The Real Number System

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1 Overview

The reals ($\mathbb{R}$) have a few properties:

1. They are a field, i.e. a groupoid with two binary operations.
2. They are ordered
3. They are also complete.

We will also look at supremum and the infimum.

We are also going to look at the extended real numbers. We are going to add two more fictitious points. $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$.

2 Properties of the Reals

We will be taking the axiomatic view point of the real numbers. No construction with Dedekind cuts or Cauchy sequences. All of these are isomorphic.

2.1 Field Properties

The real numbers are a set, $\mathbb{R}$, with two binary operations, $+$ and $\times$. They must satisfy the following axioms. So take $a, b, c \in \mathbb{R}$:

1. $a + b = b + a$ and $ab = ba$ (commutativity)
2. $(a + b) + c = a + (b + c)$ and $a(bc) = (ab)c$ (associativity)
3. $a(b + c) = ab + ac$ (distributivity)
4. There are two distinctive identities 0 (additive identity) and 1 (multiplicative identity), such that $a + 0 = 0 + a = a$ and $a1 = 1a = a$
5. We also have inverses, $-a$ (additive inverse) such that $a + -a = 0$ and if $a \neq 0$, there is a real number $\frac{1}{a}$ such that: $a(\frac{1}{a}) = 1$

2.2 Order Relation

The real numbers are ordered, that means:

1. For each pair of reals a and b, exactly one of the following is true
$$a = b \qquad a < b \qquad b < a$$
2. It is also transitive, if $a < b$ and $b < c$, then $a < c$
3. If $a < b$ then $a + c < b + c$ for any c, and if $0 < c$, then $ac < bc$

2.3 Supremum

Let $S \subset \mathbb{R}$. If there exists $b \in \mathbb{R}$ such that $x \leq b \quad \forall x \in S$ then S is bounded above and b is an upper bound of S.

If β is an upper bound of S, but no number less than β is, then β is called the supremum of S, denoted:

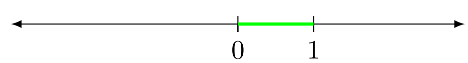
$$\beta = \sup S$$

Figure 1: *Let S be the orange set and then b is an upper bound of S and β is $\sup S$*

We also call the supremum the least upper bound.

Example 1. *$S = [\,0, 1\,]$ and prove sup $S = 1$*

Solution 1. *Take our diagram from above:*



We need to check that $x \leq 1 \quad \forall x \in S$, which is definitionally true.
Secondly we need to prove that $\forall\, b < 1, \exists\, x \in S, b < x$, which is again trivially true. So $\sup S = 1$

Example 2. *Take $T = (\,0, 1\,)$ where $\sup T = 1$*

Solution 2. *Again every number is less than 1, but if you take any number less than one you can always find another element larger.*
NB: The supremum here isn't in the set

2.4 Infimum

Similarly, if there exists an $a \in \mathbb{R}$ such that $a \leq x \quad x \in S$, then S is bounded below and a is a lower bound of S.

If α is a lower bound of S, but no number is greater than α is, then α is called the infimum of S:

$$\alpha = \inf\ S$$

Figure 2: *Let S be the orange set and then a is a lower bound of S and α is* $\inf S$

Another name for the infimum is the greatest lower bound.

2.5 Completeness Axiom

Do the supremum and the infimum actually exist? Well, not all subsets are bounded above, i.e. $\mathbb{R} \subset \mathbb{R}$ or what about the empty set? This is what the completeness axiom does:

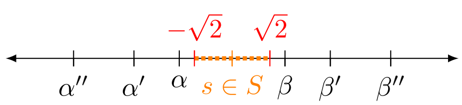
1. If a non-empty set of real numbers are bounded above, then it has a supremum.

So the reals are a **complete ordered field**

The completeness axiom is distinguishing of the reals. They are the only complete ordered field. The rationals possess everything but completeness in terms of our axioms.

