Year 3 — Topology and Metric Spaces

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Spring Term 2022

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine (especially the typos!).

Contents

1 Introduction

1.1 Motivation

In this module we will look at ways to generalise Real Analysis.

(i) Metric Spaces

(ii) Topological Spaces

(iii) Measure Spaces

A key idea in Real Analysis is continuity, a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if, given $a \in \mathbb{R}$ given $\varepsilon > 0$ there exists some $\delta > 0$ so that,

$$|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon$$

We have seen a version of this for $\mathbb{R}^n \to \mathbb{R}^m$ or $\mathbb{C} \to \mathbb{C}$. This can interpreted as a notion of a distance, we can ensure that the distance between $f(x)$ and $f(a)$ be less than ε. Here the distance between real numbers is $|x-y|$. This leads to metric spaces is a set where we have a distance function $d_X(a,b)$ for any points $a, b \in X$.

Another way to interpret the continuity of $f : \mathbb{R} \to \mathbb{R}$ is to day that for any U in $\mathbb{R}$, the set,

$$f^{-1}(U) := \{x \in \mathbb{R} : f(x) \in U\}$$

is also open.

We may ask what happens if we choose a U such that $f^{-1}(U) = \varnothing$, but we say that the empty set is open.

We can talk about continuity without talking about distances, provided that we know what we mean by the idea of open sets. Open sets may not be defined by distance. A space together with a collection of open subsets is a topological space. Metric spaces are topological spaces with a idea of distance.

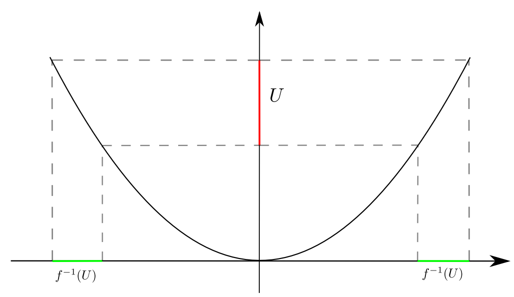

Figure 1: Image Convergence.

Measure spaces are related to length of a subset, and also integration. These are linked since if A is a subset of $\mathbb{R}$ of length ℓ, then,

$$\ell = \int_{\mathbb{R}} 1_A(x)\,dx$$

where $1_A : \mathbb{R} \to \mathbb{R}$ is the indicator function,

$$1_A \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

This unproblematic if we have $A = [a, b]$, then we can integrate this nicely,

However, if $A = \mathbb{Q}$ it is not clear that we can make sense of this 'length' of $\mathbb{Q}$, and the integral is not defined (as a Riemann Integral). Measure Theory provides the theoretical framework for assigning a length to most (but not all, the measurable ones work) subsets of $\mathbb{R}$ and making corresponding integral as Lesbegue integrals. It turns out that $\mathbb{Q}$ has 'length' of 0, so there are way more irrational numbers, and $\mathbb{Q}$ is countable.

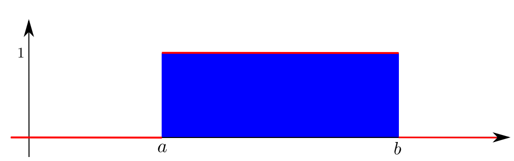

Figure 2: Image Convergence.

1.2 Review of Real Analysis

For real numbers $a \leq b$, we have the open interval,

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

and closed interval,

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$

We can also have the mixed intervals, $(a, b]$ or (a, ∞).

In general, a subset U is open, if for each $a \in U$ there is some $\varepsilon > 0$ such that $(a - \varepsilon, a + \varepsilon) \subset U$ (U does not contain it's boundary, every point is interior). A closed set is a set where it's complement is open. The empty set and $\mathbb{R}$ are clopen, open and closed.

Lemma 1.1 (Triangle Inequality)**.** For some $a, b \in R$,

$$|a + b| \leq |a| + |b|$$

and we can extend this to say $|a - b| \geq ||a| - |b||$.

Let $A \subset \mathbb{R}$. An upper bound is a number u such that $a \leq u$ for all $a \in A$. If u is an upper bound of a then it has many upper bounds, if at least one exists, the set is bounded. A least upper bound or supremum for A is a number u such that,

(i) $a \leq u$ for all $a \in A$

(ii) if $u_* < u$ then there is some $a \in A$ with $a > u_*$

If A has a least upper bound u, then u might or might not be in A. There are similar definitions for greatest lower bound or infimum. A set is bounded, if it is bounded above and below, or there is some M such that $|a| \leq M$ for all $a \in A$. An important property of the real numbers is the completeness property: every non-empty set of real numbers which is bounded above has a least upper bound.

