Note

B07201020 姚皓勻

Contents

1 Semisimplicity

1.1 Simplicity

Definition. A **division ring** K is a ring with $1 \neq 0$ such that every non-zero element is a unit.

- Every non-zero module M over K has a basis, and the cardinalities of two bases are the same. We call this cardinality the **dimension** of M over K.

  Proof. For simplicity, assume M admits a finite generating set $S = \{s_i\}_{i=1}^m$. We prove the replacement theorem: if T is a K-linearly independent subset of M, then we can find $T' \subseteq S$ with $\#T' = \#T$ such that $(S \backslash T') \cup T$ still generates.

  We prove this by induction on $n = \#T$, $n = 0$ being nothing to do. Assume $n \geqslant 1$, and write $T = \{v_1, \ldots, v_n\}$. By induction we can find $T'' \overset{\text{say}}{=} \{s_1, \ldots, s_{n-1}\} \subseteq S$ such that $(S \backslash T'') \cup (T \backslash \{v_n\})$ generates. Write $v_n = a_1 v_1 + \cdots + a_{n-1} v_{n-1} + a_n s_n + \cdots + a_m s_m$ for some $a_i \in K$. Since T is linearly independent, at least one of $a_n, \ldots, a_m$ is nonzero, say $a_n \neq 0$. Then

  $$s_n = -(a_n^{-1} a_n v_1 + \cdots + a_n^{-1} a_{n-1} v_{n-1} + a_n^{-1} v_n + a_n^{-1} a_{n+1} s_{n+1} + \cdots + a_n^{-1} a_m s_m)$$

  Take $T' = T'' \cup \{s_n\}$; then $(S \backslash T') \cup T$ generates. □

Definition. Let R be a ring. An R-module is **simple** if it is non-zero and it contains no proper trivial submodule.

Proposition 1.1 (Schur's lemma)**.** Let E, F be simple R-module. Then every non-zero R-homomorphism from E to F is an isomorphism. In particular, $\mathrm{End}_R(E)$ is a division ring.

Proof. Let $f : E \to F$ be a nonzero homomorphism. Then $\ker f \subseteq E$ and $\mathrm{Im}\, f \subseteq F$; by simplicity, we must have $\ker f = 0$ and $\mathrm{Im}\, f = F$. Thus $f : E \to F$ is an isomorphism. □

Proposition 1.2. Let $E = E_1^{n_1} \oplus \cdots \oplus E_r^{n_r}$ be a direct sum of simple modules, the E_i being non-isomorphic, and each E_i being repeated n_i times in the sum. Then, up to a permutation, $E_1, \ldots, E_r$ are uniquely determined up to isomorphisms, and the multiplicities $n_1, \ldots, n_r$ are uniquely determined.

Proof. Suppose there is an isomorphism

$$E_1^{n_1} \oplus \cdots \oplus E_r^{n_r} \longrightarrow F_1^{m_1} \oplus \cdots \oplus F_s^{m_s}$$

where the E_i are non-isomorphic, and the F_j are non-isomorphic. By Schur's lemma, we see each E_i must be isomorphic to some F_j, and vice versa. It follows that $r = s$ and after a permutation, $E_i \cong F_i$. Furthermore, the isomorphism must induce an isomorphism

$$E_i^{n_i} \longrightarrow F_i^{m_i}$$

for each i. Since $E_i \cong F_i$, we may assume $E_i = F_i$. Hence we are reduced to proving: if E is a simple module and $E^n \cong E^m$, then $n = m$. Since $\mathrm{End}_R(E^n)$ is an $\mathrm{End}_R(E) = K$-vector space isomorphic to the $n \times n$ matrix ring $M_n(E)$, which has dimension n^2 over K. Thus the multiplicity n is uniquely determined. □

1.2 Semisimplicity

Let R be a ring.

Proposition 1.3. For an R-module E, TFAE:

(i) E is a sum of a family of simple submodules.

(ii) E is the direct sum of a family of simple submodules.

(iii) Every submodule F of E is a direct summand of E.

If E satisfies one of the both condition, E is called **semisimple**.

Proof.

(i) ⇒(ii) Say $E = \sum\{E_i \mid i \in I\}$ with $E_i \leqslant E$. Let $J \subseteq I$ be a maximal subset such that the sum $E' = \sum\{E_j \mid j \in J\}$ is direct. To show (ii), it suffices to show each E_i $(i \in I)$ is contained in the sum. For each E_i, $E_i \cap E'$ is a submodule of E_i, so it is either 0 or E_i; if it is 0, then J is not maximal, a contradiction.

(ii) ⇒(iii) Say $E = \sum\{E_i \mid i \in I\}$ with $E_i \leqslant E$ and the sum being direct. Let $J \subseteq I$ be the maximal subset such that the sum $F + \sum\{E_j \mid j \in J\}$ is direct. The argument above shows (iii).

(iii) ⇒(i) We first show every nonzero submodule of E contains a simple module, and it suffices to consider the principal submodule Rv with $E \ni v \neq 0$. The kernel of the homomorphism $R \to Rv$ is a proper left ideal L of R, and thus is contained in a maximal ideal M of R. Then M/L is a maximal (proper) submodule of R/L, and hence Mv is a maximal (proper) submodule of Rv being isomorphic to M/L under the isomorphism $R/L \to Rv$. Write $E = Mv \oplus M'$ for some submodule M'. Then $Rv = Mv \oplus (M' \cap Rv)$, for $x \in Rv$ can be written as $x = mv + m'$, and $m' = x - mv \in Rv$. Since Mv is maximal, $M' \cap Rv$ is simple.

Let E' be the sum of all simple submodules of E. If $E' \neq E$, then $E = E' \oplus F$ for some $F \neq 0$, and there exists a simple submodule of F as proved above, a contradiction to the definition of E'.

□

Proposition 1.4. Every submodule or quotient module of a semisimple module is semisimple.

Proof. Let E be a semisimple module and F be a submodule of E. Let F' be the sum of all simple submodules of F and write $E = F' \oplus F''$ for some F''. Every element $x \in F$ has a unique expression $x = x' + x''$ with $x' \in F'$ and $x'' \in F''$, and so $x'' = x - x' \in F$. Hence $F = F' \oplus (F'' \cap F)$. Then we must have $F = F'$ (otherwise, $F'' \cap F$ contains a simple submodule of F).

For the quotient module, write $E = F \oplus F'''$ for some F'''; then $E/F \cong F'''$ is semisimple as shown above. □

1.3 Jacobson's Density Theorem

Let E be a semisimple R-module. Let $R' = \mathrm{End}_R(E)$. There is a R-bilinear pairing

$$\begin{aligned} R' \times E &\longrightarrow E \\ (\varphi, x) &\longmapsto \varphi(x) \end{aligned}$$

and thus a homomorphism $R' \to \mathrm{End}_R(E)$, making E an R'-module. There is also a homomorphism $R \to \mathrm{End}_{R'}(E)$, given by $R \ni r \mapsto [f_r : x \mapsto rx]$. This is due to the fact $\varphi(rx) = r\varphi(x)$ for all $\varphi \in R'$. We ask how large is the image of this homomorphism.

Theorem 1.5 (Jacobson)**.** Let E be semisimple over R and let $R' = \mathrm{End}_R(E)$. Let $f \in \mathrm{End}_{R'}(E)$. For $x_1, \ldots, x_n \in E$ there exists $r \in R$ such that $rx_i = f(x_i)$ for $i = 1, \ldots, n$. In particular, if E is finite over R', then the natural map $R \to \mathrm{End}_{R'}(E)$ is surjective.

We equip R and E with discrete topology and equip $\mathrm{End}_{R'}(E)$ with pointwise convergence topology; E being discrete, the topology on $\mathrm{End}_{R'}(E)$ is the same as the compact-open topology. The theorem above then shows that the homomorphism $R \to \mathrm{End}_{R'}(E)$ is dense.

Proof. (of Theorem 1.5) First consider the case $n = 1$. Since E is semisimple, we can write $E = Rx \oplus F$ for some F. Let $\pi : E \to Rx$ be the projection; then $\pi \in R'$, and hence $f(x) = f(\pi x) = \pi f(x)$. Thus $f(x) \in Rx$, as wanted. For general $n \geqslant 1$, consider E^n and $F := \mathrm{End}_R(E^n)$. We need a lemma.

Lemma 1.6. Let E be an R-module, $R' := \mathrm{End}_R(E)$, $n > 0$ and $F = \mathrm{End}_R(E^n)$. If $f \in \mathrm{End}_{R'}(E)$, then the homomorphism

$$\begin{aligned} f^n : E^n &\longrightarrow E^n \\ (x_1, \ldots, x_n) &\longmapsto (f(x_1), \ldots, f(x_n)) \end{aligned}$$

is F-linear.

Proof. Let $\varphi \in F$; write $\varphi = (\varphi_{ij})_{1 \leqslant i,j \leqslant n}$ with $\varphi_{ij} \in \mathrm{End}_R(E) = R'$ such that

$$\varphi(x_1, \ldots, x_n) = \left(\sum_{j=1}^n \varphi_{1j} x_j, \ldots, \sum_{j=1}^n \varphi_{nj} x_j \right)$$

Then since $f \in \mathrm{End}_{R'}(E)$, it commutes with any element of R', and thus

$$\begin{aligned} f^n(\varphi(x_1, \ldots, x_n)) = f^n \left(\sum_{j=1}^n \varphi_{1j} x_j, \ldots, \sum_{j=1}^n \varphi_{nj} x_j \right) &= \left(\sum_{j=1}^n f(\varphi_{1j} x_j), \ldots, \sum_{j=1}^n f(\varphi_{nj} x_j) \right) \\ &= \left(\sum_{j=1}^n \varphi_{1j} f(x_j), \ldots, \sum_{j=1}^n \varphi_{nj} f(x_j) \right) = \varphi(f^n(x_1, \ldots, x_n)) \end{aligned}$$

□

Return to the proof. By Lemma, $f^n \in \mathrm{End}_F(E^n)$. Since E^n is semisimple, by the first paragraph, applied to E^n, we can find $r \in R$ such that $r(x_1, \ldots, x_n) = f^n(x_1, \ldots, x_n)$, as desired. □

Corollary 1.6.1 (Burnside)**.** Let E be a finite dimension vector space over an algebraically closed field k and let R be a subalgebra of $\mathrm{End}_k(E)$. If E is a simple R-module, then $R = \mathrm{End}_{R'}(E)$.

Proof. We contend $\mathrm{End}_R(E) = k$. Since E is simple, $R' = \mathrm{End}_R(E)$ is a division ring containing k such that $k \subseteq Z(R')$. Let $\alpha \in R'$. Then $k(\alpha)$ is a field. Furthermore, R' is contained in $\mathrm{End}_k(E)$ as a k-subspace, and therefore finite dimensional over k. Hence $k(\alpha)/k$ is finite, and hence $k(\alpha) = k$ for k is algebraically closed. This proves that $R' = k$.

Now let $\{v_1, \ldots, v_n\}$ be a k-basis for E. Let $A \in \mathrm{End}_k(E)$. By Jacobson's density theorem, there exists $r \in R$ such that $rv_i = Av_i$ for $i = 1, \ldots, n$. Since the effect of A is determined by its effect on a basis, we conclude $R = \mathrm{End}_k(E)$. □

The above Corollary is used in the following situation. Let E be a finite dimensional vector space over k. Let G be a multiplicative submonoid of $\mathrm{GL}(E)$. A ***G*-invariant** subspace F of E is such that $\sigma F \subseteq F$ for all $\sigma \in F$. We say E is ***G*-simple** if it has no trivial proper G-invariant subspace. Let $R = k[G]$ be the subalgebra of $\mathrm{End}_k(E)$ generated by G over k. Since G is assumed to be a monoid, it follows that R consists of the linear combination

$$\sum a_i\sigma_i$$

with $a_i \in k$ and $\sigma_i \in G$. Then we see a subspace F of E is G-invariant if and only if it is R-invariant. Thus E is G-simple if and only if it is R-simple.

Corollary 1.6.2. Let E be a finite dimensional vector space over k and let G be a multiplicative submonoid of $\mathrm{GL}(E)$. If E is G-simple, then $k[G] = \mathrm{End}_k(E)$.

When k is not algebraically closed, we still get some result.

Definition. An R-module E is **faithful** if the structure homomorphism $R \to \mathrm{End}_{\mathbb{Z}}(E)$ is injective.

Corollary 1.6.3 (Wedderburn)**.** Let R be a ring and E a simple faithful R-module. Let $D = \mathrm{End}_R(E)$ and assume that E is finite dimensional over D. Then $R = \mathrm{End}_D(E)$.

Proof. Let $\{v_1, \dots, v_n\}$ be a D-basis for E. Given $A \in \mathrm{End}_D(E)$, by Jacobson's density theorem there exists $r \in R$ such that $rv_i = Av_i$ for $i = 1, \dots, n$. Hence $R \to \mathrm{End}_D(E)$ is surjective. Since E is faithful over R, $R \to \mathrm{End}_D(E)$ is injective, and our corollary is proved. ☐

Suppose R is a finite dimensional k-algebra, and assume R has a unit element. If R has no trivial proper two-sided ideal, then any nonzero R-module R is faithful, for the kernel of $R \to \mathrm{End}_k(E)$ is a two sided ideal not equal to R. If E is simple, then E is finite dimensional over k. Then $D = \mathrm{End}_R(E)$ is a finite dimensional division algebra over k. Wedderburn's theorem gives a representation of R as the ring of D-endomorphisms of E.

Corollary 1.6.4. Let R be a ring, finite dimensional algebra over an algebraically closed field k. Let V be a finite dimensional vector space over k with a simple faithful representation $\rho : R \to \mathrm{End}_k(V)$. Then ρ is an isomorphism; in other words, $R \cong M_n(k)$.

Proof. We apply Wedderburn's theorem with $E = V$. Note that $D = \mathrm{End}_R(V)$ is finite dimensional over k. Given $\alpha \in D$, since $k(\alpha)$ is a commutative subfield of D, so $k(\alpha) = k$ by assumption that k is algebraically closed. ☐

Theorem 1.7. Let k be a field, R a k-algebra, and $V_1, \dots, V_m$ finite dimensional k-spaces which are also simple R-module, and such that V_i is not R-isomorphic to V_j for $i \neq j$. Then there exist elements $e_i \in R$ such that e_i acts as the identity on V_i and $e_iV_j = 0$ if $j \neq i$.

Proof. Let $E = V_1 \oplus \cdots \oplus V_m$. Let $p_i : E \to V_j$ be the canonical projection. We have $p_i \in \mathrm{End}_{R'}(E)$, for if $\varphi \in R'$, then $\varphi(V_j) \subseteq V_j$ by Schur's lemma. Since the V_j are finite dimensional over k, the result follows from Jacobson's density theorem. ☐

Corollary 1.7.1 (Bourbaki)**.** Let k be a field, R be a k-algebra and E, F R-modules finite dimensional over k. Assume either

(i) k is characteristic zero and E, F are semisimple over R.

(ii) E, F are simple over R.

For each $r \in R$ let r_E and r_F be the corresponding k-endomorphisms on E and F respectively. Suppose that $\mathrm{Tr}(r_E) = \mathrm{Tr}(r_F)$ for all $\alpha \in R$. Then $E \cong F$ as R-modules.

Proof. For (ii), assume otherwise. Then by Theorem we can find $e \in R$ such that $e_E = \mathrm{id}_E$ and $e_F(F) = 0$. Then $\dim_k E = \mathrm{Tr}(e_E) = \mathrm{Tr}(e_F) = 0$, a contradiction (recall a simple module is nonzero).

For (i), let V be a simple R-module and suppose $E = V^n \oplus E'$ and $F = V^m \oplus F'$ with E' and F' contains no V. Let $e \in R$ be such that $e_V = \mathrm{id}_V$ and 0 on E' and F'. Then

$$n \dim_k V = \mathrm{Tr}(e_E) = \mathrm{Tr}(e_F) = m \dim_k V$$

It follows that $n = m$. Note that the characteristic 0 is used, because the values of the trace are in k. $\square$

In the language of representations, suppose G is a monoid, and we have two semisimple representations into finite dimensional k-spaces

$$\rho : G \to \mathrm{End}_k(E) \quad \text{and} \quad \rho' : G \to \mathrm{End}_k(F)$$

Assume that $\mathrm{Tr}\,\rho(\sigma) = \mathrm{Tr}\,\rho'(\sigma)$ for all $\sigma \in G$. Then ρ and ρ' are isomorphic. Indeed, we let $R = k[G]$, so that ρ and ρ' extend to representations of R. By linearity one has that $\mathrm{Tr}\,\rho(r) = \mathrm{Tr}\,\rho'(r)$ for all $r \in R$, so one can apply Corollary above.

2 Local ζ-integral on $\mathrm{GL}(1)$

We first set up our notation. Let $p \leqslant \infty$ be a rational prime and $\mathbb{Q}_p$ the p-adic completion of $\mathbb{Q}$. The p-adic absolute value is denoted by $|\cdot|_p : \mathbb{Q}_p \to \mathbb{R}_{\geqslant 0}$.

- $p = \infty$. Then $|\cdot|_\infty = |\cdot|$ is the usual absolute value on $\mathbb{R}$.
- $p < \infty$. Then $|\cdot|_p$ is the normalized absolute value such that $|p|_p = p^{-1}$.

Then $(\mathbb{Q}, |\cdot|_p)$ is a Banach space. If F is a finite extension of $\mathbb{Q}_p$, define $|\cdot|_F : F \to \mathbb{R}_{\geqslant 0}$ by $|a|_F := |N_{F/\mathbb{Q}_p}(a)|_p$. Then $|\cdot|_F$ is an absolute value on F and F is complete with respect to $|\cdot|_F$. Note that when $F = \mathbb{C}$, $|z|_\mathbb{C} = |z\overline{z}|$ is the square of the usual norm on $\mathbb{C}$.

Suppose $p < \infty$. Let dx be a Haar measure on $\mathbb{Q}_p$. Then $\mathrm{vol}(\mathbb{Z}_p, dx) \neq 0$, and for $a \in \mathbb{Q}_p$,

$$\mathrm{vol}(a\mathbb{Z}_p, dx) = |a|_p \,\mathrm{vol}(\mathbb{Z}_p, dx)$$

so that $d(ax) = |a|dx$, i.e.

$$\int_{\mathbb{Q}_p} f(xa^{-1})dx = |a| \int_{\mathbb{Q}_p} f(x)dx$$

for all $f \in C_c(\mathbb{Q}_p)$. This means $\dfrac{dx}{|x|}$ is a Haar measure on $\mathbb{Q}_p^\times$.

$$\int_{\mathbb{Z}_p^\times} \frac{dx}{|x|} = \int_{\mathbb{Z}_p^\times} dx = (1 - p^{-1}) \,\mathrm{vol}(\mathbb{Z}_p, dx)$$

We usually normalize dx so that $\mathrm{vol}(\mathbb{Z}_p, dx) = 1$. Similarly, we normalized the Haar measure on $\mathbb{Q}_p^\times$, denoted by $d^\times x$, so that $\mathrm{vol}(\mathbb{Z}_p^\times, d^\times x) = \dfrac{1}{1-p^{-1}} \dfrac{dx}{|x|}$. When $p = \infty$, we simply take dx to be the Lebesgue measure and $d^\times x = \dfrac{dx}{|x|}$.

If $p = \infty$, $\mathbb{Q}_p = \mathbb{R}$, let $\psi_\infty : \mathbb{R} \to \mathbb{C}$ be given by $\psi_\infty(x) = e^{2\pi i x}$. If $p < \infty$ by given by $\psi_p(x) = e^{-2\pi i \{x\}}$, where $\{\cdot\} : \mathbb{Q}_p \to \mathbb{Q}$ is the fractional part

$$\left\{ \frac{a_{-n}}{p^n} + \frac{a_{1-n}}{p^{n-1}} + \cdots + \frac{a_{-1}}{p} + a_0 + \cdots \right\} := \frac{a_{-n}}{p^n} + \cdots + \frac{a_{-1}}{p} \in \mathbb{Q}$$

These are called the standard additive characters on $\mathbb{Q}_p$.

Let $\mathcal{S}(\mathbb{Q}_p)$ be the **space of Schwartz-Bruhat functions** on $\mathbb{Q}_p$: when $p = \infty$, $\mathcal{S}(\mathbb{R})$ consists of the usual Schwartz functions on $\mathbb{R}$, and when $p < \infty$, $\mathcal{S}(\mathbb{Q}_P)$ is the space of all locally constant functions with compact support.

We define the **Fourier transform** on $\mathcal{S}(\mathbb{Q}_p)$:

$$\begin{aligned} \mathcal{S}(\mathbb{Q}) &\longrightarrow \mathcal{S}(\mathbb{Q}_p) \\ f &\longmapsto \hat{f}(x) := \int_{\mathbb{Q}_p} f(y)\psi_p(xy)dy \end{aligned}$$

Example.

1. $p = \infty$, $f(x) = e^{-\pi x^2}$. Then $\hat{f}(x) = f(x)$.
2. $p < \infty$, $f(x) = \mathbb{I}_{a\mathbb{Z}_p}(x)$, $a \in \mathbb{Q}_p$. Then $\widehat{\mathbb{I}_{a\mathbb{Z}_p}}(x) = |a|\mathbb{I}_{a^{-1}\mathbb{Z}_p}(x)$. In particular, $\widehat{\mathbb{I}_{\mathbb{Z}_p}} = \mathbb{I}_{\mathbb{Z}_p}$.

Proposition 2.1.

1. If $\varphi \in \mathcal{S}(\mathbb{Q}_p)$, then $\hat{\varphi} \in \mathcal{S}(\mathbb{Q}_p)$.
2. We have $\widehat{\hat{\varphi}}(x) = \varphi(-x)$.

In particular, the Fourier transform defines a bijection on $\mathcal{S}(\mathbb{Q}_p)$.

2.1 Functional equation for Riemann ζ-functions

Let $\chi : (\mathbb{Z}/n\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character of conductor N. We extend χ to $\mathbb{Z}$ by setting $\chi(n) = 0$ if $(n, N) > 1$. Define

$$L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

This is absolutely convergent for $\operatorname{Re} s > 1$.

Theorem 2.2.

(i) For $\operatorname{Re} s > 1$, we have

$$L(s, \chi) = \prod_{p<\infty} \frac{1}{1 - \dfrac{\chi(p)}{p^s}}$$

(ii) $L(s, \chi)$ has an analytic continuation to the whole plane $\mathbb{C}$ with the only simple pole at $s = 1$.

(iii) We have the **functional equation**: define

$$\Lambda(s, \chi) := L(s, \chi) \cdot \begin{cases} \pi^{-\frac{s}{2}} \Gamma\left(\dfrac{s}{2}\right) & \text{, if } \chi(-1) = 1 \\ \pi^{-\frac{s+1}{2}} \Gamma\left(\dfrac{s+1}{2}\right) & \text{, if } \chi(1) = 1 \end{cases}$$

where $\Gamma(s)$ is the usual Gamma function, which has a meromorphic continuation with the only simple poles at $s = 0, -1, -2, \ldots$. Then there exists a unique number $W(\chi) \in S^1$, called the **root number**, such that

$$\Lambda(1 - s, \chi^{-1}) = N^{s-\frac{1}{2}} W(\chi) \Lambda(s, \chi)$$

We prove this theorem when $\chi = 1$ is the trivial character. (i) is clear. For (ii) and (iii), we proceed as follows.

Integral representation of the ζ-function. Define the **θ-function** $\theta : \mathbb{R} \to \mathbb{C}$ by

$$\theta(t) := \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t}$$

This series converges compactly on $\mathbb{R}$. Consider the Mellin transform of $\tilde{\theta} := \frac{1}{2}\theta - 1$: for $\operatorname{Re} s > 0$

$$\begin{aligned} \mathcal{M}(\tilde{\theta})(s) &:= \int_0^\infty \tilde{\theta}(t) t^s \frac{dt}{t} = \int_0^\infty \sum_{n \geqslant 1} e^{-\pi n t^2} t^s \frac{dt}{t} = \sum_{n \geqslant 1} \int_0^\infty e^{-\pi n^2 t} t^s \frac{dt}{t} = \sum_{n \geqslant 1} \frac{1}{(\pi n^2)^s} \int_0^\infty e^{-t} t^s d^\times t \\ &= \pi^{-s} \Gamma(s) \zeta(2s) = \Lambda(2s) \end{aligned}$$

Poisson summation formula.

Theorem 2.3. If $\varphi \in \mathcal{S}(\mathbb{R})$, then

$$\sum_{n\in\mathbb{Z}} \varphi(n) = \sum_{n\in\mathbb{Z}} \hat{\varphi}(n)$$

Corollary 2.3.1. For $t > 0$, we have

$$\theta(t) = \frac{1}{\sqrt{t}}\theta\left(\frac{1}{t}\right)$$

The argument.

2.2 Local L-functions on $\mathbb{Q}_p$

Let $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ be a continuous group homomorphism.

- $p = \infty$, $\mathbb{Q}_p = \mathbb{R}$. Then $\chi = |\cdot|^r \operatorname{sign}^\varepsilon$ for some $r \in \mathbb{C}$ and $\epsilon \in \{0, 1\}$. Then we define

$$L(s,\chi) := \Gamma_{\mathbb{R}}(s + r + \epsilon)$$

  where $\Gamma_{\mathbb{R}}(s) := \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)$.

- $p < \infty$.

  - χ **unramified**, i.e., $\chi|_{\mathbb{Z}_p^\times} \equiv 1$. Then define

$$L(s,\chi) := \frac{1}{1-\chi(p)p^{-s}}$$

  - χ **ramified**, i.e., $\chi|_{\mathbb{Z}_p^\times} \not\equiv 1$. Then define

$$L(s,\chi) := 1$$

The function $L(s,\chi)$ is called the L**-function** for χ.

Definition. For $\varphi \in \mathcal{S}(\mathbb{Q}_p)$ and $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$, define (formally) the **Tate integral/local ζ-integral**

$$Z(\varphi,\chi,s) := \int_{\mathbb{Q}_p^\times} \varphi(x)\chi(x)|x|^s d^\times x, \qquad s \in \mathbb{C}$$

Example. We compute Tate integrals of some test functions.

- $p = \infty$, $\varphi(x) = e^{-\pi x^2}$ or $xe^{-\pi x^2}$.
- $p < \infty$, χ unramified, $\varphi = \mathbb{I}_{\mathbb{Z}_p}$.
- $p < \infty$, χ ramified, $\varphi = \mathbb{I}_{1+p^n\mathbb{Z}_p}$, where $n = c(\chi)$ is the conductor.

2.3 Intrinsic definition for $L(s,\chi)$

For $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$, we can find $\sigma_0 \in \mathbb{R}$ such that

$$\chi(x) = \chi^u(x)|x|^{\sigma_0}$$

where $\chi^u : \mathbb{Q}_p^\times \to S^1$ is a unitary character. Then $Z(\varphi,\chi,s) = Z(\varphi,\chi^u,s+\sigma_0)$ by definition. Thus in the study of local zeta integrals, we may assume χ is unitary.

Proposition 2.4. If χ is a unitary character, $Z(\varphi, \chi, s)$ is absolutely convergent for $\operatorname{Re} s > 0$.

Theorem 2.5.

(i) For $\varphi \in \mathcal{S}(\mathbb{Q}_p)$ and $\chi : \mathbb{Q}_p^\times \to \mathbb{C}$, the Tate integral $Z(\varphi, \chi, s)$ has a meromorphic continuation to $\mathbb{C}$.

(ii) For $\varphi \in \mathcal{S}(\mathbb{Q}_p)$,

$$\Xi(\varphi, \chi, s) := \frac{Z(\varphi, \chi, s)}{L(s, \chi)}$$

is an entire function on $\mathbb{C}$.

(iii) We have **local functional equation**:

$$\frac{Z(\hat{\varphi}, 1-s, \chi^{-1})}{Z(\varphi, s, \chi)} = \gamma(s, \chi)$$

is a constant independent of $\varphi \in \mathcal{S}(\mathbb{Q}_p)$. The constant $\gamma(s, \chi)$ is called the **γ-factor for** χ.

Remark 2.6.

1. Let $\mathcal{O}_\mathbb{C}$ be the ring of entire functions on $\mathbb{C}$. Then $L(s, \chi)$ is the gcd of local zeta integrals, i.e.,

$$\sum_{\varphi \in \mathcal{S}(\mathbb{Q}_p)} \mathcal{O}_\mathbb{C} Z(\varphi, s, \chi) = \mathcal{O}_\mathbb{C} L(s, \chi)$$

in the field $\operatorname{Frac} \mathcal{O}_\mathbb{C}$ of meromorphic functions on $\mathbb{C}$.

2. Consider $\rho : \mathbb{Q}_p^\times \to \operatorname{Aut} \mathcal{S}(\mathbb{Q}_p)$ defined by right translation: $\rho(x)\varphi(z) = \varphi(zx)$. One computes

$$Z(\rho(x)\varphi, \chi, s) = \chi^{-1}|x|^{-s} Z(\varphi, \chi, s)$$

Hence

$$Z(\cdot, \chi, s) \in \operatorname{Hom}_{\mathbb{Q}_p^\times}((\rho, \mathcal{S}(\mathbb{Q}_p)), \chi^{-1}|\cdot|^{-s})$$

and the map

$$\varphi \mapsto \left.\frac{Z(\varphi, \chi, s)}{L(s, \chi)}\right|_{s=0} \in \operatorname{Hom}_{\mathbb{Q}_p^\times}(\mathcal{S}(\mathbb{Q}_p), \chi^{-1})$$

is a non-zero intertwining operator.

Proposition 2.7. Given $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{Q}_p)$, we have

$$Z(\varphi_1, \chi, s) Z(\hat{\varphi}_2, \chi^{-1}, 1-s) = Z(\varphi_2, \chi, z) Z(\hat{\varphi}_1, \chi^{-1}, 1-s)$$

with $0 < \operatorname{Re} s < 1$.

As before we compute the ratio $\dfrac{Z(\hat{\varphi}, 1-s, \chi^{-1})}{Z(\varphi, s, \chi)}$ explicitly for some particular test function φ.

- $p = \infty$.
- $p < \infty$, χ unramified.
- $p < \infty$, χ ramified.

Definition. Define the **ϵ-factor** for $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$

$$\epsilon(s,\chi,\psi_p) = \begin{cases} i^\epsilon & \text{, if } p=\infty,\ \chi = \operatorname{sign}^\epsilon |\cdot|^n \\ 1 & \text{, if } p<\infty,\ \chi \text{ unramified} \end{cases}$$

If $p < \infty$ and χ is ramified, let $c(\chi)$ be the conductor of χ and choose any $t \in p^{c(\chi)}\mathbb{Z}_p^\times$. Define

$$\begin{aligned} \epsilon(s,\chi,\psi_p) &= \int_{t^{-1}\mathbb{Z}_p^\times} \chi^{-1}(x)|x|^{-s}\psi_p(x)dx \\ &= |t|^{s-1}\chi(t)\int_{\mathbb{Z}_p^\times} \chi^{-1}(x)\psi_p\left(\frac{x}{t}\right)dx \end{aligned}$$

Definition. Define the **γ-factor** for $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$

$$\gamma(s,\chi,\psi_p) = \frac{L(1-s,\chi^{-1})}{L(s,\chi)}\epsilon(s,\chi,\psi_p)$$

Theorem 2.8.

$$\frac{Z(\hat{\varphi},1-s,\chi^{-1})}{Z(\varphi,s,\chi)} = \gamma(s,\chi,\psi_p)$$

for $0 < \operatorname{Re} s < 1$.

Lemma 2.9. Let $t \in p^c\mathbb{Z}_p^\times$, $c = c(\chi) \geqslant 1$.

1. $\epsilon(s,\chi,\psi_p) = |t|^s\epsilon(0,\chi,\psi_p)$.
2. $\epsilon(0,\chi,\psi_p)\epsilon(0,\chi^{-1},\psi_p) = |t|^{-1}\chi(-1)$

Theorem 2.10. $Z(\varphi,\chi,s)$ has a meromorphic continuation to $\mathbb{C}$.

3 Haar measures

3.1 $\mathrm{GL}_n(\mathbb{Q}_p)$ is unimodular

Let $p \leqslant \infty$ be a prime. For $X = (x_{ij}) \in \mathrm{GL}_n(\mathbb{Q}_p)$, define

$$dX := |\det X|_p^{-n} \prod_{i,j=1}^{n} dx_{ij}$$

Then dX is a Haar measure on $\mathrm{GL}_n(\mathbb{Q}_p)$, and it is unimodular. To see this, note that $\mathrm{GL}_n(\mathbb{Q}_p)$ is generated by the matrices of the forms:

(i) $A_{\mathbf{a}} := a_1 E_{11} + \cdots + a_n E_{nn}$ for $\mathbf{a} = (a_i)_{1 \leqslant i \leqslant n} \in \left(\mathbb{Q}_p^\times\right)^n$.

(ii) $B_{i,j,a} := I_n + aE_{ij}$ for $a \in \mathbb{Z}_p$ (resp. $\mathbb{R}$) and $1 \leqslant i \neq j \leqslant n$.

(iii) $C_{i,j} := I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}$ for $1 \leqslant i \neq j \leqslant n$.

We must show for $\phi \in C_c(\mathrm{GL}_n(\mathbb{Q}_p))$ and $A \in \mathrm{GL}_n(\mathbb{Q}_p)$,

$$\int_{\mathrm{GL}_n(\mathbb{Q}_p)} \phi(X)dX = \int_{\mathrm{GL}_n(\mathbb{Q}_p)} \phi(AX)dX = \int_{\mathrm{GL}_n(\mathbb{Q}_p)} \phi(XA)dX$$

When $A = A_{\mathbf{a}}$, then doing change of variable $a_i x_{ij} = y_{ij}$, we have $dy_{ij} = d(a_i x_{ij}) = |a_i|_p dx_{ij}$ and $\det Y = \det AX = \det A \det X$, so that

$$\int_{\mathrm{GL}_n(\mathbb{Q}_p)} \phi(AX)dX = \int_{\mathrm{GL}_n(\mathbb{Q}_p)} \phi(Y) \frac{|\det A|_p^n}{|\det Y|_p^n} \prod_{i,j} \frac{dy_{ij}}{|a_i|_p} = \int_{\mathrm{GL}_n(\mathbb{Q}_p)} \phi(Y)dY = \int_{\mathrm{GL}_n(\mathbb{Q}_p)} \phi(X)dX$$

The same holds for $Y = XA$. For (ii) and (iii), note that under the open compact subgroup $\mathrm{GL}_n(\mathbb{Z}_p)$ for $p < \infty$ (resp. the unit cube when $p = \infty$) is unchanged (resp. has the same volume) under the transformation $X \mapsto B_{i,j,a}X$ and $X \mapsto C_{i,j}X$, so the Haar integral has the formula above. The same holds for the right translation.

3.2 Basic representation theory

In the following we let $p < \infty$ be a finite prime and $G = \mathrm{GL}_2(\mathbb{Q}_p)$.

Definition.

1. Let V be a $\mathbb{C}$-vector space. We say (ρ, V) is a **representation** of G if $\rho : G \to \mathrm{Aut}_{\mathbb{C}} V$ is a group homomorphism.

2. If (ρ_1, V_1) and (ρ_2, V_2) are representations of G, we define the space of **intertwining operators** to be

$$\mathrm{Hom}_G((\rho_1, V_1), (\rho_2, V_2)) := \{f \in \mathrm{Hom}_{\mathbb{C}}(V_1, V_2) \mid f(\rho_1(g)v) = \rho_2(g)f(v) \text{ for all } g \in G,\ v \in V\}$$

3. A representation (ρ, V) of G is **smooth** if for any $v \in V$, there exists an open subgroup $U \leqslant G$ such that $\rho(g)v = v$ for all $g \in U$. Equivalently, (ρ, V) is smooth if and only if

$$V = \bigcup_{n=1}^{\infty} V^{K_n}$$

where the K_n are the **standard open-compact subgroups** of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ defined by

$$K_n = \{g \in \mathrm{GL}_2(\mathbb{Z}_p) \mid g \equiv I_2 \pmod{p^n}\} = I_2 + p^n M_2(\mathbb{Z}_p)$$

4. A representation (ρ, V) of G is **admissible** if for all open compact $K \leqslant G$, we have $\dim_{\mathbb{C}} V^K < \infty$.

5. A representation (ρ, V) is **irreducible** if V does not contain any proper nontrivial G-invariant subspace of V.

In the theory of representation of finite groups G, a representation (ρ, V) of G is equivalent to a $\mathbb{C}[G]$-module V, where

$$\mathbb{C}[G] := \{f : G \to \mathbb{C}\} = \mathbb{C}^G$$

and $\mathbb{C}[G]$ acts on V by

$$\rho(f).v := \sum_{g \in G} f(g)\rho(g).v$$

for all $f \in \mathbb{C}[G]$ and $v \in V$. Here $\mathbb{C}[G]$ is a finite dimensional $\mathbb{C}$-algebra with multiplication given by the **convolution**: for $f_1, f_2 \in \mathbb{C}[G]$, define $f_1 * f_2 \in \mathbb{C}[G]$ by

$$f_1 * f_2(x) := \sum_{g \in G} f_1(xg^{-1}) f_2(g)$$

Then $(\mathbb{C}[G], *)$ is a (usually non-commutative) $\mathbb{C}$-algebra, and V is a $\mathbb{C}[G]$-module.

In algebra, $\mathbb{C}[G]$ usually denotes the **group ring** of G:

$$\mathbb{C}[G] = \bigoplus_{g \in G} \mathbb{C}[g]$$

with $[g_1].[g_2] := [g_1 g_2]$ for all $g_1, g_2 \in G$.

Lemma 3.1. $(\mathbb{C}[G], *)$ is isomorphic to the group ring of G defined above, via the map $\mathbf{1}_g \mapsto [g]$, where $\not{K}_g$ is the characteristic function of the set $\{g\}$.

3.2.1 Hecke algebra

Definition. Let $f : G = \mathrm{GL}_2(\mathbb{Q}_p) \to \mathbb{C}$ be a function.

1. For an open compact $U \leqslant G$, f is called **bi U-invariant** if $f(u_1 g u_2) = f(g)$ for all $u_1, u_2 \in U$ and $g \in G$. Equivalently, f descends to a map $f : U \backslash G / U \to \mathbb{C}$ on the set of double cosets.

2. Define

$$\mathcal{H}(G) := \left\{ f : G \to \mathbb{C} \mid \operatorname{supp} f \text{ is compact, } \exists\, U \underset{\substack{\text{cpt} \\ \text{open}}}{\leqslant} G \text{ such that } f \text{ is bi } U\text{-invariant.} \right\}$$

Fix a Haar measure dg on G. For $f_1, f_2 \in \mathcal{H}(G)$, define $f_1 * f_2 \in \mathcal{H}(G)$ by

$$f_1 * f_2(x) := \int_G f_1(xg^{-1}) f_2(g) dg$$

for all $x \in G$. Then $(\mathcal{H}(G), *)$ is an associative $\mathbb{C}$-algebra, called the **Hecke algebra** of $G = \mathrm{GL}_2(\mathbb{Q})_p$.

Note that $\mathcal{H}(G)$ has no unit element (for G is not compact). However, for every open compact $U \leqslant G$, define

$$e_U := \frac{1}{\mathrm{vol}(U, dg)} \mathbb{I}_U \in \mathcal{H}(G)$$

Lemma 3.2. Let U be an open compact subgroup of G.

1. e_U is idempotent, i.e., $e_U * e_U = e_U$.
2. Put $\mathcal{H}(G,U) := e_U \mathcal{H}(G) e_U$. Then $\mathcal{H}(G,U)$ is a $\mathbb{C}$-algebra with the identity e_U, and

$$\mathcal{H}(G,U) = \{f \in \mathcal{H}(G) \mid f \text{ is bi } U\text{-invariant}\}$$

   In particular, $e_U * f * e_U = f$ for $f \in \mathcal{H}(G,U)$.

Suppose (ρ, V) is a smooth admissible representation of G. Then we can view V as a $\mathcal{H}(G)$-module as follows. For $f \in \mathcal{H}(G)$ and $v \in V$, define

$$\rho(f)v := \int_G f(g)\rho(g)v dg \in V$$

This is in fact a finite sum. Let $U \leqslant G$ be compact open such that f is bi U-invariant and $v \in V^U$. Cover $\operatorname{supp} f$ by finitely many translations of U, say $\operatorname{supp} f = g_1 U \cup \cdots \cup g_n U$. Then

$$\rho(f).v = \sum_{i=1}^{n} f(g_i)\rho(g_i)v$$

Lemma 3.3.

(i) For ϕ_1, $\phi_2 \in \mathcal{H}(G)$ and $v \in V$, one has $\rho(\phi_1 * \phi_2)v = \rho(\phi_1)\rho(\phi_2)v$. In particular, this means V is a $\mathcal{H}(G)$-module.

(ii) For open compact $U \leqslant G$, $\rho(e_U)V = V^U$.

(iii) If V is an $\mathcal{H}(G)$-module, then V^U is an $\mathcal{H}(G,U)$-module for any open compact $U \leqslant G$.

(iv) V is simple as a $\mathcal{H}(G)$-module if and only if each V^{K_n} is a simple $\mathcal{H}(G,K_n)$-module.

Proof.

