Algebraic Geometry

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Première partie

Basics

1 Glossary

1.1 Purposes. In this section we collect some definitions that will be used in the sequel. We also serve this section as indexes.

1.1 From category

1.2 Additive category.

1.3 Abelian category. An additive category is called **abelian** if

(i) Every morphism has a kernel and cokernel.

(ii) For every morphism $f : X \to Y$, the natural map $\mathrm{coim}(f) \to \mathrm{im}(f)$ is an isomorphism.

1.3.1 Theorem. An additive category is abelian if and only if

(i) Every morphism has a kernel and cokernel.

(ii) Every monomorphism is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel.

1.4 Projector. Let $\mathcal{C}$ be a category. A **projector** in $\mathcal{C}$ is a morphism $p : A \to A$ such that $p \circ p = p$ for some object A. We also say p is **idempotent**.

1.4.1 Splitting. A projector $p : A \to A$ is said to **split** if there exists another object B and two morphisms $q : A \to B,\ s : B \to A$ such that

$$s \circ q = p, \qquad q \circ s = \mathrm{id}_B\,.$$

Such a triple (B, q, s) is called a **splitting** of p.

1.5 Pseudo-abelian category. A **pseudo-abelian category** is an additive category $\mathcal{C}$ such that all projectors split.

1.5.1 Pseudo-abelian envelop. Let $\mathcal{C}$ be an additive category. The **pseudo-abelian envelop** of $\mathcal{C}$ is the category $\mathcal{C}^\sharp$ defined as follows.

— Objects : all pairs $(A, p : A \to A)$ such that A is an object in $\mathcal{C}$ and p is a projector.

— Morphisms :

$$\mathrm{Hom}_{\mathcal{C}^\sharp}((A,p),(B,q)) := \{f \in \mathrm{Hom}_{\mathcal{C}}(A,B) \mid q \circ f = f = f \circ p\} = q \circ \mathrm{Hom}_{\mathcal{C}}(A,B) \circ p$$

1.6 Strictly full subcategory. A subcategory $\mathcal{D}$ of a category $\mathcal{C}$ is **strictly full** if it is a full subcategory and closed under isomorphism.

1.7 Serre subcategory. A nonempty full subcategory $\mathcal{D}$ of an abelian category $\mathcal{C}$ is a **Serre subcategory** whenever given any exact sequence $A \to B \to C$ in $\mathcal{C}$ with $A, C \in \mathrm{Ob}(\mathcal{D})$, we have $B \in \mathrm{Ob}(\mathcal{D})$.

1.8 Weak Serre subcategory. A nonempty full subcategory $\mathcal{D}$ of an abelian category $\mathcal{C}$ is a **weak Serre subcategory** whenever given any exact sequence $A \to B \to C \to D \to E$ in $\mathcal{C}$ with $A, B, D, E \in \mathrm{Ob}(\mathcal{D})$, we have $C \in \mathrm{Ob}(\mathcal{D})$.

2 Local-ringed spaces

2.1 Sheaves on topological spaces

2.1 Presheaf. Let X be a topological space. Define a category $\mathrm{Top}(X)$ as follows.

— An object of $\mathrm{Top}(X)$ is an open set in X.

— For two objects U, V of $\mathrm{Top}(X)$, set

$$\mathrm{Hom}_{\mathrm{Top}(X)}(U,V) = \begin{cases} \{\iota_{UV} : U \to V\} & \text{, if } U \subseteq V \text{, where } \iota_{UV} : U \to V \text{ denotes the inclusion,} \\ \varnothing & \text{, otherwise.} \end{cases}$$

Let $\mathcal{C}$ be a category. A **$\mathcal{C}$-presheaf** on X is just a contravariant functor $\mathcal{F} : \mathrm{Top}(X) \to \mathcal{C}$. When $\mathcal{C}$ is **Set** (resp. **Ab**, **Ring**, **Mod**$_R$), we say $\mathcal{F}$ is a presheaf of sets (resp. abelian groups, rings, R-modules). A morphism between two presheaves is a natural transformation. We denote by $\mathcal{C}_X^{\mathrm{pre}}$ the category of all $\mathcal{C}$-presheaves on X. For $U \subseteq V$, the morphism $\mathcal{F}(\iota_{UV}) : \mathcal{F}(V) \to \mathcal{F}(U)$ is usually called a restriction map. An element in $\mathcal{F}(X)$ is called a **global section** of $\mathcal{F}$, and we usually write $\Gamma(X, \mathcal{F}) = \mathcal{F}(X)$.

2.2 Sheaf. In the following we assume $\mathcal{C}$ is either **Set**, **Ab**,**Ring** or **Mod**$_R$. A $\mathcal{C}$-valued presheaf $\mathcal{F}$ is called a **sheaf** if for every open U and every open cover $\{U_i\}_{i\in I}$ of U there is an equalizer diagram

$$\mathcal{F}(U) \longrightarrow \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(U_i \cap U_j)$$

We denote by $\mathcal{C}_X$ the full subcategory of $\mathcal{C}_X^{\mathrm{pre}}$ consisting of all $\mathcal{C}$-sheaves on X.

It follows from the definition that the empty product is the final object in $\mathcal{C}$. By the sheaf axiom, we see that $\mathcal{F}(\varnothing)$ is the final object in $\mathcal{C}$ as long as $\mathcal{F}$ is a sheaf.

2.3 Sheaf on a basis. Let $\mathcal{B}$ be a basis of open sets of X. We can view $\mathcal{B}$ as a full subcategory of $\mathrm{Top}(X)$. We define a **$\mathcal{C}$-presheaf on the basis** $\mathcal{B}$ to be a contravariant functor $\mathcal{B} \to \mathcal{C}$. A $\mathcal{C}$-presheaf $\mathcal{F}$ on the basis $\mathcal{B}$ is called a sheaf if for any $B \in \mathcal{B}$ and its open cover $\{B_i\}_{i\in I} \subseteq \mathcal{B}$, there exists an equalizer diagram

$$\mathcal{F}(B) \longrightarrow \prod_{i\in I} \mathcal{F}(B_i) \rightrightarrows \prod_{i,j\in I} \prod_{\substack{B'\subseteq B_i\cap B_j \\ B'\in\mathcal{B}}} \mathcal{F}(B')$$

Clearly, every sheaf on X restricts to a sheaf on $\mathcal{B}$. What's more, the converse is also true. Let $\mathcal{F}$ denote a sheaf on $\mathcal{B}$. For $U \in \mathrm{Top}(X)$, define

$$\mathcal{F}'(U) := \varprojlim_{\mathcal{B}\ni B\subseteq U} \mathcal{F}(B).$$

Note that when $U = B \in \mathcal{B}$, the canonical projection $\mathcal{F}'(B) \to \mathcal{F}(B)$ is an isomorphism. For opens $V \subseteq U$, clearly we have $\mathcal{F}'(U) \to \mathcal{F}'(V)$ given by projections. This shows $\mathcal{F}'$ defines a presheaf on X. To show it is a sheaf, let $U \in \mathrm{Top}(X)$ and $\{U_i\}_{i\in I}$ be an open cover of U. We must show

$$\mathcal{F}'(U) \longrightarrow \prod_{i\in I} \mathcal{F}'(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}'(U_i \cap U_j)$$

is an equalizer diagram. A possible way to show this is to express inverse limit as a equalizer of certain arrows, and do some easy diagram chasing; we omit the proof. In fact, $\mathcal{F}$ is unique up to isomorphism, which is easy to see.

If $\mathcal{B}$ is closed under finite intersection, then for a presheaf $\mathcal{F}$ on $\mathcal{B}$, it is a sheaf if and only if for any $B \in \mathcal{B}$ and its open cover $\{B_i\}_{i\in I} \subseteq \mathcal{B}$, there exists an equalizer diagram

$$\mathcal{F}(B) \longrightarrow \prod_{i\in I} \mathcal{F}(B_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(B_i \cap B_j) \tag{$\star$}$$

This is easily checked from the definition.

Let $\mathcal{B}, B, \{B_i\}_{i\in I}$ be as above. Suppose, in addition, that $B \in \mathcal{B}$ is compact. Then to show the exactness of $(\star)$, it suffices to show the exactness of

$$\mathcal{F}(B) \longrightarrow \prod_{i\in J} \mathcal{F}(B_i) \rightrightarrows \prod_{i,j\in J} \mathcal{F}(B_i \cap B_j)$$

for all finite subsets J of I with $B = \bigcup_{i\in J} B_i$. Indeed, the injectivity is clear, and to show the exactness in the middle, consider the diagram

$$\begin{tikzcd}
\mathcal{F}(B) \arrow[r] \arrow[d] & \prod_{i\in I} \mathcal{F}(B_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \prod_{i,j\in I} \mathcal{F}(B_i \cap B_j) \arrow[d] \\
\prod_{J\subseteq I} \mathcal{F}(B) \arrow[r] & \prod_{J\subseteq I}\prod_{i\in J} \mathcal{F}(B_i) \arrow[r, shift left] \arrow[r, shift right] & \prod_{J\subseteq I}\prod_{i,j\in J} \mathcal{F}(B_i \cap B_j)
\end{tikzcd}$$

An easy diagram chasing and argument then shows the exactness of the above sequence.

2.3.1 Let X be a topological space and $\mathcal{B}$ an open basis of the topology on X. Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves on X. Suppose for any $B \in \mathcal{B}$, there exists a map $\mathcal{F}(B) \to \mathcal{G}(B)$ that is compatible with restriction. Then there exists a unique morphism $\mathcal{F} \to \mathcal{G}$ of sheaves extending those maps on $\mathcal{B}$. This is easy to construct and see its uniqueness once we regard $\mathcal{F}$ and $\mathcal{G}$ as constructed from sheaves on $\mathcal{B}$, just like we do in 2.3.

If we denote by $\mathcal{C}_{X,\mathcal{B}}$ the category of $\mathcal{C}$-sheaves on the basis $\mathcal{B}$, then our discussion shows that the restriction defines a natural equivalence

$$\mathcal{C}_X \xrightarrow{\sim} \mathcal{C}_{X,\mathcal{B}}$$

2.4 Stalk and Sheafification. Let $\mathcal{F}$ be a $\mathcal{C}$-presheaf. For each point x of X, the **stalk** of $\mathcal{F}$ at $x \in X$ is the direct limit

$$\mathcal{F}_x := \varinjlim_{\mathrm{Top}(X)\ni U\ni x} \mathcal{F}(U)$$

where the open neighborhoods of x are directed by inclusions. The **étale space** of $\mathcal{F}$ is the set-theoretic disjoint union

$$\mathrm{Et}\,\mathcal{F} = \bigsqcup_{x\in X} \mathcal{F}_x$$

If $s \in \mathcal{F}(U)$, we use either s_x, $s|_x$ or $\varinjlim_{U\supseteq V\ni x} s|_V$ to denote its image in $\mathcal{F}_x$. For each $s \in \mathcal{F}(U)$, define $s_U : U \to \mathrm{Et}\,\mathcal{F}$ by $s_U(x) := (x, s_x)$. On $\mathrm{Et}\,\mathcal{F}$ we install the final topology with respect to the collection of maps $\{s_U \mid U \in \mathrm{Top}(X), s \in \mathcal{F}(U)\}$. With this topology the natural projection $\pi : \mathrm{Et}\,\mathcal{F} \to X$ becomes a local homeomorphism.

Denote by $\mathcal{F}^\dagger$ the sheaf of continuous section of $\pi : \mathrm{Et}\,\mathcal{F} \to X$. If $\mathcal{F}$ is a $\mathcal{C}$-presheaf, then $\mathcal{F}^\dagger$ is a $\mathcal{C}$-sheaf. This sheaf $\mathcal{F}^\dagger$ is called the **sheafification** of the presheaf $\mathcal{C}$. Clearly $(\cdot)^\dagger$ defines a functor from $\mathcal{C}_X^{\mathrm{pre}}$ to $\mathcal{C}_X$. The map $s \mapsto s_U$ defines a morphism $\iota_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^\dagger$. This morphism $\iota_{\mathcal{F}}$ enjoys the universal property : there is a bifunctorial bijection

$$\mathrm{Hom}_{\mathcal{C}_X^{\mathrm{pre}}}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Hom}_{\mathcal{C}_X}(\mathcal{F}^\dagger, \mathcal{G})$$

whose inverse is given by pre-composing with $\iota_{\mathcal{F}}$. Here $\mathcal{G}$ is a $\mathcal{C}$-sheaf, and we view it as a $\mathcal{C}$-presheaf on the left. In other words, the sheafification functor is left adjoint to the forgetful functor $\mathcal{C}_X \to \mathcal{C}_X^{\mathrm{pre}}$. In particular, this shows if $\mathcal{F}$ is already a sheaf, then $\iota_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^\dagger$ is an isomorphism of sheaves.

2.4.1 A categorical caveat. The category of **Ring** is not well-behaved compared to **Ab** and $\mathbf{Mod}_R$. One point that deserves an attention is that the forgetful functor **Ring** $\to$ **Set** does not preserve arbitrary colimit : it only preserves filtered colimit. Nevertheless, the set-theoretic filtered colimit of rings has a unique structure of a ring so that it is a colimit in **Ring**. In particular, the stalk of a (pre)sheaf of rings is indeed a ring.

2.4.2 Retain the notation in (2.4). Since each section $s \in \mathcal{F}^\dagger(U)$ is necessarily injective, we obtain a canonical injection

$$\begin{aligned} \mathcal{F}^\dagger(U) &\longrightarrow \prod_{x \in U} \mathcal{F}_x \\ f &\longmapsto (f(x))_{x \in U}. \end{aligned}$$

It is easy to describe the image in $\prod_{x \in U} \mathcal{F}_x$:

$$\mathcal{F}^\dagger(U) = \left\{ (f_x)_{x \in U} \in \prod_{x \in U} \mathcal{F}_x \;\middle|\; \begin{array}{l} \text{for any } x \in U \text{ there exist an open neighborhood } V \subseteq U \text{ of } x \\ \text{and } s \in \mathcal{F}(V) \text{ such that } f_y = s|_y \text{ for all } y \in V \end{array} \right\}.$$

2.4.3 For any presheaf $\mathcal{F}$ and $x \in X$, the canonical morphism $\iota = \iota_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^\dagger$ induces a map $\iota_x : \mathcal{F}_x \to \mathcal{F}_x^\dagger$ on the stalk. This is in fact an isomorphism in $\mathcal{C}$. To see this, an element $f_x \in \mathcal{F}_x^\dagger$ is represented by some $f \in \mathcal{F}^\dagger(U)$, where U is a small neighborhood of x in X. By shrinking U further we can assume $f(U) = \{(x, s_x) \mid x \in U\}$ for some $s \in \mathcal{F}(U)$. Then $s_U = f$ since they are sections of a homeomorphism $\pi|_{f(U)}^U$, and thus $(s_U)_x = f_x$. This proves surjectivity, and injectivity can be proved in a similar way.

2.5 Define a category Et_X as follows. An object is a pair $(Y, p_Y : Y \to X)$ consisting of a topological space Y and a local homeomorphism $p_Y : Y \to X$. A morphism between objects is a continuous map compatible with their projections to the base X. Taking sheaf of continuous sections defines a functor from Et_X to $\mathbf{Set}_X$ (operations between sections are defined stalk-wise). In fact, this establishes an equivalence of categories

$$\mathrm{Et}_X \longrightarrow \mathbf{Set}_X$$

with inverse given by associating a sheaf $\mathcal{F}$ with its étale space $\mathrm{Et}\,\mathcal{F} \to X$.

2.6 Example : constant sheaf. Let X be a topological space and A a set. The presheaf $U \mapsto A$, $\varnothing \mapsto \{*\}$ is called the **constant presheaf** $\underline{A}_X^{\mathrm{pre}}$, which is usually not a sheaf. Its sheafification is called the constant sheaf, and is denoted by $\underline{A}_X$. There is an explicit description of $\underline{A}_X$: for any open U in X

$$\underline{A}_X(U) = \{f : U \to A \mid f \text{ locally constant}\}.$$

This is clear from the construction of sheafification (2.4). It can be shown that if $\#A \geqslant 2$, then $\underline{A}_X^{\mathrm{pre}}$ is already a sheaf if and only if X is irreducible (3.34).

2.7 Example : skyscraper sheaf. Let X be a topological space and A a set. Let $x \in X$ be a point. Then the presheaf

$$U \mapsto \begin{cases} A & \text{, if } x \in U \\ \{*\} & \text{, if } x \notin U \end{cases}$$

is a sheaf, and is called **the skyscraper sheaf at** x **with value** A. It is so named as if we denote this sheaf by $\mathcal{F}$, then

$$\mathcal{F}_y = \begin{cases} A & \text{, if } y \in \overline{\{x\}} \\ \{*\} & \text{, if } y \notin \overline{\{x\}} \end{cases}$$

Particularly, if $\{x\} = \overline{\{x\}}$ (e.g. if X is a T_1 space), then x is the only point at which $\mathcal{F}$ has nontrivial stalk.

If we view A as a sheaf on the one point space $\{x\}$ and denote by $\iota_x : \{x\} \to X$ the inclusion, then $\mathcal{F} = (\iota_x)_* A$ (2.9).

2.8 Example : locally constant sheaf. Let X be a topological space. A sheaf $\mathcal{F}$ is called a **locally constant sheaf** if there exists an open cover $\mathcal{U}$ of X such that $\mathcal{F}|_U$ is isomorphic to a constant sheaf for each $U \in \mathcal{U}$. Here for a presheaf $\mathcal{G}$ and an open set U of X, the presheaf $\mathcal{G}|_U$ is a presheaf on U defined by $V \mapsto \mathcal{F}(V)$. Clearly, if $\mathcal{G}$ is a sheaf on X, then $\mathcal{G}|_U$ is a sheaf on U.

2.1.1 Adjunction between f^{-1} and f_*.

2.9 Let $f : X \to Y$ be a continuous map. For a $\mathcal{C}$-presheaf $\mathcal{F}$ on X, define the **direct image presheaf** (or the push-forward presheaf) $f_*\mathcal{F}$ on Y by the formula

$$f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$$

where $V \in \mathrm{Top}(Y)$. If $\mathcal{F}$ is a $\mathcal{C}$-sheaf, then $f_*\mathcal{F}$ is also a $\mathcal{C}$-sheaf. If $\mathcal{G}$ is a $\mathcal{C}$-presheaf on Y, define the **inverse image presheaf** (or pull-back presheaf) $f^{\mathrm{pre}}\mathcal{F}$ on X by

$$f^{\mathrm{pre}}\mathcal{F}(U) = \varinjlim_{\mathrm{Top}(Y) \ni V \supseteq f(U)} \mathcal{F}(V)$$

where $U \in \mathrm{Top}(X)$. When $\mathcal{F}$ is a sheaf, $f^{\mathrm{pre}}\mathcal{F}$ may still fail to be a sheaf. Nevertheless, for a sheaf $\mathcal{F}$, we define the **inverse image sheaf** $f^{-1}\mathcal{F}$ of $\mathcal{F}$ to be the sheafification of $f^{\mathrm{pre}}\mathcal{F}$.

We compute the stalk of $f^{\mathrm{pre}}\mathcal{F}$. For $x \in X$,

$$(f^{\mathrm{pre}}\mathcal{F})_x = \varinjlim_{\mathrm{Top}(X) \ni U \ni x} \ \varinjlim_{\mathrm{Top}(Y) \ni V \supseteq f(U)} \mathcal{F}(V) \cong \varinjlim_{V \ni f(x)} \mathcal{F}(V) = \mathcal{F}_{f(x)}$$

There is no formula for $(f_*\mathcal{F})_y$ in general. Nevertheless, if X is a subspace of Y and f is the inclusion, we have $(f_*\mathcal{F})_y = \mathcal{F}_y$ if $y \in X$. If X is closed, then $(f_*\mathcal{F})_y = 0$ for $y \notin X$.

2.10 Adjunction between f^{-1} and f_*. Define a category $\mathcal{C}^{\mathrm{pre}}_{\mathbf{Top}}$ as follows.

- An object in $\mathcal{C}^{\mathrm{pre}}_{\mathbf{Top}}$ is a topological space X together with a $\mathcal{C}$-presheaf $\mathcal{F}$. We denote an object by $(X, \mathcal{F})$.
- For two objects $(X, \mathcal{F})$ and $(Y, \mathcal{G})$, a morphism between them is a continuous map $f : X \to Y$ together with a collection of maps $\left\{\mathcal{G}(V) \xrightarrow{T_{U,V}} \mathcal{F}(U)\right\}_{\substack{f(U) \subseteq V \\ U \in \mathrm{Top}(X),\, V \in \mathrm{Top}(Y)}} \subseteq \mathrm{Mor}(\mathcal{C})$ compatible with the restriction.

Let us have a careful look at the condition imposed on a morphism in $\mathcal{C}^{\mathrm{pre}}_{\mathbf{Top}}$. If $f(U) \subseteq V$, then $U \subseteq f^{-1}(V)$, the latter being open in X since f is continuous. Since a morphism is compatible with inclusions, we see that for a fixed $V \in \mathrm{Top}(Y)$, the subcollection $\{T_{U,V}\}_{\substack{f(U) \subseteq V \\ U \in \mathrm{Top}(X)}}$ is completely determined by a distinguished element, namely $T_{f^{-1}(V),V} : \mathcal{G}(V) \to \mathcal{F}(f^{-1}(V))$. On the other hand, for a fixed $U \in \mathrm{Top}(X)$, the subcollection $\{T_{U,V}\}_{\substack{f(U) \subseteq V \\ V \in \mathrm{Top}(Y)}}$ is packed to a map from the direct limit $\varinjlim_{V \supseteq f(U)} \mathcal{G}(V) \to \mathcal{F}(U)$.

To draw a conclusion, let us consider the forgetful functor $\mathcal{C}^{\mathrm{pre}}_{\mathbf{Top}} \to \mathbf{Top}$. For objects $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ in $\mathcal{C}_{\mathbf{Top}}$, the forgetful functor gives a projection map

$$\Phi : \mathrm{Hom}_{\mathcal{C}^{\mathrm{pre}}_{\mathbf{Top}}}((X, \mathcal{F}), (Y, \mathcal{G})) \longrightarrow \mathrm{Hom}_{\mathbf{Top}}(X, Y)$$

The above consideration implies the preimage under Φ of a continuous map $f : X \to Y$ has two interpretations, that is,

$$\mathrm{Hom}_{\mathcal{C}_Y^{\mathrm{pre}}}(\mathcal{G}, f_*\mathcal{F}) \xrightarrow{\ \sim\ } \Phi^{-1}(f) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathcal{C}_X^{\mathrm{pre}}}(f^{\mathrm{pre}}\mathcal{G}, \mathcal{F})$$

Ignoring the "bridge", we obtain the **adjunction** of f_* and f^{pre}. If we consider the full subcategory $\mathcal{C}_{\mathbf{Top}}$ of $\mathcal{C}_{\mathbf{Top}}^{\mathrm{pre}}$ consisting of $(X, \mathcal{F})$ with $\mathcal{F}$ a $\mathcal{C}$-sheaf on X, by composing with the sheafification we obtain the adjunction of f_* and f^{-1} :

$$\mathrm{Hom}_{\mathcal{C}_Y}(\mathcal{G}, f_*\mathcal{F}) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathcal{C}_X}(f^{-1}\mathcal{G}, \mathcal{F})$$

For two appropriate continuous map f and g, we clearly have $(fg)_* = f_*g_*$. The adjunction then shows that $(fg)^{-1}$ and $g^{-1}f^{-1}$ are "naturally isomorphic".

2.10.1 Saying that f_* and f^{-1} are adjoint to each other in the categorical sense requires them to be functors. For a continuous map $f : X \to Y$, define the **direct image functor** $f_* : \mathcal{C}_X^{\mathrm{pre}} \to \mathcal{C}_Y^{\mathrm{pre}}$ as follows. If $\mathcal{F}$ is a $\mathcal{C}$-presheaf, put $f_*\mathcal{F}$ as in (2.9), and for a morphism $T : \mathcal{F} \to \mathcal{G}$ in $\mathcal{C}_X^{\mathrm{pre}}$, define $f_*T : f_*\mathcal{F} \to f_*\mathcal{G}$ by assigning to each open set $V \in \mathrm{Top}(Y)$ a morphism $(f_*T)_V = T_{f^{-1}(V)}$. Clearly such a definition makes f_* into a functor, and it sends $\mathcal{C}$-sheaves on X to those on Y. It also yields a functorial map

$$\mathrm{Hom}_{\mathcal{C}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Hom}_{\mathcal{C}_Y}(f_*\mathcal{F}, f_*\mathcal{G}). \tag{$\spadesuit$}$$

Next, define the **inverse image functor** $f^{\mathrm{pre}} : \mathcal{C}_Y^{\mathrm{pre}} \to \mathcal{C}_X^{\mathrm{pre}}$ as follows. For a $\mathcal{C}$-presheaf $\mathcal{F}$ on Y, define $f^{\mathrm{pre}}\mathcal{F}$ as in (2.9), and for a morphism $T : \mathcal{F} \to \mathcal{G}$ in $\mathcal{C}_Y^{\mathrm{pre}}$, define $f^{\mathrm{pre}}T : f^{\mathrm{pre}}\mathcal{F} \to f^{\mathrm{pre}}\mathcal{G}$ by assigning to each open set $U \in \mathrm{Top}(X)$ a morphism

$$(f^{\mathrm{pre}}T)_U = \varinjlim_{\mathrm{Top}(Y) \ni V \supseteq f(U)} T_V : \varinjlim_{\mathrm{Top}(Y) \ni V \supseteq f(U)} \mathcal{F}(V) \to \varinjlim_{\mathrm{Top}(Y) \ni V \supseteq f(U)} \mathcal{G}(V).$$

This morphism is obtained by the universal property of the direct limit, applied to the morphisms $\mathcal{F}(V) \to \mathcal{G}(V) \to \varinjlim_{\mathrm{Top}(Y) \ni V \supseteq f(U)} \mathcal{G}(V)$. Since taking direct limit is functorial, f^{pre} is really a functor. Restricting to the full subcategory $\mathcal{C}_Y$ and post-composing with the sheafification functor defines the **inverse image functor** $f^{-1} : \mathcal{C}_Y \to \mathcal{C}_X$ on sheaves. Again, we have a functorial map

$$\mathrm{Hom}_{\mathcal{C}_Y}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Hom}_{\mathcal{C}_X}(f^{-1}\mathcal{F}, f^{-1}\mathcal{G}). \tag{$\clubsuit$}$$

2.10.2 Adjunction and stalks Let $\theta : \mathcal{G} \to f_*\mathcal{F}$ be a morphism of sheaves on Y, and let $\theta^\sharp : f^{-1}\mathcal{G} \to \mathcal{F}$ be the morphism obtained by adjunction. Let's compare the induced maps on stalks. Let $x \in X$ and $y = f(x) \in Y$. The stalk map of θ at y is

$$\mathcal{G}_y \to (f_*\mathcal{F})_y = \varinjlim_{\mathrm{Top}(Y) \ni V \ni y} \mathcal{F}(f^{-1}(V))$$

while by (2.9) the stalk map of $\theta^\sharp$ at x is

$$\mathcal{G}_y = (f^{-1}\mathcal{G})_x \to \mathcal{F}_x = \varinjlim_{\mathrm{Top}(X) \ni U \ni x} \mathcal{F}(U).$$

By the universal property of direct limits, these two are related in a diagram

$$\begin{tikzcd} \mathcal{G}_y \arrow[r, "\theta_y"] \arrow[dr, "\theta_x^\sharp"'] & (f_*\mathcal{F})_y \arrow[d] \\ & \mathcal{F}_x \end{tikzcd}$$

It follows from the very construction of adjunction that this is a commutative triangle.

2.11 Let us analyze the adjunction (2.10) further in the case $\mathcal{C} = \mathbf{Mod}$. By definition an objective in $\mathcal{C}$ is a pair $A \rightharpoondown M$, where A is a ring and M is an A-module. A morphism $A \rightharpoondown M \to B \rightharpoondown N$ in **Mod** is a ring homomorphism $r : A \to B$ and an abelian group homomorphism $f : M \to N$ with compatible ring action; that is, there is a commutative diagram.

$$\begin{tikzcd} A \times M \arrow[r] \arrow[d, "r \times f"'] & M \arrow[d, "f"] \\ B \times N \arrow[r] & N \end{tikzcd}$$

Let $(X, \mathcal{A} \rightharpoondown \mathcal{M})$ and $(Y, \mathcal{B} \rightharpoondown \mathcal{N})$ be in $\mathbf{Mod}^{\mathrm{pre}}_{\mathbf{Top}}$. Forgetful functors gives a chain of projections

$$\mathrm{Hom}_{\mathbf{Mod}^{\mathrm{pre}}_{\mathbf{Top}}}((X, \mathcal{A} \rightharpoondown \mathcal{M}), (Y, \mathcal{B} \rightharpoondown \mathcal{N})) \longrightarrow \mathrm{Hom}_{\mathbf{Ring}^{\mathrm{pre}}_{\mathbf{Top}}}((X, \mathcal{A}), (Y, \mathcal{B})) \longrightarrow \mathrm{Hom}_{\mathbf{Top}}(X, Y)$$

Let $f : X \to Y$ be a continuous map. As in (2.10), consider the following diagram.

$$\begin{tikzcd} & \mathrm{Hom}_{\mathbf{Mod}^{\mathrm{pre}}_{\mathbf{Top}}}((X, \mathcal{A} \rightharpoondown \mathcal{M}), (Y, \mathcal{B} \rightharpoondown \mathcal{N})) \arrow[d] & \\ \mathrm{Hom}_{\mathbf{Ring}^{\mathrm{pre}}_{X}}(f^{\mathrm{pre}}\mathcal{B}, \mathcal{A}) \arrow[r] & \mathrm{Hom}_{\mathbf{Ring}^{\mathrm{pre}}_{\mathbf{Top}}}((X, \mathcal{A}) & \mathrm{Hom}_{\mathrm{Ring}^{\mathrm{pre}}_{Y}}(\mathcal{B}, f_*\mathcal{A}) \arrow[l] \end{tikzcd}$$

The fibre of f in the middles gives a bijection between the leftmost set and the rightmost set. Let $\theta : \mathcal{B} \to f_*\mathcal{A}$ be in the rightmost and that $\theta^\sharp$ be the corresponding element in the leftmost; they map to the same element (f, θ) in the middle. We ask what is the fibre of (f, θ) in the upper set. A moment consideration gives the answer :

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{Mod}^{\mathrm{pre}}_{\mathcal{A}}}(\mathcal{A} \otimes_{f^{\mathrm{pre}}\mathcal{B}} f^{\mathrm{pre}}\mathcal{N}, \mathcal{M}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbf{Mod}^{\mathrm{pre}}_{\mathbf{Top}}}((X, \mathcal{A} \rightharpoondown \mathcal{M}), (Y, \mathcal{B} \rightharpoondown \mathcal{N})) \arrow[d] & \mathrm{Hom}_{\mathbf{Mod}^{\mathrm{pre}}_{\mathcal{B}}}(\mathcal{N}, (f_*\mathcal{M})^{[\theta]}) \arrow[l] \arrow[d] \\ \{\theta^\sharp : f^{\mathrm{pre}}\mathcal{B} \to \mathcal{A}\} \arrow[r] & \mathrm{Hom}_{\mathbf{Ring}^{\mathrm{pre}}_{\mathbf{Top}}}((X, \mathcal{A}) & \{\theta : \mathcal{B} \to f_*\mathcal{A}\} \arrow[l] \end{tikzcd}$$

We need to explain the notations used here.

- For a presheaf (resp. sheaf) of rings $\mathcal{A}$ on X, a presheaf (resp. sheaf) $\mathcal{M}$ of abelian groups on X is called a **presheaf of $\mathcal{A}$-modules** (resp. **sheaf of $\mathcal{A}$-modules**) if $\mathcal{M}(U)$ is an $\mathcal{A}(U)$-module for any open U and the module structure is compatible with the restriction of $\mathcal{A}$. A **morphism** between presheaves of $\mathcal{A}$-modules is a morphism of presheaves T such that $T(U)$ is an $\mathcal{O}_X(U)$-module homomorphism for every open U. We denote by $\mathbf{Mod}^{\mathrm{pre}}_{\mathcal{A}}$ the category of presheaves of $\mathcal{A}$-modules, and when $\mathcal{A}$ is sheaf, we denote by $\mathbf{Mod}_{\mathcal{A}}$ the full subcategory of $\mathbf{Mod}^{\mathrm{pre}}_{\mathcal{A}}$ consisting of sheaves of $\mathcal{A}$-modules.
- For $\mathcal{M}, \mathcal{N} \in \mathbf{Mod}^{\mathrm{pre}}_{\mathcal{A}}$, define their **tensor product** $\mathcal{M} \otimes^{\mathrm{p}}_{\mathcal{A}} \mathcal{N} \in \mathbf{Mod}^{\mathrm{pre}}_{\mathcal{A}}$ by the assignment $U \mapsto \mathcal{M}(U) \otimes_{\mathcal{A}(U)} \mathcal{N}(U)$. When in $\mathbf{Mod}_{\mathcal{A}}$, we define $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N} \in \mathbf{Mod}_{\mathcal{A}}$ by $(\mathcal{M} \otimes^{\mathrm{p}}_{\mathcal{A}} \mathcal{N})^\dagger$.
- $(f_*\mathcal{M})^{[\theta]})$ means we use $\theta : \mathcal{B} \to f_*\mathcal{A}$ to view $f_*\mathcal{M}$ as a $\mathcal{B}$-module.
- On the left-upper corner, $\mathcal{A}$ is viewed as a $f^{\mathrm{pre}}\mathcal{B}$-module via $\theta^\sharp$.

Let us replace everything by sheaves. Suppose that $(X, \mathcal{A} \rightharpoondown \mathcal{M})$ and $(Y, \mathcal{B} \rightharpoondown \mathcal{N})$ are actually in $\mathbf{Mod}_{\mathbf{Top}}$. Passing to sheafification, the upper part of the diagram above gives a bijection

$$\mathrm{Hom}_{\mathbf{Mod}_{\mathcal{A}}}\Big((\mathcal{A} \otimes_{f^{\mathrm{pre}}\mathcal{B}} f^{\mathrm{pre}}\mathcal{N})^\dagger, \mathcal{M}\Big) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Mod}_{\mathcal{B}}}(\mathcal{N}, (f_*\mathcal{M})^{[\theta]})$$

The same consideration, but with each f^{pre} replaced by f^{-1}, gives a bijection

$$\mathrm{Hom}_{\mathbf{Mod}_{\mathcal{A}}}\left(\left(\mathcal{A}\otimes_{f^{-1}\mathcal{B}} f^{-1}\mathcal{N}\right)^{\dagger}, \mathcal{M}\right) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{Mod}_{\mathcal{B}}}(\mathcal{N}, (f_*\mathcal{M})^{[\theta]})$$

As a by-product, we see $(\mathcal{A}\otimes_{f^{\mathrm{pre}}\mathcal{B}} f^{\mathrm{pre}}\mathcal{N})^{\dagger}$ and $\left(\mathcal{A}\otimes_{f^{-1}\mathcal{B}} f^{-1}\mathcal{N}\right)^{\dagger}$ are naturally isomorphic. We give it a new notation : $f^*\mathcal{N} := (\mathcal{A}\otimes_{f^{\mathrm{pre}}\mathcal{B}} f^{\mathrm{pre}}\mathcal{N})^{\dagger} \in \mathbf{Mod}_{\mathcal{A}}$. This is called the **inverse image** of $\mathcal{N}$ by (f,θ). Also, we simply put $f_*\mathcal{M} = (f_*\mathcal{M})^{[\theta]}$, and call it the **direct image** of $\mathcal{N}$ by (f,θ). In this way, the adjunction takes the form

$$\mathrm{Hom}_{\mathbf{Mod}_{\mathcal{A}}}(f^*\mathcal{N}, \mathcal{M}) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{Mod}_{\mathcal{B}}}(\mathcal{N}, f_*\mathcal{M})$$

It is easy to see $f_* : \mathbf{Mod}_{\mathcal{A}} \to \mathbf{Mod}_{\mathcal{B}}$ and $f^* : \mathbf{Mod}_{\mathcal{B}} \to \mathbf{Mod}_{\mathcal{A}}$ define functors, and the bijection above is bifunctorial in $\mathcal{M}$ and $\mathcal{N}$. Concisely, this bijection says that f^* is left adjoint to f_*.

2.12 We compute the stalk of $f_*\mathcal{M}$ and $f^*\mathcal{N}$. For the former, the stalk is the same as the one for the usual direct image, since computing stalk has nothing to do with the module structure. For the latter, by (2.4.3) we only need to compute

$$\left(\mathcal{A}\otimes^{\mathrm{p}}_{f^{\mathrm{pre}}\mathcal{B}} \mathcal{N}\right)_x$$

It follows from the following lemma and (2.9) that this is naturally isomorphic to

$$\mathcal{A}_x \otimes_{\mathcal{B}_{f(x)}} \mathcal{N}_{f(x)}$$

Lemma. Let $(A_\alpha)_\alpha$ be a directed system of rings, $(M_\alpha)_\alpha$ and $(N_\alpha)_\alpha$ be directed systems of abelian groups with M_α and N_α being A_α-modules and the transition maps being compatible with the ring homomorphisms $A_\alpha \to A_\beta$. Then there is a natural bijection

$$\varinjlim_\alpha M_\alpha \otimes_{A_\alpha} N_\alpha \cong \left(\varinjlim_\alpha M_\alpha\right) \otimes_{\varinjlim_\alpha A_\alpha} \left(\varinjlim_\alpha N_\alpha\right) .$$

Here $M_\alpha \otimes_{A_\alpha} N_\alpha$ is directed by the natural map $M_\alpha \otimes_{A_\alpha} N_\alpha \to M_\beta \otimes_{A_\beta} N_\beta$, which exists either by the explicit construction of tensor products or by the universal property.

Proof. It suffices to show $\varinjlim_\alpha M_\alpha \otimes_{A_\alpha} N_\alpha$ satisfies the obvious universal property that $\left(\varinjlim_\alpha M_\alpha\right) \otimes_{\varinjlim_\alpha A_\alpha} \left(\varinjlim_\alpha N_\alpha\right)$ enjoys. For brevity, let A, M, N stand for the limit objects $\varinjlim_\alpha A_\alpha, \varinjlim_\alpha M_\alpha, \varinjlim_\alpha N_\alpha$. Let H be an A-module, and let $T : M \times N \to H$ be an A-bilinear map. Precomposing with $M_\alpha \times N_\alpha \to M \times N$, we obtain an A_α-bilinear map $T_\alpha : M_\alpha \times N_\alpha \to H$, which by the universal property induces an A_α-linear map $M_\alpha \otimes_{A_\alpha} N_\alpha \to H$. From construction it is clear that $(M_\alpha \otimes_{A_\alpha} N_\alpha \to H)_\alpha$ is a cocone, so it gives a map $\varinjlim_\alpha M_\alpha \otimes_{A_\alpha} N_\alpha \to H$, as wanted. The uniqueness (resp. functoriality) of this map follows from the uniqueness (resp. functoriality) at each step. This finishes the proof. □

2.12.1 We extend the comparison done in (2.10.2) to this case. Let $\theta : \mathcal{N} \to f_*\mathcal{M}$ be a morphism in $\mathbf{Mod}_{\mathcal{A}}$, and let $\theta^\sharp : f^*\mathcal{N} \to \mathcal{M}$ be the corresponding morphism obtained by adjunction. Let $x \in X$ and $y = f(x) \in Y$. The stalks maps fits into a diagram

$$\begin{tikzcd} \mathcal{N}_y \arrow[r, "\theta_y"] \arrow[d] & (f_*\mathcal{M})_y \arrow[d] \\ \mathcal{A}_x \otimes_{\mathcal{B}_y} \mathcal{N}_y \cong (f^*\mathcal{N})_x \arrow[r, "\theta^\sharp_x"] & \mathcal{M}_x \end{tikzcd}$$

where the left-vertical arrow is $f \mapsto 1 \otimes f$, and the right-vertical arrow is the one in (2.10.2). It follows from (2.10.2) that this diagram commutes.

2.1.2 Gluing process

2.13 Gluing sheaves Let X_i $(i \in I)$ be a collection of topological space. On each X_i is a sheaf $\mathcal{F}_i$ of sets. Suppose for any $i \in I$ there exist open subspaces $X_{ij} \subseteq X_i$ and $X_{ji} \subseteq X_j$ and an isomorphism $f_{ij} : (X_{ij}, \mathcal{F}_i|_{X_{ij}}) \to (X_{ji}, \mathcal{F}_j|_{X_{ji}})$ in $\mathbf{Set_{Top}}$. Assume these f_{ij} satisfy

(a) $f_{ii} = \mathrm{id}_{X_i}$ for any $i \in I$;

(b) $f_{ik} = f_{jk} \circ f_{ij}$ on $X_{ij} \cap X_{ik}$ for any $i, j, k \in I$

Then there exists a topological space X containing each X_i as an open subspace with $X_i \cap X_j = X_{ij}$, a sheaf of sets $\mathcal{F}$ on X and isomorphisms $f_i : \mathcal{F}|_{X_i} \to \mathcal{F}_i$ of sheaves on X_i $(i \in I)$ satisfying $f_i = f_{ij} \circ f_j$ on $X_i \cap X_j$.

Moreover, the data $(\mathcal{F}, \{f_i\}_{i \in I})$ is unique up to a unique isomorphism, in the sense that if $(\mathcal{F}', \{f_i'\}_{i \in I})$ is another such data, then there exists a unique isomorphism $f : \mathcal{F} \to \mathcal{F}'$ of sheaves on X such that $f_i = f_i' \circ f|_{X_i}$ for any $i \in I$.

2.13.1 This can be phrased in the language of representable functors. Define a functor $F : \mathbf{Set_{Top}} \to \mathbf{Set}$ by

$$F(Y, \mathcal{G}) := \left\{ (g_i)_{i \in I} \in \prod_{i \in I} \mathrm{Hom}_{\mathbf{Set_{Top}}}((X_i, \mathcal{F}_i), (Y, \mathcal{G})) \;\middle|\; \begin{array}{l} g_i|_{X_{ij}} = g_j|_{X_{ji}} \circ f_{ij} \text{ for all } i, j \in I \\ g_k \circ f_{ik}|_{X_{ij} \cap X_{ik}} = g_i \circ f_{ji}|_{X_{ij} \cap X_{jk}} \circ f_{ij}|_{X_{ij} \cap X_{ij}} \text{ for all } i, j, k \in I \end{array} \right\}$$

Then any topological space X containing each X_i as an open subspace with $X_i \cap X_j = X_{ij}$ and a sheaf of sets $\mathcal{F}$ on X and isomorphisms $f_i : \mathcal{F}|_{X_i} \to \mathcal{F}_i$ of sheaves on X_i $(i \in I)$ satisfying $f_i = f_{ij} \circ f_j$ on $X_i \cap X_j$ represents the functor. Moreover, such a pair $(X, \mathcal{F})$ exists.

2.13.2 Proof. We claim if X is a scheme along with open subschemes isomorphic to the X_i respecting the gluing data, then X represents the functor F. We denote by $\iota_i : X_i \to X$ the open embedding. Note that our assumption implies $(\iota_i)_{i \in I} \in F(X)$. Let Y be a scheme and $(g_i)_{i \in I} \in F(Y)$. Define a map $g : X \to Y$ as follows. For $x \in X$, if $x \in \iota_i(X_i)$, then set $g(x) = g_i(\iota_i^{-1}(x))$. This is well-defined, as if $x \in \iota_j(X_j)$ as well, then $x \in \iota_i(X_i) \cap \iota_j(X_j) = \iota_i(X_{ij}) = \iota_j(X_{ji})$ and so $g_i(\iota_i^{-1}(x)) = g_j(f_{ij}(\iota_i^{-1}(x))) = g_j(\iota_j^{-1}(x))$. We must show $g : X \to Y$ is continuous. Let $U \subseteq Y$ be an open subset. It suffices to show $g^{-1}(U) \cap \iota(X_i)$ is open in X_i for each $i \in I$, and we prove this by showing $g^{-1}(U) \cap \iota_i(X_i) = \iota_i(g_i^{-1}(U))$; this is sufficient as the $g_i^{-1}(U)$ is open in X_i and ι_i is an open map. If $x \in \iota_i(g_i^{-1}(U))$, then $g_i(\iota_i^{-1}(x)) \in U$. Since $g_i(\iota_i^{-1}(x)) = g(x)$, this implies $x \in g^{-1}(U)$, or $x \in g^{-1}(U) \cap \iota_i(X_i)$. Conversely, if $x \in g^{-1}(U) \cap \iota_i(X_i)$, then $U \ni g(x) = g_i(\iota_i^{-1}(x))$ so $x \in \iota_i(g_i^{-1}(U))$.

We now turn to sheaves. We must define a sheaf map $\theta : \mathcal{O}_Y \to g_* \mathcal{O}_X$. What we have now is $g_i : \mathcal{O}_Y \to (g_i)_* \mathcal{O}_{X_i}$. By assumption the inclusion ι_i induces an isomorphism $\iota_i^{\iota(X_i)} : \mathcal{O}_X|_{\iota_i(X_i)} \cong (\iota_i)_* \mathcal{O}_{X_i}$. Since $g_i = g \circ \iota_i$, we obtain a map $\theta_i := (g_* \circ \iota_i^{\iota(X_i)})^{-1} \circ g_i : \mathcal{O}_Y \to (g_i)_* \mathcal{O}_{X_i} = g_* (\iota_i)_* \mathcal{O}_{X_i} \cong g_* \left(\mathcal{O}_X|_{\iota_i(X_i)} \right)$. We claim the maps $(g|_{\iota_i(X_i)}, \theta_i) : \iota_i(X_i) \to Y$ glue. Let $V \subseteq Y$ be an open set. Then the $(\theta_i)_V : \mathcal{O}_Y(V) \to \mathcal{O}_X(\iota_i(X_i) \cap g^{-1}(V))$ defines $\psi_V : \mathcal{O}_Y(V) \to \prod_{i \in I} \mathcal{O}_X(\iota_i(X_i) \cap g^{-1}(V))$.

To show $\mathcal{O}_Y(V)$ maps to $\mathcal{O}_X(g^{-1}(V))$, by the universal property of equalizer it suffices to show two arrows

$$\mathcal{O}_Y(V) \xrightarrow{\psi_V} \prod_{i \in I} \mathcal{O}_X(\iota_i(X_i) \cap g^{-1}(V)) \rightrightarrows \prod_{i,j \in I} \mathcal{O}_X(\iota_i(X_i) \cap \iota_j(X_j) \cap g^{-1}(V))$$

are the same. This follows from the condition $g_i|_{X_{ij}} = g_j|_{X_{ji}} \circ f_{ij}$. This finishes the construction, and defines a map

$$\begin{aligned} F(Y) &\longrightarrow \mathrm{Hom}_{\mathbf{Sch}}(X, Y) \\ (g_i)_{i \in I} &\longmapsto (g, \theta). \end{aligned}$$

The whole construction is functorial in Y, so this defines a natural transformation. It has an obvious inverse : if $g \in \mathrm{Hom}_{\mathbf{Sch}}(X, Y)$, then $(g \circ \iota_i)_{i \in I} \in F(Y)$ maps to g under the above map. Hence it is a natural isomorphism, proving that X represents F. In particular, this proves the uniqueness.

It remains to show the existence of such X. This is straightforward. Let $X' = \bigsqcup_{i\in I} X_i$ be the disjoint union of the spaces X_i, equipped with the final topology given by the inclusions $\iota_i : X_i \to X'$. Define a relation $\sim$ on X' be declaring $(i,x) \sim (j,y)$ iff $x \in X_{ij}$, $y \in X_{ji}$ and $f_{ij}(x) = y$. This is reflexive and symmetric by the third bullet, and is transitive by the cocycle condition. Let $X = X'/\sim$ and equip it with the quotient topology given by $\pi : X' \to X$. We show $\pi \circ \iota_i : X_i \to X$ is an open embedding. This is continuous by construction, and is injective as $\sim$ does not collapse $\iota_i(X_i)$. Let $U \subseteq X_i$ be open, Then $(\pi \circ \iota_i)(U)$ is open if and only if $\pi^{-1}(\pi \circ \iota_i)(U)$ is open, if and only if $\iota_j^{-1}\pi^{-1}(\pi \circ \iota_i)(U)$ is open in X_j. The latter set is exactly $f_{ij}(U \cap X_{ij})$, which is open in X_j. Observe also $(\pi \circ \iota_i)(X_{ij}) = (\pi \circ \iota_j)(X_{ji})$ for $i, j \in I$. To ease the notation, we identify X_i with its image in X; under this identification, the previous observation tells that $X_i \cap X_j = X_{ij} = X_{ji}$ as topological spaces.

Our final work is to glue together the sheaves $\mathcal{O}_{X_i}$. We leave it to the next paragraph.

2.13.3 Let X be a topological with an open cover $\mathcal{U} = \{U_i\}_{i\in I}$. Suppose that on each U_i there is a sheaf $\mathcal{F}_i$ of abelian groups. For any $i, j \in I$, suppose there is an isomorphism $\theta_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i\cap U_j}$ of sheaves on $U_i \cap U_j$. Assume these θ_{ij} satisfy

(a) θ_{ii} is the identity map for any $i \in I$;

(b) $\theta_{ik}|_{U_i\cap U_j\cap U_k} = \theta_{jk}|_{U_i\cap U_j\cap U_k} \circ \theta_{ij}|_{U_i\cap U_j\cap U_k}$.

Then there exists a unique sheaf of abelian groups $\mathcal{F}$ on X and isomorphisms $\theta_i : \mathcal{F}|_{U_i} \to \mathcal{F}_i$ of sheaves on U_i such that $\theta_j|_{U_i\cap U_j} = \theta_{ij} \circ \theta_i|_{U_i\cap U_j}$ for any $i, j \in I$.

Proof. Define a presheaf $\mathcal{F}$ on X by setting

$$\mathcal{F}(V) = \left\{ (s_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}_i(V \cap U_i) \mid (\theta_{ij})_{V\cap U_i \cap U_j}(s_i|^{V\cap U_i}_{V\cap U_i\cap U_j}) = s_j|^{V\cap U_j}_{V\cap U_i\cap U_j} \text{ for any } i, j \in I \right\}$$

By (b), this really defines a presheaf on X. We can show $\mathcal{F}$ is a sheaf of abelian groups by a way similar to (2.3). □

2.13.4 Let X, Y be two topological spaces, and $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X and Y, respectively. Suppose there are an open cover $\{U_i\}_{i\in I}$ and a collection of morphisms $f_i : (U_i, \mathcal{F}|_{U_i}) \to (Y, \mathcal{G})$ in $\mathbf{Ab_{Top}}$ such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for any $i, j \in I$. Then there exists a unique morphism $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$ in $\mathbf{Ab_{Top}}$ extending the f_i.

2.14 Let us mention some categorical limit and colimit objects in $\mathcal{C}_X^{\text{pre}}$ and $\mathcal{C}_X$.

- — Direct product. If $\{\mathcal{F}_i\}_{i\in I}$ is a family of presheaves (resp. sheaves), then $U \mapsto \prod_{i\in I} \mathcal{F}_i(U)$ defines a presheaves (resp. sheaves), and is the categorical product in either category.
- — Finite direct sum. Suppose $\mathcal{C} = \mathbf{Ab}$. Then finite direct sum coincides with finite direct product.
- — Inverse limit. Let $\{\mathcal{F}_i\}_{i\in I}$ be an inverse system of presheaves (resp. sheaves). Then the assignment $U \mapsto \varprojlim_{i\in I} \mathcal{F}_i(U)$ defines a presheaf (resp. sheaf), and is the categorical limit in either category.
- — Direct limit. Let $\{\mathcal{F}_i\}_{i\in I}$ be a direct system of presheaves (resp. sheaves). Then $U \mapsto \varinjlim_{i\in I} \mathcal{F}_i(U)$ is a presheaf, but fails to be a sheaf even if each $\mathcal{F}_i$ is a sheaf. For this, for a direct system of sheaves, we denote by $\varinjlim_{i\in I} \mathcal{F}_i$ the sheafification of the above direct limit presheaf. Both are categorical directed limit in respective category.

2.15 Let $\mathcal{C}$ be either **Ab**, **Ring** or $\mathbf{Mod}_R$. If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves (resp. sheaves), then $U \mapsto \ker \varphi(U)$ defines a presheaves (resp. sheaves), called the **kernel** of $\varphi : \mathcal{F} \to \mathcal{G}$, and is denoted by $\ker \varphi$. When $\mathcal{C}$ is **Ab** or $\mathbf{Mod}_R$, the assignment $U \mapsto \mathcal{G}(U)/\mathcal{F}(U) = \operatorname{coker} \varphi(U)$ only defines a presheaf, so if we are discussing in $\mathcal{C}_X$, we define the **cokernel** $\operatorname{coker} \varphi$ of $\varphi : \mathcal{F} \to \mathcal{G}$ to be the sheafification of the previously mentioned presheaf. Similarly, we define the **image** $\operatorname{Im} \varphi$ to be the sheafification of $U \mapsto \operatorname{Im} \varphi(U)$. Both kernel and cokernel in $\mathcal{C}_X^{\text{pre}}$ (resp. $\mathcal{C}_X$) satisfy the usual universal properties.

2.15.1 Lemma. Let X be a topological space and $\mathcal{A}$ a sheaf of rings. The categories $\mathbf{Mod}_{\mathcal{A}}^{\text{pre}}$ and $\mathbf{Mod}_{\mathcal{A}}$ are abelian. In particular, $\mathbf{Ab}_X^{\text{pre}} = \mathbf{Mod}_{\mathbb{Z}_X}^{\text{pre}}$ and $\mathbf{Ab}_X = \mathbf{Mod}_{\mathbb{Z}_X}$ are abelian.

2.16 By (2.15.1) we can talk about exactness in $\mathbf{Ab}_X^{\text{pre}}$ and $\mathbf{Ab}_X$. Explicitly, a sequence $\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$ is exact in $\mathbf{Ab}_X^{\text{pre}}$ (resp. $\mathbf{Ab}_X$) if the natural map $\operatorname{Im}\alpha \to \ker\beta$ is an isomorphism. Thus a sequence of sheaves may be exact in $\mathbf{Ab}_X$ while fails to be exact in $\mathbf{Ab}_X^{\text{pre}}$. A useful criterion for exactness is the following :

Lemma. A sequence $\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$ of sheaves of abelian groups is exact if and only if the induced map on the stalk $\mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x \xrightarrow{\beta_x} \mathcal{H}_x$ for every $x \in X$.

2.16.1 As a consequence, we see the sheafification functor $(\cdot)^\dagger : \mathbf{Ab}_X^{\text{pre}} \to \mathbf{Ab}_X$ is exact, in the sense that it sends short exact sequences to short exact sequences. To see this, let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence in $\mathbf{Ab}_X^{\text{pre}}$. By (2.4.3) and (2.16), we only need to show $0 \to \mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x \to 0$ is exact in $\mathbf{Ab}$ for every $x \in X$. This is indeed the case, which can be seen from the proof of (2.16).

2.16.2 Let us split the proof of Lemma 2.16 into several lemmas.

Lemma. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a sheaf of sets. Then ϕ is an isomorphism in $\mathbf{Set}_X$ if and only if $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism for all $x \in X$.

Proof. Only if part is clear as taking stalks is functorial. For the if part, it suffices to show $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism for every open $U \subseteq X$, for various inverses must glue to an inverse of ϕ. Consider the commutative diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\phi_U"] \arrow[d, hook] & \mathcal{G}(U) \arrow[d, hook] \\ \prod_{x\in U} \mathcal{F}_x \arrow[r, "\prod_x \phi_x"] & \prod_{x\in U} \mathcal{G}_x \end{tikzcd}$$

We see ϕ_U is injective at once. Let $s \in \mathcal{G}(U)$ and denote by $(s_x)_x$ its image in the product of stalks. Since the bottom map is an isomorphism, we can find $(t_x)_x \in \prod_{x\in U} \mathcal{F}_x$ such that $\prod_x \phi_x(t_x)_x = (s_x)_x$. For each $x \in U$ pick an open neighborhood U_x of x and $t_{U_x} \in \mathcal{F}(U_x)$ such that $(t_{U_x})_x = t_x$. Since $\phi_{U_x}(t_{U_x})_x = \phi_x(t_x) = s_x = (s|_{U_x})_x$, shrinking U_x if necessary, we can assume $\phi_{U_x}(t_{U_x}) = s|_{U_x}$. For $x, y \in U$, the injectivity of $\phi_{U_x \cap U_y}$ implies $t_{U_x}|_{U_x\cap U_y} = t_{U_y}|_{U_x\cap U_y}$, so there exists $t \in \mathcal{F}(U)$ with $t|_{U_x} = t_{U_x}$. Then $\phi_U(t) = s$, proving the surjectivity. □

2.16.3 Lemma. Let $\phi, \psi : \mathcal{F} \to \mathcal{G}$ be two morphisms of sheaves of sets. Then $\phi = \psi$ if and only if $\phi_x = \phi_x$ for all $x \in X$.

Proof. The only if part is evident, and the if part follows from the commutative diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, shift left, "\psi_U"] \arrow[r, shift right, "\phi_U"'] \arrow[d, hook] & \mathcal{G}(U) \arrow[d, hook] \\ \prod_{x\in U} \mathcal{F}_x \arrow[r, shift left, "\prod_x \psi_x"] \arrow[r, shift right, "\prod_x \phi_x"'] & \prod_{x\in U} \mathcal{G}_x \end{tikzcd}.$$

□

2.16.4 Lemma. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a sheaf of sets. Then ϕ is an epimorphism (resp. monomorphism) in $\mathbf{Set}_X$ if and only if

$\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective (resp. injective) for all $x \in X$.

Proof. This follows from a direct verification of the definition by using (2.16.3). Also, we use the fact that $(f \circ g)_x = f_x \circ g_x$, which is clear. $\square$

2.16.5 Lemma. Let I be a directed set and $A : I \to \mathbf{Ring}$ a directed system of rings. Suppose $M, N, L : I \to \mathbf{Mod}$ are directed systems of A-modules such that $M \xrightarrow{f} N \xrightarrow{g} L$ is exact. Then $\varinjlim M \to \varinjlim N \to \varinjlim L$ is exact.

Proof. This is immediate if one realizes $\varinjlim M$ as equivalence classes in the disjoint union. For the proof when one realizes it as a quotient of the direct sum, see [AM94, Exercise 2.19]. $\square$

2.16.6 Corollary. Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism in $\mathbf{Ab}_X$. Then $(\ker f)_x = \ker f_x$ and $(\operatorname{coker} f)_x = \operatorname{coker} f_x$ for all $x \in X$.

2.16.7 Proof. The lemma follows at once from we have proved.

2.17 Let $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$ be a morphism in $\mathbf{Ab}_{\mathbf{Top}}$. Suppose Y admits an open cover $\mathcal{U}$ such that the induced morphism $f|^{U}_{f^{-1}(U)} : (f^{-1}(U), \mathcal{F}|_{f^{-1}(U)}) \to (U, \mathcal{G}|_U)$ is an isomorphism for any $U \in \mathcal{U}$. Then f is an isomorphism. The map on topological spaces are clearly a homeomorphism. For the sheaf map, one can use (2.16) to show $\mathcal{G} \to f_*\mathcal{F}$ is an isomorphism.

2.2 Local-ringed spaces

2.18 Ringed space. An object in $\mathbf{Ring}_{\mathbf{Top}}$ is called a **ringed space**. For simplicity, put $\mathbf{RS} = \mathbf{Ring}_{\mathbf{Top}}$. Explicitly, a ringed space morphism from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ consists of a continuous map $f : X \to Y$ and a morphism $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y.

2.19 The morphism $f^\flat$ induces maps of stalks. To be precise, for $x \in X$ and V an open neighborhood of $y = f(x)$ in Y, we have a map $f^\flat_V : \mathcal{O}_Y(V) \to f_*\mathcal{O}_X(V) = \mathcal{O}_X(f^{-1}(V))$. Post-composing with inclusion into direct limit, we obtain a map $\mathcal{O}_Y(V) \to \varinjlim_{\mathrm{Top}(X) \ni U \ni x} \mathcal{O}_X(U) = \mathcal{O}_{X,x}$. Letting V varying and passing to limit, we obtain a map $f^\flat_x : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$.

On the other hand, write $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ to be the morphism obtained via (2.10). Let $x \in X$. By (2.9) and (2.4.3), we have a map

$$\varinjlim_{\mathrm{Top}(X) \ni U \ni x} f^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}.$$

By (2.10.2), this map coincides with $f^\flat_x$.

2.20 Local-ringed space. A ringed space $(X, \mathcal{O}_X)$ is called a **local-ringed space** (or locally ringed space) if each stalk $\mathcal{O}_{X,x}$ is a local ring. A **morphism** between local-ringed spaces is a morphism of ringed spaces such that each stalk map is a **local homomorphism** of local rings, which is to be explained. For a ring homomorphism $\varphi : (A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ of local rings, we say it is a **local homomorphism** when $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$. Note that in general we only have $\varphi^{-1}(\mathfrak{m}_B) \subseteq \mathfrak{m}_A$. The category of local-ringed spaces is denoted by **LRS**.

2.20.1 Let k be a field. A **ringed space over** k is a ringed space $(X, \mathcal{O}_X)$ such that $\mathcal{O}_X$ is a sheaf of k-algebras. The category $\mathbf{RS}_k$ of ringed spaces is the full subcategory of $(\mathbf{Alg}_k)_{\mathbf{Top}}$ whose objects consist of ringed spaces over k. A **local-ringed space over** k is a local-ringed space $(X, \mathcal{O}_X)$ that is also a ringed-space over k, and a **morphism of local-ringed spaces of** k is both a morphism of local-ringed spaces and ringed space over k. Denote by $\mathbf{LRS}_k$ the category of local-ringed space over k.

2.21 Let $(X, \mathcal{O}_X)$ be a local-ringed space. For a point $x \in X$, the quotient $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ is called the **residue field** at x. For open U, $x \in U$ and $f \in \mathcal{O}_X(U)$, we sometimes write $f(x)$ to denote the class of f in the residue field $\kappa(x)$. For an open subset U of X and $f \in \mathcal{O}_X(U)$, put

$$\begin{aligned} U_f &:= \{x \in U \mid f(x) \neq 0 \text{ in } \kappa(x)\} \\ &= \{x \in U \mid f_x \in \mathcal{O}_{X,x}^\times\}. \end{aligned}$$

Then U_f is an open set in U. For if $f_x \in \mathcal{O}_{X,x}^\times$, then we can find some neighborhood V of x and $g \in \mathcal{O}_X(V)$ such that $f_x g_x = 1$. But this means $fg = 1$ in $\mathcal{O}_X(W)$ for a smaller neighborhood $W \subseteq V$ of x, and thus $x \in U \cap W \subseteq U$. Note that since $\mathcal{O}_X$ is a sheaf, for $f \in \mathcal{O}_X(X)$, we in fact have $f \in \mathcal{O}_X(X_f)^\times$. To construct an inverse, we do it locally and patch them together to a section over X_f.

2.22 Topological embedding. Let X be a topological space and $Y \subseteq X$ an subspace. We denote by $\iota : Y \to X$ the inclusion map. For a sheaf $\mathcal{F}$ on X, we denote the inverse image sheaf $\iota^{-1}\mathcal{F}$ by $\mathcal{F}|_Y$. From the adjunction we also have a morphism $\rho : \mathcal{F} \to \iota_*(\mathcal{F}|_Y)$. We call the morphism $(\iota, \rho) : (Y, \mathcal{F}|_Y) \to (X, \mathcal{F})$ the **topological embedding**. It enjoys the following universal property similar to that of the subspace topology : if $(f, f^\sharp) : (T, \mathcal{G}) \to (X, \mathcal{F})$ is a morphism with $f(T) \subseteq Y$, then there exists a unique morphism $(T, \mathcal{G}) \to (Y, \mathcal{F}|_Y)$ making the following triangle commute

$$\begin{tikzcd} (T, \mathcal{G}) \arrow[rr, "(f, f^\sharp)"] \arrow[dr] & & (X, \mathcal{F}) \\ & (Y, \mathcal{F}|_Y) \arrow[ur, "(\iota, \rho)"'] & \end{tikzcd}$$

The only candidate for the map on topological spaces are $f|^Y$. The morphism between sheaves is best defined using the very definition of a morphism in $\mathcal{C}^{\text{pre}}_{\text{Top}}$ (2.10) then passing to sheafification. In this way the uniqueness is also evident. Another way is to define it via adjunction. To be specific, the morphism $f^\sharp : \mathcal{F} \to f_*\mathcal{G} = \iota_*(f|^Y)_*\mathcal{G}$ induces, by adjunction, another morphism $\iota^*\mathcal{F} \to (f|^Y)_*\mathcal{G}$, which is what we want. The commutativity is proved using functoriality of adjunction.

Let $\mathcal{G}$ be a sheaf on Y. By adjunction the identity morphism $\iota_*\mathcal{G} \to \iota_*\mathcal{G}$ induces a morphism $\iota^{-1}\iota_*\mathcal{G} \to \mathcal{G}$. This is in fact an isomorphism as long as ι is an embedding. It is enough to check on the stalks, and this follows from (2.9).

2.22.1 Let us consider the above situation in **LRS**. Let $(X, \mathcal{O}_X)$ be a local-ringed space and $Y \subseteq X$ a subspace. We check $(Y, \mathcal{O}_X|_Y)$ is also a local-ringed space, and the topological embedding $(Y, \mathcal{O}_X|_Y) \to (X, \mathcal{O}_X)$ is a morphism in **LRS**. The first is easy, for we have

$$(\mathcal{O}_X|_Y)_y = \varinjlim_{\mathrm{Top}(X) \ni U \supseteq W} \ \varinjlim_{\mathrm{Top}(Y) \ni W \ni y} \mathcal{O}_X(U) = \mathcal{O}_{X,y}.$$

To see the stalk map is a local homomorphism, since the stalk of a presheaf coincides with that of its sheafification (2.4.3), we can replace $(Y, \mathcal{O}_X|_Y)$ by $(Y, \iota^{\text{pre}}\mathcal{O}_X)$, and the computation above indicates the stalk map is simply the identity map.

Now let $(\psi, \theta) : (X', \mathcal{O}_{X'}) \to (X, \mathcal{O}_X)$ be a morphism in **LRS** and let Y be a subspace of X. Put $Y' = \psi^{-1}(Y)$ and give it subspace topology of X'. Then we have a cartesian square in **LRS**

$$\begin{tikzcd} (Y', \mathcal{O}_{X'}|_{Y'}) \arrow[r] \arrow[d] & (Y, \mathcal{O}_X|_Y) \arrow[d] \\ (X', \mathcal{O}_{X'}) \arrow[r, "(\psi, \theta)"] & (X, \mathcal{O}_X) \end{tikzcd}$$

The two vertical morphisms are the canonical embedding, and the upper horizontal map follows from the universal property for $(Y, \mathcal{O}_X|_Y)$; to see the so obtained map is a morphism in **LRS**, one can argue as in (2.22).

2.23 Ideal sheaf. Let $(X, \mathcal{O}_X)$ be a ringed space. An **ideal sheaf** $\mathcal{J}$ of $\mathcal{O}_X$ is a sheaf on X such that $\mathcal{J}(U)$ is an ideal of $\mathcal{O}_X(U)$ for any open U. Symbolically we write $\mathcal{J} \trianglelefteq \mathcal{O}_X$. The ringed space $(X, \mathcal{O}_X/\mathcal{J})$ is called the ringed space associated to $\mathcal{J}$. Note that $\mathcal{O}_X/\mathcal{J}$ means the sheafification of the presheaf $U \mapsto \mathcal{O}_X(U)/\mathcal{J}(U)$.

The natural map $\iota_{\mathcal{J}} : (X, \mathcal{O}_X/\mathcal{J}) \to (X, \mathcal{O}_X)$ enjoys the following universal property : if $(f, f^\sharp) : (T, \mathcal{G}) \to (X, \mathcal{O}_X)$ is a morphism in **RS** with $\mathcal{J} \subseteq \ker f^\sharp$, then there exists a unique morphism $(T, \mathcal{G}) \to (X, \mathcal{O}_X/\mathcal{J})$ making the following triangle commute

$$\begin{tikzcd}
(T, \mathcal{G}) \arrow[rr, "{(f, f^\sharp)}"] \arrow[dr] & & (X, \mathcal{O}_X) \\
& (X, \mathcal{O}_X/\mathcal{J}) \arrow[ur, "\iota_{\mathcal{J}}"'] &
\end{tikzcd}$$

2.23.1 Let $(f, \theta) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ be a morphism in **RS** and $\mathcal{J}$ an ideal sheaf of $\mathcal{O}_X$. Note that $f^*\mathcal{O}_X = \mathcal{O}_Y$ (2.11). For each $\mathcal{O}_Y$-module $\mathcal{N}$, define

$$\mathcal{J}\mathcal{N} = \operatorname{im}(f^*\mathcal{J} \otimes_{\mathcal{O}_Y} \mathcal{N} \to f^*\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{N} \cong \mathcal{N}) \in \mathbf{Mod}_{\mathcal{O}_Y}$$

When $\mathcal{N} = \mathcal{O}_Y$, we see $\mathcal{J}\mathcal{O}_Y = \operatorname{Im}(f^*\mathcal{J} \to \mathcal{O}_Y)$. Put $\theta^\sharp : f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$. Then $\theta^\sharp$ induces a map

$$f^{-1}(\mathcal{O}_X/\mathcal{J}) \cong f^{-1}\mathcal{O}_X/f^{-1}\mathcal{J} \to \mathcal{O}_Y/\mathcal{J}\mathcal{O}_Y$$

By adjunction we obtain a morphism $\overline{\theta} : \mathcal{O}_X/\mathcal{J} \to f_*(\mathcal{O}_Y/\mathcal{J}\mathcal{O}_Y)$. In sum, we obtain a map

$$(f, \overline{\theta}) : (Y, \mathcal{O}_Y/\mathcal{J}\mathcal{O}_Y) \longrightarrow (X, \mathcal{O}_X/\mathcal{J})$$

in **RS**. In fact, it fits into a Cartesian square in **RS**

$$\begin{tikzcd}
(Y, \mathcal{O}_Y/\mathcal{J}\mathcal{O}_Y) \arrow[r, "{(f, \overline{\theta})}"] \arrow[d] & (X, \mathcal{O}_X/\mathcal{J}) \arrow[d] \\
(Y, \mathcal{O}_Y) \arrow[r, "{(f, \theta)}"] & (X, \mathcal{O}_X)
\end{tikzcd}$$

2.24 Let $(X, \mathcal{O}_X)$ be a local-ringed space and $\mathcal{J} \trianglelefteq \mathcal{O}_X$ an ideal ideal. For $x \in X$, it may happen that the stalk $(\mathcal{O}_X/\mathcal{J})_x$ is not a local ring. Taking stalk is exact, so $(\mathcal{O}_X/\mathcal{J})_x \cong \mathcal{O}_{X,x}/\mathcal{J}_x$. Thus it is not a local ring if and only if it is zero, or equivalently, $\mathcal{O}_{X,x} = \mathcal{J}_x$. Define

$$V(\mathcal{J}) = \{x \in X \mid \mathcal{J}_x \subsetneq \mathcal{O}_{X,x}\}$$

This is a closed subset of X, for if $x \notin V(\mathcal{J})$, then $\mathcal{O}_{X,x} = \mathcal{J}_x$, so there is a neighborhood U of x and $f \in \mathcal{J}(U)$ such that $f_x = 1 \in \mathcal{O}_{X,x}$. Shrinking U further shows that $f|_U = 1 \in \mathcal{J}(U)$, so $U \subseteq X \backslash V(\mathcal{J})$.

Let $j : V(\mathcal{J}) \to X$ denote the inclusion. Then $V(\mathcal{J})$ together with the sheaf of rings $j^{-1}(\mathcal{O}_X/\mathcal{J}) = \mathcal{O}_X/\mathcal{J}\big|_{V(\mathcal{J})}$ becomes a local-ringed space, called the **closed local-ringed subspace of** $(X, \mathcal{O}_X)$ **associated to the ideal sheaf** $\mathcal{J}$.

Denote by $i_{\mathcal{J}} : (V(\mathcal{J}), j^{-1}(\mathcal{O}_X/\mathcal{J})) \xrightarrow{\iota_{\mathcal{J}}} (X, \mathcal{O}_X/\mathcal{J}) \to (X, \mathcal{O}_X)$ the composition (2.23). This morphism in **LRS** enjoys the following universal property : if $(f, f^\sharp) : (T, \mathcal{G}) \to (X, \mathcal{O}_X)$ is a morphism in **LRS** with $\mathcal{J} \subseteq \ker f^\sharp$, then there exists a unique morphism $(T,\mathcal{G}) \to (V(\mathcal{J}), j^{-1}(\mathcal{O}_X/\mathcal{J}))$ making the following triangle commute

$$\begin{tikzcd}
(T,\mathcal{G}) \arrow[rr, "(f,f^\sharp)"] \arrow[dr] & & (X,\mathcal{O}_X) \\
& (V(\mathcal{J}), j^{-1}(\mathcal{O}_X/\mathcal{J})) \arrow[ur, "i_{\mathcal{J}}"'] &
\end{tikzcd}$$

We first show $f(t) \in V(\mathcal{J})$. Since taking stalk at $f(t)$ is exact, we see $(\ker f^\sharp)_{f(t)} = \ker(\mathcal{O}_{X,f(t)} \to \mathcal{G}_t)$. Since the stalk map is a local homomorphism, we see $(\ker f^\sharp)_{f(t)}$ is contained in the maximal ideal of $\mathcal{O}_{X,f(t)}$, whence $\mathcal{J}_{f(t)} \subsetneq \mathcal{O}_{X,f(t)}$. Similar to (2.23), we have a morphism $\mathcal{O}_X/\mathcal{J} \to f_*\mathcal{G} = j_*\left(f|^{V(\mathcal{J})}\right)_*\mathcal{G}$, so adjunction gives $j^{-1}(\mathcal{O}_X/\mathcal{J}) \to \left(f|^{V(\mathcal{J})}\right)_*\mathcal{G}$. The resulting map is a morphism in **LRS** and makes this triangle commutative, as one can argue as in (2.22).

2.24.1 Let $(f,\theta) : (Y,\mathcal{O}_Y) \to (X,\mathcal{O}_X)$ be a morphism in **LRS**. Let $\mathcal{N} \in \mathbf{Mod}_{\mathcal{O}_Y}$ and $\mathcal{J} \trianglelefteq \mathcal{O}_X$ an ideal sheaf. By definition there exists a commutative diagram

$$\begin{tikzcd}
f^*\mathcal{J} \otimes_{\mathcal{B}} \mathcal{N} = f^{-1}\mathcal{J} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Y \otimes_{\mathcal{O}_Y} \mathcal{N} \arrow[rr] \arrow[dr, two heads] & & \mathcal{N} \\
& \mathcal{J}\mathcal{N} \arrow[ur, hook] &
\end{tikzcd}$$

Taking stalk at $y \in Y$ gives (c.f. (2.12))

$$\begin{tikzcd}
\mathcal{J}_{f(y)} \otimes_{\mathcal{O}_{X,f(y)}} \mathcal{O}_{Y,y} \otimes_{\mathcal{O}_{Y,y}} \mathcal{N}_y \arrow[rr] \arrow[dr, two heads] & & \mathcal{N}_y \\
& (\mathcal{J}\mathcal{N})_y \arrow[ur, hook] &
\end{tikzcd}$$

This proves that

$$(\mathcal{J}\mathcal{N})_y = \operatorname{Im}\left(\mathcal{J}_{f(y)} \otimes_{\mathcal{O}_{X,f(y)}} \mathcal{N}_y \to \mathcal{N}_y\right) = \mathcal{J}_{f(y)}\mathcal{N}_y.$$

2.24.2 Lemma. $V(\mathcal{J}\mathcal{O}_Y) = f^{-1}(V(\mathcal{J}))$.

Proof. Recall that for $y \in Y$, $(\mathcal{O}_Y/\mathcal{J}\mathcal{O}_Y)_y \cong \mathcal{O}_{Y,y}/\mathcal{J}_{f(y)}\mathcal{O}_{Y,y}$. Let $\mathfrak{m}_{f(y)}$ and $\mathfrak{m}_y$ be the maximal ideals of $\mathcal{O}_{X,f(y)}$ and $\mathcal{O}_y$ respectively; note that $f_y(\mathfrak{m}_{f(y)}) \subseteq \mathfrak{m}_y$.

— If $y \in V(\mathcal{J}\mathcal{O}_Y)$, then $\mathcal{J}_{f(y)}\mathcal{O}_{Y,y} \subsetneq \mathcal{O}_{Y,y}$. A fortiori we have $\mathcal{J}_{f(y)} \subsetneq \mathcal{O}_{X,x}$, so $y \in f^{-1}(V(\mathcal{J}))$.

— If $y \in f^{-1}(V(\mathcal{J}))$, then $\mathcal{J}_{f(y)} \subseteq \mathfrak{m}_{f(y)}$. But

$$\mathcal{J}_{f(y)}\mathcal{O}_{Y,y} \subseteq \mathfrak{m}_{f(y)}\mathcal{O}_{Y,y} \subseteq \mathfrak{m}_y \subsetneq \mathcal{O}_{Y,y}$$

so $y \in V(\mathcal{J}\mathcal{O}_Y)$.

□

As as corollary, we see from (2.24) and this lemma that a morphism $(Y,\mathcal{O}_Y) \to (X,\mathcal{O}_X)$ in **LRS** induces a morphism

$$(V(\mathcal{J}\mathcal{O}_Y), (\mathcal{O}_Y/\mathcal{J}\mathcal{O}_Y)|_{V(\mathcal{J}\mathcal{O}_Y)}) \to (V(\mathcal{J}), (\mathcal{O}_X/\mathcal{J})|_{V(\mathcal{J})})$$

Moreover, it fits into a Cartesian square in **LRS**.

$$\begin{tikzcd}
(V(\mathcal{J}\mathcal{O}_Y), (\mathcal{O}_Y/\mathcal{J}\mathcal{O}_Y)|_{V(\mathcal{J}\mathcal{O}_Y)}) \arrow[r] \arrow[d] & (V(\mathcal{J}), (\mathcal{O}_X/\mathcal{J})|_{V(\mathcal{J})}) \arrow[d] \\
(Y,\mathcal{O}_Y) \arrow[r] & (X,\mathcal{O}_X)
\end{tikzcd}$$

This follows from (2.23.1) and (2.24).

2.24.3 Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let us compute the pullback sheaf $j^*\mathcal{F}$. By definition (2.11)

$$j^*\mathcal{F} := \left(\mathcal{O}_{V(\mathcal{J})} \otimes^p_{j^{\mathrm{pre}}\mathcal{O}_X} j^{\mathrm{pre}}\mathcal{F}\right)^\dagger = \left(j^{\mathrm{pre}}\left(\mathcal{O}_X/\mathcal{J}\right) \otimes^p_{j^{\mathrm{pre}}\mathcal{O}_X} j^{\mathrm{pre}}\mathcal{F}\right)^\dagger$$

Define $j^{\mathrm{pre}}\left(\mathcal{O}_X/\mathcal{J}\right) \otimes^p_{j^{\mathrm{pre}}\mathcal{O}_X} j^{\mathrm{pre}}\mathcal{F} \to j^{\mathrm{pre}}(\mathcal{O}_X/\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F})$ as follows. For any open set $U \subseteq V(\mathcal{J})$, we have

$$j^{\mathrm{pre}}\left(\mathcal{O}_X/\mathcal{J}\right) \otimes^p_{j^{\mathrm{pre}}\mathcal{O}_X} j^{\mathrm{pre}}\mathcal{F}(U) \cong \varinjlim_{j(U)\subseteq V\in \mathbf{Top}(X)} \left(\left(\mathcal{O}_X/\mathcal{J}\right)(V) \otimes_{\mathcal{O}_X(V)} \mathcal{F}(V)\right).$$

Here we use the fact that direct limit commutes with tensor product. Consider the sheafification map

$$\varinjlim_{j(U)\subseteq V\in \mathbf{Top}(X)} \left(\left(\mathcal{O}_X/\mathcal{J}\right)(V) \otimes_{\mathcal{O}_X(V)} \mathcal{F}(V)\right) \xrightarrow{(2.4)} \varinjlim_{j(U)\subseteq V\in \mathbf{Top}(X)} (\mathcal{O}_X/\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F})(V)$$

The right hand side is precisely $j^{\mathrm{pre}}(\mathcal{O}_X/\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F})$. This finishes our definition. Passing to sheafification (2.4), we obtain a morphism of $\mathcal{O}_{V(\mathcal{J})}$-modules

$$j^*\mathcal{F} \longrightarrow j^{-1}(\mathcal{O}_X/\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F}) \cong j^{-1}(\mathcal{F}/\mathcal{J}\mathcal{F}).$$

The last isomorphism is the obvious one. From the construction above, we see this is in fact an isomorphism (check on stalks).

2.25 Definition. Let $(X, \mathcal{O}_X)$ be a local-ringed space.

1. For open U in X, $(U, \mathcal{O}_X|_U)$ is called an **open local-ringed subspace** of $(X, \mathcal{O}_X)$.
2. For $\mathcal{J} \trianglelefteq \mathcal{O}_X$, $(V(\mathcal{J}), (\mathcal{O}_X/\mathcal{J})|_{V(\mathcal{J})})$ is called a **closed local-ringed space** of $(X, \mathcal{O}_X)$ associated to $\mathcal{J}$.
3. For open U and $J \trianglelefteq \mathcal{O}_X|_U$, $(V(\mathcal{J}), (\mathcal{O}_X|_U/\mathcal{J})|_{V(\mathcal{J})})$ is called a **locally closed local ringed subspace** of $(X, \mathcal{O}_X)$.

A morphism $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ in **LRS** is called an **open immersion / closed immersion / immersion** if there exists an open / closed / locally closed local-ringed subspace $(Z, \mathcal{O}_Z)$ of $(X, \mathcal{O}_X)$ and an isomorphism $(Y, \mathcal{O}_\mathcal{Y}) \to (Z, \mathcal{O}_Z)$ of **LRS** making the following diagram commute

$$\begin{tikzcd}
(Y, \mathcal{O}_Y) \arrow[rr, "f"] \arrow[dr, "\sim"'] & & (X, \mathcal{O}_X) \\
& (Z, \mathcal{O}_Z) \arrow[ur] &
\end{tikzcd}$$

2.26 Proposition. Let $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ be a morphism in **LRS**.

1. f is an open immersion if and only if f is a topological open embedding and for any $y \in Y$, the stalk map $f_y : \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is an isomorphism.
2. f is a closed immersion if and only if f is a topological closed embedding and for any $y \in Y$, the stalk map $f_y : \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is surjective.
3. f is an immersion if and only if f is a topological locally closed embedding and for any $y \in Y$, the stalk map $f_y : \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is surjective.

Proof.

1. The only if part is clear. For the if part, let U be the image of f. It suffices to consider $(U, \mathcal{O}_X|_U)$.

2. To show the only if part, say

$$\begin{tikzcd}
(Y,\mathcal{O}_Y) \arrow[rr, "f"] \arrow[dr, "\sim"'] & & (X,\mathcal{O}_X) \\
& (V(\mathcal{J}), (\mathcal{O}_X/\mathcal{J})\big|_{V(\mathcal{J})}) \arrow[ur] &
\end{tikzcd}$$

Taking stalk at $y \in Y$ gives

$$\begin{tikzcd}
\mathcal{O}_{Y,y} \arrow[rr] \arrow[dr, "\sim"'] & & \mathcal{O}_{X,f(y)} \\
& \mathcal{O}_{X,f(y)}/\mathcal{J}_{f(y)} \arrow[ur] &
\end{tikzcd}$$

and this proves the surjectivity.

Now we consider the if part. Let $\mathcal{J}$ be the kernel of the morphism $\mathcal{O}_X \to f_*\mathcal{O}_Y$. By the universal property 2.24 there exists a morphism $(Y,\mathcal{O}_Y) \to (V(\mathcal{J}), (\mathcal{O}_X/\mathcal{J})\big|_{V(\mathcal{J})})$ making the triangle commute

$$\begin{tikzcd}
(Y,\mathcal{O}_Y) \arrow[rr, "f"] \arrow[dr] & & (X,\mathcal{O}_X) \\
& (V(\mathcal{J}), (\mathcal{O}_X/\mathcal{J})\big|_{V(\mathcal{J})}) \arrow[ur] &
\end{tikzcd}$$

We claim that $f(Y) = V(\mathcal{J})$, and $\mathcal{O}_Y \cong (\mathcal{O}_X/\mathcal{J})\big|_{V(\mathcal{J})}$. For $x \in X$, since f is a closed embedding, we have

$$(f_*\mathcal{O}_Y)_x = \begin{cases} \mathcal{O}_{Y,y} & \text{, if } x = f(y) \\ 0 & \text{, otherwise} \end{cases}$$

For convenience, we write $\operatorname{supp} f_*\mathcal{O}_X$ to denote the set of those points in X at which the stalk of $f_*\mathcal{O}_X$ does not vanish. Then we see $\operatorname{supp} f_*\mathcal{O}_Y = f(Y)$.

On the other hand, since f is an embedding (and by the fact mentioned in the last part of (2.22)), we have an exact sequence

$$0 \longrightarrow f^{-1}\mathcal{J} \longrightarrow f^{-1}\mathcal{O}_X \longrightarrow f^{-1}f_*\mathcal{O}_Y \cong \mathcal{O}_Y$$

We contend the last morphism is surjective. This follows from our assumption once we look at the stalks. Hence

$$(f_*\mathcal{O}_Y)_x = (f^{-1}f_*\mathcal{O}_Y)_x = \begin{cases} (\mathcal{O}_X/\mathcal{J})_x & \text{, if } x = f(y) \\ 0 & \text{, otherwise} \end{cases}$$

and this shows $\operatorname{supp} f_*\mathcal{O}_Y \subseteq V(\mathcal{J}) = \operatorname{supp} \mathcal{O}_X/\mathcal{J}$ Finally, taking stalk directly to the exact sequence $0 \to \mathcal{J} \to \mathcal{O}_X \to f_*\mathcal{O}_Y \to 0$ gives $(f_*\mathcal{O}_Y)_x \subseteq (\mathcal{O}_X/\mathcal{J})_x$, showing that $V(\mathcal{J}) \subseteq \operatorname{supp} f_*\mathcal{O}_Y$. Hence the equality $f(Y) = V(\mathcal{J})$ is proved. The above computation of stalks also proves the assertion for sheaves.

3. This follow from 1. and 2. (and perhaps their proofs).

□

2.27 The definition of locally closed subspaces seems to depend on the choice of open sets we choose. In fact it does not, in the following sense. Let U be an open subspace of X and $\mathcal{J} \trianglelefteq \mathcal{O}_X\big|_U$. Denote by $(Y,\mathcal{O}_Y)$ the associated locally closed subspace. Let U_0 be the largest open subspace of X containing Y as a closed subset and let $j : U \to U_0$ be the inclusion. Let ι_U and ι_{U_0} be the inclusion of Y into U and U_0, respectively. Let us write $\mathcal{O}_U$ and $\mathcal{O}_{U_0}$ for brevity to mean the sheaves of rings for their open local-ringed subspace structures. By definition we have an exact sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_U \longrightarrow (\iota_U)_*\mathcal{O}_Y \longrightarrow 0$$

Applying j_* this sequence (j_* is exact in this case), we obtain an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & j_*\mathcal{J} \arrow[r] & j_*\mathcal{O}_U \arrow[r] & j_*(\iota_U)_*\mathcal{O}_Y = (\iota_{U_0})_*\mathcal{O}_Y \arrow[r] & 0 \\
& & \mathcal{O}_{U_0} \arrow[u] \arrow[r] & (\iota_{U_0})_*\mathcal{O}_Y \arrow[u, equal] &
\end{tikzcd}$$

The middle vertical arrow is surjective, and hence so is the lower horizontal one. Denote by $\mathcal{I}$ the kernel of the lower horizontal arrow. Then we have a commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & j_*\mathcal{J} \arrow[r] & j_*\mathcal{O}_U \arrow[r] & j_*(\iota_U)_*\mathcal{O}_Y \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{I} \arrow[u] \arrow[r] & \mathcal{O}_{U_0} \arrow[u] \arrow[r] & (\iota_{U_0})_*\mathcal{O}_Y \arrow[u, equal] \arrow[r] & 0
\end{tikzcd}$$

We claim $V(\mathcal{I}) = V(\mathcal{J}) = Y$ and the chain on the bottom induces an isomorphism $(\mathcal{O}_{U_0}/\mathcal{I})\big|_{V(\mathcal{I})} \cong \mathcal{O}_Y$. The first is easy, as taking stalks we see $V(\mathcal{I}) = \mathrm{supp}(\iota_{U_0})_*\mathcal{O}_Y$, and for $x \in U_0$,

$$((\iota_{U_0})_*\mathcal{O}_Y)_x = \begin{cases} \mathcal{O}_{Y,x} & \text{, if } x \in V(\mathcal{J}) \\ 0 & \text{, otherwise} \end{cases} .$$

So $V(\mathcal{I}) = V(\mathcal{J})$. Now the assertion for sheaves is clear.

3 Schemes

3.1 Affine schemes

3.1 Let A be a (unital commutative) ring. The set of all prime ideals of A is called the **spectrum** of A, and is denoted by $\operatorname{Spec} A$. For a subset $S \subseteq A$, put $V(S) = \{\mathfrak{p} \in \operatorname{Spec} A \mid S \subseteq \mathfrak{p}\}$. One checks easily that for ideals I, J and I_α, we have

(i) $\bigcap_\alpha V(I_\alpha) = V(\bigcup_\alpha I_\alpha)$,

(ii) $V(I) \cup V(J) = V(I \cap J) = V(IJ)$,

(iii) $V(A) = \varnothing$, $V((0)) = \operatorname{Spec} A$,

(iv) $V(I) = V(\sqrt{I})$,

(v) $V(S) = V(\langle S \rangle)$, where $S \subseteq A$ is a subset and $\langle S \rangle$ denotes the ideal generated by S.

(vi) $V(I) \subseteq V(J)$ if and only if $J \subseteq \sqrt{I}$.

In particular, by (i), (ii), (iii), the $V(S)$ define a topology of closed sets on $\operatorname{Spec} A$. For each $f \in A$, the open set $D(f) = \operatorname{Spec} A \backslash V((f)) = \{\mathfrak{p} \mid f \notin \mathfrak{p}\}$ is called a **principal open set**. The collection of all principal open sets form a basis for this topology.

The topological space $\operatorname{Spec} A$ is compact. More generally, each principal open set $D(f)$ is compact. To see this, say $D(f) = \bigcup_i D(g_i)$. Taking complement, we see $V((f)) = \bigcap_i V((g_i)) = V((g_i)_{i \in I})$. By (vi), this means $f \in \sqrt{(g_i)_{i\in I}}$. Thus we can find $i_1, \dots, i_n \in I$ such that $f \in \sqrt{(g_{i_1}, \dots, g_{i_n})}$, and by (vi) again we see $V((f)) \supseteq V((g_{i_1}, \dots, g_{i_n}))$. Taking complements, we see $D(f) \subseteq \bigcup_{k=1}^{n} D(g_{i_k})$, and hence $D(f) = \bigcup_{k=1}^{n} D(g_{i_k})$.

3.2 Affine schemes. For a principal open set $D(f)$, define

$$\mathcal{O}_A(D(f)) = \mathcal{O}_{\operatorname{Spec} A}(D(f)) = A_f.$$

If $D(f) \subseteq D(g)$, i.e., $f \in \sqrt{(g)}$ by (3.1).(vi), then $f^n = gh$ for some $n \in \mathbb{N}$ and $h \in A$. This gives rises to a map $A_g \to A_f$, by sending $\frac{1}{g}$ to $\frac{h}{f^n}$. The resulting map is easily seen to be independent of the choice of n and h. This independence also shows that $\mathcal{O}_A$ defines a presheaf of rings on the principal open sets. To show this is a sheaf, since $D(f) \cap D(g) = D(fg)$, by (2.3) and (3.1) we need to check to exactness of the following sequence

$$0 \longrightarrow A_f \longrightarrow \prod_{i=1}^{n} A_{f_i} \longrightarrow \prod_{i,j=1}^{n} A_{f_i f_j} \tag{$\star$}$$

where $f, f_i \in A$ with $D(f) = \bigcup_{i=1}^{n} D(f_i)$. The argument is the same as the one in the next paragraph, so we defer our proof. By (2.3) we then obtain a sheaf of rings $\mathcal{O}_A = \mathcal{O}_{\operatorname{Spec} A}$ defines on the whole $\operatorname{Spec} A$. The ringed space $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is called the **affine scheme**. If no confusion will occur, we write $\operatorname{Spec} A$ to denote the pair $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$.

We compute the stalk of $\mathcal{O}_{\operatorname{Spec} A}$ at a prime $\mathfrak{p}$. Since the $D(f)$ form a basis, we have

$$\mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}} = \varinjlim_{\operatorname{Top}(\operatorname{Spec} A) \ni U \ni \mathfrak{p}} \mathcal{O}_A(U) = \varinjlim_{D(f) \ni \mathfrak{p}} \mathcal{O}_A(D(f)) = \varinjlim_{f \notin \mathfrak{p}} A_f = A_{\mathfrak{p}}.$$

The last isomorphism is given by the natural maps $A_f \to A_{\mathfrak{p}}$. Hence $\operatorname{Spec} A$ is in fact a local-ringed space.

3.3 Affine tilde. Let A be a ring and $(X, \mathcal{O}_X) = (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$. For an A-module M, we construct an $\mathcal{O}_X$-module $\widetilde{M}$ as follows. For each principal open set $D(f)$, define

$$\widetilde{M}(D(f)) = M_f = M \otimes_A A_f.$$

This defines a presheaf on principal open sets. To show this really defines a sheaf on them, and hence on X, by (2.3) we must show the sequence

$$0 \longrightarrow M_f \longrightarrow \prod_{i=1}^{n} M_{f_i} \longrightarrow \prod_{i,j=1}^{n} M_{f_i f_j} \tag{$\heartsuit$}$$

is exact for every $f, f_1, \ldots, f_n \in A$ with $D(f) = D(f_1) \cup \cdots \cup D(f_n)$. Since $D(f) = D(f_1) \cup \cdots \cup D(f_n)$, $V(f) = V((f_1, \ldots, f_n))$, so there exist some $k \geqslant 0$ and $a_1, \ldots, a_n \in A$ such that $f^k = \sum_{i=1}^{n} a_i f_i$. Raising to arbitrary powers, we see for each $m \in \mathbb{N}$ we can find $M \geqslant 0$ and $a_1, \ldots, a_n \in A$ such that

$$f^M = \sum_{i=1}^{n} a_i f_i^m$$

We first show the exactness at the first place. Suppose $\frac{a}{f^\ell} \in M_f$ is mapped to zero in each M_{f_i}. By definition this means $f_i^{r_i} a = 0$ in M for some $r_i \geqslant 0$. If we take $m \geqslant \max\{r_1, \ldots, r_n\}$, we see $f_i^m a = 0$ in M. Choose $M \geqslant 0$ and $a_1, \ldots, a_n \in A$ corresponding to m as above. Then

$$0 = a(a_1 f_1^m + \cdots + a_n f_n^m) = a f^M,$$

and this means $\frac{a}{f^\ell} = 0$ in M_f.

Next we show the exactness at the middle place. Suppose $\left(\frac{b_i}{f_i^{\ell_i}}\right) \in \prod_{i=1}^{n} M_{f_i}$ satisfies $\frac{b_i}{f_i^{\ell_i}} - \frac{b_j}{f_j^{\ell_j}} = 0$ in $M_{f_i f_j}$ for all $1 \leqslant i, j \leqslant n$; we may assume each ℓ_i is the same, say equal to ℓ. A similar argument as above show that we can find $m \geqslant 0$ such that

$$(b_i f_j^\ell - b_j f_i^\ell)(f_i f_j)^m = 0 \text{ in } M$$

for all i, j. If we put $b_i' = b_i f_i^m$, the above identities become $b_i' f_j^{\ell+m} = b_i' f_j^{\ell+m}$. Take $M \geqslant 0$ and $a_1, \ldots, a_n \in A$ with respect to $\ell + m$. We claim $\frac{a_1 b_1' + \cdots + a_n b_n'}{f^M}$ is mapped to $\left(\frac{b_i}{f_i^\ell}\right) = \left(\frac{b_i'}{f_i^{\ell+m}}\right) \in \prod_{i=1}^{n} M_{f_i}$. Indeed,

$$(a_1 b_1' + \cdots + a_n b_n') f_j^{\ell+m} = \sum_{i=1}^{n} a_i b_i' f_j^{\ell+m} = \sum_{i=1}^{n} a_i b_j' f_i^{\ell+m} = \left(\sum_{i=1}^{n} a_i f_i^{\ell+m}\right) b_j' = f^M b_j'.$$

This show $\widetilde{M}$ is really a sheaf on X. In fact, as one easily can see, $\widetilde{M}$ is an $\mathcal{O}_X$-module, and $M \mapsto \widetilde{M}$ defines a functor $\mathrm{Mod}_A \to \mathrm{Mod}_{\mathcal{O}_X}$. The computation of stalks in (3.2) implies that $\widetilde{M}_\mathfrak{p} \cong M \otimes A_\mathfrak{p} = M_\mathfrak{p}$ for all primes $\mathfrak{p}$.

3.4 Lemma. For a complex $M \to N \to L$ in $\mathbf{Mod}_A$, it is exact if and only if $\widetilde{M} \to \widetilde{N} \to \widetilde{L}$ is an exact sequence in $\mathbf{Mod}_{\mathcal{O}_X}$.

Proof. Suppose $M \to N \to L$ is exact. Since localization is an exact functor, for each $\mathfrak{p} \in \operatorname{Spec} A$, there exists an exact sequence $M_\mathfrak{p} \to N_\mathfrak{p} \to L_\mathfrak{p}$. Moreover, this is the same as the sequence $\widetilde{M}_\mathfrak{p} \to \widetilde{N}_\mathfrak{p} \to \widetilde{L}_\mathfrak{p}$, so by (2.16) we see $\widetilde{M} \to \widetilde{N} \to \widetilde{L}$ is exact. For the converse, it suffices to show $M \xrightarrow{\alpha} N \xrightarrow{\beta} L$ is exact if each localization sequence $M_\mathfrak{p} \to N_\mathfrak{p} \to L_\mathfrak{p}$ is exact. That is, we must show $\ker\beta/\operatorname{Im}\alpha = 0$ if $(\ker\beta/\operatorname{Im}\alpha)_\mathfrak{p} = 0$ for all $\mathfrak{p} \in \operatorname{Spec} A$. If $D := \ker\beta/\operatorname{Im}\alpha$ is nontrivial, take $m \in D - \{0\}$ with $\operatorname{ann}_A(m) \subsetneq A$ and consider a maximal ideal $\mathfrak{m}$ containing $\operatorname{ann}_A(m)$. The image of m in $D_\mathfrak{m}$ is then nonzero, for otherwise $ma = 0$ for some $a \in A - \mathfrak{m}$, a contradiction to our choice of $\mathfrak{m}$. Hence $D = 0$, and the sequence $M \to N \to L$ is exact. □

3.5 Definition. A **scheme** is a local-ringed space $(X, \mathcal{O}_X)$ admitting an open cover $\mathcal{U}$ such that for any $U \in \mathcal{U}$, there is an isomorphism $(U, \mathcal{O}_X|_U) \cong (\operatorname{Spec} A_U, \mathcal{O}_{\operatorname{Spec} A_U})$ in **LRS** for some ring A_U.

3.6 Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be local-ringed spaces. There exists a canonical map

$$\mathrm{Hom}_{\mathbf{LRS}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathbf{Ring}}(\mathcal{O}_Y(Y), \mathcal{O}_X(X)).$$

Moreover, this map is bifunctorial in X and Y. More generally, there is a bifunctorial map

$$\mathrm{Hom}_{\mathbf{Mod}_{\mathbf{LRS}}}((X, \mathcal{F}), (Y, \mathcal{G})) \longrightarrow \mathrm{Hom}_{\mathbf{Mod}}(\mathcal{O}_Y(Y) \rightharpoondown \mathcal{G}(Y), \mathcal{O}_X(X) \rightharpoondown \mathcal{F}(X)).$$

3.7 Theorem. Let A be a ring and $(X, \mathcal{O}_X) \in \mathbf{LRS}$. Then the map in 3.6

$$\mathrm{Hom}_{\mathbf{LRS}}(X, \operatorname{Spec} A) \longrightarrow \mathrm{Hom}_{\mathbf{Ring}}(A, \mathcal{O}_X(X)).$$

is a bijection.

Proof. Let $(f, f^\sharp) \in \mathrm{Hom}_{\mathbf{LRS}}(X, \operatorname{Spec} A)$. Then for any $x \in X$, if we put $y = f(x) = \mathfrak{p} \in \operatorname{Spec} A$, then we have a commutative diagram

$$\begin{tikzcd}
\mathcal{O}_X(X) \arrow[d, "\mathrm{res}"] & A \arrow[l, "f^\sharp_{\operatorname{Spec} A}"'] \arrow[d] \\
\mathcal{O}_{X,x} & A_{\mathfrak{p}} \arrow[l, "f_x"]
\end{tikzcd}$$

Write $\mathfrak{m}_x$ the unique maximal ideal of $\mathcal{O}_{X,x}$. Then since f_x is a local homomorphism, we see $\mathfrak{p} = (f^\sharp_{\operatorname{Spec} A})^{-1} \mathrm{res}^{-1}(\mathfrak{m}_x)$. On the other hand, for any $g \in A$, there is a commutative diagram

$$\begin{tikzcd}
\mathcal{O}_X(X) \arrow[d, "\mathrm{res}"] & A \arrow[l, "f^\sharp_{\operatorname{Spec} A}"'] \arrow[d] \\
f_*\mathcal{O}_X(D(g)) & A_g \arrow[l, "f^\sharp_{D(g)}"]
\end{tikzcd}$$

The right vertical arrow is localization, so $f^\sharp_{D(g)}$ is in fact uniquely determined by $\mathrm{res} \circ f^\sharp_{\operatorname{Spec} A}$. Since the $D(g)$ form an open basis for $\operatorname{Spec} A$, this shows $f^\sharp$ is uniquely determined by $f^\sharp_{\operatorname{Spec} A}$. This proves the injectivity.

For the surjectivity, let $\theta \in \mathrm{Hom}_{\mathbf{Ring}}(\mathcal{O}_X(X), A)$. We define a map $f : X \to \operatorname{Spec} A$ by setting $f(x) = \theta^{-1}(\mathrm{res}^X_x)^{-1}(\mathfrak{m}_x)$, where $\mathrm{res}^X_x : \mathcal{O}_X(X) \to \mathcal{O}_{X,x}$ is the restriction and $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{X,x}$. To show this is continuous, we claim

$$f^{-1}(D(g)) = X_{\theta(g)}$$

Indeed, $x \in f^{-1}(D(g)) \Leftrightarrow f(x) \in D(g) \Leftrightarrow g \notin f(x) = \theta^{-1}(\mathrm{res}^X_x)^{-1}(\mathfrak{m}_x) \Leftrightarrow \mathrm{res}^X_x(\theta(g)) \notin \mathfrak{m}_x \Leftrightarrow x \in X_{\theta(g)}$.

For $g \in A$, consider the diagram

$$\begin{tikzcd}
\mathcal{O}_X(X) \arrow[d, "\mathrm{res}^X_{X_{\theta(g)}}"] & A \arrow[l, "\theta"'] \arrow[d] \\
f_*\mathcal{O}_X(D(g)) & A_g
\end{tikzcd}$$

To construct a morphism in the bottom so that the square is commutative, we use the universal property of localization. To this end, we need to show $\mathrm{res}^X_{X_{\theta(g)}}(\theta(g)) \in (f_*\mathcal{O}_X(D(g)))^\times = \mathcal{O}_X(X_{\theta(g)})^\times$. But this is the content of (2.21), thus there exists a unique arrow $f_*\mathcal{O}_X(D(g)) \leftarrow A_g$ making the above square commuting. For arbitrary U, the map can be defined using (2.3). Finally, by construction we see each stalk map is a local homomorphism. $\square$

3.7.1 Corollary. Let A be a ring, M an A-module. and $(X, \mathcal{O}_X) \in$ **LRS**. Then the map in (3.6)

$$\mathrm{Hom}_{\mathbf{Mod}_{\mathbf{LRS}}}((X, \mathcal{F}), (\operatorname{Spec} A, \widetilde{M})) \longrightarrow \mathrm{Hom}_{\mathbf{Ring}}(A \rightharpoondown M, \mathcal{O}_X(X) \rightharpoondown \mathcal{F}(X)).$$

is a bijection.

3.7.2 Corollary. Let A be a ring. Then $M \mapsto \widetilde{M}$ defines a fully faithful exact functor $\mathbf{Mod}_A \to \mathbf{Mod}_{\mathcal{O}_{\operatorname{Spec} A}}$.

3.8 Denote by **Sch** (resp. **AffSch**) the full subcategory of **LRS** whose objects are schemes (resp. affine schemes). Then the bijection in (3.7) implies that the functor Spec : **Ring** $\to$ **AffSch** defines an equivalence of categories

$$\begin{aligned} \operatorname{Spec} : \mathbf{Ring}^{\mathrm{op}} &\longrightarrow \mathbf{AffSch} \\ A &\longmapsto \operatorname{Spec} A \\ \mathcal{O}_X(X) &\longmapsfrom X \end{aligned}$$

3.8.1 Associated affine schemes. For a local-ringed space $(X, \mathcal{O}_X)$, the identity $\mathrm{id}_{\mathcal{O}_X(X)}$ defines via (3.7) a canonical map

$$(X, \mathcal{O}_X) \longrightarrow \operatorname{Spec} \mathcal{O}_X(X)$$

universal in all arrows in **LRS** from $(X, \mathcal{O}_X)$ into affine schemes. We will see if X is an affine scheme, then this is an isomorphism. In general, this is neither surjective nor injective.

For example, let k be a ring and $n \geqslant 2$. Consider the affine n-space $X = \mathbb{A}^n_k := \operatorname{Spec} k[x_1, \ldots, x_n]$ but minus the origin :

$$\mathbb{A}^n_k - \{0\} := \mathbb{A}^n_k \backslash V(x_1, \ldots, x_n) = D(x_1) \cup \cdots \cup D(x_n).$$

Let's compute $\mathcal{O}_X(\mathbb{A}^n_k - \{0\})$. It is the kernel of the map

$$\prod_{i \in [n]} k[x_1, \ldots, x_n]_{x_i} \longrightarrow \prod_{1 \leqslant i,j \leqslant n} k[x_1, \ldots, x_n]_{x_i, x_j}.$$

Let $(f_i)_{i\in[n]}$ be in the kernel. Pick $N \gg 0$ so that $g_i := (x_1 \cdots x_n)^N f_i \in k[x_1, \ldots, x_n]$. Since $(g_i)_{i\in[n]}$ also lies in the kernel and $k[x_1, \ldots, x_n]$ embeds into every localization in the scene, it follows that $g_1 = \cdots = g_n$. Since $x_1^N f_1, x_2^N f_2 \in k[x_1, \ldots, x_n]$ already, it follows that $f_1 = \cdots = f_n \in k[x_1, \ldots, x_n]$. In sum

$$\mathcal{O}_X(X) = k[x_1, \ldots, x_n] \longrightarrow \mathcal{O}_X(\mathbb{A}^n_k - \{0\})$$

is an isomorphism, so the universal map is

$$\mathbb{A}^n_k - \{0\} \to \operatorname{Spec} k[x_1, \ldots, x_n] = \mathbb{A}^k_n$$

and it coincides with the open embedding. In particular, it cannot be surjective.

On the other hand, let $X := \mathbb{P}^1_k$ denote the projective 1-space, which is the scheme obtained by glueing $U_x := \operatorname{Spec} k[x]$ and $U_y := \operatorname{Spec} k[y]$ along $k[x, x^{-1}] \cong k[y, y^{-1}]$ where $x \mapsto y^{-1}$. We compute $\mathcal{O}_X(X)$. It is the kernel of the map

$$\begin{aligned} k[x] \times k[y] &\longrightarrow k[x, x^{-1}] \times k[y, y^{-1}] \\ (f, g) &\longmapsto (f(x) - g(x^{-1}), g(x) - f(x^{-1})) \end{aligned}$$

But $f(x) - g(x^{-1}) = 0$ only happens when f and g are constant, so

$$\mathcal{O}_X(X) = k.$$

The universal map is then $\mathbb{P}^1_k \to \operatorname{Spec} k$, which is far from being injective.

3.9 The bifunctorial maps in (3.6) fit into the commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathbf{Mod}_{\mathbf{LRS}}}((X,\mathcal{F}),(Y,\mathcal{G})) \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathbf{Mod}}(\mathcal{O}_Y(Y) \rightharpoonup \mathcal{G}(Y), \mathcal{O}_X(X) \rightharpoonup \mathcal{F}(X)) \arrow[d] \\
\operatorname{Hom}_{\mathbf{LRS}}(X,Y) \arrow[r] & \operatorname{Hom}_{\mathbf{Ring}}(\mathcal{O}_Y(Y), \mathcal{O}_X(X)).
\end{tikzcd}$$

Choosing $f \in \operatorname{Hom}_{\mathbf{LRS}}(X,Y)$ and taking its preimages of vertical maps we obtain a functorial bijection

$$\operatorname{Hom}_{\mathbf{Mod}_{\mathcal{O}_{\operatorname{Spec} A}}}(\mathcal{G}, f_*\mathcal{F}) \longrightarrow \operatorname{Hom}_{\mathcal{O}_Y(Y)}(\mathcal{G}(Y), \mathcal{F}(X)^{[f^\sharp_{\operatorname{Spec} A}]})$$

where $\mathcal{F}(X)^{[f^\sharp_Y]}$ means that we are regarding $\mathcal{F}(X)$ as an $\mathcal{O}_Y(Y)$-module via the map $f^\sharp_Y : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$. By 3.7 and 3.7.1, we know when $(Y,\mathcal{G}) = (\operatorname{Spec} A, \widetilde{M})$, this is a bijection :

$$\operatorname{Hom}_{\mathbf{Mod}_{\mathcal{O}_{\operatorname{Spec} A}}}(\widetilde{M}, f_*\mathcal{F}) \longrightarrow \operatorname{Hom}_A(M, \mathcal{F}(X)^{[f^\sharp_{\operatorname{Spec} A}]})$$

3.10 Let $\varphi : A \to B$ be a ring homomorphism. Then it induces a continuous map $\operatorname{Spec}\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ between affine schemes. We list some properties of $\operatorname{Spec}\varphi$. For brevity, we write $f = \operatorname{Spec}\varphi$.

(i) $f^{-1}(D(f)) = D(\varphi(f))$ for any $f \in A$.

(ii) $f^{-1}(V(I)) = V(IB)$ for any ideal I of A.

(iii) $\overline{f(V(J))} = V(J \cap A)$ for any ideal J of B.

(iv) $f(\operatorname{Spec} B)$ is dense in $\operatorname{Spec} A$ if and only if $\ker\varphi \subseteq \sqrt{0}$.

(v) f is a homeomorphism onto its image if for all $b \in B$ there exists some $u \in B^\times$ such that $ub \in f(A)$.

Proof.

(v) For $b \in B$, we can find $u \in B^\times$ and $a \in A$ such that $ub = \varphi(a)$. Thus

$$D(b) = D(ub) = D(\varphi(a)) = f^{-1}(D(a)).$$

Thus we are left to prove the injectivity of f. Let $\mathfrak{p} \neq \mathfrak{q} \in \operatorname{Spec} B$. Say we can pick $b \in \mathfrak{p}\backslash\mathfrak{q}$. Choose $a \in A$ with $D(b) = f^{-1}(D(a))$. Then $\mathfrak{q} \in D(b) \not\ni \mathfrak{p}$, and thus $f(\mathfrak{q}) \in D(a) \not\ni f(\mathfrak{p})$.

□

3.11 Let S be a multiplicatively closed subset of A. As a consequence of (3.10).(v), we see the map $\operatorname{Spec} S^{-1}A \to \operatorname{Spec} A$ induced by the canonical map $A \to S^{-1}A$ is a homeomorphism onto its image. Recall from algebra the image of $\operatorname{Spec} S^{-1}A \to \operatorname{Spec} A$ is $\{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \cap S = \varnothing\}$.

If we consider the case $S = \{f^n\}_{n \geqslant 0}$, we then obtain a homeomorphism $h : \operatorname{Spec} A_f \to D(f)$. For $D(g) \subseteq D(f)$, we have a map $A_f \to A_g$. Denote by $\overline{g}$ its image in A_f; we then have an isomorphism $(A_f)_{\overline{g}} \to A_g$ given by the universal property of localization. Then

$$\mathcal{O}_A(D(g)) = A_g \cong (A_f)_{\overline{g}} = \mathcal{O}_{\operatorname{Spec} A_f}(D(\overline{g})) \overset{(3.10).(i)}{=} h_*\mathcal{O}_{\operatorname{Spec} A_f}(D(g))$$

This isomorphism is compatible with the restriction of principal open sets, so by (2.3.1) we obtain an isomorphism $\mathcal{O}_{\operatorname{Spec} A}\big|_{D(f)} \cong h_*\mathcal{O}_{\operatorname{Spec} A_f}$. In sum, the natural homomorphism $A \to A_f$ induces an open immersion $\operatorname{Spec} A_f \to \operatorname{Spec} A$ of **LRS** with image being the principal open set $D(f)$.

In fact, the above computation shows that $\widetilde{M}|_{D(f)} \cong h_*\widetilde{M_f}$, where in the right hand side, M_f is viewed as an A_f-module.

3.11.1 The morphism $D(f) \to \operatorname{Spec} A_f$ is in fact the same as the morphism coming from the bijection (3.7) and id_{A_f}. To be specific, we have a bijection

$$\operatorname{Hom}_{\mathbf{LRS}}(D(f), \operatorname{Spec} A_f) \longrightarrow \operatorname{Hom}_{\mathbf{Ring}}(A_f, A_f).$$

Then the morphism corresponds to the identity map $\mathrm{id} : A_f \to A_f$ is exactly the same as the one we construct in (3.11). Indeed, this follows from the construction of the bijection (3.7).

3.11.2 The morphism $D(f) \to \operatorname{Spec} A_f$ is compatible with the restriction. Suppose $D(g) \subseteq D(f)$, so we have a map $M_f \to M_g$. Then we have a commutative diagram

$$\begin{tikzcd} D(f) \arrow[r] & \operatorname{Spec} A_f \\ D(g) \arrow[r] \arrow[u] & \operatorname{Spec} A_g \arrow[u] \end{tikzcd}$$

The map on topological spaces are clearly commutative. For sheaves, let $D(h) \subseteq D(g)$. Then we have a commutative diagram

$$\begin{tikzcd} \widetilde{M}(D(h)) = M_h & (M_f)_{\overline{h}} \arrow[l] \arrow[d] \\ \widetilde{M}(D(h)) \arrow[u, equal] & (M_g)_{\overline{h}} \arrow[l] \end{tikzcd}$$

each arrow given by the localization. This tells us the commutativity of the sheaf maps.

3.12 Closed subscheme of $\operatorname{Spec} A$ **definf by ideals.** Let A be a ring and I an ideal. Then the morphism

$$\iota : \operatorname{Spec}(A/I) \to \operatorname{Spec} A$$

induced by the quotient map $A \to A/I$ is a closed immersion in the sense of (2.25). Indeed, the map on topological spaces is obviously a homeomorphism onto the closed subset $V(I)$ of $\operatorname{Spec} A$. To see it is a closed immersion, we compute the kernel $\mathcal{I}$ of the sheaf map $\mathcal{O}_{\operatorname{Spec} A} \to \iota_*\mathcal{O}_{\operatorname{Spec}(A/I)}$. Let $f \in A$. On the open set $D(f)$, this map is $A_f \to (A/I)_{f \bmod I}$. By a similar argument as above, we see the kernel is I_f, so that $\mathcal{I}(D(f)) = I_f$. Therefore, $\mathcal{I}$ is the affine tilde $\widetilde{I}$, and ι factor through the inclusion $V(I) \subseteq \operatorname{Spec} A$, yielding a unique isomorphism $\operatorname{Spec}(A/I) \cong (V(I), \mathcal{O}_{\operatorname{Spec} A}/\widetilde{I})|_{V(I)})$ by (2.24).

3.1.1 Quasi-coherent sheaves

3.13 Definition. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F} \in \mathbf{Mod}_{\mathcal{O}_X}$.

(i) $\mathcal{F}$ is **quasi-coherent** if every point in X admits an open neighborhood U such that there exists an exact sequence

$$(\mathcal{O}_X|_U)^{\oplus I} \longrightarrow (\mathcal{O}_X|_U)^{\oplus J} \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

for some index sets I and J, depending on x.

(ii) $\mathcal{F}$ is **locally free** if every point in X admits an open neighborhood U such that $\mathcal{F}|_U = (\mathcal{O}_X|_U)^{\oplus I}$ for some index set I, depending on x.

(iii) $\mathcal{F}$ is **locally free of rank** n if the index set I in the (ii) can be chosen to be $[n]$ for any $x \in X$.

(iv) An **invertible sheaf** is a locally free sheaf of rank 1.

3.13.1 Denote by $\mathbf{Qcoh}_{\mathcal{O}_X}$ the full subcategory of $\mathbf{Mod}_{\mathcal{O}_X}$ consisting of all quasi-coherent $\mathcal{O}_X$-modules. If $(X, \mathcal{O}_X)$ is a scheme, we write $\mathbf{Qcoh}_X = \mathbf{Qcoh}(X) := \mathbf{Qcoh}_{\mathcal{O}_X}$ instead, if no confusion arises.

3.14 Lemma. Let A, B be rings and $f : \operatorname{Spec} B \to \operatorname{Spec} A$ be any morphism. We know $f = (\varphi^a, \varphi^\sharp)$ for a unique ring homomorphism $\varphi : A \to B$.

1. $f_* \widetilde{N} = \widetilde{N^{[\varphi]}}$ for any B-module N.
2. $f^* \widetilde{M} = \widetilde{B \otimes_A M}$ for any A-module M.

Proof.

1. Let $h \in A$. By (3.10), we have

$$f_* \widetilde{N}(D(h)) = \widetilde{N}(f^{-1}(D(h))) = \widetilde{N}(D(\varphi(h))) = N_{\varphi(h)} = (N^{[\varphi]})_h = \widetilde{N^{[\varphi]}}(D(h))$$

Now 1. follows from (2.3.1).

2. For any $\mathcal{O}_B$-module $\mathcal{F}$, by adjunction and (3.9)

$$\begin{aligned} \operatorname{Hom}_{\mathbf{Mod}_{\mathcal{O}_B}}(f^* \widetilde{M}, \mathcal{F}) = \operatorname{Hom}_{\mathbf{Mod}_{\mathcal{O}_A}}(\widetilde{M}, f_* \mathcal{F}) \cong \operatorname{Hom}_{\mathbf{Mod}_A}(M, \mathcal{F}(\operatorname{Spec} B)^{[\varphi]}) &\cong \operatorname{Hom}_{\mathbf{Mod}_B}(M \otimes_A B, \mathcal{F}(\operatorname{Spec} B)) \\ &\cong \operatorname{Hom}_{\mathbf{Mod}_{\mathcal{O}_B}}(\widetilde{M \otimes_A B}, \mathcal{F}). \end{aligned}$$

□

3.15 Theorem. Let $\mathcal{F}$ be a quasi-coherent sheaf on $X = \operatorname{Spec} A$. Then $\mathcal{F} \cong \widetilde{M}$ for some A-module M.

Proof. Let $U = D(f)$ be an open set such that there exist I, J and an exact sequence

$$\begin{tikzcd} (\mathcal{O}_X|_U)^{\oplus I} \arrow[r] \arrow[d, equal] & (\mathcal{O}_X|_U)^{\oplus J} \arrow[r] \arrow[d, equal] & \mathcal{F}|_U \arrow[r] \arrow[d, equal] & 0 \\ \widetilde{A_f}^{\oplus I} \arrow[r] & \widetilde{A_f}^{\oplus J} \arrow[r] & \mathcal{F}|_U \arrow[r] & 0 \end{tikzcd}$$

where $\widetilde{\ }$ is taken from $\mathbf{Mod}_{A_f}$. Consider the corresponding map $A_f^{\oplus I} \to A_f^{\oplus J}$ of A-modules and let K denote its cokernel. Then $\widetilde{K} \cong \mathcal{F}|_U$ as they are the cokernels of the same map.

Since X is compact, we can find $D(f_1), \ldots, D(f_n)$ such that $X = D(f_1) \cup \cdots \cup D(f_n)$ and

$$(D(f_i), \mathcal{F}|_{D(f_i)}) \cong (\operatorname{Spec} A_{f_i}, \widetilde{M_i})$$

for some $M_i \in \mathbf{Mod}_{A_{f_i}}$, where the isomorphism here is as in (3.11). For each open U, we have an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}(U) \arrow[r] \arrow[d, equal] & \prod_{i=1}^n \mathcal{F}(U \cap D(f_i)) \arrow[r] \arrow[d, equal] & \prod_{i,j=1}^n \mathcal{F}(U \cap D(f_i) \cap D(f_j)) \arrow[d, equal] \\ 0 \arrow[r] & \mathcal{F}(U) \arrow[r] & \prod_{i=1}^n (\iota_{D(f_i)})_* \left(\mathcal{F}|_{D(f_i)}\right)(U) \arrow[r] & \prod_{i,j=1}^n (\iota_{D(f_i) \cap D(f_j)})_* \left(\mathcal{F}|_{D(f_i) \cap D(f_j)}\right)(U) \end{tikzcd}$$

where for an open U, $\iota_U : U \to X$ denotes the inclusion, so we have an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \prod_{i=1}^{n} (\iota_{D(f_i)})_* \left(\mathcal{F}|_{D(f_i)}\right) \longrightarrow \prod_{i,j=1}^{n} (\iota_{D(f_i)\cap D(f_j)})_* \left(\mathcal{F}|_{D(f_i)\cap D(f_j)}\right)$$

Now Lemma 3.14 implies the latter two $\mathcal{O}_X$-modules arise from some A-modules, which in turns says that $\mathcal{F}$ arises from the A-module. □

3.15.1 Corollary Let A be a ring. The affine tilde $\widetilde{\cdot} : \mathbf{Mod}_A \to \mathbf{Qcoh}_{\operatorname{Spec} A}$ is an equivalence of categories with inverse $\mathcal{F} \mapsto \mathcal{F}(\operatorname{Spec} A)$.

Proof. This follows immediately from Theorem 3.15 and (3.7.2). □

3.15.2 From (3.7.1), for $\mathcal{F} \in \mathbf{Mod}_{\mathcal{O}_A}$ we have a canonical morphism $\alpha : \mathcal{F}(\widetilde{\operatorname{Spec} A}) \to \mathcal{F}$ in $\mathbf{Mod}_{\mathcal{O}_A}$ induced by the identity map $\mathrm{id}_{\mathcal{F}(X)}$. The content of Theorem 3.15 can be made more formal. That is,

Lemma. $\alpha : \mathcal{F}(\widetilde{\operatorname{Spec} A}) \to \mathcal{F}$ is an isomorphism if $\mathcal{F}$ is $\mathcal{O}_A$-quasi-coherent.

Proof. Let $U = D(f)$ be a principal affine open subset of Spec A. By construction, α_U is given by the localization. Precisely, we have a commutative triangle

$$\begin{tikzcd} & \mathcal{F}(X) \arrow[dl] \arrow[dr] & \\ \mathcal{F}(X)_f \arrow[rr, "\alpha_U"] & & \mathcal{F}(D(f)) \end{tikzcd}$$

each map being canonical. We claim α_U is an isomorphism. This amounts to show that

(i) If $s \in \mathcal{F}(X)$ restricts to 0 in $\mathcal{F}(D(f))$, then $f^n s = 0$ for some $n > 0$.

(ii) Given $t \in \mathcal{F}(D(f))$, there exists some n such that $f^n t \in \mathcal{F}(X)$.

Note that (i) and (ii) imply injectivity and surjectivity, respectively. We first show (i). Let $s \in \mathcal{F}(X)$ with $s|_{D(f)} = 0$. Let $D(f_i)$ be as in the proof of Theorem 3.15. Then

$$0 = s|_{D(f_i)\cap D(f)} \in \mathcal{F}(D(f_i) \cap D(f)) = \mathcal{F}|_{D(f_i)}(D(f_i f)) \cong (M_i)_f$$

(note that f_i acts invertibly on M_i). This means $f^{n_i}(s|_{D(f_i)}) = 0$ in M_i for some $n_i \geqslant 1$, and hence an $n \geqslant 1$ such that $(f^n s)|_{D(f_i)} = 0$ for each i. But this means $f^n s = 0$.

For (ii), let $t \in \mathcal{F}(D(f))$. Then $t|_{D(ff_i)} \in \mathcal{F}(D(ff_i)) \cong (M_i)_f$, so we can find $t_i \in M_i \cong \mathcal{F}(D(f_i))$ and $n_i \geqslant 1$ such that $f^{n_i} t = t_i$ on $D(ff_i)$. Again pick $n \gg 0$ so that $f^n t = t_i$ on $D(ff_i)$ for each i. On the intersection $D(f_i) \cap D(f_j) = D(f_i f_j)$, we have

$$t_i|_{D(ff_if_j)} = (f^n t)|_{D(ff_if_j)} = t_j|_{D(ff_if_j)}$$

so by (i) (applied to $X = D(f_i f_j)$) we can find $m_{ij} \geqslant 1$ such that $f^{m_{ij}}(t_i - t_j) = 0$. Again we can take $m \gg 0$ such that $f^m(t_i - t_j) = 0$ for any i, j. Now $f^m t_i \in \mathcal{F}(D(f_i))$ patches to a global section $s \in \mathcal{F}(X)$ whose restriction to $D(f)$ is $f^{n+m}t$. □

3.15.3 Corollary. Let X be a scheme and $\mathcal{F} \in \mathbf{Mod}_{\mathcal{O}_X}$. TFAE :

1. $\mathcal{F}$ is quasi-coherent.
2. For any affine open $U \subseteq X$, $\mathcal{F}|_U \cong \widetilde{M}$ for some $\mathcal{O}_X(U)$-module M.
3. There exists an affine open cover $\mathcal{U}$ of X such that for any $U \in \mathcal{U}$, $\mathcal{F}|_U \cong \widetilde{M}$ for some $\mathcal{O}_X(U)$-module M.

3.15.4 Corollary. Let X be a scheme and $f : \mathcal{F} \to \mathcal{G}$ be a morphism in $\mathbf{Mod}_{\mathcal{O}_X}$. If $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-quasi-coherent modules, then so are $\ker f$, $\operatorname{coker} f$ and $\operatorname{Im} f$.

3.16 Theorem Let X be a scheme. Then $\mathbf{Qcoh}_X$ is a weak Serre abelian subcategory of $\mathbf{Mod}_{\mathcal{O}_X}$. [1]

3.17 Let $f : X \to Y$ be a morphism of schemes. If $\mathcal{F}$ is an $\mathcal{O}_Y$-quasi-coherent module, then the pullback $f^*\mathcal{F}$ is also $\mathcal{O}_X$-quasi-coherent. Indeed, this follows from the definition and the fact that tensor product is right exact. For the pushforward, we have the following

Lemma. Let $\mathcal{F}$ be an $\mathcal{O}_X$-quasi-coherent module. Then $f_*\mathcal{F}$ is $\mathcal{O}_Y$-quasi-coherent if there exist

(i) an affine open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of Y, and

(ii) for each $i \in I$, a finite affine open cover $\{X_{i,j}\}_{j \in J_i}$ of $f^{-1}(U_i)$ such that $X_{i,j} \cap X_{i,j'}$ is compact for every $j, j' \in J_i^2$.

Proof. For each $i \in I$, put $f_i : f^{-1}(U_i) \to U_i$ to be the map induced by f. Then $(f_*\mathcal{F})|_{U_i} = (f_i)_*(\mathcal{F}|_{f^{-1}(S_i)})$. Since quasi-coherence is a local property, we may then assume that $Y = \operatorname{Spec} A$ for some ring A and X is covered by finite affine opens X_j $(j \in J)$ with $X_j \cap X_{j'}$ compact for any $j, j' \in J$.

For any $j, j' \in J$ by compactness we may fix a finite affine open cover $X_{j,j',k}$ $(k \in K_{j,j'})$; let $\iota_j : X_j \to X$ and $\iota_{j,j',k} : X_{j,j',k} \to X$ be the inclusions. Then we have an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \prod_{j \in J} (\iota_j)_* \left(\mathcal{F}|_{X_j}\right) \longrightarrow \prod_{j,j' \in J} \prod_{k \in K_{j,j'}} (\iota_{j,j',k})_* \left(\mathcal{F}|_{X_{j,j',k}}\right)$$

Since f_* is left exact, we have the following exact sequence

$$0 \longrightarrow f_*\mathcal{F} \longrightarrow \prod_{j \in J} (f \circ \iota_j)_* \left(\mathcal{F}|_{X_j}\right) \longrightarrow \prod_{j,j' \in J} \prod_{k \in K_{j,j'}} (f \circ \iota_{j,j',k})_* \left(\mathcal{F}|_{X_{j,j',k}}\right)$$

Note that $f \circ \iota_j$ and $f \circ \iota_{j,j',k}$ are morphisms between affine schemes, so by Lemma 3.14, the last two sheaves above are $\mathcal{O}_Y$-quasi-coherent. Thus by Corollary 3.15.4 $f_*\mathcal{F}$ is $\mathcal{O}_Y$-coherent as well. $\square$

3.18 Let X be a local ringed space. Recall in (2.21) for any $g \in \mathcal{O}_X(X)$ we defined the open subset X_g of X. Then by patching we can construct an inverse of $g|_{X_g}$, so we actually have $g|_{X_g} \in \mathcal{O}_X(X_g)^\times$. Let $\mathcal{F} \in \mathbf{Mod}_{\mathcal{O}_X}$. Then g acts on $\mathcal{F}(X_g)$ in an invertible manner, so the universal property of localization gives rise to a commutative triangle

$$\begin{tikzcd} & \mathcal{F}(X) \arrow[dl] \arrow[dr] & \\ \mathcal{F}(X)_g \arrow[rr] & & \mathcal{F}(X_g) \end{tikzcd}$$

Now let $f : X \to S$ be a morphism in $\mathbf{LRS}$ with S affine. Let $h \in \mathcal{O}_S(S)$ and $g = f^\sharp_S(h) \in \mathcal{O}_X(X)$. Suppose $f_*\mathcal{F}$ is $\mathcal{O}_S$-quasi-coherent, then the above map

$$\mathcal{F}(X)_g \longrightarrow \mathcal{F}(X_g)$$

is in fact an isomorphism. The first step to see this is the equality

$$X_g = f^{-1}(S_h).$$

1. This is not true for a general ringed space.

For $x \in X_g$ if and only if $g_x \neq 0$ in $\kappa(x)$. But $g_x = f_x(h_{f(x)})$ and $f_x : \kappa(f(x)) \hookrightarrow \kappa(x)$, so this is equivalent to saying that $h_{f(x)} \neq 0$ in $\kappa(f(x))$, which is the same as saying that $x \in f^{-1}(S_h)$. Since S is affine, by 3.15 we know $f_*\mathcal{F} = \widetilde{M}$ for some $\mathcal{O}_S(S)$-module. We have a similar commutative triangle

$$\begin{tikzcd} & (f_*\mathcal{F})(S) \arrow[dl] \arrow[dr] & \\ (f_*\mathcal{F})(S)_h \arrow[rr] & & (f_*\mathcal{F})(S_h) \end{tikzcd}$$

In this time the lower horizontal arrow is an isomorphism, for they are in fact both M_h. But

$$(f_*\mathcal{F})(S)_h = \mathcal{F}(X)_h = \mathcal{F}(X)_g$$

where h acts on $\mathcal{F}(X)$ via $f_S^\sharp : \mathcal{O}_S(S) \to \mathcal{O}_X(X)$. Thus the horizontal arrow becomes the arrow that we focus on, and this proves it is an isomorphism.

3.19 We apply (3.18) to the case $\mathcal{F} = \mathcal{O}_A$ for some scheme X. We now prove X is affine if and only if $\alpha : X \to \operatorname{Spec} \mathcal{O}_X(X)$ (cf. (3.8.1)) is an isomorphism in **LRS**. Only "only if" part needs a proof. For the map on topological space, we easily see from the construction that there is a commuting square

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\wr"] & \operatorname{Spec} \mathcal{O}_X(X) \arrow[d, "\wr"] \\ \operatorname{Spec} A \arrow[r, equal] & \operatorname{Spec} A \end{tikzcd}$$

where A is a ring such that $X \cong \operatorname{Spec} A$ as **LRS**, so α is a homeomorphism. For the sheaf map, for each $g \in \mathcal{O}_X(X)$, $\alpha_*\mathcal{O}(D(g)) \leftarrow \mathcal{O}_{\operatorname{Spec} \mathcal{O}_X(X)}(D(g))$ is induced by the localization $\mathcal{O}_X(X_g) \leftarrow \mathcal{O}_X(X)_g$ (here we use $\alpha^{-1}(D(g)) = X_g$), and thus by (3.18) it is an isomorphism. Since the $D(g)$ form a basis of the topology on $\operatorname{Spec} \mathcal{O}_X(X)$, so $\alpha_*\mathcal{O}_X \leftarrow \mathcal{O}_{\operatorname{Spec} \mathcal{O}_X(X)}$ is an isomorphism.

3.20 For another application of (3.18), we introduce

Definition. A morphism $f : X \to S$ of schemes is called **affine** if it satisfies the following equivalent conditions.

(i) S admits an affine open cover $\mathcal{V}$ such that $f^{-1}(V)$ is affine for each $V \in \mathcal{V}$.

(ii) For any affine open $V \subseteq S$, $f^{-1}(V)$ is affine.

Clearly (ii) implies (i). To see (i) implies (ii), first note that if we write $\mathcal{V} = \{V_i\}_{i\in I}$ to be a cover satisfying (i) for f, then the preimage of principal affine open subsets of each V_i is also affine. This means f admits an open basis consisting of affine opens whose preimages under f are affine. This means for any affine open V, the induced morphism $f^{-1}(V) \to V$ again satisfies (i). Thus we may replace S by $V \cong \operatorname{Spec} A$ and X by $f^{-1}(V)$, and we must show, in this case, that X is affine. By (3.7) we have a commuting square

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[d, "f"'] & \operatorname{Spec} \mathcal{O}_X(X) \arrow[d, "\operatorname{Spec}(f_S^\sharp)"] \\ S \arrow[r, "\alpha_S"'] & \operatorname{Spec} \mathcal{O}_S(S) \end{tikzcd}$$

where the horizontal arrows are canonical; note that α_S is an isomorphism by (3.19). As said above, we can find $\{h_i\}_{i\in I} \subseteq A$ such that $X_{g_i} = f^{-1}(D(h_i))$ is affine, where $g_i = f_S^\sharp(h_i)$. We see the $D(g_i) = (\operatorname{Spec} f_S^\sharp)^{-1}(D(h_i))$ covers $\operatorname{Spec} \mathcal{O}_X(X)$ and

$(\alpha_X)^{-1}(D(g_i)) = f^{-1}(D(h_i)) = X_{g_i}$. Hence

$$(\alpha_X)^{-1}(D(g_i)) = X_{g_i} \xrightarrow[\sim]{\alpha_{X_{g_i}}} \operatorname{Spec} \mathcal{O}_X(X_{g_i}) \xrightarrow[\sim]{(3.18)} \operatorname{Spec} \mathcal{O}_X(X)_{g_i} \cong D(g_i)$$

Note that this is simply $\alpha_X|_{X_{g_i}}^{D(g_i)}$ by (3.11), so this proves α_X is an isomorphism (2.17).

3.21 Closed subscheme. Let X be a scheme, and let $\mathcal{I}$ be an ideal sheaf of $\mathcal{O}_X$. Following (2.24), we may construct a closed local-ringed subspace $(V(\mathcal{I}), (\mathcal{O}_X/\mathcal{I})|_{V(\mathcal{I})})$ of $(X, \mathcal{O}_X)$. Since we are discussing schemes, a natural question is whether $(V(\mathcal{I}), (\mathcal{O}_X/\mathcal{I})|_{V(\mathcal{I})})$ is itself a scheme.

Suppose $Y = (Y, \mathcal{O}_Y) = (V(\mathcal{I}), (\mathcal{O}_X/\mathcal{I})|_{V(\mathcal{I})})$ is a scheme. Pick $y \in Y$ and let U' be an affine open neighborhood of y in X. Since we assume Y is a scheme, we can find an affine open neighborhood V' of y in Y contained in $U' \cap Y$. The topology on Y is given by subspace topology, so we can find $f \in \mathcal{O}_X(U')$ with $D(f) \cap Y \subseteq V'$. Note that we can restrict f to $Y \cap U'$, and

$$D(f) \cap Y = (U')_f \cap Y = (U' \cap Y)_{f|_{Y \cap U'}} = (V')_{f|_{V'}}$$

which is still affine in Y. In sum we find an affine neighborhood U of y in X such that $U \cap Y$ remains affine in Y. In other words, the closed immersion $Y \to X$ is an affine morphism (3.20).

Suppose $U = \operatorname{Spec} A$, $U \cap Y = \operatorname{Spec} B$ and let $\phi : A \to B$ be the homomorphism corresponding to $U \cap Y \subseteq U$. We contend that $\mathcal{I}|_U \cong \widetilde{\ker \phi}$. In fact, for any $f \in A$,

$$\widetilde{\ker \phi}(U_f) = (\ker \phi)_f \cong \ker(A_f \to B_f) \cong \ker(\mathcal{O}_X(U_f) \to \mathcal{O}_Y(Y \cap U_f)) = \mathcal{I}(U_f).$$

The isomorphisms involved are functorial in U_f, so this proves our contention. In particular, this shows $\mathcal{I}$ is quasi-coherent at $y \in Y$. Furthermore, the inclusion $\mathcal{I}|_{X \setminus Y} \to \mathcal{O}_X|_{X \setminus Y}$ is an isomorphism, which shows that $\mathcal{I}$ is in fact quasi-coherent on the whole X.

Conversely, suppose $\mathcal{I}$ is quasi-coherent. Let U be an affine open set of X. Then $\mathcal{I}|_U$ is an quasi-coherent ideal sheaf of $\mathcal{O}_X|_U$. Clearly, $V(\mathcal{I}) \cap U = V(\mathcal{I}|_U)$, and

$$(\mathcal{O}_X/\mathcal{I})|_{V(\mathcal{I})}|_{V(\mathcal{I}) \cap U} \cong (\mathcal{O}_X/\mathcal{I})|_{V(\mathcal{I}) \cap U} \cong (\mathcal{O}_X|_U/\mathcal{I}|_U)|_{V(\mathcal{I}|_U)}.$$

Thus we can assume $X = \operatorname{Spec} A$ is affine. By (3.15.2), there is a commutative diagram with vertical arrows being isomorphisms

$$\begin{tikzcd} \mathcal{I} \arrow[r] & \mathcal{O}_X \\ \widetilde{\mathcal{I}(X)} \arrow[u] \arrow[r] & \widetilde{A} \arrow[u] \end{tikzcd}$$

so that $\mathcal{O}_X/\mathcal{I} \cong \widetilde{A/\mathcal{I}(X)}$. A easy computation shows that the closed subset $V(\mathcal{I}(X))$ of $\operatorname{Spec} A$ coincides with $V(\mathcal{I})$. From (3.12) we can conclude that $(V(\mathcal{I}), (\mathcal{O}_X/\mathcal{I})|_{V(\mathcal{I})})$ is isomorphic to the affine scheme $\operatorname{Spec} A/\mathcal{I}(X)$. We summarize what we obtain and give some consequences in the following theorem.

