Year 3 — Groups, Rings and Fields

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Contents

1 Basics of Groups

We start by defining a group, it is an example of an algebraic structure. *Lecture 1*

Definition 1.1 (Group)**.** G is a nonempty set and endowed with a composition rule $(\cdot)$. We denote this $(G, \cdot)$. $(\cdot)$ is well defined, so we can associate another element $a \cdot b \in G$ and $a \cdot b$ is unique. $(\cdot)$ must be associative,

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

The brackets are irrelevant when combining more than two elements. We also have **natural element**, so,

$$c \cdot e_G = c = e_G \cdot c$$

There are also inverses, so,

$$a \cdot a^{-1} = e_G = a^{-1} \cdot a$$

So the inverse naturalises the element.

If we just have a group usually $a \cdot b \neq b \cdot a$, if $a \cdot b = b \cdot a$ are called abelian or commutative groups. This is in reference to the mathematician Abel.
If G is finite as a set, then we can say that G is a finite group and we denote the size or cardinality of G as $|G|$, sometimes this is said to be the order. The cardinality can be infinite.

Example. We know a very important group, the group of integers $\mathbb{Z}$. This set is infinite as $n \neq n + 1$ and the composition law is $+$ and we know that it's associative and natural element of 0 and each element n has an inverse of $-n$. We can also say,

$$k_1 + k_2 = k_2 + k_1$$

and so we have an infinite abelian group.

Example. We can also consider groups of integers module n, denoted,

$$\mathbb{Z}_n = \{[0]_n, [1]_n, \dots, [n-1]_n\}$$

where we have modulo classes (see Number Theory notes week 2). We can say, if $[k]_n = [l]_n$ if and only if $n \mid k - l$. Also if you have $[k_1]_n$ and $[k_2]_n$, then $[k_1]_n + [k_2]_n = [k_1 + k_2]_n$. We have to check if this addition is well defined and it is, as you can just multiply by a constant as $[k + rn]_n = [k]_n$. This is also a group with natural element of $[0]_n$ the inverse of $[k]_n$ is just $[-k]_n$ as $[k]_n + [-k]_n = [0]_n$. This is a finite abelian group and $|\mathbb{Z}_n| = n$.

There is two worlds, non-commutative and commutative. Nature is not commutative, things aren't that nice. Our best example of the non-commutative group is the group of permutations. Let $n \in \mathbb{Z}^+$ and then let there be a set $S_n = \{1, 2, \dots, n\}$ and consider all possible bijections σ from that set to itself. As these are finite sets and of the same cardinality, it suffices to check it's injective.

$$\begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n-1) & \sigma(n) \end{pmatrix}$$

saying this is a bijection says the bottom row, given they are integers from 1 to n, appear only once, they don't appear twice.

Example. Let us take S_4, then we can take an element,

$$\sigma = \begin{pmatrix} 4 & 3 & 2 & 1 \end{pmatrix}$$

and we can call this σ and is an element of the group.

New question, what is $|S_n|$, how many σ are there? It's $n!$.

Proof. Define σ and you have to consider $\sigma(1)$ and theres n possibilities, then for $\sigma(2)$ theres $n-1$ possibilities, then we can't use $\sigma(1)$ or $\sigma(2)$ and hence theres $n-2$ possibilities for $\sigma(3)$ and so on. So we have,

$$n(n-1)\cdot(n-2)\cdot(n-3)\ldots 2\cdot 1 = n!$$

$\square$

We can form a group where the composition is just $\circ$ on our set of bijections σ. If we take a $\sigma\circ\tau$ then this is also a bijection into S_n. This is associative and we get a natural element of id_{S_n}. Then every bijection has an inverse σ^{-1}, which is unique. What is σ^{-1}, just reverse the order of the rows,

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$$

This group is non-commutative if $n \geq 3$ then S_n is not commutative. If we an integer $1 \leq k \leq n$ and take k elements $\{a_1, a_2, \ldots, a_k\} \subset \{1, 2, 3, \ldots, n\}$. Then we define

Definition 1.2 (k-cycle)**.** A k cycle, $\sigma = (a_1, a_2, \ldots, a_k) \in S_n$ is a permutation,

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_{k-1} & a_k \\ a_2 & a_3 & \ldots & a_k & a_1 \end{pmatrix}$$

A k-cycle is a permutation and a bijection as you only write each number from 1 to n once. The 1-cycle is just the identity. The 2-cycle is the transposition. Then onwards it just shifts elements around. We can count the number of k-cycles, which is,

$$\frac{n(n-1)\ldots(n+k-1)}{k}$$

We can now see the dihedral group D_{2n},

Definition 1.3 (Dihedral Group)**.** Let us take the n-gon ($n \geq 3$) and depending on when n is odd or even we have a vertex along with the vertex one, you get them lying on the y-axis. Then you get all the rotations symmetries in the plane, which maps the n-gon to itself. There are $2n$ of them, the rotation clockwise with angle $\frac{2\pi}{n}$, there are n of these. Then we have the elements where we flip the shape, s, first where $s^2 = 1$.

$$D_{2n} = \{1, r, r^2, \ldots, r^{n-1}, s, sr, sr^2, \ldots, sr^{n-1}\}$$

Then this is our $2n$ elements. This is indeed a group with composition of rotations and $n \geq 3$ then the group isn't abelian. We also have the interesting rule which spits out the non-commutative behavior, *Lecture 2*

$$sr^i = r^{-i}s = r^{n-i}s$$

We can describe the group by it's elements and it's composition rule. We can define D_4 quite nicely,

$$D_4 = \{1, r, s, sr\}$$

and we find this to be commutative. Hence, D_4 is abelian.