(i) Compute directly.

$$\begin{aligned}
\rho(\phi_1 * \phi_2).v &= \int_G \phi_1 * \phi_2(g)\rho(g).v dg \\
&= \int_G \left(\int_G \phi_1(gh^{-1})\phi_2(h)dh\right)\rho(g).v dg \\
\text{(Fubini)} &= \int_G \int_G \phi_1(gh^{-1})\phi_2(h)\rho(g).v dg dh \\
\text{(invariant)} &= \int_G \int_G \phi_1(g)\phi_2(h)\rho(gh).v dg dh \\
&= \int_G \phi_1(g)\rho(g).\left(\int_G \phi_2(h)\rho(h).v dh\right) dg \\
&= \rho(\phi_1)\rho(\phi_2).v
\end{aligned}$$

(ii) This follows from (i) and Lemma 3.2.1: $\rho(e_U)\rho(e_U)V = \rho(e_U * e_U)V = \rho(e_U)V$, so $\rho(e_U)V \subseteq V^U$. Conversely, we need to show $\rho(e_U)V^U = V^U$. For $v \in V^U$,

$$\rho(e_U)v = \int_G e_U(x)\rho(g)v dg = \frac{1}{\operatorname{vol}(U, dg)} \int_U \rho(g)v dg = v$$

(iii) For $f \in \mathcal{H}(G,U)$ we have $e_U * f * e_U = f$ by Lemma 3.2.2, so that

$$\rho(f)V^U = \rho(e_U)\rho(f)\rho(e_U)V^U \subseteq \rho(e_U)V = V^U$$

(iv) Let $0 \neq W \subsetneq V^{K_n}$ be a proper submodule. Then $\mathcal{H}(G)W = V$ as V is simple. Then

$$V^{K_n} = e_{K_n}V = e_{K_n}\mathcal{H}(G)W = \mathcal{H}(G,K_n)e_{K_n}W = \mathcal{H}(G,K_n)W = W$$

a contradiction. Conversely, let $0 \leqslant W \subsetneq V$ be a proper $\mathcal{H}(G)$-module. Since $W = \bigcup_{n=1}^{\infty} W^{K_n}$, $0 \neq W^{K_n} \subsetneq V^{K_n}$ for some n, but this contradicts the simplicity of V^{K_n} as a $\mathcal{H}(G,K_n)$-module.

□

Proposition 3.4. There is a bijection

$$\{\text{smooth admissible representation of } G\} \longleftrightarrow \{\text{smooth admissible } \mathcal{H}(G)\text{-module}\}$$

where a smooth admissible $\mathcal{H}(G)$-module (ρ, V) means that $V = \bigcup_{n=1}^{\infty} \rho(e_{K_n})V$ with $\dim_{\mathbb{C}} \rho(e_{K_n})V < \infty$. Under this bijection, the irreducible representations of G correspond to simple $\mathcal{H}(G)$-modules.

3.2.2 Traces

In general, for V with $\dim_{\mathbb{C}} V = \infty$ we cannot define naive trace $\operatorname{Tr}(\rho(g))$ for $g \in G$. Nevertheless, if V is smooth admissible, then for all $f \in \mathcal{H}(G)$, f is bi U-invariant for some open compact $U \leqslant G$, so that $e_U * f * e_U = f$. Thus

$$\rho(f)V \subseteq \rho(e_U)V = V^U$$

so that $\dim_{\mathbb{C}} \rho(f)V < \infty$. Then we can define $\operatorname{Tr}\rho(f) := \operatorname{Tr}\rho(f)|_{V^U}$; this is well-defined by the following elementary lemma.

Lemma 3.5. Let $T : V \to V$ be a linear operator such that $\operatorname{Im} T \subseteq U, W$ for some finite-dimensional subspaces U, W of V. Then $\operatorname{Tr} T|_U = \operatorname{Tr} T|_W$.

Proof. It suffices to show $\operatorname{Tr} T|_U = \operatorname{Tr} T_{U \cap W}$, so we may assume $W \subseteq U$ in the first place. Let $w_1, \ldots, w_n$ be a basis for W and extend it to a basis $w_1, \ldots, w_n, u_1, \ldots, u_m$ for U. Then by writing down the matrix explicitly we easily see $\operatorname{Tr} T|_U = \operatorname{Tr} T|_W$. □

Theorem 3.6. Let (ρ_1, V_1) and (ρ_2, V_2) be irreducible smooth admissible representation of $G = \mathrm{GL}_2(\mathbb{Q}_p)$. If $\operatorname{Tr}\rho_1 = \operatorname{Tr}\rho_2$ on $\mathcal{H}(G)$, then $(\rho_1, V_1) \cong (\rho_2, V_2)$.

Proof. We first prove a lemma.

Lemma 3.7. If for all $n \in \mathbb{N}$ we have $V_1^{K_n} \cong V_2^{K_n}$ as $\mathcal{H}(G,K_n)$-modules, then $V_1 \cong V_2$ as $\mathcal{H}(G)$-modules.

Proof. Since $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$, we have

$$V^{K_1} \subseteq V^{K_2} \subseteq V^{K_3} \subseteq \cdots \subseteq V^{K_n} \subseteq \cdots$$

and $V = \bigcup_{n=1}^{\infty} V^{K_n}$ by smoothness. Fix a $\sigma_1 \in \operatorname{Isom}_{K_1}(V_1^{K_1}, V_2^{K_1})$ and let $\sigma_2 \in \operatorname{Isom}_{K_2}(V_1^{K_2}, V_2^{K_2})$. Then

$$\sigma_2|_{V_1^{K_1}} \in \operatorname{Isom}_{K_1}(V_1^{K_1}, V_2^{K_1})$$

Since each V_i is irreducible, by Lemma 3.3.(iv) each $V_i^{K_n}$ is a simple $\mathcal{H}(G,K_n)$-modules, so by Schur's lemma $\sigma_2|_{V_1^{K_1}} = \lambda\sigma_1$ for some $\lambda \in \mathbb{C}^\times$. Replacing σ_2 by $\lambda^{-1}\sigma_2$, we may assume $\sigma_2|_{V_1^{K_1}} = \sigma_1$. Continuing in this way, we can construct $\sigma \in \operatorname{Isom}_G(V_1, V_2)$ such that $\sigma_{V_1^{K_n}} = \sigma_n$ for each n. □

By this Lemma, it suffices to show $V_1^{K_n} \cong V_2^{K_n}$ as $\mathcal{H}(G,K_n)$-modules for each $n \in \mathbb{N}$. Since each $V_i^{K_n}$ is a simple $\mathcal{H}(G,K_n)$-module and $\operatorname{Tr}\rho_1 = \operatorname{Tr}\rho_2$ on $\mathcal{H}(G,K_n)$ by assumption, it follows from Jacobson's density theorem that $V_1^{K_n} \cong V_2^{K_n}$ for each $n \in \mathbb{N}$, hence the theorem. □

3.3 Contragredient representation

Let $G = \mathrm{GL}_2(\mathbb{Q}_p)$ with $p < \infty$ a finite prime, and (π, V) a smooth admissible representation of G. Put $V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ to be the **algebraic dual** of V, and define

$$\pi^\vee : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V^\vee)$$

by $\pi^\vee(g)\Lambda(v) := \Lambda(\pi(g^{-1})v)$. V^* is too big to be smooth. To fix this, define the **smooth dual**

$$V^\vee = \left\{ \Lambda \in V^* \mid \exists U \underset{\substack{\text{open}\\ \text{cpt}}}{\leqslant} G \text{ such that } \pi^\vee(g)\Lambda = \Lambda \text{ for all } g \in U \right\}$$

A linear functional $\Lambda \in V^\vee$ is the smooth dual is said to be **smooth**.

Definition. $(\pi^\vee, V^\vee) := (\pi^\vee|_{V^\vee}, V^\vee)$ is called the **contragredient representation** of (π, V).

Let

$$\begin{aligned} \langle , \rangle : V \times V^\vee &\longrightarrow \mathbb{C} \\ (v, \Lambda) &\longmapsto \langle v, \Lambda \rangle := \Lambda(v) \end{aligned}$$

be the canonical pairing.

Lemma 3.8. If $0 \to U \xrightarrow{\alpha} V \xrightarrow{\beta} W \to 0$ is an exact sequence of smooth admissible G-modules, then

$$0 \to W^\vee \xrightarrow{\beta^*} V^\vee \xrightarrow{\alpha^*} U^\vee \to 0$$

is also exact.

Proof.

- Suppose $\Lambda \in W^\vee$ such that $\beta^*\Lambda = \Lambda \circ \beta = 0$. Since $V \xrightarrow{\beta} W$ is surjective, $\Lambda = 0$.
- Let $\Lambda \in U^\vee$. Then we can find $\Lambda' \in V^*$ in the algebraic dual such that $\alpha^*\Lambda' = \Lambda$. Let $K \leqslant G = \mathrm{GL}_2(\mathbb{Q}_p)$ be a compact open subgroup such that $\pi^\vee(e_K)\Lambda = \Lambda$. Then

$$\begin{aligned} \alpha^* \left(\pi^\vee(e_K)\Lambda' \right)(v) &:= \int_G e_K(g)\Lambda'(\pi(g^{-1})\alpha v)dg = \int_G e_K(g)\Lambda'(\alpha\pi(g^{-1})v)dg \\ &= \int_G e_K(g)\alpha^*\Lambda'(\pi(g^{-1})v)dg \\ &= \pi^\vee(e_K)(\alpha^*\Lambda')(v) = \pi^\vee(e_K)\Lambda(v) = \Lambda(v) \end{aligned}$$

  Since $\pi^\vee(e_K)\Lambda' \in V^\vee$ is smooth (see Homework 2), this shows the surjectivity of α^*.
- Suppose $\Lambda \in V^\vee$ is such that $\alpha^*\Lambda = 0$ in $U^\vee$. Then we can find $\Lambda' \in W^*$ in the algebraic dual such that $\beta * \Lambda' = \Lambda$. The same argument as above says we can replace Λ' by a smooth one.

□

Proposition 3.9. Let (π, V) be a smooth admissible representation.

(i) For all compact open $K \leqslant G$, the restriction $\Lambda \mapsto \Lambda|_{V^K}$ is an isomorphism $(V^\vee)^K \to (V^K)^*$.

(ii) $(\pi^\vee, V^\vee)$ is admissible.

(iii) The pairing $\langle , \rangle : V \times V^\vee \to \mathbb{C}$ is a perfect pairing, in the sense that for all compact open $K \leqslant G$, the induced map $V^K \times (V^\vee)^K \to \mathbb{C}$ is perfect. In particular, $V \cong (V^\vee)^\vee$.

Proof.

(i) Suppose $\Lambda \in (V^\vee)^K$ such that $\Lambda|_{V^K} = 0$. Then for $v \in V$

$$\begin{aligned}\Lambda(v) &= \pi^\vee(e_K)\Lambda(v) \\ &= \int_G e_K(g)\Lambda(\pi(g^{-1})v)dg \\ &= \Lambda \int_G e_K(g)\pi(g^{-1})vdg = \Lambda \int_G e_K(g^{-1})\pi(g)vdg \\ &= \Lambda(\pi(e_K)v) = 0\end{aligned}$$

for $\pi(e_K)v \in V^K$. Hence $\Lambda = 0$, proving the injectivity.

For the surjectivity, let $\Lambda \in (V^K)^*$ and pick $\Lambda' \in V^*$ in the algebraic dual such that $\Lambda'|_{V^K} = \Lambda$. But as in the proof of Lemma 3.8, we have

$$(\pi^\vee(e_K)\Lambda')|_{V^K} = \pi^\vee(e_K)(\Lambda'|_{V^K}) = \pi^\vee(e_K)\Lambda = \Lambda$$

Since $\pi^\vee(e_K)\Lambda' \in (V^\vee)^K$, we are done.

(ii) By (i), $\dim_{\mathbb{C}}(V^\vee)^K = \dim_{\mathbb{C}}(V^K)^* = \dim_{\mathbb{C}} V^K < \infty$.

(iii) This follows from (i), (ii), the fact (iii) holds trivially in the finite dimensional case, and Lemma 3.7.

□

Remark 3.10. For $\phi \in \mathcal{H}(G)$ and $\Lambda \in V^*$, we always have $\pi^\vee(\phi)\Lambda \in V^\vee$. This is the p-adic analogue of approximation by smooth functions.

Suppose (π, V) an *irreducible* smooth admissible representation of $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Consider a new representation defined by

$$\begin{aligned}\breve{\pi} : G &\longrightarrow \mathrm{Aut}_{\mathbb{C}}(V) \\ g &\longmapsto \breve{\pi}(g) := \pi({}^t g^{-1})\end{aligned}$$

Then $(\breve{\pi}, V)$ is also irreducible smooth admissible.

Theorem 3.11. There is an isomorphism $(\breve{\pi}, V) \cong (\pi^\vee, V^\vee)$.

3.4 Td-space

Definition.

1. A topological space is a **td-space** if it admits a compact open basis. Equivalently, it is a totally disconnected locally compact space.

2. A topological group is a **td-group** if its underlying space is a td-space.

In the following let X be a td-space. We put

$$\begin{aligned}\mathcal{S}(X) &:= \{\phi : X \to \mathbb{C} \mid \phi \text{ is smooth (i.e. locally constant) with compact support}\} \ (= C_c^\infty(X)) \\ \mathcal{D}(X) &:= \mathrm{Hom}_{\mathbb{C}}(\mathcal{S}(X), \mathbb{C}) \qquad \text{(no continuity is concerned)}\end{aligned}$$

Lemma 3.12. For closed $Z \subseteq X$, we have an exact sequence

$$0 \longrightarrow \mathcal{S}(X - Z) \longrightarrow \mathcal{S}(X) \longrightarrow \mathcal{S}(Z) \longrightarrow 0$$

The first arrow is "extending by zero", and the second arrow is the restriction.

Proof. If $\phi \in \mathcal{S}(X)$ such that $\phi|_Z = 0$, then since ϕ is locally constant, we can find open W containing Z such that $\phi|_W \equiv 0$, implying $\operatorname{supp}\phi \subseteq X - W \subseteq X - Z$, i.e. $\phi \in \mathcal{S}(X - Z)$. This shows the complex is exact in the middle.

To show the complex is exact in the last position, note that by definition $\mathcal{S}(Z)$ is generated by the $\mathbf{1}_V$'s, where $V = U \cap Z$ for open U in X; then $\mathbf{1}_V = \mathbf{1}_U|_Z$, showing the exactness. $\square$

We can regard $\mathcal{S}(X)$ as a $\mathbb{C}$-algebra, with pointwise multiplication; note that $\mathcal{S}(X)$ has no identity element unless X is compact. For $x \in X$, put

$$\mathfrak{m}_x := \{\phi \in \mathcal{S}(X) \mid \phi(x) = 0\} \trianglelefteq \mathcal{S}(X)$$

Then $\mathcal{S}(X)/\mathfrak{m}_x \cong \mathbb{C}$.

Definition.

1. A $\mathcal{S}(X)$-module M is **smooth** if for all $m \in M$, there exists open compact $V \subseteq X$ such that $\mathbf{1}_V.m = m$.
2. The **fibre** of M at $x \in X$ is defined as $M_x := \dfrac{M}{\mathfrak{m}_x M}$.

Lemma 3.13. Let M be a smooth $\mathcal{S}(X)$-module.

(i) $m \in \mathfrak{m}_x M \Leftrightarrow \mathbf{1}_V.m = 0$ for all sufficiently small open compact neighborhoods V of x.

(ii) If $M_x = 0$ for all $x \in X$, then $M = 0$.

Proof.

(i) Assume $m = \phi.m'$ for some $\phi \in \mathfrak{m}_x$; then we can find an open compact neighborhood W of x such that $\phi|_W \equiv 0$. Then for all $V \subseteq W$ sufficiently small, $\mathbf{1}_V.m = \underbrace{\mathbf{1}_V\phi}_{=0}.m' = 0$.

For the converse, take an open compact V such that $\mathbf{1}_V.m = m$ by virtue of smoothness. If $x \notin V$, then $\mathbf{1}_V \in \mathfrak{m}_x$ so that $m = \mathbf{1}_V.v \in \mathfrak{m}_x M$. If $x \in V$, then by assumption, then we can find $x \in W \subseteq V$ small enough such that $\mathbf{1}_V.m = 0$. Thus

$$\mathbf{1}_{V-W}.m = \mathbf{1}_V.m - \mathbf{1}_W.m = m$$

Since $\mathbf{1}_{V-W}(x) = 0$, $m \in \mathfrak{m}_x M$.

(ii) Given $m \in M$, there exists an open compact V in X such that $\underline{\mathbf{1}_W.m = 0 \text{ for all open compact } W \subseteq V \subseteq X}$. Since $M_x = 0$ for all $x \in X$, then for each $x \in V$ we can find an open compact $x \in V_x \subseteq V$ such that $\mathbf{1}_{V_x}.m = 0$. Then

$$V = \bigcup_{x \in V} V_x = V_{x_1} \cup \cdots \cup V_{x_n}$$

for some $x_1, \ldots, x_n \in V$ by compactness. Put $V_1 = V_{x_1}$, $V_2 = V_{x_2} - V_{x_1}$, and so on; then

$$V = V_1 \sqcup \cdots \sqcup V_n$$

Thus

$$m = \mathbf{1}_V m = \sum_{i=1}^{n} \mathbf{1}_{V_i} m = 0$$

the last equality resulting from the underlined statement.

$\square$

Suppose X, Y are td-spaces and $f : Y \to X$ is a continuous map. Then $\mathcal{S}(X)$ acts on $\mathcal{S}(Y)$ via f, defined by

$$\phi.\xi(y) = \phi(f(y))\xi(y)$$

for all $\phi \in \mathcal{S}(X)$, $\xi \in \mathcal{S}(Y)$ and $y \in Y$. Then $\mathcal{S}(Y)$ is a smooth $\mathcal{S}(X)$-module. Indeed, for $\xi \in \mathcal{S}(Y)$, since $f(\operatorname{supp}\phi)$ is compact, we can cover it by a finite number of open compact sets $V_1 \ldots, V_n$; denote their union by V and $\phi = \mathbf{1}_V$. Then if $y \in \operatorname{supp}\phi$, $\phi(f(y))\xi(y) = \xi(y)$, and if $y \notin \operatorname{supp}\phi$, $\xi(y) = 0$. Thus $\phi.\xi = \xi$.

In general, $f^*(\mathcal{S}(X)) \nsubseteq \mathcal{S}(Y)$ unless f is proper.

Proposition 3.14. For $x \in X$, put $Y_x := f^{-1}(x) \underset{\text{closed}}{\subseteq} Y$. Then the restriction $\mathcal{S}(Y) \to \mathcal{S}(Y_x)$ induces an isomorphism $\mathcal{S}(Y)_x \cong \mathcal{S}(Y_x)$.

Proof. By the exact sequence

$$0 \longrightarrow \mathcal{S}(X - Z) \longrightarrow \mathcal{S}(X) \longrightarrow \mathcal{S}(Z) \longrightarrow 0$$

it suffices to show that $\mathcal{S}(Y - Y_x) = \mathfrak{m}_x\mathcal{S}(Y)$. By definition we have $\mathfrak{m}_x\mathcal{S}(Y) \subseteq \mathcal{S}(Y - Y_x)$. Conversely, suppose $\phi \in \mathcal{S}(Y - Y_x)$. Since $\operatorname{supp}\phi$ is compact, $f(\operatorname{supp}\phi)$ is compact not containing x, and thus we can find an open neighborhood U of x such that $f^{-1}(U)$ does not intersect with $\operatorname{supp}\phi$. Now consider $\mathbf{1}_U.\phi$. If $y \in \operatorname{supp}\phi$, then $\mathbf{1}_U(f(y))\mathbf{1}_{f^{-1}(U)}(y) = 0$; if $y \notin \operatorname{supp}\phi$, then $\phi(y) = 0$. From these we conclude $\mathbf{1}_U.\phi = 0$, and by Lemma 3.13.(i) we see $\phi \in \mathfrak{m}_x\mathcal{S}(Y)$. $\square$

Consider $X = G = \mathrm{GL}_2(\mathbb{Q}_p)$, and the right invariant distributions

$$\mathcal{D}(G)^G := \{\Delta \in \mathcal{D}(G) \mid \Delta(\rho_g\phi) = \Delta(\phi) \text{ for all } g \in G\}$$

where $\rho_g\phi(x) := \phi(xg)$ for all $x, g \in G$ and $\phi \in \mathcal{S}(G)$. The integral $\displaystyle\int_G dg \in \mathcal{D}(G)^G\backslash\{0\}$. Furthermore, we can show $\mathcal{D}(G)^G = \mathbb{C}\displaystyle\int_G dg$.

Proposition 3.15. $\dim_{\mathbb{C}} \mathcal{D}(G)^G \leqslant 1$.

Proof. It suffices to show that if $\Delta \in \mathcal{D}(G)^G$ is such that $\Delta(\mathbf{1}_{K_0}) = 0$ for some open compact subgroup $K_0 \leqslant G$, then $\Delta \equiv 0$. Suppose $K \leqslant K_0$ is an open compact subgroup of K_0, and put $\ell = [K_0 : K]$; the index is finite for K_0 is compact and K is open. Then

$$K = K_0g_1 \sqcup K_0g_2 \sqcup \cdots \sqcup K_0g_\ell$$

so that $\mathbf{1}_{K_0} = \rho_{g_1^{-1}}\mathbf{1}_K + \cdots + \rho_{g_\ell^{-1}}\mathbf{1}_K$. Thus

$$\Delta(\mathbf{1}_{K_0}) = \sum_{n=1}^{\ell} \Delta(\rho_{g_n^{-1}}\mathbf{1}_K) = \sum_{n=1}^{\ell} \Delta(\mathbf{1}_K) = \ell \cdot \Delta(\mathbf{1}_K)$$

Thus $\Delta(\mathbf{1}_K) = 0$ for all sufficiently small open compact subgroups K of G. Since $\mathcal{S}(G)$ is generated by the characteristic functions of all sufficiently small open compact subgroups, it follows that $\Delta \equiv 0$. $\square$

3.5 Theorem

Theorem 3.16. If $\Delta : \mathcal{H}(G) \to \mathbb{C}$ is a linear functional invariant under conjugation, then Δ is also invariant under transpose.

4 Local Whittaker Functionals

4.1 Bessel distributions

4.2 Multiplicity one of Whittaker models

Let (π, V) be an irreducible smooth admissible representation of $G = \mathrm{GL}_2(\mathbb{Q}_p)$, and $\psi : \mathbb{Q}_p \to \mathbb{C}$ a nontrivial character. The set

$$W_{\pi,\psi} := \{\Lambda \in V^* \mid \Lambda(\pi(\mathbf{n}(x))v) = \psi(x)\Lambda(v)\}$$

is called the space of **Whittaker functionals**. Note that we are considering all *algebraic duals* of V, not only the smooth ones.

Proposition 4.1. $\dim_{\mathbb{C}} W_{\pi,\psi} \leqslant 1$ (Homework 2)

Proposition 4.2. If $\dim_{\mathbb{C}} V = 1$, then $\dim_{\mathbb{C}} W_{\pi,\psi} = 0$.

Proof. Since $\dim_{\mathbb{C}} V = 1$, $\pi : G \to \mathrm{GL}(V) = \mathbb{C}^\times$ factors through the abelianization $G^{\mathrm{ab}} \overset{\det}{\cong} \mathbb{Q}_p^\times$, so that $\pi(g)v = \chi(\det g)v$ for some character $\chi : \mathbb{Q}_p^\times \to \mathbb{C}$. Then for $\Lambda \in W_{\pi,\psi}$, we have

$$\psi(x)\Lambda(v) = \Lambda(\pi(\mathbf{n}(x))v) = \Lambda(\chi(\det \mathbf{n}(x))v) = \Lambda(v)$$

Since ψ is chosen to be nontrivial, this implies $\Lambda = 0$. $\square$

Lemma 4.3. If $V^{N(\mathbb{Q}_p)} \neq 0$, then $\dim_{\mathbb{C}} V = 1$, and $\pi(g).v = \chi(\det g)v$ for some continuous character $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$.

Proof. Let $0 \neq v \in V^{N(\mathbb{Q}_p)}$ and $H \leqslant G$ the stabilizer of v. Then $H \supseteq N(\mathbb{Q}_p)$ and H is open by smoothness. By openness we see

$$\begin{pmatrix} 1 & \\ a & 1 \end{pmatrix} \in H \text{ for } a \in p^n\mathbb{Z}_p,\ n \gg 0$$

Now use the very important identity in $\mathrm{GL}_2(\mathbb{Q}_p)$:

$$\begin{pmatrix} 1 & \\ a & 1 \end{pmatrix} = \begin{pmatrix} 1 & a^{-1} \\ & 1 \end{pmatrix} \begin{pmatrix} & a^{-1} \\ -a & \end{pmatrix} \begin{pmatrix} 1 & a^{-1} \\ & 1 \end{pmatrix}$$

This implies $\begin{pmatrix} & a^{-1} \\ -a & \end{pmatrix} \in H$ for $0 \neq |a| \to 0$. Put $w_0 := \begin{pmatrix} & a^{-1} \\ -a & \end{pmatrix}$. Then

$$\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} = w_0^{-1} \begin{pmatrix} 1 & -a^2x \\ & 1 \end{pmatrix} w_0 \in H$$

for all $x \in \mathbb{Q}_p$. Thus H contains $\left\{ \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}, \begin{pmatrix} 1 & y \\ & 1 \end{pmatrix} \right\}_{x,y \in \mathbb{Q}}$, a generating set of $\mathrm{SL}_2(\mathbb{Q}_p)$. Hence $\mathrm{SL}_2(\mathbb{Q}_p) \leqslant H$, so that $V^{\mathrm{SL}_2(\mathbb{Q}_p)} \neq 0$. Since $\mathrm{SL}_2(\mathbb{Q}_p)$ is normal in G, $V^{\mathrm{SL}_2(\mathbb{Q}_p)}$ is G-invariant, and thus $V = V^{\mathrm{SL}_2(\mathbb{Q}_p)}$ by irreducibility. This means the action of G on V factor through $G/\mathrm{SL}_2(\mathbb{Q}_p) \overset{\det}{\cong} \mathbb{Q}_p^\times$ which is abelian. Thus $\dim_{\mathbb{C}} V = 1$, and the second statement follows at once. $\square$

Corollary 4.3.1. If $0 \neq \dim_{\mathbb{C}} V < \infty$, then $\dim_{\mathbb{C}} V = 1$.

Proof. Choose a basis of V and consider the intersection U of their stabilizer in G. By smoothness and finiteness, it is a nonempty open subgroup. Let $x \in \mathbb{Q}_p$ and take $a \in \mathbb{Q}_p^\times$ making $|ax| \to 0$ so small that $\mathbf{n}(ax) \in U$. Then

$$\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \begin{pmatrix} a^{-1} & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & ax \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & 1 \end{pmatrix}$$

so that $\mathbf{n}(x) \in U$. This shows $N(\mathbb{Q}_p) \subseteq U$, and thus $\dim_{\mathbb{C}} V = 1$ by Lemma 4.3. $\square$

Theorem 4.4. Suppose $\psi : \mathbb{Q}_p \to \mathbb{C}^\times$ be a nontrivial continuous homomorphism, and $\dim_{\mathbb{C}} V > 1$. Then $\dim_{\mathbb{C}} W_{\pi,\psi} = 1$.

Proof. It suffices to show $W_{\pi,\psi} \neq 0$. We proceed in the following steps.

1) Let $1 \neq \psi : \mathbb{Q}_p \to \mathbb{C}^\times$ be a continuous homomorphism. We know $\psi(x) = \psi_p(ax)$ for some $a \in \mathbb{Q}_p^\times$. We contend that $\underline{\text{if } W_{\pi,\psi_p} \neq 0, \text{ then } W_{\pi,\psi} \neq 0}$. This is easy, for if we are given $\Lambda \in W_{\pi,\psi_p}$, then the map $\Lambda_a(v) := \Lambda(\pi \begin{pmatrix} a & \\ & 1 \end{pmatrix} v)$ lies in $W_{\pi,\psi}$.

   We prove the theorem by contradiction. By 1) we then have $W_{\pi,\psi} = 0$ for all $\psi \neq 1$.

2) We equip V with another structure of smooth $\mathcal{S}(\mathbb{Q}_p)$-modules as follows: for $\phi \in \mathcal{S}(\mathbb{Q}_p)$ and $v \in V$, define

   $$\phi.v := \int_{\mathbb{Q}_p} \hat{\phi}(x)\pi(\mathbf{n}(x))v dx$$

   Here $\hat{\phi}(x) := \int_{\mathbb{Q}_p} \phi(y)\psi_p(xy)dy$ is the Fourier transform. It is clear V then becomes an $\mathcal{S}(\mathbb{Q}_p)$-module. To see the smoothness, for $v \in V$, since (π, V) is smooth, we can find $N \gg 0$ such that $\pi(\mathbf{n}(x))v = v$ for $x \in p^N\mathbb{Z}_p$. Take $\phi = \mathbf{1}_{p^{-N}\mathbb{Z}_p}$. Then

   $$\hat{\phi}(x) = \int_{p^{-N}\mathbb{Z}_p} \psi_p(xy)dy = p^N \mathbf{1}_{p^N\mathbb{Z}_p}(x)$$

   so that

   $$\phi.v = \int_{p^N\mathbb{Z}_p} p^N \pi(\mathbf{n}(x))v dx = p^N \int_{p^N\mathbb{Z}_p} v dx = v$$

   Consider the fibre of this $\mathcal{S}(\mathbb{Q}_p)$-action. For $x \in \mathbb{Q}_p$, by Lemma 3.13.1,

   $$\begin{aligned} \mathfrak{m}_x V &= \left\{ v \in V \mid \mathbf{1}_{x+p^n\mathbb{Z}_p} v = 0 \text{ for } n \gg 0 \right\} \\ &= \left\{ v \in V \mid \int_{p^{-n}\mathbb{Z}_p} \psi_p(xy)\pi(\mathbf{n}(y))v dy = 0 \text{ for } n \gg 0 \right\} \end{aligned}$$

   On the other hand, for $x \in \mathbb{Q}_p$ define

   $$\begin{aligned} \psi_x : \mathbb{Q}_p &\longrightarrow \mathbb{C}^\times \\ y &\longmapsto \psi_p(-xy) \end{aligned}$$

   and consider the subspace $V_{\psi_x}(N) := \operatorname{span}_{\mathbb{C}} \{ \pi(\mathbf{n}(a))v - \psi_x(a)v \mid v \in V, a \in \mathbb{Q}_p \}$. We contend the equality (important!!)

   $$V_{\psi_x}(N) = \mathfrak{m}_x V$$

$\subseteq$: For $v = \pi(\mathbf{n}(a))w - \psi_x(a)w$.

$$\begin{aligned}\int_{p^{-n}\mathbb{Z}_p} \psi(xy)\pi(\mathbf{n}(y))v dy &= \int_{p^{-n}\mathbb{Z}_p} \psi(xy)\left(\pi(\mathbf{n}(a))w - \psi_x(a)w\right) dy \\ &= \int_{p^{-n}\mathbb{Z}_p} \psi_p(xy)\pi(\mathbf{n}(y+a))w dy - \int_{p^{-n}\mathbb{Z}_p} \psi_p(x(y-a))\pi(\mathbf{n}(y))w dy \\ &= 0\end{aligned}$$

if $n \gg 0$ so that $a \in p^{-n}\mathbb{Z}_p$.

$\supseteq$: Let $v \in \mathfrak{m}_x V$. Then

$$0 = \int_{p^{-n}\mathbb{Z}_p} \psi_p(xy)\pi(\mathbf{n}(y))v dy$$

Take $N \gg 0$ so that $\pi(\mathbf{n}(t))v = v$ for $t \in p^N\mathbb{Z}_p$ and $xy \in \mathbb{Z}_p$ for all $y \in p^{-n}\mathbb{Z}_p$. Then

$$\begin{aligned}0 &= \sum_{y \in p^{-n}\mathbb{Z}_p/p^N\mathbb{Z}_p} \psi_p(xy)\pi(\mathbf{n}(y)v) \\ &= \sum_{y \in p^{-n}\mathbb{Z}_p/p^N\mathbb{Z}_p} \psi_p(xy)\left(\pi(\mathbf{n}(y)v) - \underbrace{\psi_p(-xy)}_{=\psi_x(y)} v\right) + \#\frac{p^{-n}\mathbb{Z}_p}{p^N\mathbb{Z}_p} v\end{aligned}$$

and hence

$$v = -\#\left(\frac{p^{-n}\mathbb{Z}_p}{p^N\mathbb{Z}_p}\right)^{-1} \sum_{y \in p^{-n}\mathbb{Z}_p/p^N\mathbb{Z}_p} \psi_p(xy)\left(\pi(\mathbf{n}(y)v) - \psi_x(y)v\right) \in V_{\psi_x}(N)$$

This proves the contention. Now $V_x := \dfrac{V}{\mathfrak{m}_x V} = \dfrac{V}{V_{\psi_x}(N)}$, so

$$V_x^* = W_{\pi,\psi_x}$$

3) Recall in 1) we are assuming $W_{\pi,\psi_x} = 0$ for all $x \neq 0$. By Lemma 3.13.2, we have an injection

$$V \hookrightarrow \prod_{x \in \mathbb{Q}_p} V_x = V_0 = \frac{V}{\mathfrak{m}_0 V}$$

This forces

$$0 = \mathfrak{m}_0 V = V_{\psi_0}(N) = \operatorname{span}_{\mathbb{C}}\{\pi(\mathbf{n}(a))v - v \mid v \in V,\, a \in \mathbb{Q}_p\}$$

so that $V = V^{N(\mathbb{Q}_p)}$. By Lemma 4.3, $\dim_{\mathbb{C}} V = 1$, a contradiction to our assumption.

□

We saw before that if V is a smooth admissible representation of $G = \mathrm{GL}_2(\mathbb{Q}_p)$, then V is a module of the Hecke algebra $\mathcal{H}(G)$. In fact,$\mathcal{H}(G) = \mathcal{S}(G)$ as sets, but with different ring multiplication:

$$(\mathcal{H}(G), *) \;:\; \phi_1 * \phi_2(x) := \int_G \phi_1(xg^{-1})\phi_2(g)dg$$
$$(\mathcal{S}(G), \cdot) \;:\; \phi_1 \cdot \phi_2(x) := \phi_1(x)\phi_2(x)$$

When $G = \mathbb{Q}_p$, we can also define $(\mathcal{H}(\mathbb{Q}_p), *)$. But in this case, they are isomorphic as rings via the Fourier transform:

$$(\mathcal{H}(G), *) \longrightarrow (\mathcal{S}(G), \cdot)$$
$$\phi \longmapsto \hat{\phi}$$

4.3 Uniqueness of Whittaker models

For a nontrivial continuous homomorphism $\psi : \mathbb{Q}_p \to \mathbb{C}^\times$, consider the space

$$W_\psi := \{W : G \to \mathbb{C} \mid W \text{ is locally constant, } W(\mathbf{n}(x)g) = \psi(x)W(g)\}$$

on which G acts by the right translation: $\rho(g)W(x) = W(xg)$.

Theorem 4.5. Let (π, V) be an irreducible smooth admissible representation with $\dim_{\mathbb{C}} V = \infty$. Then

$$\dim_{\mathbb{C}} \mathrm{Hom}_G((\pi, V), (\rho, W_\psi)) = 1$$

Proof. Consider the maps

$$\begin{array}{rcl} \mathrm{Hom}_G((\pi, V), (\rho, W_\psi)) & \xrightarrow{\quad\sim\quad} & W_{\pi,\psi} \\ f & \longmapsto & [\Lambda_f(v) = f(v)(1)] \\ [f_\Lambda(v)(g) = \Lambda(\pi(g)v)] & \longleftarrow & \Lambda \end{array}$$

The maps are well-defined and are mutually inverses. Hence the result follows from Theorem 4.4. □

Let $0 \neq f : (\pi, V) \to (\rho, W_\psi)$. Since V is irreducible, f must be injective. Let

$$\mathrm{Im}\, f := W_\psi(\pi)$$

This is called the **Whittaker model of (π, V) in (ρ, W_ψ)**. We have $(\rho, W_\psi(\pi)) \cong (\pi, V)$, and Theorem 4.5 is equivalent to the **uniqueness of the Whittaker model**, i.e.,

if $(\rho, W_\psi(\pi))$ and $(\rho, W_\psi(\pi)')$ are subrepresentations of (ρ, W_ψ), each of which isomorphic to (π, V), then $W_\psi(\pi) = W_\psi(\pi)'$ identically.

5 Jacquet module

Let (π, V) be an irreducible smooth admissible representation of $G = \mathrm{GL}_2(\mathbb{Q}_p)$. For a *continuous homomorphism* $\psi : \mathbb{Q}_p \to \mathbb{C}^\times$, put

$$V_\psi(N) = \left\{ \pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v - \psi(x)v \mid x \in \mathbb{Q}_p,\, v \in V \right\} \subseteq V$$

Then we have two spaces

$$\begin{aligned} J_\psi(V) &:= V/V_\psi(N) \\ W_{\pi,\psi} &:= \left\{ \Lambda : V \to \mathbb{C} \mid \Lambda(\pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v) = \psi(x)\Lambda(x) \right\} \\ &= J_\psi(V)^* = \mathrm{Hom}_{\mathbb{C}}(J_\psi(V), \mathbb{C}) \end{aligned}$$

Theorem 5.1. If $\psi \neq 1$ and $\dim_{\mathbb{C}} V > 1$, then

$$\dim_{\mathbb{C}} J_\psi(V) = 1$$

Proof. This follows from Theorem 4.4. □

If $\psi = 1$, we write

$$J(V) := J_1(V) = V/V(N)$$

where

$$V(N) = V_1(N) = \mathrm{span}_{\mathbb{C}} \left\{ \pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v - v \mid x \in \mathbb{Q}_p,\, v \in V \right\}$$

$J(V)$ is called the **Jacquet module** of V.

Lemma 5.2. If $\psi : \mathbb{Q}_p \to \mathbb{C}^\times$ be a continuous homomorphism, then there exists $a \in \mathbb{Q}_p$ such that $\psi(x) = \psi_p(ax)$ for all $x \in \mathbb{Q}_p$. Here ψ_p is the standard character on $\mathbb{Q}_p$:

$$\psi_p(x) = e^{-2\pi i \{x\}_p}$$

Proof. We show that ψ is trivial on $p^N\mathbb{Z}_p$ for some $N \gg 0$. Let W be an sufficiently small open disk in $\mathbb{C}$ with center 1:

$$W = \{ z \in \mathbb{C}^\times \mid |z - 1| < \varepsilon \}$$

Lemma 5.3. If ε is small enough, then W contains no nontrivial subgroup of $\mathbb{C}^\times$.

Proof. Recall that $\exp : \mathbb{C} \to \mathbb{C}^\times$ is a local diffeomorphism. Then we can find an open neighborhood U of 0 such that $\exp||_U : U \to \exp(U) = W$ is an isomorphism. If W contains a nontrivial subgroup, then there exists $U \ni z_0 \neq 0$ such that $\exp(z_0)^n \in W$ for all $n \in \mathbb{Z}$, i.e., $nz_0 \in U$ for all $n \in \mathbb{Z}$, a contradiction. □

Pick W as in the lemma. Then $\psi^{-1}(W)$ is an open set containing 0 in $\mathbb{Q}_p$, so we can find $N \gg 0$ such that $p^N\mathbb{Z}_p \subseteq \psi^{-1}(W)$. The lemma implies $\psi(p^N\mathbb{Z}_p) = \{1\}$. Then for each $n > 0$,

$$\psi|_{p^{-n}\mathbb{Z}_p} : \underbrace{\frac{p^{-n}\mathbb{Z}_p}{p^N\mathbb{Z}_p}}_{\text{a finite cyclic group}} \to \mathbb{C}$$

Lemma 5.4. The character group $\widehat{\dfrac{p^{-n}\mathbb{Z}_p}{p^N\mathbb{Z}_p}}$ is generated by $x \mapsto \psi_p(p^{-N}x)$.

Proof. We have isomorphisms

$$\frac{p^{-n}\mathbb{Z}_p}{p^N\mathbb{Z}_p} \longrightarrow \frac{\mathbb{Z}_p}{p^{n+N}\mathbb{Z}_p} \longrightarrow \frac{\mathbb{Z}}{p^{n+N}\mathbb{Z}}$$

$$x \longmapsto p^n x$$

$$x \longmapsto x \bmod p^{n+N}$$

The character group $\widehat{\frac{\mathbb{Z}}{p^{n+N}\mathbb{Z}}}$ is generated by the map $x \mapsto e^{-2\pi i x p^{-(N+n)}}$. The number $\frac{x}{p^{N+n}}$ in the exponent can be replaced by the number $\left\{\frac{x}{p^{N+n}}\right\}_p$. Thus $\widehat{\frac{p^{-n}\mathbb{Z}_p}{p^N\mathbb{Z}_p}}$ is generated by the map

$$x \mapsto e^{-2\pi i \{xp^{-N}\}_p} = \psi_p(p^{-N}x)$$

□

Thus can find $a_n \in p^{-n}\mathbb{Z}_p$ such that

$$\psi(x) = \psi_p(a_n x) \text{ for all } x \in p^{-n}\mathbb{Z}_p$$

If $x \in p^{-m}\mathbb{Z}_p$, $m > n$, then

$$\psi_p(a_m x) = \psi_p(a_n x) \text{ for all } x \in p^{-n}\mathbb{Z}_p$$

or $\psi((a_m - a_n)x) = 1$ for all $x \in p^{-n}\mathbb{Z}_p$, or $a_m - a_n \in p^n\mathbb{Z}_p$. Thus $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{Q}_p$; say $a_n \to a \in \mathbb{Q}_p$. Then $\psi(x) = \psi_p(ax)$ for all $x \in \mathbb{Q}_p$. □

Let

$$T = \left\{ \begin{pmatrix} a & \\ & d \end{pmatrix} \mid a, d \in \mathbb{Q}_p^\times \right\} \subseteq G$$

For $t \in T$, $tNt^{-1} \subseteq N$, so that $\pi(t)V(N) \subseteq V(N)$. Thus $(\pi, J(V))$ is an representation of T:

$$\pi(t)(v \bmod V(N)) := \pi(t)v \bmod V(N)$$

Since (π, V) is smooth, it is clear from definition that $(\pi, J(V))$ is smooth.

Theorem 5.5. $J(V)$ is an admissible representation of T.

Proof.

