Integration of Functions of Several Variables

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Consider a function $f(x, y)$ defined on a closed rectangle

$$R = \{a \leq x \leq b,\ c \leq y \leq d\}.$$

The graph $f(x, y)$ is a surface with the equation $z = f(x, y)$.

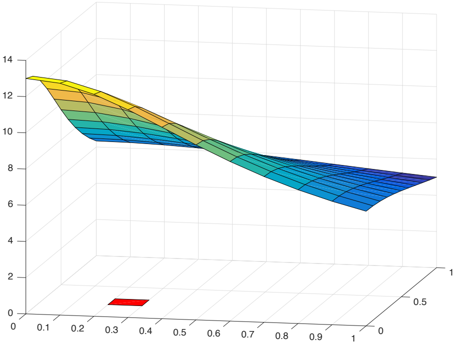

Figure: Surface of $z = f(x, y)$ - Looking to compute the volume below the surface, we borrow partition ideas from Riemann Sums

Using ideas from Riemann sums we form "rectangular partitions"

$$[a, b] : x_0 = a, \ldots, x_m = b; \qquad [c, d] : y_0 = c, \ldots, y_n = d$$

and then look to approximate the volume by

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A_{ij}, \qquad \Delta A_{ij} = (x_i - x_{i-1})(y_j - y_{j-1})$$

As with Riemann sums for scalar valued functions: (x_{ij}^*, y_{ij}^*) is any sample point in the (red) rectangle $\{x_{i-1} \le x \le x_i, y_{j-1} \le y \le y_j\}$
Then the double integral of f over the rectangle
$R = \{a \le x \le b,\ c \le y \le d\}$ is:

$$I = \int \int_R f(x, y)\, dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A_{ij}$$

when the limit exists and we take finer partitions in the limit.

In the expression

$$I = \int\int_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A_{ij}$$

- the sum is called a double Riemann sum;
- R is the domain of integration, the region of the xy-plane over which the integral is taken;
- $f(x,y)$, a function of two variables, is called the *integrand*;
- dA is the "element of area".
- As with integration of functions of one variable, we use the Riemann sum as a formal definition.
- To compute double integrals, if possible we seek to turn them in to a sequence of "usual" single integrals.

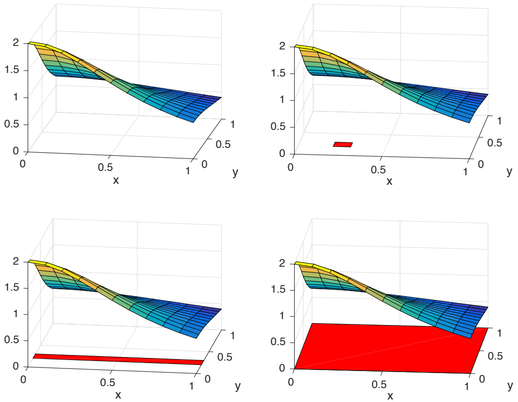

Figure: Double integrals with rectangular domains. **Top right**: Compute volume for a rectangular element. **Bottom left**: Sum up rectangular elements across x range $\to$ Compute volume for an elemental strip. **Bottom right**: Sum up volumes of elemental strips $\to$ Total volume.

From the Riemann approximation

$$V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A_{ij}, \qquad \Delta A_{ij} = (x_i - x_{i-1})(y_j - y_{j-1})$$

we can rewrite this as

$$V \approx \underbrace{\sum_{i=1}^{m} \underbrace{\left(\sum_{j=1}^{n} f(x_i, y_j)(y_j - y_{j-1})\right)}_{\to \int_c^d f(x_i,y)dy} (x_i - x_{i-1})}_{\to \int_a^b \left(\int_c^d f(x,y)dy\right)dx}$$

where we have chosen $(x_{ij}^*, y_{ij}^*) = (x_i, y_j)$ in the rectangle $\{x_{i-1} \leq x \leq x_i, y_{j-1} \leq y \leq y_j\}$

