Year MAGIC — Algebraic Topology

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine (especially the typos!).

Contents

1 Introduction

What is this course about? *Lecture 1*

Theorem 1.1 (Invariance of domain)**.** If $n \neq m$, then $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic

This is hard to prove, there is no way to directly prove this without a big deal of theory, but soon we will be able to do that. Now consider, SO(3) and $p = \{\text{trace-one projectors in } \mathbb{R}^4\} = \{A \in M_4(\mathbb{R}) : A^T = A = A^2, \operatorname{tr} A = 1\}$ and now $S^3 = \{x \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$ and $\mathbb{RP}^3 = S^3 \setminus \sim$, where $x \sim (\pm x)$, so all S^3 with opposite points glued together.

Theorem 1.2. SO(3), P, $\mathbb{RP}^3$ are homeomorphic to eachother, but not S^3.

This is similar to the first theorem that we wrote, but in this case the spaces aren't the most natural or easiest to consider. We need some more thoughts to prove this. We can prove positive statements, that is, producing a homeomorphisms. To prove a negation, that is harder and we need more theory. Here is another theorem,

Theorem 1.3 (Brouwer Fixed Point Theorem)**.** Let $f : [0,1]^n \to [0,1]^n$ be continuous. Then there exists some $a \in [0,1]^n$ with $f(a) = a$

How can we prove this? Well we need the fact that S^{n-1} is not contractible.

Theorem 1.4 (Bosak-Ulam)**.** Say $f : S^n \to S^m$ is odd (or antipodal) if $f(-x) = f(x)$ for all x. There is no odd continuous function $S^n \to S^m$ if $n > m$.

We have said we need several complicated methods to do things in Algebraic Topology. Here we are interested in cohomology. In any space X there is a ring $H^*(X)$ (the cohomology ring of X), this will be hard to construct and not instantly obvious. This is a graded ring, there are abelian groups $H^i(X)$ for $i \in \mathbb{Z}$ (but with $H^i(X) = 0$ if $i < 0$) and a product rule defining $ab \in H^{i+j}(X)$ for $a \in H^i(X)$ and $b \in H^j(X)$ and an identity $1 \in H^0(X)$. This is nearly a commutativity, we have a graded commutativity, $ba = (-1)^{ij}ab$ if $a \in H^i(X)$ and $b \in H^j(X)$. Any continuous map from $X \to Y$ provides a ring homomorphism $f^* = H^*(Y) \to H^*(X)$.

Here is the most basic example, take $X = \mathbb{R}^n \setminus 0$ with $n > 1$. There is an element $u_{n-1} \in H^{n-1}(\mathbb{R}^n \setminus 0)$ such that $H^*(\mathbb{R} \setminus 0) = \mathbb{Z} \oplus \mathbb{Z}u_{n-1}$ and so $H^0(\mathbb{R}^n \setminus 0) = \mathbb{Z}$ and $H^{n-1}(\mathbb{R} \setminus 0) = \mathbb{Z}u_{n-1}$ and $H^k(\mathbb{R}^n \setminus 0) = 0$ for $k \neq 0, n-1$. What is the ring structure? $p \times (qu_{n-1}) = (pq)u_{n-1}$, the only thing of interest is $u_{n-1}^2 \in H^{2n-2}(\mathbb{R}^n \setminus 0) = 0$ and so $u_{n-1}^2 = 0$ and so there is no space for anything 'interesting'. So, if $n \neq m$ then $H^*(\mathbb{R} \setminus 0)$ and $H^*(\mathbb{R}^m \setminus 0)$ are not isomorphic as graded rings and so therefore $\mathbb{R}^n \setminus 0$ is not homeomorphic to $\mathbb{R}^m \setminus 0$. Then after a few more steps, invariance of domain follows from this.

