Year 3 — Number Theory

Based on lectures by Professor Henri Johnston
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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Contents

1 Divisibility

1.1 Division Algorithm

Definition 1.1 (Well Ordering Principle)**.** Every non-empty subset of $\mathbb{N}_0$ contains a least element

Theorem 1.2 (Division Algorithm)**.** Given a $a \in \mathbb{Z}$ and a $b \in \mathbb{N}_1$ there exists unique integers q and r satisfying $a = bq + r$ and $0 \leq r < b$.

The proof splits into uniqueness and existence.

Proof. We shall first prove existence, define $S := \{a - xb : x \in \mathbb{Z} \quad \text{and } a - xb \geq 0\}$. We know $S \neq 0$ since,

- if $a \geq 0$, then choose $m = 0$, them $a - mb = a \geq 0$
- if $a < 0$, then let $a = m$, so $a - mb = a - ab = (-a)(b-1) \geq 0$ since $-a > 0$ and $b > 0$[1]

Hence S is non-empty subset of $\mathbb{N}_0$ and so by the well ordering principle S must contain a least element $r \geq 0$. Since $r \in S$, then we have there exists a $q \in \mathbb{Z}$ such that $a - qb = r$ and so $a = qb + r$. Now it remains to check that $r < b$, so assume for a contradiction that $r \geq b$, then let there be a $r_1 = r - b \geq 0$. Then,

$$a = qb + r = qb + (r_1 + b) = (q+1)b + r_1$$

and so $a - (q+1)b = r_1 \in S$ and is smaller than r, a contradiction.

Now let us show uniqueness, assume that there exist another pair q', r' such that $a = q'b + r'$ where $0 \leq r' < b$. Then form $a = a + qb + r = q'b + r'$ we have that, $(q - q')b = r' - r$. If $q = q'$, then we must have $r = r'$, suppose for a contradiction that this isn't true, then,

$$b \leq |q - q'||b| = |r - r'|$$

However, since $0 \leq r, r' < b$ and so $|r - r'| < b$ which gives a contradiction. $\square$

Here's a definition that I feel is useful that wasnt covered in the lectures,

Definition 1.3 (Divisible)**.** We say that some $a \in \mathbb{Z}$ is divisible by some $b \in \mathbb{Z}$ if and only is,

$$\exists\, n \in \mathbb{Z}, \text{ such that } b = na$$

and denote it, $a \mid b$

1.2 Greatest Common Divisor

Let us start with a theorem.

Theorem 1.4. Let $a, b \in \mathbb{Z}$, $\exists\, d \in \mathbb{N}_0$ and non-unique $x, y \in \mathbb{Z}$ such that,

(i) $d \mid a$ and $d \mid b$

(ii) and if $e \in \mathbb{Z}$, $e \mid a$ and $e \mid b$, then $e \mid d$

(iii) $d = ax + by$

[1] You absolute plank, there doesn't exist any numbers between 0 and 1 in $\mathbb{Z}$, so $b > 0$ is the same as $b \geq 1$

Proof. If $a = b = 0$, then $d = 0$
Suppose that $a \neq b \neq 0$, then let

$$S := \{am + bn : m, n \in \mathbb{Z} \text{ and } am + bn > 0\}$$

Now $a^2 + b^2 > 0$ so S is non-empty and a subset of $\mathbb{N}_1$. Hence, by the Well ordering principle then there must be some minimum element d. Then we can write $d = ax + by$ by definition of S.

By the division Algorithm, $a = qs + r$ for some $q, r \in \mathbb{Z}$ with $0 \leq q < d$. Suppose for a contradiction that $r \neq 0$. Then,

$$0 < r = a - qd = a - q(ax + by) = (1 - qx)a - qby$$

Hence, $r \in S$. But $r < d$, contradiciting the minimality of d in S. So we must have $r = 0$, i.e $d \mid a$. The same works for $d \mid b$.

Suppose that $e \in \mathbb{Z}$, $e \mid a$ and $e \mid b$. Then e divides any linear combination of a and b, so $e \mid d$. Suppose that $e \in \mathbb{N}_1$ also satisfies (i) and (ii). Then, $e \mid d$ and $d \mid e$ and so $d = \pm e$, but $d, e \geq 0$ and so $d = e$. Thus d is unique. $\square$

Note that this is a standard trick to prove that integers divide, by just proving that $r = 0$ by contradiction.

