Vector Calculus Week 3 - Differentiating

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Contents

1 Differentiating Scalar Fields

Definition 1.1: Partial Differentiation

Let $U \subset \mathbb{R}^n$ be an open set. The $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial^n f}{\partial x_n}$ partial derivatives of $f(x_1, \ldots, x_n)$ which at point $\underline{x}$ are defined by:

$$\frac{\partial f}{\partial x_j} =$$

$$\lim_{h \to 0} \frac{f(x_1, \ldots, x_j + h, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}$$

where the limit exists for j from 1 to n.

Example 1. *If $f(x, y) = x^2 y + y^3$, then find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$*

Solution 1. *We can simply work out that:*

$$\frac{\partial f}{\partial x} = 2xy$$

$$\frac{\partial f}{\partial y} = x^2 + 3y^2$$

To say that a partial derivative shall be evaluated at a point (x_0, y_0), we write; $\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)}$

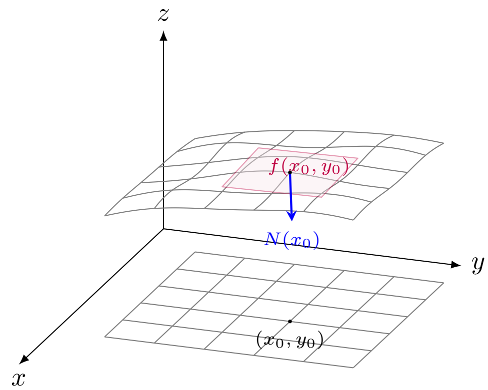
1.1 Equations of Tangent planes

Definition 1.2: Tangent Plane

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differeniable at (x_0, y_0), the plane described by:

$$z_p = f(x_0, y_0) + \left.\frac{\partial f}{\partial x_0}\right|(x - x_0) + \left.\frac{\partial f}{\partial y}\right|(y - y_0)$$

is called the tangent plane of f at (x_0, y_0).



Definition 1.3

Let f be a function $f : \mathbb{R}^2 \to \mathbb{R}$ we say that f is differentiable at (x_0, y_0), if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exists at (x_0, y_0) and if

$$\lim_{(x,y) \to (x_0, y_0)} \frac{f(x, y) - z_p}{\|(x, y) - (x_0, y_0)\|}$$

then z_p is a good approximation of f.

1.2 Gradient of a scalar field

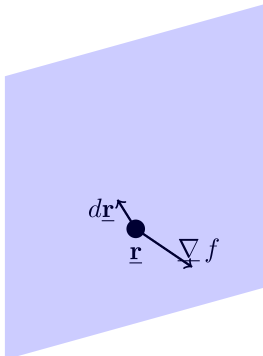
Definition 1.4

The gradient of a scalar field is a vector field with a direction that is perpendicular to the level surface and pointing in the direction of increasing f, with a magnitude equal to the rate of change of f in this direction.

$$\underline{\nabla} f = \frac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{k}}$$

Consider an infitesimal change in the position in space from $\underline{\mathbf{r}}$ to $d\underline{\mathbf{r}}$. This results in a small change in the value of f, from f to $f + df$.

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \\ &= \underline{\nabla} f \cdot d\underline{\mathbf{r}} \end{aligned}$$



Suppose that $d\underline{\mathbf{r}}$ lies in the level surface $f = C$, then $d\underline{\mathbf{f}} = \underline{\nabla} f \cdot d\underline{\mathbf{r}} = 0$ so $\underline{\nabla} f$ and $d\underline{\mathbf{r}}$ are perpendicular. To show that $\underline{\nabla} f$ has the required magnitude, let $d\underline{\mathbf{r}} = \underline{\hat{\mathbf{n}}} ds$, where $\underline{\hat{\mathbf{n}}}$ is normal to the surface and s is a distance measured along the normal.