3.21.1 Theorem. Let X be a scheme and $\mathcal{I}$ an ideal sheaf of X.

(i) Every closed immersion of schemes is an affine morphism.

(ii) The closed local-ringed subspace $(V(\mathcal{I}), (\mathcal{O}_X/\mathcal{I})|_{V(\mathcal{I})})$ is a scheme if and only if $\mathcal{I}$ is $\mathcal{O}_X$-quasi-coherent. In this case, we say $(V(\mathcal{I}), (\mathcal{O}_X/\mathcal{I})|_{V(\mathcal{I})})$ is a **closed subscheme** of X.

(iii) If $X = \operatorname{Spec} A$ is affine, then every closed subscheme of X has the form $(V(I), \widetilde{A/I}|_{V(I)})$ for some ideal I of A.

3.22 Let A be a ring. For an ideal I, we can equip the closed subset $Z := V(I)$ with a scheme structure, making it the image of the closed immersion $\operatorname{Spec} A/I \to \operatorname{Spec} A$. However, there are many possible closed subscheme structures on Z. For instance, if J is another ideal such that $\sqrt{J} = \sqrt{I}$, then $V(J) = Z$, so we can also equip Z with the scheme structure defined by J. Nevertheless, among all possible closed subscheme structures on Z we see the one determined by $\sqrt{I}$ is the most natural one. Moreover, any ideal with $V(J) = Z$ has a inclusion $J \subseteq \sqrt{I}$, which induces a closed immersion

$$(Z, \widetilde{A/\sqrt{I}}|_Z) \twoheadrightarrow (Z, \widetilde{A/J}|_Z)$$

of schemes.

For an ideal I, there is an ideal sheaf $I\mathcal{O}_{\operatorname{Spec} A}$ defined by $U \mapsto I\mathcal{O}_{\operatorname{Spec} A}(U)$. The ideal sheaf $\sqrt{I}\mathcal{O}_{\operatorname{Spec} A}$ has the following description :

$$\sqrt{I}\mathcal{O}_{\operatorname{Spec} A}(U) = \{s \in \mathcal{O}_{\operatorname{Spec} A}(U) \mid s|_x = 0 \text{ for all } x \in U \cap Z\}.$$

Indeed, this follows as $\sqrt{IA_f} = \sqrt{I}A_f$ for any $f \in A$.

3.1.2 Invertible sheaves

3.23 Sheaf hom Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}, \mathcal{G} \in \mathbf{Mod}^{\mathrm{pre}}_{\mathcal{O}_X}$. We define the **hom sheaf** as follows. For any open $U \subseteq X$, let

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) = \operatorname{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$$

and for any $V \subseteq U$, the restriction is given by the obvious arrow $\operatorname{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U) \to \operatorname{Hom}_{\mathcal{O}_X|_V}(\mathcal{F}|_V, \mathcal{G}|_V)$. This makes $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ a presheaf of abelian groups on X. We also write $\mathcal{E}nd_{\mathcal{O}_X}\mathcal{F} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$.

When $\mathcal{G}$ is a sheaf, we easily check that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is a sheaf of abelian groups. Also, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is naturally, as in the case of modules, a (left) $\mathcal{E}nd_{\mathcal{O}_X}\mathcal{G}$-module and a right $\mathcal{E}nd_{\mathcal{O}_X}\mathcal{F}$-module.

3.24 Adjunction between f^* and f_* : sheafified version Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. We saw in (2.11) that there is a functorial bijection

$$\operatorname{Hom}_{\mathcal{O}_X}(f^*\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{F}, f_*\mathcal{G})$$

Let $V \subseteq Y$ be an open subset and view it as an open ringed subspace $(V, \mathcal{O}_V = \mathcal{O}_Y|_V)$; a similar notation works for the open subset $f^{-1}(V) \subseteq X$. The morphism f restricts to $f_V := f|^V_{f^{-1}(V)} : (f^{-1}(V), \mathcal{O}_{f^{-1}(V)}) \to (V, \mathcal{O}_V = \mathcal{O}_Y|_V)$, and we obtain a functorial bijection

$$\operatorname{Hom}_{\mathcal{O}_{f^{-1}(V)}}((f_V)^*\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{O}_V}(\mathcal{F}, (f_V)_*\mathcal{G})$$

for $\mathcal{F}$ in $\mathbf{Mod}_{\mathcal{O}_V}$ and $\mathcal{G}$ in $\mathbf{Mod}_{\mathcal{O}_{f^{-1}(V)}}$.

Now let $\mathcal{F}$ be an $\mathcal{O}_Y$-module and $\mathcal{G}$ an $\mathcal{O}_X$-module. We have $(f^*\mathcal{F})|_{f^{-1}V} = (f_V)^*(\mathcal{F}|_V)$ and $(f_*\mathcal{G})|_V = (f_V)_*(\mathcal{G}|_{f^{-1}(V)})$. Various adjunctions are clearly compatible, so they give rise to an isomorphism of $\mathcal{O}_Y$-modules

$$f_*\mathcal{H}om_{\mathcal{O}_X}(f^*\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, f_*\mathcal{G}).$$

3.25 Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ be an $\mathcal{O}_X$-module. Define its **dual module**

$$\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$$

For any index set I and $\mathcal{F} \in \mathbf{Mod}_{\mathcal{O}_X}$, one has

$$\operatorname{Hom}_{\mathcal{O}_X}((\mathcal{O}_X)^{\oplus I}, \mathcal{F}) \cong \mathcal{F}(X)^{\oplus I}.$$

the map being given by evaluation at $(e_i)_{i\in I}$, where $e_i \in \mathcal{O}_X(X)^{\oplus I}$ is $e_i = \delta_{ij}(j)$. In particular, $\mathcal{E}nd_{\mathcal{O}_X}\mathcal{O}_X = \mathcal{O}_X$, and $\mathcal{L}^\vee$ is thus an $\mathcal{O}_X$-module.

There is a natural morphism

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}^\vee \longrightarrow \mathcal{O}_X$$

given by the evaluation. Explicitly, for any open U,

$$\begin{aligned} \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{O}_X|_U) &\longrightarrow \mathcal{O}_X(U) \\ (f, T) &\longmapsto T_U(f). \end{aligned}$$

This defines a morphism $\mathcal{F} \otimes^p \mathcal{F}^\vee \to \mathcal{O}_X$ (2.11), and hence a morphism $\mathcal{F} \otimes \mathcal{F}^\vee \to \mathcal{O}_X$ by passing to sheafification.

3.25.1 Lemma Let $\mathcal{L}$ be an invertible sheaf (3.13).

(i) If $\mathcal{L}_1$ and $\mathcal{L}_2$ are invertible sheaves, so is $\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2$, and $\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2 \cong \mathcal{L}_2 \otimes_{\mathcal{O}_X} \mathcal{L}_1$.

(ii) $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong \mathcal{L} \cong \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{L}$.

(iii) $\mathcal{L}^\vee$ is also an invertible sheaf.

(iv) $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee \cong \mathcal{O}_X$.

Proof.

(iv) We claim the above morphism $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee \to \mathcal{O}_X$ is an isomorphism. It suffices to check that $(\star)$ is an isomorphism for arbitrary small open set. Let U be an open set such that $\mathcal{L}|_U \cong \mathcal{O}_X|_U$. Then we have a commutative diagram

$$\begin{tikzcd}
 & \mathcal{L}(U) \otimes_{\mathcal{O}_X(U)} \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{L}|_U, \mathcal{O}_X|_U) \arrow[r] \arrow[d, "\wr"'] & \mathcal{O}_X(U) & ab \\
(f,T) \arrow[d, mapsto] & \mathcal{O}_X(U) \otimes_{\mathcal{O}_X(U)} \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{O}_X|_U, \mathcal{O}_X|_U) \arrow[ur, bend right] \arrow[d, "\wr"'] & & \\
(f, T_U(1)) & \mathcal{O}_X(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(U) \arrow[uur, bend right] & & \\
 & (a,b) \arrow[uuurr, mapsto, bend right] & &
\end{tikzcd}$$

(this is commutative since T is an $\mathcal{O}_X|_U$-module homomorphism). The arrow $(a,b) \mapsto ab$ is an isomorphism, hence so is the arrow on the top.

□

3.26 Picard group. The previous lemma shows the isomorphism classes of invertible sheaves on X form an abelian group (multiplication being tensor product). We denote by this group $\mathrm{Pic}(X)$, called the **Picard group** of the ringed space X.

3.26.1 Let $f : X \to Y$ be a morphism in **RS**. If $\mathcal{L}$ is an invertible sheaf on Y, then $f^*\mathcal{L}$ is an invertible sheaf on X. Indeed, if U is an open set in Y such that $\mathcal{L}|_U \cong \mathcal{O}_Y|_U$, we have

$$(f^*\mathcal{L})|_{f^{-1}(U)} \cong \left(f^{-1}\mathcal{L}\right)|_{f^{-1}(U)} \otimes_{(f^{-1}\mathcal{O}_Y)|_{f^{-1}(U)}} \mathcal{O}_X|_{f^{-1}(U)}$$

If we put $g = f|^{U}_{f^{-1}(U)}$, then the above is isomorphic to

$$g^{-1}(\mathcal{L}|_U) \otimes_{g^{-1}(\mathcal{O}_Y|_U)} \mathcal{O}_X|_{f^{-1}(U)} \cong \mathcal{O}_X|_{f^{-1}(U)}.$$

Thus f^* induces a map $\mathrm{Pic}(Y) \to \mathrm{Pic}(X)$. Moreover, since $\otimes$ commutes with $\varinjlim$, we see $f^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is a group homomorphism.

3.27 Twists. Let $(X, \mathcal{O}_X)$ be a ringed space. For an invertible sheaf $\mathcal{L}$ and $n \geqslant 0$, denote $\mathcal{L}^n = \mathcal{L}^{\otimes n}$, and for $n \leqslant 0$, denote $\mathcal{L}^{-n} = (\mathcal{L}^\vee)^{\otimes n}$. Then for $n, m \in \mathbb{Z}$, we have $\mathcal{L}^n \otimes \mathcal{L}^m \cong \mathcal{L}^{n+m}$.

For any open U, by definition we have a canonical bilinear map $\mathcal{L}^n(U) \times \mathcal{L}^m(U) \to (\mathcal{L}^n \otimes \mathcal{L}^m)(U) \cong \mathcal{L}^{n+m}(U)$. This makes

$$\Gamma_*(U) := \bigoplus_{n \in \mathbb{Z}} \mathcal{L}^n(U)$$

a graded ring, and $U \mapsto \Gamma_*(U)$ is a presheaf of graded rings on X.

3.27.1 Suppose $(X, \mathcal{O}_X)$ is a local-ringed space. For each $g \in \mathcal{L}(X)$, define

$$X_g = X_g^{\mathcal{L}} = \{x \in X \mid g_x \notin \mathfrak{m}_x \mathcal{L}_x\}$$

If U is an open set such that $\mathcal{L}|_U \cong \mathcal{O}_X|_U$, and if $g|_U$ corresponds to $a \in \mathcal{O}_X(U)$, then $X_g \cap U = U_a$ in the sense of (2.21). In particular, this shows X_g is an open subset of X. If $U \subseteq X_g$ is an open set such that $\mathcal{L}|_U \cong \mathcal{O}_X|_U$, then $X_g \cap U = U_a$ and we can find a unique $h \in \mathcal{L}^\vee|_U$ such that under the isomorphism $\mathcal{L}^\vee|_U \cong \mathcal{O}_X|_U$, it corresponds to the inverse of a. This means $g|_U h = 1 \in \mathcal{O}_X(U)$; let us denote $h = (g|_U)^{-1}$. If V is another open set trivializing $\mathcal{L}$ and intersecting with U nontrivially, and $U \cap V$ also trivializes $\mathcal{L}$, then clearly $(g|_V)^{-1} = (g|_{U \cap V})^{-1} = (g|_U)^{-1}$, so the $(g|_U)^{-1}$ patch to a section $(g|_{X_f})^{-1} \in \mathcal{L}^\vee(X_f)$, satisfying $g(g|_{X_f})^{-1} = 1 \in \mathcal{O}_X(X_f)$. If there is no confusion, we simply put $g^{-1} = (g|_{X_f})^{-1}$.

Moreover, the multiplication

$$\begin{array}{ccc} \mathcal{O}_X|_{U_g} & \longrightarrow & \mathcal{L}|_{U_g} \\ a & \longmapsto & ag \end{array}$$

is an isomorphism. Indeed, for $b \in \mathcal{L}|_{U_g}$, we have $b \otimes g^{-1} \in \mathcal{L}|_{U_g} \otimes \mathcal{L}|_{U_g}^{-1} \cong \mathcal{O}_X|_{U_g}$. If c is the image of $b \otimes g^{-1}$, then $cg = b$.

3.27.2 For any $\mathcal{F} \in \mathbf{Mod}_{\mathcal{O}_X}$ and any open U, define

$$\Gamma_*(\mathcal{F}, \mathcal{L})(U) = \bigoplus_{n \in \mathbb{Z}} (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n)(U)$$

This is naturally a graded $\Gamma_*(U)$-module, and $\Gamma_*(\mathcal{F}, \mathcal{L})$ is a presheaf of S-modules. Note that every $g \in \mathcal{L}^n(U)$ acts on $\Gamma_*(\mathcal{F}, \mathcal{L})(U_g)$ invertibly, for the presence of $g^{-1} \in \mathcal{L}^{-n}(U_g)$. Thus we have a canonical commuting triangle

$$\begin{tikzcd} & \Gamma_*(\mathcal{F}, \mathcal{L})(U) \arrow[dl, "\text{localization}"'] \arrow[dr, "\text{restriction}"] & \\ \Gamma_*(\mathcal{F}, \mathcal{L})(U)_g \arrow[rr, "\alpha(g)"] & & \Gamma_*(\mathcal{F}, \mathcal{L})(U_g) \end{tikzcd}$$

3.27.3 Lemma. Let X be a scheme, $\mathcal{L}$ an invertible sheaf and $\mathcal{F}$ a $\mathcal{O}_X$-quasi-coherent sheaf. Then for any $g \in \mathcal{L}^d(X)$, we have the following.

(1) If X is compact, then $\alpha(g)$ is injective.

(2) If X admits a finite affine open cover $\{U_i\}_{i \in I}$ such that $\mathcal{L}|_{U_i} \cong \mathcal{O}_X(U_i)$ and $U_i \cap U_j$ is compact for any $i, j \in I$, then $\alpha(g)$ is surjective.

Proof. This is a generalization of Lemma 3.15.2, and can be proved in a similar way. Here we use a slightly different way which is essentially the same. Cover X be a family of affine open subsets U_i $(i \in I)$ such that

$$(U_i,\ \mathcal{L}|_{U_i},\ \mathcal{F}|_{U_i},\ g|_{U_i}) \cong (\mathrm{Spec}\, A_i,\ \widetilde{A_i},\ \widetilde{M_i},\ a_i)$$

for some ring A_i, $a_i \in A_i$ and $M_i \in \mathbf{Mod}_{A_i}$. Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\Gamma_*(\mathcal{F},\mathcal{L})(X)_g \arrow[r, "\alpha(g)"] \arrow[d] & \Gamma_*(\mathcal{F},\mathcal{L})(X_g) \arrow[d] \\
\prod_{i\in I}\Gamma_*(\mathcal{F},\mathcal{L})(U_i)_g \arrow[r, "\prod_j \alpha(g|_{U_i})"] \arrow[d] & \prod_{i\in I}\Gamma_*(\mathcal{F},\mathcal{L})(X_g\cap U_j) \arrow[d] \\
\prod_{i,j\in I}\Gamma_*(\mathcal{F},\mathcal{L})(U_i\cap U_j)_g \arrow[r, "\prod_{i,j}\alpha(g|_{U_i\cap U_j})"] & \prod_{i,j\in I}\Gamma_*(\mathcal{F},\mathcal{L})(X_g\cap U_i\cap U_j)
\end{tikzcd} \tag{$\star$}$$

Note that the middle horizontal arrow is bijective, since there exists a commutative diagram

$$\begin{tikzcd}
\Gamma_*(\mathcal{F},\mathcal{L})(U_i)_g \arrow[r, "\alpha(g|_{U_i})"] \arrow[d, "\wr"'] & \Gamma_*(\mathcal{F},\mathcal{L})(X_g\cap U_i) \arrow[d, "\wr"'] \\
\Gamma_*(\widetilde{M_i},\widetilde{A_i})(\operatorname{Spec} A_i)_a \arrow[r] \arrow[d, "\wr"'] & \Gamma_*(\widetilde{M_i},\widetilde{A_i})(D(a_i)) \arrow[d, "\wr"'] \\
\bigoplus_{n\in\mathbb{Z}}(M_i\otimes_{A_i} A_i^{\otimes n})_{a_i} \arrow[r, "\sim"] & \bigoplus_{n\in\mathbb{Z}}\left((M_i)_{a_i}\otimes_{(A_i)_{a_i}}(A_i)_{a_i}^{\otimes n}\right)
\end{tikzcd}$$

For (1), since X is compact, we can assume $\#I < \infty$. In this case, there exists an exact sequence

$$0 \longrightarrow \Gamma_*(\mathcal{F},\mathcal{L})(X) \longrightarrow \prod_{i\in I}\Gamma_*(\mathcal{F},\mathcal{L})(U_i) \longrightarrow \prod_{i,j\in I}\Gamma_*(\mathcal{F},\mathcal{L})(U_i\cap U_j)$$

To be specific, this is obtained by taking direct sum of exact sequences

$$0 \to (\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}^n)(X) \to \prod_i(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}^n)(U_i) \to \prod_{i,j}(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}^n)(U_i\cap U_j).$$

The finiteness of I is used here for then the product and direct sum commute. In particular, this means the two vertical sequences in $(\star)$ are exact. In particular, the left middle arrow in $(\star)$ is injective, so $\alpha(g)$ is injective.

For (2), we can still assume $\#I < \infty$ so the above discussion is valid. Moreover, since $U_i \cap U_j$ is compact, the middle bottom arrow in $(\star)$ is injective by (1). As this stage, the surjectivity of $\alpha(g)$ follows from a simple diagram chasing. □

3.27.4 Let X be a scheme, $\mathcal{L}$ an invertible sheaf, $\mathcal{F}$ an $\mathcal{O}_X$-quasi-coherent module and $g \in \mathcal{L}(X)$. Define

$$\begin{aligned}\Gamma_*(\mathcal{F},\mathcal{L})(U)_{(g)} &\longrightarrow \mathcal{F}(U_g)\\ xg^{-n} &\longmapsto x|_{U_g}\otimes g|_{U_g}^{-n}.\end{aligned} \tag{$\spadesuit$}$$

where $\Gamma_*(\mathcal{F},\mathcal{L})(U)_{(g)}$ is the degree 0 part of the localization $\Gamma_*(\mathcal{F},\mathcal{L})(U)_g$ (c.f (3.108)). Note this is simply the degree 0 part of the map

$$\Gamma_*(\mathcal{F},\mathcal{L})(U)_g \xrightarrow{\alpha(g)} \Gamma_*(\mathcal{F},\mathcal{L})(U_g)$$

modulo the identification $\mathcal{O}_X|_{U_g} \cong \mathcal{L}|_{U_g}$ made in (3.27.1). Hence from (3.27.3) we deduce

Corollary.

1. If X is compact, then the map (♠) is injective.
2. If X admits a finite affine open cover $\{U_i\}_{i\in I}$ such that $\mathcal{L}|_{U_i} \cong \mathcal{O}_X(U_i)$ and $U_i \cap U_j$ is compact for any $i, j \in I$, then the map (♠) is surjective.

3.1.3 Coherent sheaves

3.28 Definition. Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{F} \in \mathbf{Mod}_{\mathcal{O}_X}$.

(i) $\mathcal{F}$ is **finitely generated** if every point in X admits an open neighborhood $\mathcal{U}$ such that there exists an $n \in \mathbb{Z}_{\geqslant 1}$ and a surjection $(\mathcal{O}_X|_U)^{\oplus n} \to \mathcal{F}|_U$.

(ii) $\mathcal{F}$ is **finitely presented** if every point in X admits an open neighborhood U such that there exists an exact sequence

$$(\mathcal{O}_X|_U)^{\oplus m} \longrightarrow (\mathcal{O}_X|_U)^{\oplus n} \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

for some $n, m \in \mathbb{Z}_{\geqslant 1}$, depending on x.

(iii) $\mathcal{F}$ is **coherent** if it is finitely generated and for any open set $U \subseteq X$, $n \in \mathbb{Z}_{\geqslant 1}$ and $\mathcal{O}_X|_U$-morphism $\left(\mathcal{O}_X|_U\right)^{\oplus n} \to \mathcal{F}|_U$, its kernel is of finite type.

3.28.1 Directly from the definition we see $\mathcal{O}_X$ is finitely generated and finitely presented. But it is not true that $\mathcal{O}_X$ is $\mathcal{O}_X$-coherent.

3.28.2 Denote by $\mathbf{Coh}_{\mathcal{O}_X}$ the full subcategory of $\mathbf{Mod}_{\mathcal{O}_X}$ consisting of $\mathcal{O}_X$-coherent modules. If X is a scheme, we write $\mathbf{Coh}_X = \mathbf{Coh}(X) = \mathbf{Coh}_{\mathcal{O}_X}$ instead, if no confusion arises.

3.29 Lemma. Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{F} \in \mathbf{Qcoh}_{\mathcal{O}_X}$.

(i) $\mathcal{F}$ finitely presented $\Rightarrow \mathcal{F}$ quasi-coherent.

(ii) $\mathcal{F}$ coherent $\Rightarrow \mathcal{F}$ finitely presented $\Rightarrow \mathcal{F}$ finitely generated

(iii) If $\mathcal{O}_X$ is $\mathcal{O}_X$-coherent, then $\mathcal{F}$ finitely presented $\Rightarrow \mathcal{F}$ coherent.

3.30 Theorem. Let $(X, \mathcal{O}_X)$ be a ringed space. $\mathbf{Coh}_{\mathcal{O}_X}$ is a weak Serre subcategory of $\mathbf{Mod}_{\mathcal{O}_X}$.

3.31 Lemma. Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{F} \in \mathbf{Qcoh}_{\mathcal{O}_X}$. Then $\mathcal{F}$ is finitely presented if any only if for any open $U \subseteq X$, $n \in \mathbb{Z}_{\geqslant 1}$ and any surjection $\varphi : \left(\mathcal{O}_X|_U\right)^{\oplus n} \to \mathcal{F}$, the kernel $\ker \varphi$ is finitely generated

Proof. The if part is clear. For the only if part assume $\mathcal{F}$ is finitely presented. Replacing any open set U by X, it suffices to show that if there exist an exact sequence

$$\mathcal{O}_X^{\oplus m} \longrightarrow \mathcal{O}_X^{\oplus n} \longrightarrow \mathcal{F} \longrightarrow 0$$

for some $n, m \in \mathbb{Z}_{\geqslant 1}$, then for any $p \in \mathbb{Z}_{\geqslant 1}$ and any surjection $\varphi : \mathcal{O}_X^{\oplus p} \to \mathcal{F}$, the kernel $\ker \varphi$ is finitely generated □

3.32 Definition. Let X be a scheme.

(i) X is **locally Noetherian scheme** if X admits an affine open cover $\mathcal{U}$ such that each $\mathcal{O}_X(U)$ is a Noetherian ring for each $U \in \mathcal{U}$.

(ii) X is a **Noetherian scheme** if X is locally Noetherian and compact.

3.33 Lemma. Let A be a ring. Then $\operatorname{Spec} A$ is a Noetherian scheme if and only if A is a Noetherian ring.

Proof. The if part is clear. For the only if part, assume $\operatorname{Spec} A$ is a Noetherian scheme. In other words, we can find $f_1, \ldots, f_n \in A$ such that $A = (f_1, \ldots, f_n) = A$ and A_{f_i} is Noetherian. Suppose that $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ is an increasing sequence of ideals in R. Since each A_{f_i} is Noetherian, we can find $N \gg 0$ such that $(I_N)_{f_i} = (I_{N+m})_{f_i}$ for all $m \geqslant 0$ and $i \in [n]$. To prove the result, we must show if I, J are two ideals such that $I_{f_i} = J_{f_i}$ for $i \in [n]$, then $I = J$. This follows from the exact sequence $0 \to J \to \prod_{i \in [n]} J_{f_i} \to \prod_{i,j \in [n]} J_{f_i f_j}$. □

3.33.1 Corollary. A scheme X is locally Noetherian if and only if $\mathcal{O}_X(U)$ is Noetherian for all affine opens $U \subseteq X$.

3.33.2 Corollary. Let X be a locally Noetherian scheme and $\mathcal{F} \in \mathbf{Qcoh}_X$. Then $\mathcal{F}$ coherent $\Leftrightarrow$ $\mathcal{F}$ finitely presented $\Leftrightarrow$ $\mathcal{F}$ finitely generated.

3.1.4 Irreducibility, reducedness and integrability.

3.34 Definition. Let X be a topological space.

1. X is called **irreducible** if it is nonempty and for any closed subspaces C_1, C_2 of X, if $C_1 \cup C_2 = X$, then either $C_1 = X$ or $C_2 = X$. Equivalently, X is irreducible if every nonempty open subspace is dense in X.
2. If Z is an irreducible closed subset of X, a **generic point** of Z is a point $\eta \in Z$ such that $Z = \overline{\{\eta\}}$.
3. An **irreducible component** of X is a maximal irreducible subset in X with respect to inclusion.

3.34.1 It is easy to see that the closure of an irreducible subset is again irreducible. It follows that an irreducible component is a closed subset in X. Also, a continuous image of an irreducible subset is irreducible.

3.34.2 Lemma. For a topological space X, TFAE :

(i) X is irreducible.

(ii) There exists an open cover $\mathcal{U}$ of X consisting of irreducible open subspaces such that $U \cap V \neq \varnothing$ for all $U, V \in \mathcal{U}$.

Proof. (i)⇒(ii) follows from definition. To see (ii)⇒(i), let $\mathcal{U}$ be such an open cover. We claim any nonempty open set is dense. Let $V_1, V_2 \subseteq X$ be two nonempty open subsets and let U_1, $U_2 \in \mathcal{U}$ such that $U_i \cap V_i \neq \varnothing$. By assumption $U_1 \cap U_2 \neq \varnothing$, so by irreducibility $U_1 \cap U_2 \cap V_1 \neq \varnothing$. Again, by irreducibility of U_2,

$$\varnothing \neq (U_1 \cap U_2 \cap V_1) \cap (U_2 \cap V_2) = U_1 \cap U_2 \cap V_1 \cap V_2$$

so $V_1 \cap V_2 \neq \varnothing$ particularly. □

3.34.3 Corollary. Let X be a topological space and $U \subseteq X$ an open subset. Then there is a bijection

$$\begin{aligned} \{\text{irreducible closed subsets of } U\} &\longrightarrow \{\text{irreducible closed subsets of } X \text{ that meet } U\} \\ Z &\longmapsto \overline{Z} \end{aligned}$$

where closure on the right is taken in X.

Proof.

□

3.35 Lemma. Let X be an irreducible topological space. Then the constant presheaf defined in (2.6) is already a sheaf.

3.36 Lemma. Let X be a scheme and let Z be an irreducible closed subset of X. Then Z admits a unique generic point. In particular, this establishes a bijection

$$\begin{aligned} X &\longrightarrow \{\text{irreducible closed subsets of } X\} \\ x &\longmapsto \overline{\{x\}}. \end{aligned}$$

Proof. Assume first that $X = \operatorname{Spec} A$ is affine. Then $Z = V(I)$ for some ideal I, and by irreducibility $I = \mathfrak{p}$ is a prime ideal. Then $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$. Indeed, if $\mathfrak{p}' \in V(\mathfrak{p})$, then for any $V(J)$ containing $\mathfrak{p}$, we have $J \subseteq \mathfrak{p} \subseteq \mathfrak{p}'$, so $\mathfrak{p}' \in V(J)$. This proves $\mathfrak{p}' \in \overline{\{\mathfrak{p}\}}$, so $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$.

Now consider the general case. Let U be an affine open set that meets Z nontrivially. Then $U \cap Z$ is an irreducible closed subset of U, so by the preceding case $U \cap Z$ admits a generic point η. Since $U \cap Z$ is an open dense subset of Z, it follows that $\overline{\eta} = \overline{U \cap Z} = Z$. Let η' be another generic point of Z and pick an affine open neighborhood W of η' in X. Then $W \cap Z$ is a nontrivial open set in Z, so $\eta \in W \cap Z$ as $\{\eta\}$ is dense in Z. The uniqueness in the preceding case then implies that $\eta' = \eta$. □

3.37 Definition. Let X be a scheme.

1. X is call **reduced** if $\mathcal{O}_{X,x}$ is a reduced ring for every $x \in X$.
2. X is called **irreducible** if the underlying topological space is irreducible.
3. X is called **integral** if it is both reduced and irreducible.

3.38 Associated reduced scheme. Let X be a scheme. Define the **nilradical** $\operatorname{nil}(\mathcal{O}_X)$ of $\mathcal{O}_X$ to be the ideal sheaf defined by

$$\operatorname{nil}(\mathcal{O}_X)(U) = \{f \in \mathcal{O}_X(U) \mid f|_x \in \sqrt{0_{\mathcal{O}_{X,x}}} \text{ for all } x \in U\}.$$

We claim that for $x \in X$, the equality

$$\operatorname{nil}(\mathcal{O}_X)_x = \sqrt{0_{\mathcal{O}_{X,x}}}$$

holds. To see this we can assume $X = \operatorname{Spec} A$ is affine, and we need to show $\operatorname{nil}(\mathcal{O}_{\operatorname{Spec} A})_{\mathfrak{p}} = \sqrt{0_{A_{\mathfrak{p}}}}$ for every prime ideal $\mathfrak{p}$ of A. If $a \in A_{\mathfrak{p}}$ is nilpotent, then $a^n = 0$ for some $n \geqslant 1$; take $f \notin \mathfrak{p}$ such that $a = a'|_{\mathfrak{p}}$ for some $a' \in A_f$; then $(a')^n|_{\mathfrak{p}} = 0$ so $(a')^n r = 0$ in A_f for some $r \notin \mathfrak{p}A_f$. This means $(a')^n = 0$ in $(A_f)_r = A_{fr}$, so that $a' \in \sqrt{0_{A_{fr}}}$. By definition, we have

$$\operatorname{nil}(\mathcal{O}_{\operatorname{Spec} A})_{\mathfrak{p}} \cong \varinjlim_{f \notin \mathfrak{p}} \operatorname{nil}(\mathcal{O}_{\operatorname{Spec} A})(D(f)).$$

If we can show $\operatorname{nil}(\mathcal{O}_{\operatorname{Spec} A})(D(f)) = \sqrt{0_{A_f}} \subseteq A_f$, we then may conclude $a \in \operatorname{nil}(\mathcal{O}_{\operatorname{Spec} A})_{\mathfrak{p}}$. Therefore we are led to show that

$$\operatorname{nil}(\mathcal{O}_{\operatorname{Spec} A})(A) = \sqrt{0_A}.$$

One direction is clear. For the other way around, let $f \in A$ satisfy $f|_{\mathfrak{p}} \in \sqrt{0_{A_{\mathfrak{p}}}}$ for any prime ideal $\mathfrak{p}$ of A. Recall that $\sqrt{0_{A_{\mathfrak{p}}}} = \bigcap_{\mathfrak{q} \in \operatorname{Spec} A_{\mathfrak{p}}} \mathfrak{q}A_{\mathfrak{p}} = \bigcap_{\mathfrak{q} \subseteq \mathfrak{p}} \mathfrak{q}A_{\mathfrak{p}}$. Fix $\mathfrak{q} \subseteq \mathfrak{p} \in \operatorname{Spec} A$. Then $f \in \mathfrak{q}A_{\mathfrak{p}}$ implies $p'(fp - q) = 0$ for some $p, p' \notin \mathfrak{p}$, $q \in \mathfrak{q}$, so that $fpp' = p'q \in \mathfrak{q}$. Since $pp' \in A \backslash \mathfrak{p} \subseteq A \backslash \mathfrak{q}$, it forces that $f \in \mathfrak{q}$. In sum, $f \in \bigcap_{\mathfrak{q} \subseteq \mathfrak{p} \in \operatorname{Spec} A} \mathfrak{q} = \sqrt{0_A}$, as we want.

We still need to show $\operatorname{nil}(\mathcal{O}_{\operatorname{Spec} A})_{\mathfrak{p}} \subseteq \sqrt{0_{A_{\mathfrak{p}}}}$. If $a \in \operatorname{nil}(\mathcal{O}_{\operatorname{Spec} A})_{\mathfrak{p}}$, then $a = a'|_{\mathfrak{p}}$ for some $a' \in \operatorname{nil}(\mathcal{O}_{\operatorname{Spec} A})(D(f))$ for some $f \notin \mathfrak{p}$. But we have shown $\operatorname{nil}(\mathcal{O}_{\operatorname{Spec} A})(D(f)) = \sqrt{0_{A_f}}$, so $(a')^n = 0$ for some $n \geqslant 1$, implying $a^n = 0$, i.e., a is nilpotent.

In the above discussion we also show that $\mathrm{nil}(\mathcal{O}_X)(U) = \sqrt{0_{\mathcal{O}_X(U)}} \trianglelefteq \mathcal{O}_X(U)$ for every affine open $U \subseteq X$, so the ideal sheaf $\mathrm{nil}(\mathcal{O}_X)$ is $\mathcal{O}_X$-quasi-coherent. By (3.21), together with an easy fact that $V(\mathrm{nil}(\mathcal{O}_X)) = X$, we conclude the ringed space $(X, \mathcal{O}_X/\mathrm{nil}(\mathcal{O}_X))$ is a closed subscheme of X. What we showed above implies the local ring $(\mathcal{O}_{X,x}/\mathrm{nil}(\mathcal{O}_X))_x \cong \mathcal{O}_{X,x}/\sqrt{0_{\mathcal{O}_{X,x}}}$ is reduced. The so constructed scheme is denoted by X_{red}, and is called the **reduced scheme associated to** X.

3.38.1 The construction $X \mapsto X_{\mathrm{red}}$ is clearly functorial, so it defines a functor $(\cdot)_{\mathrm{red}} : \mathbf{Sch} \to \mathbf{redSch}$ from the category of schemes to the full subcategory of reduced schemes. It is the left adjoint and left inverse of the inclusion functor $\mathbf{redSch} \to \mathbf{Sch}$.

3.38.2 Lemma. Let X be a scheme and Z a closed subset of X. Then there is a unique scheme structure on Z making it a reduced closed subscheme of X.

Proof. We begin by showing the uniqueness. If $U \subseteq X$ is affine open, then $(Z \cap U, \mathcal{O}_Z|_U)$ is a closed subscheme of U, so (3.21.1).(iii) says that $Z \cap U \cong V(I)$ as schemes for some ideal I of $\mathcal{O}_X(U)$. Then $\mathcal{O}_Z(Z \cap U) \cong \mathcal{O}_X(U)/I$. Since Z is reduced, the quotient ring $\mathcal{O}_X(U)/I$ is reduced by (3.39).(i) so that $I = \sqrt{I}$. To conclude, it suffices to note that $\sqrt{I}$ is completely determined by $V(I) \cong Z \cap U$, as $\sqrt{I} = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p}$.

For the existence, we first recollect that in (3.22) there is a natural way to equip $V(I)$ a scheme structure for an ideal I of a ring A. Moreover, one easily sees that the description there makes $V(I)$ a reduced closed subscheme of $\operatorname{Spec} A$. Now take an affine open cover $\mathcal{U}$ of X. For each $U \in \mathcal{U}$, let $I_U \trianglelefteq \mathcal{O}_X(U)$ be such that $V(I_U) = Z \cap U$, and equip $Z \cap U$ with the reduced scheme structure defined in (3.22). The structure sheaves for the $Z \cap U$'s glue by uniqueness of such structure on the intersection, as proved in the first paragraph. This earns Z a desired structure. $\square$

3.39 Lemma. Let A be a ring and let X be a scheme.

(i) $\operatorname{Spec} A$ is reduced if and only if A is reduced.

(ii) $\operatorname{Spec} A$ is irreducible if and only if the nilradical $\sqrt{0}$ of A is a prime.

(iii) $\operatorname{Spec} A$ is integral if and only if A is an integral domain.

(iv) X is reduced if and only if $\mathcal{O}_X(U)$ is a reduced ring for any open set U of X. If it is the case, $\mathcal{O}_X(U)$ is reduced for any open U in X.

(v) X is integral if and only if $\mathcal{O}_X(U)$ is an integral domain for any open set U of X.

Proof.

(i) For $n \geqslant 1$, define $[n] : A \to A$ by $[n](a) = a^n$. Then $\sqrt{0_A} = \bigcup_{n \geqslant 1} \ker[n]$, and

$$\sqrt{0_A} \otimes_A A_{\mathfrak{p}} = \left(\bigcup_{n \geqslant 1} \ker[n] \right) \otimes_A A_{\mathfrak{p}} = \bigcup_{n \geqslant 1} (\ker[n] \otimes_A A_{\mathfrak{p}}) = \sqrt{0_{A_{\mathfrak{p}}}}$$

for all $\mathfrak{p} \in \operatorname{Spec} A$. Hence $\sqrt{0_A} = 0$ if and only if $\sqrt{0_{A_{\mathfrak{p}}}} = 0$ for all $\mathfrak{p} \in \operatorname{Spec} A$.

(ii) First note that for $f \in A$, we have $D(f) = \varnothing$ if and only if $f \in \mathfrak{p}$ for any $\mathfrak{p} \in \operatorname{Spec} A$, i.e., $f \in \sqrt{0}$. Assume $\operatorname{Spec} A$ is irreducible. Let $f, g \in A$ with $fg \in \sqrt{0}$ and $f \notin \sqrt{0}$. Then $D(f) \cap D(g) = D(fg) = \varnothing$, and since $D(f)$ is dense (by irreducibility), this forces that $D(g) = \varnothing$, i.e., $g \in \sqrt{0}$. Conversely, assume $\sqrt{0}$ is a prime. Let $f \notin \sqrt{0}$. We must show $D(f)$ is dense. If $g \notin \sqrt{0}$, then $fg \notin \sqrt{0}$ as $\sqrt{0}$ is assumed to be a prime, which implies $D(f) \cap D(g) = D(fg) \neq \varnothing$. This proves that $D(f)$ is dense.

(iii) This follows from (i) and (ii).

(iv) This follows from the local natural of being reduced and (i). The last assertion follows from the sheaf axiom and the fact that a product of reduced rings is reduced.

(v) By (iv) we may assume in the first place that X is reduced. If X is irreducible, then every nonempty open set U of X is itself irreducible, and hence reduced. Replacing U by X, it suffices to prove $\mathcal{O}_X(X)$ is integral. Let $f, g \in \mathcal{O}_X(X)$ with $fg = 0$. Let U be an affine open set in X. Then $V(f|_U) \cup V(g|_U) = V((fg)|_U) = U$, so by irreducibility we have, say $V(f|_U) = U$. Let V be any other affine open set in X. Then $V \cap U \subseteq V(f|_V)$ with $V \cap U$ dense in V, so $V(f|_V) = V$, or $f|_V = 0$. This implies $f = 0$ in $\mathcal{O}_X(X)$. Conversely, assume that $\mathcal{O}_X(U)$ is an integral domain for any open in X. Let U, V be two nonempty open set in X. If $U \cap V = \varnothing$, then the sheaf axiom implies that

$$\mathcal{O}_X(U \cup V) \cong \mathcal{O}_X(U) \oplus \mathcal{O}_X(V)$$

as rings, which is a contradiction as this is not an integral domain.

□

3.1.5 Cartier divisors

3.40 Localization. Let $(X, \mathcal{O}_X)$ be a ringed space. A subsheaf $\mathcal{S}$ of $\mathcal{O}_X$ is called **multiplicatively closed** if $\mathcal{S}(U) \subseteq \mathcal{O}_X(U)$ is a submonoid for any $U \in \mathrm{Top}(X)$. In this situation, any restriction $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ naturally gives rises to a ring homomorphism $\mathcal{S}(U)^{-1}\mathcal{O}_X(U) \to \mathcal{S}(V)^{-1}\mathcal{O}_X(V)$. We denote by $\mathcal{S}^{-1}\mathcal{O}_X$ the sheafification of the presheaf $U \mapsto \mathcal{S}(U)^{-1}\mathcal{O}_X(U)$.

It is clear that for any $x \in X$, the stalk $\mathcal{S}_x$ is a multiplicatively closed subset of $\mathcal{O}_{X,x}$, so we can form the localization $\mathcal{S}_x^{-1}\mathcal{O}_{X,x}$. The natural map

$$\mathcal{S}(U)^{-1}\mathcal{O}_X(U) \to \mathcal{S}_x^{-1}\mathcal{O}_{X,x}$$

then induces a ring isomorphism $(\mathcal{S}^{-1}\mathcal{O}_X)_x \cong \mathcal{S}_x^{-1}\mathcal{O}_{X,x}$.

3.41 Total quotient sheaf. For a ring A, an element $r \in A$ is **regular** if the multiplication $A \to A$ by r is injective. Denote by A_{reg} the set of all regular elements in A, which is a submonoid of A. The localization $\mathrm{Frac}\, A := A_{\mathrm{reg}}^{-1}A$ is called the **total quotient ring** of A.

Let $(X, \mathcal{O}_X)$ be a ringed space. For any $U \in \mathrm{Top}(X)$, define

$$\mathcal{O}_{X,\mathrm{reg}}(U) := \{s \in \mathcal{O}_X(U) \mid s|_x \in (\mathcal{O}_{X,x})_{\mathrm{reg}} \text{ for any } x \in U\}.$$

It is clear from its local nature that $\mathcal{O}_{X,\mathrm{reg}}$ is a multiplicatively closed subsheaf of $\mathcal{O}_X$. The sheaf

$$\mathcal{K}_X = \mathrm{Frac}\,\mathcal{O}_X := \mathcal{O}_{X,\mathrm{reg}}^{-1}\mathcal{O}_X$$

defined as in (3.40) is called the **total quotient sheaf** of $\mathcal{O}_X$, or called the **sheaf of rational functions** of X.

3.41.1 Lemma. The map $\mathcal{O}_X \to \mathcal{K}_X$ is injective. Hence we can view $\mathcal{O}_X$ as a subsheaf of $\mathcal{K}_X$.

Proof. We must show the natural map $\mathcal{O}_{X,x} \to \mathcal{K}_{X,x} \overset{(3.40)}{\cong} \mathrm{Frac}\,\mathcal{O}_{X,x}$ is injective. If $x \in \mathcal{O}_{X,x}$ is zero in $\mathrm{Frac}\,\mathcal{O}_{X,x}$, then $sx = 0$ for some $s \in (\mathcal{O}_{X,x})_{\mathrm{reg}}$. Since s is regular, it follows that $x = 0$. □

3.42 Example : integral schemes. Let X be an integral scheme. By (3.36), X admits a unique generic point η. Let U be an affine open subset of X; note that $\eta \in U$. The natural map $\mathcal{O}_X(U) \to \mathcal{O}_{X,\eta}$ induces an isomorphism

$$\mathrm{Frac}(\mathcal{O}_X(U)) \cong \mathcal{O}_{X,\eta} \cong \kappa(\eta).$$

Indeed, if we write $U = \mathrm{Spec}\, A$, then η corresponds to the zero ideal, and $\mathrm{Frac}\, A$ is by definition $A_{(0)} \cong \mathcal{O}_{X,\eta} \cong \kappa(\eta)$. This implies that $\mathcal{K}_X$ is isomorphic to the constant sheaf $\underline{\kappa(\eta)}_X$. In this case, we call $K(X) := \kappa(\eta)$ the **(rational) function field** of X.

Generally, if U is an arbitrary open subset of X, as $\eta \in U$, we still have a natural map $\mathcal{O}_X(U) \to \mathcal{O}_{X,\eta}$. This is injective as it is injective for every affine U (and by sheaf axiom). Moreover, if we view $\mathcal{O}_X(U)$ as well as every $\mathcal{O}_{X,x}$ ($x \in U$) as subrings of $\mathcal{O}_{X,\eta}$, we have the equality

$$\mathcal{O}_X(U) = \bigcap_{x \in U} \mathcal{O}_{X,x}.$$

It suffices to show the equality for affine U. Let $f \in \operatorname{Frac} \mathcal{O}_X(U)$ lie in the right hand side. Then the ideal $I = \{g \in \mathcal{O}_X(U) \mid gf \in \mathcal{O}_X(U)\}$ is not, by definition, contained in any prime ideal of $\mathcal{O}_X(U)$. Hence $I = \mathcal{O}_X(U)$, so $f = 1.f \in \mathcal{O}_X(U)$ particularly.

3.43 Definition. A morphism $f : X \to Y$ of schemes is called **dominant** if the set-theoretic image $f(X)$ is dense in Y.

3.43.1 Suppose $f : X \to Y$ is a morphism between irreducible schemes. By (3.36), X (resp. Y) admits a unique generic point η_X (resp. η_Y). Then f is dominant if and only if $f(X)$ is dense in Y. Since $X = \overline{\{\eta_X\}}$ and f is continuous, the latter happens if and only if $\overline{\{f(\eta_X)\}} = Y$. By uniqueness, it is equivalent to saying that $f(\eta_X) = \eta_Y$, or equivalently $\eta_Y \in f(X)$.

3.43.2 Assume further that X, Y are integral. If $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$, then $f : X \to Y$ is dominant if and only if the corresponding ring homomorphism $A \to B$ is injective. This is clear for the generic point corresponds to the zero ideal. It follows that for arbitrary integral schemes X, Y, the morphism $f : X \to Y$ is dominant if and only if the sheaf map $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is injective. Note that since X is integral, by reducing to the affine cases, the restriction map $\mathcal{O}_X(U) \to \mathcal{O}_X(U')$ is injective. Thus $f^\sharp$ is injective implies that $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is injective for any open $V \subseteq Y$ and open $U \subseteq f^{-1}(V)$. It follows that if f is dominant, the sheaf map $f^\sharp$ *extends* to a field homomorphism $\kappa(\eta_Y) \to \kappa(\eta_X)$, or $\mathcal{K}_Y \to f_*\mathcal{K}_X$.

3.44 Flat morphisms. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces.

(i) A ring homomorphism $\varphi : A \to B$ is **flat** if $B^{[\varphi]}$ a flat A-module.

(ii) f is **flat at** $x \in X$ if the induced stalk map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat.

(iii) f is **flat** if it is flat at every point of X.

3.44.1 Flatness and Exactness of f^* Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a flat morphism between ringed spaces. Then the inverse image functor $f^* : \mathbf{Mod}_{\mathcal{O}_Y} \to \mathbf{Mod}_{\mathcal{O}_X}$ is exact. To see this, let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of $\mathcal{O}_Y$-module. To show $0 \to f^*\mathcal{F} \to f^*\mathcal{G} \to f^*\mathcal{H} \to 0$ is exact, we check this stalkwise (2.16). If $x \in X$, then the induced stalk map is

$$0 \to \mathcal{F}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x} \to \mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x} \to \mathcal{H}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x} \to 0$$

One easily sees that this is the stalk map induced by $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ tensored by $\mathcal{O}_{X,x}$ over $\mathcal{O}_{Y,f(x)}$. Since f is flat, $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,f(x)}$. This proves the exactness.

3.45 In (3.43.2) we see if $f : X \to Y$ is a dominant morphism between integral schemes, then the sheaf map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ extends to a morphism of $\mathcal{K}_Y \to f_*\mathcal{K}_X$. In particular, they fit into a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_Y & \longrightarrow & f_*\mathcal{O}_X \\ \downarrow & & \downarrow \\ \mathcal{K}_Y & \longrightarrow & f_*\mathcal{K}_X. \end{array}$$

It is natural to ask whether such extension can be defined for other types of morphisms. It is the case for any flat morphism $f : X \to Y$ between arbitrary schemes, which we now prove. It suffices to show for any affine open $V = \operatorname{Spec} A$ in Y and affine open $U = \operatorname{Spec} B$ in X with $f(U) \subseteq V$, the corresponding homomorphism $\varphi : A = \mathcal{O}_Y(V) \to \mathcal{O}_X(U) = B$ sends regular elements to regular elements. Let $a \in A_{\mathrm{reg}}$. The multiplication $B \to B$ by $\varphi(a)$ is obtained by tensoring with B the multiplication $A \to A$ by a. Since φ is flat, this shows $\varphi(a) \in B_{\mathrm{reg}}$.

3.46 Definition. Let $(X, \mathcal{O}_X)$ be a ringed space. A global section

$$\Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times) = (\mathcal{K}_X^\times/\mathcal{O}_X^\times)(X)$$

of the quotient sheaf $\mathcal{K}_X^\times/\mathcal{O}_X^\times$ is called a **Cartier divisor** on X. Here for a sheaf of rings $\mathcal{A}$, we denote by $\mathcal{A}^\times$ the sheaf $U \mapsto \mathcal{A}(U)^\times$ of invertible elements.

Unwinding the definition (c.f. (2.4.2)), we see a Cartier divisor is represented by a collection of pairs $\{(f_U, U)\}_{U \in \mathcal{U}}$, where $\mathcal{U}$ is an open cover of X and $f_U \in \mathcal{K}(U)^\times$ such that for any $U, V \in \mathcal{U}$ with $U \cap V \neq \varnothing$, we have $f_U f_V^{-1} \in \mathcal{O}_X(U \cap V)^\times$. Two such collections $\{(f_U, U)\}_{U \in \mathcal{U}}$ and $\{(g_V, V)\}_{V \in \mathcal{V}}$ represent the same Cartier divisor if and only if $f_U g_V^{-1} \in \mathcal{O}_X(U \cap V)$ for any $U \in \mathcal{U}$, $V \in \mathcal{V}$ with $U \cap V \neq \varnothing$.

This gives an alternative way to define a Cartier divisor : it is a maximal collection of pairs $\{(f_U, U)\}_{U \in \mathcal{U}}$, in the sense that if (g, V) with $V \underset{\text{open}}{\subseteq} X$, $g \in \mathcal{K}(V)^\times$ satisfies $f_U g^{-1} \in \mathcal{O}_X(U \cap V)^\times$ for any $U \in \mathcal{U}$ with $U \cap V \neq \varnothing$, then in fact $(g, V) \in \{(f_U, U)\}_{U \in \mathcal{U}}$.

3.46.1 Cartier class group. Taking global sections of the exact sequence

$$0 \longrightarrow \mathcal{O}_X^\times \longrightarrow \mathcal{K}_X^\times \longrightarrow \mathcal{K}_X^\times/\mathcal{O}_X^\times \longrightarrow 0$$

we obtain

$$0 \longrightarrow \mathcal{O}_X(X)^\times \longrightarrow \mathcal{K}_X(X)^\times \longrightarrow (\mathcal{K}_X^\times/\mathcal{O}_X^\times)(X).$$

An element in the image of the last arrow is called a **principal Cartier divisor**. A Cartier divisor is called **effective** if it lies in the image of $\Gamma(X, \mathcal{O}_X \cap \mathcal{K}_X^\times) \to \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$. We write $D \geqslant 0$ if D is an effective Cartier divisor.

There is an obvious abelian group structure on $\Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$, which we will write additively. We say two Cartier divisors D_1, D_2 on X is **linearly equivalent** if $D_1 - D_2$ is principal. Denote by $\mathrm{CaCl}(X)$ the group of **Cartier divisors** on X modulo the linear equivalence relation. Equivalently, it is the quotient group of $\Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$ mod out the group of principal Cartier divisor. The group $\mathrm{CaCl}(X)$ is called the **Cartier class group** of X.

3.46.2 Invertible sheaf associated to a Cartier divisor. Let $(X, \mathcal{O}_X)$ be a ringed space and let D be a Cartier divisor on X. Define the subsheaf $\mathcal{O}_X(D)$ of $\mathcal{K}_X$ as follows. Let $\{(f_U, U)\}_{U \in \mathcal{U}}$ be the maximal collection of compatible pairs that represents D. We take $\mathcal{O}_X(D)|_U = f_U^{-1}\mathcal{O}_X|_U \subseteq \mathcal{K}_X|_U$. This is well-defined, as on $U \cap V$, we have

$$(\mathcal{O}_X(D)|_U)|_{U \cap V} = f_U^{-1}(\mathcal{O}_X|_U)|_{U \cap V} = f_V^{-1}(f_V f_U^{-1})\mathcal{O}_X|_{U \cap V} = f_V^{-1}(\mathcal{O}_X|_V)|_{U \cap V} = (\mathcal{O}_X(D)|_V)|_{U \cap V}.$$

The sheaf $\mathcal{O}_X(D)$ is an invertible sheaf of X. If E be a principal Cartier divisor defined by $g \in \Gamma(X, \mathcal{K}_X^\times)$, then there is an isomorphism $\mathcal{O}_X(D) \cong \mathcal{O}_X(D + E)$ given by multiplication by g^{-1}. Precisely, on any open U of X such that D is represented by f_U, the isomorphism is given by $s \mapsto s(g|_U)^{-1}$. This isomorphism is independent of the choice of g representing E, so this gives a well-defined map

$$\begin{aligned} \mathrm{CaCl}(X) &\longrightarrow \mathrm{Pic}(X) \\ D &\longmapsto \mathcal{O}_X(D). \end{aligned}$$

If A is a ring and $f, g \in A_{\mathrm{reg}}$, then the multiplication gives an isomorphism $fA \otimes_A gA \to fgA$ of A-modules. This shows $D \mapsto \mathcal{O}_X(D)$ is also a group homomorphism (c.f. (3.25)).

3.46.3 Lemma. Let $(X, \mathcal{O}_X)$ be a ringed space. The homomorphism $\mathrm{CaCl}(X) \to \mathrm{Pic}(X)$ is injective. Moreover, the image consists of classes of invertible subsheaves of $\mathcal{K}_X$.

Proof. Let $\{(f_U, U)\}_{U \in \mathcal{U}}$ be the maximal collection of compatible pairs that represents D. Say $\mathcal{O}_X(D) \cong \mathcal{O}_X$ is trivial. By (3.25) there exists $f \in \Gamma(X, \mathcal{O}_X(D))$ such that $f\mathcal{O}_X = \mathcal{O}_X(D)$ in $\mathcal{K}_X$. Then $f\mathcal{O}_X|_U = f_U^{-1}\mathcal{O}_X|_U$ for any $U \in \mathcal{U}$. Hence $f|_U \in \mathcal{K}_X(U)^\times$ for any $U \in \mathcal{U}$, so that $f \in \mathcal{K}_X(X)^\times$. This shows D is principal.

Let $\mathcal{L} \subseteq \mathcal{K}_X$ be an invertible subsheaf, and let $\mathcal{U}$ be an affine open cover such that for each $U \in \mathcal{U}$ (c.f. (3.25)) there exists $f_U \in \mathcal{K}_X(U)$ such that $\mathcal{L}|_U = f_U\mathcal{O}_X|_U \cong \mathcal{O}_X|_U$. Since then f_U is regular in the local ring of points in U, we see $f_U \in \mathcal{K}_X(U)^\times$. Hence $\mathcal{L}$ is the invertible sheaf associated to the Cartier divisor $\{(f_U, U)\}_{U \in \mathcal{U}}$. $\square$

3.46.4 Effective Cartier divisors as closed subspaces. Suppose $(X, \mathcal{O}_X)$ is a local-ringed space, and let D be an effective Cartier divisor on X. Then $\mathcal{O}_X(-D)$ is an invertible ideal sheaf of $\mathcal{O}_X$, so it corresponds to a closed local-ringed subspace

$$j : (D, \mathcal{O}_D) := (V(\mathcal{O}_X(-D)), \mathcal{O}_X/\mathcal{O}_X(-D)|_{V(\mathcal{O}_X(-D))}) \longrightarrow (X, \mathcal{O}_X)$$

and there is a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow j_*\mathcal{O}_D \longrightarrow 0.$$

Upshot : effective Cartier divisors are those closed subspaces locally cut by a single regular function.

3.47 Lemma. For an integral scheme X, the map $\mathrm{CaCl}(X) \to \mathrm{Pic}(X)$ is an isomorphism.

Proof. We must show each invertible sheaf $\mathcal{L}$ on X has the form $\mathcal{O}_X(D)$ for some Cartier divisor D. Let $\mathcal{U}$ be an affine open cover of X such that $\mathcal{L}$ is trivial on each $U \in \mathcal{U}$. For $U \in \mathcal{U}$, we have an isomorphism $\varphi_U : \mathcal{O}_X|_U \cong \mathcal{L}|_U$; for any other $V \in \mathcal{U}$ with $U \cap V \neq \emptyset$, by (3.25) the isomorphism $\varphi_{UV} : \varphi_V^{-1} \circ \varphi_U : \mathcal{O}_X|_{U\cap V} \cong \mathcal{O}_X|_{U\cap V}$ corresponds to an element $\varphi_{UV} \in \mathcal{O}_X(U \cap V)^\times$.

Since X is integral, the function field $\mathcal{K}_X$ is the constant sheaf $\underline{K(X)}_X$. For each open U, we view $\mathcal{O}_X(U)$ as a subring of $K(X)$; if $V \subseteq U$, then $\mathcal{O}_X(U) \subseteq \mathcal{O}_X(V) \subseteq K(X)$ (3.42). Fix an $U_0 \in \mathcal{U}$. We then have a compatible collection $\{(\varphi_{U_0U}, U)\}_{U \in \mathcal{U}}$, which then defines an Cartier divisor D.

Define a map $\mathcal{O}_X(D) \to \mathcal{L}$ as follows. For $U \in \mathcal{U}$, let $\mathcal{O}_X(D)|_U = \varphi_{UU_0}\mathcal{O}_X|_U \to \mathcal{L}|_U$ be the morphism defined by

$$\varphi_{UU_0}\mathcal{O}_X|_U(V) \ni \varphi_{UU_0}x \mapsto \varphi_U(x) \in \mathcal{L}|_U(V).$$

Note that this is an isomorphism. For $U, V \in \mathcal{U}$, we have a commutative diagram

$$\begin{tikzcd}
\varphi_{UU_0}\mathcal{O}_X|_{U\cap V} \arrow[r] \arrow[d, "\varphi_{VU_0}\varphi_{UU_0}^{-1}"'] & \mathcal{L}|_{U\cap V} \arrow[d, "\varphi_V\circ\varphi_U^{-1}"] \\
\varphi_{VU_0}\mathcal{O}_X|_{U\cap V} \arrow[r] & \mathcal{L}|_{U\cap V}.
\end{tikzcd}$$

Vertical arrows are transition maps of $\mathcal{O}_X(D)$ and $\mathcal{L}$ respectively, so they glue to an isomorphism $\mathcal{O}_X(D) \cong \mathcal{L}$. This proves the surjectivity. $\square$

3.1.6 Rational maps

3.48 Definition. Let X be a scheme. For two open dense sets U, V of X, the intersection $U \cap V$ is again open dense. Partially ordered by inclusion, the collection $\mathcal{U}_X$ of dense open sets in X forms a directed set. For Y another scheme, the collection $\{\mathrm{Hom}_{\mathbf{Sch}}(U, Y)\}_{U \in \mathcal{U}_X}$ is directed by the restriction $\mathrm{Hom}_{\mathbf{Sch}}(U, Y) \to \mathrm{Hom}_{\mathbf{Sch}}(V, Y)$ with $V \subseteq U \in \mathcal{U}_X$. An element in the direct limit $\varinjlim_{U \in \mathcal{U}_X} \mathrm{Hom}_{\mathbf{Sch}}(U, Y)$ is called a **rational map**. A rational map is usually written with dashed arrow $f : X \dashrightarrow Y$.

By definition, two morphisms $f : U \to Y$ and $g : V \to Y$ with $U, V \in \mathcal{U}_X$ determine the same rational map if and only if there exists $W \subseteq U \cap V$ with $W \in \mathcal{U}_X$ such that $f|_W = g|_W$. In this case we write $f \sim g$, and use either f or g to denote the corresponding rational map.

3.49 Composition. Let $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ be rational maps. We want to define the composition $g \circ f$, but it is not always possible. For example, we certainly cannot do so if $f : U \to Y$ hits to the subset that g is not defined. Say g is represented by $g : V \to Z$ for some $V \in \mathcal{U}_Y$. What we need is the nonemptyness of the subset $V \cap f(X) \subseteq Y$. To fix this, we need some density constraint.

3.49.1 Lemma. If $f : X \dashrightarrow Y$ is a dominant rational map, then every representative of f is dominant.

Proof. Let $U, V \in \mathcal{U}_X$ and $f : U \to Y$, $g : V \to Y$ with $f \sim g$. Take $W \subseteq U \cap V$ with $W \in \mathcal{U}_X$ such that $f|_W = g|_W$. Since f is continuous, we have $f(\overline{W}) \subseteq \overline{f(W)}$. By density, we have $\overline{W} = U$, and since $f(U)$ is dense, we see $Y = \overline{f(U)} \subseteq \overline{f(W)}$ so that $f(W)$ is dense as well. This shows g is also dominant. $\square$

3.49.2 Composition of dominant rational maps. We continue the discussion in (3.49). Instead of consider the composition of two rational maps, we define the composition of two *dominant rational maps*. Let X, Y, Z be schemes, $U \in \mathcal{U}_X$ and $V \in \mathcal{U}_Y$, and $f : U \to Y$, $g : V \to Z$ be dominant morphisms. Pictorially,

$$\begin{tikzcd}
X \arrow[rr, dashed] & & Y \arrow[rr, dashed] & & Z \\
U \arrow[u, hook] \arrow[urr, bend right, "f"] & & V \arrow[u, hook] \arrow[urr, bend right, "g"] & & \\
U \cap f^{-1}(V) \arrow[u, hook] & & & &
\end{tikzcd}$$

Since $f(U)$ is dense, $V \cap f(U) \subseteq Y$, and hence $U \cap f^{-1}(V) \subseteq X$, is nonempty. We then can well-defined the composition $g \circ f|_{U \cap f^{-1}(V)} : U \cap f^{-1}(V) \to Z$. But here is still an issue : $U \cap f^{-1}(V)$ may not be dense! Nevertheless, if we assume X is irreducible, then everything goes well, i.e., $g \circ f|_{U \cap f^{-1}(V)} : U \cap f^{-1}(V) \to Z$ represents a rational map, which we denote by $g \circ f : X \dashrightarrow Z$. It is direct to see this does not depend on the representatives of f and g.

3.49.3 Definition. By the previous discussion, it makes sense to define the category **IrrSch** of irreducible schemes with morphisms being dominant rational maps.

(i) A **birational map** is an isomorphism in **IrrSch**.

(ii) Two irreducible schemes are **birational** if there is a birational map between them.

Put **IntSch** to be the full subcategory of **IrrSch** consisting of integral schemes.

3.50 Lemma. Two irreducible schemes X, Y are birational if and only if there exist open dense subsets $U \subseteq X$ and $V \subseteq Y$ such that $U \cong V$ as schemes.

Proof. The if part is clear. For the only if part, suppose there are $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow$ such that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$. Say X_1 (resp. Y_1) is the domain of definition of f (resp. g). $\square$

3.2 Functor of points

3.51 Yoneda. For a scheme X, we can associate it with a functor $h_X : \mathbf{Sch} \to \mathbf{Set}$ by $h_X(T) := \mathrm{Hom}_{\mathbf{Sch}}(T, X)$. It is the content of Yoneda's lemma that there is a functorial bijection

$$\begin{aligned} \mathrm{Hom}(h_X, h_Y) &\longrightarrow \mathrm{Hom}_{\mathbf{Sch}}(X, Y) \\ T &\longmapsto T_X(\mathrm{id}_X). \end{aligned}$$

From this we see that $X \cong Y$ if and only if $h_X \cong h_Y$ as functors. This more or less says that the scheme X is uniquely determined by the family $\{\mathrm{Hom}_{\mathbf{Sch}}(T, X)\}_{T \in \mathbf{Sch}}$. In fact, we have more.

3.51.1 Lemma. For a scheme X, let $h_X : \mathbf{Ring} \to \mathbf{Set}$ be the functor defined by $h_X(R) = \mathrm{Hom}_{\mathbf{Sch}}(\mathrm{Spec}\, R, X)$. There is a bijection

$$\mathrm{Hom}_{\mathbf{Sch}}(X, Y) \cong \mathrm{Hom}(h_X, h_Y)$$

natural in $X, Y \in \mathbf{Sch}$. This is an incarnation of the local nature of a scheme.

Proof. Let $F \in \mathrm{Hom}(h_X, h_Y)$. Let $\mathcal{U}$ be an affine open cover of X. For each $U \in \mathcal{U}$, let $\iota_U : U = \mathrm{Spec}\, A_U \to X$ be the inclusion. Then $\iota_U \in h_X(A_U)$, so applying F_U we obtain $f_U = F_U(\iota_U) \in h_Y(A_U)$. We claim the $f_U : U \to Y$ glue to a global morphism $X \to Y$. Let $U, V \in \mathcal{U}$, and let $W = \mathrm{Spec}\, B \subseteq U \cap V$ be an affine open set. Put $\alpha : W = \mathrm{Spec}\, B \to X$ be the inclusion. Applying F to the diagram

$$\begin{tikzcd} & U = \mathrm{Spec}\, A_U \arrow[dr, "\iota_U"] & \\ W = \mathrm{Spec}\, B \arrow[ur, "\text{inclusion}"] \arrow[rr, "\alpha"] \arrow[dr, "\text{inclusion}"'] & & X \\ & V = \mathrm{Spec}\, A_V \arrow[ur, "\iota_V"'] & \end{tikzcd}$$

we obtain

$$\begin{tikzcd} & U = \mathrm{Spec}\, A_U \arrow[dr, "f_U"] & \\ W = \mathrm{Spec}\, B \arrow[ur, "\text{inclusion}"] \arrow[rr, "F_W(\alpha)"] \arrow[dr, "\text{inclusion}"'] & & Y \\ & V = \mathrm{Spec}\, A_V \arrow[ur, "f_V"'] & \end{tikzcd}$$

In particular, this shows $f_U|_W = F_W(\alpha) = f_V|_W$. Hence we obtain a well-defined morphism $f : X \to Y$ extending the f_U's. It is easy to see the resulting f is independent of the choice of $\mathcal{U}$; in fact, we can use all the affine open sets in X in the first place. This defines a map $\mathrm{Hom}(h_X, h_Y) \to \mathrm{Hom}_{\mathbf{Sch}}(X, Y)$. The map the other way around is defined by composition.

We claim they are mutually inverse. One direction is clear. For the other, let $F \in \mathrm{Hom}(h_X, h_Y)$ and let f be the corresponding morphism $X \to Y$. Let R be a ring and $g \in h_X(R)$. Let U be an affine open set in X and pick $W \subseteq \mathrm{Spec}\, R$ an affine open set such that $g(W) \subseteq U$. Then

$$(f \circ g)|_W = F_V(\iota_V) \circ g|_W^V = F_W(\iota_V \circ g|_W^V) = F_W(g|_W) = F_U(g)|_W,$$

where the second equality follows from naturality of F. Hence $f \circ (\cdot) = F$. $\square$

3.52 Definition. Let S be a scheme. A scheme X together with a morphism $X \to S$, called the **structure morphism**, is called a S**-scheme / scheme over** S. A **morphism between** S**-schemes** is a morphism of schemes that commutes with the structure morphisms to S. Such a morphism is called an S-**morphism**. Denote by $\mathbf{Sch}_S$ the category of S-schemes. If $S = \operatorname{Spec} A$, we write $\mathrm{Sch}_S = \mathrm{Sch}_A$, and simply call an S-morphism as an A-morphism.

3.53 Example. The ring $\mathbb{Z}$ is the initial object in **Ring**, so by (3.7) the affine scheme $\operatorname{Spec}\mathbb{Z}$ is the final object in **Sch**. In particular, the categories Sch and $\mathrm{Sch}_{\mathbb{Z}}$ are isomorphic.

3.54 Let A be a ring. Using the forgetful functors $\mathbf{Alg}_A \to \mathbf{Ring}$ and $\mathbf{AffSch}_A \to \mathbf{AffSch}$, we deduce from the equivalence of categories $\operatorname{Spec} : \mathbf{Ring}^{\mathrm{op}} \to \mathbf{AffSch}$ in (3.8) gives rise to the equivalence of categories

$$\begin{aligned} \operatorname{Spec} : \mathbf{Alg}_A^{\mathrm{op}} &\longrightarrow \mathbf{AffSch}_A \\ R &\longmapsto \operatorname{Spec} R \end{aligned}$$

3.55 Let S be a scheme. Let X be an S-scheme and denote by $f : X \to S$ the structure morphism of X. For an open set U of S, we sometimes use $X|_U$ to denote the U-scheme $f|_{f^{-1}(U)}^{U} : f^{-1}(U) \to U$ so as to prevent ourselves from cumbersome notation.

3.56 Let S be a scheme and Y, X be two S-schemes. Denote by $f : Y \to S$ and $g : X \to S$ the structure morphisms. Consider the commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{Sch}}(Y,X) \arrow[r] \arrow[d, "g\circ(-)"] & \mathrm{Hom}_{\mathbf{Ring}}(\mathcal{O}_X(X), \mathcal{O}_Y(Y)) \arrow[d, "(-)\circ\theta_S"] \\ \mathrm{Hom}_{\mathbf{Sch}}(Y,S) \arrow[r] & \mathrm{Hom}_{\mathbf{Ring}}(\mathcal{O}_S(S), \mathcal{O}_Y(Y)) \end{tikzcd}$$

where $\theta : \mathcal{O}_S \to f_*\mathcal{O}_X$ is the sheaf map of f. Taking the preimage of f along the vertical maps we obtain

$$\mathrm{Hom}_{\mathbf{Sch}_S}(Y,X) \longrightarrow \mathrm{Hom}_{\mathbf{Alg}_{\mathcal{O}_S}}(\mathcal{O}_X(X), \mathcal{O}_Y(Y)).$$

By (3.7), this is an isomorphism if S and X are affine.

3.57 Functor of points. Let S be a scheme and let X be an S-scheme. For another S-scheme T, put

$$X(T) := \mathrm{Hom}_{\mathbf{Sch}_S}(T, X)$$

and call its element a T-**valued point** of $X_{/S}$. If $T = \operatorname{Spec} R$ (where R is an S-algebra), we call it an R-**valued point**, and simply write $X(T) =: X(R)$.

3.58 Example. Let $f_1, \ldots, f_m \in \mathbb{Z}[x_1, \ldots, x_n]$ and let $X = \operatorname{Spec}\mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Let R be a ring. Then there is a natural bijection

$$X(R) \xrightarrow{\ \sim\ } \{a = (a_1, \ldots, a_n) \in R^n \mid f_1(a) = \cdots = f_m(a) = 0\}.$$

Indeed, by (3.7) we have

$$\mathrm{Hom}_{\mathbf{Sch}}(\operatorname{Spec} R, X) \cong \mathrm{Hom}_{\mathbf{Ring}}(\mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_m), R)$$

There is a canonical inclusion

$$\mathrm{Hom}_{\mathbf{Ring}}(\mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_m), R) \xrightarrow{\circ p} \mathrm{Hom}_{\mathbf{Ring}}(\mathbb{Z}[x_1, \ldots, x_n], R) \cong R^n$$

where $p : \mathbb{Z}[x_1, \ldots, x_n] \to \mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ is the quotient map. The last isomorphism is the evaluation at $x_1, \ldots, x_n$. The image of the first set in R^n is then the common zeros of $f_1, \ldots, f_m$ in R^n.

Inspired by this bijection, we can understand the R-valued points of a scheme as a generalization of solving equations in the ring R. In this way the functor of points is quite a natural concept.

3.59 Example. Let k be a field and V a finite dimensional vector space over k. Consider the affine scheme $\overline{V} := \operatorname{Spec} \operatorname{Sym} V^\vee$, where $\operatorname{Sym} V^\vee$ means the symmetric algebra of $V^\vee$. By the universal property of symmetric algebras, we have functorial bijections

$$\overline{V}(k) = \operatorname{Hom}_{\mathbf{Sch}_k}(\operatorname{Spec} k, \overline{V}) \cong \operatorname{Hom}_{\mathbf{Alg}_k}(\operatorname{Sym} V^\vee, k) \cong \operatorname{Hom}_k(V^\vee, k) = (V^\vee)^\vee \cong V$$

Hence we can always think of a finite dimensional vector space V over k as the k-valued points of the affine space $\overline{V} \cong \mathbb{A}_k^{\dim_k V}$ (non-canonically) over k.

3.60 Points in local rings. Denote by **LocRing** the category of local rings. By definition, a morphism of local rings is a local homomorphism (2.20). Let A be a local ring and $\mathfrak{m}$ be the unique maximal ideal. We show that there is a bijection

$$\operatorname{Hom}_{\mathbf{Sch}}(\operatorname{Spec} A, X) \longrightarrow \coprod_{x \in X} \operatorname{Hom}_{\mathbf{LocRing}}(\mathcal{O}_{X,x}, A).$$

First, let $f : \operatorname{Spec} A \to X$ be a morphism of scheme and put $x = f(\mathfrak{m})$. Choose an affine open neighborhood $U = \operatorname{Spec} B$ of x in X, and let $\mathfrak{p} \in \operatorname{Spec} B$ be the point corresponding to x. Then $f(\operatorname{Spec} A) \subseteq U$. Indeed, $f^{-1}(U)$ is an open set in $\operatorname{Spec} A$ containing $\mathfrak{m}$, and there exists $h \in A$ such that $f^{-1}(U) \supseteq D(h) \ni \mathfrak{m}$, which implies $h \notin \mathfrak{m}$ and, thus, $D(h) = \operatorname{Spec} A$. In fact, this shows that $f(\operatorname{Spec} A)$ is contained in any open neighborhood of x in X. The map f then factors through $f|^U : \operatorname{Spec} A \to U = \operatorname{Spec} B$, so it gives a map $(f|^U)^\sharp : B \to A$ with $((f|^U)^\sharp)^{-1}(\mathfrak{m}) = \mathfrak{p}$. This means it factors through the localization $B \to B_\mathfrak{p}$, giving a commutative triangle

$$\begin{tikzcd}
B \arrow[rr, "(f|^U)^\sharp"] \arrow[dr] & & A \\
& B_\mathfrak{p} \arrow[ur] &
\end{tikzcd}$$

Since $B_\mathfrak{p} \cong \mathcal{O}_{X,x}$, we have a homomorphism $\theta_f : \mathcal{O}_{X,x} \to A$. Taking spec gives a morphism $\widetilde{\theta_f} : \operatorname{Spec} A \to \operatorname{Spec} \mathcal{O}_{X,x}$. At the same time, we obtain another morphism $\varphi_x : \operatorname{Spec} \mathcal{O}_{X,x} \to \operatorname{Spec} B = U \to X$.

The constructions of $\widetilde{\theta_f}$ and φ_x depend on the choice of U. Nevertheless, the morphisms themselves do not, which we now prove. Let $V = \operatorname{Spec} C$ be another affine open neighborhood of x and let $W = \operatorname{Spec} D \subseteq U \cap V$ be still another affine open neighborhood of x. To distinguish, let $\mathfrak{p}_W$, $\mathfrak{p}_U, \mathfrak{p}_V$ denote the corresponding primes of x in D, B, C. Now we have ring homomorphism $B, C \to D \to A$, and since $\mathfrak{m} \mapsto \mathfrak{p}_W \mapsto \mathfrak{p}_V, \mathfrak{p}_U$, the natural diagram

$$\begin{tikzcd}
& & A & & \\
& & & D_{\mathfrak{p}_W} \arrow[ul] & \\
C_{\mathfrak{p}_V} \arrow[uurr] \arrow[urrr, "\sim"] & & D \arrow[uu] \arrow[ur] & & B_{\mathfrak{p}_U} \arrow[ul, "\sim"'] \arrow[uull, bend right] \\
& C \arrow[ul] \arrow[ur] & & B \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

commutes everywhere. Taking spec gives

$$\begin{tikzcd}
 & & \operatorname{Spec} A \arrow[d] & & \\
 & & \operatorname{Spec} D_{\mathfrak{p}_W} \arrow[dl, "\sim"'] \arrow[d, "\wr"] \arrow[dr, "\sim"] \arrow[dd, bend left=60] & & \\
 & \operatorname{Spec} C_{\mathfrak{p}_V} \arrow[ddl] & \operatorname{Spec} \mathcal{O}_{X,x} \arrow[l, "\sim"] \arrow[r, "\sim"'] & \operatorname{Spec} B_{\mathfrak{p}_U} \arrow[ddr] & \\
 & & \operatorname{Spec} D \arrow[dll] \arrow[drr] \arrow[dd] & & \\
\operatorname{Spec} C \arrow[drr] & & & & \operatorname{Spec} B \arrow[dll] \\
 & & X & &
\end{tikzcd}$$

This tells the independence. Hence $f : \operatorname{Spec} A \to X$ admits a *canonical* factorization

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[rr, "f"] \arrow[dr, "\widetilde{\theta_f}"'] & & X \\
 & \operatorname{Spec} \mathcal{O}_{X,x} \arrow[ur, "\varphi_x"'] &
\end{tikzcd}$$

Recall that $\widetilde{\theta}_f$ corresponds to the homomorphism $\theta_f : \mathcal{O}_{X,x} \to A$.

Finally we can define the maps

$$\begin{array}{ccc}
\operatorname{Hom}_{\mathbf{Sch}}(\operatorname{Spec} A, X) & \longrightarrow & \displaystyle\coprod_{x \in X} \operatorname{Hom}_{\mathbf{LocRing}}(\mathcal{O}_{X,x}, A) \\
f & \longmapsto & \theta_f : \mathcal{O}_{X,x} \to A \text{ with } x = f(\mathfrak{m}) \\
\varphi_x \circ \operatorname{Spec} \theta & \longleftarrow\!\shortmid & \theta : \mathcal{O}_{X,x} \to A
\end{array}$$

By checking on any affine open neighborhood of x, we easily see that these are mutually inverse.

3.60.1 Example - points in a field. Let X be a scheme over a field k. Then there is a bijection

$$\begin{array}{ccc}
X(k) = \operatorname{Hom}_{\mathbf{Sch}_k}(\operatorname{Spec} k, X) & \xrightarrow{\sim} & \{x \in X \mid \kappa(x) = k\} \\
f & \longmapsto & f(\mathrm{pt})
\end{array}$$

Here pt denotes the unique point in $\operatorname{Spec} k$. Note that a scheme over k is the same as a scheme that is a local-ringed space over k (2.20.1). Each $\kappa(x)$ is naturally a k-algebra, so the equality $\kappa(x) = k$ makes sense.

More generally, if K/k is a field extension, then there is a bijection

$$X(K) \xrightarrow{\sim} \coprod_{x \in X} \operatorname{Hom}_{\mathbf{Alg}_k}(\kappa(x), K).$$

This follows from (3.60) : the kernel of a local k-homomorphism $\mathcal{O}_{X,x} \to K$ is precisely its unique maximal ideal, so such a map is the same as a k-algebra homomorphism $\kappa(x) \to K$.

3.61 Galois action. Let X be a scheme over a field k. Let K/k be a Galois extension with Galois group G. If $s : \operatorname{Spec} K \to X$ and $\sigma \in G$, it is clear that $s \circ \operatorname{Spec}\sigma$ is again a morphism of k-scheme, so that $s \circ \operatorname{Spec}\sigma \in X(K)$. This defines a (left) G-action on $X(K)$.

Consider the (set-theoretic) map

$$\begin{aligned} \phi : X(K) &\longrightarrow X \\ s &\longmapsto s(\operatorname{Spec} K). \end{aligned}$$

By (3.60.1), the fibre of each point $x \in X$ is identified as

$$\phi^{-1}(x) \cong \operatorname{Hom}_{\mathbf{Alg}_k}(\kappa(x), K).$$

It follows from the construction this bijection is G-equivariant, where G acts on $\operatorname{Hom}_{\mathbf{Alg}_k}(\kappa(x), K)$ naturally. As K/k is Galois, it follows that the G-action on $\phi^{-1}(x)$ is transitive, and hence ϕ induces an injective map

$$G\backslash X(K) \hookrightarrow X.$$

On the other hand, since the bijection

$$X(K) \xrightarrow{\sim} \coprod_{x\in X} \operatorname{Hom}_{\mathbf{Alg}_k}(\kappa(x), K).$$

is also G-equivariant, taking invariants gives

$$X(K)^G \xrightarrow{\sim} \coprod_{x\in X} \operatorname{Hom}_{\mathbf{Alg}_k}(\kappa(x), K)^G = \coprod_{x\in X} \operatorname{Hom}_{\mathbf{Alg}_k}(\kappa(x), k) \xrightarrow{\sim} X(k).$$

To summarize,

Lemma. Let X be a scheme over a field k and let K/k be a Galois extension with Galois group G.

(i) The natural map $X(K) \to X$ induces an injection $G\backslash X(K) \to X$.

(ii) The injection $X(k) \to X(K)$ gives a bijection $X(k) \cong X(K)^G$.

3.2.1 Zariski sheaves

3.62 Definitions in this subsubsection are from Tag 01JF.

3.63 Definition. A functor $F : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ is called a **Zariski sheaf** if for any scheme X and any open cover $\mathcal{U}$ of X, the sequence

$$F(X) \longrightarrow \prod_{U\in\mathcal{U}} F(U) \rightrightarrows \prod_{U,V\in\mathcal{U}} F(U \cap V)$$

is a equalizer diagram.

3.64 Definition. Let $F : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ be a functor.

(i) A functor $H : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ is called a **subfunctor** of F if $H(X) \subseteq F(X)$ for each scheme X and for each $f \in \operatorname{Hom}_{\mathbf{Sch}}(X, Y)$, the map $F(f) : F(Y) \to F(X)$ restricts to $H(f) : H(Y) \to H(X)$.

(ii) A subfunctor $H \subseteq F$ said to be **represented by open immersions** if for any scheme X and $\xi \in F(X)$, there exists an open subscheme U_ξ of X satisfying the following :

$$\text{a morphism } f \in \operatorname{Hom}_{\mathbf{Sch}}(Y, X) \text{ maps into } U_\xi \text{ if and only if } F(f)(\xi) \in H(Y).$$

(iii) A collection $(H_i)_{i\in I}$ of subfunctors of F is said to **cover** F if for every scheme X and $\xi \in F(X)$, there exists an open cover $\mathcal{U}$ of X such that $F(U \to X)(\xi) \in H_i(U)$ for each $U \in \mathcal{U}$.

3.65 Theorem A functor $F : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ is representable if it is a Zariski sheaf and there exists a collection $(F_i)_{i\in I}$ of subfunctors of F such that

(a) each F_i is representatble,

(b) each $F_i \subseteq F$ is represented by open immersions, and

(c) $(F_i)_{i\in I}$ covers F.

3.3 Fibre products

3.66 Let S be a scheme. Let $f : X \to S$ and $g : Y \to S$ be two schemes over S. A **fibre product** of f and g is a scheme $X \times_S Y$ together with two morphisms $X \times_S Y \to X$ and $X \times_S Y \to Y$ that represents the functor

$$T \mapsto \mathrm{Hom}_{\mathbf{Sch}}(T, X) \times_{\mathrm{Hom}_{\mathbf{Sch}}(T,S)} \mathrm{Hom}_{\mathbf{Sch}}(T, Y).$$

By universal property nonsense, if a fibre product exists, it is unique up to a unique isomorphism.

3.67 Let C be a ring and let A, B be two C-algebras. The tensor product ring $A \otimes_C B$ is the fibre coproduct of $A \leftarrow C \to B$ in the category of rings. This means we have a bijection

$$\mathrm{Hom}_{\mathbf{Ring}}(A \otimes_C B, D) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{Ring}}(A, D) \times_{\mathrm{Hom}_{\mathbf{Ring}}(C,D)} \mathrm{Hom}_{\mathbf{Ring}}(B, D)$$

functorial in $D \in \mathbf{Ring}$. It then follows from (3.7) that $\operatorname{Spec} A \otimes_C B$ is the fibre product of $\operatorname{Spec} A$ and $\operatorname{Spec} B$ over $\operatorname{Spec} C$ in the category of schemes (in fact, also in **LRS**).

3.68 Base change and functor of points. Let S be a scheme and X, Y be two S-schemes. By (3.66), for any S-scheme Z, there is a canonical bijection

$$\mathrm{Hom}_{\mathrm{Sch}_S}(Z, X \times_S Y) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathrm{Sch}_S}(Z, X) \times \mathrm{Hom}_{\mathrm{Sch}_S}(Z, Y)$$

By taking $Z = Y$, we obtain

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{Sch}_S}(Y, X \times_S Y) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathrm{Sch}_S}(Y, X) \times \mathrm{Hom}_{\mathrm{Sch}_S}(Y, Y) \\ & \mathrm{Hom}_{\mathrm{Sch}_S}(Y, X) \times \{\mathrm{id}_Y\}. \arrow[u, hook] \end{tikzcd}$$

Taking preimage of the subset $\mathrm{Hom}_{\mathrm{Sch}_S}(Y, X) \times \{\mathrm{id}_Y\}$, we see this induces a bijection

$$\mathrm{Hom}_{\mathrm{Sch}_Y}(Y, X \times_S Y) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathrm{Sch}_S}(Y, X)$$

3.3.1 Weil restriction

3.69 Restriction of scalars. Let $X \to T \to S$ be morphisms of schemes. Define a contravariant functor $\mathrm{Res}_{T/S} X : \mathbf{Sch}_S \to \mathbf{Set}$ as follows. For any S-scheme Z, set

$$\mathrm{Res}_{T/S} X(Z) = X(T \times_S Z) = \mathrm{Hom}_{\mathbf{Sch}_T}(T \times_S Z, X),$$

and for any morphism $f : Y \to Z$ of S-schemes,

$$\mathrm{Res}_{T/S} X(f) = \mathrm{Hom}_{\mathbf{Sch}_T}(\mathrm{id}_T \times_S f, X) : \mathrm{Res}_{T/S} X(Y) \to \mathrm{Res}_{T/S} X(Z).$$

Here $\mathrm{id}_T \times_S f : T \times_S Y \to T \times_S Z$ denotes the obvious morphism. The functor $\mathrm{res}_{T/S} X$ is called the **Weil restriction (of scalars)**. If $T = \operatorname{Spec} R$ and $S = \operatorname{Spec} A$ are affine, we write $\operatorname{Res}_{R/A} = \operatorname{Res}_{T/S}$ for simplicity.

3.70 Example : Affine spaces Let k'/k be a field extension, finite of degree d. For a k-algebra R, we have

$$\begin{aligned}\operatorname{Res}_{k'/k} \mathbb{A}^n_{k'}(\operatorname{Spec} R) &= \operatorname{Hom}_{\mathbf{Sch}_{k'}}(\operatorname{Spec} k' \times_{\operatorname{Spec} k} \operatorname{Spec} R, \mathbb{A}^n_{k'}) \\ &\cong \operatorname{Hom}_{\mathbf{Alg}_{k'}}(k'[x_1, \dots, x_n], k' \otimes_k R) \cong (k' \otimes_k R)^n \\ &\cong R^{nd} \cong \operatorname{Hom}_{\mathbf{Alg}_k}(k[\{y_{ij}\}_{\substack{i=1,\dots,n\\ j=1,\dots,d}}], R) \cong \operatorname{Hom}_{\mathbf{Sch}_k}(\operatorname{Spec} R, \mathbb{A}^{nd}_k).\end{aligned}$$

Concretely, one can understand the above bijections as follows. Let $a_1, \dots, a_d$ be a k-basis for k' and consider the substitutions

$$x_i = y_{i1} a_1 + \cdots + y_{id} a_d$$

for $i = 1, \dots, n$. Then

$$\operatorname{Res}_{k'/k} \mathbb{A}^n_{k'} = \operatorname{Res}_{k'/k} \operatorname{Spec} k'[x_1, \dots, x_n]$$

is represented by the affine scheme $\operatorname{Spec} k[\{y_{ij}\}_{\substack{i=1,\dots,n\\ j=1,\dots,d}}]$.

3.71 Example : Tori. If A is a ring, put $\mathbb{G}_{m,A} = \operatorname{Spec} A[x,y]/(xy-1) = \operatorname{Spec} A[x, x^{-1}]$. If R is any A-algebra, we have set-theoretic bijections

$$\operatorname{Hom}_{\mathbf{Sch}_A}(\operatorname{Spec} R, \mathbb{G}_{m,A}) \overset{(3.54)}{\cong} \operatorname{Hom}_{\mathbf{Alg}_A}(A[x, x^{-1}], R) \cong R^\times.$$

In other words, the scheme $\mathbb{G}_{m,A}$ represents the functor $R \mapsto R^\times$ that takes a ring to its group of units.

Consider the case $R = \mathbb{C}$ and $A = \mathbb{R}$. For each $\mathbb{R}$-algebra R, we compute

$$\operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}(\operatorname{Spec} R) = \operatorname{Hom}_{\mathbf{Sch}_\mathbb{C}}(\operatorname{Spec} \mathbb{C} \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} R, \mathbb{G}_{m,\mathbb{C}}) \overset{(3.67)}{\cong} (\mathbb{C} \otimes_\mathbb{R} R)^\times \cong \big(R[x]/(x^2+1)\big)^\times.$$

Of course, we use the natural identification $\mathbb{C} \cong \mathbb{R}[x]/(x^2+1)$. To compute the last group, note that for $a, b \in R$, one always has

$$(a + bx)(a - bx) = a^2 - bx^2 \equiv a^2 + b^2 \pmod{x^2+1}.$$

Thus if $a^2+b^2 \in R^\times$, then $a+bx \bmod x^2+1$ $\big(R[x]/(x^2+1)\big)^\times$ with inverse $(a^2+b^2)^{-1}(a-bx)$. Conversely, if $a+bx \bmod x^2+1 \in \big(R[x]/(x^2+1)\big)^\times$ with inverse $c + dx \bmod x^2+1$, then

$$1 \equiv (a+bx)(c+dx) \equiv (ac-bd) + (ad+bc)x \pmod{x^2+1}$$

and hence

$$(a-bx)(c-dx) = ac - (ad+bc)x + bdx^2 \equiv (ac-bd) - (ad+bc)x \equiv 1 \pmod{x^2+1}.$$

so that $a - bx \bmod x^2+1$ $\big(R[x]/(x^2+1)\big)^\times$. In particular, $a^2+b^2 \equiv (a+bx)(a-bx)$ is also a unit. This establishes a first bijection of the following :

$$\big(R[x]/(x^2+1)\big)^\times \cong \{(a,b) \in R^2 \mid a^2+b^2 \in R^\times\} \cong \operatorname{Hom}_{\mathbf{Alg}_\mathbb{R}}(\mathbb{R}[x,y,(x^2+y^2)^{-1}], R).$$

Therefore, the functor $\operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}$ is represented by the affine scheme $\operatorname{Spec} \mathbb{R}[x,y,(x^2+y^2)^{-1}]$. A more intrinsic way to express this scheme is

$$\operatorname{Spec} \mathbb{R}[x,y,(x^2+y^2)^{-1}] = \operatorname{Spec} (\operatorname{Sym} \operatorname{Hom}_{\mathbf{Mod}_\mathbb{R}}(\mathbb{C}, \mathbb{R}))_{N_{\mathbb{C}/\mathbb{R}}}$$

where $N_{\mathbb{C}/\mathbb{R}} : \mathbb{C} \to \mathbb{R}$ is the norm given by $N_{\mathbb{C}/\mathbb{R}}(z) = z\bar{z}$.

3.72 Let A be a ring and R a finite A-algebra which is also a projective A-module. Write $R^\vee := \operatorname{Hom}_{\mathbf{Mod}_A}(R, A)$ for its A-linear dual. We are going to show $\operatorname{Res}_{R/A} \operatorname{Spec} B$ is represented by an A-scheme for each R-algebra B by an explicit construction.

3.3.2 Separated morphisms

3.73 Graph. Let $f : X \to Y$ be a morphism of S-schemes. By the universal property (3.66), the morphisms $X \xleftarrow{\mathrm{id}_X} X \xrightarrow{f} Y$ defines a morphism $\Gamma_f : X \to X \times_S Y$ fitting into the commutative diagram

$$\begin{tikzcd}
X \arrow[dr, "\Gamma_f"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\
& X \times_S Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & Y \arrow[d] \\
& X \arrow[r] & Y
\end{tikzcd}$$

The morphism $\Gamma_f : X \to X \times_S Y$ is called the **graph of** f.

3.73.1 Lemma. The diagonal $\Gamma_f : X \to X \times_S Y$ is an immersion.

Proof. Let $\mathrm{pr}_1 : X \times_S Y \to X$ and $\mathrm{pr}_2 : X \times_S Y \to Y$ be the canonical projection. By Proposition 2.26.3, it suffices to show the restriction $\Gamma_f^{-1}(\mathrm{pr}_1^{-1}(U) \cap \mathrm{pr}_2^{-1}(V)) = U \cap f^{-1}(V) \to \mathrm{pr}_1^{-1}(U) \cap \mathrm{pr}_2^{-1}(V)$ is a closed immersion for any affine opens $U \subseteq X$ and $V \subseteq Y$, lying over some affine open $T \subseteq S$; by further shrinking, we may assume $U \subseteq f^{-1}(V)$. Let $U = \operatorname{Spec} A$, $V = \operatorname{Spec} B$, $T = \operatorname{Spec} C$; then $\mathrm{pr}_1^{-1}(U) \cap \mathrm{pr}_2^{-1}(V)$ together with the projections to U and V, respectively, represents the fibre product of U and V over T, so that it is isomorphic to $\operatorname{Spec} A \otimes_C B$. The restriction $f|^{\mathrm{pr}_1^{-1}(U) \cap \mathrm{pr}_2^{-1}(V)}_{U \cap f^{-1}(V)} : U \cap f^{-1}(V) \to \mathrm{pr}_1^{-1}(U) \cap \mathrm{pr}_2^{-1}(V)$ now takes the form $\operatorname{Spec} A \to \operatorname{Spec} A \otimes_C B$, the morphism corresponding to the C-algebra map $A \otimes_C B \to A$ defined by $a \otimes b \mapsto a\varphi(b)$, where $\varphi : B \to A$ corresponds to $f|^V_U : U \to V$. The algebra map is surjective, so $\operatorname{Spec} A \to \operatorname{Spec} A \otimes_C B$ is a closed embedding. This finishes the proof. $\square$

3.73.2 Diagonal. Let $X \to S$ be an S-scheme. The above construction applied to the identity morphism $\mathrm{id}_X : X \to X$ yields a morphism $\Delta_{X/S} : X \to X \times_S X$. This is called the **diagonal morphism** of X/S.

3.74 Definition. Let $f : X \to Y$ be a morphism.

(i) f is called **quasi-compact** if for all open compact subsets $U \subseteq Y$, the preimage $f^{-1}(U)$ is compact.

(ii) f is called **separated** (resp. **quasi-separated**) if the diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is a closed immersion (resp. quasi-compact).

Finally, a scheme X is called **separated** (resp. **quasi-separated**) if the natural morphism $X \to \operatorname{Spec} \mathbb{Z}$ is separated (resp. quasi-separated).

3.75 Base change and morphisms. Let P be a property about morphisms of schemes that holds for all isomorphisms (in **Sch**). Consider the following statements.

(i) P holds for all closed immersions.

(ii) For $X \xrightarrow{j} Y \xrightarrow{g} Z$ with j an immersion, if g verifies P, then $g \circ j$ verifies P.

(iii) For $X \xrightarrow{j} Y \xrightarrow{g} Z$ with j a closed immersion, if g verifies P, then $g \circ j$ verifies P.

(iv) If $f : X \to Y$ verifies P, then $f_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$ verifies P.

(v) Local on the base
If $f : X \to Y$ verifies P, then for every affine open $V \subseteq Y$, the morphism $f|^V_{f^{-1}(V)} : f^{-1}(V) \to V$ verifies P.

(vi) If $f \in \mathrm{Hom}_{\mathbf{Sch}_S}(X, X')$ and $g \in \mathrm{Hom}_{\mathbf{Sch}_S}(Y, Y')$ verify P, then $(f, g) : X \times_S Y \to X' \times_S Y'$ verifies P.

(vii) Stable under base change
If $f : X \to Y$ verifies P and $Y' \to Y$ is a morphism, then the base change $X \times_Y Y' \to Y$ verifies P.

(viii) Stable under fibre product
If $X \to S$ and $Y \to S$ verify P, then $X \times_S Y \to S$ verifies P.

(ix) For $X \xrightarrow{f} Y \xrightarrow{g} Z$, if $g \circ f$ verifies P, then f verifies P.

(x) For $X \xrightarrow{f} Y \xrightarrow{g} Z$ with g separated, if $g \circ f$ verifies P, then f verifies P.

(xi) Stable under composition
For $X \xrightarrow{f} Y \xrightarrow{g} Z$, if f, g verify P, then $g \circ f$ verifies P.

3.3.3 Scheme-theoretic fibre

3.76 In (3.11) we see that if A is a ring and S is a submonoid of A, the natural map $f : \operatorname{Spec} S^{-1}A \to \operatorname{Spec} A$ is a homeomorphism onto its image. In fact, this is also a topological embedding (2.22). To start with, put

$$Y = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \cap S = \varnothing\}$$

to be the image. By the universal property (2.22), we then have a commutative triangle

$$\begin{tikzcd}
(\operatorname{Spec} S^{-1}A, \mathcal{O}_{\operatorname{Spec} S^{-1}A}) \arrow[rr, "f"] \arrow[dr, "f|^Y"'] & & (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \\
& (Y, \mathcal{O}_{\operatorname{Spec} A}|_Y) \arrow[ur] &
\end{tikzcd}$$

We need to show the morphism on the left is an isomorphism in **LRS**, and it remains to show the sheaf map

$$\mathcal{O}_{\operatorname{Spec} A}|_Y \to (f|^Y)_* \mathcal{O}_{\operatorname{Spec} S^{-1}A}$$

is an isomorphism. This is clear.

3.77 Let us talk a little more about the morphism $\operatorname{Spec} \mathcal{O}_{X,x} \to X$ constructed in (3.60). Pick an affine open subset $U = \operatorname{Spec} A$ that covers the image of $\operatorname{Spec} \mathcal{O}_{X,x}$. Then the morphism $\operatorname{Spec} \mathcal{O}_{X,x} \to \operatorname{Spec} A$ corresponds to the ring homomorphism $A \to A_\mathfrak{p} \cong \mathcal{O}_{X,x}$, where $\mathfrak{p}$ is the prime corresponding to the point x. By (3.76), the morphism $\operatorname{Spec} \mathcal{O}_{X,x} \to \operatorname{Spec} A$ is then a topological embedding in **LRS**. Hence the morphism $\operatorname{Spec} \mathcal{O}_{X,x} \to X$ identifies $\operatorname{Spec} \mathcal{O}_{X,x}$ with $(Y, \mathcal{O}_X|_Y)$, where

$$Y = \{x' \in X \mid x \in \overline{\{x'\}}\}$$

3.78 Definition. Let $(A, \mathfrak{m})$ be a local ring. An ideal $I \trianglelefteq A$ is called a **defining ideal** if there exists $n \in \mathbb{Z}_{\geqslant 1}$ such that $\mathfrak{m}^n \subseteq I \subseteq \mathfrak{m}$.

In other words, I is an defining ideal if and only if the I-adic topology on A is the same as the $\mathfrak{m}$-adic topology on A.

3.79 Let $f : X \to S$ be a morphism of schemes, $s \in S$, and I a defining ideal of $\mathcal{O}_{S,s}$. Consider the fibre squares :

$$\begin{tikzcd}
X \arrow[d, "f"] & X \times_S \operatorname{Spec} \mathcal{O}_{S,s} \arrow[l] \arrow[d] & X \times_S \operatorname{Spec} \mathcal{O}_{S,s}/I \arrow[l] \arrow[d] \\
S & \operatorname{Spec} \mathcal{O}_{S,s} \arrow[l] & \operatorname{Spec} \mathcal{O}_{S,s}/I \arrow[l]
\end{tikzcd}$$

By (2.22.1) and (3.77) we see that $X \times_S \operatorname{Spec} \mathcal{O}_{S,s}$ is identified with $(Y, \mathcal{O}_X|_Y)$, where

$$Y = f^{-1}\left(\{s' \in S \mid s \in \overline{\{s'\}}\}\right).$$

Note that $\operatorname{Spec} \mathcal{O}_{S,s}/I$ is the singleton $\{\mathfrak{m}/I\}$, for if $\mathfrak{p}$ is a prime containing I, then since I is an defining ideal, taking radicals gives $\mathfrak{m} \subseteq \mathfrak{p} \subseteq \mathfrak{m}$. This implies that $X \times_S \operatorname{Spec} \mathcal{O}_{S,s}/I \to X$ is a homeomorphism onto its image which is $f^{-1}(s)$, the *set-theoretic fibre*.

3.80 Definition. For a morphism $f : X \to S$ of schemes and $s \in S$, we call the fibre product

$$X \times_S \operatorname{Spec} \kappa(s)$$

the **scheme-theoretic fibre** of f at s. In (3.79), we see that as a topological space it is homeomorphic to the set-theoretic fibre $f^{-1}(s)$.

From now on we regard $f^{-1}(s)$ as a scheme by identifying $f^{-1}(s)$ with $X \times_S \operatorname{Spec} \kappa(s)$. In this way there is always a fibre square

$$\begin{tikzcd}
X \arrow[d, "f"'] & f^{-1}(s) \arrow[l] \arrow[d] \\
S & \operatorname{Spec} \kappa(s) \arrow[l]
\end{tikzcd}$$

3.81 Underlying space of a fibre product. We can describe the underlying set of $X \times_S Y$ as follows. Let $s \in S$ and let $x \in X$, $y \in Y$ lie over s. Then we have fibre squares

$$\begin{tikzcd}
\operatorname{Spec} \kappa(x) \arrow[d] & \operatorname{Spec} \kappa(x) \times_S Y \arrow[l] \arrow[d] & \operatorname{Spec}(\kappa(x) \otimes_{\kappa(s)} \kappa(y)) \arrow[l] \arrow[d] \\
X \arrow[d] & X \times_S Y \arrow[l, "\pi_X"'] \arrow[d, "\pi_Y"] & X \times_S \operatorname{Spec} \kappa(y) \arrow[l] \arrow[d] \\
S & Y \arrow[l] & \operatorname{Spec} \kappa(y). \arrow[l]
\end{tikzcd}$$

For a scheme T, denote by $\underline{T}$ the underlying topological space. The universal property of $\underline{X} \times_{\underline{S}} \underline{Y}$, the fibre product in **Top**, gives a continuous map

$$\underline{X \times_S Y} \to \underline{X} \times_{\underline{S}} \underline{Y}.$$

The above fibre squares read that each fibre at $(s, x, y) \in \underline{X} \times_{\underline{S}} \underline{Y}$ is homeomorphic to $\operatorname{Spec}(\kappa(x) \otimes_{\kappa(s)} \kappa(y))$.

3.4 Dimension

3.82 Krull dimension - topology. Let X be a topological space. For us a **chain of irreducible closed subsets** of X is a sequence

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \subseteq X$$

of strictly increasing irreducible closed subsets $Z_0, \ldots, Z_n$ of X, and we call the integer n the **length** of the chain. The **(Krull) dimension** of X is defined as

$$\dim X := \sup\{n \mid Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \subseteq X \text{ is a chain of irreducible closed subsets of } X\} \leqslant \infty$$

if X is nonempty, and $\dim \varnothing := -\infty$ by convention.

3.83 Example. Consider the euclidean space $\mathbb{R}^n$. If $n = 0$, then $\mathbb{R}^0$ is a singleton, so $\dim \mathbb{R}^0 = 0$. For $n \geqslant 1$, we claim that a closed subset $C \subseteq \mathbb{R}^n$ is irreducible if and only if $\#C = 1$. The if part is clear. For the other way around, suppose $x \neq y \in C$. Let $r > 0$ such that $|x - y| > 2r$. Then the open subspace $B_r(x) \cap C$ has closure contained in $\overline{B_r(x)}$, while $|x - y| > 2r \geqslant r$. This shows $B_r(x) \cap C$ is not dense in C. Hence we conclude $\dim \mathbb{R}^n = 0$ for all $n \in \mathbb{Z}_{\geqslant 0}$.

3.84 Example. Let A be a ring. Any irreducible closed subset of $\operatorname{Spec} A$ has the form $V(\mathfrak{p})$ for some prime ideal $\mathfrak{p} \in \operatorname{Spec} A$. Indeed, if $V(\mathfrak{p}) = V(I) \cup V(J) = V(IJ)$ for some ideals I, J of A, then $IJ \subseteq \sqrt{\mathfrak{p}} = \mathfrak{p}$. Since $\mathfrak{p}$ is prime, this shows either $I \subseteq \mathfrak{p}$ or $J \subseteq \mathfrak{p}$, so that $V(\mathfrak{p}) \subseteq V(I)$ or $V(\mathfrak{p}) \subseteq V(J)$. On the other hand, if $V(I)$ is irreducible, then $\operatorname{Spec} A/I \cong V(I)$ is irreducible. By Lemma 3.39, we see $\sqrt{I}$ is a prime in A. But $V(I) = V(\sqrt{I})$.

Hence, any chain of irreducible closed subsets in $\operatorname{Spec} A$ has the form

$$V(\mathfrak{p}_0) \subsetneq V(\mathfrak{p}_1) \subsetneq \cdots \subsetneq V(\mathfrak{p}_n) \subseteq \operatorname{Spec} A.$$

which in turn gives an strictly decreasing sequence

$$\mathfrak{p}_n \subsetneq \cdots \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_0 \subsetneq A$$

of prime ideals in A.

3.85 Krull dimension - ring. Let A be a ring. The **(Krull) dimension** $\dim A$ of A is defined the dimension of the topological space $\operatorname{Spec} A$. In other words,

$$\dim A := \sup\{n \mid \mathfrak{p}_n \subsetneq \cdots \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_0,\ \mathfrak{p}_i \in \operatorname{Spec} A\}$$

For a prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, the **height** is defined as

$$\operatorname{ht} \mathfrak{p} := \dim A_{\mathfrak{p}} = \sup\{n \mid \mathfrak{p}_n \subsetneq \cdots \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p},\ \mathfrak{p}_i \in \operatorname{Spec} A\}.$$

With this definition, we see that

$$\dim A = \sup_{\mathfrak{p} \in \operatorname{Spec} A} \operatorname{ht} \mathfrak{p} = \sup_{\mathfrak{m} \in \operatorname{mSpec} A} A_{\mathfrak{m}}.$$

3.86 Example. A field has dimension 0. Any PID which is not a field has dimension 1, as any nonzero prime ideal is maximal. In number theory, one usually study the arithmetic of a **Dedekind domain**, which is by definition a Noetherian integrally closed domain of dimension 1. For example, the ring of integer $\mathcal{O}_F$ of a number field F, i.e. the integral closure of $\mathbb{Z}$ in a finite extension F of $\mathbb{Q}$, is a Dedekind domain. That $\dim \mathcal{O}_F = 1$ can be seen from the following lemma.

3.87 Going-up. Let A, B be rings and $\varphi : A \to B$ be an **integral homomorphism**, i.e., every element in B is integral over the subring $\varphi(A) \subseteq B$. Then

(i) For $\mathfrak{q} \in \operatorname{Spec} B$, we have $\operatorname{ht} \mathfrak{q} \leqslant \operatorname{ht} \varphi^{-1}(\mathfrak{q})$. In particular, this shows $\dim B \leqslant \dim A$.

(ii) Suppose in addition that φ is injective. Then $\operatorname{Spec} \varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ is surjective and $\dim A = \dim B$.

Proof. For (i) it suffices to show if $\mathfrak{q}_2 \subsetneq \mathfrak{q}_1 \in \operatorname{Spec} B$, then $\varphi^{-1}(\mathfrak{q}_2) \subsetneq \varphi^{-1}(\mathfrak{q}_1) \in \operatorname{Spec} A$. By replacing $\varphi : A \to B$ by the induced map $\varphi : A/\varphi^{-1}(\mathfrak{q}_2) \to B/\mathfrak{q}_2$, we may assume A, B are integral domains with $A \subseteq B$ as subrings, and we only need to show if $\mathfrak{q} \neq 0$, then $\mathfrak{q} \cap A \neq 0$.

Let $0 \neq b \in \mathfrak{q}$. Since B is integral over A, we can find a monic $f \in A[x]$ with $f(0) \neq 0$ such that $f(b) = 0$. But then $f(0) \in A \cap \mathfrak{q}$.

Next we show the first assertion of (ii). We begin with another lemma. ☐

3.87.1 Lying-over. Let $A \subseteq B$ be integral domains with B integral over A. Then A is a field if and only if B is a field.

Proof. By (3.87).(i), we have $\dim B \leqslant \dim A$, so if A is field, then $\dim B = 0$, i.e., the zero ideal is the only prime ideal, so that B is a field. Conversely, suppose B is a field. Let $0 \neq a \in A$ and let $b \in B$ be its inverse in B. Taking any monic $f = x^n + \sum_{i=0}^{n-1} a_i x^i \in A[x]$ with $a_0 \neq 0$ and $f(b) = 0$. Define $g(x) = x^n f(1/x)$; then $g(a) = a^n f(b) = 0$ so that

$$0 = g(a) = a(a_0 a^{n-1} + \cdots + a_{n-1}) + 1.$$

Hence $b = -(a_0 a^{n-1} + \cdots + a_{n-1}) \in A$. ☐

3.87.2 Resume the proof of (3.87).(ii). By the above lemma we see $\operatorname{Spec} \varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ maps closed points to closed points, and is surjective on closed point. Now let $\mathfrak{p} \in \operatorname{Spec} A$, and consider the induced map $A_\mathfrak{p} \to B_\mathfrak{p} := B \otimes_A A_\mathfrak{p}$; pictorially, we have

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A_\mathfrak{p} \arrow[r] & B_\mathfrak{p} \end{tikzcd}$$

By clearing the denominators, the bottom-horizontal arrow is again an integral homomorphism. Since $\mathfrak{p}$ is maximal in $A_\mathfrak{p}$, by (3.87.1) we can find a maximal ideal $\mathfrak{q}$ in $B_\mathfrak{p}$ lying over $\mathfrak{p}$. It is then easy to see $\mathfrak{q} \cap B \in \operatorname{Spec} B$ lies over $\mathfrak{p} \in \operatorname{Spec} A$. This proves the surjectivity.

For the last assertion, in view of (i), it remains to show $\dim A \leqslant \dim B$. Let $\mathfrak{p}_2 \subset\neq \mathfrak{p}_1$ be prime ideals in A. By surjectivity pick any $\mathfrak{q}_2 \in \operatorname{Spec} B$ lying over $\mathfrak{p}_2$. By surjectivity again, but this time applied to the map $A/\mathfrak{p}_2 \to B/\mathfrak{q}_2$, we can find a prime ideal $\mathfrak{q}_1 \in \operatorname{Spec} B$ lying over $\mathfrak{p}_1$ with $\mathfrak{q}_2 \subsetneq \mathfrak{q}_1$. This shows $\dim A \leqslant \dim B$.

3.88 Remark. A ring homomorphism $A \to B$ satisfies the **going up property** if for any prime ideals $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2$ in A and $\mathfrak{q}_1 \in \operatorname{Spec} B$ with $\mathfrak{q}_1 \cap A = \mathfrak{p}_1$, we can find $\mathfrak{q}_2 \in \operatorname{Spec} B$ with $\mathfrak{q}_2 \cap A = \mathfrak{p}_2$ and $\mathfrak{q}_1 \subsetneq \mathfrak{q}_2$. We say $A \subseteq B$ satisfies the **incomparability property** when for any prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, if $\mathfrak{q}, \mathfrak{q} \in \operatorname{Spec} B$ are two prime ideals lying over $\mathfrak{p}$, then $\mathfrak{q} \nsubseteq \mathfrak{q}'$ and $\mathfrak{q}' \nsubseteq \mathfrak{q}$.

In the proof of (3.87), we actually show that if $A \subseteq B$ is an integral extension, then it satisfies the going up property and incomparability property.

3.88.1 Lemma. Let $\varphi : A \to B$ be a ring homomorphism (not necessarily injective) that satisfies the going up property. Then $\operatorname{Spec} \varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ is a closed map.

Proof. Let $I \trianglelefteq B$ be an ideal. It suffices to show $\varphi^{-1}(V(I)) = V(\varphi^{-1}(I))$. The containment $\subseteq$ is obvious. For the other way around, we must show if $\varphi^{-1}(I) \subseteq \mathfrak{p} \subseteq A$, then we can find $I \subseteq \mathfrak{q} \subseteq B$ with $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$. It suffices to apply the going-up property to the extension $A/\varphi^{-1}(I) \to B/I$.

☐

3.89 Definition. Let X be a topological space and Z an irreducible closed subset. The **codimension of** Z **in** X is

$$\operatorname{codim}_X Z := \sup\{n \mid Z \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \subseteq X,\ Z_i \text{ is irreducible closed in } X\}$$

For a closed subspace $Y \subseteq X$, define the **codimension of** Y **in** X as

$$\operatorname{codim}_X Y := \inf \{\operatorname{codim}_X Z \mid Z \subseteq Y : \text{irreducible component of } Y\}.$$

3.89.1 If $X = \operatorname{Spec} A$ and $Z = V(\mathfrak{p})$, then by definition

$$\operatorname{codim}_X Z = \operatorname{ht} \mathfrak{p} = \dim A_{\mathfrak{p}}.$$

If $Y = V(I)$ for some ideal $I \trianglelefteq A$, then

$$\operatorname{codim}_X Y = \inf\{\operatorname{ht} \mathfrak{p} \mid I \subseteq \mathfrak{p} \in \operatorname{Spec} A\}.$$

3.89.2 Lemma. Let X be a topological space and Z an irreducible closed subset. If $U \subseteq X$ is an open set such that $U \cap Z \neq \varnothing$, then

$$\operatorname{codim}_X Z = \operatorname{codim}_U U \cap Z.$$

3.90 Lemma. Let X be a topological space and Y a closed subspace of X. Then

$$\dim Y + \operatorname{codim}_X Y \leqslant \dim X.$$

Proof. Let Z be an irreducible component of Y. Clearly from the definition, we have $\dim Z + \operatorname{codim}_X Z \leqslant \dim X$, so $\dim Z + \operatorname{codim}_X Y \leqslant \dim X$. Since every chain of irreducible closed subsets of Y is contained in an irreducible component of Y, varying Z gives $\dim Y + \operatorname{codim}_X Y \leqslant \dim X$. □

3.91 Definition. A topological space is called **catenary** if

1. $\operatorname{codim}_Y Z < \infty$ for all irreducible closed subsets $Z \subseteq Y$, and
2. for every triple of irreducible closed subsets $Z'' \subseteq Z' \subseteq Z$, we have

$$\operatorname{codim}_Z Z'' = \operatorname{codim}_Z Z' + \operatorname{codim}_{Z'} Z''.$$

3.91.1 Lemma. A topological space is catenary if and only if for any pair of irreducible closed subsets $Z \subseteq Y$, we have $\operatorname{codim}_Y Z < \infty$ and every maximal chain $Z = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n = Y$ has the same length.

Proof.

□

3.91.2 Lemma. Let X be a topological space.

1. If X is catenary, then any locally closed subset of X is catenary.
2. X is catenary if and only if X has an open cover consisting of catenary spaces.

3.92 Codimension and local ring Let X be a scheme and let Z be an irreducible closed subset with generic point z (3.36). Let $Z' \supseteq Z$ be any irreducible closed subset of X, with generic point, say z'. Let $U = \operatorname{Spec} A$ be any affine open neighborhood of z. In (3.36) we saw that $z' \in U$. Let $\mathfrak{p}$ (resp. $\mathfrak{p}'$) be the corresponding prime ideal of z (resp. z') in A. Then $z \in \overline{\{z'\}}$ implies

$\mathfrak{p} \in V(\mathfrak{p}')$, or $\mathfrak{p}' \subseteq \mathfrak{p}$, meaning that z' actually corresponds to a prime ideal $\mathfrak{p}'$ in the local ring $\mathcal{O}_{X,z}$. Since z' lies in any affine open neighborhood of z, the prime ideal $\mathfrak{p}$ is independent of the choice of U. This establishes an inclusion reversing map

$$\{\text{irreducible closed subsets } Z' \supseteq Z\} \longrightarrow \operatorname{Spec} \mathcal{O}_{X,z}$$

Conversely, if $\mathfrak{p}' \in \operatorname{Spec} \mathcal{O}_{X,z}$, take any affine open $U = \operatorname{Spec} A$ containing z and identify p' as a prime ideal of A contained in z. The closure $Z' := \overline{\{\mathfrak{p}'\}}$ is then an irreducible closed subset containing $Z = \overline{\{z\}}$. The subset Z' is independent of the choice of U, as the affine opens form a basis for the topology of X. This association $\mathfrak{p}' \mapsto Z'$ is clearly inverse to the the above map, so that it is an inclusion reversing bijection. In particular, we see

$$\operatorname{codim}_X Z = \dim \mathcal{O}_{X,z}.$$

3.93 Dimension and open cover. Let X be a topological space and $\mathcal{U}$ be an open cover. If $U \in \mathcal{U}$ and $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$ is a chain of irreducible closed in U, then taking closure in X gives a chain of irreducible closed

$$\overline{Z_0} \subsetneq \overline{Z_1} \subsetneq \cdots \subsetneq \overline{Z_n}$$

of X. Indeed, we must have $\overline{Z_i} \neq \overline{Z_j}$ for otherwise $Z_i = \overline{Z_i} \cap U = \overline{Z_j} \cap U = Z_j$, a contradiction. In particular, this shows $\dim U \leqslant \dim X$. Conversely, if $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$ is a chain of irreducible closed in X, let $U \in \mathcal{U}$ be such that $U \cap Z_0 \neq \varnothing$; then

$$Z_0 \cap U \subsetneq Z_1 \cap U \subsetneq \cdots \subsetneq Z_n \cap U$$

is a chain of irreducible closed; if $Z_i \cap U = Z_j \cap U$, then $Z_i = \overline{Z_i \cap U} = \overline{Z_j \cap U} = Z_j$, a contradiction. From this we conclude that

$$\dim X = \sup_{U \in \mathcal{U}} \dim U.$$

In fact, our argument also shows that $\dim Y \leqslant \dim X$ whenever Y is a subset of X, equipped with subspace topology.

3.94 Lemma. Let X be a topological space and Y a subspace of X. Then

$$\dim Y + \operatorname{codim}_X Y \leqslant \dim X.$$

3.95 Lemma. Let X be a scheme, and let U, V be two affine opens in X. Then there exists $f \in \mathcal{O}_X(U)$ and $g \in \mathcal{O}_X(V)$ such that $U_f = V_g$.

Proof. Take $\alpha \in \mathcal{O}_X(U)$ such that $U_\alpha \subseteq U \cap V$. Take $\in \mathcal{O}_X(V)$ such that $V_g \subseteq U_\alpha$. Then

$$V_g = V_g \cap U_\alpha = (V \cap U_\alpha)_{\beta|_{V \cap U_\alpha}} = (U_\alpha)_{\beta|_{U_\alpha}}$$

Take $f' \in \mathcal{O}_X(U_\alpha) = \mathcal{O}_X(U)_\alpha$ such that $f' = \beta|_{U_\alpha}$, and choose $N \gg 0$ such that $f := f'\alpha^N \in \mathcal{O}_X(U)$. Then

$$(U_\alpha)_{\beta|_{U_\alpha}} = (U_\alpha)_{f'} = (U_\alpha)_f = U_f$$

□

3.4.1 Artinian rings

3.96 Length. Let A be a ring and M an A-module. A **finite descending chain** of M has the form

$$0 = M_n \subsetneq M_1 \subsetneq \cdots \subsetneq M_n \subsetneq M_0 = M$$

We say such a filtration has **length** n. We define the **length of the** A**-module** M as

$$\operatorname{length} M = \operatorname{length}_A M = \sup \{\text{length of all descending chain of } M\} \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}.$$

We say M is a **simple** A**-module** if $M \neq 0$ and the only A-submodules of M are 0 and itself; in other words, $\operatorname{length}_A M = 1$. A **composition series** of M is a descending chain of M

$$0 = M_n \subsetneq M_{n-1} \subsetneq \cdots \subsetneq M_1 \subsetneq M_0 = M$$

such that each consecutive subquotient M_n/M_{n+1} is simple.

3.96.1 Lemma.

(i) If N is an A-submodule of M, then $\operatorname{length}_A N + \operatorname{length}_A M/N = \operatorname{length}_A M$.

(ii) If M has a composition series, then $\operatorname{length}_A M < \infty$.

(iii) If $\operatorname{length}_A M < \infty$, then every composition series has the same length.

(iv) If $\operatorname{length}_A M < \infty$, every finite descending chain of M can be refined into a composition series.

Proof.

(i) If $\operatorname{length}_A N$ or $\operatorname{length}_A M/N$ is infinite, then clearly so is $\operatorname{length}_A M$, and the equality holds trivially. Suppose now both $\operatorname{length}_A N$ and $\operatorname{length}_A M/N$ are finite. Clearly $\operatorname{length}_A N + \operatorname{length}_A M/N \leqslant \operatorname{length}_A M$. Conversely, let

$$0 = M_n \subsetneq M_{n-1} \subsetneq \cdots \subsetneq M_1 \subsetneq M_0 = M$$

be a finite descending chain of M. Intersecting each term with N gives

$$0 = M_n \cap N \subseteq M_{n-1} \cap N \subseteq \cdots \subseteq M_1 \cap N \subseteq M_0 \cap N = N$$

Adding N to each term gives

$$N = M_n + N \subseteq M_{n-1} + N \subseteq \cdots \subseteq M_1 + N \subseteq M_0 + N = M$$

To show $n \leqslant \operatorname{length}_A N + \operatorname{length}_A M/N$, it suffices to show that

$$M_{i+1} \cap N = M_i \cap N, \qquad M_{i+1} + N = M_i + N$$

cannot happen simultaneously. Assume these both happen to hold at the same time. Take $x \in M_i$; then there exists $y \in M_{i+1}$ and $n \in N$ such that $y = x + n$, or $y - x = n \in M_i \cap N = M_{i+1} \cap N$. This implies $x \in M_{i+1}$, and hence $M_i = M_{i+1}$, which is absurd.

(ii) Let

$$0 = M_n \subsetneq M_{n-1} \subsetneq \cdots \subsetneq M_1 \subsetneq M_0 = M$$

be a composition series of M. By (i) we have

$$\operatorname{length}_A M = \sum_{i=0}^{n-1} \operatorname{length}_A M_i/M_{i+1} = \sum_{i=0}^{n-1} 1 = n < \infty.$$

(iii) Say $0 = M_n \subsetneq M_1 \subsetneq \cdots \subsetneq M_n \subsetneq M_0 = M$ is a composition series. By (i)

$$\operatorname{length}_A M = \sum_{i=0}^{n-1} \operatorname{length}_A M_i/M_{i+1} = \sum_{i=0}^{n-1} 1 = n.$$

(iv) If $N \subseteq N' \subseteq M$ are A-submodules, by (i) $\operatorname{length}_A N'/N < \infty$, so there is a sequence $N \subsetneq N_1 \subsetneq \cdots \subsetneq N_n = N'$ with each consecutive subquotient simple. Now given a finite descending chain, if we refine each consecutive subquotient in this way, we get a composition series.

$\square$

3.96.2 Corollary. $\operatorname{length}_A M < \infty$ if and only if M is both an artinian and Noetherian A-module

Proof. Length of ascending chains and descending chains are bounded by $\operatorname{length}_A M$. For the if, since M is Noetherian we can choose a maximal proper submodule M_1 of M. Doing the same this for M_1, we then get a descending chain $M \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots$. Since M is artinian, this process must stop, yielding a composition series of M. $\square$

3.97 Lemma. Let A be an Artinian ring. Then $\#\operatorname{Spec} A < \infty$, $\dim A = 0$ and the zero ideal is a product of maximal ideals.

Proof. Consider the collection of all finite intersections of maximal ideals. Since A is Artinian, it has a minimal element, say $\mathfrak{m}_1 \cap \cdot \cap \mathfrak{m}_n$. If $\mathfrak{m}$ a maximal ideal, by minimality $\mathfrak{m}_1 \cap \cdot \cap \mathfrak{m}_n \cap \mathfrak{m} = \mathfrak{m}_1 \cap \cdot \cap \mathfrak{m}_n$, so $\mathfrak{m} \subseteq \mathfrak{m}_i$ for some $i \in [n]$. This shows $\#\text{mSpec } A < \infty$. If $\mathfrak{p} \in \operatorname{Spec} A$ and $f \notin \mathfrak{p}$, consider the descending chain

$$(f, \mathfrak{p}) \supseteq (f^2, \mathfrak{p}) \supseteq (f^3, \mathfrak{p}) \supseteq \cdots.$$

Since A is artinian, it follows that $f^n \in (f^{n+1}, \mathfrak{p})$ for $n \gg 0$, so $f^n = f^{n+1}x + y$ for some $x \in A$, $y \in \mathfrak{p}$. But then $f^n(1 - fx) = y$ and $f \notin \mathfrak{p}$, so $1 - fx \in \mathfrak{p}$. It follows that $(f, \mathfrak{p}) = (1) = A$.

Finally, let J denote the product of all maximal ideals. Since A is artinian, the chain $J \supseteq J^2 \supseteq J^3 \supseteq \cdots$ stabilizes, and hence $J^n = J^{n+1}$ for $n \gg 0$. We claim $J^n = 0$; otherwise, consider the collection $\{0 \neq I \trianglelefteq A \mid IJ^n \neq 0\}$. This is nonempty as $JJ^n = J^{n+1} = J^n \neq 0$. Since A is artinian, it has a minimal element, and by minimality it must be principal, say, generated by $0 \neq f \in A$. Then $fJ^n = fJ^{n+1} = (fJ)J^n$, so by minimality again $fJ = (f)$. It follows that $f = fr$ for some $r \in J$, or $f(1 - r) = 0$. Note that $1 - r \in A^\times$; for otherwise it is contained in some maximal ideal, which is absurd. Hence $f = 0$, which is another contradiction. This proves $J^n = 0$. $\square$

3.97.1 Lemma. For an exact sequence $0 \to M' \to M \to M'' \to 0$ of A-modules, M is Noetherian (resp. Artinian) if and only if M' and M'' are Noetherian (resp. Artinian).

3.97.2 Characterization of artinian rings. For a ring A, TFAE :

(i) $\operatorname{length}_A A < \infty$.

(ii) A is artinian.

(iii) A is Noetherian and $\dim A = 0$.

Proof. (i)⇒(ii) is (3.96.2). Assume (ii). Then $\dim A = 0$ by (3.97). Again by (3.97), $\mathfrak{m}_1 \cdots \mathfrak{m}_n = 0$ for some maximal ideals $\mathfrak{m}_i$ of A. Consider the filtration

$$A \supseteq \mathfrak{m}_1 \supseteq \mathfrak{m}_1\mathfrak{m}_2 \supseteq \cdots \supseteq \mathfrak{m}_1 \cdots \mathfrak{m}_{n_1} \supseteq \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1}\mathfrak{m}_n = 0.$$

Each successive subquotient is an $A/\mathfrak{m}_i$-module, i.e., a vector space over $A/\mathfrak{m}_i$. This quotient has finite length, so the dimension (as vector spaces) over $A/\mathfrak{m}_i$ must be finite, which then implies it is Noetherian. By (3.97.1) this tells A is Noetherian.

Finally assume (iii). Suppose $\operatorname{length}_A A = \infty$, and consider the collection $\{I \trianglelefteq A \mid \operatorname{length}_A(A/I) = \infty\}$. This is nonempty as $\operatorname{length}_A A/0 = \infty$. Since A is Noetherian, there exists a maximal element $\mathfrak{m}$. We claim $\mathfrak{m}$ is a prime. Say $xy \in \mathfrak{m}$ with $y \notin \mathfrak{m}$. Form the short exact sequence

$$0 \longrightarrow A/(\mathfrak{p} : y) \xrightarrow{\times y} A/\mathfrak{p} \longrightarrow A/(y, \mathfrak{p}) \longrightarrow 0$$

where $(\mathfrak{p} : y) = \{f \in A \mid fy \in \mathfrak{p}\}$. If $x \notin \mathfrak{p}$, then both $(\mathfrak{p} : y)$ and $(y, \mathfrak{p})$ strictly contain $\mathfrak{p}$. By maximality the corresponding quotient has finite length. By (3.96.1) this implies $A/\mathfrak{m}$ has finite length, a contradiction. Hence $x \in \mathfrak{m}$ and $\mathfrak{m}$ is a prime. Since $\dim A = 0$, $\mathfrak{m}$ is maximal. But then $A/\mathfrak{m}$ is a field, so it has finite length as A-module, which is absurd. Hence $\operatorname{length}_A A < \infty$. □

3.4.2 Hilbert polynomials

We follow the exposition in [AM94, Chapter 11].

3.98 Lemma. Let $A = \bigoplus_{n \geqslant 0} A_n$ be a graded ring. Then A is a Noetherian ring if and only if A_0 is a Noetherian ring and A is of finite type over A_0.

Proof. The if part follows from Hilbert basis theorem. For the only if part, suppose A is a Noetherian ring. Note $A_+ \trianglelefteq A$, so (3.97.1) $A_0 = A/A_+$ is a Noetherian ring. Since A is Noetherian and A_+ is a proper ideal of A, it is finite over A. By (3.116), it follows that A is of finite type over A_0. □

3.99 Lemma. Let A be a Noetherian graded ring and M a finite graded A-module. Then M_n is a finite A_0-module for each $n \geqslant 0$.

Proof. Say $A = A_0[x_1, \dots, x_m]$ with $x_i \in A_+$ homogeneous, and $M = Ay_1 + \cdots + Ay_\ell$ for homogeneous $y_i \in M$. Then each element in M has the form $\sum_{i=1}^{\ell} f_i(x_1, \dots, x_m)y_i$ with $f_i \in A_0[X_1, \dots, X_m]$. From this we see M_n is generated by the $g(x_1, \dots, x_m)y_i$ where $1 \leqslant i \leqslant \ell$ and $g \in A_0[X_1, \dots, X_m]$ runs over all homogeneous polynomials of degree $n - \deg y_i$. □

3.100 Poincaré series. Let A be a graded Noetherian ring. Let λ be a $\mathbb{Z}$-valued function on all finite A_0-modules that is additive, in the sense if

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is a short exact sequence of finite A_0-modules, then

$$\lambda(M) = \lambda(M') + \lambda(M'').$$

For a finite graded A-module M, the **Poincaré series of** M with respect to λ is the generating function of $(\lambda(M_n))_{n \geqslant 0}$:

$$P(M, t) = P_\lambda(M, t) = \sum_{n=0}^{\infty} \lambda(M_n)t^n \in \mathbb{Z}[\![t]\!]$$

3.100.1 Example. Let A be an artinian ring and $R = A[x_1, \dots, x_m]$ be the polynomial ring in m variables over A. Take $M \mapsto \operatorname{length}_A M$ to be our additive function. Then

$$\operatorname{length}_A R_n = \binom{n + m - 1}{n}$$

is the number of the degree n monomials, so

$$P_{\mathrm{length}_A}(A[x_1, \ldots, x_m], t) = \sum_{n=0}^{\infty} \binom{n+m-1}{n} t^n = (1-t)^{-m}$$

3.101 Theorem. (Hilbert; Serre) Let A be a Noetherian graded ring with $A = A_0[x_1, \ldots, x_s]$ and $x_i \in A_+$ homogeneous, and let M be a finite graded A-module. Let λ be an additive function as in (3.100). Then

$$P(M, t) = f(t) \prod_{i=1}^{s} (1 - t^{\deg x_i})^{-1},$$

for some $f \in \mathbb{Z}[t]$.

Proof. We do induction on s. When $s = 0$, then $M = A_0 m_1 + \cdots + A_0 m_t$ for some $m_i \in M$, so that $M_n = 0$ for $n \gg 0$. Hence $P(M, t)$ is a polynomial in this case. Assume $s \geqslant 1$. For each $n \geqslant 0$ multiplication by x_s yields an exact sequence

$$0 \longrightarrow K_n \longrightarrow M_n \xrightarrow{\times x_s} M_{n+\deg x_s} \longrightarrow L_{n+\deg x_s} \longrightarrow 0.$$

Define $K = \bigoplus_n K_n$ and $L = \bigoplus_n L_n$ (where $L_k = 0$ for $0 \leqslant k < \deg x_s$). By (3.97.1) K and L are finite graded A-modules. Since L and K are annihilated by x_s, we can treat them as $A[x_1, \ldots, x_{s-1}]$-modules. Applying λ to the above sequence we get

$$\lambda(K_n) - \lambda(M_n) + \lambda(M_{n+\deg x_s}) - \lambda(L_{n+\deg x_s}) = 0.$$

Multiplying by $t^{n+\deg x_s}$ and summing over $n \geqslant 0$, we see

$$t^{\deg x_s} P(K, t) - t^{\deg x_s} P(M, t) + \left(P(M, t) - \sum_{n=0}^{\deg x_s - 1} \lambda(M_n) t^n \right) - \left(P(L, t) - \sum_{n=0}^{\deg x_s - 1} \lambda(L_n) t^n \right) = 0$$

or

$$P(M, t)(1 - t^{\deg x_s}) = P(L, t) - t^{\deg x_s} P(K, t) + g(t)$$

for some $g \in \mathbb{Z}[t]$. By induction this completes the proof. □

3.101.1 Since $P(M, t)$ is then a rational function, we can put

$$d(M) = -\min\{0, \operatorname{ord}_{t=1} P(M, t)\} = \text{ order of pole of } P(M, t) \text{ at } t = 1.$$

This will be an important quantity in studying the dimension.

3.101.2 Corollary. If $x \in A_+$ is homogeneous and is not a zero-divisor of M, then $d(M) = d(M/xM) + 1$.

Proof. For each $n \geqslant 0$ consider the exact sequence

$$0 \longrightarrow M_n \xrightarrow{\times x} M_{n+\deg x} \longrightarrow M_{n+\deg x}/xM_n = (M/xM)_{n+\deg x} \longrightarrow 0.$$

Multiplying by $t^{n+\deg x}$ and summing over $n \geqslant 0$, we see

$$P(M/xM, t) = g(t) + (1 - t^{\deg x}) P(M, t)$$

for some $g \in \mathbb{Z}[t]$. Since g has no pole, $P(M/xM, t)$ has a pole if and only if $(1 - t^{\deg x}) P(M, t)$ has a pole, and have the same order. This finishes the proof. □

3.101.3 Corollary. If in (3.101) $\deg x_i = 1$, then $\lambda(M_n)$ is a polynomial in n with rational coefficients for $n \gg 0$, of degree $d(M) - 1$. Here the zero polynomial has -1 degree.

Proof. By (3.101),

$$P(M, t) = f(t)(1-t)^{-s} = \left(\sum_{i=0}^{\deg f} f_i t^i\right)\left(\sum_{j=0}^{\infty} \binom{j+s-1}{j} t^j\right).$$

Cancelling out powers of $(1-t)$, we assume $s = d(M)$ and $f(1) \neq 0$. For $n \geqslant \deg f$, we have

$$\lambda(M_n) = \sum_{i=0}^{\deg f} f_i \binom{n-i+s-1}{n-i}.$$

This is a rational polynomial with leading term $\dfrac{f(1)}{(s-1)!} n^{s-1}$ when $n \gg 0$.

$\square$

3.4.3 Noetherian local rings

3.102 I-filtration. Let A be a ring, I an ideal and M an A-module. A descending filtration

$$\cdots \subseteq M_n \subseteq \cdots \subseteq M_1 \subseteq M_0 \subseteq M$$

is said to be an I-**filtration** if $IM_n \subseteq M_{n+1}$ for $n \geqslant 0$. Consider the auxiliary graded module

$$M^* := \bigoplus_{n \geqslant 0} M_n.$$

If we denote $\mathrm{Bl}_I A = A^* := \bigoplus_{n \geqslant 0} I^n$, then M^* is a graded $\mathrm{Bl}_I A$-module. On the other hand, there is another common associated graded module

$$\mathrm{Gr}_F M = \bigoplus_{n \geqslant 0} M_n / M_{n+1}.$$

This is a $\mathrm{Gr}_I A = \mathrm{GL}_{(I^n)_{n \geqslant 0}} A$-module.

3.102.1 Lemma. Let A be a Noetherian ring, I an ideal, M a finite A-module and $F = (M_n)_{n \geqslant 0}$ an I-filtration. TFAE :

(i) M^* is a finite $\mathrm{Bl}_I A$-module.

(ii) The filtration F is **stable**, i.e., $IM_n = M_{n+1}$ for $n \gg 0$.

Proof. Since M is Noetherian, each M_n is finite over A. Let $Q_n = \bigoplus_{k=0}^{n} M_k$; then Q_n is finite over A, and generates the finite $\mathrm{Bl}_I A$-submodule

$$M_n^* = M_0 \oplus \cdots \oplus M_n \oplus IM_n \oplus I^2 M_n \oplus \cdots$$

of M^*. Then $L_0 \subseteq L_1 \subseteq \cdots$ is an ascending chain which unions to M^*. Since $\mathrm{Bl}_I A$ is Noetherian, we see M^* is finite over $\mathrm{Bl}_I A$ if and only if the chain stops, i.e. $M^* = M_n^*$ for $n \gg 0$. The last condition is the same as saying F is stable. $\square$

3.102.2 Corollary. (Artin-Rees) Let A be a Noetherian ring, I an ideal, M a finite A-module and $F = (M_n)_{n \geqslant 0}$ a stable I-filtration. If $N \subseteq M$ is an A-submodule, then $(M_n \cap N)_{n \geqslant 0}$ is a stable I-filtration of N.

Proof. Since $\mathrm{Bl}_I A$ is Noetherian, by (3.102.1) we see M^* is a Noetherian $\mathrm{Bl}_I A$-module, and we only need to show $N^* = \bigoplus_{n \geqslant 0}(M_n \cap N)$ is also a finite $\mathrm{Bl}_I A$-module. But this is clear, as N^* is a $\mathrm{Bl}_I A$-submodule of M^* and M^* is Noetherian. □

3.103 Lemma. Let $(A, \mathfrak{m})$ be a Noetherian local ring, and $\mathfrak{q}$ a $\mathfrak{m}$-primary ideal. Suppose M is a finite A-module, and $F = (M_n)_{n \geqslant 0}$ is a stable $\mathfrak{q}$-filtration. Then

(i) Each M/M_n has finite length as an A-module.

(ii) There exists some polynomial g of degree $\leqslant s$ such that $g(n) = \mathrm{length}_A\, M/M_n$ for $n \gg 0$, where s is the least number of generators of $\mathfrak{q}$.

(iii) The degree and the leading coefficient of g do not depend on the filtration F, but only on M and $\mathfrak{q}$.