Lemma 1.4. The following are true:

- The natural element is unique
- The inverse of each element is unique
- $(ab)^{-1} = b^{-1}a^{-1}$
- $au = av \implies u = v$ and $ub = vb \implies u = v$.
- Exponentiation makes sense
- Associativity means that any string of elements combined with the composition rule can be done in any order.

1.1 Subgroups and Orders

Definition 1.5 (Subgroup)**.** A subgroup, $H \subset G$, of a group $(G, \cdot)$,

- $\forall x, y \in H, x \cdot y \in H$
- $\forall x \in H, x^{-1} \in H$

This leads to also us being able to say $x \cdot x^{-1} = e_G \in H$, so the natural element must also be in H.

Example.
- $(G, \cdot)$ is a subgroup of itself.
- We can take the trivial subgroup $\{e_G\}$.
- Given a $m \in \mathbb{Z}$ the subset $m\mathbb{Z} = \{mk : k \in \mathbb{Z}\}$ of integers is a subgroup of $(\mathbb{Z}, +)$.
- If we take $\{1, r, r^2, \ldots, r^{n-1}\}$ this is a subgroup of D_{2n}.

Definition 1.6 (Order of an element)**.** Let G be a group and $a \in G$. The order of a is,

$$\operatorname{ord}(a) = \min\{n \geq 1 : a^n = e_G\}$$

If you never reach the natural element, we call $\operatorname{ord} a$ to be infinite.

Lemma 1.7. The following are true,

- $\operatorname{ord} a = 1$ if and only if $a = e_G$
- Let $0 \neq n \in \mathbb{Z}$, then $\operatorname{ord} n = \infty$
- Every element in a finite group must have finite order. As if the order was infinite, then you must have infinitely elements, namely, $\{1, a, a^2, a^3, \ldots, a^i, a^{i+1}, \ldots\}$ which are all distinct and so G cannot be finite.
- Consider some $k = \operatorname{ord} a < \infty$ and $n \geq 1$ with $a^n = e_G$, then $k \mid n$

  Proof. We have instantly that $n \geq k$ and now let $n = tk + r$ with $0 \leq r < k$. Then, $a^n = a^{tk+r} = a^{tk} \cdot a^r = (a^k)^t a^r = e_G^t a^r = a^r = e_G$. Hence, we can say that $r = 0$ as n is the smallest number such that $a^n = e_G$. □

If we consider the symmetric group, then we can say,

Lemma 1.8. Let $n \geq k \geq 1$ and $\sigma = (a_1, a_2, \ldots, a_k) \in S_n$ and is a k-cycle. Then $\operatorname{ord} \sigma = k$. Further, if $\sigma \in S_n$ then one can write $\sigma = \tau_1 \circ \tau_2 \circ \cdots \circ \tau_m$ and we can find the order of this disjoin composition of cycles. We find that this is, $\operatorname{ord}(\operatorname{lcm}(\tau_i))_{i=0}^m$

Remark. Disjoint cycles commute and the decomposition is unique.

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Lemma 1.9. If we take $\mathbb{Z}_n$, then we can take the order of say $[k]$, then we say that,

$$\operatorname{ord}[k] = \frac{n}{\gcd(n, k)}$$

Definition 1.10 (Generator)**.** If G is a group, $a \in G$, the subset $H = \{a^n : n \in \mathbb{Z}\}$ of G consisting of all powers of the element a is a subgroup, and is called the cyclic subgroup of G generated by a, and a is called a generator of H. The subgroup is denoted by $\langle a \rangle$.

Definition 1.11 (Cyclic Group)**.** A group G is called cyclic if $\exists\, a \in G$ such that $G = \langle a \rangle$ equals the (sub)group generated by a.

Lemma 1.12. If a group is generated by a, it is also generated by a^{-1}

Proof. If we have any a, then we can write this: $a = (a^{-1})^{-1}$ and so the generator is not unique. □

We notice that this works because we can cycle around n and this can be proved using Euclidean division.

Example.

- $\mathbb{Z} = \langle 1 \rangle$, is an infinite cyclic group generated by 1. NB! Here $a^n = a \cdot n$
- on a similar note, $\mathbb{Z}_n = \langle [1]_n \rangle$. However, we can go further! If $k \geq 1$, with $\gcd(k, n) = 1$, then $\mathbb{Z}_n = \langle [k]_n \rangle$ is also generated by $[k]_n$. This is proved as $\text{ord}[k]_n = \frac{n}{\gcd(k,n)} = n$ and so the order is the group and so $H = \langle k \rangle = \mathbb{Z}_n$.
- We can talk about $H = \langle (1234) \rangle$, which is a cyclic subgroup of S_4.

Definition 1.13 (Product of Groups)**.** Let $(G, \circ)$ and $(H, *)$ be two groups. We define a new group $(G \times H, \cdot)$ called the product group of G and H, as follows,

$$G \times H = \{(g, h) : g \in G, h \in H\}$$

is the set-theoretic product of G and H. The composition law $(\cdot)$ is defined by,

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 \circ g_2, h_1 * h_2)$$

The from this, the rest of the group axioms follow trivially.

