x)\overline{\chi(x)}dx = \int_A f(x)\overline{\chi(x)}\chi_0(x)\overline{\chi_0(x)}dx = \widehat{f\overline{\chi}\chi_0}(\chi_0) = \mu_0(L(f\overline{\chi}\chi_0))$$

It follows that $|m(f)| = |\mu_0(L(f\overline{\chi}\chi_0))| \leqslant \|L(f\overline{\chi}\chi_0)\|_{C^*(A)}$. So it suffices to show that for $f \in L^1(A)$, $\|L(f)\|_{C^*(A)} = \|L(f\eta)\|_{C^*(A)}$ for all $\eta \in \widehat{A}$. As C^*-norm is the operator norm in $\mathcal{B}(L^2(A))$, we consider $\phi, \psi \in L^2(A)$ and compute

$$\begin{aligned}\langle L(\eta f)\phi, \psi\rangle &= \int_A \eta(x)f(x)\langle L_x\phi, \psi\rangle dx \\ &= \int_A \eta(x)f(x)\int_A \phi(x^{-1}y)\overline{\psi(y)}dydx \\ &= \int_A \eta(x)f(x)\int_A (\overline{\eta}\phi)(x^{-1}y)\overline{(\overline{\eta}\psi)(y)}dydx \\ &= \langle L(\eta f)\overline{\eta}\phi, \overline{\eta}\psi\rangle\end{aligned}$$

Putting $\psi = L(\eta f)\phi$, we get

$$\|L(\eta f)\phi\|_2^2 = \langle L(f)(\overline{\eta}\phi), \overline{\eta}L(\eta f)\phi\rangle \leqslant \|L(f)(\overline{\eta}\phi)\|_2 \,\|\overline{\eta}L(\eta f)\phi\|_2$$

Since $\|L(\eta f)\phi\|_2 = \|\overline{\eta}L(\eta f)\phi\|_2$, it follows that $\|L(\eta f)\phi\|_2 \leqslant \|L(f)(\overline{\eta}\phi)\|_2$, and hence $\|L(\eta f)\|_{C^*(A)} \leqslant \|L(f)\|_{C^*(A)}$; by symmetry we get equality. □

Corollary 5.3.3.1. Let A be an LCA group. Then the Fourier transform $L^1(A) \to C_0(\widehat{A})$ is injective.

Proof. By functoriality, from $L^1(A) \to C^*(A)$ we obtain a commutative square

$$\begin{tikzcd} L^1(A) \arrow[r, "\mathrm{Gelfand}_1"] \arrow[d, "L"] & C_0(\Delta_{L^1(A)}) \arrow[d, "(L^*)^*"] \\ C^*(A) \arrow[r, "\mathrm{Gelfand}_2"] & C_0(\Delta_{C^*(A)}) \end{tikzcd}$$

The vertical right arrow is bijective by Theorem 5.3.3. In Theorem 5.2.3 we have a commutative triangle

$$\begin{tikzcd} L^1(A) \arrow[rr, "\mathrm{Fourier}"] \arrow[dr, "\mathrm{Gelfand}_1"'] & & C_0(\widehat{A}) \\ & C_0(\Delta_{L^1(A)}) \arrow[ur, "d^*"'] & \end{tikzcd}$$

Hence we see the Fourier transform is the composition of injective maps

$$L^1(A) \xrightarrow{L} C^*(A) \xrightarrow{((L^*)^*)^{-1}\circ\,\mathrm{Gelfand}_2} C_0(\Delta_{L^1(A)}) \xrightarrow{d^*} C_0(\widehat{A})$$

(In fact, the injectivity simply results from the commutative triangle in Theorem 5.2.3.) □

5.4 The Plancherel Theorem

The Plancherel Theorem says that the Fourier transform extends to a unitary equivalence $L^2(A) \cong L^2(\widehat{A})$. The proof of the theorem is postponed to the next section. In this section we do some preparations.

Lemma 5.4.1. Let $\phi, \psi \in L^2(A)$. Then the convolution product $\phi * \psi(x) := \int_A \phi(y)\psi(y^{-1}x)dy$ exists for every $x \in A$ and defines a continuous function in x. The convolution $\phi * \psi$ lies in $C_0(A)$ and its sup-norm satisfies $\|\phi * \psi\|_A \leqslant \|\phi\|_2 \|\psi\|_2$. Finally one has $\phi * \phi^*(1) = \|\phi\|_2^2$.

Proof. Since A is abelian (and hence unimodular), we have

$$\phi * \psi(x) = \int_A \phi(y)\psi(y^{-1}x)dy = \int_A \phi(y)\overline{L_x\psi^*(y)}dy = \langle \phi, L_x\psi^* \rangle$$

Since ϕ and ψ are L^2, $\langle \phi, L_x\psi^* \rangle$ exists for every x by Hölder's inequality. The continuity follows from Lemma 2.6.7. Next, by Cauchy-Schwarz, we obtain

$$\|\phi * \psi\|_A = \sup_{x \in A} |\langle \phi, L_x\psi^* \rangle| \leqslant \|\phi\|_2 \|\psi\|_2$$

By taking $(\phi_n)_n$ and $(\psi_n)_n$ in $C_c(A)$ such that $\|\phi_n - \phi\|_2 \to 0$ and $\|\psi_n - \psi\|_2 \to 0$, we see from the above inequality that $\phi_n * \psi_n \to \phi * \psi$ uniformly. Since $C_c(A) \subseteq C_0(A)$ and $C_0(A)$ is complete, $\phi * \psi \in C_0(A)$. Finally, $\phi * \phi^*(1) = \langle \phi, \phi \rangle = \|\phi\|_2^2$. $\square$

The space $C_0(A) \times C_0(\widehat{A})$ is a Banach space with the norm

$$\|(f, \eta)\|_0^* := \max\{\|f\|_A, \|\eta\|_{\widehat{A}}\}$$

We embed $C_0(A) \cap L^1(A)$ into this product space by

$$\begin{aligned} C_0(A) \cap L^1(A) &\longrightarrow C_0(A) \times C_0(\widehat{A}) \\ f &\longmapsto (f, \hat{f}) \end{aligned}$$

and denote its closure in the product space by $C_0^*(A)$; it is a Banach space with norm $\|\cdot\|_0^*$. We have two natural projections

$$C_0(A) \xleftarrow{\pi_0} C_0(A) \times C_0(\widehat{A}) \xrightarrow{\pi_*} C_0(\widehat{A})$$

Lemma 5.4.2. Both restrictions to $C_0^*(A)$ of π_0 and π_* are injective. Hence we can view $C_0^*(A)$ as a subspace of $C_0(A)$ as well as of $C_0(\widehat{A})$.

Proof. Let $f \in C_0^*(A)$. We must show if one of $\pi_0(f)$ and $\pi_*(f)$ is zero, then so is the other. Let $(f_n)_n \subseteq C_0(A) \cap L^1(A)$ such that $f_n \to f$; then $\pi_*(f_n) \to \pi_*(f)$ in $C_0(\widehat{A})$ and $\pi_0(f_n) \to \pi_0(f)$ in $C_0(A)$.

- Recall the isomorphism $C^*(A) \cong C_0(\widehat{A})$. Then $\pi_*(f_n) \to \pi_*(f)$ in $C_0(\widehat{A})$ if and only if $L(f_n) \to \pi_*(f)$ (view $\pi_*(f)$ as an element in $C^*(A)$), if and only if $f_n * \psi \to \pi_*(f)(\psi)$ in $L^2(A)$ for all $\psi \in C_c(A)$.
- Also, $f_n * \psi \to \pi_0(f) * \psi$ in $C_0(A)$.

Hence for $\phi \in C_c(A)$, $\langle f_n * \psi, \phi \rangle$ converges to $\langle \pi_*(f)(\psi), \phi \rangle$, and also to $\langle \pi_0(f) * \psi, \phi \rangle$, and therefore,

$$\langle \pi_*(f)(\psi), \phi \rangle = \langle \pi_0(f) * \psi, \phi \rangle$$

holds for all $\psi, \phi \in C_c(A)$. Then $\pi_*(f) = 0$ if and only if $\pi_0(f) = 0$. $\square$

In the sequel, given an element $f \in C_0^*(A)$ we will freely view it as an element of $C_0(A)$ or of $C^*(A) \cong C_0(\widehat{A})$. If we want to emphasize the distinction, we write f for the former and write $\widehat{f}$ for the latter.

Lemma 5.4.3. $f \in C_0^*(A)$. If the Fourier transform $\widehat{f}$ is real-valued, then $f(1) \in \mathbb{R}$. If $\widehat{f} \geqslant 0$, then $f(1) \geqslant 0$. Here $1 \in A$ is the identity element.

Proof. Suppose $\widehat{f}$ is real-valued. Then $\widehat{f} = \overline{\widehat{f}} = \hat{f}^*$, and since $\hat{\cdot}$ is injective, $f = f^*$ and therefore $f(1) = f^*(1) = \overline{f(1)}$. Now suppose $\hat{f} \geqslant 0$. Then there exists $g \in C_0(\widehat{A}) \cong C^*(A)$ with $g \geqslant 0$ sand $\hat{f} = g^2$. Let $\phi = \phi^* \in C_c(A)$. Then $L(g)\phi \in L^2(A)$, so

$$(L(g)\phi) * (L(g)\phi)^*(1) = \|L(g)\phi\|_2^2 \geqslant 0$$

Take $(g_n)_n \subseteq L^1(A)$ such that $L(g_n) \to g$ in $C^*(A)$. Then

$$\begin{aligned}
(L(g)\phi) * (L(g)\phi)^* &= \lim_n (L(g_n)\phi) * (L(g_n)\phi)^* \\
&= \lim_n (g_n * \phi) * (g_n * \phi)^* \\
&= \lim_n g_n * \phi * \phi * g_n^* \\
&= \lim_n g_n * g_n^* * \phi * \phi \\
&= \lim_n L(g_n * g_n^*)(\phi * \phi) = L(f)(\phi * \phi) = f * \phi * \phi
\end{aligned}$$

where the second last equality is due to $g^2 = \hat{f}$. □

Lemma 5.4.4.

(a) The space $L^1(A) * C_c(A)$ is a subspace of $C_0(A)$.

(b) Let $f \in C^*(A)$, and let $\phi, \psi \in C_c(A)$. Then $L(f)(\phi*\psi) \in C_0^*(A) \cap L^2(A)$, viewed as a subspace of $C_0(A)$. One has $\widehat{L(f)(\phi * \psi)} = \widehat{f}\widehat{\phi}\widehat{\psi}$.

Proof.

(a) For $f \in L^1(A)$, let $f_n \in C_c(A)$ such that $f_n \to f$ in L^1. Then for $\phi \in C_c(A)$,

$$|(f_n - f_m) * \phi(x)| = \left| \int_A (f_n - f_m)(y)\phi(y^{-1}x)dy \right| \leqslant \|f_n - f_m\|_1 \|\phi\|_\infty$$

Thus $f_n * \phi$ is Cauchy in $C_0(A)$, and since $C_0(A)$ is complete, their limit $f * \phi$ lies in $C_0(A)$.

(b) For $f \in C^*(A)$, we can find $f_n \in L^1(A)$ such that $L(f_n) \to f$ in $C^*(A)$. We have $L(f_n)(\phi*\psi) = f_n * \phi * \psi \in C_0(A) \cap L^1(A)$ by Lemma 5.3.1 and (a). We show that $f_n * \phi * \psi$ is Cauchy in $C_0^*(A)$, i.e. it is Cauchy in $C_0(A)$ and its Fourier transform is Cauchy is $C_0(\widehat{A})$.

Since $\widehat{f_n * \phi * \psi} = \widehat{f_n}\widehat{\phi}\widehat{\psi}$ and $\widehat{f_n}$ converges uniformly on $\widehat{A}$ (recall $C^*(A) \cong C_0(\widehat{A})$), we see $\widehat{f_n}\widehat{\phi}\widehat{\psi} \to \widehat{f}\widehat{\phi}\widehat{\psi}$ uniformly on $\widehat{A}$. On the other hand, by Lemma 5.4.1,

$$\|(f_m - f_n) * \phi * \psi\|_A \leqslant \|(f_m - f_n) * \phi\|_2 \|\psi\|_2 \leqslant \|f_m - f_n\|_1 \|\phi\|_2 \|\psi\|_2$$

so $f_n * \phi * \psi$ is Cauchy in $C_0(A)$ as well.

□

Lemma 5.4.5. Let (ϕ_U) be a Dirac net in $C_c(A)$. Then

(a) $(f * \phi_U)$ converges to f in $C^*(A)$ for all $f \in C^*(A)$.

(b) $(f * \phi_U)$ converges uniformly to f for every $f \in C_0(A)$.

(c) $(f * \phi_U)$ converges to f in $C_0^*(A)$ for every $f \in C_0^*(A)$.

(d) $(\widehat{\phi_U})$ converges locally uniformly to 1 on $\widehat{A}$.

Proof.

1. It holds when $f \in L^1(A)$ by Lemma 2.6.9. Since $L^1(A) \subseteq C^*(A)$ is dense, the result follows.
2. Similar to (a), with $L^1(A)$ replaced by $C_c(A)$, which is dense in $C_0(A)$.
3. This follows from (a) and (b).
4. Let $C \subseteq \widehat{A}$ be compact and pick positive $\psi \in C_c(\widehat{A})$ such that $\psi \equiv 1$ on C. Let $f \in C^*(A)$ such that $\hat{f} = \psi$ (see the last composition in Corollary 5.3.3.1). Then
$$\left\| \hat{\phi_U} \psi - \psi \right\|_{\widehat{A}} = \|\phi_U * f - f\|_{\mathrm{op}} \to 0$$
by (a), and the result follows. (Recall that the Gelfand transform is isometric by Gelfand-Naimark so that map $C^*(A) \cong C_0(\widehat{A})$ is isometric; see Corollary 5.3.3.1 as well.)

□

Lemma 5.4.6. Let $\eta \in C_c(\widehat{A})$ be real-valued and let $\varepsilon > 0$. Then there are $f_1, f_2 \in C_0^*(A) \cap L^2(A)$, considered as subspaces of $C_0(A)$, such that

(i) the Fourier transforms $\hat{f}_1$, $\hat{f}_2$ lie in $C_c(\widehat{A})$,

(ii) they satisfy $\hat{f}_1 \leqslant \eta \leqslant \hat{f}_2$, further $\left\| \hat{f}_1 - \hat{f}_2 \right\|_{\widehat{A}} < \varepsilon$, and $\operatorname{supp} \hat{f}_i \subseteq \operatorname{supp} \eta$ for $i = 1, 2$,

(iii) as well as $0 \leqslant f_2(1) - f_1(1) < \varepsilon$

In particular, every $\eta \in C_c(\widehat{A})$ is the uniform limit of functions of the form $\hat{f}$ with $f \in C_0^*(A)$ of support contained in $\operatorname{supp} \eta$.

Proof. For any Dirac function $\phi \in C_c(A)$ one has $\hat{\phi} \in C_0(\widehat{A})$ by Theorem 5.2.3.2 and Lemma 5.4.5 says that the ensuing function $\hat{\phi}$ can be chosen to approximate the constant 1 arbitrarily close on any compact set.

Note that the Fourier transform $\widehat{h * h^*} = \hat{h}\hat{h^*} = \hat{h}\overline{\hat{h}} \geqslant 0$. Let $K := \operatorname{supp} \eta \subseteq \widehat{A}$. Since $C_c(A)$ contains Dirac functions of arbitrary small support, we conclude that for every $\delta > 0$ there exists a function $\phi_\delta \in C_c^+(A)$ such that the function $\psi_\delta := \phi_\delta * \phi_\delta^*$ satisfies
$$1 - \delta \leqslant \hat{\psi_\delta}(\chi) \leqslant 1 + \delta \qquad \text{for every } \chi \in K$$
Fix $\phi \in C_c^*(A)$ such that $\psi := \phi * \psi^*$ satisfies $\hat{\psi}(\chi) \geqslant 1$ for all $\chi \in K$. Let $f \in C^*(A)$ with $\hat{f} = \eta$ and set
$$f_1 := f * (\psi_\delta - \delta\psi), \qquad f_2 := f * (\psi_\delta + \delta\psi)$$
By Lemma 5.4.4, $f_1, f_2 \in C_0^*(A) \cap L^2(A)$. For every $\chi \in \widehat{A}$, we have
$$\hat{f}_1(\chi) = \hat{f}(\chi)(\hat{\psi_\delta}(\chi) - \delta\hat{\psi}(\chi)) \leqslant \eta(\chi) \leqslant \hat{f}_2(\chi)$$
Further, as $\hat{f}(\chi) = \eta(\chi)$, one has $\operatorname{supp}(\hat{f}_i) \subseteq \operatorname{supp} \eta$. The other properties follow by choosing δ small enough and Lemma 5.4.3. □

Proposition 5.4.7. Let $\psi \in C_c(\widehat{A})$ be real-valued. Then

$$\sup\{f(1) \mid f \in C_0^*(A),\ \hat{f} \leqslant \psi\} = \inf\{f(1) \mid f \in C_0^*(A),\ \hat{f} \geqslant \psi\}$$

Denote this common value by $I(\psi)$. Extending linearly, we obtain a positive linear functional $I : C_c(\widehat{A}) \to \mathbb{C}$. Then I is a Haar integral. We write this integral as

$$I(\psi) = \int_{\widehat{A}} \psi(\chi) d\chi$$

Proof. By Lemma 5.4.3, if we pick $f \in \mathrm{LHS}$ and $g \in \mathrm{RHS}$, then $\widehat{g - f} \geqslant 0$ so that $g(1) \geqslant f(1)$; this implies

$$\sup\{f(1) \mid f \in C_0^*(\widehat{A}),\ \hat{f} \leqslant \psi\} \leqslant \inf\{f(1) \mid f \in C_0^*(A),\ \hat{f} \geqslant \psi\}$$

The equality holds, for their difference is arbitrarily small by Lemma 5.4.6 (particularly (ii) and (iii)).

- I linear. Clear.
- I positive. If $\psi \geqslant 0$, then $f(1) \geqslant 0$ for all $f \in C_0^*(A)$ with $\hat{f} \geqslant \psi$ by Lemma 5.4.3. Thus $I(\psi) \geqslant 0$.
- Invariance. Let $\psi \in C_c(\widehat{A})$ and $f \in C_0^*(A)$ with $\hat{f} \leqslant \psi$. For $\chi \in \widehat{A}$ we have $L_\chi \hat{f} \leqslant L_\chi \psi$. Further,

  $$L_\chi \hat{f}(\rho) = \hat{f}(\chi^{-1}\rho) = \int_A f(y)\overline{\chi^{-1}\rho(y)} dy = \int_A f\chi(y)\overline{\rho(y)} dy = \widehat{f\chi}(\rho)$$

  and $\chi f(1) = f(1)$. This shows the invariance.

□

The following will be proved in the next section as a consequence of the Pontryagin duality theorem.

Theorem 5.4.8 (Plancherel)**.** For a given Haar measure on A there exists a uniquely determined Haar measure on $\widehat{A}$, called the **Plancherel measure**, such that for $f \in L^1(A) \cap L^2(A)$ one has

$$\|f\|_2 = \left\|\hat{f}\right\|_2$$

This implies that the Fourier transform extends to an isometry from $L^2(A)$ to $L^2(\widehat{A})$. Indeed, it is also surjective, so the Fourier transform extends to a canonical unitary equivalence $L^2(A) \cong L^2(\widehat{A})$.

Corollary 5.4.8.1. Let K be a compact LCA group. Then the elements of the dual group $\widehat{K}$ form an orthonormal basis of $L^2(K)$.

Proof. We normalized the Haar measure on K so that the volume of K is 1. K being compact, every continuous function on K is L^2; particularly, every character of K is L^2. We show the characters of K form an orthonormal system, i.e. that for $\chi, \eta \in \widehat{K}$ we have

$$\langle \chi, \eta \rangle = \delta_{\chi,\eta} = \begin{cases} 1 & , \chi = \eta \\ 0 & , \chi \neq \eta \end{cases}$$

This is standard, and can be proved as in the finite group case. It follows that the Fourier transform of a character χ is the map δ_χ with $\delta_\chi(\eta) := \delta_{\chi,\eta}$.

Since K is compact, $\widehat{K}$ is discrete so that the Plancherel measure is just a multiple of the counting measure; say the constant is $c > 0$. Let S be any finite subset of $\widehat{K}$. For each $f \in L^2(\widehat{K})$, one has

$$\left\| f - \sum_{\chi \in S} f(\chi)\delta_\chi \right\|_2^2 = c \left(\sum_{\chi \in \hat{A}} |f(\chi)|^2 - \sum_{\chi \in S} |f(\chi)|^2 \right)$$

It follows from the definition of the integral (associated to the counting measure) that we can pick S large enough that the difference is arbitrarily small; hence $\{\delta_\chi\}_{\chi \in \widehat{K}}$ forms an orthonormal basis of $L^2(\widehat{K})$. By Plancherel theorem, the Fourier transform is a unitary equivalence, the characters form an orthonormal basis of $L^2(K)$. $\square$

5.4.1 Examples of Plancherel measures

Lemma 5.4.9. Let K be a compact LCA group and let dx be a Haar measure on K. Then the Plancherel measure on $\widehat{K}$ is $\dfrac{1}{\mathrm{vol}(K, dx)}$ times the counting measure on $\widehat{K}$.

Proof. For all $\chi \in \widehat{K}$,

$$\widehat{\mathbf{1}_K}(\chi) = \int_K \mathbf{1}_K(x)\overline{\chi(x)}dx = \begin{cases} \mathrm{vol}(K, dx) & , \text{ if } \chi \text{ is trivial} \\ 0 & , \text{ if } \chi \text{ is non-trivial.} \end{cases}$$

Since $\widehat{K}$ is discrete, the Plancherel measure is a multiple of the counting measure; say the constant is $c > 0$. By Plancherel theorem and the computation above,

$$\mathrm{vol}(K, dx) = \|\mathbf{1}_K\|_2^2 = \left\|\widehat{\mathbf{1}_K}\right\|_2^2 = c \sum_{\chi \in \widehat{K}} \widehat{\mathbf{1}_K(\chi)}^2 = c\,\mathrm{vol}(K, dx)^2.$$

Hence $c = \dfrac{1}{\mathrm{vol}(K, dx)}$. $\square$

Lemma 5.4.10. Let D be a discrete subgroup, and pick $c > 0$. Then the Plancherel measure on $\widehat{D}$ corresponding to $c\#$ is the measure such that the total volume of $\widehat{D}$ is c.

Proof. Denote by e the identity in D, and let $d\chi$ be the measure on $\widehat{D}$ so that $\mathrm{vol}(\widehat{D}, d\chi) = 1$. Then

$$\widehat{\mathbf{1}_{\{e\}}}(\chi) = \sum_{x \in D} \mathbf{1}_{\{e\}}(x)\chi(x) = \chi(e) = \mathrm{id}_{\widehat{D}}(\chi).$$

Say $ad\chi$ is the Plancherel measure on $\widehat{D}$ for some $a > 0$. By Plancherel theorem, we have

$$c = \left\|\mathbf{1}_{\{e\}}\right\|_2^2 = \left\|\widehat{\mathbf{1}_{\{e\}}}\right\|_2^2 = \int_{\widehat{D}} \mathrm{id}_{\widehat{D}}(\chi) a d\chi = a.$$

$\square$

Lemma 5.4.11. Let G, H be two LCA groups, and let dg, dh be any Haar measures on G, H. The Plancherel measure on $\widehat{G \times H}$ corresponding to $dg \otimes dh$ is the product of the Plancherel measures on $\widehat{G} \times \widehat{H}$, under the identification of $\widehat{G \times H} \cong \widehat{G} \times \widehat{H}$ in Lemma 5.1.9.

Proof. This follows from Fubini and the fact that $C_c(G) \otimes C_c(H) \subseteq C_c(G \times H)$ is dense by Stone-Weirestrass. $\square$

5.5 Pontryagin Duality

For an LCA group A, we see in Theorem 5.2.3 that $\widehat{A}$ is again an LCA group, so we can consider the dual group $\widehat{\widehat{A}}$ of $\widehat{A}$, and it is again an LCA group. There is a canonical association, called the **Pontryagin map**, δ which assigns each $x \in A$ to a group homomorphism $\delta_x : \widehat{A} \to S^1$ given by $\delta_x(\chi) = \chi(x)$.

Lemma 5.5.1. For each $x \in A$, the homomorphism $\delta_x : \widehat{A} \to S^1$ is continuous, and thus $\delta_x \in \widehat{\widehat{A}}$.

Proof. For each $\varepsilon > 0$, $\delta_x^{-1}(B_\varepsilon(1)) = \{\chi \in \widehat{A} \mid |\chi(x) - 1| < \varepsilon\}$ is open in $\widehat{A}$ by the definition of the compact-open topology (note that every singleton is compact). $\square$

Proposition 5.5.2. The Pontryagin map $\delta : A \to \widehat{\widehat{A}}$ is an injective continuous group homomorphism. In particular, if $1 \neq x \in A$, there exists some $\chi \in \widehat{A}$ such that $\chi(x) \neq 1$.

Proof.

- Group homomorphism. Clear.
- Continuity. We show the Pontryagin map is continuous at the identity element. Let $V \subseteq \widehat{\widehat{A}}$ be a unit-neighborhood. By Lemma 5.1.3 we can find compact $K' \subseteq \widehat{A}$ and $\varepsilon > 0$ such that

  $$B_{K',\varepsilon} = B_{K',\varepsilon}(1) = \{\alpha \in \widehat{\widehat{A}} \mid |\alpha(\chi) - 1| < \varepsilon \text{ for all } \chi \in K'\} \subseteq V$$

  Let $L \subseteq A$ be a compact unit-neighborhood. Since K' is compact, we can find $\chi_1, \ldots, \chi_n \in K'$ such that $K' \subseteq B_{L,\varepsilon/2}(\chi_1) \cup \cdots \cup B_{L,\varepsilon/2}(\chi_n)$, where

  $$B_{L,\varepsilon/2}(\chi) := \{\psi \in \widehat{A} \mid |\psi(x) - \chi(x)| < \varepsilon/2 \text{ for all } x \in L\}$$

  For $j = 1, \ldots, n$, let $U_j = \{x \in A \mid |\chi_j(x) - 1| < \varepsilon/2\}$. Finally, let

  $$U = \operatorname{int} L \cap U_1 \cap \cdots \cap U_n$$

  Then for $x \in U$ and $\chi \in K'$ (say $\chi \in B_{L,\varepsilon/2}(\chi_i)$), we have

  $$|\delta_x(\chi) - 1| = |\chi(x) - 1| \leqslant |\chi(x) - \chi_i(x)| + |\chi_i(x) - 1| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

  so that $\delta(U) \subseteq B_{K',\varepsilon} \subseteq V$.
- Injectivity. Assume $1 \neq x \in A$ with $\delta_x = \mathbf{1}_{\widehat{A}}$. Then $\chi(x) = 1$ for every $\chi \in \widehat{A}$. Choose $g \in C_c(A)$ with $g(1) = 1$ and $g(x^{-1}) = 0$. Then $L_x(g) \neq g$, but

  $$\widehat{L_x(g)}(\chi) = \overline{\chi}(x)\widehat{g}(\chi) = \widehat{g}(\chi)$$

  for all $\chi \in \widehat{A}$, a contradiction to injectivity of Fourier transform.

$\square$

Lemma 5.5.3. Let $f \in C_0^*(A)$ be such that it Fourier transform lies in $C_c(\widehat{A})$. Then for every $x \in A$ one has $f(x) = \widehat{\widehat{f}}(\delta_{x^{-1}})$.

Proof. One has for $x \in A$,

$$f(x) = L_{x^{-1}}f(1) = \int_{\widehat{A}} \widehat{L_{x^{-1}}f}(\chi)d\chi = \int_{\widehat{A}} \widehat{f}(\chi)\delta_x(\chi)d\chi = \widehat{\widehat{f}}(\delta_{x^{-1}})$$

The second equality holds for $\hat{f} \in C_c(\widehat{A})$ and by Proposition 5.4.7. $\square$

Lemma 5.5.4. For an LCA group A, the following hold.

(i) $C_c(A)$ is dense in $C_0^*(A)$.

(ii) $C_c(\widehat{A}) \cap \{\hat{f} \mid f \in C_0^*(A) \cap L^2(A)\}$ is dense in $C_0^*(\widehat{A})$.

(iii) $C_c(\widehat{A}) \cap \{\hat{f} \mid f \in C_0^*(A) \cap L^2(A)\}$ is dense in $L^2(\widehat{A})$.

Proof.

(i) By definition, $C_0(A) \cap L^1(A)$ is dense $C_0^*(A)$. It suffices to show that for a given $f \in C_0(A) \cap L^1(A)$, there exists a sequence in $C_c(A)$ converging in $\|\cdot\|_A$ and $\|\cdot\|_1$. For each $n \geqslant 1$, let $K_n := \{x \in A \mid |f(x)| \geqslant \frac{1}{n}\}$; this is compact since $f \in C_0(A)$. Choose $\chi_n \in C_c(A)$ with $\chi_n|_{K_n} \equiv 1$ and set $f_n = \chi_n f$. Then $f_n \to f$ in both norms.

(ii) and (iii) follow from (i) and Lemma 5.4.6.

$\square$

Theorem 5.5.5 (Pontryagin Duality)**.** The Pontryagin map $\delta : A \to \widehat{\widehat{A}}$ is an isomorphism of LCA groups.

Proof. We have seen that δ is an injective group homomorphism. It remains to show δ is surjective with a continuous inverse. We will show that δ is a closed map with dense image.

- δ has a dense image. Suppose otherwise. Then there exists open $U \subseteq \widehat{\widehat{A}}$ disjoint from $\delta(A)$. By Lemma 5.4.6 (with $\widehat{A}$ in place of A) together with Urysohn's lemma, we can find $\psi \in C_0^*(\widehat{A})$ which is *nonzero* and $\widehat{\psi}$ is supported in U, i.e., $\widehat{\psi}(\delta(A)) = 0$. By Lemma 5.5.4, we can find $(f_n) \subseteq C_0^*(A)$ such that $\psi_n := \widehat{f_n}$ lies in $C_c(\widehat{A})$ with $\psi_n \to \psi$ in $C_0^*(\widehat{A})$. By inversion formula, we have
$$f_n(x) = \widehat{\psi_n}(\delta_{x^{-1}}) \text{ for all } x \in A.$$
This implies (for $\widehat{\psi}(\delta(A)) = 0$) that $f_n \to 0$ uniformly on A. On the other hand, $\widehat{f_n} =: \psi_n \to \psi$ uniformly on $\widehat{A}$. These two imply that (f_n) is a Cauchy sequence in $C_0^*(A)$, so it converges in this space. Since the limit is unique, it follows from Lemma 5.4.2 that $\psi = 0$, a contradiction. Hence the image of δ must be dense in $\widehat{\widehat{A}}$.

- δ is a proper map. It suffices to show $\breve{\delta}(x) := \delta(x^{-1})$. Let $K \subseteq \widehat{\widehat{A}}$ be compact. By Lemma 5.4.6 (with Urysohn's lemma) there exists $\psi \in C_0^*(\widehat{A})$ such that $\hat{\psi}$ has compact support, $\geqslant 0$ on $\widehat{\widehat{A}}$, and $\geqslant 1$ on K. As above, there is a sequence $(f_n) \subseteq C_0^*(A)$ such that $\psi_n := \hat{f}_n \geqslant 0$ lies in $C_c(\widehat{A})$ and converges to ψ in $C_0^*(\widehat{A})$. Fix n with $\left\|\hat{\psi} - \hat{\psi}_n\right\|_{\widehat{\widehat{A}}} < \dfrac{1}{2}$. We have $f_n(x) = \hat{\psi}_n(\delta_{x^{-1}})$ for all $x \in A$, and since $f_n \in C_0(A)$, we can find compact $C \subseteq A$ such that $|f_n| < \dfrac{1}{2}$ outside C. As $\hat{\psi}_n|_K \geqslant \dfrac{1}{2}$, it follows that $\breve{\delta}^{-1}(K) \subseteq C$.

- δ is a closed map. This is Proposition A.7.2.

$\square$

Proposition 5.5.6. The Fourier transform induces an isometric isomorphism of Banach spaces

$$\mathcal{F} : C_0^*(A) \to C_0^*(\widehat{A})$$

with inverse map given by the dual Fourier transform

$$\begin{aligned}\widehat{\mathcal{F}} : C_0^*(\widehat{A}) &\longrightarrow C_0^*(A)\\ \psi &\longmapsto [x \mapsto \hat{\psi}(\delta_{x^{-1}})]\end{aligned}$$

Proof. Let $B = \{f \in C_0^*(A) \mid \widehat{f} \in C_c(\widehat{A})\}$. For $f \in B$ we have $\widehat{\mathcal{F}} \circ \mathcal{F}(f) = f$ by Lemma 5.5.3. Further, one has

$$\begin{aligned}\|f\|_0^* := \max\{\left\|\hat{f}\right\|_{\widehat{A}}, \|f\|_A\} &= \max\{\left\|\hat{f}\right\|_{\widehat{A}}, \left\|\widehat{\mathcal{F}} \circ \mathcal{F}(f)\right\|_A\}\\ &= \max\{\left\|\hat{f}\right\|_{\widehat{A}}, \left\|\widehat{\widehat{f}}\right\|_{\widehat{A}}\}\\ &= \|\mathcal{F}(f)\|_0^*\end{aligned}$$

By Lemma 5.5.4 $\mathcal{F}(B)$ is dense in $C_0^*(\widehat{A})$, so the Fourier transform defines a surjective isometry from the closure of B to $C_0^*(\widehat{A})$, and thus $\widehat{F}$ is an isometry from $C_0^*(\widehat{A})$ to $C_0^*(A)$. Since $\widehat{\mathcal{F}} = \mathcal{F}_{\widehat{A}} \circ \delta^{-1}$ (in an appropriate sense), where $\mathcal{F}_{\widehat{A}}$ denotes the Fourier transform on $\widehat{A}$, and since $\mathcal{F}_{\widehat{A}}(C_0^*(\widehat{A}))$ contains a subset of $C_c(\widehat{\widehat{A}})$ that is dense in $C_0^*(\widehat{\widehat{A}})$ by Lemma 5.5.4, it follows from Pontryagin duality that $\widehat{F}(C_0^*(\widehat{A}))$ is dense in $C_*(A)$. Since $\widehat{F}$ is isometric, it is then an isomorphism of Banach spaces. $\square$

Theorem 5.5.7 (Inversion Formula)**.** Let $f \in L^1(A)$ be such that $\widehat{f} \in L^1(\widehat{A})$. Then f is continuous, and for $x \in A$ one has

$$f(x) = \widehat{\widehat{f}}(\delta_{x^{-1}})$$

Proof. Let $f \in L^1(A)$ with $\widehat{f} \in L^1(\widehat{A})$. Then $\widehat{f} \in C_0(\widehat{A}) \cap L^1(\widehat{A}) \subseteq C_0^*(\widehat{A})$ by Theorem 5.2.3.2. and definition of $C_0^*(\widehat{A})$. By Proposition 5.5.6 we can find $g \in C_0^*(A)$ with $\widehat{g} = \widehat{f}$ and $g(x) = \widehat{\widehat{f}}(\delta_{x^{-1}})$ for every $x \in A$. Since the Fourier transform $C^*(A) \to C_0(\widehat{A})$ is injective, we see $f = g$ in $C^*(A)$. $\square$

Proof. (of Plancherel theorem) Let $f \in L^1(A) \cap L^2(A)$. By Lemma 5.4.1, we have $f * f^* \in L^1(A) \cap C_0(A)$. The continuous function $h = \widehat{f * f^*} = |\widehat{f}|^2 \in C_0(\widehat{A})$ is positive. Let $\phi \in C_c(\widehat{A})$ satisfy $0 \leqslant \phi \leqslant h$. Then by Proposition 5.4.7 and Lemma 5.4.1, we have

$$\int_{\widehat{A}} \phi(\chi) d\chi \leqslant f * f^*(1) = \|f\|_2^2 < \infty$$

Thus h is integrable, so $\widehat{f * f^*} \in L^1(A)$. By inversion formula we have

$$\|f\|_2^2 = f * f^*(1) = \widehat{\widehat{f * f^*}}(1) = \widehat{|\widehat{f}|^2}(1) = \left\|\hat{f}\right\|_2^2$$

As $L^1(A) \cap L^2(A)$ is dense in $L^2(A)$, the Fourier transform extends uniquely to an isometric linear map $L^2(A) \to L^2(\widehat{A})$. By Lemma 5.4.6 the image in $L^2(A)$ is dense, whence the map is surjective. $\square$

Proposition 5.5.8. Let $\phi, \psi \in L^1(A) \cap L^2(A)$ and let $f = \phi * \psi$. Then $f \in L^1(A)$ and $\widehat{f} \in L^1(\widehat{A})$, so the inversion formula applies to f.

Proof. We have $\widehat{f} = \widehat{\phi * \psi} = \widehat{\phi}\widehat{\psi} \in L^1(\widehat{A})$ by Hölder's inequality. $\square$

5.6 Classical applications

5.6.1 Poisson summation formula

Let A be an LCA group. Recall we have a pairing

$$\begin{aligned} \langle\,,\rangle : A \times \widehat{A} &\longrightarrow \mathbb{C} \\ (a,\chi) &\longmapsto \chi(a) = \delta_a(\chi) \end{aligned}$$

From this we introduce some notation. For $E \subseteq A$, let

$$E^{\perp} := \{\chi \in \widehat{A} \mid \langle E, \chi\rangle = 1\} = \{\chi \in \widehat{A} \mid \delta_a(\chi) = 1 \text{ for all } a \in E\}$$

and for $L \subseteq \widehat{A}$, let

$$L^{\perp} = \{a \in A \mid \langle a, L\rangle = 1\} = \{a \in A \mid \chi(a) = 1 \text{ for all } \chi \in L\}$$

By definition (and perhaps Proposition 5.5.2), $E^{\perp}$ is closed in $\widehat{A}$, and $L^{\perp}$ is closed in A.

Proposition 5.6.1. Let A be an LCA group and $B \leqslant A$ be closed. Then

(i) $$\begin{aligned} B^{\perp} &\longrightarrow \widehat{A/B} \\ \chi &\longmapsto \overline{\chi} : xB \mapsto \chi(x) \end{aligned}$$ is an isomorphism.

(ii) $(B^{\perp})^{\perp} = B$.

(iii) $$\begin{aligned} \widehat{A}/B^{\perp} &\longrightarrow \widehat{B} \\ \chi B^{\perp} &\longmapsto \chi|_B \end{aligned}$$ is an isomorphism.

Proof.

(i) Since $A \to A/B$ is surjective, we have an injection $\widehat{A/B} \to \widehat{A}$. Its image is clearly $B^{\perp}$.

(ii) From definition follows $B \subseteq (B^{\perp})^{\perp}$. Similarly, if $L \subseteq \widehat{\mathbb{A}}$, we have $L \subseteq (L^{\perp})^{\perp}$. It follows that

$$B^{\perp} \supseteq ((B^{\perp})^{\perp})^{\perp} \supseteq B^{\perp}$$

so $B^{\perp} = ((B^{\perp})^{\perp})^{\perp}$.

(iii) $(\widehat{A}/B^{\perp})^{\wedge} = (B^{\perp})^{\perp} = B$.

□

Corollary 5.6.1.1. Let A be an LCA group and $B \leqslant A$ be closed. Then we have an short exact sequence

$$1 \longrightarrow \widehat{A/B} \longrightarrow \widehat{A} \xrightarrow{(\cdot)|_B} \widehat{B} \longrightarrow 1$$

Theorem 5.6.2 (Poisson's Summation Formula)**.** Let A be an LCA group and $B \leqslant A$ be closed. For $f \in L^1(A)$, define $f^B \in L^1(A/B)$ by

$$f^B(xB) := \int_B f(xb)db$$

If we identify $\widehat{A/B}$ with $B^\perp$, we get $\widehat{f^B} = \widehat{f}|_{B^\perp}$. If, in addition, $\widehat{f}|_{B^\perp} \in L^1(B^\perp)$, we get

$$\int_B f(xb)db = \int_{B^\perp} \widehat{f}(\chi)\chi(x)d\chi$$

for almost all $x \in A$. Here the Haar measure on $B^\perp \cong \widehat{A/B}$ is the Plancherel measure with respect to the chosen Haar measure on A/B such that the quotient integral formula holds.

Proof. By the quotient integral formula, for $\chi \in B^\perp$

$$\widehat{f^B}(\chi) = \int_{A/B} f^B(xB)\overline{\chi}(xB)dxB = \int_{A/B}\int_B f(xb)\chi(xb)dbdxB = \int_A f(x)\chi(x)dx = \widehat{f}(\chi)$$

Moreover, if $\widehat{f}|_{B^\perp} \in L^1(B^\perp) = L^1(\widehat{A/B})$, then the inversion formula implies

$$\int_B f(xb)db = f^B(xB) = \widehat{\widehat{f^B}}\quad(\delta_{x^{-1}B}) = \widehat{\widehat{f}|_{B^\perp}}\quad(\delta_{x^{-1}B}) = \int_{B^\perp} \widehat{f}(\chi)\overline{\chi(x)}d\chi$$

holds almost everywhere. ◻

Example 5.6.3. Let A be the additive group $\mathbb{R}$ with euclidean topology. Then A is self-dual, where the isomorphism $A \cong \widehat{A}$ is given by $y \mapsto [\chi_y : x \mapsto e^{2\pi ixy}]$. Let B be the discrete subgroup $\mathbb{Z}$. Then $y \mapsto \chi_y$ maps B bijectively to $B^\perp$.

For $f \in L^1(\mathbb{R})$ such that $\widehat{f}|_{\mathbb{Z}} \in L^1(\mathbb{Z})$, the equality

$$\sum_{k\in\mathbb{Z}} f(x+k) = \sum_{k\in\mathbb{Z}} \widehat{f}(k)e^{2\pi ikx} \tag{$\spadesuit$}$$

holds for almost all x, where $\widehat{f}(x) = \int_{\mathbb{R}} f(y)e^{-2\pi ixy}dy$.

Define the **Schwartz space** $\mathcal{S}(\mathbb{R})$ as the space of all smooth functions $f : \mathbb{R} \to \mathbb{C}$ such that for all $m, n \in \mathbb{Z}$ the function $x^n f^{(m)}(x)$ is bounded. Then the Fourier transform is a bijection on $\mathcal{S}(\mathbb{R})$. For $f \in \mathcal{S}(\mathbb{R})$, both sums in ($\spadesuit$) converge uniformly and define continuous functions, so by taking $x = 0$ we obtain

$$\sum_{k\in\mathbb{Z}} f(k) = \sum_{k\in\mathbb{Z}} \widehat{f}(k)$$

This is the classical Poisson summation formula.

5.6.2 Mellin inversion

In this subsection we exclusively discuss $\mathbb{R}_{>0}$; see §7.1.2 for local fields. We write $d^\times x = \frac{dx}{x}$ for the Haar measure on $\mathbb{R}_{>0}$. For $\sigma \in \mathbb{R}$, define

$$L^1(\mathbb{R}_{>0}, \sigma) = \left\{f : \mathbb{R}_{>0} \to \mathbb{C} \mid f(x)x^\sigma \in L^1(\mathbb{R}_{>0})\right\}.$$

For $s = \sigma + i\tau \in \mathbb{C}$ and $f \in L^1(\mathbb{R}_{>0}, \sigma)$ define the **Mellin transform**

$$\mathcal{M}(f, s) := \int_{\mathbb{R}_{>0}} f(x)x^s d^\times x = \int_{\mathbb{R}_{>0}} f(x)x^\sigma x^{i\tau} d^\times x.$$

Keeping in mind the fact that $\widehat{\mathbb{R}_{>0}} \cong i\mathbb{R}$ (whose inverse is given by $it \mapsto [x \mapsto x^{it}]$), the Mellin transform $\mathcal{M}(f, s)$ is simply the Fourier transform of $x \mapsto f(x)x^\sigma$ evaluated at the character $x \mapsto x^{i\tau}$. Via this isomorphism, we also see the Plancherel measure on $\widehat{\mathbb{R}_{>0}}$ is given by the Lebesgue measure on $\mathbb{R}$. Hence,

Theorem 5.6.4 (Mellin inversion)**.** For $f \in L^1(\mathbb{R}_{>0}, \sigma)$ with $\mathcal{M}(f, s) \in L^1(\mathbb{R})$, we have

$$f(x)x^\sigma = \int_{\mathbb{R}} \mathcal{M}(f, \sigma + it)x^{-it}dt$$

Chapter 6

The Structure of LCA-Groups

6.1 Connectedness

Recall that for a topological space X with base point x_0, the set $\pi_0(X, x_0)$ is defined to be the collection of all connected components with a mark point being the one containing x_0.

Definition. Let G be a topological group. The connected component of G containing e is called the **identity component**, and is denoted by G°.

Proposition 6.1.1. Let G be a topological group.

(i) The identity component G° is a closed normal subgroup of G.

(ii) For any $x \in G$, the coset xG° is the connected component containing x. In particular, the natural map $G \to \pi_0(G, e)$ induces a bijection $G/G^\circ \cong \pi_0(G, e)$.

(iii) The quotient group G/G° is totally disconnected.

Proof. To show G/G° is totally disconnected, we prove that every subset $A \subseteq G/G^\circ$ with $\#A \geqslant 2$ is not connected. Let $\pi : G \to G/G^\circ$ be the projection and put $B = \pi^{-1}(A)$. Then B contains at least two distinct cosets of G°, so B is not connected. Thus we can find open $W_1, W_2 \subseteq G$ with $B \cap W_i \neq \varnothing$, $i = 1, 2$, $B \cap W_1 \cap W_2 = \varnothing$ and $B \subseteq W_1 \cup W_2$. Then $xG^\circ \cap W_1 \cap W_2 = \varnothing$ and $xG^\circ \subseteq W_1 \cup W_2$ for all $x \in B$; since xG° is connected, we have $xG^0 \subseteq W_i$ for some $i = 1, 2$. Hence, $V_i := \pi(W_i)$, $i = 1, 2$ are nonempty disjoint open sets in G/G° separating A. $\square$

Proposition 6.1.2. Let G be a topological group and let U be an open compact unit-neighborhood of G. Then U contains an open compact subgroup of G.

Proof. By compactness of U we can find a symmetric unit-neighborhood V of G such that $UV = U = VU$. The subgroup $K := \bigcup_{n \in \mathbb{N}} V^n$ generated by V is then an open, hence closed, subgroup contained in U. Since U is compact, K is also compact. $\square$

Proposition 6.1.3. Every totally disconnected LCH space X has a basis consisting open and compact subsets of X.

Proof. It suffices to show that for each $x \in X$ and each compact neighborhood U of x, there exists an open and closed subset V of X such that $x \in V \subseteq U$. For this, let

$$M := \{y \in U \mid \text{there exists a relatively open and closed subset } C_y \subseteq U \text{ with } y \in C_y,\ x \notin C_y\}$$

By definition M is the union of all such C_y, showing that M is relatively open in U. Let $A = U \backslash M$; then $x \in A$ and A is closed. We claim

- the set A is connected.

Assuming this, since X is totally disconnected, we have $A = \{x\}$, so the compact boundary $\partial U := U \backslash \operatorname{int} U$ lies in M, hence lies in the union of finitely many sets $C_{y_1}, \ldots, C_{y_n}$. Then $V := U \backslash \bigcup_{i=1}^{n} C_{y_i}$ contains x, is relatively open in U and is closed in X. Since $V \subseteq \operatorname{int} U$, V is also open in X.

It remains to show A is connected. Let B_1, B_2 be closed, hence compact in U, subsets of A such that $B_1 \cap B_2 = \varnothing$ and $A = B_1 \cup B_2$. Assuming $x \in B_1$, we must show B_2 is empty. Since B_1 and B_2 are compact, we can find $W \subseteq U$ open in U such that $B_2 \subseteq W$ and $\overline{W} \cap B_1 = \varnothing$. Since $\overline{W} \backslash W \subseteq M$ is compact in U, it is covered by finitely many of the open sets C_y; say $C \subseteq M$ is their (finite) union. Then $\overline{W} \backslash W \subseteq C$ and C is open and closed in U. Set $\tilde{W} := W \cup C$; since $W \cup C = \overline{W} \cup C$, $\tilde{W}$ is open and closed in U. Also, $\tilde{W} \cap A = W \cap A = B_2$, so $x \notin \tilde{W}$, and $\tilde{W} \subseteq M$, which implies $B_2 = \varnothing$. ☐

Theorem 6.1.4. A totally disconnected LCH group admits an open neighborhood basis of identity consisting of open compact subgroups.

6.1.1 Profinite groups

Definition. A **profinite group** is a topological group isomorphic to a projective limit of finite groups.

Theorem 6.1.5. For a topological group G, TFAE:

(i) G is profinite.

(ii) G is compact, Hausdorff and totally disconnected.

(iii) G is compact, Hausdorff and has an open neighborhood basis of the identity consisting of normal subgroups.

Proof. (i) ⇒ (ii) is clear. Assume (ii). By Theorem 6.1.4 G has an open neighborhood basis of identity consisting of open subgroups. Let U be an open subgroup of G; since G is compact, the index $[G : U]$ is finite. Consider the natural homomorphism $G \to S_{G/U}$; the fibre of g is of the form $g \bigcup_{h \in G/U} h^{-1}Uh$, an open neighborhood of g. Hence the homomorphism is continuous, so its kernel N is an open normal subgroup of G contained in U. Hence (iii).

Now assume (iii). We have a natural map $\Phi : G \to H := \varprojlim_N G/N$, where N runs over all open normal subgroups of G. Since G is compact, each G/N is a finite group. We have to check it is an isomorphism of topological groups.

- Injectivity. The kernel is the intersection of all open normal subgroups. Since they form a neighborhood basis of the identity e of G and G is Hausdorff, it follows that $\ker \Phi = \{e\}$.

- Continuity. Let S be a finite collection of open normal subgroups of G. Then the inverse image of $H_S := \left(\prod_{N \in S} \{eN\} \times \prod_{N \notin S} G/N \right) \cap \varprojlim_N G/N$ under Φ is $\bigcap_{N \in S} N$, which is open in G.

- Dense image. Let S and H_S be as in previous item. Let $g \in G$ and write $\Phi(g) = \prod_N g_N \in H$. Then $\Phi(g)H_S$ is a neighborhood of $\Phi(g)$, and if we take $g' \in G$ that is mapped under $G \to G/\bigcap_{N\in S} N$ to $g_{\bigcap_{N\in S} N}$, then $\Phi(g') \in \Phi(g)H_S$.

- Closed mapping. For G is compact.

$\square$

Example 6.1.6 (Infinite Galois theory)**.** Let Ω/k be a (finite or infinite) Galois extension, i.e, a normal separable algebraic extension. The Galois group $G := \mathrm{Gal}(\Omega/k)$ is given a topology, called the **Krull topology**, defined as follows. For $\sigma \in G$, take the cosets $\sigma \,\mathrm{Gal}(\Omega/K)$ as an open neighborhood basis of $\sigma \in G$, where K/k runs over all finite Galois subextensions of Ω/k.

- The multiplication $G \times G \to G$ is continuous, since for $(\sigma, \tau) \in G \times G$, the preimage of $\sigma\tau \,\mathrm{Gal}(\Omega/K)$ contains a neighborhood $\sigma \,\mathrm{Gal}(\Omega/\tau(K)) \times \tau \,\mathrm{Gal}(\Omega/K)$ of (σ, τ).

- The inversion $G \to G$ is continuous, for the preimage of $\sigma^{-1} \mathrm{Gal}(\Omega/K)$ is
$$\mathrm{Gal}(\Omega/K)\sigma = \sigma\sigma^{-1} \mathrm{Gal}(\Omega/K)\sigma = \sigma \,\mathrm{Gal}(\Omega/\sigma^{-1}(K))$$

In this way $G = \mathrm{Gal}(\Omega/k)$ becomes a topological group. In fact, one can easily see that this topology is just the subspace topology inherited from the pointwise convergence topology on Ω^Ω, where Ω is thought of as a discrete space. In particular, this implies G is Hausdorff. Now consider the map
$$\begin{aligned} \Phi : G &\longrightarrow H := \prod_K \mathrm{Gal}(K/k) \\ \sigma &\longmapsto (\sigma|_K)_K \end{aligned}$$

where K/k runs over all finite Galois subextensions of Ω/k. By Tychonov's theorem, H is compact, and to show G is compact it suffices to show Φ is a closed embedding.

- Injective. Clear.

- Continuity. Let K_0/k be a finite Galois subextension of Ω/k. The inverse image of the set $\prod_{K\neq K_0} \mathrm{Gal}(K/k) \times \{\sigma_0\}$ with $\sigma_0 \in \mathrm{Gal}(K_0/k)$ is $\sigma \,\mathrm{Gal}(\Omega/K_0)$, where $\sigma \in G$ is any extension of σ_0 to Ω.

- Open mapping. Let the notation be as in the previous item. Then the image of $\sigma \,\mathrm{Gal}(\Omega/K_0)$ under Φ is $\Phi(G) \cap \left(\prod_{K\neq K_0} \mathrm{Gal}(K/k) \times \{\sigma_0\}\right)$, which is open in $\Phi(G)$.

- $\Phi(G)$ is closed in H. Indeed, we have
$$\Phi(G) = \bigcap_{L\subseteq L'} \{(\sigma_K)_K \in H \mid \sigma_{L'}|_L = \sigma_L\}$$
where $L \subseteq L'$ runs over all finite subextensions of Ω/k. For such a pair $L \subseteq L'$, if we write $\mathrm{Gal}(L/k) = \{\sigma_i\}_{i=1}^n$ and $S_i \subseteq \mathrm{Gal}(L'/k)$ such that $S_i|_L = \{\sigma_i\}$, then
$$\{(\sigma_K)_K \in H \mid \sigma_{L'}|_L = \sigma_L\} = \bigcup_{i=1}^n \left(\prod_{K\neq L,L'} \mathrm{Gal}(K/k) \times S_i \times \{\sigma_i\}\right)$$
is a closed set in H, and hence $\Phi(G)$ is closed in H.

This shows G is a compact Hausdorff topological space. Moreover, since $G/\operatorname{Gal}(\Omega/K) \cong \operatorname{Gal}(K/k)$ is a group, $\operatorname{Gal}(\Omega/K)$ is normal, i.e. G admits an open neighborhood basis of the identity consisting of normal subgroups. Therefore G is profinite by Theorem 6.1.5; in particular, we have an isomorphism of topological groups

$$\operatorname{Gal}(\Omega/k) \cong \varprojlim_{K} \operatorname{Gal}(K/k)$$

The infinite Galois theory states that the association $K \mapsto \operatorname{Gal}(\Omega/K)$ gives an 1-1 correspondence between the subextensions K/k of Ω/k and the closed subgroups of $\operatorname{Gal}(\Omega/k)$, under which the open subgroups of $\operatorname{Gal}(\Omega/k)$ correspond to the finite subextensions of Ω/k.

- Each open subgroup is closed.
- If K/k is a finite subextension of Ω/k, take N/k be its Galois closure in Ω. Then for $\operatorname{Gal}(\Omega/N)$ is an open unit-neighborhood of $\operatorname{Gal}(\Omega/K)$, so $\operatorname{Gal}(\Omega/K)$ is open in G.
- If K/k is any subextension of Ω/k, then $\operatorname{Gal}(\Omega/K) = \bigcap_{K'} \operatorname{Gal}(\Omega/K')$, where K' runs over all finite subextensions of K/k, so $\operatorname{Gal}(\Omega/K)$ is closed.
- Injectivity. For a subextension K/k of Ω/k, the fixed field of $\operatorname{Gal}(\Omega/K)$ in Ω is K. Clearly, $K \subseteq \Omega^{\operatorname{Gal}(\Omega/K)}$. If $x \in \Omega^{\operatorname{Gal}(\Omega/K)}$, let $\sigma \in \operatorname{Gal}(K(x)/K)$ and extend it to an element $\sigma' \in \operatorname{Gal}(\Omega/K) \subseteq G$. Then $\sigma(x) = \sigma'(x) = x$, so $\operatorname{Gal}(K(x)/K) = 1$, i.e., $x \in K$.
- Surjectivity. Let $H \leqslant G$ be a closed subgroup. Put $K = \Omega^H$. We claim $H = \operatorname{Gal}(\Omega/K)$. Clearly, $H \subseteq \operatorname{Gal}(\Omega/K)$. Conversely, let $\sigma \in \operatorname{Gal}(\Omega/K)$. If L/K be a finite Galois subextension of Ω/K, then $\sigma \operatorname{Gal}(\Omega/L)$ is an open neighborhood of σ in $\operatorname{Gal}(\Omega/K)$. The restriction to H of $\operatorname{Gal}(\Omega/K) \to \operatorname{Gal}(L/K)$ is surjective, since K is the fixed field of the image $H|_L$ of H, so $H|_L = \operatorname{Gal}(L/K)$ by the finite Galois theory. Now choose $\tau \in H$ such that $\tau|_L = \sigma|_L$. Then $\tau \in H \cap \sigma \operatorname{Gal}(\Omega/L)$, showing that σ lies in the closure of H. Since H is closed, $\sigma \in H$.
- If H is an open subgroup of G, since it is closed, it is of the form $H = \operatorname{Gal}(\Omega/K)$. Since $G/H = \operatorname{Gal}(\Omega/k)/\operatorname{Gal}(\Omega/K) \cong \operatorname{Gal}(K/k)$ is a finite group by compactness of G, K/k has finite degree.

Example 6.1.7. If A is a discrete abelian torsion group, then the dual group $\widehat{A} = \operatorname{Hom}(A, S^1)$ is profinite. By Proposition 5.1.7, $\widehat{A}$ is compact. Consider the canonical map

$$\Phi : \widehat{A} \longrightarrow \varprojlim_{\alpha} \operatorname{Hom}(A_\alpha, S^1)$$

where A_α runs over all finite subgroups of A. Since A is torsion, $A = \bigcup A_\alpha$, so Φ is a continuous bijection. Since $\widehat{A}$ is compact, this implies Φ is a homeomorphism.

Example 6.1.8. The rings $\mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{N}$ together with the projection $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$, $m \mid n$ form a projective system. The projective limit

$$\widehat{\mathbb{Z}} := \varprojlim_{n \in \mathbb{N}} \mathbb{Z}/n\mathbb{Z}$$

is **the profinite completion of** $\mathbb{Z}$. For each $n \in \mathbb{N}$ by Chinese Remainder theorem we have $\mathbb{Z}/n\mathbb{Z} \cong \prod_p \mathbb{Z}/p^{\operatorname{ord}_p n}\mathbb{Z}$. Passing to inverse limit, we have

$$\widehat{\mathbb{Z}} = \varprojlim_{n \in \mathbb{N}} \mathbb{Z}/n\mathbb{Z} \cong \varprojlim_{n \in \mathbb{N}} \prod_p \mathbb{Z}/p^{\operatorname{ord}_p n}\mathbb{Z} = \prod_p \varprojlim_{n \in \mathbb{N}} \mathbb{Z}/p^{\operatorname{ord}_p n}\mathbb{Z} \cong \prod_p \mathbb{Z}_p$$

The third equality is canonical, and the last equality is given by the projection.

A profinite group G is **procyclic** if there exists $g \in G$ such that $G = \overline{\langle g \rangle}$. Such an element g is called a **topological generator** of G. Let G be a procyclic group. For instance, $\mathbb{Z}_p$ and $\widehat{\mathbb{Z}}$ are procyclic, and 1 is a topological generator of them.

- G is abelian. More generally, if G is a Hausdorff topological group and A an abelian subgroup, then $\overline{A}$ is abelian. To show that, consider the map $c : G \times G \to G$ defined by $(x, y) \mapsto xyx^{-1}y^{-1}$. Since A is abelian, $A \times A \subseteq \ker c$. Since G is Hausdorff, $\ker c = c^{-1}(e)$ is closed. Then $\overline{A} \times \overline{A} \subseteq \overline{A \times A} \subseteq \ker c$, so that $\overline{A}$ is abelian.

- The open subgroups of G have the forms nG, $n \in \mathbb{N}$. Since nG is a continuous image of G, nG is closed. The quotient G/nG is then Hausdorff, and is finite since it contains a dense finite subgroup $\{kg \mod nG \mid 0 \leqslant k < n\}$. Hence nG is open.

  Conversely, let H be an open subgroup of index n. Then $nG \subseteq H \subseteq G$, and $n = [G : H] = [G : nG] \leqslant n$, so that $H = nG$.

- G is a quotient of $\widehat{\mathbb{Z}}$. For each $n \in \mathbb{N}$, we have a surjective map $\mathbb{Z}/n\mathbb{Z} \to G/nG$. Passing to limits, we obtain $\widehat{\mathbb{Z}} \to G$.

Example 6.1.9. Let k be a non-archimedean local field, and let $(\mathfrak{o}, \mathfrak{p})$ be its ring of integers. The projections $\mathfrak{o} \to \mathfrak{o}/\mathfrak{p}^n$ $(n \geqslant 1)$ induces a continuous map

$$\mathfrak{o} \longrightarrow \varprojlim_{n \geqslant 1} \mathfrak{o}/\mathfrak{p}^n.$$

Since $\{\mathfrak{p}^n\}_{n \geqslant 1}$ forms a unit-neighborhood basis of $\mathfrak{o}$ and $\mathfrak{o}$ is compact, the above map is a topological group isomorphism.

Also, let F be a global field and $\mathfrak{p}$ a nonzero prime ideal of its ring of integers $\mathcal{O}_F$. If we denote by $F_{\mathfrak{p}}$ the completion with respect to the valuation corresponding to $\mathfrak{p}$ and $(\mathfrak{o}, \mathfrak{p}')$ its ring of integers, then $\mathcal{O}_F \to \mathfrak{o}$ induces bijections $\mathcal{O}_F/\mathfrak{p}^n \cong \mathfrak{o}/\mathfrak{p}'^n$ $(n \geqslant 1)$ and induces a topological group isomorphism $\varprojlim_{n \geqslant 1} \mathcal{O}_F/\mathfrak{p}^n \cong \mathfrak{o}$. This can be served as an alternative definition of the completion with respect to $\mathfrak{p}$ (essentially the same).

6.1.2 Path-connected groups

Lemma 6.1.10.

1. A connected topological group G having a path-connected neighborhood of the identity is path connected.

2. If G is a connected LCA group, then the dual $\widehat{G}$ is torsion-free.

3. If G is a compact LCA group with $\widehat{G}$ torsion-free, then G is connected.

Proof.

1. Let U be a path connected unit-neighborhood of G. Since G is connected, the subgroup generated by U is the whole G, namely, $G = \bigcup_{n=1}^{\infty} U^n$. Since each U^n is path-connected and they share a common point e, G is also path-connected.

2. We prove the contrapositive. Let $\chi \in \widehat{G}$ be nontrivial with finite order, say n. Then $\chi(G) \subseteq \mu_n(\mathbb{C})$, so $\chi : G \to \mu_n(\mathbb{C})$ and hence $\ker \chi$ is a proper open and closed subgroup of G. Thus G is not connected.

3. Suppose G is not connected and let U be a proper nontrivial closed and open unit-neighborhood. Since G is compact, so is U, so we can find a symmetric open unit-neighborhood V of G such that $VU = U = UV$. Let H be the (open) subgroup generated by V; then $HU = U$, and $H \subseteq U$ particularly. Then G/H is a nontrivial finite group, and any character of G/H gives a torsion character of G, so $\widehat{G}$ is not torsion free.

□

Lemma 6.1.11. Let $\Lambda \leqslant G$ be a discrete subgroup of a topological group G and let $\pi : G \to G/\Lambda$ be the quotient map. Suppose $\sigma : [0,1] \to G/\Lambda$ is a path starting from 0. Then there exists a path $\tilde{\sigma} : [0,1] \to G$ starting from 0 such that $\sigma = \pi \circ \tilde{\sigma}$.

Proof. The discreteness of Λ makes the projection $\pi : G \to G/\Lambda$ a covering map, and the statement now follows from the path lifting property of the covering map.

□

Theorem 6.1.12. Let K be a second countable path-connected compact Hausdorff group. Then K is isomorphic to a product of countably many circle groups S^1.

Proof. Let $D = \widehat{K}$ be its dual group. By Proposition 5.1.7 D is discrete, and by Lemma 6.1.10, D is torsion-free. Also, by Corollary 5.4.8.1, the D forms an orthonormal basis for the Hilbert space $L^2(K)$. Since $L^2(K)$ has countable orthonormal basis by Lemma D.5.6, it follows that D is countable by Proposition E.2.6.

For an abelian group A, the rank of A is defined as $\dim_{\mathbb{Q}} A \otimes_{\mathbb{Z}} \mathbb{Q}$. We show that every finite rank subgroup of D is finitely generated. Let $F \subseteq D$ be a finite rank subgroup of rank r; since D is with discrete topology, the inclusion $F \subseteq D$ is continuous. Dualizing gives a surjection $K \to \widehat{F} =: L$, so L is compact and path-connected. Now since F is of finite rank, $F \cong \varinjlim_j F_j$, where F_j runs over all finitely generated subgroups of F with full rank r. We claim the limit stops, i.e., $F_j = F$ for some j.

Dualizing gives (since each F_j is discrete) $L \cong \varprojlim_j T_j$, where $T_j = \widehat{F_j}$ and each projection in the projective system is surjective. Since $F_j \cong \mathbb{Z}^r$, $T_j \cong \mathbb{R}^r/\mathbb{Z}^r$. Fix an index ν and an isomorphism $T_\nu \cong \mathbb{R}^r/\mathbb{Z}^r$. Then by some isomorphism theorem for each $j \geqslant \nu$ there is a subgroup $\Lambda_j \subseteq \mathbb{Z}^r$ of full rank and an isomorphism $\psi_j : T_j \to \mathbb{R}^r/\Lambda_j$ such that diagram

$$\begin{tikzcd}
T_j \arrow[r, two heads, "\pi^j_\nu"] \arrow[d, "\psi_j"] & T_\nu \arrow[d, "\wr"] \\
\mathbb{R}^r/\Lambda_j \arrow[r, "p_j"] & \mathbb{R}^r/\mathbb{Z}^r
\end{tikzcd}$$

where p_j is the natural projection. Let $\Lambda = \bigcap_{j \geqslant \nu} \Lambda_j \subseteq \mathbb{R}^r$. The group $G := \mathbb{R}^r/\Lambda$ then injects into the $\varprojlim_j T_j = L$, and we claim that it is an isomorphism. For this let $x \in L$ and let p be a path connecting the identity element of L to x. Lemma 6.1.11, for each projection $\pi_j : L \to T_j \cong$ with

$j \geqslant \nu$, the path $\pi_j \circ p$ lifts to a path in G, as depicted below (the commutativity of the leftmost trapezoid is what we need)

$$
\begin{tikzcd}
 & & \mathbb{R}^r \arrow[d, two heads] \arrow[ddr] & \\
 & & G \arrow[dl, hook'] \arrow[d] & \\
{[0,1]} \arrow[uurr] \arrow[r, "p"] & L \arrow[r, "\pi_j"] & T_j \arrow[r, equal] & \mathbb{R}^r/\Lambda_j
\end{tikzcd}
$$

so the path p actually lies in G, whence $x \in G$. Thus $L \cong \mathbb{R}^r/\Lambda$, and since L is compact, Λ has full rank. So the limit stops and hence every finite rank subgroup of D is finitely generated.

Dualizing the theorem, our assertion is equivalent to saying that D is isomorphic to a direct sum of countably many cyclic groups $\mathbb{Z}$. Hence the theorem is reduced to the following purely algebraic property.

Lemma 6.1.13. Let D be a countable torsion free abelian group such that every finite rank subgroup of D is finitely generated. Then D is a direct sum of cyclic groups.

Proof. The $\mathbb{Q}$-module $D_{\mathbb{Q}} = D \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by D, hence it contains a basis of consisting of elements of D. Let $v_1, v_2, \ldots$ be a basis for $D_{\mathbb{Q}}$ with $v_j \in D$ for each j. We shall construct inductively a basis $w_1, w_2, \ldots$ of $D_{\mathbb{Q}}$ such that for each $n \in \mathbb{N}$

$$\mathbb{Q}w_1 \oplus \cdots \oplus \mathbb{Q}w_n = \mathbb{Q}v_1 \oplus \cdots \oplus \mathbb{Q}v_n$$

and

$$(\mathbb{Q}w_1 \oplus \cdots \oplus \mathbb{Q}w_n) \cap D = \mathbb{Z}w_1 \oplus \cdots \oplus \mathbb{Z}w_n$$

To start, let $F = \mathbb{Q}v_1 \cap D$. Then F has rank 1, so it is finitely generated, say $F = \mathbb{Z}w_1$ for some $w_1 \in D$. Now assume $w_1, \ldots, w_n$ have been constructed. The group $G = (\mathbb{Q}w_1 \oplus \cdots \oplus \mathbb{Q}w_n \oplus \mathbb{Q}v_{n+1}) \cap D$ has rank $n+1$. The fundamental theorem for finitely generated abelian groups guarantees the existence of $u_1, \ldots, u_{n+1} \in D$ and $a_1, \ldots, a_n \in \mathbb{Z}$ such that $G = \mathbb{Z}u_1 \oplus \cdots \oplus \mathbb{Z}u_{n+1}$ and

$$\mathbb{Z}w_1 \oplus \cdots \oplus \mathbb{Z}w_n = \mathbb{Z}a_1u_1 \oplus \cdots \oplus \mathbb{Z}a_nu_n \subseteq G$$

This equality implies that for $1 \leqslant i \leqslant n$, we have

$$u_i \in (\mathbb{Q}w_1 \oplus \cdots \oplus \mathbb{Q}w_n) \cap D = \mathbb{Z}w_1 \oplus \cdots \oplus \mathbb{Z}w_n = \mathbb{Z}a_1u_1 \oplus \cdots \oplus \mathbb{Z}a_nu_n$$

and hence $a_i = \pm 1$. Thus in effect we have $\mathbb{Z}w_1 \oplus \cdots \oplus \mathbb{Z}w_n = \mathbb{Z}u_1 \oplus \cdots \oplus \mathbb{Z}u_n$, so we may pick $w_{n+1} = u_{n+1}$. This finishes the construction, and thus

$$D = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2 \oplus \cdots = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}w_n$$

is a direct sum of cyclic groups $\mathbb{Z}$. $\square$

$\square$

6.2 The structure theorems

Theorem 6.2.1 (Open mapping theorem)**.** Suppose G and H are LCH groups and G is σ-compact. Then a continuous surjective homomorphism $\phi : G \to H$ is automatically an open map.

Proof. Let 1_G and 1_H denote the identity elements in G and H, respectively. It suffices to show $\phi(U)$ is a neighborhood of 1_H whenever U is a neighborhood of 1_G. To see this, choose a compact symmetric neighborhood of 1_G such that $V^2 \subseteq U$. As G is σ-compact, write $G = \bigcup_{n\geqslant 1} K_n$ with each K_n compact in G. For each $n \in \mathbb{N}$ find a finite subset $F_n \subseteq K_n$ such that $K_n \subseteq \bigcup_{x\in F_n} xV$. Put $F = \bigcup_{n\geqslant 1} F_n$; then F is a countable subset of G such that $G = \bigcup_{x\in F} xV$. Since ϕ is surjective, it follows that $H = \bigcup_{x\in F} \phi(xV)$. Each $\phi(xV)$ is compact, so it is closed. Since H is a Baire space by Theorem A.6.1, it follows that there exists $x \in F$ such that $\phi(xV)$ has nonempty interior. By translation it then implies $\phi(V)$ has nonempty interior. Finally, if choose a nonempty open set W in H with $W \subseteq \phi(V)$, then $W^{-1}W$ is open in H and

$$1_H \in W^{-1}W \subseteq \phi(V)^{-1}\phi(V) \subseteq \phi(V^{-1}V) = \phi(V^2) \subseteq \phi(U)$$

□

6.2.1 Statements and corollaries

Theorem 6.2.2 (First structure theorem)**.** Let A be an LCA group. Then there exist $n \in \mathbb{Z}_{\geqslant 0}$ and another LCA group H such that

(i) $A \cong \mathbb{R}^n \times H$ in **TopGp**, and

(ii) H contains a compact open subgroup.

Definition. A topological group G is called **compactly generated** if there exists a compact unit-neighborhood K of G which generates G as a group.

Theorem 6.2.3 (Second structure theorem)**.** Let A be a compactly generated LCA group. Then there exists $n, m \in \mathbb{Z}_{\geqslant 0}$ and a compact group K such that

$$A \cong \mathbb{R}^n \times \mathbb{Z}^m \times K.$$

in **TopGp**.

Definition. A topological space is **locally euclidean of dimension** n if each point admits an open neighborhood that is homeomorphic to $\mathbb{R}^n$.

Theorem 6.2.4 (Third structure theorem)**.** Let A be a locally euclidean LCA group (of a fixed dimension). Then there exist $n, m \in \mathbb{Z}_{\geqslant 0}$ and a discrete abelian group D such that

$$A \cong \mathbb{R}^n \times (S^1)^m \times D$$

in **TopGp**.

Using these structure theorems along with the Pontryagin duality, we obtain

Corollary 6.2.4.1. For an LCA group A, that A is compactly generated is equivalent of that $\widehat{A}$ is locally euclidean.

Corollary 6.2.4.2. A locally euclidean compactly generated LCA group is isomorphic to

$$\mathbb{R}^n \times (S^1)^m \times \mathbb{Z}^\ell \times F$$

for some $n, m, \ell \in \mathbb{Z}_{\geqslant 0}$ and some finite abelian group F.

Using Theorem 6.1.12, we obtain

Corollary 6.2.4.3. A second countable path-connected LCA group is isomorphic to $\mathbb{R}^n \times (S^1)^I$ for some $n \in \mathbb{Z}_{\geqslant 0}$ and some countable index set I.

6.2.2 Proofs

Part II

Harmonic analysis on $\mathrm{GL}(1)$

Chapter 7

Tate thesis : local theory

As an application to the results in the preceding chapters, we are going to dive into one of the most beautiful theory invented by Tate. In this section we assume basic algebraic number theory.

Definition. Let G be a topological group. A continuous group homomorphism $\chi : G \to \mathbb{C}^\times$ is called a **quasi-character**.

- We've defined the notion of characters: if a quasi-character takes value in S^1, it is a character. Sometimes to distinguish, we call a character by a **unitary character**.

7.1 Local theory

Let k be a local field. If Char $k = 0$, then k is a finite extension of $\mathbb{Q}_p$ ($p \leqslant \infty$). If Char $k =: p > 0$, then k is isomorphic to $\mathbb{F}_q(\!(T)\!)$, the field of Laurent series, where $q = p^n$ for some $n \in \mathbb{Z}_{\geqslant 1}$. The local field $\mathbb{R}$ and $\mathbb{C}$ are called the **archimedean** local fields. The remaining cases are called **non-archimedean / discrete**, and they are equipped with natural discrete valuations. When k is a finite extension of $\mathbb{Q}_p$ ($p < \infty$), we write $(\mathfrak{o}, \mathfrak{p})$ for the local ring which is by definition the integral closure of $\mathbb{Z}_p$ in k. For an element $a \in k$, the principal ideal (a) has the form $\mathfrak{p}^m$ for some $m \in \mathbb{Z}$; the number m is then denoted by $m = \mathrm{ord}_{\mathfrak{p}}(a)$. If $k = \mathbb{F}_q(\!(T)\!)$, the discrete valuation is the usual order ord_T for the Laurent series. The corresponding local ring $(\mathfrak{o}, \mathfrak{p})$ is $(\mathbb{F}_q[\![T]\!], (T))$.

7.1.1 Fourier analysis on local fields

Let k be a non-archimedean local field. For an ideal $I \trianglelefteq \mathfrak{o}$, the quotient $\mathfrak{o}/I$ is finite; we denote the size by NI. This is called the **norm** of the ideal I. We extend the definition naturally to a fractional ideal I. For an element $a \in k$, we put $|a| = |a|_k := N(a\mathfrak{o}) = (N\mathfrak{p})^{-\mathrm{ord}_k a}$. Then $|\cdot| : k \to \mathbb{Q}$ defines an absolute value on k, and it satisfies the strengthened triangle inequality:

$$|a - b| \leqslant \min\{|a|, |b|\}, \qquad a, b \in k.$$

Moreover, when $|a| \neq |b|$, the equality holds. When $k = \mathbb{R}$, we let $|\cdot| = |\cdot|_{\mathbb{R}}$ denote the usual absolute value on $\mathbb{R}$. When $k = \mathbb{C}$, we set $|z| = |z|_{\mathbb{C}} = z\overline{z}$, square of the usual absolute value.

In all cases, the norm function defines a *locally compact Hausdorff* topology on k. Moreover, the topology makes k a topological field. The "unit disc" $\{x \in k \mid \|x\| \leqslant 1\}$ is compact; in the non-archimedean case, the unit disc is $\mathfrak{o}$, and the maximal ideal $\mathfrak{p} = \{x \in k \mid \|x\| < 1\}$ is also compact. In the non-archimedean case, there exists a unique Haar measure dx^{std} normalized so that the unit disc has volume 1. For the archimedean case, let dx^{std} be the usual Lebesgue measure (for which the volume of the unit cube has volume 1).

Lemma 7.1.1. Let σ be an automorphism of a non-archimedean local field k. Then $\text{mod}_k\sigma = N\sigma(\mathfrak{o})^{-1}$. In particular, $d(ax)^{\text{std}} = |a|dx^{\text{std}}$ for $a \in K$.

Proof. By definition, the modulus $\text{mod}_G\sigma$ is the ratio $\dfrac{\text{vol}(\sigma\mathfrak{o}, dx^{\text{std}})}{\text{vol}(\mathfrak{o}, dx^{\text{std}})} = \text{vol}(\sigma\mathfrak{o}, dx^{\text{std}})$. Write $\sigma\mathfrak{o} = \mathfrak{p}^n$ for some $n \in \mathbb{Z}$. If $n \geqslant 0$,

$$1 = \text{vol}(\mathfrak{o}, dx^{\text{std}}) = \sum_{a\in\mathfrak{o}/\mathfrak{p}^n} \text{vol}(a + \mathfrak{p}^n, dx^{\text{std}}) = N\mathfrak{p}^n \,\text{vol}(\mathfrak{p}^n, dx^{\text{std}})$$

so that $\text{vol}(\mathfrak{p}^n, dx^{\text{std}}) = N\mathfrak{p}^{-n}$. The same holds for $n \geqslant 0$. This finishes the proof. ☐

One of our goal is to do Fourier analysis on k. To do so, we start by studying the Pontryagin dual $\widehat{k}$ of the additive group k. Let us first note the following.

Lemma 7.1.2. If k is non-archimedean, then all quasi-characters are characters.

Proof. Let $\chi : k \to \mathbb{C}^\times$ be a quasi-character. Since $\mathfrak{p}$ is a compact subgroup of k, its image $\chi(\mathfrak{p})$ is a compact subgroup of $\mathbb{C}^\times$. Hence $\chi(\mathfrak{p}) \subseteq S^1$. Similarly, each power $\mathfrak{p}^n$ ($n \in \mathbb{Z}$) is a compact subgroup, so the image lies in S^1. Since $k = \bigcup_{n\in\mathbb{Z}} \mathfrak{p}^n$, this shows $\chi(k) \subseteq S^1$. ☐

The following theorem shows that the additive group of a local field k is **self-dual**.

Theorem 7.1.3. Let k be a local field and $\chi \in \widehat{k}$ a nontrivial character. Then the map

$$\begin{aligned} k &\longrightarrow \widehat{k} \\ x &\longmapsto [\chi_x : y \mapsto \chi(xy)] \end{aligned}$$

is an isomorphism of topological groups.

Proof. We proceed as follows.

(1) We first show that if $y \in k$ is such that $\chi(xy) = 1$ for all $x \in k$, then $y = 0$. Since χ is nontrivial, we can find $z \in k$ such that $\chi(z) \neq 1$. If $y \neq 0$, then $1 = \chi((zy^{-1})y) = \chi(z) \neq 1$, a contradiction; thus $y = 0$. This shows $x \mapsto \chi_x$ is injective.

(2) We claim the set $H := \{\chi_x \mid x \in k\}$ is dense in $\widehat{k}$, or equivalently, $\widehat{k}/\overline{H} = 0$. By Proposition 5.5.2, it suffices to show

$$0 = \left(\widehat{k}/\overline{H}\right)^\wedge \overset{(5.6.1).(i)}{\cong} \overline{H}^\perp = H^\perp$$

But this follows from (1).

(3) We show the map is a topological embedding.

- Continuity. Let $N \in \mathbb{Z}$ and $\varphi > 0$. We must show the set

$$A = A_{N,\varepsilon} := \{x \in k \mid |\chi_x(\mathfrak{p}^N) - 1| < \varepsilon\}$$

is a neighborhood of 0 in k^+. Since χ is a continuous group homomorphism, we can find $M \gg 0$ such that $\chi(\mathfrak{p}^M) = 1$. Then it is clear that $\mathfrak{p}^M \subseteq A$; this shows the continuity.

- Continuous inverse. (In the following $|\cdot|$ denotes the usual euclidean distance)

  $\mathfrak{p}$ discrete. Let $n \in \mathbb{Z}$. We must show $B = B_n := \{\chi_x \mid x \in \mathfrak{p}^n\}$ is a neighborhood of the trivial character 1. Let $\xi \in \mathfrak{p}^M \backslash \mathfrak{p}^{M+1}$ such that $\chi(\xi) \neq 1$. Then we claim $\{\chi_x \mid |\chi_x(\mathfrak{p}^{M+1-n}) - 1| < |\chi(\xi) - 1|\} \subseteq B$. Indeed, if $0 \neq x \in k$ is such that $|\chi_x(\mathfrak{p}^{M+1-n}) - 1| < |\chi(\xi) - 1|$, then in particular, $\xi \notin x\mathfrak{p}^{M+1-n}$. Say $x \in \mathfrak{p}^m \backslash \mathfrak{p}^{m+1}$; then $\xi \notin \mathfrak{p}^{M+1-n+m}$, i.e., $n+1 < m$. Hence $x \in \mathfrak{p}^m \subseteq \mathfrak{p}^n$, or $x \in B$.

  $\mathfrak{p}$ archimedean. For each $r > 0$, we show that $B = B_r := \{\chi_x \mid x \in B_r(0)\}$ is a neighborhood of the trivial character 1. Let $\xi \neq 0$ such that $\chi(\xi) \neq 1$. Then we claim $\{\chi_x \mid |\chi_x(B_{|\xi|/r}(0)) - 1| < |\chi(\xi) - 1|\} \subseteq B$. For if $0 \neq x \in k$ is such that $|\chi_x(B_{|\xi|/r}(0)) - 1| < |\chi(\xi) - 1|$, then in particular, $\xi \notin xB_{|\xi|/r}(0) = B_{|x||\xi|/r}$, i.e. $\dfrac{|x||\xi|}{r} < |\xi|$, or $|x| < r$.

(4) Thus, we see $\{\chi_x \mid x \in k\}$ is a locally compact subgroup of $\widehat{k}$. By Lemma 17.1.3, this means $\{\chi_x \mid x \in k\}$ is closed in $\widehat{k}$. But $\{\chi_x \mid x \in k\}$ is dense in $\widehat{k}$ by (2), so it is the entire $\widehat{k}$.

$\square$

To fix an identification $k \cong \widehat{k}$, we explicitly construct a nontrivial character ψ_k of k. We begin with the base cases. If $k = \mathbb{R}$, define

$$\psi_{\mathbb{R}}(x) = e^{2\pi i x}$$

which is the usual additive character we use. If $k = \mathbb{Q}_p$, define

$$\psi_{\mathbb{Q}_p}(x) = e^{-2\pi i \{x\}}.$$

We explain the notation $\{x\}$. Any element x in $\mathbb{Q}_p$ has a unique expression $x = \sum_{n \gg -\infty}^{\infty} a_n p^n$ with $0 \leqslant a_n < p$. Then $\{x\}$ is defined to be the principal part: $\{x\} = \sum_{n \gg -\infty}^{-1} a_n p^n \in \mathbb{Q}$. If $k = \mathbb{F}_p(\!(T)\!)$, define

$$\psi_{\mathbb{F}_p(\!(T)\!)}(x) = e^{\frac{2\pi i \operatorname{res}_T x}{p}}$$

where for $x = \sum_{n \gg -\infty} a_n T^n$, the residue $\operatorname{res}_T x$ is defined as usual: $\operatorname{res}_T x = a_{-1}$. One should note that the definition depends on a choice of **uniformizer**, i.e., a generator of the principal ideal $\mathfrak{p}$.

Now for any local field K, from definition we see it is a finite separable extension of one of the local field k mentioned above. Being separable means that the trace map $\operatorname{Tr}_{K/k} : K \times K \to k$ is nondegenerate. Now define $\psi_K : K \to \mathbb{C}^\times$ by $\psi_K = \psi_k \circ \operatorname{Tr}_{K/k}$. In any case, it is easy to see ψ_K is non-trivial. We call this the **standard additive character** on K. Therefore, by Theorem 7.1.3 we have the isomorphism

$$\begin{aligned} \psi : K &\longrightarrow \widehat{K} \\ x &\longmapsto \psi_x : y \mapsto \psi_K(xy). \end{aligned}$$

Consider the non-archimedean case. The character ψ_x is trivial on $\mathfrak{o}_K$ if and only if $\psi_k(\operatorname{Tr}_{K/k}(xy)) = 0$ for all $y \in \mathfrak{o}_K$. In any case we see this is the same as saying $\operatorname{Tr}_{K/k}(xy) \in \mathfrak{o}_k$ for all $y \in \mathfrak{o}_K$. The set

$$\mathfrak{o}^\vee = \mathfrak{o}_K^\vee := \{x \in K \mid \operatorname{Tr}_{K/k}(x\mathfrak{o}_K) \subseteq \mathfrak{o}_k\}$$

is the $\mathfrak{o}$-module dual to $\mathfrak{o}$ with respect to the bilinear form $(x, y) \mapsto \operatorname{Tr}_{K/k}(xy)$. Its inverse ideal $\mathfrak{d} = \mathfrak{d}_K := [\mathfrak{o}_K : \mathfrak{o}_K^\vee]$ is called the **(absolute) different**; the dual ideal $\mathfrak{o}^\vee = \mathfrak{d}_K^{-1}$ is sometimes called the inverse different. Note that $\mathfrak{d}$ is an ideal of $\mathfrak{o}$.

By Plancherel theorem there exists a unique Plancherel measure on $\widehat{K}$ with respect to dx^{std} making the Fourier transform $f \mapsto \widehat{f}$ an isometry. Since $\widehat{K} \cong K$ as topological groups, we can view Plancherel measure as a measure on K. Let us denote by dx^{Plan}. Under the isomorphism $K \cong \widehat{K}$, the Fourier transform of an integrable function $f \in L^1(K)$ has the form

$$\widehat{f}(x) = \int_K f(y)\overline{\psi_x(y)}dx^{\text{std}} = \int_K f(y)\psi_K(-xy)dx^{\text{std}}.$$

Set $\text{inv}(K) := \{f \in L^1(K) \mid \widehat{f} \in L^1(K)\}$; this is the set of functions that the Fourier inversion is valid, i.e.,

$$f(x) = \int_K \widehat{f}(y)\psi_K(xy)dy^{\text{Plan}}$$

holds for all $f \in \text{Inv}(K)$. But $dx^{\text{Plan}} = \alpha dx^{\text{std}}$ for some unique $\alpha > 0$, if we set $dx^{\text{tam}} = \sqrt{\alpha}dx^{\text{std}}$ and redefine

$$\widehat{f}(x) = \int_K f(y)\overline{\psi_x(y)}dx^{\text{tam}} = \int_K f(y)\psi_K(-xy)dx^{\text{tam}},$$

then the Fourier inversion now reads off

$$f(x) = \int_K \widehat{f}(y)\psi_K(xy)dy^{\text{tam}},$$

and the Plancherel measure with respect to dx^{tam} on K is dx^{tam} itself. In this case we say dx^{tam} is a **self-dual** measure on K. Warning: such measure depends on the choice of the nontrivial character defining the isomorphism $K \cong \widehat{K}$.

Lemma 7.1.4. We have the following description of the self-dual measure on K.

(i) If $K = \mathbb{R}$, then $dx^{\text{tam}} = dx^{\text{std}} = dx$ is the usual Lebesgue measure on $\mathbb{R}$.

(ii) If $K = \mathbb{C}$, then $dx^{\text{tam}} = 2dx^{\text{std}} = 2dxdy$ is twice the usual Lebesgue measure on $\mathbb{C}$.

(iii) If K is non-archimedean, then $dx^{\text{tam}} = (N\mathfrak{d})^{-\frac{1}{2}}dx^{\text{std}}$, i.e., the unique measure against which $\mathfrak{o}$ has volume $(N\mathfrak{d})^{-\frac{1}{2}}$.

Proof. We compute the Fourier transform of a specific choice of f.

(i) K real. Take $f(x) = e^{-\pi x^2}$. Using Cauchy integral formula, we compute

$$\begin{aligned}\hat{f}(x) &= \int_{\mathbb{R}} e^{-\pi y^2}e^{-2\pi ixy}dy = \int_{\mathbb{R}} e^{-\pi(y+ix)^2-\pi x^2}dy \\ &= e^{-\pi x^2}\left(\lim_{M\to\infty}\int_0^x e^{-\pi(-M+it)^2}dt + \lim_{\substack{N\to\infty\\ M\to\infty}}\int_{-M}^N e^{-\pi y^2}dy + \lim_{N\to\infty}\int_0^x e^{-\pi(N+it)^2}dt\right) \\ &= e^{-\pi x^2}\int_{\mathbb{R}} e^{-\pi y^2}dy = e^{-\pi x^2} = f(x)\end{aligned}$$

(ii) K complex. Take $f(z) = e^{-2\pi z\overline{z}}$. Writing $z = \sigma + i\tau$, we have

$$\hat{f}(z) = 2\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-2\pi(x^2+y^2)}e^{-4\pi i(\sigma x-\tau y)}dxdy = e^{-\pi(\sqrt{2}\sigma)^2}e^{-\pi(\sqrt{2}\tau)^2} = e^{-2\pi z\overline{z}} = f(z)$$

(iii) K non-arch. Take $f(x) = \mathbf{1}_{\mathfrak{o}}(x)$, the characteristic function of $\mathfrak{o} = \mathfrak{o}_K$. Then

$$\hat{f}(x) = \int_K \mathbf{1}_{\mathfrak{o}}(y)\psi_K(-xy)dy = \int_{\mathfrak{o}} \psi_K(-xy)dy = \text{vol}(\mathfrak{o}, dx^{\text{tam}})\mathbf{1}_{\mathfrak{d}^{-1}}(x)$$

and

$$\widehat{\mathbf{1}_{\mathfrak{d}^{-1}}}(x) = \int_K \mathbf{1}_{\mathfrak{d}^{-1}}(y)\psi_K(-xy)dy = \int_{\mathfrak{d}^{-1}} \psi_K(-xy)dy = \mathrm{vol}(\mathfrak{d}^{-1}, dx^{\mathrm{tam}})\mathbf{1}_{\mathfrak{o}}(x)$$

Being self-dual is then them same as

$$\mathrm{vol}(\mathfrak{o}, dx^{\mathrm{tam}})\,\mathrm{vol}(\mathfrak{d}^{-1}, dx^{\mathrm{tam}}) = 1.$$

This follows from $\mathrm{vol}(\mathfrak{d}^{-1}, dx^{\mathrm{std}}) = N\mathfrak{d}$.

□

7.1.2 Fourier analysis on multiplicative groups of local fields

Characters

Next we consider the group $k^\times$ of nonzero elements of a local field k. Let $\mathfrak{u} = \mathfrak{u}_k := \{x \in k \mid |k| = 1\}$. Clearly,

$$\mathfrak{u} = \begin{cases} \{\pm 1\} & \text{, if } k = \mathbb{R} \\ S^1 & \text{, if } k = \mathbb{C} \\ \mathfrak{o}^\times = \mathfrak{o}\backslash\mathfrak{p} & \text{, if } k \text{ is non-arch.} \end{cases} \qquad (\spadesuit)$$

In particular, we see $\mathfrak{u}$ is compact. In the non-archimedean case, it is also open. To do Fourier analysis on $k^\times$ again we first study the quasi-characters of $k^\times$. We start by addressing the archimedean case.

Lemma 7.1.5. Let k be an archimedean local field and let $\chi : k^\times \to \mathbb{C}^\times$ be a continuous group homomorphism. Then $\chi(x) = \left(\frac{x}{|x|}\right)^n |x|^c$ for some $n \in \mathbb{Z}$ and $c \in \mathbb{C}$. Here $|\cdot|$ is the usual euclidean norm, and we can take $n \in \{0, 1\}$ if $k = \mathbb{R}$.

Proof. The polar coordinates provide an isomorphism $\mathbb{C}^\times \cong \mathbb{R}_{>0} \times S^1$ of topological groups. Recall that $\widehat{S^1} \cong \mathbb{Z}$. Then $\chi(e^{i\theta}) = e^{i\theta n}$ for some $n \in \mathbb{Z}$, and

$$\chi(x) = \chi(x|x|^{-1})\chi(|x|) = \left(\frac{x}{|x|}\right)^n \chi(|x|).$$

Now we are reduced to showing that if $\chi : \mathbb{R}_{>0} \to \mathbb{C}^\times$ is a continuous group homomorphism, then $\chi(x) = x^c$ for some $c \in \mathbb{C}$. To this end, consider $\rho := \chi \circ \exp : \mathbb{R} \to \mathbb{C}^\times$. Pick any $\delta > 0$ so that $\alpha := \int_0^\delta \rho(t)dt \neq 0$. Then

$$\alpha\rho(x) = \int_0^\delta \rho(t)\rho(x)dt = \int_0^\delta \rho(t+x)dt = \int_x^{x+\delta} \rho(t)dt.$$

In particular, this shows ρ is differentiable. From $\rho(x+y) = \rho(x)\rho(y)$ we obtain $\rho'(x+y) = \rho(x)\rho'(y)$, so $\rho'(x) = \rho(x)\rho'(0) = \rho(x)$ by plugging in $y = 0$. Then $\rho(x) = \rho(0)e^{\rho'(0)x} = e^{\rho'(0)x}$. Now $\chi(x) = \rho(\log x) = x^{\rho'(0)}$. □

In particular, we have an isomorphism of abstract groups

$$\begin{aligned} \mathrm{Hom}_{\mathbf{TopGp}}(\mathbb{C}^\times, \mathbb{C}^\times) &\longrightarrow \widehat{S^1} \times \mathbb{C} \\ \chi &\longrightarrow (\chi|_{S^1}, (\chi \circ \exp)'(0)) \end{aligned}$$

and this restricts to a version for $k = \mathbb{R}$ (with S^1 replaced by $\{\pm 1\}$).

Lemma 7.1.6. This restricts to an isomorphism

$$\mathrm{Hom}_{\mathbf{TopGp}}(\mathbb{C}^\times, S^1) \xrightarrow{\sim} \widehat{S^1} \times 2\pi i\mathbb{R}$$

of topological groups. Restricting to $\mathbb{R}^\times$ gives the isomorphism

$$\mathrm{Hom}_{\mathbf{TopGp}}(\mathbb{R}^\times, S^1) \xrightarrow{\sim} \{\pm 1\} \times 2\pi i\mathbb{R}$$

Proof. Since $\mathbb{C}^\times \cong S^1 \times \mathbb{R}_{>0}$ as topological groups, by Lemma 5.1.9 we have an isomorphism

$$\mathrm{Hom}_{\mathbf{TopGp}}(\mathbb{C}^\times, S^1) \xrightarrow{\sim} \widehat{S^1} \times \widehat{\mathbb{R}_{>0}}$$

of topological groups. To conclude it remains to show that

$$\begin{aligned} \mathbb{R} &\longrightarrow \widehat{\mathbb{R}_{>0}} \\ r &\longmapsto |\cdot|^{2\pi i r} \end{aligned}$$

is a homeomorphism. Recall $\mathbb{R} \xrightarrow{\exp} \mathbb{R}_{>0}$ as topological groups, so by Lemma 5.1.4,

$$\begin{aligned} \widehat{\mathbb{R}_{>0}} &\longrightarrow \widehat{\mathbb{R}} \\ \chi &\longmapsto \chi \circ \exp \end{aligned}$$

is a homeomorphism. Composing the above two maps, we get

$$\begin{aligned} \mathbb{R} &\longrightarrow \widehat{\mathbb{R}} \\ r &\longmapsto x \mapsto e^{2\pi i r x} \end{aligned}.$$

But this is an isomorphism by Theorem 7.1.3, so we are done. $\square$

Next we proceed to discuss the non-archimedean case. Fix a uniformizer ϖ of k; then for $x \in k^\times$, $x\varpi^{-\operatorname{ord}_k x}$ lies in $\mathfrak{u}$, and

$$\chi(x) = \chi(x\varpi^{-\operatorname{ord}_k x}) \cdot \chi(\varpi)^{\operatorname{ord}_k x}.$$

We do some magic: if we write $\chi(\varpi) = re^{2\pi i\theta}$, then

$$\chi(\varpi)^{\operatorname{ord}_k x} = \chi(\varpi)^{-\log_{N\mathfrak{p}}|x|} = r^{-\log_{N\mathfrak{p}}|x|} e^{-2\pi i\theta \log_{N\mathfrak{p}}|x|} = |x|^{-\log_{N\mathfrak{p}} r - \frac{2\pi i\theta}{\log N\mathfrak{p}}}.$$

Hence

$$\chi(x) = \chi(x\varpi^{-\operatorname{ord}_k x})|x|^s$$

for some $s \in \mathbb{C}$. The number s is only unique modulo $\frac{2\pi i}{\log N\mathfrak{p}}$, as θ is unique modulo 1. Hence we have an isomorphism of abstract groups (the modulo reflects the fact the *discreteness* of the valuation ord_k.)

$$\begin{aligned} \mathrm{Hom}_{\mathbf{TopGp}}(k^\times, \mathbb{C}^\times) &\longrightarrow \widehat{\mathfrak{u}} \times \frac{\mathbb{C}}{\frac{2\pi i}{\log N\mathfrak{p}}\mathbb{Z}} \\ \chi &\longrightarrow \left(\chi|_{\mathfrak{u}}, -\log_{N\mathfrak{p}}|\chi(\varpi)| - i\frac{\arg \chi(\varpi)}{\log N\mathfrak{p}}\right) \end{aligned}$$

Lemma 7.1.7. This restrict to an isomorphism

$$\mathrm{Hom}_{\mathbf{TopGp}}(k^\times, S^1) \longrightarrow \widehat{\mathfrak{u}} \times \frac{i\mathbb{R}}{\frac{2\pi i}{\log N\mathfrak{p}}\mathbb{Z}}$$

of topological groups.

Proof. Fixing an uniformizer ϖ amounts to choosing an isomorphism

$$\begin{aligned} \mathfrak{u} \times \mathbb{Z} &\longrightarrow k^\times \\ (u, n) &\longmapsto u\varpi^n. \end{aligned}$$

So by Lemma 5.1.9 we have an isomorphism

$$\begin{aligned} \widehat{k^\times} &\longrightarrow \widehat{\mathfrak{u}} \times \widehat{\mathbb{Z}} \\ \chi &\longmapsto (\chi|_{\mathfrak{u}}, n \mapsto \chi(\varpi)^n). \end{aligned}$$

Put $\alpha = \chi(\varpi) \in S^1$. It remains to show

$$\begin{aligned} \frac{i\mathbb{R}}{\frac{2\pi i}{\log N\mathfrak{p}}\mathbb{Z}} &\longrightarrow \widehat{\mathbb{Z}} \\ -i\frac{\arg c}{\log N\mathfrak{p}} &\longmapsto [n \mapsto c^n] \end{aligned}$$

is an isomorphism of topological groups. This is just a dilation of the isomorphism $\mathbb{R}/\mathbb{Z} \cong \widehat{\mathbb{Z}}$. $\square$

Definition. The **weight** $\mathrm{wt}(\chi)$ of a quasi-character is the unique real number such that $|\chi| = |\cdot|^{\mathrm{wt}(\chi)}$.

- The weight can be also read off from the above isomorphisms: if $\chi \mapsto (\chi|_{\mathfrak{u}}, s)$, then $\mathrm{wt}(\chi) = \mathrm{Re}(s)$. By definition, a quasi-character χ is a unitary character if and only if $\mathrm{wt}(\chi) = 0$.

Next, we define some measures on $k^\times$. Since k is a topological field, $k^\times$ is again a topological group and $k^\times \subseteq k$ is open. Hence $k^\times$ is a locally compact abelian group. Our choice of a left invariant measure on $k^\times$ is

$$d^\times x^{\mathrm{tam}} := \frac{dx^{\mathrm{tam}}}{|x|_k}.$$

when k is archimedean, and

$$d^\times x^{\mathrm{tam}} := \frac{1}{1-(N\mathfrak{p})^{-1}} \frac{dx^{\mathrm{tam}}}{|x|_k}.$$

when k is non-archimedean. These normalization will be proved to be useful. Of course, we can simply use $d^\times x^{\mathrm{std}} = \dfrac{dx^{\mathrm{std}}}{|x|_k}$. Generally, any choice of Haar measure dx on k yields a Haar measure $\dfrac{dx}{|x|_k}$ on $k^\times$ (c.f. Example 2.3.5).

Lemma 7.1.8. For non-archimedean k, we have $\mathrm{vol}(\mathfrak{u}, d^\times x^{\mathrm{tam}}) = (N\mathfrak{d})^{-\frac{1}{2}}$.

Proof. Since

$$\mathrm{vol}(\mathfrak{u}, d^\times x^{\mathrm{std}}) = \int_{\mathfrak{u}} \frac{dx^{\mathrm{std}}}{|x|_k} = \mathrm{vol}(\mathfrak{o}\backslash\mathfrak{p}, dx^{\mathrm{std}}) = 1 - (N\mathfrak{p})^{-1}.$$

we have

$$\mathrm{vol}(\mathfrak{u}, d^\times x^{\mathrm{tam}}) = \mathrm{vol}(\mathfrak{o}, dx^{\mathrm{tam}}) = (N\mathfrak{d})^{-\frac{1}{2}}.$$

□

Mellin transform

Although it won't be used in the following, we compute the Plancherel measure on the dual group $\widehat{k^\times} = \mathrm{Hom}_{\mathbf{TopGp}}(k^\times, S^1)$ corresponding to the measure $d^\times x^{\mathrm{tam}}$. To do this we keep track of each isomorphism carefully. In the following all isomorphisms are in **TopGp**.

First assume $k = \mathbb{C}$. We have an isomorphism

$$\begin{aligned} S^1 \times \mathbb{R} &\longrightarrow \mathbb{C}^\times \\ (u, r) &\longmapsto ue^r. \end{aligned}$$

We choose the measure dk on S^1 so that the map is measure-preserving. The image of $S^1 \times [0,1]$ in $\mathbb{C}^\times$ is $\{1 \leqslant |z| \leqslant e\}$ ($|\cdot|$ again denotes the usual Euclidean norm), which has measure

$$\int_{1\leqslant|z|\leqslant e} d^\times z^{\mathrm{tam}} = \int_{1\leqslant x^2+y^2\leqslant e^2} \frac{2dxdy}{x^2+y^2} = \int_0^{2\pi}\int_1^e \frac{2drd\theta}{r} = 4\pi.$$

Hence $\mathrm{vol}(S^1, dk) = 4\pi$. Now by §5.4.1, the Plancherel measure on $\widehat{S^1 \times \mathbb{R}} \cong \widehat{S^1} \times \widehat{\mathbb{R}} \cong \mathbb{Z} \times \mathbb{R}$ is given by $\dfrac{1}{4\pi}$ times the product measure of the counting measure on $\mathbb{Z}$ and the Lebesgue measure on $\mathbb{R}$, which then gives the Plancherel measure on $\mathbb{C}^\times$. Explicitly, for $g \in L^1(\widehat{\mathbb{C}^\times})$, if we identify g as a function $g : \mathbb{Z} \times \mathbb{R} \to \mathbb{C}$ by

$$g(n, x) = g([ue^r \mapsto u^n e^{2\pi i r x}])$$

then the integral of g against the Plancherel measure is

$$\frac{1}{4\pi}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} g(n,x)dx^{\mathrm{tam}}$$

In particular, by Fourier inversion we obtain

Lemma 7.1.9 (Mellin inversion formula)**.** Let $f \in L^1(\mathbb{C}^\times)$ be such that $\widehat{f} \in L^1(\widehat{\mathbb{C}^\times})$. Then

$$f(a) = \frac{1}{4\pi}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} \widehat{f}(n,x)(a/|a|)^{-n}|a|^{-2\pi i x}dx^{\mathrm{tam}}$$

for all $a \in \mathbb{C}^\times$.

Note here the Fourier transform is taken on $\mathbb{C}^\times$, not on $\mathbb{C}$.

Example 7.1.10. Take $f(z) = |z|_{\mathbb{C}}e^{-|z|_{\mathbb{C}}}$. Then for $\chi \in \widehat{\mathbb{C}^\times}$

$$\begin{aligned} \widehat{f}(\chi) = \int_{\mathbb{C}^\times} |z|_{\mathbb{C}}e^{-|z|_{\mathbb{C}}}\chi(z)d^\times z^{\mathrm{tam}} = \int_0^{2\pi}\int_0^\infty e^{-r^2}\chi(re^{i\theta})2rdrd\theta &= 2\pi\delta_{\mathbf{1}_{S^1}}(\chi|_{S^1})\int_0^\infty e^{-r^2}\chi(r)2rdr \\ &= 2\pi\delta_{\mathbf{1}_{S^1}}(\chi|_{S^1})\int_0^\infty e^{-r}\chi(r)^{\frac{1}{2}}dr. \end{aligned}$$

Write $\chi(r) = r^{2\pi i x}$ for some $x \in \mathbb{R}$, and $\chi|_{S^1}(u) = u^n$ for some $n \in \mathbb{Z}$; then

$$\widehat{f}(\chi) = \widehat{f}(n, x) = 2\pi\delta_{0n} \int_0^\infty e^{-r} r^{\pi i x} dr = 2\pi\delta_{0n}\Gamma(1 + \pi i x).$$

Now Mellin inversion tells, for each $a \in \mathbb{C}^\times$, that

$$|a|^2 e^{-|a|^2} = \frac{1}{2} \int_{\mathbb{R}} \Gamma(1 + \pi i x)|a|^{-2\pi i x} dx = \frac{1}{2\pi} \int_{\mathbb{R}} \Gamma(1 + i x)|a|^{-2 i x} dx$$

In particular, we get an interesting identity

$$\frac{2\pi}{e} = \int_{\mathbb{R}} \Gamma(1 + i x) dx$$

This identity can be seen by applying classical Mellin inversion to the function e^{-x}.

Next assume $k = \mathbb{R}$. Similarly we have

$$\begin{aligned} \{\pm 1\} \times \mathbb{R} &\longrightarrow \mathbb{R}^\times \\ (u, r) &\longmapsto u e^r. \end{aligned}$$

By choosing the counting measure to be the Haar measure on $\{\pm 1\}$, the map becomes measure-preserving. By §5.4.1, the corresponding Plancherel measure on $\widehat{\{\pm 1\} \times \mathbb{R}} \cong \{0, 1\} \times \mathbb{R}$ is given by $\frac{1}{2}$ times the product measure of the counting measure on $\{0, 1\}$ and the Lebesgue measure on $\mathbb{R}$, which again gives the Plancherel measure on $\mathbb{R}^\times$. Explicitly, for $g \in L^1(\widehat{\mathbb{R}^\times})$, if we identify g as a function $g : \{0, 1\} \times \mathbb{R} \to \mathbb{C}$ by

$$g(0, x) = g([r \mapsto |r|^{2\pi i x}]), \qquad g(1, x) = g([r \mapsto \operatorname{sign}(r)|r|^{2\pi i x}])$$

then the integral of g against the Plancherel measure is

$$\frac{1}{2} \sum_{n \in \{0,1\}} \int_{\mathbb{R}} g(n, x) dx^{\mathrm{tam}}$$

Again, by Fourier inversion we obtain

Lemma 7.1.11 (Mellin inversion formula)**.** Let $f \in L^1(\mathbb{R}^\times)$ be such that $\widehat{f} \in L^1(\widehat{\mathbb{R}^\times})$. Then

$$f(a) = \frac{1}{2} \sum_{n \in \{0,1\}} \int_{\mathbb{R}} \widehat{f}(n, x)(a/|a|)^{-n}|a|^{-2\pi i x} dx^{\mathrm{tam}}$$

for all $a \in \mathbb{R}^\times$.

Finally consider non-archimedean k. By choosing a uniformizer ϖ of k, we have an isomorphism

$$\begin{aligned} \mathfrak{u} \times \mathbb{Z} &\longrightarrow k^\times \\ (u, n) &\longmapsto u\varpi^n. \end{aligned}$$

On $\mathfrak{u}$ we still use the measure $d^\times x^{\mathrm{tam}}$, and on $\mathbb{Z}$ we use the counting measure. Then this map is measure-preserving, and by §5.4.1 and Lemma 7.1.8, the Plancherel measure on $\widehat{\mathfrak{u} \times \mathbb{Z}} \cong \widehat{\mathfrak{u}} \times \widehat{\mathbb{Z}} \cong \widehat{\mathfrak{u}} \times S^1$ is the $(N\mathfrak{d})^{\frac{1}{2}}$ times the product measure of the counting measure on $\widehat{\mathfrak{u}}$ and the Haar measure on $\mathbb{R}/\mathbb{Z}$ of total volume 1. Explicitly, for $g \in L^1(\widehat{k^\times})$, if we identify g as a function $g : \widehat{\mathfrak{u}} \times \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ by

$$g(\chi, r) = g([x \mapsto \chi(x\varpi^{-\operatorname{ord}_k x})|x|^{\frac{2\pi i}{\log N\mathfrak{p}} r}])$$

them the integral against the Plancherel measure is

$$(N\mathfrak{d})^{\frac{1}{2}} \sum_{\chi \in \widehat{\mathfrak{u}}} \int_{\mathbb{R}/\mathbb{Z}} g(\chi, r) dr = \frac{(N\mathfrak{d})^{\frac{1}{2}} \log N\mathfrak{p}}{2\pi} \sum_{\chi \in \widehat{\mathfrak{u}}} \int_{-\frac{\pi i}{\log N\mathfrak{p}}}^{\frac{\pi i}{\log N\mathfrak{p}}} g(\chi, r) dr$$

Local zeta integrals

We now begin our main subject: local zeta functions. Our notation will be slightly different from that in Tate's thesis. For any quasi-character χ, any measurable function $f : k \to \mathbb{C}$ and any $s \in \mathbb{C}$, define the **zeta integral** as

$$Z(f, \chi, s) := \int_{k^\times} f(x)\chi|\cdot|^s(x) d^\times x^{\mathrm{tam}}$$

if it exists. With χ fixed, this should be thought of as a family of distributions on certain space of functions of k. With this idea in mind, we will mainly consider the space of functions $f : k \to \mathbb{C}$ satisfying

(i) $f \in \mathrm{inv}(k)$,

(ii) $f(x)|x|^s$ and $\widehat{f}(x)|x|^s$ are in $L^1(k^\times)$ for $\mathrm{Re}\, s > 0$.

We denote by $Z(k)$ the space of functions on k satisfying (i) and (ii). By (ii), we see for (f, χ, s) with $f \in Z(k)$ and $\mathrm{wt}(\chi) + \mathrm{Re}\, s > 0$, the zeta integral $Z(f, \chi, s)$ is defined. More is true.

Lemma 7.1.12. Fix a quasi-character χ and $f \in Z(k)$. Then $s \mapsto Z(f, \chi, s)$ defines a holomorphic function in $\mathrm{Re}\, s + \mathrm{wt}(\chi) > 0$.

Proof. Replacing χ be $\chi|\cdot|^{-\mathrm{wt}(\chi)}$, we can assume χ is unitary. Notice that the integral

$$\int_{k^\times} f(x)\chi|\cdot|^s(x) \log|x| d^\times x^{\mathrm{tam}}$$

exists for $\mathrm{Re}\, s > 0$ as well. To finish the proof, consider

$$\frac{Z(f, \chi, s) - Z(f, \chi, t)}{s - t} = \int_{k^\times} f(x)\chi(x) \frac{|x|^s - |x|^t}{s - t} d^\times x^{\mathrm{tam}}$$

for $\mathrm{Re}\, s$, $\mathrm{Re}\, t > 0$. The holomorphicity is a local property, so it suffices to consider s, t within a compact convex set $K \subseteq \{\mathrm{Re}\, s > 0\}$. Then for $t, s \in K$, if we let $\gamma : [0, 1] \to K$ be the line segment joining s to t, then $\ell(\gamma) = |s - t|$ and

$$\left||x|^s - |x|^t\right| = \left|\int_0^1 \frac{d}{du}|x|^u d\gamma(u)\right| \leqslant \ell(\gamma) \cdot \sup_{u \in K} ||x|^u \log|x||,$$

or

$$\left|\frac{|x|^s - |x|^t}{s - t}\right| \leqslant \log|x| \cdot \sup_{u \in K} ||x|^u| \leqslant |x|^M \log|x|$$

for some $M \gg 0$ (e.g. we can take $M = \sup_{u \in K} \mathrm{Re}(u)$). Then for $s, t \in K$, we have

$$\left|\frac{Z(f, \chi, s) - Z(f, \chi, t)}{s - t}\right| \leqslant \int_{k^\times} |f(x)||x|^M \log|x| d^\times x^{\mathrm{tam}}.$$

The integrand of the last integral is L^1 by assumption (ii), so it follows from DCT that

$$\lim_{s \to t} \frac{Z(f, \chi, s) - Z(f, \chi, t)}{s - t} = \int_{k^\times} f(x)\chi(x) \lim_{s \to t} \frac{|x|^s - |x|^t}{s - t} d^\times x^{\mathrm{tam}} = \int_{k^\times} f(x)\chi|\cdot|^s \log|x| d^\times x^{\mathrm{tam}}.$$

In particular, this shows $Z(f, \chi, s)$ is holomorphic for $\mathrm{Re}\, s > 0$. □

Our main goal is to show $s \mapsto Z(f, \chi, s)$ admits a meromorphic continuation to the whole complex plane $s \in \mathbb{C}$. This will be done by establishing certain functional equation.

Lemma 7.1.13. Fix a quasi-character χ and $f, g \in Z(k)$. Then for s with $0 < \operatorname{Re} s + \operatorname{wt}(\chi) < 1$, we have

$$Z(f, \chi, s) Z(\widehat{g}, \chi^{-1}, 1-s) = Z(\widehat{f}, \chi^{-1}, 1-s) Z(g, \chi, s)$$

Proof. This is an easy application of Fubini:

$$\begin{aligned} Z(f,\chi,s)Z(\widehat{g},\chi^{-1},1-s) &= \int_{k^\times \times k^\times} f(x)\widehat{g}(y)\chi(xy^{-1})|x|^s|y|^{1-s} d^\times x^{\mathrm{tam}} d^\times y^{\mathrm{tam}} \\ (y \mapsto xy) &= \int_{k^\times \times k^\times} f(x)\widehat{g}(xy)\chi(y^{-1})|x|^s|xy|^{1-s}|x|^{-1} d^\times x^{\mathrm{tam}} d^\times y^{\mathrm{tam}} \\ &= \int_{k^\times \times k^\times} f(x)\widehat{g}(xy)\chi(y^{-1})|y|^{1-s} d^\times x^{\mathrm{tam}} d^\times y^{\mathrm{tam}} \\ &= \int_{k^\times \times k^\times \times k} f(x)g(z)\psi_k(-xyz)\chi(y^{-1})|y|^{1-s} d^\times x^{\mathrm{tam}} d^\times y^{\mathrm{tam}} dz^{\mathrm{tam}} \end{aligned}$$

The last expression is symmetric in f, g, so replacing the roles of f and g yields the result. $\square$

Our goal will be completed once we can find $f \in Z(k)$ such that the ratio

$$\gamma(s, \chi, \psi_k) := \frac{Z(\widehat{f}, \chi^{-1}, 1-s)}{Z(f, \chi, s)}$$

is well-defined and admits a meromorphic continuation to $\mathbb{C}$; we view ψ_k as a parameter to emphasize the dependence of the self-duality $k \cong \widehat{k}$ on the nontrivial quasi-character ψ_k. To do this we introduce a subfamily of functions in $Z(k)$. For a local field k, let $S(k)$ denote the space of **Bruhat-Schwartz functions**, which we define now. In fact, we are going to define the space $S(k^n)$.

(i) For $k = \mathbb{R}, \mathbb{C}$, a function $f \in S(k^n)$ if and only if it is a usual Schwartz function. Recall that the space $S(\mathbb{R}^n)$ of usual Schwartz functions consists of functions $f : \mathbb{R}^n \to \mathbb{C}$ such that

$$\infty > \|f\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^n} \left| x_1^{\alpha_1} \cdots x_n^{\alpha_n} \frac{\partial^{\beta_1 + \cdots \beta_n} f}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}(x) \right|$$

for all multi-indices $\alpha, \beta \in \mathbb{Z}_{\geqslant 0}^n$. We regard $\mathbb{C}^n$ as $\mathbb{R}^{2n}$ here.

(ii) For non-archimedean k, a function $f \in S(k^n)$ if and only if it is locally constant with compact support. Note that $S(k^n)$ is the linear span of all characteristic functions of the form $\mathbf{1}_{a+\mathfrak{p}^n}$ for some $a \in k$ and $n \in \mathbb{Z}$.

Lemma 7.1.14. Let k be a local field.

(i) The Fourier transform induces a bijection on $S(k)$.

(ii) $S(k) \subseteq Z(k)$.

Proof. First consider the non-archimedean case. For (i) it suffices to show $\widehat{\mathbf{1}_{a+\mathfrak{p}^n}} \in S(k)$. Compute

$$\begin{aligned} \widehat{\mathbf{1}_{a+\mathfrak{p}^n}}(x) &= \int_k \mathbf{1}_{a+\mathfrak{p}^n}(y)\psi_k(-xy) dy^{\mathrm{tam}} = \psi_k(-xa) \int_{\mathfrak{p}^n} \psi_k(-xy) dy^{\mathrm{tam}} \\ &= \psi_k(-xa) \int_{\mathfrak{o}_k} \psi_k(-x\varpi^n y)|\varpi|^n dy^{\mathrm{tam}} = \psi_k(-xa)\mathbf{1}_{\varpi^{-n}\mathfrak{d}^{-1}}(x)|\varpi|^n \operatorname{vol}(\mathfrak{o}, dx^{\mathrm{tam}}). \end{aligned}$$

Here we recall that $y \mapsto \psi_k(xy)$ is trivial on $\mathfrak{o}$ if and only if $x \in \mathfrak{d}^{-1}$. In particular, this shows $x \mapsto \psi_k(-xa)$ is locally constant (but fails to have compact support), so $\widehat{\mathbf{1}_{a+\mathfrak{p}^n}} \in S(k)$. This finishes the proof of (i) for non-archimedean k, as $S(k) \subseteq \mathrm{Inv}(k)$ so the Fourier inversion is valid. As a record, note that

$$\psi_{-1}\widehat{\mathbf{1}_{\mathfrak{p}^{-n}\mathfrak{d}^{-1}}} = \mathrm{vol}(\mathfrak{o}, dx^{\mathrm{tam}})^{-1}|\varpi|^{-n}\mathbf{1}_{1+\mathfrak{p}^n} = (N\mathfrak{d})^{\frac{1}{2}}(N\mathfrak{p})^n\mathbf{1}_{1+\mathfrak{p}^n}.$$

For (ii), we compute $Z(\mathbf{1}_{a+\mathfrak{p}^n}, \mathbf{1}, s)$. If $a \in \mathfrak{p}^n$, then

$$\begin{aligned} Z(\mathbf{1}_{a+\mathfrak{p}^n}, \mathbf{1}, s) = Z(\mathbf{1}_{\mathfrak{p}^n}, \mathbf{1}, s) &= \int_{k^\times} \mathbf{1}_{\mathfrak{p}^n}(x)|x|^s d^\times x^{\mathrm{tam}} \\ &= \sum_{m\in\mathbb{Z}} \int_{\mathfrak{p}^m\setminus\mathfrak{p}^{m+1}} \mathbf{1}_{\mathfrak{p}^n}(x)|x|^s d^\times x^{\mathrm{tam}} = \sum_{m=n}^{\infty} (N\mathfrak{p})^{-ms}\cdot \mathrm{vol}(\mathfrak{o}^\times, d^\times x^{\mathrm{tam}}) \\ &= \frac{(N\mathfrak{p})^{-ns}(N\mathfrak{d})^{-\frac{1}{2}}}{1-(N\mathfrak{p})^{-s}}. \end{aligned}$$

The computation is valid if $|(N\mathfrak{p})^{-s}| < 1$, or Re $s > 0$. If $a \notin \mathfrak{p}^n$, then $x \in a + \mathfrak{p}^n$ implies $|x| = |a|$ so that

$$Z(\mathbf{1}_{a+\mathfrak{p}^n}, \mathbf{1}, s) = \int_{a+\mathfrak{p}^n} |a|^s \cdot \frac{1}{1-(N\mathfrak{p})^{-1}}\frac{dx^{\mathrm{tam}}}{|a|} = \frac{|a|^{s-1}\,\mathrm{vol}(\mathfrak{p}^n, dx^{\mathrm{tam}})}{1-(N\mathfrak{p})^{-1}} = \frac{(N\mathfrak{p})^{-(s-1)\,\mathrm{ord}_k a}(N\mathfrak{p})^{-n}(N\mathfrak{d})^{-\frac{1}{2}}}{1-(N\mathfrak{p})^{-1}}$$

In particular, this shows $\mathbf{1}_{a+\mathfrak{p}^n}|\cdot|^s \in L^1(k^\times)$ possibly unless $1-(N\mathfrak{p})^{-s} = 0$, i.e., $s = 0$. In view of (i), this shows $S(k) \subseteq Z(k)$.

Now consider the archimedean case. Clearly, $S(\mathbb{R}^n)$ is an $\mathbb{R}[x_1, \dots, x_n]$-module invariant under differentiation. An easy manipulation show that for $f \in S(\mathbb{R}^n)$, the Fourier transform

$$\widehat{f}(x) := \int_{\mathbb{R}^n} f(y)e^{-2\pi i\langle x,y\rangle}dy$$

satisfies

$$\widehat{x_j f} = -2\pi i\partial_{x_j}\widehat{f}, \qquad \widehat{\partial_{x_j} f} = 2\pi i x_j\widehat{f}.$$

The first is easy, and we prove the second one. Note that one cannot deduce the second from the first before one knows that $S(\mathbb{R}^n) \subseteq \mathrm{Inv}(\mathbb{R}^n)$. The second follows from integration by parts:

$$\begin{aligned} \widehat{\partial_{x_j} f}(x) &= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \partial_{x_j} f(y)e^{-2\pi i\langle x,y\rangle}dx_j\right) dx_1\cdots\widehat{dx_j}\cdots dx_n \\ &= \int_{\mathbb{R}^{n-1}} \left(f(y)e^{-2\pi i\langle x,y\rangle}\Big|_{x_j=-\infty}^{x_j=\infty} - \int_{\mathbb{R}} -2\pi i x_j f(y)e^{-2\pi i\langle x,y\rangle}dx_j\right) dx_1\cdots\widehat{dx_j}\cdots dx_n \\ &= 2\pi i x_j\widehat{f}(x). \end{aligned}$$

The former term in the inner parenthesis vanishes as f is Schwartz. This two properties show that $S(\mathbb{R}^n)$ is invariant under Fourier transform. These altogether show (i), (ii). □

7.1.3 Computations

We are going to carry out explicitly computations of $\dfrac{Z(\widehat{f}, \chi^{-1}, 1-s)}{Z(f, \chi, s)}$ with special $f \in S(k)$ for each quasi-character χ. Before that, note that if χ and χ' are two quasi-characters with $\chi^{-1}\chi' = |\cdot|^{s_0}$, then formally $Z(f, \chi', s) = Z(f, \chi, s+s_0)$; in this case we call χ and χ' equivalent. Keeping this in mind, only a few computations need to be carried out.

Real

There are two equivalence classes of quasi-characters: one is $|\cdot|^s$ ($s \in \mathbb{C}$) and the other is $x \mapsto \operatorname{sign}|\cdot|^s$ ($s \in \mathbb{C}$). The trivial character $\mathbf{1}$ belongs to the first class, and sign belongs to the second class.

We consider $f(x) = e^{-\pi x^2}$ and $g(x) = xe^{-\pi x^2}$. We already saw in Lemma 7.1.4 that $f = \widehat{f}$; explicitly,

$$e^{-\pi x^2} = \int_{\mathbb{R}} e^{-\pi y^2 - 2\pi ixy} dy$$

By applying $\dfrac{d}{dx}$ to both sides, we obtain

$$-2\pi xe^{\pi x^2} = \int_{\mathbb{R}} -2\pi iye^{-\pi y^2 - 2\pi ixy} dy$$

or $g = i\widehat{g}$. Now we compute the ζ-functions.

$$\begin{aligned}
Z(f, \mathbf{1}, s) &= \int_{\mathbb{R}^\times} e^{-\pi x^2}|x|^s \frac{dx}{|x|} = 2\int_0^\infty e^{-\pi x^2} x^s \frac{dx}{x} = 2\pi^{-\frac{s}{2}} \int_0^\infty e^{-x^2} x^s \frac{dx}{x} = \pi^{-\frac{s}{2}} \int_0^\infty e^{-x} x^{\frac{s}{2}} \frac{dx}{x} = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \\
Z(g, \operatorname{sign}, s) &= \int_{\mathbb{R}^\times} xe^{-\pi x^2} \operatorname{sign}(x)|x|^s \frac{dx}{|x|} = 2\int_0^\infty xe^{-\pi x^2} x^s \frac{dx}{x} = \pi^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right) \\
Z(\widehat{f}, \mathbf{1}, 1-s) &= Z(f, \mathbf{1}, 1-s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \\
Z(\widehat{g}, \operatorname{sign}, 1-s) &= -i \cdot Z(g, \operatorname{sign}, 1-s) = i^3 \pi^{-\frac{(1-s)+1}{2}} \Gamma\left(\frac{(1-s)+1}{2}\right)
\end{aligned}$$

Let us introduce the **complete Γ-function**:

$$\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right).$$

From definition $\Gamma_{\mathbb{R}}$ is entire except the simple poles along $2\mathbb{Z}_{\leqslant 0}$. Then we have explicit expressions for γ :

$$\gamma(s, \mathbf{1}, \psi_k) = \frac{\Gamma_{\mathbb{R}}(1-s)}{\Gamma_{\mathbb{R}}(s)}, \qquad \gamma(s, \operatorname{sign}, \psi_k) = i^3 \cdot \frac{\Gamma_{\mathbb{R}}(1-s+1)}{\Gamma_{\mathbb{R}}(s+1)}$$

For a general quasi-character $\chi : \mathbb{R}^\times \to \mathbb{C}^\times$, we have $\chi = \operatorname{sign}^\epsilon|\cdot|^{s_0}$ for some $\epsilon \in \{0, 1\}$ and $s_0 \in \mathbb{C}$ (7.1.5). We set

$$L(s, \chi) := \Gamma_{\mathbb{R}}(s + s_0 + \epsilon).$$

Then

$$\gamma(s, \chi, \psi_k) = i^{3\epsilon} \cdot \frac{L(1-s, \chi^{-1})}{L(s, \chi)}$$

Complex

The characters $c_n(re^{i\theta}) = e^{in\theta}$ ($n \in \mathbb{Z}$) represent the different equivalence classes. Consider the functions

$$f_n(z) := \begin{cases} \overline{z}^{|n|} e^{-2\pi z\overline{z}} & \text{, if } n \geqslant 0 \\ z^{|n|} e^{-2\pi z\overline{z}} & \text{, if } n \leqslant 0 \end{cases}$$

We contend that

$$\hat{f}_n(z) = (-i)^{|n|} f_{-n}(z) \text{ for all } n \in \mathbb{Z}.$$

Induction on $n \geqslant 0$, $n = 0$ being shown in Lemma 7.1.4. Suppose we have proved the contention for some $n \geqslant 0$, i.e., we have established the formula

$$\int_{\mathbb{C}} \overline{s}^n e^{-2\pi s\overline{s}} e^{-2\pi i\cdot \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(sz)} ds = (-i)^n z^n e^{-2\pi z\overline{z}}$$

Applying the operator $\dfrac{\partial}{\partial \overline{z}}$ to both sides, we obtain

$$\int_{\mathbb{C}} \overline{s}^n e^{-2\pi s\overline{s}} e^{-2\pi i\cdot \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(sz)} \left(-2\pi i\overline{s}\right)\right) ds = i^n z^n(-2\pi z) e^{-2\pi z\overline{z}}$$

or

$$\int_{\mathbb{C}} \overline{s}^{n+1} e^{-2\pi s\overline{s}} e^{-2\pi i\cdot \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(sz)} = (-i)^{n+1} z^{n+1} e^{-2\pi z\overline{z}}$$

which is the contention for $n+1$. The induction step is carried out. For the case $n < 0$, the previous proof shows $\widehat{f_{-n}}(z) = (-i)^{-n} f_n(z)$, so

$$(-i)^{-n}\widehat{f_n}(z) = \widehat{\widehat{f_{-n}}}(z) = f_{-n}(-z) = (-1)^{-n} f_{-n}(z)$$

or $\widehat{f_n}(z) = (-i)^{-n} f_{-n}(z) = (-i)^{|n|} f_{-n}(z)$. This finishes the proof.

We proceed. Write $z = re^{i\theta}$; then

$$f_n(z) = r^{|n|} e^{-in\theta} e^{-2\pi r^2}$$

and

$$d^{\times} z = \frac{2dxdy}{x^2+y^2} = \frac{2rdrd\theta}{r^2}.$$

We make some normalization: let $g_n = f_n$ $(n \in \mathbb{Z})$; then

$$\begin{aligned} Z(g_n, c_n, s) &= \int_0^{\infty}\int_0^{2\pi} r^{|n|} e^{-in\theta} e^{-2\pi r^2} e^{in\theta} r^{2s} \frac{2rdrd\theta}{r^2} \\ &= 2\pi \int_0^{\infty} (r^2)^{s-1+\frac{|n|}{2}} e^{-2\pi r^2} d(r^2) = (2\pi)^{1-(s+\frac{|n|}{2})}\Gamma\left(s+\frac{|n|}{2}\right) \\ Z(\widehat{g_n}, c_n^{-1}, 1-s) &= (-i)^{|n|} Z(g_{-n}, c_{-n}, 1-s) = (-i)^{|n|}(2\pi)^{1-(1-s+\frac{|n|}{2})}\Gamma\left(1-s+\frac{|n|}{2}\right) \end{aligned}$$

so

$$\gamma(s, c_n, \psi_k) = (-i)^{|n|} \frac{(2\pi)^{1-(1-s+\frac{|n|}{2})}\Gamma\left(1-s+\frac{|n|}{2}\right)}{(2\pi)^{1-(s+\frac{|n|}{2})}\Gamma\left(s+\frac{|n|}{2}\right)}.$$

Define the **complete Γ-function over $\mathbb{C}$**:

$$\Gamma_{\mathbb{C}}(s) = (2\pi)^{1-s}\Gamma(s).$$

By Legendre duplication formula, one has $\Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1)$. With these notations,

$$\gamma(s, c_n, \psi_k) = (-i)^{|n|} \frac{\Gamma_{\mathbb{C}}\left(1-s+\frac{|n|}{2}\right)}{\Gamma_{\mathbb{C}}\left(s+\frac{|n|}{2}\right)}$$

Generally, a quasi-character $\chi : \mathbb{C}^\times \to \mathbb{C}^\times$ has the form $\chi(z) = |z|_{\mathbb{C}}^c \left(\frac{z}{|z|_{\mathbb{C}}^{\frac{1}{2}}}\right)^n$ for some $c \in \mathbb{C}$ and $n \in \mathbb{Z}$ (7.1.5). Set

$$L(s, \chi) = \Gamma_{\mathbb{C}}\left(s + c + \frac{|n|}{2}\right).$$

Then

$$\gamma(s, \chi, \psi_k) = (-i)^{|n|} \frac{L(1-s, \chi^{-1})}{L(s, \chi)}$$

Non-archimedean

Let $\chi : k^\times \to \mathbb{C}^\times$ be any quasi-character. Since $1 + \mathfrak{p}^n$ ($n \geqslant 1$) form a basis of unit-neighborhoods of $\mathfrak{u}$, and since $\mathbb{C}^\times$ has no small subgroup (c.f. (I.2.10)), it follows that if $\chi|_{\mathfrak{u}}$ is not trivial, then there exists some $n \geqslant 1$ such that $\chi|_{1+\mathfrak{p}^n} \equiv 1$. The smallest such $n \geqslant 1$ is called the **conductor** of χ, and is denoted by $c(\chi)$.

Definition. A quasi-character χ is called **unramified** if $\chi|_{\mathfrak{u}} \equiv 1$.

We proceed to compute $Z(f, \chi, s)$ case by case. Let χ_0 be an unramified quasi-character and χ_n ($n \geqslant 1$) be a quasi-character with conductor $c(\chi_n) = n$. For $n \geqslant 0$, set

$$f_n(x) = \psi_k(-x)\mathbf{1}_{\mathfrak{p}^{-n}\mathfrak{d}^{-1}}(x).$$

In Lemma 7.1.14 we've seen that

$$\widehat{f_n}(x) = (N\mathfrak{d})^{\frac{1}{2}}(N\mathfrak{p})^n \mathbf{1}_{1+\mathfrak{p}^n}.$$

Say $\mathfrak{d}^{-1} = \mathfrak{p}^d$. Compute

$$\begin{aligned}
Z(f_0, \chi_0, s) &= \int_{k^\times} \psi_k(-x)\mathbf{1}_{\mathfrak{d}^{-1}}(x)\chi_0|\cdot|^s(x)d^\times x^{\mathrm{tam}} = \int_{\mathfrak{d}^{-1}\setminus\{0\}} \chi_0|\cdot|^s(x)d^\times x^{\mathrm{tam}} \\
&= \sum_{n\geqslant 0} \int_{\mathfrak{p}^{n+d}\setminus\mathfrak{p}^{n+d+1}} \chi_0|\cdot|^s(x)d^\times x^{\mathrm{tam}} = \sum_{n\geqslant 0} \chi_0(\varpi)^{n+d}(N\mathfrak{p})^{-s(n+d)} \operatorname{vol}(\mathfrak{o}^\times, d^\times x^{\mathrm{tam}}) \\
&= (N\mathfrak{d})^{-\frac{1}{2}} \frac{\chi_0(\varpi)^d (N\mathfrak{p})^{-sd}}{1 - \chi_0(\varpi)(N\mathfrak{p})^{-s}} = \frac{\chi_0(\varpi)^d (N\mathfrak{d})^{s-\frac{1}{2}}}{1 - \chi_0(\varpi)(N\mathfrak{p})^{-s}}
\end{aligned}$$

$$\begin{aligned}
Z(\widehat{f_0}, \chi_0^{-1}, 1-s) &= (N\mathfrak{d})^{\frac{1}{2}} \int_{k^\times} \mathbf{1}_{\mathfrak{o}}(x)\chi_0^{-1}|\cdot|^{1-s}(x)d^\times x^{\mathrm{tam}} = (N\mathfrak{d})^{\frac{1}{2}} \sum_{n\geqslant 0} \int_{\mathfrak{p}^n\setminus\mathfrak{p}^{n+1}} \chi_0^{-1}|\cdot|^{1-s}(x)d^\times x^{\mathrm{tam}} \\
&= (N\mathfrak{d})^{\frac{1}{2}} \sum_{n\geqslant 0} \chi_0(\varpi)^{-n}(N\mathfrak{p})^{n(s-1)} \operatorname{vol}(\mathfrak{o}^\times, d^\times x^{\mathrm{tam}}) \\
&= \frac{1}{1 - \chi_0(\varpi)^{-1}(N\mathfrak{p})^{s-1}}
\end{aligned}$$

Therefore,

$$\gamma(s, \chi_0, \psi_k) = \frac{1}{\chi_0(\varpi)^d (N\mathfrak{d})^{s-\frac{1}{2}}} \cdot \frac{(1 - \chi_0(\varpi)^{-1}(N\mathfrak{p})^{s-1})^{-1}}{(1 - \chi_0(\varpi)(N\mathfrak{p})^{-s})^{-1}}$$

We proceed to compute the ramified cases.

$$Z(\widehat{f_n}, \chi_n^{-1}, 1-s) = (N\mathfrak{d})^{\frac{1}{2}}(N\mathfrak{p})^n \int_{1+\mathfrak{p}^n} \chi_n^{-1}|\cdot|^{1-s}(x)d^\times x^{\mathrm{tam}} = (N\mathfrak{d})^{\frac{1}{2}}(N\mathfrak{p})^n \int_{1+\mathfrak{p}^n} d^\times x^{\mathrm{tam}}$$

$$
\begin{aligned}
Z(f_n,\chi_n,s) &= \int_{\mathfrak{p}^{-n+d}} \psi_k(-x)\chi_n|\cdot|^s(x)d^\times x^{\mathrm{tam}} = \sum_{m\geqslant 0}\int_{\mathfrak{p}^{-n+d+m}\setminus\mathfrak{p}^{-n+d+m+1}} \psi_k(-x)\chi_n|\cdot|^s(x)d^\times x^{\mathrm{tam}}\\
&= \sum_{m\geqslant -n+d} \chi_n(\varpi)^m (N\mathfrak{p})^{-ms}\int_{\mathfrak{o}^\times}\psi_k(-\varpi^m x)\chi_n(x)d^\times x^{\mathrm{tam}}\\
&= \sum_{d>m\geqslant -n+d} \chi_n(\varpi)^m (N\mathfrak{p})^{-ms}\int_{\mathfrak{o}^\times}\psi_k(-\varpi^m x)\chi_n(x)d^\times x^{\mathrm{tam}}\\
&= \sum_{d>m\geqslant -n+d} \chi_n(\varpi)^m (N\mathfrak{p})^{-ms}\sum_{a\in\mathfrak{o}^\times/1+\mathfrak{p}^n}\chi_n(a)\int_{1+\mathfrak{p}^n}\psi_k(-\varpi^m ax)d^\times x^{\mathrm{tam}}\\
&= \sum_{d>m\geqslant -n+d} \chi_n(\varpi)^m (N\mathfrak{p})^{-ms}\sum_{a\in\mathfrak{o}^\times/1+\mathfrak{p}^n}\chi_n(a)\psi_k(\varpi^m a)\int_{\mathfrak{p}^n}\psi_k(-\varpi^m ax)\frac{dx^{\mathrm{tam}}}{1-(N\mathfrak{p})^{-1}}
\end{aligned}
$$

The last integral vanishes if $m>-n+d$, as $m>-n+d$ implies $m+n>d$, so that $\mathfrak{p}^n \ni x\mapsto \psi_k(-\varpi^m ax)$ is nontrivial. Hence

$$
Z(f_n,\chi_n,s) = (N\mathfrak{p})^{(n-d)s}\sum_{a\in\mathfrak{o}^\times/1+\mathfrak{p}^n}\chi_n(\varpi^{-n+d}a)\psi_k(\varpi^{-n+d}a)\cdot\int_{1+\mathfrak{p}^n}d^\times x^{\mathrm{tam}}
$$

For a ramified quasi-character $\chi: k^\times\to\mathbb{C}^\times$, introduce the **Gauss sum**

$$
\mathfrak{g}(\chi)=\mathfrak{g}(\chi,\psi_k) := \sum_{a\in\mathfrak{o}^\times/1+\mathfrak{p}^n}\chi_n(\varpi^{-n+d}a)\psi_k(\varpi^{-n+d}a).
$$

If $c\in k^\times$ is such that $\mathrm{ord}_k\, c=-n+d$, then $\varpi^{-n+d}=uc$ for some $u\in\mathfrak{o}^\times$, and

$$
\sum_{a\in\mathfrak{o}^\times/1+\mathfrak{p}^n}\chi_n(\varpi^{-n+d}a)\psi_k(\varpi^{-n+d}a) = \sum_{a\in\mathfrak{o}^\times/1+\mathfrak{p}^n}\chi_n(c\cdot ua)\psi_k(c\cdot u^{-n+d}a) = \sum_{a\in\mathfrak{o}^\times/1+\mathfrak{p}^n}\chi_n(ca)\psi_k(ca)
$$

as u is a unit. In particular, this shows $\mathfrak{g}(\chi)$ is independent of the choice of uniformizer. Hence

$$
Z(f_n,\chi_n,s) = (N\mathfrak{p})^{(n-d)s}\mathfrak{g}(\chi_n)\cdot\int_{1+\mathfrak{p}^n}d^\times x^{\mathrm{tam}}
$$

and

$$
\gamma(s,\chi_n,\psi_k) = \frac{(N\mathfrak{d})^{\frac12}(N\mathfrak{p})^n}{(N\mathfrak{p})^{(n-d)s}\mathfrak{g}(\chi_n)}.
$$

By Theorem 7.1.15 to be proved, we have $\gamma(s,\chi,\psi_k)\gamma(1-s,\chi^{-1},\psi_k)=\chi(-1)$, so

$$
\frac{(N\mathfrak{d})^{\frac12}(N\mathfrak{p})^n}{(N\mathfrak{p})^{(n-d)s}\mathfrak{g}(\chi_n)}\cdot\frac{(N\mathfrak{d})^{\frac12}(N\mathfrak{p})^n}{(N\mathfrak{p})^{(n-d)(1-s)}\mathfrak{g}(\chi_n^{-1})} = \chi_n(-1)
$$

or

$$
\mathfrak{g}(\chi_n)\mathfrak{g}(\chi_n^{-1}) = \chi_n(-1)\frac{(N\mathfrak{d})(N\mathfrak{p})^{2n}}{(N\mathfrak{p})^{n-d}} = \chi_n(-1)(N\mathfrak{p})^n.
$$

Then we can rewrite $\gamma(s,\chi_n,\psi_k)$ as

$$
\gamma(s,\chi_n,\psi_k) = (N\mathfrak{p})^{-(n-d)s}\chi(-1)(N\mathfrak{d})^{\frac12}\mathfrak{g}(\chi_n^{-1}) = (N\mathfrak{p})^{-ns}(N\mathfrak{d})^{-s+\frac12}\chi_n(-1)\mathfrak{g}(\chi_n^{-1}).
$$

Hence, for a general quasi-character χ, we have

$$
\gamma(s,\chi,\psi_k) = \begin{cases} \dfrac{1}{\chi(\varpi)^d(N\mathfrak{d})^{s-\frac12}}\cdot\dfrac{(1-\chi(\varpi)^{-1}(N\mathfrak{p})^{s-1})^{-1}}{(1-\chi(\varpi)(N\mathfrak{p})^{-s})^{-1}} & \text{, if } \chi \text{ is unramified}\\[2ex] (N\mathfrak{p})^{-ns}(N\mathfrak{d})^{-s+\frac12}\chi(-1)\mathfrak{g}(\chi^{-1}) & \text{, if } \chi \text{ is ramified with conductor } n=c(\chi).\end{cases}
$$

7.1.4 Functional equation

Theorem 7.1.15. For all $f \in Z(k)$ and quasi-characters χ, the function $s \mapsto Z(f, \chi, s)$ admits a meromorphic continuation to $\mathbb{C}$ and satisfies the functional equation

$$Z(\widehat{f}, \chi^{-1}, 1-s) = \gamma(s, \chi, \psi_k) Z(f, \chi, s).$$

The function $\gamma(s, \chi, \psi_k)$ is called the γ-factor, and it satisfies the following properties

(i) $\gamma(s, \chi|\cdot|^{s_0}, \psi_k) = \gamma(s + s_0, \chi, \psi_k)$.

(ii) $\gamma(s, \chi, \psi_k)\gamma(1-s, \chi^{-1}, \psi_k) = \chi(-1)$.

(iii) $\gamma(s, \overline{\chi}, \psi_k) = \chi(-1)\overline{\gamma(s, \chi, \psi_k)}$.

(iv) $|\gamma(s, \chi, \psi_k)| = 1$ if $s + \mathrm{wt}(\chi) = \dfrac{1}{2}$.

Proof. By Lemma 7.1.12 we see $s \mapsto Z(f, \chi, s)$ is holomorphic when $\operatorname{Re} s + \mathrm{wt}(\chi) > 0$, and $s \mapsto Z(\widehat{f}, \chi^{-1}, 1-s)$ is holomorphic when $\operatorname{Re}(1-s) + \mathrm{wt}(\chi)^{-1} > 0$, or $1 > \operatorname{Re}(s) + \mathrm{wt}(\chi)$. But by Lemma 7.1.13, when $0 < \operatorname{Re}(s) + \mathrm{wt}(\chi) < 1$, we have

$$Z(\widehat{f}, \chi^{-1}, 1-s) = \gamma(s, \chi, \psi_k) Z(f, \chi, s),$$

where $\gamma(s, \chi, \psi_k)$ is computed in previous subsections. From the explicit formula of $\gamma(s, \chi, \psi_k)$, we see it is meromorphic on $\mathbb{C}$. Hence the functional equation provides $Z(f, \chi, s)$ a meromorphic continuation to the whole plane.

It remains to show the listed properties of $\gamma(s, \chi, \psi_k)$.

(i) This follows from $Z(f, \chi|\cdot|^{s_0}, s) = Z(f, \chi, s + s_0)$.

(ii) Keeping in mind that $\widehat{\widehat{f}}(x) = f(-x)$, we have

$$Z(\widehat{f}, \chi^{-1}, 1-s) = \gamma(s, \chi, \psi_k) Z(f, \chi, s) = \gamma(s, \chi, \psi_k)\gamma(1-s, \chi^{-1}, \psi_k)\chi(-1) Z(\widehat{f}, \chi^{-1}, 1-s).$$

So $\gamma(s, \chi, \psi_k)\gamma(1-s, \chi^{-1}, \psi_k)\chi(-1) = 1$.

(iii) We may assume s is real. Then $\overline{Z(f, \chi, s)} = \displaystyle\int_{k^\times} \overline{f}(x)\overline{\chi}|\cdot|^s d^\times x^{\mathrm{tam}} = Z(\overline{f}, \overline{\chi}, s)$, so

$$\gamma(s, \overline{\chi}, \psi_k)\overline{Z(f, \chi, s)} = Z(\widehat{\overline{f}}, \overline{\chi}^{-1}, 1-s).$$

But

$$\widehat{\overline{f}}(x) = \int_{k^\times} \overline{f}(x)\psi_k(-x) dx^{\mathrm{tam}} = \overline{\int_{k^\times} f(x)\psi_k(x) d^\times x^{\mathrm{tam}}} = \overline{\widehat{f}(-x)}.$$

Then

$$Z(\widehat{\overline{f}}, \overline{\chi}^{-1}, 1-s) = \overline{\chi}^{-1}(-1) Z(\overline{\widehat{f}}, \overline{\chi}^{-1}, 1-s) = \overline{\chi}(-1)\overline{Z(\widehat{f}, \chi^{-1}, 1-s)}.$$

Hence

$$\overline{Z(\widehat{f}, \chi^{-1}, 1-s)} = \overline{\chi}(-1)\gamma(s, \overline{\chi}, \psi_k)\overline{Z(f, \chi, s)}.$$

This shows $\overline{\chi}(-1)\gamma(s, \overline{\chi}, \psi_k) = \overline{\gamma(s, \chi, \psi_k)}$.

(iv) We have $2s + \mathrm{wt}(\chi) = 1$, so $|\cdot|^{2s}\chi\overline{\chi} = |\cdot|$, or $\overline{\chi} = \chi^{-1}|\cdot|^{1-2s}$. Then

$$\gamma(s, \overline{\chi}, \psi_k) = \gamma(s, \chi^{-1}|\cdot|^{1-2s}, \psi_k) \overset{(i)}{=} \gamma(s + 1 - 2s, \chi^{-1}, \psi_k) = \gamma(1-s, \chi^{-1}, \psi_k).$$

Now (iv) follows from (ii) and (iii).

$\square$

7.2 Distributions

7.2.1 Archimedean case

Let $U \subseteq \mathbb{R}^n$ be an open set. For any compact subset $K \subseteq U$, set

$$C_K(U) := \{f \in C(U) \mid \operatorname{supp} f \subseteq K\}$$

to be the spaces of continuous functions with support contained in K. Also, set $C_K^r(U) = C_K(U) \cap C^r(U)$ $(r = 0, 1, 2, \ldots, \infty)$. Clearly, we have

$$C_c(U) = \bigcup_{\substack{K \subseteq U \\ \text{cpt}}} C_K(U), \qquad C_c^r(U) = \bigcup_{\substack{K \subseteq U \\ \text{cpt}}} C_K^r(U).$$

We topologize $C_K^r(U)$ as follows. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}_{\geqslant 0})^n$ with $|\alpha| := \alpha_1 + \cdots + \alpha_n \leqslant r$, define for each $f \in C_K^r(U)$

$$\|f\|_{\alpha,K} := \sup_{x \in K} |D^\alpha f(x)| < \infty$$

where $D^\alpha f := \dfrac{\partial^r f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ is the α-th derivative of f. Topologize $C_K^r(U)$ with the initial topology induced by all $\|\cdot\|_{\alpha,K}$, $|\alpha| \leqslant r$. If $K \subseteq K' \subseteq U$, since $\|\cdot\|_{\alpha,K'}$ extends $\|\cdot\|_{\alpha,K}$, the natural inclusion $\iota_{K,K'} : C_K^r(U) \to C_{K'}^r(U)$ is a topological embedding. The set-theoretic colimit of the $C_K^r(U)$ is exactly $C_c^r(U)$. We equip $C_c^r(U)$ with the strict locally convex colimit topology. (c.f. §E.4.2.) Note this is strict as U admits a compact exhaustion, which is cofinal in all compact sets in U. In particular

Lemma 7.2.1. If $(f_n)_n \subseteq C_c^r(U)$ is a Cauchy sequence, then there exists a compact subset $K \subseteq U$ such that $(f_n)_n \subseteq C_K^r(U)$ and $(f_n)_n$ is Cauchy in $C_K^r(U)$.

Proof. This is Corollary E.4.5.1.(ii). □

Lemma 7.2.2. The canonical map

$$\begin{array}{ccc} C_c^\infty(\mathbb{R})^{\otimes n} & \longrightarrow & C_c^\infty(\mathbb{R}^n) \\ f_1 \otimes \cdots \otimes f_n & \longmapsto & (x_1, \ldots, x_n) \mapsto f_1(x_1) \cdots f_n(x_n) \end{array}$$

has dense image.

Proof. Let $f \in C_c^\infty(\mathbb{R}^n)$; we must approximate f by a sequence in $C_c^\infty(\mathbb{R})^{\otimes n}$. For this we can assume $\operatorname{supp} f \subseteq [-1, 1]^n$. Periodize f with period $[-\pi, \pi]^d$. Then we can represent f as Fourier series

$$f(x) = \sum_{k \in \mathbb{Z}^n} e^{ik \cdot x} \widehat{f}(k).$$

Define $g_m : \mathbb{R}^n \to \mathbb{C}$ by

$$g_m(x) = \sum_{|k| \leqslant n} e^{ik \cdot x} \widehat{f}(k).$$

Then $\|g_m - f\|_\infty \to 0$ as $m \to \infty$.

By §F.5 take $\phi \in C_c^\infty(\mathbb{R}^n)$ such that $\phi(x) = 1$ for $\|x\| \leqslant 1$, and take

$$f_m(x) := \sum_{|k| \leqslant m} e^{ik \cdot x} \prod_{j=1}^n \phi(x_j) \widehat{f}(k).$$

THen $f_m \to f$ in any

□

Let $0 \leqslant r < \infty$. The initial topology on $C_K^r(U)$ is in fact the same as the initial topology induced by the map $C_K^r(U) \ni f \mapsto \|f\|_{r,K} := \sum_{|\alpha| \leqslant r} \|f\|_{\alpha,K}$. Indeed, the former topology has a subbasis consisting of sets of the form $U_{\alpha,a,b} := \{f \in C_K^r(U) \mid \|f\|_{\alpha,K} \in (a,b) \cap [0,\infty)\}$, $-\infty < a < b \leqslant \infty$, and the latter topology has a subbasis consisting of sets of the form $U_{a,b} := \{f \in C_K^r(U) \mid \|f\|_{r,K} \in (a,b) \cap [0,\infty)\}$, $-\infty < a < b < \infty$. Clearly, the following containment

$$\bigcap_{\alpha} U_{\alpha,a_\alpha,b_\alpha} \subseteq U_{a,b}$$

holds with $a = \sum_\alpha a_\alpha$, $b = \sum_\alpha b_\alpha$. On the other hand, for $p \in C_K^r(U)$, set

$$U_{\alpha,b}(p) := \{f \in C_K^r(U) \mid \|f - p\|_{\alpha,K} < b\}.$$

If $p \in U_{\alpha,a,b}$, then $U_{\alpha,\epsilon}(p) \subseteq U_{\alpha,a,b}$ with $\epsilon < \min\{\|p\|_{\alpha,K}, \|p\|_{\alpha,K} - a,\, b - \|p\|_{\alpha,K}\}$. Set also that

$$U_{r,b}(p) := \{f \in C_K^r(U) \mid \|f - p\|_{r,K} < b\}.$$

If $p \in U_{r,a,b}$, then $U_{r,\epsilon}(p) \subseteq U_{r,a,b}$ with $\epsilon < \min\{\|p\|_{r,K}, \|p\|_{r,K} - a,\, b - \|p\|_{r,K}\}$. Now clearly $U_{r,b}(p) \subseteq U_{\alpha,b}(p)$, so the two topologies on $C_K^r(U)$ are the same. Since $\|f\|_{r,K}$ is a norm, it follows that $C_K^r(U)$ is a normed vector space.

Lemma 7.2.3. For $0 \leqslant r \leqslant \infty$ and $K \subseteq U$ compact, the space $C_K^r(U)$ is a Banach space.

Let $0 \leqslant r < \infty$ and let $u : C_c^r(U) \to \mathbb{C}$ be a continuous linear functional. By Corollary E.4.3.1, this is the same as saying that each $u \circ f_K : C_K^r(U) \to \mathbb{C}$ is continuous, so it is a bounded linear operator, i.e.,

$$|u(\phi)| \leqslant C \|\phi\|_{r,K} = C \sum_{|\alpha| \leqslant r} \|\phi\|_{\alpha,K}$$

holds for all $\phi \in C_K^r(U)$.

Proposition 7.2.4. Let $u : C_c^\infty(U) \to \mathbb{C}$ be a linear functional. TFAE:

(i) u is continuous.

(ii) For each compact set $K \subseteq U$, there exists $m = m_K \in \mathbb{Z}_{\geqslant 0}$ and $C = C_K > 0$ such that

$$|u(\phi)| \leqslant C \sum_{|\alpha| \leqslant m} \|\phi\|_{\alpha,K}$$

holds for all $\phi \in C_K^\infty(U)$.

In either case, we say u is a **generalized function/distribution** on U, and we sometimes refer to elements in $C_c^\infty(U)$ as test functions on U.

Proof. For each $0 \leqslant r < \infty$, let $\rho_r : C_K^\infty(U) \to C_K^\infty(U)$ denote the identity map with the codomain $C_K^\infty(U)$ equipped with the subspace topology induced by $C_K^\infty(U) \subseteq C_K^r(U)$. By construction, ρ_r is continuous. The condition (ii) means that $u \circ f_K$ is continuous when $C_K^\infty(U)$ is equipped with the subspace topology from $C_K^m(U)$, so by precomposing with ρ_r we see $u \circ f_K$ is continuous for each K. This proves (i).

Now assume (i). Hence $u \circ f_K : C_K^\infty(U) \to \mathbb{C}$ is continuous for any compact $K \subseteq U$. In particular, it is continuous at $0 \in C_K^\infty(U)$. Observe that $U_{m,b} \cap C_K^\infty(U)\,(b > 0)$ forms a neighborhood basis of

0 in $C_K^\infty(U)$. So by fixing some $C > 0$ we can find some $m \in \mathbb{Z}_{\geqslant 0}$ and $b > 0$ such that $\|\phi\|_{m,K} < b$ implies $|u(\phi)| \leqslant C$. But for any $\phi \in C_K^\infty(U)$, as $\left\|\frac{b\phi}{2\|\phi\|_{m,K}}\right\|_{m,K} = \frac{b}{2} < b$,

$$\left|u\left(\frac{b\phi}{2\|\phi\|_{m,K}}\right)\right| \leqslant C$$

or

$$|u(\phi)| \leqslant \frac{2C}{b}\|\phi\|_{m,K} = \frac{2C}{b}\sum_{|\alpha| \leqslant m}\|\phi\|_{\alpha,K}.$$

□

Example 7.2.5.

1. Consider the space

$$L^1_{\text{loc}}(U) := \{f : U \to \mathbb{C} \mid f|_K \in L^1(K) \text{ for all } K \underset{\text{cpt}}{\subseteq} U\}.$$

   Then each $f \in L^1_{\text{loc}}(U)$ defines a distribution u_f by integration

$$u_f(\phi) = \int_U f\phi dx$$

   The integral exists if $\phi \in C_c(U)$, and for each compact $K \subseteq U$ and $\phi \in C_K(U)$, one has

$$|u_f(\phi)| = \left|\int_U f\phi dx\right| = \left|\int_K f\phi dx\right| \leqslant \|f|_K\|_{L^1}\|\phi\|_{\text{sup}}.$$

   More generally (in the sense of Radon-Nykodim), if μ is a σ-finite positive measure on U, integration against μ

$$\phi \mapsto \int_U \phi d\mu$$

   defines a distribution.

2. For any $\alpha \in (\mathbb{Z}_{\geqslant 0})^n$ and $x_0 \in U$, the linear map

$$\phi \mapsto D^\alpha\phi(x_0), \qquad \phi \in C_c^\infty(U)$$

   defines a distribution. If $|\alpha| \geqslant 1$, this is not given by any complex measure on U, as it is not a bounded linear functional of $C_0^\infty(U)$ (e.g. consider the bump functions). When $|\alpha| = 0$, it is the so-called Dirac measure concentrated at x_0.

Definition. Let $u : C_c^\infty(U) \to \mathbb{C}$ be a distribution.

(i) For any $\alpha \in (\mathbb{Z}_{\geqslant 0})^n$, define the $D^\alpha u$ to be the unique distribution satisfying

$$D^\alpha u(\phi) = (-1)^{|\alpha|}u(D^\alpha\phi), \qquad \phi \in C_c^\infty(U).$$

This is called the **α-th distributional derivative** of u.

(ii) The **support** $\operatorname{supp} u$ is the complement of the largest open set $\Omega \subseteq U$ such that $u|_{C_c^\infty(\Omega)} \equiv 0$. Here the inclusion $C_c^\infty(\Omega) \to C_c^\infty(U)$ is the extension by zero.

(iii) For each compact $K \subseteq U$, denote by $C_c^\infty(U)_K^\vee$ the subspace of distributions with support contained in K.

There is larger space containing $C_c^\infty(\mathbb{R}^n)$, i.e., the Schwartz space $S(\mathbb{R}^n)$. Recall that it is topologized by the initial topology given by the functions

$$f \mapsto \|f\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta f(x)|$$

where $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Lemma 7.2.6. The inclusion $C_c^\infty(\mathbb{R}^n) \subseteq S(\mathbb{R}^n)$ is continuous and dense.

Proof. To show the inclusion $C_c^\infty(\mathbb{R}^n) \subseteq S(\mathbb{R}^n)$ is continuous, we must show $\|\cdot\|_{\alpha,\beta} : C_c^\infty(\mathbb{R}) \to \mathbb{R}_{\geqslant 0}$ is continuous for each $\alpha, \beta \in (\mathbb{Z}_{\geqslant 0})^n$. By Lemma E.4.3 it suffices to show $\|\cdot\|_{\alpha,\beta} : C_K^\infty(\mathbb{R}^n) \to \mathbb{R}_{\geqslant 0}$ is continuous for each compact subset $K \subseteq \mathbb{R}^n$. For $f, g \in C_K^\infty(\mathbb{R}^n)$, trivially we have

$$\|f-g\|_{\alpha,\beta} \leqslant \sup_{x\in K} |x^\alpha| \cdot \|f-g\|_{\beta,K}.$$

This proves the continuity.

For density, by §F.5 take $\phi \in C_c^\infty(\mathbb{R}^n)$ such that $\phi(x) = 1$ for $\|x\| \leqslant 1$. Let $f \in S(\mathbb{R}^n)$ and for each $\varepsilon > 0$ put $f_\varepsilon(x) = f(x)\phi(\varepsilon x)$. Then $f_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ and

$$f_\varepsilon(x) - f(x) = f(x)(\phi(\varepsilon x) - 1) = 0$$

if $\|x\| < \dfrac{1}{\varepsilon}$. Then for $\alpha, \beta \in (\mathbb{Z}_{\geqslant 0})^n$

$$\|f_\varepsilon - f\|_{\alpha,\beta} \leqslant \left(\sup_{x\in\mathbb{R}^n} |\phi(x)| + 1\right) \sup_{\|x\|>\varepsilon^{-1}} |x^\alpha D^\beta f(x)| \leqslant \left(\sup_{x\in\mathbb{R}^n} |\phi(x)| + 1\right) \|f\|_{\alpha',\beta}\, \varepsilon^2$$

where $\alpha' = \alpha + (2, \ldots, 2)$. This finishes the proof. ∎

For $k \geqslant 0$, define $\|\cdot\|_k : S(\mathbb{R}^n) \to \mathbb{R}_{\geqslant 0}$ by

$$\|f\|_k := \sup_{x\in\mathbb{R}^n} (1 + \|x\|^2)^{\frac{k}{2}} \sum_{|\alpha|\leqslant k} |D^\alpha f(x)|$$

The $\|\cdot\|_k$ define norms on $S(\mathbb{R}^n)$. Trivial estimate and Cauchy inequality yield

$$(1 + \|x\|^2)^{\frac{k}{2}} \leqslant \sum_{|\alpha|\leqslant k} |x^\alpha| = (1 + |x_1| + \cdots + |x_n|)^k \leqslant (n+1)^k (1 + \|x\|^2)^{\frac{k}{2}}$$

In particular, this gives

$$(n+1)^k \|f\|_k \geqslant \sum_{|\alpha|,|\beta|\leqslant k} \|f\|_{\alpha,\beta} \geqslant \|f\|_k.$$

Hence the topology defined by $\|f\|_k$ is the same as the one defined by $\sum_{|\alpha|,|\beta|\leqslant k} \|f\|_{\alpha,\beta}$, which turns out to be the same as that induced by $\|f\|_{\alpha,\beta}$, $|\alpha|, |\beta| \leqslant k$ (this can be proved by a similar argument to the case $C_K^r(U)$). Hence the topology on $S(\mathbb{R}^n)$ is the same as the initial topology given by the norms $\|f\|_k$, $0 \leqslant k < \infty$. Note also that $\|f\|_k \leqslant \|f\|_\ell$ as long as $k \leqslant \ell$.

Let $u : S(\mathbb{R}^n) \to \mathbb{C}$ be a continuous linear functional. In particular, it is continuous at 0, so by fixing $C > 0$ there exists $k_1 < \cdots < k_\ell =: k \in \mathbb{Z}_{\geqslant 0}$ and $b_1, \ldots, b_\ell > 0$ such that if $\phi \in S(\mathbb{R}^n)$ satisfies

$\|\phi\|_{k_\ell} < b_\ell$, $i = 1, \dots, m$, then $|u(\phi)| \leqslant C$. In particular, if $\|\phi\|_{k_\ell} < b := \min_{1 \leqslant i \leqslant \ell} b_i$, then $|u(\phi)| \leqslant C$. For any ϕ, we have $\left\| \frac{b\phi}{2\|\phi\|_k} \right\|_k < \frac{b}{2} < b$, so

$$|u(\phi)| \leqslant \frac{2C}{b} \|\phi\|_k .$$

The converse holds trivially.

Definition. Let $k \in \mathbb{Z}_{\geqslant 0}$. A tempered distribution $u \in S(\mathbb{R}^n)^\vee$ is **of order** $\leqslant k$ if there exists $C > 0$ such that $|u(\phi)| \leqslant C \|\phi\|_k$ for all $\phi \in S(\mathbb{R}^n)$.

Proposition 7.2.7. The Fourier transform $\widehat{\cdot} : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ is an topological vector space isomorphism.

Proof. By Lemma 7.1.14 and Fourier inversion, it remains to show $\widehat{\cdot} : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ is continuous. In Lemma 7.1.14 we see for $\alpha, \beta \in (\mathbb{Z}_{\geqslant 0})^n$ and $f \in S(\mathbb{R}^n)$ that

$$x^\alpha D^\beta \widehat{f}(x) = (-2\pi i)^{|\alpha|} (2\pi i)^{|\beta|} D^\alpha \widehat{(x^\beta f(x))}.$$

Then

$$\begin{aligned} \left\| \widehat{f} \right\|_{\alpha,\beta} &\leqslant (2\pi i)^{|\alpha+\beta|} \int_{\mathbb{R}^n} (1 + \|x\|^2)^{-n} (1 + \|x\|^2)^n D^\alpha (x^\beta f(x)) dx \\ &\leqslant (2\pi i)^{|\alpha+\beta|} \int_{\mathbb{R}^n} (1 + \|x\|^2)^{-n} dx \cdot \sup_{x \in \mathbb{R}^n} |(1 + \|x\|^2)^n D^\alpha (x^\beta f(x))| \end{aligned}$$

The last sup is bounded by a finite sum of various $\|f\|_{\alpha',\beta'}$. This proves the continuity. $\square$

Definition. A continuous linear function $u : S(\mathbb{R}^n) \to \mathbb{C}$ is called a **tempered distribution** on $\mathbb{R}^n$.

Definition. Let $u : S(\mathbb{R}^n) \to \mathbb{C}$ be a tempered distribution.

(i) For any $\alpha \in (\mathbb{Z}_{\geqslant 0})^n$, define the **$\alpha$-th distributional derivative** of u to be the unique distribution satisfying

$$D^\alpha u(\phi) = (-1)^{|\alpha|} u(D^\alpha \phi), \qquad \phi \in S(\mathbb{R}^n).$$

(ii) The **Fourier transform** $\widehat{u}$ of u is the functional on $S(\mathbb{R}^n)$ such that

$$\widehat{u}(\phi) = u(\widehat{\phi}), \qquad \phi \in S(\mathbb{R}^n).$$

(iii) The **support** $\operatorname{supp} u$ is the complement of the largest open set $\Omega \subseteq U$ such that $u(\phi) = 0$ for all $\phi \in S(\mathbb{R})$ with $\operatorname{supp} \phi \subseteq \Omega$.

(iv) For each compact $K \subseteq U$, denote by $S(\mathbb{R}^n)^\vee_K$ the subspace of tempered distributions with support contained in K.

Lemma 7.2.8. For $p \in \mathbb{R}^n$, the tempered distributions with support contained in $\{p\}$ is

$$S(\mathbb{R}^n)^\vee_p = \left\{ \sum_{\alpha \in (\mathbb{Z}_{\geqslant 0})^n} c_\alpha D^\alpha|_p \mid c_\alpha = 0 \text{ for almost all } \alpha \in (\mathbb{Z}_{\geqslant 0})^n \right\} = \bigoplus_{\alpha \in (\mathbb{Z}_{\geqslant 0})^n} D^\alpha|_p.$$

Proof. For convenience, assume $p = 0$ is the origin. Let $u \in S(\mathbb{R}^n)_p^\vee$ and suppose u is of order $\leqslant k$ for some $k \in \mathbb{Z}_{\geqslant 0}$. Hence there exists $C > 0$ such that $|u(f)| \leqslant C \|f\|_k$ for all $f \in S(\mathbb{R}^n)$.

We claim $\{f \in S(\mathbb{R}) \mid 0 \notin \operatorname{supp} f\}$ is dense in $\{f \in S(\mathbb{R}^n) \mid D^\alpha f(0) = 0 \text{ for all } |\alpha| \leqslant k+1\}$ with respect to the topology induced by $\|\cdot\|_k$. The former set lies the latter as f/x^α is well-defined for $0 \notin \operatorname{supp} f$. Assuming this, by continuity we see $u(f) = 0$ for $f \in S(\mathbb{R}^n)$ with $D^\alpha f(0) = 0$, $|\alpha| \leqslant k+1$. For each $g \in S(\mathbb{R}^n)$, then

$$u(g) = u\left(\sum_{|\alpha|\leqslant k} \frac{D^\alpha g(0)}{\alpha!} x^\alpha\right) + u\left(g(x) - \sum_{|\alpha|\leqslant k} \frac{D^\alpha g(0)}{\alpha!} x^\alpha\right) = u\left(\sum_{|\alpha|\leqslant k} \frac{D^\alpha g(0)}{\alpha!} x^\alpha\right) = \sum_{|\alpha|\leqslant} \frac{x^\alpha}{\alpha!} D^\alpha g(0).$$

This proves $u = \sum_{|\alpha|\leqslant k} \frac{x^\alpha}{\alpha!} D^\alpha|_0$, as claimed.

It remains to show the density. Let $\phi \in C_c^\infty(\mathbb{R})$ be such that $0 \leqslant \phi \leqslant 1$, $\phi(x) = 1$ for $\|x\| \leqslant 1$ and $\phi(x) = 0$ for $\|x\| \geqslant 2$. For $0 < \varepsilon \leqslant 1$ put $\phi_\varepsilon(x) = \phi(\varepsilon^{-1}x)$. For $g \in S(\mathbb{R}^n)$ and $|\beta| \leqslant k$, compute

$$D^\beta g - D^\beta(g(1-\phi_\varepsilon)) = D^\beta(g\phi_\varepsilon) = \sum_{\gamma\leqslant\beta} \binom{\beta}{\gamma} D^\gamma g \cdot D^{\beta-\gamma}\phi_\varepsilon$$

and hence

$$\sum_{|\beta|\leqslant k} \left|D^\beta g - D^\beta(g(1-\phi_\varepsilon))\right| \leqslant \sum_{|\gamma|\leqslant k} |D^\gamma g(x)| \sum_{\substack{\gamma\leqslant\beta \\ |\beta|\leqslant k}} \binom{\beta}{\gamma} |D^{\beta-\gamma}\phi_\varepsilon(x)| \leqslant c_k \sum_{|\gamma|\leqslant k} |D^\gamma g(x)| \mathbf{1}_{B_{2\varepsilon}(0)}(x)\varepsilon^{|\gamma|-k}$$

for some constant c_k. We claim

$$\lim_{\varepsilon\to 0} \sup_{\|x\|<2\varepsilon} (1+\|x\|^2)^{\frac{k}{2}} |D^\gamma f(x)|\varepsilon^{|\gamma|-k} = 0$$

for $|\gamma| \leqslant k$ and $f \in S(\mathbb{R}^n)$ with $D^\alpha f(0) = 0$, $|\alpha| \leqslant k+1$. If $|\gamma| = k$, this follows from uniform continuity of $D^\gamma f$ and from $D^\gamma f(0) = 0$. If $|\gamma| < k$, for such f, by Taylor's expansion we have

$$|D^\gamma f(x)| \leqslant \frac{1}{(k-|\gamma|)!} \sup_{0<t<1} \left| \left(\frac{d}{dt}\right)^{k-|\gamma|} (D^\gamma f)(tx) \right|.$$

Since $k - |\gamma| > 0$, the RHS has at least a factor x, and hence the limit goes to 0. $\square$

Now we turn to our main topic. For $s \in \mathbb{C}$ with $\operatorname{Re} s > -1$, the function $\mathbf{1}_{\geqslant 0}(x)x^s$ defines a tempered distribution $I_s^+ : S(\mathbb{R}) \to \mathbb{C}$ by integration:

$$I_s^+(\phi) := \int_0^\infty x^s \phi(x)dx.$$

Fix a $\phi \in S(\mathbb{R})$ and consider the function $s \mapsto I_s(\phi)$. By Lemma 7.1.12 and Lemma 7.1.14, it is a holomorphic function. Using integration by part, for $\operatorname{Re} s > 0$, we have

$$I_s^+(\phi) = -sI_{s-1}(\phi').$$

Iterating gives

$$I_s^+(\phi) = \frac{(-1)^k}{(s+1)\cdots(s+k)} I_{s+k}(\phi^{(k)}) \qquad (\spadesuit)$$

with $\operatorname{Re} s > -1$ and $k \in \mathbb{Z}_{\geqslant 0}$. This provides a meromorphic continuation of $s \mapsto I_s^+(\phi)$ to $s \in \mathbb{C}\backslash\mathbb{Z}_{<0}$:

$$I_s^+(\phi) = \frac{(-1)^k}{(s+1)\cdots(s+k)} I_{s+k}(\phi^{(k)})$$

where k is any integer with $k > \max\{0, -1-\operatorname{Re} s\}$. The formula (♠) shows this is well-defined, and it has simple pole at each negative integer $-m$ ($m \in \mathbb{Z}_{>0}$) with residue

$$\begin{aligned}\lim_{s\to -m}(s+m)I_s^+(\phi) &= \lim_{s\to -m}\frac{(-1)^k(s+m)}{(s+1)\cdots(s+k)}I_{s+k}(\phi^{(k)})\\ &\overset{\text{pick } k=m}{=} \frac{-1}{(m-1)!}I_0(\phi^{(m)}) = \frac{1}{(m-1)!}\phi^{(m-1)}(0)\end{aligned}$$

In turn of distribution, this reads

$$\lim_{s\to -m}(s+m)I_s^+ = \frac{(-1)^{m-1}}{(m-1)!}\delta_0^{(m)}$$

where δ_0 is the Dirac measure concentrated at 0 and $\delta_0^{(m-1)}$ is its m-th distributional derivative. To eliminate the poles, define $I_s^+ : S(\mathbb{R}) \to \mathbb{C}$ by

$$\mathcal{I}_s^+(\phi) = \frac{1}{\Gamma(s+1)}I_s^+(\phi).$$

It is $s+1$ because $\displaystyle\lim_{s\to -m}(s+m)\Gamma(s) = \frac{(-1)^m}{m!}$ for $m \in \mathbb{Z}_{\geqslant 0}$. Hence, for $\phi \in S(\mathbb{R}^n)$,

$$\lim_{s\to -m}\mathcal{I}_s^+(\phi) = \lim_{s\to -m}\frac{(s+m)I_s^+(\phi)}{(s+m)\Gamma(s+1)} = \phi^{(m-1)}(0).$$

Since Γ has no zero, it follows that $s \mapsto \mathcal{I}_s^+(\phi)$ is an entire function.

Similarly, for each $\operatorname{Re} s > -1$ the function $\mathbf{1}_{<0}(x)|x|^s$ defines a tempered distribution I_s^- by integration:

$$I_s^-(\phi) = \int_{-\infty}^0 |x|^s\phi(x)dx = \int_0^\infty x^s\phi(-x)dx = I_s^+(\widetilde{\phi})$$

where $\widetilde{\phi}(x) := \phi(-x)$. It follows that $\mathcal{I}_s^- := \dfrac{1}{\Gamma(s+1)}I_s^-$ satisfies $\mathcal{I}_s^-(\phi) = \mathcal{I}_s^+(\widetilde{\phi})$, so $s \mapsto \mathcal{I}_s^-(\phi)$ defines an entire function for each fixed $\phi \in S(\mathbb{R})$.

Finally, consider the tempered distribution I_s defined by the function $|x|^s$, i.e.,

$$I_s(\phi) = \int_{\mathbb{R}}|x|^s\phi(x)dx = I_s^-(\phi) + I_s^+(\phi).$$

Unlike I_s^+ and I_s^-, it has fewer poles; indeed, $\operatorname{res}_{s=-m} I_s^+(\phi) = \dfrac{(-1)^{m-1}}{(m-1)!}\phi^{(m-1)}(0)$, while $\operatorname{res}_{s=-m} I_s^-(\phi) = \dfrac{1}{(m-1)!}\phi^{(m-1)}(0)$. Hence

$$\operatorname{res}_{s=-m} I_s(\phi) = \begin{cases} 0 & \text{, if } m = -2, -4, \ldots \\ \dfrac{2}{(m-1)!}\phi^{(m-1)}(0) & \text{, if } m = -1, -3, -5, \ldots \end{cases}$$

To eliminate the poles, this time we use $s \mapsto \Gamma\left(\dfrac{s+1}{2}\right)$, which has simple poles at $s = -(2j+1)$ ($j \in \mathbb{Z}_{\geqslant 0}$) with residue $\dfrac{2(-1)^j}{j!}$. Hence the function $s \mapsto \mathcal{I}_s$ given by

$$\mathcal{I}_s(\phi) := \frac{1}{\Gamma\left(\frac{s+1}{2}\right)}I_s(\phi)$$

is an entire family of tempered distributions. In addition, we have $\mathcal{I}_{-(2j+1)} = \dfrac{(-1)^j j!}{(2j)!}\delta_0^{(2j)}$.

We return to the local zeta integral. For each $\phi \in S(\mathbb{R})$, we have

$$Z(\phi, \mathbf{1}, s) = \int_{\mathbb{R}^\times} \phi(x)|x|^s d^\times x = I_{s-1}(\phi).$$

The result in the last paragraph says that

$$s \mapsto \frac{Z(\cdot, \mathbf{1}, s)}{\Gamma\left(\frac{s}{2}\right)}$$

defines a entire family of tempered distributions with

$$\left.\frac{Z(\cdot, \mathbf{1}, s)}{\Gamma\left(\frac{s}{2}\right)}\right|_{s=-2j} = \frac{(-1)^j j!}{(2j)!}\delta_0^{(2j)}, \qquad j \in \mathbb{Z}_{\geqslant 0}$$

as elements in $S(\mathbb{R})^\vee$. Next consider $\mathrm{sign} : \mathbb{R}^\times \to \{\pm 1\}$; we have

$$Z(\phi, \mathrm{sign}, s) = \int_0^\infty \phi(x)|x|^s d^\times x - \int_{-\infty}^0 \phi(x)|x|^s d^\times x = I_{s-1}^+(\phi) - I_{s-1}^-(\phi).$$

The function $s \mapsto I_s^+(\phi) - I_s^-(\phi)$ has simple poles at $s = -2j$ ($j \in \mathbb{Z}_{\geqslant 1}$) with residue $\dfrac{-2}{(2j-1)!}\phi^{(2j-1)}(0)$. On the other hand, the function $s \mapsto \Gamma(\frac{s}{2}+1)$ has simple poles at $s = -2j$ ($j \in \mathbb{Z}_{\geqslant 1}$) with residue $\dfrac{2(-1)^{j-1}}{(j-1)!}$, so the distribution

$$s \mapsto \frac{I_s^+ - I_s^-}{\Gamma\left(\frac{s}{2}+1\right)}$$

is entire with value at $s = -2j$ being $\dfrac{(-1)^{j-1}(j-1)!}{(2j-1)!}\delta_0^{(2j-1)}$. Applied to our case, we obtain that

$$s \mapsto \frac{Z(\cdot, \mathrm{sign}, s)}{\Gamma\left(\frac{s+1}{2}\right)}$$

is a entire family of tempered distributions with

$$\left.\frac{Z(\cdot, \mathrm{sign}, s)}{\Gamma\left(\frac{s+1}{2}\right)}\right|_{s=-(2j-1)} = \frac{(-1)^{j-1}(j-1)!}{(2j-1)!}\delta_0^{(2j-1)}.$$

Generally, let $\chi : \mathbb{R}^\times \to \mathbb{C}^\times$ be a quasi-character; then $\chi = \mathrm{sign}^\epsilon |\cdot|^{s_0}$ for some $\epsilon \in \{0,1\}$ and $s_0 \in \mathbb{C}$. Then $Z(\cdot, \chi, s) = Z(\cdot, \mathrm{sign}^\epsilon, s+s_0)$, so $\dfrac{Z(\cdot, \chi, s)}{\Gamma\left(\frac{s+s_0+\epsilon}{2}\right)}$ is a entire family of tempered distributions with value at $s = -s_0 - 2j - \epsilon$ ($j \in \mathbb{Z}_{\geqslant 0}$) being $\dfrac{(-1)^j j!}{(2j+\epsilon)!}\delta_0^{(2j+\epsilon)}$. In summary,

Theorem 7.2.9. For any quasi-character $\chi : \mathbb{R}^\times \to \mathbb{C}$, the map $s \mapsto \dfrac{Z(\cdot, \chi, s)}{L(s, \chi)}$ defines a entire family of tempered distribution with

$$\left.\frac{Z(\cdot, \chi, s)}{L(s, \chi)}\right|_{s=-s_0-2j-\epsilon} = \frac{(-1)^j \pi^j j!}{(2j+\epsilon)!}\delta_0^{(2j+\epsilon)} \qquad (j \in \mathbb{Z}_{\geqslant 0}).$$

where $\chi = \mathrm{sign}^\epsilon |\cdot|^{s_0}$ with $\epsilon \in \{0,1\}$, $s_0 \in \mathbb{C}$.

We've seen that there exists $\phi \in S(\mathbb{R})$ such that $Z(\phi, \chi, s) = L(s, \chi)$. With this in mind, the local factor $L(s, \chi)$ can be viewed as the "GCD" of all zeta integrals $Z(\phi, \chi, s)$. By the functional equation, there exists a unique function $\epsilon(s, \chi, \psi_k)$ satisfying

$$\frac{Z(\widehat{f}, \chi^{-1}, 1-s)}{L(1-s, \chi^{-1})} = \epsilon(s, \chi, \psi_k) \frac{Z(f, \chi, s)}{L(s, \chi)}.$$

This is called the **ϵ-factor**. By (7.1.3), we have $\epsilon(s, \chi, \psi_k) = i^{3\operatorname{sign}(\chi)}$, where $\operatorname{sign}(\chi) \in \{0, 1\}$ is the parity of χ, i.e., $\chi(-1) = (-1)^{\operatorname{sign}(\chi)}$.

We state without proof for the result when $k = \mathbb{C}$. For any quasi-character $\chi : \mathbb{C}^\times \to \mathbb{C}^\times$, the map $S(\mathbb{C}) \ni \phi \mapsto \dfrac{Z(\phi, \chi, s)}{L(s, \chi)}$ defines an entire family of tempered distribution. It satisfies the functional equation

$$\frac{Z(\widehat{f}, \chi^{-1}, 1-s)}{L(1-s, \chi^{-1})} = \epsilon(s, \chi, \psi_k) \frac{Z(f, \chi, s)}{L(s, \chi)},$$

with $\epsilon(s, \chi, \psi_k) = (-i)^{|n|}$, where n is the unique integer such that $\chi|_{S^1} = [x \mapsto x^n]$. The function $\epsilon(s, \chi, \psi_k)$ is again called the **ϵ-factor**.

7.2.2 Non-archimedean case I

The non-archimedean case is easier. Let k be a local field; recall that the space $S(k)$ of Bruhat-Schwartz functions on k consists of all complex-valued locally constant functions with compact support. We topologize $S(k)$ by weak topology, namely, equip it with the initial topology with respect to $S(k)^\vee = \operatorname{Hom}_{\mathbb{C}}(S(k), \mathbb{C})$. Although the topology is defined, we barely use them so one can simply ignore it.

Definition. An element in $S(k)^\vee$ is called a **(tempered) distribution** on $S(k)$.

Fix a quasi-character $\chi : k^\times \to \mathbb{C}^\times$. For $\phi \in S(k)^\vee$ and $s \in \mathbb{C}$, consider the zeta integral

$$Z(\phi, \chi, s) = \int_{k^\times} \phi(x) \chi |\cdot|^s(x) d^\times x^{\mathrm{tam}}.$$

The computation in Lemma 7.1.14 shows that for each fixed $\phi \in S(k)$, the zeta integral $Z(\phi, \chi, s)$ defines an element in $\mathbb{C}(q^{-s})$ with $q = N\mathfrak{p}$; in particular, it defines an meromorphic function in s. Consider the $\mathbb{C}$ vector subspace

$$W = \operatorname{span}_{\mathbb{C}}\{Z(\phi, \chi, s) \mid \phi \in S(k)\} \subseteq \mathbb{C}(q^{-s}).$$

Lemma 7.2.10. *W is a $\mathbb{C}[q^s, q^{-s}]$-submodule.*

Proof. Let $\phi \in S(k)$. Let ϖ be a uniformizer of k and consider the function $\phi' \in S(k)$ given by $\phi'(x) = \phi(\varpi x)$. Then

$$Z(\phi', \chi, s) = \int_{k^\times} \phi(\varpi x) \chi |\cdot|^s(x) d^\times x^{\mathrm{tam}} = \int_{k^\times} \phi(x) \chi |\cdot|^s(x\varpi^{-1}) d^\times x^{\mathrm{tam}} = \chi(\varpi^{-1}) q^{-s} Z(\phi, \chi, s),$$

so that

$$q^{-s} Z(\phi, \chi, s) = Z(\chi(\varpi)\phi', \chi, s) \in W.$$

Replacing ϖ by ϖ^n, $n \in \mathbb{Z}$ shows $q^{-ns} Z(\phi, \chi, s) \in W$. □

The computation in (7.1.3) shows that $(1-\chi(\varpi)q^{-s})Z(\phi,\chi,s) \in \mathbb{C}[q^{-s}]$ for all $\phi \in S(k)$. This implies

Corollary 7.2.10.1. W is a fractional ideal of $\mathbb{C}[q^s,q^{-s}]$.

Since $\mathbb{C}[q^s,q^{-s}]$ is a PID, there exist $P,Q \in \mathbb{C}[X]$ such that $0 \neq \dfrac{P}{Q}(q^{-s})$ is a generator of the fractional ideal W. Since q^{-s} is a unit in $\mathbb{C}[q^s,q^{-s}]$, we may assume $Q(0)=1=P(0)$ and $\gcd(P,Q)=1$; such a generator is unique. Again, it follows from (7.1.3) that $P \equiv 1$ and $Q(X) = \begin{cases} 1-\chi(\varpi)X & \text{, if } \chi \text{ is unramified} \\ 1 & \text{, if } \chi \text{ is ramified} \end{cases}$. Define the **local L-factor**

$$L(s,\chi) := Q(q^{-s})^{-1} = \begin{cases} \dfrac{1}{1-\chi(\varpi)(N\mathfrak{p})^{-s}} & \text{, if } \chi \text{ is unramified} \\ 1 & \text{, if } \chi \text{ is ramified} \end{cases}.$$

Then the map

$$S(k) \ni \phi \mapsto \frac{Z(\phi,\chi,s)}{L(s,\chi)} \in \mathbb{C}$$

defines a entire family of tempered distribution. By the functional equation, there exists a unique function $\epsilon(s,\chi,\psi_k)$, again called the **ϵ-factor**, satisfying

$$\frac{Z(\widehat{f},\chi^{-1},1-s)}{L(1-s,\chi^{-1})} = \epsilon(s,\chi,\psi_k)\frac{Z(f,\chi,s)}{L(s,\chi)}.$$

By the computation in (7.1.3), we see

$$\epsilon(s,\chi,\psi_k) = \begin{cases} \dfrac{1}{\chi(\varpi)^d(N\mathfrak{d})^{s-\frac{1}{2}}} & \text{, if } \chi \text{ is unramified} \\ (N\mathfrak{p})^{-ns}(N\mathfrak{d})^{-s+\frac{1}{2}}\chi(-1)\mathfrak{g}(\chi^{-1}) & \text{, if } \chi \text{ is ramified with conductor } n=c(\chi). \end{cases}$$

where $d \in \mathbb{Z}$ is such that $\mathfrak{d}^{-1} = \mathfrak{p}^d$.

We know $L(s,\chi) = Z(\phi,\chi,s)$ for some $\phi \in S(k)$. In fact,

Lemma 7.2.11. Let χ be an unramified quasi-character. Then

$$Z(\mathbf{1}_{\mathfrak{o}},\chi,s) = (N\mathfrak{d})^{-\frac{1}{2}}L(s,\chi), \qquad Z(\widehat{\mathbf{1}_{\mathfrak{o}}},\chi^{-1},1-s) = \chi(\varpi)^{-d}(N\mathfrak{p})^{ds}L(1-s,\chi^{-1})$$

Proof. Recall in Lemma 7.1.8 that $\mathrm{vol}(\mathfrak{u},d^\times x^{\mathrm{tam}}) = (N\mathfrak{d})^{-\frac{1}{2}}$. Hence

$$\begin{aligned} Z(\mathbf{1}_{\mathfrak{o}},\chi,s) &= \int_{k^\times} \mathbf{1}_{\mathfrak{o}}(x)\chi|\cdot|^s(x)d^\times x^{\mathrm{tam}} = \sum_{n=0}^{\infty}\int_{\varpi^n\mathfrak{u}} \chi|\cdot|^s(x)d^\times x^{\mathrm{tam}} \\ &= \sum_{n=0}^{\infty}\chi(\varpi)^n(N\mathfrak{p})^{-ns}\,\mathrm{vol}(\mathfrak{u},d^\times x^{\mathrm{tam}}) = \frac{(N\mathfrak{d})^{-\frac{1}{2}}}{1-\chi(\varpi)(N\mathfrak{p})^{-s}} = (N\mathfrak{d})^{-\frac{1}{2}}L(s,\chi). \end{aligned}$$

For the second one, recall $\widehat{\mathbf{1}_{\mathfrak{o}}} = \mathrm{vol}(\mathfrak{o},dx^{\mathrm{tam}})\mathbf{1}_{\mathfrak{d}^{-1}} = (N\mathfrak{d})^{-\frac{1}{2}}\mathbf{1}_{\mathfrak{d}^{-1}}$. Then

$$\begin{aligned} Z(\widehat{\mathbf{1}_{\mathfrak{o}}},\chi^{-1},1-s) &= (N\mathfrak{d})^{-\frac{1}{2}}\int_{k^\times}\mathbf{1}_{\mathfrak{d}^{-1}}(x)\chi^{-1}|\cdot|^{1-s}(x)d^\times x^{\mathrm{tam}} = (N\mathfrak{d})^{-\frac{1}{2}}\sum_{n=d}^{\infty}\int_{\varpi^n\mathfrak{u}}\chi^{-1}|\cdot|^{1-s}(x)d^\times x^{\mathrm{tam}} \\ &= (N\mathfrak{d})\sum_{n=d}^{\infty}\chi^{-1}(\varpi)^n(N\mathfrak{p})^{-n(1-s)} \\ &= \frac{(N\mathfrak{d})\chi(\varpi)^{-d}(N\mathfrak{p})^{-d(1-s)}}{1-\chi^{-1}(\varpi)(N\mathfrak{p})^{1-s}} = \chi(\varpi)^{-d}(N\mathfrak{d})^{-s}L(1-s,\chi^{-1}). \end{aligned}$$

□

7.2.3 Change of characters

Let k be a local field. Recall by Theorem 7.1.3 every nontrivial character $\psi \in \widehat{k}$ defines an isomorphism $k \cong \widehat{k}$. By a similar argument preceding Lemma 7.1.4, there exists a unique measure dx on k such that if we define the Fourier transform $\widehat{f}$ of an integrable function f to be

$$\widehat{f}(x) = \int_k f(y)\psi(-xy)dy,$$

then the Fourier inversion $\widehat{\widehat{f}}(x) = f(-x)$ holds for suitable functions f.

Lemma 7.2.12. Retain the notation above. If $\psi(x) = \psi_k(ax)$ for some $a \in k^\times$, then $dx = |a|^{\frac{1}{2}} dx^{\mathrm{tam}}$.

Proof. The uniqueness of Haar measures implies that $dx = \alpha dx^{\mathrm{tam}}$ for some $\alpha \in k^\times$. To distinguish different Fourier transforms, we put $\widetilde{f}$ to be the one defined by the standard character ψ_k. Then

$$\widehat{f}(x) = \int_k f(y)\psi(-xy)dy = \int_k f(y)\psi_k(-axy)\alpha dy^{\mathrm{tam}} = \alpha \widetilde{f}(ax)$$

and hence

$$f(-x) = \widehat{\widehat{f}}(x) = \int_k \widehat{f}(y)\psi(-xy)dy = \alpha^2 \int_k \widetilde{f}(ay)\psi(-axy)dy^{\mathrm{tam}} = \alpha^2|a|^{-1}\widetilde{\widetilde{f}}(x) = \alpha^2|a|^{-1}f(-x).$$

This implies $\alpha = |a|^{\frac{1}{2}}$. □

Define the Haar measure $d^\times x$ on $k^\times$ as before. Then we can consider the zeta integral

$$Z_\psi(f, \chi, s) := \int_{k^\times} f(x)\chi|\cdot|^s(x)dx^\times.$$

By Lemma 7.2.12 and its proof, we see

$$Z_\psi(f, \chi, s) = |a|^{\frac{1}{2}} Z_{\psi_k}(f, \chi, s) \tag{♠}$$

and

$$Z_\psi(\widehat{f}, \chi^{-1}, 1-s) = \int_{k^\times} |a|^{\frac{1}{2}}\widetilde{f}(ax)\chi^{-1}|\cdot|^{1-s}(x)|a|^{\frac{1}{2}}d^\times x^{\mathrm{tam}} = |a|^s\chi(a)Z_{\psi_k}(\widetilde{f}, \chi^{-1}, 1-s).$$

With the measure $d^\times x$, we also have functional equations

$$Z_\psi(\widehat{f}, \chi^{-1}, 1-s) = \gamma(s, \chi, \psi)Z_\psi(f, \chi, s).$$

Comparing to the one for Z_{ψ_k}, we obtain

$$\gamma(s, \chi, \psi) = |a|^{s-\frac{1}{2}}\chi(a)\gamma(s, \chi, \psi_k).$$

This is the reason that we regard ψ_k as an argument of the γ-factor. Because of (♠), the $\mathbb{C}$-subspace spanned by the Z_ψ is the same as the one spanned by the Z_{ψ_k}. This implies the non-archimedean local L-factor remains unchanged; they are all denoted by $L(s, \chi)$. Similarly, we have the ϵ-factor $\epsilon(s, \chi, \psi)$ characterized by

$$\frac{Z_\psi(\widehat{f}, \chi^{-1}, 1-s)}{L(1-s, \chi^{-1})} = \epsilon(s, \chi, \psi)\frac{Z_\psi(f, \chi, s)}{L(s, \chi)}.$$

This implies

$$\gamma(s, \chi, \psi) = \epsilon(s, \chi, \psi) \cdot \frac{L(1-s, \chi^{-1})}{L(s, \chi)}$$

so

$$\epsilon(s, \chi, \psi) = |a|^{s-\frac{1}{2}}\chi(a)\epsilon(s, \chi, \psi_k).$$

Lemma 7.2.13. For a quasi-character χ on a local field k, we have the identity

$$\gamma(s,\chi,\psi) = \epsilon(s,\chi,\psi)\cdot\frac{L(1-s,\chi^{-1})}{L(s,\chi)}$$

7.2.4 Non-archimedean case II: invariant distributions

Definition. Let X be a totally disconnected locally compact space. A function f is called **smooth** if f is locally constant. We denote by $C^\infty(X)$ be the space of all smooth functions on f. Also, set $C_c^\infty(X) = C^\infty(X)\cap C_c(X)$.

- When k is a non-archimedean local field, $C_c^\infty(k) = S(k)$.
- We impose no topology on $C_c^\infty(X)$. In particular, we set

$$D(X) := C_c^\infty(X)^\vee = \mathrm{Hom}_{\mathbb{C}}(C_c^\infty(X),\mathbb{C}),$$

  and call it the space of tempered distributions.

- Let G be a totally disconnected locally compact group. We let G act on $C_c^\infty(G)$ by right translation and denote by $\rho : G \to \mathrm{GL}(C_c^\infty(G))$ the resulting representation. Namely, for $x, g \in G$ and $f \in C_c^\infty(G)$, we set

$$\rho(g)f(x) := f(xg).$$

  This is called the **right regular representation** of G. We let G act on $D(G)^\vee$ and denote the resulting representation by $\rho^\vee$. Namely, for $g \in G$ and $T \in D(G)^\vee$,

$$\rho^\vee(g)T := T\circ\rho(g^{-1}).$$

  This is called the **contragredient representation** of ρ. Similarly, we define the **left regular representation** $\lambda : G \to \mathrm{GL}(D(G))$ and its contragredient $\lambda^\vee : G \to \mathrm{GL}(D(G)^\vee)$.

Lemma 7.2.14. Let G be a totally disconnected locally compact group, and $f \in C_c^\infty(G)$. Then there exists an open compact K such that

$$f(k'xk) = f(x)$$

for $x \in G$, $k, k' \in K$.

Proof. Consider the stabilizer

$$\mathrm{Stab}_\rho f := \{g \in G \mid \rho(g)f = f\}$$

of f under the regular representation ρ. This is a subgroup of G. By Theorem 6.1.4, for each point $x \in G$ we can find an open compact subgroup U_x so that $f(xg) = f(x)$ for all $g \in U_x$. Since $\operatorname{supp} f$ is compact, we can find a finite set $S \subseteq \operatorname{supp} f$ so that $\operatorname{supp} f \subseteq \bigcup_{x\in S} xU_x$. Set $U := \bigcap_{x\in S} U_x$. We claim

$$f(xg) = f(x)$$

for all $x \in \operatorname{supp} f$ and $g \in U$. Indeed, for each $x \in \operatorname{supp} f$, take any $y \in F$ so that $x \in yU_y$; say $x = yh$ for some $h \in U_y$. Then for any $g \in U$,

$$f(xg) = f(yhg) = f(y) = f(yh) = f(x)$$

as $hg \in U_y U = U_y$ and $h \in U_y$. This proves $U \subseteq \mathrm{Stab}_\rho f$, so $\mathrm{Stab}_\rho f$ is open.

Next we show

$$\mathrm{Stab}_\rho f$$

is compact. Suppose $x \in \operatorname{supp} f$; then $f(xg) = f(x)$ for $g \in \mathrm{Stab}_\rho f$, so that $xg \in \operatorname{supp} f$, or $g \in x^{-1}(\operatorname{supp} f)$. Hence $\mathrm{Stab}_\rho f$ is contained in $x^{-1}(\operatorname{supp} f)$, a compact subset. Since $\mathrm{Stab}_\rho f$ is closed (being open), this shows the compactness.

The same argument applies for the left regular representation λ, i.e.

$$\mathrm{Stab}_\lambda f := \{g \in G \mid \lambda(g)f = f\}$$

is also open compact. Take $K = \mathrm{Stab}_\lambda f \cap \mathrm{Stab}_\rho f$ to conclude the proof. □

Let (π, V) be a (continuous) representation of G. If $\chi \in \mathrm{Hom}_{\mathbf{TopGp}}(G, \mathbb{C}^\times)$, denote by $V^{\pi=\chi} = V[\chi]$ the χ**-eigenspace** of V:

$$V[\chi] := \{v \in V \mid \pi(g)v = v \text{ for all } g \in G\}.$$

Lemma 7.2.15. Let G be a totally disconnected locally compact group and $\chi \in \mathrm{Hom}_{\mathbf{TopGp}}(G, \mathbb{C}^\times)$. Then $\dim_{\mathbb{C}} D(G)^{\rho^\vee=\chi} = 1$, and a basis element is given by integration against the measure $\chi(g)dg$, where dg is a right Haar measure on G.

Similarly, $\dim_{\mathbb{C}} D(G)^{\lambda^\vee=\chi} = 1$ and has a basis element $\chi(g)dg$ with dg a left Haar measure on G.

Proof. We first show that $\chi(g)dg$ defines an element in $D(G)[\chi]$. This is easy:

$$\int_{k^\times} \rho(y^{-1})f(x)\chi(x)d^\times x = \int_{k^\times} f(x)\chi(xy)d^\times x = \chi(y)\int_{k^\times} f(x)\chi(x)d^\times x.$$

To show the first assertion, note that there is a $\mathbb{C}$-isomorphism

$$\begin{aligned} D(G)[\chi] &\longrightarrow D(G)[\mathbf{1}_G] \\ T &\longmapsto f \mapsto T(\chi^{-1}f) \end{aligned}$$

Hence we may assume χ is trivial. To show it is at most one dimensional, we claim if $T \in D(G)[\chi]$ satisfies $T(\mathbf{1}_K) = 0$ for some open compact subgroup K of G, then $T = 0$. Let $f \in C_c^\infty(G)$ and let K_0 be an open compact subgroup of $\mathrm{Stab}_\rho f \cap \mathrm{Stab}_\lambda f$. Let $x_1, \ldots, x_n \in \operatorname{supp} f$ be such that $\operatorname{supp} f \subseteq K_0x_1 \cup \cdots \cup K_0x_n$. Since f is left-invariant under K_0, it is in fact an equality, and we can write $f = \sum_{i=1}^n f(x_i)\mathbf{1}_{K_0x_i} = \sum_{i=1}^n f(x_i)\rho(x_i^{-1})\mathbf{1}_{K_0}$. Then

$$T(f) = \sum_{i=1}^n f(x_i)T(\rho(x_i^{-1})\mathbf{1}_{K_0}) = \sum_{i=1}^n f(x_i)T(\mathbf{1}_{K_0}).$$

Let $a_1, \ldots, a_m$ be a set of representatives of $K_0\backslash K$ in K; then $K = K_0a_1 \cup \cdots \cup K_0a_m$ and

$$T(\mathbf{1}_K) = \sum_{i=1}^m T(\mathbf{1}_{K_0a_i}) = mT(\mathbf{1}_{K_0}).$$

This implies $T(\mathbf{1}_{K_0}) = 0$, and hence $T(f) = 0$. □

Let X be a totally disconnected locally compact space and $Y \subseteq X$ a closed subset. Let $i : X\backslash Y \to X$ be the open embedding and $j : Y \to X$ be the closed embedding. Consider the complex

$$0 \longrightarrow C_c^\infty(X\backslash Y) \xrightarrow{i_!} C_c^\infty(X) \xrightarrow{j^*} C_c^\infty(Y) \longrightarrow 0$$

Note that the extension by zero map $i_!$ is well-defined since $X\backslash Y$ is open and we are considering functions with compact support.