1° Let

$$T_n = \left\{ \begin{pmatrix} a & \\ & d \end{pmatrix} \mid a, d \equiv 1 \pmod{p^n\mathbb{Z}_p} \right\} \subseteq T$$

$J(V)$ being smooth, we have

$$J(V) = \bigcup_{n=1}^{\infty} J(V)^{T_n}$$

so we only need to show $\dim_{\mathbb{C}} J(V)^{T_n} < \infty$. The number n is fixed throughout this proof. Consider

$$K_n^N := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) \mid a, d \equiv 1 \pmod{p^n},\ c \equiv 0 \pmod{p^N} \right\}$$

Assume $\dim_{\mathbb{C}} V^{K_n^n} = d$, and choose $x_1, \ldots, x_{d+1} \in J(V)^{T_n}$. There is a natural projection

$$\begin{aligned} V^{K_n^n} &\longrightarrow J(V)^{T_n} \\ v &\longmapsto [v] := v \bmod V(N) \end{aligned}$$

But this is not surjective. To fix this, note that for $[v] \in J(V)^{T_n}$, $v \in V$ can be replaced by

$$c \int_{1+p^n\mathbb{Z}_p} \int_{1+p^n\mathbb{Z}_p} \int_{\mathbb{Z}_p} \pi \begin{pmatrix} a_1 & b \\ & a_2 \end{pmatrix} v \, db d^\times a_1 d^\times a_2$$

for some constant $c \neq 0$. Indeed, write

$$\begin{aligned} &\int_{1+p^n\mathbb{Z}_p} \int_{1+p^n\mathbb{Z}_p} \int_{\mathbb{Z}_p} \pi \begin{pmatrix} a_1 & b \\ & a_2 \end{pmatrix} v \, db d^\times a_1 d^\times a_2 \\ &= \int_{\mathbb{Z}_p} \pi \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \left(\int_{1+p^n\mathbb{Z}_p} \int_{1+p^n\mathbb{Z}_p} \pi \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} v \, d^\times a_1 d^\times a_2 \right) db \end{aligned}$$

Since π is smooth, there exists some $M \gg 0$ such that the above sum becomes

$$\operatorname{vol}(p^M\mathbb{Z}_p) \sum_{b \in \mathbb{Z}_p/p^M} \pi \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \left(\int_{1+p^n\mathbb{Z}_p} \int_{1+p^n\mathbb{Z}_p} \pi \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} v \, d^\times a_1 d^\times a_2 \right)$$

Since $[v] = v \bmod V(N)$ is fixed by T_n, we see the above integral reduces to

$$\operatorname{vol}(p^M\mathbb{Z}_p) \sum_{b \in \mathbb{Z}_p/p^M} \pi \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \operatorname{vol}(1+p^n\mathbb{Z}_p)^2 [v] = \operatorname{vol}(p^M\mathbb{Z}_p) \#(\mathbb{Z}_p/p^M) \operatorname{vol}(1+p^n\mathbb{Z}_p)^2 [v].$$

Then $c := \operatorname{vol}(p^M\mathbb{Z}_p)\#(\mathbb{Z}_p/p^M)\operatorname{vol}(1+p^n\mathbb{Z}_p)^2 = \operatorname{vol}(1+p^n\mathbb{Z}_p)^2$ works. In particular, this shows that $[v] \in J(V)^{T_n}$ has a representative fixed by

$$B_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) \mid a, d \equiv 1 \pmod{p^n},\ b \in \mathbb{Z}_p \right\}$$

In other words,

$$V^{B_n} \longrightarrow J(V)^{T_n}$$

is surjective. Say $x_i \in J(V)^{T_n}$ is represented by some $v_i \in V^{B_n}$, $i = 1, \ldots, d+1$.

2° We have $K_n^N = B_n N^-(p^N\mathbb{Z}_p)$ for $N \geqslant n$, where

$$N^-(p^N\mathbb{Z}_p) = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \mid x \in p^N\mathbb{Z}_p \right\}$$

This is because for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_n^N$, by definition $d \in 1 + p^n\mathbb{Z}_p \subseteq \mathbb{Z}_p^\times$ so that we can write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a - bc/d & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c/d & 1 \end{pmatrix}$$

Since $J(V)$ is smooth, we can find $N \gg n$ such that each v_i is fixed by $N^-(p^N\mathbb{Z}_p)$. Thus $v_i \in V^{K_n^N}$ for $i = 1, \ldots, d+1$.

3° For open compact $K', K \leqslant G$ and $x \in G$, define

$$\begin{aligned}
[K'xK] : V^K &\longrightarrow V^{K'} \\
v &\longrightarrow \frac{1}{\mathrm{vol}(K)} \pi(\mathbf{1}_{K'xK}).v = \frac{1}{\mathrm{vol}(K)} \int_{K'xK} \pi(g) v \, dg
\end{aligned}$$

This is essentially a finite sum: if we write $K'xK = \bigsqcup_{i=1}^{m} y_i K$ for some y_i, then

$$[K'xK]v = \sum_{i=1}^{m} \pi(y_i) v$$

Take $K = K_n^N = B_n^1 N^-(p^N \mathbb{Z}_p)$, $K' = K_n^n$ and $x = \begin{pmatrix} p^m & \\ & 1 \end{pmatrix}$, where $N \gg n \geqslant 1$ and $m = N - n$; then

$$K_n^n x K_N^n = \bigsqcup_{y=0}^{p^m - 1} \begin{pmatrix} p^m & y \\ 0 & 1 \end{pmatrix} K \tag{$\spadesuit$}$$

To see this, we start with studying the double coset

$$K_N^n \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} K_{N+1}^n$$

Compute

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix} \begin{pmatrix} p & \\ & 1 \end{pmatrix} = \begin{pmatrix} p & bd^{-1} \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & d \end{pmatrix} \begin{pmatrix} 1 & \\ pe & 1 \end{pmatrix}$$

If $b \in \mathbb{Z}_p$, $a, d \in 1 + p^n \mathbb{Z}_p$ and $e \in p^N \mathbb{Z}_p$, then $bd^{-1} \in \mathbb{Z}_p$ and $pe \in p^{N+1}\mathbb{Z}_p$. Also,

$$\begin{pmatrix} p & \alpha \\ & 1 \end{pmatrix} K_{N+1}^n = \begin{pmatrix} p & \beta \\ & 1 \end{pmatrix} K_{N+1}^n$$

if and only if

$$K_n^{N+1} \ni \begin{pmatrix} p^{-1} & -\alpha p^{-1} \\ & 1 \end{pmatrix} \begin{pmatrix} p & \beta \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & p^{-1}(\beta - \alpha) \\ & 1 \end{pmatrix}$$

These show that

$$K_n^N \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} K_n^{N+1} = \bigsqcup_{y=0}^{p-1} \begin{pmatrix} p & y \\ 0 & 1 \end{pmatrix} K_n^{N+1}$$

($\spadesuit$) can be derived exactly in the same way, and thus the map $[K'xK]$ has the form

$$\begin{aligned}
V^{K_n^N} &\longrightarrow V^{K_n^n} \\
x &\longmapsto \sum_{y=0}^{p^m - 1} \pi \begin{pmatrix} p^m & y \\ 0 & 1 \end{pmatrix} v
\end{aligned}$$

Then we have a commutative diagram

$$\begin{tikzcd}
v_i \arrow[d, mapsto] & v \arrow[d, mapsto] \in & V^{K_n^N} \arrow[r, "{[K_n^n x K_n^N]}"] \arrow[d] & V^{K_n^n} \arrow[d] & {[K_n^n x K_n^N]v} \arrow[d, mapsto] \\
x_i & {[v]} & J(V)^{T_n} \arrow[r, "\Phi"] & J(V)^{T_n} & {\left[\sum_{y=0}^{p^m-1} \pi \begin{pmatrix} p^m & y \\ 0 & 1 \end{pmatrix} v\right] = p^m \pi \begin{pmatrix} p^m & \\ & 1 \end{pmatrix} [v]}
\end{tikzcd}$$

where Φ is induced by $[K_n^n x K_n^N]$. The description given above in the right shows that Φ is in fact a $\mathbb{C}$-vector space isomorphism.

4° Since $\{[K_n^n x K_n^N]v_i\}_{i=1,\ldots,d+1} \subseteq V^{K_n^n}$ and $\dim_{\mathbb{C}} V^{K_n^n} := d$, there exist $\alpha_1, \ldots, \alpha_{d+1} \in \mathbb{C}$ not all zero such that

$$\sum_{i=1}^{d+1} \alpha_i [K_n^n x K_n^N] v_i = 0$$

Then

$$\begin{aligned}
0 &= \sum_{i=1}^{d+1} \alpha_i \cdot p^m \pi \begin{pmatrix} p^m & \\ & 1 \end{pmatrix} x_i = \Phi\left(\sum_{i=1}^{d+1} \alpha_i \cdot x_i\right) && \text{in } J(V)^{T_n} \\
\Rightarrow 0 &= \sum_{i=1}^{d+1} \alpha_i \cdot x_i && \text{in } J(V)^{T_n}
\end{aligned}$$

so that any $d+1$ elements in $J(V)^{T_n}$ are linearly dependent, proving $\dim_{\mathbb{C}} J(V)^{T_n} \leqslant d$.

□

Theorem 5.6. $\dim_{\mathbb{C}} J(V) \leqslant 2$.

Proof. Suppose $J(V) \neq 0$. Since $J(V)$ is admissible as a representation of T, $(J(V)^\vee)^{T_n} \neq 0$ and $\dim_{\mathbb{C}}(J(V)^\vee)^{T_n} < \infty$ for some $n \gg 0$. Then the action of T on $(J(V)^\vee)^{T_n}$ factors through T/T_n, which is a finite abelian group. Since T is abelian, there exist $\Lambda \in J(V)^\vee \backslash \{0\}$ (in some irreducible sub T/T_n-repn of $(J(V)^\vee)^{T_n}$) and continuous homomorphism $\chi : T \to \mathbb{C}^\times$ (by Schur's lemma) such that

$$\pi^\vee(t)\Lambda = \chi^{-1}(t)\Lambda, \qquad t \in T$$

Then

$$\begin{aligned}
\Lambda : J(V) &\longrightarrow \mathbb{C} \\
\pi(t)x &\longmapsto \chi(t)\Lambda(x)
\end{aligned}$$

Extending to $B = TN$ by 0 across N, we have (recall that $J(V) = V/V(N)$)

$$\begin{aligned}
\Lambda : V &\longrightarrow \mathbb{C} \\
\pi(tn)x &\longmapsto \chi(t)\Lambda(x)
\end{aligned}$$

for $t \in T$ and $n \in N$ (we also extend $\chi : B \to \mathbb{C}^\times$ by setting $\chi|_N \equiv 1$). Then

$$0 \neq \Lambda \in \mathrm{Hom}_B((V, \pi|_B), (\mathbb{C}, \chi)) = \mathrm{Hom}_G((V, \pi), \mathrm{ind}_B^G \chi)$$

by the Frobenius reciprocity, where

$$\mathrm{ind}_B^G \chi := \left\{ f : G \to \mathbb{C} \;\middle|\; \begin{array}{c} f(bg) = \chi(b) f(g) \text{ for } b \in B \\ \exists U \underset{\substack{\text{open} \\ \text{cpt}}}{\leqslant} G \text{ such that } f(gu) = f(g) \text{ for all } g \in G,\, u \in U \end{array} \right\}$$

and G acts on $\mathrm{ind}_B^G \chi$ by $\rho : G \to \mathrm{Aut}_{\mathbb{C}} \mathrm{ind}_B^G \chi$ defined by $\rho(g) f(x) = f(xg)$. The isomorphism is given as below:

Lemma 5.7 (Frobenius reciprocity)**.** Let G be a td-group and H a closed subgroup. Suppose (V, π) and (W, ρ) be smooth representations of G and H, respectively. Then there is an isomorphism

$$\mathrm{Hom}_H((V, \pi)|_B, (W, \psi)) \cong \mathrm{Hom}_G((V, \pi), \mathrm{ind}_H^G (W, \psi))$$

where $\mathrm{ind}_H^G W$ is defined by

$$\mathrm{ind}_B^G W := \left\{ f : G \to W \;\middle|\; \begin{array}{c} f(bg) = \psi(b) f(g) \text{ for } b \in B \\ \exists U \underset{\substack{\text{open} \\ \text{cpt}}}{\leqslant} G \text{ such that } f(gu) = f(g) \text{ for all } g \in G,\, u \in U \end{array} \right\}$$

with G acts on $\mathrm{ind}_B^G \chi$ by $\rho : G \to \mathrm{Aut}_{\mathbb{C}} \mathrm{ind}_B^G W$ defined by $\rho(g) f(x) = f(xg)$.

Proof. Define

$$\begin{array}{ccc} \mathrm{Hom}_H((V, \pi)|_B, (W, \psi)) & \underset{(\cdot)_H}{\overset{(\cdot)^G}{\rightleftarrows}} & \mathrm{Hom}_G((V, \pi), (\mathrm{ind}_H^G W, \rho)) \\ T & \longmapsto & T^G(v)(g) := T(\pi(g) v) \\ T_H(v) := T(v)(1) & \longleftarrow\!\shortmid & T \end{array}$$

The only thing that needs to check is the well-definedness.

- Let $T \in$ LHS. Then for $v \in V$, $g, g' \in G$

  $$T^G(\pi(g) v)(g') = T(\pi(g') \pi(g) v) = T(\pi(g' g) v) = T^G(v)(g' g) = \rho(g) T^G(v)(g')$$

  For $v \in V$, by smoothness we can find open compact $U \leqslant G$ by which v is fixed. Then for $g \in G$ and $u \in U$,

  $$T^G(v)(gu) = T(\pi(gu) v) = T(\pi(g) \pi(u) v) = T(\pi(g) v) = T^G(v)(g)$$

  so that $T^G(v) \in \mathrm{ind}_B^G W$.

- Let $T \in$ RHS. Then for $v \in V$, $h \in H$

  $$T_H(\pi(h) v) = T(\pi(h) v)(1) = \rho(h) T(v) 1 = T(v)(h) = \psi(h) T(v)(1) = \psi(h) T_H(v)$$

  For $v \in V$, by smoothness we can find open compact $U \leqslant G$ such that $\rho(u) T(v) = T(v)$ for all $u \in U$, and thus for $g \in G$ and $h \in U \cap B$, we have

  $$\psi(h) T_H(v) = \rho(h) T(v) 1 = T(v) 1 = T_H(v)$$

  Thus $T_H(v)$ is smooth so that $T_H(v) \in W$.

$\square$

By definition, $\operatorname{ind}_B^G \chi$ is smooth, and it is also admissible by

Lemma 5.8 (Iwasawa decomposition)**.** $G = BK$, where $K = \mathrm{GL}_2(\mathbb{Z}_p)$.

Proof. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$g = \begin{pmatrix} \det g/c & a \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & d/c \end{pmatrix}$$

if $\operatorname{ord}_p c \leqslant \operatorname{ord}_p d$, and

$$g = \begin{pmatrix} \det g/d & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c/d & 1 \end{pmatrix}$$

if $\operatorname{ord}_p c > \operatorname{ord}_p d$. $\square$

To see how does this imply the admissibility, suppose generally (W, ψ) is a smooth admissible representation of B. A function $f \in \operatorname{ind}_B^G W$ is determined by $f|_K$ for$f(bk) = \rho(b)f(k)$. Let $U \leqslant G$ be open compact. Then any function $f \in (\operatorname{ind}_B^G W)^U$ induces $f : K/K \cap U \to W$. Since K is compact, $K/K \cap U$ is a finite group. At this point, if W is finite dimensional, then $(\operatorname{ind}_B^G W)^U \subseteq \operatorname{span}_{\mathbb{C}}\{f : K/K \cap U \to \mathbb{C}\}$ is also finite dimensional. In general, let $x_1, \ldots, x_n \in K$ be a complete set of representative of $K/K \cap U$. Then $f(x_i)$ is fixed by $B \cap x_i U x_i^{-1}$ so that $f(x_i) \in W^{B \cap x_i U x_i^{-1}}$ which is finite dimensional thanks to the admissibility of W. Thus $\dim_{\mathbb{C}}(\operatorname{ind}_B^G W)^U < \infty$ as well.

Then $\operatorname{Hom}_G(V, \operatorname{ind}_B^G \chi) \neq 0$, and since V is irreducible, we have $V \hookrightarrow \operatorname{ind}_B^G \chi$ is injective.

Lemma 5.9. If we have an exact sequence of admissible smooth representations of G

$$0 \longrightarrow V_1 \xrightarrow{\alpha} V_2 \xrightarrow{\beta} V_3 \longrightarrow 0$$

then

$$0 \longrightarrow J(V_1) \longrightarrow J(V_2) \longrightarrow J(V_3) \longrightarrow 0$$

is also exact.

Proof. The nontrivial part is to show $J(V_1) \to J(V_2)$ is injective. If $x = [v] \in J(V_1)$ with $\alpha(x) = 0$ in $J(V_2)$. Then $\alpha(v) \in V_2(N)$, i.e..

$$\int_{p^{-n}\mathbb{Z}_p} \pi \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \alpha(v)\, dx = 0 \text{ for } n \gg 0$$

Since the integral is in fact a finite sum (which can be seen by choosing $U \leqslant p^{-n}\mathbb{Z}_p$ that fixes v and $\alpha(v)$ simultaneously), it follows that

$$\alpha\left(\int_{p^{-n}\mathbb{Z}_p} \pi \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} v dx\right) = 0$$

Since α is injective, it follows that $\int_{p^{-n}\mathbb{Z}_p} \pi \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} v dx = 0$, i.e., $v \in V_1(N)$. $\square$

By this lemma, it suffices to show $\dim_{\mathbb{C}} J(\operatorname{ind}_B^G \chi) \leqslant 2$, or dually $\dim_{\mathbb{C}}(J(\operatorname{ind}_B^G \chi))^* \leqslant 2$.

$$J(\operatorname{ind}_B^G \chi)^* = \{L : \operatorname{ind}_B^G \chi \to \mathbb{C} \mid L(\rho(n)f) = L(f) \text{ for } n \in N\}$$

Consider the projection

$$\begin{aligned} \mathcal{S}(G) &\xrightarrow{\quad p_\chi \quad} \operatorname{ind}_B^G \chi \\ \phi &\longmapsto p_\chi(\phi)(g) := \int_B \phi(bg)\chi^{-1}(b)db \end{aligned}$$

where db is the right-invariant Haar measure on B. This is in fact surjective, for if $f \in \operatorname{ind}_B^G \chi$, let $\phi := f \cdot \mathbf{1}_K \in \mathcal{S}(G)$. Then

$$p_\chi(\phi)(g) = \int_B f(bg)\mathbf{1}_K(bg)\chi^{-1}(b)db = \int_B f(g)\mathbf{1}_K(bg)db = f(g)\operatorname{vol}(K \cap B, db)$$

For $L \in J(\operatorname{ind}_B^G \chi)^*$, put $\Delta = \Delta_L := L \circ p_\chi : \mathcal{S}(G) \to \mathbb{C}$; then $\Delta \in \mathcal{D}(G)$.

Let $B \times N$ act on B by $\tau(b,n)x = b^{-1}xn$. For $(b_1, n) \in B \times N$,

$$\begin{aligned} p_\chi(\tau(b_1,n)^*\phi)(g) &= \int_B \phi(b_1^{-1}bgn)\chi^{-1}(b)db \\ &= \int_B \phi(bgn)\chi^{-1}(b_1b)d(b_1b) \\ &= \chi^{-1}\delta_B^{-1}(b_1).\rho(n)p_X(\phi)(g) \end{aligned}$$

(where δ_B is the modular character of B.) Thus

$$\Delta(\tau(b_1,n)^*\phi) = \chi\delta_B(b_1^{-1})L(\rho(n)p_\chi(\phi)) = \chi\delta_B(b_1^{-1})\Delta(\phi)$$

Lemma 5.10 (Bruhat decomposition)**.** We have

$$G = B \sqcup BwB$$

where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

It follows that we have an exact sequence

$$0 \longrightarrow \mathcal{S}(BwB) \longrightarrow \mathcal{S}(G) \longrightarrow \mathcal{S}(B) \longrightarrow 0$$

Taking dual, we have

$$0 \longrightarrow \mathcal{D}(BwB) \longrightarrow \mathcal{D}(G) \longrightarrow \mathcal{D}(B) \longrightarrow 0$$

so

$$0 \longrightarrow \mathcal{D}(BwB)^\chi \longrightarrow \mathcal{D}(G)^\chi \longrightarrow \mathcal{D}(B)^\chi$$

where

$$\mathcal{D}(\cdot)^\chi := \{\Delta \in \mathcal{D}(\cdot) \mid \tau(b,n)_*\Delta = \chi(b^{-1})\Delta \text{ for } (b,n) \in B \times N\}$$

Since $B \times N$ acts on BwB and B respectively, we have $B = \dfrac{B \times N}{B^{B\times N}}$ and $BwB = \dfrac{B \times N}{(BwB)^{B\times N}}$ as topological spaces.

Lemma 5.11. For G a td-group and χ a continuous character of G, $\dim_{\mathbb{C}} \mathcal{D}(G)^\chi \leqslant 1$.

Proof. Let $\Delta \in \mathcal{D}(G)^\chi$ and $K_0 \leqslant G$ a compact open subgroup such that $\Delta(\chi^{-1}\mathbf{1}_{K_0}) = 0$ (note that $\chi \in \mathcal{S}(G)$ thanks to its continuity and by a no small subgroup argument). We need to show that $\Delta \equiv 0$. □

Since B and BwB a quotient of $B \times N$, we obtain

$$\dim_{\mathbb{C}} \mathcal{D}(B)^{\chi}, \dim_{\mathbb{C}} \mathcal{D}(BwB)^{\chi} \leqslant \dim_{\mathbb{C}} \mathcal{D}(B \times N)^{\chi} \leqslant 1$$

so that $\dim_{\mathbb{C}} \mathcal{D}(G)^{\chi} \leqslant 2$. Finally, since p_{χ} is injective, the pullback map

$$\begin{aligned} p_{\chi}^{*} : J(\operatorname{ind}_B^G \chi) &\longrightarrow \mathcal{D}(G)^{\chi} \\ L &\longmapsto p_{\chi}^{*} L = L \circ p_{\chi} \end{aligned}$$

is injective, so $\dim_{\mathbb{C}} J(\operatorname{ind}_B^G \chi) \leqslant 2$. $\square$

Remark 5.12. This is a general method to study the representation of $G = \mathrm{GL}_2(\mathbb{Q}_p)$. We have several important subgroup

$$\begin{tikzcd} & \begin{array}{c}\textbf{Borel subgroup}\\ B = \left\{\begin{pmatrix} a & b \\ & d \end{pmatrix}\right\} \leqslant G\end{array} & \\ \begin{array}{c} N = \left\{\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\right\} \\ \textbf{unipotent radical}\end{array} \arrow[ur, phantom, sloped, "\subseteq"] & & \begin{array}{c} T = \left\{\begin{pmatrix} a & \\ & d \end{pmatrix}\right\} \\ \textbf{maximal torus} \text{ / Levi subgroup}\end{array} \arrow[ul, phantom, sloped, "\supseteq"] \end{tikzcd}$$

Say (π, V) a representation of G, form $J(V) = V/V(N)$ and prove that $J(V)$ is a admissible representation of T. If $J(V) \neq 0$, V is a subrepresentation of $\operatorname{ind}_B^G \chi$.

6 Classification of $(\mathfrak{g}, K)$-modules

6.1 Basics on real Lie groups

Let G be a Lie group. For $x \in G$, denote $T(G)_x$ to be the tangent space of G at x, that is

$$T(G)_x := \{D : \mathcal{O}_{G,x} \to \mathbb{C} \mid D \text{ is a derivation at the point } x\}$$

where $\mathcal{O}_{G,x}$ is the real algebra of smooth functions defined around x. Then we can form the tangent bundle

$$T(G) = \bigsqcup_{x \in G} T(G)_x$$

Definition. $\mathrm{Lie}(G) = T(G)_e$ is called the **Lie algebra** of G, where e is the identity element of G.

For $g \in G$, put

$$\begin{aligned} \rho_g : G &\longrightarrow G \\ x &\longmapsto xg \end{aligned} \qquad \begin{aligned} \lambda_g : G &\longrightarrow G \\ x &\longmapsto g^{-1}x \end{aligned}$$

For $X \in \mathrm{Lie}(G)$, we can construct a right invariant vector field $\mathcal{L}_X$; namely, a smooth section

$$\mathcal{L}_X : G \to TG$$

with $\mathcal{L}_X(e) = X$ and for all $g \in G$, the diagram

$$\begin{tikzcd} G \arrow[r, "\mathcal{L}_X"] \arrow[d, "\rho_g"'] & TG \arrow[d, "\rho_{g*}"] \\ G \arrow[r, "\mathcal{L}_X"'] & TG \end{tikzcd}$$

commutes. It is clear that $\mathcal{L}_X(g) := \rho_{g*}X$ is the unique right invariant vector field with $\mathcal{L}_X(e) = X$.

Theorem 6.1. For $X \in \mathrm{Lie}(G)$, there exists a unique curve $\gamma_X : \mathbb{R} \to G$ such that

- $\gamma_X(0) = e$;
- $\gamma_X'(t_0) := (\gamma_X)_* \left(\left. \dfrac{d}{dt} \right|_{t=t_0} \right) = \mathcal{L}_X(\gamma_X(t_0))$ for all $t_0 \in \mathbb{R}$.

Such a curve is called the **integral curve** for $\mathcal{L}_X$. Moreover, the unique **local flow** $\Phi(g,t) : G \times \mathbb{R} \to G$ for $\mathcal{L}_X$ is smooth and is given by $\Phi(g,t) = g\gamma_X(t)$.

Definition. Define the **exponential map**

$$\begin{aligned} \exp : \mathrm{Lie}(G) &\longrightarrow G \\ X &\longrightarrow \gamma_X(1) \end{aligned}$$

The exp is smooth and is a local diffeomorphism at the origin.

- We have $\left. \dfrac{d}{dt} \right|_{t=0} \exp(tX) = \left. \dfrac{d}{dt} \right|_{t=0} \gamma_X(t) = \gamma_X'(0) = X$.

Example. Let $G = \mathrm{GL}_2(\mathbb{R})$ or one of its connected components $G = \mathrm{GL}_2(\mathbb{R})^+ = \{A \in \mathrm{GL}_2(\mathbb{R}) \mid \det A > 0\}$. Note that $\mathrm{GL}_2(\mathbb{R})^+ \trianglelefteq \mathrm{GL}_2(\mathbb{R})$ has index two.

With the standard coordinates x_{ij} on G,

$$\mathrm{Lie}(G) = \bigoplus_{1 \leqslant i,j \leqslant 2} \mathbb{R} X_{ij}$$

where for $f \in \mathcal{O}_{G,e}$

$$X_{ij}(f) = \frac{\partial f}{\partial x_{ij}}(e)$$

With this standard basis, $\mathrm{Lie}(G) = M_2(\mathbb{R})$. For $X \in M_2(R) = \mathrm{Lie}(G)$, we have

$$\gamma_X(t) = e^{tX} = \sum_{n=0}^{\infty} \frac{X^n t^n}{n!}$$

Then $\exp : \mathrm{Lie}(G) \to G$ has the form

$$\exp(X) = e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!}$$

When $G = \mathrm{GL}_2(\mathbb{R})$, we will write $t \mapsto e^{tX}$ to mean the integral curve for $\mathcal{L}_X$.

Definition. For $X \in \mathrm{Lie}(G)$, let $\rho(X) : \mathcal{O}_{G,e} \to \mathcal{O}_{G,e}$ be the derivation defined by

$$\rho(X) f(g) = \left.\frac{d}{dt}\right|_{t=0} f(g e^{tX})$$

Note that $\rho(X) f(e) = X(f)$.

Definition. Define the **Lie bracket** $[,] : \mathrm{Lie}(G) \times \mathrm{Lie}(G) \to \mathrm{Lie}(G)$ by

$$[X, Y] f := X(\rho(Y) f) - Y(\rho(X) f)$$

for $f \in \mathcal{O}_{G,e}$. It satisfies the **Jacobi's identity**

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$$

With the exponential map, we can show that G has *no small subgroup*, i.e., there exists an open neighborhood of e in G such that W contains no nontrivial subgroup of G. Further, we can show that if G' is a compact td-group and $f : G' \to G$ is a continuous group homomorphism, then $f(G') \subseteq G$ must be finite.

6.2 Representations

Definition. A **representation** (π, H) of $G = \mathrm{GL}_2(\mathbb{R})$ consists of a Hilbert space $(H, \langle , \rangle)$ and a homomorphism $\pi : G \to \mathrm{Aut}_{\mathbb{C}} H$ such that the action map

$$\begin{aligned} G \times H &\longrightarrow H \\ (g, v) &\longmapsto \pi(g).v \end{aligned}$$

is continuous. We say (π, H) is unitary if for all $g \in G$,

$$\langle \pi(g) v, \pi(g) w \rangle = \langle v, w \rangle$$

for all $v, w \in H$.

Let $C_c^\infty(G)$ denote the space of smooth functions on G with compact support. We define the **smooth convolution**. Let dg be the right invariant Haar measure on G. For $\phi \in C_c^\infty(G)$ and $v \in H$, define

$$\pi(\phi).v := \int_G \phi(g)\pi(g).vdg$$

In fact, $\pi(\phi).v$ is defined to be the unique vector in H such that for all $w \in H$,

$$\langle \pi(\phi)v, w\rangle = \int_G \phi(g)\langle \pi(g)v, w\rangle dg$$

The existence and the uniqueness of such vector is guaranteed by the Rieze's representation theorem.

Definition. A vector $v \in H$ is C^1 if for all $X \in \mathrm{Lie}(G)$, the limit

$$\lim_{t\to 0} \frac{\pi(e^{tX}).v - v}{t}$$

exists. If it exists, we put

$$\pi(X).v := \lim_{t\to 0} \frac{\pi(e^{tX}).v - v}{t} = \left.\frac{d}{dt}\right|_{t=0} \pi(e^{tX}).v$$

Inductively, we say $v \in C^k$ $(k \geqslant 2)$ if $\pi(X).v \in C^{k-1}$ for all $X \in \mathrm{Lie}(G)$. Put

$$H^{\mathrm{sm}} := \{v \in H \mid v \in C^k \text{ for all } k \geqslant 1\}$$

to be the subspace of **smooth vectors** in H.

- If $\phi \in C_c^\infty(G)$ and $v \in H$, then $\pi(\phi)v \in H^{\mathrm{sm}}$. Indeed,

$$\begin{aligned}\pi(X)\pi(\phi)v &= \left.\frac{d}{dt}\right|_{t=0} \pi(e^{tX})\pi(\phi).v = \left.\frac{d}{dt}\right|_{t=0} \int_G \phi(g)\pi(e^{tX}g)vdg \\ &= \left.\frac{d}{dt}\right|_{t=0} \int_G \phi(e^{-tX}g)\pi(g)vdg \\ &- \int_G \phi_X(g)\pi(g)vdg\end{aligned}$$

  where $\phi_X(g) := \left.\frac{d}{dt}\right|_{t=0} \phi(e^{-tX}g)$.

Let $\{\phi_n\}$ be an approximate of identity on G, namely,

(1) $\phi_n \in C_c^\infty(G)$ for all n,

(2) $\int_G \phi_n(g)dg = 1$ for all nn and

(3) for all open neighborhoods U of e, $\lim_{n\to\infty} \int_U \phi_n(g)dg = 1$.

Lemma 6.2. For all $v \in H$, $\lim_{n\to\infty} \pi(\phi_n)v = v$, where $\{\phi_n\}$ is an approximate of identity. In particular, H^{sm} is dense in H.

6.3 Classification

Definition. Let $G = \mathrm{GL}_2(\mathbb{R})$, $\mathfrak{g} = \mathrm{Lie}(G)$, $K = \mathrm{O}(2)$. A **($\mathfrak{g}$,K)-module (π, V)** is a $\mathbb{C}$-vector space with a Lie algebra homomorphism $\pi : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}} V$ and a group homomorphism $\pi : K \to \mathrm{Aut}_{\mathbb{C}}(V)$ such that

- for all $X \in \mathrm{Lie}\, K \subseteq \mathfrak{g}$, we have
$$\pi(X)v = \left.\frac{d}{dt}\right|_{t=0} \pi(e^{tX})v$$
- for all $X \in \mathfrak{g}$ and $k \in K$
$$\pi(\mathrm{Ad}_k X)v = \pi(k)\pi(X)\pi(k^{-1})v$$
where $\mathrm{Ad}_k := (c_k)_{*,e}$ and $c_k : G \to G$ is defined by $c_k(x) = kxk^{-1}$.

and the representation (π, V) of K is **admissible**, or ***K*-finite**, i.e.

- for all $v \in V$, the $\mathbb{C}$-span of $\{\pi(k)v \mid v \in K\}$ is finite dimensional.

In addition, we assume V is **smooth**, i.e., for all $X \in \mathrm{Lie}\, K$, $v \in V$, $\Lambda \in V^\vee$, the function
$$\mathbb{R} \ni t \mapsto \langle \pi(e^{tX})v, \Lambda \rangle \in \mathbb{C}$$
is smooth in the usual sense.

For an Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, we can define the **universal enveloping algebra** $U(\mathfrak{g})$ by the quotient $T(\mathfrak{g})/I$, where $T(\mathfrak{g})$ is the tangent algebra generated by the $\mathbb{C}$-module $\mathfrak{g}$, and I is the two-sided ideal generated by the elements $[X,Y] - X \otimes Y + Y \otimes X$. The resulting quotient $U(\mathfrak{g})$ is then a (non-commutative) $\mathbb{C}$-algebra. More precisely, if $\mathfrak{g}$ has a $\mathbb{C}$-basis $x_1, \ldots, x_d$, and $[x_i, x_j] = \sum_{\ell=1}^{d} b_{ij}^\ell x_\ell$ with $b_{ij}^\ell \in \mathbb{C}$, then the **Poincaré-Birkhoff-Witt** theorem, , or PBW theorem, says that
$$U(\mathfrak{g}) = \bigoplus_{a_1,\ldots,a_d \in \mathbb{N}_0} \mathbb{C} x_1^{a_1} \cdots x_d^{a_d}$$
with $x_i x_j = x_j x_i + \sum_{\ell=1}^{d} b_{ij}^\ell x_\ell$. In particular, if $\mathfrak{g}$ is an abelian Lie algebra, then $U(\mathfrak{g}) = \mathbb{C}[x_1, \ldots, x_d]$.

For a Lie algebra $\mathfrak{g}$, we have the **adjoint representation**
$$\begin{aligned} \mathrm{ad} : \mathfrak{g} &\longrightarrow \mathrm{End}\, \mathfrak{g} \\ X &\longmapsto \mathrm{ad}_X : Y \mapsto [X,Y] \end{aligned}$$
The Jacobi identity becomes
$$\mathrm{ad}_{[X,Y]} = \mathrm{ad}_X \mathrm{ad}_Y - \mathrm{ad}_Y \mathrm{ad}_X \;\text{ in }\; \mathrm{End}\, \mathfrak{g}$$
We have the **Killing form** on $\mathfrak{g}$, which is by definition the symmetric bilinear form $B(X,Y) := \mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Y)$ on $\mathfrak{g}$. The Jacobi identity tells
$$B(\mathrm{ad}_Z X, Y) = -B(X, \mathrm{ad}_Z Y)$$
Let us assume the Killing form B is nondegenerate. Then for a basis $x_1, \ldots, x_d$ for $\mathfrak{g}$, there exists a dual basis $y_1, \ldots, y_d$ satisfying $B(x_i, y_j) = \delta_{ij}$. The **Casimir element** is defined as
$$\Delta := x_1 y_1 + \cdots + x_d y_d = \sum_{i=1}^{d} x_i y_i \in U(\mathfrak{g})$$

Proposition 6.3.

1. The element Δ is independent of the choice of basis $x_1, \ldots, x_d$.
2. Δ lies in the center of $U(\mathfrak{g})$.

Example. Consider the $\mathfrak{sl}_2(\mathbb{R}) = \operatorname{Lie}\operatorname{SL}_2(\mathbb{R}) = \{A \in \mathrm{M}_2(\mathbb{R}) \mid \operatorname{Tr} A = 0\}$. We have $\mathfrak{sl}_2(\mathbb{R}) = \mathbb{R}H_+ \oplus \mathbb{R}R_+ \oplus \mathbb{R}L_+$, where

$$H_+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad R_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad L_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with the relations

$$[H_+, R_+] = 2R_+, \qquad [H_+, L_+] = -2L_+, \qquad [R_+, L_+] = H_+$$

Thus, with respect to the ordered basis $\{H_+, R_+, L_+\}$,

$$\operatorname{ad}_{H_+} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \qquad \operatorname{ad}_{R_+} = \begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \operatorname{ad}_{L_+} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

In matrices, the Killing form B is

$$B = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix}$$

and thus the dual basis is $\frac{1}{8}H_+$, $\frac{1}{4}L_+$, $\frac{1}{4}R_+$. The Casimir element is then $\frac{1}{8}H_+^2 + \frac{1}{4}R_+L_+ + \frac{1}{4}L_+R_+$. For convenience, let us put

$$\Delta = H_+^2 + 2R_+L_+ + 2L_+R_+ \in Z(U(\mathfrak{sl}_2(\mathbb{R})))$$

Consider $\mathfrak{g} := \operatorname{Lie}\operatorname{GL}_2(\mathbb{R})$. The Killing form B on $\mathfrak{g}$ is degenerate. To see this, note that

$$\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The element $J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ commutes with everyone, i.e, $\operatorname{ad}_J = 0$ on $\mathfrak{g}$. Thus $J \neq 0$ lies in the radical of B. Nonetheless,

$$U(\mathfrak{g}) = \mathbb{R}[J] \otimes_{\mathbb{R}} U(\mathfrak{sl}_2(\mathbb{R}))$$

so the constructed element Δ also commutes with elements in $U(\mathfrak{g})$.

Consider the action of $K = \mathrm{O}(2)$. We have $\operatorname{Ad}_g X = gXg^{-1}$ for all $g \in G = \operatorname{GL}_2(\mathbb{R})$ and $X \in \mathfrak{g}$. Then $B(\operatorname{Ad}_g X, \operatorname{Ad}_g Y) = B(X, Y)$ and thus

$$\operatorname{Ad}_g \Delta = \operatorname{Ad}_g(H_+)^2 + 2\operatorname{Ad}_g(R_+)\operatorname{Ad}_g(L_+) + 2\operatorname{Ad}_g(L_+)\operatorname{Ad}_g(R_+) = \Delta$$

In particular, $\operatorname{Ad}_k \Delta = \Delta$ for all $k \in K$. Therefore, for any $(\mathfrak{g}, K)$-module (π, V), we have $\pi(\Delta) \in \operatorname{End}_{(\mathfrak{g},K)}(V)$.

Proposition 6.4 (Schur's lemma)**.** If (π, V) is an irreducible admissible $(\mathfrak{g}, K)$-module and $X \in \mathfrak{g}$ such that $\pi(X) \in \operatorname{End}_{(\mathfrak{g},K)}(V)$, then $\pi(X)$ acts on V by a scalar.

In particular, $\pi(\Delta)$ and $\pi(J)$ acts on V as scalars, where

$$J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathfrak{g}$$

Let $G^+ = \mathrm{GL}_2(\mathbb{R})^+ = \{g \in M_2(\mathbb{R}) \mid \det g > 0\}$; then $\mathfrak{g} := \operatorname{Lie} G = \operatorname{Lie} G^+$. Put

$$K^+ := K \cap G^+ = \mathrm{SO}_2(\mathbb{R}) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mid \theta \in \mathbb{R} \right\}$$

which is an index two abelian subgroup of K^+. Let (π, V) be an admissible irreducible $(\mathfrak{g}, K^+)$-module, which is defined in a similar way as $(\mathfrak{g}, K)$-modules. Let $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ and

$$H = -i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; R = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \; L = \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$$

For each $\ell \in \mathbb{Z}$, define the **weight ℓ space**

$$V(\ell) := \left\{ v \in V \mid \pi \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} v = e^{i\ell\theta} v \right\}$$

By K^+-finiteness, together with the fact $\widehat{K^+} = \widehat{\mathbb{R}/2\pi\mathbb{Z}} = \{x \mapsto e^{i\ell x} \mid \ell \in \mathbb{Z}\}$, we have the decomposition

$$V = \bigoplus_{\ell \in \mathbb{Z}} V(\ell)$$

with each $V(\ell)$ finite dimensional. ???

We have the following formulas. For $v \in V(\ell)$,

1. $\pi(H)v = \ell v$.
2. If we put $k_\theta := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, then

$$\pi(\operatorname{Ad}_{k_\theta} L)v = \pi(k_\theta L k_\theta^{-1})v = e^{2i\theta}\pi(L)v$$
$$\pi(\operatorname{Ad}_{k_\theta} L)v = \pi(k_\theta L k_\theta^{-1})v = e^{2i\theta}\pi(L)v$$

In particular, this means $R : V(\ell) \to V(\ell+2)$ and $L : V(\ell) \to V(\ell-2)$.

Since (π, V) is irreducible, by Schur's lemma, $\pi(\Delta) = \lambda_\Delta \operatorname{id}$ and $\pi(J) = \lambda_J \operatorname{id}$ for some constant λ_Δ, $\lambda_J \in \mathbb{C}$.

Pick $0 \neq v \in V(\ell)$ and form the subspace

$$V' = \mathbb{C}v \oplus \bigoplus_{n \geqslant 1} \mathbb{C}R^n v \oplus \bigoplus_{n \geqslant 1} \mathbb{C}L^n v$$

This is a $(\mathfrak{g}, K^+)$-submodule of V, so by irreducibility of V, $V = V'$. In particular, $\dim_{\mathbb{C}} V(\ell) = 0$ or 1 for each $\ell \in \mathbb{Z}$. Put

$$\Sigma_V := \{\ell \in \mathbb{Z} \mid \dim_{\mathbb{C}} V(\ell) = 1\}$$

Then $V = \bigoplus_{\ell \in \Sigma_V} V(\ell)$, and if ℓ_1, $\ell_2 \in \Sigma_V$, then $\ell_1 \equiv \ell_2 \pmod 2$. Let $\epsilon \in \{0, 1\}$ be the **parity** of V, i.e., $\epsilon \equiv \ell \pmod 2$ for all $\ell \in \Sigma_V$.

Theorem 6.5.

1. If λ_Δ is not of the form m^2-1, $m \in \mathbb{Z}$, or $\lambda_\delta = m^2-1$ for some $m \in \mathbb{Z}$ with $m \equiv \epsilon \pmod 2$, then

$$\Sigma_V = \{\ell \in \mathbb{Z} \mid \ell \equiv_2 \epsilon\}$$

2. If $\lambda_\Delta = m^2 - 1$ with $m \equiv \epsilon + 1 \pmod 2$, then there are three possibilities of Σ_V. If we put $m = k+1$, then either

   - $\Sigma_V = \{|k|, |k|+2, \ldots\} = \{\ell \in \mathbb{Z} \mid \ell \geqslant |k|, \ell \equiv_2 \epsilon\}$,
   - $\Sigma_V = \{-|k|, -|k|+2, \ldots, |k|-2, |k|\} = \{\ell \in \mathbb{Z} \mid |\ell| \leqslant |k|,\ \ell \equiv_2 \epsilon\}$, or
   - $\Sigma_V = \{-|k|, -|k|-2, \ldots\} = \{\ell \in \mathbb{Z} \mid \ell \leqslant -|k|, \ell \equiv_2 \epsilon\}$.