We say that a sequence converges to a, if given $\varepsilon > 0$ thiere is some $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for all $n > N$. Then a is the limit of a sequence. A sequence is a bounded if $|a_n| < M$ for all n. If a_n is bounded which is monotonically increasing, then it must converge, same for monotonically decreasing. In general a sequence that is bounded, doesn't have to converge. However, a bounded sequence always has a convergence subsequence.

A function is continuous at a point $a \in \mathbb{R}$, for all $\varepsilon > 0$ there is some $\delta > 0$ so that $|f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \delta$. We say that f is continuous if it holds for every a. If $f, g : \mathbb{R} \to \mathbb{R}$, then $f \pm g$, fg, $\frac{f}{g}$ $(g \neq 0)$ are all continuous. Suppose we have a continuous function on a closed and bounded interval

Theorem 1.2 (Intermediate Value Theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be continuous, for any v between $f(a)$ and $f(b)$, there is at least one $x \in [a, b]$ with $f(x) = v$.

Theorem 1.3. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then $f(x)$ is bounded and attains its bounds, i.e. f has a (finite) maximum M and minimum m in $[a, b]$. More precisely $x_{\min}$ and $x_{\max} \in [a, b]$ so that $m = f(x_{\min}) \leq f(x) \leq f(x_{\max})$ for all $x \in [a, b]$.

2 Metric Spaces

We firstly define a metric space,

Definition 2.1 (Metric Space)**.** A metric space, (X, d) consists of a non-empty set X and a function $d : X \times X \to \mathbb{R}$ satisfying,

(i) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0 \iff x = y$

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$ (symmetry)

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

Here are a load of examples,

Example. Take, $X = \mathbb{R}$ and $d_{\mathbb{R}}(x, y) = |x - y|$. Now, we can probably see normally that the three axioms hold. The first is how we define $|\cdot|$, then $|x - y| = |(-1)(y - x)| = |y - x|$ and the third is the triangle inequality.

and now for $\mathbb{R}^m$,

Example. If we let $\mathbb{R}^m$ and $d_{\mathbb{R}^m}(x, y) = \sqrt{\sum_{j=1}^{n} |x_j - y_j|^2}$ where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$. The axioms hold, as if $d_{\mathbb{R}^m}(\mathbf{x}, \mathbf{y}) = 0$, then we require that $x_j = y_j$ for all j and so $\mathbf{x} = \mathbf{y}$. For the second, we can use a similar argument to before as $|x_j - y_j| = |y_j - x_i|$. For the triangle inequality for this metric space, we need to use the Cauchy Schwartz inequality,

$$\left| \sum_{j=1}^{n} a_j \overline{b_j} \right| \leq \left(\sum_{j=1}^{n} |a_j|^2 \right) \left(\sum_{j=1}^{n} |b_j|^2 \right)$$

that is $|\mathbf{a} \cdot \overline{\mathbf{b}}| < |\mathbf{a}|^2 |\mathbf{b}|^2$.

We now can look at the taxicab metric,

Example. Take $X = \mathbb{R}^m$ and $d'_{\mathbb{R}^m}(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^{n} |x_i - y_j|$ for $x, y \in \mathbb{R}^m$. The first two are trivial for d', but the third easier than before,

$$\sum_{j=1}^{n} |x_j - z_j| = \sum_{j=1}^{n} |x_j - y_j - (y_j - z_j)| \leq \sum_{j=1}^{n} |x_j - y_j| + \sum_{j=1}^{n} |y_j - z_j| = d'(\mathbf{x}, \mathbf{y}) + d'(\mathbf{y}, \mathbf{z})$$

For an example not $\mathbb{R}^m$,

Example. Take any X that is non-empty, then

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

The first two axioms are clear, then for the third consider $x = z$,

$$d(x, z) = 0 \leq d(x, y) + d(y, z)$$

and this is always true. If $x \neq z$, then,

$$d(x, z) = 1 \leq d(x, y) + d(y, z)$$

if $x \neq z$, then either $x \neq y$ or $y \neq z$, so the above holds.

Now for something more abstract,

Example. Consider $\mathcal{C}[0,1]$ and let the metric be, $d(f,g) = \max\{f(t) - g(t) : t \in [0,1]\}$. Does this metric make sense? Are they bounded / why does this maximum make sense. This makes sense because of a Theorem in the last lecture. The first two of the conditions follow nicely, then the third,

$$\begin{aligned} |f(t) - h(t)| &= |(f(t) - g(t)) + (g(t) - h(t))| \\ &\leq |f(t) - g(t)| + |g(t) - h(t)| \\ &= d(f,g) + d(g,h) \end{aligned}$$

and so taking the maximum, we can get that $d(f,h) \leq d(f,g) + d(g,h)$.