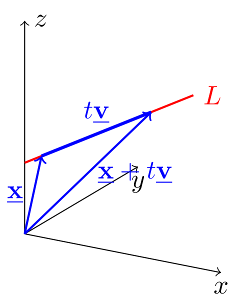
$$\begin{aligned} df &= \underline{\nabla} f \cdot d\underline{\mathbf{r}} \\ &= \underline{\nabla} f \cdot \underline{\hat{\mathbf{n}}} ds \\ &= |\underline{\nabla} f| ds \end{aligned}$$

So we know that $\underline{\nabla} f \parallel ds \implies \dfrac{df}{ds} = |\underline{\nabla} f|$.

Example 2. *Let $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, the euclidean norm.*

Solution 2. *Then we know that $\underline{\nabla} f = (\frac{x}{r}, \frac{y}{r} + \frac{z}{r}) = \frac{\underline{\boldsymbol{r}}}{r}$, where $r = \sqrt{x^2 + y^2 + z^2}$ and $\underline{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$*

2 Directional Derivative



Suppose $f : \mathbb{R}^3 \to \mathbb{R}$, let $\underline{\mathbf{v}}, \underline{\mathbf{x}} \subset \mathbb{R}^3$ be fixed vectors. Consider the function from $\mathbb{R} \to \mathbb{R}$ defined as:

$$t \mapsto f(\underline{\mathbf{x}} + t\underline{\mathbf{v}}) \tag{$\dagger$}$$

The set of points of the form $\underline{\mathbf{x}} + t\underline{\mathbf{v}}$, $t \in \mathbb{R}$ is the line L through which the point $\underline{\mathbf{x}}$ is parallel to $\underline{\mathbf{v}}$. ($\dagger$) is a function, f, restricted to L.

Definition 2.1: Directional Derivative

If $f : \mathbb{R}^3 \to \mathbb{R}$, the directional derivative of f at $\underline{\mathbf{x}}$ along a vector $\underline{\mathbf{v}}$ is given by:

$$\frac{d}{dt}\bigg|_{t=0} f(\underline{\mathbf{x}} + t\underline{\mathbf{v}})$$

if it exists.

Note that we usually choose $\underline{\mathbf{v}}$ to be of length unity.

Theorem 2.1

If $f : \mathbb{R}^3 \to \mathbb{R}$ and differentiable, then all directional derivatives exist. The directional derivative at $\underline{\mathbf{x}}$ in direction $\underline{\mathbf{v}}$ is given by:

$$\frac{d}{dt}\bigg|_{t=0} f(\underline{\mathbf{x}} + t\underline{\mathbf{v}}) = \underline{\nabla} f(\underline{\mathbf{x}}) \cdot \underline{\mathbf{v}}$$

Proof. Let $\underline{\mathbf{c}}(t) = \underline{\mathbf{x}} + t\underline{\mathbf{v}}$, $f(\underline{\mathbf{x}} + t\underline{\mathbf{v}}) = f(\underline{\mathbf{c}}(t))$ and

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=0} f(\underline{\mathbf{c}}(t)) &= \underline{\nabla} f(\underline{\mathbf{c}}(t)) \cdot \underline{\mathbf{c}}'(t) \\ &= \underline{\nabla} f \underline{\mathbf{c}}(0) \cdot \underline{\mathbf{c}}'(0) \\ &= \underline{\nabla} f(\underline{\mathbf{x}}) \cdot \underline{\mathbf{v}} \end{aligned}$$

□

Theorem 2.2

Assume that $\underline{\nabla} f \neq 0$. Then $\underline{\nabla} f(x)$ points in the direction along which f is increasing fastest

Proof. If $\underline{\hat{\mathbf{n}}}$ is a unit vector, the rate of change of f in the direction $\underline{\hat{\mathbf{n}}}$ is given by:

$$\underline{\nabla} f \cdot \underline{\hat{\mathbf{n}}} = |\underline{\nabla} f||\underline{\hat{\mathbf{n}}}| \cos\vartheta = |\underline{\nabla} f| \cos\vartheta$$

where ϑ is the angle between $\underline{\hat{\mathbf{n}}}$ and $\underline{\nabla} f$. This maximum is when $\vartheta = 0$, so $\underline{\hat{\mathbf{n}}}$ and $\underline{\nabla} f$ are parallel. If we wish to move in the direction in which f decreases the fastest, we should proceed in the direction, $-\underline{\nabla} f$. □

