Year MAGIC — Algebraic Geometry

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Contents

In this course we will study some introductory algebraic geometry, we will study Classical Algebraic Geometry and Sheaves There are three chapters,

(i) Affine Varieties

(ii) Noetherian Rings

(iii) Algebraic Varieties in general

Literature: Karen Smith's Book, has lots of examples and is very readable. We will cover chapter one and the start of chapter two of Hardshawn.

Prerequistites: Commutative Algebra, Topology.

1 Affine Varieties

Algebraic Sets in n-space, we want to study zero sets of polynomials in several variables in affine spaces. The affine spaces are k-vector spaces. We will consider algebraically closed fields k.

Definition 1.1 (Affine n-space)**.** Let k be a field. We write $\mathbb{A}^n(k)$ to be an affine n-space over k. This is the set, $\{a_1, a_2, \ldots, a_n : a_i \in k\}$

Let $k[X_1, \ldots, X_n]$ be the polynomial ring in n-variables over k where $n < \infty$.

Definition 1.2 (Vanishing Set)**.** Let $f \in k[X_1, \ldots, X_n]$ then the zero-set of f is,

$$\mathcal{V}(f) = \{(a_1, \ldots, a_n) \in \mathbb{A}^n(k) : f(a_1, \ldots, a_n) = 0\}$$

Example. Let $k = \mathbb{R}$ and $n = 1$, then $f(X) = X + 1$,

$$\mathcal{V}(f) = \{-1\} \in \mathbb{A}^1(\mathbb{R})$$

Example. Let $k = \mathbb{R}$, $n = 2$ and $f(X, Y) = X^2 + Y^2 - 1$, then,

$$\mathcal{V}(f) = \{X, Y \in \mathbb{R}^2 : X^2 + Y^2 = 1\}$$

Example. Let $k = \mathbb{R}$, $n = 3$ and $f(X, Y, Z) = Z^3 + Z^2Y^2 - X^2$, this is not as obvious. The vanishing set is just some curve, and if we intersect it with a sphere we get,

This is slightly odd, it intersects itself and so this isn't a manifold and so is slightly more complicated.

More generally: $f_1, \ldots f_m \in k[X_1, \ldots, X_n]$, we define,

$$\mathcal{V}(f_1, \ldots, f_m) = \{a \in \mathbb{A}^n : f_1(a) = f_2(a) = \cdots = f_m(a) = 0\}$$

Even more generally, we can take any $S \subset k[X_1, \ldots, X_n]$, then

$$\mathcal{V}(S) = \{a \in \mathbb{A}^n : f(a) = 0 \, \forall f \in S\}$$

This allows us to have infinitely many functions. We call S an algebraic subset of $\mathbb{A}^n$.

Example.

$$\mathcal{V}(X^2 - Y, X^3 - Z) \subset \mathbb{A}^3(\mathbb{R})$$

This defines a smooth space curve.

Example. $M_{n\times n}(\mathbb{C})$ can be identified by $\mathbb{A}^{n^2}(\mathbb{C})$ and we can look at subsets of this space. Let $V = \{A \in M_{n\times n}(\mathbb{C}) : \det A = 1\}$. $V = \mathcal{V}(S)$ is an algebraic subset of $\mathbb{A}^{n^2}$. For $\mathbb{A}^{n^2}$ we associate $k[X_{ij}]$ where $1 \leq i, j \leq n$. Let $S = \Delta - 1$ where

$$\Delta(X_{ij}) = \det \begin{pmatrix} X_{11} & \ldots & X_{1n} \\ & \ddots & \\ X_{n1} & \ldots & X_{nn} \end{pmatrix}$$

We can say slightly more than this,

Remark. (i) $\mathbb{A}^n$ is a algebraic subset, 0 is a polynomial and we can see that $\mathcal{V}(0) = \mathbb{A}^n$.

(ii) $\varnothing$ is an algebraic set, $V(1) = \{a \in \mathbb{A}^n : 1(a) = 1 = 0\} = \varnothing$.