We can remark, that this is not the only way to consider the distance between two functions, we could have integrated.

Definition 2.2 (Subspace)**.** A subspace of a metric space (X, d_X), is a non-empty subset Y together with the metric d_Y by restricting d_X to Y.

$$d_Y(y, y') = d_X(y, y') \qquad \forall y, y' \in Y$$

This is clearly a metric space as if the conditions hold for X, they will then hold for Y.

2.1 Continuity in Metric Spaces

We can talk nicely about continuity in metric space, in a rather obvious way once we realise it's all about distance,

Definition 2.3 (Limit)**.** Let (X,d) be a metric space, then let (a_n) be a sequence of points in X. For some $a \in X$ we say that (a_n) converges to a, written $a_n \to a$ as $n \to \infty$ if, for any real number $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ with $d(a_n, a) < \varepsilon$ for all $n > N$. We say that a is the limit of the sequence.

This is just a copy of the definition of a limit, just with our metric placed in. Here is an interesting quirk, if we take the discrete metric, then the sequence $\left(\frac{1}{n}\right)$ then this does not converge to zero. For, if we choose $\varepsilon > 0$ with $\varepsilon < 1$, then $d\left(\frac{1}{n}, 0\right) > \varepsilon$

Definition 2.4 (Continuity)**.** Let (X, d_X) and (Y, d_Y) be metric spaces, then $f : X \to Y$. For $a \in X$, we say that f is continuous at a if, given $\varepsilon > 0$, there is some $\delta > 0$ so that $d_Y(f(x), f(a)) < \varepsilon$ for all $x \in X$ with $d_X(x,a) < \delta$. We say f is continuous if it is continuous for every a.

We can prove that in the discrete metric then any function $f : X \to Y$ is convergent where X and Y have the discrete metric, just take $\delta = 1$.

2.2 Opens Sets

We can consider balls, as we have a distance metric we can move forwards to open sets and the required analytic tools.

Definition 2.5 (Open Ball)**.** Let (X,d) be a metric space, for any $a > 0$ and any $a \in X$, the set

$$B_\varepsilon(a) = \{x \in X : d(x,a) < \varepsilon\}$$

is called an open ball in X of radius ε and center a.

As a sanity check, when $X = \mathbb{R}$ we get an interval, $(a - \varepsilon, a + \varepsilon)$ and with $X = \mathbb{R}^2$ or $\mathbb{C}$, then we see we get an open disc

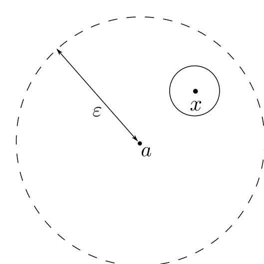

Figure 3: Open Ball.

Definition 2.6 (Open Set)**.** A subset U of a metric space X is open if, for every $x \in U$ there is some $\varepsilon > 0$ such that

$$B_\varepsilon(x) \subset U$$

A subset V is closed if $X \setminus V$ is open.

By convention, $\varnothing$ is open and now we prove that the epsilon ball is open.

Proposition 2.7. For any $a \in X$ and every $\varepsilon > 0$ the set $B_\varepsilon(a)$ is an open set in X.

Proof. Let $x \in B_\varepsilon(a)$, then we need to find a $\delta > 0$ such that $B_\delta(a) \subset B_\varepsilon(a)$. Take $\delta = \varepsilon - d(x, a)$. Then $\varepsilon > 0$ and if $y \in B_\delta(a)$ then $d(y, a) \leq d(y, x) + d(x, a) < \delta + d(x, a) = \varepsilon$. Thus $y \in B_\varepsilon(a)$. This holds for every $y \in B_\delta(a)$ and so $B_\delta(a) \subset B_\varepsilon(a)$. □

Here's a slight quirk, if we consider X and $Y \subset X$. If we consider a $U \subset Y$ which is open, this need not be open in X. Consider $Y = [0, 1] \subset \mathbb{R}$, and $B_{\frac{1}{2}}(0)$ as our open set, which is just $\{x \in [0, 1] : |x - 0| < \frac{1}{2}\}$. However, in $\mathbb{R}$ this subset is $\left[0, \frac{1}{2}\right)$.

Proposition 2.8. Let U and V be open sets in the metric space (X, d). Then $U \cap V$ is an open set.