Example 3. *Find the unique normal to $x^2 + y^2 - z = 0$ at $(1, 1, 2)$*

Solution 3. *We say that* $f(x,y,z) = x^2+y^2-z = 0$*, and that* $\underline{\nabla} f$ *is normal as* f *is a level surface. So:*

$$\underline{\nabla}\, f = 2x\hat{\boldsymbol{\imath}} + 2y\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}$$

and we can work out $\hat{\underline{\mathbf{n}}}$ *as:*

$$\hat{\underline{\mathbf{n}}} = \frac{(2x, 2y, -1)}{\sqrt{1+4(x^2+y^2)}}\bigg|_{(1,1,2)}$$

and so $\hat{\underline{\mathbf{n}}} = \frac{1}{3}(2, 2, -1)$

2.1 Properties of Gradient

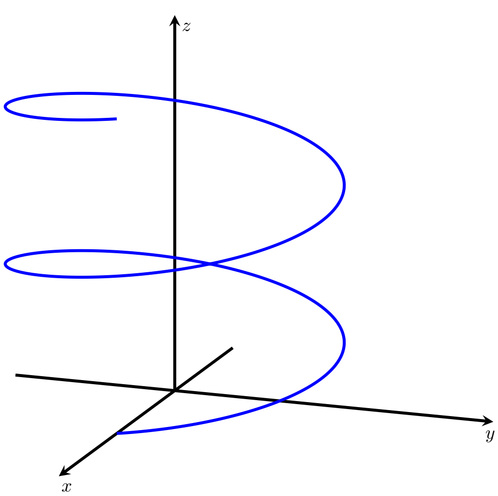
For any scalar functions of $f(x,y,z)$ and $g(x,y,z)$ and any $c \in \mathbb{R}$, we have:

$$\begin{aligned}
\underline{\nabla}(f+g) &= \underline{\nabla}\, f + \underline{\nabla} g \\
\underline{\nabla}(cf) &= c\underline{\nabla}\, f \\
\underline{\nabla}(fg) &= f\underline{\nabla} g + g\underline{\nabla}\, f \\
\underline{\nabla}(f \circ g) &= f'(g(x))\underline{\nabla} g
\end{aligned}$$

3 Parameterised Curves

We consider smooth curves in $\mathbb{R}^3$ specified in terms of rectangular cartesian coordinates (x,y,z). Such curves are generated by three smooth functions of a single parameter, t.

Example 4. *A good example is a circular helix,* $r(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}} + z(t)\hat{\boldsymbol{k}}$*, where:* $x(t) = a\cos t$*,* $y(t) = b\sin t$ *and* $z(t) = ct$



We can calculate the length of a path using an integral. Take a function that parameterised with three variables, $x(t), y(t), z(t)$ and between two points, $t_0 \leq t \leq t_1$, we can find the length, L:

$$L(\underline{\mathbf{r}}) = \int_{t_0}^{t_1} \sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)}$$

We could also parameterise a curve using an arc length parameter, s, where differential of arc-length satisfy the equation:

$$\begin{aligned}
ds^2 &= dr \cdot dr \\
&= dx^2 + dy^2 + dz^2
\end{aligned}$$

We call ds the line element of the curve. We can also write this with respect to t:

$$\dot{s}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r} \cdot \dot{r}$$

Now we have a curve in a space $\underline{\mathbf{r}}(t)$. Then we can find a tangent, $\dot{\underline{\mathbf{r}}} = (\dot{x}, \dot{y}, \dot{z})$, which then we know that $|\dot{\underline{\mathbf{r}}}| = \dot{s}$ and $\hat{\underline{\mathbf{t}}} = \dfrac{\dot{\underline{\mathbf{r}}}}{|\dot{\underline{\mathbf{r}}}|}$. We have now swapped the parameter from t to s.

