Year 3 — Mathematical Biology and Ecology

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Autumn Term 2021

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Contents

1 Continuous Models for a single species

We are going to model simply how we model population dynamics for a single species. We are going to call $N(t)$ our population size at a certain time. We are going to say that $N(t)$ is continuous. We also say $N \in \mathbb{R}$ and so it's going to be a density measure. We are going to constrain this $N \geq 0$, $\forall t$. We can measure this and create a model, *Lecture 2*

$$\frac{dN}{dt} = f(N, t, \boldsymbol{\mu})$$

We call N the variable, then $\boldsymbol{\mu}$ the parameter. We let $t \in \mathbb{R}$ and $t > 0$.

Let's start off by thinking about an actual population of individuals, we have observed that there is some sort of growth dynamics. We can write a mechanistic model, by including mechanisms that affect the population.

$$\frac{dN}{dt} = +\text{births} + \text{resources} + \text{net migration} - \text{deaths}$$

If the positive things are greater, the population will grow, otherwise if they are smaller they decline, or they stay equal.

Now we make some assumptions, so we can write down a mathematical model.

(i) Births and deaths predominate:

$$\frac{dN}{dt} = \text{births} - \text{deaths}$$

(ii) Births and deaths are proportional to N:

$$\frac{dN}{dt} = \alpha N - \beta N \qquad \alpha, \beta \in \mathbb{R}^+$$

where we call α the birthrate and β the deathrate and $\boldsymbol{\mu} = (\alpha, \beta)$.

> If we consider, as an aside,
>
> $$\frac{dN}{dt} = \alpha N - \beta N + \gamma \qquad \gamma \in \mathbb{R}$$
>
> this adds something like independent migration, where we assume migration is constant, this could be γN if it's proportional to N.

We can solve this equation nicely,

$$\begin{aligned} \frac{dN}{dt} &= N(\alpha - \beta) \\ \int \frac{1}{N} \frac{dN}{dt} \, dt &= \int (\alpha - \beta) dt \\ \ln N &= (\alpha - \beta)t + C \\ N(t) &= Ae^{(\alpha - \beta)t} \end{aligned}$$

Now assume this is an initial value problem, $N_0 = A$ and so,

$$N(t) = N_0 e^{(\alpha - \beta)t}$$

We are also interested in the long term dynamics of these models, the asymptotic dynamics. Hence, we consider $t \to \infty$.

$$\lim_{t \to \infty} N(t) = \begin{cases} \infty & \text{if } \alpha > \beta \\ 0 & \text{if } \alpha < \beta \\ N_0 & \text{if } \alpha = \beta \end{cases}$$

We call the case where $\alpha = \beta$ a steady state solution.

You don't usually see just a t in the equation, but we may want to use a forcing term, i.e. periodic migration in Cornwall.

$$\frac{dN}{dt} = \alpha N - \beta N + \cos(t)$$

We are not going to consider these non-autonomous systems. Hence we can write,

$$\frac{dN}{dt} = f(N, \boldsymbol{\mu})$$

We have a nice thing to make sure our growth doesn't go exponential. It's the logistic map,

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{k}\right) \qquad r, k > 0 \quad N \geq 0$$

where $f(N) = rN\left(1 - \frac{N}{k}\right)$ is called the logistic model. This model has a level of self regulation, so if N is high, then it will decrease later. Firstly, look at $f(N)$,

$$f(N) = rN\left(1 - \frac{N}{k}\right)$$

We start by graphing it,

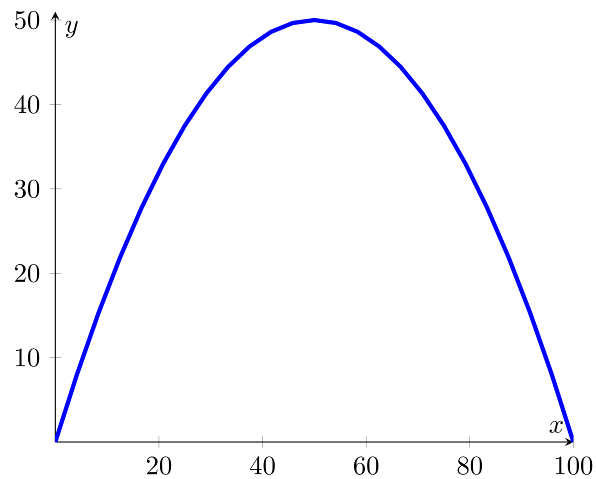

Figure 1: $y = 2x\left(1 - \frac{x}{100}\right)$

Exercise. Solve $\frac{dN}{dt} = rN(1 - \frac{N}{k})$

The solution to this equation is,

$$N(t) = \frac{N_0 k e^{rt}}{k - N_0 + N_0 e^{rt}}$$

As $t \to \infty$, we can consider it and see that it will depend on N_0. If $N_0 = 0$, then we get that $N(t) = 0, \forall t$. If we then take $N_0 > 0$, then we get,

$$\begin{aligned} N(t) &= \frac{N_0 k}{\frac{k - N_0}{e^{kt}} + N_0} \\ &= k \qquad\qquad\qquad t \to \infty \end{aligned}$$

lets try and formalise some of these ideas. So consider,

$$\frac{dN}{dt} = f(N)$$

where $N \in \mathbb{R}, t \in \mathbb{R}, f : \mathbb{R} \to \mathbb{R}$. Then we define steady states as, where $f(N^*) = 0$. These are also reffered to as fixed points or equilibrium. When a system in the first place depends on whether the state is attracting.

Definition 1.1 (Attrracting)**.** A steady state N^* is attracting if all trajectories that start close to N^* approach it as $t \to \infty$.

We consider $N = N^* + n$, recall the taylor series is an approximation to a function at a point,

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \dots$$

as we consider small values of n we can throw away higher terms. Hence, we can linearise it.

$$\begin{aligned}
\frac{dN}{dt} &= f(N) \\
\frac{dN^* + n}{dt} &= f(N^* + n) \\
\frac{dn}{dt} &= f(N^* + n) \\
\frac{dn}{dt} &\approx f(N^*) + f'(N^*)(N^* + n - N^*) + f''(a)\frac{(N^* + n - N^*)}{2!} + \dots \\
\frac{dn}{dt} &\approx 0 + f'(N^*)n + \frac{f''(N^*)}{2!}n^2 + \dots \\
\frac{dn}{dt} &\approx f'(N^*)n
\end{aligned}$$

and so we can model it by,

$$\frac{dn}{dt} = f'(N^*)n$$

which can be solved as,

$$n(t) \approx n_0 e^{f'(N^* t}$$

If $f'(N^*) > 0$, then we just have an exponential (unstable), but if $f'(N^*) < 0$ then we have a decaying exponential (stable).

Example. We shall consider,

$$\frac{dx}{dt} = rN\left(1 - \frac{N}{k}\right)$$

and so we want $f(N^*) = 0$ and hence we get that $N^* = 0$ and $N^{**} = k$. Then we find $f'(N) = r\left(1 - \frac{2N}{K}\right)$. Hence we can find $f'(0) = r$ and $r'(k) = -r$ and as $r > 0$, $f'(N^*) > 0$ hence N^* is unstable and as $f(N^{**}) < 0$ then N^{**} is stable.