Lemma 7.2.16. The above complex is exact.

Proof. It is clear that $i_!$ is injective, and if $f \in C_c^\infty(X)$ satisfies $j^*f = f|_Y \equiv 0$, then $i_!\left(f|_{X\backslash Y}\right) = f$. Let $f \in C_c^\infty(Y)$. Let $x_1, \ldots, x_n \in \operatorname{supp} f$ and let U_i be an open neighborhood of x_i in X such that f is constant on $U_i \cap Y$ and f is zero on $Y \setminus \bigcup_{i=1}^n U_i$. Define $g : U_1 \cup \cdots \cup U_n \to \mathbb{C}$ by setting $g(x) = g(x_i)$ if $x \in U_i$ and extend it to a function $X \to \mathbb{C}$ by zero. This finishes the proof. $\square$

Taking dual yields an exact sequence of distributions

$$0 \longrightarrow D(Y) \longrightarrow D(X) \longrightarrow D(X\backslash Y) \longrightarrow 0.$$

Let k be a non-archimedean local field. Taking $X = k$ and $Y = \{0\}$ in the above sequence, we obtain a short exact sequence

$$0 \longrightarrow D(k)_0 \longrightarrow D(k) \longrightarrow D(k^\times) \longrightarrow 0$$

where $D(k)_0$ is the image of $D(\{0\})$ in $D(k)$ which consists of all distributions supported at 0. The unit group $k^\times$ acts on $D(k)$ and $D(k^\times)$ by right translation, and $D(k)_0$ is stable under $k^\times$-action. Let $\chi : k^\times \to \mathbb{C}^\times$ be a quasi-character. Taking χ-eigenspaces gives an exact sequence

$$0 \longrightarrow D(k)_0[\chi] \longrightarrow D(k)[\chi] \longrightarrow D(k^\times)[\chi]$$

Let $T \in D(k)_0[\chi]$. Suppose $f \in C_c^\infty(k)$ and pick $N \gg 0$ such that $f(\mathfrak{p}^N) = f(0)$. Then for $n \geqslant N$, we have

$$T(f) = T(f - \mathbf{1}_{\mathfrak{p}^n} f(0)) + T(\mathbf{1}_{\mathfrak{p}^n} f(0)) = f(0)T(\mathbf{1}_{\mathfrak{p}^n}) = f(0)T(\rho(\varpi^{-n})\mathbf{1}_{\mathfrak{o}_k}) = f(0)\chi(\varpi)^{-n}T(\mathbf{1}_{\mathfrak{o}_k}).$$

Suppose $T(\mathbf{1}_{\mathfrak{o}_k}) \neq 0$ and take f such that $f(0) = 0$. The above identity holds for all $n \geqslant N$, implying $\chi(\varpi) = 1$. Hence $T(f) = f(0)T(\mathbf{1}_{\mathfrak{o}_k})$. Moreover, if $\chi|_{\mathfrak{o}_k^\times} \not\equiv 1$, say $\chi(u) \neq 1$ for some $u \in \mathfrak{o}_k^\times$, then $T(\mathbf{1}_{\mathfrak{o}_k}) = T(\rho(u)\mathbf{1}_{\mathfrak{o}_k}) = \chi(u)T(\mathbf{1}_{\mathfrak{o}_k})$. This contradicts to the assumption $T(\mathbf{1}_{\mathfrak{o}_k}) \neq 0$. This shows

$$D(k)_0[\chi] = \begin{cases} 0 & \text{, if } \chi \neq 1 \\ \mathbb{C}\delta_0 & \text{, if } \chi = 1 \end{cases}$$

where δ_0 is the dirac distribution at 0, i.e., the evaluation at 0.

Proposition 7.2.17. $\dim_{\mathbb{C}} D(k)[\chi] = 1$ and a basis element is given by the distribution

$$C_c^\infty(k) \ni f \mapsto \left.\frac{Z(f, \chi, s)}{L(s, \chi)}\right|_{s=0} \in \mathbb{C}.$$

Proof. We already see in (7.2.2) that $f \mapsto \dfrac{Z(f, \chi, s)}{L(s, \chi)}$ is an entire family of tempered distributions, so it makes sense to let $s = 0$. For $g \in k^\times$, we have

$$Z(\rho(g^{-1})f, \chi, s) = \int_{k^\times} f(xg^{-1})\chi|\cdot|^s(x)d^\times x^{\mathrm{tam}} = \chi|\cdot|^s(g)Z(f, \chi, s)$$

so $\rho^\vee(g)\left.\frac{Z(f,\chi,s)}{L(s,\chi)}\right|_{s=0} = \chi(g)\left.\frac{Z(f,\chi,s)}{L(s,\chi)}\right|_{s=0}$. We also know this is nonzero by, for example, the computation in (7.1.3). Hence, it remains to show $\dim_{\mathbb{C}} D(k)[\chi] \leqslant 1$.

On applying $\dim_{\mathbb{C}}$ to the exact sequence

$$0 \longrightarrow D(k)_0[\chi] \longrightarrow D(k)[\chi] \longrightarrow D(k^\times)[\chi]$$

we see

$$\dim_{\mathbb{C}} D(k)[\chi] \leqslant \dim_{\mathbb{C}} D(k^\times)[\chi] + \dim_{\mathbb{C}} D(k)_0[\chi].$$

When $\chi \neq 1$, the RHS is 1, proving that $\dim_{\mathbb{C}} D(k)[\chi] \leqslant 1$. For $\chi = 1$, we claim that $D(k)[\chi] \to D(k^\times)[\chi]$ is not surjective, which implies $\dim_{\mathbb{C}} D(k)[\chi] < 2$. This will finish the proof. We claim the distribution $T \in D(k^\times)[1]$ defined by

$$T(f) = \int_{k^\times} f(x) d^\times x^{\mathrm{tam}}$$

does not come from an element in $D(k)[1]$. Suppose otherwise that $S \in D(k)[1]$ is such an extension to $C_c^\infty(k)$. Then for any $N \geqslant 1$,

$$\begin{aligned} S(\mathbf{1}_{\mathfrak{o}_k}) = \sum_{n=0}^{N} S(\mathbf{1}_{\mathfrak{p}^n \backslash \mathfrak{p}^{n+1}}) + S(\mathbf{1}_{\mathfrak{p}^{N+1}}) &= \sum_{n=0}^{N} \mathrm{vol}(\mathfrak{p}^n \backslash \mathfrak{p}^{n+1}, d^\times x^{\mathrm{tam}}) + S(\rho(\varpi^{-N-1})\mathbf{1}_{\mathfrak{o}_k}) \\ &= \sum_{n=0}^{N} \mathrm{vol}(\mathfrak{p}^n \backslash \mathfrak{p}^{n+1}, d^\times x^{\mathrm{tam}}) + S(\mathbf{1}_{\mathfrak{o}_k}). \end{aligned}$$

This is a contradiction, as the first series on the right is nonzero. Hence such S does not exist, proving the non-surjectivity. $\square$

Corollary 7.2.17.1. Let $\chi \in \mathrm{Hom}_{\mathbf{TopGp}}(k^\times, \mathbb{C}^\times)$. For each $s \in \mathbb{C}$ there exists a number $\epsilon(s,\chi,\psi_k) \in \mathbb{C}^\times$ such that

$$\frac{Z(\widehat{f},\chi^{-1},1-s)}{L(1-s,\chi^{-1})} = \epsilon(s,\chi,\psi_k)\frac{Z(f,\chi,s)}{L(s,\chi)}.$$

The numbers $\epsilon(s,\chi,\psi_k)$ are the ϵ-factors defined in (7.2.2) and satisfies $\epsilon(s,\chi|\cdot|^t,\psi_k) = \epsilon(s+t,\chi,\psi_k)$ for $s,t \in \mathbb{C}^\times$.

Proof. We prove that $f \mapsto \left.\frac{Z(\widehat{f},\chi^{-1},1-s)}{L(1-s,\chi^{-1})}\right|_{s=0}$ is an nonzero element in $D(k)[\chi]$. For $g \in k^\times$,

$$\widehat{\rho(g^{-1})f}(x) = \int_k f(yg^{-1})\psi_k(-xy)dy^{\mathrm{tam}} = |g|_k \int_k f(y)\psi_k(-xyg)dy^{\mathrm{tam}} = |g|_k \rho(g)\widehat{f}(x)$$

so that

$$\begin{aligned} Z(\widehat{\rho(g^{-1})f},\chi^{-1},1-s) = |g|_k Z(\rho(g)\widehat{f}(x),\chi^{-1},1-s) &= |g|_k \chi^{-1}|\cdot|^{1-s}(g^{-1}) Z(\widehat{f},\chi^{-1},1-s) \\ &= \chi(g)|g|^s Z(\widehat{f},\chi^{-1},1-s). \end{aligned}$$

Hence $\rho^\vee(g)\left.\frac{Z(\widehat{f},\chi^{-1},1-s)}{L(1-s,\chi^{-1})}\right|_{s=0} = \chi(g)\left.\frac{Z(\widehat{f},\chi^{-1},1-s)}{L(1-s,\chi^{-1})}\right|_{s=0}$. The computation in (7.1.3) shows that this is nonzero. Hence by Proposition 7.2.17, there exists a unique nonzero number $\epsilon(0,\chi,\psi_k)$ such that

$$\left.\frac{Z(\widehat{f},\chi^{-1},1-s)}{L(1-s,\chi^{-1})}\right|_{s=0} = \epsilon(s,\chi,\psi_k)\left.\frac{Z(f,\chi,s)}{L(s,\chi)}\right|_{s=0}.$$

Replacing χ by $\chi|\cdot|^t$, $t \in \mathbb{C}$, we obtain a family of nonzero numbers $\epsilon(t, \chi, \psi_k)$ such that

$$\begin{aligned}\left.\frac{Z(\widehat{f}, \chi^{-1}, 1-s)}{L(1-s, \chi^{-1})}\right|_{s=t} = \left.\frac{Z(\widehat{f}, \chi^{-1}|\cdot|^{-t}, 1-s)}{L(1-s, \chi^{-1}|\cdot|^{-t})}\right|_{s=0} &= \epsilon(t, \chi, \psi_k)\left.\frac{Z(f, \chi|\cdot|^t, s)}{L(s, \chi|\cdot|^t)}\right|_{s=0} \\ &= \epsilon(t, \chi, \psi_k)\left.\frac{Z(f, \chi, s)}{L(s, \chi)}\right|_{s=t}.\end{aligned}$$

Fix an $f \in S(k)$; then both distributions we consider are polynomials in $q^{\pm s}$. The above identity of distributions then implies the claimed equation in the corollary. $\square$

Chapter 8

Valuations

Definition. Let k be a field. A **valuation** on k is a map $|\cdot| : k \to \mathbb{R}_{\geqslant 0}$ such that

(i) $|x| = 0$ if and only if $x = 0$.

(ii) $|xy| = |x||y|$ for $x, y \in k$.

(iii) There is a constant C such that $|1 + x| \leqslant C$ if $|x| \leqslant 1$.

A valuation $|\cdot|$ is called **trivial** if $|x| = 1$ for $x \in k^\times$. If $|\cdot|$ is a valuation, so are the $|\cdot|^c$ for all $c > 0$. We say two valuations $|\cdot|_1$, $|\cdot|_2$ on k are **equivalent** if $|\cdot|_2^c = |\cdot|_1$ for some $c > 0$.

- Denote by M_k the set of equivalent classes of all nontrivial valuations on k.

There is another way to speak of a valuation. An **absolute value** on k is a map $|\cdot| : k \to \mathbb{R}_{\geqslant 0}$ such that

(a) $|x| = 0$ if and only if $x = 0$.

(b) $|xy| = |x||y|$ for $x, y \in k$.

(c) $|x + y| \leqslant |x| + |y|$ for $x, y \in k$.

An absolute value $|\cdot|$ is **trivial** if $|x| = 1$ for all $x \in k^\times$.

Lemma 8.0.1. Let k be a field and $|\cdot|$ be a valuation on k.

(i) $|\cdot|$ is equivalent to a valuation with constant $C = 2$ in (iii).

(ii) If $C = 2$, then $|x + y| \leqslant |x| + |y|$ for $x, y \in k$, i.e., $|\cdot|$ is an absolute value.

In other words, every element in M_k is represented by an absolute value.

Proof. (i) is achieved by taking appropriate r-th root for some $r > 0$. For (ii), note that for $x_1, x_2 \in k$ we have $|x_1 + x_2| \leqslant 2\max\{|x_1|, |x_2|\}$, so by induction $\left|\sum_{i=1}^{2^n} x_i\right| \leqslant 2^n \max_{1 \leqslant i \leqslant 2^n} |x_i|$. Hence, for $m \geqslant 1$, if $2^{n-1} \leqslant m \leqslant 2^n$, then

$$\left|\sum_{i=1}^{m} x_i\right| \leqslant 2^n \max_{1 \leqslant i \leqslant m} |x_i| \leqslant 2m \max_{1 \leqslant i \leqslant m} |x_i|$$

by inserting some zeros. In particular, $|m| \leqslant 2m$ for $m \geqslant 1$. Now for $n \geqslant 1$, we have

$$\begin{aligned} |x+y|^n &= \left|\sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}\right| \leqslant 2(n+1) \max_{0\leqslant i\leqslant n} \left|\binom{n}{i} x^i y^{n-i}\right| \\ &\leqslant 4(n+1) \max \max_{0\leqslant i\leqslant n} \binom{n}{i} |x^i y^{n-i}| \leqslant 4(n+1)(|x|+|y|)^n. \end{aligned}$$

Taking n-th root on both sides and letting $n \to \infty$ yield $|x+y| \leqslant |x|+|y|$. □

8.1 Valuation and topology

Let $|\cdot|$ be an absolute value on k. Then $(x, y) \mapsto |x-y|$ defines a metric on k. Note that an absolute value is trivial if and only if it defines a trivial metric topology. Let $c(k)$ be the set of all Cauchy sequences on k. This is naturally a k-algebra: $(a_n)_n$ and $(b_n)_n$ are two Cauchy sequences, then there are bounded sequences and

$$|a_n b_n - a_m b_m| \leqslant |b_n||a_n - a_m| + |a_m||b_n - b_m|,$$

so that $(a_n b_n)_n$ is again Cauchy. Let $n(k)$ be the set of all null Cauchy sequences, i.e., $(a_n)_n \in n(k)$ if and only if $\lim_{n\to\infty} |a_n| = 0$. Since Cauchy sequences are bounded, $n(k)$ is an ideal of $c(k)$, and hence the quotient $\widehat{k} := c(k)/n(k)$ is again a k-algebra. The natural map

$$\begin{aligned} k &\longrightarrow \widehat{k} \\ r &\longmapsto (r)_n \end{aligned}$$

is injective; this map is called **the completion of k with respect to the absolute value** $|\cdot|$. Any continuous map from k to a complete metric space uniquely extends to a map from $\widehat{k}$. In particular, $|\cdot|$ extends to an absolute value $|\cdot| : \widehat{k} \to \mathbb{R}_{\geqslant 0}$ on $\widehat{k}$. Precisely, if $x \in \widehat{k}$ and $(a_n)_n$ is a Cauchy sequence representing x, then $|x| := \lim_{n\to\infty} |a_n|$.

Lemma 8.1.1. Let $|\cdot|_1, |\cdot|_2$ be two absolute values on k. Then they define the same (metric) topology on k if and only if they are equivalent as valuations.

Proof. The if part is clear. For the only if part, suppose $|\cdot|_i$ $(i = 1, 2)$ define the same topology. Let $x \in k^\times$ with $|x|_1 < 1$. Then $(x^n)_n$ is a null sequence with respect to $|\cdot|_1$. Since $|\cdot|_2$ defines the same metric space as $|\cdot|_1$ does, we must have $|x|_2 < 1$. Similarly, we have $|x|_1 > 1$ if and only if $|x|_2 > 1$. By law of trichotomy, $|x|_1 = 1$ if and only if $|x|_2 = 1$.

Fix an $x_0 \in k$ such that $|x_0|_1 < 1$. Then $|x_0|_2 < 1$, so $|x_0|_2 = |x_0|_1^c$ for some $c > 0$. For any other $y \in k^\times$ with $|y|_i < 1$, write $|y|_2 = |x_0|_2^r$. If $r \in \mathbb{Z}$, then $|y|_2 = |x_0^r|_2$, so that $|y/x_0^r|_2 = 1$, or $|y/x_0^r|_1 = 1$, i.e., $|y|_1 = |x_0|_1^r$. If $r = m/n$ for $m, n \in \mathbb{Z}$ with $n \neq 0$, then $|y^n|_2 = |x_0^m|_2$, so again $|y^n|_1 = |x_0^m|_1$, or $|y|_1 = |x_0|_2^r$. If $r > 0$, approximating r by rational sequences shows $|y|_1 = |x_0|_1^r$. Hence

$$|y|_2 = |x_0|_2^r = |x_0|_1^{rc} = |y|_1^c.$$

Taking inverse implies this holds for $|y|_2 > 1$ as well. Hence $|\cdot|_2 = |\cdot|_1^c$, as wanted. □

A field k together with a valuation $|\cdot|$ is called a **valued field**. A valuation is equivalent to an absolute value, and equivalent absolute values define the same metric topology on k. Hence a valued field has a natural metric topology, and is a uniform space.

Definition. A **complete valued field** is a valued field that is complete with respect to the metric topology (or complete as a uniform space). Note that a field with trivial valuation is automatically complete.

Let $(k, |\cdot|)$ be a complete valued field; here we assume $|\cdot|$ is an absolute value. For a k-vector space V, by a **(k-)norm on** V we mean a map $\|\cdot\| : V \to \mathbb{R}_{\geqslant 0}$ such that

(i) $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|rx\| = |r| \, \|x\|$ for $x \in V$, $r \in k$.

(iii) $\|x + y\| \leqslant \|x\| + \|y\|$ for $x, y \in V$.

In particular, the map $(x, y) \mapsto \|x - y\|$ defines a metric on V. The induced topology is called a **normed topology** on V, and $(V, \|\cdot\|)$ is called a $(k-)$normed space.

Theorem 8.1.2. Let $(k, |\cdot|)$ be a complete valued field. Let V be a finite dimensional k-vector space. Then all norm topologies on V are equivalent. In particular, V is complete.

Proof. Let $e_1, \ldots, e_n$ be a k-basis of V. Define a norm $\|\cdot\|_\infty : V \to \mathbb{R}_{\geqslant 0}$ by

$$\left\| \sum_{i=1}^n a_i e_i \right\|_\infty := \max_{1 \leqslant i \leqslant n} |a_i|.$$

Then V is complete with respect to $\|\cdot\|_\infty$.

Let $\|\cdot\|$ be a norm on V. We show that $\|\cdot\|$ is equivalent to $\|\cdot\|_\infty$. The last assertion will also follow. One side is very easy:

$$\left\| \sum_{i=1}^n a_i e_i \right\| \leqslant \sum_{i=1}^n \|a_i e_i\| = \sum_{i=1}^n |a_i| \, \|e_i\| \leqslant \sum_{i=1}^n \|e_i\| \cdot \max_{1 \leqslant i \leqslant n} |a_i| = \sum_{i=1}^n \|e_i\| \cdot \left\| \sum_{i=1}^n a_i e_i \right\|_\infty$$

The other side is to find $c = c_V > 0$ such that $c_V \|\cdot\|_\infty \leqslant \|\cdot\|$. We do this by induction on all subspaces of V. If $0 \neq v \in V$, then for $r \in k$, we have $\|rv\| = |r| \, \|v\| = \|rv\|_\infty \|v\|$; so $c_{kv} = \|v\|$ does the job. Suppose the equality holds for all codimension one subspaces of V. In particular, all codimension one subspaces are closed. Let $V_i := \operatorname{span}_k\{e_1, \ldots, \widehat{e_i}, \ldots, e_n\} \subsetneq V$, and consider the affine subspace $V_i + v_i$ $(i \in [n])$. Since $0 \notin \bigcup_{i=1}^n V_i + v_i$, there is a $\rho > 0$ such that $\|x + v_i\| \geqslant \rho$ for all $i \in [n]$ and $x \in V_i$. Now for $0 \neq x = \sum_{i=1}^n a_i e_i \in V$ and let $i \in [n]$ be such that $\|x\|_\infty = |a_i|$. Then $a_i^{-1} x \in V_i + e_i$, so that $\left\| a_i^{-1} x \right\| \geqslant \rho$, or $\|x\| \geqslant \rho |a_i| = \rho \|x\|_\infty$. This finishes the proof. $\square$

Corollary 8.1.2.1. Let $(k, |\cdot|)$ be a non-discrete[1] locally compact valued field and $(V, \|\cdot\|)$ a complete k-normed space. Then V is locally compact if and only if $\dim_k V < \infty$[2].

Proof. The if part follows from Theorem 8.1.2. Now assume V is locally compact. Then $\{v \mid |v| \leqslant \varepsilon\}$ is compact for some small $\varepsilon > 0$. Since k is non-discrete, we can find $x \in k$ such that $|x| > 1$. By multiplying by powers of x, we see $\{v \mid |v| \leqslant r\}$ is compact for all $r > 0$.

Let $x \in k^\times$ such that $r := |x| < 1$. Since $B_1(0) = \{x \in k \mid |x| < 1\}$ is relatively compact, we can find a finite subset $S \subseteq B_1(0)$ such that $B_1(0) \subseteq S + B_r(0)$. Let W be the k-linear span of S; then $B_1(0) \subseteq W + B_r(0)$, and iterating yields $B_1(0) \subseteq W + B_{r^n}(0)$ for $n \geqslant 1$. Since

[1] That is, the topology on k is not discrete. In this setting, this means the valuation $|\cdot|$ is not trivial.

[2] See Proposition E.1.12 for another proof when $k = \mathbb{R}$. See Corollary 2.5.3.1.(ii) for a more general setting.

$B_{r^n}(0) = r^n B_1(0)$, $n \geqslant 1$ form a unit-neighborhood basis for 0 in V, this implies W is dense in V. By Theorem 8.1.2, any finite dimensional k-vector space is closed, so in particular W is closed under the subspace topology inherited from V. Hence $V = W$, so that $\dim_k V < \infty$. $\square$

Lemma 8.1.3 (weak approximation)**.** Let k be a field and $|\cdot|_1, \ldots, |\cdot|_n$ be nontrivial valuations on k. For each $i \in [n]$, denote by k_i the space k equipped with the metric topology given by $|\cdot|_i$. Then the diagonal embedding

$$\begin{aligned} \Delta : k &\longrightarrow k_1 \times \cdots \times k_n \\ r &\longmapsto (r, \ldots, r) \end{aligned}$$

has dense image if and only if $|\cdot|_1, \ldots, |\cdot|_n$ are inequivalent valuations.

Proof. If, say $|\cdot|_1$ and $|\cdot|_2$ are equivalent, then $k \to k_1 \times k_2$ cannot be dense. For in the proof of Lemma 8.1.1 we see $|x|_1 < 1$ if and only if $|x|_2 < 1$. If we pick $a, b \in k$ with $|a|_1 < 1$ and $|b|_2 > 1$, then (a, b) cannot be approximated by elements in k. In particular, Δ does not have dense image.

Now suppose $|\cdot|_1, \ldots, |\cdot|_n$ are inequivalent valuations. The case $n = 1$ follows from the definition. For $n \geqslant 2$, we begin by finding $x \in k$ such that $|x|_1 > 1$ while $|x|_m < 1$ for $2 \leqslant m \leqslant n$. We proceed by induction on $n \geqslant 2$. For $n = 2$, we first find $x \in k$ such that $|x|_1 > 1$ while $|x|_2 \leqslant 1$. Suppose otherwise for $x \in k$, $|x|_1 < 1$ if and only if $|x|_2 < 1$. From the proof of Lemma 8.1.1, we see this would imply $|\cdot|_1$ and $|\cdot|_2$ are equivalent, contradicting to our assumption. Hence there exists $x \in k$ with $|x|_1 < 1$ and $|x|_2 \geqslant 1$. By symmetry we can find $y \in k$ such that $|y|_1 \geqslant 1$ and $|y|_2 < 1$. Then $x^{-1}y \in k$ satisfies $|x^{-1}y|_1 > 1$ and $|x^{-1}y|_2 < 1$.

For $n \geqslant 3$, by induction hypothesis we can find $y \in k$ such that $|y|_1 > 1$ while $|y|_m < 1$ for $2 \leqslant m \leqslant n-1$, and $x \in k$ such that $|x|_1 > 1$ but $|x|_n < 1$. Set $z \in k$ as

$$z = \begin{cases} y & \text{, if } |y|_n < 1 \\ y^r x & \text{, if } |y|_n = 1 \\ \dfrac{y^r}{1+y^r} x & \text{, if } |y|_n > 1 \end{cases}.$$

where $r \gg 0$. Then $|z|_1 > 1$ and $|z|_m < 1$ for $2 \leqslant m \leqslant n$. Indeed, the case when $|y|_n < 1$ is clear. If $|y|_n = 1$, take $r \gg 0$ such that $|y^r x|_m < 1$ for $2 \leqslant m \leqslant n-1$ (possible as $|y|_m < 1$). If $|y|_n > 1$, then $\lim_{r \to \infty} \dfrac{y^r}{1+y^r} = 1$ in k_1 and k_n, and $= 0$ in k_m with $2 \leqslant m \leqslant n-1$. Taking $r \gg 0$ does the job.

Now for $i \in [n]$, by the preceding construction we can find $z_i \in k$ such that $|z_i|_i > 1$ while $|z_i|_m < 1$ for $m \in [n] \backslash \{i\}$. For $(x_1, \ldots, x_n) \in k_1 \times \cdots \times k_n$, we have $\sum_{i=1}^{n} \dfrac{z_i^r}{1+z_i^r} x_i \to x_i$ in k_i as $r \to \infty$. $\square$

8.2 (Non-)archimedean valuations

Definition. Let k be a field and $|\cdot|$ a *nontrivial*[3] valuation on k.

(i) $|\cdot|$ is called **discrete** if there exists a $\delta > 0$ such that $1 - \delta < |x| < 1 + \delta$ implies $|x| = 1$.

(ii) $|\cdot|$ is called **non-archimedean** if $C = 1$, i.e., $|x + y| \leqslant \max\{|x|, |y|\}$ for $x, y \in k$.

(iii) $|\cdot|$ is called **archimedean** if it is not non-archimedean.

[3] That is, we do not say a trivial valuation is non-archimedean, archimedean or discrete. In other words, when a valuation is said to be non-archimedean, archimedean or discrete, it is assumed to be nontrivial.

If $|\cdot|$ is non-archimedean, the set $\mathfrak{o} := \{x \in k \mid |x| \leqslant 1\}$ forms a ring, called the **ring of integers of** $|\cdot|$. It is a local ring with the unique maximal ideal $\mathfrak{p} := \{x \in k \mid |x| < 1\}$.

Lemma 8.2.1. Let k be a field and $|\cdot|$ a valuation on k. Let R be the subring of k generated by 1.

(i) If $|\cdot|$ is discrete, then $\{\log |x| \mid x \in k^\times\}$ is a discrete subgroup of $\mathbb{R}$. In particular, there exists some $1 > c > 0$ such that $\{|x| \mid x \in k^\times\} = \{c^n \mid n \in \mathbb{Z}\}$.

(ii) If $|\cdot|_1$ and $|\cdot|_2$ are equivalent valuations on k, then $|\cdot|_1$ is non-archimedean (resp. discrete, archimedean) if and only if $|\cdot|_2$ is non-archimedean (resp. discrete, archimedean).

(iii) $|\cdot|$ is non-archimedean if and only if $|\cdot|$ is nontrivial and $|n| \leqslant 1$ for all $n \in R$.

(iv) Let $|\cdot|$ be non-archimedean. Then $|\cdot|$ is discrete if and only if $\mathfrak{p} = \{x \in k \mid |x| < 1\}$ is a principal ideal of the ring of integers $\mathfrak{o}$ of $|\cdot|$.

(v) If $|\cdot|_1$ and $|\cdot|_2$ is non-archimedean, they are equivalent if and only if they have the same ring of integers.

(vi) If C cannot be chosen to be 1, then $|n| \neq 1$ for some $n \in R\backslash\{0\}$.

By (ii) we can denote by $M_{k,\mathrm{a}}$ (resp. $M_{k,\mathrm{na}}$) the set of equivalence classes of archimedean (resp. non-archimedean) valuations. We have $M_k = M_{k,\mathrm{a}} \sqcup M_{k,\mathrm{na}}$.

Proof.

(i) The definition implies that 0 is an isolated point.

(ii) Clear.

(iii) If $|\cdot|$ is non-archimedean, since $|1| = 1$, we see $|2| = |1+1| \leqslant 1$. By induction we see $|n| \leqslant 1$ for all $n \in R$. For the if part, we can replace $|\cdot|$ by its power so that we can assume that we can take $C = 2$. By Lemma 8.0.1.(ii) the triangle inequality holds, so for $|x| \leqslant 1$,

$$|1+x|^n = |(1+x)^n| \leqslant \sum_{i=0}^{n} \left|\binom{n}{i} x^i\right| = \sum_{i=0}^{n} \left|\binom{n}{i}\right| |x|^i \leqslant n+1.$$

Taking n-th root and letting $n \to \infty$ yield $|1+x| \leqslant 1$, so C can be chosen to 1.

(iv) Suppose $|\cdot|$ is discrete. By (i), we can find $0 < c < 1$ and $x \in k^\times$ such that $|x| = c$ and for each $y \in k^\times$ there is some $n \in \mathbb{Z}$ such that $|y| = c^n$. For $y \in \mathfrak{p}$, say $|y| = c^n$ with $n \geqslant 1$; then $|y/x| = c^{n-1} \leqslant 1$ so that $y/x \in \mathfrak{o}$. This shows $\mathfrak{p} = \mathfrak{o}x$. Conversely, if $\mathfrak{p}$ is principal, say generated by $x \in \mathfrak{p}$, then for $y \in \mathfrak{p}$ we have $|y| \leqslant |x|$. Take $\delta > 0$ such that $|x| < 1 - \delta < 1 + \delta < |x|^{-1}$. Let $y \in k^\times$ with $1 - \delta < |y| < 1 + \delta$; replacing y with y^{-1} if necessary, we can assume $|y| \leqslant 1$. If $|y| \neq 1$, then $|y| \leqslant |x|$, which is a contradiction. Hence $|y| = 1$, proving the discreteness.

(v) Clear.

(vi) We prove the contrapositive. Suppose $|n| = 1$ for all $n \in R\backslash\{0\}$. The proof for the if part of (iii) implies that $|1+x| \leqslant 1$ for all $x \in k^\times$ with $|x| \leqslant 1$. This means C can be chosen to be 1.

□

Lemma 8.2.2. Let k be a field and $|\cdot|$ be a non-archimedean valuation on k. Let $(\mathfrak{o}, \mathfrak{p})$ be the ring of integers of $|\cdot|$.

(i) The extension of $|\cdot|$ to the completion $\widehat{k}$ with respect to $|\cdot|$ is non-archimedean.

(ii) Let $(\widehat{\mathfrak{o}}, \widehat{\mathfrak{p}})$ be the ring of integers of $(\widehat{k}, |\cdot|)$. The natural map $\mathfrak{o}/\mathfrak{p} \to \widehat{\mathfrak{o}}/\widehat{\mathfrak{p}}$ is an isomorphism.

(iii) Let $I \trianglelefteq \mathfrak{o}$ be an ideal such that the closure $\widehat{I}$ of I in $\widehat{\mathfrak{o}}$ is open. Then the natural map $\mathfrak{o}/I \to \widehat{\mathfrak{o}}/\widehat{I}$ is an isomorphism.

Proof. (i) follows from Lemma 8.2.1.(iii). Note that $\widehat{\mathfrak{o}}$ is open in $\widehat{k}$: this follows from $|x+y| = \max\{|x|, |y|\}$ if $|x| \neq |y|$. Since k is dense in $\widehat{k}$, we see $\mathfrak{o} = k \cap \widehat{\mathfrak{o}}$ is dense in $\widehat{\mathfrak{o}}$. Since $\mathfrak{p} = \widehat{\mathfrak{p}} \cap k$, the inclusion $\mathfrak{o} \to \widehat{\mathfrak{o}}$ induces an injection $\mathfrak{o}/\mathfrak{p} \to \widehat{\mathfrak{o}}/\widehat{\mathfrak{p}}$. This is surjective, as for $x \in \widehat{\mathfrak{o}}$, by density there exists $r \in \mathfrak{o}$ such that $x - r \in \widehat{\mathfrak{p}}$, or $x + \widehat{\mathfrak{p}} = r + \widehat{\mathfrak{p}}$. This proves (ii). For (iii) similarly the inclusion $\mathfrak{o} \to \widehat{\mathfrak{o}}$ induces an injection $\mathfrak{o}/I \to \widehat{\mathfrak{o}}/\widehat{I}$. The surjectivity is proved in the same way as above since $\widehat{I}$ is open in $\widehat{k}$. $\square$

Definition. For a rational prime p, define the ***p*-adic absolute value** $|\cdot|_p$ on $\mathbb{Q}$ as follows. If $n \in \mathbb{Q}^\times$, write $n = p^a \cdot \dfrac{b}{c}$ with $a, b, c \in \mathbb{Z}$, $c \neq 0$ and $(p, b, c) = 1$. Define $\operatorname{ord}_p n := a$ and set $|n|_p := p^{-\operatorname{ord}_p n}$.

Also, let $|\cdot|_\infty$ be the usual absolute value on $\mathbb{Q}$. Namely, if $x \in \mathbb{Q}$, $|x|_\infty := \begin{cases} x & \text{, if } x \geqslant 0 \\ -x & \text{, if } x < 0 \end{cases}$

Theorem 8.2.3 (Ostrowski)**.** A valuation on $\mathbb{Q}$ is equivalent either to trivial, to $|\cdot|_p$ for some rational prime p or to $|\cdot|_\infty$.

Proof. Let $|\cdot|$ be an absolute value on $\mathbb{Q}$. If $|n| \leqslant 1$ for all $n \in \mathbb{Z}$, by Lemma 8.2.1.(iii) $|\cdot|$ is non-archimedean. Consider the subring $\{x \in \mathbb{Z} \mid |x| < 1\}$. If it is trivial, then $|\cdot|$ is a trivial valuation. Otherwise, this is a nontrivial prime ideal in $\mathbb{Z}$, so it is generated by a prime $p \geqslant 2$. But then $|\cdot|$ is equivalent to $|\cdot|_p$ by Lemma 8.2.1.(v).

Now assume $|n| > 1$ for some $n \in \mathbb{Z}_{\geqslant 2}$. Let $a, b \in \mathbb{Z}_{\geqslant 1}$ and write $b = b_m a^m + \cdots + b_1 a^m + b_0$ with $0 \leqslant b_i < a$ for each i. Then $m \leqslant \log_a b$ and

$$|b| \leqslant \sum_{i=0}^{m} |b_i||a|^m \leqslant \max_{0 \leqslant i < a} |i| \cdot \max\{1, |a|^{\log_a b}\} \cdot (1 + \log_a b)$$

Plugging in $b = n^m$ and letting $m \to \infty$, we see $|n| \leqslant \max\{1, |a|^{\log_a n}\}$. Since $|n| > 1$, this shows $|a| > 1$ and $|n| \leqslant |a|^{\log_a n}$ for $a \geqslant 2$. By symmetry $|a| \leqslant |n|^{\log_n a}$, so $|a| = |n|^{\log_n a} = a^{\log_n |n|}$ for all $a \geqslant 2$. If $a \leqslant -1$, then $-a = |a|_\infty \geqslant 1$ and $|a| = |-a| = (-a)^{\log_n |n|} = |a|_\infty^{\log_n |n|}$. Hence $|\cdot|$ is equivalent to $|\cdot|_\infty$. $\square$

Corollary 8.2.3.1. If $(k, |\cdot|)$ is an archimedean valued field, then k contains $\mathbb{Q}$ and the restriction of $|\cdot|$ to $\mathbb{Q}$ is equivalent to $|\cdot|_\infty$.

Proof. If k has positive characteristic, then by Lemma 8.2.1.(iii) $|\cdot|$ is either trivial or non-archimedean. Since k is archimedean, these cannot happen. So k is of characteristic 0, and hence contains $\mathbb{Q}$. The remaining then follows from Lemma 8.2.1.(vi) and Theorem 8.2.3. $\square$

Theorem 8.2.4 (Ostrowski)**.** An archimedean complete valued field is isomorphic either to $\mathbb{R}$ or $\mathbb{C}$ as valued fields. In particular, an archimedean valued field is isomorphic to a subfield of $\mathbb{C}$ as valued field.

Proof. If $(k, |\cdot|)$ is an archimedean valued field, then $(k, |\cdot|) \to (\widehat{k}, |\cdot|)$ is an inclusion of valued field. So the last assertion follows from the first assertion.

Now let $(K, |\cdot|)$ be an archimedean complete valued field. By Corollary 8.2.3.1, K contains $\mathbb{Q}$ and the valuation $|\cdot|$ restricts to a valuation on $\mathbb{Q}$ equivalent to $|\cdot|_\infty$, so K contains $\mathbb{R}$. Replacing K by $K[x]/(x^2+1)$ and extending $|\cdot|$ by $|a+bx| := |a^2+b^2|^{\frac{1}{2}}$ if necessary, we assume $\mathbb{C} \subseteq K$. We must show it is an equality, and we do so by showing each element in K solves a real quadratic polynomial. For $x \in K$, consider the function $p_\xi : \mathbb{C} \to \mathbb{R}_{\geqslant 0}$ defined by

$$p_\xi(z) = |\xi^2 - (z+\overline{z})\xi + z\overline{z}|, \qquad z \in \mathbb{C}.$$

Since $\lim\limits_{|z|\to\infty} p_\xi(z) = \infty$, $p_\xi(z)$ has a minimum, say $m \in \mathbb{R}_{\geqslant 0}$. The set $S = \{z \in \mathbb{C} \mid p_\xi(z) = m\}$ is a nonempty compact set; pick $z_0 \in S$ such that $|z_0| \geqslant |z|$ for all $z \in S$.

It suffices to show $m = 0$. Suppose for contradiction that $m > 0$. Let $m > \varepsilon > 0$ and consider the real polynomial

$$g(x) = x^2 - (z_0 + \overline{z_0})x + z_0\overline{z_0} = \varepsilon = (x - \alpha_1)(x - \overline{\alpha_1}).$$

Then $z_0\overline{z_0} + \varepsilon = \alpha_1\overline{\alpha_1}$, so $|\alpha_1| > |z_0|$. By maximality, we see $p_\xi(\alpha_1) > m$. For each $n \in \mathbb{N}$, consider the real polynomial

$$G(x) = (g(x) - \varepsilon)^n - (-\varepsilon)^n = \prod_{i=1}^{2n}(x - \alpha_i)$$

with $\alpha_i \in \mathbb{C}$, $i \in [2n]$. Then $G(x)^2 = \prod\limits_{i=1}^{2n}(x^2 - (\alpha_i + \overline{\alpha_i})x + \alpha_i\overline{\alpha_i})$. Plugging $x = \xi$ yields

$$|G(\xi)|^2 = \prod_{i=1}^{2n} p_\xi(\alpha_i) \geqslant p_\xi(\alpha_1)m^{2n-1}.$$

Also,

$$|G(\xi)| \leqslant |\xi^2 - (z_0 + \overline{z_0})\xi + z_0\overline{z_0}|^n + |-\varepsilon|^n = p_\xi(z_0)^n + \varepsilon^n = m^n + \varepsilon^n.$$

These together imply $p_\xi(\alpha_1)m^{2n-1} \leqslant (m^n + \varepsilon^n)^2$, or

$$\frac{p_\xi(\alpha_1)}{m} \leqslant \left(1 + \left(\frac{\varepsilon}{m}\right)^n\right)^2.$$

Letting $n \to \infty$, we see $p_\xi(\alpha) \leqslant m$, a contradiction. $\square$

Let k be a field and $\sigma : k \to \mathbb{C}$ be a field homomorphism. Then $|x|_\sigma := |\sigma(x)|_{\mathbb{C}}$ defines an archimedean valuation on k (recall that $|z|_{\mathbb{C}} = z\overline{z}$). Suppose $\sigma, \tau : k \to \mathbb{C}$ are two distinct non-conjugate field homomorphisms. We claim they are not equivalent. Note that k must be characteristic 0, so k contains $\mathbb{Q}$. Since $|\cdot|_\sigma$ and $|\cdot|_\tau$ are nontrivial and coincide on $\mathbb{Q}$, it suffices to find $x \in k$ such that $|\sigma(x)|_{\mathbb{C}} \neq |\tau(x)|_{\mathbb{C}}$. Note that if $z, w \in \mathbb{C}$ with $|z|_{\mathbb{C}} = |w|_{\mathbb{C}}$, then

$$|z+1|_{\mathbb{C}} = z\overline{z} + (z + \overline{z}) + 1 = |w+1|_{\mathbb{C}} + (z+\overline{z}) - (w + \overline{w}).$$

If $z \neq w$, then $\operatorname{Re}(z) \neq \operatorname{Re}(w)$ since they have the same length. Hence $|z+1|_{\mathbb{C}} \neq |w+1|_{\mathbb{C}}$. Now pick any $x \in k^\times$ with $\sigma(x) \neq \tau(x), \overline{\tau(x)}$. If $|\sigma(x)|_{\mathbb{C}} \neq |\tau(x)|_{\mathbb{C}}$, we are done. Otherwise, $|\sigma(x)|_{\mathbb{C}} = |\tau(x)|_{\mathbb{C}}$ and this implies $|\sigma(x+1)|_{\mathbb{C}} \neq |\tau(x+1)|_{\mathbb{C}}$.

On the set $\operatorname{Hom}_{\mathbf{Field}}(k, \mathbb{C})$ we define an equivalent relation $\sim$ by declaring $\sigma \sim \tau$ if and only if $\sigma = \overline{\tau}$.

Corollary 8.2.4.1. The map $\sigma \mapsto |\cdot|_\sigma$ defines a bijection

$$\mathrm{Hom}_{\mathbf{Field}}(k, \mathbb{C})/\sim \longrightarrow M_{k,\mathrm{a}}.$$

Proof. This follows from the above discussion and Theorem 8.2.4. ☐

Definition. Let k be a field and consider the rational function field $k(t)$, where t is transcendental over k. Let $0 < c < 1$ be fixed If $p \in k[t]$ is an irreducible polynomial, define an absolute value $|\cdot|_p$ as follows. For $f \in k(t)$, write $f = p^a \dfrac{h}{g}$ with $a \in \mathbb{Z}$, $h, g \in k[t]$, $g \neq 0$, $(p, g, h) = 1$. Define $\mathrm{ord}_p f := a$ and set $|f|_p := c^{-\mathrm{ord}_p f}$.

In addition, for $f, g \in k[t]$ with $g \neq 0$, set $|f/g|_\infty := c^{\deg g - \deg f}$. If we put $s = t^{-1}$ and identify $k(t) = k(s)$, then $|\cdot|_\infty = |\cdot|_s$.

Theorem 8.2.5. A nontrivial valuation on $k(t)$ that is trivial on k is equivalent to either $|\cdot|_p$ for some irreducible $p \in k[t]$ or to $|\cdot|_\infty$.

Proof. Let $|\cdot|$ be a non-trivial valuation on $k(t)$ that is trivial on k. By Lemma 8.2.1.(iii), $|\cdot|$ is non-archimedean. Consider the subring $\{f \in k[t] \mid |f| < 1\}$. If it is nonzero, then is a prime ideal in $k[t]$, so it is generated by some irreducible $p \in k[t]$. But then $|\cdot|$ is equivalent to $|\cdot|_p$ by Lemma 8.2.1.(v). If it is zero, i.e., $|f| \geqslant 1$ for all $f \in k[t]\backslash\{0\}$. Write $f = \sum_{i=0}^{n} f_n t^n$; then $|f| \leqslant \max\{1, |t|, \ldots, |t|^n\} = |t|^n$. Since $|\cdot|$ is nontrivial, we see $|t| > 1$. But then $|t|^{n-1} < |t|^n$, so

$$|t^n| = \left|f - \sum_{i=0}^{n-1} f_n t^n\right| \leqslant \max\{|f|, \left|\sum_{i=0}^{n-1} f_n t^n\right|\} \leqslant \max\{|f|, |t|^{n-1}\} \leqslant |f|$$

so that $|f| = |t|^n = |t|^{\deg f}$. This implies $|\cdot|$ is equivalent to $|\cdot|_\infty$. ☐

Corollary 8.2.5.1. If k has positive characteristic, then all nontrivial valuations on $k(t)$ are either equivalent to $|\cdot|_p$ for some irreducible $p \in k[t]$ or to $|\cdot|_\infty$.

8.3 Classification of locally compact valued field

Theorem 8.3.1. Let $(k, |\cdot|)$ be a non-archimedean valued field. Then k is locally compact if and only if k is complete, the quotient $\mathfrak{o}/\mathfrak{p}$ is finite and $|\cdot|$ is discrete.

Proof. Suppose k is locally compact, and let K be a compact unit-neighborhood of K. Then there is some $r > 0$ such that $\{x \in k \mid |x| < r\} \subseteq K$. Pick $0 \neq \pi \in \mathfrak{p}$ so that $|\pi^n| \to 0$ as $n \to \infty$. Then $\pi^m\mathfrak{o} \subseteq \{|x| < r\}$ for $m \gg 0$, so that $\pi^m\mathfrak{o} \subseteq K$. This implies $\pi^m\mathfrak{o}$, and hence $\mathfrak{o}$, is compact. In particular, every closed ball $\{|x| \leqslant r\}\,(r > 0)$ is compact. Since a Cauchy sequence is bounded, it is contained in some ball and hence is convergent. This shows the completeness. Since $\mathfrak{p} \leqslant \mathfrak{o}$ is open, the quotient $\mathfrak{o}/\mathfrak{p}$ is finite by compactness. Finally, consider the increasing filtration $\{x \in k \mid |x| < 1 - n^{-1}\}\,(n \in \mathbb{Z}_{\geqslant 1})$; these union to $\mathfrak{p}$. Since $\mathfrak{p}$ is compact, $\mathfrak{p} = \{|x| < 1 - n^{-1}\}$ for some $n \geqslant 1$. This proves the discreteness.

Now we prove the if part. We only need to show $\mathfrak{o}$ is compact. Let S be a representative of $\mathfrak{o}/\mathfrak{p}$ in $\mathfrak{o}$. By Lemma 8.2.1.(iv) let $\pi \in \mathfrak{p}$ be a generator of $\mathfrak{p}$. Let $\{U_\alpha\}_\alpha$ be an open cover of $\mathfrak{o}$. We prove it by contradiction that $\mathfrak{o}$ admits no finite subcover.[4] Since $\mathfrak{o} = \bigsqcup_{x \in S} x + \mathfrak{p}$, there is some $x_0 \in S$ such that $x_0 + \pi\mathfrak{o}$ is not covered by finitely many $\{U_\alpha\}$. Suppose for $i \leqslant n$ we've found $x_0, \ldots, x_i \in S$

[4]The proof strikes a resemblance to the proof of Heine-Borel theorem in $\mathbb{R}^n$.

such that $x_0 + x_1\pi + \cdots + x_n\pi^n\mathfrak{o}$ is not covered by finitely many $\{U_\alpha\}_\alpha$. Take $x_{n+1} \in S$ such that $x_0 + x_1\pi + \cdots + x_n\pi^n + x_{n+1}\pi^{n+1}\mathfrak{o}$ is not covered by finitely many $\{U_\alpha\}_\alpha$. Consider the infinite series $x = \sum_{n \geqslant 0} x_n\pi^n$. Its partial sums form a Cauchy sequence in k, so by completeness $x \in k$; since $|x| \leqslant |x_0|$, $x \in \mathfrak{o}$. Hence $x \in U_\beta$ for some β. But then $x + \pi^N\mathfrak{o} \subseteq U_\beta$ for some $N \gg 0$. This is a contradiction by the construction of x. Hence $\{U_\alpha\}$ has a finite subcover of $\mathfrak{o}$. This proves the compactness. ☐

Theorem 8.3.2. Let $(k, |\cdot|)$ be a **local field**, i.e., a non-discrete locally compact valued field.

(i) If $|\cdot|$ is archimedean, then $k \cong \mathbb{R}$ or $\mathbb{C}$.

(ii) If $|\cdot|$ is non-archimedean and $\operatorname{Char} k = 0$, then k is a finite extension of $\mathbb{Q}_p$ for some rational prime p.

(iii) If $|\cdot|$ is non-archimedean and $\operatorname{Char} k = p$ for some rational prime p, then k is a finite extension of $\mathbb{F}_p(\!(t)\!)$, where t is transcendental over $\mathbb{F}_p$.

Proof. Being non-discrete locally compact, for each $r > 0$ the ball $\{x \in k \mid |x| \leqslant r\}$ is compact. Since every Cauchy sequence is bounded, it follows that $(k, |\cdot|)$ is complete. In view of Corollary 8.1.2.1, we only need to show $(k, |\cdot|)$ contains $\mathbb{R}$, $\mathbb{Q}_p$ or $\mathbb{F}_p(\!(t)\!)$ as valued subfields in each case.

(i) This is Theorem 8.2.4. But since we only need to show k contains $\mathbb{R}$, this follows immediately from Corollary 8.2.3.1.

(ii) We have $\mathbb{Q} \subseteq k$. By Theorem 8.3.1, $\mathfrak{o}/\mathfrak{p}$ is finite. Since $\mathbb{Z} \subseteq \mathfrak{o}$ is infinite, we see $n \in \mathfrak{p}$ for some $n \in \mathbb{Z}$. This implies $|\cdot||_{\mathbb{Q}}$ is nontrivial.[5] By Theorem 8.2.3, $|\cdot|$ restricts to $|\cdot|_p$ for some rational prime p by Theorem 8.2.3. Since k is complete, k contains the closure of $\mathbb{Q}$ which is $\mathbb{Q}_p$.

(iii) We have $\mathbb{F}_p \subseteq k$. Since we are assuming k is non-discrete, the possibility that k being algebraic over $\mathbb{F}_p$ is excluded: indeed, any element $0 \neq x$ of k algebraic over $\mathbb{F}_p$ lies in some finite field, so $x^n = 1$ for some $n \geqslant 1$. This implies $1 = |x^n| = |x|^n$, so that $|x| = 1$. If $k/\mathbb{F}_p$ were algebraic, $|\cdot|$ would be a trivial valuation, making k being discretely topologized, a contradiction. Hence $k/\mathbb{F}_p$ is transcendental. Let u be a transcendental element such that $|u| \neq 1$ and consider the subfield $\mathbb{F}_p(u)$. By our choice, the restriction of $|\cdot|$ to $\mathbb{F}_p(u)$ is nontrivial. By Theorem 8.2.5 the restriction to either equivalent to $|\cdot|_s$ for some irreducible $s \in \mathbb{F}_p[u]$ or to $|\cdot|_\infty$. In the former case, the completion is $\mathbb{F}_p(\!(s)\!)$, and in the latter is $\mathbb{F}_p(\!(u^{-1})\!)$. In any case, k contains $\mathbb{F}_p(\!(t)\!)$ for some transcendental $t \in k$ over $\mathbb{F}_p$.

☐

8.4 Extension of valuations

Definition. Let K/k be a field extension and $|\cdot|$ a valuation on k. An **extension of** $|\cdot|$ **to** K is a valuation $|\cdot|_K$ on K such that $|\cdot|_K|_k$ is a valuation on k equivalent to $|\cdot|$.

Theorem 8.4.1. Let $(k, |\cdot|)$ be a complete valued field, and let K/k be a finite field extension. Then there exists a unique[6] extension $|\cdot|_K$ of $|\cdot|$ to K. Precisely, $|\cdot|_K$ can be chosen to be

$$|x|_K = |N_{K/k}(x)|^{\frac{1}{[K:k]}} \qquad (x \in K)$$

[5] We can also argue as follows. If $|\cdot||_{\mathbb{Q}}$ is trivial, then $\mathbb{Z}$ is a discrete subset. Since $\mathbb{Z} \subseteq \mathfrak{o}$ and $\mathfrak{o}$ is compact, this implies $\mathbb{Z}$ is finite, which is absurd. Still alternatively, this follows from Corollary 2.5.2.1.(iii)

[6] up to an equivalence.

where $N_{K/k} : K \to k$ is the norm map.

Proof. We start with the uniqueness. Let $|\cdot|_K$ be an extension; in view of Lemma 8.0.1, replacing $|\cdot|_K$ by its power, we assume $|\cdot|_K$ is an absolute value. We also replace $|\cdot|$ by $|\cdot|_K|_k$ so that $|\cdot|_K|_k = |\cdot|$. In particular, $|\cdot|_K$ defines a (k-)norm topology on K. Now the uniqueness follows from Theorem 8.1.2 and Lemma 8.1.1.

For the existence, we deal with the archimedean case and the non-archimedean case separately. If $|\cdot|$ is archimedean, by Theorem 8.2.4, k is either $\mathbb{R}$ or $\mathbb{C}$, and so is K. In either case, the map $z \mapsto |z\overline{z}|$ defines a valuation on K, where $\bar{\cdot}$ denotes the complex conjugation.

Since we will only exclusively deal with the case when k is locally compact, we give a proof adapted to this case. For each $x \in K$ consider the k-linear map $\ell_x : K \to K$ given by multiplication: $\ell_x(y) := xy$. The map $x \mapsto \ell_x$ gives an injective k-algebra homomorphism $K \to \operatorname{End}_k K$. Define the norm $N_{K/k}$ as the composition $K \to \operatorname{End}_k K \xrightarrow{\det} k$ (this is independent of the choice of k-bases of K), and put $|\cdot|_K : K \to \mathbb{R}_{\geqslant 0}$ as $|x|_K := |N_{K/k}(x)|$. To show it is a valuation, we must show if $|x|_K \leqslant 1$, then $|1+x|_K \leqslant C$ for some $C > 0$. Let $\|\cdot\|$ be a norm on K (as a k-vector space). Then $x \mapsto |x|_K$ is a continuous map $K \to \mathbb{R}_{\geqslant 0}$ with respect to $\|\cdot\|$. Since k is locally compact, K is locally compact as well; in particular, $\{\|x\| = 1\}$ is compact, so we can find $A, B > 0$ such that $B < |x|_K \leqslant A$ for all x with $\|x\| = 1$. Then for $|x| \leqslant 1$

$$|1+x|_K = \frac{|1+x|_K}{\|1+x\|}\,\|1+x\| \leqslant A(\|1\| + \|x\|) = A\left(\|1\| + \frac{\|x\|}{|x|_K}|x|_K\right) \leqslant A(\|1\| + B^{-1}) =: C$$

For the remaining case, we need to introduce more tools. ☐

Let $|\cdot|$ be a non-archimedean valuation on a field k, and put $(\mathfrak{o}, \mathfrak{p})$ to be its ring of integers. For a polynomial $f \in \mathfrak{o}[X]$, we write $\overline{f} \in (\mathfrak{o}/\mathfrak{p})[X]$ for its image in $(\mathfrak{o}/\mathfrak{p})[X]$. We say a polynomial $f \in \mathfrak{o}[X]$ is **primitive** if $\overline{f} \neq 0$, i.e., not all coefficients of f lie in $\mathfrak{p}$.

Theorem 8.4.2 (Hensel's lemma)**.** Let $(k, |\cdot|)$ be a complete non-archimedean valued field and let $f \in \mathfrak{o}[X]$ be primitive. If $\overline{f} = \overline{g_0}\,\overline{h_0}$ for some $g_0, h_0 \in \mathfrak{o}[X]$ with $\overline{g_0} \in (\mathfrak{o}/\mathfrak{p})[X]$ monic and $\overline{g_0}$, $\overline{h_0}$ relatively prime, then there exist $g, h \in \mathfrak{o}[X]$ such that $f = gh$ with $\overline{g} = \overline{g_0}$, $\overline{h} = \overline{h_0}$, g monic and $\deg g = \deg \overline{g_0}$.

Proof. Without altering $\overline{g_0}$ and $\overline{h_0}$, we assume all nonzero coefficients of g_0 and h_0 do not lie in $\mathfrak{p}$. In particular, the leading coefficients of g_0, h_0 are units in $\mathfrak{o}$, $\deg g_0 = \deg \overline{g_0}$ and $\deg h_0 = \deg \overline{h_0}$; hence $\deg h_0 \leqslant \deg f - \deg g_0$. Also, we can assume g_0 is monic.

Since $\overline{g_0}$, $\overline{h_0}$ are relatively prime, $g_0 s + h_0 t \equiv 1 \pmod{\mathfrak{p}}$ for some $s, t \in \mathfrak{o}[X]$. Put $r_0 = f - g_0 h_0 = \sum_{i \geqslant 0} a_i X^i$ and $g_0 s + h_0 t - 1 = \sum_{i \geqslant 0} b_i X^i$; then $\{a_i, b_i\}_{i \geqslant 0} \subseteq \mathfrak{p}$ and we can choose $0 \neq c \in \mathfrak{p}$ such that $|c| = \max\{|a_i|, |b_j|\}_{i,j \geqslant 0} < 1$. Then a_i/c, $b_j/c \in \mathfrak{o}$, so that

$$f \equiv g_0 h_0 \pmod{c}, \qquad g_0 s + h_0 t \equiv 1 \pmod{c}.$$

Here mod c means all coefficients of their difference lie in $c\mathfrak{o}$.

We construct $g_n, h_n \in \mathfrak{o}[X]$ satisfying $f \equiv g_n h_n \pmod{c^{n+1}}$ as follows. Since g_0 is monic, division by g_0 is possible. Hence, there exists r_n, q, $k \in \mathfrak{o}[X]$ such that

$$f - g_n = c^{n+1} r_n, \qquad r_n t = g_0 q + k$$

with $\deg k < \deg g_0$ or $k = 0$. Consider the polynomial $h_0 q + r_n s$ and denote by $\ell \in \mathfrak{o}[X]$ the one obtained by replacing all coefficients in $h_0 q + r_n s$ divisible by c with 0, so that $h_0 q + r_n s \equiv \ell \pmod{c}$.

Set

$$g_{n+1} = g_n + c^{n+1}k, \qquad h_{n+1} = h_n + c^{n+1}\ell.$$

We prove by induction on n that

- $\overline{g_n} = \overline{g_0}$, $\overline{h_n} = \overline{h_0}$,
- $\deg h_n \leqslant \deg f - \deg g_0$,
- g_n is monic and $f \equiv g_n h_n \pmod{c^{n+1}}$.

These hold for $n = 0$ as said in the first paragraph. For $n \geqslant 0$, we have $\overline{g_{n+1}} = \overline{g_n} = \overline{g_0}$ and $\overline{h_{n+1}} = \overline{g_n} = \overline{g_0}$. Also, g_{n+1} is monic as g_n is and $\deg k < \deg g_0$ or $k = 0$. Since

$$r_n \equiv r_n(g_0 s + h_0 t) \equiv g_0(\ell - h_0 q) + h_0(g_0 q + k) = g_0\ell + h_0 k \pmod{c}$$

we have

$$\begin{aligned}
f - g_{n+1}h_{n+1} &= f - (g_n + c^{n+1}k)(h_n + c^{n+1}\ell) = (f - g_n h_n) - c^{n+1}(g_n\ell + h_n k) - c^{2n+2}k\ell \\
&\equiv c^{n+1}(r_n - (g_n\ell + h_n k)) \pmod{c^{n+2}} \\
&\equiv c^{n+1}(r_n - (g_0\ell + h_0 k)) \equiv 0 \pmod{c^{n+2}}
\end{aligned}$$

For the second item, if $\deg h_{n+1} > \deg f - \deg g_0$, then we would have $\deg \ell > \deg f - \deg g_0$. Together with $\deg h_0 k < \deg h_0 + \deg g_0 \leqslant \deg f$, we have

$$\deg g_0\ell + h_0 k = \deg g_0\ell > \deg f.$$

But $r_n \equiv g_0\ell + h_0 k \pmod{c}$ and $\deg r_n \leqslant \deg f$, the leading coefficients of $g_0\ell$ is a multiple of c. This is a contradiction to the construction of ℓ. Hence $\deg h_{n+1} \leqslant \deg f - \deg g_0$.

By construction, $(g_n)_n$ is a sequence in $\mathfrak{o}[X]$ satisfying $g_{n+1} \equiv g_n \pmod{c^{n+1}}$. Since $c \in \mathfrak{p}$, this implies $(g_n)_n$ is a Cauchy sequence in $\mathfrak{o}[X]$. By completeness, $g := \lim\limits_{n\to\infty} g_n$ exists in $\mathfrak{o}[X]$; similarly, $h := \lim\limits_{n\to\infty} h_n$ exists in $\mathfrak{o}[X]$. Since $f \equiv g_n h_n \equiv gh \pmod{c^{n+1}}$, we have $f = gh$. Also, $\overline{g} = \overline{g_0}$, $\deg g = \deg g_0$ and g is monic. Also, $\overline{h} = \overline{h_0}$ and $\deg h \leqslant \deg f - \deg g_0$. This finishes the proof. $\square$

Corollary 8.4.2.1. Let $(k, |\cdot|)$ be a completed non-archimedean valued field. Let $f = \sum\limits_{i=0}^{n} a_i X^i \in k[X]$ be an irreducible polynomial such that $a_n a_0 \neq 0$, then $\max\limits_{0 \leqslant i \leqslant n} |a_i| = \max\{|a_n|, |a_0|\}$. In particular, $a_n = 1$ and $a_0 \in \mathfrak{o}$ imply $f \in \mathfrak{o}[X]$.

Proof. Multiplying some suitable elements in k if necessary, we can assume $f \in \mathfrak{o}[X]$ and $\max\limits_{0 \leqslant i \leqslant n} |a_i| = 1$; let $0 \leqslant r \leqslant n$ be the smallest index such that $|a_r| = 1$. Then

$$f(X) \equiv X^r(a_r + a_{r+1}X + \cdots + a_n X^{n-r}) \pmod{\mathfrak{m}}.$$

If $\max\{|a_n|, |a_0|\} < 1$, then $0 < r < n$ and the congruence would contradict Theorem 8.4.2. $\square$

Proof. (of Theorem 8.4.1) Let $\mathfrak{O}$ denote the integral closure of the ring of integers $\mathfrak{o}$ in K. We claim

$$\mathfrak{O} = \{x \in K \mid N_{K/k}(x) \in \mathfrak{o}\}.$$

Let $x \in \mathfrak{O}$ and consider the tower $k \subseteq k(x) \subseteq K$. Let β be a $k(x)$-basis for K and γ be a k-basis for $k(x)$; then $\{vw \mid v \in \beta, w \in \gamma\}$ is a k-basis for K, and computing determinant with respect to this basis, we see $N_{K/k} = N_{k(x)/k} \circ N_{K/k(x)}$. Since $x \in k(x)$, we obtain $N_{K/k}(x) = N_{k(x)/k}(x^{[K:k(x)]}) = N_{k(x)/k}(x)^{[K:k(x)]}$. Since x is integral over k, the minimal polynomial of x over k is monic and lies in $\mathfrak{o}[X]$; in particular, by computing $N_{k(x)/k}$ with respect to the basis $1, x, \ldots, x^{[k(x):k]-1}$, we see $N_{k(x)/k}(x) \in \mathfrak{o}$. This proves the containment $\subseteq$. For $\supseteq$, let $x \in K$ with $N_{K/k}(x) \in \mathfrak{o}$. Let $f = \sum_{i=0}^{n} a_i X^i \in k[X]$ be the (monic) minimal polynomial of x over k. Then

$$\mathfrak{o} \ni N_{K/k}(x) = N_{k(x)/k}(x)^{[K:k(x)]} = \pm a_0^{[K:k(x)]}$$

so that $|a_0| \leqslant 1$, i.e., $a_0 \in \mathfrak{o}$. It follows from Corollary 8.4.2.1 that $f \in \mathfrak{o}[X]$, so that $x \in \mathfrak{O}$.

We finish the proof by showing $|\cdot|_K : K \to \mathbb{R}_{\geqslant 0}$ is a non-archimedean valuation. We only need to show that $|1+x|_K \leqslant 1$ if $|x|_K \leqslant 1$. This amounts to showing that $x \in \mathfrak{O}$ implies $1 + x \in \mathfrak{O}$. This is clear as $\mathfrak{O}$ is a subring of K. $\square$

Corollary 8.4.17.1. Let $(k, |\cdot|)$ be a complete valued field. Then there is a unique extension $|\cdot|_{\mathrm{alg}}$ of $|\cdot|$ to the algebraic closure $\overline{k}$ of k. Precisely, for $x \in \overline{k}$, if $x \in K$ for some finite extension K/k, then

$$|x|_{\mathrm{alg}} := |N_{K/k}(x)|^{\frac{1}{[K:k]}}.$$

Proof. Recall for a tower of finite extensions $k \subseteq K \subseteq K'$, we have $N_{K'/k} = N_{K/k} \circ N_{K'/K}$. Hence if $x \in K$, we have

$$N_{K'/k}(x) = N_{K/k}(x^{[K':K]}) = N_{K/k}(x)^{[K':K]}$$

so that $|N_{K'/k}(x)|^{\frac{1}{[K':k]}} = |N_{K/k}(x)^{\frac{[K':K]}{[K':k]}}| = |N_{K/k}(x)|^{\frac{1}{[K:k]}}$. This shows $|\cdot|_{\mathrm{alg}} : \overline{k} \to \mathbb{R}_{\geqslant 0}$ is well-defined and is an extension of $|\cdot|$. The uniqueness follows from the construction and Theorem 8.4.1. $\square$

Let $(k, |\cdot|)$ be a complete valued field. Denote by G_k the absolute Galois group of k, i.e., $G_k = \mathrm{Gal}(\overline{k}/k)$. If $\sigma \in G_k$, then $x \mapsto |\sigma(x)|_{\mathrm{alg}}$ also defines an extension of $|\cdot|$ to $\overline{k}$. By uniqueness, we obtain that $|\sigma(x)|_{\mathrm{alg}} = |x|_{\mathrm{alg}}$ for all $x \in \overline{k}$, implying that G_k acts on $\overline{k}$ by isometries. As a consequence, we can deduce the following. Let G_k have its usual topology (c.f. Example 6.1.6) so that G_k is a profinite group. We claim the action map $G_k \times \overline{k} \to \overline{k}$ is continuous. Let $\sigma \in G_k$, $x \in \overline{k}$ and let $\delta > 0$. Say $x \in L$ for some finite extension L/k. Then if $|y - x|_{\mathrm{alg}} < \delta$, then

$$|\sigma\tau(y) - \sigma(x)|_{\mathrm{alg}} = |\sigma\tau(y) - \sigma\tau(x)|_{\mathrm{alg}} = |y - x|_{\mathrm{alg}} < \delta$$

so that $\sigma\,\mathrm{Gal}(\overline{k}/L) \times B_\delta(x)$ goes to $B_\delta(\sigma(x))$. This shows the action map is not only continuous, but also open.

Let $(k, |\cdot|)$ be a complete valued field. From Corollary 8.4.17.1 we know $|\cdot|$ extends uniquely to the algebraic closure $\overline{k}$ of k, making $(\overline{k}, |\cdot|)$ a valued field. For any $\alpha \in \overline{k}$, let $c(\alpha)$ be the set consisting of conjugates of α over k, and if $\#c(\alpha) \geqslant 2$, set $r(\alpha) := \min_{\gamma \in c(\alpha)\setminus\alpha} |\gamma - \alpha| > 0$.

Lemma 8.4.20 (Krasner's lemma)**.** Let $\alpha \in \overline{k}$ with $\#c(\alpha) \geqslant 2$.

(i) For all $\beta \in \overline{k}$ with $|\beta - \alpha| < r(\alpha)$, the minimal polynomial of α over $k(\beta)$ has no root other than α.

(ii) If α is separable, then $k(\alpha) \subseteq k(\beta)$ for all $|\beta - \alpha| < r(\alpha)$.

Proof. Let $\gamma \in \overline{k}$ be a conjugate of α over $k(\alpha)$. Since $\gamma - \beta$ and $\alpha - \beta$ are conjugate, from the result above we see $|\gamma - \beta| = |\alpha - \beta|$, and hence

$$|\alpha - \gamma| = |(\alpha - \beta) - (\gamma - \beta)| \leqslant |\alpha - \beta| < r(\alpha).$$

But γ is also conjugate to α over k, so $\alpha = \gamma$ by the definition of $r(\alpha)$. This shows (i). For (ii), since α is separable, by (i) we see the minimal polynomial of α over $k(\beta)$ is separable and has only one root. In other words, $\alpha \in k(\beta)$. □

Lemma 8.4.21. If $[\overline{k} : k] = \infty$, then $\overline{k}$ is not complete.

Proof. If $|\cdot|$ is archimedean, then $k \cong \mathbb{R}$ or $\mathbb{C}$ and $[\overline{k} : k] = 1, 2$. Hence $|\cdot|$ is non-archimedean. Since $k^{\text{sep}} \subseteq \overline{k}$ is dense, we must have $[k^{\text{sep}} : k] = \infty$ (otherwise k^{sep} is complete and $k^{\text{sep}} = \overline{k}$, so $[\overline{k} : k] < \infty$). Now choose $x_0 = 1, x_1, x_2, \ldots \in k^{\text{sep}}$ that are linearly independent over k. Choose $(c_n)_{n \geqslant 1} \subseteq k^{\times}$ such that $|c_{n+1}x_{n+1}| \leqslant |c_n x_n|$, $|c_n x_n| \to 0$ as $n \to \infty$ and $|c_{n+1}x_{n+1}| < r\left(\sum_{i=1}^{n} c_i x_i\right)$ (note that the sum is not in k by linear independence, so it has a conjugate other than it self). We claim $\sum_{i=1}^{\infty} c_i x_i$ has no limit in $\overline{k}$. Suppose for contradiction that it converges to $x \in \overline{k}$. Since

$$\left| x - \sum_{i=1}^{n} c_i x_i \right| = \left| \sum_{i=n+1}^{\infty} c_i x_i \right| \leqslant |c_{n+1}x_{n+1}| < r\left(\sum_{i=1}^{n} c_i x_i\right)$$

by Krasner's lemma $\sum_{i=1}^{n} c_i x_i \in k(x)$ for all $n \geqslant 1$. Since all c_n are nonzero, this implies that $x_n \in k(x)$ for all $n \geqslant 1$, a contradiction to $[k(x) : k] < \infty$. □

Denote by $\mathbb{C}_k$ the completion of $(\overline{k}, |\cdot|)$.

Theorem 8.4.22. $\mathbb{C}_k$ is algebraically closed.

Proof. In view of Theorem 8.2.4, we only need to cope with the case when $|\cdot|$ is non-archimedean. Let $f = X^n + \sum_{i=0}^{n-1} a_i X^i \in \mathbb{C}_k[X]$ be a polynomial with $n \geqslant 1$. For each $j \geqslant 1$, since $\overline{k}$ is dense in $\mathbb{C}_k[X]$ we can find $f_j(X) = \sum_{i=0}^{n} a_{ij} X^i \in \overline{k}[X]$ such that $|a_{ij} - a_i| < \min\{|a_i|, 1/j\}$ for each $0 \leqslant i \leqslant n-1$; if $a_i = 0$, we assume $a_{ij} = 0$. Since $\overline{k}$ is algebraically closed, we choose for each j a root $r_j \in \overline{k}$ of the polynomial f_{ij}. We are going to show that $(r_j)_j$ admits a Cauchy subsequence; since $\mathbb{C}_k$ is complete, it has a limit in it and by continuity it is a root of f. Since $f_{ij}(r_j) = 0$, we have

$$|r_j^n| = \left| -\sum_{i=0}^{n-1} a_{ij} r_j^i \right| \leqslant \max_{0 \leqslant i \leqslant n-1} |a_{ij}| |r_j|^i = \max_{0 \leqslant i \leqslant n-1} |a_i| |r_j|^i$$

Then there exists $0 \leqslant i(j) \leqslant n-1$ such that $|r_j|^n \leqslant |a_i| |r_j|^{i(j)}$, so that $|r_j| \leqslant |a_{i(j)}|^{\frac{1}{n-i(j)}}$. Hence

$$|r_j| \leqslant C := \max_{0 \leqslant i \leqslant n-1} |a_i|^{\frac{1}{n-i}}$$

for all $j \geqslant 1$, and

$$|f(r_j)| = |f(r_j) - f_j(r_j)| = \max_{0 \leqslant i \leqslant n-1} |a_i - a_{ij}| |r_j^i| \leqslant \frac{\max\{1, C^{n-1}\}}{j}.$$

This shows $f(r_j) \to 0$ as $j \to \infty$.

Now let L be a finite extension of $\mathbb{C}_k$ such that f splits. Write $f(X) = \prod_{i=1}^{n}(X - \alpha_i)$. Then $\left|\prod_{i=1}^{n}(r_j - \alpha_i)\right| \to 0$ as $j \to \infty$, so by pigeonhole principle there exists an $i \in [n]$ such that $(r_j - \alpha_i)_{j\geqslant 1}$ has a subsequence converging to 0. Hence $(r_j)_{j\geqslant 1}$ has a subsequence converging to α_i in L, so it has a Cauchy subsequence. $\square$

Definition. Let K/k be a field extension, $|\cdot|$ a non-archimedean valuation on k and $|\cdot|_K$ an extension of $|\cdot|$ on K. For notational issue, we sometimes set $v = |\cdot|$ and $w = |\cdot|_K$. We also assume $w|_k = v$.

(i) The **ramification index** $e(w|v)$ is the (group-theoretic) index $[w(K^\times) : v(k^\times)]$.

(ii) The quotient $\mathfrak{o}/\mathfrak{p}$ of the ring of integers of $|\cdot|$ by its maximal ideal is called the **residue field** of $|\cdot|$, and is denoted by $\kappa(v)$.

(iii) The **inertia degree** $f(w|v)$ is the (field-theoretic) index $[\kappa(w) : \kappa(v)]$.

Theorem 8.4.23. Let K/k be a simple algebraic extension.

(i) The map

$$\begin{aligned} M_{K,\mathrm{na}} &\longrightarrow M_{k,\mathrm{na}} \\ |\cdot| &\longmapsto |\cdot||_k \end{aligned}$$

is well-defined, surjective and has finite fibre.

Let $v \in M_{k,\mathrm{na}}$ and let $w \in M_{K,\mathrm{na}}$ be an extension of v with $w|_k = v$.

(ii) $[K : k] \geqslant e(w|v)f(w|v)$ with equality if v is discrete and (k, v) is complete.

(iii) Let $w_1, \ldots, w_r$ be all the extensions of v. Then $[K : k] \geqslant \sum_{i=1}^{r} e(w_i|v)f(w_i|v)$ with equality if v is discrete and either K/k is separable or (k, v) is complete.

Proof.

(i) Say $K = k[X]/f$ for some irreducible $f \in k[X]$. Let $v = |\cdot|$ be a non-archimedean valuation on k and denote by $\widehat{k}$ the completion of k with respect to $|\cdot|$. By Chinese remainder theorem, we have an isomorphism

$$\Phi : K \otimes_k \widehat{k} = \widehat{k}[X]/(f) \xrightarrow{\sim} \widehat{k}[X]/(f_1^{e_1}) \times \cdots \times \widehat{k}[X]/(f_n^{e_n}),$$

where $f_1, \ldots, f_n \in \widehat{k}[X]$ are the irreducible factors of f in $\widehat{k}[X]$. Let us put $K_i = \widehat{k}[X]/(f_i)$ $(i \in [n])$; then $K_1 \times \cdots \times K_n$ is the reduction[7] of $K \otimes_k \widehat{k}$. Form the commutative diagram

$$\begin{tikzcd} K \arrow[r] \arrow[drr, bend right, "\Psi"'] & K \otimes_k \widehat{k} \arrow[r, "\Phi"] & \widehat{k}[X]/(f_1^{e_1}) \times \cdots \times \widehat{k}[X]/(f_n^{e_n}) \arrow[d, two heads] \\ & & K_1 \times \cdots \times K_n \end{tikzcd}$$

Put $\iota_i = \mathrm{pr}_i \circ \Psi : K \to K_i$ $(i \in [n])$; since $\mathrm{pr}_i \circ \Phi|_K$ is nontrivial, so is ι_i. Since K is a field, it follows that $\iota_i : K \to K_i$ is injective. We will identify K with its image $\iota_i(K_i)$ in K_i. Each K_i

[7] That is, it is the quotient of $\widehat{k}[X]/(f_1^{e_1}) \times \cdots \times \widehat{k}[X]/(f_n^{e_n})$ by its nilradical.

is a finite field extension of $\widehat{k}$, so by Theorem 8.4.1 K_i admits a valuation $|\cdot|_i$ extending the one on $\widehat{k}$. This makes $(K_i, |\cdot|_i)$ a complete valued field, and is non-archimedean by Lemma 8.2.1.(iii). Consider the diagram

$$\begin{tikzcd}
K \arrow[r] & K_i \arrow[r, "|\cdot|_i"] & \mathbb{R}_{\geqslant 0} \\
k \arrow[u] \arrow[r] & \widehat{k} \arrow[u] \arrow[ur, "|\cdot|"'] &
\end{tikzcd}.$$

The square on the left is commutative, and the triangle on the right is commutative up to an equivalence. Hence, if we let, by abuse of notation, $|\cdot|_i : K \to \mathbb{R}_{\geqslant 0}$ stand for the restriction of $|\cdot|_i : K_i \to \mathbb{R}_{\geqslant 0}$ to K, then $|\cdot|_i$ extends $|\cdot|$. This shows the surjectivity assertion in (i).

To show the restriction in the theorem is well-defined, we must show $|\cdot| \in M_{K,\mathrm{na}}$ restricts to a nontrivial valuation $|\cdot||_k \in M_{k,\mathrm{na}}$ on k. Let $\beta \in K \backslash k$ be any element, and write $g(X) = \sum\limits_{i=0}^{N} a_i X^i$ with $a_0, a_N \neq 0$ for its minimal polynomial over k. If $|\beta| < 1$, then

$$0 = \left| \sum_{i=0}^{N} a_i \beta^i \right| = |a_0| = 1$$

a contradiction. If $|\beta| > 1$, then

$$0 = \left| \sum_{i=0}^{N} a_i \beta^i \right| = |a_N \beta^N| |\beta|^N > 1$$

a contradiction. This forces $|\beta| = 1$; since $\beta \in K \backslash k$ is arbitrary, it follows that $|\cdot|$ is trivial, again a contradiction to our assumption.

Now we make use of the topology to facilitate our argument. Any k-basis for K is also a $\widehat{k}$-basis for $K \otimes_k \widehat{k}$; we simply fix one such basis and topologize $K \otimes_k \widehat{k}$ by the induced sup norm $\|\cdot\|_\infty$. With this basis, it is easy to see K is dense. Next, we topologize $\widehat{k}[X]/(f_i^{e_i})$ by any norm topology, and topologize their product by the product topology; note the product topology is also normable. Since Φ is a $\widehat{k}$-isomorphism, it follows from Theorem 8.1.2 that Φ is a homeomorphism. Similarly we topologize $K_1 \times \cdots \times K_n$ by the product topology. If we identify $K_1 \times \cdots \times K_n$ as a quotient, we can equip it with the quotient norm (c.f. Lemma E.1.9) and the projection becomes continuous. Again by Theorem 8.1.2 the quotient norm is equivalent to the product norm. In sum, $\Psi : K \to K_1 \times \cdots \times K_n$ and, hence, $\iota_i : K \to K_i$ have dense image. This allows us to think of K_i as the completion of K with respect to $|\cdot|_i$. As a by-product, under our identification, weak approximation implies that $|\cdot|_1, \ldots, |\cdot|_n$ are inequivalent valuations on K.

Suppose $|\cdot|_K$ is any extension of $|\cdot|$. Let $\widehat{K}$ be the completion of K with respect to $|\cdot|_K$. Consider the composition $K\widehat{k} \subseteq \widehat{K}$. Since $[K\widehat{k} : \widehat{k}] < \infty$, $K\widehat{k}$ is closed under subspace topology inherited from $\widehat{K}$ by Theorem 8.1.2. Since K is both dense in $K\widehat{k}$ and $\widehat{K}$, we see $\widehat{K} = K\widehat{k}$. In particular, we have a surjection $K \otimes_k \widehat{k} \to \widehat{K}$, and taking reduction we obtain a surjection $K_1 \times \cdots \times K_n \to \widehat{K}$. It follows that $K_i \cong \widehat{K}$ for some $i \in [n]$, and by Theorem 8.1.2 again it is a homeomorphism. By Lemma 8.1.1, $|\cdot|_K$ is equivalent to $|\cdot|_i$. This proves (i). Also, the number r in (iii) is equal to n.

(ii) Let $\{x_i\}_{i\in I} \subseteq \mathfrak{O}$ be such that $\{\overline{x_i} = x_i \bmod \mathfrak{P}\}_{i\in I}$ is a $\kappa(v)$-basis of $\kappa(w)$, and let $\{\pi_j\}_{j\in J}$ be such that $\{w(\pi_j) = |\pi_j|_K\}_{j\in J}$ is a coset representative of $w(K^\times)/v(k^\times)$; up to a multiplication by an element in $k^\times$, we can assume $\{\pi_j\}_{j\in J} \subseteq \mathfrak{O}$. We claim $\{x_i\pi_j\}_{\substack{i\in I\\ j\in J}} \subseteq \mathfrak{O}$ is k-linearly independent. Say $\sum\limits_{i,j} a_{ij}x_i\pi_j = 0$ for some $a_{ij} \in k$; clearing the denominators, we assume $a_{ij} \in \mathfrak{o}$. Suppose for contradiction that $a_{i_0j_0} \neq 0$ for some $(i_0, j_0) \in I \times J$. Put $y_j = \sum\limits_{i\in I} a_{ij}x_i$ $(j \in J)$. Let i' be such that $|a_{i'j_0}| = \max\limits_{i\in J} |a_{ij_0}| > 0$; then $y_{j_0}/a_{i'j_0} = x_{i'} + \sum_{i\neq i;} a_{ij_0}a_{i'j_0}^{-1}x_i$ with $a_{ij_0}a_{i'j_0}^{-1} \in \mathfrak{o}$. Since $\{\overline{x_i}\}$ is a $\kappa(v)$-basis for $\kappa(w)$, we see $y_{j_0}/a_{i'j_0} \bmod \mathfrak{P} \neq 0$, i.e., $y_{j_0}/a_{i'j_0} \in \mathfrak{O}\backslash\mathfrak{P}$. Hence $|y_{j_0}|_K = |a_{i'j_0}|_K = |a_{i'j_0}| \in v(k^\times)$. In fact, from the argument we see for $j \in J$ either $|y_j|_K \in v(k^\times)$ or $y_j = 0$. Let $j' \in J$ be such that $|y_{j'}\pi_{j'}|_K = \max\limits_{j\in J} |y_j\pi_j| > 0$. If $|y_{j'}\pi_{j'}|_K > |y_j\pi_j|_K$ for any other j, then since $|\cdot|_K$ is non-archimedean, we have $\left|\sum\limits_{i,j} a_{ij}x_i\pi_j\right|_K = \left|\sum\limits_{j} y_j\pi_j\right|_K = |y_{j'}\pi_{j'}|_K \neq 0$, a contradiction. Hence $|y_{j'}\pi_{j'}|_K > |y_{j''}\pi_{j''}|_K$ for some other $j'' \neq j'$. But then $|\pi_{j'}|_K v(k^\times) = |\pi_{j''}|_K v(k^\times)$, a contradiction. Hence $a_{ij} = 0$ for all i, j, and this proves the inequality part in (ii).

Now suppose v is discrete. The above inequality shows $e(w|v) < \infty$, so $w(K^\times)$ is also a discrete subgroup of $\mathbb{R}_{>0}$. In particular, w is discrete. Let $\pi \in \mathfrak{P}$ be such that $\mathfrak{P} = \pi\mathfrak{O}$. Then $\{w(\pi^j)\}_{0\leqslant j\leqslant e(w|v)-1}$ is a coset representative of $w(K^\times)/v(k^\times)$. Consider the $\mathfrak{o}$-module

$$\mathfrak{O}' := \operatorname{span}_{\mathfrak{o}}\{x_i\pi^j \mid 0 \leqslant j \leqslant e(w|v),\ 1 \leqslant i \leqslant f(w|v)\} \subseteq \mathfrak{O}.$$

Also, put $\mathfrak{O}'' = \operatorname{span}_{\mathfrak{o}}\{x_i \mid 1 \leqslant i \leqslant f(w|v)\} \subseteq \mathfrak{O}$. By definition, we have $\mathfrak{O} = \mathfrak{O}'' + \mathfrak{P} = \mathfrak{O}'' + \pi\mathfrak{O}$. Iterating, we see

$$\mathfrak{O} = \mathfrak{O}'' + \pi\mathfrak{O}'' + \cdots + \pi^{e(w|v)-1}\mathfrak{O}'' + \pi^{e(w|v)}\mathfrak{O} = \mathfrak{O}' + \pi^{e(w|v)}\mathfrak{O}.$$

Since $w(\pi^{e(w|v)})$ generates $v(k^\times)$, we have $\pi^{e(w|v)}\mathfrak{O} = \mathfrak{p}\mathfrak{O}$, so that $\mathfrak{O} = \mathfrak{O}' + \mathfrak{p}\mathfrak{O}$. Iterating again, we obtain $\mathfrak{O} = \mathfrak{O}' + \mathfrak{p}^n\mathfrak{O}$ for all $n \geqslant 1$. Since $\{\mathfrak{p}^n\mathfrak{O}\}_{n\geqslant 1}$ forms a unit-neighborhood basis for $\mathfrak{O}$, this implies $\mathfrak{O}'$ is dense in $\mathfrak{O}$.

Now we assume (k, v) is complete. Since $\mathfrak{o} \subseteq k$ is closed, if follows from Theorem 8.1.2 that $\mathfrak{O}' \subseteq \mathfrak{O}$ is closed, and hence $\mathfrak{O}' = \mathfrak{O}$. This shows the equality holds when (k, v) is complete and discrete. Also, the assertion in (iii) with equality (k, v) is complete and discrete also follows (note that $r = 1$ in this case by Theorem 8.4.1).

(iii) By the isomorphism Φ and (ii), we have

$$[K : k] = \dim_{\widehat{k}} K \otimes_k \widehat{k} = \sum_{i=1}^{r} \dim_{\widehat{k}} \widehat{k}[X]/(f_i^{e_i}) \geqslant \sum_{i=1}^{r} [K_i : \widehat{k}] \geqslant \sum_{i=1}^{r} e(w_i|v)f(w_i|v).$$

If K/k is separable, the first inequality is an equality. If v is discrete, by (ii) the second inequality is an equality. This finishes the proof.

□

8.5 Ramification

Definition. Let K/k be a finite extension, and let v be a non-archimedean valuation on k.

(i) v is **unramified** in K if $f(w|v) = [K : k]$[8] and $\kappa(w)/\kappa(v)$ is a separable extension for each extension w of v to K.

(ii) v is **ramified (ramifies)** in K if v is not unramified in K.

(iii) v is **tamely ramified** if $\operatorname{Char} k = 0$ or $\operatorname{Char} k = p \geqslant 2$ and $p \nmid e(w, v)$, and $\kappa(w)/\kappa(v)$ is a separable extension for each extension w of v to K.

(iv) v is **totally ramified** in K if v extends to a unique valuation w on K with $e(w|v) = [K : k]$.

(v) v is said to **split completely** in K if v has exactly $[K : k]$ inequivalent extensions to K.

If (k, v) is complete, by Theorem 8.4.1 K admits a unique extension of v. In this case, the extension K/k is said to be **unramified/ramified/tamely ramified/totally ramified** if v is.

Lemma 8.5.1. Let (k, v) be a complete non-archimedean valued field, and (K, w) a finite extension of (k, v). Then K/k is unramified if and only if $K = k(\alpha)$ for some $\alpha \in \mathfrak{o}_w$ such that $\overline{\alpha} \in \kappa(w)$ is separable over $\kappa(v)$ and $\kappa(w) = \kappa(v)(\overline{\alpha})$.

Proof. The if part is clear. For the only if part, suppose K/k is unramified. Then $\kappa(w)/\kappa(v)$ is finite separable, so by primitive element theorem there exists $\overline{\alpha} \in \kappa(w)$ such that $\kappa(w) = \kappa(v)(\overline{\alpha})$. Let $\alpha \in \mathfrak{o}_w$ be any lift, and $f \in \mathfrak{o}_k[X]$ be the minimal polynomial of α (here it has integral coefficients by the last part of the proof of Theorem 8.4.1). Then

$$[\kappa(w) : \kappa(v)] \leqslant \deg f = [k(\alpha) : k] \leqslant [K : k] = [\kappa(w) : \kappa(v)],$$

so $K = k(\alpha)$. ☐

8.6 Galois theory of valuations

Lemma 8.6.1. Let K/k be an algebraic field extension. Then the restrictions

$$\begin{array}{ccc} M_{K,\mathrm{na}} & \longrightarrow & M_{k,\mathrm{na}} \\ |\cdot| & \longmapsto & |\cdot||_k \end{array}, \qquad \begin{array}{ccc} M_{K,\mathrm{a}} & \longrightarrow & M_{k,\mathrm{a}} \\ |\cdot| & \longmapsto & |\cdot||_k \end{array}$$

are well-defined.

Proof. Let $|\cdot|$ be a nontrivial valuation on K. If $|\cdot|$ is archimedean, then its restriction to $\mathbb{Q}$ is already nontrivial by Lemma 8.2.1.(vi) and $|\cdot||_k$ is still archimedean. If $|\cdot|$ is non-archimedean, this is a part of Theorem 8.4.23.(i). ☐

For a field k, $v \in M_k$ and $\sigma \in \operatorname{Aut} k$, we define $\sigma v \in M_k$ to be the valuation defined by

$$\sigma v(x) := v(\sigma(x)) \qquad (x \in k)$$

This defines a right $\operatorname{Aut} k$-action on M_k. Clearly v is non-archimedean if and only if σv is non-archimedean, so $M_{k,\mathrm{na}}$ and $M_{k,\mathrm{a}}$ are stabilized by $\operatorname{Aut} k$-actions.

Let K/k be a field extension, $v \in M_k$ and $w \in M_K$ such that $w|_k = v$. For $\sigma \in \operatorname{Aut}_k K$, we see σw again restricts to $v = \sigma v$. Hence $\operatorname{Aut}_k K$ acts on the fibre of the restriction $M_K \to M_k$. It is natural to ask if $\operatorname{Aut}_k K$ acts transitively on the fibre.

[8]It is more common to require $e(w|v) = 1$, but the current definition works better for general valuations. Note $f(w|v) = [K : k]$ implies $e(w|v) = 1$ by Theorem 8.4.23.(ii), and they are equivalent when (k, v) is complete and discrete by the same theorem.

Lemma 8.6.2. Let K/k be a Galois extension. Then $\mathrm{Gal}(K/k)$ acts transitively on the fibre of $M_k \to M_k$.

Proof. Suppose K/k is finite, and suppose for contradiction that $\mathrm{Gal}(K/k)w$ and $\mathrm{Gal}(K/k)w'$ are two disjoint $\mathrm{Gal}(K/k)$-orbits in the fibre of $v \in M_k$. By weak approximation there exists $\alpha \in K$ such that $|\sigma(\alpha)|_w < 1$ for each $\sigma \in \mathrm{Gal}(K/k)$ while $|\sigma(\alpha)|_{w'} > 1$ for each $\sigma \in \mathrm{Gal}(K/k)$. But then

$$1 > \prod_{\sigma \in \mathrm{Gal}(K/k)} |\sigma(\alpha)|_w = |N_{K/k}(\alpha)|_w = |N_{K/k}(\alpha)|_v = |N_{K/k}(\alpha)|_{w'} = \prod_{\sigma \in \mathrm{Gal}(K/k)} |\sigma(\alpha)|_{w'} > 1$$

is an obvious contradiction.

Suppose that K/k is infinite, and $w, w' \in M_K$ be in the same fibre. For each finite Galois subextension M/k of K, consider the set $X_M := \{\sigma \in \mathrm{Gal}(K/k) \mid \sigma w|_M = w'|_M\}$. We've seen that X_M is nonempty (as M/k is finite), and it is closed: if $\sigma \in \mathrm{Gal}(K/k)\backslash X_M$, then $\sigma \,\mathrm{Gal}(K/M) \subseteq \mathrm{Gal}(K/k)\backslash X_M$. By compactness of $\mathrm{Gal}(K/k)$, we have $\bigcap_M X_M \neq \varnothing$, where M runs over all finite Galois subextensions M/k of K. Then any element in the intersection sends w to w'. $\square$

Definition. Let K/k be a Galois extension, and $w \in M_K$. The **decomposition group of** w is the closed subgroup

$$D(w, K/k) := \{\sigma \in \mathrm{Gal}(K/k) \mid \sigma w = w\} \leqslant \mathrm{Gal}(K/k).$$

Lemma 8.6.3. Let K/k be a finite Galois extension, $w \in M_K$ and $v = w|_k \in M_k$.

(i) The extension K_w/k_v is finite Galois.

(ii) The image of the restriction $\mathrm{Gal}(K_w/k_v) \to \mathrm{Gal}(K/k)$ is the decomposition group $D(w, K/k)$.

Hence $\mathrm{Gal}(K_w/k_v) \cong D(w, K/k)$ canonically.

8.7 Norm map

Chapter 9

Adele ring of a global field

9.1 Restricted Topological Products

9.1.1 Generality

Let $\{X_\alpha \mid \alpha \in I\}$ be a family of topological spaces. Let $J \subseteq I$ be a cofinite subset and to each $\alpha \in J$ we assign an open subspace $Y_\alpha \subseteq X_\alpha$. Consider the **restricted (topological) product of the** X_α **with respect to the** Y_α:

$$\prod_{\alpha\in I}{}' X_\alpha = \prod_{\alpha\in I}^{\{Y_\alpha\}_{\alpha\in J}} X_\alpha = \left\{ (x_\alpha)_\alpha \in \prod_{\alpha\in I} X_\alpha \;\middle|\; \begin{array}{c} \text{there exists a cofinite subset } J_0 \subseteq J \\ \text{such that } x_\alpha \in Y_\alpha \text{ for all } \alpha \in J_0 \end{array} \right\}.$$

The condition imposed on the restricted product is usually rephrased as "$x_\alpha \in Y_\alpha$ for all but finitely many $\alpha \in I$", or "$\forall'\alpha \in I[x_\alpha \in Y_\alpha]$". Hence

$$\prod_{\alpha\in I}{}' X_\alpha = \left\{ (x_\alpha)_\alpha \in \prod_{\alpha\in I} X_\alpha \mid \forall'\alpha \in I\,[x_\alpha \in Y_\alpha] \right\}.$$

We topologize $\prod_{\alpha\in I}{}' X_\alpha$ by declaring the collection $\left\{ \prod_{\alpha\in I} U_\alpha \mid U_\alpha \underset{\text{open}}{\subseteq} X_\alpha,\, \forall'\alpha \in I\,[U_\alpha = Y_\alpha] \right\}$ to be the (sub)basis for the topology; we will refer to the set of this form as a **basic open set**. It is clear that if we alter J by a finite subset of I and change finitely many Y_α, the restricted product as well as the topology remains unchanged.

Lemma 9.1.1. *A relatively compact set in $\prod_{\alpha\in I}{}' X_\alpha$ is contained in some X_S for some finite $I\backslash J \subseteq S \subseteq I$, where*

$$X_S := \prod_{\alpha\in S} X_\alpha \times \prod_{\alpha\in I\backslash S} Y_\alpha \underset{\text{open}}{\subseteq} \prod_{\alpha\in I}{}' X_\alpha$$

In particular, a relatively compact set is contained in some $\prod_{\alpha\in I} K_\alpha$, where $K_\alpha \underset{\text{cpt}}{\subseteq} X_\alpha$ and $K_\alpha = Y_\alpha\, \forall'\alpha \in I$.

Proof. Let K be a compact set in $X = \prod_{\alpha\in I}{}' X_\alpha$. Since the collection $\{X_S \mid I\backslash J \subseteq S \subseteq I\text{: finite}\}$ form an open cover of X, and is closed under finite union, we see $K \subseteq X_S$ for some S. The last assertion follows from the projection of K to an X_α is compact.

□

Lemma 9.1.2. Suppose each X_α is LCH and each Y_α is compact. Then the space $\prod'_{\alpha\in I} X_\alpha$ is LCH.

Proof. Being Hausdorff is clear. By Tychonov's theorem, the sets $\prod_{\alpha\in I} U_\alpha$ described above are locally compact and these form an open cover of $\prod'_{\alpha\in I} X_\alpha$. Hence it is locally compact. □

Lemma 9.1.3. Suppose $X_\alpha = G_\alpha$ are topological groups and $Y_\alpha = H_\alpha$ are open subgroups. Then the restricted product $G = \prod_{\alpha\in I}^{\{H_\alpha\}_{\alpha\in J}} G_\alpha$ is a topological groups.

Proof. Let $x_\alpha \in G_\alpha$ and suppose y_α, $z_\alpha \in G_\alpha$ satisfy $y_\alpha z_\alpha = x_\alpha$. Let $U_\alpha \subseteq G_\alpha$ be a neighborhood of x_α; then we can find neighborhoods $V_\alpha, W_\alpha \subseteq G_\alpha$ of y_α and z_α respectively such that $V_\alpha W_\alpha \subseteq U_\alpha$.

Suppose $(x_\alpha)_\alpha, (y_\alpha)_\alpha, (z_\alpha)_\alpha \in G$ and $U := \prod_{\alpha\in I} U_\alpha$ is a basic open set. Then for all but finitely many $\alpha \in I$, we have $x_\alpha, y_\alpha, z_\alpha \in H_\alpha$ and $U_\alpha = H_\alpha$; in this case, we can choose $V_\alpha = W_\alpha = H_\alpha$ so that $V := \prod_{\alpha\in I} V_\alpha$, $W := \prod_{\alpha\in I} W_\alpha$ are open neighborhoods of $(y_\alpha)_\alpha$ and $(z_\alpha)_\alpha$ in G respectively. Our choice satisfies $VW \subseteq U$, so the multiplication is continuous.

The inversion is clearly continuous: the image of $\prod_{\alpha\in I} U_\alpha$ is simply $\prod_{\alpha\in I} U_\alpha^{-1}$, which is again a basic open set of G. □

In the following, assume each X_α is LCH and each Y_α is compact open.

Put $X = \prod'_{\alpha\in I} X_\alpha$. We are going to construct a measure on X from measures on X_α. We recall a definition. Let $\mathcal{S}$ be the collection of finite subsets of I containing $I\backslash J$; this is a set directed by inclusion. We say a function $f : \mathcal{S} \to \mathbb{C}$, or $(f_S)_{S\in\mathcal{S}} \subseteq \mathbb{C}$, converges if it converges as a net (c.f. Example A.2.4).

Example 9.1.4 (Infinite product)**.** This will be the only practical example we encounter in this section. Let $(x_\alpha)_{\alpha\in I} \subseteq \mathbb{C}^\times$, and for $S \in \mathcal{S}$, put

$$x_S := \prod_{\alpha\in S} x_\alpha.$$

When the net $S \mapsto x_S$ converges, we write the limit as $\prod_{\alpha\in I} x_\alpha$. In this case we simply say $\prod_{\alpha\in I} x_\alpha$ converges.

For any cofinite $T \subseteq I$ if we similarly define $\prod_{\alpha\in T} x_\alpha$. If the whole infinite product $\prod_{\alpha\in I} x_\alpha$ converges, then $\prod_{\alpha\in T} x_\alpha$ converges as well and

$$\prod_{\alpha\in I} x_\alpha = \prod_{\alpha\notin T} x_\alpha \times \prod_{\alpha\in T} x_\alpha.$$

Definition. A function $f \in C_c(X)$ is called **factorizable** if there exist $f_\alpha \in C_c(X_\alpha)$ with $f_\alpha = \mathbf{1}_{Y_\alpha}$ $\forall'\alpha \in I$ such that

$$f(x) = \prod_{\alpha\in I} f_\alpha(x_\alpha)$$

for all $x = (x_\alpha)_\alpha \in X$. Note that the product is in fact a finite product. In this case we write $f = \otimes'_{\alpha\in I} f_\alpha$.

Lemma 9.1.5. Let A be the subspace of $C_0(X)$ spanned by all factorizable functions. Then A is a dense unital subalgebra of $C_0(X)$.

Proof. Of course we are going to apply Stone-Weierstrass to A. We must show A is a subalgebra that separates points, vanishes nowhere and is closed under conjugation.

- A is a subalgebra. This is clear as the product of two factorizable functions is factorizable.
- A is closed under conjugation. Clear as $\mathbf{1}_{X_\alpha}$ is a real-valued function.
- A vanishes nowhere. This is clear as $\mathbf{1}_X \in A$; note that $\mathbf{1}_X$ is factorizable with $f_\alpha = \mathbf{1}_{X_\alpha}$. In particular, A is unital.
- A separates points. Let $x = (x_\alpha)_\alpha$, $y = (y_\alpha)_\alpha \in X$ and $x \neq y$. Then $x_\alpha \neq y_\alpha$ for some $\alpha \in I$. Take $f_\alpha \in C(X_\alpha)$ with $f_\alpha(x_\alpha) \neq f_\alpha(y_\alpha)$, and set $f_\beta = \mathbf{1}_{X_\beta}$ for $\beta \neq \alpha$. Then the function $f : X \to \mathbb{C}$ defined by $f = \otimes'_{\alpha\in I} f_\alpha$ is continuous and satisfies $f(x) \neq f(y)$.

Hence by Stone-Weierstrass, A is dense in $C_0(X)$. Note that $A \subseteq C_c(X)$. □

Corollary 9.1.5.1. Let $\{(X_\alpha, \mu_\alpha) \mid \alpha \in I\}$ be a collection of compact Hausdorff spaces together with outer Radon measures. Suppose $\mu_\alpha(X_\alpha) = 1$ for almost all $\alpha \in I$. Then there exists a unique Radon measure μ on the product X such that

$$\int_X f d\mu = \prod_{\alpha\in I} \int_{X_\alpha} f_\alpha d\mu_\alpha$$

for all factorizable $f = \otimes'_{\alpha\in I} f_\alpha$. The identity holds for $f = \otimes'_{\alpha\in I} f_\alpha$[1] with $f_\alpha \in L^1(X_\alpha)$ and $f_\alpha = \mathbf{1}_{X_\alpha}$ $\forall'\alpha \in I$ as well.

Proof. Let A be as in the lemma. Define a functional $T : A \to \mathbb{C}$ as follows. If $f \in A$ is factorizable, say $f = \otimes'_{\alpha\in I} f_\alpha$, then set

$$T(f) = \prod_{\alpha\in I} \int_{X_\alpha} f_\alpha d\mu_\alpha$$

This is well-defined, as $f_\alpha = \mathbf{1}_{X_\alpha}$ $\forall'\alpha \in I$, and by assumption this is a finite prodcut. By linearity this finishes the definition of the functional $T : A \to \mathbb{C}$. Note that if $f \in A$, there exists a finite subset $S \subseteq I$ and $g \in C\left(\prod_{\alpha\in S} X_\alpha\right)$ such that $f(x) = g((x_\alpha)_{\alpha\in S}) \prod_{\alpha\in I\setminus S} \mathbf{1}_{X_\alpha}(x_\alpha)$. In this case we have

$$T(f) = \int_{\prod_{\alpha\in S} X_\alpha} g d(\otimes_{\alpha\in S}\mu_\alpha)$$

where $\otimes_{\alpha\in S}\mu_\alpha$ is the unique Radon measure on $\prod_{\alpha\in S} X_\alpha$ constructed in Theorem D.4.7 (and by induction). Note also that $\|g\|_{\prod_{\alpha\in S} X_\alpha} = \|f\|_X$.

We extend T to a functional on $C(X)$ by continuity. Precisely, for $f \in C(X)$ pick a sequence $(f_n)_{n\geqslant 1}$ in A such that $f_n \to f$ in sup norm. Define

$$T(f) := \lim_{n\to\infty} T(f_n).$$

[1] Such a function is μ-integrable by Theorem D.4.7.(b). We will also call such an integrable function **factorizable**.

We must check this is well-defined. First, we show $(T(f_n))_n$ is a Cauchy sequence. Let $m, n \geqslant 1$ and take finite S and $g_n, g_m \in C\left(\prod_{\alpha \in S} X_\alpha\right)$ chosen as in the end of the first paragraph. Then

$$|T(f_n) - T(f_m)| = \left| \int_{\prod_{\alpha \in S} X_\alpha} g_n - g_m d(\otimes_{\alpha \in S} \mu_\alpha) \right| \leqslant \|g_n - g_m\|_{\prod_{\alpha \in S} X_\alpha} = \|f_n - f_m\|_X$$

Hence $(T(f_n))_n$ is a Cauchy sequence, and hence the limit exists. Second, we must show the limit is independent of the choice of $(f_n)_n$. It suffices to show if $f_n \to 0$ uniformly, then $T(f_n) \to 0$. This is clear by the above inequality. Hence this defines a functional $T : C(X) \to \mathbb{C}$. It is clear from the construction that T is positive, so by Riesz's representation theorem there exists a unique Radon measure μ on X such that

$$\int_X f d\mu = T(f)$$

for all $f \in C(X)$. The claimed integral formula follows from the construction, and this characterizes the measure μ again by Riesz's representation theorem. The last assertion follows from Theorem D.4.7.(b). $\square$

Theorem 9.1.6. For each $\alpha \in I$ let μ_α be an outer Radon measure on X_α. For $\alpha \in J$, suppose $\mu_\alpha(Y_\alpha) = 1$. There exists a unique Radon measure $\mu = \bigotimes_{\alpha \in I}' \mu_\alpha$ on X such that if $f = \otimes'_{\alpha \in I} f_\alpha$ is factorizable (either integrable or continuous with compact support), then

$$\int_X f d\mu = \prod_{\alpha \in I} \int_{X_\alpha} f_\alpha d\mu_\alpha.$$

The measure μ is called the **restricted product measure** of $\{\mu_\alpha \mid \alpha \in I\}$ on X.

Proof. For each finite subset $S \subseteq I$ containing $I \backslash J$, set

$$Y^S = \prod_{\alpha \in I \backslash S} Y_\alpha, \qquad X_S = \prod_{\alpha \in S} X_\alpha \times Y^S.$$

By Tychonov's theorem, Y^S is compact, and hence X_S is locally compact. By Corollary 9.1.5.1 there is a unique Radon measure μ^S on Y^S satisfying the integral formula there. By Theorem D.4.7 and induction, there exists a unique Radon measure μ_S on X_S satisfying

$$\int_{X_S} f d\mu_S = \prod_{\alpha \in S} \int_{X_\alpha} f_\alpha d\mu_\alpha \times \int_{Y^S} f^S d\mu^S \tag{$\spadesuit$}$$

where $f = \otimes_{\alpha \in S} f_\alpha \otimes f^S$ satisfies $f_\alpha \in C_c(X_\alpha)$ and $f^S \in C(Y^S)$. Since $X = \bigcup_{I \backslash J \subseteq S \subseteq I:\ \text{finite}} X_S$, to show $(\mu_S)_S$ defines a Radon measure on X, we begin by showing $\mu_T|_{X_S} = \mu_S$ for $S \subseteq T$. This is clear since μ_S is characterized by the formula ($\spadesuit$).

Let A be as in Lemma 9.1.5. For $f \in C_c(X)$, pick a sequence $(f_n)_n$ in A such that $f_n \to f$ in sup norm. For each n, there is a finite $I \backslash J \subseteq S_n \subseteq I$ such that $\operatorname{supp} f \subseteq X_{S_n}$. The same argument as in Corollary 9.1.5.1 shows that

$$\left(\int_{X_{S_n}} f_n d\mu_{S_n} \right)_n \subseteq \mathbb{C}$$

forms a Cauchy sequence, so the limit $T(f) = \lim_{n \to \infty} \int_{X_{S_n}} f_n d\mu_S$ exists.

- If we change S_n to a larger finite set, the integral is unchanged.
- The limit is independent of the choice of $(f_n)_n$; this follows from an argument similar to the one in Corollary 9.1.5.1.

This well-defines a positive linear functional $T : C_c(X) \to \mathbb{C}$. It follows from Riesz's representation theorem that there is a unique Radon measure μ on X such that

$$\int_X f d\mu = T(f)$$

for all $f \in C_c(X)$. The uniqueness is clear. □

Corollary 9.1.6.1. Let $G_\alpha\,(\alpha \in I)$ be an LCH group and $H_\alpha \leqslant G_\alpha\,(\alpha \in J)$ be an open compact subgroup. Let μ_α be a left (resp. right) Haar measure on G_α, and for $\alpha \in J$ we normalize the Haar measure μ_α so that $\mu_\alpha(H_\alpha) = 1$. Then there exists a unique left (resp. right) Haar measure μ on $G = \prod_{\alpha\in I}^{\{H_\alpha\}_{\alpha\in J}} G_\alpha$ such that if $f = \otimes'_{\alpha\in I} f_\alpha$ is factorizable, then

$$\int_G f d\mu = \prod_{\alpha\in I} \int_{G_\alpha} f_\alpha d\mu_\alpha.$$

Proof. Let $\mu = \bigotimes'_{\alpha\in I} \mu_\alpha$ be the measure as in Theorem 9.1.6. To show μ is a left Haar measure, by Lemma 2.2.4 it suffices to show the corresponding function $T : C_c(G) \to \mathbb{C}$ is left invariant. Let $f = \otimes'_{\alpha\in I} f_\alpha$ be factorizable and $g = (g_\alpha)_\alpha \in G$. Let $S \subseteq I$ be a cofinite subset such that $g_\alpha \in H_\alpha$ and $f_\alpha = \mathbf{1}_{H_\alpha}$ for $\alpha \in S$. Then $\lambda(g^{-1})f$ is again factorizable and

$$\begin{aligned} T(\lambda(g^{-1})f) = \prod_{\alpha\in I\setminus S} \int_{G_\alpha} f_\alpha(g_\alpha x_\alpha) d\mu_\alpha(x_\alpha) \times \prod_{\alpha\in S} \int_{G_\alpha} \mathbf{1}_{H_\alpha}(g_\alpha x_\alpha) d\mu_\alpha(x_\alpha) \\ = \prod_{\alpha\in I\setminus S} \int_{G_\alpha} f_\alpha(x_\alpha) d\mu_\alpha(x_\alpha) \times \prod_{\alpha\in S} \int_{G_\alpha} \mathbf{1}_{H_\alpha}(x_\alpha) d\mu_\alpha(x_\alpha) = T(f) \end{aligned}$$

since each $d\mu_\alpha$ is a Haar measure and $g_\alpha \in H_\alpha$ for $\alpha \in S$. Hence $T : A \to \mathbb{C}$ is a left-invariant functional. Finally, if $f \in C_c(X)$ and $(f_n)_n \in A$ with $f_n \to f$ in $C_c(X)$, then $\lambda(g)f_n \to \lambda(g)f$ so that

$$T(\lambda(g)f) = \lim_{n\to\infty} T(\lambda(g)f_n) = \lim_{n\to\infty} T(f_n) = T(f)$$

so that T is also left-invariant. This finishes the proof. The statement for the right Haar measure is proved in the same way. □

Retain the notation in Theorem 9.1.6 and its proof. Let $f : X \to [0, \infty]$ be a measurable function with $\{x \in X \mid f(x) \neq 0\}$ being σ-finite. Then by definition and Urysohn's Lemma

$$\int_X f d\mu = \sup_{\substack{\phi\in C_c(X) \\ 0\leqslant\phi\leqslant f}} \int_X \phi d\mu = \sup_{K \underset{\text{cpt}}{\subseteq} X} \int_K f d\mu$$

By Lemma 9.1.1, this tells

$$\int_X f d\mu = \sup_{\substack{I\setminus J\subseteq S\subseteq I \\ \#S<\infty}} \int_{X_S} f d\mu \tag{$\clubsuit$}$$

where $X_S = \prod_{\alpha\in S} X_\alpha \times \prod_{\alpha\in I\setminus S} Y_\alpha \underset{\text{open}}{\subseteq} X$. In particular, ($\clubsuit$) holds for all $f \in L^1(X)$ with sup replaced by lim, in the sense of nets.

Theorem 9.1.7. For each $v \in I$ let $f_\alpha \in L^1(X_\alpha) \cap C(X_\alpha)$ and suppose $f_\alpha(Y_\alpha) = \{1\}\, \forall' \alpha \in I$. Define formally the product $f : X \to \mathbb{C}$

$$f(x) := \prod_{\alpha \in I} f_\alpha(x_\alpha).$$

(i) f is well-defined and $f \in C(X)$.

(ii) For any finite subset $I \backslash J \subseteq S \subseteq I$ such that $f_\alpha(Y_\alpha) = 1$ for all $\alpha \notin S$, the formula holds

$$\int_{X_S} f d\mu = \prod_{\alpha \in S} \int_{X_\alpha} f_\alpha d\mu_\alpha.$$

(iii) If the net $\left(\prod_{\alpha \in S} \int_{X_\alpha} |f_\alpha| d\mu_\alpha \right)_{\substack{S \subseteq I \\ \# S < \infty}}$ converges and $\{x \in X \mid f(x) \neq 0\}$ is σ-finite, then $f \in L^1(X)$ and

$$\int_X f d\mu = \prod_{\alpha \in I} \int_{X_\alpha} f_\alpha d\mu_\alpha$$

Proof.

(i) That well-defined is clear since $x = (x_\alpha)_\alpha \in X$ satisfies $x_\alpha \in Y_\alpha$ for almost all $\alpha \in I$. This is continuous since $X = \bigcup X_S$ and $f|_{X_S}$ is a finite product of continuous functions.

(ii) This follows from the formula (♠) in Theorem 9.1.6.

(iii) By the identity (♣) applied to the function $|f|$ and (ii), we see

$$\int_X |f| d\mu = \sup_{\substack{I \backslash J \subseteq S \subseteq I \\ \# S < \infty}} \prod_{\alpha \in S} \int_{X_\alpha} |f_\alpha| d\mu_\alpha.$$

This is finite by assumption, so $f \in L^1(X)$. The same reason applied to f then proves the claim identity.

□

9.1.2 Abelian case

In this subsection, suppose that G_α is abelian. Then the dual group $\widehat{G_\alpha} := \mathrm{Hom}_{\mathbf{TopGp}}(G_\alpha, S^1)$ is LCA. We begin with discussion of quasi-characters.

Lemma 9.1.8.

(i) Let $\chi \in \mathrm{Hom}_{\mathbf{TopGp}}(G, \mathbb{C}^\times)$. Then there is a cofinite subset $J' \subseteq J \subseteq I$ such that $\chi_\alpha|_{H_\alpha} \equiv 1$ for all $\alpha \in J'$.

(ii) Conversely, for each $\alpha \in I$ let $\chi_\alpha \in \mathrm{Hom}_{\mathbf{TopGp}}(G_\alpha, \mathbb{C}^\times)$ such that $\chi_\alpha|_{H_\alpha} \equiv 1\, \forall' \alpha \in I$. Then $\chi = \bigotimes'_{\alpha \in I} \chi_\alpha : G \to \mathbb{C}^\times$ is a quasi-character.

Proof.

(i) Since $\mathbb{C}^\times$ has no small subgroup by Proposition I.2.10.

(ii) χ is clearly multiplicative. To see continuity let $S \subseteq I$ be a finite subset consisting of all α with $\chi_\alpha(H_\alpha) \neq 1$ and let $s = \#S$. Given a unit-neighborhood U of $\mathbb{C}^\times$ choose a unit-neighborhood V such that $V^s \subseteq U$. Let N_α be a unit-neighborhood of G_α such that $\chi_\alpha(N_\alpha) \subseteq V$ for all $\alpha \in S$, and let $N_\alpha = H_\alpha$ for $\alpha \notin S$. Then

$$\chi\left(\prod_\alpha N_\alpha\right) \subseteq V^s \subseteq U$$

$\square$

Theorem 9.1.9. The restricted product of the groups $\widehat{G_\alpha}$ with respect to the subgroups $H_\alpha^\perp$ is naturally isomorphic to the character group $\widehat{G}$ of G as topological groups.

Proof. The isomorphism is given by

$$\begin{aligned} \prod_\alpha{}' \widehat{G_\alpha} &\longrightarrow \hat{G} \\ (\chi_\alpha) &\longmapsto \chi := \bigotimes\nolimits'_{\alpha \in I} \chi_\alpha \end{aligned}$$

The preceding lemmas show that this is an abstract group isomorphism. It remains to show it is a homeomorphism. Let K be a compact set in G and $\varepsilon > 0$. We may assume $K = \prod_\alpha K_\alpha$ as described in Lemma 9.1.1. Let $S := \{\alpha \in J \mid K_\alpha \neq H_\alpha\}$ and put $n = \#S$. Then

$$\chi \in \{\chi \in \widehat{G} \mid |\chi(K) - 1| < \varepsilon\} \Leftrightarrow \left|\prod_\alpha \chi_\alpha(B_\alpha) - 1\right| < \varepsilon$$

For $\alpha \in S$ let $V_\alpha := \{\chi \in \widehat{G_\alpha} \mid |\chi(B_\alpha) - 1| < \rho := (\varepsilon + 1)^{\frac{1}{n}} - 1\}$, and for $\alpha \notin S$ let $V_\alpha = H_\alpha^\perp$. Now for $(\chi_\alpha) \in \prod_\alpha V_\alpha$, we have

$$\left|\prod_\alpha \chi_\alpha(B_\alpha) - 1\right| < (1 + \rho)^n - 1 = \varepsilon$$

so that $(\chi_\alpha)_\alpha \mapsto \chi$ is continuous.

Conversely, let $I \backslash J \subseteq S \subseteq I$ be a finite set with $\#S = n$ and $1 > \varepsilon > 0$. For $\alpha \in S$ let K_α be a compact set in G_α and put $V_\alpha = \{\chi \in \widehat{G_\alpha} \mid |\chi(K_\alpha) - 1| < \varepsilon\}$, and for $\alpha \notin S$ put $V_\alpha = H_\alpha^\perp$. Put $S = \{\alpha_1, \ldots, \alpha_n\}$ and let $K = \left(\{1\} \times \cdots \times \{1\} \cup \bigcup_{i=1}^n \left(K_{\alpha_i} \times \prod_{j \neq i} \{1\}\right)\right) \times \prod_{\alpha \notin S} H_\alpha$ which is a compact set in G. Then for $\chi \in \widehat{G}$ with $|\chi(K) - 1| < \varepsilon$, we have the following:

- $|\chi_\alpha(H_\alpha) - 1| < \varepsilon$ for $\alpha \notin S$. This implies $\chi_\alpha(H_\alpha) = 1$ because $\chi_\alpha(H_\alpha)$ is a subgroup of S^1.

- $|\chi_{\alpha_i}(K_{\alpha_i}) - 1| < \varepsilon$ for $1 \leqslant i \leqslant n$.

Hence $(\chi_\alpha)_\alpha \in \prod_\alpha V_\alpha$, showing $\chi \mapsto (\chi_\alpha)_\alpha$ is continuous. $\square$

9.1.3 Restricted tensor products

Let $\{V_\alpha \mid \alpha \in I\}$ be a a family of vector spaces. Let $J \subseteq I$ be a cofinite subset and to each $\alpha \in J$ choose an element $e_\alpha \in V_\alpha$. Say an element $x = (x_\alpha)_\alpha \in \prod_{\alpha \in I} V_\alpha$ is **factorizable** with respect to

$\{e_\alpha\}_{\alpha\in J}$ if $x_\alpha = e_\alpha \,\forall'\alpha \in J$. The quotient of the vector space free on all factorizable elements with respect to $\{e_\alpha\}_{\alpha\in J}$ by linearity condition on each component is called the **restricted tensor product of $\{V_\alpha \mid \alpha \in I\}$ with respect to** $\{e_\alpha\}_{\alpha\in J}$, and is denoted by

$$\bigotimes_{\alpha\in I}{}' V_\alpha = \bigotimes_{\alpha\in I}^{\{e_\alpha\}_{\alpha\in J}} V_\alpha$$

The image of a factorizable element $(x_\alpha)_{\alpha\in I}$ in the tensor is denoted by $\otimes'_{\alpha\in I} x_\alpha$; sometimes such an element is called a **pure tensor**. Suppose $J' \subseteq I$ is another cofinite set and $\{e'_\alpha\}_{\alpha\in J'}$ is another set of elements such that $e'_\alpha = e_\alpha \,\forall'\alpha \in J \cap J'$. Then there is a canonical isomorphism $\bigotimes_{\alpha\in I}^{\{e_\alpha\}_{\alpha\in J}} V_\alpha \cong \bigotimes_{\alpha\in I}^{\{e'_\alpha\}_{\alpha\in J'}} V_\alpha$.

Let $\{A_\alpha \mid \alpha \in I\}$ be a family of algebras, and for each $\alpha \in J$ choose an idempotent $f_\alpha \in A_\alpha$. Then the set of factorizable elements with respect to $\{f_\alpha\}_{\alpha\in J}$ is closed under multiplication, so the restricted tensor product $\bigotimes_{\alpha\in I}^{\{f_\alpha\}_{\alpha\in J}} A_\alpha$ has a natural algebra structure. Again, altering $\{f_\alpha\}_{\alpha\in J}$ by a finite subset yields isomorphic algebras.

Example 9.1.10. With the introduced notation, the space A appeared in Lemma 9.1.5 is $\bigotimes_{\alpha\in I}^{\{\mathbf{1}_{Y_\alpha}\}_{\alpha\in J}} C_c(X_\alpha)$.

Suppose moreover each V_α is an A_α-module. Then $\bigotimes_{\alpha\in I}^{\{e_\alpha\}_{\alpha\in J}} V_\alpha$ has a naturally a $\bigotimes_{\alpha\in I}^{\{f_\alpha\}_{\alpha\in J}} A_\alpha$-module.

9.2 Adeles

Let F be a field. The arithmetic of F is believed to be encoded in its completions with respect to its various valuations. It would be best if we can do harmonic analysis on the completion. For a valuation v of F, we denote by F_v the completion of F with respect to v. If $v \in M_{F,\mathrm{a}}$, then $F_v \cong \mathbb{R}$ or $\mathbb{C}$ by Theorem 8.2.4, which is locally compact. If $v \in M_{F,\mathrm{na}}$, we see by Theorem 8.3.1 that F_v is locally compact if and only if v is discrete and $\#\kappa(v) < \infty$, in which case the ring of integers $\mathfrak{o}_v$ of (F_v, v) is compact.

In the following let us assume F is a field such that F_v is locally compact for all valuations v on F. To deal with various completion in a shot, we consider their restricted product with respect to the rings of integers:

$$\mathbb{A}_F := \prod_{v\in M_F}^{\{\mathfrak{o}_v\}_{v\in M_{F,\mathrm{na}}}} F_v.$$

This makes sense if $M_{F,\mathrm{na}}$ is assumed to be cofinite, i.e., $\#M_{F,\mathrm{a}} < \infty$. Under this assumption, $\mathbb{A}_F$ an LCA group, which is a suitable stage to play harmonic analysis. This is called the **adele ring of the field F.**

We've made the following assumptions:

(i) $\#M_{F,\mathrm{a}} < \infty$.

(ii) F_v is locally compact for each $v \in M_F$.

Suppose F has a infinite subfield k such that F is algebraic over $k(x)$ for some $x \in F$ transcendental over k. Let $|\cdot|_x$ denote the valuation on $k(x)$ defined by x. Then the residue field of $|\cdot|_x$ is k itself, so that $k(x)$ does not satisfy (ii). To deduce F does not satisfy (ii), we only need to show every valuation on $k(x)$ extends to F. Indeed, if v is a valuation on $k(x)$, then we have a homomorphism $k(x) \to \overline{k(x)_v}$, where $\overline{k(x)_v}$ denotes the algebraic closure of the completion $k(x)_v$. By Corollary 8.4.17.1 v extends to $\overline{k(x)}_v$ uniquely, and we denote the extension again by v. It is standard that $k(x) \to \overline{k(x)_v}$ can be extended to a homomorphism from any algebraic extension of $k(x)$, so we have a homomorphism $F \to \overline{k(x)_v}$ particularly. Postcomposing with v gives a valuation on F extending v.

If $\operatorname{Char} F = 0$, such k exists if $\operatorname{tr.deg}_{\mathbb{Q}} F \geqslant 1$. If $\operatorname{Char} F = p \geqslant 2$, such k exists if $\operatorname{tr.deg}_{\mathbb{F}_p} F \geqslant 2$. Hence, the condition (ii) implies that F is either

(a) an algebraic extension of $\mathbb{Q}$, or

(b) an algebraic extension of $\mathbb{F}_p(t)$ with t a variable.

In the case (a), the condition (i) further implies that $[F : \mathbb{Q}] < \infty$. In fact, the condition (i) alone together a restriction on the cardinality of F also implies $[F : \mathbb{Q}] < \infty$. To be precise,

Lemma 9.2.1. Let k be a field of characteristic 0 with $\#k \leqslant \#\mathbb{C}$. If $k/\mathbb{Q}$ is transcendental, then $\#\operatorname{Hom}_{\mathbf{Field}}(k, \mathbb{C}) = \infty$.

Proof. We have $\mathbb{Q} \subseteq k$. Let k' be the algebraic closure of $\mathbb{Q}$ in k, so that k/k' is purely transcendental. It is standard that there exists a field homomorphism $k' \to \mathbb{C}$, so we can regard k' as a subfield of $\mathbb{C}$; note that k' is countable. Say $\gamma \subseteq k$ is a transcendence basis of k/k'. Since $\#k \leqslant \#\mathbb{C}$, $\#\gamma$ is not bigger than $\operatorname{tr.deg}_{k'}\mathbb{C}$, so there exists a transcendence basis β of $\mathbb{C}/k'$ and an injection $\gamma \to \beta$. Composing with any permutation of β, we obtain infinitely many injections $\gamma \to \beta$. Each of them gives a field homomorphism $k = k'(\{x \mid x \in \gamma\}) \to k'(\{y \mid y \in \beta\}) \subseteq \mathbb{C}$. ☐

In particular, when $\operatorname{Char} F = 0$, if $\#F \leqslant \#\mathbb{C}$, then (i) implies $F/\mathbb{Q}$ is algebraic. If $[F : \mathbb{Q}] = \infty$, then choose a subextension K of degree $\#\operatorname{Hom}_{\mathbf{Field}}(F, \mathbb{C}) + 1$. It is standard that $[K : \mathbb{Q}] = \#\operatorname{Hom}_{\mathbf{Field}}(K, \mathbb{Q})$ and each embedding of K extends to that of F. This is a contradiction, and consequently $[F : \mathbb{Q}] < \infty$.

We turn to the case (b), i.e., when $\operatorname{Char} F = p$ for some rational prime p and F is an algebraic extension of $\mathbb{F}_p(t)$. In this case (i) holds automatically.

Lemma 9.2.2. [2] Let k be a perfect field of characteristic $p \geqslant 2$ and consider its rational function field $k(t)$.

(i) If $F/k(t)$ is an algebraic extension, then the field

$$F^{\frac{1}{p}} := \{x \in \overline{F} \mid x^p \in F\}$$

is F if F is perfect, and is the unique purely inseparable extension of F of degree p if F is non-perfect.

(ii) Let $K/k(t)$ be any algebraic extension. Then K is separable over the perfect closure

$$F := \{x \in K \mid x^{p^n} \in k(t) \text{ for some } n \geqslant 1\}$$

of $k(t)$ in K.

Proof.

[2] See the mathoverflow post. The lemma does not hold for fields of higher dimension.

(i) Generally, if F is a perfect field of characteristic p, then $F = F^{\frac{1}{p}}$. Indeed, for $a \in F$, the polynomial $X^p - a \in F[X]$ is not separable, so it is reducible by perfectness. But $X^p - a = (X-y)^p$, so it follows that if $y \in F^{\frac{1}{p}}$, then $y \in F$.

Now suppose $F/k(t)$ is a non-perfect algebraic extension. From the above remark, we have $k(t^{\frac{1}{p}}) = k(t)^{\frac{1}{p}}$. Since the polynomial $X^p - t \in k(t)[X]$ is irreducible, $[k(t^{\frac{1}{p}}) : k(t)] = p$. If $F/k(t)$ is finite, then

$$[F^{\frac{1}{p}} : F] = \frac{[F^{\frac{1}{p}} : k(t^{\frac{1}{p}})][k(t^{\frac{1}{p}}) : k(t)]}{[F : k(t)]}.$$

Since $x \mapsto x^p$ defines an isomorphism $F^{\frac{1}{p}} \cong F$ and $k(t^{\frac{1}{p}}) \cong k(t)$, we have $[L^{\frac{1}{p}} : k(t^{\frac{1}{p}})] = [L : k(t)]$, so $[L^{\frac{1}{p}} : L] = [k(t^{\frac{1}{p}}) : k(t)] = p$. For the uniqueness, if $L \subseteq F$ is purely inseparable of degree p, then for each $x \in L \backslash L$, $x^{p^n} \in F$ for some $n \geqslant 1$ and the minimal polynomial of x has degree p and divides $X^{p^n} - x^{p^n} = (X-x)^{p^n}$. Then the minimal polynomial is $(X-x)^p$, and hence $x^p \in F$. This shows $F^{\frac{1}{p}} \subseteq L$, and proves the uniqueness.

Assume $F/k(t)$ is infinite, and suppose the lemma is wrong. Then there exist $x, y \in F$ such that $F \subsetneq F(x^{\frac{1}{p}}) \subsetneq F(x^{\frac{1}{p}}, y^{\frac{1}{p}})$. Consider the tower $k(x,y) \subsetneq k(x^{\frac{1}{p}}, y) \subsetneq k(x^{\frac{1}{p}}, y^{\frac{1}{p}})$. By the finite case we see $k(x,y)$ has a unique purely inseparable extension of degree p. But then $k(x^{\frac{1}{p}}, y) = k(x, y^{\frac{1}{p}}) = k(x^{\frac{1}{p}}, y^{\frac{1}{p}})$ leads to a contradiction.

(ii) Let F^{sep} be the separable closure of F in K; then we have a tower $F \subseteq F^{\mathrm{sep}} \subseteq K$ with K/F^{sep} purely inseparable and F^{sep}/F separable. If $K \neq F^{\mathrm{sep}}$, then by (i) we have $F^{\mathrm{sep}} \subsetneq (F^{\mathrm{sep}})^{\frac{1}{p}} \subseteq K$.[3] Since F is not perfect, it follows from (i) and the above that $F \subsetneq F^{\frac{1}{p}} \subseteq (F^{\mathrm{sep}})^{\frac{1}{p}} \subseteq K$. But the inequality $F \subsetneq F^{\frac{1}{p}} \subseteq K$ is a contradiction to the definition of F.

□

Corollary 9.2.2.1. Let k be a perfect field of characteristic $p \geqslant 2$. Then every finite purely inseparable extension of $k(t)$ has the form $k(t^{\frac{1}{q}})$ with $q = p^n$ for some $n \geqslant 1$.

Proof. By induction on n, it suffices to show any purely inseparable extension of degree p is $k(t^{\frac{1}{p}})$. But this is Lemma 9.2.2.(i). □

Consider the purely inseparable extension $\mathbb{F}_p(t^{\frac{1}{q}})$ of $\mathbb{F}_p(t)$ with $q = p^n$ for some $n \geqslant 1$. The isomorphism $\mathbb{F}_p(t^{\frac{1}{q}}) \xrightarrow{\sim} \mathbb{F}_p(t)$ given by $x \mapsto x^q$ establishes a bijection of the valuations on them. For example, the valuation defined by $t^{\frac{1}{q}}$ is the unique extension of the one $|\cdot|_t$ defined by t, and the ramification index is q. Then $|\cdot|_t$ extends uniquely to a valuation v on the perfect closure $\mathbb{F}_p(t^{p^{-\infty}}) = \bigcup_{n\geqslant 1} \mathbb{F}_p(t^{p^{-n}})$ of $\mathbb{F}_p(t)$. Although the residue field of v is finite, it is not a discrete valuation. In particular $\mathbb{F}_p(t^{-p^{\infty}})_v$ is not locally compact.

Let K denote the perfect closure of $\mathbb{F}_p(t)$ in F. By Lemma 9.2.2.(ii), F/K is separable and $K/\mathbb{F}_p(t)$ is purely inseparable. The preceding discussion shows that $K/\mathbb{F}_p(t)$ cannot be infinite, so it is finite, having the form $K = \mathbb{F}_p(t^{\frac{1}{q}})$ with $q = p^n$ for some $n \geqslant 1$. Replacing t by $t^{\frac{1}{q}}$, we can assume $F/\mathbb{F}_p(t)$ is a separable algebraic extension. Can I show F is in fact finite over $\mathbb{F}_p(t)$?

[3] The argument here is to make sure that $F^{\frac{1}{p}} \subseteq K$.

9.2.1 Classical geometry of numbers

Definition.

(i) An **(algebraic) number field** is a finite extension of $\mathbb{Q}$.

(ii) A **global function field** is a finite extension of $\mathbb{F}_p(t)$, where p is a rational prime and t is a transcendental element over $\mathbb{F}_p$.

(iii) A **global field** is either a number field or a global function field.

A valuation on a global field is usually called a **place**. We have the following terminology.

Definition. Let F be a global field.

(i) If F is a number field, an archimedean (resp non-archimedean) valuation on F is called an **infinite** (resp. **finite**) **place** of F.

(ii) If F is a global function field, an extension of the valuation $|\cdot|_\infty$ on $\mathbb{F}_p(t)$ is called an **infinite place** of F. All the other valuations are called **finite places** of F.

Denote by $M_{F,\infty}$ (resp. $M_{F,\mathrm{fin}}$) the set of equivalence classes of infinite places (resp finite places) of F. Then

$$M_F = M_{F,\infty} \sqcup M_{F,\mathrm{fin}} = M_{F,\mathrm{a}} \sqcup M_{F,\mathrm{na}}.$$

We will see in many proofs that infinite places of a global function field have the same position as infinite places of a number field.

There is a convention that we must make clear. In a previous subsection we usually use v to stand for an absolute value that lies in the equivalence class in v. Since F_v is locally compact, in either §2.5 or §7.1 we have seen a natural normalization of the valuation v. That is,

$$|x|_v = \#\kappa(v)^{-\operatorname{ord} x}$$

where if ϖ_v is a uniformizer of F_v, then $\operatorname{ord} x = \operatorname{ord}_{F_v} x \in \mathbb{Z}$ is such that $x\varpi_v^{-\operatorname{ord} x} \in \mathfrak{o}_v^\times$. Under this normalization, $|\cdot|_v$ is not an absolute value, nor is compatible with the restriction to subfields. But this has the advantage that it is the topological modulus. In what follows we will use $|\cdot|_v$ to denote this valuation unless otherwise chosen.

If F is a number field, then there is an embedding

$$F \longrightarrow F \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{\sigma \in M_{F,\infty}} F_\sigma.$$

If σ is real, i.e., $\sigma(F) \subseteq \mathbb{R}$, then $F_\sigma = \mathbb{R}$; if it is non-real, i.e., $\sigma(F) \nsubseteq \mathbb{R}$, then $F_\sigma = \mathbb{C}$. Here we identify $M_{F,\infty}$ with $\mathrm{Hom}_{\mathbf{Field}}(F, \mathbb{C})$ up to conjugation (c.f. Corollary 8.2.4.1). It is a standard notation that we write r_1 (resp. r_2) to be the number of real embeddings (resp. half the number of non-real embeddings); then $[F : \mathbb{Q}] = r_1 + 2r_2$. In classical algebraic number theory one studies the arithmetic of F via this embedding.

Definition. For a global field F, we call

$$\mathcal{O}_F := \{x \in F \mid |x| \leqslant 1 \text{ for all } |\cdot| \in M_{F,\mathrm{fin}}\} = \bigcap_{v \in M_{F,\mathrm{fin}}} F \cap \mathfrak{o}_v$$

the **ring of integers** of F.

For example, we have $\mathcal{O}_F = \begin{cases} \mathbb{Z} & \text{, if } F = \mathbb{Q} \\ \mathbb{F}_p[t] & \text{, if } F = \mathbb{F}_p(t) \end{cases}$ in the two basic cases.

Lemma 9.2.3. Let K/F be a finite extension of global fields. Then $\mathcal{O}_K$ is the integral closure of $\mathcal{O}_F$ in K.

Proof. The key point lies in the proof of Theorem 8.4.1: if w is an extension of $v \in M_F$ to K, then $\mathfrak{o}_w \subseteq K_w$ is the integral closure of $\mathfrak{o}_v \subseteq F_v$ in K_w. If $\alpha \in K \backslash F$ is integral over $\mathcal{O}_F$, write $X^n + a_{n_1} X^{n-1} + \cdots + a_1 X + a_0 \in \mathcal{O}_F[X]$ for its integral dependence. Let $|\cdot| \in M_{K,\mathrm{fin}}$; since $a_i \in \mathcal{O}_F$, $|a_i| \leqslant 1$ for each $0 \leqslant i \leqslant n-1$ so that

$$|\alpha|^n = \left| -\sum_{i=0}^{n-1} a_i \alpha^i \right| \leqslant \max_{0 \leqslant i \leqslant n-1} |a_i||\alpha|^i \leqslant \max\{1, |\alpha|^{n-i}\}.$$

This implies $|\alpha| \leqslant 1$. Since $|\cdot|$ is arbitrary, this shows $\alpha \in \mathcal{O}_K$.

Conversely, let $\alpha \in \mathcal{O}_K$ and let $m_{\alpha,F} \in F[X]$ be the monic minimal polynomial of α over F. By Chinese remainder theorem, for each $v \in M_{F,\mathrm{fin}}$ we have

$$F(\alpha) \otimes_F F_v \cong F_v[X]/(g_1^{e_1}) \times \cdots \times F_v(g_n^{e_n})$$

where $m_{\alpha,F} = g_1^{e_1} \cdots g_n^{e_n}$ is the irreducible decomposition in $F_v[X]$. If we denote by w_i the extension of v to $F_v[X]/(g_i)$, by assumption we have $w_i(\alpha) \leqslant 1$, so by what we remark in the very beginning, α is integral over $\mathfrak{o}_v$. In particular, $g_i \in \mathfrak{o}_v[X]$, and hence $m_{\alpha,F} \in (F \cap \mathfrak{o}_v)[X]$. Letting v run over all $v \in M_{F,\mathrm{fin}}$ yields $m_{\alpha,F} \in \mathcal{O}_F[X]$. $\square$

The discussion is incomplete if we do not say anything about the algebraic aspects of the ring of integers. The ring of integers of a global field is a **Dedekind domain**, namely, a normal Noetherian domain of (Krull) dimension $\leqslant 1$. The normality follows from the previous lemma. The dimension also follows from it, since for $A \to B$ is an injective integral homomorphism of rings, we have $\dim A = \dim B$, and since $\dim \mathbb{Z} = 1 = \dim \mathbb{F}_p(t)$ (they are all PIDs). To show it is Noetherian, we make use of the trace form. If K is a number field, it is finite separable over $F := \mathbb{Q}$. If K is a global function field, then it is finite separable over $F = \mathbb{F}_p(t^{\frac{1}{q}})$ with $q = p^n$ for some $n \geqslant 1$. The separability implies the trace form $K \times K \ni (x, y) \mapsto \mathrm{Tr}_{K/F}(xy) \in F$ is non-degenerate.[4] Let $v_1, \ldots, v_n \in \mathcal{O}_K$ be an F-basis for K[5], and let $w_1, \ldots, w_n \in K$ be the dual basis with respect to the trace form. Pick $x \in \mathcal{O}_K$ and write $x = \sum_{i=1}^n \mathrm{Tr}_{K/F}(xv_i)w_i$. Since $xv_i \in \mathcal{O}_K$, it follows that $\mathrm{Tr}_{K/F}(xv_i) \in \mathcal{O}_F$. This proves $\mathcal{O}_K \subseteq \sum_{i=1}^n \mathcal{O}_F w_i$, so we only need to show $\mathcal{O}_F$ is Noetherian. This is clear: in the number field case $\mathcal{O}_F = \mathbb{Z}$ and in the global function field case $\mathcal{O}_F = \mathbb{F}_p[t^{\frac{1}{q}}] \cong \mathbb{F}_p[t]$. Since $\mathcal{O}_F$ is a PID, this also shows that $\mathcal{O}_K$ is a finite free $\mathcal{O}_F$-module of rank $[K : F]$. We record these as a lemma.

Lemma 9.2.4. Let F be a global field and $S \subseteq M_{F,\mathrm{fin}}$ a finite set.

(i) The ring of integers $\mathcal{O}_F$ is a Dedekind domain.

(ii) If K/F is a finite extension, then $\mathcal{O}_K$ is finite over $\mathcal{O}_F$.

(iii) If, moreover, in (ii) $\mathcal{O}_F$ is a PID, then $\mathcal{O}_K$ is finite free over $\mathcal{O}_F$ of rank $[K : F]$.

[4] Since taking trace is transitive, by induction we can assume K/F is a simple extension. One can show some power of the simple generator has nonzero trace.

[5] Such a basis exists as $K/\mathcal{O}_K$ is a torsion $\mathcal{O}_F$-module.

We conclude our discussion on algebraic aspects by relating the valuations and maximal ideals. A domain D is called a **discrete valuation ring**, or **DVR** for short, if there is a discrete valuation on $\operatorname{Frac} D$ such that D is its ring of integers. It is well-known that D is a DVR if and only if D is normal Noetherian local domain of dimension one. It follows that a domain is Dedekind of dimension 1 if and only if all of its localization at maximal ideals are DVRs. This allows us to establish a bijection:

$$\begin{aligned} M_{F,\mathrm{fin}} &\longrightarrow \operatorname{mSpec} \mathcal{O}_F \\ v &\longrightarrow \mathfrak{p}_v := \{x \in \mathcal{O}_F \mid |x|_v < 1\}. \end{aligned}$$

Since we assume v is nontrivial, $\mathfrak{p}_v$ is a nonzero prime ideal, and hence a maximal ideal. Conversely, if $\mathfrak{p}$ is a maximal ideal in $\mathcal{O}_F$, the localization $\mathcal{O}_{\mathfrak{p}} := (\mathcal{O}_F)_{\mathfrak{m}}$ is a DVR. By Lemma 9.2.3 we can deduce that $F = \operatorname{Frac} \mathcal{O}_F$, so we obtain a discrete valuation on F. To see these assignment are inverse to each other, it suffices to notice that $\mathcal{O}_{\mathfrak{p}_v}$ is exactly the ring of integers of (F, v).

Let us consider the number field F case.

Lemma 9.2.5. The inclusion $\mathcal{O}_F \to F_{\mathbb{R}} := F \otimes_{\mathbb{Q}} \mathbb{R}$ is discrete and cocompact.[6]

Proof. Since F is dense in $F_{\mathbb{R}}$, the linear span of $\mathcal{O}_F$ over $\mathbb{R}$ is the whole $F_{\mathbb{R}}$. Let β be a $\mathbb{Z}$-basis for $\mathcal{O}_F$. We need to show the image of β in $F_{\mathbb{R}}$ is linearly independent. The square of the determinant of these vectors is

$$\det(\sigma v)^2_{\substack{\sigma \in M_{F,\infty} \\ v \in \beta}} = \det(\operatorname{Tr}_{F/\mathbb{Q}}(vw))_{v,w\in\beta}$$

which is positive by the non-degeneracy of the trace form. This finishes the proof. □

For each $\sigma \in M_{F,\infty}$, the Haar measure dx_σ on F_σ is chosen as in Lemma 7.1.4. Namely, it is the usual Lebesgue measure of $F_\sigma \cong \mathbb{R}$ if σ is real, and is twice the Lebesgue measure of $F_\sigma \cong \mathbb{C}$ if σ is non-real. On the space $F_{\mathbb{R}} \cong \prod_{\sigma \in M_{F,\infty}} F_\sigma$ we use the product measure $\otimes_{\sigma \in M_{F,\infty}} dx_\sigma$. Since $\mathcal{O}_F$ is discrete, whence closed, in $F_{\mathbb{R}}$, it makes sense to consider the quotient measure μ of $F_{\mathbb{R}}$ by the counting measure on $\mathcal{O}_F$. This quotient measure can be described in a more concrete way as follows. Let β be a $\mathbb{Z}$-basis for $\mathcal{O}_F$ and let P be the fundamental parallelotope

$$P = P_\beta = \left\{ \sum_{v\in\beta} a_v v \mid 0 \leqslant a_v < 1 \right\}.$$

Let $\pi : F_{\mathbb{R}} \to F_{\mathbb{R}}/\mathcal{O}_F$ denote the quotient map. Then

$$\begin{aligned} T : C_c(F_{\mathbb{R}}/\mathcal{O}_F) &\longrightarrow \mathbb{C} \\ f &\longrightarrow \int_P f \circ \pi \otimes_\sigma dx_\sigma \end{aligned}$$

defines a nontrivial positive linear functional on $F_{\mathbb{R}}/\mathcal{O}_F$, and is clearly $\mathcal{O}_F$-invariant. By Theorem 2.4.7.(ii), $T = c\mu$ for some $c > 0$. Our goal is to find c. Let $x_1, \ldots, x_{r_1}, x_{r_1+1}, y_{r_1+1}, \ldots, x_{r_1+r_2}, y_{r_1+r_2}$ be the standard basis for $F_{\mathbb{R}} \cong \mathbb{R}^{\oplus r_1} \oplus \mathbb{C}^{\oplus r_2}$. Let Λ be the lattice spanned by this standard basis and equip $F_{\mathbb{R}}/\Lambda$ with the quotient measure. By Example 2.4.11 $\operatorname{vol}(F_{\mathbb{R}}/\Lambda) = 1$. Let $\phi : \Lambda \to \mathcal{O}_F$ be an abelian group isomorphism and extend it by linearity to an automorphism of $F_{\mathbb{R}}$. By Proposition 17.1.8, $\operatorname{vol}(F_{\mathbb{R}}/\mathcal{O}_F, \mu) = \operatorname{mod}_{F_{\mathbb{R}}}(\phi) \operatorname{vol}(F_{\mathbb{R}}/\Lambda) = \operatorname{mod}_{F_{\mathbb{R}}}(\phi)$. By Corollary 2.5.3.1.(iii), $\operatorname{mod}_{F_{\mathbb{R}}}(\phi) =$

[6] See Lemma 9.3.14 for an adelic proof.

$|\det \phi|$. Now by elementary calculus, we know $|\det \phi| = \mathrm{vol}(P, \otimes_\sigma dx_\sigma)$ (recall that on $\mathbb{C}$ we use twice the Lebesgue measure). In sum,

$$\mathrm{vol}(F_\mathbb{R}/\mathcal{O}_F, \mu) = \mathrm{vol}(P, \otimes_\sigma dx_\sigma).$$

By evaluating T at the constant function 1, we see $\mathrm{vol}(P, \otimes_\sigma dx_\sigma) = c\,\mathrm{vol}(F_\mathbb{R}/\mathcal{O}_F, \mu)$. What we just obtained shows $c = 1$. In conclusion,

Lemma 9.2.6. The quotient measure on $F_\mathbb{R}/\mathcal{O}_F$ is given by integration over a fundamental parallelotope P for some $\mathbb{Z}$-basis of $\mathcal{O}_F$.

By definition, the **discriminant** of the number field F is $\mathrm{disc}\ F := \det(\sigma v)^2_{\substack{\sigma\in M_{F,\infty}\\ v\in\beta}}$, where β is a $\mathbb{Z}$-basis for $\mathcal{O}_F$. It is direct to see it is independent of the choice of β, and squaring makes the order irrelevant. Hence it is well-defined, and also

$$\mathrm{disc}\ F = \det(\sigma v)^2_{\substack{\sigma\in M_{F,\infty}\\ v\in\beta}} = \det(\mathrm{Tr}_{F/\mathbb{Q}}(vw))_{v,w\in\beta} \in \mathbb{Q}_{>0}.$$

This is positive since the trace form is non-degenerate. A moment consideration tells us that $\mathrm{vol}(P, \otimes_\sigma dx_\sigma) = \sqrt{\mathrm{disc}\ F}$, so

Corollary 9.2.6.1. For a number field F, we have $\sqrt{\mathrm{disc}\ F} = \mathrm{vol}(F_\mathbb{R}/\mathcal{O}_F)$.[7]

9.2.2 Topological property of adele ring

Let us begin our discussion on adeles. In the following F will always stand for a global field.

Lemma 9.2.7. For each $r \in F$, $\#\{|\cdot| \in M_F \mid |r| > 1\} < \infty$.

Proof. Since $\# M_{F,\infty} < \infty$, we only need to consider non-archimedean valuations. Let k be either $\mathbb{Q}$ or $\mathbb{F}_p(t)$, and let $X^n + \sum_{i=0}^{n-1} a_i X^i \in k[X]$ be the minimal polynomial of r over k. For any $|\cdot| \in M_{F,\mathrm{fin}}$,

$$|r^n| = \left|\sum_{i=0}^{n-1} a_i r^i\right| \leqslant \max_{0\leqslant i\leqslant n-1} |a_i||r^i| \leqslant \max_{0\leqslant i\leqslant n-1} |a_i| \cdot \max\{1, |r|^{n-1}\}$$

so that $|r| \leqslant \max\{1, |a_0|, \ldots, |a_{n-1}|\}$. In view of this inequality, it suffices to prove the lemma for $F = k$. This follows easily from the fact that $\mathbb{Z}$ and $\mathbb{F}_p[t]$ are UFD. $\square$

By the lemma, there is a natural injection

$$\begin{aligned} F &\longrightarrow \mathbb{A}_F \\ r &\longmapsto (r)_{v\in M_F}. \end{aligned}$$

An element in $F \subseteq \mathbb{A}_F$ is sometimes called a **principal adele**. The map enjoys many properties similar to the map $\mathcal{O}_F \to F_\mathbb{R}$ we discussed above. Before we talk about this map, a digression is needed. Suppose $K \subseteq F$ is another global field. Let $v \in M_{K,\mathrm{fin}}$, and let $w_1, \ldots, w_n \in M_{F,\mathrm{fin}}$ be the extensions of v to F. From the proof of Theorem 8.4.23, we know there is an isomorphism

$$F \otimes_K K_v \xrightarrow{\sim} F_{w_1} \oplus \cdots \oplus F_{w_n}$$

[7] In classical language, there should be a factor 2^{r_2} on the right. The difference results from the choice of Haar measures for $\mathbb{C}$.

It is not hard to see from this isomorphism that if $x \in F$, then $\operatorname{Tr}_{F/K} x = \sum_{i=1}^{n} \operatorname{Tr}_{F_{w_i}/K_v} x$. Moreover, we can extend $\operatorname{Tr}_{F/K}$ by K_v-linearity to $\operatorname{Tr}_{F/K} : F \otimes_K K_v \to K_v$, and the above identity holds for all $x \in F \otimes_K K_v$ in an obvious sense.

Lemma 9.2.8. Let β be a K-basis for F. For almost all v, $\bigoplus_{e \in \beta} e\mathfrak{o}_v$ maps onto $\mathfrak{o}_{w_1} \oplus \cdots \oplus \mathfrak{o}_{w_n}$.

Proof. By Lemma 9.2.7, if we exclude those v with some extension w to F satisfying $w(e) > 1$ for some $e \in \beta$, then β is contained in each $\mathfrak{o}_{w_i}$, $i = 1, \ldots, n$.

The other way around uses the discriminant. For $x_1, \ldots, x_m \in F \otimes_K K_v$, put

$$D(x_1, \ldots, x_m) = \det(\operatorname{Tr}_{F/K}(x_i x_j))_{1 \leqslant i,j \leqslant m}.$$

Since $\operatorname{Tr}_{F/K}(x_i x_j) = \sum_{i=1}^{n} \operatorname{Tr}_{F_{w_i}/K_v}(x_i x_j)$, if $x_1, \ldots, x_m$ lie in $\mathfrak{o}_{w_1} \oplus \cdots \oplus \mathfrak{o}_{w_n}$, then $\operatorname{Tr}_{F_{w_i}/K_v}(x_i x_j) \in \mathfrak{o}_v$ so that $\operatorname{Tr}_{F/K}(x_i x_j) \in \mathfrak{o}_v$ and hence $D(x_1, \ldots, x_m) \in \mathfrak{o}_v$. Now let $x = \sum_{e \in \beta} a_e e \in \mathfrak{o}_{w_1} \oplus \cdots \oplus \mathfrak{o}_{w_n}$ with $a_e \in k_v$. Write $\beta = \{e_1, \ldots, e_m\}$ (so $m = [F : K]$). For $1 \leqslant i \leqslant m$, we have

$$\mathfrak{o}_v \ni D(e_1, \ldots, e_{i-1}, x, e_{i+1}, \ldots, e_m) = a_i^2 D(e_1, \ldots, e_m).$$

Since F/K is separable, the trace form is non-degenerate, i.e., $D(e_1, \ldots, e_m) \neq 0$. Applying Lemma 9.2.7 to $D(e_1, \ldots, e_m)$ and its inverse, we see $v(D(e_1, \ldots, e_m)) = 1$ for almost all v. This implies $a_i \in \mathfrak{o}_v$ $(i = 1, \ldots, m)$ for almost all v. $\square$

By the universal property of tensor products, there is a canonical map $\mathbb{A}_K \otimes_K F \to \mathbb{A}_F$. We claim this is an isomorphism of topological rings. Let β be a K-basis for F. It is easy to see that β defines an obvious isomorphism $\mathbb{A}_K \otimes_K F \to \mathbb{A}_K^{\oplus\beta}$ both algebraically and topologically. $\mathbb{A}_K^{\oplus\beta}$ can be viewed as the restricted product of the $K_v^{\oplus\beta}$ with respect to $\mathfrak{o}_v^{\oplus\beta}$. On the other hand, $K_v \otimes_K F \cong \bigoplus_{w|v} F_w$, where $w \mid v$ means w is an extension of v, and the previous lemma says their rings of integers match for almost all v. This proves the claim. We record this as a

Corollary 9.2.8.1. The canonical map $\mathbb{A}_F \cong \mathbb{A}_K \otimes_K F$ is a topological isomorphism. In particular, $\mathbb{A}_F \cong \mathbb{A}_K^{[F:K]}$ as additive topological groups, and F maps onto $K^{[F:K]}$.

Corollary 9.2.8.2. Let K/F be a extension of global field. Then the canonical map $\mathbb{A}_F \to \mathbb{A}_K$ is a closed embedding.

Proposition 9.2.9. Let F be a global field. The inclusion $F \to \mathbb{A}_F$ is discrete and cocompact.

Proof. In view of Corollary 9.2.8.1, we may assume F is either $\mathbb{Q}$ or $\mathbb{F}_p(t)$. We must find a unit-neighborhood U of $\mathbb{A}_F$ with $F \cap U = \{0\}$. If $F = \mathbb{Q}$, take

$$U = \{(x_p)_p \in \mathbb{A}_{\mathbb{Q}} \mid |x_p|_p \leqslant 1 \text{ for } p < \infty,\ |x_\infty|_\infty < 1\}.$$

If $F = \mathbb{F}_p(t)$, take

$$U = \{(x_g)_g \in \mathbb{A}_{\mathbb{F}_p[t]} \mid |x_g|_g \leqslant 1 \text{ for all irreducibles } g \in \mathbb{F}_p[t],\ |x_\infty|_\infty > 1\}.$$

Using unique factorization property, it is easy to check $U \cap F = \{0\}$.

To show $\mathbb{A}_F/F$ is compact, we are going to construct a compact subset $W \subseteq \mathbb{A}_F$ that surjects onto $\mathbb{A}_F/F$ under the quotient map $\mathbb{A}_F \to \mathbb{A}_F/F$. When $F = \mathbb{Q}$, take

$$W = \left\{(x_p)_p \in \mathbb{A}_{\mathbb{Q}} \mid |x_p|_p \leqslant 1 \text{ for } p < \infty,\ |x_\infty|_\infty \leqslant \frac{1}{2}\right\}.$$

When $F = \mathbb{F}_p(t)$, take

$$W = \{(x_g)_g \in \mathbb{A}_{\mathbb{F}_p[t]} \mid |x_g|_g \leqslant 1 \text{ for all irreducibles } g \in \mathbb{F}_p[t],\, |x_\infty|_\infty \leqslant p\}.$$

These choices satisfy $\mathbb{A}_F = F + W$, so W surjects (one way to see this is first clearing the "principal part" and next adjusting the "infinite part"). That W is compact follows from Tychonov's theorem. $\square$

From the proof of the previous proposition, we obtain

Corollary 9.2.9.1. There is a subset U of $\mathbb{A}_F$ defined by the inequalities of the type $|\xi_v|_v \leqslant \delta_v$ where $\delta_v = 1$ for almost all v, such that

$$\mathbb{A}_k = k + U$$

Proof. Let $\omega_1, \ldots, \omega_N$ be a basis for F/k. Then

$$\mathbb{A}_F = \mathbb{A}_k \otimes_k F = \mathbb{A}_k\omega_1 \oplus \cdots \oplus \mathbb{A}_k\omega_N$$

where $k = \mathbb{Q}$ or $\mathbb{F}_q(t)$, and F is mapped into $k\omega_1 \oplus \cdots \oplus k\omega_N$. Take $U' = W\omega_1 \oplus \cdots \oplus W\omega_N$, where W is the subset constructed in the proof of the proposition. Note that for almost all v on F, $|\omega_i|_v \leqslant 1$. Then it is clear from the definition of W that U' is contained in some U of the type described above. $\square$

9.2.3 Adelic geometry of numbers

Define the standard measure dx^{std} on $\mathbb{A}_F$ to be the restricted product Corollary 9.1.6.1 of the local ones (as defined in the beginning of §7.1). Since $F \to \mathbb{A}_F$ is discrete, the counting measure on F can be served as a Haar measure on F. These induce a finite quotient measure on the compact quotient $\mathbb{A}_F/F$, which we again denote by dx^{std}. We will find the exact value of $\mathrm{vol}(\mathbb{A}_F/F, dx^{\mathrm{std}})$ in the sequel. For now it suffices to know it is finite.

Lemma 9.2.10. There is a constant $C > 0$ depending only on the global field F with the following property: let $\alpha \in \mathbb{A}_F$ be such that $\prod_v |\alpha_v|_v > C$. Then there exists a principal adele $\beta \in F \subseteq \mathbb{A}_F$, $\beta \neq 0$ such that $|\beta|_v \leqslant |\alpha_v|_v$ for all v.

Proof. Let c_0 be the total volume of $\mathbb{A}_F/F$, and let c_1 be that of the set

$$\left\{\gamma \in \mathbb{A}_F \mid |\gamma_v|_v \leqslant \frac{1}{10} \text{ for archimedean } v,\, |\gamma_v|_v \leqslant 1 \text{ for non-archimedean } v\right\}$$

Then $0 < c_0 < \infty$ and $0 < c_1 < \infty$ for the number of archimedean places is finite. We show that $C = \dfrac{c_0}{c_1}$ will do.

The set

$$T = \left\{\tau \in \mathbb{A}_F \mid |\tau_v|_\nu \leqslant \frac{1}{10}|\gamma_v|_v \text{ for archimedean } v,\, |\tau_v|_v \leqslant |\gamma_v|_v \text{ for non-archimedean } v\right\}$$

has measure $c_1 \prod_v |\alpha_v|_v > c_1 C = c_0$. By Pigeonhole principle there must be a pair of distinct points of T which have the same image in $\mathbb{A}_F/F$, say $\tau', \tau'' \in T$ and $\tau' - \tau'' =: \beta \in F$. Then

$$|\beta|_v = |\tau_v' - \tau_v''| \leqslant |\alpha_v|_v$$

for all v, as required. $\square$

Corollary 9.2.10.1. Let v_0 be a valuation on F and let $\delta_v > 0$ be given for all $v \neq v_0$ with $\delta_v = 1$ for almost all v. Then there exists a $\beta \in F^\times$ with $|\beta|_v \leqslant \delta_v$ for all $v \neq v_0$

Proof. Choose $\alpha_v \in F_v$ with $0 < |\alpha_v|_v \leqslant \delta_v$ and $|\alpha_v|_v = 1$ if $\delta_v = 1$. We then can choose $\alpha_{v_0} \in F_{v_0}$ so that $\prod_v |\alpha_v|_v > C$, where C is as in Lemma 9.2.10. The resulting $\beta \in F$ given by the same lemma does the job. □

Theorem 9.2.11 (Strong approximation)**.** Let v_0 be any valuation of the global field F. Let V to be the restricted product of the F_v with respect to the $\mathfrak{o}_v$, where v runs through all normalized $v \neq v_0$. Then F is dense in V.

Proof. It is equivalent to proving the following statements: given $\varepsilon > 0$ and a finite set S of valuations $v \neq v_0$, together with elements $\alpha_v \in F_v$ for $v \in S$, there exists $\beta \in F$ such that $|\beta - \alpha_v|_v < \varepsilon$ for all $v \in S$ and $|\beta|_v \leqslant 1$ for all $v \notin S$, $v \neq v_0$.

Let δ_v and $U \subseteq \mathbb{A}_F$ be as in Corollary 9.2.9.1. By Corollary 9.2.10.1 there is a $\lambda \neq 0 \in F$ such that

$$\begin{aligned} |\lambda|_v &\leqslant \delta_v^{-1}\varepsilon \qquad (v \in S) \\ |\lambda|_v &\leqslant \delta_v^{-1} \qquad (v \notin S,\, v \neq v_0) \end{aligned}$$

Then we have $\mathbb{A}_F = \lambda U + F$. Let $\alpha \in \mathbb{A}_v$ have component α_v at $v \in S$ and 0 elsewhere, and write $\alpha = x + \beta$ for $x \in \lambda U$ and $\beta \in F$. Then

- for $v \in S$, $|\alpha_v - \beta|_v = |\alpha - \beta|_v = |x|_v \leqslant \varepsilon$, and
- for $v \notin S$, $v \neq v_0$, $|\beta|_v = |-x|_v \leqslant 1$.

□

Similarly we have a density result for ring of integers. Denote by $\mathbb{A}_{F,\text{fin}}$ the **ring of finite adeles**, which is by definition the restricted product with respect to the ring of integers over all finite places:

$$\mathbb{A}_{F,\text{fin}} := \prod_{v \in M_{F,\text{fin}}}^{\{\mathfrak{o}_v\}_{v \in M_{F,\text{fin}}}} F_v.$$

Observe that under the map $F \to \mathbb{A}_F \to \mathbb{A}_{F,\text{fin}}$, the ring of integers $\mathcal{O}_F$ maps into the infinite product $\prod_{v \in M_{\mathrm{F},\text{fin}}} \mathfrak{o}_v$. Recall that $\mathcal{O}_F$ is dense in $\mathfrak{o}_v$. In fact, more is true:

Corollary 9.2.11.1. $\mathcal{O}_F$ is dense in $\prod_{v \in M_{F,\text{fin}}} \mathfrak{o}_v \subseteq \mathbb{A}_{F,\text{fin}}$.

Proof. By Strong approximation, F is dense in $\mathbb{A}_{F,\text{fin}}$. That $\prod_{v \in M_{F,\text{fin}}} \mathfrak{o}_v$ is open in $\mathbb{A}_{F,\text{fin}}$ and $F \cap \prod_{v \in M_{F,\text{fin}}} \mathfrak{o}_v = \mathcal{O}_F$ prove the lemma. □

As a result, we will use the notation $\widehat{\mathcal{O}}_F$ to stand for the compact subring $\prod_{v \in M_{\mathrm{F},\text{fin}}} \mathfrak{o}_v \subseteq \mathbb{A}_{F,\text{fin}}$. Since the projection map is open, we obtain the

Corollary 9.2.11.2. For any global field F and $v \in M_{F,\text{fin}}$, the ring of integers $\mathcal{O}_F$ is dense in $\mathfrak{o}_v$.

9.2.4 Self-dualness

There is another measure on $\mathbb{A}_F$ that is frequently used, that is the self-dual measure dx^{tam}, which is also by definition the restricted product of the local ones (defined before Lemma 7.1.4). To show dx^{tam} is well-defined, we must show $\mathfrak{d}_w = \mathfrak{o}_w$ for all but finitely many places $w \in M_{F,\mathrm{fin}}$. For this, define the global analog of **(absolute) inverse different**:

$$\mathcal{O}_F^\vee := \{x \in F \mid \mathrm{Tr}_{F/k}(x\mathcal{O}_F) \subseteq \mathcal{O}_k\}$$

where k is either $\mathbb{Q}$ or $\mathbb{F}_p(t)$, and define the global **(absolute) different** $\mathfrak{D}_F$ to be the inverse ideal of $\mathcal{O}_F^\vee$.

Lemma 9.2.12. For $w \in M_{F,\mathrm{fin}}$, the closure of the image of $\mathcal{O}_F^\vee$ under $F \to F_w$ is $\mathfrak{o}_w^\vee$.

Proof. Let $v = w|_k$. Recall the global trace is a sum of local traces: $\mathrm{Tr}_{F/k} = \sum_{w'|v} \mathrm{Tr}_{F_{w'}/k_v}$ as operators on F. The proof will make use of strong approximation and the continuity of local traces to show $\mathfrak{o}_w^\vee$ contains $\mathcal{O}_F^\vee$ as a dense subspace.

Let $x \in \mathcal{O}_F^\vee$ and $y \in \mathfrak{o}_{\mathfrak{v}}$. By Corollary 9.2.11.1, take $y' \in \mathcal{O}_F$ close to y in $\mathfrak{o}_v$ and take $\varepsilon \in \mathcal{O}_F$ such that $\varepsilon \to 1$ in F_w while $\varepsilon \to 0$ in other places of F extending v. Then $\mathcal{O}_F \ni \mathrm{Tr}_{F/k}(xy'\varepsilon) = \mathrm{Tr}_{F_w/k_v}(xy'\varepsilon) + \sum_{w \neq w'|w} \mathrm{Tr}_{F_{w'}/k_v}(xy'\varepsilon)$ and each summand lies in $\mathfrak{o}_v$, so $\mathrm{Tr}_{F_w/k_v}(xy'\varepsilon) \in \mathfrak{o}_v$. Letting $y' \to y$ and $\varepsilon \to 1$ show $\mathrm{Tr}_{F_w/k_v}(xy) \in \mathfrak{o}_v$.

Let $x \in \mathfrak{o}_w^\vee$ and choose $x' \in F^\times$ such that $x' \to x$ in F_w while $x' \to 0$ in other places of F extending v and $|x'|_{w'} \leqslant 1$ for all the other places $w' \in M_{F,\mathrm{fin}}$. Then for $y \in \mathcal{O}_F$, $\mathrm{Tr}_{F/k}(x'y) = \mathrm{Tr}_{F_w/k_v}(x'y) + \sum_{w \neq w'|w} \mathrm{Tr}_{F_{w'}/k_v}(x'y) \in k \cap \mathfrak{o}_v$, and for $v \neq v' \in M_{k,\mathrm{fin}}$, $\mathrm{Tr}_{F/k}(x'y) = \sum_{w'|v'} \mathrm{Tr}_{F_{w'}/k_{v'}}(x'y) \in k \cap \mathfrak{o}_{v'}$. Hence $Tr_{F/k}(x'y) \in \mathcal{O}_F$, so that $x' \in \mathcal{O}_F^\vee$. This proves the density of $\mathcal{O}_F^\vee$ in $\mathfrak{o}_w^\vee$. $\square$

Lemma 9.2.13. Let I be an ideal of $\mathcal{O}_F$. Then for all but finitely $w \in M_{F,\mathrm{fin}}$, the closure of I in $\mathfrak{o}_v$ is $\mathfrak{o}_v$.

Proof. Recall from Lemma 9.2.4.(i) that $\mathcal{O}_F$ is Noetherian, so I is finitely generated, say $I = \langle a_1, \ldots, a_n \rangle$ with each $a_i \neq 0$. By Lemma 9.2.7, there is a finite set $S \subseteq M_{F,\mathrm{fin}}$ such that $|a_i|_v = 1$ for $v \notin S$. Then each a_i is a unit of $\mathfrak{o}_v$ for $v \notin S$. Finally, it suffices to observe that the closure of I is the same as the ideal of $\mathfrak{o}_v$ generated by $a_1, \ldots, a_n$. $\square$

Combining the two lemmas, we obtain

Corollary 9.2.13.1. Let F be a global field. Then $\mathfrak{d}_w = \mathfrak{o}_w$ for all but finitely many places $w \in M_{F,\mathrm{fin}}$.

This shows that dx^{tam} is well-defined. To explain why it is called the "self-dual" measure, we show in the following that there is a canonical isomorphism $\mathbb{A}_F \xrightarrow{\sim} \widehat{\mathbb{A}_F}$. This is just a matter of picking up things. By Theorem 9.1.9, the map

$$\begin{array}{ccc} \displaystyle\prod_{v \in M_F}^{\{\mathfrak{o}_v^\perp\}_{v \in M_{F,\mathrm{fin}}}} \widehat{k_v} & \longrightarrow & \widehat{\mathbb{A}_F} \\ (\psi_v)_v & \longmapsto & \otimes_v' \psi_v \end{array}$$

is a topological group isomorphism. Recall that local fields are self-dual Theorem 7.1.3: if $\psi_v \in \widehat{k_v}\backslash\{0\}$, then the map

$$\begin{aligned} k_v &\longrightarrow \widehat{k_v} \\ x &\longmapsto y \mapsto \psi_v(xy) \end{aligned}$$

is a topological group isomorphism. We pick ψ_v as in the discussion after Theorem 7.1.3. From there recall $y \mapsto \psi_v(xy)$ is trivial on $\mathfrak{o}_v$ if and only if $x \in \mathfrak{d}_v^{-1}$. In other words, under the duality isomorphism above, $\mathfrak{o}_v^{\perp}$ corresponds to $\mathfrak{d}_v^{-1}$. By Corollary 9.2.13.1, the latter is $\mathfrak{o}_v$ for almost all $v \in M_{F,\mathrm{fin}}$. In sum, the above isomorphisms give rises to the duality map

$$\begin{aligned} \mathbb{A}_F &\longrightarrow \widehat{\mathbb{A}_F} \\ (x_v)_v &\longmapsto (y_v)_v \mapsto \prod_{v \in M_F} \psi_v(x_v y_v) \end{aligned}$$

and this is a topological group isomorphism. Let $\psi_F = \otimes_v' \psi_v$ and call its the **standard additive character** of $\mathbb{A}_F$. By construction, we see if F/K is an extension of global fields, then $\psi_K = \psi_F \circ \mathrm{Tr}_{F/K}$. For $f \in L^1(\mathbb{A}_F)$, define the Fourier transform

$$\widehat{f}(x) = \int_{\mathbb{A}_F} f(y)\psi_F(-xy)dx^{\mathrm{tam}}.$$

It follows from the local duality that dx^{tam} is the unique measure on $\mathbb{A}_F$ such that the Fourier inversion holds:

$$f(x) = \int_{\mathbb{A}_F} \widehat{f}(y)\psi_F(xy)dx^{\mathrm{tam}}.$$

In general, if dx^{tam} is replaced by any Haar measure on $\mathbb{A}_F$ and ψ_F is replaced by any nontrivial additive character of $\mathbb{A}_F$, then the Fourier inversion holds up to a constant.

We turn to the compact quotient $\mathbb{A}_F/F$. Under the isomorphism $\mathbb{A}_{\mathbb{F}} \cong \widehat{\mathbb{A}_F}$, the character group $\widehat{\mathbb{A}_F/F}$ is isomorphic to the subspace

$$F^{\perp} = \{x \in \mathbb{A}_F \mid \psi_F(xF) = 1\}.$$

Unwinding the definition, it is not hard to see $F \subseteq F^{\perp}$. Since $\mathbb{A}_F/F$ is compact, $F^{\perp}$ is discrete (5.1.7).1. The quotient $F^{\perp}/F$ is then a discrete subgroup of the compact group $\mathbb{A}_F/F$, so $\#F^{\perp}/F < \infty$. Since F is infinite, it follows that $F = F^{\perp}$.

We compute the ratio of dx^{tam} and dx^{std} to conclude this subsection. By Lemma 7.1.4, we have

$$dx^{\mathrm{tam}} = 2^{r_2} \prod_{v \in M_{F,\mathrm{na}}} (N\mathfrak{d}_v)^{-\frac{1}{2}} \cdot dx^{\mathrm{std}}.$$

Note that every ideal in $\mathfrak{o}_v$ is open. Keeping this in mind, by Lemma 8.2.2.(iii) and Lemma 9.2.12, we see $\mathfrak{o}_v/\mathfrak{d}_v \cong (\mathcal{O}_F)_{\mathfrak{p}_v}/(\mathfrak{D}_F)_{\mathfrak{p}_v}$, where the subscript means localization at $\mathfrak{p}_v$. Note also that $(\mathcal{O}_F)_{\mathfrak{p}_v}$ is the ring of integers of the valuation $|\cdot|_v$ on F. It follows from the fact that $\mathcal{O}_F$ is Dedekind and Chinese remainder theorem that $\mathcal{O}_F/\mathfrak{D}_F \cong \prod_{v \in M_{F,\mathrm{fin}}} \mathfrak{o}_v/\mathfrak{d}_v$. Therefore,

$$2^{r_2} \prod_{v \in M_{F,\mathrm{fin}}} (N\mathfrak{d}_v)^{-\frac{1}{2}} = 2^{r_2} \left(\#\mathcal{O}_F/\mathfrak{D}_F\right)^{-\frac{1}{2}} \prod_{v \in M_{F,\infty} \cap M_{F,\mathrm{na}}} (N\mathfrak{d}_v)^{-\frac{1}{2}}.$$

The last product only appears when F is a global function field.

Lemma 9.2.14. For a global field F, we have

$$dx^{\mathrm{tam}} = 2^{r_2} \left(\#\mathcal{O}_F/\mathfrak{D}_F\right)^{-\frac{1}{2}} \prod_{v \in M_{F,\infty} \cap M_{F,\mathrm{na}}} (N\mathfrak{d}_v)^{-\frac{1}{2}} \cdot dx^{\mathrm{std}},$$

where r_2 is half the number of non-real embedding, and $\mathfrak{D}_F \trianglelefteq \mathcal{O}_F$ is the global absolute different of F.

9.2.5 Curves over finite fields

In this subsection we discuss global function fields, and put them into a geometric setting. Throughout this subsection F always stands for a global function field of characteristic p.

Lemma 9.2.15. The algebraic closure of the prime field $\mathbb{F}_p$ in F is a finite field, and is equals to

$$F \cap \prod_{v \in M_F} \mathfrak{o}_v = \{0\} \cup \left(F^\times \cap \prod_{v \in M_F} \mathfrak{o}_v^\times \right)$$

This is called the **constant (function) field** of F.

Proof. The displayed equality follows from product formula which is proved in the next section. If $0 \neq x \in F$ is algebraic over $\mathbb{F}_p$, then x lies in a finite field extension of $\mathbb{F}_p$. But any valuation on a finite field is trivial, so in particular $|x|_v = 1$ for all $v \in M_F$. Finally, since F is discrete in $\mathbb{A}_F$ and $\prod_{v \in M_F} \mathfrak{o}_v$ is compact, the set $F^\times \cap \prod_{v \in M_F} \mathfrak{o}_v^\times$ is finite. Since it is a group under multiplication, it follows

$$\mu(F) := \{x \in F \mid x^n = 1 \text{ for some } n \in \mathbb{Z}_{\geqslant 1}\} = F^\times \cap \prod_{v \in M_F} \mathfrak{o}_v^\times.$$

It is clear from the definition that each element in $\mu(F)$ is algebraic over $\mathbb{F}_p$. This finishes the proof. ☐

In the sequel we prove F determines a smooth projective curve. We can work in a slightly general setting. Henceforth assume k is an arbitrary field, and F is a finitely generated field extension of k of transcendence degree 1. Set

$$M_{F,k} = \{|\cdot| \in M_F \mid |\cdot||_k \text{ is trivial}\}.$$

If we take any $t \in F$ transcendental over k, then F is a finite extension of $k(t)$. By Theorem 8.2.5 and Theorem 8.4.1, $M_{F,k}$ consists of all discrete valuations on F that are trivial on k. We topologize $M_{F,k}$ by the cofinite topology. Namely, $U \subseteq M_{F,k}$ is open if and only if $U = \varnothing$ or $M_{F,k} \backslash U$ is finite. For each open $U \subseteq M_{F,k}$, set

$$\mathcal{O}_F(U) := F \cap \bigcap_{v \in M_{F,k}} \mathfrak{o}_v.$$

This defines a sheaf of rings on $M_{F,k}$, and $(M_{K,k}, \mathcal{O}_F)$ is a local-ringed space.

Theorem 9.2.16. $(M_{K,k}, \mathcal{O}_F)$ is isomorphic to a smooth projective integral algebraic k-scheme over of dimension 1. If k is algebraically closed in F, then it is geometrically irreducible.

Corollary 9.2.16.1. A global function field with constant field $\mathbb{F}_q$ is the function field of some smooth projective curve X over $\mathbb{F}_q$.

In fact, more is true:

Theorem 9.2.17. Any smooth projective curve ... isomorphic.

So let X be a smooth projective (geometrically irreducible) curve X over $\mathbb{F}_q$. For any $n \geqslant 1$, we have the bijection

$$X(\mathbb{F}_{q^n}) \cong \bigsqcup_{v \in M_F} \mathrm{Hom}_{\mathbf{Alg}_{\mathbb{F}_q}}(\kappa(v), \mathbb{F}_{q^n}) = \bigsqcup_{\substack{v \in M_F \\ \#\kappa(v) \leqslant q^n}} \mathrm{Gal}(\kappa(v)/\mathbb{F}_q)$$

so

$$\#X(\mathbb{F}_{q^n}) = \sum_{\substack{v \in M_F \\ \#\kappa(v) \leqslant q^n}} [\kappa(v) : \mathbb{F}_q] = \sum_{k \mid n} k \times \#\{v \in M_F \mid \#\kappa(v) = q^k\}$$

Lemma 9.2.18. Let F be a global function field with constant field $\mathbb{F}_q$. Let $k/\mathbb{F}_q$ be a finite extension. Then $F \otimes_{\mathbb{F}_q} k$ is a global function field with constant field k.

Proof. Say F is a finite separable extension of $\mathbb{F}_q(t)$. Fix an algebraic closure $\overline{F}$ of $\mathbb{F}_q(t)$ and we think of k as there. Write $F = \mathbb{F}_q(t)[X]/(f)$ for some irreducible $f \in \mathbb{F}_q[t][X]$. It suffices to show f remains irreducible in $k[t][X]$. In fact, we will prove this is even true when $k \subseteq \overline{F}$ is the algebraic closure of $\mathbb{F}_q$.

Suppose otherwise $f = f_1 f_2$ with $f_i \in k[t, X]$. In particular, $f_i \in \mathbb{F}_q(\alpha)[t, X]$ with $\alpha \in k$ algebraic over $\mathbb{F}_q$. Say $g \in \mathbb{F}_q[Y]$ is the minimal polynomial of α. Then g is also irreducible in $\mathbb{F}_q(t)[Y]$, as its roots lie in k, making such decomposition have coefficient in $k \cap \mathbb{F}_q(t) = \mathbb{F}_q$. Consider the diagram

$$\begin{tikzcd}
& F(\alpha) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{F}_q(t,\alpha) \arrow[dr, dash, "\deg g"'] & & F \arrow[dl, dash, "\deg f"] \\
& \mathbb{F}_q(t) &
\end{tikzcd}$$

Since f factors in $\mathbb{F}_q(\alpha)[t, X]$, it follows that $[F(\alpha) : \mathbb{F}_q(t, \alpha)] < \deg f$, so that $[F(\alpha) : F] < \deg g$. This means g factors in $F[Y]$. As said, any decomposition would have coefficient in $k \cap F(t) = \mathbb{F}_q$, so it follows that g factors in $\mathbb{F}_q[Y]$, a contradiction. $\square$

For any $m \in \mathbb{Z}_{\geqslant 1}$, consider the base change $X_m := X \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} \mathbb{F}_{q^m}$. Its function field is $F \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}$ with constant field $\mathbb{F}_{q^m}$. By definition, we have

$$X_m(\mathbb{F}_{q^{nm}}) = \mathrm{Hom}_{\mathbf{Sch}_{\mathbb{F}_{q^m}}}(\operatorname{Spec} \mathbb{F}_{q^{nm}}, X \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} \mathbb{F}_{q^m})$$

But RHS is the same as

$$\mathrm{Hom}_{\mathbf{Sch}_{\mathbb{F}_q}}(\operatorname{Spec} \mathbb{F}_{q^{nm}}, X \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} \mathbb{F}_{q^m}) = X(\mathbb{F}_{q^{nm}}).$$

Indeed, the map $X_m \to \operatorname{Spec} \mathbb{F}_q$ is the composition $X_m \to \operatorname{Spec} \mathbb{F}_{q^m} \to \operatorname{Spec} \mathbb{F}_q$, and any $\mathbb{F}_q$-morphism $\operatorname{Spec} \mathbb{F}_{q^{nm}} \to \operatorname{Spec} \mathbb{F}_q$ factors through $\operatorname{Spec} \mathbb{F}_{q^m} \to \operatorname{Spec} \mathbb{F}_q$ (dually, $\mathbb{F}_{q^m}$ is a subextension of $\mathbb{F}_q \subseteq \mathbb{F}_{q^{nm}}$). In sum

$$X_m(\mathbb{F}_{q^{nm}}) = X(\mathbb{F}_{q^{nm}})$$

9.3 Ideles

For a topological ring R, in general the group of units $R^\times$ is not an open subspace of R, and the inversion $x \mapsto x^{-1}$ is not a topological automorphism on $R^\times$. In other words, $R^\times$ may not be a topological group. To remedy the situation, instead of using only subspace topology on $R^\times$, we use the initial topology induced by the inclusion $R^\times \subseteq R$ and the inversion $R^\times \to R^\times$. Equivalently, $R^\times$ is equipped with the subspace induced by the twist diagonal

$$\begin{aligned} R^\times &\longrightarrow R \times R \\ x &\longmapsto (x, x^{-1}). \end{aligned}$$

where $R \times R$ is with product topology.

For a number field F, by definition the **group of ideles** is the unit group of the ring of adeles

$$\mathbb{A}_F^\times = \{(x_v)_v \in \mathbb{A}_F \mid x_v \in \mathfrak{o}_v^\times \text{ for almost all } v \in M_{F,\mathrm{fin}}\}$$

equipped with the topology described above. It is not hard to see $\mathbb{A}_F^\times$ is isomorphic to the restrict product

$$\mathbb{A}_F^\times = \prod_{v \in M_F}^{\{\mathfrak{o}_v^\times\}_{v \in M_{F,\mathrm{fin}}}} F_v^\times.$$

as topological groups. For $x = (x_v)_v \in \mathbb{A}_F$, we define the **adelic (quasi)-norm** $|\cdot|_{\mathbb{A}_F} : \mathbb{A}_F \to \mathbb{R}_{\geqslant 0}$ by

$$|x|_{\mathbb{A}_F} := \prod_{v \in M_F} |x_v|_v.$$

This is well-defined as $|x_v|_v \leqslant 1$ for all but finitely many v. If $x \in \mathbb{A}_F^\times$, then it is a finite product. Put

$$(\mathbb{A}_F^\times)^1 = \ker(|\cdot|_{\mathbb{A}_F} : \mathbb{A}_F^\times \to \mathbb{R}_{>0}) = \left\{x \in \mathbb{A}_F^\times \mid |x|_{\mathbb{A}_F} = 1\right\}$$

to be the **group of norm one ideles**.

Lemma 9.3.1. If $x \in \mathbb{A}_F$ satisfies $|x|_{\mathbb{A}_F} = 1$, then $x \in \mathbb{A}_F^\times$ and hence $x \in (\mathbb{A}_F^\times)^1$.

By Lemma 9.2.7, the diagonal map $F \to \mathbb{A}_F$ restricts to an embedding $F^\times \to \mathbb{A}_F^\times$. More is true:

Theorem 9.3.2 (Artin Product formula)**.** $F^\times \subseteq (\mathbb{A}_F^\times)^1$.

Proof. This can be proved by a direct computation by reducing to the case for $\mathbb{Q}$ and $\mathbb{F}_p(t)$. Here we use integration and unfolding. The key ingredient is that $\mathrm{mod}_{\mathbb{A}_F}(x) = |x|_{\mathbb{A}_F}$ for $x \in \mathbb{A}_F^\times$. To see this, recall this number is the unique number $c > 0$ such that $\mathrm{vol}(xM, dx^{\mathrm{std}}) = c\,\mathrm{vol}(M, dx^{\mathrm{std}})$ for all measurables $M \subseteq \mathbb{A}_F$. In proving $c = |x|_{\mathbb{A}_F}$, it suffices to take M to be the product of unit balls in various places.

Put $K = \mathbb{A}_F/F$. Let $\xi \in k$. Since $\xi k \subseteq k$, we have $\xi K = K$. By Lemma 2.4.2.(ii) we can find $f \in C_c(\mathbb{A}_F)$ such that $f^k = \mathbf{1}_K$, the characteristic function of K. Then

$$\begin{aligned} \int_K \mathbf{1}_{\xi K}(\beta) d\beta = \int_K \mathbf{1}_K(\xi^{-1}\beta) d\beta &= \int_K \int_k f(\xi^{-1}(\beta + r)) dr d\beta \\ &= \int_{\mathbb{A}_k} f(\xi^{-1}x) dx = |\xi|_{\mathbb{A}_F} \int_{\mathbb{A}_k} f(x) dx = |\xi|_{\mathbb{A}_F} \int_K \mathbf{1}_K(\beta) d\beta \end{aligned}$$

Since $\xi K = K$ and K is compact (so the integral is finite), it follows that $|\xi|_{\mathbb{A}_F} = 1$. ∎

Theorem 9.3.3. $(\mathbb{A}_F^\times)^1$ is a closed subset of $\mathbb{A}_F$, and the two subspace topologies on $(\mathbb{A}_F^\times)^1$ from $\mathbb{A}_F$ and $\mathbb{A}_F^\times$ are the same.

Proof. Let $\alpha \in \mathbb{A}_F \backslash (\mathbb{A}_F^\times)^1$; by Lemma 9.3.1 we have $|\alpha|_{\mathbb{A}_F} \neq 1$. We must find an open neighborhood W of α in $\mathbb{A}_F$ that is disjoint from $(\mathbb{A}_F^\times)^1$.

- $|\alpha|_{\mathbb{A}_F} < 1$. Then there is a finite set S of places such that
  - S contains all the places v with $|\alpha_v|_v > 1$ and
  - $\prod_{v \in S} |\alpha_v|_v < 1$.

  Now take $0 < \varepsilon < \min_{v \in S} |\alpha|_v$ and define

  $$W := \{x = (x_v) \in \mathbb{A}_k \mid |x_v - \alpha_v|_v < \varepsilon \text{ for } v \in S,\, |x_v|_v \leqslant 1 \text{ for } v \notin S\}$$

  Clearly, every element x in W has $|x| < 1$.

- $|\alpha|_{\mathbb{A}_F} > 1$. Put $C = \prod_{v\,:\,|\alpha_v|_v > 1} |\alpha_v|_v > 1$. Then there is a finite set S of places such that S contains
  - all the places v with $|\alpha_v|_v > 1$,
  - all archimedean places, and
  - all non-archimedean places v with $\#\kappa(v) \leqslant 2C$.

  For $\varepsilon > 0$ define

  $$W := \{x = (x_v) \in \mathbb{A}_F \mid |x_v - \alpha_v|_v < \varepsilon \text{ for } v \in S,\, |x_v|_v \leqslant 1 \text{ for } v \notin S\}$$

  Take $\varepsilon > 0$ small enough so that $x \in W$ implies $1 < \prod_{v \in S} |x_v| < 2C$. Then for $x \in W$, if $|x_v|_v = 1$ for all $v \notin S$, then

  $$|x|_{\mathbb{A}_F} = \prod_{v \in S} |x_v|_v > 1$$

  If $|x_v|_v < 1$ for some $v \notin S$, then $|x_v|_v \leqslant \#\kappa(v)^{-1} < (2C)^{-1}$ so that

  $$|x|_{\mathbb{A}_F} < \left(\prod_{\nu \in S} |x_v|_\nu \right) (2C)^{-1} < 1$$

It remains to show the second statement. Let $\alpha \in (\mathbb{A}_F^\times)^1$. A neighborhood basis of α in $\mathbb{A}_F$ consists of sets of the form

$$W = W_{\varepsilon,S} := \{x \in \mathbb{A}_F \mid |x_v - \alpha_v|_v < \varepsilon \text{ for } v \in S,\, |x_v|_v \leqslant 1 \text{ for } v \notin S\}$$

where $\varepsilon > 0$ and S is a finite set of places. By replacing $\leqslant$ with $=$, we see every such a set contains a neighborhood of α in $\mathbb{A}_F^\times$. Conversely, a neighborhood basis of α in $\mathbb{A}_F$ consists of sets of the form

$$H = H_{\varepsilon,S} := \{x \in \mathbb{A}_F^\times \mid |x_v - \alpha_v|_v < \varepsilon \text{ for } \nu \in S,\, |x_v|_v = 1 \text{ for } v \notin S\}$$

where $\varepsilon > 0$ and S is a finite set of places containing all archimedean places and all v with $|\alpha_v|_\nu \neq 1$. We claim for ε small enough

$$H_{\varepsilon,S} \cap (\mathbb{A}_F^\times)^1 = W_{\varepsilon,S} \cap (\mathbb{A}_F^\times)^1$$

$\subseteq$ is clear. Let $x \in W_{\varepsilon,S} \cap (\mathbb{A}_F^\times)^1$. Let ε small enough so that $|x_v|_v = |\alpha_v|_v$ for all non-archimedean places v in S. Since x is an idele, we then have $|x_v|_v = 1$ for almost all $v \notin S$. Now it follows from the discreteness of $v \notin S$ that it we take ε far smaller, then we must have $|x_v|_v = 1$ for all $v \notin S$. NEED IMPROVEMENT ☐

Theorem 9.3.4 (Fujisaki)**.** The inclusion $F^\times \to (\mathbb{A}_F^\times)^1$ has discrete image and the quotient $(\mathbb{A}_F^\times)^1/F^\times$ is compact.

Proof. The first assertion follows from the continuity of the inclusion $\mathbb{A}_F^\times \subseteq \mathbb{A}_F$ and Proposition 9.2.9. For the latter assertion, by Theorem 9.3.3 it suffices to find a compact subset W of $\mathbb{A}_F$ such that the projection $W \cap (\mathbb{A}_F^\times)^1 \to (\mathbb{A}_F^\times)^1/F^\times$ is surjective.

Let C be as in Lemma 9.2.10, and take $\alpha \in \mathbb{A}_F^\times$ such that $|\alpha| > C$. Take

$$W = \{x \in \mathbb{A}_F \mid |x_v|_v \leqslant |\alpha_v|_v \text{ for all } v\}$$

Let $y \in (\mathbb{A}_F^\times)^1$. By the same lemma there exists $r \in F^\times$ such that $|r|_v \leqslant |y_v^{-1}\alpha_v|$ for all v. Then $ry \in W$, as required. ☐

Lemma 9.3.5. Let F/K be an extension of global field.

(i) The canonical map $\mathbb{A}_K^\times \to \mathbb{A}_F^\times$ is a closed embedding.

(ii) The induced map $\mathbb{A}_K^\times/K^\times \to \mathbb{A}_F^\times/F^\times$ is a closed embedding.

Lemma 9.3.6. Let K be a global field.

(i) For any place $v \in M_K$, the inclusion $K_v^\times \to \mathbb{A}_K^\times/K^\times$ is a closed embedding.

(ii) For any finite set $S \subseteq M_K$ with $\#S \geqslant 2$, the inclusion $\prod_{v \in S} K_v^\times \to \mathbb{A}_K^\times/K^\times$ is not a closed embedding.

9.3.1 Ideal class group

For an Dedekind domain R with fraction field F, a **fractional ideal** of R is an R-submodule I of F such that $\alpha I \subseteq R$ for some $\alpha \in F$. It turns out the ideal multiplication defines a group structure on the collection $\mathfrak{I}(R)$ of all fractional ideals. Moreover, it is free on the set of maximal ideals. For an element $\alpha \in F$, the R-module $\alpha R \subseteq F$ is called the **principal fractional ideals**. All principal fractional ideals $\mathfrak{P}(R)$ form a subgroup of the fractional ideals. We denote their quotient by $\mathrm{Cl}(R)$ and call its the **(ideal) class group** of the Dedekind domain R. When the domain R itself is clear from the context, we write $\mathrm{Cl}(F)$ instead of $\mathrm{Cl}(R)$.

There is a canonical surjection

$$\begin{array}{ccccc} \mathbb{A}_F^\times & \longrightarrow & \mathfrak{I}(\mathcal{O}_F) & \longrightarrow & \mathrm{Cl}(F) := \mathrm{Cl}(\mathcal{O}_F) \\ (x_v)_v & \longmapsto & \prod_{\mathfrak{p}_v} \mathfrak{p}_v^{\mathrm{ord}_v x_v} & & \end{array}$$

Let k be either $\mathbb{Q}$ or $\mathbb{F}_p(t)$, and let ∞ denote the unique infinite place on k. For a global field F, set $F_\infty := F \otimes_k k_\infty$; when F is a number field, this is the same as $F_\mathbb{R}$. Since maximal ideals are in bijection with $M_{F,\mathrm{fin}}$, it follows that $F_\infty^\times = \{1\} \times F_\infty^\times \subseteq \mathbb{A}_F^\times$ lies in the kernel of the above surjection. Also $F^\times$ and $\widehat{\mathcal{O}}_F^\times = \prod_{v \in M_{F,\mathrm{fin}}} \mathfrak{o}_v^\times$ lie in the kernel for a trivial reason. It is not hard to see in fact that

Lemma 9.3.7. The above surjection induces a bijection

$$F^\times \backslash \mathbb{A}_F^\times / \widehat{\mathcal{O}}_F^\times F_\infty^\times \xrightarrow{\quad\sim\quad} \mathrm{Cl}(F).$$

We equip $\mathrm{Cl}(\mathcal{O}_F)$ with the discrete topology. Then the above bijection is continuous. We claim in fact that $F^\times \backslash \mathbb{A}_F^\times / \widehat{\mathcal{O}}_F^\times F_\infty$ is compact, which will imply

Corollary 9.3.7.1 (Finiteness of class group)**.** $\#\mathrm{Cl}(\mathcal{O}_F) < \infty$.

To show $F^\times \backslash \mathbb{A}_F^\times / \widehat{\mathcal{O}}_F^\times F_\infty$ is compact, we need to relate this space with the compact quotient $(\mathbb{A}_F^\times)^1 / F^\times$. For this we need to know the image of the adelic norm $|\cdot|_{\mathbb{A}_F} : \mathbb{A}_F^\times \to \mathbb{R}_{>0}$.

Lemma 9.3.8.

$$\mathrm{Im}\left(|\cdot|_{\mathbb{A}_F} : \mathbb{A}_F^\times \to \mathbb{R}_{>0}\right) = \begin{cases} \mathbb{R}_{>0} & \text{, if } F \text{ is a number field} \\ p^{m\mathbb{Z}} & \text{, if } F \text{ is a global function field} \end{cases}$$

for some $m \in \mathbb{Z}$.[8]

Proof. The absolute values on $\mathbb{R}^\times$ and $\mathbb{C}^\times$ surjects onto $\mathbb{R}_{>0}$, and the absolute values on a non-archimedean valued field (k, v) has image contained in $p^{\mathbb{Z}}$, where $p = \mathrm{Char}\,\kappa(v)$. If F is a number field, then $\mathbb{A}_F^\times$ contains a copy of $\mathbb{R}^\times$ or $\mathbb{C}^\times$. If F is a global function field, then $\mathrm{Char}\,F = p > 0$ is the characteristic of every of its residue field. In this case $\mathrm{Im}\,|\cdot|_{\mathbb{A}_F}$ is a subgroup of $p^{\mathbb{Z}}$, so it has the form $p^{m\mathbb{Z}}$ for some $m \in \mathbb{Z}_{\geqslant 1}$. ☐

More generally, we introduce the ring of S**-integers**: for a finite *nonempty*[9] set of places $S \subseteq M_F$ containing all archimedean places, set

$$\mathcal{O}_F^S := \bigcap_{v \notin S} F \cap \mathfrak{o}_v = \{x \in F \mid |x|_v \leqslant 1 \text{ for all } v \notin S\}.$$

Then $\mathcal{O}_F^S \subseteq F$, and $\mathcal{O}_F^S$ is a Dedekind domain, making sense to talk about the class group

$$\mathrm{Cl}_S(F) := \mathrm{Cl}(\mathcal{O}_F^S).$$

Adelically, on the other hand, let $F_S = \prod_{v \in S} F_v$ and $\widehat{\mathcal{O}}_F^S = \prod_{v \notin S} \mathfrak{o}_v$. There is a canonical surjection

$$\begin{array}{ccccc} \mathbb{A}_F^\times & \longrightarrow & \mathfrak{I}(\mathcal{O}_F^S) & \longrightarrow & \mathrm{Cl}(\mathcal{O}_F^S) \\ (x_v)_v & \longmapsto & \displaystyle\prod_{\mathfrak{p}_v : v \notin S} \mathfrak{p}_v^{\mathrm{ord}_v x_v} & & \end{array}$$

whose kernel contains $F^\times$ and $(\widehat{\mathcal{O}}_F^S)^\times F_S$. A similar observation gives

Lemma 9.3.9. The induced map is a bijection

$$F^\times \backslash \mathbb{A}_F^\times / (\widehat{\mathcal{O}}_F^S)^\times F_S^\times \xrightarrow{\quad\sim\quad} \mathrm{Cl}_S(F).$$

If we equip $\mathrm{Cl}_S(F)$ with the discrete topology, this is a continuous bijection.

[8] In fact, p^m is the cardinality of the constant field of F. Say X is a smooth projective geometrically connected k-curve of function field $k(X) = F$. Then it is the same as showing X has a degree 1 divisor.

This has an interesting consequence: if X moreover has genus 1, then X has a k-rational point. In particular, X is an elliptic curve.

[9] This assumption is mainly for the global function field.

Consider the image of $F_S^\times = \{1\} \times F_S^\times$ in $\mathbb{R}_{>0}$ under $|\cdot|_{\mathbb{A}_F}$. It is the entire $\mathbb{R}_{>0}$ if F is a number field, and is $p^{n\mathbb{Z}}$ for some $m \mid n \in \mathbb{Z}_{\geqslant 1}$ if F is a global function field. Then

$$\mathbb{A}_F^\times / F_S^\times \cong (\mathbb{A}_F^\times)^1 / (\mathbb{A}_F^\times)^1 \cap F_S^\times$$

if F is a number field, and there is a surjection

$$\coprod_{0 \leqslant \ell < \frac{n}{m}} (\mathbb{A}_F^\times)^{p^{m\ell}} \twoheadrightarrow \mathbb{A}_F^\times / F_S^\times$$

if F is a global function field, where $(\mathbb{A}_F^\times)^\delta = \{x \in \mathbb{A}_F^\times \mid |x|_{\mathbb{A}_F} = \delta\}$. The same proof as Theorem 9.3.4 shows that the quotient space $F^\times \backslash (\mathbb{A}_F^\times)^\delta$ is compact, so in either case $F^\times \backslash \mathbb{A}_F^\times / F_S^\times$ is compact. Hence the further quotient $F^\times \backslash \mathbb{A}_F^\times / (\widehat{\mathcal{O}}_F^S)^\times F_S^\times$ is compact. From this we conclude

Corollary 9.3.9.1. $\#\mathrm{Cl}_S(F) < \infty$. Moreover, we can pick S large enough such that $\#\mathrm{Cl}_S(F) = 1$.

Proof. It remains to show the last assertion. Pick $x_1, \ldots, x_n \in \mathbb{A}_F^\times$ such that their images in $\mathrm{Cl}(F)$ generate the group. Choose a finite $M_{F,\infty} \subseteq S \subseteq M_F$ such that $|x_i|_v \neq 1$ for some $i \in [n]$ implies $v \in S$; then $x_1, \ldots, x_n \in (\widehat{\mathcal{O}}_F^S)^\times F_S^\times$. Since $F^\times \backslash \mathbb{A}_F^\times / \widehat{\mathcal{O}}_F^\times F_\infty^\times \to F^\times \backslash \mathbb{A}_F^\times / (\widehat{\mathcal{O}}_F^S)^\times F_S^\times$ is surjective and $x_1, \ldots, x_n$ generates the former quotient, it follows that the latter quotient is trivial. ☐

As an aside, we justify the notation $\widehat{\mathcal{O}}_F^S$.

Lemma 9.3.10. The image of the inclusion $\mathcal{O}_F^S \to \mathbb{A}_F / F_S$ is dense in $\widehat{\mathcal{O}}_F^S$.

Proof. $\mathcal{O}_F$ is already dense in $\widehat{\mathcal{O}}_F^S$ by Corollary 9.2.11.1. ☐

A notational caveat: $\widehat{\mathcal{O}}_F^S$ is "smaller than" $\widehat{\mathcal{O}}_F$, while $\mathcal{O}_F \subseteq \mathcal{O}_F^S \subseteq F$.

9.3.2 Ray class group

Definition. Let F be a global field and S a *nonempty* finite set of places. An S**-modulus** $\mathfrak{m}$ is a integral formal sum

$$\mathfrak{m} = \sum_{v \notin S} a_v v$$

with the properties that $a_v = 0$ for almost all v, $a_v = 0$ if v is archimedean non-real and $a_v \in \{0, 1\}$ if v is archimedean real.

(i) We write $v \mid \mathfrak{m}$ or $v \in \mathfrak{m}$ when $a_v \neq 0$.