Lemma 1.14. Let $(G, \circ)$ and $(H, *)$ be groups. If G and H are abelian, then so is $G \times H$. If both G and H are finite, then so is $G \times H$ and $|G \times H| = |G||H|$

Proof. Assume that G, H are abelian, and $g_1, g_2 \in G$ and $h_1, h_2 \in H$ then $(g_1, h_1) \cdot (g_2, h_2) = (g_1 \circ g_2, h_1 * h_2) = (g_2 \circ g_1, h_2 * h_1) = (g_2, h_2) \cdot (g_1, h_1)$, hence abelian. If both groups are finite, then the number of elements in $G \times H$ is the same as the number of pairs of elements and so that must be $|G| \times |H|$. □

1.2 Homomophism

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Definition 1.15 (Homomophism)**.** Let there be a group $(G, \circ)$ and $(H, *)$ and define a homomophism from $G \to H$ which satisfy,

(i) For $g_1, g_2 \in G$, $f(g_1 \circ g_1) = f(g_1) * f(g_2)$

(ii) $f(e_G) = e_H$

If we take $\mathbb{Z} \to \mathbb{Z}_n$ then we define the map $f(k) = [k]_n$ and we can see this by, $f(k_1 + k_2) = [k_1 + k_2]_n = [k_1]_n + [k_2]_n = f(k_1) + f(k_2)$

$$\begin{aligned} f(k_1 + k_2) = [k_1 + k_2]_n \\ &= [k_1]_n + [k_2]_n \\ &= f(k_1) + f(k_2) \end{aligned}$$

So this is a homomophism and it's surjective. If we let $f : \mathbb{Z} \to \mathbb{Z}$ and have $m \to km$ and this is also a homomorphism.

$$\begin{aligned} f(k_1 + k_2) &= m(k_1 + k_2) \\ &= mk_1 + mk_2 \\ &= f(k_1) + f(k_2) \end{aligned}$$

Definition 1.16 (Image)**.** Let $f : G \to H$ be a homomorphism, we define the image as,

$$\operatorname{Im} f = \{h \in H \mid \exists\, g \in G, h = f(g)\}$$

Definition 1.17 (Kernel)**.** Let $f : G \to H$ be a homomorphism, we define the kernel as,

$$\operatorname{Ker} f = \{h \in H \mid f(h) = e_G\}$$

For example, consider $f : \mathbb{Z} \to \mathbb{Z}_n$ where $f(k) = [k]_n$ and so we can say $\operatorname{Ker} f = \{nz \mid z \in \mathbb{Z}\}$, we notice this is a subgroup. However, if $g : \mathbb{Z} \to \mathbb{Z}$ where $z \mapsto mz$ we say $\operatorname{Ker} g = \{0\}$ if $m \neq 0$, another subgroup. This leads us to the following lemmas,

Lemma 1.18. $\operatorname{Im} f$ is a subgroup of H and $\operatorname{Ker} f$ is a subgroup of G.

Proof. The first part, follows quite nicely from absorbing and splitting using the definition of group homomorphisms. the second part is also follows nicely, so we verify the subgroup axiom,

- Closure, $g_1, g_2 \in \operatorname{Ker} f$ and so, $f(g_1) = f(g_2) = e_H$ and show $f(g_1 \circ g_2) = f(g_1) * f(g_2) = e_H * e_H = e_H$.
- If $f(g) = e_H$ then prove $f(g^{-1}) = e_H$ and so, $e_H = f(g \circ g^{-1}) = f(e_G) = f(g) * f(g^{-1})$, hence, $f(g^{-1}) = (f(g))^{-1}$. Hence, $f(g)^{-1} \in \operatorname{Ker} f$.

$\square$

Lemma 1.19. Let $f : G \to H$ be a homomorphism.

- f is surjective if and only if $\operatorname{Im} H = f$.
- f is injective if and only if $\operatorname{Ker} f = e_G$

Proof. Assume that f is injective, so $\operatorname{Ker} f = \{e_G\}$, so if $g \in \operatorname{Ker} f$ then $g = e_G$. We also know that the kernel also always contains e_G and g and we know f is injective and so $g = e_G$ as they both map to e_H. Now suppose that $\operatorname{Ker} f = \{e_G\}$ and show that f is injective. Take $g_1, g_2 \in G$ and assume that $f(g_1) = f(g_2)$. We get $f(g_1) \circ f(g_2)^{-1} = e_H$ and so, $f(g_1 \circ g_2^{-1}) = e_H$ and hence, we must have $g_1 \circ g_2^{-1} \in \operatorname{Ker} f$. However $\operatorname{Ker} f = \{e_G\}$ and so, $g_1 \circ g_2^{-1} = e_G$ and so, $g_1 = g_2$. $\square$

2 Cosets and Normal Subgroups

Consider G be a group an consider a subgroup H of G. We want to define the left coset, but before we define a relation, *Lecture 5*

Definition 2.1 (Relation)**.** $x \sim y \implies x^{-1}y = h \in H$

This can then be proved to be an equivalence relation,

Proof. (i) Reflexive, $x \sim x$ which means $x^{-1}x = e_G \in H$ as H is a subgroup.

(ii) Symmetry, $x \sim y \implies y \sim x$. If $x \sim y$, $y = xh$ implies $yh^{-1} = x$ but $h^{-1} \in H$ and so $y \sim x$.

(iii) Transitivity, $x \sim y$ and $y \sim z$ then $x \sim z$. We have $y = xh$ and $z = yh'$ and so $z = yhh'$ and $hh' \in H$ and so $x \sim z$.

□

Now we can consider equivalence classes of elements of this relation, which is,

$$\overline{x} = \{x \sim y \mid y \in G\} = \{xh \mid h \in H\} = xH$$

Definition 2.2 (Left Coset)**.** We define the left coset as this equivalence relation.

We also know that equivalence classes form a partition,

$$G = \bigcup_{x \in G} \overline{x} = \bigcup_{x \in G} xH$$

Cosets are also not unique, we can have $x_1H = x_2H$ when $x_1 \sim x_2$.