$$\hat{\underline{\mathbf{t}}} = \frac{dx}{ds}\hat{\boldsymbol{\imath}} + \frac{dy}{ds}\hat{\boldsymbol{\jmath}} + \frac{dy}{ds}\hat{\boldsymbol{k}}$$

As we then know that $\hat{\underline{\mathbf{t}}}$ is a unit vector, $\hat{\underline{\mathbf{t}}} \cdot \hat{\underline{\mathbf{t}}} = 1$, now differentiate and $\hat{\underline{\mathbf{t}}} \cdot \dfrac{d\hat{\underline{\mathbf{t}}}}{ds} = 0$, hence $\dfrac{d\hat{\underline{\mathbf{t}}}}{ds} \perp \hat{\underline{\mathbf{t}}}$. The $\frac{d\hat{\underline{\mathbf{t}}}}{ds}$ is in the direction of the principle normal, $\underline{\mathbf{n}}$, of the curve. So $\frac{d\hat{\underline{\mathbf{t}}}}{ds} = \kappa(s)\hat{\underline{\mathbf{n}}}$

The plane spanned by $\hat{\underline{\mathbf{t}}}(s)$ and $\hat{\underline{\mathbf{n}}}(s)$ is the osculating plane.

So if we have a curve $\underline{\mathbf{r}}(t) \in \mathbb{R}^3$, then $\dfrac{d\underline{\mathbf{r}}}{dt}$, so we can now say that $\dfrac{\dot{\underline{\mathbf{r}}}(t)}{|\dot{\underline{\mathbf{r}}}(t)|} = \dfrac{d\underline{\mathbf{r}}}{ds} = \hat{\underline{\mathbf{t}}}$. Now we can take derivatives and hence:

$$\frac{d\hat{\underline{\mathbf{t}}}}{ds} = \kappa\hat{\underline{\mathbf{n}}}$$

Moving forward now, we can take $\hat{\underline{\mathbf{t}}} = \underline{\mathbf{r}}'(s)$ and then differentiating: $\hat{\underline{\mathbf{t}}} = \underline{\mathbf{r}}''(s)$, which then implies:

$$\kappa = |\underline{\mathbf{r}}''(s)|$$

and then we know that $\dot{\underline{\mathbf{r}}}(t) = \underline{\mathbf{r}}'(s)\dot{s}$ and then $\ddot{\underline{\mathbf{r}}}(t) = r''\dot{s}^2 + \frac{\ddot{s}}{\dot{s}}\dot{\underline{\mathbf{r}}}$ and hence we can say that: $\underline{\mathbf{r}}''(s) = \frac{1}{\dot{s}^2}\ddot{\underline{\mathbf{r}}} - \frac{\ddot{s}}{\dot{s}^3}\dot{\underline{\mathbf{r}}}$. So now,

$$\kappa^2(s) = \frac{1}{(\dot{\underline{\mathbf{r}}}\cdot\dot{\underline{\mathbf{r}}})^3}\left((\ddot{\underline{\mathbf{r}}}\cdot\ddot{\underline{\mathbf{r}}})(\dot{\underline{\mathbf{r}}}\cdot\dot{\underline{\mathbf{r}}}) - (\dot{\underline{\mathbf{r}}}\cdot\ddot{\underline{\mathbf{r}}})^2\right)$$

Given a unit tangent vector, $\hat{\underline{\mathbf{t}}}$ and a unit normal vector, $\hat{\underline{\mathbf{n}}}$ at a point on a curve in $\mathbb{R}^3$, we can define a third unit vector $\hat{\underline{\mathbf{b}}}$ which is the unit binormal vector.

$$\hat{\underline{\mathbf{b}}} = \hat{\underline{\mathbf{t}}} \times \hat{\underline{\mathbf{n}}}$$

They form a right handed system of unit vectors, this forms the moving trihedron as s varies.