Proof. If $x \in U \cap V$, then there are $\varepsilon_1, \varepsilon_2 > 0$ such that $B_{\varepsilon_1}(x) \subset U$ and $B_{\varepsilon_2}(x) \subset V$ and so we just choose $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. Then $B_\varepsilon(x) \subset U \cap V$. □

Then by induction we can generalise this,

Proposition 2.9. The intersection of any finite family of open sets is open, ie. if $n \geq 0$, then $U_1, \ldots, U_n$ are open sets then $U_1 \cap U_2 \cap \cdots \cap U_n$ is an open set.

We often write this to mean the above intersection,

$$\bigcap_{i=0}^{n} U_i$$

The same works for unions, but we can say more. Suppose we have a family of open sets, indexed by some set $\mathcal{I}$. This means for every $i \in \mathcal{I}$ we have an open set $U_i \subset X$. The indexing set doesn't need to be finite.

Proposition 2.10. If U_i, $i \in \mathcal{I}$ is a family of open sets $\bigcup_{i \in \mathcal{I}} U_i$ is open.

Proof. Let $U = \bigcup_{i \in \mathcal{I}} U_i$. We need to show that U is open. Let $x \in U$, then $x \in U_i$ for some $i \in \mathcal{I}$. As U_i is open, there is some $\varepsilon > 0$ with $B_\varepsilon(x) \subset U_i$. As $U_i \subset U$, we have $B_\varepsilon(x) \subset U$. Hence U is open. □

The intersection of infinitely many open sets, need not be open. Consider,

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

which is then closed.

Now let us redefine the continuity and convergence in terms of these open sets,

Definition 2.11 (Limit)**.** Let (a_n) be a sequence in a metric space, (X, d) and let $a \in X$. Then $a_n \to a$ as $n \to \infty$ if and only if the following hold,

(i) for every open set U containing a there is some $N \in \mathbb{N}$ such that $a_n \in U$ for all $n > N$

Proof. First suppose $a_n \to a$ as $n \to \infty$. We must show that the condition holds. Let $a \in U$, U is open. Then there is some ε with $B_\varepsilon(a) \subset U$. As $a_n \to a$ there exists $N \in \mathbb{N}$ with $d(a_n, a) < \varepsilon$ for all $n > N$. But then $a_n \in B_\varepsilon(a) \subset U$ for all $n > N$ as required.

Conversely, suppose the conditon holds, then we must show that $a_n \to a$. Let $\varepsilon > 0$. Then $B_\varepsilon(a)$ is an open set containing a, so by the condition there is some N with $a_n \in B_\varepsilon(a)$ is an open set containing a, so there is some N with $a_n \in B_e(a)$ for all $n > N$. Hence $d(a_n, a) < \varepsilon$ for all $n > N$. This shows $a_n \to a$. □

We can do a similar thing for continuity.

Proposition 2.12. Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$. Then f is continuous if and only if, for every open set U in Y, the set $\{x \in X : f(x) \in U\}$ is an open set in X.

We often use the notation $f^{-1}(U)$ for the set $\{x \in X : f(x) \in U\}$. This is the preimage of the set U. We use this notation even if there is no actual function f^{-1}.

Proof. Suppose f is continuous, let $U \subset Y$ be open. We must show that $f^{-1}(U)$ is open. If $f^{-1}(U) = \varnothing$, then $f^{-1}(U)$ is open. Let $x \in f^{-1}(U)$, then $f(x) \in U$. Since U is open, there is some $\varepsilon > 0$ such $B_\varepsilon^Y(f(x)) \subset U$ (with metric Y). Since f is continuous, there is some $\delta > 0$ so that $d_Y(f(x'), f(x)) < \varepsilon$ for all x' such that $d(x, x') < \delta$. If $x' \in B_\delta^X(x)$ then $f(x') \in B_\varepsilon^Y(f(x)) \subset U$ and so $x' \in f^{-1}(U)$ and $B_\delta^X(x) \subset f^{-1}(U)$. So $f^{-1}(U)$ is open.

Conversely suppose $f^{-1}(U)$ is open for all open $U \subset Y$. Let $x \in X$ and $\varepsilon > 0$. Then $U = B_\varepsilon^Y(f(x))$ is an open set in Y, then $x \in f^{-1}(U)$, which is open in X. So there is some $\delta > 0$ with $B_\delta^Y(x) \subset f^{-1}(U)$. Therefore for all $x' \in B_\delta(x)$ where $x' \in f^{-1}(U)$ and so $f(x) \in B_\varepsilon^Y(f(x))$, that is forall x' with $d^X(x', x) < \delta$ and so we have

$$d_Y(f(x'), f(x)) < \varepsilon$$

Hence f is continuous. $\square$

2.3 Equivalent Metrics

Definition 2.13 (Equivalent Metrics)**.** Let d_1 and d_2 be two metrics on the same set X.