If we consider all of the left cosets $(G/H)_{\text{left}} = \{xH : x \in G\}$. If G is finite, so there are finitely many left cosets. This is the index of $H \in G$ and denoted, $|G : H|$

Example. Consider $\mathbb{Z}$ and $n\mathbb{Z}$ as our groups, then if we consider $a \sim b$ this is just saying $-a + b \in n\mathbb{Z}$, however this just says $b - a \in n\mathbb{Z}$ which is the definition for divisibility. Let $a \in \mathbb{Z}$, then $a = kn + r$, then we can say $a \sim r$ which is equivalent to $\overline{a} = \overline{r}$. Hence,

$$\mathbb{Z}/n\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{n-1}\} = n\mathbb{Z}$$

Theorem 2.3 (Legrange's Theorem)**.** Let G be a group and H be a subgroup. Then,

$$|G| = |H||G : H|$$

Proof. Firstly, we aim to show that all left cosets have the same number of elements, more specifically $|H| = |xH|$. We aim to find a bijection $H \to xH$, we can try $x \mapsto xh$. Now prove this is a bijection, surjectivity is obvious, so prove injectivity. Hence we prove that if $\phi(h_1) = \phi(h_2)$ then $h_1 = h_2$. We have that $xh_1 = xh_2$ and so injectivity is clear. So we can say that $|H| = |xH|$, and as we know,

$$G = \bigcup_{x \in G} xH$$

then $|G| = |G : H||H|$ □

Corollary 2.4.

- Let G be a finite group and H a subgroup. Then $|H| \mid |G|$.
- Let G be a finite group and $x \in G$ then $\text{ord}(x) = |\langle x \rangle| \mid |G|$

Theorem 2.5 (Cauchy's Theorem)**.** Let G be finite group and let p be a prime, then if $p \mid |G|$, then you can find a subgroup and an element of order p

We will see sylows theorem later, which is a converse to Legranges theorem and instead of relating just to p, it related to p^n.

Suppose that H is a subgroup, we have seen a left coset, xH. We can do the same with Hx which is the right coset. In general $xH \neq Hx$ as the group law is not generally commutative, as we want $xh = h'x$. However this works for more than just commutativity, so we define a normal subgroup.

2.1 Normal Subgroups

Definition 2.6 (Normal Subgroup)**.** A subgroup H of G is called normal if,

$$xH = Hx = \{h'x : h' \in H\} \qquad \forall\, x \in G$$

Lets consider a non-example,

Example. Consider $K = \langle s \rangle$ of D_8 and we claim it's not normal, so $rK = Kr$. We have $H' = \{1, s\}$ and $rK = \{r, rs = sr^2\}$ and $Kr = \{r, sr\}$[1]. However, $Kr \neq rK$ as $sr \neq sr^2$. Hence, not normal.

Definition 2.7 (Conjugate)**.** Two elements $g, h \in G$ if we can find a $x \in G$ such that, *Lecture 6*

$$g = xhx^{-1}$$

and we call it the conjugate of g by x.

If we consider a subgroup to be normal we must have $Hx = xH$, this is equivalent to saying $H = xHx^{-1} = \{xhx^{-1} : h \in H\}$. This can be seen by writing $xh = hx$.

Lemma 2.8. If we have a group homeomorphism $\phi : G \to H$, then $\operatorname{Ker}\phi$ is a normal subgroup.

Proof. So we have to prove that any $g \in \operatorname{Ker}\phi$ and then $xgx^{-1} \in \operatorname{Ker}\phi$ and so consider $f(xgx^{-1}) = f(x)f(g)f(x^{-1}) = f(x)e_H f(x)^{-1} = f(x)f(x)^{-1} = e_H$ as so $xgx^{-1} \in \operatorname{Ker}\phi$ as required. □

Now we will consider the symmetry group. If we have some $\sigma \in S_n$, then we can decompose a σ uniquely as $\sigma = (a_1\, a_2\, \ldots\, a_{n_1}) \ldots (b_1\, b_2\, \ldots\, b_{n_k})$. The k-tuple of $(n_1\, n_2\, \ldots\, n_k)$ is called the cycle type of σ.

Example. The permutation $(1\,2)(3\,4\,5\,6)$ has type (2, 4).

Proposition 2.9. If two permutations are conjugate if and only if they have the same cycle type.

Proof. In notes □

Consider our permutation $\sigma = (1\,2)(3\,4\,5\,6)$ and another one of the same type $\widetilde{\sigma} = (3\,4)(1\,2\,5\,6)$ then there exists $\tau \in S_6$ such that $\widetilde{\sigma} = \tau\sigma\tau^{-1}$ we write out,

$$\begin{array}{ll} \sigma & (1\,2)(3\,4\,5\,6) \\ \widetilde{\sigma} & (3\,4)(1\,2\,5\,6) \\ \tau & (1\,3)(2\,4)(5)(6) \end{array}$$

The important thing is that, τ is not unique. Note, that in S_3 all three elements must be conjugate. We have two three cycles and two transpositions, and we know that a two cycle can't be conjugate to a three cycle, which shows the power of this proposition.

In S_n we have a subgroup A_n (the subgroup of even permutations). If $\sigma = (a_1\, a_2\, \ldots\, a_k)$, ie. a k-cycle.

