Vector Calculus Week 2 - More Suffix Notation

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Contents

1 Gradient, Divergence and Curl

1.1 Gradient

Assume we have a $f = f(x, y, z)$ or $f = f(x_1, x_2, x_3)$, so a scalar calued function. Then we define grad f as:

$$\underline{\nabla}\, f = \left(\frac{\partial}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial}{\partial z}\hat{\boldsymbol{k}} \right) f$$

We say grad of f is a differential operator. So:

$$\underline{\nabla}\, f = \left(\frac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{k}} \right)$$

and we can write it in suffix notation aswell:

$$[\underline{\nabla}\, f]_i = \frac{\partial}{\partial x_i} \qquad i = 1, 2, 3$$

1.2 Divergence

Assume we have a vector field, $\underline{\mathbf{u}} = \underline{\mathbf{u}}(x, y, z, t)$. We define the divergence of this vector field as;

$$\underline{\nabla} \cdot \underline{\mathbf{u}} = \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right)$$

Placing this in suffix notation, we get that:

$$[\underline{\nabla} \cdot \underline{\mathbf{u}}]_j = \frac{\partial u_j}{\partial x_j}$$

1.3 Curl

the curl of a vector field can be written as:

$$\underline{\nabla} \times \underline{\mathbf{u}}$$

To write this in suffix notation, we can just use the cross produce formula:

$$[\underline{\nabla} \times \underline{\mathbf{u}}]_i = \varepsilon_{ijk} \underline{\nabla}_j u_k$$

which then can be manipulated into:

$$[\underline{\nabla} \times \underline{\mathbf{u}}]_i = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} \qquad j, k = 1, 2, 3$$

where i is a free index and j, k are dummy suffixes, so $j, k = 1, 2, 3$

2 Combinations of $\underline{\nabla}\, f$, $\underline{\nabla} \cdot (\quad)$ and $\underline{\nabla} \times (\quad)$

If we take $\underline{\nabla} \cdot \underline{\nabla}\, f$ where $f = (x_1, x_2, x_3, t)$. We can write the div of grad as:

$$\begin{aligned}
\underline{\nabla} \cdot \underline{\nabla}\, f &= \left(\frac{\partial}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial}{\partial z}\hat{\boldsymbol{k}} \right) \cdot \left(\frac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{k}} \right) \\
&= \frac{\partial}{\partial x_1}\frac{\partial f}{\partial x_1} + \frac{\partial}{\partial x_2}\frac{\partial f}{\partial x_2} + \frac{\partial}{\partial x_3}\frac{\partial f}{\partial x_3} \\
&= \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2} + \frac{\partial^2 f}{\partial x_3} \\
&= \Delta f
\end{aligned}$$

Where the $\Delta = \underline{\nabla}^2$ is the laplacian. So how do we write this in suffix notation?

$$\begin{aligned}
\underline{\nabla} \cdot \underline{\nabla}\, f &= \underline{\nabla}_j [\underline{\nabla}\, f]_j \\
&= \frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_j} \\
&= \frac{\partial^2 f}{\partial x_j}
\end{aligned}$$