Proof. Since $\mathfrak{q}$ is finite over A, $\mathrm{Gr}_\mathfrak{q} A$ is Noetherian and $\mathrm{Gr}_F M$ is a finite $\mathrm{Gr}_\mathfrak{q} A$-module. Since $\mathrm{Gr}_F^n M = M_n/M_{n+1}$ is killed by $\mathfrak{q}$, it is a finite $A/\mathfrak{q}$-module. Since $A/\mathfrak{q}$ is artinian, it follows that M_n/M_{n+1} is Noetherian and artinian, whence of finite length (3.96.2). Now by (3.96.1).(i)

$$\mathrm{length}_A\, M/M_n = \sum_{i=0}^{n-1} \mathrm{length}_A\, M_i/M_{i+1} < \infty.$$

This proves (i). For (ii), say $\mathfrak{q} = Ax_1 + \cdots + Ax_s$; then $\mathrm{Gr}_\mathfrak{q} A = (A/\mathfrak{q})[x_1, \ldots, x_s]$. By (3.101.3), $\mathrm{length}_A(M_n/M_{n+1})$ is then a polynomial in n when $n \gg 0$, of degree $\leqslant s-1$. In particular, $\mathrm{length}_A\, M/M_n$ is a polynomial in n when $n \gg 0$ of degree $\leqslant s$.

Finally, for (iii) it suffices to compare with the filtration $(\mathfrak{q}^n M)_{n \geqslant 0}$. Since F is stable, $\mathfrak{q}^n M \subseteq M_n$ for all $n \geqslant 0$. Now take $N \gg 0$ so that $\mathfrak{q} M_n = M_{n+1}$ for all $n \geqslant N$. Then $M_{n+N} = \mathfrak{q}^n M_N \subseteq \mathfrak{q}^n M$, and $\mathfrak{q}^{n+N} M \subseteq M_{n+N} \subseteq M_n$. But then

$$\mathrm{length}_A\, M/M_{n+N} \geqslant \mathrm{length}_A\, M/\mathfrak{q}^n M, \qquad \mathrm{length}_A\, M/\mathfrak{q}^{n+N} M \geqslant \mathrm{length}_A\, M/M_n$$

for all $n \geqslant 0$. By (ii) they are polynomials when $n \gg 0$, so this implies they has the same leading term, proving (iii). □

3.103.1 Characteristic polynomial. We denote the polynomial in (3.103) associated to the filtration $(\mathfrak{q}^n M)_{n \geqslant 0}$ by

$$\chi_\mathfrak{q}^M(n) = \mathrm{length}_A\, M/\mathfrak{q}^n M \qquad n \gg 0.$$

When $M = A$, we simply write

$$\chi_\mathfrak{q}(n) = \chi_\mathfrak{q}^A(n) = \mathrm{length}_A\, A/\mathfrak{q}^n, \qquad n \gg 0$$

and call it the **characteristic polynomial of the $\mathfrak{m}$-primary ideal** $\mathfrak{q}$. The degree of $\chi_\mathfrak{q}(n)$ gives a lower bound of the number of the generators of $\mathfrak{q}$.

3.103.2 Lemma. Let $A, \mathfrak{m}, \mathfrak{q}$ be as in (3.103). Then $\deg \chi_\mathfrak{q}(n) = \deg \chi_\mathfrak{m}(n)$.

Proof. Since A is Noetherian, that $\sqrt{\mathfrak{q}} = \mathfrak{n}$ implies $\mathfrak{m}^r \subseteq \mathfrak{q} \subseteq \mathfrak{m}$ for some $r \gg 0$, and hence

$$\mathrm{length}_A\, A/\mathfrak{m}^n \leqslant \mathrm{length}_A\, A/\mathfrak{q}^n \leqslant \mathrm{length}_A\, A/\mathfrak{m}^{r+n}$$

for all $n \geqslant 0$. This proves the lemma. □

3.103.3 In view of the previous lemma, we denote by $d(A)$ the common degree of the $\chi_\mathfrak{q}(n)$. By (3.101.3), in fact

$$d(A) = d(\mathrm{Gr}_\mathfrak{m} A),$$

where the right hand side is defined as in (3.101.1).

3.104 Let $(A, \mathfrak{m})$ be a Noetherian local ring and $\mathfrak{q}$ an $\mathfrak{m}$-primary ideal. Denote by

$$\delta(A) = \delta_{\mathfrak{q}}(A) = \text{the least number of generators of } \mathfrak{q}.$$

By (3.103) we see $\delta(A) \geqslant d(A)$. Our goal is to show

$$\delta(A) = d(A) = \dim A.$$

For this we are going to prove $\delta(A) \geqslant d(A) \geqslant \dim A \geqslant \delta(A)$.

3.4.4 Weil divisors

3.105 Cycles. Let X be a topological space. An irreducible closed subset of codimension r is called a **codimension r prime cycle**. Denote by $Z^r(X)$ the free abelian group on all codimension r prime cycles. Similarly, an irreducible closed subset of dimension k is called a **dimension k prime cycle**. Denote by $Z_k(X)$ the free abelian group on all dimension k prime cycles.

3.105.1 Weil divisors. Let X be a Noetherian scheme. An element in $Z^1(X)$ is called a **Weil divisor** on X. For brevity we call a codimension 1 prime cycle a **prime divisor**.

3.5 Proj

3.106 Let $S = \bigoplus_{n \geqslant 0} S_0$ be a graded ring, i.e., $S_n \cdot S_m \subseteq S_{n+m}$ for any $n, m \geqslant 0$ and each S_n is an abelian subgroup of S. An element f in S_n $(n \geqslant 0)$ is said to be **homogeneous of degree** n, and in this case we put $n = \deg f$. For an element $x \in S$, we can write $x = \sum_{n \geqslant 0} x_n$ with $x_n \in S_n$ in a unique way; the x_n are called the **homogeneous parts** of x. An ideal I of S is called **homogeneous** if it is generated by its homogeneous elements, i.e.,

$$I = \sum_{n \geqslant 0} (I \cap S_n)$$

Put $S_+ = \bigoplus_{n \geqslant 1} S_0$. This is the ideal of elements of positive degrees. Now define

$$\operatorname{Proj} S := \{\mathfrak{p} \in \operatorname{Spec} S \mid \mathfrak{p} \not\supseteq S_+,\ \mathfrak{p} \text{ is homogeneous}\}$$

For a subset $T \subseteq S$, define $V_+(T) := \{\mathfrak{p} \in \operatorname{Proj} S \mid T \subseteq \mathfrak{p}\}$. By definition we have $V_+(T) = V(T) \cap \operatorname{Proj} S$. We equip $\operatorname{Proj} S$ with the subspace topology inherited from $\operatorname{Spec} S$. Note that $V_+(T) = V_+(I)$, where I is the homogeneous ideal generated by the homogeneous part of all $x \in T$. Thus all closed sets of $\operatorname{Proj} S$ have the form $V_+(I)$ for some homogeneous ideals I of S. For any homogeneous f, define the **principal open set**

$$D_+(f) = \{\mathfrak{p} \in \operatorname{Proj} S \mid f \notin \mathfrak{p}\} = D(f) \cap \operatorname{Proj} S = \operatorname{Proj} S \backslash V_+(f).$$

The principal open sets form a basis for the topology on $\operatorname{Proj} S$. In fact, homogeneous elements of positive degree suffices to produce a basis. Indeed, for $f \in S_0$, one has

$$D_+(f) = \bigcup_{n \geqslant 1} \bigcup_{g \in S_n} D_+(fg)$$

$\supseteq$ is clear. To see $\subseteq$ it suffices to recall $\mathfrak{p} \in \operatorname{Proj} S$ do not contain whole S_+ by definition. Because of this fact, in the following by a principal open set we always refer to the one given by homogeneous of positive degree.

3.107 Lemma. For an homogeneous ideal I, we have

$$\sqrt{I} = \bigcap_{\mathfrak{p} \in V_+(I)} \mathfrak{p}$$

Proof. $\subseteq$ is clear. For the another containment, suppose $f \notin \sqrt{I}$. Choose from the family

$$\{J : \text{homogeneous} \mid J \supseteq I,\ f^n \notin J \text{ for all } n \geqslant 1\}$$

a maximal element $\mathfrak{q}$ by Zorn's lemma. Then $\mathfrak{q}$ is a homogeneous prime not containing f. $\square$

3.108 Let $S = \bigoplus_{n \geqslant 0} S_n$ be a graded ring and let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a $\mathbb{Z}$-graded S-module, i.e., $S_n \cdot M_m \subseteq M_{n+m}$ for any $n \geqslant 0, m \in \mathbb{Z}$. For a multiplicatively closed subset T of S consisting of homogeneous elements, the localization $T^{-1}S$ is naturally a $\mathbb{Z}$-graded ring, and $T^{-1}M$ is naturally a $\mathbb{Z}$-graded $T^{-1}S$-module, i.e.,

$$(T^{-1}M)_n = \left\{ \frac{x}{t} \mid x \in M_{m+n},\ t \in T \cap S_m \text{ for some } m \geqslant 0 \right\}.$$

For any homogeneous $f \in S_+$ of positive degree, put

$$M_{(f)} = (M_f)_0$$

to be the degree 0 part of the localization M_f.

3.109 $\mathcal{O}_{\operatorname{Proj} S}$ and projective tilde. For a graded ring S and homogeneous f of positive degree, define

$$\mathcal{O}_{\operatorname{Proj} S}(D_+(f)) = S_{(f)}.$$

If $D_+(f) \supseteq D_+(g)$, then $V_+((f)) \subseteq V_+((g))$, and $(g) \subseteq \sqrt{(f)}$ by Lemma 3.107. This means $g^n = sf$ for some $n \geqslant 1$ and $s \in S$; in particular s is homogeneous. The map $S_f \to S_g$ given by $\frac{1}{f} \mapsto \frac{s}{g^n}$ is independent of the choice of n and s, and is degree-preserving, so it gives a map $S_{(f)} \to S_{(g)}$. This shows $\mathcal{O}_{\operatorname{Proj} S}$ defines a presheaf of rings on the principal open sets. To show it is a sheaf, we use (2.3), and prove more generally that there is an exact sequence

$$0 \longrightarrow M_{(f)} \longrightarrow \prod_{i=1}^{n} M_{(g_i)} \longrightarrow \prod_{i,j=1}^{n} M_{(g_i g_j)}.$$

where M is a $\mathbb{Z}$-graded S-module, and $f, g_i \in S_+$ are homogeneous with $D_+(f) = \bigcup_{i=1}^{n} D_+(g_i)$. We use the isomorphism $\varphi : D_+(f) \to \operatorname{Spec} S_{(f)}$ in the next paragraph (3.110). The identity $D_+(fg_i) = \varphi^{-1}(D(g_i^{\deg f} f^{-\deg g_i}))$ then implies

$$\operatorname{Spec} S_{(f)} = \bigcup_{i=1}^{n} D(g_i^{\deg f} f^{-\deg g_i}) \tag{$\spadesuit$}$$

The natural map $M_f \to M_g$ induces a map $M_{(f)} \to M_{(g)}$, and an isomorphism $(M_{(f)})_{g_i^{\deg f} f^{-\deg g_i}} \cong M_{(g_i)}$, where we regard them all as $S_{(f)}$-modules. Similarly $(M_{(f)})_{(g_i g_j)^{\deg f} f^{-\deg g_i g_j}} \cong M_{(g_i g_j)}$. So the sequence we are concerning becomes

$$0 \longrightarrow M_{(f)} \longrightarrow \prod_{i=1}^{n} (M_{(f)})_{g_i^{\deg f} f^{-\deg g_i}} \longrightarrow \prod_{i,j=1}^{n} (M_{(f)})_{(g_i g_j)^{\deg f} f^{-\deg g_i g_j}}.$$

In view of ($\spadesuit$), this reduces to (3.3).($\heartsuit$) (with $A = S_{(f)}$, $M = M_{(f)}$, $f = 1$, $f_i = g_i^{\deg f} f^{-\deg g_i}$). This finishes the proof.

For a $\mathbb{Z}$-graded S-algebra M, we define

$$\widetilde{M}(D_+(f)) = M_{(f)}$$

Then we have showed that this defines a sheaf on principal open sets, and hence a sheaf on the ringed space $(\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S})$.

3.110 We prove that for homogeneous $f \in S_+$ there exists an isomorphism in $\mathbf{Ring_{Top}}$

$$(D_+(f), \mathcal{O}_{\operatorname{Proj} S}|_{D_+(f)}) \longrightarrow (\operatorname{Spec} S_{(f)}, \mathcal{O}_{\operatorname{Spec} S_{(f)}}).$$

In particular, this shows $\operatorname{Proj} S$ is a scheme.

We first construction a continuous map on topological spaces. Recall in (3.11) we have a homeomorphism $D(f) \to \operatorname{Spec} S_f$. We claim the composition $\varphi : D_+(f) \subseteq D(f) \to \operatorname{Spec} S_f \to \operatorname{Spec} S_{(f)}$ is again a homeomorphism. Explicitly, $\varphi(\mathfrak{p}) = \mathfrak{p}S_f \cap S_{(f)}$. We define an inverse map $\psi : \operatorname{Spec} S_{(f)} \to D_+(f)$ by

$$\psi(\mathfrak{p}') = \bigoplus_{n \geqslant 0} \left\{ a \in S_n \mid \frac{a^{\deg f}}{f^n} \in \mathfrak{p}' \right\}$$

Each summand is an abelian subgroup of S_n, for if $a^{\deg f}f^{-n}, b^{\deg f}f^{-n} \in \mathfrak{p}'$, then $(a-b)^{2\deg f}/f^{-2n} \in \mathfrak{p}'$, and thus $(a-b)^{\deg f}/f^{-n} \in \mathfrak{p}'$ as $\mathfrak{p}'$ is a prime. This is an homogeneous ideal for if $a^{\deg f}f^{-n} \in \mathfrak{p}'$ and $s \in S_m$, then $(as)^{\deg f}f^{-(n+m)} = (a^{\deg f}f^{-n})(s^{\deg f}f^{-m}) \in \mathfrak{p}'$. This is a prime ideal, for if $a \in S_n$ and $b \in S_m$ with $(ab)^{\deg f}f^{-(n+m)} \in \mathfrak{p}'$, then either $a^{\deg f}f^{-n} \in \mathfrak{p}'$ or $b^{\deg f}f^{-m} \in \mathfrak{p}'$ as the fractions are of degree 0 and $\mathfrak{p}'$ is a prime. Lastly, $f \notin \psi(\mathfrak{p}')$ for otherwise we would have $1 = f^{\deg f}f^{-\deg f} \in \mathfrak{p}'$, which is absurd; in particular $\psi(\mathfrak{p}') \supseteq S_+$ We must show φ and ψ are mutually inverses. For $g \in S_n$, we have

$$g \in \psi(\varphi(\mathfrak{p})) \Leftrightarrow g^{\deg f}f^{-n} \in \varphi(\mathfrak{p}) \subseteq \mathfrak{p}S_f \Rightarrow g \in \mathfrak{p}S_f \cap S = \mathfrak{p} \Rightarrow g^{\deg f}f^{-n} \in \mathfrak{p}S_f \cap S_{(f)} = \varphi(\mathfrak{p}) \Leftrightarrow g \in \psi(\varphi(\mathfrak{p}))$$

so $\psi(\varphi(\mathfrak{p})) = \mathfrak{p}$. For $g \in S_{(f)}$, if $g \in \varphi(\psi(\mathfrak{p}'))$, we can write $g = \dfrac{a}{f^n}$ for some homogeneous $a \in \psi(\mathfrak{p}')$ with $\deg a = n \deg f$. But these imply $\mathfrak{p}' \ni a^{\deg f}f^{-\deg a} = g^{\deg f}$, so $g \in \mathfrak{p}'$.

Conversely, if $g \in \mathfrak{p}'$, write $g = \dfrac{a}{f^n}$ for some homogeneous $a \in S$ with $\deg a = n \deg f$. The above argument shows $a \in \psi(\mathfrak{p}')$, so $g \in \psi(\mathfrak{p}')S_f \cap S_{(f)} = \varphi(\psi(\mathfrak{p}'))$. To show φ is a homeomorphism, note that for $g^{\deg f}f^{-\deg g} \in S_{(f)}$ with $g \in S_+$ homogeneous, we have

$$\varphi^{-1}(D(g^{\deg f}f^{-\deg g})) = D_+(gf). \tag{$\star$}$$

For if $g^{\deg f}f^{-\deg g} \notin \varphi(\mathfrak{p})$, then $g^{\deg f}f^{-\deg g} \notin \mathfrak{p}S_f$, so $g \notin \mathfrak{p}$. This implies $gf \notin \mathfrak{p}$. Note that is $g^{\deg f}f^{-\deg g} \in \mathfrak{p}S_f$, then $g^{\deg f}f^{-\deg g} = af^{-m}$ for some homogeneous $a \in \mathfrak{p}$, so $g \in \mathfrak{p}$ since $\mathfrak{p}$ is a prime. Thus the above argument is reversible, so this proves the equality. To conclude our assertion, it suffices to note that $D_+(gf)$ forms an open basis of $D_+(f)$ when g runs over S_+ and is homogeneous.

To show $\mathcal{O}_{\operatorname{Spec} S_{(f)}} \cong \varphi_* \mathcal{O}_{\operatorname{Proj} S}|_{D_+(f)}$, we check there are compatible isomorphisms on basis elements of the form $(\star)$. One has

$$\mathcal{O}_{\operatorname{Spec} S_{(f)}}(D(g^{\deg f}f^{-\deg g})) = (S_{(f)})_{g^{\deg f}f^{-\deg f}} \cong S_{(gf)} = \mathcal{O}_{\operatorname{Proj} S}(D_+(gf)) = \varphi_* \mathcal{O}_{\operatorname{Proj} S}(D(g^{\deg f}f^{-\deg g})),$$

the isomorphism being given by the universal property of localization. This isomorphism is compatible with the restriction, so by (2.3.1) it patches to an isomorphism $\mathcal{O}_{\operatorname{Spec} S_{(f)}} \to \varphi_* \left(\mathcal{O}_{\operatorname{Proj} S}|_{D_+(f)}\right)$.

In fact, the same argument also shows that $\widetilde{M_{(f)}} \cong \varphi_* \left(\widetilde{M}|_{D_+(f)}\right)$, so in fact we have an isomorphism in $\mathbf{Mod_{LRS}}$

$$(D_+(f), \mathcal{O}_{\operatorname{Proj} S}|_{D_+(f)}, \widetilde{M}|_{D_+(f)}) \longrightarrow (\operatorname{Spec} S_{(f)}, \mathcal{O}_{\operatorname{Spec} S_{(f)}}, \widetilde{M_{(f)}}).$$

Be aware of the difference of two tildes : one is projective, and another is affine.

3.110.1 For a graded ring S, define the category $\mathbf{GrMod}_S$ of graded S-modules as follows. The objects of $\mathbf{GrMod}_S$ consist of all graded S-modules. A morphism in $\mathbf{GrMod}_S$ is an S-module homomorphism $\varphi : M \to N$ satisfying $\varphi(M_n) \subseteq N_n$ for each $n \in \mathbb{Z}$. With this terminology, from the last isomorphism in the previous paragraph, we see the projective tilde defines a functor

$$\widetilde{(\cdot)} : \mathbf{GrMod}_S \longrightarrow \mathbf{Qcoh}_X$$

3.110.2 The map $D_+(f) \to \operatorname{Spec} S_{(f)}$ in (3.110) is compatible with the restriction, as one can argue as in (3.11.2). Composing with $\operatorname{Spec} S_{(f)} \to \operatorname{Spec} S_0$ (coming from the natural map $S_0 \to S_{(f)}$), we obtain a family of morphisms $D_+(f) \to \operatorname{Spec} S_0$, compatible with the restriction. By (2.13.4) they give rises to a morphism $\operatorname{Proj} S \to \operatorname{Spec} S_0$.

Similarly, let A be a ring and let S be a graded A-algebra. Here we assume the image of A lies in S_0. With this assumption, the homomorphism $A \to S \to S_f$ stabilizes $S_{(f)}$, making $S_{(f)}$ an A-algebra, which gives rise to a morphism $\operatorname{Spec} S_{(f)} \to \operatorname{Spec} A$. Composing with $D_+(f) \to \operatorname{Spec} S_{(f)}$ and gluing, we obtain $\operatorname{Proj} S \to \operatorname{Spec} A$ so that $\operatorname{Proj} S$ is naturally an A-scheme.

3.110.3 Let M be a graded S-module and $f \in S_d$. Note that there is a canonical homomorphism

$$\begin{aligned} M_0 &\longrightarrow M_{(f)} = \widetilde{M}(D_+(f)) \\ m &\longmapsto \frac{m}{1}. \end{aligned}$$

The homomorphisms obtained by varying $f \in S_+$ are compatible, so this yields an S_0-homomorphism

$$M_0 \longrightarrow \widetilde{M}(\operatorname{Proj} S).$$

In particular, there is a canonical ring homomorphism $S_0 \to \widetilde{S}(\operatorname{Proj} S)$,

3.111 Let S and S' be two graded rings and $\varphi : S \to S'$ a ring homomorphism such that $\varphi(S_n) \subseteq S'_{nd}$ for any $n \in \mathbb{Z}$ and a fixed $d \in \mathbb{N}$. For any $\mathfrak{p}' \in \operatorname{Proj} S'$, we have

$$\varphi^{-1}(\mathfrak{p}') \in \operatorname{Proj} S \Leftrightarrow S_+ \nsubseteq \varphi^{-1}(\mathfrak{p}') \Leftrightarrow \varphi(S_+) \nsubseteq \mathfrak{p}'$$

or $\mathfrak{p}' \in \operatorname{Proj} S' \setminus V_+(\varphi(S_+))$. If we put $G(\varphi) = \operatorname{Proj} S' \setminus V_+(\varphi(S_+))$, we see φ induces a continuous map

$$\begin{aligned} \Phi : G(\varphi) &\longrightarrow \operatorname{Proj} S \\ \mathfrak{p}' &\longmapsto \varphi^{-1}(\mathfrak{p}'). \end{aligned}$$

Note that $G(\varphi) = \bigcup\{D_+(\varphi(f)) \mid f \in S_d,\ d \geqslant 1\}$ and $\Phi^{-1}(D_+(f)) = D_+(\varphi(f))$ for homogeneous $f \in S_+$. Observe that φ induces a homomorphism $S_f \to S'_{\varphi(f)}$ of *graded rings*, i.e., $\varphi((S_f)_n) \subseteq (S'_{\varphi(f)})_n$ for any $n \geqslant 0$, so it further induces a ring homomorphism $S_{(f)} \to S'_{(\varphi(f))}$. This gives a morphism $D_+(\varphi(f)) \to D_+(f)$ of schemes (3.110). For any homogeneous $g \in S_+$, it is easy to see there is a commuting square

$$\begin{tikzcd} S'_{\varphi(f)} \arrow[d] & S_{(f)} \arrow[l] \arrow[d] \\ S'_{\varphi(fg)} & S_{(fg)} \arrow[l] \end{tikzcd}$$

so the $D_+(\varphi(f)) \to D_+(f)$ glue to a morphism $G(\varphi) \to \operatorname{Proj} S$ of schemes. We denote this morphism by $\operatorname{Proj}\varphi$. By construction we see $\operatorname{Proj}\varphi$ is an affine morphism (3.20).

3.112 Lemma. Let A be a ring, B a graded A-algebra and C an A-algebra. Then there is a canonical isomorphism

$$\operatorname{Proj}(B \otimes_A C) \cong \operatorname{Proj} B \times_{\operatorname{Spec} A} \operatorname{Spec} C$$

Here $B \otimes_A C$ is graded via $B \otimes_A C = \bigoplus_{n=0}^{\infty} (B_n \otimes_A C)$.

Proof. Put $E = B \otimes_A C$. The canonical A-algebra homomorphism $\iota : B \to E$ satisfies $\iota(B_n) \subseteq E_n = B_n \otimes_A C$ for every n, so $\iota(B_+) \subseteq E_+$; in fact, $\iota(B_+)E = E_+$. Then $V_+(\iota(B_+)) = V_+(E_+) = \varnothing$. By (3.111) and (3.110.2) we thus obtain an A-morphism $\psi : \operatorname{Proj} E \to \operatorname{Proj} B$. Also, by (3.110.2) the natural map $C \to E$ gives an A-morphism $\operatorname{Proj} E \to \operatorname{Spec} C$. From the definition of fibre product these two morphisms give

$$\varphi : \operatorname{Proj} E \to \operatorname{Proj} B \times_{\operatorname{Spec} A} \operatorname{Spec} C$$

For homogeneous $f \in B_+$, we have

$$\varphi^{-1}(D_+(f) \times_{\operatorname{Spec} A} \operatorname{Spec} C) = \psi^{-1}(D_+(f)) = D_+(\iota(f))$$

so we only need to show $D_+(\iota(f)) \to D_+(f) \times_{\operatorname{Spec} A} \operatorname{Spec} C$ is an isomorphism for every such f. Back to algebra, we need to show $B_{(f)} \otimes_A C \to E_{(\iota(f))}$ is an isomorphism. Clearly, this map is given by

$$\frac{b}{f^n} \otimes c \mapsto \frac{b \otimes c}{\iota(f)^n}$$

which is clearly surjective. To show the injectivity, we only need to show $B_{(f)} \otimes_A C \to B_f \otimes_A C \to E_{\iota(f)}$ is injective. This is clear. □

3.113 Let A be a ring, S be a graded A-algebra and $f \in A$; again we assume the image of A is contained in S_0. The localization $S \to S_f$ preserves degree, so by (3.111) we have a morphism $\operatorname{Proj} S_f \to \operatorname{Proj} S$. Moreover, by (3.110.2) we have a diagram

$$\begin{tikzcd}
\operatorname{Proj} S_f \arrow[r] \arrow[d] & \operatorname{Proj} S \arrow[d] \\
\operatorname{Spec} A_f \arrow[r] & \operatorname{Spec} A
\end{tikzcd}$$

By checking affine locally, this is a commutative diagram. We claim this is a fibre square. This is clear from the proof of Lemma 3.112.

3.114 Closed subschemes of Proj **defined by homogeneous ideals.** Let S be a graded ring and $I \trianglelefteq S$ a homogeneous ideal. Then $\widetilde{I} \trianglelefteq \mathcal{O}_{\operatorname{Proj} S}$ is an ideal sheaf, so we can form the closed local-ringed space $\left(V(\widetilde{I}), \mathcal{O}_{\operatorname{Proj} S}/\widetilde{I}|_{V(\widetilde{I})}\right)$. We claim that

$$V(\widetilde{I}) = V_+(I) = \{\mathfrak{p} \in \operatorname{Proj} A \mid I \subseteq \mathfrak{p}\}.$$

For each homogeneous $f \in S_+$,

$$V(\widetilde{I}) \cap D_+(f) = \{\mathfrak{p} \in D_+(f) \mid \widetilde{I}_\mathfrak{p} \subsetneq \mathcal{O}_{\operatorname{Proj} S,\mathfrak{p}}\} \cong \{\mathfrak{p} \in \operatorname{Spec} S_{(f)} \mid (I_{(f)})_\mathfrak{p} \subsetneq (S_{(f)})_\mathfrak{p}\} = V(I_{(f)})$$

We must show the image of $V_+(I) \cap D_+(f)$ under the isomorphism $D_+(f) \cong \operatorname{Spec} S_{(f)}$ is also $V(I_{(f)})$. One containment is clear. For the other way around, let $\mathfrak{p}' \in V(I_{(f)})$. We must check $\psi(\mathfrak{p}') \supseteq I$, where ψ is defined as in (3.109). If $a \in I_n$, then $a^{\deg f} f^{-n} \in I_{(f)} \subseteq \mathfrak{p}'$. Hence $I_n \subseteq \psi(\mathfrak{p}'))_n$ for each $n \geqslant 0$, or $I \subseteq \psi(\mathfrak{p}')$ as claimed.

Let $\varphi : S \to S/I$ be the natural projection. Then φ preserves degree, so by (3.111) there is a natural map $\Phi : \operatorname{Proj} S/I \to \operatorname{Proj} S$. From the construction we see the image of Φ lies in $V_+(I)$. The sheaf map $\mathcal{O}_{\operatorname{Proj} S} \to \Phi_*\mathcal{O}_{\operatorname{Proj} S/I}$ is defined by gluing the spec of the maps $S_{(f)} \to (S/I)_{(\varphi(f))}$ $(f \in S_+)$, which has kernel $I_{(f)}$, so $\ker(\mathcal{O}_{\operatorname{Proj} S} \to \Phi_*\mathcal{O}_{\operatorname{Proj} S/I})$ is exactly $\widetilde{I}$. By (2.24) Φ factors through the closed immersion $\left(V(\widetilde{I}), \mathcal{O}_{\operatorname{Proj} S}/\widetilde{I}|_{V(\widetilde{I})}\right) \to \operatorname{Proj} S$, yielding a unique isomorphism

$$\operatorname{Proj} S/I \cong \left(V(\widetilde{I}), \mathcal{O}_{\operatorname{Proj} S}/\widetilde{I}|_{V(\widetilde{I})}\right).$$

3.115 Closed subschemes of $\operatorname{Proj} S$ (I). Let S be a graded ring. Let $\mathcal{I} \trianglelefteq \mathcal{O}_{\operatorname{Proj} S}$ be an ideal sheaf. By (3.21.1) the closed local-ringed subspace $(V(\mathcal{I}), \mathcal{O}_X/\mathcal{I})|_{V(\mathcal{I})})$ is a closed subscheme if and only if $\mathcal{I}$ is $\mathcal{O}_{\operatorname{Proj} S}$-quasi-coherent.

Assume $\mathcal{I}$ is $\mathcal{O}_{\operatorname{Proj} S}$-quasi-coherent. Let $f \in S_+$ be homogeneous. Then the canonical map $\widetilde{\mathcal{I}(D_+(f))} \to \mathcal{I}|_{D_+(f)}$ is an isomorphism. Define

$$J_f = \bigoplus_{n \geqslant 0} \left\{ a \in S_n \mid \frac{a^{\deg f}}{f^n} \in \mathcal{I}(D_+(f)) \right\}.$$

By construction $\mathcal{I}(D_+(f)) = (J_f)_{(f)}$. Indeed, for $x \in \mathcal{I}(D_+(f)) \subseteq S_{(f)}$, we can find $n \in \mathbb{Z}_{\geqslant 1}$ so that $xf^n \in S_{n \deg f}$. Hence $(xf^n)^{\deg f} f^{-nd} = x^{\deg f} \in \mathcal{I}(D_+(f))$, so that $xf^n \in J_f$, or $x \in (J_f)_{(f)}$.

If $g \in S_+$ is another homogeneous element, we get a similar ideal J_g. Replacing (f, g) by $(f^{\deg g}, g^{\deg f})$, we assume $\deg f = \deg g =: d$. We claim

$$(J_f \cap J_g)_{(f)} = (J_f)_{(f)}$$

and similarly for g. The containment $\subseteq$ is evident. Now let $a \in (J_f)_{(f)}$; then $b := af^n \in S_{nd} \cap J_f$ for some $n \in \mathbb{Z}_{\geqslant 0}$ and hence $b^{\deg f} f^{-nd} \in \mathcal{I}(D_+(f))$. On $\mathcal{I}(D_+(fg))$, we have

$$\frac{b^{\deg f}}{f^{nd}} = \frac{b^{\deg g}}{g^{nd}} \cdot \frac{g^{nd}}{f^{nd}}.$$

Since $S_{(gf)} = (S_{(g)})_{fg^{-1}}$ and $\mathcal{I}(D_+(fg)) = \mathcal{I}(D_+(g)) \otimes_{S_{(g)}} (S_{(g)})_{fg^{-1}}$, we see $\dfrac{b^{\deg g}}{g^{nd}} \cdot \dfrac{f^m}{g^m} \in \mathcal{I}(D_+(g))$ for some $m \in \mathbb{Z}_{\geqslant 1}$, and hence $(bf^m)^{\deg g} g^{-nd - m \deg g} \in \mathcal{I}(D_+(g))$. This implies $bf^m \in J_g$, and hence $bf^m \cap J_g \cap J_f$ with $a = \dfrac{bf^m}{f^{n+m}}$.

Suppose $\operatorname{Proj} S$ is compact, i.e., $\operatorname{Proj} S = D_+(f_1) \cup \cdots D_+(f_n)$ for some homogeneous $f_1, \dots, f_n \in S_+$; raising each f_i to some power we can assume $\deg f_1 = \cdots = \deg f_n > 0$. By the argument as above, we obtain several homogeneous ideals $J_{f_1}, \dots, J_{f_n}$ and $I := J_{f_1} \cap \cdots \cap J_{f_n}$ such that

$$I_{(f_i)} = (J_{f_i})_{(f_i)}, \qquad i \in [n].$$

In other words, $\widetilde{I}(D_+(f_i)) = (J_{f_i})_{(f_i)} = \mathcal{I}(D_+(f_i))$. By gluing we see $\widetilde{I} \cong \mathcal{I}$. We summarize the result in the next subparagraph. We will discuss this result in (3.125) in a functorial way.

3.115.1 Lemma. Let S be a graded ring and suppose $\operatorname{Proj} S$ is compact. Every closed subscheme of $\operatorname{Proj} S$ has the form $\operatorname{Proj} S/I$ for some homogeneous ideal I of S.

3.5.1 Quasi-coherent sheaves on Proj.

3.116 Let S be a graded ring. Then S can be viewed as an S_0-algebra. Let $E \subseteq S_+$ be a subset consisting of homogeneous elements.

Lemma. $S_+ = SE$ if and only if $S = S_0[E]$.

Proof. Considering the grading, we see if part holds obviously. For the only if part, we prove $S_n \subseteq S_0[E]$ inductively on $n \geqslant 1$. For $s \in S_n$, write $s = \sum_{i=1}^{m} s_i e_i$, where $s_i \in S$ and $e_i \in E \cap S_{n_i}$ for some $n_i \geqslant 1$. Writing s_i as the sum of its homogeneous part, we can assume $s_i \in S_{n-n_i}$. Since $n - n_i < n$, we can apply induction hypothesis to see $s \in S_0[E]$. □

3.117 Suppose S be a graded ring that is generated by S_1 as an S_0-algebra. Concisely, $S = S_0[S_1]$. Then $\operatorname{Proj} S$ is covered by those principal open sets of the form $D_+(f)$ with $f \in S_1$. Indeed, for any $f \in S_d$, write $f = \sum_{i=1}^{m} s_i f_i$, where f_i is a product of elements in S_1. Then $(f) \subseteq (f_1, \dots, f_m)$, so $V_+(f) \supseteq V_+(f_1) \cap \cdots \cap V_+(f_m)$, and hence $D_+(f) \subseteq D_+(f_1) \cup \cdots \cup D_+(f_m)$. But $D_+(f_i) \subseteq D_+(g_i)$ if $g_i \in S_1$ is any element appearing in the product f_i, so $D_+(f)$ is covered by those principal open sets given by elements in S_1. Since the $D_+(f)$ form a basis of topology of $\operatorname{Proj} S$, this proves the assertion.

In fact, the above argument shows that if $E \subseteq S_+$ consists of homogeneous elements such that $S = S_0[E]$, then S can be cover by those principal open sets given by elements in E.

3.118 $\mathcal{O}(n)$. Let S be a graded ring and $X = \operatorname{Proj} S$. For each $n \in \mathbb{Z}$, denote by $S(n)$ the $\mathbb{Z}$-graded S-module whose graded pieces are given by

$$S(n)_d = S_{n+d}.$$

We call this a **twist of** S **by** n. The $\mathcal{O}_X$-module $\widetilde{S(n)}$ is denoted by $\mathcal{O}_X(n)$. Particularly, the sheaf $\mathcal{O}_{\operatorname{Proj} S}(1)$ is called the **twisting sheaf of Serre**. For any $\mathcal{O}_X$-module $\mathcal{F}$, we denote by $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$, and called this $\mathcal{F}$ **twisted by** n. By (3.110.3) there are canonical S_0-homomorphism

$$S_n = S(n)_0 \longrightarrow \Gamma(X, \mathcal{O}_X(n))$$

Taking direct sum, we obtain

$$S \longrightarrow \bigoplus_{n \geqslant 0} \Gamma(X, \mathcal{O}_X(n))$$

3.118.1 For $f \in S_d$, the sheaf $O_X(nd)|_{D_+(f)}$ is in fact trivial. To see this, recall that by (3.110), there is an isomorphism

$$(D_+(f), \mathcal{O}_X(nd)|_{D_+(f)}) \longrightarrow (\operatorname{Spec} S_{(f)}, \widetilde{S(nd)_{(f)}}).$$

Consider the $S_{(f)}$-module isomorphism

$$\begin{aligned} S_{(f)} &\longrightarrow S(nd)_{(f)} \\ s &\longmapsto f^n s \end{aligned}$$

This is well-defined for any n since f is inverted and $S(nd)_{(f)}$ consists of elements of degree nd in S_f. This proves that $S(nd)_{(f)}$ is free of rank 1 over $S_{(f)}$.

3.118.2 Let M be a graded S-module and $f \in S_d$. Note that the natural isomorphism

$$\begin{aligned} M_f \otimes_{S_f} N_f &\longrightarrow (M \otimes_S N)_f \\ \frac{m}{f^a} \otimes \frac{n}{f^b} &\longmapsto \frac{m \otimes n}{f^{a+b}} \end{aligned}$$

is grading preserving. In particular, this induces $M_{(f)} \otimes_{S_{(f)}} N_{(f)} \to (M \otimes_S N)_{(f)}$. Moreover, for any other homogeneous

$g \in S_+$, we have a commutative diagram

$$\begin{tikzcd} M_{(f)} \otimes_{S_{(f)}} N_{(f)} \arrow[r] \arrow[d] & (M \otimes_S N)_{(f)} \arrow[d] \\ M_{(fg)} \otimes_{S_{(fg)}} N_{(fg)} \arrow[r] & (M \otimes_S N)_{(fg)} \end{tikzcd}$$

This implies we have a morphism $\widetilde{M} \otimes_{\mathcal{O}_X} \tilde{N} \to \widetilde{M \otimes_S N}$ in $\mathbf{Mod}_{\mathcal{O}_X}$. In particular, we obtain a morphism

$$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) \longrightarrow \mathcal{O}_X(n+m)$$

and by tensoring with an arbitrary $\mathcal{O}_X$-module $\mathcal{F}$, we obtain

$$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{F}(m) \longrightarrow \mathcal{F}(n+m)$$

3.118.3 Taking global section, we obtain a canonical bilinear map

$$\Gamma(X, \mathcal{O}_X(n)) \times \Gamma(X, \mathcal{O}_X(m)) \to \Gamma(X, \mathcal{O}_X(n+m)).$$

This defines a graded ring structure on $\bigoplus_{n \geqslant 0} \Gamma(X, \mathcal{O}_X(n))$, making the homomorphism

$$S \longrightarrow \bigoplus_{n \geqslant 0} \Gamma(X, \mathcal{O}_X(n))$$

a graded ring homomorphism (c.f. 3.27). Similarly, we have

$$\bigoplus_{n \geqslant 0} \Gamma(X, \mathcal{O}_X(n)) \times \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n)) \longrightarrow \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n))$$

so that $\bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n))$ is equipped with a graded $\bigoplus_{n \geqslant 0} \Gamma(X, \mathcal{O}_X(n))$-module structure, and hence a graded S-module structure.

3.118.4 Let M be a graded S-module. Consider the homomorphism

$$\widetilde{M}(n) := \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \to \widetilde{M \otimes_S S(n)} = \widetilde{M(n)}.$$

Taking global sections, we have

$$M_n \to \Gamma(X, \widetilde{M}(n)) \to \Gamma(X, \widetilde{M(n)})$$

and by taking direct sum, we obtain

$$M \to \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \widetilde{M}(n)) \to \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \widetilde{M(n)}).$$

Clearly these are S-graded module homomorphism.

3.118.5 In sum, we have three natural homomorphisms

$$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) \longrightarrow \mathcal{O}_X(n+m)$$

$$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{F}(m) \longrightarrow \mathcal{F}(n+m)$$

$$\widetilde{M}(n) \longrightarrow \widetilde{M(n)}$$

Let $d > 0$ and $f \in S_d$. Consider their restrictions to $D_+(f)$:

$$\mathcal{O}_X(nd)|_{D_+(f)} \otimes_{\mathcal{O}_X|_{D_+(f)}} \mathcal{O}_X(m)|_{D_+(f)} \longrightarrow \mathcal{O}_X(nd+m)|_{D_+(f)}$$

$$\mathcal{O}_X(nd)|_{D_+(f)} \otimes_{\mathcal{O}_X|_{D_+(f)}} \mathcal{F}(m)|_{D_+(f)} \longrightarrow \mathcal{F}(nd+m)|_{D_+(f)}$$

$$\widetilde{M}(nd)|_{D_+(f)} \longrightarrow \widetilde{M(nd)}|_{D_+(f)}$$

In fact, these are all isomorphisms. The first follows from (3.118.1), and the second follows from the first. For the third, it suffices to note that the map $M \to M(nd)$ defined by $m \mapsto f^n m$ induces an isomorphism $S(nd)_{(f)} \otimes_{S_{(f)}} M_{(f)} \cong M(nd)_{(f)}$.

3.119 Lemma. Let S be a graded ring and $X = \operatorname{Proj} S$. Assume that X is covered by the principal open sets $D_+(f)$ given by $f \in S_1$ (e.g. $S = S_0[S_1]$ by (3.117)).

(i) $\mathcal{O}_X(n)$ is an invertible sheaf, and $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \cong \mathcal{O}_X(n+m)$.

(ii) For any graded S-module M, $\widetilde{M}(n) \cong \widetilde{M(n)}$.

Proof. These follows from (3.118.5). □

3.120 Let $\varphi : S \to T$ be a homomorphism of graded rings. Let $U = G(\varphi)$ and $f = \operatorname{Proj} \varphi : U \to \operatorname{Proj} S$ be the morphism associated with φ (3.111). We have an analog of Lemma 3.14, namely

(i) $f^*\widetilde{M} \cong \left(\widetilde{M \otimes_S T}\right)|_U$ for any graded S-module M.

(ii) $f_*(\widetilde{N}|_U) \cong \widetilde{N^{[\varphi]}}$ for any graded T-module N.

In particular, if we put $X = \operatorname{Proj} S$ and $Y = \operatorname{Proj} T$, this shows that $f^*(\mathcal{O}_X(n)) \cong \mathcal{O}_Y(n)|_U$ and $f_*(\mathcal{O}_Y(n)|_U) \cong (f_*(\mathcal{O}_X|_U))(n)$.

Proof. Let $g \in S_+$ be homogeneous. Then $f|^{D_+(g)}_{D_+(\varphi(g))} : D_+(\varphi(g)) \to D_+(g)$ is the spec of the ring homomorphism $S_{(g)} \to T_{\varphi(g)}$. By Lemma 3.14 we have

$$(f^*\widetilde{M})|_{D_+(\varphi(g))} = (f|^{D_+(g)}_{D_+(\varphi(g))})^*(\widetilde{M}|_{D_+(g)}) \cong \widetilde{M_{(g)} \otimes_{S_{(g)}} T_{(\varphi(g))}} = \left(\widetilde{M \otimes_S T}\right)|_U(D_+(\varphi(g))).$$

where the tilde in the last second place is affine with respect to $T_{(\varphi(g))}$. Now (i) follows from (2.3.1). (ii) is proved similarly. □

3.121 Let A be a ring, B a graded A-algebra and C an A-algebra. We then have a fibre square

$$\begin{tikzcd}
\operatorname{Proj} B \times_{\operatorname{Spec} A} \operatorname{Spec} C \arrow[r, "g"] \arrow[d] & \operatorname{Proj} B \arrow[d] \\
\operatorname{Spec} C \arrow[r] & \operatorname{Spec} A
\end{tikzcd}$$

By (3.112) there is a canonical isomorphism

$$\operatorname{Proj}(B \otimes_A C) \xrightarrow{\sim} \operatorname{Proj} B \times_{\operatorname{Spec} A} \operatorname{Spec} C.$$

In fact, it gives an isomorphism

$$(\operatorname{Proj}(B \otimes_A C), \mathcal{O}_{\operatorname{Proj} B \otimes_A C}(n)) \xrightarrow{\sim} (\operatorname{Proj} B \times_{\operatorname{Spec} A} \operatorname{Spec} C, g^*\mathcal{O}_{\operatorname{Proj} B}(n))$$

in $\mathbf{Mod_{LRS}}$ for each $n \in \mathbb{Z}$.

3.122 Quasi-coherent sheaves on Proj Let S be a graded ring, $X = \operatorname{Proj} S$ and $\mathcal{F}$ be an $\mathcal{O}_X$-module. Define

$$\Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n))$$

We already see this S-graded module in (3.27).

3.122.1 Example-Lemma. Let A be a ring, $n \in \mathbb{Z}_{\geqslant 1}$, $S = A[x_0, \dots, x_n]$ and $X = \operatorname{Proj} S$. The map $S \to \Gamma_*(\mathcal{O}_X)$ in (3.118) is an isomorphism.

Proof. Cover X be the affine open $D_+(x_i)$ (3.117). For any $m \in \mathbb{Z}$, by (3.110) we have a commutative diagram with the first row exact

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X(m)(X) \arrow[r] & \prod_{i=0}^{n} \mathcal{O}_X(m)(D_+(x_i)) \arrow[r] \arrow[d, "\wr"] & \prod_{i,j=0}^{n} \mathcal{O}_X(m)(D_+(x_i x_j)) \arrow[d, "\wr"] \\
& & \prod_{i=0}^{n} S(m)_{(x_i)} \arrow[r] \arrow[d, equal] & \prod_{i,j=0}^{n} S(m)_{(x_i x_j)} \arrow[d, equal] \\
& & \prod_{i=0}^{n} S_{x_i}(m) \arrow[r] & \prod_{i,j=0}^{n} S_{x_i x_j}(m)
\end{tikzcd}$$

Since the x_i are non-zero divisors in S, the localization maps $S \to S_{x_i}$ and $S_{x_i} \to S_{x_i x_j}$ are injective, and these rings can be viewed as subrings of $S_{x_0 \cdots x_n}$. Hence the above diagram reads $\mathcal{O}_X(m)(X) = \bigcap_{i=0}^{n} S_{x_i}(m)$, and thus by degree consideration

$$\Gamma_*(\mathcal{O}_X) = \bigcap_{i=0}^{n} S_{x_i} \text{ in } S_{x_0 \cdots x_n}.$$

It remains to show this intersection is precisely S. An element in $S_{x_0 \cdots x_n}$ can be written uniquely as $x_0^{\alpha_0} \cdots x_n^{\alpha_n} f(x_0, \dots, x_n)$ with $\alpha_j \in \mathbb{Z}$ and $f \in S$ homogeneous. Such an element lies in S_{x_i} if and only if $\alpha_j \geqslant 0$ for any $j \neq i$. Now the result follows. $\square$

3.123 When $\mathcal{F}$ is quasi-coherent, there is a natural homomorphism

$$\widetilde{\Gamma_*(\mathcal{F})} \longrightarrow \mathcal{F}$$

which we now describe. Let $f \in S_d$. Recall that by (3.110), there is an isomorphism

$$(D_+(f), \widetilde{\Gamma_*(\mathcal{F})}|_{D_+(f)}) \longrightarrow (\operatorname{Spec} S_{(f)}, \widetilde{\Gamma_*(\mathcal{F})_{(f)}}).$$

Define

$$\Gamma_*(\mathcal{F})_{(f)} \longrightarrow \mathcal{F}(D_+(f))$$

$$m/f^n \longrightarrow m|_{D_+(f)} \otimes (f|_{D_+(f)})^{-n}$$

Here we implicit use the isomorphism in (3.118.5) (perhaps also (3.118.1)). Since $\mathcal{F}$ is quasi-coherent and $D_+(f)$ is affine, by (3.15.1) this gives a homomorphism

$$\widetilde{\Gamma_*(\mathcal{F})}|_{D_+(f)} \longrightarrow \mathcal{F}|_{D_+(f)}.$$

Since all maps in (3.118.5) are functorial, the homomorphisms above, when $f \in S_d$, $d > 0$ vary, glue, yielding

$$\widetilde{\Gamma_*(\mathcal{F})} \longrightarrow \mathcal{F}.$$

3.123.1 Example. We claim $\widetilde{\Gamma_*\mathcal{O}_{\operatorname{Proj} S}} \to \mathcal{O}_{\operatorname{Proj} S}$ is right inverse to the tilde of the map $S \to \Gamma_*\mathcal{O}_{\operatorname{Proj} S}$ defined in (3.118). Indeed, for homogeneous $f \in S_+$ the former map is

$$\begin{array}{ccccc} \Gamma_*(\mathcal{O}_{\operatorname{Proj} S})_{(f)} & \longrightarrow & \mathcal{O}_{\operatorname{Proj} S}(D_+(f)) & = & S_{(f)} \\ m/f^n & \longrightarrow & m|_{D_+(f)} \otimes (f|_{D_+(f)})^{-n} & \longmapsto & m|_{D_+(f)}/f^n \end{array}$$

while the latter map

$$S_{(f)} \longrightarrow \Gamma_*(\mathcal{O}_{\operatorname{Proj} S})_{(f)}$$

is given by sending s/f^n to s'/f^n, where s' is the image of s under $S_{n \deg f} = S(n \deg f)_0 \to \Gamma(X, \widetilde{S(n \deg f)})$ given in (3.110.3). By construction, $m|_{D_+(f)}/f^n$ is mapped to m/f^n by the latter map, which proves our claim.

3.123.2 When $X = \operatorname{Proj} S$ is compact, we contend that

$$\widetilde{\Gamma_*(\mathcal{F})} \longrightarrow \mathcal{F}.$$

is an isomorphism. Since X is compact, we can find homogeneous elements $f_1, \ldots, f_n$ of positive degree such that $X = \bigcup_{i=1}^{n} D_+(f_i)$ (3.106). Let d be the least common multiple of those degrees; by raising f_i to suitable power, which does not alter $D_+(f_i)$, we may assume all f_i have the same degree d. From the discussions in (3.118) and its subparagraphs, we see $\mathcal{O}_X(d)$ is an invertible sheaf such that the multiplication maps

$$\mathcal{O}_X(ad) \otimes_{\mathcal{O}_X} \mathcal{O}_X(bd) \longrightarrow \mathcal{O}_X((a+b)d)$$

are isomorphisms for any $a, b \in \mathbb{Z}$. Recall the map $S_d \to \Gamma(X, \mathcal{O}_X(d))$ stated in (3.118). Denote by s_i the image of f_i in $\Gamma(X, \mathcal{O}_X(d))$. By construction (3.110.3), one has (by restricting to the affine pieces)

$$D_+(f_i) = X_{s_i} = \{p \in X \mid s_i \notin \mathfrak{m}_p \mathcal{O}_X(d)_p\}.$$

Now we have

$$\begin{aligned} \Gamma_*(\mathcal{F})_{(f_i)} &= \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n))_{(f_i)} \cong \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(nd))_{(f_i)} \\ &= \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d)^{\otimes n})_{(f_i)} \overset{(3.27)}{=} \Gamma_*(\mathcal{F}, \mathcal{O}_X(d))(X)_{(f_i)} \\ &= \Gamma_*(\mathcal{F}, \mathcal{O}_X(d))(X)_{(s_i)} \\ &\overset{(3.27.4)}{\cong} \mathcal{F}(X_{s_i}) = \mathcal{F}(D_+(f_i)) \end{aligned}$$

The unlabelled isomorphism in the first line is given by multiplication by f_i, as f_i has degree d, and the equality in the third line results from the definition of S-action (3.118.3). Unwinding all homomorphisms, one easily checks the above isomorphism is precisely the homomorphism $\Gamma_*(\mathcal{F})_{(f)} \to \mathcal{F}(D_+(f))$ defined in (3.122). This finishes the proof.

3.124 Let S be a graded ring and $X = \operatorname{Proj} S$. Let $\mathcal{F}$, $\mathcal{G}$ be two $\mathcal{O}_X$-modules. For any $n \in \mathbb{Z}$, by twisting and taking global section, to each homomorphism $\mathcal{F} \to \mathcal{G}$ we may construct homomorphisms $\Gamma(X, \mathcal{F}(n)) \to \Gamma(X, \mathcal{G}(n))$. Taking direct sum then yields

$$\Gamma_*(\mathcal{F}) \longrightarrow \Gamma_*(\mathcal{G}).$$

The construction being natural, this means Γ_* really defines a functor

$$\Gamma_* : \mathbf{Mod}_{\mathcal{O}_X} \longrightarrow \mathbf{GrMod}_{\Gamma_*(S)}.$$

The graded ring homomorphism $S \to \Gamma_*(S)$ in (3.118.3) yields a forgetful functor $\omega : \mathbf{GrMod}_{\Gamma_*(S)} \to \mathbf{GrMod}_S$, so we also have a functor

$$\omega \circ \Gamma_* : \mathbf{Mod}_{\mathcal{O}_X} \longrightarrow \mathbf{GrMod}_S.$$

Since localization and restrictions to open sets are exact, we see $\widetilde{(\cdot)}$ is an exact functor.

3.124.1 Adjunction. Let M, N be two graded S-modules, and let $g_1, \ldots, g_\ell \in S_+$ be homogeneous such that $S = \bigcup_{i=1}^{\ell} D_+(g_i)$. Suppose $\{\varphi_i : M_{(g_i)} \to N_{(g_i)}\}_{i=1}^{n}$ be a *compatible* set of homomorphisms, in the sense that $(\varphi_i)_{g_i g_j} = (\varphi_j)_{g_j g_i}$ for any $i, j = 1, \ldots, \ell$. Then they join the exact sequences in (3.109) for M, N, i.e.,

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & \prod_{i=1}^{\ell} M_{(g_i)} \arrow[r] \arrow[d, "\prod_i \varphi_i"] & \prod_{i,j=1}^{\ell} M_{(g_i g_j)} \arrow[d, "\prod_{i,j} (\varphi_i)_{g_i g_j}"] \\
0 \arrow[r] & N \arrow[r] & \prod_{i=1}^{\ell} N_{(g_i)} \arrow[r] & \prod_{i,j=1}^{\ell} N_{(g_i g_j)}.
\end{tikzcd}$$

Hence there exists a unique graded S-module homomorphism $\varphi : M \to N$ such that $\varphi_{g_i} = \varphi_i$ for $i = 1, \ldots, \ell$. This defines a bijection from $\mathrm{Hom}_{\mathbf{GrMod}_S}(M, N)$ to the set

$$\left\{ (\varphi_f)_f \in \prod_{\substack{f \in S_+: \\ \text{homogeneous}}} \mathrm{Hom}_{\mathbf{Mod}_S}(M_{(f)}, N_{(f)}) \mid (\varphi_f)_g = \varphi_g \text{ for all homogeneous } f, g \in S_+ \text{ with } D_+(g) \subseteq D_+(f) \right\}.$$

But by construction, such compatible homomorphism $(\varphi_f)_f$ defines an $\mathcal{O}_X$-module homomorphism $\widetilde{M} \to \tilde{N}$, and vice versa. Hence there is a natural bijection

$$\mathrm{Hom}_{\mathbf{GrMod}_S}(M, N) \longrightarrow \mathrm{Hom}_{\mathbf{Mod}_{\mathcal{O}_X}}(\widetilde{M}, \tilde{N}).$$

This means the projective tilde $\widetilde{(\cdot)} : \mathbf{GrMod}_S \to \mathbf{Qcoh}_X$ is a fully faithful functor.

Now assume X is compact. If $\mathcal{F}$ is quasi-coherent, then (3.123.2) says that there is an natural isomorphism $\widetilde{\omega\Gamma_*(\mathcal{F})} \xrightarrow{\sim} \mathcal{F}$. Hence there is a bijection

$$\mathrm{Hom}_{\mathbf{GrMod}_S}(M, \omega\Gamma_*(\mathcal{F})) \longrightarrow \mathrm{Hom}_{\mathbf{Mod}_{\mathcal{O}_X}}(\widetilde{M}, \widetilde{\omega\Gamma_*(\mathcal{F})}) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{Qcoh}_X}(\widetilde{M}, \mathcal{F}).$$

functorial in M and $\mathcal{F}$. Equivalently, this means $\omega\Gamma_*$ is right adjoint to the $\widetilde{(\cdot)}$, when $X = \operatorname{Proj} S$ is compact. In particular, $\omega\Gamma_*$ and Γ_* are left exact.

3.125 Closed subschemes of Proj S (II). Let S be a graded ring, $X = \operatorname{Proj} S$ and let $j : Z \to X$ be a closed subscheme. Let $\mathcal{I} := \ker(\mathcal{O}_X \to j_*\mathcal{O}_Z)$. We then have an exact sequence

$$0 \longrightarrow \Gamma_*(\mathcal{I}) \longrightarrow \Gamma_*(\mathcal{O}_X) \longrightarrow \Gamma_*(j_*\mathcal{O}_Z).$$

By (3.123.1) and (3.123.2), the tilde of the natural map $S \to \Gamma_*(\mathcal{O}_X)$ is an isomorphism. Let $I \trianglelefteq S$ be the homogeneous ideal fitting into the pullback diagram

$$\begin{tikzcd}
\Gamma_*(\mathcal{I}) \arrow[r, hook] & \Gamma_*(\mathcal{O}_X) \\
I \arrow[r, hook] \arrow[u] & S \arrow[u]
\end{tikzcd}.$$

Taking tilde we obtain an isomorphism $\widetilde{I} \cong \widetilde{\Gamma_*(\mathcal{I})}$. By (3.123.2) again we deduce that $\widetilde{I} \cong \mathcal{I}$. This recovers (3.115.1); moreover, the ideal I is given explicitly by the kernel of the canonical map

$$S \longrightarrow \Gamma_*(\mathcal{O}_X) \longrightarrow \Gamma_*(j_*\mathcal{O}_Z) = \bigoplus_{n \geqslant 0} \Gamma(X, (j_*\mathcal{O}_Z)(n)).$$

3.126 Theorem (Serre). Let S be a graded ring and suppose $X = \operatorname{Proj} S$ is compact. For a finitely generated quasi-coherent sheaf $\mathcal{F}$ on X, there exists $n_0 \in \mathbb{Z}_{\geqslant 0}$ and $d \in \mathbb{Z}_{>0}$ such that for all $n \geqslant n_0$, the sheaf $\mathcal{F}(nd)$ is generated by finitely many global sections. If $S = S_0[S_1]$, then d can be chosen as 1.

Proof. As in (3.123.2) take homogeneous elements $f_1, \ldots, f_n$ of positive degree d such that $X = \bigcup_{i=1}^{n} D_+(f_i)$; we can take $d = 1$ if $S = S_0[S_1]$. By (3.15.1), $\mathcal{F}(D_+(f_i))$ is finite over $\mathcal{O}_X(D_+(f_i))$; let $\{s_{ij} \in \mathcal{F}(D_+(f_i))\}_{j \in [m_i]} \subseteq \mathcal{F}(D_+(f_i))$ be a finite generating set. By Lemma 3.27.3, applied to $\mathcal{L} = \mathcal{O}_X(1)$, there exists n_0 such that $f_i^{n_0} s_{ij} \in \Gamma(X, \mathcal{F}(n_0 d))$ for $i \in [n]$ and $j \in [m_i]$. Then for $n \geqslant n_0$, the set $\{f_i^n s_{ij}\} \subseteq \Gamma(X, \mathcal{F}(nd))$ generates $\mathcal{F}(nd)$. □

3.126.1 Corollary Let S be a graded ring and suppose $X = \operatorname{Proj} S$ is compact If $\mathcal{F}$ is a finitely generated quasi-coherent sheaf on X, then there exists a finite S-submodule N of $\Gamma_*(\mathcal{F})$ such that $\mathcal{F} \cong \widetilde{N}$.

Proof. By Theorem 3.126 there exists some $n \in \mathbb{Z}_{\geqslant 0}$ such that $\mathcal{F}(n)$ is generated by finitely many global sections. Let N be the S-submodule of $\Gamma_*(\mathcal{F})$ generated by these sections. Then $N \subseteq \Gamma_*(\mathcal{F})$ induces an injection $\widetilde{N} \hookrightarrow \widetilde{\Gamma_*(\mathcal{F})} \cong \mathcal{F}$. Twisting by n gives $\widetilde{N}(n) \hookrightarrow \mathcal{F}(n)$, which is an isomorphism by construction. Twisting back gives $\widetilde{N} \cong \mathcal{F}$. □

3.5.2 Projective spaces

3.127 Let A be a ring. For an integer $n \geqslant 1$, we can form the polynomial ring $S = A[x_0, x_1, \ldots, x_n]$ with n-variables and coefficient in A. If we view each variable x_i as a degree one element in S, it naturally becomes a $\mathbb{Z}_{\geqslant 0}$-graded ring generated by all degree one elements. The scheme

$$\mathbb{P}^n_A = \operatorname{Proj} A[x_0, x_1, \ldots, x_n]$$

is called the n**-dimensional projective space over** A. By (3.117), we have

$$\mathbb{P}^n_A = D_+(x_0) \cup D_+(x_1) \cup \cdots \cup D_+(x_n)$$

and each $D_+(x_i)$ is isomorphic to the affine scheme (3.110)

$$(D_+(x_i), \mathcal{O}_{\mathbb{P}^n_A}|_{D_+(x_i)}) \cong (\operatorname{Spec} A[x_0, \ldots, x_n]_{x_i}, \mathcal{O}_{\operatorname{Spec} A[x_0, \ldots, x_n]_{x_i}})$$

The variables $x_0, \ldots, x_n$ are called the **homogeneous coordinates** of $\mathbb{P}^n_A$.

3.127.1 Let k be a field and V a finite dimensional k-vector space. Define

$$\mathbb{P}(V) := \operatorname{Proj} \operatorname{Sym} V^\vee.$$

Upon picking a k-basis for V, we can identify $\operatorname{Sym} V^\vee$ with the polynomial ring over k with $\dim_k V$ variables, so $\mathbb{P}(V) \cong \mathbb{P}_k^{\dim_k V - 1}$ non-canonically. The scheme $\mathbb{P}(V)$ is called the **projectivization** of V. (compare with (3.59).)

3.128 Definition. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ be a $\mathcal{O}_X$-module.

1. $\mathcal{F}$ is **generated by global sections at** $x \in X$ if $\mathcal{F}(X) \otimes_{\mathcal{O}_X(X)} \mathcal{O}_{X,x} \to \mathcal{F}_x$ is surjective.

2. $\mathcal{F}$ is **generated by global sections** if it is so at every point of X.
3. Let $S \subseteq \mathcal{F}(X)$ be a subset. $\mathcal{F}$ is **generated by** S if $S \otimes_{\mathcal{O}_X(X)} \mathcal{O}_{X,x} \to \mathcal{F}_x$ is surjective for any $x \in X$.
4. $\mathcal{F}$ is **finitely generated at** $x \in X$ if there exists an open neighborhood U of x, an integer $n \geqslant 0$, and an exact sequence

$$(\mathcal{O}_X|_U)^n \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

3.129 Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. If $\mathcal{F}$ is an $\mathcal{O}_Y$-module, by adjunction applied to the identity $f^*\mathcal{F} \to f^*\mathcal{F}$, we obtain a canonical map $\mathcal{F} \to f_*f^*\mathcal{F}$; by abuse of notation, we denote this morphism by

$$f^* : \mathcal{F} \to f_*f^*\mathcal{F}.$$

Now for a section $s \in \mathcal{F}(U)$, its image $f^*s \in (f_*f^*\mathcal{F})(V) = (f^*\mathcal{F})(f^{-1}(V))$ is called the **pullback section** of s along f. In particular, if $s \in \mathcal{F}(Y)$ is a global section, then $f^*s \in (f^*\mathcal{F})(X)$ is a global section.

A careful computation using the construction in (2.10) and (2.12) shows that if $s \in \mathcal{F}(Y)$, then

$$(f^*s)|_x = s|_y \otimes 1 \in \mathcal{F}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \cong (f^*\mathcal{F})_x$$

for any $x \in X$ and $y = f(x) \in Y$. In particular, this implies $X_{f^*s} = f^{-1}(Y_s)$. Furthermore, we have

$$f^*s = \varinjlim_U s|_U \otimes 1 \in (f^*\mathcal{F})(X)$$

3.130 The homogeneous coordinates $x_0, \ldots, x_n$ of $\mathbb{P}^n_A$, by (3.122.1), give rise to elements in $\mathcal{O}_{\mathbb{P}^n_A}(1)(\mathbb{P}^n_A)$. In fact, the sheaf $\mathcal{O}_{\mathbb{P}^n_A}(1)$ is generated by the x_i's. To see this, note that $\mathcal{O}_{\mathbb{P}^n_A}(1)(D_+(x_i)) = S(1)_{(x_i)} = x_i S_{(x_i)}$ is a free $S_{(x_i)}$-module of rank 1. Then for any $p = \mathfrak{p} \in \operatorname{Spec} S_{(x_i)} \cong D_+(x_i)$, the stalk at p is $x_i(S_{(x_i)})_{\mathfrak{p}} = x_i \mathcal{O}_{\mathbb{P}^n_A,p}$, which is generated by x_i over $\mathcal{O}_{\mathbb{P}^n_A,p}$, as said.

3.130.1 Let A be a ring and X be a scheme over A. Let $f : X \to \mathbb{P}^n_A$ be a morphism. By (3.26.1), the pullback sheaf $f^*\mathcal{O}_{\mathbb{P}^n_A}(1)$ is an invertible sheaf; let us put $\mathcal{L} = f^*(\mathcal{O}_{\mathbb{P}^n_A}(1))$. For $0 \leqslant i \leqslant n$, let $s_i := f^*x_i \in \mathcal{L}(X)$ (3.129). The last assertion in (3.129) implies that the s_i's generate the sheaf $\mathcal{L}$. In fact, the datum $(\mathcal{L}, s_0, \ldots, s_n)$ characterizes the morphism f.

3.130.2 Theorem. Let A be a ring and X be a scheme over A.

(i) If $f : X \to \mathbb{P}^n_A$ is an A-morphism, then $f^*(\mathcal{O}_{\mathbb{P}^n_A}(1))$ is an invertible sheaf on X generated by the global sections f^*x_i $(0 \leqslant i \leqslant n)$.

(ii) If $\mathcal{L}$ is an invertible sheaf on X generated by the global sections $s_0, \ldots, s_n \in \mathcal{L}(X)$, then there exists a unique A-morphism $f : X \to \mathbb{P}^n_A$ with $\mathcal{L} \cong f^*(\mathcal{O}_{\mathbb{P}^n_A}(1))$ as $\mathcal{O}_X$-modules and $s_i = f^*x_i$ $(0 \leqslant i \leqslant n)$ under this isomorphism.

Proof. It remains to prove (ii). Let $X_i = X^{\mathcal{L}}_{s_i}$ be the open set defined as in (3.27). Since the s_i's generate $\mathcal{L}$, it follows at once that $(X_i)_{i=0}^n$ covers X. Define the morphism $f : X \to \mathbb{P}^n_A$ as follows. Define an A-algebra homomorphism

$$\begin{aligned} A_i := A\left[\frac{x_0}{x_i}, \ldots \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right] &\longrightarrow \mathcal{O}_X(X_i) \\ \frac{x_j}{x_i} &\longmapsto \frac{s_j}{s_i} \end{aligned}$$

This is well-defined as on X_i, there exists $s_i^{-1} \in \mathcal{L}^\vee(X_i)$ and so $s_j s_i^{-1} := (s_j|_{X_i}) \otimes s_i^{-1} \in (\mathcal{L} \otimes \mathcal{L}^\vee)(X_i) \cong \mathcal{O}_X(X_i)$ (3.27). By (3.56), this gives an A-morphism $f_i : X_i \to D_+(x_i) \subseteq \mathbb{P}^n_A$. These morphisms glue for an obvious reason, so we obtain a global A-morphism $f : X \to \mathbb{P}^n_A$. By construction and (3.129), we have

$$(f^*(\mathcal{O}_{\mathbb{P}^n_A}(1)))|_{X_i} = f_i^*(\mathcal{O}_{\mathbb{P}^n_A}(1))|_{D_+(x_i)}) \cong f_i^*(x_i \mathcal{O}_{\mathbb{P}^n_A}|_{D_+(x_i)}) = (f^*x_i) f_i^*(\mathcal{O}_{\mathbb{P}^n_A}|_{D_+(x_i)}) = (f^*x_i)\mathcal{O}_X|_{X_i}$$

On the other hand, we have $\mathcal{L}|_{X_i} \cong s_i\mathcal{O}_X|_{X_i}$ (see the fourth line of the proof). By viewing $\mathcal{O}_{\mathbb{P}^n_A}(1)|_{X_i\cap X_j}$ as subsheaves of $\mathcal{O}_{\mathbb{P}^n_A}(1)|_{D_+x_i}$ and $\mathcal{O}_{\mathbb{P}^n_A}(1)|_{D_+(x_j)}$ respectively, we obtain a *transition map*

$$\begin{aligned} \frac{x_i}{x_j} : x_i\mathcal{O}_{\mathbb{P}^n_A}|_{D_+(x_ix_j)} &\longrightarrow x_j\mathcal{O}_{\mathbb{P}^n_A}|_{D_+(x_ix_j)} \\ x_i &\longmapsto x_j \end{aligned}$$

On the other hand, the transition map for $\mathcal{L}|_{X_i}$ and $\mathcal{L}|_{X_j}$ is $\dfrac{s_i}{s_j} : s_i\mathcal{O}_X|_{X_i\cap X_j} \to s_j\mathcal{O}_X|_{X_i\cap X_j}$. This implies the isomorphisms $(f^*x_i)\mathcal{O}_X|_{X_i} \to s_i\mathcal{O}_X|_{X_i}$ defined by $f^*x_i \mapsto s_i$ glue to an isomorphism $f^*(\mathcal{O}_{\mathbb{P}^n_A}(1)) \cong \mathcal{L}$.

For the uniqueness, let $g : X \to \mathbb{P}^n_A$ be an A-morphism satisfying (ii). By (3.129), the morphism $g_i = g|_{X_i}^{D_+(x_i)}$ is well-defined. It suffices to show the corresponding homomorphism $\theta_i : A_i \to \mathcal{O}_X(X_i)$ satisfies $\theta_i(x_jx_i^{-1}) = s_js_i^{-1}$. Restricting, we have $\mathcal{L}|_{X_i} \cong (g^*\mathcal{O}_{\mathbb{P}^n}(1))|_{X_i} = g_i^*(\mathcal{O}_{\mathbb{P}^n_A}(1)|_{D_+(x_i)})$, and obtain an $\mathcal{O}_{\mathbb{P}^n_A}|_{D_+(x_i)}$-module morphism $G : \mathcal{O}_{\mathbb{P}^n_A}(1)|_{D_+(x_i)} \to (g_i)_*\mathcal{L}|_{X_i}$. Recall in (2.11) the latter is viewed as an $\mathcal{O}_{\mathbb{P}^n_A}|_{D_+(x_i)}$-module via the homomorphism θ_i, so

$$G_{D_+(X_i)}(x_j) = G_{D_+(X_i)}\left(x_i \cdot \frac{x_j}{x_i}\right) = \theta_i\left(\frac{x_j}{x_i}\right) G_{D_+(X_i)}(x_i).$$

Our claim will follow once we prove $G_{D_+(X_i)}(x_j) = f^*x_i \in \mathcal{L}(X_i)$ for any $0 \leqslant j \leqslant n$. This is clear from the definition (3.129). □

3.131 Functor of projective spaces. Let A be a ring and R an A-algebra. The last theorem describes the R-points of the projective n-space $\mathbb{P}^n_A$. Precisely, there is a bijection

$$\mathbb{P}^n_A(R) \cong \left\{(L, R^{n+1} \twoheadrightarrow L) \mid L \text{ is an } R\text{-module of locally free of rank } 1\right\} / \sim$$

where $\sim$ is an equivalence relation : two pairs $(L, R^{n+1} \twoheadrightarrow L)$ and $(M, R^{n+1} \twoheadrightarrow M)$ are equivalent if there is an R-isomorphism $L \to M$ fitting into the commutative diagram

$$\begin{tikzcd} R^{n+1} \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d, "\wr"] & 0 \\ R^{n+1} \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

with exact rows.

If $R \to S$ is a ring homomorphism, the resulting map $\mathbb{P}^n_A(R) \to \mathbb{P}^n_A(S)$ is given by tensoring S :

$$\begin{aligned} \mathbb{P}^n_A(R) &\longrightarrow \mathbb{P}^n_A(S) \\ (L, R^{n+1} \twoheadrightarrow L) &\longmapsto (L \otimes_R S, S^{n+1} \twoheadrightarrow L \otimes_R S). \end{aligned}$$

This can be seen by (3.130.2).(i). Note this is well-defined as tensor is right exact.

3.132 Example : projection from puncture affine space to projective space. Let A be a ring, and put

$$\mathbb{A}^{n+1}_A\backslash\{0\} := D(t_0, \dots, t_n) = \mathbb{A}^{n+1}_A\backslash V(t_0, \dots, t_n).$$

The structure sheaf $\mathcal{O}_{\mathbb{A}^{n+1}_A}$ is generated by the global section $t_0, \dots, t_n$ at each point except along $V(t_0, \dots, t_n)$. Indeed, each t_i acts a unit on $D(t_i)$, so this follows from the fact that $\mathbb{A}^{n+1}_A\backslash\{0\} = \bigcup\limits_{i=0}^{n} D(t_i)$. Hence by (3.130.2).(ii), there is a unique A-morphism

$$\pi : \mathbb{A}^{n+1}_A\backslash\{0\} \longrightarrow \mathbb{P}^n_A = \operatorname{Proj} A[x_0, \dots, x_n]$$

such that $\pi^*\mathcal{O}_{\mathbb{P}^n_A}(1) \cong \mathcal{O}_{\mathbb{A}^{n+1}_A\setminus\{0\}}$ and $t_i = \pi^* x_i$ for $0 \leqslant i \leqslant n$. By (3.26.1) it follows that

$$\pi^*\mathcal{O}_{\mathbb{P}^n_A}(m) \cong \mathcal{O}_{\mathbb{A}^{n+1}_A\setminus\{0\}}$$

for all $m \in \mathbb{Z}$. This will be useful in the computation of cohomology of $\mathcal{O}(m)$'s.

3.133 Example : Segre embedding Let A be a ring and $n, m \in \mathbb{Z}_{\geqslant 0}$. Consider the projections

$$\begin{tikzcd} & \mathbb{P}^n_A \times_{\operatorname{Spec} A} \mathbb{P}^m_A \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ \mathbb{P}^n_A & & \mathbb{P}^m_A \end{tikzcd}$$

and the invertible sheaf $\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^n_A}(1) \otimes \mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^m_A}(1)$ on the product $\mathbb{P}^n_A \times_{\operatorname{Spec} A} \mathbb{P}^m_A$. We will see in (8.21) that this invertible sheaf is generated by $\{x_i y_j\}_{0\leqslant i\leqslant n, 0\leqslant j\leqslant m}$, so by (3.130.2) it determines an A-morphism

$$\varphi : \mathbb{P}^n_A \times_{\operatorname{Spec} A} \mathbb{P}^m_A \longrightarrow \mathbb{P}^{mn+m+n}_A.$$

We claim this is an closed embedding. Say $z_{00}, \ldots, z_{ij}, \ldots, z_{nm}$ is the coordinates of $\mathbb{P}^{mn+m+n}_A$ and $\varphi^* z_{ij} = x_i y_j$. By construction, on $D_+(z_{ij})$ the map φ is given by the spec of

$$\begin{aligned} A[z_{k\ell} z_{ij}^{-1}] &\longrightarrow \mathcal{O}_X(D_+(x_i y_j)) = A[x_k x_i^{-1}] \otimes_A A[y_\ell y_j^{-1}] \\ z_{k\ell} z_{ij}^{-1} &\longmapsto x_k x_i^{-1} \otimes y_\ell y_j^{-1}. \end{aligned}$$

Since this is surjective, φ is a closed embedding over $D_+(z_{ij})$. Hence φ is a closed embedding. This is called the **Segre embedding**.

3.133.1 Lemma. The image of φ is given by the V_+ of all the 2×2 minors of $(z_{ij})_{0\leqslant i\leqslant n, 0\leqslant j\leqslant m}$.

Proof. Suppose $f \in A[z_{ij}]_{\mathrm{hom}}$ satisfies $f(x_i y_j) = 0$. We must show f lies in the ideal generated by all the 2×2 minors of $(z_{ij})_{0\leqslant i\leqslant n, 0\leqslant j\leqslant m}$. This follows from a simple induction. □

3.134 Example : Veronese embedding Let A be a ring, $n \in \mathbb{Z}_{\geqslant 0}$ and $m \in \mathbb{Z}_{\geqslant 1}$. We will see in (8.21) that the invertible sheaf $\mathcal{O}_{\mathbb{P}^n_A}(m)$ is generated by global sections which are $A[x_0, \ldots, x_n]_{\deg = m}$. By (3.130.2) this defines an A-morphism

$$\psi : \mathbb{P}^n_A \longrightarrow \mathbb{P}^{\binom{n+m}{m}-1}_A.$$

We claim this is a closed embedding. For $\alpha = (\alpha_0, \ldots, \alpha_n) \in (\mathbb{Z}_{\geqslant 0})^{n+1}$, write $x^\alpha = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$. Let $(z_\alpha)_{|\alpha|=m}$ be the coordinates of $\mathbb{P}^{\binom{n+m}{m}-1}_A$ and $\psi^* z_\alpha = x^\alpha$. On $D_+(z_\alpha)$ the map ψ is given by the spec of

$$\begin{aligned} A[z_\beta z_\alpha^{-1}] &\longrightarrow \mathcal{O}_{\mathbb{P}^n_A}(D_+(x^\alpha)) \\ z_\beta z_\alpha^{-1} &\longmapsto x^\beta (x^\alpha)^{-1}. \end{aligned}$$

Here

$$\mathcal{O}_{\mathbb{P}^n_A}(D_+(x^\alpha)) = A[x_0, \ldots, x_n]_{(x^\alpha)} = A[x^\beta (x^\alpha)^{-1}]$$

so this ring homomorphism is surjective. Hence ψ is a closed embedding. This is called the **degree m Veronese embedding.**

3.135 Nakayama's lemma Let $\mathcal{F}$ be a finitely generated quasi-coherent sheaf over a scheme X. If $x \in X$ and the images of $s_1, \dots, x_n \in \mathcal{F}_x$ in $\mathcal{F}_x \otimes \kappa(x)$ generate the $\kappa(x)$-vector space $\mathcal{F}_x \otimes \kappa(x)$, then there exists an open neighborhood U of x in X such that the s_i extends to U and define a surjection

$$\begin{aligned} (\mathcal{O}_X|_U)^n &\longrightarrow \mathcal{F}|_U \\ (x_1, \dots, x_n) &\longmapsto x_1 s_1 + \cdots + x_n s_n. \end{aligned}$$

In other words, $s_1, \dots, s_n$ generates $\mathcal{F}$ on U.

Proof. This immediately reduces to the affine case : let A be a ring, $\mathfrak{p}$ a prime ideal and M a finite A-module. If $s_1, \dots, s_n \in M$ spans $M_\mathfrak{p} \otimes_{A_\mathfrak{p}} \kappa(\mathfrak{p})$ over $\kappa(\mathfrak{p})$, then there exists $f \in A \backslash \mathfrak{p}$ such that $s_1, \dots, s_n$ generate M_f over A_f. To show this, let $x_1, \dots, x_l$ be a generating set for M and write

$$x_i \equiv r_{i1} s_1 + \cdots r_{in} s_n \pmod{\mathfrak{p}}$$

with $r_{ij} \in A_\mathfrak{p}$, so clearing the denominators we see

$$a_i x_i = \sum_{j=1}^{n} b_{ij} s_j + \sum_{j=1}^{l} c_{ij} x_j$$

for some $a_i \in A \backslash \mathfrak{p}$, $b_{ij} \in A$ and $c_{ij} \in \mathfrak{p}$. In matrix form, we see

$$(a_i \delta_{ij} - c_{ij})_{ij} \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix} = (b_{ij}) \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}$$

Multiplying both sides by adj $(a_i \delta_{ij} - c_{ij})_{ij}$ and expanding, we obtain

$$f x_i = \sum_{j=1}^{n} d_{ij} s_j$$

where $f = \det(a_i \delta_{ij} - c_{ij})_{ij} \in A \backslash \mathfrak{p}$ and $d_{ij} \in A$. This f does the job. □

3.136 Let A be a ring and X be a scheme over A. Let $\mathcal{L}$ be an invertible sheaf over X and $s_0, \dots, s_n \in \mathcal{L}(X)$ be some global section. Put

$$U := \{x \in X \mid s_0|_x, \dots, s_n|_x \text{ generate the stalk } \mathcal{L}_x \text{ as } \mathcal{O}_{X,x}\text{-module}\}$$

This is an open subset of X by Nakayama's lemma (but possibly empty). By Theorem 3.130.2 applied to the A-scheme U and the invertible sheaf $\mathcal{L}|_U$ together with the sections $s_0|_U, \dots, s_n|_U \in \mathcal{L}(U)$, we obtain a unique A-morphism $g : U \to \mathbb{P}^n_A$ with $\mathcal{L}|_U = g^* \mathcal{O}_{\mathbb{P}^n_A}(1)$ and $s_i|_U = f^* x_i$.

3.137 Let us discuss on some consequences of Nakayama's lemma. For a quasi-coherent sheaf $\mathcal{F}$ on a scheme X and $x \in X$, define

$$\operatorname{rank}_x \mathcal{F} := \dim_{\kappa(x)} \left(\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x) \right) \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}.$$

We also recall that for a topological space X, a map $f : X \to \mathbb{R}$ is called **upper** (resp. **lower**) **semicontinuous** if for all $c \in \mathbb{R}$, the preimage $f^{-1}((-\infty, c))$ (resp. $f^{-1}((c, \infty))$) is open in X.

3.137.1 If $\mathcal{F}$ is a finitely generated quasi-coherent sheaf, then the rank function

$$\begin{aligned} \operatorname{rank}\mathcal{F} : X &\longrightarrow \mathbb{Z}_{\geqslant 0} \\ x &\longmapsto \operatorname{rank}_x \mathcal{F} \end{aligned}$$

is upper semicontinuous. Explicitly, we must show for $n \geqslant 0$, the set

$$\{x \in X \mid \operatorname{rank}_x \mathcal{F} \leqslant n\}$$

is an open set in X. This is clear, as by Nakayama's lemma, we see if $\mathcal{F}_x \otimes \kappa(x)$ has dimension n, then there is an open neighborhood U of x such that for each $y \in U$, the $\kappa(y)$-vector space $\mathcal{F}_y \otimes \kappa(y)$ has a generating set of size n. Hence $\operatorname{rank}_y \mathcal{F} \leqslant n$ for all $y \in U$. This finishes the proof.

3.137.2 If $\mathcal{F}$ is locally free (of finite rank), then trivially $\operatorname{rank}\mathcal{F} : X \to \mathbb{Z}_{\geqslant 0}$ is locally constant. Conversely, assume $\operatorname{rank}\mathcal{F}$ is locally constant. Say $n = \operatorname{rank}_x \mathcal{F}$ and U is an open neighborhood of x such that $n = \operatorname{rank}_y \mathcal{F}$. By Nakayama's lemma there is an surjection

$$T : (\mathcal{O}_X|_U)^n \longrightarrow \mathcal{F}|_U$$

The constancy condition on U implies that $T_y \otimes_{\mathcal{O}_{X,y}} \mathrm{id}_{\kappa(y)} : \kappa(y)^n \to \mathcal{F}_y \otimes \kappa(y)$ is an isomorphism for all $y \in U$. In particular, $(\ker T_y) = \ker T_y \subseteq \mathfrak{m}_{X,y}\mathcal{O}_{X,y}^n$ for each $y \in U$.

3.5.3 Grassmannian

3.138 Let k be a field and V a finite dimensional k-vector space. For $0 \leqslant m \leqslant n := \dim_k V$, the set

$$\operatorname{Gr}(m, V) := \left\{ W \underset{\text{subspace}}{\subseteq} V \mid \dim_k W = m \right\}$$

is called the **Grassmannian** of m-dimensional subspaces of V. There is an injection

$$\begin{aligned} \operatorname{Gr}(m, V) &\longrightarrow \operatorname{Gr}\left(1, \textstyle\bigwedge^m V\right) \\ W &\longmapsto \textstyle\bigwedge^m W \end{aligned}$$

3.6 Relative spec

3.139 Let X be a scheme and $\mathcal{R}$ a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. We are going to construct a scheme $\mathbf{Spec}_X \mathcal{R}$, called the **relative spectrum of $\mathcal{R}$ over** X, along with a morphism $\pi : \mathbf{Spec}_X \mathcal{R} \to X$ and an isomorphism $\mathcal{R} \cong \pi_* \mathcal{O}_{\mathbf{Spec}_X \mathcal{R}}$ of $\mathcal{O}_X$-algebras satisfying the following universal property : for any morphism $f : Y \to X$ of local-ringed spaces and a morphism $\alpha : \mathcal{R} \to f_* \mathcal{O}_Y$ of $\mathcal{O}_X$-algebras, there exists a unique morphism

$$g : Y \to \mathbf{Spec}_X \mathcal{R}$$

of schemes fitting into a commutative triangle

$$\begin{tikzcd} Y \arrow[rr, "g"] \arrow[dr, "f"'] & & \mathbf{Spec}_X \mathcal{R} \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

so that α factors as

$$\mathcal{R} \cong \pi_* \mathcal{O}_{\mathbf{Spec}_X \mathcal{R}} \xrightarrow{g_*} f_* \mathcal{O}_Y .$$

3.139.1 As an example, let X be a scheme and consider $\mathcal{O}_X$, which is itself a quasi-coherent $\mathcal{O}_X$-algebra. It is tautological that the morphism $\mathrm{id} : X \to X$ and the equality $\mathcal{O}_X = \mathcal{O}_X$ satisfies the universal property in (3.139). Hence $X = \mathbf{Spec}_X \mathcal{O}_X$ as schemes.

3.139.2 Let X and $\mathcal{R}$ be as in (3.139). For a local-ringed space Y, define

$$\begin{aligned} F(Y) &= \left\{ (f, \beta) \mid f \in \mathrm{Hom}_{\mathbf{LRS}}(Y, X),\ \beta \in \mathrm{Hom}_{\mathbf{Alg}_{\mathcal{O}_X}}(\mathcal{R}, f_*\mathcal{O}_Y) \right\} \\ &= \mathrm{Hom}_{\mathbf{Alg}_{\mathbf{LRS}}}((Y, \mathcal{O}_Y), (X, \mathcal{R})) \end{aligned}$$

If $Y \to Y'$ is a morphism in **LRS** by composition we obtain a map $F(Y') \to F(Y)$. Thus F defines a contravariant functor $F : \mathbf{LRS} \to \mathbf{Set}$. A part of (3.139) says that F is a representable functor that is represented by the scheme $\mathbf{Spec}_X \mathcal{R}$. If we fix some $f \in \mathrm{Hom}_{\mathbf{LRS}}(Y, X)$, we then obtain the isomorphism

$$\mathrm{Hom}_{\mathbf{Alg}_{\mathcal{O}_X}}(\mathcal{R}, f_*\mathcal{O}_Y) \cong \mathrm{Hom}_{\mathbf{LRS}_X}(Y, \mathbf{Spec}_X \mathcal{R}).$$

where $\mathbf{LRS}_X$ is the subcategory of **LRS** consisting of local-ringed spaces over X (obviously defined). We will see in (3.140) that this generalizes the isomorphism (3.7). In particular, if we take $Y = \mathbf{Spec}_X \mathcal{T}$ for some quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{T}$ and f to be the canonical morphism $\pi' : Y \to X$, we have

$$\mathrm{Hom}_{\mathbf{Alg}_{\mathcal{O}_X}}(\mathcal{R}, \mathcal{T}) \cong \mathrm{Hom}_{\mathbf{Alg}_{\mathcal{O}_X}}(\mathcal{R}, \pi'_*\mathcal{O}_{\mathbf{Spec}_X \mathcal{T}}) \cong \mathrm{Hom}_{\mathbf{Sch}_X}(\mathbf{Spec}_X \mathcal{T}, \mathbf{Spec}_X \mathcal{R})$$

By restricting to affine opens in X (3.143), this is the same as the anti-equivalence between affine schemes and commutative rings. From this we see $\mathbf{Spec}_X$ defines a fully faithful functor from the category of quasi-coherent $\mathcal{O}_X$-algebras to $\mathbf{Sch}_X$.

3.140 Affine case. We start our construction of $\mathbf{Spec}_X \mathcal{R}$ by first considering the case X being affine. We claim that

$$\mathbf{Spec}_X \mathcal{R} = \operatorname{Spec} \mathcal{R}(X)$$

does the job. Say $X = \operatorname{Spec} A$ and $\mathcal{R} = \widetilde{R}$ for some A-algebra R. By (3.7.1), for any local-ringed space Y

$$\mathrm{Hom}_{\mathbf{Alg}_{\mathbf{LRS}}}((Y, \mathcal{O}_Y), (X, \mathcal{R})) \cong \mathrm{Hom}_{\mathbf{Alg}}(A \rightharpoonup R, \mathcal{O}_Y(Y) \rightharpoonup \mathcal{O}_Y(Y))$$

Since R and $\mathcal{O}_Y(Y)$ are unital, the last set is simply $\mathrm{Hom}_{\mathbf{Ring}}(R, \mathcal{O}_Y(Y))$, and hence

$$F(Y) = \mathrm{Hom}_{\mathbf{Alg}_{\mathbf{LRS}}}((Y, \mathcal{O}_Y), (X, \mathcal{R})) \cong \mathrm{Hom}_{\mathbf{Ring}}(R, \mathcal{O}_Y(Y)) \cong \mathrm{Hom}_{\mathbf{LRS}}(Y, \operatorname{Spec} R).$$

by (3.7). Now for any local-ringed space Y and $(f, \alpha) \in F(Y)$, there is a commutative diagram

$$\begin{tikzcd}
(f, \alpha) \arrow[r, mapsto] & \widetilde{f} \\
F(Y) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathbf{LRS}}(Y, \operatorname{Spec} R) \\
F(\operatorname{Spec} R) \arrow[r, "\sim"] \arrow[u, "\circ \widetilde{f}"'] & \mathrm{Hom}_{\mathbf{LRS}}(\operatorname{Spec} R, \operatorname{Spec} R) \arrow[u, "\circ \widetilde{f}"'] \\
(\pi, \gamma) \arrow[r, mapsto] & \mathrm{id}_{\operatorname{Spec} R}
\end{tikzcd}$$

From construction we see $\pi : \operatorname{Spec} R \to X$ is the map obtained by taking spec of the structure map $\mathcal{O}_X(X) \to R$. Also, the map $\gamma : \mathcal{R} \to \pi_*\mathcal{O}_{\operatorname{Spec} R}$ is just the one induced by $\mathrm{id}_R : R \to R$. This is an isomorphism by (3.15.2).

3.141 Local nature of $\mathbf{Spec}_X$. Let U be an open subset of X, and let $f : Y \to X$ be a morphism with image in U. Then the morphism $g : Y \to \mathbf{Spec}_X \mathcal{R}$ goes into $\pi^{-1}(U)$. Moreover, we have $\alpha|_U : \mathcal{R}|_U \to (f_*\mathcal{O}_Y)|_U = (f|^U)_*\mathcal{O}_Y$ and

$(\pi_*\mathcal{O}_{\mathbf{Spec}_X\mathcal{R}})|_U = (\pi|_{\pi^{-1}(U)})_*(\mathcal{O}_{\mathbf{Spec}_X\mathcal{R}}|_{\pi^{-1}(U)})$, which means $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to X$ solves the universal property for $\mathbf{Spec}_U\mathcal{R}|_U$. Therefore $\pi^{-1}(U) \cong \mathbf{Spec}_U\mathcal{R}|_U$, and in the notation (3.55) it reads

$$(\mathbf{Spec}_X\mathcal{R})\,|_U \cong \mathbf{Spec}_U\mathcal{R}|_U$$

If U is affine, by (3.140) this is $\operatorname{Spec}\mathcal{R}(U)$. This suggests us to construct $\mathbf{Spec}_X\mathcal{R}$ by patching $\operatorname{Spec}\mathcal{R}(U)$ together.

3.142 Lemma. Let Z be a local-ringed space representing the functor F in (3.139.2).

(i) There is a natural morphism $\pi : Z \to X$.

(ii) For any open $U \subseteq X$, the open local-ringed subspace $\pi^{-1}(U)$ represents $Y \mapsto \operatorname{Hom}_{\mathbf{Alg_{LRS}}}((Y, \mathcal{O}_Y), (U, \mathcal{R}|_U))$.

(iii) Z is naturally a scheme and $\pi_*\mathcal{O}_Z \cong \mathcal{R}$ canonically.

In particular, Z solves the universal property in (3.139).

Proof. Since Z is representing F, we have $F(Z) = \operatorname{Hom}_{\mathbf{Alg_{LRS}}}((Z, \mathcal{O}_Z), (X, \mathcal{R})) \cong \operatorname{Hom}_{\mathbf{Sch}}(Z, Z)$; let $(\pi, \gamma) \in \operatorname{Hom}_{\mathbf{Sch}}(Z, X) \times \operatorname{Hom}_{\mathbf{Alg}_{\mathcal{O}_X}}(\mathcal{R}, \pi_*\mathcal{O}_Z)$ be the pair corresponding to $\mathrm{id}_Z \in \operatorname{Hom}_{\mathbf{Sch}}(Z, Z)$. Then $\pi : Z \to X$ is the map in (i), and the map $\gamma : \mathcal{R} \to \pi_*\mathcal{O}_Z$ is the map in (iii), which will be shown to an isomorphism.

We turn to (ii). Let G be the functor in the statement, and let $\iota : U \to X$ be the open immersion. For $(f, \beta) \in G(Y)$, post-composing it with ι gives an element in $F(Y)$, which corresponds to a morphism $\tilde{f} : Y \to Z$. Then $\pi \circ \tilde{f} = \iota \circ f$, so $\tilde{f}(Y) \subseteq \pi^{-1}(U)$. By (2.22) there exists a unique $f' : Y \to \pi^{-1}(U)$ which $\tilde{f}$ factors through. This establishes a natural map $G(Y) \to \operatorname{Hom}_{\mathbf{Sch}}(Y, \pi^{-1}(U))$ with an obvious inverse. Hence $\pi^{-1}(U)$ represents G.

It remains to show (iii). If U is an affine open subspace of X, by (ii) and (3.140) $\pi^{-1}(U) \cong \operatorname{Spec}\mathcal{R}(U)$ as local-ringed spaces, and $\gamma|_U : \mathcal{R}|_U \to (\pi|^U_{\pi^{-1}(U)})_*\mathcal{O}_Z|_{\pi^{-1}(U)}$ is an isomorphism. These together show (iii). ☐

3.143 Construction. For any $U \in \mathcal{U}$, let $\pi_U : \operatorname{Spec}\mathcal{R}(U) \to U$ be the map as in (3.140). For any $U, V \in \mathcal{U}$, we must construct an isomorphism $\theta_{UV} : \pi_U^{-1}(U \cap V) \cong \pi_V^{-1}(U \cap V)$ satisfying the conditions in (2.13). By (3.15.2) and (3.14), we have

$$\mathcal{R}|_U \cong (\pi_U)_*\mathcal{O}_{\operatorname{Spec}\mathcal{R}(U)}, \qquad (\heartsuit)$$

and thus $\mathcal{R}(U \cap V) \cong \mathcal{O}_{\operatorname{Spec}\mathcal{R}(U)}((\pi_U)^{-1}(U \cap V))$. Thus from the restriction $\mathcal{R}(V) \to \mathcal{R}(U \cap V)$ we obtain a morphism $\pi_U^{-1}(U \cap V) \to \operatorname{Spec}\mathcal{R}(V)$. This morphism fits into a commutative diagram

$$\begin{tikzcd}
\pi_U^{-1}(U \cap V) \arrow[rr] \arrow[dr, "\pi_U"'] & & \operatorname{Spec}\mathcal{R}(V) \arrow[dl, "\pi_V"] \\
& U \cap V \subseteq V &
\end{tikzcd}$$

(which, as V is affine, follows easily from taking global sections), implying the image lies in $\pi_V^{-1}(U \cap V)$; denote by $\theta_{UV} : \pi_U^{-1}(U \cap V) \to \pi_V^{-1}(U \cap V)$ the resulting morphism. Note that the inclusion $\pi_U^{-1}(U \cap V) \subseteq \operatorname{Spec}\mathcal{R}(U)$ corresponds to the restriction $\mathcal{R}(U) \to \mathcal{R}(U \cap V)$ as well, and from this we easily deduce that θ_{UV} is an isomorphism and $\theta_{UV}^{-1} = \theta_{VU}$. Also, this shows the θ_{UV}'s satisfy the cocycle condition in (2.13), so we obtain a well-defined scheme, which is of course denoted by $\mathbf{Spec}_X\mathcal{R}$. By (2.13.4), the π_U's patch together to a morphism $\pi : \mathbf{Spec}_X\mathcal{R} \to X$. The way we glue $\operatorname{Spec}\mathcal{R}(U)$ and π also glue the isomorphisms $(\heartsuit)$ together to obtain a global isomorphism $\mathcal{R} \cong \pi_*\mathcal{O}_{\mathbf{Spec}_X\mathcal{R}}$.

3.143.1 Finish of construction. We still need to show that $\pi : \mathbf{Spec}_X\mathcal{R} \to X$ and $\mathcal{R} \cong \pi_*\mathcal{O}_{\mathbf{Spec}_X\mathcal{R}}$ satisfy the universal property in (3.139). Let $f : Y \to X$ be a morphism in **LRS** and $\alpha : \mathcal{R} \to f_*\mathcal{O}_Y$ a morphism of $\mathcal{O}_X$-algebras. Let $\mathcal{U}$ be an affine open cover of X. Put $f_U = f|^U_{f^{-1}(U)} : f^{-1}(U) \to U$ for any $U \in \mathcal{U}$. Note that $(f_*\mathcal{O}_Y)|_U = (f_U)_*(\mathcal{O}_Y|_{f^{-1}(U)})$. Then as in (3.140), we have a morphism $g_U : f^{-1}(U) \to \operatorname{Spec}\mathcal{R}(U)$, which is, in the level of rings, induced by $\alpha(U) : \mathcal{R}(U) \to \mathcal{O}_Y(f^{-1}(U))$. The

g_U's patch together, as for any $U, V \in \mathcal{U}$, we can cover $U \cap V$ by affine open subsets W's, and both $g_U|_{f^{-1}(W)}$ and $g_V|_{f^{-1}(W)}$ are then given by $\mathcal{R}(W) \to \mathcal{O}_Y(f^{-1}(W))$. Thus we obtain a well-defined morphism $g : Y \to \mathbf{Spec}_X\mathcal{R}$. Finally, α factors as $\mathcal{R} \cong \pi_*\mathcal{O}_{\mathbf{Spec}_X\mathcal{R}} \xrightarrow{g_*} f_*\mathcal{O}_Y$ since it does on every affine open $U \in \mathcal{U}$. The morphism g is unique as the restriction $g|_{f^{-1}(U)}^{\pi^{-1}(U)}$ must equal g_U, which is unique as shown in (3.140).

3.144 Lemma. Let X be a scheme and $\mathcal{R}$ a quasi-coherent $\mathcal{O}_X$-algebra. Let $g : X' \to X$ be a morphism of schemes. By (3.17), $g^*\mathcal{R}$ is a quasi-coherent $\mathcal{O}_{X'}$-algebra. Then there exists a canonical isomorphism

$$\mathbf{Spec}_{X'}(g^*\mathcal{R}) \cong X' \times_X \mathbf{Spec}_X\mathcal{R}$$

Proof. Let F be the functor in (3.139.2) represented by $\mathbf{Spec}_X\mathcal{R}$, and let G be that of represented by $\mathbf{Spec}_{X'}(g^*\mathcal{R})$. It suffices to construct a natural bijection

$$G(T) \cong h_{X'}(T) \times_{h_X(T)} F(T),$$

where $T \in \mathbf{LRS}$. But this is more than tautological : giving $f' : T \to X'$ with a $\beta' : g^*\mathcal{R} \to (f')_*\mathcal{O}_T$ is the same as giving $f : T \to X$ with $\beta : \mathcal{R} \to f_*\mathcal{O}_T$ and $f = g \circ f'$. Here we use adjunction (2.11) to replace β by $f^*\mathcal{R} \to \mathcal{O}_T$ and replace β' by $(f')^*g^*\mathcal{R} \to \mathcal{O}_T$. □

3.145 Proposition. Let $f : Y \to X$ be a morphism of schemes. TFAE :

(a) f is an affine morphism (3.20).

(b) There exists a quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{R}$ such that $Y \cong \mathbf{Spec}_X\mathcal{R}$ as X-schemes.

Proof. (b)⇒(a) follows from the discussion in (3.143). For (a)⇒(b), note that an affine morphism $f : Y \to X$ satisfies the conditions in Lemma 3.17, so $f_*\mathcal{O}_Y$ is a quasi-coherent $\mathcal{O}_X$-algebra. At this stage, it suffices to show that $Y \cong \mathbf{Spec}_X f_*\mathcal{O}_Y$ as X-schemes. By the last isomorphism in (3.139.2), the identity morphism on $f_*\mathcal{O}_Y$ induces a canonical morphism $g : Y \to \mathbf{Spec}_X f_*\mathcal{O}_Y$. To see this is an isomorphism, let U be an affine open subset of X such that $f^{-1}(U)$ is affine. We only need to show $g|_{f^{-1}(U)}^{\pi^{-1}(U)}$ is an isomorphism, where $\pi : \mathbf{Spec}_X f_*\mathcal{O}_Y \to X$ is the canonical morphism. From construction (3.143.1) $g|_{f^{-1}(U)}^{\pi^{-1}(U)}$ is built from the isomorphism $\mathrm{id}_{f^{-1}(U)} : f_*\mathcal{O}_Y(U) \to \mathcal{O}_Y(f^{-1}(U))$, so $g|_{f^{-1}(U)}^{\pi^{-1}(U)}$ is an isomorphism as well. □

Note that the proof also gives another way to see the equivalence in (3.20). Further, the claim in (3.19) also follows from the proposition.

3.146 Let X be a scheme. Put $\mathbf{QcohAlg}_X = \mathbf{Qcoh}_X \cap \mathbf{Alg}_{\mathcal{O}_X}$ in the obvious sense. Recall in (3.139.2) we see that $\mathbf{Spec}_X$ defines a fully faithful functor :

$$\mathbf{Spec}_X : \mathbf{QcohAlg}_X \longrightarrow \mathbf{Sch}_X$$

(3.145) shows that the essential image of $\mathbf{Spec}_X$ is the full subcategory of $\mathbf{Sch}_X$ consisting of X-schemes with structure morphism being affine; we say such an X-scheme is **affine over** X. Hence $\mathbf{Spec}_X$ establishes an anti-equivalence from $\mathbf{QcohAlg}_X$ to the category of schemes affine over X.

3.147 Lemma. Let X be a scheme and $\mathcal{A}$ a quasi-coherent $\mathcal{O}_X$-algebra. Then an $\mathcal{A}$-module $\mathcal{F}$ is $\mathcal{A}$-quasi-coherent if and only if it is $\mathcal{O}_X$-quasi-coherent.

3.148 Let $f : X \to S$ be an affine morphism. By (3.17) we see $f_*\mathcal{O}_X$ is $\mathcal{O}_S$-quasi-coherent. Recall by construction if $\mathcal{F}$ is an $\mathcal{O}_X$-module, then $f_*\mathcal{F}$ is viewed as an $\mathcal{O}_S$-module via the homomorphism $\mathcal{O}_S \to f_*\mathcal{O}_X$ (c.f. see the end of (2.11)). By (3.17) again we see f_* defines a functor $f_* : \mathbf{Qcoh}_X \to \mathbf{Qcoh}_S$. But (3.147) implies it actually factors through $\mathbf{Qcoh}_{f_*\mathcal{O}_X} \to \mathbf{Qcoh}_S$, defining a functor

$$f_* : \mathbf{Qcoh}_X \to \mathbf{Qcoh}_{f_*\mathcal{O}_X}$$

3.148.1 We claim f_* is a fully faithful functor. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-quasi-coherent. Note that there is an equality

$$\mathcal{H}om_{f_*\mathcal{O}_X}(f_*\mathcal{F}, f_*\mathcal{G}) = \mathcal{H}om_{\mathcal{O}_S}(f_*\mathcal{F}, f_*\mathcal{G})$$

as $\mathcal{O}_S$ acts on $f_*\mathcal{F}, f_*\mathcal{G}$ via $f_*\mathcal{O}_X$. If $V \subseteq S$ is affine open, together with the above identity we have

$$\begin{aligned}\mathcal{H}om_{f_*\mathcal{O}_X}(f_*\mathcal{F}, f_*\mathcal{G})(V) &= \mathrm{Hom}_{f_*\mathcal{O}_X|_V}(f_*\mathcal{F}|_V, f_*\mathcal{G}|_V)\\ &\overset{(3.15.1)}{\cong} \mathrm{Hom}_{\mathcal{O}_X(f^{-1}(V))}(\mathcal{F}(f^{-1}(V)), \mathcal{G}(f^{-1}(V)))\\ &\overset{(3.15.1)}{\cong} \mathrm{Hom}_{\mathcal{O}_X|_{f^{-1}(V)}}(\mathcal{F}|_{f^{-1}(V)}, \mathcal{G}|_{f^{-1}(V)}) = f_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(V)\end{aligned}$$

3.6.1 Inverse limit of schemes

3.149 Example. Let $(A_i)_{i\in I}$ be a direct system of rings, and put $A = \varinjlim_i A_i$. Then the affine scheme Spec A is the (inverse) limit of the schemes $(\operatorname{Spec} A_i)_i$ in the category **LRS**. Indeed, by Theorem 3.7, we have

$$\mathrm{Hom}_{\mathbf{LRS}}(X, \operatorname{Spec} A) \cong \mathrm{Hom}_{\mathbf{Ring}}(A, \mathcal{O}_X(X)) \cong \varprojlim_i \mathrm{Hom}_{\mathbf{Ring}}(A_i, \mathcal{O}_X(X)) \cong \varprojlim_i \mathrm{Hom}_{\mathbf{LRS}}(X, \operatorname{Spec} A_i).$$

Hence this shows $\operatorname{Spec} A \cong \varprojlim_i \operatorname{Spec} A_i$ in **LRS**. Of course this is not precise unless we specify the canonical maps $\{\operatorname{Spec} A \to \operatorname{Spec} A_i\}_i$, which are those induced by $\{A_i \to A\}_i$

3.150 Generally, let $(X_i, f_{ij})_{i\geqslant j\in I}$ be an inverse system of schemes such that

♣ the transition maps $f_{ij} : X_i \to X_j$ are affine for all $i \geqslant j \in I$.

Fix an index $i_0 \in I$. Then $(X_i, f_{ij})_{i\geqslant j\geqslant i_0}$ defines an inverse system in the category of schemes affine over X_{i_0}. To define $\varprojlim_{i\geqslant i_0} X_i$, by (3.146), it suffices to show that $\mathbf{QcohAlg}_{X_{i_0}}$ admits direct limits. We show this in the following subparagraph.

3.150.1 Lemma. $\mathbf{Qcoh}_{X_{i_0}}$ admits direct limits.

Proof. The question is local, so we may assume $X_{i_0} = \operatorname{Spec} A$ is affine. This then follows from (3.15.1), which says that $\mathbf{Mod}_A \cong \mathbf{Qcoh}_{\operatorname{Spec} A}$. □

3.150.2 Put $X = \varinjlim_{i\geqslant i_0} X_i$; by construction this is again a scheme affine over X_{i_0}, and there are natural maps $f_i : X \to X_i$ $(i \geqslant i_0)$ compatible with the transition maps. For general $j \in I$, pick any $i \geqslant j, i_0$ and define $f_j = f_{ij} \circ f_i : X \to X_j$; the definition does not depend on the choice of i by compatibility. We claim $(X, f_i)_{i\geqslant 0}$ is the inverse limit of $(X_i, f_{ij})_{i\geqslant j\in I}$ in **Sch**. Let $g_i : Y \to X_i$ $(i \in I)$ be a cone over $(X_i, f_{ij})_{i\geqslant j\in I}$. Let $y \in Y$ and choose an affine open neighborhood U_y of $g_{i_0}(y)$ in X_{i_0}. Consider the inverse system $(f_{ii_0}^{-1}(U_y), f_{ij}|_{f_{ii_0}^{-1}(U_y)}^{f_{ji_0}^{-1}(U_y)})_{i\geqslant j\geqslant i_0}$; since the f_{ij} are affine, this is an inverse system of affine schemes, so $\varprojlim_{i\geqslant i_0} f_{ii_0}^{-1}(U_y)$ exists (3.149); but clearly $f_{i_0}^{-1}(U_y) \subseteq X$ is also the inverse limit, so

$$f_{i_0}^{-1}(U_y) \cong \varprojlim_{i\geqslant i_0} f_{ii_0}^{-1}(U_y).$$

together with the morphisms $f_i|_{f_{i_0}^{-1}(U_y)}^{f_{ii_0}^{-1}(U_y)}$ represents the inverse limit of $(f_{ii_0}^{-1}(U_y), f_{ij}|_{f_{ii_0}^{-1}(U_y)}^{f_{ji_0}^{-1}(U_y)})_{i\geqslant j\geqslant i_0}$ in **Sch**.

Let $V_y \subseteq g_{i_0}^{-1}(U_y)$ be any affine open neighborhood of y in Y. Since $g_{i_0} = f_{ii_0} \circ g_i$ for any $i \geqslant i_0$, we see $g_i(V_y) \subseteq f_{ii_0}^{-1}(U_y)$. The above discussion shows there exists a unique morphism $g_{V_y} : V_y \to f_{i_0}^{-1}(U_y) \subseteq X$ compatible with the f_i and f_{ii_0} $(i \geqslant i_0)$. By uniqueness, the morphisms g_{V_y}, where $y \in Y$ and V_y are taken as above, glue, yielding a global morphism $g : Y \to X$. Clearly this is unique with respect to the compatibility. This demonstrates our claim. We summarize what we obtain in the next paragraph.

3.151 Theorem. Let $(X_i, f_{ij})_{i \geqslant j \in I}$ be an inverse system of schemes such that the transition maps $f_{ij} : X_i \to X_j$ are affine for all $i \geqslant j \in I$. Then $X = \varprojlim_{i \in I} X_i$ exists in the category of schemes. Moreover,

(i) The canonical morphism $f_i : X \to X_i$ is affine for every $i \in I$.

(ii) For any index $i_0 \in I$ and any open subspace $U \subseteq X_{i_0}$, one has

$$f_{i_0}^{-1}(U) = \varprojlim_{i \geqslant i_0} f_{ii_0}^{-1}(U).$$

in the category of schemes.

3.6.2 Vector bundles

3.152 Definition. Let X be a scheme and $n \in \mathbb{Z}_{\geqslant 0}$. An X-scheme $p : Y \to X$ is called a **vector bundle of rank n over** X if Y admits an open cover $\mathcal{U}$ such that

(i) for any $U \in \mathcal{U}$, there is an isomorphism $\psi_U : p^{-1}(U) \cong \mathbb{A}_U^n$ of U-schemes, and

(ii) for any $U, V \in \mathcal{U}$ and any affine open subset $W = \operatorname{Spec} A \subseteq U \cap V$, the isomorphism $\psi_V^{-1} \circ \psi_U$ of $\mathbb{A}_W^n = \operatorname{Spec} A[x_1, \ldots, x_n]$ is given by a linear automorphism θ of $A[x_1, \ldots, x_n]$, namely, $\theta|_A = \mathrm{id}_A$ and $\theta(x_i) = \sum_{j=1}^{n} a_{ij} x_j$ for some $a_{ij} \in A$.

We shall call the datum $\{\psi_U\}_{U \in \mathcal{U}}$ satisfying (i) and (ii) a **vector bundle structure over** X on Y. A **morphism between vector bundles** (Y, ψ_U) of rank n and (Y', ψ'_V) of rank n' over X is a morphism $g : Y \to Y'$ of X-schemes such that for any $U \in \mathcal{U}$, $V \in \mathcal{V}$ and $W = \operatorname{Spec} A \subseteq U \cap V$, the restriction $g|_W : Y|_W \to Y'|_W$ comes from a linear homomorphism $A[y_1, \ldots, y_{n'}] \to A[x_1, \ldots, x_n]$.

3.153 Symmetric algebra. Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Put $\operatorname{Sym}^0 \mathcal{F} = \mathcal{O}_X$, $\operatorname{Sym}^1 \mathcal{F} = \mathcal{F}$, and for integers $n \geqslant 2$,

$$\operatorname{Sym}^n \mathcal{F} = \mathcal{F}^{\otimes n} / \mathcal{I}_n$$

where $\mathcal{I}_n$ is the $\mathcal{O}_X$-module generated by the local sections $s_1 \otimes \cdots \otimes s_n - s_{\sigma(1)} \otimes \cdots \otimes s_{\sigma(n)} \in \mathcal{F}^{\otimes n}$ $(\sigma \in S_n)$. This is called the **n-th symmetric power** of $\mathcal{F}$, and it is an $\mathcal{O}_X$-module. Here the tensor product is always over $\mathcal{O}_X$. The direct sum

$$\operatorname{Sym} \mathcal{F} = \bigoplus_{n \geqslant 0} \operatorname{Sym}^n \mathcal{F}$$

is called the **symmetric algebra** of $\mathcal{F}$, which is an $\mathcal{O}_X$-algebra.

3.154 For any integer $n \geqslant 0$, the n-th symmetric power $\operatorname{Sym}^n \mathcal{F}$ is isomorphic to the sheafification of the presheaf $U \mapsto \operatorname{Sym}^n \mathcal{F}(U)$. To see this, for $n \geqslant 2$, the presheaf $U \mapsto \operatorname{Sym}^n \mathcal{F}(U)$ is the quotient of $\mathcal{F} \otimes^p \cdots \otimes^p \mathcal{F}$ by the subsheaf $\mathcal{I}'_n$ generated by the local sections $s_1 \otimes \cdots \otimes s_n - s_{\sigma(1)} \otimes \cdots \otimes s_{\sigma(n)} \in \mathcal{F}^{\otimes n}$ $(\sigma \in S_n)$. By definition, $(\mathcal{F} \otimes^p \cdots \otimes^p \mathcal{F})^\dagger = \mathcal{F}^{\otimes n}$ and $(\mathcal{I}'_n)^\dagger = \mathcal{I}_n$. At this stage the claim follows from (2.16.1).

3.154.1 Let $f : Y \to X$ be a morphism of ringed spaces and let $\mathcal{F}$ be a $\mathcal{O}_Y$-module. Then there exists a canonical isomorphism

$$\operatorname{Sym}^n f^* \mathcal{F} \cong f^*(\operatorname{Sym}^n \mathcal{F}).$$

It is clear for $n = 0, 1$. For $n \geqslant 2$, since f^* is right exact, there is an exact sequence

$$f^*\mathcal{I}_n \longrightarrow f^*(\mathcal{F}^{\otimes n}) \longrightarrow f^*(\operatorname{Sym}^n \mathcal{F}) \longrightarrow 0$$

Now it suffices to show that image of $f^*\mathcal{I}_n$ in $f^*(\mathcal{F}^{\otimes n}) \cong (f^*\mathcal{F})^{\otimes n}$ is isomorphic to the ideal defining $\operatorname{Sym}^n f^*\mathcal{F}$. This is clear in the level of presheaves, and by (2.16.1) the same holds in the level of sheaves as well.

3.154.2 Lemma. Let A be a ring and M an A-module. Then $\operatorname{Sym}^n \widetilde{M} \cong \widetilde{\operatorname{Sym}^n M}$.

Proof. The cases $n = 0, 1$ are obvious. Let $n \geqslant 2$, and let $f \in A$. Consider the composition

$$\widetilde{\operatorname{Sym}^n M}(D(f)) = (\operatorname{Sym}^n M)_f \cong \operatorname{Sym}^n M_f = \operatorname{Sym}^n \widetilde{M}(D(f)) \to (\operatorname{Sym}^n \widetilde{M})(D(f))$$

where the isomorphism results from the same argument in (3.154.1), and the last arrow follows from (3.154). Since the arrows involved are natural, by (2.3.1) this defines a morphism $\alpha : \widetilde{\operatorname{Sym}^n M} \to \operatorname{Sym}^n \widetilde{M}$ of $\mathcal{O}_{\operatorname{Spec} A}$-modules. For any $\mathfrak{p} \in \operatorname{Spec} A$, there is an isomorphism $(\operatorname{Sym}^n M)_\mathfrak{p} \cong \operatorname{Sym}^n M_\mathfrak{p}$ compatible with the ones above, which coincides with the stalk map $\alpha_\mathfrak{p}$. Hence α is an isomorphism by (2.16). □

3.154.3 Corollary. Let X be a scheme and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. Then for any affine open subset $U = \operatorname{Spec} A$ of X, we have

$$(U, (\operatorname{Sym}^n \mathcal{F})|_U) \cong (\operatorname{Spec} A, \widetilde{\operatorname{Sym}^n \mathcal{F}(U)}).$$

In particular, $(\operatorname{Sym}^n \mathcal{F})(U) \cong \operatorname{Sym}^n \mathcal{F}(U)$ for any affine open subset U, and

$$(\operatorname{Sym} \mathcal{F})|_U \cong \widetilde{\operatorname{Sym} \mathcal{F}(U)}$$

Proof. By (3.154.1), (3.154.2) and (3.15.2), we have

$$(\operatorname{Sym}^n \mathcal{F})|_U \cong \operatorname{Sym}^n(\mathcal{F}|_U) \cong \operatorname{Sym}^n \widetilde{\mathcal{F}(U)}.$$

For the last assertion,

$$\left(\bigoplus_{n \geqslant 0} \operatorname{Sym}^n \mathcal{F}\right)|_U \cong \bigoplus_{n \geqslant 0} \operatorname{Sym}^n(\mathcal{F}|_U) \cong \bigoplus_{n \geqslant 0} \operatorname{Sym}^n \widetilde{\mathcal{F}(U)} \cong \bigoplus_{n \geqslant 0} \widetilde{\operatorname{Sym}^n \mathcal{F}(U)}.$$

The first and the last isomorphisms result from the facts that $(\cdot)|_U$ and $\widetilde{(\cdot)}$ are exact functors (2.16) (3.4).

□

3.155 Total space of a locally free sheaf. Let X be a scheme and let $\mathcal{F}$ be a locally free $\mathcal{O}_X$-module of rank $n < \infty$. We can find an affine open cover $\mathcal{U}$ of X consisting such that $\mathcal{F}|_U \cong (\mathcal{O}_X|_U)^{\oplus n}$ for any $U \in \mathcal{U}$. Fix a $U \in \mathcal{U}$ and write $(U, \mathcal{O}_X|_U) \cong (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$. Then

$$(\operatorname{Sym} \mathcal{F})|_U \cong \widetilde{\operatorname{Sym} \mathcal{F}(U)} \cong \widetilde{\operatorname{Sym} A^{\oplus n}} \cong A[x_1, \ldots, x_n].$$

This means $\operatorname{Sym} \mathcal{F}$ is locally free and is locally of finite type. The same argument works for the dual $\mathcal{F}^\vee$; in particular, $\operatorname{Sym}(\mathcal{F}^\vee)$ is a quasi-coherent $\mathcal{O}_X$-algebra, so we can apply the relative spec construction. The X-scheme $\mathbb{V}(\mathcal{F}) := \mathbf{Spec}_X \operatorname{Sym}(\mathcal{F}^\vee)$ is called the **total space** of $\mathcal{F}$.

3.155.1 Retain the notations in (3.155). Let $U \in \mathcal{U}$. Then

$$(\mathbf{Spec}_X \operatorname{Sym}(\mathcal{F}^\vee))|_U \cong \operatorname{Spec}\operatorname{Sym}(\mathcal{F}^\vee)(U) \cong \operatorname{Spec}\mathcal{O}_X(U)[y_1,\dots,y_n] = \mathbb{A}^n_U.$$

Let $V \in \mathcal{U}$ and pick $W \subseteq U \cap V$ with W affine. Then $\mathcal{F}|_W$ is free. The transition function on W is obtained by applying Sym to the isomorphism

$$(\mathcal{O}_X|_W)^n \cong (\mathcal{F}|_U)|_W = \mathcal{F}|_W = (\mathcal{F}|_V)|_W \cong (\mathcal{O}_X|_W)^n.$$

In particular, this shows the transition function is a linear automorphism (for it is degree-preserving). Hence, we have shown that $\mathbb{V}(\mathcal{F}) \to X$ is a vector bundle of rank n over X.

3.155.2 If $\mathcal{F} \to \mathcal{G}$ is a morphism of finite locally free $\mathcal{O}_X$-modules, first taking dual and next applying $\mathbf{Spec}_X \operatorname{Sym}$ gives a morphism $g : \mathbb{V}(\mathcal{F}) \to \mathbb{V}(\mathcal{G})$. Locally on an affine open, it is the morphism $g|_U : \operatorname{Spec}\operatorname{Sym}\widetilde{\mathcal{F}^\vee(U)} \to \operatorname{Spec}\operatorname{Sym}\widetilde{\mathcal{G}^\vee(U)}$, and it is induced by the morphism $\mathcal{F}|_U \to \mathcal{G}|_U$. Thus $g|_U$ corresponds to a linear homomorphism in the sense of (3.152), and this shows that g is a vector bundle morphism.

3.156 Sheaf of sections. Let Y be an X-scheme and denote by $f : Y \to X$ the structure morphism. Let U be an open set of X. An element in $\operatorname{Hom}_{\mathbf{Sch}_U}(U, Y|_U)$ is called a **section of** f **over** U. Let us put

$$\Gamma(U, Y/X) = \operatorname{Hom}_{\mathbf{Sch}_U}(U, Y|_U) = \{s \in \operatorname{Hom}_{\mathbf{Sch}}(U, Y) \mid f \circ s = \mathrm{id}_U\}$$

By (2.13.4), the presheaf $U \mapsto \Gamma(U, Y/X)$ of sections is a sheaf of sets. We denote this sheaf by $\Gamma(\cdot, Y/X)$, and it is a subsheaf of $\operatorname{Hom}_{\mathbf{Sch}}(\cdot, Y)$. It is clear that $\Gamma(\cdot, Y/X)$ defines a functor from $\mathbf{Sch}_X$ to $\mathbf{Set}_X$.

3.157 Let $p : Y \to X$ be a vector bundle of rank n over a scheme X. We show that $\Gamma(\cdot, Y/X)$ has a natural $\mathcal{O}_X$-module structure. First consider the case $X = \operatorname{Spec} A$ being affine and Y is a trivial bundle, i.e., $Y = \mathbb{A}^n_X$. Then $\Gamma(X, Y/X) \cong \operatorname{Hom}_{\mathbf{Alg}_A}(A[x_1,\dots,x_n], A) \cong A^{\oplus n}$, so $\Gamma(X, Y/X)$ has a naturally an A-module (note that the resulting module structure is independent of the last isomorphism). For general case, let $U = \operatorname{Spec} A$ be an affine open set of X trivializing Y and let $W = \operatorname{Spec} B \subseteq U$ be another such affine open in X. We then know $\Gamma(U, Y/X)$ (resp. $\Gamma(W, Y/X)$) is a A- (resp. B-) module. Choose a free basis of $\mathcal{O}_{\mathbb{A}^n_W}(W)$ in a way that $\mathcal{O}_{\mathbb{A}^n_W}(W) = B[x_1,\dots,x_n]$ and $\mathcal{O}_{\mathbb{A}^n_U}(U) = A[x_1,\dots,x_n]$. Then there is a commutative diagram

$$\begin{tikzcd}
\Gamma(U, Y/X) = \operatorname{Hom}_{\mathbf{Sch}_U}(U, \mathbb{A}^n_U) \arrow[r, "\sim"] \arrow[d, "\text{restriction}"] & \operatorname{Hom}_{\mathbf{Alg}_A}(A[x_1,\dots,x_n], A) \arrow[r, "\sim"] \arrow[dd, "\circ\varphi"] & A^{\oplus n} \arrow[dd, "\varphi"] \\
\{s \in \operatorname{Hom}_{\mathbf{Sch}}(W, \mathbb{A}^n_U) \mid p \circ s = \mathrm{id}_W\} \arrow[d, "\wr\ (2.22)"] & & \\
\Gamma(W, Y/X) = \operatorname{Hom}_{\mathbf{Sch}_W}(W, \mathbb{A}^n_W) \arrow[r, "\sim"] & \operatorname{Hom}_{\mathbf{Alg}_B}(B[x_1,\dots,x_n], B) \arrow[r, "\sim"] & B^{\oplus n}
\end{tikzcd}$$

where we denote by $\varphi : A \to B$ the restriction $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$. From this we see the module structures are compatible with the restriction, provided the affine open subsets are concerned. Finally, we extend the action to all open sets by (2.3.1). Moreover, the argument above also shows that $\Gamma(\cdot, Y/X)$ is locally free of rank n.

3.158 Let X be a scheme. Denote by $\mathbf{FinLoc}_X$ the full subcategory of $\mathbf{Mod}_{\mathcal{O}_X}$ consisting of finite locally free $\mathcal{O}_X$-modules, and denote by $\mathbf{VB}_X$ the category of vector bundles over X (3.152). So far we have two functors

$$\mathbb{V} : \mathbf{FinLoc}_X \longrightarrow \mathbf{VB}_X\ , \qquad \Gamma(\cdot, \cdot/X) : \mathbf{VB}_X \longrightarrow \mathbf{FinLoc}_X$$

In fact, these are mutually inverse to each other, and hence define an anti-equivalence between $\mathbf{FinLoc}_X$ and $\mathbf{VB}_X$.

3.159 Lemma. Let $\mathcal{G}$ be a locally free $\mathcal{O}_X$-module of rank n, and let $\mathcal{F} = \Gamma(\cdot, \mathbb{V}(\mathcal{G})/X)$ be the sheaf of sections of the vector bundle $\mathbb{V}(\mathcal{G})$. Then $\mathcal{F} \cong \mathcal{G}$.

3.6.3 Relative Proj

3.160 Definition. Let X be a scheme. A **graded $\mathcal{O}_X$-algebra** is an $\mathcal{O}_X$-algebra $\mathcal{R}$ together with a $\mathbb{Z}_{\geqslant 0}$-gradation

$$\mathcal{R} = \bigoplus_{n \geqslant 0} \mathcal{R}_n$$

with each $\mathcal{R}_n$ an $\mathcal{O}_X$-module. We say $\mathcal{R}$ is **quasi-coherent** if it is quasi-coherent as an $\mathcal{O}_X$-module and each $\mathcal{R}_n$ is quasi-coherent.

3.161 Construction of relative Proj. Let X be a scheme and let $\mathcal{R}$ be a quasi-coherent graded $\mathcal{O}_X$-algebra. For an affine open U in X, denote by $\pi_U : \operatorname{Proj} \mathcal{R}(U) \to U$ the natural projection (3.110.2). Note here that $\mathcal{R}(U) = \bigoplus_{n \geqslant 0} \mathcal{R}_n(U)$ since U is affine (for example, by (3.7.2)). If $V \subseteq U$ is an affine open subset, then

$$\pi_U^{-1}(V) \cong \operatorname{Proj} \mathcal{R}(U) \times_U V \cong \operatorname{Proj}(\mathcal{R}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V)) \cong \operatorname{Proj} \mathcal{R}(V).$$

The first isomorphism is canonical. The second is due to (3.112), and the third is because $\mathcal{R}$ is quasi-coherent and it is true for principal affines in U by (3.15.2). Hence, for two affine opens $U, V \subseteq X$, there is an isomorphism

$$\theta_{UV} : \pi_U^{-1}(U \cap V) \xrightarrow{\sim} \pi_V^{-1}(U \cap V).$$

To glue the $\pi^{-1}(U)$ $(U \underset{\text{affine}}{\subseteq} X)$, we must show the θ_{UV} satisfy the cocycle condition in (2.13). Say $U, V, W \subseteq X$ are three affine opens. To check $\theta_{UW} = \theta_{VW} \circ \theta_{UV}$ on $\pi_U^{-1}(U \cap V \cap W)$, it suffices to check this on $\pi_U^{-1}(T)$ for all affine open subsets $T \subseteq U \cap V \cap W$. This is then clear, as all $\pi_U^{-1}(T)$, $\pi_V^{-1}(T)$, $\pi_W^{-1}(T)$ are naturally isomorphic to $\operatorname{Proj} \mathcal{R}(T)$. Hence there exists an X-scheme $\pi : \mathbf{Proj}_X \mathcal{R} \to X$ such that $\pi^{-1}(U) \cong \operatorname{Proj} \mathcal{R}(U)$ for all affine opens $U \subseteq X$, and is unique in the sense of (2.13). This is called the **relative Proj of $\mathcal{R}$ over** X.

3.161.1 $\mathcal{O}(n)$ of $\mathbf{Proj}_X \mathcal{R}$. For an affine open $U \subseteq X$, the U-scheme $\operatorname{Proj} \mathcal{R}(U)$ naturally admits the quasi-coherent sheaves $\mathcal{O}_{\operatorname{Proj} \mathcal{R}(U)}(n)$ $(n \in \mathbb{Z})$ (3.118). We aim to glue these together to obtain a relative version : $\mathcal{O}_{\mathbf{Proj}_X \mathcal{R}}(n)$.

We must of course apply (2.13.3). Similar to the construction of $\mathbf{Proj}_X \mathcal{R}$, let $V \subseteq U$ be affine opens in X. By (3.113) (and glueing), we have a fibre square

$$\begin{tikzcd} \operatorname{Proj} \mathcal{R}(V) \arrow[r, "r"] \arrow[d, "\pi_V"] & \operatorname{Proj} \mathcal{R}(U) \arrow[d, "\pi_U"] \\ V \arrow[r, "\iota"] & U. \end{tikzcd}$$

By (3.120), we have $r^* \mathcal{O}_{\operatorname{Proj} \mathcal{R}(U)}(n) \cong \mathcal{O}_{\operatorname{Proj} \mathcal{R}(V)}(n)$. By a similar argument as in (3.161), we see they glue.

3.161.2 Homogeneous algebras. If we want $\mathcal{O}_{\mathbf{Proj}_X \mathcal{R}}(n)$ to be an invertible sheaf on $\mathbf{Proj}_X \mathcal{R}$, by (3.119) we must at least assume $\mathcal{R}$ is generated in degree 1, i.e., the canonical map $\operatorname{Sym}_{\mathcal{R}_0} \mathcal{R}_1 \to \mathcal{R}$ is surjective (where $\operatorname{Sym}_{\mathcal{R}_0} \mathcal{R}_1$ is the symmetric algebra of $\mathcal{R}_1$ with $\mathcal{R}_1$ viewed as an $\mathcal{R}_0$-module).

For convenience, we shall call such quasi-coherent graded $\mathcal{O}_X$-algebra $\mathcal{R}$ **homogeneous**.

3.162 Base change. Let X be a scheme and suppose $\mathcal{R}$ is a quasi-coherent graded $\mathcal{O}_X$-algebra. For any morphism $f : Z \to X$

of schemes, consider the fibre square

$$\begin{tikzcd}
Z \times_X \mathbf{Proj}_X\mathcal{R} \arrow[r] \arrow[d] & \mathbf{Proj}_X\mathcal{R} \arrow[d, "\pi"] \\
Z \arrow[r, "f"] & X
\end{tikzcd}$$

Let $U \subseteq X$, $V \subseteq Z$ be affine opens such that $f(V) \subseteq U$. Then locally the above fibre square becomes

$$\begin{tikzcd}
V \times_U \operatorname{Proj}\mathcal{R}(U) \arrow[r] \arrow[d] & \operatorname{Proj}\mathcal{R}(U) \arrow[d, "\pi_U"] \\
V \arrow[r, "f|_V^U"] & U.
\end{tikzcd}$$

On the other hand, $f^*\mathcal{R}$ is a quasi-coherent graded $\mathcal{O}_Z$-algebra, so we can form $\pi' : \mathbf{Proj}_Z f^*\mathcal{R} \to Z$. By (3.112)

$$V \times_U \operatorname{Proj}\mathcal{R}(U) \cong \operatorname{Proj}\left(\mathcal{R}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_Z(V)\right) \cong \operatorname{Proj}\left((f|_V^U)^*\mathcal{R}|_U(V)\right) = \operatorname{Proj}(f^*\mathcal{R})(V) = \pi'^{-1}(V).$$

Via gluing Better write down a map explicitly, this establishes an isomorphism $Z \times_X \mathbf{Proj}_X\mathcal{R} \cong \mathbf{Proj}_Z f^*\mathcal{R}$ of Z-schemes.

3.163 Closed subschemes of Proj. Let S be a scheme and let $\mathcal{R}$ be quasi-coherent graded $\mathcal{O}_S$-algebra. Let $\pi : \mathbf{Proj}_S\mathcal{R} \to S$ be the structure map. Suppose $j : Z \to \mathbf{Proj}_S\mathcal{R}$ be a closed subscheme and let $\mathcal{I}$ be the kernel of the sheaf map $\mathcal{O}_{\mathbf{Proj}_S\mathcal{R}} \to j_*\mathcal{O}_Z$. Then there is a natural morphism

$$\mathcal{R} \longrightarrow \bigoplus_{n \geqslant 0} \pi_*(\mathcal{O}_{\mathbf{Proj}_S\mathcal{R}}(n)) \longrightarrow \bigoplus_{n \geqslant 0} \pi_*((j_*\mathcal{O}_Z)(n))$$

described as follows. The second arrow is clear. The first arrow is from the gluing : if $U \subseteq S$ is an affine open set, by (3.118) we have

$$\mathcal{R}(U) \longrightarrow \bigoplus_{n \geqslant 0} \mathcal{O}_{\mathbf{Proj}_S\mathcal{R}(U)}(n) = \bigoplus_{n \geqslant 0} \pi_*(\mathcal{O}_{\mathbf{Proj}_S\mathcal{R}}(n))(U).$$

An argument as in (3.161.2) shows they glue, which finishes the definition of the first arrow. Note that this is simply the relative version of the map in (3.125).

Denote by $\mathcal{J}$ the kernel of this morphism. By (3.125) there is a natural morphism $\mathcal{J} \to \mathcal{I}$ and is an isomorphism when $\pi : \mathbf{Proj}_S\mathcal{R} \to S$ is quasi-compact (so that each $\operatorname{Proj}\mathcal{R}(U)$ is compact).

3.163.1 Lemma. Let S be a scheme and let $\mathcal{R}$ be quasi-coherent graded $\mathcal{O}_S$-algebra. Suppose the structure morphism $\pi : \mathbf{Proj}_S\mathcal{R} \to S$ is quasi-compact. Then every closed subscheme of $\mathbf{Proj}_S\mathcal{R}$ has the form $\mathbf{Proj}_S\mathcal{R}/\mathcal{J}$ for some quasi-coherent graded ideal sheaf $\mathcal{J}$ of $\mathcal{R}$.

3.164 Projective morphisms. A morphism $f : X \to Y$ of schemes is called **projective** if there is a commutative triangle

$$\begin{tikzcd}
X \arrow[r, hook, "\text{closed immersion}"] \arrow[dr, "f"'] & \mathbf{Proj}_Y \operatorname{Sym}\mathcal{F} \arrow[d] \\
& Y
\end{tikzcd}$$

where $\mathcal{F}$ be a finitely generated quasi-coherent sheaf on Y. In this case we also say X **is projective over** Y.

3.164.1 Example. Let Y be a scheme and let $\mathcal{R}$ be a finitely generated homogeneous quasi-coherent graded $\mathcal{O}_Y$-algebra. Then there is a degree-preserving surjection $\operatorname{Sym}\mathcal{R}_1 \to \mathcal{R}$. By (3.114), this gives a closed immersion $\mathbf{Proj}_Y\mathcal{R} \to \mathbf{Proj}_Y \operatorname{Sym}\mathcal{R}_1$ and a commutative triangle

$$\begin{tikzcd}
\mathbf{Proj}_Y\mathcal{R} \arrow[rr, hook, "\substack{\text{closed}\\ \text{immersion}}"] \arrow[dr] & & \mathbf{Proj}_Y \operatorname{Sym}\mathcal{R}_1 \arrow[dl] \\
& Y &
\end{tikzcd}$$

The converse holds as well : if X is a closed subscheme of $\mathbf{Proj}_Y \operatorname{Sym}\mathcal{F}$ for some $\mathcal{F}$ is a finitely generated homogeneous quasi-coherent graded $\mathcal{O}_Y$-algebra, then $X \cong \mathbf{Proj}_Y(\operatorname{Sym}\mathcal{F})/\mathcal{I}$ for some quasi-coherent ideal sheaf $\mathcal{I}$ of $\operatorname{Sym}\mathcal{F}$, by virtue of (3.163.1).

3.164.2 Lemma. A morphism $f : X \to Y$ of schemes is projective if and only if there exists a commutative triangle

$$\begin{tikzcd}
X \arrow[rr, "\sim"] \arrow[dr, "f"'] & & \mathbf{Proj}_Y\mathcal{R} \arrow[dl] \\
& Y &
\end{tikzcd}$$

for some quasi-coherent graded $\mathcal{O}_Y$-algebra $\mathcal{R}$ generated by degree 1 elements.

3.165 Projective over an affine base. Suppose $Y = \operatorname{Spec} A$ is affine. Then X is projective over $\operatorname{Spec} A$ if and only if X is a closed subscheme of $\mathbb{P}^n_A$ for some $n \in \mathbb{Z}_{\geqslant 1}$.

Proof. Since $\mathbb{P}^n_A = \mathbf{Proj}_{\operatorname{Spec} A} \operatorname{Sym} \mathcal{O}^{\oplus n+1}_{\operatorname{Spec} A}$, the if part holds. For the only if part, assume X is projective over $\operatorname{Spec} A$; then X is an A-scheme and is closed subscheme of $\mathbf{Proj}_{\operatorname{Spec} A} \operatorname{Sym}\mathcal{F}$ for some finitely generated quasi-coherent sheaf $\mathcal{F}$ on A. But then $\mathcal{F} \cong \widetilde{M}$ for some finite A-module M, so $\mathbf{Proj}_{\operatorname{Spec} A} \operatorname{Sym}\mathcal{F} \cong \operatorname{Proj}\operatorname{Sym} M$ by (3.161) and (3.154.3). Take an surjection $A^{\oplus n} \to M$ for some $n \in \mathbb{Z}_{\geqslant 2}$. Then $\operatorname{Sym} A^{\oplus n} \to \operatorname{Sym} M$ is a degree-preserving surjection, so we have a closed immersion $\operatorname{Proj}\operatorname{Sym} M \hookrightarrow \operatorname{Proj}\operatorname{Sym} A^{\oplus n} = \mathbb{P}^{n-1}_A$. □

3.166 Base change. Let $f : X \to Y$ be a projective morphism, and $g : Z \to Y$ be any morphism of schemes. Then $f_Z : Z \times_Y X \to Z$ is projective.