Example. A continuous character $\chi : \mathbb{R}^\times \to \mathbb{C}^\times$ has the form $\chi = |\cdot|^s \mathrm{sign}^\varepsilon$ with $s \in \mathbb{C}$, $\varepsilon \in \{0,1\}$. Now pick $s_1, s_2 \in \mathbb{C}$, $\epsilon_1, \epsilon_2 \in \{0,1\}$ and put $\chi_i = |\cdot|^{s_i}\mathrm{sign}^{\varepsilon_i}$. Form the unitary induction $\mathrm{ind}_B^G(\chi_1, \chi_2)$

$$I(\chi_1,\chi_2) = \left\{ f : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{C} \mid f \text{ is smooth and } K\text{-finite, } f\left(\begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix} g\right) = \chi_1(a_1)\chi_2(a_2)\left|\frac{a_1}{a_2}\right|^{\frac{1}{2}} f(g)\right\}$$

Then $V = I(\chi_1, \chi_2)$ is a $(\mathfrak{g}, K)$-module, and in particular a $(\mathfrak{g}, K^+)$-module. For $\ell \in \mathbb{Z}$, we have

$$V(\ell) = \left\{ f \in I(\chi_1,\chi_2) \mid f(gk_\theta) = e^{i\ell\theta} f(g) \right\}$$

The Iwasawa decomposition $G = BK^+$ implies $\dim_{\mathbb{C}} V(\ell) \leqslant 1$, with equality if and only if $\ell \equiv \epsilon_1 + \epsilon_2 \pmod 2$. To see the equality, if $f \in V(\ell)$, then

$$(-1)^\ell f(e) = e^{i\ell\pi} f(e) = f\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = (-1)^{\epsilon_1+\epsilon_2}$$

Thus $f \neq 0$ if and only if $f(e) \neq 0$, if and only if $\ell \equiv \epsilon_1 + \epsilon_2 \pmod 2$. Then

$$\Sigma_V = \{\ell \in \mathbb{Z} \mid \ell \equiv \epsilon := \epsilon_1 + \epsilon_2 \pmod 2\}$$

For $\ell \equiv_2 \epsilon$, let $\varphi_\ell \in V(\ell)$ be the unique function with $\varphi_\ell(e) = 1$. If we put $s = s_1 - s_2$, then

1. $\rho(R)\varphi_\ell = \dfrac{s+1+\ell}{2}\varphi_{\ell+2}$.
2. $\rho(L)\varphi_\ell = \dfrac{s+1-\ell}{2}\varphi_{\ell-2}$.
3. $\rho(R_+)\varphi_\ell(e) = 0$.
4. $\rho(H_+)\varphi_\ell(e) = s+1$.
5. $\rho(\Delta) = (s^2-1)\varphi_\ell$, so that $\lambda_\Delta = s^2 - 1$.
6. $\rho(J)\varphi_\ell = (s_1+s_2)\varphi_\ell$, so that $\lambda_J = s_1 + s_2$.

7 Kirillov Model

Let $\psi : \mathbb{Q}_p \to \mathbb{C}^\times$ be the standard additive character $\psi = \psi_p$, and (π, V) an irreducible smooth admissible representation of $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Recall we have Whittaker functional

$$\Lambda : V \to \mathbb{C}$$

associated with ψ satisfying

$$\Lambda(\pi(\mathbf{n}(x))v) = \psi(x)\Lambda(v)$$

Define

$$C_0(\mathbb{Q}_p^\times) := \{\phi : \mathbb{Q}_p^\times \to \mathbb{C} \mid \operatorname{supp}\phi \text{ is bounded in } \mathbb{Q}_p\}$$

Clearly, both $\mathcal{S}(\mathbb{Q}_p)$, $\mathcal{S}(\mathbb{Q}_p) \subseteq C_0(\mathbb{Q}_p^\times)$. Let

$$B_1 = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{Q}_p^\times,\, b \in \mathbb{Q}_p \right\} \leqslant G$$

and let $(K_\psi, C_0(\mathbb{Q}_p^\times))$ be the representation of B_1 given by

$$K_\psi \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \phi(x) := \psi(bx)\phi(xa)$$

Then $K_\psi : B_1 \to \mathrm{GL}(C_0(\mathbb{Q}_p^\times))$ is called the **Kirillov representation.**

Consider the map

$$\begin{aligned} (\pi, V) &\longrightarrow C_0(\mathbb{Q}_p^\times) \\ v &\longmapsto \xi_v(a) := \Lambda\left(\pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} v\right) \end{aligned}$$

Note that this association $[v \mapsto \xi_v]$ is an intertwining operator: for $g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in B_1$ and $x \in \mathbb{Q}_p^\times$

$$\begin{aligned} \xi_{\pi(g)v}(x) &= \Lambda\left(\pi\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} v\right) \\ &= \Lambda\left(\pi\begin{pmatrix} 1 & bx \\ 0 & 1 \end{pmatrix}\pi\begin{pmatrix} ax & 0 \\ 0 & 1 \end{pmatrix} v\right) = \psi(bx)\Lambda\left(\pi\begin{pmatrix} ax & 0 \\ 0 & 1 \end{pmatrix} v\right) \end{aligned}$$

and

$$K_\psi(g)\xi_v(x) = \psi(bx)\xi_v(ax) = \psi(bx)\Lambda\left(\pi\begin{pmatrix} ax & 0 \\ 0 & 1 \end{pmatrix} v\right)$$

so that $\xi_{\pi(g)v}(x) = K_\psi(g)\xi_v(x)$ as claimed.

Proposition 7.1. $v \mapsto \xi_v$ is injective if $\dim V = \infty$.

Proof. Let $v \in V$ and $\xi_v = 0$. Recall the space

$$V_\psi(N) = \operatorname{span}_{\mathbb{C}}\left\{\pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v - \psi(x)v \mid x \in \mathbb{Q}_p,\, v \in V\right\} \subseteq V$$

Recall Theorem 5.1 that $\dim_{\mathbb{C}} J_\psi(V) = 1$ with $J_\psi(V) := V/V_\psi(N)$. In this setting, $\Lambda : J_\psi(V) \to \mathbb{C}$ is an isomorphism (note $\psi \neq 1$). Then

$$\begin{aligned}
\xi_v = 0 &\Leftrightarrow \Lambda\left(\pi\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} v\right) = 0 \text{ for all } a \in \mathbb{Q}_p^\times \\
&\Leftrightarrow \pi\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} v \in V_\psi(N) \text{ for all } a \in \mathbb{Q}_p^\times \\
&\Rightarrow v \in V_{\psi_a}(N) \text{ for all } a \in \mathbb{Q}_p^\times
\end{aligned}$$

where $\psi(x) := \psi(ax)$. The last implication is because that if we write

$$\pi\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} v = \sum_{x,w}\left(\pi\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} w - \psi(x)w\right)$$

then

$$\begin{aligned}
v &= \sum_{x,w}\left(\pi\begin{pmatrix} a^{-1} & 0 \\ 0 & 1\end{pmatrix}\pi\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} w - \psi(x)\pi\begin{pmatrix} a^{-1} & 0 \\ 0 & 1\end{pmatrix} w\right) \\
&= \sum_{x,w}\left(\pi\begin{pmatrix} 1 & a^{-1}x\,{\scriptstyle(=:y)} \\ 0 & 1\end{pmatrix}\pi\begin{pmatrix} a^{-1} & 0 \\ 0 & 1\end{pmatrix} w - \psi(x)\underbrace{\pi\begin{pmatrix} a^{-1} & 0 \\ 0 & 1\end{pmatrix} w}_{=:w'}\right) \\
&= \sum_{x',w'}\left(\pi\begin{pmatrix} 1 & y \\ 0 & 1\end{pmatrix} w' - \psi(ay)w'\right)
\end{aligned}$$

We view V as a smooth $\mathcal{S}(\mathbb{Q}_p)$-module, where the action is given by

$$\phi.v := \pi(\widehat{\phi})v$$

for $\phi \in \mathcal{S}(\mathbb{Q}_p)$, $v \in V$, where $\widehat{\phi}$ is the Fourier transform of ϕ (with respect to the standard character ψ_p). For $a \in \mathbb{Q}_p^\times$, put $V_a := J_{\psi_a}(V)$, which is the stalk of V at $a \in \mathbb{Q}_p^\times$. Then

$$v \in V_{\psi_a}(N) \text{ for all } a \in \mathbb{Q}_p^\times \Rightarrow v = 0 \text{ in } V_a \text{ for all } a \in \mathbb{Q}_p^\times \tag{$\spadesuit$}$$

By Lemma 3.13, we have an injective map

$$V \longhookrightarrow \prod_{a\in\mathbb{Q}_p} V_a$$

Suppose for contradiction that $v \neq 0$. Then ($\spadesuit$) and the injectivity of the above map force that $v \neq 0$ in the Jacquet module $V_0 = J(V) = V/V(N)$. Denote

$$K := \{v \in V \mid \xi_v = 0\}$$

Then the above map induces an injective map

$$K \hookrightarrow V_0 = J(V)$$

For $v \in K$, we have $\pi\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} v \in K$ for all $x \in \mathbb{Q}_p$. Indeed, we have $\xi_{\pi(g)v} = K_\psi(g)\xi_v = 0$ with $g = \begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}$. Then

$$K \ni v - \pi\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} v \equiv 0 \pmod{V(N)}$$

so that the injectivity implies that

$$v = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v$$

for all $x \in \mathbb{Q}_p$. This (by a lemma in the class) implies $\dim V = 1$ since $0 \neq v \in K$, a contradiction. $\square$

Suppose (π, V) is an irreducible smooth admissible representation with $\dim V = \infty$. The proposition shows we have an injective operator

$$\begin{aligned} (\pi, V) &\longrightarrow C_0(\mathbb{Q}_p^\times) \\ v &\longmapsto \xi_v(a) := \Lambda\left(\pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} v\right) \end{aligned}$$

Let $K_\psi(\pi) \subseteq C_0(\mathbb{Q}_p^\times)$ be the image; then

$$V \cong K_\psi(\pi) = \{\xi_v \mid v \in V\} \subseteq C_0(\mathbb{Q}_p^\times)$$

The action of G on V is transferred to an action on $K_\psi(\pi)$ via this map, namely,

$$\begin{aligned} K_\psi : G &\longrightarrow \mathrm{GL}(K_\psi(\pi)) \\ g &\longmapsto [K_\psi(g) : \xi_v \mapsto \xi_{\pi(g).v}] \end{aligned}$$

$(K_\psi, K_\psi(\pi))$ is called the **Kirillov model** of (π, V). In general, it is difficult to write down explicitly the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $K_\psi(\pi)$, but we know

$$K_\psi \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \xi_v(x) = \psi(bx)\xi_v(xa)$$

Recall the Kirillov representation $(K_\psi, C_0(\mathbb{Q}_p^\times))$ of B_1 defined above. Consider its subrepresentation $(K_\psi, \mathcal{S}(\mathbb{Q}_p^\times))$.

Theorem 7.2. $(K_\psi, \mathcal{S}(\mathbb{Q}_p^\times))$ is an irreducible representation of B_1.

Proof. For any $a \in \mathbb{Q}_p^\times$ and a continuous homomorphism $\nu : \mathbb{Z}_p^\times \to \mathbb{C}^\times$, define $\phi_{a,\nu} \in \mathcal{S}(\mathbb{Q}_p^\times)$ by

$$\phi_{a,\nu}(x) := \nu(ax)\mathbf{1}_{\mathbb{Z}_p^\times}(ax)$$

Lemma 7.3.

$$\mathcal{S}(\mathbb{Q}_p^\times) = \mathrm{span}_{\mathbb{C}}\{\, \phi_{a,\nu} \mid a \in \mathbb{Q}_p^\times,\, \nu : \mathbb{Z}_p^\times \to \mathbb{C}^\times\}$$

Proof. Let $\phi \in \mathcal{S}(\mathbb{Q}_p^\times)$. Then $\phi(x) = \sum_{n\in\mathbb{Z}} \phi(x)\mathbf{1}_{\mathbb{Z}_p^\times}(p^n x)$. We first show a smooth function φ supported on $\mathbb{Z}_p^\times$ can be written as a sum of characters. Let H be a subgroup of $\mathbb{Z}_p^\times$ such that on each coset of H, φ is a constant; this is possible, for $\mathbb{Z}_p^\times$ is compact (and totally disconnected). Then φ descends to the quotient $\varphi' : \mathbb{Z}_p^\times/H \to \mathbb{C}$. Since $\mathbb{Z}_p^\times/H$ is a finite abelian group, $\varphi' = \sum_{\nu \in \widehat{\mathbb{Z}_p^\times/H}} a_\nu \cdot \nu$, and hence so is φ.

Each $\phi(x)\mathbf{1}_{\mathbb{Z}_p^\times}(p^n x)$ can be viewed (under suitable dilation) as a smooth function on $\mathbb{Z}_p^\times$, so the above argument proves the lemma. $\square$

Suppose $0 \neq W \subseteq \mathcal{S}(\mathbb{Q}_p^\times)$ is B_1-invariant. We want to show $W = \mathcal{S}(\mathbb{Q}_p^\times)$.

1) There exists $\phi_{a,\nu} \in W$ for some a, ν. To show this, take $0 \neq \phi \in W$. Since ϕ is compactly supported, we can find $n \in \mathbb{Z}$ such that $\phi|_{p^n\mathbb{Z}_p^\times} \neq 0$ while $\phi|_{p^m\mathbb{Z}_p^\times} = 0$ for all $m < n$. Write

$$\phi(p^n u) = \sum_{\nu \in \widehat{\mathbb{Z}_p^\times}} a_\nu \cdot \nu(u)$$

where $u \in \mathbb{Z}_p^\times$ and $\widehat{\mathbb{Z}_p^\times}$ denotes the continuous dual, and

$$a_\nu := \int_{\mathbb{Z}_p^\times} \phi(p^n u)\nu^{-1}(u) d^\times u$$

This is in fact a finite sum, as said in the above lemma.

Since $\phi \neq 0$, we have $a_\nu \neq 0$ for some $\nu \in \widehat{\mathbb{Z}_p^\times}$. Define

$$\begin{aligned}
\phi_\nu(x) &:= \int_{\mathbb{Z}_p^\times} \phi(p^n u x)\nu^{-1}(u) d^\times u \\
&= \int_{\mathbb{Z}_p^\times} K_\psi \begin{pmatrix} p^n u & \\ & 1 \end{pmatrix} \phi(x) \nu^{-1}(u) d^\times u
\end{aligned}$$

Then $[x \mapsto \phi_\nu(x)]$ lies in W, for $\phi \in W$ and W is B_1-invariant. Note that $\phi_\nu(xu) = \nu(y)\phi_\nu(x)$ for all $u \in \mathbb{Z}_p^\times$. Define

$$\begin{aligned}
\phi^+_{p^n,\nu}(x) &:= \int_{\mathbb{Z}_p} K_\psi \begin{pmatrix} 1 & \frac{z}{p^n} \\ & 1 \end{pmatrix} \phi_\nu(x) dz \in W \\
&= \int_{\mathbb{Z}_p^\times} \psi\left(\frac{zx}{p^n}\right) \phi_\nu(x) dz = \phi_\nu(x) \mathbb{I}_{p^n\mathbb{Z}_p}(x)
\end{aligned}$$

The last equality is because $\psi = \psi_p$ is the standard additive character. Then

$$\phi_{p^n,\nu}(x) = \phi^+_{p^n\nu}(x) - \phi^+_{p^{n+1},\nu}(x) = \phi_\nu(x)\mathbb{I}_{p^n\mathbb{Z}_p^\times}(x) \in W$$

2) For $\mu \in \widehat{\mathbb{Z}_p^\times} \backslash \{\nu\}$, let $c := p^n$ be the conductor of μ and consider

$$\begin{aligned}
&\int_{\mathbb{Z}_p^\times} \mu^{-1}(u) K_\psi \begin{pmatrix} 1 & \frac{au}{c} \\ & 1 \end{pmatrix} \phi_{a,\nu}(x) d^\times u \in W \\
&= \int_{\mathbb{Z}_p^\times} \mu^{-1}(u) \psi_p\left(\frac{aux}{c}\right) \phi_{a,\nu}(x) d^\times u \\
&= \epsilon(0, \mu^{-1}) \mu\left(\frac{ax}{c}\right) \phi_{a,\nu}(x) \\
&= \underbrace{\epsilon(0, \mu^{-1}) \mu^{-1}(c)}_{\neq 0} \phi_{a,\mu\nu}(x)
\end{aligned}$$

where we have extended μ to be a character on $\mathbb{Q}_p^\times$ by setting $\mu(p) := 1$, and

$$\epsilon(0, \mu^{-1}) := \int_{\mathbb{Z}_p^\times} \mu^{-1}(u) \psi_p(u) d^\times u$$

Thus $\phi_{a,\mu\nu} \in W$ for all $\mu \neq \nu$, so that $\phi_{a,\mu} \in W$ for all $\mu \in \widehat{\mathbb{Z}_p^\times}$. Finally,

$$K_\psi \begin{pmatrix} a' & \\ & 1 \end{pmatrix} \phi_{a,\nu} = \phi_{aa',\nu}$$

so that $\phi_{a,\mu} \in W$ for all $a \in \mathbb{Q}_p^\times$, $\mu \in \widehat{\mathbb{Z}_p^\times}$. Thus $W = \mathcal{S}(\mathbb{Q}_p^\times)$.

$\square$

Lemma 7.4. For all $v \in V(N)$, we have $\xi_v \in \mathcal{S}(\mathbb{Q}_p^\times)$. Further we have a commutative diagram

$$\begin{array}{ccccc} V & \xrightarrow{\sim} & K_\psi(\pi) & \hookrightarrow & C_0(\mathbb{Q}_p^\times) \\ \cup\!\vert & & & & \cup\!\vert \\ V(N) & & \xrightarrow{\quad\sim\quad} & & \mathcal{S}(\mathbb{Q}_p^\times) \end{array}$$

Proof. Recall

$$V(N) = \mathrm{span}_{\mathbb{C}} \left\{ \pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v - v \mid x \in \mathbb{Q}_p,\, v \in V \right\}$$

For $v = \pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} w - w$ with $x \neq 0$,

$$\xi_v(y) = \pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \xi_w(y) - \xi_w(y) = (\psi(xy) - 1)\xi_w(y)$$

If $y \in x^{-1}\mathbb{Z}_p$, then $\psi(xy) = 1$ so that $\xi_v(y) = 0$; in particular, $\xi_v(y) \in \mathcal{S}(\mathbb{Q}_p^\times)$.

On the other hand, $V(N)$ is a B_1-module for

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & ax \\ & 1 \end{pmatrix} \begin{pmatrix} a & b \\ & 1 \end{pmatrix}$$

so by the theorem we have either $V(N) = 0$ or $V(N) \cong S(\mathbb{Q}_p^\times)$. But if $V(N) = 0$, then $V^N \neq \varnothing$ so that $\dim_{\mathbb{C}} V = 1$ by Lemma 4.3, a contradiction. $\square$

Conclusion. For (π, V) admissible smooth irreducible representation of $G = \mathrm{GL}_2(\mathbb{Q})$ with $\dim V = \infty$, we have

$$\mathcal{S}(\mathbb{Q}_p^\times) \subseteq K_\psi(\pi) \subseteq C_0(\mathbb{Q}_p^\times)$$

with

$$\frac{K_\psi(\pi)}{\mathcal{S}(\mathbb{Q}_p^\times)} \cong \frac{V}{V(N)} = J(V)$$

and (by Theorem 5.6)

$$\dim_{\mathbb{C}} \frac{K_\psi(\pi)}{\mathcal{S}(\mathbb{Q}_p^\times)} \leqslant 2$$

Now recall the space

$$W_\psi = \{W : G \to \mathbb{C} \mid W \text{ is smooth, } W(\mathbf{n}(x)g) = \psi(x)W(g)\}$$

and the map

$$\begin{aligned} V &\longrightarrow W_\psi \\ v &\longmapsto W_v(g) := \Lambda(\pi(g)v) \end{aligned}$$

Let $W_\psi(\pi) \subseteq W_\psi$ denote the image of V under this map, and let G act on $W_\psi(\pi)$ by right translation $\rho : G \to \mathrm{GL}(W_\psi(\pi))$, namely, $\rho(g)W(x) := W(xg)$. Then $(\rho, W_\psi(\pi))$ is called the **Whittaker model** of (π, V). We have a commutative triangle

$$\begin{tikzcd}
& & v \arrow[rr, mapsto] & & W_v & & \\
v \arrow[dr, mapsto] & & (\pi, V) \arrow[rr, "\sim"] \arrow[dr, "\sim"'] & & (\rho, W_\psi(\pi)) \arrow[dl, "\sim"] & & W \arrow[dl, mapsto] \\
& \xi_v & & (K_\psi, K_\psi(\pi)) & & \xi_W(a) := W\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right) &
\end{tikzcd}$$

8 Classification of Irreducible Representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

8.1 Weil representation

For two characters $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to \mathbb{C}^\times$, define $\chi : B \to \mathbb{C}^\times$ by

$$\chi \begin{pmatrix} a_1 & b \\ & a_2 \end{pmatrix} = \chi_1(a_1)\chi_2(a_2)$$

and

$$I(\chi_1, \chi_2) = \mathrm{Ind}_B^G \chi := \left\{ f : G \underset{\text{smooth}}{\to} \mathbb{C} \mid f(bg) = \chi(b)\delta_B(b)^{\frac{1}{2}} f(g) \right\} = \mathrm{ind}_B^G \chi \delta_B^{\frac{1}{2}}$$

where

$$\begin{aligned} \delta_B : B &\longrightarrow \mathbb{R}_+ \\ \begin{pmatrix} a_1 & b \\ & a_2 \end{pmatrix} &\longmapsto \left| \frac{a_1}{a_2} \right|_p \end{aligned}$$

is the modular character of B. Now let G act on $I(\chi_1, \chi_2)$ by right translation:

$$\begin{aligned} \rho : G &\longrightarrow \mathrm{GL}(I(\chi_1, \chi_2)) \\ g &\longmapsto \rho(g)f(x) := f(xg) \end{aligned}$$

By Lemma 5.8 (and the argument below there), $I(\chi_1, \chi_2)$ is an admissible smooth representation of G.

Definition. The space of **Bruhat-Schwartz functions** is defined as

$$\mathcal{S}(\mathbb{Q}_p^2) = \mathcal{S}(\mathbb{Q}_p) \otimes_{\mathbb{C}} \mathcal{S}(\mathbb{Q}_p) := \mathrm{span}_{\mathbb{C}}\{\varphi_1 \otimes \varphi_2(x, y) := \varphi_1(x)\varphi_2(y) \mid \varphi_i \in \mathcal{S}(\mathbb{Q}_p)\}$$

on which G acts by right translation:

$$\begin{aligned} \rho : G &\longrightarrow \mathrm{GL}(\mathcal{S}(\mathbb{Q}_p^2)) \\ g &\longmapsto \rho(g)\Phi(x, y) := \Phi((x\, y)g) \end{aligned}$$

Definition. On $\mathcal{S}(\mathbb{Q}_p^2)$ we define the **partial Fourier transform**

$$\begin{aligned} \mathcal{S}(\mathbb{Q}_p^2) &\longrightarrow \mathcal{S}(\mathbb{Q}_p^2) \\ \Phi &\longmapsto \Phi^\sim \end{aligned}$$

Here $\Phi^\sim$ is defined by the integral

$$\Phi^\sim(x, y) := \int_{\mathbb{Q}_p} \Phi(x, a)\psi_p(ay)da$$

where da is the self-dual Haar measure on $\mathbb{Q}_p$ (in this case, da is chosen so that $\mathrm{vol}(\mathbb{Z}_p, da) = 1$).

When $\Phi = \varphi_1 \otimes \varphi_2$ is a simple tensor, then

$$(\varphi_1 \otimes \varphi_2)^\sim = \varphi_1 \otimes \widehat{\varphi_2}$$

Since $\varphi \mapsto \widehat{\varphi}$ is an isomorphism on $\mathcal{S}(\mathbb{Q}_p)$, the partial Fourier transform is an isomorphism

$$\begin{aligned} \mathcal{S}(\mathbb{Q}_p^2) &\xrightarrow{\sim} \mathcal{S}(\mathbb{Q}_p^2) \\ \Phi &\longmapsto \Phi^\sim \end{aligned}$$

and this induces a new action of G on $\mathcal{S}(\mathbb{Q}_p^2)$:

$$\omega_\psi : G \longrightarrow \mathrm{GL}(\mathcal{S}(\mathbb{Q}_p^2))$$

such that

$$(\omega_\psi(g)\Phi)^\sim := \rho(g)\Phi^\sim$$

$(\omega_\psi, \mathcal{S}(\mathbb{Q}_p^2))$ is called the **Weil representation** of $G = \mathrm{GL}_2(\mathbb{Q}_p)$. By definition,

$$(\cdot)^\sim \in \mathrm{Isom}_G((\omega_\psi, \mathcal{S}(\mathbb{Q}_p^2)), (\rho, \mathcal{S}(\mathbb{Q}_p^2)))$$

and ω_ψ is smooth (for ρ is smooth).

Formulas. For $\Phi \in \mathcal{S}(\mathbb{Q}_p^2)$ and $\psi = \psi_p$, we have the following:

(i) $\omega_\psi \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \Phi(x, y) = |a|\Phi(xa, ya)$.

(ii) $\omega_\psi \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \Phi(x, y) = \psi(bxy)\Phi(x, y)$.

(iii) $\omega_\psi \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \Phi(x, y) = \int_{\mathbb{Q}_p^2} \Phi(a, b)\psi(ay + bx)dadb$.

(iv) $\omega_\psi \begin{pmatrix} a & \\ & 1 \end{pmatrix} \Phi(x, y) = \Phi(ax, y)$.

Proof. The first step to prove these formulas is to take $^\sim$ and prove the corresponding identities.

(i) We need to show

$$\rho \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \Phi^\sim(x, y) =: \left(\omega_\psi \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \Phi(x, y)\right)^\sim (x, y) = \left(|a|\rho \begin{pmatrix} a & \\ & a \end{pmatrix} \Phi\right)^\sim (x, y)$$

Now just compute

$$\left(|a|\rho \begin{pmatrix} a & \\ & a \end{pmatrix} \Phi\right)^\sim (x, y) = \int_{\mathbb{Q}_p} |a|\Phi(ax, at)\psi(yt)dt$$
$$= \int_{\mathbb{Q}_p} \Phi(ax, t)\psi(ya^{-1}t) = \Phi^\sim(ax, a^{-1}y) = \rho \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \Phi^\sim(x, y)$$

(ii)

$$\Phi^\sim(x, bx + y) = \rho \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \Phi^\sim(x, y) = \int_{\mathbb{Q}_p} \psi(bxt)\Phi(x, t)\psi(yt)dt = \int_{\mathbb{Q}_p} \Phi(x, t)\psi((bx + y)t)dt$$

(iii) We need to show

$$\Phi^\sim(-y, x) = \rho \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \Phi^\sim(x, y) = \int_{\mathbb{Q}_p} \left(\int_{\mathbb{Q}_p^2} \Phi(a, b)\psi(at + bx)\psi(yt)dadb\right) dt$$

Let $\Phi = \varphi_1 \otimes \varphi_2$. Expanding, we have

$$\int_{\mathbb{Q}_p} \left(\int_{\mathbb{Q}_p^2} \Phi(a, b)\psi(at + bx)\psi(yt)dadb\right) dt = \int_{\mathbb{Q}_p} \widehat{\varphi_1}(t)\widehat{\varphi_2}(x)\psi(yt)dt = \varphi_1(-y)\widehat{\varphi_2}(x) = \Phi^\sim(-y, x)$$

(iv)

$$\Phi^{\sim}(ax,y)=\rho\begin{pmatrix}a&\\&1\end{pmatrix}\Phi^{\sim}(x,y)=\left(\rho\begin{pmatrix}a&\\&1\end{pmatrix}\Phi\right)^{\sim}(x,y)=\int_{\mathbb{Q}_p}\Phi(ax,t)\psi(ty)dt$$

□

8.2 Construction of Whittaker functional

Given $\chi=(\chi_1,\chi_2):\mathbb{Q}_p^\times\to\mathbb{C}^\times$ and $\Phi\in\mathcal{S}(\mathbb{Q}_p^2)$, define

$$\begin{aligned}W_{\Phi,\chi}:G&\longrightarrow\mathbb{C}\\ g&\longmapsto\chi_1|\cdot|^{\frac{1}{2}}(\det g)\int_{\mathbb{Q}_p^\times}\omega_\psi(g)\Phi(t,t^{-1})\chi_1\chi_2^{-1}(t)d^\times t\end{aligned}$$

The integral really takes place on a compact set, so it is absolutely convergent. To see this, since Φ has compact support, so does $\omega_\psi(g)\Phi$. Then $\omega_\psi(g)\Phi(t,t^{-1})\neq 0$ if and only if $|t|\leqslant C_1$ and $|t^{-1}|\leqslant C_2$ for some $C_1,C_2>0$, i.e.,

$$0<C_2^{-1}\leqslant|t|\leqslant C_1$$

The map $W_{\Phi,\chi}$ is a Whittaker functional of ψ, i.e., $W_{\Phi,\chi}$ is smooth and satisfies

$$W_{\Phi,\chi}(\mathbf{n}(x)g)=\psi(x)W_{\Phi,\chi}(g)$$

for all $x\in\mathbb{Q}_p$ and $g\in G$.

- Smoothness. This follows from that ω_ψ is smooth.
- Expanding the LHS, we see

$$\begin{aligned}W_{\Phi,\chi}(\mathbf{n}(x)g)&=\chi_1|\cdot|^{\frac{1}{2}}(\det(\mathbf{n}(x)g))\int_{\mathbb{Q}_p^\times}\omega_\psi(\mathbf{n}(x)g)\Phi(t,t^{-1})\chi_1\chi_2^{-1}(t)d^\times t\\&\overset{\text{For.(ii)}}{=}\chi_1|\cdot|^{\frac{1}{2}}(\det g)\int_{\mathbb{Q}_p^\times}\psi(x)\omega_\psi(g)\Phi(t,t^{-1})\chi_1\chi_2^{-1}(t)d^\times t\\&=\psi(x)W_{\Psi,\chi}(g)\end{aligned}$$

The map $[\Phi\mapsto W_{\Phi,\chi}]\in\mathrm{Hom}_G((\rho,\mathcal{S}(\mathbb{Q}_p^2)),(\rho,W_\psi))$ is NOT intertwining. Nevertheless, formally we have

$$\begin{aligned}\chi_1^{-1}|\cdot|^{-\frac{1}{2}}(a)W_{\rho(g)\Phi^\sim,\chi}\left(\begin{pmatrix}a&0\\0&1\end{pmatrix}\right)&=\int_{\mathbb{Q}_p^\times}(\omega_\psi(g)\Phi)^{\sim}(at,t^{-1})\chi_1\chi_2^{-1}(t)d^\times t\\&=\int_{\mathbb{Q}_p^\times}\int_{\mathbb{Q}_p}\omega_\psi(g)\Phi(at,x)\psi(t^{-1}x)\chi_1\chi_2^{-1}(t)dxd^\times t\\&=\int_{\mathbb{Q}_p^\times}\int_{\mathbb{Q}_p}\omega_\psi(g)\Phi(t,x)\psi(at^{-1}x)\chi_1\chi_2^{-1}(a^{-1}t)dxd^\times t\\&=\chi_1^{-1}\chi_2(a)\int_{\mathbb{Q}_p^\times}\int_{\mathbb{Q}_p}\omega_\psi(g)\Phi(t,tx)\psi(ax)\chi_1\chi_2^{-1}|\cdot|(t)dxd^\times t\end{aligned}$$

Changes of variables are valid if $\mathrm{wt}(\chi_1\chi_2^{-1}) > 0$ is assumed. If we write $(t, tx) = (0, t)w\mathbf{n}(x)$, where $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then

$$\begin{aligned}
\chi_2^{-1}|\cdot|^{-\frac{1}{2}}(a)W_{\rho(g)\Phi^\sim,\chi}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) &= \int_{\mathbb{Q}_p^\times}\int_{\mathbb{Q}_p} \omega_\psi(g)\Phi((0\ t)w\mathbf{n}(x))\psi(ax)\chi_1\chi_2^{-1}(t)|t|dxd^\times t \\
&= \int_{\mathbb{Q}_p} f_{\omega_\psi(g)\Phi,\chi}(w\mathbf{n}(x))\psi(ax)dx
\end{aligned}$$

where $f_{\Phi,\chi}$ is the function defined by

$$\begin{aligned}
f_{\Phi,\chi} : G &\longrightarrow \mathbb{C} \\
g &\longmapsto \chi_1|\cdot|^{\frac{1}{2}}(\det g)\int_{\mathbb{Q}_p^\times}\Phi((0\ t)g)\chi_1\chi_2^{-1}|\cdot|(t)d^\times t
\end{aligned}$$

This is a local zeta integral, or a Tate integral, on $\mathrm{GL}(1)$, and it converges absolutely when $\mathrm{wt}(\chi_1\chi_2^{-1}) > -1$. (Recall the weight of a character χ is the unique real number $\mathrm{wt}(\chi)$ such that $|\chi| = |\cdot|^{\mathrm{wt}(\chi)}$.) When $\mathrm{wt}(\chi_1\chi_2^{-1}) > -1$, we check that $f_{\Phi,\chi} \in I(\chi_1, \chi_2)$. For $b = \begin{pmatrix} a_1 & * \\ & a_2 \end{pmatrix} \in B$,

$$\begin{aligned}
f_{\Phi,\chi}(bg) &= \chi_1|\cdot|^{\frac{1}{2}}(\det bg)\int_{\mathbb{Q}_p^\times}\Phi((0\ t)bg)\chi_1\chi_2^{-1}|\cdot|(t)d^\times t \\
(t \mapsto a_2^{-1}t) &= \chi_1|\cdot|^{\frac{1}{2}}(a_1a_2\det g)\int_{\mathbb{Q}_p^\times}\Phi((0\ t)g)\chi_1\chi_2^{-1}|\cdot|(a_2^{-1}t)d^\times t \\
&= \chi_1(a_1)\chi_2(a_2)\left|\frac{a_1}{a_2}\right|^{\frac{1}{2}}\int_{\mathbb{Q}_p^\times}\Phi((0\ t)g)\chi_1\chi_2^{-1}|\cdot|(t)d^\times t \\
&= \chi(b)\delta_B(b)^{\frac{1}{2}}f_{\Phi,\chi}(g)
\end{aligned}$$

Again, $\Phi \mapsto f_{\Phi,\chi}$ is NOT intertwining. Nevertheless, we have $\rho(g)f_{\Phi,\chi} = \chi_1|\cdot|^{\frac{1}{2}}(\det g)f_{\rho(g)\Phi,\chi}$; indeed, by definition,

$$\begin{aligned}
\rho(g)f_{\Phi,\chi}(x) &= \chi_1|\cdot|^{\frac{1}{2}}(\det xg)\int_{\mathbb{Q}_p^\times}\Phi((0\ t)xg)\chi_1\chi_2^{-1}|\cdot|(t)d^\times t \\
&= \chi_1|\cdot|^{\frac{1}{2}}(\det x\det g)\int_{\mathbb{Q}_p^\times}\rho(g)\Phi((0\ t)x)\chi_1\chi_2^{-1}|\cdot|(t)d^\times t \\
&= \chi_1|\cdot|^{\frac{1}{2}}(\det g)f_{\rho(g)\Phi,\chi}(x)
\end{aligned}$$

$$\boxed{\text{In the following, we always assume } \mathrm{wt}(\chi_1\chi_2^{-1}) > 1.}$$

Consider the diagram

$$\begin{tikzcd}
\Phi \arrow[rr, mapsto] \arrow[dd, mapsto] & & f_{\Phi^\sim,\chi} \\
& \mathcal{S}(\mathbb{Q}_p^2) \arrow[r] \arrow[d] & I(\chi_1,\chi_2) \\
W_{\Phi,\chi} & W_\psi &
\end{tikzcd}$$

On each space G act by right translation.

Proposition 8.1.

1. If $f_{\Phi^\sim,\chi} = 0$, then $W_{\Phi,\chi} = 0$.

2. The map $\Phi \mapsto f_{\Phi^\sim,\chi}$ is surjective onto $I(\chi_1, \chi_2)$.

By this proposition, we obtain a (colored) arrow

$$\begin{tikzcd} \mathcal{S}(\mathbb{Q}_p^2) \arrow[r] \arrow[d] & I(\chi_1, \chi_2) \arrow[dl, red] \\ W_\psi & \end{tikzcd}$$

making this triangle commutative. To show this proposition, we need the following.

Lemma 8.2. For all $x \in \mathbb{Q}_p$, we have the identity

$$\int_{\mathbb{Q}_p} W_{\Phi,\chi} \begin{pmatrix} a & \\ & 1 \end{pmatrix} \chi_2^{-1}|\cdot|^{-\frac{1}{2}}(a)\psi(ax)da = f_{\Phi^\sim,\chi}(w\mathbf{n}(x))$$

where $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in G = \mathrm{GL}_2(\mathbb{Q}_p)$ is the Weyl element.

Proof. Define $\xi^* : \mathbb{Q}_p^\times \to \mathbb{C}$ by

$$\begin{aligned} \xi^*(a) &:= W_{\Phi,\chi} \begin{pmatrix} a & \\ & 1 \end{pmatrix} \chi_2^{-1}|\cdot|^{-\frac{1}{2}}(a) \\ &\overset{\text{For.(iv)}}{=} \chi_1\chi_2^{-1}(a) \int_{\mathbb{Q}_p^\times} \Phi(at, t^{-1})\chi_1\chi_2^{-1}(t)d^\times t \end{aligned}$$

Then $\xi^* \in L^1(\mathbb{Q}_p)$, for (first replace t by t^{-1} in the definition of ξ^*)

$$\begin{aligned} \int_{\mathbb{Q}_p} |\xi^*(a)|da &\leqslant \int_{\mathbb{Q}_p} |\chi_1\chi_2^{-1}(a)| \int_{\mathbb{Q}_p^\times} |\Phi(at^{-1}, t)||\chi_1\chi_2^{-1}(t^{-1})|d^\times t da \\ &= \int_{\mathbb{Q}_p^\times \times \mathbb{Q}_p} |\Phi(at^{-1}, t)||\chi_1\chi_2^{-1}(at^{-1})|d^\times t da \\ (a \mapsto at) &= \int_{\mathbb{Q}_p^\times \times \mathbb{Q}_p} |\Phi(a, t)||\chi_1\chi_2^{-1}(a)||t|d^\times t da \\ (|t|d^\times t da = |a|dt d^\times a) &= \int_{\mathbb{Q}_p \times \mathbb{Q}_p^\times} |\Phi(a, t)||\chi_1\chi_2^{-1}|\cdot|(a)|dt d^\times a < \infty \end{aligned}$$

because $\operatorname{supp}\Phi$ is compact and $\mathrm{wt}(\chi_1\chi_2^{-1}|\cdot|) > 0$. Now for the sake of absolute convergence, we have

$$\begin{aligned} \int_{\mathbb{Q}_p} \xi^*(a)\psi(ax)da &= \int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p^\times} \Phi(at, t^{-1})\chi_1\chi_2^{-1}(at)\psi(ax)d^\times t da \\ (t \mapsto ta^{-1}) &= \int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p^\times} \Phi(t, t^{-1}a)\chi_1\chi_2^{-1}(t)\psi(ax)d^\times t da \\ (a \mapsto at) &= \int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p^\times} \Phi(t, a)\chi_1\chi_2^{-1}(t)\psi(atx)|t|d^\times t da \\ &= \int_{\mathbb{Q}_p^\times} \Phi^\sim(t, tx)\chi_1\chi_2^{-1}|\cdot|(t)d^\times t \\ &= f_{\Phi^\sim,\chi}(w\mathbf{n}(x)) \end{aligned}$$

The last equality holds because of $\det(w\mathbf{n}(x)) = 1$ and

$$(t\ tx) = (0\ t)\begin{pmatrix} & -1 \\ 1 & x \end{pmatrix} = (0\ t)w\mathbf{n}(x)$$

$\square$

Remark 8.3. If $\mathrm{wt}(\chi_1\chi_2^{-1}) > 0$, then

$$\int_{\mathbb{Q}_p} f_{\Phi^\sim,\chi}(w\mathbf{n}(x))\psi(-ax)dx = W_{\Phi,\chi}\begin{pmatrix} a & \\ & 1 \end{pmatrix}\chi_2^{-1}|\cdot|^{-\frac{1}{2}}(a)$$

for all $a \in \mathbb{Q}_p^\times$. This is a kind of Fourier inversion formula.

Proof. (of Proposition 8.1.1) Suppose $f_{\Phi^\sim,\chi} = 0$; in particular, $f_{\Phi^\sim,\chi}(w\mathbf{n}(x)) = 0$ for all $x \in \mathbb{Q}_p$. Let

$$\xi^*(a) = W_{\Phi,\chi}\begin{pmatrix} a & \\ & 1 \end{pmatrix}\chi_2^{-1}|\cdot|^{-\frac{1}{2}}(a)$$

be the same as in Lemma 8.2. Then by the same lemma, we have

$$\int_{\mathbb{Q}_p} \xi^*(a)\psi(ax)da = 0 \text{ for all } x \in \mathbb{Q}_p$$

Integrating, for $N \gg 0$ and $x \in \mathbb{Q}_p^\times$, we have

$$\begin{aligned} 0 &= \int_{p^{-N}\mathbb{Z}_p}\int_{\mathbb{Q}_p} \xi^*(a)\psi(ab)\psi(-bx)dadb \\ &= \int_{\mathbb{Q}_p} \xi^*(a)\int_{p^{-N}\mathbb{Z}_p} \psi(b(a-x))dbda \\ (\psi = \psi_p) &= \int_{\mathbb{Q}_p} \xi^*(a)\mathbb{I}_{x+p^N\mathbb{Z}_p}(a)da \\ &= \xi^*(x)\,\mathrm{vol}(p^n\mathbb{Z}_p) \end{aligned}$$

since ξ^* is smooth (and if $N \gg 0$, x and a are sufficiently close). This proves $\xi^*(x) = 0$; putting $x = 1$, this gives $W_{\Phi,\chi}(e) = 0$.