Here is an outline programme for the course,

(i) Define $H^*(X)$ and prove key properties (Eilenberg-Steener Axioms)

(ii) Use ES axioms to calulate $H^*(\mathbb{R} \setminus 0)$ and $H^*(S^{n-1})$

(iii) Deduce invariance of domain. Brouwers fixed point, fundemental theorem of algebra.

(iv) Calculate $H^*(X)$ for various interesting spaces X developing additional techniques along the way.

As an example of (iv), consider $\mathbb{C}^n$ and $F_n(\mathbb{C}) = \{(z_1, z_2, \ldots, z_n) : z_i \neq z_j \text{ for } i \neq j\} \subset \mathbb{C}^n$, now define $f_{ij} : F_n(\mathbb{C}) \to S^1$ by $f_{ij}(z) = \frac{z_i - z_j}{|z_i - z_j|}$ and this makes sense as $z_i \neq z_j$. Recall, $H^1(S^1) = H^1(\mathbb{C} \setminus 0) = \mathbb{Z}u_1$ and so put $a_{ij} = f_{ij}^*(u_1)$ (where is a ring homomorphism as cohomologies are a thing). Here are the facts,

- $a_{ij} = a_{ji}$
- $a_{ij}^2 = 0$

– $a_{ij}a_{jk} + a_{jk}a_{ki} + a_{ki}a_{ij} = 0$ for all distinct i, j, k.

– $H^*(F_n(\mathbb{C}))$ is the free graded-commutative ring generated by the elements a_{ij} subject to these relations (and give a basis for $H^*(F_n(\mathbb{C}))$ over $\mathbb{Z}$).

Exercise. Understand $H^*(F_3(\mathbb{C}))$ and $H^*(F_4(\mathbb{C}))$ from this description, for example $H^1(F_3(\mathbb{C})) = \mathbb{Z}[a_{12}, a_{13}, a_{23}] \cong \mathbb{Z}^3$. We look at the second cohomology group we can take products of these so we guess $a_{12}a_{13}, a_{12}a_{23}, a_{13}a_{23}$ as generators, but $a_{12}a_{23} + a_{23}a_{31} + a_{31}a_{12} = a_{12}a_{23} - a_{13}a_{23} - a_{12}a_{13} = 0$ and so we can remove $a_{13}a_{23}$ as it's just $a_{12}a_{13} - a_{12}a_{23}$ and so $H^2(F_3(\mathbb{C})) = \mathbb{Z}[a_{12}a_{13}, a_{12}a_{23}]$. Furthermore $H^k(F_3(\mathbb{C})) = 0$ for $k > 2$.

2 Defining Cohomology

We now define these cohomology groups,

- For every space X, we define a differential graded ring $C^*(X)$
- For every DGR, A^*, we define a cohomology ring $H^*(A^*)$
- We define $H^*(X) = H^*(C^*(X))$.

We start with (ii),

Definition 2.1 (Cochain Complex)**.** A cochain complex is a sequence of abelian groups U^k (for $k \in \mathbb{Z}$, but often $U^k = 0$ for $k < 0$) together with some homomorphisms $d : U^k \to U^{k+1}$ satisfying $D^2 = 0$, $(U^{k-1} \to U^k \to U^{k+1} = 0)$

Definition 2.2 (Differential Graded Ring)**.** A DGR is a cochain complex A^* with a product rule $ab \in A^{i+j}$ for $a \in A^i$ and $b \in A^j$ and an element $1 \in A^0$ satisfying $1a = a1 = a$, $(ab)c = (ab)c$, $a(b+c) = ab + ac$, $(a+b)c = ac + bc$ and satisfies some Liebnitz rule $d(a,b) = d(a)b + (-1)^i a \cdot d(b)$ for $a \in U^i$ and $b \in U^j$ (The $(-1)^j$ is the dizul sign rule. It says if you swap two odd things a minus sign pops out.)