[1] Check this

Definition 2.10 (Signature)**.** If we consider $\varepsilon : S_n \to \{\overline{0}, \overline{1}\}$ and consider a new map, $\sigma \mapsto \varepsilon(\sigma)$ where we define,

$$\varepsilon(\sigma) = \begin{cases} 0 & \text{if } \sigma \text{ is even} \\ 1 & \text{if } \sigma \text{ is odd} \end{cases}$$

A k-cycle can be considered as a product of transpositions is to start with $\sigma = (a_k\, a_{k-1})\,(a_{k-2})(a_{k-3})\dots(a_1\, a_0)$. We can also say that A_n is normal as if we consider ε we really have $\mathbb{Z}/2\mathbb{Z}$ and we have a homomorphism, ie. $\varepsilon(\sigma_1\sigma_2) = \varepsilon(\sigma_1)\varepsilon(\sigma_2)$. The kernel is just the even permutations, A_n. Hence, A_n is normal.

Take two $\sigma_1, \sigma_2 \in A_n$, when are they conjugate in A_n? Hence find, $\tau \in A_n$ such that $\sigma_2 = \tau\sigma_1\tau^{-1}$. We need them to find two of the same cycle type, but we see that this τ doesn't exist. Consider $A_4 = \{e,\, (1\,2\,3),\, (a\,b)(c\,d)\}$, if we look to the product of transpositions, they are conjugate, but if we look at the three cycles, $(1\,2\,3)(1\,3\,2)$ there doesn't exist a $\tau \in A_4$.

2.2 Quotient Groups

We are going to consider a factor group, so we are going to start with H, a normal subgroup of G. *Lecture 7*

Definition 2.11 (Quotient Group Law)**.** We define a composition law $(\cdot)$ on the set of left cosets G/H by,

$$\begin{gathered}(\cdot) : G/H \times G/H \to G/H \\ (xH, yH) \mapsto xH \cdot yH = xyH\end{gathered}$$

This is well defined as H is normal, $x'H = xH$ and $y' = yH \implies x'y'H = xyH$.

Proposition 2.12. $(G/H, \cdot)$ is a group and it is called the quotient group of G by H

Proof. Associativity can be checked quickly, then $e_{G/H}$ is just $e_G H = H$, we can see this by $e_G H \cdot xH = e_g xH = xH$. The inverse, is just $x^{-1}H$, then we see, $xHx^{-1}H = xx^{-1}H = e_G H = H$ $\square$

Now consider $\phi : G \to G/H$ and get $\phi(g) = gH$. This is a group homomorphism.

Proposition 2.13. The map ϕ is a group homomorphism and $\operatorname{Ker}\phi = H$.

Proof. The fact that ϕ is surjective is clear as $gH = \phi(g)$. It is a homomorphism as,

$$\phi(g_1 g_2) = g_1 g_2 H = (g_1 H) \cdot (g_2 H) = \phi(g_1)\phi(g_2)$$

We now show that $\operatorname{Ker}\phi = H$, first $H \subset \operatorname{Ker}\phi$ since if $g \in H$, then $e_G^{-1}g = g \in H$ and $e_G \sim g$ hence $\phi g = gH = e_G H$. Conversely let $g \in \operatorname{Ker}\phi$ meaning $\phi g = gH = e_G H$, then $e_G \sim g$ and $e_G^{-1}g = g \in H$. $\square$

2.2.1 First Isomorphism Theorem

Theorem 2.14 (First Isomorphism Theorem)**.** Suppose $f : G \to H$ is a group homomorphism. The quotient group $G/\operatorname{Ker}(f) \cong \operatorname{Im}(f)$

Proof. Consider $\pi : G/\operatorname{Ker}(f) \to \operatorname{Im}(f)$ defined by $\pi(g\operatorname{Ker}(f)) = f(g)$ and we show π is a group isomorphism. Firstly, check π is well defined. Assume $g\operatorname{Ker}(\pi) = g'\operatorname{Ker}(\pi)$ meaning $g'^{-1}g = \widetilde{g} \in \operatorname{Ker}(\pi)$. Then,

$$\begin{aligned} f(g) &= f(g'\widetilde{g}) \\ &= f(g')f(\widetilde{g}) \\ &= f(g')e_H \\ &= f(g') \end{aligned}$$

since $\widetilde{g} \in \operatorname{Ker}(f)$. Further π is a homomorphism:

$$\begin{aligned}
\pi(g \operatorname{Ker}(f) \cdot g' \operatorname{Ker}(f)) &= \pi(gg' \operatorname{Ker}(f)) \\
&= f(gg') \\
&= f(g)f(g') \\
&= \pi(g \operatorname{Ker}(f))\pi(g' \operatorname{Ker}(f))
\end{aligned}$$

The homomorphism is surjective, if $f(g) \in \operatorname{Im}(f)$, $g \in G$, then $f(g) = \pi(g \operatorname{Ker}(f))$. It is also injective, assume $f(g) = \pi(g \operatorname{Ker}(f)) = \pi(g' \operatorname{Ker}(f)) = f(g')$, then $f(g')^{-1}f(g) = f(g'^{-1}g) = e_H$ and $g'^{-1}g \in \operatorname{Ker}(f)$and so $g \operatorname{Ker}(f) = g' \operatorname{Ker}(f)$. □

Corollary 2.15. Suppose G is finite, and we have a group homomorphism, $f : G \to H$, then,

$$\frac{|G|}{|\operatorname{Ker}(f)|} = |\operatorname{Im}(f)|$$

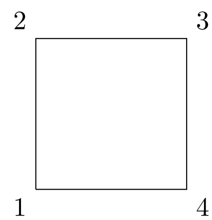
Proof. As $G/\operatorname{Ker}(f) \cong \operatorname{Im}(f)$ then if G is finite, then everything is finite. Further, we can say $|G/H| = |\operatorname{Im} f|$, now applying Legranges Theorem, we get the result,

$$\frac{|G|}{|\operatorname{Ker} f|} = |\operatorname{Im} f|$$

□

3 Group Actions

Groups acts on sets and so we can focus our attention to something called group actions. Let's start with a motivating example. Consider D_8, which is linked to the four vertices of a square. We can consider a rotation of $\frac{\pi}{2}$ and s which is just the symmetry. D_8 acts on the vertices $1, 2, 3, 4$ *Lecture 8*



What it does to this square is just a group action.