Hence for rectangular domains we can express and compute the double integral as two single integrals:

$$\int\int_R f(x,y)\,dA \;=\; \int_a^b \overbrace{\left(\int_c^d f(x,y)dy\right)}^{\text{function of } x \text{ only}} dx$$

integrate first w.r.t. y, then w.r.t. x

OR

$$\int\int_R f(x,y)\,dA \;=\; \int_c^d \overbrace{\left(\int_a^b f(x,y)dx\right)}^{\text{function of } y \text{ only}} dy$$

integrate first w.r.t. x, then w.r.t. y

Example: Find

$$I = \int_2^3 \int_0^1 x \sin \pi y \, dy \, dx$$

So here we mean the outer x limits are from 2 to 3, whilst the inner y limits are from 0 to 1.
We are integrating over the rectangle

$$\{2 \leq x \leq 3, 0 \leq y \leq 1\}$$

Thus

$$\begin{aligned} I &= \int_2^3 x \left(\int_0^1 \sin \pi y \, dy \right) \, dx \\ &= \int_2^3 x \left[-\tfrac{1}{\pi} \cos \pi y \right]_0^1 dx \\ &= \int_2^3 x \tfrac{2}{\pi} dx \\ &= \left[\tfrac{1}{\pi} x^2 \right]_2^3 = \tfrac{1}{\pi} \left(3^2 - 2^2 \right) = \tfrac{5}{\pi} \end{aligned}$$

Example: Find

$$I = \int_0^3 \left[\int_1^2 x^2 y \, dy \right] \, dx$$

First, regarding x as a constant, integrating w.r.t. y gives:

$$I = \int_0^3 \left[x^2 (y^2/2) \right]_1^2 \, dx = \int_0^3 \left[x^2 (2^2/2) - x^2 (1^2/2) \right]_1^2 \, dx = \int_0^3 \left[(3/2) x^2 \right] \, dx$$

$$I = \int_0^3 \left[(3/2) x^2 \right] \, dx = (3/2) \left[(1/3) x^3 \right]_0^3 = (3/2)(1/3) 3^3 = 27/2$$

We can also integrate with respect to x first (regarding y as a constant)

$$I = \int_1^2 \left[\int_0^3 x^2 y \, dx \right] \, dy = \int_1^2 \left[(x^3/3) y \right]_0^3 \, dy$$

$$= \int_1^2 \left[(3^3/3) y \right] \, dy = \left[(3^3/3)(y^2/2) \right]_1^2 = 3^2 (2^2 - 1^2)/2 = 27/2$$

- To define multiple integrals we first start with rectangular domains
- We partition the rectangle into small rectangles using a regular grid
- We approximate the double integral as a volume made from summing up blocks

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A_{ij}, \qquad \Delta A_{ij} = (x_i - x_{i-1})(y_j - y_{j-1})$$

- We define the double integral as a limit of sum of blocks as we make the partition (grid) finer and finer
- To compute double integrals over rectangular domains we integrate first w.r.t. y (or x) and then w.r.t. x (or y):

$$\int \int_R f(x,y)\, dA \quad = \quad \int_a^b \overbrace{\left(\int_c^d f(x,y) dy \right)}^{\text{function of } x \text{ only}} dx$$

integrate first w.r.t. y, then w.r.t. x

We must (i) identify the domain R in the xy-plane, especially the boundary curves and (ii) decide the order of integration — x first or y first.

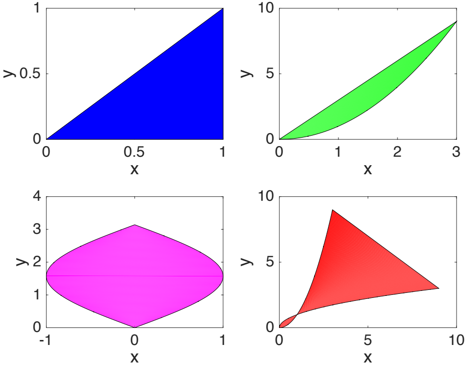

Figure: Top Left: Triangular domain with vertices $(0,0),(1,0),(1,1)$.
Top right: Region between the $y = x^2$ and $y = 3x$.
Bottom left: Region between $x = sin(y)$ and $x = -sin(y)$, $0 \leq y \leq \pi$.
Bottom right: Region between $y = x^2$, $y = \sqrt{x}$ and $x + y = 12$.