Proof. Say X is a closed subscheme of $\mathbf{Proj}_R \operatorname{Sym}\mathcal{F}$ for some finitely generated quasi-coherent sheaf $\mathcal{F}$ on Y. We then have a commutative diagram

$$\begin{tikzcd}
Z \times_Y X \arrow[r] \arrow[d] \arrow[dd, bend right=80, "f_Z"'] & X \arrow[d] \arrow[dd, bend left=80, "f"] \\
\mathbf{Proj}\operatorname{Sym} g^*\mathcal{F} \arrow[r] \arrow[d] & \mathbf{Proj}\operatorname{Sym}\mathcal{F} \arrow[d] \\
Z \arrow[r, "g"'] & Y.
\end{tikzcd}$$

The lower square is cartesian by (3.162) and (3.154.1). Since the outer rectangle is cartesian, it follows that the upper square is cartesian. Hence $Z \times_Y X \to \mathbf{Proj}\operatorname{Sym} g^*\mathcal{F}$ is a closed embedding and the half circle on the left is commutative, it follows that $f_Z : Z \times_Y X \to Z$ is projective.

□

4 Varieties

4.1 Classical varieties

4.1.1 Affine varieties

4.1 In this subsection, let k be a field. For any positive integer $n \geqslant 1$, we write $\mathbb{A}^n_k$ for the product space k^n, and called the **n-dimensional affine space over** k. If the field involved is clear from the context, we usually omit the subscript and simply write $\mathbb{A}^n$. For a subset S of the polynomial ring $k[x_1, \ldots, x_n]$, we put $V(S)$ to denote the common zeroes in $\mathbb{A}^n$ of the polynomials in S, i.e.,

$$V(S) := \{p \in \mathbb{A}^n \mid f(p) = 0 \text{ for all } f \in S\}.$$

If $I \trianglelefteq k[x_1, \ldots, x_n]$ is the ideal generated by S, then clearly $V(I) = V(S)$. A set of the form $V(S)$ is called an **(affine) algebraic set**. There are some properties of the assignment $S \mapsto V(S)$ that can be observed directly from the definition :

(i) If $\{I_\alpha\}_\alpha$ is a family of ideals in $k[x_1, \ldots, x_n]$, then $V\left(\bigcup_\alpha I_\alpha\right) = \bigcap_\alpha V(I_\alpha)$.

(ii) If $I_1, I_2 \trianglelefteq k[x_1, \ldots, x_n]$ are two ideals, then $V(I_1) \cup V(I_2) = V(I_1 \cap I_2) = V(I_1 I_2)$.

(iii) $\varnothing = V(1)$ and $\mathbb{A}^n = V(0)$.

(iv) If $I \trianglelefteq k[x_1, \ldots, x_n]$ is an ideal, then $V(I) = V(\sqrt{I})$.

(v) If $S_1 \subseteq S_2 \subseteq k[x_1, \ldots, x_n]$, then $V(S_2) \subseteq V(S_1)$.

By (i), (ii) and (iii), the sets of the form $V(I)$, $I \trianglelefteq k[x_1, \ldots, x_n]$ define a topology (of closed sets) on $\mathbb{A}^n$. The defined topology is called the **Zariski topology** on the affine space $\mathbb{A}^n$.

It is easy to write down an open basis for the Zariski topology. To start with, since $k[x_1, \ldots, x_n]$ is Noetherian, every ideal I is finitely generated ; say $I = \langle f_1, \ldots, f_m \rangle$. Then by (i),

$$V(I) = V((f_1)) \cap \cdots \cap V((f_m)).$$

For $f \in k[x_1, \ldots, x_n]$, put $D(f) = \{p \in \mathbb{A}^n \mid f(p) \neq 0\}$; a set of this form is called a **principal affine open set**. Then taking complement, the above equation becomes

$$\mathbb{A}^n \backslash V(I) = D(f_1) \cup \cdots \cup D(f_m).$$

Every open set in $\mathbb{A}^n$ has the form as the left hand side. For $f, g \in k[x_1, \ldots, x_n]$, by (ii) we have $D(f) \cap D(g) = D(fg)$. Thus the sets of the form $D(f)$ are a basis for the Zariski topology.

4.2 For a subset $S \subseteq \mathbb{A}^n$, put $I(S)$ to denote the set of all polynomials that vanish on S, i.e.,

$$I(S) = \{f \in k[x_1, \ldots, x_n] \mid f(p) = 0 \text{ for all } p \in S\}.$$

This is a radical ideal in $k[x_1, \ldots, x_n]$. There are some formal properties of the assignment $S \mapsto I(S)$:

(i) If $S_1, S_2 \subseteq \mathbb{A}^n$, then $I(S_1 \cup S_2) = I(S_1) \cap I(S_2)$.

(ii) If $S_1 \subseteq S_2 \subseteq \mathbb{A}^n$, then $I(S_2) \subseteq I(S_1)$.

(iii) If $V \subseteq \mathbb{A}^n$ is an algebraic set, then $V(I(V)) = V$.

(iv) For $S \subseteq \mathbb{A}^n$, we have $V(I(S)) = \overline{S}$, the Zariski closure of S in $\mathbb{A}^n$.

(v) $I(\varnothing) = k[x_1, \ldots, x_n]$, and if $\#k = \infty$, $I(\mathbb{A}^n) = 0$.

For an algebraic set $V \subseteq \mathbb{A}^n$, the quotient

$$k[V] := k[x_1, \dots, x_n]/I(V)$$

is called the **coordinate ring** of V. Since $I(V)$ is radical, the quotient ring $k[V]$ is reduced.

4.3 Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be algebraic sets. A map $\varphi : V \to W$ is called a **morphism** if for any integer $1 \leqslant i \leqslant m$, we have $\mathrm{pr}_i \circ \varphi \in k[x_1, \dots, x_n]$, where pr_i denotes the projection to the i-th component.

4.4 If $\varphi : V \to W$ is a morphism, then the pullback $f \mapsto f \circ \varphi$ defines a k-algebra homomorphism $\varphi^* : k[W] \to k[V]$. In fact, this defines a bijection

$$\begin{aligned} \{\text{morphisms } V \to W\} &\longrightarrow \mathrm{Hom}_{\mathbf{Alg}_k}(k[W], k[V]) \\ \varphi &\longmapsto \varphi^* \end{aligned}$$

4.5 Lemma. Let $\varphi : V \to W$ be a morphism between algebraic sets. Then

(a) $\ker \varphi^* = I(\varphi(V))$.

(b) $\overline{\varphi(V)} = V(\ker \varphi^*)$.

In particular, $\overline{\varphi(I)}$ is an algebraic set with coordinate ring $k[W]/\ker \varphi^*$.

4.1.2 Nullstellensatz.

4.6 Noether's normalization lemma. Let A be a k-algebra of finite type. Then there exist $y_1, \dots, y_d \in A$ $(0 \leqslant d \leqslant m)$ such that the y_i are algebraically independent and A is finite over $k[y_1, \dots, y_d]$.

Proof. Write $A = k[r_1, \dots, r_m]$. We prove this by induction on m.

1° $m = 1$: Say $A = k[r]$. If r is transcendental over k, pick $y_1 = r$. Otherwise, r is algebraic over k so that A is finite over k.

2° $m > 1$: If the $r_1, \dots, r_m$ are algebraically independent over k, then done. Otherwise, there's a nonzero $f \in k[x_1, \dots, x_m]$ such that $f(r_1, \dots, r_m) = 0$. Renumbering the subscripts, if necessary, we assume $f(x_1, \dots, x_m)$ is not a constant in the variable x_m. Let $d = \deg f$, the maximum of the total monomial degrees. For $j = 1, \dots, m-1$, define

$$X_j := x_j - x_m^{(1+d)^j}$$

For each monomial $x_1^{e_1} \cdots x_m^{e_m}$, we have

$$\begin{aligned} x_1^{e_1} \cdots x_m^{e_m} &= (X_1 + x_m^{1+d})^{e_1} \cdots (X_{m-1} + x_m^{(1+d)^{m-1}})^{e_{m-1}} x_m^{e_m} \\ &= x_m^{e_m + e_1(1+d) + \cdots + e_{m-1}(1+d)^{m-1}} + \cdots \end{aligned}$$

Note that different $(e_1, \dots, e_m)$ give polynomials in $X_1, \dots, X_{m-1}, x_m$ with the different highest degrees of x_m.
Now write

$$g(X_1, \dots, X_{m-1}, x_m) = f(X_1 + x_m^{e_1(1+d)^1}, \dots, x_m^{e_m}) = c x_m^N + \sum_{j=0}^{N-1} h(X_1, \dots, X_{m-1}) x_m^j$$

for non-zero $c \in k$. For $j = 1, \dots, m-1$, let $s_j = r_j - r_m^{(1+d)^j}$. Then $\frac{1}{c} g(s_1, \dots, s_{m-1}, r_m) = \frac{1}{c} f(r_1, \dots, r_m) = 0$, i.e, r_m is integral over $B := k[s_1, \dots, s_{m-1}]$. By induction hypothesis, there exists $y_1, \dots, y_d$, $(0 \leqslant d \leqslant m-1)$ such that $y_1, \dots, y_d$ are algebraically independent over k and B is finite over $k[y_1, \dots, y_d]$ and thus A is finite over $k[y_1, \dots, y_d]$.

□

4.7 Zariski's lemma. Let K/k be a field extension. If K is of finite type over k, it's finite over k.

Proof. By normalization lemma, $k \subseteq k[y_1, \ldots, y_d] \subseteq K$ with K finite over $k[y_1, \ldots, y_d]$ for some algebraically independent elements $y_1, \ldots, y_d$ over k. Since K is a field, it follows that $k[y_1, \ldots, y_d]$ is a field, and thus $d = 0$, i.e, K is algebraic over k. Since K is finitely generated as k-algebra, $[K : k]$ is finite. □

4.8 Hilbert's Nullstellensatz. In what follows, we assume k is an algebraically closed field.

(i) (Weak form) There is a bijection

$$\begin{aligned} \mathbb{A}_k^n &\longrightarrow \operatorname{mSpec} k[x_1, \ldots, x_n] \\ (a_1, \ldots, a_n) &\longmapsto (x_1 - a_1, \ldots, x_n - a_n) \end{aligned}.$$

In particular, if $I \trianglelefteq k[x_1, \ldots, x_n]$ is a proper ideal, then $V(I) \neq \varnothing$ in $\mathbb{A}_k^n$.

(ii) For any ideal I of $k[x_1, \ldots, x_n]$, we have the equality

$$I(V(I)) = \sqrt{I}.$$

Proof.

(i) Clearly, $(x_1 - a_1, \ldots, x_n - a_n)$ is a maximal ideal. Conversely, let $\mathfrak{m} \in \operatorname{mSpec} k[x_1, \ldots, x_n]$. Then $K := k[x_1, \ldots, x_n]/\mathfrak{m}$ is a field of finite type over k, so by (4.7) K is finite over k. In particular, K/k is algebraic. But k is algebraically closed, this implies $K = k$. For each $1 \leqslant i \leqslant n$, we have $x_i - a_i \in \mathfrak{m}$ for some $a_i \in k$, and thus $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$.

(ii) The nontrivial part if $I(V(I)) \subseteq \sqrt{I}$. Assume $g \in I(V(I))$ and $I = (f_1, \ldots, f_m)$. Introduce a new indeterminate x_{m+1}, and consider the ideal

$$I' = (f_1, \ldots, f_m, gx_{n+1} - 1) \trianglelefteq k[x_1, \ldots, x_n, x_{n+1}]$$

Then $V(I') = \varnothing$, so by (i), it must be the case $(f_1, \ldots, f_m, gx_{m+1} - 1) = k[x_1, \ldots, x_n, x_{n+1}]$. But note

$$\frac{k[x_1, \ldots, x_n]_g}{I_g} \cong \frac{k[x_1, \ldots, x_n, x_{n+1}]}{I'},$$

so that $1 \in I_g$. This means

$$1 = \frac{a_1 f_1 + \cdots + a_m f_m}{g^\ell}$$

for some $a_i \in k$ and $\ell \in \mathbb{Z}_{\geqslant 0}$, which exactly means $g \in \sqrt{I}$.

□

4.9 Let A be a k-algebra of finite type. Say $\{y_1, \ldots, y_s\}$ is a generating set of A. Then there is a surjection

$$\begin{aligned} \Phi : k[x_1, \ldots, x_s] &\longrightarrow A = k[y_1, \ldots, y_s] \\ x_i &\longmapsto y_i \end{aligned}$$

Since $k[x_1, \ldots, x_s]$ is Noetherian, $\ker(\Phi) = (f_1, \ldots, f_m)$ for some $f_1, \ldots, f_m \in k[x_1, \ldots, x_s]$, and we can form the algebraic set

$$V = V(\ker(\Phi)) = V((f_1, \ldots, f_m)) \subseteq \mathbb{A}_k^s.$$

The coordinate ring of V is by definition $k[V] = k[x_1, \ldots, x_s]/I(V)$. By Nullstellensatz, we have $I(V) = \sqrt{\ker(\Phi)}$. Hence if A is a reduced ring, we have $k[V] \cong A$ as k-algebras.

Suppose $\{z_1, \ldots, z_r\}$ is another generating set of A. Then there is a similarly defined surjection $\Psi : k[x_1, \ldots, x_r] \to A$ and the algebraic set $V' = V(\ker(\Psi)) \subseteq \mathbb{A}^r_k$. We have the isomorphism

$$k[V'] \cong A \cong k[V]$$

as k-algebras, so $V \cong V'$ as algebraic sets over k.

Let $\mathbf{AffVar}_k$ be the category of affine algebraic sets over k whose morphisms are defined as (4.3), and let $\mathbf{redfgAlg}_k$ be the full subcategory of $\mathbf{Alg}_k$ consisting of reduced k-algebra of finite type. Together with (4.4), we then have shown that there is an anti-equivalence of categories

$$\begin{aligned} \mathbf{AffVar}_k &\longrightarrow \mathbf{redfgAlg}_k \\ V &\longmapsto k[V] \end{aligned}$$

4.10 Let A be a reduced k-algebra of finite type. Let $\{y_1, \ldots, y_n\}$ be a generating set of A and form the surjection $\Phi : k[x_1, \ldots, x_n] \to A$. Let $V = V(\ker(\Phi)) \subseteq \mathbb{A}^n_k$. Let $p = (a_1, \ldots, a_n) \in V$ and consider the maximal ideal $\mathfrak{m}_p = (x_1 - a_1, \ldots, x_n - a_n)$ of $k[x_1, \ldots, x_n]$. By (4.2).(ii), we have $\ker(\Phi) \subseteq \mathfrak{m}_p$. Conversely, if $\mathfrak{m}$ is a maximal ideal of $k[V]$, regarding it as a maximal ideal in $k[x_1, \ldots, x_n]$ containing $\ker(\Phi)$, by Nullstellensatz we obtain a point $p \in \mathbb{A}^n_k$ with $\mathfrak{m}_p = \mathfrak{m}$. The point lies in V as $\{p\} = V(\mathfrak{m}_p) \subseteq V(\ker(\Phi)) = V$. This establishes a bijection between V and mSpec $k[V]$, and by the isomorphism $k[V] \cong A$, we have a bijection

$$\begin{aligned} V &\longrightarrow \operatorname{mSpec} A \\ p &\longmapsto \Phi(\mathfrak{m}_p). \end{aligned}$$

Now let $a \in A$ and let $f \in k[V]$ be the corresponding element. Then the value of f at a point $p \in V$ is the same as the class of a in the residue field $\kappa(\Phi(\mathfrak{m}_p))$, i.e., the image of A in $A/\Phi(\mathfrak{m}_p)$. Indeed, we have $A/\Phi(\mathfrak{m}_p) \cong k[V]/\mathfrak{m}_p \cong k$ as k-algebras.

4.11 Retain the notations in (4.10). For a ring R, equip mSpec R with the subspace topology from the topology on the affine scheme Spec R. Then clearly mSpec $k[V] \cong$ mSpec A as topological spaces. Also, it follows from the very definition that the bijection

$$\begin{aligned} \mathbb{A}^n_k &\longrightarrow \operatorname{mSpec} k[x_1, \ldots, x_n] \\ (a_1, \ldots, a_n) &\longmapsto (x_1 - a_1, \ldots, x_n - a_n) \end{aligned}.$$

in Nullstellensatz is a homeomorphism. For an ideal I, we have a commutative diagram

$$\begin{tikzcd} \mathbb{A}^n_k \arrow[r, "\sim"] & \operatorname{mSpec} k[x_1, \ldots, x_n] \\ V(I) \arrow[u] \arrow[r, "\sim"] & \operatorname{mSpec} k[V] \arrow[u] \end{tikzcd}$$

The vertical maps are all closed embeddings, so the horizontal map on the bottom is a homeomorphism as well. Altogether we see the bijection

$$V \longrightarrow \operatorname{mSpec} A$$

in (4.10) is a homeomorphism.

Let $f : B \to A$ be a k-algebra homomorphism of reduced k-algebra of finite types. Let W be any affine algebraic set constructed as in (4.10) with A replaced by B. Then we have a k-algebra homomorphism $f' : k[W] \cong B \xrightarrow{f} A \cong k[V]$, and by (4.4) it gives a morphism $F : V \to W$ of algebraic sets. We claim the diagram

$$\begin{tikzcd} W \arrow[r, "\sim"] & \operatorname{mSpec} k[W] \\ V \arrow[u, "F"] \arrow[r, "\sim"] & \operatorname{mSpec} k[V] \arrow[u, "f'^{-1}"'] \end{tikzcd}$$

commutes. This is clear from the construction of F (4.4).

4.12 Consider the projective space $\mathbb{P}^n$ over k with homogeneous coordinates $x_0, \dots, x_n$. For a point $p \in \mathbb{P}^n$, it lies in some affine pieces $D_+(x_i)$ which is isomorphic to the affine space $\mathbb{A}^n_k$. By Nullstellensatz it corresponds to a maximal ideal in $k[x_0, \dots, x_n]_{x_i}$, which can be viewed as a maximal ideal in $k[x_0, \dots, x_n]$ not containing x_i; call this maximal ideal $\mathfrak{m}_p$. If p also lies in $D_+(x_j)$, it will then corresponds to a maximal ideal in $k[x_0, \dots, x_n]_{x_j}$. Nevertheless, since $p \in D_+(x_i) \cap D_+(x_j)$, it gives a maximal ideal in $k[x_0, \dots, x_n]_{x_i x_j}$, and it can be obtained by localizing $\mathfrak{m}_p$ at x_j. By symmetry we see $\mathfrak{m}_p \in \operatorname{mSpec} k[x_0, \dots, x_n]$ is well-defined. Thus we obtain a well-defined map

$$\begin{aligned} \mathbb{P}^n_k &\longrightarrow \{\mathfrak{m} \in \operatorname{mSpec} k[x_0, \dots, x_n] \mid (x_0, \dots, x_n) \nsubseteq \mathfrak{m}\} \\ p &\longmapsto \mathfrak{m}_p. \end{aligned}$$

The procedure of obtaining $\mathfrak{m}_p$ from p above also shows that this is a bijection. It is this bijection that motivates the Proj construction in (3.106).

4.13 Projective Hilbert's Nullstellensatz. Let I be a homogeneous ideal of $k[x_0, \dots, x_n]$.

4.1.3 Sheaf of regular functions

Recall that we are assuming k is algebraically closed.

4.14 Let Y be an algebraic set and $V \subseteq Y$ a Zariski open set. A **regular function on** V is a map $g : V \to \mathbb{A}^1$ such that any point $y \in V$ admits an open neighborhood $W \subseteq Y$ and $u, v \in k[Y]$ with u nonvanishing on W such that $g|_W = \dfrac{v|_W}{u|_W}$. A regular function on V is continuous in the Zariski topology.

We put $\mathcal{O}_Y(V)$ to be the set of regular functions on V. The assignment $V \mapsto \mathcal{O}_Y(V)$ is clearly a sheaf of k-algebras on Y, and $(Y, \mathcal{O}_Y)$ is a local-ringed space. Indeed, if a regular function defined near a point $y \in Y$ is nonzero at y, then by continuity it is nonvanishing on an open neighborhood of y, making it an invertible element in the stalk $\mathcal{O}_{Y,y}$.

4.15 Let Y be an algebraic set. There is an inclusion $k[Y] \to \mathcal{O}_Y(Y)$. Then for $f \in k[Y]$, we can form the open set Y_f as in (2.21), and we see there that $f \in \mathcal{O}_Y(Y_f)^\times$. We can compare Y_f with $D(f)$. In fact, $D(f) = Y_f$. Indeed, $y \in Y_f$ if and only if $f_y \in \mathcal{O}^\times_{Y,y}$, if and only if $f(y) \neq 0$, or $y \in D(f)$.

By the universal property of localization, the composition $k[Y] \to \mathcal{O}_Y(Y) \to \mathcal{O}_Y(Y_f)$ induces a homomorphism $\theta : k[Y]_f \to \mathcal{O}_Y(Y_f)$. This is injective, for if $\theta(gf^{-m}) = 0$, then $g|_{Y_f} = 0$, or $Y_f \subseteq V(g)$, which implies $\varnothing = Y_f \cap Y_g = D(fg)$, or $fg = 0 \in k[Y]$. This exactly means $gf^{-m} = 0$ in $k[Y]_f$. Moreover,

Theorem. The canonical map $\theta : k[Y]_f \to \mathcal{O}_Y(Y_f)$ is a k-algebra isomorphism.

Proof. It remains to show the surjectivity. The proof is similar to (3.3), with some extra topological concern. Let $g \in \mathcal{O}_Y(Y_f)$. By definition, each $y \in Y_f$ admits an open neighborhood $W_y \subseteq Y_f$ and $v_y, u_y \in k[Y]$ with u_y nonvanishing on W_y such that $g|_{W_y} = \dfrac{v_y|_{W_y}}{u_y|_{W_y}}$. Since $\{Y_g \mid g \in k[Y]\}$ is a basis for Y, we can find $u'_y \in k[Y]$ such that $y \in Y_{u'_y} \subseteq W_y \subseteq Y_{u_y}$. Taking complement gives $V(u'_y) \supseteq V(u_y)$, and by Nullstellensatz, $\sqrt{(u'_y)} \subseteq \sqrt{(u_y)}$. Thus $u'^m_y = u_y u''_y$ for some $u''_y \in k[Y]$ and $m \in \mathbb{Z}_{\geqslant 1}$; then

$$\frac{v_y}{u_y} = \frac{v_y u''_y}{u_y u''_y} = \frac{v_y u''_y}{(u'_y)^m}.$$

Since k[Y] is Noetherian, Y is Noetherian, which implies Y_f is Noetherian, and hence compact. Thus we can find a finite set $\{y_i\}_{i\in I} \subseteq Y_f$ such that $\{Y_{u'_{y_i}} = Y_{(u'_{y_i})^m}\}_{i\in I}$ covers Y_f. Replacing v_y by $v_y u''_y$ and u_y by $(u'_y)^m$, we find $u_1, \ldots, u_n, v_1, \ldots, v_n \in k[Y]$ such that $g|_{Y_{u_j}} = \dfrac{v_j|_{Y_{u_j}}}{u_j|_{Y_{u_j}}}$ with $Y_f = Y_{u_1} \cup \cdots \cup Y_{u_n}$.

For each $i \neq j$, we have

$$\frac{v_i|_{Y_{u_i}\cap Y_{u_j}}}{u_i|_{Y_{u_i}\cap Y_{u_j}}} = g|_{Y_{u_i}\cap Y_{u_j}} = \frac{v_j|_{Y_{u_i}\cap Y_{u_j}}}{u_j|_{Y_{u_i}\cap Y_{u_j}}}$$

so $(v_i u_j - v_j u_i)|_{Y_{u_i}\cap Y_{u_j}} = 0$. This implies $u_i u_j (v_i u_j - v_j - u_i) = 0$ on Y. Further replacing v_j by $v_j u_j$ and u_j by u_j^2, we may assume $u_i v_j - u_j v_i = 0$ on Y for any i, j. Since $Y_f = \bigcup_{i=1}^{n} Y_{u_i}$, we have $\sqrt{(f)} = \sqrt{(u_1, \ldots, u_n)}$ so that

$$f^m = a_1 u_1 + \cdots + a_n u_n$$

for some $a_1, \ldots, a_n \in k[Y]$ and $m \in \mathbb{Z}_{\geqslant 1}$. Define $v = a_1 v_1 + \cdots + a_n v_n$. We claim $g f^m = v$, which will imply $\theta(v f^{-m}) = g$, completing the proof. This is easy, as for any $1 \leqslant j \leqslant n$,

$$g f^m|_{Y_{u_j}} = \sum_{i=1}^{n} a_i|_{Y_{u_j}} (g u_i)|_{Y_{u_j}} = \sum_{i=1}^{n} a_i|_{Y_{u_j}} (\frac{v_j}{u_j} u_i)|_{Y_{u_j}} = \sum_{i=1}^{n} a_i|_{Y_{u_j}} v_i|_{Y_{u_j}} = v|_{Y_{u_j}}.$$

□

4.15.1 Corollary. Let Y be an algebraic set.

(a) $\mathcal{O}_Y(Y) = k[Y]$.

(b) For any $y \in Y$, we have $\mathcal{O}_{Y,y} \cong k[Y]_{I(\{y\})}$.

Note these strike a resemblance with the results in (3.2).

Proof. (a) follows directly from Theorem 4.15. For (b), we compute

$$\mathcal{O}_{Y,y} = \varinjlim_{y\in U} \mathcal{O}_Y(U) \cong \varinjlim_{f\in k[Y],\, f(y)\neq 0} \mathcal{O}_Y(Y_f) \cong \varinjlim_{f\notin I(\{y\})} k[Y]_f \cong k[Y]_{I(\{y\})}.$$

The third isomorphism results from the functoriality of the isomorphism in Theorem 4.15, and the final isomorphism follows from the same reason as in (3.2). □

4.16 Let X be a topological space. The assignment $U \mapsto \mathrm{Hom}_{\mathbf{Set}}(U, k)$ defines a sheaf of k-algebras on X, which we denote by k^X. A local-ringed space $(X, \mathcal{O}_X)$ is called a **basic k-space** if $\mathcal{O}_X$ is a subsheaf of k-algebras of k^X and for any $x \in X$, the unique maximal ideal of $\mathcal{O}_{X,x}$ is

$$\mathfrak{m}_{X,x} = \{g_x \in \mathcal{O}_{X,x} \mid g(x) = 0 \in k\}.$$

A **morphism of basic k-spaces** is a continuous map $f : X \to Y$ such that for any open $V \subseteq Y$ and $g \in \mathcal{O}_Y(V)$, we have $g \circ f \in \mathcal{O}_X(f^{-1}(V))$. In other words, a morphism of basic k-space is a morphism in **LRS** such that the morphism on sheaves is given by function pullback. The category of basic k-spaces is denote by $\mathbf{bSp}_k$.

As an example, if Y is an algebraic set, we see in (4.14) that $(Y, \mathcal{O}_Y)$ is a basic k-space.

4.17 Definition. Let $(X, \mathcal{O}_X)$ be a basic k-space.

1. X is called an **affine variety over** k if there exist an algebraic set Y over k such that $(X, \mathcal{O}_X) \cong (Y, \mathcal{O}_Y)$ as basic k-spaces.
2. X is called an **(algebraic) variety over** k if it admits a finite open cover $\mathcal{U}$ such that $(U, \mathcal{O}_X|_U)$ is an affine variety over k for any $U \in \mathcal{U}$.

Denote by $\mathbf{AffVar}_k$ and $\mathbf{Var}_k$ the full subcategories of $\mathbf{bSp}_k$ whose objects consist of affine varieties and varieties over k, respectively.

4.18 Note that by (4.4), $\mathbf{AffVar}_k$ defined in (4.17) is equivalent to the one defined in (4.9), and by (4.15.1) the anti-equivalence there now takes the form

$$\begin{array}{ccc} \mathbf{AffVar}_k & \longrightarrow & \mathbf{redfgAlg}_k \\ X & \longmapsto & \mathcal{O}_X(X) \end{array}$$

A coordinate-free description of the inverse is given by $A \mapsto \operatorname{mSpec} A$.

4.19 Theorem. Let X be a basic k-space and Y an affine variety over k. There exists a bijection

$$\begin{array}{ccc} \operatorname{Hom}_{\mathbf{bSp}_k}(X, Y) & \longrightarrow & \operatorname{Hom}_{\mathbf{Alg}_k}(k[Y], \mathcal{O}_X(X)) \\ f : X \to Y & \longmapsto & k[Y] = \mathcal{O}_Y(Y) \ni g \mapsto g \circ f \end{array}$$

Proof. The proof is similar to that of Theorem 3.7. In fact, the proof for injectivity is exactly the same. For surjectivity, let $\theta \in \operatorname{Hom}_{\mathbf{Alg}_k}(k[Y], \mathcal{O}_X(X))$ and define $f : X \to \operatorname{Spec} k[Y]$ by setting $f(x) = \theta^{-1}(\operatorname{res}^X_x)^{-1}(\mathfrak{m}_{X,x})$. We claim $f(x)$ is in fact a maximal ideal. In fact, from the definition, the composition $k \to \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$ is an isomorphism. Then the homomorphism $\operatorname{res}^X_x \theta : k[Y] \to \mathcal{O}_{X,x}$ gives rise to an isomorphism $k[Y]/(\operatorname{res}^X_x \theta)^{-1}(\mathfrak{m}_{X,x}) \cong \mathcal{O}_{X,x}/\mathfrak{m}_{X,x} \cong k$, which implies $f(x) = (\operatorname{res}^X_x \theta)^{-1}(\mathfrak{m}_{X,x})$ is a maximal ideal. Thus f is in fact a map $f : X \to \operatorname{mSpec} k[Y]$, which by Nullstellensatz gives a map $f : X \to Y$ in turn. This is a continuous map, as shown in Theorem 3.7. It remains to show

1. for any open $V \subseteq Y$ and $g \in \mathcal{O}_Y(V)$, $g \circ f \in \mathcal{O}_X(f^{-1}(V))$, and
2. for any $g \in k[Y]$, $\theta(g) = g \circ f \in \mathcal{O}_X(X)$.

For 2., if $x \in X$, then the value $\theta(g)(x)$ is the same as the class of $\theta(g)$ in $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x} \cong k$. On the other hand, $g(f(x))$ is the same as the class of g in $\mathcal{O}_{Y,f(x)}/\mathfrak{m}_{Y,f(x)}$, which is isomorphic to $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$ as k-algebras. Hence $\theta(g)(x) = g(f(x)) \in k$. For 1., we use the argument in Theorem 3.7 to obtain, for any $h \in k[Y]$, a map $\theta_h : \mathcal{O}_Y(Y_h) \to \mathcal{O}_X(X_{\theta(h)})$. Explicitly,

$$\theta_h\left(\frac{g}{h^m}\right) = \frac{\theta(g)}{\theta(h)^m}.$$

By 2. we have $\theta_h(gh^{-m}) = \dfrac{g \circ f}{(h \circ f)^m} = (gh^{-m}) \circ f$, so 1. holds for open sets of the form $Y_h = D(h)$. The general case follows from covering V with open sets of this form and the sheaf axiom. □

4.20 Remark. A k**-space** is a local-ringed space $(X, \mathcal{O}_X)$ over k (2.20.1) such that the composition $k \to \mathcal{O}_{X,x} \to \kappa(x)$ is a field isomorphism for any $x \in X$. Denote by $\mathbf{Sp}_k$ the full subcategory of $\mathbf{LRS}_k$ whose objects are all k-spaces.

It is straightforward to check that a basic k-space is a k-space, and there is a faithful functor $\mathbf{bSp}_k \to \mathbf{Sp}_k$. There is also a bijection

$$\operatorname{Hom}_{\mathbf{Sp}_k}(X, Y) \longrightarrow \operatorname{Hom}_{\mathbf{Alg}_k}(k[Y], \mathcal{O}_X(X))$$

defined as in Theorem 3.7. The proof is the same as (4.19).

4.1.4 Dimension

4.21 Definition. Let X be a topological space.

1. A subset $C \subseteq X$ is called **locally closed** if it is an intersection of an open set and a closed set in X.
2. A subset of X is called **constructible** if it is a finite union of locally closed subsets.

4.22 Theorem (Chevalley). Let $f : X \to Y$ be a morphism of algebraic varieties. If $C \subseteq X$ is constructible, then $f(C) \subseteq Y$ is constructible.

4.22.1 Lemma Let $f : X \to Y$ be a morphism of algebraic varieties. Then $f(X)$ contains a non-empty open subset of its closure $\overline{f(X)}$.

Proof. Using affine opens to cover Y, we may assume Y is affine. Also we can assume X is affine. Since X only has finitely many irreducible components, we can further assume X is irreducible. If we replace Y by $\overline{f(X)}$, then f induces a k-algebra injective homomorphism $f^* : k[Y] \to k[X]$ with $k[X]$ an integral domain. We pick $s \in k[Y]$ as in Corollary 4.49.(i). We claim $D(s) = f(D(f^*s))$. If $y \in f(D(f^*s))$, then $y = f(x)$ for some x such that $f^*s(x) \neq 0$, or $s(y) = s(f(x)) \neq 0$. Conversely, if $y \in D(s)$, then evaluation at y defines a homomorphism $\mathrm{ev}_y : k[Y] \to k$ with $\mathrm{ev}_y(s) = s(y) \neq 0$. The corollary implies there exists $\phi : k[X] \to k$ extending $\mathrm{ev}_y(s)$, so $\phi(s) \neq 0$ and $\phi \circ f^* = \mathrm{ev}_y$. Since ϕ is nonzero, it corresponds to a point in $k[X]$, say $x \in X$. Then $f(x) = y$ and $f^*s(x) = s(y) \neq 0$. This proves the claim, and in particular $D(s) \subseteq f(X)$. ☐

4.22.2 Proof of Chevalley theorem. First assume $C = X$. We may replace Y by $\overline{f(X)}$. By a previous lemma, there is an open set $U \subseteq Y$ with $U \subseteq f(X)$. If $U = f(X)$, we are done. Otherwise, let $X' := X \backslash f^{-1}(U) \subsetneq X$; then $\dim X' < \dim X$ and by induction on dimension on the domain we see $f(X')$ is constructible. Then $f(X) = U \cup f(X')$ is construtible. It remains to check the case $\dim X = 0$. In this case X is a finite set of points with discrete topology, so it suffices to show a singleton is construtible. This is clear as a point is closed.

For the general case, we can assume C is locally closed in X. If C is closed, then by the previous case applied to $f|_C : C \to Y$, we see $f|_C(C) = f(C)$ is construtible in Y. It remains to deal with the case when C is open.

4.1.5 Associated complex analytic spaces

In this subsubsection by a variety we mean an irreducible algebraic variety.

4.23 Now we consider $\mathbb{C}^n$ as the usual euclidean space. Let U be an open subset of $\mathbb{C}^n$ and denote by $\mathcal{O}_U$ the sheaf of holomorphic functions on U. Let $f_1, \dots, f_m \in \mathcal{O}_U(U)$ be holomorphic functions and let $Y = V(f_1, \dots, f_m)$ be the common zero locus of these $f_1, \dots, f_m$. Put $\mathcal{O}_Y = (\mathcal{O}_U/(f_1, \dots, f_m))|_Y$, where $(f_1, \dots, f_m) \trianglelefteq \mathcal{O}_U$ is the ideal sheaf generated by $f_1, \dots, f_m$. It is clear that $(Y, \mathcal{O}_Y)$ is a local-ringed space, and the sheaf $\mathcal{O}_Y$ is a sheaf of $\mathbb{C}$-algebra.

4.24 Definition. A **complex analytic space** is a local-ringed space $(X, \mathcal{O}_X)$ over $\mathbb{C}$ that admits an open cover $\mathcal{U}$ such that for any $U \in \mathcal{U}$, $(U, \mathcal{O}_X|_U) \cong (Y, \mathcal{O}_Y)$ in $\mathbf{LRS}_{\mathbb{C}}$ for some Y defined as in (4.23).

4.25 Let X be a variety over $\mathbb{C}$. We cover X with affine open subsets Y_i. Each Y_i is isomorphic to some algebraic set, so we can write $Y_i = V(f_1, \dots, f_m) \subseteq \mathbb{A}^n_{\mathbb{C}}$ for some polynomials $f_1, \dots, f_m \in \mathbb{C}[x_1, \dots, x_n]$. If we regard the f_i as holomorphic functions defined on $\mathbb{C}^n$, then Y_i is an complex analytic space, which we denote by $(Y_i)^{an}$. To distinguish the sheaf of holomorphic functions from that of regular functions, we denote the former by $\mathcal{O}^{an}_{Y_i}$. The gluing data for $(X, \mathcal{O})$ allows us to glue $(Y_i, \mathcal{O}^{an}_{Y_i})$ together, by (2.13), to a complex analytic space. We denote the resulting space by $(X^{an}, \mathcal{O}^{an}_X)$.

The underlying spaces of X and X^{an} are the same; the only difference is the topology. We refer to the topology on X^{an} the **classical topology/analytic topology** on X.

4.25.1 Let A be a reduced $\mathbb{C}$-algebra of finite type. By picking a finite generating set $\{y_1, \ldots, y_s\}$ of A, we obtain a closed embedding $\mathrm{mSpec}(A) \to \mathbb{A}^s_{\mathbb{C}}$ with image $V(\ker(\Phi))$, where $\Phi : k[x_1, \ldots, x_s] \to A$ is given by $\Phi(x_i) = y_i$ (4.9). In this way the affine variety $\mathrm{mSpec}(A)$ has a analytic topology inherited from $\mathbb{A}^s_{\mathbb{C}}$. If $\{z_1, \ldots, z_r\}$ is another finite generating set of A, via $\Psi : \mathbb{C}[X_1, \ldots, X_r] \to A$ the affine variety $\mathrm{mSpec}(A)$ has another analytic topology from $\mathbb{A}^r_{\mathbb{C}}$. We show these topologies on $\mathrm{mSpec}(A)$ are homeomorphic in analytic topology. But this is rather clear, for a morphism $V(\ker(\Phi)) \to V(\ker(\Psi))$ comes from a polynomial map $\mathbb{A}^s_{\mathbb{C}} \to \mathbb{A}^r_{\mathbb{C}}$, and a polynomial is continuous.

4.26 Lemma. Let X be a variety over $\mathbb{C}$. If $U \subseteq X$ is a nonempty Zariski open set, then U is classically dense.

Proof. Covering X by affine open sets, we may assume X is affine. Since $D(f) \subseteq U$ for some $0 \neq f \in \mathbb{C}[X]$, to prove the lemma we may assume $U = D(f)$. Taking complement, we need to show $V(f) \subseteq X$ is classically nowhere dense, and it suffices to show $V(f)$ has empty classical interior. If $V(f)$ contains a nonempty open set V, then f is trivial on V, which by identity principle $f \equiv 0$ throughout, a contradiction to $f \neq 0$. □

4.27 Lemma. Let X be a variety over $\mathbb{C}$ and Y a construtible subset of X. Then the Zariski closure of Y in X coincides with the classical closure in X.

Proof. We may assume Y is nonempty Zariski locally closed. Denote by $\overline{Y}^Z$ and $\overline{Y}^C$ the Zariski closure of Y and the classical closure of Y in X, respectively; we always have $\overline{Y}^C \subseteq \overline{Y}^Z$. By definition, Y is Zariski open in $\overline{Y}^Z$, so by (4.26) Y is classically dense in $\overline{Y}^Z$, which means $\overline{Y}^Z \subseteq \overline{Y}^C$. □

4.28 Lemma. Let $U \subseteq \mathbb{A}^n_{\mathbb{C}}$ be a Zariski open set. Then U is classically path-connected.

Proof. Let $x \neq y \in U$ and pick any affine line $L \subseteq \mathbb{A}^n_{\mathbb{C}}$ connecting x, y. Since $U \cap L \subseteq L$ is Zariski open and $L \cong \mathbb{A}^1_{\mathbb{C}}$, $\#(L \backslash U) < \infty$. But $L \cong \mathbb{R}^2$ is a plane, so $U \cap L = L \backslash (L \backslash U)$ is still path-connected. In particular, there is a (smooth) path in $U \cap L$ connecting x and y. □

4.29 Theorem. A variety X over $\mathbb{C}$ is separated if and only if the complex analytic space X^{an} is Hausdorff.

Proof. Since the diagonal $\Delta : X \to X \times_{\mathbb{C}} X$ is always an immersion, we only need to show $\Delta(X)$ is Zariski closed if and only if it is classically closed. But $\Delta(X)$ is construtible by Chevalley theorem, so the result follows from Lemma 4.27.

□

4.30 Theorem. A variety X over $\mathbb{C}$ is complete if and only if the complex analytic space X^{an} is compact.

4.31 Theorem. A morphism $X \to Y$ of varieties over $\mathbb{C}$ is proper if and only if the continuous map $X^{an} \to Y^{an}$ is classically proper.

4.2 Schemes and varieties

4.2.1 Generalities on topology

4.32 Definition. A topological space is called **sober** if every irreducible closed subset has a unique generic point.

4.33 Let X be a sober space. For example, a scheme is sober (3.36). In fact, from the proof there we see a topological space with an open cover consisting of sober spaces is again sober. We have a bijection

$$\begin{aligned} X &\longrightarrow \{\text{irreducible closed subset of } X\} \\ x &\longmapsto \overline{\{x\}}. \end{aligned}$$

Not every topological space is sober. Nevertheless, with each topological space we may associate a sober space in a functorial way, which we will discuss in the next paragraph.

4.34 Soberification. Comparing the topology on $\operatorname{Spec} A$ (3.1) and that on an affine variety (4.1), we are naturally led to the following construction. For a topological space, define $\operatorname{Sob}(X)$ to be the set of all irreducible closed subsets in X. For a closed subset C of X, define

$$V(C) = V^X(C) := \{Z \in \operatorname{Sob}(X) \mid Z \subseteq C\}.$$

Then $\left\{V(C) \mid C \underset{\text{closed}}{\subseteq} X\right\}$ forms a topology on $\operatorname{Sob}(X)$. Namely,

(i) $V(C_1) \cup V(C_2) = V(C_1 \cup C_2)$ for any closed subsets C_1, C_2 of X.

(ii) $V\left(\bigcap_\alpha C_\alpha\right) = \bigcap_\alpha V(C_\alpha)$ for any family $\{C_\alpha\}$ of closed subsets of X.

(iii) $V(\varnothing) = \varnothing$, and $V(X) = \operatorname{Sob}(X)$.

These are clear. Recall that by definition an irreducible space is nonempty (3.34). The resulting topological space $\operatorname{Sob}(X)$ is called the **soberification** of X. There is a natural inclusion $\iota : X \to \operatorname{Sob}(X)$ defined by $x \mapsto \overline{\{x\}}$. This is a continuous map, for

$$\iota^{-1}(V(C)) = \{x \in X \mid \overline{\{x\}} \in C\} = \{x \in X \mid x \in C\} = C$$

for any closed subset C of X. In fact, this shows the inverse ι^{-1} induces a bijection between $\operatorname{Top}(\operatorname{Sob}(X))$ and $\operatorname{Top}(X)$.

As its name indicates, the space $\operatorname{Sob}(X)$ is a sober topological space. Indeed, the aforementioned bijection implies that C is irreducible if and only if $V(C)$ is irreducible. If $Z \in \operatorname{Sob}(X)$ and $Z \in V(C)$ for some closed C', it is obvious that $V(Z) \subseteq V(C)$. This implies that $V(Z) = \overline{\{Z\}}$ in $\operatorname{Sob}(X)$. If $V(Z) = V(Z')$ for two $Z, Z' \in \operatorname{Sob}(X)$, then clearly $Z = Z'$. This finishes the proof that $\operatorname{Sob}(X)$ is sober. We will see in the following paragraph that $\iota = \iota_X : X \to \operatorname{Sob}(X)$ is universal among all the other sober spaces. To give an intuition, we first observe that ι is a homeomorphism if and only if X is sober. Indeed, if X is sober, then x is uniquely determined by $\overline{\{x\}}$, so ι is a bijection. A continuous bijection whose inverse ι^{-1} induces a bijection between topologies is by definition a homeomorphism. In particular, this shows $\iota : X \to \operatorname{Sob}(X)$ is a homeomorphism.

4.35 Universality of soberification. Denote by **Sob** the full subcategory of **Top** consisting of sober topological spaces. Let X be a topological space, and define a functor $\mathbf{Sob} \to \mathbf{Set}$ by $Y \mapsto \operatorname{Hom}_{\mathbf{Top}}(X, Y)$. Then the soberification $\iota : X \to \operatorname{Sob}(X)$ represents this functor.

As a first step, we show Sob actually defines a functor from **Top** to **Sob**. If $f : X \to Y$ is a continuous map, define $\operatorname{Sob}(f) : \operatorname{Sob}(X) \to \operatorname{Sob}(Y)$ by sending Z to $\overline{f(Z)}$. This is continuous, as for any $C' \underset{\text{closed}}{\subseteq} Y$, we have

$$\operatorname{Sob}(f)^{-1}(V(C')) = \{Z \in \operatorname{Sob}(X) \mid \overline{f(Z)} \in V(C')\} = \{Z \in \operatorname{Sob}(X) \mid f(Z) \subseteq C'\} = V(f^{-1}(C')).$$

If f, g are continuous maps with $g \circ f$ being defined, then $\overline{g(\overline{f(Z)})} = \overline{g(f(Z))}$. Also, $\operatorname{Sob}(\mathrm{id}_X) = \mathrm{id}_{\operatorname{Sob}(X)}$. These prove that $\operatorname{Sob} : \mathbf{Top} \to \mathbf{Sob}$ is really a functor. Additionally, for continuous $f : X \to Y$, since $\overline{f(x)} = \overline{f(\overline{\{x\}})}$ for any $x \in X$, there is a commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\iota_X"'] & Y \arrow[d, "\iota_Y"] \\ \operatorname{Sob}(X) \arrow[r, "\operatorname{Sob}(f)"'] & \operatorname{Sob}(Y) \end{tikzcd}$$

In particular, if Y is sober, the map ι_Y is a homeomorphism, and the diagram gives a functorial map

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Top}}(X,Y) &\longrightarrow \mathrm{Hom}_{\mathbf{Sob}}(\mathrm{Sob}(X),Y) \\ f &\longmapsto \iota_Y^{-1}\circ \mathrm{Sob}(f). \end{aligned}$$

In fact, this is a bijection with inverse $f' \mapsto f'\circ \iota_X$. To see this, let $f' \in \mathrm{Hom}_{\mathbf{Sob}}(\mathrm{Sob}(X),Y)$ and put $f = f'\circ\iota_X$. The identity $\mathrm{Sob}(f) = \iota_Y \circ f'$ is a tautology. This exactly tells what we claim in the beginning of this paragraph.

4.35.1 We can also describe open sets for the soberification of a space X. Let U be an open subspace of X and C its complement in X. Clearly for $Z \in \mathrm{Sob}(X)$, we have $Z \nsubseteq C$ if and only if $Z \cap U \neq \varnothing$. Thus, the set of the form

$$D(U) = D^X(U) = \mathrm{Sob}(X)\backslash V(C) = \{Z \in \mathrm{Sob}(X) \mid Z\cap U \neq \varnothing\}$$

is precisely an open set in $\mathrm{Sob}(X)$.

We record a fact that will be used later. Let $\mathcal{U}$ be an open cover of X. Then $\{D^X(U) \mid U \in \mathcal{U}\}$ also covers $\mathrm{Sob}(X)$. This is tautological. Let $Z \in \mathrm{Sob}(X)$. Then $Z\cap U \neq \varnothing$ for some $U \in \mathcal{U}$, so that $Z \in D(U)$.

Denote by $\alpha : U \to X$ the inclusion. Then we have a map $\mathrm{Sob}(\alpha) : \mathrm{Sob}(U) \to \mathrm{Sob}(X)$. This is in fact an open embedding with image precisely $D(U)$. Indeed, by definition, for $Z \in \mathrm{Sob}(U)$, we have

$$\mathrm{Sob}(\alpha)(Z) = \overline{\alpha(Z)} = \overline{Z} \in \mathrm{Sob}(X).$$

If $Z, Z' \in \mathrm{Sob}(U)$ satisfy $\overline{Z} = \overline{Z}'$ in X, then $Z\cap Z'$ and $Z \cap (\overline{Z}\backslash Z')$ are two closed subsets of Z whose union is Z. Since Z is irreducible and $Z\cap Z' \neq \varnothing$, we must have $Z\cap Z' = Z$, or $Z \subseteq Z'$. By symmetry we then obtain $Z = Z'$, proving that $\mathrm{Sob}(\alpha)$ is injective. For a closed subset $C \subseteq U$, we have

$$\mathrm{Sob}(\alpha)(V_U(C)) = \{\overline{Z} \mid Z \in \mathrm{Sob}(U),\, Z\subseteq C\} = V_X(\overline{C}) \cap \mathrm{Im}\,\mathrm{Sob}(\alpha).$$

The last equality holds, as $\overline{Z} \subseteq \overline{C}$ implies $Z = \overline{Z}\cap U \subseteq \overline{C}\cap U = C$. This shows $\mathrm{Sob}(\alpha)$ is an embedding, and it remains to show $\mathrm{Im}\,\mathrm{Sob}(\alpha) = D(U)$. The containment $\subseteq$ is clear. For $\supseteq$, if $Z \cap U \neq \varnothing$, since Z is irreducible, $Z\cap U$ is dense in Z, showing that $Z = \overline{Z\cap U}$ with $Z \cap U \in \mathrm{Sob}(U)$.

There is counterpart for a closed subspace $C \subseteq X$, stating that $\mathrm{Sob}(C) \to \mathrm{Sob}(X)$ is a closed embedding, and the proof is much easier.

4.36 By definition, a topological space X is $\mathbf{T_0}$ / **Kolmogorov** if for any $x \neq y \in X$ there exists a closed set in X containing exactly either x or y. We note here that the image $\iota(X)$ of X under $\iota : X \to \mathrm{Sob}(X)$ is T_0. To see this, let $x, y \in X$ with $\overline{\{x\}} \neq \overline{\{y\}}$. If every closed subset in $\iota(X)$ that contains, say, x also contains y, then $y \in \overline{\{x\}}$. By symmetry, we have $x \in \overline{\{y\}}$, which altogether gives that $\overline{\{x\}} = \overline{\{y\}}$, a contradiction.

Denote by $\mathbf{Top}_{T_0}$ the full subcategory of $\mathbf{Top}$ consisting of T_0 spaces. Let X be a topological space. We show that the map $i : \iota|^{\iota(X)} : X \to \iota(X)$ represents the functor $\mathrm{Hom}_{\mathbf{Top}}(X, -) : \mathbf{Top} \to \mathbf{Top}_{T_0}$. To see this, we only need to show that if X is T_0, then $i : X \to \iota(X)$ is a homeomorphism. Once this is proven, the rest of the proof follows exactly the same as that in (4.35). Since i^{-1} induces a bijection between topologies, just as what ι^{-1} does, it suffices to show $i : X \to \iota(X)$ is bijective, and it remains to show injectivity. This is clear as X is T_0.

4.37 Jacobson space. Let k be an algebraically closed field. By Nullstellensatz, the closed points of the affine scheme $\mathrm{Spec}\, k[x_1,\dots,x_n]$ are in bijection with the affine variety k^n. Under this bijection, we see k^n is a topological subspace of $\mathrm{Spec}\, k[x_1,\dots,x_n]$, and for $I \trianglelefteq k[x_1,\dots,x_n]$ we have

$$V^{\mathrm{var}}(I) = V^{\mathrm{sch}}(I) \cap \{\text{closed points in } \mathrm{Spec}\, A\}$$

In view of this relation, we are led to the following definition.

Let X be a topological space. Denote by X_{cl} the set of closed points in X, and equip X_{cl} with the subspace topology. We say the space X is **Jacobson** if for any closed subset Z of X, the subset Z_{cl} is dense in Z. Note that $Z_{\mathrm{cl}} = Z \cap X_{\mathrm{cl}}$ as Z is closed.

4.38 Example. Let A be a ring. The closed points in $\operatorname{Spec} A$ are exactly the maximal ideals in A; in other words, $(\operatorname{Spec} A)_{\mathrm{cl}} = \operatorname{mSpec} A$. For an ideal $I \trianglelefteq A$, we see that $V(I)_{\mathrm{cl}}$ is the set of maximal ideals containing A. For $V(I)_{\mathrm{cl}}$ to be dense in $V(I)$, it is sufficient and necessary that $\sqrt{I} = \operatorname{Jac}(I)$, where $\operatorname{Jac}(I)$ is the intersection of all maximal ideals in A containing I, and it called the **Jacobson radical** of I. To see this, first note that for $f \in A$, $D(f) \cap V(I) \neq \varnothing$ if and only if $f \notin \sqrt{I}$. For such f, clearly we have $D(f) \cap V(I)_{\mathrm{cl}} \neq \varnothing$ if and only if $f \notin \operatorname{Jac}(I)$. This implies the claim. The ring A with $\operatorname{Spec} A$ being Jacobson is called a **Jacobson ring**. The Jacobson radical of the zero ideal is denoted by $\operatorname{rad}(A)$, and is called the **Jacobson radical of the ring** A.

For example, Nullstellensatz implies that every algebra of finite type over an algebraically closed field k is Jacobson. Indeed, if I is an ideal of $k[x_1, \ldots, x_n]$, then

$$\begin{aligned} I(V(I)) &= \{f \in k[x_1, \ldots, x_n] \mid f(p) = 0 \text{ for all } p \in V(I)\} \\ &= \{f \in k[x_1, \ldots, x_n] \mid f \in \mathfrak{m}_p \text{ for all } p \in V(I)\} = \operatorname{Jac}(I) \end{aligned}$$

where the second equality results from (4.10). This shows $k[x_1, \ldots, x_n]$ is Jacobson, and clearly this implies what we say.

4.39 Definition. A continuous map $f : X \to Y$ is called a **quasi-homeomorphism** if the inverse f^{-1} establishes a bijection $\operatorname{Top}(Y) \to \operatorname{Top}(X)$ between topologies. Equivalently, it establishes a bijection between closed sets in X and Y.

As an example, we see in (4.34) that for a topological space X, the natural map $\iota_X : X \to \operatorname{Sob}(X)$ into its soberification is a quasi-homeomorphism.

4.39.1 The inverse f^{-1} of a function f is well-behaved with arbitrary intersection and arbitrary union. If $f : X \to Y$ is a quasi-homeomorphism, many identities happening in $\operatorname{Top}(Y)$ can be pulled back to identities in $\operatorname{Top}(X)$, and vice versa. Thus those topological properties that are defined or can be checked only using intersection and union are preserved under quasi-homeomorphism. We list some in the following lemma.

Lemma. Let $f : X \to Y$ be a quasi-homeomorphism between topological spaces.

(i) Let $\mathcal{U}$ be a collection of open sets in Y. Then $\mathcal{U}$ is a cover (resp. a basis) of Y if and only if $f^{-1}\mathcal{U} := \{f^{-1}(U)\}_{U \in \mathcal{U}}$ is a cover (resp. a basis) of X.

(ii) X is compact if and only if Y is compact.

(iii) X is connected (resp. irreducible) if and only if Y is connected (resp. irreducible).

4.39.2 If $f : X \to Y$ is a quasi-homeomorphism, then the pushforward f_* of presheaves on X to Y induces an equivalence of categories

$$\begin{aligned} \mathcal{C}_X^{\mathrm{pre}} &\longrightarrow \mathcal{C}_Y^{\mathrm{pre}} \\ \mathcal{F} &\longmapsto f_*\mathcal{F} \end{aligned}$$

where $\mathcal{C} = \mathbf{Set}, \mathbf{Ab}, \mathbf{Ring}, \mathbf{Mod}_R$ etc. This is obviously a fully faithful functor. Now given a presheaf $\mathcal{G}$ on Y, consider the pullback presheaf $f^{\mathrm{pre}}\mathcal{G}$. We claim $f_* f^{\mathrm{pre}}\mathcal{G} \cong \mathcal{G}$ as presheaves canonically. Indeed, for any open set V in Y,

$$f_* f^{\mathrm{pre}}\mathcal{G}(V) = \varinjlim_{V' \supseteq f(f^{-1}(V))} \mathcal{G}(V') = \varinjlim_{V' \supseteq V} \mathcal{G}(V') \cong \mathcal{G}(V)$$

Passing to sheafification, we see $f_* : \mathcal{C}_X \to \mathcal{C}_Y$ is again an equivalence of categories with inverse f^{-1}.

4.40 Lemma. Let X be a topological space. TFAE :

(a) X is Jacobson.

(b) The inclusion $X_{cl} \to X$ is a quasi-homeomorphism.

(c) Every nonempty locally closed subset of X meets nontrivially with X_{cl}.

Proof. The equivalence (a)⇔(b) is simply a paraphrase of the very definition, and (a)⇒(c) is clear. To see (c)⇒(a), note that for any closed $Z \subseteq X$, if $\overline{Z_{cl}} \subsetneq Z$, then $Z \setminus \overline{Z_{cl}}$ is a nonempty locally closed set in X. However, we have $Z \setminus \overline{Z_{cl}} \subseteq Z \setminus X_{cl}$, which is a contradiction to (c). □

4.41 Corollary. Let A be a ring. Assume A satisfies the property that for any $f \in A$ and $\mathfrak{p} \in D(f)$, $\mathfrak{p} \in D(f)_{cl}$ implies $\mathfrak{p} \in \operatorname{mSpec} A$. Then A is Jacobson.

Proof. We use (4.40).(c). Assume that $A \neq 0$. Let $I \trianglelefteq A$ and $f \in A \setminus \sqrt{I}$. It suffices to show that $D(f) \cap V(I)$ contains a closed point in $\operatorname{Spec} A$. The space $D(f) \cap V(I)$ is homeomorphic to the affine scheme $\operatorname{Spec} A_f/I_f$, so it contains a closed point. By hypothesis, it is also a closed point in $\operatorname{Spec} A$. □

Note that the proof is essentially based on the fact that $\operatorname{Spec} A$ has a closed point as long as $A \neq 0$.

4.42 Lemma.

1. Any locally closed subset of a Jacobson space is Jacobson.
2. A topological space that admits an open cover consisting of Jacobson spaces is Jacobson.

Proof.

1. Let X be a Jacobson. It follows from definition that every closed subset of X is Jacobson, so we only need to show every open subset of X is Jacobson. Let $U \subseteq X$ be nonempty open, and let $Z \subseteq U$ be nonempty closed. By (4.40).(c), Z contains a closed point in X which is, a fortiori, a closed point in U. By (4.40).(c) again, U is Jacobson.
2. Let X be a space and $\mathcal{U}$ be an open cover of X with each $U \in \mathcal{U}$ Jacobson. We use (4.40).(c). Let $W \subseteq X$ be a nonempty locally closed subset. Then $W \cap U \neq \varnothing$ for some $U \in \mathcal{U}$, and by (4.40).(c) $W \cap U$ contains a closed point x in U. In particular, $\{x\}$ is locally closed in X. If $x \in V$ for some $V \in \mathcal{U}$, then $x \in V_{cl}$ for $\{x\}$ is locally closed in V and by (4.40).(c). Thus $\{x\}$ is closed in X, for $\{x\} \cap V$ is closed for any $V \in \mathcal{U}$.

□

4.43 Example. Let A be a ring. The universal property of soberification gives rise to a continuous map

$$\operatorname{Sob}(\operatorname{mSpec} A) \longrightarrow \operatorname{Spec} A.$$

We are going to write down this map explicitly under some additional conditions imposed on A. A closed set of the form $V^{\text{mspec}}(I)$ is irreducible if and only if $\operatorname{Jac}(I)$ is a prime ideal. For such I, the image of $V^{\text{mspec}}(I)$ in $\operatorname{Spec} A$ would be the generic point of $\overline{V^{\text{mspec}}(I)} \subseteq \operatorname{Spec} A$. For another ideal I, we have $V^{\text{mspec}}(\operatorname{Jac}(I)) \subseteq V(J)$ if and only if $J \subseteq \operatorname{Jac}(I)$, and this implies $V(\operatorname{Jac}(I)) \subseteq V(J)$, which further implies $J \subseteq \sqrt{\operatorname{Jac}(I)} = \sqrt{I}$. If A is Jacobson (4.38), we then see $\overline{V^{\text{mspec}}(I)} = V(\sqrt{I})$ whose generic point is $\sqrt{I} = \operatorname{Jac}(I)$. For a prime $\mathfrak{p}$, if A is Jacobson, we have $\operatorname{Jac}(\mathfrak{p}) = \sqrt{\mathfrak{p}} = \mathfrak{p}$. Hence, if A is a Jacobson ring, every element in $\operatorname{Sob}(\operatorname{mSpec} A)$ has the form $V^{\text{mspec}}(\mathfrak{p})$ with $\mathfrak{p}$ prime and it has image $V(\mathfrak{p})$ in $\operatorname{Spec} A$.

Assume A is Jacobson. The map is then clearly bijective. A closed set in $\operatorname{Sob}(\operatorname{mSpec} A)$ has the form $\{V^{\text{mspec}}(\mathfrak{p}) \mid I \subseteq \mathfrak{p}\}$, where I is an ideal. Its image in $\operatorname{Spec} A$ is then $V(I)$, which is closed. Thus the canonical inclusion $\operatorname{mSpec} A \to \operatorname{Spec} A$ is the soberification when A is Jacobson.

Let $f : A \to B$ be a homomorphism between Jacobson rings with $f^{-1}(\operatorname{mSpec} B) \subseteq \operatorname{mSpec} A$. Let $\theta = \operatorname{Spec}(f)$ and $\theta' = \theta|_{\operatorname{mSpec} B}^{\operatorname{mSpec} A}$.

Then there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Sob}(\mathrm{mSpec}\,A) \arrow[r, "\sim"] & \mathrm{Spec}\,A \\
\mathrm{Sob}(\mathrm{mSpec}\,B) \arrow[r, "\sim"] \arrow[u, "\mathrm{Sob}(\theta')"] & \mathrm{Spec}\,B \arrow[u, "\theta"']
\end{tikzcd}$$

Indeed, this follows from (3.10).(iii).

4.44 By definition, every point in a T_1 **topological space** is a closed point. Denote by $\mathbf{Top}_{T_1}$ the full subcategory of **Top** consisting of T_1 spaces. Let X be a T_1 topological space and consider the soberification $\iota : X \to \mathrm{Sob}(X)$. Since every point $x \in X$ is a closed point, we have $\iota(x) = \{x\} \in \mathrm{Sob}(X)$; in particular, ι is injective, and $X \subseteq \mathrm{Sob}(X)_{\mathrm{cl}}$. A point $Z \in \mathrm{Sob}(X)$ is closed if and only if $\{Z\} = \overline{\{Z\}} = V(Z)$ (4.34), which means that Z is a minimal closed irreducible subset in X with respect to inclusion. But $X = X_{\mathrm{cl}}$, we see $Z \subseteq X$ is a singleton. This proves $X = \mathrm{Sob}(X)_{\mathrm{cl}}$. Since ι is a quasi-homeomorphism (4.34), by (4.40) we see $\mathrm{Sob}(X)$ is a Jacobson space.

Denote by **JacSob** the subcategory of **Top** whose objects consist of Jacobson sober spaces and whose morphisms consist of continuous maps $f : X \to Y$ with $f(X_{\mathrm{cl}}) \to Y_{\mathrm{cl}}$. Then the functor $\mathbf{Sob} : \mathbf{Top} \to \mathbf{Sob}$ restricts to a functor $\mathrm{Sob} : \mathbf{Top}_{T_1} \to \mathbf{JacSob}$. Furthermore, this is an equivalence of categories. We state it as a theorem in the next paragraph.

4.45 Theorem. The functor

$$\mathrm{Sob} : \mathbf{Top}_{T_1} \to \mathbf{JacSob}$$

is an equivalence of categories with inverse $Y \mapsto Y_{\mathrm{cl}}$.

Proof. We already show in (4.45) that for a T_1 space X, the map $\iota : X \to \mathrm{Sob}(X)$ is an embedding with $\iota(X) = \mathrm{Sob}(X)_{\mathrm{cl}}$. Conversely, let Y be a Jacobson sober space. By definition Y_{cl} is T_1. By the universal property of soberification (4.35), there exists a continuous map $f : \mathrm{Sob}(Y_{\mathrm{cl}}) \to Y$ with $f \circ \iota_{Y_{\mathrm{cl}}}$ equal the inclusion $Y_{\mathrm{cl}} \to Y$. For any closed irreducible $Z \subseteq Y_{\mathrm{cl}}$, we have $\overline{Z} \cap Y_{\mathrm{cl}} = Z$, where the closure is taken in Y. In particular, this implies f is injective. Conversely, if $y \in Y$, since Y is Jacobson, $Z_y := \overline{\{y\}} \cap Y_{\mathrm{cl}} \neq \varnothing$ is a closed irreducible subset of Y_{cl}. Since $\overline{Z_y} \cap Y_{\mathrm{cl}} = Z_y$, we have $f(Z_y) = y$. To show f is a homeomorphism, at this stage it suffices to show f is a quasi-homeomorphism. This follows as $Y_{\mathrm{cl}} \to Y$ and $Y_{\mathrm{cl}} \to \mathrm{Sob}(Y_{\mathrm{cl}})$ are quasi-homeomorphisms.

The way we define the morphisms in **JacSob** is a sufficient condition for $Y \mapsto Y_{\mathrm{cl}}$ to be functor $\mathbf{JacSob} \to \mathbf{Top}_{T_1}$. If $f : X \to Y$ is a morphism of T_1 spaces, by (4.35) we have a commuting square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\iota_X"'] & Y \arrow[d, "\iota_Y"] \\
\mathrm{Sob}(X) \arrow[r, "\mathrm{Sob}(f)"'] & \mathrm{Sob}(Y).
\end{tikzcd}$$

Since $X = \mathrm{Sob}(X)_{\mathrm{cl}}$ and similar for Y, we see $\mathrm{Sob}(f) \in \mathrm{Hom}_{\mathbf{JacSob}}(\mathrm{Sob}(X), \mathrm{Sob}(Y))$. Also from this diagram we conclude that Sob is an equivalence of categories with inverse $Y \mapsto Y_{\mathrm{cl}}$. $\square$

4.2.2 Jacobson rings

4.46 In this subsubsection we follow [AM94, exercises in chapter 5] to introduce some properties of Jacobson rings.

4.47 Lemma. Let A be a subring of a ring B such that B is integral over A, and let $f : A \to \Omega$ be a homomorphism of A into an algebraically closed field Ω. Then f can be extended to a homomorphism of B into Ω.

Proof. One has $\ker f \in \operatorname{Spec}(A)$, for $A/\ker f$ is isomorphic to a subring of Ω, hence an integral domain. By going-up, there exists $\mathfrak{q} \in \operatorname{Spec}(B)$ such that $\mathfrak{q} \cap A = \ker f$. Now f can be extended to a homomorphism $f : \operatorname{Frac}(A/\ker f) \to \Omega$. Since $B/\mathfrak{q}$ is integral over $A/\ker f$, the field $\operatorname{Frac}(B/\mathfrak{q})$ is algebraic over $\operatorname{Frac}(A/\ker f)$ so that f can be extended to a map $f : \operatorname{Frac}(B/\mathfrak{q}) \to \Omega$. Finally, restricting to $B/\mathfrak{q}$ and pre-composing with the quotient $B \to B/\mathfrak{q}$, we obtain a homomorphism $B \to \Omega$. □

4.47.1 The essence for the proof is that for a field homomorphism $f : K \to \Omega$ with Ω algebraically closed, we can extend to a homomorphism $L \to \Omega$ for any algebraic extension L/K. For completeness we give a proof below. First consider the case $L = K(\alpha)$ being a simple extension of K with $\alpha \notin K$. Say $m_\alpha(T) = \sum_{i=0}^{n} a_i T^i \in K[T]$ is the minimal polynomial of α. Let $\beta \neq 0$ be a root of the polynomial $\sum_{i=0}^{n} f(a_i)T^i$ in Ω. Now define $K[T] \to \Omega$ by extending f and sending T to β. $(m_\alpha(T))$ lies in ins kernel, so it induces a map $K[T]/(m_\alpha(T)) \cong K(\alpha) \to \Omega$.

A finite extension L/K is composed of a series of simple extensions, so one can extend $f : K \to \Omega$ along them as we have done. For the general algebraic extension L/K, consider the set

$$S := \{(M, g) \mid M/K \text{ is an algebraic subextension of } L/K,\ g : M \to \Omega \text{ extends } f\}$$

with a partial order $\preceq$ defined by

$$(M_1, g_1) \preceq (M_2, g_2) \Leftrightarrow M_1 \subseteq M_2 \text{ and } g_2|_{M_1} \equiv g_1$$

Now it follows easily from Zorn's lemma that S admits a maximal element, say (M, g). It must be the case $M = L$, for otherwise we can pick $\alpha \in L - M$ and extend g to a map $M(\alpha) \to \Omega$, contradicting to the maximality.

4.48 Lemma. Let A be a subring of an integral domain B such that B is of finite type over A. Then that there exists $s \neq 0$ in A and elements $y_1, \ldots, y_n$ in B, algebraically independent over A and such that B_s is integral over B'_s, where $B' = A[y_1 ..., y_n]$.

Proof. Let $S = A - \{0\}$ and $K = S^{-1}A$ be the fraction field of A. Then $S^{-1}B$ is a finitely generated K-algebra. By Noether normalization there exists $x_1, \ldots, x_n \in S^{-1}B$ such that $C := K[x_1, \ldots, x_n]$ is purely transcendental and $C \hookrightarrow S^{-1}B$ is finite.

Write $B = A[z_1, \ldots, z_m]$. Since $z_i \in S^{-1}B$ is integral over C, each z_i verifies a integral dependence $T^{r_i} + a_{i,r_i-1}T^{r_i-1} + \cdots + a_{i,1}T + a_{i,0} \in C[T]$; we may assume the r_i are the same. Let $s \in S$ be such that $y_i := sx_i \in B$ and $sa_{i,j} \in B' := A[y_1, \ldots, y_n]$; for example, take s to be the products of denominators of the x_i and all coefficients of the $a_{i,j}$ and raise s to a large power. From what we have done it is easy to see that the sz_i are integral over B', and hence over B'_s. Finally, since $s^{-1} \in B'_s$ is of course integral over B'_s, the z_i are integral B'_s, i.e., B_s is integral over B'_s, as wanted. □

4.49 Corollary. Let A be a subring of an integral domain B such that B is of finite type over A.

(i) There exists $s \neq 0$ in A such that, if Ω is an algebraically closed field and $f : A \to \Omega$ is a homomorphism for which $f(s) \neq 0$, then f can be extended to a homomorphism $B \to \Omega$.

(ii) If the Jacobson radical of A is zero, then so is the Jacobson radical of B.

Proof.

(i) Take s as in Lemma 4.48; we also use the notation therein. Then we can extend f to $f : B' \to \Omega$ by setting each y_i to 0. Since $f(s) \neq 0$, it induces a homomorphism $f : B'_s \to \Omega$. By Lemma 4.47 f extends to a map $f : B_s \to \Omega$. Restricting to B gives a desired map.

(ii) Let $v \in B$ be a nonzero element. We claim there exists a maximal ideal of B not containing v. By applying (i) to the ring B_v and its subring A, we obtain a nonzero element $s \in A$. Let $\mathfrak{m}$ be a maximal ideal of A not containing s and put $k = A/\mathfrak{m}$. Then the projection $A \to k$ extends to a homomorphism $g : B_v \to \Omega$ where Ω is an algebraic closure of k. Since v is invertible in B_v, $g(v) \neq 0$ so that $v \notin \ker g \cap B$. Now it suffices to show $\ker g \cap B$ is a maximal ideal of B. But note that $\mathfrak{m} \subseteq \ker g \cap B$; thus since B is integral over A, $B/\ker g \cap B$ is integral over $k = A/\mathfrak{m}$ ($B \cap \ker g \cap A = \mathfrak{m}$ because $s \notin \mathfrak{m}$), and hence $B/\ker g \cap B$ is a field, namely $\ker g \cap B$ is maximal.

□

4.50 In (4.38) we say A is Jacobson if $\sqrt{I} = \mathrm{Jac}(I)$ for any ideal $I \trianglelefteq A$. In practice, we only want to check as few conditions as possible. It turns out it suffices to check the equality for prime ideals. We prove this as a

Lemma. Let A be a ring. Then TFAE :

(i) Every prime ideal in A is an intersection of maximal ideals.

(ii) In every homomorphic image of A the nilradical is equal to the Jacobson radical.

(iii) Every prime ideal in A which is not maximal is equal to the intersection of the prime ideals which contain it strictly.

Proof. Clearly (i) ⇔ (ii) ⇒ (iii). It remains to show (iii) ⇒ (ii). Suppose (ii) does not hold. Then there exists a prime ideal of A which is not a intersection of maximal ideals. Passing to quotient we may assume A is an integral domain where the Jacobson radical is not zero. Let $f \in \mathrm{Jac}(A)$ be a nonzero element. Then $A_f \neq 0$ and hence it admits a maximal ideal whose contraction in A is a prime ideal $\mathfrak{p}$ not containing f and is maximal with respect to this property (here we use the correspondence of prime ideals between A and A_f). Since $f \in \mathrm{Jac}(A) - \mathfrak{p}$, $\mathfrak{p}$ is not maximal. Also, $\mathfrak{p}$ is not the intersection of the prime ideals strictly containing $\mathfrak{p}$, for any prime strictly containing $\mathfrak{p}$ must contain f, and so must their intersection. □

4.51 There is another characterization of Jacobson rings that is of geometric taste.

Lemma. Let A be a ring. Then A is Jacobson if and only if every finite type A-algebra that is a field is finite over A.

Proof. Suppose A is Jacobson, and let B be a finite type A-algebra that is a field. Let $\mathfrak{p}$ be the kernel of the structure map $A \to B$. To show B is finite over A, it suffices to show B is finite over $A/\mathfrak{p}$. Hence we can assume A is a subring of B. We take $0 \neq s \in A$ as in Corollary 4.49. Let $\mathfrak{m}$ be a maximal ideal not containing s. By that lemma, the homomorphism $A \to A/\mathfrak{m} =: k$ extends to a homomorphism $\phi : B \to \bar{k}$. Since B is a field, it follows ϕ is injective so that B is algebraic over k. Finite generation of B over A implies $\dim_k B < \infty$, which shows that B is finite over A.

Now we turn to if part. We use Lemma 4.50.(iii). Let $\mathfrak{p}$ be a non-maximal prime ideal of A, and consider the quotient map $A \to A/\mathfrak{p} =: B$. For $0 \neq f \in B$, since B_f is of finite type over A, if it is a field, then B_f is finite over A by assumption, and a fortiori it is finite over B. By Lemma 3.87.1 B is a field. But $\mathfrak{p}$ is non-maximal, this implies B_f is not a field. Then B_f has a nonzero prime ideal, whose contraction $\mathfrak{p}'$ to B is a prime ideal not containing f. Letting f vary finishes the proof. □

4.51.1 Immediate from the lemma, if A is a Jacobson ring, B a finite type A-algebra and $\mathfrak{m}$ a maximal ideal of B, then $B/\mathfrak{m}$ is a finite type A-algebra that is a field, so $B/\mathfrak{m}$ is finite over A. Let $\mathfrak{m}' := A \cap \mathfrak{m}$ so that $A/\mathfrak{m}' \to B/\mathfrak{m}$ is injective and finite. By Lemma 3.87.1 $A/\mathfrak{m}'$ is a field, so $\mathfrak{m}'$ is maximal. This implies that the natural map $\operatorname{Spec} B \to \operatorname{Spec} A$ restricts to

$$\operatorname{mSpec} B \longrightarrow \operatorname{mSpec} A.$$

4.52 Nullstellansatz. Let A be a Jacobson ring and B an A-algebra. Assume either that B is integral over A or is of finite type over A. Then B is Jacobson.

Proof. Assume B is integral over A. Let $\mathfrak{q}$ be a prime in B and put $\mathfrak{p} = \mathfrak{q} \cap A$. Since A is Jacobson, $\mathfrak{p}$ is the intersection of all maximal ideals $\mathfrak{m}_i$ containing $\mathfrak{p}$. By going-up we can find maximal $\mathfrak{n}_i \supseteq \mathfrak{q}$ such that $\mathfrak{n}_i \cap A = \mathfrak{m}_i$. Let I be the intersection of the $\mathfrak{n}_i$. If $I = \mathfrak{q}$ we are done. If $\mathfrak{q} \subsetneq I$. By localizing A and B at $\mathfrak{p}$ we can find a maximal ideal $\mathfrak{q}'$ of B containing I whose contraction to A is $\mathfrak{p}$. But incomparability tells $\mathfrak{q} = \mathfrak{q}'$, which is absurd. Hence $\mathfrak{q} = I$.

Assume B is of finite type over A. We may assume $A \subseteq B$ are integral domains, and our goal is to show $\operatorname{Jac}(B) = 0$. Since A is Jacobson, so the $\operatorname{Jac}(A) = 0$, and it follows from Corollary 4.49.(ii) that $\operatorname{Jac}(B) = 0$. □

4.52.1 Suppose k is a field and A is a finite type k-algebra. By (4.52) A is Jacobson. If $\mathfrak{m}$ is a maximal ideal of A, then Lemma 4.51 tells that $\dim_k A/\mathfrak{m} < \infty$. Conversely, if $\mathfrak{m}$ is a prime ideal of A such that $k \to A/\mathfrak{m}$ is finite, then Lemma 3.87.1 shows that $\mathfrak{m}$ is maximal. In conclusion, we've proved the equality

$$\operatorname{mSpec} A = \{\mathfrak{p} \in \operatorname{Spec} A \mid [\kappa(\mathfrak{p}) : k] < \infty\}.$$

In the case k is algebraically closed, this recovers the Hilbert's Nullstellensatz.

4.2.3 Morphisms of finite type

4.53 Definition. A morphism $f : X \to Y$ of schemes is called **locally of finite type** if Y admits an affine open cover $\mathcal{V}$ such that for any $V = \operatorname{Spec} A \in \mathcal{V}$, the open set $f^{-1}(V)$ ($V \in \mathcal{V}$) admits an affine open cover $\mathcal{U}_V$ such that for any $U = \operatorname{Spec} B \in \mathcal{U}_V$, the corresponding ring homomorphism $A \to B$ is of finite type.

If each $\mathcal{U}_V$ can be chosen to be a finite set, $f : X \to Y$ is called **of finite type**.

4.54 Let $A \to B$ be a ring homomorphism of finite type. For any $g \in B$, the composition $A \to B \to B_g$ is also of finite type, as $B_g \cong B[x]/(xg - 1)$ is a B-algebra of finite type. If $A \to B$ factors through some localization $A \to A_f$ at $f \in A$, then $A_f \to B$ is still of finite type.

These imply that for a morphism $f : X \to Y$ of schemes, being locally of finite type is really local on both X and Y. Precisely, let $f : X \to Y$ be locally of finite type. Then

(i) For any open $U \subseteq X$, $f|_U : U \to Y$ is locally of finite type.

(ii) For any open $V \subseteq Y$ and open $U \subseteq f^{-1}(V)$, $f|_U^V : U \to V$ is locally of finite type.

Also, let $f : X \to Y$ be a morphism of schemes.

(iii) If Y admits an open cover $\mathcal{V}$ such that $f|_{f^{-1}(V)}^V : f^{-1}(V) \to V$ is locally of finite type for any $V \in \mathcal{V}$, then f is locally of finite type.

(iv) If X admits an open cover $\mathcal{U}$ such that $f|_U : U \to Y$ is locally of finite type for any $U \in \mathcal{U}$, then f is locally of finite type.

4.55 Lemma. Consider morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ of schemes.

(i) If f and g are locally of finite type, then so is $g \circ f$.

(ii) If $g \circ f$ is locally of finite type, then so is f.

Proof. Only (ii) deserves a proof. Let $y \in Y$ and $x \in f^{-1}(y)$. Pick an affine open neighborhood $W = \operatorname{Spec} A$ of $g(y)$ in Z and an affine open neighborhood $U = \operatorname{Spec} C$ of x in X such that $A \to C$ induced by $(g \circ f)|_U^W$ is of finite type. Choose an affine neighborhood $V = \operatorname{Spec} B$ of y such that $g(V) \subseteq W$. Pick $h \in C$ with $f(U_h) \subseteq V$. Then $U_h \to V \to W$ gives $A \to B \to C_h$. Since $A \to C$ is of finite type, so is $A \to C_h$. In particular, $B \to C_h$ is of finite type. □

4.56 Lemma. A morphism $f : X \to Y$ is locally of finite type if and only if for every affine open $V = \operatorname{Spec} A$ in Y and every affine open $U = \operatorname{Spec} B$ in $f^{-1}(V)$, the corresponding ring homomorphism $A \to B$ is of finite type.

In particular, a morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ of affine schemes is locally of finite type if and only $A \to B$ is of finite type.

Proof. We need to prove the only if part. We see in (4.54) that $f|_U^V$ is locally of finite type. By definition, for any $h \in A$ we can find $g \in B$ such that $f(U_g) \subseteq V_h$ with $A_h \to B_g$ of finite type. In particular, B_g is of finite type over A. All such U_g cover U, and since U is compact, we can find $g_1, \ldots, g_n \in B$ with $B = (g_1, \ldots, g_n)$ and each B_{g_i} of finite type over A.

Let $b_i \in B$ be such that $1 = b_1 g_1 + \cdots + b_n g_n$. Raising to arbitrary power, for any $N \in \mathbb{Z}_{\geqslant 1}$, we can find $b_{iN} \in B$ with $1 = \sum_{i=1}^{n} b_{iN} g_i^N$. Say $B_{g_i} = A[x_{ij}]_{j=1}^{a_i}$ for some $x_{i1}, \ldots, x_{ia_i} \in B_{g_i}$. Let $N_i \in \mathbb{Z}_{\geqslant 1}$ be such that $x_{ij} g_i^{N_i} \in B$ for any i, j. Let $N = \max_{1 \leqslant i \leqslant n} N_i$. Then $B = A[x_{ij} g_i^N]_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant a_i}}$. To see this, for $b \in B$, let $f_i \in A[X_{ij}]_{j=1}^{a_i}$ be such that $b = f_i(x_{ij})$. Let $M = \max_{1 \leqslant i \leqslant n} \deg f_i$ and put $L = MN$. Then

$$b = b1 = \sum_{i=1}^{n} b_{iL} f_i(x_{ij}) g_i^{MN} \in A[x_{ij} g_i^N]_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant a_i}}$$

□

4.2.4 Equivalence of categories

In this subsubsection, we fix an algebraically closed field k.

4.57 We paraphrase the results in (4.11) in terms of the languages in the preceding subsubsection. For an affine variety V over k, its soberification $\mathrm{Sob}(V)$ is homeomorphic to that of $\mathrm{mSpec}\, k[V]$, which is homeomorphic to $\mathrm{Spec}\, k[V]$ by (4.38) and (4.43). Precisely, we have a commutative diagram with horizontal maps being isomorphisms

$$\begin{tikzcd} \mathrm{Sob}(V) \arrow[r, "\sim"] & \mathrm{Spec}\, k[V] \\ V \arrow[u, "\iota_V"] \arrow[r, "\sim"] & \mathrm{mSpec}\, k[V]. \arrow[u, "\text{inclusion}"'] \end{tikzcd}$$

Denote by α the bottom horizontal isomorphism and by ι the inclusion on the right. By Theorem 4.15, the sheaf $\iota_* \alpha_* \mathcal{O}_V$ on $\mathrm{Spec}\, k[V]$ is naturally isomorphic to the structure sheaf $\mathcal{O}_{\mathrm{Spec}\, k[V]}$. In this way, we see $(\mathrm{Sob}(V), (\iota_V)_* \mathcal{O}_V)$ is isomorphic to $\mathrm{Spec}\, k[V]$ in $\mathbf{LRS}_k$, and we obtain a functor

$$\begin{aligned} \mathbf{AffVar}_k &\longrightarrow \mathbf{Sch}_k \\ V &\longmapsto (\mathrm{Sob}(V), (\iota_V)_* \mathcal{O}_V), \end{aligned}$$

where for a morphism $f : V \to W$, we define $(\mathrm{Sob}(V), (\iota_V)_* \mathcal{O}_V) \to (\mathrm{Sob}(W), (\iota_W)_* \mathcal{O}_W)$ as follows. The map on topological spaces is certainly $\mathrm{Sob}(f)$. To define a sheaf map $\theta : (\iota_W)_* \mathcal{O}_W \to \mathrm{Sob}(f)_* (\iota_V)_* \mathcal{O}_V = (\iota_W)_* f_* \mathcal{O}_V$, we only need to choose a map $\mathcal{O}_W \to f_* \mathcal{O}_V$. We already have one : the function pullback. In fact, under the isomorphism $(\mathrm{Sob}(V), (\iota_V)_* \mathcal{O}_V) \cong \mathrm{Spec}\, k[V]$, we see the morphism $(\mathrm{Sob}(V), (\iota_V)_* \mathcal{O}_V) \to (\mathrm{Sob}(W), (\iota_W)_* \mathcal{O}_W)$ coincides $\mathrm{Spec}\, f^* : \mathrm{Spec}\, k[V] \to \mathrm{Spec}\, k[W]$. The maps on spaces are the same by (4.43). The maps on sheaves are the same as well, for both are induced by $f^* : k[W] \to k[V]$.

Similarly, we can define a functor

$$\begin{aligned} \mathbf{Var}_k &\longrightarrow \mathbf{Sch}_k \\ X &\longmapsto (\mathrm{Sob}(X), (\iota_X)_* \mathcal{O}_X). \end{aligned}$$

The only issue is to show $(\mathrm{Sob}(X), (\iota_X)_* \mathcal{O}_X)$ is really a scheme. Let V be an affine open subset of X. By (4.35.1) we can regard $U = \mathrm{Sob}(V)$ as an open subset of $\mathrm{Sob}(X)$. We show that

$$(\mathrm{Sob}(X), (\iota_X)_* \mathcal{O}_X)|_U \cong (U, (\iota_V)_* (\mathcal{O}_X|_V)).$$

For this we only need to notice that $((\iota_X)_*\mathcal{O}_X)|_U = (\iota_V)_*(\mathcal{O}_X|_V)$.

4.58 Our goal is to show the functor

$$\begin{aligned} \mathbf{Var}_k &\longrightarrow \mathbf{Sch}_k \\ X &\longmapsto (\mathrm{Sob}(X), (\iota_X)_*\mathcal{O}_X). \end{aligned}$$

is fully faithful and describe its essential image. If V is an affine k-variety, by construction we see $\mathrm{Sob}(V)$ is a reduced affine k-scheme locally of finite type over $\operatorname{Spec} k$. Hence for a general k-variety X, we see $\mathrm{Sob}(X)$ is a reduced k-scheme locally of finite type over $\operatorname{Spec} k$.

To proceed, first note that an affine k-variety is a T_1 topological space. Next, suppose X is a topological space that admits an open cover $\mathcal{U}$ consisting of T_1 space. Then X is itself T_1. Indeed, for $x \in X$, if $\overline{\{x\}} \cap U \neq \varnothing$ for some $U \in \mathcal{U}$, then $x \in U$, as U is open. Since each $U \in \mathcal{U}$ is T_1, the intersection $\overline{\{x\}} \cap U$ is either $\varnothing$ or $\{x\}$, proving that $x \in X$ is a closed point. As a result, we see that a k-variety is T_1. By (4.44) and (3.36), the scheme $\mathrm{Sob}(X)$ is a Jacobson sober space.

If $f : V \to W$ is a morphism of affine k-varieties, in (4.57) we also see that the morphism

$$(\mathrm{Sob}(V), (\iota_V)_*\mathcal{O}_V) \to (\mathrm{Sob}(W), (\iota_W)_*\mathcal{O}_W)$$

is the same as $\operatorname{Spec} f^* : \operatorname{Spec} k[V] \to \operatorname{Spec} k[W]$. Since $f^* : k[W] \to k[V]$ is automatically of finite type, we see $(\mathrm{Sob}(V), (\iota_V)_*\mathcal{O}_V) \to (\mathrm{Sob}(W), (\iota_W)_*\mathcal{O}_W)$ is locally of finite type. Thus, the image of a morphism between k-varieties is a morphism of k-scheme locally of finite type.

Now let $f : X \to Y$ be a morphism of k-varieties. We see $\mathrm{Sob}(X)$ and $\mathrm{Sob}(Y)$ are Jacobson sober spaces. A natural question is whether for any morphism

$$(\mathrm{Sob}(X), (\iota_X)_*\mathcal{O}_X) \to (\mathrm{Sob}(Y), (\iota_Y)_*\mathcal{O}_Y)$$

of schemes, the underlying continuous map sends closed points to closed points, i.e., a morphism of Jacobson sober spaces. Such a morphism is necessarily locally of finite type by (4.55), so we are now in the situation stated in the following, which is essentially the Nullstellensatz.

4.58.1 Lemma Let $f : X \to Y$ be a morphism of schemes locally of finite type with Y Jacobson. Then X is Jacobson with $f(X_{\mathrm{cl}}) \subseteq Y_{\mathrm{cl}}$.

Proof. Observe by (4.52.1) $Y_{\mathrm{cl}} \cap V = V_{\mathrm{cl}}$ for any open $V \subseteq Y$. Indeed, $\subseteq$ is clear. For $\supseteq$, if $x \in V_{\mathrm{cl}}$, then $\{x\}$ is locally closed in Y. By (4.40).(c) $x \in Y_{\mathrm{cl}}$. From this observation together with (4.42).(ii), we can assume X and Y are affine. Now the lemma follows from (4.52) and (4.51.1). $\square$

4.58.2 Let us continue the discussion in (4.58). Let $(f, \theta) : (\mathrm{Sob}(V), (\iota_V)_*\mathcal{O}_V) \to (\mathrm{Sob}(W), (\iota_W)_*\mathcal{O}_W)$ be a k-morphism. Since $\mathrm{Sob}(V)$ is locally of finite type over k, by (4.55).(ii) we see (f, θ) is locally of finite type, and (4.58.1).(ii) implies that $f(\mathrm{Sob}(V)_{\mathrm{cl}}) \subseteq \mathrm{Sob}(W)_{\mathrm{cl}}$, i.e., f is a morphism in **JacSob** (4.44). Put $g = f_{\mathrm{cl}} : V \to W$ be the restriction of f to the closed points (c.f. (4.45)). Then we have a commutative diagram in **Top**

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, "\iota_V"'] & W \arrow[d, "\iota_W"] \\ \mathrm{Sob}(V) \arrow[r, "f"'] & \mathrm{Sob}(W). \end{tikzcd}$$

If W is an affine k-variety, by (3.7) and (4.19), we have functorial bijections

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathbf{Sch}_k}(\mathrm{Sob}(V),\mathrm{Sob}(W)) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathbf{Alg}_k}(\mathcal{O}_W(W),\mathcal{O}_V(V)) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathbf{Var}_k}(V,W) \\ (f,\theta) & \longmapsto & \theta_{\mathrm{Sob}(W)} & & \\ & & g^* & \longleftarrow & g \end{array}$$

From their proofs (and as ι_V is an embedding and (2.9)), we see $\theta_{\mathrm{Sob}(W)} = g^*$ if and only if $g = f_{\mathrm{cl}}$. In particular, this shows θ is given by pullback of functions by g. The general case follows by covering W by affine opens. These altogether prove that

$$\begin{array}{ccc} \mathbf{Var}_k & \longrightarrow & \mathbf{Sch}_k \\ X & \longmapsto & (\mathrm{Sob}(X),(\iota_X)_*\mathcal{O}_X). \end{array}$$

is fully faithful.

4.59 Inspired by (4.58), we say a k-scheme is **algebraic** if it is of finite type over $\mathrm{Spec}\,k$. Thus, if X is a k-variety, its soberification $\mathrm{Sob}(X)$ is a reduced algebraic k-scheme. We are now going to show any such a scheme comes from a k-variety, and then prove that the essential image of the functor

$$\begin{array}{ccc} \mathbf{Var}_k & \longrightarrow & \mathbf{Sch}_k \\ X & \longmapsto & (\mathrm{Sob}(X),(\iota_X)_*\mathcal{O}_X). \end{array}$$

is the full subcategory of $\mathbf{Sch}_k$ consisting of reduced algebraic k-schemes.

4.60 For a starter, we show if A is a reduced k-algebra of finite type, then the local-ringed space $(\mathrm{mSpec}\,A, \mathcal{O}_{\mathrm{Spec}\,A}|_{\mathrm{mSpec}\,A})$ is isomorphism to an affine variety in $\mathbf{LRS}_k$. To see this, we are back to the situation in the first paragraph of (4.57), and we need to show there is an isomorphism

$$(V,\mathcal{O}_V) \cong (\mathrm{mSpec}\,k[V], \mathcal{O}_{\mathrm{Spec}\,k[V]}|_{\mathrm{mSpec}\,k[V]})$$

The map on topological space is clearly given by $\alpha : V \to \mathrm{mSpec}\,k[V]$ there. For the sheaf map, put $\iota : \mathrm{mSpec}\,k[V] \to \mathrm{Spec}\,k[V]$ to be the inclusion. For $f \in k[V]$, since ι is a quasi-homeomorphism (4.43), we have

$$\begin{aligned} \iota^{\mathrm{pre}}\mathcal{O}_{\mathrm{Spec}\,k[V]}(D^{\mathrm{mspec}}(f)) &\cong \varinjlim_{D^{\mathrm{spec}}(g)\supseteq\iota(D^{\mathrm{mspec}}(f))} \mathcal{O}_{\mathrm{Spec}\,k[V]}(D^{\mathrm{spec}}(g)) \\ &= \varinjlim_{D^{\mathrm{spec}}(g)\supseteq D^{\mathrm{spec}}(f)} k[V]_g \\ &\cong k[V]_f \\ &\overset{(4.15)}{\cong} \mathcal{O}_V(D^{\mathrm{Var}}(f)) = \alpha_*\mathcal{O}_V(D^{\mathrm{mspec}}(f)). \end{aligned}$$

Every isomorphism is functorial, so this defines an isomorphism $\iota^{\mathrm{pre}}\mathcal{O}_{\mathrm{Spec}\,k[V]} \to \alpha_*\mathcal{O}_V$ of presheaves on the principal open sets. But $\alpha_*\mathcal{O}_V$ is a sheaf, this means $\iota^{\mathrm{pre}}\mathcal{O}_{\mathrm{Spec}\,k[V]}$ is a sheaf on principal open sets. By (2.3.1) this extends to an isomorphism $\mathcal{O}_{\mathrm{Spec}\,k[V]}|_{\mathrm{mSpec}\,k[V]} \to \alpha_*\mathcal{O}_V$ of sheaves on $\mathrm{mSpec}\,k[V]$.

4.61 Let X be an algebraic k-scheme. Put $V = X_{\mathrm{cl}}$ and denote by $\iota = \iota_V : V \to X$ the inclusion. By (4.58.1), X is Jacobson and $V = \{x \in X \mid k = \kappa(x)\}$. For a section $f \in \mathcal{O}_X(U)$ on some open set U of X, we can regard it as a function on $U \cap V$ by means of

$$\begin{array}{ccc} \iota_* f : U \cap V & \longrightarrow & k \\ x & \longmapsto & \text{class of } f \text{ in } k = \kappa(x) \end{array}$$

That $k = \kappa(x)$ follows from (4.58.1).(ii). In this way we have defined a morphism $\alpha = \alpha_V : \mathcal{O}_X \to \iota_* k^V$ of sheaves, where k^V denotes the sheaves $U \mapsto \mathrm{Hom}_{\mathbf{Set}}(U, k)$. In other words, we obtain a morphism $(\iota, \alpha) : (V, k^V) \to (X, \mathcal{O}_X)$ in $\mathbf{RS}_k$. But it follows from definition that this is a morphism in $\mathbf{LRS}_k$. By adjunction (2.10) α gives a morphism $\iota^{-1}\mathcal{O}_X \to k^V$. Denote by $\mathcal{O}_V$ the image sheaf, which is a subsheaf of k^V. Since X is Jacobson, ι is a quasi-homeomorphism (4.40), which implies that the sheaf $\mathcal{O}_V$ is the unique sheaf of k^V such that $\iota_*\mathcal{O}_V$ is the image of α (4.39.2). By definition we have a chain of morphisms $\iota^{-1}\mathcal{O}_X \twoheadrightarrow \mathcal{O}_V \hookrightarrow k^V$. Computing stalks (c.f. (2.19)) reads that $(V, \mathcal{O}_V)$ is a basic k-space (4.16).

Let $f : X \to Y$ be a morphism between algebraic k-schemes. Let $V = X_{\mathrm{cl}}$, $U = Y_{\mathrm{cl}}$ and ι_V, ι_U be the respective inclusions. By (4.58.1), we have $f(V) \subseteq U$. Let us denote by $g = f^U_V : V \to U$. There is a commutative diagram in $\mathbf{LRS}_k$

$$\begin{tikzcd}
(V, k^V) \arrow[r, "(\iota_V, \alpha_V)"] \arrow[d, "(g, g^*)"'] & (X, \mathcal{O}_X) \arrow[d, "(f, f^\sharp)"] \\
(U, k^U) \arrow[r, "(\iota_U, \alpha_U)"'] & (Y, \mathcal{O}_Y).
\end{tikzcd}$$

To see this, for open $W \subseteq Y$, $g \in \mathcal{O}_Y(W)$ and $x \in f^{-1}(W) \cap V$, we must show the class of $f^\sharp_W(g)$ in $k = \kappa(x)$ is the same as that of g in $k = \kappa(f(x))$. This is clear (c.f. (4.19)). This implies that $f : X \to Y$ induces a morphism $(g, g^*) : (V, \mathcal{O}_V) \to (U, \mathcal{O}_U)$ in $\mathbf{bSp}_k$. The construction is entirely functorial, and this shows the assignment $X \mapsto (V, \mathcal{O}_V)$ defines a functor from the category of algebraic k-schemes to $\mathbf{bSp}_k$.

We shall expect $\iota_*\mathcal{O}_V$ is isomorphic to the structure sheaf $\mathcal{O}_X$ of X. The morphism α induces a surjective morphism $\alpha : \mathcal{O}_X \to \iota_*\mathcal{O}_V$, so it would be nice if α is injective. Let $U = \operatorname{Spec} A$ be an affine open set of X. By (4.58.1) we have $U \cap V = \operatorname{mSpec} A$. For $\alpha_V(U)$ to be injective, it is the same as saying that if $f \in A$ is such that $f \in \mathfrak{m}_\mathfrak{m} \subseteq A_\mathfrak{m}$ for all maximal $\mathfrak{m}$, then $f = 0$ in A, i.e., $\mathrm{rad}(A) = 0$. Since A is Jacobson by (4.42) and (4.38), this is equivalent to saying that $0 = \sqrt{0}$, i.e., A is reduced. Hence, if X is a reduced algebraic k-scheme, X can be recovered from the local-ringed space $(V, \mathcal{O}_V)$, in the sense that $\iota_*\mathcal{O}_V \cong \mathcal{O}_X$ canonically.

Therefore, we assume X is reduced in the following. We are going to show $(V, \mathcal{O}_V)$ is a k-variety. Let U be an affine open set in X; we see $U_{\mathrm{cl}} = U \cap V$ in the previous paragraph. Put $U' = U \cap V$ and let $j : U' \to U$ be the inclusion. Since U is also an algebraic k-scheme, we may construct $(U', \mathcal{O}_{U'})$. By functoriality we have a morphism $(U', \mathcal{O}_{U'}) \to (V, \mathcal{O}_V)$ in $\mathbf{bSp}_k$. By (2.22) this gives $(U', \mathcal{O}_{U'}) \to (U', \mathcal{O}_V|_{U'})$ in $\mathbf{LRS}_k$, which is necessarily in $\mathbf{bSp}_k$. This morphism fits into a similarly obtained commutative diagram

$$\begin{tikzcd}
(\mathcal{O}_X|_U)|_{U'} \arrow[r, two heads] & \mathcal{O}_{U'} \arrow[r, hook] & k^{U'} \\
(\mathcal{O}_X|_V)|_{U'} \arrow[r, two heads] \arrow[u] & \mathcal{O}_V|_{U'} \arrow[r, hook] \arrow[u] & k^V|_{U'} \arrow[u]
\end{tikzcd}$$

The vertical arrows except the middle one are clearly isomorphisms, and thus so is the middle one. Hence $(U', \mathcal{O}_{U'}) \to (U', \mathcal{O}_V|_{U'})$ in $\mathbf{bSp}_k$.

We are now reduced to the case X being affine. Let A be a reduced k-algebra of finite type. Let $\iota : \operatorname{mSpec} A \to \operatorname{Spec} A$ be the inclusion. The third paragraph shows that $\iota_*\mathcal{O}_{\operatorname{mSpec} A} \cong \mathcal{O}_{\operatorname{Spec} A}$ in $\mathbf{LRS}_k$. Since ι is a quasi-isomorphism (4.43), it follows from (4.39.2) that we have an isomorphism $\mathcal{O}_{\operatorname{mSpec} A} \cong \mathcal{O}_{\operatorname{Spec} A}|_{\operatorname{mSpec} A}$. By (4.60), if W is an affine k-variety constructed from A by means of (4.9), we then have an isomorphism

$$(W, \mathcal{O}_W) \cong (\operatorname{mSpec} A, \mathcal{O}_{\operatorname{mSpec} A})$$

in $\mathbf{LRS}_k$, where the map on topological spaces is given by $\varphi : W \cong \operatorname{mSpec} k[W] \cong \operatorname{mSpec} A$. It remains to show the map on sheaves are given by pullback of functions by φ. This is easily checked on every principal open set. Therefore we see that the isomorphism is in fact an isomorphism in $\mathbf{bSp}_k$, proving that $(\operatorname{mSpec} A, \mathcal{O}_{\operatorname{mSpec} A})$ is an affine k-variety.

4.62 We summarize what have been done so far. Denote by $\mathbf{AlgSch}_k$ (resp. $\mathbf{redAlgSch}_k$) the full subcategory of $\mathbf{Sch}_k$ consisting of algebraic (resp. reduced algebraic) k-schemes. Categorically speaking, (4.58.1) implies that $\mathbf{AlgSch}_k$ admits a forgetful functor $\mathbf{AlgSch}_k \to \mathbf{JacSob}$, given by sending schemes to its underlying topological spaces.

In (4.57) we constructed a functor

$$\begin{aligned} \mathrm{Sob} : \mathbf{Var}_k &\longrightarrow \mathbf{redAlgSch}_k \\ X &\longmapsto (\mathrm{Sob}(X), (\iota_X)_* \mathcal{O}_X). \end{aligned}$$

which is proved to be fully faithful (4.58.2). This functor fits into a commutative diagram

$$\begin{tikzcd} \mathbf{Var}_k \arrow[r] \arrow[d] & \mathbf{AlgSch}_k \arrow[d] \\ \mathbf{Top}_{T_1} \arrow[r] & \mathbf{JacSob} \end{tikzcd}$$

with the bottom horizontal arrow $X \mapsto \mathrm{Sob}(X)$ being the equivalence (4.45) with inverse $Y \mapsto Y_{\mathrm{cl}}$.

In (4.61) we constructed a functor

$$\begin{aligned} \mathrm{cl} : \mathbf{AlgSch}_k &\longrightarrow \mathbf{bSp}_k \\ X &\longmapsto (X_{\mathrm{cl}}, \mathcal{O}_{X_{\mathrm{cl}}}) \end{aligned}$$

and showed that it restricts to a functor $\mathbf{redAlgSch}_k \to \mathbf{Var}_k$ that is injective and inverse to the functor Sob on objects. From the discussion there, we easily see that it is also inverse to Sob on morphisms. Thus they are really inverse to each other. Pictorially,

$$\begin{tikzcd} \mathbf{Var}_k \arrow[r, bend left, "\mathrm{Sob}"] & \mathbf{redAlgSch}_k \arrow[l, bend left, "\mathrm{cl}"] \end{tikzcd}$$

4.63

4.3 Some Birational geometry

In this subsection, let k be a field.

4.64 Definition. An integral algebraic k-scheme is called **rational** if it is birational to $\mathbb{P}^n_k$ for some n.

4.65 Example : Pythagorean triples. $\mathrm{Proj}\,\dfrac{k[x,y,z]}{(x^2+y^2-z^2)}$ is birational to $\mathbb{P}^1_k$.

4.4 Galois group

4.66 Let X be a scheme of finite type over $\mathbb{C}$. Then there exists a subring $R \subseteq \mathbb{C}$ of finite type over $\mathbb{Z}$, and a scheme X_0 of finite type over R such that

$$X \cong X_0 \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, \mathbb{C}.$$

Proof. Let $\mathcal{U} = \{U_i\}_{i=1}^n$ be a finite affine open cover and write

$$\mathcal{O}_X(U_i) = R_i = \mathbb{C}[x_1, \ldots, x_{t_i}]/(f_{i,1}, \ldots, f_{i,n_i}).$$

For each i, j, cover $U_i \cap U_j$ by affine open subsets in U_i and U_j; then each subset defines an isomorphism $\phi_{ij,l} : (R_i)_{g_{ij,l}} \cong (R_j)_{g_{ji,l}}$. For any i, j, k, l, l', we have

$$(U_j)_{g_{ji,l} \cdot g_{jk,l'}} = (U_i)_{g_{ij,l} \cdot \phi_{ij,l}^{-1}(g_{jk,l'})} \subseteq U_i \cap U_j \cap U_k \subseteq \bigcup_{l''} (U_i)_{g_{ik,l''}}.$$

This means

$$\left(g_{ij,l} \cdot \phi_{ij,l}^{-1}(g_{jk,l'})\right)^N = \sum_{l''} a_{ijkll'l''} g_{ikl''}$$

for some $N \geqslant 1$ and $a_{ijkll'l''} \in R_i$. Lift a's and g's to $\mathbb{C}[X]$, and let R be the subring generated by the coefficients of the f_{ij}'s, g's, a's and of the polynomials defining the $\phi_{ij,l}$'s. Put

$$R_{i,0} = R[x_1, \ldots, x_{t_i}]/I_i,$$

where $I_i = \ker(R[x_1, \ldots, x_{t_i}] \to R_i)$, and put $U_{i,0} = \operatorname{Spec} R_{i,0}$. The element $g_{ij,l}$ lies in $R_{i,0}$, and $\phi_{ij,l}$ restricts to an isomorphism $(R_{i,0})_{g_{ij,l}} \cong (R_{j,0})_{g_{ji,l}}$. Now use (2.13) to glue all $U_{i,0}$. □

4.67 Base change. Let k, K be fields and $\sigma \in \operatorname{Hom}_{\mathrm{Field}}(k, K)$. For a k-scheme X, consider the fibre product

$$\begin{CD}
X \times_{\operatorname{Spec} k} \operatorname{Spec} K @>{\sigma_X}>> X \\
@VVV @VVV \\
\operatorname{Spec} K @>{\operatorname{Spec} \sigma}>> \operatorname{Spec} k
\end{CD}$$

We denote by X^σ the scheme $X \times_{\operatorname{Spec} k} \operatorname{Spec} K$. Note that if $X = \operatorname{Spec} k[x]/(f)$, then

$$X^\sigma = \operatorname{Spec}\left(k[x]/(f) \otimes_{k,\sigma} K\right) = \operatorname{Spec} K[x]/(f^\sigma)$$

where $f^\sigma \in K[x]$ is the polynomial obtained by applying σ to the coefficients of f. If the embedding σ is obvious from the context, we usually write X_K instead of X^σ.

4.67.1 Let $U \subseteq X$ be an open subspace, and $x \in X^\sigma$. Consider the commutative diagram

$$\begin{CD}
\mathcal{O}_X(U) @>{\sigma_X^\sharp}>> \mathcal{O}_{X^\sigma}(\sigma_X^{-1}(U)) \\
@VVV @VVV \\
\mathcal{O}_{X,\sigma_X(x)} @>{\sigma_{X,x}}>> \mathcal{O}_{X^\sigma,x}.
\end{CD}$$

Then for $f \in \mathcal{O}_X(U)$, we have

$$(\sigma_X^\sharp f)(x) = \sigma^{-1}(f(\sigma_X(x))) \text{ in } \kappa(x).$$

Here we use the notation in (2.21). To see this, note that

$$\begin{aligned}
f - f(\sigma_X(x)) \in \mathfrak{m}_{X,\sigma_X(x)} &\Leftrightarrow \sigma_{X,x}(f - f(\sigma_X(x))) \in \mathfrak{m}_{X^\sigma,x} \\
&\Leftrightarrow \sigma_X^\sharp f - \sigma_{X,x}(f(\sigma_X(x))) \in \mathfrak{m}_{X^\sigma,x} \\
&\Leftrightarrow (\sigma_X^\sharp f)(x) = \sigma_{X,x}(f(\sigma_X(x))) = \sigma^{-1}(f\sigma_X(x)) \text{ in } \kappa(x)
\end{aligned}$$

4.68 Let k be a field and let $\overline{k}$ be an algebraic closure of k. If $X \in \mathbf{Sch}_k$, we can consider the $\overline{k}$-scheme

$$X_{\overline{k}} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}.$$

Put $p = \mathrm{pr}_1 : X_{\overline{k}} \to X$.

5 Smoothness

5.1 Normality

5.1 Recall that a ring A is called a **normal domain**, or simply **normal**, if it is an integral domain that is integrally closed in its fraction field. For example, every UFD is a normal domain. If $S \subseteq A$ is a multiplicatively closed set, by clearing the denominators we quickly see the localization $S^{-1}A$ is also a normal domain. Conversely, if A is an integral domain with the property that the localization $A_{\mathfrak{m}}$ is normal for every maximal ideal $\mathfrak{m}$, then A is normal.

In fact, let C be the integral closure of A in $\operatorname{Frac} A$. By clearing the denominators, we see $S^{-1}C$ is the integral closure of $S^{-1}A$ in $\operatorname{Frac} A$. If we write $f : A \to C$ for the natural inclusion, then $A_{\mathfrak{m}}$ is normal implies the localization $f_{\mathfrak{m}} : A_{\mathfrak{m}} \to C_{\mathfrak{m}}$ of f at $\mathfrak{m}$ is surjective (or identity). Varying $\mathfrak{m}$ yields that f is surjective, i.e., $C = A$.

5.2 Definition. Let X be a scheme.

1. X is **normal** at a point $x \in X$ if the local ring $\mathcal{O}_{X,x}$ is a normal domain.
2. X is called the **normal scheme** if it is normal at each point.

5.3 Let X be an irreducible normal scheme. In particular, X is integral, so $\mathcal{O}_X(U)$ is an integral domain for any open set U of X. If $f \in K(X)$ is integral over $\mathcal{O}_X(U)$, then f is integral over $\mathcal{O}_{X,x}$ for every $x \in U$, as $\mathcal{O}_X(U) \subseteq \mathcal{O}_{X,x}$. By assumption we then have $f \in \mathcal{O}_{X,x}$ for every $x \in U$. The equality $\mathcal{O}_X(U) = \bigcap_{x \in U} \mathcal{O}_{X,x}$ in (3.42) then shows that $f \in \mathcal{O}_X(U)$. This proves $\mathcal{O}_X(U)$ is also a normal domain. Conversely, if $\mathcal{O}_X(U)$ is normal for every open $U \subseteq X$, then the local natural of normality stated in (5.1) implies that X is normal.

5.3.1 Lemma. Let X be an irreducible scheme. TFAE :

(i) X is normal.

(ii) $\mathcal{O}_X(U)$ is a normal domain for every open $U \subseteq X$.

5.2 Tangent spaces and differential forms

5.4 Definition. Let $(X, \mathcal{O}_X)$ be a local-ringed space. For a point $x \in X$, the **(Zariski) cotangent space** $T^*_{X,x}$ of X at the point x is defined as the quotient $\mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x}$. The **(Zariski) tangent space** $T_{X,x}$ is defined as the linear dual

$$T_{X,x} := \operatorname{Hom}_{\kappa(x)}(\mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x}, \kappa(x))$$

of the cotangent space $T^*_{X,x}$.

5.5 Example. Let M be a smooth manifold. Write $C^\infty := C^\infty_M$ for the sheaf of real-valued smooth functions on M. Then (M, C^∞) is a local-ringed space over $\mathbb{R}$. For each $p \in M$, the stalk C^∞_p is the collection of smooth functions defined near p. The unique maximal ideal $\mathfrak{m}_p$ of C^∞_p is the kernel of the evaluation map

$$\begin{aligned} C^\infty_p &\longrightarrow \mathbb{R} \\ f &\longmapsto f(p) \end{aligned}$$

Denote by $D_p = \operatorname{Der}(C^\infty_p)$ the space of all point derivations on C^∞_p. Then there is an isomorphism

$$\begin{aligned} D_p &\longrightarrow \operatorname{Hom}_{\mathbb{R}}(\mathfrak{m}_p/\mathfrak{m}_p^2, \mathbb{R}) \\ X &\longmapsto f \mapsto X(f). \end{aligned}$$

This is well-defined, as for $f, g \in \mathfrak{m}_p$, clearly $X(fg) = X(f)g(p) + f(p)X(g) = 0$. For $f \in C_p^\infty$, $f - f(p)$ lies in $\mathfrak{m}_p$, so $X(f) = X(f - f(p)) + X(f(p)) = X(f - f(p))$; this shows the injectivity. For $T \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{m}_p/\mathfrak{m}_p^2, \mathbb{R})$, define $X_T : C_p^\infty \to \mathbb{R}$ by $X_T(f) = T(f - f(p))$. This is clearly linear, and for $f, g \in C_p^\infty$,

$$\begin{aligned} X_T(fg) &= T(fg - f(p)g(p)) \\ &= T((f - f(p))(g - g(p)) + (f - f(p))g(p) + f(p)(g - g(p))) \\ &= T(f - f(p))g(p) + f(p)T(g - g(p)) = X_T(f)g(p) + f(p)X_T(g) \end{aligned}$$

showing that $X_T \in D_p$, whence the surjectivity. Now since M is smooth, D_p is naturally identified as the tangent space of M at p. This somewhat gives the intuition that why (co)tangent spaces of local-ringed spaces are defined so.

5.6 Let X be a scheme over a field k and $x \in X$ a k-valued point, i.e., $\kappa(x) = k$. Then

$$T_{X,x} \cong \{f \in \mathrm{Hom}_{\mathbf{Sch}_k}(\mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2), X) \mid f((\varepsilon)) = x\}$$

To see this, recall in (3.60) that the right hand side is in bijection with $\mathrm{Hom}_{\mathbf{LocAlg}_k}(\mathcal{O}_{X,x}, k[\varepsilon]/(\varepsilon^2))$, where $\mathbf{LocAlg}_k$ denotes the subcategory of **LocRing** consisting k-algebras. For $a \in \mathcal{O}_{X,x}$, denote by $a(x)$ its class in the residue field $\kappa(x) = k \subseteq \mathcal{O}_{X,x}$; then $a - a(x) \in \mathfrak{m}_{X,x}$, which defines a k-isomorphism

$$\begin{aligned} \mathcal{O}_{X,x}/\mathfrak{m}_{X,x} &\longrightarrow k \oplus \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2 \\ a &\longmapsto a(x) + (a - a(x)) \end{aligned}$$

Similarly (but for much trivial reason), we have $k[\varepsilon]/(\varepsilon^2) \cong k \oplus k\varepsilon$. Then obviously

$$\begin{aligned} \mathrm{Hom}_{\mathbf{LocAlg}_k}(\mathcal{O}_{X,x}, k[\varepsilon]/(\varepsilon^2)) &\longrightarrow \mathrm{Hom}_k(\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2, k\varepsilon) \\ f &\longmapsto f|_{\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2} \end{aligned}$$

is a bijection (f maps $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ to $k\varepsilon$ as it is a local homomorphism). But the latter set is simply $T_{X,x}$.

5.7 Module of Kälher differentials Let A be a ring, R an A-algebra and M an R-module. An A**-linear derivation** $d : R \to M$ is a A-linear map satisfying the **Leibniz rule** : $d(fg) = fd(g) + d(f)g$ for any $f, g \in R$. Denote by $\mathrm{Der}_A(R, M)$ the set of all A-linear derivations $R \to M$. This is naturally an A-module.

Let $\Omega_{R/A}$ be the R-module free on the symbols $\{df \mid f \in R\}$ modulo the relations $d(fg) = f(dg) + (df)g\,(f, g \in R)$ and $d(af + bg) = a(df) + b(dg)\,(a, b \in A,\ f, g \in R)$. This is called the **module of Kähler differential** of R over A. There is a natural map $d : R \to \Omega_{R/A}$ given by $f \mapsto df$, called the **universal** A**-derivation**. The pair $(\Omega_{R/A}, d)$ represents the functor $\mathbf{Mod}_R \to \mathbf{Mod}_A$ defined by $M \mapsto \mathrm{Der}_A(R, M)$, i.e., there are functorial bijection

$$\begin{aligned} \mathrm{Hom}_R(\Omega_{R/A}, M) &\longrightarrow \mathrm{Der}_A(R, M) \\ \phi &\longmapsto \phi \circ d \end{aligned}$$

5.8 Conormal module. Retain the notations in the previous paragraph. We shall give another description of $\Omega_{R/A}$. Denote by $\mu : R \otimes_A R \to R$ the multiplication map and put $I = \ker \mu$.

Lemma. I is the ideal of $R \otimes_A R$ generated by $f \otimes 1 - 1 \otimes f\ (f \in R)$.

Proof. Suppose $\sum_{i=1}^{n} f_i \otimes g_i \in R \otimes_A R$ satisfies $\sum_{i=1}^{n} f_i g_i = 0$ in R. Write $f_i \otimes g_i = (f_i \otimes 1)(1 \otimes g_i - g_i \otimes 1) + f_i g_i \otimes 1$. Then by assumption

$$\sum_{i=1}^{n} f_i \otimes g_i = \sum_{i=1}^{n} (f_i \otimes 1)(1 \otimes g_i - g_i \otimes 1) + \sum_{i=1}^{n} f_i g_i \otimes 1 = \sum_{i=1}^{n} (f_i \otimes 1)(1 \otimes g_i - g_i \otimes 1)$$

□

5.8.1 Let

$$\begin{aligned} e : R &\longrightarrow I/I^2 \\ f &\longmapsto f \otimes 1 - 1 \otimes f \pmod{I^2} \end{aligned}.$$

It is easy to see e is an A-derivation. Since I/I^2 is an $R \otimes_A R$-module killed by I, we can view it as an R-module. From the lemma we see e is surjective.

Define the map

$$\begin{aligned} \mathrm{Hom}_R(I/I^2, M) &\longrightarrow \mathrm{Der}_A(R, M) \\ \phi &\longmapsto e \circ \phi. \end{aligned}$$

We claim this is bijective. Since e is surjective, $\phi \mapsto e \circ \phi$ is injective. For the surjectivity, for a derivation $D : R \to M$, define $\phi = \phi_D : R \otimes_A R \to M$ by

$$\phi(f \otimes g) = fe(g).$$

One easily checks that

$$\phi(xy) = \mu(x)\psi(y) + \mu(y)\psi(x),$$

so that ϕ is trivial on I^2, and whence defined a map $\phi : I/I^2 \to M$. It is easy to see ϕ is R-linear, and $\phi(f \otimes 1 - 1 \otimes f) = -e(f)$, so $D = e \circ (-\phi)$.

Clearly the bijection is functorial in M, so the pair $(I/I^2, e)$ represents the functor $M \mapsto \mathrm{Der}_A(R, M)$ as well. It follows that there is a unique isomorphism $(I/I^2, e) \cong (\Omega_{R/A}, d)$.

5.9 Let X be a topological space, $\mathcal{A}$ a sheaf of rings on X and $\mathcal{R}$ an $\mathcal{A}$-algebra. For an $\mathcal{R}$-module $\mathcal{M}$, an **$\mathcal{A}$-derivation** of $\mathcal{R}$ with valued in $\mathcal{M}$ is an $\mathcal{A}$-morphism $d : \mathcal{R} \to \mathcal{M}$ such that $d_U \in \mathrm{Der}_{\mathcal{A}(U)}(\mathcal{R}(U), \mathcal{M}(U))$ for each open $U \subseteq X$. Denote by $\mathrm{Der}_{\mathcal{A}}(\mathcal{R}, \mathcal{M})$ the set of all $\mathcal{A}$-derivations $\mathcal{R} \to \mathcal{M}$. In this way we obtain a functor

$$\begin{aligned} \mathbf{Mod}_{\mathcal{R}} &\longrightarrow \mathbf{Set} \\ \mathcal{M} &\longmapsto \mathrm{Der}_{\mathcal{A}}(\mathcal{R}, \mathcal{M}). \end{aligned}$$

5.9.1 Lemma. The above functor is representable.

5.9.2 Definition. The object that represents the functor $\mathcal{M} \mapsto \mathrm{Der}_{\mathcal{A}}(\mathcal{R}, \mathcal{M})$ is called the **sheaf of Kähler differentials / 1-forms** of the $\mathcal{A}$-algebra $\mathcal{R}$, and is denoted by $\Omega_{\mathcal{R}/\mathcal{A}}$. The universal element $d : \mathcal{R} \to \Omega_{\mathcal{R}/\mathcal{A}}$ is called the **universal $\mathcal{A}$-derivation**.

5.10 Lemma. Let $f : Y \to X$ be a continuous map, $\mathcal{A}$ a sheaf of rings on X and $\mathcal{R}$ an $\mathcal{A}$-algebra. Then there is an natural isomorphism

$$f^{-1}\Omega_{\mathcal{R}/\mathcal{A}} \cong \Omega_{f^{-1}\mathcal{R}/f^{-1}\mathcal{A}}.$$

compatible with universal derivations.

Proof. This follows from the definition and the adjunction (2.9).

□

5.11 Let $f : X \to S$ be an S-scheme. Recall from Lemma 3.73.1 that the diagonal embedding $\Delta_{X/S} : X \to X \times_S X$. Let $U \subseteq X \times_S X$ be any open subset containing the image $\Delta(X)$ as a closed subset, and let $\mathcal{I}$ be the defining ideal of $\Delta(X)$ in U, i.e., $V(\mathcal{I}) = \Delta(X)$.

Definition. The **sheaf of Kälher differentials** of $f : X \to S$ is defined as

$$\Omega_{X/S} = \Delta_{X/S}^*(\mathcal{I}/\mathcal{I}^2).$$

Clearly the right hand side is independent of the choice of the open set U. This is also called the **sheaf of** 1**-forms**.

6 Homological algebra

6.1 Triangulated category

6.1 Triangle. Let $\mathcal{T}$ be an additive category and and let $[1] : \mathcal{T} \to \mathcal{T}$ be an additive automorphism. A **triangle** in $\mathcal{T}$ is a complex in $\mathcal{T}$ of the form $X \to Y \to Z \to X[1]$. A **morphism between two triangles** $X \to Y \to Z \to X[1]$ and $X' \to Y' \to Z' \to X'[1]$ is a triple $(f, g, h) \in \mathrm{Hom}_{\mathcal{T}}(X, X') \times \mathrm{Hom}_{\mathcal{T}}(Y, Y') \times \mathrm{Hom}_{\mathcal{T}}(Z, Z')$ fitting into a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

Under the obvious composition of morphisms, the class of triangles in $\mathcal{T}$ then forms a category.

6.2 Pre-triangulated category. A **pre-triangulated category** is an additive category $\mathcal{T}$ together with an additive automorphism $[1] : \mathcal{T} \to \mathcal{T}$, called the **shift functor**, and a collection of triangles, called the **distinguished triangles**, satisfying the following three axioms

(TR 0)
- $X \xrightarrow{\mathrm{id}} X \to 0 \to X[1]$ is distinguished.
- Distinguished triangles are closed under isomorphism.

(TR 1) Any morphism in $\mathcal{T}$ can be completed to a distinguished triangle. Namely, or any $f \in \mathrm{Hom}_{\mathcal{T}}(X, Y)$, there exist $Z \in \mathrm{Ob}\,\mathcal{T}$ and $(g, h) \in \mathrm{Hom}_{\mathcal{T}}(Y, Z) \times \mathrm{Hom}_{\mathcal{T}}(Z, X[1])$ such that $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is distinguished.

(TR 2) A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is distinguished if and only if $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ is distinguished.

(TR 3) For any commutative diagram of distinguished triangles

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] & X[1] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

there exists a morphism $h \in \mathrm{Hom}_{\mathcal{T}}(Z, Z')$ making the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

commutative.

We often write $[0] = \mathrm{id}_{\mathcal{T}}$, $[-1] = [1]^{-1}$, $[n] = \underbrace{[1] \circ \cdots \circ [1]}_{n\text{-times}}$ $(n \geqslant 1)$ and $[n] = [-n]^{-1}$ $(n \leqslant 0)$.

6.2.1 A consequence. If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a distinguished triangle, then $g \circ f = 0$ and $h \circ g = 0$.

Proof. By (TR 2), it suffices to show $g \circ f = 0$. For this, consider the diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, equal] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] & X[1] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & X[1] \end{tikzcd}$$

By (TR 3), the unique map $0 \to Z$ completes the above commutative diagram. In particular, $g \circ f = 0$. □

6.3 Example : homotopy category. Let $\mathcal{A}$ be an additive category, and let $\mathrm{Kom}(\mathcal{A})$ denote the category of complexes of objects in $\mathcal{A}$. For a complex $X = (X^\bullet, d_X^\bullet)$, define $X[1] = X^\bullet[1]$ by

$$(X^\bullet[1])^n = X^{n+1}, \qquad d^n_{X[1]} = -d_X^{n+1}.$$

Aware of the minus sign here. For a morphism $f : X \to Y$, the morphism $f[1] : X[1] \to Y[1]$ is defined as $f[1]^n = f^{n+1} : X^{n+1} \to Y^{n+1}$. Then $[1] : \mathrm{Kom}(\mathcal{A}) \to \mathrm{Kom}(\mathcal{A})$ is an additive automorphism.

For a morphism $f : X \to Y$, the **mapping cone** $cone(f)$ is a complex defined by

$$cone(f)^\bullet = Y^\bullet \oplus X^\bullet[1], \qquad d^n_{cone(f)} = \begin{pmatrix} d_Y^n & f^{n+1} \\ 0 & -d_X^{n+1} \end{pmatrix}$$

where we view an element in $Y^\bullet \oplus X^\bullet[1]$ as a column vector. Then there is a natural diagram $X \xrightarrow{f} Y \to cone(f) \to X[1]$.

For two complexes X and Y, define the $\mathrm{Hom}^\bullet(X, Y)$ by

$$\mathrm{Hom}^n(X, Y) = \prod_{m \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(X^m, Y^{m+n})$$

and $d^n_{\mathrm{Hom}^\bullet(X,Y)} : \mathrm{Hom}^n(X, Y) \to \mathrm{Hom}^{n+1}(X, Y)$ by setting for $(u^m)_m \in \mathrm{Hom}^n(X, Y)$ that

$$d^n_{\mathrm{Hom}^\bullet(X,Y)}(u^m) = d_Y^{n+m} \circ u^m + (-1)^{n+1} u^{m+1} \circ d_X^m.$$

Note that $\mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}(X, Y) = \ker d^n_{\mathrm{Hom}^\bullet(X,Y)}$.

We say two morphism $f, g : X \to Y$ of complexes are **homotopic** if $f - g \in \mathrm{Im}\, d^{-1}_{\mathrm{Hom}^\bullet(X,Y)}$, i.e, there exists $h \in \mathrm{Hom}^{-1}(X, Y)$ such that $f - g = d_Y \circ h + h \circ d_X$. In this case we write $f \sim g$. Note that $f \sim g$ if and only if $f - g \sim 0$, and $\sim$ is an equivalence relation. Define the **homotopy category of complexes** $K(\mathcal{A})$ by

$$\begin{aligned} \mathrm{Ob}\, K(\mathcal{A}) = \mathrm{Ob}\ \mathrm{Kom}(\mathcal{A}), \qquad \mathrm{Hom}_{K(\mathcal{A})}(X, Y) &= \mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}(X, Y)/\sim \\ &= \mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}(X, Y)/\,\mathrm{Im}\, d^{-1}_{\mathrm{Hom}^\bullet(X,Y)} \end{aligned}$$

Define $\mathrm{Kom}^+(\mathcal{A})$ (resp. $\mathrm{Kom}^-(\mathcal{A})$, $\mathrm{Kom}^b(\mathcal{A})$) to be the full subcategory of $\mathrm{Kom}(\mathcal{A})$ consisting of complexes X such that $X^n = 0$ for $n \ll 0$ (resp. $n \gg 0$, $|n| \gg 0$). We then can similarly define the homotopy categories

$$K^\square(\mathcal{A}) \qquad \text{for } \square = +, -, b$$

of complexes that are bounded below (resp. bounded above, bounded) in the same way.

6.3.1 Lemma. (TR 2) Given $f : X \to Y$ a morphism of complexes, consider the diagram

$$\begin{tikzcd} Y \arrow[r, "\tau"] \arrow[d, equal] & cone(f) \arrow[r, "\rho"] \arrow[d, equal] & X[1] \arrow[r, "-f"] \arrow[d, dashed] & Y[1] \arrow[d, equal] \\ Y \arrow[r, "\tau"] & cone(f) \arrow[r, "\theta"] & cone(\tau) \arrow[r] & Y[1]. \end{tikzcd}$$

There exists a morphism $g : X[1] \to cone(\tau)$ completing the above diagram, making it commutative in $K(\mathcal{A})$. Also, g is an isomorphism in $K(\mathcal{A})$.

Proof. Define $g : X[1] \to \mathrm{cone}(\tau)$ by

$$\begin{aligned} g^n : X^{n+1} &\longrightarrow \mathrm{cone}(\tau)^n = Y^n \oplus X^{n+1} \oplus Y^{n+1} \\ x &\longmapsto (0, x, -fx). \end{aligned}$$

Define $h : \mathrm{cone}(\tau) \to X[1]$ simply by projecting down to $X[1]$ component. Then $h \circ g = \mathrm{id}_{X[1]}$, and

$$(g^n \circ h^n - \mathrm{id})(y, x, y') = (-y, 0, -fx - y')$$

If we put $\psi \in \mathrm{Hom}^{-1}(\mathrm{cone}(\tau), \mathrm{cone}(\tau))$ by setting $\psi(y, x, y') = (0, 0, -y)$, then $g \circ h - \mathrm{id}_{\mathrm{cone}(\tau)} = d_{\mathrm{cone}(\tau)} \circ \psi + \psi \circ d_{\mathrm{cone}(\tau)}$, i.e., $g \circ h \sim \mathrm{id}_{\mathrm{cone}(\tau)}$. Hence g is inverse to h in $K(\mathcal{A})$.

It is direct to see that the rightmost square is commutative in $\mathrm{Kom}(\mathcal{A})$. For the middle, we compute

$$(g \circ \rho - \theta)(y, x) = (-y, 0, -fx).$$

If we put $\phi : \mathrm{Hom}^{-1}(\mathrm{cone}(f), \mathrm{cone}(\tau))$ by setting $\phi(y, x) = (0, 0, -y)$, then $g \circ \rho - \theta = d \circ \phi + \phi \circ d$, i.e., $g \circ \rho \sim \theta$. $\square$

6.3.2 Lemma. (TR 0) $X \xrightarrow{\mathrm{id}_X} X \to 0 \to X[1]$ is isomorphic to $X \xrightarrow{\mathrm{id}_X} X \to \mathrm{cone}(\mathrm{id}_X) \to X[1]$ as triangles in $K(\mathcal{A})$.

6.3.3 As a result, if we declare a triangle to be distinguished in $K(\mathcal{A})$ if and only if it is isomorphic to $X \xrightarrow{f} Y \to \mathrm{cone}(f) \to X[1]$ as triangles, then $K(\mathcal{A})$ becomes a pre-triangulated category. We shall always equip $K(\mathcal{A})$ with this structure of pre-triangulated categories. Similarly $K^\square(\mathcal{A})$ ($\square = +, -, b$) are pre-triangulated in this way.

6.3.4 Sanity check. The composition $X \xrightarrow{f} Y \to \mathrm{cone}(f)$ is homotopic to 0.

6.3.5 It should be noticed that $K^\square(\mathcal{A})$ is not abelian in general. In fact,

Lemma. [2] Every monomorphism in a pre-triangulated category split.

More precisely, if $f : X \to Y$ is a monomorphism and if we complete it by (TR 1) to a distinguished triangle $X \to Y \to Z \to X[1]$, then $X \oplus Z \to Y$ with f being the natural inclusion.

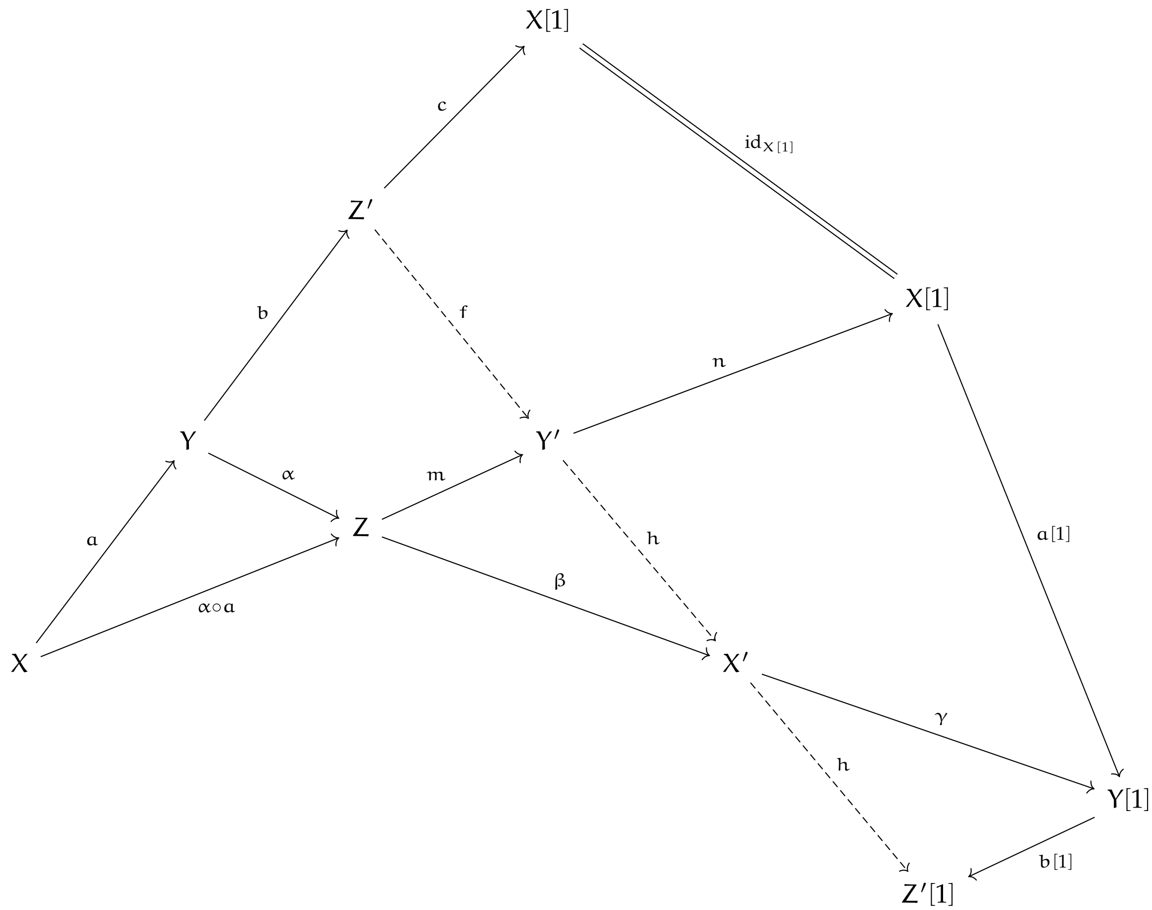
Consider $\mathcal{A} = \mathbf{Ab}$ and p a prime. Consider the map $[p] : \mathbb{Z}/p^2 \to \mathbb{Z}/p$ in $\mathbf{Ab}$. Since $\mathbb{Z}/p^2$ is an indecomposible [3] abelian group, one can show that $\mathbb{Z}/p^2$ is an indecomposible object in $K(\mathbf{Ab})$ as well [4]. In particular, this means $[p]$ cannot have a kernel; otherwise, a kernel is monic, which would, by the lemma, imply that $\mathbb{Z}/p^2$ decomposes.

6.4 Triangulated category. A pre-triangulated category $\mathcal{T}$ is said to be **triangulated** if it satisfies the following axiom

(TR 4) For any distinguished triangles depicted below as solid arrows, there exists a distinguished triangle $Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{h}$

2. From the mathoverflow post.
3. not expressible as a direct sum of abelian groups
4. See another post.

$Z'[1]$ completing the following commutative diagram



6.5 Theorem. Let $\mathcal{A}$ be an additive category. Then $\mathcal{K}(\mathcal{A})$ with the shift functor $[1]$ is a triangulated category.

Proof. It remains to verify (TR 4). Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two morphisms. There are natural maps

$$\mathit{cone}(f) \to \mathit{cone}(g \circ f) \to \mathit{cone}(g) \to \mathit{cone}(f)[1]$$

induced by

$$Y^n \oplus X^{n+1} \xrightarrow{(g,\mathrm{id})} Z^n \oplus X^{n+1} \xrightarrow{(\mathrm{id},f)} Z^n \oplus Y^{n+1} \to Y^{n+1} \oplus X^{n+2}$$

It is routine and tedious to check $\mathit{cone}(g)$ is homotopic equivalent to $\mathit{cone}(\mathit{cone}(f) \to \mathit{cone}(g \circ f))$, and the resulting quadrilateral commutes up to homotopy. □

6.6 Triangulated functor. Let $\mathcal{T}$, $\mathcal{S}$ be pre-triangulated categories. A functor $\Phi : \mathcal{T} \to \mathcal{S}$ is a **triangulated functor** if

- there is a natural equivalence $\Phi \circ [1]_{\mathcal{T}} \cong [1]_{\mathcal{S}} \circ \Phi$, and
- $X \to Y \to Z \to X[1]$ being distinguished in $\mathcal{T}$ implies $\Phi(X) \to \Phi(Y) \to \Phi(Z) \to \Phi(X)[1]$ being distinguished in $\mathcal{S}$

6.1.1 Homological functors

6.7 (Co)homological functor. Let $\mathcal{A}$ be an abelian category and $\mathcal{T}$ a pre-triangulated category. An additive functor (resp. additive contravariant functor) $H : \mathcal{T} \to \mathcal{A}$ is **homological** (resp. **cohomological**) if for any distinguished triangle $X \to Y \to Z \to X[1]$, the sequence $H(X) \to H(Y) \to H(Z)$ (resp. $H(Z) \to H(Y) \to H(X)$) is exact in $\mathcal{A}$.

6.7.1 Associated long exact sequence. Let $H : \mathcal{T} \to \mathcal{A}$ be homological. For $X \in \mathrm{Ob}(\mathcal{T})$ and $n \in \mathbb{Z}$, put

$$H^n(X) = H(X[n]).$$

If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is distinguished, we have an exact sequence $H(X) \xrightarrow{H(f)} H(Y) \xrightarrow{H(g)} H(Z)$. But by TR 2, $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ is also distinguished, so we have another exact sequence $H(Y) \xrightarrow{H(g)} H(Z) \xrightarrow{H(h)} H(X[1]) = H^1(X)$. Similarly we have an exact sequence $H^{-1}(Z) = H(Z[-1]) \xrightarrow{H(-h[1])} H(X) \xrightarrow{H(f)} H(Y)$. Combining these gives a long exact sequence

$$H^{-1}(Z) \xrightarrow{H(-h[1])} H^0(X) \xrightarrow{H(f)} H^0(Y) \xrightarrow{H(g)} H^0(Z) \xrightarrow{H(h)} H^1(X)$$

6.8 Example : representable functors. Let $\mathcal{T}$ be a pre-triangulated category, and $A \in \mathrm{Ob}(\mathcal{T})$. Then the functor $X \mapsto \mathrm{Hom}_{\mathcal{T}}(A, X) \in \mathbf{Ab}$ is homological. Similarly, $X \mapsto \mathrm{Hom}_{\mathcal{T}}(X, A)$ is cohomological.