(iii) Algebraic sets are closed under intersection. Let $V(S_i)_{i\in\mathcal{I}}$ be a collection of algebraic sets in $\mathbb{A}^n$, then,

$$\bigcap_{i\in\mathcal{I}} V(S_i) = V\left(\bigcup_{i\in\mathcal{I}} S_i\right)$$

Proof. Exercise □

(iv) Algebraic sets are closed under **finite** unions. We want to show that the union of two algebraic sets is algebraic. Let $V(S), V(T)$ be algebraic sets in $\mathbb{A}^n$, let $S.T = \{fg : f \in S, g \in T\}$. Then we claim that $V(S) \cup V(T) = V(S.T)$. We aim to show both inclusions,

Proof. ($\subset$): Suppose $a \in V(S)$, then $f(a) = 0$ for all $f \in S$, but, $(f \cdot g)(a) = f(a) \cdot g(a) = 0$ for all $g \in T$. Therefore $a \in V(S.T)$.
($\supset$) Suppose $a \in V(S.T) \setminus V(S)$. Then there is some $f \in V(S)$ such that $f(a) \neq 0$, but then, for any $g \in T$ $fg(a) = f(a) \cdot g(a) = 0$ as $a \in V(S.T)$ and as we are in a field, and as $f(a) \neq 0$, then $g(a) = 0$ for all $g \in T$. Therefore $a \in V(T)$. □

Proposition 1.3. The collection of algebraic subsets of $\mathbb{A}^n(k)$ form the closed sets of a topology on $\mathbb{A}^n$. This topology is called the Zariski Topology on $\mathbb{A}^n$.

Here are some examples of closed sets,

Example. If $a \in \mathbb{A}^n$ is a point then $\{a\} = V(X_1 - a_1, X_2 - a_2, \dots, X_n - a_n)$ and so points are closed in the Zariski Topology.

Example. If $n = 1$ and $S = 0$, then $V(S) = \mathbb{A}^n$, but if $S \subset \mathbb{A}^n$ is algebraic, and if $\exists\, f \neq 0 \in S$ then since we have every polynomial in $k[X]$ has finitely many zeros. Then $\mathcal{V}(f)$ is finite. However $\mathcal{V}(S) \subset \mathcal{V}(f)$ and so $\mathcal{V}(S)$ must be finite. Therefore the Zariski Topology is cofinite, the sets are finite or the whole space.

We defined the most important algebraic variety last time and then we defined an algebraic set, $\mathcal{V}(S)$. We start with some remarks from last time. There is some issues about k, we assumed that we could take any k. If k is finite, then $p = p^e$, then for $a \in \mathbb{F}_p$ then $a^q = a$ by Euler Fermat Theorem. Then $f(X) = X^q - X \in \mathbb{F}_q[X]$, evaluates to 0 forall $a \in k$, but f is not the zero polynomial. So we have problems with finite fields.

The other issues, is more geometric. If we have $\mathbb{R}$, then $X^2 + 1 \in k[X]$ doesn't have any zeros. Hence, we need to work with algebraically closed sets (another example is the Whitney Umbrella).

From now on, we consider $k = \overline{k}$, here k is algebraically closed.
Also, all rings in the course are commutative, and contain 1 and all ring homomorphisms take 1 to 1.

2 Affine Varieties

Today, we give a definition of an affine variety that is dependant of the embedding in $\mathbb{A}^n$. Therefore, we need to define an algebra,

Definition 2.1 (Algebra)**.** Let k be a field and A be a ring, that is also a k-vector space. Then A is a k-algebra if $\lambda \cdot (ab) = (\lambda \cdot a) \cdot b$, for all $\lambda \in k$ and $a, b \in A$

The trivial example is k is a k-algebra. The second example is $A = k[X]$. The third is that k being any field and V a set, let $A := \mathrm{Map}(V, K)$, this is a k-algebra as A is a ring with $(f + g)(v) = f(v) + g(v)$ and $(f \cdot g)(v) = f(v) \cdot g(v)$. A is a k-vector space as $(\lambda \cdot f)(v) = \lambda \cdot f(v)$ for all $\lambda \in k$, for all $v \in V$.

We now need morphisms,

Definition 2.2 (k-algebra homomorphism)**.** Let A, B be k-algebras. A map $\phi : A \to B$ is a morphism between k-algebras if it is a ring homomorphism and a k-linear map. We write,

$$\hom_{k-alg}(A, B) = \{k\text{-alg homom from} A \to B\}$$

Definition 2.3 (Subalgebra)**.** Let $C \subset A$, C is a subalgebra if C is a subring and a k-subspace.