(i) We say that d_1 and d_2 are topologically equivalent if the open sets with respect to d_1 are the same as the open sets with repsect to d_2

(ii) We say that d_1 and d_2 are Lipschitz equivalent if there are constants $A \geq B > 0$ such that,

$$Bd_1(x, y) \leq d_2(x, y) \leq Ad_1(x, y) \qquad \forall x, y \in X$$

Proposition 2.14. If d_1 and d_2 are Lipschitz equivalent metrics on X then they are topologically equivalent.

Proof. Let $B_\varepsilon^{d_1}(a)$ and $B_\varepsilon^{d_2}(a)$ be the open balls with respect to d_1 and d_2 respectively. By hypothesis, there are constants such that,

$$Bd_1(x, y) \leq d_2(x, y) \leq Ad_1(x, y) \qquad \forall x, y \in X$$

Let U be an open set with respect to d_1. Given an $a \in U$ there is some $\varepsilon > 0$ with $B_\varepsilon^{d_1}(a) \subset U$. Now if $d_2(x, a) < B\varepsilon$ then $Bd_1(x, a) \leq d_2(x, a) < B\varepsilon$ so $d_1(x, a) < \varepsilon$. Hence $B_{B\varepsilon}^{d_2}(a) \subset B_\varepsilon^{d_2}(a) \subset U$. This shows that U is an open set with respect to d_2. $\square$

Example. Let $X = \mathbb{R}$ with d_1 is the usual metric and d_2 is the taxi-cab metric. Then d_1 and d_2 are Lischitz equivalent. This is because, if $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ in $\mathbb{R}^2$. Then, for some $A \geq B > 0$,

$$Bd_1(\mathbf{x}, \mathbf{y}) \leq d_2(\mathbf{x}, \mathbf{y}) \leq Ad_1(\mathbf{x}, \mathbf{y})$$

that is,

$$\begin{aligned} B\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} &\leq |x_1 - y_1| + |x_2 - y_2| \\ &\leq A\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \end{aligned}$$

Let $u_1 = |x_1 - y_1|$ and $u_2 = |x_2 - y_2|$, and then squaring,

$$\begin{aligned} B^2(u_1^2 + u_2^2) &\leq (u_1 + u_2)^2 \\ &\leq A^2(u_1^2 + u_2^2) \end{aligned}$$

for all $u_1, u_2 \geq 0$. We now want to find such A and B. For B, we let $B = 1$ as $u_1^2 + u_2^2 \leq (u_1 + u_2)^2$. For A, $u_1^2 + u_2^2 - 2u_1u_2 \geq 0$ and so $u_1^2 + u_2^2 \geq 2u_1u_2$ and so $(u_1 + u_2)^2 \leq 2(u_1^2 + u_2^2)$, so $A = \sqrt{2}$.

Consider $X = \mathbb{R}_{>0}$ and d_1 be the usual metric and $d'(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$, it can be proved that this d' is a metric. Now let $x = \frac{1}{n}$ and $y = \frac{1}{n+1}$ and we can see that our normal distance, $d\left(\frac{1}{n}, \frac{1}{n+1}\right) = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)}$ and $d'\left(\frac{1}{n}, \frac{1}{n+1}\right) = 1$ and so we can pick points close together in d but not in d'. Now consider,

$$\frac{d'(x,y)}{d(x,y)} = n(n+1)$$

and so we can make this whatever we want and so we cannot have these as Lipschitz equivalent. However, they are topologically equivalent because $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ where $x \mapsto \frac{1}{x}$ is continuous.

3 Topological Spaces

We are going to mainly start by focussing on defintions and examples.

Definition 3.1 (Topological Space)**.** A topological space $(X, \mathcal{T})$ is a non-empty set X along with a family $\mathcal{T}$ of subsets X satisfying,

(i) $X \in \mathcal{T}$ and $\varnothing \in \mathcal{T}$

(ii) If $U_1, U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$

(iii) If $U_1 \in \mathcal{T}$ are any collection of sets in $\mathcal{T}$, indexed by $i \in \mathcal{I}$ for some set $\mathcal{I}$, then

$$\bigcup_{i \in \mathcal{I}} U_i \in \mathcal{T}$$

We call a collection $\mathcal{T}$ of subsets satisfying these axioms a topology on X and we call the elements of $\mathcal{T}$ the open sets of X in the topology $\mathcal{T}$.