Example 3. *We restrict to the $\mathbb{Q}$, $S = \{r \in \mathbb{Q} : r^2 < 2\}$. Find the supremum and infimum.*

Solution 3. *If we take the example below;*



we can say that we won't reach $\sqrt{2}$ in the supremum or $-\sqrt{2}$ in the infimum. This is because we are using rationals and $\sqrt{2}$ is an irrational. We can go either way and there is always a number closer to $\sqrt{2}$.

This proves that rationals are not complete.

3 Extended Real Numbers

It is convenient to attach ∞ and $-\infty$ to the reals. How do they fit in? Firstly lets look at orders. Take $x \in \mathbb{R}$, then:

$$-\infty < x < \infty$$

Now if a set S is unbounded above or below, we can write:

$$\sup S = \infty \quad \inf S = -\infty$$

Example 4. *Find the infimum of $S = \{x \in \mathbb{R} : x : 2\}$*

Solution 4. *As there is technically no lower bound, it is $-\infty$*

We usually denote the extended reals with the symbol, $\overline{\mathbb{R}}$ or $[-\infty, \infty]$ or $\mathbb{R} \cup \{-\infty, \infty\}$

3.1 Arithmetic

If $a \in \mathbb{R}$,

1. Then:

$$a + \infty = \infty + a = \infty$$
$$a - \infty = -\infty + a = -\infty$$
$$\frac{a}{\infty} = \frac{a}{-\infty} = 0$$

2. and $0 < a$, then:

$$a\infty = \infty a = \infty$$
$$a(-\infty) = (-\infty)a = -\infty$$

3. and $a < 0$, then:

$$a\infty = \infty a = -\infty$$
$$a(-\infty) = (-\infty)a = \infty$$

We also define:

1. $\infty + \infty = \infty\infty = (-\infty)(-\infty) = \infty$
2. and also $-\infty - \infty = \infty(-\infty) = (-\infty)\infty = -\infty$
3. and finally, $|\infty| = |-\infty| = \infty$

We say it isn't useful to define; $\infty - \infty$, $0 \cdot \infty$, $\dfrac{\infty}{\infty}$ and $\dfrac{0}{0}$. We call them indeterminate forms.

4 Triangle Inequality

As we can use the ordered relation of the reals we can produce something known as the triangle inequality.

Theorem 1. *It states for any $a, b \in \mathbb{R}$, we have:*

$$|a+b| \leq |a| + |b|$$

Proof. There are four possibilities:

1. If $0 \leq a$ and $0 \leq b$, then $0 \leq a+b$, so $|a+b| = a+b = |a|+|b|$.
2. If $a \leq 0$ and $b \leq 0$, then $a+b \leq 0$, so $|a+b| = -a+(-b) = |a|+|b|$.
3. If $0 \leq a$ and $b \leq 0$, then $a+b = |a|-|b|$.
4. If $a \leq 0$ and $0 \leq b$, then $a+b = -|a|+|b|$.

It holds in cases (c) and (d), since

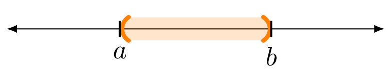
$$|a+b| = \begin{cases} |a|-|b| & \text{if } |b| \leq |a|, \\ |b|-|a| & \text{if } |a| \leq |b|. \end{cases}$$

□

5 Open and Closed Sets

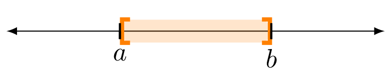
Definition 1. *We define an open interval between a and b, $a, b \in \overline{\mathbb{R}}$, as such:*

$$(a,\, b) = \{x : a < x < b\}$$



Definition 2. *We define a closed interval between a and b, $a, b \in \overline{\mathbb{R}}$, as such:*

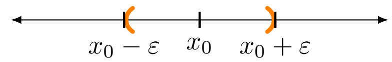
$$[a,\, b] = \{x : a \leq x \leq b\}$$



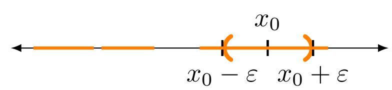
5.1 Neighbourhoods

A neighborhood is used to talk about closeness of points. We are now going to go through a load of set definitions!