Example. $\Omega^*(M)$ is the de Rham complex of smooth differential forms on a smooth manifold. *Lecture 2*

Example. Let $A^* = \mathbb{Z}[x] \oplus \mathbb{Z}[x]a$ with $x \in A^2$ and $a \in A^1$. This means that $A^{2n} = \mathbb{Z}x^n$ and $A^{2n+1} = \mathbb{Z}x^n a$ (for $n \geq 0$). We also need some differential, we let $d(a) = x$, this only gives a differential of one class but if forces everything else, as $d^2(a) = d(d(a)) = d(x) = 0$ and this forces $d(x^n) = 0$ by Liebnitz rule, what about $d(x^n a) = 0 + x^n x = x^{n+1}$.

Now we define the homology of the DGR. Given a DGR ring A^*, put $Z^* = Z^*(A^*) = \operatorname{Ker} d$, or more specifically $Z^i = \operatorname{Ker}(d : A^i \to A^{i+1}) = \{a \in A^i : da = 0\}$ and $B^*(A^*) = \operatorname{im} A^*$. We claim that,

Claim. Z^* is a subring of A^* and B^* is a two-sided ideal in Z^*.

Proof. We need to check it's a subset of Z^*. If $b \in B^*$ then $b = d(x)$ for some x, therefore $d(b) = d^2(x) = 0$ and so $b \in Z^*$. Hence $B^* \subset Z^*$.

The Liebnitz rule gives $d(1) = d(1 \times 1) = d(1) \times 1 + 1 \times d(1) = 2d(1)$ and so $d(1) = 0$. Hence $1 \in Z^0$. Suppose that $a, b \in Z^*$. Then $d(ab) = d(a)b + ad(b) = 0 + 0 = 0$. So Z^* contains 1, closed under multiplication and closed under addition and subtraction. Therefore it is a subgroup.

Suppose $a \in Z^*$ and $b \in B^*$ so $b = d(y)$ for some y. Then $d(ay) = d(a)y \pm ad(y) = 0 \pm ab$ and so $abin \operatorname{im}(d) = B^*$. So B^* is closed under left multiplication by Z^*, so it is a left ideal. Similarly if $a \in B^*$ and $b \in Z^*$ then $ab \in B^*$. □

Definition 2.3 (Cohomology)**.** $H^* = H^*(A^*) = Z^*(A^*) \setminus B^*(A^*) = \operatorname{Ker} d \setminus \operatorname{im} d$

Terminology: Z^* is the subring of cocycles, B^* is the ideal of coboundaries and H^* is the cohomology ring, for $f \in Z^i$ write $[f]$ =coset $f + B^i \in H^i$. (So $[f_0] = [f_1]$ if and only if $f_0 - f_1 \in B^i$ if and only if $f_0 - f_1 = d(x)$ for some x)

Example. Let $A^* = \mathbb{Z}[x] \oplus \mathbb{Z}[x]a$ with $d(x^n) = 0$ and $d(x^n a) = x^{n+1}$. Then $Z^* = \mathbb{Z}[x]$ and $B^* = \mathbb{Z}[x]x = \operatorname{span}\{x^{n+1} : n \geq 0\}$. Therefore, $H^* = \mathbb{Z}[x] \setminus (x\mathbb{Z}[x]) = \mathbb{Z}$ ie. $H^0 = \mathbb{Z}$ and $H^i = 0$ for $i \neq 0$

Example. If $A^* = \Omega^*(M)$ then $H^*(A^*)$ is the de Rahm cohomology of M.

The next step for each space X, define a DGR $C^*(X)$. Then define $H^*(X) = H^*(C^*(X))$. For a bit of motivation we define $H^0(X)$. Introduce an equivalence relation as X by $x \sim y$ if and only if there is a continuous path $u : [0,1] \to X$ with $u(0) = x$ and $u(1) = y$.