In general, for all $g \in G$, we have

$$f_{\Phi^\sim,\chi} = 0 \Rightarrow 0 = \rho(g)f_{\Phi^\sim,\chi} = f_{\rho(g)\Phi^\sim,\chi} = f_{(\omega_\psi(g)\Phi)^\sim,\chi} \Rightarrow 0 = W_{\omega_\psi(g)\Phi,\chi}(e) \Rightarrow W_{\Phi,\chi}(g) = 0$$

The third implication follows from the case we prove above, and the last implication follows from the definition of $W_{\Phi,\chi}$:

$$\begin{aligned} W_{\omega_\psi(g)\Phi,\chi}(e) &= \chi_1|\cdot|^{\frac{1}{2}}(\det e)\int_{\mathbb{Q}_p^\times} \omega_\psi(e)\omega_\psi(g)\Phi(t,t^{-1})\chi_1\chi_2^{-1}(t)d^\times t \\ &= \chi_1|\cdot|^{\frac{1}{2}}(\det g)^{-1}W_{\Phi,\chi}(g) \end{aligned}$$

$\square$

Proof. (of Proposition 8.1.2) For $f \in I(\chi_1,\chi_2)$, f is completely determined by $f|_K$ by Iwasawa decomposition, where $K = \mathrm{GL}_2(\mathbb{Z}_p)$. Now define $\Phi \in \mathcal{S}(\mathbb{Q}_p^2)$ by

$$\Phi(x,y) = \begin{cases} \chi_1^{-1}|\cdot|^{-\frac{1}{2}}(\det k)f(k) & \text{, if } (x\ y) = (0\ 1)k \text{ for some } k \in K \\ 0 & \text{, otherwise} \end{cases}$$

We have $\operatorname{supp}\Phi \subseteq (0\ 1)K$ is compact, and

$$\begin{aligned} f_{\Phi,\chi}(k) &= \chi_1|\cdot|^{\frac{1}{2}}(\det k)\int_{\mathbb{Q}_p^\times}\Phi((0\ t)k)\chi_1\chi_2^{-1}|\cdot|(t)d^\times t \\ &= \chi_1|\cdot|^{\frac{1}{2}}(\det k)\int_{\mathbb{Z}_p^\times}\chi_1^{-1}|\cdot|^{-\frac{1}{2}}(t\det k)f(\begin{pmatrix}1 & \\ & t\end{pmatrix}k)\chi_1\chi_2^{-1}|\cdot|(t)d^\times t \\ &= \chi_1|\cdot|^{\frac{1}{2}}(\det k)\int_{\mathbb{Z}_p^\times}\chi_1^{-1}|\cdot|^{-\frac{1}{2}}(t\det k)\chi_2|\cdot|^{-\frac{1}{2}}(t)f(k)\chi_1\chi_2^{-1}|\cdot|(t)d^\times t \\ &= \int_{\mathbb{Z}_p^\times}f(k)d^\times t = f(k) \end{aligned}$$

Since $\Phi \mapsto \Phi^\sim$ is bijective, we are done. $\square$

Therefore, we obtain an operator

$$\begin{aligned} I(\chi_1,\chi_2) &\longrightarrow W_\psi \\ f_{\Phi^\sim,\chi} &\longmapsto W_{\Phi,\chi} \end{aligned}$$

We show this is intertwining; denote this operator by Θ temporarily. We must show

$$\Theta(\rho(g)f_{\Phi^\sim,\chi}) = \rho(g)\Theta(f_{\Phi^\sim,\chi})$$

We have seen that $\rho(g)f_{\Phi,\chi} = \chi_1|\cdot|^{\frac{1}{2}}(\det g)f_{\rho(g)\Phi,\chi}$; in other words,

$$\rho(g)f_{\Phi^\sim,\chi} = \chi_1|\cdot|^{\frac{1}{2}}(\det g)f_{\rho(g)\Phi^\sim,\chi} = \chi_1|\cdot|^{\frac{1}{2}}(\det g)f_{(\omega_\psi(g)\Phi)^\sim,\chi}$$

On the other hand,

$$W_{\omega_\psi(g)\Phi,\chi}(x) = \chi_1|\cdot|^{\frac{1}{2}}(\det x)\int_{\mathbb{Q}_p^\times}\omega_\psi(x)\omega_\psi(g)\Phi(t,t^{-1})\chi_1\chi_2^{-1}(t)d^\times t = \chi_1^{-1}|\cdot|^{-\frac{1}{2}}(\det g)W_{\Phi,\chi}(xg)$$

Thus

$$\begin{aligned} \Theta(\rho(g)f_{\Phi^\sim,\chi}) &= \Theta(\chi_1|\cdot|^{\frac{1}{2}}(\det g)f_{(\omega_\psi(g)\Phi)^\sim,\chi}) = \chi_1|\cdot|^{\frac{1}{2}}(\det g)W_{\omega_\psi(g)\Phi,\chi} \\ &= \chi_1|\cdot|^{\frac{1}{2}}(\det g)\chi_1^{-1}|\cdot|^{-\frac{1}{2}}(\det g)\rho(g)W_{\Phi,\chi} \\ &= \rho(g)W_{\Phi,\chi} = \rho(g)\Theta(f_{\Phi^\sim,\chi}) \end{aligned}$$

as desired. We will use this map to study the irreducibility of $I(\chi_1,\chi_2)$.

8.3 Classification

Recall $N = \left\{\mathbf{n}(x) := \begin{pmatrix}1 & x \\ & 1\end{pmatrix} \mid x \in \mathbb{Q}_p\right\}$

Lemma 8.4.

$$I(\chi_1,\chi_2)^N \neq 0 \Leftrightarrow \chi_1\chi_2^{-1} = |\cdot|^{-1}$$

If either holds, then $\dim_{\mathbb{C}} I(\chi_1,\chi_2)^N = 1$.

Proof. By Bruhat decomposition, we have $G = B \sqcup BwB = B \sqcup BwN$. Then $f \in I(\chi_1,\chi_2)^N$ is uniquely determined by $f(e)$ and $f(w)$. Recall the very important identity that holds for all $x \in \mathbb{Q}_p^\times$:

$$\begin{pmatrix}1 & \\ x & 1\end{pmatrix} = \begin{pmatrix}x^{-1} & 1 \\ & x\end{pmatrix}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix}1 & x^{-1} \\ & 1\end{pmatrix}$$

Then

$$f\begin{pmatrix}1 & \\ x & 1\end{pmatrix} = f\left(\begin{pmatrix}x^{-1} & 1 \\ & x\end{pmatrix} w \begin{pmatrix}1 & x^{-1} \\ & 1\end{pmatrix}\right) = \chi_1^{-1}\chi_2|\cdot|^{-1}(x)f(w)$$

For $|x|$ sufficiently small, since f is smooth, we have

$$f(e) = \chi_1^{-1}\chi_2|\cdot|^{-1}(x)f(w)$$

This implies either $\chi_1\chi_2^{-1} = |\cdot|^{-1}$ or $f(e) = f(w) = 0$ (i.e. $f \equiv 0$), and f is uniquely determined by $f(e)$. This shows $\dim_{\mathbb{C}} I(\chi_1, \chi_2)^N = 1$. □

Proposition 8.5. Consider the pairing

$$\langle\,,\rangle : I(\chi_1, \chi_2) \times I(\chi_1^{-1}, \chi_2^{-1}) \longrightarrow \mathbb{C}$$

defined by

$$\langle f_1, f_2\rangle := \int_K f_1(k)f_2(k)dk$$

Then

(i) The pairing is perfect, i.e., for all compact open $U \leqslant G$, the induced pairing

$$I(\chi_1, \chi_2)^U \times I(\chi_1^{-1}, \chi_2^{-1})^U \to \mathbb{C}$$

is perfect. In particular, $I(\chi_1^{-1}, \chi_2^{-1}) \cong I(\chi_1, \chi_2)^\vee$.

(ii) The pairing is G-equivariant, i.e.,

$$\langle\rho(g)f_1, \rho(g)f_2\rangle = \langle f_1, f_2\rangle$$

for all $g \in G = \mathrm{GL}_2(\mathbb{Q}_p)$.

Proof. By Iwasawa decompsotion $G = BK$ elements in $I(\chi_1, \chi_2)$ are uniquely determined by their restriction to $K = \mathrm{GL}_2(\mathbb{Z}_p)$, i.e.,

$$I(\chi_1, \chi_2) \cong \{f : K \to \mathbb{C} \mid f(bg) = \chi(b)f(g) \text{ for all } b \in K \cap B,\ g \in K\}$$

We show that if $f \in I(\chi_1, \chi_2)$ is such that $\langle f, g\rangle = 0$ for all $g \in I(\chi_1^{-1}, \chi_2^{-1})$, then $f = 0$. For a fixed $k_1 \in K$, let $U \leqslant G$ be compact open such that $f(k_1U) = f(k_1)$. Define $g : G \to \mathbb{C}$ such that

$$g(x) := \begin{cases} \chi^{-1}\delta_B^{\frac{1}{2}}(b) & \text{, if } x = bk_1u \text{ for some } b \in B,\ u \in U \\ 0 & \text{, otherwise} \end{cases}$$

To see g is well-defined, suppose $bk_1u = b'k_1u'$ for some other $b' \in B$, $u' \in U$. Then $b'^{-1}b = k_1u'u^{-1}k_1^{-1} \in k_1Uk_1^{-1}$. We now take U smaller so that $k_1Uk_1^{-1}$ is contained in the conductor of $\chi\delta_B^{-\frac{1}{2}}$. Then $\chi^{-1}\delta_B^{\frac{1}{2}}(b'^{-1}b) = 1$, or $\chi^{-1}\delta_B^{\frac{1}{2}}(b) = \chi^{-1}\delta_B^{\frac{1}{2}}(b')$, as wanted. It is clear that $g \in I(\chi_1^{-1}, \chi_2^{-1})$. Now

$$0 = \langle f, g\rangle = f(k_1)\int_{K\cap Bk_1U} \delta_B(b)dk$$

which implies $f(k_1) = 0$. Thus $f \equiv 0$.

To show the pairing is G-equivariant, we use the integration formula

$$\int_G f(g)dg = \int_B\int_K f(bg)dkd_Lb$$

where $d_L b$ is the left invariant Haar measure on B. On the other hand, consider

$$p_\chi : \mathcal{S}(G) \longrightarrow I(\chi_1, \chi_2)$$

$$\phi \longmapsto p_\chi(\phi)(g) = \int_B \phi(bg)\chi^{-1}\delta_B^{-\frac{1}{2}}(b)db$$

where db is a chosen right invariant Haar measure on B.

- p_χ is surjective. The proof is similar to that of Proposition 8.1.2. For $f \in I(\chi_1, \chi_2)$, define $\phi \in \mathcal{S}(G)$ by

$$\phi(g) = \begin{cases} f(g) & \text{, if } g \in K \\ 0 & \text{, otherwise} \end{cases}$$

  Then

$$\begin{aligned} p_\chi(\phi)(g) &= \int_B \phi(bg)\chi^{-1}\delta_B^{-\frac{1}{2}}(b)db \\ &= \int_{B\cap Kg^{-1}} f(bg)\chi^{-1}\delta_B^{-\frac{1}{2}}(b)db \\ &= \int_{B\cap Kg^{-1}} \chi(b)\delta_B(b)^{\frac{1}{2}} f(g)\chi^{-1}\delta_B^{-\frac{1}{2}}(b)db \\ &= f(g)\,\mathrm{vol}(B\cap Kg^{-1}, db) \end{aligned}$$

- p_χ is intertwining. For

$$p_\chi(\rho(g)\phi)(x) = \int_B \rho(g)\phi(bx)\chi^{-1}\delta_B^{-\frac{1}{2}}(b)db = \int_B \phi(bxg)\chi^{-1}\delta_B^{-\frac{1}{2}}db = p_\chi(\phi)(xg) = \rho(g)p_\chi(\phi)(x)$$

Now for $f_1 \in I(\chi_1, \chi_2)$ and $f_2 \in I(\chi_1^{-1}, \chi_2^{-1})$, choose $\phi_1 \in \mathcal{S}(G)$ such that $p_\chi(\phi_1) = f_1$

$$\begin{aligned} \int_K f_1(k)f_2(k)dk &= \int_K \left(\int_B \phi_1(bk)\delta_B^{-\frac{1}{2}}\chi^{-1}(b)db\right) f_2(k)dk \\ &= \int_K\int_B \phi_1(bk)f_2(bk)dbdk \\ &= \int_G \phi_1(g)f_2(g)dg \end{aligned}$$

Let us write the last integral as (ϕ_1, f_2). Then

$$\langle \rho(g)f_1, \rho(g)f_2\rangle = (\rho(g)\phi_1, \rho(g)f_2) = (\phi_1, f_2) = \langle f_1, f_2\rangle$$

for $p_\chi(\rho(g)\phi) = \rho(g)p_\chi(\phi) = \rho(g)f_1$ and dg is right-invariant. □

Theorem 8.6.

(i) $I(\chi_1, \chi_2)$ is irreducible if $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm}$.

(ii) $I(\chi_1, \chi_2)$ has a unique irreducible (infinite dimensional) subrepresentation, denoted by $I(\chi_1, \chi_2)_s$, if $\chi_1\chi_2^{-1} = |\cdot|$, and the sequence is exact

$$0 \longrightarrow I(\chi_1, \chi_2)_S \longrightarrow I(\chi_1, \chi_2) \longrightarrow \mathbb{C}\chi_1|\cdot|^{-\frac{1}{2}} \circ \det \longrightarrow 0$$

(iii) $I(\chi_1, \chi_2)$ has a unique one-dimensional subrepresentation if $\chi_1\chi_2^{-1} = |\cdot|^{-1}$.

$$0 \longrightarrow \mathbb{C}\chi_1|\cdot|^{\frac{1}{2}} \circ \det \longrightarrow I(\chi_1, \chi_2) \longrightarrow I(\chi_1, \chi_2)_Q \longrightarrow 0$$

Proof. Taking dual, if necessary, we can always assume that $\mathrm{wt}(\chi_1\chi_2^{-1}) > -1$. Consider the composition (which is well-defined by Proposition 8.1)

$$\begin{array}{ccccc} I(\chi_1,\chi_2) & \longrightarrow & W_\psi & \hookrightarrow & C_0(\mathbb{Q}_p^\times) \\ f_{\Phi^\sim,\chi} & & \longmapsto & & \xi_{\Phi,\chi}(a) := W_{\Phi,\chi}\begin{pmatrix} a & \\ & 1 \end{pmatrix} \end{array}$$

This map is injective. To see this, assume $\xi_{\Phi,\chi} = 0$. By Lemma 8.2, this implies $f_{\Phi^\sim,\chi}(w\mathbf{n}(x)) = 0$ for all $x \in \mathbb{Q}_p$. By Bruhat decomposition $G = B \sqcup BwN$, to see $f = 0$, it suffices to show $f(e) = 0$, but this follows from the smoothness of $\xi_{\Phi,\chi}$ and that BwN is dense in G. Let V be the image of $I(\chi_1,\chi_2)$ in W_ψ; then $V \cong I(\chi_1,\chi_2)$.

Suppose V contains a proper nontrivial invariant subspace $0 \neq U \subsetneq V$. Consider

$$U(N) = \mathrm{span}_{\mathbb{C}}\{\rho(\mathbf{n}(x))u - u \mid u \in U,\, x \in \mathbb{Q}_p\} \subseteq U$$

- $\underline{U(N) = 0.}$ Then $U = U^N \neq 0$, and by Lemma 8.4 we see $\chi_1\chi_2^{-1} = |\cdot|^{-1}$.
- $\underline{U(N) \neq 0.}$ Then $U(N) = V(N) (= \mathcal{S}(\mathbb{Q}_p^\times))$ by Theorem 7.2 and Lemma 7.4, so

  $$V(N) = U(N) \subseteq U \subseteq V$$

  thus $(V/U)^\vee \subseteq (V/V(N))^\vee = (V^\vee)^N = I(\chi_1^{-1},\chi_2^{-1})^N$ by Proposition 8.5. Since U is proper, this implies $0 \neq I(\chi_1^{-1},\chi_2^{-1})^N$, hence $\chi_1^{-1}\chi_2 = |\cdot|^{-1}$ by Lemma 8.4.

Hence, if $I(\chi_1,\chi_2)$ is irreducible, we must have $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm}$ by our discussion, whence (i).

(ii) $\underline{\chi_1\chi_2^{-1} = |\cdot|.}$ Since U is chosen arbitrary, it follows $\dim_{\mathbb{C}} V/U = 1$ and that U is the unique irreducible subrepresentation. Thus we have the exact sequence

$$0 \longrightarrow U \longrightarrow V \cong I(\chi_1,\chi_2) \longrightarrow \mathbb{C}\chi_1|\cdot|^{-\frac{1}{2}} \circ \det \longrightarrow 0$$

We must explain why $V/U \cong \mathbb{C}\chi_1|\cdot|^{-\frac{1}{2}} \circ \det$. We have

$$(V/U)^\vee = I(\chi_1^{-1},\chi_2^{-1})^N = \mathbb{C}\chi_1^{-1}|\cdot|^{\frac{1}{2}} \circ \det$$

By Proposition 3.9.(iii), we have

$$V/U = ((V/U)^\vee)^\vee = (\mathbb{C}\chi_1^{-1}|\cdot|^{\frac{1}{2}} \circ \det)^\vee = \mathbb{C}\chi_1|\cdot|^{-\frac{1}{2}} \circ \det$$

The last isomorphism results from the definition of contragredient action.

(iii) $\underline{\chi_1\chi_2^{-1} = |\cdot|^{-1}.}$ Since $V^N = 1$ in this case, it follows from the above argument that V^N is the unique irreducible subrepresentation which is one dimensional. That V/V^N is irreducible also follows from the uniqueness.

□

Definition. Consider the induced module $(\rho, I(\chi_1,\chi_2))$ and Theorem 8.6.

1. For $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm}$, let $\pi(\chi_1,\chi_2)$ denote the isomorphism class of $(\rho, I(\chi_1,\chi_2))$. This is called the **principal series**.

2. Denote by St the unique irreducible subrepresentation of $I(|\cdot|^{\frac{1}{2}}, |\cdot|^{-\frac{1}{2}})$, and call it the **standard Steinberg representation.** For $\chi_0 : \mathbb{Q}_p^\times \to \mathbb{C}^\times$, we have

$$\mathrm{St} \otimes \chi_0 = (\rho, I(\chi_0|\cdot|^{\frac{1}{2}}, \chi_0|\cdot|^{-\frac{1}{2}})_S)$$

This is called the the **Steinberg representation**, or the **special / degenerate principal series**.

- We have $\pi(\chi_1, \chi_2)^\vee = \pi(\chi_1^{-1}, \chi_2^{-1})$.
- Put $\chi_1 = \chi_0|\cdot|^{\frac{1}{2}}$ and $\chi_2 = \chi_0|\cdot|^{-\frac{1}{2}}$. Then $\chi_1\chi_2^{-1} = |\cdot|$, so the Steinberg representation $\mathrm{St} \otimes \chi_0$ is the unique irreducible subrepresentation of $I(\chi_1, \chi_2)$, and we have the following commutative digram

$$\begin{tikzcd}
0 \arrow[r] & I(\chi_1, \chi_2)_S \arrow[r] \arrow[d, equal] & I(\chi_1, \chi_2) \arrow[r] \arrow[d, equal] & \mathbb{C}\chi_1|\cdot|^{-\frac{1}{2}} \circ \det \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{St} \otimes \chi_0 \arrow[r] & I(\chi_0|\cdot|^{\frac{1}{2}}, \chi_0|\cdot|^{-\frac{1}{2}}) \arrow[r] & \mathbb{C}\chi_0 \circ \det \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Taking contragredient, and with the identification $I(\chi_1, \chi_2)^\vee = I(\chi_1^{-1}, \chi_2^{-1})$, we have

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C}\chi_1^{-1}|\cdot|^{\frac{1}{2}} \circ \det \arrow[r] \arrow[d, equal] & I(\chi_1^{-1}, \chi_2^{-1}) \arrow[r] \arrow[d, equal] & I(\chi_1^{-1}, \chi_2^{-1})_Q \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{C}\chi_0^{-1} \circ \det \arrow[r] & I(\chi_0|\cdot|^{\frac{1}{2}}, \chi_0|\cdot|^{-\frac{1}{2}}) \arrow[r] & (\mathrm{St} \otimes \chi_0)^\vee \arrow[r] & 0
\end{tikzcd}$$

so that $(\mathrm{St} \otimes \chi_0)^\vee \cong I(\chi_0^{-1}|\cdot|^{-\frac{1}{2}}, \chi_0^{-1}|\cdot|^{\frac{1}{2}})_Q$. We will prove in the following that, in fact,

$$(\mathrm{St} \otimes \chi_0)^\vee \cong \mathrm{St} \otimes \chi_0^{-1} = I(\chi_0^{-1}|\cdot|^{\frac{1}{2}}, \chi_0^{-1}|\cdot|^{-\frac{1}{2}})_S$$

Definition. Let (π, V) be a representation of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ a character. Define

$$\pi \otimes \chi : G \longrightarrow \mathrm{GL}(V)$$

by $(\pi \otimes \chi)(g).v = \chi(\det g)\pi(g)v$. The new representation $(\pi \otimes \chi, V)$ is called **(π, V) twisted by χ.**

- We have $(\rho \otimes \mu, I(\chi_1, \chi_2)) \cong (\rho, I(\chi_1\mu, \chi_2\mu))$, given by

$$\begin{aligned}
I(\chi_1, \chi_2) &\longrightarrow I(\chi_1\mu, \chi_2\mu) \\
f &\longmapsto f \otimes (\mu \circ \det) : g \mapsto f(g)\mu(\det g)
\end{aligned}$$

Indeed, for $x, g \in G$, we have

$$\begin{aligned}
\rho(g)(f \otimes (\mu \circ \det))(x) &= f(xg)\mu(\det xg) \\
&= \mu(\det g)(\rho(g)f \otimes (\mu \circ \det))(x) = (\rho \otimes \mu)(g)f \otimes (\mu \circ \det)(x)
\end{aligned}$$

Then $\pi(\chi_1, \chi_2) \otimes \mu = \pi(\chi_1\mu, \chi_2\mu)$ in the principal series case.

Definition. Let (π, V) be an *irreducible* representation of $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Let $a \in \mathbb{Q}_p^\times$ and consider $\begin{pmatrix} a & \\ & a \end{pmatrix}$; being in the center of G, we have $\pi\begin{pmatrix} a & \\ & a \end{pmatrix} \in \mathrm{End}_G(V, V)$. Let U be an compact open subgroup of G such that $V^U \neq 0$. Then $\pi\begin{pmatrix} a & \\ & a \end{pmatrix} \in \mathrm{End}_G(V^U, V^U)$, and since $\dim_\mathbb{C} V^U < \infty$, $\pi\begin{pmatrix} a & \\ & a \end{pmatrix}$ has an eigenvalue. By Schur's lemma we can find $\omega(a) \in \mathbb{C}$ such that $\pi\begin{pmatrix} a & \\ & a \end{pmatrix} v = \omega(a)v$ for all $v \in V$. The resulting character $\omega : \mathbb{Q}_p^\times \to \mathbb{C}$ is called the **central character** of π.

Proposition 8.7. For (π, V) irreducible, we have $\pi^\vee \cong \pi \otimes \omega^{-1}$.

Proof. From Theorem 3.11 we have an isomorphism $(\pi^\vee, V^\vee) \cong (\breve{\pi}, V)$, where $\breve{\pi}(g) := \pi({}^t g^{-1})$. It suffices to show $\breve{\omega} \cong \pi \otimes \omega^{-1}$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$ {}^t g^{-1} \cdot \det g = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = wgw^{-1} $$

where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Now define

$$ \begin{aligned} \theta : (\breve{\pi}, V) &\longrightarrow (\pi \otimes \omega^{-1}, V) \\ v &\longmapsto \theta(v) = \pi(w^{-1})v \end{aligned} $$

Compute

$$ \theta(\breve{\pi}(g)v) = \pi(w^{-1})\pi({}^t g^{-1})v = \pi(w^{-1}wgw^{-1}\det g^{-1})v = \omega^{-1}(\det g)\pi(gw^{-1})v = \pi \otimes \omega^{-1}(g)\theta(v) $$

□

Corollary 8.7.1.

1. For $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm}$, we have $\pi(\chi_1, \chi_2)^\vee = \pi(\chi_2, \chi_1)$.
2. For $\chi_0 : \mathbb{Q}_p^\times \to \mathbb{C}^\times$, we have $(\mathrm{St} \otimes \chi_0)^\vee = \mathrm{St} \otimes \chi_0^{-1}$.

Proof.

1. The central character of $(\rho, I(\chi_1, \chi_2))$ is $\omega = \chi_1\chi_2$. Thus
$$ \pi(\chi_1^{-1}, \chi_2^{-1}) = (\pi(\chi_1, \chi_2))^\vee = \pi(\chi_1, \chi_2) \otimes (\chi_1\chi_2)^{-1} = \pi(\chi_2^{-1}, \chi_1^{-1}). $$
2. $(\mathrm{St} \otimes \chi_0)^\vee = (\mathrm{St} \otimes \chi_0) \otimes \chi_0^{-2} \cong \mathrm{St} \otimes \chi_0^{-1}$.

□

Let (π, V) be an irreducible representation of $G = \mathrm{GL}_2(\mathbb{Q}_p)$. We will consider the Whittaker model $W_\psi(\pi)$ of π. Recall the space

$$ W_\psi := \{W : G \to \mathbb{C} \mid W \text{ is smooth, } W(\mathbf{n}(x)g) = \psi(x)W(g)\} $$

Let ω be the central character of π. For $W \in W_\psi$, define

$$ W \otimes \omega^{-1}(g) := W(g)\omega^{-1}(\det g) $$

Then $W \otimes \omega^{-1} \in W_\psi$, as $\det \mathbf{n}(x) = 1$ for all $x \in \mathbb{Q}_p$. Then

$$ (\rho, W_\psi(\pi) \otimes \omega^{-1}) \cong (\rho \otimes \omega^{-1}, W_\psi(\pi)) \cong (\pi \otimes \omega^{-1}, V) \cong (\pi^\vee, V^\vee) $$

where the first isomorphism is defined by $W \otimes \omega^{-1} \mapsto W$, and hence

$$ W_\psi(\pi^\vee) = W_\psi(\pi) \otimes \omega^{-1} $$

by the uniqueness of Whittaker models.

8.4 Useful integration formulas

Let $G = \mathrm{GL}_2(\mathbb{Q}_p)$, $K = \mathrm{GL}_2(\mathbb{Z}_p)$, $N = \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}$, $B = \left\{\begin{pmatrix} * & * \\ & * \end{pmatrix}\right\}$, $T = \left\{\begin{pmatrix} * & \\ & * \end{pmatrix}\right\}$, where $* \in \mathbb{Q}_p$. Then for $f \in \mathcal{S}(G)$, we have the following integration formulas.

$$\int_G f(g)dg = \int_B \int_K f(bk)dkd_Lb \qquad (\spadesuit)$$
$$= \int_B \int_N f(bwn)dnd_Lb \qquad (\clubsuit)$$

where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. $(\spadesuit)$ results from the Iwasawa decomposition, and $(\clubsuit)$ results from the Bruhat decomposition together with the fact that $\mathrm{vol}(B, dg) = 0$. (proofs to be filled) Also,

$$\int_B f(b)d_Lb = \int_T \int_N f(tn)dndt$$

Note that the formulas above hold up to a positive scalar, due to the uniqueness of Haar measures. We will determine the scalar when we really need it.

Recall in the proof of Proposition 8.5 we showed the map

$$\begin{aligned} \mathcal{S}(G) &\longrightarrow I(|\cdot|^{\frac{1}{2}}, |\cdot|^{-\frac{1}{2}}) \\ f &\longmapsto \overline{f}(g) := \int_B f(bg)d_Lb \end{aligned}$$

is surjective; take $\chi = (|\cdot|^{\frac{1}{2}}, |\cdot|^{-\frac{1}{2}})$ so that $\chi\delta_B^{\frac{1}{2}} = \delta_B$, and thus $\chi^{-1}\delta_B^{-\frac{1}{2}}db = \delta_B^{-1}db = d_Lb$. Hence for $\overline{f} \in I(|\cdot|^{\frac{1}{2}}, |\cdot|^{-\frac{1}{2}})$, take any $\mathcal{S}(G) \ni f \mapsto \overline{f}$ and compute

$$\int_K \overline{f}(k)dk \overset{(\spadesuit)}{=} \int_B \int_K f(bk)dkd_Lb$$
$$\overset{(\clubsuit)}{=} \int_B \int_N f(bwn)dnd_Lb = \int_B \overline{f}(wn)dn$$

Consider the pairing $\langle\,,\rangle : I(\chi_1, \chi_2) \times I(\chi_1^{-1}, \chi_2^{-1}) \to \mathbb{C}$ defined in Proposition 8.5. For (f_1, f_2) in the domain, we have $f_1 f_2 \in I(|\cdot|^{\frac{1}{2}}, |\cdot|^{-\frac{1}{2}})$, and hence

$$\langle f_1, f_2\rangle = \int_K f_1(k)f_2(k)dk = \int_N f_1(wn)f_2(wn)dn \qquad (\heartsuit)$$

The first integral takes place on a compact set, so we can easily know its convergence. The second integral takes place on an abelian group, so the computation is rather easy.

8.5 Whittaker models for Steinberg representations

Let (π, V) be an irreducible smooth admissible representation of $G = \mathrm{GL}_2(\mathbb{Q}_p)$.

Principal series. $(\pi, V) \cong \pi(\chi_1, \chi_2)$ for some $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to \mathbb{C}$ such that $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm}$. Then the Whittaker model of V is

$$W_\psi(\pi) = \left\{W_{\Phi,\chi} \mid \Phi \in \mathcal{S}(\mathbb{Q}_p^2)\right\}$$

This follows from Proposition 8.1.2 and the uniqueness of Whittaker models.

Steinberg representation. $(\pi, V) = \mathrm{St} \otimes \chi_0 \subsetneq I(\chi_0|\cdot|^{\frac{1}{2}}, \chi_0|\cdot|^{-\frac{1}{2}})$, where $\chi_0 : \mathbb{Q}_p^\times \to \mathbb{C}^\times$. Put $\chi = \chi_0|\cdot|^{\frac{1}{2}}, \chi_0|\cdot|^{-\frac{1}{2}}$. Then

$$W_\psi(\pi) \subsetneq \{W_{\Phi,\chi} \mid \Phi \in \mathcal{S}(\mathbb{Q}_p^2)\}$$

We want to characterize the subspace $W_\psi(\pi)$.

Proposition 8.8. For $\pi = \mathrm{St} \otimes \chi_0$,

$$W_\psi(\pi) = \left\{ W_{\Phi,\chi} \mid \Phi \in \mathcal{S}(\mathbb{Q}_p^2), \int_{\mathbb{Q}_p} \Phi(x,0)dx = 0 \right\}$$

Proof. Our assumption is $(\pi, V) = (\rho, I(\chi_1, \chi_2)_S)$, where $\chi_1 = \chi_0|\cdot|^{\frac{1}{2}}$ and $\chi_2 = \chi_0|\cdot|^{-\frac{1}{2}}$. Note that

$$I(\chi_1,\chi_2)_S = \{f \in I(\chi_1,\chi_2) \mid \langle f, \chi_0^{-1} \circ \det \rangle = 0\}$$

where $\langle , \rangle$ is the pairing defined in Proposition 8.5. To see this, the same proposition says

$$\begin{aligned} I(\chi_1,\chi_2)_S &= \left(\frac{I(\chi_1^{-1},\chi_2^{-1})}{\mathbb{C}\chi_0^{-1} \circ \det}\right)^\vee = \{T \in I(\chi_1^{-1},\chi_2^{-1})^\vee \mid T(\chi_0^{-1} \circ \det) = 0\} \\ &= \{f \in I(\chi_1,\chi_2) \mid \langle f, \chi_0^{-1} \circ \det \rangle = 0\} \end{aligned}$$

By Proposition 8.1.2, each $f \in I(\chi_1,\chi_2)$ has the form $f_{\Phi^\sim,\chi}$ for some $\Phi \in \mathcal{S}(\mathbb{Q}_p^2)$. Then $f_{\Phi^\sim,\chi} \in I(\chi_1,\chi_2)_S$ if and only if

$$\begin{aligned} 0 = \langle f_{\Phi^\sim,\chi}, \chi_0^{-1} \circ \det \rangle &= \int_K f_{\Phi^\sim,\chi}(k)\chi_0^{-1}(\det k)dk \\ &\overset{(\heartsuit)}{=} \int_N f_{\Phi^\sim,\chi}(wn)\chi_0^{-1}(\det wn)dn \\ &= \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p^\times} \Phi^\sim((0\ t)w\mathbf{n}(x))|t|^2 d^\times t dx \\ &= \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p^\times} \Phi^\sim(-t,-tx)|t|^2 d^\times t dx \\ &= \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} \Phi^\sim(t,x)dtdx = \int_{\mathbb{Q}_p} \Phi(t,0)dt \end{aligned}$$

where the last equality follows from definition: since $\Phi^\sim(x,y) = \int_{\mathbb{Q}_p} \Phi(x,y)\psi(ay)da$, letting $y = 0$ yields $\Phi^\sim(x,0) = \int_{\mathbb{Q}_p} \Phi(x,a)da$. $\square$

8.6 Summary

Let (π, V) be an irreducible smooth admissible representation of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ with $\dim_{\mathbb{C}} V = \infty$. Consider the Jacquet module $J(V)$.

- $\boldsymbol{J(V) = 0}$. In this case, (π, V) is called **supercuspidal**.
- $\boldsymbol{J(V) \neq 0}$. As in the first paragraph of the proof of Theorem 5.6, we can find $\chi : T \to \mathbb{C}^\times$ and $0 \neq \Lambda \in \mathrm{Hom}_G(V, \mathrm{ind}_B^G \chi)$. Since V is irreducible, Λ embeds V into $\mathrm{ind}_B^G \chi = \mathrm{Ind}_B^G \chi\delta_B^{-\frac{1}{2}}$. Denote $\chi\delta_B^{-\frac{1}{2}} = (\chi_1, \chi_2)$, so $\mathrm{ind}_B^G \chi = I(\chi_1, \chi_2)$.
  - $\boldsymbol{\chi_1\chi_2^{-1}} \neq |\cdot|^{\pm}$. Then $(\pi, V) = \pi(\chi_1, \chi_2) = I(\chi_1, \chi_2)$, and it is called the **principal series**.
  - $\boldsymbol{\chi_1\chi_2^{-1}} = |\cdot|^{\pm}$. Then we can find χ_0 such that $\pi = \mathrm{St} \otimes \chi_0$, and (π, V) is called the **Steinberg representation**.

9 Theory of L-functions on $\mathrm{GL}_2(\mathbb{Q}_p)$

Let (π, V) be an irreducible representation of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ with $\dim V = \infty$. Consider the Whittaker model $W_\psi(\pi)$ of (π, V).

Definition. For $W \in W_\psi(\pi)$ and $s \in \mathbb{C}$, define formally the local ζ-integral

$$\Psi(W, s) := \int_{\mathbb{Q}_p^\times} W\begin{pmatrix} a & \\ & 1 \end{pmatrix} |a|^{s-\frac{1}{2}} d^\times a$$

where $d^\times a$ is the normalized Haar measure such that $\mathrm{vol}(\mathbb{Z}_p^\times, d^\times a) = 1$. In general, if $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ is a character, we define

$$\Psi(W, \chi, s) := \int_{\mathbb{Q}_p^\times} W\begin{pmatrix} a & \\ & 1 \end{pmatrix} \chi(a)|a|^{s-\frac{1}{2}} d^\times a$$

Theorem 9.1.

1. $\Psi(W, s)$ converges absolutely for $\mathrm{Re}\, s \gg 0$, and has a meromorphic continuation to $\mathbb{C}$.
2. There exists a unique L-factor $L(s, \pi)$ such that

   $$\Xi(W, s) := \frac{\Psi(W, s)}{L(s, \pi)}$$

   is entire for all $W \in W_\psi(\pi)$, and exists $W_0 \in W_\psi(\pi)$ such that $\Xi(W_0, s) = 1$. In other words, $L(s, \pi)$ is the gcd of $\{\Psi_{W,s}\}_{W \in W_\psi(\pi)}$.

   In general, a function $L(s, \pi)$ is called an ***L*-factor** if $L(s, \pi)^{-1} = Q(p^{-s})$ where $Q \in \mathbb{C}[X]$ with $Q(0) = 1$, i.e.,

   $$L(s, \pi)^{-1} = \prod_{i=1}^{*} (1 - \alpha_i p^{-s})$$

   for some $\alpha_i \in \mathbb{C}^\times$.
3. We have the functional equation: for $W \in W_\psi(\pi)$, define

   $$\widehat{W}(g) := W(gw)\omega^{-1}(\det g) = \rho(w)W \otimes \omega^{-1}(g) \in W_\psi(\pi^\vee)$$

   where $w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ and ω is the central character. Then there exists an epsilon factor $\epsilon(s, \chi, \psi)$ such that

   $$\frac{\Psi(\widehat{W}, 1-s)}{L(1-s, \pi^\vee)} = \frac{\Psi(W, s)}{L(s, \pi)} \cdot \epsilon(s, \pi, \psi)$$

If $(\pi, V) = \pi(\chi_1, \chi_2)$, then

$$L(s, \pi) = L(s, \chi_1)L(s, \chi_2)$$
$$\epsilon(s, \pi, \psi) = \epsilon(s, \chi_1, \psi)\epsilon(s, \chi_2, \psi)$$

If $(\pi, V) = \mathrm{St} \otimes \chi_0 \subseteq I(\chi_0|\cdot|^{\frac{1}{2}}, \chi_0|\cdot|^{-\frac{1}{2}})$, write $\chi_1 = \chi_0|\cdot|^{\frac{1}{2}}$ and $\chi_2 = \chi_0|\cdot|^{-\frac{1}{2}}$; then

$$L(s, \pi) = L(s, \chi_1)$$
$$\epsilon(s, \pi, \psi) = \epsilon(s, \chi_1, \psi)\epsilon(s, \chi_2, \psi)\frac{L(1-s, \chi_1^{-1})}{L(s, \chi_2)}$$

If (π, V) is supercuspidal, then

$$L(s, \pi) = 1$$
$$\epsilon(s, \pi, \psi) = \text{complicated}$$

Similar to the GL(1), we define the **γ-factor** for π to be

$$\gamma(s, \pi, \psi) := \frac{L(1-s, \pi^\vee)}{L(s, \pi)} \epsilon(s, \pi, \psi)$$

Then the functional equation takes the form

$$\frac{\Psi(\widehat{W}, 1-s)}{\Psi(W, s)} = \gamma(s, \pi, \psi)$$

9.1 Principal Series

Let $(\pi, V) \cong \pi(\chi_1, \chi_2)$, $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm}$ be a principal series. Put $\chi = (\chi_1, \chi_2)$. Then

$$W_\psi(\pi) := \{W_{\Phi,\chi} \mid \Phi \in \mathcal{S}(\mathbb{Q}_p^2)\}$$

We may assume $\Phi = \varphi_1 \otimes \varphi_2$ with $\varphi_i \in \mathcal{S}(\mathbb{Q}_p)$. Compute

$$\begin{aligned}
\Psi(W_{\Phi,\chi}, s) &= \int_{\mathbb{Q}_p^\times} W_{\Phi,\chi}\begin{pmatrix} a & \\ & 1 \end{pmatrix} |a|^{s-\frac{1}{2}} d^\times a \\
&= \int_{\mathbb{Q}_p^\times} \chi_1|\cdot|^{\frac{1}{2}}(a) \int_{\mathbb{Q}_p^\times} \Phi(at, t^{-1})\chi_1\chi_2^{-1}(t) d^\times t |a|^{a-\frac{1}{2}} d^\times a \\
&= \int_{\mathbb{Q}_p^\times}\int_{\mathbb{Q}_p^\times} \Phi(at, t^{-1})\chi_1\chi_2^{-1}(t)\chi_1(a)|a|^s d^\times a d^\times t \\
(a \mapsto at^{-1},\ t \mapsto t^{-1}) &= \int_{\mathbb{Q}_p^\times}\int_{\mathbb{Q}_p^\times} \Phi(a, t)\chi_1(a)|a|^s\chi_2(t)|t|^s d^\times t d^\times a \\
(\Phi = \varphi_1 \otimes \varphi_2) &= \left(\int_{\mathbb{Q}_p^\times} \varphi_1(a)\chi_1(a)|a|^s d^\times a\right)\left(\int_{\mathbb{Q}_p^\times} \varphi(t)\chi_2(t)|t|^s d^\times\right) \\
&= Z(\varphi_1, \chi_1, s) Z(\varphi_2, \chi_2, s)
\end{aligned}$$

which is a product of two Tate integrals. From the theory of L-functions on GL(1), we find the function

$$\Psi(W_{\Phi,\chi}, s) = Z(\varphi_1, \chi_1, s) Z(\varphi_2, \chi_2, s)$$

has analytic continuation

$$\frac{\Psi(W_{\Phi,\chi}, s)}{L(s, \chi_1)L(s, \chi_2)} = \frac{Z(\varphi_1, \chi_1, s)}{L(s, \chi_1)} \cdot \frac{Z(\varphi_2, \chi_2, s)}{L(s, \chi_2)}$$

so that

$$L(s, \pi) = L(s, \chi_1)L(s, \chi_2)$$

From the formula.(iv), we know

$$\begin{aligned}
\omega_\psi(w)\Phi(x, y) &= \int_{\mathbb{Q}_p^2} \Phi(a, b)\psi(ay + bx) da db \\
(\Phi = \varphi_1 \otimes \varphi_2) &= \widehat{\varphi_2} \otimes \widehat{\varphi_1}(x, y)
\end{aligned}$$

Then

$$W_{\Phi,\chi}(gw) = \chi_1|\cdot|^{\frac{1}{2}}(t)\int_{\mathbb{Q}_p^\times}\omega_\psi(gw)\Phi(t,t^{-1})\chi_1\chi_2^{-1}(t)d^\times t$$
$$= \chi_1|\cdot|^{\frac{1}{2}}(t)\int_{\mathbb{Q}_p^\times}\omega_\psi(g)\widehat{\varphi_2}\otimes\widehat{\varphi_1}(t,t^{-1})\chi_1\chi_2^{-1}(t)d^\times t = W_{\widehat{\varphi_2}\otimes\widehat{\varphi_1},\chi}(g)$$

Consequently,

$$\Psi(\widehat{W_{\Phi,\chi}},1-s) = \Psi(W_{\widehat{\varphi_2}\otimes\widehat{\varphi_1},\chi}(g),\omega^{-1},1-s)$$
$$= Z(\widehat{\varphi_2},\chi_1\omega^{-1},1-s)Z(\widehat{\varphi_1},\chi_2\omega^{-1},1-s)$$

Recall that the central character of $(\pi,V)=(\rho,I(\chi_1,\chi_2))$ is $\omega=\chi_1\chi_2$. Thus

$$\Psi(\widehat{W_{\Phi,\chi}},1-s) = Z(\widehat{\varphi_2},\chi_2^{-1},1-s)Z(\widehat{\varphi_1},\chi_1^{-1},1-s)$$

and

$$L(1-s,\pi^\vee) = L(1-s,\pi\otimes\omega^{-1}) = L(1-s,\chi_1\omega^{-1})L(1-s,\chi_2\omega^{-1}) = L(1-s,\chi_2^{-1})L(1-s,\chi_1^{-1})$$

From the theory of L-functions on GL(1) we deduce that

$$\epsilon(s,\pi,\psi) = \epsilon(s,\chi_1,\psi)\epsilon(s,\chi_2,\psi)$$

9.2 Steinberg Representation

Assume $(\pi,V)=\mathrm{St}\otimes\chi_0$ for some character $\chi_0:\mathbb{Q}_p^\times\to\mathbb{C}^\times$. Put $\chi_1=\chi_0|\cdot|^{\frac{1}{2}}$ and $\chi_2=\chi_0|\cdot|^{-\frac{1}{2}}$. We know its Whittaker model is

$$W_\psi(\pi) = \left\{W_{\Phi,\chi} \mid \Phi\in\mathcal{S}(\mathbb{Q}_p^2),\ \int_{\mathbb{Q}_p}\Phi(x,0)dx = 0\right\}$$

Assume $\Phi=\varphi_1\otimes\varphi_2$ with $\varphi_i\in\mathcal{S}(\mathbb{Q}_p)$. The the imposed condition on elements of $W_\psi(\pi)$ means $\widehat{\varphi_1}(0)\varphi_2(0)=0$, i.e., $\widehat{\varphi_1}(0)=0$ or $\varphi_2(0)=0$, i.e., $\widehat{\varphi_1}\in\mathcal{S}(\mathbb{Q}_p^\times)$ or $\varphi_2\in\mathcal{S}(\mathbb{Q}_p^\times)$. The computation in the principal series case shows

$$\Psi(W_{\Phi,\chi},s) = Z(\varphi_1,\chi_1,s)Z(\varphi_2,\chi_2,s)$$

If $\varphi_2\in\mathcal{S}(\mathbb{Q}_p^\times)$, then $Z(\varphi_2,\chi_2,s)\in\mathbb{C}[p^s,p^{-s}]$, so the ratio

$$\frac{\Psi(W_{\Phi,\chi},s)}{L(s,\chi_1)} = \frac{Z(\varphi_1,\chi_1,s)}{L(s,\chi_1)}\cdot Z(\varphi_2,\chi_2,s)$$

is entire. If $\widehat{\varphi_1}\in\mathcal{S}(\mathbb{Q}_p^\times)$, then

$$\frac{\Psi(W_{\Phi,\chi},s)}{L(s,\chi_1)} = \frac{Z(\varphi_1,\chi_1,s)}{L(s,\chi_1)}\cdot Z(\varphi_2,\chi_2,s)$$
$$= \frac{Z(\widehat{\varphi_1},\chi_1^{-1},1-s)}{L(1-s,\chi_1^{-1})}\epsilon(s,\chi_1,\psi)\cdot Z(\varphi_2,\chi_2,s)$$
$$= Z(\widehat{\varphi_1},\chi_1^{-1},1-s)\epsilon(s,\chi_1,\psi)\cdot\frac{Z(\varphi_2,\chi_2,s)}{L(1-s,\chi_1^{-1})}$$

Recall that $\chi_1\chi_2^{-1}=|\cdot|$. Then

$$L(1-s,\chi_1^{-1})^{-1} = 1-\chi_1^{-1}(p)|p|^{1-s} = 1-\chi_2^{-1}(p)|p|^{-s} = -\chi_2^{-1}(p)|p|^{-s}L(x,\chi_2)^{-1}$$

and therefore

$$\frac{\Psi(W_{\Phi,\chi},s)}{L(s,\chi_1)} = Z(\widehat{\varphi_1},\chi_1^{-1},1-s)\epsilon(s,\chi_1,\psi)\cdot\frac{Z(\varphi_2,\chi_2,s)}{L(x,\chi_2)}\cdot(-\chi_2^{-1}(p)|p|^{-s})$$

is entire. Now the theorem follows from the theory of L-functions on GL(1).