(ii) Set

$$U_{\mathfrak{m}}^S = F_S^\times \times \prod_{\substack{v \in \mathfrak{m} \cap M_{F,\mathrm{a}} \\ a_v = 0}} F_v^\times \times \prod_{\substack{v \in \mathfrak{m} \cap M_{F,\mathrm{a}} \\ a_v = 1}} (F_v^\times)_{>0} \times \prod_{v \in \mathfrak{m} \cap M_{F,\mathrm{na}}} 1 + \mathfrak{p}_v^{a_v} \times \prod_{\substack{v \in M_{F,\mathrm{na}} \\ v \notin \mathfrak{m} \\ v \notin S}} \mathfrak{o}_v^\times.$$

where $(F_v^\times)_{>0} = \mathbb{R}_{>0}$ when $a_v = 1$.

(iii) The quotient

$$\mathrm{Cl}_{\mathfrak{m}}(F) := F^\times \backslash \mathbb{A}_F^\times / U_{\mathfrak{m}}$$

is called the S**-ray class group of conductor** $\mathfrak{m}$.

Lemma 9.3.11. $\mathrm{Cl}_{\mathfrak{m}}(F) < \infty$, and there is a surjective homomorphism $\mathrm{Cl}_{\mathfrak{m}}(F) \to \mathrm{Cl}_S(F)$.

Proof. This follows as $U_{\mathfrak{m}}^S \leqslant (\widehat{\mathcal{O}}_F^S)^\times F_S^\times$ is a closed subgroup of finite index. ☐

9.3.3 Unit theorem

The Dirichlet unit theorem tells about the group structure of the unit group $\mathcal{O}_F^\times$ of the ring of integers. To state the theorem, we introduce some notation. Denote

$$\mu(F) = \{r \in F \mid r^n = 1 \text{ for some } n \in \mathbb{Z}_{\geqslant 1}\}.$$

This is the group of roots of unity in F. In fact

$$\mu(F) = F^\times \cap \widehat{\mathcal{O}}_F^\times = \bigcap_{v \in M_{F,\mathrm{fin}}} F^\times \cap \mathfrak{o}_v^\times.$$

Clearly $\subseteq$ holds. For $\supseteq$ note that the right hand side is finite since it is an intersection of a discrete closed set and a compact set. In particular, every element in it has finite order, i.e. is a root of unity. Now we can state the S-version of the Dirichlet unit theorem.

Theorem 9.3.12. Let $M_{F,\infty} \subseteq S \subseteq M_F$ be a finite set of places. We have a group isomorphism

$$(\mathcal{O}_F^S)^\times \cong \mu(F) \times \mathbb{Z}^{\#S-1}.$$

Proof. By the same argument as above we have $\mathrm{Tor}(\mathcal{O}_F^S)^\times = \mu(F)$. To show $\mathrm{rank}(\mathcal{O}_F^S)^\times = \#S - 1$, we use logarithm. Put

$$\begin{aligned}(\mathbb{A}_F^\times)_S &= (\widehat{\mathcal{O}}_F^S)^\times F_S \\ &= \{(x_v)_v \in \mathbb{A}_F^\times \mid |x_v|_v = 1 \text{ for all } v \notin S\} \underset{\text{open}}{\leqslant} \mathbb{A}_F^\times \\ (\mathbb{A}_F^\times)_S^1 &= (\mathbb{A}_F^\times)_S \cap (\mathbb{A}_F^\times)^1\end{aligned}$$

Consider the log map

$$\begin{aligned}\log : (\mathbb{A}_F^\times)_S &\longrightarrow \mathbb{R}^S \\ (x_v)_v &\longmapsto (\log |x_v|_v)_{v \in S}.\end{aligned}$$

which is a continuous homomorphism. Immediately we have

$$\begin{aligned}\log(\mathbb{A}_F^\times)_S &= \prod_{\nu \in S \cap M_{F,\mathrm{fin}}} \mathbb{Z} \log \#\kappa(v) \times \prod_{\nu \in S \cap M_{F,\infty}} \mathbb{R} \\ \log(\mathbb{A}_F^\times)_S^1 &= \left\{ (y_\nu)_{\nu \in S} \in \log(\mathbb{A}_F^\times)_S \mid \sum_{\nu \in S} y_\nu = 0 \right\}\end{aligned}$$

so that $\dim_{\mathbb{R}} \mathbb{R} \otimes_{\mathbb{Z}} \log(\mathbb{A}_F^\times)_S^1 = \#S - 1$. Since $(\mathcal{O}_F^S)^\times = (\mathbb{A}_F^\times)_S \cap F^\times$, we have a closed embedding

$$(\mathcal{O}_F^S)^\times \backslash (\mathbb{A}_F^\times)_S \cap (\mathbb{A}_F^\times)^1 \subseteq F^\times \backslash (\mathbb{A}_F^\times)^1$$

The latter group is compact by Theorem 9.3.4. Since log is continuous, it follows that

$$\log\,(\mathcal{O}_F^S)^\times \backslash (\mathbb{A}_F^\times)_S^1 = \log(\mathcal{O}_F^S)^\times \backslash \log(\mathbb{A}_F^\times)_S^1$$

is compact as well. From this we see $\dim_{\mathbb{R}} \mathbb{R} \otimes_{\mathbb{Z}} \log(\mathcal{O}_F^S)^\times = \dim_{\mathbb{R}} \mathbb{R} \otimes_{\mathbb{Z}} \log(\mathbb{A}_F^\times)_S^1 = \#S - 1$.

To conclude, note that every bounded subset in $\log(\mathcal{O}_F^S)^\times$ is finite. Indeed, every subset is the intersection of F, discrete, and a compact set in $\mathbb{A}_F$. In particular, Lemma 9.3.13 applies, proving

$$\mathrm{rank}(\mathcal{O}_F^S)^\times = \dim_{\mathbb{R}} \mathbb{R} \otimes_{\mathbb{Z}} (\mathcal{O}_F^S)^\times = \#S - 1.$$

□

Lemma 9.3.13. Let $G \leqslant \mathbb{R}^n$ such that every bounded subset of G is finite. Then $\operatorname{rank} G = \dim_{\mathbb{R}} \mathbb{R} \otimes_{\mathbb{Z}} G$.

Proof. For any subgroup $H \leqslant \mathbb{R}^n$, set $H_{\mathbb{R}} = \operatorname{span}_{\mathbb{R}} H \subseteq \mathbb{R}^n$. Let Λ be a subgroup of G such that $\infty > \operatorname{rank} \Lambda = \dim_{\mathbb{R}} \Lambda_{\mathbb{R}}$; assume $\operatorname{rank} \Lambda$ is maximal. For any $v \in G$, by maximality $\operatorname{rank} \Lambda + \mathbb{Z}v = \operatorname{rank} \Lambda$, so $v \in \Lambda_{\mathbb{R}}$. This shows $G \subseteq \Lambda_{\mathbb{R}}$. We replace $\mathbb{R}^n$ by $\Lambda_{\mathbb{R}}$, and let P be a fundamental parallelotope of Λ. Each coset $v + \Lambda\,(v \in G)$ has a representative in P, so by assumption $\#G/\Lambda < \infty$. In particular, $\operatorname{rank} G = \operatorname{rank} \Lambda = \dim_{\mathbb{R}} \Lambda_{\mathbb{R}} \leqslant \dim_{\mathbb{R}} G_{\mathbb{R}} \leqslant \operatorname{rank} G$, which finishes the proof. $\square$

This lemma also gives a quick adelic proof of Lemma 9.2.5:

Lemma 9.3.14. The inclusion $\mathcal{O}_F \to F_{\mathbb{R}} := F \otimes_{\mathbb{Q}} \mathbb{R}$ is discrete and cocompact.

Proof. Let $\iota : \mathcal{O}_F \to F_{\mathbb{R}}$ be the inclusion, and pick K be any compact set in $F_{\mathbb{R}}$. Then $\iota^{-1}(K) = F \cap \left(K \times \widehat{\mathcal{O}_F}\right)$ is an intersection of discrete and compact, so $\#\iota^{-1}(K) < \infty$. By Lemma 9.3.13, this shows $\iota : \mathcal{O}_F \to F_{\mathbb{R}}$ is discrete and cocompact. $\square$

More generally, we have

Lemma 9.3.15. For F a global field and for any finite set $S \subseteq M_F$, the inclusion $\mathcal{O}_F^S \to F_S$ is discrete and cocompact.

Proof. This is a consequence of Theorem 9.2.9. Consider the commutative diagram

$$\begin{tikzcd}
F_S & F_S \times \prod_{v \notin S} \mathfrak{o}_v \arrow[l] \arrow[r] & \mathbb{A}_F \\
\mathcal{O}_F^S \arrow[u] \arrow[r, equal] & \mathcal{O}_F^S \arrow[u] \arrow[r] & F. \arrow[u]
\end{tikzcd}$$

Notice the square on the right is a fibre square. The set $F_S \times \prod_{v \notin S} \mathfrak{o}_v$ is open in $\mathbb{A}_F$, so it follows that $\mathcal{O}_F^S$ is discrete in $F_S \times \prod_{v \notin S} \mathfrak{o}_v$. Since the projection $F_S \times \prod_{v \notin S} \mathfrak{o}_v \to F_S$ is open, we conclude that $\mathcal{O}_F^S$ is discrete in F_S. For cocompactness, since the square on the right is a fibre square, we have a closed (and open) embedding $\left(F_S \times \prod_{v \notin S} \mathfrak{o}_v\right)/\mathcal{O}_F^S \to \mathbb{A}_F/F$. Hence $\left(F_S \times \prod_{v \notin S} \mathfrak{o}_v\right)/\mathcal{O}_F^S$ is compact. Since $\left(F_S \times \prod_{v \notin S} \mathfrak{o}_v\right)/\mathcal{O}_F^S$ surjects onto $F_S/\mathcal{O}_F^S$, we deduce that $F_S/\mathcal{O}_F^S$ is compact. $\square$

9.3.4 Regulator and fundamental domain

In this subsection we introduce a fundamental domain of the idele class group. Assume F is a number field and $S = M_{F,\infty} = \infty$. The restriction to $\mathcal{O}_F^\times = F^\times \cap (\mathbb{A}_F^\times)^1_\infty$ of the log map $\log : (\mathbb{A}_F^\times)_\infty \longrightarrow \mathbb{R}^\infty$ is usually denoted by

$$\operatorname{reg} : \mathcal{O}_F^\times \longrightarrow \mathbb{R}^\infty$$

and is called the **regulator map** of F. It has the properties that

$$\ker \operatorname{reg} = \mu(F)$$

and that $\operatorname{Im}\operatorname{reg} = \log(\mathcal{O}_F^\times)$ is a full rank lattice of the hyperplane

$$H := \left\{ (y_v)_v \in \mathbb{R}^\infty \mid \sum_{v \in \infty} y_v = 0 \right\}.$$

Let $\{\varepsilon_i\}_{1 \leqslant i \leqslant r}$ be a $\mathbb{Z}$-basis for $\mathcal{O}_F^\times$ modulo its torsion. Then $\{\operatorname{reg}(\varepsilon_i)\}_{1 \leqslant i \leqslant r}$ is a basis for H and we can form their fundamental parallelotope P in H:

$$P := \left\{ \sum_{i=1}^{r} t_i \operatorname{reg}(\varepsilon_i) \in H \mid 0 \leqslant t_i < 1,\ 1 \leqslant i \leqslant r \right\}$$

Let $h = \#\operatorname{Cl}(F)$ be the class number of F, and take a set $c_1, \ldots, c_h \in (\mathbb{A}_k^\times)^1$ of complete representatives of $\operatorname{Cl}(F)$. Let $w = \#\mu(F)$ be the number of roots of unity in F. Finally, fix some $v_0 \in M_{F,\infty}$ and set

$$E_{v_0} := \left\{ (x_v)_v \in (\mathbb{A}_F^\times)^1_\infty \cap \log^{-1}(P) \mid 0 \leqslant \operatorname{Arg} x_{v_0} < \frac{2\pi}{w} \right\}$$

We define the **multiplicative fundamental domain** E for $(\mathbb{A}_F^\times)^1$ mod $F^\times$ to be

$$E = E_{v_0} c_1 \cup \cdots \cup E_{v_0} c_h$$

The name is justified by the

Lemma 9.3.16.

$$(\mathbb{A}_F^\times)^1 = \bigsqcup_{r \in F^\times} rE$$

Moreover, $E = E_{v_0} c_1 \cup \cdots \cup E_{v_0} c_h$ is a disjoint union.

Proof. Let $x \in (\mathbb{A}_F^\times)^1$. There exists a unique c_i and $r \in F^\times$ such that

$$rxc_i^{-1} \in (\mathbb{A}_F^\times)^1 \cap (F_\infty^\times \times \widehat{\mathcal{O}}_F^\times)$$

There is a unique $u \in \langle \varepsilon_i \mid 1 \leqslant i \leqslant r \rangle_{\mathbb{Z}}$ so that $rxc_i^{-1}u \in (\mathbb{A}_F^\times)^1 \cap \log^{-1}(P)$. Finally take some $w \in \mu(F)$ so that $wrxc_i^{-1}u \in E_{v_0}$. This proves the first assertion.

For the moreover part, recall $(\mathbb{A}_F^\times)^1 \subseteq F_\infty^\times \times \widehat{\mathcal{O}}_F^\times$. Then $E_{v_0} c_i \cap E_{v_0} c_j \neq \varnothing$ would imply that the ideals in $\mathcal{O}_F$ that c_i and c_j represent are the same. This is a contradiction. $\square$

9.3.5 Norm map

Theorem 9.3.17. Let F/K be an extension of global fields. The norm map $N_{L/K} : \mathbb{A}_L^\times / L^\times \to \mathbb{A}_K^\times / K^\times$ is continuous, proper and open.

9.3.6 Identity component of idele class group

Let K be a number field. The identity component of $K_\infty^\times$ is

$$(K_\infty^\times)^\circ \cong (\mathbb{R}_{>0})^{r_1} \times (\mathbb{C}^\times)^{r_2}.$$

Denote by D_K the closure of the image of $(K_\infty^\times)^\circ$ in $\mathbb{A}_K^\times / K^\times$.

Lemma 9.3.18. One has the equalities

$$D_K = \{\text{divisible elements in } \mathbb{A}_K^\times/K^\times\} = (\mathbb{A}_K^\times/K^\times)^\circ.$$

Moreover, the quotient $(\mathbb{A}_K^\times/K^\times)/D_K$ is profinite.

Proof. Consider the map

$$(K_\infty^\times)^\circ \times \widehat{\mathcal{O}_K^\infty}^\times \longrightarrow (\mathbb{A}_K^\times)_\infty \longrightarrow \mathbb{A}_K^\times/K^\times.$$

where ∞ denotes $M_{K,\infty}$.

A profinite group has only trivial divisible element. $\square$

Artin-Tate

Chapter 10

Tate thesis : global theory

10.1 Hecke characters

Definition. For a global field F, a **Hecke character** is a quasi-character $\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$. [1]

By Lemma 9.1.8, a quasi-character $\chi : \mathbb{A}_F^\times \to \mathbb{C}^\times$ has the form $\chi = \otimes'_{v \in M_F} \chi_v$ with $\chi_v \in \mathrm{Hom}_{\mathbf{TopGp}}(k_v^\times, \mathbb{C}^\times)$ and there exists a finite subset $S \subseteq M_F$ containing $M_{F,\mathrm{a}}$ such that χ_v is unramified, i.e., $\chi_v|_{\mathfrak{o}_v^\times} \equiv 1$ for all $v \notin S$.

Since a Hecke character χ is trivial on $F^\times$ by definition, from Theorem 9.3.4 we see $\chi((\mathbb{A}_F^\times)^1)$ is a compact subgroup of $\mathbb{C}^\times$, so $\chi((\mathbb{A}_F^\times)^1) \subseteq S^1$. Then $|\chi| : \mathbb{A}_F^\times \to \mathbb{R}_{>0}$ is trivial on $(\mathbb{A}_F^\times)^1$. If F is a number field, we have the diagonal embedding

$$\begin{aligned} \mathbb{R}_{>0} &\longrightarrow F_\infty = \{1\} \times F_\infty \subseteq \mathbb{A}_F^\times \\ r &\longmapsto (r, \dots, r) \end{aligned}$$

Let us denote temporarily the image by Δ. Assume F is a global function field, and pick $z \in \mathbb{A}_F^\times$ such that $|z|_{\mathbb{A}_F} \geqslant 1$ generates the image $\mathrm{Im}\, |\cdot|_{\mathbb{A}_F}$. Let $\Delta = \Delta_z$ denote the subgroup in $\mathbb{A}_F^\times$ generated by z. In either case, $\Delta \leqslant \mathbb{A}_F^\times$ and the multiplication provides a splitting

$$\mathbb{A}_F^\times \xrightarrow{\sim} (\mathbb{A}_F^\times)^1 \times \Delta$$

If F is a number field, then this map is given by

$$x = (x_f, x_\infty) \mapsto ((x_f, x_\infty |x|_{\mathbb{A}_F}^{-1}), |x|_{\mathbb{A}_F}).$$

where x_f (resp. x_∞) denotes the component of x in $\mathbb{A}_{F,\mathrm{fin}}$ (resp. F_∞). If F is a global function field, then this is

$$x \mapsto (xz^{-\log_{|z|_{\mathbb{A}_F}} |x|_{\mathbb{A}_F}}, z^{\log_{|z|_{\mathbb{A}_F}} |x|_{\mathbb{A}_F}})$$

Since $|\chi|$ is trivial on $(\mathbb{A}_F^\times)^1$, it can be regarded as a continuous homomorphism $\Delta \to \mathbb{R}_{>0}$. If F is a number field, then $\Delta = \mathbb{R}_{>0}$ and any such map has the form $x \mapsto x^\sigma$ for some $\sigma \in \mathbb{R}$. Hence

$$|\chi(x)| = |\chi(|x|_{\mathbb{A}_F})| = ||x|_{\mathbb{A}_F}|^\sigma = |x|_{\mathbb{A}_F}^\sigma.$$

[1] There is no different in writing $F^\times \backslash \mathbb{A}_F^\times$ and $\mathbb{A}_F^\times / F^\times$, as $\mathbb{A}_F^\times$ is an abelian group.

If F is a global function field, then $\Delta = z^{\mathbb{Z}}$ and any such function is determined by the value of z; if z is mapped to λ, then z^n is mapped to $\lambda^n = |z|_{\mathbb{A}_F}^{n \log_{|z|_{\mathbb{A}_F}} \lambda} =: |z^n|_{\mathbb{A}_F}^{\sigma}$. So such function has the form $x \mapsto |x|_{\mathbb{A}_F}^{\sigma}$. Hence

$$|\chi(x)| = |\chi(z^{\log_{|z|_{\mathbb{A}_F}} |x|_{\mathbb{A}_F}})| = |z|_{\mathbb{A}_F}^{\sigma \log_{|z|_{\mathbb{A}_F}} |x|_{\mathbb{A}_F}} = |x|_{\mathbb{A}_F}^{\sigma}.$$

The number $\sigma \in \mathbb{R}$ does not depend on the choice of $z \in \mathbb{A}_F^{\times}$. In any case, we write $\sigma = \operatorname{wt} \sigma$ and call it the **weight** of the Hecke character χ. We then have a bijection

$$\begin{aligned} \operatorname{Hom}_{\mathbf{TopGp}}(F^{\times} \backslash \mathbb{A}_F^{\times}, \mathbb{C}^{\times}) &\longrightarrow \widehat{F^{\times} \backslash \mathbb{A}_F^{\times}} \times \mathbb{R} \\ \chi &\longmapsto (\chi | \cdot |_{\mathbb{A}_F}^{-\operatorname{wt} \chi}, \operatorname{wt} \chi). \end{aligned}$$

Assume χ is a Hecke character that is trivial on $(\mathbb{A}_F^{\times})^1$. By the splitting above, χ is then determined alone by its restriction to Δ. If F is a number field, by Lemma 7.1.5 we see any continuous homomorphism $\mathbb{R}_{>0} \to \mathbb{C}^{\times}$ has the form $x \mapsto x^s$ for some $s \in \mathbb{C}$. Hence $\chi(x) = |x|_{\mathbb{A}_F}^{s}$. If F is a global function field, by the same trick the same holds as well. Note that $\operatorname{Re} s = \operatorname{wt} \chi$.

10.2 Poisson summation formula

The key tool in the global theory is the Poisson summation formula applied to the pair $(A, B) = (\mathbb{A}_F, F)$. For this, recall the $\psi_F : \mathbb{A}_F \to \mathbb{C}^{\times}$ is the standard additive character, and it induces a topological isomorphism $\mathbb{A}_F \cong \widehat{\mathbb{A}_F}$ such that dx^{tam} corresponds to the Plancherel measure. Also,

$$\widehat{\mathbb{A}_F/F} \cong F^{\perp} = \{x \in \mathbb{A}_F \mid \psi_F(xF) = 1\} = F.$$

For $f \in L^1(\mathbb{A}_F)$ such that $\widehat{f}|_F \in L^1(F)$, the Poisson summation formula tells

$$\sum_{r \in F} f(x + r) = \frac{1}{\operatorname{vol}(\mathbb{A}_F/F, dx^{\text{tam}})} \sum_{r \in F} \widehat{f}(r) \psi_F(rx) \qquad (\spadesuit)$$

holds for almost all $x \in \mathbb{A}_F$. Here the factor appears thanks to Lemma 5.4.9.

Theorem 10.2.1 (Poisson summation formula)**.** Suppose $f \in C(\mathbb{A}_F) \cap L^1(\mathbb{A}_F)$,

(i) $\sum_{r \in F} |f(r + x)|$ converges compactly in $x \in \mathbb{A}_F$, and

(ii) $\sum_{r \in F} |\widehat{f}(r)| < \infty$.

Then both sides of $(\spadesuit)$ define continuous functions and equality holds everywhere. In particular

$$\sum_{r \in F} f(r) = \frac{1}{\operatorname{vol}(\mathbb{A}_F/F, dx^{\text{tam}})} \sum_{r \in F} \widehat{f}(r).$$

Corollary 10.2.1.1. For a global field F, $\operatorname{vol}(\mathbb{A}_F/F, dx^{\text{tam}}) = 1$.

Proof. If $f \in C(\mathbb{A}_F) \cap L^1(\mathbb{A}_F)$ is such that $\sum_{r \in F} |f(r + x)|$ and $\sum_{r \in F} |\widehat{f}(r + x)|$ converge compactly in $x \in \mathbb{A}_F$, then since dx^{tam} is self-dual, by applying Poisson summation formula twice we get

$$\sum_{r \in F} f(r) = \frac{1}{\operatorname{vol}(\mathbb{A}_F/F, dx^{\text{tam}})} \sum_{r \in F} \widehat{f}(r) = \frac{1}{\operatorname{vol}(\mathbb{A}_F/F, dx^{\text{tam}})^2} \sum_{r \in F} f(r).$$

If $\sum_{r\in F} f(r) \neq 0$, then $\mathrm{vol}(\mathbb{A}_F/F, dx^{\mathrm{tam}})^2 = 1$, which finishes the proof. It remains to find such f, and this is done in the next subsubsection.

$\square$

Corollary 10.2.1.2. For a global field F, $\mathrm{vol}(\mathbb{A}_F/F, dx^{\mathrm{std}}) = 2^{-r_2}(\#\mathcal{O}_F/\mathfrak{D}_F)^{\frac{1}{2}} \prod_{v\in M_{F,\infty}\cap M_{F,\mathrm{na}}} (N\mathfrak{d}_v)^{-\frac{1}{2}}$.

Proof. This follows from Example 2.4.15 and Lemma 9.2.14. $\square$

Corollary 10.2.1.3 (Riemann-Roch)**.** Let $f \in C(\mathbb{A}_F) \cap L^1(\mathbb{A}_F)$ and $a \in \mathbb{A}_F^\times$. Suppose that

(i) $\sum_{r\in F} |f(a(x+r))|$ converges compactly in $x \in \mathbb{A}_F$, and

(ii) $\sum_{r\in F} |\widehat{f}\left(\frac{r}{a}\right)| < \infty$.

Then we have

$$\sum_{r\in F} f(ar) = \frac{1}{|a|_{\mathbb{A}_F}} \sum_{r\in F} \widehat{f}\left(\frac{r}{a}\right).$$

Proof. Define $g \in C(\mathbb{A}_F) \cap L^1(\mathbb{A}_F)$ by $g(x) := f(ax)$. Then

$$\widehat{g}(x) = \int_{\mathbb{A}_F} f(ay)\psi(-xy)dy = \frac{1}{|a|_{\mathbb{A}_F}} \int_{\mathbb{A}_F} f(y)\psi\left(-y\cdot\frac{x}{a}\right)dy = \frac{1}{|a|_{\mathbb{A}_F}} \widehat{f}\left(\frac{r}{a}\right).$$

The proof is complete upon applying Poisson summation formula to the function g. $\square$

10.2.1 Adelic Schwartz functions

Recall for each local field k we define a special family of functions $S(k)$ called Schwartz functions that is well-behaved under Fourier transform (c.f. Lemma 7.1.14). For archimedean local fields it is the classical Schwartz functions. For non-archimedean local fields it collects all locally constant functions with compact supports. In particular it contains $\mathbf{1}_{\mathfrak{o}}$ where $\mathfrak{o}$ stands for the ring of integers of k.

For a global field F, we define the space $S(\mathbb{A}_F)$ of **adelic Schwartz functions** as the restricted tensor products of the local Schwartz functions:

$$S(\mathbb{A}_F) := \bigotimes_{v\in M_F}^{\{\mathbf{1}_{\mathfrak{o}_v}\}_{v\in M_{F,\mathrm{na}}}} S(F_v).$$

Generally we define for each $n \geqslant 1$ that the space $S(\mathbb{A}_F^n)$ of Schwartz functions on the adelic affine n-space as

$$S(\mathbb{A}_F^n) = \bigotimes_{v\in M_F}^{\{\mathbf{1}_{\mathfrak{o}_v^n}\}_{v\in M_{F,\mathrm{na}}}} S(F_v^n).$$

Note $S(\mathbb{A}_F^n) \cong S(\mathbb{A}_F)^{\oplus n}$ canonically. Also, for a finite set $S \subseteq M_F$, we put $S(F_S) := \bigotimes_{v\in S} S(F_v)$. We define $S(\mathbb{A}_{F,\mathrm{fin}}^n)$ in the same way; then

$$S(\mathbb{A}_F^n) = S(F_\infty^n) \otimes S(\mathbb{A}_{F,\mathrm{fin}}^n)$$

By definition, functions of the form $\otimes_{v\in M_{F,\mathrm{a}}} f_v \otimes \otimes_{v\in S}\mathbf{1}_{U_v}\otimes\otimes_{v\in M_{F,\mathrm{na}}\setminus S}\mathbf{1}_{\mathfrak{o}_v}$ with $f_v \in S(F_v)$, $S\subseteq M_{F,\mathrm{na}}$ finite and $U_v \subseteq k_v$ compact open spans $S(\mathbb{A}_F)$; functions of this form are called **factorizable** or **pure tensors**. For such function, by our construction of dx^{tam} we have

$$\otimes_{v|\infty} f_v \otimes \otimes_{v\in S}\widehat{\mathbf{1}_{U_v}\otimes}\otimes_{v\in M_{F,\mathrm{fin}}\setminus S}\mathbf{1}_{\mathfrak{o}_v} = \otimes_{v|\infty}\widehat{f}_v\otimes\otimes_{v\in S}\widehat{\mathbf{1}_{U_v}}\otimes\otimes_{v\in M_{F,\mathrm{fin}}\setminus S}\widehat{\mathbf{1}_{\mathfrak{o}_v}}$$

Since $\widehat{\mathbf{1}_{\mathfrak{o}_v}} = \mathrm{vol}(\mathfrak{o}_v, dx_v^{\mathrm{tam}})\mathbf{1}_{\mathfrak{d}_v^{-1}} = \mathbf{1}_{\mathfrak{o}_v}$ for almost all v by Lemma 9.2.13.1, it follows by Lemma 7.1.14 that the Fourier transform leaves $S(\mathbb{A}_F)$ invariant and defines a bijection on it. Notice that $S(\mathbb{A}_F)\subseteq L^1(\mathbb{A}_F)$ by Theorem 9.1.6.

Lemma 10.2.2. Each function in $S(\mathbb{A}_F)$ satisfies the conditions in Poisson summation formula.

Proof. Since $S(\mathbb{A}_F)$ is invariant under Fourier transform, it suffices to show (i) is satisfied. Let $f\in S(\mathbb{A}_F)$. By linearity we assume $f = f'\otimes f''$ with $f'\in S(F_\infty)$ and $f''\in S(\mathbb{A}_{F,\mathrm{fin}})$. For each $v\in M_{F,\mathrm{na}}$, let $n_v\in\mathbb{Z}$ be the maximal integer such that the projection of $\operatorname{supp} f$ to F_v is contained in $\mathfrak{p}_v^{n_v}$; by construction $n_v = 0$ for almost all $v\in M_{F,\mathrm{na}}$. If F is a global function field, then $M_F = M_{F,\mathrm{na}}$ and $\operatorname{supp} f\subseteq \prod_{v\in M_F}\mathfrak{p}_v^{n_v}$. Then the sum in (i) is over $F\cap\left(\prod_{v\in M_F}\mathfrak{p}_v^{n_v} - x\right)$ which is finite, being an intersection of discrete and compact. Hence it is a finite sum which obviously converges compactly. If F is a number field, then the sum in (i) is over $F\cap\left(\prod_{v\in M_{F,\mathrm{na}}}\mathfrak{p}_v^{n_v} - x\right) =: \Lambda_x$. If K is a compact set containing x, then we can find $\mathbb{Z}\ni m_v\leqslant n_v$ with $m_v = n_v$ $\forall' v$ such that $\Lambda_x\subseteq\{r\in F \mid \mathrm{ord}_{F_v} x\geqslant m_v \text{ for all } v\in M_{F,\mathrm{fin}}\} =: \Lambda$. Now Λ is a fractional ideal of $\mathcal{O}_F$, so its image in F_∞ is a lattice. Hence

$$\sum_{r\in F}|f(r+x)| \leqslant \sup_{y\in\mathbb{A}_{F,\mathrm{fin}}}|f''(y)|\cdot\sum_{r\in\Lambda}|f'(r+x_\infty)|$$

where x_∞ is the component of x on F_∞. We are then reduced to the classical case:

Lemma 10.2.3. Let $\Lambda\in\mathbb{R}^n$ be a lattice and $f\in S(\mathbb{R}^n)$. The sum $\sum_{r\in\Lambda} f(r+x)$ converges compactly in $x\in\mathbb{R}^n$.

Proof. Let $r = r(x_1,\dots,x_n) = x_1^2+\cdots+x_n^2$, and let $C := \sup_{x\in\mathbb{R}^n}|r(x)^{n+2}f(x)| < \infty$; then $|f(x)|\leqslant \dfrac{C}{r(x)^{n+2}}$ for all $x\in\mathbb{R}^n$. It is known that $\int_{r(x)\geqslant 1}\dfrac{1}{r(x)^{n+2}}dx < \infty$. Hence for a fixed compact set K, for each $\varepsilon>0$ we can find a compact set K_ε such that $\sum_{\alpha\in\Lambda\setminus K_\varepsilon}\dfrac{1}{r(\alpha+x)^{n+2}} < \dfrac{\varepsilon}{C}$ for all $x\in K$. Hence for $x\in K$

$$\left|\sum_{\alpha\in\Lambda\setminus K_\varepsilon} f(\alpha+x)\right| \leqslant \sum_{\alpha\in\Lambda\setminus K_\varepsilon}\frac{C}{r(\alpha+x)^{n+2}} < \varepsilon.$$

□

10.2.2 Riemann-Roch for curves over finite fields

We explain the name of the Riemann-Roch. Let F be a global function field with constant field $\mathbb{F}_q$. Denote by $\mathrm{Div}(F)$ the abelian group free on M_F; an element in $\mathrm{Div}(F)$ is called a **divisor**. For a divisor $D = \sum_{n\in M_F} n_v v$, the **degree** (over $\mathbb{F}_q$) is the sum

$$\deg D = \sum_{v\in M_F} n_v\deg v := \sum_{v\in M_F} n_v[\kappa(v):\mathbb{F}_q].$$

Clearly degree defines a group homomorphism $\deg : \mathrm{Div}(F) \to \mathbb{Z}$. We denote the kernel by $\mathrm{Div}^0(F)$, the group of **divisors of degree** 0. Each $f \in F^\times$ there is an associated divisor

$$\mathrm{div}(f) := \sum_{v \in M_F} \mathrm{ord}_{F_v}(f) v$$

which is well-defined by Lemma 9.2.7, and $\mathrm{div}(f) \in \mathrm{Div}^0(F)$ by Product formula. This also defines a group homomorphism $\mathrm{div} : F^\times \to \mathrm{Div}(F)$. Define the **Picard group** of F to be

$$\mathrm{Pic}(F) := \mathrm{Div}(F)/\mathrm{div}(F^\times).$$

Similarly define **Picard group of degree** 0 as the quotient $\mathrm{Pic}^0(F) := \mathrm{Div}^0(F)/\mathrm{div}(F^\times)$. This is the kernel of the induced degree map $\deg : \mathrm{Pic}(F) \to \mathbb{Z}$ on the Picard group. There exists a canonical map $\mathbb{A}_F^\times \to \mathrm{Div}(F)$ which restricts to $(\mathbb{A}_F^\times)^1 \to \mathrm{Div}^0(F)$. We omit the definition of the map and the obvious properties.

We define a partial order on $\mathrm{Div}(F)$: say $D = \sum n_v v \geqslant D' = \sum n'_v v$ if $n_v \geqslant n'_v$ for all $v \in M_F$. Then for each divisor D there is an associated $\mathbb{F}_q$-vector space

$$L(D) := \{0\} \cup \{f \in F^\times \mid \mathrm{div}(f) \geqslant -D\}.$$

This is called the **linear system** of D. We write $\ell(D) = \dim_{\mathbb{F}_q} L(D) \leqslant \infty$.

Lemma 10.2.4.

(i) For $f \in F^\times$, we have $L(D + \mathrm{div}(f)) = L(D)$.

(ii) $L(0) = \#\mu(F) = q$.

This space can be understood adelically. For a divisor $D = \sum n_v v$, let

$$\mathbb{A}_F(D) = \{x = (x_v)_v \in \mathbb{A}_F \mid \mathrm{ord}_{F_v} x_v \geqslant -n_v\} = \prod_{v \in M_F} \mathfrak{p}_v^{-n_v},$$

which is a compact subgroup of $\mathbb{A}_F$. Then $L(D) = F \cap \mathbb{A}_F(D)$; in particular, being the intersection of discrete and compact sets, $\#L(D) < \infty$ and so $\ell(D) < \infty$. We have

$$\mathrm{vol}(\mathbb{A}_F(D), dx^{\mathrm{std}}) = \prod_{v \in M_F} \mathrm{vol}(\mathfrak{p}_v^{-n_v}, dx_v^{\mathrm{std}}) = \prod_{v \in M_F} (N\mathfrak{p}_v)^{n_v} = q^{\deg D}.$$

For any nontrivial additive character $\psi : F\backslash\mathbb{A}_F \to S^1$, define

$$K_\psi = -\sum_{v \in M_F} m_v v$$

where m_v is the integer such that $\mathfrak{d}_v^{-1} = \mathfrak{p}_v^{m_v}$. We have $m_v = 0$ for almost $v \in M_F$, so K_ψ is a divisor on F. If ψ' is any other character, then $\psi'(x) = \psi(ax)$ for a unique $a \in F^\times$, and hence

$$K_{\psi'} = K_\psi + \mathrm{div}(a).$$

We call K_ψ a **canonical divisor**, and its class in $\mathrm{Pic}(F)$ the **canonical class**.

Recall ψ defines a perfect pairing

$$\begin{aligned} F_v \times F_v &\longrightarrow S^1 \\ (x, y) &\longmapsto \psi_v(xy) \end{aligned}$$

for each $v \in M_F$. Under this pairing, one has

$$\mathbb{A}_F(D)^\perp = \mathbb{A}_F(K_\psi - D)$$

and hence

$$(F + \mathbb{A}_F(D))^\perp = F^\perp \cap \mathbb{A}_F(K_\psi - D) = F \cap \mathbb{A}_F(K_\psi - D).$$

By Proposition 5.6.1, we see

$$F \cap \mathbb{A}_F(K - D) = (F + \mathbb{A}_F(D))^\perp \cong \mathbb{A}_F/\widehat{F + \mathbb{A}_F}(D)$$

so that

$$F^\perp \cap \widehat{\mathbb{A}_F(K} - D) \cong \mathbb{A}_F/F + \mathbb{A}_F(D)$$

LHS is a finite abelian group, so it has the same size as $L(K - D)$. Since $\mathbb{A}_F(D) \subseteq \mathbb{A}_F$ is open, by Example 2.4.12 we conclude that

$$\mathrm{vol}(\mathbb{A}_F/F, dx) = \#L(K-D)\,\mathrm{vol}(F + \mathbb{A}_F(D)/F, dx) = q^{\ell(K-D)}\,\mathrm{vol}(F + \mathbb{A}_F(D)/F, dx).$$

On the other hand, the projection $\mathbb{A}_F \to \mathbb{A}_F/F$ restricts to a surjection $\mathbb{A}_F(D) \to F + \mathbb{A}_F(D)/F$ with finite kernel $F \cap \mathbb{A}_F(D) = L(D)$. By Lemma 17.1.7, we see

$$\mathrm{vol}(\mathbb{A}_F(D), dx) = \#L(D)\,\mathrm{vol}(F + \mathbb{A}_F(D)/F, dx) = q^{\ell(D)}\,\mathrm{vol}(F + \mathbb{A}_F(D)/F, dx).$$

LHS is simply $q^{-\deg D}$ if we take $dx = dx^{\mathrm{std}}$, so we get

$$\mathrm{vol}(\mathbb{A}_F/F, dx^{\mathrm{std}}) = q^{\ell(K-D)-\ell(D)-\deg D}.$$

In particular, $\mathrm{vol}(\mathbb{A}_F/F, dx^{\mathrm{std}}) \in q^{\mathbb{Z}}$. Define the **genus of** F to be the integer such that

$$\mathrm{vol}(\mathbb{A}_F/F, dx^{\mathrm{std}}) =: q^{g-1}$$

By taking $D = 0$, we see

$$g - 1 = \ell(K) - \ell(0) - \deg 0 = \ell(K) - 1$$

so that $g = \ell(K) \in \mathbb{Z}_{\geqslant 0}$. By taking $D = K$ this time, we see $\deg K = 2g - 2$. We've proved

Theorem 10.2.5 (Riemann-Roch)**.** Let F be a global function field with constant field $\mathbb{F}_q$. There exist $g \in \mathbb{Z}_{\geqslant 0}$ and a divisor K of degree $2g - 2$ such that

$$\ell(D) - \ell(K - D) = \deg D - g + 1$$

for any divisor D.

Alternatively, we can argue using Poisson summation formula, which will explain the name for Corollary 10.2.1.3. Start with

$$q^{\ell(D)} = \#L(D) = \sum_{x\in F} \mathbf{1}_{\prod_{v\in M_F} \mathfrak{o}_v}(xD)$$

where we view $D = \sum n_v v$ as any idele element whose order at v is n_v. By Corolloary 10.2.1.3 and Lemma 7.1.4

$$\sum_{x\in F} \mathbf{1}_{\prod_{v\in M_F} \mathfrak{o}_v}(xD) = \frac{\prod_{v\in M_F} (N\mathfrak{d}_v)^{-\frac{1}{2}}}{|D|_{\mathbb{A}_F}} \sum_{x\in F} \mathbf{1}_{\prod_{v\in M_F} \mathfrak{d}_v^{-1}}\left(\frac{x}{D}\right)$$

Since $\mathfrak{d}_v^{-1} = \mathfrak{p}_v^{m_v}$, we see

$$\sum_{x \in F} \mathbf{1}_{\prod_{v \in M_F} \mathfrak{d}_v^{-1}} \left(\frac{x}{D}\right) = 1 + \#\{x \in F^\times \mid \operatorname{ord}_v x - n_v \geqslant m_v \text{ for all } v \in M_F\} = \#L(K-D)$$

Also $|D|_{\mathbb{A}_F} = \prod_{v \in M_F} (N\mathfrak{p}_v)^{-n_v} = q^{-\deg D}$ and

$$\prod_{v \in M_F} (N\mathfrak{d}_v)^{-\frac{1}{2}} = \prod_{v \in M_F} (N\mathfrak{p}_v)^{\frac{m_v}{2}} = \prod_{v \in M_F} q^{\frac{m_v \deg v}{2}} = q^{-\frac{\deg K}{2}}$$

In sum, we obtain

$$\ell(D) - \ell(K-D) = \deg D - \frac{\deg K}{2} = \deg D - g + 1.$$

Lemma 10.2.6. For $m \geqslant 1$, the genus of $F \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}$ is the same as that of F.

Proof. Let V be an n-dimensional vector space over F, viewed as an F-scheme. It suffices to show $\operatorname{vol}(V(\mathbb{A}_F)/V(F), dx^{\text{std}}) = q^{n(g-1)}$. □

10.3 Measures on ideles

We set up the measures on the group of ideles. In the proof of Product formula we see $\operatorname{mod}_{\mathbb{A}_F}(x) = |x|_{\mathbb{A}_F}$ for $x \in \mathbb{A}_F^\times$. Hence, any Haar measure dx on $\mathbb{A}_F$ induces a Haar measure

$$d^\times x := \frac{dx}{|x|_{\mathbb{A}_F}}$$

on $\mathbb{A}_F^\times$. In particular, we have defined two Haar measures $d^\times x^{\text{std}}$ and $d^\times x^{\text{tam}}$. Since $F^\times \to \mathbb{A}_F^\times$ is discrete, so we can form the quotient measure of $d^\times x$, which we will still denote by $d^\times x$, on $\mathbb{A}_F^\times/F^\times$ by the the counting measure on $F^\times$.

We turn to the measure on $(\mathbb{A}_F^\times)^1$. Let $d^\times x$ be any Haar measure on $\mathbb{A}_F^\times$. If F is a number field, equip Δ with the measure corresponding to the standard measure $d^\times t$ on $\mathbb{R}_{>0}$. If F is a global function field, we fix some $z \in \mathbb{A}_F^\times$ as above and equip $\Delta = \Delta_z$ with the counting measure which for convenience we also write as $d^\times t$. Finally, let $d^1 x$ denote the Haar measure on $(\mathbb{A}_F^\times)^1$ satisfying the integral formula

$$\int_{\mathbb{A}_F^\times} f(x) d^\times x = \int_\Delta \int_{(\mathbb{A}_F^\times)^1} f(tx) d^1 x d^\times t.$$

valid for all $f \in L^1(\mathbb{A}_F^\times)$; the existence is by Theorem 2.4.7. Also, since $F^\times \subseteq (\mathbb{A}_F^\times)^1$ is discrete, we treat the counting measure on $F^\times$ as a Haar measure and form the quotient measure of $F^\times \backslash (\mathbb{A}_F^\times)^1$, which we denote by $d^1 x$ again.

10.3.1 Volume of fundamental domain

Lemma 10.3.1. For a number field F, we have

$$\operatorname{vol}(F^\times \backslash (\mathbb{A}_F^\times)^1, d^1 x^{\text{tam}}) = \frac{2^{r_1} (2\pi)^{r_2} \# \operatorname{Cl}(F) R_F}{\#\mu(F) (\#\mathcal{O}_F/\mathfrak{D}_F)^{\frac{1}{2}}}$$

Here R_F is the **regulator of** F, defined by

$$R_F = \frac{\operatorname{vol}(P, d\lambda)}{\operatorname{vol}(Q \cap H, d\lambda)}$$

where P, H as in §9.3.4, Q the unit cube in $\mathbb{R}^\infty$ and $d\lambda$ is any Haar measure on H.

Proof. We write d^1x for d^1x^{tam} for brevity. Let us use the notations in §9.3.4. We compute first $\mathrm{vol}(E, d^1x)$. Since $E = \bigsqcup_{i\in[h]} E_{v_0}c_i$,

$$\mathrm{vol}(E, d^1x) = h\,\mathrm{vol}(E_{v_0}, d^1x)$$

and it remains to compute $\mathrm{vol}(E_{v_0}, d^1x)$. Since

$$\mathrm{vol}((\mathbb{A}_F^\times)^1_\infty \cap \log^{-1}(P), d^1x) = \frac{1}{\#\mu(F)} \times \mathrm{vol}(E_{v_0}, d^1x)$$

it suffices to compute $\mathrm{vol}((\mathbb{A}_F^\times)^1_\infty \cap \log^{-1}(P), d^1x)$. Pick any Haar measure $d\lambda$ on H, and let Q be the unit cube in $\mathbb{R}^\infty$. By Proposition 2.4.30, we have

$$\frac{\mathrm{vol}((\mathbb{A}_F^\times)^1_\infty \cap \log^{-1}(P), d^1x)}{\mathrm{vol}(\mathbb{A}_F^\times)^1_\infty \cap \log^{-1}(Q), d^1x)} = \frac{\mathrm{vol}(P, d\lambda)}{\mathrm{vol}(Q \cap H, d\lambda)}.$$

Since $\log^{-1}(Q) = \{(x_v)_v \in (\mathbb{A}_F^\times)_\infty \mid 1 \leqslant |x_v|_v \leqslant e\}$, we can compute

$$\begin{aligned}\mathrm{vol}(\log^{-1}(Q), d^\times x) = \int_\Delta \int_{(\mathbb{A}_F^\times)^1} \mathbf{1}_{\log^{-1}(Q)}(tx) d^1x d^\times t &= \int_{(\mathbb{A}_F^\times)^1_\infty \cap \log^{-1}(Q)} \int_1^e \frac{dt}{t} d^1x \\ &= \mathrm{vol}(\mathbb{A}_F^\times)^1_\infty \cap \log^{-1}(Q), d^1x).\end{aligned}$$

To compute $\mathrm{vol}(\log^{-1}(Q), d^\times x)$, since

$$\log^{-1}(Q) = \prod_{v\in\infty}\{1 \leqslant |x|_v \leqslant e\} \times \widehat{\mathcal{O}}_F^\times$$

by Lemma 7.1.8 we have

$$\mathrm{vol}(\log^{-1}(Q), d^\times x^{\mathrm{tam}}) = 2^{r_1}(2\pi)^{r_2} \prod_{v\in M_{F,\mathrm{fin}}} (N\mathfrak{d})^{-\frac{1}{2}} = 2^{r_1}(2\pi)^{r_2}(\#\mathcal{O}_F/\mathfrak{D}_F)^{-\frac{1}{2}}.$$

To conclude the proof, we only need to show

$$\mathrm{vol}(F^\times\backslash(\mathbb{A}_F^\times)^1, d^1x^{\mathrm{tam}}) = \mathrm{vol}(E, d^1x).$$

Indeed,

$$\mathrm{vol}(E, d^1x) = \int_{F^\times\backslash(\mathbb{A}_F^\times)^1} \sum_{r\in F^\times} \mathbf{1}_E(rx) d^1x$$

For each $x \in F^\times\backslash(\mathbb{A}_F^\times)^1$ there exists a unique $r \in F^\times$ with $rx \in E$. Hence the integral equals

$$\int_{F^\times\backslash(\mathbb{A}_F^\times)^1} d^1x = \mathrm{vol}(F^\times\backslash(\mathbb{A}_F^\times)^1, d^1x^{\mathrm{tam}}).$$

□

Lemma 10.3.2. For F a global function field with constant field $\mathbb{F}_q$, we have

$$\mathrm{vol}(F^\times\backslash(\mathbb{A}_F^\times)^1, d^1x^{\mathrm{tam}}) = \frac{\#\mathrm{Pic}^0(F)}{q-1} q^{1-g}$$

Proof. Consider the surjection

$$\begin{aligned}\mathrm{div} : (\mathbb{A}_F^\times)^1 &\longrightarrow \mathrm{Div}^0(F) \\ (x_v)_v &\longmapsto \textstyle\sum_{v\in M_F} \mathrm{ord}_v\, x_v v.\end{aligned}$$

It has degree 0 as

$$-\sum_{v\in M_F}\operatorname{ord}_v x_v[\#\kappa(v):\mathbb{F}_q]=\log_q\prod_{v\in M_F}(N\mathfrak{p}_v)^{-\operatorname{ord}_v x_v}=\log_q|x|_{\mathbb{A}_F}=0.$$

This coincides with the usual divisor map div on $F^\times$, which induces a surjection $F^\times\backslash(\mathbb{A}_F^\times)^1\to\mathrm{Pic}^0(F)$ with kernel $F^\times\cap\prod_{v\in M_F}\mathfrak{o}_v^\times\backslash\prod_{v\in M_F}\mathfrak{o}_v^\times$. The denominator is $\mathbb{F}_q^\times$ by Lemma 9.2.15. In sum, we have the following short exact sequence

$$1\longrightarrow\mathbb{F}_q^\times\backslash\prod_{v\in M_F}\mathfrak{o}_v^\times\longrightarrow F^\times\backslash(\mathbb{A}_F^\times)^1\longrightarrow\mathrm{Pic}^0(F)\longrightarrow 1.$$

By Example 2.4.12, 2.4.13 and Lemma 17.1.7, we conclude

$$\mathrm{vol}(F^\times\backslash(\mathbb{A}_F^\times)^1,d^1x^{\mathrm{tam}})=\#\mathrm{Pic}^0(F)\times\mathrm{vol}\left(\prod_{v\in M_F}\mathfrak{o}_v^\times,d^1x^{\mathrm{tam}}\right)=\frac{\#\mathrm{Pic}^0(F)}{q-1}\prod_{v\in M_F}(N\mathfrak{d}_v)^{-\frac{1}{2}}$$

It remains to see from §10.2.2 that

$$\prod_{v\in M_F}(N\mathfrak{d}_v)^{-\frac{1}{2}}=q^{1-g}.$$

□

10.4 Functional equations

With Riemann-Roch in mind, we introduce the global counterpart of the local space $Z(k)$: define $Z(\mathbb{A}_F)$ to be the space of functions $f:\mathbb{A}_F\to\mathbb{C}$ satisfying

(i) $f\in\mathrm{inv}(\mathbb{A}_F):=\{g\in L^1(\mathbb{A}_F)\mid\widehat{g}\in L^1(\mathbb{A}_F)\}$,

(ii) the series $\sum_{r\in F}|f(a(x+r))|$ and $\sum_{r\in F}|\widehat{f}(a(x+r))|$ converge compactly in $x\in\mathbb{A}_F$ for all ideles $a\in\mathbb{A}_F^\times$, and

(iii) $f(x)|x|_{\mathbb{A}_F}^s$ and $\widehat{f}(x)|x|_{\mathbb{A}_F}^s$ are in $L^1(\mathbb{A}_F)$ for $\mathrm{Re}\,s>1$.

In view of Theorem 5.5.7, we see (i) and (ii) guarantee that Riemann-Roch is applicable for functions in $Z(\mathbb{A}_F)$. The purpose of (iii) is to introduce the

Definition. For $f\in Z(\mathbb{A}_F)$, a Hecke character $\chi:F^\times\backslash\mathbb{A}_F^\times\to\mathbb{C}^\times$ and $s\in\mathbb{C}$, define the **(global) zeta integral**

$$Z(f,\chi,s):=\int_{\mathbb{A}_F^\times}f(x)\chi|\cdot|_{\mathbb{A}_F}^s(x)d^\times x^{\mathrm{tam}}.$$

By (iii) this is absolutely convergent for wt $\chi+\mathrm{Re}\,s>1$.

Theorem 10.4.1. For all $f\in Z(\mathbb{A}_F)$ and Hecke-characters χ, the function $s\mapsto Z(f,\chi,s)$ admits a meromorphic continuation to $\mathbb{C}$ and satisfies the functional equation

$$Z(f,\chi,s)=Z(\widehat{f},\chi^{-1},1-s).$$

The continuation is entire unless $\chi = |\cdot|_{\mathbb{A}_F}^t$ for some $t \in \mathbb{C}$, in which case it has possible simple poles at $s = 1-t$ and $-t$. We have

$$\operatorname{res}_{s=1-t} Z(f,\chi,s) = \begin{cases} \operatorname{vol}(F^\times\backslash(\mathbb{A}_F^\times)^1, d^1x^{\mathrm{tam}})\widehat{f}(0) & \text{, if } F \text{ is a number field} \\ \operatorname{vol}(F^\times\backslash(\mathbb{A}_F^\times)^1, d^1x^{\mathrm{tam}})\dfrac{\widehat{f}(0)}{\log|z|_{\mathbb{A}_F}} & \text{, if } F \text{ is a global function field.} \end{cases}$$

and

$$\operatorname{res}_{s=-t} Z(f,\chi,s) = \begin{cases} -\operatorname{vol}(F^\times\backslash(\mathbb{A}_F^\times)^1, d^1x^{\mathrm{tam}})f(0) & \text{, if } F \text{ is a number field} \\ -\operatorname{vol}(F^\times\backslash(\mathbb{A}_F^\times)^1, d^1x^{\mathrm{tam}})\dfrac{f(0)}{\log|z|_{\mathbb{A}_F}} & \text{, if } F \text{ is a global function field.} \end{cases}$$

where $z \in \mathbb{A}_F^\times$ is such that $|z|_{\mathbb{A}_F} \geqslant 1$ generates the value group $|\mathbb{A}_F^\times|_{\mathbb{A}_F}$.

Proof. To ease the notation, we put $dx = dx^{\mathrm{tam}}$. Let $f \in Z(\mathbb{A}_F)$, a Hecke character χ and $s \in \mathbb{C}$. In this proof, we will first proceed formally, and next justify each step. Write

$$Z(f,\chi,s) = \int_{\mathbb{A}_F^\times} f(x)\chi(x)|x|_{\mathbb{A}_F}^s d^\times x = \int_\Delta \left(\int_{(\mathbb{A}_F^\times)^1} f(tx)\chi(tx)|tx|_{\mathbb{A}_F}^s d^1x\right) d^\times t =: \int_\Delta Z_t(f,\chi,s)d^\times t.$$

Since this integral is finite for $\operatorname{wt}\chi + \operatorname{Re}s > 1$, within such s, the map $s \mapsto Z_t(f,\chi,s)$ is finite for almost all $t \in \Delta$. In fact it exists for all χ and s; indeed

$$\int_{(\mathbb{A}_F^\times)^1} \left|f(tx)\chi(tx)|tx|_{\mathbb{A}_F}^s\right| d^1x = \chi(t)|t|_{\mathbb{A}_F}^s \int_{(\mathbb{A}_F^\times)^1} |f(tx)|d^1x$$

and this is bounded for all χ and s since it is already bounded for some χ and s. Now write

$$Z(f,\chi,s) = \int_{\Delta\cap\{|t|_{\mathbb{A}_F}>1\}} Z_t(f,\chi,s)d^\times t + \int_{\Delta\cap\{|t|_{\mathbb{A}_F}<1\}} Z_t(f,\chi,s)d^\times t + \int_{\Delta\cap\{|t|_{\mathbb{A}_F}=1\}} Z_t(f,\chi,s)d^\times t.$$

The last integral vanishes unless F is a global function field, and it is equal to $Z_1(f,\chi,s)$. The first integral converges for all s, since we already know it exists for $\operatorname{wt}\chi + \operatorname{Re}s > 1$, since if $\operatorname{Re}s$ is smaller, then $|x|_{\mathbb{A}_F}^{\operatorname{Re}s}$ is smaller in the domain of integration. To tackle the second integral, we exploit the integrand $s \mapsto Z_t(f,\chi,s)$. Write

$$\begin{aligned} Z_t(f,\chi,s) &= \sum_{\alpha\in F^\times} \int_{F^\times\backslash(\mathbb{A}_F^\times)^1} f(\alpha tx)\chi(\alpha tx)|\alpha tx|_{\mathbb{A}_F}^s d^1x \\ &= \int_{F^\times\backslash(\mathbb{A}_F^\times)^1} \chi(tx)|tx|_{\mathbb{A}_F}^s \sum_{\alpha\in F^\times} f(\alpha tx)d^1x \\ &= \int_{F^\times\backslash(\mathbb{A}_F^\times)^1} \chi(tx)|tx|_{\mathbb{A}_F}^s \left(\sum_{\alpha\in F} f(\alpha tx)\right) d^1x - f(0)\int_{F^\times\backslash(\mathbb{A}_F^\times)^1} \chi(tx)|tx|_{\mathbb{A}_F}^s d^1x \end{aligned}$$

By Riemann-Roch, this equals

$$\begin{aligned} &\int_{F^\times\backslash(\mathbb{A}_F^\times)^1} \chi(tx)|tx|_{\mathbb{A}_F}^s \left(\sum_{\alpha\in F} \frac{1}{|tx|_{\mathbb{A}_F}}\widehat{f}\left(\frac{\alpha}{tx}\right)\right) d^1x - f(0)\chi(t)|t|_{\mathbb{A}_F}\int_{F^\times\backslash(\mathbb{A}_F^\times)^1} \chi(x)|x|_{\mathbb{A}_F}^s d^1x \\ &= \int_{F^\times\backslash(\mathbb{A}_F^\times)^1} \sum_{\alpha\in F^\times} \chi(tx)|tx|_{\mathbb{A}_F}^{s-1}\widehat{f}\left(\frac{\alpha}{tx}\right) d^1x + \left(\widehat{f}(0)\chi(t)|t|_{\mathbb{A}_F}^{s-1} - f(0)\chi(t)|t|_{\mathbb{A}_F}^s\right)\int_{F^\times\backslash(\mathbb{A}_F^\times)^1} \chi(x)d^1x \end{aligned}$$

We unfold the first integral:

$$\begin{aligned} \int_{F^\times\backslash(\mathbb{A}_F^\times)^1} \sum_{\alpha\in F^\times} \chi(tx)|tx|_{\mathbb{A}_F}^{s-1}\widehat{f}\left(\frac{\alpha}{tx}\right) d^1x &= \int_{(\mathbb{A}_F^\times)^1} \chi(tx)|tx|^{s-1}\widehat{f}\left(\frac{1}{tx}\right) d^1x \\ &\overset{x\mapsto x^{-1}}{=} \int_{(\mathbb{A}_F^\times)^1} \chi^{-1}(t^{-1}x)|t^{-1}x|^{1-s}\widehat{f}\left(\frac{x}{t}\right) d^1x = Z_{\frac{1}{t}}(\widehat{f},\chi^{-1},1-s) \end{aligned}$$

For the error term, since $F^\times\backslash(\mathbb{A}_F^\times)^1$ is a compact group and χ is a quasi-character,

$$\int_{F^\times\backslash(\mathbb{A}_F^\times)^1}\chi(x)d^1x=\begin{cases}\mathrm{vol}(F^\times\backslash(\mathbb{A}_F^\times)^1,d^1x) & \text{, if } \chi|_{(\mathbb{A}_F^\times)^1}\equiv 1\\ 0 & \text{, otherwise.}\end{cases}$$

We denote this value by δ_χ. Hence

$$Z_1(f,\chi,s)=\frac{1}{2}(Z_1(f,\chi,s)+Z_1(\widehat{f},\chi^{-1},1-s))+\frac{\delta_\chi}{2}(\widehat{f}(0)-f(0)).$$

and

$$\begin{aligned}&\int_{\Delta\cap\{|t|_{\mathbb{A}_F}<1\}}Z_t(f,\chi,s)d^\times t\\&=\int_{\Delta\cap\{|t|_{\mathbb{A}_F}<1\}}Z_{\frac{1}{t}}(\widehat{f},\chi^{-1},1-s)d^\times t+\delta_\chi\int_{\Delta\cap\{|t|_{\mathbb{A}_F}<1\}}\left(\widehat{f}(0)\chi(t)|t|_{\mathbb{A}_F}^{s-1}-f(0)\chi(t)|t|_{\mathbb{A}_F}^{s}\right)d^\times t\\&\overset{t\mapsto t^{-1}}{=}\int_{\Delta\cap\{|t|_{\mathbb{A}_F}>1\}}Z_t(\widehat{f},\chi^{-1},1-s)d^\times t+\delta_\chi\int_{\Delta\cap\{|t|_{\mathbb{A}_F}<1\}}\left(\widehat{f}(0)\chi(t)|t|_{\mathbb{A}_F}^{s-1}-f(0)\chi(t)|t|_{\mathbb{A}_F}^{s}\right)d^\times t\end{aligned}$$

Suppose $\delta_\chi\neq 0$. Then $\chi=|\cdot|_{\mathbb{A}_F}^{s_\chi}$ for some $s_\chi\in\mathbb{C}$. If F is a number field, the last integral is then

$$\int_0^1\left(\widehat{f}(0)t^{s-1+s_\chi}-f(0)t^{s+s_\chi}\right)d^\times t=\frac{\widehat{f}(0)}{s+s_\chi-1}-\frac{f(0)}{s+s_\chi};$$

note everything makes sense if $\mathrm{Re}(s+s_\chi)=\mathrm{wt}\ \chi+\mathrm{Re}\, s>1$. If F is a global function field, this is

$$\widehat{f}(0)\sum_{n<0}|z^n|_{\mathbb{A}_F}^{s-1+s_\chi}-f(0)\sum_{n<0}|z^n|_{\mathbb{A}_F}^{s+s_\chi}=\frac{\widehat{f}(0)}{|z|_{\mathbb{A}_F}^{s-1+s_\chi}-1}-\frac{f(0)}{|z|_{\mathbb{A}_F}^{s+s_\chi}-1}.$$

Putting thing together, we get

$$\begin{aligned}Z(f,\chi,s)=&\int_{|x|_{\mathbb{A}_F}>1}f(x)\chi|\cdot|_{\mathbb{A}_F}^{s}(x)d^\times x+\int_{|x|_{\mathbb{A}_F}>1}\widehat{f}(x)\chi^{-1}|\cdot|_{\mathbb{A}_F}^{1-s}(x)d^\times x\\&+\begin{cases}\delta_\chi\left(\dfrac{\widehat{f}(0)}{s+s_\chi-1}-\dfrac{f(0)}{s+s_\chi}\right)\\ \dfrac{Z_1(f,\chi,s)+Z_1(\widehat{f},\chi^{-1},1-s)}{2}+\delta_\chi\left(\dfrac{\widehat{f}(0)-f(0)}{2}+\dfrac{\widehat{f}(0)}{|z|_{\mathbb{A}_F}^{s-1+s_\chi}-1}-\dfrac{f(0)}{|z|_{\mathbb{A}_F}^{s+s_\chi}-1}\right)\end{cases}\end{aligned}\tag{$\spadesuit$}$$

The whole expression is invariant under the change of variable $(f,\chi,s)\mapsto(\widehat{f},\chi^{-1},1-s)$, so

$$Z(f,\chi,s)=Z(\widehat{f},\chi^{-1},1-s)$$

for $\mathrm{wt}\ \chi+\mathrm{Re}\, s>1$. Note the integral

$$\int_{|x|_{\mathbb{A}_F}>1}\widehat{f}(x)\chi^{-1}|\cdot|_{\mathbb{A}_F}^{1-s}(x)d^\times x$$

is absolutely convergent for $\mathrm{wt}\ \chi^{-1}+\mathrm{Re}(1-s)>1$, or $\mathrm{wt}\ \chi+\mathrm{Re}\, s<0$. Since $|x|_{\mathbb{A}_F}^{\mathrm{Re}(1-s)}$ is smaller if $\mathrm{Re}\, s$ is larger, it follows that this is absolutely convergent for all χ and s. Hence ($\spadesuit$) provides a meromorphic continuation of $s\mapsto Z(f,\chi,s)$ to the whole complex plane. From ($\spadesuit$) we can also read off the possible poles and residues. This finishes the proof. $\square$

10.4.1 Hecke L-functions

Definition. Let F be a global field, and $\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ a Hecke character. The **Hecke L-function associated to** χ is the infinite product

$$L(s,\chi) := \prod_{v\in M_F} L(s,\chi_v),$$

where $\chi_v : F_v^\times \to \mathbb{C}^\times$ is the restriction of χ to $F_v^\times \subseteq \mathbb{A}_F^\times$. Also, for a set of places S, write

$$L_S(s,\chi) = \prod_{v\in S} L(s,\chi_v), \qquad L^S(s,\chi) = \prod_{v\notin S} L(s,\chi_v).$$

Fix a Hecke character χ of F. Let S be a finite set of places containing $M_{F,\mathrm{a}}$ such that χ_v is unramified at $v \notin S$; in this case we also say that χ **is unramified outside** S. Then by definition (c.f. §7.2.2)

$$L(s,\chi) = L_S(s,\chi)\cdot L^S(s,\chi) = L_S(s,\chi)\cdot \prod_{v\notin S} \frac{1}{1-\chi_v(\varpi_v)(N\mathfrak{p}_v)^{-s}}$$

where $N\mathfrak{p}_v = \#\kappa(v)$ and ϖ_v is a uniformizer of the non-archimedean local field $(F_v, |\cdot|_v)$. Recall that $|\chi| = |\cdot|_{\mathbb{A}_F}^{\mathrm{wt}\,\chi}$, so that $|\chi_v(\varpi_v)(N\mathfrak{p}_v)^{-s}| = (N\mathfrak{p}_v)^{-\operatorname{Re} s - \mathrm{wt}\,\chi}$.

Lemma 10.4.2. For any global field F, the infinite product

$$\prod_{v\in M_{F,\mathrm{fin}}} \frac{1}{1-(N\mathfrak{p}_v)^{-s}}$$

converges compactly and absolutely for $\sigma = \operatorname{Re}(s) > 1$. Moreover, it has limit 1 when $\sigma \to \infty$, uniformly in $\operatorname{Im}(s)$.

Proof. Let k be either $\mathbb{Q}$ or $\mathbb{F}_p(t)$ and suppose F is a finite separable extension of k. Then

$$\prod_{v\in M_{F,\mathrm{fin}}} \frac{1}{1-(N\mathfrak{p}_v)^{-s}} = \prod_{v\in M_{k,\mathrm{fin}}} \sum_{w|v} \frac{1}{1-(N\mathfrak{p}_v)^{-sf(w|v)}}$$

By Theorem 8.4.23 (and primitive element theorem), for $\sigma > 0$, we have

$$\left|\sum_{w|v} \frac{1}{1-(N\mathfrak{p}_v)^{-sf(w|v)}}\right| \leqslant \left(\frac{1}{1-(N\mathfrak{p}_v)^{-\sigma}}\right)^{[F:k]}$$

so it suffices to show prove the lemma for $F = k$.

Assume first $k = \mathbb{F}_p(t)$. By Corollary 8.2.5.1 and the unique factorization, we have

$$\prod_{v\in M_{k,\mathrm{fin}}} \frac{1}{1-(N\mathfrak{p}_v)^{-s}} = \prod_{v\in M_{k,\mathrm{fin}}} \sum_{n\geqslant 0} (N\mathfrak{p}_v)^{-ns} = \sum_f \frac{1}{p^{s\deg f}}$$

where the sum is over all monic nonconstant polynomials f in $k[t]$. Keeping in mind that there are p^d monic polynomial of degree d. Then the sum equals

$$\sum_{d=0}^{\infty} \frac{1}{p^{sd}} = \sum_{d=0}^{\infty} \frac{1}{p^{d(s-1)}}$$

Being a geometric series, this converges when $\sigma > 1$, and

$$\lim_{\sigma\to\infty} \sum_{d=0}^{\infty} \frac{1}{p^{d(s-1)}} = \lim_{\sigma\to\infty} \frac{1}{1-p^{1-s}} = 1.$$

Next assume $k = \mathbb{Q}$. Again using the unique factorization, the product is simply

$$\sum_p \frac{1}{1-p^{-s}} = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

If $s = \sigma > 1$ is real, we have

$$1 \leqslant \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \leqslant 1 + \int_1^\infty \frac{1}{x^\sigma} dx = 1 + \frac{1}{\sigma - 1}.$$

This proves the abosolute and compact convergence, and the limit goes to 1 as $\sigma \to \infty$ uniformly in $\mathrm{Im}(s)$.

$\square$

It follows that the infinite product $L^S(s,\chi)$, and hence $L(s,\chi)$, converges absolutely for $\operatorname{Re} s + \operatorname{wt} \chi > 1$.

Lemma 10.4.3. $S(\mathbb{A}_F) \subseteq Z(\mathbb{A}_F)$.

Proof. By Lemma 10.2.2 each function in $S(\mathbb{A}_F)$ satisfies (ii). Since Fourier transform defines a bijection on $S(\mathbb{A}_F)$, (i) is fulfilled, and it remains to show $f(x)|x|_{\mathbb{A}_F}^s \in L^1(\mathbb{A}_F)$ for all $\operatorname{Re} s > 1$. For this we can assume $f = \otimes_v f_v$ is factorizable; then $f(x)|x|_{\mathbb{A}_F}^s = \prod_{v \in M_F} f_v(x)|x|_v^s$. The collection $(x \mapsto f_v(x)|x|_v^s)_{v \in M_F}$ satisfies the assumption in Theorem 9.1.7, and the preceding discussion shows that the assumption in Theorem 9.1.7.(iii) holds; hence $f(x)|x|_{\mathbb{A}_F}^s \in L^1(\mathbb{A}_F)$. $\square$

Enlarge S such that $\mathfrak{d}_v = \mathfrak{o}_v$ for all $v \notin S$; in this case we say ψ_F **is unramified outside** S**.** By Lemma 7.2.11,

$$Z(\mathbf{1}_{\mathfrak{o}_v}, \chi_v, s) = L(s, \chi_v), \qquad Z(\widehat{\mathbf{1}_{\mathfrak{o}}}, \chi_v^{-1}, 1-s) = L(1-s, \chi_v^{-1})$$

for $v \notin S$. Let $f = \otimes_{v \in S} f_v \otimes \otimes_{v \notin S} \mathbf{1}_{\mathfrak{o}_v} \in S(\mathbb{A}_F)$ with $f_v \in S(F_v)$ $(v \in S)$. Then for $\operatorname{Re} s + \operatorname{wt} \chi > 1$

$$Z(f, \chi, s) = \prod_{v \in M_F} Z(f_v, \chi_v, s) = \prod_{v \in S} \frac{Z(f_v, \chi_v, s)}{L(s, \chi_v)} \times L(s, \chi)$$

holds by Theorem 9.1.7.(iii). Now choose $(f_v)_{v \in S}$ such that $Z(f_v, \chi_v, s) \neq 0$; this exists by the computation in local theory. By Theorem 10.4.1 and Theorem 7.1.15 we see the above identity defines a meromorphic continuation of $L(s,\chi)$. Similarly we have

$$Z(\widehat{f}, \chi^{-1}, 1-s) = \prod_{v \in S} \frac{Z(\widehat{f}_v, \chi_v^{-1}, 1-s)}{L(1-s, \chi_v^{-1})} \times L(1-s, \chi^{-1}).$$

Note that $Z(\widehat{f}_v, \chi_v^{-1}, 1-s) \neq 0$ by Theorem 7.1.15. It now follows from §7.2.2 and Theorem 10.4.1 that

$$L(1-s, \chi^{-1}) = \prod_{v \in S} \epsilon(s, \chi_v, \psi_{F_v}) \times L(s, \chi).$$

Since $\mathfrak{d}_v = \mathfrak{o}_v$ for all $v \notin S$, the explicit formula for ϵ-factor in §7.2.2 gives $\epsilon(s, \chi, \psi_{F_v}) = 1$ if $v \notin S$. Hence if we define the **global ϵ-factor** as

$$\epsilon(s, \chi, \psi_F) := \prod_{v \in M_F} \epsilon(s, \chi, \psi_{F_v}),$$

we obtain the first part of the

Theorem 10.4.4. For every Hecke character χ on a global field F, the Hecke L-function $L(s,\chi)$ admits a meromorphic continuation to $\mathbb{C}$ and satisfies the functional equation

$$L(1-s,\chi^{-1}) = \epsilon(s,\chi,\psi_F)L(s,\chi).$$

The function $L(s,\chi)$ is entire unless $\chi = |\cdot|_{\mathbb{A}_F}^t$ for some $t \in \mathbb{C}$, in which case it has simple poles at $s = 1-t$ and $-t$ with residues

$$\operatorname{res}_{s=1-t} L(s,\chi) = \begin{cases} \operatorname{vol}(F^\times\backslash(\mathbb{A}_F^\times)^1, d^1x^{\mathrm{tam}}) & \text{, if } F \text{ is a number field} \\ \operatorname{vol}(F^\times\backslash(\mathbb{A}_F^\times)^1, d^1x^{\mathrm{tam}})\dfrac{1}{\log|z|_{\mathbb{A}_F}} & \text{, if } F \text{ is a global function field.} \end{cases}$$

and

$$\operatorname{res}_{s=-t} L(s,\chi) = \begin{cases} -\operatorname{vol}(F^\times\backslash(\mathbb{A}_F^\times)^1, d^1x^{\mathrm{tam}})(\#\mathcal{O}_F/\mathfrak{D}_F)^{-\frac{1}{2}} & \text{, if } F \text{ is a number field} \\ -\operatorname{vol}(F^\times\backslash(\mathbb{A}_F^\times)^1, d^1x^{\mathrm{tam}})\dfrac{\prod_{v\in M_{F,\mathrm{na}}}(N\mathfrak{d}_v)^{\frac{1}{2}}}{\log|z|_{\mathbb{A}_F}} & \text{, if } F \text{ is a global function field.} \end{cases}$$

where $z \in \mathbb{A}_F^\times$ is such that $|z|_{\mathbb{A}_F} \geqslant 1$ generates the value group $|\mathbb{A}_F^\times|_{\mathbb{A}_F}$.

Proof. For the last assertion, take

$$f = \bigotimes_{v\in M_{F,\mathrm{a}}} f_{\chi_v} \otimes \bigotimes_{v} \widehat{f_{c(\chi_v)}} \in S(\mathbb{A}_F).$$

where $c(\chi_v)$ is the conductor of χ_v and $f_{c(\chi_v)}$ is defined in §7.1.3, and f_{χ_v} is chosen so that $Z(f_{\chi_v},\chi_v,s) = L(s,\chi_v)$. Collecting the results from §7.1.3 and Lemma 7.2.11, we see

$$\begin{aligned} Z(f,\chi,s) &= \prod_{v\in M_{F,\mathrm{a}}} L(s,\chi_v) \times \prod_{v:\ \chi_v \text{ ramified}} \frac{(N\mathfrak{p}_v)^{c(\chi_v)}}{\#\left(\mathfrak{o}_v^\times/1+\mathfrak{p}_v^{c(\chi_v)}\right)} \times \prod_{v:\ \chi_v \text{ unramified}} L(s,\chi_v) \\ &= L(s,\chi) \times \prod_{v:\ \chi_v \text{ ramified}} \frac{1}{1-(N\mathfrak{p}_v)^{-1}} \end{aligned}$$

and

$$\begin{aligned} f(0) &= (\#\mathcal{O}_F/\mathfrak{D}_F)^{-\frac{1}{2}} \prod_{v\in M_{F,\infty}\cap M_{F,\mathrm{na}}} (N\mathfrak{d})^{\frac{1}{2}} \times \prod_{v:\ \chi_v \text{ ramified}} (N\mathfrak{p}_v)^{c(\chi_v)} \\ \widehat{f}(0) &= 1. \end{aligned}$$

In view of these identities, the last assertion follows at once from Theorem 10.4.1. ☐

The L-functions are independent of the choice of the additive character ψ_F, while the local ϵ-factors do depend (§7.2.3). However, from the functional equation above, we see in fact the global ϵ-factor is irrelevant to the character; we thus can drop the argument ψ_F and simply write

$$\epsilon(s,\chi) := \epsilon(s,\chi,\psi_F).$$

Comparing the global functional equation and the local functional equation, we get

Theorem 10.4.5. For every Hecke character χ on a global field F, we have the identity

$$1 = \prod_{v\in M_F} \gamma(s,\chi_v,\psi_v).$$

We say some words on the partial L-function $L^S(s,\chi)$, where S always stands for a finite set of places in F including all archimedean places. By definition, we have

$$L^S(s,\chi) = L(s,\chi)\prod_{v\in S} L(s,\chi_v)^{-1}$$

Each $L(s,\chi_v)^{-1}$ is entire, so the singularities of $L^S(s,\chi)$ are at worst as $L(s,\chi)$. As a result, $L^S(s,\chi)$ has poles only when $\chi = |\cdot|_{\mathbb{A}_F}^t$ for some $t\in\mathbb{C}$. In fact, $L^S(s,\chi)$ still has a simple pole at $s=1-t$. To see this, if χ_v is unramified, then

$$L(s,\chi_v)^{-1} = 1-\chi(\varpi_v)(N\mathfrak{p}_v)^{-s} = 1-(N\mathfrak{p}_v)^{-t-s}$$

which is nonzero when $s=1-t$. When v is archimedean real,

$$L(s,\chi_v) = \pi^{-\frac{s+t}{2}}\Gamma\left(\frac{s+t}{2}\right)$$

which has poles along $-t+2\mathbb{Z}_{\leqslant 0}$. When v is archimedean non-real,

$$L(s,\chi_v) = (2\pi)^{1-(s+t)}\Gamma(s+t)$$

which has poles along $-t+\mathbb{Z}_{\leqslant 0}$. None of these touches $1-t$, so this proves our claim. Moreover,

$$\begin{aligned}\mathrm{Res}_{s=1-t}L^S(s,\chi) &= \mathrm{Res}_{s=1-t}L(s,\chi)\times\prod_{\substack{v\in S\\ \chi_v\text{ unramified}}}(1-(N\mathfrak{p}_v)^{-1})\times\pi^{\frac{r_1}{2}}\Gamma\left(\tfrac{1}{2}\right)^{-r_1}\Gamma(1)^{-r_2}\\ &= \mathrm{Res}_{s=1-t}L(s,\chi)\times\prod_{\substack{v\in S\\ \chi_v\text{ unramified}}}(1-(N\mathfrak{p}_v)^{-1})\end{aligned}$$

This proves

Theorem 10.4.6. Let F be a global field and S a finite set of places of F including all infinite places. For any Hecke character χ on F, the partial L-function $L^S(s,\chi)$ admits a meromorphic continuation to $\mathbb{C}$ and satisfies the functional equation

$$L^S(1-s,\chi^{-1}) = \gamma_S(s,\chi,\psi_F)L^S(s,\chi).$$

where

$$\gamma_S(s,\chi,\psi_F) = \prod_{v\in S}\gamma(s,\chi_v,\psi_{F_v}).$$

The function $L^S(s,\chi)$ is entire unless $\chi = |\cdot|_{\mathbb{A}_F}^t$ for some $t\in\mathbb{C}$, in which it has a simple pole at $s=1-t$ with residue

$$\mathrm{res}_{s=1-t}\,L^S(s,\chi) = \mathrm{vol}(F^\times\backslash(\mathbb{A}_F^\times)^1, d^1x^{\mathrm{tam}})\times\prod_{\substack{v\in S\\ \chi_v\text{ unramified}}}(1-(N\mathfrak{p}_v)^{-1})$$

When F is a global function field, it also has a simple pole at $s=-t$.

Proof. The assertion apart from the poles and residues follows from Theorem 10.4.5 and Theorem 10.4.4. When F is a number field, that $s=-t$ is not a pole of $L^S(s,\chi)$ can be seen from the above computation: the archimedean L-factors cancel out the pole. When F is a global function field, the pole $s=-t$ remains. □

10.4.2 Dedekind zeta function

For a global field F and a finite set S of places containing all archimedean places, the **S-Dedekind zeta function** $\zeta_F^S(s)$ is defined as

$$\zeta_F^S(s) = \prod_{v \notin S} \frac{1}{1-(N\mathfrak{p}_v)^{-s}}$$

for $\mathrm{Re}(s) > 1$. When $S = M_{F,\mathrm{a}}$ we simply write $\zeta_F(s) = \zeta_F^{M_{F,\mathrm{a}}}(s)$ and call its the **Dedekind zeta function**. Directly from the definition, we see

$$L^S(s, \mathbf{1}) = \zeta_F^S(s).$$

The full L-function $L(s, \mathbf{1})$ is then called the **completed Dedekind zeta function**. We have

$$L(s, \mathbf{1}) = L_{M_{F,\mathrm{a}}}(s, \mathbf{1}) \cdot \zeta_F(s) = \Gamma_{\mathbb{R}}(s)^{r_1} \Gamma_{\mathbb{C}}(s)^{r_2} \zeta_F(s).$$

Notice when F is a global function field, $L(s, \mathbf{1}) = \zeta_F(s)$.

Theorem 10.4.7 (Class number formula)**.** For a number field F, one has

$$\mathrm{Res}_{s=1}\zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2} \# \mathrm{Cl}(F) R_F}{\#\mu(F)(\#\mathcal{O}_F/\mathfrak{D}_F)^{\frac{1}{2}}}$$

where R_F is the regulator of F defined in Lemma 10.3.1.

Proof. This follows from Lemma 10.3.1 and Theorem 10.4.6 applied to $S = M_{F,\mathrm{a}}$. □

To an algebraic scheme over a finite field $\mathbb{F}_q$ we can attach a formal power series

$$\zeta(X;t) := \exp\left(\sum_{n=1}^{\infty} \frac{\#X(\mathbb{F}_{q^n})}{n} t^n\right) \in \mathbb{Q}[\![t]\!]$$

Lemma 10.4.8. Let F be a global function field with constant field $\mathbb{F}_q$ and X the corresponding smooth projective curve over $\mathbb{F}_q$. Then

$$\zeta(X; q^{-s}) = \zeta_F(s)$$

when $\mathrm{Re}(s) > 1$.

Proof. Formally, taking log of both sides it suffices to prove

$$\sum_{n=1}^{\infty} \frac{\#X(\mathbb{F}_{q^n})}{n} q^{-ns} = \sum_{v \in M_F} \log(1-(N\mathfrak{p}_v)^{-s})^{-1} = \sum_{v \in M_F} \sum_{m \geqslant 1} \frac{1}{m}(N\mathfrak{p}_v)^{-ms}.$$

We rearrange RHS according to $[\#\kappa(v) : \mathbb{F}_q]$ so that it equals

$$\sum_{m\geqslant 1}\sum_{k=1}^{\infty} \frac{\#\{v \in M_F \mid N\mathfrak{p}_v = q^k\}}{n} q^{-kns} = \sum_{n \geqslant 1} \frac{q^{-ns}}{n} \sum_{k|n} k \times \#\{v \in M_F \mid N\mathfrak{p}_v = q^k\}.$$

We've seen the last sum equals $\#X(\mathbb{F}_{p^n})$. □

Theorem 10.4.9. Let F be a global function field with constant field $\mathbb{F}_q$ and let g be its genus. Then

$$\zeta_F(s) = \frac{P(q^{-s})}{(1-q^{-s})(1-q^{1-s})}$$

where $P \in \mathbb{Z}[t]$ has degree $2g$ satisfying

$$P(t) = q^g t^{2g} P(1/qt).$$

Moreover, $P(0) = 1$ and $P(1) = \#\mathrm{Pic}^0(F)$.

Proof. By Theorem 10.4.4, $\zeta_F(s)$ has simple poles at $s = 1$ and 0. Put $\ell = \log |z|_{\mathbb{A}_F}$. An easy computation show that

$$\frac{1}{1-\ell^{-s}}$$

has a simple pole at $s = 0$ with residue $\dfrac{1}{\log \ell}$. Hence we can write

$$\zeta_F(s) = \frac{P(\ell^{-s})}{(1-\ell^{-s})(1-\ell^{1-s})}$$

with $P(\ell^{-s})$ entire such that

$$\begin{aligned} P(\ell^{-1}) &= (1-\ell)\operatorname{vol}(F^\times\backslash(\mathbb{A}_F^\times)^1, d^1x^{\mathrm{tam}}) \\ P(\ell^0) = P(1) &= -(1-\ell)\operatorname{vol}(F^\times\backslash(\mathbb{A}_F^\times)^1, d^1x^{\mathrm{tam}}) \prod_{v\in M_{F,\mathfrak{na}}} (N\mathfrak{d})^{\frac{1}{2}} \end{aligned}$$

From §10.2.2 we see $\displaystyle\prod_{v\in M_{F,\mathfrak{na}}} (N\mathfrak{d})^{\frac{1}{2}} = q^{g-1}$, and by Lemma 10.3.2 we then see

$$P(\ell^{-1}) = \frac{\#\mathrm{Pic}^0(F)}{q-1}(1-\ell)q^{1-g}, \qquad P(1) = \frac{\#\mathrm{Pic}^0(F)}{q-1}(\ell-1).$$

Also from Lemma 10.4.2 we see $\displaystyle\lim_{s\to+\infty} P(\ell^{-s}) = 1 < \infty$, so this shows $z \mapsto P(z)$ is entire and meromorphic at ∞. From the functional equation and collecting the epsilon factors from §7.2.2, we have

$$\zeta_F(1-s) = \epsilon(s,\mathbf{1})\zeta_F(s) = q^{(1-g)(2s-1)}\zeta_F(s)$$

or

$$P(\ell^{s-1}) = q^{(1-g)(2s-1)}\frac{(1-\ell^{-s})(1-\ell^{1-s})}{(1-\ell^{s})(1-\ell^{s-1})}P(\ell^{-s}) = \ell^{1-2s}q^{(1-g)(2s-1)}P(\ell^{-s}).$$

To proceed, we must show

Lemma 10.4.10. One has $\ell = q$. Unwinding the definition, this means

$$|\mathbb{A}_F^\times|_{\mathbb{A}_F} = q^{\mathbb{Z}}$$

where $q = \#\mu(F)$.

Proof. Say q^m is the gcd of the $\#\kappa(v)$, $v \in M_F$. We must show $m = 1$. Fix an algebraic closure of $\mathbb{F}_q$. Each $\kappa(v)$ is a finite extension of $\mathbb{F}_q$. Since q^m is the gcd of $\#\kappa(v)$, it follows the unique degree m extension $\mathbb{F}_{q^m}$ of $\mathbb{F}_q$ is contained in every $\kappa(v)$. Consider the global function field $K = F \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}$. For each $v \in M_F$, we then have

$$F_v \otimes_F K = F_v \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m} = F_v \otimes_{\kappa(v)} (\kappa(v) \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}) \cong F_v \otimes_{\kappa(v)} \prod_{i \in [m]} \kappa(v)^{\oplus m} \cong \prod_{i \in [m]} F_v.$$

This means v splits completely in K (c.f. the proof of Theorem 8.4.23). By Corollary 10.4.19.2, this shows $K = F$, i.e. $m = 1$. ☐

By the lemma, we see

$$P(q^{s-1}) = q^{1-2s} q^{(1-g)(2s-1)} P(q^{-s}) = q^{g(2s-1)} P(q^{-s})$$

or

$$P(z) = q^g z^{2g} P(1/qz).$$

This in particular implies P is a polynomial of degree $2g$. It remains to see $P \in \mathbb{Z}[z]$. This is clear from the expression

$$P(z) = (1-z)(1-qz) \prod_{v \in M_F} (1 - z^{\deg v})^{-1} \in \mathbb{Z}[\![z]\!].$$

Finally, we've seen $P(0) = 1$ above and

$$P(1) = \frac{\#\mathrm{Pic}^0(F)}{q-1}(\ell - 1) = \#\mathrm{Pic}^0(F).$$

This complete the proof. ☐

10.4.3 A nonvanishing result

Theorem 10.4.11. Let χ be a nontrivial unitary Hecke character on a global field F, and S a finite set of places on F containing all archimedean places such that χ is unramified outside S. Then

$$L^S(1, \chi) \neq 0.$$

We give two proofs, each of which only provides partial proof to the theorem. Nevertheless, they altogether prove the theorem.

Proof using class field theory, assuming χ has finite order. It suffices to consider the S having the property that χ_v is ramified if and only if $v \in S$ for all nonarchimedean v. By the global class field theory, there exists a class field K to the finite index open subgroup $\ker \chi$. In particular, $F^\times \backslash \mathbb{A}_F^\times / \ker \chi \cong \mathrm{Gal}(K/F)$. We have

$$\zeta_K^S(s) = \zeta_F^S(s) \prod_{\chi' \neq \mathbf{1}} L^S(s, \chi').$$

where χ' runs over all Hecke characters trivial on $\ker \chi$ (or equivalently, all characters of $\mathrm{Gal}(K/F)$). Since $\mathrm{ord}_{s=1} \zeta_K^S = \mathrm{ord}_{s=1} \zeta_F^S = -1$, we conclude $L^S(1, \chi) \neq 0$.

To see the displayed identity, it suffices to see the local factors match. That is, for a place $v \notin S$, we must show

$$\prod_{w | v} \frac{1}{1 - (N\mathfrak{q}_w)^{-s}} = \prod_{\chi'} L(s, \chi'_v).$$

☐

Elementary proof when F is a number field. Since χ is nontrivial unitary, the expression $L^S(1,\chi)$ makes sense by Theorem 10.4.6. First assume χ^2 is not trivial. Consider the product

$$L^S(s,\mathbf{1})^3L^S(s,\chi)^4L^S(s,\chi^2)=\prod_{v\notin S}\frac{1}{(1-(N\mathfrak{p}_v)^{-s})^3(1-\chi(\varpi_v)(N\mathfrak{p}_v)^{-s})^4(1-\chi^2(\varpi_v)(N\mathfrak{p}_v)^{-s})}$$

Lemma 10.4.12. For $t,\lambda\in\mathbb{C}$, put

$$\varphi(\lambda,t)=(1-t)^3(1-\lambda t)^4(1-\lambda^2t).$$

Then $|\varphi(\lambda,t)|<1$ for $0<t<1$ and $\lambda\in S^1$.

Proof. Indeed,

$$\begin{aligned}\log|\varphi(\lambda,t)|^2=\log\left(\varphi(\lambda,t)\varphi(\overline{\lambda},t)\right)&=-\sum_{n=1}^{\infty}\frac{t^n}{n}(6+4\lambda^n+4\overline{\lambda}^n+\lambda^{2n}+\overline{\lambda}^{2n})\\&=-\sum_{n=1}^{\infty}\frac{t^n}{n}(2+\lambda^n+\overline{\lambda}^n)^2<0\end{aligned}$$

□

By the lemma, we see

$$\left|L^S(s,\mathbf{1})^3L^S(s,\chi)^4L^S(s,\chi^2)\right|>1.$$

Since we assume χ^2 is not trivial, $L^S(1,\chi^2)<\infty$ by Theorem 10.4.6. Hence

$$0\geqslant\operatorname{ord}_{s=1}\left(L^S(s,\mathbf{1})^3L^S(s,\chi)^4L^S(s,\chi^2)\right)\geqslant-3+4\operatorname{ord}_{s=1}L^S(s,\chi),$$

so that $\operatorname{ord}_{s=1}L^S(s,\chi)\leqslant 0$. This proves $L^S(1,\chi)\neq 0$ when χ^2 is not trivial. So far this part of the proof works for any global field.

Assume $\chi^2=\mathbf{1}$. Consider the product

$$\zeta_F^S(s)L^S(s,\chi).$$

For $\mathrm{Re}(s)>1$, we have

$$F(s):=\log\zeta_F^S(s)L^S(s,\chi)=\sum_{v\notin S}\sum_{n\geqslant 1}\frac{1+\chi(\varpi_v)^n}{(N\mathfrak{p})^{ns}}.$$

Since $\chi(\varpi_v)\in\{0,\pm1\}$, this is a Dirichlet series with non-negative coefficient. An important property is

Lemma 10.4.13. Let $f(z)=\sum a_ne^{-\lambda_nz}$ be a Dirichlet series whose coefficients a_n are non-negative. Suppose that f converges for $\operatorname{Re}z>\rho$ with $\rho\in\mathbb{R}$, and that the function admits a holomorphic continuation to a neighborhood of $z=\rho$. Then there exists a $\epsilon>0$ such that f converges for $\operatorname{Re}z>\rho-\epsilon$.

(In other terms, the domain of convergence of f is bounded by a *singularity* of f located on the real axis.)

Proof. Replacing z by $z-\rho$, we can assume $\rho=0$. Since f holomorphic for $\operatorname{Re}z>0$ and in a neighborhood of 0, f is holomorphic in a disc $|z-1|\leqslant 1+\epsilon$ with $\epsilon>0$. In particular, its Taylor series converges in this disc, so the p-the derivatives of f is given by the formula

$$f^{(p)}(z)=\sum_{n=0}^{\infty}a_n(-\lambda_n)^pe^{-\lambda_nz}$$

hence

$$f^{(p)}(1) = (-1)^p \sum \lambda_n^p a_n e^{-\lambda_n}$$

so that its Taylor series can be written as

$$f(z) = \sum_{p=0}^{\infty} \left(\sum_{n=0}^{\infty} \lambda_n^p a_n e^{-\lambda_n} \right) \frac{(-1)^p}{p!} (z-1)^p, \quad |z-1| \leqslant 1+\epsilon$$

Particularly for $z = -\epsilon$, we have

$$f(-\epsilon) = \sum_{p=0}^{\infty} \left(\sum_{n=0}^{\infty} \lambda_n^p a_n e^{-\lambda_n} \right) \frac{1}{p!} (1+\epsilon)^p$$

Since this is a double series with positive term, Tonelli theorem applies. Hence

$$f(-\epsilon) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n} \sum_{p=0}^{\infty} \frac{1}{p!} (1+\epsilon)^p \lambda_n^p = \sum_n a_n e^{-\lambda_n} e^{\lambda_n(1+\epsilon)} = \sum_n a_n e^{\lambda_n \epsilon}$$

This shows the Dirichlet series converges for $z = -\epsilon$, thus also for $\operatorname{Re} z > -\epsilon$. □

Suppose for contradiction that $L^S(1,\chi) = 0$. Then $\lim_{s \to 1^+} F(s)$ is finite. Indeed, since $\operatorname{ord}_{s=1} \zeta_F^S(s) = -1$, we already know $\operatorname{ord}_{s=1} \zeta_F^S(s) L^S(s,\chi) \geqslant 0$. But $F(s)$ is nonnegative on the real line right to $s = 1$, so $\lim_{s \to 1^+} F(s) \neq -\infty$.

Now extend $F(s)$ to the left. Say there exists a singularity $\sigma_0 \in \mathbb{R}$ of F on the real line. Since its coefficients are non-negative, the previous lemma shows that the series F converges compactly and absolutely for $\operatorname{Re}(s) > \sigma_0$. In particular, $\lim_{s \to \sigma_0^+} F(s) = \infty$. This is absurd, as this would imply F has a pole at $s = \sigma_0$, while ζ_F^S and L^S have no pole other than $s = 1$. This is the place we need F to be a number field.

Hence the series $F(s)$ converges everywhere. But notice $\zeta_F^S(s)$ has a zero at $s = -2$, as $L(s, \mathbf{1})$ does not have a pole there while the archimedean local factors have a pole there. This means $\lim_{s \to -2} F(s) = -\infty$, a contradiction. □

10.4.4 Density

Definition. Let $S \subseteq M_{F,\mathrm{na}}$ be a set of non-archimedean places in a global field F. The **(Dirichlet) upper density** and **lower density** of S are

$$\delta^+(S) := \limsup_{s \to 1^+} \frac{\sum_{v \in S} (N\mathfrak{p}_v)^{-s}}{\sum_{v \in M_{F,\mathrm{na}}} (N\mathfrak{p}_v)^{-s}}, \qquad \delta^-(S) := \liminf_{s \to 1^+} \frac{\sum_{v \in S} (N\mathfrak{p}_v)^{-s}}{\sum_{v \in M_{F,\mathrm{na}}} (N\mathfrak{p}_v)^{-s}}.$$

When these two are finite and agree, the common value is called the **(Dirichlet) density** of S, and is denoted by $\delta(S)$.

Lemma 10.4.14. For any global field F, the sum

$$\sum_{v \in M_{F,\mathrm{fin}}} \frac{1}{(N\mathfrak{p}_v)^s}$$

converges compactly and absolutely for $\sigma = \operatorname{Re}(s) > 1$.

Proof. Let k be either $\mathbb{Q}$ or $\mathbb{F}_p(t)$ and suppose F is a finite separable extension of k. Arrange the sum so that

$$\sum_{v \in M_{F,\mathrm{fin}}} \frac{1}{(N\mathfrak{p}_v)^s} = \sum_{v \in M_{k,\mathrm{fin}}} \sum_{w|v} \frac{1}{(N\mathfrak{p}_v)^{sf(w|v)}}$$

Applying the same argument in Lemma 10.4.2, it suffices to show the series

$$\sum_{v \in M_{k,\mathrm{fin}}} \frac{1}{(N\mathfrak{p}_v)^s}$$

converges compactly and absolutely for $\sigma > 1$.

Assume first $k = \mathbb{F}_p(t)$. By Corollary 8.2.5.1,

$$M_{k,\mathrm{fin}} = \{|\cdot|_p \mid p \in k[t] \text{ is monic irreducible}\}$$

so that

$$\sum_{v \in M_{k,\mathrm{fin}}} \frac{1}{(N\mathfrak{p}_v)^s} = \sum_{d=1}^{\infty} \frac{\#\{p \in k[t] \mid p \text{ is monic irreducible of degree } d\}}{p^{ds}}.$$

The numerator is bounded by p^d, so the whole series is bounded above by

$$\sum_{d=1}^{\infty} \frac{1}{p^{d(\sigma-1)}}.$$

This is a geometric series, so it converges when $\sigma > 1$. The compact convergence is proved in the same way.

Next assume $k = \mathbb{Q}$. Then the sum is simply

$$\sum_{v \in M_{F,\mathrm{fin}}} \frac{1}{(N\mathfrak{p}_v)^s} = \sum_{p} \frac{1}{p^s}.$$

The same argument in Lemma 10.4.2 shows the asserted convergences.

□

By taking log of $\zeta_F(s)$ within $\mathrm{Re}(s) > 1$, we obtain

$$\log \zeta_F(s) = \sum_{v \in M_{F,\mathrm{na}}} \sum_{n=1}^{\infty} \frac{1}{n(N\mathfrak{p}_v)^{ns}} = \sum_{v \in M_{F,\mathrm{na}}} \frac{1}{(N\mathfrak{p}_v)^s} + \sum_{v \in M_{F,\mathrm{na}}} \sum_{n \geqslant 2} \frac{1}{n(N\mathfrak{p}_v)^{ns}}$$

The latter sum converges compactly and absolutely for $\sigma := \mathrm{Re}(s) > \dfrac{1}{2}$; indeed,

$$\left| \sum_{v \in M_{F,\mathrm{na}}} \sum_{n \geqslant 2} \frac{1}{n(N\mathfrak{p}_v)^{ns}} \right| \leqslant \frac{1}{2} \sum_{v \in M_{F,\mathrm{na}}} \sum_{n \geqslant 2} \frac{1}{(N\mathfrak{p}_v)^{n\sigma}} = \frac{1}{2} \sum_{v \in M_{F,\mathrm{na}}} \frac{1}{(N\mathfrak{p}_v)^{2\sigma}(1-(N\mathfrak{p}_v)^{-\sigma})}$$

Since $N\mathfrak{p}_v \geqslant 2$, we have $1-(N\mathfrak{p}_v)^{-\sigma} \geqslant 1 - \frac{1}{\sqrt{2}}$ so

$$\left| \sum_{v \in M_{F,\mathrm{na}}} \sum_{n \geqslant 2} \frac{1}{n(N\mathfrak{p}_v)^{ns}} \right| \leqslant \frac{1}{2-\sqrt{2}} \sum_{v \in M_{F,\mathrm{na}}} \frac{1}{(N\mathfrak{p})^{2\sigma}} < \infty$$

by Lemma 10.4.14. This show absolute convergence, and the compact convergence is proven in the same way.

Definition. For two meromorphic functions f, g defined in a neighborhood of 1, we write

$$f \sim g$$

if $f - g$ is holomorphic at $s = 1$.

With this notation, we see

$$\sum_{v \in M_{F,\mathrm{na}}} \frac{1}{(N\mathfrak{p}_v)^s} \sim \log \zeta_F(s).$$

Since ζ_F has a simple pole at $s = 1$, we have $\log \zeta_F(s) \sim \log(s-1)^{-1}$.

Lemma 10.4.15. For $S \subseteq M_{F,\mathrm{na}}$, we have

$$\delta^+(S) := \limsup_{s \to 1^+} \frac{\sum_{v \in S} (N\mathfrak{p}_v)^{-s}}{\log \dfrac{1}{s-1}}, \qquad \delta^-(S) := \liminf_{s \to 1^+} \frac{\sum_{v \in S} (N\mathfrak{p}_v)^{-s}}{\log \dfrac{1}{s-1}}.$$

Lemma 10.4.16. Let $P \subseteq M_{F,\mathrm{na}}$ be a set with $\delta(P) = 1$. Then

$$\delta(S) = \delta(P \cap S)$$

for any subset $S \subseteq M_{F,\mathrm{na}}$ when $\delta(S)$ is defined.

Proof. Since $\delta(S) = \delta(P \cap S) + \delta(S \backslash P)$, it suffices to prove $\delta(S \backslash P)$. But this is clear: $S \backslash P \subseteq M_{F,\mathrm{na}} \backslash P$ and the latter set has density 0.

$\square$

Lemma 10.4.17. Let F/k be an extension of number field. Then the set

$$\{w \in M_{F,\mathrm{fin}} \mid f(w|v) = 1 \text{ where } v \in M_{k,\mathrm{fin}} \text{ lies below } w\}$$

has density 1.

Proof. We have

$$\begin{aligned} \sum_{w \in M_{F,\mathrm{fin}}} \frac{1}{(N\mathfrak{p}_v)^s} &= \sum_{v \in M_{k,\mathrm{fin}}} \sum_{w|v} \frac{1}{(N\mathfrak{p}_v)^{f(w|v)s}} \\ &= \sum_{v \in M_{k,\mathrm{fin}}} \sum_{\substack{w|v \\ f(w|v)=1}} \frac{1}{(N\mathfrak{p}_v)^{f(w|v)s}} + \sum_{v \in M_{k,\mathrm{fin}}} \sum_{\substack{w|v \\ f(w|v)\geqslant 2}} \frac{1}{(N\mathfrak{p}_v)^{f(w|v)s}} \end{aligned}$$

It suffices to prove the latter series defines a holomorphic function near $s = 1$. The proof goes along the same line as our previous discussion. $\square$

Theorem 10.4.18. Let F be a global field, S be a finite set of places on F and let $\mathfrak{m}$ be an S-modulus. For any $c \in \mathrm{Cl}_{\mathfrak{m}}(F)$, the set

$$S_c := \left\{ v \in M_{F,\mathrm{na}} \mid v \notin S \cup \mathfrak{m},\ \varpi_v \in F^\times c U_{\mathfrak{m}} \right\}$$

has density $\dfrac{1}{\# \mathrm{Cl}_{\mathfrak{m}}(F)}$.

Here $\varpi_v = (\ldots, 1, \ldots, \varpi_v, \ldots, 1, \ldots)$ is the idele with all components 1 except at v, and ϖ_v at v, and ϖ_v is a uniformizer of F_v. Notice the condition does not depend on a specific choice of ϖ_v, as for $v \notin S \cup \mathfrak{m}$, the v-th component of $U_{\mathfrak{m}}$ is $\mathfrak{o}_v^\times$. We say in this case the place v **is contained in the ray idele class represented by c.**

Proof. Let χ be a Hecke character trivial on $U_{\mathfrak{m}}$; then it is naturally identified as a character of $\mathrm{Cl}_{\mathfrak{m}}(F)$. Let $T = S \cup \mathfrak{m}$. Since χ is unramified outside T, we have

$$\log L^T(s,\chi) = \sum_{v \notin T} \sum_{n=1}^{\infty} \frac{\chi(\varpi_v)}{n(N\mathfrak{p}_v)^{ns}} \sim \sum_{v \notin T} \frac{\chi(\varpi_v)}{(N\mathfrak{p}_v)^s} = \sum_{c \in \mathrm{Cl}_{\mathfrak{m}}(F)} \chi(c) \sum_{v \in S_c} \frac{1}{(N\mathfrak{p}_v)^s}.$$

Fix some $c_0 \in \mathrm{Cl}_{\mathfrak{m}}(F)$. By orthogonality, we have

$$\sum_{\chi \in \widehat{\mathrm{Cl}_{\mathfrak{m}}(F)}} \chi(c_0^{-1}) \sum_{c \in \mathrm{Cl}_{\mathfrak{m}}(F)} \chi(c) \sum_{v \in S_c} \frac{1}{(N\mathfrak{p}_v)^s} = \#\,\mathrm{Cl}_{\mathfrak{m}}(F) \sum_{v \in S_{c_0}} \frac{1}{(N\mathfrak{p}_v)^s}.$$

Finally, by virtue of Theorem 10.4.11 and 10.4.6, we conclude

$$\log \frac{1}{s-1} \sim \log \zeta^T(s) \sim \sum_{\chi \in \widehat{\mathrm{Cl}_{\mathfrak{m}}(F)}} \chi(c_0^{-1}) \log L^T(s,\chi) \sim \#\,\mathrm{Cl}_{\mathfrak{m}}(F) \sum_{v \in S_{c_0}} \frac{1}{(N\mathfrak{p}_v)^s}.$$

$\square$

For convenience, we set some notations. Let K/F be a finite Galois extension of global fields.

- Denote by $\mathrm{Spl}(K/F)$ the set of places in F that split completely in K.
- Denote by $S_{\mathrm{ram}}(K/F)$ the set of places in F that ramify in K.

Theorem 10.4.19 (Chebotarev)**.** Let F be a global field and let K/F be a finite Galois extension. For $\sigma \in \mathrm{Gal}(K/F)$, let $C = C_\sigma$ denote the conjugacy class of σ in $\mathrm{Gal}(K/F)$. Then the set

$$\{v \in M_{F,\mathrm{na}} \mid v \text{ unramified in } L \text{ with } \mathrm{Frob}_v \in C\}$$

has density $\dfrac{\#C}{[K:F]}$.

Proof. As a very special case, we consider the case when $\sigma = 1$. In this case $C = \{1\}$, and the set equals $\mathrm{Spl}(K/F)$. The value of density follows from the computation:

$$\sum_{v \in \mathrm{Spl}(K/F)} \frac{1}{(N\mathfrak{p}_v)^s} \sim \frac{1}{[K:F]} \sum_{\substack{w \in M_{K,\mathrm{fin}} \\ f(w|v)=1}} \frac{1}{(N\mathfrak{p}_w)^s} \sim \frac{1}{[K:F]} \sum_{v \in M_{F,\mathrm{fin}}} \frac{1}{(N\mathfrak{p}_v)^s}$$

The first $\sim$ is due to Theorem 8.4.23 and the fact that K/F is unramified almost everywhere. The second $\sim$ follows from Lemma 10.4.17.

The proofs of general cases for number fields and global function fields are different in nature[2], so we only prove when F is a number field. Assume first that L/K is finite abelian.

$\square$

Corollary 10.4.19.1. For an extension K/F of global fields, the set $\mathrm{Spl}(K/F)$ has density $\dfrac{1}{[K:F]}$.

Corollary 10.4.19.2. For two extensions of global fields L/F and K/F, we have $K \subseteq L$ if and only if $\mathrm{Spl}(L/F) \subseteq \mathrm{Spl}(K/F)$.

Proof. The only if part is clear. For the if part, consider the compositum KL. We have

$$\mathrm{Spl}(KL/F) = \mathrm{Spl}(K/F) \cap \mathrm{Spl}(L/F) = \mathrm{Spl}(L/F).$$

By the previous corollary, this implies $[KL:F] = [L:F]$, or $K \subseteq L$. $\square$

[2]The proof in the global function field case use algebraic geometry. To be slightly precise, the proof uses Riemann hypothesis for curves over finite fields. See https://lii4.github.io/Etale_Cohomology.pdf. The reason that the argument for number fields would fail for global function fields is that there is no archimedean place, and we need a fixed place to make the idelic quotient compact. Of course we could have given under a strange condition.

10.5 Weil conjecture for curves over finite fields

The Weil conjecture for curves over finite fields is following

Theorem 10.5.1. Let F be a global function field with constant field $\mathbb{F}_q$ and let $P = P_F$ be the polynomial in Theorem 10.4.9. Write

$$P_F(t) = \prod_{i=1}^{2g}(1 - \alpha_i t).$$

Then the α_i's are algebraic integers with $|\alpha_i| = q^{\frac{1}{2}}$. (Here $|\cdot|$ is the usual euclidean norm.)

This statement is also called the Riemann hypothesis for curves, as explained by the

Corollary 10.5.1.1. Let F be a global function field. Then the Dedekind zeta function $\zeta_F(s)$ has $2g$ zeros, all lying on the line $\mathrm{Re}(s) = \dfrac{1}{2}$.

Proof. This is a direct consequence of the previous theorem, using the identity

$$\zeta_F(s) = \frac{P(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})}.$$

□

We have an equality of formal power series:

$$\exp\left(\sum_{k\geqslant 1} \frac{\#X(\mathbb{F}_{q^k})}{k} t^k\right) = \frac{\prod_{i=1}^{2g}(1 - \alpha_i t)}{(1 - t)(1 - qt)}.$$

Taking log and comparing the coefficients, we get

$$\#X(\mathbb{F}_{q^k}) = 1 + q^k - \sum_{i\in[2g]} \alpha_i^k.$$

Lemma 10.5.2. With the notation in Theorem, TFAE:

(i) $|\alpha_i| = q^{\frac{1}{2}}$ for all $i \in [2g]$.

(ii) $|\alpha_i| \leqslant q^{\frac{1}{2}}$ for all $i \in [2g]$.

(iii) The inequality

$$|\#X(\mathbb{F}_{q^n}) - (q^n + 1)| \leqslant 2gq^{\frac{n}{2}}$$

holds for any $n \in \mathbb{Z}_{\geqslant 1}$.

(iv) There exist some $m \geqslant 1$, a constant γ_m and $N = N_m \in \mathbb{Z}_{\geqslant 1}$ such that

$$\left|\#X(\mathbb{F}_{q^{2mn}}) - (q^{2mn} + 1)\right| \leqslant \gamma_m q^{nm}$$

holds for all $n \geqslant N$.

Proof. In view of the last equality, clearly (i)⇒(ii)⇒(iii)⇒(iv). That (ii)⇒(i) follows from the functional equation

$$\zeta_F(1-s) = q^{(1-g)(2s-1)}\zeta_F(s).$$

Indeed, if $|\alpha_i| \leqslant q^{\frac{1}{2}}$, then all roots of $\zeta_F(s)$ are left to the line $\mathrm{Re}(s) = \frac{1}{2}$, which in turns means the roots of $\zeta_F(1-s)$ are right to the same line. Since $q^{(1-g)(2s-1)}$ is nowhere vanishing, it follows that all roots must lie on $\mathrm{Re}(s) = \frac{1}{2}$. This proves (i). Finally, for (iv)⇒(ii), recall the identity

$$\sum_{i\in[2g]} \frac{\alpha_i^{2m}t}{1-\alpha_i^{2m}t} = \sum_{k=1}^{\infty}\left(\sum_{i\in[2g]}\alpha_i^{2mk}\right)t^k$$

By assumption we have

$$\left|\sum_{i\in[2g]}\alpha_i^{2mn}\right| \leqslant \gamma_m q^{nm}$$

for $n \geqslant N$. This implies the power series on the RHS converges for $|t| < q^{-m}$. Hence LHS cannot has pole within this range. In particular, this implies

$$|\alpha_i^{2m}q^{-m}| \leqslant 1$$

for all $i \in [2g]$, or $|\alpha_i| \leqslant q^{\frac{1}{2}}$. □

Lemma 10.5.3. Let F be a global function field with constant field $\mathbb{F}_q$ such that q is a square and $q > (g+1)^4$. Then

$$\#X(\mathbb{F}_q) < 1 + q + 2gq^{\frac{1}{2}}.$$

Proof. □

10.6 Mellin inversion

Chapter 11

More on L-functions

11.1 L functions

We follow [ANT, Iwaniec]

Definition. A Dirichlet series

$$L(f,s) := \sum_{n\geqslant 1} \lambda_f(n) n^{-s}$$

with $\lambda_f(1) = 1$ that converges absolutely for $\mathrm{Re}(s) > 1$ is called an **L-function** if it satisfies the following properties:

(i) $L(f,s)$ admits an Euler product of degree $d \geqslant 1$:

$$L(f,s) = \prod_p \prod_{i\in[d]} (1 - \alpha_i(p)p^{-s})^{-1}$$

where RHS converges absolutely for $\mathrm{Re}(s) > 1$ with $|\alpha_i(p)| < p$ for all p and $i \in [d]$. We call the $\alpha_i(p)$ the **local roots/local parameters of f at p.**

(ii) There exists a gamma factor

$$\gamma(f,s) = \pi^{\frac{-ds}{2}} \prod_{j=1}^{d} \Gamma\left(\frac{s+\kappa_j}{2}\right).$$

The κ_j's are either real or come in conjugate pairs, and $\mathrm{Re}(\kappa_j) > -1$. They are the **local roots/local parameters of f at ∞**

(iii) There exists an integer $q(f) \geqslant 1$, called the **conductor of f**, such that

$$p \nmid q(f) \implies \alpha_i(p) \neq 0 \text{ for all } i \in [d].$$

(iv) The **completed L-function**

$$\Lambda(f,s) := q(f)^{\frac{s}{2}} \gamma(f,s) L(f,s)$$

admits an analytic continuation to a meromorphic function on $\mathbb{C}$ of order 1 with at most poles at $s = 0, 1$ and satisfies the functional equation

$$\Lambda(f,s) = \varepsilon(f)\Lambda(\overline{f}, 1-s).$$

Here $\varepsilon(f) \in S^1$ is called the root number of f, and $\overline{f}$ is the "dual" of f for which $\lambda_{\overline{f}}(n) = \overline{\lambda_f(n)}$, $\gamma(\overline{f},s) = \gamma(f,s)$, $q(\overline{f}) = q(f)$.

We want to bound various analytic quantities related to $L(f,s)$ in a uniform way. The **analytic conductor of** f, which we define below, is usually a nice quantity that provides such a bound. Put

$$\mathfrak{q}_\infty(s) := \prod_{j\in[d]} (|s+\kappa_j|+3).$$

The analytic conductor is by definition

$$\mathfrak{q}(f,s) := q(f)\mathfrak{q}_f(s).$$

Also denote $\mathfrak{q}(f) = \mathfrak{q}(f,0)$

11.2 Summary of global class field theory

Let F be a global field. The global class field theory provides a continuous homomorphism with dense image

$$\mathrm{rec}_F : F^\times\backslash\mathbb{A}_F^\times \longrightarrow \mathrm{Gal}_F^{\mathrm{ab}}$$

satisfying the following properties:

(i) (Existence) Any open subgroup of finite index in $F^\times\backslash\mathbb{A}_F^\times$ arises as the kernel of the composition

$$\mathrm{rec}_{K/F} : F^\times\backslash\mathbb{A}_F^\times \longrightarrow \mathrm{Gal}_F^{\mathrm{ab}} \longrightarrow \mathrm{Gal}(K/F)$$

for some finite abelian extension K/F of F.

(ii) (Local-global compatibility) If K/F is a finite abelian extension and $v \in M_{F,\mathrm{na}}$, then the composition

$$F_v^\times \longrightarrow F^\times\backslash\mathbb{A}_F^\times \longrightarrow \mathrm{Gal}_F^{\mathrm{ab}} \longrightarrow \mathrm{Gal}(K/F)$$

annihilates $\mathfrak{o}_{F_v}^\times$ if and only if v is unramified in K, in which case any uniformizer of v maps to the Frobenius $\mathrm{Frob}_v \in \mathrm{Gal}(L/K)$.

Lemma 11.2.1. The property (ii) alone determines the map $F^\times\backslash\mathbb{A}_F^\times \to \mathrm{Gal}_F^{\mathrm{ab}}$.

Lemma 11.2.2. The association

$$\begin{aligned} \{\text{finite abelian extensions of } F\} &\longrightarrow \left\{\text{finite index open subgroups of } F^\times\backslash\mathbb{A}_F^\times\right\} \\ K &\longmapsto \ker \mathrm{rec}_{K/F} \end{aligned}$$

is an inclusion reversing bijection.

Definition. For an open subgroup H of finite index in $F^\times\backslash\mathbb{A}_F^\times$, the finite abelian extension K/F with $H = \ker \mathrm{rec}_{K/F}$ is called the **class field to** H.

11.3 Artin L-functions

Part III

Representation theory

Chapter 12

Operators on Hilbert Spaces

The space $\mathcal{B}(H)$ of all bounded linear operators on a Hilbert space H is a Banach space with the operator norm, and is even a C^*-algebra as seen in Example 3.2.1.1. We will write

$$\sigma(T) := \sigma_{\mathcal{B}(H)}(T)$$

for the spectrum of T with respect to the C^*-algebra $\mathcal{B}(H)$, and simply call it **the spectrum of the operator T.**

12.1 Functional Calculus

Let H be a Hilbert space and T a bounded **normal operator** on H, i.e., T commutes with its adjoint T^*, or, equivalently, T is normal as an element of the C^*-algebra $\mathcal{B}(H)$. We then can apply the results of Section 3.3, which for any continuous function $f : \sigma(T) \to \mathbb{C}$ gives a unique element $f(T) \in \mathcal{B}(H)$ that commutes with T and satisfies

$$\widehat{f(T)} = f \circ \widehat{T}$$

where the has means the Gelfand transform with respect to the unital C^*-algebra generated by T. Recall by Lemma 3.3.2 the spectrum of a normal operator T does not depend on the C^*-algebra. The map

$$\begin{aligned} C(\sigma(T)) &\longrightarrow \mathcal{B}(H) \\ f &\longmapsto f(T) \end{aligned}$$

is the **continuous functional calculus**. We summarize some important properties in the next proposition.

Proposition 12.1.1. Let T be a bounded normal operator on the Hilbert space H and let $\mathcal{A} = C^*(T, 1)$ be the unital C^*-algebra generated by T.

(a) The map $f \mapsto f(T)$ is a unital isometric C^*-isomorphism from $C(\sigma(T))$ to $\mathcal{A}$ which sends the identity map $\mathrm{id}_{\sigma(T)}$ to T.

(b) Let $V \subseteq H$ be a closed subspace stable under T and T^*. Then V is stable under $\mathcal{A}$, and $f(T)|_V = f(T|_V)$.

(c) Let $V = \ker f(T)$. Then V is stable under T and T^*, and the spectrum of $T|_V$ is contained in the zero locus of f.

(d) If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a power series that converges for $z = \|T\|$, then $f(T) = \sum_{n=0}^{\infty} a_n T^n$.

Proof.

(a) This follows from Theorem 3.3.3.(i).

(b) V is then stable under $\mathcal{A}$ for the linear combinations of operators of the form $(T)^k(T^*)^\ell$ are dense in $\mathcal{A}$. We then obtain a well-defined *-homomorphism

$$\Psi : \mathcal{A} \longrightarrow \mathcal{B}(V)$$

$$S \longmapsto S|_V$$

The result now follows from Corollary 3.3.31.1.(b).

(c) The first assertion is clear, and the second is obtained by Corollary 3.3.31.1

$$f(\sigma(T|_V)) = \sigma(f(T|_V)) = \sigma(f(T)|_V) = 0$$

(d) This follows from Theorem 3.3.3.(iv).

□

There are two important classes of normal operators.

(i) **Self-adjoint operators**, i.e., $T = T^*$. By Corollary 3.3.31.1.(a), $\sigma(T) \subseteq \mathbb{R}$.

(ii) **Unitary operators**, i.e., $UU^* = U^*U = 1$. Note that a normal operator $U \in \mathcal{B}(H)$ is unitary if and only if $\sigma(U) \subseteq \mathbb{T}$. This follows from Theorem 3.3.3, for $U^*U = 1$ if and only if $\overline{\mathrm{id}_{\sigma(U)}}\,\mathrm{id}_{\sigma(U)} = 1$, i.e., $\sigma(U) \subseteq \mathbb{T}$.

Recall the **Schwartz space** $\mathcal{S}(\mathbb{R})$ consists of all smooth functions $f : \mathbb{R} \to \mathbb{C}$ such that for any two integers $m, n \geqslant 0$ the function $x^n f^{(m)}(x)$ is bounded, i.e., those smooth functions f with all its derivatives **rapidly decreasing**. An element in $\mathcal{S}(\mathbb{R})$ is called an **Schwartz function**.

For $f \in \mathcal{S}(\mathbb{R})$, the **Fourier inversion formula** says that

$$f(x) = \int_{\mathbb{R}} \hat{f}(y) e^{2\pi i x y} dy$$

where $\hat{f}(y) = \int_{\mathbb{R}} f(x) e^{-2\pi i x y} dx$ is the **Fourier transform**.

Proposition 12.1.2. Let H be a Hilbert space and $T \in \mathcal{B}(H)$ self-adjoint. Then for every $f \in \mathcal{S}(\mathbb{R})$,

$$f(T) = \int_{\mathbb{R}} \hat{f}(y) e^{2\pi i y T} dy$$

where $e^{2\pi i y T}$ is a unitary operator defined by the continuous functional calculus, and the integral should be viewed as a Bochner integral with value in $\mathcal{B}(H)$.

Proof. Put $g(x) = e^{2\pi i y x}$ and let

$$\Phi : C(\sigma(T)) \longrightarrow \mathcal{B}(H)$$

$$f \longmapsto f(T)$$

be the continuous functional calculus of T. Then $e^{2\pi iyT} = g(T) = \Phi(g)$. To see $e^{2\pi iyT}$ is unitary, we compute

$$(e^{2\pi iyT})^* = \Phi(g)^* = \Phi(\overline{g}) = \Phi(g^{-1}) = \Phi(g)^{-1} = (e^{2\pi iyT})^{-1}$$

The second equality holds for Φ is *-invariant, by Theorem 3.3.3, and the third equality holds for $g(x) \in \mathbb{T}$. The function

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathcal{B}(H) \\ y &\longmapsto \hat{f}(y)e^{2\pi iyT} \end{aligned}$$

then satisfies the condition Proposition D.7.3.1.2., so the integral $\int_{\mathbb{R}} \hat{f}(y)e^{2\pi iyT}dy$ exists. To see the equality, note that the Fourier inversion formula implies that

$$f|_{\sigma(T)} = \int_{\mathbb{R}} \hat{f}(y)e^{2\pi iy\,\mathrm{id}_{\sigma(T)}}dy$$

so by the continuity of Φ (and the very definition of Lebesgue integral) we have

$$f(T) = \Phi(f|_{\sigma(T)}) = \Phi\left(\int_{\mathbb{R}} \hat{f}(y)e^{2\pi iy\,\mathrm{id}_{\sigma(T)}}dy\right) = \int_{\mathbb{R}} \hat{f}(y)\Phi(e^{2\pi iy\,\mathrm{id}_{\sigma(T)}})dy = \int_{\mathbb{R}} \hat{f}(y)e^{2\pi iyT}dy$$

□

12.1.1 Positive Operators

Definition. A self-adjoint operator $T \in \mathcal{B}(H)$ is called **positive** (or positive semi-definite), if

$$\langle Tv, v\rangle \geqslant 0$$

for all $v \in H$.[1]

Theorem 12.1.3. Let $T \in \mathcal{B}(H)$ be self-adjoint. TFAE:

(a) T is positive;

(b) The spectrum $\sigma(T) \subseteq [0, \infty)$, i.e., T is positive in the sense of C^*-algebras;

(c) There exists an operator $R \in \mathcal{B}(H)$ such that $T = R^*R$;

(d) There exists a unique positive operator S with $T = S^2$. In this case we write $S = \sqrt{T}$.

Proof. That (d) ⇒ (c) ⇒ (a) is trivial, and (b) ⇒ (d) follows from Proposition 4.1.2, so it remains to show (a) ⇒ (b).

For (a) ⇒ (b), WLOG we assume $\|T\| = 1$. Since T is self-adjoint, $\sigma(T) \subseteq \mathbb{R} \cap \overline{B_1(0)} = [-1, 1]$. We contend that $T_\mu := T + \mu 1$ is invertible for every $\mu > 0$, which implies that T has no negative spectral value. By assumption, we have

$$\|T_\mu v\|\,\|v\| \geqslant \langle T_\mu v, v\rangle = \langle Tv, v\rangle + \mu\langle v, v\rangle \geqslant \mu\,\|v\|^2$$

which implies that $\|T_\mu n\| \geqslant \mu\,\|v\|$ for every $v \in H$. It follows that T_μ is injective. (Note that the proof ends here if H is finite dimensional.) Since T_μ is self-adjoint, we also get

$$(T_\mu(H))^\perp = \ker T_\mu = 0$$

[1]We don't need to assume T is self-adjoint in advance: the condition $\langle Tv, v\rangle \in \mathbb{R}$ for all $v \in H$ merely guarantees that T is self-adjoint.

for $w \in (T_\mu(H))^\perp \Leftrightarrow 0 = \langle T_\mu v, w\rangle = \langle v, T_\mu w\rangle$ for every $v \in H \Leftrightarrow T_\mu w = 0$. By Lemma E.2.2 we then see $\overline{T_\mu(H)} = H$. Thus for each $w \in H$ we can find $(v_n)_n \subseteq H$ with $Tv_n \to w$. Since

$$\|v_n - v_m\| \leqslant \frac{1}{\mu}\|T_\mu v_n - T_\mu v_m\|$$

for $m, n \in \mathbb{N}$, it follows that $(v_n)_n$ is a Cauchy sequence in H and hence converges to, say, $v \in H$. Thus $T_\mu v = w$, showing that T_μ is surjective. By Open Mapping Theorem, T_μ^{-1} is continuous, so T_μ is invertible in $\mathcal{B}(H)$.

□

Definition. For $T \in \mathcal{B}(H)$, define the operator $|T| := \sqrt{T^*T}$. This is well-defined by the Theorem above.

Theorem 12.1.4 (Polar decomposition)**.** Let $T \in \mathcal{B}(H)$.

(i) $\||T|v\| = \|Tv\|$ for every $v \in H$. In particular $\||T|\|_{\mathrm{op}} = \|T\|_{\mathrm{op}}$.

(ii) There is an isometry $U : \overline{\operatorname{Im}|T|} \to \overline{\operatorname{Im}T}$ such that $T = U|T|$.

(iii) The decomposition in (ii) is unique, in the sense that if $T = U'P$ with P positive and $U' : \overline{\operatorname{Im}P} \to H$ isometric, then $U' = U$ and $P = |T|$.

Proof. For $v \in H$,

$$\||T|v\|^2 = \langle |T|v, |T|v\rangle = \langle |T|^2 v, v\rangle = \langle T^*Tv, v\rangle = \langle Tv, Tv\rangle = \|Tv\|^2$$

so (i) holds. For $v \in H$, define

$$\begin{aligned} U : \operatorname{Im}|T| &\longrightarrow \operatorname{Im}T \\ |T|v &\longmapsto Tv \end{aligned}$$

This is well-defined, for if $|T|v = |T|w$, then $0 = \||T|(v-w)\| = \|T(v-w)\|$ so that $Tv = Tw$. By (i) this is an isometry. Also, U extends to a bounded operator $U : \overline{\operatorname{Im}|T|} \to \overline{\operatorname{Im}T}$.

- Suppose $(v_n)_n \subseteq H$ is such that $|T|v_n$ is Cauchy. Then $U(|T|v_n) = Tv_n$ is also Cauchy by (i), so $Tv_n \to v$ for some $v \in \overline{\operatorname{Im}T}$. For $w = \lim_{n\to\infty} |T|v_n \in \overline{\operatorname{Im}|T|}$, define $Uw := v$. This is again well-define by (i), and $\|w\| = \|v\|$ by continuity of norm.

This proves (ii). For (iii), extend U to a bounded operator on H by setting $U \equiv 0$ on $\overline{\operatorname{Im}|T|}^\perp$ and do likewise for U'. Then U^*U is the orthogonal projection to $\overline{\operatorname{Im}|T|}$ and U'^*U' is the orthogonal projection to $\overline{\operatorname{Im}P}$. Note

$$|T| = \sqrt{T^*T} = \sqrt{(U'P)^*U'P} = \sqrt{P^*U'^*U'P} = \sqrt{P^*P} = \sqrt{P^2} = P$$

where the last equality is the uniqueness part of Theorem 12.1.3.(d). This also proves $U = U'$. □

Lemma 12.1.5. Let $T \in B(H)$ and $T = U|T|$ be its polar decomposition. We (always) extend U by setting $U|_{\overline{\operatorname{Im}|T|}^\perp} \equiv 0$.

(i) $U^*T = |T|$.

(ii) $U^*|_{\overline{\operatorname{Im}T}}$ is an isometry and $U^*|_{\overline{\operatorname{Im}T}^\perp} \equiv 0$.

(iii) U^*U is an orthogonal projection onto $\overline{\operatorname{Im}|T|}$, and UU^* is an orthogonal projection on $\overline{\operatorname{Im}T}$.

(iv) $U|T|U^* = |T^*|$

Proof.

(i) For $y \in (\operatorname{Im}|A|)^\perp$

$$\langle U^*Ax, y\rangle = \langle Ax, Uy\rangle = 0 = \langle |A|x, y\rangle$$

and for $y \in H$

$$\langle U^*Ax, |A|y\rangle = \langle Ax, Ay\rangle = \langle A^*Ax, y\rangle = \langle |A|x, |A|y\rangle.$$

These show $U^*Ax = |A|x$ for all $x \in H$.

(ii) For $x \in H$, by (i) $U^*Tx = |T|$, so

$$\langle U^*Tx, U^*Ty\rangle = \langle |T|x, |T|y\rangle = \langle |T|^2x, y\rangle = \langle T^*Tx, Ty\rangle = \langle Tx, Ty\rangle.$$

This shows U^* is an isometry on $\operatorname{Im} T$, and hence on $\overline{\operatorname{Im} T}$.

If $x \in (\operatorname{Im} A)^\perp$, then for $y \in H$ if we write $y = v + w$ with $v \in \overline{\operatorname{Im}|A|}$ and $w \in (\operatorname{Im}|A|)^\perp$

$$\langle U^*x, y\rangle = \langle x, Uy\rangle = \langle x, Uw\rangle = \lim_{n\to\infty}\langle x, U|A|w_n\rangle = \lim_{n\to\infty}\langle x, Aw_n\rangle = 0$$

where $(w_n)_n \subseteq H$ is such that $|A|w_n \to w$. This shows $U^*x = 0$ for $x \in (\operatorname{Im} A)^\perp$, and hence $\ker U^* \supseteq (\operatorname{Im} A)^\perp$. Now for $x \in \ker U^*$, write $x = v + w$ with $v \in (\operatorname{Im} A)^\perp$ and $w \in \overline{\operatorname{Im} A}$. Then

$$0 = U^*x = U^*v + U^*w = U^*w$$

so by injectivity of U^* on $\overline{\operatorname{Im} T}$ we see $w = 0$. Hence $x = v \in (\operatorname{Im} A)^\perp$.

(iii) This is shown in Theorem 12.1.4, and UU^* is proved similarly (using (ii)).

(iv) Both sides of the desired identity being positive, by Proposition 4.1.2 it suffices to show their squares are the same, i.e.,

$$U|T|U^*U|T|U^* = |T^*|^2 = TT^*.$$

In Theorem 12.1.4 we see U^*U is the orthogonal projection onto $\overline{\operatorname{Im}|T|}$, so $U^*U|T| = |T|$ and hence

$$U|T|U^*U|T|U^* = U|T||T|U^* = TT^*.$$

□

Lemma 12.1.6. Let $T \in \mathcal{B}(H)$ and write $T = U|T|$ for its polar decomposition. Then T is normal if and only if U is normal and commutes with $|T|$.

Proof. The if part is clear, given that $|T|$ is self-adjoint. Now assume T is normal. Extend U to H by setting $U|_{(\operatorname{Im}|T|)^\perp} \equiv 0$. We prove the following.

(a) T and $|T|$ commutes.

(b) U and $|T|$ commutes, and U^* and T commutes.

□

12.1.2 Schur's Lemma

Lemma 12.1.7. Let H be a Hilbert space and $T \in \mathcal{B}(H)$ be normal. If $\sigma(T) = \{\lambda\}$ is a singleton, then $T = \lambda\,\mathrm{id}_H$.

Proof. By Lemma 3.2.4, for then we have $\|T - \lambda\,\mathrm{id}_H\| = r(T - \lambda\,\mathrm{id}_H) = 0$. Alternatively, this can be proved by continuous functional calculus as follows. $\sigma(T) = \{\lambda\} \Leftrightarrow \mathrm{id}_{\sigma(T)} = \lambda \mathbf{1}_{\sigma(T)}$, and therefore $T = \mathrm{id}_{\sigma(T)}(T) = \lambda \mathbf{1}_{\sigma(T)}(T) = \lambda\,\mathrm{id}_H$. ☐

Theorem 12.1.8 (Schur)**.** Suppose $A \subseteq \mathcal{B}(H)$ is a self-adjoint set of bounded operators (namely, $S \in A$ implies $S^* \in A$). TFAE:

(a) A is **topological irreducible**, i.e., if $0 \neq L \subseteq H$ is any A-invariant *closed* subspace, then $L = H$.

(b) If $T \in \mathcal{B}(H)$ commutes with all $S \in A$, then $T = \mu\,\mathrm{id}$ for some $\mu \in \mathbb{C}$.

Proof. Assume (b) holds and that $0 \neq L \subseteq H$ is any A-invariant closed subspace. Then the orthogonal complement $L^\perp$ is also A-invariant, for with $v \in L$, $u \in L^\perp$ and $S \in A$, we have

$$\langle v, Su \rangle = \langle \underbrace{S^* v}_{\in L}, u \rangle = 0$$

So the orthogonal projection $P_L : H \to L$ commutes with A, and thus P_L is multiple of the identity. But $P_L|_L = \mathrm{id}_L$, this implies $P_L = \mathrm{id}_H$ and $L = H$.

Assume (a) holds, and let $T \in \mathcal{B}(H)$ commute with A. Then also T^* commutes with A since A is self-adjoint. By writing $T = \dfrac{1}{2}(T + T^*) + \dfrac{1}{2i}(T - T^*)$, we may assume $T \neq 0$ is self-adjoint. We will show that $\sigma(T)$ is a singleton. Note that every operator commuting with T also commutes with $f(T)$ for every $f \in C(\sigma(T))$. Now suppose there exist $x \neq y \in \sigma(T)$. Then there are two functions $f, g \in C(\sigma(T))$ with $f(x) \neq 0 \neq g(y)$ and $fg = 0$. Then $f(T) \neq 0 \neq g(T)$ and $f(T)g(T) = fg(T) = 0$. Since $g(T)$ commutes with A, the space $L := \overline{g(T)H}$ is a nonzero A-invariant subspace of H. By (a) we get $L = H$, but then $0 \neq f(T)H = f(T)\overline{g(T)H} \subseteq \overline{f(T)g(T)H} = 0$, a contradiction. ☐

12.2 Compact Operators

Definition. Let $T \in \mathcal{B}(H)$.

1. T is a **finite rank operator** if $\dim_{\mathbb{C}} \operatorname{Im} T < \infty$.
2. T is a **compact operator** if T sends bounded subsets to relatively compact subsets.

- It follows from definition that if T is compact and S is bounded, then both TS and ST is compact.
- T is compact if and only if for a given bounded sequence $v_j \in H$, the sequence Tv_j has a convergent subsequence.

Proposition 12.2.1. For $T \in \mathcal{B}(H)$, TFAE:

(a) T is compact.

(b) For every orthonormal sequence e_j, the sequence Te_j has a convergent subsequence.

(c) For every orthonormal sequence e_j, the sequence $Te_j \to 0$.

(d) There exists a sequence F_n of finite rank operators such that $\|T - F_n\|_{\text{op}} \to 0$ as $n \to \infty$.

Proof. (a) ⇒ (b) is trivial. (d) ⇒ (a) follows from a diagonal argument. For (b) ⇒ (c), we only need to show every subsequence of Te_j admits a subsequence converging to 0. Passing to subsequence, we assume $Te_j \to v \in H$. As said, we must show $v = 0$.

Lemma 12.2.2. Every orthonormal sequence in an inner product space converges weakly to 0.

Proof. Let v_n be an orthonormal sequence. For $y \in H$, by Bessel's inequality one has $\sum_{n=1}^{\infty} \langle v_n, y\rangle^2 \leqslant \|y\|^2 < \infty$. In particular, $\langle v_n, y\rangle \to 0 = \langle 0, y\rangle$ as $n \to \infty$, as claimed. □

By this lemma, we then have $\langle T^*v, e_j\rangle \to 0$ as $j \to \infty$. Now for $\varepsilon > 0$, we can find $N \gg 0$ such that $\|T^*Te_j - T^*v\| < \varepsilon$ and $|\langle T^*v, e_j\rangle| < \varepsilon$ whenever $j \geqslant N$. Then $\|Te_j\|^2 = \langle T^*Te_j, e_j\rangle \leqslant 2\varepsilon$.

For (c) ⇒ (d), we will construct an adequate orthonormal sequence e_n, as follows. Choose $e_1 \in H$ such that $\|Te_1\| \geqslant \frac{1}{2}\|T\|$. Assume that $e_1, \ldots, e_n\,(n \geqslant 2)$ are constructed, and let $U_n := \operatorname{span}_{\mathbb{C}}\{e_1, \ldots, e_n\}$. Let $P_n : H \to U_n$ be the orthogonal projection. If $\|T - TP_n\|_{\text{op}} = 0$, then $T = TP_n$ is of finite rank. Otherwise, choose

$$e_{n+1} \in U_n^{\perp} \cap \{x \in H \mid \|x\| = 1\}$$

with $\|Te_{n+1}\| \geqslant \frac{1}{2}\|T(\mathrm{id} - P_n)\|_{\text{op}}$. This is possible. For $\|T - TP_n\|_{\text{op}} > \frac{1}{2}\|T - TP_n\|$, so we can find $\|v\| = 1$ such that $\|(T - TP_n)v\| \geqslant \frac{1}{2}\|T - TP_n\|_{\text{op}} > 0$. If we put $w' = (\mathrm{id} - P_n)v \in U_n^{\perp}$, then

$$\|(T - TP_n)v\| = \|T(\mathrm{id} - P_n)v\| = \|Tw'\| \leqslant \|T(w'/\|w'\|)\|\,\|w'\| \leqslant \|T(w'/\|w'\|)\|$$

Thus $e_{n+1} := \dfrac{w'}{\|w'\|}$ works. Now, by (c), we have

$$\|T - TP_n\| = \|T(\mathrm{id} - P_n)\| \leqslant 2\|Te_{n+1}\| \to 0 \text{ as } n \to \infty$$

so that $\|T - TP_n\|_{\text{op}} \to 0$ as $n \to \infty$.

□

Corollary 12.2.2.1. For a bounded operator T, TFAE:

(a) T is compact.

(b) T^*T is compact.

(c) TT^* is compact.

(d) T^* is compact.

Proof. Suppose (a) and let v_n is a bounded sequence. Then we can find a convergent subsequence of Tv_n, and by continuity of T^*, T^*Tv_n also has a convergent subsequence. Also, by continuity of T^* we know T^*v_n is again bounded, so by compactness of T we can find a convergent subsequence of TT^*v_n; this shows (b) and (c). Assume (b) and let e_n be an orthonormal sequence in H. Then $T^*Te_n \to 0$ by Proposition 12.2.1.(c), and by Cauchy-Schwarz, we have

$$\|Te_n\|^2 = \langle T^*Te_n, e_n\rangle \leqslant \|T^*Te_n\|\,\|e_n\| = \|T^*Te_n\| \to 0$$

so $Te_n \to 0$. By Proposition 12.2.1.(c) again we deduce T is compact.

By symmetry we see (d) ⇒ (b),(c) and (c) ⇒(d), and hence the proof is finished. □

Proposition 12.2.3. A bounded operator T on a Hilbert space H is compact if and only if the image of the closed unit ball is compact.

Proof. Suppose $T(\overline{B_1(0)})$ is compact. Then for each $r > 0$, $T(\overline{B_r(0)}) = T(r\overline{B_1(0)}) = rT(\overline{B_1(0)})$ is still compact, so T is compact by its very definition. Now suppose T is compact. By definition we have $T(\overline{B_1(0)})$ is relatively compact, so it is enough to show it is closed. Let $(v_n)_n$ be a sequence in $\overline{B_1(0)}$ with $Tv_n \to w$ for some $w \in H$. By Banach Alaoglu, we can find $\alpha \in H^\vee$ with $\|\alpha\| \leqslant 1$ such that, by passing to subsequence, $\langle v_n, x\rangle \to \alpha(x)$ as $n \to \infty$ for all $x \in H$. By Riesz's Representation theorem, we can find $v \in H$ such that $\alpha = \langle v, \cdot\rangle$ with $\|v\| = \|\alpha\| \leqslant 1$. Finally, take $\varepsilon > 0$ and pick $n \gg 0$ so that $\|Tv_n - w\| < \varepsilon$ and $|\langle v - v_n, T^*(Tv - w)\rangle| < \varepsilon$. Then

$$\begin{aligned}\|Tv - w\|^2 &= \langle v, T^*(Tv - w)\rangle - \langle v_n, T^*(Tv - w)\rangle + \langle Tv_n, Tv - w\rangle - \langle w, Tv - w\rangle \\ &\langle v - v_n, T^*(Tv - w)\rangle + \langle Tv_n - w, Tv - w\rangle \\ &< \varepsilon + \|Tv_n - w\| \|Tv - w\| < \varepsilon(1 + \|Tv - w\|)\end{aligned}$$

Since ε is arbitrary, we see $\|Tv - w\| = 0$, i.e., $Tv = w$. ☐

12.2.1 Spectral Theorem for Compact Normal Operators

Theorem 12.2.4. Let T be a compact normal operator on a Hilbert space. Then there exists a sequence λ_n of non-zero complex numbers, which is either finite or tends to zero, such that one has an orthogonal decomposition

$$H = \ker T \oplus \overline{\bigoplus_n \operatorname{Eig}(T, \lambda_n)}$$

Each eigenspace $\operatorname{Eig}(T, \lambda_n) := \{v \in H \mid Tv = \lambda_n v\}$ is finite dimensional, and the eigenspaces are pairwise orthogonal.

Proof. The proof will go as follows.

1° Show a compact normal operator $T \neq 0$ has a nonzero eigenvalue by reducing to the case T being self-adjoint.

2° Show H is a direct sum of eigenspaces of T.

3° Show every eigenspace corresponding to a nonzero eigenvalue is finite dimensional, and the eigenvalues do not accumulate away from zero.

We first show that we can reduce to the case T being self-adjoint. Write

$$T = \frac{1}{2}(T + T^*) - \frac{i}{2}(iT - iT^*) := T_1 + iT_2$$

as a linear combination of commuting compact self-adjoint operators. If $T_2 = 0$, we are done. Otherwise, T_2 has a nonzero real eigenvalue ν. The corresponding eigenspace is T_1-invariant (for the T_i commute), so the restriction of T_1 on it also compact self-adjoint, and hence has an eigenvalue $\mu \in \mathbb{R}$. Then $\lambda := \mu + i\nu$ is a nonzero eigenvalue of T.

Now we have to show that a compact self-adjoint operator $T \neq 0$ has a nonzero eigenvalue.

Lemma 12.2.5. For a bounded self-adjoint operator T on a Hilbert space H, we have

$$\|T\| = \sup\{|\langle Tv, v\rangle| \mid \|v\| = 1\}$$

Proof. Let C be the RHS. By Cauchy-Schwarz, we have $C \leqslant \|T\|$. On the other hand, for $v, w \in H$ with $\|v\|, \|w\| \leqslant 1$, one has

$$\begin{aligned} C &\geqslant \frac{1}{2}C(\|v\|^2 + \|w\|^2) = \frac{1}{4}C(\|v+w\|^2 + \|v-w\|^2) \\ &\geqslant \frac{1}{4}|\langle T(v+w), v+w\rangle - \langle T(v-w), v-w\rangle| \\ &= \frac{1}{2}|\langle Tv, w\rangle + \langle Tw, v\rangle| = \frac{1}{2}|\langle Tv, w\rangle + \langle w, Tv\rangle| \\ &= |\operatorname{Re}\langle Tv, w\rangle| \end{aligned}$$

Replacing v with θv for some $\theta \in S^1$ we get $C \geqslant |\langle Tv, w\rangle|$ for all $\|v\|, \|w\| \leqslant 1$, and so $\|T\| \leqslant C$. □

We proceed to show a compact self-adjoint operator $T \neq 0$ has a nonzero eigenvalue. In fact, we will show either $\|T\|$ or $-\|T\|$ is an eigenvalue. By the lemma, there is a sequence $v_n \in H$ with $\|v_n\| = 1$ such that $\langle Tv_n, v_n\rangle \to \pm\|T\|$. Passing to a subsequence and replacing T with $-T$ if necessary, we assume $\langle Tv_n, v_n\rangle \to \|T\|$. Since T is compact, passing to subsequence again we can assume $Tv_n \to u \in H$. Then $\|u\| \leqslant \|T\|$, and we get

$$0 \leqslant \|Tv_n, \|T\| v_n\|^2 = \|Tv_n\|^2 - 2\|T\|\langle Tv_n, v_n\rangle + \|T\|^2\|v_n\|^2 \to \|u\|^2 - \|T\|^2 \leqslant 0$$

which implies $\|Tv_n - \|T\| v_n\| \to 0$. Thus $v := \lim_{n\to\infty} v_n = \frac{1}{\|T\|}u$ exists and $Tv = \lim_n Tv_n = u = \|T\| v$. This finishes 1°.

Let $0 \neq U \subseteq V$ be the closure of the (direct) sum of all eigenspaces of T corresponding to nonzero eigenvalues. Since every eigenvector for T is also an eigenvector for its adjoint T^*, U is stable under T and T^*, and hence the orthogonal complement $U^\perp$ is stable under T and T^* as well. The operator T restricts to a compact normal operator on $U^\perp$. Since $T|_{U^\perp}$ cannot have nonzero eigenvalue, $T|_{U^\perp} \equiv 0$ and $U^\perp = \ker T$. This shows 2°, and it is clear that the eigenspaces are pairwise orthogonal.

Need more explanation!! It remains to show 3°. For $\lambda \in \mathbb{C}^\times$, put $E_\lambda := \operatorname{Eig}(T, \lambda) = \ker(T - \lambda\,\mathrm{id})$. We show $\dim_{\mathbb{C}} E_\lambda < \infty$. Suppose otherwise; then there exists an orthonormal sequence f_n in E_λ. Then $\|f_n - f_m\| = \sqrt{2}$ for $n \neq m$, and $\|Tf_n - Tf_m\| = \sqrt{2}|\lambda| \neq 0$; in particular, this shows no subsequence of Tf_n is Cauchy, a contradiction to the compactness of T. Let $\mu > 0$ and consider the space $V_\mu := \sum\{E_\lambda \mid |\lambda| \geqslant \mu, E_\lambda \neq 0\}$. We claim $\dim_{\mathbb{C}} V_\mu < \infty$, and this will imply no spectral values of T can accumulate away from zero. Suppose otherwise. Then there exists an infinite sequence λ_n of distinct eigenvalues of T; let v_n be a corresponding eigenvector with norm 1. Then v_n is an orthonormal sequence, and

$$\|Tv_n - Tv_m\|^2 = \|\lambda_n v_n - \lambda_m v_m\|^2 = |\lambda_n|^2 + |\lambda_m|^2 \geqslant 2\mu^2 > 0$$

again a contradiction to the compactness of T.

□

12.2.2 Singular Values

Lemma 12.2.6. For a compact operator T, $\sqrt{T^*T}$ is also compact.

Proof. By Corollary 12.2.2.1.(b) we know T^*T is compact. To see $\sqrt{T^*T}$ is compact, let e_n be an orthonormal sequence. Then

$$\left\|\sqrt{T^*T}e_n\right\|^2 = \langle T^*Te_n, e_n\rangle \leqslant \|T^*Te_n\|\,\|e_n\| = \|T^*Te_n\| \to 0$$

by Proposition 12.2.1.(c). ☐

Definition. Let T be a compact operator. Then by the lemma we see $|T|$ is a positive compact operator. Let $s_j(T)$ be the family of nonzero eigenvalues of $|T|$ repeated with multiplicities and such that $s_{j+1}(T) \leqslant s_j(T)$ for all j. The $s_j(T)$ are called the **singular values** of T.

Proposition 12.2.7. Let T be a compact operator.

1. We have $s_1(T) = \|T\|$ and

$$s_{j+1}(T) = \inf_{v_1,\ldots,v_j \in H} \sup\{\|Tw\| \mid w \perp v_1,\ldots,v_j,\ \|w\| = 1\}$$

   where the vectors $v_1,\ldots,v_j$ are unit eigenvectors for the eigenvalues $s_1(T),\ldots,s_j(T)$, respectively.

2. For any bounded operator S on H one has $s_j(ST) \leqslant \|S\|\, s_j(T)$.

Proof. To be filled

☐

12.3 Hilbert-Schmidt and Trace Class

12.3.1 Hilbert-Schmidt Operators

Let H, H' be two Hilbert spaces and $T : H \to H'$ a bounded operator. For an orthonormal basis $(e_\alpha)_\alpha$ of H, consider the sum

$$\sum_\alpha \langle Te_\alpha, Te_\alpha\rangle_{H'} = \sum_\alpha \|Te_\alpha\|_{H'}^2 \in [0,+\infty].$$

If $(f_\beta)_\beta$ is an orthonormal basis of H', by Parseval's identity we have

$$\begin{aligned}\sum_\alpha \|Te_\alpha\|_{H'}^2 &= \sum_\alpha\left(\sum_\beta \langle Te_\alpha, f_\beta\rangle_{H'}\langle f_\beta, Te_\alpha\rangle_{H'}\right) = \sum_\alpha\left(\sum_\beta \langle e_\alpha, T^* f_\beta\rangle_H \langle T^* f_\beta, e_\alpha\rangle_H\right)\\ &= \sum_\beta\left(\sum_\alpha \langle e_\alpha, T^* f_\beta\rangle_H \langle T^* f_\beta, e_\alpha\rangle_H\right) = \sum_\alpha \|T^* f_\beta\|_H^2 .\end{aligned}$$

The same computation works if we replace $(e_\alpha)_\alpha$ by another orthonormal basis of H. This shows the sum is independent of the choice of orthonormal bases of H. We define the **Hilbert-Schmidt norm** $\|T\|_{\mathrm{HS}} \in [0,+\infty]$ of the operator T by

$$\|T\|_{\mathrm{HS}}^2 := \sum_\alpha \langle Te_\alpha, Te_\alpha\rangle_{H'} = \sum_\alpha \|Te_\alpha\|_{H'}^2 .$$

Definition. A bounded operator $T : H \to H'$ between Hilbert spaces is **Hilbert-Schmidt** if it has finite Hilbert-Schmidt norm. Put

$$\mathcal{HS}(H,H') := \{T \in \mathcal{B}(H,H') \mid \|T\|_{\mathrm{HS}} < \infty\}.$$

When $H = H'$, we put $\mathcal{HS}(H) = \mathcal{HS}(H,H)$ for brevity.

It is clear that $(\mathcal{HS}(H,H'), \|\cdot\|_{\mathrm{HS}})$ is a normed linear space over $\mathbb{C}$, and the above computation shows that the adjoint defines an isometry $\mathcal{HS}(H,H') \xrightarrow{T\mapsto T^*} \mathcal{HS}(H',H)$. In fact, for $T,S \in \mathcal{HS}(H,H')$ and an orthonormal basis $(e_\alpha)_\alpha$ for H, consider the sum

$$(T,S) := \sum_\alpha \langle Te_\alpha, Se_\alpha\rangle_{H'}.$$

By Cauchy-Schwartz, we have

$$\sum_\alpha |\langle Te_\alpha, Se_\alpha\rangle_{H'}| \leqslant \sum_\alpha \|Te_\alpha\|_{H'} \|Se_\alpha\|_{H'} \leqslant \left(\sum_\alpha \|Te_\alpha\|_{H'}^2\right)^{\frac{1}{2}} \left(\sum_\alpha \|Se_\alpha\|_{H'}^2\right)^{\frac{1}{2}} = \|T\|_{\mathrm{HS}} \|S\|_{\mathrm{HS}} < \infty$$

so (T,S) is absolutely convergent. Polarization of hermitian inner products gives

$$(B,C) = \frac{1}{4}\sum_{k=1}^4 i^k \left\|B + i^k C\right\|_{\mathrm{HS}}^2$$

where $i = \sqrt{-1}$. This shows (B,C) is independent of the choice of orthonormal bases for H. With this pairing, $\mathcal{HS}(H,H')$ becomes an inner product space. More is true.

Theorem 12.3.1. Let H, H' be Hilbert spaces.

(i) $\mathcal{HS}(H,H')$ is complete, hence is a Hilbert space.

(ii) The map

$$\begin{aligned} H^\vee \widehat{\otimes} H' &\longrightarrow \mathcal{HS}(H,H') \\ t\otimes v &\longmapsto [T_{t,v} : w \mapsto t(w)v] \end{aligned}$$

is well-defined and is a Hilbert space isomorphism.

Proof.

(i) Let $(e_\alpha)_{\alpha\in A}$ (resp. $(f_\beta)_{\beta\in B}$) be an orthonormal basis for H (resp for H'). Consider the map

$$\begin{aligned} \mathcal{HS}(H,H') &\longrightarrow \ell^2(A\times B) \\ T &\longmapsto M_T : (\alpha,\beta) \mapsto \langle Te_\alpha, f_\beta\rangle_{H'}. \end{aligned}$$

To see this is well-defined, by Parseval's identity we have

$$\sum_\alpha\sum_\beta |\langle Te_\alpha, f_\beta\rangle_{H'}|^2 = \sum_\alpha \|Te_\alpha\|_{H'}^2 = \|T\|_{\mathrm{HS}}^2 < \infty.$$

Moreover, the identity tells that $T \mapsto M_T$ is norm-preserving. It is not hard to construct an inverse of $T \mapsto M_T$: for $g \in \ell^2(A\times B)$, define $T_g : H \to H'$ by $T_g(e_\alpha) = \sum_\beta g(\alpha,\beta) f_\beta$. This is well-defined as g is square-summable. It is clear that $g \mapsto T_g$ is inverse to $T \mapsto M_T$, so $\mathcal{HS}(H,H')$ is isomorphic to $\ell^2(A\times B)$ as normed spaces. It is standard that the latter space is complete, whence $\mathcal{HS}(H,H')$ is also complete.

(ii) For any $t\otimes v \in H^\vee \widehat{\otimes} H'$,

$$\|T_{t,v}\|_{\mathrm{HS}}^2 = \sum_\alpha \langle T_{t,v}e_\alpha, T_{t,v}e_\alpha\rangle_{H'} = \sum_\alpha |t(e_\alpha)|^2 \langle v,v\rangle_{H'} = \|v\|_{H'}^2 \|t\|_{H^\vee}^2 = \|t\otimes v\|^2$$

By polarization this shows the described map is a Hilbert space homomorphism. In particular, it is injective. For the surjectivity, let $T \in \mathcal{HS}(H, H')$. For each $w \in H$, write $w = \sum_\alpha \langle w, e_\alpha \rangle_H e_\alpha = \sum_\alpha \widehat{e_\alpha}(w) e_\alpha$. Then $Tw = \sum_\alpha \widehat{e_\alpha}(w) T e_\alpha$ so that T is mapped to by $\sum_\alpha \widehat{e_\alpha} \otimes T e_\alpha \in H^\vee \widehat{\otimes} H'$. Alternatively, if f_β is an orthonormal basis for H', then $Tw = \sum_\beta \langle Tw, f_\beta \rangle_{H'} f_\beta = \sum_\beta \langle w, T^* f_\beta \rangle_H f_\beta = \sum_\beta \widehat{T^* f_\beta}(w) f_\beta$, so that T is also mapped to by $\sum_\beta \widehat{T^* f_\beta} \otimes f_\beta$

□

Lemma 12.3.2. Let $S \in \mathcal{B}(H', H''), T \in \mathcal{B}(H, H')$. Then

(a) $\|T\|_{\mathrm{HS}} = \|T^*\|_{\mathrm{HS}}$.

(b) $\|ST\|_{\mathrm{HS}}, \|TS\|_{\mathrm{HS}} \leqslant \|S\|_{\mathrm{op}} \|T\|_{\mathrm{HS}}$.

(c) $\|T\|_{\mathrm{op}} \leqslant \|T\|_{\mathrm{HS}}$.

(d) For any isometry U on H and isometry U' on H', one has $\|U'T\|_{\mathrm{HS}} = \|TU\|_{\mathrm{HS}} = \|T\|_{\mathrm{HS}}$.

Proof. Let $(e_\alpha)_\alpha$ be an orthonormal basis for H.

(a) This is shown in the beginning of this subsection.

(b) $\|ST\|_{\mathrm{HS}}^2 = \sum_\alpha \|STe_n\|^2 \leqslant \sum_\alpha \|S\|_{\mathrm{op}}^2 \|Te_\alpha\|^2 = \|S\|_{\mathrm{op}}^2 \|T\|_{\mathrm{HS}}^2$. The second follows from (a) and the fact that $\|S^*\|_{\mathrm{op}} = \|S\|_{\mathrm{op}}$.

(c) Let $v \in H$ with $\|v\| = 1$. Then we can find an orthonormal basis $(\phi_\beta)_\beta$ with $\phi_{\beta_0} = v$. Then

$$\|Tv\|^2 = \|T\phi_{\beta_0}\|^2 \leqslant \sum_\beta \|T\phi_\beta\|^2 = \|T\|_{\mathrm{HS}}^2$$

(d) This is clear for $(Ue_\alpha)_\alpha$ is also an orthonormal basis when U is unitary.

□

Proposition 12.3.3. An operator T is Hilbert-Schmidt if and only if it is compact and its singular values satisfy $\sum_n s_n(T)^2 < \infty$. In this case, one has $\|T\|_{\mathrm{HS}}^2 = \sum_n s_n(T)^2$.

Proof. For a bounded operator T, we have

$$\|T\|_{\mathrm{HS}}^2 = \sum_j \langle Te_j, Te_j \rangle = \sum_j \langle T^*Te_j, e_j \rangle = \sum_j \langle |T|^2 e_j, e_j \rangle = \sum_j \langle |T|e_j, |T|e_j \rangle = \||T|\|_{\mathrm{HS}}^2$$

so that T is Hilbert-Schmidt if and only if $|T|$ is.

Let T be Hilbert-Schmidt. To see T is compact, we use Proposition 12.2.1.(c). Let $\{e_j\}_{j=1}^\infty$ be an orthonormal sequence and let A be an orthonormal basis for T containing the e_j. Then

$$\infty > \|T\|_{\mathrm{HS}}^2 = \sum_{h \in H} \|Th\|^2 \geqslant \sum_{j=1}^\infty \|Te_j\|^2$$

implies that $\lim_{j \to \infty} Te_j = 0$.

Now assume T is compact; in particular, $|T|$ is compact. By using an orthonormal basis consisting eigenvectors of $|T|$, which exists by Spectral Theory, to compute $\||T|\|_{\mathrm{HS}}$, we can see $|T|$ is Hilbert-Schmidt if and only if $\sum_j s_j(T)^2 < \infty$, and the sum equals $\||T|\|_{\mathrm{HS}}^2$. The last assertion follows from the computation in the first paragraph. □

12.3.2 Integral kernel

In this subsection, let $(X,\mu),(Y,\nu)$ be either two σ-finite measure spaces, or two LCH spaces with μ,ν Radon. We study the space $\mathcal{HS}(H,H')$ with $H=L^2(X,\mu)$ and $H'=L^2(Y,\nu)$.

Combining Lemma E.2.9 and Theorem 12.3.1, we get

$$\mathcal{HS}(L^2(X,\mu),L^2(Y,\nu)) \xrightarrow[\sim]{} L^2(X,\mu)^\vee \widehat{\otimes} L^2(Y,\nu) \xrightarrow[\sim]{} L^2(X,\mu)\widehat{\otimes} L^2(Y,\nu) \xrightarrow[\sim]{} L^2(X\times Y,\mu\otimes\nu) .$$

Let $T\in\mathcal{HS}(L^2(X,\mu),L^2(Y,\nu))$. Then its image in $L^2(X,\mu)\widehat{\otimes} L^2(Y,\nu)$ is

$$\sum_\alpha \overline{e_\alpha}\otimes Te_\alpha = \sum_\beta \overline{T^*f_\beta}\otimes f_\beta$$

where $(e_\alpha)_\alpha$ and $(f_\beta)_\beta$ are orthonormal bases for $L^2(X,\mu)$ and $L^2(Y,\nu)$ respectively. Here we twist it by complex conjugation so as to make it $\mathbb{C}$-linear. The last isomorphism is given by multiplication. All these together with a swapping[2] in the end give the isomorphism

$$\begin{aligned} k:\mathcal{HS}(L^2(X,\mu),L^2(Y,\nu)) &\longrightarrow L^2(Y\times X,\mu\otimes\nu)\\ T &\longmapsto k_T:(y,x)\mapsto \sum_\alpha \overline{e_\alpha(x)}Te_\alpha(y)=\sum_\beta \overline{T^*f_\beta(x)}f_\beta(y). \end{aligned}$$

The function $k_T\in L^2(X\times Y,\mu\otimes\nu)$ is called the **(integral) kernel** of the operator T, as it satisfies

$$Tf(y)=\int_X k(y,x)f(x)d\mu(x)$$

for all $f\in L^2(X,\mu)$. In general, we refer to a function $k\in L^2(Y\times X,\mu\otimes\nu)$ as an L^2**-kernel**. We record the above result as the following

Proposition 12.3.4. Let $(X,\mu),(Y,\nu)$ be either two σ-finite measure spaces, or two LCH spaces with μ,ν Radon. Then there is a Hilbert space isomorphism

$$k:\mathcal{HS}(L^2(X,\mu),L^2(Y,\nu)) \longrightarrow L^2(Y\times X,\mu\otimes\nu)$$

such that the identity

$$Tf(y)=\int_X k_T(y,x)f(x)d\mu(x)$$

holds for all $T\in\mathcal{HS}(L^2(X,\mu),L^2(Y,\nu))$ and $f\in L^2(X,\mu)$. In particular,

$$\|T\|_{\mathrm{HS}}^2=\int_Y\int_X |k_T(y,x)|^2 d\nu(y)d\mu(x).$$

For notational simplicity, we suppress the measures μ, ν and simply write dx and dy for $d\mu(x)$ and $d\nu(y)$. Suppose $T:L^2(X)\to L^2(Y)$ is Hilbert-Schmidt. Then the adjoint $T^*:L^2(Y)\to L^2(X)$ is also Hilbert-Schmidt, so it has a kernel function

$$k_{T^*}(x,y)=\sum_\beta \overline{f_\beta(y)}T^*f_\beta(x)=\sum_\alpha \overline{Te_\alpha(y)}e_\alpha(x)=\overline{k_T(y,x)}.$$

In other words, the adjoint of T has an integral representation

$$T^*g(x)=\int_Y \overline{k_T(y,x)}g(y)dy$$

[2]This convention makes the formula for convolution of kernels which appears below nicer (at least for me).

valid for all $g \in L^2(Y)$.

Next we consider the composition. Suppose Z is another space of the same type as X and Y, and suppose $T : L^2(X) \to L^2(Y)$ and $S : L^2(Y) \to L^2(Z)$ are Hilbert-Schmidt. By Lemma 12.3.2 the composition $ST : L^2(X) \to L^2(Z)$ is again Hilbert-Schmidt, so it has a kernel function $k_{ST} \in L^2(Z \times X)$. For $f \in L^2(X)$, by Fubini we have

$$STf(z) = \int_Y k_S(z,y)Tf(y)dy = \int_Y k_S(z,y)\left(\int_X k_T(y,x)f(x)dx\right)dy = \int_X \left(\int_Y k_S(z,y)k_T(y,x)dy\right)f(x)dx$$

which implies

$$k_{ST}(z,x) = \int_Y k_S(z,y)k_T(y,x)dy$$

We call this the **convolution** of the kernels k_S and k_T. Under these operations, the map

$$k : \mathcal{HS}(L^2(X,\mu)) \longrightarrow L^2(X \times X, \mu \otimes \nu)$$

becomes a C^*-algebra isomorphism. more detail

12.3.3 Trace Class Operators

Lemma 12.3.5. Let $T \in B(H)$ and $(e_\alpha)_\alpha$ be an orthonormal basis. The sum

$$\|T\|_{\text{tr}} := \sum_\alpha \langle |T|e_\alpha, e_\alpha\rangle \in [0, +\infty]$$

is independent of the choice of orthonormal bases $(e_\alpha)_\alpha$.

(a) $\|T\|_{\text{tr}} = \||T|\|_{\text{tr}} = \left\||T|^{\frac{1}{2}}\right\|_{\text{HS}}^2$.

(b) $\|T\|_{\text{op}} = \||T|\|_{\text{op}} \leqslant \|T\|_{\text{HS}} \leqslant \|T\|_{\text{tr}}$.

(c) If $\|T\|_{\text{tr}} < \infty$, then T is Hilbert-Schmidt.

Proof. Since $|T| = |T|^{\frac{1}{2}}|T|^{\frac{1}{2}}$ and $|T|^{\frac{1}{2}}$ is self-adjoint, we see

$$\sum_\alpha \langle |T|e_\alpha, e_\alpha\rangle = \sum_\alpha \langle |T|^{\frac{1}{2}}e_\alpha, |T|^{\frac{1}{2}}e_\alpha\rangle = \left\||T|^{\frac{1}{2}}\right\|_{\text{HS}}^2.$$

It follows from the corresponding result for $\|\cdot\|_{\text{HS}}$ that the sum is independent of the choice of $(e_\alpha)_\alpha$, and this also proves (a).

(b) The first follows from Theorem 12.1.4. The second follows from Lemma 12.3.2 and Proposition 12.3.3.

Next we show $\||T|\|_{\text{op}} \leqslant \|T\|_{\text{tr}}$. Let $v \in H$ with $\|v\| = 1$ and pick any orthonormal basis $\{e_\alpha\}_\alpha$ with $e_{\alpha_0} = v$. Then

$$\langle |T|v, v\rangle \leqslant \sum_\alpha \langle |T|e_\alpha, e_\alpha\rangle = \|T\|_{\text{tr}}.$$

It then follows from Lemma 12.2.5 that $\||T|\|_{\text{op}} \leqslant \|T\|_{\text{tr}}$. Finally,

$$\||T|\|_{\text{HS}} \leqslant \left\||T|^{\frac{1}{2}}\right\|_{\text{op}} \left\||T|^{\frac{1}{2}}\right\|_{\text{HS}} = \||T|\|_{\text{op}}^{\frac{1}{2}} \|T\|_{\text{tr}}^{\frac{1}{2}} \leqslant \|T\|_{\text{tr}}$$

(c) From (a) we see $|T|^{\frac{1}{2}}$ is Hilbert-Schmidt, so $|T|$ is also Hilbert-Schmidt. By Proposition 12.3.3, T is Hilbert-Schmidt.

□

Definition. The value $\|T\|_{\mathrm{tr}}$ in Lemma 12.3.5 is called the **trace norm** of T. Denote by

$$B^1(H) = \{T \in B(H) \mid \|T\|_{\mathrm{tr}} < \infty\}$$

the space of **trace-class operators** on H.

By Lemma 12.3.5 we see trace-class operators are Hilbert-Schmidt. By Proposition 12.3.3 we see

$$\|T\|_{\mathrm{tr}} := \sum_j s_j(T)$$

where the $s_j(T)$ are singular values of T.

Lemma 12.3.6. $(B^1(H), \|\cdot\|_{\mathrm{tr}})$ is a normed linear space, and

$$B^1(H) = \mathcal{HS}(H)^2 = \mathrm{span}_{\mathbb{C}}\{TS \mid T, S \in \mathcal{HS}(H)\} \subseteq B(H).$$

In addition, $B^1(H)$ contains all finite rank operators.

Proof. We must show $\|\cdot\|_{\mathrm{tr}}$ satisfies the triangle inequality. Let $T, S \in B^1(H)$, and write $T = U|T|$, $S = V|S|$, $T + S = W|T + S|$ for the respective polar decompositions. By Lemma 12.1.5 we have

$$W^*(T + S) = |T + S|.$$

Let $\{e_\alpha\}_\alpha$ be any orthonormal basis. Then by Lemma 12.3.2

$$\begin{aligned}
\|T+S\|_{\mathrm{tr}} &= \sum_\alpha \langle |T+S| e_\alpha, e_\alpha\rangle = \sum_\alpha \langle (T+S)e_\alpha, We_\alpha\rangle \leqslant \sum_\alpha |\langle Te_\alpha, We_\alpha\rangle| + \sum_\alpha |\langle Se_\alpha, We_\alpha\rangle| \\
&= \sum_\alpha |\langle |T|^{\frac{1}{2}} e_\alpha, |T|^{\frac{1}{2}} U^* W e_\alpha\rangle| + \sum_\alpha |\langle |S|^{\frac{1}{2}} e_\alpha, |S|^{\frac{1}{2}} V^* W e_\alpha\rangle| \\
&\leqslant \left\| |T|^{\frac{1}{2}} \right\|_{\mathrm{HS}} \left\| |T|^{\frac{1}{2}} U^* W \right\|_{\mathrm{HS}} + \left\| |S|^{\frac{1}{2}} \right\|_{\mathrm{HS}} \left\| |S|^{\frac{1}{2}} V^* W \right\|_{\mathrm{HS}} \\
&\leqslant \left\| |T|^{\frac{1}{2}} \right\|_{\mathrm{HS}}^2 + \left\| |S|^{\frac{1}{2}} \right\|_{\mathrm{HS}}^2 = \|T\|_{\mathrm{tr}} + \|S\|_{\mathrm{tr}}.
\end{aligned}$$

This proves the first statement. To see $B^1(H) \subseteq \mathcal{HS}(H)^2$, for $T \in B^1(H)$ write $T = U|T|$ for its polar decomposition. Then $T = U|T|^{\frac{1}{2}} \circ |T|^{\frac{1}{2}}$ and both $U|T|^{\frac{1}{2}}$ and $|T|^{\frac{1}{2}}$ are Hilbert-Schmidt by Lemma 12.3.5. For $\mathcal{HS}(H)^2 \subseteq B^1(H)$, for $T, S \in \mathcal{HS}(H)$ write $T = T_1 + iT_2$, $S = S_1 + iS_2$ with T_i, S_j self-adjoint. Since T_i, S_j are linear combinations of T, S, they are Hilbert-Schmidt. Note that

$$T_i S_j = \frac{1}{2}\left((T_i + S_j)^*(T_i + S_j) - T_i^* T_i - S_j^* S_j\right).$$

If $A \in \mathcal{HS}(H)$ is self-adjoint, then $\|A^*A\|_{\mathrm{tr}} = \|A\|_{\mathrm{HS}}^2 < \infty$ so that A^*A is trace-class. It follows that $T_i S_j$ is trace class, implying TS is trace class.

The last assertion remains. By linearity it suffices to show any rank 1 operator is trace class. A rank 1 operator has the form $f \otimes x$ with $0 \neq f \in H^*$ and $0 \neq x \in H$. A direct computation shows that $(f \otimes x)^* = \langle \cdot, x\rangle \otimes v_f$, where $0 \neq v_f \in H$ is such that $f = \langle \cdot, v_f\rangle$. Then $(f \otimes x)^*(f \otimes x) = \|x\|^2 f \otimes v_f$, so the only singular value for $f \otimes x$ is $\|x\| \|f\|_{\mathrm{op}}$, showing $\|f \otimes x\|_{\mathrm{tr}} = \|x\| \|f\|_{\mathrm{op}} < \infty$. □

Lemma 12.3.7. Let $T, S \in \mathcal{B}(H)$

(i) $\|T\|_{\mathrm{tr}} = \|T^*\|_{\mathrm{tr}}$.

(ii) $\|TS\|_{\mathrm{tr}}, \|ST\|_{\mathrm{tr}} \leqslant \|S\|_{\mathrm{op}} \|T\|_{\mathrm{tr}}$.

Proof.

(i) Write $T = U|T|$ for its polar decomposition. By Lemma 12.1.5 we have $|T^*| = U|T|U^*$, so by Lemma 12.3.5

$$\|T^*\|_{\mathrm{tr}} = \left\||T^*|^{\frac{1}{2}}\right\|_{\mathrm{HS}}^2 = \left\|(U|T|U^*)^{\frac{1}{2}}\right\|_{\mathrm{HS}}^2.$$

Since $U|T|U^* = (U|T|^{\frac{1}{2}})(U|T|^{\frac{1}{2}})^* = ||T|^{\frac{1}{2}}U^*|^2$, we see $(U|T|U^*)^{\frac{1}{2}} = ||T|^{\frac{1}{2}}U^*|$, and hence

$$\|T^*\|_{\mathrm{tr}} = \left\|||T|^{\frac{1}{2}}U^*|\right\|_{\mathrm{HS}}^2 = \left\||T|^{\frac{1}{2}}U^*\right\|_{\mathrm{HS}}^2 \leqslant \left\||T|^{\frac{1}{2}}\right\|_{\mathrm{HS}}^2 = \|T\|_{\mathrm{tr}}$$

by Lemma 12.3.5, Proposition 12.3.3 and Lemma 12.3.2.

(ii) By (ii) it suffices to prove for TS. Write $TS = V|TS|$ for its polar decomposition and let $\{e_\alpha\}_\alpha$ be any orthonormal basis. Then by Lemma 12.1.5, Lemma 12.3.2 and Lemma 12.3.5

$$\|TS\|_{\mathrm{tr}} = \sum_\alpha \langle |TS|e_\alpha, e_\alpha\rangle = \sum_\alpha \langle |T|^{\frac{1}{2}}Se_\alpha, |T|^{\frac{1}{2}}Ve_\alpha\rangle \leqslant \left\||T|^{\frac{1}{2}}S\right\|_{\mathrm{HS}} \left\||T|^{\frac{1}{2}}\right\|_{\mathrm{HS}} \leqslant \|S\|_{\mathrm{op}} \|T\|_{\mathrm{tr}}$$

□

Lemma 12.3.8. If T is a compact operator, then

$$\|T\|_{\mathrm{tr}} = \sup_{(e_i),\,(h_i)} \sum_i |\langle Te_i, h_i\rangle|$$

where (e_i) and (h_i) run over all orthonormal bases.

Proof. For simplicity, write $s_j = s_j(T)$. Let f_j be an orthonormal sequence consisting of eigenvectors of $|T|$. Then for each $v \in H$, we have

$$|T|v = \sum_j s_j \langle v, f_j\rangle f_j$$

Use polar decomposition to obtain a partial isometry $U : \overline{\operatorname{Im}|T|} \to \overline{\operatorname{Im} T}$ such that $T = U|T|$. If we put $g_j := Uf_j$, then for $v \in H$,

$$Tv = U\left(\sum_j s_j \langle v, f_j, \rangle f_j\right) = \sum_j s_j \langle v, f_j\rangle g_j$$

Now for any two orthonormal bases e_i, h_j, by Cauchy-Schwarz and Parseval's identity, we have

$$\begin{aligned}
\sum_i |\langle Te_i, h_i\rangle| &= \sum_i \left|\sum_j s_j \langle e_i, f_j\rangle\langle g_j, h_i\rangle\right| \\
&\leqslant \sum_j s_j \sum_i |\langle e_i, f_j\rangle\langle g_j, h_i\rangle| \\
&\leqslant \sum_j s_j \left(\sum_i |\langle e_i, f_j\rangle|^2\right)^{\frac{1}{2}} \left(\sum_i |\langle g_j, h_i\rangle|^2\right)^{\frac{1}{2}} \\
&= \sum_j s_j \|f_j\| \|g_j\| = \sum_j s_j
\end{aligned}$$

For the reversed inequality, take e_j to be any orthonormal basis that prolongs the orthonormal sequence f_j and h_j to be any orthonormal basis prolonging g_j. Then

$$\sum_i |\langle Te_i, h_i\rangle| = \sum_i \left|\sum_j s_j \langle e_i, f_j\rangle\langle g_j, h_i\rangle\right| = \sum_i \left|\sum_j s_j \delta_{ij}\right| = \sum_i \sum_j s_j \delta_{ij} = \sum_j s_j$$

□

Theorem 12.3.9. There exists a unique linear functional with operator norm 1 called the **trace**

$$\mathrm{tr} : B^1(H) \longrightarrow \mathbb{C}$$

such that if T has finite rank, then $\mathrm{tr}\, T$ is the usual trace operator. Explicitly,

$$\mathrm{tr}\, T = \sum_\alpha \langle Te_\alpha, e_\alpha\rangle,$$

where the sum is absolutely convergent and is independent of the choice of an orthonormal basis (e_α). If T is trace class and normal, we have

$$\mathrm{tr}\, T = \sum_n \lambda_n \dim \mathrm{Eig}\,(T, \lambda_n)$$

where the sum runs over the sequence of non-zero eigenvalues (λ_n) of T, and the sum converges absolutely.[3]

Proof. By Lemma 12.3.6, write $T = S_1 S_2$ with $S_i \in \mathcal{HS}(H)$. Then

$$\sum_\alpha \langle Te_\alpha, e_\alpha\rangle = \sum_\alpha \langle S_2 e_\alpha, S_1^* e_\alpha\rangle = (S_2, S_1^*)$$

where $(\,,)$ is the pairing on the space of Hilbert-Schmidt operators; this shows the sum converges (absolutely) and is independent of choice of orthonormal basis $\{e_\alpha\}_\alpha$. When T is normal, use a basis consisting of eigenvectors to compute trace.

Next we claim $B^1(H) \ni T \mapsto \sum_\alpha \langle Te_\alpha, e_\alpha\rangle$ is bounded. Write $T = U|T|$ for its polar decomposition. This is clear by Lemma 12.3.5:

$$\left|\sum_\alpha \langle Te_\alpha, e_\alpha\rangle\right| \leqslant \sum_\alpha |\langle |T|^{\frac{1}{2}} e_\alpha, |T|^{\frac{1}{2}} U^* e_\alpha\rangle| \leqslant \left\| |T|^{\frac{1}{2}} \right\|_{\mathrm{HS}} \left\| |T|^{\frac{1}{2}} U^* \right\|_{\mathrm{HS}} \leqslant \left\| |T|^{\frac{1}{2}} \right\|_{\mathrm{HS}}^2 = \|T\|_{\mathrm{tr}}.$$

This is an equality if T is positive, so this shows the operator norm is 1. Finally, the uniqueness follows from the following density result.

Lemma 12.3.10. The space of finite rank operators is dense in $(B^1(H), \|\cdot\|_{\mathrm{tr}})$.

Proof. Let $T \in B^1(H)$ and $\{e_\alpha\}_\alpha$ be an orthonormal basis. Since $\sum_\alpha \left\| |T|^{\frac{1}{2}} e_\alpha \right\|^2 = \|T\|_{\mathrm{tr}} < \infty$, the sum is supported on a countable subset of $\{e_\alpha\}_\alpha$. Let $(f_n)_n = \{e_\alpha \mid |T|^{\frac{1}{2}} e_\alpha \neq 0\}$ and let $P_n : H \to H$ be the orthogonal projection onto $\mathrm{span}_{\mathbb{C}}\{f_1, \ldots, f_n\}$. We claim $\|T - TP_n\|_{\mathrm{tr}} \to 0$.

Note that $|T - TPn|^2 = (1 - P_n)T^*T(1 - P_n) = ||T| - |T|P_n|^2$, so $|T - TP_n| = ||T| - |T|P_n|$ by Proposition 4.1.2 and

$$\|T - TP_n\|_{\mathrm{tr}} = \||T| - |T|P_n\|_{\mathrm{tr}} \leqslant \left\| |T|^{\frac{1}{2}} \right\|_{\mathrm{HS}} \left\| |T|^{\frac{1}{2}} (1 - P_n) \right\|_{\mathrm{HS}}.$$

[3] This holds without normality. See Lidskii's Theorem.

It remains to show $\left\| |T|^{\frac{1}{2}}(1-P_n) \right\|_{\mathrm{HS}} \to 0$ as $n \to \infty$. Indeed,

$$\left\| |T|^{\frac{1}{2}}(1-P_n) \right\|_{\mathrm{HS}}^2 = \sum_{\alpha} \left\| |T|^{\frac{1}{2}}(1-P_n)e_\alpha \right\|^2 = \sum_{m=n+1}^{\infty} \left\| |T|^{\frac{1}{2}}(1-P_n)f_m \right\|^2 \to 0$$

□

Proposition 12.3.11.

(a) For a trace class operator T, we have $\operatorname{tr} T^* = \overline{\operatorname{tr} T}$.

(b) For Hilbert-Schmidt S, T, we have $\operatorname{tr}(ST) = \operatorname{tr}(TS)$.

Proof.

(a) $\operatorname{tr} T^* = \sum_j \langle T^* e_j, e_j \rangle = \sum_j \overline{\langle T e_j, e_j \rangle} = \overline{\operatorname{tr} T}$.

(b) If T is unitary, then

$$\operatorname{tr}(ST) = \operatorname{tr}(T^*(TS)T) = (TST, T) = (TS, \mathrm{id}) = \operatorname{tr} TS$$

for the pairing is independent of the choice of orthonormal basis. The general case follows at once, since the desired identity is linear in T, and every bounded operator is a linear combination of unitary operators by Proposition 4.1.3.(iv).

□

12.4 Spectral Theorem for Normal Operators

12.4.1 Resolution of the Identity

Definition. Let $(X, \mathcal{A})$ be a measurable space and H a Hilbert space. A **resolution of the identity** on $\mathcal{A}$ is a map $E : \mathcal{A} \to \mathcal{B}(H)$ with the following properties:

(a) $E(\varnothing) = 0$, $E(\Omega) = \mathrm{id}_H$.

(b) Each $E(\omega)$ is an orthogonal projection.

(c) $E(\omega' \cap \omega'') = E(\omega')E(\omega'')$.

(d) If $\omega' \cap \omega'' = \varnothing$, then $E(\omega' \cup \omega'') = E(\omega') + E(\omega'')$.

(e) For every $v, w \in H$, the set function $E_{v,w} : \mathcal{A} \to \mathbb{C}$ defined by

$$E_{v,w}(\omega) = \langle E(\omega)v, w \rangle$$

is a complex measure on $(X, \mathcal{A})$.

When $(X, \mathcal{A})$ is the Borel σ-algebra on a locally compact Hausdorff space, we further require each $E_{v,w}$ to be regular.

We derive some formal properties of a resolution of the identity.

(i) Since each $E(\omega)$ is an orthogonal projection, for each $v \in H$, we have

$$E_{v,v}(\omega) = \langle E(\omega)v, v\rangle = \|E(\omega)v\|^2 .$$

Thus each $E_{v,v}$ is a positive measure on $(X, \mathcal{A})$ with total variation $\|E_{v,v}\| = E_{v,v}(X) = \|v\|^2$.

(ii) The complex measure $E_{v,w}$ has total variation less than $\|v\|\,\|w\|$. Indeed, let $(\omega_n)_{n=1}^N$ be a finite measurable partition of X. Write $|E_{v,w}(\omega_n)| = \alpha_n E_{v,w}(\omega_n)$ for some $|\alpha_n| = 1$. Then

$$\sum_{n=1}^{N} |E_{v,w}(\omega_n)| = \sum_{n=1}^{N} \alpha_n E_{v,w}(\omega_n) = \left\langle \sum_{n=1}^{N} \alpha_n E(\omega_n)v, w \right\rangle \leqslant \left\| \sum_{n=1}^{N} \alpha_n E(\omega_n)v \right\| \|w\|$$

The $E(\omega_n)v$ are pairwise orthogonal, so

$$\left\| \sum_{n=1}^{N} E(\omega_n)\alpha_n v \right\|^2 = \sum_{n=1}^{N} \|E(\omega_n)v\|^2 = \left\| \sum_{n=1}^{N} E(\omega_n)v \right\|^2 = \|E(X)v\|^2 = \|v\|^2$$

This proves the claim.

(iii) If $(\omega_n)_n$ is a sequence such that $E(\omega_n) = 0$ for all $n \geqslant 1$, then $E\left(\bigcup_{n\geqslant 1} \omega_n\right) = 0$. Indeed, if we put ω to be the union, then for each $x \in H$, we have

$$E_{x,x}(\omega) = \sum_{n=1}^{\infty} E_{x,x}(\omega_n) = \sum_{n=1}^{\infty} \langle E(\omega_n)x, x\rangle = 0.$$

Since $\|E(\omega)x\|^2 = E_{x,x}(\omega)$, we deduce $E(\omega) = 0$.

(iv) By (c), any two of the projections $E(\omega)$ commute with each other.

(v) By (a), (c), if $\omega' \cap \omega''$, then the ranges of $E(\omega')$ and $E(\omega'')$ are orthogonal.

(vi) By (b), we have $\overline{E_{v,w}(\omega)} = E_{w,v}(\omega)$.

(vii) By (d), the set function E is finitely additive.

It is natural to ask whether E is countably additive in the norm topology. Let $(\omega_n)_n$ be a sequence of disjoint sets in $\mathcal{A}$. Since $\|E(\omega_n)\|$ is either 0 or 1, partial sums of the series $\sum_{n=1}^{\infty} E(\omega_n)$ cannot form a Cauchy sequence unless $E(\omega_n) = 0$ for all but finitely many n.

However, E is in fact countably additive in strong operator topology. Let $(\omega_n)_n$ be as above and $v \in H$. By (e), we have

$$\sum_{n=1}^{\infty} \langle E(\omega_n)v, w\rangle = \langle E(\omega)v, w\rangle$$

for all $w \in H$, where ω denotes the union of the ω_n. Note that the vectors $E(\omega_n)v$ are pairwise orthogonal by (vi), so Proposition E.2.7 implies

$$\sum_{n=1}^{\infty} E(\omega_n)v = E(\omega)v$$

In other words, $\sum_{n=1}^{\infty} E(\omega_n) = E(\omega)$ in strong operator topology. In a fancier term, for each $v \in H$, the map $\omega \mapsto E(\omega)v$ is a countably additive H-valued measure on $(X, \mathcal{A})$.

Recall that $B = B(X, \mathcal{A}, \mathbb{C})$ be the set of all bounded $\mathcal{A}$-measurable functions on X; this is a unital commutative C^*-algebra with respect to the sup norm $\|\cdot\|_{\sup}$. Let $E : \mathcal{A} \to \mathcal{B}(H)$ be a resolution of the identity. Let $\{D_n\}_n$ be a countable collection of open discs which forms a basis for the topology of $\mathbb{C}$. For $f \in B$, let $V = V_f$ be the union of those D_n with $E(f^{-1}(D_n)) = 0$; then $E(f^{-1}(V)) = 0$ (by (iv)) and V is the largest open subspace of $\mathbb{C}$ with this property (use (c)).

For $f \in B$, the **essential image** $\operatorname{ess.im}(f)$ of f is the complement of V; this is the smallest closed subset of $\mathbb{C}$ that contains $f(p)$ for almost every $p \in X$ (with respect to the H-valued measure E). Explicitly,

$$\operatorname{ess.im}(f) = \{z \in \mathbb{C} \mid E(\{x \in X \mid |f(x) - z| < \varepsilon\}) \neq 0 \text{ for all } \varepsilon > 0\}$$

We say f is **essentially bounded** if its essential image is bounded in $\mathbb{C}$ (hence compact). In this case, we define the **essential supremum**

$$\|f\|_\infty := \sup_{\lambda \in \operatorname{ess.im}(f)} |\lambda| = \inf_{E(N)=0} \sup_{x \notin N} |f(x)|$$

Let $N := \{f \in B \mid \|f\|_\infty = 0\}$; then N is a *-ideal, and a similar argument to the proof of Theorem D.5.5 and (iv) shows N is closed. Then B/N is a Banach algebra, and we denote it by $L^\infty(E)$.

Lemma 12.4.1.

1. Every $f \in B$ is a uniform limit of simple functions.
2. The quotient norm for a coset $f + N$ is $\|f\|_\infty$.
3. The spectrum $\sigma_{L^\infty(E)}(f + N)$ is the essential image of f.

Proof.

1. This is due to boundedness.
2. Let $f_0 \in N$ and $N = f_0^{-1}(V_{f_0})$. Then $\|f + f_0\|_{\sup} \geqslant \sup_{x \notin N} |f(x) + f_0(x)|$. Since $f_0(x)$ lies in the essential image of f_0 for $x \notin N$, $\|f_0\|_\infty = 0$ implies $f_0(x) = 0$ for all $x \notin N$, whence,

$$\|f + f_0\|_{\sup} \geqslant \sup_{x \notin N} |f(x)| \geqslant \|f\|_\infty$$

The reverse estimate is given by

$$\|f\|_\infty = \inf_{E(N)=0} \sup_{x \notin N} |f(x)| = \inf_{E(N)=0} \left\| f + \|f\|_{X \setminus N} \mathbf{1}_N \right\|_{\sup} \geqslant \inf_{f_0 \in N} \|f + f_0\|_{\sup} = \|f + N\|_{\sup}.$$

3.

$\square$

The use of this lemma will be implicit. Say a subalgebra A of $\mathcal{B}(H)$ **normal** if it is commutative and self-adjoint.

Theorem 12.4.2. If E is a resolution of the identity as above, then there exists an isometric *-homomorphism

$$\Psi : L^\infty(E) \longrightarrow \mathcal{B}(H)$$

onto a closed unital normal commutative subalgebra A of $\mathcal{B}(H)$ given by the formula

$$\langle \Psi(f)v, w\rangle = \int_X f dE_{v,w}$$

for all $v, w \in H$ and $f \in B$. Symbolically we will write

$$\Psi(f) = \int_X f dE.$$

Moreover,

1. $\|\Psi(f)v\|^2 = \int_X |f|^2 dE_{v,v}$.
2. $C_{\mathcal{B}(H)}(\{E(\omega) \mid \omega \in \mathcal{A}\}) = C_{\mathcal{B}(H)}(\{\Psi(f) \mid f \in L^\infty(E)\})$.

In particular, we have

$$\|f\|_\infty^2 = \sup_{\|v\| \leqslant 1} \int_X |f|^2 dE_{v,v}$$

Proof. For v, w and $f \in B$, define $B_f(v, w) := \int_X f dE_{v,w}$ (integration against a complex measure). Then by (iii) $|B_f(v, w)| \leqslant \|f\|_\infty \|E_{v,w}\| \leqslant \|f\|_\infty \|v\| \|w\|$, so by Riesz's Representation theorem we find a unique operator $\Psi(f) \in \mathcal{B}(H)$ such that

$$\langle \Psi(f)v, w\rangle = \int_X f dE_{v,w}$$

for all $v, w \in H$. The estimate and uniqueness at once show that $\Psi : B \to \mathcal{B}(H)$ is a continuous linear map.

If s is a simple function, say $s = \sum_{i=1}^n \alpha_i \mathbf{1}_{\omega_i}$, where $\{\omega_1, \ldots, \omega_n\}$ is a measurable partition of X, then

$$B_s(v, w) = \sum_{i=1}^n \alpha_i E_{v,w}(\omega_i) = \left\langle \sum_{i=1}^n \alpha_i E(\omega_i)v, w \right\rangle$$

The uniqueness shows $\Psi(s) = \sum_{i=1}^n \alpha_i E(\omega_i)$. Now

$$\Psi(s)^* = \sum_{i=1}^n \overline{\alpha_i} E(\omega_i) = \Psi(\overline{s}).$$

Since every function in B is a uniform limit of simple functions, this shows $\Psi(f)^* = \Psi(\overline{f})$. If $t = \sum_{j=1}^m \beta_j \mathbf{1}_{\omega'_j}$ is another simple function, where $\{\omega'_j\}$ is a measure partition of X, then

$$st = \sum_{i,j} \alpha_i \beta_j \mathbf{1}_{\omega_i \cap \omega'_j}$$

On the other hand, by (c)

$$\Psi(s)\Psi(t) = \sum_{i,j} \alpha_i \beta_j E(\omega_i) E(\omega'_j) = \sum_{i,j} \alpha_i \beta_j E(\omega_i \cap \omega'_j).$$

This show $\Psi(st) = \Psi(s)\Psi(t)$, thus $\Psi(fg) = \Psi(f)\Psi(g)$ for all $f, g \in B$. This proves Ψ is a continuous *-homomorphism. Also,

$$\|\Psi(s)v\|^2 = \langle \Psi(s)^*\Psi(s)v, v\rangle = \langle \Psi(|s|^2)v, v\rangle = \int_X |s|^2 dE_{v,v},$$

so 1. holds and $\|\Psi(s)v\| \leqslant \|s\|_\infty \|v\|$ by (ii). On the other hand, if v lies in the image of $E(\omega_i)$ for some i, then by (vi)

$$\Psi(s)v = \alpha_i E(\omega_i)v = \alpha_i v.$$

If we choose i so that $|\alpha_i| = \|s\|_\infty$, then together with a previous estimate we obtain $\|s\|_\infty = \|\Psi(s)\|$. This shows Ψ is isometric. Finally, Q commutes with every $E(\omega)$ if and only Q commutes with every $\Psi(s)$ with s simple. A limit process shows 2. holds. $\square$

12.4.2 Spectral theorem

In the following let H be a Hilbert space. Recall from Lemma 3.3.2 that if A is a closed unital *-subalgebra of $\mathcal{B}(H)$, then $\sigma_A(T) = \sigma_{\mathcal{B}(H)}(T)$ for all $T \in A$.

Theorem 12.4.3. Let A be closed unital normal commutative subalgebra of $\mathcal{B}(H)$.

(I) There exists a unique resolution of the identity E on the Borel subsets of the structure space $\Delta = \Delta_A$ which satisfies

$$T = \int_\Delta \widehat{T} dE$$

for every $T \in A$, where $\widehat{T}$ is the Gelfand transform.

(II) The inverse of the Gelfand transform extends to an isometric *-homomorphism $\Phi : L^\infty(E) \to \mathcal{B}(H)$ onto the closed subalgebra B of $\mathcal{B}(H)$ containing A, given by

$$\Phi f = \int_\Delta f dE.$$

(III) B is the norm closure of the set of all finite linear combinations of the projections $E(\omega)$.

(IV) If $\omega \subseteq \Delta$ is nonempty and open, then $E(\omega) \neq 0$.

(V) $C_{\mathcal{B}(H)}A = C_{\mathcal{B}(H)}(\{E(\omega) \mid \omega \subseteq \Delta : \text{Borel}\})$

Proof. Recall by Gelfand-Naimark Theorem, Δ is compact and the Gelfand transform

$$\begin{aligned} A &\longrightarrow C(\Delta) \\ T &\longmapsto \widehat{T} \end{aligned}$$

is an isometric *-isomorphism. We first prove the uniqueness part in 1. Suppose E is a resolution of identity on Δ satisfying

$$\langle Tv, w\rangle = \int_\Delta \widehat{T} dE_{v,w}$$

for all $v, w \in H$, $T \in A$. Since $\widehat{T}$ runs over $C(\Delta)$ and $E_{v,w}$ is complex regular, so the uniqueness part of Riesz's representation theorem shows that each $E_{v,w}$ is uniquely determined by the above identity. Since $E_{v,w}(\omega) = \langle E(\omega)v, w\rangle$, each projection $E(\omega)$ is also uniquely determined.

We now consider the existence. For any $v, w \in H$, by Gelfand-Naimark Theorem, the functional

$$\begin{aligned} C(\Delta) &\longrightarrow \mathbb{C} \\ \widehat{T} &\longmapsto \langle Tv, w\rangle \end{aligned}$$

is a bounded linear function on $C(\Delta)$ of norm $\leqslant \|v\|\,\|w\|$. So by Riesz's representation theorem we can find a unique regular complex Borel measure $\mu_{v,w}$ on Δ such that

$$\langle Tv, w\rangle = \int_\Delta \widehat{T} d\mu_{v,w}$$

holds for all $v, w \in H$ and $T \in A$. The right hand side still defines a bounded sesquilinear functional on H if $\widehat{T}$ is replaced by bounded Borel functions f. To each such f it corresponds to a unique operator $\Phi f \in \mathcal{B}(H)$ (by Riesz's representation theorem) such that

$$\langle \Phi f v, w\rangle = \int_\Delta f d\mu_{v,w}.$$

Comparing, we see $\Phi\widehat{T} = T$, so it extends the inverse of Gelfand transform. The uniqueness part shows that Φ is linear. Note that T is self-adjoint if and only if $\widehat{T}$ is real-valued, and for such T, we have

$$\int_\Delta \widehat{T} d\mu_{v,w} = \langle Tv, w\rangle = \langle v, Tw\rangle = \overline{\langle Tw, v\rangle} = \overline{\int_\Delta \widehat{T} d\mu_{w,v}}$$

implying $\mu_{v,w} = \overline{\mu_{w,v}}$, thus

$$\langle \Phi\overline{f} v, w\rangle = \int_\Delta \overline{f} d\mu_{v,w} = \overline{\int_\Delta f d\mu_{w,v}} = \overline{\langle \Phi f w, v\rangle} = \langle v, \Phi f w\rangle$$

for all $v, w \in H$. This shows $\Phi\overline{f} = (\Phi f)^*$. Next we prove $\Phi(fg) = (\Phi f)(\Phi g)$ for all bounded Borel f, g on Δ. For $S, T \in A$, we have $\widehat{ST} = \widehat{S}\widehat{T}$, so

$$\int_\Delta \widehat{S}\widehat{T} d\mu_{v,w} = \langle STv, w\rangle = \int_\Delta \widehat{S} d\mu_{Tv,w}.$$

This holds for all $\widehat{S} \in C(\Delta)$, so it remains true if $\widehat{S}$ is replaced by any bounded Borel f. Thus

$$\int_\Delta f\widehat{T} d\mu_{v,w} = \int_\Delta f d\mu_{Tv,w} = \langle \Phi f(Tv), w\rangle = \langle Tv, u\rangle = \int_\Delta \widehat{T} d_{v,u}$$

where we put $u = (\Phi f)^* w$. Again, this remains true if $\widehat{T}$ is replaced by any g, giving

$$\langle \Phi(fg)v, w\rangle = \int_\Delta fg d\mu_{v,w} = \int_\Delta g d\mu_{v,u} = \langle \Phi g v, u\rangle = \langle \Phi g v, (\Phi f)^* w\rangle = \langle \Phi(f)\Phi(g)v, w\rangle.$$

Now we define E: for $\omega \subseteq \Delta$ Borel, define $E(\omega) := \Phi(\mathbf{1}_\omega)$.

(c) Since Φ is multiplicative, this shows $E(\omega' \cap \omega'') = E(\omega')E(\omega'')$.

(b) Since Φf is self-adjoint when f is real, this shows $E(\omega)$ is self-adjoint, and by (c) (with $\omega' = \omega'' = \omega$) we see $E(\omega)$ is a projection.

(a) $E(\varnothing) = \Phi(0) = 0$, and $E(\Delta) = \Phi(\mathrm{id}_\Delta) = \mathrm{id}_H$.

(d) The integral representation $\langle \Phi f v, w\rangle = \int_\Delta f d\mu_{x,y}$ show that E is finitely additive.

(e) For $v, w \in H$, we have

$$E_{v,w}(\omega) = \alpha E(\omega)v, w\rangle = \int_\Delta \mathbf{1}_\omega d\mu_{v,w} = \mu_{v,w}(\omega)$$

so $E_{v,w}$ is a complex measure, and thus $\Phi f = \int_\Delta f dE$.

To finish (II) we must show Φ is an isometry, but this follows from Theorem 12.4.2. (III) is clear for every function in $L^\infty(E)$ is a uniform limit of simple functions. For (IV), let ω be open with $E(\omega) = 0$. Let $T \in A$ with $\operatorname{supp} \widehat{T} \subseteq \omega$. Then by (I)

$$T = \int_\Delta \widehat{T} dE = \int_\omega \widehat{T} dE = 0$$

whence $\widehat{T} = 0$. From Urysohn's Lemma we see $\omega = \varnothing$. This proves (IV). To prove (V), choose $S \in \mathcal{B}(H)$, $v, w \in H$, and put $u = S^* w$. For any $T \in A$ and Borel $\omega \subseteq \Delta$, we have

$$\begin{aligned}
\langle STv, w\rangle &= \langle Tv, u\rangle = \int_\Delta \widehat{T} dE_{v,u} \\
\langle Tsv, w\rangle &= \int_\Delta \widehat{T} dE_{Sv,w} \\
\langle SE(\omega)v, w\rangle &= \langle E(\omega)v, u\rangle = E_{v,u}(\omega) \\
\langle E(\omega)Sv, w\rangle &= E_{Sv,w}(\omega).
\end{aligned}$$

Thus $S \in C_{\mathcal{B}(H)}(A)$ if and only if $E_{Sv,w} = E_{v,u}$, if and only if $SE(\omega) = E(\omega)S$ for all Borel ω. □

Now we specialize to a single normal operator T.

Theorem 12.4.4. If $T \in \mathcal{B}(H)$ is normal, then there exists a unique resolution of the identity E on the Borel subsets of $\sigma(T) = \sigma_{\mathcal{B}(H)}(T)$ which satisfies

$$T = \int_{\sigma(T)} \lambda dE(\lambda).$$

Furthermore, every projection $E(\omega)$ commutes with every $S \in \mathcal{B}(H)$ that commutes with T.

We refer to this E as the **spectral decomposition of** T. Sometimes it is convenient to extend E to all Borel sets in $\mathbb{C}$, by setting $E(\omega) = 0$ if $\omega \cap \sigma(T) = \varnothing$.

Proof. Let $A = C^*(\mathrm{id}_H, T, T^*) \subseteq \mathcal{B}(H)$ be the smallest unital commutative C^*-algebra generated by T in $\mathcal{B}(H)$. Now Theorem 12.4.3 applies. In the proof of continuous functional calculus we see Δ_A can be identified with $\sigma(T)$ via the Gelfand transform $\widehat{T}$; explicitly, we have $\widehat{T}(\lambda) = \lambda$ for all $\lambda \in \sigma(T)$. The proves the existence of E.

On the other hand, if E exists so that $T = \int_{\sigma(T)} \lambda dE(\lambda)$ holds, then Theorem 12.4.2 implies

$$p(T, T^*) = \int_{\sigma(T)} p(\lambda, \overline{\lambda}) dE(\lambda)$$

holds for all $p \in \mathbb{C}[X, Y]$. The uniqueness then follows from Stone-Weierstrass (and argue as in the first paragraph of the proof of Theorem 12.4.2).