Proof. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ be a distinguished triangle. We must show

$$\mathrm{Hom}_{\mathcal{T}}(A, X) \xrightarrow{f\circ} \mathrm{Hom}_{\mathcal{T}}(A, Y) \xrightarrow{g\circ} \mathrm{Hom}_{\mathcal{T}}(A, Z)$$

is exact. Suppose $h \in \mathrm{Hom}_{\mathcal{T}}(A, Y)$ satisfies $g \circ h : A \to Z$ is zero. Consider the diagram

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}_A"] \arrow[d, dashed] & A \arrow[r] \arrow[d, "h"] & 0 \arrow[r] \arrow[d, "0"] & A[1] \arrow[d, dashed] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & X[1]
\end{tikzcd}$$

By (TR 0), (TR 2) and (TR 3), there exists a morphism $h' \in \mathrm{Hom}_{\mathcal{T}}(A, X)$ making the diagram commute. In other words, $f \circ h' = h$. This proves the exactness. ☐

6.9 Example : (co)homology. Let $\mathcal{A}$ be an abelian category. For a complex $X = (X^\bullet, d_X^\bullet)$, its **cohomology complex** $H^\bullet(X) = H^\bullet(X^\bullet, d_X^\bullet)$ is the quotient

$$H^n(X) = \ker d^n / \operatorname{Im} d^{n-1}$$

with zero differential. Note that $H^n(X) = H^0(X[n])$. A morphism $f : X \to Y$ in $\mathrm{Kom}(\mathcal{A})$ induces a morphism between cohomology $H^\bullet(f) : H^\bullet(X) \to H^\bullet(Y)$ in a natural way.

Lemma. $H^0 : X \mapsto H^0(X)$ is a homological functor.

Proof. Let $f : X \to Y$ be a morphism in $\mathcal{A}$. We must the sequence

$$H^0(X) \longrightarrow H^0(Y) \longrightarrow H^0(\mathrm{cone}(f))$$

is exact. Suppose $y \in Y^0$ such that $dy = 0$ and there exists $(y', x') \in \mathrm{cone}(f)^{-1}$ such that $(y, 0) = (dy' + fx', -dx')$. Then $dx' = 0$, i.e., $x' \in H^0(X)$ and $fx' = y - dy' = y$ in $H^0(Y)$. ☐

Hence by (6.7.1), for each morphism $f : X \to Y$ we have an induced long exact sequence on cohomology :

$$\cdots \longrightarrow H^{n-1}(\mathrm{cone}(f)) \longrightarrow H^n(X) \xrightarrow{H^n(f)} H^n(Y) \longrightarrow H^n(\mathrm{cone}(f)) \longrightarrow H^{n+1}(X) \longrightarrow .$$

6.9.1 Let $0 \to X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \to 0$ be a short exact sequence in $\mathrm{Kom}(\mathcal{A})$. We've seen the long exact sequence induced by $X \to Y \to \mathrm{cone}(f) \to X[1]$ above. On the other hand,

Lemma. Let $0 \to X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \to 0$ be a short exact sequence in $\mathrm{Kom}(\mathcal{A})$. Then there exists a quasi-isomorphism ((6.20) $\mathrm{cone}(f)^\bullet \xrightarrow{\sim} Z^\bullet$ fitting in a commutative triangle

$$\begin{tikzcd} \mathrm{cone}(f) \arrow[rr] & & Z \\ & Y \arrow[ul, "\tau"] \arrow[ur, "g"'] & \end{tikzcd}$$

where $\tau : Y \to \mathrm{cone}(f) = Y \oplus X[1]$ is the inclusion.

Proof. The map $\varphi^\bullet : \mathrm{cone}(f)^\bullet \xrightarrow{\sim} Z^\bullet$ can only be given by $(y, x) \mapsto g^\bullet(y)$. To see this induces an isomorphism $H^0(\mathrm{cone}(f)) \xrightarrow{\sim} H^0(Z)$, assume $(y, x) \in \ker d^0_{\mathrm{cone}(f)}$ gets sent to $g(y) = 0$ in $H^0(Z)$. Then there exists $z' \in Z^{-1}$ such that $g(y) = d_Z^{-1}(z')$. By exactness we can find $y' \in Y^{-1}$ with $g(y') = z'$. But $g \circ d_Y^{-1}(y') = d_Z^{-1} \circ g(y') = d_Z^{-1}(z') = g(y)$, so $g(y - d_Y^{-1}(y')) = 0$. By exactness again we can find $x'' \in X^0$ such that $f(x'') = y - d_Y^{-1}(y')$. But then

$$d^{-1}_{\mathrm{cone}(f)}(y', x'') = (d_Y^{-1}y' + f(x''), -d_X^0 x'') = (y, -d_X^0 x'')$$

Since $(y, x) \in \ker d^0_{\mathrm{cone}(f)}$, by definition $d_Y^0 y + f(x) = 0 = d_X^1 x$. We must show $-d_X^0 x'' = x$; indeed,

$$f(x + d_X^0 x'') = -d_Y^0 y + d_Y^0 f(x'') = -d_Y^0 y + d_Y^0(y - d_Y^{-1}(y')) = 0.$$

so $x + d_X^0 x'' = 0$ by injectivity of f. □

By the lemma, we can replace $H^\bullet(\mathrm{cone}(f))$ by $H^\bullet(Z)$ everywhere to get a long exact sequence

$$\cdots \longrightarrow H^{n-1}(Z) \longrightarrow H^n(X) \xrightarrow{H^n(f)} H^n(Y) \longrightarrow H^n(Z) \longrightarrow H^{n+1}(X) \longrightarrow .$$

familiar in the usual homological algebra.

6.1.2 t-structures

6.10 t-structure. Let $\mathcal{T}$ be a pre-triangulated category. A **t-structure** of $\mathcal{T}$ consists of a pair $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$ of strictly full subcategories (1.6) of $\mathcal{T}$ such that by setting $\mathcal{T}^{\leqslant n} := \mathcal{T}^{\leqslant 0}[-n]$ and $\mathcal{T}^{\geqslant n} := \mathcal{T}^{\geqslant 0}[-n]$ we have

(TS 1) $\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}) = 0$,

(TS 2) $\mathcal{T}^{\leqslant 0} \subseteq \mathcal{T}^{\leqslant 1}$ and $\mathcal{T}^{\geqslant 1} \subseteq \mathcal{T}^{\geqslant 0}$, and

(TS 3) for any object X in $\mathcal{T}$ there exists a distinguished triangle $Y \to X \to Z \to Y[1]$ with $Y \in \mathcal{T}^{\leqslant 0}$ and $Z \in \mathcal{T}^{\geqslant 1}$.

The **heart** of a t-structure is given by

$$\mathcal{T}^{\heartsuit} := \mathcal{T}^{\leqslant 0} \cap \mathcal{T}^{\geqslant 0}$$

which is a full subcategory of $\mathcal{T}$.

6.10.1 Truncations.

1. The inclusion $\mathcal{T}^{\leqslant 0} \to \mathcal{T}$ admits a right adjoint, denoted by $\tau_{\leqslant 0} : \mathcal{T} \to \mathcal{T}^{\leqslant 0}$
2. The inclusion $\mathcal{T}^{\geqslant 1} \to \mathcal{T}$ admits a left adjoint, denoted by $\tau_{\geqslant 1} : \mathcal{T} \to \mathcal{T}^{\geqslant 1}$.

Proof. Let X be an object in $\mathcal{T}$. By (TS 3) we can find $Y \in \mathcal{T}^{\leqslant 0}$, $Z \in \mathcal{T}^{\geqslant 1}$ and a distinguished triangle $Y \to X \to Z \to Y[1]$, or $Z[-1] \to Y \to X \to Z$. Since $Z[-1] \in \mathcal{T}^{\geqslant 1}[-1] \subseteq \mathcal{T}^{\geqslant 1}$ by (TS 2), by (TS 1) and (6.8) we have an isomorphism

$$\mathrm{Hom}_{\mathcal{T}^{\leqslant 0}}(A, Y) \cong \mathrm{Hom}_{\mathcal{T}}(A, X).$$

functorial in $A \in \mathcal{T}^{\leqslant 0}$. Similarly, we have

$$\mathrm{Hom}_{\mathcal{T}^{\geqslant 1}}(Z, B) \cong \mathrm{Hom}_{\mathcal{T}}(X, B).$$

functorial in $B \in \mathcal{T}^{\geqslant 1}$. For each object X fix Y, Z satisfying (TS 3), and define $\tau_{\leqslant 0}(X) := Y$, $\tau_{\geqslant 1}(X) := Z$. □

6.10.2 In particular, the distinguished triangle in (TS 3) takes the form

$$\tau_{\leqslant 0}X \to X \to \tau_{\geqslant 1}X \to (\tau_{\leqslant 0}X)[1].$$

For any $n \in \mathbb{Z}$ we can similarly define

$$\tau_{\leqslant n} : \mathcal{T} \to \mathcal{T}^{\leqslant n}$$

as the right adjoint of $\mathcal{T}^{\leqslant n} \to \mathcal{T}$, and

$$\tau_{\geqslant n} : \mathcal{T} \to \mathcal{T}^{\geqslant n}$$

as the left adjoint of $\mathcal{T}^{\geqslant n} \to \mathcal{T}$. Clearly

$$\tau_{\leqslant n} \circ [1] = [1] \circ \tau_{\leqslant n+1}$$
$$\tau_{\geqslant n} \circ [1] = [1] \circ \tau_{\geqslant n+1}$$

and there is a distinguished triangle

$$\tau_{\leqslant n}X \to X \to \tau_{\geqslant n+1}X \to (\tau_{\leqslant n}X)[1].$$

For psychological reason, we also put

$$\tau_{>n} := \tau_{\geqslant n+1}, \qquad \tau_{<n} := \tau_{\leqslant n-1}.$$

6.10.3 Lemma. $\mathcal{T}^{\geqslant n} = \{X \in \mathcal{T} \mid \tau_{<n}X = 0\}$ [5], and $\mathcal{T}^{\leqslant n} = \{X \in \mathcal{T} \mid \tau_{>n}X = 0\}$. In particular,

$$\mathcal{T}^{\geqslant n} = \{X \in \mathcal{T} \mid \mathrm{Hom}(\mathcal{T}^{\leqslant n-1}, X) = 0\}$$
$$\mathcal{T}^{\leqslant n} = \{X \in \mathcal{T} \mid \mathrm{Hom}(X, \mathcal{T}^{\geqslant n+1}) = 0\}$$

Proof. It suffices to assume $n = 0$. Assume $X \in \mathcal{T}^{\leqslant 0}$. Then the natural map $\tau_{\leqslant 0}X \to X$ is an isomorphism. By (TS 3) and (TR 0), this means $\tau_{>0}X = 0$. Conversely, assume $\tau_{>0}X = 0$. But by (6.8) applied to $\tau_{\leqslant 0}X \to X \to \tau_{>0}X = 0 \to (\tau_{\leqslant 0}X)[1]$ and Yoneda's lemma, we see $\tau_{\leqslant 0}X \cong X$. This shows $X \in \mathcal{T}^{\leqslant 0}$.

The other assertion is proved in a similar way. The last assertion follows from (t-i) and (6.10.1). □

6.10.4 Lemma. Let $X \to Y \to Z \to X[1]$ be a distinguished triangle.

1. If $X, Z \in \mathcal{T}^{\leqslant n}$, then $Y \in \mathcal{T}^{\leqslant n}$.
2. If $X \in \mathcal{T}^{\leqslant n+1}$ and $Y \in \mathcal{T}^{\leqslant n}$, then $Z \in \mathcal{T}^{\leqslant n}$

Similarly,

3. If $X, Z \in \mathcal{T}^{\geqslant n}$, then $Y \in \mathcal{T}^{\geqslant n}$.

5. Something being 0 simply means it is isomorphic to 0. No confusion arises as there is only one isomorphism between 0 and any object.

4. If $Z \in \mathcal{T}^{\geqslant n-1}$ and $Y \in \mathcal{T}^{\geqslant n}$, then $X \in \mathcal{T}^{\geqslant n}$

Proof. This follows from Lemma 6.10.3 and (6.8). $\square$

6.11 Theorem. Let $\mathcal{T}$ be a pre-triangulated category and $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$ a t-structure. Then its heart $\mathcal{T}^{\heartsuit}$ is an abelian category. Moreover, $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is a short exact sequence if and only if there exists a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1]$.

6.11.1 Existence of kernels and cokernels We split the proofs into several parts. We begin by showing every morphism in $\mathcal{T}^{\heartsuit}$ admits kernels and cokernels. Let $f : X \to Y$ be a morphism in $\mathcal{T}^{\heartsuit}$, and extend it by (TR 1) to a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1].$$

Since $Y \in \mathcal{T}^{\leqslant -1} \subseteq \mathcal{T}^{\geqslant 0}$ by (TS 2), from Lemma 6.10.4.2 (witg $n = -1$) we see $Z \in \mathcal{T}^{\leqslant -1}$. By (TR 2) $Y \to Z \to X[1] \to Y[1]$ is distinguished. Since $X[1] \in \mathcal{T}^{\leqslant 0}[1] \subseteq \mathcal{T}^{\leqslant 1}[1] = \mathcal{T}^{\leqslant 0}$ by (TS 2), so by Lemma 6.10.4 (with $n = 0$) we see $Z \in \mathcal{T}^{\leqslant 0}$. In sum

$$Z \in \mathcal{T}^{\leqslant 0} \cap \mathcal{T}^{\geqslant -1}.$$

By (6.8) and (6.10.1), for any object $A \in \mathcal{T}^{\heartsuit}$ we have a diagram with exact rows

$$\begin{tikzcd}
\mathrm{Hom}(A, Y[-1]) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(A, Z[-1]) \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}(A, X) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(A, Y) \arrow[d, equal] \\
\mathrm{Hom}(A, Y[-1]) \arrow[r] & \mathrm{Hom}(A, \tau_{\leqslant 0}(Z[-1])) \arrow[r] & \mathrm{Hom}(A, X) \arrow[r] & \mathrm{Hom}(A, Y)
\end{tikzcd}$$

Since $Y[-1] \in \mathcal{T}^{\geqslant 0}[-1] = \mathcal{T}^{\geqslant 1}$, we see $\mathrm{Hom}(A, Y[-1]) = 0$ by (TS 1). This implies the sequence

$$0 \longrightarrow \mathrm{Hom}(A, \tau_{\leqslant 0}(Z[-1])) \longrightarrow \mathrm{Hom}(A, X) \longrightarrow \mathrm{Hom}(A, Y)$$

induced by $\tau_{\leqslant 0}(Z[-1]) \to X \to Y$ is exact for any $A \in \mathcal{T}^{\heartsuit}$. To conclude that $\tau_{\leqslant 0}(Z[-1]) \to X$ is a kernel, it suffices to note $\tau_{\leqslant 0}(Z[-1]) \in \mathcal{T}^{\heartsuit}$.

A similar argument shows that $\tau_{\geqslant 0} Z \in \mathcal{T}^{\heartsuit}$ and the natural map $Y \to \tau_{\geqslant 0} Z$ is a cokernel of f.

6.11.2 By (1.3.1) we must show every monomorphism is the kernel of its cokernel. But $f : X \to Y$ is a monomorphism if and only if the unique map $0 \to X$ is the kernel of f. By the last subparagraph, this is equivalent of saying $\tau_{\leqslant 0}(Z[-1]) = 0$, or $Z[-1] \in \mathcal{T}^{\geqslant 1}$, or $Z \in \mathcal{T}^{\geqslant 0}$. By the last subparagraph again, this is nothing but

$$Z \in \mathcal{T}^{\heartsuit}.$$

Suppose $f : X \to Y$ is a monomorphism. Its cokernel is $Y \to \tau_{\geqslant 0} Z = Z$ which is simply $g : Y \to Z$. By (TR 2) the triangle $Y \to Z \to X[1] \to Y[1]$ is distinguished, so the kernel of $g : Y \to Z$ is $X = \tau_{\leqslant 0}(X[1][-1]) \to Y$ which is simply f. This proves the claim. The other statement for epimorphism can be similarly proved.

6.11.3 Finish of the proof. We've seen in the last subparagraph that $T^{\heartsuit}$ is indeed a abelian category. The last statement follows from the construction of kernels and cokernels. $\square$

6.12 Compatibility. Let $\mathcal{T}$ be a pre-triangulated category and $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$ a t-structure. For any $a \leqslant b$, there are natural isomorphisms

$$\tau_{\leqslant a} \circ \tau_{\leqslant b} \cong \tau_{\leqslant a}$$

$$\tau_{\geqslant b} \circ \tau_{\geqslant a} \cong \tau_{\geqslant b}$$

If $\mathcal{T}$ is triangulated, i.e., satisfies (TR 4), then there is an natural isomorphism

$$\tau_{\geqslant a} \circ \tau_{\leqslant b} \cong \tau_{\leqslant b} \circ \tau_{\geqslant a}.$$

Proof. For the first one, for $A \in \mathcal{T}^{\leqslant a}$ and $B \in \mathcal{T}$, by (6.10.1) we have

$$\mathrm{Hom}(A, \tau_{\leqslant a} B) \cong \mathrm{Hom}(A, B)$$

Since $\mathcal{T}^{\leqslant a} \subseteq \mathcal{T}^{\leqslant b}$, by (6.10.1) again we have

$$\mathrm{Hom}(A, B) \cong \mathrm{Hom}(A, \tau_{\leqslant b} B) \cong \mathrm{Hom}(A, \tau_{\leqslant a}\tau_{\leqslant b} B).$$

Then the first isomorphism follows from the Yoneda lemma. The second one is proved similarly.

For the last one, upon applying (6.10.2) to $X \to \tau_{\geqslant a} X$, we obtain a commutative diagram

$$\begin{tikzcd}
\tau_{\leqslant b} X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \tau_{\geqslant b+1} X \arrow[r] \arrow[d, "\wr"] & (\tau_{\leqslant b} X)[1] \arrow[d] \\
\tau_{\leqslant b}\tau_{\geqslant a} X \arrow[r] & \tau_{\geqslant a} X \arrow[r] & \tau_{\geqslant b+1}\tau_{\geqslant a} X \arrow[r] & (\tau_{\leqslant b}\tau_{\geqslant a} X)[1]
\end{tikzcd}$$

where the third vertical arrow is an isomorphism by what we've shown above. On the other hand, by (TR 4) applied to the incomplete quadrilateral

$$\begin{tikzcd}
\tau_{\geqslant a}\tau_{\leqslant b} X & & \\
\tau_{\leqslant b} X \arrow[u] \arrow[dr] & & \tau_{\geqslant a} X \\
 & X \arrow[ur] \arrow[dr] & \\
\tau_{\leqslant a-1}\tau_{\leqslant b} X \cong \tau_{\leqslant a-1} X \arrow[uu] \arrow[ur] & & \tau_{\geqslant b+1} X
\end{tikzcd}$$

there exists a distinguished triangle

$$\tau_{\geqslant a}\tau_{\leqslant b} X \to \tau_{\geqslant a} X \to \tau_{\geqslant b+1} X \to$$

fitting into the above commutative diagram

$$\begin{tikzcd}
\tau_{\geqslant a}\tau_{\leqslant b} X \arrow[r] & \tau_{\geqslant a} X \arrow[r] & \tau_{\geqslant b+1} X \arrow[r] & (\tau_{\geqslant a}\tau_{\leqslant b} X)[1] \\
\tau_{\leqslant b} X \arrow[r] \arrow[d] \arrow[u] & X \arrow[r] \arrow[d] \arrow[u] & \tau_{\geqslant b+1} X \arrow[r] \arrow[d, "\wr"] \arrow[u, equal] & (\tau_{\leqslant b} X)[1] \arrow[d] \arrow[u] \\
\tau_{\leqslant b}\tau_{\geqslant a} X \arrow[r] & \tau_{\geqslant a} X \arrow[r] & \tau_{\geqslant b+1}\tau_{\geqslant a} X \arrow[r] & (\tau_{\leqslant b}\tau_{\geqslant a} X)[1]
\end{tikzcd}$$

By (TR 3), there then exists an isomorphism

$$\alpha : \tau_{\geqslant a}\tau_{\leqslant b} X \cong \tau_{\leqslant b}\tau_{\geqslant a} X$$

connected the first terms in the bottom and upper row. To see this is functorial in X, note the bijection

$$\mathrm{Hom}(\tau_{\geqslant a}\tau_{\leqslant b} X, \tau_{\leqslant b}\tau_{\geqslant a} X) \cong \mathrm{Hom}(\tau_{\geqslant a}\tau_{\leqslant b} X, \tau_{\geqslant a} X) \cong \mathrm{Hom}(\tau_{\leqslant b} X, \tau_{\geqslant a} X) \cong \mathrm{Hom}(\tau_{\leqslant b} X, \tau_{\leqslant b}\tau_{\geqslant a} X)$$

and the map $\mathrm{Hom}(X, \tau_{\geqslant a}X) \to \mathrm{Hom}(\tau_{\leqslant b}X, \tau_{\leqslant b}\tau_{\geqslant a}X)$. It follows from the above commutative diagram that α is uniquely determined by $X \to \tau_{\geqslant a}X$. $\square$

6.12.1 Remark. The use of (TR 4) is to guarantee that $\tau_{\geqslant a}\tau_{\leqslant b}X \to \tau_{\geqslant a}X \to \tau_{\geqslant b+1}X \to$ is distinguished. Any map appearing in the proof can be constructed using adjunction without invoking (TR 4).

6.13 t-cohomological functor. Let $\mathcal{T}$ be a triangulated category and $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$ a t-structure. The **0-th t-cohomological functor** ${}^{t}H : \mathcal{T} \to \mathcal{T}^{\heartsuit}$ is defined by

$$ {}^{t}H := \tau_{\leqslant 0} \circ \tau_{\geqslant 0}. $$

As its name suggests, this is a homological functor. We break the proof into several parts.

6.13.1 Let $X \to Y \to Z \to X[1]$ be a distinguished triangle. Assume that $Z = \tau_{\geqslant 0}Z \in \mathcal{T}^{\geqslant 0}$. Then for any $A \in \mathcal{T}$, by (6.8) and (6.10.2) we have a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & {} & {} & \\
 & \mathrm{Hom}(A, (\tau_{\leqslant -1}X)[1]) \arrow[u] \arrow[r] & \mathrm{Hom}(A, (\tau_{\leqslant -1}Y)[1]) \arrow[u] & \\
 & \mathrm{Hom}(A, \tau_{\geqslant 0}X) \arrow[u] \arrow[r] & \mathrm{Hom}(A, \tau_{\geqslant 0}Y) \arrow[u] \arrow[dr] & \\
\mathrm{Hom}(A, Z[-1]) \arrow[r] \arrow[ur] & \mathrm{Hom}(A, X) \arrow[u] \arrow[r] & \mathrm{Hom}(A, Y) \arrow[u] \arrow[r] & \mathrm{Hom}(A, Z) \\
 & \mathrm{Hom}(A, \tau_{\leqslant -1}X) \arrow[u] \arrow[r] & \mathrm{Hom}(A, \tau_{\leqslant -1}Y) \arrow[u] &
\end{tikzcd}$$

Here the map $\mathrm{Hom}(A, \tau_{\geqslant 0}X) \to \mathrm{Hom}(A, Z)$ is induced from the map $Y \to \tau_{\geqslant 0}Y \to \tau_{\geqslant 0}Z = Z$. If we let A run over objects in $\mathcal{T}^{\leqslant -1}$, it follows from (TS 1) that $\mathrm{Hom}(A, Z[-1]) = 0 = \mathrm{Hom}(A, Z)$. By adjunction (6.10.1) we see

$$ \mathrm{Hom}(A, \tau_{\leqslant -1}X) \cong \mathrm{Hom}(A, X) \cong \mathrm{Hom}(A, Y) \cong \mathrm{Hom}(A, \tau_{\leqslant -1}Y). $$

By Yoneda's, this means $\tau_{\leqslant -1}X \cong \tau_{\leqslant -1}Y$. Now we claim

Lemma. The sequence

$$ \mathrm{Hom}(A, Z[-1]) \longrightarrow \mathrm{Hom}(A, \tau_{\geqslant 0}X) \longrightarrow \mathrm{Hom}(A, \tau_{\geqslant 0}Y) \longrightarrow \mathrm{Hom}(A, Z) $$

is exact.

Proof. Exactness at $\mathrm{Hom}(A, \tau_{\geqslant 0}X)$ follows from a diagram chasing using the fact that flat row and columns are exact and that $\tau_{\leqslant -1}X \cong \tau_{\leqslant -1}Y$. For the exactness at $\mathrm{Hom}(A, \tau_{\geqslant 0}Y)$, look at the piece

$$\begin{tikzcd}
{} & {} & \\
\mathrm{Hom}(A, (\tau_{\leqslant -1}X)[1]) \arrow[u, "a"] \arrow[r, "b", "\sim"'] & \mathrm{Hom}(A, (\tau_{\leqslant -1}Y)[1]) \arrow[u] & \\
\mathrm{Hom}(A, \tau_{\geqslant 0}X) \arrow[u] \arrow[r] & \mathrm{Hom}(A, \tau_{\geqslant 0}Y) \arrow[u, "c"] \arrow[dr, "d"] & \\
 & & \mathrm{Hom}(A, Z)
\end{tikzcd}$$

We claim $b^{-1}c(\ker d) \subseteq \ker a$. Assuming this, it is straightforward to prove the exactness.

To prove the claim, consider the commutative diagram

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, equal] & \tau_{\geqslant 0}Y \arrow[r] \arrow[d] & (\tau_{\leqslant -1}Y)[1] \arrow[r] & Y[1] \arrow[d, equal] \\
Y \arrow[r] & Z \arrow[r] & X[1] \arrow[r] & Y[1]
\end{tikzcd}$$

By (TR 2), (TR 3) and (6.10.2) there exists an arrow $f : (\tau_{\leqslant -1}Y)[1] \to X[1]$ completing the commuting diagram above. By (6.8) we get

$$c(\ker d) \subseteq \ker(\mathrm{Hom}(A, f)) \subseteq \mathrm{Hom}(A, (\tau_{\leqslant -1}Y)[1]).$$

It remains to show $b^{-1}(\ker(\mathrm{Hom}(A, f))) \subseteq \ker a$. Unwinding the definition, we must show that diagram

$$\begin{tikzcd}
X[1] \arrow[r] & Y[1] \\
(\tau_{\leqslant -1}X)[1] \arrow[u] \arrow[r] & (\tau_{\leqslant -1}Y)[1] \arrow[u] \arrow[ul, "f"]
\end{tikzcd}$$

commutes. The outer rectangle and the upper triangle commute by construction. For the lower triangle, since $(\tau_{\leqslant -1}Y)[1] = \tau_{\leqslant -2}(Y[1])$, we see f factors as $\tau_{\leqslant -2}(Y[1]) = \tau_{\leqslant -2}\tau_{\leqslant -2}(Y[1]) \xrightarrow{\tau_{\leqslant -2}(f)} \tau_{\leqslant -2}(X[1]) = (\tau_{\leqslant -1}X)[1] \to X[1]$ by adjunction (6.10.1). If we apply $\tau_{\leqslant -2}$ to the whole square, the vertical arrows collapse. In particular, $\tau_{\leqslant -2}f$ is inverse to $(\tau_{\leqslant -1}X)[1] \to (\tau_{\leqslant -1}Y)[1]$ by the commutativity of the upper triangle. With the factorization above in mind, this finishes the proof. □

6.13.2 Corollary. For any distinguished triangle $X \to Y \to Z \to X[1]$ with $Z \in \mathcal{T}^{\geqslant 0}$, one has the exact sequence

$$0 \to {}^{t}H(X) \to {}^{t}H(Y) \to {}^{t}H(Z)$$

in $\mathcal{T}^{\heartsuit}$. Dually for any distinguished triangle $X \to Y \to Z \to X[1]$ with $X \in \mathcal{T}^{\leqslant 0}$, one has the distinguished triangle

$${}^{t}H(X) \to {}^{t}H(Y) \to {}^{t}H(Z) \to 0$$

in $\mathcal{T}^{\heartsuit}$

Proof. The first statement follows from the lemma with A varying in $\mathcal{T}^{\heartsuit}$ and the adjunction (6.10.1). The last assertion follows from a dual argument for $\tau_{\geqslant 0} \circ \tau_{\leqslant 0}$, which naturally isomorphic to ${}^{t}H$ by (6.12). □

6.13.3 Finish of the proof. Now drop the boundedness condition on Z. For any distinguished triangle $X \to Y \to Z \to X[1]$, we must show

$${}^{t}H(X) \to {}^{t}H(Y) \to {}^{t}H(Z)$$

is exact. By (TR 2) and (TR 4), there exists a commuting quadrilateral with each side distinguished :

$$\begin{tikzcd}
{} \\
\tau_{\geqslant 1}(Z[-1]) \arrow[u] \arrow[drr] & & & {} \\
Z[-1] \arrow[u] \arrow[dr] & & {*} \arrow[ur] \arrow[ddr] \\
& X \arrow[ur] \arrow[drr] \\
\tau_{\leqslant 0}(Z[-1]) \arrow[uu] \arrow[ur] & & & Y \arrow[d] \arrow[dr] \\
& & & \tau_{\geqslant 0}Z & Z \arrow[l]
\end{tikzcd}$$

The exactness follows from the previous corollary and ${}^tH(Z) = {}^tH(\tau_{\geqslant 0}Z)$. □

6.14 Definition. Let $\mathcal{T}$ be a triangulated category and $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$ a t-structure. The t-structure is said to be **non-degenerate** if

$$\bigcap_n \mathcal{T}^{\leqslant n} = \bigcap_n \mathcal{T}^{\geqslant n} = 0.$$

6.14.1 Lemma. Suppose $\mathcal{T}$ is a triangulated category with a non-degenerated t-structure.

(i) $f : X \to Y$ is an isomorphism in $\mathcal{T}$ if and only if ${}^tH(f[n])$ is an isomorphism in $\mathcal{T}^{\heartsuit}$ for any n.

(ii) One has

$$\mathcal{T}^{\leqslant n} = \{X \in \mathcal{T} \mid {}^tH(X[m]) = 0 \text{ for any } m > n\}$$
$$\mathcal{T}^{\geqslant n} = \{X \in \mathcal{T} \mid {}^tH(X[m]) = 0 \text{ for any } m < n\}$$

Proof.

(i) First we claim if X is an object such that ${}^tH(X[n]) = 0$ for any n, then $X = 0$. By (6.10.2) we must show $\tau_{\leqslant 0}X = \tau_{\geqslant 1}X = 0$. Put $Y = \tau_{\leqslant 0}X$. Then ${}^tH(X) = 0$ implies $\tau_{\geqslant 0}Y = 0$, or $Y \in \mathcal{T}^{\leqslant -1}$, so that $\tau_{\leqslant -1}Y = Y$. Iterating this procedure will tell $Y \in \bigcap_{n \geqslant 0} \mathcal{T}^{\leqslant -n} = 0$, i.e. $\tau_{\leqslant 0}X = 0$. Similarly we can show $\tau_{\geqslant 1}X = 0$ as well.
For (i), complete $f : X \to Y$ into a distinguished triangle $X \to Y \to Z \to X[1]$. Then f is an isomorphism if and only if $Z = 0$. This is equivalent to saying ${}^tH(Z[n]) = 0$ by what we've just proven. By (6.13) and (6.7.1), this is the same as saying ${}^tH(f[n])$ is an isomorphism in $\mathcal{T}^{\heartsuit}$ for any n.

(ii) This follows from (i), (6.10.3) and (6.10.2).

□

6.15 t-exact functors. Let $\mathcal{T}_i$ be a pre-triangulated category with a t-structure $(\mathcal{T}_i^{\leqslant 0}, \mathcal{T}_i^{\geqslant 0})$. A triangulated functor $F : \mathcal{T}_1 \to \mathcal{T}_2$ is called **right t-exact** (resp. **left t-exact**) if $F(\mathcal{T}_1^{\leqslant 0}) \subseteq \mathcal{T}_2^{\leqslant 0}$ (resp. if $F(\mathcal{T}_1^{\geqslant 0}) \subseteq \mathcal{T}_2^{\geqslant 0}$). It is called **t-exact** if it is both left and right t-exact.

6.2 Derived category

6.16 Localization. Let $\mathcal{C}$ be a category and S a collection of morphisms in $\mathcal{C}$. The **localization of** $\mathcal{C}$ **at** S is a category $S^{-1}\mathcal{C}$ together with a natural functor $\Phi : \mathcal{C} \to S^{-1}\mathcal{C}$ satisfying the following universal property : if $\Psi : \mathcal{C} \to \mathcal{D}$ is a functor such that $\Psi(s)$ is an isomorphism in $\mathcal{D}$ for each $s \in S$, then there exists a unique functor $\Psi' : S^{-1}\mathcal{C} \to \mathcal{D}$ such that $\Psi' \circ \Phi = \Psi$.

6.17 Localizing system. For a category $\mathcal{C}$, a collection S of morphisms in $\mathcal{C}$ is called a **left localizing system** if

(i) $\mathrm{id}_X \in S$ for all $X \in \mathrm{Ob}(\mathcal{C})$ and $S \circ S \subseteq S$ in the obvious sense.

(ii) (Extension property) For $X \leftarrow Y \xrightarrow{s} Z$ with $s \in S$, there exists $X \xrightarrow{t} W \leftarrow Z$ with $t \in S$ completing the diagram

$$\begin{tikzcd} W & Z \arrow[l] \\ X \arrow[u, "t"] & Y. \arrow[l] \arrow[u, "s"'] \end{tikzcd}$$

(iii) For all $f, g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, if $f \circ s = g \circ s$ for some $s \in S$, then $t \circ f = t \circ g$ for some $t \in S$.

Similarly, S is called a **right localizing system** if

(i) $\mathrm{id}_X \in S$ for all $X \in \mathrm{Ob}(\mathcal{C})$ and $S \circ S \subseteq S$ in the obvious sense.

(ii) (Extension property) For each diagram $X \to Y \xleftarrow{s} Z$ with $s \in S$, there exists $X \xleftarrow{t} W \to Z$ with $t \in S$ completing the commutative diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d, "t"] & Z \arrow[d, "s"] \\ X \arrow[r] & Y. \end{tikzcd}$$

(iii) For all $f, g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, if $t \circ f = t \circ g$ for some $t \in S$, then $f \circ s = g \circ s$ for some $s \in S$.

We say S is a **localizing system** if it is both a left localizing system and right localizing system.

6.18 Roof category. Let $\mathcal{C}$ be a category and S a right localizing system. Define the **roof category** $\mathrm{Roof}_S\mathcal{C}$ as follows. Set $\mathrm{Ob}(\mathrm{Roof}_S\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$, and

$$\mathrm{Hom}_{\mathrm{Roof}(\mathcal{C})}(X, Y) = \left\{ X \xleftarrow{s} Z \xrightarrow{f} Y \mid Z \in \mathrm{Ob}(\mathcal{C}), f \in \mathrm{Hom}_{\mathcal{C}}(Z, Y),\ s \in S \right\} / \sim$$

where $\sim$ is an equivalence relation such that $X \xleftarrow{s} Z \xrightarrow{f} Y \sim X \xleftarrow{s'} Z' \xrightarrow{f'} Y$ if there exists another diagram $X \xleftarrow{t} Z'' \to Y$ with $t \in S$, and two morphisms $Z'' \to X$, $Z'' \to Y$ in $\mathcal{C}$ fitting into the commutative diagram

$$\begin{tikzcd} & & Z'' \arrow[dl] \arrow[dr] \arrow[dll, bend right, "t"'] \arrow[drr, bend left] & & \\ & Z \arrow[dl, "s"'] \arrow[drrr, "f" near end] & & Z' \arrow[dlll, "s'"' near end] \arrow[dr, "f'"] & \\ X & & & & Y \end{tikzcd}$$

For two morphisms $X \xleftarrow{s} Z \xrightarrow{f} Y$, $Y \xleftarrow{t} Z' \xrightarrow{g} X'$ in $\mathrm{Roof}_S\mathcal{C}$, their composition is defined to be the equivalence class of $X \xleftarrow{s \circ u} W \xrightarrow{g \circ h} Z'$, where $Z \xleftarrow{u} W \xrightarrow{h} Z'$ is some morphism $\mathrm{Roof}_S\mathcal{C}$, fitting into the commutative diagram

$$\begin{tikzcd} & & W \arrow[dl, "u"'] \arrow[dr, "h"] & & \\ & Z \arrow[dl, "s"'] \arrow[dr, "f"] & & Z' \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & & Y & & X' \end{tikzcd}$$

Note that such $Z \xleftarrow{u} W \xrightarrow{h} Z'$ exists by the extension property of S.

6.18.1 $\sim$ is an equivalence relation. Reflexivity and symmetry are clear. For transitivity, suppose we have two commutative roofs

$$
\begin{tikzcd}
 & & W \arrow[dl] \arrow[dr] \arrow[dll, bend right, "t"'] \arrow[drr, bend left] & & \\
 & Z \arrow[dl, "s"'] \arrow[drrr] & & Z' \arrow[dlll, "s'"] \arrow[dr] & \\
X & & & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & W' \arrow[dl] \arrow[dr] \arrow[dll, bend right, "t'"'] \arrow[drr, bend left] & & \\
 & Z' \arrow[dl, "s'"'] \arrow[drrr] & & Z'' \arrow[dlll, "s''"] \arrow[dr] & \\
X & & & & Y
\end{tikzcd}
$$

with adorned arrows lying in S.

6.18.2 Composition is well-defined.

6.18.3 Composition is associative and unital. Clearly the composition is unital, where the identity morphism of an object X is represented by the trivial roof $X \xleftarrow{\mathrm{id}_X} X \xrightarrow{\mathrm{id}_X} X$.

6.19 Theorem. Let $\mathcal{C}$ be a category and S a right localizing system. Then $\mathrm{Roof}_S\mathcal{C}$ is a localization of $\mathcal{C}$ at S.

Proof. There is a natural inclusion $\Phi : \mathcal{C} \to \mathrm{Roof}_S\mathcal{C}$, which is identity on objects and sends a morphism $f : X \to Y$ is $\mathcal{C}$ to the roof $X \xleftarrow{\mathrm{id}_X} X \xrightarrow{f} Y$. Let $\Psi : \mathcal{C} \to \mathcal{D}$ be any functor sending morphisms in S to isomorphisms.

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "\Phi"] \arrow[dr, "\Psi"'] & \mathrm{Roof}_S\mathcal{C} \arrow[d, dashed] \\
 & \mathcal{D}
\end{tikzcd}
$$

Suppose $\Psi' : \mathrm{Roof}_S\mathcal{C} \to \mathcal{D}$ is a functor making the above triangle commutative. Clearly Ψ' must be the same as Ψ on objects. Now let $\varphi = (X \xleftarrow{s} Z \xrightarrow{f} Y)$ be a morphism in $\mathrm{Roof}_S\mathcal{C}$. Upon applying Ψ' to the identity $\varphi \circ \Phi(s) = \Phi(f)$, we get

$$\Psi'(\varphi) \circ \Psi(s) = \Psi'(\varphi) \circ \Psi'(\Phi(s)) = \Psi'\Phi(f) = \Psi(f).$$

Since $\Psi(s)$ is invertible in $\mathcal{D}$, we see $\Psi'(\varphi) = \Psi(f) \circ \Psi(s)^{-1}$. This proves the uniqueness.

For the existence, define $\Psi' : \mathrm{Roof}_S\mathcal{C} \to \mathcal{D}$ by being identical to Ψ on objects, and

$$\Psi'(X \xleftarrow{s} Z \xrightarrow{f} Y) = \Psi(f) \circ \Psi(s)^{-1}.$$

We must check this is a well-defined functor. Say we have a commutative diagram

$$
\begin{tikzcd}
 & & Z'' \arrow[dl, "a"'] \arrow[dr, "b"] \arrow[dll, bend right, "t"'] \arrow[drr, bend left, "g"] & & \\
 & Z \arrow[dl, "s"'] \arrow[drrr, "f"] & & Z' \arrow[dlll, "s'"'] \arrow[dr, "f'"] & \\
X & & & & Y
\end{tikzcd}
$$

We need to show $\Psi(f') \circ \Psi(s')^{-1} = \Psi(f) \circ \Psi(s)^{-1}$. Indeed, $\Psi(f)\Psi(s)^{-1}\Psi(t) = \Psi(f)\Psi(a) = \Psi(g)$, so $\Psi(f)\Psi(s)^{-1} = \Psi(g)\Psi(t)^{-1}$. Similarly, $\Psi(f')\Psi(s')^{-1} = \Psi(g)\Psi(t)^{-1}$, so they are equal. Next, it is clear that Ψ' preserves identity. Finally, say we have a commutative roof

$$\begin{tikzcd}
 & & W \arrow[dl, "u"'] \arrow[dr, "h"] & & \\
 & Z \arrow[dl, "s"'] \arrow[dr, "f"] & & Z' \arrow[dl, "t"'] \arrow[dr, "g"] & \\
X & & Y & & X'.
\end{tikzcd}$$

Then

$$\Psi(gh)\Psi(su)^{-1} = \Psi(g)\Psi(h)\Psi(u)^{-1}\Psi(u)^{-1} = \Psi(g)\Psi(t)^{-1}\Psi(f)\Psi(u)^{-1},$$

which is what we want. □

6.20 Quasi-isomorphisms. Let $\mathcal{A}$ be an abelian category. For a complex $X = (X^\bullet, d_X^\bullet)$, its cohomology complex $H^\bullet(X) = H^\bullet(X^\bullet, d_X^\bullet)$ is the quotient

$$H^n(X) = \ker d^n / \operatorname{Im} d^{n-1}$$

with zero differential. A morphism $f : X \to Y$ in $\mathrm{Kom}(\mathcal{A})$ induces a morphism between cohomology $H^\bullet(f) : H^\bullet(X) \to H^\bullet(Y)$ in a natural way. We say f is a **quasi-isomorphism**, or **qis** for brevity, if the induced map $H^\bullet(f)$ is an isomorphism.

6.20.1 Lemma. If $f \sim g$, then $H(f) = H(g)$.

6.20.2 Lemma. A morphism $f : X \to Y$ is a qis if and only if $H^\bullet(\mathrm{cone}(f)) = 0$.

Proof. This follows from the long exact sequence on cohomology in (6.9). □

6.21 Lemma. Let $\mathcal{A}$ be an abelian category. The collection of quasi-isomorphisms in $K^\square(\mathcal{A})$ ($\square = \varnothing, +, -, b$) is a localizing system.

Proof. We must verify (i), (ii) and (iii) in (6.17). We only prove qis' are right localizing; the other is proved similarly. (i) is obvious. For (ii), let $Z \xrightarrow{s} Y \xleftarrow{f} X$ be a diagram in $K^\square(\mathcal{A})$. Let $\tau : Y \to \mathrm{cone}(s)$ be the inclusion. Consider the commutative diagram

$$\begin{tikzcd}
Z \arrow[r, "s"] \arrow[d, "\wr"'] & Y \arrow[r] \arrow[d, equal] & \mathrm{cone}(s) \arrow[r, "\tau"] \arrow[d, equal] & Z[1] \arrow[d, "\wr"'] \\
\mathrm{cone}(\tau)[-1] \arrow[r] & Y \arrow[r, "\tau"] & \mathrm{cone}(s) \arrow[r] & \mathrm{cone}(\tau) \\
\mathrm{cone}(\tau f)[-1] \arrow[r] \arrow[u] & X \arrow[r, "\tau f"] \arrow[u, "f"] & \mathrm{cone}(s) \arrow[r] \arrow[u, equal] & \mathrm{cone}(\tau f) \arrow[u]
\end{tikzcd}$$

Here the first row is isomorphic to the second row by TR 2. By (6.20.2), s is a qis if and only if $H^\bullet(\mathrm{cone}(s)) = 0$, if and only if $\mathrm{cone}(\tau f)[-1] \to X$ is a qis. Now two commutative squares on the left merge to the commutative square we want. For (iii), it suffices to show if $f : X \to Y$ is such that $tf \sim 0$ for some qis $t : Y \to Y'$, then $fs \sim 0$ for some qis s. Write $sf = dh + hd$ for some $h \in \mathrm{Hom}^{-1}(X, Y)$ and define

$$\begin{aligned} g : X &\longrightarrow \mathrm{cone}(s)[-1] = Y'[-1] \oplus Y \\ x &\longmapsto (-hx, fx). \end{aligned}$$

Then we have a commutative diagram with row being distinguished triangles

$$\begin{tikzcd} \mathrm{cone}(g) & \mathrm{cone}(s)[-1] \arrow[l] \arrow[d, equal] & X \arrow[l, "g"'] \arrow[d, "f"] & \mathrm{cone}(g)[-1] \arrow[l, "t"'] \\ \mathrm{cone}(s)[-1] \arrow[r] & Y \arrow[r, "s"] & Y' \arrow[r] & \mathrm{cone}(s) \end{tikzcd}$$

By (6.20.2), s is a qis implies $H^\bullet(\mathrm{cone}(s)) = 0$, so t is a qis. Since $gt \sim 0$, we see $ft \sim 0$. $\square$

6.22 Derived category. Let $\mathcal{A}$ be an abelian category. The **derived category of** $\mathcal{A}$, denoted by $D(\mathcal{A})$, is the localization of $K(\mathcal{A})$ at quasi-isomorphisms. By (6.21) and (6.19),

$$D(\mathcal{A}) = \mathrm{Roof}_{\mathrm{qis}} K(\mathcal{A}).$$

Similarly, we define $D^\square(\mathcal{A}) = \mathrm{Roof}_{\mathrm{qis}} K^\square(\mathcal{A})$ for $\square = +, -, b$.

6.22.1 Isomorphism. A morphism $X \to Y$ in $D^\square(\mathcal{A})$ is an isomorphism if and only if it is represented by a roof $X \xleftarrow{s} Z \xrightarrow{t} Y$ with t, s qis.

Proof.

$\square$

6.23 Localization of subcategory. Let $\mathcal{C}$ be a category and S a right localizing system. For a full subcategory $\mathcal{B}$ of $\mathcal{C}$, put $S_{\mathcal{B}}$ to be the subcollection of morphisms in S that are morphisms in $\mathcal{B}$. By the universal property of localization (6.16), there is a natural functor $S_{\mathcal{B}}^{-1}\mathcal{B} \to S^{-1}\mathcal{C}$.

Lemma. Suppose $S_{\mathcal{B}}$ is a right localizing system such that for all $X \in \mathrm{Ob}(\mathcal{B})$ and $s : X' \to X$ in S, there exists $X'' \in \mathrm{Ob}(\mathcal{B})$ and a morphism $X'' \to X'$ in $\mathcal{C}$ such that the composition $X'' \to X' \xrightarrow{s} X$ lies in $S_{\mathcal{B}}$. Then $S_{\mathcal{B}}^{-1}\mathcal{B} \to S^{-1}\mathcal{C}$ is fully faithful.

Proof. By our assumption and (6.19), we can instead show the canonical map $\mathrm{Roof}_{S_{\mathcal{B}}}\mathcal{B} \to \mathrm{Roof}_S\mathcal{C}$ is fully faithful. This is straightforward. $\square$

Hence, under the assumption of the lemma, the natural functor $S_{\mathcal{B}}^{-1}\mathcal{B} \to S^{-1}\mathcal{C}$ realizes $S_{\mathcal{B}}^{-1}\mathcal{B}$ as a full subcategory of $S^{-1}\mathcal{C}$.

6.23.1 Canonical truncation. Let $\mathcal{A}$ be an additive category and X a complex. For $n \in \mathbb{Z}$, define the **canonical truncations** :

$$\begin{aligned} \tau_{\leqslant n} X &= \left(\cdots \to X^{n-2} \to X^{n-1} \to \ker d_X^n \to 0 \to 0 \to \cdots\right) \\ \tau_{> n} X &= \left(\cdots \to 0 \to 0 \to \mathrm{Im}\, d_X^{n-1} \to X^{n+1} \to X^{n+2} \to \cdots\right). \end{aligned}$$

The reason for their names and the subscripts is

$$H^m(\tau_{\leqslant n} X) = \begin{cases} H^m(X) & \text{, if } m \leqslant n \\ 0 & \text{, if } m > n \end{cases}, \qquad H^m(\tau_{> n} X) = \begin{cases} H^m(X) & \text{, if } m > n \\ 0 & \text{, if } m \leqslant n \end{cases}$$

Using these truncations we immediately verify that quasi-isomorphisms satisfy the assumption in Lemma 6.23. Hence

6.23.2 Corollary. Let $\mathcal{A}$ be an abelian category. Then the natural functors $D^b(\mathcal{A}) \to D^\pm(\mathcal{A}) \to D(\mathcal{A})$ are fully faithful embeddings of categories. The essential image of $D^b(\mathcal{A})$ is

$$\{X \in D(\mathcal{A}) \mid H^n(X) = 0 \text{ if } n < b \text{ or } n > a \text{ for some } a < b.\}$$

The essential images of $D^{\pm}(\mathcal{A})$ is

$$\{X \in D(\mathcal{A}) \mid H^n(X) = 0 \text{ if } n \ll 0 \text{ (resp. } n \gg 0)\}.$$

(To avoid confusion, $+ \leftrightsquigarrow \ll$ and $- \leftrightsquigarrow \gg$.)

6.24 Localization and triangles. Let $\mathcal{T}$ be a pre-triangulated category and S a collection of morphisms. We say S **is compatible with triangulation** if

(i) $f \in S$ if and only if $f[1] \in S$.

(ii) For any commutative diagram of distinguished triangles

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

with $f, g \in S$, there exists a morphism $h \in \mathrm{Hom}_{\mathcal{T}}(Z, Z') \cap S$ making the above diagram commutative.

6.24.1 Theorem. Let $\mathcal{T}$ be a (pre-)triangulated category and S a right localizing system compatible with triangulation. Then $S^{-1}\mathcal{T}$ is (pre-)triangulated as follows.

- The shift functor on $S^{-1}\mathcal{T}$ is induced by the one on $\mathcal{T}$.
- A triangle in $S^{-1}\mathcal{T}$ is distinguished if it is isomorphic in $S^{-1}\mathcal{T}$ to the image of a distinguished triangle of $\mathcal{T}$ under the inclusion $\mathcal{T} \to S^{-1}\mathcal{T}$

Proof.

$\square$

6.25 Natural t-structure. Let $\mathcal{A}$ be an abelian category. Define the **natural t-structure** on $D^{\square}(\mathcal{A})$ by

$$D^{\square}(\mathcal{A})^{\geqslant 0} := \{X \in D^{\square}(\mathcal{A}) \mid H^n(X) = 0 \text{ for all } n < 0\}$$
$$D^{\square}(\mathcal{A})^{\leqslant 0} := \{X \in D^{\square}(\mathcal{A}) \mid H^n(X) = 0 \text{ for all } n > 0\}$$

Then $(D^{\square}(\mathcal{A})^{\leqslant 0}, D^{\square}(\mathcal{A})^{\geqslant 0})$ defines a t-structure on $D^{\square}(\mathcal{A})$, and the corresponding truncation functors are given in (6.23.1), and the natural map

$$\mathcal{A} \longrightarrow D(\mathcal{A})$$

that realizes an object as a complex concentrated at 0-th term defines an equivalence of categories $\mathcal{A} \cong D^{\square}(\mathcal{A})^{\heartsuit}$.

Proof. By construction we see

$$D^{\square}(\mathcal{A})^{\geqslant m} = \{X \in D^{\square}(\mathcal{A}) \mid H^n(X) = 0 \text{ for all } n < m\}.$$

Let $X \in D^{\square}(\mathcal{A})^{\leqslant 0}$, $Y \in D^{\square}(\mathcal{A})^{\geqslant 1}$ and $X \xleftarrow{s} Z \xrightarrow{f} Y$ be a roof representing a morphism $X \to Y$ in $D^{\square}(\mathcal{A})$. Since s is a qis, $Z \in D^{\square}(\mathcal{A})^{\leqslant 0}$ as well. By (6.23.1), we can form the diagram

$$\begin{tikzcd} \tau_{\leqslant 0} Z \arrow[d] & \tau_{\geqslant 1} Y \\ Z \arrow[r, "f"'] & Y \arrow[u] \end{tikzcd}$$

with all vertical maps being qis. But this composition is necessarily a zero map in $\mathcal{K}^{\square}(\mathcal{A})$. In particular, $f : Z \to Y$ is the zero morphism in $D^{\square}(\mathcal{A})$. This proves (6.10).(i). (ii) there is automatic. For (iii) we must show the triangle

$$\tau_{\leqslant 0}X \to X \to \tau_{\geqslant 1} \to (\tau_{\leqslant 0}X)[1]$$

is distinguished. Since the sequence

$$0 \to \tau_{\leqslant 0}X \to X \to \tau_{\geqslant 1} \to 0$$

is a short exact sequence in $K^{\square}(\mathcal{A})$, by Lemma 6.9.1 and the definition of the triangulated structure on $K^{\square}(\mathcal{A})$ (6.3.3) the claim follows. The last two assertions are obvious. □

6.3 Derived functors

6.26 δ-functor. Let $\mathcal{A}$ be an abelian category and $\mathcal{T}$ a pre-triangulated category. A **δ-functor** is a functor $T : \mathcal{A} \to \mathcal{T}$ together with a rule which assigns to each short exact sequence $0 \to A \to B \to C \to 0$ a morphism $\delta_{A\to B\to C} : T(C) \to T(A)[1]$ such that

- $T(A) \to T(B) \to T(C) \xrightarrow{\delta_{A\to B\to C}} T(A)[1]$ is distinguished, and
- for any morphism $(0 \to A \to B \to C \to 0) \to (0 \to A' \to B' \to C' \to 0)$ of short exact sequences, the diagram

$$\begin{tikzcd} T(C) \arrow[r, "\delta_{A\to B\to C}"] \arrow[d] & T(A)[1] \arrow[d] \\ T(C') \arrow[r, "\delta_{A'\to B'\to C'}"] & T(A')[1] \end{tikzcd}$$

is commutative.

6.26.1 Lemma. Let $\mathcal{A}$ be an abelian category. The embedding $\mathcal{A} \to D^{\square}(\mathcal{A})$ has a natural structure of δ-functors.

Proof. Let $0 \to X \xrightarrow{f} Y \to Z \to 0$ be a short exact sequence in $\mathcal{A}$. By (6.9.1) the cone $Y \to \mathrm{cone}(f)$ is quasi-isomorphic to $Y \to Z$. In particular, there is a map $Z \to X[1]$ in $D^{\square}(\mathcal{A})$ and the triangle $X \to Y \to Z \to X[1]$ is distinguished. The map $Z \to X[1]$ is functorial in the datum $0 \to X \xrightarrow{f} Y \to Z \to 0$ by construction. This finishes the proof. □

6.26.2 Lemma. Let $\mathcal{A}$ be an abelian category and $\mathcal{T}, \mathcal{S}$ be two pre-triangulated category. If $T : \mathcal{A} \to \mathcal{T}$ is a δ-functor and $F : \mathcal{T} \to \mathcal{S}$ is a triangulated functor, then $F \circ T$ is a δ-functor.

Proof. Immediate.

□

6.27 Derived functors. Let $\mathcal{A}, \mathcal{B}$ be two abelian categories, and $F : K^{\square}(\mathcal{A})^{?} \to K^{*}(\mathcal{B})$ be a triangulated functor. Here $\square, * \in \{b, \pm, \varnothing\}$ and $? \in \{\varnothing, \mathrm{op}\}$.

A **right derived functor** $RF : D^{\square}(\mathcal{A})^{?} \to D^{*}(\mathcal{B})$ is a triangulated functor together with a natural transformation

$$\epsilon : \begin{tikzcd} K^{\square}(\mathcal{A})^{?} \arrow[r, "F"] \arrow[d, "\Phi_{\mathcal{A}}"'] & K^{*}(\mathcal{B}) \arrow[d, "\Phi_{\mathcal{B}}"] \arrow[dl, phantom, "\Swarrow"] \\ D^{\square}(\mathcal{A})^{?} \arrow[r, "RF"'] & D^{*}(\mathcal{B}) \end{tikzcd}$$

satisfying the universal property : if $G : D^{\square}(\mathcal{A})^{?} \to D^{\square}(\mathcal{B})$ is another triangulated functor together with a natural transformation

$$\epsilon' : \begin{tikzcd} K^{\square}(\mathcal{A})^{?} \arrow[r, "F"] \arrow[d] \arrow[dr, phantom, "\swarrow"] & K^{*}(\mathcal{B}) \arrow[d] \\ D^{\square}(\mathcal{A})^{?} \arrow[r, "G"'] & D^{*}(\mathcal{B}) \end{tikzcd}$$

then there exists a unique natural transformation $\eta : RF \to G$ such that

$$\epsilon' = (\eta\Phi_{\mathcal{A}}) \circ \epsilon.$$

where $\Phi_{\mathcal{A}} : K^{\square}(\mathcal{A})^{?} \to D^{\square}(\mathcal{A})^{?}$ is the canonical functor, and $\eta\Phi_{\mathcal{A}} : RF \circ \Phi_{\mathcal{A}} \to G \circ \Phi_{\mathcal{A}}$ is the horizontal composition. One can similarly define a **left derived functor** $LF : D^{\square}(\mathcal{A})^{?} \to D^{*}(\mathcal{B})$ by a dual version of the universal property.

6.27.1 Associated long exact sequence. Assume the existence of $RF : D^{\square}(\mathcal{A}) \to D^{*}(\mathcal{B})$. By (6.26.2) and (6.26.1), the restriction to $\mathcal{A}$ of $RF : \mathcal{A} \to D^{*}(\mathcal{B})$ is a δ-functor. In particular, for a short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ there is an associated distinguished triangle in $D^{*}(\mathcal{B})$:

$$RX \to RY \to RZ \to (RX)[1].$$

For $n \in \mathbb{Z}$, put

$$R^{n}F := H^{n} \circ RF : D^{\square}(\mathcal{A})^{?} \to \mathcal{B}$$

Clearly, $R^{n}F = R^{0}F \circ [n]$. Using (6.7.1) we obtain a long exact sequence

$$\cdots \to R^{n-1}FZ \to R^{n}FX \to R^{n}FY \to R^{n}FZ \to R^{n+1}FX \to \cdots \qquad (n \in \mathbb{Z}).$$

This is clearly functorial in $0 \to X \to Y \to Z \to 0$. Similar remark holds for the left derived functor LF.

6.28 Exact functor. Let $\mathcal{A}$, $\mathcal{B}$ be two abelian categories. An additive functor $F : \mathcal{A} \to \mathcal{B}$ is called **right exact** (resp. **left exact**) if for any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, the sequence

$$F(X) \to F(Y) \to F(Z) \to 0$$

is exact in $\mathcal{B}$ (resp. the sequence

$$0 \to F(X) \to F(Y) \to F(Z)$$

is exact in $\mathcal{B}$). An **exact functor** is an additive functor that is left and right exact.

6.28.1 Lemma. If $F : \mathcal{A} \to \mathcal{B}$ is exact, then there exists a triangulated functor $\overline{F} : D(\mathcal{A}) \to D(\mathcal{B})$ with a natural isomorphism $\overline{F} \circ \Phi_{\mathcal{A}} \cong \Phi_{\mathcal{B}} \circ F$. In particular, $\overline{F}$ is a left and right derived functor of F.

Proof. Since F is exact, there is a natural isomorphism $F \circ H^{0} \cong H^{0} \circ F$, where H^{0} is the cohomology functor (6.9). In particular, F preserves qis, so it induces a functor $\overline{F} : D(\mathcal{A}) \to D(\mathcal{B})$ on the level of derived categories (6.16). The remaining assertions are straightforward. $\square$

6.29 Adapted class. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories. A full subcategory $\mathcal{I}_{F} \subseteq \mathcal{A}$ is called a **right F-adapted class for F** if

(i) for any object $X \in \mathcal{A}$, there exists an injection $X \to X'$ with $X' \in \mathcal{I}_{F}$,

(ii) if $0 \to X \to Y \to Z \to 0$ is a short exact sequence in $\mathcal{A}$ with $X, Y \in \mathcal{I}_F$, then $Z \in \mathcal{I}_F$, and

(iii) for any short exact sequence $0 \to X \to Y \to Z \to 0$ with $X \in \mathcal{I}_F$, the sequence

$$0 \to F(X) \to F(Y) \to F(Z) \to 0$$

is exact.

Dually, if F is a right exact functor, a **left** F**-adapted class for** F is a full subcategory $\mathcal{J}_F$ such that

(i) for any object $X \in \mathcal{A}$, there exists a surjection $X' \to X$ with $X' \in \mathcal{J}_F$,

(ii) if $0 \to X \to Y \to Z \to 0$ is a short exact sequence in $\mathcal{A}$ with $Y, Z \in \mathcal{J}_F$, then $X \in \mathcal{J}_F$, and

(iii) for any short exact sequence $0 \to X \to Y \to Z \to 0$ with $Z \in \mathcal{J}_F$, the sequence

$$0 \to F(X) \to F(Y) \to F(Z) \to 0$$

is exact.

7 Sheaf Cohomology

7.1 Definition

Recall that if R is a unital ring (not necessarily commutative), then the abelian category ${}_R\mathbf{Mod}$ of left R-modules has enough injectives. Now let $(X, \mathcal{O}_X)$ be a ringed space. Recall from Lemma 2.15.1 that the category $\mathbf{Mod}_{\mathcal{O}_X}$ of $\mathcal{O}_X$-modules is abelian.

Lemma 7.1. $\mathbf{Mod}_{\mathcal{O}_X}$ has enough injectives.

Proof. Let $\mathcal{F} \in \mathbf{Mod}_{\mathcal{O}_X}$. For each $x \in X$, the stalk $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module, so there exists an injective $\varphi_x : \mathcal{F}_x \to I_x$ for some injective $\mathcal{O}_{X,x}$-module I_x. For each $x \in X$, denote by $\iota_x : \{x\} \to X$ the inclusion and consider the sheaf

$$\mathcal{I} := \prod_{x \in X} (\iota_x)_*(I_x)$$

Here we consider I_x as a sheaf on the one point space $\{x\}$, and view $(\{x\}, \mathcal{O}_{X,x})$ as a ringed space. Explicitly, for each open $U \subseteq X$,

$$\mathcal{I}(U) = \prod_{x \in U} I_x.$$

Then $\mathcal{I} \in \mathbf{Mod}_{\mathcal{O}_X}$, and there is a natural morphism $\mathcal{F} \to \mathcal{I}$ given as follows.

$$\begin{array}{ccccc} \mathcal{F}(U) & \longrightarrow & \prod_{x \in U} \mathcal{F}_x & \longrightarrow & \prod_{x \in U} I_x = \mathcal{I}(U) \\ f & \longmapsto & (f_x)_{x \in U} & \longmapsto & (\varphi_x(f_x))_{x \in U} \end{array}$$

This is injective, for $\mathcal{F}$ is a sheaf (so the first arrow is injective) and each φ_x is injective.

It remains to show $\mathcal{I}$ is an injective $\mathcal{O}_X$-module. Let $\mathcal{G} \in \mathbf{Mod}_{\mathcal{O}_X}$. The universal property of direct products gives an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{I}) \xrightarrow[\sim]{} \prod_{x \in X} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, (\iota_x)_*(I_x)).$$

As the direct product preserves exactness, it suffices to show each factor on the right is exact. But the adjunction (or argue directly) gives

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, (\iota_x)_*(I_x)) \xrightarrow[\sim]{} \mathrm{Hom}_{\mathcal{O}_{X,x}}((\iota_x)^*\mathcal{G}, I_x) = \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x).$$

Since the stalk functor $\mathcal{G} \mapsto \mathcal{G}_x$ is exact, we conclude that $\mathrm{Hom}_{\mathcal{O}_X}(\cdot, \mathcal{I})$ is an exact functor, i.e., $\mathcal{I}$ is injective. ☐

Corollary 7.1.1. The abelian category of sheaves of abelian groups on a topological space X has enough injectives.

Proof. Regard X with the ringed space $(X, \mathbb{Z}_X)$ where $\mathbb{Z}_X$ is the constant sheaf with values in $\mathbb{Z}$. ☐

Let X be a topological space. There is a global section functor $\Gamma(X, \cdot) : \mathbf{Mod}_{\mathbb{Z}_X} \to \mathbf{Mod}_{\mathbb{Z}}$, which is left exact but not right exact in general. We then define the **sheaf cohomology functors** $H^p(X, \cdot)$ to be the right derived functors of $\Gamma(X, \cdot)$, i.e., $H^p(X, \cdot) := R^p\Gamma(X, \cdot)$.

It should be noted that even if X or $\mathcal{F}$ has some additional structure (e.g. X a manifold or scheme, and $\mathcal{F}$ a quasi-coherent sheaf), we always take cohomology by regarding X merely as a topological space and $\mathcal{F}$ as a sheaf of abelian groups over X.

Definition. A sheaf $\mathcal{F}$ on a space X is **flasque** if for every open U, the restriction $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective.

— Equivalently, $\mathcal{F}$ is flasque if and only if for all opens $V \subseteq U$, the restriction $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

Before continuing our discussion on cohomology, we talk about some operation on a sheaf. Let Z be a closed subset of X and $U := X \backslash Z$. Denote by $i : Z \to X$ and $j : U \to X$ the canonical inclusions.

1. If $\mathcal{F}$ is a sheaf on Z, the direct image $i_*\mathcal{F}$ is a sheaf on X. Then

$$(i_*\mathcal{F})_x = \begin{cases} 0 & \text{, if } x \notin Z \\ \mathcal{F}_x & \text{, if } x \in Z \end{cases}$$

   This follows from the definition of stalks, and the fact that the topology on Z is subspace topology. Because of this, we speak of $i_*\mathcal{F}$ the sheaf obtained by **extending $\mathcal{F}$ by zero outside** Z**.**

2. If $\mathcal{F}$ is a sheaf on U, denote by $j_!\mathcal{F}$ the sheafification of the presheaf $\mathcal{F}'$ defined by $V \mapsto \mathcal{F}(V)$ if $V \subseteq U$ and $\mapsto 0$ otherwise. Then

$$(j_!\mathcal{F})_x = \begin{cases} 0 & \text{, if } x \notin U \\ \mathcal{F}_x & \text{, if } x \in U \end{cases}$$

   This follows from the description of $\mathcal{F}'$, and the fact that stalk remain unchanged after sheafification. Moreover, $j_!\mathcal{F}$ is the unique sheaf on X such that the above identities hold and whose restriction to U to $\mathcal{F}$. Indeed, we can easily construct a morphism $\mathcal{F}' \to \mathcal{G}$ of presheaves by extending the isomorphism $\mathcal{G}|_U \cong \mathcal{F}$ by "zero outside U", so we obtain a morphism $j_!\mathcal{F} \to \mathcal{G}$ of sheaves. Taking stalks shows this is an isomorphism. We call $j_!\mathcal{F}$ is the sheaf obtained by **extending $\mathcal{F}$ by zero outside** U**.**

3. Let $\mathcal{F}$ be a sheaf on X. Then there exists an exact sequence

$$0 \longrightarrow j_!(\mathcal{F}|_U) \longrightarrow \mathcal{F} \longrightarrow i_*(\mathcal{F}|_Z) \longrightarrow 0$$

   The first arrow is easily constructed, and the second is from the adjunction. Exactness is easily checked on the stalks. Another important property is that $j_!$ is left adjoint to j^{-1}, that is, we have a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(j_!\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}, j^{-1}\mathcal{G})$$

Lemma 7.2. Let $(X, \mathcal{O}_X)$ be a ringed space.

1. Any injective $\mathcal{O}_X$-module is flasque.
2. The restriction $\mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_X|_U}$ preserves injective objects for any open set $U \subseteq X$.
3. If $0 \to \mathcal{F}_1 \xrightarrow{\alpha} \mathcal{F}_2 \xrightarrow{\beta} \mathcal{F}_3 \to 0$ is an exact sequence of sheaves with $\mathcal{F}_1$ flasque, then

$$0 \longrightarrow \mathcal{F}_1(X) \xrightarrow{\alpha_X} \mathcal{F}_2(X) \xrightarrow{\beta_X} \mathcal{F}_3(X) \longrightarrow 0$$

   is an exact sequence of abelian groups.
4. If $0 \to \mathcal{F}_1 \xrightarrow{\alpha} \mathcal{F}_2 \xrightarrow{\beta} \mathcal{F}_3 \to 0$ is an exact sequence of sheaves with $\mathcal{F}_1$, $\mathcal{F}_2$ flasque, then $\mathcal{F}_3$ is also flasque.
5. Any flasque sheaf $\mathcal{F}$ on X is **acyclic**, i.e, $H^i(X, \mathcal{F}) = 0$ for $i \geqslant 1$.

Proof.

1. Let $\mathcal{J}$ be an injective $\mathcal{O}_X$-module. Let $U \subseteq X$ be an open set and denote by $j : U \to X$ the inclusion. Applying $\mathrm{Hom}_{\mathcal{O}_X}(\cdot, \mathcal{J})$ to the inclusion $0 \to j_!(\mathcal{O}_X|_U) \to \mathcal{O}_X$, we obtain a surjection $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{J}) \to \mathrm{Hom}_{\mathcal{O}_X}(j_!(\mathcal{O}_X|_U), \mathcal{J}) \to 0$, which is equal to the map $\mathcal{J}(X) \to \mathcal{J}(U)$.
2. Let $\mathcal{J}$ be an injective $\mathcal{O}_X$-module. Let $U \subseteq X$ be an open set and denote by $j : U \to X$ the inclusion. Suppose $0 \to \mathcal{F} \to \mathcal{G}$ is an injective morphism in $\mathbf{Mod}_{\mathcal{O}_X|_U}$. By 3. above, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{G}, j^{-1}\mathcal{J}) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{O}_X}(j_!\mathcal{G}, \mathcal{J}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}, j^{-1}\mathcal{J}) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{O}_X}(j_!\mathcal{F}, \mathcal{J}) \end{array}$$

   By computing the stalk, we see $j_!\mathcal{F} \to j_!\mathcal{G}$ is injective. Since $\mathcal{J}$ is injective, the map on the right is surjective. Hence the map on the left is surjective, and this proves the injectivity of $j^{-1}\mathcal{J}$.

3. It suffices to show $\beta_X : \mathcal{F}_2(X) \to \mathcal{F}_3(X)$ is surjective. Fix an element in $h \in \mathcal{F}_3(X)$ and consider the set
$$S = \left\{ (U, g) \mid g \in \mathcal{F}_2(U), U \underset{\text{open}}{\subseteq} X,\ \beta_U(g) = h|_U \right\}.$$
Partially ordered S by the rule : $(U_1, g_1) \leqslant (U_2, g_2)$ if and only if $U_1 \subseteq U_2$ and $g_2|_{U_1} = g_1$. By exactness S is nonempty. By Zorn's lemma S admits a maximal element (U_0, g_0).
We claim $U_0 = X$. Suppose otherwise and pick $x \in X \backslash U_0$. By looking at the stalk at x we can find a neighborhood W of X and $g_W \in \mathcal{F}_2(W)$ with $\beta_W(g_W) = h|_W$. Then $g_0|_{U_0\cap W} = g_W|_{U_0\cap W}$, so by exactness there exists an $f' \in \mathcal{F}(U_0 \cap W)$ such that
$$\alpha_{U_0\cap W}(f') = g_0|_{U_0\cap W} - g_W|_{U_0\cap W}$$
Since $\mathcal{F}_1$ is flasque, $f' = f|_{U_0\cap W}$ for some $f \in \mathcal{F}_1(X)$. Then $\alpha_X(f)|_{U_0\cap W} = g_0|_{U_0\cap W} - g_W|_{U_0\cap W}$, or
$$g_0|_{U_0\cap W} = \alpha_X(f)|_{U_0\cap W} + g_W|_{U_0\cap W} = (\alpha_X(f)|_W + g_W)|_{U_0\cap W}.$$
This means g_0 and $\alpha_X(f)|_W + g_W$ glue to a section $g \in \mathcal{F}_2(U_0 \cup W)$, and $\beta_{U_0\cup W}(g) = h|_{U_0\cup W}$. This contradicts the maximality.
4. This follows from 2. and snake lemma.
5. Choose an embedding $\mathcal{F} \to \mathcal{J}$ into an injective object $\mathcal{J}$ in $\mathbf{Mod}_{\mathbb{Z}_X}$. Put $\mathcal{G} = \mathcal{J}/\mathcal{F}$; then we have an exact sequence
$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{J} \longrightarrow \mathcal{G} \longrightarrow 0$$
of sheaves. By 2. we have an exact sequence
$$0 \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{J}(X) \longrightarrow \mathcal{G}(X) \longrightarrow 0$$
Since $\mathcal{J}$ is injective, it is acyclic, so in view of the long exact sequence of sheaf cohomology, we obtain $H^1(X, \mathcal{F}) = 0$ and $H^i(X, \mathcal{F}) = H^{i-1}(X, \mathcal{G})$ for each $i \geqslant 2$. But by 3. $\mathcal{G}$ is also flasque, so 4. follows from induction. on i.

□

Corollary 7.2.1. Let $(X, \mathcal{O}_X)$ be a ringed space. The derived functor of $\Gamma(X, \cdot) : \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathbb{Z}}$ coincide with the sheaf cohomology functors $H^i(X, \cdot)$.

Proof. This follows from the previous lemma and the fact that we can compute $H^\bullet(X, \mathcal{F})$ by taking a acyclic resolution of $\mathcal{F}$. □

Corollary 7.2.2. Let X be a topological space and $j : Z \to X$ a closed subspace. Then $H^i(Y, \mathcal{F}) \cong H^i(X, j_*\mathcal{F})$ canonically for any sheaf $\mathcal{F}$ of abelian groups on Z.

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be a flasque resolution of $\mathcal{F}$ in $\mathbf{Ab}_Z$. Clearly $j_*\mathcal{I}^\bullet$ is flasque. Since j is a closed embedding, $j_*\mathcal{F} \to j_*\mathcal{I}^\bullet$ remains a flasque resolution by (2.9) and (2.16). Since $j_*\mathcal{I}^\bullet(X) = \mathcal{I}^\bullet(Z)$, we get the result. □

We conclude this subsection by introducing a functorial flasque resolution of an abelian sheaf. Let $\mathcal{F}$ be a sheaf of abelian groups on X. Define a sheaf $\mathcal{C}^0(\mathcal{F})$ on X by (c.f. (7.1))
$$\mathcal{C}^0(\mathcal{F})(U) = \prod_{x\in U} \mathcal{F}_x.$$
This is clearly a flasque sheaf on X, and there is a natural morphism $\iota_{\mathcal{F}} : \mathcal{F} \to \mathcal{C}^0(\mathcal{F})$ given by $s \mapsto (s_x)_{x\in U}$ (c.f. (2.4)), which is injective as $\mathcal{F}$ is a sheaf. If we denote by $Z^1(\mathcal{F})$ the cokernel of ι_F, we can obtain a morphism $\delta^0_{\mathcal{F}}\mathcal{C}^0(\mathcal{F}) \to \mathcal{C}^1(\mathcal{F}) := \mathcal{C}^0(Z^1(\mathcal{F}))$ by injecting $Z^1(\mathcal{F})$ into $\mathcal{C}^1(\mathcal{F})$. Continuing this process, we obtain a flasque resolution of $\mathcal{F}$
$$0 \longrightarrow \mathcal{F} \xrightarrow{\iota_{\mathcal{F}}} \mathcal{C}^0(\mathcal{F}) \xrightarrow{\delta^0_{\mathcal{F}}} \mathcal{C}^1(\mathcal{F}) \xrightarrow{\delta^1_{\mathcal{F}}} \cdots.$$

This is called the **Godement resolution** of $\mathcal{F}$. It is clear from the construction that the resolution is functorial in $\mathcal{F}$. Additionally, if $\mathcal{F}$ is an $\mathcal{O}_X$-module for some sheaf of rings $\mathcal{O}_X$ on X, then each $\mathcal{C}^i(\mathcal{F})$ has a natural structure of $\mathcal{O}_X$-modules. Particularly, we see

Lemma 7.3. Let $(X, \mathcal{O}_X)$ be a ringed space. Every $\mathcal{O}_X$-module admits a flasque resolution in $\mathbf{Mod}_{\mathcal{O}_X}$. In particular, cohomology of an $\mathcal{O}_X$-module is a naturally an $\mathcal{O}_X(X)$-module.

7.2 Fine sheaves

The main reference of this section is [Voi02].

Definition. Let X be a topological space. A **fine sheaf** $\mathcal{F}$ on X is a $\mathcal{A}$-module, where $\mathcal{A}$ is a sheaf of rings satisfying the following property : For every open cover $\mathcal{U}$ of X, there exists a partition of unity $(f_U)_{U \in \mathcal{U}}$ subordinate to $\mathcal{U}$.

By a partition of unity we mean the family $(f_U)_{U \in \mathcal{U}}$ satisfies

(i) $f_U \in \mathcal{A}(X)$ with $\operatorname{supp} f_U \subseteq U$.

(ii) $\{\operatorname{supp} f_U\}_{U \in \mathcal{U}}$ is a locally finite family of subsets of X, and $\sum_{U \in \mathcal{U}} f_U = 1$.

Here $\operatorname{supp} f_U := \{x \in X \mid (f_U)_x \neq 0 \in \mathcal{A}_x\}$, and it is a closed set in X.

Proposition 7.4. A fine sheaf $\mathcal{F}$ on X is acyclic.

Proof. Say $\mathcal{A}$ is the sheaf of ring on X making $\mathcal{F}$ a fine sheaf. Choose any acyclic resolution $\mathcal{I}^\bullet$ of $\mathcal{F}$ in $\mathbf{Mod}_{\mathcal{A}}$. Then

$$H^k(X, \mathcal{F}) \cong \frac{\ker(\mathcal{I}^k(X) \to \mathcal{I}^{k+1}(X))}{\operatorname{image}(\mathcal{I}^{k-1}(X) \to \mathcal{I}^k(X))}.$$

Let $k \geqslant 1$ and $\alpha \in \mathcal{I}^k(X)$ be in the kernel. By exactness we can find an open cover $\mathcal{U}$ such that $\alpha|_U$ comes from some $\beta_U \in \mathcal{I}^{k-1}(U)$ for every $U \in \mathcal{U}$. Let $(f_U)_{U \in \mathcal{U}}$ be a partition of unity subordinate to $\mathcal{U}$, and put

$$\beta = \sum_{U \in \mathcal{U}} f_U \beta_U.$$

Note that the sum is locally finite and we view $f_U\beta_U \in \mathcal{I}^{k-1}(X)$ by setting $(f_U\beta_U)_x = 0$ for $x \in X \backslash U$ and $(f_U\beta_U)_x = (f_U)_x(\beta_U)_x$ for $x \in U$ (c.f. (2.4.2)). Thus $\beta \in \mathcal{I}^{k-1}(X)$ is well-defined and is mapped to α as $\alpha = \sum_{U \in \mathcal{U}} f_U \alpha|_U$. Then $\alpha = 0$ in $H^k(X, \mathcal{F})$, and since α is arbitrary, this proves $H^k(X, \mathcal{F}) = 0$. □

Corollary 7.4.1. Let M be a (paracompact) smooth manifold and $\mathcal{C}^\infty_M$ the sheaf of (real-valued) smooth functions on M. Then every $\mathcal{C}^\infty_M$-module is fine, and hence acyclic by (7.4).

7.2.1 de Rham cohomology

Let M be a smooth manifold and $p \geqslant 0$. Let $\mathcal{A}^p_{M,\mathbb{R}}$ denote sheaf of smooth sections of the p-th cotangent bundle $\bigwedge^p(TM)^\vee \to M$; note that $\mathcal{A}^0_{M,\mathbb{R}} = \mathcal{C}^\infty_M$. Together with the exterior derivatives d, we obtain the **de Rham complex**

$$0 \longrightarrow \mathcal{A}^0_{M,\mathbb{R}} \xrightarrow{d} \mathcal{A}^1_{M,\mathbb{R}} \xrightarrow{d} \mathcal{A}^2_{M,\mathbb{R}} \longrightarrow \cdots$$

Note that $\ker(\mathbb{A}^0_{M,\mathbb{R}} \xrightarrow{d} \mathbb{A}^1_{M,\mathbb{R}}) = \mathbb{R}_M$ consists of locally constant functions on M. We define the **p-th de Rham cohomology group** $H^p_{dR}(M, \mathbb{R})$ as

$$H^p_{dR}(M, \mathbb{R}) := H^p(\mathcal{A}^\bullet_{M,\mathbb{R}}(M), d).$$

A fundamental result is that every closed form on M is locally exact. Precisely,

Lemma 7.5 (Poincaré lemma). $H^p_{dR}(\mathbb{R}^n, \mathbb{R}) = 0$ for $p \geqslant 1$.

Proof. We are going the construct an integration operator $L_p : \mathcal{A}^p_{\mathbb{R}^n,\mathbb{R}}(\mathbb{R}^n) \to \mathcal{A}^{p-1}_{\mathbb{R}^n,\mathbb{R}}(\mathbb{R}^n)$ such that $L_{p+1}d + dL_k = \mathrm{id}$.

□

It follows from Poincaré lemma and (7.4.1) that

$$0 \longrightarrow \underline{\mathbb{R}}_M \longrightarrow \mathcal{A}^0_{M,\mathbb{R}} \xrightarrow{d} \mathcal{A}^1_{M,\mathbb{R}} \xrightarrow{d} \mathcal{A}^2_{M,\mathbb{R}} \longrightarrow \cdots$$

is an acyclic resolution of $\underline{\mathbb{R}}_M$.

Theorem 7.6. For any smooth manifold M, there is a natural isomorphism

$$H^\bullet(M, \underline{\mathbb{R}}_M) \cong H^\bullet_{dR}(M, \mathbb{R}).$$

Assume M is a complex smooth manifold. Similarly we define $\mathcal{A}^p_{M,\mathbb{C}}$ to be the sheaf of smooth sections of the p-th complexified cotangent bundle $\bigwedge^p(TM \otimes_{\mathbb{R}} \mathbb{C})^\vee \to M$, and there is a de Rham complex over $\mathbb{C}$

$$0 \longrightarrow \mathcal{A}^0_{M,\mathbb{C}} \xrightarrow{d} \mathcal{A}^1_{M,\mathbb{C}} \xrightarrow{d} \mathcal{A}^2_{M,\mathbb{C}} \longrightarrow \cdots$$

which resolves the locally constant sheaf $\underline{\mathbb{C}}_M$. If we put

$$H^p_{dR}(M, \mathbb{C}) := H^p(\mathcal{A}^\bullet_{M,\mathbb{C}}(M), d),$$

the same reason then shows that there is a natural isomorphism

$$H^\bullet(M, \underline{\mathbb{C}}_M) \cong H^\bullet_{dR}(M, \mathbb{C}).$$

7.2.2 Dolbeault cohomology

Let M be a complex (smooth) manifold. Multiplication by $\sqrt{-1}$ on local coordinates of TM gives rise to a smooth vector bundle isomorphism $J : TM \to TM$ satisfying $J^2 = -\mathrm{id}_{TM}$. This is called the **complex structure** on M. Consider the complexified tangent bundle $TM \otimes_{\mathbb{R}} \mathbb{C}$. On each fibre, the linear map J has eigenvalues $\pm\sqrt{-1}$, and this gives a global decomposition

$$TM \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}_M \oplus T^{0,1}_M$$

of $TM \otimes_{\mathbb{R}} \mathbb{C}$, where $T^{1,0}_M$ and $T^{0,1}_M$ are complex subbundles corresponding to the eigenvalue $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. Both $T^{1,0}_M$ and $T^{0,1}_M$ are isomorphic to TM as real vector bundles. This decomposition induces a decomposition

$$\mathcal{A}^k_{M,\mathbb{C}} = \bigoplus_{p+q=k} \mathcal{A}^{p,q}_M \tag{♣}$$

on differential forms. Explicitly, in a local chart U of M with local coordinates $z_1, \ldots, z_n$, $\mathcal{A}^{p,q}_M(U)$ is the subspace of $\mathcal{A}^k_M(U)$ generated by the k-forms of the form

$$f dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$$

with $f : U \to \mathbb{C}$ a smooth function. An element in $\mathcal{A}^{p,q}_M(U)$ is called a (p, q)**-form**.

Define two differentials ∂, $\bar{\partial}$ by

$$\partial : \mathcal{A}^{p,q}_M \hookrightarrow \mathcal{A}^{p+q}_{M,\mathbb{C}} \xrightarrow{d} \mathcal{A}^{p+q+1}_{M,\mathbb{C}} \twoheadrightarrow \mathcal{A}^{p+1,q}_M$$

$$\bar{\partial} : \mathcal{A}^{p,q}_M \hookrightarrow \mathcal{A}^{p+q}_{M,\mathbb{C}} \xrightarrow{d} \mathcal{A}^{p+q+1}_{M,\mathbb{C}} \twoheadrightarrow \mathcal{A}^{p,q+1}_M$$

where the last arrows are the projections from (♣). It is straightforward to see that

(i) a smooth function $f : U \to \mathbb{C}$ is holomorphic if and only if $\overline{\partial} f = 0$,

(ii) $d = \partial + \overline{\partial}$ and $\partial\overline{\partial} + \overline{\partial}\partial = 0$,

(iii) $(\mathcal{A}^{\bullet,q}, \partial)$ and $(\mathcal{A}^{p,\bullet}, \overline{\partial})$ are complexes,

(iv) for any k-form α and k'-form β, we have

$$\partial(\alpha \wedge \beta) = (\partial\alpha) \wedge \beta + (-1)^k \alpha \wedge (\partial\beta)$$

and the same for $\overline{\partial}$.

In what follows we put $\mathcal{C}^\infty_M$ to be the sheaf of complex-valued smooth functions on M. Let E be a holomorphic vector bundle over M and let $\mathcal{E}$ be the sheaf of *holomorphic* sections of $E \to M$. Define

$$\mathcal{A}^{p,q}_M(E) := \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{A}^{p,q}_M.$$

Explicitly, if $U \subseteq M$ is a local chart with holomorphic coordinates $z_1, \dots, z_n$ that trivializes the bundle E, and if $\sigma_1, \dots, \sigma_r \in \mathcal{E}(U)$ such that $\sigma_1(x), \dots, \sigma_r(x)$ form a basis of the fibre E_x for each $x \in U$, then

$$\mathcal{A}^{p,q}_M(E)(U) = \left\{ \sum_{i=1}^{r} \sum_{\#I=p,\#J=q} f_{I,J,i} \cdot \sigma_i \otimes dz_I \wedge d\bar{z}_J \mid f_{I,J,i} \in \mathcal{C}^\infty_M(U) \right\}$$

where I, J run over all ordered multi indices in $[n]$ of length p and q, respectively. With this notation, define

$$\overline{\partial}_{E,U}(f_{I,J,i} \cdot \sigma_i \otimes dz_I \wedge d\bar{z}_J) = \sigma_i \otimes (\overline{\partial} f_{I,J,i} \wedge dz_I \wedge d\bar{z}_J).$$

This extends by linearity to a map $\overline{\partial}_{E,U} : \mathcal{A}^{p,q}_M(E)(U) \to \mathcal{A}^{p,q+1}_M(E)(U)$.

Lemma 7.7. Let $V \subseteq M$ be another local trivialization of E. Then $\overline{\partial}_{E,U}$ and $\overline{\partial}_{E,V}$ agree on $\mathcal{A}^{p,q}_M(E)(U \cap V)$.

This means we can glue $\overline{\partial}_{E,U}$ to a global morphism

$$\overline{\partial}_E : \mathcal{A}^{p,q}_M(E) \to \mathcal{A}^{p,q+1}_M(E).$$

As above, we can show that

(a) a smooth section σ of $E \to M$ is holomorphic if and only if $\overline{\partial}_E \sigma = 0$,

(b) $(\mathcal{A}^{p,\bullet}_M(E), \overline{\partial}_E)$ is a complex,

(c) for any k-form α and any local section σ of $\mathcal{A}^{p,q}_M(E)$, we have

$$\overline{\partial}_E(\alpha \wedge \sigma) = (\overline{\partial}\alpha) \wedge \sigma + (-1)^k \alpha \wedge (\overline{\partial}_E \sigma).$$

Definition. Let $E \to M$ be a holomorphic vector bundle on a complex manifold M and $\mathcal{E}$ the sheaf of holomorphic sections of $E \to M$. For $p, q \in \mathbb{Z}_{\geqslant 0}$, the (p,q)**-th Dolbeault cohomology group** of E is defined as

$$H^{p,q}_{\overline{\partial}}(X, E) = \frac{\ker(\overline{\partial}_E : \mathcal{A}^{p,q}_M(E)(X) \to \mathcal{A}^{p,q+1}_M(E)(X))}{\mathrm{image}(\overline{\partial}_E : \mathcal{A}^{p,q-1}_M(E)(X) \to \mathcal{A}^{p,q}_M(E)(X))}$$

Note that each $H^{p,q}(X, E)$ is a $\mathbb{C}$-vector space, and $H^{0,0}(X, E) = \mathcal{E}(M)$ by (a).

To relate it with the sheaf cohomology, we need a following $\overline{\partial}$-analogue of Poincaré lemma.

Lemma 7.8 (Dolbeault-Grothendieck lemma)**.** The complex

$$\mathcal{A}^{p,q-1}_M \xrightarrow{\overline{\partial}} \mathcal{A}^{p,q}_M \xrightarrow{\overline{\partial}} \mathcal{A}^{p,q+1}_M$$

is exact for $p \geqslant 0,\ q \geqslant 1$.

It follows that the complex

$$\mathcal{A}_M^{p,q-1}(E) \xrightarrow{\bar\partial_E} \mathcal{A}_M^{p,q}(E) \xrightarrow{\bar\partial_E} \mathcal{A}_M^{p,q+1}(E)$$

is also exact for $q \geqslant 1$. By (i) we see the kernel of $\bar\partial : \mathcal{A}_M^{p,0} \to \mathcal{A}_M^{p,q}$ is the sheaf Ω_M^p of holomorphic p-forms on M. Hence by (7.4.1)

$$0 \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_M} \Omega_M^p \longrightarrow \mathcal{A}_M^{p,0}(E) \xrightarrow{\bar\partial_E} \mathcal{A}_M^{p,1}(E) \xrightarrow{\bar\partial_E} \cdots$$

is an acyclic resolution of $\mathcal{E} \otimes_{\mathcal{O}_M} \Omega_M^p$, where $\mathcal{O}_M$ is the sheaf of holomorphic functions on M. This proves

Theorem 7.9 (Dolbeault theorem)**.** There is a natural isomorphism

$$H^q(M, \mathcal{E} \otimes_{\mathcal{O}_M} \Omega_M^p) \cong H^{p,q}(X, E).$$

7.2.3 Holomorphic de Rham complex

Again let M be a complex manifold and Ω_M^p the sheaf of holomorphic p-form on M. If ω is a holomorphic form, then $d\omega = \partial\omega$ is again holomorphic, so $(\Omega_M^\bullet, \partial)$ forms a subcomplex of the complex de Rham complex $(\mathcal{A}_{M,\mathbb{C}}^\bullet, d)$. The former complex is called the holomorphic de Rham complex :

$$0 \longrightarrow \mathcal{O}_M \xrightarrow{\partial} \Omega_M^1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \cdots \Omega_M^n \to 0.$$

We show this is an acyclic resolution of the locally constant sheaf $\mathbb{C}_M$ by means of double complexes. Consider the double complex $(\mathcal{A}_M^{p,q}, \partial, (-1)^p\bar\partial)$:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \Omega_M^2 \arrow[r, "\bar\partial"] \arrow[u, "\partial"] & \mathcal{A}_M^{2,0} \arrow[r, "\bar\partial"] \arrow[u, "\partial"] & \mathcal{A}_M^{2,1} \arrow[r, "\bar\partial"] \arrow[u, "\partial"] & \mathcal{A}_M^{2,2} \arrow[r, "\bar\partial"] \arrow[u, "\partial"] & \cdots \\
0 \arrow[r] & \Omega_M^1 \arrow[r, "-\bar\partial"] \arrow[u, "\partial"] & \mathcal{A}_M^{1,0} \arrow[r, "-\bar\partial"] \arrow[u, "\partial"] & \mathcal{A}_M^{1,1} \arrow[r, "-\bar\partial"] \arrow[u, "\partial"] & \mathcal{A}_M^{1,2} \arrow[r, "-\bar\partial"] \arrow[u, "\partial"] & \cdots \\
0 \arrow[r] & \mathcal{O}_M \arrow[r, "\bar\partial"] \arrow[u, "\partial"] & \mathcal{A}_M^{0,0} \arrow[r, "\bar\partial"] \arrow[u, "\partial"] & \mathcal{A}_M^{0,1} \arrow[r, "\bar\partial"] \arrow[u, "\partial"] & \mathcal{A}_M^{0,2} \arrow[r, "\bar\partial"] \arrow[u, "\partial"] & \cdots
\end{tikzcd}$$

Each row is exact by Dolbeault lemma, so the inclusion $\Omega_M^\bullet \to \mathrm{tot}(\mathcal{A}_M^{\bullet,\bullet})^\bullet$ is a quasi-isomorphism, by virtue of Lemma 8.12. But $\mathrm{tot}(\mathcal{A}_M^{\bullet,\bullet})^\bullet = \mathcal{A}_{M,\mathbb{C}}^\bullet$, this shows

$$(\Omega_M^\bullet, \partial) \to (\mathcal{A}_{M,\mathbb{C}}^\bullet, d)$$

is a quasi-isomorphism. In particular, our assertion follows.

7.2.4 Logarithmic de Rham complex

In this subsubsection we write X for the complex manifold (instead of M), and let D be a **hypersurface** in X, i.e., it is locally defined as the zero locus of a single holomorphic function. We say D is a **normal crossing divisor** if near each point of D we can find a local coordinate $(U, z^1, \ldots, z^n)$ of X such that $D \cap U = V(z^1 \cdots z^r)$ for some $1 \leqslant r \leqslant n$ (depending on U) in U. Denote by $\Omega_X^p(*D)$ the sheaf of meromorphic p-forms which are holomorphic on $X \backslash D$. For each open U, define

$$\Omega_X^p(\log D)(U) := \{\omega \in \Omega_X^p(*D)(U) \mid \mathrm{ord}_p\, \omega, \mathrm{ord}_p\, d\omega \geqslant -1 \text{ for each } p \in D\}$$

Then $\Omega_X^p(\log D)$ is the subsheaf of $\Omega_X^p(*D)$ consisting of forms with log poles along the divisor D.

7.3 Grothendieck vanishing theorem

Theorem 7.10. Let X be a noetherian topological space of dimension n. Then for all $i > n$ and all sheaves of abelian groups $\mathcal{F}$ on X, we have $H^i(X, \mathcal{F}) = 0$.

7.4 Serre's theorem on affineness

Let X be a topological space and $\mathcal{F}$ a sheaf of abelian groups on X. For an open U, if we denote by $\iota : U \to X$ the natural inclusion, then we have a canonical map $\mathcal{F} \to \iota_* (\mathcal{F}|_U)$, and hence a map $H^i(X, \mathcal{F}) \to H^i(X, \iota_* (\mathcal{F}|_U))$ on sheaf cohomology groups . The image of an element $x \in H^i(X, \mathcal{F})$ in the latter group will be denoted by $x|_U$.

We start with a general topological lemma.

Lemma 7.11. Let X be a compact topological space and $\mathcal{B}$ a basis of the topology of X. For a sheaf $\mathcal{F}$ of abelian groups and $i \geqslant 0$, we say $\mathcal{F}$ satisfies (P_i) if

(P_i) for all $\gamma \in H^i(X, \mathcal{F})$ there exists a finite cover $U_1, \dots, U_r$ of X by elements in $\mathcal{B}$ such that $\alpha|_{U_i} = 0$ for any $1 \leqslant i \leqslant r$.

Then we have the following :

(i) (P_1) holds for all sheaves of abelian groups on X.

(ii) Suppose in addition $\mathcal{B}$ is closed under finite intersection. Let $\mathcal{F}$ be a sheaf and $i > 1$. If $H^p(U, \mathcal{F}|_U) = 0$ for all $0 < p < i$ and $U \in \mathcal{B}$, then (P_i) holds for $\mathcal{F}$.

Proof. Let $\mathcal{F}$ be a sheaf of abelian groups on X. Let $0 \to \mathcal{F} \to \mathcal{I}$ be an embedding into a flasque sheaf $\mathcal{I}$. If we write $\mathcal{G}$ to be the quotient sheaf $\mathcal{I}/\mathcal{F}$, we get an exact sequence of sheaves :

$$0 \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{I} \xrightarrow{\psi} \mathcal{G} \longrightarrow 0$$

Taking cohomology, we have a long (but quite short) exact sequence

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{I}(X) \longrightarrow \mathcal{G}(X) \longrightarrow H^1(X, \mathcal{F}) \longrightarrow 0$$

Let $\gamma \in H^1(X, \mathcal{F})$ and lift it to $\alpha \in \mathcal{G}(X)$. Since $\mathcal{I} \to \mathcal{G}$ is surjective (as sheaves) and since X is compact, we can find a finite cover $U_1, \dots, U_r$ of X by $\mathcal{B}$ such that $\alpha|_{U_i} = \psi_{U_i}(s_i)$ for some $s_i \in \mathcal{I}(U_i)$.

Let ι_i denote the canonical inclusion $U_i \to X$. Then we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{I} \arrow[r, "\psi"] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\iota_i)_* (\mathcal{F}|_{U_i}) \arrow[r] & (\iota_i)_* (\mathcal{I}|_{U_i}) \arrow[r] & (\iota_i)_* (\mathcal{G}|_{U_i}) &
\end{tikzcd}$$

Replacing $(\iota_i)_* (\mathcal{G}|_{U_i})$ by $\mathcal{G}_i := \operatorname{coker}\big((\iota_i)_* (\mathcal{F}|_{U_i}) \to (\iota_i)_* (\mathcal{I}|_{U_i})\big)$, we obtain a commutative diagram with exact rows (note that $(\cdot)|_{U_i}$ is exact!)

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{I} \arrow[r, "\psi"] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\iota_i)_* (\mathcal{F}|_{U_i}) \arrow[r] & (\iota_i)_* (\mathcal{I}|_{U_i}) \arrow[r] & \mathcal{G}_i \arrow[r] & 0.
\end{tikzcd}$$

Taking cohomology, by functoriality we have (note that $(\iota_i)_* (\mathcal{I}|_{U_i})$ is flasque)

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}(X) \arrow[r, "\varphi"] \arrow[d] & \mathcal{I}(X) \arrow[r, "\psi"] \arrow[d] & \mathcal{G}(X) \arrow[r] \arrow[d] & H^1(X, \mathcal{F}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}(U_i) \arrow[r] & \mathcal{I}(U_i) \arrow[r] & \mathcal{G}_i(X) \arrow[r] & H^1(X, (\iota_i)_* (\mathcal{F}|_{U_i})) \arrow[r] & 0
\end{tikzcd}$$

The image of α in $\mathcal{G}_i(X) \subseteq (\iota_i)_* (\mathcal{G}|_{U_i})(X) = \mathcal{G}(U_i)$ is $\alpha|_{U_i}$, and it comes from $s_j \in \mathcal{I}(U_i)$ via ψ_{U_j}. Thus $\gamma|_{U_i}$ is the image of s_j in $H^1(X, (\iota_i)_* (\mathcal{F}|_{U_i}))$, which is zero. This proves (i).

Next we prove (ii). Proceed by induction on $i > 1$. Let $U_1, \dots, U_r$ be an arbitrary finite open cover of X by $\mathcal{B}$, and put $\iota_j : U_j \to X$ to be the natural inclusion. If $U \in \mathcal{B}$, then $U_j \cap U \in \mathcal{B}$ and hence $H^1(U_j \cap U, \mathcal{F}|_{U_j \cap U})$ by our assumption. The sequence $0 \to \mathcal{F}|_{U_j \cap U} \to \mathcal{I}_{U_j \cap U} \to \mathcal{G}|_{U_j \cap U} \to 0$ is still exact, so taking cohomology gives

$$0 \longrightarrow \mathcal{F}(U_j \cap U) \longrightarrow \mathcal{I}(U_j \cap U) \longrightarrow \mathcal{G}(U_j \cap U) \longrightarrow 0 \ ,$$

or equivalently

$$0 \longrightarrow (\iota_j)_* \left(\mathcal{F}|_{U_j}\right)(U) \longrightarrow (\iota_j)_* \left(\mathcal{I}|_{U_j}\right)(U) \longrightarrow (\iota_j)_* \left(\mathcal{G}|_{U_j}\right)(U) \longrightarrow 0$$

On the other hand, we have an exact sequence $0 \to (\iota_j)_* \left(\mathcal{F}|_{U_j}\right) \to (\iota_j)_* \left(\mathcal{I}|_{U_j}\right) \to \mathcal{G}_j \to 0$ (with $\mathcal{G}_j$ being the cokernel of the former map). Restricting to U and taking cohomology gives

$$0 \longrightarrow (\iota_j)_* \left(\mathcal{F}|_{U_j}\right)(U) \longrightarrow (\iota_j)_* \left(\mathcal{I}|_{U_j}\right)(U) \longrightarrow \mathcal{G}_j(U) \longrightarrow H^1(U, (\iota_j)_* \left(\mathcal{F}|_{U_j}\right)|_U) \longrightarrow 0.$$

(Note that $(\iota_j)_* \left(\mathcal{I}|_{U_j}\right)$ is flasque, and so is its restriction to U.) If $0 \to \mathcal{F} \to \mathcal{C}^\bullet$ is a flasque resolution of $\mathcal{F}$, then $\mathcal{C}^\bullet|_{U_j \cap U}$ is also a flasque resolution of $\mathcal{F}|_{U_j \cap U}$, and $(\iota_j)_*(\mathcal{C}^\bullet|_{U_j})|_U$ is also a flasque resolution of $(\iota_j)_*(\mathcal{F}|_{U_j})|_U$. Since

$$\Gamma(U, (\iota_j)_*(\mathcal{C}^n|_{U_j})|_U) = \Gamma(U_j \cap U, \mathcal{C}^n|_{U_j \cap U})$$

we see particularly that $H^1(U, (\iota_j)_*(\mathcal{F}|_{U_j})|_U) = H^1(U_j \cap U, \mathcal{F}|_{U_j \cap U}) = 0$, and hence

$$0 \longrightarrow (\iota_j)_* \left(\mathcal{F}|_{U_j}\right)(U) \longrightarrow (\iota_j)_* \left(\mathcal{I}|_{U_j}\right)(U) \longrightarrow \mathcal{G}_j(U) \longrightarrow 0$$

is exact. Regarding $\mathcal{G}_j$ naturally as a subsheaf of $(\iota_j)_* \left(\mathcal{G}|_{U_j}\right)$, since we see that these two sheaves agrees on every basis element $U \in \mathcal{B}$, they are the same. In particular, we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{I} \arrow[r, "\psi"] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\iota_j)_* \left(\mathcal{F}|_{U_j}\right) \arrow[r] & (\iota_j)_* \left(\mathcal{I}|_{U_j}\right) \arrow[r] & (\iota_j)_* \left(\mathcal{G}|_{U_j}\right) \arrow[r] & 0
\end{tikzcd}$$

Taking cohomology and using the fact that $\mathcal{I}$ and $(\iota_j)_* \left(\mathcal{I}|_{U_j}\right)$ are flasque, for each $i \geqslant 2$ we have a commuting square

$$\begin{tikzcd}
H^{i-1}(X, \mathcal{G}) \arrow[r] \arrow[d] & H^i(X, \mathcal{F}) \arrow[d] \\
H^{i-1}(X, (\iota_j)_* \left(\mathcal{G}|_{U_j}\right)) \arrow[r] & H^i(X, (\iota_j)_* \left(\mathcal{F}|_{U_j}\right))
\end{tikzcd}$$

with two horizontal arrows being isomorphisms. By (i) $\mathcal{G}$ satisfies (P_1), so if we choose those $U_1, \dots, U_r$ to be as in (P_1) for $\mathcal{G}$, this says that $\mathcal{F}$ satisfies (P_2). Now suppose (ii) holds for all i with $2 \leqslant i < n$. To show (ii) in n, it suffices to show $\mathcal{G}$ satisfies (P_{n-1}). By induction hypothesis it is sufficient to show $H^p(U, \mathcal{G}|_U) = 0$ for all $U \in \mathcal{B}$ and $0 < p < n-1$. This of course holds, as by the exact sequence

$$0 \longrightarrow \mathcal{F}|_U \longrightarrow \mathcal{I}|_U \longrightarrow \mathcal{G}|_U \longrightarrow 0$$

and as $H^p(U, \mathcal{I}|_U) = 0$ and $H^{p+1}(U, \mathcal{F}|_U) = 0$ (this is true because $p + 1 < n$). $\square$

Theorem 7.12. Let A be a ring and $\mathcal{F}$ a quasi-coherent sheaf on $X = \operatorname{Spec} A$. Then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

Proof. We use Lemma 7.11 with $\mathcal{B}$ the collection of all affine open subsets of X, which is a basis for the Zariski topology on X. Note also that X is (quasi-)compact. We prove this by induction on $i \geqslant 1$.

Assume $i = 1$. Let $\gamma \in H^1(X, \mathcal{F})$ By Lemma 7.11.(i) we can find affine opens $U_1, \dots, U_r$ such that γ maps to 0 in all $H^1(X, (\iota_j)_* (\mathcal{F}|_{U_j}))$, where $\iota_j : U_j \to X$ denotes the inclusion. Now each $(\iota_j)_* (\mathcal{F}|_{U_j})$ is quasi-coherent, and so is the cokernel $\mathcal{G} := \operatorname{coker}\left(\mathcal{F} \to \prod_{j=1}^r (\iota_j)_* (\mathcal{F}|_{U_j})\right)$. Since X is affine, we obtain a short exact sequence

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow \prod_{j=1}^r (\iota_j)_* (\mathcal{F}|_{U_j})(X) \longrightarrow \mathcal{G}(X) \longrightarrow 0. \tag{$*$}$$

In view of long exact sequence of cohomology, we see $H^1(X, \mathcal{F}) \to \prod_{j=1}^r H^1(X, (\iota_j)_* (\mathcal{F}|_{U_j}))$ is injective. This shows $\gamma = 0$.

For $i > 1$, by induction hypothesis, $H^p(U, \mathcal{F}|_U) = 0$ for each $0 < p < i$ and $U \in \mathcal{B}$. Now let $\gamma \in H^i(X, \mathcal{F})$ be given. By Lemma 7.11.(ii) we can find affine opens $U_1, \dots, U_r$ that covers X such that γ maps to zero in each $H^i(X, (\iota_j)_* (\mathcal{F}|_{U_j}))$. By induction hypothesis, $H^{i-1}(X, \mathcal{G}) = 0$, so $(*)$ implies $H^i(X, \mathcal{F}) \to \prod_{j=1}^r H^i(X, (\iota_j)_* (\mathcal{F}|_{U_j}))$ is injective. This again shows $\gamma = 0$. $\square$

7.5 Higher direct image

Let X, Y be spaces and $f : X \to Y$ be a continuous map. The direct image of f defines a functor $f_* : \mathbf{Ab}_X \to \mathbf{Ab}_Y$. This is a left exact functor, as it is a right adjoint functor (2.11). Since $\mathbf{Ab}_X$ has enough injective, we then can consider the right derived functors $R^i f_*$ of f_*.

Definition. The right derived functors $R^i f_* : \mathbf{Ab}_X \to \mathbf{Ab}_Y$ of the direct image f_* are called the **higher direct image** functors of f.

Lemma 7.13. For $i \geqslant 0$ and any sheaf $\mathcal{F}$ of abelian groups on X, the sheaf $R^i f_* \mathcal{F}$ is isomorphic to the sheafification of the presheaf $V \mapsto H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$ on Y.

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$ in $\mathbf{Ab}_X$. By definition, $R^i f_* \mathcal{F}$ is the i-th cohomology sheaf of the complex $f_* \mathcal{I}^0 \to f_* \mathcal{I}^1 \to f_* \mathcal{I}^2 \to \cdots$, hence it is the sheafification of the presheaf

$$V \mapsto \frac{\ker(f_*\mathcal{I}^i(V) \to f_*\mathcal{I}^{i+1}(V))}{\operatorname{Im}(f_*\mathcal{I}^{i-1}(V) \to f_*\mathcal{I}^i(V))} = \frac{\ker(\mathcal{I}^i(f^{-1}(V)) \to \mathcal{I}^{i+1}(f^{-1}(V)))}{\operatorname{Im}(\mathcal{I}^{i-1}(f^{-1}(V)) \to \mathcal{I}^i(f^{-1}(V)))}.$$

Since restriction to $f^{-1}(V)$ is exact and by (7.2).2., $\mathcal{F}|_{f^{-1}(V)} \to \mathcal{I}^\bullet|_{f^{-1}(V)}$ is an injective resolution of $\mathcal{F}|_{f^{-1}(V)}$. Hence the last quotient computes $H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$, and the proof is completed. $\square$

Corollary 7.13.1. If $U \subseteq Y$ is an open subspace, there is a unique isomorphism of δ-functor

$$(R^i f_* \mathcal{F})|_U \xrightarrow{\sim} R^i (f|^V_{f^{-1}(U)})_* (\mathcal{F}|_U) \qquad (i \geqslant 0)$$

extending the $i = 0$ case $(f_*\mathcal{F})|_U \cong (f|^V_{f^{-1}(U)})_* (\mathcal{F}|_U)$.

Corollary 7.13.2. Let $f \in \operatorname{Hom}_{\mathbf{Sch}}(X, Y)$ be an affine morphism and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. Then $R^p f_* \mathcal{F} = 0$ for $p \geqslant 1$.

Proof. This follows from Lemma 7.13 and (7.12). $\square$

In particular, if $\mathcal{F}$ is a flasque sheaf, then all $R^i f_* \mathcal{F}$ vanish by Lemma 7.2.4 and Lemma 7.13. Therefore, if $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are ringed space and $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, then the higher direct image functor $R^i f_* : \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_Y}$ coincides with that computed as a functor $\mathbf{Ab}_X \to \mathbf{Ab}_Y$. Also, Lemma 7.13 holds if f is replaced by a morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces and f_* is viewed as a functor $f_* : \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_Y}$.

7.5.1 Quasi-coherency

Let X, Y be schemes, and $f \in \mathrm{Hom}_{\mathbf{Sch}}(X, Y)$. In this subsubsection we prove the quasi-coherency of $R^i f_* \mathcal{F}$ when $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module under some mild condition on f. In (3.17), we see if f is, for example, quasi-compact and quasi-separated, then $R^0 f_* \mathcal{F} = f_* \mathcal{F}$ is quasi-coherent. Generally, we have

Theorem 7.14. Let $f \in \mathrm{Hom}_{\mathbf{Sch}}(X, Y)$ be quasi-separated and quasi-compact. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, then all $R^i f_* \mathcal{F}$ are quasi-coherent $\mathcal{O}_Y$-modules.

Proof. By Corollary 7.13.1 we can assume $Y = \operatorname{Spec} A$ is affine. This then follows from Lemma 7.16 below. □

The key ingredient is the following categorical property of $\mathbf{Qcoh}_X$.

Lemma 7.15. Let A be a ring and $f : X \to \operatorname{Spec} A$ be a quasi-compact A-scheme. If $\mathcal{F}$ is quasi-coherent $\mathcal{O}_X$-module, then there exists a quasi-coherent flasque sheaf $\mathcal{I}$ and an injection $\mathcal{F} \to \mathcal{I}$. [6]

Proof. By compactness of X, take a finite affine open cover $\mathcal{U}$ of X. For each $U \in \mathcal{U}$ let $j_U : U \to X$ denote the inclusion. Since $\mathcal{F}$ is quasi-coherent, for each $U \in \mathcal{U}$ say $\mathcal{F}|_U \cong \widetilde{M_U}$ for some $\mathcal{O}_X(U)$-module M_U. Embed M_U into an injective $\mathcal{O}_X(U)$-module I_U. Set

$$\mathcal{I} := \bigoplus_{U \in \mathcal{U}} (j_U)_* \widetilde{I_U}.$$

By adjunction, from $\mathcal{F}|_U \to \widetilde{I_U}$ we get a morphism $\mathcal{F} \to (j_U)_* \widetilde{I_U}$. Taking direct sum yields $\mathcal{F} \to \mathcal{I}$. Using (2.10.2) and (2.16) we easily see this is an injection. By (7.2).2. each $\widetilde{I_U}$ is flasque and quasi-coherent on U, so $\mathcal{U}$ is flasque and quasi-coherent on X. □

Lemma 7.16. Let A be a ring and let $f : X \to \operatorname{Spec} A$ be a quasi-compact and quasi-separated A-scheme. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, then $\widetilde{H^p(X, \mathcal{F})} \cong R^p f_* \mathcal{F}$ canonically for $p \geqslant 0$, where affine tilde is with respect to A. (c.f. (7.3))

Proof. For $p = 0$, since $R^0 f_* \mathcal{F} = f_* \mathcal{F}$ is quasi-coherent, by (3.15.2) the canonical morphism

$$\widetilde{\mathcal{F}(X)} = f_* \widetilde{\mathcal{F}(\operatorname{Spec} A)} \to f_* \mathcal{F} = R^0 f_* \mathcal{F}$$

is an isomorphism. In general, since $\widetilde{\ }$ is exact, the functor $\mathbf{Qcoh}_X \ni \mathcal{F} \mapsto \widetilde{H^p(X, \mathcal{F})}$ is a δ-functor. By Lemma 7.15 functors on both sides are effaceable for $p > 0$. Hence there is a unique isomorphism of δ-functors $\widetilde{H^p(X, \mathcal{F})} \xrightarrow{\sim} R^p f_* \mathcal{F}$ extending the $p = 0$ case. □

7.5.2 Leray spectral sequence

Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism between ringed spaces. We have three functors

$$f_* : \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_Y}, \qquad \Gamma(Y, \cdot) : \mathbf{Mod}_{\mathcal{O}_Y} \to \mathbf{Ab}, \qquad \Gamma(X, \cdot) : \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Ab}$$

Clearly from the definition that $\Gamma(Y, \cdot) \circ f_* = \Gamma(X, \cdot)$. We want to apply Grothendieck spectral sequence. For this we need

Lemma 7.17. If $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, then $H^p(Y, f_* \mathcal{I}) = 0$ for $p \geqslant 1$.

6. See [Har66, Theorem 2.7.18] for the locally Noetherian case.

Proof. By (8.13.3) it suffices to show $\widetilde{H}^p(\mathcal{U}, f_*\mathcal{I}) = 0$ for $p \geqslant 1$, any open $U \subseteq Y$ and any open cover $\mathcal{U}$ of U. Since

$$C^\bullet(\mathcal{U}, f_*\mathcal{I}) = C^\bullet(f^{-1}\mathcal{U}, \mathcal{I})$$

and $\mathcal{I}|_{f^{-1}(U)}$ is flasque, it follows from (8.16) that $\widetilde{H}^p(\mathcal{U}, f_*\mathcal{I}) = 0$ for $p \geqslant 1$. □

Hence by Grothendieck spectral sequence, there is a biregular spectral sequence

$$E_2^{pq} = H^p(Y, R^q f_*\mathcal{F}) \Rightarrow H^{p+q}(X, f_*\mathcal{F}).$$

This is called the **Leray spectral sequence**.

Lemma 7.18. The edge homomorphism $E_2^{n0} = H^n(Y, f_*\mathcal{F}) \to H^n(X, \mathcal{F})$ is the natural map extending $f_*\mathcal{F}(U) \xrightarrow{=} \mathcal{F}(f^{-1}(U))$.

Corollary 7.18.1. If $f \in \mathrm{Hom}_{\mathbf{Sch}}(X, Y)$ is an affine morphism and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, then $H^n(Y, f_*\mathcal{F}) \xrightarrow{\sim} H^n(X, \mathcal{F})$ is an isomorphism for all $n \geqslant 0$.

Proof. This follows from (7.13.2).

□

7.5.3 Base change morphisms

Let $f : Y \to X$ be a continuous map between topological spaces, and let $\mathcal{F}$ be a sheaf of abelian groups on X. Since $f^{-1} : \mathbf{Ab}_X \to \mathbf{Ab}_Y$ is exact, $\mathcal{F} \mapsto H^i(Y, f^{-1}\mathcal{F})\,(i \geqslant 0)$ remains a δ-functor. By the universality there is a unique map of δ-functors

$$H^i(X, \mathcal{F}) \longrightarrow H^i(Y, f^{-1}\mathcal{F})$$

such that when $i = 0$ it is the natural map $\mathcal{F}(X) \to (f_*f^{-1}\mathcal{F})(X) = (f^{-1}\mathcal{F})(Y)$. This is the **topological base change map**. Explicitly, let $\mathcal{F} \to \mathcal{I}^\bullet$ and $f^{-1}\mathcal{F} \to \mathcal{J}^\bullet$ be injective resolutions. Since f^{-1} is exact, by Lemma 8.14 there exists $f^{-1}\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ extending the identity map $f^{-1}\mathcal{F} \xrightarrow{\mathrm{id}} f^{-1}\mathcal{F}$. By adjunction this gives $\mathcal{I}^\bullet \to f_*\mathcal{J}^\bullet$. Now taking global section and then taking cohomology gives the topological base change map.

If in addition $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are ringed spaces and f is a morphism between ringed spaces, then there is a natural map $f^{-1}\mathcal{F} \to f^*\mathcal{F}$ (defined again in the level of presheaves), so it induces a map $H^i(Y, f^{-1}\mathcal{F}) \to H^i(Y, f^*\mathcal{F})$. Composing yields the so-called **base change map**

$$H^i(X, \mathcal{F}) \longrightarrow H^i(Y, f^*\mathcal{F}) \qquad (i \geqslant 0)$$

Now turn to the relative version. Consider a commutative diagram in **RS**

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\psi"'] & X \arrow[d, "\varphi"] \\ T \arrow[r, "g"'] & S \end{tikzcd}$$

Let $U \subseteq S$ be an open set. From the commutativity and the base change map above, we have a map

$$H^i(\varphi^{-1}(U), \mathcal{F}|_{\varphi^{-1}(U)}) \longrightarrow H^i(\psi^{-1}(g^{-1}(U)), (f|^{\varphi^{-1}(U)}_{\psi^{-1}(g^{-1}(U))})^*(\mathcal{F}|_{\varphi^{-1}(U)})) = H^i(\psi^{-1}(g^{-1}(U)), (f^*\mathcal{F})_{\psi^{-1}(g^{-1}(U))})$$

By the definition, there is a natural map

$$H^i(\psi^{-1}(g^{-1}(U)), (f^*\mathcal{F})_{\psi^{-1}(g^{-1}(U))}) \to (R^i\psi_*f^*\mathcal{F})(g^{-1}(U)) = g_*(R^i\psi_*f^*\mathcal{F})(U)$$

so composing these two gives

$$H^i(\varphi^{-1}(U), \mathcal{F}|_{\varphi^{-1}(U)}) \longrightarrow g_*(R^i\psi_* f^*\mathcal{F})(U)$$

It is easy to check this map is functorial in $U \underset{\text{open}}{\subseteq} S$, so it defines a morphism of presheaves on S. By sheafification this gives the morphism of $\mathcal{O}_S$-modules

$$R^i\varphi_*\mathcal{F} \longrightarrow g_*(R^i\psi_* f^*\mathcal{F})$$

and by adjunction we obtain the desired **base change morphism**

$$g^*R^i\varphi_*\mathcal{F} \longrightarrow R^i\psi_* f^*\mathcal{F} \qquad (i \geqslant 0).$$

We compute the stalk of the base change morphism. Let $t \in T$ and $s = g(t) \in S$. Then it is

$$\mathcal{O}_{T,t} \otimes_{\mathcal{O}_{S,s}} (R^i\varphi_*\mathcal{F})_s \longrightarrow (R^i\psi_* f^*\mathcal{F})_t$$

By (2.12.1) this is induced by

$$(R^i\varphi_*\mathcal{F})_s \longrightarrow (g_*(R^i\psi_* f^*\mathcal{F}))_s \longrightarrow (R^i\psi_* f^*\mathcal{F})_t$$

which by (7.13) is

$$\varinjlim_{\mathrm{Top}(S)\ni U\ni s} H^i(\varphi^{-1}(U), \mathcal{F}|_{\varphi^{-1}(U)}) \longrightarrow \varinjlim_{\mathrm{Top}(T)\ni V\ni t} H^i(\psi^{-1}(V), (f^*\mathcal{F})|_{\psi^{-1}(V)}).$$

It follows from construction that this is induced by the base change maps

$$H^i(\varphi^{-1}(U), \mathcal{F}|_{\varphi^{-1}(U)}) \longrightarrow H^i(\psi^{-1}(g^{-1}(U)), (f^*\mathcal{F})_{\psi^{-1}(g^{-1}(U))})$$

In sum, the stalk of the base change morphism gives the base change maps.

7.6 Formal function theorem

Let $f : X \to \operatorname{Spec} A$ be a morphism of schemes, and let I be an ideal of A. For an $\mathcal{O}_X$-module $\mathcal{F}$, we can form the subsheaf $I\mathcal{F}$ of $\mathcal{F}$ (2.23.1) : by definition,

$$I\mathcal{F} = \operatorname{Im}\left(f^*\tilde{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to f^*\mathcal{O}_A \otimes_{\mathcal{O}_X} \mathcal{F} \cong \mathcal{F}\right).$$

For $n \geqslant 0$, we similarly form $I^n\mathcal{F}$ (here $I^0 = A$). Consider the short exact sequences of $\mathcal{O}_X$-modules

$$\begin{tikzcd}
0 \arrow[r] & I^{n+1}\mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d, equal] & \mathcal{F}/I^{n+1}\mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^n\mathcal{F} \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}/I^n\mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

It then induces long exact sequences on cohomology

$$\begin{tikzcd}
H^i(X, I^{n+1}\mathcal{F}) \arrow[r, "\beta_{n+1}"] \arrow[d] & H^i(X, \mathcal{F}) \arrow[r, "\alpha_{n+1}"] \arrow[d, equal] & H^i(X, \mathcal{F}/I^{n+1}\mathcal{F}) \arrow[r] \arrow[d] & H^{i+1}(X, I^{n+1}\mathcal{F}) \arrow[r, "\gamma_{n+1}"] \arrow[d] & H^{i+1}(X, \mathcal{F}) \arrow[d, equal] \\
H^i(X, I^n\mathcal{F}) \arrow[r, "\beta_n"] & H^i(X, \mathcal{F}) \arrow[r, "\alpha_n"] & H^i(X, \mathcal{F}/I^n\mathcal{F}) \arrow[r] & H^{i+1}(X, I^n\mathcal{F}) \arrow[r, "\gamma_n"] & H^{i+1}(X, \mathcal{F})
\end{tikzcd}$$

Let us look at the maps $\alpha_n : H^i(X, \mathcal{F}) \to H^i(X, \mathcal{F}/I^n\mathcal{F})$. Note that the latter cohomology group has a structure of A/I^nA-module; one can see this by resolving $\mathcal{F}/I^n\mathcal{F}$ with Godement resolution. Hence α_n induces a map

$$\alpha'_n : H^i(X, \mathcal{F}) \otimes_A A/I^n \longrightarrow H^i(X, \mathcal{F}/I^n\mathcal{F}).$$

The commutative diagram above shows that various α'_n form an inverse system of abelian groups, yielding

$$\alpha' : \varprojlim_{n \geqslant 0} \left(H^i(X, \mathcal{F}) \otimes_A A/I^n\right) \longrightarrow \varprojlim_{n \geqslant 0} H^i(X, \mathcal{F}/I^n\mathcal{F})$$

On the other hand, we can trim the long exact sequences, obtaining

$$\begin{tikzcd}
0 \arrow[r] & H^i(X, \mathcal{F})/\operatorname{Im}\beta_{n+1} \arrow[r, "\alpha''_{n+1}"] \arrow[d] & H^i(X, \mathcal{F}/I^{n+1}\mathcal{F}) \arrow[r] \arrow[d] & \ker\gamma_{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^i(X, \mathcal{F})/\operatorname{Im}\beta_n \arrow[r, "\alpha''_n"] & H^i(X, \mathcal{F}/I^n\mathcal{F}) \arrow[r] & \ker\gamma_n \arrow[r] & 0
\end{tikzcd}$$

Taking limits gives

$$0 \longrightarrow \varprojlim_{n \geqslant 0} \left(H^i(X, \mathcal{F})/\operatorname{Im}\beta_n\right) \xrightarrow{\alpha''} \varprojlim_{n \geqslant 0} H^i(X, \mathcal{F}/I^n\mathcal{F}) \longrightarrow \varprojlim_{n \geqslant 0} \ker\gamma_n \longrightarrow 0$$

Note that this is exact at the third place as the transition maps of $\left(H^i(X, \mathcal{F})/\operatorname{Im}\beta_n\right)_n$ are surjective. In fact, α' and α'' are compatible, in the sense we now describe. From the long exact sequences we read $I^nH^i(X, \mathcal{F}) \subseteq \ker\alpha_n = \operatorname{Im}\beta_n$, so we have a projection

$$p_n : H^i(X, \mathcal{F})/\operatorname{Im}\beta_n \longrightarrow H^i(X, \mathcal{F})/I^nH^i(X, \mathcal{F}) = H^i(X, \mathcal{F}) \otimes_A A/I^n.$$

Passing to limit, we obtain $p : \varprojlim_{n \geqslant 0} \left(H^i(X, \mathcal{F})/\operatorname{Im}\beta_n\right) \to \varprojlim_{n \geqslant 0} \left(H^i(X, \mathcal{F}) \otimes_A A/I^n\right)$. It is clear that

$$\alpha'' = \alpha' \circ p,$$

so we have a larger commutative diagram with exact row

$$\begin{tikzcd}
0 \arrow[r] & \varprojlim_{n \geqslant 0} \left(H^i(X, \mathcal{F})/\operatorname{Im}\beta_n\right) \arrow[r, "\alpha''"] \arrow[d, "p"] & \varprojlim_{n \geqslant 0} H^i(X, \mathcal{F}/I^n\mathcal{F}) \arrow[r] & \varprojlim_{n \geqslant 0} \ker\gamma_n \arrow[r] & 0 \\
& \varprojlim_{n \geqslant 0} \left(H^i(X, \mathcal{F}) \otimes_A A/I^n\right) \arrow[ur, "\alpha'"'] & & &
\end{tikzcd}$$

Theorem 7.19 (Formal Function Theorem)**.** Let $f : X \to \operatorname{Spec} A$ be a proper morphism of schemes with A Noetherian, $\mathcal{F}$ a coherent $\mathcal{O}_X$-module, and I an ideal of A. Then for each $i \geqslant 0$, the natural homomorphism

$$\alpha' : \varprojlim_{n \geqslant 0} \left(H^i(X, \mathcal{F}) \otimes_A A/I^n\right) \longrightarrow \varprojlim_{n \geqslant 0} H^i(X, \mathcal{F}/I^n\mathcal{F})$$

is an isomorphism of topological $\varprojlim_{n \geqslant 0} A/I^n$-modules.

To prove this, we claim that under these circumstances, we have the followings.

(1) There exists $n_0 \in \mathbb{Z}_{\geqslant 0}$ such that $\operatorname{Im}\beta_n \subseteq I^{n-n_0}H^i(X, \mathcal{F})$ for all $n \geqslant n_0$.

(2) There exists $d \in \mathbb{Z}_{\geqslant 0}$ such that then transition maps $\ker\gamma_{n+d} \to \ker\gamma_n$ are trivial for $n \gg 0$.

These will complete the proof. Indeed, (1) says that p is an isomorphism, and (2) says that $\varprojlim_{n\geqslant 0} \ker\gamma_n = 0$.

To show (1) and (2), we make use of the blow-up algebra

$$B_\bullet = \bigoplus_{n\geqslant 0} I^n = A \oplus I \oplus I^2 \oplus \cdots$$

which is a graded algebra. Since A is Noetherian, I is finitely generated. Since $B_\bullet$ is generated by I as an A-algebra, this shows $B_\bullet$ is Noetherian. Our proof is based on the following

Lemma 7.20. Let A be a Noetherian ring, $I \trianglelefteq A$ an ideal, M a finite A-module, and $(M_n)_{n\geqslant 0}$ a descending I-filtration of M. Put $B_\bullet = \bigoplus_{n\geqslant 0} I^n$. TFAE :

(i) The graded $B_\bullet$-module $\bigoplus_{n\geqslant 0} M_n$ is finitely generated.

(ii) $IM_n = M_{n+1}$ for $n \gg 0$.

To utilize this lemma, we consider

$$\bigoplus_{n\geqslant 0} H^i(X, I^n\mathcal{F}) \twoheadrightarrow \bigoplus_{n\geqslant 0} \operatorname{Im}\beta_n \subseteq \bigoplus_{n\geqslant 0} H^i(X,\mathcal{F}).$$

For each $a \in I^m$, the multiplication by a gives a map $I^m\mathcal{F} \to I^{n+d}\mathcal{F} \subseteq \mathcal{F}$, inducing homomorphisms on cohomology groups

$$\begin{tikzcd} H^i(X, I^n\mathcal{F}) \arrow[r] \arrow[dr] & H^i(X, I^{n+m}\mathcal{F}) \arrow[d] \\ & H^i(X,\mathcal{F}). \end{tikzcd}$$

By considering various m and $a \in I^m$, we can equip $\bigoplus_{n\geqslant 0} H^i(X, I^n\mathcal{F})$ and $\bigoplus_{n\geqslant 0} H^i(X,\mathcal{F})$ with $B_\bullet$-module structures. With such structures, the map $\bigoplus_{n\geqslant 0} H^i(X, I^n\mathcal{F}) \to \bigoplus_{n\geqslant 0} H^i(X,\mathcal{F})$ becomes a $B_\bullet$-module homomorphism. In particular, $\bigoplus_{n\geqslant 0} \operatorname{Im}\beta_n$ is a $B_\bullet$-submodule of $\bigoplus_{n\geqslant 0} H^i(X,\mathcal{F})$, and $\bigoplus_{n\geqslant 0} \ker\gamma_n$ is a $B_\bullet$-submodule of $\bigoplus_{n\geqslant 0} H^{i+1}(X, I^n\mathcal{F})$. We claim

(3) The graded $B_\bullet$-module $\bigoplus_{n\geqslant 0} H^i(X, I^n\mathcal{F})$ is finite over $B_\bullet$ for each $i \geqslant 0$.

Once this is proved, (1) and (2) will follow immediately. Indeed, since $B_\bullet$ is Noetherian, (3) implies the submodules $\bigoplus_{n\geqslant 0} \ker\gamma_n$ and the homomorphic image $\bigoplus_{n\geqslant 0} \operatorname{Im}\beta_n$ are finite over $B_\bullet$ as well. The lemma above then applies, showing that there exist $n_0, d \geqslant 0$ satisfying

(1)′ $\operatorname{Im}\beta_{n+1} = I \operatorname{Im}\beta_n$ for $n \geqslant n_0$, and

(2)′ $\ker\gamma_{n+1} = I\ker\gamma_n$ for $n \geqslant d$.

From (1)′, we see for all $n \geqslant n_0$ that

$$\operatorname{Im}\beta_n = I^{n-n_0} \operatorname{Im}\beta_{n_0} \subseteq I^{n-n_0} H^i(X,\mathcal{F})$$

proving (1). Similarly, from (2)′ we see

$$\ker\gamma_{n+d} = I^d \ker\gamma_n$$

for all $n \geqslant d$. In particular, we see $\operatorname{Im}(\ker\gamma_{n+d} \to \ker\gamma_n)$ is the $B_\bullet$-span of the homomorphisms

$$\ker\gamma_n \subseteq H^i(X, I^n\mathcal{F}) \xrightarrow{\ a\ } H^i(X, I^{n+d}\mathcal{F}) \longrightarrow H^i(X, I^n\mathcal{F})$$

with a running over I^d. Hence it suffices to show each element in I^d kills $\ker\gamma_n = I^{n-d}\ker\gamma_d$, or kills $\ker\gamma_d$. This is obvious, as from the long exact sequence we see

$$\ker\gamma_d = \mathrm{Im}(H^i(X, \mathcal{F}/I^d\mathcal{F}) \to H^{i+1}(X, I^d\mathcal{F}))$$

and $H^i(X, \mathcal{F}/I^d\mathcal{F})$ is an A/I^d-module.

Now we only need to show (3). We need a variant of Grothendieck coherency theorem :

Theorem 7.21. Let $f : X \to \operatorname{Spec} A$ be a proper morphism of schemes with A Noetherian. Let B be an A-algebra of finite type and set $\mathcal{B} := f^*\widetilde{B}$. If $\mathcal{F}$ is a coherent $\mathcal{B}$-module, then $H^i(X, \mathcal{F})$ is a finite B-module for each $i \geqslant 0$.

We apply this theorem to the sheaf $\bigoplus_{n\geqslant 0} I^n\mathcal{F}$ which is clearly $\mathcal{B} = f^*\widetilde{B_\bullet} = \bigoplus_{n\geqslant 0}(f^*\widetilde{I})^n$-coherent. This shows

$$H^i\left(X, \bigoplus_{n\geqslant 0} I^n\mathcal{F}\right) \cong \bigoplus_{n\geqslant 0} H^i(X, I^n\mathcal{F})$$

is a finite $B_\bullet$-module. Caution : one needs to check that this $B_\bullet$-structure is the same as the one defined above, but this is clear. This finishes the proof of (3), and hence the proof of formal function theorem.

7.6.1 Reformulation

Let $f : X \to Y$ be a morphism of schemes and $y \in Y$ a point. Consider the base changes

$$\begin{tikzcd}
X_n = X \times_Y \operatorname{Spec} \mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}^n \arrow[r] \arrow[d, "f_n"] & X_1 = f^{-1}(y) \arrow[r] \arrow[d, "f_1"] & X \arrow[d, "f"] \\
\operatorname{Spec} \mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}^n \arrow[r] & \operatorname{Spec} \mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y} \arrow[r] & Y
\end{tikzcd}$$

Each X_n is the same as topological spaces : they are all the fibre $f^{-1}(y)$. What's different is the structure sheaf ; X_n captures more nilpotents when n grows. In some sense we are "thickening the fibre $f^{-1}(y)$".

7.6.2 Consequences

Theorem 7.22 (Zariski connectedness principle). Let $\pi : X \to Y$ be a proper morphism between locally Noetherian schemes with $\pi_*\mathcal{O}_X \cong \mathcal{O}_Y$. Then $\pi^{-1}(q)$ is connected for each $q \in Y$.

Proof. □

7.7 Cohomology of base change

Theorem 7.23. Let $f : X \to Y$ be a proper morphism of Noetherian schemes with $Y = \operatorname{Spec} A$. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module that is flat over Y, i.e., each stalk $\mathcal{F}_x$ is flat over $\mathcal{O}_{Y,f(x)}$. Then there exist a finite complex $K^\bullet : 0 \to K^0 \to K^1 \to \cdots \to K^n \to 0$ of finitely generated projective A-modules and a natural isomorphism

$$H^\bullet(X \times_Y \operatorname{Spec} B, \mathcal{F} \otimes_A B) \cong H^\bullet(K^\bullet \otimes_A B)$$

on the category of A-algebras B.

Proof. First we make a reduction so that this becomes a purely homological problem. Choose a finite affine open cover $\mathcal{U}$ of X. By Corollary 8.15.1 (and since X is separated) and Theorem 8.7, we can compute the cohomology by the ordered Čech complex $C^\bullet = (C^\bullet_{\mathrm{ord}}(\mathcal{U}, \mathcal{F}), d^\bullet)$, which is a finite complex as $\#\mathcal{U} < \infty$, and consists of A-flat modules.

Also, for any A-algebra B, $\{U \times_Y \operatorname{Spec} B \mid U \in \mathcal{U}\}$ is an finite affine cover of $X \times_Y \operatorname{Spec} B$, and $C^\bullet \otimes_A B$ is the Čech complex of $\mathcal{F} \otimes_A B$ for this cover. Therefore,

$$H^\bullet(X \times_Y \operatorname{Spec} B, \mathcal{F} \otimes_A B) \cong H^\bullet(C^\bullet \otimes_A B)$$

for all B, which is in fact functorial in B.

$\square$

8 Čech Cohomology

8.1 Definitions

Let X be a topological space, $\mathcal{U} = \{U_i\}_{i\in I}$ an open cover of X and $\mathcal{F}$ a *presheaf* of abelian groups on X. For $p \geqslant 0$, define a complex $(C^\bullet(\mathcal{U},\mathcal{F}), d^\bullet)$ as follows. For each $p \geqslant 0$, define the **Čech p-th cochain group with values in $\mathcal{F}$** by

$$C^p(\mathcal{U},\mathcal{F}) = \prod_{i_0,\dots,i_p\in I} \mathcal{F}(U_{i_0}\cap\cdots\cap U_{i_p}).$$

For a p-th cochain $s \in C^p(\mathcal{U},\mathcal{F})$, we write $s(i_0,\dots,i_p)$ to denote its component in $\mathcal{F}(U_{i_0}\cap\cdots\cap U_{i_p})$.

For $p \geqslant 0$, define the p-th coboundary map $d^p : C^p(\mathcal{U},\mathcal{F}) \to C^{p+1}(\mathcal{U},\mathcal{F})$ by the formula

$$(ds)(i_0,\dots,i_{p+1}) = \sum_{j=0}^{p+1}(-1)^j s(i_0,\dots,\widehat{i_j},\dots,i_{p+1})|_{U_{i_0}\cap\cdots\cap U_{i_{p+1}}}.$$

Lemma 8.1. For $p \geqslant 0$, the composition

$$C^p(\mathcal{U},\mathcal{F}) \xrightarrow{d^p} C^{p+1}(\mathcal{U},\mathcal{F}) \xrightarrow{d^{p+1}} C^{p+2}(\mathcal{U},\mathcal{F})$$

is zero.

Proof. Let $s \in C^p(\mathcal{U},\mathcal{F})$. Then

$$\begin{aligned} d(ds)(i_0,\dots,i_{p+2}) &= \sum_{j=0}^{p+2}(-1)^j (ds)(i_0,\dots,\widehat{i_j},\dots,i_{p+2})|_{U_{i_0}\cap\cdots\cap U_{i_{p+2}}} \\ &= \sum_{j=0}^{p+2}\Bigg(\sum_{0\leqslant k<j}(-1)^{j+k}s(i_0,\dots,\widehat{i_k},\dots,\widehat{i_j},\dots,i_{p+2})|_{U_{i_0}\cap\cdots\cap U_{i_{p+2}}} \\ &\qquad + \sum_{j<k\leqslant p+2}(-1)^{j+k-1}s(i_0,\dots,\widehat{i_j},\dots,\widehat{i_k},\dots,i_{p+2})|_{U_{i_0}\cap\cdots\cap U_{i_{p+2}}}\Bigg) = 0. \end{aligned}$$

□

In this way we define a cochain complex $(C^\bullet(\mathcal{U},\mathcal{F}), d)$ of abelian groups, called the **Čech complex**. The **p-th Čech-cohomology group $\check{H}^p(\mathcal{U},\mathcal{F})$ with respect to the cover $\mathcal{U}$** is then defined to be the p-th cohomology group of the Čech complex :

$$\check{H}^p(\mathcal{U},\mathcal{F}) := H^p(C^\bullet(\mathcal{U},\mathcal{F}), d^\bullet).$$

Lemma 8.2. The maps $C^p(\mathcal{U},\mathcal{F}) \to C^p(\mathcal{U},\mathcal{F}^\dagger)$ $(p \geqslant 0)$ induced by sheafification commute with coboundaries.