In the diagram $a_0 \sim a_1$, $c_0 \sim c_1$ but $a_0 \not\sim c_0$. We define $\pi_1(X) = X \backslash \sim [A, B, C]$. Then $H^0(X)$ will be map$(\pi_1(X), \mathbb{Z})$ (which is $\mathbb{Z}^3$). OR, $H^0(X) = \{f : X \to \mathbb{Z} : f(u_1) = f(u_0) \text{ for all paths} u \in X\}$. Let us tidy this definition $S_0(X) = X$ and $S_1(X)$ is the set of paths in X, $C^0(X)$ is the set of maps from $S_0(X)$ to $\mathbb{Z}$ and $C^1(X)$ is the set of maps from $S_1(X)$ to $\mathbb{Z}$. Given some $f \in C^0(X)$, that is some $f : X \to \mathbb{Z}$ we want to define $df \in C^1(X)$, that is $df : S_1(X) \to \mathbb{Z}$. For every $u \in S_1(X)$ we need to define $(df)(u) \in \mathbb{Z}$. We put $(df)(u) = f(u(1)) - f(u(0))$. Now

$$\begin{aligned} Z^0 &= \operatorname{Ker}(d : C^0(X) \to C^1(X)) \\ &= \{f : X \to \mathbb{Z} : (df)(u) = 0 \ \forall u\} \\ &= \{f : X \to \mathbb{Z} : f(u(1)) = f(u(0)) \ \forall u\} \end{aligned}$$

Also $C^k(X) = 0$ for all $k < 0$ therefore $B^0(X) = 0$ and $H^0 = Z^0 \setminus B^0 = Z^0 = \text{Map}(\pi_1(X), \mathbb{Z})$ as before.

We have only defined two of the C^k's so we need the rest of them and the differentials.

Definition 2.4 (Standard k-simplex)**.** $\Delta_k = \{(t_0, t_1, \ldots, t_k) \in \mathbb{R}^k : t_i \geq 0 \, t_0 + t_1 + \cdots + t_k = 1\}$

Example.

$$\Delta_0 = \{t_0 : t_0 \geq 0 \, t_0 = 1\} = \{1\}$$

$$\Delta_1 = \{(t_0, t_1) : t_0, t_1 \geq 0 \, t_1 = 1 - t_0\}$$

We see that Δ_1 is a line, the Δ_2 is just a plane and Δ_3 is a tetrahedron and so on.

NB! For $0 \leq i \leq k$ let e_i be the i^{th} basis representation so $e_i \in \Delta_k$

Definition 2.5. $S_k(X)$ is the set of all continuous maps $u : \Delta_k \to X$

Definition 2.6 (Cochain Group)**.** $C^*(X) = \text{Map}(S_k(X), \mathbb{Z})$

We now want to make this into a DGR and so we want to define a differential and then a product. We now need to consider the faces the simplices. We define $\delta_1, \delta_2, \delta_3 : \Delta_1 \to \Delta_2$ by $\delta_1(t_0, t_1) = (0, t_0, t_1)$, $\delta_2(t_0, t_1) = (t_0, 0, t_1)$ and $\delta_3(t_0, t_1) = (t_0, t_1, 0)$. They insert 0 into position i. Recall that $e_0 = (1, 0, 0)$, $e_1 = (0, 1, 0)$ and $e_2 = (0, 0, 1)$ and so it's the oppposite case. Hence the δ_i's give the edge opposite e_i. More generally, let us define,

$$\delta_i : \Delta_k \to \Delta_{k+1} \quad 0 \leq i \leq k+1 \qquad \delta_i = t \text{ with 0 in posiiton} i$$

Here is a formula, to be explained next time,

Definition 2.7. $d : C^k(X) \to C^{k+1}(X)$ where

$$(df)(u) = \sum_{i=0}^{n+1} (-1)^i f(u \circ \delta_i)$$

for all $u : \Delta_{k+1} \to X$.

Suppose $f \in C^k X$, so $f : S_k X \to \mathbb{Z}$, that is, $f(u) \in \mathbb{Z}$ for all $u : \Delta_k \to X$/ Suppose we have $S_{k+1}X$ that is $\Delta_{k+1} \to X$ so $v \circ S_i : \Delta_k \to X$ and so $f(v \circ S_i) \in \mathbb{Z}$, put $(df)(v)$ as we defined in the last lecture. So $df \in C^{k+1}(X)$.