Type I: A plane region R lies between the graphs of two continuous functions of x (green figure above)

$$a \leq x \leq b,\ y_1(x) \leq y \leq y_2(x) \text{ with } y_1(a) = y_2(a), y_1(b) = y_2(b)$$

where we assume that $y_2(x) \geq y_1(x)$ for all $x \in [a, b]$.
In this case we fix x between a and b and integrate first w.r.t. y from $y = y_1(x)$ to $y = y_2(x)$, and then w.r.t. x from a to b.
In this case, the double integral is

$$I = \int_a^b \int_{y_1(x)}^{y_2(x)} f(x, y)\, dy\, dx.$$

Note we can easily modify this to regions sandwiched between two curves

$$y = y_1(x) \quad \text{and} \quad y_2(x)$$

without the assumption $y_2(x) \geq y_1(x)$ by piecing the region together.

Example Find

$$I = \int\int_R xy\,dy\,dx$$

where R is the region in the positive quadrant bounded by

$$y = 0, x = 0 \text{ and } y = 1 - x^2\,.$$

First sketch the region R.

Extreme values of x are $x = 0$ and $x = 1$, so these are the limits of the outer integral.
At fixed x, vertical strips go from $y = 0$ to $y = 1 - x^2$
So the integral is

$$
\begin{aligned}
I &= \int_0^1 \int_0^{1-x^2} xy\,dy\,dx \\
&= \int_0^1 \left[xy^2/2\right]_{y=0}^{y=1-x^2} dx \\
&= \int_0^1 \frac{x(1-x^2)^2}{2}\,dx = \left[\frac{x^2}{4} - \frac{x^4}{4} + \frac{x^6}{12}\right]_0^1 = \frac{1}{12}.
\end{aligned}
$$

Type II A plane region R lies between the graphs of two continuous functions of $x_1(y)$ and $x_2(y)$ (magenta figure above)

$$c \le y \le d,\ x_1(y) \le x \le x_2(y).$$

- In this case, we do the x integral first.
- We identify the extreme values $y = c$ and $y = d$.
- These are the limits for the outer y integral.
- We identify the boundaries $x = x_1(x)$ and $x = x_2(x)$ of the domain R.
- The double integral is then

$$I = \int_c^d \int_{x_1(y)}^{x_2(y)} f(x,y)\,dx\,dy\,,$$

meaning we: (i) first integrate $f(x,y)$ with respect to x holding y fixed; (ii) evaluate the limits by setting $x = x_2(y)$ and $x = x_1(y)$, leaving a function of y only; (iii) integrate w.r.t. y, using c and d as the limits.

Example: Find

$$I = \int \int_R xy \, dy \, dx$$

where R is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.
First sketch the region R (Type II).

The Type II region is

$$-2 \le y \le 4,\ y^2/2 - 3 \le x \le y + 1$$

with intersections between the line $y = x - 1$ and the parabola $y^2 = 2x + 6$ at $(-1, -2)$ and $(5, 4)$.
Hence, the extreme values for y are $y = -2$ and $y = 4$, so these are the limits of the outer integral.
At fixed y, horizontal strips go from $x = y^2/2 - 3$ to $x = y + 1$, so the integral is

$$\begin{aligned} I &= \int\int_R xy\,dy\,dx = \int_{-2}^{4}\int_{y^2/2-3}^{y+1} xy\,dx\,dy \\ &= \int_0^1 \left[x^2y/2\right]_{x=y^2/2-3}^{x=y+1} dy \\ &= \tfrac{1}{2}\int_{-2}^{4} y\left[(y+1)^2 - (y^2/2-3)^2\right]\,dy \\ &= \tfrac{1}{2}\int_{-2}^{4}\left[-y^5/4 + 4y^3 + 2y^2 - 8y\right]\,dy \\ &= \tfrac{1}{2}\left[-y^6/24 + y^4 + 2y^3/3 - 4y^2\right]_{-2}^{4} = 36. \end{aligned}$$

Further Example: Find

$$I = \int\int_R xy^{1/2}\,dx\,dy$$

where R is the region in the positive quadrant bounded by the parabolas

$$y = \sqrt{x} \text{ and } y = x^2.$$

This is a Type I region: We sketch the region: $0 \leq x \leq 1, x^2 \leq y \leq \sqrt{x}$.