Definition 3.1 (Group Action)**.** Let $(G, *)$ be a group and a set A. A group action is a map,

$$(\cdot) : G \times A \to A$$

$$(g, a) \mapsto g \cdot a$$

satisying,

$$(g_1 * g_2) \cdot a = g_1 \cdot (g_2 \cdot a) \quad \forall\, g_1, g_2 \in G, \quad a \in A \tag{1}$$

$$e_G \cdot a = a \quad \forall\, a \in A \tag{2}$$

A group can act on itself, in two ways; by left multiplication and conjugation.

Definition 3.2 (Action by left multiplication)**.** Consider $(\cdot) : G \times G \to G$ and define $(h, g) \mapsto h \cdot g = h * g$. Axiom (1) is satisfied,

$$(h_1 * h_2) \cdot g = (h_1 * h_2) * g = h_1 * (h_2 * g) = h_1.(h_2.g)$$

and axiom (2) is also satisfied.

Definition 3.3 (Action by conjugation)**.** A group $(G, *)$ acts on itself defined by $(h, g) \mapsto (h \cdot g) = h * g * h^{-1}$. Now check the axioms,

$$\begin{aligned}(h_1 * h_2) \cdot g &= (h_1 * h_2) * g * (h_1 * h_2)^{-1} \\ &= (h_1 * h_2) * g * (h_2^{-1} * h_1^{-1}) \\ &= h_1 * (h_2 * g * h_2^{-1}) * h_1^{-1} \\ &= h_1 \cdot (h_2 \cdot g)\end{aligned}$$

The second axiom is also satisfied.

We are now going to consider a permutation action, if we have a map, $\tau_g : A \to A$ such that $\tau_g(a) = g \cdot a$ and this is a bijection. It has an inverse, $t_{g^{-1}} : A \to A$,

$$\tau_{g^{-1}} \circ \tau_g = \tau_g \circ \tau_{g^{-1}} = \mathrm{id}_A$$

Or more precisely,

$$\begin{aligned}(\tau_{g^{-1}} \circ \tau_g)(a) &= \tau_{g^{-1}}(\tau_g(a)) \\ &= \tau_{g^{-1}}(g \cdot a) \\ &= g^{-1} \cdot (g \cdot a) \\ &= (g^{-1} * g) \cdot a \\ &= e_G \cdot a \\ &= a\end{aligned}$$

Definition 3.4 (Permutation Representation)**.** Let $(S_A, \circ)$ be the group of all bijections from $A \to A$; S_A is the group of symmetries of A, the group law is just composition of bijections. The map,

$$\tau : G \to S_A$$

is defined by,

$$\tau(g) = \tau_g$$

is a group homomorphism,

$$\begin{aligned}\tau(g_1 * g_2)(a) &= (g_1 * g_2) \cdot a \\ &= g_1 \cdot (g_2 \cdot a) \\ &= \tau_{g_1}(\tau_{g_2}(a)) \\ &= (\tau(g_1) \circ \tau(g_2))(a)\end{aligned}$$

and we call τ the permutation representation associated to the action $(\cdot)$.

If A is finite, say $|A| = n$, then we can list the elements of $A = \{a_1, \ldots, a_n\}$ and label them. This isn't unique, but then what is the group of bijections? It's just S_n.

We now define the kernel of a representation,

Definition 3.5 (Kernel of representation)**.** The kernel of $\tau : G \to S_A$

$$\operatorname{Ker} \tau = \{g \in G : \tau_g = \mathrm{id}_A\} = \{g \in G : g \cdot a = a\}$$

is just the kernel of the representation τ. If we find $\operatorname{Ker} \tau = \{e_G\}$, or τ is injective, we say $(\cdot)$ is faithful.

Lecture 9

3.1 Stabilisers and Orbits

Consider a group G acting on a set A,

Definition 3.6 (Stabiliser)**.** We define the following set called the stabiliser

$$\operatorname{Stab}(a) = \{g \in G : g \cdot a = a\}$$

Remark. These are the elements that when acted on g doesn't change it. They fix g.

The interesting thing is that

Proposition 3.7. $\operatorname{Stab}(a)$ is a subgroup of G.

Proof. We begin by seeing that $e_G \cdot a = a$ and so $e_G \in \operatorname{Stab}(a)$. Then we can prove that $g^{-1} \in \operatorname{Stab}(a)$,

$$a = e_G \cdot a = (g^{-1} \cdot g) \cdot a = g^{-1} \cdot (g \cdot a) = g^{-1} \cdot a$$

Furthermore, let $g_1, g_2 \in \operatorname{Stab}(a)$ and then,

$$(g_1 * g_2) \cdot a = g_1 \cdot (g_2 \cdot a) = g_1 \cdot a = a$$

□

Let us define a relation amoung elements of a non-empty set, $a \sim b \iff \exists\, g \in G : a = g \cdot b$

Proposition 3.8. This relation is an equivalence relation.

Proof. Simple. □

Definition 3.9 (Orbit)**.** Let $a \in A$. The equivalence class of a for the relation $\sim$ is,

$$\overline{a} = \{b \in A : \exists\, g \in G, b = g \cdot a\} = \{g \cdot a : g \in G\}$$

is called the orbit of a, for the given action, ad is denoted $\operatorname{orb}(a)$.