Recall we spoke briefly about chain complexes and cochain complexes, for these we need $d^2 = 0 : C^{k-1}(X) \to C^{k+1}(X)$. Note

$$(d^2 f)(v) = \sum_{j=0}^{k+1} (-1)^j (df)(v \circ \delta_j) = \sum_{i=0}^{k} \sum_{j=0}^{k+1} (-1)^{i+j} f(v \circ \delta_j \circ \delta_i)$$

Now we look at these terms and find the following diagram. Consider a tetrahedron, the δ_i give ud the faces and the $\delta_i \circ \delta_j$ give us the edges. We can look at the arrows and the orientations of the faces. If we consider the ways we go around the faces, we find they go in opposite ways around and so they cancel out.

Lemma 2.8. if $0 \leq j \leq i \leq k$, then $\Delta_{k-1} \xrightarrow{\delta_i} \Delta_k \xrightarrow[\delta_j]{} \Delta_{k+1}$ is the same as $\Delta_{k-1} \xrightarrow{\delta_j} \Delta_k \xrightarrow[\delta_{i+1}]{} \Delta_{k+1}$ or $\delta_j \delta_i = \delta_{i+1} \delta_j$.

Proof. Consider $x = (x_0, \ldots, x_{k-1}) \in \Delta_{k-1}$. Then $\delta_i x = x$ with 0 inserted in position i. Therefore $\delta_j \delta_i x = \delta_i x$ with a 0 inserted into position $j \leq i$ which pushes the first zero to position $i + 1$. $\delta_j x = x$ with 0 inserted in position j and $\delta_{i+1} \delta_j x$ with 0 inserted in position $i + 1 > j$. Therefore the first 0 doesn't move, so it's just x with zeros at positions at j and $i + 1$ which is just $\delta_i \delta_j x$. $\square$

Example. Let $k = 6$, $j = 2$ and $i = 4$. Then

$$\begin{aligned} \delta_2 \delta_4 (x_0, \ldots, x_5) &= \delta_2(x_0, x_1, x_2, x_3, 0, x_4, x_5) \\ &= (x_0, x_1, 0, x_2, x_3, 0, x_4, x_5) \end{aligned}$$

But also,

$$\begin{aligned} \delta_5 \delta_2 &= \delta_5(x_0, x_1, 0, x_2, x_3, x_4, x_5) \\ &= (x_0, x_1, 0, x_2, x_3, 0, x_4, x_5) \end{aligned}$$

which are the same.

Note: in $(d^2 f)(v)$, then $f(v \circ \delta_j \circ \delta_i)$ occurs with a sign $(-1)^{i+j}$ and $f(v \circ \delta_{i+1} \circ \delta_j)$ occurs withw $-(-1)^{i+j}$ and these two terms are the same and so they cancel with their difference in sign. Thus, $d^2 f = 0$. Thus, the abelian group $C^k X$ and the maps $d : C^k X \to C^{k+1} X$ form a cochain complex.

Recall the definition of a cohomology of a cochain complex, U^*,

- $Z^k = \mathrm{Ker}(d : U^k \to U^{k+1})$
- $B^k = \mathrm{im}(d : U^{k-1} \to U^k)$

and $d^2 = 0$ implies that $B^k \leq Z^k$ and so we can form a quotient $H^k = Z^k / B^k = H^*(U^*)$. We define $H^k(X) = H^k(C^k(X)) = \frac{\mathrm{Ker}(d : U^k \to U^{k+1})}{\mathrm{im}(d : U^{k-1} \to U^k)}$. This has a ring structure and so $1 \in C^0(X) = \mathrm{Map}(S_0(X), \mathbb{Z}) = \mathrm{Map}(X, \mathbb{Z})$ and so we let 1 be the constant map sending everything in X to 1. Then $(d1)(u) = 1(u \circ \delta_0) - 1(u \circ \delta_1) = 0$ for $u : \Delta_1 \to X$. So $1 \in Z^0(X)$ and so $[1] = 1 + B^0(X) \in H^0(X)$ (but $B^0(X) = 0$).