For the last statement, if $ST = TS$, then $ST^* = T^*S$ by Corollary 3.3.31.3; hence S commutes with A. By Theorem 12.4.3.(V), this shows $SE(\omega) = E(\omega)S$ for all Borel $\omega \subseteq \sigma(T)$. □

This theorem is clearly an extension of the continuous functional calculus of normal operators. Let T be a normal operator and A is closed normal subalgebra generated by T. Let E be the spectral decomposition of T. Then Theorem 12.4.3 gives an isometric *-homomorphism

$$\begin{aligned}
\Psi : L^\infty(E) &\longrightarrow \mathcal{B}(H) \\
f &\longmapsto \Psi f
\end{aligned}$$

with

$$\Psi f = \int_{\sigma(T)} f dE.$$

Let us write $f(T) := \Psi f$ for any bounded Borel function f. Then $f \mapsto f(T)$ satisfies

1. $\overline{f}(T) = f(T)^*$,
2. $\|f(T)\| = \|f\|_\infty \leqslant \sup_{\lambda \in \sigma(T)} |f(\lambda)|$ with equality when f is continuous.

Thus $f \mapsto f(T)$ gives an isometric *-homomorphism from $C(\sigma(T))$ to $\mathcal{B}(H)$ (with image A) satisfying

$$\|f(T)v\|^2 = \int_{\sigma(T)} |f|^2 dE_{x,x}.$$

The image of Borel functional calculus is the closure of finite linear combinations of projections $E(\omega)$ with $\omega \subseteq \sigma(T)$ Borel.

Corollary 12.4.4.1. Let $T \in \mathcal{B}(H)$ be normal. Then there exists a subset $X \subseteq H$, a Radon measure on $\sigma(T) \times X$ (topologize X discretely) and an isometric isomorphism $U : H \to L^2(\sigma(T) \times X, \mu)$ such that

$$m_f = U f(T) U^{-1}$$

for all $f \in C(\sigma(T))$, where $m_f \in \mathcal{B}(L^2(\sigma(T) \times X, \mu))$ is given by $g \mapsto [(t, x) \mapsto g(t, x) f(t)]$. If H is separable, then X is countable and μ can be made finite.

Proof. Put $A = \{f(T) \mid f \in C(\sigma(T))\} \subseteq \mathcal{B}(H)$, the image of the continuous functional calculus of T. Let E be the unique resolution of the identity on $\sigma(T)$ given in Theorem 12.4.4. For $x \in H$, put $E_x = E_{x,x}$. Then

$$\langle f(T)x, x\rangle_H = \int_{\sigma(T)} f dE_x.$$

Since $f \mapsto f(T)$ is a $*$-homomorphism, it follows that

$$\|f(T)x\|^2 = \langle |f|^2(T)x, x\rangle_H = \int_{\sigma(T)} |f|^2 dE_x = \|f\|^2_{L^2(\sigma(T), E_x)}.$$

Put $H_x := \overline{Ax}^{\text{norm}} \subseteq H$. Then $f \mapsto f(T)x$ extends to an isometric isomorphism $U_x : L^2(\sigma(T), E_x) \cong H_x$. For $f, g \in C(\sigma(T))$, we have

$$U_x^{-1} f(T) U_x g = U_x^{-1} f(T) g(T) x = U_x^{-1}(fg)(T)x = fg.$$

Notice for each $x \in H$, the closed subspace H_x is A-invariant. Since A is normal, it follows that its orthogonal complement $H_x^\perp$ is also A-invariant. By a Zorn's lemma argument, we can find a subset $X \subseteq H$ such that

$$H = \widehat{\bigoplus_{x \in X}} H_x.$$

Then the isometries U_x, $x \in X$ give rise to an isometry $U : H \to \widehat{\bigoplus_{x \in X}} L^2(\sigma(T), E_x)$.

Equip X with discrete topology and equip $\sigma(T) \times X$ with product topology. Let μ be the Radon measure on $\sigma(T) \times X$ induced, via Riesz's representation theorem, by the functional

$$C_c(\sigma(T) \times X) \ni f \mapsto \sum_{x \in X} \int_{\sigma(T)} f|_{\sigma(T) \times \{x\}} dE_x$$

Then, as Hilbert spaces,

$$\widehat{\bigoplus_{x \in X}} L^2(\sigma(T), E_x) \cong L^2(\sigma(T) \times X, \mu)$$

The second last assertion follow at once from what we've established. For the last assertion, note that X is necessarily countable when H is separable, as $L^2(\sigma(T) \times X, \mu)$ is cannot be separable unless X is countable. To see μ can be made finite, enumerate $X = \{x_i\}_{i \geqslant 1}$ and replace x_i by $2^{-i} \frac{x_i}{\|x_i\|}$ in the above construction. Then

$$\sum_{x \in X} \int_{\sigma(T)} dE_x = \sum_{i \geqslant 1} \int_{\sigma(E)} dE_X = \sum_{i \geqslant 1} \|x_i\| = \sum_{i \geqslant 1} 2^{-i} < \infty$$

as claimed. □

Corollary 12.4.4.2. Let T be a densely defined closed normal operator on H. Then there exists a subset $X \subseteq H$, a compact subset $\Delta \subseteq \overline{B_1}(0) \subseteq \mathbb{C}$, a Radon measure μ on $\Delta \times X$ (topologize X discretely) and an isometric isomorphism $U : H \to L^2(\Delta \times X, \mu)$ such that

$$m_f = UTU^{-1}$$

where $f : z \mapsto \dfrac{z}{(1 - |z|^2)^{\frac{1}{2}}}$, and m_f is an operator on $L^2(\Delta \times X, \mu)$ given by $g \mapsto [(t, x) \mapsto g(t, x) f(t)]$. If T is self-adjoint, then $\Delta \subseteq \mathbb{R}$. If H is separable, then X is countable and μ can be made finite.

Proof. Applying Corollary 12.4.4.1 to the bounded operator Z_T defined near Lemma E.5.9, there exists a subset $X \subseteq H$, a Radon measure μ on $\sigma(Z_T) \times X$ and an isometric isomorphism $U : H \to L^2(\sigma(Z_T) \times X, \mu)$ such that

$$m_h = Uh(Z_T)U^{-1}.$$

for all $h \in C(\sigma(Z_T))$. Take $\Delta = \sigma(Z_T)$. Since $\|Z_T\|_{\text{op}} \leqslant 1$, we have $\Delta \subseteq \overline{B_1}(0)$. The last assertion follows from the last assertion in Corollary 12.4.4.1. The second last follows from Lemma E.5.9.(iv).

Next we claim $\mu(S^1 \times X) = 0$. It suffices to show $\mu(S^1 \times \{x\}) = 0$ for each $x \in X$. To this end, by taking $h(z) = 1 - |z|^2$, we see

$$m_{1-|z|^2} = UI_T U^{-1}$$

where I_T is defined near Lemma E.5.9. Since I_T is injective, so is $m_{1-|z|^2}$ as an operator on $L^2(\sigma(Z_T) \times \{x\}, \mu)$. This implies $\{1 - |z|^2 = 0\} \times \{x\} = S^1 \times \{x\}$ is null, as we claimed (hit $m_{1-|z|^2}$ by $\mathbf{1}_{\{1-|z|^2=0\} \times \{x\}}$).

Since $f \in C(\Delta \backslash S^1)$ and $\mu(S^1 \times X) = 0$, the operator m_f given by $g \mapsto [(t, x) \mapsto g(t, x) f(t)]$ is a densely defined closed operator on $L^2(\Delta \times X, \mu)$. We claim $m_f = UTU^{-1}$. For this put $S = U^{-1} m_f U$; then S is a densely defined closed normal operator on H and

$$Z_S = U^{-1} Z_{m_f} U = U^{-1} m_z U = Z_T$$

so that $S = T$ by Lemma E.5.9.

□

Chapter 13

Representation of C^*-algebras

Definition. Let $\mathcal{A}$ be a C^*-algebra.

(i) A ***-representation** of $\mathcal{A}$ is a *-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(H)$ to the C^*-algebra of bounded linear operators on a Hilbert space H.

(ii) A *-representation $\pi : \mathcal{A} \to \mathcal{B}(H)$ is **non-degenerate** if $\pi(\mathcal{A})H$ is dense in H.

(ii) is not a trivial notion as $\mathcal{A}$ is not unital in general.

We collect some results for C^*-algebras specialized in this context.

Lemma 13.0.1. Let $\pi : \mathcal{A} \to \mathcal{B}(H)$ be a *-representation of a C^*-algebra $\mathcal{A}$. Then the following are true.

(i) $\|\pi(a)\|_{\mathrm{op}} \leqslant \|a\|$ for all $a \in \mathcal{A}$. In particular, π is continuous in norm topology.

(ii) π is injective if and only if π is isometric. In particular, π has closed range, and $\pi(\mathcal{A})$ is a C^*-algebra of $\mathcal{B}(H)$.

Proof. Corollary 4.4.2.1. ☐

13.1 Positive Linear Functionals

Definition.

(i) A $\mathbb{C}$-linear map $\phi : \mathcal{A} \to \mathcal{B}$ between C^*-algebras is called **positive** if $\phi(\mathcal{A}_+) \subseteq \mathcal{B}_+$.

(ii) A positive linear map $\phi : \mathcal{A} \to \mathbb{C}$ is called a **positive linear functional**.

(iii) A positive linear functional ϕ is called a **state** if $\|\phi\| = 1$.

(iv) A positive linear functional ϕ is called a **trace** if $\phi(ab) = \phi(ba)$ for all $a, b \in \mathcal{A}$

(v) A **tracial state** is a state that is also a trace.

(vi) A state ϕ is **faithful** if $\phi(a^*a) = 0$ implies $a = 0$.

The set of all states (reps. tracial states) on $\mathcal{A}$ is denoted by $S(\mathcal{A})$ (resp. $T(\mathcal{A})$).

Lemma 13.1.1. Any positive linear functional on a C^*-algebra $\mathcal{A}$ is bounded.

Proof. Let $\phi : \mathcal{A} \to \mathbb{C}$ by a positive linear functional. First note that if $M := \sup_{a \in \mathcal{A}_+,\, \|a\|=1} \phi(a) < \infty$, then ϕ is bounded. Indeed, for an element $x \in \mathcal{A}$, write $x = \operatorname{Re} x + i \operatorname{Im} x$; then $\operatorname{Re} \phi(x) = \phi(\operatorname{Re} x)$ and $\operatorname{Im} \phi(x) = \phi(\operatorname{Im} x)$ since $\phi(\mathcal{A}_{sa}) \subseteq \mathbb{R}$. So to show ϕ is bounded it suffices to show ϕ is bounded on $\mathcal{A}_{sa}$. But for $x \in \mathcal{A}_{sa}$, we can write $x = x^+ - x^-$ with $x^\pm$ positive and $\|x^\pm\| \leqslant \|x\|$; the inequality holds since continuous functional calculus is isometric. Then $\|\phi(x)\| = \|\phi(x^+) - \phi(x^-)\| \leqslant M \|x^+\| + M \|X^-\| \leqslant 2M \|x\|$.

Now suppose for contradiction that ϕ is unbounded. The previous paragraph implies $\sup_{a \in \mathcal{A}_+,\, \|a\|=1} |\phi(a)| = \infty$. In particular, we can find a sequence $(a_n)_{n \in \mathbb{N}}$ with $a_n \in \mathcal{A}_+$ and $\|a_n\| = 1$ such that $\phi(a_n) \geqslant 2^n$. But if we put $a := \sum_{n=1}^{\infty} 2^{-n} a_n \in \mathcal{A}_+$ (Proposition 4.2.1), then for each $N \in \mathbb{N}$ we have $a - \sum_{n=1}^{N-1} 2^{-n} a_n \geqslant 0$, so

$$\phi(a) \geqslant \phi\left(\sum_{n=1}^{N-1} 2^{-n} a_n\right) = \sum_{n=1}^{N-1} 2^{-n} \phi(a_n) \geqslant N - 1$$

a contradiction. □

Lemma 13.1.2. Let $\phi : \mathcal{A} \to \mathbb{C}$ be any linear map such that $f(\mathcal{A}_+) \subseteq \mathbb{R}$. Then $f(x^*) = \overline{f(x)}$ for all $x \in \mathcal{A}$.

Proof. The claim is true obvious for $x \in \mathcal{A}$. Now the general case follows from Proposition 4.1.3.(iii). □

Proposition 13.1.3. Let $\mathcal{A}$ be a C^*-algebra and $\phi : \mathcal{A} \to \mathbb{C}$ a linear functional. Then the pairing $(\,,)_\phi : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ defined by

$$(a, b)_\phi := \phi(ab^*)$$

is a nonnegative Hermitian form on $\mathcal{A}$ if and only if ϕ is positive.

Proof. Let ϕ be a linear functional. To say $(\,,)_\phi$ is nonnegative Hermitian, it is equivalent to the following conditions.

(i) $(ra + b, c)_\phi = r(a, c)_\phi + (b, c)_\phi$ for all $a, b, c \in \mathcal{A}$ and $r \in \mathbb{C}$.

(ii) $\overline{(a, b)_\phi} = (b, a)_\phi$ for all $a, b \in \mathcal{A}$.

(iii) $(a, a)_\phi \geqslant 0$ for all $a \in \mathcal{A}$.

Since ϕ is linear, (i) always holds. By Proposition 4.2.2, (iii) holds if and only if ϕ is positive. When ϕ is positive, (ii) holds automatically by Proposition 4.1.3. □

Corollary 13.1.3.1. Let $\mathcal{A}$ be a C^*-algebra and $\phi : \mathcal{A} \to \mathbb{C}$ a positive linear functional. Then

$$|\phi(ab^*)|^2 \leqslant \phi(aa^*)\phi(bb^*)$$

for all $a, b \in \mathcal{A}$.

Proof. By the previous proposition, the statement is reduced to a linear algebra lemma.

Lemma 13.1.4. Let V be a complex vector space and $f : V \times V \to \mathbb{C}$ a nonnegative Hermitian form. Then for $v, w \in V$, we have $|f(v, w)|^2 \leqslant f(v, v)f(w, w)$.

Proof. We may assume $f(v,w) \neq 0$. Since f is nonnegative, for all $\lambda \in \mathbb{C}$

$$0 \leqslant f(\lambda v + w, \lambda v + w) = |\lambda|^2 f(v,v) + 2\,\mathrm{Re}(\lambda f(v,w)) + f(w,w)$$

In particular, this holds for $\lambda = t\dfrac{|f(v,w)|}{f(v,w)}$ for any $t \in \mathbb{R}$, so that (note that $|\lambda| = |t|$)

$$t^2 f(v,v) + 2t|f(v,w)| + f(w,w) \geqslant 0$$

If $f(v,v) = 0$, then $2t|f(v,w)| + f(w,w) \geqslant 0$ for all $t \in \mathbb{R}$, which is impossible since we are assuming $f(v,w) \neq 0$. Therefore $f(v,v) \neq 0$, whence

$$4|f(v,w)|^2 - 4f(v,v)f(w,w) \leqslant 0$$

as wanted. □

It suffices to take $f = (\,,)_\phi$ in the lemma. □

Corollary 13.1.4.1. Let $\mathcal{A}$ be a C^*-algebra and $\phi : \mathcal{A} \to \mathbb{C}$ a positive linear functional. Then

$$|\phi(a)|^2 \leqslant \|\phi\|_{\mathrm{op}}\, \phi(a^*a)$$

for all $a \in \mathcal{A}$. Moreover,

$$\lim_{\lambda} \phi(u_\lambda) = \|\phi\|_{\mathrm{op}}$$

where $(u_\lambda)_\lambda$ is an approximate identity in Theorem 4.3.2. In particular, $\phi(1_{\mathcal{A}}) = \phi$ when $\mathcal{A}$ is unital.

Proof. Let $(u_\lambda)_\lambda$ be the approximate identity in Theorem 4.3.2. Since ϕ is positive and u_λ is increase of norm less than 1, we see $(\phi(u_\lambda))_\lambda$ is an increasing net bounded above by $\|\phi\|_{\mathrm{op}}$. Let r denote the limit.

Then for $a \in \mathcal{A}\backslash\{0\}$, Cauchy-Schwartz

$$|\phi(au_\lambda)|^2 \leqslant \phi(aa^*)\phi(u_\lambda^* u_\lambda) \leqslant \|\phi\|_{\mathrm{op}}\, \|a^*a\|\, \phi(u_\lambda) \leqslant \|\phi\|_{\mathrm{op}}\, \|a\|^2\, r$$

Here we use $u_\lambda^* u_\lambda = u_\lambda^2 \leqslant u_\lambda$, which follows from continuous functional calculus and $\|u_\lambda\| \leqslant 1$. Taking $\lim\limits_{\lambda}$ we get

$$|\phi(a)|^2 \leqslant \|\phi\|_{\mathrm{op}}\, \|a\|^2\, r$$

which implies $\|\phi\|_{\mathrm{op}}^2 \leqslant \|\phi\|_{\mathrm{op}}\, r$, or $\|\phi\|_{\mathrm{op}} \leqslant r$. This proves $\|\phi\|_{\mathrm{op}} = r$. □

Lemma 13.1.5. Let $\mathcal{A}$ be a C^*-algebra and let $\mathcal{A}^e$ be its unitization. If $\phi : \mathcal{A} \to \mathbb{C}$ is a positive linear functional, then $\phi^e : \mathcal{A}^e \to \mathbb{C}$ given by

$$\phi^e(a,\lambda) = \phi(a) + \lambda\, \|\phi\|_{\mathrm{op}}$$

is positive. Moreover, $\|\phi^e\|_{\mathrm{op}} = \|\phi\|_{\mathrm{op}}$.

Proof. There is no harm in assuming $\|\phi\|_{\mathrm{op}} = 1$. Compute

$$\begin{aligned}
\phi^e(((a,\lambda)^*(a,\lambda)) = \phi^e(a^*a + \lambda a^* + \overline{\lambda} a, \lambda\overline{\lambda}) &= \phi(a^*a) + \lambda\phi(a*) + \overline{\lambda}\phi(a) + |\lambda|^2. \\
&= \phi(a^*a) + 2\Re(\overline{\lambda}\phi(a)) + |\lambda|^2 \\
&\geqslant |\phi(a)|^2 + 2\Re(\overline{\lambda}\phi(a)) + |\lambda|^2 \\
&= |\phi(a) + \overline{\lambda}|^2 > 0.
\end{aligned}$$

For the last assertion, by Corollary 13.1.4.1 we have

$$\|\phi^e\|_{\mathrm{op}} = \phi^e(1_{\mathcal{A}}) = \|\phi\|_{\mathrm{op}}$$

by definition. □

13.2 The Gelfand-Naimark-Segal construction

Let $\mathcal{A}$ be a C^*-algebra throughout this section.

Theorem 13.2.1. Let $\phi : \mathcal{A} \to \mathbb{C}$ be a positive linear functional. Then there exists a *-representation (π_ϕ, H_ϕ) of $\mathcal{A}$ and $\xi \in H_\phi$ such that $\pi(\mathcal{A})\xi \subseteq H_\phi$ is dense and

$$\phi(x) = \langle \pi(x)\xi, \xi \rangle$$

for all $x \in \mathcal{A}$. If ϕ is a state, then $\|\xi\| = 1$. Moreover, if (σ, V) is another *-representation of $\mathcal{A}$ and if $v \in V$ is such that $\sigma(\mathcal{A})v \subseteq V$ is dense and $\phi(x) = \langle \sigma(x)v, v \rangle$ for all $x \in \mathcal{A}$, then (σ, H) and (π_ϕ, H_ϕ) are unitarily equivalent.

Proof. Let $\phi : \mathcal{A} \to \mathbb{C}$ be a positive linear functional. Put

$$N_\phi := \{a \in \mathcal{A} \mid \phi(a^*a) = 0\}$$

which is a closed left ideal in $\mathcal{A}$. Indeed, closedness is clear, and if $x \in N_\phi$ and $a \in \mathcal{A}$, then by Cauchy-Schwarz, we have

$$\phi((ax)^*(ax))^2 = \phi(x^*(a^*ax))^2 \leqslant \phi(x^*x)\phi(x^*a^*aa^*ax) = 0$$

Now consider the vector space quotient $\mathcal{A}/N_\phi$. Define a pairing on the quotient:

$$\begin{aligned} \langle , \rangle_\phi : \mathcal{A}/N_\phi \times \mathcal{A}/N_\phi &\longrightarrow \mathbb{C} \\ (a + N_\phi, b + N_\phi) &\longmapsto \phi(b^*a) \end{aligned}$$

By Proposition 13.1.3, we see this pairing defines an inner product on $\mathcal{A}/N_\phi$. Let H_ϕ denote the completion of $\mathcal{A}/N_\phi$ with respect to this inner product.

For $a \in \mathcal{A}$, define a linear map $\pi_\phi(a) \in \operatorname{End}_{\mathbb{C}} \mathcal{A}/N_\phi$ by

$$\pi_\phi(a)(b + N_\phi) = ab + N_\phi$$

(this is well-defined for N_ϕ is a left ideal). Let $b + N_\phi \in \mathcal{A}/N_\phi$ be such that $\phi(b^*b) \leqslant 1$. Then

$$\|\pi_\phi(a)b\|^2 = \langle ab + N_\phi, ab + N_\phi \rangle_\phi = \phi(b^*a^*ab) \leqslant \phi(b^* \|a\|^2 b) \leqslant \|a\|^2$$

by Proposition 4.2.3.(i) (note that $a^*a \leqslant \|a^*a\| = \|a\|^2$). This shows $\pi_\phi(a)$ is bounded, whence extends to an operator on H_ϕ. Finally, $\pi_\phi : \mathcal{A} \to \mathcal{B}(H_\phi)$ is a $*$-homomorphism. Indeed,

$$\langle b + N_\phi, \pi_\phi(a^*)(c + N_\phi) \rangle_\phi = \phi((a^*c)^*b) = \phi(c^*ab) = \langle \pi_\phi(a)(b + N_\phi), c + N_\phi \rangle_\phi.$$

If $\mathcal{A}$ is unital, then take $\xi = 1 + N_\phi$. Indeed,

$$\phi(x) = \phi(1^*x) = \langle x + N_\phi, 1 + N_\phi \rangle_\phi = \langle \pi_\phi(x)\xi, \xi \rangle_\phi.$$

If $\mathcal{A}$ is not unital, we argument as follows. Let $(u_\lambda)_\lambda$ be the approximate identity in Theorem 4.3.2. For $\lambda < \lambda'$, we have

$$\|(u_{\lambda'} + N_\phi) - (u_\lambda + N_\phi)\|_\phi^2 = \phi((u_{\lambda'} - u_\lambda)^*(u_{\lambda'} - u_\lambda)) \leqslant \phi(u_{\lambda'} - u_\lambda).$$

Since $(\phi(u_\lambda))_\lambda$ is convergent in $\mathbb{R}_{\geqslant 0}$, we see $(u_\lambda + N_\phi)_\lambda$ is Cauchy. Let ξ denote the limit in H_ϕ. For $a \in \mathcal{A}$,

$$\pi(a)\xi = \lim_\lambda \pi(a)(u_\lambda + N_\phi) = \lim_\lambda au_\lambda + N_\phi = a + N_\phi.$$

This proves $\pi(\mathcal{A})\xi$ is dense. Moreover, for $a \in \mathcal{A}$,

$$\langle \pi(a^*a)\xi, \xi\rangle_\phi = \langle \pi(a)\xi, \pi(a)\xi\rangle_\phi = \langle a + N_\phi, a + N_\phi\rangle_\phi = \phi(a^*a).$$

By Proposition 4.1.3.(iii), this proves $\phi(a) = \langle \phi(a)\xi, \xi\rangle$ for all $a \in \mathcal{A}$.

Suppose $\|\xi\| = 1$ when $\|\phi\|_{\text{op}} = 1$, consider its extension ϕ^e to the unitization $\mathcal{A}^e$. Clearly $N_{\phi^e} \cap \mathcal{A} = N_\phi$ so we obtain an isometric embedding $H_\phi \subseteq H_{\phi^e}$. The operator $\pi_{\phi^e}(1)$ is the identity on H_{ϕ^e}. In particular, $\pi_{\phi^e}(1)$ acts as identity on H_ϕ. This shows

$$\|\xi\|^2 = \langle \xi, \xi\rangle_\phi = \langle \xi, \xi\rangle_{\phi^e} = \langle \pi_{\phi^e}(1)\xi, \xi\rangle_{\phi^e} = \phi^e(1) = \|\phi\|_{\text{op}}$$

where the last equality is by definition in Lemma 13.1.5.

It remains to prove the last assertion. If $x \in N_\phi$, then

$$0 = \phi(x^*x) = \langle \sigma(x^*x)v, v\rangle = \|\sigma(x)v\|^2.$$

Hence $\sigma(x)v = 0$. This shows the map

$$\begin{aligned} \mathcal{A}/N_\phi &\longrightarrow V \\ x &\longmapsto \sigma(x)v \end{aligned}$$

is well-defined. It is straightforward to see this map intertwines $\mathcal{A}$-actions, and extends to a unitary equivalence.

□

Definition. For each positive linear functional ϕ, the resulting *-representation (π_ϕ, H_ϕ) is called the **Gelfand-Naimark-Segal representation**, or simply the **GNS representation**, associated to ϕ.

Now given a family $(\pi_\lambda, H_\lambda)_{\lambda\in\Lambda}$ of *-representation of a C^*-algebra $\mathcal{A}$, consider the homomorphism

$$\begin{aligned} \oplus_\lambda \pi_\lambda : \mathcal{A} &\longrightarrow B\left(\bigoplus_{\lambda\in\Lambda} H_\lambda\right) \\ a &\longmapsto \oplus_\lambda \pi_\lambda(a) \end{aligned}$$

This again gives a *-representation of $\mathcal{A}$ on the Hilbert space direct sum $\bigoplus_{\lambda\in\Lambda} H_\lambda$.

Recall that $S(\mathcal{A})$ consists of all states of $\mathcal{A}$, i.e., positive linear functional with norm 1. Note that for a positive linear functional ϕ and $\alpha > 0$, the GNS constructions π_ϕ and $\pi_{\alpha\phi}$ are equivalent. So we may confine ourselves to those associated to states. The representation

$$\left(\bigoplus_{\phi\in S(\mathcal{A})} \pi_\phi, \bigoplus_{\phi\in S(\mathcal{A})} H_\phi\right)$$

is called the **universal representation** of $\mathcal{A}$.

Theorem 13.2.2 (Gelfand-Naimark)**.** The universal representation is faithful.

Proof. It suffices to show for each $a \in \mathcal{A}\backslash\{0\}$, we can find some state ϕ such that $\pi_\phi(a)$ is a nontrivial operator on H_ϕ, i.e., we can find $b \in \mathcal{A}$ such that $\phi((ab)^*ab) \neq 0$.

□

Chapter 14

Representation

Let V be an F-space over $\mathbb{C}$. Put

$$\mathrm{GL}_{\mathrm{cts}}(V) := \{T \in \mathrm{Aut}_{\mathbb{C}}(V) \mid T \text{ is continuous}\}$$

It follows from the Open Mapping theorem that $\mathrm{GL}_{\mathrm{cts}}(V)$ is a group.

Definition. Let G be a topological group.

1. An **(F-space) representation** of G is a group homomorphism $\pi : G \to \mathrm{GL}_{\mathrm{cts}}(V)$ with V an F-space such that the action map

$$\begin{aligned} G \times V &\longrightarrow V \\ (g, v) &\longmapsto \pi(g)v \end{aligned}$$

   is continuous. If V is Fréchet (resp. Banach, Hilbert), we say π is a Fréchet (resp. Banach, Hilbert) space representation.

2. For two representations (π, V_π), (η, V_η) of G, a continuous linear operator $T : V_\pi \to V_\eta$ is a **G-homomorphism**, or is called **G-equivariant/intertwining**, if $T\pi(g) = \eta(g)T$ for all $g \in G$. The set of all G-intertwining operators from V_π to V_η is denoted by $\mathrm{Hom}_G(V_\pi, V_\eta)$, or $\mathrm{Hom}_G(\pi, \eta)$.

Lemma 14.0.1. Let G be a topological group, V be a Banach space and $\pi : G \to \mathrm{GL}_{\mathrm{cts}}(V)$ be a group homomorphism. Then π is a representation if and only if

(a) the map $g \mapsto \pi(g)v$ is continuous at $g = 1$ for every $v \in V$, and

(b) the map $g \mapsto \|\pi(g)\|_{\mathrm{op}}$ is bounded in a neighborhood of the unit of G.

Proof. Suppose π is a representation. Then (a) is obvious. For (b), let $R > 0$ be any real number. Then by continuity, we can find $r > 0$ and a unit neighborhood U of G such that $\pi(U)B_r(0) \subseteq B_R(0)$. Then for $g \in U$,

$$\|\pi(g)\|_{\mathrm{op}} = \sup_{\|v\|=1} \|\pi(g)v\| = \frac{2}{r} \sup_{\|v\|=1} \left\|\pi(g)\frac{rv}{2}\right\| \leqslant \frac{2R}{r}$$

Conversely, we write

$$\begin{aligned} \|\pi(g)v - \pi(h)w\| &= \left\|\pi(h)(\pi(h^{-1}g)v - w)\right\| \leqslant \|\pi(h)\| \\ &\leqslant \|\pi(h)\| \left\|\pi(h^{-1}g)v - \pi(h^{-1}g)w + \pi(h^{-1}g)w - w\right\| \\ &\leqslant \|\pi(h)\| \left(\left\|\pi(h^{-1}g)\right\| \|v - w\| + \left\|\pi(h^{-1}g)w - w\right\|\right) \end{aligned}$$

By assumption, as $(g, v) \to (h, w)$, the last term tends to 0, hence the continuity. $\square$

Definition. A representation (π, V_π) is called a **subrepresentation** of a representation (η, V_η) if

- V_π is a closed subspace of V_π, and
- $\eta|_{V_\pi} = \pi$.

In other words, a subrepresentation of (η, V_η) is a closed subspace invariant under η.

Definition. Let H be a Hilbert space. A representation π of G on H is called a **unitary representation** if

$$\langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle$$

for all $g \in G$ and $v, w \in H$; in other words, all $\pi(g)$, $g \in G$ are unitary operators on H.

- For a Hilbert space H, denote by $\mathcal{U}(H) \subseteq \mathrm{GL}_{\mathrm{cts}}(H)$ the subgroup of unitary operators.
- By Lemma 14.0.1, a homomorphism $\pi : G \to \mathcal{U}(H)$ is a representation of G if and only if $g \mapsto \pi(g)v$ is continuous at $g = 1$ for all $v \in V$.

Lemma 14.0.2. A representation π of the group G on a Hilbert space V is unitary if and only if $\pi(g^{-1}) = \pi(g)^*$ holds for every $g \in G$.

Proof. For $g \in G$, $\pi(g)$ is unitary if and only if $\pi(g)$ is invertible and $\pi(g)^* = \pi(g^{-1})$. $\square$

Example 14.0.3.

1. For a continuous group homomorphism $\chi : G \to \mathbb{C}^\times$, we can define a representation π_χ of G on $V = \mathbb{C}$ by $\pi_\chi(g)v = \chi(g)v$, and π_χ is unitary if and only if χ maps into the unit circle $\mathbb{T} = S^1$.

2. Let G be an LCH group. Then one can consider the **left regular representation**

$$\begin{aligned} \pi_{\mathrm{reg}} : G &\longrightarrow \mathrm{GL}_{\mathrm{cts}}(L^2(G)) \\ x &\longmapsto L_x \end{aligned}$$

This is really a representation by Lemma 14.0.1, Lemma 2.6.7 and

$$\begin{aligned} \langle L_x\phi, L_x\psi \rangle &= \int_G L_x\phi(y)\overline{L_x\psi(y)}dy \\ &= \int_G \phi(x^{-1}y)\overline{\psi(x^{-1}y)}dy \\ &= \int_G \phi(y)\overline{\psi(y)}dy = \langle \phi, \psi \rangle \end{aligned}$$

so $\|L_x\| = 1\,(x \in G)$ and π_{reg} is unitary (essentially by the left-invariant of Haar measure).

Theorem 14.0.4. Let H be a Hilbert space, G a locally compact group and $\pi : G \to \mathrm{GL}_{\mathrm{cts}}(H)$ a homomorphism. Then π is a representation if and only if

(i) for each compact set K in G there exists a constant $C_K > 0$ such that $\|\pi(g)\| \leqslant C_K$ for each $g \in K$, and

(ii) there is a dense subspace $V \subseteq H$ such that the function $g \mapsto c_{v,w}(g) := \langle \pi(g)v, w \rangle$ is continuous for each $v, w \in V$.

Proof. Assume π is a representation. Since the action map $G \times H \to H$ is continuous, (ii) is clear. Let K be any compact subset of G. Then for each $v \in V$, the function $K \ni g \mapsto \|\pi(g)v\|$ attains its maximum. By Uniform boundedness principle, there exists $C_K > 0$ such that $\|\pi(g)\| \leqslant C_K$ for each $g \in K$. This shows (i).

Conversely, suppose π satisfies (i) and (ii). For a relatively compact open set U in G, we put

$$L^1(U) = \{f \in L^1(G) \mid \operatorname{supp} f \subseteq U\}$$

If $f \in C_c(G) \cap L^1(U)$, then for $v, w \in$

□

14.1 Construction

In this section let G be a topological group.

Definition. Let (π_i, H_i) be unitary representations of G on Hilbert spaces $H_i\,(i \in I)$. We define the **direct sum representation** $\bigoplus_{i\in I} \pi_i : G \to \mathrm{GL}_{\mathrm{cts}}(\widehat{\bigoplus_{i\in I}} H_i)$ by

$$\left(\bigoplus_{i\in I} \pi_i\right)(g) \sum_{i\in I} v_i := \sum_{i\in I} \pi_i(g) v_i$$

Example 14.1.1. Let $G = \mathbb{R}/\mathbb{Z}$ and $V = L^2(\mathbb{R}/\mathbb{Z})$. Let π be the left regular representation of $\mathbb{R}/\mathbb{Z}$. By the Plancherel theorem, $\widehat{G} = \{e_k(x) := e^{2\pi i k x} \mid k \in \mathbb{Z}\}$ forms a orthonormal basis of $L^2(\mathbb{R}/\mathbb{Z})$, so π admits a direct sum decomposition

$$V = \widehat{\bigoplus_{k\in\mathbb{Z}}} \mathbb{C} e_k$$

and G acts on $\mathbb{C}e_k$ through the character e_k.

Definition. Let G_1, G_2 be two topological group, and (π_i, H_i) be a representation of G_i on the Hilbert space H_i. Define the **tensor product representation** $\pi_1 \otimes \pi_2 : G_1 \times G_2 \to \mathrm{GL}_{\mathrm{cts}}(H_1 \widehat{\otimes} H_2)$ by

$$(\pi_1 \otimes \pi_2)(g_1, g_2)(v_1 \otimes v_2) = \pi_1(v_1) \otimes \pi_2(v_2)$$

Definition. Let (π_i, V_i) be two representations of G on Banach spaces. Then $\mathrm{Hom}_G(V_1, V_2)$ is a closed subspace of the Banach space $\mathrm{Hom}_{\mathrm{cts}}(V_1, V_2)$. Define a representation $\pi : G \to \mathrm{GL}_{\mathrm{cts}}(\mathrm{Hom}_G(V_1, V_2))$ of G by

$$(\pi_L(g)T)(v) = \pi_2(g)T(v) = T(\pi_1(g)v)$$

14.2 Schur's Lemma

Definition. A representation (π, V_π) is called a **subrepresentation** of a representation (η, V_η) if

- V_π is a closed subspace of V_π, and
- $\eta|_{V_\pi} = \pi$.

In other words, a subrepresentation of (η, V_η) is a closed subspace invariant under η.

Definition. Let (π, V) be a representation on a group G.

1. (π, V) is called **irreducible** if it does not contain any proper nontrivial subrepresentation.
2. A vector $v \in V$ is called a **cyclic vector** if the linear span of the G-orbit $Gv := \{\pi(g)v \mid g \in G\}$ of v is dense in V.

- A representation is irreducible if and only if every nonzero vector is cyclic.

Example 14.2.1.

1. Every one-dimensional representation is irreducible.
2. Consider the unitary group $\mathrm{U}(n)$ and its natural action on $\mathbb{C}^n$. Then this representation is irreducible. The case $n = 1$ is trivial. When $n \geqslant 2$, we show for any two vectors $v, u \in \mathbb{C}^n$ with $\|v\| = \|u\| = 1$, we can find $A \in \mathrm{U}(n)$ such that $Av = u$. Consider the plane spanned by v, u; up to a (unitary) rotation, we can assume $v = (1, 0, 0, \ldots, 0)$ and $u = (a, b, 0, \ldots, 0)$ with $|a|^2 + |b|^2 = 1$. Then $A := \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \oplus I_{n-2}$ does the job. Hence a nontrivial $\mathrm{U}(n)$-invariant subspace of $\mathbb{C}^n$ contains all directions, and hence it is the whole space.

Lemma 14.2.2 (Schur)**.** Let (π, H) be a unitary representation of the topological group G. TFAE

(a) (π, H) is irreducible.

(b) $\mathrm{Hom}_G(H, H) = \mathbb{C}\,\mathrm{id}_H$.

Proof. Since $\pi(g^{-1}) = \pi(g)^*$, the set $\{\pi(g) | g \in G\} \subseteq \mathcal{B}(H)$ is self-adjoint. Then the result follows from Theorem 12.1.8.[1] $\square$

Definition. For two unitary representations (π, V_π), (η, V_η), they are called **unitarily equivalent** if there exists a unitary G-intertwining operator $T : V_\pi \to V_\eta$.

Example 14.2.3. Let $G = \mathbb{R}$ and let $V_\pi = V_\eta = L^2(\mathbb{R})$. The representation π is given by $\pi(x)\phi(y) = \phi(x + y)$, and η is given by $\eta(x)\phi(y) = e^{2\pi i x y}\phi(y)$. The Fourier transformation is then unitary equivalence $V_\pi \to V_\eta$. This follows from the Plancherel theorem and a direct computation.

Corollary 14.2.6.1. Let (π, V_π) and (η, V_η) be two irreducible unitary representations. Then a G-homomorphism T from V_π to V_η is either zero or invertible with continuous inverse. In the latter case there exists $c > 0$ such that cT is unitary. Hence the space $\mathrm{Hom}_G(\pi, \eta)$ is zero unless π and η is unitary equivalent, in which case the space is of dimension 1.

[1] For the implication (a)⇒(b), there is an alternative derivation using spectral theorem. Note that if $T \in \mathrm{Hom}_G(H, H)$, so is $T^* \in \mathrm{Hom}_G(H, H)$. Since $T = \frac{T + T^*}{2} + i\frac{T - T^*}{2i}$, it suffices to show if $T \in \mathrm{Hom}_G(H, H)$ is self-adjoint, then T is scalar. By spectral theorem, there exists a unique resolution of the identity E on the Borel algebra of $\sigma(T)$. Since T commutes with each $\pi(g)$, each $E(\omega)$ also commutes with $\pi(g)$; in other words, $E(\omega) \in \mathrm{Hom}_G(H, H)$ for every Borel $\omega \subseteq \sigma(T)$. By irreducibility, we have $E(\omega) = 0$ or id_H for each Borel ω. Since $\sigma(T) \subseteq \mathbb{R}$ is compact, we then can find a closed interval $I = [-a, a] \supseteq \sigma(T)$ such that $E(J) = \mathrm{id}_H$. Bisect J into two halves $J = J_1 \sqcup J_1'$; then $E(J_1)E(J_1') = E(J_1')E(J_1) = 0$ and $E(J_1) + E(J_1') = \mathrm{id}_H$. WLOG, assume $E(J_1) \neq 0$, so that $E(J_1) = \mathrm{id}_H$ by irreducibility, and hence $E(J_1') = 0$. Continuing this bisecting process, we find a decreasing sequence of closed intervals $J_0 \supseteq J_1 \supseteq \cdots$ converging to a singleton p, such that $E(J_i) = \mathrm{id}_H$. It follows that the measure E is supported at the singleton p, so using the definition of the integration, we see

$$T = \int_{-a}^{a} x dE(x) = p\,\mathrm{id}_H .$$

Proof. Let $T : V_\pi \to V_\eta$ be a G-homomorphism. Then its adjoint $T^* : V_\eta \to V_\pi$ is also a G-homomorphism:

$$\begin{aligned}\langle v, T^*\eta(g)w\rangle &= \langle Tv, \eta(g)w\rangle = \langle \eta(g^{-1})Tv, w\rangle \\ &= \langle T\pi(g^{-1})v, w\rangle = \langle v, \pi(g)T^*w\rangle\end{aligned}$$

Thus $T^*T \in \operatorname{End}_G(V_\pi)$, so by Schur's lemma $T^*T = \lambda \operatorname{id}_{V_\pi}$ for some $\lambda \in \mathbb{C}$. If $T \neq 0$, then T^*T is nonzero and positive, so $\lambda > 0$. If we let $c = \sqrt{\lambda^{-1}} > 0$, then $(cT)^*(cT) = \operatorname{id}_{V_\pi}$; this shows cT is unitary. A similar argument shows TT^* is bijective, which implies cT is bijective, whence T is a homeomorphism by Open Mapping theorem. □

Definition. For an LCH group G, we denote by $\widehat{G}$ the set of all equivalence classes the irreducible unitary representations of G. We call $\widehat{G}$ the **unitary dual** of G.

- We need to explain why $\widehat{G}$ forms a set. First, we show that there exists a cardinality α depending on G such that every irreducible representation (π, V_π) of G satisfies $\dim V_\pi \leqslant \alpha$. Indeed, every irreducible representation has a cyclic vector, so we can take $\alpha = \#G$. Secondly, a representation of G is just a vector space V together with an appropriate function $V \times V \to \mathbb{C}$ and a map $G \to \mathrm{GL}(V)$, and for each cardinality $\beta \leqslant \alpha$ there exists a unique vector space V_β of dimension β. Hence every equivalence class of irreducible representations of G has a representation in the set $\bigcup\limits_{\beta \leqslant \alpha} \mathbb{C}^{V_\beta \times V_\beta} \times \mathrm{GL}(V_\beta)^G$.

Example 14.2.7. If G is LCA, then every irreducible representation is one-dimensional, so the unitary dual of G coincides with the Pontryagin dual of G. To see this, let (π, V_π) be any nonzero irreducible unitary representation of G. Then $\pi(x)\pi(y) = \pi(xy) = \pi(yx) = \pi(y)\pi(x)$ for all $x, y \in G$, so that $\pi(x) \in \operatorname{Hom}_G(V_\pi, V_\pi)$. By Schur's lemma, for all $x \in G$ we have $\pi(x) = \lambda(x) \operatorname{id}_{V_\pi}$ for some $\lambda(x) \in \mathbb{C}$. But this means every closed subspace of V_π is G-invariant, so $\dim_{\mathbb{C}} V_\pi = 1$ particularly. By Lemma 14.0.2 we have $\lambda(x) \in S^1$ for all $x \in G$, so λ is a unitary character of G. The converse is clear.

There is a version of Schur's lemma for the densely defined operators.

Corollary 14.2.6.1. Let (π, V_π) be an irreducible unitary representation. Let $D \leqslant H$ be a G-invariant dense subspace, and suppose $T \in \operatorname{Hom}_G(D, H)$ (here D has no topology). Assume there exists another dense subspace $D' \leqslant H$ and $S \in \operatorname{Hom}_{\mathbb{C}}(D', H)$ (again, no topology on D') such that

$$\langle Tv, w\rangle = \langle v, Sw\rangle$$

for $v \in D$, $w \in D'$. Then $T = \lambda \operatorname{id}_D$ for some $\lambda \in \mathbb{C}$.

Proof. Denote by A the subalgebra of $\mathcal{B}(H)$ generated by $\{\pi(g) \mid g \in G\}$; note that A is unital, and since π is unitary, A is self-adjoint. By Schur's lemma, $C_{\mathcal{B}(H)}(A) = \operatorname{Hom}_G(H, H) = \mathbb{C} \operatorname{id}_H$, so $C_{\mathcal{B}(H)}(C_{\mathcal{B}(H)}(A)) = \mathcal{B}(H)$.

Let $v \in D$ and assume v and Tv are linear independent. By Bicommutant theorem (applied to the operator $\operatorname{id}_H \in \mathcal{A}$ and the vectors v, Tv), we can find $U_j \in A$ such that $\lim_j U_j v = v$ and $\lim_j U_j Tv = Tv$. Now, for $w \in D'$,

$$\langle v, w\rangle = \lim\langle U_j Tv, w\rangle = \lim\langle TU_j v, w\rangle = \lim\langle U_j v, Sw\rangle = \langle v, Sw\rangle = \langle Tv, w\rangle.$$

Since D' is dense, this implies $v = Tv$, a contradiction. Hence v and Tv are linear dependent for all $v \in D$, proving that T is a scalar on D. □

14.3 Representation of $L^1(G)$

Proposition 14.3.1. Let (π, V_π) be a unitary representation of an LCH group G. For each $f \in L^1(G)$ there exists a unique bounded operator $\pi(f)$ on V_π such that

$$\langle \pi(f)v, w\rangle = \int_G f(x)\langle \pi(x)v, w\rangle dx$$

holds for all $v, w \in V_\pi$. The induced map $\pi : L^1(G) \to \mathcal{B}(V_\pi)$ is a continuous homomorphism of Banach *-algebras. Also, we have $\pi(L_x f) = \pi(x)\pi(f)$ for each $x \in G$ and $f \in L^1(G)$.

Proof. We invoke Riesz's Representation theorem. Taking complex conjugation, it suffices to show

$$\langle w, \pi(f)v\rangle = \int_G \overline{f(x)}\langle w, \pi(w)v\rangle dx$$

Now the map $w \mapsto \displaystyle\int_G \overline{f(x)}\langle w, \pi(w)v\rangle dx$ is $\mathbb{C}$-linear and bounded, since

$$\begin{aligned}\left|\int_G \overline{f(x)}\langle w, \pi(w)v\rangle dx\right| &\leqslant \int_G |f(x)\langle w, \pi(w)v\rangle|\, dx \\ &\leqslant \int_G |f(x)|\, \|w\|\, \|\pi(x)v\|\, dx \\ &= \|f\|_1\, \|w\|\, \|v\|\end{aligned}$$

Here we use π is unitary. Therefore there exists a unique vector, denoted by $\pi(f)v \in V_\pi$, such that the equality holds. The same theorem shows $v \mapsto \pi(f)v$ is linear. To show it is bounded, note that the above computation gives

$$\|\pi(f)v\|^2 = \langle \pi(f)v, \pi(f)v\rangle \leqslant \|f\|_1\, \|\pi(f)v\|\, \|v\|$$

so that $\|\pi(f)v\| \leqslant \|f\|_1\, \|v\|$. It remains to show $\pi(f * g) = \pi(f)\pi(g)$ and $\pi(f)^* = \pi(f^*)$. For the former,

$$\begin{aligned}\langle \pi(f * g)v, w\rangle &= \int_G \left(\int_G f(y)g(y^{-1}x)dy\right)\langle \pi(x)v, w\rangle dx \\ &\overset{\text{Fubini}}{=} \int_G \int_G f(y)g(y^{-1}x)\langle \pi(x)v, w\rangle dx dy \\ (x \mapsto yx) &= \int_G \int_G f(y)g(x)\langle \pi(yx)v, w\rangle dx dy \\ &= \int_G \int_G f(y)g(x)\langle \pi(x)v, \pi(y^{-1})w\rangle dx dy \\ &= \int_G f(y)\langle \pi(g)v, \pi(y^{-1})w\rangle dy \\ &= \int_G f(y)\langle \pi(y)\pi(g)v, w\rangle dy = \langle \pi(f)\pi(g)v, w\rangle\end{aligned}$$

and for the latter,

$$\begin{aligned}
\langle \pi(f^*)v, w\rangle &= \int_G f^*(x)\langle \pi(x)v, w\rangle dx \\
&= \int_G \Delta_G(x^{-1})\overline{f(x^{-1})}\langle v, \pi(x^{-1})w\rangle dx \\
&= \overline{\int_G \Delta_G(x^{-1})f(x^{-1})\langle \pi(x^{-1})w, v\rangle dx} \\
&= \overline{\int_G f(x)\langle \pi(x)w, v\rangle dx} \\
&= \overline{\langle \pi(f)w, v\rangle} = \langle v, \pi(f)w\rangle
\end{aligned}$$

For the last assertion,

$$\langle \pi(L_x f)v, w\rangle = \int_G f(x^{-1}g)\langle \pi(g)v, w\rangle dg = \int_G f(g)\langle \pi(xg)v, w\rangle dg = \langle \pi(x)\pi(f)v, w\rangle$$

□

Remark 14.3.2. Alternatively, one can define $\pi(f)$ as a Bochner integral

$$\pi(f) := \int_G f(x)\pi(x)dx$$

in the Banach space $\mathcal{B}(V_\pi)$. To see why this integral exists, we use Proposition D.7.3.2. By linearity, we may assume $f \in L^1(G)$ is nonnegative. Consider the finite positive measure $d\mu(x) = f(x)dx$ on G. Then $\pi(f) = \int_G \pi(x)d\mu(x)$. By Corollary 2.2.2.1.4 the measure $d\mu$ is supported on a σ-compact open subset K of G, so the integral $\int_G \pi(x)d\mu(x)$ really takes place on a σ-compact set. Since π is continuous, by Lemma D.7.2 π is separable when restricting to K, and thus $\int_G \pi(x)d\mu(x)$ exists by Proposition D.7.3.2.

Lemma 14.3.3. Let (π, V_π) be a representation of G. Then for every $v \in V_\pi$ and every $\varepsilon > 0$ there exists a unit-neighborhood U such that for every Dirac function ϕ_U with support in U one has $\|\pi(\phi_U)v - v\| < \varepsilon$. In particular, for every Dirac net $(\phi_U)_U$, the net $(\pi(\phi_U)v)_U$ converges to v in norm topology.

Proof. For any open set U and ϕ_U a Dirac function supported in U, we have

$$\|\pi(\phi_U)v - v\| = \left\|\int_G \phi_U(x)(\pi(x)v - v)dx\right\| \leqslant \int_G \phi_U(x)\, \|\pi(x)v - v\|\, dx$$

The first equality is clear, and the second is Proposition D.7.1.(b). For given $\varepsilon > 0$ there exists a unit-neighborhood U_0 such that $\|\pi(x)v - v\| < \varepsilon$ whenever $x \in U_0$. Then $U = U_0$ does the job. □

Definition. A *-representation $\pi : L^1(G) \to \mathcal{B}(V)$ of $L^1(G)$ on a Hilbert space V is **non-degenerate** if the subspace

$$\pi(L^1(G))V := \operatorname{span}\{\pi(f)v \mid f \in L^1(G),\, v \in V\}$$

is dense in V.

- It follows from Lemma 14.3.3 that every representation of $L^1(G)$ constructed from a unitary representation (π, V_π) as in Proposition 14.3.1 is non-degenerate.

Proposition 14.3.4. Let $\pi : L^1(G) \to \mathcal{B}(V)$ be a non-degenerate *-representation of $L^1(G)$ on a Hilbert space V. Then there exists a unique unitary representation $(\tilde{\pi}, V)$ of G such that

$$\langle \pi(f)v, w\rangle = \int_G f(x)\langle \tilde{\pi}(x)v, w\rangle dx$$

holds for all $f \in L^1(G)$ and all $v, w \in V$.

Proof. Note that π is continuous by Lemma 3.3.1. Let $x \in G$. We first define an operator $\tilde{\pi}(x)$ on the dense subspace $\pi(L^1(G))V$. Each element in $\pi(L^1(G))V$ has the form $\sum_{i=1}^n \pi(f_i)v_i,\ f_i \in L^1(G),\ v_i \in V$. Define

$$\tilde{\pi}(x)\sum_{i=1}^n \pi(f_i)v_i := \sum_{i=1}^n \pi(L_x f_i)v_i$$

We must show this is well-defined, which amounts to show that $\sum_{i=1}^n \pi(f_i)v_i = 0$ implies $\sum_{i=1}^n \pi(L_x f_i)v_i = 0$. For $f, g \in L^1(G)$, we have $g^* * L_x f = (L_{x^{-1}}g)^* * f$. Indeed,

$$\begin{aligned} g^* * L_x f(y) &= \int_G \Delta_G(z^{-1})\overline{g(z^{-1})}f(x^{-1}z^{-1}y)dz \\ &\overset{2.3.1.4.}{=} \int_G \overline{g(z)}f(x^{-1}zy)dz \\ (z \mapsto xz) &= \int_G \overline{g(xz)}f(zy)dz \\ &\overset{2.3.1.4.}{=} \int_G \Delta_G(z^{-1})\overline{g(xz^{-1})}f(z^{-1}y)dz = (L_{x^{-1}}g)^* * f(y) \end{aligned}$$

Then for $g, f_1, \ldots, f_n \in L^1(G)$, $v_1, \ldots, v_n, w \in V$, $x \in G$

$$\begin{aligned} \left\langle \sum_{i=1}^n \pi(L_x f_i)v_i, \pi(g)w \right\rangle &= \sum_{i=1}^n \langle \pi(g^* * L_x f_i)v_i, w\rangle \\ &= \sum_{i=1}^n \langle \pi((L_{x^{-1}}g)^* * f_i)v_i, w\rangle = \left\langle \sum_{i=1}^n \pi(f_i)v_i, \pi(L_{x^{-1}}g)w \right\rangle \end{aligned}$$

Now assume $\sum_{i=1}^n \pi(f_i)v_i = 0$. Then the above shows that $\sum_{i=1}^n \pi(L_x f_i)v_i$ is orthogonal to $\pi(L^1(G))V$, which is dense in V, so $\sum_{i=1}^n \pi(L_x f_i)v_i = 0$. Thus $\tilde{\pi}(x)$ is a well-defined operator on $\pi(L^1(G))V$, and the above computation also shows $\tilde{\pi}(x)$ is unitary on $\pi(L^1(G))V$. Therefore $\tilde{\pi}(x)$ extends to a unique unitary operator on V with inverse $\tilde{\pi}(x^{-1})$, and clearly $\tilde{\pi}(xy) = \tilde{\pi}(x)\tilde{\pi}(y)$ for all $x, y \in G$. By Lemma 2.6.7 for each $f \in L^1(G)$ the map $x \mapsto L_x f$ is continuous, so $x \mapsto \tilde{\pi}(x)v$ is continuous for every $v \in V$. Thus $(\tilde{\pi}, V)$ is a unitary representation of G by Lemma 14.0.1.

It remains to show $\pi(f) = \tilde{\pi}(f)$ for all $f \in L^1(G)$. By continuity we may assume $f \in C_c(G)$ and it suffices to show $\langle \tilde{\pi}(f)\pi(g)v, w\rangle = \langle \pi(f)\pi(g)v, w\rangle$ for all $g \in C_c(G)$, $v, w \in V$.

$$\begin{aligned} \langle \tilde{\pi}(f)\pi(g)v, w\rangle &= \int_G f(x)\langle \tilde{\pi}(x)\pi(g)v, w\rangle dx \\ &= \int_G f(x)\langle \pi(L_x g)v, w\rangle dx \\ &= \int_G \langle \pi(f(x)L_x g)v, w\rangle dx \\ &\overset{D.7.1.(c)}{=} \left\langle \pi\left(\int_G f(x)L_x g dx\right)v, w \right\rangle \\ &\overset{D.7.4}{=} \langle \pi(f * g)v, w\rangle = \langle \pi(f)\pi(g)v, w\rangle \end{aligned}$$

□

Example 14.3.5. Let G be an LCH group, and consider the left regular representation $L : G \to \mathcal{U}(L^2(G))$. By Proposition 14.3.1 we then have a continuous *-homomorphism $L : L^1(G) \to \mathcal{B}(L^2(G))$ defined by

$$L(f)g = f * g \in L^2(G).$$

By Lemma 2.6.9, $L : L^1(G) \to \mathcal{B}(L^2(G))$ is injective.

In particular, the case G being LCA recovers Lemma 5.3.2. Also, using the Plancherel isomorphism $L^2(G) \cong L^2(\widehat{G})$ we obtain a map $L^1(G) \to \mathcal{B}(L^2(\widehat{G}))$ given by $f \mapsto [g \mapsto \widehat{f} \cdot g]$. Hence

$$\|L(f)\| = \left\|g \mapsto \widehat{f}g\right\| = \left\|\widehat{f}\right\|_{\sup} = \sup_{\chi \in \widehat{G}} |\widehat{f}(\chi)|.$$

14.4 Square Integrable Representations

Definition. Let G be a topological group and (π, H) be a representation on a Hilbert space H.

(i) For $v, w, \in H$, the function $c_{v,w} : G \to C(G)$ defined as

$$c_{v,w}(g) := \langle \pi(g)v, w \rangle$$

is called a **matrix coefficient of** π.

(ii) The representation π is called **square integrable** if (π, H) is irreducible, unitary, and it has a nonzero square integrable matrix coefficient.

Lemma 14.4.1. Let (π, H) be a unitary representation of an LCH group G. For $v \in H$, define

$$D_v := \{w \in H \mid c_{w,v} \in C(G) \cap L^2(G)\}.$$

Define a linear map $T : D_v \to L^2(G)$ defined by $Tw = c_{w,v}$.

(i) T is a closed operator, in the sense that the graph

$$\Gamma_T := \{(w, Tw) \mid w \in D_v\} \subseteq D_v \times L^2(G)$$

is a closed subset.

(ii) D_v is complete with respect to the graph norm.

Proof. Let $(w_n)_n$ be a sequence in D_v such that $w_n \to w \in H$ and $Tw_n \to \phi \in L^2(G)$. Passing to a subsequence (D.5.5), we may assume $Tw_n \to \phi$ pointwise almost everywhere. But then for $g \in G$,

$$|\phi(g) - Tw_n(g)| = |Tw(g) - Tw_n(g)| = |\langle \pi(g)(w - w_n), v \rangle| \leqslant \|w - w_n\| \, \|v\| \to 0$$

as $n \to \infty$. This implies $\phi = Tw$ almost everywhere. □

Theorem 14.4.2. Let G be a unimodular LCH group, and let (π, H) be a square integrable representation.

(i) All matrix coefficients of π is square integrable.

(ii) There exists an isometry $T \in \mathrm{Hom}_G(H, L^2(G))$ with $T(H) \leqslant C(G) \cap L^2(G)$.

Proof. Take $v', w' \in H$ such that $c_{v',w'} : G \to C(G)$ is nonzero and square integrable; we may assume v', w' have norm 1. Define

$$D = \{v \in H \mid c_{v,w'} \in L^2(G)\}.$$

which is a G-invariant subspace of H. Since D' contains $\operatorname{span}_{\mathbb{C}}\{\pi(g)v' \mid g \in G\}$, D' is dense by assumption. Define

$$\begin{aligned} T : D &\longrightarrow L^2(G) \\ v &\longmapsto c_{v,w'}. \end{aligned}$$

For $g, h \in G$, we have $c_{\pi(g)v,w'}(h) = \langle \pi(hg)v, w' \rangle = c_{v,w'}(hg)$, so that T intertwines π and the right regular representation on $L^2(G)$. Our goal is to show $D = H$.

Let $(,) : D \times D \to \mathbb{C}$ denote the inner product induced by the graph of T:

$$(v, w) = \langle v, w \rangle_H + \langle Tv, Tw \rangle_{L^2(G)}.$$

Let $\iota : D \to H$ be the inclusion of D; by Lemma 14.4.1, ι is a bounded operator. Consider the adjoint map $\iota^* : H \to D$ and the composition $\iota \circ \iota^* : H \to H$. Since H is unimodular, the right regular representation on $L^2(G)$ is unitary, so the inner product on D is unitary as well. In particular, ι^* is G-equivariant, showing that $\iota \circ \iota^* \in \operatorname{Hom}_G(H, H)$. Now Schur's lemma implies that $\iota \circ \iota^* = \lambda \operatorname{id}_H$ for some $\lambda \in \mathbb{C}$; since $\iota \circ \iota^*$ is positive and nonzero, $\lambda > 0$. But then $D \ni \iota^* v = \lambda v$, so $v \in D$ for all $v \in H$, proving that $H = D$.

The equality $\iota^* = \lambda \operatorname{id}_H$ also implies that

$$\langle \iota v, w \rangle_H = (v, \iota^* w) = \lambda(v, w) = \lambda(\langle v, w \rangle_H + \langle Tv, Tw \rangle_{L^2(G)})$$

so that

$$(1 - \lambda)\langle v, w \rangle_H = \lambda \langle Tv, Tw \rangle_{L^2(G)}$$

for all $v, w \in H$. Hence $0 < \lambda < 1$, and the operator $\sqrt{\dfrac{\lambda}{1-\lambda}} T \in \operatorname{Hom}_G(H, L^2(G))$ is unitary; this proves the first statement in (ii). The last containment in (ii) is clear, as each matrix coefficient is continuous. For (i), let $v, w \in H$ and we must show $c_{v,w} \in L^2(G)$. Since $D = H$, we see $c_{v,w'} \in L^2(G)$. Since $\overline{c_{v,w'}(g^{-1})} = c_{w',v}(g)$ for all $g \in G$, we see $c_{w',v} \in L^2(G)$. If we form the subspace $\{x \in H \mid c_{x,v} \in L^2(G)\}$ as in the beginning, the proof implies this coincides with H, whence $c_{w,v} \in L^2(G)$. ☐

Corollary 14.4.2.1 (Schur's orthogonality)**.** Let (π, V_π) and (τ, V_τ) be two square-integrable representations of a unimodular LCH group G.

(i) If π and τ are not unitarily equivalent, then

$$\int_G \langle \pi(g)v, w \rangle \overline{\langle \tau(g)v', w' \rangle} dg = 0$$

for all $v, w \in V_\pi$, $v', w' \in V_\tau$.

(ii) There exists a constant $d(\pi) > 0$ such that

$$\int_G \langle \pi(g)v, w \rangle \overline{\langle \pi(g)v', w' \rangle} dg = \frac{\langle v, v' \rangle \langle w', w \rangle}{d(\pi)}$$

for all $v, w, v', w' \in V_\pi$.

The constant $d(\pi)$ is called the **formal degree** of the representation π.

Proof. Fix $w \in V_\pi$ and $w' \in V_\tau$. Define $T : V_\pi \to L^2(G)$ and $S : V_\tau \to L^2(G)$ by

$$T(v)(g) = \langle \pi(g)v, w\rangle, \qquad S(v')(g) = \langle \tau(g)v', w'\rangle.$$

By Theorem 14.4.2 these are well-defined, and we can find $t, s > 0$ such that $t^{-1}T$, $s^{-1}S$ are unitary. Then

$$|\langle Tv, Sv'\rangle_{L^2(G)}| \leqslant \|Tv\|_{L^2(G)} \left\|Sv'\right\|_{L^2(G)} = ts\,\|v\|\,\left\|v'\right\|$$

for all $v \in V_\pi$, $v' \in V_\tau$, so the operator $T^*S : V_\tau \to V_\pi$ is bounded. Clearly T^*S is G-equivariant, so $T^*S \in \mathrm{Hom}_G(\pi, \tau)$; by Schur's lemma it is zero if π and τ are not unitarily equivalent, and this proves (i). Consider the case $\pi = \tau$. By Schur's lemma $T^*S = a(w, w')\,\mathrm{id}_{V_\pi}$ for some $a(w, w') \in \mathbb{C}$. Hence

$$\int_G \langle \pi(g)v, w\rangle\overline{\langle \pi(g)v', w'\rangle}dg = \langle Tv, Sv'\rangle_{L^2(G)} = \langle v, T^*Sv'\rangle = a(w, w')\langle v, v'\rangle.$$

Changing the variable $g \mapsto g^{-1}$, we obtain a similar identity

$$\int_G \langle \pi(g)v, w\rangle\overline{\langle \pi(g)v', w'\rangle}dg = a(v', v)\langle w', w\rangle$$

Pick any $x \in V_\pi$ such that $Tx \neq 0$, which always exists by assumption; then $a(x, x) > 0$ as $T = S$ in this case, and hence $T^*S = T^*T$ is positive and nonzero. Then

$$a(w, w') = \frac{\langle x, x\rangle}{a(x, x)}\langle w', w\rangle,$$

so that

$$\int_G \langle \pi(g)v, w\rangle\overline{\langle \pi(g)v', w'\rangle}dg = \frac{\langle x, x\rangle}{a(x, x)}\langle w', w\rangle\langle v, v'\rangle$$

∎

14.5 Smooth Vectors

Let G be a Lie group and (π, V) a Fréchet space representation of G. For $X \in \mathrm{Lie}(G)$ and $v \in V$, we define

$$\pi(X)v := \left.\frac{d}{dt}\right|_{t=0} \pi(\exp(tX))v = \lim_{t\to 0} \frac{\pi(\exp(tX))v - v}{t}$$

if the limit exists. Note that if $\pi(X)v$ and $\pi(X)w$ exist, so does $\pi(X)(av + bw)$ for $a, b \in \mathbb{C}$.

Definition. Let $\pi : G \to \mathrm{GL}_{\mathrm{cts}}(V)$ be a representation of a Lie group G in a Banach space V.

(i) A vector $v \in V$ is called $\mathbf{C^1}$ if the limit $\pi(X)v$ exists for all $X \in \mathrm{Lie}(G)$.

(ii) Inductively, a vector $v \in V$ is called $\mathbf{C^k}$ $(\mathbf{k > 1})$ if v is C^1 and $\pi(X)v$ is C^{k-1} for all $X \in \mathrm{Lie}(G)$.

(iii) A vector $v \in V$ is called **smooth** if $v \in C^k$ for each $k \geqslant 1$.

Denote by V^∞ the subspace of all smooth vectors in the representation space V. Hence we obtain a linear map $\pi : \mathrm{Lie}(G) \to \mathrm{End}_{\mathbb{C}} V^\infty$.

Lemma 14.5.1. The subspace V^∞ is G-invariant. More precisely, the subspace of C^k-vectors in V is G-invariant.

Proof. Let v be C^1. For $g \in G$ and $X \in \mathrm{Lie}(G)$, we have

$$\pi(\exp(tX))\pi(g)v = \pi(g)\pi(g^{-1}\exp(tX)g)v = \pi(g)\pi(\exp(t\,\mathrm{Ad}(g^{-1})X))v.$$

Since $\pi(g)$ is bounded and $\mathrm{Ad}(g^{-1})X \in \mathrm{Lie}(G)$, it follows that $\pi(X)\pi(g)v$ exists for each $X \in \mathrm{Lie}(G)$. Hence $\pi(g)v$ is C^1 for each $g \in G$.

Assume $k > 1$ and let v be C^k. Hence $\pi(X)v$ is C^{k-1} for each $X \in \mathrm{Lie}(G)$, and by induction on k we see $\pi(g)\pi(X)v$ is C^{k-1} for each $g \in G$. But the above identity says that

$$\pi(X)\pi(g)v = \pi(g)\pi(\mathrm{Ad}(g^{-1})X)v$$

so $\pi(X)\pi(g)v$ is C^{k-1}. Hence $\pi(g)v$ is C^k as long as v is C^k. Taking intersection proves that V^∞ is G-invariant. ☐

For a topological vector space V, we write $\langle\, , \rangle : V \times V^\vee \to \mathbb{C}$ for the canonical pairing:

$$\langle v, \varphi \rangle := \varphi(v).$$

Lemma 14.5.2. The linear map $\pi : \mathrm{Lie}(G) \to \mathrm{End}_{\mathbb{C}} V^\infty$ is a Lie algebra homomorphism.

Proof. It comes down to showing that for $X, Y \in \mathrm{Lie}(G)$ and $v \in V^\infty$,

$$\pi(X)\pi(Y)v - \pi(Y)\pi(X)v = \pi([X,Y])v.$$

For this, for $\varphi \in V^\vee$ we introduce the map

$$\begin{aligned} I = I_\varphi : V^\infty &\longrightarrow C(G) \\ v &\longmapsto c_{v,\varphi} : g \mapsto \langle \pi(g)v, \varphi \rangle \end{aligned}$$

Recall $\mathrm{Lie}(G)$ acts on $C^\infty(G)$ by differentiation:

$$(Xf)(g) := \left.\frac{d}{dt}\right|_{t=0} f(g \exp_G(tX)).$$

We claim the followings:

(i) For $v \in V^\infty$, the function $c_{v,\varphi}$ is smooth. Hence I goes into $C^\infty(G)$.

(ii) I intertwines $\mathrm{Lie}(G)$-action.

For (i), let $X \in \mathrm{Lie}(G)$ and compute

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} c_{v,\varphi}(g \exp tX) &= \left.\frac{d}{dt}\right|_{t=0} \varphi(\pi(g)\pi(\exp tX)v) \\ &= (\varphi \circ \pi(g)) \left(\left.\frac{d}{dt}\right|_{t=0} \pi(\exp(tX))v \right) = (\varphi \circ \pi(g))(\pi(X)v) = c_{\pi(X)v,\varphi}(g). \end{aligned}$$

By Lemma I.2.19, a function $f \in C(G)$ is smooth if and only if it is C^1 and Xf exists and is smooth for all $X \in \mathrm{Lie}(G)$. Hence the above computation shows that $I(v) \in C^\infty(G)$. The above identity simultaneously proves (ii).

Now to show $\pi(X)\pi(Y)v - \pi(Y)\pi(X)v = \pi([X,Y])v$, by Hahn-Banach's theorem it suffices to show

$$\varphi(\pi(X)\pi(Y)v - \pi(Y)\pi(X)v) = \varphi(\pi([X,Y])v),$$

or $I_\varphi(\pi(X)\pi(Y)v) - I_\varphi(\pi(Y)\pi(X)v) = I_\varphi(\pi([X,Y])v)$ for all $\varphi \in V^\vee$. By (ii) it suffices to show

$$(XY - YX)I_\varphi(v) = [X,Y]I_\varphi(v).$$

This follows directly from Lemma I.2.20. ☐

Lemma 14.5.3. For $f \in C_c^\infty(G)$ and $v \in V$, we have $\pi(f)v \in V^\infty$. Moreover, the subspace V^∞ is dense in V.

Proof. For $X \in \mathrm{Lie}(G)$, we have

$$\pi(\exp(tX))\pi(f)v = \pi(\exp(tX))\int_G f(g)\pi(g)vdg = \int_G f(g)\pi(\exp(tX)g)vdg = \int_G f(\exp(-tX)g)\pi(g)vdg.$$

Put $f_X(g) := \left.\frac{d}{dt}\right|_{t=0} f(\exp(-tX)g)$, which exists and is smooth with compact support as f is. Since f has compact support, differentiation and integral commute, which implies

$$\pi(X)\pi(f)v = \left.\frac{d}{dt}\right|_{t=0}\pi(\exp(tX))\pi(f)v = \int_G \left.\frac{d}{dt}\right|_{t=0} f(\exp(-tX)g)\pi(g)vdg = \pi(f_X)v.$$

This shows $\pi(f)v$ is C^1, and induction shows $\pi(f)v \in V^\infty$. The density result follows from Lemma 14.3.3 and the fact that we can pick ϕ_U there to be smooth, using smooth Urysohn's lemma. ☐

Now let V be Banach. Define a semi-norm $p_{X_1,\ldots,X_n} : V^\infty \to \mathbb{R}_{\geqslant 0}$ by

$$p_{X_1,\ldots,X_n}(v) := \|\pi(X_1) \circ \cdots \circ \pi(X_n)(v)\|$$

When $n = 0$, the semi-norm $p_\varnothing$ is the original norm on V. We equip V^∞ with the topology defined by these semi-norms. Since $\dim \mathrm{Lie}\, G < \infty$, we only need countably many semi-norms to generate the topology. Since $p_\varnothing$ is a norm, the topology is Hausdorff.

Lemma 14.5.4. V^∞ is a Fréchet space.

Proof. It remains to show V^∞ is complete. Let $(v_m)_{m\geqslant 1}$ be a Cauchy sequence in the Fréchet topology. In particular, for any $n \geqslant 1$ and $X_1, \ldots, X_n$ (possibly repeated) in $\mathrm{Lie}(G)$, then

$$(\pi(X_1) \circ \cdots \circ \pi(X_n)v_m)_{m\geqslant 0}$$

is a Cauchy sequence in the norm topology of V. Set

$$v_{X_1,\ldots,X_n} := \lim_{m\to\infty} \pi(X_1) \circ \cdots \circ \pi(X_n)v_m.$$

In particular, put $v := v_\varnothing$.

☐

14.6 Restriction to compact subgroups

Let G be an LCH group, and $K \leqslant G$ a compact subgroup.

Lemma 14.6.1. Let $\pi : G \to \mathrm{GL}_{\mathrm{cts}}(H)$ be a Hilbert space representation. Then there exists a K-invariant inner product on H inducing an equivalent norm on H.

Proof. Denote by $(\,,)$ be the inner product on H. Define the pairing $\langle\,,\rangle : H \times H \to \mathbb{C}$ by

$$\langle v, w\rangle := \int_K (\pi(k)v, \pi(k)w)dk$$

where dk is the Haar measure on K so that $\mathrm{vol}(K, dk) = 1$. This clearly defines a K-invariant inner product on H, and $\pi|_K$ becomes unitary as dk is unimodular.

It remains to show $(\,,)$ and $\langle\,,\rangle$ induce the equivalent norms. Let $\|\cdot\|$ denote the norm induced by $\langle\,,\rangle$. For any $v \in H$, the map $K \ni k \mapsto \|\pi(k)v\| \in \mathbb{R}$ is continuous, so it attains a maximum by compactness of K. It follows by uniform boundedness principle that there exists $C > 0$ such that $\|\pi(k)\|_{\mathrm{op}} \leqslant C\,\|v\|$ for any $k \in K$, $v \in H$. In particular, we have $(v, v) \leqslant C^2(\pi(k)v, \pi(k)v)$ for any $k \in K$, $v \in H$. Then

$$C^{-2}(v, v) \leqslant \langle v, v\rangle \leqslant C^2\langle v, v\rangle$$

as $\mathrm{vol}(K, dk) = 1$. This finishes the proof. ☐

Lemma 14.6.2. Let (π_1, H_1) and (π_2, H_2) be two representations of K with π_2 unitary. If m_1, m_2 are two matrix coefficients for π_1, π_2 respectively that are not orthogonal in $L^2(K)$, then there exists a nontrivial intertwiner $I : H_1 \to H_2$.

Proof. Say $m_i = [k \mapsto \langle\pi_i(k)x_i, y_i\rangle]$ for some $x_i, y_i \in H_i$. Define $I : H_1 \to H_2$ by

$$I(x) = \int_K \langle\pi_1(k)x, y_1\rangle\pi_2(k)^{-1}y_2dk$$

☐

14.7 Functions of positive type

Definition. Let G be an LCH group and μ a left Haar measure. A **function of positive type** on G is a function $\phi \in L^\infty(G)$ such that the integration

$$L^1(G) \ni f \mapsto \int_G (f^* * f)\phi d\mu$$

defines a positive linear functional on $L^1(G)$.

- If ϕ is of positive type, then so is $\overline{\phi}$. Indeed,

  $$\int_G (f^* * f)\overline{\phi}d\mu = \overline{\int_G ((\overline{f}^*) * \overline{f})\phi d\mu}$$

  and the latter is positive as ϕ is of positive type.

In the following, to ease notational cumbersome we use dx (dy, dz and so on) to denote a choice of a left Haar measure on G.

Lemma 14.7.1. A function $\phi \in L^\infty(G)$ is of positive type if and only if

$$\int_{G\times G} f(x)\overline{f(y)}\phi(y^{-1}x)dx \otimes dy \geqslant 0.$$

Proof. This is an application of Fubini: write

$$\int_G (f^* * f)\phi dx = \int_G \left(f^*(y)f(y^{-1}x)dy\right)\phi(x)dx = \int_{G\times G} \Delta_G(y^{-1})\overline{f(y^{-1})}f(y^{-1}x)\phi(x)dx \otimes dy.$$

Performing $y \mapsto y^{-1}$ and then $x \mapsto y^{-1}x$ finishes the proof. ☐

Lemma 14.7.2. If (π, V_π) is a unitary representation and $v \in V_\pi$, then the matrix coefficient $c_{v,v} : x \mapsto \langle \pi(x)v, v\rangle$ is of positive type.

Proof. Since π is unitary, we have

$$c_{v,v}(y^{-1}x) = \langle \pi(y^{-1}x)v, v\rangle = \langle \pi(x)v, \pi(y)v\rangle.$$

Inserting this into the integral in Lemma 14.7.1, we see

$$\int_{G\times G} f(x)\overline{f(y)}c_{v,v}(y^{-1}x)dx \otimes dy = \langle \pi(f)v, \pi(f)v\rangle = \|\pi(f)\|^2 \geqslant 0.$$

☐

Let $\phi \in L^\infty(G)$ be a function of positive type. By definition, the pairing

$$L^1(G)^2 \ni (f, g) \mapsto (f, g)_\phi := \int_G (g^* * f)\phi d\mu$$

defines a hermitian pairing and satisfies the inequality

$$|(f, g)_\phi| \leqslant \|\phi\|_\infty \|f\|_1 \|g\|_1 .$$

Denote by $N = N_\phi = \{f \in L^1(G) \mid (f, f)_\phi = 0\}$. By Lemma 13.1.4, we have

$$|(f, g)_\phi|^2 \leqslant (f, f)_\phi (g, g)_\phi$$

so $f \in N$ if and only if $(f, g)_\phi = 0$ for every $g \in L^1(G)$. Hence, $(\cdot, \cdot)_\phi$ descends to a non-degenerate hermitian pairing on the quotient space $L^1(G)/N$. Denote

$$H_\phi = \text{ the Hilbert space completion of } L^1(G)/N \text{ with respect to } (\cdot, \cdot)_\phi$$

and put $\|f\|_\phi := (f, f)_\phi^{\frac{1}{2}}$.

Lemma 14.7.3. One has $L_g(N) \subseteq N$ for each $g \in G$. Hence the left translation L descends an unitary representation

$$\pi_\phi : G \longrightarrow \mathrm{GL}_{\mathrm{cts}}(H_\phi).$$

The corresponding algebra representation $\pi_\phi : L^1(G) \to \mathcal{B}(H_\phi)$ is given by

$$\pi_\phi(f)(g) = f * g \bmod N$$

for $f, g \in L^1(G)/N$.

Proof. For $g \in G$, we have

$$(L_g f, L_g f)_\phi = \int_{G\times G} f(g^{-1}x)\overline{f(g^{-1}y)}\phi(y^{-1}x)dx \otimes dy.$$

By a change of variables $(x, y) \mapsto (gx, gy)$, this becomes $(f, f)_\phi$. This proves $L_g(N) \subseteq N$. The last assertion more or less follows from Lemma D.7.4 add an example. ☐

Theorem 14.7.4. Let ϕ be a function of positive type on G, and let $\pi_\phi : G \to \mathcal{B}(H_\phi)$ be the unitary representation constructed above. Then there exists a vector $v \in H_\phi$ such that

$$\pi_\phi(f)v = f \text{ mod } N_\phi$$

for each $f \in L^1(G)$, and $\phi = c_{v,v}$ locally a.e.

14.8 Group C^*-algebras

Let G be an LCH group. For each irreducible unitary representation (π, V_π), we have a representation $\pi : L^1(G) \to \mathcal{B}(V_\pi)$ of Banach *-algebra. Define

$$\begin{aligned} \|\cdot\|_\pi : L^1(G) &\longrightarrow \mathbb{R}_{\geqslant 0} \\ f &\longmapsto \|\pi(f)\|_{\mathrm{op}}. \end{aligned}$$

This is a seminorm on $L^1(G)$, and if π and τ are unitarily equivalent, then $\|f\|_\pi = \|f\|_\tau$ clearly. In this way we get a collection of seminorms

$$\{\|\cdot\|_\pi\}_{\pi\in\widehat{G}}$$

indexed by the unitary dual $\widehat{G}$ of G.

Chapter 15

Compact Groups

15.1 Finite Dimensional Representations

In this section, let K be a compact (Hausdorff topological) group and (τ, V_τ) a finite dimensional (complex) representation of K.

Lemma 15.1.1. On V_τ there exists an inner product making τ a unitary representation. If τ is irreducible, this inner product is determined up to a positive scalar.

Proof. Let $(\,,)$ be any inner product on V_τ, and define a map $\langle\,,\rangle : V_\tau \times V_\tau \to \mathbb{C}$ by the formula

$$\langle v, w\rangle := \int_K (\tau(k)v, \tau(k)w)dk$$

where dk is the normalized Haar measure on K so that $\mathrm{vol}(K, dk) = 1$. This clearly defines an inner product on V_τ, and τ becomes a unitary representation because dk is unimodular.

Now assume τ is irreducible and let $\langle\,,\rangle_1, \langle\,,\rangle_2$ be two inner products making τ unitary. For $i = 1, 2$, denote by (τ_i, V_i) the representation (τ, V_τ) when equipped with the inner product $\langle\,,\rangle_i$. Since V_τ is finite dimensional, $\mathrm{id} : V_1 \to V_2$ is bounded nonzero intertwining operator from τ_1 to τ_2. Since V_τ is irreducible, by Corollary 14.2.6.1 (in which Schur's lemma can be replaced by the usual Schur's lemma) there exists $c > 0$ such that $c \cdot \mathrm{id}$ is unitary. But this implies that $c^2\langle v, w\rangle_2 = \langle v, w\rangle_1$ for all $v, w \in V_\tau$. ☐

Remark 15.1.2. In fact, the first paragraph of the proof does not use the finite dimensionality of V_τ: it holds for any Hilbert space representation of K. The second part of this lemma does not holds in general, but when V_τ is a Hilbert space, we can show that the constructed unitary representation defines an equivalent norm to the original one.

To prove this, let V_τ be a Hilbert space with inner product $(\,,\,)$. For any $g \in K$, define $T_g : V_\tau \to V_\tau$ by $T_g(v) = \pi(g)v$. For any $v \in V$, the map $K \to \mathbb{R}$, $g \mapsto \|\pi(g)v\|$ is continuous, so it attains a maximum by compactness of K. It follows by uniform boundedness principle that there exists $C > 0$ such that $\|T_g(v)\| \leqslant C\,\|v\|$ for any $g \in K$, $v \in V_\tau$. In particular, we have $(v, v) \leqslant C^2(\pi(g)v, \pi(g)v)$ for any $g \in K$, $v \in V_\tau$. The inner product $\langle,\,\rangle : V_\tau \times V_\tau \to \mathbb{C}$ defined by

$$\langle v, w\rangle := \int_K (\pi(g)v, \pi(g)w)dg$$

then satisfies $C^{-2}(v, v) \leqslant \langle v, v\rangle \leqslant C^2\langle v, v\rangle$ (as $\mathrm{Vol}(K) = 1$). This shows the equivalence (as Banach spaces).

Proposition 15.1.3. A finite dimensional representation of a compact group is a direct sum of irreducible representations.

Proof. We prove this by induction on the dimension of representations; the cases dim $= 0, 1$ are evident. Let (τ, V) be a representation with $\dim_{\mathbb{C}} V \geqslant 2$. If τ is irreducible, we are done. Otherwise, V admits a nonzero proper invariant subspace U. By the previous lemma, we equip V with an inner product so that τ is unitary. Let W be the orthogonal complement of U. If we can show W is also invariant, the result will follow from the induction. Now let $w \in W$ and $k \in K$. Then for all $u \in U$

$$\langle \tau(k)w, u\rangle \overset{\tau \text{ unitary}}{=} \langle w, \underbrace{\tau(k^{-1})u}_{\in U}\rangle = 0$$

so that $\tau(k)w \in W$. This proves W is invariant under K. $\square$

Definition. Let (τ, V) be a finite dimensional complex representation of a compact group K. The dual space $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ carries a natural representation of K, called the **contragredient representation** τ^*, defined by

$$\tau^*(k)\alpha(v) = \alpha(\tau(k^{-1})v)$$

for all $k \in K$, $\alpha \in V^*$, $v \in V$.

Suppose V is equipped with an inner product so that τ is unitary. By Riesz's Representation theorem for every $\alpha \in V^*$ there exists a unique vector v_α such that

$$\alpha(w) = \langle w, v_\alpha\rangle$$

for all $w \in V$. One installs an inner product on V^* by

$$\langle \alpha, \beta\rangle := \langle v_\beta, v_\alpha\rangle$$

Lemma 15.1.4. If (τ, V_τ) is irreducible (resp. unitary), then so is the contragredient τ^*. Install V_τ with the inner product making τ unitary. Then for $x \in K$ and $\alpha \in V_\tau^*$, one has

$$v_{\tau^*(x)\alpha} = \tau(x)v_\alpha$$

so the map $\alpha \mapsto v_\alpha$ is an antilinear K-intertwining operator from V_τ^* to V_τ.

Proof. Let $U \subseteq V_\tau^*$ be a K-invariant subspace; then $U^\perp := \{v \in V_\tau \mid \alpha(v) = 0 \text{ for all } \alpha \in U\}$ is a K-invariant subspace of V. Thus if V_τ is irreducible, then $U^\perp$ is either 0 or V_τ, whence $U = 0$ or V_τ^*.

Next we show the intertwining relation. For $w \in V_\tau$,

$$\langle w, v_{\tau^*(x)\alpha}\rangle = \tau^*(x)\alpha(w) = \alpha(\tau(x^{-1})w) = \langle \tau(x^{-1})w, v_\alpha\rangle = \langle w, \tau(x)v_\alpha\rangle$$

here we use τ is unitary. Hence $v_{\tau^*(x)\alpha} = \tau(x)v_\alpha$ holds for all $v \in V_\tau$. Finally,

$$\langle \tau^*(x)\alpha, \tau^*(x)\beta\rangle = \langle v_{\tau^*(x)\beta}, v_{\tau^*(x)\alpha}\rangle = \langle \tau(x)v_\beta, \tau(x)v_\alpha\rangle = \langle v_\beta, v_\alpha\rangle = \langle \alpha, \beta\rangle$$

so that τ^* is unitary as well. $\square$

Definition. Let (τ, V_τ) and (γ, V_γ) be finite dimensional representations of a compact group K. There is a natural representation of the group $K \times K$ on the tensor product space $V_\tau \otimes V_\gamma$ given by

$$(\tau \otimes \gamma)(k_1, k_2) = \tau(k_1) \otimes \gamma(k_2) \in \mathrm{GL}(V_\tau \otimes V_\gamma)$$

15.2 The Peter-Weyl Theorem

Let K be a compact group and let $\widehat{K}$ denote its unitary dual. Let $\widehat{K}_{\text{fin}}$ be the subset of $\widehat{K}$ consisting of all finite dimensional irreducible representations. For convenience, we normalize dk so that $\text{vol}(K, dk) = 1$.

Theorem 15.2.1. Let K be a compact group.

(a) $\widehat{K} = \widehat{K}_{\text{fin}}$, i.e., every irreducible unitary representation a compact group is finite dimensional.

(b) Every unitary representation of K is an orthogonal direct sum of irreducible representations.

Proof. Let (π, V_π) be a unitary representation of K. We will show that V_π can be written as a direct sum

$$V_\pi = \widehat{\bigoplus_{i\in I}} V_i$$

with each V_i a finite dimensional irreducible subrepresentation of V_π. This proves (b), and if we apply this to a given irreducible unitary representation, it also implies (a).

So let (π, V_π) be a unitary representation of K. Consider the collection S of all families $(V_i)_{i\in I}$, where each V_i is a finite dimensional irreducible subrepresentation of V_π and $V_i \perp V_j$ whenever $i \neq j \in I$. Introduce a partial order $\leqslant$ on S by

$$(V_i)_{i\in I} \leqslant (W_\alpha)_{\alpha\in A} \Leftrightarrow I \subseteq A \text{ and } V_i = W_i \text{ for all } i \in I$$

By Zorn's lemma, S admits a maximal element, denoted by $(V_i)_{i\in I}$. We contend that the orthogonal (algebraic) direct sum $\bigoplus_{i\in I} V_i$ is dense in V, which is equivalent to the orthogonal space $W := \left(\bigoplus_{i\in I} V_i\right)^{\perp}$ being the zero space.

Assume for contradiction that $W \neq 0$. By maximality, it suffices to show the following.

Lemma 15.2.2. Every nonzero unitary representation (π, V_π) contains a finite dimensional irreducible subrepresentation.

Proof. Let (π, V_π) be a unitary representation of K. Let $T_0 \in \text{Hom}_{\mathbb{C}}(V_\pi, V_\pi)$ be the orthogonal projection onto a nonzero finite dimensional subspace W of V. Define $T \in \text{Hom}_K(V_\pi, V_\pi)$ by

$$Tv = \int_K \pi(g)^{-1} T_0 \pi(g) v dg.$$

The integrand lands in a finite dimensional subspace W, so T is of finite rank. Moreover, T is self-adjoint as

$$\langle Tv, w\rangle = \int_K \langle T_0\pi(g)v, \pi(g)w\rangle dg = \int_K \langle \pi(g)v, T_0\pi(g)w\rangle dg = \langle v, Tw\rangle$$

To see this map is nonzero, suppose for contradiction that $Tv = 0$ for all $v \in V_\pi$. Then

$$0 = \langle Tv, v\rangle = \int_K \langle T_0\pi(g)v, \pi(g)v\rangle dg = \int_K \|T_0\pi(g)v\|^2 \, dg.$$

This implies $T_0\pi(g)v = 0$ for all $v \in V_\pi$ and $g \in K$, as the map $g \mapsto T_0\pi(g)v$ is continuous. This is absurd, as $T_0 v \neq 0$ for $v \in \text{Im}\, T_0 \backslash \{0\}$.

Now by Spectral theorem for compact normal operators, there exists $\lambda \in \mathbb{C}^\times$ such that $0 < \dim\ker(T - \lambda\,\text{id}) < \infty$. In other words, we have find a nonzero finite dimensional subrepresentation of V_π, namely $\ker(T - \lambda\,\text{id})$. ∎

$\square$

Definition. For a unitary representation (τ, V_τ) of K, a **matrix coefficient** is a function on K of the form $k \mapsto \langle \tau(k)v, w\rangle$ for some $v, w \in V_\tau$.

- A matrix coefficient is continuous, so it lies in the Hilbert space $L^2(K)$.
- The set of matrix coefficients of all finite dimensional representations is closed under complex conjugation. To see this, let $v, v' \in V_\tau$ and $\alpha, \beta \in V_\tau^*$ be their Riesz's dual, i.e., $v = v_\alpha$, $v' = v_\beta$. Then by Lemma 15.1.4,

$$\overline{\langle \tau(x)v, v'\rangle} = \langle v_\beta, \tau(x)v_\alpha\rangle = \langle v_\beta, v_{\tau^*(x)\alpha}\rangle = \langle \tau^*(x)\alpha, \beta\rangle$$

Lemma 15.2.3 (Schur's orthogonality)**.** For $\pi, \tau \in \widehat{K}$, one has

$$\int_K \langle \pi(k)v, w\rangle\overline{\langle \tau(k)x, y\rangle}dk = \frac{\delta_{\pi,\tau}\langle v, x\rangle\langle y, w\rangle}{\dim V_\pi}$$

for $v, w \in V_\pi$ and $x, y \in V_\tau$. Here

$$\delta_{\pi,\tau} = \begin{cases} 1 & \text{, if } \pi = \tau \text{ in } \widehat{K} \\ 0 & \text{, otherwise} \end{cases}$$

and we take $V_\pi = V_\tau$ when $\pi \cong \tau$ so the inner products on RHS makes sense.

Proof. This is a direct consequence of Corollary 14.4.2.1.(i), modulo the computation $d(\pi) = \dim V_\pi$. However, it is better to give a direct proof here.

Define $T : V_\pi \to V_\tau$ by the formula

$$Tv = \int_K \langle \pi(k)v, w\rangle\tau(k)^{-1}y dk.$$

Since

$$T(\pi(g)v) = \int_K \langle \pi(kg)v, w\rangle\tau(k)^{-1}y dk = \int_K \langle \pi(k)v, w\rangle\tau(kg^{-1})^{-1}y dk = \tau(g)Tv$$

the map T is K-intertwining. If $\pi \not\cong \tau$, the certainly $T = 0$. But then

$$\begin{aligned} 0 = \langle Tv, x\rangle = \left\langle \int_K \langle \pi(kg)v, w\rangle\tau(k)^{-1}y dk, x\right\rangle &= \int_K \langle \pi(kg)v, w\rangle\langle \tau(k)^{-1}y, x\rangle dk \\ &= \int_K \langle \pi(k)v, w\rangle\overline{\langle \tau(k)x, y\rangle}dk. \end{aligned}$$

Now suppose $\pi = \tau$. Pick an orthonormal basis $\{e_i\}$ for V_π. Then

$$\begin{aligned} \operatorname{tr} T = \sum_i \langle Te_i, e_i\rangle &= \int_K \sum_i \langle \pi(k)e_i, w\rangle\overline{\langle \pi(k)e_i, y\rangle}dk \\ &= \int_K \sum_i \langle e_i, \pi(k)^{-1}w\rangle\overline{\langle e_i, \pi(k)^{-1}y\rangle}dk \\ &= \int_K \langle \pi(k)^{-1}y, \pi(k)^{-1}w\rangle dk = \operatorname{vol}(K, dk)\langle y, w\rangle = \langle y, w\rangle \end{aligned}$$

where we use Parseval's identity in the fourth equality. If $T = 0$, then $\langle y, w\rangle = 0$. If $T \neq 0$, then by Schur's lemma that $T = c\,\mathrm{id}$ for some $c \in \mathbb{C}^\times$. But then

$$c \dim V_\pi = \operatorname{tr} T = \langle y, w\rangle$$

so $\langle y, w\rangle \neq 0$ and $c = \frac{\langle y,w\rangle}{\dim V_\pi}$. Hence

$$\int_K \langle \pi(k)v, w\rangle\overline{\langle \pi(k)x, y\rangle}dk = \langle Tv, x\rangle = \frac{\langle y,w\rangle\langle v,x\rangle}{\dim V_\pi}.$$

$\square$

For each $\pi \in \widehat{K}$, consider the map

$$\begin{aligned} \pi^* \otimes \pi &\longrightarrow L^2(K) \\ \phi \otimes v &\longmapsto c_{\phi,v} : g \mapsto \phi(\pi(g)v) \end{aligned}$$

Let $K \times K$ act on $L^2(K)$ by $(k_1, k_2)f(x) := f(k_1^{-1}xk_2)$, and let $K \times K$ act on $\pi^* \otimes \pi$ separately. Then the above map is $K \times K$-intertwining. Indeed

$$c_{\phi,v}(k_1^{-1}gk_2) = \phi(\pi(k_1^{-1}gk_2)v) = \pi^*(k_1)\phi(\pi(g)\pi(k_2)v) = c_{\pi^*(k_1)\phi,\pi(k_2)v}(g).$$

Let $\{e_i\}$ be an ONB of V_π and let $e_i^\vee$ denote the unique vector in V_π^* such that $e_j^\vee(e_i) = \langle e_i, e_j\rangle = \delta_{ij}$. Then

$$\left\|\sum_{ij} a_{ij}e_j^\vee \otimes e_i\right\|^2_{\pi^*\otimes\pi} = \sum_{i,j,k,l} a_{ij}\overline{a_{kl}}\langle e_j^\vee \otimes e_i, e_l^\vee \otimes e_k\rangle = \sum_{ij}|a_{ij}|^2$$

while by Lemma 15.2.3

$$\left\|\sum_{ij} a_{ij}c_{e_j^\vee,e_i}\right\|^2_{L^2(K)} = \sum_{i,j,k,l} a_{ij}\overline{a_{kl}}\int_K \langle \pi(x)e_i, e_j\rangle\overline{\langle \pi(x)e_k, e_l\rangle}dx = \frac{1}{\dim V_\pi}\sum_{i,j,k,l} a_{ij}\overline{a_{kl}}\langle e_i, e_k\rangle\langle e_j, e_l\rangle$$

Hence, if we normalize the above map

$$\begin{aligned} \pi^* \otimes \pi &\longrightarrow L^2(K) \\ \phi \otimes v &\longmapsto (\dim V_\pi)^{\frac{1}{2}}c_{\phi,v} \end{aligned}$$

then this is an isometry. Varying $\pi \in \widehat{K}$, we obtain an $K \times K$-intertwining map

$$\bigoplus_{\pi\in\widehat{K}} \pi^* \otimes \pi \longrightarrow L^2(K)$$

By Lemma 15.2.3 this is again an isometry.

Theorem 15.2.4 (Peter-Weyl)**.** The map

$$\bigoplus_{\pi\in\widehat{K}} \pi^* \otimes \pi \longrightarrow L^2(K)$$

has dense image. Consequently,

$$\widehat{\bigoplus_{\pi\in\widehat{K}}} \pi^* \otimes \pi \longrightarrow L^2(K)$$

is an Hilbert space $K \times K$-isomorphism.

Proof. Denote by M the image. We must show $M^\perp = 0$. Suppose otherwise that $M^\perp \neq 0$. Then it is a nonzero unitary subrepresentation of $L^2(K \times K)$. By Lemma 15.2.2 it contains some irreducible subrepresentation, say $(\tau, M_\tau) \subseteq M^\perp$ under right regular representation. For $f, g \in M_\tau$, the corresponding matrix coefficient $c_{f,g} : (k) \mapsto \langle \tau(k) f, g \rangle_{L^2(K)}$ lies in $M^\perp$. Indeed,

$$c_{f,g}(k) = \int_K \tau(k) f(x) g(x) dx = \int_K f(xk) g(x) dx = \int_K f(x^{-1}k)\overline{g^*(x)} dx = L(\overline{g^*}) f(k).$$

Since $M^\perp$ is $K \times K$-invariant, it is a subrepresentation of K under left regular representation. Since $f \in M_\tau \subseteq M^\perp$, this proves $c_{f,g} \in M^\perp$. But being a matrix coefficient, it must lie in the image of $\tau^* \otimes \tau \to L^2(K)$, i.e., lie in M. This forces $c_{f,g} = 0$ for all $g, f \in M^\perp$, which is absurd. Hence $M^\perp = 0$, as we claim. □

Corollary 15.2.4.1. We have

$$L \cong \widehat{\bigoplus_{\tau \in \widehat{K}}} \tau^* \otimes 1_{V_\tau} \qquad \text{and} \qquad R \cong \widehat{\bigoplus_{\tau \in \widehat{K}}} 1_{V_{\tau^*}} \otimes \tau$$

for the left and right regular representations of K. Here 1_{V_τ} means the trivial representation on V_τ.

15.2.1 Characters

Definition. Let π be a finite dimensional representation of a compact group K. The function $\chi_\pi : K \to \mathbb{C}$ defined by

$$\chi_\pi(k) := \operatorname{tr} \pi(k)$$

is called the **character** of the representation π.

- We can write χ_π as a sum of matrix coefficients. Let (e_i) be the chosen orthonormal basis for V_π. Then

$$\operatorname{tr} \pi(k) = \sum_{i=1}^{\dim(\pi)} \langle \pi(k) e_i, e_i \rangle = \sum_{i=1}^{\dim(\pi)} \pi_{ii}(k)$$

- If π and τ are unitarily equivalent, then $\chi_\pi = \chi_\tau$. To see this, say $\alpha : \pi \to \tau$ is a unitary K-isomorphism. If e_i is an orthonormal basis for π, then for all $k \in K$

$$\operatorname{tr} \tau(k) = \sum \langle \tau(k) \alpha e_i, \alpha e_i \rangle = \sum \langle \alpha \pi(k) e_i, \alpha e_i \rangle = \sum \langle \pi(k) e_i, e_i \rangle = \operatorname{tr} \pi(k)$$

Theorem 15.2.5. Let π, η be two irreducible unitary representations of the compact group K. For their characters, we have

$$\langle \chi_\pi, \chi_\eta \rangle = \delta_{\pi\eta} = \begin{cases} 1 & \text{, if } \pi = \tau \text{ in } \widehat{K} \\ 0 & \text{, else} \end{cases}$$

Here the inner product is the one of $L^2(K)$.

Proof. Follow from Lemma 15.2.3. □

Let π be any finite dimensional representation of K. By Proposition 15.1.3 we can write π as a direct sum of K-subrepresentations. This shows χ_π is a finite sum of characters of representations appearing in π. In particular,

Corollary 15.2.5.1. Let π be a finite dimensional representation of K. Then π is irreducible if and only if $\langle \chi_\pi, \chi_\pi \rangle = 1$.

15.2.2 Fourier Transform

We now give a reformulation of the Peter-Weyl theorem.

Lemma 15.2.6. Let $\pi \in \widehat{K}$. Then the map

$$\begin{aligned} \pi^* \otimes \pi &\longrightarrow \operatorname{End} V_\pi = \mathcal{HS}(V_\pi) \\ \phi \otimes v &\longmapsto [w \mapsto \phi(w)v] \end{aligned}$$

is an isometry. Moreover, the map intertwines $K \times K$-action, where for $T \in \operatorname{End} V_\pi$, $(k_1, k_2)T = \pi(k_2) \circ T \circ \pi(k_1)^{-1}$.

Proof. The first statement follows from Theorem 12.3.1 or any elementary computation. The second assertion follows at once:

$$(k_1, k_2)[w \mapsto \phi(w)v] = \phi(\pi(k_1)^{-1}w)\pi(k_2)v = [w \mapsto \pi^*(k_1)\phi(w)\pi(k_2)v].$$

□

Let $\pi \in \widehat{K}$, $v, w \in V_\pi$ and $f = c_{v,w} : k \mapsto \langle \pi(k)v, w\rangle$ be a matrix coefficient. For $\tau \in \widehat{K}$, we consider the operator $\tau(f) \in \operatorname{End} V_\tau$. Using orthogonality, we can compute this explicitly. For $x, y \in V_\tau$, one have

$$\langle \tau(f)x, y\rangle = \int_K f(k)\langle \tau(k)x, y\rangle = \int_K \langle \tau(k)x, y\rangle\langle \pi(k)v, w\rangle dk = \frac{\delta_{\tau,\pi}\langle x, v\rangle\langle w, y\rangle}{\dim V_\pi}.$$

For $u \in V_\pi$, denote $u^\vee \in V_{\pi*}$ be the unique vector such that $u^\vee(u') = \langle u', u\rangle$. One has $\pi^*(k)u^\vee(u') = u^\vee(\pi(k)^{-1}u') = \langle u', \pi(k)u\rangle$, so the map $u \mapsto u^\vee$ is conjugate-linear K-equivariant. Then

$$\langle \pi(k)v, w\rangle = \langle w^\vee, (\pi(k)v)^\vee\rangle = \langle w^\vee, \pi^*(k)v^\vee\rangle$$

and

$$\langle \tau(f)x, y\rangle = \int_K \langle \tau(k)x, y\rangle\langle w^\vee, \pi^*(k)v^\vee\rangle dk = \frac{\delta_{\tau,\pi*}\langle x, w^\vee\rangle\langle v^\vee, y\rangle}{\dim V_{\pi*}}.$$

This implies $\tau(f) = 0$ for $\tau \ncong \pi^*$. For $\tau = \pi^*$, one has

$$\langle \pi^*(f)x, y\rangle = \frac{\langle x, w^\vee\rangle\langle v^\vee, y\rangle}{\dim V_\pi}$$

so that

$$\pi^*(f) = \frac{1}{\dim V_\pi}\langle \cdot, w^\vee\rangle v^\vee.$$

Let $\{e_i\}$ be an ONB of V_π, so that $\{e_i^\vee\}$ is an ONB of $V_{\pi*}$. Then

$$\begin{aligned} \|\pi^*(f)\|_{\mathrm{HS}}^2 &= \sum_i \langle \pi^*(f)e_i^\vee, \pi^*(f)e_i^\vee\rangle = \frac{1}{\dim V_\pi}\sum_i \langle e_i^\vee, w^\vee\rangle\langle v^\vee, \pi^*(f)e_i^\vee\rangle \\ &= \frac{1}{\dim V_\pi}\sum_i \langle w, e_i\rangle\langle \pi(\overline{f})e_i, v\rangle = \frac{1}{\dim V_\pi}\langle w, \pi(\overline{f})^* v\rangle \\ &= \frac{1}{\dim V_\pi}\langle \pi(\overline{f})w, v\rangle = \frac{1}{\dim V_\pi}\langle w, w\rangle\langle v, v\rangle = \frac{1}{V_\pi}\left\|w^\vee \otimes v\right\|^2 = \|f\|_{L^2(K)}^2 \end{aligned}$$

Since $(g \mapsto \langle \pi(g)e_i, e_j\rangle)_{\pi,i,j}$ forms an ONB for $L^2(K)$, we conclude that

Lemma 15.2.7. For $f \in L^2(K)$,

$$\|f\|^2_{L^2(K)} = \sum_{\pi \in \widehat{K}} \|\pi^*(f)\|^2_{\mathrm{HS}},$$

where $\pi(f) \in \operatorname{End} V_\pi = \mathcal{HS}(V_\pi)$ is as usual:

$$\pi(f)v := \int_K f(x)\pi(x)v dx.$$

Corollary 15.2.7.1 (Peter-Weyl)**.** The map

$$\begin{aligned} L^2(K) &\xrightarrow{\sim} \widehat{\bigoplus_{\pi \in \widehat{K}}} \mathcal{HS}(V_\pi) \\ f &\longmapsto (\pi(f))_\pi \end{aligned}$$

is an isometry.

Unfortunately, the map is not $K \times K$-invariant unless we define another action on RHS by $(k_1, k_2)^\flat :\mapsto \pi(k_1) \circ T \circ \pi(k_2)^{-1}$. Indeed

$$\pi((k_1, k_2).f) = \int_K f(k_1^{-1} x k_2)\pi(x) = \int_K f(x)\pi(k_1 x k_2^{-1})dx = (k_1, k_2)^\flat \pi(f).$$

In particular, this isometry is not inverse to the isomorphism in Theorem **??**. Nevertheless, from the previous computation, the restriction to $\mathcal{H}\int(V_\pi)$ of the inverse of $f \mapsto (\pi(f))_\pi$ is

$$\phi \otimes v \mapsto [g \mapsto \dim V_\pi \langle \pi^*(g)v^\vee, \phi\rangle]$$

Definition. For a given $f \in L^2(K)$, we define the **(operator-valued) Fourier transform**

$$\begin{aligned} \widehat{f} : \widehat{K} &\longrightarrow \widehat{\bigoplus_{\pi \in \widehat{K}}} \operatorname{End} V_\tau \\ \pi &\longmapsto \pi(f) \end{aligned}$$

It satisfies

(i) $\widehat{f * g}(\pi) = \pi(f) \circ \pi(g)$ for $f, g \in L^2(G)$ and $\tau \in \widehat{K}$,

(ii) $\pi(f)^* = \pi(f^*)$ for $\tau \in \widehat{K}$, $f \in L^2(G)$ (where $f^*(x) := \overline{f(x^{-1})}$, c.f. Proposition 3.2.2),

(iii) $\|f\| = \left\|\widehat{f}\right\|$ for every $f \in L^2(K)$.

In this way the Peter-Weyl theorem presents itself as a generalization of the Plancherel formula to the compact groups.

Definition. A function $f : K \to \mathbb{C}$ is called a **class function** if $f(k^{-1}xk) = f(x)$ for all $x, k \in K$. In other words, f descends to a function on the conjugacy class of K.

- Denote by $L^2(K/\mathrm{conj})$ the space of all square-integrable class functions of K.

Corollary 15.2.7.2. $L^2(K/\mathrm{conj})$ is a closed subspace of $L^2(K)$, and $(\chi_\pi)_{\pi \in \widehat{K}}$ forms an orthonormal basis for $L^2(K/\mathrm{conj})$.

Proof. $L^2(K/\mathrm{conj}) = \bigcap_{k\in K} \ker(R_k L_k - \mathrm{id})$, so it is a closed subspace of $L^2(K)$. For the second assertion, since each trace χ_π is a class function, it suffices to show if $f \in L^2(K/\mathrm{conj})$ such that $\langle f, \chi_\pi\rangle = 0$ for all $\pi \in \widehat{K}$, then $f = 0$. By Peter-Weyl theorem we must show the Fourier transform $\widehat{f}$ vanishes, i.e., $\pi(f) = 0$ for all $\pi \in \widehat{K}$.

We first show $\pi(f) \in \operatorname{End} V_\pi$ is K-intertwining. For all $k \in K$, we compute

$$\pi(k)\pi(f) = \int_K f(x)\pi(kx)dx = \int_K f(k^{-1}x)\pi(x)dx = \int_K f(xk^{-1})\pi(x)dx = \int_K f(x)\pi(xk)dx = \pi(f)\pi(k)$$

By Schur's lemma, there exists $\lambda \in \mathbb{C}$ such that $\pi(f) = \lambda\,\mathrm{id}_{V_\pi}$. Now taking trace, one sees

$$\lambda \dim(\pi) = \operatorname{tr}\pi(f) = \int_K f(x)\operatorname{tr}\pi(x)dx = \langle f(x), \chi_{\pi^*}\rangle = 0$$

so that $\lambda = 0$, i.e., $\pi(f) = 0$. □

Corollary 15.2.7.3. For $f \in \operatorname{span}(L^2(K) * L^2(K)) \subseteq C(K)$ (c.f. Lemma 5.4.1), we have

$$f(k) = \sum_{\pi\in\widehat{K}} \dim(\pi)\widehat{R_k f}(\pi).$$

Proof. It suffices to consider $f = f_1 * f_2$ with $f_i \in L^2(K)$. For $k \in K$, by Lemma 2.6.3 we have $R_k f = f_1 * R_k f_2$. Put $f_3(g) = \overline{(R_k f_2)(g^{-1})} = (R_k f_2)^*(g)$ Then

$$\begin{aligned} f(k) = R_k f(e) &= \int_K f_1(g)(R_k f_2)(g^{-1})dg = \int_K f_1(g)\overline{f_3(g)}dg = \langle f_1, f_3\rangle_{L^2(K)} = \langle \widehat{f}_1, \widehat{f}_3\rangle \\ &= \sum_{\tau\in\widehat{K}} \dim(\tau)\operatorname{tr}(\tau(f_1)\circ\tau(f_3)^*) = \sum_{\tau\in\widehat{K}} \dim(\tau)\operatorname{tr}(\tau(f_1 * f_3^*)) = \sum_{\tau\in\widehat{K}} \dim(\tau)\operatorname{tr}(\tau(f_1 * R_k f_2)) \\ &= \sum_{\tau\in\widehat{K}} \dim(\tau)\operatorname{tr}(\tau(R_k(f_1 * f_2))) = \sum_{\tau\in\widehat{K}} \dim(\tau)\widehat{R_k f}(\tau) \end{aligned}$$

□

15.2.3 K-finite vectors and matrix coefficients

Definition. Let G be a group and V a representation of G. A vector $v \in V$ is called G**-finite** if its orbit Gv is contained in some finite dimensional subspace of V. Denote by V^{fin} the subspace of all G-finite vectors.

- It is clear that V^{fin} is invariant under G action.

Let K be a compact group. Consider the subspace $C(K)^{\mathrm{fin}}$ of finite vectors in $C(K)$. We let K acts on $C(K)^{\mathrm{fin}}$ from the left. If $\pi \in \widehat{K}$, then the matrix coefficient $k \mapsto \langle \pi(k)v, w\rangle$ is clearly K-finite, as its lands in the image of $\pi^* \otimes \pi$ in $L^2(K)$, which is a finite dimensional K-subrepresentation of $C(K)$. In fact,

Lemma 15.2.8. $C(K)^{\mathrm{fin}}$ consists exactly of matrix coefficients of K.

Proof. Let $f \in C(K)^{\mathrm{fin}}$ and V a finite dimensional K-subrepresentation of $C(K)$ containing f. By Riesz's Representation theorem, let $\alpha \in \overline{V}$ be the unique element such that $\langle h, \alpha\rangle = h(e)$ for all $h \in \overline{V}$. Then

$$\overline{f}(k) = (\ell_{g^{-1}}\overline{f})(e) = \langle \ell_{k^{-1}}\overline{f}, \alpha\rangle = \langle \overline{f}, \ell_g\alpha\rangle$$

so that $f(k) = \langle \ell_g\alpha, \overline{f}\alpha\rangle$ is a matrix coefficient of $\overline{V}$. □

The same argument shows that matrix coefficients are also those K-finite vectors when the action of K acts from the right. In particular, this shows

Corollary 15.2.8.1. View $C(K)$ as an $K \times K$-representation. Then the matrix coefficients are exactly those $K \times K$-finite vectors in $C(K)$. In particular,

$$C(K)^{\text{fin}} \cong \bigoplus_{\tau \in \widehat{K}} \tau^* \otimes \tau$$

we $K \times K$-representations.

In the rest of this subsection, we change our notation: let G be a compact group, and put

$$R(G, \mathbb{C}) = C(G)^{\text{fin}}$$

to be the $\mathbb{C}$-subalgebra of $C(G)$ consisting of all matrix coefficients. Similarly, put

$$R(G, \mathbb{R}) = R(G, \mathbb{C}) \cap C(G, \mathbb{R})$$

to be the $\mathbb{R}$-subalgebra of $R(G, \mathbb{C})$ consisting of real-valued functions. Since the complex conjugation leaves $R(G, \mathbb{C})$ invariant, we have

$$R(G, \mathbb{C}) = R(G, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$$

under the natural identification $C(G) = C(G, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$.

15.3 Isotypes

Let (π, V_π) be a unitary representation of the compact group K. For $(\tau, V_\tau) \in \widehat{K}$, define the **isotype** of τ, or the **isotypical component** of τ in π as the subspace

$$V_\pi[\tau] := \sum_{\substack{U \subseteq V_\pi \\ U \cong V_\tau}} U$$

This is the sum of all invariant subspaces U which are K-isomorphic to V_τ. Alternatively one can describe $V_\pi[\tau]$ as follows. There is a canonical map

$$\begin{aligned} T_\tau : \operatorname{Hom}_K(V_\tau, V_\pi) \widehat{\otimes} V_\tau &\longrightarrow V_\pi \\ \alpha \otimes v &\longmapsto \alpha(v) \end{aligned}$$

By definition we have $T_\tau \in \operatorname{Hom}_K(1 \otimes \tau, \pi)$. We claim the image of T_τ in V_π is the whole $V_\pi[\tau]$.

- Let $0 \neq \alpha \in \operatorname{Hom}_K(V_\tau, V_\pi)$; since V_τ is irreducible, α is injective. Therefore $T_\tau|_{\alpha \otimes V_\tau} = \alpha$ is a K-homomorphism mapping onto a subspace of V_π that is K-isomorphic to V_τ.
- Conversely, let $U \subseteq V_\pi$ be a subspace with $\pi|_U \cong \tau$ via the map $\alpha : V_\tau \to U$. Then $U = T_\tau(\alpha \otimes V_\tau)$.

Note also that if (τ, V_τ) and (σ, V_σ) are non-equivalent irreducible representations of K, then $V_\pi[\tau] \perp V_\pi[\sigma]$. This follows from the fact that if $U, U' \subseteq V_\pi$ are subspaces with $U \cong V_\tau$, $U' \cong V_\sigma$, then the orthogonal projection $P : V_\pi \to U'$ restricts to a K-homomorphism $P|_U : U \to U'$, which must be zero.

Lemma 15.3.1. On the vector space $\mathrm{Hom}_K(V_\tau, V_\pi)$ there exists an inner product making it a Hilbert space such that T_τ is an isometry.

Proof. Let $v_0 \in V_\tau$ be a norm-one vector. For $\alpha, \beta \in \mathrm{Hom}_K(V_\tau, V_\pi)$, define

$$\langle \alpha, \beta \rangle := \langle \alpha(v_0), \beta(v_0) \rangle$$

Since an element in $\mathrm{Hom}_K(V_\tau, V_\pi)$ is either zero or injective, it follows at once that $\langle\,,\rangle$ defines an inner product on $\mathrm{Hom}_K(V_\tau, V_\pi)$. We claim with this inner product, the space $\mathrm{Hom}_K(V_\tau, V_\pi)$ is complete. For this let $(\alpha_n)_n$ be a Cauchy sequence in $\mathrm{Hom}_K(V_\tau, V_\pi)$. By definition $(\alpha_n(v_0))_n$ is a Cauchy sequence in V_π; say $\alpha_n(v_0) \to w_0 \in V_\pi$. Then for all $k \in K$, the sequence $\alpha_n(\tau(k)v_0) = \pi(k)\alpha_n(v_0) \to \pi(k)w_0$. Likewise, for each $f \in L^1(K)$, we have

$$\alpha_n(\tau(f)v_0) = \int_K f(k)\alpha_n(\tau(k)v_0)dk = \int_K f(k)\pi(k)\alpha_n(v_0)dk = \pi(f)\alpha_n(v_0) \to \pi(f)w_0 \qquad (\spadesuit)$$

Consider the annihilator $\mathrm{ann}\, v_0 = \mathrm{ann}_{L^1(K)}(v_0) \subseteq L^1(K)$ of v_0, i.e,

$$\mathrm{ann}_{L^1(K)}(v_0) := \{f \in L^1(K) \mid \tau(f)v_0 = 0\}$$

Since V_τ is irreducible and $\dim_{\mathbb{C}} V_\tau < \infty$, the map $\begin{array}{ccc} L^1(K) & \longrightarrow & V_\tau \\ f & \longmapsto & \tau(f)v_0 \end{array}$ induces a K-isomorphism $V_\tau \cong L^1(K)/\mathrm{ann}\, v_0$. By $(\spadesuit)$ we have $\mathrm{ann}\, v_0 \subseteq \mathrm{ann}\, w_0$, and therefore the map

$$\begin{array}{rcl} \alpha : V_\tau \cong L^1(K)/\mathrm{ann}\, v_0 & \longrightarrow & V_\pi \\ \tau(f)v_0 & \longmapsto & \pi(f)w_0 \end{array}$$

is well-defined and is K-intertwining. It follows that $\alpha_n \to \alpha$ in $\mathrm{Hom}_K(V_\tau, V_\pi)$.

We now show T_τ is an isometry. For a fixed $\alpha \in \mathrm{Hom}_K(V_\tau, V_\pi)$, the inner product on V_τ defined by $(v, w) := \langle \alpha(v), \alpha(w) \rangle$ is K-invariant, so by Schur's lemma there exists $c(\alpha) > 0$ such that $(v, w) = c(\alpha)\langle v, w \rangle$ for all $v, w \in V_\tau$. Thus

$$\langle T_\tau(\alpha \otimes v), T_\tau(\alpha \otimes v) \rangle = (v, v) = c(\alpha)\langle v, v \rangle$$

Setting $v = v_0$ we conclude that $c(\alpha) = \langle \alpha, \alpha \rangle$, which proves T_τ is an isometry. $\square$

It follows from the above lemma that $V_\pi[\tau]$ is isometrically isomorphic to Hilbert space tensor product $\mathrm{Hom}_K(V_\tau, V_\pi) \widehat{\otimes} V_\tau$, and that $\pi|_{V_\pi(\tau)}$ is unitarily equivalent to the representation $1 \otimes \tau$ on the tensor product. If we choose an orthonormal basis $(\alpha_i)_{i \in I}$ of $\mathrm{Hom}_K(V_\tau, V_\pi)$, then we get an isomorphism

$$\begin{array}{rcl} \mathrm{Hom}_K(V_\tau, V_\pi) \widehat{\otimes} V_\tau & \xrightarrow{\ \sim\ } & \widehat{\bigoplus_{i \in I}} V_\tau \\ \alpha \otimes v & \longmapsto & \sum_{i \in I} \langle \alpha, \alpha_i \rangle v \end{array}$$

Thus we see that $V_\pi(\tau)$ is unitarily equivalent to a direct sum of V_τ's with multiplicity $\#I = \dim_{\mathbb{C}} \mathrm{Hom}_K(V_\tau, V_\pi)$.

Theorem 15.3.2. Let K be a compact group.

(a) $V_\pi[\tau]$ is a closed invariant subspace of V_π.

(b) $V_\pi[\tau]$ is K-isomorphic to a Hilbert space direct sum of copies of V_τ.

(c) V_π is the Hilbert space direct sum of the isotypes $V_\pi[\tau]$, where τ runs over $\widehat{K}$.

Proof. Since $V_\pi[\tau]$ is an isometric image of a complete space, it is itself complete, whence closed in V_π. Since $V_\pi[\tau]$ is a sum of invariant subspaces, it is invariant as well. Now let $V_\pi = \widehat{\bigoplus}_i V_i$ be any decomposition into irreducibles. Set

$$\tilde{V_\pi}(\tau) := \widehat{\bigoplus_{i:V_i \cong V_\tau}} V_i \subseteq V_\pi$$

It follows that $\tilde{V_\pi}(\tau) \subseteq V_\pi[\tau]$ since the latter contains the direct sum and is closed. Now clearly V_π is the Hilbert space direct sum of all spaces $\tilde{V_\pi}(\tau)$, and hence it is also that of all spaces $V_\pi[\tau]$, as the latter are pairwise orthogonal. This implies (c), and a fortiori $\tilde{V_\pi}(\tau) = V_\pi[\tau]$, and thus (b). $\square$

By the lemma, for a unitary representation (π, V_π) of a compact groups K, there are K-equivariant isomorphisms

$$V_\pi \cong \widehat{\bigoplus_{(\tau,V_\tau)\in\widehat{K}}} V_\pi[\tau] \cong \widehat{\bigoplus_{(\tau,V_\tau)\in\widehat{K}}} \operatorname{Hom}_K(V_\tau, V_\pi) \widehat{\otimes} V_\tau$$

The next proposition gives an explicit formula for the first isomorphism.

Proposition 15.3.3. Let (π, V_π) be a unitary representation of the compact group K. For $\tau \in \widehat{K}$, the orthogonal projection $P : V_\pi \to V_\pi[\tau]$ is given by the formula

$$P(v) = \dim(\tau) \int_K \overline{\chi_\tau}(x)\pi(x)v dx$$

Proof. We need to show for each $v, w \in V_\pi$ we have

$$\langle Pv, w\rangle = \dim(\tau) \int_K \overline{\chi_\tau}(x)\langle \pi(x)v, w\rangle\, dx$$

Denote the right hand side by (v, w). Write $v = v_0 + v_1$ where $v_0 \in V_\pi[\tau]$, $v_1 \in V_\pi[\tau]^\perp$. Decompose $w = w_0 + w_1$ likewise. Then $\langle Pv, w\rangle = \langle v_0, w_0\rangle$. We claim

$$(v_0, w_0) = \langle v_0, w_0\rangle$$
$$(v_i, w_j) = 0 \text{ for } (i, j) \neq (0, 0)$$

Then since $(\,,)$ is bilinear, it follows $(v, w) = (v_0, w_0) = \langle v_0, w_0\rangle = \langle Pv, w\rangle$, as wanted.

To show the claim, decompose $V_\pi[\tau]$ into irreducibles, each equivalent to V_τ. Since the sum is orthogonal, we may assume v_0, w_0 lie in the same component. If we let (e_i) be an orthonormal basis for V_τ, then by orthogonality,

$$\begin{aligned}(v_0, w_0) &= \dim(\tau) \int_K \overline{\chi_\tau(x)}\langle \pi(x)v_0, w_0\rangle dx \\ &= \dim(\tau) \sum_{i,j} \langle v_0, e_i\rangle \overline{\langle w_0, e_j\rangle} \int_K \sum_k \overline{\tau_{kk}(x)} (\pi|_{V_\tau})_{ij}(x) dx \\ &= \sum_i \langle v_0, e_i\rangle \overline{\langle w_0, e_i\rangle} = \langle v_0, w_0\rangle\end{aligned}$$

For the remaining equalities, since $V_\pi[\tau]$ and $V_\pi[\tau]^\perp$ are invariant under K, we have $(v_1, w_0) = 0 = (v_0, w_1)$, and since $V_\pi[\tau]^\perp$ is an orthogonal direct sum of isotypes other than τ, orthogonality shows $(v_1, w_1) = 0$. $\square$

Example 15.3.4. Let us consider the regular representation of a compact group K. By Peter-Weyl theorem, we see there is a direct sum decomposition

$$L^2(K) \cong \widehat{\bigoplus_{\pi \in \widehat{K}}} \pi^* \otimes \pi$$

where for each $\pi \in \widehat{K}$, the finite dimensional subspace M_π is the linear span of all matrix coefficients of π and is invariant under left and right translations. It is then easy to see that $L^2(K)_R[\tau] \cong \pi^* \otimes \pi$ and $L^2(K)_L[\tau] = \pi \otimes \pi^*$ under the above isomorphism. Alternatively, one can verify this by the last proposition. For instance, for $f \in L^2(K)$, we compute

$$\int_K \overline{\chi_\tau}(x) L_x f(k) dx = \int_K \overline{\chi_\tau}(kx^{-1}) f(x) dx = \sum_{i,j=1}^{\dim(\tau)} \overline{\tau_{ji}}(k) \left(\int_K \overline{\tau_{ij}}(x^{-1}) f(x) dx \right) = \sum_{i,j=1}^{\dim(\tau)} \langle f, \overline{\tau_{ij}} \rangle \overline{\tau_{ij}}(k)$$

so $L^2(K)_L[\tau]$ is spanned by the $\overline{\tau_{ij}}$.

Example 15.3.5 (Irreducible representations of product groups)**.** Let G_1, G_2 be compact groups. Let W be a unitary representation of $G_1 \times G_2$. If we think it of a representation of $\{1\} \times G_2 \cong G_2$, there is a decomposition

$$W = \widehat{\bigoplus_{(\tau, V_\tau) \in \widehat{G_2}}} \mathrm{Hom}_{G_2}(V_\tau, W) \widehat{\otimes} V_\tau$$

For $T \in \mathrm{Hom}_{G_2}(V_\tau, W)$ and $g \in G_1$, define $g.T \in \mathrm{Hom}_{G_2}(V_\tau, W)$ by $(g.T)(x) = (g, 1)T(x)$; this is indeed G_2-equivariant, as for $g' \in G_2$ and $v \in V_\tau$,

$$(g.T)(g'v) = (g, 1)T(g'.v) = (g, 1)(1, g')T(v) = (1, g')(g.1)T(v) = (1, g')(g.T)(v).$$

If we equip each hom set with such G_1-actions, this decomposition is in fact $G_1 \times G_2$-equivariant. Indeed, for $\alpha \otimes v \in \mathrm{Hom}_{G_2}(V_\tau, W) \widehat{\otimes} V_\tau$,

$$(g_1.\alpha)(g_2.v) = (g_1, 1)\alpha(g_2.v) = (g_1, 1)(1, g_2)\alpha(v) = (g_1, g_2)\alpha(v).$$

In sum, we obtain a $G_1 \times G_2$-equivariant isomorphism

$$W = \widehat{\bigoplus_{(\tau, V_\tau) \in \widehat{G_2}}} \mathrm{Hom}_{G_2}(V_\tau, W) \widehat{\otimes} V_\tau.$$

Recall on $\mathrm{Hom}_{G_2}(V_\tau, W)$ we defined a Hilbert space structure in the proof of Lemma 15.3.1: if $v_0 \in V_\tau$ is a norm-one vector, for $\alpha, \beta \in \mathrm{Hom}_{G_2}(V_\tau, W)$, we put

$$\langle \alpha, \beta \rangle := \langle \alpha(v_0), \beta(v_0) \rangle.$$

For $g \in G_1$, since W is unitary

$$\langle g.\alpha, g.\beta \rangle := \langle (g, 1)\alpha(v_0), (g, 1)\beta(v_0) \rangle = \langle \alpha(v_0), \beta(v_0) \rangle = \langle \alpha, \beta \rangle.$$

This means the just defined G_1-action on $\mathrm{Hom}_{G_2}(V_\tau, W)$ is unitary. Hence we have a G_1-equivariant decomposition

$$\mathrm{Hom}_{G_2}(V_\tau, W) = \widehat{\bigoplus_{(\sigma, V_\sigma) \in \widehat{G_1}}} \mathrm{Hom}_{G_1}(V_\sigma, \mathrm{Hom}_{G_2}(V_\tau, W)) \widehat{\otimes} V_\sigma$$

The usual adjunction gives a G_1-equivariant isomorphism

$$\operatorname{Hom}_{G_1}(V_\sigma, \operatorname{Hom}_{G_2}(V_\tau, W)) \cong \operatorname{Hom}_{G_1\times G_2}(V_\sigma \otimes V_\tau, W).$$

Putting things together yields

$$W = \widehat{\bigoplus_{(\sigma,\tau)\in\widehat{G_1}\times\widehat{G_2}}} \operatorname{Hom}_{G_1\times G_2}(V_\sigma \otimes V_\tau, W) \mathbin{\widehat{\otimes}} (V_\sigma \otimes V_\tau)$$

It is easy to see that $V_\sigma \otimes V_\tau$ is irreducible as a $G_1 \times G_2$ representation: indeed, $\chi_{V_\sigma\otimes V_\tau} = \chi_\sigma \otimes \chi_\tau$, so

$$\int_{G_1\times G_2} |\chi_{V_\sigma\otimes V_\tau}(g_1, g_2)|^2 dg_1 \otimes dg_2 = \int_{G_1} |\chi_\sigma(g_1)|^2 dg_1 \cdot \int_{G_2} |\chi_\tau(g_2)|^2 dg_2 = 1 \cdot 1 = 1.$$

The irreducibility then follows from Corollary 15.2.5.1. Putting ourselves in the case $W = L^2(G_1 \times G_2)$, Peter-Weyl shows that $V_\sigma \otimes V_\tau$ exhausts $\widehat{G_1 \times G_2}$, demonstrating the bijection

$$\widehat{G_1} \times \widehat{G_2} \cong \widehat{G_1 \times G_2}$$

By this, the natural inclusion $C(G_1, \mathbb{R}) \otimes_{\mathbb{R}} C(G_2, \mathbb{R}) \to C(G_1 \times G_2, \mathbb{R})$ restricts to an algebra isomorphism

$$R(G_1, \mathbb{R}) \otimes_{\mathbb{R}} R(G_2, \mathbb{R}) \cong R(G_1 \times G_2, \mathbb{R}).$$

Clearly the above remains valid with all $\mathbb{R}$ replaced by $\mathbb{C}$.

15.4 Induced Representations

Let K be a compact group and $M \leqslant K$ a closed subgroup. Let (σ, V_σ) be a unitary representation of M. We are going to define the **induced representation** $\pi_\sigma = \operatorname{Ind}_M^K \sigma$. First, define the space

$$L^2(K, V_\sigma) := \left\{ f : K \to V_\sigma \mid f \text{ is measurable with } \int_K \|f(x)\|_\sigma^2 \, dk < \infty \right\} / \sim$$

where $f \sim g$ if and only if $f - g$ is a null function, where $\|\cdot\|_\sigma$ is the norm on V_σ. On $L^2(K, V_\sigma)$ there is an inner product

$$\langle f, g \rangle := \int_K \langle f(k), g(k) \rangle_\sigma dk$$

where $\langle\,,\rangle_\sigma$ is the inner product on V_σ. By taking an orthonormal basis (e_i) of V_σ, we see there is an isometric isomorphism $L^2(K, V_\sigma) \cong L^2(K) \mathbin{\widehat{\otimes}} V_\sigma$; in particular, this shows $L^2(K, V_\sigma)$ is a Hilbert space.

The representation space of π_σ is the subspace

$$\operatorname{Ind}_M^K V_\sigma := \left\{ f \in L^2(K, V_\sigma) \mid \text{for all } m \in M,\ f(mk) = \sigma(m) f(k) \text{ a.e. in } k \in K \right\}$$

This is a closed subspace of $L^2(K, V_\sigma)$, as if for each $m \in M$ we define $T_m \in \mathcal{B}(L^2(K, V_\sigma))$ by $T_m(f) = L_{m^{-1}} f - \sigma(m) f$, then

$$\operatorname{Ind}_M^K V_\sigma = \bigcap_{m\in M} \ker T_m$$

The representation π_σ is defined as the right translation:

$$\pi_\sigma(y) f(x) = f(xy)$$

Since K is unimodular, π_σ is clearly unitary. The following theorem allows us to compute the irreducible components in the induced representation $\operatorname{Ind}_M^K V_\sigma$. We begin with an easy lemma.

Lemma 15.4.1. The subspace $C(K, V_\sigma)$ of all continuous functions $K \to V_\sigma$ is dense in $L^2(K, V_\sigma)$.

Proof. Obviously we have $C(K, V_\sigma) \cong C(K) \otimes V_\sigma$, so the assertion follows. Explicitly, since the action π_σ on $L^2(K, V_\sigma)$ is isomorphic to the tensor $R \otimes 1$ on $L^2(K) \otimes V_\sigma$, for $\phi \in C(K)$, $f \in L^2(K)$, $v \in V_\sigma$, we have

$$\pi_\sigma(\phi)(f \otimes v) = \int_K \phi(k)\pi_\sigma(k)(f \otimes v)dk = \left(\int_K \phi(k) R_k f dk\right) \otimes v = (f * \phi) \otimes v \in C(K, V_\sigma)$$

Since $f * \phi \to f$ in $L^2(K)$ as ϕ runs over all Dirac functions, we have $\pi_\sigma(\phi)(f \otimes v) \to f \otimes v$ in $L^2(K, V_\sigma)$. □

Theorem 15.4.2 (Frobenius reciprocity)**.** If $\sigma \in \widehat{M}$ is irreducible, for every irreducible representation $(\tau, U) \in \widehat{K}$, there is a canonical isomorphism

$$\mathrm{Hom}_K(U, \mathrm{Ind}_M^K V_\sigma) \xrightarrow{\sim} \mathrm{Hom}_M(U|_M, V_\sigma)$$

where $U|_M$ is the representation $\pi|_M$ of M.

Proof. Before writing down the isomorphism, we make some preparation. Put $V^c := C(K, V_\sigma) \cap \mathrm{Ind}_M^K V_\sigma$. The space V^c is K-invariant, and is dense in $\mathrm{Ind}_M^K V_\sigma$ by the previous lemma. Let $\alpha \in \mathrm{Hom}_K(U, \mathrm{Ind}_M^K V_\sigma)$. We show that the image of α lies in V^c. For this recall $L^2(K)$ decomposes into a direct sum of isotypes $L^2(K)[\gamma]$ for $\gamma \in \widehat{K}$ (here we consider the right regular representation) with each $L^2(K)[\gamma]$ finite dimensional consisting of continuous functions (matrix coefficients). Consider the composition

$$\alpha : U \xrightarrow{\alpha} \mathrm{Ind}_M^K V_\sigma \hookrightarrow L^2(K, V_\sigma) \xrightarrow{\sim} L^2(K) \otimes V_\sigma$$

The map α (by abuse of notation) intertwines τ and $R \otimes 1$, so $\alpha(U) \subseteq (L^2(K) \otimes V_\sigma)[\tau] \subseteq L^2(K)[\tau] \otimes V_\sigma$ consists of continuous functions, and this implies $\alpha(U) \subseteq V^c$.

Define

$$\begin{aligned} \mathrm{Hom}_K(U, \mathrm{Ind}_M^K V_\sigma) &\longrightarrow \mathrm{Hom}_M(U|_M, V_\sigma) \\ \alpha &\longmapsto [u \mapsto \alpha(u)(1)] \end{aligned}$$

and

$$\begin{aligned} \mathrm{Hom}_M(U|_M, V_\sigma) &\longrightarrow \mathrm{Hom}_K(U, \mathrm{Ind}_M^K V_\sigma) \\ \beta &\longmapsto [(u, k) \mapsto \beta(\tau(k)u)] \end{aligned}$$

It is direct to check these two maps are mutually inverse and intertwine. □

Example 15.4.3. Let K be a compact group and $M \leqslant K$ a closed subgroup. Then K/M carries a unique Radon measure μ that is invariant under left translation of K and normalized so that $\mu(K/M) = 1$. The group K acts on the Hilbert space $L^2(K/M, \mu)$ by left translation, and this is a unitary representation. This representation is K-isomorphic to the representation $\mathrm{Ind}_M^K \mathbb{C}$ induced from the trivial representation via the map

$$\begin{aligned} L^2(K/M, \mu) &\longrightarrow \mathrm{Ind}_M^K \mathbb{C} \\ \psi &\longmapsto [k \mapsto \psi(k^{-1}M)] \end{aligned}$$

By Frobenius reciprocity for any $\tau \in \widehat{K}$, the multiplicity in $L^2(K/M, \mu)$ is

$$\mathrm{Hom}_K(V_\tau, L^2(K/M, \mu)) \cong \mathrm{Hom}_M(V_\tau|_M, \mathbb{C}) \cong V_\tau^M.$$

Hence, we obtain

$$L^2(K/M) \cong \widehat{\bigoplus_{\tau \in \widehat{K}}} V_\tau^M \otimes V_\tau.$$

An immediate consequence of this example is

Corollary 15.4.3.1. Let K be a compact group with $\#K \geqslant 2$ and M a proper closed subgroup. Then there exists a nontrivial irreducible representation of K whose restriction to M possesses a trivial representation of M.

Proof. Since M is proper, the quotient K/M has at least two points. Consider the decomposition in Example 15.4.3:

$$L^2(K/M) \cong \widehat{\bigoplus_{\tau \in \widehat{K}}} V_\tau^M \otimes V_\tau.$$

Since $\#K/M \geqslant 2$, LHS has dimension $\geqslant 2$. The trivial irreducible representation of K only contributes one dimension to RHS, so there must be some nontrivial $(\tau, V_\tau) \in \widehat{K}$ with $V_\tau^M \neq 0$. But then $V_\tau|_M$ contains a copy of the trivial representation. ☐

Corollary 15.4.3.2. Retain the notation. There exists a nontrivial $(\pi, V_\pi) \in \widehat{K}$ and $v \in V_\pi$ such that the stabilizer subgroup $K_v = \{g \in K \mid \pi(g)v = v\}$ is a proper subgroup of K containing M.

Proof. Let (π, V_π) be the nontrivial irreducible representation found in the previous corollary. Pick any nonzero $v \in V_\pi^M$; then $M \subseteq K_v$. Since π is nontrivial, K_v cannot be the whole K, so that $K_v \subsetneq K$, as wanted.

☐

15.5 Compact Lie groups

15.5.1 Characterization among compact groups

Theorem 15.5.1. For any unit-neighborhood U of K, there exists a finite dimensional representation τ of K such that $\ker \tau \subseteq U$.

Proof. Let $k \in K$ with $k \neq e$. Since $C(K)$ separates points and the space of matrix coefficients is dense in $L^2(K)$ by Peter-Weyl, it follows that $m(k) \neq m(e)$ for some matrix coefficient m of a finite dimensional (unitary) representation τ of K. In particular, this implies $\tau(k) \neq \tau(e)$. Hence the open sets $G \backslash \ker \tau$, where τ runs over all finite dimensional representations of K, form an open cover of $G \backslash \{e\}$, and hence an open cover of the compact set $G \backslash U$. We then can find some finite dimensional representations $\tau_1, \ldots, \tau_n$ of K with $G \backslash U \subseteq \bigcup_{i=1}^n G \backslash \ker \tau_i$. If we write τ for the direct sum of the τ_i, we see that $\ker \tau = \bigcap_{i=1}^n \ker \tau_i$ and thus $\ker \tau \subseteq U$. ☐

Corollary 15.5.1.1. Every compact group is isomorphic, as topological groups, to a projective limit of compact Lie groups.

Proof. Denote by F the set of finite dimensional (unitary) representations of a compact group K. If $(\tau, V) \in F$, then $\tau(K) \subseteq \mathrm{GL}(V)$. Since $\tau(K)$ is compact and $\mathrm{GL}(V)$ is a Lie group, it follows from Theorem I.3.2 that $\tau(K)$ is itself a compact Lie group. Since $K/\ker \tau \cong \tau(K)$, we see $K/\ker \tau$ is isomorphic to a compact Lie group, as topological groups.

The canonical projections $K \to K/\ker\tau\,(\tau \in F)$ give rises to a continuous homomorphism $\phi : K \to \varprojlim_{\tau\in F} G/\ker\tau$ with dense image. Since K is compact, ϕ is surjective. To show ϕ is a homeomorphism, it remains to show ϕ is injective. Note that $\ker\phi = \bigcap_{\tau\in F}\ker\tau$, so we have to show the intersection is trivial. From Theorem 15.5.1 that $\{\ker\tau \mid \tau \in F\}$ forms a neighborhood basis of identity of K. Now our claim follows as K is Hausdorff. ☐

Corollary 15.5.1.2. For a compact group K, TFAE:

(i) There exists a unit-neighborhood of K containing no nontrivial closed normal subgroup.

(ii) K is isomorphic to a closed subgroup of $\mathrm{GL}(V)$ for some finite dimensional complex vector space V.

(iii) K is a Lie group.

Proof. (i)⇒(ii) follows from Theorem 15.5.1, which implies that there exists a finite dimensional representation (τ, V) of K with $\ker\tau = \{e\}$, i.e., $\tau : K \to \mathrm{GL}(V)$ is injective. (ii)⇒(iii) follows from Theorem I.3.2. (iii)⇒(i) is Proposition I.2.10. ☐

Corollary 15.5.1.3. For every proper closed subgroup of a compact Lie group K, there exists a finite dimensional representation (π, V_π) of K and a vector $v \in V_\pi$ such that $M = K_v = \{g \in K \mid \pi(g)v = v\}$.

Proof. Let M be a closed subgroup of K.

☐

15.5.2 Matrix coefficients

Now let K be a compact group with a faithful representation $\pi : K \to \mathrm{GL}(V)$ in a finite dimensional complex vector space V. Choose any basis $e_1, \ldots, e_n$ for V and define the matrix coefficients $\pi_{ij} : K \to \mathbb{C}$ by

$$\pi_{ij}(g) = \langle \pi(g)e_i, e_j\rangle.$$

These $\pi_{ij}\,(1 \leqslant i, j \leqslant n)$ vanish nowhere as $\pi(K) \subseteq \mathrm{GL}(V)$. Since π is injective, these $\pi_{ij}\,(1 \leqslant i, j \leqslant n)$ separate points. Hence, by Stone-Weierstrass the subalgebra of $C(K)$ generated by the π_{ij}, $\overline{\pi_{ij}}\,(1 \leqslant i, j \leqslant n)$ is dense in $C(G)$.

In fact, by Lemma 15.1.1 we can choose a K-invariant inner product on V so that (π, V) is unitary. Notice the functions $\overline{\pi_{ij}}$ are the matrix coefficients of the contragredient representation (π^*, V^*) of π. Moreover, the "monimials" $\pi_{i_1j_1}\cdots\pi_{i_nj_n}\overline{\pi_{k_1l_1}}\cdots\overline{\pi_{k_ml_m}}$ are the matrix coefficients of the representation $V^{\otimes n} \otimes (V^*)^{\otimes m}$.

Lemma 15.5.2.

(i) Every irreducible unitary representation of K appears in some $V^{\otimes n} \otimes (V^*)^{\otimes m}$.

(ii) π_{ij}, $\overline{\pi_{ij}}\,(1 \leqslant i, j \leqslant n)$ generate $C(K)^{\mathrm{fin}}$. In particular, the algebra $C(K)^{\mathrm{fin}}$ is of finite type over $\mathbb{C}$.

Proof. For a finite dimensional representation τ of K, denote by M_τ the space of matrix coefficients for τ. If we denote by A the unital subalgebra of $C(K)$ generated by the π_{ij}, $\overline{\pi_{ij}}$ $(1 \leqslant i, j \leqslant n)$, the discussion above shows that

$$A = \sum_{n,m \geqslant 0} M_{V^{\otimes n} \otimes (V^*)^{\otimes m}}.$$

Let S be the subset of $\widehat{K}$ consisting of those representations appearing in $V^{\otimes n} \otimes (V^*)^{\otimes m}$ $(n, m \geqslant 0)$; decomposing those tensor product representations into irreducibles shows

$$A = \bigoplus_{\tau \in S} M_\tau \subseteq \bigoplus_{\tau \in \widehat{K}} M_\tau = C(K)^{\mathrm{fin}}.$$

The last subspace is dense in $L^2(K)$ by Peter-Weyl, and the subalgebra A is dense as said before. Since the direct sum is orthogonal, A cannot be dense unless $S = \widehat{K}$. This shows (i) and (ii) simultaneously. □

15.5.3 Compact Lie groups as real algebraic groups

In this subsection let G be compact Lie group. Lemma 15.5.2 shows that $R(G, \mathbb{C})$ is of finite type over $\mathbb{C}$. In fact, if we view V as a real vector space in the above discussion, the same argument shows that $R(G, \mathbb{R})$ is of finite type over $\mathbb{R}$. The affine scheme

$$X = \operatorname{Spec} R(G, \mathbb{R})$$

is then a (possibly reducible) algebraic variety over $\mathbb{R}$. We are going to show this is an algebraic group whose $\mathbb{R}$-point $X(\mathbb{R})$ is isomorphic to the original compact Lie group G. Let k be either $\mathbb{R}$ or $\mathbb{C}$. To give it a group scheme structure it is the same as giving $R(G, k)$ a commutative Hopf algebra structure. Define

$$\begin{aligned} \Delta : R(G, k) &\longrightarrow R(G \times G, k) = R(G, k) \otimes R(G, k) \\ f &\longmapsto [(g, g') \mapsto f(gg')] \end{aligned}$$

where the last isomorphism is from Example 15.3.5. This is well-defined as matrix coefficients are $G \times G$-finite by Corollary 15.2.8.1. Define

$$\begin{aligned} \iota : R(G, k) &\longrightarrow R(G, k) \\ f &\longmapsto [g \mapsto f(g^{-1})] \end{aligned}$$

and

$$\begin{aligned} \varepsilon : R(G, k) &\longrightarrow k \\ f &\longmapsto f(e) \end{aligned}$$

Finally let $m : R(G, k) \otimes R(G, k) \to R(G, k)$ and $e : k \to R(G, k)$ be the multiplication and the inclusion of constant maps; they are the structure maps for the k-algebra $R(G, k)$. These just defined morphisms satisfy three commutative diagrams

$$\begin{tikzcd} & R(G,k) \otimes R(G,k) \arrow[r, "\mathrm{id} \otimes \Delta"] & R(G,k) \otimes R(G,k) \otimes R(G,k) \arrow[d, equal] \\ R(G,k) \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & \\ & R(G,k) \otimes R(G,k) \arrow[r, "\Delta \otimes \mathrm{id}"] & R(G,k) \otimes R(G,k) \otimes R(G,k) \end{tikzcd}$$

$$\begin{tikzcd}
 & R(G,k)\otimes R(G,k) \arrow[r, "\varepsilon\otimes\mathrm{id}"] & k\otimes R(G,k) \arrow[d, "m\circ(e\otimes\mathrm{id})"] \\
R(G,k) \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rr, equal] & & R(G,k) \\
 & R(G,k)\otimes R(G,k) \arrow[r, "\mathrm{id}\otimes\varepsilon"] & R(G,k)\otimes k \arrow[u, "m\circ(\mathrm{id}\otimes e)"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & R(G,k)\otimes R(G,k) \arrow[rr, "\mathrm{id}\otimes\iota"] & & R(G,k)\otimes R(G,k) \arrow[dr, "m"] & \\
R(G,k) \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rr, "\varepsilon"] & & k \arrow[rr, "e"] & & R(G,k) \\
 & R(G,k)\otimes R(G,k) \arrow[rr, "\iota\otimes\mathrm{id}"] & & R(G,k)\otimes R(G,k) \arrow[ur, "m"'] &
\end{tikzcd}$$

Taking spec shows that $X = \operatorname{Spec} R(G,\mathbb{R})$ is really an affine group scheme over $\mathbb{R}$. Each point $g \in G$ determines an evaluation map $\mathrm{ev}_g : R(G,\mathbb{R}) \to \mathbb{R}$, giving rise to a map

$$\begin{aligned} \Phi : G &\longrightarrow X(\mathbb{R}) = \operatorname{Hom}_{\mathbf{Alg}_{\mathbb{R}}}(R(G,\mathbb{R}),\mathbb{R}) \\ g &\longmapsto \mathrm{ev}_g \end{aligned}$$

This is injective as $R(G,\mathbb{R})$ separates points. Equip $X(k)$ with the initial topology with respect to the maps $X(k) = \operatorname{Hom}_{\mathbf{Alg}_k}(R(G,k),k) \ni p \mapsto p(f) \in k$ ($f \in R(G,k)$), where k is with its usual euclidean topology. If $f_1,\dots,f_n$ is a generating set for $R(G,k)$, the topology just defined is the same as the subspace topology inherited from the injection

$$\begin{aligned} X(k) &\longrightarrow k^n \\ p &\longmapsto (p(f_1),\dots,p(f_n)) \end{aligned}$$

In particular, this shows $X(k)$ is Hausdorff under this topology.

Lemma 15.5.3.

(i) $X(k)$ is a topological group.

(ii) $\Phi : G \to X(\mathbb{R})$ is a continuous injective group homomorphism.

In particular, since G is compact and $X(\mathbb{R})$ is Hausdorff, via Φ we can view G as a closed subgroup of $X(\mathbb{R})$

Proof. The topology on $X(k)$ enjoys the universal property: if Y is another topological space with a set-theoretic map $\varphi : Y \to X(k)$, then φ is continuous if and only if $Y \xrightarrow{\varphi} X(k) \xrightarrow{p\mapsto p(f)} k$ is continuous for every $f \in R(G,k)$. It is then direct to see $X(k)$ is a topological group and Φ is continuous.

It remains to show Φ is a group homomorphism. For $f \in R(G,k)$, write

$$\Delta(f) = \sum f_i \otimes f_i'$$

for some $f_i, f_i' \in R(G,k)$. Then for $g,h \in G$,

$$(\mathrm{ev}_g \otimes \mathrm{ev}_h) \circ \Delta(f) = \sum f_i(g) f_i'(h) = f(gh) = \mathrm{ev}_{gh}(f)$$

where the third identity results from the definition that $\Delta(f)(g,h) = f(gh)$. Varying $f \in R(G,k)$ shows $(\mathrm{ev}_g \otimes \mathrm{ev}_h) \circ \Delta = \mathrm{ev}_{gh}$, which is exactly what we want. ∎

Let $\pi : G \to \mathrm{GL}_n(k)$ be a representation of G. Then the matrix coefficients π_{ij} lie in $R(G,k)$. Consider the k-algebra homomorphism

$$\begin{aligned} k[x_{ij}] &\longrightarrow R(G,k) \\ x_{ij} &\longrightarrow \pi_{ij} \end{aligned}$$

Since $(\pi_{ij})_{ij} \in \mathrm{GL}_n(k)$, the determinant $\det(x_{ij})$ is invertible in $R(G,k)$. This induces a homomorphism $k[x_{ij}, \det(x_{ij})^{-1}] \to R(G,k)$ on the localization. Taking spec gives a k-morphism

$$X \to \mathrm{GL}_{n,k}$$

of varieties. It is straightforward to verify that $k[x_{ij}, \det(x_{ij})^{-1}] \to R(G,k)$ is in fact a Hopf algebra homomorphism, so $X \to \mathrm{GL}_{n,k}$ is an algebraic group homomorphism. Taking k-points yields a group homomorphism

$$\begin{aligned} \pi_k : X(k) &\longrightarrow \mathrm{GL}_n(k) \\ p &\longmapsto (p(\pi_{ij}))_{ij}. \end{aligned}$$

This is continuous if and only if each $p \mapsto p \circ \pi_{ij}$ is, which is the case by definition. If we choose π to be faithful (possible by Corollary 15.5.1.2), by Lemma 15.5.2.(ii) the homomorphism $k[x_{ij}, \det(x_{ij})^{-1}] \to R(G,k)$ is surjective, meaning that π_k is a closed embedding. It is worth-noting that, since $\Phi(g) = \mathrm{ev}_g$, we have a commuting triangle

$$\begin{tikzcd} X(\mathbb{R}) \arrow[r, "\pi_{\mathbb{R}}"] & \mathrm{O}(n) \\ G \arrow[u, "\Phi"] \arrow[ur, "\pi"'] & \end{tikzcd}$$

A similar argument implies that if $k = \mathbb{R}$ and $\pi(G) \subseteq \mathrm{O}(n)$, then $\pi_{\mathbb{R}}(X(\mathbb{R})) \subseteq \mathrm{O}(n)$.

Theorem 15.5.4. $X(\mathbb{R})$ is a compact Lie group and $\Phi : G \to X(\mathbb{R})$ is a Lie group isomorphism.

Proof. Pick any faithful representation $\pi : G \to \mathrm{GL}_n(\mathbb{R})$ of π. By Lemma 15.1.1, the representation space $\mathbb{R}^n$ admits a G-invariant inner product, so up to a conjugation by an element in $\mathrm{GL}_n(\mathbb{R})$ we may assume $\pi(G) \subseteq \mathrm{O}(n)$. Now consider the homomorphism $\pi_{\mathbb{R}} : X(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$. It has image in $\mathrm{O}(n)$ as mentioned above, and is a closed embedding. By Theorem I.3.2, $X(\mathbb{R})$ is a compact Lie group.

It remains to show Φ is surjective. The inclusion $\Phi : G \to X(\mathbb{R})$ induces an algebra homomorphism $\Phi^* : R(X(\mathbb{R}), \mathbb{R}) \to R(G, \mathbb{R})$ on the algebras of matrix coefficients. On the other hands, if $\rho : G \to \mathrm{GL}_n(\mathbb{R})$ is a representation and f is its matrix coefficient, then the evaluation $\lambda_f : X(\mathbb{R}) \ni p \mapsto p(f) \in \mathbb{R}$ is a matrix coefficient for $\rho_{\mathbb{R}} : X(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$. This follows from the construction of $\rho_{\mathbb{R}}$ in the above. Hence we obtain an algebra homomorphism

$$\begin{aligned} \lambda : R(G, \mathbb{R}) &\longrightarrow R(X(\mathbb{R}), \mathbb{R}) \\ f &\longmapsto \lambda_f. \end{aligned}$$

Since π is faithful, so is $\pi_{\mathbb{R}}$, and by Lemma 15.5.2.(ii), the algebra $R(X(\mathbb{R}), \mathbb{R})$ is generated by the $(\pi_{\mathbb{R}})_{ij}$. By construction $\lambda(\pi_{ij}) = (\pi_{\mathbb{R}})_{ij}$ and $\Phi^*((\pi_{\mathbb{R}})_{ij}) = \pi_{ij}$, so λ and Φ^* are mutually inverses, whence isomorphisms. Consider the commutative square

$$\begin{tikzcd}
R(X(\mathbb{R}),\mathbb{R}) \arrow[r, "\Phi^*"] \arrow[d, hook] & R(G,\mathbb{R}) \arrow[d, hook] \\
C(X(\mathbb{R}),\mathbb{R}) \arrow[r, "\Phi^*"] & C(G,\mathbb{R})
\end{tikzcd}$$

Since the upper horizontal arrow is an isomorphism and all vertical maps have dense images with sup-norm topology, the lower horizontal arrow is also an isomorphism. (Note that Φ^* is continuous as Φ is injective.) This forces $\Phi : G \to X(\mathbb{R})$ to be surjective by an Urysohn lemma argument. $\square$

We turn to the affine group scheme $X_{\mathbb{C}} = \operatorname{Spec} R(G,\mathbb{C})$ over $\mathbb{C}$. The argument that precedes Theorem 15.5.4 shows that $X_{\mathbb{C}}(\mathbb{C}) = X(\mathbb{C})$ is a closed subgroup of $\mathrm{GL}_n(\mathbb{C})$. A general fact in algebraic geometry says that $X(\mathbb{C})$ contains at least a smooth point, so by translation we see $X(\mathbb{C})$ is a smooth algebraic group over $\mathbb{C}$, and hence a complex Lie group.

15.6 Examples

15.6.1 SU(2)

Consider the n-dimensional sphere $S^n \subseteq \mathbb{R}^{n+1}$. We equip S^n with the subspace topology from $\mathbb{R}^{n+1}$ with euclidean topology.

Lemma 15.6.1. For a subset $A \subseteq S^n$, A is Borel-measurable in S^n if and only if $IA := \{ta \mid 0 \leqslant t \leqslant 1,\ a \in A\}$ is Borel-measurable in $\mathbb{R}^{n+1}$.

Proof. The Borel sets in S^n is the intersection of S^n and those of $\mathbb{R}^{n+1}$, so if IA is Borel, so is $A = IA \cap S^n$. For the converse, if A is closed, since the map $[0,1]\times S^n \to \mathbb{R}^{n+1}$ defined by $(t,a) \mapsto ta$ is closed (the domain being compact), IA is closed. If $A \neq \varnothing$ is open, then $IA = (\overline{B_1(0)}\backslash IA^c) \cup \{0\}$ is Borel. It remains to show the family $\mathcal{S} := \{A \subseteq S^n \mid IA \text{ is Borel}\}$ is a σ-algebra.

- $S^n \in \mathcal{S}$ and $\varnothing \in \mathcal{S}$.
- If $\varnothing \neq A \in \mathcal{S}$, then $IA^c = (\overline{B_1(0)}\backslash IA) \cup \{0\}$ is Borel.
- If $A_i \in \mathcal{S}$ with $\bigcup_i A_i \neq \varnothing$, then $I\bigcup_i A_i = \bigcup_i IA_i$ is Borel.
- If $A_i \in \mathcal{S}$ with $\bigcap_i A_i \neq \varnothing$, then $I\bigcap_i A_i = (\overline{B_1(0)}\backslash I\bigcup_i A_i^c) \cup \{0\}$ is Borel.

$\square$

Let λ be the Lebesgue measure on $\mathbb{R}^{n+1}$. For a measurable set $A \subseteq S^n$, define the normalized Lebesgue measure as

$$\mu(A) := \frac{\lambda(IA)}{\lambda(IS^n)} = \frac{\lambda(IA)}{\lambda(B_1(0))}$$

The Lebesgue measure λ is invariant under the action of the orthogonal group $\mathrm{O}(n+1)$, and it follows that μ is also invariant under $\mathrm{O}(n+1)$.

Lemma 15.6.2. Let $n \in \mathbb{N}$ and $e_1 = (1, 0, \ldots, 0)^t \in \mathbb{R}^{n+1}$. The matrix multiplication $g \mapsto ge_1$ gives an identification

$$S^n \cong \mathrm{O}(n+1)/\mathrm{O}(n) \cong \mathrm{SO}(n+1)/\mathrm{SO}(n)$$

This map is invariant under left translation, and the normalized Lebesgue measure on S^n is the unique normalized invariant measure on this quotient space.

Proof. Here we embed $\mathrm{O}(n)$ into $\mathrm{O}(n+1)$ by the map $g \mapsto \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}$. Then $\mathrm{O}(n)$ is the stabilizer of e_1 under the action of $\mathrm{O}(n+1)$; this gives an injective continuous map $\mathrm{O}(n+1)/\mathrm{O}(n) \to S^n$. It is then easy to see this is a homeomorphism, and similarly $S^n \cong \mathrm{SO}(n+1)/\mathrm{SO}(n)$. The normalized Lebesgue is transferred to the quotient $\mathrm{O}(n+1)/\mathrm{O}(n)$, and as said above it is invariant under $\mathrm{O}(n+1)$. Now the uniqueness follows the uniqueness part in Theorem 2.4.7.

□

Recall that SU(2) consists of 2×2 unitary matrices with determinant 1. In other words,

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \mid \begin{pmatrix} a \\ b \end{pmatrix} \in S^3 \subseteq \mathbb{C}^2 \right\}$$

From this description and the fact that the normalized Lebesgue measure on S^3 is invariant under O(4) as well as the uniqueness of the invariant measures, we obtain the following lemma.

Lemma 15.6.3. The map $\mathrm{SU}(2) \to S^3$, mapping the matrix $g \in \mathrm{SU}(2)$ to its first column, is a homeomorphism. Via this homeomorphism, the normalized Lebesgue measure on S^3 coincides with the normalized Haar measure on SU(2).

Lemma 15.6.4. For measurable $A \subseteq S^n$, define $\sigma(A) = (n+1)\lambda(IA)$. Then for any Borel-measurable function $f : \mathbb{R}^{n+1} \to [0, \infty]$, we have

$$\int_{\mathbb{R}^{n+1}} f(x) d\lambda(x) = \int_0^\infty r^n \left(\int_{S^n} f(ru) d\sigma(u) \right) dr$$

Lemma 15.6.5. Let $f : S^n \to \mathbb{C}$ be integrable and for $n \in \mathbb{N}_0$ let $F_m : \mathbb{R}^{n+1} \to \mathbb{C}$ be defined by

$$F_m(rx) = r^m f(x)$$

for all $x \in S^n$ and $r \geqslant 0$. Then

$$\int_{S^n} f(x) d\mu(x) = c_{n,m} \int_{\mathbb{R}^{n+1}} F_m(x) e^{-\|x\|^2} d\lambda(x)$$

where $c_{n,m}$ will be specified in the proof.

Proof. By the previous lemma, we have

$$\begin{aligned} \int_{\mathbb{R}^{n+1}} F_m(x) e^{-\|x\|^2} d\lambda(x) &= c \int_0^\infty r^n \int_{S^n} F_m(rx) e^{-r^2} d\mu(x) dr \\ &= c \int_0^\infty r^{n+m} e^{-r^2} dr \int_{S^n} f(x) d\mu(x) \\ (r \mapsto r^2) &= \frac{c}{2} \Gamma\left(\frac{n+m}{2} + 1 \right) \int_{S^3} f(x) d\mu(x) \end{aligned}$$

for some constant c. Let $f \equiv 1$ and $m = 0$ in the above identity; then

$$\frac{c}{2}\Gamma\left(\frac{n}{2}+1\right) = \int_{\mathbb{R}^{n+1}} e^{-\|x\|^2} d\lambda(x) = \left(\int_{\mathbb{R}} e^{-x^2} dx\right)^{n+1} = \pi^{\frac{n+1}{2}}$$

so that

$$c_{n,m} := \left(\frac{c}{2}\Gamma\left(\frac{n+m}{2}+1\right)\right)^{-1} = \pi^{-\frac{n+1}{2}} \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n+m}{2}+1\right)}$$

□

For $n \in \mathbb{N}_0$, denote by $\mathcal{P}_n$ the set of homogeneous polynomials on $\mathbb{C}^2$ of degree n. Then

$$\mathcal{P}_n = \bigoplus_{k=0}^{n} \mathbb{C} z_1^k z_2^{n-k}$$

For $p, q \in \mathcal{P}_n$, we define

$$\langle p, q\rangle_n := \langle p|_{S^3}, q|_{S^3}\rangle_{L^2(S^3)} = \int_{S^3} p(x)\overline{q(x)} d\mu(x)$$

It follows from the previous lemma that

$$\langle p, q\rangle_n = c_{3,2n} \int_{\mathbb{C}^2} p(z)\overline{q(z)} e^{-\|z\|^2} d\lambda(z)$$

We define a representation π_n of $\mathrm{SU}(2)$ on $\mathcal{P}_n$ by

$$\pi_n(g)p(z) := p(g^{-1}z)$$

As a preparation, we compute the inner product $\langle z_1^k z_2^{n-k}, z_1^r z_2^{n-r}\rangle_n$ for $0 \leqslant k, r \leqslant n$.

$$\int_{\mathbb{C}^2} (z_1^k z_2^{n-k})\overline{z_1^r z_2^{n-r}} e^{-\|z\|^2} d\lambda(z) = \left(\int_{\mathbb{C}} z^k \overline{z}^r e^{-|z|^2} dz\right)\left(\int_{\mathbb{C}} z^{n-k} \overline{z}^{n-r} e^{-|z|^2} dz\right)$$

Compute

$$\int_{\mathbb{C}} z^k \overline{z}^r dz = \int_0^{2\pi} y \int_0^{\infty} (\rho e^{i\theta})^k (\rho e^{-i\theta})^r e^{-\rho^2} \rho d\rho d\theta = \delta_{kr} \pi \Gamma(k+1) = \delta_{kr}\pi k!.$$

Hence

$$\langle z_1^k z_2^{n-k}, z_1^r z_2^{n-r}\rangle_n = c_{3,2n}\delta_{kr}\pi^2 k!(n-k)! = \delta_{kr}\frac{k!(n-k)!}{n!} = \delta_{kr}\binom{n}{k}^{-1}.$$

In view of the computation, we find that $\left\{\binom{n}{k}^{\frac{1}{2}} z_1^k z_2^{n-k}\right\}_{k=0}^{n}$ forms an orthonormal basis for $\mathcal{P}_n$.

Lemma 15.6.6. The representation $\pi_n : \mathrm{SU}(2) \to \mathrm{GL}(\mathcal{P}_n)$ is unitary and irreducible

Proof. By Lemma 15.6.3, we choose the measure μ on S^3 respecting the homeomorphism $\mathrm{SU}(2) \cong S^3$; the homeomorphism is explicitly defined by $g \mapsto g.e_1$, where $e_1 = (1,0,0,0) \in \mathbb{C}^2$. Then

$$\begin{aligned}\langle \pi_n(h)p, \pi_n(h)q\rangle_n &= \int_{S^3} p(h^{-1}x)\overline{q(h^{-1}x)} d\mu(x) \\ &= \int_{\mathrm{SU}(2)} p(h^{-1}g.e_1)\overline{q(h^{-1}g.e_1)} dg \overset{\text{inv}}{=} \int_{\mathrm{SU}(2)} p(g.e_1)\overline{q(g.e_1)} dg = \langle p, q\rangle_n\end{aligned}$$

for any $h \in \mathrm{SU}(2)$ and $p, q \in \mathcal{P}_m$.

To show irreducibility, □

Theorem 15.6.7. $\widehat{\mathrm{SU}(2)} = \{[\mathcal{P}_n, \pi_n] \mid n \in \mathbb{N}_0\}$, where $[\mathcal{P}_n, \pi_n]$ denotes the equivalence class of $(\mathcal{P}_n, \pi_n)$.

Proof. Regard S^1 as a subgroup of SU(2) by $e^{i\theta} \mapsto \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}$. Let $\chi_n = \operatorname{tr} \pi_n$ be the character of the representation π_n. We have

$$\chi_n(e^{i\theta}) = \sum_{k=0}^{n} \langle \pi(e^{i\theta}) p_k, p_k \rangle_n = \sum_{k=0}^{n} e^{i\theta(n-2k)}$$

where the $p_k = \binom{n}{k}^{\frac{1}{2}} z_1^k z_2^{n-k}$ are the orthonormal bases for $\mathcal{P}_n$. The key observation is that

$$\operatorname{span}_{\mathbb{C}}\{\chi_n(e^{i\theta}) \mid n \in \mathbb{N}_0\} = \operatorname{span}_{\mathbb{C}}\{\cos n\theta \mid n \in \mathbb{N}_0\}$$

which is more-or-less obvious (as $e^{i\theta} + e^{-i\theta} = 2\cos\theta$). Since each element in SU(2) is conjugate to S^1, by classical Fourier analysis on S^1 we see the restriction

$$\begin{aligned} L^2(\mathrm{SU}(2)/\mathrm{conj}) \cap C(\mathrm{SU}(2)) &\longrightarrow C_{\mathrm{even}}(S^1) \\ f &\longmapsto f|_{S^1} \end{aligned}$$

is a linear isomorphism, where

$$C_{\mathrm{even}}(S^1) = \{f \in C(S^1) \mid f(x) = f(x^{-1}) \text{ for all } x \in S^1\}$$

and it contains $\operatorname{span}_{\mathbb{C}}\{\cos n\theta \mid n \in \mathbb{N}_0\}$ as a dense subset. Moreover, it is (almost) norm-preserving:

$$\langle z^k \overline{z}^{n-k}, z^r \overline{z}^{n-r} \rangle_{L^2(S^1)} = \delta_{kr}.$$

Hence $\operatorname{span}_{\mathbb{C}}\{\chi_n(e^{i\theta}) \mid n \in \mathbb{N}_0\}$ is a dense subset of $L^2(\mathrm{SU}(2)/\mathrm{conj}) \cap C(\mathrm{SU}(2))$, and hence of $L^2(\mathrm{SU}(2)/\mathrm{conj})$. It follows from Corollary 15.2.7.2 that the χ_n are the only characters of SU(2). In view of Theorem 15.2.5, the proof is completed. ∎

Corollary 15.6.7.1. The SU(2) representation on $L^2(S^3)$ is isomorphic to the Hilbert space direct sum $\widehat{\bigoplus\limits_{n \geq 0}} (n+1)\mathcal{P}_n$.

Proof. Since $S^3 \cong \mathrm{SU}(2)$ and $\dim \mathcal{P}_n = n+1$, this follows at once from Example 15.4.3. ∎

15.6.2 SO(n)

Let $n, m \geq 0$, and put

$$V_{n,m} = \mathbb{C}[x_1, \dots, x_n]_m = \bigoplus_{\substack{1 \leq k_1, \dots, k_n \\ k_1 + \cdots + k_n = m}} \mathbb{C} x_1^{k_1} \cdots x_n^{k_n}$$

to be the space of n-variable homogeneous polynomials of degree m over $\mathbb{C}$. The space $\mathcal{P}_n$ that appeared in the previous subsection is now the space $V_{2,n}$. The general linear group $\mathrm{GL}_n(\mathbb{R})$ acts on $V_{n,m}$ naturally on the left:

$$g.p(x) := p(g^{-1}x)$$

In particular, the orthogonal group $\mathrm{O}(n)$ acts on $V_{n,m}$; we denote the representation it affords by $\pi_{n,m} : \mathrm{O}(n) \to \mathrm{GL}(V_{n,m})$. Let $\Delta \in \operatorname{Der} C^\infty(\mathbb{R}^n)$ be the Laplacian on $\mathbb{R}^n$:

$$\Delta = \sum_{i=1}^{n} \partial_i^2 = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

Lemma 15.6.8. For $p \in \mathbb{C}[x_1, \ldots, x_n]$ and $g \in \mathrm{O}(n)$, one has $g.\Delta(p) = \Delta(g.p)$.

Proof. For $g \in \mathrm{GL}_n(\mathbb{C})$, by chain rules one computes

$$\Delta(g.p) = \sum_{i,j=1}^{n} ((gg^t)^{-1})_{ij} \cdot g. \left(\frac{\partial^2}{\partial x_i \partial x_j} p \right).$$

Hence, for $g \in \mathrm{O}(n)$, we have

$$\Delta(g.p) = \sum_{i,j=1}^{n} \delta_{ij} \cdot g. \left(\frac{\partial^2}{\partial x_i \partial x_j} p \right) = g.\Delta(p)$$

□

Write $H_{n,m} := \ker \Delta \cap V_{n,m}$ for the space of all **harmonic polynomial of degree** m **in** n**-variables**. The previous lemma then show that $g.H_{n,m} \subseteq H_{n,m}$ for $g \in \mathrm{O}(n)$, so $H_{n,m}$ is actually an $\mathrm{O}(n)$-representation.

As before, define an inner product on $V_{n,m}$ by

$$\langle p, q \rangle := \langle p|_{S^{n-1}}, q|_{S^{n-1}} \rangle_{L^2(S^{n-1})} = \int_{S^{n-1}} p(x)\overline{q(x)} d\mu(x) \overset{(15.6.5)}{=} c_{n-1,2m} \int_{\mathbb{R}^n} p(x)\overline{q(x)} e^{-\|x\|^2} d\lambda(x)$$

where the measure μ is chosen as in (and before) Lemma 15.6.2. Here we view complex polynomials as functions on $\mathbb{R}$. In particular, this inner product is $\mathrm{O}(n)$-invariant.

To proceed further, we introduce the Laplacian on S^{n-1}. First, we associate with $f : S^{n-1} \to \mathbb{C}$ a function $F : \mathbb{R}^n \to \mathbb{C}$ given by $F(x) = f(x/\|x\|)$. Then for $f \in C^\infty(S^{n-1})$, define

$$\Delta_{S^{n-1}} f := (\Delta F)|_{S^{n-1}}$$

Let $f \in V_{n,m}$. Recall the Euler identity

$$\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} f = mf.$$

To see this, differentiate the identity $f(tx) = t^m f(x)$ with respect to t; we have

$$\sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} f \right)(tx) \cdot x_i = mt^{m-1} f(x).$$

Taking $t = 1$ yields the identity.

Let $f \in V_{n,m}$. Its extension $F : \mathbb{R}^n \to \mathbb{C}$ to $\mathbb{R}^n$ is $F(x) = f(x/\|x\|) = \|x\|^{-m} f(x)$. A tedious computation along with the Euler identity yields that

$$\Delta_{S^{n-1}} f = -m(m+n-2)f + (\Delta f)|_{S^{n-1}}$$

Lemma 15.6.9. For $f \in V_{n,m}$, f is harmonic if and only if f is an eigenfunction of $\Delta_{S^{n-1}}$, in which case f has eigenvalue $-m(m+n-2)$.

Proof. It is clear from the above identity when f is harmonic. Conversely if $\Delta_{S^{n-1}} f = \lambda f$ for some $\lambda \in \mathbb{C}$, then

$$\lambda f = -m(m+n-2)f + (\Delta f)|_{S^{n-1}},$$

or

$$(\lambda + m(m+n-2))\, f = (\Delta f)|_{S^{n-1}}.$$

LHS is homogeneous of degree m, while RHS is homogeneous of degree $m-2$. Hence $(\Delta f)|_{S^{n-1}} = 0$, and thus $\Delta f = 0$. $\square$

15.6.3 Spin(n)

Recall that $\mathrm{Spin}(n)$ is a double cover $\mathrm{SO}(n)$ and it fits into a short exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n) \longrightarrow 1.$$

In particular, every representation of $\mathrm{SO}(n)$ results in a representation of $\mathrm{Spin}(n)$. Precisely, for every topological group G, there is a bijection

$$\{\pi \in \mathrm{Hom}_{\mathbf{TopGp}}(\mathrm{Spin}(n), G) \mid \pi(-1) = e \in G\} \longrightarrow \mathrm{Hom}_{\mathbf{TopGp}}(\mathrm{SO}(n), G)$$

Definition. A representation $\pi : \mathrm{Spin}(n) \to \mathrm{GL}(V)$ is called **spinorial** if $\pi(-1) \neq \mathrm{id}_V \in \mathrm{GL}(V)$.

In this subsection we are going to construct two spinorial representations, called the **spin** (resp. **half-spin**) **representation**.

Let $(\ ,\ )$ be the symmetric bilinear form on $\mathbb{C}^n$ given by the (real) inner product. Let $m = \left\lfloor \frac{n}{2} \right\rfloor$; then $n = 2m$ if $2 \mid n$, and $n = 2m+1$ if $2 \nmid n$. Take

$$\begin{aligned} W &= \{(z_1, \ldots, z_m, iz_1, \ldots, iz_m(,0)) \mid z_k \in \mathbb{C}\} \\ W' &= \{(z_1, \ldots, z_m, -iz_1, \ldots, -iz_m(,0)) \mid z_k \in \mathbb{C}\} \end{aligned}$$

to be maximal isotropic (complex) subspaces and $e_0 = (0, \ldots, 0, 1)$. Then

$$\mathbb{C}^n = \begin{cases} W \oplus W' & , n \text{ even} \\ W \oplus W' \oplus \mathbb{C}e_0 & , n \text{ odd.} \end{cases}$$

Recall the Clifford algebra C_n associated to the real quadratic space $(\mathbb{R}^n, \|\cdot\|^2)$. Put $C_n(\mathbb{C}) := C_n \otimes_{\mathbb{R}} \mathbb{C}$; then by definition $C_n(\mathbb{C}) = T(\mathbb{C}^n)/\langle z \otimes z + (z,z) \mid z \in \mathbb{C}^n \rangle$. In fact,

Lemma 15.6.10. There are $\mathbb{C}$-algebra isomorphisms

$$C_n(\mathbb{C}) \cong \begin{cases} \mathrm{End}_{\mathbb{C}} \bigwedge W & , \text{if } n \text{ even} \\ \mathrm{End}_{\mathbb{C}} \bigwedge W \oplus \mathrm{End}_{\mathbb{C}} \bigwedge W & , \text{if } n \text{ odd.} \end{cases}$$

Proof. We begin with the case n being even. For $z = w + w' \in \mathbb{C}^n$ with $w \in W$, $w' \in W'$, define $\Phi : \mathbb{C}^n \to \mathrm{End}_{\mathbb{C}} \bigwedge W$ by

$$\Phi(z) = \epsilon(w) - 2\iota(w').$$

Here, $\epsilon(w) \in \mathrm{End}_{\mathbb{C}} \bigwedge W$ is given by $x \mapsto w \wedge x$, and $\iota(w') \in \mathrm{End}_{\mathbb{R}} \bigwedge W$ by

$$\iota(w')(x_1 \wedge \cdots \wedge x_\ell) = \sum_{i=1}^{\ell} (-1)^{i+1} (w', x_i) x_1 \wedge \cdots \wedge \widehat{x_I} \wedge \cdots \wedge x_\ell.$$

A basic property for ϵ and ι is that $\epsilon(w)^2 = 0 = \iota(w')^2$. We claim that $\Phi(z)^2 = -2(w, w')\,\mathrm{id} = -(z, z)\,\mathrm{id}$, i.e.,

$$\epsilon(w)\iota(w') + \iota(w')\epsilon(w) = (w, w').$$

This follows from an easy calculation. In particular, this show that Φ, extending multiplicatively, defines a map

$$\Phi : C_n(\mathbb{C}) \to \mathrm{End}_{\mathbb{C}} \bigwedge W.$$

Since $\dim_{\mathbb{R}} C_n = 2^n$ and $\dim_{\mathbb{C}} W = m = \dfrac{n}{2}$, both sides have complex dimension 2^n, so to show Φ is an isomorphism it suffices to show it is surjective. To this end, let $w_1, \ldots, w_m$ be a basis for W and let $w_1', \ldots, w_n'$ be its dual basis with respect to $(\,,\,)$. Then for $1 \leqslant i_1 < \cdots < i_k \leqslant m$, a moment's thought shows that the operator

$$\Phi(w_{i_1} \cdots w_{i_k} w_{i_1}' \cdots w_{i_k}') : \bigwedge W \to \bigwedge W$$

kills $\bigwedge^p W$ for $p < k$, maps $\bigwedge^k W$ onto $\mathbb{C}w_{i_1} \wedge \cdots \wedge w_{i_k}$, and stabilizes $\bigwedge^p W$ for $p > k$; in fact, it kills every pure k-form except $w_{i_1} \wedge \cdots \wedge w_{i_k}$. Easy manipulation shows that the image of Φ contains all maps on $\bigwedge W$ that maps $w_{i_1} \wedge \cdots \wedge w_{i_k}$ to $w_{i_1} \wedge \cdots \wedge w_{i_k}$ and all the other pure forms to 0. For $1 \leqslant j_1 < \cdots < j_l \leqslant m$, the operator $\Phi(w_{i_1} \cdots w_{i_k} w_{j_1}' \cdots w_{j_l}')$ maps $w_{j_1} \wedge \cdots \wedge w_{j_l}$ to $\pm w_{i_1} \wedge \cdots \wedge w_{i_k}$, and any other pure form to the others. Composing shows that the image of Φ contains all maps sending $w_{j_1} \wedge \cdots \wedge w_{j_l}$ to $w_{i_1} \wedge \cdots \wedge w_{i_k}$ and sending other basis elements to 0. This shows the surjectivity.

Now we consider the odd n case. This times, for $z = w + w' + ce_0 \in \mathbb{C}^n$ with $w \in W$, $w' \in W'$, define $\Phi^{\pm} : \mathbb{C}^n \to \mathrm{End}_{\mathbb{C}} \bigwedge W$ by

$$\Phi^{\pm}(z) = \epsilon(w) - 2\iota(w') \pm (-1)^{\deg} ic\,\mathrm{id}\,.$$

The factor $(-1)^{\deg}$ is added so that $\Phi^{\pm}(z)^2 = -2(w, w') - c^2 = -(z, z)$. Again this implies that $\Phi^{\pm}$ defines an algebra homomorphism $\Phi^{\pm} : C_n(\mathbb{C}) \to \mathrm{End}_{\mathbb{C}} \bigwedge W$, and hence

$$\Phi : C_n(\mathbb{C}) \to \mathrm{End}_{\mathbb{C}} \bigwedge W \oplus \mathrm{End}_{\mathbb{C}} \bigwedge W$$

by $z \mapsto (\Phi^+ z, \Phi^- z)$. Both sides having the same dimension 2^n, we can prove Φ is an isomorphism by showing Φ is surjective. This is similar to the proof in the even case; to "separate" the action, one makes use of use $\Phi(ie_0) = ((-1)^{\deg}, -(-1)^{\deg})$. $\square$

Corollary 15.6.10.1. There are $\mathbb{C}$-algebra isomorphisms

$$C_n^0(\mathbb{C}) := C_n^0 \otimes_{\mathbb{R}} \mathbb{C} \cong \begin{cases} \mathrm{End}_{\mathbb{C}} \bigwedge^{\mathrm{even}} W \oplus \mathrm{End}_{\mathbb{C}} \bigwedge^{\mathrm{odd}} W & \text{, if } n \text{ even} \\ \mathrm{End}_{\mathbb{C}} \bigwedge W & \text{, if } n \text{ odd.} \end{cases}$$

Here $\mathrm{End}_{\mathbb{C}} \bigwedge^{\mathrm{even}} W, \mathrm{End}_{\mathbb{C}} \bigwedge^{\mathrm{odd}} W$ stand for their obvious meanings.

Proof. For n even, it is clear that $\Phi(C_n^0(\mathbb{C}))$ stabilizes $\bigwedge^{\mathrm{even}} W$ and $\bigwedge^{\mathrm{odd}} W$, so restricting to them yields an injective algebra homomorphism

$$C_n^0(\mathbb{C}) \longrightarrow \mathrm{End}_{\mathbb{C}} \bigwedge^{\mathrm{even}} W \oplus \mathrm{End}_{\mathbb{C}} \bigwedge^{\mathrm{odd}} W$$

Since both sides have dimension 2^{n-1}, this is an isomorphism.

For n odd, restricting Φ^+ to $C_n^0(\mathbb{C})$ yields

$$\Phi^+ : C_n^0(\mathbb{C}) \longrightarrow \operatorname{End}_{\mathbb{C}} \bigwedge W.$$

In the proof of surjectivity of Φ in the even case, note that the crucial operator $\Phi(w_{i_1} \cdots w_{i_k} w'_{i_1} \cdots w'_{i_k})$ lies in the image of $C_n^0(\mathbb{C})$, and the operator $\Phi(w_{i_1} \cdots w_{i_k} w'_{j_1} \cdots w'_{j_l})$ in following argument can be modified, by adding some $\Phi(e_0)$, so that it also lies in the image of $C_n^0(\mathbb{C})$. To sum up, this means Φ^+ remains surjective. To show it is an isomorphism it suffices to notice that both sides have dimension 2^{n-1}. We remark that we could also use Φ^+ to establish an isomorphism. $\square$

Now we can construct the spin and half-spin representation. The isomorphism Φ (resp. Φ^+ or Φ^-) from the last lemma allows as to define a representation of $\operatorname{Spin}(n) \subseteq C_n^0(\mathbb{C})$ on $\bigwedge W$:

$$\Phi \,(\text{resp. } \Phi^\pm) : \operatorname{Spin}(n) \to \operatorname{GL}(\bigwedge W)$$

This is the desired **spin representation**. When n is even, we can further restrict Φ to $\bigwedge^{\text{even}} W$ or $\bigwedge^{\text{odd}} W$, yielding

$$\Phi : \operatorname{Spin}(n) \to \operatorname{GL}(\bigwedge^{\text{even}} W) \,(\text{resp. } \operatorname{GL}(\bigwedge^{\text{odd}} W)).$$

They are called the **half-spin representation**. These constructed representations are clearly spinorial, as $\Phi(-1) = -\operatorname{id}$.

Lemma 15.6.11. For n even, the half-spin representations are irreducible. For n odd, the spin representation is irreducible.

Proof. Let $\{e_i\}_{i=1}^n$ be the standard basis for $\mathbb{C}^n$. In $C_n(\mathbb{C})$, compute

$$(e_j \pm ie_{j+m})(e_k \pm ie_{k+m}) = e_j e_k \pm i(e_j e_{k+m} + e_{j+m} e_k) - e_{j+m} e_{k+n}.$$

Note that each summand on the right lies in $\operatorname{Spin}(n)$. This implies the operators $\epsilon(e_j)\epsilon(e_k)$ and $\iota(e_j)\iota(e_k)$ lie in the span of $\Phi(\operatorname{Spin}(n))$ in $\operatorname{End}_{\mathbb{C}} \bigwedge W$. Hence a $\operatorname{Spin}(n)$-invariant subspace is also invariant under $\epsilon(e_j)\epsilon(e_k)$ and $\iota(e_j)\iota(e_k)$. It is then clear that when n is even, two half spin representations are irreducible.

For n odd, compute (note that $e_n = e_0$ in the previous notation)

$$(e_j \pm ie_{j+m})e_n = e_j e_n \pm ie_{j+m} e_n$$

so that $\epsilon(e_j)(-1)^{\deg}$ and $\iota(e_j)(-1)^{\deg}$ lie in the span of $\Phi(\operatorname{Spin}(n))$ in $\operatorname{End}_{\mathbb{C}} \bigwedge W$. It is then clear that the spin representation is irreducible.

$\square$

Chapter 16

Direct Integrals

16.1 Von Neumann Algebras

Let H be a Hilbert space. For a subset $M \subseteq \mathcal{B}(H)$, we define the **commutant**, or the centralizer, of M as

$$M' = C_{\mathcal{B}(H)}(M) := \{T \in \mathcal{B}(H) \mid Tm = mT \text{ for all } m \in M\}$$

Clearly, for $N \subseteq M \subseteq \mathcal{B}(H)$, we have $M' \subseteq N'$. The commutant of M' is called the **bi-commutant** or **double commutant** of M. Also, for a subset $M \subseteq \mathcal{B}(H)$, define the **adjoint set** $M^* = \{m^* \mid m \in M\}$. A subset M is called **self-adjoint** if $M = M^*$.

Definition. A **von Neumann algebra** is a *-subalgebra $\mathcal{A}$ of $\mathcal{B}(H)$ with $\mathcal{A}'' = \mathcal{A}$.

- A von Neumann algebra is necessarily norm closed, so it is itself a C^*-algebra. The converse need not hold.
- For a subset $M \subseteq \mathcal{B}(H)$, we have $M \subseteq M''$, so $M''' \subseteq M'$. On the other hand $M' \subseteq (M')'' = M'''$, so $M' = M'''$. Thus, if $M^* = M$, then M' is a von Neumann algebra.
- For self-adjoint $M \subseteq \mathcal{B}(H)$, its double commutant M'' is the smallest von Neumann algebra in $\mathcal{B}(H)$ containing M, and it is called the von Neumann algebra generated by M.

For a von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(H)$, $Z(\mathcal{A}) = A \cap A'$ is called the **center** of $\mathcal{A}$. A von Neumann algebra $\mathcal{A}$ is called a **factor** if its center is trivial, i.e., $Z(\mathcal{A}) = \mathbb{C}\,\mathrm{id}_H$.

Example 16.1.1.

1. $\mathcal{B}(H)$ is a factor; this is called a **type-I factor**.

16.2 Weak and Strong Topologies

Let H be a Hilbert space. On $\mathcal{B}(H)$ we can define three topologies.

1. The topology induced by the operator norm. This is called the **norm topology**.
2. The topology induced by the family of seminorms $T \mapsto \|Tv\|$, where v runs over H. This is called the **strong operator topology (SOT)**.

3. The topology induced by the family of seminorms $T \mapsto |\langle Tv, w\rangle|$, where v, w run over H. This is called the **weak operator topology (WOT)**.

It is clear that for a subset $\mathcal{A} \subseteq \mathcal{B}(H)$, we have the inclusions

$$\mathcal{A} \subseteq \overline{\mathcal{A}}^{\text{NORM}} \subseteq \overline{\mathcal{A}}^{\text{SOT}} \subseteq \overline{\mathcal{A}}^{\text{WOT}} \subseteq \mathcal{A}''$$

Theorem 16.2.1 (von Neumann's Bi-commutant Theorem)**.** Let H be a Hilbert space, and let $\mathcal{A}$ be a unital *-subalgebra of $\mathcal{B}(H)$. Then $\overline{\mathcal{A}}^{\text{SOT}} = \overline{\mathcal{A}}^{\text{WOT}} = \mathcal{A}''$.

Proof. It suffices to show $\mathcal{A}'' \subseteq \overline{\mathcal{A}}^{\text{SOT}}$. Let $T \in \mathcal{A}''$. A fundamental system of neighborhoods of $0 \in \mathcal{B}(H)$ in the strong operator topology consists of $\{S \in \mathcal{B}(H) \mid \|Sv_j\| < \varepsilon,\, j = 1, \dots, n\}$, $v_1, \dots, v_n \in H$, $\varepsilon > 0$, $n \in \mathbb{N}$. To show $T \in \overline{\mathcal{A}}^{\text{SOT}}$ it suffices to show for $v_1, \dots, v_n \in H$, $\varepsilon > 0$ there exists $a \in \mathcal{A}$ such that

$$\|Tv_j - av_j\| < \varepsilon \text{ for all } j \in [n].$$

To this end, let $\mathcal{B}(H)$ act on H^n diagonally. If we put $v = (v_1 \cdots v_n)^t \in H^n$, for each $\varepsilon > 0$ we only need to find $a \in \mathcal{A}$ such that $\|Tv - av\| < \varepsilon$, or $Tv \in \overline{\mathcal{A}v}$. It then suffices to show T leaves $\overline{\mathcal{A}v}$ invariant.

First, we have $C_{\mathcal{B}(H^n)}(\mathcal{A}) = M_n(\mathcal{A}') \subseteq \mathcal{B}(H^n)$. Since $\mathcal{A}'$ is unital, it follows that $C_{\mathcal{B}(H^n)}(M_n(\mathcal{A}')) = \mathcal{A}'' I_n$. Since $\mathcal{A}$ is *-closed in $\mathcal{B}(H)$, the orthogonal complement $(\mathcal{A}v)^\perp$ is $\mathcal{A}$-stable. Hence the orthogonal projection P onto $\overline{\mathcal{A}v}$ commutes with $\mathcal{A}$, i.e., $P \in C_{\mathcal{B}(H^n)}(\mathcal{A})$. It follows that $T \in \mathcal{A}'' I_n$ commutes with P and thus leaves $\overline{\mathcal{A}v}$ stable. This finishes the proof.

□

16.3 Representations

Definition. Let G be an LCH group. An unitary representation π of G is called a **factor representation** if the von Neumann algebra $\mathrm{VN}(\pi)$ generated by $\pi(G) \subseteq \mathcal{B}(V_\pi)$ is a factor.

- Since $\pi(G)$ is self-adjoint, we have $\mathrm{VN}(\pi) = \pi(G)''$, and the center of $\mathrm{VN}(\pi)$ is $\pi(G)'' \cap \pi(G)''' = \pi(G)'' \cap \pi(G)'$. Hence π is a factor representation if and only if $\pi(G)' \cap \pi(G)'' = \mathbb{C}\, \mathrm{id}_{V_\pi}$.

Lemma 16.3.1. Every irreducible unitary representation is a factor representation.

Proof. Let $\pi \in \widehat{G}$. Then $\pi(G)' = \mathrm{Hom}_G(V_\pi, V_\pi) = \mathbb{C}\, \mathrm{id}_{V_\pi}$ by Schur's lemma, so $\mathrm{VN}(\pi) = \mathcal{B}(V_\pi)$, which is a factor. □

Definition. Two unitary representations π_1, π_2 of G are called **quasi-equivalent** if there is a $*$-algebra isomorphism

$$\phi : \mathrm{VN}(\pi_1) \longrightarrow \mathrm{VN}(\pi_2)$$

such that $\phi \circ \pi_1 = \pi_2$.

Lemma 16.3.2. Tow irreducible unitary representations of a locally compact gorpu G are quasi-equivalent if and only if they are unitarily equivalent.

16.4 Direct Integrals

Let $(X, \mathcal{D})$ be a measurable space throughout this section. Without otherwise specified, all Hilbert spaces are separable.

Definition. A family $\{H_x \mid x \in X\}$ of Hilbert spaces indexed by X is called a **field of Hilbert spaces over** X. A set-theoretic section to $\bigsqcup_{x\in X} H_x \to X$ is called a **section over** X.

Definition. A **Hilbert bundle over** X is a field of Hilbert spaces $\{H_x \mid x \in X\}$ together with a countable collection $\{e_n\}_{n\geqslant 1}$ of sections such that

(i) the function $x \mapsto \langle e_n(x), e_m(x)\rangle_{H_x}$ is measurable for all n, m, and

(ii) the linear span $\operatorname{span}_{\mathbb{C}}\{e_n(x)\}_{n\geqslant 1}$ is dense in H_x for each $x \in X$.

Given a Hilbert bundle $\{H_x \mid x \in X\}$ over X with countable section $\{e_n\}_{n\geqslant 1}$, we can stratify the base space X as follows. For a fixed $x \in X$, define $n_1(x) = \inf\{n \geqslant 1 \mid e_n(x) \neq 0\}$ and inductively for $k \geqslant 2$

$$n_k(x) := \begin{cases} \inf\{n \geqslant 1 \mid e_n(x) \text{ and } e_n(x) \notin \operatorname{span}_{\mathbb{C}}\{e_{n_1(x)}(x), \dots, e_{n_{k-1}(x)}(x)\}\} & \text{, if the inf exists} \\ 0 & \text{, otherwise} \end{cases}$$

Define $f_k(x) := e_{n_k(x)}(x)$; clearly $\operatorname{span}\{f_n(x)\}_n = \operatorname{span}\{e_n(x)\}_n$. For $n \in [1, \infty]$ set

$$X_n = \{x \in X \mid \dim H_x \leqslant n\} = \{x \in X \mid f_n(x) \neq 0 \text{ but } f_{n+1}(x) = 0\}.$$

From the last expression we see X_n is measurable. We apply Gram-Schmidt to the sections $\{f_k\}$ pointwise to obtain sections $\{u_k\}$ that are pointwise orthonormal sequences (once having 0 deleted). Then $u_k(x) = 0$ for x with $k > \dim H_x$.

Lemma 16.4.1. For each $k \geqslant 1$, there exists a measurable partition $\{X_m^k\}_{m\geqslant 1}$ of X such that for all m and all $j \leqslant k$, one has either $f_j(x) = 0$ for all $x \in X_m^k$, or $f_j(x) = e_{m(j)}(x) \neq 0$ for all $x \in X_m^k$ where $m(j) \geqslant 1$ is independent of x. In particular, on X_m^k the section u_k is a linear combinations $e_{m(1)}, \dots, e_{m(k)}$ with coefficients varying measurably.

Proof. The set X_m^k is constructed by induction. Suppose $k = 1$. Define

$$\begin{aligned} X_m^1 &= \{x \in X \mid e_m(x) \neq 0\} \setminus \bigcup_{j=1}^{m-1} \{x \in X \mid e_j(x) \neq 0\} \\ &= \{x \in X \mid e_m(x) \neq 0 = e_1(x) = \cdots = e_{m-1}(x)\} \end{aligned}$$

Since $\operatorname{span}\{e_m(x)\}_m$ is dense in H_x for each x, we must have $X = \bigcup_{m\geqslant 1} X_m^1$. Also,

$$\{x \in X \mid e_j(x) = 0\} = \{x \in X \mid \langle e_j(x), e_j(x)\rangle_{H_x} = 0\}$$

is measurable by the very assumption, so each X_m^1 is measurable. This proves the claim when $k = 1$.

Suppose $k > 1$ and that the sets X_m^j have been constructed for all m and $j \leqslant k-1$. It suffices to prove the lemma for $Y = X_m^{k-1}$ for all m. Indeed, one then just combines those partitions to acquire the desired partitions. Now on Y, by hypothesis either $f_{k-1} = 0$, in which forces $f_k = 0$ and we can take the trivial partition of Y, or $f_j = e_{m(j)}$ for $j \leqslant k-1$. In the latter case, for each $m \geqslant 1$ define

$$Y_m := \{x \in Y \mid f_k(x) = e_m(x)\}$$

Notice that

$$Y_m = \left\{ x \in Y \mid \begin{array}{c} e_{m(1)}(x) \wedge \cdots \wedge e_{m(k-1)}(x) \wedge e_j(x) = 0 \text{ for all } j < m, \\ e_{m(1)}(x) \wedge \cdots \wedge e_{m(k-1)}(x) \wedge e_m(x) \neq 0 \end{array} \right\}$$

so Y_m is measurable. Then $\{Y_m\}_{m \geqslant 1}$ together with $Y_0 := \{x \in Y \mid f_k(x) = 0\}$ give the required partition of Y. $\square$

Definition. Given a Hilbert bundle $\{H_x \mid x \in X\}$ with countable sections $\{e_n\}_{n \geqslant 1}$, a section σ over X is called **measurable** if $x \mapsto \langle \sigma(x), e_n(x) \rangle_{H_x}$ is measurable for all $n \geqslant 1$.

In particular, the sections $\{u_k\}$ constructed above is measurable. From the last lemma and the fact that $\{u_n(x)\} \backslash \{0\}$ is an orthonormal basis for H_x, we obtain

Lemma 16.4.2. A section σ is a measurable if and only if $x \mapsto \langle \sigma(x), u_n(x) \rangle_{H_x}$ is measurable for all $n \geqslant 1$.

Corollary 16.4.2.1. If σ, τ are two measurable sections, then $x \mapsto \langle \sigma(x), \tau(x) \rangle_{H_x}$ are measurable.

Proof. By Parseval, one has

$$\langle \sigma(x), \tau(x) \rangle_{H_x} = \sum_{n \geqslant 1} \langle \sigma(x), u_n(x) \rangle_{H_x} \langle u_n(x), \tau(x) \rangle_{H_x}.$$

$\square$

Definition. Let μ be a measure on $(X, \mathcal{D})$.

(i) A measurable section s is called a **null-section** if it vanishes outside a null set.

(ii) The **direct Hilbert integral** is the vector space of all measurable sections σ satisfying

$$\|\sigma\|^2 := \int_X \|\sigma(x)\|^2_{H(x)} \, d\mu(x) < \infty$$

modulo the space of null-sections. We denote the space by

$$H = \int_X^{\oplus} H_x d\mu(x).$$

Loosely speaking, it is the space of all square-integrable sections of the Hilbert bundle.

Lemma 16.4.3. For $\sigma, \tau \in H$, define

$$\langle \sigma, \tau \rangle := \int_X \langle \sigma(x), \tau(x) \rangle_{H_x} \, d\mu(x)$$

This defines an inner product on H, and H becomes a Hilbert space.

Proof. We must show the completeness. For $n \geqslant 1$, set

$$X_n := \{x \in X \mid u_n(x) \neq 0\} = \{x \in X \mid \dim H_x \geqslant n\}$$

and $P_n : H \to L^2(X_n, \mu)$ by $P_n(\sigma) : x \mapsto \langle \sigma(x), u_n(x) \rangle_{H_x}$. We claim this is surjective. Indeed, for $f \in L^2(X_n, \mu)$, define $\sigma_f : x \mapsto f(x) u_n(x)$. Since $\langle \sigma_f(x), u_m(x) \rangle = f(x) \delta_{ij}$, we see σ_f is measurable. Moreover,

$$\|\sigma_f\|^2 = \int_X \|\sigma_f(x)\|^2 \, d\mu(x) = \int_{X_n} |f(x)|^2 d\mu(x) = \|f\|_2^2 < \infty$$

so $\sigma_f \in H$.

Thus the P_n define a map with dense image

$$H = \int_X^{\oplus} H_x \, d\mu(x) \longrightarrow \widehat{\bigoplus_{n \geqslant 1}} L^2(X_n)$$

Now it suffices to show this map is isometric. For $\sigma \in H$, by Parseval

$$\begin{aligned} \|\sigma\|^2 &= \int_X \|\sigma(x)\|^2 \, d\mu(x) = \int_X \left(\sum_{n \geqslant 1} |\langle \sigma(x), u_n(x) \rangle|^2 \right) d\mu(x) \\ &= \sum_{n \geqslant 1} \int_X |\langle \sigma(x), u_n(x) \rangle|^2 d\mu(x) \\ &= \sum_{n \geqslant 1} \int_{X_i} |P_n(\sigma)(x)|^2 d\mu(x) = \sum_{n \geqslant 1} \|P_n(\sigma)\|_2^2 \end{aligned}$$

□

Example 16.4.4 (Constant field)**.** Let H be a Hilbert space with orthonormal basis $\{e_n\}_{n \geqslant 1}$. If we take $H_x = H$ and $e_n(x) := e_n$, then we get a Hilbert bundle over X. We call this a **constant field**.

In this case, the direct integral is just all H-valued measurable functions $f : X \to H$ that are square integrable with respect to μ. We shall set

$$L^2(X, \mu, H) = \int_X^{\oplus} H_x d\mu(x).$$

Example 16.4.5 (Direct sum)**.** Suppose $(X, \mathcal{D})$ is discrete and $\mu = \#$ is the counting measure. Let $\{H_x \mid x \in X\}$ be a field of Hilbert spaces. For each $x \in X$, let $\{e_n(x)\}_{n \geqslant 1}^{\dim H_x}$ be an orthonormal basis for H_x. If we set $e_n(x) = 0$ for $n > \dim H_x$, then $\{e_n\}_{n \geqslant 1}$ makes $\{H_x \mid x \in X\}$ into a Hilbert bundle.

In this case, $\int_X^{\oplus} H_x \, d\mu(x)$ is simply $\widehat{\bigoplus}_{x \in X} H_x$.

Example 16.4.6 (Vector bundle)**.** Let X be a second countable (topological or smooth) manifold and V a (locally trivial) vector bundle over X whose fibers are Hilbert spaces. By a partition of unity argument, one shows there exists a countable continuous or smooth sections that makes $\{V_x \mid x \in X\}$ a Hilbert bundle.

Equip X with its Borel σ-algebra and μ a Borel measure. In this case, the direct integral is the space of all square-integrable sections of the bundle $V \to X$ with respect to μ.