9.3 Supercuspidal

Let (π, V) be supercuspidal and identify V with its Kirillov model $K_\psi(\pi)$. Since $J(V) = 0$ by definition, we have $K_\psi(\pi) = \mathcal{S}(\mathbb{Q}_p^\times)$. Then the Whittaker model is

$$\begin{array}{ccc} V & \longrightarrow & W_\psi(\pi) \\ \xi & \longmapsto & W_\xi \begin{pmatrix} a & \\ & 1 \end{pmatrix} := \xi(a) \end{array}$$

and the local zeta integral

$$\Psi(W_\xi, s) = \int_{\mathbb{Q}_p^\times} \xi(a)|a|^{s-\frac{1}{2}} d^\times a \in \mathbb{C}[s, s^{-1}]$$

is entire. Thus $L(s, \pi) = 1$.

We proceed to prove the existence of epsilon factor $\epsilon(s, \chi, \psi)$ and the functional equation. For $\xi \in V = \mathcal{S}(\mathbb{Q})_p^\times)$, $\nu \in \widehat{\mathbb{Z}_p^\times}$ and $n \in \mathbb{Z}$, put

$$\widehat{\xi_n}(\nu) = \xi_n^\wedge(\nu) := \int_{\mathbb{Z}_p^\times} \xi(p^n u)\nu(u) d^\times u \in \mathbb{C}$$

and

$$\widehat{\xi}(\nu, t) = \xi^\wedge(\nu, t) := \sum_{n \in \mathbb{Z}} \widehat{\xi_n}(\nu) \cdot t^n$$

This is a polynomial in t, t^{-1} since $\xi \in \mathcal{S}(\mathbb{Q}_p^\times)$ the support of ξ is bounded above and below.

Put $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and for $\nu \in \widehat{\mathbb{Z}_p^\times}$ define

$$\varphi_\nu(a) := \mathbf{1}_{\mathbb{Z}_p^\times}(a)\nu(a) \in \mathcal{S}(\mathbb{Q}_p^\times)$$

Then

- $\mathcal{S}(\mathbb{Q}_p^\times)$ is spanned by the $\pi \begin{pmatrix} p^n & \\ & 1 \end{pmatrix} \varphi_\nu$. See the lemma in Theorem 7.2.
- $\widehat{\varphi_\nu}(\mu, t) = \begin{cases} 1 & \text{if } \mu\nu = \mathbf{1} \\ 0 & \text{if } \mu\nu \neq \mathbf{1} \end{cases}$, where $\mathbf{1}$ denotes the trivial character.

Define $C(\pi, \nu, t) \in \mathbb{C}[t, t^{-1}]$ by

$$C(\pi, \nu, t) := \widehat{\pi(w)\varphi_{\nu\omega}}(\nu, t)$$

where $\nu \in \widehat{\mathbb{Z}_p^\times}$ and ω is the central character of π.

Lemma 9.2. Let $z_0 = \omega(p)$. For any $\nu \in \widehat{\mathbb{Z}_p^\times}$ we have

$$\widehat{\pi(w)\xi}(\nu, t) = C(\pi, \nu, t) \cdot \widehat{\xi}(\nu^{-1}\omega^{-1}, z_0^{-1}t^{-1})$$

Proof. For any $\xi \in V = \mathcal{S}(\mathbb{Q}_p^\times)$, ptu

$$\Delta(\xi) = \widehat{\pi(w)\xi}(\nu, t) = -(\pi, \nu, t) \cdot \widehat{\xi}(\nu^{-1}\omega^{-1}, z_0^{-1}t^{-1})$$

We need to show $\Delta(\xi) = 0$, and we verify $\Delta(\varphi_\mu) = 0$ first.

- $\mu = \nu\omega$. Then

$$\Delta(\varphi_{\nu\omega}) = \widehat{\pi(w)\varphi_{\nu\omega}}(\nu, t) - C(\pi, \nu, t) \cdot \underbrace{\widehat{\varphi_{\nu\omega}}(\nu^{-1}\omega^{-1}, z_0^{-1}t^{-1})}_{=1}$$

$$= C(\pi, \nu, t) - C(\pi, \nu, t) = 0$$

- $\mu \neq \nu\omega$. Then

$$\Delta(\varphi_{\nu\omega}) = \widehat{\pi(w)\varphi_{\nu\omega}}(\nu, t) - 0 = \widehat{\pi(w)\varphi_{\nu\omega}}(\nu, t)$$

Observe that $\pi(w)\varphi_\mu$ is the eigenfunction of $\mathbb{Z}_p^\times$ with eigencharacter $\omega\mu^{-1}$: for $a \in \mathbb{Z}_p^\times$,

$$\begin{aligned}\pi\begin{pmatrix} a & \\ & 1 \end{pmatrix}\pi(w)\varphi_\mu = \pi(w)\pi\begin{pmatrix} 1 & \\ & a \end{pmatrix}\varphi_\mu = \pi(w)\omega(a)\pi\begin{pmatrix} a^{-1} & \\ & 1 \end{pmatrix}\varphi_\mu \\ = \omega\mu^{-1}(a)\pi(w)\varphi_\mu\end{aligned}$$

Thus

$$\widehat{\pi(w)\varphi_\mu}_n(\nu) = \int_{\mathbb{Z}_p^\times} \pi(w)\varphi_\mu(p^n u)\nu(u)d^\times u = \pi(w)\varphi_\mu(p^n)\int_{\mathbb{Z}_p^\times} \omega\mu^{-1}\nu(u)d^\times u = 0$$

if $\omega\mu^{-1}\nu \neq \mathbf{1} \Leftrightarrow \mu \neq \omega\nu$, so that

$$\Delta(\varphi_{\nu\omega}) = \widehat{\pi(w)\varphi_{\nu\omega}}(\nu, t) = \sum_{n\in\mathbb{Z}} \widehat{\pi(w)\varphi_\mu}_n(\nu)t^n = 0$$

Next we show $\Delta(\pi\begin{pmatrix} p^n & \\ & 1 \end{pmatrix}\varphi_\mu) = 0$, from which we can conclude the lemma.

$$\begin{aligned}\Delta(\pi\begin{pmatrix} p^n & \\ & 1 \end{pmatrix}\varphi_\mu) &= \left(\pi(w)\pi\begin{pmatrix} p^n & \\ & 1 \end{pmatrix}\varphi_\mu\right)^\wedge(\nu, t) - C(\pi, \nu, t)\left(\pi\begin{pmatrix} p^n & \\ & 1 \end{pmatrix}\varphi_\mu\right)^\wedge(\nu^{-1}\omega^{-1}, z_0^{-1}t^{-1}) \\ &= \left(\pi\left(p^n\begin{pmatrix} p^{-n} & \\ & 1 \end{pmatrix}\right)\pi(w)\varphi_\mu\right)^\wedge(\nu, t) - C(\pi, \nu, t)(z_0^{-1}t^{-1})^{-n}\widehat{\varphi_\mu}(\nu^{-1}\omega^{-1}, z_0^{-1}t^{-1}) \\ &= z_0^n t^n \widehat{\pi(w)\varphi_\mu}(\nu, t) - C(\pi, \nu, t)(z_0^{-1}t^{-1})^{-n}\widehat{\varphi_\mu}(\nu^{-1}\omega^{-1}, z_0^{-1}t^{-1}) \\ &= z_0^n t^n \Delta(\varphi_\mu) = 0\end{aligned}$$

□

Now we use this lemma twice and the fact $w^2 = -1$.

$$\begin{aligned}\widehat{\xi}(\nu, t) &= \widehat{\pi(-w^2)\xi}(\nu, t) = \omega(-1)\widehat{\pi(w)\pi(w)\xi}(\nu, t) \\ &= \omega(-1)C(\pi, \nu, t) \cdot \widehat{\pi(w)\xi}(\nu^{-1}\omega^{-1}, z_0^{-1}t^{-1}) \\ &= \omega(-1)C(\pi, \nu, t)C(\pi, \nu^{-1}\omega^{-1}, z_0^{-1}t^{-1}) \cdot \widehat{\xi}(\nu, t)\end{aligned}$$

Hence

$$C(\pi, \nu, t)C(\pi, \nu^{-1}\omega^{-1}, z_0^{-1}t^{-1}) = \omega(-1)$$

Since $C(\pi, \nu, t) \in \mathbb{C}[t, t^{-1}]$, this implies $C(\pi, \nu, t) = At^n$ for some $A \in \mathbb{C}^\times$ and $n \in \mathbb{Z}$. Finally,

$$\Psi(W_\xi, s) = \int_{\mathbb{Q}_p^\times} \xi(a)|a|^{s-\frac{1}{2}}d^\times a = \sum_{n\in\mathbb{Z}} |p^n|^{s-\frac{1}{2}}\int_{\mathbb{Z}_p^\times} \xi(p^n u)d^\times u = \widehat{\xi}(\mathbf{1}, p^{\frac{1}{2}-s})$$

Recall that $\widehat{W}(g) := W(gw)\omega^{-1}(\det g)$. Then

$$\begin{aligned}
\Psi(\widehat{W}_\xi, 1-s) &= \int_{\mathbb{Q}_p^\times} W_\xi\left(\begin{pmatrix} a & \\ & 1\end{pmatrix} w\right)\omega^{-1}(a)|a|^{1-s-\frac{1}{2}} d^\times a \\
&= \sum_{n\in\mathbb{Z}} |p^n|^{1-s-\frac{1}{2}} \int_{\mathbb{Z}_p^\times} W_\xi\left(\begin{pmatrix} p^n u & \\ & 1\end{pmatrix} w\right))\omega^{-1}(p^n u) d^\times u \\
&= \sum_{n\in\mathbb{Z}} p^{n(s-\frac{1}{2})} z_0^{-n} \int_{\mathbb{Z}_p^\times} \pi(w)\xi(p^n u)\omega^{-1}(u) d^\times u \\
&= \widehat{\pi(w)\xi}(\omega^{-1}, p^{s-\frac{1}{2}} z_0^{-1})
\end{aligned}$$

By the lemma we have

$$\widehat{\pi(w)\xi}(\omega^{-1}, p^{s-\frac{1}{2}} z_0^{-1}) = C(\pi, \omega^{-1}, p^{s-\frac{1}{2}} z_0^{-1})\widehat{\xi}(\mathbf{1}, p^{\frac{1}{2}-s})$$

Now we define our dreamed epsilon factor:

$$\epsilon(s,\pi,\psi) := C(\pi, \omega^{-1}, p^{s-\frac{1}{2}} z_0^{-1}) = Ap^{ns} \text{ for some } A \in \mathbb{C}^\times,\ n \in \mathbb{Z}$$

Then we attain the functional equation

$$\Psi(\widehat{W}_\xi, 1-s) = \epsilon(s,\pi,\psi)\Psi(W_\xi, s) \text{ for all } \xi \in V = \mathcal{S}(\mathbb{Q}_p^\times)$$

9.4 Archimedean Case

Let (π, V) be an irreducible $(\mathfrak{g}, K)$-module, where $\mathfrak{g} = \mathrm{Lie}(\mathrm{GL}_2(\mathbb{R}))$ and $K = \mathrm{O}(2)$. Then $(\pi, V) \subseteq I(\chi_1, \chi_2)$ with $\chi_1\chi_2^{-1} = |\cdot|^s \mathrm{sign}^\epsilon$, $s \in \mathbb{C}$, $\epsilon \in \{0, 1\}$.

- $s - \epsilon \notin 1 + 2\mathbb{Z}$. Then $\pi \cong \pi(\chi_1, \chi_2)$ is the principal series and

  $$V = \bigoplus_{\ell\equiv\epsilon \pmod 2} V(\ell)$$

  with $\dim_{\mathbb{C}} V(\ell) = 1$.

- $s - \epsilon \in 1 + 2\mathbb{Z}$ and $s = k - 1 \geqslant 0$, where k is the minimal weight of π. Let $\sigma_k \subseteq I(|\cdot|^{\frac{k-1}{2}}, |\cdot|^{\frac{1-k}{2}}\mathrm{sign}^k)$ be the unique irreducible subrepresentation. Then $\pi = \sigma_k \otimes \chi_0 \subseteq I(\chi_1, \chi_2)$ is the discrete series of weight k when $k \geqslant 2$, and is the limit discrete series when $k = 1$. In this case,

  $$V = \bigoplus_{\substack{\ell\geqslant k,\, \ell\leqslant -k \\ \ell\equiv k \pmod 2}} V(\ell)$$

  For $\pi \cong \pi(\chi_1, \chi_2)$, we have

$$W_\psi(\pi) = \left\{W_{\Phi,\chi} \mid \Phi(x,y) = p(x,y)e^{-\pi(x^2+y^2)},\ p \in \mathbb{C}[x,y]\right\}$$

where $\psi = \psi_\infty$ is the standard additive character. If $\pi = \sigma_k \otimes \chi_0$, then

$$W_\psi(\pi) = \left\{W_{\Phi,\chi} \mid \Phi(x,y) = p(x,y)e^{-\pi(x^2+y^2)},\ p \in \mathbb{C}[x,y],\ \int_{\mathbb{R}} x^i \frac{\partial^j \Phi}{\partial y^j}(x,y)dx = 0 \text{ for } i+j = k-2\right\}$$

where $\chi = (\chi_0|\cdot|^{\frac{k-1}{2}}, \chi_0|\cdot|^{\frac{1-k}{2}}\mathrm{sign}^k)$. To see this, put $\chi = (\chi_1, \chi_2)$ for brevity. Consider the pairing $\langle , \rangle : I(\chi_1, \chi_2) \times I(\chi_1^{-1}, \chi_2^{-1}) \to \mathbb{C}$ defined by

$$\langle f_1, f_2\rangle = \int_{\mathrm{O}(2)} f_1(x)f_2(x)dx$$

This pairing is $\operatorname{Lie}\mathrm{GL}_2(\mathbb{R})$-invariant, in the sense that for all $X \in \operatorname{Lie}\mathrm{GL}_2(\mathbb{R})$ we have

$$\langle X f_1, f_2 \rangle = -\langle f_1, X f_2 \rangle$$

and is $\mathrm{O}(2)$-invariant, in the sense that

$$\langle \rho(g) f_1, f_2 \rangle = \langle f_1, \rho(g^{-1}) f_2 \rangle$$

To be filled

10 Intertwining Operators

Let $p \leqslant \infty$ be a rational prime. For $s \in \mathbb{C}$ and $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to \mathbb{C}^\times$, put $I(\chi_1, \chi_2, s) := I(\chi_1|\cdot|^s, \chi_2|\cdot|^{-s})$. Define $\ell_N : I(\chi_1, \chi_2, s) \to \mathbb{C}$ by

$$\ell_N(f) := \int_{\mathbb{Q}_p} f\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) dx$$

Formally, we write

$$\begin{aligned} \ell_N(f) &= \int_{|x|\leqslant 1} f\begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix} dx + \int_{|x|>1} f\left(\begin{pmatrix} x^{-1} & -1 \\ 0 & x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}\right) dx \\ (x \mapsto x^{-1}) &= \int_{|x|\leqslant 1} f\begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix} dx + \int_{|x|<1} \chi_1\chi_2^{-1}|\cdot|^{2s}(x) f\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} d^\times x \end{aligned}$$

The first term always exists, and the second term is a Tate integral. Thus $\ell_N(f)$ converges absolutely when $\mathrm{wt}(\chi_1\chi_2^{-1}) + 2\,\mathrm{Re}(s) > 0$, and by Theorem 2.5.(i), $\ell_N : I(\chi_1, \chi_2, s) \to \mathbb{C}$ has a "meromorphic continuation" to $\mathbb{C}$.

For $\mathrm{Re}(s) \gg 0$, define $M(\chi_1, \chi_2, s) : I(\chi_1, \chi_2, s) \to I(\chi_2, \chi_1, -s)$ by

$$M(\chi_1, \chi_2, s)f(g) := \ell_N(\rho(g)f)$$

To see $g \mapsto \ell_N(\rho(g)f) \in I(\chi_2, \chi_1, -s)$, compute

$$\begin{aligned} M(\chi_1, \chi_2, s)f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) &= \int_{\mathbb{Q}_p} f\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) dx \\ &= \int_{\mathbb{Q}_p} f\left(\begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & a^{-1}xd + a^{-1}b \\ 0 & 1 \end{pmatrix} g\right) dx \\ &= \int_{\mathbb{Q}_p} \chi_1(d)\chi_2(a)\left|\frac{d}{a}\right|^{s+\frac{1}{2}} \left|\frac{a}{d}\right| f\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx \\ &= \chi_2(a)\chi_1(d)\left|\frac{a}{d}\right|^{-s+\frac{1}{2}} \ell_N(\rho(g)f) \end{aligned}$$

Introduce the normalized intertwining operator

$$M^*(\chi_1, \chi_2, s) := L(2s, \chi_1\chi_2^{-1})^{-1} M(\chi_1, \chi_2, s)$$

By Theorem 2.5.(ii), this is a well-defined map for all $s \in \mathbb{C}$.

To proceed, we first extend the modular function $\delta_B : B \to \mathbb{R}_{>0}$ to a function on $\mathrm{GL}_2(\mathbb{Q}_p)$, by setting $\delta_B(g) = \delta_B(b)$ if $g = bk$, $b \in B$, $k \in K$. To see this is well-defined, if $bk = b'k'$ with $b, b' \in B$, $k, k' \in K$, then $b^{-1}b' = kk'^{-1} \in B \cap K = B(\mathbb{Z}_p)$. Since $B(\mathbb{Z}_p) \leqslant B$ is compact, $\delta_B(B(\mathbb{Z}_p))$ is a compact subgroup of $\mathbb{R}_{>0}$, so $\delta_B(B(\mathbb{Z}_p)) = \{1\}$. Consequently, $\delta_B(b^{-1}b') = 1$, or $\delta_B(b') = \delta_B(b)$. In conclusion, we obtain a well-defined map $\delta_B : \mathrm{GL}_2(\mathbb{Q}_p) \to \mathbb{R}_{>0} \subseteq \mathbb{C}^\times$.

For $f \in I(\chi_1, \chi_2)$ and $s \in \mathbb{C}$, we see $f\delta_B^s \in I(\chi_1, \chi_2, s)$; this is called a **flat section**, which can be viewed as a section of the bundle $\bigsqcup_{s\in\mathbb{C}} I(\chi_1, \chi_2, s) \to \mathbb{C}$, and "flat" means $f\delta_B^s|_K = f|_K$ is independent of s. Consider the composition

$$\begin{array}{ccccccc} M(\chi_1, \chi_2) : I(\chi_1, \chi_2) & \longrightarrow & I(\chi_1, \chi_2, s) & \xrightarrow{M^*(\chi_1, \chi_2, s)} & I(\chi_2, \chi_1, -s) & \xrightarrow{(\cdot)|_{s=0}} & I(\chi_2, \chi_1) \\ f & \longmapsto & f\delta_B^s & & & & \end{array}$$

Definitely, for $f \in I(\chi_1, \chi_2)$, we define

$$M(\chi_1, \chi_2)f := M^*(\chi_1, \chi_2, s)(f\delta_B^s)|_{s=0}$$

We now study the action of $M(\chi_1, \chi_2)$ on the Godement section:

$$f_{\Phi,\chi,s}(g) = \chi_1|\cdot|^{s+\frac{1}{2}}(\det g)\int_{\mathbb{Q}_p^\times} \Phi((0\ t)g)\chi_1\chi_2^{-1}|\cdot|^{2s+1}(t)d^\times t$$

where $\chi = (\chi_1, \chi_2)$. For this, we introduce the **sympletic Fourier transform**. For $\Phi \in \mathcal{S}(\mathbb{Q}_p^2)$, define

$$\begin{aligned}\widehat{\Phi}(x, y) &:= \int_{\mathbb{Q}_p^2} \Phi(u, v)\psi_p(-vx + uy)dudy \\ &= \int_{\mathbb{Q}_p^2} \Phi(u, v)\psi_p\left(\begin{pmatrix} u & v\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix}\right)dudv\end{aligned}$$

Clearly, if $\Phi = \varphi_1 \otimes \varphi_2 \in \mathcal{S}(\mathbb{Q}_p^2)$ is a pure tensor, then

$$\widehat{\Phi}(x, y) = \widehat{\varphi_2}(-x)\widehat{\varphi_1}(y)$$

Proposition 10.1. For $\Phi \in \mathcal{S}(\mathbb{Q}_p^2)$, we have

$$M(\chi_1, \chi_2, s)f_{\Phi,\chi,s} = \gamma(2s, \chi_1\chi_2^{-1}, \psi)^{-1}f_{\widehat{\Phi},\chi^{\mathrm{sw}},-s}$$

where $\chi = (\chi_1, \chi_2)$ and $\chi^{\mathrm{sw}} = (\chi_2, \chi_1)$, and γ is the γ-factor.

Proof. By linearity, we may assume $\Phi = \varphi_1 \otimes \varphi_2$ is a pure tensor. Further, using the formulas

$$\begin{aligned}f_{\Phi,\chi,s}(g) &= \chi_1|\cdot|^{s+\frac{1}{2}}(\det g)f_{\rho(g)\Phi,\chi,s}(e) \\ f_{\widehat{\Phi},\chi,s}(g) &= \chi_2|\cdot|^{-s+\frac{1}{2}}(\det g)f_{\widehat{\rho(g)\Phi},\chi,s}(e)\end{aligned}$$

we only need to show

$$M(\chi_1, \chi_2, s)f_{\Phi,\chi,s}(e) = \gamma(2s, \chi_1\chi_2^{-1}, \psi)^{-1}f_{\widehat{\Phi},\chi^{\mathrm{sw}},-s}(e)$$

First, we have

$$f_{\widehat{\Phi},\chi^{\mathrm{sw}},-s}(e) = \widehat{\varphi_2}(0)Z(\widehat{\varphi_1}, \chi_2\chi_1^{-1}, 1 - 2s)$$

Secondly, compute the left hand side.

$$\begin{aligned}M(\chi_1, \chi_2, s)f_{\Phi,\chi,s}(e) &= \int_{\mathbb{Q}_p} f_{\Phi,\chi,s}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\right)dx \\ &= \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p^\times} \Phi(t, tx)\chi_1\chi_2^{-1}|\cdot|^{2s+1}(t)d^\times tdx \\ (x \mapsto xt^{-1}) &= \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p^\times} \Phi(t, x)\chi_1\chi_2^{-1}|\cdot|^{2s}(t)d^\times tdx \\ &= \widehat{\varphi_2}(0)\int_{\mathbb{Q}_p^\times} \varphi_1(t)\chi_1\chi_2^{-1}|\cdot|^{2s}(t)d^\times t \\ &= \widehat{\varphi_2}(0)Z(\varphi_1, \chi_1\chi_2^{-1}, 2s)\end{aligned}$$

Thus from Theorem 2.5.(iii) we obtain

$$\frac{M(\chi_1, \chi_2, s)f_{\Phi,\chi,s}(e)}{f_{\widehat{\Phi},\chi^{\mathrm{sw}},-s}(e)} = \frac{Z(\varphi_1, \chi_1\chi_2^{-1}, 2s)}{Z(\widehat{\varphi_1}, \chi_2\chi_1^{-1}, 1 - 2s)} = \gamma(2s, \chi_1\chi_2^{-1}, \psi)^{-1}$$

□

11 Local Jacquet-Langlands Correspondence

Assume that $(V, \langle\,,\rangle)$ is a finite dimensional nondegenrate quadratic space over $\mathbb{Q}_p$, $p \leqslant \infty$. Let $\psi : \mathbb{Q}_p \to \mathbb{C}^\times$ be a nontrivial additive character. Then we have the **Weil representation**

$$\omega_\psi : \mathrm{SL}_2(\mathbb{Q}_p) \times \mathrm{O}(V) \longrightarrow \mathrm{GL}(\mathcal{S}(V))$$

defined by the following formulas

(i) $\omega_\psi(1,h)\Phi(x) = \Phi(h^{-1}x)$ for $x \in V$.

For simplicity, define $r_V : \mathrm{SL}_2(\mathbb{Q}_p) \to \mathrm{GL}(\mathcal{S}(V))$ by $r_V(g) := \omega_\psi(g,1)$ and assume $m := \dim V$ is even.

(ii) $r_V \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \Phi(x) = ((-1)^{\frac{m(m-1)}{2}} \det V, a)_p \cdot |a|^{\frac{m}{2}} \cdot \Phi(xa)$, where $(\cdot,\cdot)_p$ is the Hilbert symbol, and $\det V$ is the determinant of the bilinear form $\langle\,,\rangle$.

(iii) $r_V \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \Phi(x) = \psi\left(\dfrac{b\langle x,x\rangle}{2}\right)\Phi(x)$.

(iv) $r_V \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Phi(x) = \gamma_\psi(V)\widehat{\Phi}(x)$, where $\gamma_\psi(V)$ is the **Weil index**, and

$$\widehat{\Phi}(x) := \int_V \Phi(y)\psi(\langle x,y\rangle)dy$$

is the **Fourier transform** in which dy is chosen so that the inversion formula holds.

Denote by $q_V : V \to \mathbb{Q}_p$ the associated quadratic form; then we have

$$q_V(x) = \frac{1}{2}\langle x,x\rangle$$
$$\langle x,y\rangle = q_V(x+y) - q_V(x) - q_V(y)$$

The Weil index depends on the form q_V, so we also write $\gamma_\psi(V) = \gamma_\psi(q_V)$. For $a \in \mathbb{Q}_p^\times$, put

$$\gamma_\psi(a) := \gamma_\psi(ax^2)$$

Then one can prove that $\gamma_\psi(a) \in \mu_8(\mathbb{C})$. We list some properties of the Weil index. By definition, $\gamma_\psi(V)$ is the unique number such that

$$\int_V \Phi(y)\psi(q_V(y))dy = \gamma_\psi(V)\int_V \widehat{\Phi}(y)\psi(-q_V(y))dy$$

holds for all $\Phi \in \mathcal{S}(V)$, where dy is the self-dual measure with respect to (ψ, q_V). We have

- $\gamma_\psi(V_1 \oplus V_2) = \gamma_\psi(V_1)\gamma_\psi(V_2)$.
- $\gamma_\psi(-q_V) = \gamma_\psi(q_V)^{-1}$.
- $\gamma_\psi(a)\gamma_\psi(b) = \gamma_\psi(1)\gamma_\psi(ab) \cdot (a,b)_p$ for all $a, b \in \mathbb{Q}_p^\times$.

11.1 Quaternion algebras

For $a, b \in \mathbb{Q}_p^\times$, define a four dimensional $\mathbb{Q}_p$-algebra

$$D := D_{a,b} = \mathbb{Q}_p \oplus \mathbb{Q}_p\alpha \oplus \mathbb{Q}_p\beta \oplus \mathbb{Q}_p\alpha\beta$$

with relation $\alpha^2 = a$, $\beta^2 = b$, $\alpha\beta = -\beta\alpha$. On D there is a natural **involution**:

$$\begin{array}{ccc} D & \longrightarrow & D \\ z = x_1 + x_2\alpha + x_3\beta + x_4\alpha\beta & \longmapsto & \overline{z} := x_1 - x_2\alpha - x_3\beta - x_4\alpha\beta \end{array}$$

Then one has $\overline{z_1 \cdot z_2} = \overline{z_2} \cdot \overline{z_1}$. We use this to define the **reduced trace**

$$\mathrm{Tr}_{D/\mathbb{Q}_p}(z) := z + \overline{z} = 2x_1 \in \mathbb{Q}_p$$

and the **reduced norm**

$$\nu(z) = N_{D/\mathbb{Q}_p}(z) := z\overline{z} = x_1^2 - x_2^2 a - x_3^2 b + x_4^2 ab \in \mathbb{Q}_p$$

Then (D, ν) is a quadratic space: define $\langle\, , \rangle_D : D \times D \to \mathbb{Q}_p$ by

$$\langle z, w \rangle_D := \nu(z + w) - \nu(z) - \nu(w) = \mathrm{Tr}_{D/\mathbb{Q}_p}(z\overline{w})$$

In terms of the ordered basis $\{1, \alpha, \beta, \alpha\beta\}$, the matrix representation of this pairing is

$$\begin{pmatrix} 2 & & & \\ & -2a & & \\ & & -2b & \\ & & & 2ab \end{pmatrix}$$

so $\det D = 16a^2b^2$. Consider the Weil representation $r_D : \mathrm{SL}_2(\mathbb{Q}_p) \to \mathrm{GL}(\mathcal{S}(D))$. We first compute the Weil index:

$$\begin{aligned} \gamma_\psi(D) &= \gamma_\psi(\mathbb{Q}_p \oplus \mathbb{Q}_p(-a) \oplus \mathbb{Q}_p(-b) \oplus \mathbb{Q}_p ab) \\ &= \gamma_\psi(1)\gamma_\psi(-a)\gamma_\psi(-b)\gamma_\psi(ab) \\ &= \gamma_\psi(a)\gamma_\psi(b)\,(a,b)_p\gamma_\psi(-a)\gamma_\psi(-b) \\ &= (a,b)_p \end{aligned}$$

Thus $\gamma_\psi(D) = 1$ if and only if $(a,b)_p = 1$, if and only if $D \cong M_2(\mathbb{Q}_p)$. In this case, we have $\nu(x) = \det x$.

Suppose $\gamma_\psi(D) = (a,b)_p = -1$; then D is the unique division algebra over $\mathbb{Q}_p$ with $\dim D = 4$. Consider the group of norm one elements:

$$D_1 := \{z \in D \mid \nu(z) = z\overline{z} = 1\}$$

It is a compact group. Let $\Omega : D^\times \to \mathrm{GL}(U)$ be a finite dimensional complex irreducible representation of the group $D^\times$. Consider the space

$$\mathcal{S}(D, \Omega) := \{\Phi \in \mathcal{S}(D) \otimes_{\mathbb{C}} U \mid \Phi(xz_1) = \Omega(z_1^{-1})\Phi(x) \text{ for all } z_1 \in D_1\}$$

where $\Omega(z_1^{-1})\Phi(x)$ really means $(\mathrm{id}_{\mathbb{C}} \otimes \Omega(z_1^{-1}))\Phi(x)$. We let $\mathrm{SL}_2(\mathbb{Q}_p)$ act on $\mathcal{S}(D, \Omega)$ by Weil representation:

$$r_D(g)\Phi(x) := (r_D(g) \otimes \mathrm{id}_U)\Phi(x)$$

Extend the action of $\mathrm{SL}_2(\mathbb{Q}_p)$ to

$$G^+ := \{g \in \mathrm{GL}_2(\mathbb{Q}_p) \mid \det g \in \nu(D^\times)\}$$

by

$$r_D \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \Phi(x) := |a|^{\frac{m}{4}} \Omega(z)\Phi(xz) = |a|\Omega(z)\Phi(xz)$$

if $a = \nu(z) \in \nu(D^\times)$, where $m = \dim D = 4$. Then we obtain a representation of G^+

$$r_D : G^+ \to \mathrm{GL}(\mathcal{S}(D, \Omega))$$

If $p < \infty$, one can find an unramified quadratic extension contained in D so that $\nu(D^\times) = \mathbb{Q}_p^\times$, implying that $G^+ = \mathrm{GL}_2(\mathbb{Q}_p)$. If $p = \infty$, then $G^+ = \mathrm{GL}_2(\mathbb{R})^+$ is the index two subgroup consisting of matrices with positive determinant.

11.1.1 Non-archimedean cases

Assume $p < \infty$. As said above, the extended Weil representation $r_D : \mathrm{GL}_2(\mathbb{Q}_p) \to \mathrm{GL}(\mathcal{S}(D, \Omega))$ is a representation of the whole group $\mathrm{GL}_2(\mathbb{Q}_p)$. Consider a sequence of maps

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{S}(\mathbb{Q}_p^\times) \otimes U & \longrightarrow & \mathcal{S}(D, \Omega) & \xrightarrow{\quad \ell \quad} & U \\ & & & & \Phi & \longmapsto & \Phi(0) \\ & & \xi & \longmapsto & \Phi_\xi : z \mapsto |\nu(z)|^{-1}\Omega(z^{-1})\xi(\nu(z)) & & \end{array}$$

We claim this is an exact sequence. If $\Phi_\xi = 0$, then since $\nu(D^\times) = \mathbb{Q}_p^\times$, this means $\xi = 0$ itself. Now suppose $\Phi(0) = 0$. Since Φ is locally constant, this means $\Phi \in \mathcal{S}(D^\times) \otimes \Omega$. But $\Omega(xz)\Phi(xz) = \Omega(x)\Phi(x)$ for all $z \in D_1$, so the map

$$\Omega(x)\Phi(x) : D^\times \to U$$

factors through $D^\times/D_1$, which is isomorphic to $\mathbb{Q}_p^\times$ via the reduced norm map $\nu : D^\times \to \mathbb{Q}_p^\times$. Thus we can find $\xi \in \mathbb{Q}_p^\times \to U$ such that $\Omega(x)\Phi(x) = \xi(\nu(x))$ for each $x \in D^\times$. Since Φ is locally constant, we must have $\xi \in \mathcal{S}(\mathbb{Q}_p^\times) \otimes U$, and $\Phi = \Phi_{x \mapsto |x|\xi(x)}$.

Since $\Phi(xz_1) = \Omega(z_1)^{-1}\Phi(x)$ for all $z \in D_1$, in particular $\Phi(0) \in U^{\Omega(D_1)}$.

- $\dim U = 1$. Then $\Omega(D_1) = \{1\}$, because D_1 is the commutator subgroup of $D^\times$. To see this, if $x = \overline{x}$, then $x^2 = x\overline{x} = 1$ so that $x = \pm 1$. Otherwise, $\mathbb{Q}_p(x)$ is a quadratic extension of $\mathbb{Q}_p$. In any case, as long as $x\overline{x} = 1$, there exists a quadratic subfield L of D containing x. By Hilbert's theorem 90, there exists $y \in L$ such that $x = y\overline{y}^{-1}$. Moreover by Noether-Skolem theorem we can find $\sigma \in D^\times$ such that $\sigma z \sigma^{-1} = \overline{z}$ for all $z \in L$. Thus $x = y\sigma y^{-1}\sigma$ lies in the commutator subgroup. Thus Ω factors through $D^\times/D_1$, and we can find $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ such that $\Omega = \chi \circ \nu$.
- $\dim U > 1$. Since $D_1 \subsetneq D^\times$ is a normal subgroup, $U^{\Omega(D_1)}$ is also stable under $D^\times$. Since U is irreducible, we must have $U^{\Omega(D_1)} = 0$ or $U^{\Omega(D_1)} = U$. But if the latter were to occur, Ω would factor through $D^\times/D_1$, which is an abelian group, implying $\dim U = 1$, a contradiction. Thus in this case we must have $U^{\Omega(D_1)} = 0$.