Definition 3. *If x_0 is a real number and $\varepsilon > 0$, then the open interval $(x_0-\varepsilon, x_0+\varepsilon)$ is an ε-neighbourhood of x_0.*



Definition 4. *If a set S contains an ε-neighbourhood of x_0, then S is a neighborhood of x_0. i.e. we need $(x_0-\varepsilon,\, x_0+\varepsilon) \subset S$*



Definition 5. *If S is a neighbourhood of x_0, then x_0 is an interior point of S.*

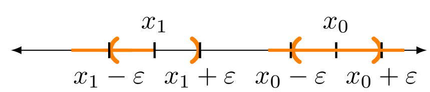


Figure 3: x_0 is an interior point, however x_1 is not

Definition 6. *The set of interior points of S is the interior of S, denoted by S^0*

Definition 7. *If every point of S is an interior point, $(S^0 = S)$, then S is open .*

Definition 8. *A set is closed if S^c is open.*

Example 5. *Any open interval $S = (a,\, b)$ is open.*

Solution 5. *Need to show that $\forall x_0 \in (a,\, b),\ \exists \varepsilon > 0 : (x_0 - \varepsilon, x_0 + \varepsilon) \subset (a,\, b)$ Assume that $a, b \in \mathbb{R}$. Let $x_0 \in (a,\, b)$ and let $\varepsilon = \min(x_0 - a, b - x_0)$. Then clearly $(x_0 - \varepsilon,\, x_0 + \varepsilon) \subset (a,\, b)$.*

The rest of the proof is left as an exercise. (Where $a = -\infty$ or $b = \infty$).

Now we know that $\mathbb{R}$ is open and S^c, where $S^c = (-\infty,\, a) \cup [b, \infty)$, and $\varnothing$ is closed

We also note that because of a vacuity argument $\varnothing$ is also open, hence $\mathbb{R}$ is also closed. So $\mathbb{R}$ and $\varnothing$ are both open and closed.

5.2 Unions and Intersections

Theorem 2. *1. The union of open sets is open*

2. The intersection of closed sets is closed

These apply to abtritary collections (finite or infinite of open and closed sets).

Proof. First lets prove (1), so let $\mathcal{G}$ be a collection of open sets.
Let $S = \bigcup_{G \in \mathcal{G}} G$, If $x_0 \in S$, then $x_0 \in G_0$ for some $G_0 \in \mathcal{G}$. Since G_0 is open, it must contain an ε- neighborhood of x_0. The ε-neighborhood, $(x_0 - \varepsilon, x_0 + \varepsilon)$, is in S, hence S is a neighborhood of x_0 and x_0 is an interior point of S.
Since x_0 was arbitrary, then all points in S are interior points and hence, S is open.
Now for part (2) of the theorem. Let $\mathcal{F}$ be a collection of closed sets and let $T = \bigcap_{F \in \mathcal{F}} F$. Then $T^c = \bigcup_{F \in \mathcal{F}} F^c$.
Since each F^c is open, that means T^c is open by (1).
Therefore T is closed $\square$

Example 6. *For $a, b \in \mathbb{R}$, the sets $[a,\, b]$ is closed.*

Solution 6. *Since $[a,\, b]^c = (-\infty,\, a) \cup (b, \infty)$. Since its a union of open intervals, it is open. Hence making $[a,\, b]$ closed.*

Example 7. *What about $[a, b)$, or $(a, b]$ for $a, b \in \mathbb{R}$?*

Solution 7. *These are half-open or half-closed intervals. These are neither open nor closed. Take $[a,\, b)$, then a isn't an interior point of the set, hence it's not open. Now take the compliment of the set $[a,\, b)^c = (-\infty,\, a) \cup [b,\, \infty)$ and now b is no longer an interior point. Hence, not closed.*

Example 8. *What about: $(-\infty,\, a]$ or $[a,\, \infty)$*

Solution 8. *Exercise*

Now, what about the intersection of open sets and union of closed sets. Well, it can be proved that the intersection of finitely many open sets is open and union of finitely many closed sets is closed. However the infinite versions of these statements need not be the same.

The concept of open and closed sets, doesn't form a dichotomy (A set is partitioned into two. i.e. odd or even naturals). A set can be neither open or closed or both.