We note that also,

$$A = \bigcup_{a \in A} \operatorname{orb}(a)$$

meaning A is equal to the disjoint union of its orbits under the given action of G.
The action is transitive if there is only one orbit in which case this orbit necessarily contains all elements of A. In this case, $A = \operatorname{orb}(a)$ for every $a \in A$.

Example. For example consider the action,

$$\begin{aligned} S_n \times \{1, 2, \ldots, n \ &\to \{1, 2, \ldots, n\} \\ (\sigma, i) &\mapsto \sigma(i) \end{aligned}$$

Then this is transitive as we just have to find that for any i and j we can map i to j. Hence, if $i = j$, then take the identity. Otherwise, take the transposition (ij).

Theorem 3.10 (The Orbit-Stabiliser Theorem)**.** Assume $(G, *)$ is a group acting on a set A and G is finite. Then the orbit $\operatorname{orb}(a)$ of an element $a \in A$ is finite and,

$$|\operatorname{orb}(a)| = \frac{|G|}{|\operatorname{Stab}(a)|}$$

Proof. Consider the map,

$$f : \operatorname{orb}(a) to (G / \operatorname{Stab}(a))$$

defined by,

$$f(g \cdot a) = g \cdot \operatorname{Stab}(a)$$

We check that this map is well defined, it is. Then we prove that this is injective and it is surjective. Then f is a bijection. Hence, we get nicely the result. □

Recall the left regular representation of G on itself by left multiplication. Assume G is of finite cardinality n. If we lavel the elements of G as $\{g_1, \ldots, g_n\}$ the regular representation defined a faithful permutation representation (an injective homomorphism), *Lecture 10*

$$\rho : G \to S_n$$

called the regular permutation representation defined as followed,

Definition 3.11 (Regular permutation representation)**.** If $g \in G$, $\rho(g)$ is the permutation defined for $i, j \in \{1, \ldots, n\}$ by,

$$\rho(g)(i) = j \qquad \text{if } g * g_i = g_j$$

The permutation representation ρ depends on the give labelling of the elements of G. In particular, since $\operatorname{Ker}\rho = \{e_G\}$ we obtain by the FIT that G is isomorphic to it's image $\rho(G)$; a subgroup of S_n. Hence, we obtain,

Theorem 3.12 (Cayley's Theorem)**.** A finite group of cardinality n is isomorphic to a subgroup of S_n.

Next, we define the left action of a group G on the set of left cosets of a given subgroup. Let H be a subgroup of G and $A = (G/H)_{left}$ the set of left cosets of H. The group G acts on A by

$$g \cdot (g'H) = (g * g')H$$

This action is called the action of G on the left cosets of H by left multiplication. If $|G : H| = m$ is finite, and we lavel the elementsof A as $\{g_1H, \ldots, g_mH\}$, then the above representation defines a homomorphism

$$\tau : G \to S_m$$

as follows: if $g \in G$, $\tau(g)$ is the permutation defined for $i, j \in \{1, \ldots, m\}$ by

$$\tau(g)(i) = j \qquad \text{if } g \cdot g_jH = (g * g_i)H = g_jH$$

The permutation representation τ depends on the given labelling of the elements of A.

Let $\tau_H : G \to S_{G/H}$ be the permutation representation associated to the action of G by left multiplication on the left cosets of H. Thus if $g \in G$,

$$\tau_H(g) : G/H \to G/H$$

is the bijection defined by,

$$\tau_H(g)(g'H) = (g * g')H$$

Theorem 3.13. The following hold,

- G acts transitively on G/H.
- The stabiliser of e_GH is the subgroup H.
- $\operatorname{Ker}(\tau_H) = \bigcap_{x \in G} xHx^{-1}$, and $\operatorname{Ker}(\tau_H)$ is the largest normal subgroup of G contained in H.

Proof.

- Let $aH, bH \in G/H$ and $g = b * a^{-1}$. Then

$$\begin{aligned} g \cdot (aH) &= (b * a^{-1}) \cdot aH \\ &= (b * a * a^{-1})H \\ &= bH \end{aligned}$$

- The stabiliser of e_GH is

$$\begin{aligned} \{g \in G : g.(e_GH) = gH = H\} &= \{g \in G : gH = H\} \\ &= H \end{aligned}$$

– By definition,

$$\begin{aligned}
\mathrm{Ker}(\pi_H) &= \{g \in G : g \cdot (xH) = xH, \forall x \in G\} \\
&= \{g \in G : (g * x)H = xH, \forall x \in G\} \\
&= \{g \in G : (x^{-1} * g * x)H = H, x \in G\} \\
&= \{g \in G : x^{-1} * g * x \in H, \forall x \in G\} \\
&= \{g \in G : g \in xHx^{-1}, \forall x \in G\} \\
&= \bigcap_{x \in G} xHx^{-1}
\end{aligned}$$

Further $\mathrm{Ker}(\pi_H)$ is a normal subgroup of both G and H. Now let N be a normal subgroup of G contained in H then $N = xHx^{-1} \subset xHx^{-1}, \forall x \in G$ hence, $N \subset \bigcap_{x\in G} xHx^{-1} = \mathrm{Ker}(\pi_H)$. This shows that $\mathrm{Ker}(\pi_H)$ is the largest subgroup of G contained in H.

□

Corollary 3.14. Let G be a finite group of cardinality n and p the smallest prime number dividing $n = |G|$, then any subgroup of G of index p is normal. In particular, if G has a subgroup of index 2 then this subgroup must be normal.