Let us assume $\dim U > 1$. Then the above discussion shows $\xi \mapsto \Phi_\xi$ is an isomorphism $\mathcal{S}(\mathbb{Q}_p^\times) \otimes U \to \mathcal{S}(D, \Omega)$. We claim

$$\Phi_{K_\psi(g)\xi} = r_D(g)\Phi_\xi$$

for all $g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. Indeed, for $x \in D^\times$

$$\begin{aligned} r_D \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \Phi_\xi(x) &= \psi(b\nu(x)) r_D \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \Phi_\xi(x) \\ (a = \nu(z)) &= \psi(b\nu(x))|a|\Omega(z)\Phi_\xi(xz) \\ &= \psi(b\nu(x))|\nu(z)|\Omega(z)|\nu(xz)|^{-1}\Omega(z^{-1}x^{-1})\xi(\nu(xz)) \\ &= \psi(b\nu(x))|\nu(x)|^{-1}\Omega(x^{-1})\xi(\nu(x)a) \\ &= |\nu(x)|^{-1}\Omega(x^{-1}) K_\psi \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \xi(\nu(x)) = \Phi_{K_\psi(g)\xi}(x) \end{aligned}$$

Thus $(r_D, \mathcal{S}(D,\Omega))$ and $(K_\psi, \mathcal{S}(\mathbb{Q}_p^\times)) \otimes U$ are isomorphic as B_1-representations. If we use this isomorphism to transfer the action of r_D to $\mathcal{S}(\mathbb{Q}_p^\times)$, we see $(r_D, \mathcal{S}(\mathbb{Q}_p^\times))$ is an irreducible supercuspidal representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ by Theorem 7.2 and Lemma 7.4. Let us put

$$\mathrm{JL}(\Omega) := (r_D, \mathcal{S}(\mathbb{Q}_p^\times))$$

Next assume $\dim U = 1$. Then we have seen $\Omega = \chi \circ \nu$ for some $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$. In this case

$$\mathcal{S}(D,\Omega) = \{\Phi \in \mathcal{S}(D) \mid \Phi(xz_1) = \Phi(x) \text{ for all } z_1 \in D_1\}$$

and we have an exact sequence

$$0 \longrightarrow \mathcal{S}(\mathbb{Q}_p^\times) \otimes U \longrightarrow \mathcal{S}(D,\Omega) \xrightarrow{\ \ell\ } \mathbb{C}$$

Consider the map $\Phi_0(x) := \mathbb{I}_{\mathbb{Z}_p^\times}(\nu(x))$. Since D_1 is compact, it is clear that $\Phi_0 \in \mathcal{S}(D,\Omega)$. We have $\ell(\Phi_0) = \Phi_0(0) = 0$ but

$$\ell(r_D(w)\Phi_0) = r_D(w)\Phi_0(0) = -\int_D \Phi_0(x)dx \neq 0 \tag{$\spadesuit$}$$

This means $\ell : \mathcal{S}(D,\Omega) \to \mathbb{C}$ is surjective, so we have an short exact sequence

$$0 \longrightarrow \mathcal{S}(\mathbb{Q}_p^\times) \otimes U \longrightarrow \mathcal{S}(D,\Omega) \xrightarrow{\ \ell\ } \mathbb{C} \longrightarrow 0$$

We claim

$$\ell\left(r_D \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \Phi \right) = \chi(ad) \left| \frac{a}{d} \right| \ell(\Phi)$$

It follows from definition that

$$r_D \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \Phi(0) = |a|\chi(a)\Phi(0), \qquad r_D \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \Phi(0) = \Phi(0)$$

and

$$\begin{aligned} r_D \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \Phi(0) &= r_D \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix} r_D \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \Phi(0) \\ &= |a^2|\chi(a^2) r_D \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \Phi(0) = |a^2|\chi(a^2)|a^{-1}|^2\Phi(0) = \chi(a^2)\Phi(0) \end{aligned}$$

If for each $\Phi \in \mathcal{S}(D, \Omega)$ we define

$$f_\Phi(g) := \ell(r_D(g)\Phi)$$

then f defines a map $\mathcal{S}(D, \Omega) \to I(\chi|\cdot|^{\frac{1}{2}}, \chi|\cdot|^{-\frac{1}{2}})$. Let us show that $\mathcal{S}(D, \Omega)$ is irreducible. Suppose V is an invariant proper subspace of $\mathcal{S}(D, \Omega)$. By definition for each $\Phi \neq 0 \in \mathcal{S}(D, \Omega)$ we can find $b \in \mathbb{Q}_p$ such that

$$r_D \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \Phi - \Phi \neq 0$$

This then implies $0 \neq V(N) \subseteq \mathcal{S}(\mathbb{Q}_p^\times) \cap V$. If $V \neq 0$, then by irreducibility of $(K_\psi, \mathcal{S}(\mathbb{Q}_p)^\times)$, this forces $\mathcal{S}(\mathbb{Q}_p^\times) = V(N) \subseteq V$, and hence $V = \mathcal{S}(\mathbb{Q}_p^\times)$ as $\mathcal{S}(\mathbb{Q}_p^\times)$ has codimension 1 in $\mathcal{S}(D, \Omega)$. But $\mathcal{S}(\mathbb{Q}_p^\times)$ is not invariant under the action of r_D as seen in (♠), this leads to a contradiction, and thus $V = 0$, showing the irreducibility of $\mathcal{S}(D, \Omega)$. Since $f : \mathcal{S}(D, \Omega) \to I(\chi|\cdot|^{\frac{1}{2}}, \chi|\cdot|^{-\frac{1}{2}})$ is nontrivial, we must have $\mathcal{S}(D, \Omega) \cong \mathrm{St} \otimes \chi$. In this case we define

$$\mathrm{JL}(\Omega) := (r_D, \mathcal{S}(S, \Omega)) \cong \mathrm{St} \otimes \chi$$

In both cases ($\dim U = 1$ or > 1), $\mathrm{JL}(\Omega)$ is an irreducible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ satisfying

$$S(D, \Omega) \cong \mathrm{JL}(\Omega) \otimes_{\mathbb{C}} U$$

The association

$$\mathrm{JL} : \mathrm{Rep}(D^\times) \longrightarrow \mathrm{Rep}(\mathrm{GL}_2(\mathbb{Q}_p))$$

is called the **Jacquet-Langlands correspondence** of Ω.

11.2 Quadratic extensions

Suppose $K/\mathbb{Q}_p$ is a quadratic field extension. Denote by $z \mapsto \overline{z}$ the nontrivial element in the Galois group $\mathrm{Gal}(K/\mathbb{Q}_p)$. Then $(K, N = N_{K/\mathbb{Q}_p})$ is a quadratic space of dimension 2. If $K = \mathbb{Q}_p(\sqrt{D})$, then $\det K = -4D$, so for each $a \in \mathbb{Q}_p^\times$, we have $(-\det K, a)_p = 1$ if $a \in NK^\times$, and -1 otherwise. For convenience, write $\tau_{K/\mathbb{Q}_p}(a) = (-\det K, a)_p$.

Let $\lambda : K^\times \to \mathbb{C}$ be a character and define

$$\mathcal{S}(K, \lambda) = \left\{\Phi \in \mathcal{S}(K) \mid \Phi(xz_1) = \lambda(z_1)^{-1}\Phi(x) \text{ for all } z_1 \in K_1\right\}$$

where K_1 is the set of norm one element in K. As before, we let the Weil representation r_K act on $\mathcal{S}(K, \lambda)$, and extend it to a representation of the subgroup $G^+ := \{g \in \mathrm{GL}_2(\mathbb{Q}_p) \mid \det g \in N_{K/\mathbb{Q}}K^\times\}$ by means of the formula

$$r_K \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \Phi(x) := |a|^{\frac{1}{2}} \lambda(z) \Phi(xz)$$

if $a = N_{K/\mathbb{Q}_p}(z) \in \mathbb{Q}_p^+ := N_{K/\mathbb{Q}_p}K^\times \subseteq \mathbb{Q}_p^\times$. Again, consider the maps

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{S}(\mathbb{Q}_p^+) & \longrightarrow & \mathcal{S}(K, \lambda) & \xrightarrow{\quad \ell \quad} & \mathbb{C} \\ & & & \Phi & \longmapsto & \Phi(0) \\ & \xi & \longmapsto & \Phi_\xi : z \mapsto |N(z)|^{-\frac{1}{2}} \lambda(z^{-1}) \xi(N(z)) & & \end{array}$$

This is an exact sequence, which can be proved in the same way as in the quaternion case. Define the subgroup $B_1^+ = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{Q}_p^+,\, b \in \mathbb{Q} \right\} \leqslant B_1$. Then, as a subspace of $\mathcal{S}(K, \lambda)$, the space $\mathcal{S}(\mathbb{Q}_p^+)$ is invariant under the action of G^+, and $(r_K|_{B_1^+}, \mathcal{S}(\mathbb{Q}_p^+)) = (K_\psi|_{B_1^+}, \mathcal{S}(\mathbb{Q}_p^+))$.

Define $\widetilde{\mathcal{S}}(K, \lambda) := \operatorname{Ind}_{G^+}^{G}(\mathcal{S}(K, \lambda), r_K)$. Consider the map

$$\begin{aligned} \operatorname{Ind}_{G^+}^{G}(\mathcal{S}(\mathbb{Q}_p^+), r_K) &\longrightarrow (K_\psi, \mathcal{S}(\mathbb{Q}_p^\times)) \\ f : G \to \mathcal{S}(\mathbb{Q}_p^+) &\longmapsto \xi_f(a) := f\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)(1) \end{aligned}$$

We claim this is well-defined and is an isomorphism as B_1^+ representations. Let $L : \mathcal{S}(\mathbb{Q}_p^+) \to \mathbb{C}^\times$ be the evaluation map at 1. Then

$$\xi_f(a) = L\left(\rho\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} f(e)\right)$$

and for $\alpha \in \mathbb{Q}_p^+$, we have

$$\begin{aligned} f\left(\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}\right)(\alpha) &= K_\psi\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right) f\left(\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}\right)(1) = r_K\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right) f\left(\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}\right)(1) \\ &= L\left(f\left(\begin{pmatrix} a\alpha & x\alpha \\ 0 & 1 \end{pmatrix}\right)\right) = L\left(K_\psi\begin{pmatrix} 1 & x\alpha \\ 0 & 1 \end{pmatrix} f\left(\begin{pmatrix} a\alpha & 0 \\ 0 & 1 \end{pmatrix}\right)\right) = \psi(x\alpha)\xi_f(a\alpha) \end{aligned}$$

This shows $\xi_f \in \mathcal{S}(\mathbb{Q}_p^\times)$, and since $\mathbb{Q}_p^\times/\mathbb{Q}_p^+$ is finite, we find $f \mapsto \xi_f$ is injective. Also, $f \mapsto \xi_f$ is B_1-intertwining, so the irreducibility of $(K_\psi, \mathcal{S}(\mathbb{Q}_p)^\times)$ implies this is a B_1-isomorphism. In particular, this shows

If $\lambda|_{K_1} \neq \mathbf{1}$, then since $\lambda(z_1)\Phi(x) = \Phi(xz_1)$, we find $\Phi(0) = 0$ for all $\Phi \in \mathcal{S}(K, \lambda)$. Thus $\mathcal{S}(\mathbb{Q}_p^+) \cong \mathcal{S}(K, \lambda)$, and $\mathcal{S}(\mathbb{Q}_p^\times) \cong \operatorname{Ind}_{G^+}^{G} \mathcal{S}(\mathbb{Q}_p^+) \cong \widetilde{\mathcal{S}}(K, \lambda)$ as B_1-representations. In this case, we find $\widetilde{\mathcal{S}}(K, \lambda)$ is supercuspidal.

Suppose $\lambda|_{K_1} = \mathbf{1}$. Then we can find a character $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ with $\lambda = \chi \circ N_{K/\mathbb{Q}_p}$. We are to construct a non-trivial $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant map from $\widetilde{\mathcal{S}}(K, \lambda)$ to $I(\chi, \chi\tau_{K/\mathbb{Q}_p})$. For this, pick any $\delta \in \mathbb{Q}_p^\times \backslash \mathbb{Q}_p^+$ and define

$$\begin{aligned} \widetilde{\ell} : \widetilde{\mathcal{S}}(K, \lambda) &\longrightarrow \mathbb{C} \\ \widetilde{\Phi} &\longmapsto \widetilde{\ell}(\widetilde{\Phi}) := \chi(\delta)\ell(\widetilde{\Phi}(1)) + \ell\left(\widetilde{\Phi}\left(\begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix}\right)\right) \end{aligned}$$

Then $\widetilde{\ell}$ is not trivial on $\mathcal{S}(K, \lambda)$, and the map $\widetilde{\Phi} \mapsto [g \mapsto \widetilde{\ell}(\rho(g)\widetilde{\Phi})]$ is what we want. It remains to show $\widetilde{\mathcal{S}}(K, \lambda)$ is irreducible as $\mathrm{GL}_2(\mathbb{Q}_p)$ representations.

12 Global Theory

Lemma 12.1. Let $p < \infty$ and (π, V) be an irreducible representation of $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Put $K_p = \mathrm{GL}_2(\mathbb{Z}_p)$. If $V^{K_p} \neq 0$, then $\dim_{\mathbb{C}} V^{K_p} = 1$.

Proof. Recall that we have the algebra

$$\mathcal{H}(G, K_p) = \{\phi \in \mathcal{S}(G) \mid \phi(k_1 g k_2) = \phi(g) \text{ for } k_i \in K_p,\, g \in G\}$$

By Lemma 3.3.(iii) and (iv), V^{K_p} is a simple $\mathcal{H}(G, K_p)$-module.

Lemma 12.2 (Cartan decompsition)**.** We have

$$\mathrm{GL}_2(\mathbb{Q}_p) = \bigsqcup_{x \geqslant y} K_p \begin{pmatrix} p^x & 0 \\ 0 & p^y \end{pmatrix} K_p$$

Then $\mathcal{H}(G, K_p)$ is spanned by the characteristic functions of $K_p \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} K_p$ over $\mathbb{C}$, and hence for $\phi \in \mathcal{H}(G, K_p)$, we have $\phi^t(g) := \phi(g^t) = \phi(g)$ for all $g \in G$, i.e., $\phi^t = \phi$.

On the other hand, since G is unimodular, a direct computation shows $(\phi_1 * \phi_2)^t = \phi_2^t * \phi_1^t$ for all $\phi_i \in \mathcal{S}(G)$. Hence,

$$\phi_1 * \phi_2 = (\phi_1 * \phi_2)^t = \phi_2^t * \phi_1^t = \phi_2 * \phi_1$$

that is, $\mathcal{H}(G, K_p)$ is a commutative ring. Since V^{K_p} is a simple module over a commutative ring, we must have $\dim_{\mathbb{C}} V^{K_p} = 1$. ◻

12.1 Representations of $\mathrm{GL}_2(\mathbb{A})$

Denote by $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$ the ring of adeles over $\mathbb{Q}$. Define

$$\mathrm{GL}_2(\mathbb{A}) := \left\{ (g_p) \in \prod_{p \leqslant \infty} \mathrm{GL}_2(\mathbb{Q}_p) \mid g_p \in \mathrm{GL}_2(\mathbb{Z}_p) \text{ for all finitely many } p \right\}$$

For finite prime p, put $K_p = \mathrm{GL}_2(\mathbb{Z}_p)$ and let (π_p, V_p) be an irreducible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. For $p = \infty$, let (π_∞, V_∞) be an irreducible $(\mathfrak{g}_\infty, K_\infty)$-module, where $\mathfrak{g}_\infty = \mathrm{Lie}(\mathrm{GL}_2(\mathbb{R}))$ and $K_\infty = \mathrm{O}(2)$.

(♠) Assume that $V^{K_p} \neq 0$ for all but finitely many p. Define

$$V := \bigotimes_{p \leqslant \infty}{}' V_p = \varinjlim_{\substack{S \subseteq M_{\mathbb{Q}} \\ \# S < \infty}} \left(\bigotimes_{p \in S} V_p \otimes \bigotimes_{p \notin S} V_p^{K_p} \right)$$

Let S_0 be a finite set of primes containing ∞. For $p \notin S_p$, since we are assuming $V_p^{K_p} \neq 0$, by Lemma 12.1, we have $V_p^{K_p} = \mathbb{C} \cdot \xi_p^\circ$. Then

$$V = \mathrm{span}_{\mathbb{C}} \left\{ \otimes_{p \in S} v_p \otimes_{p \notin S} \xi_p^\circ \mid v_p \in V_p,\, S \supseteq S_0,\, \# S < \infty \right\}$$

Then

$$\pi := \otimes' \pi_p : \mathrm{GL}_2(\mathbb{A}) \to \mathrm{GL}(V)$$

is an representation of $\mathrm{GL}_2(\mathbb{A})$, or more precisely, a representation of $(\mathfrak{g}_\infty, K_\infty) \times \prod_{p < \infty}{}' \mathrm{GL}_2(\mathbb{Q}_p)$.

Definition. We say (π, V) is an **irreducible representation of** $\mathrm{GL}_2(\mathbb{A})$ if $(\pi, V) \cong \left(\otimes'\pi_p, \bigotimes_{p\leqslant\infty}' V_p\right)$ with each (π_p, V_p) irreducible and $\{(\pi_p, V_p)\}_{p\leqslant\infty}$ satisfying (♠).

For $(g_p) = (a_{ij}) \in \mathrm{GL}_2(\mathbb{Q}_p)$, define

$$\|g_p\|_p := \begin{cases} \sum_{i,j} |a_{ij}|_\infty^2 & , \text{ if } p = \infty \\ \max_{i,j} |a_{ij}|_p & , \text{ if } p < \infty \end{cases}$$

For $g = (g_p) \in \mathrm{GL}_2(\mathbb{A})$, define

$$\|g\| = \prod_{p\leqslant\infty} \|g_p\|_p$$

which is well-defined since for all but finitely many g_p, we have $\|g_p\|_p \leqslant 1$.

Definition. A function $\phi : \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$ is called an **automorphic form** on $\mathrm{GL}_2(\mathbb{A})$ if

(i) ϕ is K-finite, where $K = \prod_{p\leqslant\infty} K_p$;

(ii) ϕ is **smooth**, i.e. there exists an open compact $U \subseteq \prod_{p<\infty} K_p$ such that

- $\phi(gu) = \phi(g)$ for all $u \in U$, and
- for all $g_f \in \mathrm{GL}_2(\mathbb{A}_f) = \prod_{p<\infty}' \mathrm{GL}_2(\mathbb{Q}_p)$, the map

$$\begin{aligned} \mathrm{GL}_2(\mathbb{R}) &\longrightarrow \mathbb{C} \\ g_\infty &\longmapsto \phi(g_\infty g_f) \end{aligned}$$

is smooth;

(iii) ϕ is **slowing increasing**, i.e. there exist $M_1, M_2 > 0$ such that

$$|\phi(g)| \leqslant M_2 \|g\|^{M_1}$$

for all $g \in \mathrm{GL}_2(\mathbb{A})$;

(iv) $\phi(rg) = \phi(g)$ for all $r \in \mathrm{GL}_2(\mathbb{Q})$ (this is why ϕ is called automorphic);

(v) ϕ is $\mathcal{Z}$-finite, where $\mathcal{Z} = \mathbb{C}[J, \Delta] \subseteq U(\mathfrak{g}_\mathbb{C})$, $J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and Δ is the Casimir element of $\mathfrak{sl}_2(\mathbb{R})$.

We denote by $\mathcal{A}(\mathrm{GL}_2(\mathbb{A}))$ the space of automorphic forms on $\mathrm{GL}_2(\mathbb{A})$. Then $\mathcal{A}(\mathrm{GL}_2(\mathbb{A}))$ is a representation of $\mathrm{GL}_2(\mathbb{A})$ under the right translation.

In the following, let us put $G = \mathrm{GL}_2$, and $\mathcal{A}(\mathrm{GL}_2(\mathbb{A})) = \mathcal{A}(G)$.

Definition. An irreducible representation (π, V) of $\mathrm{GL}_2(\mathbb{A})$ is **automorphic** if $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{A})}(\pi, \mathcal{A}(G)) \neq 0$.

Definition. A continuous character $\omega : \mathbb{Q}^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ is called a **Hecke character** of $\mathbb{Q}$.

We write

$$\mathcal{A}(G, \omega) = \left\{\phi \in \mathcal{A}(G) \mid \phi(zg) = \omega(z)\phi(g) \text{ for all } z \in \mathbb{A}^\times\right\}$$

to be the space of automorphic forms of $\mathrm{GL}_2(\mathcal{A})$ with central character ω. Then

$$\mathcal{A}(G) = \bigoplus_{\omega\ :\ \text{Hecke}} \mathcal{A}(G, \omega)???$$

and a smooth function $\phi : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$ with central character ω is automorphic if and only if ϕ is K-finite, $\mathcal{Z}$-finite and slowly decreasing. A representation $\pi = \otimes'\pi_p$ is automorphism if and only if $\mathrm{Hom}_{G(\mathbb{A})}(\pi, \mathcal{A}(G, \omega)) \neq 0$ for some Hecke character ω of $\mathbb{Q}$.

12.2 Siegel Set

If $(x,y) \in \mathbb{Q}_p^2$, define

$$\|(x,y)\|_p := \begin{cases} \max\{|x|_p, |y|_p\} & \text{, if } p < \infty \\ \sqrt{|x|_\infty^2 + |y|_\infty^2} & \text{if } p = \infty \end{cases}$$

and for $(x,y) \in \mathbb{A}^2$, define

$$\|(x,y)\| := \prod_{p \leqslant \infty} \|(x_p, y_p)\|_p$$

Then $\|\cdot\| : \mathbb{A}^\times \to \mathbb{R}_{>0}$ is a continuous map.

We list some facts.

- For $\alpha \in \mathbb{Q}^\times \subseteq \mathbb{A}^\times$, we have

$$|\alpha| := \prod_{p \leqslant \infty} |\alpha|_p = 1$$

  This is the **product formula**. In other words, $|\cdot| : \mathbb{A}^\times \to \mathbb{R}_{>0}$ factors through $\mathbb{Q}^\times \backslash \mathbb{A}^\times$.

- $\mathbb{A} = \mathbb{Q} + [0,1] \times \prod_{p<\infty} \mathbb{Z}_p$.

- Put $(\mathbb{A}^\times)^0 = \{x \in \mathbb{A}^\times \mid |x| = 1\}$. Then $(\mathbb{A}^\times)^0 = \mathbb{Q}^\times \left(\{\pm 1\} \times \prod_{p<\infty} \mathbb{Z}_p^\times \right)$.

- $\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{Q}) \left(\mathrm{GL}_2(\mathbb{R}) \times \prod_{p<\infty} \mathrm{GL}_2(\mathbb{Z}_p) \right)$.

Lemma 12.3. There exists $c_0 > 0$ such that for any $g \in \mathrm{GL}_2(\mathbb{A})$ there exists $\gamma \in \mathrm{GL}_2(\mathbb{Q})$ such that

$$\|(0,1)\gamma g\| < c_0 |\det g|^{\frac{1}{2}}$$

where $\det : \mathrm{GL}_2(\mathbb{A}) \to \mathbb{A}^\times$.

Put

$$B^0(\mathbb{A}) = \left\{ \begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} \mid a_i \in (\mathbb{A}^\times)^0,\, x \in \mathbb{A} \right\}$$

By product formula, we have $B(\mathbb{Q}) \subseteq B^0(\mathbb{A})$. Since $\mathbb{Q}\backslash\mathbb{A}$ and $\mathbb{Q}^\times \backslash (\mathbb{A}^\times)^0$ are compact, $B(\mathbb{Q})\backslash B^0(\mathbb{A})$ is compact as well. In particular, we can find compact $\Omega_0 \subseteq B^0(\mathbb{A})$ such that

$$B^0(\mathbb{A}) = B(\mathbb{Q})\Omega_0$$

In fact, we can take

$$\Omega_0 = \left\{ \begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} \mid a_i \in \{\pm 1\} \times \prod_{p<\infty} \mathbb{Z}_p^\times,\, x \in [-1,1] \times \prod_{p<\infty} \mathbb{Z}_p \right\}$$

For $c > 0$, we define the **Siegel set** to be

$$\mathfrak{S}(\Omega_0, c) := \left\{ b \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k \mid b \in \Omega_0,\, a \in \mathbb{R}^\times,\, |a| > c,\, k \in K \right\}$$

where $K = \mathrm{O}(2) \times \prod_{p<\infty} \mathrm{GL}_2(\mathbb{Z}_p)$.

Theorem 12.4. There exists $c > 0$ such that

$$\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{Q})\mathbb{R}_+\mathfrak{S}(\Omega_0, c)$$

where $\mathbb{R}_+ \subseteq \mathrm{GL}_2(\mathbb{R}) \subseteq \mathrm{GL}_2(\mathbb{A})$.

Lemma 12.5. Take $c > 0$ be as in Theorem 12.4. The set

$$\left\{r \in \mathbb{Q}^\times \backslash \mathrm{GL}_2(\mathbb{Q}) \mid r\mathfrak{S} \cap \mathbb{A}^\times \mathfrak{S} \neq \varnothing\right\}$$

is finite, where $\mathfrak{S} = \mathfrak{S}(\Omega_0, c)$ is the Siegel set.

Corollary 12.5.1. Let $\omega : \mathbb{Q}^\times \backslash \mathbb{A}^\times \to S^1$ be a unitary Hecke character of $\mathbb{Q}$ and $\phi \in \mathcal{A}(G, \omega)$. If there exists $m < 1$ and c_1 such that

$$|\phi(g)| \leqslant c_1 \|g\|^m$$

for all $g \in \mathrm{GL}_2(\mathbb{A})$, then $\phi \in L^1(\mathbb{A}^\times G(\mathbb{Q})\backslash G(\mathbb{A}))$, i.e.,

$$\int_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} |\phi(g)| dg < \infty$$

That $|\phi|$ is a function on $Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})$ results from that ϕ is automorphic and ω is unitary.

Proof. Let $\mathfrak{S}$ be the Siegel set as in the previous lemma, and $\pi_1 : G \to Z(\mathbb{A})\backslash G(\mathbb{A})$ be the projection. Put

$$\mathfrak{S}' = \pi_1(\mathfrak{S}) \subseteq Z(\mathbb{A})\backslash G(\mathbb{A})$$

and for $g \in Z(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})$, define

$$A_{\mathfrak{S}}(g) := \sum_{r \in \mathbb{Q}^\times \backslash G(\mathbb{Q})} \mathbb{I}_{\mathfrak{S}'}(rg)$$

Formally, it descents to a map for $g \in Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})$. To see this sum is well-defined, by Theorem 12.4, the projection $\mathfrak{S} \to Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})$ is surjective, so for $g \in Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})$ we can choose $x \in \mathfrak{S}$ with $A_{\mathfrak{S}}(g) = A_{\mathfrak{S}}(x)$. Then

$$\{r \in \mathbb{Q}^\times\backslash G(\mathbb{Q}) \mid rx \in \mathfrak{S}'\} = \{r \in \mathbb{Q}^\times\backslash G(\mathbb{Q}) \mid \mathbb{A}^\times\mathfrak{S} \cap \mathbb{A}^\times rx \neq \varnothing\} \subseteq \{r \in \mathbb{Q}^\times\backslash G(\mathbb{Q}) \mid \mathbb{A}^\times\mathfrak{S} \cap r\mathfrak{S} \neq \varnothing\}$$

The last set above is finite by the previous lemma, so the sum $\sum_{r \in \mathbb{Q}^\times\backslash G(\mathbb{Q})} \mathbb{I}_{\mathfrak{S}'}(rg)$ is actually a finite sum; this shows $A_{\mathfrak{S}}(g)$ is well-defined. Now $A_{\mathfrak{S}}(g) \geqslant \mathbb{I}_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})}(g)$, so

$$\begin{aligned}
\int_{Z(\mathbb{A})\backslash G(\mathbb{A})} |\phi(g)|\mathbb{I}_{\mathfrak{S}'}(g) dg &= \int_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} \sum_{r \in \mathbb{Q}^\times\backslash G(\mathbb{Q})} |\phi(rg)|\mathbb{I}_{\mathfrak{S}'}(rg) dg \\
&= \int_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} |\phi(g)| A_{\mathfrak{S}}(g) dg \\
&\geqslant \int_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} |\phi(g)| dg
\end{aligned}$$

It suffices to show $\int_{Z(\mathbb{A})\backslash G(\mathbb{A})} |\phi(g)|\mathbb{I}_{\mathfrak{S}'}(g) dg < \infty$. By assumption, we have

$$\int_{Z(\mathbb{A})\backslash G(\mathbb{A})} |\phi(g)|\mathbb{I}_{\mathfrak{S}'}(g) dg \leqslant c_1 \int_c^\infty |t|^{m-1} d^\times t \,\mathrm{vol}(\Omega_0 K) \text{ ????}$$

The last integral is finite if $m < 1$, so the result follows. $\square$

12.3 Cusp forms

Definition. Let $N = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$. For $\phi \in \mathcal{A}(G)$, define

$$\phi_N(g) := \int_{\mathbb{Q}\backslash\mathbb{A}} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx$$

This is called the **constant term** of ϕ (along N). Here dx is the quotient measure of the Haar measure on $\mathbb{A}$ normalized so that $\mathrm{vol}([0,1] \times \prod_{p<\infty} \mathbb{Z}_p) = 1$ by the counting measure on $\mathbb{Q}$. An automorphic form ϕ is called **cuspidal**, or a **cusp form** if $\phi_N = 0$.

Proposition 12.6. If ϕ is a cusp form, then ϕ is **rapidly decreasing**, i.e., for all $m \in \mathbb{Z}$ there exists c_m such that

$$|\phi(g)| \leqslant c_m \|g\|^m$$

12.4 Poisson summation formula

For each $p \leqslant \infty$, let $\psi_p : \mathbb{Q}_p \to \mathbb{C}^\times$ be the standard additive character. Define

$$\psi_{\mathbb{A}} = \prod_{p \leqslant \infty} : \mathbb{A} \to \mathbb{C}^\times$$

By definition, one can show $\psi_{\mathbb{A}}(x + \alpha) = \psi_{\mathbb{A}}(x)$ for all $\alpha \in \mathbb{Q}$, so it induces a map on the quotient $\psi_{\mathbb{A}} : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$.

For each $p < \infty$ we fix the element $\mathbb{I}_{\mathbb{Z}_p} \in \mathcal{S}(\mathbb{Q}_p)$. Form the restricted tensor product $\mathcal{S}(\mathbb{A}) = \bigotimes_{p \leqslant \infty}' \mathcal{S}(\mathbb{Q}_p)$. For each $\Phi \in \mathcal{S}(\mathbb{A})$, define its **Fourier transform**

$$\widehat{\Phi}(x) := \int_{\mathbb{A}} \Phi(y) \psi_{\mathbb{A}}(xy) dy$$

The Fourier transform induces a bijection on $\mathcal{S}(\mathbb{A})$.

Theorem 12.7. For $\Phi \in \mathcal{S}(\mathbb{A})$, we have

$$\sum_{\alpha \in \mathbb{Q}} \Phi(\alpha) = \sum_{\alpha \in \mathbb{Q}} \widehat{\Phi}(\alpha)$$

Proof. Define $f : \mathbb{A} \to \mathbb{C}$ by

$$f(x) = \sum_{\alpha \in \mathbb{Q}} \Phi(\alpha + x)$$

This series converges absolutely and compactly, so it defines a continuous function on $\mathbb{A}$. To see this, let us assume $\Phi(x) = \Phi_\infty(x_\infty)\Phi_f(x_f)$ with $\Phi_\infty \in \mathcal{S}(\mathbb{R})$, $\Phi_f \in \mathcal{S}(\mathbb{A}_{\mathrm{fin}})$. Since Φ_f has compact support, by prime factorization there exists a discrete subgroup $\Lambda \leqslant \mathbb{R}$ such that if $\alpha \in \mathbb{Q}$, then $\Phi_f(\alpha_f) = 0$ unless $\alpha_\infty \in \Lambda$. Now it suffices to show $\sum_{\alpha \in \Lambda} \Phi_\infty(\alpha_\infty + x_\infty)$ converges absolutely and compactly in x_∞. This is easy.

Since it is periodic, it induces a continuous map $f : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}$, by abuse of notation. Since $\widehat{\mathbb{Q}\backslash\mathbb{A}} = \{\psi_\alpha : x \mapsto \psi_{\mathbb{A}}(\alpha x) \mid \alpha \in \mathbb{Q}\}$ and $\mathcal{A}\backslash\mathbb{Q}$ is compact abelian, we have the Fourier expansion

$$f(x) = \sum_{\alpha \in \mathbb{Q}} a_\alpha \psi_\alpha(x)$$

with $a_\alpha = \int_{\mathbb{Q}\backslash\mathbb{A}} f(x)\psi_\alpha(-x)dx$. We compute the coefficients a_α.

$$\begin{aligned}
a_\alpha &= \int_{\mathbb{Q}\backslash\mathbb{A}} f(x)\psi_\alpha(-x)dx = \int_{\mathbb{Q}\backslash\mathbb{A}} f(x)\psi_{\mathbb{A}}(-\alpha x)dx \\
&= \int_{\mathbb{Q}\backslash\mathbb{A}} \sum_{\beta\in\mathbb{Q}} \Phi(x+\beta)\psi_{\mathbb{A}}(-\alpha x)dx \\
&= \int_{\mathbb{Q}\backslash\mathbb{A}} \sum_{\beta\in\mathbb{Q}} \Phi(x+\beta)\psi_{\mathbb{A}}(-\alpha(x+\beta))dx \\
&= \int_{\mathbb{A}} \Phi(x)\psi_{\mathbb{A}}(-\alpha x)dx = \widehat{\Phi}(-\alpha)
\end{aligned}$$

Thus

$$f(x) = \sum_{\alpha\in\mathbb{Q}} \widehat{\Phi}(-\alpha)\psi_\alpha(x)$$

The right hand side defines a continuous function as well, so this equality holds everywhere in $x \in \mathbb{A}$. Taking $x = 0$, we obtain

$$\sum_{\alpha\in\mathbb{Q}} \Phi(\alpha) = f(0) = \sum_{\alpha\in\mathbb{Q}} \widehat{\Phi}(-\alpha)$$

□

For $\Phi \in \mathcal{S}(\mathbb{A}^n)$, we can similarly define $\widehat{\Phi} : \mathbb{A}^n \to \mathbb{C}$ by

$$\widehat{\Phi}(x) = \int_{\mathbb{A}^n} \Phi(y)\psi_{\mathbb{A}}(x\cdot y)dy$$

where $x \cdot y = x_1y_1 + \cdots + x_ny_n$ if $x = (x_n)$, $y = (y_n)$. In this way we still have the Poisson summation formula

$$\sum_{\alpha\in\mathbb{Q}^n} \Phi(\alpha) = \sum_{\alpha\in\mathbb{Q}^n} \widehat{\Phi}(\alpha)$$

Let $\Phi \in \mathcal{S}(\mathbb{A}^n)$ and $a \in \mathbb{A}^\times$. Define $\Phi_a \in \mathcal{S}(\mathbb{A}^n)$ by $\Phi_a(x) := \Phi(ax)$. We compute its Fourier transform.

$$\begin{aligned}
\widehat{\Phi_a}(x) &= \int_{\mathbb{A}^n} \Phi_a(y)\psi_{\mathbb{A}}(x\cdot y)dy = \int_{\mathbb{A}^n} \Phi(ay)\psi_{\mathbb{A}}(x\cdot y)dy \\
(y \mapsto a^{-1}y) &= \int_{\mathbb{A}^n} \Phi(y)\psi(a^{-1}xy)|a|^{-n}dy = |a|^{-n}\widehat{\Phi}(a^{-1}x)
\end{aligned}$$

Thus we have the following (slight) generalization of Poisson summation formula.

Theorem 12.8. For $\Phi \in \mathcal{S}(\mathbb{A}^n)$ and $a \in \mathbb{A}^\times$, we have

$$\sum_{\alpha\in\mathbb{Q}^n} \Phi(a\alpha) = \frac{1}{|a|^n} \sum_{\alpha\in\mathbb{Q}^n} \widehat{\Phi}(a^{-1}\alpha)$$

12.5 Eisenstein series

Let $\chi_1, \chi_2 : \mathbb{Q}^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ be two Hecke characters of $\mathbb{Q}$; then they together define a character $\chi = (\chi_1, \chi_2) : B(\mathbb{A}) \to \mathbb{C}$ For $\Phi \in \mathcal{S}(\mathbb{A}^2)$, define the **Godement section** $f_{\Phi,\chi,s} : G(\mathbb{A}) \to \mathbb{C}$ by the formula

$$f_{\Phi,\chi,s}(g) := \chi_1|\cdot|^{s+\frac{1}{2}}(\det g) \int_{\mathbb{A}^\times} \Phi((0,t)g)\chi_1\chi_2^{-1}|\cdot|^{2s+1}(t)d^\times t$$

where $d^\times t = \prod_{p\leqslant\infty} d^\times t_p$. If $\Phi = \bigotimes'_{p\leqslant\infty} \Phi_p$ (with $\Phi_p = \mathbb{I}_{\mathbb{Z}_p\times\mathbb{Z}_p}$ for almost all $p < \infty$), we have

$$f_{\Phi,\chi,s}(g) = \prod_{p\leqslant\infty} f_{\Phi_p,\chi_p,s}(g_p)$$

For $\Phi \in \mathcal{S}(\mathbb{A}^2)$, $s \in \mathbb{C}$, $g \in G(\mathbb{A})$, define the **Eisenstein series**

$$E_\chi(\Phi, s, g) := \sum_{r\in B(\mathbb{Q})\backslash G(\mathbb{Q})} f_{\Phi,\chi,s}(rg)$$

Ignoring the convergence issue, we see that $g \mapsto E_\chi(\Phi, s, g)$ is automorphic, i.e., $E_\chi(\Phi, s, rg) = E_\chi(\Phi, s, g)$ for all $r \in G(\mathbb{Q})$.

Theorem 12.9. Suppose $|\chi_1\chi_2^{-1}| = |\cdot|^\rho$ for some $\rho \in \mathbb{R}$.

(i) The series $E_\chi(\Phi, s, g)$ converges absolutely if $\mathrm{Re}(s) > \dfrac{1-\rho}{2}$.

(ii) $E_\chi(\Phi, s, g)$ has a meromorphic continuation to $\mathbb{C}$ and satisfies the functional equation

$$E_\chi(\Phi, s, g) = E_{\chi^{\mathrm{sw}}}(\widehat{\Phi}, -s, g)$$

where $\chi^{\mathrm{sw}} = (\chi_2, \chi_1)$.

(iii) $E_\chi(\Phi, s, g)$ is entire if $\chi_1\chi_2^{-1}$ is not of the form $|\cdot|^{s_0}$ for some $s_0 \in \mathbb{C}$, and has only a simple pole at $s = \dfrac{-\rho \pm 1}{2}$??? if $\chi_1\chi_2^{-1} = |\cdot|^{\rho+it}$ for some $t \in \mathbb{R}$.

In fact, one can show $E_\chi(\Phi, s, g) \in \mathcal{A}(G, \chi_1\chi_2)$ is an automorphic form.

Proof. We have the Bruhat decomposition

$$G(\mathbb{Q}) = B(\mathbb{Q}) \bigsqcup_{\alpha\in\mathbb{Q}} B(\mathbb{Q}) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

so

$$B(\mathbb{Q})\backslash G(\mathbb{Q}) = \left\{ e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \mid \alpha \in \mathbb{Q} \right\}$$

Since $E_\chi(\Phi,s,g)=E_\chi(\rho(g)\Phi,s,e)$, we may assume $g=e$. Then formally

$$\begin{aligned}
E_\chi(\Phi,s,e) &= f_{\Phi,\chi,s}(e)+\sum_{\alpha\in\mathbb{Q}} f_{\Phi,\chi,s}\left(\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}1&\alpha\\0&1\end{pmatrix}\right)\\
&= \int_{\mathbb{A}^\times}\Phi(0,t)\chi_1\chi_2^{-1}|\cdot|^{2s+1}(t)d^\times t+\sum_{\alpha\in\mathbb{Q}}\int_{\mathbb{A}^\times}\Phi(t,t\alpha)\chi_1\chi_2^{-1}|\cdot|^{2s+1}(t)d^\times\\
&= \int_{\mathbb{Q}^\times\backslash\mathbb{A}^\times}\sum_{\beta\in\mathbb{Q}^\times}\Phi(0,\beta t)\chi_1\chi_2^{-1}|\cdot|^{2s+1}(\beta t)d^\times t+\sum_{\alpha\in\mathbb{Q}}\int_{\mathbb{Q}^\times\backslash\mathbb{A}^\times}\sum_{\beta\in\mathbb{Q}^\times}\Phi(t\beta,t\beta\alpha)\chi_1\chi_2^{-1}|\cdot|^{2s+1}(t\beta)d^\times t\\
&= \int_{\mathbb{Q}^\times\backslash\mathbb{A}^\times}\sum_{0\neq\xi\in\mathbb{Q}^2}\Phi(t\xi)\chi_1\chi_2^{-1}|\cdot|^{2s+1}(t)d^\times t\\
&= \int_{|t|>1}\sum_{0\neq\xi\in\mathbb{Q}^2}\Phi(t\xi)\chi_1\chi_2^{-1}|\cdot|^{2s+1}(t)d^\times t\\
&\quad+\int_{|t|<1}\sum_{\xi\in\mathbb{Q}^2}\Phi(t\xi)\chi_1\chi_2^{-1}|\cdot|^{2s+1}(t)d^\times t-\int_{|t|<1}\Phi(0)\chi_1\chi_2^{-1}|\cdot|^{2s+1}(t)d^\times t\\
&\overset{12.8}{=} \int_{|t|>1}\sum_{0\neq\xi\in\mathbb{Q}^2}\Phi(t\xi)\chi_1\chi_2^{-1}|\cdot|^{2s+1}(t)d^\times t\\
&\quad+\int_{|t|<1}\sum_{\xi\in\mathbb{Q}^2}\widehat{\Phi}(t^{-1}\xi)|t|^{-2}\chi_1\chi_2^{-1}|\cdot|^{2s+1}(t)d^\times t-\int_{|t|<1}\Phi(0)\chi_1\chi_2^{-1}|\cdot|^{2s+1}(t)d^\times t\\
(t\mapsto t^{-1}) &= \underbrace{\int_{|t|>1}\left(\sum_{0\neq\xi\in\mathbb{Q}^2}\Phi(t\xi)\right)\chi_1\chi_2^{-1}|\cdot|^{2s+1}(t)d^\times t}_{\text{(A)}}+\underbrace{\int_{|t|>1}\left(\sum_{0\neq\xi\in\mathbb{Q}^2}\widehat{\Phi}(t^{-1}\xi)\right)\chi_1^{-1}\chi_2|\cdot|^{1-2s}(t)d^\times t}_{\text{(B)}}\\
&\quad\underbrace{-\Phi(0)\int_{|t|<1}\chi_1\chi_2^{-1}|\cdot|^{2s+1}(t)d^\times t}_{\text{(C)}}+\underbrace{\widehat{\Phi}(0)\int_{|t|<1}\chi_1\chi_2^{-1}|\cdot|^{2s-1}(t)d^\times t}_{\text{(D)}}
\end{aligned}$$

For (A) and (B), the parenthetical terms are rapidly decreasing in t, so the integrals converge absolutely for all $s\in\mathbb{C}$ (note that $\int_{|t|>1}=\int_1^\infty\int_{\mathbb{Q}^\times\backslash(\mathbb{A}^\times)^0}$ and recall that $\mathbb{Q}^\times\backslash(\mathbb{A}^\times)^0$ is compact). For (C)

$$\Phi(0)\int_{|t|<1}\chi_1\chi_2^{-1}|\cdot|^{2s+1}(t)d^\times t=\Phi(0)\int_0^1\int_{\mathbb{Q}^\times\backslash(\mathbb{A}^\times)^0}\chi_1\chi_2^{-1}|\cdot|^{2s+1}(tx)d^\times td^\times x$$

Since $\mathbb{Q}^\times\backslash(\mathbb{A}^\times)^0$ is compact, the integral vanishes if $\chi_1\chi_2^{-1}|_{(\mathbb{A}^\times)^0}\neq 1$, and if $\chi_1\chi_2^{-1}|_{(\mathbb{A}^\times)^0}=1$, it is

$$\Phi(0)\operatorname{vol}(\mathbb{Q}^\times\backslash(\mathbb{A}^\times)^0,d^\times t)\int_0^1\chi_1\chi_2^{-1}|\cdot|^{2s+1}(x)d^\times x$$

Similarly, (D) vanishes if $\chi_1\chi_2^{-1}|_{(\mathbb{A}^\times)^0}\neq 1$, and if $\chi_1\chi_2^{-1}|_{(\mathbb{A}^\times)^0}=1$, it is

$$\widehat{\Phi}(0)\operatorname{vol}(\mathbb{Q}^\times\backslash(\mathbb{A}^\times)^0,d^\times t)\int_0^1\chi_1\chi_2^{-1}|\cdot|^{2s-1}(x)d^\times x$$

Now recall that a continuous character $\chi:\mathbb{R}^\times\to\mathbb{C}^\times$ has the form $\chi=|\cdot|^r\operatorname{sign}^\varepsilon$ for some $r\in\mathbb{C}$ and $\varepsilon\in\{0,1\}$; this $\chi_1\chi_2^{-1}|_{\mathbb{R}_{>0}}=|\cdot|^{\rho+it_0}$ for some $t_0\in\mathbb{R}$. Thus if $2\operatorname{Re}(s)-1+\rho>0$ and $\chi_1\chi_2^{-1}|_{(\mathbb{A}^\times)^0}=1$, we have

$$\text{(C)}-\text{(D)}=\operatorname{vol}(\mathbb{Q}^\times\backslash(\mathbb{A}^\times)^0,d^\times t)\left(\frac{\Phi(0)}{2s+1+\rho+it_0}-\frac{\widehat{\Phi}(0)}{2s-1+\rho+it_0}\right)$$

Then $E_\chi(\Phi,s,e)$ satisfies all desired properties. $\square$

12.5.1 Fourier Expansion

For $\phi \in \mathcal{A}(G)$, define $W_\phi : G(\mathbb{A}) \to \mathbb{C}$ by

$$W_\phi(g) := \int_{\mathbb{Q}\backslash\mathbb{A}} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(-x) dx$$

where $\psi = \psi_{\mathbb{A}} : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$ is the standard additive character. W_ϕ is called the **Whittaker function** of ϕ, and it satisfies

$$W_\phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \psi(x) W_\phi(g)$$

for all $g \in G(\mathbb{A})$, $x \in \mathbb{A}$. Then for all $\phi \in \mathcal{A}(G)$, we have the **Fourier expansion** of ϕ:

$$\phi(g) = \phi_N(g) + \sum_{\alpha \in \mathbb{Q}^\times} W_\phi\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

To see this, since the function $x \mapsto \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)$ is continuous on the compact abelian group $\mathbb{Q}\backslash\mathbb{A}$, it has the expansion $\sum_{\alpha \in \mathbb{Q}} \phi_\alpha \psi(\alpha x)$ with

$$\phi_\alpha = \int_{\mathbb{Q}\backslash\mathbb{A}} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(-\alpha x) dx$$

For $\alpha = 0$, by definition we have $\phi_\alpha = \phi_N$. For $\alpha \neq 0$, compute

$$\begin{aligned} \phi_\alpha &= \int_{\mathbb{Q}\backslash\mathbb{A}} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(-\alpha x) dx \overset{x \mapsto \alpha^{-1} x}{=} \int_{\mathbb{Q}\backslash\mathbb{A}} \phi\left(\begin{pmatrix} 1 & \alpha^{-1} x \\ 0 & 1 \end{pmatrix} g\right) \psi(-x) |\alpha^{-1}| dx \\ &= \int_{\mathbb{Q}\backslash\mathbb{A}} \phi\left(\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right) \psi(-x) dx = W_\phi\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right) \end{aligned}$$

Taking $x = 0$ proves the desired identity.

