

Vector Calculus Week 2 - More Suffix Notation

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September 29, 2020

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1 Gradient, Divergence and 2 Combinations of gradient, divergence and curl

1.1 Gradient

Assume we have a $f = f(x, y, z)$ or $f = f(x_1, x_2, x_3)$, so a scalar valued function. Then we define grad f as:

$$\underline{\nabla} f = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) f$$

We say grad of f is a differential operator. So:

$$\underline{\nabla} f = \left(\frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} \right)$$

and we can write it in suffix notation aswell:

$$[\underline{\nabla} f]_i = \frac{\partial}{\partial x_i} \quad i = 1, 2, 3$$

1.2 Divergence

Assume we have a vector field, $\underline{u} = \underline{u}(x, y, z, t)$. We define the divergence of this vector field as:

$$\underline{\nabla} \cdot \underline{u} = \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right)$$

Placing this in suffix notation, we get that:

$$[\underline{\nabla} \cdot \underline{u}]_j = \frac{\partial u_j}{\partial x_j}$$

1.3 Curl

the curl of a vector field can be written as:

$$\underline{\nabla} \times \underline{u}$$

To write this in suffix notation, we can just use the cross produce formula:

$$[\underline{\nabla} \times \underline{u}]_i = \varepsilon_{ijk} \underline{\nabla}_j u_k$$

which then can be manipulated into:

$$[\underline{\nabla} \times \underline{u}]_i = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} \quad j, k = 1, 2, 3$$

where i is a free index and j, k are dummy suffixes, so $j, k = 1, 2, 3$

2.1 Divergence of Gradient

If we take $\underline{\nabla} \cdot \underline{\nabla} f$ where $f = f(x_1, x_2, x_3, t)$. We can write the div of grad as:

$$\begin{aligned} \underline{\nabla} \cdot \underline{\nabla} f &= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot \left(\frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} \right) \\ &= \frac{\partial}{\partial x_1} \frac{\partial f}{\partial x_1} + \frac{\partial}{\partial x_2} \frac{\partial f}{\partial x_2} + \frac{\partial}{\partial x_3} \frac{\partial f}{\partial x_3} \\ &= \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} \\ &= \Delta f \end{aligned}$$

Where the $\Delta = \underline{\nabla}^2$ is the laplacian. So how do we write this in suffix notation?

$$\begin{aligned} \underline{\nabla} \cdot \underline{\nabla} f &= \underline{\nabla}_j [\underline{\nabla} f]_j \\ &= \frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_j} \\ &= \frac{\partial^2 f}{\partial x_j^2} \end{aligned}$$

2.2 Curl of Gradient

We can write the curl of gradient as:

$$\begin{aligned} [\underline{\nabla} \times \underline{\nabla} f]_i &= \varepsilon_{ijk} \underline{\nabla}_j \underline{\nabla}_k f \\ &= \varepsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_k} \\ &= \varepsilon_{ikj} \frac{\partial}{\partial x_k} \frac{\partial f}{\partial x_j} \\ &= -\varepsilon_{ijk} \frac{\partial}{\partial x_k} \frac{\partial f}{\partial x_j} \\ &= -\varepsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_k} \quad \text{if } f \in C^2 \\ &\implies \underline{\nabla} \times \underline{\nabla} f = 0 \end{aligned}$$