Example 16.4.7 (Direct integral of representations)**.** Let $\{H_x \mid x \in X\}$ be a Hilbert bundle with section $\{e_n\}_n$ and μ a measure on X. Let G be an LCH group, and for every $x \in X$ suppose H_x affords a unitary representation π_x of G_x such that the map $x \mapsto \langle \pi_x(g) e_n(x), e_m(x) \rangle_{H_x}$ is measurable for all $g \in G$, $n, m \geqslant 1$. Then

$$\pi : G \longrightarrow \mathrm{GL}_{\mathrm{cts}} \left(\int_X^{\oplus} H_x d\mu(x) \right)$$

given by $(\pi(g)\sigma)(x) := \pi_x(g)\sigma(x)$ defines a unitary representations of G on $\int_X^{\oplus} H_x d\mu(x)$.

Example 16.4.8 (LCA group)**.** Let A be an LCA group with a chosen Haar measure dx. Hence on the Pontryagin dual $\widehat{A}$ there exists a corresponding Plancherel measure $d\chi$.

Each $\chi \in \widehat{A}$ determines a one-dimensional representation $\chi : A \to \mathrm{GL}_1(\mathbb{C}_\chi)$. Then $\{\mathbb{C}_\chi \mid \chi \in \widehat{A}\}$ equipped with the constant section $e : \chi \mapsto 1$ forms a Hilbert bundle. By Example 16.4.4, the direct integral is nothing but $L^2(\widehat{A}, d\chi)$, so we get an equality

$$\int_{\widehat{A}}^{\oplus} \mathbb{C}_\chi d\chi = L^2(\widehat{A}, d\chi).$$

From the last example we get a unitary representation π of A on $\int_{\widehat{A}}^{\oplus} \mathbb{C}_\chi d\chi$. By Plancherel theorem, the regular representation on $L^2(\widehat{A}, d\chi)$ is unitarily equivalent the regular representation on $L^2(A)$. We claim the resulting Hilbert space isomorphism

$$L^2(A) \xrightarrow{\sim} \int_{\widehat{A}}^{\oplus} \mathbb{C}_\chi d\chi$$

is A-intertwining. In fact, on $L^1(A) \cap L^2(A)$, the map is given by sending $f \in L^2(A)$ to the section

$$\sigma_f : \chi \mapsto \widehat{f}(\chi) \in \mathbb{C}_\chi.$$

For $g \in A$, one has $(\pi(g)\sigma_f)(\chi) = \pi(g)\sigma_f(\chi) = \chi(g)\widehat{f}(\chi)$. On the other hand,

$$\widehat{R(g)f}(\chi) = \int_A f(xg)\overline{\chi}(x)dx = \chi(g)\widehat{f}(\chi).$$

This proves the claim.

16.5 The Plancherel Theorem

Definition. A locally compact group G is called a **type-I group** if every factor representation of G is of type I.

Theorem 16.5.1. Let G be a second countable, unimodular, locally compact group of type I. There is a unique measure μ on $\widehat{G}$ such that for $f \in L^1(G) \cap L^2(G)$ one has

$$\|f\|_2^2 = \int_{\widehat{G}} \|\pi(f)\|_{\mathrm{HS}}^2 \, d\mu(\pi)$$

The map $f \mapsto (\pi(f))_\pi$ extends to a unitary $G \times G$ equivariant map

$$L^2(G) \cong \int_{\widehat{G}}^{\oplus} \mathcal{H}(V_\pi) d\mu(\pi)$$

where the representation of η_π of $G \times G$ on the space of Hilbert-Schmidt operators $\mathcal{H}(V_\pi)$ is given by $\eta_\pi(x, y)T = \pi(x)T\pi(y^{-1})$ for each $\pi \in \widehat{G}$ and $x, y \in G$.

Chapter 17

Trace Formula

17.1 Cocompact Groups and Lattices

Definition. Let G be a topological group.

1. A subgroup $H \leqslant G$ is called **cocompact** if the quotient G/H is a compact space.

2. A subgroup $\Lambda \leqslant G$ is called **discrete** if the subspace topology on Λ is the discrete topology.

Proposition 17.1.1. Let G be an LCH group. If G admits a unimodular closed cocompact subgroup, then G is unimodular itself.

Proof. Let $H \leqslant G$ be a unimodular closed cocompact subgroup. Since $G/H \cong H\backslash G$ via $gH \mapsto Hg^{-1}$, $H\backslash G$ is compact as well and the map $C_c(G) \to C(H\backslash G)$ given by $g \mapsto^H g$ is surjective by Lemma 2.4.2, where ${}^H g(x) := \int_H g(hx)dh$. We install a Radon measure μ on the right coset space $H\backslash G$ by Riesz's representation theorem as follows. For $f \in C(H\backslash G)$, choose $g \in C_c(G)$ with ${}^H g = f$, and define

$$\int_{H\backslash G} f(x)d\mu(x) := \int_G g(x)dx$$

To show this is well-define, suppose ${}^H g = 0$. Note that since H is unimodular, $\int_H g(h^{-1}x)dh = \int_H g(hx)dh = 0$. Let $\phi \in C_c(G)$ with ${}^H\phi = 1$. One gets

$$\int_G g(x)dx = \int_G \int_H \phi(hx)g(x)dhdx = \int_H \int_G \phi(hx)g(x)dxdh = \int_H \int_G \phi(x)g(h^{-1}x)dxdh = 0$$

Now, for $f \in C_c(H\backslash G)$, if we take $g \in C_c(G)$ with ${}^H g = f$, then

$$\int_{H\backslash G} f(xy)d\mu(x) = \int_G g(xy)dx = \Delta(y^{-1}) \int_G g(x)dx = \Delta(y^{-1}) \int_{H\backslash G} f(x)d\mu(x)$$

In particular, for $f \equiv 1 \in C(H\backslash G)$, we have

$$0 < \int_{H\backslash G} f(x)d\mu(x) = \int_{H\backslash G} f(xy)d\mu(x) = \Delta(y^{-1}) \int_{H\backslash G} f(x)d\mu(x)$$

so that $\Delta(y^{-1}) = 1$ for all $y \in G$. $\square$

Lemma 17.1.2. Let G be a topological group.

1. A subgroup $\Gamma \leqslant G$ is discrete if and only if there exists a unit-neighborhood $U \subseteq G$ with $\Gamma \cap U = \{1\}$.

2. A discrete subgroup is closed in G if G is Hausdorff.

Proof.

1. Clear.

2. We prove the following generalization.

   Lemma 17.1.3. Let G be a Hausdorff topological group and $H \leqslant G$ be a locally compact subgroup. Then H is closed in G.

   Proof. Replacing G with the closure of H in G, we may assume H is dense in G. Let $x \in H$ and choose a neighborhood U of x in H with compact closure C. Write $U = V \cap H$ for some open $V \subseteq G$. Since C is compact and G is Hausdorff, C is closed in G, and thus $V \backslash C$ is open in G. But for $V \cap H = U \subseteq C$, it forces $(V \backslash C) \cap H = \varnothing$, and since H is dense in G, it must be the case $V \subseteq C$; in particular, $V \subseteq H$. This shows H is open in G, and since they are topological groups, H is closed in G. ☐

   A discrete space is automatically locally compact and Hausdorff, so the lemma applies.

☐

Definition. Let G an LCH group. A discrete subgroup $\Gamma \leqslant G$ such that G/Γ carries an invariant Radon measure μ with $\mu(G/\Gamma) < \infty$ is called a **lattice** in G. A cocompact lattice is called a **uniform lattice**.

- The quotient G/Γ admits a invariant Radon measure, so by Theorem 2.4.7, $\Delta_G|_\Gamma = \Delta_\Gamma \equiv 1$.
- When we speak of the Haar measure on a lattice, we always measure the counting measure, and on the quotient G/Γ we always equip it with the measure so that the quotient measure formula holds.

Proposition 17.1.4. Let G be an LCH group. A discrete cocompact subgroup Γ is a uniform lattice.

Proof. By Proposition 2.3.3, Γ is unimodular, so Proposition 17.1.1 together with Lemma 17.1.2.2. implies G is unimodular. By Theorem 2.4.7 G/Γ admits an invariant Radon measure, and since G/Γ is compact, it has finite volume, so it is a lattice. ☐

Proposition 17.1.5. Let G be an LCH group. If G admits a lattice, then G is unimodular.

Proof. Let Δ be the modular function of G and let $H = \ker \Delta$; then H is unimodular by Corollary 2.4.8.2.2. Let $\Gamma \leqslant G$ be a lattice. As observed above, we have $\Delta_G|_\Gamma = 1$, so $\Gamma \leqslant H$. Since $H/\Gamma \leqslant G/\Gamma$, by Theorem 2.4.7 we have

$$\infty > \operatorname{vol}(G/\Gamma) = \int_{G/\Gamma} 1dx = \int_{G/H} \int_{H/\Gamma} 1dxdy = \operatorname{vol}(G/H) \operatorname{vol}(H/\Gamma)$$

In particular, $\operatorname{vol}(G/H) < \infty$, so G/H is compact by Proposition 2.3.4. Now it follows from Proposition 17.1.1 that G is unimodular. ☐

We conclude this section with some interesting facts about lattices.

Proposition 17.1.6 (Pigeonhole principle)**.** Let G be an LCH group and Γ a cocompact lattice. If $X \subseteq G$ is a measurable subset such that $\mathrm{vol}(X) > \mathrm{vol}(G/\Gamma)$, then there are two distinct elements $x, x' \in X$ satisfying $x^{-1}x' \in \Gamma$.

Proof. Let $\pi : G \to G/\Gamma$ be the canonical projection. We prove that for each measurable $X \subseteq G$, we have $\mathrm{vol}(X) \geqslant \mathrm{vol}(\pi(X))$ with equality if π is injective on X. The proposition follows by taking contrapositive.

Let $X \subseteq G$ be measurable. Then

$$\begin{aligned}\mathrm{vol}(X) &= \int_G \mathbf{1}_X(x)dx = \int_{G/\Gamma} \sum_{\gamma\in\Gamma} \mathbf{1}_X(x\gamma)dx \\ &= \int_{G/\Gamma} \#\{\gamma \in \Gamma \mid x\gamma \in X\}dx = \int_{G/\Gamma} \#\{y \in X \mid x^{-1}y \in \Gamma\}dx\end{aligned}$$

On the other hand, we have $x\Gamma \in \pi(X)$ if and only if $\#\{y \in X \mid x^{-1}y \in \Gamma\} \geqslant 1$, which implies

$$\mathrm{vol}(\pi(X)) = \int_{G/\Gamma} \mathbf{1}_{\pi(X)}(x)dx \leqslant \int_{G/\Gamma} \#\{y \in X \mid x^{-1}y \in \Gamma\}dx = \mathrm{vol}(X)$$

When $\pi|_X$ is injective, $x\Gamma \in \pi(X)$ if and only $\#\{y \in X \mid x^{-1}y \in \Gamma\} = 1$, so the equality holds. $\square$

Proposition 17.1.7. Let G be an LCH group and Γ a cocompact lattice. If Γ' is another lattice of G containing Γ, then $\mathrm{vol}(G/\Gamma) = [\Gamma' : \Gamma]\,\mathrm{vol}(G/\Gamma')$.

Proof. Since G/Γ is compact, by Lemma 2.4.2 we can find $f_0 \in C_c^+(G)$ such that

$$f_1(x) := f_0^{\Gamma}(x) = \sum_{\gamma\in\Gamma} f_0(x\gamma) = 1$$

for all $x \in G$. If we form $f_2(x) = f_0^{\Gamma'}(x)$, then $f_2(x) = [\Gamma' : \Gamma]$ for all $x \in G$. Now by quotient integral formula we have

$$\int_G f_0(x)dx = \int_{G/\Gamma} \sum_{\gamma\in\Gamma} f_0(x\gamma)dx = \int_{G/\Gamma} 1dx = \mathrm{vol}(G/\Gamma)$$

and

$$\int_G f_0(x)dx = \int_{G/\Gamma'} \sum_{\gamma\in\Gamma'} f_0(x\gamma)dx = \int_{G/\Gamma'} [\Gamma' : \Gamma]dx = [\Gamma' : \Gamma]\,\mathrm{vol}(G/\Gamma')$$

$\square$

Proposition 17.1.8. Let G be an LCH group, Γ is cocompact lattice and $\sigma : G \to G$ an topological group automorphism. Then $\mathrm{vol}(G/\sigma(\Gamma)) = \mathrm{mod}_G(\sigma)\,\mathrm{vol}(G/\Gamma)$.

Proof. Let f_0 and f_1 be as in the proof of the previous proposition. Then

$$\int_G f_0(\sigma^{-1}(x))dx = \mathrm{mod}_G(\sigma) \int_G f_0(x)dx = \mathrm{mod}_G(\sigma) \int_{G/\Gamma} \sum_{\gamma\in\Gamma} f_0(x\gamma)dx = \mathrm{mod}_G(\sigma)\,\mathrm{vol}(G/\Gamma)$$

On the other hand,

$$\int_G f_0(\sigma^{-1}x)dx = \int_{G/\sigma(\Gamma)} \sum_{\gamma\in\sigma(\Gamma)} f_0(\sigma^{-1}(x\gamma))dx = \int_{G/\sigma(\Gamma)} \sum_{\gamma\in\Gamma} f_0(\sigma^{-1}(x)\gamma)dx = \mathrm{vol}(G/\sigma(\Gamma))$$

$\square$

17.2 Discreteness of the Spectrum

Let G be an LCH group and $\Gamma \leqslant G$ a cocompact lattice. Then the right coset space $\Gamma\backslash G$ admits a right-invariant Radon measure μ with $\mu(\Gamma\backslash G) < \infty$. Therefore the action of G on $L^2(\Gamma\backslash G)$ by right translation R gives rise to a unitary representation of G.

Theorem 17.2.1. Let G be an LCH group and $\Gamma \leqslant G$ a uniform lattice. The representation R on $L^2(\Gamma\backslash G)$ decomposes as a direct sum of irreducible representations with finite multiplicities, i.e.,

$$L^2(\Gamma\backslash G) \cong \widehat{\bigoplus_{\pi\in\widehat{G}}} N_\Gamma(\pi)\pi$$

where $\widehat{G}$ is the unitary dual of G, and $N_\Gamma(\pi) \in \mathbb{N}_0$.

The proof of this theorem will occupy the rest of this section. First, we need a lemma to tell us that for $f \in C_c(G)$ the operator $R(f)$ is given a continuous integral kernel. For later use we will extend this to a greater class of functions f. Let U be a compact unit-neighborhood in G. For a continuous function f on G, let $f_U : G \to [0,\infty)$ be defined by

$$f_U(y) := \sup_{x,z\in U} |f(xyz)|$$

Lemma 17.2.2. The function f_U is continuous.

Proof. It suffices to show that for all $a \geqslant 0$, the sets $f^{-1}((a,\infty))$ and $f_U^{-1}([0,a))$ are open. For the former, assume $f_U(x) > a$. Then we can find $u_1, u_2 \in U$ with $|f(u_1xu_2)| > a$. As the function $y \mapsto f(u_1yu_2)$ is continuous, we can find a neighborhood V of x such that $|f(u_1vu_2)| > a$ for all $v \in V$. This implies $f_U(v) > a$ for every $v \in V$.

For the latter, let assume $a > f_U(y) \geqslant 0$ and let $\varepsilon > 0$ be given so that $a - \varepsilon > f_U(y)$. The function $(u,x,v) \mapsto |f(uxv)|$ is continuous, so for all $(u,v) \in U \times U$, we can find open neighborhood U_{uv} of u, v in U and open neighborhood V_{uv} of y in G such that $|f(u_1xu_2)| < a-\varepsilon$ for all $u_1, u_2 \in U_{uv}$ and $x \in V_{uv}$. The family $\{U_{uv} \times U_{uv}\}_{u,v\in U}$ of open sets covers $U \times U$, so there exist $u_i, v_j \in U$ $(1 \leqslant i,j \leqslant n < \infty)$ such that $U \times U = \bigcup_{i=1}^n U_{u_iv_i} \times U_{u_iv_i}$. Let $V = \bigcap_{i=1}^n V_{u_iv_i} \underset{\text{open}}{\subseteq} G$. Then for $x \in V$ and $u, v \in U$, we have $(u,v) \in U_{u_iv_i} \times U_{u_iv_i}$ for some $1 \leqslant i \leqslant n$, so $|f(uxv)| < a - \varepsilon$. Thus $f_U(x) \leqslant a - \varepsilon < a$ for all $x \in V$.

□

Definition. A continuous function $f : G \to \mathbb{C}$ is called **uniformly integrable** if there exists a compact unit-neighborhood U such that $f_U \in L^1(G)$. Denote by $C_{\text{unif}}(G)$ the set of all uniformly integrable continuous function on G.

- If $f \in C_{\text{unif}}(G)$, then $|f| \leqslant f_U$, so $f \in L^1(G)$.

Lemma 17.2.3. Let G be unimodular. Then $C_{\text{unif}}(G) \subseteq C_0(G)$ and it is an algebra under convolution.

Proof. Let $f \in C_{\text{unif}}(G)$ and let U be a compact symmetric unit-neighborhood of G such that $f_U \in L^1(G)$. If f does not vanish at infinity, then we can find $\varepsilon > 0$ such that for all compact set K of G there exists $x \in G\backslash K$ with $|f(x)| \geqslant \varepsilon$. Let $x_1 \in G$ such that $|f(x_1)| \geqslant \varepsilon$, and let $x_2 \notin x_1U^2$ such that $|f(x_2)| \geqslant \varepsilon$. Since U is symmetric, $x_1U \cap x_2U = \varnothing$. Next, pick $x_3 \notin x_1U \cup x_2U$ with $|f(x_3)| \geqslant \varepsilon$. Continuing in this way, we construct a sequence $(x_n)_n$ in G such that $x_nU \cap x_mU = \varnothing$

whenever $n \neq m$, and $|f(x_n)| \geqslant \varepsilon$ for every n. But then $f_U \geqslant \varepsilon$ on $x_n U$, which contradicts the integrability of f_U. Note that in this paragraph the unimodularity of G is not used.

Since $L^1(G) \cap C_0(G) \subseteq L^2(G)$, we have $C_{\text{unif}}(G) \subseteq L^2(G)$. Let $f, g \in C_{\text{unif}}(G)$. We can write $f * g(x) = \langle f, L_x g^* \rangle$. Since $x \mapsto L_x g^*$ is continuous by Lemma 2.6.7 and the inner product is continuous, the convolution $f * g(x)$ is also continuous in x. Finally, take a compact unit-neighborhood U such that $f_U,\, g_U \in L^1(G)$. Then

$$\begin{aligned}
(f * g)_U(y) &= \sup_{x,z \in U} \left| \int_G f(\xi) g(\xi^{-1} x y z) d\xi \right| \\
&= \sup_{x,z \in U} \left| \int_G f(x\xi) g(\xi^{-1} y z) d\xi \right| \\
&\leqslant \sup_{x,z \in U} \int_G |f(x\xi) g(\xi^{-1} y z)| d\xi \\
&\leqslant \int_G f_U(\xi) g_U(\xi^{-1} y) d\xi = f_U * g_U(y)
\end{aligned}$$

This implies that $(f * g)_U \in L^1(G)$, so $f * g \in C_{\text{unif}}(G)$. □

Note that we have a sequence of inclusions

$$C_c(G) \subseteq C_{\text{unif}}(G) \subseteq L^1(G) \cap C_0(G) \subseteq L^2(G).$$

Lemma 17.2.4. For $f \in C_{\text{unif}}(G)$ and $\phi \in L^2(\Gamma \backslash G)$ one has

$$R(f)\phi(x) = \int_{\Gamma \backslash G} k(x, y) \phi(y) dy$$

where $k(x, y) := \sum_{\gamma \in \Gamma} f(x^{-1} \gamma y)$. The kernel is continuous on $\Gamma \backslash G \times \Gamma \backslash G$.

Proof. Let $f \in L^1(G)$. For $\phi \in L^2(\Gamma \backslash G)$, by quotient integral formula one computes

$$\begin{aligned}
R(f)\phi(x) &= \int_G f(y) R(y) \phi(x) dy = \int_G f(y) \phi(xy) dy \\
&= \int_G f(x^{-1} y) \phi(y) dy \\
&= \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} f(x^{-1} \gamma y) \phi(\gamma y) dy = \int_{\Gamma \backslash G} k(x, y) \phi(y) dy
\end{aligned}$$

The obtained formula is valid almost everywhere in $x \in G$. In particular, for $f \in C_{\text{unif}}(G)$ this argument works with f replaced by f_U for a suitable compact symmetric unit-neighborhood U to get a kernel k_U. Now choose $g \in C_c(G)$ with $g \geqslant 0$ and ${}^{\Gamma} g(x) := \sum_{\gamma \in \Gamma} g(\gamma x) = 1$ for all $x \in G$ (c.f. Lemma 2.4.2) and use quotient integral formula to get

$$\begin{aligned}
\int_{\Gamma \backslash G \times \Gamma \backslash G} k_U(x, y) dx dy &= \int_{\Gamma \backslash G} \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} g(\gamma x) \sum_{\tau \in \Gamma} f_U(x^{-1} \gamma^{-1} \tau y) dx dy \\
&= \int_{\Gamma \backslash G} \int_G g(x) \sum_{\tau \in \Gamma} f_U(x^{-1} \tau y) dx dy \\
&= \int_G \int_G g(x) f_U(x^{-1} y) dx dy = \|g * f_U\|_1 < \infty
\end{aligned}$$

Thus $\sum_{\gamma \in \Gamma} f(x^{-1} \gamma y)$ converges almost everywhere in (x, y), so it converges on a dense set of (x, y). Let (x_0, y_0) be such a point of convergence. We are going to show $k(x, y)$ is continuous on the subset

$x_0U \times y_0U$. Let $\varepsilon > 0$ be given and choose a finite subset $S \subseteq \Gamma$ so that $\sum_{\gamma \notin S} f_U(x_0^{-1}\gamma y_0) < \frac{\varepsilon}{2}$. This means for $(x, y) \in x_0U \times y_0U$, one has $\sum_{\gamma \notin S} |f(x^{-1}\gamma y)| < \frac{\varepsilon}{2}$, which implies that for $(x, y), (x', y') \in x_0U \times y_0U$

$$|k(x, y) - k(x', y')| \leqslant \sum_{\gamma \in S} |f(x^{-1}\gamma y) - f(x'^{-1}\gamma y')| + \varepsilon$$

Letting $(x', y') \to (x, y)$, the continuity of k follows from that of f. $\square$

As $k \in C(\Gamma\backslash G \times \Gamma\backslash G) \subseteq L^2(\Gamma\backslash G \times \Gamma\backslash G)$, the operator $R(f)$ is Hilbert-Schmidt by Proposition 12.3.4, whence compact by Proposition 12.3.3. The theorem follows from the next lemma.

Lemma 17.2.5. Let A be a *-closed subspace of $C_c(G)$. Let (η, V_η) be a unitary representation of G such that for every $f \in A$, the operator $\eta(f)$ is compact and such that for every nonzero $v \in V_\eta$, the space $\eta(A)v$ is nonzero. Then η is a direct sum of irreducible representations with finite multiplicities.[1]

Notice here $C_c(G)$ is viewed as a *-subalgebra of $L^1(G)$.

Proof. By a Zorn's lemma argument we can find a subspace $E \leqslant V$ maximal with the property that it decomposes as a sum of irreducibles. The assumption of the lemma also holds for the orthogonal complement $E^\perp$ of E in $V = V_\eta$. By maximality of E this orthogonal complement cannot contain any irreducible subspace. We must show $E^\perp = 0$. Equivalently, we show that a representation η as in the assumption always contains an irreducible subspace.

The space A is generated by the self-adjoint elements. Let $f \in A$ be self-adjoint; then the operator $\eta(f)$ is self-adjoint as well. By assumption, $\eta(f)$ is nonzero and compact. Therefore by the Spectral Theorem $\eta(f)$ has a nonzero eigenvalue, say μ. Let V_μ be the corresponding eigenspace. Consider the collection

$$\{E \cap V_\mu \mid E \leqslant V \text{ is closed and invariant, } E \cap V_\mu \neq 0\}$$

This is nonempty. Indeed, if we choose $0 \neq v \in V_\mu$, then $\overline{\eta(L^1(G))v}$ is closed and invariant with $0 \neq \mu v \in \overline{\eta(L^1(G))v} \cap V_\mu$. Among this collection we pick W with minimal dimension. Let

$$E^1 := \bigcap \{E \leqslant V \mid E \text{ is closed and invariant, } E \cap V_\mu = W\} \supseteq W$$

Then $E^1 \leqslant V$ is closed and invariant. To conclude it suffices to show E^1 is irreducible. Let $F \leqslant E$ be a closed and invariant subspace. Note the invariance implies $\eta(f)E \subseteq E$ and $\eta(f)F \subseteq F$. Replacing F by its orthogonal complement in E, we may assume $F \cap V_\mu \neq 0$. But $0 \neq F \cap V_\mu \subseteq E \cap V_\mu = W$, the minimality forces $F \cap V_\mu = W$, and thus $E^1 \subseteq F$, or $F = E^1$. This shows the irreducibility.

It remains to show the multiplicities are finite. For this note first that if τ and σ are unitarily equivalent representations, then λ is an eigenvalue of $\tau(f)$ if and only if it is one for $\sigma(f)$. Thus any collection of orthogonal subspaces of V that are mutually unitarily equivalent must all have nontrivial intersection with the same eigenspaces of some $f \in A$. But each eigenspace is finite dimensional, the finiteness of multiplicity follows. $\square$

We include a lemma similar to the above one.

[1] If G is compact, we can take $\Gamma = \{1\}$. Then Lemma 17.2.4 implies that $R(f) \in \mathcal{B}(L^2(G))$ is compact for each $f \in C(G)$. Then this lemma implies $L^2(G)$ is a direct sum of irreducible representations with finite multiplicities. Hence this lemma can be viewed as a generalization of Peter-Weyl theorem.

Lemma 17.2.6. Let G be an LCH group and (π, V) a unitary representation of G. Suppose that for each unit-neighborhood U of G there exists an integrable function f on G such that

(i) f is nonnegative and symmetric with support in U,

(ii) $\int_G f(g)dg = 1$, and

(iii) $\pi(f) \in \mathcal{B}(V)$ is compact.

Then π is a direct sum of irreducibles with finite multiplicities.

Proof. Similar to the proof of Lemma 17.2.5, it suffices to show such a representation always admits an irreducible subspace. Let $v \in V$ be of norm one. By continuity, we may choose a unit-neighborhood U so small that $\|v - \pi(g)v\| < \frac{1}{2}$ for all $g \in U$. Pick f as in the statement of the lemma; then

$$\|\pi(f)v - v\| \leqslant \int_U f(g)\,\|\pi(g)v - v\|\,dg < \frac{1}{2}\int_U f(g)dg = \frac{1}{2}$$

Note that $f^*(x) := \Delta_G(x^{-1})\overline{f(x^{-1})}$ also satisfies those conditions in the theorem, with the same estimate $\|\pi(f^*)v - v\| \leqslant \frac{1}{2}$, so

$$\|\pi(f + f^*)v - 2v\| \leqslant \|\pi(f)v - v\| + \|\pi(f^*)v - v\| \leqslant 1$$

In particular, if we put $h = f + f^*$, then $\pi(h)v \neq 0$. In sum, $\pi(h)$ is a nonzero self-adjoint compact operator. The rest of the proof is the same as that of the previous lemma. $\square$

17.3 The Trace Formula

Definition. Let X be an LCH space and μ a Radon measure on X. A continuous L^2-kernel k on X is called **admissible** if there exists a function $g \in C(X) \cap L^2(X)$ such that $|k(x,y)| \leqslant g(x)g(y)$ for all $x, y \in X$.

- If X is compact, then every continuous kernel is admissible.

An operator $S : L^2(X) \to L^2(X)$ is called an **admissible operator** if there exists an admissible kernel k such that

$$S\phi(x) = \int_X k(x,y)\phi(y)d\mu(y)$$

for all $\phi \in L^2(X)$, $x \in X$.

- Note that an integral operator on $L^2(X)$ with L^2-kernel is necessarily a bounded operator.

Proposition 17.3.1. Let X be an LCH space equipped with a Radon measure dx. Assume X is either first countable or compact. Let T be an integral operator with continuous L^2-kernel k on X. Assume that there exists admissible operators S_1, S_2 with $T = S_1S_2$. Then T is of trace class and[2]

$$\mathrm{tr}(T) = \int_X k(x,x)dx.$$

[2]The formula holds as long as T is of trace class when the diagonal $k(x,x)$ is defined in a correct way. See [Bri91] and also mathoverflow post

Proof. Let us replace S_1 by S_1^*, Since S_1, S_2 are Hilbert-Schmidt, $T = S_1^* S_2$ is trace class by Lemma 12.3.6 and

$$\mathrm{tr}(T) = \sum_\alpha \langle Te_\alpha, e_\alpha\rangle = \sum_\alpha \langle S_2 e_\alpha, S_1 e_\alpha\rangle = (S_2, S_1)$$

where $(e_\alpha)_\alpha$ is an orthonormal basis for $L^2(X)$, and $(\,,)$ is the inner product on $\mathcal{HS}(L^2(X))$. Let k_i be the admissible kernel of S_i ($i \in [2]$). By the result in §12.3.2, the operator T has kernel

$$k_T(x,y) = k_{S_1^* S_2}(x,y) = \int_X \overline{k_1(y,z)} k_2(x,z) dz$$

so by Proposition 12.3.4

$$\int_X k_T(x,x)dx = \int_X \int_X \overline{k_1(x,z)} k_2(x,z) dx dz = (S_2, S_1) = \mathrm{tr}(T).$$

The proof won't be complete until we proof $k_T(x,x)$ is measurable. For this we claim $k_T(x,y)$ is continuous.

- If X is first countable, the continuity can be checked with sequences. Since $z \mapsto \overline{k_1}(z,y)k_2(x,z)$ is integrable over X for each (x,y) with a common integrable upper bound by virtue of admissibility, the continuity then follows from DCT.
- If X is compact, then k_i is uniformly continuous, in the sense that for all $x_0 \in X$ and $\varepsilon > 0$ there exists a neighborhood U of x_0 such that for all $(x,z) \in U \times X$ we have $|k_i(x,z) - k_i(x_0,z)| < \varepsilon$. Together with $\mathrm{vol}(X, dx) < \infty$, this implies the continuity.

□

Recall that for a uniform lattice Γ of G and $\pi \in \widehat{G}$, the number $N_\Gamma(\pi) \geqslant 0$ is the multiplicity of π as a subrepresentation of $(R, L^2(\Gamma\backslash G))$. Denote by $\widehat{G}_\Gamma$ the set of all $\pi \in \widehat{G}$ with positive multiplicity.

Write $C_{\mathrm{unif}}(G)^2 = C_{\mathrm{unif}}(G) * C_{\mathrm{unif}}(G)$ for the linear span of functions of the form $g * h$ with $g, h \in C_{\mathrm{unif}}(G)$.

If Γ is a lattice in G and $\gamma \in \Gamma$, denote by $[\gamma]$ the conjugacy class of γ in Γ, $G_\gamma = C_G(\gamma)$ the centralizer of γ in G, and $\Gamma_\gamma = G_\gamma \cap \Gamma$ the centralizer of γ in Γ. Note that the map $\Gamma \ni \nu \mapsto \nu^{-1}\gamma\nu \in [\gamma]$ induces a bijection $\Gamma_\gamma\backslash\Gamma \cong [\gamma]$.

Lemma 17.3.2 (Pretrace formula)**.** For $f \in C_{\mathrm{unif}}(G)$, we have

$$\sum_{\gamma\in\Gamma} f(x^{-1}\gamma y) = \sum_\pi \sum_{\phi\in B_\pi} R(f)\phi(x)\overline{\phi(y)}$$

where $\pi \in \widehat{G}_\Gamma$ and B_π is an orthonormal basis of the π-isotypic part $L^2(\Gamma\backslash G)$ of π in $L^2(\Gamma\backslash G)$.

Proof. By Lemma 17.2.4, LHS is the kernel of the operator $R(f)$ on the space $L^2(\Gamma\backslash G)$. On the other hand, if we rewrite the decomposition in Theorem 17.2.1 as

$$L^2(\Gamma\backslash G) = \widehat{\bigoplus_\pi} L^2(\Gamma\backslash G)[\pi]$$

then the kernel of $R(f)$ equals, in L^2 sense, the sum of the kernel what does this kernel mean? of $R(f)|_{L^2(\Gamma\backslash G)[\pi]}$ with π varying over $\pi \in \widehat{G}_\Gamma$. We contend that each of this kernel is

$$\sum_{\phi\in B_\pi} R(f)\phi(x)\overline{\phi(y)}$$

□

Theorem 17.3.3 (Trace Formula)**.** Let G be an LCH group and $\Gamma \leqslant G$ a uniform lattice. Then for every $\pi \in \widehat{G}_\Gamma$ and $f \in C_{\mathrm{unif}}(G)^2$, the operator $\pi(f)$ is of trace class and

$$\sum_{\pi \in \widehat{G}_\Gamma} N_\Gamma(\pi) \operatorname{tr} \pi(f) = \sum_{[\gamma]} \operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) \mathcal{O}_\gamma(f)$$

where the summation on the right runs over all conjugacy classes $[\gamma]$ in the group Γ, and $\mathcal{O}_\gamma(f)$ denotes the **orbital integral**

$$\mathcal{O}_\gamma(f) := \int_{G_\gamma \backslash G} f(x^{-1}\gamma x) dx$$

We shall see in the sequel that the centralizer G_γ is unimodular and that $\Gamma_\gamma \backslash G_\gamma$ is of finite measure for all $\gamma \in \Gamma$. The expression $\operatorname{vol}(\Gamma_\gamma \backslash G_\gamma)\mathcal{O}_\gamma(f)$ is therefore well-defined, in the sense that it is independent of the choice of Haar measure on G_γ.
The left hand side is called the **spectral side**, while the right hand side is called the **geometric side**.

Proof. The algebra $C_{\mathrm{unif}}(G)^2$ consists of all finite linear combinations of functions of the form $g * h^*$ with $g, h \in C_{\mathrm{unif}}(G)$, so it suffices to prove the trace formula for $f = g * h^*$. By Lemma 17.2.4, the operators $R(g)$ and $R(h)$ are integral operators with continuous kernels. Since $\Gamma \backslash G$ is compact, $R(g)$ and $R(h)$ are admissible operators. By Proposition 17.3.1, the operator $R(f) = R(g)R(h)^*$ is of trace class with trace $\operatorname{tr} R(f) = \int_{\Gamma \backslash G} k_f(x,x) dx$. By Theorem 17.2.1 and the definition of trace, all restriction of $R(f)$ to subrepresentations are of trace class and

$$\sum_{\pi \in \widehat{G}_\Gamma} N_\Gamma(\pi) \operatorname{tr} \pi(f) = \operatorname{tr} R(f) = \int_{\Gamma \backslash G} k_f(x,x) dx = \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} f(x^{-1}\gamma x) dx$$

We order the sum in accordance with the conjugacy classes $[\gamma]$ in Γ, interchange integration and summation (valid as $\Gamma \backslash G$ is compact), and quotient integral formula to obtain

$$\begin{aligned}
\operatorname{tr} R(f) &= \int_{\Gamma \backslash G} \sum_{[\gamma]} \sum_{\sigma \in \Gamma_\gamma \backslash \Gamma} f(x^{-1}\sigma^{-1}\gamma\sigma x) dx \\
&= \sum_{[\gamma]} \int_{\Gamma \backslash G} \sum_{\sigma \in \Gamma_\gamma \backslash \Gamma} f(x^{-1}\sigma^{-1}\gamma\sigma x) dx = \sum_{[\gamma]} \int_{\Gamma_\gamma \backslash G} f(x^{-1}\gamma x) dx
\end{aligned}$$

Lemma 17.3.4. For every $\gamma \in \Gamma$, the centralizer G_γ is unimodular and $\Gamma_\gamma \backslash G_\gamma$ has finite invariant measure.

Proof. The above computation shows that for $f \in C_{\mathrm{unif}}(G)^2$ with $f \geqslant 0$, one has $\int_{\Gamma_\gamma \backslash G} f(x^{-1}\gamma x) dx < \infty$ for all $\gamma \in \Gamma$. Since G is unimodular by Lemma 17.1.5, the space $\Gamma_\gamma \backslash G$ carries an invariant Radon measure ν. Consider the projection $p : \Gamma_\gamma \backslash G \to G_\gamma \backslash G$ and let $\mu = p_* \nu$ be the pushforward measure of ν via p; in other word, for $f \in C_c(G_\gamma \backslash G)$, we have

$$\int_{G_\gamma \backslash G} f d\mu = \int_{\Gamma_\gamma \backslash G} f \circ p \, d\nu$$

We show μ is a Radon measure, i.e., the integral above is finite. For this let $0 \leqslant f \in C_c(G_\gamma \backslash G)$ and let $\Phi : G_\gamma \backslash G \to G$ be given by $\Phi(G_\gamma x) = x^{-1}\gamma x$. Then the subset $\Phi(\operatorname{supp} f) \subseteq G$ is compact, so by Tietze's extension theorem we have find $0 \leqslant \tilde{f} \in C_c(G)$ such that $\tilde{f}(y^{-1}\gamma y) = f(G_\gamma y)$ for every $y \in G$ with $f(G_\gamma y) > 0$. Choose $0 \leqslant F \in C_c(G)^2$ such that $\tilde{f} \leqslant F$.

- To show the existence of F, let $g \in C_c^+(G)$ be such that $g > 0$ in a neighborhood of $\operatorname{supp} \tilde{f}$. There exists a unit-neighborhood U such that $(\phi_U * g)|_{\operatorname{supp} \tilde{f}} > 0$, where ϕ_U is a Dirac function supported in U. Set $F := c\phi_U * g$ for $c \gg 0$ so that $F \geqslant \tilde{f}$.

Since $F \in C_c(G)^2$, as mentioned in the beginning of the proof of this lemma, we have

$$\int_{G_\gamma \backslash G} f d\mu = \int_{\Gamma_\gamma \backslash G} f \circ p \, d\nu = \int_{\Gamma_\gamma \backslash G} \tilde{f}(y^{-1}\gamma y) d\nu(y) \leqslant \int_{\Gamma_\gamma \backslash G} F(y^{-1}\gamma y) d\nu(y) < \infty$$

This show μ is a Radon measure. μ is G-invariant since so is ν, and by Theorem 2.4.7 it follows that G_γ is unimodular. Finally, for $f \in C_c(G_\gamma \backslash G)$, we have

$$\begin{aligned} \infty > \int_{\Gamma_\gamma \backslash G} f(p(y)) d\nu(y) &= \int_{G_\gamma \backslash G} \int_{\Gamma_\gamma \backslash G_\gamma} f(p(x\sigma)) d\sigma d\mu(x) \\ &= \int_{G_\gamma \backslash G} \int_{\Gamma_\gamma \backslash G_\gamma} f(x) d\sigma d\mu(x) = \operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) \int_{\Gamma_\gamma \backslash G} f(x) d\mu(x) \end{aligned}$$

so that $\operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) < \infty$. □

To conclude the proof, we continue our computation with the help of the lemma.

$$\begin{aligned} \operatorname{tr} R(f) &= \sum_{[\gamma]} \int_{\Gamma_\gamma \backslash G} f(x^{-1}\gamma x) dx \\ &= \sum_{[\gamma]} \int_{G_\gamma \backslash G} \int_{\Gamma_\gamma \backslash G_\gamma} f((\sigma x)^{-1}\gamma(\sigma x)) d\sigma dx = \sum_{[\gamma]} \operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) \mathcal{O}_\gamma(f) \end{aligned}$$

□

Example 17.3.5. Consider the case $G = \mathbb{R}$ and $\Gamma = \mathbb{Z}$. By Plancherel theorem we have the decomposition

$$L^2(\mathbb{Z} \backslash \mathbb{R}) \cong \widehat{\bigoplus_{k \in \mathbb{Z}}} \mathbb{C}[t \mapsto e^{2\pi i t k}]$$

so if we identify $\widehat{G}$ with $\mathbb{R}$ via $x \mapsto [t \mapsto e^{2\pi i x}]$, we see $\widehat{G}_\Gamma$ is mapped onto $\mathbb{Z}$, and the multiplicities are all equal to one. Thus the spectral side equals $\sum_{k \in \mathbb{Z}} \widehat{f}(k)$, and the geometric side is $\sum_{k \in \mathbb{Z}} f(k)$. Thus in this case, the trace formula is the Poisson summation formula in disguise.

Lemma 17.3.6. Let G be an LCH group and A a *-subalgebra of $C_c(G)$ stable under left translation L_y ($y \in G$) which contains a Dirac net $(\phi_U)_U$. Let (π, V) and (σ, W) be two unitary representations of G such that for each $f \in A$ the operators $\pi(f)$ and $\sigma(f)$ are of trace class. If

$$\operatorname{tr} \pi(f)^* \pi(f) \geqslant \operatorname{tr} \sigma(f)^* \sigma(f)$$

for all $f \in A$, then σ is a subrepresentation of π. If the equality holds for all $f \in A$, then σ is isomorphic to π.

Proof. By Lemma 14.3.3 the orbit $\pi(A)v$ is nonzero for each nonzero $v \in V$. Thus by Lemma 17.2.5 both π and σ decomposes into irreducibles with finite multiplicity. By a Zorn's lemma argument there exists a maximal subrepresentation of σ with the property of being isomorphic to a subrepresentation of π. Restricting to their orthogonal complements, respectively, we assume π and σ have no isomorphic subrepresentations. To show the first assertion, we must show $\sigma = 0$. If this is shown, the second assertion follows once we reverse the roles of π and σ.

Let $V = \bigoplus_{\alpha \in I} V_\alpha$ be a decomposition into pairwise orthogonal subrepresentations and let $(v_{\alpha,\mu})_\mu \in V_\alpha$ be such that

$$\sum_\alpha \sum_\mu \|\pi(f)v_{\alpha,\mu}\|^2 < \infty$$

for every $f \in A$. This is possible for each $\pi(f)$ is of trace class, and thus Hilbert Schmidt. We can simply take $(v_{\alpha,\mu})_{\alpha,\mu}$ to be an orthonormal basis. Choose any nonzero vector $w \in W$. We claim that for every $\varepsilon > 0$ we can find $f \in A$ such that

$$\sum_\alpha \sum_\mu \|\pi(f)v_{\alpha,\mu}\|^2 < \varepsilon \|\sigma(f)w\|^2$$

Suppose otherwise. For each μ let $V_{\alpha,\mu}$ be a copy of V_α, and define the space

$$L := \overline{\left\{ \sum_\alpha \sum_\mu \pi(f)v_{\alpha,\mu} \mid f \in A \right\}} \subseteq V' := \bigoplus_\alpha \bigoplus_\mu V_{\alpha,\mu}$$

Then for every $f \in A$ the map

$$\sum_\alpha \sum_\mu \pi(f)v_{\alpha,\mu} \mapsto \sigma(f)w$$

would define a bounded unitary G-equivariant operator T from L to W. It is nontrivial as said in the very beginning of the proof. Extend T to the whole V' by setting $T|_{L^\perp} = 0$. Consider the restriction of T to $\bigoplus_\mu V_{\alpha,\mu} \subseteq V'$. Since $V_{\alpha,\mu}$ does not occur in W, this restriction must be trivial. Hence $T = 0$, a contradiction.

Now assume $\sigma \neq 0$. As said in the very beginning of the proof, there exists $h \in A$ with $\sigma(h) \neq 0$. If we put $f = h * h^*$, then $\sigma(f)$ is of trace class and positive. Therefore $\sigma(f)$ possesses a largest eigenvalue and we can scale h in such a way that this eigenvalue is 1. Let $w \in W$ be such that $\|w\| = 1$ and $\sigma(f)w = w$. Let $\lambda > 0$ be the largest eigenvalue of $\pi(f)$. For every α let $(v_{\alpha,\mu})_\mu$ be an orthonormal basis of V_α consisting of eigenvectors of $\pi(f)$, and write $\pi(f)v_{\alpha,\mu} = \lambda_{\alpha,\mu}v_{\alpha,\mu}$. Then for every $g \in A$,

$$\sum_\alpha \sum_\mu \|\pi(g)v_{\alpha,\mu}\|^2 = \|\pi(g)\|_{\mathrm{HS}}^2 < \infty$$

and by the second paragraph we can find $g \in A$ with

$$\sum_\alpha \sum_\mu \|\pi(g)v_{\alpha,\mu}\|^2 < \frac{1}{\lambda^2} \|\sigma(g)w\|^2$$

Then

$$\begin{aligned} \operatorname{tr} \pi(g * f)^* \pi(g * f) &= \sum_{\alpha,\mu} \|\pi(g * f)v_{\alpha,\mu}\|^2 = \sum_{\alpha,\mu} \lambda_{\alpha,\mu}^2 \|\pi(g)v_{\alpha,\mu}\|^2 \leqslant \lambda^2 \sum_{\alpha,\mu} \|\pi(g)v_{\alpha,\mu}\|^2 < \|\sigma(g)w\|^2 \\ &= \|\sigma(g * f)w\|^2 = \operatorname{tr} \sigma(g * f)^* \sigma(g * f) \leqslant \operatorname{tr} \pi(g * f)^* \pi(g * f) \end{aligned}$$

a contradiction. This implies $\sigma = 0$. □

17.4 Locally Constant Functions

Definition. Let X be a topological space.

1. A function $f : X \to \mathbb{C}$ is called **locally constant** if each point $x \in X$ admits an neighborhood U to which the restriction of f is constant.

Suppose, in addition, $X = G$ is a topological group.

2. A function $f : G \to \mathbb{C}$ is called **uniformly locally constant** if there exists a unit-neighborhood U such that f is constant on every set of the form UxU, $x \in G$.

- If f is locally constant with compact support, then f is uniformly locally constant.

Proposition 17.4.1. Let G be a totally disconnected LCH group and f a uniformly locally constant and integrable function on G. Then $f \in C_{\text{unif}}(G)^2$. In particular, the trace formula is valid for f.

Proof. Let U denote the open compact symmetric unit neighborhood of G so that f is constant on UxU for every $x \in G$. Then $f_U = |f|$, and therefore $f \in C_{\text{unif}}(G)$. The same holds for the function $e_U := \dfrac{1}{\text{vol}(U)}\mathbf{1}_U$. We have

$$e_U * f(g) = \frac{1}{\text{vol}(U)} \int_G \mathbf{1}_U(h) f(h^{-1}g)dh = \frac{1}{\text{vol}(U)} \int_U f(g)dh = f(g)$$

for every $g \in G$. Thus $f = f * e_U \in C_{\text{unif}}(G)^2$. ☐

17.5 Lie Groups

Theorem 17.5.1. Let G be a Lie group of dimension n and let $\Gamma \leqslant G$ be a cocompact lattice. Let $f \in C(G) \cap L^1(G)$ such that the kernel

$$k(x, y) := \sum_{\gamma \in \Gamma} f(x^{-1}\gamma y)$$

converges uniformly and is $2r$-times continuously differentiable in the first argument, where $r = \left\lceil \frac{n}{2} \right\rceil$. Then the trace formula is valid for f.

In particular, the trace formula holds for every $f \in C_c^{2r}(G)$.

To show this theorem we need a partition of unity with a smooth square root. The following lemma is proved in Appendix F.5.

Lemma 17.5.2. Let M be a smooth manifold and let $(U_i)_{i \in I}$ be an open covering of M. There there are smooth functions $u_i : M \to [0, 1]$, such that $\operatorname{supp} u_i \subseteq U_i$ and that

$$\sum_{i \in I} u_i \equiv 1$$

where the sum is locally finite. Moreover, one can choose the u_i in a way that for each $i \in I$ the function $\sqrt{u_i}$ is smooth as well.

A Borel measure ν on $\mathbb{R}^n$ is called a **smooth measure** if the Radon-Nikodym derivative of ν with respect to the Lebesgue measure dx is smooth and non-negative, i.e., there exists a smooth function $h : \mathbb{R}^n \to [0, \infty)$ such that

$$\nu(A) = \int_A h(x)dx$$

for all Borel A. A measure μ on a smooth manifold M is called a **smooth measure** if for every chart $\phi : U \to \mathbb{R}^n$, the pushforward measure $\phi_* \mu$ on $\mathbb{R}^n$ is smooth.

Proposition 17.5.3. Let M be a compact smooth manifold of dimension n with a smooth measure μ. Let $k : M \times M \to \mathbb{C}$ be continuous and $2r$-times continuously differentiable in the first argument, where $r = \left\lceil \frac{n}{2} \right\rceil$. Then the induced integral operator $T_k : L^2(M) \to L^2(M)$

$$T_k(\phi)(x) := \int_M k(x, x)\phi(y) d\mu(y)$$

is of trace class and

$$\operatorname{tr} T_k = \int_M k(x, x) dx.$$

Proof. We first prove the assertion in the case $M = \mathbb{R}^n/\mathbb{Z}^n$ and μ being the Lebesgue measure (the Haar measure with total volume 1). In this case we define

$$l(x, y) = \sum_{k \in \mathbb{Z}^n} \left(\frac{1}{1 + 4\pi^2 \|k\|^2} \right)^r e_k(x)\overline{e_k(y)}$$

where $e_k(x) = e^{2\pi i \langle x,k \rangle}$. By integral test we see this sum converges absolutely and uniformly, and the kernel $l(x, y)$ is therefore continuous. Let Δ be the Laplacian

$$\Delta = -\frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2}.$$

Lemma 17.5.4. For $\phi \in C^{2r}(\mathbb{R}^n/\mathbb{Z}^n)$ one has

$$T_l(1 + \Delta)^r \phi = \phi.$$

Proof. Both sides being continuous functions, it suffices to show they are equal in L^2-sense. Since $\{e_k \mid k \in \mathbb{Z}^n\}$ forms an orthonormal basis for $L^2(\mathbb{R}^n/\mathbb{Z}^n)$, it is enough to show $T_l(1 + \Delta)^r e_k = e_k$ for each $k \in \mathbb{Z}^n$. This follows from the computations

$$(1 + \Delta)^r e_k = (1 + 4\pi^2 \|k\|^2)^r e_k$$

and

$$T_l e_k(x) = \sum_{\ell \in \mathbb{Z}^n} \left(\frac{1}{1 + 4\pi^2 \|\ell\|^2} \right)^r e_\ell(x) \int_M \overline{e_\ell(y)} e_k(y) dy = \left(\frac{1}{1 + 4\pi^2 \|k\|^2} \right)^r e_k(x)$$

□

Now let k be as in the theorem. Then for $\phi \in L^2(M)$, the function $T_k\phi$ is $C^{2r}(M)$ (for M is compact we can differentiate under the integral sign), so by the lemma

$$T_l(1 + \Delta)^r T_k \phi = T_k \phi.$$

By the same reason one has $(1 + \Delta)^r T_k = T_{k'}$, where

$$k'(x, y) := (1 + \Delta_x)^r k(x, y)$$

so $T_k = T_l T_{k'}$ is a product of two Hilbert Schmidt operator, hence of trace class. As both T_l and $T_{k'}$ are admissible (as M is compact), the theorem follows from Proposition 17.3.1.

Next let M be an arbitrary smooth compact manifold of dimension n. Let $((U_i, \psi_i))_{i=1}^s$ be an open cover of M consisting of charts $\psi_i : U_i \to \mathbb{R}^n/\mathbb{Z}^n$; so for each i, the map ψ_i is a homeomorphism of U_i into some open set $V_i \subseteq \mathbb{R}^n/\mathbb{Z}^n$. We choose $V_1, \ldots, V_s$ in a way that they are pairwise disjoint.

Let $(u_i)_i$ be a smooth partition of unity with smooth square root subordinate to the cover $(U_i)_i$. For $1 \leqslant i, j \leqslant s$ let

$$k_{i,j}(x,y) = \sqrt{u_i(x)}k(x,y)\sqrt{u_j(y)} \in C(M \times M)$$

and define a continuous kernel on $\mathbb{R}^n/\mathbb{Z}^n \times \mathbb{R}^n/\mathbb{Z}^n$

$$\widetilde{k}_{i,j}(x,y) := \begin{cases} \sqrt{d_i(x)}k_{i,j}(\psi_i^{-1}(x), \psi_j^{-1}(y)\sqrt{d_j(y)} & \text{, if } (x,y) \in V_i \times V_j \\ 0 & \text{, otherwise} \end{cases}$$

Here d_j denotes the Radon-Nikodym derivative of $(\psi_j)_*\mu$ with respect to the Lebesgue measure. We define $\widetilde{k} := \sum_{i,j} \widetilde{k}_{i,j}$ and for $\phi \in L^2(M)$, we set

$$\phi_j(x) := \phi(x)\sqrt{u_j(x)}.$$

Define $\widetilde{\phi}_j \in L^2(\mathbb{R}^n/\mathbb{Z}^n)$ by

$$\widetilde{\phi}_j(x) := \begin{cases} \phi_j(\psi_j^{-1}(x))\sqrt{d_j(x)} & \text{, if } x \in V_j \\ 0 & \text{, otherwise} \end{cases}$$

Finally set $\widetilde{\phi} = \sum_j \widetilde{\phi}$.

Lemma 17.5.5. The map

$$\Psi : L^2(M) \longrightarrow L^2(\mathbb{R}^n/\mathbb{Z}^n)$$

$$\phi \longmapsto \widetilde{\phi}$$

is a linear isometry, and

$$\Psi(T_k\phi) = T_{\widetilde{k}}\Psi(\phi)$$

for every $\phi \in L^2(M)$. The operator $T_{\widetilde{k}}$ equals $PT_{\widetilde{k}}P$, where P is the orthogonal projection $L^2(\mathbb{R}^n/\mathbb{Z}^n) \to \operatorname{Im}\Psi$. Finally, we have

$$\int_M k(x,x)d\mu(x) = \int_{\mathbb{R}^n/\mathbb{Z}^n} \widetilde{k}(x,x)dx$$

Proof. The map Ψ is linear. For $\phi \in L^2(M)$, we compute

$$\begin{aligned}\|\Psi(\phi)\|^2 &= \int_{\mathbb{R}^n/\mathbb{Z}^n} |\widetilde{\phi}(x)|^2 dx = \sum_j \int_{V_j} |\widetilde{\phi_j}(x)|^2 dx = \sum_j \int_{V_j} |\phi_j(\psi_j^{-1}(x))|^2 d_j(x)dx \\ &= \sum_j \int_{U_j} |\phi_j(x)|^2 d\mu(x) = \sum_j \int_M |\phi(x)|^2 u_j(x)d\mu(x) = \int_M |\phi(x)|^2 d\mu(x) = \|\phi\|^2\end{aligned}$$

so Ψ is an isometry. For the second, compute

$$\begin{aligned}T_{\widetilde{k}}\Psi(\phi)(x) &= T_{\widetilde{k}}\widetilde{\phi}(x) = \sum_j T_{\widetilde{k}}\widetilde{\phi}_j(x) = \sum_j \int_{V_j} \widetilde{k}(x,y)\widetilde{\phi_j}(y)dy = \sum_{i,j} \int_{V_j} \widetilde{k}_{i,j}(x,y)\widetilde{\phi}_j(y)dy \\ &= \sum_i \sqrt{d_i(x)}\sqrt{u_i(\psi_i^{-1}(x))} \int_M k(\psi_i^{-1}(x), y)\phi(y)d\mu(y) = \Psi T_{\widetilde{k}}(\phi)(x)\end{aligned}$$

To show $T_{\widetilde{k}} = PT_{\widetilde{k}}P$ one has to show for all $g \in (\operatorname{Im}\Psi)^\perp$,

$$T_{\widetilde{k}}(g) = 0 \text{ and } \langle T_{\widetilde{k}}h, g\rangle = 0$$

for all $h \in L^2(\mathbb{R}^n/\mathbb{Z}^n)$. These follow from the facts that $\operatorname{Im}(y \mapsto \widetilde{k}(x,y)) \subseteq \operatorname{Im}\Psi$ and $\operatorname{Im}(x \mapsto \widetilde{k}(x,y)) \subseteq \operatorname{Im}\Psi$ if the other argument is fixed. Finally,

$$\begin{aligned}\int_{\mathbb{R}^n/\mathbb{Z}^n} \widetilde{k}(x,x)dx &= \sum_i \int_{V_i} k_{i,i}(\psi_i^{-1}(x), \psi_i^{-1}(x))d_i(x)dx = \sum_i \int_{U_i} k_{i,i}(x,x)d\mu(x) \\ &= \sum_i \int_M k(x,x)u_i(x)d\mu(x) = \int_M k(x,x)d\mu(x)\end{aligned}$$

□

We use this lemma to show T_k is compact, and next of trace class with trace $\operatorname{tr} T_k = \operatorname{tr} T_{\widetilde{k}}$. If these are shown, then the last assertion of the lemma implies

$$\operatorname{tr} T_k = \operatorname{tr} T_{\widetilde{k}} = \int_{\mathbb{R}^n/\mathbb{Z}^n} \widetilde{k}(x,x)dx = \int_M k(x,x)d\mu(x)$$

and the proposition follows.

For the compactness, we use Proposition 12.2.1.(c). Let (e_j) be an orthonormal sequence in $L^2(M)$. Then

$$\|T_k e_j\| = \|\Psi T_k e_j\| = \left\|T_{\widetilde{k}} \Psi e_j\right\| \to 0$$

as $T_{\widetilde{k}}$ is compact and $(\Psi e_j)_j$ is orthonormal. To show its trace norm is finite, we use Lemma 12.3.8. Let (e_i) and (h_j) be two orthonormal bases of $L^2(M)$. Then

$$\sum_i |\langle T_k e_i, h_i\rangle| = \sum_i |\langle \Psi T_k e_i, \Psi h_i\rangle| = \sum_i |\langle T_{\widetilde{k}} \Psi e_i, \Psi h_i\rangle| \leqslant \left\|T_{\widetilde{k}}\right\|_{\mathrm{tr}}$$

so that $\|T_k\|_{\mathrm{tr}}$ is finite. Finally, let e_i be any orthonormal basis of $L^2(M)$. Then

$$\operatorname{tr} T_k = \sum_i \langle T_k e_i, e_i\rangle = \sum_i \langle T_{\widetilde{k}} \Psi e_i, \Psi e_i\rangle$$

On the other hand, $(\Psi e_i)_i$ forms an orthonormal basis of $\operatorname{Im}\Psi$. Let $(e'_j)_j$ be any orthonormal basis prolonging $(\Psi e_i)_i$; then

$$\operatorname{tr} T_{\widetilde{k}} = \sum_j \langle T_{\widetilde{k}} e'_j, e'_j\rangle = \sum_j \langle T_{\widetilde{k}} P e'_j, P e'_j\rangle = \sum_i \langle T_{\widetilde{k}} \Psi e_i, \Psi e_i\rangle = \operatorname{tr} T_k$$

This concludes the proof of the proposition. □

Now we deduce Theorem 17.5.1 from Proposition 17.5.3. The proof is the same as that of trace formula. Let ρ be the right translation by G on $L^2(\Gamma\backslash G)$. Let f be as in the theorem. Then $\rho(f)$ is of trace class, and

$$\operatorname{tr} \rho(f) = \sum_{\pi \in \widehat{G}} N_\Gamma(\pi) \operatorname{tr} \pi(f)$$

where $N_\Gamma(\pi)$ is the multiplicity of π occurring in $L^2(\Gamma\backslash G)$. On the other hand, we have

$$\begin{aligned}\operatorname{tr} \rho(f) &= \int_{\Gamma\backslash G} k(x,x)dx = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma x)dx = \int_{\Gamma\backslash M} \sum_{[\gamma]} \sum_{\sigma \in \Gamma_\gamma\backslash\Gamma} f(x^{-1}\sigma^{-1}\gamma\sigma x)dx \\ &= \sum_{[\gamma]} \int_{\Gamma_\gamma\backslash G} f(x^{-1}\gamma x)dx = \sum_{[\gamma]} \int_{G_\gamma\backslash G} \int_{\Gamma_\gamma\backslash G_\gamma} f((\sigma x)^{-1}\gamma(\sigma x))d\sigma dx = \sum_{[\gamma]} \operatorname{vol}(\Gamma_\gamma\backslash G_\gamma)\mathcal{O}_\gamma(f)\end{aligned}$$

where $[\gamma]$ runs over all conjugacy classes of Γ.

Chapter 18

Heisenberg Groups

Chapter 19

$\mathbf{SL}_2(\mathbb{R})$

19.1 The special linear group

Theorem 19.1.1 (Iwasawa Decomposition)**.** Let $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a > 0 \right\}$, $N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}$ and $K = \mathrm{SO}(2)$. Then the map

$$\begin{aligned} \psi : A \times N \times K &\longrightarrow G \\ (a, n, k) &\longmapsto ank \end{aligned}$$

is a homeomorphism.

Proof. Let $g \in G$, and let $gi = x + yi$. If we put

$$a = \begin{pmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix}, \qquad n = \begin{pmatrix} 1 & xy^{-1} \\ & 1 \end{pmatrix},$$

then $gi = ani$, so that $g^{-1}an \in K$. This implies that $g = ank$ for some $k \in K$. Now define the inverse map

$$\begin{aligned} \phi : G &\longrightarrow A \times N \times K \\ g &\longrightarrow (\underline{a}(g), \underline{n}(g), \underline{k}(g)) \end{aligned}$$

where

$$\begin{aligned} \underline{a}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} \frac{1}{\sqrt{c^2 + d^2}} & \\ & \sqrt{c^2 + d^2} \end{pmatrix} \\ \underline{n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} 1 & ac + bd \\ & 1 \end{pmatrix} \\ \underline{k}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \frac{1}{\sqrt{c^2 + d^2}} \begin{pmatrix} d & -c \\ c & d \end{pmatrix}. \end{aligned}$$

It is clear that $\phi \circ \psi = \mathrm{id}$ and $\psi \circ \phi = \mathrm{id}$. $\square$

We shall keep the notation $\underline{a}(g)$, $\underline{n}(g)$ and $\underline{k}(g)$ used in the above proof. Also, for x, t, $\theta \in \mathbb{R}$, put

$$a_t = \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix} \in A \qquad n_x = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \in N \qquad k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in K$$

Say a function $f : G \to \mathbb{C}$ is a **smooth** if the map $\mathbb{R}^3 \to \mathbb{C}$ given by $(t, x, \theta) \mapsto f(a_t n_x k_\theta)$ is smooth. This is the same as saying $f : G \to \mathbb{C}$ is smooth when G is equipped with the usual manifold structure. We denote the space of smooth functions by $C^\infty(G)$, and $C_c^\infty(G) = C^\infty(G) \cap C_c(G)$.

Theorem 19.1.2. The group $G = \mathrm{SL}_2(\mathbb{R})$ is unimodular.

Proof. Let $\phi : G \to \mathbb{R}_{>0}$ be a continuous homomorphism. We claim $\phi \equiv 1$. Since K is compact, we have $\phi(K) = 1$. Since $\phi(a_{t+s}) = \phi(a_t a_s) = \phi(a_t)\phi(a_s)$ and ϕ is continuous, there exists $x \in \mathbb{R}$ such that $\phi(a_t) = e^{tx}$ for any $t \in \mathbb{R}$. Consider the element $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in G$; we have $w a_t w^{-1} = a_{-t}$. Since $e^{-tx} = \phi(a_{-t}) = \phi(w a_t w^{-1}) = \phi(a_t) = e^{tx}$, and it follows that $x = 0$, i.e., $\phi(A) = 0$. Similarly, there exists $y \in \mathbb{R}$ such that $\phi(n_t) = e^{ty}$ for any $t \in \mathbb{R}$. As $a_r n_t a_r^{-1} = n_{e^{2r}t}$, it follows that $e^{ty} = e^{e^{2r}ty}$, which implies $t = 0$, i.e., $\phi(N) = 1$.

A quick proof: any homomorphism ϕ factors through the abelianization $\mathrm{SL}_2(\mathbb{R})^{\mathrm{ab}}$, which is trivial. Hence ϕ is trivial. □

Theorem 19.1.3. For any given Haar measures on three of the four groups G, A, N K, there exists a unique Haar measure on the fourth such that for any $f \in L^1(G)$, the integration formula

$$\int_G f(x)dx = \int_A \int_N \int_K f(ank)dkdnda$$

holds. For $\phi \in L^1(K)$ and $x \in G$ one has

$$\int_K \phi(k)dk = \int_K \phi(\underline{k}(kx))e^{2\underline{t}(kx)}dk$$

where $\underline{t}(g)$ for the unique $t \in \mathbb{R}$ with $\underline{a}(g) = a_t$.

Put $A^+ := \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \geqslant 1 \right\}$.

Theorem 19.1.4 (Cartan decomposition)**.** The multiplication induces a surjection

$$K \times A^+ \times K \longrightarrow \mathrm{SL}_2(\mathbb{R})$$

Moreover, if (k_1, a, k_2) and (k_1', a', k_2') have the same image in $\mathrm{SL}_2(\mathbb{R})$, then $a = a'$. If in addition $a \neq \mathrm{id}$, then $(k_1, k_2) = \pm(k_1', k_2')$.
For $f \in L^1(G)$, the following integral formula holds:

$$\int_G f(x)dx = 2\pi \int_K \int_0^\infty \int_K f(k_1 a_t k_2)dk_1 dt dk_2.$$

Proof. The surjectivity follows from the singular value decomposition of a matrix. The uniqueness is a straightforward computation. For the integral formula, consider the map

$$\begin{aligned} \phi : K/\{\pm \mathrm{id}\} \times A^+ &\longrightarrow AN \\ (\pm k, a) &\longmapsto \underline{a}(ka)\underline{n}(ka) \end{aligned}$$

□

19.2 Geometry of upper half plane

Denote by $\mathbb{H} = \{z \in \mathbb{C} \mid \text{im } z > 0\}$ the upper half plane. The group $\mathrm{GL}_2(\mathbb{R})^+$ acts on $\mathbb{H}$ by Möbius transformation:

$$T_g z := \frac{az+b}{cz+d}.$$

The upper half plane $\mathbb{H}$ is naturally a Riemannian manifold with the metric:

$$ds^2 = \frac{dx^2+dy^2}{y^2} = \frac{dzd\overline{z}}{y^2}$$

where $z = x+iy$. In other words, the metric tensor $g = (g_{ij})$ is given by $g_{ij} = \dfrac{\delta_{ij}}{y}$. The complex structure J on $\mathbb{H}$ is given by $J(\partial_x) = -\partial_y$, $J(\partial_y) = \partial_x$. Then

$$J^T gJ = g$$

i.e., g is a hermitian metric on $\mathbb{H}$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})^+$, we compute the Jacobian

$$\frac{dT_g}{dz} = \frac{a(cz+d)-c(az+b)}{(cz+d)^2} = \frac{\det g}{(cz+d)^2}$$

Note that $\operatorname{Im} T_g(z) = \dfrac{(\det g)\operatorname{Im} z}{|cz+d|^2}$. One then easily computes that T_g preserves the metric g:

$$\begin{pmatrix} \dfrac{\det g}{(cz+d)^2} & 0 \\ 0 & \dfrac{\det g}{(c\overline{z}+d)^2} \end{pmatrix} \begin{pmatrix} 0 & \dfrac{1}{2\operatorname{Im} T_g(z)^2} \\ \dfrac{1}{2\operatorname{Im} T_g(z)^2} & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\det g}{(cz+d)^2} & 0 \\ 0 & \dfrac{\det g}{(c\overline{z}+d)^2} \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{1}{2y^2} \\ \dfrac{1}{2y^2} & 0 \end{pmatrix}$$

Here we write g in terms of the coordinates $(z, \overline{z})$.

19.2.1 Hyperbolic geodesic

For a piecewise differentiable path $\gamma : [0,1] \to \mathbb{H}$, the hyperbolic length is given by

$$h(\gamma) := \int_0^1 \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}dt = \int_0^1 \sqrt{\left(\frac{d(x\circ\gamma)}{dt}\right)^2 + \left(\frac{d(y\circ\gamma)}{dt}\right)^2} \frac{dt}{(y\circ\gamma)(t)} = \int_0^1 \left|\frac{d(z\circ\gamma)}{dt}\right| \frac{dt}{\operatorname{Im}\gamma(t)}$$

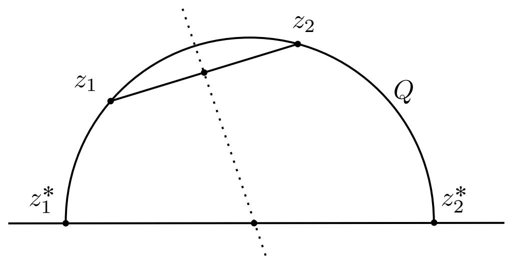
Since T_g preserves metric for each $g \in \mathrm{GL}_2(\mathbb{R})^+$, we see $h(T_g\gamma) = h(\gamma)$.

Let us find the geodesic connecting $r+ia$ and $r+ib$ ($b > a > 0$, $r \in \mathbb{R}$). Let $\gamma(t) = (x(t), y(t))$ be any piecewise differentiable path from $r+ia$ to $r+ib$. Then

$$h(\gamma) = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \frac{dt}{y} \geqslant \int_0^1 \left|\frac{dy}{dt}\right| \frac{dt}{y} \geqslant \int_0^1 \frac{dy}{dt}\frac{dt}{y} = \int_a^b \frac{dy}{y} = \log\frac{b}{a}$$

with equality if and only if $\dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} \geqslant 0$, in which case γ is the usual (Euclidean-)line segment connecting $r+ia$ and $r+ib$.

Suppose that z_1, z_2 do not have the same real part. Let Q be the unique circle connecting z_1, z_2 whose center is the intersection bisector of the segment z_1z_2 and the real axis. Suppose Q intersects the real axis at $z_1^* < z_2^*$, as depicted in the picture below.



Take $g \in \mathrm{SL}_2(\mathbb{R})$ with $T_g(z_1^*) = 0$, $T_g(z_2^*) = \infty$. Since T_g is conformal, $T_g(Q)$ must be the imaginary axis, and thus the geodesic connecting $T_g(z_1)$, $T_g(z_2)$ is the Euclidean segment connecting them. Since T_g preserves metric, the arc of Q joining z_1 and z_2 is hence the geodesic joining z_1 and z_2.

19.2.2 Formula for hyperbolic distance

For $z \neq w \in \mathbb{H}$, let $\rho(z, w)$ denote the length of the geodesic connecting z and w. We already see $\rho(ia, ib) = \log \frac{b}{a}$ for $b > a > 0$. In general, let Q be the geodesic joining z, w which intersects $\mathbb{P}^1(\mathbb{R})$ at z^* and w^*; we label them in the way that z lies between z^* and w. There exists a unique element $g \in \mathrm{PSL}_2(\mathbb{R})$ such that $T_g(z^*) = 0$, $T_g(w^*) = \infty$ and $T_g(z) = i$. Since $T_g(w)$ lies between $T_g(z) = i$ and $T_g(w^*) = \infty$, we have $T_g(w) = ir$ for some $r > 1$, and hence

$$\rho(z, w) = \rho(T_g(z), T_g(w)) = \rho(i, ir) = \log r.$$

To find r explicitly in terms of z and w, consider the function $\tau : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ defined by

$$\tau(z, w) := \left| \frac{z - w}{z - \overline{w}} \right|$$

We claim $\tau(T_g z, T_g w) = \tau(z, w)$ for $g \in \mathrm{PSL}_2(\mathbb{R})$. This will follow from the identity

$$|T_g(z) - T_g(w)| = |z - w| \cdot |(T_g)'(z)(T_g)'(w)|^{1/2}$$

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$. Then

$$T_g(z) - T_g(w) = \frac{az + b}{cz + d} - \frac{aw + b}{cw + d} = \frac{(z - w)(ad - cb)}{(cz + d)(cw + d)} = \frac{z - w}{(cz + d)(cw + d)}$$

and hence

$$|T_g(z) - T_g(w)| = \frac{|z - w|}{|cz + d||cw + d|} = |z - w||(T_g)'(z)(T_g)'(w)|^{1/2}.$$

(recall that $(T_g)'(z) = |cz + d|^{-2}$.)

We can use τ to find r. Indeed, let $g \in \mathrm{PSL}_2(\mathbb{R})$ be as in the beginning. Then

$$\tau(z, w) = \tau(T_g z, T_g w) = \tau(i, ir) = \left| \frac{i - ir}{i + ir} \right| = \frac{r - 1}{r + 1},$$

or

$$r = \frac{1 + \tau(z, w)}{1 - \tau(z, w)}.$$

Hence

$$\rho(z, w) = \log r = \log \frac{|z - \overline{w}| + |z - w|}{|z - \overline{w}| - |z - w|}.$$

We can also express $\rho(z,w)$ using sinh, the hyperbolic sine function. Recall that

$$\tanh\frac{u}{2} = \frac{e^u - 1}{e^u + 1} \qquad \sinh^2\frac{u}{2} = \frac{\tanh^2(u/2)}{1-\tanh^2(u/2)}$$

Then

$$\sinh^2\frac{\rho(z,w)}{2} = \frac{\tau(z,w)^2}{1-\tau(z,w)^2} = \frac{|z-w|^2}{|z-\overline{w}|^2 - |z-w|^2}$$

Since

$$|z-\overline{w}|^2 - |z-w|^2 = -(z-\overline{z})(w-\overline{w}) = 4\operatorname{Im}(z)\operatorname{Im}(w)$$

therefore

$$\sinh^2\frac{\rho(z,w)}{2} = \frac{|z-w|^2}{4\operatorname{Im}(z)\operatorname{Im}(w)}$$

As an application of the last formula, we find the hyperbolic circle with center i and radius δ. This is (with $z = x+iy$)

$$\begin{aligned}\left\{z\in\mathbb{H} \mid \sinh^2\frac{\rho(z,i)}{2} = \sinh^2\frac{\delta}{2}\right\} &= \left\{z\in\mathbb{H} \mid |z-i|^2 = 4y\sinh^2\frac{\delta}{2}\right\}\\ &= \left\{z\in\mathbb{H} \mid x^2+y^2+1 = 2y\left(2\sinh^2\frac{\delta}{2}+1\right) = 2y\cosh\delta\right\}\\ &= \left\{z\in\mathbb{H} \mid x^2+(y-\cosh\delta)^2 = \cosh^2\delta - 1 = \sinh^2\delta\right\}\end{aligned}$$

19.2.3 Hyperbolic measure on $\mathbb{H}$

The Riemannian volume form on $(\mathbb{H}, g)$ is given by

$$d\text{vol} = \sqrt{\det g}\,dx\wedge dy = \frac{dx\wedge dy}{y^2}.$$

Let μ be the corresponding outer Radon measure on $\mathbb{H}$. Then for a measurable $E\subseteq\mathbb{H}$, we have

$$\mu(E) := \int_{\mathbb{E}}\frac{dxdy}{y^2} = \int_{\mathbb{H}}\mathbf{1}_E(x,y)\frac{dxdy}{y^2}$$

Again, since T_g preserves metric, it follows that T_g preserves measure: $\mu(T_g(E)) = \mu(E)$.

We compute the Gaussian curvature K on $\mathbb{H}$, using Christoffel symbols. Recall

$$\Gamma^k_{ij} = \frac{1}{2}g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{\ell i} - \partial_\ell g_{ij})$$

Now $g_{11} = g_{22} = \dfrac{1}{y^2}$, $g_{12} = g_{21} = 0$, so $g^{11} = g^{22} = y^2$, $g^{12} = g^{21} = 0$. Then

$$\Gamma^1_{11} = \frac{1}{2}g^{11}(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}) = 0$$

$$\Gamma^2_{11} = \frac{1}{2}g^{22}(\partial_1 g_{12} + \partial_1 g_{21} - \partial_2 g_{11}) = \frac{1}{y}$$

$$\Gamma^1_{12} = \frac{1}{2}g^{11}(\partial_1 g_{21} + \partial_2 g_{11} - \partial_2 g_{12}) = \frac{-1}{y}$$

$$\Gamma^2_{12} = \frac{1}{2}g^{22}(\partial_1 g_{22} + \partial_2 g_{21} - \partial_2 g_{12}) = 0$$

$$\Gamma^1_{22} = \frac{1}{2}g^{11}(\partial_2 g_{12} + \partial_2 g_{12} - \partial_1 g_{22}) = 0$$

$$\Gamma_{22}^2 = \frac{1}{2}g^{22}(\partial_2 g_{22} + \partial_2 g_{22} - \partial_2 g_{22}) = \frac{-1}{y}$$

and thus

$$R_{1221} = \left(\partial_1 \Gamma_{22}^k - \partial_2 \Gamma_{12}^k + \Gamma_{22}^p \Gamma_{1p}^k - \Gamma_{12}^p \Gamma_{2p}^k\right) g_{k1} = \frac{-1}{y^4}$$

Finally, $\det(g_{ij}) = \dfrac{1}{y^4}$, so the Gaussian curvature is

$$K = \frac{R_{1221}}{\det(g_{ij})} = -1$$

Now, using Gauss-Bonnet theorem, we see if Δ is a geodesic triangle in $\mathbb{H}$ with angles α, β, γ,

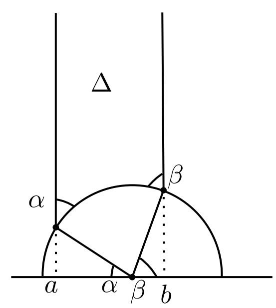
$$0 = (\alpha + \beta + \gamma) - \pi - \int_\Delta -1 d\mu$$

namely,

$$\mu(\Delta) = \pi - (\alpha + \beta + \gamma)$$

We can directly compute the geodesic triangles Δ on $\mathbb{H}$ without the Gauss-Bonnet theorem. We divide the computation into three cases.

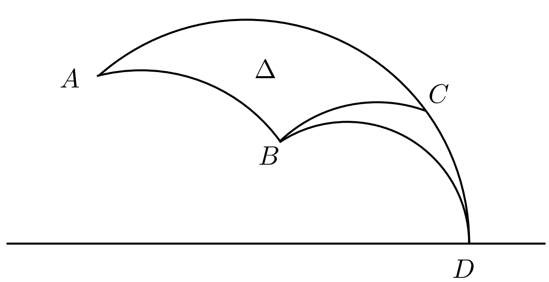
- Two sides are vertical geodesics. By applying $z \mapsto z + a\,(a \in \mathbb{R})$ and $z \mapsto \lambda z\,(\lambda > 0)$, we can assume the base of Δ is an arc of the semicircle centered at 0 with radius 1, depicted below.



  Then
$$\mu(\Delta) = \int_a^b \int_{\sqrt{1-x^2}}^\infty \frac{dxdy}{y^2} = \int_a^b \frac{dx}{\sqrt{1-x^2}} = \int_{\pi-\alpha}^\beta d\theta = \pi - \alpha - \beta$$

- One vertex of Δ lies on $\mathbb{R}$. Then we can send this vertex to ∞ by an element of $\mathrm{PSL}_2(\mathbb{R})$ without altering the area and the angles. Then the result follows from the first case.

- $\Delta = ABC$ has no vertex in $\mathbb{R} \cup \{\infty\}$. By applying an element in $\mathrm{PSL}_2(\mathbb{R})$ we can assume no side is vertical. Let D be the intersection of AC and $\mathbb{R}$, as below.



  Then the result follows from the second case. (Note that the angle at D is 0.)

We can extend our computation to a slightly general domain. Call a subset $C \subseteq \mathbb{H}$ **hyperbolically starlike** if there is a point O in the interior of C such that for all $P \in C$, the geodesic segment joining O and P lies in C. Then if Π is an n-sided hyperbolically starlike polygon with angles $\alpha_1, \ldots, \alpha_n$. Then

$$\mu(\Pi) = (n-2)\pi - (\alpha_1 + \cdots + \alpha_n)$$

19.3 Hecke algebras

19.3.1 Gelfand pair

Let G be an LCH group and K a compact subgroup of G. Define

$$C_c(G/\!/K) := \{f \in C_c(G) \mid f(kgk') = f(g) \text{ for all } g \in G,\ k, k' \in K\}$$

This is an algebra under convolution: for $f, g \in C_c(G/\!/K)$

$$f * g(kyk') = \int_G f(x)g(x^{-1}kyk')dx = \int_G f(x)g((k^{-1}x)^{-1}y)dx = \int_G f(kx)g(x^{-1}y)dx = f * g(y)$$

Definition. The pair (G, K) is called a **Gelfand pair** if $C_c(G/\!/K)$ is commutative.

Lemma 19.3.1. Let G be an LCH group and K a compact subgroup. If $\iota : G \to G$ is a continuous involution such that $\iota(x) \in KxK$ for all $x \in G$, then G is unimodular and (G, K) is a Gelfand pair.

Proof. Note that the pullback $\iota_* : C_c(G) \to C_c(G)$ by ι restricts to the identity on $C_c(G/\!/K)$ as it preserves (K, K)-double cosets. For $f \in C_c(G/\!/K)$ and $y \in G$ we then have

$$\int_G f(xy)dx = \int_G f(\iota(xy))dx = \int_G f(xy)\iota_*(dx).$$

Since ι is an anti-automorphism, $\iota_*(dx)$ is a right Haar measure on G, so RHS of the above equals

$$\int_G f(x)\iota_*(dx) = \int_G f(\iota(x))dx = \int_G f(x)dx.$$

If we can find $f \in C_c(G/\!/K)$ with $\int_G f(x)dx \neq 0$, varying $y \in G$ will prove G is unimodular. This is easy: pick any $g \in C_c(G)$ with nonvanishing integral, choose any Haar measure on K, and simply take $f = {}^K g^K$ (c.f. §2.4.4). Since $\Delta_G|_K \equiv 1$, by Fubini we compute

$$\begin{aligned}\int_G f(x)dx &= \int_G \left(\int_K \int_K g(kxh)dkdh\right) dx = \int_K \int_K \left(\int_G g(kxh)dx\right) dkdh \\ &= \int_K \int_K \left(\int_G g(x)dx\right) dkdh = \mathrm{vol}(K)^2 \int_G g(x)dx \neq 0\end{aligned}$$

It remains to see $C_c(G/\!/K)$ is commutative. Since G is unimodular, it follows that $\iota_*(dx)$ is a multiple of dx. Since ι_* acts trivially on $C_c(G/\!/K)$ and dx is nonzero when restricts to $C_c(G/\!/K)$ as we've shown above, it follows that $\iota_*(dx) = dx$. To conclude, for $f, g \in C_c(G/\!/K)$ and $y \in G$, we compute

$$\begin{aligned}f * g(y) &= \int_G f(x)g(x^{-1}y)dx \\ &= \int_G f(x)g(x^{-1}y)\mathrm{inv}_*\iota_*(dx) \\ &= \int_G f(x^{-1})g(\iota(x)y)dx = \int_G g(x)f(x^{-1}\iota(y))dx = g * f(\iota(y)) = g * f(y)\end{aligned}$$

□

Example 19.3.2 (Archimedean GL_n)**.** Take $(G, K) = (\mathrm{GL}_n(\mathbb{C}), \mathrm{U}(n))$, $(\mathrm{SL}_n(\mathbb{C}), \mathrm{SU}(n))$, $(\mathrm{GL}_n(\mathbb{R}), \mathrm{O}(n))$ or $(\mathrm{SL}_n(\mathbb{R}), \mathrm{SO}(n))$. For $g \in G$, the singular value decomposition allows us to write $g = kak'$ with $k, k' \in K$ and a diagonal and positive. It we denote by g^* its hermitian transpose, it follows that $g^* = (k')^* a k^* \in KgK$, so $g \mapsto g^*$ satisfies the condition in Lemma 19.3.1. Hence (G, K) is a Gelfand pair in these cases.

Example 19.3.3 (Non-archimedean GL_n)**.** Let k be a non-archimedean local field and let $\mathfrak{o}_k$ denote its ring of integers. We claim

$$\mathrm{GL}_n(k) = \mathrm{GL}_n(\mathfrak{o}_k) \left\{ \begin{pmatrix} \varpi^{r_1} & & \\ & \ddots & \\ & & \varpi^{r_n} \end{pmatrix} \mid r_1, \ldots, r_n \in \mathbb{Z} \right\} \mathrm{GL}_n(\mathfrak{o}_k)$$

where ϖ is a uniformizer of k. Assuming this, we see taking transpose $g \mapsto g^t$ satisfies the condition in Lemma 19.3.1, so $(\mathrm{GL}_n(k), \mathrm{GL}_n(\mathfrak{o}_k))$ is a Gelfand pair.

The claim is proved using the structure theorem of finitely generated modules over PID. For $g \in \mathrm{GL}_n(k)$, take $N \gg 0$ so that $\varpi^N g \in \mathrm{GL}_n(\mathfrak{o}_k)$. Now the structure theorem implies that

$$\varpi^N g = k \begin{pmatrix} \varpi^{s_1} & & \\ & \ddots & \\ & & \varpi^{s_n} \end{pmatrix} k'$$

for some $k, k' \in \mathrm{GL}_n(\mathfrak{o}_k)$ and $s_1, \ldots, s_n \in \mathbb{Z}_{\geqslant 0}$. Multiplying both sides by ϖ^{-N} proves the claim.

Example 19.3.4. Take $(G, K) = (\mathrm{SO}(n+1), \mathrm{SO}(n))$, where the latter is embedded in the former as in Example I.7.9.1.

Lemma 19.3.5. Let G be an LCH group and K a compact subgroup.

(i) If $(\pi, V_\pi) \in \widehat{G}$, then $V_\pi^K = \{v \in V \mid \pi(k)v = v \text{ for all } k \in K\}$ is either trivial, or a topologically irreducible $C_c(G/\!\!/K)$-module (i.e. contains no proper nontrivial closed submodule)

(ii) If (G, K) is a Gelfand pair, then $\dim_{\mathbb{C}} V_\pi^K \leqslant 1$ for all $(\pi, V_\pi) \in \widehat{G}$.

Proof.

(i) Suppose $V_\pi^K \neq 0$, and take $v \in V_\pi^K$. Fix a Haar measure on K with total mass 1. Since π is irreducible, $C_c(G)v$ is dense in V_π. For $w \in V_\pi^K$, take $(f_n)_n \subseteq C_c(G)$ so that $\pi(f_n)v \to w$ in V_π. We claim there exists $(g_n)_n \subseteq C_c(G/\!\!/K)$ such that $\pi(g_n)v \to w$. Put $g_n = {}^K f_n^K$. Then

$$\begin{aligned} \pi(g_n)v - w &= \int_{K \times K} \left(\int_G f_n(kxk')\pi(x)v dx - w \right) dk dk' \\ &= \int_{K \times K} \pi(k^{-1}) \left(\int_G f_n(x)\pi(x)v dx - w \right) dk dk' \end{aligned}$$

so that

$$\|\pi(g_n)v - w\| \leqslant \int_{K \times K} \left\|\pi(k^{-1})(\pi(f_n)v - w)\right\| dk dk' = \int_{K \times K} \|(\pi(f_n)v - w)\| \, dk dk' = \|\pi(f_n)v - w\| \, .$$

The second equality holds as π is unitary. This shows $\pi(g_n)v \to w$ as we want.

(ii) This follows from the commutativity of $C_c(G/\!\!/K)$ and Schur's lemma.

∎

19.3.2 Unitary representations of $\mathbf{SL}_2(\mathbb{R})$

Take $(G, K) = (\mathrm{SL}_2(\mathbb{R}), \mathrm{SO}(2))$ in this subsection. For $\lambda \in \mathbb{C}$, define

$$V_\lambda := \left\{\phi : G \to \mathbb{C} \mid \phi|_K \in L^2(K),\ \phi(\pm a_t n x) = e^{t(2\lambda+1)}\phi(x) \text{ for all } t \in \mathbb{R},\ n \in N,\ x \in G\right\}$$

19.4 Explicit Plancherel

19.5 Trace formula

Theorem 19.5.1. Let Γ be a torsion free uniform lattice in $\mathrm{SL}_2(\mathbb{R})$. Let $\varepsilon > 0$, and let h be a holomorphic function on the strip $\{|\operatorname{Im}(z)| < \frac{1}{2} + \varepsilon\}$. Suppose h is even, and $h(z) = O(|z|^{-2-\varepsilon})$ as $|z| \to \infty$. Then

$$\sum_{j=0}^{\infty} h(r_j) = \frac{\mathrm{vol}(\Gamma\backslash G)}{4\pi} \int_{\mathbb{R}} r h(r) \tanh(\pi r) dr + \sum_{[\gamma]\neq 1} \frac{l(\gamma_0)}{e^{l(\gamma)/2} - e^{-l(\gamma)/2}} g(l(r))$$

where

$$g(t) := \frac{1}{2\pi} \int_{\mathbb{R}} h(r) e^{-irt} dr$$

and for $\gamma \neq 1$, γ_0 denotes the primitive element underlying γ.

19.6 Weyl Law

19.7 Selberg zeta function

Part IV

Topology

Appendix A

Topology

Definition. A **topological space** is a set X together with a subcollection $\mathcal{T} \subseteq 2^X$ called the **topology** that contains $\varnothing$ and X, closed under arbitrary union and closed under finite intersection. A map $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is called **continuous** if taking preimage induces a map $f^{-1} : \mathcal{T}_Y \to \mathcal{T}_X$ between topologies.

Denote by **Top** the category of topological spaces with morphisms being continuous maps. **Top** admits a forgetful functor

$$\begin{aligned} \omega_{\mathbf{Top}} : \mathbf{Top} &\longrightarrow \mathbf{Set} \\ f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y) &\longmapsto f : X \to Y. \end{aligned}$$

$\omega_{\mathbf{Top}}$ admits both left adjoint and right adjoint, given respectively by $X \mapsto (X, 2^X)$ and $X \mapsto (X, \{\varnothing, X\})$. The former is called the **discrete topology** on X, while the latter is called the **trivial topology** on X. In particular,

Lemma A.0.1. The forgetful functor $\omega_{\mathbf{Top}} : \mathbf{Top} \to \mathbf{Set}$ preserves limits and colimits.

On a set X there are various topologies. We partially order the set of topologies on X by inclusion $\subseteq$ on 2^X. For two topologies $\mathcal{S}, \mathcal{T}$ on X, we say $\mathcal{T}$ is **finer** than S, or S is **coarser** than $\mathcal{T}$, if $\mathcal{S} \subseteq \mathcal{T}$. If $\{T_\alpha\}_\alpha$ is a collection of topologies on X, then $\bigcap_\alpha T_\alpha$ is also a topology on X. Hence the coarsest topology on X containing a fixed subcollection $S \subseteq 2^X$ is the intersection of all topologies containing S.

Let $S \subseteq 2^X$. If $\mathcal{T}$ is a topology on X containing S, then $\mathcal{T}$ contains

$$S^\cap := \{X\} \cup \{S_1 \cap \cdots \cap S_n \mid \{S_i\}_{i \in [n]} \subseteq S,\, 1 \leqslant n < \infty\}$$

and

$$\mathcal{T}_S := \left\{ \bigcup_{U \in A} U \mid A \subseteq S^\cap \right\}.$$

Hence any topology containing S contains $S^\cap$ and $\mathcal{T}_S$. In fact,

Lemma A.0.2. $\mathcal{T}_S$ is the coarsest topology on X containing S. In this case, we say S **generates** $\mathcal{T}_S$, and S is the **subbase** for the topology $\mathcal{T}_S$.

Definition. Let X be a set. A subcollection $B \subseteq 2^X$ is called a **base** for a topology on X if

(i) $X = \bigcup_{U \in B} U$, and

(ii) for all $U_1, U_2 \in B$ and $x \in U_1 \cap U_2$, there exists $U_3 \in B$ such that $x \in U_3 \subseteq U_1 \cap U_2$.

Lemma A.0.3.

1. If B is a base for a topology on X, then the coarsest topology on X containing B is given by
$$\left\{ \bigcup_{U \in A} U \mid A \subseteq B \right\}.$$

2. For any $S \subseteq 2^X$, the collection $S^\cap$ is a base.

Let X be a set. Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_\alpha$ be a collection of topological spaces and $f_\alpha : X_\alpha \to X$ be a collection of maps. The **final topology** on X induced by the maps f_α is the finest topology $\mathcal{T}$ on X making each $f_\alpha : (X_\alpha, \mathcal{T}_\alpha) \to (X, \mathcal{T})$ continuous. Explicitly, the final topology is given by
$$\{U \subseteq X \mid f_\alpha^{-1}(U) \in \mathcal{T}_\alpha \text{ for all } \alpha\}.$$
By definition, for any topological space Y, a map $f : X \to Y$ is continuous if and only if $f \circ f_\alpha : X_\alpha \to Y$ is continuous.

Likewise, let $\{g_\alpha : X \to X_\alpha\}_\alpha$ is a collection of maps. The **initial topology** on X is the coarsest topology $\mathcal{S}$ on X making each $g_\alpha : (X, \mathcal{S}) \to (X_\alpha, \mathcal{T}_\alpha)$ continuous. Explicitly, the initial topology has a subbase given by
$$\{g_\alpha^{-1}(U) \mid U \in \mathcal{T}_\alpha\}_\alpha$$
By definition, for any topological space Y, a map $f : Y \to X$ is continuous if and only if $f_\alpha \circ f : Y \to X_\alpha$ is continuous.

Let $F : J \to \mathbf{Top}$ be a small diagram. Since **Set** is complete, the small diagram $\omega_{\mathbf{Top}} \circ F : J \to \mathbf{Set}$ admits a limit $(\pi_\alpha : \lim \omega_{\mathbf{Top}} \circ F \to \omega_{\mathbf{Top}}(F(\alpha)))_\alpha$. Let's endow $\lim \omega_{\mathbf{Top}} \circ F$ with the initial topology induced by the π_α; denote the resulting topological space by $\lim F$, and denote again by $\pi_\alpha : \lim F \to F(\alpha)$ the projection. Then $(\pi_\alpha : \lim F \to F(\alpha))_\alpha$ represents the limit of F.

Similarly, since **Set** is cocomplete, the small diagram $\omega_{\mathbf{Top}} \circ F : J \to \mathbf{Set}$ admits a colimit $(\iota_\alpha : \omega_{\mathbf{Top}}(F(\alpha)) \to \operatorname{colim} \omega_{\mathbf{Top}} \circ F)_\alpha$. Let's endow $\operatorname{colim} \omega_{\mathbf{Top}} \circ F$ with the final topology induced by the ι_α; denote the resulting topological space by $\operatorname{colim} F$, and denote again by $\iota_\alpha : F(\alpha) \to \operatorname{colim} F$ the inclusion. Then $(\iota_\alpha : F(\alpha) \to \operatorname{colim} F)_\alpha$ represents the colimit of F.

Lemma A.0.4. The category **Top** is complete and cocomplete.

We name several common topologies. The limit topology on the product space is usually referred to as the **product topology**. If X is a topological space and $Y \subseteq X$ be a subset, the **subspace topology** on Y inherited from X is the initial topology on Y induced by the inclusion $Y \hookrightarrow X$.

For an equivalence relation $\sim$ on a set X, the quotient set $X/\sim$ is the set of $\sim$ equivalence classes. If X is a topological space, we topologize $X/\sim$ by the final topology induced by the projection $X \to X/\sim$. This is called the **quotient topology** on $X/\sim$, and is a quotient object in **Top**. If we regard $\iota : \sim \subseteq X \times X$ and equip $\sim$ with the subspace topology from the product topology $X \times X$, then $X/\sim$ is the coequalizer of $\mathrm{pr}_i \circ \iota : \sim \to X$, $i = 1, 2$ in **Top**.

A.1 Filters

Definition. Let X be a set. A family $\mathfrak{F} \subseteq 2^X$ of sets is called a **filter** on X if it is closed under finite intersection, $\varnothing \notin \mathfrak{F}$, and for all $S \in 2^X$, $S \in \mathfrak{F}$ whenever S contains an element of $\mathfrak{F}$.

We partially order the set of all filters on X by the inclusion in 2^X. That said, given two filters $\mathfrak{F}, \mathfrak{F}'$ on X, we say $\mathfrak{F}$ **is finer than** $\mathfrak{F}'$ (or $\mathfrak{F}'$ **is coarser than** $\mathfrak{F}$) if $\mathfrak{F}' \subseteq \mathfrak{F}$.

Example A.1.1.

(a) Let X be a set and $\varnothing \neq A \subseteq X$. The collection of all subsets in X containing A forms a filter. This is called the **principal filter** generated by A.

(b) Let X be a infinite set. A subset A in X is called **cofinite** if $\#(X \backslash A) < \infty$. The collection of all cofinite sets in X forms a filter. Particularly, the cofinite sets in $\mathbb{N} = \mathbb{Z}_{\geqslant 1}$ forms a filter, called the **Fréchet filter**.

(c) Let X be a topological space and $\varnothing \neq A \subseteq X$. The collection of all neighborhoods of A in X is a filter, called the **neighborhood filter** of A.

Let $S \subseteq 2^X$ be a collection of sets. If S is contained in some filter, then it is necessary that any two set in S has nonempty intersection. In other words,

$$S^{\cap} := \left\{ \bigcap_{A \in I} A \mid I \subseteq S,\ \#I < \infty \right\}$$

does not include the empty set. Conversely, if $\varnothing \notin S^{\cap}$, then the collection

$$\mathfrak{F} = \mathfrak{F}_S := \{U \mid A \subseteq U \text{ for some } A \in S^{\cap}\}$$

is a filter on X. Moreover, every filter containing S must contain $\mathfrak{F}$, i.e., $\mathfrak{F}$ is the coarsest filter that contains the collection S. We say $\mathfrak{F}$ is the **filter generated by** S, and S is a **subbase** of $\mathfrak{F}$. Using this construction, we easily obtain

Lemma A.1.2. Let $\mathfrak{F}$ be a filter on X, and let A be a subset of X. There exists a filter on X finer than $\mathfrak{F}$ and containing A if and only if A meets every element in $\mathfrak{F}$ nontrivially.

Let $S \subseteq 2^X$ be a collection of sets. The **upward closure** of S, which is defined as

$$S^{\subseteq} := \{U \mid A \subseteq U \text{ for some } A \in S\}$$

need not be a filter. But, obviously, it is a filter if and only if $S \neq \varnothing$, $\varnothing \notin S$, and the intersection of any two sets in S contains a set in S. In this situation, $S = S^{\cap}$, and $S^{\subseteq} = \mathfrak{F}_S$. We then say S is a **base** of $S^{\subseteq}$. From the definition it follows that

Lemma A.1.3. Let $\mathfrak{F}$ be a filter on X. A collection $B \subseteq \mathfrak{F}$ of sets in $\mathfrak{F}$ is a base of $\mathfrak{F}$ if and only if any set in $\mathfrak{F}$ contains a set in B.

Definition. Let X be a set. An **ultrafilter** on X is a filter maximal in the poset of all filters.

Let $\mathfrak{F}$ be a filter, and consider the collection

$$\mathcal{F} = \{\mathfrak{F}' \subseteq 2^X \mid \mathfrak{F} \subseteq \mathfrak{F}',\ \mathfrak{F}' \text{ is a filter on } X\}.$$

Let $\{\mathfrak{F}_\alpha\}_\alpha \subseteq \mathcal{F}$ be a chain in $\mathcal{F}$, and put $\mathfrak{F}' = \bigcup_\alpha \mathfrak{F}_\alpha$. This is a collection of subsets of 2^X, and clearly any two set in $\mathfrak{F}'$ has nonempty intersection. Hence $\mathfrak{F}'$ generates a filter, which is finer than each $\mathfrak{F}_\alpha$. Hence every chain in $\mathcal{F}$ has an upper bound in $\mathcal{F}$, so by Zorn's lemma $\mathcal{F}$ admits a maximal element. Hence

Lemma A.1.4. Let X be a set and $\mathfrak{F}$ a filter on X. Then there exists an ultrafilter on X containing $\mathfrak{F}$.

Lemma A.1.5. Let X be a set.

(i) If $\mathfrak{F}$ is an ultrafilter on X, then for all $A, B \in 2^X$, if $A \cup B \in \mathfrak{F}$, then either $A \in \mathfrak{F}$ or $B \in \mathfrak{F}$[1].

(ii) Let S be a subbase of some filter on X. If for any $A \in 2^X$ we have either $A \in S$ or $X \backslash A \in S$, then S is an ultrafilter.

(iii) Any filter $\mathfrak{F}$ on X is the intersection of all ultrafilters finer than $\mathfrak{F}$[2].

Proof.

(i) Suppose for contradiction that there exists $A, B \in 2^X$ with $A, B \notin \mathfrak{F}$ but $A \cup B \in F$. Consider the collection $\mathfrak{F}' := \{U \in 2^X \mid U \cup A \in \mathfrak{F}\}$. Then $\mathfrak{F}'$ is a filter: $\varnothing \notin \mathfrak{F}'$ as $A \notin \mathfrak{F}$; if $U, U' \in \mathfrak{F}'$, then $(U \cap U') \cup A = (U \cup A) \cap (U' \cup A) \in F$ as $\mathfrak{F}$ is a filter; if $U \in \mathfrak{F}'$ and $U' \subseteq U$, then $U' \cap A \subseteq U \cap A \in F$, so $U' \cap A \in F'$. Note that $\mathfrak{F} \subseteq \mathfrak{F}'$ while $\mathfrak{F} \not\ni B \in \mathfrak{F}'$, so $\mathfrak{F}'$ is strictly finer than $\mathfrak{F}$, a contradiction.

(ii) Let $\mathfrak{F}$ be any filter containing S. For $A \in \mathfrak{F}$, we have $X \backslash A \notin F$ so that $X \backslash A \notin S$. Hence $A \in S$ by assumption. This shows $\mathfrak{F} = S$.

(iii) Let $A \notin \mathfrak{F}$. By definition it follows that A contains no set in $\mathfrak{F}$. If we put $A' = X \backslash A$, we see A' meets every set in $\mathfrak{F}$. By Lemma A.1.3 there exists a filter $\mathfrak{F}'$ finer than $\mathfrak{F}$ and containing A'. Now any ultrafilter finer than F' cannot contain A (otherwise it would contain $\varnothing = A \cap A'$).

□

Consider a subset $Y \subseteq X$ of a given set X. If $\mathfrak{F}$ is a filter on X, when is the collection

$$\mathfrak{F}|_Y = \{S \cap Y \mid S \in \mathfrak{F}\} \subseteq 2^Y$$

a filter on Y? Clearly a sufficient and necessary condition is that Y meets every set in $\mathfrak{F}$ nontrivially. To see this, note that $\mathfrak{F}|_Y$ is automatically closed under finite intersection. If $S \in \mathfrak{F}$ and $S \cap Y \subseteq S' \subseteq Y$, then $S' = (S \cup S') \cap Y \in \mathfrak{F}$. Hence $\mathfrak{F}|_Y$ is a filter if and only if it does not contain empty set, as we claim. In this case, we say $\mathfrak{F}|_Y$ is the **induced filter** of $\mathfrak{F}$ on Y. The following properties are clear:

(a) If $\mathfrak{F}$ is a filter and $Y \in \mathfrak{F}$, then $\mathfrak{F}|_Y$ is a filter on Y.

(b) If $\mathfrak{F}$ induces a filter on $Y \subseteq X$, then any base S of $\mathfrak{F}$ restricts a base $\mathfrak{F}_Y := \{A \cap Y \mid A \in S\}$ for $\mathfrak{F}_Y$.

(c) Let $\mathfrak{F}$ be an ultrafilter on X and $Y \in 2^X$. Then $\mathfrak{F}$ induces a filter on Y if and only if $Y \in \mathfrak{F}$, in which case $\mathfrak{F}|_Y$ is an ultrafilter on Y.

We explain (c). If $\mathfrak{F}|_Y$ is a filter on Y, then Y meets every set in $\mathfrak{F}$ nontrivially. The filter generates by $\mathfrak{F}$ and Y is then finer than $\mathfrak{F}$, so by maximality $Y \in \mathfrak{F}$. This proves the only if part, and the if part is (a). To see $\mathfrak{F}|_Y$ is an ultrafilter on Y, let $A \in 2^Y$ and suppose $A \notin \mathfrak{F}|_Y$. This means A contains no set in $\mathfrak{F}$ (otherwise $A \in F$, and $A = A \cap Y \in \mathfrak{F}|_Y$). Taking complement, we see $X \backslash A$

[1]Interestingly, this means that an ultrafilter behaves like a prime ideal.
[2]This means $\mathfrak{F}$ equals its Jacobson radical.

meets every set in $\mathfrak{F}$. By maximality $X\backslash A \in \mathfrak{F}$, so that $Y\backslash A \in \mathfrak{F}|_Y$. Hence $\mathfrak{F}|_Y$ is an ultrafilter by Lemma A.1.5.(ii).

We turn to the relative case. Let $f : X \to X'$ be a map. If B is a filter base on X, then

$$f(B) := \{f(S) \mid S \in B\}$$

is a filter base on X': $f(B)$ does not contain empty set, and $f(S \cap S') \subseteq f(S) \cap f(S')$ so the intersection of two sets in $f(B)$ contains a set in $f(B)$. This induces a map

$$\begin{aligned} \{\text{filter bases on } X\} &\longrightarrow \{\text{filter bases on } X'\} \\ B &\longmapsto f(B). \end{aligned}$$

Clearly, this is order-preserving. What's important is that

Lemma A.1.6. If B is an ultrafilter base on X, then $f(B)$ is an ultrafilter base on X'.

Proof. Let $\mathfrak{F}$ (resp. $\mathfrak{F}'$) the filter generated by B (resp. $f(B)$). Let $A' \in 2^{X'}$. If $f^{-1}(A') \in \mathfrak{F}$, then $f^{-1}(A')$ contains a set A in B, so that $f(A) \subseteq A'$. This means $A' \in \mathfrak{F}'$. If $f^{-1}(A') \notin \mathfrak{F}$, then $X\backslash f^{-1}(A') \in \mathfrak{F}$ by Lemma A.1.5.(i), so $X\backslash f^{-1}(A')$ contains a set A in B. But then $f(A) \subseteq f(X\backslash f^{-1}(A')) \subseteq X\backslash A'$, so that $X\backslash A' \in \mathfrak{F}'$. Hence $\mathfrak{F}'$ is an ultrafilter by Lemma A.1.5.(ii) $\square$

Let's consider the other way around. Let B' be a filter base on X'. For $S, T \in B'$, we have $f^{-1}(S) \cap f^{-1}(T) = f^{-1}(S \cap T)$, so

$$f^{-1}(B') := \{f^{-1}(S) \mid S \in B'\}$$

is a filter base on X if and only if $f^{-1}(S) \neq \varnothing$ for each $S \in B'$. This induces a map

$$\begin{aligned} \{\text{filter bases on } X' \text{ which every set within meet } f(X)\} &\longrightarrow \{\text{filter bases on } X\} \\ B' &\longmapsto f^{-1}(B'). \end{aligned}$$

For B' in LHS, the filter base $f(f^{-1}(B'))$ is finer than B'. On the other hand, if B is a filter base on X, then $f(X)$ meets every set in $B' = f(B)$ so that $f^{-1}(B')$ is a filter base on X, which is coarser than B. If Y is a subset of X and $f : Y \to X$ is the inclusion, then the above result recovers that condition under which a filter on X induces a filter on Y.

Example A.1.7 (Product filters)**.** Let $\{X_i\}_{i \in I}$ be a collection of set, and let B_i be a filter base on X_i. On the product $X := \prod_{i \in I} X_i$, the collection B of the subset of X of the form

$$\prod_{i \in J} S_i \times \prod_{i \in I\backslash J} X_i$$

where $J \subseteq I$ is finite and $S_i \in B_i$ for $i \in J$, forms a filter base on X. If we denote by $\mathrm{pr}_i : X \to X_i$ the i-th projection, it is clear that the filter generated by B is also generated by the collection

$$\{\mathrm{pr}_i^{-1}(S_i) \mid S_i \in B_i,\, i \in I\}.$$

If we denote by F_i the filter generated by F_i, the resulting filter on X generated by B is called the **product filter** of F_i, and is denoted by $\prod_{i \in I} F_i$. Still another way: the product filter $\prod_{i \in I} F_i$ is the coarsest filter F on X satisfying the equalities $\mathrm{pr}_i(F) = F_i$ for each $i \in I$.

Example A.1.8 (Elementary filters)**.** Let X be a set and $(x_n)_{n\geqslant 1}$ be a sequence in X. The sequence defines a map $f : \mathbb{N} \to X$ by $f(n) := x_n$. The image of the Fréchet filter on $\mathbb{N}$ under f is called the **elementary filter** associated to the sequence. Unwinding the definition, it is the collection of subsets $S \in 2^X$ such that $(x_n)_{n\geqslant k} \subseteq S$ for some $k \geqslant 1$. The sets $\{x_m \mid m \geqslant n\}\,(n \geqslant 1)$ form a base for the elementary filter. If $(x_{n_k})_{k\geqslant 1}$ is a subsequence of $(x_n)_{n\geqslant 1}$, then the elementary filter associated to (x_{n_k}) is finer than that to $(x_n)_{n\geqslant 1}$.

Lemma A.1.9. Let X be a set and $\mathfrak{F}$ be a filter with a countable base. Then $\mathfrak{F}$ is the intersection of all elementary filters finer than $\mathfrak{F}$.

Proof. Let $(A_n)_{n\geqslant 1}$ be a countable base of $\mathfrak{F}$. If we put $B_n = \bigcap\limits_{1\leqslant m\leqslant n} A_m$, then $(B_n)_{n\geqslant 1}$ is again a countable base of $\mathfrak{F}$, and it satisfies $B_{n+1} \subseteq B_n$ for $n \geqslant 1$. For each $n \geqslant 1$ if we pick $a_n \in B_n$, then the elementary filter associated to the sequence $(a_n)_{n\geqslant 1}$ is a filter finer than $\mathfrak{F}$. Hence the intersection $\mathfrak{F}'$ of all elementary filters finer than $\mathfrak{F}$ exists, and is finer than $\mathfrak{F}$. If $\mathfrak{F} \subsetneq \mathfrak{F}'$, pick $S \in \mathfrak{F}'\backslash\mathfrak{F}$. Then $B_n \nsubseteq S$ for all $n \geqslant 1$, so that $B_n \cap (X\backslash S) \neq \varnothing$. Choose any $b_n \in B_n \cap (X\backslash S)$; the elementary filter associated to $(b_n)_n$ is then finer than F but does not contain S (if so, it would contain $\varnothing = (X\backslash S) \cap S$). This contradicts to the definition of $\mathfrak{F}'$. ☐

A.2 Filters and limits in topology

Definition. Let X be a topological space and $x \in X$.

1. The collection N_x of all neighborhoods of x in X is a filter, called the **neighborhood filter of** x.

2. x is called a **limit point of the filter** F on X if F is finer than the neighborhood filter N_x of x, i.e., $N_x \subseteq F$. We also say **the filter** F **converges to** x in this case.

3. x is called a **limit point of the filter base** B (or B **converges to** x) if x is a limit point of the filter generated by B.

Let Φ be a collection of filters on X, all of which converge to the same point x. In other words, the neighborhood filter N_x of x is coarser than any filter in Φ. Hence N_x is still coarser than the intersection $F = \bigcap\limits_{F'\in\Phi} F'$ of all filters in Φ. Recall that a filter is the intersection of all finer ultrafilters (Lemma A.1.5.(iii)). Hence

Lemma A.2.1. A filter F on X converges to a point $x \in X$ if and only if every ultrafilter finer than F converges to x.

Another related notion to limit points is

Definition. Let X be a topological space and B a filter base on X. A point $x \in X$ is called the **cluster point of** B if x lies in the closure of all sets in B.

By definition, $x \in X$ is a cluster point of the filter base B on X if and only if $U \cap S \neq \varnothing$ for all $U \in N_x$ and $S \in B$. Hence if $x \in X$ is a cluster point of B, there exists a filter F finer both than the one with base B and N_x. The converse holds by definition. Summary:

Lemma A.2.2. A point x is a cluster point of a filter F if and only if there exists a filter finer than F that converges to x.

In particular, every limit point is a cluster point. Also, if F is a ultrafilter, it follows from maximality that x is a cluster point of F if and only if it is a limit point of F.

Let B be a filter base. By definition x is a cluster point of B if and only if $x \in \bigcap_{S \in B} \overline{S}$. Hence the set of cluster points of B is precisely

$$\bigcap_{S \in B} \overline{S}$$

which is a closed set in X. If $Y \subseteq X$ is a subspace and B is a filter base on Y, then every cluster point of B lies in the closure $\overline{Y}$. Conversely, if $x \in \overline{Y}$, then the induced filter on Y of the neighborhood filter N_x of x in X is a filter on Y converging to x. This characterizes the closed sets in X in terms of filters.

We turn to the relative case.

Definition. Let $f : X \to Y$ be a map from a set X to a topological space Y, and let F be a filter on X. A point $y \in Y$ is called a **limit point** (resp. a **cluster point**) **of** f **with respect to the filter** F if y is a limit point (resp. a cluster point) of the filter base $f(F)$.

- The relation "y is a limit point of f with respect to the filter F" is written as

$$\lim_F f, \qquad \lim_{x,F} f(x).$$

- By definition a point $y \in Y$ is a limit point of f with respect to the filter F if and only if for all neighborhoods V of y there exists $S \in F$ such that $f(S) \subseteq V$, if and only if $f^{-1}(V) \in F$ for all $V \in N_y$.

- A point $y \in Y$ is a cluster point of f with respect to the filter F if and only if for all neighborhoods V of y and $S \in F$, there exists $x \in S$ such that $f(x) \in V$.

Example A.2.3 (Sequences)**.** Let $(x_n)_{n \geqslant 1}$ be a sequence in a topological space X, which defines a map $f : \mathbb{N} \to X$. A limit point $y \in X$ of f with respect to the Fréchet filter on $\mathbb{N}$ is called a **limit point of the sequence** $(x_n)_n$ **as** $n \to \infty$, and we write $y = \lim_{n\to\infty} x_n$ in this case. Similarly a cluster point of f with respect to the Fréchet filter is called a **cluster point of the sequence** $(x_n)_n$. In other words, a point is a limit (resp. cluster point) of the sequence $(x_n)_n$ if and only if it is a limit (resp. cluster point) of the elementary sequence associated to $(x_n)_n$.

In a more familiar formulation, a point $y \in X$ is a limit of the sequence $(x_n)_n$ if and only if for all $V \in N_y$ there exists $k \in \mathbb{N}$ such that $(x_n)_{n \geqslant k} \subseteq V$ (eventually), and is a cluster point if and only if for all $V \in N_y$ and $k \in \mathbb{N}$, there exists $l \geqslant k$ such that $x_l \in V$ (infinitely often).

Example A.2.4 (Nets)**.** A poset $(D, \geqslant)$ is called **directed** if any two element has a common upper bound. Let $(D, \geqslant)$ be a directed set. For $a \in D$, the set $D_a := \{x \in D \mid x \geqslant a\}$ is called a section of D relative to a. The collection $\{D_a \mid a \in D\}$ of all sections then form a filter on $(D, \geqslant)$, by virtue of the directedness. This is called the **section filter** on $(D, \geqslant)$. For example, the Fréchet filter on $\mathbb{N}$ is the section filter on $\mathbb{N}$, directed by the usual relation $\geqslant$.

Now let $f : D \to X$ be a map from the directed set D to a topological space X. A **limit** (resp. a **cluster point**) of f is a limit (resp. a cluster point) of f with respect to the section filter on D. A map $f : D \to X$ is usually called a **net** in X.

Definition. Let X, Y be two topological spaces and $f : X \to Y$ a map. A point $y \in Y$ is called **a limit of** f **at the point** $a \in X$ if y is a limit point of f with respect to the neighborhood filter N_a

of a in X. In this case we write

$$y = \lim_{x \to a} f(x)$$

instead, which is more classical. Similarly y is a **cluster point of f at a** if y is a cluster point of f with respect to N_a.

Lemma A.2.5. A map $f : X \to Y$ between topological spaces is continuous at $a \in X$ if and only if $\lim_{x \to a} f(x) = f(a)$.

Proof. By definition, f is continuous at a if and only if for all $V \in N_{f(a)}$ there exists $U \in N_a$ such that $f(U) \subseteq V$, i.e., the filter base $f(N_a)$ is finer than $N_{f(a)}$. This is precisely the same thing as $f(a)$ is a limit point of f at a. $\square$

Corollary A.2.5.1. Let $f : X \to Y$ be a map between topological spaces.

(i) If f is continuous at $a \in X$, then for every filter base B on X that converges to a, $f(B)$ converges to $f(a)$.

(ii) If for every ultrafilter F on X that converges to a, the ultrafilter base $f(F)$ converges to $f(a)$, then f is continuous at a.

Proof.

(i) By Lemma A.2.5 the filter base $f(N_a)$ is finer than $N_{f(a)}$. Hence if F is any filter on X with $N_a \subseteq F$, then $f(N_a) \subseteq f(F)$, so that $f(F)$ is finer than $N_{f(a)}$, i.e., $f(F)$ converges to $f(a)$.

(ii) Suppose f is not continuous at a, so that there exists $V \in N_{f(a)}$ such that $f^{-1}(V) \notin N_a$. Since $f^{-1}(V)$ contains no set in N_a, $A := X \backslash f^{-1}(V)$ meets every set in N_a, so that there exists an ultrafilter F finer than N_a and containing A. Since $f(A) \cap W = \varnothing$, so that $W \notin f(F)$. It follows that $f(F)$ does not converge to $f(a)$.

$\square$

Corollary A.2.5.2. Let $f : X \to Y$ be a map between topological spaces such that f is continuous at a point $a \in X$. If Z is a set, F a filter on Z, and $g : Z \to X$ a map that has a limit a with respect to F, then $f \circ g$ has a limit $f(a)$ with respect to F.

Proof. By assumption the filter base $g(F)$ is finer than N_a, so the filter base $(f \circ g)(F)$ is finer than $f(N_a)$. By Lemma A.2.5 $f(N_a)$ is finer than $N_{f(a)}$, so $(f \circ g)(F)$ is finer than $N_{f(a)}$. $\square$

Let X, Y be two topological spaces and $A \subseteq X$ a subset. Let $a \in \overline{A}$ and let F be the filter on A induced by the neighborhood filter N_a of a in X. If $f : A \to Y$ is a map, we write

$$y = \lim_{x \in A,\, x \to a} f(x)$$

if $y \in Y$ is a limit of f with respect to the filter F, and say y **is a limit of f at a relative to the subspace A.** Note that $y \in \overline{f(A)} \subseteq Y$. In the special case where $\{a\}$ is not open and $A = X \backslash \{a\}$, we write

$$y = \lim_{x \neq a,\, x \to a} f(x)$$

instead of $y = \lim_{x \in A,\, x \to a} f(x)$. We make analogous definition for cluster points.

If $f : X \to Y$ is a map, we say f has a limit (resp. cluster point) $y \in Y$ at $a \in \overline{A}$ relative to A if y is a limit (resp. cluster point) of $f|_A : A \to Y$ at a relative to A. If $\{a\}$ is not open (so that $a \in \overline{X \backslash \{a\}}$), then a map $f : X \to Y$ is continuous at a if and only if $f(a) = \lim_{x \neq a,\, x \to a} f(x)$.

Lemma A.2.6. Let X be a set, $\{Y_i\}_{i\in I}$ be a family of topological spaces and $f_i : X \to Y$ be maps. Equip X with the initial topology with respect to $\{f_i\}_{i\in I}$. Then a filter F on X converges to a if and only if for each $i \in I$, the filter base $f_i(F)$ converges to $f_i(a)$.

Proof. The only if part follows from Corollary A.2.5.2. For the if part, let $V \in N_a$; by construction there exists a finite subset $J \subseteq I$ and for each $j \in J$ an open neighborhood U_j of $f_j(a)$ such that $a \in \bigcap_{j\in J} f_j^{-1}(U_j) \subseteq V$. By assumption we can find $S \in F$ such that $f_j(S) \subseteq U_j$, so that $S \subseteq f_j^{-1}(f_j(S)) \subseteq f_j^{-1}(U_j)$, i.e., $U_j \in F$. Since J is finite, it follows that $\bigcap_{j\in J} f_j^{-1}(U_j) \in F$. Hence F is finer than N_a. □

A.3 Separation axiom.

Theorem A.3.1. Let X be a topological space. TFAE:

(i) Any two distinct points of X have disjoint neighborhoods.

(ii) For every index set I, the diagonal embedding $X \to X^I$ has closed image.

(iii) The diagonal in $X \times X$ is closed.

(iv) The intersection of all closed neighborhoods of a point $x \in X$ solely consists of x itself.

(v) If a filter F on X converges to x, then x is the only cluster point of F.

(vi) Every filter on X has at most one limit point.

If either holds, X is called a **Hausdorff space**, or a **$\mathbf{T_2}$-space**.

Proof. The implications (i)⇒(ii)⇒(iii)⇒(i) are clear.

(i)⇒(iv) Let $y \neq x$, and pick $V \in N_y$, $U \in N_x$ with $V \cap U = \varnothing$; by shrinking we may assume V is open. Then $C := X\backslash V$ is closed, $y \notin C$ and $C \in N_x$ as $U \subseteq C$.

(iv)⇒(v) By definition F is finer than N_x. Hence the set of all cluster points of F is $\bigcap_{S\in F} \overline{S} = \bigcap_{S\in N_x} \overline{S}$, which is $\{x\}$ by assumption.

(v)⇒(vi) Clear since every limit point is a cluster point.

(vi)⇒(i) Let $x \neq y \in X$, and suppose every $U \in N_x$ meet every $V \in N_y$. Then $\{U \cap V \mid U \in N_x, V \in N_y\}$ forms a base of filter which is finer than N_x and N_y, a contradiction.

□

Denote by **Haus** the full subcategory of **Top** consisting of Hausdorff spaces. Products and equalizers in **Haus** are the same as those in **Top**. In particular, this shows **Haus** is complete and the inclusion functor ι : **Haus** → **Top** preserves limit. It also admits a left adjoint, which can be proved by the general adjoint functor theorem as follows. It remains to verify the solution set condition. For each topological space X, the equivalence classes of continuous surjections from X to a Hausdorff space Y form a set as Y varies, since each such Y is homeomorphic to a quotient of X with some topology. Since each continuous map $f : X \to Y$ factors through its image with subspace topology inherit from Y, this shows the solution set condition. We denote by $H : \mathbf{Top} \to \mathbf{Haus}$ its left adjoint. For $X \in \mathbf{Top}$ and $Y \in \mathbf{Haus}$ we then have a functorial bijection

$$\mathrm{Hom}_{\mathbf{Haus}}(H(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Top}}(X, \iota(Y))$$

Taking $Y = H(X)$ gives a continuous map $\pi : X \to H(X)$, and the above map is then given by $f \mapsto f \circ \pi$. It follows that π is surjective. We then say $H(X)$ is the **maximal Hausdorff quotient** of X.

If X is already a Hausdorff space, we take $H(X) = X$, $\pi = \mathrm{id}_X$. Hence H is both a left-adjoint and a left inverse of $\iota : \mathbf{Haus} \to \mathbf{Top}$. Since $\mathbf{Top}$ is cocomplete, it follows that $\mathbf{Haus}$ is cocomplete. Explicitly, a coproduct in $\mathbf{Haus}$ is the one in $\mathbf{Top}$ (by construction), and a coequalizer in $\mathbf{Haus}$ is the maximal Hausdorff quotient of the coequalizer in $\mathbf{Top}$.

On other other hand, the inclusion $\iota : \mathbf{Haus} \to \mathbf{Top}$ has no right adjoint. This follows from the following characterization of epimorphisms in $\mathbf{Haus}$ (actually if part suffices).

Lemma A.3.2. A continuous map $X \to Y$ between Hausdorff spaces is epic if and only if it has dense image.

Proof. The if part follows from Lemma A.3.3. For the only if part, suppose $f : X \to Y$ is a morphism in $\mathbf{Haus}$ with $\overline{f(X)} \subsetneq Y$. Consider the adjunction $Y \sqcup_{\overline{f(X)}} Y$, i.e., gluing two copies of Y along the subspace $\overline{f(X)}$; it is topologized by the final topology of the two inclusions $\iota_1, \iota_2 : Y \to Y \sqcup_{\overline{f(X)}} Y$. This clearly Hausdorff. Since $\overline{f(X)} \neq Y$, we have $\iota_1 \neq \iota_2$, while $\iota_1 \circ f = \iota_2 \circ f$. Hence f is not an epimorphism. ☐

Lemma A.3.3. Let $f, g : X \to Y$ be two continuous maps to a Hausdorff space Y. If the equalizer $\mathrm{eq}(f, g)$ is dense in X, then $f = g$.

Proof. Let $h = (f, g) : X \to Y \times Y$, which is a continuous map. Let Δ_Y be the image of Y in $Y \times Y$ under diagonal embedding. Then $\Delta_Y \subseteq Y \times Y$ is closed and $\mathrm{eq}(f, g) = h^{-1}(\Delta_Y)$, so $\mathrm{eq}(f, g) \subseteq X$ is closed. Since it is dense, it follows that $\mathrm{eq}(f, g) = X$, i.e. $f = g$.

☐

Proposition A.3.4. Let X be a topological space. TFAE:

(i) For each point $x \in X$, the set of all closed neighborhoods of x is a fundamental system of neighborhoods of x.

(ii) For each $x \in X$ and each closed set $F \subseteq X$ with $x \notin F$, there exist a neighborhood of x and a neighborhood of F that do not intersect.

If either holds and X is Hausdorff, we call X a **regular space**.

Theorem A.3.5. Let X be a topological space, $A \subseteq X$ a dense subset, Y a regular space, and $f : A \to Y$ a map. f extends to a continuous map $\overline{f} : X \to Y$ if and only if for each $x \in X$, the limit $\lim\limits_{y \to x,\, y \in A} f(y)$ exists. In this case, the extension $\overline{f}$ is unique.

Proof. The uniqueness follows from Lemma A.3.3. For the existence, define $\overline{f} : X \to Y$ simply by

$$\overline{f}(x) = \lim_{y \to x,\, y \in A} f(y).$$

To show the continuity, let $x \in X$ and V a closed neighborhood of $\overline{f}(x)$ in Y. By definition there exists an open neighborhood U of x such that $f(U \cap A) \subseteq V$. Since U is a neighborhood of each of its points, we have

$$\overline{f}(z) = \lim_{y \to z,\, y \in U \cap A} f(y)$$

for all $z \in U$, whence $\overline{f}(z) \in \overline{f(U \cap A)} \subseteq V$, as V is closed. Since Y is regular, all closed neighborhoods of a point forms a fundamental system of neighborhoods, and hence the proof is completed. ☐

Corollary A.3.5.1. Let F_i be a filter on a set X_i $(i = 1, 2)$, and let $F = F_1 \times F_2$ be the product filter on the product set $X = X_1 \times X_2$. Let Y be a regular space and $f : X \to Y$ a map. If

1. $\lim\limits_{F_1 \times F_2} f$ exists and
2. $\lim\limits_{x_2, F_2} f(x_1, x_2) =: g(x_1)$ exists for all $x_1 \in X$,

then $\lim\limits_{x_1, F_1} g(x_1)$ exists and equals $\lim\limits_{F_1 \times F_2} f$.

A.4 Compactness and countability

Lemma A.4.1. Let X be a topological space. TFAE:

(i) Every filter on X has at least one cluster point.

(ii) Every ultrafilter on X is convergent.

(iii) Every family of closed sets of X whose intersection is empty contains a finite subfamily whose intersection is empty.

(iv) Every open cover of X admits a finite subcover.

A topological space satisfying either above condition is said to be **(covering) compact**.

Proof. (iii)⇔(iv) follow from taking complement. Assume (i). Recall by maximality a cluster point of an ultrafilter is a limit point, so (ii) is fulfilled. (ii)⇒(i) is obvious. For (i)⇒(iii), let $\mathcal{S}$ be a family of closed sets in X with empty intersection. If every finite subfamily of $\mathcal{S}$ has nonempty intersection, then $\mathcal{S}$ generates a filter, so it has a cluster point by (i). This is a contradiction as any cluster point lies in the intersection of all sets in $\mathcal{S}$. For (iii)⇒(i), let F be a filter on X without cluster point. Then $\{\overline{S} \mid S \in F\}$ is a family of closed sets contradicting to (iii). ☐

Corollary A.4.1.1 (Tychonov)**.** A product of compact spaces is compact.

Proof. This follows directly from Lemma A.2.6, Lemma A.1.6 and Lemma A.4.1.(ii). ☐

Definition. Let X be a set. A collection $\mathcal{S} \subseteq 2^X$ of subsets of X is said to have the **finite intersection property** if $\bigcap\limits_{A \in F} A \neq \varnothing$ for any finite subcollection $F \subseteq \mathcal{S}$.

Corollary A.4.1.2. Let X be a space. Then X is compact if and only if any collection of closed subsets in X that has the finite intersection property has nonempty intersection.

Proof. This is a reformulation of Lemma A.4.1.(iii).

☐

Definition. Let X be a topological space.

1. A sequence $(a_n)_n$ in X is called **convergent** if there exists a point $a \in X$ such that for any open neighborhood U of a there exists $N \geqslant 0$ such that $a_n \in U$ for $n \geqslant N$.

2. For a subset S of X, a point $x \in X$ is called a **limit point** of S if for every neighborhood U of x in X, $(U \cap S)\backslash\{x\}$ is nonempty.

Definition. Let X be a topological space.

1. X is called **limit point compact** if every infinite subset of X has a limit point in X.

2. X is called **sequentially compact** if every sequence in X has a convergent subsequence in X.

3. X is called **Lindelöf** if every open cover of X admits a countable subcover.

4. X is called **countably compact** if every countable open cover of X admits a finite subcover.

5. X is called **second countable** if the topology on X admits a countable basis.

6. X is called **first countable** if every point of X admits a countable neighborhood basis.

7. X is called **separable** if X admits a countable dense subset.

Theorem A.4.2. Let X be a topological space.

(i) If X is compact, then it is limit point compact.

(ii) If X is sequentially compact, then X is countably compact.

(iii) If X is first countable and compact, then X is sequentially compact.

(iv) If X is first countable Hausdorff and limit point compact, then X is sequentially compact.

(v) If X is Lindelöf and sequentially compact, then X is compact.

(vi) If X is second countable, then it is first countable, separable and Lindelöf.

Proof.

(v) Suppose for contrary that $\mathcal{U}$ is an open cover of X that does not admit a finite subcover.. Since X is Lindelöf, $\mathcal{U}$ admits a countable subcover, say $\mathcal{V} = \{U_n\}_{n=1}^{\infty}$. By assumption, we can find a sequence $(p_n)_n$ such that $p_n \in X\backslash \bigcup_{i=1}^{n} U_i$. Since X is sequentially compact, $(p_n)_n$ admits a convergent subsequence $(p_{n_k})_k$ converging to $p \in X$. Since $\mathcal{V}$ covers X, we can find $m \geqslant 1$ such that $p \in U_m$, and by convergence we can find $N \geqslant 0$ such that $p_{n_k} \in U_m$ whenever $k \geqslant N$. But for $\ell \geqslant m + N$, we have

$$p_{n_\ell} \in U_m \subseteq \bigcup_{i=1}^{m} U_i \subseteq \bigcup_{i=1}^{\ell} U_i \subseteq \bigcup_{i=1}^{n_\ell} U_i$$

a contradiction to our choice of $(p_n)_n$.

□

Lemma A.4.3. Let $f : X \to Y$ be a map between topological space. Suppose $x \in X$ has a countable neighborhood basis N. Then f is continuous at x if and only if for all sequences $(x_n)_n$ converging to x, we have $\lim_{n\to\infty} f(x_n) = f(x)$.

Proof. The only if part holds by Lemma A.2.5. For the in part, write $N = \{U_n\}_n$; we may assume $U_n \supseteq U_{n+1}$ for all $n \geqslant 1$. Suppose f is not continuous at x, so that there exists $V \in N_{f(x)}$ such that $f^{-1}(V) \notin N_x$. Then $f^{-1}(V) \not\supseteq U_n$ for all $n \geqslant 1$. Take $x_n \in U_n \backslash f^{-1}(V)$; then $x_n \to x$ and $f(x_n) \notin V$ for all $n \geqslant 1$, so $f(x_n)$ cannot converges to $f(x)$. ☐

Definition. A map $f : X \to Y$ between topological spaces is called **sequentially continuous** if for all $x \in X$ and all sequences $(x_n)_n$ converging to x, we have $\lim_{n\to\infty} f(x_n) = f(x)$.

Corollary A.4.3.1. Let $f : X \to Y$ be a map between topological space with X first countable. Then f is continuous if and only if f is sequentially continuous.

A.5 Metric spaces

Definition. Let X be a set. A function $d : X \times X \to \mathbb{R}_{\geqslant 0}$ satisfies

(i) $d(x, y) = d(y, x)$ for any $x, y \in X$,

(ii) $d(x, y) \leqslant d(x, z) + d(z, y)$ for any $x, y, z \in X$, and

(iii) $d(x, y) = 0$ if and only if $x = y$

is called a metric on X. A pair (X, d) with d a metric on X is called a **metric space**.

For $x \in X$ and $r > 0$, the set

$$B_r(x) := \{y \in X \mid d(x, y) < r\}$$

is called an **open ball** centered at x with radius r. It is easy to see that the collection $\{B_r(x) \mid x \in X,\ r > 0\}$ defines a topology on (X, d), making X a topological space.

Lemma A.5.1. Let (X, d) be a metric space. TFAE:

1. X is second countable.
2. X is separable.
3. X is Lindelöf.

Lemma A.5.2. Let (X, d) be a metric space.

(i) If X is compact, then X is second countable.

(ii) If X is sequentially compact, then X is second countable.

Corollary A.5.2.1. A metric space (X, d) is compact if and only if it is sequentially compact.

Proof. This follows from Lemma A.5.1, Lemma A.5.2 and Theorem A.4.2. ☐

Lemma A.5.3. Let $A \subseteq X$ be a subset. The closure $\overline{A}$ is compact if and only if every sequence in A admits a convergent subsequence.

Corollary A.5.3.1. Let $(a_n)_n \subseteq X$ be a sequence. Then $(a_n)_n$ has compact closure if and only if every subsequence of $(a_n)_n$ admits a convergent subsequence.

Proof. Let $A = \{a_n \mid n \in \mathbb{Z}_{\geqslant 1}\}$. A subsequence of $(a_n)_n$ is a sequence in A, so the only if part follows directly from Lemma A.5.3.

For the if part, assume that every subsequence of $(a_n)_n$ admits a convergent subsequence. By Lemma A.5.3, it is enough to show every sequence in A admits a convergent subsequence. Let $(b_n)_n$ be a sequence in A. We construct a subsequence of $(b_n)_n$ that is also a subsequence on $(a_n)_n$. Define $\psi, \phi : \mathbb{N} \to \mathbb{N}$ as follows. Let $\psi(1) = 1$ and $\phi(1)$ be any number such that $b_1 = a_{\phi(1)}$. For $n \geqslant 2$, define inductively that

$$\psi(n) = \min\{k \in \mathbb{Z} \mid k > \psi(n-1),\, b_k = a_m \text{ for some } m > \phi(n-1)\}$$

If the set on the right is empty, then for $k > \psi(n-1)$, we have $b_k = a_m$ implies $m \leqslant \phi(n-1)$, so that $(b_k)_{k>\psi(n-1)} \subseteq \{a_1, \dots, a_{\phi(n-1)}\}$ a finite set. In this case it is possible to construct a subsequence of $(b_n)_n$ converging to some a_i, where $1 \leqslant i \leqslant \phi(n-1)$. Hence we may assume the set of the right is nonempty (for every n), so that $\psi(n) < \infty$ is well-defined. Define $\phi(n)$ to be any index $m > \phi(n-1)$ such that $b_{\psi(n)} = a_{\phi(n)}$. By construction ψ and ϕ are strictly increasing, so $(b_{\psi(n)})_n = (a_{\phi(n)})_n$ is a subsequence, and hence $(b_{\psi(n)})_n$ admits a convergent subsequence by assumption. ☐

A.6 Baire's Category

Definition. A topological space X is called a **Baire space** if every countable intersection of open and dense subspaces of X is again dense. Equivalently, if X is a countable union of closed sets in X, then at least one of the closed set has nonempty interior.

Theorem A.6.1. Every LCH space and every complete metric space is a Baire space.

Proof. Let X be either an LCH space or a complete metric space. Let U be an open subset of X and suppose $A_i \subseteq X$ is nowhere dense for $i = 0, 1, \dots$. Let $V_0 = U$ and define recursively that $\overline{V}_{i+1} \subseteq V_i \backslash \overline{A_i}$.

- In the LCH case, we may find an open set V_{i+1} such that $V_{i+1} \subseteq \overline{V}_{i+1} \subseteq V_i \backslash \overline{A}_i$ with $\overline{V}_{i+1}$ compact. Note that the $\overline{V}_i$ satisfy the finite intersection property, so $\varnothing \neq \bigcap_i \overline{V}_i \subseteq U \backslash \bigcup_i \overline{A}_i$.
- In the complete metric case, we take $V_{i+1} = B_{\frac{1}{i+1}}(x)$ for some $x \in V_i \backslash \overline{A}_i$ such that $B_{\frac{2}{i+1}}(x) \subseteq V_i \backslash \overline{A}_i$. Thus a sequence $x_i \in V_i$ is Cauchy, and the completeness shows that the limit exists; moreover, $x = \lim_n x_n \in \overline{V}_i$ for all i, i.e, $x \in \bigcap_i \overline{V}_i \subseteq U \backslash \bigcup_i \overline{A}_i$.

In both cases we've shown $U \nsubseteq \bigcup_i \overline{A}_i$. Since U is arbitrary open set, we conclude $\operatorname{int} \bigcup_i \overline{A}_i = \varnothing$. ☐

A.7 Proper Map

Definition. Let X, Y be spaces. A map $f : X \to Y$ is called **proper** if for every compact $K \subseteq Y$, the preimage $f^{-1}(K)$ is compact.

Proposition A.7.1. If $f : X \to Y$ is a continuous closed map with compact fibre, then f is proper.

Proof. Let K be a compact set in Y and $\{U_\alpha\}_{\alpha \in I}$ an open cover of $f^{-1}(K)$. For each $y \in K$, let $I_y \subseteq I$ be a finite index set such that $\{U_\alpha\}_{\alpha \in I_y}$ covers $f^{-1}(y)$ and put $V_y := \bigcup_{\alpha \in I_y} U_\alpha$. Since f is closed, $W_y := Y \setminus f(X \setminus V_y)$ is an open neighborhood of y. By compactness we can find a finite subset $F \subseteq K$ such that $K \subseteq \bigcup_{y \in F} W_y$. Then $f^{-1}(K) \subseteq \bigcup_{y \in F} V_y = \bigcup_{y \in F,\, \alpha \in I_y} U_\alpha$. Indeed, for $x \in X$ with $f(x) \in K$, take $y \in F$ such that $f(x) \in W_y$; this means $x \in V_y$. $\square$

Proposition A.7.2. Let Y be LCH and $f : X \to Y$ a continuous proper map. Then f is closed.

Proof. Let $C \subseteq X$ be closed and $L \subseteq Y$ be compact. We first show $f(C) \cap L$ is closed. As $C \cap f^{-1}(L)$ is compact,

$$f(C \cap f^{-1}(L)) = f(C) \cap L \subseteq Y$$

is compact, and hence closed in Y. Now for $y \in \overline{f(C)}$, we can find a compact neighborhood $L_y \subseteq Y$ of y. Since all neighborhood of y meet $L_y \cap f(C)$, we have

$$y \in \overline{L_y \cap f(C)} = L_y \cap f(C) \subseteq f(C)$$

$\square$

Corollary A.7.2.1. Let the setting be as above. For every $B \subseteq Y$ and every open neighborhood U of $f^{-1}(B)$ in X, there exists an open neighborhood V of B in Y such that $f^{-1}(V) \subseteq U$.

Proof. $V = Y \setminus f(X \setminus U)$ does the job.

$\square$

A.8 Urysohn's Lemma

Definition. Let X be a topological space.

1. X is called **normal** if for any two disjoint closed subspaces $C_1, C_2 \subseteq X$ there are open neighborhoods U, V of C_1, C_2, respectively such that $U \cap V = \varnothing$.

2. X is called **completely regular** if for each point $x \in X$ and a closed subspace $C \subseteq X$ not containing x, there exists a continuous map $f : X \to [0, 1]$ such that $f(x) = 0$ and $f \equiv 1$ on C.

Lemma A.8.1. Let X be an LCH space. If K is compact and U is open with $K \subseteq U$, then we can find a relatively compact open set V with $K \subseteq V \subseteq \overline{V} \subseteq U$.

Proof. Cover K by compact neighborhoods of points in K, and by compactness finitely many of them suffices to do so; let G be their union. If $U = X$, then $V = X$ does the job. If $U \subsetneq X$, let C be the complement of U. For each $p \in C$, let W_p be an open set such that $p \notin \overline{W_p}$ and $K \subseteq W_p$; such a W_p exists by compactness of K. Then $\{C \cap \overline{G} \cap \overline{W_p}\}_{p \in C}$ is a collection of compact sets with empty intersection. The finite intersection property assures that there are points $p_1, \ldots, p_n \in C$ such that

$$C \cap \overline{G} \cap \overline{W_{p_1}} \cap \cdots \cap \overline{W_{p_n}} = \varnothing$$

Then $V = G \cap W_{p_1} \cap \cdots \cap W_{p_n}$ does the job. $\square$

Lemma A.8.2. Let X be a topological space. Let $(r_n)_{n\geqslant 0}$ be an enumeration of $[0,1]\cap\mathbb{Q}$ with $r_0=0$ and $r_1=1$. Suppose $(V_n)_{n\geqslant 0}$ is a family of open sets with the property that $r_n\leqslant r_m$ if and only if $\overline{V_m}\subseteq V_n$. Then the function

$$f(x) := \begin{cases} 0 & \text{, if } x\notin V_0 \\ \sup\{r_n \mid x\in\overline{V_n}\} & \text{, if } x\in V_0 \end{cases}$$

is continuous.

Proof. For $r>s$, one has

$$f^{-1}(s,r) = \bigcup_{s<s'<s''<r} V_{s'}\backslash\overline{V_{s''}}$$

which is open. Similarly, $f^{-1}[0,s)$ and $f^{-1}(r,1]$ are open. Since such intervals generate the topology of $[0,1]$, this shows f is continuous. □

Lemma A.8.3. Let X be space and U an open set. Suppose either

- X is LCH and $K\subseteq U$ is compact, or
- X is normal and $K\subseteq U$ is closed in X.

Then we have the following.

(i) There is a continuous map $f: X\to[0,1]$ with $f|_K\equiv 1$ and $\operatorname{supp} f\subseteq U$. In the case X is LCH, f can be constructed so that $\operatorname{supp} f$ is compact.

(ii) Let $A\subseteq X$ be closed and $h\in C^+(A)$. If X is LCH we assume h vanishes at infinity. Suppose $h\geqslant 1$ on $K\cap A$. Then there exists $f\in C(X)$ as in (i) with additional property that $f(a)\leqslant h(a)$ for every $a\in A$.

Proof.

(i) Put $r_1=0$, $r_2=1$ and $\{r_n\}_{n\geqslant 3}$ an enumeration of $(0,1)\cap\mathbb{Q}$. Pick V_0,V_1 be open such that

$$K\subseteq V_1\subseteq\overline{V_1}\subseteq V_0\subseteq\overline{V_0}\subseteq U$$

This is possible by normality if X normal, and by Lemma A.8.1 if X is LCH.

Suppose $n\geqslant 2$ and $V_1,\dots,V_n$ are constructed in a way that $r_i<r_j$ implies $\overline{V_j}\subseteq V_i$. Now we construct V_{n+1}. Say $r_i<r_{n+1}<r_j$ with i maximal and j minimal among $\{1,\dots,n\}$. By normality or Lemma A.8.1 we can find V_n such that

$$\overline{V_j}\subseteq V_{n+1}\subseteq\overline{V_{n+1}}\subseteq V_i$$

Doing this indefinitely, we obtain a collection $\{V_n\}_{n\geqslant 0}$ of open sets such that

(a) $K\subseteq V_1\subseteq\overline{V_0}\subseteq U$ with $\overline{V_0}$ compact if X is LCH,
(b) $i>j$ implies $\overline{V_i}\subseteq V_j$.

Define f as in Lemma A.8.2 with the $(V_n)_{n\geqslant 0}$ constructed above. Then f is continuous and $f|_K\equiv 1$.

(ii) At each stage of constructing V_{n+1} in (ii) (including $n=1$), replace V_i by $V_i\cap\{a\in A \mid h(a)>r_{n+1}\}$.

□

Corollary A.8.3.1. normality ⇒ completely regularity; LCH ⇒ completely regularity.

Theorem A.8.4 (Tietze's)**.**

A.9 Paracompact Space

We begin with a series of definitions.

Definition. Let X be a topological space.

1. An open cover $\mathcal{V}$ of X is a **refinement** of another open cover $\mathcal{U}$ of X if each element of $\mathcal{V}$ is a subset of some element of $\mathcal{U}$.

2. A collection $\mathcal{U} \subseteq 2^X$ is called **locally finite** if each point $x \in X$ admits a neighborhood N which meets only a finite number of the members of $\mathcal{U}$.

3. The space X is called **paracompact** if X is Hausdorff and every open cover of X has a locally finite open refinement.

Theorem A.9.1. A metric space (X, d) is paracompact.

Proof. (Due to M.E. Rudin) Given an open cover U_α $(\alpha \in A)$ of X, we aim to find a locally finite open refinement $V_{\alpha,n}$ $((\alpha, n) \in A \times \mathbb{N})$ of U_α. By AC, A has a well-ordering, say $\leqslant$. For each $(\alpha, n) \in A \times \mathbb{N}$, define

$$V_{\alpha,n} = \bigcup_x B_{2^{-n}}(x)$$

where the union is taken over all $x \in X$ such that

(i) α is the smallest index with $x \in U_\alpha$. (such α exists since $\leqslant$ is a well-order)

(ii) $x \notin V_{\beta,j}$ for $j < n$ and $\beta \in A$.

(iii) $B_{3 \cdot 2^{-n}}(x) \subseteq U_\alpha$.

We claim the $V_{\alpha,n}$ is an locally finite open refinement of the U_α.

First, we show the $V_{\alpha,n}$ cover X. For each $x \in X$, let $\alpha \in A$ be the smallest index such that $x \in U_\alpha$ and pick n so large that $B_{3 \cdot 2^{-n}}(x) \subseteq U_\alpha$; thus (i) and (iii) holds. If (ii) holds for x, then $x \in V_{\alpha,n}$; otherwise, $x \in V_{\alpha,j}$ for some $j < n$. In a nutshell, $x \in V_{\alpha,n}$ for $j \leqslant n$.

Second, we show the $V_{\alpha,n}$ is locally finite. Pick $x \in X$ and let $\alpha \in A$ be the smallest such that $x \in V_{\alpha,n}$ for some $n \in \mathbb{N}$ (such pair exists since we've shown the $V_{\alpha,n}$ cover X), and choose j so large that $B_{2^{-j}}(x) \subseteq V_{\alpha,n}$.

1. If $i \geqslant n + j$, then $B_{2^{-n-j}}(x) \cap D_{\beta,i} = \varnothing$: Since $i > n$, every ball $B_{2^{-i}}(y)$ that constitutes $V_{\beta,i}$ has its center y outside $V_{\alpha,n}$ (for otherwise that $y \in V_{\alpha,n} \cap V_{\beta,i}$ and $i > n$ would be in breach of (ii)). Also, since $B_{2^{-j}}(x) \subseteq V_{\alpha,n}$ (by our choice), $d_X(x, y) \geqslant 2^{-j}$. But $i \geqslant j + 1$ and $n + j \geqslant j + 1$,
$$2^{-n-j} + 2^{-i} \leqslant 2^{-j-1} + 2^{-j-1} = 2^{-j}$$
so that $B_{2^{-n-j}}(x) \cap B_{2^{-i}}(y) = \varnothing$.

2. If $i < n + j$, then $\#\{\beta \in A \mid B_{2^{-n-j}}(x) \cap V_{\beta,i} \neq \varnothing\} \leqslant 1$: Suppose $p \in V_{\beta,i}$, $q \in V_{\gamma,i}$ and $\beta < \gamma$; we want to show $d(p, q) > 2^{-n-j+1}$ (for if it holds and $p \in B(x, 2^{-n-j})$, then
$$d_X(q, x) \geqslant |d(q, p) - d(p, x)| = 2^{-n-j+1} - 2^{-n-j} = 2^{-n-j}$$
so that $q \notin B_{2^{-n-j}}(x)$). By definition there are points y, z such that $p \in B_{2^{-i}}(y)$ and $q \in B_{2^{-i}}(z)$. By (iii), $B_{3 \cdot 2^{-i}}(y) \subseteq U_\beta$, and by (i), $z \notin U_\beta$. Hence $d_X(y, z) \geqslant 3 \cdot 2^{-i}$ and thus
$$d_X(p, q) \geqslant d(y, z) - d(y, p) - d(z, q) > 2^{-i} \geqslant 2^{-n-j+1}.$$

□

Remark. Why $n+j$? By definition, if $B_r(x)\cap V_{\beta,i}\neq\varnothing$, there exists y such that $B_r(x)\cap B_{2^{-i}}(y)\neq\varnothing$, i.e, $y\in B_{r+2^{-i}}(x)$. If r is picked so small that $r+2^{-i}\leqslant 2^{-j}$, then

$$y\in B(x,r+2^{-i})\subseteq B_{2^{-j}}(x)\subseteq V_{\alpha,n}$$

(ii) then forces $n\geqslant i$.

Hence, if $n<i$ and $j<i$, we can choose an $r\in(0,2^{-j}-2^{-i}]$ such that $B_r(x)\cap V_{\beta,i}=\varnothing$. Therefore for all $i\geqslant n+j+1$, we can choose r such that $B_r(x)\cap V_{\beta,i}=\varnothing$. Nevertheless, it's possible to choose other lower bound of i, $\max\{n,j\}+1$ for instance.

We list some properties of a paracompact space.

Proposition A.9.2. Let X be a paracompact space.

1. A closed subspace of X is paracompact.
2. X is normal.

Another notion that is closely related to the paracompactness is the following.

Definition. Let $(U_i)_{i\in I}$ be an open cover of the space X. A **partition of unity** subordinate to this cover is a collection continuous map $\{f_j : X\to[0,1]\mid j\in J\}$ such that

1. the collection $\{\operatorname{supp} f_j\}_j$ is a locally finite closed refinement of $(U_i)_i$, and
2. $\sum\limits_j f_j(x)=1$ for each $x\in X$.

Theorem A.9.3. A space is paracompact if and only if for every open cover there exists a partition of unity subordinate to it.

Theorem A.9.4. An LCH space X is paracompact if and only if it is a disjoint union of open σ-compact subspace.

Proof. We first prove the if part. It is clear that a disjoint union of paracompact spaces is paracompact, so we may assume X itself is σ-compact. Suppose $X=\bigcup\limits_{n\geqslant 1}K_n$ with each K_n compact. By some modification we may assume $K_n\subseteq\operatorname{int}K_{n+1}$. Define compact sets $L_1=K_1$ and $L_n=K_n\backslash\operatorname{int}K_{n-1}$ for $n\geqslant 2$. Then L_i can only intersect with L_{i-1} and L_{i+1} nontrivially. By Lemma A.8.1 we can find compact sets M_i whose interiors contain L_i and only intersect with M_{i-1} and M_{i+1}.

Now let an open cover of X be given. Consider the induced cover on L_i, and pick a finite open refinement such that each cover set is contained in M_i. Then these finite covers taken together provide a locally finite open refinement of the original cover. □

Theorem A.9.5. Let X be an second countable LCH space. Then the one-point compactification X^+ is second countable. In particular, X is σ-compact, and is paracompact by Theorem A.9.4.

Proof. Let $\mathcal{B}$ be a countable basis for X and let $K\subseteq X$ be compact. Each point x in K has a compact neighborhood N, and there exists $U_x\in\mathcal{B}$ with $x\in U_x\subseteq N$. By compactness there exists a finite subset I of K such that $K\subseteq\bigcup\limits_{x\in I}U_x$. Put $V=X\backslash\bigcup\limits_{x\in I}\overline{U_x}$. Then $V\cup\{\infty\}$ is a neighborhood of ∞ in X^+ contained in the arbitrary neighborhood $X^+\backslash K$. This shows X^+ is second countable, and particularly X is σ-compact. □

A.10 The Stone-Weierstrass Theorem

Theorem A.10.1. Let X be LCH and suppose $A \subseteq C_0(X)$ is a $\mathbb{C}$-subalgebra of $C_0(X)$ such that

(i) A separates points, i.e., for $x \neq y \in X$ there exists $f \in A$ with $f(x) \neq f(y)$,

(ii) A is nowhere vanishing, i.e., for $x \in X$ there exists $f \in A$ with $f(x) \neq 0$, and

(iii) A is closed under complex conjugation.

Then A is dense in $C_0(X)$ in sup norm.

We begin the proof by making two reductions. By (iii), the $\mathbb{R}$-subspace

$$A' = \{f \in C_0(X) \mid \overline{f} = f\} = C_0(X, \mathbb{R}) \cap A \subseteq A$$

is an $\mathbb{R}$-rational structure of A, i.e., the canonical map $A' \otimes_{\mathbb{R}} \mathbb{C} \to A$ is a $\mathbb{C}$-isomorphism. It is clear that A' still satisfies (i) and (ii). Note that the $\mathbb{C}$-isomorphism

$$C_0(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} C_0(X)$$

is a topological isomorphism. Indeed, this is continuous by triangle inequality, and hence an isomorphism as it is surjective and by open mapping theorem; alternatively, one can simply construct an inverse and show it is continuous. Hence it suffices to show A' is dense in $C_0(X, \mathbb{R})$. It is then reduced to the

Theorem A.10.2. Let X be LCH and suppose $A \subseteq C_0(X, \mathbb{R})$ is a $\mathbb{R}$-subalgebra of $C_0(X, \mathbb{R})$ such that

(i) A separates points, i.e., for $x \neq y \in X$ there exists $f \in A$ with $f(x) \neq f(y)$ and

(ii) A is nowhere vanishing, i.e., for $x \in X$ there exists $f \in A$ with $f(x) \neq 0$.

Then A is dense in $C_0(X, \mathbb{R})$ in sup norm.

To make further reduction, let $X^+ = X \cup \{\infty\}$ be the one-point compactification of X. The inclusion $X \to X^+$ induces an identification

$$C_0(X, \mathbb{R}) = \{f \in C(X^+, \mathbb{R}) \mid f(\infty) = 0\}.$$

Since the evaluation map $C(X^+, \mathbb{R}) \to \mathbb{R}$ at ∞ is continuous, $C_0(X, \mathbb{R})$ is closed. Let A' be the linear span of A along with all constant functions. Then A' is a subalgebra of $C(X^+, \mathbb{R})$, and clearly (i) and (ii) hold for A'. If A' is dense in $C(X^+, \mathbb{R})$, then

$$\overline{A} = \overline{A' \cap C_0(X, \mathbb{R})} = \overline{A'} \cap \overline{C_0(X, \mathbb{R})} = C_0(X, \mathbb{R})$$

so that A is dense in $C_0(X, \mathbb{R})$. Hence we only need to show

Theorem A.10.3. Let X be compact suppose $A \subseteq C(X, \mathbb{R})$ is a $\mathbb{R}$-subalgebra of $C(X, \mathbb{R})$ such that

(i) A separates points, i.e., for $x \neq y \in X$ there exists $f \in A$ with $f(x) \neq f(y)$ and

(ii) A is nowhere vanishing, i.e., for $x \in X$ there exists $f \in A$ with $f(x) \neq 0$.

Then A is dense in $C(X, \mathbb{R})$ in sup norm.

Tensoring with $\mathbb{C}$ yields

Corollary A.10.3.1. Let X be compact and suppose $A \subseteq C(X)$ is a $\mathbb{C}$-subalgebra of $C(X)$ such that

(i) A separates points, i.e., for $x \neq y \in X$ there exists $f \in A$ with $f(x) \neq f(y)$,

(ii) A is nowhere vanishing, i.e., for $x \in X$ there exists $f \in A$ with $f(x) \neq 0$, and

(iii) A is closed under complex conjugation.

Then A is dense in $C(X)$ in sup norm.

The version that we are going to prove the Theorem A.10.3. We break the proof into several lemmas.

Lemma A.10.4 (Dini's)**.** Let X be a compact space and let $(f_n)_n \subseteq C(X, \mathbb{R})$ be an increasing sequence that pointwise converges to a continuous function $f : X \to \mathbb{R}$. Then $f_n \to f$ uniformly.

Proof. Let $\varepsilon > 0$ be fixed. For each point $x \in X$ pick an N_x such that $|f(x) - f_{N_x}(x)| = f(x) - f_{N_x}(x) < \varepsilon$. For each x, define $U_x := \{y \in X \in f(y) - \varepsilon < f_{N_x}(y)\} = (f - f_{N_x})^{-1}(-\infty, \varepsilon)$. Then $(U_x)_{x \in X}$ forms an open cover of X, and by compactness we can find $x_1, \ldots, x_n \in X$ such that $X = U_{x_1} \cup \cdots \cup U_{x_n}$. If we let $N = N_\varepsilon = \max\limits_{1 \leqslant m \leqslant n} N_{x_m}$, we have $f(x) - \varepsilon < f_N(x)$ for all $x \in X$, so that $\|f - f_n\|_X < \varepsilon$ for $n \geqslant N$. This shows the uniformity. $\square$

Lemma A.10.5. Let $A \subseteq C(X, \mathbb{R})$ be an $\mathbb{R}$-algebra. If $f \in \overline{A}$, then $|f| \in \overline{A}$. Consequently, if $f, g \in \overline{A}$, then $\max\{f, g\}$, $\min\{f, g\} \in \overline{A}$.

Proof. Replacing A with $\overline{A}$, we assume A is closed. Let $0 \neq f \in A$; replacing f with $f / \|f\|_X$, we can assume $-1 \leqslant f \leqslant 1$. Then $0 \leqslant f^2 \leqslant 1$.

Define inductively a sequence $(p_n)_{n \geqslant 1}$ of polynomials on $[0, 1]$ such that $p_1 \equiv 0$ and

$$p_{n+1}(x) = p_n(x) - \frac{1}{2}(p_n(x)^2 - x), \qquad x \in [0, 1].$$

We claim $(p_n)_{n \geqslant 1}$ is an increasing sequence that converges pointwise to $\sqrt{x}$ on $[0, 1]$. We have

$$p_{n+1}(x) - \sqrt{x} = (p_n(x) - \sqrt{x}) \left(1 - \frac{1}{2}\left(p_n(x) + \sqrt{x}\right)\right)$$

Since $0 \leqslant p_1 \leqslant \sqrt{x}$, by induction we see $0 \leqslant p_n \leqslant \sqrt{x}$ for all $n \geqslant 1$. In particular, we see $(p_n)_{n \geqslant 1}$ is increasing. For each $x \in [0, 1]$, the sequence $(p_n(x))_{n \geqslant 1}$ is increasing and bounded above by $\sqrt{x}$, so it converges to a value, say, $g(x)$. This defines a function $g : X \to [0, 1]$, and since

$$0 = g(x) - g(x) = \lim_{n \to \infty}(p_{n+1}(x) - p_n(x)) = -\frac{1}{2}\left(\lim_{n \to \infty} p_n(x)^2 - x\right) = -\frac{1}{2}(g(x)^2 - x)$$

we see $g(x) = \sqrt{x}$. It follows from Lemma A.10.4 that $p_n \to \sqrt{x}$ uniformly. Define $f_n = p_n(f(x)^2)\,(n \geqslant 1)$. Then $f_n \to g(f(x)^2) = \sqrt{f(x)^2} = |f|$ uniformly. Since each f_n is a linear combination of powers of f, $f_n \in \overline{A}$, and hence $|f| \in \overline{A}$. $\square$

Lemma A.10.6. Let $x \neq y \in X$ and $a, b \in \mathbb{R}$. There exists $f \in A$ such that $f(x) = a$, $f(y) = b$.

Proof. Since A separates points, we can find $h \in A$ such that $h(x) \neq h(y)$. Since A vanishes nowhere, we can find g_1, $g_2 \in A$ such that $g_1(x) \neq 0 \neq g_2(y)$. Put $g = g_1^2 + g_2^2 \in A$ and for each $\alpha, \beta \in \mathbb{R}$ consider the function $k_{\alpha,\beta} := \alpha g + \beta g h \in A$. Since $\det \begin{pmatrix} g(x) & g(x)h(x) \\ g(y) & g(y)h(y) \end{pmatrix} = g(x)g(y)(h(y) - h(x)) \neq 0$, we can find $\alpha, \beta \in \mathbb{R}$ such that $k_{\alpha,\beta}(x) = a$ and $k_{\alpha,\beta}(y) = b$. Then $f = k_{\alpha,\beta}$ does the job. $\square$

Proof. (of Theorem A.10.3.) Let $F \in C(X, \mathbb{R})$ and $\varepsilon > 0$. Let $p \in X$ be fixed. By Lemma A.10.6 for $q \neq p$ we can find $h_{p,q} \in A$ such that $h_{p,q}(p) = F(p)$, $h_{p,q}(q) = F(q)$. If $q = p$, pick $h_{p,q} = F$. Set

$$h'_{p,q} = h_{p,q} - F + \varepsilon;$$

then $h'_{p,q}(q) = \varepsilon$ for each $q \in X$. Since it is continuous, we can find a neighborhood V_q of q such that $h'_{p,q}(x) > 0$ for $x \in V_q$. The collection $\{V_q \mid q \in X\}$ forms an open cover of X, so by compactness we can find $q_1, \ldots, q_n \in X$ such that $X = V_{q_1} \cup \cdots \cup V_{q_n}$. Put $h_p = \max\{h_{p,q_1}, \ldots, h_{p,q_n}\}$, which lies in $\overline{A}$ by Lemma A.10.5. Then for $x \in X$, we have

$$h_p(x) - F(x) + \varepsilon \geqslant h_{p,q_i}(x) - F(x) + \varepsilon > 0$$

where $i \in [n]$ satisfies $x \in V_{q_i}$. Also, we have $h_p(p) = F(p)$. For each $p \in X$, set

$$h'_p = F - h_p + \varepsilon.$$

Then $h'_p(p) = \varepsilon > 0$, so by continuity we can find a neighborhood U_p of p such that $h'_p(x) > 0$ for $x \in U_p$. Again $\{U_p \mid p \in X\}$ forms an open cover of X, so there exists $p_1, \ldots, p_m$ such that $X = U_{p_1} \cup \cdots \cup U_{p_m}$. If we put $h = \min\{h_{p_1}, \ldots, h_{p_m}\}$, which lies in $\overline{A}$ by Lemma A.10.5. Then for $x \in X$, we have

$$F(x) - h(x) + \varepsilon \geqslant F(x) - h_{p_i}(x) + \varepsilon > 0$$

where $i \in [m]$ satisfies $x \in U_{p_i}$. Since

$$h(x) - F(x) + \varepsilon = \min_{i \in [m]} h_{p_i}(x) - F(x) + \varepsilon > 0$$

for each $x \in X$, these together show that $\|F - h\|_X < \varepsilon$. Since $h \in \overline{A}$ and $\varepsilon > 0$ is arbitrary, this proves the density of A in $C(X, \mathbb{R})$. ☐

A.11 Isometry of locally compact metric spaces

In this section, fix a metric space (X, d).

Definition. A map $f : X \to X$ is called an **isometry** if it is bijective and $d(f(x), f(y)) = d(x, y)$ for any $x, y \in X$.

Denote by $I(X) = I(X, d)$ the set of all isometries on (X, d). It follows from the definition that $I(X) \subseteq \mathrm{Isom}_{\mathbf{Top}}(X, X)$, and $I(X)$ is a subgroup under function composition. We topologize $I(X)$ using the subspace topology inherited from the compact-open topology on X^X (c.f. Section 5.1). Hence it has a subbasis consisting of sets of the form

$$L(K, U) := \{f \in I(X) \mid f(K) \subseteq U\}$$

where K (resp. U) runs over all compact (resp. open) subsets of X.

In the following we further assume that (X, d) is connected locally compact.

Lemma A.11.1. (X, d) is separable.

Proof. By Theorem A.9.1 and Theorem A.9.4, X is σ-compact. But a compact metric space is already separable, being a countable union of separable spaces, X is itself a separable space. ☐

Lemma A.11.2. $I(X)$ is a second countable Hausdorff topological space.

Proof. That $I(X)$ is Hausdorff follows from the fact that X is Hausdorff, and that $I(X)$ is second countable follows from Lemma 5.1.2 and Lemma A.11.1. $\square$

Lemma A.11.3. Let $(f_n)_n$ be a sequence in $I(X)$. If $(f_n)_n$ is pointwise Cauchy on a subset $A \subseteq X$, then $(f_n)_n$ also converges pointwise on the closure $\overline{A}$ of A.

Proof. Let $p \in \overline{A}$ and pick $r > 0$ such that the open ball $B_r(p)$ has compact closure. Let $0 < \varepsilon < r$ be given, pick a point $p' \in B_{\varepsilon/3}(p) \cap A$ and $N \gg 0$ such that $d(f_n(p'), f_m(p')) < \dfrac{\varepsilon}{3}$ whenever $n, m \geqslant N$. Then for $n, m \geqslant N$

$$\begin{aligned} d(f_n(p), f_m(p)) &\leqslant d(f_n(p), f_n(p')) + d(f_n(p'), f_m(p')) + d(f_m(p'), f_m(p)) \\ &< d(p, p') + \frac{\varepsilon}{3} + d(p', p) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence $(f_n(p))_{n \geqslant N} \subseteq B_\varepsilon(f_N(p))$. Since f_N is an isometry, $B_\varepsilon(f_N(p))$ has compact closure as $B_r(p)$ does. It follows that $(f_n(p))_{n \geqslant N}$ has a convergent subsequence in $B_\varepsilon(f_N(p))$ converging to a point $p^* \in \overline{B_\varepsilon(f_N(p))}$. The above estimate then shows that $f_n(p) \to p^*$ as well. $\square$

Lemma A.11.4. Let $(f_n)_n$ be a sequence in $I(X)$ such that there exists a point $x \in X$ such that $(f_n(x))_n \subseteq X$ is convergent. Then for any $p \in X$, the sequence $(f_n(p))_n \subseteq X$ has compact closure.

Proof. Put

$$S = \{p \in X \mid (f_n(p))_n \subseteq X \text{ has compact closure.}\} \subseteq X$$

Since $(f_n(x))_n$ is convergent and X is LCH, it is easy to see $x \in S$. We are going to show S is closed and open, so that $S = X$ by connectedness of X.

- S is closed. Let $(p_n)_n \subseteq X$ be a sequence. By a diagonal argument, we may find a subsequence $(f_{n_k})_k$ of $(f_n)_n$ such that $(f_{n_k}(p_n))_k$ is convergent for any n. By Lemma A.11.3, $(f_{n_k})_k$ also converges pointwise on any limit point on $(p_n)_n$.

- S is open. Let $p \in S$ and pick $r > 0$ such that $\overline{B_r(p)}$ is compact. We claim $B_{r/4}(p) \subseteq S$. Let $q \in B_{r/4}(p)$, and we must show $(f_n(q))_n$ has compact closure.

  By Corollary A.5.3.1, it is enough to show that every subsequence of $(f_n(q))_n$ admits a convergent subsequence. Let $(\varphi_n(q))_n$ be any subsequence of $(f_n(q))_n$. Since $(\varphi_n(p))_n$ has compact closure, we can find a convergent subsequence $(\varphi_{n_k}(p))_k$ converging to, say p^*; by passing to a subsequence, we may assume $(\varphi_{n_k}(p))_k \subseteq B_{r/4}(p^*)$. Then

  $$d(\varphi_{n_k}(q), p^*) \leqslant d(\varphi_{n_k}(q), \varphi_{n_k}(p)) + d(\varphi_{n_k}(p), p^*) < d(q, p) + \frac{r}{4} < \frac{r}{4} + \frac{r}{4} = \frac{r}{2}$$

  so that $(\varphi_{n_k}(q))_k \subseteq B_{r/2}(p^*)$. The latter set has compact closure, as $\varphi_{n_k}(p) \in B_{r/4}(p^*)$ implies $B_{r/2}(p^*) \subseteq B_r(\varphi_{n_k}(p)) = \varphi_{n_k}(B_r(p))$, which has compact closure as $B_r(p)$ does. Hence $(\varphi_{n_k}(q))_k$, and thus $(\varphi_n(q))_n$, has a convergent subsequence.

$\square$

Lemma A.11.5. Let $(f_n)_n$ be a sequence in $I(X)$ pointwise converging to a function $f : X \to X$. Then $f \in I(X)$ and $f_n \to f$ in the compact-open topology of $I(X)$.

Proof. Since the metric $d : X \times X \to \mathbb{R}$ is continuous, for $x, y \in X$, we have

$$d(f(x), f(y)) = \lim_{n\to\infty} d(f_n(x), f_n(y)) = \lim_{n\to\infty} d(x, y) = d(x, y).$$

This shows f preserves metric. Let K be a compact subset of X and $\varepsilon > 0$. Pick a finite subset $P \subseteq K$ such that $K \subseteq \bigcup_{p\in P} B_{\varepsilon/3}(p)$. Let $x \in K$ and pick $p \in F$ such that $d(x, p) < \frac{\varepsilon}{3}$. Then

$$d(f_n(p), f(p)) \leqslant d(f_n(p), f_n(x)) + d(f_n(x), f(x)) + d(f(x), f(p)) = d(p, x) + d(f_n(x), f(x)) + d(x, p) < \varepsilon$$

for $n \geqslant N$, where N is chosen so that $d(f_n(x), f(x)) < \frac{\varepsilon}{3}$ whenever $n \geqslant N$. By Lemma 5.1.3, this shows $f_n \to f$ in $\mathrm{Hom}_{\mathbf{Top}}(X, X)$. It remains to show $f \in I(X)$, i.e., $f(X) = X$.

Let $p \in X$ and $q = f(p) \in X$. Then

$$0 = d(q, f(p)) = \lim_{n\to\infty} d(q, f_n(p)) = \lim_{n\to\infty} d(f_n^{-1}(q), p)$$

implying that $(f_n^{-1}(q))_n$ converges to $p \in X$. By Lemma A.11.4, the sequence $(f_n^{-1}(p))_n$ has compact closure for any $p \in X$. Since X is separable, by a diagonal argument we may find a subsequence $(f_{n_k}^{-1})_k$ converging pointwise on a countable dense subset of X, whence on the whole X by Lemma A.11.3. Let $p' = \lim_{k\to\infty} f_{n_k}^{-1}(p)$; then

$$\lim_{k\to\infty} d(f_{n_k}(p'), p) = \lim_{k\to\infty} d(p', f_{n_k}^{-1}(p)) = d(p', p') = 0$$

so that $f(p') = \lim_{k\to\infty} f_{n_k}(p') = p$. Since p is arbitrary, this shows $f(X) = X$. ☐

Lemma A.11.6. Let $(f_n)_n$ be a sequence in $I(X)$ such that there exists a point $x \in X$ such that $(f_n(x))_n \subseteq X$ is convergent. Then $(f_n)_n$ admits a convergent subsequence in the compact-open topology of $I(X)$.

Proof. By Lemma A.11.4 and the fact that X is separable, we may use a diagonal argument to obtain a subsequence $(f_{n_k})_k$ of $(f_n)_n$ that converges pointwise on a countable dense subset of X, whence on the whole X by Lemma A.11.3. But then Lemma A.11.5 implies that $(f_{n_k})_k$ is convergent in the compact-open topology of $I(X)$. ☐

Theorem A.11.7. Let (X, d) be a connected locally compact metric space.

(i) The group of isometries $I(X)$ on X, equipped with the compact-open topology, is an LCH group, and its action on X is continuous.

(ii) For any $x \in X$, the isotropy subgroup of x is a compact subgroup of $I(X)$.

Proof. Let $(f_n)_n$ and $(g_n)_n$ be two sequences in $I(X)$ converging to f and g in compact-open topology. For any $x \in X$,

$$\begin{aligned} d(f_n(g_n(x)), f(g(x))) &\leqslant d(f_n(g_n(x)), f_n(g(x))) + d(f_n(g(x)), f(g(x))) \\ &= d(g_n(x), g(x)) + d(f_n(g(x)), f(g(x))) \to 0 \end{aligned}$$

as $n \to \infty$. Hence $f_n \circ g_n \to f \circ g$ pointwise, and thus in compact-open topology by Lemma A.11.5. This show the multiplication is continuous. Also,

$$d(f_n^{-1}(x), f^{-1}(x)) = d(x, f_n(f^{-1}(x))) = d(f(f^{-1}(x)), f_n(f^{-1}(x))) \to 0$$

as $n \to \infty$, so $f_n^{-1} \to f$ in compact-open topology again by Lemma A.11.5. This shows the inversion is continuous. In conclusion, $I(X)$ is a topological group.

Next, we show the action map $I(X) \times X \to X$ is continuous. Let $(f_n)_n$ be a sequence in $I(X)$ converging to $f \in I(X)$ and $(p_n)_n$ a sequence to X converging to $p \in X$. Then

$$d(f_n(p_n), f(p)) \leqslant d(f_n(p_n), f_n(p)) + d(f_n(p), f(p)) = d(p_n, p) + d(f_n(p), f(p)) \qquad (\spadesuit)$$

Let $\varepsilon > 0$ be given. Pick $N \gg 0$ such that $d(p_n, p) < \dfrac{\varepsilon}{2}$ for $n \geqslant N$. Put $K = \{p_n, p \mid n \geqslant N\}$, which is a compact set by Corollary A.5.3.1. Consider the subbase element $L(K, B_{\varepsilon/2}(f(p)))$ of $I(X)$; then $f \in L(K, B_{\varepsilon/2})(f(p))$, as $d(f(p_n), f(p)) = d(p_n, p) < \dfrac{\varepsilon}{2}$. Take $N' \gg 0$ such that $f_n \in L(K, B_{\varepsilon/2}(f(p)))$ for $n \geqslant N'$. From the estimate $(\spadesuit)$, we see $d(f_n(p_n), f(p)) < \varepsilon$ for $n \geqslant N + N'$. Hence $f_n(p_n) \to f(p)$, proving the continuity.

It remains to show $I(X)$ is locally compact. Let $x \in X$ and take an relatively compact open neighborhood U of x. By Theorem A.4.2.(v), Lemma A.11.2 and Lemma A.11.6, the open set $L(\{x\}, U)$ has compact closure. Finally, since the stabilizer $\{f \in I(X) \mid f(x) = x\}$ is contained in $L(\{p\}, U)$, to show it is compact it suffices to show it is closed. If $f \in I(X)$ is such that $f(x) \neq x$, pick any open neighborhood U of $f(x)$ not containing x and any compact neighborhood K of x contained in $f^{-1}(U)$. Then $f \in L(K, U)$, and for any $g \in L(K, U)$, we have $g(x) \subseteq U \notin x$, so $g(x) \neq x$. $\square$

Appendix B

Uniformity

B.1 Basics

Definition. A **uniform structure**, or **uniformity**, on a set X consists of a collection $\mathcal{U} \subseteq 2^{X\times X}$ satisfying the following axioms

1. Every subset of $X \times X$ containing an element of $\mathcal{U}$ lies in $\mathcal{U}$
2. $\mathcal{U}$ is closed under finite intersection.
3. Every element in $\mathcal{U}$ contains the diagonal Δ of X.
4. If $V \in \mathcal{U}$, the $V^{-1} = \{(y,x) \in X \times X \mid (x,y) \in V\} \in \mathcal{U}$.
5. For each $V \in \mathcal{U}$, there exists $W \in \mathcal{U}$ such that

$$W^2 = W \circ W := \{(x,z) \in X \times X \mid \exists\, y \in X\, [(x,y) \in W \wedge (y,z) \in W]\} \subseteq V$$

An element of $\mathcal{U}$ is called an **entourage**. A set together with a uniformity is called a **uniform space**.

- If X is nonempty, then 3. implies $\varnothing \notin \mathcal{U}$, so that $\mathcal{U}$ is a filter on $X \times X$.
- The conjunction of 4. and 5. is equivalent to the following axiom:

6. For each $V \in \mathcal{U}$ there exists $W \in \mathcal{U}$ such that $W \circ W^{-1} \subseteq V$.

If V is an entourage of a uniformity on X, we say x and x' are V**-close** if $(x, x') \in V$.

Definition. A **fundamental system of entourages** of a uniformity is a collection $\mathcal{B}$ of entourages such that every entourage contains an element in $\mathcal{B}$.

- An entourage V such that $V = V^{-1}$ is called **symmetric**. The symmetric entourage form a fundamental system of entourages. This follows from 2. and 4.
- If $\mathcal{B}$ is a fundamental system of entourages, then so is $\{V^n \mid V \in \mathcal{B}\}$ for each $n \in \mathbb{N}$. This follows from 5.

Proposition B.1.1. A collection $\mathcal{B} \subseteq 2^{X\times X}$ is a fundamental system of entourages of a uniformity on X if and only if $\mathcal{B}$ satisfies the following axioms

1. The intersection of two elements of $\mathcal{B}$ contains an element of $\mathcal{B}$
2. Every element of $\mathcal{B}$ contains the diagonal Δ of X.
3. For each $V \in \mathcal{B}$ there exists $V' \in \mathcal{B}$ such that $V' \subseteq V^{-1}$.
4. For each $V \in \mathcal{B}$ there exists $W \in \mathcal{B}$ such that $W^2 \subseteq V$.

In particular, if X is nonempty, $\mathcal{B}$ forms a fundamental system of the filter formed by the entourages of this structure.

Proof. Given $\mathcal{B} \subseteq 2^{X \times X}$, we have to define a uniformity $\mathcal{U}$ on X such that $\mathcal{B}$ is a fundamental system of $\mathcal{U}$. Let

$$\mathcal{U} := \{U \subseteq X \times X \mid \exists\, B \in \mathcal{B}\, [B \subseteq U]\}$$

We check $\mathcal{U}$ defines a uniformity on X.

1. Let $U \subseteq X \times X$ with $V \subseteq U$ for some $V \in \mathcal{U}$. Then $B \subseteq V \subseteq U$ for some $B \in \mathcal{B}$.
2. Let $U_1, \ldots, U_n \in \mathcal{U}$. Then $B_i \subseteq U_i$ for some B_i. We can find $B \in \mathcal{B}$ such that $B \subseteq B_1 \cap \cdots \cap B_n$, and thus $B \subseteq U_1 \cap \cdots \cap U_n$.
3. Every element in $\mathcal{U}$ contains Δ for every element of $\mathcal{B}$ does.
4. For each $V \in \mathcal{U}$, we can find $B_1, B_2 \in \mathcal{B}$ such that $B_2 \subseteq V$ and $B_1 \subseteq B_2^{-1}$, so that $B_1 \subseteq V^{-1}$.
5. For each $V \in \mathcal{U}$, take $B \in \mathcal{B}$ such that $B \subseteq V$ and $B' \in \mathcal{B}$ such that $B'^2 \subseteq B$. Then $B'^2 \subseteq V$.

□

Example B.1.2.

1. Let (X, d_X) be a metric space. For each $\varepsilon > 0$, put

   $$V_\varepsilon := \{(x, y) \in X \mid d_X(x, y) < \varepsilon\}$$

   When $X = \mathbb{R}$ with euclidean distance, we call the uniformity generated by the V_ε the **additive uniformity** on $\mathbb{R}$.

2. Let X be a set and $R \subseteq X \times X$ be an equivalence relation. Then $\Delta \subseteq R$ and $R^2 = R^{-1} = R$, so R alone is a fundamental system of entourages of a uniformity on X. In particular, if we take $R = \Delta$, then the entourages of the corresponding uniformity are all subsets of $X \times X$ containing Δ; this uniformity is called the **discrete uniformity** on X, and X equipped with this uniformity is called a **discrete uniformity space**.

3. For each rational prime p, we can define a uniformity on $\mathbb{Z}$: for each $n \in \mathbb{N}$, define

   $$W_n := \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x \equiv y \pmod{p^n}\}$$

   One easily checks that the W_n form a fundamental system of entourages of a uniformity on $\mathbb{Z}$, called the **p-adic uniformity**.

B.1.1 Topology of a Uniform Space

Proposition B.1.3. Let X be a set with a uniform structure $\mathcal{U}$. For each $x \in X$ let $\mathcal{B}(x) = \{V(x) \mid V \in \mathcal{U}\}$, where $V(x) := \{y \in X \mid (x, y) \in V\}$. Then there is a unique topology on X such that for each $x \in X$, $\mathcal{B}(x)$ is all neighborhoods of x in this topology.

Proof. Let $\mathcal{T} := \{U \subseteq X \mid \forall\, x \in U\, [U \in \mathcal{B}(x)]\}$. We check $\mathcal{T}$ defines a topology on X.

1. Let $(U_i)_{i \in I} \subseteq \mathcal{T}$ and $x \in U := \bigcup_{i \in I} U_i$. Write $U_i = V_i(x)$ for some $V \in \mathcal{U}$ for those U_i containing x. Then
$$W := (\{x\} \times U) \cup ((X \backslash \{x\}) \times X)) \subseteq X \times X$$
contains a (in fact, every) V_i, so $W \in \mathcal{U}$. By construction $W(x) = U$, so that $U \in \mathcal{T}$.

2. Let $U_1, U_2 \in \mathcal{T}$ and $x \in U := U_1 \cap U_2$. Write $U_1 = V_1(x)$ and $U_2 = V_2(x)$. Then $W := (\{x\} \times U) \cup ((X \backslash \{x\}) \times X) \subseteq X \times X$ contains $V_1 \cap V_2 \in \mathcal{U}$, so that $W \in \mathcal{U}$. By construction $W(x) = U$, so that $U \in \mathcal{T}$.

3. $X \in \mathcal{T}$, for $X \times X \in \mathcal{U}$.

Hence $\mathcal{T}$ defines a topology on X, and it remains to show that $\mathcal{B}(x)$ is the set of all neighborhoods of x. We must show if $U \subseteq \mathcal{B}(x)$ for some x, then U contains an element of $\mathcal{T}$. In fact, we claim that
$$x \in V := \{y \in X \mid U \in \mathcal{B}(y)\} \in \mathcal{T}$$
and is contained in U.

- $x \in V$ for $U \in \mathcal{B}(x)$.
- Let $y \in V$. Then $U \in \mathcal{B}(y)$ so that $y \in U$. Hence $V \subseteq U$.
- Let $y \in V$. We need to show $V \in \mathcal{B}(y)$. We contend the following property:

  if $V \in \mathcal{B}(a)$, then there is a set $W \in \mathcal{B}(a)$ such that $V \in \mathcal{B}(b)$ for each $b \in W$.

  Indeed, write $V = V'(a)$ for some $V' \in \mathcal{U}$. Let $V'' = V' \cap (V')^{-1} \in \mathcal{U}$ and put $W := V''(a)$. For each $b \in W$, define $V''' := (\{b\} \times V) \cup ((X - \{b\}) \times X)$. Since $V'' \subseteq V'''$, $V''' \in \mathcal{U}$, and by construction $V = V'''(b)$, so $V \in \mathcal{B}(b)$. With help of this property, we can find $W \in \mathcal{B}(y)$ such that $U \in \mathcal{B}(z)$ for all $z \in W$; it follows by definition that $y \in W \subseteq V$, and thus $V \in \mathcal{T}$.

$\square$

Definition. The topology on X defined above is called **the topology induced by the uniform structure $\mathcal{U}$.**

- A uniform space is said to be Hausdorff, compact, or locally compact, etc., if the induced topology has this property.

Proposition B.1.4. Let X be a uniform space. Let $M \subseteq X \times X$.

1. For every symmetric entourage V, VMV is a neighborhood of M in the product space $X \times X$.

2. The closure of M in $X \times X$ is given by
$$\overline{M} = \bigcap_V VMV$$
where V runs over all symmetric entourages of X.

Proof. $(x, y) \in VMV$ means there exists $(p, q) \in M$ such that $(x, p) \in V$, $(q, r) \in V$, or $(x, y) \in V(p) \times V(q)$ since V is symmetric. This shows 1. The relation $(x, p) \in V$, $(q, r) \in V$ also can be written as $(p, q) \in V(x) \times V(y)$. As V runs over all symmetric entourages, $V(x) \times V(y)$ forms a fundamental system of neighborhoods of (x, y) in $X \times X$; for if U, U' are any two entourages, there is always a symmetric $V \subseteq U \cap U'$ so that $V(x) \times V(y) \subseteq U(x) \times U'(y)$. Thus $V(x) \times V(y)$ meets M for each V if and only if $(x, y) \in \overline{M}$, and 2. follows. $\square$

Corollary B.1.4.1. If $A \subseteq X$ is any subset and V is any symmetric entourage of X, then

$$V(A) := \{x \in X \mid \exists\, a \in A\, [(x, a) \in V]\}$$

is a neighborhood of A in X, and

$$\overline{A} = \bigcap_{V} V(A) = \bigcap_{U \in \mathcal{U}} U(A)$$

where V runs over all symmetric entourages of X and $\mathcal{U}$ denotes the set of all entourages in X.

Proof. It is clear that if $M = A \times A$, then $VMV = V(A) \times V(A)$. The results follow from the above proposition and that fact that $\overline{A \times A} = \overline{A} \times \overline{A}$. $\square$

Corollary B.1.4.2. The interiors (resp. the closures) of the entourages of X in $X \times X$ form a fundamental system of entourages of X.

Proof. If V is an entourage, there is a symmetric entourage W such that $W^3 \subseteq V$. By proposition W^3 is a neighborhood of W, so the interior of V contains W and it is therefore an entourage. Furthermore, we have $W \subseteq \overline{W} \subseteq W^3 \subseteq V$ by proposition, so V contains the closure of some entourage. $\square$

Corollary B.1.4.3. In a uniform space, the sets of closed neighborhoods of a point form a fundamental system of neighborhoods of the point.

Proof. The above corollary tells that for each $x \in X$, the sets $V(x)$ form a fundamental system of neighborhood of x, where V runs over all closed entourages of X, and each $V(x)$ is closed in X. $\square$

Corollary B.1.4.4. A Hausdorff uniform space is regular.

Proposition B.1.5. A uniform space is Hausdorff if and only if the intersection of all the entourages is the diagonal Δ of X.

Proof. Since the closed entourages form a fundamental systems of entourages, it follows that if their intersection is Δ, the Δ is closed, and hence X is Hausdorff. Conversely, suppose X is Hausdorff. Then for any $x \neq y$, there exists an entourage such that $y \notin V(x)$, i.e., $(x, y) \notin V$. Hence Δ is the intersection of all the entourages. $\square$

B.1.2 Uniformly Continuous Maps

Definition. Let X, X' be two uniform spaces. A map $f : X \to X'$ is **uniformly continuous** if for each entourage V' of X', there exists an entourage V of X such that the relation $(x, y) \in V$ implies $(f(x), f(y)) \in V'$.

- If we put $g = f \times f$, then f being uniformly continuous means that $g^{-1}(V')$ is an entourage of X whenever V' is an entourage of X'.

A uniformly continuous map $f : X \to X'$ is called an **isomorphism** if f is bijective and $g(\mathcal{U}) = \mathcal{U}'$, where $\mathcal{U}$ (resp. $\mathcal{U}'$) is the set of all entourages of X (resp. of X').

Example B.1.6.

1. The identity map of a uniform space is uniformly continuous.
2. A constant map of a uniform space into a uniform space is uniformly continuous.
3. Every map from a discrete uniform space into a uniform space is uniformly continuous.

Proposition B.1.7. Every uniformly continuous map is continuous. In particular, every isomorphism of uniform spaces is a homeomorphism.

Proof. Let X, X' be uniform spaces and $f : X \to X'$ be uniformly continuous. Let $U \subseteq X'$ be open and $x \in f^{-1}(U)$. We need to show $f^{-1}(U)$ is a neighborhood of x. Since $f(x) \in U$, $U = V(f(x))$ for some entourage V of X'. Then $(f \times f)^{-1}(V)$ is an entourage of X, and

$$(f\times f)^{-1}(V)(x) = \{y \in X \mid (x,y) \in (f\times f)^{-1}(V)\} = \{y \in X \mid (f(x), f(y)) \in V\} = f^{-1}(V(f(x))) = f^{-1}(U)$$

so $f^{-1}(U)$ is really a neighborhood of x. □

Proposition B.1.8.

1. If $f : X \to X'$ and $g : X' \to X''$ are two uniformly continuous maps, then $g \circ f : X \to X''$ is uniformly continuous.
2. A bijection $f : X \to X'$ is an isomorphism if and only if f and the inverse of f are uniformly continuous.

B.1.3 Comparison of Uniformities

Definition. If $\mathcal{U}_1$ and $\mathcal{U}_2$ are two uniform structures on the same set X, we say $\mathcal{U}_1$ is **finer** than $\mathcal{U}_\in$ (and $\mathcal{U}_\in$ is **coarser** than $\mathcal{U}_1$) if the identity map from $(X,\mathcal{U}_1)$ to $(X,\mathcal{U}_2)$ is uniformly continuous.

- Equivalently, $\mathcal{U}_1$ is finer than $\mathcal{U}_2$ if and only if $\mathcal{U}_2 \subseteq \mathcal{U}_1$.
- If $\mathcal{U}_1$ is finer than $\mathcal{U}_2$, then the topology on X induced by $\mathcal{U}_1$ is finer than that of induced by $\mathcal{U}_2$.

If $\mathcal{U}_1$ is finer than $\mathcal{U}_2$ and $\mathcal{U}_1 \neq \mathcal{U}_2$, we say $\mathcal{U}_1$ is **strictly finer** than $\mathcal{U}_2$. Two uniformities are **comparable** if one is finer than the other.

Example B.1.9.

1. On a set the finest uniformity is the discrete uniformity, and the coarsest uniformity is that in which the set of entourages consists of a single set, namely the whole product set.
2. It can happen that a uniformity $\mathcal{U}_1$ is strictly finer than a uniformity $\mathcal{U}_2$, but their induced topology are identical. Let X be a nonempty set, for each finite partition $\varpi = (A_i)_{1\leqslant i\leqslant n}$ of X, put
$$V_\varpi = \bigcup_{i=1}^{n} A_i \times A_i$$
The set V_ϖ form a fundamental system of entourages of a uniformity $\mathcal{U}$ on X, called the **uniformity of finite partitions** on X. The induced topology on X is the discrete topology, but if X is infinite, it is clear that $\mathcal{U}$ is strictly coarser than the discrete uniformity.

3. If $f : X \to X'$ is uniformly continuous, f remains uniformly continuous if we replace X' by a coarser uniformly or replace X by a finer uniformity.

B.2 Initial Uniformities

Proposition B.2.1. Let X be a set, $(Y_i)_{i\in I}$ a family of uniform spaces and $f_i : X \to Y_i$ $(i \in I)$. For each $i \in I$ let $g_i := f_i \times f_i$. Let $\mathfrak{S} = \left\{g_i^{-1}(V_i) \mid i \in I,\, V_i \text{ is an entourage of } Y_i\right\}$ and B be the set of all finite intersections

$$\mathbf{U}(V_{i_1}, \ldots, V_{i_n}) = g_{i_1}^{-1}(V_{i_1}) \cap \cdots \cap g_{i_1}^{-1}(V_{i_n})$$

of sets of $\mathfrak{S}$. Then B is a fundamental system of entourages of a uniformity $\mathcal{U}$ on X.

1. It is the coarsest uniformity on X for which all f_i are uniformly continuous.

2. A map $h : Z \to X$ of a uniform space Z into X is uniformly continuous if and only if each of the maps $f_i \circ h : Z \to Y_i$ is uniformly continuous.

Proof. B is closed under finite intersection and each element contains the diagonal of X; if $W_i = g_i^{-1}(V_i)$, then $W_i^{-1} = g_i^{-1}(V_i^{-1})$ and $W_i^2 = g_i^{-1}(V_i^2)$; hence B is a fundamental system of entourages of a uniformity $\mathcal{U}$ on X. It follows from definition that each f_i is uniformly continuous and the only if part of 2. holds. Suppose that $f_i \circ h$ is uniformly continuous for each $i \in I$. Consider a set $U := \mathbf{U}(V_{i_1}, \ldots, V_{i_n})$; by hypothesis, for each $1 \leqslant k \leqslant n$, the set $W_k := ((f_{i_k} \circ h) \times (f_{i_k} \circ h))^{-1}(U)$ is an entourage of Z. Put $W = \bigcap_{k=1}^{n} W_k$; then $(h(z), h(z')) \in U$ if $(z, z') \in W$. ☐

Corollary B.2.1.1. The topology on X induced by the initial uniformity $\mathcal{U}$ on X with respect to the maps $(f_i)_{i\in I}$ is also the initial topology on X with respect to the maps $(f_i)_{i\in I}$.

Proposition B.2.2 (Transitivity)**.** Let X be a set, let $(Z_i)_{i\in I}$ be a family of uniform spaces, let $(J_\lambda)_{\lambda\in\Lambda}$ be a partition of I and let $(Y_\lambda)_{\lambda\in\Lambda}$ be a family of sets. For each $\lambda \in \Lambda$, let $h_\lambda \in Y_\lambda^X$; for each $\lambda \in \Lambda$ and each $j \in J_\lambda$, let $g_{j\lambda} \in Z_j^{Y_\lambda}$ and put $f_j = g_{j\lambda} \circ h_\lambda$. Let each Y_λ carry the initial uniformity with respect to $(g_{j\lambda})_{j\in J_\lambda}$. Then the initial uniformity on X with respect to $(f_i)_{i\in I}$ is the same as that with respect to $(h_\lambda)_{\lambda\in\Lambda}$.

Subspace uniformity

Definition. Let X be a uniform space and $A \subseteq X$ a subset. The initial uniformity on A with respect the the natural inclusion $A \subseteq X$ is called the **uniformity induced on A by the uniformity of X**, and A with this uniformity is called the **uniform subspace** of X.

- If $f : X \to Y$ is uniformly continuous, then so is $f|_A : A \to Y$. If $B \subseteq Y$ is a uniform subspace with $f(X) \subseteq B$, then $f|^B : X \to B$ is uniformly continuous.

Proposition B.2.3. Let A be a dense subset of a uniform space X. Then the closures, in $X \times X$, of the entourages of the uniform subspace A form a fundamental system of entourages of X.

Proof. Note that $A \times A$ is dense in $X \times X$. Let V be an open entourage of A; $V = (A \times A) \cap U$ for some open entourage U of X. Then $U \subseteq \overline{V} \subseteq \overline{U}$. ☐

Product uniformity

Definition. If $(X_i)_{i\in I}$ is a family of uniform spaces, the **product uniform space** of this family is the product set $X = \prod_{i\in I} X_i$ endowed with the initial uniformity with respect to the projection $p_i : X \to X_i$.

Proposition B.2.4. Let $f = (f_i) : Y \to \prod_{i\in I} X_i$ be a map from a uniform space to a product uniform space. Then f is uniformly continuous if and only if each f_i is uniformly continuous.

Corollary B.2.4.1. Let $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$ be two families of uniform spaces. For each $i \in I$, let $f_i : X_i \to Y_i$ be a map. If each f_i is uniformly continuous, then so is the product map $f : (x_i) \mapsto (f_i(x_i))$. Conversely, if the X_i are nonempty and f is uniformly continuous, then each f_i is uniformly continuous.

Proposition B.2.5. Let X be a set, let $(Y_i)_{i\in I}$ be two families of uniform spaces, and for each $i \in I$, let $f_i : X_i \to Y_i$ be a map. Let $f : (x) \mapsto (f_i(x))$ be a map from X to $Y = \prod_{i\in I} Y_i$, and let $\mathcal{U}$ be the initial uniformity on X with respect to $(f_i)_{i\in I}$. Then $\mathcal{U}$ is the initial uniformity with respect to $f|^{f(X)} : X \to f(X)$, where $f(X)$ is viewed as a uniform subspace of Y.

Corollary B.2.5.1. For each $i \in I$, let A_i be a subspace of Y_i. Then the uniformity induced on $A = \prod_{i\in I} A_i$ by the product uniformity on $\prod_{i\in I} Y_i$ is the same as the product uniformity of the A_i.

Corollary B.2.5.2. Let $f : X_1 \times X_2 \to Y$ be a uniformly continuous maps from a product uniform space to a uniform space. Then every partial mapping $x_2 \mapsto f(x_1, x_2)$ of X_2 to Y is uniformly continuous.

Inverse limits of uniform spaces

Let $(I, \leqslant)$ be a partially ordered set. For each $\alpha \in I$ let X_α be a uniform space, and for each pair of indices $\alpha \leqslant \beta$, let $f_{\alpha\beta} : X_\beta \to X_\alpha$ be a map.

Definition. Let the notation be as above. $(X_\alpha, f_{\alpha\beta})$ is called an **inverse system of uniform spaces** if it is an inverse system of sets and for each $\alpha \leqslant \beta$, $f_{\alpha\beta}$ is uniformly continuous. The initial uniformity on $X := \varprojlim_{\alpha\in I} X_\alpha$ with respect to the projections $f_\alpha : X \to X_\alpha$ $(\alpha \in I)$ is called the **inverse limit of the uniformities** of the X_α, and is called the **inverse limit of the inverse system of uniform spaces** $(X_\alpha, f_{\alpha\beta})$.

Proposition B.2.6. Let the notation be as above, and let J be a cofinal subset of I. For each $\alpha \in I$ put $g_\alpha := f_\alpha \times f_\alpha$. Then the family of sets $g_\alpha^{-1}(V_\alpha)$, where α runs over J and for each $\alpha \in J$, V_α runs over a fundamental system of entourages of X_α, is a fundamental system of entourages of X.

B.3 Complete Spaces

Definition. Let X be a uniform space.

1. Let V be an entourage. A subset $A \subseteq X$ is called V**-small** if every part of points in A is V-close (i.e., $A \times A \subseteq V$).

2. A filter F on X is called a **Cauchy filter** if for each entourage V of X there exists $A \in F$ which is V-small.

Lemma B.3.1. Let X be a uniform space and V an entourage. If A and B intersect and are V-small, then $A \cup B$ is V^2-small.

Proof. Let $x, y \in A \cup B$ and $z \in A \cap B$. Then $(x, z) \in V$ and $(z, y) \in V$ so that $(x, y) \in V^2$. $\square$

Example B.3.2. Let X be a uniform space.

1. We say an (infinite) sequence $(a_n)_{n \geqslant 1} \subseteq X$ is a **Cauchy sequence** if the associated elementary filter is a Cauchy filter. In other words, $(a_n)_n$ is Cauchy if for each entourage V of X there exists $N \in \mathbb{N}$ such that $(a_n, a_m) \in V$ whenever $n, m \geqslant N$.

2. Every convergent filter F is a Cauchy filter.

   Proof. If x is a point and V is an symmetric entourage of X, then $V(x) \in N_x$ is V^2-small. If F converges to x, we can find $A \in F$ such that $A \subseteq V(x)$. In particular, A is V^2-small. (Note that this is enough as $\{V^2\}$ also forms a fundamental system of entourage.) $\square$

Proposition B.3.3. If $f : X \to X'$ is a uniformly continuous map, then the image under f of any Cauchy filter base on X is a Cauchy filter base on X'.

Proof. Let $g = f \times f$. If V' is an entourage of X', then $g^{-1}(V')$ is an entourage of X, and the image under f of a $g^{-1}(V')$-small set is V'-small; hence the result. $\square$

Proposition B.3.4. Let X be a set, $(Y_i)_{i \in I}$ a family of uniform spaces and for each $i \in I$ let $f_i : X \to Y_i$ be a map. Endow X with the initial uniformity with respect to the f_i. Then a filter base B on X is a Cauchy filter base if and only if $f_i(B)$ is a Cauchy filter base for each $i \in I$.

Proof. The only if part is the proposition above. Suppose $f_i(B)$ is a Cauchy filter base for each $i \in I$. Let $U := \mathbf{U}(V_{i_1}, \ldots, V_{i_n})$ be an entourage of X. By hypothesis for each $1 \leqslant k \leqslant n$ there exists $A_{i_k} \in B$ such that $f_{i_k}(A_{i_k})$ is V_{i_k}-small. Let $M \in B$ be a set contained in all A_{i_k}, $1 \leqslant k \leqslant n$; then M is U-small. $\square$

Corollary B.3.4.1. If a Cauchy filter on a uniform space X induces a filter on a subset A, then this filter is a Cauchy filter on the uniform subspace A.

Corollary B.3.4.2. A filter base B on a product uniform space $\prod_{i \in I} X_i$ is a Cauchy filter if and only if $p_i(B)$ is a Cauchy filter for each $i \in I$, where p_i is the canonical projection.

B.3.1 Minimal Cauchy filters

Definition. The minimal elements (with respect to inclusion) of the set of Cauchy filters on a uniform space are called **minimal Cauchy filters**.

Proposition B.3.5. Let X be a uniform space, and F a Cauchy filter. Then there exists a unique minimal Cauchy filter F_0 coarser than F. If B is a base of F and $\mathfrak{S}$ the fundamental system of symmetric entourages of X, then the sets $V(M)$ with $(M, V) \in B \times \mathfrak{S}$ form a base of F_0.

Proof. The sets $V(M)$ clearly form a base of a filter F_0 on X. Moreover, if M is V-small, then $V(M)$ is V^3-small, so that F_0 is a Cauchy filter and is clearly coarser than F. It remains to show if F' is a Cauchy filter coarser than F, then F' is finer than F_0. For each $M \in B$ and $V \in \mathfrak{S}$, there exists a V-small set $N \in F'$; since $N \in F$ as well, $N \cap M \neq \varnothing$, and thus $N \subseteq V(M)$ so that $V(M) \in F'$. $\square$

In particular, if we take F to be the principal filter generated by $B = \{x\}$, then the resulting minimal Cauchy filter F_0 has a base consisting of the sets $V(x)$, which are precisely the neighborhoods of x in the topology induced by the uniform structure. In sum,

Corollary B.3.5.1. In a uniform space X, for each $x \in X$ the neighborhood filter $\mathcal{B}(x)$ is a minimal Cauchy filter.