The integral is

$$
\begin{aligned}
I &= \int_0^1 \int_{x^2}^{\sqrt{x}} x y^{1/2}\, dy\, dx \\
&= \int_0^1 x \left[2y^{3/2}/3 \right]_{y=x^2}^{y=\sqrt{x}} dx \\
&= \int_0^1 \frac{2}{3} x (x^{3/4} - x^3)\, dx = \frac{2}{3} \left[\frac{4x^{11/4}}{11} - \frac{x^5}{5} \right]_0^1 \\
&= \frac{2}{3} \left[\frac{4}{11} - \frac{1}{5} \right] = \frac{6}{55}.
\end{aligned}
$$

Example: If $f(x)$ is continuous on $[0, 1]$ and

$$\int_0^1 f(x)\, dx = \alpha,$$

then find the value of

$$I = \int_0^1 \int_x^1 f(x) f(y)\, dy\, dx.$$

First sketch the region R: $0 \leq x \leq 1, x \leq y \leq 1$.

R is triangular. So we first change the order of the integration

$$\int_0^1 \left[\int_x^1 f(x)f(y)\,dy\right]\,dx = \int_0^1 \left[\int_0^y f(x)f(y)\,dx\right]\,dy.$$

If we interchange x and y, then the second version of the integral can be written:

$$I = \int_0^1 \left[\int_0^x f(y)f(x)\,dy\right]\,dx$$

Then adding

$$\int_0^1 \left[\int_x^1 f(x)f(y)\,dy\right]\,dx \text{ and } \int_0^1 \left[\int_0^x f(y)f(x)\,dy\right]\,dx$$

we see that

$$2I = \int_0^1 \int_0^1 f(x)f(y)dydx = \int_0^1 f(x)dx \int_0^1 f(y)dy = \alpha^2$$

so that

$$I = \frac{\alpha^2}{2}\,.$$

To find the area of a region we simply compute the double integral

$$\text{Area of } R = \int\int_R 1\,dxdy$$

that is where the integrand is the constant function $f(x, y) = 1$

- To compute integrals over non-rectangular domains we first draw a picture of the domain
- Then we look to see if the domain is sandwiched between curves:

$$y = y_1(x) \text{ and } y = y_2(x) \quad \text{OR} \quad x = x_1(y) \text{ and } x = x_2(y)$$

- If the former or latter, then the integral is

$$I = \int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx \qquad \text{OR} \qquad I = \int_c^d \int_{x_1(y)}^{x_2(y)} f(x,y)\,dx\,dy$$

- For double integrals over more complicated domains, we look to break the domain up into several portions, each sandwiched between curves $(y = y_1(x), y = y_2(x)$ or $x = x_1(y), x = y_2(x))$.

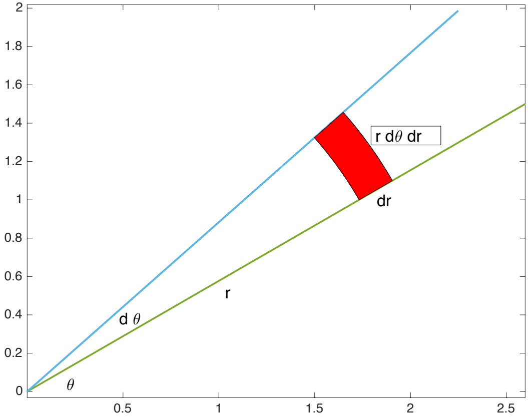

Figure: Polar coordinates. Elemental area (in red) has area $rd\theta \times dr$

In polar coordinates r, θ:

- r is the distance from the origin,
- θ the angle, measured counter-clockwise from the x-axis.

Polar coordinates are related to Cartesian coordinates by:

$$\begin{aligned} x &= r\cos\theta, \quad y = r\sin\theta, \\ r &= \sqrt{x^2+y^2} \quad \theta = \tan^{-1} y/x \end{aligned}$$

In polar coordinates the area element is

$$dA = rdrd\theta$$

This can be seen using a geometrical argument: in Figure **??** the "rectangular" element (in red) has area $dr \times rd\theta$.

If f is continuous on a polar rectangle

$$R = \{0 \le a \le r \le b,\ \alpha \le \theta \le \beta\},$$

then

$$\int\int_R f(x,y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\ dr\ d\theta.$$

Example: Find

$$I = \int\int_R (x+y)\,dA$$

where R is the semicircle $\{0 \le r \le 2,\ 0 \le \theta \le \pi\}$.