Proof. Recall from (2.4) that for each open $U \subseteq X$ the map $\mathcal{F}(U) \to \mathcal{F}^\dagger(U)$ is given by $s \mapsto [s_U : x \mapsto (x, s_x)]$.

□

Suppose $\mathcal{V} = \{V_j\}_{j\in J}$ is another open cover X that refines $\mathcal{U}$. Pick any map $\sigma : J \to I$ such that $V_j \subseteq U_{\sigma(j)}$ for all $j \in J$. The restriction induces a map on cochain group

$$\mathrm{ref}^{\mathcal{U},p}_{\mathcal{V},\sigma} : C^p(\mathcal{U},\mathcal{F}) \longrightarrow C^p(\mathcal{V},\mathcal{F})$$

given by

$$\mathrm{ref}^{\mathcal{U},p}_{\mathcal{V},\sigma}(s)(j_0,\dots,j_p) = s(\sigma(j_0),\dots,\sigma(j_p))|_{V_{j_0}\cap\cdots\cap V_{j_p}}.$$

Lemma 8.3. One has

$$d^{p+1} \circ \mathrm{ref}^{\mathcal{U},p}_{\mathcal{V},\sigma} = \mathrm{ref}^{\mathcal{U},p+1}_{\mathcal{V},\sigma} \circ d^p.$$

This shows $\mathrm{refs}^{\mathcal{U},\bullet}_{\mathcal{V},\sigma}$ induces a map on the Čech cohomology group, still denoted by

$$\mathrm{ref}^{\mathcal{U},\bullet}_{\mathcal{V},\sigma} : \check{H}^\bullet(\mathcal{U},\mathcal{F}) \longrightarrow \check{H}^\bullet(\mathcal{V},\mathcal{F}).$$

Proof.

□

The construction of $\mathrm{ref}^{\mathcal{U},\bullet}_{\mathcal{V},\sigma}$ depends on the map σ between index sets, but it turns out that the resulting map on cohomology levels is independent of σ. Suppose $\tau : J \to I$ is another map with $V_j \subseteq U_{\tau(j)}$. In particular, we have

$$V_j \subseteq U_{\sigma(j)} \cap U_{\tau(j)}.$$

Suppose s is a 1-cocycle (i.e. $ds = 0$) for the cover $\mathcal{U}$. Then

$$\begin{aligned}
\mathrm{ref}^{\mathcal{U},1}_{\mathcal{V},\sigma} s(j_0,j_1) &= s(\sigma(j_0),\sigma(j_1))\big|_{V_{j_0}\cap V_{j_1}} \\
\text{(omit the restriction)} &= s(\sigma(j_0),\tau(j_1)) - s(\sigma(j_1),\tau(j_1)) \\
&= \big(s(\sigma(j_0),\tau(j_0)) + s(\tau(j_0),\tau(j_1))\big) - s(\sigma(j_1),\tau(j_1)) \\
&= \mathrm{ref}^{\mathcal{U},1}_{\mathcal{V},\tau} s(j_0,j_1) + s(\sigma(j_0),\tau(j_0)) - s(\sigma(j_1),\tau(j_1)).
\end{aligned}$$

If we define $t \in C^1(\mathcal{V},\mathcal{F})$ by $t(j) := s(\sigma(j),\tau(j)) \in \mathcal{F}(V_j)$, the above computation shows

$$\mathrm{ref}^{\mathcal{U},1}_{\mathcal{V},\sigma} s - \mathrm{ref}^{\mathcal{U},1}_{\mathcal{V},\tau} s = d^0 t$$

i.e., two different ref differ only from a 1-coboundary. In general

Lemma 8.4. If s is a p-cocycle for the cover $\mathcal{U}$, then

$$\mathrm{ref}^{\mathcal{U},p}_{\mathcal{V},\sigma} s - \mathrm{ref}^{\mathcal{U},p}_{\mathcal{V},\tau} s = d^{p-1} t$$

where $t \in C^{p-1}(\mathcal{V},\mathcal{F})$ is given by

$$t(j_0,\dots,j_p) = \sum_{k=0}^{p-1} (-1)^k s(\sigma(j_0),\dots,\sigma(j_k),\tau(j_k),\dots,\tau(j_{p-1}))\big|_{V_{j_0}\cap\cdots\cap V_{j_p}}.$$

Proof.

□

This shows $\mathrm{ref}^{\mathcal{U},\bullet}_{\mathcal{V},\sigma}$ is in fact independent of the choice of σ in the cohomology level, and hence we can drop the subscript when mentioning the cohomology map :

$$\mathrm{ref}^{\mathcal{U},\bullet}_{\mathcal{V}} : \check{H}^\bullet(\mathcal{U},\mathcal{F}) \longrightarrow \check{H}^\bullet(\mathcal{V},\mathcal{F}).$$

This also shows that all $\check{H}^\bullet(\mathcal{U},\mathcal{F})$, with $\mathcal{U}$ ranging over all open covers of X, forms a directed system under the partial order given by the refinement. Now we can define the p**-th Čech cohomology group** by

$$\check{H}^\bullet(X,\mathcal{F}) = \varinjlim_{\mathcal{U}} \check{H}^\bullet(\mathcal{U},\mathcal{F}).$$

Introduce the **alternating Čech cochain group** :

$$C^p_{\text{alt}}(\mathcal{U},\mathcal{F}) = \left\{ s \in C^p(\mathcal{U},\mathcal{F}) \mid \begin{array}{l} s(i_0,\dots,i_p) = 0 \text{ if } i_n = i_m \text{ for some } n \neq m \\ s(i_{\sigma(0)},\dots,s_{\sigma(p)}) = \operatorname{sgn}(\sigma) s(i_0,\dots,i_p) \text{ for all } \sigma \in \mathfrak{S}_{p+1}. \end{array} \right\}$$

When $p = 1$, every 1-cocycle is automatically alternating : if $s \in C^1(\mathcal{U},\mathcal{F})$ with $ds = 0$, then $s(i_0,i_1) = s(i_0,i_2) - s(i_1,i_2)$, and in particular, if $i_0 = i_1 = i$, choosing $i_2 = i$ shows $s(i,i) = 0$; if we choose $i_2 = i_0$, then $s(i_0,i_1) = s(i_0,i_0) - s(i_1,i_0) = -s(i_1,s_0)$. This is no longer holds for $p > 1$.

Lemma 8.5. For $p \geqslant 0$, one has $d^p C^p_{\text{alt}}(\mathcal{U},\mathcal{F}) \subseteq C^{p+1}_{\text{alt}}(\mathcal{U},\mathcal{F})$.

Proof. Let $s \in C^p_{\text{alt}}(\mathcal{U},\mathcal{F})$. Then

$$(ds)(i_0,\dots,i_{p+1}) = \sum_{j=0}^{p+1} (-1)^j s(i_0,\dots,\widehat{i_j},\dots,i_{p+1})|_{U_{i_0}\cap\cdots\cap U_{i_{p+1}}}.$$

If $i_n = i_m$ for some $n \neq m$, this formula readily implies that $(ds)(i_0,\dots,i_{p+1})$. To show ds is alternating, it suffices to check

$$ds(i_0,\dots,i_n,\dots,i_m,\dots,i_{p+1}) = -ds(i_0,\dots,i_m,\dots,i_n,\dots,i_{p+1}).$$

for all $n < m$. □

Thus $(C^\bullet_{\text{alt}}(\mathcal{U},\mathcal{F}), d^\bullet)$ forms a subcomplex of $(C^\bullet(\mathcal{U},\mathcal{F}), d^\bullet)$. A significant result is the following

Theorem 8.6. The cohomology of $(C^\bullet_{\text{alt}}(\mathcal{U},\mathcal{F}), d^\bullet)$ is isomorphic to that of $(C^\bullet(\mathcal{U},\mathcal{F}), d^\bullet)$.

Proof. Let $<$ be a total order on I. For each $k \geqslant 0$, define $D^p_{(k)} \leqslant C^p(\mathcal{U},\mathcal{F})$ by

$$D^p_{(k)} = \{ s \in C^p(\mathcal{U},\mathcal{F}) \mid s(i_0,\dots,i_p) = 0 \text{ if } i_0 < \cdots < i_{p-k} \}.$$

Clearly we have $0 = D^p_{(p)} \subseteq D^p_{(1)} \subseteq \cdots \subseteq D^p_{(0)}$ and $d^p D^p_{(k)} \subseteq D^{(p+1)}_{(k)}$. Moreover, the inclusion induces an isomorphism

$$D^p_{(0)} \oplus C^p_{\text{alt}}(\mathcal{U},\mathcal{F}) \cong C^p(\mathcal{U},\mathcal{F}).$$

To see this, let $s \in C^p(\mathcal{U},\mathcal{F})$, and define $s_a \in C^p_{\text{alt}}(\mathcal{U},\mathcal{F})$ by

$$s_a(i_0,\dots,i_p) = \begin{cases} 0 & \text{, if } i_n = i_m \text{ for some } n \neq m \\ \operatorname{sgn}(\pi) s(i_{\pi(0)},\dots,i_{\pi(p)}) & \text{, if } \pi \in \mathfrak{S}_{p+1} \text{ is the unique element such that } i_{\pi(0)} < \cdots < i_{\pi(p)}. \end{cases}$$

It is clear from the construction that s_a is alternating and $s - s_a \in D^p_{(0)}$. This proves the above map is an isomorphism. In fact, we have $D^\bullet_{(0)} \oplus C^\bullet_{\text{alt}}(\mathcal{U},\mathcal{F}) \cong C^\bullet(\mathcal{U},\mathcal{F})$, and taking cohomology gives

$$\check{H}^\bullet(\mathcal{U},\mathcal{F}) \cong H^\bullet(C^\bullet_{\text{alt}}(\mathcal{U},\mathcal{F}), d^\bullet) \oplus H^\bullet(D^\bullet_{(0)}, d^\bullet).$$

We claim $H^\bullet(D^\bullet_{(0)}, d^\bullet) = 0$, i.e., $(D^\bullet_{(0)}, d^\bullet)$ is exact. It suffices to show the quotient complex $(D^\bullet_{(k)}/D^\bullet_{(k+1)}, d^\bullet)$ is exact for each k.

There is an isomorphism induced by the projection

$$D^p_{(k)}/D^p_{(k+1)} \cong \left\{ s \in \prod_{\substack{i_0,\dots,i_p \in I \\ i_0 < \cdots < i_{p-k-1}}} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p}) \mid s(i_0,\dots,i_p) = 0 \text{ if } i_0 < \cdots < i_{p-k} \right\}$$

The right-handed side also defines a complex with coboundary map defined as before, and this isomorphism in fact an isomorphism of these two complexes. From now on we identify $(D^\bullet_{(k)}/D^\bullet_{(k+1)}, d^\bullet)$ with the right-handed side. Define

$$h^p : D^p_{(k)}/D^p_{(k+1)} \longrightarrow D^{p-1}_{(k)}/D^{p-1}_{(k+1)}$$

as follows. For $i_0, \dots, i_{p-1} \in I$ with $i_0 < \cdots < i_{p-k-2}$, set

$$\begin{aligned} &h^p s(i_0, \dots, i_{p-1}) \\ &= \begin{cases} 0 & \text{, if } i_{p-k-1} = i_n \text{ for some } 0 \leqslant n \leqslant p-k-2 \\ (-1)^{n+1} s(i_0, \dots, i_n, i_{p-k-1}, i_{n+1}, \dots, i_{p-1}) & \text{, if } i_n < i_{p-k-1} < i_{n+1} \text{ for some } 0 \leqslant n \leqslant p-k-3 \\ s(i_{p-k-1}, i_0, \dots, i_{p-1}) & \text{, if } i_{p-k-1} < i_0 \\ 0 & \text{, if } i_{p-k-1} > i_{p-k-2} \end{cases} \end{aligned}$$

We show $d^{p-1} \circ h^p + h^{p+1} \circ d^p = \mathrm{id}$, proving the exactness of $(D^\bullet_{(k)}/D^\bullet_{(k+1)}, d^\bullet)$. Let $i_0, \dots, i_p \in I$ with $i_0 < \ldots < i_{p-k-1}$. We have

$$\begin{aligned} d^{p-1} \circ h^p(s)(i_0, \dots, i_p) &= \sum_{j=0}^{p} (-1)^j h^p s(i_0, \dots, \widehat{i_j}, \dots, i_p) \\ &= \sum_{0 \leqslant j \leqslant p-k-1} (-1)^j h^p s(i_0, \dots, \widehat{i_j}, \dots, i_{p-k-1}, \dots, i_p) \\ &\quad + \sum_{p-k-1 < j \leqslant p} (-1)^j h^p s(i_0, \dots, i_{p-k-1}, \dots, \widehat{i_j}, \dots, i_p) \\ &= \sum_{0 \leqslant j \leqslant p-k-1} (-1)^j h^p s(i_0, \dots, \widehat{i_j}, \dots, i_{p-k-1}, \dots, i_p). \end{aligned}$$

There are four cases.

— $i_{p-k} = i_n$ for some $0 \leqslant n \leqslant p-k-1$. Then

$$h^{p+1} \circ d^p(s)(i_0, \dots, i_p) = 0$$

and

$$d^{p-1} \circ h^p(s)(i_0, \dots, i_p) = (-1)^n (-1)^n s(i_0, \dots, i_{n-1}, i_{p-k}, i_{n+1}, \dots, i_p) = s(i_0, \dots, i_p).$$

So in this case the homotopy relation holds.

— $i_n < i_{p-k} < i_{n+1}$ for some $0 \leqslant n \leqslant p-k-2$. Then

$$\begin{aligned} d^{p-1} \circ h^p(s)(i_0, \dots, i_p) &= \sum_{0 \leqslant j \leqslant p-k-1} (-1)^j h^p s(i_0, \dots, \widehat{i_j}, \dots, i_{p-k-1}, \dots, i_p) \\ &= \sum_{0 \leqslant j \leqslant n} (-1)^j (-1)^n s(i_0, \dots, \widehat{i_j}, \dots, i_n, i_{p-k}, i_{n+1}, \dots,) + \\ &\quad + \sum_{n+1 \leqslant j \leqslant p-k-1} (-1)^j (-1)^{n+1} s(i_0, \dots, i_n, i_{p-k}, i_{n+1}, \dots, \widehat{i_j}, \dots, i_{p-k-1}, \dots) \end{aligned}$$

$$
\begin{aligned}
h^{p+1} \circ d^p(s)(i_0, \ldots, i_p) &= (-1)^{n+1} d^p s(i_0, \ldots, i_n, i_{p-k}, i_{n+1}, \ldots, i_{p-k-1}, \ldots, i_p) \\
&= (-1)^{n+1} \left(\sum_{0 \leqslant j \leqslant n} (-1)^j s\left(i_0, \ldots \widehat{i_j}, \ldots, i_n, i_{p-k}, i_{n+1}, \ldots\right) + (-1)^{n+1} s(i_0, \ldots, i_p) \right. \\
&\quad \left. + \sum_{n+1 \leqslant j \leqslant p} (-1)^{j+1} s(i_0, \ldots, i_n, i_{p-k}, i_{n+1}, \ldots, \widehat{i_j}, \ldots) \right) \\
&= s(i_0, \ldots, i_p) + (-1)^{n+1} \sum_{0 \leqslant j \leqslant n} (-1)^j s\left(i_0, \ldots \widehat{i_j}, \ldots, i_n, i_{p-k}, i_{n+1}, \ldots\right) \\
&\quad + (-1)^{n+1} \sum_{n+1 \leqslant j \leqslant p-k-1} (-1)^{j+1} s(i_0, \ldots, i_n, i_{p-k}, i_{n+1}, \ldots, \widehat{i_j}, \ldots, i_{p-k-1}, \ldots, i_p)
\end{aligned}
$$

Hence the homotopy relation holds as well.

— $i_{p-k} > i_{p-k-1}$. Each term is zero.

— $i_{p-k} < i_0$. Then

$$
d^{p-1} \circ h^p(s)(i_0, \ldots, i_p) = \sum_{0 \leqslant j \leqslant p-k-1} (-1)^j s(i_{p-k}, i_0, \ldots, \widehat{i_j}, \ldots, i_p)
$$

$$
\begin{aligned}
h^{p+1} \circ d^p(s)(i_0, \ldots, i_p) &= d^p(s)(i_{p-k}, i_0, \ldots, i_p) \\
&= s(i_0, \ldots, i_p) - \sum_{0 \leqslant j \leqslant p-k-1} (-1)^j s(i_{p-k}, i_0, \ldots, \widehat{i_j}, \ldots, i_{p-k-1}, \ldots, i_p).
\end{aligned}
$$

Thus the homotopy relation holds.

In any case we see $d^{p-1} \circ h^p + h^{p+1} \circ d^p = \mathrm{id}$ holds, and this completes the proof. □

There is an equivalent formulation of the alternating Čech complex. Let $<$ be a total order on I. Define the **ordered Čech complex** $(C^\bullet_{\mathrm{ord}}(\mathcal{U}, \mathcal{F}), d^\bullet)$ as follows : set

$$
C^p_{\mathrm{ord}}(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p})
$$

and set d^p as before. There is a natural projection

$$
\pi^p : C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^p_{\mathrm{ord}}(\mathcal{U}, \mathcal{F}).
$$

Its restriction to $C^p_{\mathrm{alt}}(\mathcal{U}, \mathcal{F})$ is an isomorphism of complexes with inverse

$$
\psi^p : C^p_{\mathrm{ord}}(\mathcal{U}, \mathcal{F}) \longrightarrow C^p_{\mathrm{alt}}(\mathcal{U}, \mathcal{F})
$$

given by

$$
\psi^p(s)(i_0, \ldots, i_p) = \begin{cases} 0 & \text{, if } i_n = i_m \text{ for some } n \neq m \\ \mathrm{sgn}(\pi) s(i_{\pi(0)}, \ldots, i_{\pi(p)}) & \text{, if } \pi \in \mathfrak{S}_{p+1} \text{ is the unique element such that } i_{\pi(0)} < \cdots < i_{\pi(p)}. \end{cases}
$$

The last theorem together with the above discussion leads to

Theorem 8.7. Let X be a topological space, $\mathcal{U}$ an open cover of X and $\mathcal{F}$ a presheaf on X. Then all arrows below are quasi-isomorphisms.

$$
C^p_{\mathrm{alt}}(\mathcal{U}, \mathcal{F}) \xrightarrow{\text{inclusion}} C^p(\mathcal{U}, \mathcal{F}) \xrightarrow{\text{projection}} C^p_{\mathrm{ord}}(\mathcal{U}, \mathcal{F}).
$$

8.2 Basic properties

8.2.1 Long exact sequence for cohomology

An easy argument shows that Čech cohomology is a δ-functor from the category of *presheaves* of abelian groups over X to that of abelian groups. Explicitly, for every short exact sequence

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

of presheaves, and for every open cover $\mathcal{U}$ of X, there exists a long exact sequence on the cohomology groups :

$$0 \longrightarrow \check{H}^0(\mathcal{U}, \mathcal{F}_1) \longrightarrow \check{H}^0(\mathcal{U}, \mathcal{F}_2) \longrightarrow \check{H}^0(\mathcal{U}, \mathcal{F}_3) \xrightarrow{\delta} \check{H}^1(\mathcal{U}, \mathcal{F}_1) \longrightarrow \check{H}^1(\mathcal{U}, \mathcal{F}_2) \longrightarrow \cdots$$

Moreover, by passing to refinement we obtain (recall that $\varinjlim$ is exact in the category of $\mathbb{Z}$-modules)

$$0 \longrightarrow \check{H}^0(X, \mathcal{F}_1) \longrightarrow \check{H}^0(X, \mathcal{F}_2) \longrightarrow \check{H}^0(X, \mathcal{F}_3) \xrightarrow{\delta} \check{H}^1(X, \mathcal{F}_1) \longrightarrow \check{H}^1(X, \mathcal{F}_2) \longrightarrow \cdots$$

But in applications, we are interested in the cohomology of *sheaves*, and a short exact sequence of sheaves is barely exact as a sequence of presheaves. But the long exact cohomology sequence continues to hold in reasonable cases. Say

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

is an exact sequence of *sheaves*. Define a subpresheaf $\mathcal{F}_3^*$ of $\mathcal{F}_3$ by

$$\mathcal{F}_3^*(U) := \operatorname{Im}\left(\mathcal{F}_2(U) \to \mathcal{F}_3(U)\right).$$

Then we obtain a short exact sequence of presheaves

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3^* \longrightarrow 0$$

whence a long exact sequence

$$\cdots \longrightarrow \check{H}^i(X, \mathcal{F}_1) \longrightarrow \check{H}^i(X, \mathcal{F}_2) \longrightarrow \check{H}^i(X, \mathcal{F}_3^*) \xrightarrow{\delta} \check{H}^{i+1}(X, \mathcal{F}_1) \longrightarrow \cdots$$

Now $\mathcal{F}_3$ is the sheafification of $\mathcal{F}_3^*$, so we can replace $\mathcal{F}_3^*$ by $\mathcal{F}_3$ in the above sequence if we can prove the following assertion

- $\diamond$ for all presheaves $\mathcal{F}$ over X, the canonical maps (c.f. (8.2))

$$\check{H}^i(X, \mathcal{F}) \longrightarrow \check{H}^i(X, \mathcal{F}^\dagger)$$

are isomorphisms.

The statement $(\diamond)$ will follow if the following assertion holds :

- $\diamond\diamond$ If $\mathcal{F}$ is a presheaf over X such that $\mathcal{F}^\dagger = 0$, then $\check{H}^i(X, \mathcal{F}) = 0$.

Proposition 8.8. $(\diamond\diamond)$ holds when X is a Hausdorff paracompact space.

Proof. Since X is paracompact, locally finite covers of X form a cofinal system among all open covers of X, and hence it suffices to compute Čech cohomology group with respect to every locally finite cover. So let $\mathcal{U} = \{U_i\}_i$ be such a cover. Each point $x \in X$ is contained in some U_i, and by regularity (a paracompact space is normal, and hence regular) we can find an open $W_x \ni x$ with $W_x \subseteq \overline{W_x} \subseteq U_i$. Pick a locally finite open refinement $\mathcal{W}$ of $\{W_x\}$. Define $f : \mathcal{W} \to I$ by requiring $W \subseteq U_{f(W)}$ for each $W \in \mathcal{W}$, and set

$$V_i := \bigcup_{f(W)=i} W$$

Then $V_i \subseteq \overline{V_i} \subseteq U_i$. We claim $\mathcal{V} = \{V_i\}_{i\in I}$ is locally finite. Let $x \in X$ and pick a neighborhood N of x such that $\#\{W \in \mathcal{W} \mid W \cap N \neq \varnothing\} < \infty$. If $N \cap V_i$ is nonempty, then N intersects nontrivially with some W with $f(W) = i$, and vice versa. Hence

$$\#\{i \in I \mid V_i \cap N = \varnothing\} \leqslant \#\{W \in \mathcal{W} \mid W \cap N \neq \varnothing\} < \infty$$

and this proves the locally finiteness of $\mathcal{V}$.

Now we prove $(\diamond\diamond)$. Let $s \in C^p(\mathcal{U}, \mathcal{F})$. Let $x \in X$; by locally finiteness we can find a neighborhood N_x such that

(i) $x \in V_i$ implies $N_x \subseteq V_i$,

(ii) $N_x \cap V_i \neq \varnothing$ implies $N_x \subseteq V_i$, and

(iii) $x \in U_{i_0} \cap \cdots \cap U_{i_p}$ implies $N_x \subseteq U_{i_0} \cap \cdots \cap U_{i_p}$ and $s(i_0, \ldots, i_p)|_{N_x} = 0$.

Indeed, (i) and (iii) follow from locally finiteness and assumption. For (ii), we can first pick N'_x so that it intersects only with finitely many V_i and then choose

$$N_x \subseteq N'_x \cap \bigcup_{x\in V_i} V_i \cap \bigcup_{x\notin V_i} (N'_x \backslash \overline{V}_i).$$

Now take $\mathcal{W} = \{N_x\}_{x\in X}$. Then $\mathcal{W}$ refines $\mathcal{V}$ and $\mathrm{ref}^{\mathcal{U},p}_{\mathcal{W}}(s) = 0$ as a cochain. Indeed, suppose $N_{x_0} \cap \cdots \cap N_{x_p} \neq \varnothing$ and pick $i_0, \ldots, i_p \in I$ with $N_{x_n} \subseteq V_{i_n}$. We must show $s(i_0, \ldots, i_p)|_{N_{x_0}\cap\cdots\cap N_{x_p}} = 0$. But $N_{x_0} \cap \cdots \cap N_{x_p} \subseteq N_{x_n} \subseteq V_{i_n}$ implies $N_{x_0} \cap V_{i_n} \neq \varnothing$, so by (ii) $N_{x_0} \subseteq V_{i_n}$ for each $0 \leqslant n \leqslant p$. Hence

$$s(i_0, \ldots, i_p)|_{N_{x_0}\cap\cdots\cap N_{x_p}} = \big(s(i_0, \ldots, i_p)|_{N_{x_0}}\big)\big|_{N_{x_0}\cap\cdots\cap N_{x_p}} = 0$$

by (iii). $\square$

It is sad that we do not have a long exact sequence for Čech cohomology for sheaves in general. Nevertheless, this is true in low dimension.

Proposition 8.9. Let X be a topological space and

8.2.2 Comparison between refinement

Proposition 8.10. Let X be a topological space, $\mathcal{F}$ a *sheaf* of abelian groups on X, and $\mathcal{U} = \{U_i\}_{i\in I}$ and $\mathcal{V} = \{V_j\}_{j\in I}$ two open covers of X. Suppose $\mathcal{V}$ refines $\mathcal{U}$. For every finite subset $I_0 \subseteq I$, put $U_{I_0} = \bigcap_{i\in I_0} U_i$, and denote by $\mathcal{V}|_{U_{S_0}}$ the cover of U_{S_0} induced by $\mathcal{V}$. Assume

$$\check{H}^p(\mathcal{V}|_{U_{S_0}}, \mathcal{F}|_{U_{S_0}}) = 0 \text{ for all } S_0 \text{ and } p > 0.$$

Then $\mathrm{ref}^{\mathcal{U}\bullet}_{\mathcal{V}} : \check{H}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{H}^\bullet(\mathcal{V}, \mathcal{F})$ is an isomorphism.

Proof. We compare two cohomologies by double complexes. For $p, q \geqslant 0$, define

$$C^{p,q} = \prod_{i_0,\ldots,i_p\in I} \prod_{j_0,\ldots,j_q\in J} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p} \cap V_{j_0} \cap \cdots \cap V_{j_q}).$$

Define two differentials $d_1 : C^{p,q} \to C^{p+1,q}$ and $d_2 : C^{p,q} \to C^{p,q+1}$ by the usual differentials but ignoring either $i_0, \ldots, i_p$ or $j_0, \ldots, j_q$. Then $d_i^2 = 0$, $i = 1, 2$ and $d_1 d_2 = d_2 d_1$, and we obtain a commutative first quadrant double complex

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
C^{0,2} \arrow[r, "d_1"] \arrow[u, "d_2"] & C^{1,2} \arrow[r, "d_1"] \arrow[u, "d_2"] & C^{2,2} \arrow[r] \arrow[u, "d_2"] & \cdots \\
C^{0,1} \arrow[r, "d_1"] \arrow[u, "d_2"] & C^{1,1} \arrow[r, "d_1"] \arrow[u, "d_2"] & C^{2,1} \arrow[r] \arrow[u, "d_2"] & \cdots \\
C^{0,0} \arrow[r, "d_1"] \arrow[u, "d_2"] & C^{1,0} \arrow[r, "d_1"] \arrow[u, "d_2"] & C^{2,0} \arrow[r] \arrow[u, "d_2"] & \cdots
\end{tikzcd}$$

Put

$$C^{(n)} = \mathrm{tot}(C^{\bullet,\bullet})^n = \sum_{\substack{p+q=n \\ p,q\geqslant 0}} C^{p,q}$$

to be the total complex of this double complex, and put

$$d := \sum_{\substack{p+q=n \\ p,q\geqslant 0}} (d_1 + (-1)^p d_2) : C^{(n)} \longrightarrow C^{(n+1)}$$

to be the differential of $C^{(n)}$. By our assumption, the d_2-cohomology of columns vanish for all $p \geqslant 0$ and $q > 0$. On the other hand, the d_1-cohomology of rows also vanish for all $p > 0$ and $q \geqslant 0$. This is because the cover used in that Čech complex contains the whole space, and the following lemma witnesses the vanishing.

Lemma 8.11. Let X be a topological space, $\mathcal{F}$ a sheaf of abelian groups, and $\mathcal{U}$ an open cover of X with $X \in \mathcal{U}$. Then $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$.

Proof. Say $X = U_\zeta$ for some $\zeta \in I$. If s is a p-cocycle, define $t \in C^{p-1}(\mathcal{U}, \mathcal{F})$ by

$$t(i_0, \dots, i_{p-1}) = s(\zeta, i_0, \dots, i_{p-1}).$$

Then $dt = s$. Indeed,

$$dt(i_0, \dots, i_p) = \sum_{j=0}^{p} (-1)^j s(\zeta, i_0, \dots, \widehat{i_j}, \dots, i_p).$$

On the other hand,

$$0 = ds(\zeta, i_0, \dots, i_p) = s(i_0, \dots, i_p) - \sum_{j=0}^{p} (-1)^j s(\zeta, i_0, \dots, \widehat{i_j}, \dots, i_p).$$

so summing these two up gives $dt = s$. $\square$

To finish the proof, we use the following homological lemma whose proof is omitted.

Lemma 8.12. Consider the following double complex

$$\begin{array}{ccccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F^3 & \longrightarrow & K^{0,3} & \longrightarrow & K^{1,3} & \longrightarrow & K^{2,3} & \longrightarrow & K^{3,3} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F^2 & \longrightarrow & K^{0,2} & \longrightarrow & K^{1,2} & \longrightarrow & K^{2,2} & \longrightarrow & K^{3,2} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F^1 & \longrightarrow & K^{0,1} & \longrightarrow & K^{1,1} & \longrightarrow & K^{2,1} & \longrightarrow & K^{3,1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F^0 & \longrightarrow & K^{0,0} & \longrightarrow & K^{1,0} & \longrightarrow & K^{2,0} & \longrightarrow & K^{3,0} & \longrightarrow & \cdots
\end{array}$$

with all rows exact. Then the natural map $F \to \mathrm{tot}(K)$ is a quasi-isomorphism.

Since $\mathcal{F}$ is a sheaf, the kernel $d_1 : C^{0,q} \to C^{1,q}$ is exactly $C^q(\mathcal{V}, \mathcal{F})$, and similarly $\ker(d_2 : C^{p,0} \to C^{p,1}) = C^p(\mathcal{U}, \mathcal{F})$. These natural inclusions give rise to the isomorphisms

$$\check{H}^\bullet(\mathcal{V}, \mathcal{F}) \xrightarrow{\sim} H^\bullet(C^{(\bullet)}, d)$$

and

$$\check{H}^\bullet(\mathcal{U}, \mathcal{F}) \xrightarrow{\sim} H^\bullet(C^{(\bullet)}, d).$$

It remains to show the composition

$$\check{H}^n(\mathcal{U}, \mathcal{F}) \longrightarrow H^n(C^{(\bullet)}, d) \longrightarrow \check{H}^n(\mathcal{U}, \mathcal{F})$$

coincides with $\mathrm{ref}^{\mathcal{U},n}_{\mathcal{V}}$. Let s be an n-cocycle for $\mathcal{U}$. Then $s \in C^{n,0}$ and $\mathrm{ref}^{\mathcal{U},n}_{\mathcal{V}}(s) \in C^{0,n}$. It is enough to show they are cohomologous in the total complex. To be precise, we have $\mathrm{ref}^{\mathcal{U},n}_{\mathcal{V}}(s) - s \in C^{(n)}$, and we show there exists $t \in C^{(n-1)}$ such that $dt = \mathrm{ref}^{\mathcal{U},n}_{\mathcal{V}}(s) - s$. Define $t \in C^{(n-1)}$ by setting its $(\ell, n-1-\ell)$-th component t_ℓ to be

$$t_\ell(i_0, \ldots, i_\ell, j_0, \ldots, j_{n-1-\ell}) = (-1)^\ell s(i_0, \ldots, i_\ell, \sigma(j_0), \ldots, \sigma(j_{n-1-\ell}))\big|_{U_{\{i_0,\ldots,i_\ell\}} \cap V_{\{j_0,\ldots,j_{n-1-\ell}\}}}$$

where $\sigma : J \to I$ is a map such that $V_j \subseteq U_{\sigma(j)}$. Then

$$\begin{aligned}
d_1 t_{n-1}(i_0, \ldots, i_n, j_0) &= \sum_{k=0}^{n} (-1)^k t_{n-1}(i_0, \ldots, \widehat{i_k}, \ldots, i_n, j_0) \\
&= \sum_{k=0}^{n} (-1)^k (-1)^{n-1} s(i_0, \ldots, \widehat{i_k}, \ldots, i_n, \sigma(j_0)) \\
\text{(cocycle condition)} &= -s(i_0, \ldots, i_n)
\end{aligned}$$

for $0 \leqslant \ell \leqslant n-2$

$$\begin{aligned}
&d_1 t_\ell(i_0, \ldots, i_\ell, i_{\ell+1}, j_0, \ldots, j_{n-1-\ell}) + (-1)^{\ell+1} d_2 t_{\ell+1}(i_0, \ldots, i_\ell, i_{\ell+1}, j_0, \ldots, j_{n-1-\ell}) \\
&\quad = \sum_{k=0}^{\ell+1} (-1)^k (-1)^\ell s(i_0, \ldots, \widehat{i_k}, \ldots, i_{\ell+1}, \sigma(j_0), \ldots, \sigma(j_{n-1-\ell})) \\
&\qquad + (-1)^{\ell+1} \sum_{k=0}^{n-1-\ell} (-1)^k (-1)^{\ell+1} s(i_0 \ldots, i_{\ell+1}, \sigma(j_0), \ldots, \widehat{\sigma(j_k)}, \ldots, \sigma(j_{n-1-\ell})) \\
&\quad = (-1)^\ell ds(i_0, \ldots, i_{\ell+1}, \sigma(j_0), \ldots, \sigma(j_{n-1-\ell})) = 0
\end{aligned}$$

and

$$\begin{aligned} d_2 t_0(i_0, j_0, \dots, j_n) &= \sum_{k=0}^{n} (-1)^k s(i_0, \sigma(j_0), \dots, \widehat{\sigma(j_k)}, \dots, \sigma(j_n)) \\ \text{(cocycle condition)} &= s(\sigma(j_0), \dots, \sigma(j_n)) = \mathrm{ref}^{\mathcal{U},n}_{\mathcal{V},\sigma}(s)(j_0, \dots, j_n) \end{aligned}$$

The above computation shows $dt = \mathrm{ref}^{\mathcal{U},n}_{\mathcal{V},\sigma}(s) - s$, and the proof is completed. □

8.2.3 Sheafified Čech complex

Let X be a topological space, $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover and $\mathcal{F}$ a *presheaf* of abelian groups. For $p \geqslant 0$, we define a presheaf $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$ by

$$\mathcal{C}^p(\mathcal{U}, \mathcal{F})(V) = \prod_{i_0,\dots,i_p \in I} \mathcal{F}(V \cap U_{i_0} \cap \dots \cap U_{i_p}).$$

and for two open $V_1 \subseteq V_2 \subseteq X$, two groups are connected with restriction :

$$\begin{tikzcd} \mathcal{C}^p(\mathcal{U},\mathcal{F})(V_2) \arrow[rr] \arrow[d, equal] \arrow[drr, phantom, "\circlearrowleft"] & & \mathcal{C}^p(\mathcal{U},\mathcal{F})(V_1) \arrow[d, equal] \\ \prod_{i_0,\dots,i_p \in I} \mathcal{F}(V_1 \cap U_{i_0} \cap \dots \cap U_{i_p}) \arrow[rr, "\text{restriction}"] & & \prod_{i_0,\dots,i_p \in I} \mathcal{F}(V_2 \cap U_{i_0} \cap \dots \cap U_{i_p}) \end{tikzcd}$$

Equivalently, if for each tuple $(i_0, \dots, i_p) \in I^{p+1}$ we put $\iota_{i_0,\dots,i_p} : U_{i_0} \cap \dots \cap U_{i_p} \to X$ to be the natural inclusion, then

$$\mathcal{C}^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0,\dots,i_p \in I} (\iota_{i_0,\dots,i_p})_* \left(\mathcal{F}|_{U_{i_0} \cap \dots \cap U_{i_p}} \right)$$

The Čech differential d on each $\mathcal{C}^p(\mathcal{U}, \mathcal{F})(V)$ is packed together to a morphism

$$d^p : \mathcal{C}^p(\mathcal{U}, \mathcal{F}) \longrightarrow \mathcal{C}^{p+1}(\mathcal{U}, \mathcal{F})$$

and we obtain a complex of presheaves :

$$0 \longrightarrow \mathcal{F} \xrightarrow{\ \epsilon\ } \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\ d^0\ } \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\ d^1\ } \mathcal{C}^2(\mathcal{U}, \mathcal{F}) \longrightarrow \cdots$$

where $\epsilon : \mathcal{F} \to \mathcal{C}^0(\mathcal{U}, \mathcal{F})$ is the product of maps $\mathcal{F} \to (\iota_i)_* (\mathcal{F}|_{U_i})$ $(i \in I)$. Observe the following properties.

— If $\mathcal{F}$ is a sheaf (resp. flasque sheaf), so is each $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$.

— $\mathcal{F}$ is a sheaf if and only if the above complex is exact at the first two places.

— $\Gamma(X, \mathcal{C}^p(\mathcal{U}, \mathcal{F})) = C^p(\mathcal{U}, \mathcal{F})$.

In fact, more is true.

Lemma 8.13. For any sheaf $\mathcal{F}$ of abelian groups on X, the sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{\ \epsilon\ } \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\ d^0\ } \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\ d^1\ } \mathcal{C}^2(\mathcal{U}, \mathcal{F}) \longrightarrow \cdots$$

is exact as sheaves.

Proof. The first two places are already exact. To show the exactness at remaining places, we check them on the stalks. To this end, we construct $k^0 : \mathcal{C}^0(\mathcal{U}, \mathcal{F})_x \to \mathcal{F}_x$ and for each $p \geqslant 1$

$$k^p : \mathcal{C}^p(\mathcal{U}, \mathcal{F})_x \longrightarrow \mathcal{C}^{p-1}(\mathcal{U}, \mathcal{F})_x$$

and prove $\epsilon \circ k^0 + k^1 \circ d^0 = \mathrm{id}$ and $d^{p-1} \circ k^p + k^{p+1} \circ d^p = \mathrm{id}$.

Let $x \in X$ be given and an index $j \in I$ such that $x \in U_j$. Given $s_x \in \mathcal{C}^p(\mathcal{U}, \mathcal{F})_x$, it is represented by some $s \in \Gamma(V, \mathcal{C}^p(\mathcal{U}, \mathcal{F})_x)$ for some neighborhood V of x; choose V so small that $V \subseteq U_j$. Now for $i_0, \ldots, i_{p-1} \in I$, put

$$k^p s(i_0, \ldots, i_{p-1}) = s(j, i_0, \ldots, i_{p-1})\big|_x.$$

This makes sense as $V \cap U_{i_0} \cap \cdots U_{i_{p-1}} = V \cap U_j \cap U_{i_0} \cap \cdots U_{i_{p-1}}$, and this map is independent of the choice of $V \subseteq U_j$. Now compute

$$k^{p+1} \circ d^p s(i_0, \ldots, i_p) = d^p s(j, i_0, \ldots, i_p)|_x$$

$$\begin{aligned} d^{p-1} \circ k^p s(i_0, \ldots, i_p) &= \sum_{k=0}^{p} (-1)^k k^p s(i_0, \ldots, \widehat{i_k}, \ldots, i_p) \\ &= \sum_{k=0}^{p} (-1)^k s(j, i_0, \ldots, \widehat{i_k}, \ldots, i_p) = s(i_0, \ldots, i_p) - d^p s(j, i_0, \ldots, i_p) \end{aligned}$$

and

$$\epsilon \circ k^0 s(i) = k^0 s(i) = s(j, i).$$

These proves the homotopy relations, and the proof is completed. □

Corollary 8.13.1. Let X denote a topological space, $\mathcal{U}$ an open cover of X and $\mathcal{F}$ a sheaf of abelian groups on X. Then for any $p \geqslant 0$ there exists a natural map $\check{H}^p(\mathcal{U}, \mathcal{F}) \to H^p(X, \mathcal{F})$ functorial in $\mathcal{F}$.

Proof. Let $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ denote an injective resolution of $\mathcal{F}$ in $\mathbf{Mod}_{\mathbb{Z}_X}$. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, "\mathrm{id}"] & \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \arrow[r] & \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \arrow[r] & \cdots \\ 0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{I}^0 \arrow[r] & \mathcal{I}^1 \arrow[r] & \cdots \end{tikzcd}$$

A general result says that there exists a morphism of complexes $C^\bullet(\mathcal{U}, \mathcal{F}) \to \mathcal{I}^\bullet$ making the above diagram commute, and the morphism is unique up to homotopy. Now the result follows once we taking global sections and cohomology. The functoriality follows again from the fact above. □

Corollary 8.13.2. Let X denote a topological space and $\mathcal{U}$ any open cover of X. If $\mathcal{F}$ is a flasque sheaf, then $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p \geqslant 1$.

Proof. Since $\mathcal{F}$ is flasque, the sheafified Čech complex is a flasque, hence acyclic, resolution of $\mathcal{F}$, and thus it computes the sheaf cohomology. But since $\mathcal{F}$ is flasque, by Lemma 7.2 $\mathcal{F}$ is acyclic, and the result follows. □

Corollary 8.13.3. Let X be a topological space and $\mathcal{F}$ a sheaf of abelian groups. If for *any* open U and any open cover $\mathcal{U}$ of U, we have $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for $p \geqslant 1$, then $H^p(U, \mathcal{F}) = 0$ for all opens $U \subseteq X$.

Proof. Take an embedding $j : \mathcal{F} \to \mathcal{I}$ into some flasque sheaf $\mathcal{I}$, and take their quotient $\mathcal{Q}$ in $\mathbf{Ab}_X$; then we have a short exact sequence in $\mathbf{Ab}_X$

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{Q} \longrightarrow 0$$

By our assumption, this is in fact an exact sequence in $\mathbf{Ab}_X^{\mathrm{pre}}$. Indeed, for any open $U \subseteq X$ and $s \in \mathcal{Q}(U)$, take an open cover $\mathcal{U}$ of U such that $s|_V$ comes from an element t_V in $\mathcal{I}(V)$ for any $V \in \mathcal{U}$. On $V \cap W$ with $V, W \in \mathcal{U}$, the element $t_V|_{V \cap W} - t_W|_{V \cap W}$ maps to 0 in $\mathcal{Q}(V \cap W)$, so $t_V|_{V \cap W} - t_W|_{V \cap W}$ comes from an element $u_{VW} \in \mathcal{F}(V \cap W)$. Since $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$ by assumption, there exists $(u_V)_V \in C^1(\mathcal{U}, \mathcal{F})$ such that $u_V|_{V\cap W} - u_W|_{V\cap W} = u_{VW}$ for any $V, W \in \mathcal{U}$. Then $(j_V(u_V) - t_V)_V$ glues to $t \in \mathcal{I}(U)$ which maps down to s. (Alternatively this can be deduced by combining (8.16) and the long exact sequence of sheaf cohomology.) Hence we have a short exact sequence

$$0 \longrightarrow C^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow C^\bullet(\mathcal{U}, \mathcal{I}) \longrightarrow C^\bullet(\mathcal{U}, \mathcal{Q}) \longrightarrow 0$$

and therefore a long exact sequence in Čech cohomology. It follows from Corollary 8.13.2 that $\check{H}^p(\mathcal{U}, \mathcal{Q}) = 0$ for $p \geqslant 1$, for any open U and any open cover $\mathcal{U}$ of U.

Now for any open $U \subseteq X$ and $p \geqslant 0$ consider the exact sequence of sheaf cohomology

$$H^p(U, \mathcal{F}) \longrightarrow H^p(U, \mathcal{I}) \longrightarrow H^p(U, \mathcal{Q}) \longrightarrow H^{p+1}(U, \mathcal{F}) \longrightarrow 0 = H^{p+1}(U, \mathcal{I}).$$

Here $H^{p+1}(U, \mathcal{I}) = 0$ as $\mathcal{I}$ is flasque. When $p = 0$ we have just proved the map $H^0(U, \mathcal{I}) \to H^0(U, \mathcal{Q})$ is surjective, so $H^1(U, \mathcal{F}) = 0$. Since $\mathcal{F}$ is arbitrary, $H^1(U, \mathcal{Q}) = 0$ as well. But then $H^2(U, \mathcal{F}) = 0$, hence $H^2(U, \mathcal{Q}) = 0$. Continuing this process we deduce $H^p(U, \mathcal{F}) = 0$ for $p \geqslant 1$. □

For the sake of completeness and future use, the statement and the proof of the homological lemma used in (8.13.1) is included here.

Lemma 8.14. Let $\mathcal{C}$ be an abelian category with enough injective. Let $\varphi : M \to N$ be a morphism in $\mathcal{C}$. Suppose $N \to I^\bullet$ is an injective resolution of N and $M \to E^\bullet$ is an arbitrary resolution. Then there exists a map $E^\bullet \to I^\bullet$ of chain complexes making the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\varphi"] & E^0 \arrow[r] \arrow[d] & E^1 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & N \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots
\end{tikzcd}$$

commutes. Moreover, any such two maps of chain complexes are homotopic.

8.3 Comparison to sheaf cohomology

Theorem 8.15. Let X be a topological space, $\mathcal{U} = \{U_i\}_{i\in I}$ an open cover of X and $\mathcal{F}$ a sheaf of abelian groups on X. Suppose that for any $p > 0$ and any finite intersection $V = U_{i_0} \cap \cdots \cap U_{i_k}$ of elements in $\mathcal{U}$, we have $H^p(V, \mathcal{F}|_V) = 0$. Then for any $p \geqslant 0$, the natural map in Corollary 8.13.1 is an isomorphism.

Proof. When $p = 0$, the natural map $\check{H}^p(\mathcal{U}, \mathcal{F}) \to H^0(X, \mathcal{F})$ is induced by the identity map $\mathrm{id} : \mathcal{F}(X) \to \mathcal{F}(X)$. It follows from definition that $\check{H}^p(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X) = H^0(X, \mathcal{F})$, so the theorem holds for $p = 0$.

For $p > 0$, embed $\mathcal{F}$ into a flasque sheaf $\mathcal{I}$ and consider the resulting short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{G} \longrightarrow 0.$$

By assumption for any $k \geqslant 0$ and $i_0, \ldots, i_k \in I$, the sequence

$$0 \longrightarrow \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p}) \longrightarrow \mathcal{I}(U_{i_0} \cap \cdots \cap U_{i_p}) \longrightarrow \mathcal{G}(U_{i_0} \cap \cdots \cap U_{i_p}) \longrightarrow 0$$

is exact, and taking product over all $(k+1)$-tuples gives an exact sequence of Čech complexes :

$$0 \longrightarrow C^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow C^\bullet(\mathcal{U}, \mathcal{I}) \longrightarrow C^\bullet(\mathcal{U}, \mathcal{G}) \longrightarrow 0$$

This gives a long exact sequence of Čech cohomology groups ; in view of Corollary 8.13.2, we have an exact sequence

$$0 \longrightarrow \check{H}^0(\mathcal{U}, \mathcal{F}) \longrightarrow \check{H}^0(\mathcal{U}, \mathcal{I}) \longrightarrow \check{H}^0(\mathcal{U}, \mathcal{G}) \longrightarrow \check{H}^1(\mathcal{U}, \mathcal{F}) \longrightarrow 0 \tag{$\star$}$$

and isomorphisms

$$\check{H}^p(\mathcal{U},\mathcal{G}) \cong \check{H}^{p+1}(\mathcal{U},\mathcal{F}) \tag{$\star\star$}$$

for $p \geqslant 1$ (given by connecting homomorphisms). Since the maps $\check{H}^\bullet(\mathcal{U},\mathcal{F}) \to H^\bullet(X,\mathcal{F})$ are functorial, by the case $p=0$, the sequence $(\star)$ implies $\check{H}^1(\mathcal{U},\mathcal{F}) \to H^1(X,\mathcal{F})$ is an isomorphism.

Let V be any finite intersection of elements in $\mathcal{U}$. The restriction $\mathcal{F} \mapsto \mathcal{F}|_V$ being exact, we have an exact sequence

$$0 \longrightarrow \mathcal{F}|_V \longrightarrow \mathcal{I}|_V \longrightarrow \mathcal{G}|_V \longrightarrow 0$$

with $\mathcal{I}|_V$ still flasque. The resulting long exact sequence on cohomology shows that $H^k(V,\mathcal{G}|_V) = 0$ for any $k > 0$. Now the result follows from induction on p and the isomorphisms $(\star\star)$. □

Corollary 8.15.1. Let X be a separated scheme, $\mathcal{U} = \{U_i\}_{i\in I}$ an open affine cover of X, and let $\mathcal{F}$ a quasi-coherent sheaf on X. Then for $p \geqslant 0$, the natural map in Corollary 8.13.1 is an isomorphism.

Proof. It suffices to take $\mathcal{U}$ to be an affine cover of X. Then the assumption in Theorem 8.15 holds by Theorem 7.12 (that X is separated is use here!). □

In general, Čech cohomology does not coincide with sheaf cohomology. But in the degree 1, two cohomology groups are always isomorphic. Precisely,

Proposition 8.16. Let X be a topological space and $\mathcal{F}$ a sheaf of abelian groups on X.

1. For open covers $\mathcal{U}$ and $\mathcal{V}$ of X, if $\mathcal{V}$ refines $\mathcal{U}$, we have a commutative triangle

$$\begin{tikzcd} \check{H}^\bullet(\mathcal{U},\mathcal{F}) \arrow[d, "\mathrm{ref}_{\mathcal{V}}^{\mathcal{U},\bullet}"'] \arrow[dr] & \\ & H^\bullet(X,\mathcal{F}) \\ \check{H}^\bullet(\mathcal{V},\mathcal{F}) \arrow[ur] & \end{tikzcd}$$

   where the slanted arrows are natural maps in Corollary 8.13.1.

By 1. we can pass to limit, obtaining a map $\check{H}^\bullet(X,\mathcal{F}) \to H^\bullet(X,\mathcal{F})$.

2. The map $\check{H}^1(X,\mathcal{F}) \to H^1(X,\mathcal{F})$ is an isomorphism.

Proof. Using the same map on index sets for induced covers, various refinement maps are packed to a morphism of sheafified Čech complexes $\mathrm{ref} : \mathcal{C}^\bullet(\mathcal{U},\mathcal{F}) \to \mathcal{C}^\bullet(\mathcal{V},\mathcal{F})$. We then have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, "\mathrm{id}"] & \mathcal{C}^0(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d, "\mathrm{ref}"] & \mathcal{C}^1(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d, "\mathrm{ref}"] & \cdots \\ 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, "\mathrm{id}"] & \mathcal{C}^0(\mathcal{V},\mathcal{F}) \arrow[r] & \mathcal{C}^1(\mathcal{V},\mathcal{F}) \arrow[r] & \cdots \\ 0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{I}^0 \arrow[r] & \mathcal{I}^1 \arrow[r] & \cdots \end{tikzcd}$$

Now 1. follows from Lemma 8.14.

For 2., embed $\mathcal{F}$ into a flasque sheaf $\mathcal{I}$ and take the quotient $\mathcal{G} = \mathcal{I}/\mathcal{F}$. Define a complex $D^\bullet(\mathcal{U})$ by the exact sequence

$$0 \longrightarrow C^\bullet(\mathcal{U},\mathcal{F}) \longrightarrow C^\bullet(\mathcal{U},\mathcal{G}) \longrightarrow D^\bullet(\mathcal{U}) \longrightarrow 0$$

The natural map $D^\bullet(\mathcal{U}) \to C^\bullet(\mathcal{U},\mathcal{G})$ and this exact sequence are compatible with refinement. Now taking cohomology, together with Corollary 8.13.2, gives

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}(X) & \longrightarrow & \mathcal{I}(X) & \longrightarrow & H^0(D^\bullet(\mathcal{U})) & \longrightarrow & \check{H}^1(\mathcal{U},\mathcal{F}) & \longrightarrow & 0 \\ & & \| & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}(X) & \longrightarrow & \mathcal{I}(X) & \longrightarrow & \mathcal{G}(X) & \longrightarrow & H^1(X,\mathcal{F}) & \longrightarrow & 0 \end{array}$$

Each map is natural as in Corollary 8.13.1. Moreover, the third map is the composition $H^0(D^\bullet(\mathcal{U})) \to \check{H}^0(\mathcal{U},\mathcal{G}) \to H^0(X,\mathcal{G})$. Being compatible with refinement, we can pass the first sequence to the limit, obtaining

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}(X) & \longrightarrow & \mathcal{I}(X) & \longrightarrow & \varinjlim_{\mathcal{U}} H^0(D^\bullet(\mathcal{U})) & \longrightarrow & \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U},\mathcal{F}) & \longrightarrow & 0 \\ & & \| & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}(X) & \longrightarrow & \mathcal{I}(X) & \longrightarrow & \mathcal{G}(X) & \longrightarrow & H^1(X,\mathcal{F}) & \longrightarrow & 0 \end{array}$$

To show that last vertical arrow is an isomorphism, it suffices to show the third vertical one is an isomorphism. The injectivity is clear; in fact, it is already injective before taking limit. To see surjectivity, for $x \in \mathcal{G}(X)$, by surjectivity of stalks we can find an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ such that $x|_{U_i} = s_i$ for some $s_i \in \mathcal{I}(U_i)$. This means x lies in $\mathrm{Im}(C^0(\mathcal{U},\mathcal{I}) \to C^0(\mathcal{U},\mathcal{G})) \cap \mathcal{G}(X) = H^0(D^\bullet(\mathcal{U}))$, demonstrating the surjectivity.

□

8.3.1 Base change morphisms

Let $f : Y \to X$ be a continuous map between topological spaces and let $\mathcal{F}$ be a sheaf of abelian groups on X. Let $\mathcal{U}$ be an open cover of X, and put $f^{-1}\mathcal{U} := \{f^{-1}(U) \mid U \in \mathcal{U}\}$ which is an open cover of Y. For each $p \geqslant 0$, we have a map

$$C^p(\mathcal{U},\mathcal{F}) \longrightarrow C^p(f^{-1}\mathcal{U}, f^{-1}\mathcal{F})$$

which is induced by $\mathcal{F}(U) \to (f^{-1}\mathcal{F})(f^{-1}(U))$. This is the **topological base change map** on the level of Čech complexes. In fact, this gives a morphism

$$\mathcal{C}^p(\mathcal{U},\mathcal{F}) \longrightarrow f_*\mathcal{C}^p(f^{-1}\mathcal{U}, f^{-1}\mathcal{F})$$

Lemma 8.17. The maps $C^p(\mathcal{U},\mathcal{F}) \to C^p(f^{-1}\mathcal{U}, f^{-1}\mathcal{F})$ $(p \geqslant 0)$ commute with coboundary maps.

This shows the topological base change maps defines a map of complexes $C^\bullet(\mathcal{U},\mathcal{F}) \to C^\bullet(f^{-1}\mathcal{U}, f^{-1}\mathcal{F})$, so it induces a map between cohomologies

$$\check{H}^p(\mathcal{U},\mathcal{F}) \longrightarrow \check{H}^p(f^{-1}\mathcal{U}, f^{-1}\mathcal{F})$$

This is the **topological base change map** in terms of Čech cohomology for the cover $\mathcal{U}$. One checks at once that this commutes with refinement map, so it further passes to a map between genuine Čech cohomologies :

$$\check{H}^p(X,\mathcal{F}) \longrightarrow \varinjlim_{\mathcal{U}} \check{H}^p(f^{-1}\mathcal{U}, f^{-1}\mathcal{F}) \to \check{H}^p(Y, f^{-1}\mathcal{F}).$$

The goal of this subsubsection is to show the Čech-sheaf (c.f. §7.5.3) compatibility of base change maps.

Lemma 8.18. The diagram

$$\begin{tikzcd} \check{H}^p(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d] & \check{H}^p(f^{-1}\mathcal{U},f^{-1}\mathcal{F}) \arrow[d] \\ H^p(X,\mathcal{F}) \arrow[r] & H^p(Y,f^{-1}\mathcal{F}) \end{tikzcd}$$

commutes. Here the horizontal arrows are topological base change maps, and the vertical arrows are as in (8.13.1).

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ and $f^{-1}\mathcal{F} \to \mathcal{J}^\bullet$ be injective resolutions. By Lemma 8.14, we have a map of complexes $\mathcal{C}^\bullet(\mathcal{U},\mathcal{F}) \to \mathcal{I}^\bullet$ extending the identity map $\mathcal{F} \xrightarrow{\text{id}} \mathcal{F}$ and

$$\begin{tikzcd} & \mathcal{C}^\bullet(f^{-1}\mathcal{U},f^{-1}\mathcal{F}) \arrow[dr] & \\ & f^{-1}\mathcal{I}^\bullet \arrow[r] & \mathcal{J}^\bullet \end{tikzcd}$$

both extending the identity map $f^{-1}\mathcal{F} \xrightarrow{\text{id}} f^{-1}\mathcal{F}$. We complete this corner into a diagram

$$\begin{tikzcd} f^{-1}\mathcal{C}^\bullet(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d] & \mathcal{C}^\bullet(f^{-1}\mathcal{U},f^{-1}\mathcal{F}) \arrow[dr] & \\ f^{-1}\mathcal{I}^\bullet \arrow[r, equal] & f^{-1}\mathcal{I}^\bullet \arrow[r] & \mathcal{J}^\bullet \end{tikzcd} .$$

The upper-horizontal is the adjunction of the topological base change map. Every vertex is pointed by an arrow starting from $f^{-1}\mathcal{F}$, and the resulting cone has commutative faces by construction, so this trapezoid commutes by Lemma 8.14 (each vertex is an exact complex since f^{-1} is exact). By adjunction, this gives a commutative trapezoid

$$\begin{tikzcd} \mathcal{C}^\bullet(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d] & f_*\mathcal{C}^\bullet(f^{-1}\mathcal{U},f^{-1}\mathcal{F}) \arrow[dr] & \\ \mathcal{I}^\bullet \arrow[r] & f_*f^{-1}\mathcal{I}^\bullet \arrow[r] & f_*\mathcal{J}^\bullet \end{tikzcd} .$$

Taking global section and then taking cohomology give the desired commutative diagram. □

Now assume X, Y are ringed spaces and $f \in \mathrm{Hom}_{\mathbf{RS}}(Y, X)$. Then the natural map $f^{-1}\mathcal{F} \to f^*\mathcal{F}$ induces, for each $p \geqslant 0$, a map

$$C^p(\mathcal{V}, f^{-1}\mathcal{F}) \longrightarrow C^p(\mathcal{V}, f^*\mathcal{F})$$

with $\mathcal{V}$ an arbitrary open cover of Y. Again, it induces a morphism $\mathcal{C}^p(\mathcal{V}, f^{-1}\mathcal{F}) \to \mathcal{C}^p(\mathcal{V}, f^*\mathcal{F})$ of sheafified Čech complexes. It follows at once (and by Lemma 8.2) that these commute with coboundary maps, so it induces a corresponding map between cohomologies

$$\check{H}^p(\mathcal{V}, f^{-1}\mathcal{F}) \longrightarrow \check{H}^p(\mathcal{V}, f^*\mathcal{F}) .$$

Lemma 8.19. The diagram

$$\begin{tikzcd} \check{H}^p(\mathcal{V},f^{-1}\mathcal{F}) \arrow[r] \arrow[d] & \check{H}^p(\mathcal{V},f^*\mathcal{F}) \arrow[d] \\ H^p(Y,f^{-1}\mathcal{F}) \arrow[r] & H^p(Y,f^*\mathcal{F}) \end{tikzcd}$$

commutes. Here horizontal arrows are those induced by $f^{-1}\mathcal{F} \to f^*\mathcal{F}$, and the vertical arrows are again as in (8.13.1).

Proof. This follows from the functoriality part of (8.13.1). □

We are ready to define the base change map for Čech-cohomology and establishes the Čech-sheaf comparison. Retain the notation above. The composition

$$\check{H}^p(\mathcal{U},\mathcal{F}) \longrightarrow \check{H}^p(f^{-1}\mathcal{U}, f^{-1}\mathcal{F}) \longrightarrow \check{H}^p(f^{-1}\mathcal{U}, f^*\mathcal{F})$$

is the **base change map** for Čech-cohomology. By the compatibility proved above, we obtain the following

Theorem 8.20. The diagram

$$\begin{array}{ccc} \check{H}^p(\mathcal{U},\mathcal{F}) & \longrightarrow & \check{H}^p(f^{-1}\mathcal{U}, f^*\mathcal{F}) \\ \downarrow & & \downarrow \\ H^p(X,\mathcal{F}) & \longrightarrow & H^p(Y, f^*\mathcal{F}) \end{array}$$

commutes. Here the horizontal arrows are base change maps, and the vertical arrows are as in (8.13.1).

8.4 Cohomology of projective spaces

Let A be a ring, $n \in \mathbb{Z}_{\geqslant 0}$ and let $\mathcal{U} = \{D_+(x_i)\}_{i=0}^n$ be the standard affine cover of $\mathbb{P}^n_A$. The goal of this subsection is to compute the Cech cohomology

$$\check{H}^p(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n_A}(m))$$

of the invertible sheaves $\mathcal{O}_{\mathbb{P}^n_A}(m)$ $(m \in \mathbb{Z})$.

Put $X = \mathcal{A}^{n+1}_A \backslash V(t_0 \cdots t_n)$. Recall from Example 3.132 there is a A-morphism

$$\pi : X \longrightarrow \mathbb{P}^n_A = \operatorname{Proj} A[x_0, \dots, x_n]$$

such that $\pi^*\mathcal{O}_{\mathbb{P}^n_A}(1) \cong \mathcal{O}_X$ and $t_i = \pi^* x_i$ for $0 \leqslant i \leqslant n$ under this isomorphism. Consider the morphism

$$C^p_{\mathrm{ord}}(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n_A}(m)) \longrightarrow C^p_{\mathrm{ord}}(\pi^{-1}\mathcal{U}, \pi^*\mathcal{O}_{\mathbb{P}^n_A}(m)) \cong C^p_{\mathrm{ord}}(\pi^{-1}\mathcal{U}, \mathcal{O}_X). \tag{$\spadesuit$}$$

induced by the pullback by π. Notice $\pi^{-1}\mathcal{U} = \{D(t_i)\}_{i=0}^n$ is the standard cover of $X = \mathbb{A}^{n+1}_A \backslash V(t_0 \cdots t_n)$. Let's look at a particular piece : if $I \subseteq \{0, 1, \dots, n\}$ with $\#I = p+1$ as multisets, the corresponding I-th part is simply the inclusion Check!

$$\begin{array}{ccc} A[x_0, \dots, x_n][\{x_i^{-1}\}_{i \in I}]_{\deg = m} & \longrightarrow & A[t_0, \dots, t_n][\{t_i^{-1}\}_{i \in I}] \\ x_j & \longmapsto & t_j. \end{array} \tag{$\clubsuit$}$$

In particular, this shows ($\clubsuit$) is a split injection of complexes, whence inducing an injection on the level of cohomology

$$\check{H}^p(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n_A}(m)) \hookrightarrow \check{H}^p(\pi^{-1}\mathcal{U}, \mathcal{O}_X). \ .$$

By using the ordered Čech complex (c.f. (8.7)), we see

$$\check{H}^p(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n_A}(m)) = 0$$

whenever $p > n$ or $p < 0$. For the other case, we first compute $\check{H}^p(\pi^{-1}\mathcal{U}, \mathcal{O}_X)$. Again we use ordered Čech complexes to facilitate our computation. In this case the complex to be considered has the form

$$0 \longrightarrow R \longrightarrow \prod_{0\leqslant i\leqslant n} R[t_i^{-1}] \longrightarrow \prod_{0\leqslant i<j\leqslant n} R[t_i^{-1}, t_j^{-1}] \longrightarrow \cdots \longrightarrow R[t_0^{-1}, \dots, t_n^{-1}] \longrightarrow 0$$

where $R = A[t_0, \dots, t_n]$. We claim this is exact except at the last nonzero place. This complex is, modulo some signs, an $n+1$-fold tensor product, i.e., it is

$$\bigotimes_{0\leqslant i\leqslant n} [0 \to R \to R[t_i^{-1}] \to 0]$$

where the position R is degree 0. We prove more generally that

$$0 \longrightarrow R \longrightarrow \prod_{\substack{0\leqslant i\leqslant n\\ i\in I}} R[t_i^{-1}] \longrightarrow \prod_{\substack{0\leqslant i<j\leqslant n\\ i,j\in I}} R[t_i^{-1}, t_j^{-1}] \longrightarrow \cdots \longrightarrow R[\{t_i^{-1}\}_{i\in I}] \longrightarrow 0$$

is exact except at the last place for any $I \subseteq \{0, 1, \dots, n\}$ by induction on $\#I$. The case where $\#I = 1$ is clear. Let $n+1 \geqslant \#I \geqslant 2$; without loss of generality we assume $n \in I$. Consider

$$\begin{tikzcd}
 & 0 & & & & & & \\
 & R[t_n^{-1}] \arrow[u] & & & & & & \\
0 \arrow[r] & R \arrow[r] \arrow[u] & \prod\limits_{\substack{0\leqslant i\leqslant n-1\\ i\in I}} R[t_i^{-1}] \arrow[r] & \prod\limits_{\substack{0\leqslant i<j\leqslant n-1\\ i,j\in I}} R[t_i^{-1}, t_j^{-1}] \arrow[r] & \cdots \arrow[r] & R[\{t_i^{-1}\}_{n\neq i\in I}] \arrow[r] & C \arrow[r] & 0 \\
 & 0 \arrow[u] & & & & & &
\end{tikzcd}$$

where C is the cokernel of the last place; introducing C makes the horizontal complex exact. Since $R \to R[t_n^{-1}]$ is flat, by Lemma 8.12, the tensor complex

$$0 \longrightarrow R \longrightarrow \prod_{\substack{0\leqslant i\leqslant n\\ i\in I}} R[t_i^{-1}] \longrightarrow \prod_{\substack{0\leqslant i<j\leqslant n\\ i,j\in I}} R[t_i^{-1}, t_j^{-1}] \longrightarrow \cdots \longrightarrow R[\{t_i^{-1}\}_{i\in I}] \oplus C \longrightarrow C \otimes R[t_n^{-1}] \longrightarrow 0$$

is exact. Truncating at the last second place and projecting down, we still need to prove the exactness of

$$\prod_{i<j,\, i,j\in I} R[\{t_k^{-1}\}_{k\in I,\, k\neq i,j}] \longrightarrow \prod_{i\in I} R[\{t_k^{-1}\}_{i\neq k\in I}] \longrightarrow R[\{t_k^{-1}\}_{k\in I}] \tag{$\spadesuit$}$$

Since each arrow preserves degrees of each t_i, it suffices to check the exactness monomial by monomial. Consider the diagram[7]

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \prod\limits_{i<j\neq n,\, i,j\in I} R[\{t_k^{-1}\}_{k\in I,\, k\neq i,j}] & \longrightarrow & \prod\limits_{n\neq i\in I} R[\{t_k^{-1}\}_{i\neq k\in I}] & \longrightarrow & R[\{t_k^{-1}\}_{k\in I}] & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \prod\limits_{i<j,\, i,j\in I} R[\{t_k^{-1}\}_{k\in I,\, k\neq i,j}] & \longrightarrow & \prod\limits_{i\in I} R[\{t_k^{-1}\}_{i\neq k\in I}] & \longrightarrow & R[\{t_k^{-1}\}_{k\in I}] & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \prod\limits_{i<n,\, i\in I} R[\{t_k^{-1}\}_{k\in I,\, k\neq i,n}] & \longrightarrow & R[\{t_k^{-1}\}_{n\neq k\in I}] & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

By construction each column is a short exact sequence. By induction hypothesis the first row is exact at the middle. Also by induction hypothesis, if we restrict the third row to a single monomial, we see it is exact at the middle except when all exponents on t_k, $k \in I\backslash\{n\}$ are all negative. Taking long exact sequence on cohomologies, we see (♠) is exact for monomials with at least one exponent of t_k ($k \in I\backslash\{n\}$) is non-negative. Playing the same game for another index in $I\backslash[n]$, we see (♠) is exact.

In view of Corollary 8.15.1, we have almost proved the following

Theorem 8.21. Let A be a ring, $n \in \mathbb{Z}_{\geqslant 0}$ and $m \in \mathbb{Z}$. Then

$$H^p(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m)) = \begin{cases} 0 & ,\text{if } p \neq 0, n \\ A[x_0, \dots, x_n]_{\deg = m} & ,\text{if } p = 0 \\ \big((x_0 \cdots x_n)^{-1} A[x_0^{-1}, \dots, x_n^{-1}]\big)_{\deg = m} & ,\text{if } p = n. \end{cases}$$

Proof. It remains to compute $\check{H}^n(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n_A}(m))$, and by (♣) we only need to compute the cokernel of

$$\begin{array}{ccc}
\prod\limits_{0\leqslant i\leqslant n} A[t_0, \dots, t_n][t_0^{-1}, \dots, \widehat{t_i^{-1}}, \dots, t_n^{-1}] & \longrightarrow & A[t_0^{\pm}, \dots, t_n^{\pm}] \\
(f_0, \dots, f_n) & \longmapsto & \sum\limits_{i=0}^{n} (-1)^i f_i.
\end{array}$$

The image is the A-span of monomials of the form $t_0^{r_0} \cdots t_n^{r_n}$, where $r_i \in \mathbb{Z}$ with at least one r_i non-negative. Hence the cokernel is isomorphic to the A-span of the monomials of the form $t_0^{r_0} \cdots t_n^{r_n}$ with $r_i \in \mathbb{Z}_{<0}$, i.e., isomorphic to $(t_0 \cdots t_n)^{-1} A[t_0^{-1}, \dots, t_n^{-1}]$. Taking degree m part finishes the proof. □

8.4.1 Coherency theorem

Let S be a Noetherian ring, $n \in \mathbb{Z}_{\geqslant 0}$ and $X = \mathbb{P}^n_S = \operatorname{Proj} S[x_0, \dots, x_n]$. By Serre's theorem we see if $\mathcal{F}$ is a coherent sheaf on X, then there exist $a_0, b_0 \in \mathbb{Z}_{\geqslant 0}$ and a surjection $\mathcal{O}_X^{b_0} \twoheadrightarrow \mathcal{F}(a_0)$. Twisting down a then gives $\mathcal{O}_X(-a_0)^{b_0} \twoheadrightarrow \mathcal{F}$. Iterating this process yields an exact sequence

$$\cdots \longrightarrow \mathcal{O}_X(-a_1)^{b_1} \longrightarrow \mathcal{O}_X(-a_0)^{b_0} \longrightarrow \mathcal{F} \longrightarrow 0.$$

This is an resolution of $\mathcal{F}$ by locally free sheaves of finite rank.

Theorem 8.22 (Serre). Let $S, X, \mathcal{F}$ be as above.

7. We could have used this trick to show the ordered Cech complex is exact except at the last place.

(i) $H^p(X, \mathcal{F}(m))$ is a finite S-modules for all $p \geqslant 0$, $m \in \mathbb{Z}$.

(ii) There exists $m_0 \in \mathbb{Z}$ such that $H^p(X, \mathcal{F}(m)) = 0$ for all $p \geqslant 1$, $m \geqslant m_0$. (This holds without Noetherian condition.)

(iii) $\mathcal{F} \cong \widetilde{M}$ for some finite graded $S[x_0, \dots, x_n]$-module M, and there exists m_1 such that $M_m \to H^0(X, \mathcal{F}(m))$ is an isomorphism for all $m \geqslant m_1$.

Proof. (i) and (ii) are proved by descending induction on p. Using ordered Cech complex with respect to the standard affine cover, we see $H^p(X, \mathcal{F}(m)) = 0$ for $p > n$. Now suppose (i) and (ii) hold for all $\mathcal{F} \in \mathbf{Coh}_X$ and $p > p_0 \geqslant 1$. As above there is a short exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_X(a)^b \longrightarrow \mathcal{F} \longrightarrow 0$$

for some a, b and $\mathcal{G} \in \mathbf{Coh}_X$. For each $m \in \mathbb{Z}$ this gives

$$0 \longrightarrow \mathcal{G}(m) \longrightarrow \mathcal{O}_X(a+m)^b \longrightarrow \mathcal{F}(m) \longrightarrow 0.$$

Taking cohomology gives an exact sequence

$$H^{p_0}(X, \mathcal{O}_X(a+m)^b) \longrightarrow H^{p_0}(X, \mathcal{F}(m)) \longrightarrow H^{p_0+1}(X, \mathcal{G}(m)) \longrightarrow H^{p_0+1}(X, \mathcal{O}_X(a+m)^b)$$

By Theorem 8.21 $H^{p_0}(X, \mathcal{O}_X(a+m)^b)$ and $H^{p_0+1}(X, \mathcal{O}_X(a+m)^b)$ are finite over S and $= 0$ for $m \gg 0$. By induction $H^{p_0+1}(X, \mathcal{G}(m))$ is finite over S and $= 0$ for $m \gg 0$. Hence the same holds for $H^{p_0}(X, \mathcal{F}(m))$. This proves (i) and (ii).

The first statement of (iii) is Corollary 3.126.1. ☐

Corollary 8.22.1. Let $f : X \to Y$ be a projective morphism with Y locally Noetherian. If $\mathcal{F}$ is $\mathcal{O}_X$-coherent, then $R^p f_* \mathcal{F}$ is $\mathcal{O}_Y$-coherent for all $p \in \mathbb{Z}_{\geqslant 0}$.

Proof. The problem is local in the base Y, so we can assume $Y = \operatorname{Spec} A$ for some Noetherian ring A and there is a commutative triangle for some $n \in \mathbb{Z}_{\geqslant 0}$

$$\begin{tikzcd} X \arrow[rr, "j", "\text{closed immersion}"'] \arrow[dr, "f"'] & & \mathbb{P}^n_A \arrow[dl] \\ & \operatorname{Spec} A & \end{tikzcd}$$

But then $R^p f_* \mathcal{F} \cong \widetilde{H^p(X, \mathcal{F})}$ by (7.16) and $H^p(X, \mathcal{F}) \cong H^p(\mathbb{P}^n_A, j_* \mathcal{F})$ by (7.2.2), we see from Theorem 8.22.(i) that $R^p f_* \mathcal{F}$ is coherent. ☐

Theorem 8.23. Let $f : X \to Y$ be a proper morphism between locally Noetherian schemes. If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then $R^i f_* \mathcal{F}$ is $\mathcal{O}_Y$-coherent for all $i \geqslant 0$.

8.4.2 Grothendieck coherency

9 Singular (co)homology

9.1 Definitions

9.1 Singular chain groups. Let $\mathbb{R}^\infty$ have the standard basis $e_0, e_1, \ldots$. For $n \in \mathbb{Z}_{\geqslant 0}$, the **standard** p**-simplex** is the set

$$\Delta_n := \left\{ \sum_{i=0}^{p} a_i e_i \mid 0 \leqslant a_i \leqslant 1, \sum_{i=0}^{p} a_i = 1 \right\}.$$

For $v_0, \ldots, v_p \in \mathbb{R}^N$, define

$$\begin{aligned} [v_0, \ldots, v_p] : \Delta_n &\longrightarrow \mathbb{R}^N \\ \sum_{i=0}^{p} a_i e_i &\longmapsto \sum_{i=0}^{p} a_i v_i. \end{aligned}$$

Let X be a topological space and $p \in \mathbb{Z}_{\geqslant 0}$. By definition, a **singular** p**-simplex** in X is a continuous map $\Delta_p \to X$. For $p \geqslant 0$, define the **singular** p**-chain group** $S_p(X)$ to be the free abelian group based on all the singular p-simplices in X. For convenience, we put $S_p(X) = 0$ for $p < 0$.

For a singular p-simplex $\sigma : \Delta_p \to X$ and $0 \leqslant i \leqslant p$, the i-th face of σ is defined as 0 if $p \leqslant 0$, and

$$\sigma|_{[e_0, \ldots, \widehat{e_i}, \ldots, e_n]} := \sigma \circ [e_0, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n] : \Delta_{n-1} \to X$$

if $p \geqslant 1$. Define the **boundary** map $\partial_p : S_p(X) \to S_{p-1}(X)$ by the formula

$$\partial_p \sigma = \sum_{i=0}^{p} (-1)^i \sigma|_{[e_0, \ldots, \widehat{e_i}, \ldots, e_p]}.$$

In particular, $\partial_p = 0$ for $p \leqslant 0$.

9.1.1 Lemma. For $n \in \mathbb{Z}_{\geqslant 0}$, one has $\partial_n \circ \partial_{n+1} = 0$. Thus there is a chain complex

$$\cdots \longrightarrow S_{p+1}(X) \xrightarrow{\partial_{p+1}} S_p(X) \xrightarrow{\partial_p} S_{p-1}(X) \xrightarrow{\partial_{p-1}} \cdots$$

9.2 Singular homology. Let G be an abelian group. Tensoring the chain complex in Lemma 9.1.1 with G, we obtain

$$\cdots \longrightarrow S_{p+1}(X) \otimes_{\mathbb{Z}} G \xrightarrow{\partial_{p+1} \otimes \mathrm{id}_G} S_p(X) \otimes_{\mathbb{Z}} G \xrightarrow{\partial_p \otimes \mathrm{id}_G} S_{p-1}(X) \otimes_{\mathbb{Z}} G \xrightarrow{\partial_{p-1} \otimes \mathrm{id}_G} \cdots$$

The p**-th singular homology group** $H_p(X; G)$ **of the space** X **with coefficient in** G is defined as

$$H_p^{\mathrm{sing}}(X; G) := H_p(S_\bullet(X) \otimes_{\mathbb{Z}} G, \partial_\bullet \otimes \mathrm{id}_G).$$

For simplicity, we write $S_p(X; G)$ for the group $S_p(X) \otimes_{\mathbb{Z}} G$ and write $\partial_\bullet$ for the boundary map $\partial_\bullet \otimes \mathrm{id}_G$.

9.3 Singular cohomology. Let G be an abelian group. Applying $\mathrm{Hom}_{\mathbb{Z}}(\cdot, G)$ to the complex in Lemma 9.1.1, we obtain a cochain complex

$$\cdots \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(S_{p-1}(X), G) \xrightarrow{\delta_p} \mathrm{Hom}_{\mathbb{Z}}(S_p(X), G) \xrightarrow{\delta_{p+1}} \mathrm{Hom}_{\mathbb{Z}}(S_{p+1}(X), G) \longrightarrow \cdots$$

The p**-th singular cohomology group** $H^p_{\mathrm{sing}}(X; G)$ **of the space** X **with coefficient in** G is defined as

$$H^p_{\mathrm{sing}}(X; G) := H^p(\mathrm{Hom}_{\mathbb{Z}}(S_\bullet(X), G), \delta_\bullet).$$

For simplicity, we write $S^p(X;G)$ for the group $\mathrm{Hom}_{\mathbb{Z}}(S_p(X),G)$.

9.4 Universal coefficient theorems for cohomology. The relation between singular homology and singular cohomology is established by the universal coefficient theorem, which is purely algebraic. A definition : a ring R is **left** (resp. **right**) **hereditary** if any R-submodule of a projective left (resp. ring) R-module is projective.

Theorem. Let R be a left hereditary ring, $\mathcal{C}^\bullet$ a complex of projective left R-modules, and M any left R-module. For every $n \in \mathbb{Z}$ there is an exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_R(H_{n-1}(\mathcal{C}^\bullet),M) \longrightarrow H^n(\mathrm{Hom}_R(\mathcal{C}^\bullet,M)) \longrightarrow \mathrm{Hom}_R(H_n(\mathcal{C}^\bullet),M) \longrightarrow 0$$

natural in $\mathcal{C}^\bullet$ and M. Moreover, it splits by a homomorphism that is natural in M (but not in $\mathcal{C}^\bullet$).

9.4.1 Universal coefficients theorem for homology. There is also a universal coefficient theorem for homology, which states that the singular homology with coefficient in G is determined by that with coefficient in $\mathbb{Z}$.

Theorem. Let R be a right hereditary ring, $\mathcal{C}^\bullet$ a complex of projective right R-modules and M any left R-module. For every $n \in \mathbb{Z}$ there is an exact sequence

$$0 \longrightarrow H_n(\mathcal{C}^\bullet)\otimes_R M \longrightarrow H_n(\mathcal{C}^\bullet \otimes_R M) \longrightarrow \mathrm{Tor}_1^R(H_{n-1}(\mathcal{C}^\bullet),M) \longrightarrow 0$$

natural in $\mathcal{C}^\bullet$ and M. Moreover, it splits by a homomorphism that is natural in M (but not in $\mathcal{C}^\bullet$).

9.2 Relation to sheaf cohomology

9.5 Definitions. Let X, Y be topological spaces.

1. Two continuous maps $f, g : X \to Y$ are called **homotopic** if there exists a continuous map $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for any $x \in X$. Such an F is called a **homotopy from** f **to** g, and we write $f \sim g$.
2. Let A be a subspace of X. A continuous map $F : X \times [0,1] \to Y$ is called a **homotopy rel(ative to)** A if $F(a,t) = a$ for any $a \in A$, $t \in [0,1]$.
3. A continuous map $f : X \to Y$ is a **homotopy equivalence** if there exists another continuous map $g : Y \to X$ such that $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$. If such f exists, we say X and Y are **homotopic equivalent**, and write $X \sim Y$.
4. X is called **contractible** if $\mathrm{id}_X : X \to X$ is homotopic to a constant map $X \to X$. Equivalently, $X \sim \{*\}$ for some $* \in X$.
5. X is called **locally contractible** if every point in X admits a neighborhood basis consisting of contractible open sets.

9.6 Induced maps on (co)homology groups. Let $f : X \to Y$ be a continuous map. For a singular p-simplex $\sigma : \Delta_p \to X$, the composition $f_*\sigma = f \circ \sigma : \Delta_p \to Y$ is a singular p-simplex in Y. Extending by linearity, we obtain a homomorphism $f_* : S_p(X) \to S_p(Y)$. The map f_* is a chain map, i.e., $f_*\partial = \partial f_*$, so it induces maps on homology groups $f_* : H_p^{\mathrm{sing}}(X;G) \to H_p^{\mathrm{sing}}(X;G)$ and cohomology groups $f^* : H^p_{\mathrm{sing}}(Y;G) \to H^p_{\mathrm{sing}}(X;G)$.

9.6.1 Lemma. If $f, g : X \to Y$ are two continuous map homotopic to each other, then

$$f_* = g_* : H_p^{\mathrm{sing}}(X;G) \to H_p^{\mathrm{sing}}(Y;G).$$

In particular, if $f : X \to Y$ is an homotopy equivalence, $f_* : H_p^{\mathrm{sing}}(X;G) \to H_p^{\mathrm{sing}}(Y;G)$ is an isomorphism. Similarly, $f^* = g^* : H^p_{\mathrm{sing}}(Y;G) \to H^p_{\mathrm{sing}}(X;G)$.

Proof. Let $F : X \times [0,1] \to Y$ be a homotopy from f to g. We are going to utilize F to show f_* is chain homotopic to g_* as maps $S_p(X) \to S_p(Y)$. Precisely, define $P : S_p(X) \to S_{p+1}(Y)$ by

$$P(\sigma) = \sum_{i=0}^{p} (-1)^i F \circ (\sigma \times \mathrm{id}_{[0,1]})|_{[v_0,\dots,v_i,w_i,\dots,w_p]}.$$

Here $v_0, \dots, v_p$ (resp. $w_0, \dots, w_p$) are the vertices of $\Delta^p \times \{0\}$ (resp. $\Delta^p \times \{1\}$) such that v_i and w_i have the same image under the projection $\Delta^p \times [0,1] \to \Delta^p$. We claim

$$g_* - f_* = \partial P + P\partial.$$

$\square$

9.7 Theorem. Let X be a locally contractible topological space and R a ring. Then there is a canonical isomorphism

$$H^p_{\mathrm{sing}}(X; R) \cong H^p(X, \underline{R}_X)$$

9.3 Relative homology groups

Let G be an abelian group, X be a topological space and A be its subspace. For $n \in \mathbb{Z}$, define

$$S_n(X, A; G) := S_n(X; G)/S_n(A; G).$$

The boundary map $\partial_n : S_n(X; G) \to S_{n-1}(X; G)$ takes $S_n(A; G)$ and $S_{n-1}(A; G)$, so it induces a boundary map

$$\partial_n : S_n(X, A; G) \longrightarrow S_{n-1}(X, A; G)$$

on the quotient groups. The homology groups of this chain complex are called the **relative homology groups**, and are denoted as

$$H^{\mathrm{sing}}_\bullet(X, A; G) := H_\bullet(S_\bullet(X, A; G), \partial_\bullet)$$

Similarly, put $S^n(X, A; G) = \mathrm{Hom}_{\mathbb{Z}}(S_n(X, A), G)$ and define the **relative cohomology group**

$$H^\bullet_{\mathrm{sing}}(X, A; G) := H^\bullet(S^\bullet(X, A; G), \delta^\bullet)$$

where $\delta^\bullet = \mathrm{Hom}_{\mathbb{Z}}(\partial_\bullet, G)$.

By definition, we have an exact sequence of chain complexes

$$0 \to S_\bullet(A; G) \to S_\bullet(X; A) \to S_\bullet(X, A; G) \to 0$$

so it induces a long exact sequence on homology groups :

$$\cdots \longrightarrow H^{\mathrm{sing}}_n(A; G) \longrightarrow H^{\mathrm{sing}}_n(X; G) \longrightarrow H^{\mathrm{sing}}_n(X, A; G) \longrightarrow H^{\mathrm{sing}}_{n-1}(A, G) \longrightarrow \cdots.$$

For the cohomology, note that the exact sequence

$$0 \to S_n(A) \to S_n(X) \to S_n(X, A) \to 0$$

splits : $S_n(X, A)$ is naturally identified as the subgroup of $S_n(X)$ free over the singular n-simplices $\Delta_n \to X$ with image not contained in A. It follows that the dual sequence

$$0 \to S^n(X, A; G) \to S^n(X; G) \to S^n(A; G) \to 0$$

is exact, so it induces a long exact sequence of cohomology groups

$$\cdots \longrightarrow H^n_{\mathrm{sing}}(X, A; G) \longrightarrow H^n_{\mathrm{sing}}(X; G) \longrightarrow H^n_{\mathrm{sing}}(A; G) \longrightarrow H^{n+1}_{\mathrm{sing}}(X, A; G) \longrightarrow \cdots.$$

Definition.

1. A pair (X, A) consists of a space X and a subspace $A \subseteq X$.
2. An arrow $f : (X, A) \to (Y, B)$ is a continuous map $f : X \to Y$ such that $f(A) \subseteq B$.

A map $f : (X, A) \to (Y, B)$, as usual, induces maps on homology groups $f_* : H_n^{\mathrm{sing}}(X, A; G) \to H_n^{\mathrm{sing}}(Y, B; G)$ and cohomology groups $f^* : H^n_{\mathrm{sing}}(Y, B; G) \to H^n_{\mathrm{sing}}(X, A; G)$. The map f_* is natural in the sense that there exists a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & H_n^{\mathrm{sing}}(A;G) \arrow[r] \arrow[d, "f_*"] & H_n^{\mathrm{sing}}(X;G) \arrow[r] \arrow[d, "f_*"] & H_n^{\mathrm{sing}}(X,A;G) \arrow[r] \arrow[d, "f_*"] & H_{n-1}^{\mathrm{sing}}(A,G) \arrow[r] \arrow[d, "f_*"] & \cdots \\ \cdots \arrow[r] & H_n^{\mathrm{sing}}(B;G) \arrow[r] & H_n^{\mathrm{sing}}(Y;G) \arrow[r] & H_n^{\mathrm{sing}}(Y,B;G) \arrow[r] & H_{n-1}^{\mathrm{sing}}(B,G) \arrow[r] & \cdots . \end{tikzcd}$$

The similar statement holds for f^*. Also, a careful look of the proof of Lemma 9.6.1 gives

Lemma 9.1. If $f, g : (X, A) \to (Y, B)$ are continuous map homotopic through maps of pairs $(X, A) \to (Y, B)$, then $f_* = g_* : H_n^{\mathrm{sing}}(X, A; G) \to H_n^{\mathrm{sing}}(Y, B; G)$ and $f^* = g^* : H^n_{\mathrm{sing}}(Y, B; G) \to H^n_{\mathrm{sing}}(X, A; G)$.

Theorem 9.2 (Excision)**.** Given a chain $Z \subseteq A \subseteq X$ of subspaces with $\overline{Z} \subseteq \operatorname{int} A$, the inclusion $(X\backslash Z, A\backslash Z) \to (X, A)$ induces an isomorphisms

$$H_n(X\backslash Z, A\backslash Z) \xrightarrow{\sim} H_n(X, A)$$

for all $n \in \mathbb{Z}$. Equivalently, for subspaces $A, B \subseteq X$ with $X = \operatorname{int} A \cup \operatorname{int} B$, the inclusion $(B, A \cap B) \to (X, A)$ induces isomorphisms $H_n(B, A \cap B) \to H_n(X, A)$ for all $n \in \mathbb{Z}$.

9.4 Algebraic Künneth formula

Theorem 9.3. Let R be a ring, and let $\mathcal{A}^\bullet$ a complex of flat right R-modules such that the subcomplex of boundaries are R-flat. Given any complex $\mathcal{C}^\bullet$ of left R-modules, for each $n \geqslant 1$ there exists a exact sequence of abelian groups

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(\mathcal{A}^\bullet \otimes_R H_q(\mathcal{C}^\bullet) \longrightarrow H_n(\mathcal{A} \otimes_R \mathcal{C}^\bullet) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(\mathcal{A}), H_q(\mathcal{C}^\bullet)) \longrightarrow 0$$

9.5 Cup product

Let X be a topological space and R be a ring. Here a ring is not necessarily commutative nor unital. Define the **cup product**

$$\begin{aligned} S^p(X; R) \times S^q(X; R) &\longrightarrow S^{p+q}(X; R) \\ (\varphi, \psi) &\longmapsto \varphi \smile \psi \end{aligned}$$

by

$$(\varphi \smile \psi)(\sigma) = \varphi(\sigma|_{[e_0,\ldots,e_p]})\psi(\sigma|_{[e_p,e_{p+1},\ldots,e_{p+q}]}) \in R$$

It is associative in the obvious sense, and it is distributive. If R is unital, then the 0-cocycle defined by $\sigma \in S^0(X) \mapsto 1 \in R$ is the identity element for $\smile$.

Lemma 9.4. $\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^p \varphi \smile \delta\psi$ for $\varphi \in S^p(X, R)$ and $\psi \in S^q(X, R)$.

Proof. This is a direct computation. For $\sigma \in S_{p+q+1}(X)$,

$$\begin{aligned}
\delta(\varphi \smile \psi))(\sigma) &= \sum_{i=0}^{p+q+1} (-1)^i (\varphi \smile \psi)(\sigma|_{[e_0,\ldots,\widehat{e_i},\ldots,e_{p+q+1}]}) \\
&= \sum_{i=0}^{p} (-1)^i \varphi(\sigma|_{[e_0,\ldots,\widehat{e_i},\ldots,e_{p+1}]})\psi(\sigma|_{[e_{p+1},e_{p+2},\ldots,e_{p+q+1}]}) \\
&\quad + \sum_{i=p+1}^{p+q+1} (-1)^i \varphi(\sigma|_{[e_0,\ldots,e_p]})\psi(\sigma|_{[e_p,e_{p+1},\ldots,\widehat{e_i},\ldots,e_{p+q+1}]}) \\
&= \sum_{i=0}^{p} (-1)^i \varphi(\sigma|_{[e_0,\ldots,\widehat{e_i},\ldots,e_{p+1}]})\psi(\sigma|_{[e_{p+1},e_{p+2},\ldots,e_{p+q+1}]}) + (-1)^{p+1}\varphi(\sigma|_{[e_0,\ldots,e_p]})\psi(\sigma|_{[e_{p+1},e_{p+2},\ldots,e_{p+q+1}]}) \\
&\quad + (-1)^p \varphi(\sigma|_{[e_0,\ldots,e_p]})\psi(\sigma|_{[e_{p+1},\ldots,e_{p+q+1}]}) + \sum_{i=p+1}^{p+q+1} (-1)^i \varphi(\sigma|_{[e_0,\ldots,e_p]})\psi(\sigma|_{[e_p,e_{p+1},\ldots,\widehat{e_i},\ldots,e_{p+q+1}]}) \\
&= (\delta\varphi \smile \psi)(\sigma) + (-1)^p(\varphi \smile \delta\psi)(\sigma)
\end{aligned}$$

□

From this lemma we see the cup product descends the level of cohomology groups :

$$H^p_{\text{sing}}(X;R) \times H^q_{\text{sing}}(X;R) \xrightarrow{\quad\smile\quad} H^{p+q}_{\text{sing}}(X;R).$$

If we form the direct sum

$$H^*_{\text{sing}}(X;R) := \bigoplus_{p\geqslant 0} H^p_{\text{sing}}(X;R),$$

the cup product then makes $H^*_{\text{sing}}(X;R)$ into an (associative and distributive) $\mathbb{Z}_{\geqslant 0}$-graded R-algebras (there is a canonical map $R \to S^0(X;R) \to H^0_{\text{sing}}(X;R)$). This is called the **singular cohomology ring**. It is unital as long as R is unital.

If $f : X \to Y$ is a continuous map, the induced map $f^* : H^*(Y;R) \to H^*(X;R)$ is a ring homomorphism. Indeed, for $\varphi \in S^p(Y;R)$ and $\psi \in S^q(Y;R)$,

$$\begin{aligned}
(f^*\varphi \smile f^*\psi)(\sigma) &= f^*\varphi(\sigma|_{[e_0,\ldots,e_p]})f^*\psi(\sigma|_{[e_p,\ldots,e_{p+q}]}) \\
&= \varphi(f\circ\sigma|_{[e_0,\ldots,e_p]})\psi(f\circ\sigma_{[e_p,\ldots,e_{p+q}]}) = (\varphi \smile \psi)(f\circ\sigma) = f^*(\varphi \smile \psi)(\sigma).
\end{aligned}$$

Theorem 9.5. If R is commutative, then $\alpha \smile \beta = (-1)^{pq}\beta \smile \alpha$ for all $\alpha \in H^p_{\text{sing}}(X;R)$ and $\beta \in H^q_{\text{sing}}(X;R)$.

Proof. For a singular n-simplex $\sigma : \Delta_n \to X$, define

$$\begin{array}{rcccc}
\overline{\sigma} : \Delta_n & \xrightarrow{\text{linear}} & \Delta_n & \xrightarrow{\sigma} & X \\
e_i & \longmapsto & e_{n-i} & &
\end{array}$$

and define

$$\begin{aligned}
\rho : S_n(X) &\longrightarrow S_n(X) \\
\sigma &\longmapsto \varepsilon_n\overline{\sigma}
\end{aligned}$$

where $\varepsilon_n := (-1)^{\frac{n(n+1)}{2}}$. We are going to show ρ is a chain map homotopic to the identity. This, together with the discussion preceding the theorem, at once shows the desired identity.

That ρ is a chain map follows from a direct computation and a trivial fact that $\varepsilon_n = (-1)^n \varepsilon_{n-1}$. To construct a chain homotopy from ρ to id, we use the notation in Lemma 9.6.1 and define $Q : S_n(X) \to S_{n+1}(X)$ by

$$Q(\sigma) = \sum_{i=0}^{n} (-1)^i \varepsilon_{n-i} (\sigma \circ \pi)|_{[v_0,\dots,v_i,w_n,\dots,w_i]}$$

where $\pi = \mathrm{pr}_1 : \Delta_n \times [0,1] \to \Delta_n$ is the projection to the first component. We claim $\partial Q + Q\partial = \rho - \mathrm{id}$. □

10 Curves

Let k be a field. Conceptually, a **curve** over k is an integral scheme of dimension 1 separated and of finite type over k. There are various definitions in the literature. We will not specify a definition for curves; instead, we will qualify our schemes appropriately in different context.

10.1 Compact Riemann surface

10.1 Definition. A **Riemann surface** X is a complex manifold of dimension 1. We denote by $\mathcal{O}_X$ (resp. C^∞_X) the sheaf of holomorphic (resp. smooth) functions on X. Note that C^∞_X is a fine sheaf.

10.2 Various tangent bundles. Let M be a complex manifold. As a real manifold, we can form the **(real) tangent bundle** $T_{M,\mathbb{R}} \to M$. As M is a complex manifold, $T_{M,\mathbb{R}}$ admits a holomorphic structure. With this structure, we call it the **holomorphic tangent bundle** of M, and denote it by $T_M \to M$. Multiplication by $\sqrt{-1}$ on local coordinates of $T_{M,\mathbb{R}}$ gives a smooth bundle isomorphism $J : T_{M,\mathbb{R}} \to T_{M,\mathbb{R}}$ that satisfies $J^2 = -\operatorname{id}$. The function J is called the **complex structure** of M.

Put $T_{M,\mathbb{C}} = T_{M,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \to M$ to be the complexified tangent bundle, which also has a holomorphic structure inherited from T_M. The complex structure J induces a complex bundle isomorphism $J_{\mathbb{C}} : T_{M,\mathbb{C}} \to T_{M,\mathbb{C}}$, and on each fibre $J_{\mathbb{C}}$ has eigenvalues $\pm\sqrt{-1}$. This gives a global decomposition

$$T_{M,\mathbb{C}} = T^{1,0}_M \oplus T^{0,1}_M$$

The bundle homomorphism $\operatorname{Re} : T_{M,\mathbb{C}} \to T_{M,\mathbb{R}}$ defined by taking real part maps $T^{1,0}_M$ and $T^{0,1}_M$ isomorphically onto $T_{M,\mathbb{R}}$ as smooth vector bundles.

10.3 Differential form. Let M be a complex manifold, U an open subset of M and $k \geqslant 0$. The sheaf $\mathcal{A}^k_{M,\mathbb{C}}$ of smooth sections of $\bigwedge^k T^\vee_{M,\mathbb{C}} \to M$ is called the **sheaf of smooth k-forms on M with coefficient $\mathbb{C}$.** The decomposition in (10.2) induces a decomposition

$$\mathcal{A}^k_{M,\mathbb{C}} = \bigoplus_{p+q=k} \mathcal{A}^{p,q}_M$$

Explicitly, in a local chart U of M with local coordinates $z_1, \ldots, z_n$, $\mathcal{A}^{p,q}_M(U)$ is the subspace of $\mathcal{A}^k_M(U)$ generated by the k-forms of the form

$$f dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$$

with $f : U \to \mathbb{C}$ a smooth function. An element in $\mathcal{A}^{p,q}_M(U)$ is called a (p,q)**-form**.

The sheaf Ω^k_M of holomorphic sections of $\bigwedge^k T_M \to M$ is called the **sheaf of holomorphic k-forms on M**. By definition, we see a holomorphic k-form can be viewed as a smooth $(k,0)$-form.

10.2 Serre duality

10.2.1 Traces and residues

In this subsubsection, we do not suppose an algebra is commutative. Still, we assume an algebra is unital.

10.4 Traces. Let k be a field and V a k-vector space. We say $\theta \in \operatorname{End}_k V$ is **finite potent** if $\dim_k \theta^n V < \infty$ for some integer $n \geqslant 0$. For a finite potent $\theta \in \operatorname{End}_k V$, a **trace** $\operatorname{tr}_V \theta \in k$ can be defined, having the following properties :

(i) If $\dim_k V < \infty$, then $\operatorname{tr}_V \theta$ is the usual trace.

(ii) If W is a subspace of V with $\theta W \subseteq W$, then

$$\mathrm{tr}_V\,\theta = \mathrm{tr}_W\,\theta + \mathrm{tr}_{V/W}\,\theta.$$

(iii) If θ is nilpotent, then $\mathrm{tr}_V\,\theta = 0$.

These properties also characterizes traces. In fact, if $\theta \in \mathrm{End}_k V$ is finite potent, there exists $n \gg 0$ such that the subspace $W := \theta^n V$ satisfies $\theta W \subseteq W$ and $\dim_k W < \infty$. Then

$$\mathrm{tr}_V\,\theta = \mathrm{tr}_W\,\theta + \mathrm{tr}_{V/W}\,\theta = \mathrm{tr}_W\,\theta$$

for θ in nilpotent on the quotient V/W. This also gives a well-defined definition of traces. There are some additional properties of traces.

(iv) If $F \subseteq \mathrm{End}_k V$ is a finite potent subspace (i.e. there exists some $n \geqslant 0$ such that $\dim_k \theta_1 \cdots \theta_n V < \infty$ for any $\theta_1, \ldots, \theta_n \in F$), then $\mathrm{tr}_V : F \to k$ is k-linear.

Indeed, we may assume F is finite dimensional, and compute the traces of all elements of F on the finite dimensional subspace

$$W = F^n V = \sum_{\sigma \in S_n} \theta_{\sigma(1)} \cdots \theta_{\sigma(n)} V$$

where $\{\theta_1, \ldots, \theta_n\}$ is a k-basis for F.

(v) If $\varphi : V' \to V$ and $\psi : V \to V'$ are k-linear with $\varphi\psi$ and $\psi\varphi$ finite potent, then

$$\mathrm{tr}_V(\varphi\psi) = \mathrm{tr}_{V'}(\psi\varphi).$$

Indeed, for $n \gg 0$ the maps φ and ψ induce mutually inverse isomorphisms between the subspace $W' = (\psi\varphi)^n V'$ and $W = (\varphi\psi)^n V$, under which the endomorphisms $(\psi\varphi)|_{W'}$ and $(\varphi\psi)|_W$ correspond.

10.5 Definition. Let V be a k-vector space and A, B be two k-subspaces of V.

1. A is **not much bigger** than B if $\dim_k(A+B)/B < \infty$, or equivalently, $A \subseteq B + W$ for some finite dimensional subspace $W \subseteq V$. In this case we write $A < B$.
2. A is **about the same size** as B if $A < B$ and $B < A$. In this case we write $A \sim B$.

It follows from definition at once that the relation $<$ is transitive and $\sim$ is an equivalence relation. Note that $<$ is preserved under $\mathrm{End}_k V$. Namely, if $A < B$, then $\varphi A < \varphi B$ for any $\varphi \in \mathrm{End}_k V$. Also, we have

$$\sum_{i=1}^m A_i < \bigcap_{j=1}^n B_j \text{ if and only if } A_i < B_j \text{ for all } i, j.$$

10.6 Let V be a k-vector space and fix a k-subspace A of V. Define

$$\begin{aligned} E &= \{\theta \in \mathrm{End}_k V \mid \theta A < A\} \\ E_1 &= \{\theta \in \mathrm{End}_k V \mid \theta V < A\} \\ E_2 &= \{\theta \in \mathrm{End}_k V \mid \theta A < 0\} = \{\theta \in \mathrm{End}_k V \mid \dim_k \mathrm{Im}\,\theta < \infty\} \\ E_0 &= E_1 \cap E_2 = \{\theta \in \mathrm{End}_k V \mid \theta V < A,\ \theta A < 0\}. \end{aligned}$$

Proposition.

(a) E is a k-subalgebra of $\mathrm{End}_k V$.

(b) The E_i's are two sided ideals in E, $E_1 + E_2 = E$ and E_0 is finite potent.

(c) E and the E_i's depend only on the equivalence class of A under $\sim$.

(d) If $(\varphi, \psi) \in E_0 \times E \cup E_1 \times E_2$, the commutator $[\varphi, \psi] = \varphi\psi - \psi\varphi$ lies in E_0 and has zero trace.

Proof. (a) is clear, and the first statement in (b) follows from the facts that $\psi V < V$ for any $\psi \in \mathrm{End}_k V$ and $<$ is transitive. That E_0 if finite potent is clear. Write $V = A \oplus A'$ for some k-subspace A' of V. If we denote by π and π' the projections to A and A', respectively, then $\mathrm{id}_V = \pi + \pi'$ with $\pi \in E_1$, $\pi' \in E_2$. This implies $E = E_1 + E_2$, so (b) is proved. (c) is clear. (d) follows from definition and (10.4).(v).

□

10.7 Definition-Theorem. Let K be a commutative k-algebra, V a K-module, and $A \subseteq V$ a k-subspace such that $fA < A$ for all $f \in K$. Let E and E_i be as in (10.6). The condition on A implies that the image $K \to \mathrm{End}_k V$ lies in E; for an element $f \in K$, we shall denote its image in E by the same letter f.

Then, there exists a unique k-linear map, the **residue map**

$$\mathrm{res}_A^V : \Omega_{K/k} \to k$$

such that for any pair $f, g \in K$, we have

$$\mathrm{res}_A^V(f\,dg) = \mathrm{tr}_V[f_1, g_1]$$

for $f_1, g_1 \in E$ such that

(a) $f \equiv f_1 \pmod{E_2}$, $g \equiv g_1 \pmod{E_2}$, and

(b) either $f_1 \in E_1$ or $g_1 \in E_1$.

Proof. Let $f, g \in K$. Since $E = E_1 + E_2$ (10.6), f_1 and g_1 that satisfy (a) and (b) always exist. Then $[f_1, g_1] \in E_1$ by (b), and $[f_1, g_1] \equiv [f, g] = 0 \pmod{E_2}$ by (b) and the commutativity of K. Hence $[f_1, g_1] \in E_1 \cap E_2 = E_0$, so the trace $\mathrm{tr}_V[f_1, g_1] \in k$ is defined. By (10.6).(d), the value of $\mathrm{tr}_V[f_1, g_1]$ is unaltered if f_1 or g_1 is changed by an element in E_2, provided that the other is in E_1. Moreover, by (10.4).(iv), $(f, g) \mapsto \mathrm{tr}_V[f_1, g_1]$ is a k-bilinear map on K, so it gives rise to a k-linear map

$$\begin{aligned} r : K \otimes_k K &\longrightarrow k \\ f \otimes g &\longmapsto \mathrm{tr}_V[f_1, g_1]. \end{aligned}$$

Recall that by the very definition of $\Omega_{K/k}$ there is a surjective k-linear map

$$\begin{aligned} c : K \otimes_k K &\longrightarrow \Omega_{K/k} \\ f \otimes g &\longmapsto f\,dg \end{aligned}$$

such that $\ker c$ is generated over k by elements of the form $f \otimes gh - fg \otimes h - fh \otimes g$. By surjectivity of c, the residue res_A^V, if it exists, is the only map r' such that $r' \circ c = r$. Such r' exists if and only if r vanishes on $\ker c$. This is the case, for if $f, g, h \in K$, we can choose suitable $f_1, g_1, h_1 \in E_1$ and let $(fg)_1 = f_1 g_1$, $(gh)_1 = g_1 h_1$, $(fh)_1 = f_1 h_1$. Now the identity

$$[f_1, g_1 h_1] = [f_1 g_1, h_1] + [f_1 h_1, g_1]$$

implies that r vanishes on $\ker c$, proving the existence and the uniqueness res_A^V. □

10.8 Retain the notation in (10.7). For $f, g \in K$, let

$$\begin{aligned} B &= A + gA \\ C &= B \cap f^{-1}(A) \cap (fg)^{-1}(A) = \{v \in B \mid fv \in A,\ fgv \in A\}. \end{aligned}$$

Then $\dim_k B/C < \infty$, and

$$\mathrm{res}_A^V(f\,dg) = \mathrm{tr}_{B/C}[\pi f, g],$$

where $\pi : V \to A$ is a k-linear projection.

10.9 Retain the notation in (10.7). We list some properties for the residue map res_A^V.

(R1) If V' is a K-submodule of V with $V \supseteq V' \supseteq A$, then $\mathrm{res}_A^V = \mathrm{res}_A^{V'}$. For this, we usually suppress the superscript V and simply write res_A.

(R2) If $fA + fgA + fg^2A \subseteq A$, then $\mathrm{res}_A(f\,dg) = 0$. In particular, this is so if $fA,\ gA \subseteq A$. Thus res_A is identically zero if A is a K-submodule of V.

(R3) For $g \in K$, $\mathrm{res}_A(g^n\,dg) = 0$ for all integers $n \geqslant 0$, and if g is invertible in K, then the same holds for $n \leqslant -2$. In particular, $\mathrm{res}_A(dg) = 0$ for all $g \in K$.

(R4) If $g \in K^\times$ and $h \in K$ such that $hA \subseteq A$, then

$$\mathrm{res}_A(hg^{-1}dg) = \mathrm{tr}_{A/(A\cap gA)}(h) - \mathrm{tr}_{gA/(A\cap gA)}(h).$$

In particular, $\mathrm{res}_A(g^{-1}dg) = \dim_k A/gA$.

(R5) Suppose $B \subseteq V$ is another k-subspace such that $fB < B$ for all $f \in K$. Then

$$f(A+B) < A+B \text{ and } f(A\cap B) < A\cap B \text{ for all } f \in K$$

and $\mathrm{res}_A + \mathrm{res}_B = \mathrm{res}_{A+B} + \mathrm{res}_{A\cap B}$.

(R6) Let K' be a commutative K-algebra which is a free K-module of finite rank. Let $V' = K' \otimes_K V$ and let $A' = \sum_i x_i \otimes A \subseteq V'$, where $\{x_i\}$ is a K-basis for K'. Then $f'A' < A'$ for all $f' \in K'$, the $\sim$-equivalence class of A' depends only on that of A, not on the choice of basis $\{x_i\}$, and we have

$$\mathrm{res}_{A'}(f'dg) = \mathrm{res}_A((\mathrm{tr}_{K'/K} f)dg)$$

for $f' \in K'$ and $g \in K$.

10.2.2 Residues on algebraic curves

Deuxième partie

Group schemes

11 Affine group schemes

11.1 Group functors. Let S be a scheme. A **group functor over** S/S**-group functor** is a functor $G : \mathbf{Sch}_S^{\mathrm{op}} \to \mathbf{Gp}$. A **morphism between group functors over** S is simply a natural transformation. Denote by $\mathbf{GpFun}_S$ the category of group functors over S.

11.2 Group schemes. A **group scheme over** S/S**-group scheme** is a representable group functor over S. Denote by $\mathbf{GpSch}_S$ the full subcategory of $\mathbf{GpFun}_S$ consisting of group schemes over S. If $S = \operatorname{Spec} A$ is affine, we simply write $\mathbf{GpSch}_A = \mathbf{GpSch}_{\operatorname{Spec} A}$, and $\mathbf{GpSch} = \mathbf{GpSch}_{\mathbb{Z}}$.

11.3 Example - classical groups. Let k be a ring and V a k-module. We can form the group functor $\mathrm{GL}_V : \mathbf{Alg}_k \to \mathbf{Gp}$ by

$$\mathrm{GL}_V(R) = \mathrm{Aut}_{\mathbf{Mod}_k}(V \otimes_k R).$$

When k is a field and V is finite over k, this is represented by the algebra

$$k[(X_{ij})_{1 \leqslant i,j \leqslant n}, (\det X)^{-1}].$$

In the sequel we shall stick to the following list of symbols :

k	unital ring
G	group functor over k
A	k-algebra that represents G if G is an affine group scheme
R	k-algebra
V	k-module

11.4 Let k be a ring. Recall in (3.54) we have an equivalence

$$\begin{aligned} \operatorname{Spec} : \mathbf{Alg}_k^{\mathrm{op}} &\longrightarrow \mathbf{AffSch}_k \\ A &\longmapsto \operatorname{Spec} A \end{aligned}.$$

If $G = \operatorname{Spec} A$ is a group scheme, the multiplication, identity and inversion are transformed to k-homomorphisms :

$$\Delta : A \to A \otimes_k A, \qquad \iota : A \to A, \qquad \varepsilon : A \to k$$

These are called the **comultiplication**, **antipode**, and **counit** on A, respectively.

11.4.1 Definition. In general, a **Hopf algebra over** k is a unital commutative k-algebra A with three maps

$$\Delta : A \to A \otimes_k A, \qquad \iota : A \to A, \qquad \varepsilon : A \to k$$

called **comultiplication**, **antipode**, and **counit** respectively such that the following diagrams commute :

(i) Coassociativity :

$$\begin{tikzcd} A \otimes_k (A \otimes_k A) \arrow[d, dash, "\text{canonical}"', "\wr"] & A \otimes_k A \arrow[l, "\mathrm{id}_A \times \Delta"'] & \\ & & A \arrow[ul, "\Delta"'] \arrow[dl, "\Delta"] \\ (A \otimes_k A) \otimes_k A & A \otimes_k A \arrow[l, "\Delta \times \mathrm{id}_A"] & \end{tikzcd}$$

(ii)

$$
\begin{tikzcd}
A \otimes_k k & & A \otimes_k A \arrow[ll, "\mathrm{id}_A \times \varepsilon"'] & \\
A \arrow[u, no head, "\wr"'] \arrow[d, no head, "\wr"] & & & A \arrow[lll, no head, "\mathrm{id}_A"'] \arrow[ul, "\Delta"'] \arrow[dl, "\Delta"] \\
k \otimes_k A & & A \otimes_k A \arrow[ll, "\varepsilon \times \mathrm{id}_G"] &
\end{tikzcd}
$$

(iii)

$$
\begin{tikzcd}
& A \otimes_k A \arrow[dl, "\mathrm{mult}"'] & & A \otimes_k A \arrow[ll, "\mathrm{id}_A \times \iota"'] & \\
A & & k \arrow[ll, "1"'] & & A \arrow[ll, "\varepsilon"'] \arrow[ul, "\Delta"'] \arrow[dl, "\Delta"] \\
& A \otimes_k A \arrow[ul, "\mathrm{mult}"] & & A \otimes_k A \arrow[ll, "\iota \times \mathrm{id}_A"'] &
\end{tikzcd}
$$

A **morphism between two Hopf algebras over** k is a unital k-algebra homomorphism $\phi : A \to B$ such that $\Delta_B \circ \phi = (\phi \times \phi) \circ \Delta_A$. Denote by $\mathbf{HopfAlg}_k$ the category of Hopf algebras over k.

11.4.2 Lemma. If $\phi : A \to B$ is a morphism between Hopf algebras over k, then ϕ preserves counits and antipodes.

11.4.3 Equivalence. If we denote by $\mathbf{AffGpSch}_k$ the full subcategory of $\mathbf{GpSch}_k$ consisting of affine group schemes, then Spec restricts to an equivalence

$$\mathrm{Spec} : \mathbf{HopfAlg}_k^{\mathrm{op}} \longrightarrow \mathbf{AffGpSch}_k$$

A similar definition and results hold for the relative spec $\mathbf{Spec}_X$ over a scheme X.

11.5 Linear representations. For a group functor $G : \mathbf{Alg}_k \to \mathbf{Gp}$, a **(linear) representation of** G is a k-homomorphism $G \to GL_V$ for some k-module. A **morphism between two linear representations** of G is a k-linear map $V \to W$ that intertwines the G-action. Denote by Rep_G the category of linear representations of G.

11.5.1 Comodules. Suppose G is an affine group scheme represented by Spec A. By Yoneda's lemma for each k-module we have a bijection

$$
\begin{aligned}
\mathrm{Hom}(G, GL_V) &\longrightarrow GL_V(A) \\
\varphi &\longmapsto \varphi_A(\mathrm{id}_A)
\end{aligned}
$$

The element $\varphi_A(\mathrm{id}_A)$ is an A-automorphism $V \otimes_k A \to V \otimes_k A$ which is uniquely determined by its restriction

$$\rho = \rho_\varphi : V \to V \otimes_k A$$

to V. That $\varphi : G \to GL_V$ is a homomorphism amounts to saying that ρ satisfies the two commutative diagrams

$$
\begin{tikzcd}
V \arrow[r, "\rho"] \arrow[d, "\rho"'] & V \otimes_k A \arrow[d, "\mathrm{id}_V \otimes \Delta"] \\
V \otimes_k A \arrow[r, "\rho \otimes \mathrm{id}_A"'] & V \otimes_k A \otimes_k A
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "\rho"] \arrow[dr, "\sim"'] & V \otimes_k A \arrow[d, "\mathrm{id}_V \otimes \varepsilon"] \\
& V \otimes_k k
\end{tikzcd}
$$

In general,

Definition. Let A be a Hopf algebra. A **right A-comodule** is a k-module V along with a k-linear map $\rho : V \to V \otimes_k A$ that satisfies the above commutative diagram. A **homomorphism between two comodules** $\rho : V \to V \otimes_k A$ and $\theta : W \to W \otimes_k A$ is a k-linear map $T : V \to W$ such that

$$\theta \circ T = (T \otimes \mathrm{id}_A) \circ \rho.$$

Denote by $\mathbf{Comod}_A$ the category of right A-comodules.

11.5.2 Theorem. If G is an affine group scheme represented by the algebra A, the assignment

$$[\varphi : G \to GL_V] \mapsto [\rho : V \to V \otimes_k A]$$

defines an equivalence of category

$$\mathbf{Rep}_G \cong \mathbf{Comod}_A$$

11.6 Stabilizers. Let G an affine k-group scheme represented by A and $\varphi : G \to GL_V$ a representation over k. For a k-subspace $W \subseteq V$, we can define the **stabilizer** of W as the group functor

$$\mathrm{Stab}_G(W) : R \mapsto \{g \in G(R) \mid \varphi_R(g)(W \otimes_k R) \subseteq W \otimes_k R\}.$$

11.6.1 Lemma. If k is a field, then $\mathrm{Stab}_G(W)$ is an affine closed subgroup scheme of G.

Proof. Let $\rho : V \to V \otimes_k A$ be the comodule map. Let $(e_i)_{i \in I}$ be a k-basis for W and extend it to a k-basis $(e_i)_{i \in I \sqcup J}$ for V. For $i \in I$ write

$$\rho(e_i) = \sum e_j \otimes a_{ij}$$

so that

$$\varphi_R(g)e_i = \sum e_j \otimes g(a_{ij})$$

for all $g \in G(R) = \mathrm{Hom}_{\mathbf{Alg}_k}(A, R)$. Then $g \in \mathrm{Stab}_G(W)(R)$ if and only if

$$g(a_{ij}) = 0 \text{ for all } (i, j) \in I \times J.$$

Hence $\mathrm{Stab}_G(W)$ is the closed subscheme of $G = \mathrm{Spec}\, A$ cut off by the ideal generated by $(a_{ij})_{i \in I, j \in J}$. □

11.7 Theorem (Chevalley). Let G be an affine algebraic group. Every algebraic subgroup of G is the stabilizer of a one-dimensional subspace in some finite dimensional representation of G.

11.8 Trigonalizable algebraic groups. An affine algebraic k-group G is said to be **trigonalizable** if every simple representation of G is one-dimensional. This term is justified by the

Lemma. For an affine algebraic k-group, TFAE :

(i) G is trigonalizable.

(ii) Every finite dimensional representation of G is trigonalizable.

(iii) G is isomorphic to a subgroup of groups of upper triangular matrices.

(iv) There exists a normal unipotent algebraic subgroup U of G such that G/U is diagonalizable.

11.9 Maximal affine quotient. Let G be an algebraic group over k. The group operation on G turns $\mathcal{O}_G(G)$ into a Hopf algebra over k, so

$$G^{\mathrm{aff}} := \operatorname{Spec} \mathcal{O}_G(G)$$

is an affine group scheme. The assignment $G \mapsto G^{\mathrm{aff}}$ defines a functor $\mathbf{AlgGp}_k \to \mathbf{AffGpSch}_k$. By (3.7) there is a bijection

$$\operatorname{Hom}_{\mathbf{Sch}_k}(G, H) \cong \operatorname{Hom}_{\mathbf{Sch}_k}(G^{\mathrm{aff}}, H)$$

for an affine scheme H, functorial in both arguments.

11.9.1 Lemma. Let G be an algebraic group over k. Then $\mathcal{O}_G(G)$ is a finitely generated k-algebra.

11.9.2 Hence G^{aff} is an affine algebraic group. Every homomorphism from G into an affine algebraic group factors through the natural map

$$G \longrightarrow G^{\mathrm{aff}}.$$

In view of this, we call G^{aff} the **maximal affine quotient** of G.

12 Lie algebras

12.1 Root datum

12.1 Reflection. Let F be a field of characteristic 0 and V be a finite dimensional F-vector space. Write $\langle , \rangle : V \times V^\vee \to F$ for the duality pairing

$$\langle x, f \rangle := f(x), \qquad (x, f) \in V \times V^\vee.$$

A **reflection** on V is an operator $s : V \to V$ that fixes an hyperplane and acts as -1 on a complementary line. If s is a reflection and $\alpha \in V\backslash\{0\}$ is such that $s(\alpha) = -\alpha$, we then say s is a **reflection with vector** α.

12.1.1 Lemma. If $\alpha^\vee \in V^\vee$ is such that $\langle \alpha, \alpha^\vee \rangle = 2$, then the operator $s_{\alpha,\alpha^\vee} : V \to V$ defined by

$$s_{\alpha,\alpha^\vee}(x) = x - \langle x, \alpha^\vee \rangle \alpha$$

is a reflection with vector v. Moreover, every reflection with vector α is for this form for a unique $\alpha^\vee$.

Proof. Compute

$$\begin{aligned} s_{\alpha,\alpha^\vee}(s_{\alpha,\alpha^\vee}(x)) &= s_{\alpha,\alpha^\vee}(x) - \langle s_{\alpha,\alpha^\vee}(x), \alpha^\vee \rangle \alpha \\ &= x - \langle x, \alpha^\vee \rangle \alpha - \langle x - \langle x, \alpha^\vee \rangle \alpha, \alpha^\vee \rangle \alpha = x - 2\langle x, \alpha^\vee \rangle \alpha + \langle x, \alpha^\vee \rangle \langle \alpha, \alpha^\vee \rangle \alpha = x, \end{aligned}$$

and

$$s_{\alpha,\alpha^\vee}(\alpha) = \alpha - 2\alpha = -\alpha.$$

Conversely, suppose s is a reflection with vector α, and let H be the hyperplane fixed by s. Define $\alpha^\vee : V \to F$ by $\alpha^\vee(H) = 0$ and $\alpha^\vee(\alpha) = 2$. This defines a functional as $V = H \oplus F\alpha$ satisfying $s = s_{\alpha,\alpha^\vee}$. $\square$

12.1.2 Lemma. Let R be a finite generating set of V. For $\alpha \in V\backslash\{0\}$, there is at most one reflection s with vector α such that $s(R) \subseteq R$.

Proof. Suppose s and s' are two such reflection, and put $t = ss'$. Then t acts on $F\alpha$ and $V/F\alpha$ all by 1, so $(t-1)^2 V \subseteq (t-1)F\alpha = 0$. This shows the minimal polynomial of t divides $(T-1)^2$. On the other hand, since $s(R) \subseteq R$ and $\#R < \infty$, there exists $n \in \mathbb{Z}_{\geqslant 1}$ such that $t^n(x) = x$ on R, and hence on V. This shows the minimal polynomial of t divides $T^n - 1$. Since $\gcd((T-1)^2, T^n - 1) = T - 1$, we deduce that $t = \mathrm{id}_V$.

$\square$

12.2 Root datum. A **root datum** is a 4-tuple $(X, R, X^\vee, R^\vee)$ consisting of a pair of finite rank free abelian groups $X, X^\vee$ with a perfect pairing $\langle , \rangle : X \times X^\vee \to \mathbb{Z}$ and a pair of finite subsets $R \subseteq X$, $R^\vee \subseteq X^\vee$ such that there is a bijection $R \ni \alpha \mapsto \alpha^\vee \in R^\vee$ satisfying

(i) $\langle \alpha, \alpha^\vee \rangle = 2$ for all $\alpha \in R$, and

(ii) the reflections $s_{\alpha,\alpha^\vee} : X \to X$ and $s_{\alpha^\vee,\alpha} : X^\vee \to X^\vee$ defined by

$$s_{\alpha,\alpha^\vee}(x) = x - \langle x, \alpha^\vee \rangle \alpha, \qquad s_{\alpha^\vee,\alpha}(x^*) = x^* - \langle \alpha, x^* \rangle \alpha^\vee$$

satisfy $s_{\alpha,\alpha^\vee}(R) \subseteq R$ and $s_{\alpha^\vee,\alpha}(R^\vee) \subseteq R^\vee$ for all $\alpha \in R$.

Clearly from the definition, we see $(X^\vee, R^\vee, X, R)$ also forms a root datum; this is the **dual root datum** of $(X, R, X^\vee, R^\vee)$. From (i) and (ii) we see $-\alpha \in R$ for all $\alpha \in R$. We say the root datum $(X, R, X^\vee, R^\vee)$ is **reduced** if the only multiples of α in R is $\pm\alpha$ for all $\alpha \in R$. The **Weyl group** $W = W(X, R, X^\vee, R^\vee)$ of the root datum is by definition

$$W := \langle s_{\alpha,\alpha^\vee} \mid \alpha \in R \rangle \leqslant \operatorname{Aut} X.$$

12.2.1 Lemma. For a root datum $(X, R, X^\vee, R^\vee)$, the bijection $R \ni \alpha \mapsto \alpha^\vee \in R^\vee$ is unique.

Proof. Put $V = \mathbb{Q}R \subseteq X_{\mathbb{Q}} := X \otimes_{\mathbb{Z}} \mathbb{Q}$. Then there is a quotient map $X_{\mathbb{Q}}^\vee \to V^\vee$; denote by f' the image of $f \in X_{\mathbb{Q}}^\vee$ in $V^\vee$. By condition (ii), (12.1.1) and (12.1.2), $(\alpha^\vee)' \in V^\vee$ is the unique vector such that $x \mapsto x - \langle x, (\alpha^\vee)' \rangle \alpha$ is the unique reflection with vector α on V that leaves R invariant. To conclude, it suffices to show the quotient $X_{\mathbb{Q}}^\vee \to V^\vee$ is injective on $R^\vee$. Indeed, say $\alpha, \beta \in R$ satisfy $\langle \gamma, \alpha^\vee \rangle = \langle \gamma, \beta^\vee \rangle$ for all $\gamma \in R$. Then for $x \in R$,

$$\begin{aligned} s_{\alpha,\alpha^\vee} s_{\beta,\beta^\vee}(x) = s_{\alpha,\alpha^\vee}(x - \langle x, \beta^\vee \rangle \beta) &= x - \langle x, \alpha^\vee \rangle \alpha - \langle x, \beta^\vee \rangle (\beta - \langle \beta, \alpha^\vee \rangle \alpha) \\ &= x + 2\langle x, \beta^\vee \rangle \alpha - \langle x, \beta^\vee \rangle (\alpha + \beta). \end{aligned}$$

When $x = \beta$, we get $s_{\alpha,\alpha^\vee} s_{\beta,\beta^\vee}(\beta) = 2\alpha - \beta = 2(\alpha - \beta) + \beta$. Also,

$$s_{\alpha,\alpha^\vee} s_{\beta,\beta^\vee}(\alpha - \beta) = (3\alpha - 2\beta) - (2\alpha - \beta) = \alpha - \beta.$$

Iterating gives $(s_{\alpha,\alpha^\vee} s_{\beta,\beta^\vee})^n(\beta) = 2n(\alpha - \beta) + \beta$. But $\#R < \infty$, so condition (ii) implies $\alpha = \beta$. $\square$

12.2.2 Lemma. For $\alpha \in R$, $x \in X$, $y \in X^\vee$, we have

$$\langle s_{\alpha,\alpha^\vee}(x), y \rangle = \langle x, s_{\alpha^\vee,\alpha}(y) \rangle.$$

In particular, $\langle s_{\alpha,\alpha^\vee}(x), s_{\alpha^\vee,\alpha} y \rangle = \langle x, y \rangle$.

Proof. This is a direct computation :

$$\langle s_{\alpha,\alpha^\vee}(x), y \rangle = \langle x - \langle x, \alpha^\vee \rangle \alpha, y \rangle = \langle x, y \rangle - \langle x, \alpha^\vee \rangle \langle \alpha, y \rangle = \langle x, y - \langle \alpha, y \rangle \alpha^\vee \rangle = \langle x, s_{\alpha^\vee,\alpha}(y) \rangle.$$

$\square$

In view of this lemma and (12.2.1), it shall causes no confusion to write

$$s_\alpha := s_{\alpha,\alpha^\vee}, \qquad s_\alpha^\vee := s_{\alpha^\vee,\alpha}$$

for each $\alpha \in R$.

12.3 Let $(X, R, X^\vee, R^\vee)$ be a root datum. Consider the homomorphism

$$\begin{aligned} p : X &\longrightarrow X^\vee \\ x &\longmapsto \sum_{\alpha \in R} \langle x, \alpha^\vee \rangle \alpha^\vee. \end{aligned}$$

For $x \in X$, one computes that

$$\langle x, p(x) \rangle = \sum_{\alpha \in R} \langle x, \alpha^\vee \rangle^2 \geqslant 0.$$

When $x \in R$, this is a strictly positive integer. From this inequality we see

$$\ker p = \{x \in X \mid \langle x, R^\vee \rangle = 0\} =: X_0.$$

Also, for $\beta \in R$, since $s_\beta^\vee(R^\vee) = R^\vee$, by (12.2.2) we compute

$$\langle s_\beta x, p(s_\beta x) \rangle = \sum_{\alpha \in R} \langle s_\beta x, \alpha^\vee \rangle^2 = \sum_{\alpha \in R} \langle x, s_\beta^\vee \alpha^\vee \rangle^2 = \sum_{\alpha \in R} \langle x, \alpha^\vee \rangle^2 = \langle x, p(x) \rangle.$$

Hence $\langle wx, p(wx) \rangle = \langle x, p(x) \rangle$ for all $x \in X$, $w \in W$.

12.3.1 Lemma. For $\alpha \in R$, one has $\langle \alpha, p(\alpha) \rangle \alpha^\vee = 2p(\alpha)$.

Proof. For $\alpha, \beta \in R$, by (12.2.2) we compute

$$\begin{aligned}\langle \alpha, \beta^\vee \rangle^2 \alpha^\vee = \langle \alpha, \beta^\vee \rangle (\alpha^\vee - s_\alpha^\vee(\beta^\vee)) = \langle \alpha, \beta^\vee \rangle \alpha^\vee + \langle -\alpha, \beta^\vee \rangle s_\alpha^\vee(\beta^\vee) &= \langle \alpha, \beta^\vee \rangle \alpha^\vee + \langle s_\alpha(\alpha), \beta^\vee \rangle s_\alpha^\vee(\beta^\vee) \\ &= \langle \alpha, \beta^\vee \rangle \alpha^\vee + \langle \alpha, s_\alpha^\vee(\beta^\vee) \rangle s_\alpha^\vee(\beta^\vee).\end{aligned}$$

Summing over $\beta \in R$, since $s_\alpha^\vee(R^\vee) = R^\vee$, we see

$$\langle \alpha, p(\alpha) \rangle \alpha^\vee = \sum_{\beta \in R} \langle \alpha, \beta^\vee \rangle^2 \alpha^\vee = 2p(\alpha).$$

□

12.3.2 Lemma. The homomorphism $p : X \to X^\vee$ induces an homomorphism $p : \mathbb{Z}R \to \mathbb{Z}R^\vee$ with $[\mathbb{Z}R^\vee : \operatorname{Im} p] < \infty$. In particular, it induces an isomorphism

$$1 \otimes p : \mathbb{Q}R \longrightarrow \mathbb{Q}R^\vee.$$

Proof. By (12.3.1), we see $[\mathbb{Z}R : 2 \operatorname{Im} p] < \infty$, so $[\mathbb{Z}R^\vee : \operatorname{Im} p] < \infty$. Tensoring with $\mathbb{Q}$ kills the torsion, implying $1 \otimes p : \mathbb{Q}R \to \mathbb{Q}R^\vee$ is surjective. In particular, $\dim \mathbb{Q}R \geqslant \dim \mathbb{Q}R^\vee$. Applying this to the dual root datum, we get the reverse inequality. Hence $\dim \mathbb{Q}R = \dim \mathbb{Q}R^\vee$, and $1 \otimes p$ is an isomorphism. □

12.3.3 Lemma. $\mathbb{Z}R \cap X_0 = \{0\}$, and $[X : \mathbb{Z}R + X_0] < \infty$. Hence there is an exact sequence

$$0 \longrightarrow \mathbb{Z}R \oplus X_0 \longrightarrow X \longrightarrow F \longrightarrow 0$$

for some finite abelian group F.

Proof. The map $1 \otimes p : X_\mathbb{Q} \to \mathbb{Q}R^\vee$ has kernel $(X_0)_\mathbb{Q}$, and maps $\operatorname{span}_\mathbb{Q} R$ isomorphically onto $\mathbb{Q}R^\vee$ by (12.3.2). Hence

$$X_\mathbb{Z} = \mathbb{Q}R \oplus (X_0)_\mathbb{Q},$$

and this finishes the proof. □

12.3.4 Toral/semisimple root datum. A root datum $(X, R, X^\vee, R^\vee)$ is **toral** if $R = \varnothing$, and is **semisimple** if R spans $X_\mathbb{Q}$. By the previous lemma, we see $(X, R, X^\vee, R^\vee)$ is "isogenous" to a direct sum of semisimple root datum $(\mathbb{Z}R, R, \mathbb{Z}R^\vee, R^\vee)$ and a toral root datum $(X_0, \varnothing, X_0^\vee, \varnothing)$.

12.4 Finiteness of Weyl group. Let $(X, R, X^\vee, R^\vee)$ be a root datum. The pairing $\langle , \rangle : X \times X^\vee \to \mathbb{Z}$ induces a pairing

$$\langle , \rangle : \mathbb{Q}R \times \mathbb{Q}R^\vee \longrightarrow \mathbb{Q}.$$

This is again a perfect pairing. Indeed, if $x \in \mathbb{Q}R$ is such that $\langle x, R^\vee \rangle = 0$, then $x \in (X_0)_\mathbb{Q} \cap \mathbb{Q}R = \{0\}$. Since $\dim \mathbb{Q}R = \dim \mathbb{Q}R^\vee$, this implies it is perfect on both sides.

12.4.1 Lemma. For $x \in X$ and $w \in W = W(X, R, X^\vee, R^\vee)$, one has $w(x) - x \in \mathbb{Z}R$.

Proof. Let $\alpha \in R$ and $w \in W$. Then

$$s_\alpha w(x) - x = s_\alpha(w(x) - x) + (s_\alpha(x) - x).$$

By induction it suffices to show $s_\alpha(x) - x \in \mathbb{Z}R$, which is clear :

$$s_\alpha(x) - x = -\langle x, \alpha^\vee \rangle \alpha \in \mathbb{Z}R.$$

□

12.4.2 The Weyl group W acts on R by condition (ii) of a root datum. We can also let W act on $X^\vee$ as follows : for $w \in W$ and $y \in R^\vee$, let $wy \in X^\vee$ be the unique element such that

$$\langle x, wy \rangle = \langle wx, y \rangle$$

for all $x \in X$. This is well-defined as the pairing is perfect. By (12.2.2) we have

$$s_\alpha.y = s_\alpha^\vee(y)$$

for all $\alpha \in R$ and $y \in X^\vee$. In particular, by condition (ii) we see W leaves $R^\vee$ invariant.

Lemma. For $\alpha, \beta \in R$, one has $(s_\alpha(\beta))^\vee = s_\alpha.\beta^\vee$.

Proof. By definition, $\langle s_\alpha(\beta), s_\alpha.\beta^\vee \rangle = \langle \beta, \beta^\vee \rangle = 2$, so by (12.1.1)

$$x \mapsto x - \langle x, s_\alpha.\beta^\vee \rangle s_\alpha(\beta)$$

is a reflection with vector $s_\alpha(\beta)$. For $x \in X$, we have

$$x - \langle x, s_\alpha.\beta^\vee \rangle s_\alpha(\beta) = s_\alpha\left(s_\alpha(x) - \langle s_\alpha(x), \beta^\vee \rangle \beta\right) = s_\alpha(s_\beta(s_\alpha(x))),$$

so the reflection leaves R invariant. By (12.1.2) and the proof of (12.2.1), this proves the lemma. □

Hence, the bijection $R \ni \alpha \mapsto \alpha^\vee \in R^\vee$ preserves the action of the Weyl group W.

12.4.3 Proposition. The Weyl group W acts on R faithfully. In particular, $\#W < \infty$.

Proof. By (12.4.2) it suffices to show W acts on $R^\vee$ faithfully. Suppose $w \in W$ is such that $w.\alpha^\vee = \alpha^\vee$ for all $\alpha \in R$. Then

$$\langle w(x) - x, \alpha^\vee \rangle = \langle x, w.\alpha^\vee \rangle - \langle x, \alpha^\vee \rangle = 0.$$

By (12.4.1) and (12.4), this shows $w(x) = x$ for all $x \in X$, i.e., $w = \mathrm{id}_X$. □

12.5 Root system. Let F be a field of characteristic 0 and V a finite dimensional F-vector space. A subset $R \subseteq V$ is called a root system of V if

(i) $\#R < \infty$, $0 \notin R$ and R spans V,

(ii) for each $\alpha \in R$ there is a reflection s_α with vector α such that $s_\alpha(R) \subseteq R$, and

(iii) for all $\alpha, \beta \in R$, the vector $s_\alpha(\beta) - \beta$ is an integral multiple of α.

An element in R is called a **root**.

12.5.1 It follows from (12.1.2) that s_α in (ii) is unique. By (12.1.1) there is a unique element $\alpha^\vee \in V^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$ and $s_\alpha = s_{\alpha,\alpha^\vee}$. The unique vector $\alpha^\vee$ is called the **coroot** of $\alpha \in R$.

12.5.2 Rational model. Let V_0 be the $\mathbb{Q}$-span of R in V. Then clearly R is a root system of V_0, and the canonical map $V_0 \otimes_{\mathbb{Q}} F \to V$ is an isomorphism. This shows that root systems over the field F are the same as root systems over the field $\mathbb{Q}$.

12.5.3 For $\alpha, \beta \in R$, the number $\langle \beta, \alpha^\vee \rangle$ is the number n such that $\beta - s_\alpha(\beta) = n\alpha$; by (iii) we have $n \in \mathbb{Z}$. Now if $\alpha \in R$ and $c\alpha \in R$ for some $c \in F$, then

$$2c = \langle c\alpha, \alpha^\vee \rangle \in \mathbb{Z}$$

so $c \in \dfrac{1}{2}\mathbb{Z}$. Also,

12.6 Category of root data. Let k be a field. Set

$$p = \begin{cases} 1 & \text{, if } \operatorname{Char} k = 0 \\ p & \text{, if } \operatorname{Char} k = p \end{cases}.$$

An **isogeny of the root data** $(X, R, X^\vee, R^\vee) \to (Y, S, Y^\vee, S^\vee)$ **defined over** k consists of a injective homomorphism $f : Y \to X$ with finite cokernel, a bijection $\iota : R \to S$ and a map $q : S \to p^{\mathbb{Z}_{\geqslant 0}}$ satisfying

$$f(\iota(\alpha)) = q(\alpha)\alpha, \qquad f^\vee(\alpha^\vee) = q(\alpha)\iota(\alpha)^\vee$$

for all $\alpha \in R$. We say an isogeny is **central** if $q \equiv 1$ and is an **isomorphism** if it is central and f is an isomorphism of abelian groups.