It follows from (T2) by induction that the intersection of finitely many open sets is an open set. This (T1) - (T3) say that the open sets in a topology on X must satisfy,

- $\varnothing$ and X are open
- the intersection of finitely many open sets is open
- The union of an arbitrary collection of open sets is open.

Moreover, any collection of subsets of X with these properties form a topology on X. Note that the intersections and unions behave differently, the union of infinitely many open sets must be open but their intersection need not be. That's a definition, here are some examples,

Example. Let (X, d) be any metric space and let $\mathcal{T}$ be the collection of open sets defined with respect to d. We have seen these satisfy the axioms of a topological space. In particular, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{R}^n$ are topological spaces with the topology given by the usual metric. We call this the **usual topology**.

Here's (potentially) a different example,

Example. Let X be any non-empty set and $\mathcal{T}$ be the powerset of X. Clearly the axioms hold, so this is a topology on X, which we call the **discrete topology**. It is the topology with the most open sets, every subset of X is open. In fact, this is a special case of the previous example, with the discrete example.

Example. Let X be a set. Then $\mathcal{T} = \{\varnothing, X\}$ is a topology on X, called the **indescrete topology** on X.

Example. The **Sierpinski space** is the two-point set $\{0, 1\}$ with the open sets $\varnothing$, $\{0\}$, $\{0, 1\}$.

Example. Let X be a non-empty set and let $\mathcal{T}$ consist of all $U \subset X$ whose complement $(X \setminus U)$ is finite, together with the empty set, $\varnothing$. Then $\mathcal{T}$ is a topology on X, called the **cofinite topology**. WE check (T1) - (T3),

(i) $\varnothing \in \mathcal{T}$ follows from the definition, also $X^c = \varnothing \in \mathcal{T}$.

(ii) Let $U, V \in \mathcal{T}$. We must show that $U \cap V \in \mathcal{T}$. If $U = \varnothing$ or $V = \varnothing$, then $U \cap V = \varnothing$. Otherwise $X \setminus U$ and $X \setminus V$ are finite. So $X \setminus (U \cap V) = (X \setminus U) \cap (X \setminus V)$ is finite, again $U \cap V \in \mathcal{T}$.

(iii) Let U_i for $i \in \mathcal{I}$ be a family of sets in $\mathcal{T}$. We must show $V := \bigcup_{i \in \mathcal{I}} U_i \in \mathcal{T}$. If $U_i = \varnothing$ for all i then $V = \varnothing$ and we are done. Otherwise. we can choose a $j \in \mathcal{I}$ such that $U_j \neq \varnothing$. As $U_j \in \mathcal{T}$, we have $X \setminus U_j$ is finite. As $U_j \subset V$ we have $X \setminus V \subset X \setminus U_j$ so $X \setminus V$ is also finite. Hence $V \in \mathcal{T}$.

3.1 Basis of a topology

Next we talk about how we start to adapt the definition such that we can define the sets in terms of building blocks, like in $\mathbb{R}$ where we talk about intervals and epsilon neighbourhoods. In fact, in a matric space, not every open set is from one open ball, but if we know of all the open balls we know of all the open sets. We can do something similar for topological spaces.

Definition 3.2 (Basis)**.** Given a topological space $(X, \mathcal{T})$, a basis of $\mathcal{T}$ is a subset $\mathcal{B}$ of $\mathcal{T}$ such that every open set is a union of sets from $\mathcal{B}$.

Remark. If $\mathcal{B}$ is a basis of $\mathcal{T}$, then every $B \in \mathcal{B}$ is open (since $\mathcal{B} \subset \mathcal{T}$) and hence every union of sets from $\mathcal{B}$ is open. So $\mathcal{T}$ consists exactly of the subsets of X which can be written as the unions of sets of $\mathcal{B}$.

Example. A basis for $\mathbb{R}$ is

$$\mathcal{B} = \{(a, b) : a, b \in \mathbb{R} \text{ with } a < b\}$$

the collection of all open intervals in $\mathbb{R}$. For id U is an open set, then for each $x \in U$ we can find $\varepsilon_x > 0$ so that the open interval $B_x = (x - \varepsilon_x, x + \varepsilon_x) \subset U$ and then,

$$U = \bigcup_{x \in U} B_x$$

and now a lemma,

Lemma 3.3. If $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on X, then,

(i) For each $x \in X$, there is some $B \in \mathcal{B}$ with $x \in B$

(ii) If $x \in B_1$ and $x \in B_2$ with $B_1, B_2 \in \mathcal{B}$ then there exists a B_3 such that $B_3 \in \mathcal{B}$ with $x \in B_3 \subset B_1 \cap B_2$.