In polar coordinates, $x + y = r\cos\theta + r\sin\theta$, so

$$\begin{aligned} I &= \int_0^\pi \int_0^2 r(\cos\theta + \sin\theta)\ r\,dr\ d\theta \\ &= \int_0^\pi \left[r^3/3\right]_0^3 (\cos\theta + \sin\theta)\,d\theta && r \text{ integral first} \\ &= (8/3)\Big[\sin\theta - \cos\theta\Big]_0^\pi && \text{then the } \theta \text{ integral} \\ &= (8/3)[1+1] = \tfrac{16}{3}. \end{aligned}$$

Example: Find the area enclosed by one petal of the four-petalled rose given by the polar curve $r = \cos 2\theta$:

$$-\pi/4 \leq \theta \leq \pi/4,\ 0 \leq r \leq \cos 2\theta.$$

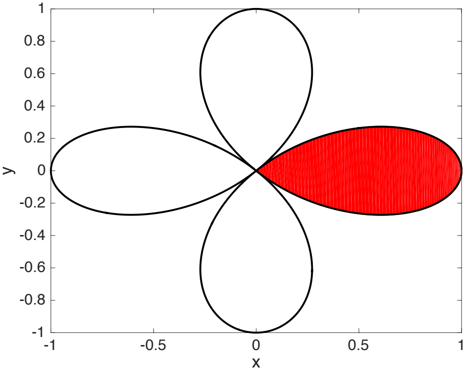

Figure: Region shaded in red is $\{-\pi/4 \leq \theta \leq \pi/4,\ 0 \leq r \leq \cos 2\theta\}$.

The area is

$$\begin{aligned}
\int\int_{\text{petal}} &= \int_{-\pi/4}^{\pi/4}\int_0^{\cos 2\theta} r dr d\theta \\
&= \int_{-\pi/4}^{\pi/4}\left[r^2/2\right]_0^{\cos 2\theta}\ d\theta \\
&= \int_{-\pi/4}^{\pi/4}\tfrac{1}{2}\cos^2 2\theta\ d\theta \\
&= \int_{-\pi/4}^{\pi/4}\tfrac{1}{4}(1+\cos 4\theta)\ d\theta \\
&= \tfrac{1}{4}\left[\theta+\tfrac{1}{4}\sin 4\theta\right]_{-\pi/4}^{\pi/4} = \tfrac{\pi}{8}
\end{aligned}$$

Find

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx\,.$$

We use a little trick:

$$\begin{aligned}
I^2 &= \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy \\
&= \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2} r dr d\theta \\
&= \int_{0}^{2\pi} \int_{0}^{\infty} \frac{d}{dr} \frac{1}{2} e^{-r^2} dr d\theta \\
&= \frac{1}{2} \int_{0}^{2\pi} 1 d\theta = \pi
\end{aligned}$$

Hence

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

We seek to compute integrals for 3D domains:

$$I = \int\int\int_V f(x,y,z)\, dV$$

Now instead of a region R in the plane, the region of integration is a region V of three-dimensional space.
As in 2D, rectangular boxes are the easiest to deal with.
Then the **element of volume** is

$$dV = dx\, dy\, dz$$

and the box is the region

$$\{x_1 \le x \le x_2,\ y_1 \le y \le y_2,\ z_1 \le z \le z_2\}\,.$$

So

$$I = \int_{z_1}^{z_2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x,y,z)\, dx\, dy\, dz.$$

As in two dimensions, the integrations can be done in any order.

If $f(x,y,z) = 1$, then the triple integral represents the volume of V:

$$V = \int\int\int_V dV.$$

Handling non-rectangular domains in 3D is a tricky (because it is more difficult to visualise that the 2D counterpart).

- Decide the order of integration
- Suppose the z integral is the outermost integral
- Find the extreme values of z, say z_1 and z_2 — these are the limits of the outermost integral.
- Find the limits on the middle y integral, take z as fixed and find the extreme values of y as x varies — it may help to sketch in the xy-plane the boundary of a $z = const.$ slice of V.
- These extreme values of y will be a function of z only — in general the limits on y will be $y = f(z)$ and $y = g(z)$, some f and g.
- Then finally the limits on x are determined from the bounding surface of V, which has to be put in the form $x = u(y, z)$,$x = v(y, z)$ where u and v are the lower and upper values of x at fixed y and z.