Let F be a Cauchy filter and x a cluster point of F. By Lemma A.2.2 there is a filter F' finer than F converging to x. Since F is Cauchy, so is F'. Let F_0 be the unique minimal Cauchy filter coarser than F. Since $\mathcal{B}(x)$ and F_0 are both minimal Cauchy filters coarser than F', it forces that $F_0 = \mathcal{B}(x)$. In particular, $\mathcal{B}(x) \subseteq F$.

Corollary B.3.5.2.

1. Every cluster point of a Cauchy filter F is a limit point of F.

2. Every Cauchy filter that is coarser than a filter converging to x also converges to x.

Let F be a minimal Cauchy filter. If V is an entourage of X, by Corollary B.1.4.2 we can find an open entourage $U \subseteq V$. For each $M \subseteq X$, the set $U(M)$ is open, and is contained in $V(M)$. In view of the proposition, this shows

Corollary B.3.5.3. Let F be a minimal Cauchy filter. Then every set in F has nonempty interior, so that F has a base consisting of open sets.

B.3.2 Complete spaces

Definition. A **complete space** is a uniform space in which every Cauchy filter converges.

- By Proposition B.3.5 and Corollary B.3.5.2, a uniform space is complete if and only if all minimal Cauchy filter converges.

Proposition B.3.6 (Cauchy's criterion)**.** Let F be a filter on a set X, and $f : X \to X'$ be a map to a complete uniform space X'. Then $\lim_F f$ exists if and only if $f(F)$ is a Cauchy filter base.

Proof. Since X' is complete, by (B.3.2) we deduce that $f(F)$ is Cauchy if and only if $f(F)$ is convergent, which is the same as saying that $\lim_F f$ exists. ☐

Let $\mathcal{U}_i$ $(i = 1, 2)$ be a uniformity on a set X, and let $\mathcal{T}_i$ $(i = 1, 2)$ be the induced topology on X. Suppose

(i) $\mathcal{U}_1$ is finer than $\mathcal{U}_2$, and

(ii) there is a fundamental system of entourages of $\mathcal{U}_1$ which are closed in $X \times X$ in the topology $\mathcal{T}_2 \times \mathcal{T}_2$ (for example, this is satisfied when $\mathcal{T}_1 = \mathcal{T}_2$).

Let F be a filter on X. Then F converges in the topology $\mathcal{T}_1$ if and only if it is Cauchy in the uniformity $\mathcal{U}_1$ and converges in the topology in $\mathcal{T}_2$. The only if part is clear, as $\mathcal{T}_2$ is coarser than $\mathcal{T}_1$. Conversely, suppose the if part and let $x \in X$ be a limit of F in the topology $\mathcal{T}_2$. We claim it is also a limit in $\mathcal{T}_1$. Let $V \in \mathcal{U}_1$ be a symmetric entourage that is closed in the topology $\mathcal{T}_2 \times \mathcal{T}_2$. By assumption F contains a V-small set M. Hence, if $x' \in M$, then $M \subseteq V(x')$. Since $V(x')$ is closed in the topology $\mathcal{T}_2$, we see $x \in \overline{M} \subseteq V(x')$, and hence $M \subseteq V^2(x)$. This finishes the proof. In particular,

Corollary B.3.6.1. In the conditions above, if $\mathcal{U}_2$ is a uniformity of a complete space, so is $\mathcal{U}_1$.

Lemma B.3.7. Let X be a uniform space.

1. If X is complete and $A \subseteq X$ is closed, then A is also a complete uniform space.

2. If X is Hausdorff and $A \subseteq X$ is a complete subspace, then A is closed.

Lemma B.3.8. Let X be a uniform space and $A \subseteq X$ a dense subspace such that every Cauchy filter base on A converges in X. Then X is complete.

B.4 Uniformity of Uniform Convergence

Definition. Let X, Y be two sets.

1. If $H \subseteq Y^X$, for each $x \in X$ put

$$H(x) := \{u(x) \in Y \mid u \in H\}$$

2. If Φ is a filter base on Y^X, for each $x \in X$ put

$$\Phi(x) := \{H(x) \mid H \in \Phi\}$$

3. For each $H \subseteq Y^X$ and $A \subseteq X$, put

$$H|_A := \{u|A : A \to Y \mid u \in H\} \subseteq Y^A$$

Definition. Let X be a set and Y a uniform space. For each entourage V of Y, let

$$\mathbf{W}(V) := \big\{(u, v) \in Y^X \times Y^X \mid (u(x), v(x)) \in V \text{ for all } x \in X\big\}$$

As V runs over the set of entourages of Y, the sets $\mathbf{W}(V)$ form a fundamental system of entourages of a uniformity on Y^X, called the **uniformity of uniform convergence**. The topology it induces is called the **topology of uniform convergence**.

- If $V \subseteq V'$ are two entourages of Y, then $\mathbf{W}(V) \subseteq \mathbf{W}(V')$; $\mathbf{W}(V)$ contains the diagonal of Y^X for V contains that of Y; $\mathbf{W}(V)^{-1} = \mathbf{W}(V^{-1})$ for each entourage V of Y; for each entourage V of Y, one has $\mathbf{W}(V)^2 \subseteq \mathbf{W}(V^2)$.

Denote by $(Y^X)_u$ the uniform space Y^X with the uniformity of uniform convergence. If a filter Φ converges to an element u in the topology of uniform convergence on Y^X, Φ is said to **converge uniformly to** u_0.

Definition. Let X be a set, Y a uniform space, and $\mathfrak{S} \subseteq 2^X$. The **uniformity of $\mathfrak{S}$-convergence** is the initial uniformity on Y^X with respect to the restrictions

$$\begin{aligned} Y^X &\longrightarrow (Y^A)_u \\ u &\longmapsto u|_A \end{aligned}$$

where A runs over $\mathfrak{S}$. The uniform space obtained by endowing Y^X with the uniformity of $\mathfrak{S}$-convergence is denoted by $(Y^X)_{\mathfrak{S}}$.

- The topology on Y^X induced by the uniformity of $\mathfrak{S}$-convergence is called the **topology of $\mathfrak{S}$-convergence**; it is the initial topology on Y^X such that the restrictions above are all continuous.

- By definition, a fundamental system of entourages of $(Y^X)_{\mathfrak{S}}$ may be obtained as follows: let B be a fundamental system of V. Then for $(A, V) \in \mathfrak{S} \times B$, define

$$\mathbf{W}(A, V) := \left\{(u, v) \in Y^X \times Y^X \mid (u(x), v(x)) \in V \text{ for all } x \in A\right\}$$

  The finite intersections of the sets $\mathbf{W}(A, V)$ with $(A, V) \in \mathfrak{S} \times B$ form a fundamental system of entourages of $(Y^X)_{\mathfrak{S}}$.

- The uniformity of $\mathfrak{S}$-convergence is unaltered by replacing $\mathfrak{S}$ by the set

$$\mathfrak{S}' := \{U \subseteq X \mid U \subseteq A_1 \cap \cdots \cap A_n \text{ for some } A_i \in \mathfrak{S} \text{ and } n \geqslant 0\}$$

  Thus in study of $\mathfrak{S}$-convergence, we can restrict ourselves to the case where the set $\mathfrak{S}$ satisfies the following two conditions

  (a) Every subset of a set in $\mathfrak{S}$ belongs to $\mathfrak{S}$.

  (b) Every finite union of sets of $\mathfrak{S}$ belongs to $\mathfrak{S}$.

  If (b) is satisfied, we obtained a fundamental system of entourages of $(Y^X)_{\mathfrak{S}}$ by taking all the sets $\mathbf{W}(A, V)$ with $(A, V) \in \mathfrak{S} \times B$, where B is a fundamental system of V.

Appendix C

Algebraic Topology

Definition. Let X, Y be two spaces, and $A \subseteq X$ a subspace.

(i) A **homotopy rel(ative to)** A **from** X **to** Y is a continuous map $F : X \times [0,1] \to Y$ whose restriction $F|_{A\times[0,1]}$ is independent of the second argument. If $A = \varnothing$, we simply say F is a **homotopy**.

(ii) Two continuous maps $f, g : X \to Y$ are **homotopic rel** A if there exist a homotopy rel A $F : X \times [0,1] \to Y$ such that $f = F(\cdot, 0)$ and $g = F(\cdot, 1)$. In this case we write $f \sim g$ rel A.

(iii) Let $f, g : X \to Y$ be continuous maps. If $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$, we say f is a **homotopy equivalence**, and say g is a **homotopy inverse** of f.

C.1 Fundamental groupoids

Definition. A category G is called a **groupoid** if any morphism in G is invertible.

Let X be a topological space, and $x_0, x_1 \in X$. We say two paths $\alpha, \beta : [0,1] \to X$ with $\alpha(i) = x_i = \beta(i)$ $(i = 0,1)$ are **homotopic** if α, β are homotopic rel $\{0,1\}$, say write $\alpha \sim \beta$. Clearly $\sim$ is an equivalence relation, and we write $\pi_1(X; x_0, x_1)$ for the set of homotopy classes of paths from x_0 to x_1. For a path α from x_0 to x_1, we write $[\alpha]$ for its class in $\pi_1(X; x_0, x_1)$.

Let $x_2 \in X$ be still another point. If $\alpha_i : [0,1] \to X$ $(i = 0,1)$ is a path from x_i to x_{i+1}, define $\alpha_1 \bullet \alpha_0 : [0,1] \to X$ by

$$\alpha_1 \bullet \alpha_0(t) = \begin{cases} \alpha_0(2t) & \text{, if } 0 \leqslant t \leqslant \frac{1}{2} \\ \alpha_1(2t-1) & \text{, if } \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

which is a path from x_0 to x_1. Such operation descends to homotopy classes, inducing a well-defined map

$$\begin{aligned} \pi_1(X; x_1, x_2) \times \pi_1(X; x_0, x_1) &\longrightarrow \pi_1(X; x_0, x_2) \\ ([\beta], [\alpha]) &\longmapsto [\beta \bullet \alpha] \end{aligned}$$

If $x_3 \in X$ is still still another point, and $\alpha_2 : [0,1] \to X$ a path from x_2 to x_3, then the path $\alpha_2 \bullet (\alpha_1 \bullet \alpha_0)$ and $(\alpha_2 \bullet \alpha_1) \bullet \alpha_0$ are homotopic via the map $F : [0,1]^2 \to X$ defined by

$$F(s,t) = \begin{cases} \alpha_0\left(\frac{4s}{t+1}\right) & \text{, if } 0 \leqslant s \leqslant \frac{t+1}{4} \\ \alpha_1(4s - t - 1) & \text{, if } \frac{t+1}{4} \leqslant s \leqslant \frac{t+2}{4} \\ \alpha_2\left(\frac{4s-t-2}{2-t}\right) & \text{, if } \frac{t+2}{4} \leqslant s \leqslant 1 \end{cases}$$

This implies the operations defined above is associative. For each point $x \in X$, let c_x denote the constant path at x, i.e., $c_x(t) = x$ for all $t \in [0,1]$. For any path γ from x to $y \in X$, we have

$$[\gamma \bullet c_x] = [\gamma] = [c_y \bullet \gamma]$$

via the maps F, G defined by

$$F(s,t) = \begin{cases} x & \text{, if } 0 \leqslant s \leqslant \frac{1-t}{2} \\ \gamma(\frac{2s-1+t}{1+t}) & \text{, if } \frac{1-t}{2} \leqslant s \leqslant 1 \end{cases}, \qquad G(s,t) = \begin{cases} \gamma(\frac{2s}{t+1}) & \text{, if } 0 \leqslant s \leqslant \frac{t+1}{2} \\ y & \text{, if } \frac{t+1}{2} \leqslant s \leqslant 1 \end{cases}$$

We can define a category $\pi_1(X)$ as follows. The objects of $\pi_1(X)$ are points of X. For $x, y \in X$, set

$$\mathrm{Hom}_{\pi_1(X)}(x,y) := \pi_1(X; x, y).$$

The properties proved above shows this really defines a category. The category $\pi_1(X)$ is called the **fundamental groupoid** of the space X. To see it deserves the name "groupoid", for a path γ from x to y, define $\gamma^{-1} : [0,1] \to X$ by $\gamma^{-1}(t) = \gamma(1-t)$; then

$$[\gamma^{-1} \bullet \gamma] = [c_x], \qquad [\gamma \bullet \gamma^{-1}] = [c_y]$$

via the maps F_γ, $F_{\gamma^{-1}}$ defined by

$$F_\gamma(s,t) = \begin{cases} \gamma(s) & \text{, if } 0 \leqslant s \leqslant \frac{1-t}{2} \\ \gamma(\frac{1-t}{2}) & \text{, if } \frac{1-t}{2} \leqslant s \leqslant \frac{1+t}{2} \\ \gamma(1-s) & \text{, if } \frac{1+t}{2} \leqslant s \leqslant 1 \end{cases}$$

Hence each element in $\pi_1(X; x, y)$ is invertible, so that $\pi_1(X)$ is a groupoid. In particular, for each $x \in X$, the set $\pi_1(X, x) := \pi_1(X; x, x)$ is a group, called the **fundamental group of X with basepoint x**.

To study the fundamental groupoid of a space, we first discuss some formal property of groupoids.

Definition. Let G be a groupoid.

(i) A subcategory of G is called a **subgroupoid** of G if it is itself a groupoid.

(ii) A morphism between groupoids is a functor. Denote by **Grpd** the category of groupoids.

(iii) G is called **connected** if $\mathrm{Hom}_G(x,y) \neq \varnothing$ for any objects x, y in G.

(iv) G is called **totally disconnected** if $\mathrm{Hom}_G(x,y) = \varnothing$ whenever $x \neq y$.

(v) G is called a **tree groupoid** if $\#\,\mathrm{Hom}_G(x,y) = 1$ for any objects x, y in G.

It is clear that the fundamental groupoid defines a functor

$$\begin{aligned} \pi_1 : \mathbf{Top} &\longrightarrow \mathbf{Grpd} \\ X &\longmapsto \pi_1(X); \end{aligned}$$

if $f : X \to Y$ is a continuous maps, we define $\pi_1(f) : \pi_1(X) \to \pi_1(Y)$ by $x \mapsto f(x)$ and $[\gamma] \mapsto f_*[\gamma] = [f \circ \gamma]$. For a subset $A \subseteq X$, denote by $\pi_1(X, A)$ the full subcategory of $\pi_1(X)$ whose objects consists of points in A. Then $\pi_1(X, A)$ is a full subgroupoid of $\pi_1(X)$.

The category **Grpd** admits finite (co)products: the product (resp. coproduct) of two groupoids is the usual product (resp. coproduct/disjoint union) of two categories. They enjoy the usual universal properties of (co)products. The similar hold in the category **Top**. In fact,

Lemma C.1.1. The fundamental groupoid $\pi_1 : \mathbf{Top} \to \mathbf{Grpd}$ preserves finite (co)products.

Denote by $\mathbf{I}$ the tree groupoid with two objects 0, 1. Note that $\mathbf{I} \cong \pi_1([0,1], \{0,1\})$ as groupoids.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **homotopy** (of functors) is a functor $F : \mathcal{C} \times \mathbf{I} \to \mathcal{D}$. We then say the functor $F(\cdot, 0)$ and $F(\cdot, 1)$ are **homotopic** via F.

Let $f, g : \mathcal{C} \to \mathcal{D}$ be functors and $F : \mathcal{C} \times \mathbf{I} \to \mathcal{D}$ a homotopy with $F(\cdot, 0) = f$, $F(\cdot, 1) = g$. For each $x \in \mathcal{C}$ we consider the functor $F(x, \cdot) : \mathbf{I} \to \mathcal{D}$. Let $\iota \in \mathrm{Hom}_{\mathbf{I}}(0,1)$. Then $F(x, \cdot)$ is uniquely determined by the value $\theta(x) := F(x, \iota) \in \mathrm{Hom}_{\mathcal{D}}(F(x,0), F(x,1)) = \mathrm{Hom}_{\mathcal{D}}(f(x), g(x))$. For any $\gamma \in \mathrm{Hom}_{\mathcal{C}}(x, y)$, by functoriality we have $g(\gamma)\theta(x) = \theta(y) f(\gamma)$; as $\theta(x)$ is invertible for each x, the functor g is then completely determined by f and θ. Conversely, if $f : \mathcal{C} \to \mathcal{D}$ is a functor, and θ is an assignment from $x \in \mathrm{Ob}(\mathcal{C})$ to an invertible morphism in $\mathcal{D}$ with domain $f(x)$, and there exists a homotopy $f \cong g$, where g is defined by $g(\gamma) := \theta(y) f(\gamma) \theta(x)^{-1}$. As for the topological spaces we can define **homotopy inverse** of a functor, and **homotopy equivalence** of categories.

Lemma C.1.2.

1. Homotopy of functors defines an equivalence relation.

2. A homotopy equivalence is fully faithful.

3. If $f \sim g : X \to Y$ are homotopic, then $\pi_1(f), \pi_1(g) : \pi_1(X) \to \pi_1(Y)$ are homotopic.

Now we discuss the van Kampen theorem. Let X be a space, and $X_1, X_2 \subseteq X$ be subspaces such that $\mathrm{int}\, X_1 \cup \mathrm{int}\, X_2 = X$. Put $X_0 = X_1 \cap X_2$. Then the diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r] & X \end{tikzcd}$$

is a fibre coproduct.

Theorem C.1.3. Let A be a subspace of X that meets every path-connected components of X_0, X_1, X_2. Then the diagram

$$\begin{tikzcd} \pi_1(X_0, X_0 \cap A) \arrow[r] \arrow[d] & \pi_1(X_1, X_1 \cap A) \arrow[d] \\ \pi_1(X_2, X_2 \cap A) \arrow[r] & \pi_1(X, A) \end{tikzcd}$$

is a fibre coproduct.

C.2 Covering spaces

C.3 Δ-complexes

Let $\mathbb{R}^\infty$ be a separable Hilbert space with standard orthonormal basis $e_0, e_1, \ldots$. For $n \in \mathbb{Z}_{\geqslant 0}$, define the **standard n-simplex**

$$\Delta_n := \left\{ \sum_{k=0}^{n} t_k e_k \mid \sum_{k=0}^{n} t_k = 1,\ t_k \geqslant 0 \text{ for } 0 \leqslant k \leqslant n \right\}.$$

For any points $v_0, \ldots, v_n$, define an n**-simplex** as

$$\begin{aligned}
[v_0, \ldots, v_n] : \Delta_n &\longrightarrow \mathbb{R}^\infty \\
\sum_{k=0}^{n} t_k e_k &\longmapsto \sum_{k=0}^{n} t_k v_k
\end{aligned}$$

Each point v_k is called a **vertex** of the simplex. The $n-1$-simplex $[v_0, \ldots, \widehat{v_k}, \ldots, v_n]$ with one vertex deleted is called a **face** of $[v_0, \ldots, v_n]$. For the standard simplex Δ_n, the union of all its faces is called the **boundary** of Δ_n, denoted as $\partial\Delta_n$, and the complement $\Delta_n \backslash \partial\Delta_n$ is called the **open simplex**, denoted by Δ_n°. We identify each face of Δ_n with Δ_n via the canonical linear isomorphism that preserves the ordering of the vertices.

Definition. Let X be a topological space. A **Δ-complex structure** on X is a collection of continuous maps

$$S = \{\sigma_\alpha : \Delta_{n(\alpha)} \to X\}_\alpha$$

(assume two maps with distinct label are distinct maps) such that

(i) S is closed under restriction to faces, and

(ii) the restriction $\sigma_\alpha|_{\Delta^\circ_{n(\alpha)}}$ is injective, and each point of X lies in the image of precisely one such restriction, $X = \bigcup_\alpha \sigma_\alpha(\Delta^\circ_{n(\alpha)})$,

(iii) a set $A \subseteq X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in $\Delta_{n(\alpha)}$ for each α.

Let X be a topological space with a Δ-complex structure $\{\sigma_\alpha\}_{\alpha \in I}$. For each $n \in \mathbb{Z}_{\geqslant 0}$, put $I^n = \{\alpha \mid n(\alpha) = n\}$. Define $X^0 = \bigsqcup_{\alpha \in I^0} \Delta_0^{(\alpha)}$. There is a natural injection $\varphi_0 : X^0 \to X$ by $\varphi_0(\alpha, x) = \sigma_\alpha(x)$ (this is injective as $\Delta_0^\circ = \Delta_0$ and (ii)). For $n \geqslant 1$, define inductively that

$$X^n = X^{n-1} \sqcup \bigsqcup_{\alpha \in I^n} \Delta_n^{(\alpha)} / \sim$$

with $X^{n-1} \ni x \sim (\alpha, v) \in \bigsqcup_{\alpha \in I^n} \Delta_n^{(\alpha)}$ if and only if $v \in \partial\Delta_n^{(\alpha)}$ and $\varphi_{n-1}(x) = \sigma_\alpha(v)$. Define $\varphi_n : X^n \to X$ by $\varphi_n|_{X^{n-1}} = \varphi_{n-1}$, and for $(\alpha, p) \in \Delta_n^{(\alpha)}$, set $\varphi_n(\alpha, p) = \sigma_\alpha(p)$. This is well-defined as we quotienting out the relation $\sim$, and is injective by (ii) again. In this way we obtain a filtration

$$X^0 \subseteq X^1 \subseteq \cdots \subseteq X^n \subseteq \cdots$$

and a collection of compatible injections $\varphi_n : X^n \to X$. Taking direct limit yields an injection

$$\varphi : \varinjlim_{n \geqslant 0} X^n \to X,$$

which is surjective by (ii), and is a homeomorphism by (iii).

More abstractly, define a category Δ as follows. Objects of Δ are nonnegative integers $\mathbb{Z}_{\geqslant 0}$ (more commonly, objects are the sets $[n] := \{0, 1, \ldots, n\}$), and for $p, q \in \mathbb{Z}_{\geqslant 0}$,

$$\mathrm{Hom}_\Delta(p, q) := \{f : [p] \to [q] \mid i \leqslant j \Rightarrow f(i) \leqslant f(j)\} \subseteq \mathrm{Hom}_{\mathbf{Set}}([p], [q]).$$

Definition. A simplicial set is a functor $X \in [\Delta^{\mathrm{op}}, \mathbf{Set}]$.

Let $h_\Delta : \Delta \to [\Delta^{\mathrm{op}}, \mathbf{Set}]$ be the Yoneda embedding of the category Δ. Put $\Delta_n = h_\Delta(n) = \mathrm{Hom}_\Delta(\cdot, n)$. To a simplicial set $X : \Delta \to \mathbf{Set}$ we associate a category whose objects are all natural transformations $\Delta_n \to X$ ($n \geqslant 0$) and morphisms are all natural transformations $\Delta_n \to \Delta_m$ over X; this is the comma category $h_\Delta \downarrow X$.

Define a functor $F_X : h_\Delta \downarrow X \to [\Delta^{\mathrm{op}}, \mathbf{Set}]$ by the composition $F_X := h_\Delta \circ \omega$, where $\omega : h_\Delta \downarrow X \to \Delta$ is the forgetful functor (a morphism $\Delta_n \to \Delta_m$ corresponds to, by Yoneda lemma, an element $f \in \Delta_m(n) = \mathrm{Hom}_\Delta(n, m)$). We have

$$X \cong \varinjlim_{\Delta_n \to X} \Delta_n := \mathrm{colim} F_X$$

by the following abstract nonsense.

Theorem C.3.1. Let $\mathcal{C}$ be a (locally small) category, $h : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ its Yoneda embedding, $F \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ a presheaf, and $\omega : h \downarrow F \to \mathcal{C}$ the forgetful functor. Then F is isomorphic to the colimit of the diagram $h \downarrow F \xrightarrow{\omega} \mathcal{C} \xrightarrow{h} [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$.

Proof. Put $J = h \downarrow F$. Before we actually start the proof, we describe the category J and its forgetful functor $\omega : J \to \mathcal{C}$ in an another but equivalent way. By definition, the category J consists of

- objects: $(A, h(A) \to F)$, where $A \in \mathrm{Ob}(\mathcal{C})$ and $h(A) \to F$ is a natural transformation.
- morphisms: $f : (A, h(A) \to F) \to (B, h(B) \to F)$ is a natural transformation $f : h(A) \to h(B)$ such that $(h(B) \to F) \circ f = (h(A) \to F)$.

By Yoneda lemma, giving an object $(A, h(A) \to F)$ amounts to giving a pair $(A, a) \in \mathrm{Ob}(\mathcal{C}) \times F(A)$, and a morphism $f : (A, a) \to (B, b)$ is a morphism $f : A \to B$ simply satisfying $F(f)(b) = a$. The forgetful functor $\omega : J \to \mathcal{C}$ is rather clear in this setting.

Let $\overline{F} : J \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ be the functor defined in the theorem. For each $(A, h(A) \to F) \in \mathrm{Ob}(J)$, the natural transformation in the second component gives a natural transformation $\overline{F}(A, h(A) \to F) = h(A) \to F$. A morphism $(A, h(A) \to F) \to (B, h(B) \to F)$ in J gives a commuting triangle. In sum, this defines a cocone from $\overline{F}$ to F.

To show F is the colimit of the diagram $\overline{F}$, let G be any contravariant functor and $(\overline{F}(A, h(A) \to F) \to G)_{(A, h(A) \to F)}$ be a cocone. By definition, $\overline{F}(A, h(A) \to F) = h(A)$, and by Yoneda lemma, an object $(\overline{F}(A, h(A) \to F) \to G$ is the same as a triple $(A, a, \overline{a})$, where (A, a) is the description mentioned in the first paragraph, and $\overline{A} \in G(A)$. Define a natural transformation $T : F \to G$ as follows. For $A \in \mathrm{Ob}(\mathcal{C})$, define the map $T_A : F(A) \to G(A)$ by $T_A(a) = \overline{a}$, where for $a \in F(A)$, the element $\overline{a} \in G(A)$ is the unique element such that $(A, a, \overline{a})$ represents a element in the cocone $(\overline{F}(A, h(A) \to F) \to G)_{(A, h(A) \to F)}$. The commutativity condition for the cocone implies that T is a natural transformation. By construction this is the unique natural transformation $S : F \to G$ such that $S \circ (h(A) \to F) = (h(A) \to G)$, and this finishes the proof.

□

Define the **geometric realization** functor $|\cdot| : [\Delta^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Top}$ as follows. For the representable functor Δ_n, define

$$|\Delta_n| := [e_0, \ldots, e_n] = \Delta_n \subseteq \mathbb{R}^\infty.$$

If $f : \Delta_n \to \Delta_m$ is a natural transformation corresponding to a non-decreasing map $f : [n] \to [m]$, define $|f| : |\Delta_n| \to |\Delta_m|$ by $|f|(e_i) := e_{f(i)}$ and extending linearly. For any presheaf $X : \Delta \to \mathbf{Set}$, define the functor $|F_X| : h_\Delta \downarrow X \to \mathbf{Top}$ by $|F_X|(f : \Delta_n \to \Delta_m) := |f|$, and define

$$|X| = \varinjlim_{\Delta_n \to X} |\Delta_n| := \mathrm{colim}|F_X|.$$

If $T : X \to Y$ is a natural transformation of presheaves on Δ, it induces a morphism $h_\Delta \downarrow X \to h_\Delta \downarrow Y$ and hence a continuous map $|T| : |X| \to |Y|$ in an obvious way. It is clear that two definitions for $|\Delta_n|$ and the morphisms $|\Delta_n \to \Delta_m|$ coincide.

In fact, the geometric realization functor is has a right adjoint. For $p \in \mathbb{Z}_{\geqslant 0}$ and a topological space X, we define

$$\widetilde{S}_p(X) := \mathrm{Hom}_{\mathbf{Top}}(\Delta_n, X)$$

which is the set of all singular p-simplexes. Each non-decreasing map $[n] \to [m]$ defines canonically a linear map $\Delta_n \to \Delta_m$, which then induces a map $\widetilde{S}_m(X) \to \widetilde{S}_n(X)$. Hence $\widetilde{S}X$ is a contravariant functor from Δ to $\mathbf{Set}$, so $\widetilde{S}X$ is a simplicial set. This defines a functor $\widetilde{S} : \mathbf{Top} \to [\Delta^{\mathrm{op}}, \mathbf{Set}]$.

Lemma C.3.2. There exists a bijection

$$\mathrm{Hom}_{\mathbf{Top}}(|X|, Y) \xrightarrow{\quad \sim \quad} \mathrm{Hom}_{[\Delta^{\mathrm{op}}, \mathbf{Set}]}(X, \widetilde{S}Y)$$

functorial in X, Y.

Proof. By the universal property of colimit,

$$\mathrm{Hom}_{\mathbf{Top}}(|X|, Y) \cong \lim_{\Delta^n \to X} \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, Y).$$

By a previous theorem, it suffices to find a functorial bijection

$$\mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, Y) \cong \mathrm{Hom}_{[\Delta^{\mathrm{op}}, \mathbf{Set}]}(\Delta_n, \widetilde{S}Y).$$

This follows from Yoneda's lemma:

$$\mathrm{Hom}_{[\Delta^{\mathrm{op}}, \mathbf{Set}]}(\Delta_n, \widetilde{S}Y) \cong \widetilde{S}Y(n) = \widetilde{S}_n(Y) = \mathrm{Hom}_{\mathbf{Top}}(|\Delta_n|, Y).$$

□

Let X be a space with Δ-complex structure. We define a simplicial set $\Delta X : \Delta \to \mathbf{Set}$ as follows. For $n \geqslant 0$, let $\Delta X(n)$ be the set of all n-simplices; in a previous term, $\Delta X(n) = I^n$. For a non-decreasing map $f : [n] \to [m]$ and a m-simplices $\sigma : \Delta_m \to X$, just define $f^*\sigma : \Delta_n \to X$, where $f^* : \Delta_n \to \Delta_m$ is the canonical linear map. This defines a morphism $\Delta X(f) : X_m \to X_n$ in a compatible way.

We describe simplicial sets alternatively. For $n \geqslant 1$ and $i \in [n]$, define $\varepsilon^i = \varepsilon^i_n : [n-1] \to [n]$ by

$$\varepsilon^i(j) = \begin{cases} j & , \text{if } j < i \\ j+1 & , \text{if } j \geqslant i \end{cases}$$

For $n \geqslant 0$ and $i \in [n]$, define $\eta^i = \eta^i_n : [n+1] \to [n]$ by

$$\eta^i(j) = \begin{cases} j & , \text{if } j \leqslant i \\ j-1 & , \text{if } j > i \end{cases}$$

They satisfy the following identities:

(a) $\varepsilon^j_{n+1}\varepsilon^i_n = \varepsilon^i_{n+1}\varepsilon^j_n$ for $i < j$.

(b) $\eta^j_n\eta^i_{n+1} = \eta^i_n\eta^{j+1}_{n+1}$ for $i \leqslant j$.

(c) $\eta^j_{n-1}\varepsilon^i_n = \begin{cases} \varepsilon^i_{n-1}\eta^{j-1}_{n-2} & \text{, if } i < j \\ \text{id} & \text{, if } i = j \text{ or } j+1 \\ \varepsilon^{i-1}_{n-1}\eta^j_{n-2} & \text{, if } i > j+1 \end{cases}$

Lemma C.3.3. Every non-decreasing map $f : [n] \to [m]$ can be uniquely factorized as

$$f = \varepsilon^{i_1} \cdots \varepsilon^{i_s}\eta^{j_1} \cdots \eta^{j_t}$$

with $m \geqslant i_1 > \cdots > i_s \geqslant 0 \leqslant j_1 < \cdots < j_t < n$ and $n - t + s = m$.

Proof. Write $i_1 > \cdots > i_s$ for the elements in the set $[m]\backslash f([n])$, and $j_1 < \cdots < j_t$ for the elements in $j \in [n]$ such that $f(j) = f(j+1)$. □

Theorem C.3.4. A simplicial set $X : \Delta \to \mathbf{Set}$ is exactly a family of sets X_n ($n \geqslant 0$) together with the maps $d^i_n : X_n \to X_{n-1}$, $s^i_n : X_n \to X_{n+1}$ satisfying

(a) $d^i_n d^j_{n+1} = d^j_n d^i_{n+1}$ for $i < j$.

(b) $s^i_{n+1}s^j_n = s^{j+1}_{n+1}s^i_n$ for $i \leqslant j$.

(c) $d^i_n s^j_{n-1} = \begin{cases} s^{j-1}_{n-2}d^i_{n-1} & \text{, if } i < j \\ \text{id} & \text{, if } i = j \text{ or } j+1 \\ s^j_{n-2}d^{i-1}_{n-1} & \text{, if } i > j+1 \end{cases}$

The maps $d^i = d^i_n$ are called the i-th face operators, and $s^j = s^j_n$ are called the j-th degeneracy operators.

C.3.1 $K(G,1)$-space

Definition. Let G be an abstract group. A topological space X is called a $K(G,1)$-space if it has a contractible universal covering and has fundamental group isomorphic to G.

Let G be a group. Define a functor $EG : \Delta \to \mathbf{Set}$ by

$$EG(n) = \mathrm{Hom}_{\mathbf{Set}}([n], G)$$

(rules for morphisms are clear). For $g_0, \ldots, g_n \in G$, we write $[g_0, \ldots, g_n]$ for the element in $EG(n)$ defined by $i \mapsto g_i$. This defines a simplicial set, so we can form its geometric realization $|EG|$. The group G acts on each $EG(n)$: each $g \in G$ sends an n-simplex $[g_0, \ldots, g_n]$ to $[gg_0, \ldots, gg_n]$. This defines a natural isomorphism $g : EG \to EG$, whence inducing a homeomorphism $g : |EG| \to |EG|$. Clearly $g \circ h = gh$, so this defines a G-action on $|EG|$; this is a free G-action.

Lemma C.3.5. Let X be a Δ-complex on which a group G acts in the way that its sends each simplex of X onto another simplex via linear homeomorphism. If the G-action is free, then it is a covering space action, i.e., $X \to X/G$ is a covering space.

Proof. Let $p \in X$ and $\sigma : \Delta_n \to X$ be the unique simplex of X whose interior $\sigma(\Delta^\circ_n)$ contains p. Take U be any open neighborhood of p in $\sigma(\Delta^\circ_n)$ such that $\overline{U} \subseteq \sigma(\Delta^\circ_n)$. Then the translation of U by G is pairwise disjoint.

We now proceed to build an open neighborhood of U in X whose translations by G are pairwise disjoint. □

The space $|EG|$ is contractible: define a homotopy $h_t : EG \to EG$ by sliding each $x \in [g_0, \ldots, g_n]$ along the ling segment in $[e, g_0, \ldots, e_n]$ to $[e]$. Note this is not a deformation retract, as h_t carries $[e]$ around the loop $[e, e] \subseteq |EG|$.

Define $BG = G\backslash|EG|$ and topologized by quotient topology. Previous discussion implies that $EG \to BG$ is a universal cover map. In particular, the fundamental group of BG is isomorphic to G, so BG is $K(G, 1)$. Moreover, BG inherits a Δ-complex structure from $|EG|$, which we describe below. Note that each simplex in $|EG|$ can be written as

$$[g_0, g_0g_1, \ldots, g_0g_1 \cdots g_n] = g_0[e, g_1, g_1g_2, \ldots, g_1 \cdots g_n]$$

In particular, we have

$$BG(n) = \{[e, g_1, g_1g_2, \ldots, g_1 \cdots g_n] \mid g_1, \ldots, g_n \in G\} = \{f \in \mathrm{Hom}_{\mathbf{Set}}([n], G) \mid f(0) = e\}.$$

Alternatively, we can define the functor BG as above, form the geometric realization $|BG|$, and prove that $|EG| \to |BG|$ is a cover map. For simplicity, we put

$$[g_1|\cdots|g_n] := [e, g_1, g_1g_2, \ldots, g_1 \cdots g_n].$$

By construction the boundary of $[g_1|\cdots|g_n]$ consists of

$$[g_2|\cdots|g_n], \;\; [g_1|\cdots|g_ig_{i+1}|\cdots|g_n], \;\; [g_1|\cdots|g_{n-1}].$$

Theorem C.3.6. The homotopy type of a $K(G, 1)$ CW-complex is uniquely determined by the group G.

The theorem is easily deduced from the following theorem.

Theorem C.3.7. Let X be a connected CW-complex and let Y be a $K(G, 1)$. Then each homomorphism $\pi_1(X, x_0) \to \pi_1(Y, y_0)$ is induced by a map $(X, x_0) \to (Y, y_0)$ unique up to a homotopy fixing x_0.

C.4 Singular (co)homology with local coefficients

Definition. Let X be a space. A **bundle of groups** is a functor $G : \Pi_1(X) \to \mathbf{Gp}$ from the fundamental groupoid to the category of groups.

C.5 de Rham theorem with local system

Part V

Real analysis

Appendix D

Measure and Integration

D.1 Basics

Definition. Let X be a set, and $\mathcal{A} \subseteq 2^X$.

(i) $\mathcal{A}$ is called an **algebra (of sets)** if $X \in \mathcal{A}$ and it is closed under complement, finite union and finite intersection.

(ii) $\mathcal{A}$ is called a **σ-algebra** if $X \in \mathcal{A}$ and it is closed under complement, countable union and countable intersection.

(iii) $\mathcal{A}$ is called an **increasing class** (resp. **decreasing class**) if $\bigcup\limits_{n \geqslant 1} A_n \in \mathcal{A}$ (resp. $\bigcap\limits_{n \geqslant 1} B_n$) when $(A_n)_{n \geqslant q} \subseteq \mathcal{A}$ is an increasing sequence (resp. $(B_n)_{n \geqslant 1} \subseteq \mathcal{A}$ is a decreasing sequence).

(iv) $\mathcal{A}$ is called a **monotone class** if it is both an increasing class and a decreasing class.

(v) $\mathcal{A}$ is called a **π-system** if it is closed under finite intersection.

(vi) $\mathcal{A}$ is called a **Dynkin class/d-system/λ-system** if

- $X \in \mathcal{A}$,
- $A \backslash B \in \mathcal{A}$ if $B \subseteq A$ and $A, B \in \mathcal{A}$, and
- $\bigcup\limits_{n \geqslant 1} A_n \in \mathcal{A}$ if $(A_n)_{n \geqslant 1} \subseteq \mathcal{A}$ is an increasing sequence in $\mathcal{A}$.

The following property is obvious.

Lemma D.1.1. Let $S \subseteq 2^X$ and suppose $\{\mathcal{A}_\alpha \mid \alpha \in I\}$ is a collection of algebras (resp. σ-algebras, monotone classes, π-systems, d-systems) that contains S. Then $\bigcap_{\alpha \in I} \mathcal{A}_\alpha \subseteq 2^X$ is an algebra (resp. σ-algebra, monotone class, π-system, d-system).

Definition. Let $S \subseteq 2^X$. We denote by $a(S)$, $\sigma(S)$, $mc(S)$, $\pi(S)$, $d(S)$ the smallest algebra, σ-algebra, monotone class, π-system, d-system, respectively, containing S. They are said to be **generated by** S.

Lemma D.1.2. Let X be a set and $\mathcal{A} \subseteq 2^X$ be an algebra. If $\mathcal{A}$ is either an increasing class or a decreasing class, then $\mathcal{A}$ is a σ-algebra.

Proof. Suppose $\mathcal{A}$ is increasing, and let $(A_n)_{n\geqslant 1}$ be a sequence in $\mathcal{A}$. For $n \geqslant 1$, define $A'_n = \bigcup_{m\leqslant n} A_m$. Then $(A'_n)_n$ is an increasing sequence in $\mathcal{A}$, and hence $\bigcup_{n\geqslant 1} A_n = \bigcup_{n\geqslant 1} A'_n \in \mathcal{A}$. This shows $\mathcal{A}$ is closed under countable union. Next, for $n \geqslant 1$, define $A''_n = \bigcap_{m\leqslant n} A_m$. Then $(A''_n)_n$ is a decreasing sequence so that $\bigcap_{n\geqslant 1} A_n = \bigcap_{n\geqslant 1} A''_n \in \mathcal{A}$. This proves $\mathcal{A}$ is a σ-algebra.

Suppose $\mathcal{A}$ is decreasing. We prove $\mathcal{A}$ is a σ-algebra by showing it is increasing. Let $(A_n)_{n\geqslant 1}$ be an increasing sequence in $\mathcal{A}$. Define $B_n = X\backslash A_n$ $(n \geqslant 1)$; then B_n is a decreasing sequence, so that $\bigcap_{n\geqslant 1} B_n \in \mathcal{A}$. But then $\bigcup_{n\geqslant 1} = X\backslash \bigcap_{n\geqslant 1} B_n \in \mathcal{A}$, this shows $\mathcal{A}$ is increasing.

$\square$

Theorem D.1.3 (Dynkin's π-λ theorem)**.** Let X be a set and P a π-system. Then $\sigma(P) = d(\mathcal{P})$.

Proof. By definition, we have $d(P) \subseteq \sigma(P)$. For the reverse inclusion, we must show $d(P)$ is a σ-algebra. $\square$

D.1.1 Measure and Integration

Definition. Let $(X, \mathcal{A})$ be a measurable space. A **(positive) measure** on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu(\varnothing) = 0$ and **countably additive**, i.e.,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

if $(A_n)_{n\geqslant 1} \subseteq \mathcal{A}$ is a disjoint sequence of measurable sets. The triple $(X, \mathcal{A}, \mu)$ is called a **(positive) measure space**.

Lemma D.1.4. Let $(X, \mathcal{A}, \mu)$ be a measure space.

(i) For $A, B \in \mathcal{A}$, $A \subseteq B$ implies $\mu(A) \leqslant \mu(B)$.

(ii) If $(A_n)_{n\geqslant 1} \subseteq \mathcal{A}$ is an increasing sequence, then $\lim_{n\to\infty} \mu(A_n) = \mu\left(\bigcup_{n\geqslant 1} A_n\right)$.

(iii) If $(A_n)_{n\geqslant 1} \subseteq \mathcal{A}$ is a decreasing sequence such that $\mu(A_1) < \infty$, then $\lim_{n\to\infty} \mu(A_n) = \mu\left(\bigcap_{n\geqslant 1} A_n\right)$.

Definition. Let $(X, \mathcal{A})$ be a measurable space and μ be a measure on $(X, \mathcal{A})$.

(i) μ is **finite** if $\mu(X) < \infty$.

(ii) A measurable set $M \in \mathcal{A}$ is called **σ-finite** if there exists an ascending sequence $(A_n)_{n\geqslant 1} \subseteq \mathcal{A}$ with $\mu(A_n) < \infty$ $(n \geqslant 1)$ and $M = \bigcup_{n\geqslant 1} A_n$.

(iii) μ is **σ-finite** if X is σ-finite.

Definition. Let X be a set.

(i) A **simple function on** X is a function $s : X \to \mathbb{C}$ with finite image.

Suppose in addition $(X, \mathcal{A}, \mu)$ is a measure space.

(ii) A **μ-simple function on** X is a simple measurable function s on X with $\mu(\{x \in X \mid s(x) \neq 0\}) < \infty$.

We now come to the definition of the **integration on a measure space** $(X, \mathcal{A}, \mu)$. If s is a non-negative μ-simple function on X, write

$$s = \sum_{\alpha \in s(X)} \alpha \mathbf{1}_{s^{-1}(y)}.$$

This is a finite sum, and $\mu(s^{-1}(y)) < \infty$ for each $y \in s(X)$. For each $E \in \mathcal{A}$, define the **integral of** s **on** E **against by** μ as

$$\int_E s d\mu := \sum_{\alpha \in s(X)} \alpha \cdot \mu(E \cap s^{-1}(y)) < \infty.$$

For a non-negative measurable function $f : X \to [0, \infty]$, define the **integral of** f **on** E **against** μ as

$$\int_E f d\mu = \sup_{0 \leqslant s \leqslant f} \int_E s d\mu$$

where s runs over all non-negative μ-simple functions with $s \leqslant f$. We say f is **(μ-)integrable** if the integral $\displaystyle\int_X f d\mu$ is finite.

Lemma D.1.5. If $f : X \to [0, \infty]$ is integrable, then $\{x \in X \mid f(x) = \infty\}$ has measure zero and $\{x \in X \mid f(x) < \infty\}$ is σ-finite.

Proof. Let $A_\infty = \{x \in X \mid f(x) = \infty\}$ and for each $n \geqslant 1$ let $A_n = \left\{x \in X \mid f(x) > \dfrac{1}{n}\right\}$.

□

Remark. It might happen that the domain of supremum is empty. If we relieve the condition to allow each $s^{-1}(y)$ has infinite measure, then it is nonempty by

Lemma D.1.6. Let $f : X \to [0, \infty]$ be a measurable function. Then there exists an increasing non-negative simple measurable functions $(s_n)_n$ with $s_n \leqslant f$ such that $s_n \to f$ pointwise.

For convenience, denote by $\displaystyle\int_E^i f d\mu$ the integral defined by approximation of arbitrary non-negative simple measure functions. Then $\displaystyle\int_E^i f d\mu = \int_E f d\mu$ if μ is σ-finite. Indeed, if $X = \displaystyle\bigcup_{n=1}^{\infty} X_n$ with $X_n \subseteq X_{n+1}$ and $\mu(X_n) < \infty$, then for any measurable E, F, by Lemma D.1.4.(ii)

$$\sup_n \int_E \mathbf{1}_{F \cap X_n} d\mu = \lim_{n \to \infty} \mu(E \cap F \cap X_n) = \mu(E \cap F) = \int_E^i \mathbf{1}_F d\mu$$

Moreover, we have

Lemma D.1.7. Let $f \geqslant 0$ be measurable. Then $\{x \in X \mid f(x) \neq 0\}$ is σ-finite if $\displaystyle\int_X^i f d\mu$ is finite.

Proof. For $n \geqslant 1$, consider the set $A_n := \left\{x \in X \mid f(x) \geqslant \dfrac{1}{n}\right\}$. By definition

$$\infty > \int_X^i f d\mu \geqslant \int_{A_n}^i f d\mu \geqslant \frac{\mu(A_n)}{n}.$$

This implies $\mu(A_n) < \infty$. Since $\{x \in X \mid f(x) \neq 0\} = \bigcup_{n \geqslant 1} A_n$, this completes the proof. $\square$

Hence, if $\int_X^i f d\mu$ is finite, we have $\int_X^i f d\mu = \int_X f d\mu$. However, the converse needs not hold as in general it might happen for a measurable set E with $\mu(E) = \infty$, every measurable subset of E has zero measure.

In general, a measurable function $f : X \to \mathbb{C}$ is called **(μ-)integrable** if $|f| : X \to [0, \infty)$ is integrable. The set of all complex-valued integrable functions is denoted by $\mathcal{L}^1(X, \mu)$. For $f \in \mathcal{L}^1(X, \mu)$, write $f = u + iv$ with u, v real-valued and write $u^{\pm}, v^{\pm}$ for their positive (resp. negative) parts. Define the **integral of f on E against μ** as

$$\int_E f d\mu = \int_E u^+ d\mu - \int_E u^- d\mu + i \int_E v^+ d\mu - i \int_E v^- d\mu.$$

D.2 Riesz's Representation Theorem

Definition. Let X be a topological space.

1. The σ-algebra generated by the topology on X is denoted by $\mathcal{B}$, and an element in $\mathcal{B}$ is called a **Borel set**.

2. A measure on the measurable space $(X, \mathcal{B})$ is called a **Borel measure**.

Let μ be a Borel measure on X.

(i) μ is **outer regular** if $\mu(M) = \inf \left\{ \mu(U) \mid M \subseteq U \underset{\text{open}}{\subseteq} X \right\}$ for all $M \in \mathcal{B}$.

(ii) μ is **weakly inner regular** if $\mu(U) = \sup \left\{ \mu(K) \mid X \underset{\text{cpt}}{\supseteq} K \subseteq U \right\}$ for all open sets U.

(iii) μ is **inner regular** if $\mu(M) = \sup \left\{ \mu(K) \mid X \underset{\text{cpt}}{\supseteq} K \subseteq M \right\}$ for all $M \in \mathcal{B}$.

(iv) μ is **regular** if it is outer regular and inner regular.

(v) μ is **locally finite** if every point $x \in X$ admits an open neighborhood U with $\mu(U) < \infty$.

(vi) μ is **(outer) Radon** if it is locally finite, outer regular and weakly inner regular.

Lemma D.2.1. For an outer Radon measure μ on a topological space X and every measurable $A \subseteq X$ with $\mu(A) < \infty$, one has

$$\mu(A) = \sup_{\substack{K \subseteq A \\ K \,:\, \text{compact}}} \mu(K)$$

In particular, if $\mu(X) < \infty$, then μ is regular.

Theorem D.2.2 (Riesz's Representation Theorem)**.** Let X be an LCH space and let $\Lambda : C_c(X) \to \mathbb{C}$ be a positive linear functional. Then there exists a unique outer Radon measure

$$\Lambda f = \int_X f d\mu$$

for all $f \in C_c(X)$.

Theorem D.2.3. Let X be an LCH space. Then every bounded linear functional Φ on $C_0(X)$ is represented by a unique regular complex Borel measure μ, in the sense that

$$\Phi f = \int_X f d\mu$$

for every $f \in C_0(X)$. Moreover, the norm of Φ is the total variation of μ.

Proof. For a regular complex Borel measure μ on X, define $\Phi_\mu : C_0(X) \to \mathbb{C}$ by the formula

$$\Phi_\mu(f) := \int_X f d\mu$$

where $f \in C_0(X)$. Clearly, we have $|\Phi_\mu(f)| \leqslant \|f\|_\infty \|\mu\|$ for all $f \in C_0(X)$, so Φ_μ is a bounded operator. Thus we obtain a well-defined association

$$\begin{aligned} M_r(X, \mathbb{C}) &\longrightarrow C_0(X)^\vee \\ \mu &\longmapsto \Phi_\mu \end{aligned}$$

The above estimate show $\|\Phi_\mu\| \leqslant \|\mu\|$, and to show this is norm-preserving, we must show the reversed inequality. Let $\varepsilon > 0$ and suppose $\|\mu\| \neq 0$. By definition of $\|\mu\|$ we can find a finite Borel-measurable partition $\{A_n\}_{n=1}^N$ of X such that $\sum_{n=1}^N |\mu(A_n)| > \|\mu\| - \varepsilon$. By regularity of μ we can find compact $K_n \subseteq A_n$ such that $|\mu(A_n) - \mu(K_n)| \leqslant N^{-1}\varepsilon$, so

$$\|\mu\| - \varepsilon < \sum_{n=1}^N |\mu(A_n)| \leqslant \sum_{n=1}^N |\mu(K_n)| + \varepsilon \leqslant \sum_{n=1}^N |\mu|(K_n) + \varepsilon$$

or, $\|\mu\| - 2\varepsilon < \sum_{n=1}^N |\mu|(K_n)$; we may assume $\mu(K_n) \neq 0$ for each n. Now choose $f \in C_c(X)$ such that $\|f\|_\infty \leqslant 1$ and $f(x) = \dfrac{\overline{\mu(K_n)}}{|\mu(K_n)|}$ for each n (this is possible for there are only finitely many K_n and since X is LCH, we have Urysohn Lemma in hand). If we put $K = \bigcup_{n=1}^N K_n$, then

$$\int_K f d\mu = \sum_{n=1}^N |\mu(K_n)| > \|\mu\| - 2\varepsilon \text{ and } \left|\int_{K^c} f d\mu\right| \leqslant |\mu|(K^c) < 2\varepsilon$$

It follows that

$$\|\Phi_\mu\| \geqslant |\Phi_\mu(f)| = \left|\int_X f d\mu\right| > \|\mu\| - 4\varepsilon$$

Since ε is arbitrary, this proves $\|\Phi_\mu\| \geqslant \|\mu\|$. Being norm-preserving, this also show $\mu \mapsto \Phi_\mu$ is injective.

What is left to do is show that $\mu \mapsto \Phi_\mu$ is surjective. For this, we first the real case:

$$\begin{aligned} M_r(X, \mathbb{R}) &\longrightarrow C_0(X, \mathbb{R})^\vee \\ \mu &\longmapsto \Phi_\mu \end{aligned}$$

Let $\Phi : C_0(X) \to \mathbb{R}$ be a positive linear functional. The restriction of Φ to $C_c(X)$ is positive, so by Theorem D.2.2 there exists an outer Radon measure μ such that $\Phi(f) = \int_X f d\mu$ for all $f \in C_c(X)$. By weak inner regularity we have $\mu(X) = \sup_{\substack{f \in C_c(X) \\ 0 \leqslant f \leqslant 1}} \Phi(f) \leqslant \|\Phi\| < \infty$, so μ is finite. Now since Φ and Φ_μ are continuous and $C_c(X)$ is dense in $C_0(X)$, we have $\Phi = \Phi_\mu$ in $C_0(X, \mathbb{R})^\vee$. The proof of real case is completed by virtue of the following lemma.

Lemma D.2.4. Let X be an LCH space. Then for all $\Lambda \in C_0(X, \mathbb{R})^\vee$ we have the decomposition $\Lambda = \Lambda^+ - \Lambda^-$ with $\Lambda^\pm \in C_0(X, \mathbb{R})^\vee$ and positive.

The complex case follows from expressing a functional by the sum of its real and imaginary part, and the proof is finished. $\square$

D.3 Signed and Complex Measures

D.3.1 Signed Measures

Definition. Let $(X, \mathcal{A})$ be a measurable space and $\mu : \mathcal{A} \to [-\infty, \infty]$ be a function.

1. μ is called **finitely additive** if for a finite disjoint family $\{A_i\}_{i=1}^n \subseteq \mathcal{A}$, we have
$$\mu\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n \mu(A_i)$$
2. μ is called **countably additive** if for a sequence of disjoint measurable sets $\{A_i\}_{i=1}^\infty$, we have
$$\mu\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty \mu(A_i)$$
3. μ is called a **signed measure** if μ is countably additive and $\mu(\varnothing) = 0$.
4. A signed measure is called **finite** if it takes values in $\mathbb{R}$.

- Suppose μ is a signed measure on $(X, \mathcal{A})$. Then for $A \in \mathcal{A}$, the sum $\mu(A) + \mu(A^c)$ is defined, and is equal to $\mu(X)$. Hence, if $\mu(A) = \infty$ for some $A \in \mathcal{A}$, then $\mu(X) = \infty$; if $\mu(A) = -\infty$ for some $A \in \mathcal{A}$, then $\mu(X) = -\infty$. Hence a signed measure can only attain one of the values $\pm\infty$.

  In the same way we see if $\mu(A)$ is finite, then $\mu(B)$ is finite for all measurable subsets B of A.
- If no possible confusion occurs, we will omit the σ-algebra $\mathcal{A}$ and simply say X is a measurable space and an element in $\mathcal{A}$ a measurable set. Moreover, when we say "μ is a signed measure on X", we implicitly mean that X is a measurable space.

Lemma D.3.1. Let X be a measurable space and μ a signed measure on X. If $\{A_i\}$ is an increasing sequences of measurable sets, then
$$\mu\left(\bigcup_{i=1}^n A_i\right) = \lim_{i\to\infty} \mu(A_i)$$
If $\{A_i\}$ is a decreasing sequence of measurable sets with $\mu(A_n)$ finite for some n, then
$$\mu\left(\bigcap_{i=1}^n A_i\right) = \lim_{i\to\infty} \mu(A_i)$$

Definition. Let μ be a signed measure on a measurable space X.

1. A subset $A \subseteq X$ is called a **positive set** if it is measurable and for all measurable subset E of A, $\mu(E) \geqslant 0$.
2. A subset $A \subseteq X$ is called a **negative set** if it is measurable and for all measurable subset E of A, $\mu(E) \leqslant 0$.

Lemma D.3.2. Let μ be a signed measure on X and let $A \subseteq X$ be a measurable set such that $-\infty < \mu(A) < 0$. Then there exists a negative set B contained in A such that $\mu(B) \leqslant \mu(A)$.

Proof. We delete some subset with positive measure from A and take B to be the remaining part. Let

$$\delta_1 := \sup\{\mu(E) \mid E \subseteq A,\, E \text{ measurable}\} \geqslant 0 \qquad (\text{for } \mu(\varnothing) = 0)$$

and choose a measurable subset A_1 of A such that $\mu(A_1) \geqslant \min\left\{\frac{1}{2}\delta_1, 1\right\}$. Construct inductively the sequences $\{\delta_n\}$ and $\{A_n\}$ such that

$$\delta_n := \sup\left\{\mu(E) \mid E \text{ measurable},\, E \subseteq A \backslash \bigcup_{i=1}^{n-1} A_i\right\}$$

and A_n is a measurable subset of $A \backslash \bigcup_{i=1}^{n-1} A_i$ such that $\mu(A_n) \geqslant \min\left\{\frac{1}{2}\delta_n, 1\right\}$. Now define $A_\infty := \bigcup_{n=1}^{\infty} A_n$ and $B = A \backslash A_\infty$. Since each A_n is disjoint with non-negative measure, $\mu(A_\infty) \geqslant 0$ by countable additivity, and hence

$$\mu(A) = \mu(A_\infty) + \mu(B) \geqslant \mu(B)$$

It remains to show B is a negative set. Since $\mu(A)$ is finite, so is $\mu(A_\infty) = \sum_{n=1}^{\infty} \mu(A_n)$, implying $\delta_n \to 0$. For any measurable subset E of B, we have $\mu(E) \leqslant \delta_n$ for all $n \in \mathbb{N}$, so $\mu(E) \leqslant 0$. $\square$

Theorem D.3.3 (Hahn Decomposition Theorem)**.** Let X be a measurable space and let μ be a signed measure on X. Then we can find two disjoint subsets P and N of X such that P is a positive set, N is a negative set and $X = P \sqcup N$.

The pair (P, N) is called a **Hahn decomposition** for the signed measure μ.

Proof. Since μ cannot take $\pm\infty$ at the same time, for definiteness we assume μ does not take $-\infty$ as its values. Let

$$L := \inf\{\mu(A) \mid A \text{ is a negative set for } \mu\}$$

The set on the right is nonempty since $\varnothing$ is a negative set. Choose a sequence $\{A_n\}$ of negative sets for which $L = \lim_{n\to\infty} \mu(A_n)$, and put $N = \bigcup_{n=1}^{\infty} A_n$. Every measurable subset of N can be decomposed as a disjoint union of measurable subsets, each of which contained in some A_n, so N is negative as well. Hence $L \leqslant \mu(N) \leqslant \mu(A_n)$ for all $n \in \mathbb{N}$, and therefore $L = \mu(N)$. Since μ does not attain $-\infty$, $\mu(N)$ is finite.

Let $P = N^c$. We claim P is positive. If P contained some measurable set A with $\mu(A) < 0$, then A would contain some negative set B with $\mu(B) < 0$ by Lemma D.3.2, and $N \sqcup B$ would be a negative set such that

$$\mu(N \sqcup B) = \mu(N) + \mu(B) < \mu(N) = L$$

a contradiction to definition of L. Hence P must be positive for μ. $\square$

Corollary D.3.3.1 (Jordan Decomposition Theorem)**.** Every signed measure is the difference of two positive measures, at least one of which is finite.

Proof. Let μ be a signed measure on X, and (P, N) a Hahn decomposition for μ. Define μ^+ and μ^- on X by

$$\mu^+(A) := \mu(A \cap P), \qquad \mu^-(A) := -\mu(A \cap N)$$

It follows from definition that μ^+ and μ^- are positive measure, and since μ cannot attain $\pm\infty$ simultaneously, at least one of the values $\mu(P)$ and $\mu(N)$ is finite, and hence at least one of the measures μ^+ and μ^- is finite. □

Let (P, N) be a Hahn decomposition for the signed measure μ on X, and let μ^+ and μ^- be the positive measures constructed in the last proof. For a measurable set A and any of its measurable subset B, we have

$$\mu(B) = \mu^+(B) - \mu^-(B) \leqslant \mu^+(B) \leqslant \mu^+(A)$$

Since $\mu^+(A) = \mu(A \cap P)$, this implies

$$\mu^+(A) = \sup\{\mu(B) \mid B \text{ measurable}, B \subseteq A\}$$

Likewise

$$\mu^-(A) = \sup\{-\mu(B) \mid B \text{ measurable}, B \subseteq A\}$$

This says that μ^+ and μ^- are independent of the choice of Hahn decomposition (P, N) for μ. The measures μ^+ and μ^- are called the positive part and the negative part of μ, and the expression $\mu = \mu^+ - \mu^-$ is called the **Jordan decomposition** of μ. Moreover, they are minimal in the sense that if $\mu = \lambda_1 - \lambda_2$ with each λ_i positive, then

$$\mu^+(A) = \mu(A \cap P) \leqslant \lambda_1(A \cap P) \leqslant \lambda_1(A)$$
$$\mu^-(A) = \mu(A \cap N) \leqslant \lambda_2(A \cap N) \leqslant \lambda_2(A)$$

Definition. The **variation** of a signed measure μ on X is the positive measure $|\mu|$ on X defined by $|\mu| := \mu^+ + \mu^-$. The **total variation** $\|\mu\|$ is defined by $\|\mu\| := |\mu|(X)$.

- It is easy to see $|\mu(A)| \leqslant |\mu|(A)$ for all measurable A. In fact, $|\mu|$ is the smallest positive measure possessing this property. Let ν be another positive measure such that $|\mu(A)| \leqslant \nu(A)$ for all measurable A. Let (P, N) be a Hahn decomposition for μ. Then for measurable A, we have $\mu^+(A) = |\mu(A \cap P)| \leqslant \nu(A \cap P)$ and $\mu^-(A) \leqslant \nu(A \cap N)$, so

$$|\mu|(A) = \mu^+(A) + \mu^-(A) \leqslant \nu(A \cap P) + \nu(A \cap N) = \nu(A)$$

D.3.2 Complex Measures

Definition. Let $(X, \mathcal{A})$ be a measurable space. A **complex measure** on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to \mathbb{C}$ that is countably additive, namely, for a sequence of disjoint measurable sets $\{A_i\}_{i=1}^{\infty}$, we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

For a complex measure μ, define its **variation** $|\mu|$ by

$$|\mu|(A) := \sup\left\{\sum_{n=1}^{\infty} |\mu(A_n)| \,\Big|\, \{A_n\} \text{ is a measurable partition of } A\right\}$$

and the **total variation** $\|\mu\|$ is defined by $\|\mu\| := |\mu|(X)$.

- For a complex measure μ, we must have $\mu(\varnothing) = 0$ by countable additivity, and thus $|\mu|(\varnothing) = 0$.
- We can write $\mu = \mu' + i\mu''$, where μ', μ'' are finite signed measures on X, so by Jordan decomposition we have

$$\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$$

where the μ_i are the finite positive measures on X. This is called the **Jordan decomposition** of μ if $\mu_1 - \mu_2$ and $\mu_3 - \mu_4$ are the Jordan decomposition of the real and imaginary parts of μ. Since each μ_i is finite, this implies $|\mu|$ only takes finite values. In particular, $\|\mu\| < \infty$.

- It can be shown that in the definition of $|\mu|(A)$, it is enough to go through all *finite* measurable partitions of A. To see this, denote

$$|\mu|_f(A) := \sup\left\{ \sum_{n=1}^{N} |\mu(A_n)| \,\Big|\, \{A_n\}_{n=1}^{N} \text{ is a finite measurable partition of } A \right\}$$

Then clearly $|\mu(A)| \leqslant |\mu|_f(A) \leqslant |\mu|(A)$ for all measurable A. Using the same argument in the next proposition, we can prove $|\mu|_f$ is a measure. But then for any measurable partition $\{A_n\}$ of A

$$\sum_{n=1}^{\infty} |\mu(A_n)| \leqslant \sum_{n=1}^{\infty} |\mu|_f(A_n) = |\mu|_f(A)$$

so by definition, $|\mu|(A) \leqslant |\mu|_f(A)$. These proves $|\mu|_f = |\mu|$.

Proposition D.3.4. Let μ be a complex measure on X. Then the variation $|\mu|$ of μ is a finite positive measure on X.

Proof. It remains to show $|\mu|$ is countably additive. Let E be a measurable set and $\{E_n\}$ be any measurable partition of E. If $\{A_n\}$ is any other measurable partition of E, then

$$\sum_n |\mu(A_n)| = \sum_n \left| \sum_m \mu(A_n \cap E_m) \right| \leqslant \sum_n \sum_m |\mu(A_n \cap E_m)| = \sum_m \sum_n |\mu(A_n \cap E_m)| \leqslant \sum_m |\mu|(E_m)$$

Conversely, let $t_n \in \mathbb{R}$ such that $t_n < |\mu|(E_n)$ and $\{A_{nm}\}_m$ a measurable partition of E_n such that $\sum_m |\mu(A_{nm})| > t_n$ for each n. Then

$$\sum_n t_n \leqslant \sum_{n,m} |\mu(A_{nm})| \leqslant |\mu|(E)$$

so that $\sum_n |\mu|(E_n) \leqslant |\mu|(E)$. This shows $|\mu|$ is countably additive. □

Let $(X, \mathcal{A})$ be a measurable space. Denote by $M(X, \mathcal{A}, \mathbb{R})$ the collection of all finite signed measures on $(X, \mathcal{A})$, and $M(X, \mathcal{A}, \mathbb{C})$ the collection of all complex measures. It is clear that $M(X, \mathcal{A}, \mathbb{R})$ and $M(X, \mathcal{A}, \mathbb{C})$ are vector spaces over $\mathbb{R}$ and $\mathbb{C}$, respectively, and the total variation defines a norm on each of them. For convenience, let call a finite signed measure a **real measure**.

Lemma D.3.5. Let μ be a finite signed measure. We can also think of μ as a complex measure. Then two variation coincide.

Proof. By $|\mu|$ we mean the variation defined by the Jordan decomposition. Let A be measurable and $\{A_n\}$ a measurable partition of A. Then

$$|\mu(A)| = \left|\sum_{n=1}^{\infty} \mu(A_n)\right| \leqslant \sum_{n=1}^{\infty} |\mu(A_n)| \leqslant \sum_{n=1}^{\infty} |\mu|(A_n) = |\mu|(A)$$

so that

$$|\mu(A)| \leqslant \sup\left\{\sum_{n=1}^{\infty} |\mu(A_n)| \,\middle|\, \{A_n\} \text{ is a measurable partition of } A\right\} \leqslant |\mu|(A)$$

We know the middle term defines a positive measure, and since $|\mu|$ is the smallest positive measure ν satisfying $|\mu(A)| \leqslant \nu(A)$ for all measurable A, this forces the second inequality to be an equality. ☐

Proposition D.3.6. Let $(X, \mathcal{A})$ be a measurable space. Then the space $M(X, \mathcal{A}, \mathbb{R})$ and $M(X, \mathcal{A}, \mathbb{C})$ are complete under the total variation norm.

Proof. It suffices to show every absolutely convergent series converges in $M(X, \mathcal{A}, \mathbb{R})$ or $M(X, \mathcal{A}, \mathbb{C})$. Let $\{\mu_n\}$ be a sequence of real or complex measures such that $\sum_{n=1}^{\infty} \|\mu_n\| < \infty$.

For a complex measure μ, we can write $\mu = \mu' + i\mu''$ with μ' and μ'' finite real, and we have $|\mu(A)| \geqslant |\mu'(A)|$ and $|\mu(A)| \geqslant |\mu''(A)|$, so by minimality we have $\|\mu\| \geqslant \|\mu'\|$ and $\|\mu\| \geqslant \|\mu''\|$. Hence it suffices to deal with the real case. Still, write the Jordan decomposition $\mu = \mu^+ - \mu^-$, so that $|\mu| = \mu^+ + \mu^-$. This implies $\|\mu\| \geqslant \|\mu^{\pm}\|$, so we can further reduce to the positive measure case.

Now assume each μ_n is a finite positive measure such that $\sum_{n=1}^{\infty} \mu_n(X) < \infty$.

Lemma D.3.7. Let $\{\mu_n\}$ be a sequence of positive measures on X. Then the function μ defined by $\mu(A) := \sum_{n=1}^{\infty} \mu_n(A)$ is a positive measures on X.

Proof. We need to show μ is countably additive. Let $\{A_n\}$ be a disjoint sequence of measurable sets on X. What we must prove is

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \mu_n(A_m) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \mu_n(A_m)$$

Since each summand is non-negative, this clearly holds. ☐

By this lemma, we see the formula $\mu(A) := \sum_{n=1}^{\infty} \mu_n(A)$ defines a positive measure on X, and since $\mu(X) < \infty$, it is a finite measure. ☐

Finally we discuss the integration with respect to a real or complex measure. Let $(X, \mathcal{A})$ be a measurable space, and denote by $B(X, \mathcal{A}, \mathbb{R})$ and $B(X, \mathcal{A}, \mathbb{C})$ the space of bounded real-valued and complex-valued, respectively, $\mathcal{A}$-measurable functions on X. If μ is a real measure on $(X, \mathcal{A})$ and $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ, then the integral for $f \in B(X, \mathcal{A}, \mathbb{R})$ with respect to μ is defined by

$$\int f d\mu := \int f d\mu^+ - \int f d\mu^-$$

and this defines a linear functional on $B(X, \mathcal{A}, \mathbb{R})$. If $\mu = \nu_1 - \nu_2$ for some finite positive ν_i, then $\nu_1 + \mu^- = \nu_2 + \mu^+$, so

$$\int_X f d\nu_1 + \int_X f d\mu^- = \int_X f d\nu_2 + \int_X f d\mu^+$$

or

$$\int f d\mu = \int f d\mu^+ - \int f d\mu^- = \int f d\nu_1 - \int f d\nu_2$$

If $A \in \mathcal{A}$ and $\mu \in M(X, \mathcal{A}, \mathbb{R})$, then $\int \mathbf{1}_A d\mu = \mu(A)$ holds, so the formula

$$\mu \mapsto \int f d\mu$$

defines a linear functional on $M(X, \mathcal{A}, \mathbb{R})$ when f is a step function. By DCT it follows that this holds for every $f \in B(X, \mathcal{A}, \mathbb{R})$. In sum, we have a bilinear pairing

$$\begin{aligned} B(X, \mathcal{A}, \mathbb{R}) \times M(X, \mathcal{A}, \mathbb{R}) &\longrightarrow \mathbb{R} \\ (f, \mu) &\longmapsto \int_X f d\mu \end{aligned}$$

Similarly if μ is a complex measure on $(X, \mathcal{A})$, we can write the Jordan decomposition of μ to define its integral, and the above statements remain valid in the complex case. Now we define a norm on $B(X, \mathcal{A}, \mathbb{R})$ and $B(X, \mathcal{A}, \mathbb{C})$ by

$$\|f\|_\infty := \sup\{|f(x)| \mid x \in X\}$$

If μ is a real or complex measure on $(X, \mathcal{A})$ and $f = \sum a_i \mathbf{1}_{A_i}$ is a simple function with each A_i disjoint, then

$$\left|\int_X f d\mu\right| = \left|\sum a_i \mu(A_i)\right| \leqslant \sum |a_i||\mu(A_i)| \leqslant \sum |a_i||\mu|(A_i) = \int_X |f| \, d|\mu|$$

Since each function in $B(X, \mathcal{A}, \mathbb{R})$ and $B(X, \mathcal{A}, \mathbb{C})$ is the uniform limit of a sequence of simple functions, it follows that the inequality holds for every bounded measurable function f. In particular, we have

$$\left|\int_X f d\mu\right| \leqslant \|f\|_\infty \|\mu\|$$

D.3.3 *-algebra of complex measures

Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two measurable spaces and let μ and ν be real measures on $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, respectively. Write the Jordan decomposition of $\mu = \mu^+ - \mu^-$, $\nu = \nu^+ - \nu^-$. We then define the product measure $\mu \otimes \nu$ by the formula

$$\mu \otimes \nu = \mu^+ \otimes \nu^+ - \mu^+ \otimes \nu^- - \mu^- \otimes \nu^+ + \mu^- \otimes \nu^-$$

Then for $A \in \mathcal{A}$ and $B \in \mathcal{B}$, the identity $(\mu \otimes \nu)(A \times B) = \mu(A)\nu(B)$ holds, and for $E \in \mathcal{A} \otimes \mathcal{B}$, we have the following integration formula

$$\mu \otimes \nu(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y)$$

All of these follow from Theorem D.4.4. To see the integration formulas above, for example, we have

$$\begin{aligned}
\mu \otimes \nu(E) &= \mu^+ \otimes \nu^+(E) - \mu^+ \otimes \nu^-(E) - \mu^- \otimes \nu^+(E) + \mu^- \otimes \nu^-(E) \\
&= \int_X \nu^+(E_x) d\mu^+(x) - \int_X \nu^-(E_x) d\mu^+(x) - \int_X \nu^+(E_x) d\mu^-(x) + \int_X \nu^-(E_x) d\mu^-(x) \\
&= \int_X \nu^+(E_x) d\mu(x) - \int_X \nu^-(E_x) d\mu(x) \\
&= \int_X \nu(E_x) d\mu(x)
\end{aligned}$$

If μ and μ' are two real measures, then

$$(\mu + \mu') \otimes \nu(E) = \int_X \nu(E_x) d(\mu + \mu')(x) = \int_X \nu(E_x) d\mu(x) + \int_X \nu(E_x) d\mu'(x)$$

so $(\mu + \mu') \otimes \nu = \mu \otimes \nu + \mu' \otimes \nu$. A similar formula holds for the second argument. In sum, the map

$$\begin{aligned}
M(X, \mathcal{A}, \mathbb{R}) \times M(Y, \mathcal{B}, \mathbb{R}) &\longrightarrow M(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mathbb{R}) \\
(\mu, \nu) &\longmapsto \mu \otimes \nu
\end{aligned}$$

is a bilinear map. If μ and ν are complex measures, we define similarly $\mu \otimes \nu \in M(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mathbb{C})$ by using Jordan decomposition for complex measures and linearity, and all the above results hold as well.

Let G be a topological group and consider the Banach space of (finite) complex Borel measures $M(G, \mathbb{C})$ on G. For $\mu, \nu \in M(G, \mathbb{C})$, we define their convolution

$$\mu * \nu(A) := \int_{G \times G} \mathbf{1}_A(xy) d(\mu \otimes \nu)(x, y)$$

In other word, if we write $\mathrm{mult} : G \times G \to G$ to be the multiplication of G, then $\mu * \nu$ is the pushforward measure of the product measure $\mu \otimes \nu$ by mult. It follows from DCT that for all bounded Borel-measurable function f we have

$$\int_G f d(\mu * \nu) := \int_{G \times G} f(xy) d(\mu \otimes \nu)(x, y)$$

When μ and ν are finite positive, by Fubini's theorem, we have

$$\begin{aligned}
\int_{G \times G} \mathbf{1}_A(xy) d(\mu \otimes \nu)(x, y) &= \int_G \left(\int_G \mathbf{1}_A(xy) d\nu(y) \right) d\mu(x) = \int_G \nu(x^{-1}A) d\mu(x) \\
&= \int_G \mu(Ay^{-1}) d\nu(y)
\end{aligned}$$

By linearity this holds for arbitrary complex measures μ and ν, and this implies $*$ defines a bilinear map on $M(G, \mathbb{C})$ particularly. Similarly, we have

$$\int_G f d(\mu * \nu) = \int_G \left(\int_G f(xy) d\mu(x) \right) d\nu(y) = \int_G \left(\int_G f(xy) d\nu(y) \right) d\mu(x)$$

for all bounded Borel-measurable functions f. For $\nu_i \in M(G)$, $i = 1, 2, 3$, we have

$$\begin{aligned}
\nu_1 * (\nu_2 * \nu_3)(A) &= \int_G (\nu_2 * \nu_3)(x^{-1}A) d\nu_1(x) = \int_G \int_G \nu_3(y^{-1}x^{-1}A) d\nu_2(y) d\nu_1(x) \\
(\nu_1 * \nu_2) * \nu_3(A) &= \int_G \nu_3(u^{-1}A) d(\nu_1 * \nu_2)(u) = \int_G \int_G \nu_3((xy)^{-1}A) d\nu_1(x) d\nu_2(y)
\end{aligned}$$

It follows that $*$ is associative. Let μ_e be the Dirac measure at the identity $e \in G$; that is, for all Borel sets A of G, define $\mu_e(A) := \begin{cases} 1 & \text{, if } e \in A \\ 0 & \text{, if } e \notin A \end{cases}$. Then μ_e is the identity element of the operation $*$. Indeed, for $\mu \in M(G)$,

$$\mu * \mu_e(A) = \int_G \mu(Ay^{-1}) d\mu_e(y) = \mu(Ay^{-1})|_{y=e} = \mu(A)$$

and similarly $\mu_e * \mu(A) = \mu(A)$. Finally, if $\{A_i\}$ is a measurable partition of X, then

$$\begin{aligned}
\sum_{i=1}^{\infty} |\mu * \nu(A_i)| &= \sum_{i=1}^{\infty} \left| \int_X \nu(x^{-1}A_i) d\mu(x) \right| \\
&\leqslant \sum_{i=1}^{\infty} \int_X |\nu(x^{-1}A_i)| d|\mu|(x) = \int_X \left(\sum_{i=1}^{\infty} |\nu(x^{-1}A_i)| \right) d|\mu|(x) \\
&\leqslant \int_X \|\nu\| \, d|\mu|(x) = \|\nu\| \, \|\mu\|
\end{aligned}$$

Letting $\{A_i\}$ run over all possible measurable partitions of X, we obtain $\|\mu * \nu\| \leqslant \|\mu\| \, \|\nu\|$. In sum,

Proposition D.3.8. $(M(G, \mathbb{C}), *)$ is a unital Banach algebra.

Let $\mathrm{inv} : G \to G$ be the inversion on G. For $\mu \in M(G, \mathbb{C})$, define a new measure $\mu^* \in M(G, \mathbb{C})$

$$\mu^*(A) = \overline{(\mathrm{inv}_* \mu)(A)} = \overline{\mu(A^{-1})}$$

where $\bar{\cdot}$ is the complex conjugation; this is well-defined since inv is a homeomorphism. It is clear that $\mu \mapsto \mu^*$ defines an involution on $M(G, \mathbb{C})$, and by definition of total variation we have $\|\mu\| = \|\mu^*\|$; this makes $M(G, \mathbb{C})$ a unital Banach *-algebra.

Now assume G is a locally compact Hausdorff group. Choose a left Haar measure dx once and for all. There is a map

$$\begin{aligned}
L^1(G) &\longrightarrow M(G, \mathbb{C}) \\
f &\longmapsto d\mu_f(x) := f(x)dx
\end{aligned}$$

In fact, this is an algebra *-homomorphism. Indeed, if $f, g \in L^1(G)$, then

$$\begin{aligned}
\mu_{f*g}(A) &\overset{\text{Fubini}}{=} \int_G \int_G \mathbf{1}_A(x) f(y) g(y^{-1}x) dx dy \\
(x \mapsto yx) &= \int_G \left(\int_G \mathbf{1}_A(yx) g(x) dx \right) f(y) dy \\
&= \int_G \mu_g(y^{-1}A) d\mu_f(y) \\
&= \mu_f * \mu_g(A)
\end{aligned}$$

and

$$\mu_{f^*}(A) = \int_G \mathbf{1}_A(x) \Delta_G(x^{-1}) \overline{f(x^{-1})} dx \overset{2.3.1.4}{=} \int_G \mathbf{1}_A(x^{-1}) \overline{f(x)} dx = \overline{\int_G \mathbf{1}_{A^{-1}}(x) d\mu_f(x)} = \mu_f^*(A)$$

It is also norm-preserving, for by Corollary D.6.5.1 we have $|\mu_f| = \mu_{|f|}$, so

$$\|\mu_f\| = \int_X d|\mu_f| = \int_X |f| dx = \|f\|_1$$

D.3.4 Regular measure

In what follows let X denote a locally compact Hausdorff space and $\mathcal{B}$ be the σ-algebra of Borel sets in X. For simplicity, we put $M(X, \mathbb{F}) = M(X, \mathcal{B}, \mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Definition. For a topological space, a complex Borel measure μ on it is called **regular** if its total variation $|\mu|$ is a **regular** measure, in the sense that it is finite on compact sets and every measurable set can be approximated by open sets from above, and every open set can be approximated by compact sets from below.

- In our previous term, a positive measure is regular if and only if it is finite on compact sets, outer regular, and weakly inner regular. If X is LCH, then a positive regular measure is the same as a outer Radon measure.
- If μ is regular, for Borel A and $\varepsilon > 0$ we can find a compact set $K \subseteq A$ such that for all Borel B with $K \subseteq B \subseteq A$ we have $|\mu(A) - \mu(B)| < \varepsilon$. Indeed, since $|\mu|$ is regular, we can find compact K such that $|\mu|(A \backslash K) < \varepsilon$. Then for any such B, we have
$$|\mu(A) - \mu(B)| = |\mu(A \backslash B)| \leqslant |\mu|(A \backslash B) \leqslant |\mu|(A \backslash K) < \varepsilon$$

Proposition D.3.9. Let μ be a complex Borel measure on X. TFAE:

(i) μ is regular;

(ii) the positive measures appearing in the Jordan decomposition are regular;

(iii) μ is a linear combination of finite positive regular Borel measures.

Proof. Let μ be complex regular and let μ' be any positive finite measure appearing in the Jordan decomposition of μ. Then $\mu' \leqslant |\mu|$. Let $\varepsilon > 0$ be given.

- If A is Borel and $U \supseteq A$ is open such that $|\mu|(A) < |\mu|(U) + \varepsilon$, then $\mu'(U \backslash A) \leqslant |\mu|(U \backslash A) < \varepsilon$ so that
$$\mu'(U) = \mu'(A) + \mu'(U \backslash A) < \mu'(A) + \varepsilon$$
- If A is Borel and $K \subseteq A$ is compact such that $|\mu|(A) - \varepsilon < |\mu|(K)$, then $\mu'(A \backslash K) \leqslant |\mu|(A \backslash K) < \varepsilon$ so that
$$\mu'(K) = \mu'(A) - \mu'(A \backslash K) > \mu'(A) - \varepsilon$$

This shows μ' is regular, so (ii) holds. Clearly (ii) implies (iii). For (iii) $\Rightarrow$ (i), the proof is similar to that of (i) $\Rightarrow$ (ii) and use the fact that if $\mu = \alpha_1 \mu_1 + \cdots + \alpha_n \mu_n$ where $\alpha_i \in \mathbb{C}$ and the μ_i are positive, then $|\mu| \leqslant |\alpha_1| \mu_1 + \cdots + |\alpha_n| \mu_n$. ☐

Let us denote by $M_r(X, \mathbb{F}) \subseteq M(X, \mathbb{F})$ the set of all $\mathbb{F}$-valued regular Borel measures, where $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$.

Lemma D.3.10. $M_r(X, \mathbb{F})$ is a closed subspace of $M(X, \mathbb{F})$.

Proof. By the previous proposition we see $M_r(X, \mathbb{F})$ is a linear subspace. It remains to show $M_r(X, \mathbb{F})$ is closed. First note that if $\mu \in M_r(X, \mathbb{F})$ and $\nu \in M(X, \mathbb{F})$ with $\|\mu - \nu\| < \varepsilon$ for some $\varepsilon > 0$, then for any Borel A and open $U \supseteq A$ with $|\mu|(U \backslash A) < \varepsilon$, we have
$$|\nu|(U \backslash A) \leqslant \|\nu - \mu\| + |\mu|(U \backslash A) < 2\varepsilon$$

This shows the limit of sequences of outer regular measures is again outer regular. Similarly for weakly inner regular. This concludes the proof. ☐

Proposition D.3.11. Let G be an LCH group. The subspace $M_r(G, \mathbb{F})$ is *-closed. Hence $M_r(G, \mathbb{F})$ is a (non-unital) Banach *-algebra.

Proposition D.3.12. Let μ be a regular positive Borel measure on X and $f \in L^1(\mu)$. Then the complex measure μ_f is regular.

Proposition D.3.13. Let μ be a regular positive Borel measure on X and $\nu \in M_r(X, \mathbb{C})$. TFAE:

(i) There exists $f \in L^1(\mu)$ such that $\nu = \mu_f$.

(ii) $\nu \ll \mu$.

(iii) Each compact subset K of X that satisfies $\mu(K) = 0$ also satisfies $\nu(K) = 0$.

Proof. By regularity (ii)$\Leftrightarrow$(iii), and clearly (i)$\Rightarrow$(ii). It remains to show (ii)$\Rightarrow$(i). Since $|\nu|$ is finite regular, we can find an increasing sequence of compact subsets K_n in X such that $|\nu|(K_n) \to |\nu|(X)$. Locally finiteness of μ implies $\mu(K_n)$ is finite, so the measure μ_0 defined y $\mu_0(A) := \mu\left(A \cap \bigcup_{n \geqslant 1} K_n\right)$ is σ-finite. Now $\nu \ll \mu$ implies $\nu \ll \mu_0$, so by Radon-Nikodym theorem there exists $f \in L^1(\mu_0)$ such that $\nu = (\mu_0)_f$. By redefining f so that it vanishing outside $\bigcup_{n \geqslant 1} K_n$, we have $\nu = \mu_f$. ☐

Corollary D.3.13.1. Let X be an LCH space and μ a regular positive Borel measure on X. Then the map

$$\begin{aligned} L^1(\mu) &\longrightarrow M(X, \mathbb{C}) \\ f &\longmapsto d\mu_f(x) := f(x)dx \end{aligned}$$

induces a linear isometry onto the subspace $M_r(X, \mathbb{C})$ consisting of those ν with $\nu \ll \mu$.

D.4 Fubini's Theorem

D.4.1 Product Measure

Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces. Define $\mathcal{A} \otimes \mathcal{B}$ to be the smallest σ-algebra on $X \times Y$ which contains $\{A \times B \mid (A, B) \in \mathcal{A} \times \mathcal{B}\}$. For $E \subseteq X \times Y$, $x \in X$, $y \in Y$, define

$$E_x := \{y \in Y \mid (x, y) \in E\} \qquad E^y := \{x \in X \mid (x, y) \in E\}$$

These are called the x**-section** and y**-section** of E, respectively.

Lemma D.4.1. If $E \in \mathcal{A} \otimes \mathcal{B}$, then $E_x \in \mathcal{B}$ and $E^y \in \mathcal{A}$ for all $x \in X$ and $y \in Y$.

Similarly, for $f \in X \times Y \to \mathbb{C}$, $x \in X$, $y \in Y$, define

$$\begin{aligned} f_x : Y &\longrightarrow \mathbb{C} \\ y &\longmapsto f(x, y) \end{aligned} \qquad \begin{aligned} f^y : X &\longrightarrow \mathbb{C} \\ x &\longmapsto f(x, y) \end{aligned}$$

Lemma D.4.2. Let f be a $\mathcal{A} \otimes \mathcal{B}$-measurable function on $X \times Y$. Then

1. f_x is $\mathcal{B}$-measurable for each $x \in X$.

2. f^y is $\mathcal{A}$-measurable for each $y \in Y$.

Now comes to measure. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces.

Lemma D.4.3. For $E \in \mathcal{A} \otimes \mathcal{B}$, $[x \mapsto \nu(E_x)]$ is $\mathcal{A}$-measurable and $[y \mapsto \mu(E^y)]$ is $\mathcal{B}$-measurable.

Theorem D.4.4. Let X, Y be as above. Then there exists a unique measure $\mu \otimes \nu$ on the σ-algebra $\mathcal{A} \otimes \mathcal{B}$ such that

$$(\mu \otimes \nu)(A \times B) = \mu(A)\nu(B)$$

for all $(A, B) \in \mathcal{A} \times \mathcal{B}$. Further, for $E \in \mathcal{A} \otimes \mathcal{B}$,

$$(\mu \otimes \nu)(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y)$$

D.4.2 Fubini's in σ-finite case

Theorem D.4.5 (Fubini's Theorem)**.** Let (X, μ) and (Y, ν) be two σ-finite measure spaces, and let f be a measurable function on $X \times Y$.

(a) If $f \geqslant 0$, then the two partial integrals $\int_X f(x,y) d\mu(x)$ and $\int_Y f(x,y) d\nu(y)$ define measurable functions such that Fubini's formula holds:

$$\int_{X \times Y} f(x,y) d(\mu \otimes \nu)(x,y) = \int_X \int_Y f(x,y) d\nu(y) d\mu(x) = \int_Y \int_X f(x,y) d\mu(x) d\nu(y)$$

(b) If f is complex valued and one of the iterated integrals

$$\int_X \int_Y |f(x,y)| d\nu(y) d\mu(x) \text{ or } \int_Y \int_X |f(x,y)| d\mu(x) d\nu(y)$$

is finite, then f is integrable and the Fubini's formula holds.

D.4.3 Fubini's in LCH cases

Lemma D.4.6. Let μ be an outer Radon measure on X. Suppose $\mathcal{F}$ is a subfamily of $C_c(X)$ consisting of functions $\phi \geqslant 0$ such that for $\phi, \psi \in \mathcal{F}$ there exists a function $\eta \in \mathcal{F}$ with $\eta \geqslant \phi$, ψ. Then

(i) $x \mapsto \sup_{\phi \in \mathcal{F}} \phi(x)$ is measurable.

(ii) If $\left\{ x \in X \mid \sup_{\phi \in \mathcal{F}} \phi(x) \neq 0 \right\}$ is σ-finite, then $\sup_{\phi \in \mathcal{F}} \int_X \phi(x) dx = \int_X \sup_{\phi \in \mathcal{F}} \phi(x) dx$.

Proof. Put $g(x) := \sup_{\phi \in \mathcal{F}} \phi(x)$.

(i) Since each ϕ is measurable, so is g.

(ii) By assumption $\{x \in X \mid g(x) \neq 0\}$ is σ-finite. This implies the integration of g can be computed by simple functions with finite support.

That $\geqslant$ is obvious. For $\leqslant$, let $s = \sum_{i=1}^{m} a_i \mathbf{1}_{A_i}$ be a simple function with $s \leqslant g$ and $\mu(A_i) < \infty$. By weakly regularity for a given $\varepsilon > 0$ we can find compact $K_i \subseteq A_i$ such that

$$\int_X s d\mu < \sum_{i=1}^{m} a_i \mu(K_i) + \varepsilon$$

Let $K = \bigcup_{i=1}^{m} K_i$ and write $s_0 = \sum_{i=1}^{m} a_i \mathbf{1}_{K_i}$. For given $0 < \delta < 1$ one has $(1-\delta)s_0(x) < g(x)$ for all $x \in K$. In particular, for each $x \in K$ we can find $\phi_x \in \mathcal{F}$ such that $(1-\delta)s_0(x) < \phi_x(x)$. Consider the open sets

$$U_x := \{y \in X \mid (1-\delta)s_0(y) < \phi_x(y)\}$$

These form an open cover of K, so by compactness we can find $x_1, \ldots, x_n \in K$ such that $K \subseteq U_{x_1} \cup \cdots \cup U_{x_n}$. By assumption there exists $\phi \in \mathcal{F}$ such that $\phi \geqslant \phi_{x_1}, \ldots, \phi_{x_n}$. Then $\phi > (1-\delta)s_0$, so that

$$\int_X s d\mu < \int_X s_0 d\mu + \varepsilon < \frac{1}{1-\delta} \int_X \phi d\mu + \varepsilon$$

Varying $\phi \in \mathcal{F}$ first, then $s \leqslant g$, we obtain (by the remark in the beginning)

$$\int_X g d\mu \leqslant \frac{1}{1-\delta} \sup_{\phi \in \mathcal{F}} \int_X \phi d\mu + \varepsilon$$

Letting $\delta, \varepsilon \to 0^+$ concludes the proof.

□

Theorem D.4.7 (Fubini's Theorem for Radon measures)**.** Let μ and ν be outer Radon measures on the LCH spaces X and Y, respectively. Then there exists a unique outer Radon measure $\mu \otimes \nu$ on $X \times Y$ such that

(a) If $f : X \times Y \to \mathbb{C}$ is $\mu \otimes \nu$-integrable, then the partial integrals $\int_X f(x,y) d\mu(x)$ and $\int_Y f(x,y) d\nu(y)$ define integrable functions such that Fubini's formula holds:

$$\int_{X \times Y} f(x,y) d(\mu \otimes \nu)(x,y) = \int_X \int_Y f(x,y) d\nu(y) d\mu(x) = \int_Y \int_X f(x,y) d\mu(x) d\nu(y)$$

(b) If $f : X \times Y \to \mathbb{C}$ is measurable such that $A = \{(x,y) \in X \times Y \mid f(x,y) \neq 0\}$ is σ-finite, then if one of the iterated integrals

$$\int_X \int_Y |f(x,y)| d\nu(y) d\mu(x) \text{ or } \int_Y \int_X |f(x,y)| d\mu(x) d\nu(y)$$

is finite, then f is integrable and the Fubini's formula holds.

Proof. The uniqueness follows from the Riesz's Representation Theorem, since the Fubini's formula determines the values of the integral on $C_c(X \times Y)$.

For the existence, observe that on each compact set $K = K_1 \times K_2$ there is a unique product measure $(\mu \otimes \nu)_K$ on K such that integration with respect to $(\mu \otimes \nu)_K$ is given by Fubini's formula, by the classical Fubini's. Since $(\mu \otimes \nu)_L$ restricts to $(\mu \otimes \nu)_K$ whenever $K \subseteq L$ with $L = L_1 \times L_2$ compact, these measures $(\mu \otimes \nu)_K$ determines a well-defined positive functional on $C_c(X \times Y)$. Let $\mu \otimes \nu$ denote the resulting outer Radon measure from the Riesz's Representation Theorem.

To show (a), a standard argument reduces us to the case $f = \mathbf{1}_A$ for some measurable $A \subseteq X \times Y$

with finite measure. If $A = U$ is open (with finite measure), a repeated use of Lemma D.4.6 shows

$$\begin{aligned}\int_X \int_Y \mathbf{1}_U(x,y)dydx &= \int_X \int_Y \sup_{0\leqslant\phi\leqslant\mathbf{1}_U} \phi(x,y)dydx = \sup_{0\leqslant\phi\leqslant\mathbf{1}_U} \int_X \int_Y \phi(x,y)dydx \\ &= \sup_{0\leqslant\phi\leqslant\mathbf{1}_U} \int_{X\times Y} \phi(x,y)d(\mu\otimes\nu)(x,y) \\ &= \int_{X\times Y} \sup_{0\leqslant\phi\leqslant\mathbf{1}_U} \phi(x,y)d(\mu\otimes\nu)(x,y) \\ &= \int_{X\times Y} \mathbf{1}_U(x,y)d(\mu\otimes\nu)(x,y)\end{aligned}$$

If $A = K$ is compact, let V be a relatively compact open neighborhood of K; then $\mathbf{1}_K = \mathbf{1}_V - \mathbf{1}_{V\setminus K}$, so (a) holds in this case as well. It follows from MCT that (a) holds for A being σ-compact. But by weakly inner regularity a measurable set with finite measure is a disjoint union of σ-compact set and a null set, it remains to consider the case $A = N$ being null. Let $\varepsilon > 0$ and U an open set containing N with measure less than ε. Then

$$\int_X \int_Y \mathbf{1}_N(x,y)dydx \leqslant \int_X \int_Y \mathbf{1}_U(x,y)dydx = \int_{X\times Y} \mathbf{1}_U < \varepsilon$$

Letting $\varepsilon \to 0$ gives (a).

For (b), it suffices to show $|f|$ is integrable. Write $A = \bigcup_{n=1}^{\infty} A_n$ with $(A_n)_n$ increasing and each A_n being of finite measure. Define $f_n : X \times Y \to \mathbb{C}$ by $f_n = \min\{|f|\mathbf{1}_{A_n}, n\}$. Then $(f_n)_n$ is an increasing sequence of integral functions converging pointwise to $|f|$. By (a), we have

$$\int_{X\times Y} f_n(x,y)d(\mu\otimes\nu)(x,y) = \int_Y \int_X f_n(x,y)dxdy \leqslant \int_Y \int_X |f(x,y)|dxdy$$

for all $n \in \mathbb{N}$. Now (b) follows from MCT. □

D.5 L^p-space and the Riesz-Fischer Theorem

Definition. Let $(X, \mathcal{A}, \mu)$ be a measure space.

1. For $1 \leqslant p < \infty$, let $\mathcal{L}^p(X)$ be the set of all measurable functions $f : X \to \mathbb{C}$ such that

$$\|f\|_p := \left(\int_X |f|^p d\mu\right)^{\frac{1}{p}} < \infty$$

2. Let $\mathcal{L}^\infty(X)$ be the set of measurable functions $f : X \to \mathbb{C}$ such that f is bounded outside a measure zero set. For such f, put

$$\begin{aligned}\|f\|_\infty &= \inf\{c > 0 \mid \mu\{|f(x)| > c\} = 0\} \\ &= \inf_{\mu(N)=0} \sup_{x\notin N} |f(x)|\end{aligned}$$

In other words, $\mathcal{L}^\infty(X)$ consists of measurable functions $f : X \to \mathbb{C}$ with finite $\|f\|_\infty < \infty$.

- We show two definitions for $\|\cdot\|_\infty$ coincide. Let $c > 0$ be such that $N_c := \{x \in X \mid |f(x)| > c\}$ is null. Then for $x \notin N_c$, we have $|f(x)| \leqslant c$, so that

$$\inf_{\mu(N)=0} \sup_{x\notin N} |f(x)| \leqslant c.$$

On the other hand, let N be a null set such that f is bounded outside N and put $c_N := \sup_{x \notin N} |f(x)|$. Then $\{x \in X \mid |f(x)| > c_N\} \subseteq N$ is null, and

$$\inf\{c > 0 \mid \mu\{|f(x)| > c\} = 0\} \leqslant c_N$$

- Let $f \in \mathcal{L}^\infty(X)$. The set $\{x \in X \mid |f(x)| \geqslant \|f\|_\infty\}$ is μ-null, since

$$\{x \in X \mid |f(x)| > \|f\|_\infty\} = \bigcap_{n \geqslant 1} \left\{x \in X \mid |f(x)| > \|f\| + \frac{1}{n}\right\}$$

  and each set involved is null by definition.

Proposition D.5.1 (Jensen)**.** Let $(X, \mathcal{A}, \mu)$ be a positive measure space with $\mu(X) = 1$. Suppose $\varphi : (a, b) \to \mathbb{R}$ is a convex function, $f \in \mathcal{L}^1(\mu)$ with $a \leqslant f(x) \leqslant b$ for $x \in X$. Then

$$\varphi\left(\int_X f d\mu\right) \leqslant \int_X \varphi \circ f d\mu.$$

Proposition D.5.2 (Young)**.** Let $a, b \geqslant 0$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}$$

with equality if and only if $a^p = b^q$.

Proof. This follows from Jensen's inequality: since $-\log$ is convex, we have

$$-\log\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \leqslant \frac{-\log a^p}{p} + \frac{-\log b^q}{q} = -\log ab.$$

□

Proposition D.5.3 (Hölder)**.** Let $1 \leqslant p_1, \ldots, p_n \leqslant \infty$ be such that $\sum_{i=1}^n \frac{1}{p_i} = 1$ and $f_i \in \mathcal{L}^{p_i}$, $i = 1, \ldots, n$. Then $\prod_{i=1}^n f_i \in \mathcal{L}^1$ and

$$\left\|\prod_{i=1}^n f_i\right\|_1 \leqslant \prod_{i=1}^n \|f_i\|_{p_i}$$

Proof. First assume $1 \leqslant p_1, \ldots, p_n < \infty$. By normalizing we may assume $\|f_i\|_{p_i} = 1$; then we must show $\left\|\prod_{i=1}^n f_i\right\| \leqslant 1$. To this end, we invoke the Jensen's inequality: for all $a_i > 0$, one has

$$\sum_{i=1}^n \frac{1}{p_i} \log a_i \leqslant \log \sum_{i=1}^n \frac{1}{p_i} a_i$$

Apply this inequality with $a_i = |f_i|^{p_i}$; we obtain

$$\log |f_1 \cdots f_n| \leqslant \log\left(\frac{|f_1|^{p_1}}{p_1} + \cdots + \frac{|f_n|^{p_n}}{p_n}\right)$$

or $|f_1 \cdots f_n| \leqslant \frac{|f_1|^{p_1}}{p_1} + \cdots + \frac{|f_n|^{p_n}}{p_n}$. This shows $f_1 \cdots f_n$ is integrable. Integrating both sides we see

$$\|f_1 \cdots f_n\|_1 = \int_X |f_1 \cdots f_n| d\mu \leqslant \int_X \left(\frac{|f_1|^{p_1}}{p_1} + \cdots + \frac{|f_n|^{p_n}}{p_n}\right) d\mu = \frac{1}{p_1} + \cdots + \frac{1}{p_n} = 1$$

Let us deal with the case $p_1 = \cdots = p_s = \infty$. Then $p_{s+1}^{-1} + \cdots + p_n^{-1} = 1$ and the result above shows $\|f_{s+1} \cdots f_n\|_1 \leqslant \|f_{s+1}\|_{p_{s+1}} \cdots \|f_n\|_{p_n}$. By induction we are reduced to the situation that if $\|f\|_1 < \infty$ and $\|g\|_\infty < \infty$, then $\|fg\|_1 < \infty$. This is trivial, for $|fg| \leqslant |f| \, \|g\|_\infty$ a.e., and thus

$$\int_X |fg| d\mu \leqslant \int_X |f| d\mu \, \|g\|_\infty = \|f\|_1 \, \|g\|_\infty$$

□

Proposition D.5.4 (Minkowski)**.** Let $1 \leqslant p \leqslant \infty$. Then for $f, g \in \mathcal{L}^p$, we have $f + g \in \mathcal{L}^p$ with

$$\|f + g\|_p \leqslant \|f\|_p + \|g\|_p$$

Thus $\|\cdot\|_p$ is a semi-norm on $\mathcal{L}^p$ for every $1 \leqslant p \leqslant \infty$.

Let $1 \leqslant p \leqslant \infty$. Since $\|\cdot\|_p$ is a semi-norm in $\mathcal{L}^p$, it follows that the zero set $N_p := \|\cdot\|_p^{-1}(0)$ of the semi-norm forms a $\mathbb{C}$-vector subspace of the $\mathcal{L}^p$. Define

$$L^p := \mathcal{L}^p / N_p$$

Note that $\|f\|_p = 0$ if and only if f is a **null function**, i.e., $f|_{X \setminus N} \equiv 0$ for some set N of measure zero. Then $\|\cdot\|_p$ is a norm on L^p. We ask whether L^p is a Banach space.

Theorem D.5.5 (Riesz-Fischer)**.** Let $1 \leqslant p \leqslant \infty$ and suppose (f_n) is a sequence in $\mathcal{L}^p(X)$ that is Cauchy with respect to $\|\cdot\|_p$. Then there exists $f \in \mathcal{L}^p(X)$ and a subsequence $(f_{n_k})_k$ of (f_n) such that $f_{n_k}(x) \to f(x)$ for every x outside a set of measure zero.

Proof. We distinguish the cases $p = \infty$ and $1 \leqslant p < \infty$. First consider the easier case, namely when $p = \infty$. Suppose $(f_n)_n$ is a Cauchy sequence in $\mathcal{L}^\infty$. Put

$$\begin{aligned} A_n &= \{x \in X \mid |f_n(x)| > \|f_n\|_\infty\} \qquad (n \in \mathbb{N}) \\ B_{n,m} &= \{x \in X \mid |f_n(x) - f_m(x)| > \|f_n - f_m\|_\infty\} \qquad (n, m \in \mathbb{N}) \end{aligned}$$

All A_n and $B_{n,m}$ are null, so if we put E to be their (countable) union, then E is null, and on the complement of E, the sequence $(f_n)_n$ converges uniformly to a bounded function f on $X \backslash E$. Extend f across E by zero; then $f \in \mathcal{L}^\infty$ and

$$\|f_n - f\|_\infty \leqslant \sup_{x \notin E} |f(x) - f_n(x)| \to 0 \text{ as } n \to \infty.$$

□

Lemma D.5.6.

1. A Hilbert space is separable if and only if it has a countable orthonormal basis.

2. If X is a second countable Hausdorff space with a finite Radon measure μ, then the Hilbert space $L^2(X, \mu)$ is separable.

Proof.

1.

2. Let C be a countable basis of X. Any element in C has finite measure. By adding every finite intersection of sets in C to itself, we may assume C is closed under finite intersection and C is still countable. Consider the subspace H spanned by all $\mathbf{1}_U$ with $U \in C$. By exclusion-inclusion principal, H contains all characteristic functions of finite unions of sets in C.

   Let U be any open subset of X and say $U = \bigcup_{n \in \mathbb{N}} U_n$ for $U_n \in C$. Then $\mathbf{1}_{U_1 \cup \cdots \cup U_n} \to \mathbf{1}_U$ in L^2. It follows that $\overline{H} \subseteq L^2(X, \mu)$ contains all characteristic function of open sets. Since μ is Radon, $\overline{H}$ contains all simple functions. Since the space of simple functions are dense in $L^2(X, \mu)$, it follows that $\overline{H} = L^2(X, \mu)$.

□

Lemma D.5.7. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two finite positive measure spaces. Then for $1 \leqslant p < \infty$, the map

$$L^p(X, \mu) \otimes_{\mathbb{C}} L^p(Y, \nu) \longrightarrow L^p(X \times Y, \mu \otimes \nu)$$

is injective with dense image.

Proof. By the definition of product measures and integration, it suffice to show that each $\mathbf{1}_C$ ($C \in \mathcal{A} \otimes \mathcal{B}$) can be approximated by $\mathbf{1}_{A \times B}$ ($(A, B) \in \mathcal{A} \times \mathcal{B}$). Consider

$$S = \left\{ C \underset{\text{meas}}{\subseteq} \Omega \times \Omega \mid \text{for all } \varepsilon > 0 \text{ there exist } A, B \underset{\text{meas}}{\subseteq} \Omega \text{ with } |\mathbf{1}_C - \mathbf{1}_{A \times B}| < \varepsilon \right\}.$$

□

D.5.1 Duality

Let μ be a positive measure on X, and suppose $1 \leqslant p \leqslant \infty$ and let q be the exponent conjugate of p, i.e., $\dfrac{1}{p} + \dfrac{1}{q} = 1$. By Hölder's inequality, if $g \in L^q(\mu)$, then the map $\Phi_g : L^p(\mu) \to \mathbb{C}$ defined by

$$\Phi_g(f) := \int_X f g d\mu$$

is a bounded linear functional with norm at most $\|g\|_q$. In sum, the map

$$\begin{aligned} \Phi : L^q(\mu) &\longrightarrow L^p(\mu)^\vee \\ g &\longmapsto \Phi_g \end{aligned}$$

is a norm-decreasing linear functional.

Proposition D.5.8. For $1 \leqslant p < \infty$, the map Φ is norm-preserving.

Proof. □

D.6 Radon-Nikodym Theorem

In this section we fix a measurable space $(X, \mathcal{A})$. By a (unadorned) measure on $(X, \mathcal{A})$ we mean either the positive or the complex measure.

Definition. Let μ be a positive measure on $(X, \mathcal{A})$. A measure λ is **absolutely continuous** with respect to μ if $\lambda(E) = 0$ for $E \in \mathcal{A}$ whenever $\mu(E) = 0$. In this case we write $\lambda \ll \mu$.

Definition.

1. Let λ be a measure on $(X, \mathcal{A})$. If there exists $A \in \mathcal{A}$ such that $\lambda(E) = \lambda(E \cap A)$ for all $E \in \mathcal{A}$, we say λ is **concentrated on** A**.**

2. Two measures λ_1 and λ_2 on $(X, \mathcal{A})$ are called **mutually singular** if there exists $A \in \mathcal{A}$ such that λ_1 is concentrated on A and λ_2 is concentrated on A^c. In this case we write $\lambda_1 \perp \lambda_2$.

Proposition D.6.1. Let $\lambda, \lambda_1, \lambda_2$ be measures on $(X, \mathcal{A})$ and μ be a positive measure.

(i) If λ is concentrated on A, so is $|\lambda|$.

(ii) If $\lambda_1 \perp \lambda_2$, then $|\lambda_1| \perp |\lambda_2|$.

(iii) If $\lambda_1 \perp \lambda$ and $\lambda_2 \perp \lambda$, then $\lambda_1 + \lambda_2 \perp \lambda$.

(iv) If $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, then $\lambda_1 + \lambda_2 \ll \mu$.

(v) If λ is signed, then $\lambda \ll \mu$ if and only if $\lambda^{\pm} \ll \mu$.

(vi) If $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 \perp \lambda_2$.

(vii) If $\lambda \ll \mu$ and $\lambda \perp \mu$, then $\lambda = 0$.

Proof.

(i) For all measurable E, $|\lambda|(E) = \sup\limits_{\{E_n\}} \sum |\lambda(E \cap E_n)| = \sup\limits_{\{E_n\}} \sum |\lambda(E \cap E_n \cap A)| = |\lambda|(E \cap A)$

(ii) This follows from (i).

(iii) Say λ_i is concentrated on A_i. Then $\lambda_1 + \lambda_2$ is concentrated on $A := A_1 \cup A_2$ and λ is concentrated on A^c.

(iv) Obvious.

(v) Let (P, N) be a Hahn decomposition of λ. Let $A \in \mathcal{A}$ be such that $\mu(A) = 0$. Then $\mu(A \cap P) = 0 = \mu(A \cap N)$. If $\lambda \ll \mu$, then $\lambda^+(A) = \lambda(A \cap P) = 0 = \lambda(A \cap N) = \lambda^-(A)$, so $\lambda^{\pm} \ll \mu$. The converse follows from (iv).

(vi) Say λ_2 is concentrated on A. Then for all measurable $E \subseteq A$, since μ is concentrated on A^c, $\mu(E) = 0$, so that $\lambda_1(E) = 0$ by absolute continuity. Hence λ_1 is concentrated on A^c.

(vii) By (vi) we have $\lambda \perp \lambda$, so $\lambda = 0$ obviously.

$\square$

Proposition D.6.2. Let μ be a positive measure on $(X, \mathcal{A})$. Let $F = \mathbb{R}$ or $\mathbb{C}$.

1. The subspace $M_{\mu,s} := \{\nu \in M(X, \mathcal{A}, F) \mid \nu \perp \mu\}$ is closed in $M(X, \mathcal{A}, F)$.

2. The subspace $M_{\mu,a} := \{\nu \in M(X, \mathcal{A}, F) \mid \nu \ll \mu\}$ is closed in $M(X, \mathcal{A}, F)$.

Proof.

1. Let $(\nu_n)_n \subseteq M_{\mu,s}$ be a sequence converging to ν. Say ν_n is concentrated on A_n and μ is concentrated on B_n with $A_n \cap B_n = \varnothing$. Put $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcap_{n=1}^{\infty} B_n$. Then μ is clearly concentrated on B. If E is measurable with $E \cap A = \varnothing$. Then in particular, $E \cap A_n = \varnothing$ for each n, so $\nu_n(E) = \nu_n(E \cap A_n) = 0$ for all n, whence $\nu(E) = \lim_{n\to\infty} \nu_n(E) = 0$. This shows ν is concentrated on A, so $\nu \perp \mu$.

2. Let $(\nu_n)_n \subseteq M_{\mu,a}$ be a sequence converging to ν. If $\mu(A) = 0$, then $\nu_n(A) = 0$ for all n, so $\nu(A) = \lim_{n\to\infty} \nu_n(A) = 0$. Thus $\nu \ll \mu$.

□

Theorem D.6.3. Let μ be a positive σ-finite measure on $(X, \mathcal{A})$, and λ be a positive σ-finite or complex measure. If $\lambda \ll \mu$, then there exists a unique function h which is

- finite nonnegative $\mathcal{A}$-measurable if λ is positive σ-finite, or
- complex μ-integrable if λ is complex

such that

$$\lambda(A) = \int_A h d\mu$$

for all $A \in \mathcal{A}$. The function h is called the **Radon-Nikodym derivative** of λ with respect to μ, and we write $h = \dfrac{d\lambda}{d\mu}$ and $d\lambda = hd\mu$.

Proof. If λ is complex, we can use Jordan decomposition for complex measures to write λ as a sum of positive finite measures which are absolutely continuous with respect to μ. If λ is positive σ-finite, say $X = \bigcup_{n=1}^{\infty} X_n$ with $\lambda(X_n) < \infty$ and $\mu(X_n) < \infty$ for each n, then by restricting to each X_n we can assume λ and μ are both finite. Hence we are reduced to show the following statement. If μ and λ are positive finite measures with $\lambda \ll \mu$, then there exists a real valued μ-integrable function h such that $\lambda(A) = \int_A hd\mu$ for all measurable A.

As said in the previous paragraph, we assume μ and λ are finite positive. Set $\tau = \mu + \lambda$. For every $\phi \in L^2(\tau)$, by Cauchy Schwarz we have

$$\left|\int_X \phi d\mu\right| \leqslant \int_X |\phi| d\mu \leqslant \int_X |\phi| d\tau \leqslant \left(\int_X |\phi|^2 d\tau\right)^{\frac{1}{2}} \tau(X)^{\frac{1}{2}}$$

Hence the map

$$\begin{aligned} L^2(\tau) &\longrightarrow \mathbb{C} \\ \phi &\longmapsto \int_X \phi d\mu \end{aligned}$$

is a bounded linear functional. The same is true for the map $\phi \mapsto \int_X \phi d\lambda$. Since $L^2(\tau)$ is a Hilbert space, by Riesz's Representation theorem there exist unique τ-square-integrable functions f and g such that

$$\int_X \phi d\lambda = \int_X f\phi d\tau, \qquad \int_X \phi d\mu = \int_X g\phi d\tau$$

for all $\phi \in L^2(\tau)$.

Let ϕ be a nonnegative $\mathcal{A}$-measurable function. Let $(\phi_n)_n$ be an increasing sequence of positive step functions such that $\phi_n \to \phi$ pointwise. By MCT applied to the identity

$$\int_X \phi_n d\lambda = \int_X f\phi_n d\tau$$

we see $\phi \in L^1(\lambda)$ if and only if $f\phi \in L^1(\tau)$. Similarly, $\phi \in L^1(\mu)$ if and only if $g\phi \in L^1(\tau)$.

Let $N = \{x \in X \mid g(x) = 0\}$. Then $\mu(N) = \int_X g\mathbf{1}_N d\tau = 0$, so $\lambda(N) = 0$ as well by absolute continuity $\lambda \ll \mu$. Define

$$h(x) = \begin{cases} \dfrac{f(x)}{g(x)} & \text{, if } x \in X\backslash N \\ 0 & \text{, if } x \in N \end{cases}$$

For every measurable $\phi : X \to \mathbb{C}$, $\phi \in L^1(\lambda)$ if and only if $f\phi = gh\phi \in L^1(\tau)$ (since $\lambda(N) = 0$), if and only if $h\phi \in L^1(\mu)$ (since $\mu(N) = 0$). Hence

$$\int_X \phi d\lambda = \int_X f\phi d\tau = \int_X gh\phi d\tau = \int_X h\phi d\mu$$

for all $\phi \in L^1(\lambda)$. In particular, by taking $\phi = g \in L^2(\tau) \subseteq L^1(\lambda)$ we obtain $h \in L^1(\mu)$.

It remains to show h is unique (up to a null function) in each case. Suppose $g, h : X \to [0, \infty)$ are $\mathcal{A}$-measurable and λ is finite such that

$$\lambda(A) = \int_A g d\mu = \int_A h d\mu$$

Then $\int_A (g-h)d\mu = 0$ for all measurable A. By taking $A = \{g \geqslant h\}$ and $A = \{g \leqslant h\}$, we see $(g-h)^+ = 0$ and $(g-h)^- = 0$ μ-almost everywhere, whence $g = h$ μ-almost everywhere. If λ is σ-finite, write $X = \bigcup_{n=1}^{\infty} X_n$ with $\lambda(X_n) < \infty$, then the preceding arguments show $g = h$ μ-almost everywhere on each X_n, whence on X. The case λ being complex can be dealt with in a similar fashion. $\square$

Proposition D.6.4. Let μ be a complex measure on $(X, \mathcal{A})$. Then there exists a measurable function h such that $|h(x)| = 1$ for all $x \in X$ such that $d\mu = hd|\mu|$.

Proof. Since $\mu \ll |\mu|$, by Radon-Nikodym there exists $h \in L^1(|\mu|)$ such that $d\mu = hd|\mu|$. For each $r > 0$, put $A_r = \{x \in X \mid |h(x)| < r\}$, and let $\{E_n\}$ be a measurable partition of A_r. Then

$$\sum_n |\mu(E_n)| = \sum_n \left| \int_{E_n} hd|\mu| \right| \leqslant \sum_n r|\mu|(E_n) = r|\mu|(A_r)$$

so that $|\mu|(A_r) \leqslant r|\mu|(A_r)$. If $r < 1$, this forces $|\mu|(A_r) = 0$. Thus $|h| \geqslant 1$ a.e.

On the other hand, if $|\mu|(E) > 0$, we have

$$\left| \frac{1}{|\mu|(E)} \int_E hd|\mu| \right| = \frac{|\mu(E)|}{|\mu|(E)} \leqslant 1$$

Lemma D.6.5. Let $(X, \mathcal{A}, \mu)$ be a positive finite measure space and $f \in L^1(\mu)$. If $S \subseteq \mathbb{C}$ is a closed subset such that the average

$$A_E(f) = \frac{1}{\mu(E)} \int_E f d\mu$$

lie in S for every $E \in \mathcal{A}$ with $\mu(E) > 0$. Then $f(x) \in S$ for almost all $x \in X$.

Proof. The complement of S can be covered by countably closed ball, so it suffices to show $\mu(f^{-1}(D)) = 0$ for all closed ball $D \subseteq S^c$. If $\mu(f^{-1}(D)) > 0$,

$$\left|A_{f^{-1}(D)}(f) - \alpha\right| = \frac{1}{\mu(f^{-1}(D))}\left|\int_{f^{-1}(D)} (f-\alpha)d\mu\right| \leqslant \frac{1}{\mu(f^{-1}(D))}\int_{f^{-1}(D)} |f-\alpha|d\mu \leqslant r$$

where α is the center of D and r the radius of D. This is impossible for $A_{f^{-1}(D)}(f) \in S$ and $D \subseteq S^c$. ☐

By this lemma, we deduce that $|h| \leqslant 1$ a.e.

In sum, we have proved that $|h| = 1$ a.e. The proposition follows once we redefine h on $\{x \in X \mid |h(x)| \neq 1\}$ so that $h(x) = 1$ throughout the whole X. ☐

Corollary D.6.5.1. Let μ be a positive measure on $(X, \mathcal{A})$ and $f \in L^1(\mu)$. Consider the finite measure μ_f given by

$$\mu_f(A) = \int_A f d\mu$$

Then $|\mu_f| = \mu_{|f|}$.

Proof. By the previous proposition, there exists a measurable function h of absolute value 1 such that $d\mu_f = hd|\mu_f|$. Then

$$hd|\mu_f| = d\mu_f = fd\mu$$

so that $d|\mu_f| = \overline{h}fd\mu$, where $\bar{\cdot}$ is complex conjugation. Since $|\mu_f| \geqslant 0$ and $\mu \geqslant 0$, it follows that $\overline{h}f \geqslant 0$ μ-almost everywhere, whence $\overline{h}f = |f|$ μ-almost everywhere, i.e., $|\mu_f| = |f|d\mu = \mu_{|f|}$. ☐

Theorem D.6.6. Let μ be a positive measure. Let λ be either a real, complex or σ-finite positive measure. Then there are unique real, complex, or σ-finite measures λ_a and λ_s on $(X, \mathcal{A})$ such that

- $\lambda_a \ll \mu$ and $\lambda_s \perp \mu$, and
- $\lambda = \lambda_a + \lambda_s$.

The expression $\lambda = \lambda_a + \lambda_s$ is called the **Lebesgue decomposition** of λ. λ_a is called the **absolutely continuous part** of λ, and λ_s is called the **singular part** of λ.

Proof. First assume λ is positive finite. Define

$$\mathcal{N}_\mu := \{B \in \mathcal{A} \mid \mu(B) = 0\}$$

and let $\{B_n\}_n \subseteq \mathcal{N}_\mu$ be a sequence such that $\lambda(B_n) \to \sup\limits_{B \in \mathcal{N}_\mu} \lambda(B)$ as $n \to \infty$. Let $N = \bigcup\limits_{n \geqslant 1} B_n$ and define the measures λ_a, λ_s by the formulas

$$\lambda_a(A) = \lambda(A \cap N^c), \qquad \lambda_s = \lambda(A \cap N)$$

for all $A \in \mathcal{A}$. Then $\lambda = \lambda_a + \lambda_s$. Since $\mu(N) = 0$, we have $\lambda_s \perp \mu$. Since $\lambda(N) = \sup\limits_{B \in \mathcal{N}_\mu} \lambda(B)$, if $B \subseteq N^c$ is $\mathcal{A}$-measurable with $\mu(B) = 0$, then $\lambda(B) = 0$, for otherwise $N \cup B \in N_\mu$ and $\lambda(N \cup B) > \lambda(N)$. Hence $\lambda_a \ll \mu$.

Suppose λ is real or complex. Apply the preceding construction to the positive finite measure $|\lambda|$. Then we have a μ-null set N such that the Lebesgue decomposition $|\lambda| = |\lambda|_a + |\lambda|_s$ is given by

$$|\lambda|_a(A) := |\lambda|(A \cap N^c), \qquad |\lambda|_s(A) := |\lambda|(A \cap N)$$

Then the measures λ_a, λ_s defined by

$$\lambda_a(A) = \lambda(A \cap N^c), \qquad \lambda_s = \lambda(A \cap N)$$

clearly give the Lebesgue decomposition of λ.

Assume λ is σ-finite positive and let $\{X_n\}$ be a measurable partition of X such that $\lambda(X_n) < \infty$ for each n. Consider the restriction of λ and μ to each X_n; then we obtain a sequence $\{N_n\}$ of μ-null sets with $N_n \subseteq X_n$. Let $N = \bigcup_{n \geqslant 1} N_n$. Then the measures λ_a, λ_s defined by

$$\lambda_a(A) = \lambda(A \cap N^c), \qquad \lambda_s = \lambda(A \cap N)$$

give the Lebesgue decomposition of λ.

It remains to show the uniqueness. Let $\lambda = \lambda_a + \lambda_s = \lambda'_a + \lambda'_s$ be Lebesgue decompositions of λ. If λ is finite, then

$$\lambda_a - \lambda'_a = \lambda'_s - \lambda_s.$$

Since $\lambda_a - \lambda'_a \ll \mu$ and $\lambda'_s - \lambda_s \perp \mu$, it follows that $\lambda_a - \lambda'_a = \lambda'_s - \lambda_s = 0$, as wanted. The case λ being positive σ-finite is proved as usual by choosing a measurable partition of X with each piece finite under λ. $\square$

D.7 Bochner Integral

Let V be a Banach space. For a function $f : X \to V$ we aim to define an integral $\int_X f d\mu \in V$ such that for every continuous linear functional α on V we have

$$\alpha\left(\int_X f d\mu\right) = \int_X \alpha(f) d\mu$$

Let $(V, \|\cdot\|)$ be a Banach space and $(X, \mathcal{A}, \mu)$ a (positive) measure space. A **simple function** is a function $s : X \to V$ that can be written as

$$s = \sum_{j=1}^{n} \mathbf{1}_{A_j} b_j \tag{$\spadesuit$}$$

for some pairwise disjoint measurable sets $A_j \in \mathcal{A}$ of finite measure and some $b_j \in V$. Define

$$\int_X s d\mu := \sum_{j=1}^{n} \mu(A_j) b_j \in V$$

This is clearly independent of the expression ($\spadesuit$), and it satisfies

- $\left\|\int_X s d\mu\right\| \leqslant \int_X \|s\| \, d\mu$;
- for every linear functional $\alpha : V \to \mathbb{C}$, one has

$$\alpha\left(\int_X s d\mu\right) = \int_X \alpha(s) d\mu$$

Equip V with the Borel σ-algebra. A measurable function $f : X \to V$ is called **Bochner integrable** if there exists a sequence s_n of simple functions such that

$$\lim_{n\to\infty} \int_X \|f - s_n\| \, d\mu = 0$$

We call such (s_n) an **approximating sequence** of f.

Proposition D.7.1.

(a) If f is Bochner integrable and (s_n) is an approximating sequence, then the sequence of vectors $\displaystyle\int_X s_n d\mu$ converges. Its limit does not depend on the choice of the approximating sequence.

Define the **integral** of f to be this limit

$$\int_X f d\mu := \lim_{n\to\infty} \int_X s_n d\mu$$

(b) For every Bochner integrable function f one has

$$\left\| \int_X f d\mu \right\| \leqslant \int_X \|f\| \, d\mu < \infty$$

(c) Let f be Bochner integrable. For every continuous linear operator $T : V \to W$ to a Banach space W one has

$$T\left(\int_X f d\mu \right) = \int_X T(f) d\mu$$

(d) If $V = \mathbb{C}$, then the Bochner integral coincides with the usual integral.

Proof.

(a)

$$\begin{aligned}\left\| \int_X s_n d\mu - \int_X s_m d\mu \right\| &\leqslant \int_X \|s_n - s_m\| \, d\mu = \int_X \|s_n - f + f - s_m\| \, d\mu \\ &\leqslant \int_X \|s_n - f\| \, d\mu + \int_X \|f - s_m\| \, d\mu\end{aligned}$$

so this value tends to 0 as $n, m \to \infty$. Let (r_n) be another approximating sequence of f. Then

$$\begin{aligned}\left\| \int_X s_n d\mu - \int_X r_n d\mu \right\| &\leqslant \int_X \|s_n - r_n\| \, d\mu = \int_X \|s_n - f + f - r_n\| \, d\mu \\ &\leqslant \int_X \|s_n - f\| \, d\mu + \int_X \|f - r_n\| \, d\mu \to 0 \text{ as } n \to 0\end{aligned}$$

(b) Let (s_n) be an approximating sequence of f. Then $\|s_n\| \to \|f\|$ in $L^1(X)$, for

$$0 \leqslant \lim_{n\to\infty} \int_X | \, \|f\| - \|s_n\| \, | \leqslant \lim_{n\to\infty} \int_X \|f - s_n\| = 0$$

so that

$$\int_X \|f\| = \int_X \|f - s_n + s_n\| \leqslant \int_X \|f - s_n\| + \int_X \|s_n\| < \infty$$

Then

$$\left\| \int_X f \right\| = \lim_{n\to\infty} \left\| \int_X s_n \right\| \leqslant \lim_{n\to\infty} \int_X \|s_n\| = \int_X \|f\| < \infty$$

(c) By continuity, we have

$$T\left(\int_X f\right) = T\left(\lim_{n\to\infty}\int_X s_n\right) = \lim_{n\to\infty}\int_X T(s_n)$$

Since T is continuous, we can find $C > 0$ such that $\|T(v)\| \leqslant C\|v\|$ for all $v \in V$, and so $\|T(f)\| \leqslant C\|f\|$. In particular, $T(f)$ is integrable. Estimate

$$\left\|\int_X T(f) - \int_X T(s_n)\right\| \leqslant \int_X \|T(f) - T(s_n)\| = \int_X \|T(f - s_n)\| \leqslant C\int_X \|f - s_n\| \to 0$$

(d) It follows from the last estimate with $T = \mathrm{id}$.

□

Definition.

(i) A function $f : X \to V$ is called **separable** if there exists a countable subset $C \subseteq V$ such that $f(X) \subseteq \overline{C}$, where $\overline{C}$ is the closure of C in V.

- If V is separable (as topological spaces), then any function is automatically separable in the sense above.

(ii) A function $f : X \to V$ is **essentially separable** if there exists a measurable zero set $N \subseteq X$ such that $f|_{X\setminus N}$ is separable.

Lemma D.7.2. Let X be a topological space and $f : X \to V$ a continuous function with σ-compact support. Then f is separable.

Proof. Being σ-compact, let $\operatorname{supp} f = \bigcup_{n=1}^{\infty} K_n$ with compact $K_n \subseteq X$. Then $f(X) \subseteq \bigcup_{n=1}^{\infty} f(K_n)$. Since f is continuous, $f(K_n)$ is compact; being a compact metric space, $f(K_n)$ is itself separable. □

Proposition D.7.3. For a measurable function $f : X \to V$ TFAE:

(i) f is Bochner integrable.

(ii) f is essentially separable and $\int_X \|f\|\, d\mu < \infty$.

Proof. Suppose f is Bochner integrable. Then by Lemma D.7.1.(b), $\int_X \|f\|\, d\mu < \infty$. To show that f is essentially separable, let (s_n) be an approximating sequence of f. Since each s_n has finite image, the Banach subspace of V generated by the $s_n(X)$, $n \in \mathbb{N}$ is separable. The set $N := f^{-1}(V\setminus E)$ is a countable union $N = \bigcup_{n\geqslant 1} N_n$, where

$$N_n := \left\{x \in X \mid \|f(x) - e\| \geqslant \frac{1}{n} \text{ for all } e \in E\right\}$$

Since $\lim_{n\to\infty}\int_X \|f - s_n\| = 0$, each N_n has zero measure, and so does N.

Conversely, suppose f is essentially separable and $\int_X \|f\| < \infty$. Up to a measure zero set we may assume f is separable. Let $C = \{c_n\}_{n\geqslant 1} \subseteq V$ be countable such that $f(X) \subseteq \overline{C}$. For each $(n, \delta) \in \mathbb{N} \times \mathbb{R}_{>0}$, put

$$A_n^\delta := \{x \in X \mid \|f(x)\| \geqslant \delta,\ \|f(x) - c_n\| < \delta\}$$

Since f is measurable, each A_n^δ is measurable. We make some modification:

$$D_n^\delta := A_n^\delta \backslash \bigcup_{k<n} A_k^\delta$$

Then for a fixed $\delta > 0$, the D_n^δ are pairwise disjoint, and

$$\bigcup_{n\geqslant 1} A_n^\delta = \bigsqcup D_n^\delta = f^{-1}(f(X)\backslash B_\delta(0))$$

for C is dense in $f(X)$. Since $\|f\|$ is integrable, $f^{-1}(f(X)\backslash B_\delta(0))$ has finite measure. Let

$$s_n = \sum_{j=1}^n \mathbf{1}_{D_j^{\frac{1}{n}}} c_j$$

Then s_n is a simple function. We content that $s_n \to f$ pointwise. Let $x \in X$.

- $f(x) = 0$. Then $s_n(x) = 0$ for every n. Great.
- $f(x) \neq 0$. Then $\|f(x)\| \geqslant \dfrac{1}{n}$ for some $n \in \mathbb{N}$, and for $m \geqslant n$ one has

  $$x \in \bigcup_{\nu \geqslant 1} D_\nu^{\frac{1}{m}}$$

  so that for each $m \geqslant n$ there exists a unique, by disjointness, ν_0 with $D_{\nu_0}^{\frac{1}{m}}$, and hence $s_m(x) = c_{\nu_0}$, and $\|f(x) - c_{\nu_0}\| < \dfrac{1}{m}$.

This shows $s_n \to f$. Also, by construction we have $\|s_n\| \leqslant 2\,\|f\|$. Since

- $\|f - s_n\| \to 0$ pointwise, and
- $\|f - s_n\| \leqslant \|f\| + \|s_n\| \leqslant 3\,\|f\|$,

by DCT we obtain $\displaystyle\int_X \|f - s_n\|\, d\mu \to 0$ as wanted. ☐

Corollary D.7.3.1. Let X be an LCH space and μ a Radon measure. Then every continuous function $f : X \to V$ with compact support is Bochner integrable.

Proof. Since $\|f\| \in C_c(X)$, it follows from Proposition and Lemma above that f is integrable. ☐

Lemma D.7.4. Let G be an LCH group. If $f \in C_c(G)$ and $g \in L^1(G)$, then the Bochner integral

$$\int_G f(x) L_x g\, dx$$

exists in the Banach algebra $L^1(G)$, and equals the convolution product $f * g$.

Proof. Consider the function

$$\begin{aligned} \phi : G &\longrightarrow L^1(G) \\ x &\longmapsto f(x) L_x g \end{aligned}$$

This is continuous by Lemma 2.6.7, and since it has compact support (as f does), by Corollary D.7.3.1 it is Bochner integrable. Thus $\displaystyle\int_G \phi(x)\, dx \in L^1(G)$, and to see it coincides with $f * g$, it suffices to show

$$h * \int_G \phi(x) L_x g\, dx = h * f * g$$

for all $h \in C_c(G)$ by Lemma 2.6.9.

Let $h \in C_c(G)$. For $\phi \in L^1(G)$ and $y \in G$, the integral $h * \phi(y)$ exists, and

$$\begin{aligned} |h * \phi(y)| &\leqslant \int_G |h(z)|\phi(z^{-1}y)|dz \\ &\overset{2.3.1.4}{=} \int_G \Delta(z^{-1})|h(yz^{-1})||\phi(z)|dz \\ &\leqslant C \|\phi\|_1 \end{aligned}$$

with $C \geqslant 0$ for the function $z \mapsto \Delta(z^{-1})|h(yz^{-1})|$ is continuous. This implies the linear functional

$$\begin{aligned} \alpha : L^1(G) &\longrightarrow \mathbb{C} \\ \phi &\longmapsto h * \phi(y) \end{aligned}$$

is continuous. Then for $\phi = \int_G f(x)L_x g\, dx$, it follows

$$\begin{aligned} h * \phi(y) = \alpha(\phi) &= \alpha\left(\int_G f(x)L_x g\, dx\right) \\ &= \int_G f(x)\alpha(L_x g)dx \\ &= \int_G \int_G g(x)h(z)g(x^{-1}z^{-1}y)dzdx \\ &= h * f * g(y) \end{aligned}$$

□

D.7.1 Cauchy's Integral Formula

Definition. Let $\Omega \subseteq \mathbb{C}$ be an open set, $f : \Omega \to V$ be holomorphic in the sense of Definition 3.1.1 and $\gamma : [0,1] \to \Omega$ be C^1. The **path integral** is defined as

$$\int_\gamma f(z)dz := \int_{[0,1]} f(\gamma(t))\gamma'(t)dt$$

Theorem D.7.5 (Cauchy's Integral Formula)**.** Let $\Omega \subseteq \mathbb{C}$ be an open set and $f : \Omega \to V$ be holomorphic. Suppose $D \subseteq \Omega$ is an open disc with $\overline{D} \subseteq \Omega$, then for every $z \in D$, we have

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\xi)}{\xi - z}d\xi$$

Proof. It follows from Hahn-Banach theorem, Proposition D.7.1.(c) and the usual Cauchy's Integral Formula in complex analysis. □

Corollary D.7.5.1. Let the situation be as in the theorem, and let D be an open disc around the origin such that $\overline{D} \subseteq \Omega$. Then there exist $v_n \in V$ such that

$$f(z) = \sum_{n=0}^{\infty} v_n z^n$$

holds for every $z \in D$, and the sum converges uniformly on every closed subset of D.

Proof. If $z \in D$ and $\xi \in \partial D$, then $\left|\frac{z}{\xi}\right| < 1$, which means that the geometric series

$$\sum_{n=0}^{\infty}\left(\frac{z}{\xi}\right)^n = \frac{1}{1-z/\xi}$$

converges uniformly for (z,ξ) in $D \times \partial D$. Applying the Cauchy's formula, we obtain

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\partial D}\frac{f(\xi)}{\xi - z}d\xi = \frac{1}{2\pi i}\int_{\partial D}\frac{1}{\xi}\frac{f(\xi)}{1-z/\xi}d\xi \\ &= \frac{1}{2\pi i}\int_{\partial D}\frac{f(\xi)}{\xi}\sum_{n=0}^{\infty}\left(\frac{z}{\xi}\right)^n d\xi \\ &= \frac{1}{2\pi i}\sum_{n=0}^{\infty} z^n \int_{\partial D}\frac{f(\xi)}{\xi^{n+1}}d\xi \end{aligned}$$

The last step is justifies by the uniform convergence. ☐

D.8 Gelfand-Pettis integral

Let V be a topological vector space such that the continuous dual $V^\vee$ separates points, in the sense that for each $v \in V$ there exists $\ell \in V^\vee$ with $\ell(x) \neq 0$. Consider the evaluation pairing

$$\begin{aligned} V^\vee \times V &\longrightarrow \mathbb{C} \\ (\ell, x) &\longmapsto \langle \ell, x\rangle := \ell(x) \end{aligned}$$

A subset $S \subseteq V^\vee$ is said to be **separating** if $x \neq 0$, then $\ell(x) \neq 0$ for some $\ell \in S$. Equivalently, if $\ell(x) = 0$ for all $\ell \in S$, then $x = 0$.

Let $(X, \mathcal{A}, \mu)$ be a measure space, and $f : X \to V$ be a function. For $\ell \in V^\vee$, the composition $\ell \circ f : X \to \mathbb{C}$ is complex-valued, and it makes sense to talk about the measurablity and integrablity in the usual sense.

Definition. Suppose $V^\vee$ is separating. $f : X \to V$ is **Dunford integrable** if $\ell \circ f$ is measurable for all $\ell \in V^\vee$, and for each $A \in \mathcal{A}$ there exists an element $d_A \in V^{\vee\vee}$ such that

$$d_A(T) = \int_A \ell \circ f d\mu$$

for all $\ell \in V^\vee$. We say f is **Pettis integrable** if $d_A \in V$ for each $A \in \mathcal{A}$. In any case, we write

$$d_A =: \int_A f d\mu$$

and call it the **Dunford** (resp. **Pettis**) **integral** of f.

Appendix E

Functional Analysis

E.1 Banach Space Basics

In the following let $(D, |\cdot|)$ be a non-discrete valued division ring. We assume $|\cdot|$ is an absolute value on D, namely, $|\cdot|$ satisfies the triangle inequality.

Definition. Let X be a left vector space over D.

(i) A **(D-)norm** on X is a function $\|\cdot\| : X \to \mathbb{R}_{\geqslant 0}$ satisfying

  (a) $\|x\| = 0$ if and only if $x = 0$.

  (b) $\|rx\| = |r| \, \|x\|$ for $r \in D$ and $x \in X$.

  (c) $\|x + y\| \leqslant \|x\| + \|y\|$ for $x, y \in X$.

  The pair $(X, \|\cdot\|)$ is called a **normed linear space** over D.

(ii) Let $\|\cdot\|$ be a norm on X. The pair $(X, \|\cdot\|)$ is called a **Banach space** if D is complete and the underlying metric space is complete.

For two normed linear spaces $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, consider the space

$$B(X, Y) = \mathrm{Hom}_D(X, Y) \cap \mathrm{Hom}_{\mathbf{Top}}(X, Y)$$

of all continuous D-linear functions from X to Y. It is naturally a left D-module. This is naturally equipped with a norm, called the **operator norm**. For any $T \in \mathrm{Hom}_D(X, Y)$ define

$$\|T\|_{\mathrm{op}} = \sup_{0 \neq x \in D} \frac{\|Tx\|_Y}{\|x\|_X} \leqslant \infty.$$

It follows formally that $\|\cdot\|_{\mathrm{op}}$ is a D-norm, except we do not on which space the norm is finite. The following lemma finds the space.

Lemma E.1.1. $B(X, Y) = \{T \in \mathrm{Hom}_D(X, Y) \mid \|T\|_{\mathrm{op}} < \infty\}$.

Proof. If $\|T\|_{\mathrm{op}} < \infty$, then by definition $\|Tx\|_Y \leqslant \|T\|_{\mathrm{op}} \|x\|_X$ for all x, so T is Lipschitz continuous. In particular, T is continuous.

For the other way around, note since D is non-discrete, we can (and we do) choose $a \in D$ with $|a| > 1$. Suppose $T : X \to Y$ is a D-linear map. If T is continuous, in particular it is continuous at

0, so there exists $\delta > 0$ such that $\|Tx\|_Y < 1$ for $\|x\|_X < \delta$. Now for $x \in X$, there exists $n \in \mathbb{Z}$ such that $|a^n| \leqslant \frac{\|x\|_X}{\delta} \leqslant |a^{n-1}|$. Then $\left\|\frac{x}{a^{n-1}}\right\|_X < \delta$ so $1 > \left\|T\left(\frac{x}{a^{n-1}}\right)\right\|_Y = \frac{\|Tx\|_Y}{|a^{n-1}|}$, or

$$\|Tx\|_Y < |a^{n-1}| \leqslant \frac{\|x\|_X}{\delta|a|}.$$

This is true for all $x \in X$, so $\|T\|_{\text{op}} < \infty$. □

This shows $(B(X,Y), \|\cdot\|_{\text{op}})$ is a normed linear space. A element in $B(X,Y)$ is usually called a **bounded operator**.

Lemma E.1.2. If Y is Banach, then $B(X,Y)$ is Banach. The converse holds if $B(X,D) \neq 0$.

Proof. Assume Y is Banach. Let $(T_n)_n \subseteq B(X,Y)$ be a Cauchy sequence. Then $(T_n(x))_n \subseteq Y$ is Cauchy for each $x \in X$. Now by completeness of Y, we can define a D-linear map $T : X \to Y$ by setting $T(x) := \lim_{n\to\infty} T_n(x)$. We prove T is continuous by showing $\|T\|_{\text{op}} < \infty$. Since $(T_n)_n$ is Cauchy, there exists $N \geqslant 0$ such that $\|T_n(x) - T_m(x)\|_Y \leqslant \|x\|_X$ for all $n, m \geqslant N$ and $x \in X$. Since $\|\cdot\|_Y : Y \to \mathbb{R}$ is continuous, letting $m \to \infty$ gives $\|T_n(x) - T(x)\|_Y \leqslant \|x\|_X$, so

$$\|T(x)\|_Y \leqslant \|T_n(x)\|_Y + \|x\|_X = (\|T_n\|_{\text{op}} + 1)\|x\|_X.$$

This proves $\|T\|_{\text{op}} \leqslant \|T_n\|_{\text{op}} + 1 < \infty$. The proof for $T_n \to T$ in $B(X,Y)$ is the same.

For the converse, assume $B(X,Y)$ is Banach and $B(X,D) \neq 0$. Take $0 \neq \alpha \in B(X,D)$. Let $(y_n)_n$ be a Cauchy sequence in Y and define $T_n \in B(X,Y)$ by $T_n(x) = \alpha(x)y_n$. Then for $x \in X$ and $n, m \geqslant 1$

$$\|T_n(x) - T_m(x)\|_Y = \|\alpha(x)y_n - \alpha(x)y_m\|_Y = |\alpha(x)|\,\|y_n - y_n\|_Y \leqslant \|\alpha\|_{\text{op}}\|x\|_X\|y_n - y_m\|_Y.$$

This says that $\|T_n - T_m\|_{\text{op}} \leqslant \|\alpha\|_{\text{op}}\|y_n - y_m\|_Y$ for all $n, m \geqslant 1$. Since $(y_n)_n$ is Cauchy and α is bounded, this shows that $(T_n)_n$ is Cauchy in $B(X,Y)$. Since $B(X,Y)$ is complete, $T := \lim_{n\to\infty} T_n$ exists. Let $x \notin \ker\alpha$ and define $y = \alpha(x)^{-1}T(x) \in Y$. We claim $y_n \to y$ as $n \to \infty$. This is clear:

$$\|y_n - y\|_Y = \left\|\alpha(x)^{-1}T_n(x) - \alpha(x)^{-1}T(x)\right\|_Y \leqslant |\alpha(x)|^{-1}\|T_n - T\|_{\text{op}} \to 0$$

as $n \to \infty$. □

Definition. Let X be left vector space over D.

(i) $D^* := \mathrm{Hom}_D(X,D)$ is called the **algebraic dual** of D.

(ii) $D^\vee := B(X,D)$ is called the **continuous dual** of D. Without otherwise stated, we always equip $D^\vee$ with the operator norm.

E.1.1 Compact operators

Definition. Let X, Y be two normed linear spaces over D. A bounded operator $T : X \to Y$ is called compact if it sends bounded subsets to precompact subsets.

Lemma E.1.3. Let X, Y be two normed linear spaces over D and $T : X \to Y$ a bounded operator. TFAE:

(i) T is compact.

(ii) If $(a_n)_n \subseteq X$ is a bounded sequence, then $(Ta_n)_n \subseteq Y$ admits a convergent subsequence.

Proof. Since Y is a metric space, the equivalence follows from Lemma A.5.3. □

E.1.2 Hahn-Banach

Let D be a non-discrete valued division ring, and let X be a normed linear space over D. We try to extend a bounded functinoal $T : M \to D$ on a subspace M of X to a larger space without increasing the operator norm. Let $x \in X \backslash M$. We have to find $y \in D$ such that the extension $T' : M \oplus Dy \to D$ of T given by $T'(x) = y$ satisfies $\|T'\|_{\text{op}} \leqslant \|T\|_{\text{op}}$. It suffices to find y such that $|T'(m - x)| \leqslant \|T\|_{\text{op}} \|m - x\|$ for all $m \in M$, or

$$|T(m) - y| \leqslant \|T\|_{\text{op}} \|m - x\| \qquad (m \in M)$$

or

$$y \in \bigcap_{m \in M} \overline{B}^D_{\|T\|_{\text{op}}\|m-x\|}(T(m))$$

We consider the case where D is non-archimedean. For this introduce

Definition. A normed linear space $(X, \|\cdot\|)$ is called **non-archimedean** if $\|\cdot\|$ satisfies the ultrametric inequality: $\|x + y\| \leqslant \max\{\|x\|, \|y\|\}$ for all $x, y \in X$.

If X is non-archimedean, then for $m, m' \in M$

$$|T(m) - T(m')| \leqslant \|T\|_{\text{op}} \left\|m - m'\right\| \leqslant \|T\|_{\text{op}} \max\{\|m - x\|, \left\|m' - x\right\|\}$$

In particular, we see if $\|m - x\| \leqslant \|m' - x\|$, then $\overline{B}^D_{\|T\|_{\text{op}}\|m-x\|}(T(m)) \subseteq \overline{B}^D_{\|T\|_{\text{op}}\|m'-x\|}(T(m'))$, so

$$\left\{\overline{B}^D_{\|T\|_{\text{op}}\|m-x\|}(T(m)) \mid m \in M\right\}$$

is a collection of closed balls totally ordered by set inclusion. If D is locally compact, then this is non-empty by finite intersection property. More generally,

Definition. A metric space is called **spherically complete** if each collection of decreasing closed balls has a common point.

The same property shows that a locally compact non-archimedean division ring is spherically complete. The above result together with a Zorn's lemma argument shows

Theorem E.1.4 (Ingleton)**.** Let D be a spherically complete non-archimedean division ring, X a non-archimedean normed linear space over D and $M \subseteq X$ a subspace. Then the restriction

$$\begin{aligned} X^\vee &\longrightarrow M^\vee \\ T &\longmapsto T|_M \end{aligned}$$

is surjective, and each fibre contains a functional with the same bound.

We now turn to the case when D is archimedean. The case $D = \mathbb{R}$ or $\mathbb{C}$ is contained in §E.3.3. The case $D = \mathbb{H}$ need more assumption. See `https://arxiv.org/abs/math/0609160v1`

Theorem E.1.5 (Hahn-Banach's)**.** Let M be s subspace of a Banach space V and let $\alpha : M \to \mathbb{C}$ be linear with $|\alpha(x)| \leqslant \|x\|$ for all $x \in M$. Then α extends to a linear function $V \to \mathbb{C}$ such that $|\alpha(x)| \leqslant \|x\|$ holds for all $x \in V$.

Corollary E.1.5.1. Let $E \subseteq V$ be a Banach subspace of V.

1. The restriction map $V^\vee \to E^\vee$ is surjective, and for $\alpha \in E^\vee$, its fibre contains $\alpha' \in V^\vee$ with $\|\alpha'\| = \|\alpha\|$.

2. $V^\vee$ separates points. In other words, if $v, w \in V$ such that $\alpha(v) = \alpha(w)$ for all $\alpha \in V^\vee$, then $v = w$.

E.1.3 Corollaries of Baire category

Theorem E.1.6 (Uniform Boundedness Principle)**.** Let V be a Banach space and E a normed linear space. Let $\{\Lambda_\alpha\}_{\alpha\in A}$ be a collection of bounded linear maps from V to E. Then either there exists $M < \infty$ such that $\|\Lambda_\alpha\| \leqslant M$ for all $\alpha \in A$, or $\sup_{\alpha\in A} \|\Lambda_\alpha v\| = \infty$ for all v in some dense G_δ set in X.

Theorem E.1.7 (Open Mapping Theorem)**.** A bounded surjective linear operator between Banach spaces is an open map.

Theorem E.1.8 (Closed Graph Theorem)**.** Suppose that $T : V \to W$ is a linear map between Banach spaces such that the graph

$$\Gamma_T := \{(v, Tv) \mid v \in V\} \subseteq V \times W$$

is a closed in the product topology. Then T is bounded.

Let $T : V \to W$ be a linear map between normed spaces. The projection $\mathrm{pr}_1 : \Gamma_T \to V$ is always an linear isomorphism, so we can use this to define the **graph norm** $\|\cdot\|_T : V \to \mathbb{R}_{\geqslant 0}$:

$$\|v\|_T := \|v\|_V + \|Tv\|_W$$

The identity map $(V, \|\cdot\|_T) \to (V, \|\cdot\|_V)$ is clearly continuous, and for this to be open it is equivalent to saying that T is bounded. The closed graph theorem amounts to saying that $\Gamma_T \subseteq V \times W$ is closed if and only if $\|\cdot\|_T$ is complete.

Lemma E.1.9. Let V be a Banach space and W a closed subspace. Then the quotient space V/W is also a Banach space equipped with the **quotient norm**

$$\|a + W\| := \inf\{\|a + w\| \mid w \in W\}$$

Proof. We must show it is V/W is complete with respect to the quotient norm. Let $(a_n + W)_n$ be a Cauchy sequence. Passing to a subsequence we may assume $\sum_{n=1}^{\infty} \|a_{n+1} - a_n + w_n\| < \infty$ for some $w_n \in W$. Since V is complete, $a := \sum_{n=1}^{\infty} (a_{n+1} - a_n + w_n)$ defines an element of V. Since the quotient map $v \mapsto v + W$ is norm-decreasing, we see $a_n + W \to a + W$ with respect to the quotient norm. □

Lemma E.1.10. Suppose F be a complete subspace of a normed vector space E. If F has finite codimension in E, then E is complete.

Proof. It at once reduces to the case E/F is one dimensional. Fix an $x \in E - F$. Suppose $(w_n)_n$ is Cauchy in E. Since the quotient map $E \to E/F$ is norm-decreasing, if we write $w_n = v_n + a_n x$ with $v_n \in F$ and $a_n \in \mathbb{C}$ for each n, then $(a_n x + F)_n$ is Cauchy in E/F. Recall that any finite dimensional normed space over $\mathbb{C}$ is complete. Thus $a_n x + F \to ax + F$ for some $a \in \mathbb{C}$ and by definition we can find $u_n \in F$ such that $\|a_n x + u_n - a\| \leqslant \|ax - a_n x + F\| + n^{-1}$ for each n. Therefore

$$\begin{aligned}\|(v_n - u_n) - (v_m - u_m)\| &\leqslant \|w_n - w_m\| + \|(a_n x + u_n) - (a_m x + u_m)\| \\ &\leqslant \|w_n - w_m\| + \|ax - a_n x + F\| + n^{-1} + \|ax - a_m x + F\| + m^{-1}\end{aligned}$$

and so $(v_n - u_n)_n$ is Cauchy. Since F is complete, $v_n - u_n \to v \in F$. Now define $w = v + a$.

$$\|w_n - w\| \leqslant \|(v_n - u_n) - v\| + \|(a_n x + u_n) - a\| \leqslant \|(v_n - u_n) - v\| + \|ax - a_n x + F\| + n^{-1}$$

The right hand side $\to 0$ as $n \to \infty$, from which we conclude $w_n \to w$. □

Lemma E.1.11. Let B be a Banach space and E be a proper closed subspace. For each $\varepsilon > 0$ there exists $x \in B$ with $\|x\| = 1$ such that $\inf_{y \in E} |x - y| > 1 - \varepsilon$.

Proof. If $\varepsilon \geqslant 1$, the assertion is obvious. Assume $0 < \varepsilon < 1$. Pick $x' \in B \backslash E$. Since E is closed, $d := \inf_{y \in E} \|x - y\| > 0$. Pick $y' \in E$ with $d \leqslant \|x' - y'\| \leqslant d + \eta$, where $\eta > 0$ is to be chosen. Put $x = \dfrac{x' - y'}{\|x' - y'\|}$. Then for $y \in E$, we have

$$\|x - y\| = \left\| \frac{x' - (y' + \|x' - y'\| \, y)}{\|x' - y'\|} \right\| \geqslant \frac{d}{\|x' - y'\|} \geqslant \frac{d}{d + \eta} = 1 - \varepsilon$$

once we choose $\eta = \dfrac{d\varepsilon}{1 - \varepsilon} > 0$. ☐

Proposition E.1.12. A Banach space is locally compact if and only if it is finite dimensional.

Proof. The if part is obvious. Suppose B is a locally compact Banach space. In particular, the unit ball B_1 is compact. Let $v_1 \in \partial B_1$. For $n \geqslant 2$, use the previous lemma to choose $v_1, \ldots, v_n \in \partial B_1$ inductively so that

$$\operatorname{disc}(v_i; \operatorname{span}\{v_1, \ldots, v_{i-1}\}) > \frac{1}{2}$$

If B is infinite dimensional, then we obtain an infinite sequence $(v_n)_n \subseteq \partial B_1$ with $\|v_i - v_j\| > \frac{1}{2}$ whenever $i \neq j$. But ∂B_1 is (sequentially) compact, this is a contradiction. ☐

Proposition E.1.13. A normed space is separable if and only if it admits a sequence of linearly independent vectors whose linear span is dense.

Proof. Let E be a separable normed space. Let C be a countable dense subset, and from C we choose a maximal linearly independent subset C'. By construction, $C \subseteq \operatorname{span} C'$, so $\operatorname{span} C'$ is dense. For the only if part, their linear span over $\mathbb{Q}$ is a countable dense subset. ☐

E.2 Hilbert Space Basics

E.2.1 Riesz's Representation Theorem

In the section, let $(H, \langle , \rangle)$ be a complex Hilbert space, i.e., a complete inner product space over $\mathbb{C}$.

Definition.

1. Let S be a subset of H. The **orthogonal complement** of S is

$$S^\perp := \{v \in H \mid \langle v, w \rangle = 0 \text{ for all } w \in S\}$$

2. A projection $p : V \to V$ is called an **orthogonal projection** if it verifies the following equivalent conditions:

   (i) For all $v \in V$, $v - pv \in (\operatorname{Im} p)^\perp$.

   (ii) $(\operatorname{Im} p)^\perp = \ker p$ and $(\ker p)^\perp = (\operatorname{Im} p)$.

   (iii) p is self-adjoint.

We will show the equivalence between 2.(i),(ii),(iii) in the sequel. For this moment, by orthogonal projection we mean 2.(i).

Theorem E.2.1. Let $W \subseteq H$ be a closed convex subset and $x \in H$.

1. There exists a unique element $y \in W$ minimizing the distance $\|x - w\|$ $(w \in W)$.
2. If W is a subspace, then $x - y \in W^{\perp}$.

Hence, the orthogonal projection onto a closed subspace always exists.

Proof. The essence of this proof is the parallelogram law:

$$\|v + w\|^2 + \|v - w\|^2 = 2(\|v\|^2 + \|w\|^2),$$

which is valid for every inner product space.

1. We begin with the uniqueness. Suppose $y, y' \in W$ are two elements that attain the minimum d. Then

   $$\left\|(y - x) + (y' - x)\right\|^2 + \left\|(y - x) - (y' - x)\right\|^2 = 2(\|y - x\|^2 + \left\|y' - x\right\|^2) = 4d^2$$

   or

   $$4d^2 \geqslant 4\left\|x - \frac{y + y'}{2}\right\| + \left\|y - y'\right\|^2 \geqslant 4d^2 + \left\|y - y'\right\|^2 \geqslant 0$$

   by minimality, or $0 \geqslant \|y - y'\|^2 \geqslant 0$. This forces $y = y'$.

   For existence, $d := \inf_{w \in W} \|x - w\|$; by definition, we can find a sequence $(w_n)_{n \geqslant 1}$ such that $d \leqslant \|x - w_n\| \leqslant d + \dfrac{1}{n}$ for each $n \geqslant 1$. We have

   $$\|w_m - x + w_n - x\|^2 + \|w_m - x - w_n + x\|^2 = 2(\|w_m - x\|^2 + \|w_n - x\|^2).$$

   Then

   $$\begin{aligned} \|w_m - w_n\|^2 &= 2(\|w_m - x\|^2 + \|w_n - x\|^2) - 4\left\|\frac{w_n + w_m}{2} - x\right\|^2 \\ &\leqslant 2\left(\left(d + \frac{1}{m}\right)^2 + \left(d + \frac{1}{n}\right)^2\right) - 4d^2 \\ &= 4d\left(\frac{1}{m} + \frac{1}{n}\right) + 2\left(\frac{1}{m^2} + \frac{1}{n^2}\right) \end{aligned}$$

   The estimate in the end of the second line holds as $\dfrac{w_n + w_m}{2} \in W$ by convexity. This shows $(w_n)_n$ is a Cauchy sequence; since W is closed, we can find its limit $y \in W$ in W. Then $\|x - y\| = d$ as we want.

2. Let $w \in W$ and consider the pencil $x - y - tw$ $(t \in \mathbb{C})$. Then $\|x - y - tw\| r \geqslant \|x - y\|^2 = d^2$, or

   $$|t|^2 \|w\|^2 - 2\operatorname{Re}(\bar{t}\langle x - y, w\rangle) \geqslant 0.$$

   If $\langle x - y, w\rangle \neq 0$, say $\langle x - y, w\rangle = ru$ for some $r > 0$, $u \in S^1$. Take $t = \varepsilon u$; then

   $$\varepsilon^2 \|w\|^2 - 2r\varepsilon \geqslant 0.$$

   This is absurd as ε is arbitrary. Hence $\langle x - y, w\rangle = 0$.

□

Lemma E.2.2. If S is a subset of H, then $S^\perp$ is a closed subspace of H and $\overline{S} = (S^\perp)^\perp$.

Proof. It is clear that $S^\perp$ is a subspace of H. Let $(v_n)_n \subseteq S^\perp$ be a sequence converging to $v \in S$. Then for $w \in S$,

$$|\langle v, w\rangle| = |\langle v - v_n + v_n, w\rangle| = |\langle v - v_n, w\rangle| \leqslant \|v - v_n\| \, \|w\|$$

so that $\langle v, w\rangle = 0$ for all $w \in S$, i.e., $v \in S^\perp$. For the last assertion, note $S \subseteq (S^\perp)^\perp$ is clear, and since the latter is closed, it follows $\overline{S} \subseteq (S^\perp)^\perp$. For the reverse inclusion, we use Theorem E.2.1. First, note that $((S^\perp)^\perp)^\perp = S^\perp$. Indeed, for subsets $S \subseteq T$, one has $S^\perp \supseteq T^\perp$. Thus $S^\perp \supseteq ((S^\perp)^\perp)^\perp \supseteq S^\perp$ and we have equality throughout. Second, by (E.2.1) for $v \in (S^\perp)^\perp$ we can write $v = x + y$ with $x \in \overline{S}$ and $y \in \overline{S}^\perp \subseteq S^\perp$. We must show $y = 0$, and proving $\langle y, w\rangle = 0$ for all $w \in H$ suffices.

- $w \in \overline{S}$. Then $\langle y, w\rangle = 0$ for $y \in S^\perp$ and w can be approximated by elements in S.
- $w \in \overline{S}^\perp \subseteq S^\perp$. Then
$$\langle y, w\rangle = \underbrace{\langle v, w\rangle}_{=0 \text{ since } v \in (S^\perp)^\perp} - \underbrace{\langle x, w\rangle}_{=0 \text{ since } x \in \overline{S}} = 0$$

This proves $y = 0$, and thus $v \in \overline{S}$. □

Theorem E.2.3 (Riesz's Representation Theorem)**.** For $T \in H^\vee$, there exists a unique $y \in H$ such that $T(x) = \langle x, y\rangle$. In other words, the inner product $H \times H \to \mathbb{C}$ induces a conjugate-linear isomorphism $H \cong H^\vee$.

Proof. The uniqueness is clear. Let $N := \ker T$. If $N = H$, then take $y = 0$. Otherwise, since N is closed, by Theorem E.2.1 we have $H = N \oplus N^\perp$, and T restricts to an isomorphism $T|_{N^\perp} : N^\perp \to \mathbb{C}$.

Let $v_0 \in N^\perp$ be of norm one and set $w = \overline{T(v_0)} v_0$. For each $v \in V$, we can write $v = x + \lambda v_0$ with $x \in N$, $\lambda \in \mathbb{C}$. Then

$$\langle v, w\rangle = \langle \lambda v_0, \overline{T(v_0)} v_0\rangle = \lambda T(v_0) = T(v)$$

□

Corollary E.2.3.1. For $T \in \mathcal{B}(H)$, there exists a unique $T^* \in \mathcal{B}(H)$, called the **adjoint** of T, such that

$$\langle Tv, w\rangle = \langle v, T^* w\rangle$$

for all $v, w \in H$.

E.2.2 Orthonormal Basis

Definition. Let H be a Hilbert space.

1. A subset $A \subseteq H$ is **orthonormal** if $\|u\| = 1$ for all $u \in A$ and $\langle u, v\rangle = 0$ for all $u \neq v \in A$.
2. A maximal (with respect to the inclusion) orthonormal subset A of H is called an **orthonormal basis** of H.

- A Zorn's lemma argument shows that every orthonormal subset of H is contained in an orthonormal basis of H. In particular, a nontrivial Hilbert space has an orthonormal basis.

Let S be an arbitrary set. For a function $\varphi : S \to [0, \infty]$, define

$$\sum_{s \in S} \varphi(s) := \sup \left\{ \sum_{s \in S'} \varphi(s) \mid S' \subseteq S,\ \#S' < \infty \right\}$$

A moment consideration shows that this exactly means the integral of φ with respect to the counting measure $\#$ on S. In this situation, we write $\ell^p(S) = L^p(\#)$. Note that for $\varphi \in \ell^2(S)$, the set $\{x \in S \mid \varphi(x) \neq 0\}$ is at most countable.

Theorem E.2.4. Let A be an orthonormal set in H, and let $P = \mathrm{span}_{\mathbb{C}} A$.

1. We have the **Bessel's inequality**

$$\sum_{\alpha \in A} |\hat{x}(\alpha)|^2 \leqslant \|x\|^2$$

   for all $x \in H$, where $\hat{x} : A \to \mathbb{C}$ is defined by $\hat{x}(\alpha) := \langle x, \alpha \rangle$.

2. The association

$$\begin{aligned} \Phi : H &\longrightarrow \ell^2(A) \\ x &\longmapsto \hat{x} \end{aligned}$$

   is a surjective continuous linear map whose restriction to $\overline{P}$ is an isometry onto $\ell^2(A)$.

Proof. 1. holds with A replaced by any of its finite subset, so it holds for A as well by definition. In particular, this shows the map Φ defined in 2. is well-defined. It is clear that Φ is linear, and for continuity, one notes that for $x, y \in H$, by 1. we have

$$\|\Phi(y) - \Phi(x)\|_2 = \|\hat{y} - \hat{x}\|_2 \leqslant \|y - x\|$$

Then $\Phi|_P : P \to \ell^2(A)$ is an isometry whose image consists of those functions with finite support; clearly, $\Phi(P)$ is dense in $\ell^2(A)$. $\Phi|_P$ being isometric, we see that $\Phi|_{\overline{P}} : \overline{P} \to \ell^2(A)$ is then a surjective isometry. $\square$

Theorem E.2.5. Let A be an orthonormal set in H. TFAE:

(a) A is an orthonormal basis of H.

(b) The linear span of A is dense in H.

(c) The equality

$$\sum_{\alpha \in A} |\hat{x}(\alpha)|^2 = \|x\|^2$$

holds for all $x \in H$.

(d) The **Parseval's identity**

$$\sum_{\alpha \in A} \hat{x}(\alpha)\overline{\hat{y}(\alpha)} = \langle x, y \rangle$$

holds for all $x, y \in H$.

Proof. (a) $\Rightarrow$ (b) follows from Lemma E.2.1, and (b) $\Rightarrow$ (c) follows from Theorem E.2.4 by noting that LHS of (c) is $\|\hat{x}\|_2^2$. (c) $\Rightarrow$ (d) follows from the polarization identity

$$4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2 + i\,\|x+iy\|^2 - i\,\|x-iy\|^2$$

Finally, if A is not maximal, then we can find $0 \neq u \in H \backslash A$ such that $\hat{u}(\alpha) = 0$ for all $\alpha \in A$. Then (d) leads to (with $x = y = u$) $0 = \|u\|^2 > 0$, a contradiction. □

Proposition E.2.6. Let H be a Hilbert space and A, B be two orthonormal basis. Then $\#A = \#B$.

Proof. Let $A = \{e_i\}_{i \in I}$ and $B = \{f_j\}_{j \in J}$. By Parseval's identity, for every $f_j \in B$ we have

$$\sum_{i \in I} |\langle f_j, e_i\rangle|^2 = \|f_j\|^2 = 1$$

Let $I_j^n := \left\{i \in I \mid \langle f_j, e_i\rangle \geqslant \dfrac{1}{n}\right\}$. The above identity implies $\#I_j^n < \infty$, so $I_j := \bigcup\limits_{n \in \mathbb{N}} I_j^n$ is countable. By Parseval's identity again, we have

$$\sum_{j \in J} |\langle f_j, e_i\rangle|^2 = \|e_i\|^2 = 1$$

and this implies that $I = \bigcup\limits_{j \in J} I_j$. Hence $\#I \leqslant \#(J \times \mathbb{N}) = \#J$. Symmetrically we have $\#J \leqslant \#I$, so $\#A = \#I = \#J = \#B$. □

Proposition E.2.7. Let $(x_n)_n$ be an orthogonal sequence in a Hilbert space H. Then the following are equivalent.

(i) $\sum\limits_{n=1}^{\infty} x_n$ converges in norm topology of H.

(ii) $\sum\limits_{n=1}^{\infty} \|x_n\|^2 < \infty$.

(iii) $\sum\limits_{n=1}^{\infty} \langle x_n, y\rangle$ converges for all $y \in H$.

Proof. We have $\left\|\sum\limits_{n=N}^{M} x_n\right\|^2 = \sum\limits_{n=N}^{M} \|x_n\|^2$ by our assumption, so (ii) implies (i). By Cauchy-Schwarz, (i) implies (iii). Now assume (iii). For each $N \in \mathbb{N}$, define an operator Λ_N on H by

$$\Lambda_N y = \sum_{n=1}^{N} \langle x_n, y\rangle.$$

By (iii), the sequence $(\Lambda_N y)_N$ converges for all $y \in H$, so by uniform boundedness principal, the sequence $(\|\Lambda_N\|)_N$ is bounded. But

$$\|\Lambda_N\| = \|\langle x_1 + \cdots + x_N, \cdot\rangle\| = \|x_1 + \cdots + x_N\| = \left(\|x_1\|^2 + \cdots + \|x_N\|^2\right)^{\frac{1}{2}}$$

we see (iii) implies (ii). □

E.2.3 Constructions

Proposition E.2.8. For a Hilbert space H, its continuous dual $H^\vee$ is also a Hilbert space, with the inner product

$$\langle f, g\rangle := \sum_{\alpha\in A} f(\alpha)\overline{g(\alpha)}$$

where A is an orthonormal basis of H. The sum is independent of the choice of orthonormal basis. In this way, the isomorphism described in Theorem E.2.3 is an antilinear norm-preserving isomorphism.

Proof. The sum always converges by Bessel's inequality and Riesz's Representation theorem, and it is clear that it defines an inner product on $H^\vee$. Let B be another orthonormal basis of H. Then we can find $a_{\alpha\beta} \in \mathbb{C}\,(\alpha \in A,\ \beta \in B)$ such that $\alpha = \sum\limits_{\beta\in B} a_{\alpha\beta}\beta.$, and they satisfy $\sum\limits_{\beta\in B} a_{\alpha\beta}\overline{a_{\alpha'\beta}} = \delta_{\alpha\alpha'}$ as well as $\sum\limits_{\alpha\in A} a_{\alpha\beta}\overline{a_{\alpha\beta'}} = \delta_{\beta\beta'}$; thus

$$\sum_{\alpha\in A} f(\alpha)\overline{g(\alpha)} = \sum_{\alpha\in A}\sum_{\beta,\beta'\in B} a_{\alpha\beta}\overline{a_{\alpha\beta'}} f(\beta)\overline{g(\beta')} = \sum_{\beta\in B} f(\beta)\overline{g(\beta)}$$

For the last assertion, we use the notation in the proof of Theorem E.2.3. We use an orthonormal basis of N together with v_0 to compute $\|T\|$; then

$$\|T\|^2 = T(v_0)\overline{T(v_0)} = T(w) = \langle w, w\rangle = \|w\|^2$$

In fact, the norm defined by this inner product is the same as the operator norm. Keep the notation as above; then

$$\|T\|_{\mathrm{op}}^2 = \left(\sup_{\|v\|=1} |T(v)|\right)^2 = |T(v_0)|^2 = T(v_0)\overline{T(v_0)} = \|T\|^2$$

as wanted. □

Definition. Let $\{H_i\}_{i\in I}$ be a family of Hilbert spaces. The algebraic direct sum has a natural inner product:

$$\langle v, w\rangle := \sum_{i\in I}\langle v_i, w_i\rangle$$

for $v = (v_i)$, $w = (w_i)$. The completion of $\bigoplus\limits_{i\in I} H_i$ is called the **Hilbert direct sum**, and is denoted by $\widehat{\bigoplus\limits_{i\in I}} H_i$. If there is no confusion, the hat is usually omitted.

- One can identity $\widehat{\bigoplus\limits_{i\in I}} H_i$ with the space $\left\{(v_i) \in \prod\limits_{i\in I} H_i \mid \sum\limits_{i\in I} \|v_i\|^2 < \infty\right\}$.

Definition. For two Hilbert spaces V, W, their algebraic tensor products (over the base field) $V \otimes W$ has a natural inner product defined by

$$\langle v\otimes w, v'\otimes w'\rangle = \langle v, v'\rangle\langle w, w'\rangle$$

and extend by linearity in the first component and by conjugate-linearity in the second argument. The completion of $V \otimes W$ is called the **Hilbert tensor product**, and is denoted by $V \widehat{\otimes} W$.

- To show the formula actually defines an inner product, we must show it is positive definite. Let $x = \sum_{i=1}^{n} v_i \otimes w_i \in V \otimes W$. By a Gram-schmidt process we can assume $(w_i)_{i=1}^{n}$ is an orthonormal sequence. Then

$$\langle x, x \rangle = \sum_{1 \leqslant i \leqslant j \leqslant n} \langle v_i, v_j \rangle \langle w_i, w_j \rangle = \sum_{i=1}^{n} \|v_i\|^2 \geqslant 0$$

so $x = 0$ iff $v_1 = \ldots = v_n = 0$, iff $\langle x, x \rangle = 0$. This proves the positive definiteness.

Let's see an example of tensor products of Hilbert spaces. Let $(X, \mu), (Y, \nu)$ be either two σ-finite measure spaces, or two LCH spaces with μ, ν Radon measures. In these two cases one can define the product measure $\mu \otimes \nu$ on $X \times Y$. Consider the bilinear map

$$\begin{aligned} L^2(X, \mu) \times L^2(Y, \nu) &\longrightarrow L^2(X \times Y, \mu \otimes \nu) \\ (f, g) &\longmapsto f \otimes g : (x, y) \mapsto f(x)g(y). \end{aligned}$$

This is well-defined by Fubini's theorem, and the universal property of tensor products implies it descends to a homomorphism

$$L^2(X, \mu) \otimes L^2(Y, \nu) \longrightarrow L^2(X \times Y, \mu \otimes \nu).$$

Again by Fubini it preserves inner product. Since the codomain is complete, it further passes to a homomorphism from the Hilbert space tensor product

$$L^2(X, \mu) \hat{\otimes} L^2(Y, \nu) \longrightarrow L^2(X \times Y, \mu \otimes \nu).$$

Let $(\varphi_\alpha)_\alpha$ and $(\psi_\beta)_\beta$ be orthonormal bases for $L^2(X, \mu)$ and $L^2(Y, \nu)$. We claim $(\varphi_\alpha \otimes \psi_\beta)_{\alpha,\beta}$ is an orthonormal basis for $L^2(X \times Y, \mu \otimes \nu)$. The orthonormality is clear. To show it is an orthonormal basis, let $f \in L^2(X \times Y, \mu \otimes \nu)$ satisfy

$$\int_{X \times Y} (\varphi_\alpha \otimes \psi_\beta) f d(\mu \otimes \nu) = 0$$

for all α, β. We need to show $f = 0$ in $L^2(X \times Y, \mu \otimes \nu)$. By Fubini,

$$0 = \int_X \left(\int_Y \psi_\beta(y) f(x, y) d\nu(y) \right) \varphi_\alpha(x) d\mu(x).$$

Since $(\varphi_\alpha)_\alpha$ is an orthonormal basis, this implies the function $x \mapsto \int_Y \psi_\beta(y) f(x, y) d\nu(y)$ is zero in $L^2(X, \mu)$. Again, since $(\psi_\beta)_\beta$ is an orthonormal basis, it follows that $f(x, y) = 0$ in $L^2(X \times Y, \mu \otimes \nu)$. We record this as a

Lemma E.2.9. Let $(X, \mu), (Y, \nu)$ be either two σ-finite measure spaces, or two LCH spaces with μ, ν Radon. Then the canonical map

$$L^2(X, \mu) \hat{\otimes} L^2(Y, \nu) \longrightarrow L^2(X \times Y, \mu \otimes \nu).$$

is a Hilbert space isomorphism.

E.3 Topological vector spaces

For the definition of a topological vector space, see §2.5. Let D be a non-discrete valued division ring (i.e., with a nontrivial absolute value $|\cdot|$).

Definition. Let V be a left vector space over D (without topology).

1. A subset $M \subseteq V$ is called **balanced** if $\lambda M \subseteq M$ for all $|\lambda| \leqslant 1$.
2. For a subset $S \subseteq V$, the **balance hull** of S is the smallest balanced set containing S, and the **balanced core** is the largest balanced subset of S.
3. For two sets $S, T \subseteq V$, we say S **absorbs** T if there exists $\alpha > 0$ such that $T \subseteq \lambda S$ for all $|\lambda| \geqslant \alpha$.
4. A subset is called **absorbent** if it absorbs every singleton in V.
5. A subset is called **bounded** if it is absorbed by any unit-neighborhood of V.

Lemma E.3.1. Let V be a left topological vector space over D.

1. The balanced core of a unit-neighborhood of V remains a unit-neighborhood.
2. Every unit-neighborhood is absorbent.
3. V admits a unit-neighborhood basis consisting of balanced absorbent subset and is stable under dilation.

Proof.

1. Let U be a unit-neighborhood and U_0 its balanced core. By continuity, we can find $\alpha > 0$ and a unit-neighborhood W such that $\lambda W \subseteq U$ for all $|\lambda| \leqslant \alpha$. Since D is non-discrete, we can take $\mu \in D^\times$ with $|\mu| \leqslant \alpha$. Then $\mu W \subseteq U$, and for any $\nu \in D$ with $|\nu| \leqslant 1$, that $|\mu\nu| \leqslant \alpha$ implies that $\nu\mu W \subseteq U$. But this means $\mu W \subseteq U_0$.
2. This follows from the continuity of the multiplication map $D \times V \to V$.
3. This follows from the previous two.

□

Lemma E.3.2. Let V be a first countable Hausdorff left topological vector space over D. Then there exists a metric $d : V \times V \to \mathbb{R}_{\geqslant 0}$ such that

1. the topology defined by d is the one on V,
2. all open balls centred at 0 are balanced, and
3. d is invariant in the sense that $d(x + z, y + z) = d(x, y)$ for all $x, y, z \in V$.

In particular, every metrizable left vector space over D admits an invariant metric.

Proof. Let $\{U_n\}_n$ be a balanced unit-neighborhood basis of V such that

$$U_{n+1} + U_{n+1} + U_{n+1} + U_{n+1} \subseteq U_n$$

for all $n \geqslant 1$. Put

$$B = \left\{ r = \sum_{n=1}^{\infty} c_n(r) 2^{-n} \mid (c_n(r))_n \subseteq \{0,1\} \text{ and all but finitely many are } 0 \right\};$$

note that $B \subseteq [0,1) \cap \mathbb{Q}$. For $r \geqslant 1$, put $A(r) := V$, and for $r \in B$,

$$A(r) := c_1(r)U_1 + c_2(r)U_2 + \cdots = \sum_{i=1}^{\infty} c_i(r)U_i.$$

Note that this is a finite sum. Finally, for $v \in V$, define $f(v) := \inf\{r \mid v \in A(r)\}$ and for $v, w \in V$, define $d(v,w) = f(v-w)$.

To show d is a metric, we first show that

$$A(r) + A(s) \subseteq A(r+s), \qquad r, s \in D.$$

If $r + s \geqslant 1$, then $A(r+s) = V$ and the containment is trivial. Assume $r + s < 1$. If $c_n(r) + c_n(s) = c_n(r+s)$ for all $n \geqslant 1$, then clearly $A(r) + A(s) = A(r+s)$. Otherwise, let N be the smallest n such that $c_n(r) + c_n(s) \neq c_n(r+s)$. Then $c_N(r) = c_N(s) = 0$ and $c_N(r+s) = 1$, so

$$\begin{aligned} A(r) &\subseteq c_1(r)U_1 + \cdots + c_{N-1}(r)U_{N-1} + U_{N+1} + U_{N+2} + \cdots \\ &\subseteq c_1(r)U_1 + \cdots + c_{N-1}(r)U_{N-1} + U_{N+1} + U_{N+1} \end{aligned}$$

and likewise

$$\begin{aligned} A(s) &\subseteq c_1(s)U_1 + \cdots + c_{N-1}(s)U_{N-1} + U_{N+1} + U_{N+2} + \cdots \\ &\subseteq c_1(s)U_1 + \cdots + c_{N-1}(s)U_{N-1} + U_{N+1} + U_{N+1}. \end{aligned}$$

These two show that $A(r) + A(s) \subseteq c_1(r+s)U_1 + \cdots + c_{N-1}(r+s)U_{N-1} + U_N \subseteq A(r+s)$ as $c_N(r+s) = 1$.

From the containment, we see if $r, s \in D$ with $r < s$, then

$$A(r) \subseteq A(r) + A(r-s) \subseteq A(s).$$

Hence $\{A(r)\}_{r \in D}$ is totally ordered. We claim

$$f(v+w) \leqslant f(v) + f(w)$$

for all $v, w \in V$. Indeed, we may assume $f(v) + f(w) < 1$, and for a fixed $\varepsilon > 0$ we can find $r, s \in D$ such that $f(v) < r$, $f(w) < s$ while $r + s < f(v) + f(w) + \varepsilon$. Then $v \in A(r)$, $w \in A(s)$ so that $v + w \in A(r) + A(s) \subseteq A(r+s)$, implying

$$f(v+w) \leqslant r + s < f(v) + f(w) + \varepsilon.$$

Since this is true for all $\varepsilon > 0$, it follows that $f(v+w) \leqslant f(v) + f(w)$.

Since each $A(r)$ is balanced, $f(x) = f(-x)$. Also, $f(0) = 0$. If $v \neq 0$, then $x \notin U_n = A(2^{-n})$ for some n, and so $f(x) \geqslant 2^{-n} > 0$. This show 3. For $\delta > 0$,

$$B_\delta := \{v \in V \mid d(v,0) < \delta\} = \{f(v) < \delta\} = \bigcup_{r<\delta} A(r).$$

If $\delta < 2^{-n}$, then $B_\delta \subseteq U_n$, so $\{B_\delta\}_\delta$ is a unit-neighborhood for V, proving 1. Since each $A(r)$ is balanced, so is B_δ and 2. is proved. $\square$

If V is a left topological vector spaces over D and $M \subseteq V$ is any D-subspace (without topology), the quotient space V/M with quotient topology is still a left topological vector spaces over D.

Denote by $\mathbf{TVS}_D$ the category of left topological vector spaces over D with morphisms being continuous D-linear maps. Denote by $\mathbf{HausTVS}_D$ the full subcategory of $\mathbf{TVS}_D$ consisting of Hausdorff left topological vector spaces over D. The inclusion functor $\iota : \mathbf{HausTVS}_D \to \mathbf{TVS}_D$, like the inclusion functor $\mathbf{Haus} \to \mathbf{Top}$ (c.f §A.3), has a left adjoint. Explicitly, define $H : \mathbf{TVS}_D \to \mathbf{HausTVS}_D$ by $H(V) := V/\overline{\{0\}}$

$$H(T : V \to W) = H(T) : V/\overline{\{0\}} \to W/\overline{\{0\}}.$$

That $V/\overline{\{0\}}$ is Hausdorff follows from Lemma 1.1.1.

If W is Hausdorff, then $H(W) = W$. Any continuous D-linear map $T : V \to W$ to a Hausdorff left topological D-vector space W uniquely factors through $V \to V/\overline{\{0\}}$; in fact, it factorizes as $V \to V/\overline{\{0\}} \xrightarrow{H(f)} H(W) = W$. Hence H is a left adjoint and a left inverse of the inclusion $\iota : \mathbf{HausTVS}_D \to \mathbf{TVS}_D$.

Lemma E.3.3. $\mathbf{TVS}_D$ is complete and cocomplete.

Proof. Construct the underlying vector spaces of (co)limits as in $\mathbf{Vec}_D$, and topologize them by initial (resp. final) topologies induced by canonical morphisms. $\square$

Corollary E.3.3.1. $\mathbf{HausTVS}_D$ is complete and cocomplete.

Proof. Product topology of Hausdorff spaces is Hausdorff, and subspace topology of a Hausdorff space is Hausdorff. Hence $\mathbf{HausTVS}_D$ is complete. $\square$

E.3.1 Metrizable TVS

Lemma E.3.4. Let V be a metrizable vector space over D and W a closed subspace. Then the quotient V/W is metrizable. Further, if V is complete, so is V/W.

Proof. Let f, d be as in the proof of Lemma E.3.2. Define $f' : V/W \to \mathbb{R}_{\geqslant 0}$ by

$$f'(v + W) := \inf_{w \in W} f(v + w).$$

It is clear that f' is well-defined. If $f'(v + W) = 0$, pick a sequence $(v_n)_n \subseteq v + W$ such that $f(v_n) \to 0$ as $n \to \infty$. But $f(v_n) = d(v_n, 0)$, this means $v_n \to 0$ as $n \to \infty$. Since W is closed, $v + W$ is closed and hence $0 \in v + W$. This shows $(v + W, v' + W) \mapsto f'((v - v') + W)$ defines a metric on V/W. To show this defines the same topology on V/W, note that

$$\{v \in V \mid f(v) < r\} + W = \{v + W \in V/W \mid f'(v + W) < r\}$$

as $0 \in W$. Since $V \to V/W$ is open, it sends a unit-neighborhood basis of V to a unit-neighborhood basis of V/W. This shows what we want.

Suppose V is complete. Let $(a_n + W)_n$ be a Cauchy sequence. Passing to a subsequence we may assume $\sum_{n=1}^{\infty} f(a_{n+1} - a_n + w_n) < \infty$ for some $(w_n)_n \in W$. Since V is complete, $a := \sum_{n=1}^{\infty} (a_{n+1} - a_n + w_n)$ defines an element of V. Since $f(v + W) \leqslant f(v)$ for all $v \in V$, we see $a_n + W \to a + W$ in V/W. $\square$

Definition. Let V, W be two left topological vector spaces over D. A continuous D-linear map $T : V \to W$ is called **strict** if the induced map $V/\ker T \to \operatorname{Im} T$ is an D-isomorphism.

Definition. A complete metrizable vector space over D is called an **F-space** over D.

Theorem E.3.5 (Open mapping theorem)**.** Let V, W be two metrizable vector spaces over D, and let $T : V \to W$ be a continuous D-linear map. Suppose V is complete. Then TFAE:

(i) T is strict and surjective.

(ii) W is complete and T is surjective.

(iii) For every unit-neighborhood U of V, the closure $\overline{T(U)}$ is a unit-neighborhood of W.

Proof. If (i) holds, then $W = \operatorname{Im} T \cong V/\ker T$. Since V is complete and $\ker T$ is closed, $V/\ker T$ is complete. Hence W is complete, showing (ii).

Assume (ii) and let U be a unit-neighborhood of V. Pick a balanced unit-neighborhood U' of V such that $U' + U' \subseteq U$. If $r \in D$ with $|r| > 1$, then $V = \bigcup\limits_{n\geqslant 1} r^n U'$. Indeed, if $x \in V$, then by continuity $x \in sU'$ for some $s \in D$. Pick $n \geqslant 1$ such that $|s| \leqslant |r|^n$. Since U' is balanced, $sU' = \dfrac{s}{r^n} r^n U' \subseteq r^n U'$ and hence $x \in r^n U'$. Consequently $W = \bigcup\limits_{n\geqslant 1} T(r^n U') = \bigcup\limits_{n\geqslant 1} r^n T(U')$. Now by Baire Category theorem, at least one $r^n \overline{T(U')}$ has non-empty interior, so $\overline{T(U')}$ has non-empty interior. Finally, if $y \in \overline{T(U')}$ is an interior point, then $0 = y + (-y)$ implies that 0 is an interior point of $\overline{T(U')} + \overline{T(U')} \subseteq \overline{T(U') + T(U')} \subseteq \overline{T(U)}$.

Assume (iii). We need the following

Lemma E.3.6. Let X, Y be two metric spaces with X complete. Let $f : X \to Y$ be a continuous map such that for each $r > 0$ there exists $\rho(r) > 0$ such that for all $x \in X$,

$$B_{\rho(r)}(f(x)) \subseteq \overline{f(B_r(x))}$$

holds. Then for all $a > r$, the containment $B_{\rho(r)}(f(x)) \subseteq f(B_a(x))$ holds.

Proof. Let $(r_n)_{n\geqslant 1} \subseteq \mathbb{R}_{>0}$ such that $r_1 = r$ and $a = \sum\limits_{i=1}^{\infty} r_n$. Also, let $(\rho_n)_{n\geqslant 1} \subseteq \mathbb{R}_{>0}$ satisfy $\rho_1 = \rho(r)$ and $B_{\rho_n}(f(x)) \subseteq \overline{f(B_{r_n}(x))}$ for all $x \in X$. We assume that $\rho_n \to 0$ as $n \to \infty$.

Let $x_0 \in X$ and $y \in B_{\rho(r)}(f(x_0))$. We are going to show that $y \in f(B_a(x_0))$. Define $(x_n)_{n\geqslant 1} \subseteq X$ inductively such that $x_n \in B_{r_n}(x_{n-1})$ and $T(x_n) \in B_{\rho_{n+1}}(y)$. If $x_0, \ldots, x_{n-1}$ have been constructed as we want, then $y \in B_{\rho_n}(f(x_{n-1})) \subseteq \overline{f(B_{r_n}(x_{n-1}))}$ so that there exists $x_n \in B_{r_n}(x_{n-1})$ with $f(x_n) \in B_{\rho_{n+1}}(y)$. Since $d(x_n, x_{n+m}) \leqslant r_{n+1} + \cdots + r_{n+m}$ is arbitrarily small if $n \gg 0$, $(x_n)_n$ is Cauchy in X. Since X is complete, it has a limit, say, $x \in X$. Since $d(x, x_0) < \sum_{n=1}^{\infty} r_n = a$, $x \in B_a(x_0)$. Since f is continuous, $f(x_n) \to f(x)$. But $f(x_n) \in B_{\rho_{n+1}}(y)$ with $\rho_n \to 0$, it forces that $f(x) = y$. This finishes the proof. $\square$

Return to the theorem. Equip V and W with invariant metrics inducing their topologies. By (iii) and translation, the map T satisfies the assumption of the lemma, so for $a > r > 0$, we have $B_{\rho(r)}(0) \subseteq T(B_a(0))$ for some $\rho(r)$. This shows T is open, and hence T is strict.

$\square$

Corollary E.3.6.1. Let V and W be two F-spaces over D.

1. Any bijective continuous D-linear map $V \to W$ is an isomorphism of topological vector spaces.

2. A continuous D-linear map $T : V \to W$ is strict if and only if $\operatorname{Im} T \subseteq W$ is closed.

Proof.

1. This follows from Theorem E.3.5.(ii)⇒(i).

2. The only if part is clear. Now if $\operatorname{Im} T \subseteq W$ is closed, then $\operatorname{Im} T$ is a complete metrizable vector space so $T : V \to \operatorname{Im} T$ is strict by Theorem E.3.5.(ii)⇒(i).

□

Corollary E.3.6.2 (Closed graph theorem)**.** Let V and W be two F-spaces over D. A D-linear map $T : V \to W$ is continuous if and only if the graph $\Gamma_T \subseteq V \times W$ is closed.

Proof. The only if part is clear. If $\Gamma_T \subseteq V \times W$ is closed, then it is complete metrizable. Since the projection $\mathrm{pr}_1 : \Gamma_T \to V$ is bijective and continuous, it is an isomorphism by Corollary E.3.6.1.1. Then $T = \mathrm{pr}_2 \circ \mathrm{pr}_1^{-1} : V \to W$ is continuous. □

E.3.2 Semi-norms

Definition. Let V be a left vector space over D. A map $p : V \to \mathbb{R}_{\geqslant 0}$ is called a **semi-norm** if $p(rv) = |r|p(v)$ for all $r \in D$, $v \in V$ and $p(v + w) \leqslant p(v) + p(w)$ for all $v, w \in V$.

- It follows from the last inequality that $|p(v) - p(w)| \leqslant p(v - w)$ holds for all $v, w \in V$.

Lemma E.3.7. Let V be a topological left vector space over D and $p : V \to \mathbb{R}_{\geqslant 0}$ a semi-norm. TFAE:

1. p is continuous.

2. p is continuous at 0.

3. p is uniformly continuous.

4. For all $r > 0$, the set $W(p, r) := \{v \in V \mid p(v) < r\}$ is open in V.

5. There exists $r > 0$ such that $W(p, r)$ is a unit-neighborhood of V.

6. For all $r > 0$, the set $V(p, r) := \{v \in V \mid p(v) \leqslant r\}$ is a unit-neighborhood in V.

Proof. 3 ⇒1 ⇒2 ⇒4 ⇒5 ⇒6 ⇒3 is clear. □

Corollary E.3.7.1. If p is a continuous semi-norm on V and q is another semi-norm on V such that $q \leqslant p$, then q is continuous.

Let p be a semi-norm on a left vector space V over D. For each $\alpha > 0$, the subset

$$V(p, \alpha) := \{v \in V \mid p(v) \leqslant \alpha\}$$

is balanced, absorbent and satisfies $V(p, \alpha/2) + V(p, \alpha/2) \subseteq V(p, \alpha)$. It is direct to see $\{V(p, \alpha)\}_{\alpha > 0}$ forms a fundamental system of unit-neighborhoods of a unique vector space topology on V. We say the topology is **defined by the semi-norm** p, and say V equipped with this topology a **semi-normed space**.

Let S be a collection of semi-norms on V. The upper bound of the topologies defined by the semi-norms $p \in S$ has a fundamental system of unit-neighborhoods of the form $\bigcap_{p \in S'} V(p, \alpha_p)$, where $\alpha_p > 0$ and $S' \subseteq S$ is finite. The topology is the coarsest topology that is invariant under translations and making all $p \in S$ continuous. We say this topology is defined by S, and it defines a vector space topology over D on V. Conversely, if V is a topological left vector space over D and S is a collection of semi-norms on V which induces the topology on V, we say S is a **fundamental system of semi-norms**.

Definition. A topological left vector space over D whose topology in defined by a collection of semi-norms is called a **locally convex space**.

Denote by $\mathbf{LCS}_D$ the full subcategory of $\mathbf{TVS}_D$ consisting of locally convex spaces over D. The inclusion functor $\mathbf{LCS}_D \to \mathbf{TVS}_D$ admits a left adjoint constructed as follows. For $X \in \mathbf{TVS}_D$, let S_X collect all continuous semi-norms on X. Denote by $LC(X)$ the left D-vector space X equipped the topology induced by S_X. If $T : X \to Y$ is a morphism in $\mathbf{TVS}_D$ and $p \in S_Y$, we have $p \circ T \in S_X$, so $T : LC(X) \to LC(Y)$ is continuous; we then define $LC(T) = T$. Moreover, $LC(X) = X$ if X is locally convex. It follows that LC is a left adjoint left inverse of the inclusion $\mathbf{LCS}_D \to \mathbf{TVS}_D$. By Lemma E.3.3 we conclude that

Lemma E.3.8. $\mathbf{LCS}_D$ is complete and cocomplete.

Proof. Product topology of locally convex spaces is locally convex, and a subspace topology of a locally convex space is locally convex. Hence $\mathbf{LCS}_D$ is complete. ☐

We will discuss more in detail the locally convex colimit in §E.4.2.

Lemma E.3.9. Let V be a topological left vector space over D whose topology is defined by a collection of semi-norms S.

1. The closure of 0 in V is $\{v \in V \mid p(v) = 0 \text{ for all } p \in S\}$.

2. If V is Hausdorff and S is countable, then V is metrizable.

Proof. 1. follows from the definition, and 2. follows from Lemma E.3.2. ☐

Definition. A complete Hausdorff topological left vector space over D whose topology is defined by a countable collection of semi-norms is called a **Fréchet space** over D.

Example E.3.10. Let M be a smooth manifold with volume form ω. For $f, g \in C_c(M)$, set

$$\langle f, g\rangle := \int_M f\overline{g}\omega.$$

This defines an inner product on $C_c(M)$. We denote by $L^2(M, \omega)$ its Hilbert space completion. For $f \in L^2(M, \omega)$, put $\|f\|_2 := \sqrt{\langle f, f\rangle}$ to be the associated norm.

Assume there exists smooth vector fields $X_1, \dots, X_m$ such that $\{(X_1)_p, \dots, (X_m)_p\}$ generates T_pM for each $p \in M$. For $i_1, \dots, i_k \in [m]$, define a function $p_{i_1,\dots,i_k} : C^\infty(M) \to [0, \infty]$ by

$$p_{i_1,\dots,i_k}(f) := \|X_{i_1} \cdots X_{i_k} f\|_2 \leqslant \infty.$$

Denote by $V \subseteq C^\infty(M)$ the subspace consisting of all $f \in C^\infty(M)$ with all $p_{i_1,\dots,i_k}(f)$ finite. Then each $p_{i_1,\dots,i_k}$ defines a semi-norm on V. We equip V with the topology induced by these semi-norms; then V is a first countable Hausdorff topological vector space over $\mathbb{C}$. We claim V is complete, so it is a Fréchet space. We begin by showing that for each $\varphi \in C_c^\infty(M)$ and $f \in V$, we have $\varphi f \in V$ and the map

$$\begin{aligned} V &\longrightarrow V \\ f &\longmapsto \varphi f \end{aligned}$$

is continuous. Indeed, by Leibniz's rule we have

$$X_{i_1} \cdots X_{i_k}(\varphi f) = \sum a_{j_1,\dots,j_p,l_1,\dots,l_{k-p}} (X_{j_1} \cdots X_{j_p}\varphi)(X_{l_1} \cdots X_{l_{k-p}} f)$$

with $a_{j_1,\dots,j_p,l_1,\dots,k_{k-p}} \in \mathbb{R}$ independent of φ and f. This implies

$$\|X_{i_1}\cdots X_{i_k}(\varphi f)\|_2 \leqslant \max|a_{j_1,\dots,j_p,l_1,\dots,k_{k-p}}| \cdot \max\left\|X_{j_1}\cdots X_{j_p}\varphi\right\|_M \cdot \sum \left\|X_{l_1}\cdots X_{l_{k-p}} f\right\|_2$$

proving the continuity.

Definition. Let V be a left vector space over D. A collection S of semi-norms on V is **separating** if

$$\bigcap_{p\in S}\{v \in V \mid p(v) = 0\} = \{0\}$$

Lemma E.3.11. Let V be a left vector space over D. The collection S of semi-norms is separating if and only if the topology defined by S is Hausdorff.

Proof. Assume S is separating. Say $v \neq w$. Then there exists $p \in S$ such that $p(v-w) \neq 0$. Then $\{x \in V \mid p(v-x) < 2^{-1}\cdot p(v-w)\}$ and $\{x \in V \mid p(w-x) < 2^{-1}\cdot p(v-w)\}$ are neighborhoods of v, w respectively that are disjoint, indeed, were x lying in the intersection, we would have

$$p(v-w) \leqslant p(v-x) + p(x-w) < p(v-w)$$

a contradiction. For the other direct, suppose the topology induced by S is Hausdorff. If $v \neq 0$, then there exist $r > 0$ and a finite subset $S' \subseteq S$ such that $v \notin \bigcap_{p\in S'} V(p,r)$. But then for $p \in S'$ we have $p(v) > r > 0$. ☐

In the rest of this subsection we assume all vector spaces are over $\mathbb{R}$.

Definition. Let V be a vector space over $\mathbb{R}$.

1. A subset S is called **convex** if $tS + (1-t)S \subseteq S$ for all $0 \leqslant t \leqslant 1$.
2. For a subset A, the intersection of all convex subsets containing A is called the **convex hull** $\operatorname{conv} A$ of A.

Lemma E.3.12. Let V be a topological vector space over $\mathbb{R}$.

(i) If $p : V \to \mathbb{R}_{\geqslant 0}$ is a continuous semi-norm, for all $r > 0$, the sets $W(p,r)$ and $V(p,r)$ are convex balanced unit-neighborhood.

(ii) If C is a convex balanced open unit-neighborhood in V, then the **Minkowski functional** $p_C : V \to \mathbb{R}_{\geqslant 0}$ defined by

$$p_C(v) = \inf\{t \geqslant 0 \mid v \in tC\}$$

is a continuous semi-norm on V.

Proof.

(i) That $V(p,r)$ and $W(p,r)$ are neighborhoods is shown in Lemma E.3.7. For $v, w \in V(p,r)$ and $0 \leqslant t \leqslant 1$,

$$p(tv + (1-t)w) \leqslant p(tv) + p((1-t)w) \leqslant tp(v) + (1-t)p(w) < r,$$

and $p(tv) \leqslant tp(v) < r$. This shows $V(p,r)$ is balanced and convex. The assertion for $W(p,r)$ is proved in the same way.

(ii) We first show p_C is a semi-norm. Let $0 \neq v \in V$. For $r \in \mathbb{R}^\times$,

$$\begin{aligned} p_C(rv) = \inf\{t \geqslant 0 \mid rv \in tC\} &= \inf\{t \geqslant 0 \mid |r|v \in tC\} \\ &= \inf\{t \geqslant 0 \mid v \in |r|^{-1}tC\} = |r| \cdot \inf\{t \geqslant 0 \mid v \in tC\} = |r|p_C(v). \end{aligned}$$

The second equality uses that tC is balanced.

For $v, w \in V$, say $t, s > 0$ is such that $v \in tC$ and $w \in sC$. Then

$$v + w \in tC + sC = (t+s)\left(\frac{t}{t+s}C + \frac{s}{t+s}C\right) \subseteq (t+s)C$$

by convexity of C. Hence $p_C(v+w) \leqslant p_C(v) + p_C(w)$.

It remains to show p_C is continuous. By Lemma E.3.7 we must show $V(p_C, r)$ is a unit-neighborhood for any $r > 0$. This is clear, as for $v \in rC$ we have $p_C(v) \leqslant r$.

∎

Corollary E.3.12.1. Let V be a topological vector space over $\mathbb{R}$ admitting a unit-neighborhood basis $\mathcal{U}$ convex balanced open sets. Then the topology on V is induced by those Minkowski functionals associated to $\mathcal{U}$.

Proof. We claim for $U \in \mathcal{U}$ that

$$U = \{v \in V \mid p_U(v) < 1\}.$$

If $p_U(v) < 1$, then $v \in tU$ for some $0 \leqslant t < 1$, and hence $v \in U$. Conversely, for $v \in U$, by continuity of action of $\mathbb{R}$ on V, there exists $\delta > 0$ such that $(1+r)v \in U$ as long as $0 < r < \delta$. But then $v \in (1+r)^{-1}U$, or $p_U(v) \leqslant (1+r)^{-1} < 1$.

For $t \in \mathbb{R}$, we have $\{v \in V \mid p_U(v) < |t|\} = tU$, so this finishes the proof. ∎

Lemma E.3.13. Let X be a locally convex space, and $Y \subseteq X$ a subspace. If $p : Y \to \mathbb{R}_{\geqslant 0}$ is a continuous semi-norm on W, there exists a continuous semi-norm p' on X such that $p'|_Y = p$.

Proof. This amounts to showing that every convex balanced open unit-neighborhood of Y is the intersection of Y with a convex balanced open unit-neighborhood of X. Let U be a convex balanced open unit-neighborhood of Y; then there exists an open unit-neighborhood V_1 of X such that $V_1 \cap Y = U$. Pick any convex balanced open unit-neighborhood V of X contained in V_1.

We claim $\operatorname{conv}(V \cup U)$ is a convex balanced open unit-neighborhood of X such that $Y \cap \operatorname{conv}(V \cup U) = U$. That $\operatorname{conv}(V \cup U)$ is convex and balanced are clear. For openness, write

$$\operatorname{conv}(V \cup U) = \bigcup_{0 \leqslant t \leqslant 1} (tV + (1-t)U).$$

Each set $tV + (1-t)U$ $(0 < t \leqslant 1)$ is open in X, while U $(t = 0)$ is not. For $v \in U$, $\varepsilon v \in V$ for some $\varepsilon > 0$. Then the vector $\dfrac{1-\varepsilon+t\varepsilon}{t}v \in Y$ $(0 < t < 1)$ tends to v as $t \to 1$. Since U is open, $\dfrac{1-\varepsilon+t\varepsilon}{t}v \in U$ for some $1 > t > 0$. Then $(1-\varepsilon+t\varepsilon)v \in tU$, or $v \in tU + (1-t)\varepsilon v \subseteq tU + (1-t)V$. This proves $\operatorname{conv}(V \cap U) \subseteq X$ is open.