Conversely, let $\mathcal{B}$ be a collection of subsets of a non-empty set X. If $\mathcal{B}$ satisfies (B1), (B2) then there exists unique topology $\mathcal{T}$ on X such that $\mathcal{B}$ is a topology for $\mathcal{T}$.

Proof. **(B1):** $\mathcal{T}$ consists of all possible unions of sets in $\mathcal{B}$. $X \in \mathcal{T}$ so X is a union of sets in $\mathcal{B}$ therefore given a $x \in X$, so $x \in B$ for some $B \in \mathcal{B}$. Hence (B1) holds.

(B2): If $x \in B_1$ and $x \in B_2$ with $B_1, B_2 \in \mathcal{B}$, so B_1, B_2 are open sets as they are in $\mathcal{T}$, therefore $B_1 \cap B_2 \in \mathcal{T}$, so $x \in B_1 \cap B_2$ and $B_1 \cap B_2$ is a union of sets in $\mathcal{B}$. So there is a $B_3 \in \mathcal{B}$ with $x \in B_3 \subset B_1 \cap B_2$. So (B2) holds.

Converse: Uniqueness is easy, if $\mathcal{B}$ is a basis, then the topology is just all the union of $B \in \mathcal{B}$. This is the only possible topology. We need to check that $\mathcal{T}$ satisfies (T1)-(T2):

(T1): To get the empty set, take no elements of $\mathcal{B}$ and take the union of them. For X, it is just $X = \bigcup_{B \in \mathcal{B}} B$.

(T2): If $U, V \in \mathcal{T}$ and $x \in U \cap V$ then there is a $B, C \in \mathcal{B}$ with $x \in B \subset U$ and $x \in C \subset V$, by (B2) there is some $W_x \in \mathcal{B}$ with $x \in W_x \subset B \cap C$. Then $U \cap V = \bigcup_{x \in U \cap V} W_x$. We have written $U \cap V$ as a union of sets in $\mathcal{B}$, hence $U \cap V \in \mathcal{T}$.

(T3): If $U_i \in \mathcal{T}$ for some $i \in \mathcal{I}$. Each U_i is a union of sets in $\mathcal{B}$, so $\bigcup_{i \in \mathcal{I}} U_i$ is a union of sets in $\mathcal{B}$. Therefore $\bigcup_{i \in \mathcal{I}} U_i \in \mathcal{T}$.

Hence we have a topology. □

We can compare two topologies on the same set X.

Definition 3.4 (Coarse/Fine)**.** Let $\mathcal{T}_1, \mathcal{T}_2$ be topologies on X. We say $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$ (or weaker) if every open set of $\mathcal{T}_1$ is an open set in $\mathcal{T}_2$. We also say that $\mathcal{T}_2$ is finer than $\mathcal{T}_1$.

On any X, the coarsest topology is the indiscrete topology and the finest is the discrete topology.

Example. Let $X = \{1, 2\}$, we can ask what are the topologies on X? The subsets of X are $\varnothing$, $\{1\}$, $\{2\}$ and $\{1, 2\}$. Any topology of X contains $\varnothing$ and $\{1, 2\}$ so the possible topologies are $\mathcal{T}_1 = \{\varnothing, \{1, 2\}\}$ (indiscrete topology), $\mathcal{T}_2 = \{\varnothing, \{1\}, \{1, 2\}\}$, $\mathcal{T}_3 = \{\varnothing, \{2\}, \{1, 2\}\}$, $\mathcal{T}_2 = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$ (discrete topology).

We can say that $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$, $\mathcal{T}_3$ and $\mathcal{T}_4$. $\mathcal{T}_2$ is finer than $\mathcal{T}_1$ and coarser than $\mathcal{T}_4$, similarly for $\mathcal{T}_4$. We say $\mathcal{T}_4$ is finer than $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$. $\mathcal{T}_2$ and $\mathcal{T}_3$ are not comparable as neither is coarser than the other.

3.2 Closed Sets in a TS

Definition 3.5 (Closed)**.** A subset A of a topological space X is closed if its complement $X \setminus A$ is open.

Note that $\varnothing$ and X are closed. So a set can be both open and closed. It is also to have a set that is neither. USing demorgans laws for sets,

$$\bigcup_{i\in\mathcal{I}} X \setminus U_i = X \setminus \left(\bigcap_{i\in\mathcal{I}} U_i\right) \qquad \bigcap_{i\in\mathcal{I}} X \setminus U_i = X \setminus \left(\bigcup_{i\in\mathcal{I}} U_i\right)$$

and the properties of open sets, we can show

Proposition 3.6. (i) An arbitrary intersection of closed sets is closed

(ii) A finite union of closed sets is closed.