Example: Find the x, y, z limits when V is the sphere

$$x^2 + y^2 + z^2 \leq 1.$$

- Extreme values of z are ± 1.
- Now fix z and look for extreme values of y. $y = \pm\sqrt{1 - z^2 - x^2}$.
- The biggest value of y as x varies is at $x = 0$ when $y = \sqrt{1 - z^2}$. The smallest value of y is $y = -\sqrt{1 - z^2}$. so the limits on y are $\pm\sqrt{1 - z^2}$.
- When z and y are fixed, $x = \pm\sqrt{1 - y^2 - z^2}$ on the boundary of the sphere, so these are the limits on x.

In this case

$$\int\int\int_V f(x, y, z)\, dx\, dy\, dz = \int_{-1}^{1} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-z^2-y^2}}^{\sqrt{1-z^2-y^2}} f(x, y, z)\, dx\, dy\, dz.$$

Cylindrical polar coordinates. A point P has coordinates (s, ϕ, z) where

- s is the distance from the z-axis,
- ϕ is the usual polar coordinate angle of P from the xz-plane, and
- z is the same as the Cartesian z

Cartesian and cylindrical polar coordinates are linked by:

$$x = s\cos\phi,\ y = s\sin\phi,\ z = z$$

The element of volume is $dV = s\,ds\,d\phi\,dz$.

Example: Evaluate

$$\int\int\int_V s^3\, dV,$$

where V is the cylindrical region

$$\{0 \leq s \leq 2, -1 \leq z \leq 1, 0 \leq \phi \leq 2\pi\}.$$

$$\begin{aligned} I &= \int_{-1}^{1} \int_0^{2\pi} \int_0^2 s^4\, ds\, d\phi\, dz \\ &= \int_{-1}^{1} \int_0^{2\pi} \left[\frac{s^5}{5}\right]_0^2 d\phi\, dz \\ &= \int_{-1}^{1} \left[\frac{32}{5}\phi\right]_0^{2\pi} dz \\ &= \int_{-1}^{1} \frac{64\pi}{5}\, dz = \frac{128\pi}{5}. \end{aligned}$$

Spherical polar coordinates. A point P has coordinates (r, θ, ϕ).

- ϕ is the longitude angle, sometimes called the azimuthal angle, measured from the xz-plane
- r is the distance from the origin
- θ is the angle from the north pole (the positive z-axis). θ is sometimes called the co-latitude because it is 90°-latitude.
- The equator (latitude 0°) has $\theta = \pi/2$. The South Pole has $\theta = \pi$.

Cartesian and spherical polar coordinates are linked by:

$$x = r\cos\phi\sin\theta, \quad y = r\sin\phi\sin\theta, \quad z = \cos\theta$$

The element of volume is $dV = dx\,dy\,dz = r^2 \sin\theta\,dr\,d\theta\,d\phi$.

Example: Evaluate

$$\int\int\int_V r^3\,dV,$$

where V is the hemispherical region

$$\{0 \le r \le 2, 0 \le \theta \le \pi/2, 0 \le \phi \le 2\pi\}\,.$$

$$\begin{aligned}
I &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 r^5 \sin\theta\,dr\,d\theta\,d\phi \\
&= \int_0^{2\pi}\int_0^{\pi/2}\left[\frac{r^6}{6}\right]_0^2 d\theta\,d\phi \\
&= \int_0^{2\pi}\left[-\frac{32}{3}\cos\theta\right]_0^{\pi/2} d\phi \\
&= \int_0^{2\pi}\frac{32}{3}\,d\phi = \frac{64\pi}{3}.
\end{aligned}$$

- Sometimes the domain of integration looks simpler in alternative coordinates
- For example a circular domain $x^2 + y^2 \leq 4$ is more easily described as a rectangular domain $r \leq 2, \theta \in [0, 2\pi]$.
- So we change coordinates, remembering to change $dx\, dy$ in to the relevant elemental area, e.g. in polar coordinates

$$dx\, dy \mapsto r\, dr\, d\theta$$

- Triple integrals are defined similarly to double integrals
  - Start with rectangular domains; Make small rectangular boxes; Add up all the boxes and take the limit
  - In practice, we compute the triple integral as a sequence of three single integrals.