If $A = k[X]$ and $B = k$ are k-algebras, then

$$\hom_{\text{k-alg}}(k[X], k) \ni \phi$$

Then ϕ is determined by $\phi(X) = a \in k$. Have a bijection $a \in k$, then we can associate a $\phi : k[X] \to K$ to it. We can associate $a \mapsto (\phi_a : X \mapsto a)$. We will see that in more generalaity that

$$\hom_{k-alg}(k[X], k) = \mathbb{A}^1(k)$$

In this course we will see that considering all algebras is too much, but there is one that is enough to describe what we want. We want to look at the right type of algebras, more specifically the finitely generated k-algebra

Definition 2.4 (Finitely generated k-algebra)**.** A is finitely generated if $A = k[a_1, a_2, \dots a_n]$ for some finite set $S = \{a_1, a_2, \dots, a_n\} \subset A$.

Then we define a morphism $\phi : k[X_1, \dots, X_n] \to A = k[a_1, \dots, a_n]$, we define $X_i \mapsto a_i$ for all i. Now we see that ϕ is surjective and so by the First Isomorphism Theorem for k-algebras we get $k[X_1, \dots, X_n]/\operatorname{Ker}\phi \cong A$. We know $\operatorname{Ker}\phi$ is an ideal in $k[X_1, \dots, X_n]$ and so finitely generated k-algebras are the same, in a bijection of rings $k[X_1, \dots, X_n]/I$.

If $A \subset \mathrm{Map}(V, K)$ be a subalgebra and $x \in V$, then there is always a k-algebra homomorphism $\varepsilon_x : A \to k$ where $\varepsilon_x(f) \mapsto f(x)$. This ε_x is the evaluation homomorphism at the element x. Now assume $k = \overline{k}$ (algebraically closed), then,

Definition 2.5 (Affine k-variety)**.** An affine k-variety is a pair (V, A), where V is a set and $A \subset \mathrm{Map}(V, K)$ is a finitely generated sub-algebra such that

$$V \to \hom_{k-alg}(A, k)$$

$$x \mapsto \varepsilon_x$$

is a bijection.

This means, the elements of V correspond one to one with k-algebra homomorphisms from $A \to k$. Here is an example, Consider the pair $(\mathbb{A}^n(k), k[X_1, \ldots, X_n])$ this an affine variety. A is finitely generated by $X_1, \ldots, X_n$. The X_i are defined coordinate function $X_i(x_1, x_2, \ldots x_n) = x_i$, we now show this is a bijection. Assume we have $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n) \in \mathbb{A}^n$ and $\varepsilon_x = \varepsilon_y$. Then, $\varepsilon_x(X_i) = \varepsilon_y(Y_i)$ for all i. Then by Exercise 1, $(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$. Hence, it is injective. We now show surjectivity, $\phi \in \hom_{k-alg}(A, k)$. Set $x_i := \phi(X_i)$ for all i and $X \in A^n$. Then $\phi(X_i) = x_i = \varepsilon_x(X_i)$ for all i. Since, X_i generate A, we must have $\phi(f) = \varepsilon_x(f)$ for all $f \in A$, and hence $\phi = \varepsilon_x$. Therefore we have surjectivity. In total, $\mathbb{A}^n \to \hom_{k-alg}(k[X_1, \ldots, X_n], k)$ is a bijection and $(\mathbb{A}^n, k[X_1, \ldots, X_n])$ is an affine variety.

We say V is an affine variety to mean, we consider the pair (V, A) for $A := k[V]$ is the coordinate algebra of A. We can now do the Zariski Topology of an affine variety.

2.1 Zariski Topology of an affine variety

Let (V, A) be an affine variety. Then $S \subset A$ define

$$\mathcal{V}(S) := \{x \in V : f(x) = 0 \,\forall f \in S\}$$

Exercise. Show that $\mathcal{V}(S)$ $S \subset A$ form the closed sets of a topology on V, the Zariski Topology.

Proposition 2.6. Let $w \subset V$, where (V, A) is an affine variety, and W is closed. Then W itself is an affine variety.