For convenience, write $E(g) = E_\chi(\Phi, s, g)$ and $f = f_{\Phi,\chi,s}$, where $s \in \mathbb{C}$, $\Phi \in \mathcal{S}(\mathbb{A}^2)$, $\chi = (\chi_1, \chi_2)$. We discuss its Fourier expansion. Firstly, the constant term

$$\begin{aligned} E_N(g) &:= \int_{\mathbb{Q}\backslash\mathbb{A}} E\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx \\ &= \int_{\mathbb{Q}\backslash\mathbb{A}} \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} f\left(\gamma \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx \\ &= \int_{\mathbb{Q}\backslash\mathbb{A}} f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) + \sum_{\alpha \in \mathbb{Q}} f\left(w^{-1} \begin{pmatrix} 1 & x+\alpha \\ 0 & 1 \end{pmatrix} g\right) dx \\ &= f(g) + Mf(g) \end{aligned}$$

The third equality is the Bruhat decomposition

$$G(\mathbb{Q}) = B(\mathbb{Q}) \bigsqcup_{\alpha \in \mathbb{Q}} B(\mathbb{Q}) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

and $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. For the last equality, note that $\mathrm{vol}(\mathbb{Q}\backslash\mathbb{A}, dx) = 1$, and define

$$Mf(g) := \int_{\mathbb{A}} f\left(w^{-1}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx$$

If $\Phi = \bigotimes_{p\leqslant\infty}' \Phi_p \in \mathcal{S}(\mathbb{A}^2)$, then

$$Mf(g) = \prod_{p\leqslant\infty} \int_{\mathbb{Q}_p} f_{\Phi_p,\chi_p,s}\left(w^{-1}\begin{pmatrix} 1 & x_p \\ 0 & 1 \end{pmatrix} g_p\right) dx_p = \prod_{p\leqslant\infty} Mf_{\Phi_p,\chi_p,s}(g_p)$$

Here $M = \ell_N$ is the intertwining operator defined before. Secondly, the Whittaker function

$$\begin{aligned} W_E(g) &:= \int_{\mathbb{Q}\backslash\mathbb{A}} E\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(-x) dx \\ &= \underbrace{\int_{\mathbb{Q}\backslash\mathbb{A}} f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(-x) dx}_{=0} + \int_{\mathbb{Q}\backslash\mathbb{A}} \sum_{\alpha\in\mathbb{Q}} f\left(w^{-1}\begin{pmatrix} 1 & x+\alpha \\ 0 & 1 \end{pmatrix} g\right) \psi(-x) dx \\ &= \int_{\mathbb{A}} f\left(w^{-1}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(-x) dx = \prod_p \int_{\mathbb{Q}_p} f_{\Phi_p,\chi_p,s}\left(w^{-1}\begin{pmatrix} 1 & x_p \\ 0 & 1 \end{pmatrix} g_p\right) \psi_p(-x_p) dx_p \end{aligned}$$

The last equality holds when $\Phi = \bigotimes_{p\leqslant\infty}' \Phi_p$. If we define the local Whittaker function

$$W_{f_p}(g) := \int_{\mathbb{Q}_p} f_p\left(w^{-1}\begin{pmatrix} 1 & x_p \\ 0 & 1 \end{pmatrix} g_p\right) \psi_p(-x_p) dx_p$$

with $f_p = f_{\Phi_P,\chi_p,s} \in I(\chi_{1,p}|\cdot|^s, \chi_{2,p}|\cdot|^{-s})$ (c.f. Remark 8.3), we have

$$W_E(g) = \prod_{p\leqslant\infty} W_{f_p}(g_p)$$

whenever $\Phi = \bigotimes_{p\leqslant\infty}' \Phi_p$ and $\chi = \prod_{p\leqslant\infty} \chi_p$.

Example.

(1) Let $p < \infty$, $\Phi = \mathbb{I}_{\mathbb{Z}_p\times\mathbb{Z}_p}$ and $\chi_p = (\chi_{1,p}, \chi_{2,p})$ with $\chi_{i,p}$ unramified. Write $f = f_{\Phi_p,\chi_p,s}$ for brevity. For

$a \in \mathbb{Q}_p^\times$,

$$
\begin{aligned}
W_f\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) &= \int_{\mathbb{Q}_p} f\left(w^{-1}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\psi_p(-x)dx \\
&= \int_{\mathbb{Q}_p} \chi_{1,p}|\cdot|^{s+\frac{1}{2}}(a)\int_{\mathbb{Q}_p^\times} \Phi_p\left((0,t)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}\right)\chi_{1,p}\chi_{2,p}^{-1}|\cdot|^{2s+1}(t)d^\times t\,\psi_p(-x)dx \\
&= \chi_{1,p}|\cdot|^{s+\frac{1}{2}}(a)\int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p^\times}\Phi_p(ta,tx)\chi_{1,p}\chi_{2,p}^{-1}|\cdot|^{2s+1}(t)\psi_p(-x)d^\times t dx \\
&= \chi_{1,p}|\cdot|^{s+\frac{1}{2}}(a)\int_{\mathbb{Q}_p}\mathbb{I}_{\mathbb{Z}_p}(ta)\chi_{1,p}\chi_{2,p}^{-1}|\cdot|^{2s+1}(t)\int_{\mathbb{Q}_p}\mathbb{I}_{\mathbb{Z}_p}(tx)\psi_p(-x)dxd^\times t \\
(x\mapsto t^{-1}x) &= \chi_{1,p}|\cdot|^{s+\frac{1}{2}}(a)\int_{\mathbb{Q}_p^\times}\mathbb{I}_{\mathbb{Z}_p}(ta)\chi_{1,p}\chi_{2,p}^{-1}|\cdot|^{2s+1}(t)|t^{-1}|\widehat{\mathbb{I}_{\mathbb{Z}_p}}(-t^{-1})d^\times t \\
(t\mapsto t^{-1}) &= \chi_{1,p}|\cdot|^{s+\frac{1}{2}}(a)\int_{\mathbb{Q}_p^\times}\mathbb{I}_{\mathbb{Z}_p}(t^{-1}a)\mathbb{I}_{\mathbb{Z}_p}(-t)(t^{-1}a)\chi_{1,p}\chi_{2,p}^{-1}|\cdot|^{2s}(t^{-1})d^\times t \\
&= \chi_{1,p}|\cdot|^{s+\frac{1}{2}}(a)\int_{0\leqslant \mathrm{ord}_p t\leqslant \mathrm{ord}_p a}\chi_{1,p}\chi_{2,p}^{-1}|\cdot|^{2s}(t^{-1})d^\times t \\
&= \chi_{1,p}|\cdot|^{s+\frac{1}{2}}(a)\sum_{m=0}^{\mathrm{ord}_p a}\chi_{1,p}^{-1}\chi_{2.p}|\cdot|^{-2s}(p^m)
\end{aligned}
$$

In particular, $W_f\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 1$ under this situation.

(2) Let $p=\infty$, $\Phi_\infty(x,y) = e^{-\pi(x^2+y^2)}$, $\chi_{1,p} = \chi_{2,p} = 1$. Then for $a \in \mathbb{R}^\times = \mathbb{Q}_\infty^\times$,

$$
\begin{aligned}
W_f\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) &= |a|^{s+\frac{1}{2}}\int_{\mathbb{R}^\times} e^{-\pi t^2a^2}|t|^{2s+1}|t^{-1}|\widehat{e^{-\pi x^2}}(-t^{-1})d^\times t \\
&= |a|^{s+\frac{1}{2}}\int_{\mathbb{R}^\times} e^{-\pi(t^2a^2+t^{-2})}|t|^{2s}d^\times \\
(t\mapsto |a|^{-\frac{1}{2}}t) &= |a|^{\frac{1}{2}}\int_{\mathbb{R}^\times} e^{-\pi|a|(t^2+t^{-2})}|t|^{2s}d^\times t = |a|^{\frac{1}{2}}\mathcal{K}_s(\pi|a|)
\end{aligned}
$$

where $\mathcal{K}_s(y) := \int_{\mathbb{R}^\times} e^{-y(t+t^{-1})}|t|^s d^\times = 2\int_0^\infty e^{-y(t+t^{-1})}t^s d^\times t$ is the K-**Bessel function**.

12.5.2 Application to Prime Number Theorem

Theorem 12.10. $\zeta(1+it) \neq 0$ for all $t \in \mathbb{R}^\times$, where

$$
\zeta(s) = \prod_{p\leqslant\infty} L(s, 1_p) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\prod_{p<\infty}(1-p^{-s})^{-1}
$$

Proof. Define

$$
\Phi_p^\circ := \begin{cases} e^{-\pi(x^2+y^2)} & \text{, if } p=\infty \\ \mathbb{I}_{\mathbb{Z}_p\times\mathbb{Z}_p} & \text{, if } p<\infty \end{cases}
$$

and put $\Phi^\circ = \bigotimes_{p\leqslant\infty}' \Phi_p^\circ$; then $\widehat{\Phi^\circ} = \Phi^\circ$. Put $\chi = (1,1)$, and form the **Epstein Eisenstein series**

$$
E(s,g) := E_\chi(\Phi^\circ, s, g)
$$

We compute its constant term; we have

$$E_N(s,g) = f_{\Phi^\circ,\chi,s}(g) + Mf_{\Phi^\circ,\chi,s}(g)$$

and by Proposition 10.1,

$$Mf_{\Phi^\circ,\chi,s}(g) = \prod_p \gamma(2s, 1_p, \psi_p)^{-1} f_{\widehat{\Phi^\circ},\chi^{\mathrm{sw}},-s} = \prod_p \frac{L(2s,1_p)}{L(1-2s,1_p)} f_{\widehat{\Phi^\circ},\chi,-s} = \frac{\zeta(2s)}{\zeta(1-2s)} f_{\Phi^\circ,\chi^{\mathrm{sw}},-s}$$

Compute

$$\begin{aligned} f_{\Phi^\circ,\chi,s}(e) &= \int_{\mathbb{A}^\times} \Phi^\circ(0,t)|\cdot|^{2s+1}(t)d^\times t \\ &= \int_{\mathbb{R}^\times} e^{-\pi t^2}|\cdot|^{2s+1}(t)d^\times t \cdot \prod_{p<\infty} \int_{\mathbb{Q}_p^\times} \mathbb{I}_{\mathbb{Z}_p}(t_p)|\cdot|^{2s+1}(t_p)d^\times t_p \\ &= \zeta(2s+1) \end{aligned}$$

Thus

$$E_N(s,e) = f_{\Phi^\circ,\chi,s}(e) + Mf_{\Phi^\circ,\chi,s}(e) = \zeta(2s+1) + \zeta(2s)$$

To be filled. □

12.6 L-functions of cuspidal automorphic representations

Recall that (π, V) is an irreducible representation of $\mathrm{GL}_2(\mathbb{A})$ if $\pi = \bigotimes'_{p\leqslant\infty} \pi_p$ with each π_p an irreducible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, and π is called automorphic if $\mathrm{Hom}_{G(\mathbb{A})}(\pi, \mathcal{A}(G)) \neq 0$.

Definition. An irreducible representation (π, V) is called **cuspidal** if $\mathrm{Hom}_{G(\mathbb{A})}(\pi, \mathcal{A}_0(G)) \neq 0$.

Suppose π is an automorphic cuspidal irreducible representation of $\mathrm{GL}_2(\mathbb{A})$. Since $\pi = \bigotimes'_{p\leqslant\infty} \pi_p$, for each $p \leqslant \infty$ we can form the local L-functions $L(s, \pi_p)$ of π_p. Define the **global L-function**

$$L(s,\pi) = \prod_{p\leqslant\infty} L(s,\pi_p) \qquad ????$$

Proposition 12.11. Suppose π is an automorphic cuspidal irreducible representation of $\mathrm{GL}_2(\mathbb{A})$ with central character ω of weight ρ (i.e., $|\omega| = |\cdot|^\rho$). Then $L(s,\pi)$ is absolutely convergent for $\mathrm{Re}(s) > \dfrac{3-\rho}{2}$.

Let $p < \infty$. Then π_p is spherical if and only if $\pi_p^{\mathrm{GL}_2(\mathbb{Z}_p)} \neq 0$, if and only if $\dim_{\mathbb{C}} \pi_p^{\mathrm{GL}_2(\mathbb{Z}_p)} = 1$. By Homework 5 , we see $\pi_p \cong \pi(\chi_1, \chi_2)$ with $\chi_i : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ unramified. Since $\mathcal{H}_p := \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p), \mathrm{GL}_2(\mathbb{Z}_p))$ is commutative, we can find $\lambda_{\pi_p} : \mathcal{H}_p \to \mathbb{C}$ such that $\pi_p(f)v = \lambda_{\pi_p}(f)v$ for all nonzero spherical vector v and $f \in \mathcal{H}_p$. (c.f. Lemma 12.1.)

Lemma 12.12. Suppose there exists $C > 0$ such that

$$|\lambda_{\pi_p}(f)| \leqslant C \int_{G(\mathbb{Q}_p)} f(g)dg$$

for all $f \in \mathcal{H}_p$. Then $|\chi_1\chi_2(p)| = 1$ and

$$p^{-\frac{1}{2}} = |p|^{\frac{1}{2}} \leqslant |\chi_i(p)| \leqslant |p|^{-\frac{1}{2}} = p^{\frac{1}{2}}$$

for $i = 1, 2$.

Proof. Define

$$
\begin{aligned}
T_n &= \mathbb{I}_{K\begin{pmatrix}p^n & \\ & 1\end{pmatrix}K}, \qquad n \in \mathbb{N}_0 \\
R_n &= \mathbb{I}_{K\begin{pmatrix}p^n & \\ & p^n\end{pmatrix}K} = \mathbb{I}_{\begin{pmatrix}p^n & \\ & p^n\end{pmatrix}K}, \qquad n \in \mathbb{Z}
\end{aligned}
$$

Then T_n, $R_n \in \mathcal{H}_p$, and if we put $\alpha_i = \chi_i(p)$, $i = 1, 2$, we have

$$
\begin{aligned}
\lambda_{\pi_p}(T_n) &= |p|^{-\frac{n}{2}}(\alpha_1^n + \alpha_2^n) \\
\lambda_{\pi_p}(R_n) &= (\alpha_1\alpha_2)^n
\end{aligned}
$$

To check this, we take $(\pi, V) = (\rho, I(\chi_1, \chi_2))$, $I(\chi_1, \chi_2)^K = \mathbb{C}f_0$, where $f_0 \in I(\chi_1, \chi_2)$ is the unique element such that $f_0(bk) = \chi\delta_B^{\frac{1}{2}}(b)$ for all $b \in B(\mathbb{Q}_p)$, $k \in K = \mathrm{GL}_2(\mathbb{Z}_p)$. Since $\pi(T_n)f_0(e) = \lambda_{\pi_p}(T_n)f_0(e)$, we have

$$
\begin{aligned}
\lambda_{\pi_p}(T_n) &= \int_{G(\mathbb{Q}_p)} T_n(g)f_0(g)dg \\
&= \sum_{x \in \mathbb{Z}_p/p^n\mathbb{Z}_p} f_0\begin{pmatrix}p^n & x \\ 0 & 1\end{pmatrix} + f_0\begin{pmatrix}1 & \\ & p^n\end{pmatrix} \\
&= \alpha_1^n|p^n|^{\frac{1}{2}}p^n + \alpha_2^n|p^n|^{-\frac{1}{2}} \\
&= |p^n|^{-\frac{1}{2}}(\alpha_1^n + \alpha_2^n)
\end{aligned}
$$

Here the measure dg is normalized so that $\mathrm{vol}(K, dg) = 1$, and we use the decomposition (c.f. Homework 5)

$$
K\begin{pmatrix}p^n & \\ & 1\end{pmatrix}K = \bigsqcup_{x \in \mathbb{Z}_p/p^n\mathbb{Z}_p} \begin{pmatrix}p^n & x \\ 0 & 1\end{pmatrix}K \sqcup \begin{pmatrix}1 & 0 \\ 0 & p^n\end{pmatrix}K
$$

The identity for R_n can be proved similarly. Now by assumption, we have

$$
\begin{aligned}
|\lambda_{\pi_p}(T_n)| &\leqslant C\int_{G(\mathbb{Q}_p)} T_n(g)dg = C(p^n + 1) \\
|\lambda_{\pi_p}(R_n)| &\leqslant C\int_{G(\mathbb{Q}_p)} R_n(d)gd = C
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|\alpha_1^n + \alpha_2^n| &\leqslant C(p^{\frac{n}{2}} + p^{-\frac{n}{2}}) \text{ for } n \in \mathbb{N}_0 \\
|\alpha_1\alpha_2|^n &\leqslant C \text{ for } n \in \mathbb{Z}
\end{aligned}
$$

The second inequalities imply $|\alpha_1\alpha_2| = 1$. We claim the first imply $p^{-\frac{1}{2}} \leqslant |\alpha_i| \leqslant p^{\frac{1}{2}}$. For this, form the formal power series

$$
f(z) = \sum_{n=0}^{\infty}(\alpha_1^n + \alpha_2^n)z^n = \frac{1}{1 - \alpha_1 z} + \frac{1}{1 - \alpha_2 z}
$$

The first inequalities imply the power series is absolutely convergent for $|z| < p^{-\frac{1}{2}}$, and the last expression implies $|\alpha_i| \leqslant p^{\frac{1}{2}}$. Since $|\alpha_1\alpha_2| = 1$, this proves the claim. $\square$

Proof. (of Proposition 12.11) Say $\pi \cong \bigotimes_{p\leqslant\infty}' \pi_p$. Let S be a finite set of primes such that π_p is NOT spherical. Replacing π by $\pi \otimes |\det|^{-\frac{\rho}{2}}$, we may assume ω is unitary.

Since π is automorphic, π has a realization $(\rho, V) \subseteq \mathcal{A}_0(G)$. Choose $0 \neq \phi \in V$ that is fixed by $\mathrm{GL}_2(\mathbb{Z}_p)$ for all $p \notin S$. Then for all $f \in \mathcal{H}_p$, $p \notin S$, $\pi(f)\phi = \lambda_{\pi_p}(f)\phi$. Choose $g_0 \in G(\mathbb{A})$ with $\phi(g_0) \neq 0$. Then

$$\lambda_{\pi_p}(f)\phi(g_0) = \int_{G(\mathbb{Q}_p)} \phi(g_0 g_p) f(g_p) dg_p$$

Since ϕ is a cusp form, ϕ is bounded on $G(\mathbb{A})$ by Proposition 12.6 (the case $m = 0$) so that

$$|\lambda_{\pi_p}(f)| \leqslant C \int_{G(\mathbb{Q}_p)} f(g_p) dg_p$$

for some C. By Lemma 12.12, for $p \notin S$ if we write $\pi_p \cong \pi(\chi_{1,p}, \chi_{2,p})$, then $p^{-\frac{1}{2}} \leqslant |\chi_{i,p}(p)| \leqslant p^{\frac{1}{2}}$ $(i = 1, 2)$. For $p \notin S$,

$$L(s, \pi_p) = \frac{1}{(1 - \chi_{,p}(p)p^{-s})(1 - \chi_{2,p}(p)p^{-s})}$$

so that

$$L(s, \pi) = \prod_{p\in S} L(s, \pi_p) \cdot \prod_{p\notin S} \prod_{i=1}^{2} \frac{1}{1 - \chi_{i,p}(p)p^{-s}}$$

Note that $\prod_{p\notin S} \frac{1}{1 - \chi_{i,p}(p)p^{-s}}$ converges absolutely if $|\chi_{i,p}|p^{-\operatorname{Re}(s)} < p^{-1}$. For $p \notin S$, $p^{-\frac{1}{2}} \leqslant |\chi_{i,p}(p)| \leqslant p^{\frac{1}{2}}$ $(i = 1, 2)$ implies $|\chi_{i,p}|p^{-\operatorname{Re}(s)} < p^{\frac{1}{2}-\operatorname{Re}(s)}$. Thus if $\operatorname{Re}(s) > \frac{3}{2}$, the product $L(s, \pi)$ converges absolutely. ☐

12.7 Zeta function for cusp forms

Let (π, V_π) be an irreducible automorphic cuspidal representation of $G(\mathbb{A})$ with central character ω; we assume $V_\pi \subseteq \mathcal{A}_0(G)$. For $\phi \in V_\pi$, define the **zeta integral**

$$Z(\phi, s) = \int_{\mathbb{Q}^\times \backslash \mathbb{A}^\times} \phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) |a|^{s-\frac{1}{2}} d^\times a$$

and $\widehat{\phi}(g) := \phi(gw)\omega^{-1}(\det g)$, where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in G(\mathbb{Q})$; then $\widehat{\phi} \in V_{\pi^\vee}$. ???

Proposition 12.13.

1. $Z(\phi, s)$ converges absolutely for $\operatorname{Re}(s) \gg 0$, has analytic continuation to an entire function and is bounded in every vertical strip.

2. $Z(\phi, s)$ satisfies the functional equation

$$Z(\phi, s) = Z(\widehat{\phi}, 1 - s)$$

Proof.

$$\begin{aligned} Z(\phi, s) &= \int_{\mathbb{Q}^\times \backslash \mathbb{A}^\times} \phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) |a|^{s-\frac{1}{2}} d^\times a \\ &= \int_{|a|>1} \phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) |a|^{s-\frac{1}{2}} d^\times a + \int_{|a|<1} \phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) |a|^{s-\frac{1}{2}} d^\times a \end{aligned}$$

On the other hand,

$$\phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = \phi\left(w^{-1}\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} w\right) = \phi\left(\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} w\right) = \omega(a)\phi\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} w\right) = \widehat{\phi}\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} w\right)$$

so

$$\int_{|a|<1} \phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) |a|^{s-\frac{1}{2}} d^\times a = \int_{|a|>1} \widehat{\phi}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) |a|^{\frac{1}{2}-s} d^\times a$$

In sum, we obtain

$$Z(\phi, s) = \int_{|a|>1} \phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) |a|^{s-\frac{1}{2}} d^\times a + \int_{|a|>1} \widehat{\phi}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) |a|^{\frac{1}{2}-s} d^\times a$$

Since ϕ and $\widehat{\phi}$ are cuspidal, they are rapidly decreasing Proposition 12.6. Thus

$$\left|\int_{|a|>1} \phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) |a|^{s-\frac{1}{2}} d^\times a\right| \leqslant C \int_1^\infty t^{s-n-\frac{1}{2}} d^\times t$$

for some $n \gg 0$ and $C = C_n > 0$. Similar for the second integral. In conclusion, both integral converges absolutely and define entire functions for $s \in \mathbb{C}$, and thus $Z(\phi, s)$ is entire and verifies the functional equation. $\square$

12.8 Whittaker functions

Let (π, V_π) be as in the last subsection. For all $p \leqslant \infty$, fix a nonzero Whittaker functional $\ell_p : V_{\pi_p} \to \mathbb{C}$. Let S be the finite set of primes such that π_p is not spherical. For $p \notin S$, we require $\ell_p(\xi_p^\circ) = 1$, where ξ_p° is a fixed basis element of $V_{\pi_p}^{\mathrm{GL}_2(\mathbb{Z}_p)}$.

Lemma 12.14. If $\ell : V_\pi \to \mathbb{C}$ is a global Whittaker function, then $\ell = C \prod_{p\leqslant\infty} \ell_p$ for some $C \in \mathbb{C}$.

Corollary 12.14.1. Let $\pi \cong \bigotimes_{p\leqslant\infty}' \pi_p$ be cuspidal irreducible. For all $p \leqslant \infty$ we have the isomorphism

$$\begin{aligned} V_{\pi_p} &\longrightarrow W(\pi_p, \psi_p) \\ \xi_p &\longmapsto W_{\xi_p} \end{aligned}$$

where $W(\pi_p, \psi_p)$ is the Whittaker model of π_p. Then there exist an isomorphism

$$\begin{aligned} \bigotimes_{p\leqslant\infty}' V_{\pi_p} &\longrightarrow V_\pi \subseteq \mathcal{A}_0(G) \\ \otimes_p \xi_p &\longleftarrow\!\shortmid \phi \end{aligned}$$

such that $W_\phi(g) = \prod_{p\leqslant\infty} W_{\xi_p}(g_p)$ for all $g = (g_p)_p \in G(\mathbb{A})$.

Now for $\phi \in \mathcal{A}_0(G)$, since $\phi_N = 0$, we have

$$\begin{aligned} Z(\phi, s) &= \int_{\mathbb{Q}^\times\backslash\mathbb{A}^\times} \phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) |a|^{s-\frac{1}{2}} d^\times a = \int_{\mathbb{Q}^\times\backslash\mathbb{A}^\times} \sum_{\alpha\in\mathbb{Q}^\times} W_\phi\left(\begin{pmatrix} \alpha a & 0 \\ 0 & 1 \end{pmatrix}\right) |a\alpha|^{s-\frac{1}{2}} d^\times a \\ &= \int_{\mathbb{A}^\times} W_\phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) |a|^{s-\frac{1}{2}} d^\times a \\ &= \prod_{p\leqslant\infty} \int_{\mathbb{Q}_p^\times} W_{\xi_p}\left(\begin{pmatrix} a_p & 0 \\ 0 & 1 \end{pmatrix}\right) |a_p|^{s-\frac{1}{2}} d^\times a_p = \prod_{p\leqslant\infty} Z(W_{\xi_p}, s) \end{aligned}$$

For each $p \leqslant \infty$ we can find $W_{\xi_p} \in W(\pi_p, \psi_p)$ such that $Z(W_{\xi_p}, s) = L(s, \pi_p)$. Thus there exists $\phi \in V_\pi$ such that $Z(\phi, s) = L(s, \pi)$, and consequently $L(s, \pi)$ admits an analytic continuation to $s \in \mathbb{C}$ with functional equation

$$L(1 - s, \pi^\vee) = \epsilon(s, \pi)L(s, \pi)$$

where $\epsilon(s, \pi) := \prod_{p \leqslant \infty} \epsilon(s, \pi_p, \psi_p)$ is the product of all local ϵ-factors.

12.9 The Converse Theorem

Let F be a number field and let $\pi \cong \bigotimes_\nu \pi_\nu$ be an irreducible admissible representation of $\mathrm{GL}_2(\mathbb{A}_F)$ with each π_ν infinite dimensional. Suppose the central character of π is a Hecke character $\omega : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ of weight $\rho \in \mathbb{R}$.

Theorem 12.15. Suppose there exists $r \in \mathbb{R}$ such that for almost all places ν with $\pi\nu = \pi(\chi_{1,\nu}, \chi_{2,\nu})$ we have

$$|\pi_\nu|^{-r} \leqslant |\chi_{i,\nu}(\pi)| \leqslant |\pi_\nu|^r \ (i = 1, 2)$$

where π_ν is a uniformizer in F_ν. Suppose that for all unitary Hecke characters $\chi : F^\times \backslash \mathbb{A}_F^\times \to S^1$ the infinite product

$$L(s, \pi \otimes \chi) = \prod_\nu L(s, \pi_\nu \otimes \chi_\nu)$$

converges absolutely for $\operatorname{Re} s \gg 0$, EBV and satisfies the functional equation

$$L(s, \pi \otimes \chi) = \epsilon(s, \pi \otimes \chi)L(1 - s, \pi^\vee \otimes \chi^{-1})$$

Then π is cuspidal.

For each Whittaker function $W \in W_\psi(\pi)$, define the series

$$\varphi_1(g) = \varphi_W(g) := \sum_{\alpha \in F^\times} W\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

We will show later that φ_1 converges absolutely and compactly on $\mathrm{GL}_2(\mathbb{A}_F)$, and the map

$$a \mapsto \varphi_1\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

is slowly decreasing for each fixed $g \in G(\mathbb{A}_F)$. Taking these for granted, we then see for each $g \in G(\mathbb{A}_F)$, the zeta integral

$$Z(\varphi_1, s, g) := \int_{F^\times \backslash \mathbb{A}_F^\times} \varphi_1\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) |a|^s d^\times a$$

converges absolutely for $\operatorname{Re} s \gg 0$. We proceed to show φ_1 is an automorphic form. Since the standard character ψ is trivial on F, we have

$$\varphi_1\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \sum_{\alpha \in F^\times} W\left(\begin{pmatrix} 1 & \alpha x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right) = \sum_{\alpha \in F^\times} \psi(\alpha x) W\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right) = \varphi_1(g)$$

By construction, φ_1 is invariant under the left translation by the $\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$, $\alpha \in F^\times$. For $a \in \mathbb{A}_F^\times$, since

$$\varphi_1\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) = \sum_{\alpha \in F^\times} W\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) = \omega(a)\varphi_1(g)$$

if $a \in F^\times$, then $\varphi_1\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) = \varphi_1(g)$. So far we have shown that $\varphi_1(bg) = \varphi_1(g)$ for all $b \in B(F)$. It remains to show $\varphi_1(wg) = \varphi_1(g)$, where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. For this we put $\varphi_2(g) = \varphi_1(wg)$ and define

$$f_1(a) := \varphi_1\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right), \qquad f_2(a) := \varphi_2\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

Let χ be a unitary Hecke character of F and consider the zeta integrals

$$Z(f_i, \chi, s) := \int_{F^\times \backslash \mathbb{A}^\times} f_i(a)\chi(a)|a|^{s-\frac{1}{2}} d^\times a$$

We have

$$f_2(a) = \varphi_1\left(w\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) = \omega(a)\varphi_1\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} wg\right)$$

and thus

$$Z_1(s) = Z(f_1, \chi, s) = \int_{F^\times \backslash \mathbb{A}^\times} \varphi_1\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right)\chi(a)|a|^{s-\frac{1}{2}} d^\times a$$

$$Z_2(s) = Z(f_2, \chi, s) = \int_{F^\times \backslash \mathbb{A}^\times} \varphi_1\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} wg\right)\omega^{-1}\chi^{-1}(a)|a|^{\frac{1}{2}-s} d^\times a$$

Unfolding, we have

$$Z(f_1, \chi, s) = \int_{\mathbb{A}^\times} W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right)\chi(a)|a|^{s-\frac{1}{2}} d^\times a$$

??? We can find $c \gg 0$ such that $Z(f_1, \chi, s)$ (resp. $Z(f_2, \omega^{-1}\chi^{-1}, 1-s)$) converges absolutely whenever $\operatorname{Re} s > c$ (resp. $\operatorname{Re} s < -c$), and are bounded in vertical strips in $\operatorname{Re} s > c$ (resp. $\operatorname{Re} s < c$).

Lemma 12.16. Let ν be a finite place of F such that π_ν is spherical principal and the additive character ψ_ν is unramified. If W_ν° is the unique spherical Whittaker function normalized so that $W_\nu^\circ(e) = 1$, then for each unitary character $\chi_\nu : F_\nu^\times \to S^1$, we have

$$Z(W_\nu^\circ, \chi_\nu, s) = L(s, \pi_\nu \otimes \chi_\nu)$$

Let us assume $W = \prod_\nu W_\nu$, and let S be a finite set of finite places such that $\pi_\nu, \chi_\nu, \psi_\nu$ are unramified, $g_\nu \in K_\nu$ and $W_\nu = W_\nu^\circ$ for $\nu \notin S$. For $\operatorname{Re} s > c$,

$$Z_1(s) = \prod_\nu Z(W_\nu, \chi_\nu, s) = L(s, \pi \otimes \chi)\prod_\nu \frac{Z(W_\nu, \chi_\nu, s)}{L(s, \pi_\nu \otimes \chi_\nu)} = L(s, \pi \otimes \chi)\prod_{\nu \in S} \frac{Z(W_\nu, \chi_\nu, s)}{L(s, \pi_\nu \otimes \chi_\nu)}$$

and for $\operatorname{Re} s < -c$,

$$Z_2(s) = L(1-s, \pi^\vee \otimes \chi^{-1})\prod_\nu \frac{Z(W_\nu, \omega^{-1}\chi_\nu^{-1}, 1-s)}{L(1-s, \pi_\nu^\vee \otimes \chi_\nu^{-1})} = L(1-s, \pi^\vee \otimes \chi^{-1})\prod_{\nu \in S} \frac{Z(W_\nu, \omega^{-1}\chi_\nu^{-1}, 1-s)}{L(1-s, \pi_\nu^\vee \otimes \chi_\nu^{-1})}$$

By assumptions on L-functions, it follows that Z_1 has an analytic continuation to some entire function in s. Recall for $\nu \notin S$, the epsilon factor $\epsilon(s, \pi_\nu \otimes \chi_\nu, \psi_\nu) = 1$. By the functional equation

$$\frac{Z(W_\nu, \chi_\nu^{-1}\omega_\nu^{-1}, 1-s, wg_\nu)}{L(1-s, \pi_\nu^\vee \otimes \chi_\nu^{-1})} = \epsilon(s, \pi_\nu \otimes \chi_\nu, \psi_\nu) \frac{Z(W_\nu, \chi_\nu, s, g_\nu)}{L(s, \pi_\nu \otimes \chi_\nu)}$$

we have

$$\begin{aligned} Z_2(s) &= L(1-s, \pi^\vee \otimes \chi^{-1})\epsilon(s, \pi \otimes \chi, \psi) \prod_{\nu \in S} \frac{Z(W_\nu, \chi_\nu, s)}{L(s, \pi_\nu \otimes \chi_\nu)} \\ &= L(s, \pi \otimes \chi) \prod_{\nu \in S} \frac{Z(W_\nu, \chi_\nu, s)}{L(s, \pi_\nu \otimes \chi_\nu)} = Z_1(s) \end{aligned}$$

Therefore Z_1 and Z_2 extend to the same entire function Z, and Z is bounded in vertical strips for $\operatorname{Re} s > c$ or $\operatorname{Re} s < -c$. We have

$$Z(s) = L(s, \pi \otimes \chi) \prod_{\nu \in S} \frac{Z(W_\nu, \chi_\nu, s)}{L(s, \pi_\nu \otimes \chi_\nu)}$$

which is valid for every $s \in \mathbb{C}$. $L(s, \pi \otimes \chi)$ is assumed to be EBV, and for each finite place ν in S, the ratio is a polynomial in $(\#\kappa(\nu))^{\pm s}$, so it is also EBV. As for the infinite place ν in S, the ratio is a product of polynomials and Gamma functions, so by Stirling's formula and the Phragmen-Lindelöf principle, Z is bounded in vertical strips for $-c \leqslant \operatorname{Re} s \leqslant c$.

Note that f_1 and f_2 descend to a map on $\mathbb{A}_F^\times / F^\times$. To show $f_1 = f_2$, it suffices to show that $f_1(tx) = f_2(tx)$ for all $t \in (\mathbb{A}_F^\times)^0 / F^\times$ and $x \in \mathbb{A}_F^\times$. Since $(\mathbb{A}_F^\times)^0 / F^\times$ is compact, it suffices to show $t \mapsto f_1(tx)$ and $t \mapsto f_2(tx)$ have same Fourier expansions. To show this, for each character $\chi : (\mathbb{A}_F^\times)^0 / F^\times \to \mathbb{C}^\times$, put

$$g_i(x) = \widehat{f}_i(x, \chi) = \chi(x) \int_{(\mathbb{A}_F^\times)^0 / F^\times} f_i(tx)\chi(t) d^\times t \, (i = 1, 2)$$

g_1 and g_2 are functions on $\mathbb{A}_F^\times / (\mathbb{A}_F^\times)^0 \cong \mathbb{R}_{>0} \cong \mathbb{R}$, and we need to show $g_1 = g_2$. Since $Z_1(f_1, \chi, s) = Z_2(f_2, \chi, s)$, we have

$$\int_{\mathbb{R}} h_1(x) e^{sx} dx = \int_{\mathbb{R}} h_1(x) e^{sx} dx \qquad (= Z(s))$$

where $h_i(x) := g_i(e^x)$. Pick $g \in C_c^\infty(\mathbb{R})$ and consider the convolution $g * h_i$. Then $\widehat{g * h_i}^{\mathrm{La}} = \widehat{g}^{\mathrm{La}} \widehat{h_i}^{\mathrm{La}}$, and by the inversion formula we have

$$g * h_i(x) = \frac{1}{2\pi i} \int_{b - i\infty}^{b + i\infty} \widehat{h_i}^{\mathrm{La}}(s) \widehat{g}^{\mathrm{La}}(s) e^{-sx} ds \tag{$\spadesuit$}$$

where $b > c$ if $i = 1$ and $b < -c$ if $i = 2$. Look at $\widehat{g}(s)$. If we write $s = \sigma + it$, then

$$\widehat{g}^{\mathrm{La}}(\sigma + it) = \int_{\mathbb{R}} g(x) e^{x(\sigma + it)} dx = \int_{\mathbb{R}} g(x) e^{x\sigma} e^{itx} dx = \widehat{g(x)e^{x\sigma}}^{\mathrm{Fourier}}(t)$$

It follows from Riemann-Lebesgue lemma that as σ lies in a fixed compact interval, the function $\widehat{g}^{\mathrm{La}}(\sigma + it)$ decays faster than any polynomial as $t \to \pm\infty$. Along with the fact that $\widehat{h_i}^{\mathrm{La}}$ is EBV, the Cauchy's integral formula implies that the integral in ($\spadesuit$) is independent of b. As a consequence, we have $g * h_1 = g * h_2$ for all $g \in C_c^\infty(\mathbb{R})$, whence $h_1 = h_2$. So $g_1 = g_2$, and since $\widehat{f}_1(x, \chi) = \widehat{f}_2(x, \chi)$ for all $\chi \in \widehat{(\mathbb{A}_F^\times)^0 / F^\times}$, we obtain $f_1 = f_2$. Therefore,

$$\varphi_1(wg) = \varphi_2(g) = f_2(1) = f_1(1) = \varphi_1(g)$$

In sum, we have proved that

$$\varphi_1(g) = \sum_{\alpha \in F^\times} W\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

is an automorphic form. We claim φ_1 is in fact cuspidal, so we obtain a map

$$\begin{aligned} W_\psi(\pi) &\longrightarrow \mathcal{A}_0(G) \\ W &\longmapsto \varphi_W \end{aligned}$$

that intertwines the G-action by right translation. Indeed, the constant term of φ_W is

$$\int_{F\backslash \mathbb{A}_F} \varphi_W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx = \sum_{\alpha \in F^\times} W\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right) \int_{F\backslash \mathbb{A}_F} \psi(\alpha x) dx = 0$$

(recall that $F\backslash \mathbb{A}_F$ is compact) so φ_W is cuspidal. Finally, we have

$$\frac{1}{\mathrm{vol}(F\backslash \mathbb{A}_F)} \int_{F\backslash \mathbb{A}_F} \varphi_W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(-\beta x) dx = W\left(\begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

if $\beta \in F^\times$, so $W = 0$ if $\varphi_W = 0$.

It remains to show

$$\varphi_1(g) = \sum_{\alpha \in F^\times} W\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

converges absolutely and compactly, and the map

$$a \mapsto \varphi_1\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

is slowly decreasing for $g \in \Omega$, where Ω is any compact set in $\mathrm{GL}_2(\mathbb{A}_F)$.

References

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