Differential Equations Week 5 - Series solutions of ODEs

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1 Power Series Method

1. Inifnite series around center x_0,

$$\sum_{m=0}^{x} a_m x^m = a_0 + a_1(x - x_0) + \dots \quad m \in \mathbb{Z}^+$$

For a given homogenous ODE: $y'' + p(x)y' + q(x) = 0$, the coefficients are a power series. Then the solution of the ODE can be obtained as a power series.

2. Replace powers of power series and collect same powers of x.

1.1 Convergence interval and Radius of Convergence

1. Useless case, always $x = x_0$

2. It converges for $|x - x_0| < R$, where

$$R = \left(\lim_{m\to\infty} \sqrt[p]{|a_m|}\right)^{-1} \qquad R = \left(\lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right|\right)^{-1}$$

if R is infinite the only converges at x_0

3. Best Case, interval is infinite, is when the two limits are 0.

1.2 Operation

1. Termwise Differention: if a power series y can be termwise differention and y'' and y' also converge w/ radius R

2. Termwise Addition: two power series y and w w/ $R > 0$ and then $y + w$ also converges w/ radius R.

3. Termwise Multiplication: two power series y and w w/ $R > 0$ and then $y \cdot w$ also converges w/ radius R.

4. Vanishing of all coefficients: if for y, a power series and $y = 0$, then $a_m = 0 \quad \forall m$.

2 Legendres Equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$

and hence $y'' - \frac{2x}{1-x^2}y' + \frac{n(n+1)}{1-x^2} = 0$. Now we can substitute in the power series:

$$= \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + \sum_{m=2}^{\infty} m(m-1)x^m a_m$$

$$- \sum_{m=1}^{\infty} 2m a_m x^m + n(n+1)\sum_{m=0}^{\infty} a_m x^m = 0$$

Let $m = s + 2$

$$a_{s+2} = \frac{(n-s)(n+s+1)}{(s+2)(s+1)} a_s \quad n \in \mathbb{N}$$

It converges for $|x| < 1$, since y_1 contains even powers and y_2 odd powers. Hence, y_1, y_2 are non-constant and linearly independent. Doing some more calculations, the $n^t h$ term is:

$$a_n = \frac{(2n)!}{2^n (n!)^2}$$

and the polynomial:

$$P_n(x) = \sum_{n=0}^{\infty} \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}$$

3 Forbenius Method

Theorem 3.1

Let $b(x)$ and $c(x)$ be analytic at x_0. Then, $y'' + \frac{1}{x}by' + \frac{1}{x^2}cy = 0$ has at least one solution that can be represented as:

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m \quad r \in \mathbb{C},\ a_0 \neq 0$$

Definition 3.1: Regular Point

A regular point is a point, x_0, where p, q are analytic. So a power series method can be applied. A non-regular point is singular

Definition 3.2: Indicial Equation

Let $y = \sum_{m=0}^{\infty} a_m x^{m+r}$ and hence plug back into the ODE, then you get an equation of the form:

$$r(r-1) + b_0 r + c_0 = 0$$

This is the indicial equation

If the roots of the indicial equation are:

1. Distinct Roots, then the solutions are:

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} a_m x^m \quad y_2 = x^{r_2} \sum_{m=0}^{\infty} A_m x^m$$

2. Double Roots, then the solutions are: $(r = \frac{1}{2}(1 - b_0))$

$$y_1 = x^r \sum_{m=0}^{\infty} a_m x^m \quad y_2 = y_1 \ln x + x^r \sum_{m=0}^{\infty} A_m x^m$$

3. Roots differing by an integer, then the solutions are: $(r_1 - r_2 > 0)$

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} a_m x^m \quad y_2 = k y_1 \ln x + x^{r_2} \sum_{m=0}^{\infty} A_m x^m$$

4 Bessel Equations / Functions

$$x^2 y'' + x y' + (x^2 - v^2) y = 0$$

Applying forbenius method, we have:

$$a_{2m} = (-1)^m \left(\frac{1}{2^{m+n} m!(n+m)!} \right)$$

and hence

$$\mathcal{J}_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^2 m}{2^{2m+n} m!(n+m)!}$$

4.1 Bessel and Gamma

If we take $\mathcal{J}_n$ where $v \geq 0$,

$$a_0 = \frac{1}{2^n \Gamma(n+1)}$$

and hence:

$$\mathcal{J}_n(x) = x^v \sum_{m=0}^{\infty} \frac{(-1)^m x^2 m}{2^{2m+v} m! \Gamma(1+v+m)!}$$

and now we can extend the bessel of first kind to all real x:

$$\mathcal{J}_n(x) = x^{-v} \sum_{m=0}^{\infty} \frac{(-1)^m x^2 m}{2^{2m-v} m! \Gamma(1-v+m)!} \quad v \notin \mathbb{Z}$$

If $v \in \mathbb{Z}$, then $\mathcal{J}_v = \mathcal{J}_{-v}(-1)^v$, otherwise the basis of the ODE is:

$$y(x) = c_1 \mathcal{J}_v + c_2 \mathcal{J}_{-v}$$

We know the following:

$$[x^v \mathcal{J}_v]' = x^v \mathcal{J}_{v-1} \text{ and } [x^{-v} \mathcal{J}_{-v}]' = -x^{-v} \mathcal{J}_{v+1}$$

Theorem 4.1: Derivatives

$$\mathcal{J}_{v-1} + \mathcal{J}_{v+1} = \frac{2v}{x} \mathcal{J}_v \qquad \text{(a)}$$

$$\mathcal{J}_v + x^v \mathcal{J}_v' = -x^{-v} \mathcal{J}_{v+1} \qquad \text{(b)}$$

Proof.

$$v x^{v-1} + x^v \mathcal{J}_v = x^v \mathcal{J}_{v-1} - v x^{v-1} \qquad (a*)$$

$$\mathcal{J}_v + x^v \mathcal{J}_v' = -x^v \mathcal{J}_{v+1} \qquad (b*)$$

Add and subtract $(a*)$ and $(b*)$ □

For half order v we also know the bessel function, as the gamma function is valid:

$$\mathcal{J}_{\frac{1}{2}} = \sqrt{\frac{2}{\pi x}} \sin x \qquad \mathcal{J}_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi x}} \cos x$$

When $v \in \mathbb{Z}$, we can find a second linearly independent solution to the bessel equation, this solution is the Bessel Equation of the second kind, $Y_n(x)$. The indicial equation has two solutions with root $r = 0$, this is case 2.

$$\therefore y_2 = \mathcal{J}_0 \ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} h_m}{2^{2m} (m!)^2} x^m$$

where we let h_m be the m^{th} partial sum of the harmonic series. We also have a second:

$$\frac{2}{\pi}(y_1 + (\gamma - \ln 2) \mathcal{J}_0)$$