Now to define the product, suppose $f \in C^n(X)$ and $g \in C^m(X)$, ie. $f(u), g(v) \in \mathbb{Z}$ for all $u : \Delta_n \to X$ and $v : \Delta_m \to X$. We want to find $fg \in C^{n+m}(X)$ and so we need to define $(fg)(w) \in \mathbb{Z}$ for all $w : \Delta_{m+n} \to \mathbb{Z}$. We define $\Delta_n \xrightarrow{\lambda} \Delta_{n+1} \xleftarrow{\rho} \Delta_m$ where,

$$\lambda(x_0, x_1, \ldots, x_n) = (x_0, x_1, \ldots, x_n, 0, \ldots, 0)$$

$$\rho(y_1, y_2, \ldots, y_m) = (0, \ldots, 0, y_0, y_1, \ldots, y_m)$$

So $w \circ \lambda : \Delta_n \to X$ and $w \circ \rho : \Delta_m \to X$, so $f(w \circ \lambda) \in \mathbb{Z}$ and $g(w \circ \rho) \in \mathbb{Z}$. Therefore $(fg)(w) = f(w \circ \lambda) g(w \circ \rho) \in C^{n+m}(X)$. This is easily seen to distribute over addition.

Exercise. This product is associative, with 1 as a 2-sided unit $1f = f = f1$ and also $d(fg) = d(f)g + (-1)^n f d(g) \in C^{n+m+1}(X)$ (so the Liebnitz formula holds)

Therefore $C^*(X)$ is a differential graded ring. Therefore, $H^*(X)$ has a well defined ring structure. This isn't hard, what is substantially harder is $ab = (-1)^{nm} ba$ for $a \in H^n(X)$ and $b \in H^m(X)$, we will prove this later. We now have a definition of a cohomology ring, this isn't useful though. We can't calculate the cohomology of each space. We proved that $H^0(X) = \mathrm{Map}(\pi_1(X), \mathbb{Z})$. Now let us focus on $H^1(X)$.

Suppose X has a basepoint x_0, the loops at x_0 are maps $u : \Delta_1 \to X$ with $u(e_0) = u(e_1) = x_0$ so $u \in S_1 X$. Suppose $f \in C^1 X$ so $f(u) \in \mathbb{Z}$ for any such u. Suppose $f \in B^1 X$, that is $f = dg$ for some $g \in C^0 X$, ie $g : X \to \mathbb{Z}$. Then for any $u : \Delta_1 \to X$ we have $f(u) = (dg)(u) = g(u(e_1)) - g(u(e_0))$, if u is a loop, then $f(u) = g(x_0) - g(x_0) = 0$. So for $a \in H^1 X$ we have a well-defined $a(u) \in \mathbb{Z}$ given by $a(u) = f(u)$ for any $f \in Z^1 X$ with $a = [f]$. But $\pi_1(X)$ is loops up to homotopy relation and points, suppose $h : [0,1]^2 \to X$ is a homotopy relative to endpoints between u and b, divide this into two triangles $p, q : \Delta_2 \to X$ suppose $f \in Z^1(X)$ so $df = 0$ in $C^2 X$ so $df(p) = df(q) = 0$ and using this we ca check that $f(v) = f(u)$. So $a \in H^1 X$ the integer $a(u)$ depends only on the homotopy class of u, the corresponding element of $\pi_1(X)$. Using this we have a well-defined map from $\alpha : H_1 X \to \mathrm{Hom}(\pi_1(X), \mathbb{Z})$ and it's a fact that if X is path connected, then α is an isomorphism.