Corollary 1.5. If $a, b \in \mathbb{Z}$ then there exists a unique $d \in \mathbb{N}_1$ such that.

(i) $d \mid a$ and $d \mid b$

(ii) if $e \in \mathbb{Z}$, then $e \mid a$ and $e \mid b$ then $e \mid d$

Proof. The existence of a d is given by the theorem. In the proof of uniqueness we only use (i) and (ii). $\square$

Definition 1.6 (Greatest Common Divisor)**.** Let $a, b \in \mathbb{Z}$. Them d of the previous corollary is just the greatest common divisor of a and b, written $\gcd(a, b)$. Also sometimes seen as $\text{hcf}(a, b)$.

If $\gcd(a, b) = 1$, then a and b are coprime.

Identity (Bezouts Identity)**.** Given $a, b \in \mathbb{Z}$ there exist $x, y \in \mathbb{Z}$ such that $\gcd(a, b) = ax + by$.

Proposition 1.7. Let $a, b, c \in \mathbb{Z}$, then,

(i) $\gcd(a, b) = \gcd(b, a)$

(ii) $\gcd(a, \gcd(b, c)) = \gcd(\gcd(a, b), c)$

(iii) $\gcd(ac, bc) = |c| \gcd(a, b)$

(iv) $\gcd(1, a) = \gcd(a, 1) = a$

(v) $\gcd(0, a) = \gcd(a, 0) = |a|$

(vi) $c \mid \gcd(a, b)$ if and only if $c \mid a$ and $c \mid b$

(vii) $\gcd(a + cb, b) = \gcd(a, b)$

Then we can consider the following remark,

Remark. Note that $\gcd(a, b) = 0$ if and only if, $a = b = 0$. Otherwise, $\gcd(a, b) \geq 1$.

Proof. Checking these properties are pretty simple, for (vi) just use Bezouts.

We shall prove (iii), so let $d = \gcd(a, b)$ and $e = \gcd(ac, bc)$. By (vi), $cd \mid e = gcd(ac, bc)$ since $cd \mid ac$ and $cd \mid bc$. Then by Bezouts, there exists $x, y \in \mathbb{Z}$ such that $d = ax + by$. Then,

$$cd = acx + bcy$$

and as $e \mid ac$ and $e \mid bc$ and so by linearity we have $e \mid cd$. Therefore, $|e| = |cd|$ and so, $e = |c|d$.

Now, let's prove (vii), let $e = \gcd(a + bc, b)$ and $f = \gcd(a, b)$. Then $e \mid (a + bc)$ and $e \mid b$. Thus by linearity, we have $e \mid a$. Hence, $e \mid a$ and $e \mid b$ so by property (vi), we have $e \mid f$. Similarly we can get that $f \mid a + bc$ and $f \mid b$ and so again my (vi) we have $e = f$ as $f, e \geq 0$. ☐

Lemma 1.8 (Euclids Lemma)**.** Let $a, b, c \in \mathbb{Z}$. If $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$.

Proof. Suppose that $a \mid bc$ and $\gcd(a, b) = 1$. By Bezouts, we get that for some $x, y \in \mathbb{Z}$ we get $1 + ax + by$. Hence, $c = acx + bcy$, but $a \mid acx$ and $a \mid bcy$, so $a \mid c$ by linearity. ☐

Theorem 1.9 (Solubility of linear equations in $\mathbb{Z}$)**.** Let $a, b, c \in \mathbb{Z}$. The equation,

$$ax + by = c$$

is soluble with $x, y \in \mathbb{Z}$ if and only if $\gcd(a, b) = c$

Proof. Let $d = \gcd(a, b)$. Then $d \mid a$ and $d \mid b$ so if there exists $x, y \in \mathbb{Z}$ such that $c = ax + by$ then $d \mid c$ by linearity of divisibility. Now, suppose that $d \mid c$. Then we can write $c = qd$ for some $q \in \mathbb{Z}$. By Bezouts, there exists some $x', y' \in \mathbb{Z}$ such that $d = ax' + by'$. Hence, $c = qd = aqx' + bqy'$ and so $x = qx'$ and $y = qy'$ gives a suitable solution. ☐