Proof. Let $H \leq G$ and then π_H is the permutation representation by the multiplication of left cosets of H in G. Let $K = \mathrm{Ker}\, pi_H$ and so $|G : K| = |G : H||H : K| = pm$.

Since H has p left cosets, G/K is isomorphic to a subgroup of S_p ($\pi_H(G)$) by the FIT. By Lagrange's Theorem, $pm = |G/K| \mid |S_p| = p!$. Thus $m \mid \frac{p!}{p} = (p-1)!$. But all prime factors of $(p-1)! < p$ and by the minimality of p, every possible prime divisor of $m \geq p$. This forces $m = 1$ as $m \mid |G|$, so $H = K$ is a normal subgroup of G (since K is normal): the equality $|H : K| = 1$ just means that $H = K$. □

4 Class Equation

4.1 Normalisers, Centralisers and Centers

Lecture 11

We are going to consider the class equation which relates to conjuagtion. We are going to consider the subsets of G, $\mathcal{S}(G) = \{A \subset G\}$, these arent necessarily subgroups, they are just subsets. Then we have the following action,

$$(\cdot) : G \times \mathcal{S}(G) \to \mathcal{S}(G)$$
$$(g,\, A) \mapsto gAg^{-1} = \{gag^{-1} : a \in A\}$$

If you have a group action, you have a stabiliser and an orbit.

$$\mathrm{Stab}(A) = \{g \in G : gAg^{-1} = A\}$$

and this is the normaliser.

Definition 4.1 (Normaliser)**.** The stabiliser of the above group action,

$$N_G(A) = \{g \in G : gAg^{-1} = A\}$$

The normaliser is a subgroup of G as it is just a stabiliser. The normaliser of a acts on A itself,

$$\phi_A : N_G(A) \times A \to A$$
$$(g, a) \mapsto gag^{-1}$$

this is a group action and we are interested in this group action. We can go deeper and find the stabiliser of ϕ_A,

$$\mathrm{orb}(a) = \{gag^{-1} : g \in N_G(A)\}$$
$$\mathrm{Stab}(a) = \{g \in N_G : gag^{-1} = a \iff ga = ag\}$$

Hence the stabiliser is just the commuting elements of this. Hence, we now look towards the kernel of ϕ_A and we say that,

Definition 4.2 (Centraliser)**.** We say that the kernel of the ϕ_A is the centraliser,

$$C_G(A) = \mathrm{Ker}\, \phi_A = \bigcap_{a \in A} \mathrm{Stab}(a) = \{g \in N_A(a) L gag^{-1} = a, \forall\, a \in A\}$$

and these are just all the commuting elements.

and we know that

Lemma 4.3. The $C_G(A)$ is always a normal subgroup of $N_G(A)$.

Now we ask, what happens when $A = G$ and so we ask, what is $N_G(G)$? G, and what is $C_G(G)$? Well we write it out,

$$Z(G) = \{g \in G : gh = hg, \forall\, h \in G\}$$

and this is called the center of G.

Definition 4.4 (Center of G)**.** The center is a normal abelian subgroup of G such that,

$$Z(G) = \{g \in G : gh = hg, \forall\, h \in G\}$$

Example. If G is abelian, then $Z(G) = G$, so we are only interested when G is not abelian.

We also note that the center is contained in the centraliser of every subset of A. The center is the intersection of all subsets of $A \in G$.

4.2 The Class Equation

Let us consider $g \in G$ and the subset $\{g\} \subset G$, then,

$$N_G(\{g\}) = C_G(\{g\}) = \{h \in G : hgh^{-1} = g\} = \{h \in G : hg = gh\}$$

This is then the subgroup of elements that commute with g. We note that $C_G(g) = \mathrm{Stab}(g)$ is precisely the stabiliser of g under the conjugation action of G onto itself. The orbit of $\{g\}$ under conjuagtion is,

$$\mathrm{orb}(g) = \{hgh^{-1} : h \in G\}$$

and consists of all elements of G which are conjugate to g.
We note that $\mathrm{orb}(g) = \{g\} \iff hgh^{-1} = g$ for all $h \in G$ and this is equivalent to $g \in Z(G)$. Thus,

$$|\,\mathrm{orb}(g)| = 1 \iff g \in Z(G)$$

Now assume that G is finite. The orbit-stabiliser theorem states that,

$$|\,\mathrm{orb}(g)| = \frac{|G|}{|C_G(g)|}$$

The congugacy classes of elements of G form a partition of G

$$G = \bigcup_{g \in G} \mathrm{orb}(g)$$

where the union is disjoint. By the above discussion, we have,

$$Z(G) = \bigcup_{g \in G,\, |\,\mathrm{orb}(g)|=1} \mathrm{orb}(g)$$

where the union is over all of these elements of G with $|\,\mathrm{orb}(g)| = 1$. Let $\{\mathrm{orb}(g_1), \ldots, \mathrm{orb}(g_r)\}$ be the distinct conjuagacy classes of G that are **not** contained in $Z(G)$. Then,

$$G = Z(G) \bigcup \left(\bigcup_{i=1}^{r} \mathrm{orb}(g_i) \right)$$

Counting the number of elements of G, and soncisdering the relation $|\,\mathrm{orb}(g_i)| = |G : C_G(g_i)|$, we find the class equation:

Theorem 4.5 (The Class Equation)**.** Let G be a finite group and $\{\mathrm{orb}(g_1), \ldots, \mathrm{orb}(g_r)\}$ be the distinct conjuagacy classes of G which are **not** contained in $Z(G)$, then,

$$|G| = |Z(G)| + \sum_{i=1}^{r} |G : C_G(g_i)|$$