Troisième partie

Étale cohomology

13 Sites

13.1 Presheaves

13.1 Presheaves. Let $\mathcal{C}, \mathcal{A}$ be categories. An **$\mathcal{A}$-valued presheaf on** $\mathcal{C}$ is a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{A}$. Denote by

$$\mathrm{PShv}(\mathcal{C}, \mathcal{A})$$

be the category of all $\mathcal{A}$-valued presheaves where the morphisms between two presheaves are simply natural transformations of functors. When $\mathcal{A} = \mathbf{Set}$, we simply write

$$\mathrm{PShv}(\mathcal{C}) = \mathrm{PShv}(\mathcal{C}, \mathbf{Set})$$

and refer this to the category of presheaves of sets.

13.1.1 Set-theoretic issue. In order for $\mathrm{PShv}(\mathcal{C}, \mathcal{A})$ to be a (locally small) category, we must and do assume $\mathcal{C}$ is small.

13.1.2 Representables. There is a Yoneda's embedding

$$\begin{aligned} h : \mathcal{C} &\longrightarrow \mathrm{PShv}(\mathcal{C}) \\ X &\longmapsto \underline{X} = h_X := \mathrm{Hom}_{\mathcal{C}}(\cdot, X) \end{aligned}$$

By Yoneda lemma, there is a functorial bijection

$$\begin{aligned} \mathrm{Hom}_{\mathrm{PShv}(\mathcal{C})}(\underline{X}, F) &\xrightarrow{\ \sim\ } F(X) \\ T &\longmapsto T(X)(\mathrm{id}_X). \end{aligned}$$

13.1.3 Lemma. A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ in $\mathrm{PShv}(\mathcal{C})$ is a monomorphism (resp. epimorphism) if and only if $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective (resp. surjective) as sets for any $U \in \mathcal{C}$. In particular, φ is an isomorphism if and only if φ_U is bijective for each $U \in \mathcal{C}$.

Proof. The last assertion is clear. Both if directions are obvious, so it remains to prove the only if directions. Suppose $\varphi : \mathcal{F} \to \mathcal{G}$ is a monomorphism and $U \in \mathcal{C}$. To see $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective, by Yoneda lemma this is the same as showing

$$\begin{aligned} \mathrm{Hom}_{\mathrm{PShv}(\mathcal{C})}(\underline{U}, \mathcal{F}) &\longrightarrow \mathrm{Hom}_{\mathrm{PShv}(\mathcal{C})}(\underline{U}, \mathcal{G}) \\ T &\longmapsto \varphi \circ T \end{aligned}$$

is injective. But this follows from the definition of a monomorphism.

Suppose $\varphi : \mathcal{F} \to \mathcal{G}$ is an epimorphism. Define a presheaf $\mathcal{H} \in \mathrm{PShv}(\mathcal{C})$ by setting

$$U \mapsto \mathcal{H}(U) := \mathcal{G}(U) \cup_{\mathcal{F}(U)} \mathcal{G}(U)$$

where the rightmost denote the pushout in **Set**. Since formation of pushouts is functorial, this clearly defines a presheaf. There are two obvious inclusion maps $\mathcal{G} \rightrightarrows \mathcal{H}$ which precompose with φ to the same map. Since φ is an epimorphism, this forces the two inclusions to be the same, i.e. $\varphi_U(\mathcal{F}(U)) = \mathcal{G}(U)$. This proves the surjectivity. □

13.1.4 Subpresheaves. A presheaf of sets $\mathcal{G}$ is a called **subpresheaf** of $\mathcal{F}$ if $\mathcal{G}(U)$ is a subset of $\mathcal{F}(U)$ for each $U \in \mathcal{C}$. This defines a monomorphism $\mathcal{G} \to \mathcal{F}$.

13.1.5 Image presheaves. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves. The **image presheaf** $\operatorname{Im}\varphi$ is the unique subpresheaf $\mathcal{H}$ of $\mathcal{G}$ so that φ factors as a composition $\mathcal{F} \to \mathcal{H} \to \mathcal{G}$ with $\mathcal{F} \to \mathcal{H}$ an epimorphism.

Proof. For the existence, simply set $(\operatorname{Im}\varphi)(U) := \varphi_U(\mathcal{F}(U))$. The uniqueness is clear. $\square$

13.1.6 Limits and colimits. For any $U \in \mathcal{C}$, set

$$\begin{aligned} \mathrm{ev}_U : \mathrm{PShv}(\mathcal{C}) &\longrightarrow \mathbf{Set} \\ \mathcal{F} &\longmapsto \mathcal{F}(U) \end{aligned}$$

For any diagram $F : \mathcal{I} \to \mathrm{PShv}(\mathcal{C})$, if for each $U \in \mathcal{C}$ the limit of the diagram $\mathrm{ev}_U \circ F : \mathcal{I} \to \mathbf{Set}$ exists, then $\lim F$ exists in $\mathrm{PShv}(\mathcal{C})$, and is given by

$$\lim F : U \mapsto \lim(\mathrm{ev}_U \circ F) = \lim F(U).$$

The same statement holds for colimits.

13.2 Generators. A set of objects F of $\mathcal{C}$ is called a **generating set** if for any pair of distinct morphisms $f, g : X \rightrightarrows Y$ in $\mathcal{C}$, there exists some $Z \in F$ and a morphism $h : Z \to X$ such that $f \circ h \neq g \circ h$. If F is a singleton, the only element in F is then called a **generator** of $\mathcal{C}$. Dually we have a notion of **cogenerating sets** and **cogenerators**.

13.2.1 Comma category. Let $F : \mathcal{D} \to \mathcal{C}$ be a functor and $X \in \mathcal{C}$. The **comma category** $F \downarrow X$ is defined as follow :

- objects : $(a, \phi : F(a) \to X)$ where $a \in \mathcal{D}$ and $\phi \in \mathrm{Hom}_{\mathcal{C}}(F(a), X)$.
- morphisms : $(a, \phi) \to (b, \psi)$ consisting of $f \in \mathrm{Hom}_{\mathcal{D}}(a, b)$ such that $\psi \circ F(f) = \phi$.

There is a forgetful functor $\omega : F \downarrow X \to \mathcal{C}$ given by

$$\omega(f : (a, \phi) \to (b, \psi)) = [F(f) : F(a) \to F(b)].$$

Let Δ_X denote the constant diagram $\Delta_X : F \downarrow X \to \mathcal{C}$ of value X. For each $(a, \phi) \in F \downarrow X$, there is a map $\omega(a, \phi) = F(a) \xrightarrow{\phi} X$. Being functorial, this gives a cocone $\omega \to \Delta_X$.

13.2.2 Lemma. Let F be a generating set of $\mathcal{C}$. Suppose $\mathcal{C}$ admits all small colimit. Then any object $X \in \mathcal{C}$ admits an epimorphism from the colimit of the forgetful functor $\omega : F \downarrow X \to \mathcal{C}$.

Here we view F as the inclusion functor $F \subseteq \mathcal{C}$ of the full subcategory consisting of objects in F.

Proof. Since $\omega \to \Delta_X$ is a cocone, this gives a morphism $\operatorname{colim}\omega \to X$. By definition of a generating set, it is straightforward to see this is an epimorphism. $\square$

13.3 Lemma. The set of representable functors in $\mathrm{PShv}(\mathcal{C})$ is a generating set.

Proof. Two morphisms of presheaves $f, g : \mathcal{F} \rightrightarrows \mathcal{G}$ are equal if and only if $f_A = g_A$ for any $A \in \mathcal{C}$. By Yoneda lemma, this is the same as saying the induced maps

$$\begin{aligned} \mathrm{Hom}(h_A, \mathcal{F}) &\rightrightarrows \mathrm{Hom}(h_A, \mathcal{G}) \\ T &\longmapsto f \circ T \\ T &\longmapsto g \circ T \end{aligned}$$

are equal for any $A \in \mathcal{C}$. Taking contrapositive gives the result. $\square$

13.3.1 Yoneda. Any presheaf $\mathcal{F} \in \mathrm{PShv}(\mathcal{C})$ is isomorphic to the colimit of the forgetful functor $\omega : h \downarrow \mathcal{F} \to \mathrm{PShv}(\mathcal{C})$

Proof. It remains to show $\Phi : \omega \to \Delta_{\mathcal{F}}$ is a universal cocone. For this, let $\Psi : F \to \Delta_{\mathcal{G}}$ be any other cocone. Define a natural transformation $T : \mathcal{F} \to \mathcal{G}$ as follows. For each $A \in \mathcal{C}$ the map $T_A : \mathcal{F}(A) \to \mathcal{G}(A)$ is the one that corresponds, by Yoneda lemma, to the map

$$\begin{aligned} \mathrm{Hom}(h_A, \mathcal{F}) &\longrightarrow \mathrm{Hom}(h_A, \mathcal{G}) \\ \phi : h_A \to \mathcal{F} &\longmapsto \Psi_{(A,\phi)}. \end{aligned}$$

The commutativity condition for the cocone implies that T is a natural transformation. By construction this is the unique morphism such that $T \circ \Phi = \Psi$. This finishes the proof. □

13.4 Functoriality of presheaves. Let $f : \mathcal{C} \to \mathcal{D}$ be a functor. Then there is a pullback functor

$$\begin{aligned} f^p : \mathrm{PShv}(\mathcal{D}) &\longrightarrow \mathrm{PShv}(\mathcal{C}) \\ \mathcal{F} &\longmapsto \mathcal{F} \circ f. \end{aligned}$$

This admits a left adjoint $f_p : \mathrm{PShv}(\mathcal{C}) \to \mathrm{PShv}(\mathcal{D})$. For the construction, for each $V \in \mathcal{D}$ consider the comma category $V \downarrow f$:

- objects : (U, ϕ) with $U \in \mathcal{C}$ and $\phi \in \mathrm{Hom}_{\mathcal{D}}(V, f(U))$.
- morphisms : $\mathrm{Hom}_{V\downarrow f}((U,\phi),(U',\phi')) = \{\psi \in \mathrm{Hom}_{\mathcal{C}}(U,U') \mid \phi' = f(\psi) \circ \phi\}$.

Let ${}_V\omega : V \downarrow f \to \mathcal{C}$ be the forgetful functor. For any $\mathcal{F} \in \mathrm{PShv}(\mathcal{C})$ consider the diagram

$${}_V\mathcal{F} := \mathcal{F} \circ {}_V\omega : (V \downarrow f)^{\mathrm{op}} \to \mathbf{Set}$$

For $V \in \mathcal{D}$ define

$$(f_p\mathcal{F})(V) := \operatorname{colim} {}_V\mathcal{F}$$

where the colim is taken in **Set**. This exists as $\mathcal{I}_V$ is small (c.f. (13.1.1)). Each $g : V \to V'$ in $\mathcal{D}$ induces a functor $g' : V' \downarrow f \to V \downarrow f$ by sending (U, ϕ) to $(U, \phi \circ g)$ and its obvious action on morphisms. One has ${}_{V'}\mathcal{F} \circ g' = {}_V\mathcal{F}$. Passing to colimit, this shows $V \mapsto (f_p\mathcal{F})(V)$. A moment consideration shows that $\mathcal{F} \mapsto f_p\mathcal{F}$ is a functor.

13.4.1 Lemma. f_p is left adjoint to f^p.

Proof. We must show there exist bifunctorial bijections

$$\mathrm{Hom}_{\mathrm{PShv}(\mathcal{D})}(f_p\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{PShv}(\mathcal{C})}(\mathcal{F}, f^p\mathcal{G})$$

Notice for any $U \in \mathcal{C}$, by construction there exists a map $\mathcal{F}(U) \to (f_p\mathcal{F})(f(U))$. This produces a morphism $i : \mathcal{F} \to f^p f_p \mathcal{F}$ functorial in $\mathcal{F} \in \mathrm{PShv}(\mathcal{C})$. Define

$$\begin{aligned} \mathrm{Hom}_{\mathrm{PShv}(\mathcal{D})}(f_p\mathcal{F}, \mathcal{G}) &\longrightarrow \mathrm{Hom}_{\mathrm{PShv}(\mathcal{C})}(\mathcal{F}, f^p\mathcal{G}) \\ T &\longmapsto f^p(T) \circ i \end{aligned}$$

To construct an inverse, let $S : \mathcal{F} \to f^p\mathcal{G}$ by given. For each U this produces a map $S_U : \mathcal{F}(U) \to (f^p\mathcal{G})(U) = \mathcal{G}(f(U))$. Fix an $V \in \mathcal{D}$. For each $(U, \phi) \in V \downarrow f$ we obtain a map

$${}_V\mathcal{F}(U, \phi) = \mathcal{F}(U) \to \mathcal{G}(f(U)) \xrightarrow{\mathcal{G}(\phi)} \mathcal{G}(V).$$

Passing to colimit this gives $(f_p\mathcal{F})(V) \to \mathcal{G}(V)$. Varying V gives a morphism $f_p\mathcal{F} \to \mathcal{G}$. It is straightforward to see this defines an inverse to the map $T \mapsto f^p(T) \circ i$.

$\square$

13.4.2 Lemma. For any object $U \in \mathcal{C}$, one has $f_p(\underline{U}) = \underline{f(U)}$.

Proof. This follows from adjunction and Yoneda lemma :

$$\mathrm{Hom}(f_p(\underline{U}), F) = \mathrm{Hom}(\underline{U}, f^p F) = (f^p F)(U) = F(f(U)) = \mathrm{Hom}(\underline{f(U)}, F).$$

$\square$

13.5 Right adjoint of f^p. Since f^p commutes with all small colimits in $\mathrm{PShv}(\mathcal{C})$ (c.f. (13.1.6)), we can expect f^p has a right adjoint ${}_p f : \mathrm{PShv}(\mathcal{C}) \to \mathrm{PShv}(\mathcal{D})$.

To construct a right adjoint, for $V \in \mathcal{D}$ consider the comma category $f \downarrow V$, and let $\omega_V : f \downarrow V \to \mathcal{C}$ be the forgetful functor. For any $\mathcal{F} \in \mathrm{PShv}(\mathcal{C})$ consider the diagram

$$\mathcal{F}_V := \mathcal{F} \circ \omega_V : (f \downarrow V)^{op} \to \mathbf{Set}.$$

Define

$$({}_p f\mathcal{F})(V) := \lim \mathcal{F}_V.$$

The same reasoning in (13.4) shows that $\mathcal{F} \mapsto {}_p f\mathcal{F}$ defines a functor

$${}_p f : \mathrm{PShv}(\mathcal{C}) \to \mathrm{PShv}(\mathcal{D})$$

By construction there is a natural transformation $j : f^p \circ {}_p f\mathcal{F} \to \mathcal{F}$. This defines a map

$$\begin{aligned} \mathrm{Hom}(\mathcal{G}, {}_p f\mathcal{F}) &\longrightarrow \mathrm{Hom}(f^p\mathcal{G}, \mathcal{F}) \\ T &\longmapsto j \circ f^p(T) \end{aligned}$$

Lemma. ${}_p f$ is right adjoint to f^p.

Proof. Let $S : f^p\mathcal{G} \to \mathcal{F}$ be given; this gives for each $U \in \mathcal{C}$ that $S_U : \mathcal{G}(f(U)) \to \mathcal{F}(U)$. For any $f(U) \to V$, this gives $\mathcal{G}(V) \to \mathcal{F}(U)$. Running over $(U, f(U) \to V) \in f \downarrow V$, this produces a map $\mathcal{G}(V) \to ({}_p f\mathcal{F})(V)$ on the limit object. It is straightforward that this defines a natural transformation $\mathcal{G} \to {}_p f\mathcal{F}$. This shows the map other way around, and it is routine to show it defines an inverse. $\square$

13.6 Interactions. Let $f : \mathcal{C} \to \mathcal{D}$ and $g : \mathcal{D} \to \mathcal{C}$ be functors such that f is left adjoint to g. Then

(i) $f^p \circ h_{\mathcal{D}} = h_{\mathcal{C}} \circ g$;

(ii) $U \downarrow g$ has an initial object for any $U \in \mathcal{C}$;

(iii) $f \downarrow V$ has a terminal object for any $V \in \mathcal{D}$;

(iv) ${}_p f = g^p$;

(v) $f^p = g_p$.

13.2 Sheaves

13.7 Sieve. For a category $\mathcal{C}$ and an object X, a **sieve of** X is a subset S of the objects in comma category $\mathrm{id}_{\mathcal{C}} \downarrow X$ that is closed under precomposition : if $(Y, Y \to X) \in S$ and $Z \to Y$ is any other morphism in $\mathcal{C}$, then $(Z, Z \to Y \to X) \in S$.

13.7.1 Subfunctor from a sieve. Let S be a sieve of X. Define a subfunctor F_S of $\mathrm{Hom}_{\mathcal{C}}(\cdot, X)$ by

— $F_S(Y) := \{\phi \in \mathrm{Hom}_{\mathcal{C}}(Y, X) \mid \phi \in S\}$.

— $F_S(Y \to Z) := \mathrm{Hom}_{\mathcal{C}}(Y \to Z, X)$.

In fact, any subfunctor F of $\mathrm{Hom}_{\mathcal{C}}(\cdot, X)$ gives rise to a sieve

$$S_F := \bigcup_{Y \in \mathcal{C}} F(Y).$$

The assignments $F \to F_S$ and $S \to S_F$ are mutually inverses.

13.7.2 Sieve from arbitrary subset. Let $S' \subseteq \mathrm{id}_{\mathcal{C}} \downarrow X$ be a collection of morphisms to X. Define $S \subseteq \mathrm{id}_{\mathcal{C}} \downarrow X$ by

$$S = \{(Y, \phi : Y \to X) \mid \text{there exists some } (U, \phi' : U \to X) \in S' \text{ and } \psi \in \mathrm{Hom}_{\mathcal{C}}(Y, U) \text{ such that } \phi = \phi' \circ \psi\}.$$

By construction this is a sieve, and is the smallest sieve that contains S'.

13.7.3 Let $S' = \{X_i \to X\}_i$ be a collection of morphisms to X, and let S denote the smallest sieve containing S'. We assume the fibre products $X_i \leftarrow X_i \times_X X_j \to X_j$ exist for all i, j. For a presheaf $\mathcal{F} \in \mathrm{PShv}(\mathcal{C})$, denote

$$\mathcal{F}(S') = \text{equalizer}\left(\prod_i \mathcal{F}(X_i) \rightrightarrows \prod_{ij} \mathcal{F}(X_i \times_X X_j)\right).$$

There is a canonical map

$$\mathcal{F}(S') \longrightarrow \mathrm{Hom}(F_S, \mathcal{F})$$

constructed as follows. Let $(x_i)_i \in \mathcal{F}(S')$. For any $(U \to X) \in S$, pick any i so that $U \to X$ factors through $X_i \to X$. Let $\phi_U(U \to X)$ denote the image of $x_i \in \mathcal{F}(X_i)$ in $\mathcal{F}(U)$. To see this is well-defined, suppose $U \to X$ also factors through $X_j \to X$. In view of the commutative diagram

$$\begin{tikzcd}
U \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X_j \times_X X_i \arrow[r] \arrow[d] & X_i \arrow[d] \\
& X_j \arrow[r] & X
\end{tikzcd}$$

it is easy to see the independence. Running over all $(U, U \to X)$, we obtain a natural transformation $\phi : F_S \to \mathcal{F}$.

Lemma. The above map is bijective.

Proof. Let $T \in \mathrm{Hom}(F_S, \mathcal{F})$ be a natural transformation. For each i, the map $X_i \to X$ is an element in $F_S(X_i)$. Hence

$$(T_{X_i}(X_i \to X))_i \in \prod_i \mathcal{F}(X_i)$$

We claim this lands in $\mathcal{F}(S')$. Since T is a natural transformation, there is a commutative diagram

$$\begin{tikzcd}
F_S(X_i) \arrow[r, "T_{X_i}"] \arrow[dr] & \mathcal{F}(X_i) \arrow[drr] & & \\
& F_S(X_i \times_X X_j) \arrow[rr, "T_{X_i \times_X X_j}"] & & \mathcal{F}(X_i \times_X X_j) \\
F_S(X_j) \arrow[r, "T_{X_j}"] \arrow[ur] & \mathcal{F}(X_j) \arrow[urr] & &
\end{tikzcd}$$

This implies

$$\begin{aligned}T_{X_i}(X_i \to X)|_{X_i\times_X X_j} &= T_{X_i\times_X X_j}((X_i \to X)|_{X_i\times_X X_j}) \\ &= T_{X_i\times_X X_j}((X_j \to X)|_{X_i\times_X X_j}) = T_{X_j}((X_j \to X)|_{X_i\times_X X_j})\end{aligned}$$

We then obtain a map

$$\mathrm{Hom}(F_S, \mathcal{F}) \longrightarrow \mathcal{F}(S')$$

It is routine to check this defines an inverse to $\mathcal{F}(S') \to \mathrm{Hom}(F_S, \mathcal{F})$. □

13.8 Grothendieck topology. A **Grothendieck topology** on $\mathcal{C}$ is an assignment to each object U a collection of sieves of U, called the **covering sieves**, satisfying the following :

— If F is a covering sieve of U and $g : V \to U$ is any morphism, then the **pullback sieve**

$$g^*F : x \mapsto \{\phi \in \mathrm{Hom}_{\mathcal{C}}(x, V) \mid g \circ \phi \in F(x)\}$$

is a covering sieve of V.

— If F is a sieve of U such that the sieve $\bigcup_{V\in\mathcal{C}}\{g \in \mathrm{Hom}_{\mathcal{C}}(V, U) \mid g^*F \text{ covers } V\}$ contains a covering sieve of U, then F is a covering sieve.

— The maximal sieve $\mathrm{Hom}_{\mathcal{C}}(\cdot, U)$ covers U for any object U.

13.8.1 Sheaves. Denote by J a Grothendieck topology on $\mathcal{C}$. We say a presheaf $\mathcal{F} \in \mathrm{PShv}(\mathcal{C})$ is a sheaf for J if the map

$$\mathrm{Hom}(h_U, \mathcal{F}) \longrightarrow \mathrm{Hom}(F_S, \mathcal{F})$$

is bijective for any object U and its covering sieve S. Here the map is the one induced by the inclusion $F_S \to h_U = \mathrm{Hom}_{\mathcal{C}}(\cdot, U)$. Denote the

$$\mathrm{Shv}(\mathcal{C}, J)$$

the full subcategory of $\mathrm{PShv}(\mathcal{C})$ consisting of sheaves for J. This is called the **topos** defined by the Grothendieck topology J.

13.8.2 Separated presheaves. Similarly, we say a presheaf $\mathcal{F} \in \mathrm{PShv}(\mathcal{C})$ is a **separated presheaf** for J if the map

$$\mathrm{Hom}(h_U, \mathcal{F}) \longrightarrow \mathrm{Hom}(F_S, \mathcal{F})$$

is injective for any object U and its covering sieve S.

13.8.3 Pretopology. A **Grothendieck pretopology** on the category $\mathcal{C}$ is a *set* T of families of morphisms $\{U_i \to U\}_i$ in $\mathcal{C}$ satisfying the following :

(i) If $V \to U$ is an isomorphism in $\mathcal{C}$, then $\{V \to U\} \in T$.

(ii) If $\{U_i \to U\}_i \in T$ and $\{U_{ij} \to U_i\} \in T$ for each i, then $\{U_{ij} \to U\}_{ij} \in T$.

(iii) If $\{U_i \to U\}_i \in T$ and $V \to U$ is any morphism in $\mathcal{C}$, then $U_i \times_U V$ exists for all i, then $\{U_i \times_U V \to V\}_i \in T$.

A family $\{U_i \to U\} \in T$ will be called a **cover(ing)** of U.

13.8.4 Site. A **site** is a category together with a Grothendieck pretopology.

13.8.5 Sheaves. Let T be a pretopology of $\mathcal{C}$. A presheaf $\mathcal{F} \in \mathrm{PShv}(\mathcal{C})$ is called a **sheaf for** T if for any covering $\{U_i \to U\}$, the sequence

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{ij} \mathcal{F}(U_i \times_U U_j)$$

is a equalizer diagram. Denote the

$$\mathrm{Shv}(\mathcal{C}, T)$$

the full subcategory of $\mathrm{PShv}(\mathcal{C})$ consisting of sheaves for T. This is called the **topos** defined by the site $(\mathcal{C}, T)$.

13.8.6 Separated presheaves. Let T be a pretopology of $\mathcal{C}$. A presheaf $\mathcal{F} \in \mathrm{PShv}(\mathcal{C})$ is called **separated for** T if for any covering $\{U_i \to U\}$, the map

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i)$$

is injective.

13.8.7 Lemma. A presheaf $\mathcal{F}$ is a sheaf (resp. a separated presheaf) for T if and only if for any cover $S' = \{U_i \to U\}_i$, the map

$$\mathrm{Hom}(h_U, \mathcal{F}) \to \mathrm{Hom}(F_S, \mathcal{F})$$

is bijective (resp. injective). Here S and F_S are defined as in (13.7.3).

Proof. By Yoneda, LHS is simply $\mathcal{F}(U)$. This then follows from (13.7.3). □

13.8.8 Generation. Let T be a pretopology of $\mathcal{C}$. For each object U we declare a sieve F of U to be a coverage sieve if it contains a covering of U. Akin to the point-set topology, we say T **generates/is a basis for this Grothendieck topology**. We shall denote this topology by J_T.

13.8.9 Comparison. Let T be a pretopology of $\mathcal{C}$. We claim

$$\mathrm{Shv}(\mathcal{C}, J_T) = \mathrm{Shv}(\mathcal{C}, T)$$

as full subcategories of $\mathrm{PShv}(\mathcal{C})$. The containment $\subseteq$ follows from (13.7.3). For $\supseteq$, let $\mathcal{F} \in \mathrm{Shv}(\mathcal{C}, T)$, F a covering sieve of U and $S' := \{U_i \to U\}_i$ be any covering contained in F. Let S be the smallest sieve containing S'. Consider the composition

$$\mathrm{Hom}(h_U, \mathcal{F}) \to \mathrm{Hom}(F, \mathcal{F}) \to \mathrm{Hom}(F_S, \mathcal{F})$$

where $\mathcal{F}(S')$ is defined in (13.7.3). By loc cit, this is bijective. To see the first arrow is surjective, it suffices to show the

Lemma. The map $\mathrm{Hom}(F, \mathcal{F}) \to \mathrm{Hom}(F_S, \mathcal{F})$ is injective if $\mathcal{F}$ is a separated presheaf.

Proof. Suppose $\phi, \psi : F \to \mathcal{F}$ are two natural transformations that restricts to the same one on F_S. Let $f : V \to U$ be in F. We must show $\phi(V \to U) = \psi(V \to U)$ as elements in $\mathcal{F}(V)$.

Consider the covering $\{V \times_U U_i \to V\}_i$ of V. Since $\{V \times_U U_i \to V \xrightarrow{g} U\}_i$ lies $F_S(V \times_U U_i)$, we have $\phi(V \times_U U_i \to U) = \psi(V \times_U U_i \to U)$ for each i, as elements in $\mathcal{F}(V \times_U U_i)$. Since the map $\mathcal{F}(V) \to \prod_i \mathcal{F}(V \times_U U_i)$ is injective, this shows $\phi(V \to U) = \psi(V \to U)$. □

The above argument also shows that for a presheaf, being separated for T is the same as begin separated for J_T.

13.9 Limits in $\mathrm{Shv}(\mathcal{C}, J_T)$**.** Let $\phi : I \to \mathrm{Shv}(\mathcal{C}, J_T)$ be a diagram such that the limit of $\phi' : I \xrightarrow{\phi} \mathrm{Shv}(\mathcal{C}, J_T) \to \mathrm{PShv}(\mathcal{C})$ exists. Then $\lim \phi$ exists equals $\lim \phi'$.

Proof. We must show $\lim \phi'$ is a sheaf. Let U be an object in $\mathcal{C}$ and F any covering sieve of U. Compute

$$\mathrm{Hom}(h_U, \lim \phi') = (\lim \phi')(U) = \lim_V (\phi'(V)(U)) = \lim_V \mathrm{Hom}(h_U, \phi'(V)) \cong \lim_V \mathrm{Hom}(F, \phi'(V)) = \mathrm{Hom}(F, \lim \phi')$$

where $\cong$ holds as each $\phi'(V)$ is a sheaf. It is straightforward to this is coincides the natural map $\mathrm{Hom}(h_U, \lim \phi') \to \mathrm{Hom}(F, \lim \phi')$. $\square$

13.10 Canonical topology. For a category $\mathcal{C}$, consider the Yoneda embedding $\mathcal{C} \to \mathrm{PShv}(\mathcal{C})$. We define a Grothendieck topology on $\mathcal{C}$ by declaring a sieve S of U to be a covering sieve if and only if for any representable h_X, the map $\mathrm{Hom}(h_U, h_X) \to \mathrm{Hom}(S, h_X)$ is bijective. This is called the **canonical topology** on $\mathcal{C}$.

13.10.1 Subcanonical topology. A Grothendieck topology on $\mathcal{C}$ is called **subcanonical** if every representable is a sheaf. One can interpret the canonical topology as the *largest* subcanonical topology on $\mathcal{C}$.

13.11 Example : the site Top(X)**.** Let X be a topological space. On the category $\mathrm{Top}(X)$ we declare a sieve on an open subspace U to be a covering sieve if it is an open cover of U in the usual language. We shall always equip $\mathrm{Top}(X)$ with this topology. In this way we've subsumed (2.2) into this theory.

13.12 Example : the big site Top. For a topological space X, we say a collection of continuous maps $\{X_i \to X\}_i$ is a covering if each $X_i \to X$ is an open immersion and their image covers X. This defines a Grothendieck pretopology on Top.

13.12.1 Lemma. The topology in Top is subcanonical.

13.12.2 Set-theoretic issue. The category Top is not small. We choose to ignore any set-theoretic issue.

13.13 +-construction. Let $(\mathcal{C}, T)$ be a site. For each object U, let $J_T(U)$ denote the set of all covering sieves of U. This is naturally a poset : say $F \leqslant F'$ if $F \subseteq F'$ as subfunctors of h_U. For a presheaf $\mathcal{F}$, define

$$\mathcal{F}^+(U) := \varinjlim_{F \in J_T(U)} \mathrm{Hom}(F, \mathcal{F})$$

Since $h_U \in J_T(U)$, this comes with a map

$$\theta_U : \mathcal{F}(U) \cong \mathrm{Hom}(h_U, \mathcal{F}) \to \mathcal{F}^+(U).$$

If $g : V \to U$ is any morphism, for each $x \in \mathcal{C}$ there is a map

$$\begin{aligned} g^*F(x) &\longrightarrow F(x) \\ \phi &\longrightarrow g \circ \phi. \end{aligned}$$

where g^*F is defined in (13.8). This defines a map $\mathrm{Hom}(F, \mathcal{F}) \to \mathrm{Hom}(g^*F, \mathcal{F})$. By universal property, this gives rise to

$$\mathcal{F}^+(U) \longrightarrow \mathcal{F}^+(V).$$

compatible with $\mathcal{F}(U) \to \mathcal{F}(V)$ (for $g^* h_U = h_V$). This proves $\mathcal{F}^+$ is a functor that comes with a natural transformation $\theta : \mathcal{F} \to \mathcal{F}^+$.

13.13.1 Lemma. For any object $U \in \mathcal{C}$ and $s \in \mathcal{F}^+(U)$, there exists a covering $\{U_i \to U\}_i \in T$ such that $s|_{U_i}$ lies in the image of $\theta_{U_i} : \mathcal{F}(U_i) \to \mathcal{F}^+(U_i)$.

Proof. Say $s \in \mathrm{Hom}(F, \mathcal{F})$ for some covering sieve F. Pick any covering $S' = \{U_i \to U\}_i$ contained in F and let F_S be the smallest sieve it generates. Consider the composition (13.7.3)

$$\mathrm{Hom}(F, \mathcal{F}) \to \mathrm{Hom}(F_S, \mathcal{F}) \xrightarrow{\sim} \mathcal{F}(S') \subseteq \prod_i \mathcal{F}(U_i) \to \mathcal{F}(U_i)$$

Denote by s_i the image of s in $\mathcal{F}(U_i)$. It is straightforward to see $\theta_{U_i}(s_i) = s|_{U_i}$. □

13.13.2 Lemma.

(i) The presheaf $\mathcal{F}^+$ is separated.

(ii) If $\mathcal{F}$ is separated, then $\mathcal{F}^+$ is a sheaf and $\mathcal{F} \to \mathcal{F}^+$ is a monomorphism.

(iii) If $\mathcal{F}$ is a sheaf, then $\mathcal{F} \to \mathcal{F}^+$ is an isomorphism.

Proof. clean up the proof.

(i) Let F be any covering sieve of U. We must show $\mathcal{F}^+(U) \cong \mathrm{Hom}(h_U, \mathcal{F}^+) \to \mathrm{Hom}(F, \mathcal{F}^+)$ is injective. Let $s, t \in \mathcal{F}^+(U)$ be two elements mapped to the same element in $\mathrm{Hom}(F, \mathcal{F}^+)$. Pick a covering sieve F' such that $F' \leqslant F$ and $s, t \in \mathrm{Hom}(F', \mathcal{F})$; we assume F' is the smallest sieve generated by a covering $\{U_i \to U\}_i$. Then s, t map to the same element in $\mathrm{Hom}(F', \mathcal{F}^+) \subseteq \prod_i \mathcal{F}^+(U_i)$ as well, implying $s|_{U_i} = t|_{U_i}$. Since we're assuming $s, t \in \mathrm{Hom}(F', \mathcal{F}) \subseteq \prod_i \mathcal{F}(U_i)$, this means $\theta_{U_i}(s|_{U_i}) = \theta_{U_i}(t|_{U_i})$ for any i (where we view $s|_{U_i}$, $t|_{U_i}$ as elements in $\mathcal{F}(U_i)$). But then we can find a further covering $\{W_{ij} \to U_i\}$ such that $s|_{W_{ij}} = t|_{W_{ij}}$ (as they are equal in the colimit). Let F'' be the smallest sieve containing $\{W_{ij} \to U\}_{ij}$; then $s = t$ as elements in $\mathrm{Hom}(F'', \mathcal{F})$.

(ii) Since $\mathcal{F}$ is separated, each $\mathrm{Hom}(h_U, \mathcal{F}) \to \mathrm{Hom}(F, \mathcal{F})$ $(F \in J_T(U))$ is injective. By passing to colimit, we see $\mathcal{F} \to \mathcal{F}^+$ is monomorphic (c.f. (13.1.3)).

Now let $\{U_i \to U\}_i$ be a covering of U, and suppose $(s_i)_i$ satisfies $s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}$ for any i, j. Choose a covering $\{U_{ik} \to U_i\}_k$ such that $s_i|_{U_{ik}} = \theta_{U_{ik}}(s_{ik})$ for some (unique) $s_{ik} \in \mathcal{F}(U_{ik})$. Moreover,

$$\theta(s_{ik}|_{U_{ik} \times_U U_{jl}}) = \theta(s_{ik})|_{U_{ik} \times_U U_{jl}} = s_i|_{U_{ik} \times_U U_{jl}}) = s_j|_{U_{ik} \times_U U_{jl}}) = \theta(s_{jl}|_{U_{ik} \times_U U_{jl}})$$

so by injectivity $s_{ik}|_{U_{ik} \times_U U_{jl}} = s_{jl}|_{U_{ik} \times_U U_{jl}}$. Hence $(s_{ik})_{ik} \in \mathcal{F}(\{U_{ik} \to U\}_{ik})$ defines an element s in $\mathcal{F}^+(U)$. It is direct to see $s|_{U_i} = s_i$.

(iii) Since $\mathcal{F}$ is a sheaf, each $\mathrm{Hom}(h_U, \mathcal{F}) \to \mathrm{Hom}(F, \mathcal{F})$ $(F \in J_T(U))$ is bijective.

□

13.13.3 Corollary. For each presheaf $\mathcal{F}$, the presheaf $\mathcal{F}^{++}$ is sheaf. Moreover, the assignment $\mathcal{F} \mapsto \mathcal{F}^{++}$ is a left adjoint of the inclusion functor $\mathrm{Shv}(\mathcal{C}, T) \subseteq \mathrm{PShv}(\mathcal{C})$.

Proof. The first assertion is clear. Since $\mathcal{F} \mapsto \mathcal{F}^+$ is a functor, we have a diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{PShv}(\mathcal{C})}(\mathcal{F}, \mathcal{G}) \arrow[r] & \mathrm{Hom}_{\mathrm{PShv}(\mathcal{C})}(\mathcal{F}^+, \mathcal{G}^+) \arrow[r] & \mathrm{Hom}_{\mathrm{PShv}(\mathcal{C})}(\mathcal{F}^{++}, \mathcal{G}^{++}) \\
& & \mathrm{Hom}_{\mathrm{Shv}(\mathcal{C},T)}(\mathcal{F}^{++}, \mathcal{G}) \arrow[u, "\theta_{\mathcal{G}^+} \circ \theta_{\mathcal{G}} \circ"', "\wr"] \arrow[ull, bend left, "\circ \theta_{\mathcal{F}^+} \circ \theta_{\mathcal{F}}"]
\end{tikzcd}$$

It is straightforward to see this is commutative everywhere, proving the last assertion.

□

13.13.4 Sheafification. For a presheaf $\mathcal{F}$, the sheaf $\mathcal{F}^\dagger := \mathcal{F}^{++}$ is called the **sheafification** of $\mathcal{F}$.

13.14 Colimits in $\mathrm{Shv}(\mathcal{C}, J_T)$**.** Let $\phi : I \to \mathrm{Shv}(\mathcal{C}, J_T)$ be a diagram such that the limit of $\phi' : I \xrightarrow{\phi} \mathrm{Shv}(\mathcal{C}, J_T) \to \mathrm{PShv}(\mathcal{C})$ exists. Then colim ϕ exists equals the sheafification of colim ϕ'.

Proof. Since sheafification is a left adjoint, it commutes with all existing colimits. □

13.15 Exactness of sheafification. The sheafification functor $\mathrm{PShv}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{C}, J_T)$ is exact.

Proof. Since it is a left adjoint, it is right exact. Conversely, note the limit in $\mathrm{Shv}(\mathcal{C}, J_T)$ is computed in $\mathrm{PShv}(\mathcal{C})$. Since in the definition of $\mathcal{F}^+$, the colimit is actually a filtered colimit. The left exactness follows at once from the following lemma. □

13.15.1 Lemma. In **Set**, filtered colimits commute with finite limits.

Proof.

□

13.16 Lemma. Let $\varphi : \mathcal{F} \to \mathcal{G}$ in $\mathrm{Shv}(\mathcal{C}, J_T)$.

(i) φ is a monomorphism if and only if $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective as sets for any $U \in \mathcal{C}$.

(ii) φ is an epimorphism if and only if for any $U \in \mathcal{C}$ and $s \in \mathcal{G}(U)$ there exists a covering $\{U_i \to U\}_i$ such that $s|_{U_i} \in \operatorname{Im} \varphi_{U_i}$.

Proof.

(i) The if part is clear. Suppose φ is a monomorphism. For any presheaf $\mathcal{H}$, we have a sequence of bijections

$$\mathrm{Hom}(\mathcal{H}, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{H}^\dagger, \mathcal{F}) \xrightarrow{\varphi\circ} \mathrm{Hom}(\mathcal{H}\dagger, \mathcal{G}) \cong \mathrm{Hom}(\mathcal{H}, \mathcal{G}).$$

This implies φ is a monomorphism in $\mathrm{PShv}(\mathcal{C})$. The result then follows from (13.1.3).

(ii) Suppose the if holds. Let $f, g : \mathcal{G} \rightrightarrows \mathcal{H}$ be two morphisms of sheaves such that $f \circ \varphi = g \circ \varphi$. Let $U \in \mathcal{C}$ and $y \in \mathcal{G}(U)$. We must show $f_U(x) = g_U(y)$. By hypothesis we can find a covering $\{U_i \to U\}_i$ and $x_i \in \mathcal{F}(U_i)$ such that $\varphi_{U_i}(x_i) = x|_{U_i}$. Then $f_U(x)|_{U_i} = f_{U_i}(x|_{U_i}) = f_{U_i}(\varphi_{U_i}(x_i)) = g_{U_i}(\varphi_{U_i}(x_i)) = g_U(x)|_{U_i}$. Since $\mathcal{H}$ is separated, this shows $f_U(x) = g_U(x)$, as claimed.

Before proving the only if part, we need some preparation. Notice by if part and (13.13.1), for any presheaf $\mathcal{H}$ the canonical map $\mathcal{H} \to \mathcal{H}^\dagger$ is epimorphic. Note also

13.16.1 Lemma. If $\phi : \mathcal{F} \to \mathcal{G}$ is a monomorphism of presheaves, then $\phi^\dagger : \mathcal{F}^\dagger \to \mathcal{G}^\dagger$ is a monomorphism of sheaves.

Proof. This follows from Lemma 13.15.1. □

13.16.2 Finish of the proof. Suppose φ is an epimorphism. Form $\operatorname{Im}\varphi$ as in (13.1.5). By the lemma and the remark above, the sheafification $(\operatorname{Im}\varphi)^\dagger$ is a subsheaf of $\mathcal{G}$ and φ factorizes as $\mathcal{F} \to (\operatorname{Im}\varphi)^\dagger \to \mathcal{G}$ with the first arrow epimorphic.

We claim $(\operatorname{Im}\varphi)^\dagger = \mathcal{G}$. Form the presheaf pushout $f_1, f_2 \mathcal{G} \rightrightarrows \mathcal{G} \cup_{(\operatorname{Im}\varphi)^\dagger} \mathcal{G}$. By construction $f_1 \circ \varphi = f_2 \circ \varphi$. By passing to the sheafification $\theta : \mathcal{G} \cup_{(\operatorname{Im}\varphi)^\dagger} \mathcal{G} \to (\mathcal{G} \cup_{(\operatorname{Im}\varphi)^\dagger} \mathcal{G})^\dagger$, we see $\theta \circ f_1 = \theta \circ f_2$ as φ is epic. Let $U \in \mathcal{C}$ and $x \in \mathcal{G}(U_i)$. Then

$\theta \circ f_1(x) = \theta \circ f_2(x)$. Since θ factors through the monomorphism $(\mathcal{G} \cup_{(\mathrm{Im}\,\varphi)^\dagger} \mathcal{G})^+ \to (\mathcal{G} \cup_{(\mathrm{Im}\,\varphi)^\dagger} \mathcal{G})^\dagger$ by (13.13.1) we can find a covering $\{U_i \to U\}_i$ such that $f_1(x)|_{U_i} = f_2(x)|_{U_i}$. But $f_j(x)|_{U_i} = f_j(x|_{U_i})$, this means $x|_{U_i} \in (\mathrm{Im}\,\varphi)^\dagger(U_i)$. Gluing, this proves $x \in (\mathrm{Im}\,\varphi)^\dagger(U)$.

To finish the proof, it suffices to apply (13.13.1) to the sheaf $(\mathrm{Im}\,\varphi)^\dagger(U)$. □

13.16.3 Corollary. A morphism of sheaves is an isomorphism if and only if it is a monomorphism and an epimorphism.

Proof. The only if part holds true without any constraint. Conversely, suppose $\varphi : \mathcal{F} \to \mathcal{G}$ is a monomorphism and an epimorphism. Let $U \in \mathcal{U}$. We define an inverse ψ_U to φ_U as follows. Let $y \in \mathcal{G}(U)$. By Lemma 13.16 we can find a covering $\{U_i \to U\}_i$ and unique $x_i \in \mathcal{F}(U_i)$ such that $\varphi(x_i) = y|_{U_i}$. Moreover

$$\varphi(x_i|_{U_i \times_U U_j}) = y|_{U_i \times_U U_j} = \varphi(x_j|_{U_i \times_U U_j})$$

so by injectivity of $\varphi_{U_i \times_U U_j}$ we see $x_i|_{U_i \times_U U_j} = x_j|_{U_i \times_U U_j}$ for any i, j. Since $\mathcal{F}$ is a sheaf, this glues to a unique element $x \in \mathcal{F}(U)$ such that $x|_{U_i} = x_i$. Define $\psi_U(y) = x$. It is clear that ψ_U is well-defined and is inverse to φ_U. This show φ_U is bijective for every U, proving φ is an isomorphism. □

13.17 Lemma. Suppose $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism in $\mathrm{PShv}(\mathcal{C})$, and $B \subseteq \mathcal{C}$ is a collection of objects such that φ_U is bijective in **Set** for all $U \in B$. If every object in $\mathcal{C}$ has a covering by elements in B, then $\varphi^\dagger$ is an isomorphism in $\mathrm{Shv}(\mathcal{C}, T)$.

13.18 Representables. Let $(\mathcal{C}, T)$ be a site. In general a representable h_U is not a sheaf. Nevertheless, for any sheaf $\mathcal{F}$ we still have

$$\mathcal{F}(U) \cong \mathrm{Hom}_{\mathrm{PShv}(\mathcal{C})}(h_U, \mathcal{F}) \cong \mathrm{Hom}_{\mathrm{Shv}(\mathcal{C},T)}(h_U^\dagger, \mathcal{F}).$$

Lemma. If $\{U_i \to U\}$ is a covering, then the morphism of presheaves

$$\coprod_i h_{U_i} \longrightarrow h_U$$

sheafifies to an epimorphism of sheaves.

Proof. For each sheaf $\mathcal{F}$, we must show the map

$$\mathrm{Hom}(h_U^\dagger, \mathcal{F}) \longrightarrow \mathrm{Hom}\left(\left(\coprod_i h_{U_i}\right)^\dagger, \mathcal{F}\right)$$

is injective. But this is the same as the map

$$\mathrm{Hom}(h_U, \mathcal{F}) \cong \mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \cong \mathrm{Hom}\left(\coprod_i h_{U_i}, \mathcal{F}\right)$$

where the middle arrow is given by restriction. Since $\mathcal{F}$ is a sheaf, this is injective. □

13.18.1 Yoneda. Let $\mathcal{F}$ be a sheaf. By (13.3.1), there is an isomorphism of presheaves

$$\operatorname{colim} \omega = \operatorname*{colim}_{(U, h_U \to \mathcal{F})} h_U \cong \mathcal{F}.$$

By Passing to sheafification, we obtain an isomorphism of sheaves

$$\left(\operatorname*{colim}_{(U, h_U \to \mathcal{F})} h_U\right)^\dagger \cong \mathcal{F}.$$

Since sheafification is a left adjoint, it commutes with colimit, whence

$$\mathcal{F} \cong \operatorname*{colim}_{(U, h_U \to \mathcal{F})} h_U^\dagger,$$

where the colimit is taken in $\mathrm{Shv}(\mathcal{C}, T)$. In other words, $\mathcal{F}$ is the colimit of the diagram $h \downarrow \mathcal{F} \to \mathrm{PShv}(\mathcal{C}) \xrightarrow{(\cdot)^\dagger} \mathrm{Shv}(\mathcal{C}, T)$.

13.3 Morphisms of topoi

13.19 Continuous functors. Let $(\mathcal{C}, T)$, $(\mathcal{D}, S)$ be sites. A functor $f : \mathcal{C} \to \mathcal{D}$ is called **continuous** if for any $\{U_i \to U\}_i \in T$ we have

— $\{f(U_i) \to f(U)\}_i \in S$, and

— for any $V \to U$ in $\mathcal{C}$, the map $f(V \times_U U_i) \to f(V) \times_{f(U)} f(U_i)$ is an isomorphism.

13.19.1 Lemma. Let $f : (\mathcal{C}, T) \to (\mathcal{D}, S)$ be continuous.

(i) f^p sends sheaves on $\mathcal{D}$ to sheaves on $\mathcal{C}$.

(ii) The composition $\mathrm{Shv}(\mathcal{C}, T) \to \mathrm{PShv}(\mathcal{C}) \xrightarrow{f_p} \mathrm{PShv}(\mathcal{D}) \xrightarrow{(\cdot)^\dagger} \mathrm{Shv}(\mathcal{D}, T)$ is left adjoint to $f^p : \mathrm{Shv}(\mathcal{D}, S) \to \mathrm{Shv}(\mathcal{C}, T)$.

We denote the composition in (ii) simply by $f_s : \mathrm{Shv}(\mathcal{C}, T) \to \mathrm{Shv}(\mathcal{D}, S)$.

Proof. (i) is clear. For (ii), compute

$$\mathrm{Hom}((f_p\mathcal{F})^\dagger, \mathcal{G}) \cong \mathrm{Hom}(f_p\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, f^p\mathcal{G}).$$

□

13.19.2 Lemma. For any presheaf $\mathcal{F}$, there is a natural isomorphism $(f_p\mathcal{F})^\dagger \cong (f_p(\mathcal{F})^\dagger)^\dagger$.

Proof. This follows from a formal computation

$$\mathrm{Hom}((f_p\mathcal{F})^\dagger, \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, f^p\mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}^\dagger, f^p\mathcal{G}) \cong \mathrm{Hom}((f_p(\mathcal{F})^\dagger)^\dagger, \mathcal{G}).$$

□

13.19.3 Corollary. One has $f_s(h_U^\dagger) \cong h_{f(U)}^\dagger$.

Proof. $f_s(h_U^\dagger) \cong (f_p h_U)^\dagger = h_{f(U)}^\dagger$.

□

13.20 Cocontinuous functors. Let $(\mathcal{C}, T)$, $(\mathcal{D}, S)$ be sites. A functor $f : \mathcal{C} \to \mathcal{D}$ is called **cocontinuous** if for any $U \in \mathcal{C}$ and any covering sieve F of $f(U)$, there exists a covering $\{U_j \to U\}_j \in S$ such that $\{f(U_j) \to f(U)\}_j$ is contained in F.

13.20.1 Lemma. Let $f : (\mathcal{C}, T) \to (\mathcal{D}, S)$ be cocontinuous.

(i) ${}_pf$ sends sheaves on $\mathcal{C}$ to sheaves on $\mathcal{D}$.

(ii) The composition $\mathrm{Shv}(\mathcal{D}, S) \to \mathrm{PShv}(\mathcal{D}) \xrightarrow{f^p} \mathrm{PShv}(\mathcal{C}) \xrightarrow{(\cdot)^\dagger} \mathrm{Shv}(\mathcal{C}, S)$ is left adjoint to ${}_sf := {}_pf : \mathrm{Shv}(\mathcal{C}, S) \to \mathrm{Shv}(\mathcal{D}, T)$, and is exact.

Proof. Assuming (i), the first assertion of (ii) is clear. Since it is left adjoint, it is right exact. For left exactness, note the inclusion $\mathrm{Shv}(\mathcal{D}, S) \to \mathrm{PShv}(\mathcal{D})$ preserves limits (13.9), so it is left exact. f^p is left exact on presheaf categories as it is right adjoint to f_p. Finally, the sheafification is exact (13.15). In particular, the whole composition is left exact.

It remains to show ${}_pf$ preserves sheaf property. □

13.4 Abelian (pre)sheaves

13.21 Abelian presheaves. Let $\mathcal{C}$ be a category. For a ring R, put

$$\mathrm{PShv}(\mathcal{C}, R) := \mathrm{PShv}(\mathcal{C}, \mathbf{Mod}_{\mathbb{R}})$$

to be the category of presheaves of R-modules. An **abelian presheaf** is a presheaf of $\mathbb{Z}$-modules.

13.21.1 Lemma. $\mathrm{PShv}(\mathcal{C}, R)$ is an abelian category that is complete and cocomplete. The forgetful functor $\mathrm{PShv}(\mathcal{C}, R) \to \mathrm{PShv}(\mathcal{C})$ creates limits and colimits.

Proof. Let $\omega : \mathbf{Mod}_R \to \mathbf{Set}$ denote the forgetful functor. It is known that ω creates limits and colimits. Using the fact that $\mathbf{Mod}_R$ is abelian, it is routine to derive the lemma. □

13.22 $\mathcal{A}$-valued sheaves. Let $\mathcal{A}$ be a category, possibly large. A presheaf $\mathcal{F} \in \mathrm{PShv}(\mathcal{C}, \mathcal{A})$ is called a **sheaf** (resp. **separated presheaf**) for a topology J if for any object $a \in \mathcal{A}$, the presheaf $U \mapsto \mathrm{Hom}(a, \mathcal{F}(U))$ is a sheaf (resp. separated presheaf). Denote by

$$\mathrm{Shv}(\mathcal{C}, J, \mathcal{A})$$

the full subcategory of $\mathcal{A}$-valued sheaves for J.

13.23 Abelian sheaves. Let $\mathcal{C} = (\mathcal{C}, T)$ be a (small) site. For a ring R, put

$$\mathrm{Shv}(\mathcal{C}, R) := \mathrm{Shv}(\mathcal{C}, J_T, \mathbf{Mod}_R)$$

to be the full subcategory of sheaves of R-modules. An abelian sheaf is a sheaves of $\mathbb{Z}$-module.

13.23.1 Lemma. A presheaf of R-modules is a sheaf if the underlying presheaf of sets is a sheaf.

Proof. Let $\omega : \mathbf{Mod}_R \to \mathbf{Set}$ be the forgetful functor, and $\omega_* : \mathrm{PShv}(\mathcal{C}, R) \to \mathrm{PShv}(\mathcal{C})$ be given by $\mathcal{F} \mapsto \omega \circ \mathcal{F}$. Our goal is to show $\mathcal{F} \in \mathrm{PShv}(\mathcal{C}, R)$ is a sheaf if and only if $\omega_* \mathcal{F}$ is a sheaf.

By definition, $\mathcal{F}$ is a sheaf if and only if ${}_M h \circ \mathcal{F}$ is a sheaf of sets for all $M \in \mathbf{Mod}_R$. In other words, for any covering $\{U_i \to U\}_i$, the sequence

$$\mathrm{Hom}(M, \mathcal{F}(U)) \to \prod_i \mathrm{Hom}(M, \mathcal{F}(U_i)) \rightrightarrows \prod_{ij} \mathrm{Hom}(M, \mathcal{F}(U_i \times_U U_j))$$

is a equalizer diagram in **Set** for any $M \in \mathbf{Mod}_R$. By Yoneda, this is the same as saying

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{ij} \mathcal{F}(U_i \times_U U_j)$$

is a equalizer diagram in $\mathbf{Mod}_R$. But the equalizer in $\mathbf{Mod}_R$ is computed exactly as the equalizer in **Set**. □

13.23.2 Lemma. The inclusion $\mathrm{Shv}(\mathcal{C}, R) \to \mathrm{PShv}(\mathcal{C}, R)$ admits a left adjoint, again called the **sheafification**.

Proof. For a presheaf $\mathcal{F}$ of R-modules and F a covering sieve, the set $\mathrm{Hom}(F, \mathcal{F})$ has a natural R-module structure. In particular, $\mathcal{F}^\dagger$ is a presheaf of R-modules. It is a sheaf as the underlying presheaf of sets is a sheaf. The adjunction is clear. □

13.24 Theorem. $\mathrm{Shv}(\mathcal{C}, R)$ is abelian.

Proof. This follows from the following abstract nonsense.

13.24.1 Lemma. Let $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{A}$ be additive functors between additive categories. Suppose

(i) $\mathcal{B}$ is abelian,

(ii) g is left adjoint to f, and

(iii) $g \circ f \cong \mathrm{id}_{\mathcal{A}}$.

Then $\mathcal{A}$ is abelian.

Proof. Tag 03A3. □

13.25 Grothendieck abelian category. An abelian category $\mathcal{A}$ (possibly large) is called a **Grothendieck abelian category** if

(AB 3) $\mathcal{A}$ admits all small colimits (equivalently, all small coproducts),

(AB 5) small filtered colimits in $\mathcal{A}$ are exact, and

(G) $\mathcal{A}$ has a generator.

Though labeled as above, the (AB 5) condition is usually referred to as the conjunction of (AB 3) and (AB 5) above.

13.26 Free abelian presheaves. There is a left adjoint functor to the forgetful $\mathrm{PShv}(\mathcal{C}, R) \to \mathrm{PShv}(\mathcal{C})$, called the **free presheaves of R-modules on a presheaf of sets**.

The construction is really straightforward. Let $F : \mathbf{Set} \to \mathbf{Mod}_R$ denote the left adjoint of the forgetful $\omega : \mathbf{Mod}_R \to \mathbf{Set}$. Then the left adjoint is given by the composition

$$\mathrm{PShv}(\mathcal{C}) \xrightarrow{F\circ} \mathrm{PShv}(\mathcal{C}, R).$$

For $\mathcal{F}$, we shall denote its image by $R^p[\mathcal{F}]$. Showing this satisfies the universal property is a formal computation : we have bijections

$$\mathrm{Hom}_{\mathbf{Mod}_R}(R^p[\mathcal{F}](X), \mathcal{G}(X)) \cong \mathrm{Hom}_{\mathbf{Set}}(\mathcal{F}(X), \mathcal{G}(X))$$

functorial in X. This glues to an isomorphism $\mathrm{Hom}_{\mathrm{PShv}(\mathcal{C},R)}(R^p[\mathcal{F}], \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{PShv}(\mathcal{C})}(\mathcal{F}, \mathcal{G})$ immediately.

For a representable h_U, we shall denote

$$R^p[h_U] = R^p_U$$

13.26.1 Free abelian sheaves. Similarly, there is a left adjoint functor to the forgetful $\mathrm{Shv}(\mathcal{C}, R) \to \mathrm{Shv}(\mathcal{C})$, called the **free sheaves of R-modules on a sheaf of sets**. For a sheaf $\mathcal{F}$, it is constructed by

$$R[\mathcal{F}] := (R^p[\mathcal{F}])^\dagger.$$

It is a formal computation to see it enjoys the universal property. Again, for a representable h_U, we shall denote

$$R[h_U] = R_U.$$

13.27 Theorem. For a (small) cite $(\mathcal{C}, T)$ and a ring R, the category $\mathrm{Shv}(\mathcal{C}, R)$ is a Grothendieck abelian category.

Proof. (AB 5) hold as they hold for $\mathbf{Mod}_R$. It remains to prove (G).

We claim $\{R^p_X \mid X \in \mathcal{C}\}$ is a generating set of $\mathrm{PShv}(\mathcal{C}, R)$. Indeed, two morphisms of presheaves $f, g : \mathcal{F} \rightrightarrows \mathcal{G}$ are equal if and only if $f_A = g_A$ for any $A \in \mathcal{C}$. By Yoneda, this is the same as saying the induced maps

$$\begin{aligned} \mathrm{Hom}(h_A, \mathcal{F}) &\rightrightarrows \mathrm{Hom}(h_A, \mathcal{G}) \\ T &\longmapsto f \circ T \\ T &\longmapsto g \circ T \end{aligned}$$

are equal for any $A \in \mathcal{C}$. Now

$$\mathrm{Hom}_{\mathrm{PShv}(\mathcal{C})}(h_A, \mathcal{F}) \cong \mathrm{Hom}_{\mathrm{PShv}(\mathcal{C},R)}(R^p_A, \mathcal{F})$$

Substitute this into the above equivalence; the claim then follows by taking the contrapositive. By passing to sheafification, we see $\{R_X \mid X \in \mathcal{C}\}$ is a generating set of $\mathrm{Shv}(\mathcal{C}, R)$. A generator of $\mathrm{Shv}(\mathcal{C}, R)$ is now given by

$$\bigoplus_{X \in \mathcal{C}} R_X$$

which exists in $\mathrm{Shv}(\mathcal{C}, R)$ by (AB 3) and that $\mathcal{C}$ is small. □

13.27.1 Lemma. For any presheaf $\mathcal{F}$ of R-modules, there is a functorial bijection

$$\mathrm{colim}(R^h \downarrow \mathcal{F} \to \mathrm{PShv}(\mathcal{C}, R)) \cong \mathcal{F}.$$

Proof. Same as (13.3.1). □

13.28 Theorem. A Grothendieck abelian category has enough injectives. Moreover, it admits an injective cogenerator.

Proof. :)

□

13.29 Grothendieck AB 6. An abelian category is said to satisfy **(AB 6)** if it satisfies (AB 3) and all small filter colimits commutes with small products.

13.29.1 Lemma. For any ring R, the abelian category $\mathbf{Mod}_R$ satisfies (AB 6).

Proof.

□

13.30 Finitary topology. The site $(\mathcal{C}, T)$ is called **finitary** if for every covering $\{U_i \to U\}_{i \in I}$ there exists a finite subset $I_0 \subseteq I$ such that $\{U_i \to U\}_{i \in I_0}$ remains a covering.

13.30.1 Lemma. Let $(\mathcal{C}, T)$ be a finitary site.

(i) For a presheaf to be a sheaf, it suffices to check the sheaf property for all finite coverings.

(ii) $\mathrm{Shv}(\mathcal{C}, R)$ satisfies **(AB 6)** for any ring R.

Proof. (AB 6) is certainly satisfied for the abelian category $\mathrm{PShv}(\mathcal{C}, R)$. Hence it suffices to show for a filtered diagram $\omega : \mathcal{I} \to \mathrm{Shv}(\mathcal{C}, R)$, the presheaf colimit of ω, which we denote by $\mathrm{colim}^p\, \omega$ temporarily, is already a sheaf. Let $\{U_i \to U\}_{i \in I}$ be a covering. We must show

$$\mathrm{colim}^p\, \omega(U) \to \prod_i \mathrm{colim}^p\, \omega(U_i) \rightrightarrows \prod_{ij} \mathrm{colim}^p\, \omega(U_i \times_U U_j)$$

is an equalizer diagram. Since T is finitary, there exists a finite index subset $I_0 \subseteq I$ so that $\{U_i \to U\}_{i \in I_0}$ is still a covering. Consider the diagram

$$\begin{tikzcd}
\mathrm{colim}^p\, \omega(U) \arrow[r] \arrow[d, equal] & \prod_i \mathrm{colim}^p\, \omega(U_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \prod_{ij} \mathrm{colim}^p\, \omega(U_i \times_U U_j) \arrow[d] \\
\mathrm{colim}^p\, \omega(U) \arrow[r] & \prod_{i \in I_0} \mathrm{colim}^p\, \omega(U_i) \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j \in I_0} \mathrm{colim}^p\, \omega(U_i \times_U U_j)
\end{tikzcd}$$

Since I_0 is finite, it commutes with filtered colimits (13.15.1). Hence the bottom row is the filtered colimit of the diagram

$$\omega(U) \to \prod_{i \in I_0} \omega(U_i) \rightrightarrows \prod_{i,j \in I_0} \omega(U_i \times_U U_j).$$

This is an equalizer diagram by the sheaf property. By (AB 5) so is its colimit. Hence the bottom row is an equalizer diagram. Hence given $(x_i)_i$ in the top middle term that lies in the equalizer, we can find $x \in \mathrm{colim}^p\, \omega(U)$ so that $x|_{U_i} = x_i$ for any $i \in I_0$. To see $x|_{U_i} = x_i$ for general i, form the finite covering $\{U_i \times_U U_j \to U_i\}_{j \in I_0}$. The above argument shows that $\mathrm{colim}^p\, \omega$ satisfies the sheaf property for this covering as well. Since

$$(x|_{U_i})|_{U_i \times_U U_j} = x|_{U_i \times_U U_j} = (x|_{U_j})|_{U_i \times_U U_j} = x_j|_{U_i \times_U U_j} = x_i|_{U_i \times_U U_j}$$

for each $j \in I_0$, by sheaf property this proves the claim. (i) is proved by the same argument. □

13.5 Topologies in algebraic geometry

13.31 Overcategory. Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$. The **over category / slice category** of X is the comma category $\mathrm{id}_{\mathcal{C}} \downarrow X$. We denote this category by

$$\mathcal{C}_{/X}.$$

There is an obvious forgetful $j_U : \mathcal{C}_{/X} \to \mathcal{C}$ which creates colimits.

13.31.1 Lemma. Let $\mathcal{C}$ be a site and $X \in \mathcal{C}$. By declaring a collection of morphisms $\{f_i : Y_i \to Y\}$ in $\mathcal{C}_{/X}$ a covering of Y if and only if it is a covering of Y in $\mathcal{C}$, we make $\mathcal{C}_{/X}$ into a site.

13.31.2 Lemma. If $\mathcal{C}$ is subcanonical, so is any overcategory $\mathcal{C}_{/X}$. Moreover, the Yoneda embedding $よ : \mathcal{C} \to \mathrm{Shv}(\mathcal{C})$ induces an equivalence of categories

$$\mathrm{Shv}(\mathcal{C}_{/X}) \xrightarrow{\sim} \mathrm{Shv}(\mathcal{C})_{/h_X}.$$

13.32 Set-theoretic issue. We don't cope with the set-theoretic issue for the size of a site.

13.33 Example : Top Let X be a topological space. The **big site of** X is the overcategory $\mathbf{Top}_{/X}$. The **small site of** X is the full subcategory $\mathbf{top}_{/X}$ consisting of spaces over X whose structure maps are open embeddings.

13.33.1 Lemma. For a topological space X, there is an equivalence of category

$$\mathrm{Shv}(\mathbf{Top}(X)) \cong \mathrm{Shv}(\mathbf{top}_{/X}).$$

13.34 Zariski topology. A **Zariski covering** of the scheme X is a collection of morphisms $\{f_i : X_i \to X\}_i$ such that each f_i is an open immersion and $X = \bigcup_i f_i(X_i)$.

13.34.1 Zariski sites. The **big Zariski site** $\mathbf{Sch}_{\mathrm{Zar}}$ is the category of schemes with Zariski topology. For a scheme X, the **big Zariski site of** X is the overcategory

$$\mathrm{Zar}_{/X} := (\mathbf{Sch}_{\mathrm{Zar}})_{/X}.$$

The **small Zariski site of** X is the full subcategory $\mathrm{zar}_{/X}$ consisting of schemes over X whose structure maps are open immersions.

13.34.2 Lemma. For a scheme X, there is an equivalence of categories

$$\mathrm{Shv}(\mathrm{zar}_{/X}) \cong \mathrm{Shv}(\mathbf{top}_{/X})$$

13.35 Étale topology. An **étale covering** of the scheme X is a collection of morphisms $\{f_i : X_i \to X\}_i$ such that each f_i is étale and $X = \bigcup_i f_i(X_i)$.

13.35.1 Étale sites. The **big Eťale site** $\mathbf{Sch}_{\text{Ét}}$ is the category of schemes with étale topology. For a scheme X, the **big Eťale site of** X is the overcategory

$$\text{Ét}_{/X} := (\mathbf{Sch}_{\text{Ét}})_{/X}.$$

The **small Eťale site of** X is the full subcategory $\text{ét}_{/X}$ consisting of schemes over X whose structure maps are étale.

13.36 Nisnevich topology. A **Nisnevich covering** of the scheme X is an étale covering $\{f_i : X_i \to X\}_i$ of X such that for each point $x \in X$ there exist an i and $y \in f_i^{-1}(x)$ such that the induced map $\kappa(x) \to \kappa(y)$ is a field isomorphism.

13.37 fppf topology. An **fppf covering** of the scheme X is a collection of morphisms $\{f_i : X_i \to X\}_i$ such that each f_i is flat, locally of finite presentation and $X = \bigcup_i f_i(X_i)$.

13.37.1 Lemma. A flat morphism locally of finite presentation is universally open.

Proof. It reduces at once to show that if a ring homomorphism $R \to S$ is flat and locally of finite presentation, then the induced map $\operatorname{Spec} S \to \operatorname{Spec} R$ is open. See Tag 00I1. □

13.37.2 Lemma. Zariski $\Rightarrow$ Nisnevich $\Rightarrow$ Étale $\Rightarrow$ fppf.

13.38 fpqc covering. Let X be a scheme. A set of morphisms of schemes $\{f_i : U_i \to X\}_{i \in I}$ is called an **fpqc covering** if

(i) each f_i is flat and $X = \bigcup_i f_i(U_i)$, and

(ii) for each affine open $U \subseteq X$ there exists an $n \geqslant 0$, a map $\alpha : [n] \to I$ and affine opens $V_j \subseteq U_{\alpha(j)}$ such that $U = \bigcup_{j \in [n]} f_{\alpha(j)}(V_j)$.

13.38.1 Vital set-theoretic issue. Recall we require the underlying category of a site to be small. There is no way to turn **Sch** with fpqc into a site in the above sense. See Tag 0BBK and nlab : fpqc site and the reference therein. But this does not stop us from considering **fpqc sheaves** in the obvious sense.

13.38.2 Lemma. An fppf covering is an fpqc covering.

Proof. (1) is clear, and (ii) follows from Lemma 13.37.1 and an affine scheme is compact. □

13.39 Standard fpqc covering. Let X be an affine scheme. A **standard fpqc covering** on X is a finite collection of morphisms $\{f_i : U_i \to X\}_{i \in [n]}$ with each U_i affine, flat over X and $X = \bigcup_i f_i(U_i)$.

13.39.1 Fact. Let T be an affine scheme. Let $\{T_i \to T\}_{i \in I}$ be an fqpc covering of T, and let F denote the smallest sieve it generates. Then F contains a standard fpqc covering of T.

13.40 Faithfully flat. A ring map $R \to S$ is called **faithfully flat** if the functor $(\cdot) \otimes_R S : \mathbf{Mod}_R \to \mathbf{Mod}_S$ is exact and faithful.

13.40.1 Lemma. Let $\varphi : R \to S$ be a flat ring map. TFAE :

(i) φ is faithfully flat.

(ii) The map $M \to M \otimes_R S$ is injective for any $M \in \mathbf{Mod}_R$.

(iii) $M \otimes_R S = 0$ if and only if $M = 0$ for any $M \in \mathbf{Mod}_R$.

(iv) If $M \to N$ is an R-module homomorphism such that $M \otimes_R S \to N \otimes_R S$ is injective, then so is $M \to N$.

13.40.2 Lemma. Let $\varphi : R \to S$ be a ring map. TFAE :

(i) φ is faithfully flat.

(ii) $\varphi_{\mathfrak{p}}$ is faithfully flat for any $\mathfrak{p} \in \operatorname{Spec} R$.

(iii) $\varphi_{\mathfrak{p}}$ is faithfully flat for any $\mathfrak{p} \in \operatorname{mSpec} R$.

13.40.3 Lemma. Let $\varphi : R \to S$ be a flat ring map. TFAE :

(i) $R \to S$ is faithfully flat.

(ii) $\mathfrak{p}S \neq S$ for any $\mathfrak{p} \in \operatorname{mSpec} R$.

(iii) $\operatorname{Spec} S \to \operatorname{Spec} R$ is surjective.

13.41 Lemma. Let $\mathcal{F} : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ be a presheaf. Then $\mathcal{F}$ is an fpqc sheaf if and only if

(i) $\mathcal{F}$ is a Zariski sheaf, and

(ii) $\mathcal{F}$ satisfies the sheaf property for every standard fpqc covering.

When (i), the condition (ii) is equivalent to

(ii)′ $\mathcal{F}$ satisfies the sheaf property for $\{V \to U\}$ with V, U affine and $V \to U$ faithfully flat.

Proof. Suppose (i) and (ii) hold. Let $S' = \{f_i : X_i \to X\}_{i \in I}$ be an fpqc covering. Let $(s_i)_i \in \mathcal{F}(S')$ and $U \subseteq X$ be an affine open. We claim there exists $s \in \mathcal{F}(U)$ such that $s|_{f_i^{-1}(U)} = s_i|_{f_i^{-1}(U)}$ for all $i \in I$. By (i) this will finish the proof.

Let S denote the sieve generated by S', and $\iota : U \to X$ denote the inclusion. Then ι^*S is the sieve generated by $\iota^*S' := \{X_i \times_X U \to U\}_{i \in I}$ and $s|_U \in \mathcal{F}(\iota^*S')$. By (13.39.1) ι^*S contains a standard fpqc covering $\{U_j \to U\}_j$. By (ii) there exists a unique $s \in \mathcal{F}(U)$ such that $s|_{U_j} = s_{i_j}|_{U_j}$ where for each j we pick some $U_j \to X_{i_j} \times_X U$.

If $V \to U$ is another affine scheme, again we can find unique $s_V \in \mathcal{F}(V)$ with $s_V|_{V \times_U U_j} = s_j|_{V \times_U U_j}$. For general $V \to U$, cover V by affine and use (i) to obtain $s_V \in \mathcal{F}(V)$ such that

$$s_V|_{V \times_U U_j} = s_j|_{V \times_U U_j}.$$

In particular, $s_V = s|_V$. Now if we take $V = U \times_X X_i$, this means

$$s_{U \times_X X_i} = s|_{U \times_X X_i}$$

Since for each j that

$$s|_{U \times_X X_i \times_U U_j} = s_{i_j}|_{U \times_X X_i \times_U U_j} = s_i|_{U \times_X X_i \times_U U_j}$$

by (ii) we have

$$s|_{U \times_X X_i} = s_i|_{U \times_X X_i}$$

This proves our claim.

Assume (i). To show (ii)′ implies (ii), let $\{X_i \to X\}$ be a standard fpqc covering. Then $\coprod_i X_i \to X$ is a faithfully flat morphism of affine schemes. By (i) we have $\mathcal{F}(\coprod_i X_i) = \prod_i \mathcal{F}(X_i)$ and similarly for the fibre products. This proves the sheaf condition for $\{X_i \to X\}_i$ is the same as that for $\coprod_i X_i \to X$. □

13.42 Descent. Let $R \to A$ be faithfully flat. Then $R \to A \rightrightarrows A \otimes_R A$ is an equalizer diagram.

Proof. By faithful flatness, it suffices to prove

$$A \to A \otimes_R A \rightrightarrows A \otimes_R A \otimes_R A$$

is an equalizer diagram. Here

— the first map is $a \mapsto 1 \otimes a$,

— the top map is $a \otimes b \mapsto a \otimes 1 \otimes b$, and

— the bottom map is $a \otimes b \mapsto 1 \otimes a \otimes b$.

Define $\varphi : A \otimes_R A \otimes_R A \to A \otimes_R A$ by

$$\varphi(a \otimes b \otimes c) = a \otimes bc.$$

Let $\sum m_i a_i \otimes b_i$ lie in the equalizer; then

$$\varphi\left(\sum m_i a_i \otimes 1 \otimes b_i\right) = \varphi\left(\sum m_i \otimes a_i \otimes b_i\right)$$

or

$$\sum m_i a_i \otimes b_i = \sum m_i \otimes a_i b_i = 1 \otimes \sum m_i a_i b_i.$$

This means $\sum m_i a_i \otimes b_i$ lie in the image of $A \to A \otimes_R A$. □

13.42.1 Theorem. Every representable of **Sch** is an fpqc sheaf.

Proof. Let X be a scheme. By (13.41) and (13.12.1), it remains to show h_X is a sheaf for the covering $\{V \to U\}$ with V, U affine and $V \to U$ faithfully flat, i.e. the sequence

$$\mathrm{Hom}_{\mathbf{Sch}}(U, X) \to \mathrm{Hom}_{\mathbf{Sch}}(V, X) \rightrightarrows \mathrm{Hom}_{\mathbf{Sch}}(V \times_U V, X)$$

is an equalizer diagram. Let $f : V \to X$ lie in the equalizer. By (3.81), we have a sequence of maps

$$\mathrm{Hom}_{\mathbf{Sch}}(V \times_U V, X) \to \mathrm{Hom}_{\mathbf{Top}}(\underline{V \times_U V}, X) \to \mathrm{Hom}_{\mathbf{Top}}(\underline{V} \times_{\underline{U}} \underline{V}, X)$$

with the last one being injective. Since h_X is a sheaf in **Top**, there exists a continuous map $g : U \to X$ such that the composition $V \to U \to X$ coincides with f set-theoretically. It remains to upgrade this into a morphism in **LRS**.

Let $R \to A$ be a faithfully flat ring homomorphism that corresponds to $\pi : V \to U$. So far we obtained a continuous map $g : \operatorname{Spec} R \to X$. Let $y \in U$ be a point and let $\operatorname{Spec} B$ be an affine open neighborhood of $g(y)$ in X. By continuity of g we can find $r \in R$ such that $y \in D(r) \subseteq g^{-1}(\operatorname{Spec} B)$. By restricting $V \times_U V \to V \to X$ to $\pi^{-1}(D(r)) \subseteq V$, we obtain a sequence of ring maps

$$(A \otimes_R A)_r \leftleftarrows A_r \leftarrow B.$$

with $A_r \to B$ inducing $f|^{\operatorname{Spec} B}_{\pi^{-1}(D(r))}$. By assumption $B \to A_r$ goes in the equalizer of $A_r \to (A \otimes_R A)_r$; by (13.42), $R_r \to A_r$ is the equalizer. Hence $B \to A_r$ factorizes uniquely through $R_r \to A_r$, obtaining a ring map $B \to R_r$. This in turn gives a morphism of schemes $D(r) \to \operatorname{Spec} B \to X$. Topologically this coincides with $g|_{D(r)}$. By varying $y \in U$, we upgrade g into a morphism of schemes. It follows from the construction that $V \to U \xrightarrow{g} X$ is exactly f in **Sch**. □

13.6 Examples of topoi

13.43 Subobjects. Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$. The set of **subobjects** $\mathrm{Sub}(X)$ of X is set of isomorphism classes of monomorphisms in $\mathrm{Ob}\,(\mathrm{id}_{\mathcal{C}} \downarrow X)$. This comes with a partial order : say $X_1 \subseteq X_2$ if there exists a morphism $X_1 \to X_2$ in $\mathrm{id}_{\mathcal{C}} \downarrow X$. Note the morphism $X_1 \to X_2$ is automatically monic.

13.44 Effective epimorphism. Let $\mathcal{C}$ be a category with fibre products. A morphism $X \to Y$ is an **effective epimorphism** if the sequence

$$X \times_Y X \rightrightarrows X \to Y$$

is a coequalizer diagram in $\mathcal{C}$.

13.44.1 Lemma. Let $\mathcal{C}$ be a category with fibre products. Then an effective epimorphism is epimorphic.

13.45 Coherent category. A category $\mathcal{C}$ is called **coherent** if

(i) $\mathcal{C}$ has all finite limits,

(ii) every morphism $X \to Z$ in $\mathcal{C}$ factorizes as a composition $X \to Y \to Z$ with $X \to Y$ effectively epic and $Y \to Z$ monic,

(iii) for any object $X \in \mathcal{C}$, the poset $\mathrm{Sub}(X)$ has joins and a least element, and

(iv) for any morphism $X \to Y$ in $\mathcal{C}$, the induced map on $\mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ by pullbacks preserves joins and least elements.

13.45.1 Coherent topology. Let $\mathcal{C}$ be a coherent category. Declare a collection of morphisms $\{f_i : U_i \to U\}_{i \in I}$ to be a covering if there is a finite subset $I_0 \subseteq I$ such that $\bigwedge_{i \in I_0} \mathrm{Im}(f_i) = U$ in $\mathrm{Sub}(U)$. This is a Grothendieck pretopology, called the **coherent topology** in $\mathcal{C}$.

Proof. Lurie.

□

13.45.2 Lemma. The coherent topology is subcanonical.

Proof. Let $X \in \mathcal{C}$. By Lemma 13.30.1 it suffices to check the sheaf property of h_X for any finite covering $\{U_i \to U\}$.

□

14 Étale cohomology

14.1 Results on étale morphisms

14.1 Notations. In the following k always denote a field. An unadorned scheme is simply just a scheme in its great generality.

14.2 Flat morphism. A morphism of scheme $f : X \to Y$ is **flat at** x if the induced map $f^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,f(y)}$ is flat. A morphism is **flat** if it is flat everywhere.

14.3 Unramified morphism. A morphism of schemes $f : X \to Y$ is called **unramified at** $x \in X$ if

(i) $f(\mathfrak{m}_{Y,f(x)})\mathcal{O}_{X,x} = \mathfrak{m}_{X,x}$, and

(ii) $\kappa(x)$ is a finite separable extension of $\kappa(f(x))$.

A morphism is called **unramified** if it is locally of finite type and unramified everywhere.

14.4 Lemma. All morphisms in the statement are assumed to be locally of finite presentation [8].

(i) Unramified morphisms are quasi-finite [9].

(ii) Immersions are unramified.

(iii) Unramified morphisms are stable under composition and base change.

(iv) If $X \to Y \to Z$ is unramified, so is $X \to Y$.

14.5 Étale morphism. A morphism of schemes $f : X \to Y$ is **étale at** $x \in X$ if it is flat and unramified at $x \in X$. A morphism is called **étale** if it is locally of finite presentation and étale everywhere.

14.5.1 Corollary.

(i) Open immersions are étale.

(ii) Étale morphisms are stable under composition and base change.

(iii) If $X \to Y \to Z$ is étale with $Y \to Z$ unramified, then $X \to Y$ is étale.

14.6 Étale locus. Let $f : X \to Y$ be a morphism locally of finite presentation. Then the set of points in X where f is étale is open.

14.7 Étale $\Rightarrow$ open. Étale morphisms are open maps.

14.7.1 If a morphism is étale and universally injective, then it is an open immersion.

8. This shit doesn't matter if we assume our schemes are Noetherian.
9. Recall a morphism is quasi-finite if each of the fibre is a finite set.

14.2 Étale fundamental group

14.8 Geometric points. A **geometric point** of a k-scheme X is a morphism $\operatorname{Spec}\Omega \to X$ for some separably closed field Ω. We shall denote $\Omega = \kappa(s)$.

14.8.1 Pointed schemes. A pair (X, s) with s a geometric point of X is called a **pointed scheme**. We shall call s the **base point** of (X, s). A morphism between two pointed schemes is a morphism of scheme that preserves the base points. For any morphism $\pi : Y \to X$, a geometric point s' of Y such that $\pi \circ s' = s$ is said to be **lying over** s.

14.9 Finite étale cover. Let FEt_X denote the category of schemes over X whose structure map to X is finite étale. An object $Y \to X$ in FEt_X might be called a **finite étale cover(ing) of** X. Let

$$\mathrm{Aut}(Y/X) = \mathrm{Isom}_{\mathbf{Sch}_X}(Y, Y)$$

14.9.1 Let (X, s) be a pointed scheme and Y a scheme finite étale over X. For any geometric point s' of Y lying over Y, we say (Y, s') is a **finite étale cover of** (X, s).

14.9.2 Lemma. Let s be a geometric point of X, and let (Y_i, s_i) be two pointed schemes with Y_i finite étale over (X, s).

(i) If Y_1 is connected, there is at most only morphism $(Y_1, s_1) \to (Y_2, s_2)$.

(ii) There exists a connected pointed scheme (Y_3, s_3) finite étale over (X, s) such that (Y_3, s_3) dominates [10] (Y_i, s_i) $i \in [2]$.

Proof. [Fu15, Proposition 2.4].

□

14.10 Galois cover.

14.11 Fibre functor. Let X be a k-scheme and s a geometric point. Define a functor $F : \mathrm{FEt}_X \to \mathbf{Set}$ by

$$F(\pi : Y \to X) := \pi^{-1}(s) = Y_s := \mathrm{Hom}_{\mathbf{Sch}_X}(\operatorname{Spec}\kappa(s), Y).$$

This is the set of all geometric points of Y lying over s. The functor F is called the **fibre functor** of the pointed scheme (X, s).

14.11.1 Lemma. Let I denote the opposite category of the category of pointed connected Galois finite étale covers over (X, s). Then

(i) I is directed.

(ii) For any (Y, s') lying over (X, s), the map

$$\begin{aligned} \operatorname*{colim}_{Y' \in I} \mathrm{Hom}_{\mathbf{Sch}_X}(Y', Y) &\longrightarrow F(Y) \\ f &\longmapsto f(s'') \end{aligned}$$

is bijective.

Proof. [Fu15, p. 127].

□

10. This simply means there exists morphisms $(Y_3, s_3) \to (Y_i, s_i)$.

14.12 Fundamental group. The **étale fundamental group** of (X, s) is define to be the limit

$$\pi_1(X, s) := \lim_{(Y', s') \in I} \mathrm{Aut}(Y'/X)$$

where the category I is as in Lemma 14.11.1.

14.13 Example - Galois theory of fields. For a field k, we have

$$\pi_1(\mathrm{Spec}\, k, \mathrm{pt}) = \mathrm{Gal}(k^{\mathrm{sep}}/k).$$

14.14 Equivalence of categories. By the map in Lemma 14.11.1, the action of $\pi_1(X, s)$ on each $Y' \in I$ defines an action of $\pi_1(X, s)$ on $F(Y)$. In fact,

Theorem 14.1. The fibre functor F defines an equivalence of categories of FEt_X and the category of finite discrete $\pi_1(X, s)$-sets.

14.14.1 Lemma. $\pi_1(X, s) \cong \mathrm{Aut}(F)$, the automorphism group of the fibre functor.

14.3 Étale sheaf

14.15 Étale sites. Let X be a scheme and Et_X be the category of schemes over X whose structure map to X is étale. The **étale site** is then the category Et_X equipped with the étale topology.

14.16 Presheaves. Let $\mathcal{C}$ be a category. A **$\mathcal{C}$-valued étale presheaf** is a functor $\mathcal{F} : \mathrm{Et}_X^{\mathrm{op}} \to \mathcal{C}$. Denote by

$$\mathrm{PShv}(X, \mathcal{C})$$

the category of étale presheaves, where the morphisms are simply natural transformations between functors.

14.16.1 Čech complex. Let $\mathcal{E}$ denote an étale cover of X, and let $\mathcal{F}$ be a presheaf. The Čech complex $\mathcal{C}^\bullet(\mathcal{E}, \mathcal{F})$ is defined as usual.

14.16.2 Sheaves. Assume $\mathcal{C}$ admits all limits. An étale presheaf is called an **étale sheaf** if it is a sheaf over the étale site. Denote by $\mathrm{Shv}(X, \mathcal{C})$ the full subcategory of étale sheaves. When $\mathcal{C} = \mathbf{Mod}_\Lambda$ for some ring Λ, we write

$$\mathrm{Shv}(X, \Lambda) := \mathrm{Shv}(X, \mathbf{Mod}_\Lambda)$$

In the following, put

$$\mathcal{C} = \mathbf{Set} \text{ or } \mathbf{Mod}_\Lambda.$$

14.16.3 Lemma. [Mil13, Proposition 6.6]

14.16.4 Lemma. $\mathrm{Shv}(X, \Lambda)$ is complete and cocomplete.

14.16.5 Lemma. $\mathrm{Shv}(X, \Lambda)$ is abelian and has enough injective.

14.16.6 Lemma. The global section functor $\mathrm{Shv}(X, \Lambda) \ni \mathcal{F} \mapsto \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$ is left exact.

14.17 Sheafification. The forgetful functor $\mathrm{PShv}(X, \mathcal{C}) \to \mathrm{Shv}(X, \mathcal{C})$ admits a left adjoint

$$(\cdot)^\dagger : \mathrm{Shv}(X, \mathcal{C}) \to \mathrm{PShv}(X, \mathcal{C})$$

called the **sheafification functor**. For a presheaf $\mathcal{F}$, the sheaf $\mathcal{F}^\dagger$ is sometimes called the **sheaf associated to** $\mathcal{F}$.

14.18 Constant sheaves. Let M be a set or a Λ-module. The **constant sheaf** M_X is the sheafification of the presheaf defined by $\mathrm{Et}_X^{\mathrm{op}} \ni U \mapsto M$. Explicitly,

$$M_X(U \to X) = \{\text{locally constant functions } U \to M\}$$

In particular,

$$\Gamma(X, M_X) = X^{\pi_0(X)}$$

14.19 Representable sheaves. Let X be a scheme and U be a scheme étale over X. Let $\widetilde{U}$ denote the sheaf represented by U, namely

$$\mathrm{Et}_X \ni (Y \to X) \mapsto \mathrm{Hom}_{\mathbf{Sch}_X}(Y, U)$$

Then this is an étale sheaf of sets.

Proof.

□

14.20 Stalk. Let $\mathcal{F}$ be a $\mathcal{C}$-valued presheaf of sets and s a geometric point of X. The **stalk** of $\mathcal{F}$ at s is the colimit

$$\mathcal{F}_s := \mathrm{colim}_U \mathcal{F}(U)$$

where U runs over all étale neighborhood of s. In the case of $\mathcal{F} = \mathcal{O}_X$, the resulting stalk is called the **strict local ring** of X, and is denote by $\mathcal{O}_{X_{et},s}$ to differentiate from the usual stalk.

14.20.1 Canonicity. Let $x \in X$ denote the image of s. For a choice of separable closure $\kappa(x)^{sep}$ of $\kappa(x)$, we have a geometric point $\overline{x} : \operatorname{Spec} \kappa(x)^{sep} \to X$ naturally attached to x.

Lemma 14.2. One has a non-canonical isomorphism $\mathcal{F}_s \cong \mathcal{F}_{\overline{x}}$.

14.21 Direct image. Let $f : X \to Y$ be a morphism of schemes. Then the **direct image functor**

$$f_* : \mathrm{Shv}(X, \mathcal{C}) \longrightarrow \mathrm{Shv}(Y, \mathcal{C})$$

is given by

$$f_*\mathcal{F}(U \to Y) := \mathcal{F}(X \times_Y U).$$

This is a left exact functor.

14.22 Inverse image. As usual, the direct image functor f_* admits a left adjoint. Its left adjoint is denoted by

$$f^* : \mathrm{Shv}(Y, \mathcal{C}) \longrightarrow \mathrm{Shv}(X, \mathcal{C})$$

and is called the **inverse image functor**.

Proof.

□

14.22.1 Lemma. One has $\mathcal{F}_s \cong \Gamma(\operatorname{Spec} \kappa(s), s^*\mathcal{F})$.

14.22.2 Lemma. For a geometric point s of X and a morphism $f : X \to Y$, the canonical map

$$\mathcal{F}_{f \circ s} \longrightarrow (f^*\mathcal{F})_s$$

is an isomorphism.

14.23 Finite morphism. Let $f : X \to Y$ be a finite morphism, $y : \operatorname{Spec} \kappa(y) \to Y$ a geometric point of Y and $\mathcal{F} \in \mathrm{Shv}(X, \mathcal{C})$. If $\kappa(y)$ is large enough so that it coincides with the residue fields of all the point in the geometric fibre $\operatorname{Spec} \kappa(y) \times_Y X \to \operatorname{Spec} \kappa(y)$, then the natural map

$$(f^*\mathcal{F})_y \longrightarrow \prod_{x \in X_s} \mathcal{F}_x.$$

is an isomorphism.

14.24 Purely inseparable morphism.

14.24.1 Equivalence. If $f : X \to Y$ is a purely inseparable morphism, then the adjunction maps

$$\mathcal{G} \to f_* f^* \mathcal{G}, \qquad \mathcal{F} \to f^* f_* \mathcal{F}$$

are isomorphisms. In particular, f_* and f^* establish equivalences of categories between $\mathrm{Shv}(X, \mathcal{C})$ and $\mathrm{Shv}(Y, \mathcal{C})$.

14.4 Construtible sheaves

14.25 Theorem. Let k be a separably closed field and X a finitely generated k-scheme. Let $\mathcal{F}$ be a constructible sheaf whose sections have order relatively prime to $\operatorname{Char} k$. Then :

(i) The cohomology groups $H^q(X, \mathcal{F})$ are finite.

(ii) $H^q(X, \mathcal{F}) = 0$ for $q \geqslant 3$.

(iii) If X is affine, then $H^q(X, \mathcal{F}) = 0$ for $q \geqslant 2$.

The assertions (ii) and (iii) can be extended to torsion sheaves by passage to limits.

Quatrième partie

Toric varieties

15 Convex Geometry

15.1 Convex polyhedral cones

In this subsection, let V be a finite dimensional real vector space with dual space $V^\vee$. Then the evaluation $V^\vee \times V \xrightarrow{\langle,\rangle} \mathbb{R}$ is a perfect pairing. We identify $(V^\vee)^\vee$ with V naturally.

Definition.

1. A **convex polyhedral cone** in V is a set of the form
$$\sigma = \mathrm{cone}(S) := \left\{ \sum_{u \in S} a_u u \mid a_u \geqslant 0 \right\} \subseteq V$$
for some finite subset $S \subseteq V$. We also say σ is the **cone generated by** S. By convention, $\mathrm{cone}(\varnothing) = \{0\}$.
2. The **dimension** of a convex polyhedral cone σ, denoted by $\dim \sigma$, is the dimension of the linear span of σ over $\mathbb{R}$; in other words, $\dim \sigma = \dim_{\mathbb{R}}(\sigma + (-\sigma))$.
3. The **dual** $\sigma^\vee$ of some set $\sigma \subseteq V$ is
$$\sigma^\vee := \{u \in V^\vee \mid \langle u, v\rangle \geqslant 0 \text{ for all } v \in \sigma\}.$$
Also, we put
$$\sigma^\perp = \{u \in V^\vee \mid \langle u, v\rangle = 0 \text{ for all } v \in \sigma\}.$$

- If $\sigma = \mathrm{cone}(S)$ for some finite subset $S \subseteq V$, then
$$\sigma^\vee = \bigcap_{v \in S} \{u \in V^\vee \mid \langle u, v\rangle \geqslant 0\}.$$

Lemma 15.1 (Farkas'). If σ is a convex polyhedral cone and $v_0 \notin \sigma$, then there exists $u_0 \in \sigma^\vee$ such that $\langle u_0, v_0\rangle < 0$. In particular, this implies $(\sigma^\vee)^\vee = \sigma$.

Proof. By choosing a basis for V, we do not distinguish V, $V^\vee$ and $\mathbb{R}^n$, and under this assumption, $\langle\,,\,\rangle$ becomes the standard inner product. Let B be the closed ball centered at 0 with radius $\|v_0\|$. Then $\sigma \cap B$ is nonempty and compact, so the function $w \mapsto \|w - v_0\|$ attains its minimal at some $w_0 \in \sigma \cap B \subseteq \sigma$. Put $u_0 = w_0 - v_0$. We claim $u_0 \in \sigma^\vee$ and $\langle u_0, v_0\rangle < 0$.

- Let $w \in \sigma$. Then $w_0 + tw \in \sigma$ for all $t \geqslant 0$, and
$$\|u_0\|^2 = \|w_0 - v_0\|^2 \leqslant \|w_0 + tw - v_0\|^2 = \|u_0 + tw\|^2 = \|u_0\|^2 + 2t\langle u_0, w\rangle + t^2\|w\|^2.$$
This implies $\langle u_0, w\rangle \geqslant 0$, and since $w \in \sigma$ is arbitrary, we see $u_0 \in \sigma^\vee$.
- Since $\langle u_0, v_0\rangle = \langle w_0, v_0\rangle - \|v_0\|^2$, if $w_0 = 0$, then we are done. Suppose now $w_0 \neq 0$; then $u_0 \neq 0$ as well. Consider
$$\|tw_0 - v_0\|^2 = t^2\|w_0\| - 2t\langle w_0, v_0\rangle + \|v_0\|^2.$$
By our choice of w_0, the quadratic function on the right minimizes at $t = 1$. This implies $\|w_0\|^2 = \langle w_0, v_0\rangle$, and hence
$$0 < \|u_0\|^2 = \|w_0\|^2 - 2\langle w_0, v_0\rangle + \|v_0\|^2 = -\langle w_0, v_0\rangle + \|v_0\|^2 = -\langle u_0, v_0\rangle.$$

$\square$

Definition. Let σ be a convex polyhedral cone in V.

1. A **supporting hyperplane** of σ is the hyperplane in V determined by some vector in $\sigma^\vee$.

2. A **face** of σ is the intersection of σ with some support hyperplane. In particular, σ is a face of itself. The faces of σ other than itself are called **proper faces**.

— For $u \in V^\vee$, we put

$$u^\perp := \{v \in V \in \langle u, v\rangle = 0\}.$$

Then a supporting hyperplane is of the form $u^\perp$ for some $u \in \sigma^\vee$.

— If τ is a face of σ, we write $\tau \preceq \sigma$. We call $\preceq$ the **face relation**.

Lemma 15.2. Let σ be a convex polyhedral cone in V.

1. Any linear subspace of σ is contained in every face of σ.
2. Every face of σ is a convex polyhedral cone.
3. An intersection of two faces of σ is a face of σ.
4. A face of a face of σ is a face of σ.

Proof.

1. If $v \in \sigma$ is such that $-v \in \sigma$, then for any $u \in \sigma^\vee$, we have $\langle u, v\rangle \geqslant 0 \leqslant \langle u, -v\rangle$, proving that $\langle u, v\rangle = 0$.
2. In fact, if $\sigma = \mathrm{cone}(S)$ for some finite subset $S \subseteq V$ and $\tau = \sigma \cap u^\perp$ for some $u \in \sigma^\vee$, then $\tau = \mathrm{cone}(S \cap u^\perp)$. Clearly, we have $\mathrm{cone}(S \cap u^\perp) \subseteq \tau$. Conversely, if $v \in \tau$, write $v = \sum\limits_{s \in S} a_s s$ with $a_s \geqslant 0$. By the following lemma, we see $a_s s \in \tau$ for each s, so $s \in S \cap \tau = S \cap u^\perp$, implying $v \in \mathrm{cone}(S \cap u^\perp)$.

Lemma 15.3. Let $\tau \preceq \sigma$. If $v, w \in \sigma$ are such that $v + w \in \tau$, then $v, w \in \tau$.

Proof. Say τ is cut off by the hyperplane determined by $u \in \sigma^\vee$. Then $\langle u, v\rangle \geqslant 0 \leqslant \langle u, w\rangle$. By assumption we have $\langle u, v + w\rangle = 0$, so this forces that $\langle u, v\rangle = \langle u, w\rangle = 0$. This shows $v, w \in \tau$. □

3. Let u_1, $u_2 \in \sigma^\vee$. Then

$$(\sigma \cap u_1^\perp) \cap (\sigma \cap u_2^\perp) = \sigma \cap (u_1 + u_2)^\perp.$$

This identity is proved as before, and clearly implies 3.

4. Let $\tau = \sigma \cap u^\perp$ for some $u \in \sigma^\vee$ and $\gamma = \tau \cap v^\perp$ for some $v \in \tau^\vee$. Consider the vector $v + tu$; since σ has a finite set of generators, we see $v + tu \in \sigma^\vee$ for $t \gg 0$. Indeed, if $v \in S$, we have either $\langle u, v\rangle > 0$, or $\langle u, v\rangle = 0$ and $\langle u', v\rangle \geqslant 0$. Thus

$$\tau \cap (v + tu)^\perp = (\tau \cap v^\perp) \cap (\tau \cap u^\perp) = \gamma \cap \tau = \gamma.$$

□

The converse of Lemma 15.3 holds in the following sense. Note that this lemma is not used in this section.

Lemma 15.4. Let σ be a convex polyhedral cone in V and $\tau \subseteq \sigma$ a convex cone such that if $v, w \in \sigma$ and $v + w \in \tau$, then $v, w \in \tau$. Then τ is a face of σ.

Proof. Replacing V by $\mathrm{span}_{\mathbb{R}} \sigma$, we may assume σ has maximal dimension. Further, by replacing σ by a face of which, we may assume τ is not contained in any proper face of σ. Under these assumptions and the condition of the lemma, we prove $\tau = \sigma$.

By Corollary 15.8, $\sigma^\vee$ is a convex polyhedral cone, so we may write $\sigma^\vee = \mathrm{cone}(u_1, \dots, u_s)$ for some minimal generating set $\{u_1, \dots, u_s\}$ of $\sigma^\vee$. By Lemma 15.1, $\sigma = \mathrm{cone}(u_1, \dots, u_s)^\vee$. Fix an integer $1 \leqslant i \leqslant s$. If $\langle u_i, \tau\rangle = 0$, then $\tau \subseteq \sigma \cap (u_i)^\perp$, contradicting to our assumption that τ is not contained in any proper face. Thus we can find $v_i \in \tau$ such that $\langle u_i, v_i \rangle > 0$. If we take $v_0 = v_1 + \cdots + v_s$, we see $\langle u_i, v_0 \rangle > 0$ for any i, so that $v_0 \in \operatorname{int}(\sigma) \cap \tau$.

If $\tau \subsetneq \sigma$, then for $v \in \sigma \backslash \tau$, consider the vector $tv_0 - v$; if $t \gg 0$, then $\langle u_i, tv_0 - v \rangle \geqslant 0$ for any $1 \leqslant i \leqslant s$, so that $tv_0 - v \in (\sigma^\vee)^\vee = \sigma$. This implies $tv_0 = (tv_0 - v) + v \in \tau$ with $tv_0 - v$, $v \in \sigma$, and our condition implies that $tv_0 - v$, $v \in \sigma$, a contradiction.

□

Definition. Let σ be a convex polyhedral cone in V.

1. A **facet** of σ is a face τ of codimension one, namely, $\dim \tau = \dim \sigma - 1$.
2. An **edge** of σ is a face of dimension one.

Lemma 15.5. Any proper face of a convex polyhedral cone σ in V is contained in some facet.

Proof. Put $\sigma = \mathrm{cone}(S)$ for some finite subset S of V. Let $u \in \sigma^\vee$ be such that $\tau = \sigma \cap u^\perp$ has codimension > 1 in σ. We saw that $\tau = \mathrm{cone}(S \cap u^\perp)$. By replacing V with $\sigma + (-\sigma)$, we may assume V is spanned by σ. Also put $W = \tau + (-\tau)$. The image of S in V/W is contained in the closed half-space determined by u. Since $\dim_{\mathbb{R}} V/W \geqslant 2$, if we rotate (in V/W) this half-space about the origin, we can obtain a closed half space containing $0 + W$ and at least one nonzero image of S in V/W. In other words, we can find $u_0 \in \sigma^\vee$ such that $W \subsetneq \sigma \cap u_0^\perp$. □

Corollary 15.5.1.

1. Any face of codimension 2 is an intersection of two facets.
2. Any proper face is the intersection of all facets containing it.

Proof. Let σ be a convex polyhedral cone.

1. Retain the notation in the proof of the last lemma. Under our assumption, we have $\dim_{\mathbb{R}} V/W = 2$, i.e., V/W is a plane. This means the image of σ in W has two supporting lines, and the result is clear.
2. Let τ be a proper face of σ. If the codimension of τ in σ is > 1, then by the last lemma, τ is contained in the facet γ of σ. Since $\sigma^\vee \subseteq \gamma^\vee$, we may regard τ as a face of γ, so by induction we see τ is the intersection of all facets of γ containing τ. But each facet of γ, by 1., is an intersection of two facets in σ, so τ is an intersection of facets.

□

Lemma 15.6. Let σ be a convex polyhedral cone in V such that $V = \sigma + (-\sigma)$. Then $\partial\sigma$ is the union of all facets of σ.

Proof. A facet is an intersection of σ with some hyperplane determined by some vector in $\sigma^\vee$, so any neighborhood of a point p in a facet intersects nontrivially with the complement of σ. Since σ has nonempty interior and is convex, the segment connecting p and any interior point entirely lies in σ. This shows any neighborhood of p meets the interior of σ. This shows $p \in \partial\sigma$.

Conversely, let $p \in \partial\sigma$ and let (w_n) be a sequence outside σ such that $w_n \to p$. By Lemma 15.1, we can find $u_n \in \sigma^\vee$ such that $\langle u_n, w_n \rangle < 0$. Rescaling, we assume $\|u_n\| = 1$. Since V is finite dimensional, the unit sphere is compact; by passing the convergent subsequence, we may assume u_n converges to some point $q \in \sigma^\vee$. Then $0 \leqslant \langle q, p \rangle \leqslant 0$, so $p \in \sigma \cap q^\perp$. □

Lemma 15.7. If σ is a convex polyhedral cone spanning the whole vector space V with $\sigma \neq V$, then

$$\sigma = \bigcap_{\tau} \{v \in V \mid \langle u_\tau, v \rangle \geqslant 0\}$$

where τ runs over all facets of σ and u_τ is the unique vector (up to multiplication by a positive scalar) such that $\tau = \sigma \cap u_\tau^\perp$.

Proof. By definition we always have the containment $\subseteq$. If v were in the intersection but not in σ, take any vector v' in the interior of σ and denote by w the last point in σ on the segment from v' to v. Then $w \in \partial\sigma$, so w lies in some facet τ of σ by Lemma 15.6. We have $\langle u_\tau, v' \rangle > 0$ and $\langle u_\tau, w \rangle = 0$, so $\langle u_\tau, v \rangle = 0$, a contradiction. □

Lemma 15.8. The dual of a convex polyhedral cone σ is a convex polyhedral cone.

Proof. If σ spans V, then $\sigma^\vee$ is generated by the u_τ in Lemma 15.8. Indeed, if $u \in \sigma^\vee \backslash \mathrm{cone}\{u_\tau\}_\tau$, by Lemma 15.1, we can find $v \in V$ with $\langle u_\tau, v\rangle \geqslant 0$ for all facets τ and $\langle u, v\rangle < 0$; the former implies $v \in \sigma$ by the last lemma, and the latter leads to a contradiction.

If σ spans a smaller space W, then $\sigma^\vee$ is generated by lifts of generators of its image in $V^\vee / W^\perp$ along with vectors u and $-u$, where u runs over a basis of $W^\perp$.

□

Corollary 15.8.1. A subset σ in V is a convex polyhedral cone if and only if it is a finite intersection of closed half spaces in V.

Proof. The only if part follows from the last lemma and the fact $\sigma^\vee = \bigcap_{v \in S}\{u \in V^\vee \mid \langle u, v\rangle \geqslant 0\}$. For the if part, say $\sigma = u_1^\vee \cap \cdots \cap u_\ell^\vee$ with $u_i \in V^\vee$. Put $\gamma = \mathrm{cone}\{u_1, \ldots, u_\ell\} \subseteq V^\vee$; then $\gamma^\vee = \sigma$. If $\gamma \subsetneq \sigma^\vee$, say $u \in \sigma^\vee \backslash \gamma$, then by Lemma 15.1 we can find $v \in \gamma^\vee = \sigma$ such that $\langle u, v\rangle < 0$, a contradiction to the facts $u \in \sigma^\vee$ and $v \in \sigma$. Hence $\sigma^\vee = \gamma$ is a convex polyhedral cone, so $\sigma = \gamma^\vee$ is also a convex polyhedral cone by the last lemma.

□

Definition.

1. A **lattice** in V is a $\mathbb{Z}$-submodule of V of rank $\dim_{\mathbb{R}} V$.
2. A convex polyhedral cone σ in V is N**-rational** if N is a lattice in V and $\sigma = \mathrm{cone}(S)$ for some finite subset $S \subseteq N$.

- Suppose N and M are lattices of V and $V^\vee$, respectively, so that N and M are dual to each other under the pairing $\langle\, , \rangle$. Then
  (i) Any face of σ is N-rational. For if $\tau = \sigma \cap u^\perp$ and $\sigma = \mathrm{cone}(S)$ for some $S \subseteq N$, then $\tau = \mathrm{cone}(S \cap u^\perp)$.
  (ii) If σ is N-rational, then $\sigma^\vee$ is M-rational. This can be seen from the 1. and the proof of Lemma 15.8.

Lemma 15.9 (Gordan's)**.** Suppose N and M are lattices of V and $V^\vee$, respectively, so that N and M are dual to each other under the pairing $\langle\, , \rangle$. If σ is a N-rational convex polyhedral cone, then

$$S_\sigma := \sigma^\vee \cap M$$

is a finitely generated semigroup.

Proof. Say $\sigma^\vee = \mathrm{cone}(S)$ for some $S \subseteq M$. Let $K = \{\sum_{u \in S} a_u u \mid 0 \leqslant a_u \leqslant 1 \text{ for all } u \in S\}$ be the fundamental parallelotope. Since K is compact and M is discrete, $K \cap M$ is finite. Then $K \cap M$ generates $\sigma^\vee \cap M$. Indeed, if $v \in \sigma^\vee \cap M$, write

$$v = \sum_{u \in S} a_u u = \sum_{u \in S} \lfloor a_u \rfloor u + \sum_{u \in S} \{a_u\} u$$

One sees that the latter two terms lie in $K \cap M$. □

Definition. The **relative interior** $\mathrm{relint}(\sigma)$ of a convex polyhedral cone σ is the interior of σ in the topological space $\mathrm{span}_{\mathbb{R}} \sigma$.

- Say $\sigma = \mathrm{cone}(S)$, and let $T \subseteq S$ be the independent subset of size $\dim \sigma$. Then every positive linear combination of vectors in T lies in $\mathrm{relint}(\sigma)$. In particular, if $S \neq \varnothing$, then $\mathrm{relint}(\sigma) \neq \varnothing$. Also, if σ is N-rational, then $\mathrm{relint}(\sigma) \cap N \neq \varnothing$.
- We have $v \in \mathrm{relint}(\sigma)$ if and only if $\langle u, v\rangle > 0$ for all $u \in \sigma^\vee \backslash \sigma^\perp$. To see this, first consider the case $V = \mathrm{span}_{\mathbb{R}} \sigma$. Then $\mathrm{relint}(\sigma)$ is just the interior of σ in V. By Lemma 15.6 we have $v \in \mathrm{relint}(\sigma)$ if and only if $\langle u_\tau, v\rangle > 0$ for all facets τ of σ. Each element in $\sigma^\vee \backslash \sigma^\perp$ gives a proper face of σ, and each proper face is contained in some facet. Now the result follows.

If σ spans a smaller space W, then the preceding paragraph shows that

$$\operatorname{relint}(\sigma) = \bigcap_{u \in W^\vee} \{v \in \sigma \mid \langle u, v \rangle > 0\}$$

But $W^\vee \cong V^\vee / W^\perp = V^\vee/\sigma^\perp$ in a way that $\langle\, ,\, \rangle$ is preserved, the result follows.

- Every vector in σ is contained in the relative interior of some face of σ. This can be seen by induction on dimensions of cones, by virtue of Lemma 15.6.

Lemma 15.10. Let σ be a convex polyhedral cone in V.

1. If τ is a face of σ, then $\tau^* := \sigma^\vee \cap \tau^\perp$ is a face of $\sigma^\vee$ with $\dim \tau + \dim \tau^* = \dim_{\mathbb{R}} V$.
2. There is an inclusion-reversing bijection

$$\begin{array}{ccc} \{\text{faces of } \sigma\} & \longrightarrow & \{\text{faces of } \sigma^\vee\} \\ \tau & \longmapsto & \tau^* \end{array}$$

3. The smallest face of σ is $\sigma \cap (-\sigma)$.

Proof. First note that every face of $\sigma^\vee$ has the form $\sigma^\vee \cap v^\perp$ for some $v \in (\sigma^\vee)^\vee = \sigma$. If v is contained in the relative interior of some face τ, then

$$\sigma^\vee \cap v^\perp = \sigma^\vee \cap \tau^\vee \cap v^\perp = \sigma^\vee \cap \tau^\perp.$$

The first equality results from $\sigma^\vee \subseteq \tau^\vee$, and the second follows from $\langle u, v\rangle > 0$ for any $u \in \tau^\vee \backslash \tau^\perp$. This shows the map 2. is surjective. The map is clearly inclusion-reversing, and it follows formally from $\tau \subseteq (\tau^*)^*$ that $\tau^* = ((\tau^*)^*)^*$, so the map is bijective. In particular, the smallest face is

$$(\sigma^\vee)^\vee \cap (\sigma^\vee)^\perp = (\sigma^\vee)^\perp = \sigma \cap (-\sigma)$$

and $\dim \sigma \cap (-\sigma) + \dim \sigma^\vee = \dim V$ (for $\langle\, ,\, \rangle$ is nondegenerate). In general, if τ is a proper face, find a sequence

$$\sigma \cap (-\sigma) = \tau_0 \subsetneq \tau_1 \subsetneq \cdots \subsetneq \tau_\ell = \tau \subsetneq \cdots \subsetneq \tau_m \subsetneq \sigma$$

of faces of σ such that τ_{i-1} has codimension one in τ_i and τ_m is a facet. Taking $*$ and computing dimensions gives $\dim \tau + \dim \tau^* = \dim_{\mathbb{R}} V$. ☐

Lemma 15.11. Let σ be a convex polyhedral cone in V. TFAE :

(a) $\{0\}$ is a face of σ.

(b) σ contains no positive dimensional subspace of V.

(c) $\sigma \cap (-\sigma) = \{0\}$.

(d) $\dim \sigma^\vee = \dim_{\mathbb{R}} V$.

If either condition holds, we say σ is **strongly convex**.

Proof. We saw in Lemma 15.2.1. that any linear subspace of σ is contained in every face of σ. Since $\sigma \cap (-\sigma)$ is the smallest face of σ, we see (b)⇔(c). Also, Lemma 15.10 shows (c)⇔(d). If $\{0\}$ is a face, it must be the smallest one, so (a)⇔(c). ☐

Corollary 15.11.1. If σ is a strongly convex polyhedral cone, then σ is generated by its edges.

Proof. For an edge ρ of σ, by Lemma 15.11.(b), we see ρ is a ray, i.e., $\rho = \mathbb{R}_{\geqslant 0} e_\rho$ for some vector $e_\rho \in \sigma$. By saying σ is generated by its edges we actually mean that $\sigma = \mathit{cone}\{e_\rho\}$, where ρ runs over all edges of σ.

Say $\sigma = \mathit{cone}(S)$ for some finite subset $S \subseteq V$. We assume that S is minimal among all generating sets of σ. We claim that $\{\mathbb{R}_{\geqslant 0} v \mid v \in S\}$ is the collection of all edges of σ.

— Let $v \in S$. Applying Lemma 15.1 to $cone(S\backslash\{v\})$, we obtain a vector $u \in cone(S\backslash\{v\})^\vee$ with $\langle u, v\rangle < 0$. Take $u' \in \operatorname{relint}(\sigma^\vee)$; then $\langle u', v'\rangle > 0$ for any $v' \in (\sigma^\vee)^\vee\backslash(\sigma^\vee)^\perp = \sigma\backslash\{0\}$. Let $\lambda > 0$ be the unique number such that $\langle u+\lambda u', v\rangle = 0$. Then $u + \lambda u' \in \sigma^\perp$, and

$$\sigma \cap (u + \lambda u')^\perp = \mathbb{R}_{\geqslant 0} v$$

for $\langle u + \lambda u', v'\rangle > 0$ for any $v' \in S\backslash\{v\}$. This shows $\mathbb{R}_{\geqslant 0} v$ is an edge.

— Let $\tau = \sigma \cap u^\perp$ be an edge of σ. Then $\tau = cone(S \cap u^\perp)$ is one dimensional, so $S \cap u^\perp$ must be a singleton $\{v\}$, and $\tau = \mathbb{R}_{\geqslant 0} v$ by Lemma 15.11.(b).

□

Remark. Let σ be a strongly convex N-rational polyhedral cone. If ρ is an edge of σ, then ρ is a ray. Since ρ is also N-rational (and N is discrete), the semigroup $\rho \cap N$ is generated by a unique element $u_\rho \in \rho \cap N$. This vector u_ρ is called the **ray generator** of ρ. Thus Corollary 15.11.1 implies that σ is generated by the ray generators of its edges.

Lemma 15.12. Let σ be a convex polyhedral cone. If $\tau = \sigma \cap u^\perp$ for some $u \in \sigma^\vee$, then $\tau^\vee = \sigma^\vee + \mathbb{R}_{\geqslant 0}(-u)$.

Proof. Both sides of the identity being convex polyhedral cones, we only need to show their duals are equal. But

$$(\sigma^\vee + \mathbb{R}_{\geqslant 0}(-u))^\vee = (\sigma^\vee)^\vee \cap (-u)^\vee = \sigma \cap (-u)^\vee = \sigma \cap u^\perp = \tau,$$

we are done. □

Lemma 15.13 (Separation lemma)**.** If σ and σ' are convex polyhedral cones whose intersection is a face τ of each, then $\tau = \sigma \cap u^\perp = \sigma' \cap u^\perp$ for any $u \in \operatorname{relint}(\sigma^\vee \cap (-\sigma')^\vee)$.

In the following we fix a lattice N in V and denote by $M \subseteq V^\vee$ the dual lattice of N. By a rational convex polyhedral cone in V (resp. in $V^\vee$) we always mean a convex polyhedral cone that is N-rational (resp. M-rational).

15.1.1 Hilbert bases

Definition. Let σ be a rational convex polyhedral cone in V and $S_\sigma := \sigma^\vee \cap M$. An element $x \in S_\sigma$ is called **irreducible** if $x = x' + x''$ for some $x', x'' \in S_\sigma$ implies $x' = 0$ or $x'' = 0$.

Lemma 15.14. Let σ be a strongly convex rational polyhedral cone of maximal dimension in V and $S_\sigma := \sigma^\vee \cap M$. Put

$$\mathcal{H} := \{x \in S_\sigma \mid x \text{ is irreducible}\}.$$

Then

(a) $\mathcal{H}$ is finite and generates S_σ.

(b) $\mathcal{H}$ contains the ray generators of the edges of $\sigma^\vee$.

(c) $\mathcal{H}$ is the minimal generating set of S_σ with respect to inclusion.

The generating set $\mathcal{H}$ is called the **Hilbert basis** of the semigroup S_σ.

Proof. Since $\sigma = (\sigma^\vee)^\vee$, Lemma 15.11.(d) implies that $\sigma^\vee$ is strongly convex, so $\{0\}$ is a face of $\sigma^\vee$ by Lemma 15.11.(a), i.e., there exists $v \in \sigma \cap N\backslash\{0\}$ such that $\langle u, v\rangle \in \mathbb{Z}_{\geqslant 0}$ for any $u \in S_\sigma$, and for $u \in S_\sigma$, $\langle u, v\rangle = 0$ if and only if $u = 0$.

Suppose $u \in S_\sigma$ is not irreducible. Then $u = u' + u''$ for some $u', u'' \in S_\sigma\backslash\{0\}$. It follows that

$$\langle u, v\rangle = \langle u', v\rangle + \langle u'', v\rangle$$

with $\langle u', v\rangle$, $\langle u'', v\rangle \in \mathbb{Z}_{>0}$, so both of them are strictly smaller than $\langle u, v\rangle$. By induction on $\langle u, v\rangle$ we conclude that every element in S_σ is a finite sum of irreducible elements, meaning that $\mathcal{H}$ generates S_σ. By Lemma 15.9, $\mathcal{H}$ is finite. This proves (a).

For (b), let ρ be an edge of $\sigma^\vee$ and $u_\rho \in \rho \cap M$ be its ray generator. We must show that $u_\rho \in S_\sigma$ is irreducible. Write $\rho = \sigma^\vee \cap v^\perp$ for some $v \in \sigma \cap N \backslash \{0\}$. If $u_\rho = u_1 + u_2$ for some $u_i \in S_\sigma \subseteq \sigma^\vee$, then

$$0 = \langle u_\rho, v \rangle = \langle u_1, v \rangle + \langle u_2, v \rangle$$

with each $\langle u_i, v \rangle$ non-negative. This forces that $\langle u_1, v \rangle = 0 = \langle u_2, v \rangle$, so u_1, $u_2 \in \rho \cap M$. But u_ρ is the unique generator of $\rho \cap M$, so either $u_1 = 0$ or $u_2 = 0$, demonstrating the irreducibility of the ray generator u_ρ. Finally, (c) results from the fact that every element in $\mathcal{H}$ is irreducible.

□

15.2 Convex polytopes

We retain the notation in the last subsection.

Definition.

1. For any subset $S \subseteq V$, the **convex hull** of S is the smallest convex set $conv(S)$ in V containing S.
2. A **convex polytope** in V is the convex hull of a finite subset of V
3. An **affine subspace** of V is a subset of the form $v + W$, where $v \in V$ and $W \subseteq V$ is a subspace. The **dimension** of this affine subspace is the dimension of W.
4. For any subset $S \subseteq V$, the **affine span** of S is the smallest affine subspace $aff(S)$ of V containing S. In fact, we have

$$aff(S) = \left\{ \sum_{v \in S} a_v v \mid \sum_{v \in S} a_v = 1,\ a_v = 0 \text{ for all but finitely many } v \in S \right\}$$

5. The **dimension** $\dim P$ of a polytope P is $\dim aff(P)$.

- For a finite subset S of V, we have

$$conv(V) = \left\{ \sum_{v \in S} a_v v \mid 0 \leqslant a_v,\ \sum_{v \in S} a_v = 1 \right\}$$

- Let $S \subseteq V$ be a finite subset. Consider $V' = V \oplus \mathbb{R}$. If we identify V as $V \times \{1\} \subseteq V'$, then

$$conv(S) = cone(S \times \{1\}) \cap V.$$

  Clearly, $conv(S) \subseteq cone(S \times \{1\}) \cap V$. Conversely, if $v \in cone(S \times \{1\}) \cap V$, write $v = \sum_{s \in S} a_s(s, 1)$ with $a_s \geqslant 0$. Since $v \in V \times \{1\}$, $\sum_{s \in S} a_s = 1$, so $v \in conv(S)$ by the last property.

- If S is a finite subset, then $cone(S \times \{1\})$ is a strongly convex polyhedral cone. To see this, let $u' \in V^\vee$ be such that $\langle u', S \rangle > 0$. Let $u = (u', 0) \in V^\vee \oplus \mathbb{R}$; then $u \in cone(S \times \{1\})^\vee$ and $cone(S \times \{1\}) \cap u^\perp = \{0\}$, implying that $\{0\}$ is a face of $cone(S \times \{1\})$.

Definition. Let P be a convex polytope in V. A **face** of P is a subset of the form

$$P \cap \{v \in V \mid \langle u, v \rangle = b\}$$

for some $u \in V^\vee$ and $b \in \mathbb{R}$ with $P \subseteq H^+_{u,b} := \{v \in V \mid \langle u, v \rangle \geqslant b\}$. In this situation, $H_{u,b} := \{v \in V \mid \langle u, v \rangle = b\}$ is called the **supporting affine hyperplane**.

- A **facet**, **edge**, and **vertex** of P is a face of dimension $\dim P - 1$, 1 and 0, respectively.

Lemma 15.15. If $P = \mathrm{conv}(S)$ for some finite subset $S \subseteq V$ and $Q = H_{u,b} \cap P$ is a face, then $Q = \mathrm{conv}(S \cap H_{u,b})$. In particular, any face of a convex polytope is a convex polytope.

Proof. Clearly we have $\subseteq$. Conversely, if $q \in Q$, write $q = \sum_{v \in S} a_v v$ with $0 \leqslant a_v$ and $\sum_{v \in S} a_v = 1$. Since $q \in H_{u,b}$, we have $\sum_{v \in S} a_v \langle u, v \rangle = b$. But $P \subseteq H^+_{u,b}$, we have

$$b = \sum_{v \in S} a_v \langle u, v \rangle \geqslant \sum_{v \in S} a_v b = b$$

and hence the equality $\langle u, v \rangle = b$ for any $v \in S$ with $a_v \neq 0$. This proves $q \in \mathrm{conv}(S \cap H_{u,b})$. □

Lemma 15.16. There is a bijection

$$\begin{array}{ccc} \{\text{faces of } P\} & \longrightarrow & \{\text{faces of } \mathrm{cone}(P \times \{1\})\} \\ Q & \longmapsto & \mathrm{cone}(Q \times \{1\}) \end{array}$$

with empty face of P corresponding to $\{0\}$.

Proof. Write $P = \mathrm{conv}(S)$ for some finite subset $S \subseteq V$; then $\mathrm{cone}(P \times \{1\}) = \mathrm{cone}(S)$, where we identify S with $S \times \{1\}$. Let $u \in \mathrm{cone}(S)^\vee$ and write $u = (u', r)$ for some $u' \in V^\vee$ and $r \in \mathbb{R}$. Then

$$\mathrm{cone}(S) \cap u^\perp \cap V = P \cap H_{u',-r}$$

and $P \subseteq H^+_{u',-r}$. Indeed, $v = (v, 1) \in \mathrm{LHS}$ if and only if $\langle u', v \rangle + r = 0$, if and only if $v \in \mathrm{RHS}$. Since $u \in \mathrm{cone}(S)^\vee$, for any $v \in P$, we have $\langle u', v \rangle + r \geqslant 0$, i.e., $v \in H^+_{u',-r}$.

If $u \in V^\vee$ and $b \in \mathbb{R}$, then $(u, -b) \in V^\vee \oplus \mathbb{R}$, and the condition $P \subseteq H^+_{u,-b}$ implies $(u, -b) \in \mathrm{cone}(S)^\vee$. Thus, the map in the lemma is well-defined, and it is clear that it is bijective. □

Corollary 15.16.1. Let $P = \mathrm{conv}(S)$ be a polytope in V.

1. Every vertex of P lies in S.
2. P is the convex hull of its vertices.
3. If Q is a face of P, the faces of Q are exactly the faces of P contained in Q.
4. Every proper face Q of P is the intersection of facets of F containing Q.

Proof.

1. Say $H_{u,b} \cap P = \mathrm{conv}(S \cap H_{u,b})$ is a vertex of P. This means $\mathrm{conv}(S \cap H_{u,b})$ is a singleton, so $\mathrm{conv}(S \cap H_{u,b}) = S \cap H_{u,b} \subseteq S$.
2. A vertex of P corresponds to an edge of $\mathrm{cone}(P \times \{1\})$. Now the result follows from Corollary 15.11.1.
3. Lemma 15.2.4.
4. Corollary 15.5.1.2.

□

Lemma 15.17. A subset P of V is a polytope if and only if it is bounded and is a finite intersection of affine half-spaces.

Proof. The only if part follows from Corollary 15.8.1. For the if part, write $P = \bigcap_{i=1}^{\ell} H^+_{u_i,b_i}$ for some $u_i \in V^\vee$ and $b_i \in \mathbb{R}$. Then

$$C(P) = \bigcap_{i=1}^{\ell} \{(v, \lambda) \in V \oplus \mathbb{R}_{\geqslant 0} \mid \langle u_i, v \rangle \geqslant \lambda b_i\} = \bigcap_{i=1}^{\ell} (u_i, -b_i)^\vee$$

is a convex polyhedral cone, so $C(P) = \mathit{cone}(S)$ for some finite subset $S \subseteq V \oplus \mathbb{R}_{\geqslant 0}$. We assume $S = \{v_i\}_{i=1}^{a} \cup \{w_j\}_{j=1}^{b}$ with, by abuse of notation, $v_i = (v_i, 1) \in V \times \{1\}$ and $w_j = (w_j, 0) \in V \times \{0\}$. Clearly $C(P) \cap V = P$, so $P = \mathit{conv}\{v_i\}_{i=1}^{a} + \mathit{cone}\{w_j\}_{k=1}^{b}$. By boundedness of P we must have $b = 0$, i.e., $P = \mathit{conv}(S)$. $\square$

Remark. A set of the form $\mathit{conv}(S) + \mathit{cone}(T)$ with $S, T \subseteq V$ finite is called a **polyhedron**. By definition, a convex polytope is exactly a bounded polyhedron. One can show that a subset of V is a polyhedron if and only if it is a finite intersection of affine half-spaces.

When $\dim P = \dim_{\mathbb{R}} V$, by Lemma 15.7 we can write P as the intersection of the half spaces determined its facets, and each facet F has a unique supporting affine hyperplane

$$H_F := \{v \in V \mid \langle u_F, v\rangle = -a_F\}$$

for some pair $(u_F, a_F) \in V^\vee \oplus \mathbb{R}$ unique up to a multiplication of a positive scalar. The vector u_F is called an **inward-pointing facet normal** of F. If we put $H_F^+ = H_{u_F, -a_F}^+$, we have

$$P = \bigcap_F H_F^+ = \{v \in V \mid \langle u_F, v\rangle \geqslant -a_F \text{ for all facets } F\},$$

where F runs over all facets of P. This is called the **facet presentation** of P.

Definition. Convex polytopes P_1 and P_2 are **combinatorially equivalent** if there is a bijection

$$\{\text{faces of } P_1\} \longrightarrow \{\text{faces of } P_2\}$$

that preserves dimensions, intersections, and the face relation (this means if Q' is a face of Q with Q, Q' being faces of P_1, then under this bijection, the image of Q' is still a face of the image of Q).

Definition. Let P be a convex polytope of dimension d in V.

1. P is a **simplex**/**d-simplex** if it has $d+1$ vertices.
2. P is **simplicial** if every facet of P is a simplex.
3. P is **simple** if every vertex is the intersection of precisely d facets.

Definition. Let $P = \mathit{conv}(S)$ be a convex polytope in V.

1. For $r \geqslant 0$, put $rP = \mathit{conv}(rS) = \mathit{conv}\{rv \mid v \in S\}$, which is again a convex polytope.
2. If $P' = \mathit{conv}(S')$ is another convex polytope, the Minkowski sum $P + P' = \mathit{conv}(S + S')$ is still a convex polytope.
3. When $\dim P = \dim_{\mathbb{R}} V$ and $0 \in \mathrm{int}(P)$, define the **polar**/**polar set** of P to be

$$P^\circ = \{u \in V^\vee \mid \langle u, v\rangle \geqslant -1 \text{ for all } v \in P\} \subseteq V^\vee.$$

- Note that the distribution law holds : for $r, t \geqslant 0$, we have

$$rP + tP = (r+t)P.$$

— The equality $P + P' = \mathit{conv}(S + S')$ needs a verification. The containment $\mathit{conv}(S + S') \subseteq P + P'$ is relatively clear. The other can be dealt with greedy.

Lemma 15.18. Let P be a convex polytope in V with $\dim P = \dim_{\mathbb{R}} V$ and $0 \in \mathrm{int}(P)$.

1. P° is convex polytope, and $(P^\circ)^\circ = P$.
2. If Q is a face of P,

$$Q^* = \{u \in P^\circ \mid \langle u, v\rangle = -1 \text{ for all } v \in Q\}$$

is a face of P°, and this establishes an inclusion-reversing bijection

$$\begin{aligned} \{\text{faces of } P\} &\longrightarrow \{\text{faces of } P^\circ\} \\ Q &\longmapsto Q^* \end{aligned}$$

with $\dim Q + \dim Q^* = \dim V - 1$.

3. If $P = \{v \in V \mid \langle u_F, v\rangle \geqslant -a_F \text{ for all facets } F\}$, then $P^\circ = \mathit{conv}\{a_F^{-1} u_F\}_{F \text{ facets}}$. Note that $a_F > 0$ since $0 \in \operatorname{int}(P)$.

Proof. Let $\sigma = \mathit{cone}(P \times \{1\})$. Then $\sigma^\vee = \mathit{cone}(P^\circ \times \{1\})$, so P° is a convex polytope. Now $(P^\circ)^\circ = P$ follows from $(\sigma^\vee)^\vee = \sigma$. For a face Q of P, $\tau = \mathit{cone}(Q \times \{1\})$ is a face of σ, and $\tau^* = \sigma^\vee \cap \tau^\perp = \mathit{cone}(Q^* \times \{1\})$. Now 2. follows from Lemma 15.10, and 3. follows from the first paragraph of the proof of Lemma 15.8. □

15.2.1 Normal lattice polytopes

Definition. Let M be a lattice in V. An M**-lattice**/M**-rational polytope** is a convex polytope $\mathit{conv}(S)$ with $S \subseteq M$.

- A convex polytope is M-rational if and only if its vertices all lie in M.
- Faces of an M-rational polytope is also M-rational. Also, every M-rational polytope is an intersection of affine half-spaces $H^+_{u,b}$ defined over M, i.e., $u \in N$ and $b \in \mathbb{Z}$.
- Let $N \subseteq V^\vee$ be the dual lattice of M. If P is an M-rational polytope with $\dim P = \dim_{\mathbb{R}} V$ and $0 \in \operatorname{int}(P)$, then P° is an N-rational polytope in $V^\vee$.
- Let N be as above and P be M-rational and $\dim P = \dim_{\mathbb{R}} V$. For any facet F of P, the inward-pointing facet normal lies on a rational ray in $V^\vee$; let $u_F \in N$ be the unique ray generator. If v is a vertex contained in F, then $\langle u_F, v\rangle = -a_F \in \mathbb{Z}$ is integral.

In the following, fix an lattice M in V and let N be the dual lattice in $V^\vee$. By a lattice polytope in V (resp. in $V^\vee$) we always mean an M-rational (resp. N-rational) convex polytope. Notice that in the last subsection N denoted a lattice in V, while here we use M to denote a lattice in V.

Definition. A lattice polytope P in V is called **normal** if

$$(kP) \cap M + (\ell P) \cap M = ((k+\ell)P) \cap M$$

for all $k, \ell \in \mathbb{Z}_{\geqslant 0}$.

- The inclusion $\subseteq$ is automatic, so the normality actually means that all lattice points of $(k+\ell)P$ come from lattice points of kP and ℓP.
- A lattice polytope P is normal if and only if

  $$\underbrace{P \cap M + \cdots + P \cap M}_{k\text{-copies}} = (kP) \cap M$$

  for all $k \in \mathbb{Z}_{\geqslant 0}$.
- Lattices polytopes of dimension one are normal.

Definition. A lattice polytope P of V is a **basic simplex**/**unimodular simplex** if P is a simplex and has a vertex m_0 such that $\{m - m_0 \mid m \neq m_0 : \text{vertices of } P\}$ forms a subset of a $\mathbb{Z}$-basis of M.

- The definition is independent of the choice of the vertex $m_0 \in P$.
- The standard simplex Δ_d $(0 \leqslant d \leqslant n)$ in $\mathbb{R}^n$ is basic.
- Any basic simplex is normal.

Theorem 15.19. Let $P \subseteq V$ be a full dimensional lattice polytope of dimension $n \geqslant 2$. Then kP is normal for all $k \geqslant n-1$.

Proof. We will prove that

$$(kP) \cap M + P \cap M = ((k+1)P) \cap M$$

for $k \geqslant n-1$. This implies kP is normal for all $k \geqslant n-1$.

First consider the case where P is a simplex with no interior lattice point. Let $v_0, \dots, v_n$ be vertices of P and take $k \geqslant n-1$. Then $(k+1)P$ has vertices $(k+1)v_0, \dots, (k+1)v_n$, so that any point $v \in ((k+1)P) \cap M$ has the form

$$v = \sum_{i=0}^{n} \mu_i (k+1) v_i$$

with $\mu_i \geqslant 0$ and $\sum_{i=0}^{n} \mu_i = 1$. Put $\lambda_i = (k+1)\mu_i$; we have $\sum_{i=0}^{n} \lambda_i = k+1$.

(i) $\lambda_i \geqslant 1$ for some $i = 0, \dots, n$. Then $v - v_i \in (kP) \cap M$, so $v = (v - v_i) + v_i \in (kP) \cap M + P \cap M$, as we desire.

(ii) $\lambda_i < 1$ for all $i = 0, \dots, n$. Then

$$n = (n-1) + 1 \leqslant k + 1 = \sum_{i=0}^{n} \lambda_i < n + 1$$

so $k = n - 1$ and $\sum_{i=0}^{n} \lambda_i = n$. Consider the point

$$\tilde{v} := (v_0 + \cdots + v_n) - v = \sum_{i=0}^{n} (1 - \lambda_i) v_i$$

The coefficients are positive, and their sum is $\sum_{i=0}^{n} (1-\lambda_i) = (n+1) - n = 1$. Thus $\tilde{v}$ is a lattice point, and since $1 - \lambda_i > 0$ for each i, $\tilde{v}$ lies in the interior of P. This is a contradiction.

For general P, it suffices to show P is a finite union of n-dimensional lattice simplices with no interior lattice points. For this, we invoke

Lemma 15.20 (Carathéodory's)**.** For a finite set $S \subseteq V$, we have

$$\mathrm{conv}(S) = \bigcup_{T} \mathrm{conv}(T)$$

where T runs over all subsets of S consisting of $\dim \mathrm{conv}(S) + 1$ affinely independent vectors.

Proof. Let $x \in \mathrm{conv}(S)$ and write $x = \sum_{v \in S} a_v v$ with $0 \leqslant a_v$ and $\sum_{v \in S} a_v = 1$. Among all such presentations of x, we choose a minimal one, i.e., the one so that

$$T := \{v \in S \mid a_v \neq 0\}$$

is minimal. We claim $\#T \leqslant \dim \mathrm{conv}(S) + 1$. Suppose for contradiction that $\#T > \dim \mathrm{conv}(S) + 1$. Pick any $t_0 \in T$. Then $\{t - t_0 \mid t \neq t_0,\ t \in T\}$ is linear dependent, so $0 = \sum_{T \ni t \neq t_0} b_t (t - t_0)$ for some nonzero sequence $(b_t)_{t \neq t_0}$. If we define $b_{t_0} = -\sum_{t \neq t_0} b_t$, we see $0 = \sum_{t \in T} b_t t$ and $0 = \sum_{t \in T} b_t$. Since $(b_t)_t$ is nonzero, $\#\{t \in T | b_t > 0\} \geqslant 1$. For any $r \in \mathbb{R}$, we may write

$$x = \sum_{v \in T} a_v v - r \sum_{v \in T} b_v v = \sum_{v \in T} (a_v - r b_v) v.$$

If we choose $r = \min\left\{\frac{a_v}{b_v} \mid v \in T,\ b_v > 0\right\} > 0$, then $a_v - rb_v \geqslant 0$ for any $v \in T$ and $\sum_{v \in T}(a_v - rb_v) = 1$. But $r = \frac{a_{v'}}{b_{v'}}$ for some $v' \in T$, so $a_{v'} - rb_{v'} = 0$, contradicting the minimality of T. In addition, we also proved that T is affinely independent. The proof is completed once we notice that every affinely independent subset of S can be extended to an affinely independent subset of S of size $\dim \mathrm{conv}(S) + 1$. □ Thus we can write P as a finite union of n-dimensional lattice simplices. If $Q = \mathrm{conv}\{w_0, \dots, w_n\}$ is an n-dimensional lattice simplex with an interior lattice point v, then

$$Q = \bigcup_{i=0}^{n} \mathrm{conv}\{w_0, \dots, \widehat{w_i}, \dots, w_n, v\}$$

with each $\mathrm{conv}\{w_0, \dots, \widehat{w_i}, \dots, w_n, v\}$ an n-dimensional lattice simplex having fewer interior lattice points. We repeat this process so that Q, and hence P, is written as a finite union of n-dimensional lattice simplices with no interior lattice points. □

Corollary 15.20.1. Every lattice polygon in $\mathbb{R}^2$ is normal.

Lemma 15.21. Let P be a lattice polytope in V. Then P is normal if and only if $(P \cap M) \times \{1\}$ generates the semigroup $C(P) \cap (M \times \mathbb{Z})$, where $C(P) = \mathrm{cone}(P \times \{1\})$.

Proof. Note that

$$C(P) \cap (M \times \mathbb{Z}) = \{0\} \cup \bigcup_{k \geqslant 1} ((kP) \cap M) \times \{k\}$$

Thus $(P \cap M) \times \{1\}$ generates $C(P) \cap (M \times \mathbb{Z})$ if and only if for any $k \geqslant 1$ and $v \in (kP) \cap M$, we can find $v_1, \dots, v_\ell \in P \cap M$ such that $v = v_1 + \cdots + v_\ell$ and $k = \ell$, which is the same as saying that P is normal. □

Remark. Elements in $(P \cap M) \times \{1\}$ are clearly irreducible, so $(P \cap M) \times \{1\}$ is contained in the Hilbert basis of the semigroup $C(P) \cap (M \times \mathbb{Z})$. This lemma simply says that P is normal if and only if $(P \cap M) \times \{1\}$ is exactly the Hilbert basis of $C(P) \cap (M \times \mathbb{Z})$.