1.3 Euclids Algorithm

Theorem 1.10 (Euclids Algorithm)**.** Let $a, b \in \mathbb{N}_1$ with $a > b > 0$ and $b \nmid a$. Let $r_0 = a$, $r_1 = b$ and apply the division Algorithm repeatedly to obtain a sequence of remainders defined sucessively,

$$\begin{aligned} r_0 &= r_1 q_1 + r_2 & \quad & 0 < r_2 < r_1 \\ r_1 &= r_2 q_2 + r_3 & & 0 < r_3 < r_2 \\ &\vdots \\ r_{n-2} &= r_{n-1} q_{n-1} + r_n & & 0 < r_n < r_{n-1} \\ r_{n-1} &= r_n q_n + r_{n+1} & & r_{n+1} = 0 \end{aligned}$$

Then the last non-zero remainder, r_n is the $\gcd(a, b)$.

Proof. There is a stage at which $r_{n+1} = 0$ because the r_i are strictly decreasing non-negative integers. We have,

$$\begin{aligned} \gcd(r_i, r_{i+1}) &= \gcd(r_{i+1} q_{i+1} + r_{i+2} r_{i+1}) \\ &= \gcd(r_{i+2} r_{i+1}) \\ &= \gcd(r_{i+1}, r_{i+2}) \end{aligned}$$

Applying this result repeatedly,

$$\begin{aligned} \gcd(a, b) &= \gcd(r_0, r_1) \\ &= \gcd(r_2, r_3) \\ &= \dots \\ &= \gcd(r_{n-1}, r_n) \\ &= r_n \end{aligned}$$

Where the last equality is because $r_n \mid r_{n-1}$ □

Remark. One can also use Euclids Algorithm to find the $x, y \in \mathbb{Z}$ Bezouts Identity state to exist by working backwards. These aren't unique.

1.4 Extended Euclidean Algorithm

Instead of doing Euclids, and working backwards we can compute our bezouts x, y during euclids. This is the extended Euclids Algorithm. This time we are going to define sequnces of integers x_i and y_i, such that $r_i = ax_i + by_i$. Recall that r_n is the last non-zero remainder and that $r_n = \gcd(a, b)$. Therefore $\gcd(a, b) = r_n = ax_n + by_n$ and so $(x, y) := (x_n, y_n)$.

We have that $r_0 = a$ and $r_1 = b$. Hence, we see $r_0 = 1 \times a + 0 \times b$ and $r_1 = 0 \times a + 1 \times b$, and so we set $(x_0, y_0) := (1, 0)$ and $(x_1, y_1) := (0, 1)$. So, now we consider for $i \geq 2$ we have a pair (x_j, y_j) for $j < i$. Then $r_{i-2} = r_{i-1}q_{i-1} + r_i$ and so,

$$\begin{aligned} r_i &= r_{i-2} - r_{i-1}q_{i-1} \\ &= (ax_{i-2} + by_{i-2}) + (ax_{i-1} + by_{i-1})q_{i-1} \\ &= a(x_{i-2} - x_{i-1}q_{i-1}) + b(y_{i-2} - y_{i-1}q_{i-1}) \end{aligned}$$

Thus we set $x_i := x_{i-2} - x_{i-1}q_{i-1}$ and $y_i := y_{i-2} - y_{i-1}q_{i-1}$. These can be defined recursively this way.

$$(x_i, y_i) := (x_{i-2}, y_{i-2}) - q_{i-1}(x_{i-1}, y_{i-1})$$

Example. We compute $\gcd(841, 160)$ use Extended Euclidean Algorithm.

i	r_{i-2}		r_{i-1}		q_{i-1}		r_i	x_i	y_i
0							841	1	0
1							160	0	1
2	841	=	160	×	5	+	41	1	-5
3	160	=	41	×	3	+	37	-3	16
4	41	=	37	×	1	+	4	4	-21
5	37	=	4	×	9	+	1	-39	205
6	4	=	1	×	4	+	0		

Therefore, $\gcd(841, 160) = 1 = 841 \times (-39) + 160 \times 205$.