It remains to show $Y \cap \operatorname{conv}(V \cup U) = U$. The containment $\supseteq$ is clear. If $y \in Y \cap \operatorname{conv}(V \cup U)$, then $y = tv + (1-t)u$ for some $v \in V$, $u \in U$, $0 \leqslant t \leqslant 1$. If $t = 0, 1$, then $y \in Y \cap (V \cup U) \subseteq U$. For $0 < t < 1$, we have then $v = \dfrac{y-(1-t)u}{t} \in V \cap Y \subseteq U$, so $y \in U$. ∎

E.3.3 Hahn-Banach

In this subsection, all vector spaces are over $\mathbb{R}$.

Definition. Let V be a vector space.

1. A subset C is called a **cone with vertex** x_0 if $x_0 + t(C - x_0) \subseteq C$ for all $t \in \mathbb{R}_{>0}$. A cone is a cone with vertex 0.
2. A cone C is called **pointed** (resp. **non-pointed**) if $0 \in C$ (resp. $0 \notin C$).
3. A cone C is called **proper** if C does not contain any line passing through 0.
4. For a subset A, the intersection of all convex cones containing A is called the **convex cone generated by** A.

Lemma E.3.14. A pointed convex cone C is proper if and only if the non-pointed cone $C' := C\backslash\{0\}$ is convex.

Proof. If C contains a line passing through 0, then C' obviously cannot be convex. Conversely, if C is proper, then for $x, y \in C'$, the closed segment $tx + (1-t)y \in C$, $0 \leqslant t \leqslant 1$. If the segment contains 0, then $x = \lambda y$ for some $\lambda < 0$. This implies C contains a line passing through 0 and x, a contradiction.

□

Definition. A vector space V together with a relation $\leqslant$ such that

1. $x \leqslant y$ implies $x + z \leqslant y + z$ for all $z \in V$
2. $x \geqslant 0$ implies $rx \geqslant 0$ for all $r \in \mathbb{R}_{\geqslant 0}$

is called a **pre-ordered vector space**. If $\leqslant$ is a partial order, then we say V is a **ordered vector space**.

Definition. An **ordered topological vector space** is a ordered vector space V together with a vector space topology such that the subset $\{x \in V \mid x \geqslant 0\}$ is closed in V.

Lemma E.3.15. Let V be an ordered topological vector space.

1. The subset $\{x \in V \mid x \leqslant 0\}$ is closed.
2. V is Hausdorff.

Proof. 1. follows from the fact that $x \mapsto -x$ is a homeomorphism. 2. follows since $\{0\} = \{x \geqslant 0\} \cap \{x \leqslant 0\}$. □

Lemma E.3.16. Let V be a vector space.

1. If V is a pre-ordered vector space, the subset $\{x \in V \mid x \geqslant 0\}$ is a pointed convex cone.
2. If C is a pointed convex cone, then the relation $x \leqslant y \Leftrightarrow y - x \in C$ makes V a pre-ordered vector space and is the unique one such that $C = \{x \in V \mid x \geqslant 0\}$.
3. The relation in 2. is a partial order if and only if C is proper.

Proof.

1. Clearly the subset is a pointed cone. For $x, y \geqslant 0$ and $0 \leqslant t \leqslant 1$, since t, $1-t \geqslant 0$, we have $tx, (1-t)y \geqslant 0$ and hence $tx + (1-t)y \geqslant 0$. This shows the convexity.

2. Clear.

3. C is proper if and only if $C \cap -C = \{0\}$, i.e. $\leqslant$ is a partial order.

□

Lemma E.3.17. Let V be a pre-ordered vector space and W a subspace of V such that every element in V is bounded by some element in W. Then for any positive linear functional $f : W \to \mathbb{R}$ (i.e., $f(\{x \geqslant 0\}) \subseteq \mathbb{R}_{\geqslant 0}$) there exists a positive linear functional $\widetilde{f} : V \to \mathbb{R}$ extending f. Moreover, for any such extension $\widetilde{f}$ and $x \in V$, we have

$$\sup_{y \in W,\, y \leqslant x} f(y) \leqslant \widetilde{f}(x) \leqslant \inf_{y \in W,\, x \leqslant y} f(y).$$

Proof. We consider the special case when $V = W + \mathbb{R}x$. If $x \in W$ then the lemma is trivial. Assume $x \notin W$. By assumption there is $y \in W$ with $x \leqslant y$, and also $z \in W$ with $-x \leqslant -z$, or $z \leqslant x$. Then $z \leqslant x \leqslant y$ and hence $f(z) \leqslant f(y)$. In particular, the two numbers in the moreover part are finite and $\sup_{y \in W,\, y \leqslant x} f(y) \leqslant \inf_{y \in W,\, x \leqslant y} f(y)$ Any extension $\widetilde{f} : V \to \mathbb{R}$ of f is uniquely determined by its value at x, and it is positive if and only if $w + rx \geqslant 0$ implies $f(w) + r\widetilde{f}(x) \geqslant 0$ for all $w \in W$, $r \in \mathbb{R}$. To show it is possible, we only need to consider the case $r = 0, 1, -1$. The case $r = 0$ is ok since f is positive. If $r = 1$, then $w + x \geqslant 0 \Rightarrow f(w) + \widetilde{f}(x) \geqslant 0$ means for $-w \leqslant x$ we have $\widetilde{f}(x) \geqslant f(-w)$. For this we only need to take $\widetilde{f}(x) \geqslant \sup_{y \in W,\, y \leqslant x} f(y)$. From $r = -1$, we likewise see we need to take $\widetilde{f}(x) \leqslant \inf_{y \in W,\, x \leqslant y} f(y)$. We already see such $\widetilde{f}(x)$ exists.

For general case, consider the set

$$\{(W', g) \mid W \subseteq W' \subseteq V,\ g \text{ is a positive linear functional extending } f\}$$

is nonempty and is clearly partially ordered. By Zorn's lemma there exists a maximal element (W_0, g_0), and we must show $W_0 = V$. For any $x \in V$, by the first paragraph we can extend g_0 to $W_0 + \mathbb{R}x$, so by maximality $x \in W_0$. The moreover part also follows from the first paragraph. □

Theorem E.3.18 (Hahn-Banach, analytic form)**.** Let $p : V \to \mathbb{R}$ be a sub-linear function, i.e., $p(x + y) \leqslant p(x) + p(y)$ and $p(rx) = rp(x)$ for all $x, y \in V$, $r \in \mathbb{R}_{\geqslant 0}$. Let W be a subspace of V and $f : W \to \mathbb{R}$ be a linear functional such that $f \leqslant p|_W$. Then there exists a functional $h : V \to \mathbb{R}$ extending f such that $h \leqslant p$.

Proof. The set $C = \{(x, a) \in V \times \mathbb{R} \mid p(x) \leqslant a\}$ is a pointed convex cone in $V \times \mathbb{R}$. Define a functional $F : W \times \mathbb{R} \to \mathbb{R}$ by setting

$$F(x, a) = -f(x) + a.$$

Then F is positive with respect to the pre-order structure $\leqslant$ defined by $C \cap (W \times \mathbb{R})$. Indeed, if $(x, a) \in C \cap (W \times \mathbb{R})$, then $p(x) \leqslant a$ so that $-f(x) + a \geqslant -p(x) + a \geqslant 0$. Now for any $(x, a) \in V \times \mathbb{R}$, we have $(y, b) \geqslant (x, a)$ (with $\leqslant$ defined by C) if and only if $p(y - x) \leqslant b - a$. We then see $(0, a') \in W \times \mathbb{R}$ with $a \geqslant p(-x) + a$ satisfies $(0, a') \geqslant (x, a)$. Hence all assumptions in Lemma E.3.17 are met, so there exists a positive linear function $\widetilde{F} : V \times \mathbb{R} \to \mathbb{R}$ extending F. Now

$$\widetilde{F}(x, a) = \widetilde{F}(x, 0) + a\widetilde{F}(0, 1) =: -h(x) + aF(0, 1) = -h(x) + a$$

so h extends f. Finally, for all $x \in E$ and $a \in \mathbb{R}$ with $p(x) \leqslant a$, we have $0 \leqslant \widetilde{F}(x, a)$ so that $h(x) \leqslant a$. Varying a yields $h(x) \leqslant p(x)$. □

Corollary E.3.18.1. Let V be a vector space and $p : V \to \mathbb{R}_{\geqslant 0}$ a semi-norm.

1. If W is a subspace and $f : W \to \mathbb{R}$ is a functional such that $|f| \leqslant p$, then f has an extension $h : V \to \mathbb{R}$ with $|h| \leqslant p$.

2. For any $x \in V$, there exists a functional $f : V \to \mathbb{R}$ such that $f(x) = p(x)$ and $|f| \leqslant p$.

Corollary E.3.18.2. Let V be a vector space over $\mathbb{C}$, $W \subseteq V$ a complex subspace and $f : W \to \mathbb{C}$ a $\mathbb{C}$-linear functional. If $p : V \to \mathbb{R}_{\geqslant 0}$ is a semi-norm such that $|f| \leqslant p|_W$, then there exists a $\mathbb{C}$-linear functional $h : V \to \mathbb{C}$ extending f such that $|h| \leqslant p$.

Proof. The real part $f_r := \operatorname{Re} f : W \to \mathbb{R}$ satisfies $|f_r| \leqslant p|_W$, so by Corollary E.3.18.1.1. there exists an extension $h_r : V \to \mathbb{R}$ with $|h_r| \leqslant p$. Define $h : V \to \mathbb{C}$ by setting $h(x) = h_r(x) - ih_r(ix)$; then $\operatorname{Re} h = h_r$, and

$$|h(x)| = |\operatorname{Re} e^{i\theta} h(x)| = |\operatorname{Re} h(e^{i\theta} x)| = |h_r(e^{i\theta} x)| \leqslant p(e^{i\theta} x) = p(x)$$

for some $\theta \in \mathbb{R}$, i.e., $|h| \leqslant p$. □

E.4 Locally convex spaces

E.4.1 Boundedness

Lemma E.4.1. Let X be a left topological vector space over D with topology induced by the collection S of semi-norms on X.

(i) A subset $B \subseteq X$ is bounded if and only if $p(B) \subseteq \mathbb{R}_{\geqslant 0}$ is bounded for all $p \in S$.

(ii) A precompact subset of X is bounded.

(iii) A Cauchy sequence in X is bounded.

Proof.

(i) Note a subset is bounded if and only if it is absorbed by sets in a unit-neighborhood basis of X. Also, for $p \in S$, $r > 0$ and $\alpha \in D$, we have $\alpha V(p, r) = V(p, |\alpha| r)$. These two observations prove (i).

(ii) It suffices to show a compact set is bounded. This is clear as each $p \in S$ is continuous on X.

(iii) It can be shown that points in a Cauchy sequence is precompact, whence bounded by (ii). We argue directly. Let $(a_n)_n$ be a Cauchy sequence. Let $p \in S$ and $r > 0$. Then there exists $N > 0$ such that $a_n - a_N \in V(p, r)$ whenever $n \geqslant N$, or $(a_n)_{n \geqslant N} \subseteq a_N + V(p, r) \subseteq V(p, r + p(a_N))$. Then $(a_n)_n \subseteq V(p, r + p(a_1) + \cdots + p(a_N))$. Since $p \in S$ is arbitrary, by (i) this shows $(a_n)_n$ is bounded.

□

E.4.2 Locally convex colimits

In this subsection all vector spaces are over $\mathbb{R}$. Denote also by $\mathbf{HausLCS}_{\mathbb{R}}$ the full subcategory of $\mathbf{LCS}_{\mathbb{R}}$ consisting of Hausdorff locally convex spaces. Then there are inclusions

$$\begin{tikzcd} & \mathbf{LCS}_{\mathbb{R}} \arrow[dr] & \\ \mathbf{HausLCS}_{\mathbb{R}} \arrow[ur] \arrow[dr] & & \mathbf{TVS}_{\mathbb{R}} \\ & \mathbf{HausTVS}_{\mathbb{R}} \arrow[ur] & \end{tikzcd}$$

In §A.3 we see the inclusion $\mathbf{Haus} \to \mathbf{Top}$ does not preserve epimorphisms, so we cannot expect colimits are preserves under these inclusions.

Let $F : J \to \mathbf{LCS}_{\mathbb{R}}$ be a small diagram. Let $(f_\alpha : F(\alpha) \to \operatorname{colim} F)_\alpha$ be the colimit of F viewed as a $\mathbf{Vec}_{\mathbb{R}}$-valued diagram. Our goal is to equip a locally convex topology on $\operatorname{colim} F$ making it a colimit of F in $\mathbf{LCS}_{\mathbb{R}}$. Let $\mathcal{U}_\alpha$ be the collection of all convex unit-neighborhoods of $F(\alpha)$. Let

$$\mathcal{U} = \{\operatorname{conv}\{f_\alpha(U_\alpha)\}_\alpha \subseteq \operatorname{colim} F \mid U_\alpha \in \mathcal{U}_\alpha \text{ for all } \alpha\}\,.$$

We declare this to be a unit-neighborhood basis of $\operatorname{colim} F$ and topologize $\operatorname{colim} F$ with the unique vector space topology induced by $\mathcal{U}$.

Lemma E.4.2. With this topology, the cocone $(f_\alpha : F(\alpha) \to \operatorname{colim} F)_\alpha$ is the colimit of F in $\mathbf{LCS}_{\mathbb{R}}$.

Proof. We must show $\operatorname{colim} F$ is a locally convex space. That it is locally convex follows from construction. Continuity of $\mathbb{R}$-action on $\operatorname{colim} F$ follows from those on the $F(\alpha)$. Take $U_\alpha \in \mathcal{U}_\alpha$ for each α; by continuity of addition on $F(\alpha)$ we can find $V_\alpha \in \mathcal{U}_\alpha$ such that $V_\alpha + V_\alpha \subseteq U_\alpha$. Then $\operatorname{conv}\{f_\alpha(V_\alpha)\}_\alpha + \operatorname{conv}\{f_\alpha(V_\alpha)\}_\alpha \subseteq \operatorname{conv}\{f_\alpha(U_\alpha)\}_\alpha$, proving the continuity.

That $F(\alpha) \to \operatorname{colim} F$ is continuous follows from construction. Suppose $(g_\alpha : F(\alpha) \to X)_\alpha$ is a cocone in $\mathbf{LCS}_{\mathbb{R}}$. By the universal property in $\mathbf{Vec}_{\mathbb{R}}$, there exists a unique linear map $g : \operatorname{colim} F \to X$ compatible with maps in the cocones. Take a convex unit-neighborhood $V \subseteq X$, and take $U_\alpha \in \mathcal{U}_\alpha$ such that $g_\alpha(U_\alpha) \subseteq X$. Then

$$g(\operatorname{conv}\{f_\alpha(U_\alpha)\}_\alpha) \subseteq \operatorname{conv}\{g_\alpha(U_\alpha)\}_\alpha \subseteq X,$$

which proves the continuity of $g : \operatorname{colim} F \to X$. ☐

Definition. We call $\operatorname{colim} F$ with this locally convex topology the **locally convex colimit** of the small diagram $F : J \to \mathbf{LCS}_{\mathbb{R}}$.

Lemma E.4.3. A semi-norm $p : \operatorname{colim} F \to \mathbb{R}_{\geqslant 0}$ is continuous if and only if $p \circ f_\alpha : F(\alpha) \to \mathbb{R}_{\geqslant 0}$ is continuous for each α.

Proof. Only if part is clear. For the if part, assume $p \circ f_\alpha : F(\alpha) \to \mathbb{R}_{\geqslant 0}$ is continuous for each α. Fix an $r > 0$. Then $(p_\alpha, r) = \{v \in F(\alpha) \mid p_\alpha(v) < r\}$ is a convex unit-neighborhood of $F(\alpha)$, and hence $\operatorname{conv}\{V(p_\alpha, r)\}_\alpha$ is a convex unit-neighborhood of $\operatorname{colim} F$ by construction. For $x_\alpha \in V(p_\alpha, r)$ and $a_\alpha \in [0,1]$ with $a_\alpha = 0$ for all but finitely many α and $\sum_\alpha a_\alpha = 1$, we have

$$p\left(\sum_\alpha a_\alpha x_\alpha\right) \leqslant \sum_\alpha a_\alpha p(x_\alpha) = \sum_\alpha a_\alpha p_\alpha(x_\alpha) < \sum_\alpha a_\alpha r = r.$$

This shows $\operatorname{conv}\{V(p_\alpha, r)\}_\alpha \subseteq V(p, r)$, so by Lemma E.3.7 p is continuous. ☐

Corollary E.4.3.1. A linear map $T : \operatorname{colim} F \to X$ to a locally convex space is continuous if and only if $T \circ f_\alpha : F(\alpha) \to X$ is continuous for each α.

Proof. Say the topology on X is induced by the collection S of the semi-norms. Then T is continuous if and only if $p \circ T$ is continuous for each $p \in S$. By Lemma E.4.3 this amounts to saying $p \circ T \circ f_\alpha$ is continuous for each $p \in S$ and α, which is the same as saying $T \circ f_\alpha : F(\alpha) \to X$ is continuous for each α. $\square$

Next we consider a special kind of diagrams. By abuse of notations we view $\mathbb{N}$ as a category with morphisms induced by the well-order $\leqslant$. A diagram $F : \mathbb{N}^{\mathrm{op}} \to \mathbf{LCS}_{\mathbb{R}}$ of type $\mathbb{N}$ is called **strict** if for each $n > m$, the map $F_{n>m} : F(m) \to F(n)$ is a topological embedding with proper image (i.e., not surjective). The locally convex colimit of F is then called the **strict (locally convex) colimit**.

Lemma E.4.4. Let $(f_n : F(n) \to \operatorname{colim} F)_n$ be a strict colimit.

(i) For any continuous semi-norm p_n on $F(n)$, there exists a continuous semi-norm p on $\operatorname{colim} F$ such that $p \circ f_n = p_n$.

(ii) Each f_n is a topological embedding.

(iii) If each $F(n)$ is Hausdorff, then $\operatorname{colim} F$ is Hausdorff.

Proof. That f_n is injective already holds in $\mathbf{Vec}_{\mathbb{R}}$. Also, we have an ascending filtration

$$f_1(F(1)) \subsetneq \cdots \subsetneq f_n(F(n)) \subsetneq \cdots \subseteq \operatorname{colim} F$$

of $\operatorname{colim} F$ such that $\bigcup_{n \geqslant 1} f_n(F(n)) = \operatorname{colim} F$. For (i), let p_n be a continuous semi-norm on $F(n)$. By Lemma E.3.13 and induction, for each $m > n$ there exists a continuous semi-norm p_m on $F(m)$ such that $p_m \circ f_{m>m-1} = p_{m-1}$. We then can define $p : \operatorname{colim} F \to \mathbb{R}_{\geqslant 0}$ by setting $p(x) := p_n(x)$ if $x \in f_n(F(n))$. This is clearly a semi-norm. For $r > 0$, the set $V(p_m, r) \subseteq F(m)$ is open and convex. By construction $\operatorname{conv}\{V(p_m, r)\}_{m \geqslant n}$ is an convex open set in $\operatorname{colim} F$. But then $\operatorname{conv}\{V(p_m, r)\}_{m \geqslant n} \subseteq V(p, r)$, so by Lemma E.3.7 p is continuous.

For (ii), if we denote by $g : f_m(F(m)) \to F(m)$ the inverse of f_m, then we must show g is continuous. Let p be a continuous semi-norm on $F(m)$. By (i) there exists a continuous semi-norm p' on $\operatorname{colim} F$ such that $p' \circ f_n = p$. Then $p \circ g = p'|_{f_m(F(m))}$, which is continuous. Since p is arbitrary, this proves the continuity of g.

For (iii), if $v \neq 0$ and, say, $v \in f_m(F(m))$, from Lemma E.3.11 we have find a continuous semi-norm p_m such that $p_m(v) \neq 0$. By (i) there is then a continuous semi-norm p on $\operatorname{colim} F$ such that $p(v) \neq 0$. Hence $\operatorname{colim} F$ is Hausdorff by Lemma E.3.11. $\square$

Lemma E.4.5. Let $F : \mathbb{N}^{\mathrm{op}} \to \mathbf{LCS}_{\mathbb{R}}$ be strict. A subset $B \subseteq \operatorname{colim} F$ is bounded if and only if there exists some n such that $B \subseteq F(n)$ and B is bounded in $F(n)$.

Proof. Assume $B \subseteq F(n)$ and is bounded in $F(n)$ for some n. For each m take a convex open unit-neighborhood $U_n \subseteq F(n)$; we must show $\operatorname{conv}\{U_m\}_m$ absorbs B. But $B \subseteq tU_n$ for some $t > 0$ as B is bounded in $F(n)$, so $B \subseteq t \operatorname{conv}\{U_m\}_m$.

We turn to the only if part. Let $B \subseteq \operatorname{colim} F$ be a bounded subset. If $B \subseteq F(n)$ for some $n \geqslant 1$, using an argument similar to that in the first paragraph we see B is bounded in $F(n)$. Hence it remains to show B is indeed contained in some $F(n)$. Suppose it is not the case. Our goal is then to construct a continuous semi-norm on $\operatorname{colim} F$ that is not bounded on B, which will lead to a contradiction to Lemma E.4.1.(i).

By Lemma E.4.4.(ii) there is no loss in identifying $F(n)$ with its image $f_n(F(n))$ in $\operatorname{colim} F$. By replacing $F(m)$ with its subsequence, we can find a sequence $(x_n)_n \subseteq B$ such that $x_1 \in B \cap F(1)$ and $x_n \in (B \cap F(n)) \backslash f_{n>n-1}(F(n-1))\,(n \geqslant 2)$. Pick any continuous semi-norm p_1 on $F(1)$ such that $p_1(x_1) = 1$. We are going to construct $(p_n)_n$ inductively in a way that p_n is a continuous semi-norm on $F(n)$ such that $p_n \circ f_{n>n-1} = p_{n-1}$ and $p_n(x_n) = n$ for $n \geqslant 2$. Once this is done, we may define $p : \operatorname{colim} F \to \mathbb{R}_{\geqslant 0}$ by setting $p(x) = p_n(x)$ if $x \in F(n)$. This is a semi-norm, and is continuous by Lemma E.4.3. Moreover, $p(x_n) = n \to \infty$ as $n \to \infty$, so it is not bounded on B, as desired.

Suppose p_n is constructed. Put $H = f_{n+1>n}(F(n)) + \mathbb{R}x_{n+1}$. In fact, this is a direct sum in $\mathbf{TVS}_{\mathbb{R}}$. Define $q : H \to \mathbb{R}_{\geqslant 0}$ by $q(f_{n+1>n}(x) + \alpha x_{n+1}) = p_n(x) + |\alpha|(n+1)$, where $\alpha \in \mathbb{R}$ and $x \in F(n)$. Then q is a continuous semi-norm on H such that $q \circ f_{n+1>n} = p_n$. By Lemma E.3.13 we can find a continuous semi-norm p_{n+1} on $F(n+1)$ such that $p_{n+1}|_H = q$. Then $p_{n+1} \circ f_{n+1>n} = p_n$ and $p_{n+1}(x_{n+1}) = n + 1$ as wanted. □

Corollary E.4.5.1. Let $F : \mathbb{N}^{\mathrm{op}} \to \mathbf{LCS}_{\mathbb{R}}$ be strict.

(i) A subset K is compact in $\operatorname{colim} F$ if and only if there exists some n such that $K \subseteq F(n)$ and K is compact in $F(n)$

(ii) If $(a_n)_n$ is a Cauchy sequence in $\operatorname{colim} F$, then there exists some N such that $(a_n)_n \subseteq F(N)$ and $(a_n)_n$ is Cauchy in $F(N)$.

We turn to colimits in $\mathbf{HausLCS}_{\mathbb{R}}$. Recall the inclusion $\mathbf{HausTVS}_{\mathbb{R}} \to \mathbf{TVS}_{\mathbb{R}}$ admits a left adjoint left inverse $H : \mathbf{TVS}_{\mathbb{R}} \to \mathbf{HausTVS}_{\mathbb{R}}$, given by sending V to its maximal Hausdorff quotient $V/\overline{\{0\}}$. These all restrict to $\mathbf{LCS}_{\mathbb{R}}$ and $\mathbf{HausLCS}_{\mathbb{R}}$, i.e., $H : \mathbf{LCS}_{\mathbb{R}} \to \mathbf{HausLCS}_{\mathbb{R}}$ is still a left adjoint and a left inverse of $\iota : \mathbf{HausLCS}_{\mathbb{R}} \to \mathbf{LCS}_{\mathbb{R}}$. In particular $\mathbf{HausLCS}_{\mathbb{R}}$ is complete and cocomplete. Explicitly,

Lemma E.4.6. If $F : \mathbf{LCS}_{\mathbb{R}} \to \mathbf{Set}$ is a functor represented by $X \in \mathbf{LCS}$, then $F \circ \iota : \mathbf{HausLCS}_{\mathbb{R}} \to \mathbf{LCS}$ is represented by $H(X)$.

Proof. By assumption we have a bijection

$$F(Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{LCS}}(X, Y)$$

functorial in $Y \in \mathbf{LCS}$. Hence

$$(F \circ \iota)(Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{LCS}}(X, \iota(Y)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{HausLCS}}(H(X), Y)$$

functorial in $Y \in \mathbf{HausLCS}_{\mathbb{R}}$. □

Hence, if $F : J \to \mathbf{HausLCS}_{\mathbb{R}}$ is a small diagram, then $H(\operatorname{colim} \iota \circ F)$ with those canonical morphisms represents the colimit $\operatorname{colim} F$ of F.

E.4.3 Fréchet spaces and LF spaces

Definition. A locally convex colimit (resp. strict colimit) of Fréchet spaces is called an **LF space** (resp. **strict LF space**).

E.5 Unbounded operators

Let k be a unital ring, and X, Y be two left k-modules. Giving a k-linear map $X \to Y$ amounts to specifying a k-submodule Γ of $X \times Y$ such that the projection $\mathrm{pr}_X|_\Gamma : \Gamma \to X$ is bijective. In fact, a function $X \to Y$ is nothing but a subset of $X \times Y$ whose projection to X is bijective. Following these rudimental ideas, we introduce

Definition. Let $\Gamma \subseteq X \times Y$ be a k-submodule.

(i) For $y \in Y$, put

$$\Gamma^{-1}(y) := \{x \in X \mid (x, y) \in \Gamma\}.$$

For $x \in X$, put

$$\Gamma(x) := \{y \in X \mid (x, y) \in \Gamma\}.$$

(ii) The **domain** of Γ is

$$D(\Gamma) = \mathrm{pr}_X(\Gamma),$$

and the **image** of Γ is

$$\mathrm{Im}(\Gamma) = \mathrm{pr}_Y(\Gamma).$$

(iii) The **kernel** of Γ is

$$\ker \Gamma = \Gamma^{-1}(0).$$

(iv) The **inverse of** Γ is the k-submodule

$$\Gamma^{-1} = \{(y, x) \mid (x, y) \in \Gamma\} \subseteq Y \times X.$$

(v) If Z is another left k-module and $\Gamma' \subseteq Y \times Z$ is a k-submodule, then the **composition** $\Gamma' \circ \Gamma \subseteq X \times Z$ is

$$\Gamma' \circ \Gamma = \{(x, z) \mid (x, y) \in \Gamma \text{ and } (y, z) \in \Gamma' \text{ for some } y \in Y\}.$$

(vi) If Γ' is another k-submodule of $X \times Y$, then the **sum** $\Gamma + \Gamma' \subseteq X \times Y$ is

$$\Gamma + \Gamma' := \{(x, y + z) \mid (x, y) \in \Gamma,\ (x, z) \in \Gamma'\}.$$

For $r \in k$,

$$r\Gamma := \{(x, ry) \mid (x, y) \in \Gamma\}.$$

Not confuse $\Gamma + \Gamma'$ with the module generation.

(vii) An **extension** of Γ is a k-submodule Γ' of $X \times Y$ containing Γ.

Lemma E.5.1. A k-submodule $\Gamma \subseteq X \times Y$ is the graph of a k-linear map $D(\Gamma) \to Y$ if and only if $\Gamma(0) = 0$.

Proof. The only if part is clear as a linear map sends 0 to 0. Conversely, we must show if $(x, y), (x, z) \in \Gamma$, then $y = z$. Since Γ is a submodule, we have $(0, y - z) = (x, y) - (x, z) \in \Gamma$, and hence $y - z \in \Gamma(0)$. The assumption forces $y - z = 0$, or $y = z$. $\square$

Definition. A k-submodule $\Gamma \subseteq X \times Y$ with $\Gamma(0) = \{0\}$ is called an **(unbounded) operator from X to Y**. In the case $X = Y$, we say Γ is an **operator on X.**

We further assume k is a topological unital ring, and X, Y are two left topological k-modules.

Definition. Let $\Gamma \subseteq X \times Y$ be a k-submodule.

(i) The **closure** of Γ is simply the closure $\overline{\Gamma}$ of Γ in the product topology of $X \times Y$. We say Γ is **closed** if it is its closure.

Assume in addition that Γ is an operator.

(ii) Γ is **closable** if $\overline{\Gamma}(0) = \{0\}$.

(iii) Γ is **densely defined** if $D(\Gamma)$ is dense in X.

E.5.1 Self-adjoint operators

Now assume $k = \mathbb{R}$ or $\mathbb{C}$, and X, Y are two Hilbert spaces.

Definition. Let $\Gamma \subseteq X \times Y$ be a k-linear subspace. The **adjoint** Γ^* of Γ is the k-linear subspace of $Y \times X$ given by

$$\Gamma^* = \{(y, x) \in Y \times X \mid \langle y, v\rangle_Y = \langle x, u\rangle_X \text{ for all } (u, v) \in \Gamma\}.$$

Lemma E.5.2. Let $\Gamma \subseteq X \times Y$ be a k-linear subspace.

(i) $\Gamma^*(0) = D(\Gamma)^\perp$.

(ii) $\Gamma^* = J(\Gamma)^\perp$, where

$$J(\Gamma) = \{(y, -x) \mid (x, y) \in \Gamma\}.$$

In particular, Γ^* is always closed.

(iii) $(\Gamma^*)^* = \overline{\Gamma}$.

(iv) $\overline{\Gamma}(0) = D(\Gamma^*)^\perp$.

(v) If $\Gamma' \subseteq X \times Y$ is a k-linear subspace containing Γ, then $(\Gamma')^* \subseteq \Gamma^*$.

Proof.

(i) If $y \in D(\Gamma)^\perp$, then $\langle u, y\rangle_Y = 0$ for all $u \in D(\Gamma)$. Then $\langle u, y\rangle_Y = 0 = \langle v, 0\rangle_X$ for all $(u, v) \in \Gamma$, so $(0, y) \in \Gamma^*$. If $(0, y) \in \Gamma^*$, then $\langle u, y\rangle_Y = \langle v, 0\rangle_X = 0$ for all $(u, v) \in \Gamma$, so $y \in D(\Gamma)^\perp$.

(ii) $(x, y) \in \Gamma^*$ if and only if $\langle v, x\rangle_X = \langle u, y\rangle_Y$ for all $(u, v) \in \Gamma$, or $\langle (x, y), (v, -u)\rangle_{X\times Y} = 0$, if and only if $(x, y) \in J(\Gamma)^\perp$.

(iii) $(\Gamma^*)^* = J(\Gamma^*)^\perp = J(J(\Gamma)^\perp)^\perp = J(J(\Gamma))^{\perp\perp} = \Gamma^{\perp\perp} = \overline{\Gamma}$.

(iv) This follows from (i) and (iii).

(v) Immediate from the definition and $\Gamma \subseteq \Gamma'$.

$\square$

Corollary E.5.2.1. For an operator T from X to Y, the adjoint T^* is an operator from Y to X if and only if $D(T)$ is dense in X.

Proof. T^* is an operator if and only if $T^*(0) = 0$, if and only if $D(T)^\perp = 0$ by Lemma E.5.2.(i), or $D(T)$ is dense. $\square$

Definition. A densely defined operator Γ on X is called **symmetric** if $\Gamma^* \supseteq \Gamma$, is called **self-adjoint** if $\Gamma^* = \Gamma$, and is called **positive** if it is self-adjoint and $\langle \Gamma x, x\rangle \geqslant 0$ for all $x \in D(\Gamma)$.

Lemma E.5.3. If T is a symmetric operator on X, then T^{**} is a closed symmetric operator on X.

Proof. Since $D(T) \subseteq D(T^*)$ and $D(T)$ is dense, $D(T^*)$ is dense. Hence T^{**} is an operator by Corollary E.5.2.1. By Lemma E.5.2.(ii) T^{**} is closed. By Lemma E.5.2.(iii), $D(T^{**}) = D(\overline{T})$. Since $D(T) \subseteq D(\overline{T})$, it follows that T^{**} is densely defined. Finally, by Lemma E.5.2.(v), that $T \subseteq T^*$ implies $T^{**} \subseteq T^{***}$. $\square$

Corollary E.5.3.1. A self-adjoint operator on X is closed.

Lemma E.5.4. If T is an injective self-adjoint operator on X, then T^{-1} is self-adjoint.

Proof. Note that $(-T)^* = -T^*$. Since T is self-adjoint, $(-T)^* = -T$. Also, $T^{-1} = J(-T)$ and $J(T^{-1}) = -T$.

By Lemma E.5.2 and T is closed, it follows $-T$ is closed and we have

$$T^{-1} = J(-T) = J(-T)^{\perp\perp} = ((-T)^*)^\perp = (-T)^\perp = J(T^{-1})^\perp = (T^{-1})^*$$

It remains to show T^{-1} is densely defined. This follows from the following lemma. $\square$

Lemma E.5.5. Let T be a densely defined operator on X. Then $(\operatorname{Im} T)^\perp = \ker T^*$. If T is further closed, then $(\operatorname{Im} T^*)^\perp = \ker T$.

Proof. $y \in (\operatorname{Im} T)^\perp$ if and only if $\langle Tx, y\rangle = 0$ for all $x \in D(T)$, or $\langle x, T^*y\rangle = 0$. Since $D(T)$ is dense, it's the same as saying $\langle x, T^*y\rangle = 0$ for all $x \in X$, or $T^*y = 0$.

Assume further T is closed. Applying what we've shown to T^*, we obtain $(\operatorname{Im} T^*)^\perp = \ker T^{**}$. By Lemma E.5.2.(iii), we have $T^{**} = \overline{T} = T$, so $(\operatorname{Im} T^*)^\perp = \ker T$. $\square$

Definition. For a k-linear subspace $\Gamma \subseteq X \times Y$, a **core** $C \subseteq D(\Gamma)$ of Γ is a subspace such that the closure of $\Gamma \cap (C \times Y)$ is $\overline{\Gamma}$.

Theorem E.5.6. Let T be a closed and densely defined operator on X. Then $\operatorname{id}_X + T^*T : D(\operatorname{id}_X + T^*T) \to X$ is densely defined, bijective and the inverse $(\operatorname{id}_X + T^*T)^{-1}$ has operator norm $\leqslant 1$. Both $(\operatorname{id}_X + T^*T)^{-1}$ and T^*T are positive and self-adjoint, and $D(T^*T)$ is a core for T

Proof. For $v \in X$, by Lemma E.5.2.(ii) and (iii) and closedness of T, there exist $(x, y) \in D(T^*) \times D(T)$ such that $(v, 0) = (T^*x, -x) + (y, Ty)$. Hence $y \in D(T^*T) = D(\operatorname{id}_X + T^*T)$ and $v = T^*x + y = (\operatorname{id}_X + T^*T)y$. This proves the surjectivity. For injectivity, let $y \in D(\operatorname{id}_X + T^*T) = D(T^*T) \subseteq D(T)$ and $v := (\operatorname{id}_X + T^*T)y$. Compute

$$\langle y, v\rangle = \|y\|^2 + \langle y, T^*Ty\rangle = \|y\|^2 + \|Ty\|^2 .$$

If $v = 0$, then $y = 0$, proving the injectivity. Since

$$\|v\|^2 = \langle v, v\rangle = \|y\|^2 + 2\|Ty\|^2 + \|T^*Ty\|^2 \geqslant \|y\|^2,$$

this shows the inverse $(\mathrm{id}_X + T^*T)^{-1}$ has operator norm $\leqslant 1$. In particular, it is bounded. The above computation also shows that if $v \perp D(T^*T)$ and $v = (\mathrm{id}_X + T^*T)y$ for some $y \in D(T^*T)$, then $0 = \langle y, v\rangle$ implies $y = 0$ and hence $v = 0$. This shows $D(T^*T)$ is dense in X. It is clear that T^*T is positive and symmetric, so $\mathrm{id}_X + T^*T$ is also positive and symmetric. Since it is bijective, $\mathrm{id}_X + T^*T$ must be self-adjoint and hence closed with self-adjoint inverse by Corollary E.5.3.1 and Lemma E.5.4. It is easy to see the inverse is also positive.

It remains to show $D(T^*T)$ is a core for T. It suffices to show the orthogonal complement of $T \cap (D(T^*T) \times X)$ in T is 0. Let $x \in D(T)$ and $y \in D(T^*T)$. Then

$$\langle (x, Tx), (y, Ty)\rangle = \langle x, y\rangle + \langle Tx, Ty\rangle = \langle x, (\mathrm{id}_X + T^*T)y\rangle.$$

If (x, Tx) lies in the orthogonal complement of $D(T^*T) \times X$, then $\langle x, (\mathrm{id}_X + T^*T)y\rangle = 0$ for all $y \in D(T^*T)$. Since $\mathrm{id}_X + T^*T$ is surjective, it follows that $x = 0$. □

Definition. A densely defined operator T on X is **normal** if $T^*T = TT^*$.

Lemma E.5.7. If T is a normal operator on X, then $D(T) = D(T^*)$. In particular, $D(T^*T) = D(TT^*) = D(T^2) = D((T^*)^2)$.

Proof. For $u \in D(T^*T) = D(TT^*) \subseteq D(T) \cap D(T^*)$, we have

$$\begin{aligned} \|u\|^2 + \|Tu\|^2 = \langle u, u\rangle + \langle Tu, Tu\rangle &= \langle (\mathrm{id}_X + T^*T)u, u\rangle \\ &\langle (\mathrm{id}_X + TT^*)u, u\rangle = \|u\|^2 + \|T^*u\|^2. \end{aligned}$$

By Theorem E.5.6 and normality, $D(T^*T) = D(TT^*)$ is a core for both T and T^*, so the above equality implies $D(T) = D(T^*)$. □

Lemma E.5.8. If T is a normal operator on X, then $(\mathrm{id}_X + T^*T)^{-1}T \subseteq T(\mathrm{id}_X + T^*T)^{-1}$.

Bounded transforms

Definition. For a closed and densely defined operator T on X, set

$$I_T := (\mathrm{id}_X + T^*T)^{-1} \text{ and } Z_T := TI_T^{\frac{1}{2}}.$$

Note that $I_T \in \mathcal{B}(X)$ is positive by Theorem E.5.6, and its square root is well-defined by Theorem 12.1.3. We call Z_T the **bounded transform** of T.

Lemma E.5.9. Let T be a closed and densely defined operator on X. Then

(i) Z_T is bounded with $\|Z_T\|_{\mathrm{op}} \leqslant 1$ and

$$I_T = \mathrm{id}_X - Z_T^* Z_T.$$

(ii) If T' is another such operator like T and $Z_T = Z_{T'}$, then $T = T'$.

(iii) If T is normal, then $Z_T^* = Z_{T^*}$.

(iv) If T is normal (resp. self-adjoint, positive), then so is Z_T.

Proof.

(i)

(ii) By (i), we have

$$I_T = \mathrm{id}_X - Z_T^* Z_T = \mathrm{id}_X - Z_{T'}^* Z_{T'} = I_{T'},$$

so $T^*T = (T')^*T'$ and $TI_T = Z_T I_T^{\frac{1}{2}} = Z_{T'} I_{T'}^{\frac{1}{2}} = T' I_{T'} = T' I_T$. Hence $T = T'$ on $D(T^*T) = D((T')^*T')$. Since $D(T^*T)$ and $D((T')^*T')$ are respectively cores of T and T' by Theorem E.5.6, so $T = T'$.

□

Cayley transforms

Lemma E.5.10. Let T be a symmetric operator on X. Then for $z \in \mathbb{C}\backslash\mathbb{R}$, the operator $T + z\,\mathrm{id}_X : D(T) \to X$ is injective and has a continuous inverse.

Proof. Write $z = x + iy$ with $x, y \in \mathbb{R}$. If T is symmetric, so is $T + x\,\mathrm{id}_X$. Hence we can assume $x = 0$. By symmetry, we have

$$\langle Tv, iv\rangle = -i\langle v, Tv\rangle = -\overline{\langle Tv, iv\rangle},$$

so $\langle Tv, iv\rangle$ is imaginary. Hence

$$\|(T + iy\,\mathrm{id}_X)v\|^2 = \|Tv\|^2 + \|yv\|^2 = \|Tv\|^2 + y^2\|v\|^2.$$

This implies $T + iy\,\mathrm{id}_X : D(T) \to X$ is injective and bounded below, and hence has a continuous inverse. □

Lemma E.5.11. Let T be a closed symmetric operator on X. Then

$$U_T := (T - i\,\mathrm{id}_X)(T + i\,\mathrm{id}_X)^{-1} : D(U_T) := D((T + i\,\mathrm{id}_X)^{-1}) \to X$$

is a well-defined closed isometric operator. Moreover, $\mathrm{id}_X - U_T : D(U_T) \to X$ is injective and

$$T = i(\mathrm{id}_X + U_T)(\mathrm{id}_X - U_T)^{-1}.$$

The operator U_T is called the **Cayley transform** of T.

Proof. By Lemma E.5.10 and its proof, $(T + i\,\mathrm{id}_X)^{-1} : D((T + i\,\mathrm{id}_X)^{-1}) \to X$ is well-defined and continuous, and

$$\|(T \pm i\,\mathrm{id}_X)v\|^2 = \|Tv\|^2 + \|v\|^2 =: \|(v, Tx)\|^2_{X\times X} \tag{♠}$$

Then for $v \in D(U_T)$, we have

$$\begin{aligned}\|v\|^2 &= \left\|T(T + i\,\mathrm{id}_X)^{-1}v\right\|^2 + \left\|(T + i\,\mathrm{id}_X)^{-1}v\right\|^2\\ &= \left\|T(T + i\,\mathrm{id}_X)^{-1}v\right\|^2 + \left(\|U_Tv\|^2 - \left\|T(T + i\,\mathrm{id}_X)^{-1}v\right\|^2\right) = \|U_Tv\|^2.\end{aligned}$$

This proves U_T is an isometry.

To show U_T is closed, let $(x_n)_n \subseteq X$ be such that $(T + i\,\mathrm{id}_X)x_n =: y_n \to y$ and $(T - i\,\mathrm{id}_X)x_n =: z_n \to z$. We must show $U_Ty = z$. To this end, since T is closed, by (♠) we see $(x_n, Tx_n) \to (x, Tx)$ for some $x \in X$. Hence $y = (T + i\,\mathrm{id}_X)x$ and $z = (T - i\,\mathrm{id}_X)x$ by closedness, and so $U_Ty = z$.

Suppose $v \in D(U_T)$ such that $(\mathrm{id}_X - U_T)v = 0$. Write $v = (T + i\,\mathrm{id}_X)w$ for some $w \in D(T)$; then $U_T v = (T - i\,\mathrm{id}_X)w$, and hence

$$0 = (T + i\,\mathrm{id}_X)w - (T - i\,\mathrm{id}_X)w = 2iw.$$

This implies $w = 0$, whence $v = 0$. Finally,

$$Tw = 2^{-1}(\mathrm{id}_X + U_T)v = i(\mathrm{id}_X + U_T)(\mathrm{id}_X - U_T)^{-1}w$$

for $w \in D(T)$, so that $T = i(\mathrm{id}_X + U_T)(\mathrm{id}_X - U_T)^{-1}$ as we claimed. □

Lemma E.5.12. Let U be a closed isometric operator on X such that $\mathrm{Im}(\mathrm{id}_X - U)$ is dense in X. Then there exists a unique closed symmetric operator T on X with $U = U_T$.

Proof. First we show $\mathrm{id}_X - U$ is injective. For $y, z \in D(U)$, compute

$$\langle y, (\mathrm{id}_X - U)z\rangle = \langle y, z\rangle - \langle y, Uz\rangle = \langle Uy, Uz\rangle - \langle y, Uz\rangle = \langle (U - \mathrm{id}_X)y, Uz\rangle.$$

If $(\mathrm{id}_X - U)y = 0$, then $\langle y, (\mathrm{id}_X - U)z\rangle = 0$ for all $z \in D(U)$ so that $y \in \mathrm{Im}(\mathrm{id}_X - U)^{\perp}$. By hypothesis this implies $y = 0$.

Define $T = i(\mathrm{id}_X + U)(\mathrm{id}_X - U)^{-1}$. Then $D(T) = D((\mathrm{id}_X - U)^{-1}) = \mathrm{Im}(\mathrm{id}_X - U)$ is dense in X.

- <u>T is symmetric</u>. Let $x, y \in D(U)$. Then

$$\begin{aligned}\langle T(\mathrm{id}_X - U)x, (\mathrm{id}_X - U)y\rangle &= i\langle (\mathrm{id}_X + U)x, (\mathrm{id}_X - U)y\rangle \\ &= i\langle x, y\rangle + i\langle Ux, y\rangle - i\langle x, Uy\rangle - i\langle Ux, Uy\rangle \\ &= i\left(\langle Ux, y\rangle - \langle x, Uy\rangle\right).\end{aligned}$$

  On the other hand,

$$\langle (\mathrm{id}_X - U)x, T(\mathrm{id}_X - U)y\rangle = -i\langle (\mathrm{id}_X - U)x, (\mathrm{id}_X + U)y\rangle = i\left(\langle Ux, y\rangle - \langle x, Uy\rangle\right).$$

  Equating these two shows that T is symmetric.

- <u>$U = U_T$</u>. On $D(T) = D((\mathrm{id}_X - U)^{-1})$ we have

$$T + i\,\mathrm{id}_X = i(\mathrm{id}_X + U)(\mathrm{id}_X - U)^{-1} + i(\mathrm{id}_X - U)(\mathrm{id}_X - U)^{-1} = 2i(\mathrm{id}_X - U)^{-1}.$$

  We deduce $D(U) = \mathrm{Im}((\mathrm{id}_X - U)^{-1}) = \mathrm{Im}(T + i\,\mathrm{id}_X) = D((T + i\,\mathrm{id}_X)^{-1}) = D(U_T)$. We also compute

$$T - i\,\mathrm{id}_X = i(\mathrm{id}_X + U)(\mathrm{id}_X - U)^{-1} - i(\mathrm{id}_X - U)(\mathrm{id}_X - U)^{-1} = 2iU(\mathrm{id}_X - U)^{-1}$$

  so that

$$U_T = (T - i\,\mathrm{id}_X)(T + i\,\mathrm{id}_X)^{-1} = U(\mathrm{id}_X - U)^{-1}(\mathrm{id}_X - U) = U$$

- <u>T is closed</u>. This is clear: for $(x_n)_n \subseteq D(T) = \mathrm{Im}(\mathrm{id}_X - U)$, let $(y_n)_n \subseteq D(U)$ be such that $x_n = (\mathrm{id}_X - U)y_n$. Then $Tx_n = i(\mathrm{id}_X + U)y_n$. If $(x_n, Tx_n) \to (x, z)$ converges, then both y_n and Uy_n converge. Since U is closed, we have $(y_n, Uy_n) \to (y, Uy)$, so that $(x_n, Tx_n) \to ((\mathrm{id}_X - U)y, i(\mathrm{id}_X + U)y) = (x, Tx)$.

□

Theorem E.5.13. Let T be a closed symmetric operator on X. Then

$$D(U_T)^\perp = \{x \in D(T^*) \mid T^*x = ix\}$$
$$\mathrm{Im}(U_T)^\perp = \{x \in D(T^*) \mid T^*x = -ix\}$$

and

$$D(T^*) = D(T) \oplus D(U_T)^\perp \oplus \mathrm{Im}(U_T)^\perp$$

Proof. Suppose $x \in D(T^*)$ and $T^*x = ix$. For $y \in D(U_T) = \mathrm{Im}(T + i\,\mathrm{id}_X)$, write $y = (T + i\,\mathrm{id}_X)z$ for some $z \in D(T)$. Then

$$\langle x, y\rangle = \langle x, Tz\rangle - i\langle x, z\rangle = \langle ix, z\rangle - i\langle x, z\rangle = 0.$$

This proves $x \in D(U_T)^\perp$. Conversely, suppose $x \in D(U_T)^\perp = \mathrm{Im}(T + i\,\mathrm{id}_X)^\perp$. Then $\langle x, (T + i\,\mathrm{id}_X)y\rangle = 0$ for all $y \in D(T)$, so that

$$\langle x, Ty\rangle = \langle ix, y\rangle.$$

This implies $(x, ix) \in T^*$, so that $T^*x = ix$. The second equality is proved in the same way.

Since U_T is isometric, we have $\|(x, U_Tx)\|_{X\times X}^2 = 2\|x\|^2 = 2\|U_Tx\|^2$. Since U_T is closed, it follows that $D(U_T)$ and $\mathrm{Im}(U_T)$ are closed in X. In particular, by Theorem E.2.1 we deduce

$$X = D(U_T) \oplus D(U_T)^\perp = \mathrm{Im}(T + i\,\mathrm{id}_X) \oplus D(U_T)^\perp.$$

Let $x \in D(T^*)$. By the decomposition applied to $(T^* + i\,\mathrm{id}_X)x$, we see

$$(T^* + i\,\mathrm{id}_X)x = (T + i\,\mathrm{id}_X)x_1 + x_2$$

for $x_1 \in D(T)$ and $x_2 \in D(U_T)^\perp$. Since $T \subseteq T^*$, we have $(T + i\,\mathrm{id}_X)x_1 = (T^* + i\,\mathrm{id}_X)x_1$. Since $T^*x_2 = ix_2$, we have $x_2 = (T^* + i\,\mathrm{id}_X)((2i)^{-1}x_2)$. Putting these together shows

$$(T^* + i\,\mathrm{id}_X)x = (T + i\,\mathrm{id}_X)x_1 + (T^* + i\,\mathrm{id}_X)((2i)^{-1}x_2)$$

so that $x - x_1 - (2i)^{-1}x_2 \in \ker(T^* + i\,\mathrm{id}_X) = \mathrm{Im}(U_T)^\perp$. This proves

$$D(T^*) = D(T) + D(U_T)^\perp + \mathrm{Im}(U_T)^\perp.$$

Suppose $0 = x_1 + x_2 + x_3$ for $x_1 \in D(T)$, $x_2 \in D(U_T)^\perp$, $x_3 \in \mathrm{Im}(U_T)^\perp$. Then

$$0 = (T^* + i\,\mathrm{id}_X)(x_1 + x_2 + x_3) = (T + i\,\mathrm{id}_X)x_1 + 2ix_2.$$

Since $(T + i\,\mathrm{id}_X)x_1 \in \mathrm{Im}(T + i\,\mathrm{id}_X) = D(U_T)$ and $x_2 \in D(U_T)^\perp$, it follows that $x_2 = 0$ and $(T + i\,\mathrm{id}_X)x_1 = 0$. Since $T + i\,\mathrm{id}_X$ is injective, $x_1 = 0$. Hence $x_3 = 0$ as well, proving the sum is direct. □

Corollary E.5.13.1. A closed symmetric operator T on X is self-adjoint if and only if U_T is unitary.

Proof. T is self-adjoint if and only if $D(T) = D(T^*)$, if and only if $D(U_T)^\perp = \mathrm{Im}(U_T)^\perp = \{0\}$ by the theorem. Since U_T is closed and isometric, this implies $D(U_T) = \mathrm{Im}(U_T) = X$. □

E.5.2 Spectrum

Let $k = \mathbb{C}$ and X a complex topological vector space.

Definition. Let T be an operator on X. For $\lambda \in \mathbb{C}$, put

$$T_\lambda = \lambda \operatorname{id}_X - T : D(T) \to X.$$

The **resolvent set** of T is

$$\rho(T) = \{\lambda \in \mathbb{C} \mid T_\lambda : D(T) \to \operatorname{Im} T_\lambda \text{ has dense image and a continuous inverse.}\}$$

For $\lambda \in \rho(T)$, the operator $R_\lambda(T) := (\lambda \operatorname{id}_X - T)^{-1} : \operatorname{Im} T_\lambda \to D(T) \subseteq X$ is called the **resolvent operator of T (at λ)**. The complement

$$\sigma(T) := \mathbb{C} \setminus \rho(T)$$

is called the **spectrum** of T. Define

$$\sigma_p(T) := \{\lambda \in \mathbb{C} \mid \lambda \operatorname{id}_X - T \text{ is not injective.}\}$$
$$\sigma_c(T) := \{\lambda \in \mathbb{C} \mid \lambda \operatorname{id}_X - T \text{ is injective with dense image, but } R_\lambda(T) \text{ is not continuous.}\}$$
$$\sigma_r(T) := \{\lambda \in \mathbb{C} \mid \lambda \operatorname{id}_X - T \text{ is injective with non-dense image.}\}$$

σ_p is the **point spectrum**, σ_c is the **continuous spectrum**, and σ_r is the **residual spectrum**. Immediately from the definition,

$$\sigma(T) = \sigma_p(T) \sqcup \sigma_c(T) \sqcup \sigma_r(T).$$

An element in $\sigma_p(T)$ is called an **eigenvalue** of T.

Lemma E.5.14. Let X be a complex Banach space and T a closed linear operator on X. For $\lambda \in \mathbb{C}$, TFAE:

(i) $\lambda \in \rho(T)$.

(ii) $T_\lambda : D(T) \to X$ is bounded below and has dense image.

(iii) $T_\lambda : D(T) \to X$ is bijective.

In particular, $R_\lambda(T)$ is a *genuine* continuous linear map $X \to X$.

Proof. Assume (i). For (ii) we must show there exists $C > 0$ such that $\|T_\lambda x\| \geqslant C\,\|x\|$ for all $x \in D(T)$. Write $x = R_\lambda y$ for some $y \in \operatorname{Im} T_\lambda$; then $y = T_\lambda x$ and

$$\|x\| = \|R_\lambda y\| \leqslant \|R_\lambda\|_{\mathrm{op}}\,\|y\| = \|R_\lambda\|_{\mathrm{op}}\,\|T_\lambda x\|.$$

This implies $\|T_\lambda x\| \geqslant C\,\|x\|$ with $C = \|R_\lambda\|_{\mathrm{op}}^{-1}$.

Assume (ii). To show $\operatorname{Im} T_\lambda = X$, let $y \in X$ and take $(x_n)_n \subseteq X$ such that $T_\lambda x_n \to y$ in X; this is possible as T_λ has dense image. Since T_λ is bounded below, it follows that $(x_n)_n$ is Cauchy; let $x = \lim_{n\to\infty} x_n$. Since T is *closed*, $((x_n, Tx_n))_n \to (x, y)$ in $X \times X$ implies $y = Tx$.

Assume (iii). Since T is closed, T_λ is closed and hence R_λ is closed. It follows from the closed graph theorem that R_λ is continuous. □

Theorem E.5.15. Let T be an operator on X. Then the resolvent $\rho(T)$ is open in $\mathbb{C}$, and the function

$$\begin{aligned} \rho(T) &\longrightarrow B(X) \\ \lambda &\longmapsto R_\lambda(T) \end{aligned}$$

is holomorphic.

Part VI

Geometry

Appendix F

Smooth manifolds

F.1 Inverse function theorem

We start with some classical results in calculus.

Theorem F.1.1 (Implicit function theorem)**.** Let $\begin{aligned} g : \mathbb{R}^n \times \mathbb{R}^m &\longrightarrow \mathbb{R}^m \\ (x, y) &\longmapsto g(x, y) \end{aligned}$ be C^1 and $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}^m$ be given that $g(\eta, \xi) = 0$. Assume that the det $\left(\frac{\partial g}{\partial y}(\xi, \eta)\right) \neq 0$. Then there exist positive numbers a, b such that there exists a unique C^1 function $\phi : B_a(\xi) \to B_b(\eta)$ with $\phi(\xi) = \eta$ such that

$$g^{-1}(0) \cap (B_a(\xi) \times B_b(\eta)) = \{(x, \phi(x)) \mid x \in B_a(\xi)\}$$

Furthermore, if g is C^p, so is ϕ (including $p = \infty$).

Theorem F.1.2 (Inverse function theorem)**.** Let $\theta : \mathbb{R}^m \to \mathbb{R}^m$ be C^1 with $\theta'(\eta)$ be invertible for some $\eta \in \mathbb{R}^m$. Then there exists an open neighborhood U of η such that

(i) $\theta(U)$ is open in $\mathbb{R}^m$

(ii) $\theta|_U : U \to \theta(U)$ is bijective, with inverse ϕ

(iii) $\phi : \theta(U) \to U$ is C^1

Theorem F.1.3 (Constant rank theorem)**.** Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be C^1 and $p \in \mathbb{R}^m$. Suppose rank $f'(x) = r$ in a neighborhood U of p. Then under suitable (smooth) changes of coordinates near $p \in U$ and $f(p) \in \mathbb{R}^n$, the map f assumes the form

$$(x^1, \ldots, x^m) \mapsto (x^1, \ldots, x^r, 0, \ldots, 0)$$

Proof. Write $f = (f_1, \ldots, f_n)$. Since rank $f'|_U \equiv r$ and $p \in U$, we can replace U by a smaller neighborhood so that, WLOG, the $r \times r$ principal minor $\left(\frac{\partial f_i}{\partial x^j}\right)_{1 \leqslant i,j \leqslant r}$ is invertible. Now consider the map

$$\begin{aligned} \Phi : U &\longrightarrow \mathbb{R}^m \\ x = (x^1, \ldots, x^m) &\longmapsto (f_1(x), \ldots, f_r(x), x^{r+1}, \ldots, x^m) \end{aligned}$$

By our choice, the Jacobian of this map is nonvanishing at p, so by restricting to a smaller neighborhood again, we may assume this map is a C^1-diffeomorphism. Now consider the commutative triangle

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\Phi", "\wr"'] & \mathbb{R}^n \\ \Phi(U) \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

One easily sees that $\overline{f}(x^1, \ldots, x^m) = (x^1, \ldots, x^r, h(x^1, \ldots, x^m))$ for some C^1 map h. But rank $\overline{f}'$ is constant on $\Phi(U)$, by considering the Jacobian of $\overline{f}$ we see $\dfrac{\partial h}{\partial y} \equiv 0$, where $y = (x^{r+1}, \ldots, x^m)$. Hence h is independent of y. Finally, let $F : \mathbb{R}^n \to \mathbb{R}^n$ be the diffeomorphism defined by $F(u, v) = (u, v - h(u))$. Then $F \circ \overline{f}(x, y) = (x^1, \ldots, x^r, 0, \ldots, 0)$, as wanted. $\square$

F.2 Basic definitions

Definition.

1. An n**-dimensional smooth manifold** M^n is a second countable Hausdorff space together with a (smooth) **atlas** $\{(U_\alpha, \varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \underset{\text{open}}{\subseteq} \mathbb{R}^n)\}$ such that the U_α covers M and the **transition map**
$$\varphi_{\alpha\beta} := \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$
is smooth (in the usual sense) for each α, β. An element in the atlas is called a **chart**. We also write $n = \dim_{\mathbb{R}} M$.

2. Let the notation be as above. For a chart $\{U, \varphi\}$, write $\varphi(p) = (x^1(p), \ldots, x^n(p))$ for each $p \in U$. The functions $x^1, \ldots, x^n$ are called the **local coordinates**.

3. Let M, N be two smooth manifolds and $f : M \to N$ a continuous map. f is said to be $\mathbf{C^k}$ / **smooth** if for each chart (U, φ) of M and (V, ψ) of N with $f(U) \subseteq V$, the composition
$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \to U \to V \to \psi(V)$$
is C^k / smooth.

Next we review the concepts of tangent vectors and tangent spaces.

Definition. Let M be a smooth manifold.

1. For $U \underset{\text{open}}{\subseteq} M$, denote by $C^\infty(U) = C^\infty(U, \mathbb{R})$ the space of smooth real-valued functions on U.

2. For $p \in M$, denote by $C_p^\infty(M)$ the **stalk** of smooth functions at p, i.e.
$$C_p^\infty(M) = \varinjlim_{p \in U \underset{\text{open}}{\subseteq} M} C^\infty(U)$$
where the $C^\infty(U)$ are directed by restriction. An element in $C_p^\infty(M)$ is called a **germ**.

3. For a smooth curve $\gamma : \mathbb{R} \to M$ with $\gamma(0) = p$, we define $D_\gamma : C_p^\infty(M) \to \mathbb{R}$ by
$$D_\gamma f = \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t))$$
Each D_γ is called the **tangent vector** to γ at p. Denote by T_pM the space of all tangent vectors of p at M, and call it the **tangent space** of M at p.

4. For each $p \in M$, a **derivation at** p is an $\mathbb{R}$-linear map $D : C_p^\infty(M) \to \mathbb{R}$ satisfying the **Leibniz rule**:
$$D(fg) = f(p)D(g) + g(p)D(f)$$

Proposition F.2.1. Let M be an n-dimensional smooth manifold and $p \in M$.

1. T_pM is an n-dimensional real vector space. To be precise, let $x^1, \ldots, x^n$ be a local coordinate around p. Then $T_pM = \operatorname{span}_{\mathbb{R}} \left\{ \dfrac{\partial}{\partial x^i} \right\}_{i=1,\ldots,n}$.
2. Every derivation at p is a tangent vector at p.

Example F.2.2. $\mathbb{R}^n$ is an n-dimensional smooth manifold. For each $p \in \mathbb{R}^n$ and a vector $v \in \mathbb{R}^n$, define $\gamma_v = p + tv$. Then there is a linear isomorphism

$$\begin{aligned} \mathbb{R}^n &\longrightarrow T_p\mathbb{R}^n \\ v &\longmapsto D_{\gamma_v} \end{aligned}$$

Definition. Let M, N be smooth manifolds, $p \in M$ and $f : M \to N$ a smooth map. The **differential of** f **at** p is a linear map $f_* = f_{*p} : T_pM \to T_{f(p)}N$ defined by

$$f_* D_\gamma = D_{f \circ \gamma}$$

- If D is a derivation at p, we have $(f_*D)g = D(g \circ f)$.
- Let $\gamma : \mathbb{R} \to M$ be a smooth curve. Then $D_\gamma = \gamma_* \left(\left.\dfrac{d}{dt}\right|_{t=0} \right)$.

1. $f : M \to N$ is an **immersion** if f_* is injective at all points of M.
2. $f : M \to N$ is a **submersion** if f_* is surjective at all points of M.
3. If f is injective and immersive, then (M, f) is called an **immersed submanifold** of N.
4. $f : M \to N$ is an **embedding** if f is injective, immersive and *homeomorphic onto its image* $f(M)$, where $f(M)$ is equipped with the subspace topology from N. In this case, we say (M, f) is an **embedded/regular submanifold** of N.

For a later use, let us mention the famous Sard's theorem.

Definition. Let $f : M \to N$ be a smooth map between smooth manifolds.

1. A point $p \in M$ is called a **critical point of** f if $f_{*,p} : T_pM \to T_{f(p)}N$ is not surjective.
2. The image of a critical point under f is called a **critical value of** f.
3. A point $q \in N$ is called a **regular value of** f if it is not a critical value of f.

- Note that $q \in N$ is regular if and only if $q \notin f(M)$ or $q \in f(M)$ and f is submersive on the fibre $f^{-1}(q)$.

Theorem F.2.3 (Sard)**.** If $f : M \to N$ is smooth, then the set of critical values of f has measure zero[1].

[1] On a smooth manifold there is a natural concept of null sets, which can be defined without having a measure.

Proposition F.2.4. Let M^m be a smooth manifold and $r \in \mathbb{N}_0$. If S is a subspace of M such that for each $p \in S$ we can find a chart (U, φ) of M near p such that $\varphi(U \cap S) = \{x^1 = \cdots = x^r = 0\} \subseteq \varphi(U)$, then S is itself a manifold of dimension $m - r$ (codimension r) and $(S, \iota : S \subseteq M)$ is a regular submanifold of M.

Proof. Let $\pi : \mathbb{R}^m \to \mathbb{R}^{m-r}$ be the projection to the last $m - r$ coordinates. Near each p we find a chart (U_p, φ_p) such that $\varphi_p(U \cap S) = \{x^1 = \cdots = x^r = 0\}$. Put $V_p := U_p \cap S$ and $\psi_p = \pi \circ \varphi_p : V_p \to \mathbb{R}^{m-r}$. We claim $\{(V_p, \psi_p)\}_{p \in M}$ forms a smooth atlas of S. It suffices to check the transition map is smooth. To this end, let $p \neq q \in M$ with $V_p \cap V_q \neq \varnothing$. If we write $\varphi_q \circ \varphi_p^{-1}(x^1, \ldots, x^m) = (h_1(x), \ldots, h_m(x))$, then $\psi_q \circ \psi_p^{-1}(x^{r+1}, \ldots, x^m) = (h_{r+1}(0, y), \ldots, h_m(0, y))$, where $y = (x^{r+1}, \ldots, x^m)$, which is clearly smooth. Finally, $\iota : S \subseteq M$ is clearly injective and immersive since in local coordinates it is an inclusion, and it is a topological embedding since S is with subspace topology. $\square$

Theorem F.2.5 (Constant rank theorem)**.** Let M^m, N^n be smooth manifolds and $p \in M$. Let $f : M \to N$ a smooth map such that $\operatorname{rank} f_{*,x} = r$ in an open neighborhood U of p. Then we can find a chart φ near p and a chart ψ near $f(p)$ such that f assumes the form

$$\begin{tikzcd}
M \arrow[r, "f"] & N \\
\mathbb{R}^m \arrow[u, "\varphi^{-1}"] \arrow[r, "f"] & \mathbb{R}^n \arrow[u, "\psi^{-1}"'] \\
(x^1, \ldots, x^m) \arrow[r, mapsto] & (x^1, \ldots, x^r, 0, \ldots, 0)
\end{tikzcd}$$

Corollary F.2.5.1. Let $f : M^m \to N^n$ be a smooth map and $q \in N$. Suppose f_* has constant rank r in a neighborhood of $f^{-1}(q)$, then $f^{-1}(q)$ is a regular submanifold of M of codimension r.

Proof. Let $p \in f^{-1}(q)$. By Constant rank theorem we can find a chart (U, φ) near p and a chart (V, ψ) near $f(p) = q$ with $f(U) \subseteq V$ and $\psi(q) = 0$ such that we have the commutative diagram

$$\begin{tikzcd}
U \arrow[r, "f"] & V \\
\varphi(U) \arrow[u, "\varphi^{-1}"] \arrow[r, "\overline{f}"] & \psi(V) \arrow[u, "\psi^{-1}"'] \\
(x^1, \ldots, x^m) \arrow[r, mapsto] & (x^1, \ldots, x^r, 0, \ldots, 0)
\end{tikzcd}$$

Then $\varphi(U \cap f^{-1}q) = \overline{f}^{-1}(0) = \{x^1 = \cdots = x^r = 0\}$, so that $f^{-1}(q)$ is a regular submanifold by Proposition F.2.4. $\square$

Corollary F.2.5.2. Let M^m, N^n be smooth manifolds and $f : M \to N$ a smooth map.

1. If f is immersive at $p \in M$, then f is locally an embedding near p, i.e., we can find an open neighborhood U of p and such that $f|_U : U \to N$ is an embedding.

2. If $q \in f(M)$ is a regular value of f, then $f^{-1}(q)$ is a regular submanifold of M.

3. If f is an embedding, then $(f(M), \iota : f(M) \subseteq N)$ is the regular submanifold of the type in Proposition F.2.4, i.e., it is locally the zero locus of some local coordinates.

Proof.

1. By Constant rank theorem, we can find a chart (U, φ) of M near p and a chart (V, ψ) near $f(p)$ with $f(U) \subseteq V$ such that $f : U \to V$ is of the form $(x^1, \ldots, x^m) \mapsto (x^1, \ldots, x^m, 0, \ldots, 0)$. This shows $f|_U \to N$ is an embedding.

2. This is just Corollary F.2.5.1 with $r = n$.

3. For each $p \in M$, let (U_p, φ_p) and (V_p, ψ_p) be the charts found as in 1. Since f is a *topological embedding*, we can find an open neighborhood W_p of $f(p)$ such that $f(U_p) = W_p \cap f(M)$. Replacing V_p by $V_p \cap W_p$, we are then in the situation as Proposition F.2.4.

□

Lemma F.2.6. Let $M^m \subseteq \mathbb{R}^n$ be a embedded submanifold. Let $p \in M$ and assume there exists a smooth function $f : U \to \mathbb{R}$ defined on an open neighborhood U of p in $\mathbb{R}^n$ such that $f|_{U \cap M}$ is constant. Then the vector $\left(\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n}\right)\Big|_p$ is perpendicular to $T_pM \subseteq T_p\mathbb{R}^n = \mathbb{R}^n$.

Proof. Choose a local coordinate system $y^1, \ldots, y^n$ of $\mathbb{R}^n$ near p such that M is defined by $y^{m+1} = \cdots = y^n = 0$. Put $x^i = \theta_i(y^1, \ldots, y^n)$, so that $\frac{\partial}{\partial y^i} = \sum_{j=1}^n \frac{\partial \theta_j}{\partial y^i} \frac{\partial}{\partial x^j}$ near p. If we write $g(y^1, \ldots, y^n) = f(x^1, \ldots, x^n)$, then the constancy condition on f implies

$$0 = \frac{\partial g}{\partial y^i} = \sum_{j=1}^n \frac{\partial f}{\partial x^j} \frac{\partial \theta_j}{\partial y^i}$$

holds for any $1 \leqslant i \leqslant m$. These equations give exactly the result that we want. □

Proposition F.2.7. Let $f : M \to N$ be a smooth map and $\varphi : S \to N$ be an immersed submanifold. Suppose $f(M) \subseteq \varphi(S)$ and let $f_0 : M \to S$ be the unique map such that $f = \varphi \circ f_0$.

(i) If f_0 is continuous, then f_0 is smooth.

(ii) If φ is an embedding, then f_0 is continuous.

Proof. (ii) is clear. For (i), assume f_0 is continuous. Let $p \in S$; by Corollary F.2.5.2.1. we can find an open neighborhood V of p in S such that $\varphi|_V : V \to N$ is an embedding. Then

$$f_0|_{f_0^{-1}(V)} = \left(\varphi|_V^{\varphi(N)}\right)^{-1} \circ f|_{f_0^{-1}(V)}^{\varphi(V)}$$

is smooth. □

F.3 Vector fields

F.3.1 Tangent bundles

Let M^m be a smooth manifold. We form the (set-theoretic) disjoint union

$$TM := \bigsqcup_{p \in M} T_pM = \{(p, v) \mid p \in M,\, v \in T_pM\}$$

This is called the **tangent bundle** of M. We are going to topologize TM and gives it a smooth structure.

We have the canonical projection $\pi : TM \to M$, sending (p, v) to p. Let (U, φ) be a chart on M and write $\varphi = (x^1, \ldots, x^m)$. Then for each $p \in U$, T_pM is spanned by the $\left.\frac{\partial}{\partial x^i}\right|_p$. By identifying the $\frac{\partial}{\partial x^i}$ with the standard basis of $\mathbb{R}^m$, we have a set-theoretic bijection:

$$\begin{aligned} \pi^{-1}(U) &\longrightarrow U \times \mathbb{R}^m \\ \left(p, \sum_{i=1}^m a_i \left.\frac{\partial}{\partial x^i}\right|_p\right) &\longmapsto (p, a_1, \ldots, a_m) \end{aligned}$$

We now use the cover $\{\pi^{-1}(U)\}$ to define a topology on TM. Say a subset $S \subseteq TM$ is open iff for every chart U on M, the image of $\pi^{-1}(U) \cap S$ in $U \times \mathbb{R}^n$ is open in the product topology. Then each $\pi^{-1}(U)$ is open in TM because if V is another chart, then $\pi^{-1}(U) \cap \pi^{-1}(V) = \pi^{-1}(U \cap V)$, and in $V \times \mathbb{R}^m$, it is $(U \cap V) \times \mathbb{R}^m$, an open subset. One checks easily that this topology is Hausdorff and second countable.

For a chart (U, φ) on M, let Φ be the composition $\pi^{-1}(U) \to U \times \mathbb{R}^m \to \varphi(U) \times \mathbb{R}^m$. Then $\{(U, \Phi)\}$ forms a smooth atlas on TM. We must show the transition map is smooth. Suppose $(V, \psi = (y^1, \ldots, y^m))$ is another chart that meets U. Let $f = \psi \circ \varphi^{-1}$ be the transition map. By definition, $f_{*,p} \left.\frac{\partial}{\partial x^i}\right|_p = \sum_j \left.\frac{\partial f_j}{\partial x^i}\right|_p \left.\frac{\partial}{\partial y^j}\right|_p$. If we write $J_p = \left(\left.\frac{\partial f_i}{\partial x^j}\right|_p\right)_{ij}$, then there is a smooth diffeomorphism

$$\begin{aligned} (U \cap V) \times \mathbb{R}^m &\longrightarrow (U \cap V) \times \mathbb{R}^m \\ (p, a_1, \ldots, a_m) &\longmapsto (p, (a_1 \ \cdots \ a_m) J_p^T) \end{aligned}$$

This is clearly a smooth map, for the transition map f is smooth so that the Jacobian J_p is smooth in p as well. This finishes our construction, and this atlas makes TM into a real $2m$-dimensional smooth manifold.

Definition. Let M be a smooth manifold. A **smooth vector bundle of rank** r **over** M is a pair $(E, \pi_E : E \to M)$ enjoying the following properties

(a) $\pi_E : E \to M$ is a surjective smooth map.

(b) Each fibre $\pi^{-1}(p)$ of $p \in M$ has a structure of real r-dimensional vector space.

(c) π_E is **locally trivial of rank** r, i.e., for each $p \in M$ there is an open neighborhood U of p and a smooth diffeomorphism $\phi : \pi_E^{-1}(U) \to U \times \mathbb{R}^r$ such that the triangle

$$\begin{tikzcd} \pi_E^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "\pi_E"'] & & U \times \mathbb{R}^r \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

commutes, and for each $q \in U$, the restriction $\phi|_{\pi_E^{-1}(q)} : \pi_E^{-1}(q) \to \{q\} \times \mathbb{R}^r$ is a linear isomorphism. Such a (U, ϕ) is called a **local trivialization** of E.

- In this language, the tangent bundle TM is a smooth vector bundle of rank m over M^m.
- We say $M \times \mathbb{R}^d$ is the **trivial bundle** of rank d over M.

Definition. Let M be a smooth manifold and (E, π_E) a vector bundle over M.

1. For an open set $U \subseteq M$, a **section** of E over U is a continuous map $s : U \to E$ with $\pi \circ s = \mathrm{id}_U$.

2. A **vector field** on M is a section over M of the tangent bundle TM.

For an open set $U \subseteq M$, denote by $\Gamma(U, E)$ the set of all continuous sections of E over U. Also denote by $\Gamma^\infty(U, E)$ (resp. $\Gamma_c(U, E)$, $\Gamma_c^\infty(U, E)$) the subspace of smooth (resp. compactly supported, compactly supported smooth) sections of E over U. Each space forms a real vector space. For short hand, put

$$\mathfrak{X}(M) = \Gamma^\infty(M, TM).$$

3. If $f : M \to N$ is a smooth diffeomorphism and $X \in \mathfrak{X}(M)$, the **pushforward of** X **by** f is the vector field f_*X defined by $(f_*X)_q := f_{*,p}X_p$, where $q = f(p)$.

F.3.2 Local flows

Definition. Let X be a smooth vector field on a smooth manifold M.

1. An **integral curve** of X is a smooth curve $\gamma : (a, b) \to M$ such that $\gamma_{*,s}\left(\left.\frac{d}{dt}\right|_{t=t} s\right) = X_{\gamma(s)}$ for each $a < s < b$. We always assume $0 \in (a, b)$.

2. Let the notation be as in 1. If $p = \gamma(0)$, we say γ is an integral curve of X **starting from** p. γ is called **maximal** if its domain cannot be extended further.

3. A **local flow** is a smooth map $F : (a, b) \times M \to M$ (with $0 \in (a, b)$) such that $F(0, p) = p$ for all $p \in M$ and $F(s + t, p) = F(s, F(t, p))$ for all $p \in M$, $s, t \in (a, b)$ with $s + t \in (a, b)$.

4. A **local flow generated by** X is a local flow $F : (a, b) \times M \to M$ such that for each $p \in M$, $F(\cdot, p) : (a, b) \to M$ is an integral curve of X. The vector field X is called the **infinitesimal generator** of the flow F.

Theorem F.3.1. Let X be a smooth vector field on M, $p \in M$ and U an open neighborhood of p.

1. There exists a unique maximal integral curve of X starting from p.

2. There exists a local flow $F : (-\varepsilon, \varepsilon) \times W \to U$ generated by X for some small $\varepsilon > 0$ and $p \in W \underset{\text{open}}{\subseteq} U$.

Lemma F.3.2. Let M, N be smooth manifolds and $f : M \to N$ be a continuous map.

1. f is smooth if and only if $f^*g := g \circ f \in C^\infty(M)$ for all $g \in C^\infty(N)$.

2. A vector field X on M is smooth if and only if $Xf \in C^\infty(M)$ for all $f \in C^\infty(M)$.

3. If $N = \mathbb{R}$, then f is smooth if and only if f is C^1 and $Xf \in C^\infty(M)$ for all $X \in \mathfrak{X}(M)$.

Proof.

1. The only part is clear. For the if part, let $(V, \psi, y^1, \ldots, y^n)$ be a chart of N. Let $p \in V$ and choose a compact neighborhood K of p in V. By Urysohn's lemma we can find a smooth function $\theta \in C^\infty(N)^+$ such that $\theta|_K \equiv 1$ and $\psi|_{N \setminus V} = 0$. Then $\psi_K := \psi(\theta, \theta, \ldots, \theta) : N \to \mathbb{R}^n$ is a well-defined smooth function that $\psi_K|_K = \psi|_K$. Since $y^k \circ \psi_K : N \to \mathbb{R}$ is smooth, by

assumption $f^*(y^k \circ \psi_K) = y^k \circ \psi_K \circ f$ is smooth. Let (U, φ) be any chart such that $f(U) \subseteq K$. Then

$$y^k \circ \psi \circ f \circ \varphi^{-1} = y^k \circ \psi_K \circ f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$$

is smooth for each $k \in [n]$, so that $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(K)$ is smooth. Running over all such U and K shows f is smooth.

2. Locally near a point $p \in X$, write $X = \sum a_i \frac{\partial}{\partial x^i}$. As in 1. we can extend the functions x^j, a_i to global functions which are unchanged near p. Then $X(x^j) = a_j$. Now 2. is clear.

3. The only if part is clear. For the if part, by extension using bump functions as before, the question becomes a purely local one. Then the result becomes obvious, as a function on $\mathbb{R}^m$ is smooth if and only if all its partial derivatives exist and are smooth.

□

For $X, Y \in \mathfrak{X}(M)$, $XY := X \circ Y : C^\infty(M) \to C^\infty(M)$ is an $\mathbb{R}$-linear map, but fails to be a derivation in general. Indeed, for $f, g \in C^\infty(M)$,

$$XY(fg) = X(Y(f)g + Y(g)f) = XY(f)g + XY(g)f + Y(f)X(g) + Y(g)X(f)$$

To cancel out the last to term, we subtract $YX(fg)$ so that

$$\begin{aligned} XY(fg) - YX(fg) &= XY(f)g + XY(g)f - (YX(f)g + YX(g)f) \\ &= (XY - YX)(f)g + (XY - YX)(g)f \end{aligned}$$

Then $XY - YX$ is a derivation on $C^\infty(M)$.

Definition. For $X, Y \in \mathfrak{X}(M)$, the **Lie bracket** $[X, Y]$ is a smooth vector field given by the derivation $XY - YX$. In other words, for $p \in M$ and $f \in C_p^\infty(M)$,

$$[X, Y]_p f := X_p(Yf) - Y_p(Xf)$$

Proposition F.3.3. For a smooth manifold M, the space $\mathfrak{X}(M)$ of smooth vector fields on M together with the Lie bracket defined above is a real Lie algebra.

Proposition F.3.4. Let $f : M \to N$ be a smooth diffeomorphism. For $X, Y \in \mathfrak{X}(M)$, we have

$$f_*[X, Y] = [f_*X, f_*Y]$$

Proof. This is a direct computation. Let $q = f(p)$, $p \in M$.

$$\begin{aligned} (f_*[X, Y])_q g &= f_{*,p}[X, Y]_p g = [X, Y]_p(g \circ f) \\ &= X_p(Y(g \circ f)) - Y_p(X(g \circ f)) \\ &= X_p((f_*Y)g \circ f) - Y_p((f_*X)g \circ f) \\ &= (f_*X)_q(f_*Y)g - (f_*Y)_q(f_*X)g = [f_*X, f_*X]_q g \end{aligned}$$

where we have used the equality $(f_*X)g \circ f = X(g \circ f)$ □

F.3.3 Lie derivatives of vector fields

Definition. Let X be a vector field and $\varphi : (-\varepsilon, \varepsilon) \times W \to U$ be a local flow generated by X. For $t \in (-\varepsilon, \varepsilon)$, put $\varphi_t = \varphi(t, \cdot) : W \to U$. If Y is a vector field and $p \in W$, define the **Lie derivative of Y in X** by[2]

$$(\mathcal{L}_X Y)_p = \lim_{t \to 0} \frac{(\varphi_{-t})_{*,\varphi_t(p)} Y_{\varphi_t(p)} - Y_p}{t} = \frac{d}{dt}\bigg|_{t=0} (\varphi_{-t})_{*,\varphi_t(p)} Y_{\varphi_t(p)} \in T_p M$$

Here note that $\varphi_0(p) = p$ for any $p \in W$, so $(\varphi_0)_{*,p} = \mathrm{id}$ on $T_p M$.

Proposition F.3.5. Let X be a vector field and φ the local flow generated by X.

(i) For $f \in C^\infty(M)$, we have $X_p(f) = \dfrac{d}{dt}\bigg|_{t=0} f(\varphi_t(p))$.

(ii) For $Y \in \mathfrak{X}(M)$, we have $\mathcal{L}_X Y = [X, Y]$.

Proof.

(i) The question is local, we may write $X = \sum\limits_{i=1}^n a_i \dfrac{\partial}{\partial x^i}$ and $\varphi_t = (\varphi_t^1, \dots, \varphi_t^n)$ with $\varphi_0^i = x^i$ and $\dfrac{d}{dt}\varphi_t^i(p) = a_i(\varphi_t(p))$. Then

$$\frac{d}{dt}\bigg|_{t=0} f(\varphi_t(p)) = \frac{d}{dt}\bigg|_{t=0} f(\varphi_t^1(p), \dots, \varphi_t^n(p)) = \sum_{i=1}^n \frac{\partial f}{\partial x^i}\bigg|_p \frac{d}{dt}\bigg|_{t=0} \varphi_t^i(p) = \sum_{i=1}^n a_i(p) \frac{\partial f}{\partial x^i}\bigg|_p = X_p(f).$$

(ii) Let $f \in C^\infty(M)$. Then

$$(\mathcal{L}_X Y)_p(f) = \lim_{t \to 0} \frac{(\varphi_{-t})_{*,\varphi_t(p)} Y_{\varphi_t(p)}(f) - Y_p(f)}{t} = \lim_{t \to 0} \frac{Y_{\varphi_t(p)}(f \circ \varphi_{-t}) - Y_p(f)}{t}$$

where we use the identity $(f_* X) g \circ f = X(g \circ f)$ again. Write $f \circ \varphi_{-t} = f + t h_t$ with $h_0 = -X(f)$. Applying Y, we obtain

$$Y_q(f \circ \varphi_{-t}) = Y_q(f) + Y_q(t h_t) = Y_q(f) + t Y_q(h_t).$$

Plugging into the expression above, we see

$$\begin{aligned} (\mathcal{L}_X Y)_p(f) &= \lim_{t \to 0} \frac{Y_{\varphi_t(p)}(f) + t Y_{\varphi_t(p)}(h_t) - Y_p(f)}{t} \\ &= \lim_{t \to 0} \frac{Y_{\varphi_t(p)}(f) - Y_p(f)}{t} + Y_p(h_0) = X_p(Y(f)) - Y_p(X(f)). \end{aligned}$$

Here the last identity results from (i).

□

Corollary F.3.5.1. Let X, Y be two vector fields and φ_t, ψ_s the respective local flows defined on U. TFAE:

1. $[X, Y] = 0$ on U.
2. $(\varphi_t)_* Y = Y$ for any small enough t.
3. For any $p \in U$, $\varphi_t \circ \psi_s(p) = \psi_s \circ \varphi_t(p)$ for any small t, s.

[2] The tangent space $T_p M$ is finite dimensional over $\mathbb{R}$, so it has a unique norm topology, which allows us to talk about limit.

F.3.4 Integral submanifolds

Definition. Let M^n be a smooth manifold and let $0 \leqslant k \leqslant n$.

1. A **rank k distribution of** TM is a subset $\mathcal{H} \subseteq TM$ satisfying
   (a) $\mathcal{H}_p := \mathcal{H} \cap T_pM$ is a k-dimensional subspace for any $p \in M$, and
   (b) $\mathcal{H}$ is smooth, in the sense that $\mathcal{H}_p = \operatorname{span}_{\mathbb{R}} \left\{ \sum_{j=1}^n a_{ij} \left.\frac{\partial}{\partial x^j}\right|_p \right\}_{i=1}^k$ holds in a small neighborhood p for any $p \in M$, where the a_{ij} are smooth.
2. A distribution $\mathcal{H}$ is **involutive** if for any $X, Y \in \mathfrak{X}(M)$ such that $X_p, Y_p \in \mathcal{H}$ for any $p \in M$, we have $[X,Y]_p \in \mathcal{H}$ for any $p \in M$.

Theorem F.3.6 (Frobenius')**.** Let M^n be a smooth manifold and $\mathcal{H}$ a rank k distribution on TM. If $\mathcal{H}$ is involutive, then for any $p \in M$, there exists a local chart $(U, x^1, \ldots, x^n)$ about p such that $\mathcal{H} = \operatorname{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial x^i} \right\}_{i=1}^k$ in U.

Proof. We prove this by induction on $k = \operatorname{rank} \mathcal{H}$; note that when $k = 1$, $\mathcal{H}$ is automatically involutive. Let $p \in M$ and choose a local chart $(U, x^1, \ldots, x^n)$ about p (where the image of p is the origin). Then $\mathcal{H} = \operatorname{span}_{\mathbb{R}} \left\{ \sum_{i=1}^n a_i \frac{\partial}{\partial x^i} \right\}$ in U. Let φ_t be the local flow in U generated by $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x^i}$. Consider the function

$$F : (x^1, \ldots, x^{n-1}, t) \mapsto \varphi_t(x^1, \ldots, x^{n-1}, 0),$$

defined on a reasonable domain. We compute the Jacobian of F at the origin:

$$F'(0) = \begin{pmatrix} & a_1(0) \\ I_{n-1} & a_2(0) \\ & \vdots \\ O_{1,n-1} & a_n(0) \end{pmatrix}$$

Up to a rearrangement and shrinking U, we may assume a_n is nonvanishing in U, implying that $F'(0)$ is invertible. This means we can use $(x^1, \ldots, x^{n-1}, t)$ as a new coordinates. By construction, we have $F_{*,p}\left(\left.\frac{\partial}{\partial t}\right|_p \right) = X_{F(p)}$, or $F_* \frac{\partial}{\partial t} = X$, so that $(F^{-1})_* X = \frac{\partial}{\partial t}$. This means in the chart $(x^1, \ldots, x^{n-1}, t)$, we have $\mathcal{H} = \mathbb{R}\frac{\partial}{\partial t}$.

For general $k \geqslant 2$, take a small neighborhood around p so that $\mathcal{H} = \operatorname{span}_{\mathbb{R}}\{V_1, \ldots, V_k\}$, where the V_i are vector fields defined near p. By the $k = 1$ case applied to $\mathbb{R}V_k$, we can find a coordinate chart $(y^1, \ldots, y^n)$ such that $\mathbb{R}V_k = \mathbb{R}\frac{\partial}{\partial y^k}$. Write $[V_i, V_j] = V_{ij} + a_{ij} V_k$ $(1 \leqslant i, j \leqslant k-1)$ for some $V_{ij} \in \operatorname{span}_{\mathbb{R}}\{V_1, \ldots, V_{k-1}\}$. By replacing V_i with $V_i - V_i(y^k) V_k$, we obtain

$$0 = V_{i,p}(V_j(y^k)) - V_{j,p}(V_i(y^k)) = [V_i, V_j]_p(y^k) = V_{ij,p}(y^k) + a_{ij}(p) V_{k,p}(y^k) = a_{ij}(p)$$

i.e., $a_{ij} = 0$. Thus $\operatorname{span}_{\mathbb{R}}\{V_1, \ldots, V_{k-1}\}$ is involutive, so by induction we can find $(z^1, \ldots, z^n)$ such that $\operatorname{span}_{\mathbb{R}}\{V_1, \ldots, V_{k-1}\} = \operatorname{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial z^i} \right\}_{i=1}^{k-1}$. So far we have arrived at

$$\mathcal{H} = \operatorname{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^{k-1}}, V_k \right\},$$

and we have found a function h such that $V_k(h) = 1$ while $\dfrac{\partial}{\partial z^i}h = 0$ for $1 \leqslant i \leqslant k-1$. In particular, we may assume $V_k \in \operatorname{span}_{\mathbb{R}} \left\{ \dfrac{\partial}{\partial z^i} \right\}_{i=k}^n$ Write $V_k = \sum_{i=k}^n b_i \dfrac{\partial}{\partial z^i}$. Since $\mathcal{H}$ is involutive,

$$\mathcal{H} \ni \sum_{j=1}^{k-1} a_j \frac{\partial}{\partial z^i} + c_i V_k = \left[\frac{\partial}{\partial z^i}, V_k \right] = \sum_{j=k}^n \frac{\partial b_j}{\partial z^i} \frac{\partial}{\partial z^j}.$$

Plugging h into the first expressing gives $c_i \equiv 0$, so the bracket actually lies in $\operatorname{span}_{\mathbb{R}}\{V_1, \ldots, V_{k-1}\}$, whence from the second expression we read off $\dfrac{\partial b_j}{\partial z^i} \equiv 0$ for $j \geqslant k$. It follows that the b_j only depend on the variables $z^k, \ldots, z^n$.

Using the coordinate $(z^1, \ldots, z^{k-1}, z^k \ \ldots, z^n)$, we identify $\mathbb{R}^n$ with $\mathbb{R}^{k-1} \times \mathbb{R}^{n-k+1}$. What we obtained above simply says that V_k is a vector field that entirely lies in $\{0\} \times \mathbb{R}^{n-k+1}$. By the case $k = 1$ applied the $\mathbb{R}V_k$ regarded as in the manifold $\mathbb{R}^{n-k+1}$, we can find $x^k, \ldots, x^n$ such that $\mathbb{R}V_k = \mathbb{R}\dfrac{\partial}{\partial x^k}$. Now the coordinates $(z^1, \ldots, z^{k-1}, x^k, \ldots, x^n)$ about p in M does the job for us, and the induction step is completed. ☐

Definition. Let M be a smooth manifold and $\mathcal{H} \subseteq TM$ a rank k distribution. An immersed submanifold $f : N \to M$ is called an **integral submanifold for $\mathcal{H}$** if for every $p \in N$, the image of $f_{*,p} : T_pN \to T_{f(p)}M$ is precisely $\mathcal{H}_{f(p)}$.

Let $\mathcal{H} \subseteq TM$ be an involutive distribution of rank k. By Theorem F.3.6, for each $p \in M$ we can find a connected chart $(U_p, x_p^1, \ldots, x_p^n)$ such that $\mathcal{H} = \operatorname{span}_{\mathbb{R}} \left\{ \dfrac{\partial}{\partial x_p^i} \right\}_{i=1}^k$ in U. Then the slices

$$S_p(\underline{c}) := \{x \in U_p \mid (x_p^{k+1}, \ldots, x_p^n) = \underline{c}\}, \ \underline{c} \in \mathbb{R}^{n-k}$$

are closed submanifolds of U_p such that $T_qS_p(\underline{c}) = \mathcal{H}_q$ for $q \in S_p(\underline{c})$. With our new language, each slice $S_p(\underline{c}) \to M$ is an integral submanifold for $\mathcal{H}$. In fact, if $f : N \to M$ is a *connected* integral submanifold for $\mathcal{H}$ with image $f(N) \subseteq U_p$, then $f(N) \subseteq S_p(\underline{c})$ for some $\underline{c} \in \mathbb{R}^{n-k}$. Indeed, if we denote by $\pi = (x_p^{k+1}, \ldots, x_p^n) : \mathbb{R}^n \to \mathbb{R}^{n-k}$, then $(\pi \circ f)_* = 0$ identically, so $\pi \circ f$ is a constant, as N is connected.

We are going to construct a maximal integral submanifold for $\mathcal{H}$ passing through a fixed point $p \in M$. Let $p \in M$. Define

$$K = \left\{ x \in M \;\middle|\; \begin{array}{l} \text{there exists a piecewise smooth } \gamma : (-\varepsilon, 1+\varepsilon) \to M \text{ such that } \gamma(0) = p,\ \gamma(1) = p, \\ \text{and all of its smooth pieces are integral curves for } \mathcal{H} \end{array} \right\}.$$

We topologize K as follows. By second countability of M, we can find a countable subset $F \subseteq M$ such that $\{U_q \mid q \in F\}$ covers M; we assume $p \in F$. For each $x \in K$, there exists $q_x \in F$ such that $x \in U_{q_x}$, so $x \in S_{q_x} := S_{q_x}(\underline{c})$ for some $\underline{c} \in \mathbb{R}^{n-k}$. Note that $S_{q_x} \subseteq K$. We use the cover $\{(S_{q_x}, x_{q_x}^1|_{S_{q_x}}, \ldots, x_{q_x}^k|_{S_{q_x}}) \mid x \in K\}$ to give K a smooth structure. If we can show this topology is second countable, then K is then a (path-)connected k-dimensional smooth manifold. For this, fix some $q \in F$, and we only need to show the set

$$\{\underline{c} \in \mathbb{R}^{n-k} \mid S_q(\underline{c}) \subseteq K\}$$

is at most countable. Each point $x \in K$ is joined to p by a piecewise smooth curve γ. Take $(q_i)_{i=0}^n \subseteq F$ with $q_0 = p$ so that $\gamma|_{[0,1]}$ passes through U_{q_i} in order. Since each smooth part of γ is an integral curve, it starts with p in U_0, and passes through U_{q_1} in some slices, and then passes through U_{q_2} in some slices, and so on. To show the countability, it suffices to show that for $q, r \in F$ and $\underline{c} \in \mathbb{R}^{n-k}$, the set

$$\{\underline{d} \in \mathbb{R}^{n-k} \mid S_q(\underline{c}) \cap S_r(\underline{d}) \neq \varnothing\}$$

is at most countable. For this, notice that $S_q(\underline{c}) \cap U_r$ is an open submanifold of U_r, so it has (at most) countably many connected components, each being an integral submanifold for $\mathcal{H}$ in U_r. Hence $S_q(\underline{c}) \cap U_r$ is contained only in a countably union of the slices $S_r(\underline{d})$ in U_r.

Hence, the inclusion $f : K \to M$ is then a connected integral submanifold for $\mathcal{H}$ passing through p. We claim it is maximal. If $g : N \to M$ is a connected integral submanifold passing through p, then any point in N can be joined to $g^{-1}(p)$ by a piecewise smooth curve, as N is path-connected. Hence every point in $g(N)$ is connected to p by a piecewise smooth curve in $g(N)$, whence by a piecewise integral curve for $\mathcal{H}$. In conclusion, $g(N) \subseteq K$.

Corollary F.3.6.1. Let M^n be a smooth manifold and $\mathcal{H}$ a rank k involutive distribution on TM. Then for each $p \in M$, there exists a unique maximal connected integral submanifold $K \to M$ for $\mathcal{H}$ passing through p.

Integral submanifolds behave well categorically, in the following sense.

Lemma F.3.7. Let $f : N \to M$ be a smooth map and $\varphi : P \to M$ an integral submanifold for an involutive distribution $\mathcal{H} \subseteq TM$ of rank k. Suppose $f(N) \subseteq \varphi(P)$, and let $f_0 : N \to P$ be the unique map such that $f = \varphi \circ f_0$. Then f_0 is continuous. In particular, f_0 is smooth by Corollary F.2.7.

Proof. Let U be an open set in P, $p \in U$ and $x \in f_0^{-1}(p)$. Take a chart $(V, y^1, \ldots, y^m)$ centred at $\varphi(p)$ such that $S(\underline{c}) := \{y \in V \mid (y^{k+1}, \ldots, y^m) = \underline{c}\}$ are integral submanifolds for $\mathcal{H}$ in V, and pick an neighborhood $p \in U'$ in U such that $\varphi(U') = S(\underline{0})$. Take W to be the connected component of $f^{-1}(V)$ containing x; then W is open in N. To show the continuity it suffices to show $f_0(W) \subseteq U'$. Since φ is injective, it suffices to show $f(W) \subseteq S(\underline{0})$. Furthermore, since $f(x) \in f(W) \cap S(\underline{0})$ and $f(W)$ lies in a component of $V \cap \varphi(P)$, it suffices to show each component C of $V \cap \varphi(P)$ is contained in some $S(\underline{c})$.

Let $\pi = (y^{k+1}, \ldots, y^m) : V \to \mathbb{R}^{m-k}$. Since P is separable and $V \cap \varphi(P)$ is a disjoint union of the slices $S(\underline{c})$, the image $\pi(V \cap \varphi(P))$ is at most countable. Since C is connected, $\pi(C)$ is a connected countable set in $\mathbb{R}^{m-k}$, whence $\#\pi(C) = 1$. $\square$

F.4 Complex manifolds

We start with the definition of holomorphic maps in higher dimensional cases. A general point in $\mathbb{C}^n$ is denoted by $(z_1, \ldots, z_n)$, and we write $z_k = x_k + iy_k$ with $x_k, y_k \in \mathbb{R}$. Introduce the differential operators

$$\partial_k = \frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right), \qquad \overline{\partial_k} = \frac{\partial}{\partial \overline{z_k}} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right).$$

Note that this makes sense as a function in z_k can be viewed as a function in x_k and y_k. These operators define sections of the **complexified tangent bundle of** $\mathbb{C}^n$:

$$\partial_k, \overline{\partial_k} : \mathbb{C}^n \to (T\mathbb{C}^n) \otimes_{\mathbb{R}} \mathbb{C}^n := \bigsqcup_{p \in \mathbb{C}^n} T_p\mathbb{C}^n \otimes_{\mathbb{R}} \mathbb{C}.$$

Definition. Let U be an open set in $\mathbb{C}^n$.

(i) A smooth function $f : U \to \mathbb{C}$ is **holomorphic** if $\overline{\partial_k} f : U \to \mathbb{C}$ is a zero function for all $k = 1, \dots, n$.

(ii) A smooth function $g : U \to \mathbb{C}^n$ is **holomorphic** if each component $\mathrm{pr}_\ell \circ g : U \to \mathbb{C}$ of g is holomorphic in the sense of (i).

Each tangent space $T_p\mathbb{C}^n$ is by definition a real vector space of dimension $2n$. Multiplication by i defines an $\mathbb{R}$-linear isomorphism $I : \mathbb{C}^n \to \mathbb{C}^n$:

$$I(\dots, x_k, y_k, \dots) = (\dots, -y_k, x_k, \dots)$$

satisfying $I^2 = -\mathrm{id}$. The usual identification $\mathbb{C}^n \cong T_p\mathbb{C}^n$ allows I to act on $T_p\mathbb{C}^n$, in the way that $I\left(\dfrac{\partial}{\partial x_k}\right) = \dfrac{\partial}{\partial y_k}$, and $I\left(\dfrac{\partial}{\partial y_k}\right) = -\dfrac{\partial}{\partial x_k}$. We then have the eigendecomposition of $T_p\mathbb{C}^n \otimes_{\mathbb{R}} \mathbb{C}$ with respect to $I \otimes_{\mathbb{R}} \mathrm{id}_{\mathbb{C}}$:

$$T_p\mathbb{C}^n \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{span}_{\mathbb{C}}\{\partial_1, \dots \partial_n\} \oplus \mathrm{span}_{\mathbb{C}}\{\overline{\partial_1}, \dots, \overline{\partial_n}\}$$

The former is the i-eigenspace, and the latter is the $-i$-eigenspace. We shall write

$$\begin{aligned} T_p^{1,0}\mathbb{C}^n &:= \mathrm{span}_{\mathbb{C}}\{\partial_1, \dots \partial_n\} \\ T_p^{0,1}\mathbb{C}^n &:= \mathrm{span}_{\mathbb{C}}\{\overline{\partial_1}, \dots, \overline{\partial_n}\} \end{aligned}$$

and form

$$T^{1,0}\mathbb{C}^n = \bigsqcup_{p \in \mathbb{C}^n} T_p^{1,0}\mathbb{C}^n, \qquad T^{0,1}\mathbb{C}^n = \bigsqcup_{p \in \mathbb{C}^n} T_p^{0,1}\mathbb{C}^n.$$

The real vector space $T_p\mathbb{C}^n$ together with the $\mathbb{R}$-automorphism becomes a complex vector space: for $x, y \in \mathbb{R}$ and $v \in \mathbb{C}^n$, we can define

$$(x + iy)v := xv + yI(v).$$

The usual identification then defines a $\mathbb{C}$-isomorphism from $(T_p\mathbb{C}^n, I)$ to $\mathbb{C}^n$.

Lemma F.4.1. Let $f : \mathbb{C}^n \to \mathbb{C}$ be a smooth function. TFAE:

1. f is holomorphic.

2. $f_{*,p} : T_p\mathbb{C}^n \to T_{f(p)}\mathbb{C} = \mathbb{C}$ is $\mathbb{C}$-linear for each $p \in \mathbb{C}^n$.

Definition.

(i) An **n-dimensional complex manifold** is an $2n$-dimensional smooth manifold such that each transition map is holomorphic (we regard the image of a chart as an open set in $\mathbb{C}^n \cong \mathbb{R}^{2n}$).

(ii) A smooth map $f : X \to Y$ between complex manifolds is called **holomorphic** if it is locally is a holomorphic map.

(iii) A **complex vector bundle** over a complex manifold is a smooth vector bundle such that each fibre is a complex vector space.

(iv) A **holomorphic vector bundle** over a complex manifold is a complex manifold that is also a complex vector bundle such that the bundle projection and each local trivialization are holomorphic.

In contrast, we will refer to smooth vector bundles as **real vector bundles**. An atlas $\{(U_\alpha, \varphi_\alpha)\}$ of X making X a complex manifold is called a **holomorphic atlas** of X. Locally we write $\varphi_\alpha(p) = (z^1(p), \ldots, z^n(p)) \in \varphi_\alpha(U_\alpha) \underset{\text{open}}{\subseteq} \mathbb{C}^n$ for each $p \in U_\alpha$.

Let X be a complex manifold of dimension n. By construction, the tangent bundle TX is a real vector bundle of rank $2n$. To emphasize, we write it as $T_{\mathbb{R}}X$ instead. The complexified tangent bundle

$$T_{\mathbb{C}}X = T_{\mathbb{R}}X \otimes_{\mathbb{R}} \mathbb{C} := \bigsqcup_{p \in X} T_pX \otimes_{\mathbb{R}} \mathbb{C}$$

can be easily shown to be a complex vector bundle over X. Recall the construction of $T_{\mathbb{R}}X$. Since we use an holomorphic atlas on X, the transition map of $T_{\mathbb{R}}X$ is holomorphic fholo=> dfholo. This makes $T_{\mathbb{R}}X$ a holomorphic vector bundle over X. To emphasize the holomorphic structure, we denote by T_X the real tangent bundle $T_{\mathbb{R}}X$ together with this holomorphic structure.

The multiplication by i on the second component of charts of $T_{\mathbb{R}}X$ are compatible with the transition maps, obtaining a global real bundle automorphism $J : T_{\mathbb{R}}X \to T_{\mathbb{R}}X$.

F.5 Smooth partition of unity

In this section, for a topological space X, we use $C_c(X)$ to denote the space of real-valued continuous functions with compact support (rather than the complex-valued ones).

Let M be a smooth manifold of dimension n. By Theorem A.9.5, M is paracompact (the second countability is imposed to a manifold in our definition). In this subsection we construct a "smooth" partition of unity subordinate to any given open cover of M, by modifying the proof of Theorem A.9.4.

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(t) = \mathbf{1}_{\geqslant 0}(t)e^{-\frac{1}{t}}$. A direct computation of derivatives of f at $t = 0$ shows that $f \in C^\infty(\mathbb{R})$. For $0 < a < b$, the function $f_1(x) := f((b-a)(x-a))$ is then a smooth function with $f_1(x) = 0$ if and only if $x \leqslant a$ or $x \geqslant b$. Define $f_2 : \mathbb{R} \to \mathbb{R}$ by

$$f_2(x) := \int_x^b f_1(t)dt \Big/ \int_a^b f_1(t)dt$$

Then $f_2 \in C^\infty(\mathbb{R})^+$ and $f_2(x) = \begin{cases} 1 & \text{, if } x \leqslant a \\ 0 & \text{, if } x \geqslant b \end{cases}$ Define $\phi : \mathbb{R}^n \to \mathbb{R}$ by

$$\phi(x) = f_2(\|x\|^2)$$

We have $\phi \in C^\infty(\mathbb{R}^n)^+$ and $\phi(x) = \begin{cases} 1 & \text{, if } \|x\|^2 \leqslant a \\ 0 & \text{, if } \|x\|^2 \geqslant b \end{cases}$. Using this function, we first prove a smooth version of Urysohn's lemma:

Lemma F.5.1. Let M^n be a smooth manifold, K a compact subset and U an open neighborhood of K in M. Then we can find $\psi \in C^\infty(M)^+$ such that $\psi|_K = 1$ and $\psi|_{M \setminus U} = 0$.

Proof. Let $p \in K$ and choose an open chart $\varphi_p : U_p \to \mathbb{R}^n$ of p in U with $\varphi_p(p) = 0$. The preceding discussion allows us to find $\phi_p \in C_c^\infty(U_p)^+$ and $0 < a < b$ with $\overline{B_b(0)} \subseteq \varphi_p(U_p)$ with $\phi_p(x) = \begin{cases} 1 & \text{, if } \|\varphi_p(x)\|^2 \leqslant a \\ 0 & \text{, if } \|\varphi_p(x)\|^2 \geqslant b \end{cases}$ Let us put $V_p := \varphi_p^{-1}(\overline{B_a(0)})$. By compactness of K, we can find

$p_1, \ldots, p_n \in K$ such that

$$K \subseteq \bigcup_{i=1}^{n} V_i \subseteq \bigcup_{i=1}^{n} \operatorname{supp} \phi_i \subseteq U$$

where $V_i := V_{p_i}$ and $\phi_i := \phi_{p_i}$. Since $\operatorname{supp} \phi_i \subseteq U_{p_i}$, we can extend ϕ_i to the whole M by setting $\phi_i|_{M \setminus U_{p_i}} = 0$, so that we can view $\phi_i \in C_c^\infty(M)^+$. Finally define

$$\psi = 1 - (1 - \phi_1)(1 - \phi_2) \cdots (1 - \phi_n) \in C^\infty(M).$$

As $K \subseteq \bigcup_{i=1}^{n} V_i$, ψ is identically 1 on K, and as $\bigcup_{i=1}^{n} \operatorname{supp} \phi_i \subseteq U$, ψ vanishes outside U. In addition, for $x \in U \setminus K$, by construction (of ϕ_p) we have $\psi(x) \in [0, 1]$, so in fact $\psi \in C^\infty(M, [0, 1])$. $\square$

We proceed to construct a smooth partition of unity. Let $p \in M$ and let U be an open neighborhood of p. By the preceding discussion, we can find $g_p \in C_c^\infty(U)^+$ with $g_p(p) > 0$ and $\operatorname{supp} g_p$ having nonempty interior. That $\operatorname{supp} g_p \subseteq U$ implies that g_p extends (by zero) to a global smooth function, which we still denote by $g_p \in C_c^\infty(M)^+$.

We start our construction of a smooth partition of unity subordinate to the given open cover $(U_\alpha)_\alpha$ of M. By Theorem A.9.5, M is σ-compact, so there exists an increasing sequence $(E_n)_n$ of relatively compact open sets of M with $\overline{E_n} \subseteq E_{n+1}$. For each $n \in \mathbb{N}$, the annulus $K_n := \overline{E_n} \setminus E_{n-1}$ is compact, and it contained in the open annulus $A_n := E_{n+1} \setminus \overline{E_{n-2}}$. For each $p \in K_n$, let W_p be an chart about p contained in $U_\alpha \cap A_n$, and let $g_p \in C_c^\infty(M)^+$ be such that $g_p(p) > 0$ and $\operatorname{supp} g_p$ has nonempty interior V_p and is contained in W_p. We can find a finite subcollection of $\{V_p\}$ that covers K_n. Collecting these V_p, we obtain a locally finite open refinement $(V_n)_{n \in \mathbb{N}}$ of $(U_\alpha)_\alpha$.

Let g_n be the corresponding function to V_n. For each $p \in M$, we see $\{n \in \mathbb{N} \mid g_n(p) \neq 0\}$ is nonempty and finite. Thus it is legal to define

$$\psi_n(p) := \frac{g_n(p)}{\sum_m g_m(p)} \in [0, 1]$$

for each $p \in M$ and $n \in \mathbb{N}$. Clearly, $\psi_n \in C^\infty(M)$, $\operatorname{supp} \psi_n = \operatorname{supp} g_n = \overline{V_n}$ is compact, and $\sum_n \psi_n \equiv 1$. This finishes our construction.

In addition, if we define $\psi_\alpha := \sum_{V_n \subseteq U_\alpha} \psi_n$ ($\psi_\alpha = 0$ if the range of summation is empty), then $(\psi_\alpha)_\alpha$ is a partition of unity with the same index set as U_α, but each $\operatorname{supp} \psi_\alpha$ fails to be compact.

Let $\ell \in \mathbb{N}$. If we define $\phi_n := \dfrac{g_n^\ell}{\sum_m g_m^\ell}$, then $(\phi_n)_n$ is a smooth partition of unity subordinate to $(U_\alpha)_\alpha$ with smooth ℓ-th root.

F.6 Density and integration

Let V be an n-dimensional real vector space. A **frame** on V is an ordered basis $\beta = (\beta_1 \cdots \beta_n)$ for V. Denote by $F(V)$ the set of all frames on V. There is a canonical bijection $F(V) \cong \operatorname{Isom}_{\mathbb{R}}(\mathbb{R}^n, V)$, by sending β to a linear map T_β defined by $e_i \mapsto \beta_i$. The general linear group $\mathrm{GL}_n(\mathbb{R})$ acts on the set $F(V)$ on the right, and the action is free and transitive.

Let M be an n-dimensional smooth manifold. Define the **(tangent) frame bundle** of M

$$FM = F(TM) := \bigsqcup_{p \in M} F(T_pM).$$

This is a smooth principal $\mathrm{GL}_n(\mathbb{R})$-bundle (c.f. §I.7.2), which we show now. Let us put $\pi : FM \to M$ to be the projection. Let (U, φ) be a local coordinates of M; then $\varphi_{*,p} : T_pU \to \mathbb{R}^n$ is a linear isomorphism. The fibrewise bijections $F(T_pM) \to \mathrm{Isom}(\mathbb{R}^n, V) \cong \mathrm{GL}_n(\mathbb{R})$ glues to a bijection

$$\begin{aligned} \pi^{-1}(U) &\longrightarrow U \times \mathrm{GL}_n(\mathbb{R}) \\ (p, \beta) &\longmapsto (p, \varphi_{*,p} \circ T_\beta) \end{aligned}$$

The general linear group $\mathrm{GL}_n(\mathbb{R})$ acts on the bundle FM from the right in a natural way, making this bijection $\mathrm{GL}_n(\mathbb{R})$-equivariant. We now equip FM with the final topology induced by the "inclusions" $U \times \mathrm{GL}_n(\mathbb{R}) \to \pi^{-1}(U) \subseteq FM$. It is clear that this topology and the above local trivialization make FM into a smooth principal $\mathrm{GL}_n(\mathbb{R})$-bundle as we promised. For convenience, we denote the section $U \to U \times \{\mathrm{id}\} \subseteq U \times \mathrm{GL}_n(\mathbb{R}) \cong \pi^{-1}(U)$ by $\sigma_\varphi = \sigma_{(U,\varphi)}$; explicitly, $\sigma(p) = (p, \varphi_{*,p}^{-1}(e_1), \ldots, \varphi_{*,p}^{-1}(e_n))$.

Let $s \in \mathbb{R}$ and define the one-dimensional representation $\rho_s : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^\times = \mathrm{GL}(\mathbb{R})$ by $\rho_s(A) = |\det A|^{-s}$. The associated bundle $FM \times_{\mathrm{GL}_n(\mathbb{R})} (\mathbb{R}, \rho_s)$ (c.f. §I.7.2) of $FM \to M$ intertwining ρ_s is called the **s-density bundle** of X, and is denoted by $\mathrm{Vol}_s M$. A **continuous** (resp. **smooth**) **s-density** is then a continuous (resp. smooth) global section of the s-density bundle $\mathrm{Vol}_s M$. Particularly, a **density** is referred to as a 1-density, and we simply write $\mathrm{Vol}(M) := \mathrm{Vol}_1 M$. Also, we put $\Gamma(U, \mathrm{Vol}_s M)$ (resp. $\Gamma^\infty(U, \mathrm{Vol}_s M)$) be the set of all continuous (resp. smooth) s-densities defined on $U \underset{\text{open}}{\subseteq} M$.

Let (U, φ) be a local chart of M, and let $\sigma = \sigma_\varphi : U \to \pi^{-1}(U) \subseteq FM$ be the associated local section. Then a local trivialization of $\mathrm{Vol}_s M$ is given by

$$\begin{aligned} \varphi(U) \times \mathbb{R} \longrightarrow U \times \mathbb{R} &\longrightarrow \pi_d^{-1}(U) \\ (p, v) &\longmapsto [\sigma(p), v] \end{aligned}$$

where $\pi_d : \mathrm{Vol}_s M \to M$ denotes the projection, and $[\sigma(p), v]$ denotes the class of $(\sigma(p), v)$ in $\mathrm{Vol}_s M$. We describe its inverse: if $[(p, \beta), w] \in \pi_d^{-1}(U)$, pick $A \in \mathrm{GL}_n(\mathbb{R})$ such that $\beta = (\varphi_{*,p}^{-1}(e_i))_{1 \leqslant i \leqslant n} A$, and send $[(p, \beta), w]$ to $(p, |\det A|^{-s} w)$. If (V, ψ) is another local chart with $U \cap V \neq \varnothing$, then the corresponding transition map on $\mathrm{Vol}_s M$ is then

$$\begin{aligned} \varphi(U \cap V) \times \mathbb{R} &\longrightarrow \psi(U \cap V) \times \mathbb{R} \\ (\varphi(p), v) &\longmapsto (\psi(p), |\det(\theta_{UV})_{*,\varphi(p)}|^{-s} v) \end{aligned} \tag{$\clubsuit$}$$

where $\theta_{UV} = \psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is the transition map on M.

The representation ρ_s can be realized in terms of functions. Consider the set $\mathrm{Vol}_s(\mathbb{R}^n)$ of functions $\omega : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ satisfying $\omega(Av_1, \ldots, Av_n) = |\det A|^s \omega(v_1, \ldots, v_n)$. It is clear that $\mathrm{Vol}_s(\mathbb{R}^n)$ is one-dimensional, and $\mathrm{GL}_n(\mathbb{R})$ acts on $\mathrm{Vol}_s(\mathbb{R}^n)$ on the left by

$$(A^*\omega)(v_1, \ldots, v_n) := \omega(A^{-1}v_1, \ldots, A^{-1}v_n) = |\det A|^{-s} \omega(v_1, \ldots, v_n).$$

Thus, via the local chart (U, φ), we can understand each element $v \in \mathbb{R}$ in the fibre of $\mathrm{Vol}_s M$ at p as a function $\omega_p : T_pM \times \cdots \times T_pM \to \mathbb{R}$ satisfying $\omega_p(\varphi_{*,p}^{-1}(e_1), \ldots, \varphi_{*,p}^{-1}(e_n)) = v$ and

$$\omega_p(AX_1, \ldots, AX_n) = |\det A| \omega_p(X_1, \ldots, X_n).$$

where $A \in \mathrm{GL}(T_pM)$ and $\det A := \det(\varphi_{x,p}^{-1} \circ A \circ \varphi_{x,p})$. Such a function is called an **s-density** on the vector space T_pM. For a section ω of $\mathrm{Vol}_s M$, we shall always think of ω_p as an s-density on

T_pM. We say an s-density ω is **positive** if ω_p is a non-negative function on $T_pM \times \cdots \times T_pM$; this is a well-defined notion independent of the choice of φ due to the transition rule (♣) of $\mathrm{Vol}_s M$.

Let $f : M \to N$ be a smooth map of smooth manifolds. For a section ω of $\mathrm{Vol}_s N$ defined on V, we define the **pullback density** $f^*\omega : f^{-1}(V) \to \mathrm{Vol}_s M$ by means of

$$(f^*\omega)_p(X_1, \ldots, X_n) = \omega_{f(p)}(f_{*,p}X_1, \ldots, f_{*,p}X_n).$$

It is easy to see $f^*\omega$ defines a section of $\mathrm{Vol}_s M$ on $f^{-1}(V)$, and it is smooth as long as ω is.

Let $U \subseteq \mathbb{R}^n$ be an open set. On $\mathbb{R}^n$ there is a natural s-density λ determined by $\lambda(e_1, \ldots, e_n) = 1$. If $\omega : U \to \mathrm{Vol}_s(U)$ is a continuous (resp. smooth) section of compact support, there exists a continuous (resp. smooth) function $c : U \to \mathbb{R}$ such that $\omega_p = c(p)\lambda$. Indeed, $c(p)$ is the image of p under the composition $U \to \mathrm{Vol}_s(U) \cong U \times \mathbb{R} \xrightarrow{\mathrm{pr}_2} \mathbb{R}$, where we use id_U as the chart on U. Now define

$$\int_U \omega := \int_{\mathbb{R}^n} c(p) d\lambda(p)$$

where $d\lambda$ is the usual Lebesgue measure on $\mathbb{R}^n$.

Now let (U, φ) be a local chart of M, $\omega \in \mathrm{Vol}(M)$ with $\operatorname{supp}\omega \subseteq U$. Define

$$\int_M \omega := \int_{\varphi(U)} (\varphi^{-1})^*\omega.$$

To see this is well-defined let (V, ψ) be another local chart with $\operatorname{supp}\omega \subseteq V$. We must show

$$\int_{\varphi(U)} (\varphi^{-1})^*\omega = \int_{\psi(V)} (\psi^{-1})^*\omega.$$

Put $\theta = \psi \circ \varphi^{-1}$ to be the transition map. By definition we have $\theta^*(\psi^{-1})^*\omega = (\varphi^{-1})^*\omega$. Thus it suffices to show

$$\int_W \theta^*\eta = \int_{\theta(W)} g\eta$$

where $W, \theta(W)$ are opens in $\mathbb{R}^n$, $\theta : W \to \theta(W)$ is a diffeomorphism, $g \in C_c(\theta(W))$ and η is a density on $\theta(W)$. Write $\eta_p = c(p)\lambda$. Then

$$(\theta^*\eta)_q(X_1, \ldots, X_n) = c(\theta(q))\eta_{\theta(q)}(\theta_{*,q}X_1, \ldots, \theta_{*,q}X_n) = c(\theta(q))|\det \theta_{*,q}|\eta_{\theta(q)}(X_1, \ldots, X_n)$$

so that the desired identity now follows from from the change of variables rule on $\mathbb{R}^n$.

By a partition of unity argument, we obtain a unique linear functional

$$\int_M : \Gamma_c(M, \mathrm{Vol}(M)) \longrightarrow \mathbb{R}$$

satisfying $\displaystyle\int_M \omega = \int_{\mathbb{R}^n} cd\lambda$ as long as $\operatorname{supp}\omega$ lies in a local chart (U, φ) of M and $(\varphi^{-1})^*\omega = c\lambda$. In addition, if ω is a positive density, we see from the construction that $\displaystyle\int_M \omega \geqslant 0$.

If ω is a positive density, the composition

$$\begin{array}{ccccc} C_c(M) & \longrightarrow & \Gamma_c(M, \mathrm{Vol}(M)) & \xrightarrow{\int_M} & \mathbb{R} \\ f & \longmapsto & f\omega & & \end{array}$$

defines a positive linear functional on $C_c(M)$, so by Riesz's representation theorem, there exists a unique outer Radon measure $\mu = \mu_\omega$ on M such that

$$\int_M f d\mu = \int_M f\omega$$

for any $f \in C_c(M)$. Generally, if ω is an arbitrary density, we can define a positive density $|\omega|$ by setting $|\omega|_p := |\omega_p|$ for any $p \in M$, which is well-defined because of (♣), so it gives rise to a outer Radon measure $\mu_{|\omega|}$. If $|\omega|$ is smooth (e.g. ω is smooth and positive), then $|\omega|$ is a smooth measure in the sense of 17.5. Conversely, if μ is a smooth measure on M, then the Radon Nikodym derivatives of μ (with respect to the Lebesgue measure on any chart) glue to a smooth positive density on M.

F.7 Differential forms

Let V an n-dimensional vector space over a field k. Consider the space $A^p(V)$ of all alternating p-linear map on V. Clearly, $A^p(V) \cong \mathrm{Hom}_k(\bigwedge^p V, F) \cong \bigwedge^k V^\vee$. To put it explicitly, let $x^1, \ldots, x^n$ be a k-basis for V and put $dx^1, \ldots, dx^n$ to be the dual basis in $V^\vee$. For $I = \{i_1, \ldots, i_p\} \subseteq [n]$ with $1 \leqslant i_1 < i_2 < \cdots < i_p \leqslant n$, define $dx^I \in A^p(V)$ by $dx^I(x_{j_1} \wedge \cdots \wedge x_{j_p}) = \delta_{IJ}$ for all $1 \leqslant j_1 < \cdots < j_p \leqslant n$ and extending linearly. Then for $f \in A^p(V)$, we have $f = \sum\limits_{I:i_1<\cdots<i_p} f(x^{i_1}, \ldots, x^{i_p}) dx^I$. Thus

$$A^p(V) = \mathrm{span}_F\{dx^I \mid I : i_1 < \cdots < i_p\} \cong \bigwedge^p V^\vee$$

We then transfer the wedge product on $\bigwedge^p V^\vee$ to $A^p(V)$ via this bijection. Under this bijection, we have $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_p}$.

Let M be an n-dimensional smooth manifold. We have constructed its tangent bundle TM. Define the **cotangent bundle** $(TM)^\vee = T^*M$ to be the dual bundle of TM. Set-theoretically, $T^*M = \bigsqcup\limits_{p\in M} (T_pM)^\vee$ is the disjoint union of the **cotangent spaces**, and we have a natural projection $\pi : T^*M \to M$. Let U be a chart on M and $x^1, \ldots, x^n$ its local coordinates. Then for $p \in U$, $dx^1, \ldots, dx^n$ is a basis for $(T_pM)^\vee$ and we have a bijection $\pi^{-1}(U) \cong U \times \mathbb{R}^n$. In this way we turn T^*M into a $2n$-dimensional smooth manifold. In a similar fashion we construct the **k-th exterior power** of the cotangent bundle $\bigwedge^k T^*M := \bigsqcup\limits_{p\in M} \bigwedge^k (T_pM)^\vee$.

Denote by

$$\Omega^k(M) = \Gamma^\infty(M, \bigwedge^k T^*M)$$

the smooth global sections of the bundle $\bigwedge^k T^*M \to M$. An element in $\Omega^k M$ is called a **(smooth) k-form**. Locally, a k-form ω has the form $\sum a_I dx^I$, where the a_I are smooth functions. From this one sees $\Omega^k M$ is a module over the ring $C^\infty(M)$ of real-valued smooth functions on M. An element in $\Omega^n M$, where $n = \dim M$, is called a **top form** on M.

For $\omega \in \Omega^k M$ and $X_1, \ldots, X_k$ vector fields on M, define $\omega(X_1, \ldots, X_k)$ to be a (continuous) function $M \to \mathbb{R}$ by

$$\omega(X_1, \ldots, X_k)(p) := \omega_p(X_{1,p}, \ldots, X_{k,p})$$

The wedge product defined fibrewise gives a globally defined wedge product:

$$\wedge : \Omega^k M \times \Omega^\ell M \longrightarrow \Omega^{k+\ell} M$$

This makes $\bigoplus\limits_{k\geqslant 0} \Omega^k M$ an unital noncommutative associative $C^\infty(M)$-algebra.

For a smooth map $\varphi : M \to N$, define the **pullback map** $\varphi^* : \Omega^k N \to \Omega^k M$ by the formula

$$(\varphi^*\omega)_p(X_1, \ldots, X_k) = \omega_{\varphi(p)}(\varphi_{*,p}X_1, \ldots, \varphi_{*,p}X_k).$$

When $k = 0$ it is just the pullback map $C(N) \to C(M)$ given by $f \mapsto f \circ \varphi$. Let U, V be open sets in $\mathbb{R}^n$ and $\varphi : U \to V$ a smooth map. Then for $\omega = f dx^1 \wedge \cdots \wedge dx^n$, we have

$$(\varphi^*\omega)_p = f(\varphi(p)) \det \varphi_{*,p} \; dx^1 \wedge \cdots \wedge dx^n$$

Concisely, $\varphi^*\omega = (f \circ \varphi) \det \varphi_* \; dx^1 \wedge \cdots \wedge dx^n$

Let ω be a top form on M. For each $p \in M$, ω_p is a function $T_pM \times \cdots \times T_pM \to \mathbb{R}$ satisfying $\omega_p(AX_1, \ldots, AX_n) = (\det A)\omega_p(X_1, \ldots, X_n)$ for any $X_i \in T_pM$ and $A \in \mathrm{GL}(T_pM)$, where $\det A := \det(\varphi_{*,p} \circ A \circ \varphi_{*,p}^{-1})$ for any local chart φ about p. Clearly, the map $|\omega| : M \to \bigsqcup_{p \in M} \{T_pM \times \cdots \times T_pM \to \mathbb{R}\}$ defined by

$$|\omega_p|(X_1, \ldots, X_n) := |\omega_p(X_1, \ldots, X_n)|$$

is a continuous section of $\mathrm{Vol}(M)$, and it defines a positive density $|\omega| \in \Gamma(M, \mathrm{Vol}(M))$.

F.7.1 Exterior derivative

Let M be an n-dimensional smooth manifold. Define the **exterior derivative**

$$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

as follows. For $f \in \Omega^0(M) = C^\infty(M)$, define $df \in \Omega^1(M)$ by

$$df(X) := Xf.$$

For $k \geqslant 1$ and $\omega \in \Omega^k(M)$, locally write $\omega = f dx^I$, where $x^1, \ldots, x^n$ is a local coordinates and $I \subseteq [n]$ has size k. Then

$$d\omega := df \wedge dx^I.$$

Lemma F.7.1. Let $U \subseteq \mathbb{R}^n$ be an open set. Then $d : \Omega^k(U) \to \Omega^{k+1}(U)$ the only linear operator that satisfies the following properties.

(i) For $\omega \in \Omega^p(U)$, $\eta \in \Omega^q(U)$, we have $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta$.

(ii) For $f \in \Omega^0(U)$, we have $df(X) = Xf$.

(iii) $d \circ d = 0$.

In particular, the lemma shows that the exterior derivative is well-defined, i.e., independent of the choice of the local charts.

F.7.2 Lie derivatives of forms

Let M be a smooth manifold. For a vector field X and a k-form ω on M, the **Lie derivative** $\mathcal{L}_X\omega$ at $p \in M$ is defined by

$$(\mathcal{L}_X\omega)_p = \lim_{t \to 0} \frac{\varphi_t^*(\omega_{\varphi_t(p)}) - \omega_p}{t} = \frac{d}{dt}\bigg|_{t=0} (\varphi_t^*\omega)_p \in \bigwedge^k (T_pM)^\vee$$

where φ_t is a flow of X defined in a neighborhood of p.

Lemma F.7.2. For $f \in C^\infty(M)$ and a vector field X, we have

$$\mathcal{L}_X f = Xf.$$

Proof. Fix a $p \in M$ and a flow φ_t of X. Then

$$(\mathcal{L}_X f)_p = \left.\frac{d}{dt}\right|_{t=0} (\varphi_t^* f)_p = \left.\frac{d}{dt}\right|_{t=0} f(\varphi_t(p)) = X_p f$$

where the last equality is Proposition F.3.5. □

Lemma F.7.3. Let X be a smooth vector field.

(i) We have $\mathcal{L}_X \circ d = d \circ \mathcal{L}_X$.

(ii) For $\omega \in \Omega^p(M)$ and $\eta \in \Omega^q(M)$, we have

$$\mathcal{L}_X(\omega \wedge \eta) = \mathcal{L}_X \omega \wedge \eta + \omega \wedge \mathcal{L}_X \eta.$$

In other words, $\mathcal{L}_X$ defines a derivation on the algebra $\bigoplus_{p \geqslant 0} \Omega^p(M)$.

Theorem F.7.4. For $\omega \in \Omega^k(M)$ and $X, Y_1, \ldots, Y_k \in \mathfrak{X}(M)$, we have

$$\mathcal{L}_X(\omega(Y_1, \ldots, Y_k)) = (\mathcal{L}_X \omega)(Y_1, \ldots, Y_k) + \sum_{i=1}^{k} \omega(Y_1, \ldots, Y_{i-1}, \mathcal{L}_X Y_i, Y_{i+1}, \ldots, Y_k).$$

F.7.3 Interior multiplication

Let V be an n-dimensional vector space over a field k. For $v \in V$, define the operator

$$\iota_v : A^p(V) \longrightarrow A^{p-1}(V)$$

by the formula

$$\iota_v \omega(v_1, \ldots, v_{p-1}) = \omega(v, v_1, \ldots, v_{p-1}).$$

when $p \geqslant 1$, and $\iota_v f :\equiv 0$ for any $f \in A^1(V) = V^\vee$. This is called the **interior multiplication by** v, and this gives a linear map

$$\begin{aligned} \iota : V &\longrightarrow \operatorname{End} \textstyle\bigwedge V^\vee \\ v &\longmapsto \iota_v. \end{aligned}$$

Lemma F.7.5. For $v \in V$ and $\alpha^1, \ldots, \alpha^p \in V^\vee$, we have

$$\iota_v(\alpha^1 \wedge \cdots \wedge \alpha^p) = \sum_{i=1}^{p} (-1)^{i-1} \alpha^i(v) \alpha^1 \wedge \cdots \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^k.$$

Theorem F.7.6 (Cartan formula)**.** For a smooth vector field X on a manifold, we have

$$\mathcal{L}_X = d \circ \iota_X + \iota_X \circ d.$$

Corollary F.7.6.1. Let $k \geqslant 1$. For $\omega \in \Omega^k(M)$ and $X_0, \ldots, X_k \in \mathfrak{X}(M)$, we have

$$\begin{aligned} d\omega(X_0, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i X_i(\omega(X_0, \ldots, \widehat{X_i}, \ldots, X_k)) \\ + \sum_{0 \leqslant i < j \leqslant k} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k), \end{aligned}$$

Corollary F.7.6.2. Let ω be a form, $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$. Then

$$\mathcal{L}_{fX} \omega = f \mathcal{L}_X \omega + df \wedge \iota_X \omega.$$

F.8 Orientations and integration

Let V be an n-dimensional real vector space. There is a natural map $F(V) \to \bigwedge^n V^\vee \backslash \{0\}$ given by $\beta \mapsto d\beta := d\beta_1 \wedge \cdots \wedge d\beta_n$. We say two frames β and γ are equivalent, and write $\beta \sim \gamma$, if $d\beta$ and $d\gamma$ lies in the same connected component in $\bigwedge^n V^\vee \backslash \{0\}$. An **orientation** is then an equivalence class of $\sim$ on $F(V)$, or equivalently, a connected component of $\bigwedge^n V^\vee \backslash \{0\}$. It follows that V admits precisely two orientations.

Let M be an n-dimensional smooth manifold. A continuous local section of the tangent frame bundle FM defined over U is called a **(continuous) frame** on U. For an open set $U \subseteq M$, we say two frames X and Y are equivalent, and write $X \sim Y$, if $X_p \sim Y_p$ for any $p \in M$. Note that if U is a local chart on M, then $X = (X_1, \ldots, X_n)$ for some continuous vector fields X_i on U. An **orientation** on M is a collection $\mu = (\mu_p)_{p \in M}$ such that each μ_p is an orientation on T_pM and for each $p \in M$, there are an open neighborhood U and a frame X on U on U satisfying $X_p \in \mu_p$. In other words, we require an orientation on M is locally represented by a continuous frame.

Definition.

(i) A smooth manifold that admits an orientation is called **orientable.**

(ii) A smooth manifold with an orientation is called **oriented**.

Suppose M is connected, and let μ, ν be two orientations on M. Define $f : M \to \{\pm 1\}$ by $f(p) = \begin{cases} 1 & \text{, if } \mu_p = \nu_p \\ -1 & \text{, if } \mu_p \neq \nu_p \end{cases}$. Let $(U, x^1, \ldots, x^n)$ be a connected local chart of M on which, say, μ and ν are represented by frames X and Y, respectively. Then on U, we can write $d(Y_p) = \alpha_Y(p) dx^1 \wedge \cdots \wedge dx^n$ and $d(X_p) = \alpha_X(p) dx^1 \wedge \cdots \wedge dx^n$ for some non-vanishing continuous functions α_X, α_Y on U. Hence $d(Y_p) = (\alpha_Y \alpha_X^{-1})(p) d(X_p)$ with $\alpha_Y \alpha_X^{-1}$ continuous and non-vanishing. Hence $p \mapsto \operatorname{sgn}(\alpha_Y \alpha_X^{-1})(p)$ is constant on U, which implies that f is locally constant on M. As we assume M is connected, f is constant on the whole M. Hence either $\mu = \nu$ or $\mu = -\nu$. This proves

Lemma F.8.1. There are precisely two orientations on a connected orientable smooth manifold.

Suppose M is oriented by μ. Let $(U, x^1, \ldots, x^n)$ be a connected local coordinates of M, and suppose μ is represented by $(X_1, \ldots, X_n)$ on μ. Write $X_i = \sum\limits_{j=1}^n a_{ij} \dfrac{\partial}{\partial x^j}$ so that $dx^1 \wedge \cdots \wedge dx^n(X_1, \ldots, X_n) = \det(a_{ij})$. Replacing x^1 by $-x^1$ if necessary, we may assume $\det(a_{ij}) > 0$. Let $\{(U_\alpha, x_\alpha^1, \ldots, x_\alpha^n)\}_\alpha$ be such a collection of local charts with $\{U_\alpha\}$ being a cover of M. Let $(\psi_\alpha)_\alpha$ be a (smooth) partition of unity subordinate to $\{U_\alpha\}$ as in F.5, and define

$$\omega = \sum_\alpha \psi_\alpha dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n.$$

This is a well-defined smooth top form on M. For $p \in M$, we have $\psi_\alpha(p) \geqslant 0$ for any α and > 0 for at least one α, and thus

$$\omega_p(X_{1,p}, \ldots, X_{n,p}) = \sum_\alpha \psi_\alpha(p)(dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n)_p(X_{1,p}, \ldots, X_{n,p}) > 0$$

i.e., ω is a (smooth) volume form on M.

Conversely, let ω be a (smooth) volume form on M. At each point $p \in M$ choose an ordered basis $(X_{1,p}, \ldots, X_{n,p})$ of T_pM such that $\omega_p(X_{1,p}, \ldots, X_{n,p}) > 0$. Let $(U, \varphi, x^1, \ldots, x^n)$ be a connected local coordinates of M, and write $(\varphi^{-1})^*\omega = f dx^1 \wedge \cdots \wedge dx^n$ for a smooth nonvanishing function

f; by suitable replacement we may assume $f > 0$ on U. Write $X_i = \sum_{j=1}^{n} a_{ij}\dfrac{\partial}{\partial x^j}$ on U; then by our choice, we have $f(\varphi(p))\det(a_{ij}(\varphi(p))) > 0$ for every $p \in U$, or $\det(a_{ij}(\varphi(p))) > 0$, so $(X_{1,p}, \ldots, X_{n,p}) \sim \left(\dfrac{\partial}{\partial x^1}\Big|_p, \ldots, \dfrac{\partial}{\partial x^n}\Big|_p\right)$ for $p \in U$. Hence $p \mapsto (X_{1,p}, \ldots, X_{n,p})$ defines a orientation on M. Thus

Lemma F.8.2. A smooth manifold is orientable if and only if it admits a (smooth) volume form.

An easy argument shows that if ω and ω' are two smooth volume forms on M, then $\omega = f\omega'$ for a unique nowhere vanishing smooth function f on M. Say $\omega \sim \omega'$ if $\omega = f\omega'$ for some smooth $f > 0$ on M. This sets up an equivalence relations on the set of smooth volume forms on M, and it is bijection with the orientations on M. Thus an oriented smooth manifold can be described as a pair $(M, [\omega])$, where $[\omega] = \{f\omega \mid f \in C^\infty(M),\ f > 0\}$ is a equivalence class containing the volume form ω.

Definition. A diffeomorphism $\varphi : (M, [\omega]) \to (N, [\eta])$ of smooth manifolds is **orientation-preserving** (resp. **orientation-reversing**) if $[\varphi^*\omega] = [\eta]$ (resp. $[\varphi^*\omega] = [-\eta]$).

Let $(M, [\omega])$ be an oriented smooth manifold. Let $(U, x^1, \ldots, x^n)$ and $(V, y^1, \ldots, y^n)$ be two local charts on M intersecting nontrivially. Assume $[dx^1 \wedge \cdots \wedge dx^n] = [\omega|_U]$ and $[dy^1 \wedge \cdots \wedge dy^n] = [\omega|_V]$. If we put θ to be the transition map from U to V, then $dx^1 \wedge \cdots \wedge dx^n = \det(\theta_*)dy^1 \wedge \cdots \wedge dy^n$ on $U \cap V$. A moment consideration shows that we must have $\det(\theta_*) > 0$ on $U \cap V$. Conversely, if M is a smooth manifold that admits an **oriented atlas**, i.e., a smooth atlas such that the Jacobian of every transition map is positive, one may construct (smooth) volume form ω on M by a (smooth) partition of unity argument (we always orient $\mathbb{R}^n$ by $dx^1 \wedge \cdots \wedge dx^n$) in a way that $[\omega|_U] = [dx^1 \wedge \cdots \wedge dx^n]$ for any oriented chart $(U, x^1, \ldots, x^n)$. Given an oriented atlas on M, there is a unique maximal oriented atlas (with respect to inclusion) containing it, and the set of all maximal oriented atlas is in bijection with the equivalence classes of smooth volume forms. Hence we can *specify* an orientation of M by a maximal oriented atlas.

For an oriented manifold M, we usually denote by $-M$ the same manifold but with *opposite orientation*. Precisely, if $(U, x^1, \ldots, x^n)$ is an oriented chart on M, then $(U, -x^1, x^2 \ldots, x^n)$ is set to be an oriented chart on $-M$.

Finally, we discuss the theory of integration via forms. Assume M is an oriented manifold, and we only use oriented charts on M. Denote by

$$\Omega_c^n(M) = \Gamma_c(M, \bigwedge\nolimits^n T^*M)$$

the space of continuous top forms with compact support. Similar to the argument for densities with compact support, we obtain

Proposition F.8.3. Let M^n be an oriented manifold. Then there exists a unique linear map

$$\int_M : \Omega_c^n M \longrightarrow \mathbb{R}$$

$$\alpha \longmapsto \int_M \alpha$$

such that if $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ is an oriented chart of M and if $\alpha \in \Omega_c^n M$ with $\operatorname{supp}\alpha \subseteq U$, then

$$\int_M \alpha = \int_{\mathbb{R}^n} (\varphi^{-1})^*\alpha := \int_{\mathbb{R}^n} a(x)dx^1 \cdots dx^n$$

where $(\varphi^{-1})^*\alpha = a(x)dx^1 \wedge \cdots \wedge dx^n$ and the last expression is the usual integration on $\mathbb{R}^n$.

Hence, for any top form ω on M, there is a linear functional

$$\begin{aligned} I_\omega : C_c(M) &\longrightarrow \Omega_c^n M \xrightarrow{\int_M} \mathbb{R} \\ f &\longmapsto f\omega \end{aligned}$$

On the other hand, recall that we can construct a positive density $|\omega|$ for any top form ω, which induces an outer Radon measure $\mu_{|\omega|}$ on M. So we have two theories for integration on M:

$$\int_M f\omega =: I_\omega(f) \qquad \text{and} \qquad \int_M f|\omega| =: \int_M f d\mu_{|\omega|}.$$

In the oriented case, there is another way to define $|\omega|$ in terms of forms but not densities. If $(U, \varphi, x^1, \ldots, x^n)$ is an oriented local chart, then $(\varphi^{-1})^*\omega = f dx^1 \wedge \cdots \wedge dx^n$ for some continuous function f defined on U. Define a form $|\omega|$ by requiring $(\varphi^{-1})^*|\omega| := |f| dx^1 \wedge \cdots \wedge dx^n$. This is well-defined since we only use oriented charts on M. Although two defined $|\omega|$ are totally different, it is clear that $I_{|\omega|} = \int_M d\mu_{|\omega|}$. Particularly, if ω is a volume form representing the orientation of M, then $|\omega| = \omega$ (this holds as we only use oriented charts), and the two ways to integrating functions coincide.

F.9 Stokes' theorem

F.9.1 Manifold with boundary

F.9.2 Stokes' theorem

Theorem F.9.1 (Stokes')**.** If M is an oriented smooth n-manifold with boundary ∂M and ω is a smooth $(n-1)$-form on M with compact support, then

$$\int_M d\omega = \int_{\partial M} \omega$$

F.9.3 Applications

In this subsection we redefine

$$\Omega_c^p(M) = \Gamma_c^\infty(M, \bigwedge^p T^*M)$$

to be the space of smooth p-forms with compact support.

Lemma F.9.2 (Integration by parts)**.** Let M be an oriented smooth n-manifold with boundary, X a smooth vector field, $\omega \in \Omega_c^p(M)$ and $\eta \in \Omega_c^q(M)$ with $p + q = n$. Then

$$\int_M \mathcal{L}_X \omega \wedge \eta = \int_{\partial M} \iota_X(\omega \wedge \eta) - \int_M \omega \wedge \mathcal{L}_X \eta.$$

Proof. By Lemma F.7.3.(ii), it suffices to show

$$\int_M \mathcal{L}_X(\omega \wedge \eta) = \int_{\partial M} \iota_X(\omega \wedge \eta).$$

By Cartan formula, we have

$$\mathcal{L}_X(\omega \wedge \eta) = \iota_X \circ d(\omega \wedge \eta) + d \circ \iota_X(\omega \wedge) = d \circ \iota_X(\omega \wedge \eta).$$

Hence

$$\int_M \mathcal{L}_X(\omega \wedge \eta) = \int_{\partial M} \iota_X(\omega \wedge \eta)$$

by Stokes' theorem. □

Corollary F.9.2.1. Let M be an oriented smooth n-manifold with boundary, X a smooth vector field, $\omega \in \Omega^n(M)$ a top form and $f, g \in C_c^\infty(M)$. Then

$$\int_M (Xf)g\omega = \int_{\partial M} fg \cdot \iota_X\omega - \int_M f(Xg)\omega - \int_M fg \cdot \mathcal{L}_X\omega$$

Appendix G

Riemannian geometry

G.1 Tensors

Let M^n be a smooth manifold. For $p, q \in \mathbb{Z}_{\geqslant 0}$, we can form the bundle

$$T^{p,q}M = \underbrace{TM \otimes \cdots \otimes TM}_{p \text{ times}} \otimes \underbrace{TM^\vee \otimes \cdots \otimes TM^\vee}_{q \text{ times}}$$

A **(smooth)** (p,q)**-tensor** defined on an open subspace $U \subseteq M$ is then a (smooth) section $t : U \to T^{p,q}M$ of the bundle projection $\pi^{p,q} : T^{p,q}M \to M$.

- A $(0,0)$-tensor is simply a smooth function on M.
- We call a $(0,k)$-tensor a **covariant** k**-tensor**.
- We call a $(k,0)$-tensor a **contravariant** k**-tensor**.

Let ω be a covariant k-tensor with $k > 0$. For each $p \in M$, $\omega_p \in T_pM^\vee \otimes \cdots \otimes T_pM^\vee$ can be viewed canonically as a linear functional

$$\omega_p : T_pM \otimes \cdots \otimes T_pM \to \mathbb{R}$$

We say ω is a **symmetric** (resp. **alternating**) tensor if ω_p is symmetric (resp. alternating) for each $p \in M$. For each covariant k-tensor ω, we define

$$\mathrm{sym}^k\omega(v_1, \ldots, v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

which is a symmetric k-tensor. Likewise, we define

$$\mathrm{alt}^k\omega(v_1, \ldots, v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

G.2 Affine connections

Recall that for a smooth manifold M, the symbol $\mathfrak{X}(M)$ denotes the set of all smooth vector fields on M. For $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$, we can define $fX \in \mathfrak{X}(M)$ by $(fX)_p(g) = f(p)X_p(g)$ for any $g \in C^\infty(M)$ and $p \in M$. In this way $\mathfrak{X}(M)$ becomes a $C^\infty(M)$-module.

Definition. Let M be a smooth manifold. An **affine connection** is a $C^\infty(M)$-module homomorphism $\nabla : \mathfrak{X}(M) \to \operatorname{End}_{\mathbb{R}} \mathfrak{X}(M)$ satisfying the **Leibniz's rule**:

$$\nabla_X(fY) = f\nabla_X(Y) + (Xf)Y$$

for $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, where $\nabla_X \in \operatorname{End}_{\mathbb{R}} \mathfrak{X}(M)$ usually denotes the image of X under ∇.

Let U be any open subspace of M. An affine connection ∇ on M induces an affine connection ∇_U on U: let $X, Y \in \mathfrak{X}(U)$ and $p \in U$. Pick any $X', Y' \in \mathfrak{X}(M)$ such that X', Y' agree with X, Y on a neighborhood V of p in U. For $q \in V$, define

$$((\nabla_U)_X(Y))_q := (\nabla_{X'}(Y'))_q$$

The following lemma guarantees that the right hand side is independent of the choice of X', Y'.

Lemma G.2.1. Let ∇ be an affine connection on M, U an open subspace and $X, Y \in \mathfrak{X}(M)$. If $X|_U = 0$ or $Y|_U = 0$, then $\nabla_X(Y)|_U = 0$.

Proof. Let $p \in U$ and $g \in C^\infty(M)$. By Urysohn's lemma, we can find $f \in C^\infty(M)$ such that $f(p) = 0$ and $f|_{M \setminus U} = 1$. If $X|_U = 0$, then $fX = X$, and

$$\nabla_X(Y)g(p) = \nabla_{fX}(Y)g(p) = f(p)\nabla_X(Y)g(p) = 0.$$

Similarly, if $Y|_U = 0$, then $fY = Y$, and

$$\nabla_X(Y)g(p) = \nabla_X(fY)g(p) = f(p)\nabla_X(Y)g(p) + (X_p f)(Y_p g) = 0$$

$\square$

Take any local chart $(U, x^1, \dots, x^n)$ of M. Then on U,

$$(\nabla_U)_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{k=1}^n {\Gamma_{ij}}^k \frac{\partial}{\partial x^k} \tag{$\clubsuit$}$$

for some (uniquely defined) smooth functions ${\Gamma_{ij}}^k \in C^\infty(U)$. If $y^1, \dots, y^n$ is another local coordinates of U, we obtain another collection of smooth functions $\{{\Gamma'_{ij}}^k\}$ by

$$(\nabla_U)_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = \sum_{k=1}^n {\Gamma'_{ij}}^k \frac{\partial}{\partial y^k}.$$

Using Leibniz's rule and chain rule, a tedious computation shows that

$${\Gamma'_{\alpha\beta}}^\gamma = \sum_{i,j,k} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta} \frac{\partial y^\gamma}{\partial x^k} {\Gamma_{ij}}^k + \sum_j \frac{\partial^2 x^j}{\partial y^\alpha \partial y^\beta} \frac{\partial y^\gamma}{\partial x^j}. \tag{$\spadesuit$}$$

On the other hand, if $\mathcal{U}$ is an open cover of M and on each $U \in \mathcal{U}$ is a collection of function ${\Gamma_{ij}}^k$ satisfying ($\spadesuit$) whenever two open sets in $\mathcal{U}$ overlap, we can define an affine connection ∇_U on each $U \in \mathcal{U}$ by ($\clubsuit$), and hence an affine connection ∇ on M by

$$\nabla_X(Y)_p = (\nabla_U)_{X'}(Y')_p$$

where $p \in M$, $U \in \mathcal{U}$ is any open set containing p, and X', Y' are vector fields on U obtained from restriction of X, Y to U.

Lemma G.2.2. Let $X, Y \in \mathfrak{X}(M)$ and suppose $X_p = 0$ for some $p \in M$. Then $\nabla_X(Y)_p = 0$ as well.

Proof. Let $(U, x^1, \ldots, x^n)$ be a local chart near p, and write $X = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x^i}$. Then $f_i(p) = 0$, and

$$\nabla_X(Y)_p = \sum_{i=1}^{n} f_i(p) \nabla_{\frac{\partial}{\partial x^i}}(Y)_p = 0.$$

□

In particular, if $v \in T_pM$ and $Y \in \mathfrak{X}(M)$, we may define

$$\nabla_v(Y) := \nabla_X(Y)_p \in T_pM$$

where X is any smooth vector field on X with $X_p = v$ (this always exists by, for example Urysohn's lemma). In other words, for each p, an affine connection ∇ on M induces a well-defined $\mathbb{R}$-linear map

$$\nabla : T_pM \longrightarrow \mathrm{Hom}_{\mathbb{R}}(\mathfrak{X}(M), T_p(M))$$

Let M be a smooth manifold and ∇ an affine connection on M. We can regard ∇ as a $\mathfrak{X}(M)$-valued $\mathbb{R}$-linear pairing:

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

that is $C^\infty(M)$-linear in the first argument and satisfies the Leibniz's rule in the second argument. In other word, ∇ is an $\mathbb{R}$-linear map

$$\nabla : \Gamma(TM) \longrightarrow \Gamma((TM)^\vee \otimes TM)$$

that satisfies the Leibniz's rule in the sense that

$$\nabla(fX) = df \otimes X + f\nabla(X)$$

for any $f \in C^\infty(M)$ and $X \in \Gamma(TM) = \mathfrak{X}(M)$. Here for a smooth vector bundle $E \to M$, we denote by $\Gamma(E)$ its smooth global sections. Generally,

Definition. Let $E \to M$ be a smooth vector bundle. A **bundle connection** is an $\mathbb{R}$-linear map

$$\nabla : \Gamma(E) \longrightarrow \Gamma((TM)^\vee \otimes E)$$

satisfying

$$\nabla(fs) = df \otimes s + f\nabla(s)$$

for any $f \in C^\infty(M)$ and $s \in \Gamma(E)$.

In this language, an affine connection on M is simply a bundle connection on the tangent bundle TM of M. Similar to the situation above, a bundle connection can also be seen as a map $\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$, or $\nabla : \Gamma(TM) \to \mathrm{End}_{\mathbb{R}} \Gamma(E)$.

Suppose $f : M \to N$ is a diffeomorphism and ∇ is an affine connection on N. We can pullback ∇ to an affine connection $f^*\nabla$ on M by

$$(f^*\nabla)_X(Y) := (f^{-1})_*(\nabla_{f_*X}(f_*Y)).$$

If $M = N$, i.e., f is a diffeomorphism on M, we say f is an **affine transformation** if $f^*\nabla = \nabla$, i.e., ∇ is ***f*-invariant**.

Definition. Let G be a Lie group. An affine connection ∇ is called **left-invariant** if for each $g \in G$, the left translation $\ell_g : G \to G$ by g is an affine transformation.

- Let $X_1, \ldots, X_n$ be a basis of L_G, the space of left-invariant vector fields on G. Then if ∇ is left-invariant, each $\nabla_{X_i} X_j$ is left-invariant as well. In fact, for each $g \in G$,

$$(\ell_g)_* \nabla_{X_i} X_j = (\ell_g)_*(\nabla_{(\ell_{g^{-1}})_* X_i} (\ell_{g^{-1}})_* X_j) = ((\ell_g)^* \nabla)_{X_i}(X_j) = \nabla_{X_i}(X_j)$$

- We can define an affine connection ∇ on G by requiring $\nabla_{X_i}(X_j)$ to be any left invariant vector fields. For any vector fields X, Y, write $X = \sum_{i=1}^{n} f_i X_i$ and $Y = \sum_{i=1}^{n} g_i X_i$ for some smooth functions $f_i, g_i \in C^\infty(G)$. Then

$$\nabla_X Y := \sum_{i,j} f_i \nabla_{X_i}(g_j X_j) := \sum_{i,j} (f_i X_i(g_j) X_j + g_j \nabla_{X_i} X_j)$$

so for any $g \in G$,

$$\nabla_{(\ell_g)_* X}((\ell_g)_* Y) = (\ell_g)_*(\nabla_X Y)$$

Hence, such defined affine connection ∇ is left-invariant. In particular, if we recall $\mathrm{Lie}(G) \cong L_G$, it follows that there is a bijection:

$$\begin{aligned} \{\text{left-invariant affine connections on } G\} &\longrightarrow \mathrm{Hom}_{\mathbb{R}}(\mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathrm{Lie}(G), \mathrm{Lie}(G)) \\ \nabla &\longmapsto [(X, Y) \mapsto \nabla_{X'}(Y')_e] \end{aligned}$$

where $X', Y' \in L_G$ are the left-invariant vector fields with $X'_e = X$, $Y'_e = Y$.

G.3 Parallelism

Definition. Let M be a smooth manifold with an affine connection ∇. A vector field $X \in \mathfrak{X}(M)$ is said to be **parallel along a curve** $\gamma : I \to M$ if

$$\nabla_{\gamma'(t)} X = 0 \in T_{\gamma(t)} M$$

for any $t \in I$.

Let X, Y be two vector fields and $(U, x^1, \ldots, x^n)$ a local chart. Write $X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}$ and $Y = \sum_{i=1}^{n} Y^i \frac{\partial}{\partial x^i}$. Then

$$\begin{aligned} \nabla_X Y = \sum_{j=1}^{n} \nabla_X Y^j \frac{\partial}{\partial x^j} &= \sum_{j=1}^{n} X(Y^j) \frac{\partial}{\partial x^j} + \sum_{j=1}^{n} Y^j \nabla_X \frac{\partial}{\partial x^j} \\ &= \sum_{j=1}^{n} X(Y^j) \frac{\partial}{\partial x^j} + \sum_{i,j,k=1}^{n} X^i Y^j {\Gamma_{ij}}^k \frac{\partial}{\partial x^k} \\ &= \sum_{k=1}^{n} \left(X(Y^k) + \sum_{i,j=1}^{n} X^i Y^j {\Gamma_{ij}}^k \right) \frac{\partial}{\partial x^k} \end{aligned}$$

In particular, for a fixed vector field X and $p \in U$, the vector $(\nabla_X Y)_p$ only depends on the $X_p(Y^j)$ and $Y^j(p)$.

Let $\gamma : I \to M$ be a curve, and write $\gamma'(t) = \sum_{i=1}^{n} a_i(t) \left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)}$ in U. Fix $s \in I$. We can always find smooth functions $f_1, \ldots, f_n$ defined near $p = \gamma(s)$ satisfying

$$(x^i \circ \gamma)(s) = f_i(s) \text{ and } X_p f_i = a_i(s) \text{ for } 1 \leqslant i \leqslant n.$$

Form a vector field $Y = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x^i}$. Then

$$\begin{aligned}
(\nabla_X Y)_p &= \sum_{k=1}^{n} \left(X_p(f_k) + \sum_{i,j=1}^{n} X^i(p) f_j(p) {\Gamma_{ij}}^k(p) \right) \left.\frac{\partial}{\partial x^k}\right|_p \\
&= \sum_{k=1}^{n} \left(a_i(s) + \sum_{i,j=1}^{n} X^i(p) (x^j \circ \gamma)(s) {\Gamma_{ij}}^k(p) \right) \left.\frac{\partial}{\partial x^k}\right|_p .
\end{aligned}$$

The right hand side is independent of the choice of the auxiliary vector field Y. By abuse of notation, denote this value by $(\nabla_X \gamma')_{\gamma(s)}$.

Definition. A curve $\gamma : I \to M$ is called a **geodesic** if

$$(\nabla_{\gamma'} \gamma')_{\gamma(t)} = 0 \in T_{\gamma(t)} M$$

for any $t \in I$.

Appendix H

Lie Algebras

Definition. A **Lie algebra over a field** k is a k-vector space L together with an alternating k-bilinear pairing $[,] : L \times L \to L$, called the **Lie bracket**, satisfying the **Jacobi's identity**:

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$

for all $x, y, z \in L$.

- If A is an associative k-algebra, then $[,] : A \times A \to A$ defined by $[x,y] := xy - yx$ makes A a Lie algebra over k. Hence an associative algebra is naturally a Lie algebra.

Definition. Let L, L' be two Lie algebras over a field k. A **Lie algebra homomorphism** $\varphi : L \to L'$ is a k-linear map that preserves the Lie brackets:

$$\varphi([x,y]) = [\varphi(x), \varphi(y)]$$

for all $x, y \in L$.

- Thus we have constructed a category $\mathbf{LieAlg}_k$ of Lie algebras over k. A Lie algebra isomorphism is then an isomorphism in this category, which is precisely a Lie algebra homomorphism that is also a k-linear isomorphism.
- For each $x \in L$ we associate it with an endomorphism $\mathrm{ad}_L(x) : L \to L$ defined by $\mathrm{ad}_L(x)y := [x,y]$. The map

$$\begin{aligned} \mathrm{ad}_L : L &\longrightarrow \mathrm{End}_k L \\ x &\longmapsto \mathrm{ad}_L(x) \end{aligned}$$

  is called the **adjoint representation** of the Lie algebra L. By Jacobi's identity, ad is a Lie algebra homomorphism. If the Lie algebra L is clear from the context, we also write ad_x and $\mathrm{ad}(x)$ for $\mathrm{ad}_L(x) = \mathrm{ad}_L x$.

Definition. Let L be a Lie algebra over a field k.

1. A **Lie subalgebra** of L is a k-linear subspace H of L such that $[x,y] \in H$ whenever $x, y \in H$.
2. A **Lie ideal** of L is a Lie subalgebra I such that $[x,y] \in I$ whenever $x \in L$, $y \in I$.

- If I is a Lie ideal of L, we write $I \trianglelefteq L$.
- If I is a Lie ideal, then the quotient space L/I is naturally a Lie algebra over k.

Proposition H.0.1. The usual isomorphism theorems hold in $\mathbf{LieAlg}_k$. Precisely:

1. Let $\varphi \in \mathrm{Hom}_{\mathbf{LieAlg}_k}(L, L')$. Then $\ker\varphi \trianglelefteq L$, and for $N \trianglelefteq L$ contained in $\ker\varphi$, the canonical map $L/N \to L'$ is a Lie algebra homomorphism.

2. If H is a Lie subalgebra of L and $I \trianglelefteq L$, then $H + I$ is a Lie subalgebra of L, $H \cap I$ is a Lie ideal of H, and the natural homomorphism $H/H \cap I \cong H + I/I$ is a Lie algebra isomorphism.

3. If $I, J \trianglelefteq L$ with $I \subseteq J$, then J/I is a Lie ideal of L/I, and the natural homomorphism $L/J \cong (L/I)/(J/I)$ is a Lie algebra isomorphism.

4. If $I \trianglelefteq L$, then the set of Lie ideals in L containing I is in bijection with the set of Lie ideals in L/I.

Definition. Let A be a (possibly nonassociative) k-algebra. A (k-linear) **derivation** is a k-linear map $D : A \to A$ satisfying the Leibniz's rule: $D(xy) = xD(y) + D(x)y$ for all $x, y \in A$. Write $\mathrm{Der}_k A$ for the space of derivations on A, which is a subspace of $\mathrm{End}_k A$.

- If L is a Lie algebra over k, then

$$\mathrm{Der}_k L := \{D \in \mathrm{End}_k L \mid D([x, y]) = [Dx, Dy] \text{ for all } x, y \in L\}.$$

  In fact, $\mathrm{Der}_k L$ is a Lie subalgebra of $\mathrm{End}_k L$.

- Thanks to Jacobi's identity we see the image $\mathrm{ad}(L)$ of the adjoint representations $\mathrm{ad} : L \to \mathrm{End}_k L$. An element in $\mathrm{ad}(L)$ is called an **inner derivation**.

- If $\delta \in \mathrm{Der}_k L$ and $x \in L$, then $\mathrm{ad}_L(x) = \mathrm{ad}_L(\delta(x))$. In particular, $\mathrm{ad}(L) \subseteq \mathrm{Der}_k L$ is a Lie ideal.

Definition. Let L be a Lie algebra over k.

1. $Z(L) = \{x \in L \mid [x, L] = 0\}$ is the **center** of L.

2. For a subset $S \subseteq L$, $C_L(S) = \{x \in L \mid [x, S] = 0\}$ is the **centralizer** of S in L, which is a Lie subalgebra of L (by Jacobi's identity).

3. For a subset $S \subseteq L$, $N_L(S) = \{x \in L \mid [x, S] \subseteq S\}$ is the **normalizer** of S in L, which is a Lie subalgebra (also by Jacobi's identity).

4. $[L, L] = \mathrm{span}_k\{[x, y] \mid x, y \in L\}$ is called the **commutator / derived subalgebra** of L.

5. Put $L^{(1)} = L$ and $L^{(n)} = [L^{(n-1)}, L^{(n-1)}]\,(n \geqslant 2)$. The sequence

$$L = L^{(1)} \supseteq L^{(2)} \supseteq \cdots \supseteq L^{(n)} \supseteq \cdots \supseteq \{0\}$$

   is the **derived series** of L. If $L^{(n)} = 0$ for some $n \geqslant 1$, then L is called **solvable**.

6. Put $L^1 = L$ and $L^n = [L, L^{n-1}]\,(n \geqslant 2)$. The sequence

$$L = L^1 \supseteq L^2 \supseteq \cdots \supseteq L^n \supseteq \cdots \supseteq \{0\}$$

   is the **lower central series** of L. If $L^n = 0$ for some $n \neq 1$, then L is called **nilpotent**.

7. L is called **abelian** if $[L, L] = 0$.

8. L is called **simple** if $[L, L] \neq 0$ and L contains no proper nonzero Lie ideal.

Lemma H.0.2. Let L be a Lie algebra over k.

1. If L is nilpotent, then L is solvable.
2. If L is solvable (resp. nilpotent), then every homomorphic image of L is solvable (resp. nilpotent).
3. If $I \trianglelefteq L$ is a solvable (resp. nilpotent) Lie ideal and $K \subseteq L$ is a solvable (resp. nilpotent) Lie subalgebra, then $I + K$ is solvable (resp. nilpotent).
4. If $L/Z(L)$ is nilpotent, then L is nilpotent.

Definition. Let L be a Lie algebra over k. Since the sum of two solvable Lie ideals of L is again a solvable Lie ideal, there exists a maximal solvable Lie ideal in L. This is called the radical

H.1 Universal enveloping algebras

In this subsection, let k be any field and L an (possibly infinite dimensional) Lie algebra over k. Notice that a associative algebra A is naturally a Lie algebra: $(a, b) \times A \times A \mapsto ab - ba$ is clearly a Lie bracket.

Definition. The universal enveloping algebra of L is an associative unital algebra $U(L)$ over k together with a k-linear map $\iota : L \to U(L)$ representing the functor $\mathbf{Alg}_k \ni A \mapsto \mathrm{Hom}_{\mathbf{LieAlg}_k}(L, A)$. In other words, the map

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Alg}_k}(U(L), A) &\longrightarrow \mathrm{Hom}_{\mathbf{LieAlg}_k}(L, A) \\ \phi &\longmapsto \phi \circ \iota \end{aligned}$$

is a bijection functorial in A. Here $\mathbf{Alg}_k$ denotes the category of unital associative k-algebras.

The existence can be easily established. Indeed, let J be the two sided ideal of the tensor algebra TL generated by the elements $x \otimes y - y \otimes x - [x, y]\,(x, y \in L)$. Put

$$U(L) = TL/J$$

and denote by $\iota : L \to U(L)$ the composition of natural maps $L \to TL \to U(L)$. One readily checks that $(U(L), \iota)$ really represents the functor described above.

Let $\pi : TL \to U(L)$ be the natural projection. Put $T_m L = \bigoplus_{0 \leqslant n \leqslant m} T^n L$ and $U_m = \pi(T_m)$; for convenience, put $U_{-1} := 0$. In this way we obtain a filtration

$$0 = U_{-1} \subseteq k = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_m \subseteq U_{m+1} \subseteq \cdots \subseteq U(L)$$

of the universal enveloping algebra $U(L)$. The filtration respects the multiplication: $U_p U_q \subseteq U_{p+q}$, which makes the associated $\mathbb{Z}_{\geqslant 0}$-graded abelian group

$$G = \bigoplus_{m \geqslant 0} G^m := \bigoplus_{m \geqslant 0} U_m/U_{m-1}$$

a unital associative k-algebra.

Since $\pi(T^m L) \subseteq \pi(T_m) = U_m$, the composition $\phi_m : T^m L \to U_m \to G^m$ of natural maps is well-defined. As $\pi(T_m \backslash T_{m-1}) = U_m \backslash U_{m-1}$, the map ϕ_m is surjective. The maps $\phi_m\,(m \geqslant 0)$ then together define a surjective linear map $\phi : TL \to G$ that sends 1 to 1. It is easy to see that ϕ is an algebra homomorphism. Moreover,

Lemma H.1.1. $\phi(x \otimes y - y \otimes x) = 0 \in G$ for any $x, y \in L$.

Proof. By definition, $\pi(x \otimes y - y \otimes x) \in U_2$. On the other hand,

$$\pi(x \otimes y - y \otimes x) = \pi(x)\pi(y) - \pi(y)\pi(x) = [\pi(x), \pi(y)] = \pi([x, y]) \in U_1,$$

so that $\phi(x \otimes y - y \otimes x) \in U_1/U_1 = 0$. $\square$

By the lemma, ϕ factors through the projection $TL \to \text{Sym } L$, inducing a unital k-algebra homomorphism $\omega : \text{Sym } L \to G$.

Theorem H.1.2 (Poincaré-Birkhoff-Witt; PBW)**.** The homomorphism

$$\omega : \text{Sym } L \to G$$

is an isomorphism of (graded) k-algebras.

Appendix I

Lie Groups

I.1 Lie Groups and Examples

Definition. A **Lie group** G is a smooth manifold G which is also an abstract group such that the multiplication $\mu : G \times G \to G$ and the inversion $\mathrm{inv} : G \to G$ are smooth.

- For $g \in G$, let $\ell_g : G \to G$ be defined by $\ell_g(x) = gx$. Similarly, define $r_g : G \to G$ by $r_g(x) = xg$.

A **homomorphism of two Lie groups** $G \to H$ is a smooth map that is also a group homomorphism.

Lemma I.1.1. Let G be a Lie group and e be the identity element of G.

1. $\mu_{*,(e,e)} : T_eG \times T_eG \to T_eG$ is just addition, and $\mathrm{inv}_{*,e} : T_eG \to T_eG$ is multiplication by -1.
2. In the definition of a Lie group, we do not need to assume the smoothness of inv.

Proof.

1. Let $\gamma : (-\varepsilon, \varepsilon) \to G$ be a curve with $\gamma(0) = e$ and put $X := \gamma_{*,e}\left(\left.\frac{d}{dt}\right|_{t=0}\right) \in T_eG$. We compute $\mu_*(X, 0)$, and the result will ensue by linearity. Define $\tilde{\gamma}(t) := (\gamma(t), e)$. We have
$$\mu_*(X,0)f = \mu_*\tilde{\gamma}_*\left(\left.\frac{d}{dt}\right|_{t=0}\right)f = \left.\frac{d}{dt}\right|_{t=0} f((\mu \circ \gamma)(t)) = \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) = \gamma_*\left(\left.\frac{d}{dt}\right|_{t=0}\right)f = Xf$$
so that $\mu_*(X, 0) = X$.

   The second statement follows from chain rules. We have $\mu(\gamma(t), \mathrm{inv}(\gamma(t))) = e$, so taking differential, we have
$$0 = \left.\frac{d}{dt}\right|_{t=0} \mu(\gamma(t), \mathrm{inv}(\gamma(t))) = \mu_*(\mathrm{id}, \mathrm{inv})_*\gamma_*\left(\left.\frac{d}{dt}\right|_{t=0}\right) = \mu_*(\mathrm{id}, \mathrm{inv}_*)X = X + \mathrm{inv}_*X$$
so that $\mathrm{inv}_*X = -X$.

2. It suffices to show there exists a unit-neighborhood U of G such that $\mathrm{inv}|_U$ is smooth. Indeed, for $g \neq 1 \in G$, the diagram
$$\begin{array}{ccc} gU & \xrightarrow{\mathrm{inv}|_{gU}} & G \\ \downarrow{\scriptstyle \ell_{g^{-1}}} & & \downarrow{\scriptstyle r_g} \\ U & \xrightarrow{\mathrm{inv}|_U} & G \end{array}$$

commutes, and since ℓ_g and r_g are diffeomorphisms, it follows that $\mathrm{inv}|_{gU}$ is smooth as well. Letting g run over G proves the global smoothness of inv.

To find U, we use Implicit function theorem. By 1, μ_* is an isomorphism in the second argument, so we can find open unit-neighborhoods U, V and a smooth map $\phi : U \to V$ such that $\mu(g, \phi(g)) = e$ for $g \in U$. But then ϕ coincides with inv on U, so inv is smooth as well.

□

Example I.1.2 (0-dimensional Lie groups)**.** All finite groups with discrete topology are zero-dimensional Lie groups. Some important examples are the symmetric groups S_n, the alternating groups A_n and the cyclic groups $C_n = \mathbb{Z}/n$.

Example I.1.3 (Direct product)**.** If G, H are Lie groups, so is their direct product $G \times H$ with usual group structure and smooth structure. For instance, the n-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n \cong (\mathbb{R}/\mathbb{Z})^n \cong (S^1)^n$ is a Lie group.

Example I.1.4 (Identity component)**.** Let G be a Lie group and G^0 be the **identity component** of G, i.e., the connected component containing the identity element e. Then

- G^0 is a normal subgroup of G. Since $\mathrm{inv} : G \to G$ is a homeomorphism, $\mathrm{inv}\, G^0$ is a connected subset of G containing e, so $\mathrm{inv}\, G^0 \subseteq G^0$. Now for $g \in G^0$, gG^0 is connected and contains e as well, so $gG^0 \subseteq G^0$. The normality is shown in the same fashion.

- G^0 is open. Indeed, this follows from the fact that G^0 is locally connected.

Hence G^0 is an open (and closed) subgroup of G, so it is itself a Lie group.

Example I.1.5 (Matrix groups)**.** Let V be a finite dimensional real/complex vector space. The group of linear isomorphisms $\mathrm{Aut}\, V$ is an open subspace of $\mathrm{End}\, V$, and hence it has a smooth structure. Taking a basis for V, we see that the multiplication on $\mathrm{Aut}\, V$ is just a polynomial map, so $\mathrm{Aut}\, V$ is a Lie group. Hence $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$ are Lie groups.

- $\mathrm{GL}_n(\mathbb{R})$ has two connected components. The identity component is
$$\mathrm{GL}_n(\mathbb{R})^+ := \{g \in \mathrm{GL}_n(\mathbb{R}) \mid \det g > 0\}$$
and the other component is $\{g \in \mathrm{GL}_n(\mathbb{R}) \mid \det g < 0\}$. To see $\mathrm{GL}_n(\mathbb{R})^+$ is connected, note that it is generated (as groups) by elementary matrices with positive determinant.

- $\mathrm{GL}_n(\mathbb{C})$ is connected.

- The tangent space of $\mathrm{GL}_n(k)$ at the identity for $k = \mathbb{R}, \mathbb{C}$ can be regarded as the matrix ring $M_n(k)$.

Consider the special linear groups $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{C})$. We show they are Lie groups. Let $k = \mathbb{R}, \mathbb{C}$.

- 1 is a regular value of $\det : \mathrm{GL}_n(k) \to k^\times$. To show this, we compute the differential of det. For $A = (a_{ij}) \in \mathrm{GL}_n(k)$, by definition we have
$$\det A = (-1)^i a_{i1} m_{i1} + \cdots + (-1)^{ij} a_{ij} m_{ij} + \cdots + (-1)^{in} a_{in} m_{in}$$
where the m_{ij} are determinant of the matrix obtained by deleting the i-th row and the j-th column of A. Hence $\det' A = ((-1)^{ij} m_{ij})_{1 \leqslant i,j \leqslant n}$, so a matrix $A \in \mathrm{GL}_n(k)$ is a critical point of det if and only all $m_{ij} = 0$. For $A \in \mathrm{SL}_n(k)$, since $\det A = 1$, it follows every matrix in $\mathrm{SL}_n(k)$ is a regular point of det. By regular level set theorem, $\mathrm{SL}_n(k)$ is then a regular submanifold of $\mathrm{GL}_n(k)$ of codimension 1.

- $\mathrm{SL}_n(k)$ is a Lie group. It remains to show the multiplication is smooth, and this follows easily from Proposition F.2.7.

Example I.1.6 (Orthogonal groups)**.** Let k be a field and B a k-bilinear form on a finite dimensional k-vector space V. Then we have an associated group

$$\{T \in \operatorname{Aut}_k V \mid B(Tv, Tw) = B(v, w) \text{ for all } v, w \in V\}$$

- $k = \mathbb{R}$, B the standard inner product on $\mathbb{R}^n$. Then we have the **orthogonal group**[1]

  $$\mathrm{O}_n(\mathbb{R}) = \mathrm{O}(n) := \{A \in \mathrm{GL}_n(\mathbb{R}) \mid A^T A = I_n\}$$

  As $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{O}_n(\mathbb{R})$ also splits into two connected components, and the identity component is the **special orthogonal groups**

  $$\mathrm{SO}_n(\mathbb{R}) = \mathrm{SO}(n) := \{A \in \mathrm{O}_n(\mathbb{R}) \mid \det A = 1\}$$

- $k = \mathbb{C}$, B the standard Hermitian product on $\mathbb{C}^n$. Then we have the **unitary groups**

  $$\mathrm{U}_n(\mathbb{R}) = \mathrm{U}(n) := \{A \in \mathrm{GL}_n(\mathbb{C}) \mid A^* A = I_n\}$$

  We can similar define the **special unitary group**

  $$\mathrm{SU}_n(\mathbb{R}) := \{A \in \mathrm{U}_n(\mathbb{C}) \mid \det A = 1\}$$

These groups are compact, for they are closed and bounded in the affine space k^{n^2}. We show $\mathrm{O}(n)$ is a regular submanifold of $\mathrm{GL}_n(\mathbb{R})$.

- Define $f : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ by $f(A) = A^T A$. We prove f has constant rank on $\mathrm{GL}_n(\mathbb{R})$, so that $\mathrm{O}(n) = f^{-1}(I_n)$ is a regular submanifold by constant level set theorem.
- For $A, B \in \mathrm{GL}_n(\mathbb{R})$, we have $f(AB) = (AB)^T AB = B^T f(A) B$, so that

  $$f \circ r_B = \ell_{(B^T)^{-1}} \circ r_B \circ f$$

  Taking differential at A, we have

  $$f_{*,AB} \circ (r_B)_{*,A} = (\ell_{(B^T)^{-1}})_{*,f(A)B} \circ (r_B)_{*,f(A)} \circ f_{*,A}$$

  Since the left and right translation are diffeomorphisms, we see $\operatorname{rank} f_{*,AB} = \operatorname{rank} f_{*,A}$ for all $A, B \in \mathrm{GL}_n(\mathbb{R})$. Since B is arbitrary, this proves f has constant rank.
- We determine $\operatorname{rank} f_{*,I_n}$. Let $X \in M_n(\mathbb{R})$ and $\gamma : \mathbb{R} \to \mathrm{GL}_n(\mathbb{R})$ a curve such that $\gamma(0) = I_n$ and $\gamma_{*,0} \left.\dfrac{d}{dt}\right|_{t=0} = X$. Then

  $$f_{*,I_n} X = f_{*,I_n} \gamma_{*,0} \left.\frac{d}{dt}\right|_{t=0} = \left.\frac{d}{dt}\right|_{t=0} \gamma(t)^T \gamma(t) = X^T + X$$

  Hence $\operatorname{rank} f_{*,I_n} = \dfrac{n^2+n}{2}$, and thus $\dim_{\mathbb{R}} \mathrm{O}_n(\mathbb{R}) = n^2 - \dfrac{n^2-n}{2} = \dfrac{n^2+n}{2}$.

[1] The notation $\mathrm{O}_n(\mathbb{R})$ implicitly tells that O_n is a real algebraic group and we are taking its real points $\mathrm{O}_n(\mathbb{R}) = \mathrm{O}(n)$. Same remark for the incoming groups.

I.1.1 Symplectic groups

Recall the **real quaternion algebra** is a 4-dimensional real algebra

$$\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}ij$$

with $i^2 = j^2$, $ij = -ji$. Concretely, it is the $\mathbb{R}$-algebra of 2×2 complex matrices of the form

$$\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} = \begin{pmatrix} a & \\ & \overline{a} \end{pmatrix} + \begin{pmatrix} & b \\ -\overline{b} & \end{pmatrix} = a + bj$$

with matrix multiplication and addition. Here we identify $z \in \mathbb{C}$ with $\begin{pmatrix} z & \\ & \overline{z} \end{pmatrix} \in \mathbb{H}$, and $j := \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$. The multiplication rule is given by

$$zj = j\overline{z} \text{ for } z \in \mathbb{C}, \qquad j^2 = -1$$

If $a + bj \neq 0$, then its determinant $|a|^2 + |b|^2$ is nonzero, and its multiplicative inverse is $\dfrac{\overline{a} - bj}{|a|^2 + |b|^2}$. This show $\mathbb{H}$ is a division ring, and we can think $\mathbb{C}$ as a subfield of $\mathbb{H}$.

- The center Z of $\mathbb{H}$ is $\mathbb{R}$. Clearly, $\mathbb{R} \subseteq Z$, and Z is a finite field extension of $\mathbb{R}$. If $\mathbb{R} \subsetneq Z$, then $Z = \mathbb{C}$. However, if we take $z \in \mathbb{H} \backslash \mathbb{C}$, we have $Z(x) = \mathbb{C}(x) = \mathbb{C}$ which is absurd. Hence $\mathbb{R} = Z$.

- $\mathbb{H}$ is a $\mathbb{C}$-vector space with $\mathbb{C}$ acting by the left multiplication. $\{1, j\}$ forms a $\mathbb{C}$-basis for $\mathbb{H}$.

- There is an involution, called **conjugation**, on $\mathbb{H}$ given by

$$\begin{aligned} \mathbb{H} &\longrightarrow \mathbb{H} \\ h = a + bj &\longmapsto \overline{h} := \overline{a} - bj \end{aligned}$$

  This is the restriction of the conjugate transpose $g \mapsto g^*$ on $\mathrm{GL}(2, \mathbb{C})$ to $\mathbb{H}$. Hence, it is an anti-$\mathbb{R}$-automorphism, which coincides with complex conjugation on $\mathbb{C}$. The **norm** for h is defined as $N(h) := h\overline{h} = \det h$, so if $h \neq 0$, the multiplicative inverse of h is $N(h)^{-1}\overline{h}$.

- As a real vector space, $\mathbb{H}$ has a basis consisting of

$$1 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \qquad i = \begin{pmatrix} i & \\ & -i \end{pmatrix}, \qquad j = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \qquad k = \begin{pmatrix} & i \\ i & \end{pmatrix} = ij$$

  The quaternion of the form $ai + bj + ck$ with $a, b, c \in \mathbb{R}$ is called a **pure quaternion**. Every $h \in \mathbb{H}$ has a unique decomposition $h = r + q$ with r real and q pure. We have $\overline{r + q} = r - q$, and therefore $N(r + q) = r^2 - q^2$. Hence if q is pure, $N(q) = -q^2$ so that $q^2 \leqslant 0$.

  - The subspace of pure quaternions can be characterized only using the ring structure on $\mathbb{H}$: If $h = r + q$ with r real and q pure, then $h^2 = r^2 + q^2 + 2rq$ is real if and only if $r = 0$ or $q = 0$, and is non-positive if and only if $r = 0$.

- The standard isomorphisms $\mathbb{H} \cong \mathbb{R}^4$ and $\mathbb{H} \cong \mathbb{C}^2$ are norm-preserving. The group

$$\mathrm{Sp}_1(\mathbb{R}) := \{h \in \mathbb{H} \mid N(h) = 1\}$$

is called the **quaternion group**, or group of unit quaternions. In terms of matrices, we see

$$\mathrm{Sp}_1(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \mid a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1 \right\} = \mathrm{SU}(2)$$

and the isomorphism $\mathbb{H} \cong \mathbb{R}^4$ identifies $\mathrm{Sp}(1)$ with the unit sphere S^3.

The basic statements of linear algebra may be formulated for modules over division rings. For example,

$$\mathrm{End}_{\mathbb{H}}(\mathbb{H}^n) = \{f \in \mathrm{End}_{\mathbb{R}}(\mathbb{H}^n) \mid f(zx) = zf(x) \text{ for all } z \in \mathbb{H},\ x \in \mathbb{H}^n\}$$

can be identified with the matrix ring $M_n(\mathbb{H})$. Under the identification, we see

$$\mathrm{GL}_n(\mathbb{H}) = \mathrm{Aut}_{\mathbb{H}}(\mathbb{H}^n) = \mathrm{End}_{\mathbb{H}}(\mathbb{H}^n) \cap \mathrm{Aut}_{\mathbb{R}}(\mathbb{H}^n).$$

We topologize $\mathbb{H}$ by the norm defined above; equivalently, since $\mathbb{H}$ is a finite dimensional real vector space, it is automatically a real Banach space. Then $\mathrm{GL}_n(\mathbb{H})$ is an open subspace of $M_n(\mathbb{H})$, and is a $4n^2$-dimensional Lie groups.

- The standard isomorphism $\mathbb{H} = \mathbb{C} + \mathbb{C}j \cong \mathbb{C}^2$ induces a standard $\mathbb{C}$-isomorphism $\mathbb{H}^n = \mathbb{C}^n + \mathbb{C}^n j \cong \mathbb{C}^{2n}$. Left multiplication by j induces an $\mathbb{R}$-endomorphism

  $$\begin{aligned} j : \mathbb{C}^{2n} \cong \mathbb{H}^n &\longrightarrow \mathbb{H}^n \cong \mathbb{C}^{2n} \\ (u, v) = u + vj &\longmapsto j(u + vj) = -\overline{v} + \overline{u}j = (-\overline{v}, \overline{u}) \end{aligned}$$

  An $\mathbb{H}$-endomorphism $\varphi : \mathbb{H}^n \to \mathbb{H}^n$ is the same of a $\mathbb{C}$-endomorphism $\varphi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ that commutes with j. Thus in matrix, φ assumes the form

  $$\begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} \in M_{2n}(\mathbb{C})$$

  with $A, B \in M_n(\mathbb{C})$.

- There is a inner product on $\mathbb{H}^n$, the **standard symplectic product**:

  $$\langle h, k \rangle := \sum_{i=1}^{n} h_i \overline{k_i}$$

  for $h = (h_1, \ldots, h_n)$, $k = (k_1, \ldots, k_n) \in \mathbb{H}^n$. The corresponding norm is $\langle h, h \rangle = \sum_{i=1}^{n} N(h_i) \geqslant 0$. The **(compact) symplectic group** is defined to be the norm-preserving $\mathbb{H}$-automorphisms:

  $$\begin{aligned} \mathrm{Sp}_n(\mathbb{R}) :=& \{\varphi \in \mathrm{GL}_n(\mathbb{H}) \mid N(\varphi(h)) = N(h) \text{ for all } h \in \mathbb{H}\} \\ =& \{\varphi \in \mathrm{GL}_n(\mathbb{H}) \mid \langle \varphi h, \varphi k \rangle = \langle h, k \rangle \text{ for all } h, k \in \mathbb{H}\} \end{aligned}$$

  Under the standard (norm-preserving) isomorphism $\mathbb{H}^n \cong \mathbb{C}^{2n}$, we have

  $$\mathrm{Sp}_n(\mathbb{R}) = \left\{ \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} \in \mathrm{U}(2n) \mid A, B \in M_n(\mathbb{C}) \right\}$$

  An element in $\mathrm{Sp}_n(\mathbb{R})$ is called a **symplectic matrix**.

- Left multiplication by j is not $\mathbb{C}$-linear. However, right multiplication by j is $\mathbb{C}$-linear:

$$\begin{aligned} J : \mathbb{C}^{2n} \cong \mathbb{H}^n &\longrightarrow \mathbb{H}^n \cong \mathbb{C}^{2n} \\ (u, v) = u + vj &\longmapsto (u + vj)j = -v + uj = (-v, u) \end{aligned}$$

In matrix form, $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \in \mathrm{GL}_n(\mathbb{C})$. Denote by $c : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ the complex conjugation. Then $j = c \circ J$, and a unitary matrix A is symplectic if and only if $AcJ = cJA$. Sine $Ac = c\overline{A}$, it becomes $c\overline{A}J = cJA$, i.e., $JA = \overline{A}J = (A^T)^{-1}J$, or $A^T JA = J$. Dropping the condition that A being unitary, we obtain the **complex symplectic group**

$$\mathrm{Sp}_{2n}(\mathbb{C}) = \{A \in \mathrm{GL}_{2n}(\mathbb{C}) \mid A^T JA = J\}$$

Then $\mathrm{Sp}_n(\mathbb{R}) = \mathrm{Sp}_{2n}(\mathbb{C}) \cap \mathrm{U}_{2n}(\mathbb{R})$.

- That $\mathrm{Sp}(2n, \mathbb{C})$ and $\mathrm{Sp}(n)$ are Lie groups can be shown in the same way as for $\mathrm{O}(n)$. We have the following lattices

$$\begin{tikzcd}
 & & \mathrm{GL}_{2n}(\mathbb{R})^+ & & \\
\mathrm{GL}_n(\mathbb{R})^+ \arrow[r] & \mathrm{GL}_n(\mathbb{R}) \arrow[r] & \mathrm{GL}_n(\mathbb{C}) \arrow[r] \arrow[u] & \mathrm{GL}_n(\mathbb{H}) \arrow[r] & \mathrm{GL}_{2n}(\mathbb{C}) \\
\mathrm{SO}(n) \arrow[r] \arrow[u] & \mathrm{O}(n) \arrow[r] \arrow[u] & \mathrm{U}(n) \arrow[r] \arrow[u] \arrow[d] & \mathrm{Sp}(n) \arrow[r] \arrow[u] \arrow[d] & \mathrm{Sp}(2n, \mathbb{C}) \arrow[u] \\
 & & \mathrm{SO}(2n) & \mathrm{U}(2n) &
\end{tikzcd}$$

- Consider the "adjoint" action of $\mathbb{H}^\times$ on $\mathbb{H}$:

$$\begin{aligned} \mathrm{Ad} : \mathbb{H}^\times \times \mathbb{H} &\longrightarrow \mathbb{H} \\ (q, x) &\longmapsto \mathrm{Ad}(q)(x) := qxq^{-1}. \end{aligned}$$

Note that $N(qxq^{-1}) = N(q)N(x)N(q)^{-1} = N(x)$. We claim if x is pure quaternion, then so is qxq^{-1}. Indeed, if we write $qxq^{-1} = r + s$ with r real and s pure, then

$$0 \geqslant -N(x) = -qN(x)q^{-1} = qx^2q^{-1} = (qxq^{-1})^2 = r^2 + s^2 + 2rs.$$

Since rs is nonreal unless $r = 0$ or $s = 0$ and $s^2 \leqslant 0$, we must have $qxq^{-1} = s$ is pure. Hence the adjoint action of $\mathbb{H}^\times$ on $\mathbb{H}$ stabilizes the subspace of pure quaternions $\mathrm{Im}\,\mathbb{H}$.

The subspace $\mathrm{Im}\,\mathbb{H}$ is clearly isomorphic to $\mathbb{R}^3$ as real vector spaces:

$$\begin{aligned} \psi : \mathrm{Im}\,\mathbb{H} &\longrightarrow \mathbb{R}^3 \\ ai + bj + ck &\longmapsto (a, b, c). \end{aligned}$$

Actually it gives more: it is norm-preserving, and for $v, u \in \mathrm{Im}\,\mathbb{H}$, we have

$$vu = -\psi(v) \cdot \psi(u) + \psi^{-1}(\psi(v) \times \psi(u)). \qquad (\spadesuit)$$

In view of this formula, it is natural to define an inner product $(\ ,\ )$ on $\operatorname{Im}\mathbb{H}$ by $(u,v) := \operatorname{Re}(u\overline{v})$; this turns ψ into a linear isometry. Now consider the restriction of Ad to $\mathrm{Sp}(1) \times \operatorname{Im}\mathbb{H}$; since $N(\mathrm{Ad}(q)x) = N(x)$, it induces a map

$$\begin{aligned} \mathrm{Ad} : \mathrm{Sp}(1) &\longrightarrow \mathrm{O}(\operatorname{Im}\mathbb{H}) \\ q &\longmapsto [\mathrm{Ad}(q) : x \mapsto qxq^{-1} = qx\overline{q}]. \end{aligned}$$

Identifying $\mathrm{O}(\operatorname{Im}\mathbb{H})$ with $\mathrm{O}(3)$, this yields a continuous homomorphism $\mathrm{Ad} : \mathrm{Sp}(1) \to \mathrm{O}(3)$. Since $\mathrm{Sp}(1) \cong S^3$ is connected and $\mathrm{Ad}(1) = \mathrm{id} \in \mathrm{SO}(3)$, we see $\mathrm{Ad}(\mathrm{Sp}(1)) \subseteq \mathrm{SO}(3)$. The kernel $\ker \mathrm{Ad} \leqslant \mathrm{Sp}(1)$ is $\{\pm 1\}$, which can be easily seen from (♠). We contend $\mathrm{Ad} : \mathrm{Sp}(1) \to \mathrm{SO}(3)$ is surjective by showing its image contains all rotations.

Restricting ψ to the norm one elements, we obtain $\psi : \operatorname{Im}\mathbb{H} \cap \mathrm{Sp}(1) \cong S^2$. From the formula (♠) again, we see each element $v \in \operatorname{Im}\mathbb{H} \cap \mathrm{Sp}(1)$ can be completed to a basis $\{v, w, u\}$ such that $v, w, u \in \operatorname{Im}\mathbb{H} \cap \mathrm{Sp}(1)$ with $vw = u$, $wu = v$, $uv = w$ and $v^2 = w^2 = u^2 = -1$.

Pick $v \in \operatorname{Im}\mathbb{H} \cap \mathrm{Sp}(1)$ and w, u as above. We claim that

$$\mathrm{Ad}(\cos\theta + v\sin\theta)$$

fixes v and acts as rotation by 2θ on $\operatorname{span}\{w, u\} \subseteq \operatorname{Im}\mathbb{H}$. To begin with, we observe that

$$\overline{\cos\theta + v\sin\theta} = \cos\theta - v\sin\theta.$$

This can be seen as follows: the natural identification $\mathbb{H} \cong \mathbb{R}^4$ preserves norm, so

$$\|\cos\theta + v\sin\theta\|^2 = \cos^2\theta + \sin^2\theta - v\cos\theta\sin\theta + v\sin\theta\cos\theta = 1$$

while $(\cos\theta + v\sin\theta)(\cos\theta - v\sin\theta) = 1$. This verifies our observation. Now compute

$$\begin{aligned} \mathrm{Ad}(\cos\theta + v\sin\theta)v &= (\cos\theta + v\sin\theta)v(\cos\theta - v\sin\theta) \\ &= (-\sin\theta + v\cos\theta)(\cos\theta - v\sin\theta) = v \\ \mathrm{Ad}(\cos\theta + v\sin\theta)u &= (\cos\theta + v\sin\theta)u(\cos\theta - v\sin\theta) \\ &= (u\cos\theta + w\sin\theta)(\cos\theta - v\sin\theta) = u\cos 2\theta + w\sin 2\theta \\ \mathrm{Ad}(\cos\theta + v\sin\theta)w &= (\cos\theta + v\sin\theta)w(\cos\theta - v\sin\theta) \\ &= (w\cos\theta - u\sin\theta)(\cos\theta - v\sin\theta) = w\cos 2\theta - u\sin 2\theta \end{aligned}$$

This proves the claim.

In conclusion, we obtain a short exact sequence of Lie groups:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Sp}(1) \xrightarrow{\mathrm{Ad}} \mathrm{SO}(3) \longrightarrow 1$$

so that $\mathrm{Sp}(1)$ is a double cover of $\mathrm{SO}(3)$.

I.1.2 Spin

Definition. Let (V, Q) be a finite dimensional real quadratic space. The **Clifford algebra** $C(Q)$ is a unital $\mathbb{R}$-algebra together with an $\mathbb{R}$-linear map $i = i_Q : V \to C(Q)$ such that

(i) $i(x)^2 = -Q(x).1$ in $C(Q)$ for all $x \in V$, and

(ii) if A is any unital $\mathbb{R}$-algebra with an $\mathbb{R}$-linear map $j : V \to A$ such that $j(x)^2 = -Q(x).1$, then there exists a unique unital $\mathbb{R}$-algebra homomorphism $\phi_j : C(Q) \to A$ fitting in the commutative triangle:

$$\begin{tikzcd} C(Q) \arrow[rr] & & A \\ & V \arrow[ul, "i"] \arrow[ur, "j"'] & \end{tikzcd}$$

By universal property nonsense, a Clifford algebra is unique up to a unique isomorphism. It is easy to construct a Clifford algebra $C(Q)$ as a quotient of the tensor algebra: define I to be the two-sided ideal of IV generated by $\{x \otimes x + Q(x).1 \mid x \in V\}$, and define

$$C(Q) = TV/I.$$

An obvious candidate for $i : V \to C(Q)$ is the composition $V \to TV \to TV/I = C(Q)$.

Example I.1.7. Let $Q : \mathbb{R}^n \to \mathbb{R}$ be the usual norm function: $Q(x) = \|x\|^2$. Denote by $C_n = C_n(\mathbb{R}) = C(\|\cdot\|^2)$ the corresponding Clifford algebra. Let $\{e_i\}_{i=1}^n$ be the standard basis for $\mathbb{R}^n$. Then their image in C_n satisfies

$$e_i^2 = -1, \qquad e_i e_j = -e_j e_i \text{ if } i \neq j.$$

If $n = 0$, by construction we have $C_0 = \mathbb{R}$. If $n = 1$, we have an algebra isomorphism

$$\begin{aligned} C_1 &\longrightarrow \mathbb{C} \\ a + be_1 &\longmapsto a + bi \end{aligned}$$

and if $n = 2$, we have

$$\begin{aligned} C_2 &\longrightarrow \mathbb{H} \\ a + be_1 + ce_2 + de_1e_2 &\longmapsto a + bi + cj + dk \end{aligned}$$

We introduce two canonical maps on $C(Q)$. The opposite algebra $C(Q)^{\mathrm{op}}$ satisfies the universal properties (ii) for any opposite unital $\mathbb{R}$-algebras, and hence for all unital $\mathbb{R}$-algebras. Hence there exists a unique unital $\mathbb{R}$-algebra anti-isomorphism

$$(\cdot)^t : C(Q) \longrightarrow C(Q)$$

satisfying $(xy)^t = y^t x^t$ and $(x^t)^t = x$ for any $x, y \in C(Q)$; it is uniquely determined by $x^t = x$ for $x \in i(V)$.

By the universal property, the $\mathbb{R}$ isomorphism $V \to C(Q)$ given by $x \mapsto -x$ extends uniquely to a unique $\mathbb{R}$-algebra automorphism

$$\alpha : C(Q) \longrightarrow C(Q)$$

satisfying $\alpha \circ \alpha = \mathrm{id}$ and $\alpha(x) = -x$ for $x \in i(V)$. For $\nu \in \mathbb{N}_{\geqslant 0}$

$$C(Q)^\nu = \{x \in C(Q) \mid \alpha(x) = (-1)^\nu x\}$$

to be the eigenspace of α with eigenvalue $(-1)^\nu$; we have $C(Q) = C(Q)^0 \oplus C(Q)^1$. This is a $\mathbb{Z}/2\mathbb{Z}$-grading on $C(Q)$ in the sense that $xy \in C(Q)^{\nu+\mu}$ whenever $x \in C(Q)^\nu$ and $y \in C(Q)^\mu$.

In general, if $A = A^0 \oplus A^1$ and $B = B^0 \oplus B^1$ are two $\mathbb{Z}/2\mathbb{Z}$-graded algebras, their (vector space) tensor product $A \otimes B$ naturally inherits a $\mathbb{Z}/2\mathbb{Z}$-grading given by

$$(A \otimes B)^0 = (A^0 \otimes B^0) \oplus (A^1 \otimes B^1)$$
$$(A \otimes B)^1 = (A^0 \otimes B^1) \oplus (A^1 \otimes B^0)$$

with multiplication

$$(a' \otimes b)(a \otimes b') = (-1)^{\nu\mu}(a'a) \otimes (bb')$$

where $a \in A^\mu$ and $b \in B^\nu$. To stress this extra structure, we denote such tensor product as $A \otimes^2 B$.

If (V, P) and (W, Q) are real quadratic spaces, we can form the orthogonal direct sum $(V \oplus W, P \oplus Q)$ by $(P \oplus Q)(v, w) = P(v) + Q(w)$. Define

$$\begin{aligned} f : V \oplus W &\longrightarrow C(P) \otimes^2 C(Q) \\ (v, w) &\longmapsto i_P v \otimes 1 + 1 \otimes i_Q w. \end{aligned}$$

We check $f(v, w)^2 = -(P(v) + Q(w))(1 \otimes 1)$, so that by universal property this extends to a map $f : C(P \oplus Q) \to C(P) \otimes^2 C(Q)$ on Clifford algebras. Indeed,

$$\begin{aligned} f(v, w)^2 &= (i_P v \otimes 1 + 1 \otimes i_Q w)^2 \\ &= (i_P v \otimes 1)^2 + (i_P v \otimes 1)(1 \otimes i_Q w) + (1 \otimes i_Q w)(i_P v \otimes 1) + (1 \otimes i_Q w)^2 \\ &= -P(v).1 \otimes 1 + i_P v \otimes i_Q w - i_P v \otimes i_Q w - 1 \otimes Q(w).1 \\ &= -(P(v) + Q(w))(1 \otimes 1). \end{aligned}$$

We claim f is an isomorphism by constructing its inverse. This is easy. The inclusions $V \to V \oplus W$, $W \to V \oplus W$ induces homomorphisms $\varphi : C(P) \to C(P \oplus Q)$, $\psi : C(Q) \to C(P \oplus Q)$. Then

$$\begin{aligned} C(P) \otimes^2 C(Q) &\longrightarrow C(P) \oplus C(Q) \\ v \otimes w &\longmapsto \varphi(v)\psi(w) \end{aligned}$$

is the inverse of f.

Recall that every real quadratic space (V, Q) admits an orthogonal decomposition $(V, Q) \cong \bigoplus_{i=1}^{n} (V_i, Q_i)$ with $\dim V_i = 1$ (so that $\dim V = n$). The above isomorphism implies that

$$C(Q) = C(Q_1) \otimes^2 C(Q_2) \otimes^2 \cdots \otimes^2 C(Q_n)$$

We compute $C(Q)$ when $\dim V = 1$. By construction this is simply

$$C(Q) \cong \frac{\mathbb{R}[X]}{X^2 + Q(e_1)}$$

where e_1 is any nonzero vector in V. In particular, $\dim C(Q) = 2$. For general V, we conclude from the above decomposition that

$$\dim_{\mathbb{R}} C(Q) = 2^n.$$

Moreover, if $e_1, \ldots, e_n$ are a basis for V, then

$$C(Q) = \operatorname{span}_{\mathbb{R}}\{e_{i_1} \cdots e_{i_k} \mid 1 \leqslant i_1 < \cdots < i_k \leqslant n\}.$$

In particular, the canonical map $i : V \to C(Q)$ is injective, so we may view V as a subspace of $C(Q)$ without loss of generality.

We introduce more terms. Composing α and $(\cdot)^t$, we obtain an algebra anti-automorphism

$$\begin{aligned}(\cdot)^* : C(Q) &\longrightarrow C(Q)\\ x &\longmapsto \alpha(x)^t = \alpha(x^t).\end{aligned}$$

Define the **norm** $N : C(Q) \to C(Q)$ by $N(x) = xx^*$. For $x \in V$, we have $N(x) = x(-x) = Q(x).1$. Also, put

$$\Gamma(Q) = \{x \in C(Q)^\times \mid \alpha(x).v.x^{-1} \in V \text{ for all } v \in V\}.$$

This is a subgroup of the unit group $C(Q)^\times$, called the **Clifford group** of Q.

Lemma I.1.8. $\alpha : C(Q) \to C(Q)$ and $(\cdot)^t : C(Q) \to C(Q)$ induces an automorphism and anti-automorphism of the group $\Gamma(Q)$ respectively.

