Vector Calculus Week 4 - Vector Operators

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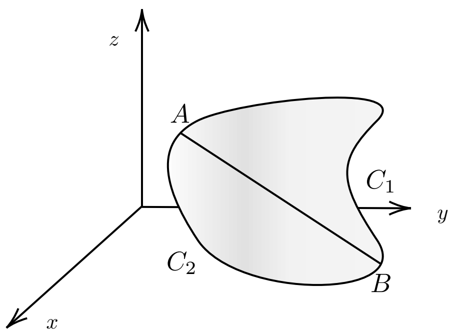
1 Conservative Fields

1.1 Gradients and Conservative Field



Definition 1.1: Conservative Vector Field

A conservative vector field is one which the line integral along a curve connecting two points does not depend on the path taken.

What this says, is that we can write:

$$\int_C \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} = \int_{C_1} \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} = \int_{C_2} \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}}$$

Theorem 1.1

Suppose that a vector field $\underline{\mathbf{F}}$ is related to a scalar field $\Phi(\underline{\mathbf{x}})$ by $\underline{\mathbf{F}} = \underline{\nabla}\Phi$ and $\underline{\nabla}\Phi$ exists everywhere in some region D. Conversely, if $\underline{\mathbf{F}}$ is conservative, then $\underline{\mathbf{F}}$ can be written as the gradient of a scalar field, $\underline{\mathbf{F}} = \underline{\nabla}\Phi$

Proof. Suppose that $\underline{\mathbf{F}} = \underline{\nabla}\Phi$, then F is conservative on D. So we can write;

$$\begin{aligned}
\int_C \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} &= \int_C \underline{\nabla}\Phi \cdot d\underline{\mathbf{r}} \\
&= \int_C \left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z}\right) \cdot (dx, dy, dz) \\
&= \int_C \frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy + \frac{\partial \Phi}{\partial z} dz \\
&= \int_C d\Phi \\
&= \Phi\Big|_A^B \\
&= \Phi(B) - \Phi(A)
\end{aligned}$$

So as this result only matters about the end points, $\underline{\mathbf{F}}$ is conservative. Now assume that $\underline{\mathbf{F}}$ is conservative, then a scalar field $\Phi(\underline{\mathbf{x}})$ can be defined as the line integral of $\underline{\mathbf{F}}$ from the origin to the point $\underline{\mathbf{x}}$:

$$\begin{aligned}
\Phi(\underline{\mathbf{x}}) &= \int_{\underline{\mathbf{0}}}^{\underline{\mathbf{x}}} \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} \\
d\Phi &= \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} \\
&= \underline{\nabla}\Phi \cdot \underline{\mathbf{r}} \\
&= \frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy + \frac{\partial \Phi}{\partial z} dz
\end{aligned}$$

and we can now say that $\underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} = \underline{\nabla}\Phi \cdot d\underline{\mathbf{r}}$ and hence, $F = \underline{\nabla}\Phi$ □

If a vector field is conservative, $\Phi(\underline{\mathbf{x}})$ which satisfies $\underline{\mathbf{F}} = \underline{\nabla}\Phi$ is called the potential of the vector field.

1.2 Curl and conservative vector fields

Suppose that $\underline{\mathbf{u}} = \underline{\nabla}\Phi$, then,

$$\begin{aligned}
\underline{\nabla} \times \underline{\mathbf{u}} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (u_1, u_2, u_3) \\
&= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_1 & u_2 & u_3 \end{vmatrix} \\
&= \left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right)\hat{\boldsymbol{\imath}} + \left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial z}\right)\hat{\boldsymbol{\jmath}} + \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)\hat{\boldsymbol{k}} \\
&= \left(\frac{\partial^2 \Phi}{\partial y \partial z} - \frac{\partial^2 \Phi}{\partial z \partial y}\right)\hat{\boldsymbol{\imath}} \left(\frac{\partial^2 \Phi}{\partial z \partial x} - \frac{\partial^2 \Phi}{\partial x \partial z}\right)\hat{\boldsymbol{\jmath}} \\
&\quad + \left(\frac{\partial^2 \Phi}{\partial x \partial y} - \frac{\partial^2 \Phi}{\partial y \partial x}\right)\hat{\boldsymbol{k}} \\
&= \underline{\mathbf{0}} \qquad \text{As } \Phi \in C^2
\end{aligned}$$

So for any vector $\underline{\mathbf{u}}$ that can be written as the gradient of a vector field is irrotational. Conversely, any irrotational vector field is conservative.

1.3 Laplacian of a scalar field

Suppose that a scalar field Φ, is twice dofferenctiable. Then $\underline{\nabla}\Phi$ is a differentiable vector field, so we can tak divergence of $\underline{\nabla}\Phi$ and obtain another scalar field

Definition 1.2: Laplacian

The scalar field $\underline{\nabla} \cdot \underline{\nabla} \Phi$ is called the Laplacian of Φ and is denoted, ∇^2 or Δ

The Laplacian can also act on a vector field, which results in another vector field.

$$\nabla^2 \underline{\mathbf{u}} = \nabla^2 u_1 \hat{\boldsymbol{\imath}} + \nabla^2 u_2 \hat{\boldsymbol{\jmath}} + \nabla^2 u_3 \hat{\boldsymbol{k}}$$

If we have $\Delta\Phi = 0$, this is a known PDE known as the laplace equation.