Proof. (i) Let C_i for $i \in \mathcal{I}$ be an arbitrary collection of closed sets in X. Then,

$$X \setminus \left(\bigcap_{i\in\mathcal{I}} U_i\right) = bigcup_{i\in\mathcal{I}} X \setminus U_i$$

Since the sets $X \setminus C_i$ are open, so is their union. Hence $\bigcap_{i\in\mathcal{I}} C_i$ is closed.

(ii) **Exercise** □

Again, the union of an infinite family of closed sets need not be closed.

3.3 Convergence and Continuity

Definition 3.7 (Limit of a sequence)**.** Let a_n, $n \geq 1$ be a sequence of points in a topological space X. We say that a_n converges to a point $a \in X$, written $a_n \to a$ as $n \to \infty$, if, for every open set U of X with $a \in U$, there is some $N \in \mathbb{N}$ such that $a_n \in U$ for all $n > N$.

Example. Let X be a topological space with the indiscrete topology (the only open sets are $\varnothing$ and X). Then every sequence (a_n) in X converges to every point $a \in X$. For, given an open set U containing a, we must have $U = X$, and then $a_n \in X$ for all n.

Remark. If X is a metric space, viewed as a topological space with topology given by it's metric, then the two definitions agree.

Definition 3.8 (Continuous)**.** A function $f : X \to Y$ between topological spaces is continuous if, for every open set U of Y, the subset $f^{-1}(U)$ is an open subset X.

Example. Let $f : \mathbb{R} \to \mathbb{R}$,

$$\begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

is not continuous since, for the open set $U = \left(\frac{1}{2}, \frac{3}{2}\right)$ we have $f^{-1}(U) = [0, \infty)$

Here's a slightly more interesting example,

Example. Let $X = (\mathbb{R}, \mathcal{T}_d)$ and let $Y = (\mathbb{R}, \mathcal{T}_u)$ where $\mathcal{T}_d$ is the discrete topology and $\mathcal{T}_u$ is the usual topology on $\mathbb{R}$. Let $f : X \to Y$ and $f : Y \to X$ be the identity map on $\mathbb{R}$.

Then f is continuous, for if $U \subset Y$ is open then $f^{-1}(U) = U$ is certainly open in X. However g is not continuous, the set $V = \{0\}$ is open in Y (because every set is open in Y) but $f^{-1}(V)$ is not open in X (since $\{0\}$ is not an open set in the usual topology.)

Lemma 3.9. If $f : X \to Y$ and $g : Y \to Z$ are continuous maps between topological spaces, then $g \circ f : X \to Z$ is continuous

Proof. Let U be an open set in Z. Then $g^{-1}(U)$ is an open set in A since g is continuous, and therefore $f^{-1}(g^{-1}(U))$ is an open set in X since f is continuous. But,

$$\begin{aligned} f^{-1}(g^{-1}(U)) &= \{x \in X : f(x) \in g^{-1}(U)\} \\ &= \{x \in X : g(f(x)) \in U\} = (g \circ f)^{-1}(U) \end{aligned}$$

Hence $g \circ f$ is continuous. □

Continuous functions should be thought of as the structure-preserving functions between topological spaces, in the same as we have homomorphisms between groups, and linear maps between vector spaces. An isomorphism of topological spaces is called a homeomorphism.

Definition 3.10 (Homeomorphism)**.** A homeomorphism between topological dpaces X and Y as a continuous function $f : X \to Y$ which is bijective and whose inverse function $f^{-1} : Y \to X$ is also continuous. We say that X and Y are homeomorphic if there is a homeomorphism between them.

Example. The intervals $(0, 1)$ and $(0, \infty)$ in $\mathbb{R}$ (usual topology) are homeomorphic. Indeed, consider $f : (0, 1) \to (0, \infty)$ with

$$f(x) = \frac{1-x}{x}$$

This is well defined and continuous, and is bijective with continuous inverse $g : (0, \infty) \to (0, 1)$ with,

$$g(y) = \frac{y}{1+y}$$

The inverse of a homeomorphism is again, a homeomorphism, but a continuous bijection is not necessarily a homeomorphism.

Example. We have seen that $(\mathbb{R}, \mathcal{T}_d) \to (\mathbb{R}, \mathcal{T}_u)$ is a continuous bijection whose inverse is not continuous. So it is not a homeomorphism.