Let us turn to the case the Clifford algebra $C_n = C(\|\cdot\|^2)$ of the euclidean space $(\mathbb{R}^n, \|\cdot\|^2)$. Then

$$C_n = \bigotimes_{i=1}^{n}(\mathbb{R} + \mathbb{R}e_i) \supseteq \mathbb{R}^n$$

with a basis $e_{i_1} \cdots e_{i_k}$ $(1 \leqslant i_1 < \cdots < i_k \leqslant n)$ and relations $e_i^2 = -1$, $e_ie_j = -e_je_i$ $(i \neq j)$. Put $\Gamma_n = \Gamma(\|\cdot\|^2)$. From the definition of Γ_n, it admits a representation

$$\rho : \Gamma_n \longrightarrow \operatorname{Aut}\mathbb{R}^n$$

defined as $\rho(x)v = \alpha(x)vx^{-1}$ $(x \in \Gamma_n,\, v \in \mathbb{R}^n)$.

Lemma I.1.9. $\ker\rho = \mathbb{R}^\times.1 \in C_n$.

Proof. Let $x \in \ker\rho$; then $\alpha(x)v = vx$ for all $v \in \mathbb{R}^n$. Write $x = x^0 + x^1$ with $x^i \in C_n^i$. Then

$$vx^0 + vx^1 = vx = \alpha(x)v = (x^0 - x^1)v$$

so that $x^0v = vx^0$ and $-x^1v = vx^1$ for all $v \in \mathbb{R}^n$. Write

$$x^0 = a^0 + e_1b^1$$

with $a^0 \in C_n^0$, $b^1 \in C_n^1$ such that they do not contain any e_1 in their basis expression. Taking $v = e_1$ yields

$$a^0 + e_1b^1 = x^0 = e_1^{-1}x^0e_1 = -e_1(a^0 + e_1b^1)e_1 = a^0 + b^1e_1 = a^0 - e_1b^1$$

so that $e_1b^1 = 0$. Hence x^0 contains no monomial with a factor e_1. Repeating this argument shows that $x^0 \in \mathbb{R}.1$. Next write $x^1 = a^1 + e_1b^0$ with $a^1 \in C_n^1$, $b^0 \in C_n^0$ such that they do not contain any e_1 in their basis expression. Then

$$a^1 + e_1b^0 = x^1 = -e_1^{-1}x^1e_1 = e_1(a^1 + e_1b^0)e_1 = a^1 - e_1b^0$$

so that $e_1b^0 = 0$. The same argument implies $x^1 \in \mathbb{R}.1 \subseteq C_n^0$, whence $x^1 \in C_n^0 \cap C_n^1 = \{0\}$, i.e., $x^1 = 0$. Hence $x = x^0 \in \mathbb{R} \cap \Gamma_n = \mathbb{R}^\times$. ◻

Lemma I.1.10. $N(\Gamma_n) \subseteq \mathbb{R}^\times$, and $N|_{\Gamma_n} : \Gamma_n \to \mathbb{R}^\times$ is a homomorphism with $N(\alpha(x)) = N(x)$.

Proof. Let $x \in \Gamma_n$. We claim $N(x) \in \ker \rho$ so that $N(x) \in \mathbb{R}^\times.1$. For $v \in \mathbb{R}^n$, we compute

$$\rho(N(x))v = \alpha(N(x))vN(x)^{-1} = \alpha(x)x^t v(x^*)^{-1}x^{-1} = \alpha(x)(\alpha(x^{-1})vx)^t x^{-1} = \alpha(x)\alpha(x)^{-1}vxx^{-1} = v.$$

Hence $N(x) \in \ker \rho$. For $x, y \in \Gamma_n$, since $N(y) \in \mathbb{R}^\times$, we have

$$N(xy) = xy(xy)^* = xyy^*x^* = xN(y)x^* = xx^*N(y) = N(x)N(y).$$

Last,

$$N(\alpha(x)) = \alpha(x)\alpha(x)^* = \alpha(x)x^t = \alpha(xx^*) = \alpha(N(x)) = N(x)$$

as $N(x) \in \mathbb{R}$. □

Lemma I.1.11. $\mathbb{R}^n\backslash\{0\} \subseteq \Gamma_n$, and if $x \in \mathbb{R}^n\backslash\{0\}$, then $\rho(x)$ is the reflection in the hyperplane orthogonal to x. Also, $\rho(\Gamma_n) \subseteq \mathrm{O}(n)$.

Proof. Let $x \in \mathbb{R}^n\backslash\{0\}$. Choose a basis $\{e_i\}_{i=1}^n$ such that $x = \|x\|\, e_1$. Since $\mathbb{R}^\times.1 \subseteq \ker \rho$, we have $\rho(x) = \rho(\|x\|\, e_1) = \rho(e_1)$; we may assume $e_1 = x$. Then we have

$$\begin{aligned} \rho(e_1)e_1 &= \alpha(e_1)e_1e_1^{-1} = -e_1 \\ \rho(e_1)e_j &= \alpha(e_1)e_je_1^{-1} = e_1e_je_1 = e_j \ (j \neq 1). \end{aligned}$$

This proves the first part. For $x \in \Gamma_n$, $v \in \mathbb{R}^n\backslash\{0\}$,

$$N(\rho(x)v) = N(\alpha(x)vx^{-1}) = N(\alpha(x))N(v)N(x)^{-1} = N(v),$$

so that $\rho(x) \in \mathrm{O}(n)$ (as $N(v) = \|v\|^2.1$). □

Definition. For $n \geqslant 1$, define

$$\mathrm{Pin}(n) = \ker N|_{\Gamma_n} = \{x \in C(\|\cdot\|^2)^\times \mid xx^* = 1,\, \alpha(x)vx^{-1} \in \mathbb{R}^n \text{ for all } v \in \mathbb{R}^n\}$$

Theorem I.1.12. The sequence

$$\{1\} \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Pin}(n) \xrightarrow{\rho} \mathrm{O}(n) \longrightarrow \{1\}$$

is a short exact sequence of groups.

Proof. By Cartan-Dieudonne theorem and Lemma I.1.11, $\rho : \mathrm{Pin}(n) \to \mathrm{O}(n)$ is surjective. Finally,

$$\ker(\rho|_{\mathrm{Pin}(n)}) = \ker \rho \cap \ker N = (\mathbb{R}^\times.1) \cap \ker N = \{\pm 1\}.$$

□

Via this exact sequence, we can topologize the set $\mathrm{Pin}(n)$, making it a topological group. With this topology, $\rho : \mathrm{Pin}(n) \to \mathrm{O}(n)$ becomes a double cover. Moreover, since $\mathrm{O}(n)$ is a Lie group, $\mathrm{Pin}(n)$ then admits a unique smooth structure so that $\mathrm{Pin}(n)$ is a Lie group and ρ is a smooth homomorphism.

Define the **spin group**

$$\mathrm{Spin}(n) := \rho^{-1}(\mathrm{SO}(n))$$

Then we have a short exact sequence of Lie groups:

$$\{1\} \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}(n) \xrightarrow{\rho} \mathrm{SO}(n) \longrightarrow \{1\}$$

When $n = 1$, we have $C_1 = \mathbb{C}$ and

$$\mathrm{Pin}(1) = \{z \in S^1 \mid \overline{z}(i\mathbb{R})z^{-1} \subseteq i\mathbb{R}\} = \langle i \rangle = \mathbb{Z}/4\mathbb{Z},$$

so that $\mathrm{Spin}(1) = \{\pm 1\} = \mathbb{Z}/2\mathbb{Z}$. For $n \geqslant 2$, we claim

$$\rho : \mathrm{Spin}(n) \to \mathrm{SO}(n)$$

is a nontrivial double cover. To see this we claim $\mathrm{Spin}(n)$ is path-connected. To show this it suffices to show there is a path connecting 1 and -1. Define $\gamma : [0, \pi] \to C_n$ by

$$\gamma(t) = \cos(t) + \sin(t)e_1e_2$$

The two endpoints are exactly ± 1, and this is so constructed that

$$\gamma(t)^{-1} = \cos(t) - \sin(t)e_1e_2 = \gamma(t)^*$$

An easy computation shows that $\gamma(t) \in \mathrm{Pin}(n)$. It follows from an argument that $\gamma(t) \in \mathrm{Spin}(n)$ for all t. Since $\mathrm{SO}(n)$ is connected, the preceding result also implies that $\mathrm{Spin}(n)$ is connected for $n \geqslant 2$. Since $\mathrm{SO}(n)$ is the identity component of $\mathrm{O}(n)$, we conclude that $\mathrm{Spin}(n)$ is the identity component of $\mathrm{Pin}(n)$.

We consider the case $n = 3$. We have an injective algebra homomorphism

$$\begin{aligned} \kappa : \mathbb{H} &\longrightarrow C_3 \\ a + bi + cj + dk &\longmapsto a + be_2e_3 + ce_3e_1 + de_1e_2 \end{aligned}$$

It follows from the definition of $(\cdot)^t$ that this map is *-equivariant and hence norm-preserving, and it restricts to an injective homomorphism $\kappa : \mathbb{H}^\times \to \Gamma_3$. There is a commutative square:

$$\begin{tikzcd} \mathbb{H}^\times \times \operatorname{Im}\mathbb{H} \arrow[r, "\mathrm{Ad}"] \arrow[d, "\kappa \times \psi"] & \operatorname{Im}\mathbb{H} \arrow[d, "\psi"] \\ \Gamma_3 \times \mathbb{R}^3 \arrow[r, "\rho"] & \mathbb{R}^3 \end{tikzcd}$$

which can be easily checked. Restricting κ to norm one elements, we obtain $\kappa : \mathrm{Sp}(1) \to \mathrm{Spin}(3)$. Since both sides have the same dimension and are connected, this is an isomorphism.

There is another description of $\mathrm{Spin}(n)$:

Lemma I.1.13. $\mathrm{Spin}(n) = \mathrm{Pin}(n) \cap C_n^0$.

I.2 Left Invariant Vector Fields and Exponential Maps

Proposition I.2.1. Let G be a Lie group. Then G is parallelizable, i.e., the tangent bundle TG is isomorphic to the trivial bundle $G \times T_eG$.

Proof. The isomorphism is given by

$$\begin{tikzcd}
(g, X) \arrow[rr, mapsto] & & (g, (\ell_g)_{*,e} X) \\
G \times T_e G \arrow[rr] \arrow[dr, "\mathrm{pr}_1"'] & & TG \arrow[dl, "\pi"] \\
& G &
\end{tikzcd}$$

□

Definition. For a Lie group G, the vector space $\mathrm{Lie}(G) := T_e G$ is called the **Lie algebra** of G.

1. A smooth vector field $X \in \mathfrak{X}(G)$ is **left invariant** if for all $g \in G$, the diagram

$$\begin{tikzcd}
TG \arrow[r, "(\ell_g)_*"] & TG \\
G \arrow[u, "X"] \arrow[r, "\ell_g"] & G \arrow[u, "X"']
\end{tikzcd}$$

commutes, i.e, $(\ell_g)_{*,h} X_h = X_{gh}$, or simply, $(\ell_g)_* X = X$, i.e, X is invariant under pushforward by ℓ_g. Denote by $L_G \subseteq \mathfrak{X}(G)$ the space of left-invariant smooth vector field on X.

- For $X \in \mathrm{Lie}(G)$ via the trivialization $G \times T_e G \cong TG$ we associate it with a left-invariant vector field $g \mapsto (\ell_g)_{*,e} X$. This establishes a one-one correspondence between $\mathrm{Lie}(G)$ and L_G.
- To justify the name of $\mathrm{Lie}(G)$, recall in Proposition F.3.3 we saw that $\mathfrak{X}(G)$ is a Lie algebra. We restrict the bracket to L_G and transfer it to $\mathrm{Lie}(G)$. To show it is well-defined, we must show if $X, Y \in \mathfrak{X}(G)$ is left-invariant, then so is $[X, Y]$; but this follows from Proposition F.3.4.
- Explicitly, let $X, Y \in T_e G$ and put $X', Y' \in \mathfrak{X}(G)$ be the associated left-invariant vector fields. Then $[X, Y] \in T_e G$ is defined by $[X', Y']_e$.

Proposition I.2.2. For $f : G \to H$ a homomorphism of Lie groups, the differential $f_{*,e} : T_e G \to T_e H$ is a Lie algebra homomorphism.

Proof. For $X \in \mathrm{Lie}(G)$, let $X' \in L_G$ be its associated left-invariant vector field. Then we must show the identity in $\mathrm{Lie}(H)$

$$f_{*,e}[X', Y']_e = [(f_{*,e} X)', (f_{*,e} Y)']_e$$

For $h \in C^\infty(H)$, compute

$$\begin{aligned}
[(f_{*,e} X)', (f_{*,e} Y)']_e h &= (f_{*,e} X)'_e((f_{*,e} Y)' h) - (f_{*,e} Y)'_e((f_{*,e} X)' h) \\
&= f_{*,e} X((f_{*,e} Y)' h) - f_{*,e} Y((f_{*,e} X)' h) \\
&= X((f_{*,e} Y)' h \circ f) - Y((f_{*,e} X)' h \circ f)
\end{aligned}$$

On the other hand,

$$f_{*,e}[X', Y']_e h = [X', Y']_e(h \circ f) = X(Y'(h \circ f)) - Y(X'(h \circ f))$$

Hence, it suffices to show $Y'(h \circ f) = (f_{*,e}Y)'h \circ f$. For $g \in G$,

$$\begin{aligned}
(f_{*,e}Y)'h \circ f|_g &= (f_{*,e}Y)'_{f(g)}h \\
&= (\ell_{f(g)})_{*,e} f_{*,e} Y h \\
&= Y(h \circ \ell_{f(g)} \circ f) = Y(h \circ f \circ \ell_g) \\
&= (\ell_g)_{*,e} Y(h \circ f) \\
&= Y'_g(h \circ f)
\end{aligned}$$

Here the identity $\ell_{f(g)} \circ f = f \circ \ell_g$ holds due to the fact that f is a group homomorphism. □

I.2.1 Exponential Maps

Suppose X is a left-invariant vector field on a Lie group G and α is an integral curve of X starting from e. Then $g\alpha = \ell_{g^{-1}}\alpha$ is an integral curve of X starting from g for every $g \in G$; indeed,

$$(\ell_{g^{-1}}\alpha)_{*,s}\left(\left.\frac{d}{dt}\right|_{t=t} s\right) = (\ell_{g^{-1}})_{*,\alpha(s)} X_{\alpha(s)} = X_{g\alpha(s)}$$

This shows $\alpha(s+t) = \alpha(s)\alpha(t)$ whenever $s, t, s+t$ lie in the domain of α, for $t \mapsto \alpha(s+t)$ and $t \mapsto \alpha(s)\alpha(t)$ are both integral curves for X starting from $\alpha(s)$. Hence the maximal integral curve α for X starting from e is defined over $\mathbb{R}$, and the local flow generated by X is actually global, and is given by $F(t,g) := \ell_{g^{-1}}\alpha(t) = g\alpha(t)$.

A homomorphism $\mathbb{R} \to G$ of Lie groups is called a **one-parameter subgroup** of G. The above discussion shows that the integral curve of X starting from e is a one-parameter subgroup. In fact, we have a correspondence

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{LieGp}}(\mathbb{R}, G) & \longrightarrow & \mathrm{Lie}(G) \\
\alpha & \longmapsto & \alpha_{*,0}\left(\left.\frac{d}{dt}\right|_{t=0}\right) \\
\alpha^X & \longleftarrow & X
\end{array}$$

where in the second map we identify $\mathrm{Lie}(G)$ with L_G, and α^X is the integral curve for X starting from e.

Proposition I.2.3. The **exponential map** defined by

$$\begin{aligned}
\exp : \mathrm{Lie}(G) &\longrightarrow G \\
X &\longmapsto \alpha^X(1)
\end{aligned}$$

is smooth, and its differential at 0 is the identity map.

Proof. The map

$$\begin{aligned}
\mathbb{R} \times G \times \mathrm{Lie}(G) &\longrightarrow G \times \mathrm{Lie}(G) \\
(t, g, X) &\longmapsto (g\alpha^X(t), X)
\end{aligned}$$

is a flow of the smooth vector field $(g, X) \mapsto (X_g, 0) \in T_{(g,X)}(G \times \mathrm{Lie}(G)) \cong T_gG \times \mathrm{Lie}(G)$, so it is smooth. Thus its restriction to $\{1\} \times \{e\} \times \mathrm{Lie}(G)$, which is $X \mapsto (\alpha^X(1), X)$, is smooth, and so is the map $\exp : X \mapsto \alpha^X(1)$.

For the second statement, note that both $s \mapsto \alpha^{tX}(s)$ and $s \mapsto \alpha^X(ts)$ are integral curves of tX starting from e, so they are the same map. In particular,

$$\exp(tX) = \alpha^{tX}(1) = \alpha^X(t)$$

so that $\left.\frac{d}{dt}\right|_{t=0} \exp(tX) = \left.\frac{d}{dt}\right|_{t=0} \alpha^X(t) = X$. Hence $\exp_{*,0} = \mathrm{id}_{\mathrm{Lie}(G)}$. ☐

Proposition I.2.4 (Naturality)**.** A homomorphism $f : G \to H$ of Lie groups induces a commutative diagram

$$\begin{tikzcd} \mathrm{Lie}(G) \arrow[r, "f_{*,e}"] \arrow[d, "\exp_G"] & \mathrm{Lie}(H) \arrow[d, "\exp_H"] \\ G \arrow[r, "f"'] & H \end{tikzcd}$$

Proof. Let $X \in \mathrm{Lie}(G)$. The one parameter subgroup $t \mapsto f(\alpha^X(t))$ has differential

$$(f \circ \alpha^X)_{*,0} \left(\left.\frac{d}{dt}\right|_{t=0} \right) = f_{*,e} \alpha^X_{*,0} \left(\left.\frac{d}{dt}\right|_{t=0} \right) = f_{*,e} X$$

at the origin, so

$$\exp(f_{*,e} X) = f(\alpha^X(1)) = f(\exp(X))$$

☐

I.2.2 Adjoint Representations

With the group structure of G, we have an alternative description of the Lie bracket on $\mathrm{Lie}(G)$.

Lemma I.2.5. For each $g \in G$, let $c(g) : G \to G$ be defined by $c(g)x := gxg^{-1}$. The homomorphism

$$\begin{aligned} \mathrm{Ad} : G &\longrightarrow \mathrm{Aut}\,\mathrm{Lie}(G) \\ g &\longmapsto c(g)_{*,e} \end{aligned}$$

is smooth, called the **adjoint representation** of G.

Proof. Locally, Ad sends g to the Jacobian matrix of $c(g)$. Since $c(g)$ is smooth, so is each entry of its Jacobian. ☐

Proposition I.2.6. The differential $\mathrm{Ad}_{*,e} : \mathrm{Lie}(G) \to \mathrm{End}\,\mathrm{Lie}(G)$ coincides with the adjoint representation of the Lie algebra $\mathrm{Lie}(G)$.

Proof. For each $X, Y \in \mathrm{Lie}(G)$, we must show $\mathrm{Ad}_*(X)Y = \mathrm{ad}_X\, Y = [X, Y]$. For $f \in C^\infty(G)$, we have

$$\begin{aligned}
\mathrm{Ad}_*(X)Yf &= \frac{d}{ds}\Big|_{s=0} \mathrm{Ad}(\alpha^X(s))Yf = \frac{d}{ds}\Big|_{s=0} c(\alpha^X(s))_*Yf \\
&= \frac{d}{ds}\Big|_{s=0} Y(f\circ c(\alpha^X(s))) \\
&= \frac{d}{ds}\Big|_{s=0} \left(\alpha^Y(t)_* \frac{d}{dt}\Big|_{t=0}\right)(f\circ c(\alpha^X(s))) \\
&= \frac{d}{ds}\Big|_{s=0} \frac{d}{dt}\Big|_{t=0} f(\alpha^X(s)\alpha^Y(t)\alpha^X(-s)) \\
&= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} f(\alpha^X(s)\alpha^Y(t)\alpha^X(-s)) \\
&= \frac{d}{dt}\Big|_{t=0} \left(\frac{d}{ds}\Big|_{s=0} f(\alpha^X(s)\alpha^Y(t)) + \frac{d}{ds}\Big|_{s=0} f(\alpha^Y(t)\alpha^X(-s))\right) \\
&= XYf - YXf = [X,Y]f
\end{aligned}$$

Here we used chain rule: for real-valued $f(x,y)$, we have $\frac{d}{dt}\Big|_{t=0} f(t,t) = f_x(0,0) + f_y(0,0)$. ☐

Example I.2.7. A finite dimensional real vector space V, as a Lie group, coincides with its Lie algebra T_0V, and for $v \in T_0V = V$, $\alpha^v(t) := tv$ is the integral curve (one parameter subgroup) of v. T_0V is an abelian Lie algebra, for its adjoint representation is a trivial map. Generally, an abelian Lie group has abelian Lie algebra.

The torus $\mathbb{R}^n/\mathbb{Z}^n$ has Lie algebra $\mathbb{R}^n$, and the one parameter subgroup for $v \in \mathbb{R}^n$ is $\alpha^v(t) = tv \bmod \mathbb{Z}^n$.

Example I.2.8 (General linear groups)**.** Let V be a finite dimensional real/complex/quaternionic vector space. The group $\mathrm{Aut}\,V$ of automorphisms on V has Lie algebra $\mathrm{End}\,V$. The one parameter subgroup for $X \in \mathrm{End}\,V$ is

$$\alpha^X(t) := e^{tX} = \sum_{n=0}^{\infty} \frac{(tX)^n}{n!}$$

In the proof of Proposition I.2.6, we see

$$\mathrm{ad}_X Y = \frac{\partial}{\partial s}\Big|_{s=0} \frac{\partial}{\partial t}\Big|_{t=0} \alpha^X(s)\alpha^Y(t)\alpha^X(-s)$$

We use this formula to compute the Lie bracket on $\mathrm{End}\,V$. We have

$$\alpha^X(s)\alpha^Y(t)\alpha^X(-s) \equiv (1+sX)(1+tY)(1-sX) \equiv 1 + tY + st(XY - YX) \pmod{(s^2, t^2)}$$

Taking derivatives yields $\mathrm{ad}_X Y = XY - YX$.

Also, the adjoint representation $\mathrm{Ad}(g) : \mathrm{End}\,V \to \mathrm{End}\,V$ is given by $\mathrm{Ad}(g)Y = gYg^{-1}$. To see this,

$$c(g)\alpha^Y(t) = ge^{tY}g^{-1} = e^{tgYg^{-1}} = \alpha^{gYg^{-1}}(t)$$

Differentiating at $t = 0$, we obtain $\mathrm{Ad}(g)Y = gYg^{-1}$.

Example I.2.9 (Matrix groups)**.** By Proposition I.2.4, if $G \subseteq \mathrm{Aut}\,V$ is a closed Lie subgroup, then the exponential map on G is the same as that of on $\mathrm{Aut}\,V$. Hence for $X \in \mathrm{Lie}(G) \leqslant \mathrm{End}\,V$, $\exp_G X = \sum_{n=0}^{\infty} \frac{(tX)^n}{n!}$

- $\mathrm{Lie}(\mathrm{SL}(n)) = \mathfrak{sl}(n) := \{A \in \mathfrak{gl}(n) \mid \mathrm{tr}\, A = 0\}$.

- $\mathrm{Lie}(\mathrm{SO}(n)) = \mathfrak{so}(n) := \{A \in \mathfrak{gl}_n(\mathbb{R}) \mid A + A^T = 0\}$.
- $\mathrm{Lie}(\mathrm{U}(n)) = \mathfrak{u}(n) := \{A \in \mathfrak{gl}_n(\mathbb{C}) \mid A + A^* = 0\}$.
- $\mathrm{Lie}(\mathrm{SU}(n)) = \mathfrak{su}(n) := \{A \in \mathfrak{u}(n) \mid \operatorname{tr} A = 0\}$.
- $\mathrm{Lie}(\mathrm{Sp}(n)) = \mathfrak{sp}(n) := \{A \in \mathfrak{gl}_n(\mathbb{H}) \subseteq \mathfrak{gl}_{2n}(\mathbb{C}) \mid A + A^* = 0\} = \left\{ \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} \in \mathfrak{u}(2n) \mid A, B \in M_n(\mathbb{C}) \right\}$

Note that $\exp_{\mathrm{SO(n)}}$ and $\exp_{\mathrm{U(n)}}$ are surjective, while $\exp_{\mathrm{SL}_2(\mathbb{R})}$ is not.

I.2.3 Applications

No small subgroup argument

Proposition I.2.10 (No small subgroup)**.** Let G be a Lie group. Then G admits a unit-neighborhood U such that U contains no nontrivial subgroup of G.

Proof. Let V, U be unit neighborhoods of $\mathrm{Lie}(G)$ and G, respectively, such that $\exp_G : V \to U$ is a homeomorphism. If $x \in U$ is such that $x^n \in U$ for all $n \in \mathbb{Z}$, say $x = \exp a$ for $a \in V$, then $x^n = (\exp a)^n = \exp(na)$, and thus $na \in V$ for all $n \in \mathbb{Z}$. If we pick V to a bounded set, then this forces $a = 0$, and thus $x = e$. ☐

Commutativity and adjoint representation

Lemma I.2.11. A connected topological group G is generated (as groups) by any unit-neighborhood.

Proof. Let U be an open unit-neighborhood of G; replacing $U \cap U^{-1} \subseteq U$, we may assume U is symmetric. Then the subgroup $\langle U \rangle = \bigcup_{n=1}^{\infty} U^n$ generated by U is open, and hence closed. G being connected, it follows that $G = \langle U \rangle$. ☐

Proposition I.2.12. A homomorphism of connected Lie groups is determined by its differential at the identity.

Proof. Let $f : G \to H$ be a homomorphism of Lie groups. Consider the diagram

$$\begin{tikzcd} \mathrm{Lie}(G) \arrow[r, "f_{*,e}"] \arrow[d, "\exp_G"] & \mathrm{Lie}(H) \arrow[d, "\exp_H"] \\ G \arrow[r, "f"'] & H \end{tikzcd}$$

Since $\exp_G$ is a local diffeomorphism, we can find an open unit-neighborhood U of $\mathrm{Lie}(G)$ such that $\exp_G|_U$ is bijective and $\exp_G(U) \subseteq G$ is open. Since G is connected, $G = \langle \exp_G(U) \rangle$, and the result follows. ☐

Corollary I.2.12.1. Let G be a connected Lie group. TFAE:

1. G is abelian.
2. $\mathrm{Ad} : G \to \operatorname{Aut} \mathrm{Lie}(G)$ is trivial.
3. $\mathrm{ad} : \mathrm{Lie}(G) \to \operatorname{End} \mathrm{Lie}(G)$ is trivial.

Proposition I.2.13. For a Lie group G, its identity component G^0 is abelian if and only if $\exp : \mathrm{Lie}(G) \to G$ is a homomorphism.

Proof. WLOG we assume $G = G^0$ is connected. If $\exp : \mathrm{Lie}(G) \to G$ is a homomorphism, then since $\mathrm{Lie}(G)$ is an abelian group, its image is abelian, and since $G = \langle \exp_G \mathrm{Lie}(G) \rangle$, G is abelian. Suppose G is abelian. Then the multiplication $\mu : G \times G \to G$ is a Lie group homomorphism, and the result follows from the naturality. $\square$

Corollary I.2.13.1. If G^0 is abelian, then $\exp : \mathrm{Lie}(G^0) = \mathrm{Lie}(G) \to G^0$ is surjective.

Some classifications

Lemma I.2.14. Let G, H be Lie groups and $f : G \to H$ a bijective (abstract) group homomorphism. If f is a local diffeomorphism, then $f : G \to H$ is a Lie group isomorphism.

Theorem I.2.15. A connected abelian Lie group G is isomorphic to $\mathbb{T}^t \times \mathbb{R}^r$.

Proof. The exponential map $\exp_G : \mathrm{Lie}(G) \to G$ is a surjective homomorphism, so we have an abstract group isomorphism $G \cong \mathrm{Lie}(G)/\ker \exp_G$. Since $\exp_G$ is a local diffeomorphism, $\ker \exp_G \subseteq \mathrm{Lie}(G)$ is discrete; in particular, this shows $G \cong \mathrm{Lie}(G)/\ker \exp_G$ is a local diffeomorphism, if we view $\mathrm{Lie}(G)/\ker \exp_G$ as a product of a torus and a vector space. Hence $G \cong \mathrm{Lie}(G)/\ker \exp_G$ is a Lie group isomorphism.

Here we use the fact that every discrete subgroup of $\mathbb{R}^n$ is a finite rank abelian subgroup. $\square$

Corollary I.2.15.1. A compact abelian Lie group G is isomorphism to $\mathbb{T}^t \times F$, where F is a finite abelian group.

Proof. The identity component G^0 of G is compact connected abelian, so $G^0 \cong \mathbb{T}^t$, and since G is compact, $G/G^0 =: F$ is finite abelian. So far we have an short exact sequence of abelian groups

$$0 \longrightarrow G^0 \longrightarrow G \xrightarrow{p} F \longrightarrow 0$$

Since G^0 is divisible, this short exact sequence splits, so we can find a section $s : F \to G$ of p. Consider the maps

$$\begin{aligned} G^0 \times F &\longrightarrow G \\ (g, f) &\longmapsto g + s(f) \\ (g - (s \circ p)(g), p(g)) &\longleftarrow\!\shortmid\; g \end{aligned}$$

They are mutually inverses, so we have an abstract group isomorphism. Again, this is a local diffeomorphism, so $G^0 \times F \cong G$ as Lie groups. $\square$

Topological generators of tori

Definition. An element x of a topological group G is called a **topological generator** if $G = \overline{\langle x \rangle}$.

Theorem I.2.16 (Kronecker)**.** A vector $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ represents a topological generator of $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ if and only if $1, v_1, \ldots, v_n$ are $\mathbb{Q}$-linearly independent.

Proof. The exact sequence

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{R}^n \longrightarrow \mathbb{T}^n \longrightarrow 0$$

defining the torus canonically identifies $\mathbb{R}^n = \operatorname{Lie}\mathbb{R}^n = \operatorname{Lie}\mathbb{T}^n$, so the projection $\mathbb{R}^n \to \mathbb{T}^n$ is the same as the exponential map by naturality, so a Lie group homomorphism $f : \mathbb{T}^n \to S^1$ induces a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}^n \arrow[r] \arrow[d, "f_*|_{\mathbb{Z}^n}"] & \mathbb{R}^n \arrow[r] \arrow[d, "f_*"] & \mathbb{T}^n \arrow[d, "f"] \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{R} \arrow[r] & S^1
\end{tikzcd}$$

so that $f_*(v_1, \ldots, v_n) = \alpha_1 v_1 + \cdots + \alpha_n v_n$ for some $\alpha_i \in \mathbb{Z}$. Now the following statements are equivalent:

(i) $1, v_1, \ldots, v_n$ are linearly dependent over $\mathbb{Q}$.

(ii) $\sum_{i=1}^{n} a_i v_i \in \mathbb{Q}$ for some non-all-zero $a_i \in \mathbb{Q}$.

(iii) $\sum_{i=1}^{n} a_i v_i \in \mathbb{Z}$ for some non-all-zero $a_i \in \mathbb{Z}$.

(iv) $v \bmod \mathbb{Z}^n$ is in the kernel of some nontrivial homomorphism $f : \mathbb{T}^n \to S^1$.

(v) $v \bmod \mathbb{Z}^n$ is not a topological generator.

(i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) are clear. (iii)$\Leftrightarrow$(iv) follows from the first paragraph. (iv)$\Leftrightarrow$(v) is obvious, so it remains to show (v)$\Leftrightarrow$(iv). A nongenerator $[v] := v \mod \mathbb{Z}^n$ is contained is a proper closed subgroup $H \leqslant \mathbb{T}^n$, and the quotient $\mathbb{T}^n/H$ is a nontrivial compact connected abelian Lie group, so $\mathbb{T}^n/H \cong \mathbb{T}^k$ for some $k \in \mathbb{N}$, and $[v]$ lies in the kernel of the nontrivial homomorphism

$$\mathbb{T}^n \to \mathbb{T}^n/H \cong \mathbb{T}^k = S^1 \times \cdots \times S^1 \xrightarrow{\mathrm{pr}_1} S^1$$

$\square$

Corollary I.2.16.1. A compact Lie group contains a dense cyclic subgroup if and only if the group is isomorphic to $\mathbb{T}^n \cong \mathbb{Z}/\ell\mathbb{Z}$ for some $k \in \mathbb{N}_0$, $\ell \in \mathbb{N}$.

Proof. Take a topological generator $t \in \mathbb{T}^n$ and pick $x \in \mathbb{T}^n$ such that $\ell x = t$. Then the subgroup generated by $(x, 1)$ is dense in $\mathbb{T}^n \cong \mathbb{Z}/\ell\mathbb{Z}$. Conversely, if a compact Lie group G contains a dense cyclic subgroup, say $\langle a \rangle$, then G is abelian, so $G \cong \mathbb{T}^n \times F$ for some finite F. But $\mathrm{pr}_2(a)$ generates F, so F is cyclic.

$\square$

I.2.4 Dynkin's formula

In this subsection let G be a Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$. Recall that $\dfrac{1 - e^{-x}}{x} = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{(n+1)!} x^n$. Using this power series identity, for $X \in \operatorname{Lie}(G)$ we can define

$$\frac{1 - e^{-\operatorname{ad} X}}{\operatorname{ad} X} := \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} (\operatorname{ad} X)^n \in \operatorname{End}(\operatorname{Lie}(G))$$

Theorem I.2.17. Let $\gamma : \mathbb{R} \to \mathrm{Lie}(G)$ be a smooth curve. Then

$$\begin{aligned} \frac{d}{dt} e^{\gamma(t)} &= e^{\gamma(t)} \left(\frac{1 - e^{-\operatorname{ad} X}}{\operatorname{ad} X} \right) (\gamma'(t)) \\ &= \left(\left(\frac{e^{\operatorname{ad} X} - 1}{\operatorname{ad} X} \right) (\gamma'(t)) \right) e^{\gamma(t)} \end{aligned}$$

Proof. Define $\varphi : \mathbb{R}^2 \to M_n(\mathbb{R})$ by

$$\varphi(s,t) = e^{-s\gamma(t)} \frac{\partial}{\partial t} e^{s\gamma(t)}.$$

To prove the theorem, we must show $\varphi(1,t) = \left(\frac{1 - e^{-\operatorname{ad} X}}{\operatorname{ad} X} \right) (\gamma'(t))$. Firstly, since $\varphi(0,t) = 0$ we have

$$\varphi(1,t) = \int_0^1 \frac{\partial}{\partial s} \varphi(s,t) ds.$$

Now compute

$$\begin{aligned} \frac{\partial}{\partial s} \varphi(s,t) &= -\gamma(t) e^{-s\gamma(t)} \frac{\partial}{\partial t} e^{s\gamma(t)} + e^{-s\gamma(t)} \frac{\partial}{\partial t} (\gamma(t) e^{s\gamma(t)}) \\ &= e^{-s\gamma(t)} \left(-\gamma(t) \frac{\partial}{\partial t} e^{s\gamma(t)} + \gamma'(t) e^{s\gamma(t)} + \gamma(t) \frac{\partial}{\partial t} e^{s\gamma(t)} \right) \\ &= e^{-s\gamma(t)} \gamma'(t) e^{s\gamma(t)} = \mathrm{Ad}(e^{-s\gamma(t)}) \gamma'(t) = e^{-s \operatorname{ad}(\gamma(t))} \gamma'(t). \end{aligned}$$

Thus

$$\begin{aligned} \varphi(1,t) &= \int_0^1 e^{-s \operatorname{ad}(\gamma(t))} \gamma'(t) ds = \int_0^1 \sum_{n=0}^{\infty} \frac{(-s)^n}{n!} (\operatorname{ad} \gamma(t))^n \gamma'(t) ds \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n s^{n+1}}{(n+1)!} (\operatorname{ad} \gamma(t))^n \gamma'(t) \Big|_0^1 = \left(\frac{1 - e^{-\operatorname{ad} X}}{\operatorname{ad} X} \right) (\gamma'(t)). \end{aligned}$$

This shows the first equality. For the second equality, it suffices to note

$$l_{e^{\gamma(t)}} = r_{e^{\gamma(t)}} \circ \mathrm{Ad}(e^{\gamma(t)}) = r_{e^{\gamma(t)}} \circ e^{\operatorname{ad} \gamma(t)}.$$

□

Corollary I.2.17.1. For $X \in \mathrm{Lie}(G)$, the exponential map $\exp_G : \mathrm{Lie}(G) \to G$ is a local diffeomorphism near X if and only if $2\pi i \mathbb{Z} \backslash \{0\}$ are not the eigenvalues of $\operatorname{ad} X$ on $\mathrm{Lie}(G)$.

Proof. Let $Y \in \mathrm{Lie}(G)$. Consider the curve $\gamma : \mathbb{R} \to \mathrm{Lie}(G)$ defined by $\gamma(t) := X + tY$. Then by Theorem I.2.17, we have

$$(\exp_G)_{*,0}(Y) = e^X \left(\frac{1 - e^{-\operatorname{ad} X}}{\operatorname{ad} X} \right) (Y).$$

Hence we only need to show $\frac{1 - e^{-\operatorname{ad} X}}{\operatorname{ad} X}$ is invertible if and only if the eigenvalues of $\operatorname{ad} X$ are away from $2\pi i \mathbb{Z} \backslash \{0\}$. For this we may work in $\mathbb{C}$. If we put $\operatorname{ad} X$ in its Jordan form, then we see the eigenvalues of $\frac{1 - e^{-\operatorname{ad} X}}{\operatorname{ad} X}$ are of the form $\frac{1 - e^{-\lambda}}{\lambda}$, where λ runs over all eigenvalues of $\operatorname{ad} X$; if $\lambda = 0$, it is 1. Hence $\frac{1 - e^{-\operatorname{ad} X}}{\operatorname{ad} X}$ is invertible if and only if $\frac{1 - e^{-\lambda}}{\lambda} \neq 0$ for all λ, i.e., $\lambda \notin 2\pi i \mathbb{Z} \backslash \{0\}$. □

To formulate the **Dynkin's formula**, we set some notations, only for the sake of convenience. If $\mathfrak{g}$ is a Lie algebra and $X_i \in \mathfrak{g}$, put

$$[X_n, \ldots, X_3, X_2, X_1] = [X_n, \ldots, [X_3, [X_2, X_1]], \ldots]$$

and

$$[X_n^{(i_n)}, \ldots, X_1^{(i_n)}] = [\overbrace{X_n, \ldots, X_n}^{i_n \text{ copies}}, \ldots, \overbrace{X_1, \ldots, X_1}^{i_1 \text{ copies}}]$$

Theorem I.2.18 (Dynkin's formula)**.** For $X, Y \in \mathrm{Lie}(G)$ sufficiently small, the vector $Z \in \mathrm{Lie}(G)$ defined by

$$e^X e^Y = e^Z$$

is explicitly given by the formula

$$Z = \sum \frac{(-1)^{n+1}}{n} \frac{1}{(i_1 + j_1) + \cdots + (i_n + j_n)} \frac{[X^{(i_1)}, Y^{(j_1)}, \ldots, X^{(i_n)}, Y^{(j_n)}]}{i_1! j_1! \cdots i_n! j_n!}$$

where the sum runs over all $(i_1, \ldots, i_n, j_1, \ldots, j_n) \in (\mathbb{Z}_{\geqslant 0})^{2n}$ with $i_k + j_k \geqslant 1$ for all $n \in \mathbb{N}$.

Proof. Pick an open neighborhood U of $0 \in \mathrm{Lie}(G)$ on which $\exp : U \to \exp(U)$ is invertible with inverse $\log : \exp(U) \to U$. Here log is the usual logarithm for matrices given by the power series. Pick an open neighborhood $V \subseteq U$ of 0 so that $\exp(V)^2 \exp(V)^{-2} \subseteq \exp U$. For $X, Y \in V$, define $\gamma : [0,1] \to \mathrm{Lie}(G)$ by $\gamma(t) = e^{tX} e^{tY}$. Then $Z(t) := \log(\gamma(t))$ is the unique smooth curve in U such that $e^{Z(t)} = \gamma(t)$. Differentiating, with the aid of Theorem I.2.17, we obtain

$$\left(\left(\frac{e^{\mathrm{ad}\, Z(t)} - 1}{\mathrm{ad}\, Z(t)} \right) (Z'(t)) \right) e^{Z(t)} = X e^{Z(t)} + e^{Z(t)} Y.$$

Since exp is a local diffeomorphism near $Z(t)$, from the proof of Corollary I.2.17.1 we see $\dfrac{1 - e^{-\mathrm{ad}\, Z(t)}}{\mathrm{ad}\, Z(t)}$ is invertible on $\mathrm{Lie}(G)$, and hence

$$Z'(t) = \left(\frac{e^{\mathrm{ad}\, Z(t)} - 1}{\mathrm{ad}\, Z(t)} \right)^{-1} (X + e^{Z(t)} Y e^{-Z(t)}) = \frac{\mathrm{ad}\, Z(t)}{e^{\mathrm{ad}\, Z(t)} - 1} (X + \mathrm{Ad}(e^{Z(t)}) Y).$$

Since $e^{Z(t)} = e^{tX} e^{tY}$, we have

$$e^{\mathrm{ad}\, Z(t)} = \mathrm{Ad}(e^{Z(t)}) = \mathrm{Ad}(e^{tX}) \,\mathrm{Ad}(e^{tY}) = e^{t\, \mathrm{ad}\, X} e^{t\, \mathrm{ad}\, Y},$$

so that

$$Z'(t) = \frac{\mathrm{ad}\, Z(t)}{e^{\mathrm{ad}\, Z(t)} - 1} (X + e^{t\, \mathrm{ad}\, X} e^{t\, \mathrm{ad}\, Y} Y) = \frac{\mathrm{ad}\, Z(t)}{e^{\mathrm{ad}\, Z(t)} - 1} (X + e^{t\, \mathrm{ad}\, X} Y).$$

From the power series identity $x = \log(1 + (e^x - 1)) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (e^x - 1)^n = (e^x - 1) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (e^x - 1)^{n-1}$, we obtain, by inserting $x = \mathrm{ad}\, Z(t)$,

$$\frac{\mathrm{ad}\, Z(t)}{e^{\mathrm{ad}\, Z(t)} - 1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (e^{t\, \mathrm{ad}\, X} e^{t\, \mathrm{ad}\, Y} - 1)^{n-1}$$

Hence,

$$\begin{aligned}
Z'(t) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (e^{t\,\mathrm{ad}\,X} e^{t\,\mathrm{ad}\,Y} - 1)^{n-1} (X + e^{t\,\mathrm{ad}\,X} Y) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\sum_{(i,j)\in \mathbb{Z}^2_{\geqslant 0} - \{(0,0)\}} \frac{t^{i+j}}{i!j!} (\mathrm{ad}\,X)^i (\mathrm{ad}\,Y)^j \right)^{n-1} \left(X + \left(\sum_{i=0}^{\infty} \frac{t^i}{i!} (\mathrm{ad}\,X)^i \right) Y \right).
\end{aligned}$$

The proof follows from expanding the integrating (note that $Z(0) = 0$). ☐

I.2.5 Representation on smooth functions

Let G be a Lie group. For $X \in \mathrm{Lie}(G) = L_G$ and $f \in C^\infty(G)$, we have

$$Xf(g) = X_g f = \left.\frac{d}{dt}\right|_{t=0} f(g\alpha^X(t)) = \left.\frac{d}{dt}\right|_{t=0} f(g \exp_G(tX)).$$

Lemma I.2.19. For $f \in C(G)$, we have $f \in C^\infty(G)$ if and only if f is C^1 and $Xf \in C^\infty(G)$ for each $X \in \mathrm{Lie}(G)$.

Proof. THe only if part is clear. For the if part, we claim this implies $Xf \in C^\infty(G)$ for each $X \in \mathfrak{X}(G)$. Let $p \in G$ and K a compact neighborhood of p in some chart $(U, x^1, \ldots, x^n)$ about p. It suffices to show Xf is smooth in K. Let $X_1, \ldots, X_n$ be a basis for $\mathrm{Lie}(G)$; then $X_1|_K, \ldots, X_n|_K$ is also a basis for T_qG for $q \in K$. By shrinking K if necessary, by linear algebra we can write $\left.\frac{\partial}{\partial x^i}\right|_K = \sum_{j=1}^n a_j X_j|_K$ for some smooth functions $a_j \in C^\infty(K)$, so that $\frac{\partial}{\partial x^i} f$ is smooth in K. Hence Xf is smooth in K. Now the result follows from Lemma F.3.2.3. ☐

Clearly, each $X \in \mathrm{Lie}(G)$ yields an operator on $C^\infty(G)$. Hence we obtain a linear map

$$\begin{aligned}
\mathrm{Lie}(G) &\longrightarrow \mathrm{End}_{\mathbb{R}}\, C^\infty(G) \\
X &\longrightarrow [f \mapsto Xf].
\end{aligned}$$

This is the restriction to $\mathrm{Lie}(G)$ of the canonical action $\mathfrak{X}(G) \to \mathrm{End}\, C^\infty(G)$. The Lie algebra structure on $\mathfrak{X}(G)$ is so defined that it is a Lie algebra homomorphism, and L_G is a Lie subalgebra of $\mathfrak{X}(G)$. In particular, this shows

$$\mathrm{Lie}(G) \longrightarrow \mathrm{End}_{\mathbb{R}}\, C^\infty(G)$$

is a Lie algebra homomorphism. We record this as a lemma.

Lemma I.2.20. For a Lie group G, the map $\mathrm{Lie}(G) \to \mathrm{End}_{\mathbb{R}}\, C^\infty(G)$ is a Lie algebra homomorphism.

I.3 Lie Subgroups

Definition. A **Lie subgroup** of a Lie group G is an injective homomorphism of Lie groups $f : H \to G$.

- H needs not be a regular submanifold of G. Nevertheless, H is always an immersed submanifold of G since such an f must have constant rank. One way to see this is using naturality of exps.

Lemma I.3.1. Let G be a Lie subgroup and $f : H \to G$ a Lie subgroup. Then

$$f_{*,e}(\mathrm{Lie}(H)) = \{X \in \mathrm{Lie}(G) \mid \exp_G(tX) \in f(H) \text{ for all } t \in \mathbb{R}\}.$$

Proof. Let $X \in \mathrm{Lie}(H)$. By naturality we have $f(\exp_H(tX)) = \exp_G(f_{*,e}(tX))$ for all $t \in \mathbb{R}$, so $f_{*,e}(X)$ lies in RHS. Conversely, let $X \in \mathrm{Lie}(G)$ such that $\exp_G(tX) \in f(H)$ for all $t \in \mathbb{R}$. Define $\gamma : \mathbb{R} \to H$ by $\gamma(t) := f^{-1}(\exp_G(tX))$, and put $Y = \gamma_{*,0}\left(\left.\frac{d}{dt}\right|_{t=0}\right) \in \mathrm{Lie}(H)$. Then $\gamma(t) = \exp_H(tY)$, and by naturality

$$\exp_G(tf_{*,e}(Y)) = f(\exp_H(tY)) = f(\gamma(t)) = \exp_G(tX).$$

Since $\exp_G$ is a local diffeomorphism, this implies $X = f_{*,e}(Y)$. □

Theorem I.3.2 (Cartan)**.** An abstract subgroup H of a Lie group G is a regular submanifold of G if and only if H is closed in G.

Proof. Note that a regular submanifold is locally closed. Thus we can find an open unit-neighborhood U of G such that $H \cap U$ is closed in U. We claim $H = \overline{H}$. For $h \in \overline{H}$, by definition we have $g \in hU \cap H$, so $h^{-1}g \in U \cap \overline{H} = U \cap H$, hence $h \in H$. Hence H is closed.

Conversely, we assume H is a closed abstract subgroup of G. By translation, it suffices to find an open unit-neighborhood of G on which H is a submanifold.

1° Let V, U be open unit-neighborhoods of $\mathrm{Lie}(G)$ and G, respectively, such that $\exp_G : V \to U$ is a smooth diffeomorphism. Put $\log : U \to V$ to be its inverse.

2° Consider the set $B = \{X \in \mathrm{Lie}(G) \mid \exists (h_n) \subseteq \log(H \cap U) \text{ with } h_n \to 0 \text{ such that } \frac{h_n}{|h_n|} \to X\}$. Then $\exp_G tX \in H$ for all $t \in \mathbb{R}$ and $X \in B$.

Since $|h_n| \to 0$, if we take $m_n = \left\lfloor \frac{t}{h_n} \right\rfloor \in \mathbb{Z}$, we have $m_n|h_n| \to t$, so that

$$\exp m_n h_n = \exp\left(m_n|h_n|\frac{h_n}{|h_n|}\right) \to \exp tX$$

On the other hand, $\exp m_n h_n = (\exp h_n)^{m_n} \in H$. Since H is closed, $\exp tX \in H$.

3° The set $\mathfrak{h} = \{tX \mid t \in \mathbb{R}, X \in B\}$ is a linear subspace of $\mathrm{Lie}(G)$.

Let $X, Y \in \mathfrak{h}$ and consider $h(t) := \log(\exp(tX)\exp(tY))$. Then $\lim_{t\to 0} \frac{h(t)}{t} = h_{*,0}\left(\left.\frac{d}{dt}\right|_{t=0}\right) = X + Y$, and $\frac{h(t)}{|h(t)|} = \frac{h(t)}{t}\frac{t}{|h(t)|} \to \frac{X+Y}{|X+Y|}$. Thus $\frac{X+Y}{|X+Y|} \in B$ and $X + Y \in \mathfrak{h}$.

4° $\exp \mathfrak{h}$ is a unit-neighborhood of H.

Let D be any linear complement of $\mathfrak{h}$ in $\mathrm{Lie}(G)$. Consider the map

$$\begin{aligned} \psi : D \oplus \mathfrak{h} &\longrightarrow G \\ (X, Y) &\longmapsto \exp X \exp Y \end{aligned}$$

Since $\exp_* = \mathrm{id}_{\mathrm{Lie}(G)}$, ψ is a local diffeomorphism. We prove 4° by contradiction. Choose $(X_n, Y_n) \in D \oplus \mathfrak{h}$ with $\psi(X_n, Y_n) \in H$, $X_n \neq 0$ and $(X_n, Y_n) \to 0$. Since D is closed, we can find $X \in D$ such that, by passing to subsequences, $\frac{X_n}{|X_n|} \to X$; since $|X| = 1$, $X \neq 0$. Because $\exp Y_n \in H$ by 2°, $\exp X_n \in H$ as well, so $X \in B \subseteq \mathfrak{h}$ by 2°, a contradiction.

This completes the proof because by 4° we can find an open unit-neighborhood W of G such that $W \cap H = U \cap \exp \mathfrak{h}$ and on it exp is invertible, so we obtain a chart of H around the identity. In addition, by 4°and Lemma I.3.1,

5° The equality holds:

$$\mathrm{Lie}(H) = \mathfrak{h} = \{X \in \mathrm{Lie}(G) \mid \exp_G(tX) \in H \text{ for all } t \in \mathbb{R}\}$$

□

Corollary I.3.2.1. Let G, H be Lie groups and $f : G \to H$ be a continuous abstract group homomorphisms. Then f is automatically smooth, so f is a Lie group homomorphism.

Proof. Consider the graph of f:

$$\Gamma_f := \{(g, f(g)) \mid g \in G\} \subseteq G \times H$$

By assumption Γ_f is a closed subgroup of $G \times H$, so Γ_f is a closed Lie subgroup of $G \times H$. Now consider the first projection $\pi = \mathrm{pr}_1|_{\Gamma_f} : \Gamma_f \to G$; it is clear that π is a smooth homeomorphism. We claim $\pi_{*,e}$ is an isomorphism. Assuming this, we see π^{-1} is a local diffeomorphism, and hence π^{-1} is smooth everywhere. Hence $f = \mathrm{pr}_2 \circ \pi^{-1}$ is smooth.

View $\mathrm{Lie}(\Gamma_f) \subseteq \mathrm{Lie}(G) \times \mathrm{Lie}(H)$. Then $\mathrm{Lie}(\Gamma_f) \cap \mathrm{Lie}(H) = 0$. For if $X \in \mathrm{Lie}(\Gamma_h) \cap \mathrm{Lie}(H)$, then $\exp tX \in \Gamma_f \cap H = \{e\}$, i.e., $X = 0$. This shows $\pi_* : \mathrm{Lie}(\Gamma_f) \to \mathrm{Lie}(G)$ is injective. Hence $\dim \mathrm{Lie}(\Gamma_f) \leqslant \dim \mathrm{Lie}(G)$ with equality if and only if π_* is an isomorphism. But π is a homeomorphism, by invariance of dimension, we must have $\dim \Gamma_f = \dim G$ as smooth manifolds, and hence equality holds. □

Remark. Alternatively, we can show $\dim \mathrm{Lie}(\Gamma_f) = \dim \mathrm{Lie}(G)$ by measure theory. If $\dim \mathrm{Lie}(\Gamma_f) < \dim \mathrm{Lie}(G)$, then the (smooth) image of Γ_f in G is of measure zero, a contradiction to the bijectivity of π. Note that here the second countability of Γ_f and G are used. However, if we do not impose the second countability on the definition of manifolds, the proof still works with slight modification. To show injectivity of π_* we may replace Γ_f by its identity component, and thus we may assume G and Γ_f are connected; but this automatically forces G and Γ_f to be second countable, thanks to the fact euclidean spaces are second countable.

I.4 Correspondence between Lie Groups and Lie Algebras

Definition. Let $p : \tilde{X} \to X$ be a continuous map between topological space. p is called a **covering map** if each $x \in X$ admits an open neighborhood U such that $f^{-1}(U)$ is a union of disjoint open sets in $\tilde{X}$ each of which being mapped homeomorphically onto U by p.

Lemma I.4.1. Let $f : G \to H$ be a continuous group homomorphism between topological groups. If H is connected and there exists a unit-neighborhood U of G such that $f(U)$ is open and $f|_U^{f(U)} : U \to f(U)$ is a homeomorphism, then f is a covering map.

Proof. We have $f^{-1}(f(U)) = \bigsqcup_{x \in \ker f} Ux$. Hence the covering property is verified at the identity of H. Since H is connected and $f(U)$ is open, f is surjective. Then for $h \in H$ and any $g \in f^{-1}(h)$, we have $f^{-1}(f(gU)) = \bigsqcup_{x \in \ker f} gUx$.

□

Theorem I.4.2. Let $f : G \to H$ be a homomorphism of Lie groups.

(i) The differential $f_* : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ is a Lie algebra homomorphism.

(ii) $\ker f_* = \mathrm{Lie}(\ker f)$. Thus f_* is injective if and only if $\ker f$ is discrete.

(iii) If H is connected, f_* is surjective if and only if f is surjective, and f_* is an isomorphism if and only if f is a covering map.

Proof.

(i) This is Proposition I.2.2.

(ii) For $X \in \mathrm{Lie}(G)$, by naturality $f_* X = 0$ implies $f(\exp(tX)) = 0$ for all $t \in \mathbb{R}$, and vice versa. By Lemma I.3.1, the last condition is equivalent to $X \in \mathrm{Lie}(\ker f)$. For the second assertion, note that for a Lie group G, $\mathrm{Lie}(G) = 0$ if and only if G is discrete.

(iii) If f_* is surjective, then by Constant rank theorem the image of f contains a unit neighborhood U of H. Since H is connected, $H = \langle U \rangle \subseteq f(G) \subseteq H$. If f is surjective and f is nowhere submersive, then by Sard's theorem, $f(G) \subseteq H$ is of measure zero, a contradiction. Hence f is submersive at some point of G, and by translation f_* is surjective.

If f_* is an isomorphism, f is a local diffeomorphism, and hence by Lemma f is a covering map. Conversely, if f is a covering map, then $\ker f$ is discrete and f is surjective, so f_* is bijective by (ii) and the first statement of (iii).

□

Proposition I.4.3. Let G be a Lie group. For $X, Y \in \mathrm{Lie}(G)$, $[X, Y] = 0$ if and only if $e^{tX} e^{sY} = e^{sY} e^{tX}$.

Proof.

$$\mathrm{ad}_X Y = \mathrm{Ad}_{*,e}(X) Y = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}(e^{tX}) Y = \frac{d}{dt}\bigg|_{t=0} c(e^{tX})_* Y = \frac{d}{dt}\bigg|_{t=0} (c(e^{tX}) e^{sY})_*$$

Hence $\mathrm{ad}_X Y = 0$ if and only if $(c(e^{tX}) e^{sY})_*$ is constant in t. But it is

$$(c(e^{tX}) e^{sY})_* = (e^{tX} e^{sY} e^{-tX})_* = e^{sY}_* = Y$$

so $c(e^{tX}) e^{sY}$ is an integral curve of Y starting from e, and thus $c(e^{tX}) e^{sY} = e^{sY}$. □

Corollary I.4.3.1. For a connected Lie group G, we have $Z(\mathrm{Lie}(G)) = \mathrm{Lie}\, Z(G)$.

Proof. Note that $Z(G)$ is a closed subgroup of G, so it is a Lie group by Theorem I.3.2. For $X \in \mathrm{Lie}(G)$, by Proposition I.4.3 we have $X \in Z(\mathrm{Lie}(G))$ if and only if e^{tX} commutes with elements of the form e^Y. But such elements generate G by Lemma I.2.11, so e^{tX} commutes with G, i.e., $e^{tX} \in Z(G)$. Taking differential yields $X \in \mathrm{Lie}(Z(G))$. □

Lemma I.4.4. For a Lie group G and $g \in G$, we have $\mathrm{Lie}\, C_G(g) = \{X \in \mathrm{Lie}(G) \mid \mathrm{Ad}(g) X = X\}$.

Proof. Note that $C_G(g)$ is a closed subgroup of G, so it is a Lie group by Theorem I.3.2. If $X \in \mathrm{Lie}\, C_C(g)$, then $e^{tX} g = g e^{tX}$ for all $t \in \mathbb{R}$, or

$$e^{tX} = g e^{tX} g^{-1} = c(g) e^{tX} = e^{t\, \mathrm{Ad}(g) X}$$

by Proposition I.2.6 and naturality. Taking differential gives $X = \mathrm{Ad}(g) X$. Reversing the argument gives the other containment. □

Lemma I.4.5. Let G be a Lie group and $H \leqslant G$ a connected closed subgroup. Then $\operatorname{Lie} C_G(H) = C_{\operatorname{Lie}(G)}(\operatorname{Lie}(H))$.

Proof. If $X \in \operatorname{Lie} C_G(H)$ and $Y \in \operatorname{Lie} H$, then $e^{tX}e^{sY} = e^{sY}e^{tX}$ for all $t, s \in \mathbb{R}$. By Proposition I.4.3 this means $[X, Y] = 0$, i.e., $X \in C_{\operatorname{Lie}(G)}(\operatorname{Lie}(H))$. Conversely, if $X \in \operatorname{Lie} C_(\operatorname{Lie}(G))(\operatorname{Lie}(H))$, then $e^{tX}e^{sY} = e^{sY}e^{tX}$ for all $t, s \in \mathbb{R}$ and $Y \in \operatorname{Lie}(H)$. By Lemma I.2.11 this implies $e^{tX}h = he^{tX}$ for all $h \in H$. Hence $e^{tX} \in C_G(H)$, and hence $X \in \operatorname{Lie} C_G(H)$. $\square$

Theorem I.4.6. Let G be a connected Lie group. Then the map

$$\begin{aligned} \{H \leqslant G : \text{Lie subgroup}\} &\longrightarrow \{\mathfrak{h} \subseteq \operatorname{Lie}(G) : \text{Lie subalgebra}\} \\ H &\longmapsto \operatorname{Lie}(H) \end{aligned}$$

is an inclusion-preserving bijection with inverse $\mathfrak{h} \mapsto \langle \exp_G \mathfrak{h} \rangle \leqslant G$.

Proof. The injectivity is clear. Let $\mathfrak{h} \subseteq \operatorname{Lie}(G)$ be a Lie subalgebra. By regarding $\operatorname{Lie}(G)$ as the space of left-invariant vector fields L_G of G, the subalgebra $\mathfrak{h}$ corresponds to an involutive distribution of TG of rank $\dim_{\mathbb{R}} \mathfrak{h}$. By Frobenius integrability theorem, there exists a unique connected (immersed) submanifold $f : H \to G$ such that $f_{*,h}(T_h H) = (\ell_{f(h)})_{*,e}\mathfrak{h}$ for all $h \in H$. Define $\operatorname{mult}_H : H \times H \to G$ and $\operatorname{inv}_H : H \to G$ by $\operatorname{mult}_H = \operatorname{mult}_G \circ (f \times f)$ and $\operatorname{inv}_H = \operatorname{inv}_G \circ f$. We claim their image lie in H, and by Lemma F.3.7 they are automatically smooth with respect to the smooth structure of H. It suffices to show (we use usual notations for groups) $hk^{-1}H = H$ for $h, k \in H$. Note that $e \in hk^{-1}H$ and it is connected, so we only need to show it is an integral submanifold for $\mathfrak{h}$. This follows from the uniqueness part of integrability theorem, as $T_{hk^{-1}p} hk^{-1}H = (\ell_{hk^{-1}p})_{*,e}\mathfrak{h} = T_{hk^{-1}p}H$. Hence H is a Lie subgroup of G with Lie algebra $\mathfrak{h}$. This shows the surjectivity.

$\square$

Theorem I.4.7. Let G, H be two connected Lie groups with G simply connected. If $\varphi : \operatorname{Lie}(G) \to \operatorname{Lie}(H)$ is a Lie algebra homomorphism, then there exists a Lie group homomorphism $f : G \to H$ such that $f_{*,e} = \varphi$.

Proof. Define

$$\mathfrak{a} = \{(X, \varphi(X)) \in \operatorname{Lie}(G) \oplus \operatorname{Lie}(H) \mid X \in \operatorname{Lie}(G)\} \subseteq \operatorname{Lie}(G \times H).$$

Note the Lie bracket on $\operatorname{Lie}(G \times H)$ is the same as that on the Lie algebra direct sum $\operatorname{Lie}(G) \oplus \operatorname{Lie}(H)$. Since φ preserves Lie bracket, $\mathfrak{a}$ is a Lie subalgebra of $\operatorname{Lie}(G \times H)$. By Theorem I.4.6, there exists a connected Lie subgroup $D \leqslant G \times H$ with Lie algebra $\mathfrak{a}$. The composition $g : D \subseteq G \times H \xrightarrow{\operatorname{pr}_1} G$ is a Lie group homomorphism whose differential is $\mathfrak{a} \xrightarrow{\operatorname{pr}_1} \operatorname{Lie}(G)$, an isomorphism. By Theorem I.4.2.(iii), g is a covering map. Since we assume G is simply connected, g is actually an isomorphism. Then the homomorphism $f := \operatorname{pr}_2 \circ g^{-1} : G \to H$ does the job.

$\square$

Definition. A **complex Lie group** is a Lie group G together with a holomorphic structure such that the multiplication and inversion are holomorphic.

Suppose G is a complex Lie group. The real tangent space $\operatorname{Lie}(G)$ admits a complex structure given by multiplication by i, so $\operatorname{Lie}(G)$ is automatically a complex Lie algebra. In particular, every left invariant vector field is holomorphic, and $\exp : \operatorname{Lie}(G) \to G$ is holomorphic. If $f : G \to H$ is a smooth Lie group homomorphism between complex Lie groups such that $f_{*,e} : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ is complex linear, then f is holomorphic, which is a consequence of naturality.

Suppose G is a connected Lie group and $f : H \to G$ a connected real Lie subgroup. Suppose $\mathrm{Lie}(H)$, as a subspace of $\mathrm{Lie}(G)$, is closed under multiplication by i. We claim H admits a complex structure, making its a complex Lie group. Indeed, let $X_1, \dots, X_m$ be a basis for $\mathrm{Lie}(H)$. For $h \in H$, consider the map

$$\begin{aligned} \mathrm{Lie}(H) &\longrightarrow H \\ \sum_{i=1}^{m} t_i X_i &\longmapsto h \exp_H(t_1 X_1 + \cdots + t_m X_m) \end{aligned}$$

This is a local diffeomorphism, so it provides a chart for H about h. This is called the **canonical coordinates of the first kind**. In view of the naturality

$$\begin{tikzcd} \mathrm{Lie}(H) \arrow[r] \arrow[d] & H \arrow[d] \\ \mathrm{Lie}(G) \arrow[r] & G \end{tikzcd}$$

all such coordinates provide H a holomorphic structure, and make the structure map $f : H \to G$ holomorphic. Moreover, if $g : M \to G$ is any holomorphic map with $g(M) \subseteq f(H)$, then the unique map $g_0 : M \to H$ with $g = f \circ g_0$ is holomorphic.

I.4.1 Adjoint group

Let $F = \mathbb{R}$, $\mathbb{C}$, and let $\mathfrak{g}$ be a finite dimensional Lie algebra over F. If $F = \mathbb{C}$ is complex, we denote by $\mathfrak{g}^{\mathbb{R}}$ the underlying real Lie algebra. Define the **automorphism group**

$$\mathrm{Aut}_F \mathfrak{g} = \{T \in \mathrm{GL}_F(\mathfrak{g}) \mid [Tv, Tw] = [v, w] \text{ for all } v, w \in \mathfrak{g}\}.$$

Here $\mathrm{GL}_F(\mathfrak{g})$ denotes the group of invertible elements in $\mathrm{End}_F \mathfrak{g}$. In any case, $\mathrm{Aut}_F \mathfrak{g}$ is a closed subgroup of $\mathrm{GL}_F(\mathfrak{g})$, so it is a Lie group itself. If $F = \mathbb{C}$, by definition

$$\mathrm{Aut}_{\mathbb{C}} \mathfrak{g} = \mathrm{GL}_{\mathbb{C}}(\mathfrak{g}) \cap \mathrm{Aut}_{\mathbb{R}} \mathfrak{g}^{\mathbb{R}} \subseteq \mathrm{GL}_{\mathbb{R}}((\mathfrak{g}^{\mathbb{R}}).$$

For a (possibly non-associative non-unital non-commutative) algebra A over F, a derivation $D : A \to A$ is an F-linear map satisfying the Leibniz rule: $D(xy) = D(x)y + xD(y)$. Denote $\mathrm{Der}_F A \subseteq \mathrm{End}_F A$ the space of all derivations on A. In the case of a Lie algebra $\mathfrak{g}$,

$$\mathrm{Der}_F \mathfrak{g} := \{D \in \mathrm{End}_F A \mid D([v, w]) = [Dv, w] + [v, Dw] \text{ for } v, w \in \mathfrak{g}\}.$$

Due to the Jacobi identity, we have $\mathrm{ad}(\mathfrak{g}) \subseteq \mathrm{Der}_F \mathfrak{g}$.

Lemma I.4.8. If $\mathfrak{g}$ is real, then $\mathrm{Lie}(\mathrm{Aut}_{\mathbb{R}} \mathfrak{g}) = \mathrm{Der}_{\mathbb{R}} \mathfrak{g}$. If $\mathfrak{g}$ is complex, then $\mathrm{Lie}(\mathrm{Aut}_{\mathbb{C}} \mathfrak{g}) = \mathrm{Der}_{\mathbb{C}} \mathfrak{g}$.

If $F = \mathbb{R}$, denote by $\mathrm{Int}\, \mathfrak{g}$ the unique connected Lie subgroup of $\mathrm{Aut}_{\mathbb{R}} \mathfrak{g}$ with Lie algebra $\mathrm{ad}\, \mathfrak{g}$. If $F = \mathbb{C}$, the definition is unaffected by using $\mathrm{Aut}_{\mathbb{C}} \mathfrak{g}$ instead of $\mathrm{Aut}_{\mathbb{R}} \mathfrak{g}$ as the ambient group, since $\mathrm{ad}\, \mathfrak{g} = \mathrm{ad}\, \mathfrak{g}^{\mathbb{R}}$. We call $\mathrm{Int}\, \mathfrak{g}$ the **adjoint group** of the Lie algebra $\mathfrak{g}$.

If G is a Lie group with Lie algebra $\mathfrak{g}$, differentials of conjugation induce a representation $\mathrm{Ad} : G \to \mathrm{Aut}_{\mathbb{R}} \mathfrak{g}$. The image $\mathrm{Ad}(G)$ is called the **adjoint group** of the Lie group G, which is a Lie subgroup of $\mathrm{Aut}_{\mathbb{R}} \mathfrak{g}$. Thanks to Corollary I.4.3.1 (or Corollary I.7.7.3), we see $\mathrm{Lie}(\mathrm{Ad}(G)) = \mathrm{ad}\, \mathfrak{g}$. Then $\mathrm{Ad}(g)^{\circ} = \mathrm{Int}\, \mathfrak{g}$. Hence one may thinks of $\mathrm{Int}\, \mathfrak{g}$ as a universal version of $\mathrm{Ad}(G)$ that can be defined without any reference to a particular group G.

I.5 Compact Lie Groups

I.5.1 Compact Lie Algebras

Definition. Let $\mathfrak{g}$ be a real Lie algebra.

1. $\mathfrak{g}$ is a **compact Lie algebra** if the adjoint group $\operatorname{Int}\mathfrak{g}$ is compact.
2. A Lie subalgebra $\mathfrak{k}$ is **compactly embedded in** $\mathfrak{g}$ if the unique connected Lie subgroup $\operatorname{Int}_{\mathfrak{g}}\mathfrak{k}$ of $\operatorname{Aut}\mathfrak{g}$ with Lie algebra $\operatorname{ad}_{\mathfrak{g}}\mathfrak{k} \subseteq \operatorname{End}\mathfrak{g}$ is compact.

Lemma I.5.1. Let G be a Lie group with Lie algebra $\mathfrak{g}$. If K is a compact subgroup of G with Lie algebra $\mathfrak{k}$, then $\mathfrak{k}$ is compactly embedded in $\mathfrak{g}$. In particular, the Lie algebra of a compact Lie group is a compact Lie algebra.

Proof. Since K is compact, so is its identity component K°. Since $\operatorname{Ad}_{\mathfrak{g}}(K^\circ)$ also has Lie algebra $\operatorname{ad}_{\mathfrak{g}}\mathfrak{k}$ (Corollary I.7.7.3), it coincides with $\operatorname{Int}_{\mathfrak{g}}\mathfrak{k}$. In particular, $\operatorname{Int}_{\mathfrak{g}}\mathfrak{k}$ is compact. □

Suppose G is a compact Lie group with Lie algebra $\mathfrak{g}$. On $\mathfrak{g}$ there always exists an inner product invariant under $\operatorname{Ad}(G)$-action. Indeed, let $\langle\,,\,\rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be any inner product, and define

$$(v, w) := \int_G \langle \operatorname{Ad}(g)v, \operatorname{Ad}(g)w\rangle dg$$

where dg is the normalized Haar measure on G so that $\operatorname{vol}(G, dg) = 1$. Since $(v, v) > 0$, this is still an inner product, and since dg is left-invariant, $(\operatorname{Ad}(g)v, \operatorname{Ad}(g)w) = (v, w)$ for any $g \in G$, $v, w \in \mathfrak{g}$. Furthermore, for $X \in \mathfrak{g}$, differentiating the equation $(\operatorname{Ad}(e^{tX})v, \operatorname{Ad}(e^{tX})w) = (v, w)$ yields

$$(\operatorname{ad}(X)v, w) + (v, \operatorname{ad}(X)w) = 0.$$

That is $\operatorname{ad}(X)$ is skew-symmetric with respect to this inner product.

Lemma I.5.2. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$.

(i) Then $\mathfrak{g}$ is **reductive**, i.e., $\mathfrak{g} = Z(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$ with $[\mathfrak{g}, \mathfrak{g}]$ semisimple.

(ii) The Killing form on $\mathfrak{g}$ is negative semidefinite.

Proof. Pick an $\operatorname{Ad}(G)$-invariant inner product $(\,,\,)$ on $\mathfrak{g}$.

(i) If $\mathfrak{a}$ is a Lie ideal of $\mathfrak{g}$, then so is its orthogonal complement $\mathfrak{a}^\perp$ (note that Lie ideals are precisely $\operatorname{ad}\mathfrak{g}$-invariant subspaces), and we have $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$. This shows $\mathfrak{g}$ is $\operatorname{ad}\mathfrak{g}$ completely reducible, i.e., reductive.

(ii) Since $\operatorname{ad} X$ is skew-symmetric, the eigenvalue of $\operatorname{ad} X$ is purely imaginary. Hence $\operatorname{tr}(\operatorname{ad} X)^2 \leqslant 0$.

□

Lemma I.5.3. Let $\mathfrak{g}$ be a real Lie algebra. If the Killing form on $\mathfrak{g}$ is negative definite, then $\mathfrak{g}$ is a compact Lie algebra.

Proof. By Cartan criterion, $\mathfrak{g}$ is semisimple, so $\operatorname{ad}\mathfrak{g} = \operatorname{Der}\mathfrak{g}$. By Lemma I.4.8, we then have $\operatorname{Int}\mathfrak{g} = (\operatorname{Aut}\mathfrak{g})^\circ$, and thus $\operatorname{Int}\mathfrak{g} \leqslant \operatorname{Aut}\mathfrak{g}$ is closed. Since the Killing form is an inner product such that $\operatorname{ad}\mathfrak{g}$ acts by skew-symmetric operators, we see $\operatorname{Int}\mathfrak{g} \subseteq O(\mathfrak{g})$. Thanks to the negative definiteness, $O(\mathfrak{g})$ is compact, and hence $\operatorname{Int}\mathfrak{g}$ is compact.

□

Theorem I.5.4. Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. Let Z_G be it center and let G_{ss} be the connected Lie subgroup of G with Lie algebra $[\mathfrak{g},\mathfrak{g}]$. Then G_{ss} has finite center, $(Z_G)^\circ$ and G_{ss} are closed subgroup, and $G = (Z_G)^\circ G_{\mathrm{ss}}$.

Proof. Let $\widetilde{(Z_G)^\circ}$ and $\widetilde{G_{\mathrm{ss}}}$ be the universal covers of $(Z_G)^\circ$ and G_{ss}, respectively. Then $\mathrm{Lie}(\widetilde{(Z_G)^\circ}) = Z(\mathfrak{g})$ and $\mathrm{Lie}(\widetilde{G_{\mathrm{ss}}}) = [\mathfrak{g},\mathfrak{g}]$. By Lemma I.5.2.(i), we see that $\widetilde{(Z_G)^\circ} \times \widetilde{G_{\mathrm{ss}}}$ is a simply connected Lie group with Lie algebra $Z(\mathfrak{g}) \oplus [\mathfrak{g},\mathfrak{g}] \cong \mathfrak{g}$. By Theorem I.4.7 and Theorem I.4.2.(iii), there exists a covering map $\pi : \widetilde{(Z_G)^\circ} \times \widetilde{G_{\mathrm{ss}}} \to G$. By looking at the Lie algebras, we see that π maps $\widetilde{(Z_G)^\circ}$ onto $(Z_G)^\circ$ and maps $\widetilde{G_{\mathrm{ss}}}$ onto G_{ss}. In particular, this shows $G = (Z_G)^\circ G_{\mathrm{ss}}$.

Next we show the center Z_{ss} of G_{ss} is finite. Pick a faithful representation $\tau : G \to \mathrm{GL}(V)$ on a finite dimensional complex vector space (15.5.1.2), and write $V = V_1 \oplus \cdots \oplus V_r$ for its irreducible decomposition (15.1.3). Put $\tau_i(g) := \tau(g)|_{V_i}$. Since Z_{ss} is central in G, by Schur's lemma each $\tau_i(g)$ is a scalar for $g \in Z_{\mathrm{ss}}$. But $[\mathfrak{g},\mathfrak{g}]$ is semisimple, this implies every one dimensional representation of $[\mathfrak{g},\mathfrak{g}]$ is trivial. By naturality this implies every one dimensional representation of G_{ss} is trivial. In particular, $\det \tau_i(g) = 1$ for all $g \in G_{\mathrm{ss}}$. Hence $\tau_i(g)$ acts by a d_i-root of unity, where $d_i = \dim V_i$. Since τ is injective, we've obtained an injection

$$Z_{\mathrm{ss}} \longrightarrow \prod_{i=1}^{r} \mu_{d_i}(\mathbb{C}).$$

Since RHS is finite, so is Z_{ss}.

Finally we tackle with the closedness. Since Z_G is clearly closed, $(Z_G)^\circ$ is also closed. By Lemma I.5.2 the Killing form on $\mathfrak{g}$ is negative semidefinite. Since $[\mathfrak{g},\mathfrak{g}]$ is a Lie ideal of $\mathfrak{g}$, the Killing form on $[\mathfrak{g},\mathfrak{g}]$ is the restriction of that on $\mathfrak{g}$ to $[\mathfrak{g},\mathfrak{g}]$. Since $[\mathfrak{g},\mathfrak{g}]$ is semisimple, by Cartan's criterion the Killing form is nondegenrate. Hence the Killing form on $[\mathfrak{g},\mathfrak{g}]$ is negative definite. Hence $\mathrm{Int}[\mathfrak{g},\mathfrak{g}]$ is compact by Lemma I.5.3. But $\mathrm{Int}[\mathfrak{g},\mathfrak{g}] \cong \mathrm{Ad}(G_{\mathrm{ss}})$, and $\mathrm{Ad} : G_{\mathrm{ss}} \to \mathrm{Ad}(G_{\mathrm{ss}})$ is a finite cover (as the kernel Z_{ss} is finite), we conclude that G_{ss} is compact as well. In particular, it is a closed subgroup of G. □

I.5.2 Maximal Tori

In this subsection, let G be a connected compact Lie group with Lie algebra $\mathfrak{g}_0$ and denote by $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ its complexification.

Definition. A **maximal torus** in G is a compact connected Lie subgroup of G that is maximal (with respect to inclusion) among all such subgroups.

- By Corollary I.2.15.1, tori are precisely those Lie groups isomorphic to $(S^1)^k$.
- A torus T in G is maximal if and only if $\mathrm{Lie}(T) \subseteq \mathfrak{g}_0$ is maximal abelian. Indeed, this follows from Theorem I.4.6 and the fact that the closure of a torus is again a torus.

Let T be a maximal torus in G with Lie algebra $\mathfrak{t}_0$. By Lemma I.5.2 $\mathfrak{g}_0$ is (real) reductive, so $\mathfrak{g}_0 = Z(\mathfrak{g}_0) \oplus [\mathfrak{g}_0,\mathfrak{g}_0]$. Since $\mathfrak{t}_0$ is maximal abelian, $\mathfrak{t}_0 = Z(\mathfrak{g}_0) \oplus \mathfrak{t}_0'$ for some maximal abelian $\mathfrak{t}_0'$ in $[\mathfrak{g}_0,\mathfrak{g}_0]$. Complexifying, we obtain

$$\mathfrak{g} = Z(\mathfrak{g}) \oplus [\mathfrak{g},\mathfrak{g}]$$

with $[\mathfrak{g},\mathfrak{g}]$ semisimple, $\mathfrak{t}' := \mathfrak{t}_0' \otimes_{\mathbb{R}} \mathbb{C}$ maximal abelian in $[\mathfrak{g},\mathfrak{g}]$, and $\mathfrak{t}_0 \otimes_{\mathbb{R}} \mathbb{C} =: \mathfrak{t} = Z(\mathfrak{g}) \oplus \mathfrak{t}'$.

Lemma I.5.5. $\mathfrak{t}'$ is a maximal toral subalgebra in $[\mathfrak{g},\mathfrak{g}]$.

Proof. Elements in $\mathrm{ad}_{\mathfrak{g}_0}(\mathfrak{t}_0)$ act by skew symmetric operators on $\mathfrak{g}_0$, so they are diagonalizable over $\mathbb{C}$. In particular, the same holds for $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{t})$ and hence for $\mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]}(\mathfrak{t}')$. So $\mathfrak{t}$ consists of semisimple elements. $\square$

Using the root space decomposition for $[\mathfrak{g}, \mathfrak{g}]$, we have

$$\begin{aligned}\mathfrak{g} &= Z(\mathfrak{g}) \oplus \mathfrak{t}' \oplus \bigoplus_{\alpha \in (\mathfrak{t}')^\vee} [\mathfrak{g}, \mathfrak{g}]_\alpha \\ &=: \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})} \mathfrak{g}_\alpha\end{aligned}$$

where $\mathfrak{g}_\alpha$ is **the α-weight space**

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{t}\}$$

and $\Delta(\mathfrak{g}, \mathfrak{t})$ is the set of **roots**:

$$\Delta(\mathfrak{g}, \mathfrak{t}) = \{\alpha \in \mathfrak{t}^\vee \backslash \{0\} \mid \mathfrak{g}_\alpha \neq 0\}.$$

The decomposition here

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})} \mathfrak{g}_\alpha$$

is the eigenspace space decomposition of $\mathfrak{g}$ with respect to $\mathrm{ad}\,\mathfrak{t}$.

Recall on $\mathfrak{g}_0$ there is an $\mathrm{Ad}(G)$-invariant inner product. In particular $\mathrm{Ad}(T)$ acts on $\mathfrak{g}_0$ by orthogonal operators. Extending the inner product to an hermitian inner product on $\mathfrak{g}_0$, we see again that $\mathrm{Ad}(T)$ acts on $\mathfrak{g}$ by unitary operators. Since T is abelian, the $\mathrm{Ad}(T)$-action is simultaneously diagonalizable, and the above root space decomposition is the eigenspace decomposition. Indeed, recall T is generated by $\exp \mathfrak{t}_0$, so it suffices to check $\mathfrak{g}_\alpha$ is a common eigenspace for $\{\exp H \mid H \in \mathfrak{t}_0\}$. But for $X \in \mathfrak{g}_\alpha$, $H \in \mathfrak{t}_0$, we have

$$\mathrm{Ad}(\exp H)X = e^{\mathrm{ad}\, H} X = e^{\alpha(H)} X.$$

In general we have $\mathrm{Ad}(t)X = \xi_\alpha(t)X$ ($t \in T$) for a unique character $\xi_\alpha \in \mathrm{Hom}_{\mathbf{TopGp}}(T, S^1)$. From above we see $\xi_\alpha(\exp H) = e^{\alpha(H)}$ and the differential of ξ_α is $\alpha|_{\mathfrak{t}_0}$. As a consequence, we see $\alpha|_{\mathfrak{t}_0}$ is purely imaginary.

Lemma I.5.6. Let T be a maximal torus in G, $\mathfrak{t} = \mathrm{Lie}(T)_\mathbb{C} \subseteq \mathfrak{g} = \mathrm{Lie}(G)_\mathbb{C}$, and $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$. If $H \in \mathfrak{t}$ satisfies $\alpha(H) \neq 0$ for all $\alpha \in \Delta$, then $C_\mathfrak{g}(H) = \mathfrak{t}$.

Proof. Say $X \in C_\mathfrak{g}(H)$, and write $X = H' + \sum_{\alpha \in \Delta} X_\alpha$ for its root space decomposition. Then

$$0 = [H, X] = 0 + \sum_{\alpha \in \Delta} [H, X_\alpha] = \sum_{\alpha \in \Delta} \alpha(H) X_\alpha.$$

Since $\alpha(H) \neq 0$, it follows that $X_\alpha = 0$ and hence $X = H' \in \mathfrak{t}$. $\square$

Theorem I.5.7. For a compact connected Lie group G, any two maximal abelian subalgebras of $\mathfrak{g}_0$ are conjugate via $\mathrm{Ad}(G)$.

Proof. Let T and T' be two maximal tori of G, and $\mathfrak{t}_0 = \mathrm{Lie}(T)$, $\mathfrak{t}_0' = \mathrm{Lie}(T')$ be their Lie algebras. Since $\#\Delta(\mathfrak{g}, \mathfrak{t}') < \infty$. by Lemma I.5.6 and the fact that a real vector space cannot be a finite union of its proper subspaces, we can find $X \in \mathfrak{t}_0'$ such that $C_\mathfrak{g}(X) = \mathfrak{t}'$. Similarly, we can find $Y \in \mathfrak{t}_0$ such that $C_\mathfrak{g}(Y) = \mathfrak{t}$

Let $(,)$ be an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}_0$. Since G is compact, the function $g \mapsto (\mathrm{Ad}(g)X, Y) \in \mathbb{R}$ attains its minimum; say $g_0 \in G$ minimizes this function. Then for any $Z \in \mathfrak{g}_0$, the function $\mathbb{R} \ni r \mapsto (\mathrm{Ad}(\exp rZ)\,\mathrm{Ad}(g_0)X, Y)$ is a smooth function which is minimized at $r = 0$. In particular,

$$0 = \frac{d}{dr}\bigg|_{r=0} (\mathrm{Ad}(\exp rZ)\,\mathrm{Ad}(g_0)X, Y) = (\mathrm{ad}(Z)\,\mathrm{Ad}(g_0)X, Y) = ([Z, \mathrm{Ad}(g_0)X], Y) = (Z, [\mathrm{Ad}(g_0)X, Y]).$$

Since Z is arbitrary, we conclude that $[\mathrm{Ad}(g_0)X, Y] = 0$, i.e., $\mathrm{Ad}(g_0)X \in C_{\mathfrak{g}_0}(Y) = \mathfrak{t}_0$. Then

$$\mathfrak{t}_0 = C_{\mathfrak{g}_0}(\mathrm{Ad}(g_0)X) = \mathrm{Ad}(g_0)C_{\mathfrak{g}_0}(X) = \mathrm{Ad}(g_0)\mathfrak{t}_0',$$

and the equality holds by maximality. □

Corollary I.5.7.1. For a compact connected Lie group G, any two maximal tori are conjugate.

Theorem I.5.8. If G is a connected compact Lie group and T is a maximal torus, then each element in G is conjugate to an element in T.

Corollary I.5.8.1. Let G be a connected compact Lie group.

1. Every element of G lies in a maximal torus of G.
2. The center $Z(G)$ lies in every maximal torus of G.

Corollary I.5.8.2. If G is a connected compact Lie group, then the exponential map $\exp_G : \mathrm{Lie}(G) \to G$ is surjective.

Theorem I.5.9. Let G be a connected compact Lie group, and $S \leqslant G$ be a torus in G. Then $g \in C_G(S)$, then there exists a torus S' in G containing S and g.

Corollary I.5.9.1. Let G be a connected compact Lie group.

(i) The centralizer of a torus in G is connected.

(ii) A maximal torus of G equals its centralizer in G.

I.5.3 Analytic Weyl group

Let G be a connected compact Lie group and let T be a maximal torus. The **analytic Weyl group** $W = W(G,T)$ of G is defined as the quotient

$$W(G,T) = N_G(T)/C_G(T) = N_G(T)/T.$$

The second equality is Corollary I.5.9.1.(ii). The Weyl group W acts on T by conjugations, and hence acts on $\mathfrak{t}_0 = \mathrm{Lie}(T)$. Note W acts faithfully on T, and hence faithfully on $\mathfrak{t}_0$.

Lemma I.5.10. Let G be a connected compact Lie group and T a maximal torus.

(i) T meets every conjugacy classes of G, and two elements in T are conjugate in G if and only if they are conjugate via $W(G,T)$. Hence, the conjugacy classes of G are in bijection with the quotient $T/W(G,T)$.

(ii) A continuous function $f \in C(T)$ extends to a conjugate-invariant continuous function on G if and only if it is invariant under $W(G,T)$.

Proof.

(i) The first part of (i) is Theorem I.5.8. Suppose $t, s \in T$ are conjugate, i.e., $t = gsg^{-1}$ for some $g \in G$. Consider the centralizer $C_G(t)$; the Lie algebra of $C_G(t)$ is

$$\operatorname{Lie} C_G(t) = \{X \in \operatorname{Lie}(G) \mid \operatorname{Ad}(t)X = X\}$$

by Lemma I.4.4. Since $\operatorname{Lie}(T)$ and $\operatorname{Ad}(g)\operatorname{Lie}(T)$ are maximal toral in $C_G(t)_0$, by Theorem I.5.7 there exists $g' \in C_G(t)$ such that $\operatorname{Lie}(T) = \operatorname{Ad}(g'g)\operatorname{Lie}(T)$. Exponentiating gives $g'g \in N_G(T)$. Since $(g'g)s(g'g)^{-1} = g'tg'^{-1} = t$, we are done.

(ii) Since $W(G,T)$ acts on T by conjugation, the only if part is clear. Conversely, suppose $f \in C(T)$ is invariant under Weyl group action. Composing with the map $G \to G/\text{conj} \cong T/W(G,T)$, we obtain a map $f : G \to \mathbb{C}$ invariant under conjugation. It remains to show it is continuous. Suppose $(g_n)_n \subseteq G$ is a sequence converging to g, and let $(x_n)_n \subseteq G$ be such that $x_n g_n x_n^{-1} \in T$ for each n. Since G and T are compact, by passing to a subsequence we can assume $(x_n)_n$ and $(x_n g_n x_n^{-1})_n$ are convergent, say to x and t. Then $xgx^{-1} = t$ by continuity and

$$\lim_{n\to\infty} f(g_n) = \lim_{n\to\infty} f(x_n g_n x_n^{-1}) = \lim_{n\to\infty} f(t_n) = f(t) = f(g)$$

by continuity of f on T. Hence $f \in C(G)$.

□

Recall that roots $\Delta(\operatorname{Lie}(G)_{\mathbb{C}}, \operatorname{Lie}(T)_{\mathbb{C}})$ on $\operatorname{Lie}(T)$ are purely imaginary. In view of this, put

$$\mathfrak{t}_{\mathbb{R}} = i\operatorname{Lie}(T) \subseteq \operatorname{Lie}(T)_{\mathbb{C}}.$$

Then all roots on $\mathfrak{t}_{\mathbb{R}}$ are real, and there is an injective homomorphism $\Delta(\operatorname{Lie}(G)_{\mathbb{C}}, \operatorname{Lie}(T)_{\mathbb{C}}) \to \mathfrak{t}_{\mathbb{R}}^{\vee}$. The negative of an invariant inner product on $\operatorname{Lie}(T)$ induces an invariant inner product B on $\mathfrak{t}_{\mathbb{R}}$. Then B induces an isomorphism

$$\begin{aligned} H : \mathfrak{t}_{\mathbb{R}}^{\vee} &\longrightarrow \mathfrak{t}_{\mathbb{R}} \\ \lambda &\longmapsto H_\lambda. \end{aligned}$$

Let us denote by $\langle , \rangle$ the inner product on $\mathfrak{t}_{\mathbb{R}}^{\vee}$ induced by B; we have

$$\langle \lambda, \mu \rangle = B(H_\lambda, H_\mu) = \lambda(H_\mu) = \mu(H_\lambda)$$

Denote by $(iZ(\operatorname{Lie}(G)))^{\vee}$ the image of $iZ(\operatorname{Lie}(G))$ under this isomorphism.

Lemma I.5.11. $\Delta(\operatorname{Lie}(G)_{\mathbb{C}}, \operatorname{Lie}(T)_{\mathbb{C}})$ spans the orthogonal complement of $(iZ(\operatorname{Lie}(G)))^{\vee}$. In particular, it defines a root system in the orthogonal complement.

Proof. Let $\lambda \in \Delta(\operatorname{Lie}(G)_{\mathbb{C}}, \operatorname{Lie}(T)_{\mathbb{C}})$ and $\mu \in (iZ(\operatorname{Lie} G))^{\vee}$. Then

$$\langle \lambda, \mu \rangle = \lambda(H_\mu) = 0.$$

(Recall $\Delta(\operatorname{Lie}(G)_{\mathbb{C}}, \operatorname{Lie}(T)_{\mathbb{C}})$ is constructed in a way that it is trivial on $Z(\operatorname{Lie}(G))$.) Also, recall that $\Delta(\operatorname{Lie}(G)_{\mathbb{C}}, \operatorname{Lie}(T)_{\mathbb{C}})$ is a root system on $[\operatorname{Lie}(G), \operatorname{Lie}(G)]$ and $\operatorname{Lie}(G) = Z(\operatorname{Lie}(G)) \oplus [\operatorname{Lie}(G), \operatorname{Lie}(G)]$, so the dimension matches, proving that $\Delta(\operatorname{Lie}(G)_{\mathbb{C}}, \operatorname{Lie}(T)_{\mathbb{C}})$ generates the orthogonal complement of $(iZ(\operatorname{Lie}(G)))^{\vee}$. □

For each $\alpha \in \Delta(\mathrm{Lie}(G)_{\mathbb{C}}, \mathrm{Lie}(T)_{\mathbb{C}})$ there is a reflection s_α on $\mathfrak{t}_{\mathbb{R}}$; explicitly

$$s_\alpha(x) := x - \frac{2\langle x, \alpha\rangle}{\|\alpha\|^2}\alpha.$$

Clearly s_α acts as identity on $(iZ(\mathrm{Lie}\, G))^\vee$, and is the usual reflection on its orthogonal complement. Define the (algebraic) Weyl group $W(\Delta(\mathrm{Lie}(G)_{\mathbb{C}}, \mathrm{Lie}(T)_{\mathbb{C}}))$ by

$$W(\Delta(\mathrm{Lie}(G)_{\mathbb{C}}, \mathrm{Lie}(T)_{\mathbb{C}})) := \langle s_\alpha \mid \alpha \in \Delta(\mathrm{Lie}(G)_{\mathbb{C}}, \mathrm{Lie}(T)_{\mathbb{C}})\rangle \leqslant \mathrm{GL}(\mathfrak{t}_{\mathbb{R}}^\vee).$$

On the other hand, since the analytic Weyl group $W(G,T)$ acts on $\mathrm{Lie}(T)$, it also acts on $\mathfrak{t}_{\mathbb{R}} = i\,\mathrm{Lie}(T)$ and hence on $\mathfrak{t}_{\mathbb{R}}^\vee$. This induces a homomorphism $W(G,T) \to \mathrm{GL}(\mathfrak{t}_{\mathbb{R}}^\vee)$.

Theorem I.5.12. The image of $W(G,T) \to \mathrm{GL}(\mathfrak{t}_{\mathbb{R}}^\vee)$ coincides with $W(\Delta(\mathrm{Lie}(G)_{\mathbb{C}}, \mathrm{Lie}(T)_{\mathbb{C}}))$.

Proof. To avoid cumbersome notations, set

$$\mathfrak{g}_0 = \mathrm{Lie}(G),\ \mathfrak{t}_0 = \mathrm{Lie}(T),\ \mathfrak{g} = \mathrm{Lie}(G)_{\mathbb{C}},\ \mathfrak{t} = \mathrm{Lie}(T)_{\mathbb{C}}.$$

In view of Theorem I.5.4 and Lemma I.5.2, we may assume G is semisimple, i.e., $\mathfrak{g}_0$ is semisimple. Let us identify $W(G,T)$ with its image in $\mathrm{GL}(\mathfrak{t}_{\mathbb{R}}^\vee)$.

To see $W(\Delta(\mathfrak{g},\mathfrak{t})) \subseteq W(G,T)$, let $\alpha \in \Delta(\mathfrak{g},\mathfrak{t})$. Denote by $\bar{\cdot}$ the complex conjugation on $\mathfrak{g}$ with respect to $\mathfrak{t}_0$, and extend B, an invariant inner product on $\mathfrak{g}_0$, to an hermitian inner product on $\mathfrak{g}$. Recall values of $\alpha \in \Delta(\mathfrak{g},\mathfrak{t})$ on $\mathfrak{t}_0$ are purely imaginary, so $\overline{\alpha(\overline{H})} = -\alpha(H)$ for $H \in \mathfrak{t}$. Hence if $E_\alpha \in \mathfrak{g}_\alpha$, then $\overline{E_\alpha} \in \mathfrak{g}_{-\alpha}$; if we write $E_\alpha = X_\alpha + iY_\alpha$ with X_α, $Y_\alpha \in \mathfrak{g}_0$, then

$$[X_\alpha, H] = -\frac{1}{2}[H, E_\alpha + \overline{E_\alpha}] = -\frac{1}{2}\alpha(H)(E_\alpha - \overline{E_\alpha}) = -i\alpha(H)Y_\alpha.$$

Also, since $[g_\beta, g_\beta] = 0$ for every root β, we have

$$[X_\alpha, Y_\alpha] = \frac{1}{4i}[E_\alpha + \overline{E_\alpha}, E_\alpha - \overline{E_\alpha}] = -\frac{1}{2i}[E_\alpha, \overline{E_\alpha}] = \frac{1}{2i}B(E_\alpha, \overline{E_\alpha})H_\alpha.$$

Since $B(E_\alpha, \overline{E_\alpha}) > 0$, define

$$r := \frac{\sqrt{2}\pi}{\|\alpha\|\sqrt{B(E_\alpha, \overline{E_\alpha})}}.$$

Since $X_\alpha \in \mathfrak{g}_0$, $g := e^{rX_\alpha} \in G$. For $H \in \mathfrak{t}_{\mathbb{R}}$, compute

$$\mathrm{Ad}(g)H = e^{rX_\alpha}H = \sum_{k\geqslant 0}\frac{r^k}{k!}(\mathrm{ad}\, X_\alpha)^k H.$$

If $\alpha(H) = 0$, then $\mathrm{Ad}(g)H = H$. If $H = H_\alpha$, then

$$r^2(\mathrm{ad}\, X_\alpha)^2 H_\alpha = r^2[X_\alpha, [X_\alpha, H_\alpha]] = -\frac{r^2}{2}\alpha(H_\alpha)B(E_\alpha, \overline{E_\alpha})H_\alpha = -\pi^2 H_\alpha.$$

Thus

$$\begin{aligned}
\mathrm{Ad}(g)H_\alpha &= \sum_{k\geqslant 0}\frac{r^{2k}}{(2k)!}(\mathrm{ad}\, X_\alpha)^{2k}H_\alpha + \sum_{k\geqslant 0}\frac{r^{2k+1}}{(2k+1)!}(\mathrm{ad}\, X_\alpha)^{2k+1}H_\alpha \\
&= \sum_{k\geqslant 0}\frac{(-1)^k\pi^{2k}}{(2k)!}H_\alpha + \sum_{k\geqslant 0}\frac{(-1)^k\pi^{2k}}{(2k+1)!}r[X_\alpha, H_\alpha] \\
&= (\cos\pi)H_\alpha + r\pi^{-1}(\sin\pi)[X_\alpha, H_\alpha] = -H_\alpha.
\end{aligned}$$

Hence $\mathrm{Ad}(g)$ leaves $\mathfrak{t}_{\mathbb{R}}$ invariant and acts as s_α on $\mathfrak{t}_{\mathbb{R}}$. This shows $W(\Delta(\mathfrak{g}, \mathfrak{t})) \subseteq W(G, T)$.

Next, note that $W(G, T)$ permutes roots. Indeed, for $H \in \mathfrak{t}$, $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$, $E_\alpha \in \mathfrak{g}_\alpha$ and $g \in N_G(T)$,

$$[H, \mathrm{Ad}(g)E_\alpha] = \mathrm{Ad}(g)[\mathrm{Ad}(g^{-1})H, E_\alpha] = \mathrm{Ad}(g)\alpha(\mathrm{Ad}(g^{-1})H)E_\alpha = (g\alpha)(H)\,\mathrm{Ad}(g)E_\alpha.$$

This shows $g\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$.

Let Π be a base for $\Delta(\mathfrak{g}, \mathfrak{t})$, and let $g \in W(G, T)$. Then $g\Pi$ is also a base for $\Delta(\mathfrak{g}, \mathfrak{t})$, so there exists $w \in W(\Delta(\mathfrak{g}, \mathfrak{t}))$ such that $wg\Pi = \Pi$. We claim wg acts as identity on $\mathfrak{t}_{\mathbb{R}}$. If so, then $g = w^{-1} \in W(\Delta(\mathfrak{g}, \mathfrak{t}))$, as wanted.

To this end, let $w' \in N_G(T)$ represent w and $g' \in N_G(T)$ represent g. Let Δ^+ be the positive roots in Π, and let $\delta = \dfrac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$. Then $wg\delta = \delta$ why, so $\mathrm{Ad}(w'g')H_\delta = H_\delta$. If $S = \overline{\langle \exp(irH_\delta) \mid r \in \mathbb{R} \rangle} \leqslant G$, then S is a torus and $w'g' \in C_G(S)$. We claim

$$C_{\mathfrak{g}_0}(\mathrm{Lie}\, S) = \mathfrak{t}_0.$$

If so, by Lemma I.4.5, Corollary I.5.9.1.(i) and Theorem I.4.6, then $C_G(S) = T$, i.e., $w'g' \in T$. This shows $\mathrm{Ad}(w'g')$ is an identity on T, and hence on $\mathfrak{t}_{\mathbb{R}}$.

It remains to prove our contention. Note that if α is a positive root, then $\langle \delta, \alpha \rangle > 0$. Hence $\alpha(H_\delta) \neq 0$ for all $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$, and hence $C_{\mathfrak{g}}(H_\delta) = \mathfrak{t}$ by Lemma I.5.6. Now

$$C_{\mathfrak{g}_0}(\mathrm{Lie}\, S) = \mathfrak{g}_0 \cap C_{\mathfrak{g}}(\mathrm{Lie}\, S) = \mathfrak{g}_0 \cap C_{\mathfrak{g}}(H_\delta) = \mathfrak{g}_0 \cap \mathfrak{t} = \mathfrak{t}_0$$

as claimed. □

I.5.4 Integral forms

Let G be a connected compact Lie group and T a maximal torus.

Lemma I.5.13. For a $\lambda \in \mathrm{Lie}(T)^\vee_{\mathbb{C}}$, TFAE:

(i) For $H \in \mathrm{Lie}(T)$ that satisfies $\exp(H) = 1$, we have $\lambda(H) \in 2\pi i\mathbb{Z}$.

(ii) There exist $\xi \in \mathrm{Hom}_{\mathbf{TopGp}}(T, S^1)$ such that $\xi(\exp H) = e^{\lambda(H)}$ for all $H \in \mathrm{Lie}(T)$.

Such an element λ is called an **analytically integral form**. In particular, every root in $\Delta(\mathrm{Lie}(G)_{\mathbb{C}}, \mathrm{Lie}(T)_{\mathbb{C}})$ is analytically integral.

Proof. The in particular part is clear. Assume (ii). If $H \in \mathrm{Lie}(T)$ satisfies $\exp(H) = 1$, then $1 = \xi(\exp H) = e^{\lambda(H)}$ so that $\lambda(H) \in 2\pi i\mathbb{Z}$. This proves (i). Now assume (i). Recall $\exp : \mathrm{Lie}(T) \to T$ is a continuous surjective homomorphism, so it induces a homeomorphism $\mathrm{Lie}(T)/\ker\exp \xrightarrow{\sim} T$. By (i) the continuous map $\mathrm{Lie}(T) \ni H \mapsto e^{\lambda(H)}$ descends to a continuous homomorphism $\xi : T \to \mathbb{C}^\times$. Since T is compact, $\xi(T) \subseteq S^1$ and (ii) is proved. □

Lemma I.5.14. If $\lambda \in \mathrm{Lie}(T)^\vee_{\mathbb{C}}$ is analytically integral, then $\dfrac{2\langle \lambda, \alpha \rangle}{\|\alpha\|^2} \in \mathbb{Z}$ for all $\alpha \in \Delta(\mathrm{Lie}(G)_{\mathbb{C}}, \mathrm{Lie}(T)_{\mathbb{C}})$. A functional satisfying this integral condition is called an **algebraically integral form**.

Proof. Let $\bar{\cdot}$ denote the complex conjugation on $\mathrm{Lie}(G)_{\mathbb{C}}$ with respect to $\mathrm{Lie}(G)$, and extend an invariant inner product on $\mathrm{Lie}(G)$ to a hermitian inner product B on $\mathrm{Lie}(G)_{\mathbb{C}}$. Fix $\alpha \in \Delta(\mathrm{Lie}(G)_{\mathbb{C}}, \mathrm{Lie}(T)_{\mathbb{C}})$

and pick a nonzero $E_\alpha \in (\mathrm{Lie}(G)_{\mathbb{C}})_\alpha$; normalize E_α so that $B(E_\alpha, \overline{E_\alpha}) = \dfrac{2}{\|\alpha\|^2}$. Write $E_\alpha = X_\alpha + iY_\alpha$ with $X_\alpha, Y_\alpha \in \mathrm{Lie}(G)$ and put $Z_\alpha = -i\,\|\alpha\|^{-2} H_\alpha \in \mathrm{Lie}(G)$. Then

$$\begin{aligned}
[Z_\alpha, X_\alpha] &= \frac{i}{\|\alpha\|^2}[X_\alpha, H_\alpha] = Y_\alpha \\
[X_\alpha, Y_\alpha] &= Z_\alpha \\
[Z_\alpha, Y_\alpha] &= X_\alpha,
\end{aligned}$$

by the computation in Theorem I.5.12. The assignment

$$(X_\alpha, Y_\alpha, Z_\alpha) \leftrightarrow \left(\frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right)$$

defines a Lie algebra isomorphism

$$\mathbb{R}X_\alpha \oplus \mathbb{R}Y_\alpha \oplus \mathbb{R}Z_\alpha \cong \mathfrak{su}(2).$$

Since $\mathrm{SU}(2)$ is simply connected, by Theorem I.4.7 there exists a Lie group homomorphism $\varphi : \mathrm{SU}(2) \to G$ whose differential is the above Lie algebra map. Under the complexification of the above Lie algebra map, $h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{su}(2)_{\mathbb{C}}$ maps to $2iZ_\alpha = 2\,\|\alpha\|^{-2} H_\alpha$. Hence

$$\varphi_{*,e}(ih) = -2Z_\alpha = 2i\,\|\alpha\|^{-2} H_\alpha$$

and

$$1 = \varphi(1) = \varphi(e^{2\pi i h}) = e^{2\pi i \varphi_{*,e}(h)} = e^{2\pi i (2\|\alpha\|^{-2} H_\alpha)}.$$

Since λ is analytically integral, by Lemma I.5.13.(i), we have $\lambda(2\pi i(2\,\|\alpha\|^{-2} H_\alpha)) \in 2\pi i\mathbb{Z}$, and hence $\dfrac{2\langle \lambda, \alpha\rangle}{\|\alpha\|^2} \in \mathbb{Z}$. $\square$

Hence we have the inclusions

$$\mathbb{Z}\Delta(\mathrm{Lie}(G)_{\mathbb{C}}, \mathrm{Lie}(T)_{\mathbb{C}}) \subseteq \{\text{analytically integral forms}\} \subseteq \{\text{algebraically integral forms}\}$$

I.6 Semisimple Lie groups

Definition. A Lie group is called **nilpotent/solvable/semisimple** if it is connected and its Lie algebra is nilpotent/solvable/semisimple.

I.7 Lie Group Actions

I.7.1 Topological Group Actions

For G an abstract group, a **(left) G-set** is a nonempty set X together with a group homomorphism $G \to S_X$, where S_X denotes the group of all bijections on X. For two G-sets X, Y, a set-theoretic map $f : X \to Y$ is called G-equivariant if $f(gx) = gf(x)$ for all $x \in X$ and $g \in G$. In this way we obtain the category of G-sets.

Let G be a topological group. A **(left) G-space** is a nonempty topological space X on which G acts on the left such that the action map $G \times X \to X$ is continuous. For two G-spaces X, Y, a continuous map $f : X \to Y$ is called **G-equivariant** if $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$.

- For $x \in X$, the subset $Gx := \{gx \mid g \in G\} \subseteq X$ is called the **G-orbit** of x, and the subgroup $G_x := \{g \in G \mid gx = x\}$ is called the **isotropy group** of x, or the **stabilizer** of x.

- The set of all G-orbits is denoted by $G\backslash X$, and is called the **orbit space**. We equip $G\backslash X$ with the final topology with respect to the projection $X \to G\backslash X$.

- With the above topology, $G\backslash X$ is a G-space and the projection $X \to G\backslash X$ is open and G-equivariant.

- If G is a Lie group, a **smooth G-space** is required to be a smooth manifold on which G acts smoothly.

Definition. Let G be a group and X a G-set.

1. The action is **faithful/effective** if the associated map $G \to S_X$ is injective, where S_X is the group of all bijections on X.

2. The action is **free** if all stabilizers are trivial.

3. The action is **transitive** if $\#G\backslash X = 1$. In this case, X is called a **homogeneous** G-set.

4. If the action is free and transitive, then X is called a **principal homogeneous G-set**, or a **G-torsor**.

- For a subgroup $H \leqslant G$, the left coset space G/H is a homogeneous G-set. Conversely, if X is a homogeneous G-set, any point $x \in X$ gives a set-theoretic G-equivariant bijection $G/G_x \cong X$.

- For X a homogeneous space and $x \in X$, the map $G/G_x \to X$ is always continuous, but it may not be a homeomorphism.

- For a subgroup H, the G-action on G/H is faithful if and only if H contains no trivial normal subgroup.

Example I.7.1. Consider the real line $(\mathbb{R}, +)$ with usual euclidean topology and the torus $(\mathbb{R}^2/\mathbb{Z}^2, +)$ with usual topology. Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and consider the topological subgroup $F := \{(x, \alpha x) \mid x \in \mathbb{R}\} \leqslant \mathbb{R}^2/\mathbb{Z}^2$. This is isomorphic to $\mathbb{R}$ as abstract groups, and the identity map $\mathbb{R} \to F$ is continuous, but it is not open.

Proposition I.7.2. Let G a topological group and $H \leqslant G$ a subgroup. Equip the left coset space G/H with the quotient topology.

1. The projection $\pi : G \to G/H$ is an open map. If H is compact, then π is also a closed map.

2. If H is normal, G/H is a topological group.

3. The natural G-action on G/H is continuous, and G/H is a homogeneous G-space.

4. If G is Hausdorff, then G/H is Hausdorff if and only if H is closed.

5. If H and G/H are compact, then so is G.

6. If H and G/H are connected, then so is G.

Proof.

1. For $U \subseteq G$ open, $\pi^{-1}\pi(U) = \bigcup_{g \in H} gU$ is open. If H is compact, for $C \subseteq G$ closed, $\pi^{-1}\pi(C) = CH$ is closed.

4. If G/H is Hausdorff, then $H = \pi^{-1}(eH)$ is closed. Suppose H is closed. We must show the diagonal $\Delta = \Delta_{G/H}$ is closed in $G/H \times G/H$. It suffices to show $(\pi \times \pi)^{-1}(\Delta)$ is closed in $G \times G$. Note that $(g, g') \in (\pi \times \pi)^{-1}(\Delta)$ if and only if $gH = g'H$, i.e, $g^{-1}g' \in H$. Hence $(\pi \times \pi)^{-1}(\Delta) = q^{-1}(H)$ is a closed set, where $q : G \times G \to G$ is defined by $q(gg') = g^{-1}g'$.

5. $\pi : G \to G/H$ is closed (by 1.) with compact fibre (each of which homeomorphic to H), so it is proper by Proposition A.7.1. Since G/H is compact by assumption, $G = \pi^{-1}(G/H)$ is compact.

6. We prove if G/H is connected and G is not connected, then H is not connected. Let U, V be disjoint nonempty open sets in G with $U \cup V = G$. Since π is open, by connectivity of G/H we have $\pi(U) \cap \pi(V) \neq \varnothing$, i.e., we can find a coset gH with $U \cap gH \neq \varnothing \neq V \cap gH$. But then $gH = (U \cup V) \cap gH = (U \cap gH) \cup (V \cap gH)$ so $gH \cong H$ is not connected.

□

Lemma I.7.3. Let X be a homogeneous G-space. Suppose there exists $x_0 \in X$ such that every open unit-neighborhood U of G the subspace $Ux_0 \subseteq X$ contains x_0 in its interior. Then the natural map $G/G_x \to X$ is open for all $x \in X$, and hence a homeomorphism.

Proof. By translation, it suffice to show $G/G_{x_0} \to X$ is open. We must show for every open subset V of G, $Vx_0 \subseteq X$ is open. Let $g \in V$; then $g^{-1}V$ is an open unit neighborhood of G, so x_0 lies in the interior of $g^{-1}Vx_0$. Hence gx_0 lies in the interior of Vx_0 for all $g \in V$, and the proof is done. □

Theorem I.7.4. Let X be a homogeneous G-space. Suppose

- G is LCH and separable, and
- X is Hausdorff and a Baire space.

Then $G/G_x \to X$ are homeomorphisms for all $x \in X$.

Proof. Let $x_0 \in X$ be any point and U an open unit-neighborhood of G. By the last lemma, it suffices to show x_0 lies in the interior of Ux_0. Let $W \subseteq U$ be a compact symmetric unit-neighborhood such that $W^2 \subseteq U$. Let $S \subseteq G$ be a countable dense subset. Then it is easy to see $G = \bigcup_{s \in S} sW$, so $X = \bigcup_{s \in S} sWx_0$ is a countable union of compact sets which are closed since X is Hausdorff. Since X is Baire, $s'Wx_0$ has nonempty interior for some $s' \in S$, i.e., $s'gx_0$ lies in the interior of $s'Wx_0$ for some $g \in W$. Hence x_0 lies in the interior of $g^{-1}Wx_0 = Wx_0 \subseteq Ux_0$, as wanted. □

I.7.2 Principal Bundles

Definition. A **fibre bundle** (E, B, π, F) consisting of three topological spaces E, B, F together with a continuous surjection $\pi : E \to B$ such that each point $x \in B$ has an open neighborhood U for which there is a homeomorphism $\phi : \pi^{-1}(U) \to U \times F$ making the triangle

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

commute, where $\pi^{-1}(U)$ is given the subspace topology from E.

- E is called the **total space**, B the **base space**, F the **fibre space** and π the **bundle projection**. The homeomorphism ϕ in the definition is called a **local trivialization**. A fibre bundle (E, B, π, F) is sometimes written as $F \to E \xrightarrow{\pi} B$, or simply $E \xrightarrow{\pi} B$.

- ϕ induces a homeomorphism on the fibre $\pi^{-1}(b) =: E_b \to \{b\} \times F$ for each $b \in U$. Also, since pr_1 is open, the bundle projection π is an open map.

- A cover map $p : Y \to X$ with X connected is a fibre bundle $(Y, X, p, p^{-1}(x))$, where $x \in X$ is an arbitrary point and $p^{-1}(x)$ is a discrete subspace of Y. Moreover, a fibre bundle (E, B, π, F) is a cover if and only if F is discrete.

A **morphism between fibre bundles** (E, B, π, F) and (E', B', π', F') consists of two continuous maps $\phi : E \to E'$ and $\psi : B \to B'$ making the diagram

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "\pi"] & E' \arrow[d, "\pi'"] \\ B \arrow[r, "\psi"] & B' \end{tikzcd}$$

commute.

- Since π is surjective, ψ is determined by ϕ. We say ϕ is **the bundle map covering ψ.**

Definition. Let G be a topological group. A **principal G-bundle** is a fibre bundle $G \to P \xrightarrow{\pi} X$ together with a continuous right G-action $P \times G \to P$ such that each local trivialization $\pi^{-1}(U) \to U \times G$ is G-equivariant, where G acts on $U \times G$ by right translation on G.

- G is called the **structure group** of the principal bundle. Each local trivialization induces a homeomorphism $P_x \to G$ with inverse $G \to P_x$ given by $g \mapsto xg$.

- By definition, π induces a continuous bijection $P/G \to X$, and each fibre is a G-torsor.

- For a Lie group G, a principal G-bundle $\pi : P \to X$ is **smooth** if all maps and actions involved are smooth. Then the bundle map π is submersive since locally it is $\pi^{-1}(U) \to U \times G \xrightarrow{\mathrm{pr}_1} U$.

A **morphism from a principal G-bundle $P \xrightarrow{\pi} X$ to a principal H-bundle $Q \xrightarrow{p} Y$** is a morphism (ϕ, ψ) of fibre bundles $P \xrightarrow{\pi} X$ to $Q \xrightarrow{p} Y$ together with a continuous group homomorphism $\theta : G \to H$ such that $\phi(yg) = \phi(p)\theta(g)$ for all $y \in P$ and $g \in G$.

Let $P \xrightarrow{\pi} X$ be a principal G-bundle, and let $\phi : \pi^{-1}(U) \to U \times G$ be a local trivialization. Consider the continuous map $\sigma = \sigma_\phi : U = U \times \{1\} \xrightarrow{\phi^{-1}} \pi^{-1}(U) \subseteq P$; clearly, $\pi \circ \sigma = \mathrm{id}_U$, so $\sigma : U \to P$ is a section to the bundle projection $\pi : P \to X$. For $g \in G$, since ϕ is equivariant, we have

$$\phi(\sigma(x)g) = \phi(\sigma(x))g = (x, 1)g = (x, g)$$

so that ϕ is completely determined by σ.

Let $\phi : \pi^{-1}(U) \to U \times G$ and $\psi : \pi^{-1}(V) \to V \times G$ be two local trivializations with $U \cap V \neq \varnothing$. On the intersection we have the transition map:

$$\psi \circ \phi^{-1} : (U \cap V) \times G \to (U \cap V) \times G$$

This induces a map $\theta : U \cap V \to G$ satisfying $(\psi \circ \phi^{-1})(x, 1) = (x, \theta(x))$. Since ψ and ϕ are G-equivariant, we have

$$(\psi \circ \phi^{-1})(x, g) = (\psi \circ \phi^{-1})(x, 1)g = (x, \theta(x))g = (x, \theta(x)g)$$

On the other hands, $(\psi \circ \phi^{-1})(x, 1) = (x, \theta(x)) = (x, 1)\theta(x)$ implies $\sigma_\phi(x) = \sigma_\psi(x)\theta(x)$, where σ_ϕ and σ_ψ are their associated local sections.

Let $\phi_i : \pi^{-1}(U_i) \to U_i \times G$ be three local trivializations with $U_1 \cap U_2 \cap U_3 \neq \varnothing$. Then we have three transition maps $\theta_{ij} := \phi_j \circ \phi_i^{-1}$ which satisfy

$$(x, \theta_{13}(x)) = \phi_3\phi_1^{-1}(x, 1) = \phi_3\phi_2^{-1}\phi_2\phi_1^{-1}(x, 1) = \phi_3\phi_2^{-1}(x, \theta_{12}(x)) = (x, \theta_{23}(x)\theta_{12}(x))$$

i.e., the **cocycle condition**

$$\theta_{13}(x) = \theta_{23}(x)\theta_{12}(x)$$

Conversely,

Theorem I.7.5. Let X be a space with an open cover $\{U_\alpha\}$ and G a topological group. Suppose for each pair (α, β), there exists $\theta_{\alpha,\beta} : U_{\alpha\beta} \to G$ such that each triple (α, β, γ) satisfies the cocycle condition. Then there exists a unique (up to isomorphism) principal G-bundle $P \xrightarrow{\pi} X$ with local trivializations ϕ_α such that the transition maps are given by the $\theta_{\alpha\beta}$.

Definition. Let G be a topological group, $P \xrightarrow{\pi} X$ a principal G-bundle and F a left G-space. Let G act on the product space $P \times F$ diagonally, i.e., $(p, v)g := (pg, g^{-1}v)$. The (right) coset space $(P \times F)/G$ is called the **associated bundle of** P **with typical fibre** F **and base** X, and is denoted by $E = P \times^G F$.

- We topologize $P \times^G F$ as usual so that $P \times G \to P \times^G F$ is an open continuous map.
- The bundle projection $\pi : P \to X$ gives rise to a continuous surjection $P \times F \xrightarrow{\mathrm{pr}_1} P \xrightarrow{\pi} X$. Since the G-action on P preserves fibres, the map induces a continuous surjection $\pi_E : P \times^G F \to X$.
- For $(p, v) \in P \times^G F$, denote by $[p, v]$ its equivalence class in $P \times^G F$. For $w, v \in F$, $[p, v] = [p, w]$ if and only if $p = pg$, $v = g^{-1}w$ for some $g \in G$. Since G acts freely on P, we have $g = 1$, so $v = w$. Hence for each $p \in P$, the map

$$\begin{aligned} \rho_p : F &\longrightarrow E_{\pi(p)} \\ v &\longmapsto [p, v] \end{aligned}$$

  is a homeomorphism into. This is also surjective: for $[q, v] \in E_{\pi(p)}$, we have $q = pg$ for some $g \in G$, so $[q, v] = [pg, v] = [p, gv]$. Hence *each* $p \in P$ *parametrizes the fibre* $E_{\pi(p)}$ *by the typical fibre* F.
- Let $\sigma : U \to P$ be a local section of π. The map

$$\begin{aligned} \psi_\sigma : U \times F &\longrightarrow \pi_E^{-1}(U) \\ (x, v) &\longmapsto [\sigma(x), v] \end{aligned}$$

  is a homeomorphism (note that the local section σ is open). Thus the associated bundle $E \xrightarrow{\pi_E} X$ is a fibre bundle.

- Suppose $\sigma : U \to P$ and $\tau : V \to P$ are two local sections, and $\theta : U \cap V \to G$ be the transition map so that $\tau(x) = \sigma(x)\theta(x)$ for $x \in U \cap V$. If we denote by ψ_σ, ψ_τ the associated trivializations, then

$$(\psi_\sigma \circ \psi_\tau^{-1})(x, v) = \psi_\sigma^{-1}([\tau(x), v]) = \psi_\sigma^{-1}([\sigma(x), \theta(x)v]) = (x, \theta(x)v)$$

Let $\sigma : X \to E = P \times^G X$ be a section to the associated bundle $\pi : E \to X$. Take any $v \in F$ and define $f_\sigma : P \to F$ by

$$f_\sigma(p) = \rho_p^{-1}(\sigma(\pi(p, v))) \in F$$

where $\rho_p : F \to E_{\pi(p)}$ is the map $v \mapsto [p, v]$ defined as above. The map clearly is independent of the choice of v. This map is P-equivariant, i.e., $f_\sigma(pg) = g^{-1}f(p)$ for all $p \in P$, $g \in G$. Indeed,

$$f_\sigma(pg) = \rho_{pg}^{-1}(\sigma(\pi(pg, v))) = \rho_{pg}^{-1}\sigma(\pi(p, gv)) = g^{-1}\rho_p\sigma(\pi(p, gv)) = g^{-1}f(p),$$

where we use the identity $\rho_{pg}(v) = \rho_p(gv)$ here. Conversely, let $f : P \to F$ be equivariant and define $f' : P \to E$ by $f'(p) = \rho_p f(p) \in E_{\pi(p)}$. For $g \in G$,

$$f'(pg) = \rho_{pg}f(pg) = \rho_p(gg^{-1}f(p)) = \rho_p f(p) = f'(p)$$

so f' is constant along pG. Hence it descends to a map $\sigma_f : X \to E$ (recall that $P/G \cong X$).

Next we see how $\sigma \mapsto f_\sigma$ behaves under local trivialization. Let $\chi : U \to P$ be a section to $P \to X$; then $\phi_\chi : U \times G \to P$ given by $(x, g) \mapsto \chi(x)g$ is a local trivialization, and $\psi_\chi : U \times F \to \pi^{-1}(U)$ given by $(x, v) \mapsto [\chi(x), v]$ is a local trivialization. Write $\psi_\chi \circ^{-1} \sigma|_U : U \to \pi^{-1}(U) \cong U \times F$ as $x \mapsto (x, \tau(x))$ for some $\tau : U \to F$; in other word,

$$\sigma(x) = \psi_\chi(x, \tau(x)) = [\chi(x), \tau(x)].$$

Then $f_\sigma(\chi(x)) = \rho_{\chi(x)}^{-1}\sigma(\pi(\chi(x), v)) = \rho_{\chi(x)}^{-1}[\chi(x), \tau(x)] = \tau(x)$, and hence

$$f_\sigma \circ \phi_\chi(x, g) = f_\sigma(\chi(x)g) = g^{-1}f_\sigma(\chi(x)) = g^{-1}\tau(x) \in F.$$

From which we see f_σ is smooth if and only if σ is smooth. This defines a bijection

$$\Gamma^\infty(X, P \times^G X) \longrightarrow \{f \in C^\infty(P, F) \mid f(pg) = g^{-1}f(p) \text{ for all } g \in G,\, p \in P\}$$

I.7.3 Homogeneous Spaces

Proposition I.7.6. Let G be a Lie group and $H \leqslant G$ a closed subgroup. Then G/H has a unique smooth structure that makes the projection $G \to G/H$ a smooth principal bundle with the structure group H (with H acting on G by right translation).

Proof. Choose a Euclidean metric on $\mathrm{Lie}(G)$, and let $\mathrm{Lie}(G) = V \oplus \mathrm{Lie}(H)$ be an orthogonal decomposition. Let $V_\varepsilon := \{X \in V \mid |X| < \varepsilon\}$ and $D_\varepsilon := \exp V_\varepsilon$. Take $\varepsilon > 0$ so small that $\exp : V_\varepsilon \to D_\varepsilon$ is a diffeomorphism; this makes D_ε is smooth manifold with $T_e D_\varepsilon = V$.

1) For $\varepsilon > 0$ small enough, the map

$$\begin{aligned} \mu : D_\varepsilon \times H &\longrightarrow G \\ (g, h) &\longmapsto gh \end{aligned}$$

is an open embedding. The differential of μ at (e, e) is addition $V \oplus \mathrm{Lie}(H) \to \mathrm{Lie}(G)$, so we can find $\varepsilon > 0$ small enough and some open unit-neighborhood U of H such that $\mu : D_\varepsilon \times U \to D_\varepsilon U$ is diffeomorphism. Translating shows $\mu : D_\varepsilon \times H \to G$ is a local diffeomorphism everywhere.

It remains to show we can make μ injective by choosing ε small enough. Let $d_1, d_2 \in D_\varepsilon$ and $h_1, h_2 \in H$ such that $d_1 h_1 = d_2 h_2$. Then $h_1 h_2^{-1} = d_1^{-1} d_2$. Let $V \underset{\text{open}}{\subseteq} G$ such that $U = V \cap H$ and let $\varepsilon > 0$ so small that $D_\varepsilon^{-1} D_\varepsilon \subseteq V$. Then $h := h_1 h_2^{-1} = d_1^{-1} d_2 \in V \cap H = U$. Since μ is injective on $D_\varepsilon \times U$, that $\mu(d_1, h) = \mu(d_2, e)$ implies $d_1 = d_2$ and $h = e$, or $h_1 = h_2$, showing the global injectivity.

2) The sets $U_g := g D_\varepsilon H$, $g \in G$ are invariant under right H-action. For $g \in G$, define $h_g : U_g / H \to D_\varepsilon$ by

$$h_g^{-1} : D_\varepsilon \xrightarrow[\sim]{} D_\varepsilon \times \{e\} \hookrightarrow D_\varepsilon \times H \xrightarrow[\sim]{\mu} D_\varepsilon H \xrightarrow[\sim]{\ell_g} g D_\varepsilon H =: U_g \xrightarrow{\pi} U_g / H$$

h_g^{-1} is a continuous bijection, and is open since it is a composition of open maps, so it is a homeomorphism. Let $g, g' \in G$ such that we can find $x, y \in D_\varepsilon$ with $h_g^{-1}(x) = h_{g'}^{-1}(y)$. This means $gx = g' y h$ for some $h \in H$, so that $yh = g'^{-1} g x$, which implies $y = \mathrm{pr}_1(y, h) = \mathrm{pr}_1 \mu^{-1}(g'^{-1} g x)$, which is smooth in x. Hence G/H is made into a smooth manifold with the atlas h_g, $g \in G$.

For $g \in G$, define $\varphi_g : U_g \to U_g / H \times H$ by

$$\varphi_g^{-1} : U_g / H \times H \xrightarrow{h_g \times \mathrm{id}} D_\varepsilon \times H \xrightarrow{\mu} D_\varepsilon H \xrightarrow{\ell_g} g D_\varepsilon H = U_g$$

In the course of the proof we also see that $\dim G/H = \dim G - \dim H$ and $T_e(G/H) \cong \mathrm{Lie}(G) / \mathrm{Lie}(H)$ given by the splitting $\mathrm{Lie}(G) = V \oplus \mathrm{Lie}(H)$.

3) It remains to show G/H is Hausdorff and second countable. The second follows from the openness of $G \to G/H$, and the first is Proposition I.7.2.4.

4) The uniqueness follows from the following lemma.

Lemma I.7.7. Let X, Y, Z be smooth manifolds with a commutative diagram with each map continuous

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Y \\ & Z \arrow[ur, "g"'] & \end{tikzcd}$$

If f is smooth and π is submersive, then g is smooth on $\pi(X)$.

Proof. Since π is submersive, $\pi(X)$ is open in Z. Let $z \in \pi(X)$ and $x \in \pi^{-1}(z)$. By Constant rank theorem we can find a chart U about x such that $\pi : U \to \pi(U)$ takes the form $(x^1, \ldots, x^n) \mapsto (x^1, \ldots, x^s)$, where $n = \dim X$, $s = \dim Z$. Define $\psi : \pi(U) \to U$ by $(x^1, \ldots, x^s) \mapsto (x^1, \ldots, x^s, 0, \ldots, 0)$. Then ψ is smooth on $\pi(U)$, and on $\pi(U)$ we have $g = g \circ \pi \circ \psi = f \circ \psi$ is smooth. $\square$

To conclude the proof, it suffices to take $X = G$, $Y = Z = G/H$, $f = \pi$ the canonical projection $G \to G/H$ and $g = \mathrm{id}_{G/H}$.

$\square$

Corollary I.7.7.1. If $N \trianglelefteq G$ is a closed normal subgroup of a Lie group G, then G/N with canonical smooth and group structures is a Lie group.

Corollary I.7.7.2. Let $f : G \to H$ be a Lie group homomorphism and $N \trianglelefteq G$ be a normal closed subgroup contained in $\ker f$. Then the induced abstract group homomorphism $G/N \to H$ is a Lie group homomorphism.

Corollary I.7.7.3. Let $f : G \to H$ be a Lie group homomorphism. Then $f(G)$ is a Lie subgroup of H with Lie algebra $\mathrm{Lie}(f(G)) = f_{*,e}(\mathrm{Lie}(G)) \subseteq \mathrm{Lie}(H)$.

Proof. By the last corollary, the canonical map $G/\ker f \to H$ is a Lie group homomorphism. Since it is injective, this is a Lie subgroup of H. Since $G/\ker f \cong f(G)$ as groups, this proves the first statement, and the Lie algebra part follows from Lemma I.3.1.

□

Proposition I.7.8. Let G be a Lie group and M be a smooth homogeneous left G-space. Then the map

$$\begin{aligned} f : G/G_x &\longrightarrow M \\ gG_x &\longmapsto gx \end{aligned}$$

is a diffeomorphism. (See also Theorem I.7.4.) In particular, $G_x \to G \to M$ is a smooth principal bundle.

Proof. It is clear that f is a continuous bijection. Also, f is smooth, since locally it is $U \to U \times G_x \cong \pi^{-1}(U) \to M$ where $\pi : G \to G/G_x$ is the projection. It remains to show f is immersive (then f is automatically a diffeomorphism). Since f is G-equivariant, it has constant rank. Now since f is injective, it must be immersive by Constant rank theorem. □

Example I.7.9.

1. $\mathrm{SO}(n)$ acts smoothly on $\mathbb{R}^n$ on the left, and when $n \geqslant 2$ the orbits are the spheres $\{x \in \mathbb{R}^n \mid |x| = r\}$, $r \geqslant 0$. To see this, let $x \neq y \in \mathbb{R}^n$ such that $|x| = |y| = r > 0$. Consider the plane spanned by x and y and let y' be a unit vector lying on the plane that is orthonormal to x. Extending x/r and y' to be orthonormal basis of $\mathbb{R}^n$, we may assume $x = (r, 0, 0, \ldots, 0)$ and $y = (a, b, 0, \ldots, 0)$ with $a^2 + b^2 = r^2$. Now we can easily find $A \in \mathrm{SO}(2)$ such that $A := A \oplus I_{n-2} \in \mathrm{SO}(n)$ takes x to y.

   Particularly, $\mathrm{SO}(n)$ acts transitively on the $n-1$-sphere S^{n-1}. The unit vector $e_1 = (1, 0, \ldots, 0)$ has stabilizer consisting of matrices of the form

   $$\begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix}$$

   with $C \in \mathrm{SO}(n-1)$, so we have a diffeomorphism $\mathrm{SO}(n)/\mathrm{SO}(n-1) \cong S^{n-1}$, and the $\mathrm{SO}(n-1)$-principal bundle

   $$\mathrm{SO}(n-1) \to \mathrm{SO}(n) \to S^{n-1}$$

   Similarly, the $\mathrm{O}(n)$-action on S^{n-1} gives rise to the $\mathrm{O}(n-1)$-principal bundle

   $$\mathrm{O}(n-1) \to \mathrm{O}(n) \to S^{n-1}$$

2. Similarly, the natural $\mathrm{SU}(n)$-action on $\mathbb{C}^n$ has orbits $\{z \in \mathbb{C}^n \mid |z| = r\}$, $r \geqslant 0$. In particular, $\mathrm{SU}(n)$ acts transitively on S^{2n-1}, and the stabilizer of $e_1 \in \mathbb{C}^n$ is $\mathrm{SU}(n-1) \subseteq \mathrm{SU}(n)$. This gives a diffeomorphism $\mathrm{SU}(n)/\mathrm{SU}(n-1) \cong S^{2n-1}$ and an $\mathrm{SU}(n-1)$-principal bundle

$$\mathrm{SU}(n-1) \to \mathrm{SU}(n) \to S^{2n-1}$$

Similarly, there is a $\mathrm{U}(n-1)$-principal bundle

$$\mathrm{U}(n-1) \to \mathrm{U}(n) \to S^{2n-1}$$

A similar argument shows that letting $\mathrm{Sp}(n)$ act on $\mathbb{H}^n$ gives rise to a principal bundle

$$\mathrm{Sp}(n-1) \to \mathrm{Sp}(n) \to S^{4n-1}$$

3. The group $\mathrm{SU}(n)$ acts on $\mathbb{C}^n$, and the induced action on $\mathbb{C}\mathbf{P}^{n-1} = \mathbb{P}^{n-1}_{\mathbb{C}} = \mathbf{P}(\mathbb{C}^n)$ is transitive. The stabilizer of the line $\mathbb{C}e_1$ consists of matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & C \end{pmatrix} \in \mathrm{SU}(n)$$

and this is the image of the embedding

$$\begin{aligned} \mathrm{U}(n-1) &\longrightarrow \mathrm{SU}(n) \\ C &\longmapsto \begin{pmatrix} \det C^{-1} & 0 \\ 0 & C \end{pmatrix} \end{aligned}$$

In this way we get a principal bundle

$$\mathrm{U}(n-1) \to \mathrm{SU}(n) \to \mathbf{P}(\mathbb{C}^n) \tag{$\spadesuit$}$$

Similarly we have an embedding $\mathrm{O}(n-1) \to \mathrm{SU}(n)$ and a principal bundle

$$\mathrm{O}(n-1) \to \mathrm{SO}(n) \to \mathbf{P}(\mathbb{R}^n)$$

4. In the case ($\spadesuit$)($n = 2$), we have $\mathrm{U}(1) = S^1$, $\mathrm{SU}(2) = S^3$ and $\mathbf{P}(\mathbb{C}^2) = S^2$ the Riemann sphere. Hence we have a principal bundle

$$S^1 \to S^3 \to S^2$$

This is called the **Hopf fibration**. Explicitly, the bundle projection is

$$\begin{aligned} S^3 &\longrightarrow S^2 \\ (a,b) &\longmapsto [a:b] = \frac{a}{b} \end{aligned}$$

In polar coordinates, it is

$$(r_0 e^{i\theta_0}, r_1 e^{i\theta_1}) \mapsto \frac{r_0}{r_1} e^{i(\theta_0 - \theta 1)},\ r_0^2 + r_1^2 = 1$$

5. Let U an n-dimensional real/complex/quaternionic inner product space. An **orthonormal k-frame** is an ordered orthonormal set in U. The **(compact) Stiefel manifold** $V_k(U)$ is the set of all orthonormal k-frames of U. Topologize $V_k(U)$ as a (closed) subspace of the

product of k-copies of the unit sphere in U. The orthogonal group $\mathrm{O}(U)$ acts $V_k(U)$ on the left continuously and transitively.

Let us consider the case $U = \mathbb{R}^n$. Then the stabilizer of $(e_1, \ldots, e_k)$ is $\mathrm{O}(n-k)$, and thus by Theorem I.7.4 we have a homeomorphism $\mathrm{O}(n)/\mathrm{O}(n-k) \cong V_k(\mathbb{R}^n)$ (of course one can use this to topologize $V_k(\mathbb{R}^n)$). We use this bijection to give a smooth structure on $V_k(\mathbb{R}^n)$, so we have a principal bundle

$$\mathrm{O}(n-k) \to \mathrm{O}(n) \to V_k(\mathbb{R}^n)$$

Similarly, we have

$$\mathrm{U}(n-k) \to \mathrm{U}(n) \to V_k(\mathbb{C}^n)$$

$$\mathrm{Sp}(n-k) \to \mathrm{Sp}(n) \to V_k(\mathbb{H}^n)$$

In fact, for $k < m \leqslant n$, we have a fibre bundle

$$V_{m-k}(\mathbb{R}^{n-k}) \to V_m(\mathbb{R}^n) \to V_k(\mathbb{R}^n)$$

where the last projection sends an m-frame onto the k-frame formed by its first k-vectors.

6. Let U an n-dimensional real/complex/quaternionic vector space. The **Grassmannian** $\mathrm{Gr}_k(U)$ is the set of all k-planes in U. There is a natural surjection $\pi : V_k(U) \to \mathrm{Gr}_k(U)$ sending an k-frame to the plane it spans, and we use this to topologize $\mathrm{Gr}_k(U)$; this coincides with the usual topology imposed on $\mathrm{Gr}_k(U)$. Consider the case $U = \mathbb{R}^n$. Then the natural surjection is a principal $\mathrm{O}(k)$-bundle:

$$\mathrm{O}(k) \to V_k(\mathbb{R}^n) \to \mathrm{Gr}_k(\mathbb{R}^n)$$

The orthogonal group $\mathrm{O}(U)$ also acts on $\mathrm{Gr}_k(U)$ transitively. The stabilizer of the k-plane $\mathbb{R}^k = \mathbb{R}^k \times \{0\} \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k}$ is $\mathrm{O}(k) \times \mathrm{O}(n-k) \subseteq \mathrm{O}(n)$ via the embedding

$$(A, B) \mapsto \begin{pmatrix} A & \\ & B \end{pmatrix}$$

Hence we have a homeomorphism $\mathrm{Gr}_k(\mathbb{R}^n) \cong \dfrac{\mathrm{O}(n)}{\mathrm{O}(k) \times \mathrm{O}(n-k)}$, and we can use this map to define a smooth structure on $\mathrm{Gr}_k(\mathbb{R}^n)$, and hence we obtain a principal bundle

$$\mathrm{O}(k) \times \mathrm{O}(n-k) \to \mathrm{O}(n) \to \mathrm{Gr}_k(\mathbb{R}^n)$$

Of course, we can reverse the process: use the usual smooth structure on $\mathrm{Gr}_k(\mathbb{R}^n)$ and show the $\mathrm{O}(n)$-action on is smooth. Similarly, we have

$$\mathrm{U}(k) \times \mathrm{U}(n-k) \to \mathrm{U}(n) \to \mathrm{Gr}_k(\mathbb{C}^n)$$

$$\mathrm{Sp}(k) \times \mathrm{Sp}(n-k) \to \mathrm{Sp}(n) \to \mathrm{Gr}_k(\mathbb{H}^n)$$

On the other hand, $\mathrm{GL}_n(\mathbb{R})$ acts transitively on $\mathrm{Gr}_k(\mathbb{R}^n)$, and the stabilizer of $\mathbb{R}^k$ is the subgroup P consisting matrices of the form

$$\begin{pmatrix} A & * \\ & C \end{pmatrix}, \; A \in \mathrm{GL}_k(\mathbb{R}),\; C \in \mathrm{GL}_{n-k}(\mathbb{R})$$

so we have diffeomorphisms

$$\mathrm{GL}_n(\mathbb{R})/P \cong \mathrm{Gr}_k(\mathbb{R}^n) \cong \frac{\mathrm{O}(n)}{\mathrm{O}(k) \times \mathrm{O}(n-k)}$$

7. We have a natural action of $\mathrm{SL}_2(\mathbb{R})$ on the upper half plane $\mathbb{H}$ (not confused with the quaternion). It is smooth and transitive, and the stabilizer of i is $\mathrm{SO}(n)$, so we have a diffeomorphism $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2) \cong \mathbb{H}$.

   Also, $\mathrm{SL}_2(\mathbb{C})$ acts on $\mathbb{P}^1_{\mathbb{C}}$ smoothly and transitively, and the stabilizer of ∞ is the subgroup P of upper triangular matrices with determinant 1. Hence $\mathbb{P}^1_{\mathbb{C}} \cong \mathrm{SL}_2(\mathbb{C})/P$.

I.7.4 Proper Free Actions

Definition. Let G be a topological group and M a right G-space. We say the G-action on M is **proper** if the continuous map

$$\begin{aligned} M \times G &\longrightarrow M \times M \\ (m, g) &\longmapsto (m, mg) \end{aligned}$$

is proper.

Theorem I.7.10. Let G be a Lie group and M a smooth right G-space on which the action is proper and free. Then the orbit space M/G has a unique smooth structure of dimension $\dim M - \dim G$ such that the canonical projection $M \to M/G$ is a smooth principal G-bundle.

I.8 Integration on Lie groups

In this section, for a topological space X, we use $C_c(X)$ to denote the space of real-valued continuous functions with compact support (rather than the complex-valued ones).

I.8.1 Left Invariant densities and forms

Let M be a smooth manifold with a smooth Lie group action G. For $g \in G$, we write ℓ_g for the diffeomorphism $p \mapsto g.p$ on M.

Definition. An s-density (resp. form) ω on M is called G**-invariant** if $\ell_g^* \omega = \omega$ for any $g \in G$.

- Denote by $\mathrm{Vol}_s(M)^G$ (resp. $(\Omega^k M)^G$) the set of G-invariant s-densities (resp. k-forms) on M.
- As the G-action is smooth, we easily see that any G-invariant s-density (resp. k-form) is automatically smooth.

Assume either ω is a G-invariant density on M, or M is oriented, ω is G-invariant top form and G acts on M in an orientation-preserving way. Then for any $f \in C_c(M)$,

$$\int_M (\ell_g^* f)\omega = \int_M (\ell_g^* f)(\ell_g^* \omega) = \int_M \ell_g^*(f\omega) = \int_M f\omega. \tag{$\spadesuit$}$$

The last equality holds for the first case since ω is a density, and holds for the second case as ℓ_g is orientation-preserving.

Now consider the Lie group G of dimension n acting on itself. There are two bijections, one being canonical:

$$\begin{aligned} (\Omega^n G)^G &\longrightarrow \textstyle\bigwedge^n \mathrm{Lie}(G)^\vee \\ \omega &\longmapsto \omega_e \end{aligned}$$

and the other canonical up to a choice of local charts about the identity $e \in G$:

$$\begin{aligned} \mathrm{Vol}(G)^G &\longrightarrow \mathrm{Vol}(\mathrm{Lie}(G)) \\ \eta &\longmapsto \eta_e \end{aligned}$$

Since $\dim \mathrm{Vol}(\mathrm{Lie}(G)) = 1$, it follows that G admits a left-invariant smooth positive density η, and such η is unique up to a positive scalar. Also, as $\dim_{\mathbb{R}} \bigwedge^n \mathrm{Lie}(G) = 1$, G admits a left-invariant smooth volume form ω such that ω_e represents a to-be-given orientation of $\mathrm{Lie}(G)$, and such ω is unique up to a positive scalar as well. Clearly, $|\omega|$ is a left-invariant smooth positive density as long as ω is a left-invariant smooth volume form.

Let η be a left-invariant positive density on G. By (♠), we see the outer Radon measure μ_η is left-invariant. In other words μ_η is a left Haar measure on G.

We compute the modular function Δ_G of G. Let $f \in C_c(G)$ and $g \in G$. Then

$$\int_G r_g^* f\eta = \int_G (r_g)^*(f r_{g^{-1}}^* \eta) = \int_G f r_{g^{-1}}^* \eta = \int_G f c_g^* \eta = \int_G f\,|\det \mathrm{Ad}(g)|\eta.$$

By Theorem 2.3.1.3., we deduce that

$$\Delta_G(g) = |\det \mathrm{Ad}(g^{-1})|$$

for every $g \in G$. Similarly, for a left-invariant volume form ω on G, we have

$$r_g^* \omega = (\det \mathrm{Ad}(g))\omega.$$

Hence, a left-invariant volume form on G is right invariant if and only if $\det \mathrm{Ad}(g) : G \to \mathbb{R}^\times$ is trivial.

Next, let H be a closed subgroup of G. By Theorem I.3.2, H is a regular submanifold of G, so it is a Lie group as well. We discuss the integration on the homogeneous space G/H. We already see in Theorem 2.4.7 that a G-invariant exists on G/H if and only if $\Delta_G|_H = \Delta_H$. By the result above, this is equivalent to saying $|\det \mathrm{Ad}_G(h)| = |\det \mathrm{Ad}_H(h)|$ for any $h \in H$.

For $g \in G$, denote by $\ell_g : G/H \to G/H$ the map given by $g'H \mapsto gg'H$. For $h \in H$, the conjugation $c(h) : G \to G$ defined by $c(h)g := hgh^{-1}$ descends to a map $c(h) : G/H \to G/H$, which coincides with ℓ_h. Indeed $c(h)(gH) = hgh^{-1}H = hgH = \ell_h(gH)$. Define

$$\begin{aligned} \mathrm{Ad}_{G/H} : H &\longrightarrow \mathrm{Aut}\, T_{eH}G/H \\ h &\longmapsto (\ell_h)_{*,eH} \end{aligned}$$

It is easy to see that $\det \mathrm{Ad}_{G/H}(h) = \dfrac{\det \mathrm{Ad}_G(h)}{\det \mathrm{Ad}_H(h)}$ (c.f. Proposition I.7.6).

A density η on G/H is G-invariant if and only if $\eta = \ell_g^* \eta$. In particular, we have $\eta = \ell_h^* \eta$ for any $h \in H$. Evaluating at $H = eH$, we must have

$$\eta_H(X_1, \ldots, X_n) = \eta_{hH}((\ell_h)_{*,hH} X_1, \ldots, (\ell_h)_{*,hH} X_n) = |\det(\ell_h)_{*,eH}|\eta_H(X_1, \ldots, X_n)$$

for any $X_i \in T_H(G/H)$, so that $1 = |\det(\ell_h)_{*,eH}| = |\det \mathrm{Ad}_{G/H}(h)|$ for any $h \in H$. Conversely, if $|\det \mathrm{Ad}_{G/H}(h)| = 1$ for any $h \in H$, it is easy to show G/H admits a nontrivial G-invariant positive density on G/H. By the same computation one shows that G/H admits a G-invariant volume form if and only if $\det \mathrm{Ad}_{G/H}(h) = 1$ for any $h \in H$.

In summary:

Proposition I.8.1. Let G be a Lie group and H a closed subgroup of G.

1. Any left Haar measure on G is a smooth measure in the sense of 17.5.
2. $\Delta_G(g) = |\det \mathrm{Ad}_G(g^{-1})|$ for any $g \in G$.
3. G/H admits a nontrivial G-invariant positive density if and only if
$$|\det \mathrm{Ad}_G(h)| = |\det \mathrm{Ad}_H(h)|$$
for any $h \in H$. This is the case if H is compact.
4. G/H admits a G-invariant volume form if and only if
$$\det \mathrm{Ad}_G(h) = \det \mathrm{Ad}_H(h)$$
for any $h \in H$.
5. If H is connected, then $\det \mathrm{Ad}_{G/H}(h) = |\det \mathrm{Ad}_{G/H}(h)|$ for any $h \in H$.

Proof. The function $h \mapsto |\det \mathrm{Ad}_{G/H}(h)|$ is a continuous homomorphism $H \to \mathbb{R}_{>0}$, so if H is compact, it must be trivial as $\{1\}$ is the only compact subgroup of $\mathbb{R}_{>0}$. Similarly, the function $h \mapsto \det \mathrm{Ad}_{G/H}(h)$ is a continuous homomorphism $H \to \mathbb{R}^\times$, so if H is connected, its image must lie entirely in $\mathbb{R}_{>0}$, hence 4. ☐

Let us conclude this subsection by a discussion on integration on principal G-bundles, where G is a LCH group or a Lie group. Let $\pi : M \to X$ be a principal G-bundle with M, X LCH; in fact, it suffices to assume X is Hausdorff, as $M/G \cong X$ and M/G is always locally compact. Let dg be a left Haar measure on G. For $f \in C_c(M)$, define a function $\displaystyle\int_G f : M \to \mathbb{R}$ by

$$\left(\int_G f\right)(x) := \int_G f(xg)dg$$

The integral exists as the domain of integration is actually $\pi^{-1}(xG) \cap \operatorname{supp} f$, which is compact. Since dg is left-invariant, the integral is G-invariant, so $\displaystyle\int_G f \in C_c(X)$. By the way, this shows that $\displaystyle\int_G$ defines a map $C_c(M) \to C_c(X)$, and this is usually called the **integration along the fibre**, and is denoted by $f \mapsto \pi_* f$ (pushforward f along the bundle projection π).

Now take any outer Radon measure μ on X, and form the linear functional $C_c(M) \to \mathbb{R}$ by

$$f \mapsto \int_X \left(\int_G f\right) d\mu$$

This is positive linear functional on $C_c(M)$, so by Riesz's representation theorem it is uniquely determined by a outer Radon measure on M, which we denote by $d\mu \otimes dg$. Hence

$$\int_M f d\mu \otimes dg = \int_X \left(\int_G f\right) d\mu = \int_X \left(\int_G f(xg)dg\right) d\mu(x)$$

The measure $d\mu \otimes dg$ is G-invariant if and only if

$$\int_X \left(\int_G f(xg)dg\right) d\mu(x) = \int_X \left(\int_G f(xgg')dg\right) d\mu(x) = \Delta_G((g')^{-1}) \int_X \left(\int_G f(xg)dg\right) d\mu(x)$$

for any $g' \in G$, i.e., G is unimodular.

Conversely, suppose we are given an outer Radon measure ν on M. A similar argument in Lemma 2.4.2, together with a version of Lemma 2.4.3, shows that $\pi_* : C_c(M) \to C_c(X)$ is surjective. Define a linear functional $I : C_c(X) \to I$ as follows. For $g \in C_c(X)$, take any $f \in C_c(M)$ with $\pi_* f = g$ and put $I(g) := \displaystyle\int_M f d\nu$. The same argument in Theorem 2.4.7.(i) shows I is positive and well-defined as long as we assume ν satisfies

$$\int_M f(xg) d\nu(x) = \Delta_G(g^{-1}) \int_M f(x) d\nu(x) \ (g \in G).$$

By Riesz's representation theorem again I is uniquely determined by an outer Radon measure, which we usually denote by $d\mu = \dfrac{d\nu}{dg}$ (not confused with Radon Nikodym derivative!). Again, it satisfies

$$\int_M f d\nu = \int_X \left(\int_G f(xg) dg \right) d\mu(x)$$

Let us go to the smooth world. Let G be a Lie group and assume $\pi : M \to X$ is a smooth principal G-bundle. We claim if the measure μ on X is from some positive density, then the measure $d\mu \otimes dg$ on M is also from a positive density. Since G is a Lie group, dg is a smooth measure, and we identify it with a left-invariant positive density on G. Define $d\mu \otimes dg$ as a density on G as follows. For $p \in M$, take a local trivialization $U \times G$ around p, and say p corresponds to $(x, g) \in U \times G$. Then $T_pM \cong T_xU \times T_gG$, and define

$$(d\mu \otimes dg)_p(X_1, \ldots, X_n, Y_1, \ldots, Y_m) = (d\mu)_p(X_1, \ldots, X_n)(dg)_g(Y_1, \ldots, Y_m).$$

To show this is well-defined, if $V \times G$ is another trivialization about p, then there exists $\theta : U \cap V \to G$ such that the transition map from $U \times G$ to $V \times G$ is given by $(x', g') \mapsto (x', \theta(x')g')$. Using the fact that dg is left-invariant, we see $d\mu \otimes dg$ is a well-defined density on M. It follows from Fubini's the integration against the density $d\mu \otimes dg$ is really the same as what we define above.

I.8.2 Invariant de Rham cohomology

In this subsection, for a smooth manifold M, we use $\Omega^k M$ denote the smooth global sections of $\bigwedge^k (TM)^\vee \to M$ instead. There is a **de Rham complex**

$$0 \longrightarrow \Omega^0 M \xrightarrow{d} \Omega^1 M \xrightarrow{d} \Omega^2 M \longrightarrow \cdots \longrightarrow \Omega^m M \xrightarrow{d} 0$$

where $m = \dim M$, and $d : \Omega^k M \to \Omega^{k+1} M$ is the **exterior derivative**. The cohomology of the de Rham complex is called the **de Rham cohomology** of M, and is denoted by $H^\bullet_{\mathrm{dR}}(M, \mathbb{R})$.

Suppose G is a Lie group acting on M smoothly; denote by $\alpha : G \to \mathrm{Diffeo}(M)$ the action map. A smooth k-form $\omega \in \Omega^k M$ is called **(G-)invariant** if $\alpha(g)^*\omega = \omega$ for any $g \in G$. Put $(\Omega^k M)^G$ to be the subspace of invariant k-forms. Since $d\alpha(g)^* = \alpha(g)^* d$, the sequence $((\Omega^\bullet M)^G, d)$ forms a subcomplex of the de Rham complex $(\Omega^\bullet M, d)$. The inclusion then induces a map on cohomology groups $H^k((\Omega^\bullet M)^G, d) \to H^k_{\mathrm{dR}}(M, \mathbb{R})$. Again, as $d\alpha(g)^* = \alpha(g)^* d$, the group G acts on $H^k_{\mathrm{dR}}(M, \mathbb{R})$ naturally, and the above cohomology map goes into $H^k_{\mathrm{dR}}(M, \mathbb{R})^G$. Hence we have a map

$$H^k((\Omega^\bullet M)^G, d) \to H^k_{\mathrm{dR}}(M, \mathbb{R})^G.$$

We are going to show that this is an isomorphism when G is a compact Lie group.

Assume G is compact; we normalize the Haar measure so that the total volume of G is 1. Define $I : \Omega^k M \to \Omega^k M$ by

$$I(\omega)_p(X_1, \ldots, X_k) = \int_G (\alpha(g)^*\omega)_p(X_1, \ldots, X_k) dg = \int_G \omega_{\alpha(g)p}(\alpha(g)_{*,p} X_1, \ldots, \alpha(g)_{*,p} X_k) dg.$$

The integral is well-defined as the integrand is continuous in g and G is compact. Taking a chart about p, we can see $I(\omega)$ is really a smooth k-form, so I is well-defined. In fact, $I(\omega) \in (\Omega^k M)^G$ because dg is a left-invariant Haar measure. Thus I actually defines a map $I : \Omega^k M \to (\Omega^k M)^G$.

Lemma I.8.2. $I : \Omega^k M \to (\Omega^k M)^G$ restricts to the identity on $(\Omega^k M)^G$, and is a chain map.

Proof. The first is clear. For the second, we apply Corollary F.7.6.1 to compute

$$\begin{aligned} d(I\omega)(X_0, \dots, X_k)(p) &= \sum_{i=0}^{k} (-1)^i X_i((I\omega)(X_0, \dots, \widehat{X_i}, \dots, X_k))(p) \\ &\quad + \sum_{0 \leqslant i < j \leqslant k} (-1)^{i+j} (I\omega)_p([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k) \\ &= \sum_{i=0}^{k} (-1)^i X_{i,p} \int_G \omega_p(\alpha(g)_{*,p} X_0, \dots, \widehat{\alpha(g)_{*,p} X_{i,p}}, \dots, \alpha(g)_{*,p} X_{k,p}))dg \\ &+ \sum_{0 \leqslant i < j \leqslant k} (-1)^{i+j} \int_G \omega_p(\alpha_{*,p}[X_i, X_j]_p, X_{0,p}, \dots, \widehat{\alpha(g)_{*,p} X_{i,p}}, \dots, \widehat{\alpha(g)_{*,p} X_{j,p}}, \dots, \alpha(g)_{*,p} X_{k,p})dg \end{aligned}$$

Since G is compact, $X_{i,p}$ and $\displaystyle\int$ interchange, giving

$$d(I\omega)(X_0, \dots, X_k)(p) = \int_G (d\omega)_p(\alpha(g)_{*,p} X_{0,k}, \dots, \alpha(g)_{*,p} X_{k,p})dg = I(d\omega)(X_0, \dots, X_k)(p)$$

$\square$

Theorem I.8.3. The inclusion $((\Omega^\bullet M)^G, d) \to (\Omega^\bullet M, d)$ induces an isomorphism

$$H^k((\Omega^\bullet M)^G, d) \cong H^k_{\mathrm{dR}}(M, \mathbb{R})^G.$$

Proof. Denote by J the inclusion. We saw right before Lemma I.8.2 that $I \circ J = \mathrm{id}$. In particular, this shows $J^* : H^k((\Omega^\bullet M)^G, d) \to H^k_{\mathrm{dR}}(M, \mathbb{R})$ is injective and $I^* : H^k_{\mathrm{dR}}(M, \mathbb{R}) \to H^k((\Omega^\bullet M)^G, d)$ is surjective. We need to show J^* is surjective onto $H^k_{\mathrm{dR}}(M, \mathbb{R})^G$.

Let $\omega \in \Omega^k M$ be a closed form represent a class in $H^k_{\mathrm{dR}}(M, \mathbb{R})^G$. Let $g \in G$. Since ω represents a class invariant under G, we have $\omega - \alpha(g)^*\omega = d\eta$ for some $\eta = \eta_g \in \Omega^{k-1} M$. Therefore, for any smooth p-cycle $\gamma \in \Delta_p(M)$, by Stokes' theorem we have

$$\int_\gamma \omega - \int_\gamma \alpha(g)^*\omega = \int_\gamma d\eta = \int_{\partial\gamma} \eta = 0,$$

and hence by Fubini,

$$\int_\gamma I(\omega) = \int_\gamma \left(\int_G \alpha(g)^*\omega dg \right) = \int_G (\alpha(g)^*\omega)\, dg = \int_G \left(\int_\gamma \omega \right) dg = \int_\gamma \omega.$$

By de Rham theorem, $I(\omega)$ and ω represent the same cohomology class, and this finishes the proof.

$\square$

Assume further G is connected. Then $\alpha(g)$ is homotopic to $\alpha(e) = \mathrm{id}_M$, implying the G-action on $H^\bullet_{\mathrm{dR}}(M, \mathbb{R})$ is trivial. Hence taking $M = G$ in Theorem I.8.3 yields

Corollary I.8.3.1. Let G be a compact connected Lie group. Then $H^\bullet_{\mathrm{dR}}(G, \mathbb{R})$ is naturally isomorphic to $H^\bullet(\bigwedge^\bullet \mathrm{Hom}_{\mathbb{R}}(\mathrm{Lie}(G), \mathbb{R}), d)$, where d is given by

$$d\omega(X_0, \dots, X_k) = \sum_{0 \leqslant i < j \leqslant k} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k).$$

I.8.3 Integration by parts on Lie groups

Lemma I.8.4. Let G be a Lie group and ω be a top form. If ω is left-invariant, then $\mathcal{L}_X\omega = 0$ for all $X \in \operatorname{Lie}(G)$. The converse holds if G is connected.

Proof. The flow generated by X can be given by $\varphi_t(g) = \exp_G(tX)g$. We then have

$$\varphi_t^*\omega = \ell^*_{\exp_G(tX)^{-1}}\omega.$$

Hence $\mathcal{L}_X\omega = 0$ if and only if $t \mapsto \ell^*_{\exp_G(tX)^{-1}}\omega$, if and only if $\ell^*_{\exp_G(tX)^{-1}}\omega = \omega$. The latter holds for every $X \in \operatorname{Lie}(G)$ if ω is left-invariant.

Suppose G is connected; by Lemma I.2.11 we have $G = \langle \exp_G(t)X \mid t \in \mathbb{R}, X \in \operatorname{Lie}(G)\rangle$. Consequently, $\mathcal{L}_X\omega = 0$ for all $X \in \operatorname{Lie}(G)$ if and only if $\ell_g^*\omega = \omega$ for all $g \in G$, i.e., ω is left-invariant. $\square$

Corollary I.8.4.1. Let G be a Lie group, ω a left-invariant top form and $X \in \operatorname{Lie}(G)$. Then for $f, g \in C_c^\infty(G)$, we have

$$\int_G (Xf)g\omega = -\int_G fX(g)\omega.$$

Proof. This follows from integration by parts and Lemma I.8.4.

$\square$

Part VII

Algebra

Appendix J

Central simple algebra

In this chapter, rings are unital and algebras are only assumed to be unital and associative. All modules are referred to as left modules. For a ring R, denote by R^{op} the opposite ring. Hence an R^{op}-module is naturally identified as a right R-module.

Phrase the terminology in terms of abelian category?

J.1 Semisimplicity

Definition. Let R be a ring. An R-module is called **simple** if it is nonzero and contains no proper nontrivial R-submodules.

Lemma J.1.1 (Schur)**.** Let R be a ring and M, N be two simple R-modules. Then $\mathrm{Hom}_R(M, N) = \{0\} \cup \mathrm{Isom}_R(M, N)$.

Proof. If $f : M \to N$ be a nonzero R-homomorphisms, then $\ker f \neq M$ and $\mathrm{Im}\, f \neq 0$. Hence $\ker f = 0$ and $\mathrm{Im}\, f = N$ by simplicity, so f is an isomorphism. $\square$

Corollary J.1.1.1. Let R be a ring and M a simple R-module. Then $\mathrm{End}_R M$ is a division ring.

Definition. Let R be a ring. An R-module is called **semisimple** if any of its submodule is its direct summand.

Lemma J.1.2. Let R be a ring and M a semisimple R-module. Then every nonzero submodule of M contains a simple submodule.

Proof. It suffices to show for each $0 \neq v \in M$, the submodule Rv contains a simple module. Consider the homomorphism $f : R \to M$ given by $f(r) = rv$. Since $v \neq 0$, $\ker f \neq R$, so it is contained in a maximal ideal $\mathfrak{m}$ of R by Zorn's lemma. Now $\mathfrak{m}v \subseteq M$ is a submodule, so by semisimplicity $M = \mathfrak{m}v \oplus N$ for some submodule N. Then $Rv = \mathfrak{m}v \oplus (N \cap Rv)$, for if $rv = mv + n$ for some $m \in \mathfrak{m}$, and $n \in N$, then $n = (r - m)v \in Rv$. Since $\mathfrak{m}$ is maximal, it follows that $N'' \cap Rv$ is a simple R-module. $\square$

Lemma J.1.3. Let R be a ring and M a nonzero R-module. TFAE:

(i) M is semisimple.

(ii) M is a sum of its all simple submodules.

(iii) M is the direct sum of a collection of its simple submodules.

Proof. For (i)⇒(ii), consider the collection of submodules

$$\left\{ \sum_{i \in I} M_i \subseteq M \mid (M_i)_{i \in I} \text{ is a family of simple submodules of } M \right\}.$$

This set is partially ordered by inclusion, and contains joins. By Zorn's lemma, it contains a maximal element; call it N. If it were not M, by semisimplicity $M = N \oplus N'$ with $N' \neq 0$. By Lemma J.1.2 N' would contain a simple submodule, which contradicts to the maximality of N. Hence $M = N$.

For (ii)⇒(iii), say $M = \sum_{i \in I} M_i$ with each M_i being a simple submodule of M. Let $J \subseteq I$ be a maximal index subset such that the sum $N := \sum_{i \in J} M_i$ is direct. But then $N = M$, for either $M_i \cap M_j = 0$ or $M_i = M_j$ holds by simplicity. The direction (iii)⇒(i) is proved by the same argument. ☐

Lemma J.1.4. Every subquotient of a semisimple module is again semisimple. Arbitrary direct sum of semisimple modules is semisimple.

Lemma J.1.5. Let R be a ring and M a semisimple R-module. Suppose M admits a decomposition

$$M = M_1^{\oplus r_1} \oplus \cdots \oplus M_n^{r_n}$$

with $0 < r_i < \infty$, M_i being its simple submodule and each M_i and M_j being non-isomorphic. If

$$M = L_1^{\oplus s_1} \oplus \cdots \oplus L_k^{s_k}$$

is another decomposition of the same type, then $k = n$, and up to permutation $M_i \cong L_i$, $s_i = r_i$ for $i \in [n]$.

Let R be a ring and M an R-module. There is a natural pairing

$$\begin{aligned} \operatorname{End}_R M \times M &\longrightarrow M \\ (\varphi, m) &\longmapsto \varphi(m). \end{aligned}$$

With this map we turn M into an $\operatorname{End}_R M$-module which extends its R-module structure. Notice that

$$\operatorname{End}_{\operatorname{End}_R M} M = Z(\operatorname{End}_R M)$$

the center of the ring $\operatorname{End}_R M$, and the natural map $R \to \operatorname{End}_R M$ has image lying in $Z(\operatorname{End}_R M)$.

Theorem J.1.6 (Jacobson density)**.** Let R be a ring and M a semisimple R-module. For each $f \in \operatorname{End}_{\operatorname{End}_R M} M$ and a finite subset $S \subseteq M$, there exists $r \in R$ such that $rv = f(v)$ for $v \in S$.

Proof. Let $s = \#S$. We begin by proving the special case when $S = \{v\}$ is a singleton. By semisimplicity, write $M = Rv \oplus N$, and denote by $\pi : M \to Rv \subseteq M$ the projection, which lies in $\operatorname{End}_R M$. Then

$$f(v) = f(\pi(v)) = \pi(f(v))$$

so $f(v) = rv$ for some $r \in R$.

For $s > 1$, consider the map $f^{\oplus s} : M^{\oplus s} \to M^{\oplus s}$. We claim that $f^{\oplus s}$ commutes with all $\operatorname{End}_R(M^{\oplus s})$. Assuming this, by the previous case we've proven, by write $S = (v_i)_{i \in [s]}$ we see there exists $r \in R$ such that $f^{\oplus}(v_1, \ldots, v_s) = r(v_1, \ldots, v_s)$, which is what we want.

It suffices to show $f^{\oplus s}$ commutes with any $\varphi \in \operatorname{End}_R(M^{\oplus s})$. Write $\varphi = (\varphi_{ij})_{1 \leqslant i,j \leqslant n}$ with $\varphi_{ij} \in \operatorname{End}_R M$ such that

$$\varphi(x_1, \ldots, x_n) = \left(\sum_{j=1}^{n} \varphi_{1j} x_j, \ldots, \sum_{j=1}^{n} \varphi_{nj} x_j \right)$$

Then since $f \in \operatorname{End}_{\operatorname{End}_R M} M$, it commutes with all φ_{ij}'s, whence

$$\begin{aligned} f^n(\varphi(x_1, \ldots, x_n)) = f^n \left(\sum_{j=1}^{n} \varphi_{1j} x_j, \ldots, \sum_{j=1}^{n} \varphi_{nj} x_j \right) &= \left(\sum_{j=1}^{n} f(\varphi_{1j} x_j), \ldots, \sum_{j=1}^{n} f(\varphi_{nj} x_j) \right) \\ &= \left(\sum_{j=1}^{n} \varphi_{1j} f(x_j), \ldots, \sum_{j=1}^{n} \varphi_{nj} f(x_j) \right) = \varphi(f^n(x_1, \ldots, x_n)). \end{aligned}$$

This proves the claim. □

Equip M with discrete topology, and equip $\operatorname{End}_R M$ with pointwise convergence topology (or compact-open topology.) The theorem can be restated as saying that the image of the natural map

$$R \longrightarrow \operatorname{End}_R M$$

is dense in $\operatorname{End}_{\operatorname{End}_R M} M$.

Definition. A ring R is called **primitive** if it admits a faithful simple R-module.

Lemma J.1.7. Let D be a division ring and M a D-module. Assume $S \subseteq \operatorname{End}_D M$ is a transitive subring, in the sense that for any $x, y \in M$ with $x \neq 0$, there exists $\varphi \in S$ with $\varphi x = y$. Then S is primitive.

Proof. Clearly M has a S-module structure, and it is faithful. For any $0 \neq x \in M$, by transitivity of S we see $Sx = M$. This finishes the proof. □

Corollary J.1.7.1. A ring R is primitive if and only if it is isomorphic to a dense subring of $\operatorname{End}_D V$ for some division ring D and a (discrete) D-module V.

Proof. Assume R is primitive, and say M is the faithful simple R-module. By Schur's lemma, $D := \operatorname{End}_R M$ is a simple R-module, and the natural map $R \to D$ is injective by faithfulness. Now R is a dense subring of $\operatorname{End}_D M$ by Theorem J.1.6.

Now assume R is a dense subring of $\operatorname{End}_D V$. By Lemma J.1.7 it suffices to show R is transitive. But any nonzero element x in V is contained in a D-basis for V, so for any $y \in V$ we can find $\varphi \in \operatorname{End}_D V$ with $\varphi(x) = y$. By density of S we can choose $\varphi \in S$. This proves the transitivity, and concludes the proof. □

Lemma J.1.8. Let D be a division ring. Then $M_n(D)$ is left artinian and left noetherian for each $n \geqslant 1$.

Proof. Denote

$$I_j := \{T e_j \mid T \in M_n(D)\}$$

where $e_j = (\delta_{ij})_{i \in [n]}$. By construction this is a left ideal of $M_n(D)$ and clearly

$$M_n(D) = I_1 \oplus \cdots \oplus I_n.$$

We claim each I_j is a simple $M_n(D)$-submodule. Assuming this, we see $M_n(D)$ admits a finite composition series, which proves the lemma.

This amounts to show D^n is a simple $M_n(D)$-module. But this is immediate as $M_n(D) = \operatorname{End}_D(D^n)$ and by linear algebra (over D). $\square$

Theorem J.1.9. Let R be a primitive ring and M a faithful simple R-module. Put $D = \operatorname{End}_R M$, which is a division ring.

(i) The ring R is left artinian if and only if $n := \dim_D M < \infty$. If it is the case, $R \cong M_n(D)$.

(ii) If R is not left artinian, then for any $n \geqslant 1$ there exists a subring R_n of R with a surjective homomorphism $R_n \to M_n(D)$.

Proof.

(i) Let β be a D-basis for M. For any finite subset $\gamma \subseteq \beta$, define

$$I_\gamma = \operatorname{ann}_R(\gamma) := \{r \in R \mid rx = 0 \text{ for all } r \in \gamma\}.$$

This is an ideal of R, and $I_\gamma \supseteq I_{\gamma'}$ if $\gamma \subseteq \gamma'$. When $\gamma \subsetneq \gamma'$, the containment $I_\gamma \supseteq I_{\gamma'}$ is strict, as we can always find a D-linear map φ so that $\varphi|_{\gamma' \setminus \gamma} = \mathrm{id}$ while $\varphi|_\gamma \equiv 0$. By density we can find such φ in R. Hence, if R is left artinian, then β must be finite.

As long as β is finite, by density the natural map $R \to \operatorname{End}_D M$ is surjective. This proves $R \cong \operatorname{End}_D M \cong M_n(D)$. Along with Lemma J.1.8, this proves (i).

(ii) Let β be a D-basis for M. For each finite subset $\gamma \subseteq \beta$, denote by M_γ the D-submodule generated by γ. Then $R_\gamma := \{r \in R \mid rM_\gamma \subseteq M_\gamma\}$ is a subring of R and

$$I_\gamma := \operatorname{ann}_R(\gamma) = \operatorname{ann}_{R_\gamma}(\gamma)$$

is a two-sided ideal of R_γ. Then M_γ is naturally an R_γ/I_γ-module. Moreover, the natural map $R_\gamma \to R_\gamma/I_\gamma \to \operatorname{End}_D M_\gamma$ is surjective. Indeed, any D-linear map of M_γ extends to a linear map on M. Since $\#\gamma$ is finite, by density we can interpolate that extension of γ by an element in $r \in R$. This concludes the proof of (ii).

$\square$

Definition. Let R be a ring.

1. R is called **simple** if it is nonzero and contains no proper nontrivial *two-sided* ideal.

2. R is called **semisimple** if every R-module is semisimple.

Lemma J.1.10. A simple ring is primitive.

Proof. Let R be a simple ring. Since $R \neq 0$, by Zorn's lemma there exists a left maximal ideal $\mathfrak{m}$. Then the quotient $R/\mathfrak{m}$ is a simple R-module. Consider its annihilator

$$\operatorname{ann}_R(R/\mathfrak{m}) := \{r \in R \mid rx \in \mathfrak{m} \text{ for all } x \in R\}.$$

This is clearly a proper left ideal. In fact, it is also a right ideal: if $rx \in \mathfrak{m}$ for all $x \in R$, then $r(sx) \in \mathfrak{m}$ for any $s, x \in R$ particularly. Now by simplicity it follows that $\operatorname{ann}_R(R/\mathfrak{m})$ is trivial, proving that $R/\mathfrak{m}$ is faithful. $\square$

Lemma J.1.11. A ring R is semisimple if and only if it is semisimple as a left R-module. In this case, R is a finite direct sum of simple left ideals of R.

Proof. The only if part follows from definition. The if part follows from Lemma J.1.4. The last assertion follows from Lemma J.1.3 and apply the decomposition there to the identity 1. $\square$

Corollary J.1.11.1. Let R be a ring. TFAE:

(i) $R \cong M_n(D)$ for some $n \geqslant 1$ and a division ring D.

(ii) R is simple and semisimple.

(iii) R is simple and left artinian.

(iv) R is primitive and left artinian.

In this case, all simple R-modules are isomorphic (to D^n).

Proof.

(i)⇒(ii) A two sided ideal of $M_n(D)$ has the form $M_n(I)$ with I a two sided ideal of D. Since D is division, I is either 0 or the whole D. This proves $M_n(D)$ is simple. That $M_n(D)$ is semisimple follows from the proof of Lemma J.1.8 and Lemma J.1.3.

(ii)⇒(iii) Follows from Lemma J.1.11 (so that it has a composition series).

(iii)⇒(iv) Follows from Lemma J.1.10.

(iv)⇒(i) Follows from Theorem J.1.9.

To see the last assertion, write $M_n(D) = \bigoplus_{1 \leqslant j \leqslant n} I_j$ as in Lemma J.1.8. If M is a simple $M_n(D)$-module, then $I_j M \neq 0$ for some $j \in [n]$, whence $I_j M = M$ by simplicity. Pick any $x \in M$ such that $I_j x \neq 0$. Then the map $r \mapsto rx$ provides an isomorphism $I_j \cong M$. Finally, each I_j is isomorphic to D^n. $\square$

Corollary J.1.11.2 (Artin-Wedderburn)**.** A ring is semisimple if and only if it is isomorphic to $M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$ for some $r \in \mathbb{Z}_{\geqslant 1}$, $n_i \in \mathbb{Z}_{\geqslant 1}$ and some division rings D_i.

Proof. The if part is clear by Corollary J.1.11.1.(i)⇒(ii), Lemma J.1.11 and Lemma J.1.4. For the only if part, by Lemma J.1.11 a semisimple ring R is R-isomorphic to a finite sum of simple R-modules. Write $R = \bigoplus_{1 \leqslant i \leqslant r} M_i^{n_i}$ for some $n_i \geqslant 1$, where each M_i is mutually non-isomorphic. By Schur's lemma, we have

$$\operatorname{End}_R R = \bigoplus_{1 \leqslant i \leqslant r} \operatorname{End}_R(M_i^{n_i}) \cong \bigoplus_{1 \leqslant i \leqslant r} M_{n_i}(\operatorname{End}_R M_i).$$

But $\operatorname{End}_R R \cong R^{\mathrm{op}}$, so

$$R \cong \bigoplus_{1 \leqslant i \leqslant r} M_{n_i}(D_i)$$

where $D_i = (\operatorname{End}_R M_i)^{\mathrm{op}}$. $\square$

Corollary J.1.11.3. The opposite ring of a semisimple ring is semisimple. Every semisimple ring is two-sided artinian and Noetherian.

Lemma J.1.12. Let $n, n' \geqslant 1$ and D, D' be division rings. If $M_n(D) \cong M_{n'}(D')$, then $n = n'$ and $D = D'$.

Proof. Regard $M_n(D)$ as $\mathrm{End}_D(D^n)$. Then by density, the injection $D \to \mathrm{End}_D(D^n)$ maps onto the center. Hence $D \cong D'$. That $n = n'$ follows at once if we take the dimension. □

J.2 Some representation theory

Definition. Let R be a ring. The **Jacobson radical** $J(R)$ is the intersection of all maximal left ideals of R.

There are some properties that immediately follow from the definition.

Lemma J.2.1. Let R be a ring. Then $J(R/J(R)) = 0$.

Lemma J.2.2. If $R_1, \ldots, R_k$ are rings, then

$$J(R_1 \times \cdots \times R_k) = J(R_1) \times \cdots \times J(R_k).$$

Lemma J.2.3. One has the following descriptions for $J(R)$.

(i) $J(R)$ is the intersection of all the annihilators of simple R-modules. In particular $J(R)$ is a two-sided ideal.

(ii) $J(R)$ is the largest two-sided ideal J in R such that $1 + x \in R^\times$ for all $x \in J$. In particular $J(R) = J(R^{\mathrm{op}})$ is also the intersection of all maximal right ideals of R.

Proof.

(i) If $I \subseteq R$ is a maximal left ideal, then R/I is a simple R-module whose annihilator is precisely I. On the other hand, let $r \in J(R)$ and M a simple R-module. If $0 \neq x \in M$, then $M = Rx \cong R/\mathrm{ann}_R(x)$. Then $\mathrm{ann}_R(x)$ is a maximal left ideal. In particular, $r \in \mathrm{ann}_R(x)$ so $rx = 0$, or $rM = 0$.

For the last assertion, notice that $\mathrm{ann}_R(M)$ is a two-sided ideal of R.

(ii) Let $x \in R$. We first show $x \in J(R)$ if and only if $1 + tx$ has a left-inverse for any $t \in R$. If $x \notin J(R)$, then $x \notin I$ for some maximal left ideal I. Since $R = Rx + I$, we have $1 = rx + y$ for some $r \in R$ and $y \in I$. But $1 - rx \in I$ has no left inverse. Conversely, if $1 + tx$ does not have a left-inverse for some t, then $1 + tx$ is contained in some maximal left ideal I of R. Since $1 \notin I$, we have $tx \notin I$, which implies $x \notin I$.

Now let $x \in J(R)$, so that $1 + x$ has a left inverse, say, y. Then $1 = y(1 + x) = y + yx$, so $y = 1 - yx$. But again this implies y has a left inverse. All these force $1 + x$ to be a right inverse of y as well, i.e., $1 + x \in R^\times$.

It remains to show $J(R)$ is the largest among all such two-sided ideal. But this follows from the equivalence established in the first paragraph.

□

Definition. A ring is called **Jacobson semisimple** if its Jacobson radical is trivial.

Theorem J.2.4. Let R be a ring. TFAE:

(i) R is semisimple.

(ii) R is left artinian and Jacobson semisimple.

(iii) R is left artinian and no nonzero nilpotent left or right ideal.

(i)⇒(ii) By Artin-Wedderburn, R is semisimple if and only if $R \cong \prod_{i \in [r]} M_{n_i}(D_i)$ for some $r \in \mathbb{Z}_{\geqslant 1}$, $n_i \in \mathbb{Z}_{\geqslant 1}$ and division rings D_i. Each $M_{n_i}(D_i)$ is left artinian (c.f. Theorem J.1.9), and

$$J(R) \cong \prod_{i \in [r]} J(M_{n_i}(D_i)).$$

We must show $J(M_{n_i}(D_i)) = 0$, which follows as $M_{n_i}(D_i)$ is simple.

(ii)⇒(i) Suppose R is left-artinian. We will see $J(R)$ is the intersection of finitely many maximal left ideals $I_1, \ldots, I_m$. Assuming this, we consider the induced map

$$R \longrightarrow R/I_1 \times \cdots \times R/I_m.$$

This map has kernel $I_1 \cap \cdots \cap I_m = J(R)$. If $J(R) = 0$, this implies the above map is injective. By Lemma J.1.4, R is semisimple.

It remains to show on assertion on $J(R)$. Indeed, let S be the set of all finite intersections of maximal left ideals in R. Since R is left-artinian, S admits a minimal elements J, say. If I is any maximal left ideal of R, then $J \supseteq J \cap I \in S$, so $J = J \cap I$. This proves $J = J(R) \in S$, as claimed.

(ii)⇒(iii) NOT NEEDED

Definition. Let R be a finite dimensional k-algebra and $\pi : R \to \operatorname{End}_k M$ an R-module that is finite dimensional over k. The **character** $\operatorname{Tr} \pi : R \to k$ of the M is given by the composition

$$R \xrightarrow{\pi} \operatorname{End}_k M \xrightarrow{\operatorname{Tr}} k$$

where $\operatorname{Tr} : \operatorname{End}_k M \to k$ is the usual matrix trace (well-defined as $\dim_k M < \infty$).

Lemma J.2.5. Let R be a finite dimensional k-algebra and $\pi : R \to \operatorname{End}_k M$, $\rho : R \to \operatorname{End}_k N$ be two simple R-modules that are finite dimensional over k. Then $(\pi, M) \cong (\rho, N)$ in as R-modules if and only if $\operatorname{Tr} \pi = \operatorname{Tr} \rho$.

Proof. Let $T : M \to N$ be an R-isomorphism; then $T \circ \pi(a) = \rho(a) \circ T$ for any $a \in R$, which implies

$$\operatorname{Tr} \pi(a) = \operatorname{Tr} T^{-1} \circ \rho(a) \circ T = \operatorname{Tr} \rho(a) \circ T \circ T^{-1} = \operatorname{Tr} \rho(a).$$

This proves the only if part. For the if part, since the Jacobson radical $J(R)$ of R acts trivially on both M and N by Lemma J.2.5.(i), by replacing R by $R/J(R)$ we can assume $J(R) = 0$. Being finite dimensional over k, Theorem J.2.4 implies R is semisimple, whence $R \cong \prod_{i \in [r]} M_{n_i}(D_i)$ for some $r \in \mathbb{Z}_{\geqslant 1}$, $n_i \in \mathbb{Z}_{\geqslant 1}$ and division k-algebras D_i (see also Lemma J.3.1 below).

□

J.3 Central simple algebra

In this section fix a field k. We assume every k-algebra is finite dimensional over k, and k is contained in the center.

Definition. An k-algebra R is **central** if $k = Z(R)$ exactly.

Definition. An k-algebra is **simple** if it is simple as a ring.

Lemma J.3.1. Let R be a k-algebra. Then R is simple if and only if $R \cong M_n(D)$ for some $n \geqslant 1$ and some division k-algebra D. Moreover, n is unique and D is unique up to k-isomorphism. If R is central, so is D.

Proof. Since $\dim_k R < \infty$, by Corollary J.1.11.1, we see R is simple if and only if $R \cong M_n(D)$ for some $n \geqslant 1$ and division ring D. Taking the center gives an injection $k \to Z(R) \cong Z(D) \subseteq D$, so D is a k-algebra. If R is central, so is D. Now the moreover part follows from Lemma J.1.12. Since M and N are simple, we can find $i, j \in [r]$ so that M is simple over $M_{n_i}(D_i)$ and N is simple over $M_{n_j}(D_j)$. But the condition $\operatorname{Tr} \pi = \operatorname{Tr} \rho$ forces $i = j$. Hence we are reduced to the case when $R = M_n(D)$ is a matrix ring over a division k-algebra D. But the last assertion from Corollary J.1.11.1 claims $M \cong N$ as R-modules. □

Lemma J.3.2. Let R be a central k-algebra. Then R is simple if and only if the map

$$\begin{aligned} R \otimes_k R^{\mathrm{op}} &\longrightarrow \operatorname{End}_k R \\ (r, r') &\longmapsto [x \mapsto r x r'] \end{aligned}$$

is an k-algebra isomorphism.

Proof. Notice both sides have the same dimension over k. If R contains a nonzero proper two-sided ideal I, then every k-linear map in the image of $R \otimes_k R^{\mathrm{op}}$ leaves I stable. In particular, when R is not simple, the map is not surjective, whence not an isomorphism.

Now assume R is simple. This means R is simple as a left $C := R \otimes_k R^{\mathrm{op}}$-module. Since R is central, we have a canonical map $k \to \operatorname{End}_C R$. Since $\operatorname{End}_C R$ is division by Schur's lemma, it follows that $\operatorname{End}_C R = k$. But Jacobson density then implies the natural map $C \to \operatorname{End}_{\operatorname{End}_C R} R = \operatorname{End}_k R$ is surjective (recall $\dim_k R < \infty$ by our convention). □

Corollary J.3.2.1. Let R, S be two simple k-algebras. If R is central over k, then $R \otimes_k S$ is simple. If moreover S is central over k, then $R \otimes_k S$ is central.

Proof. Let $n = \dim_k R$. By Lemma J.3.2,

$$R^{\mathrm{op}} \otimes_k R \otimes_k S \cong (\operatorname{End}_k R^{\mathrm{op}}) \otimes_k S \cong M_n(k) \otimes_k S \cong M_n(S).$$

If $R \otimes_k S$ contains a nontrivial proper two-sided ideal, so does $R^{\mathrm{op}} \otimes_k R \otimes_k S$. But $M_n(S)$ is simple (as S is), so this $R \otimes_k S$ must be simple.

Assume in addition that S is central. Then the center of $M_n(S)$ is k. If $R \otimes_K S$ is not central, then the center of $R^{\mathrm{op}} \otimes_k R \otimes_k S$ will be strictly larger than k, a contradiction. □

Corollary J.3.2.2. Let R be a central k-algebra.

(i) For any field extension k'/k, $R \otimes_k k'$ is central simple over k' if and only if R is simple.

(ii) R is simple if and only if $R \otimes_k \overline{k} \cong M_n(\overline{k})$ for some algebraic closure of $\overline{k}$ and some $n \geqslant 1$.

(iii) If R is simple, then $\dim_k R$ is a square.

Proof.

1. Assume R is simple. By Lemma J.3.2 we have

$$R^{\mathrm{op}} \otimes_k \otimes R \otimes_k k' \cong M_n(k').$$

Since $M_n(k')$ is central over k', it forces $R \otimes_k k'$ to be central over k'. Likewise the simplicity follows (or by Corollary J.3.2.1).

Assume $R' := R \otimes_k k'$ is central simple. By Lemma J.3.2, $R' \otimes_{k'} R'^{\mathrm{op}} \to \operatorname{End}_{k'} R'$ is an isomorphism. Notice

$$R' \otimes_{k'} R'^{\mathrm{op}} = R \otimes_k R^{\mathrm{op}} \otimes_k k',$$

$\operatorname{End}_{k'} R' = (\operatorname{End}_k R) \otimes_k k'$, and the map $R' \otimes_{k'} R'^{\mathrm{op}} \to \operatorname{End}_{k'} R'$ is the base change of $R \otimes_k R^{\mathrm{op}} \to \operatorname{End}_k R$ to k'. Hence $R \otimes_k R^{\mathrm{op}} \to \operatorname{End}_k R$ is also an isomorphism, whence the simplicity follows by Lemma J.3.2 again.

2. Since $M_n(\overline{k})$ is central simple, that if part follows from (i). For the other way around, we know $R \otimes_k \overline{k} \cong M_n(D)$ for some division $\overline{k}$-algebra D. It suffices to show $D = \overline{k}$. Indeed, for any $x \in D$, the subalgebra $\overline{k}(x) \subseteq D$ is an algebraic extension of $\overline{k}$, so $x \in \overline{k}$.

3. Since $\dim_k R = \dim_{\overline{k}} R \otimes_k \overline{k}$, this follows from (ii).

∎

Corollary J.3.2.3. let k'/k be a finite extension of degree n. Suppose R is a central simple k-algebra of dimension n^2 that contains a subfield k-isomorphic to k'. Then $R \otimes_k k' \cong M_n(k')$.

Proof. Assume simply that $k' \subseteq R$. Then R is naturally a left k'-module and $\dim_{k'} R = n$. Now we have a nonzero k-linear $R \to \operatorname{End}_{k'} R$ given $r \mapsto [x \mapsto xr]$. Extending k'-linear we get a nonzero k'-linear map $R \otimes_k k' \to \operatorname{End}_{k'}(R)$. By simplicity this is then an isomorphism. ∎

Theorem J.3.3 (Noether-Skolem)**.** Suppose R is a simple k-algebra and S a central simple k-algebra. Then $S^\times$ acts on $\operatorname{Hom}_{\mathbf{Alg}_k}(R, S)$ transitively by conjugation on S.

Proof. Let $f, g \in \operatorname{Hom}_{\mathbf{Alg}_k}(R, S)$. Consider two $R \otimes_k S^{\mathrm{op}}$-modules structures on S:

$$(r, s)_1 v := g(r)vs$$
$$(r, s)_2 v := f(r)vs.$$

Since $R \otimes_k S^{\mathrm{op}}$ is a simple and finite dimensional over k, by Corollary J.1.11.1 it is semisimple and there is only one isomorphism class of simple modules. A dimension consideration shows that these two $R \otimes_k S^{\mathrm{op}}$ must be isomorphic; denote

$$\varphi : S_1 \to S_2$$

an $R \otimes_k S^{\mathrm{op}}$-isomorphism, where the subscript indicates its module structure. Then

$$\varphi(g(r)vs) = \varphi((r, s)_1 v) = (r, s)_2 \varphi(v) = f(r)\varphi(v)s.$$

Taking $r = 1$ yields $\varphi(vs) = \varphi(v)s$, so φ is an S^{op}-homomorphism. Since S is simple an an S^{op}-module (as we are assuming $\dim_k S < \infty$), it follows that $\varphi(v) = \beta v$ for some $\beta \in S^\times$. But then

$$\beta g(r) = \varphi(g(r)) = \varphi((r, 1)_1 1) = (r, 1)_2 \varphi(1) = f(r)\varphi(1) = f(r)\beta$$

so that $g(r) = \beta^{-1} f(r) \beta$ for any $r \in R$, as we claim. ∎

Definition. Let k'/k be a field extension and R a k-algebra. We say R **splits over** k' if $R \otimes_k k' \cong M_n(k')$ for some $n \geqslant 1$. If R splits over k, we simply say R **splits**. When R splits over k', we also say k' **splits** R.

For two central simple algebras A, B over k, we write $A \sim B$ if $A \otimes_k B^{\mathrm{op}}$ splits.

Lemma J.3.4. $\sim$ is an equivalence relation.

Proof. Reflexivity is clear. For symmetry, we claim for two k-algebras A, B, we have

$$(A \otimes_k B)^{\mathrm{op}} \cong B^{\mathrm{op}} \otimes_k A^{\mathrm{op}}.$$

This is clear: the map is given by $a \otimes b \mapsto b \otimes a$. To see this is an algebra homomorphism, note that

$$(a \otimes b) \cdot_{\mathrm{op}} (x \otimes y) = (xa \otimes yb) \mapsto (yb \otimes xa) = (b, a) \cdots (y, x).$$

Finally, assume $A \sim B$ and $B \sim C$. We must show $A \otimes_k C^{\mathrm{op}}$ splits. By assumption

$$(A \otimes_k B^{\mathrm{op}}) \otimes_k (B \otimes_k C^{\mathrm{op}}) = A \otimes_k (\mathrm{End}_k B^{\mathrm{op}}) \otimes_k C^{\mathrm{op}}$$

splits over k. Say LHS $\cong M_n(k)$ and $\mathrm{End}_k B^{\mathrm{op}} \cong M_b(k)$, so

$$M_n(k) \cong A \otimes_k M_b(k) \otimes_k C^{\mathrm{op}}.$$

But

$$A \otimes_k M_b(k) \cong M_b(A) \cong M_b(k) \otimes_k A$$

so

$$M_n(k) \cong M_b(k) \otimes_k A \otimes_k C^{\mathrm{op}}.$$

Since $A \otimes_k C^{\mathrm{op}}$ is central simple, $A \otimes_k C^{\mathrm{op}} \cong M_c(D)$ for some $c \geqslant 1$ and a central division k-algebra. Then

$$M_n(k) \cong M_b(k) \otimes_k M_c(D) \cong M_b(k) \otimes_k M_c(k) \otimes_k D \cong M_{bc}(D).$$

By Lemma J.1.12, we see $k = D$, proving $A \sim C$. $\square$

Lemma J.3.5. Say $A \cong M_n(D)$ and $B \cong M_m(D')$ are two central simple algebras over k. Then $A \sim B$ if and only if $D \sim D'$, if and only if $D \cong D'$.

Proof. We have

$$A \otimes_k B^{\mathrm{op}} \cong M_n(D) \otimes_k M_m(D'^{\mathrm{op}}) \cong M_{nm}(D \otimes_k D'^{\mathrm{op}}).$$

If we write $D \otimes_k D'^{\mathrm{op}} = M_l(D'')$ for some division D'', by Lemma J.1.12 we see $A \sim B$ if and only if $D'' = k$. The last equivalence follows from Lemma J.1.12 and Theorem J.3.2. $\square$

Definition. The **Brauer group** of k is

$$\mathrm{Br}(k) := \{\text{central simple algebras over } k\}/\sim.$$

The multiplication is induced by $(A, B) \mapsto A \otimes_k B$. The identity is the equivalence class consisting of split algebras. For a central simple algebra A, the inverse is, by Lemma J.3.2, given by its opposite algebra A^{op}.

For any extension k'/k, tensoring with k' induces a group homomorphism

$$\begin{aligned} \mathrm{Br}(k) &\longrightarrow \mathrm{Br}(k') \\ A &\longmapsto A \otimes_k k' \end{aligned}$$

on Brauer groups. This is well-defined by Corollary J.3.2.2. Denote by $\mathrm{Br}(k'/k)$ its kernel so that there is an exact sequence

$$1 \longrightarrow \mathrm{Br}(k'/k) \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}(k')$$

By construction

$$\mathrm{Br}(k'/k) = \left\{ A \in \mathrm{Br}(k') \mid A \text{ splits over } k' \right\}.$$

Lemma J.3.6 (Double commutant)**.** Let A be a central simple k-algebra and B a simple k-subalgebra of A. Then $C_A(C_A(B)) = B$ and

$$(\dim_k B)(\dim_k C_A(B)) = \dim_k A.$$

Proof. Let $R = A \otimes_k B^{\mathrm{op}}$, which is simple by Corollary J.3.2.1 and semisimple by Corollary J.1.11.1, as it is finite dimensional over k. Under the isomorphism $\mathrm{End}_A A \cong A^{\mathrm{op}}$, it is direct to see $\mathrm{End}_R A$ maps onto $C_A(B)^{\mathrm{op}}$.

Let M be the "the" faithful simple R-module, and put $D = \mathrm{End}_R M$. Since A is an R-module, we can write $A \cong M^a$ for some $a \geqslant 1$. Hence $\mathrm{End}_R(A) \cong M_a(D)$ so that

$$C_A(B) \cong M_a(D^{\mathrm{op}})$$

implying

$$\dim_k C_A(B) = a^2 \dim_k D.$$

On the other hand, by Jacobson density we see $R \cong \mathrm{End}_{\mathrm{End}_R M} M = \mathrm{End}_D M$. If we write $M \cong D^m$ as D-modules, then $R \cong M_m(D)$, whence

$$m^2 \dim_k D = \dim_k R = (\dim_k A)(\dim_k B)$$

Since $A \cong M^a \cong D^{am}$, it follows that $\dim_k A = am \dim_k D$, and

$$(\dim_k B)(\dim_k C_A(B)) = a^2(\dim_k D) \times \frac{m^2 \dim_k D}{\dim_k A} = \dim_k A.$$

To conclude, notice that

$$C_A(C_A(C_A(B))) = C_A(B)$$

holds unconditionally. Applying the equality with B replaced by $C_A(B)$, we see

$$\dim_k B = \dim_k C_A(C_A(B)).$$

Since $B \subseteq C_A(C_A(B))$ tautologically, it follows that $C_A(C_A(B)) = B$. When B is commutative, then $B \subseteq C_A(B)$ so $(\dim_k B)^2 \leqslant \dim_k A$. ☐

Corollary J.3.6.1. Let D be a central simple division k-algebra of dimension n^2. If K is a subfield of D containing k, then $\dim_k K \leqslant n$ with equality if and only if $K = C_D(K)$.

Proof. Since K is commutative, $K \subseteq C_D(K)$ so that

$$(\dim_k K)^2 \leqslant (\dim_k K)(\dim_k C_D(K)) = \dim_k D$$

by Lemma J.3.6. This proves $\dim_k K \leqslant n$. If $\dim_k K < n$, then $\dim_k C_D(K) = (\dim_k D)/(\dim_k K) > n$ so that $K \subsetneq C_D(K)$. Adjoining any element in $C_D(K) \backslash K$ to K will yield a subfield of D strictly larger than K. This proves the equality part. □

Theorem J.3.7. Let D be a central simple *division* algebra over k of dimension n^2.

(i) Maximal subfields of D are exactly those k-subalgebras of D which are their own centralizers in D, and they are degree n extension of k. Every subfield of D containing k is contained in a maximal subfield.

(ii) Every non-maximal subfield of D containing k admits a proper separable extension in D.

(iii) D has a maximal subfield separable over k.

(iv) Every maximal subfield of D splits D.

(v) If a degree N extension k'/k splits D, then $n \mid N$ and there exists a k-embedding $k' \to M_{N/n}(D)$ whose image is its own centralizer in $M_{N/n}(D)$.

In particular, a degree n extension of k that splits D is k-isomorphic to maximal subfield of D.

Proof. (i) follows from Corollary J.3.6.1. For (ii) and (iii), we claim D contains a proper separable extension of k when $n > 1$. Suppose otherwise there exists a $q \geqslant 1$ such that $x^q \in k$ for every $x \in D$. The number q then must be a prime and is a power of $\operatorname{Char} k$, so that $x \mapsto x^q$ is a k-endomorphism on D. Hence $x \mapsto x^q$ has its image in the center of D. But this continues to be true when we base change to an algebraic closure $\overline{k}$ of k. Since $D \otimes_k \overline{k} \cong M_n(\overline{k})$, we must have $n = 1$. This proves our claim.

Now let K be a subfield of D containing k with $[K : k] < n$. Consider the K-subalgebra $C_D(K)$ of D. This is division, and is central by Lemma J.3.6. Hence $C_D(K)$ contains a proper separable extension of K by what we've proven. This proves (ii) and (iii).

Let's prove (iv). Let K be a maximal subfield of D, and put $R = D \otimes_k K$. Let R act on D as usual so that D is a simple R-module. The kernel of the map $R \to \operatorname{End}_k D$ is a two-sided ideal of R not containing 1, so it is injective, i.e., D is an faithful R-module. So by Jacobson density

$$R \cong \operatorname{End}_{\operatorname{End}_R D}(D).$$

In the proof of Lemma J.3.6 we saw $\operatorname{End}_R(D) \cong C_D(K)^{\mathrm{op}} = K$, so $R \cong \operatorname{End}_K D \cong M_d(K)$ for some $d \geqslant 1$.

Finally we prove (v). By assumption

$$D \otimes_k K \cong M_n(K).$$

RHS has a simple module of K-dimension n, e.g. $M = K^n$, so it is also a D-module and its has dimension

$$\dim_D M = \frac{\dim_k M}{\dim_k D} = \frac{nN}{n^2} = N/n.$$

This shows $n \mid N$, and we have a natural embedding $K \to \operatorname{End}_D M \cong M_{N/n}(D)$. By Lemma J.3.6, we have

$$\dim_k C_{M_{N/n}(D)}(K) = \frac{\dim_k M_{N/n}(D)}{\dim_k K} = \frac{N^2}{N} = N.$$

Since $K \subseteq C_{M_{N/n}(D)}(K)$, this proves $K = C_{M_{N/n}(D)}(K)$.

$\square$

Bibliography

[Wei64] André Weil. "Sur certains groupes d'opérateurs unitaires". In: *Acta math* 111.143-211 (1964), p. 14.

[JL70] Hervé Jacquet and Robert P Langlands. *Automorphic Forms on GL (2): Part 1*. Springer Berlin, Heidelberg, 1970.

[War83] Frank W Warner. *Foundations of differentiable manifolds and Lie groups*. Springer New York, NY, 1983.

[BT85] Theodor Bröcker and Tammo Tom Dieck. *Representations of compact Lie groups*. Springer Berlin, Heidelberg, 1985.

[Bri91] Christopher Brislawn. "Traceable integral kernels on countably generated measure spaces". In: *Pacific Journal of Mathematics* 2 (1991), pp. 229–240. URL: https://msp.org/pjm/1991/150-2/pjm-v150-n2-p03-p.pdf.

[Rud91] Walter Rudin. "Functional analysis 2nd ed". In: (1991).

[Bre93] Glen E Bredon. *Topology and geometry*. Springer New York, NY, 1993.

[Bou95] Nicolas Bourbaki. *General Topology: Chapters 1-4*. Springer Berlin, Heidelberg, 1995.

[Bou03] Nicolas Bourbaki. *Topological Vector Spaces: Chapters 1-5*. Springer Berlin, Heidelberg, 2003.

[Rud06] Walter Rudin. *Real and Complex Analysis*. 2006.

[Con07] John B. Conway. *A Course in Functional Analysis*. 2nd ed. Springer New York, NY, 2007.

[Wei13] André Weil. *Basic number theory*. Springer Berlin, Heidelberg, 2013.

[DE14] Anton Deitmar and Siegfried Echterhoff. *Principles of harmonic analysis*. Springer Cham, 2014.

[Fol16] Gerald B Folland. *A course in abstract harmonic analysis*. CRC press, 2016.

[Has18] Markus Hasse. *Lectures on Functional Calculus*. 2018. URL: https://www.mathematik.tu-darmstadt.de/media/analysis/lehrmaterial_anapde/hallerd/ISem21complete.pdf.

[San21] Luiz AB San Martin. *Lie Groups*. Springer Cham, 2021.

[SP+21] Karen R Strung, Francesc Perera, et al. *An Introduction to C*-algebras and the Classification Program*. Springer, 2021.

Index