Lemma 15.22. Let P be a full dimensional lattice polytope in V of dimension $n \geqslant 2$ and let k_0 be the maximal height of an element in the Hilbert basis of $C(P)$. Then $k_0 \leqslant n - 1$.

Proof. This follows from the identity

$$(kP) \cap M + P \cap M = ((k+1)P) \cap M \qquad (k \geqslant n - 1)$$

proved in Theorem 15.19. □

15.2.2 Very ample polytopes

Definition. A lattice polytope P in V is called **very ample** if for every vertex v of P, the semigroup

$$S_{P,v} := \mathbb{Z}_{\geqslant 0}(P \cap M - v)$$

generated by the set $\{v' - v \mid v' \in P \cap M\}$ is saturated in M.

- Recall that an affine semigroup $S \subseteq M$ is **saturated** if for all $k \in \mathbb{Z}_{>0}$ and $v \in M$, $kv \in S$ implies $v \in S$.

Lemma 15.23. A normal lattice polytope P is very ample.

Proof. Fix a vertex v_0 of P and take $w \in M$ such that $kw \in S_{P,v_0}$ for some $k \in \mathbb{Z}_{>0}$. Write

$$kw = \sum_{v' \in P \cap M} a_{v'}(v' - v_0)$$

for some $a_{v'} \in \mathbb{Z}_{\geqslant 0}$. Take $d \in \mathbb{Z}_{\geqslant 0}$ so that $kd \geqslant \sum_{v' \in P \cap N} a_{v'}$. Then

$$kw + kdv_0 = \sum_{v' \in P \cap M} a_{v'} v' + \left(kd - \sum_{v' \in P \cap M} a_{v'} \right) v_0 \in kdP$$

Dividing out k, we see $w + dv_0 \in dP$. By normality we see $w + dv_0 = \sum_{m \in T} m$ for some $T \subseteq P \cap M$ with $\#T = d$. Thus

$$w = \sum_{m \in T} (m - v_0) \in S_{P,v_0}$$

as desired. $\square$

Corollary 15.23.1. Let $P \subseteq V$ be a full dimensional lattice polytope. If $\dim P \geqslant 2$, then kP is very ample for all $k \geqslant n-1$. In particular, P is very ample if $\dim P = 2$.

Example. There exists non-normal very ample polytope. Given $1 \leqslant i < j < k \leqslant 6$, let $[ijk]$ denote the vector in $\mathbb{Z}^6$ with 1 in positions i, j, k and 0 elsewhere. Let

$$A = \{[123], [124], [135], [146], [156], [236], [245], [256], [345], [346]\} \subseteq \mathbb{Z}^6.$$

The lattice polytope $P = conv(A)$ lies in the affine hyperplane $\sum_{i=1}^{6} x_i = 3$ in $\mathbb{R}^6$. It is straightforward to see that $A = P \cap \mathbb{Z}^6$, and A is the set of vertices of P. Label the points in A as $m_1, \ldots, m_{10}$. Then

$$(1,1,1,1,1,1) = \frac{1}{5} \sum_{i=1}^{10} m_i = \sum_{i=1}^{10} \frac{1}{10}(2m_i)$$

which shows that $v = (1,1,1,1,1,1) \in 2P$. Since v is not a sum of lattices point of P (when $[ijk] \in A$, the vector $v - [ijk] \notin A$), we conclude that P is not a normal polytope.

To show P is very ample, firstly, by computer we see $A \times \{1\} \cup \{(v,2)\} \subseteq \mathbb{R}^6 \times \mathbb{R}$ is a Hilbert basis of the semigroup $C(P) \cap \mathbb{Z}^7$, where $C(P) = cone(P \times \{1\}) \subseteq \mathbb{R}^6 \times \mathbb{R}$. Next fix i and let S_{P,m_i} be the semigroup generated by the $m_j - m_i$. Take $m \in \mathbb{Z}^6$ such that $km \in S_{P,m_i}$. As in the proof of the last lemma, this implies $m + dm_i \in dP$ for some $d \in \mathbb{Z}_{\geqslant 0}$, and thus $(m + dm_i, d) \in C(P) \cap \mathbb{Z}^7$. Expressing this in terms of the above Hilbert basis easily implies that

$$m = a(v - 2m_i) + \sum_{j=1}^{10} a_j (m_j - m_i)$$

for some $a, a_j \in \mathbb{Z}_{\geqslant 0}$. If we can show $v - 2m_i \in S_{P,m_i}$, then $m \in S_{P,m_i}$ follows and this proves S_{P,m_i} is saturated. When $i = 1$, one checks that

$$v + [123] = [124] + [135] + [236]$$

which implies that

$$v - 2m_1 = (m_2 - m_1) + (m_3 - m_1) + (m_6 - m_1) \in S_{P,m_1}.$$

One computes this for $i = 2, \ldots, 10$ similarly, and this proves that P is very ample.

15.2.3 Normal fans

Let P be a full dimensional lattice polytope. We have the facet presentation of P

$$P = \{x \in V \mid \langle u_F, x \rangle \geqslant -a_F \text{ for all facets } F\}.$$

A vertex v of P gives cones

$$C_v = \mathrm{cone}(P \cap M - v) \subseteq V \text{ and } \sigma_v = C_v^\vee \subseteq V^\vee.$$

where $P \cap M - v = \{w - v \mid w \in P \cap M\}$. There is a bijection

$$\begin{aligned}
\{\text{faces of P containing } v\} &\longrightarrow \{\text{faces of } C_v\} \\
Q &\longmapsto Q_v = \mathrm{cone}(Q \cap M - v) \\
Q = (Q_v + v) \cap P &\longleftarrow\!\!\shortmid\; Q_v
\end{aligned}$$

that preserves dimensions, inclusions and intersections. In particular, all facets of C_v come from facets of P containing v, so that Lemma 15.7 gives

$$C_v = \{x \in V \mid \langle u_F, x\rangle \geqslant 0 \text{ for all facets F of P containing } v\}.$$

It follows that the proof of Lemma 15.8 that

$$\sigma_v = \mathrm{cone}\,\{u_F \mid F : \text{facets of P containing } v\}.$$

For any face Q of P, we set

$$\sigma_Q = \mathrm{cone}\,\{u_F \mid F : \text{facets of P containing Q}\}.$$

In particular, $\sigma_F = \mathbb{R}_{\geqslant 0} u_F$ and $\sigma_P = \mathrm{cone}\,\varnothing = \{0\}$.

Theorem 15.24. Let P be a full dimensional lattice polytope and set $\Sigma_P = \{\sigma_Q \mid Q \text{ is a face of P}\}$. Then :

1. For all $\sigma_Q \in \Sigma_P$, each face of σ_Q also lies in Σ_P.
2. The intersection $\sigma_Q \cap \sigma_{Q'}$ of any two cones in Σ_P is a face of each.

We call Σ_P the **(inner) normal fan** of P.

To avoid cumbersome sentences, we will always write Q, F, v to denote faces, facets, vertices of P.

Lemma 15.25. Let Q be a face of P and let $H_{u,b}$ be a supporting affine hyperplane of P. Then $u \in \sigma_Q$ if and only if $Q \subseteq H_{u,b} \cap P$.

Proof. Suppose $u \in \sigma_Q$ and write $u = \sum_{Q \subseteq F} \lambda_F u_F$, $\lambda_F \geqslant 0$. By setting $b_0 = -\sum_{Q \subseteq F} \lambda_F a_F$, we see

$$\langle u, x\rangle = \sum_{Q \subseteq F} \lambda_F \langle u_F, x\rangle \geqslant -\sum_{Q \subseteq F} \lambda_F a_F = b_0$$

for any $x \in P$, and the middle inequality is an equality for $x \in Q$. In other words, $P \subseteq H^+_{u,b_0}$ and $Q \subseteq H_{u,b_0} \cap P$, so H_{u,b_0} is a supporting hyperplane of P. By assumption $H_{u,b}$ is a support hyperplane of P, so we must have $b = b_0$, and $Q \subseteq H_{u,b} \cap P$ in turn.

Conversely, suppose that $Q \subseteq H_{u,b} \cap P$. Take a vertex $v \in Q$. Then $P \subseteq H^+_{u,b}$ and $v \in H_{u,b}$ imply that $C_v \subseteq H^+_{u,0}$. Hence $u \in C_v^\vee = \sigma_v$, so that

$$u = \sum_{v \in F} \lambda_F u_F$$

for some $\lambda_F \geqslant 0$. Let F_0 be a facets of P containing v but not Q, and pick $p \in Q$ with $p \notin F_0$. Then $p, v \in Q \subseteq H_{u,b}$ imply that

$$\begin{aligned}
b &= \langle u, p\rangle = \sum_{v \in F} \lambda_F \langle u_F, p\rangle \\
b &= \langle u, v\rangle = \sum_{v \in F} \lambda_F \langle u_F, v\rangle = -\sum_{v \in F} \lambda_F a_F,
\end{aligned}$$

so that $\sum_{v \in F} \lambda_F \langle u_F, p \rangle = -\sum_{v \in F} \lambda_F a_F$. But $p \notin F_0$ gives $\langle u_{F_0}, p \rangle > -a_F$, and since $\langle u_F, p \rangle \geqslant -a_F$ for all F, it forces that $\lambda_{F_0} = 0$ whenever $Q \nsubseteq F_0$. Thus $u \in \sigma_Q$, and the proof is completed. □

Corollary 15.25.1. If Q is a face of P and F is a facet of P, then $u_F \in \sigma_Q$ if and only if $Q \subseteq F$.

Proof. The if part follows from definition, and the only if part follows from the last lemma with $u = u_F$ and $b = -a_F$, where $F = P \cap H_{u_F, -a_F}$. □

Theorem 15.24 is an immediate consequence of the following

Lemma 15.26. Let Q and Q' be faces of a full dimensional lattice polytope P in V. Then

(a) $Q \subseteq Q'$ if and only if $\sigma_{Q'} \subseteq \sigma_Q$.

(b) If $Q \subseteq Q'$, then $\sigma_{Q'}$ is a face of σ_Q, and all faces of σ_Q have this form.

(c) $\sigma_Q \cap \sigma_{Q'} = \sigma_{Q''}$, where Q'' is the smallest face of P containing Q and Q'.

Proof.

(a) The only if part is obvious, and the if part follows from Corollary 15.25.1 and Corollary 15.5.1.2.

(b) Recall that a vertex $v \in Q$ of P determines a face Q_v of C_v. By Lemma 15.10.2,

$$Q_v^* := C_v^\vee \cap Q_v^\perp = \sigma_v \cap Q_v^\perp$$

is a face of σ_v. Since $Q_v \subseteq C_v = \sigma_v^\vee$, for $u \in Q_v^*$, if we write $u = \sum_{v \in F} \lambda_F u_F$ with $\lambda_F \geqslant 0$, then for any $x \in Q_v$, we have

$$0 = \langle u, x \rangle = \sum_{v \in F} \lambda_F \langle u_F, x \rangle$$

with each $\langle u_F, x \rangle \geqslant 0$. Thus $Q_v \subseteq H_{u_F,0}$ if $\lambda_F > 0$, proving that $u \in \mathrm{cone}\{u_F \mid v \in F,\ Q_v \subseteq H_{u_F,0}\}$. Thus $Q_v^* \subseteq \mathrm{cone}\{u_F \mid v \in F,\ Q_v \subseteq H_{u_F,0}\}$. The reversed inclusion is clear, so we obtain

$$Q_v^* = \mathrm{cone}\{u_F \mid v \in F,\ Q_v \subseteq H_{u_F,0}\}.$$

Since $v \in Q$, the inclusion $Q_v \subseteq H_{u_F,0}$ is the same as $Q \subseteq H_{u_F,-a_F}$, which is equivalent to $Q \subseteq F$. It follows that

$$Q_v^* = \mathrm{cone}\{u_F \mid Q \subseteq F\} = \sigma_Q,$$

so σ_Q is a face of σ_v, and all faces of σ_v arise in this way (Lemma 15.10.2).

In particular, $Q \subseteq Q'$ implies that $\sigma_{Q'}$ is a face of σ_v, and since $\sigma_{Q'} \subseteq \sigma_Q$ by (a), $\sigma_{Q'}$ is thus a face of σ_Q. Furthermore, every face of σ_Q is a face of σ_v by Lemma 15.2.4, and hence has the form $\sigma_{Q'}$ for some face Q'; by (a) again we see $Q' \subseteq Q$, and (b) follows.

(c) Let Q'' be the smallest face of P containing Q and Q' (whose existence is assured by Lemma 15.2.3). By (b) $\sigma_{Q''}$ is a face of both σ_Q and $\sigma_{Q'}$, so $\sigma_{Q''} \subseteq \sigma_Q \cap \sigma_{Q'}$.

It remains the prove the reversed inclusion. If $\sigma_Q \cap \sigma_{Q'} = \{0\} = \sigma_P$, then $Q'' = P$ and we are done. If $\sigma_Q \cap \sigma_{Q'} \neq \{0\}$, pick a nonzero vector u in the intersection. If we take $b = \min_{v \in P} \langle u, v \rangle$, then $P \subseteq H_{u,b}^+$ and $v \in H_{u,b}$ for at least one vertex v of P, implying that $H_{u,b}$ is a supporting hyperplane. By Lemma 15.25, $u \in \sigma_Q \cap \sigma_{Q'}$ implies $Q \subseteq H_{u,b} \cap P \supseteq Q'$, so $Q'' \subseteq H_{u,b} \cap P$ by minimality of Q''. Using the same lemma again, we obtain $u \in \sigma_{Q''}$, as desired.

□

Lemma 15.27. Let P be a full dimensional lattice polytope in V of dimension n. Then

1. $\dim Q + \dim \sigma_Q = n$ for all faces Q of P.

2. $V^\vee = \bigcup_{v \in P} \sigma_v = \bigcup_{\sigma_Q \in \Sigma_P} \sigma_Q$.

We will use the sentence "the normal fan Σ_P is **complete**" to refer to the statement 2.

Proof.

1. This follows from the proof of Lemma 15.26.(b) and Lemma 15.10.1.
2. In the proof of Lemma 15.26.(c) we saw for every nonzero element $u \in V^\vee$, we can find $b \in \mathbb{R}$ making $H_{u,b}$ a supporting hyperplane of P. If $v \in H_{u,b}$, then Lemma 15.25 gives $u \in \sigma_v$. The last equality follows from Lemma 15.26.(a).

□

Lemma 15.28. Let P be a full dimensional lattice polytope in V. For any lattice point $m \in M$ and any integer $k \geqslant 1$, the translation $P + m$ and the dilation kP have the same normal fan as P.

16 Toric Varieties

In this section, a variety over k is defined as in (4.17).

Definition. A **toric variety** is an irreducible variety X over $\mathbb{C}$ containing a torus $T_N \cong (\mathbb{C}^\times)^n$ as a Zariski open subset such that the action of T_N on itself extends to an algebraic action of T_N on X.

16.1 Affine toric varieties

16.1 We recall some properties about tori. Denote by **AlgGp** the category of algebraic groups over $\mathbb{C}$. A **torus** (over $\mathbb{C}$) is an affine algebraic group T isomorphic to $(\mathbb{C}^\times)^n$ in **AlgGp**. The set $\mathrm{Hom}_{\mathbf{AlgGp}}(T, \mathbb{C}^\times)$, which is an abstract group, is called the **character group** of T, and its element is called a **character** of T. From the theory of tori, the character group is isomorphic to $\mathbb{Z}^n$. Explicitly, for $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$, define $\chi^m : T \to \mathbb{C}^\times$ by

$$\chi^m(x_1, \dots, x_n) = x_1^{m_1} \cdots x_n^{m_n}$$

where $(x_1, \dots, x_n) \in (\mathbb{C}^\times)^n \cong T$. The assignment $m \mapsto \chi^m$ is an group isomorphism between $\mathbb{Z}^n$ and $\mathrm{Hom}_{\mathbf{AlgGp}}(T, \mathbb{C}^\times)$. Also, if we put $M = \mathrm{Hom}_{\mathbf{AlgGp}}(T, \mathbb{C}^\times)$, then the coordinate ring $\mathbb{C}[T]$ of T is the same as the group algebra $\mathbb{C}[M]$.

On the other hand, the set $\mathrm{Hom}_{\mathbf{AlgGp}}(\mathbb{C}^\times, T)$, which is also an abstract group, is called the **cocharacter group** of T, and its element is called a **cocharacter**. It is also isomorphic to $\mathbb{Z}^n$. Explicitly, for $u = (u_1, \dots, u_n) \in \mathbb{Z}^n$, define $\lambda^u : \mathbb{C}^\times \to T$ by

$$\lambda^u(t) = (t^{u_1}, \dots, t^{u_n}) \in (\mathbb{C}^\times)^n \cong T.$$

The assignment $u \mapsto \lambda^u$ is an group isomorphism between $\mathbb{Z}^n$ and $\mathrm{Hom}_{\mathbf{AlgGp}}(\mathbb{C}^\times, T)$.

There is a pairing defined as follows.

$$\begin{aligned} \mathrm{Hom}_{\mathbf{AlgGp}}(T, \mathbb{C}^\times) \times \mathrm{Hom}_{\mathbf{AlgGp}}(\mathbb{C}^\times, T) &\longrightarrow \mathrm{Hom}_{\mathbf{AlgGp}}(\mathbb{C}^\times, \mathbb{C}^\times) \\ (\chi, \lambda) &\longmapsto \chi \circ \lambda \end{aligned}$$

The last group is isomorphic to $\mathbb{Z}$, so $\chi \circ \lambda(x) = x^\ell$ for some unique $\ell \in \mathbb{Z}$. We set $\ell = \langle \chi, \lambda \rangle$. If we identify both the character group and the cocharacter group with $\mathbb{Z}^n$, this becomes the inner product on $\mathbb{Z}^n$:

$$\begin{aligned} \mathbb{Z}^n \times \mathbb{Z}^n &\longrightarrow \mathbb{Z} \\ (m = (m_i), u = (u_i)) &\longmapsto m \cdot n = \sum_{i=1}^n m_i u_i. \end{aligned}$$

In particular, this shows the pairing $\langle\, , \rangle$ is perfect.

16.2 Let T be a torus with character group M. Every point $t \in T$ induces an evaluation $\phi_t : M \to \mathbb{C}^\times$ given by $\phi_t(\chi) = \chi(t)$. The map

$$\begin{aligned} T &\longrightarrow \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^\times) \\ t &\longmapsto \phi_t \end{aligned}$$

is an group isomorphism. Indeed, if we identify T with $(\mathbb{C}^\times)^n$, then

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^\times) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{C}^\times) &\longrightarrow (\mathbb{C}^\times)^n \\ \phi &\longmapsto (\phi(e_1), \dots, \phi(e_n)) \end{aligned}$$

is an group isomorphism whose inverse is exactly the map given above. Thus

$$T \cong \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^\times) \cong \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^\times \cong N \otimes_{\mathbb{Z}} \mathbb{C}^\times$$

The map is explicitly given by $N \otimes_{\mathbb{Z}} \mathbb{C}^\times \ni u \otimes t \mapsto \lambda^u(t) \in T$.

16.3 We quote, without proof, some facts about tori that we will use in the sequel.

(i) If $\varphi : T_1 \to T_2$ is a morphism of algebraic groups between tori, then the image of φ is closed in T_2, and is a torus.

(ii) If T is a torus and $H \subseteq T$ is an irreducible subvariety that is also an abstract subgroup, then H is a torus.

Suppose that a torus T acts linearly on a finite dimensional $\mathbb{C}$-vector space V. For each $m \in M$, set

$$V_m := \{v \in V \mid t.v = \chi^m(t)v \text{ for all } t \in T\}$$

This is the eigenspace of T with eigencharacter χ^m. Then we always have

(iii) $V = \bigoplus_{m \in M} V_m$.

16.4 Affine toric variety from lattice points. a Let T_N be a torus with character group M and cocharacter group N. Let $A = \{m_1, \dots, m_s\} \subseteq M$ be a finite subset of M. Using A, we construct a morphism $\Phi_A : T_N \to (\mathbb{C}^\times)^s$ by

$$\Phi_A(t) = (m_1(t), \dots, m_s(t)) \in (\mathbb{C}^\times)^s$$

Denote by Y_A the Zariski closure in $\mathbb{A}^s_{\mathbb{C}} = \mathbb{C}^s$ of the image of Φ_A. Then Y_A is an affine toric variety whose torus has character group $\mathbb{Z}A \leqslant M$, the abelian group generated by A.

To see this, by (16.3).(i), the image $T = \Phi_A(T)$ of Φ_A in $(\mathbb{C}^\times)^s$ is closed and is a torus. So by definition we have $Y_A \cap (\mathbb{C}^\times)^s = T$, which means that T is open in Y_A. This shows Y_A contains the torus T as a Zariski open subset. To show the action of T on itself extends to an action of T on Y_A, note that for $t \in T$, we have $T = t.T \subseteq t.Y_A$, so taking closure gives $Y_A \subseteq t.Y_A$. Replacing t by t^{-1} shows $Y_A = t.Y_A$, so the action of T on itself really induces an action on Y_A. This shows Y_A is an affine toric variety.

We are left to show the character group of T is $\mathbb{Z}A$. By definition of T, we have a commutative diagram

$$\begin{tikzcd}
T_N \arrow[rr] \arrow[dr, two heads] & & (\mathbb{C}^\times)^s \\
& T \arrow[ur, hook, "\text{closed}"'] &
\end{tikzcd}$$

Taking coordinate rings gives

$$\begin{tikzcd}
\mathbb{C}[M] & & \mathbb{C}[x_1^\pm, \dots, x_s^\pm] \arrow[ll] \arrow[dl, two heads] \\
& \mathbb{C}[T] \arrow[ul, hook] &
\end{tikzcd}$$

The upper horizontal map is given by $x_i \mapsto m_i$, so $\mathbb{C}[T] \cong \mathbb{C}[m_1^\pm, \dots, m_s^\pm] = \mathbb{C}[\mathbb{Z}A]$. Note that the m_i are viewed as characters of T in a way that for $t = (t_1, \dots, t_s) \in T \subseteq (\mathbb{C}^\times)^s$,

$$m_i(t) = x_i(t_1, \dots, t_s) = t_i.$$

Let us put M' to be the character group of T temporarily. Then $\mathbb{Z}A \leqslant M'$ (written additively). But $\mathbb{C}[\mathbb{Z}A] = \mathbb{C}[M']$, we must have $\mathbb{Z}A = M'$, as we desire.

16.5 Toric ideals Let $A = \{m_1, \dots, m_s\} \subseteq M$ and Y_A be the affine toric variety constructed in (16.4). The map Φ_A there gives a homomorphism on character groups :

$$\begin{aligned}\widehat{\Phi}_A : \mathbb{Z}^s &\longrightarrow M \\ e_i &\longmapsto m_i\end{aligned}$$

Let L be the kernel of this map, so that we have an exact sequence of abelian groups

$$0 \longrightarrow L \longrightarrow \mathbb{Z}^s \xrightarrow{\widehat{\Phi}_A} M$$

For an element $\ell = (\ell_1, \dots, \ell_s) \in L$, write

$$\ell_+ = \sum_{i:\ell_i>0} \ell_i e_i \text{ and } \ell_- = -\sum_{i:\ell_i<0} \ell_i e_i.$$

Then $\ell = \ell_+ - \ell_-$ with $\ell_\pm \in (\mathbb{Z}_{\geqslant 0})^s$. The binomial

$$X^{\ell_+} - X^{\ell_-} = \prod_{i:\ell_i>0} x_i^{\ell_i} - \prod_{i:\ell_i<0} x_i^{-\ell_i}$$

vanishes on the image of Φ_A, and hence on Y_A. Indeed, by definition of L we have $\sum_{i=1}^s \ell_i m_i = 0$, so $\sum_{i:\ell_i>0} \ell_i m_i = -\sum_{i:\ell_i<0} \ell_i m_i$, which implies our assertion. Hence

$$\langle X^{\ell_+} - X^{\ell_-} \mid \ell \in L\rangle \subseteq I(Y_A).$$

In fact, we have the equalities :

$$I(Y_A) = \langle X^{\ell_+} - X^{\ell_-} \mid \ell \in L\rangle = \langle X^a - X^b \mid a, b \in (\mathbb{Z}_{\geqslant 0})^s,\ a - b \in L\rangle.$$

The last equality is clear, so it remains to show the first is contained in the second. For this, fix an isomorphism $T_N \cong (\mathbb{C}^\times)^n$ and identify $M = \mathbb{Z}^n$; then the map $\Phi_A : (\mathbb{C}^\times)^n \to (\mathbb{C}^\times)^s$ has the form

$$\Phi_A(t_1, \dots, t_n) = (t^{m_1}, \dots, t^{m_s})$$

where for $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$, we write $t^a := t_1^{a_1} \cdots t_n^{a_n}$. Fix a monomial order $>$ on $\mathbb{C}[x_1, \dots, x_s]$. If $I_L := \langle X^{\ell_+} - X^{\ell_-} \mid \ell \in L\rangle \subsetneq I(Y_A)$, pick $f \in I(Y_A)\backslash I_L$ with minimal leading monomial $x^\alpha = \prod_{i=1}^s x_i^{\alpha_i}$, where $\alpha = (\alpha_1, \dots, \alpha_s) \in (\mathbb{Z}_{\geqslant 0})^s$. Rescaling, if necessary, we may assume x^α is the leading term of f (that is, its coefficient is one).

Since $f(t^{m_1}, \dots, t^{m_s}) = 0$ identically, there must be some cancellation involving the term x^α. In other words, there must exist $\beta = (\beta_1, \dots, \beta_s) \in (\mathbb{Z}_{\geqslant 0})^s$ such that

$$\prod_{i=1}^s t^{\alpha_i m_i} = \prod_{i=1}^s t^{\beta_i m_i}.$$

which implies $\sum_{i=1}^s \alpha_i m_i = \sum_{i=1}^s \beta_i m_i$. It follows that $\alpha - \beta \in L$ and $X^\alpha - X^\beta \in I_L$, so $f - (X^\alpha - X^\beta) \in I(Y_A)\backslash I_L$ has smaller leading monomial, a contradiction.

The ideal $I(Y_A)$ is then an example of *toric ideals*, a notion defined in the next paragraph.

16.6 Definition. Let $L \subseteq \mathbb{Z}^s$ be a lattice.

1. The ideal $I_L := \langle X^a - X^b \mid a, b \in (\mathbb{Z}_{\geqslant 0})^s,\ a - b \in L\rangle$ is called a **lattice ideal**.

2. A prime lattice ideal is called a **toric ideal**.

16.7 Proposition. An ideal $I \trianglelefteq \mathbb{C}[x_1, \dots, x_s]$ is toric if and only if it is prime and is generated by some binomials of the form $X^\alpha - X^\beta$ $(\alpha, \beta \in (\mathbb{Z}_{\geqslant 0})^s)$.

Proof. The only if part is shown in (16.5). For the if part, say I is a prime ideal generated by $X^{\alpha_j} - X^{\beta_j}$ for some α_j, $\beta_j \in (\mathbb{Z}_{\geqslant 0})^s$. Then $(1, \dots, 1) \in V(I) \cap (\mathbb{C}^\times)^s$ and $V(I) \cap (\mathbb{C}^\times)^s$ is a subgroup of $(\mathbb{C}^\times)^s$. Indeed, if $t = (t_1, \dots, t_s)$, $r = (r_1, \dots, r_s)$ lies in the intersection, then

$$\prod_{i=1}^{s} t_i^{\alpha_{ji}} = \prod_{i=1}^{s} t_i^{\beta_{ji}} \text{ and } \prod_{i=1}^{s} r_i^{\alpha_{ji}} = \prod_{i=1}^{s} r_i^{\beta_{ji}}$$

so $\prod_{i=1}^{s}(r_i t_i)^{\alpha_{ji}} = \prod_{i=1}^{s}(r_i t_i)^{\beta_{ji}}$, implying $ts \in V(I) \cap (\mathbb{C}^\times)^s$. By (16.3).(ii), the intersection $T := V(I) \cap (\mathbb{C}^\times)^s$ is a torus, so $V(I)$ is an affine toric variety.

Denote by m_i the induced character on T from the projection to the i-th component of $(\mathbb{C}^\times)^s$. Then clearly $V(I) = Y_A$ with $A = \{m_1, \dots, m_s\} \subseteq M$ (here we take $T_N = T$, so M is the character group of T). Since I is a prime, by Hilbert's Nullstellensatz we have $I = I(Y_A)$, and hence I is toric by (16.5). □

16.8 Affine monoids. By definition, a **monoid** is a set S equipped with an associative unital binary operation. We say a monoid S is an **affine monoid** if

(i) S is commutative and finitely generated, and

(ii) S is embedded in some lattice M of a finite dimensional vector space.

For an affine monoid S, the **monoid algebra** $\mathbb{C}[S]$ is a $\mathbb{C}$-algebra with a $\mathbb{C}$-basis S and with multiplication induced from the binary operation on S. If, say, S is embedded in M and M is the character group of a torus T_N, then

$$\mathbb{C}[S] = \left\{ \sum_{m \in S} a_m \chi^m \mid a_m \in \mathbb{C},\ a_m = 0 \text{ for all but finitely many } m \in S \right\}$$

with $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$ for any $m, m' \in S$. If S is generated by $A = \{m_1, \dots, m_s\}$, then

$$\mathbb{C}[S] = \mathbb{C}[\chi^{m_1}, \dots, \chi^{m_s}].$$

16.9 Proposition. Let S be an affine monoid embedded in the lattice M.

1. The monoid algebra $\mathbb{C}[S]$ is an integral domain and is of finite type over $\mathbb{C}$.
2. The affine variety V corresponding to $\mathbb{C}[S]$ is an affine toric variety whose torus has character group $\mathbb{Z}S$, and if S is generated by some finite set $A \subseteq M$, then $V \cong Y_A$.

Proof. Say S is generated by $A = \{m_1, \dots, m_s\} \subseteq M$. Then $\mathbb{C}[S] = \mathbb{C}[\chi^{m_1}, \dots, \chi^{m_s}]$, so $\mathbb{C}[S]$ is finitely generated. Since $\mathbb{C}[S] \subseteq \mathbb{C}[M]$ and $\mathbb{C}[M] \cong \mathbb{C}[\mathbb{Z}^n] \cong \mathbb{C}[t_1^\pm, \dots, t_n^\pm]$ is an integral domain, so is the monoid algebra $\mathbb{C}[S]$.

There is a $\mathbb{C}$-algebra homomorphism

$$\begin{aligned} \pi : \mathbb{C}[x_1, \dots, x_s] &\longrightarrow \mathbb{C}[M] \\ x_i &\longmapsto \chi^{m_i} \end{aligned}$$

with image $\mathbb{C}[S]$. If M is the character group of the torus T_N, then π corresponds to the morphism $\Phi_A : T_N \to \mathbb{C}^s$ constructed in 16.4. The image of π corresponds to the closure of the image of Φ in $\mathbb{C}^s$, so the coordinate ring of Y_A is $\operatorname{Im} \pi = \mathbb{C}[S]$. The assertion on the character group is clear. □

16.10 Let T_N be a torus with character group M. Then $\mathbb{C}[M]$ is the coordinate ring of T_N. We let T_N acts on $\mathbb{C}[M]$ by left translation : for $t \in T_N$ and $f : T_N \to \mathbb{C}$ in $\mathbb{C}[M]$, define $t.f \in \mathbb{C}[M]$ to be

$$t.f(p) = f(t^{-1}.p)$$

for any $p \in T_N$. We have the following result.

Lemma. Let $W \subseteq \mathbb{C}[M]$ be a subspace stable under the action of T_N. Then

$$W = \bigoplus_{\substack{m \in M \\ \chi^m \in W}} \mathbb{C}\chi^m$$

Proof. The right hand side is of course contained in W, so it suffices to prove the opposite inclusion. For $0 \neq f \in W \subseteq \mathbb{C}[M]$, write $f = \sum_{m \in M} a_m \chi^m$. Let $B = \operatorname{span}_{\mathbb{C}}\{\chi^m \mid m \in M \text{ with } a_m \neq 0\}$, which is a finite dimensional subspace of $\mathbb{C}[M]$. Since $t.\chi^m = \chi^m(t^{-1})\chi^m$, it follows that B is invariant under T_N-action, and hence so is $B \cap W$. By (16.3).(iii), $B \cap W$ is spanned by simultaneous eigenvectors of T_N. An easy computation shows that any simultaneous vector is a character of T_N, so $B \cap W$ is spanned by characters. It follows that $\chi^m \in W$ for those m with $a_m \neq 0$, and hence f lies in the right hand side. □

16.11 Theorem. For an affine variety V, TFAE :

(i) V is an affine toric variety.

(ii) $V = Y_A$ for a finite set A in some lattice (of a finite dimensional vector space).

(iii) $I(V)$ is a toric ideal.

(iv) The coordinate ring of V is $\mathbb{C}[S]$ for some affine monoid S.

Proof. (ii)⇔(iii) follows from (16.5) and (the proof of) Proposition (16.7), and (ii)⇔(iv)⇒(i) follows from (16.9.2). It remains to show (i)⇒(iv). Let V be an affine toric variety containing the torus T_N with character group M. The inclusion $T_N \subseteq V$ induces a homomorphism $\mathbb{C}[V] \to \mathbb{C}[M]$, which is injective for T_N is Zariski dense in V. Thus we can think of $\mathbb{C}[V]$ as a subalgebra of $\mathbb{C}[M]$. The T_N-action on V being given by a morphism $T_N \times V \to V$, it follows that $\mathbb{C}[V]$ is a T_N-invariant subspace of $\mathbb{C}[V]$. By Lemma 16.10

$$\mathbb{C}[V] = \bigoplus_{\substack{m \in M \\ \chi^m \in \mathbb{C}[V]}} \mathbb{C}\chi^m$$

so $\mathbb{C}[V] = \mathbb{C}[S]$ for the monoid $S = \{m \in M \mid \chi^m \in \mathbb{C}[V]\}$. Finally, since $\mathbb{C}[V]$ is finitely generated, it follows that S is a finitely generated monoid, and hence an affine monoid. □

16.12 Affine toric varieties from rational convex polyhedral cones. Let N be a free $\mathbb{Z}$-module of rank n. Let $\sigma \subseteq N_{\mathbb{R}}$ be an N-rational convex polyhedral cone and set $S_\sigma = \sigma^\vee \cap M$, where M is the lattice in $(N_{\mathbb{R}})^\vee$ dual to N with respect to the evaluation pairing $(N_{\mathbb{R}})^\vee \times N_{\mathbb{R}} \to \mathbb{R}$. By Gordan's lemma, the set S_σ is an affine monoid so by (16.11) it gives an affine toric variety, which we denote by U_σ.

16.13 Theorem. Retain the notation in (16.12). TFAE :

(i) $\dim U_\sigma = n$.

(ii) The torus of U_σ is $T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^\times$.

(iii) σ is strongly convex.

Proof. We see in (16.9) that the character lattice of the torus of U_σ is $\mathbb{Z}S_\sigma \subseteq M$. To proceed, we first show the quotient group $M/\mathbb{Z}S_\sigma$ is torsion-free. Let $m \in M$ and suppose $km \in \mathbb{Z}S_\sigma$ for some $k \in \mathbb{Z}_{\geqslant 1}$. Then $km = m_1 - m_2$ for some $m_1, m_2 \in S_\sigma$. Since $\sigma^\vee$ is convex,

$$m + m_2 = \frac{1}{k} m_1 + \frac{k-1}{k} m_2 \in \sigma^\vee$$

and thus $m = (m + m_2) - m_2 \in \mathbb{Z}S_\sigma$.

Since $M/\mathbb{Z}S_\sigma$ is finitely generated and torsion-free, it follows that the torus of U_σ is T_N if and only if $\mathbb{Z}S_\sigma = M$, if and only if $\operatorname{rank} \mathbb{Z}S_\sigma = n$, or $\dim U_\sigma = n$, proving (i)⇔(ii). Since σ is strongly convex if and only if $\dim \sigma^\vee = n$ by Lemma 15.11.(d), it remains to show $\operatorname{rank} \mathbb{Z}S_\sigma = n$ if and only if $\dim \sigma^\vee = n$. But $\sigma^\vee$ is rational, we see $\sigma^\vee = \mathit{cone}(\sigma^\vee \cap M) = \mathit{cone}(S_\sigma)$, and consequently

$$\operatorname{rank} \mathbb{Z}S_\sigma = \dim \mathbb{R}S_\sigma = \dim \mathit{cone}(S_\sigma) = \dim \sigma^\vee.$$

□

16.14 Definition. Let A, B be two monoids. A map $\varphi : A \to B$ is called a **monoid homomorphism** if $\varphi(1) = 1$ and $\varphi(xy) = \varphi(x)\varphi(y)$ for any $x, y \in A$.

16.15 Points of affine toric varieties. Let S be an affine monoid and let V be the affine toric variety with coordinate ring $\mathbb{C}[S]$ (16.9). By Nullstellensatz, there is a bijection $V \cong \operatorname{mSpec} \mathbb{C}[S]$. In the case of an affine toric variety V, there is a more useful bijection

$$\begin{aligned} V &\longrightarrow \operatorname{Hom}_{\mathbf{Monoid}}(S, \mathbb{C}) \\ p &\longmapsto \gamma_p : m \mapsto \chi^m(p) \end{aligned}$$

We describe its inverse as follows. Let $\gamma \in \operatorname{Hom}_{\mathbf{Monoid}}(S, \mathbb{C})$. It extends to a nontrivial $\mathbb{C}$-algebra homomorphism $\gamma : \mathbb{C}[S] \to \mathbb{C}$, so $\ker \gamma \in \operatorname{mSpec} \mathbb{C}[S]$, which corresponds to a point $p_\gamma \in V$ in turn.

We can write down p_γ more explicitly in terms of the embedding $V = Y_A \subseteq \mathbb{A}^s_{\mathbb{C}}$, where $A = \{m_1, \dots, m_s\}$ is a finite generating set of S. Put $q = (\gamma(m_1), \dots, \gamma(m_s)) \in \mathbb{C}^s$. We claim $q = p_\gamma$. First we must show $q \in Y_A$, and by (16.5) it amounts to showing that all the binomials $X^\alpha - X^\beta$ with $\alpha, \beta \in (\mathbb{Z}_{\geqslant 0})^s$ and $\sum_{i=1}^s \alpha_i m_i = \sum_{i=1}^s \beta_i m_i$ vanishes at q. This is clear, as

$$\prod_{i=1}^s \gamma(m_i)^{\alpha_i} = \gamma\left(\sum_{i=1}^s \alpha_i m_i\right) = \gamma\left(\sum_{i=1}^s \beta_i m_i\right) = \prod_{i=1}^s \gamma(m_i)^{\beta_i}$$

Next, we prove $q = p_\gamma$ by showing $\ker \gamma = \ker \gamma_q$. Let $f \in \mathbb{C}[S] = \mathbb{C}[\chi^{m_1}, \dots, \chi^{m_s}]$. Write $f = \sum_{\alpha \in (\mathbb{Z}_{\geqslant 0})^s} c_\alpha X^\alpha$; then

$$\gamma(f) = \sum_{\alpha \in (\mathbb{Z}_{\geqslant 0})^s} c_\alpha \gamma(X^\alpha) = \sum_{\alpha \in (\mathbb{Z}_{\geqslant 0})^s} c_\alpha \prod_{i=1}^s \gamma(m_i)^{\alpha_i} = f(q)$$

This implies our contention that $\ker \gamma = \ker \gamma_q$. In fact, it tells more; for any $m \in S$, one has

$$\gamma(m) = \chi^m(p_\gamma),$$

and from this we easily see that $\gamma \mapsto p_\gamma$ is inverse to $p \mapsto \gamma_p$.

Suppose T is the torus of V with character group M. In (16.2) we see there is a bijection

$$\begin{aligned} T &\longrightarrow \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{C}^\times) \\ t &\longmapsto [m \mapsto m(t) = \chi^m(t)] \end{aligned}$$

Observe that $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^\times) = \mathrm{Hom}_{\mathbf{Monoid}}(M, \mathbb{C})$, so this isomorphism is compatible with the bijection above, i.e., it fits into the commutative diagram

$$\begin{tikzcd} V \arrow[r] & \mathrm{Hom}_{\mathbf{Monoid}}(S, \mathbb{C}) \\ T \arrow[u, "\text{inclusion}"] \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^\times) \arrow[u, "\text{restriction}"'] \end{tikzcd}$$

16.16 Let T_N be a torus with character group M. If $A = \{m_1, \dots, m_s\}$ is a finite set in M, then we have an affine toric variety $V = Y_A \subseteq \mathbb{A}^s_{\mathbb{C}}$. If $p = (p_1, \dots, p_s) \in Y_A \subseteq \mathbb{A}^s_{\mathbb{C}}$, from (16.4) we see the action of $t \in T_N$ on p is

$$t.p = (m_1(t), \dots, m_s(t))(p_1, \dots, p_s) = (m_1(t)p_1, \dots, m_s(t)p_s).$$

We can also write this action in terms of monoid homomorphism (16.15). The action maps

$$\begin{tikzcd} T_N \times V \arrow[r] & V \\ T_N \times T_N \arrow[u] \arrow[r] & T_N \arrow[u] \end{tikzcd}$$

corresponds to the diagram (here $S = \mathbb{Z}_{\geqslant 0}A \subseteq M$)

$$\begin{tikzcd} \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^\times) \times \mathrm{Hom}_{\mathbf{Monoid}}(S_\sigma, \mathbb{C}) \arrow[r] & \mathrm{Hom}_{\mathbf{Monoid}}(S, \mathbb{C}) \\ \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^\times) \times \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^\times) \arrow[u] \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^\times) \arrow[u] \end{tikzcd}$$

where the horizontal maps are given by multiplication on $\mathbb{C}$.

16.17 Proposition. Let V be an affine toric variety with coordinate ring $\mathbb{C}[S]$ for some affine monoid.

(a) The torus action on V has a fixed point if and only if S is **pointed**, i.e., $S \cap (-S) = \{0\}$, in which case the unique fixed point is given by the monoid homomorphism $\gamma_0 : S \to \mathbb{C}$ defined by

$$m \mapsto \begin{cases} 1 & \text{, if } m = 0 \\ 0 & \text{, if } m \neq 0 \end{cases}$$

(b) If $V = Y_A \subseteq \mathbb{A}^s_{\mathbb{C}}$ for some $A \subseteq S \backslash \{0\}$, then the torus action has a fixed point if and only if $0 \in Y_A$, in which case the unique fixed point is 0.

Proof. Let $p \in V$ be a point and let $\gamma : S \to \mathbb{C}$ be the corresponding monoid homomorphism. Then by (16.16), p is a fixed point if and only if $\gamma(m) = \chi^m(t)\gamma(m)$ for every $t \in T_N$ and $m \in S$. This equation is satisfied for $m = 0$, for $\gamma(0) = 1$. If $m \neq 0$, we can find $t \in T_N$ such that $\chi^m(t) \neq 1$ so that $\gamma(m) = 0$. Thus if a fixed point exists, it is unique and is given by the homomorphism γ_0 described in the proposition. To finish the proof of (a), it suffices to note that γ_0 is a monoid homomorphism if and only if S is pointed. Indeed, if γ_0 is a homomorphism, then for $m, n \in S$ such that $m = -n \in S \cap (-S)$, we have $m + n = 0$ so that $\gamma(m)\gamma(n) = 1$, and thus $\gamma(m) \neq 0 \neq \gamma(n)$. By definition we have $m = n = 0$. Conversely, if $S \cap (-S) \neq \{0\}$, say $m = -n \in (S \cap (-S))\backslash\{0\}$, then $\gamma(m + n) = 1 \neq 0 = \gamma(m)\gamma(n)$.

For (b), 0 is clearly a fixed point under the $(\mathbb{C}^\times)^s$-action, so if $0 \in Y_A$, 0 is a fixed point under the T_N-action. Conversely, if the torus action has a fixed point, by (a) the fixed point is given by γ_0, and from the proof of (16.16) we see γ_0 corresponds to the origin 0 of $\mathbb{A}^s_{\mathbb{C}}$. Thus $0 \in Y_A$; indeed, the corresponding point is $(\gamma_0(m_1), \dots, \gamma_0(m_s))$ if we write $A = \{m_i\}_{i=1}^s$, and all components vanish since $0 \notin A$. $\square$

16.18 Corollary. Let U_σ be the affine toric variety of a strongly convex rational polyhedral cone $\sigma \subseteq N_\mathbb{R}$. Then the torus action on U_σ has a fixed point if and only if $\dim \sigma = \dim_\mathbb{R} N_\mathbb{R}$, in which case the fixed point is unique and is given by the maximal ideal

$$\langle \chi^m \mid m \in S_\sigma \backslash \{0\} \rangle \subseteq \mathbb{C}[S_\sigma]$$

where as usual $S_\sigma = \sigma^\vee \cap M$.

Proof. By Proposition 16.17, The torus action on U_σ has a fixed point if and only if S_σ is pointed. But

$$S_\sigma \cap (-S_\sigma) = \sigma^\vee \cap (-\sigma)^\vee \cap M$$

and since $\sigma^\vee \cap (-\sigma)^\cap$ is rational, we see S_σ is pointed if and only if $\sigma^\vee \cap (-\sigma)^\vee = \{0\}$ (if $\sigma^\vee \cap (-\sigma)^\vee \neq \{0\}$, it would contain a rational ray). By Lemma 15.11, this is equivalent of saying that $\dim \sigma = \dim_\mathbb{R} N_\mathbb{R}$. Finally, it is straightforward to verify that maximal ideal is the kernel of $\gamma_0 : \mathbb{C}[S] \to \mathbb{C}$, where γ_0 is as in Proposition 16.17.(a). □

16.19 Theorem. Let V be an affine toric variety with torus T_N. TFAE :

(i) V is normal.

(ii) $V = \operatorname{mSpec} \mathbb{C}[S]$ for some saturated affine monoid S.

(iii) $V = \operatorname{mSpec} \mathbb{C}[S_\sigma]$, where $S_\sigma = \sigma^\vee \cap M$ and $\sigma \subseteq N_\mathbb{R}$ is a strongly convex rational polyhedral cone.

16.20 Definition. Let V be a finite dimensional real vector space with a lattice N and let σ be a strongly convex N-rational polyhedral cone.

1. σ is called **smooth/regular** if its ray generators of the edges forms a subset of some $\mathbb{Z}$-basis of N.
2. σ is called **simplicial** if its ray generators of the edges are linearly independent over $\mathbb{R}$.

16.21 Let M be the dual lattice of N in $V^\vee$. Let σ be a strongly convex N-rational polyhedral cone of maximal dimension. Then $\sigma^\vee$ is also strongly convex.

Lemma. The cone σ is smooth if and only if its dual $\sigma^\vee$ is smooth.

Proof. In the proof of Corollary 15.11.1, we saw that the ray generators of σ is exactly the minimal generating set of the cone σ; denote by $\{e_1, \dots, e_n\}$ the minimal generating set. If σ is smooth, then $\{e_1, \dots, e_n\}$ is a $\mathbb{Z}$-basis of N. We have $\sigma^\vee = \operatorname{cone}\{e_1^*, \dots, e_n^*\}$, where the e_j^* are the dual basis of the e_i, and $\{e_j^*\}$ is a $\mathbb{Z}$-basis of M. This must be a minimal generating set of $\sigma^\vee$, so $\{e_j^*\}$ consists of ray generators of $\sigma^\vee$. This proves $\sigma^\vee$ is smooth. The other implication follows from $\sigma = (\sigma^\vee)^\vee$. □

16.22 Smooth affine toric varieties. Let the notation be as in (16.12). Then the variety U_σ is smooth if and only if σ is smooth. Furthermore, all smooth affine toric varieties are of this form.

16.23 Faces and affine open subsets. Let N be a free $\mathbb{Z}$-module of rank n and M be the lattice in $(N_\mathbb{R})^\vee$ dual to N with respect to the evaluation pairing. Let $\sigma \subseteq N_\mathbb{R}$ be a strongly convex rational polyhedral cone. Let τ be a face of σ and write $\tau = \sigma \cap m^\perp$ for some $m \in M \cap \sigma^\vee$. Since $\tau \subseteq \sigma$, we have $S_\sigma \subseteq S_\tau = \tau^\vee \cap M$. In fact, we have

$$S_\tau = S_\sigma + \mathbb{Z}_{\geqslant 0}(-m)$$

To see this, since $\langle m, u \rangle = 0$ for any $u \in \tau$, we have $\pm m \in \tau^\vee$, and thus the containment $\supseteq$. For the opposite inclusion, take a finite set $S \subseteq N$ with $\operatorname{cone}(S) = \sigma$ and pick $m' \in S_\tau$. If we set

$$C = \max_{u \in S} |\langle m', u \rangle| \in \mathbb{Z}_{\geqslant 0},$$

then $m' + Cm \in S_\sigma$. Indeed, if $u \in \tau$, then $\langle m', u\rangle \geqslant 0 \leqslant \langle m, u\rangle$, and if $u \in \sigma\backslash\tau$, then $\langle m, u\rangle \geqslant 1$, so $\langle m', u\rangle + \langle Cm, u\rangle \geqslant 0$. This finishes the proof.

From this equality, we obtain $\mathbb{C}[S_\tau] = \mathbb{C}[S_\sigma, \chi^{\pm m}] = \mathbb{C}[S_\sigma]_{\chi^m}$. This implies we may view U_τ as the affine open set $D(\chi^m)$ of U_σ.

16.2 Projective toric varieties

16.24 Affine pieces of projective toric varieties.

16.25 Projective toric variety from a very ample polytope. Let T_N be a torus with character group M and cocharacter group N, and let $P \subseteq M_\mathbb{R}$ be a full dimensional very ample M-rational convex polytope. Put $n = \dim P$. If we write $P \cap M = \{m_1, \dots, m_s\}$, then we have a projective toric variety $X_{P\cap M}$, which is by definition the Zariski closure of the image of the map $T_N \to \mathbb{P}^{s-1}$ given by

$$t \longmapsto [m_1(t) : \cdots : m_s(t)] \in \mathbb{P}^{s-1}.$$

For any $m_i \in P \cap M$, let $S_i = \mathbb{Z}_{\geqslant 0}(P \cap M - m_i)$ be the monoid generated by $\{m - m_i \mid m \in P \cap M\}$. Fix a homogeneous coordinates $x_1, \dots, x_s$ of $\mathbb{P}^{s-1}$, and let $U_i = \mathbb{P}^{s-1}\backslash V(x_i) \cong \mathbb{C}^{s-1}$. In (16.24), we saw that intersection $X_{P\cap M} \cap U_i$ is an affine subvariety of U_i, with coordinate ring

$$\mathbb{C}[X_{P\cap M} \cap U_i] \cong \mathbb{C}[S_i]$$

and

$$X_{P\cap M} = \bigcup_{m_i:\,\text{vertex of } P} X_{P\cap M} \cap U_i$$

For such a projective toric variety from a very ample polytope, we have the following results.

(i) For each vertex m_i of P, the affine piece $X_{P\cap M} \cap U_i$ is the affine toric variety with coordinate ring $\mathbb{C}[\sigma_i^\vee \cap M]$, where

$$\sigma_i = \mathrm{cone}(P \cap M - m_i)^\vee \subseteq N_\mathbb{R}$$

is a strongly convex rational polyhedral cone with dimension $n = \dim P$.

(ii) The torus of $X_{P\cap M}$ has character group M, and hence is T_N.

Proof.

(i) Since m_i is a vertex, $\{0\}$ is a face of $\mathrm{cone}(P \cap M - m_i)$, so $\mathrm{cone}(P \cap M - m_i)$ is strongly convex by Lemma 15.11.(a). Clearly, $\mathrm{cone}(P \cap M - m_i)$ has the same dimension as P, so $\sigma_i = \mathrm{cone}(P \cap M - m_i)^\vee$ is strongly convex by Lemma 15.11.(d). Thus both $\mathrm{cone}(P \cap M - m_i)$ and σ_i are strongly convex rational polyhedral cones of dimension n.

We have $S_i \subseteq \mathrm{cone}(P \cap M - m_i) \cap M = \sigma_i^\vee \cap M$. Since P is very ample, by definition S_i is saturated (in M). In the proof of Theorem 16.19, we see this implies $S_i = \sigma_i^\vee \cap M$, so $\mathbb{C}[S_i] = \mathbb{C}[\sigma_i^\vee \cap M]$

(ii) From Theorem 16.12 (applicable as σ_i is strongly convex) we see the torus of $U_{\sigma_i} = X_{P\cap M} \cap U_i$ is T_N. Since $T_N \subseteq U_{\sigma_i} \subseteq X_{P\cap M}$, it follows that T_N is also the torus of $X_{P\cap M}$.

□

16.26 Intersection of affine pieces of $X_{P\cap M}$. Let T_N, N, M, P be as in (16.25), and put $s = \#(P \cap M)$. We can construct the normal fan Σ_P of the polytope P as in Theorem 15.24. If $X_{P\cap M} \cap U_v$ is the affine piece of $X_{P\cap M}$ corresponding to a vertex v of P, then (16.25) shows that $X_{P\cap M} \cap U_v$ is the affine toric variety of the cone σ_v in the normal fan Σ_P.

Next we study the intersection of affine pieces. Suppose $v \neq w$ are two vertices of P. We want to describe the coordinate ring of $X_{P\cap M} \cap U_v \cap U_w$. To this end, let Q be the smallest face in P containing v and w. The result is that

$$X_{P\cap M} \cap U_v \cap U_w = U_{\sigma_Q}$$

where $\sigma_Q \in \Sigma_P$, i.e., the coordinate ring of the intersection of $\mathbb{C}[\sigma_Q^\vee \cap M]$.

Proof. From (16.24) we see the coordinate ring of $X_{P\cap M} \cap U_v \cap U_w$ is $\mathbb{C}[\sigma_v^\vee \cap M]_{\chi^{w-v}} = \mathbb{C}[\sigma_w^\vee \cap M]_{\chi^{v-w}}$, so we only need to show

$$\mathbb{C}[\sigma_v^\vee \cap M]_{\chi^{w-v}} = \mathbb{C}[\sigma_Q^\vee \cap M]$$

But $w - v \in \mathrm{cone}(P \cap M - v) = \sigma_v^\vee$, from (16.23) we see

$$\mathbb{C}[\sigma_v^\vee \cap M]_{\chi^{w-v}} = \mathbb{C}[U_\tau] = \mathbb{C}[\tau^\vee \cap M]$$

where $\tau = (w-v)^\perp \cap \sigma_v$ is a face of σ_v. Thus we only need to show $\tau = \sigma_Q$, or

$$(w-v)^\perp \cap \sigma_v = \sigma_Q = \sigma_v \cap \sigma_w$$

where the last equality is by Lemma 15.24.(c). Let $u \in (w-v)^\perp \cap \sigma_v$. If $u \neq 0$, there exists $b \in \mathbb{R}$ such that $H_{u,b}$ is a supporting hyperplane of P, so by Lemma 15.25, $u \in \sigma_v$ implies $v \in H_{u,b} \cap P$. As $u \in (w-v)^\perp$, we obtain $w \in H_{u,b}$, and using Lemma 15.25 again we see $u \in \sigma_w$. Conversely, let $u \in \sigma_v \cap \sigma_w$. If $u \neq 0$, pick b as above; then by Lemma 15.25, $u \in \sigma_v \cap \sigma_w$ implies $v, w \in H_{u,b} \cap P$, which implies $u \in (w-v)^\perp$ in turn. □

In this proof, we see

$$\mathbb{C}[U_v]_{\chi^{w-v}} = \mathbb{C}[U_\tau] = \mathbb{C}[U_w]_{\chi^{v-w}}$$

for any vertices $v \neq w$. Thus the toric variety $X_{P\cap M}$ can be glued from the affine toric varieties given by $\sigma_v \in \Sigma_P$ along those given by $\sigma_\tau \in \Sigma_P$. Concisely, the normal fan Σ_P completely determines the toric variety $X_{P\cap M}$.

16.27 Projective toric variety from a polytope. Let N and M be as in 16.12, and let $P \subseteq M_\mathbb{R}$ be a full dimensional lattice polytope. By Corollary 15.23.1, for $k \gg 0$, the multiple kP of P becomes very ample, so from 16.25 we can construct the projective variety $X_{kP\cap M}$. In (16.26) we see $X_{kP\cap M}$ is completely determined by the normal fan Σ_{kP}. By Lemma 15.28, $\Sigma_{kP} = \Sigma_P$ for any $k \geqslant 1$, so the resulting toric variety $X_{kP\cap M}$ are all isomorphic as $\mathbb{C}$-varieties. This suggests us to put

$$X_P := X_{kP\cap M}$$

for any $k \gg 0$ such that kP is very ample. This is the toric variety determined by the polytope P.

16.28 Example; toric variety from the standard simplex Δ_n. By definition, $\Delta_n = \mathrm{cone}\{0, e_1, \dots, e_n\} \subseteq \mathbb{R}^n$. Clearly, $k\Delta_n$ is a normal, and hence very ample by Lemma 15.23, lattice polytope for any $k \in \mathbb{Z}_{\geqslant 1}$. We then can construct X_{Δ_n} using $k\Delta_n$ for any integer $k \geqslant 1$. The lattice points of $k\Delta_n$ are all the integer points $(x_1, \dots, x_n) \in (\mathbb{Z}_{\geqslant 0})^n$ with $x_1 + \cdots + x_n \leqslant k$, which corresponds to the monomials in $\mathbb{C}[x_1, \dots, x_n]$ with total degree $\leqslant k$. These monomials are in number $s_k = \binom{n+k}{k}$, so each kP gives a projective embedding $X_{\Delta_n} \subseteq \mathbb{P}^{s_k - 1}$. When $k = 1$, $\Delta_n \cap \mathbb{Z}^n = \{0, e_1, \dots, e_n\}$, which implies

$$X_{\Delta_n} = \mathbb{P}^n.$$

From the definition of $X_{k\Delta_n \cap \mathbb{Z}^n}$, for any integer $k \geqslant 1$, the embedding $X_{\Delta_n} \subseteq \mathbb{P}^{s_k - 1}$ is given by the morphism

$$\nu_k : \mathbb{P}^n \longrightarrow \mathbb{P}^{s_k - 1}$$

defined using all the monomial of total degree $k \leqslant 1$ in $\mathbb{C}[x_0, x_1, \ldots, x_n]$. The morphism ν_k is really an embedding, usually called the **Veronese embedding**.

16.29 Example; rational normal scrolls. For $a, b \in \mathbb{Z}$ with $1 \leqslant a \leqslant b$, define the polygon

$$P_{a,b} = \operatorname{conv}\{0, ae_1, e_2, be_1 + e_2\}.$$

The toric variety of $P_{a,b}$ is denoted by $S_{a,b}$, and is called the **rational normal scroll**. By Corollary 15.23.1, $P_{a,b}$ is very ample, so $S_{a,b} = X_{P_{a,b} \cap \mathbb{Z}^2}$ is the closure of the morphism

$$\begin{aligned} (\mathbb{C}^\times)^2 &\longrightarrow \mathbb{P}^{a+b+1} \\ (s,t) &\longmapsto [1 : s : \cdots : s^a : t : st : \cdots : s^b t] \end{aligned}$$

To describe the image, we write this morphism as

$$\begin{aligned} \mathbb{C} \times \mathbb{P}^1 &\longrightarrow \mathbb{P}^{a+b+1} \\ (s, [\lambda : \mu]) &\longmapsto [\lambda : \lambda s : \cdots : \lambda s^a : \mu : s\mu : \cdots : s^b \mu] \end{aligned}$$

When $[\lambda : \mu] = [1 : 0]$, it maps to a rational normal curve C_a, and when $[\lambda : \mu] = [0 : 1]$, it maps to another rational normal curve C_b. One thinks C_a and C_b as edges of $S_{a,b}$. By fixing an $s \in \mathbb{C}$ we obtain points on C_a and C_b respectively, and varying $[\lambda : \mu] \in \mathbb{P}^1$ gives a segment connecting the two points.

Note that the normal fan of $P_{a,b}$ only depends on the difference $b - a$. Thus for any integer $r \geqslant 1$, we have

$$X_{P_{1,r}} = X_{P_{2,r+1}} = X_{P_{3,r+2}} = \cdots$$

as $\mathbb{C}$-varieties. If we consider $S_{a,b}$ as a projective variety embedded in $\mathbb{P}^{a+b+1}$, the roles of integers $a \leqslant b$ can be seen from the defining equations of $S_{a,b}$: if we let $x_0, \ldots, x_a, y_0, \ldots, y_b$ be the homogeneous coordinates of $\mathbb{P}^{a+b+1}$, then $I(S_{a,b})$ is generated by the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{a-1} & y_0 & y_1 & \cdots & y_{b-1} \\ x_1 & x_2 & \cdots & x_a & y_1 & y_2 & \cdots & y_b \end{pmatrix}$$

16.30 Normality. Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional lattice polytope. Then

(i) X_P is normal.

(ii) X_P is projectively normal under the embedding given by kP if and only if kP is normal.

Proof.

1. This follows from Theorem 16.19.(ii) and the definition of very ample.
2. The affine cone of $X_{kP \cap M}$ is $Y_{((kP) \cap M) \times \{1\}}$, and by Theorem 16.19, $Y_{kP \cap M \times \{1\}}$ is normal if and only if the monoid $S := \mathbb{Z}_{\geqslant 0}(((kP) \cap M) \times \{1\})$ is saturated in $M \times \mathbb{Z}$. But $((kP) \cap M) \times \{1\}$ generates the cone $C(kP)$, so it is equivalent to saying that $((kP) \cap M) \times \{1\}$ generates the monoid $C(kP) \cap (M \times \mathbb{Z})$: if $(m, \ell) \in M \times \mathbb{Z}$ and $(nm, n\ell) \in S$, then $nm \in n\ell((kP) \cap M)$ so that $m \in \ell((kP) \cap M)$, or $(m, \ell) \in S$. Conversely, for $x \in C(kP) \cap (M \times \mathbb{Z})$, let $kP \cap M = \{m_1, \ldots, m_s\}$ and write $x = \left(\sum_{i=1}^s a_i m_i, \sum_{i=1}^s a_i \right)$ for some $a_i \geqslant 0$. Since $x \in M \times \mathbb{Z}$, every a_s is rational, so there exists $r \in \mathbb{Z}_{\geqslant 1}$ such that $ra_s \in \mathbb{Z}_{\geqslant 0}$. But this implies $rx \in S$, and thus $x \in S$ as S is saturated. Finally, by Lemma 15.21, the last equivalent statement is the same as saying kP is normal.

$\square$

16.31 Definition. Let $P \subseteq M_{\mathbb{R}}$ be a lattice polytope.

1. Given a vertex v of P and an edge E containing v, denote by w_E the unique lattice point on E such that $w_E - v$ is the ray generator of the ray $\mathit{cone}(E - v)$.
2. P is called **smooth/unimodular** if for every vertex v, the vectors $w_E - v$, where E runs over all edges containing v, form a subset of some $\mathbb{Z}$-basis of M. In particular, if $\dim P = \dim M_{\mathbb{R}}$, then $\{w_E - v\}_E$ forms a $\mathbb{Z}$-basis of M.

16.32 Smoothness. Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional lattice polytope. TFAE :

(i) X_P is a smooth projective variety.

(ii) Σ_P is a smooth fan, in the sense that every cone in Σ_P is smooth (16.20).

(iii) P is a smooth polytope.

Proof. Being smooth is a local property, so X_P is smooth if and only if for any vertex v of P, the affine toric variety U_{σ_v} is smooth. By (16.22), this is true if and only if for any vertex v of P, the cone $\sigma_v \in \Sigma_P$ is smooth. Clearly faces of a smooth cone are again smooth, so by Lemma 15.26.(a) this is the same as saying that every cone in Σ_P is smooth. Thus (i)$\Leftrightarrow$(ii).

For (ii)$\Leftrightarrow$(iii), since σ_v is of maximal dimension, in (16.21) we see σ_v is smooth if and only if $C_v = \sigma_v^\vee$ is smooth. Recall that $C_v = \mathit{cone}(P \cap M - v)$, so the ray generators of C_v are exactly $w_E - v$, where E runs over all edges of P containing v. Thus P is smooth if and only if C_v, and hence σ_v, is smooth for every vertex v. $\square$

16.33 Lemma. Every smooth full dimensional lattice polytope $P \subseteq M_{\mathbb{R}}$ is very ample.

Proof. This follows from the fact

$$\mathbb{Z}_{\geqslant 0}\{w_E - v\}_{v \in E} = \mathbb{Z}_{\geqslant 0}(P \cap M - v)$$

for any vertex v. $\square$

16.34 Let V_1, V_2 be two finite dimensional real vector spaces and let $P_i \subseteq V_i$ $(i = 1, 2)$ be a convex polytope. Consider the cartesian product $P_1 \times P_2 \subseteq V_1 \times V_2$. If $P_i = \mathit{conv}(S_i)$ for some finite subset $S_i \subseteq V_i$, then

$$P_1 \times P_2 = \mathit{conv}(S_1 \times S_2) \subseteq V_1 \times V_2.$$

In particular, this shows $P_1 \times P_2$ is also a convex polytope.

Now let S_i be the set of vertices of P_i. The above equality shows that the vertex of $P_1 \times P_2$ is contained in $S_1 \times S_2$. Conversely, if $v_i \in S_i$ $(i = 1, 2)$, then $v_i = P_i \cap H_{u_i, b_i}$ for some supporting hyperplane H_{u_i, b_i} of P_i. Then $(v_1, v_2) \in H_{u_1, b_2} \times H_{u_2, b_2} \subseteq H_{(u_1, u_2), b_1 + b_2}$ and $P_1 \times P_2 \subseteq H^+_{u_1, b_2} \times H^+_{u_2, b_2} \subseteq H^+_{(u_1, u_2), b_1 + b_2}$, so $H_{(u_1, u_2), b_1 + b_2}$ is a supporting hyperplane of $P_1 \times P_2$ containing (v_1, v_2). If $(x, y) \in (P_1 \times P_2) \cap H_{(u_1, u_2), b_1 + b_2}$, then $b_1 + b_2 = \langle u_1, x \rangle + \langle u_2, y \rangle$. But $P_i \subseteq H^+_{u_i, b_i}$ $(i = 1, 2)$, so $\langle u_1, x \rangle \geqslant b_1$ and $\langle u_2, y \rangle \geqslant b_2$, which forces these two to be equalities in turn. Thus $(x, y) = (v_1, v_2)$, so that $\{(v_1, v_2)\} = (P_1 \times P_2) \cap H_{(u_1, u_2), b_1 + b_2}$, i.e., (v_1, v_2) is a vertex of $P_1 \times P_2$.

16.35 Let $i = 1, 2$ and N_i be a free $\mathbb{Z}$-module of rank n_i. Let M_i be the dual lattice of N_i in $(N_i)_{\mathbb{R}}$ with respect to the evaluation pairing. Let $P_i \subseteq (M_i)_{\mathbb{R}}$ be a full dimensional lattice polytope. The product $P_1 \times P_2$ is then a full dimensional lattice polytope in $(M_1 \times M_2)_{\mathbb{R}} = (M_1)_{\mathbb{R}} \times (M_2)_{\mathbb{R}}$, so we may consider the normal fan $\Sigma_{P_1 \times P_2}$. A result is that

$$\Sigma_{P_1 \times P_2} = \Sigma_{P_1} \times \Sigma_{P_2}.$$

To see this, by virtue of Lemma 15.26, we only need to show :

(i) For any vertices $v_i \in P_i$ $(i = 1, 2)$, we have $\sigma_{(v_1, v_2)} = \sigma_{v_1} \times \sigma_{v_2}$.

(ii) Suppose $\sigma_i \subseteq (N_i)_{\mathbb{R}}$ are convex polyhedral cones and $\tau_i \subseteq \sigma_i$ $(i = 1, 2)$ are faces. Then $\tau_1 \times \tau_2$ is a face of $\sigma_1 \times \sigma_2$, and all faces of $\sigma_1 \times \sigma_2$ arise in this way.

(ii) is clear, and to show (i), we prove $C_{(v_1, v_2)} = C_{v_1} \times C_{v_2}$. This follows from $\mathit{cone}(S_1) \times \mathit{cone}(S_2) = \mathit{cone}(S_1 \times S_2)$ for finite subsets $S_i \subseteq (M_i)_{\mathbb{R}}$.

16.36 Let $P_i \subseteq (M_i)_{\mathbb{R}}$ be full dimensional lattice polytopes of dimension n_i $(i = 1, 2)$. Put $s_i = \#(P_i \cap M_i)$. Replacing P_i by their multiples, we assume the P_i are very ample, so there are projective embeddings $X_{P_i} \to \mathbb{P}^{s_i - 1}$. We obtain a projective embedding by the composition

$$X_{P_1} \times X_{P_2} \longrightarrow \mathbb{P}^{s_1 - 1} \times \mathbb{P}^{s_2 - 1} \xrightarrow{\text{Segre embedding}} \mathbb{P}^{s_1 s_2 - 1}$$

Theorem.

(i) $P_1 \times P_2 \subseteq (M_1 \times M_2)_{\mathbb{R}}$ is very ample with lattice points

$$(P_1 \times P_2) \cap (M_1 \times M_2) = (P_1 \cap M_1) \times (P_2 \cap M_2),$$

which has cardinality $s = s_1 s_2$.

(ii) The image of the embedding $X_{P_1 \times P_2} \subseteq \mathbb{P}^{s-1}$ determined by $P_1 \times P_2$ equals that of $X_{P_1} \times X_{P_2} \to \mathbb{P}^{s-1}$ above.

(iii) $X_{P_1 \times P_2} \cong X_{P_1} \times X_{P_2}$.

Proof. (i) is clear, and (iii) is an immediate consequence of (ii). For (ii), let T_{N_i} be the torus of X_{P_i}. The torus $T_{N_1} \times T_{N_2}$ is Zariski dense in $X_{P_1} \times X_{P_2}$, so by composing with the Segre embedding, we see $X_{P_1} \times X_{P_2}$ is the closure of the image of $X_{P_1} \times X_{P_2} \to \mathbb{P}^{s-1}$, which is given by the characters $\chi^m \chi^{m'}$, where $(m, m') \in (P_1 \times P_2) \cap (M_1 \times M_2)$. When we identify $T_{N_1} \times T_{N_2}$ with $T_{N_1 \times N_2}$, the characters $\chi^m \chi^{m'}$ becomes $\chi^{(m, m')}$, so the above map coincides with $T_{N_1 \times N_2} \to \mathbb{P}^{s-1}$ coming from the polytope $P_1 \times P_2$. This finishes the proof. □

In view of the theorem, we have

$$X_{P_1} \times X_{P_2} = \left(\bigcup_{v_1} U_{\sigma_{v_1}} \right) \times \left(\bigcup_{v_2} U_{\sigma_{v_2}} \right) = \bigcup_{v_1, v_2} U_{\sigma_{v_1}} \times U_{\sigma_{v_2}} = \bigcup_{(v_1, v_2)} U_{\sigma_{(v_1, v_2)}} = X_{P_1 \times P_2}.$$

In the third equality we use

$$\mathbb{C}[\sigma_1^\vee \cap M_1] \otimes_{\mathbb{C}} \mathbb{C}[\sigma_2^\vee \cap M_2] = \mathbb{C}[(\sigma_1 \times \sigma_2)^\vee \cap (M_1 \times M_2)].$$

and (16.35).(i).

16.3 Fans and toric varieties

16.37 Definition. Let N be a free $\mathbb{Z}$-module of rank n. A **fan** Σ in $N_{\mathbb{R}}$ is a finite collection of strongly convex N-rational polyhedral cones σ in $N_{\mathbb{R}}$ such that

1. for any $\sigma \in \Sigma$, all faces of σ lie in Σ;
2. for any $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of each.

The **support** of a fan Σ is defined as

$$|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma \subseteq N_{\mathbb{R}}.$$

For $r \in \mathbb{Z}_{\geqslant 0}$, let $\Sigma(r)$ be the set of all r-dimensional cones in Σ.

The normal fan defined in Theorem 15.24 is certainly a fan in the above sense.

16.38 Lemma. Let Σ be a fan and $\sigma_1, \sigma_2 \in \Sigma$. If $\tau = \sigma_1 \cap \sigma_2$, then

$$S_\tau = S_{\sigma_1} + S_{\sigma_2}.$$

Proof. First note that

$$\sigma_1^\vee + \sigma_2^\vee = (\sigma_1 \cap \sigma_2)^\vee = \tau^\vee.$$

This implies $S_{\sigma_1} + S_{\sigma_2} \subseteq S_\tau$. For the opposite inclusion, take $p \in S_\tau$ and pick $m \in \sigma_1^\vee \cap (-\sigma_2)^\vee \cap M$ such that $\sigma_1 \cap m^\perp = \tau = \sigma_2 \cap m^\perp$, which exists by Lemma 15.13. Then (16.23) implies that $S_\tau = S_{\sigma_1} + \mathbb{Z}_{\geqslant 0}(-m)$, so $p = q + \ell(-m)$ for some $q \in S_{\sigma_1}$ and $\ell \in \mathbb{Z}_{\geqslant 0}$. But $-m \in \sigma_2^\vee$ implies $-m \in S_{\sigma_2}$, so $p \in S_{\sigma_1} + S_{\sigma_2}$. $\square$

16.39 Toric variety from a fan. Let Σ be a fan in $N_\mathbb{R}$. By (16.12), each cone σ in Σ gives rise to an affine toric variety U_σ with coordinate ring $\mathbb{C}[\sigma^\vee \cap M]$. If $\tau = \sigma \cap m^\perp$ is a face of σ, then in (16.23) we see that $U_\tau = (U_\sigma)_{\chi^m}$ is an affine open subset of U_σ. If σ_1, $\sigma_2 \in \Sigma$ with $\tau = \sigma_1 \cap \sigma_2$, then Lemma 15.13 says that

$$\sigma_1 \cap m^\perp = \tau = \sigma_2 \cap m^\perp$$

for some $m \in \sigma_1^\vee \cap (-\sigma_2)^\vee \cap M$, which implies that

$$\mathbb{C}[\sigma_1^\vee \cap M]_{\chi^m} = \mathbb{C}[\tau^\vee \cap M] = \mathbb{C}[\sigma_2^\vee \cap M]_{\chi^{-m}}.$$

Let $g_{\sigma_1,\sigma_2} : (U_{\sigma_1})_{\chi^m} \to (U_{\sigma_2})_{\chi^{-m}}$ be the resulting isomorphism of affine varieties. We check $\{g_{\sigma_1,\sigma_2}\}_{\sigma_1,\sigma_2 \in \Sigma}$ satisfies the condition in (2.13). In fact, it suffices to check the conditions on topological spaces, for the morphisms on sheaves are completely determined by the maps of topological spaces. At this stage it is clear that both conditions in (2.13) hold. Hence $\{U_\sigma\}_{\sigma \in \Sigma}$ glues to a well-defined variety, denoted by X_Σ. We claim that X_Σ is a normal separated toric variety.

Proof. Since each cone σ in Σ is strongly convex, by (16.13) the torus in each affine toric variety U_σ is $T_N = U_{\{0\}}$ for any $\sigma \in \Sigma$. These T_N in U_σ are identified by the gluing, so that $T_N \subseteq X_\Sigma$ as a dense open subspace. In particular, X_Σ is irreducible. By (2.13.4), the action maps $T_N \times U_\sigma \to U_\sigma$ glue to an algebraic action $T_N \times X_\Sigma \to X_\Sigma$. Indeed, the action map corresponds to the homomorphism $\mathbb{C}[S_\sigma] \to \mathbb{C}[S_\sigma] \otimes \mathbb{C}[M]$, and on $U_{\sigma_1} \cap U_{\sigma_2} = U_\tau$, the two action maps agree (or are compatible with the g_{σ_1,σ_2}), for they are all $\mathbb{C}[S_\tau] \to \mathbb{C}[S_\tau] \otimes \mathbb{C}[M]$. Thus, this proves X_Σ is a toric variety.

Since each σ is strongly convex, each U_σ is normal by Theorem 16.19.(iii), and thus X_Σ is a normal variety. To see X_Σ is separated, we must show the diagonal morphism $X_\Sigma \to X_\Sigma \times X_\Sigma$ is a closed immersion, and in view of Proposition 2.26, it suffices to show that for any cones σ_1, $\sigma_2 \in \Sigma$, the diagonal morphism

$$\Delta : U_\tau \to U_{\sigma_1} \times U_{\sigma_2}$$

(where $\tau = \sigma_1 \cap \sigma_2$) is a closed immersion. On coordinate rings the diagonal morphism reads

$$\Delta^* : \mathbb{C}[S_{\sigma_1}] \times \mathbb{C}[S_{\sigma_2}] \longrightarrow \mathbb{C}[S_\tau]$$

$$\chi^m \otimes \chi^{m'} \longmapsto \chi^{m+m'}.$$

By Lemma 16.38, Δ^* is surjective, showing that Δ is a closed immersion in turn. $\square$

For a full dimensional lattice polytope $P \subseteq M_{\mathbb{R}}$, the description of the projective toric variety X_P in (16.26) shows that X_P is isomorphic to the toric variety X_{Σ_P} associated to the normal fan Σ_P.

16.40 Product of fans. Let N_i $(i = 1, 2)$ be a free $\mathbb{Z}$-module of rank n_i and let Σ_i be a fan in $(N_i)_{\mathbb{R}}$. Then the product

$$\Sigma_1 \times \Sigma_2 = \{\sigma_1 \times \sigma_2 \mid \sigma_i \in \Sigma_i,\ i = 1,\ 2\}$$

is a fan in $(N_1)_{\mathbb{R}} \times (N_2)_{\mathbb{R}} = (N_1 \times N_2)_{\mathbb{R}}$. Indeed, each $\sigma_1 \times \sigma_2$ is strongly convex $(N_1 \times N_2)$-rational polyhedral cone, and (16.35).(ii) implies $\Sigma_1 \times \Sigma_2$ is a fan. Therefore we may construct the toric variety $X_{\Sigma_1 \times \Sigma_2}$. A result is that

$$X_{\Sigma_1 \times \Sigma_2} \cong X_{\Sigma_1} \times X_{\Sigma_2}.$$

16.41 Example; the blowup of $\mathbb{C}^n$ at the origin. Let $N = \mathbb{Z}^n$ and $e_1, \ldots, e_n$ be the standard basis. Set $e_0 = e_1 + \cdots + e_n$. Consider the fan Σ of $N_{\mathbb{R}}$ consisting of the cones generated by all subset of $\{e_0, \ldots, e_n\}$ not containing $\{e_1, \ldots, e_n\}$. Then the toric variety X_Σ is isomorphic to the blowup $\mathrm{Bl}_0(\mathbb{C}^n)$ of $\mathbb{C}^n$ at the origin.

To see this, for $1 \leqslant i \leqslant n$, let $\sigma_i = \mathrm{cone}\{e_0, e_1, \ldots, \widehat{e_i}, \ldots, e_n\} \subseteq N_{\mathbb{R}}$. Its dual in $M_{\mathbb{R}}$ is

$$\sigma_i^\vee = \mathrm{cone}\{e_i, e_1 - e_i, \ldots, e_{i-1} - e_i, e_{i+1} - e_i, \ldots, e_n - e_i\} \subseteq M_{\mathbb{R}}.$$

Note that the generating set of $\sigma_i^\vee$ entirely lies in M, so

$$S_{\sigma_i} = \sigma_i^\vee \cap M = \mathbb{Z}_{\geqslant 0}\{e_i, e_1 - e_i, \ldots, e_{i-1} - e_i, e_{i+1} - e_i, \ldots, e_n - e_i\}$$

and thus

$$U_{\sigma_i} = \mathrm{mSpec}\,\mathbb{C}\left[x_i, \frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right].$$

The separating hyperplane of σ_i and σ_j, where $1 \leqslant i < j \leqslant n$, is the hyperplane with normal vector $e_i - e_j$; note that $e_i - e_j \in \sigma_j^\vee$ and $e_j - e_i \in \sigma_i^\vee$. Thus the gluing data is the isomorphism

$$\mathbb{C}\left[x_i, \frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right]_{\frac{x_j}{x_i}} = \mathbb{C}\left[x_j, \frac{x_1}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_n}{x_j}\right]_{\frac{x_i}{x_j}}$$

By the change of coordinates $x_i \mapsto s_i$ and $\dfrac{x_j}{x_i} \mapsto \dfrac{t_j}{t_i}$, we see

$$U_{\sigma_i} = \mathrm{mSpec}\,\mathbb{C}\left[\frac{t_1}{t_i}, \ldots, \frac{t_{i-1}}{t_i}, \frac{t_{i+1}}{t_i}, \ldots, \frac{t_n}{t_i}, s_i\right]$$

and the gluing isomorphism becomes the map defined by $s_i \mapsto \dfrac{t_i}{t_j} s_j$ and $\dfrac{t_k}{t_i} \mapsto \dfrac{t_k}{t_i} = \dfrac{t_k}{t_j}\dfrac{t_j}{t_i}$. This is the same as the blowup

$$\mathrm{Bl}_0(\mathbb{C}^n) = V(t_i s_j - t_j s_i \mid 1 \leqslant i, j \leqslant n) \subseteq \mathbb{P}^{n-1} \times \mathbb{C}^n$$

at the origin; on the product we use the coordinates $([t_1 : \cdots : t_n], s_1, \ldots, s_n)$.

16.42 Example; Hirzebruch surfaces. Let $r \in \mathbb{Z}_{\geqslant 0}$ and consider the fan consisting of the four cone

$$\sigma_1 = \mathrm{cone}\{(1, 0), (0, 1)\} \qquad \sigma_2 = \mathrm{cone}\{(1, 0), (0, -1)\}$$
$$\sigma_3 = \mathrm{cone}\{(-1, r), (0, -1)\} \qquad \sigma_4 = \mathrm{cone}\{(-1, r), (0, 1)\}$$

together with all of their faces. In (16.29) we consider the rational normal scroll $S_{a,b}$ constructed from the polytope $P_{a,b}$ ($1 \leqslant a \leqslant b$). The normal fan $\Sigma_{P_{a,b}}$ is the fan Σ_{b-a} defines above, so $X_{\Sigma_{b-a}} \cong S_{a,b}$ as $\mathbb{C}$-varieties. Generally, the toric variety X_{Σ_r} is called the **Hirzebruch surface**, and by definition it is covered by the four affine varieties

$$U_{\sigma_1} = \operatorname{mSpec} \mathbb{C}[x,y] \qquad U_{\sigma_2} = \operatorname{mSpec} \mathbb{C}[x,y^{-1}]$$
$$U_{\sigma_3} = \operatorname{mSpec} \mathbb{C}[x^{-1},x^{-r}y^{-1}] \qquad U_{\sigma_4} = \operatorname{mSpec} \mathbb{C}[x^{-1},x^{r}y]$$

each isomorphic to the affine space $\mathbb{A}^2_{\mathbb{C}}$.

16.43 Definition. Let Σ be a fan in $N_{\mathbb{R}}$.

1. Σ is **smooth** if every cone in Σ is smooth.
2. Σ is **simplicial** if every cone in Σ is simplicial (16.20).
3. Σ is **complete** if its support $|\Sigma|$ is the whole $N_{\mathbb{R}}$.

16.44 Theorem. Let X_Σ be the toric variety defined by a fan $\Sigma \subseteq N_{\mathbb{R}}$.

(i) X_Σ is a smooth variety if and only if Σ is smooth.

(ii) X_Σ is a orbifold if and only if Σ is simplicial.

(iii) X_Σ is compact in the classical topology (4.25) if and only if Σ is complete.

Proof. Being smooth is a local property, so X_Σ is smooth if and only if each U_σ is smooth. At this stage (i) simply follows from (16.22). We will not prove (ii) here. The proof of (iii) will be given in the sequel. □

16.3.1 The orbit-cone correspondence

16.45 Retain the notations in (16.12). Consider the map $\gamma_\sigma : S_\sigma \to \mathbb{C}$ defined by

$$m \mapsto \begin{cases} 1 & \text{, if } m \in S_\sigma \cap \sigma^\perp = \sigma^\perp \cap M \\ 0 & \text{, otherwise} \end{cases}$$

This is a monoid homomorphism, as $\sigma^\vee \cap \sigma^\perp = \sigma^*$ is a face of $\sigma^\vee$ (15.10) : if $m + m' \in S_\sigma$ and $m+m' \in S_\sigma \cap \sigma^\perp$, then $m,\ m' \in S_\sigma \cap \sigma^\perp$ by (15.3). The corresponding point (16.15) of γ_σ is again denoted by $\gamma_\sigma \in U_\sigma$. We call this the **distinguished point corresponding to** σ. By (16.18), γ_σ is fixed under the T_N-action if and only if $\dim \sigma = \dim_{\mathbb{R}} N_{\mathbb{R}}$.

If τ is a face of σ, then $\gamma_\tau \in U_\tau \subseteq U_\sigma$. More precisely, there is a commutative diagram

$$\begin{array}{ccc} U_\tau & \xrightarrow{\text{inclusion}} & U_\sigma \\ \downarrow\wr & & \downarrow\wr \\ \operatorname{Hom}_{\mathbf{Monoid}}(\tau^\vee \cap M, \mathbb{C}) & \xrightarrow{\text{restriction}} & \operatorname{Hom}_{\mathbf{Monoid}}(\sigma^\vee \cap M, \mathbb{C}) \end{array}$$

and $\gamma_\tau|_{\sigma^\vee \cap M} \in \operatorname{Hom}_{\mathbf{Monoid}}(\sigma^\vee \cap M, \mathbb{C})$.

16.46 Lemma. Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone and $u \in N$. Then

$$u \in \sigma \quad \Leftrightarrow \quad \lim_{s \to 0} \lambda^u(s) \text{ exists in } U_\sigma.$$

Here the limit is taken in the classical topology of U_σ (4.25). Moreover, $u \in \operatorname{Relint}(\sigma) \Rightarrow \lim_{s\to 0} \lambda^u(s) = \gamma_\sigma$.

Proof. Say $U_\sigma = Y_A$ for some finite set $A = \{m_1, \dots, m_s\} \subseteq S_\sigma \backslash \{0\}$. Since σ is strongly convex, the torus of Y_A is T_N (16.13). If we view T_N as a subgroup of $(\mathbb{C}^\times)^s$ by $t \mapsto (m_1(t), \dots, m_s(t))$, the characters $m_i \in A$ are simply the projections to i-th component.

$$\begin{aligned}
\lim_{s\to 0} \lambda^u(s) \text{ exists in } U_\sigma &\Leftrightarrow \lim_{s\to 0} \chi^m(\lambda^u(s)) \text{ exists in } \mathbb{C} \text{ for any } m \in S_\sigma \\
&\Leftrightarrow \lim_{s\to 0} s^{\langle m,u\rangle} \text{ exists in } \mathbb{C} \text{ for any } m \in S_\sigma \\
&\Leftrightarrow \langle m,u\rangle \geqslant 0 \text{ for any } m \in \sigma^\vee \cap M \\
&\Leftrightarrow u \in (\sigma^\vee)^\vee = \sigma.
\end{aligned}$$

This proves the first part of the lemma. If $u \in \sigma \cap N$, since $m_i = \chi^{m_i}$ is simply the i-th projection as said above, from the explicit expression in (16.15) we see $\lim_{s\to 0} \lambda^u(t)$ is the point corresponding to the monoid homomorphism $S_\sigma \to \mathbb{C}$ defined by

$$\sigma^\vee \cap M \ni m \mapsto \lim_{s\to\infty} s^{\langle m,u\rangle}.$$

If $u \in \mathrm{Relint}(\sigma)$, then $\langle m,u\rangle > 0$ for all $m \in \sigma^\vee \backslash \sigma^\perp$, and $\langle m,u\rangle = 0$ if and only if $m \in \sigma^\vee \cap \sigma^\perp$. Thus this monoid homomorphism is the same as γ_σ defined in (16.45). $\square$

16.47 Let σ be a strongly convex rational polyhedral cone in $N_\mathbb{R}$. Let N_σ be the sublattice of N generated by $\sigma \cap N$, and let $N(\sigma) = N/N_\sigma$. The perfect pairing $\langle\,,\,\rangle : M \times N \to \mathbb{Z}$ induces a nondegenerate pairing

$$\langle\,,\,\rangle : (\sigma^\perp \cap M) \times N(\sigma) \to \mathbb{Z}.$$

In fact, this is a perfect pairing. To see this, from the short exact sequence

$$0 \longrightarrow N_\sigma \longrightarrow N \longrightarrow N(\sigma) \longrightarrow 0$$

we obtain

$$\mathrm{Hom}_\mathbb{Z}(N(\sigma), \mathbb{Z}) = \{\varphi \in \mathrm{Hom}_\mathbb{Z}(N, \mathbb{Z}) \mid \varphi|_{N_\sigma} = 0\}$$

Using the identification $M = \mathrm{Hom}_\mathbb{Z}(N, \mathbb{Z})$, it becomes

$$\mathrm{Hom}_\mathbb{Z}(N(\sigma), \mathbb{Z}) = \{m \in M \mid \langle m, N_\sigma\rangle = 0\} = (N_\sigma)^\perp = \sigma^\perp \cap M$$

16.48 The torus orbits abstractly. Let Σ be a fan in $N_\mathbb{R} \cong \mathbb{R}^n$. Recall that $\tau \preceq \sigma$ means τ is a face of σ. For $\sigma \in \Sigma$, define the torus

$$O(\sigma) = T_{N(\sigma)} := N(\sigma) \otimes_\mathbb{Z} \mathbb{C}^\times.$$

Clearly, $\dim O(\sigma) = n - \dim \sigma$, and T_N acts on $T_{N(\sigma)}$ transitively via the projection $T_N \to T_{N(\sigma)}$.

Define the **star** of a cone $\tau \in \Sigma$ to be

$$\mathrm{Star}(\tau) = \{\overline{\sigma} = \sigma \bmod (N_\tau)_\mathbb{R} \subseteq N(\tau)_\mathbb{R} \mid \tau \preceq \sigma \in \Sigma\}.$$

This is the collection of cones $\sigma \in \Sigma$ that contain τ as a face.

Lemma. $\mathrm{Star}(\tau)$ is a fan in $N(\tau)_\mathbb{R}$

Proof. It is clear that each $\overline{\sigma}$ is a $N(\tau)$-rational convex polyhedral cone. The dual $\overline{\sigma}^v ee = \sigma^\vee \cap \tau^\perp \subseteq \tau^\perp$ has maximal dimension, as $\sigma^\vee \subseteq M_\mathbb{R}$ has, so $\overline{\sigma}$ is strongly convex.

If $\tau \preceq \sigma' \preceq \sigma$, then $\overline{\sigma'}$ is a face of $\overline{\sigma}$. Indeed, if $\sigma' = \sigma \cap u^{\perp}$ for some $u \in \sigma^{\vee}$, then $\tau \subseteq u^{\perp}$ so that $u \in \tau^{\perp} \cap \sigma^{\vee} = \overline{\sigma}^{\vee}$. We claim that

$$\overline{\sigma'} = \overline{\sigma} \cap u^{\perp} \subseteq N(\sigma)_{\mathbb{R}}$$

If $x \in \overline{\sigma'}$, then $x - x' \in (N_{\sigma})_{\mathbb{R}}$ for some $x' \in \sigma'$, and thus

$$0 = \langle x - x', u \rangle = \langle x, u \rangle - \langle x', u \rangle = \langle x, u \rangle.$$

If $x \in \overline{\sigma} \cap u^{\perp}$, write $x - x' \in (N_{\sigma})_{\mathbb{R}}$ for some $x' \in \sigma$, and $0 = \langle x' - x, u \rangle = \langle x', u \rangle$ so that $x' \in \sigma \cap u^{\perp} = \sigma'$.

Now let C be a face of $\overline{\sigma}$. Then $C^{\perp} \cap \overline{\sigma}^{\vee}$ is a face of $\overline{\sigma}^{\vee} = \sigma^{\vee} \cap \tau^{\perp}$, and thus by (15.10) it has the form $\sigma^{\vee} \cap (\sigma')^{\perp}$ for some face $\tau \preceq \sigma' \preceq \sigma$. In sum, $C^{\perp} \cap \overline{\sigma}^{\vee} = \overline{\sigma'}^{\perp} \cap \overline{\sigma}^{\vee}$, and since $\overline{\sigma'} \preceq \overline{\sigma}$, so by (15.10) again we have $C = \overline{\sigma}$. This shows every face of $\overline{\sigma}$ has the form $\overline{\sigma'}$ for some $\tau \preceq \sigma' \preceq \sigma$.

Let $\tau \preceq \sigma, \sigma'$. Then

$$\overline{\sigma \cap \sigma'}^{\vee} = (\sigma \cap \sigma')^{\vee} \cap \tau^{\perp} = \sigma^{\vee} \cap (\sigma')^{\vee} \cap \tau^{\perp} = \overline{\sigma}^{\vee} \cap \overline{\sigma'}^{\vee} \subseteq \tau^{\perp}$$

so by duality we have $\overline{\sigma \cap \sigma'} = \overline{\sigma} \cap \overline{\sigma'}$ in $N(\sigma)_{\mathbb{R}}$. $\square$

Put $V(\tau) = X_{\mathrm{Star}(\tau)}$. Note that the torus of $V(\tau)$ is $T_{N(\tau)} = O(\tau)$. By (16.47), the toric variety $V(\tau)$ has an affine open cover consisting of

$$U_{\tau}(\sigma) := \operatorname{mSpec} \mathbb{C}[\sigma^{\vee} \cap \tau^{\perp} \cap M]$$

where σ runs over all faces in Σ with $\tau \preceq \sigma$. Note here that $\sigma^{\vee} \cap \tau^{\perp}$ is a face of $\sigma^{\vee}$ (15.10), and $U_{\tau}(\tau) = O(\tau)$ is dense in $V(\tau)$.

16.49 Lemma. There exists a closed immersion $V(\tau) \subseteq X_{\Sigma}$.

Proof. We prove this by constructing compatible closed immersion $U_{\tau}(\sigma) \subseteq U_{\sigma}$ for any $\tau \preceq \sigma \in \Sigma$. On coordinate rings there is an obvious projection

$$\mathbb{C}[\sigma^{\vee} \cap M] \longrightarrow \mathbb{C}[\sigma^{\vee} \cap \tau^{\perp} \cap M]$$

defined by $\chi^m \mapsto \begin{cases} \chi^m & \text{, if } m \in \sigma^{\vee} \cap \tau^{\perp} \cap M \\ 0 & \text{, otherwise} \end{cases}$. This gives a closed immersion $U_{\tau}(\sigma) \subseteq U(\sigma)$. There morphisms are compatible, for if $\tau \preceq \sigma \preceq \sigma'$, the diagram

$$\begin{tikzcd} U_{\tau}(\sigma) \arrow[r] \arrow[d] & U_{\tau}(\sigma') \arrow[d] \\ U_{\sigma} \arrow[r] & U_{\sigma'} \end{tikzcd}$$

corresponds to the diagram on coordinate rings

$$\begin{tikzcd} \mathbb{C}[(\sigma')^{\vee} \cap \tau^{\perp} \cap M]_{\chi^m} \arrow[r, equal] & \mathbb{C}[\sigma^{\vee} \cap \tau^{\perp} \cap M] & \mathbb{C}[(\sigma')^{\vee} \cap \tau^{\perp} \cap M] \arrow[l] \\ \mathbb{C}[(\sigma')^{\vee} \cap M]_{\chi^m} \arrow[r, equal] & \mathbb{C}[\sigma^{\vee} \cap M] \arrow[u] & \mathbb{C}[(\sigma')^{\vee} \cap M] \arrow[l] \arrow[u] \end{tikzcd} .$$

where $m \in (\sigma')^{\vee}$ is such that $\sigma = \sigma' \cap m^{\perp}$, and this is clearly commuting. Thus, these $U_{\tau}(\sigma) \to U(\sigma)$ glue to a closed immersion $V(\tau) \subseteq X_{\Sigma}$. $\square$

For the later use, we can also write $U_{\tau}(\sigma) \subseteq U_{\sigma}$ in terms of monoid homomorphisms. It corresponds to

$$\mathrm{Hom}_{\mathbf{Monoid}}(\sigma^{\vee} \cap \tau^{\perp} \cap M, \mathbb{C}) \longrightarrow \mathrm{Hom}_{\mathbf{Monoid}}(\sigma^{\vee} \cap M, \mathbb{C})$$

defined by extending $\gamma \in \mathrm{Hom}_{\mathbf{Monoid}}(\sigma^\vee \cap \tau^\perp \cap M, \mathbb{C})$ to $\sigma^\vee \cap M$ by zero, i.e, by setting $\gamma(m) = 0$ if $m \in (\sigma^\vee \cap M)\backslash\tau^\perp$. In particular, the inclusion $T_{N(\tau)} \subseteq U_\tau$ corresponds to

$$\mathrm{Hom}_{\mathbb{Z}}(\tau^\perp \cap M, \mathbb{C}^\times) \longrightarrow \mathrm{Hom}_{\mathbf{Monoid}}(\tau^\vee \cap M, \mathbb{C}).$$

16.50 If $\tau \preceq \tau' \in \Sigma$, we have a closed immersion $V(\tau') \subseteq V(\tau)$, which on each affine U_σ, $\sigma \in \mathrm{Star}(\tau')$ is given by the projection

$$\mathbb{C}[\sigma^\vee \cap (\tau')^\perp \cap M] \longleftarrow \mathbb{C}[\sigma^\vee \cap \tau^\perp \cap M]$$

Also, from the proof (16.49) we see the defining ideal of $U_\tau(\sigma) = V(\tau) \cap U_\sigma \subseteq U_\sigma$ is

$$\langle \chi^m \mid m \in S_\sigma \backslash \tau^\perp \rangle \subseteq \mathbb{C}[\sigma^\vee \cap M].$$

16.51 The torus orbits. The domain and codomain of the bijection

$$T_{N(\sigma)} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\tau^\perp \cap M, \mathbb{C}^\times)$$

admit T_N-actions; the action on the left is from the projection $T_N \to T_{N(\sigma)}$, and the action on the right is from the inclusion

$$\mathrm{Hom}_{\mathbb{Z}}(\tau^\perp \cap M, \mathbb{C}^\times) \longrightarrow \mathrm{Hom}_{\mathbf{Monoid}}(\tau^\vee \cap M, \mathbb{C}).$$

and (16.16). In fact, this bijection intertwines the T_N-actions. To see this, let $N \otimes_{\mathbb{Z}} \mathbb{C}^\times \ni u \otimes s = t \in T_N$ and $u' \otimes s' \in N(\sigma) \otimes_{\mathbb{Z}} \mathbb{C}^\times$. The element $u' \otimes s'$ maps to $\gamma' \in \mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^\times)$ defined by $\gamma'(m) = s'^{\langle m, u' \rangle}$. The element $t.\gamma' = (u \otimes s)(u' \otimes s')$ then corresponds to (c.f. (16.16) and (16.2))

$$m \mapsto s^{\langle m,u \rangle}\gamma'(m) = \chi^m(\lambda^u(s))\gamma'(m) = \chi^m(t)\gamma'(m) = t.\gamma'(m),$$

so the T_N-action on both sides are compatible.

In other word, the inclusion $O(\tau) \subseteq U_\tau$ is T_N-equivariant. Since T_N acts on $O(\tau)$ transitively, it follows that $O(\tau)$ is an T_N-orbit in U_τ. Moreover, the distinguished point $\gamma_\tau \in U_\tau$ lies in $O(\tau)$, so that

$$O(\tau) = T_N.\gamma_\tau \subseteq X_\Sigma.$$

It suffices to show that $\gamma_\tau \in \mathrm{Hom}_{\mathbf{Monoid}}(\tau^\vee \cap M, \mathbb{C})$ restricts to a nonvanishing map on $\tau^\perp \cap M$, and it is clear from its definition (16.45).

16.52 Lemma. Let σ be a rational convex polyhedral cone in $M_{\mathbb{R}}$, and $\gamma \in \mathrm{Hom}_{\mathbf{Monoid}}(\sigma^\vee \cap M, \mathbb{C})$. Then there exists a face τ of σ such that

$$\gamma^{-1}(\mathbb{C}^\times) = \sigma^\vee \cap \tau^\perp \cap M.$$

Proof. Put $T = \gamma^{-1}(\mathbb{C}^\times)$. This is a finitely generated sub monoid of S_σ, so $\mathrm{cone}(T) \subseteq \sigma^\vee$ is a convex polyhedral cone. Note that we have

$$\gamma^{-1}(\mathbb{C}^\times) = \mathrm{cone}(T) \cap M$$

To see $\mathrm{cone}(T) \cap M \subseteq T$, we may replace $\mathrm{cone}(T)$ by $\mathrm{cone}_{\mathbb{Q}}(T) = \mathrm{cone}(T) \cap M_{\mathbb{Q}}$. If $v \in \mathrm{cone}_{\mathbb{Q}}(T) \cap M \subseteq S_\sigma$, we can find an integer $n \gg 0$ so that $nv \in T \cap M \subseteq S_\sigma$ and thus $0 \neq \gamma(nv) = \gamma(v)^n$.

For $x, y \in \sigma^\vee \cap M_{\mathbb{Q}}$ such that $x + y \in \mathrm{cone}_{\mathbb{Q}}(T)$ for some $\theta \in (0,1) \cap \mathbb{Q}$, take an integer $n \gg 0$ such that $nx, ny \in S_\sigma$ (possible as $\sigma^\vee$ is rational), and $nx + ny = n(x+y) \in T$. Then nx, $ny \in T$, and thus $x, y \in \mathrm{cone}_{\mathbb{Q}}(T)$. By continuity we see

$\mathrm{cone}(T) \subseteq \sigma^\vee$ is a convex subset satisfying the condition in Lemma 15.4. Thus $\mathrm{cone}(T)$ is a face of $\sigma^\vee$, which must be of the form $\mathrm{cone}(T) = \sigma^\vee \cap \tau^\perp$ for some face τ of σ by Lemma 15.10.

□

16.53 Orbit-Cone correspondence. Let X_Σ be the toric variety of the fan Σ in $N_\mathbb{R}$.

(a) There is a bijection

$$\begin{aligned} \Sigma &\longrightarrow T_N \backslash X_\Sigma \\ \sigma &\longmapsto O(\sigma) = \mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^\times). \end{aligned}$$

(b)

$$U_\sigma = \bigcup_{\tau \preceq \sigma} O(\tau),$$

(c) $\tau \preceq \sigma$ if and only if $O(\sigma) \subseteq V(\tau)$, and

$$V(\tau) = \bigcup_{\tau \preceq \sigma \in \Sigma} O(\sigma) = \overline{O(\tau)}^C.$$

Here $\overline{O(\tau)}^C$ denotes the closure in the classical topology.

Proof. Let O be a T_N-orbit in X_Σ. Since each U_σ is T_N-invariant, O is contained in U_σ for some $\sigma \in \Sigma$. Moreover, as $U_{\sigma_1} \cap U_{\sigma_2} = U_{\sigma_1 \cap \sigma_2}$ for any $\sigma_1, \sigma_2 \in \Sigma$, there exists a unique minimal cone $\sigma \in \Sigma$ with $O \subseteq U_\sigma$. We contend that $O = O(\sigma)$. Note that (a) follows from this contention.

To prove $O = O(\sigma)$, it suffices to show $O \cap O(\sigma) \neq \varnothing$. Let $\gamma \in O$. By Lemma 16.52, there exists a face τ of σ such that

$$\gamma^{-1}(\mathbb{C}^\times) = \sigma^\vee \cap \tau^\perp \cap M.$$

Write $\tau = \sigma \cap m^\perp$ for some $m \in \sigma^\vee \cap M$; then $U_\tau = (U_\sigma)_{\chi^m}$ (16.23). As in (16.15), we have $\chi^m(\gamma) = \gamma(m)$, and since $m \in \sigma^\vee \cap \tau^\perp \cap M$, we see $\chi^m(\gamma) \neq 0$, or $\gamma \in U_\tau$. By minimality of σ it forces that $\tau = \sigma$, and hence

$$\gamma^{-1}(\mathbb{C}^\times) = \sigma^\perp \cap M.$$

so that $\gamma|_{\sigma^\perp \cap M} \in \mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^\times) = O(\sigma)$. Hence $(\gamma \in) O \cap O(\sigma)$ is nonempty.

For (b), since U_σ is T_N-invariant, it is a union of T_N-orbits. If O is an orbit, the contention above implies that $O = O(\tau)$, where τ is the minimal cone in Σ such that $O \subseteq U_\tau$. It follows that $\tau \preceq \sigma$, since $\sigma \cap \tau$ is a face of each with $U_{\sigma\cap\tau} = U_\tau \cap U_\sigma$, by minimality it must be the case $\sigma \cap \tau = \tau$, implying $\tau \preceq \sigma$. This proves (b).

It remains to prove (c). We prove the first statement with $V(\tau)$ replaced by $\overline{O(\tau)}^C$ first. Since $\overline{O(\tau)}^C$ is T_N-invariant, it is a union of T_N-orbits. If $O(\sigma) \in \overline{O(\tau)}^C$, then $O(\tau) \subseteq U_\sigma$, since for otherwise $O(\tau) \cap U_\sigma = \varnothing$, which implies $\overline{O(\tau)}^C \cap U_\sigma = \varnothing$ as U_σ is classically open. With $O(\tau) \subseteq U_\sigma$, it follows from (b) that $\tau \preceq \sigma$. Conversely, suppose $\tau \preceq \sigma$. Let $u \in \mathrm{Relint}(\sigma)$ and consider the curve $\gamma : \mathbb{C}^\times \to X_\Sigma$ defined by $\gamma(s) = \lambda^u(s).\gamma_\tau$; note that $\gamma(s) \in O(\tau) \subseteq U_\tau \subseteq U_\sigma$. As a monoid homomorphism, we have, for $m \in \tau^\vee \cap M$,

$$\gamma(s)(m) = \chi^m(\lambda^u(s))\gamma_\tau(m) = s^{\langle m,u\rangle}\gamma_\tau(m)$$

Since $u \in \mathrm{Relint}(\sigma)$, we have $\langle m, u\rangle > 0$ if $m \in \sigma^\vee \backslash \sigma^\perp$ and $\langle m, u\rangle = 0$ if $m \in \sigma^\perp$; it follows that

$$\lim_{s\to 0} \gamma(s)(m) = \begin{cases} 1 & \text{, if } m \in (\sigma^\vee \cap M)\backslash\sigma^\perp \\ 0 & \text{, if } m \in \sigma^\perp \cap M \end{cases} = \gamma_\sigma(m)$$

and thus $\lim_{s\to 0}\gamma(s)$ exists in U_σ and is equal to $\gamma_\sigma \in O(\sigma)$. In particular, this implies $\overline{O(\tau)}^C \cap O(\sigma) \neq \varnothing$, i.e., $O(\sigma) \subseteq \overline{O(\tau)}^C$. The equality

$$\overline{O(\tau)}^C = \bigcup_{\tau\preceq\sigma\in\Sigma} O(\sigma)$$

follows immediately from the first statement in (c), so it remains to show $V(\tau) = \overline{O(\tau)}^C$. From (b) and (c) we see

$$\overline{O(\tau)}^C \cap U_\sigma = \bigcup_{\tau\preceq\sigma'} O(\sigma') \cap \bigcup_{\sigma'\preceq\sigma} O(\sigma') = \bigcup_{\tau\preceq\sigma'\preceq\sigma} O(\sigma') \subseteq U_\sigma.$$

On the other hand we have the closed subvariety $U_\tau(\sigma)$, which, as monoid homomorphisms, is given by (16.49)

$$\mathrm{Hom}_{\mathbf{Monoid}}(\sigma^\vee \cap \tau^\perp \cap M, \mathbb{C}) \longrightarrow \mathrm{Hom}_{\mathbf{Monoid}}(\sigma^\vee \cap M, \mathbb{C})$$

Since $U_\tau(\sigma)$ is the Zariski closure of $O(\tau)$ in U_σ, we see $\overline{O(\tau)}^C \cap U_\sigma \subseteq U_\tau(\sigma)$. Conversely, if $\gamma \in \mathrm{Hom}_{\mathbf{Monoid}}(\sigma^\vee \cap \tau^\perp \cap M, \mathbb{C})$, by Lemma 16.52 there exists $\sigma' \preceq \sigma$ with, viewed γ as a map $\gamma : \sigma^\vee \cap M \to \mathbb{C}$,

$$\gamma^{-1}(\mathbb{C}^\times) = \sigma^\vee \cap (\sigma')^\perp \cap M$$

which implies $\sigma^\vee \cap \tau^\perp \supseteq \sigma^\vee \cap (\sigma')^\vee$, so that by Lemma 15.10 we obtain $\tau \preceq \sigma'$. Thus $\gamma \in \overline{O(\tau)}^C \cap U_\sigma$, proving the opposite inclusion. □

16.3.2 Toric morphisms

16.54 Definition. Let N_1, N_2 be two free $\mathbb{Z}$-modules of finite rank and let $\Sigma_i \subseteq (N_i)_\mathbb{R}$ $(i = 1, 2)$ be a fan.

1. A $\mathbb{Z}$-linear morphism $\overline{\phi} : N_1 \to N_2$ is **compatible** with the fans Σ_1 and Σ_2 if for every cone $\sigma_1 \in \Sigma_1$, there exits a cone $\sigma \in \Sigma_2$ such that $\overline{\phi}_\mathbb{R}(\sigma_1) \subseteq \sigma_2$.
2. A morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ is **toric** if $\phi(T_{N_1}) \subseteq T_{N_2}$ and $\phi|_{T_{N_1}} : T_{N_1} \to T_{N_2}$ is a group homomorphism.

16.55 Theorem. Let N_1, N_2 be two free $\mathbb{Z}$-modules of finite rank and let $\Sigma_i \subseteq (N_i)_\mathbb{R}$ $(i = 1, 2)$ be a fan.

(i) If $\overline{\phi} : N_1 \to N_2$ is a $\mathbb{Z}$-linear map compatible with Σ_1 and Σ_2, then there is a toric morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ such that $\phi|_{T_{N_1}}$ is the map

$$\overline{\phi} \otimes \mathrm{id}_{\mathbb{C}^\times} : N_1 \otimes_\mathbb{Z} \mathbb{C}^\times \longrightarrow N_2 \otimes_\mathbb{Z} \mathbb{C}^\times.$$

(ii) If $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ is a toric morphism, then ϕ induces a $\mathbb{Z}$-linear map $\overline{\phi} : N_1 \to N_2$ compatible with the fans Σ_1 and Σ_2.

16.56 Proposition. Let $N' \leqslant N$ be a sublattice with $\dim_\mathbb{R} N_\mathbb{R} = n$ and $\dim_\mathbb{R} N'_\mathbb{R} = k$. Let Σ be a fan in $N'_\mathbb{R}$, which can be viewed as a fan in $N_\mathbb{R}$. Then

(i) If N' is spanned by a subset of basis of N, then there is an isomorphism

$$X_{\Sigma,N} \cong X_{\Sigma,N'} \times T_{N/N'}.$$

(ii) In general, a basis for N' can be extended to a sublattice $N'' \leqslant N$ of finite index. Then $X_{\Sigma,N}$ is isomorphic to the quotient of

$$X_{\Sigma,N''} \cong X_{\Sigma'} \times T_{N''/N'}$$

by the finite abelian group N/N''.

16.57 Fibre bundle. Let N, N' be two free $\mathbb{Z}$-modules of finite rank and let $\overline{\phi} : N \to N'$ be a surjective $\mathbb{Z}$-linear map. Let Σ, Σ' be fans in $N_{\mathbb{R}}$ and $N'_{\mathbb{R}}$, respectively, that are compatible with $\overline{\phi}$. Then there is a corresponding toric morphism

$$\phi : X_{\Sigma} \longrightarrow X_{\Sigma'}.$$

Let $N_0 = \ker \overline{\phi}$; note that N_0 is spanned by a subset of basis of N. The collection

$$\Sigma_0 = \{\sigma \in \Sigma \mid \sigma \subseteq (N_0)_{\mathbb{R}}\}$$

is a fan in $(N_0)_{\mathbb{R}}$. By (16.56).(i) we have an isomorphism

$$X_{\Sigma_0,N} \cong X_{\Sigma_0,N_0} \times T_{N'}$$

as $N' \cong N/N_0$. Furthermore, $\overline{\phi}$ is compatible with Σ_0 in $N_{\mathbb{R}}$ and the trivial fan $\{0\}$ in $N'_{\mathbb{R}}$, so it gives the toric morphism

$$\phi|_{X_{\Sigma_0,N}} : X_{\Sigma_0,N} \longrightarrow T_{N'}.$$

Moreover,

$$\phi^{-1}(T_{N'}) = X_{\Sigma_0,N} \cong X_{\Sigma_0,N_0} \times T_{N'}$$

so X_{Σ} is a fibre bundle over $T_{N'}$ with fibre X_{Σ_0,N_0}.

16.58 Lemma. Let V_1, V_2 be two finite dimensional real vector spaces and let $\sigma_i \subseteq V_i$ be a convex polyhedral cone. Let $\phi : V_1 \to V_2$ be an $\mathbb{R}$-linear map such that $\phi(\sigma_1) \subseteq \sigma_2$. If

$$\phi(\mathrm{Relint}(\sigma_1)) \cap (\sigma_2 \backslash \mathrm{Relint}(\sigma_2)) \neq \varnothing,$$

then $\phi(\sigma_1)$ is contained in a proper face of σ_2.

Proof. Let $v_0 \in \mathrm{Relint}(\sigma_1)$ such that $\phi(v_0) \in \sigma_2 \backslash \mathrm{Relint}(\sigma_2)$. By Lemma 15.6, we see $\phi(v_0) \in \tau'$ for some proper face τ' of σ_2. Take any $v \in \sigma_1$; as $v_0 \in \mathrm{Relint}(\sigma_1)$, we have $v_0 - \varepsilon v \in \sigma_1$ if $\varepsilon > 0$ is small enough. Then

$$\phi(v_0) = \phi(v_0 - \varepsilon v) + \phi(\varepsilon v)$$

with the two summands in σ_2. Since τ' is a face, Lemma 15.3 implies $\phi(\varepsilon v) \in \tau'$, and thus $\phi(v) \in \tau'$. □

16.59 Images of Distinguished points. Let $\phi : X_{\Sigma} \to X_{\Sigma'}$ be a toric morphism coming from a map $\overline{\phi} : N \to N'$ compatible with the fans Σ, Σ'. Let σ be a given cone in Σ, and let σ' be the minimal cone in Σ' containing $\overline{\phi}_{\mathbb{R}}(\sigma)$. Then

(i) $\phi(\gamma_{\sigma}) = \gamma_{\sigma'}$.

(ii) $\phi(O(\sigma)) \subseteq O(\sigma')$, and $\sigma(V(\sigma)) \subseteq V(\sigma')$.

(iii) The induced map $\phi|_{V(\sigma)} : V(\sigma) \to V(\sigma')$ is toric.

Proof.

(i) By Lemma 16.58 and the minimality of σ', we have $\phi(\mathrm{Relint}(\sigma)) \subseteq \mathrm{Relint}(\sigma')$.

□

16.3.3 Complete and proper

16.60 Theorem. Let X_{Σ} be the toric variety of the fan Σ in $N_{\mathbb{R}}$. TFAE :

(i) X_{Σ} is a complete variety.

(ii) X_Σ is compact in the classical topology.

(iii) The limit $\lim_{s\to 0} \lambda^u(s)$ exists in X_Σ for all $u \in N$.

(iv) Σ is a **complete fan**, i.e., $|\Sigma| = N_\mathbb{R}$.

Proof. (i)⇔(ii) is a general result (4.30) on varieties over $\mathbb{C}$. For the others, note that by (16.39) and (4.29), X_Σ is Hausdorff in the classical topology. Since the classical topology on each U_σ is metrizable, X_Σ is compact if and only if it is sequentially compact (both classically).

We first show (ii)⇒(iii). Assume X_Σ is compact and fix a $u \in N$. Let $(t_k)_k$ be a sequence in $\mathbb{C}^\times$ converging to 0. Applying λ^u, we obtain a sequence $(\lambda^u(t_k))_k$ in X_Σ. Since X_Σ is compact, by passing to a convergent subsequence we may assume $\lambda^u(t_k) \to \gamma \in X_\Sigma$. Since X_Σ is covered by the U_σ, and these are classically open, we may assume $\lambda^u(t_k) \in U_\sigma \ni \gamma$ for any k. Take $m \in \sigma^\vee \cap M$. Since χ^m is a morphism, it is a classical continuous function on U_σ, so that

$$\chi^m(\gamma) = \lim_{k\to\infty} \chi^m(\lambda^u(t_k)) = \lim_{k\to\infty} t_k^{\langle m,u\rangle}.$$

Since $t_k \to 0$, it must be $\langle m, u\rangle \geqslant 0$ for any $m \in \sigma^\vee \cap M$. This implies $\langle m, u\rangle \geqslant 0$ for any $m \in \sigma^\vee$, so that $u \in (\sigma^\vee)^\vee = \sigma$. Now Lemma 16.46 implies that $\lim_{s\to 0} \lambda^u(s)$ exists in U_σ, and hence in X_Σ.

Now consider (iii)⇒(iv). Take $u \in N$ and consider the limit $\lim_{s\to 0} \lambda^u(s)$. This lies in some U_σ, so Lemma 16.46 implies $u \in \sigma \cap N$. Thus $N \subseteq |\Sigma|$, and hence $N_\mathbb{R} = |\Sigma|$.

Finally we show (iv)⇒(ii). We prove this by induction on $n = \dim_\mathbb{R} N_\mathbb{R}$. For $n = 1$, assume $N = \mathbb{Z}$. The only complete fan in $\mathbb{R}$ is $\Sigma = \{\{0\}, \mathbb{R}_{\geqslant 0}, \mathbb{R}_{\leqslant 0}\}$. The corresponding toric variety is $\mathbb{P}^1_\mathbb{C}$, which is classically homeomorphic to S^2, the unit sphere in $\mathbb{R}^3$, and hence is compact.

Assume the statement is true for all complete fans of dimension $< n$, and let Σ be a complete fan in $N_\mathbb{R} \cong \mathbb{R}^n$. Let $(\gamma_k)_k$ be a sequence in X_Σ. We are going to show that $(\gamma_k)_k$ has a convergent subsequence.

Since X_Σ is a finite union of the $O(\tau)$ $(\tau \in \Sigma)$ (16.53), by passing to a subsequence we may assume $(\gamma_k)_k \subseteq O(\tau)$ for some $\tau \in \Sigma$. If $\tau \neq \{0\}$, then the orbit closure $V(\tau) = X_{\mathrm{Star}(\tau)}$ is a toric variety of dimension $\leqslant n-1$.

Lemma. If Σ is a complete fan, then for any $\tau \in \Sigma$, then $\mathrm{Star}(\tau)$ is a complete fan in $N(\tau)_\mathbb{R}$.

Proof. This amounts to show that for $x \in N_\mathbb{R}\backslash(N_\tau)_\mathbb{R}$, there exists a cone $\tau \preceq \sigma \in \Sigma$ such that $x \in \sigma + (N_\tau)_\mathbb{R}$. Consider the set

$$U = \{\sigma \in \Sigma \mid \dim \sigma \text{ maximal}, (x + (N_\tau)_\mathbb{R}) \cap \sigma \neq \varnothing\} = \{\sigma_1, \ldots, \sigma_r\}.$$

We contend that $\tau \preceq \sigma_i$ for some $1 \leqslant i \leqslant r$. Suppose not; take any $u \in \tau$. Take $(m_{ij})_{ij} \subseteq M_\mathbb{R}$ such that $\sigma_i = \bigcap_j m_{ij}^\vee$. Since $u \npreceq \sigma_i$, we can find $j = j(i)$ such that $\langle u, m_{ij}\rangle < 0$. Take $\lambda \gg 0$ so that

$$\langle m_{ij(i)}, x + \lambda u\rangle < 0$$

for each i. But then $x + \lambda u \notin \sigma_i$ for any $1 \leqslant i \leqslant n$, a contradiction to the completeness of Σ. Hence $\sigma \preceq \sigma_i$ for some $1 \leqslant i \leqslant r$, and $x \in \sigma_i + (N_\tau)_\mathbb{R}$ as wanted. ☐

Since Σ is complete, the fan $\mathrm{Star}(\tau)$ is complete in $N(\tau)_\mathbb{R}$. By induction hypothesis, $(\gamma_k)_k$ has a convergent subsequence in $V(\tau)$, and we are done. It remains to deal with the case $\tau = \{0\}$; in other words, we may assume $(\gamma_k)_k$ lies in $O(\{0\}) \subseteq T_N \subseteq X_\Sigma$. To proceed, we introduce a key tool in the next paragraph. ☐

16.61 In this paragraph we introduce the **logarithm map** $L : T_N \to N_\mathbb{R}$ and discuss some of its important properties. Let $p \in T_N$ and let $\gamma : M \to \mathbb{C}^\times$ be the corresponding (semi)group homomorphism. Define $L(\gamma) : M \to \mathbb{R}$ by the formula

$$L(\gamma)(m) = \log|\gamma(m)|$$

Then $L(\gamma) \in \mathrm{Hom}_\mathbb{Z}(M, \mathbb{R}) \cong N_\mathbb{R}$, and we define $L(p)$ to be the corresponding vector in $N_\mathbb{R}$.

Lemma. If $L(p) \in -\sigma$ for some $\sigma \in \Sigma$, then $|\gamma(m)| \leqslant 1$ for all $m \in \sigma^\vee \cap M$.

Proof. Let $p \in T_N$ be such that $L(p) \in -\sigma$ for some $\sigma \in \Sigma$. If $m \in \sigma^\vee \cap M$, from the definition we have

$$\log|\gamma(m)| = L(\gamma)(m) = \langle m, L(p)\rangle.$$

Since $L(p) \in -\sigma$, it follows that $\log|\gamma(m)| \leqslant 0$, i.e. $|\gamma(m)| \leqslant 1$. $\square$

16.62 Finish of the proof. We apply the logarithm map $L : T_N \to N_\mathbb{R}$ to our sequence $(\gamma_k)_k \subseteq T_N$, so we obtain a sequence $(L(\gamma_k))_k$ in $N_\mathbb{R}$. Since Σ is complete, we have $N_\mathbb{R} = \bigcup_{\sigma\in\Sigma}(-\sigma)$, so by passing to subsequence we may assume there is a $\sigma \in \Sigma$ with $L(\gamma_k) \in -\sigma$ for any k. By (16.61) we conclude $|\gamma_k(m)| \leqslant 1$ for all $m \in \sigma^\vee \cap M$ and k. Let A be a finite generating set of the monoid $\sigma^\vee \cap M$. Denote by $\mathbb{D}$ the closed unit ball in $\mathbb{C}$. Then there is an injection

$$\begin{aligned} B := \mathrm{Hom}_{\mathbf{Monoid}}(\sigma^\vee \cap M, \mathbb{D}) &\longrightarrow \mathbb{D}^A \\ \gamma &\longmapsto (\gamma(m))_{m\in A}. \end{aligned}$$

We topologize the left hand side by the subspace topology inherited from $\mathbb{D}^A$. Then B is a closed subspace of $\mathbb{D}^A$. Since $\mathbb{D}^A$ is compact, so is B. Thus by passing to a convergent subsequence, we may assume that $\gamma_k \to \gamma \in B = \mathrm{Hom}_{\mathbf{Monoid}}(\sigma^\vee \cap M, \mathbb{D})$. This corresponds to a point γ in U_σ, and it remains to show $\gamma_k \to \gamma$ in the classical topology of U_σ. For this it suffices to note that $\chi^m(\gamma_k) \to \chi^m(\gamma)$ in $\mathbb{C}$ for all $m \in \sigma^\vee \cap M$.

16.63 Next we prove the relative version of Theorem 16.60.

Theorem. Let $\phi : X_\Sigma \to X_{\Sigma'}$ be the toric morphism corresponding to a homomorphism $\overline{\phi} : N \to N'$ that is compatible with fans Σ in $N_\mathbb{R}$ and Σ' in $N'_\mathbb{R}$. TFAE :

(i) ϕ is classically proper.

(ii) ϕ is a proper morphism.

(iii) If $u \in N$ and $\lim_{s\to 0} \lambda^{\overline{\phi}(u)}(s)$ exists in $X_{\Sigma'}$, then $\lim_{s\to 0} \lambda^u(s)$ exists in X_Σ.

(iv) $(\overline{\phi}_\mathbb{R})^{-1}(|\Sigma'|) = |\Sigma|$.

Proof. (i)⇔(ii) is a general result (4.31) for varieties over $\mathbb{C}$. To prove (ii)⇒(iii), let $u \in N$ and suppose $\lim_{s\to 0} \lambda^{\overline{\phi}(u)}(s) = \gamma' \in X_{\Sigma'}$. First consider the case that $\overline{\phi}(u) \neq 0$. By Theorem 4.22, the image $\lambda^u(\mathbb{C}^\times)$ is construtible in X_Σ. By (4.27), the Zariski closure $\overline{\lambda^u(\mathbb{C}^\times)}$ in X_Σ is the same as the classical closure. Since ϕ is Zariski proper, $\phi(\overline{\lambda^u(\mathbb{C}^\times)}) \subseteq X_{\Sigma'}$ is Zariski closed, hence classically, closed. It follows that

$$\overline{\lambda^{\overline{\phi}(u)}(\mathbb{C}^\times)} \subseteq \phi(\overline{\lambda^u(\mathbb{C}^\times)})$$

and thus we can find $\gamma \in \overline{\lambda^u(\mathbb{C}^\times)}$ such that $\phi(\gamma) = \gamma'$. Take $(t_k)_k \subseteq \mathbb{C}^\times$ with $\lambda^u(t_k) \to \gamma$. Then

$$\lim_{s\to 0} \lambda^{\overline{\phi}(u)}(s) = \gamma' = \phi(\gamma) = \lim_{k\to\infty} \phi(\lambda^u(t_k)) = \lim_{k\to\infty} \lambda^{\overline{\phi}(u)}(t_k)$$

Assume $\gamma' \in U_{\sigma'}$ for some $\sigma' \in \Sigma'$. Since $U_{\sigma'}$ is classically open, we can further assume that $\lambda^{\overline{\phi}(u)}(t_k) \in U_{\sigma'}$ for all k and $\lambda^{\overline{\phi}(u)}(s) \in U_{\sigma'}$ for s sufficiently small. As $\overline{\phi}(u) \neq 0$, the cocharacter $\lambda^{\overline{\phi}(u)} : \mathbb{C}^\times \to T_{N'} \subseteq U_{\sigma'}$ is nontrivial, so there must be $m \in (\sigma')^\vee \cap M'$ with $\langle m, \overline{\phi}(u)\rangle \neq 0$. But

$$\lim_{s\to 0} s^{\langle m,\overline{\phi}(u)\rangle} = \chi^m(\gamma') = \lim_{k\to\infty} t_k^{\langle m,\overline{\phi}(u)\rangle}$$

exists, we must have $\langle m, \overline{\phi}(u)\rangle > 0$ and thus $t_k \to 0$. Now the argument in (16.60).(ii)⇒(iii) shows that $\lim_{s\to 0} \lambda^u(s)$ exists in X_Σ.

Now drop the restriction $\overline{\phi}(u) \neq 0$. Consider the map $(\phi, \mathrm{id}_{\mathbb{C}}) : X_\Sigma \times \mathbb{A}^1_{\mathbb{C}} \to X_{\Sigma'} \times \mathbb{A}^1_{\mathbb{C}}$, which is proper as ϕ is. This is the toric morphism corresponding to $(\phi, \mathrm{id}_{\mathbb{Z}}) : N \times \mathbb{Z} \to N' \times \mathbb{Z}$. Now applying the argument above to $(u, 1) \in N \times \mathbb{Z}$ shows that $\lim_{s \to 0} \lambda^u(s)$ exists in X_Σ.

For (iii)⇒(iv), first note that since $\overline{\phi}$ is compatible with Σ and Σ', the inclusion

$$|\Sigma| \subseteq \overline{\phi}_{\mathbb{R}}^{-1}(|\Sigma'|)$$

is automatic. For the opposite inclusion, let $u \in \overline{\phi}_{\mathbb{R}}^{-1}(|\Sigma'|)$. Then $\overline{\phi}(u) \in |\Sigma'|$, so by Lemma 16.46 the limit $\lim_{s \to 0} \lambda^{\overline{\phi}(u)}(s)$ exists in $X_{\Sigma'}$. By assumption, $\lim_{s \to 0} \lambda^u(s)$ exists in X_Σ, and Lemma 16.46 again implies $u \in \sigma \cap N$ for some $\sigma \in \Sigma$. Since all the cones are rational, this implies $\overline{\phi}_{\mathbb{R}}^{-1}(|\Sigma'|) \subseteq |\Sigma|$.

Finally we prove (iv)⇒(i). We begin with two special cases. Suppose that a toric morphism $\phi : X_\Sigma \to T_{N'}$ satisfies (iv) and has the additional property that $\overline{\phi} : N \to N'$ is surjective. The fan of $T_{N'}$ consists of the trivial cone $\{0\}$, so that (iv) implies

$$|\Sigma| = (\overline{\phi}_{\mathbb{R}})^{-1}(0) = \ker(\overline{\phi}_{\mathbb{R}}) = (\ker \overline{\phi})_{\mathbb{R}}.$$

We can think of Σ as a fan Σ'' in $\ker(\overline{\phi}_{\mathbb{R}})$. As in (16.57), there is an isomorphism

$$X_\Sigma \cong X_{\Sigma''} \times T_{N'},$$

and ϕ corresponds to the projection $X_{\Sigma''} \times T_{N'} \to T_{N'}$. The fan Σ'' is complete in $\ker(\overline{\phi}_{\mathbb{R}})$, so by (16.60) $X_{\Sigma''}$ is compact, implying $X_{\Sigma''} \times T_{N'} \to T_{N'}$ is proper. The second case is discussed in the next paragraph. □

16.64 Suppose that a homomorphism of tori $\phi : T_N \to T_{N'}$ has the additional property that $\overline{\phi} : N \to N'$ is injective. The condition (iv) is then clear for ϕ. We prove directly that ϕ is proper.

Let $\overline{\phi}^* : M' \to M$ be the adjoint of ϕ; note that $(\overline{\phi}^*)_{\mathbb{R}} = (\overline{\phi}_{\mathbb{R}})^*$. Since $\overline{\phi}$ is injective, $(\overline{\phi}^*)_{\mathbb{R}}$ is surjective, and thus $\overline{\phi}^*(M') \leqslant M$ has finite index. Pick any integer $d > 0$ such that $dM \subseteq \overline{\phi}^*(M')$.

Let $(\gamma_k)_k \subseteq T_N$ be a sequence such that $(\phi(\gamma_k))_k$ converges in $T_{N'}$. We are going to show $(\gamma_k)_k$ admits a convergent subsequence in T_N. Since $\chi^{\overline{\phi}*(m')} = \chi^{m'} \circ \phi$ for any $m' \in M'$, we see $\chi^m(\gamma_k)$ converges in T_N for all $m \in \overline{\phi}^*(M')$, and hence $\chi^m(\gamma_k^d)$ converges in T_N for all $m \in M$. Choose a basis of $M \cong \mathbb{Z}^n$ and identify T_N with $(\mathbb{C}^\times)^n$. If we write $\gamma_k = (\gamma_{1k}, \ldots, \gamma_{nk})$, then each γ_{1k}^d converges to some point $\widetilde{\gamma_1} \in \mathbb{C}^\times$, and hence

$$\gamma_k^d = (\gamma_{1k}^d, \ldots, \gamma_{nk}^d) \to (\widetilde{\gamma_1}, \ldots, \widetilde{\gamma_n}) \in (\mathbb{C}^\times)^n.$$

As each $\widetilde{\gamma}_i \in \mathbb{C}^\times$, it follows that we can choose the d-th roots $\widetilde{\gamma}_i^{1/d}$ in a way that, by passing to subsequences, $\gamma_{ik} \to \widetilde{\gamma}_i^{1/d}$. This proves γ_k converges in T_N.

16.65 Proof of (iv)⇒(i). Consider a general toric morphism $\phi : X_\Sigma \to X_{\Sigma'}$ satisfying (iv). Assume $(\gamma_k)_k$ is a sequence in X_Σ such that $(\phi(\gamma_k))_k$ converges in $X_{\Sigma'}$. Our goal is to show that $(\gamma_k)_k$ admits a convergent subsequence in X_Σ.

Since X_Σ has only finitely many T_N-orbits, we may assume $(\gamma_k)_k \subseteq O(\sigma)$ for some $\sigma \in \Sigma$. Let σ' be the minimal cone of Σ' containing $\overline{\phi}_{\mathbb{R}}(\sigma)$. By (16.59).(iii), the restriction $\phi|_{V(\sigma)} : V(\sigma) \to V(\sigma')$ is toric, and it corresponds to the map $N(\sigma) \to N(\sigma')$ induced by $\overline{\phi}$. This map satisfies the condition (iv) with the fans $\mathrm{Star}(\sigma)$, $\mathrm{Star}(\sigma')$. By (16.59).(ii), we thus can assume $\gamma_k \in T_N$ and $\phi(\gamma_k) \in T_{N'}$ for all k.

The limit $\gamma' = \lim_{k \to \infty} \phi(\gamma_k)$ lies in some $O(\tau')$ $(\tau' \in \Sigma)$; we then can assume the $\phi(\gamma_k)$ and γ' lie in $U_{\tau'}$. Note that (iv) implies

$$(\overline{\phi}_{\mathbb{R}})^{-1}(\tau') = \bigcup_{\overline{\phi}_{\mathbb{R}}(\sigma) \subseteq \tau'} \sigma$$

(If $u \in (\overline{\phi}_{\mathbb{R}})^{-1}(\tau')$, let σ be the minimal cone in Σ containing u, and σ' the minimal cone in Σ' containing $\overline{\phi}_{\mathbb{R}}(\sigma)$. Since σ is minimal, we have $u \in \mathrm{Relint}(\sigma)$. Since σ' is minimal, by Lemma 16.58 we have $\overline{\phi}_{\mathbb{R}}(u) \in \mathrm{Relint}(\sigma') \cap \tau'$. Since Σ' is a fan, $\sigma' \cap \tau'$ is a face of σ', so $\mathrm{Relint}(\sigma')$ does not meet $\sigma' \cap \tau$ unless $\sigma' \cap \tau' = \sigma'$, i.e., $\sigma' \preceq \tau'$. Thus $\overline{\phi}_{\mathbb{R}}(\sigma) \subseteq \sigma' \subseteq \tau'$). Also, the construction of ϕ from $\overline{\phi}_{\mathbb{R}}$ (16.55) implies that $\phi^{-1}(U_{\tau'})$ is the toric variety given by the fan $\{\sigma \in \Sigma \mid \overline{\phi}_{\mathbb{R}}(\sigma) \subseteq \tau'\}$. Therefore, we may assume $X_\Sigma = U_{\tau'}$ and $(\overline{\phi}_{\mathbb{R}})^{-1}(\tau') = |\Sigma|$.

If $\tau' = \{0\}$, then $O(\tau') = U_{\tau'} = T_{N'}$. By writing $\overline{\phi}$ as the composition $N \to \overline{\phi}(N) \to N'$, we see $\phi : X_\Sigma \to T_{N'}$ factors as

$$X_\Sigma \longrightarrow T_{\overline{\phi}(N)} \longrightarrow T_{N'}.$$

Since the composition of proper maps remains proper, we conclude ϕ is proper from the above two special cases.

It remains to deal with the case $\tau' \neq \{0\}$. Recall

$$\gamma' \in O(\tau') = \mathrm{Hom}_{\mathbb{Z}}((\tau')^\perp \cap M', \mathbb{C}^\times) \subseteq \mathrm{Hom}_{\mathbf{Monoid}}((\tau')^\vee \cap M', \mathbb{C}).$$

As $\phi(\gamma_k) \to \gamma'$ and $(\tau')^\vee \cap M'$ is a finitely generated monoid, by passing to a subsequence we may assume

$$|\phi(\gamma_k)(m')| \leqslant 1 \text{ for all } k \text{ and } m' \in (\tau')^\vee \cap M \backslash (\tau')^\perp$$

The logarithm map defined in (16.61) gives maps $L_N : T_N \to N_{\mathbb{R}}$ and $L_{N'} : T_{N'} \to N'_{\mathbb{R}}$, and they fit into a commutative square

$$\begin{tikzcd} T_N \arrow[r, "L_N"] \arrow[d, "\phi|_{T_N}"'] & N_{\mathbb{R}} \arrow[d, "\overline{\phi}_{\mathbb{R}}"] \\ T_{N'} \arrow[r, "L_{N'}"'] & N'_{\mathbb{R}}. \end{tikzcd}$$

This is commutative by (16.55).(i). Let $\overline{\phi}^* : M' \to M$ be the adjoint of $\overline{\phi} : N \to N'$. Then for $m' \in (\tau')^\vee \cap M \backslash (\tau')^\perp$ and all k, we have

$$\begin{aligned} \langle \overline{\phi}^*(m'), L_N(\gamma_k) \rangle &= \langle m', \overline{\phi}_{\mathbb{R}}(L_N(\gamma_k)) \rangle \\ &= \langle m', L_{N'}(\phi(\gamma_k)) \rangle = \log |\phi(\gamma_k)(m')| \leqslant 0. \end{aligned} \tag{$\clubsuit$}$$

On the other hand, we have the equivalences :

$$\begin{aligned} u \in (\overline{\phi}_{\mathbb{R}})^{-1}(\tau') &\Leftrightarrow \overline{\phi}_{\mathbb{R}}(u) \in \tau' \\ &\Leftrightarrow \langle m', \overline{\phi}_{\mathbb{R}}(u) \rangle \geqslant 0 \text{ for all } m' \in (\tau')^\vee \cap M' \\ &\Leftrightarrow \langle \overline{\phi}^*(m'), u \rangle \geqslant 0 \text{ for all } m' \in (\tau')^\vee \cap M' \end{aligned}$$

To proceed further, we need a lemma.

16.66 Lemma. Let $\sigma \neq \{0\}$ be a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$. Then

$$u \in \sigma \text{ if and only if } \langle m, u \rangle \geqslant 0 \text{ for all } m \in \sigma^\vee \cap M \backslash \sigma^\perp$$

16.67 Finish of the proof. The preceding lemma establishes the equivalence

$$u \in (\overline{\phi}_{\mathbb{R}})^{-1}(\tau') \Leftrightarrow \langle \overline{\phi}^*(m'), u \rangle \geqslant 0 \text{ for all } m' \in (\tau')^\vee \cap M' \backslash (\tau')^\perp.$$

From ($\clubsuit$) we conclude that $-L_N(\gamma_k) \in (\overline{\phi}_{\mathbb{R}})^{-1}(\tau')$ for all k. But we are assuming $(\overline{\phi}_{\mathbb{R}})^{-1}(\tau') = |\Sigma|$, so

$$-L_N(\gamma_k) \in |\Sigma|$$

for all k. By passing to a subsequence we may assume $L_N(\gamma_k) \in -\sigma$ for some $\sigma \in \Sigma$. The argument in (16.62) then implies γ_k admits a convergent subsequence in $U_\sigma \subseteq X_\Sigma$. This finishes the proof.

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