Theorem 1.2: Divergence of curl

For any $\mathcal{C}^2$ vector field, $\underline{\mathbf{F}}$,

$$\underline{\nabla} \cdot \underline{\nabla} \times \underline{\mathbf{F}} = 0$$

Proof.

$$\begin{aligned}
\underline{\nabla} \times \underline{\mathbf{F}} &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\
&= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \hat{\boldsymbol{\imath}} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial z} \right) \hat{\boldsymbol{\jmath}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{\boldsymbol{k}} \\
\underline{\nabla} \cdot \underline{\nabla} \times \underline{\mathbf{F}} &= \frac{\partial F_3}{\partial x \partial y} - \frac{\partial F_2}{\partial x \partial z} + \frac{\partial F_1}{\partial y \partial z} - \frac{\partial F_3}{\partial x \partial y} + \frac{\partial F_2}{\partial x \partial z} - \frac{\partial F_1}{\partial y \partial z} \\
&= \underline{\mathbf{0}}
\end{aligned}$$

□

1.4 Vector Operators Identities

Let Φ, f, g be scalar fields and $\underline{\mathbf{F}}, \underline{\mathbf{G}}$ be vector fields, then:

$$\underline{\nabla} \cdot (\underline{\nabla} \times \underline{\mathbf{F}}) = 0 \tag{1}$$
$$\underline{\nabla} \times \underline{\nabla} \Phi = \underline{\mathbf{0}} \tag{2}$$
$$\underline{\nabla}(f + g) = \underline{\nabla} f + \underline{\nabla} g \tag{3}$$
$$\underline{\nabla} \cdot (\underline{\mathbf{F}} + \underline{\mathbf{G}}) = \underline{\nabla} \cdot \underline{\mathbf{F}} + \underline{\nabla} \cdot \underline{\mathbf{G}} \tag{4}$$
$$\underline{\nabla} \times (\underline{\mathbf{F}} + \underline{\mathbf{G}}) = \underline{\nabla} \times \underline{\mathbf{F}} + \underline{\nabla} \times \underline{\mathbf{G}} \tag{5}$$
$$\underline{\nabla}(fg) = f \underline{\nabla} g + g \underline{\nabla} f \tag{6}$$
$$\underline{\nabla} \cdot (\Phi \underline{\mathbf{F}}) = \Phi \underline{\nabla} \cdot \underline{\mathbf{F}} + \underline{\mathbf{F}} \cdot \underline{\nabla} \Phi \tag{7}$$
$$\underline{\nabla} \times (\Phi \underline{\mathbf{F}}) = \Phi \underline{\nabla} \times \underline{\mathbf{F}} - \underline{\mathbf{F}} \times \underline{\nabla} \Phi \tag{8}$$
$$\underline{\nabla}(\underline{\mathbf{F}} \cdot \underline{\mathbf{G}}) = \underline{\mathbf{F}} \times (\underline{\nabla} \times \underline{\mathbf{G}}) + \underline{\mathbf{G}} \times (\underline{\nabla} \times \underline{\mathbf{F}}) \tag{9}$$
$$+ (\underline{\mathbf{F}} \cdot \underline{\nabla}) \underline{\mathbf{G}} + (\underline{\mathbf{G}} \cdot \underline{\nabla}) \underline{\mathbf{F}} \tag{10}$$
$$\tag{11}$$
$$\underline{\nabla} \cdot (\underline{\mathbf{F}} \times \underline{\mathbf{G}}) = \underline{\mathbf{G}} \cdot (\underline{\nabla} \times \underline{\mathbf{F}}) - \underline{\mathbf{F}}(\underline{\nabla} \times \underline{\mathbf{G}}) \tag{12}$$
$$\underline{\nabla} \times (\underline{\mathbf{F}} \times \underline{\mathbf{G}}) = \underline{\mathbf{F}}(\underline{\nabla} \cdot \underline{\mathbf{G}}) - \underline{\mathbf{G}}(\underline{\nabla} \cdot \underline{\mathbf{F}}) \tag{13}$$
$$+ (\underline{\mathbf{G}} \cdot \underline{\nabla}) \underline{\mathbf{F}} - (\underline{\mathbf{F}} \cdot \underline{\nabla}) \underline{\mathbf{G}} \tag{14}$$
$$\underline{\nabla} \times (\underline{\nabla} \times \underline{\mathbf{F}}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{\mathbf{F}}) - \underline{\nabla}^2 \underline{\mathbf{F}} \tag{15}$$

2 Orthoginal Curvilinear Coordinate Systems