$$\hat{\underline{\mathbf{b}}} = \hat{\underline{\mathbf{t}}} \times \hat{\underline{\mathbf{n}}}, \quad \hat{\underline{\mathbf{n}}} = \hat{\underline{\mathbf{b}}} \times \hat{\underline{\mathbf{t}}}, \quad \hat{\underline{\mathbf{t}}} = \hat{\underline{\mathbf{n}}} \times \hat{\underline{\mathbf{b}}}$$

3.1 Deriving Frenet-Serret Equations

We can now differentiate the other two equations, and get; $\frac{d\hat{\underline{\mathbf{b}}}}{ds} \perp \hat{\underline{\mathbf{b}}}$ and

$$\begin{aligned}
\frac{d\hat{\underline{\mathbf{b}}}}{ds} &= \frac{d\hat{\underline{\mathbf{t}}}}{ds} \times \hat{\underline{\mathbf{n}}} + \hat{\underline{\mathbf{t}}} \times \frac{d\hat{\underline{\mathbf{n}}}}{ds} \\
&= \kappa\hat{\underline{\mathbf{n}}} \times \hat{\underline{\mathbf{n}}} + \hat{\underline{\mathbf{t}}} \times \frac{d\hat{\underline{\mathbf{n}}}}{ds} \\
&= \hat{\underline{\mathbf{t}}} \times \frac{d\hat{\underline{\mathbf{n}}}}{ds}
\end{aligned}$$

which also tells us that:

$$\frac{d\hat{\underline{\mathbf{b}}}}{ds} \perp \hat{\underline{\mathbf{t}}}, \frac{d\hat{\underline{\mathbf{n}}}}{ds}$$

and hence $\frac{d\hat{\underline{\mathbf{n}}}}{ds} \parallel \hat{\underline{\mathbf{n}}}$ and so,

$$\frac{d\hat{\underline{\mathbf{b}}}}{ds} = -\tau\hat{\underline{\mathbf{n}}}$$

we call, τ the torsion of the curve.

Example 5. *We shall take the helix again,*

$$\begin{aligned}
d\underline{\boldsymbol{r}} &= -a\sin t dt\hat{\boldsymbol{\imath}} + a\cos t dt\hat{\boldsymbol{\jmath}} + c dt\hat{\boldsymbol{k}} \\
ds^2 &= (a^2(\sin^2 t + \cos^2 t) + c^2)dt^2 \\
ds &= (a^2 + c^2)^{\frac{1}{2}} dt \\
&\implies t = (a^2 + c^2)^{-\frac{1}{2}} s
\end{aligned}$$

Now we can find the tangent to any point.

$$\underline{\boldsymbol{r}}'(s) = \frac{1}{\sqrt{a^2+c^2}}\left(-a\sin\frac{s}{\sqrt{a^2+c^2}}\hat{\boldsymbol{\imath}} + a\cos\frac{s}{\sqrt{a^2+c^2}}\hat{\boldsymbol{\jmath}} + c\hat{\boldsymbol{k}}\right)$$

and now for $\hat{\underline{\boldsymbol{t}}}'(s)$

$$\hat{\underline{\boldsymbol{t}}}' = \underline{\boldsymbol{r}}''(s) = \frac{a}{a^2+c^2}\left(-\cos\frac{s}{\sqrt{a^2+c^2}}\hat{\boldsymbol{\imath}} - \sin\frac{s}{\sqrt{a^2+c^2}}\hat{\boldsymbol{\jmath}}\right)$$

comapring both sides, we can say that: $\kappa(s) = \frac{a}{a^2+c^2}$. *Finally, we find* $\hat{\underline{\boldsymbol{b}}}(s)$ *as:*

$$\hat{\underline{\boldsymbol{b}}} = \hat{\underline{\boldsymbol{t}}}\times\hat{\underline{\boldsymbol{n}}} = \frac{1}{\sqrt{a^2+c^2}}\left(-c\sin\frac{s}{\sqrt{a^2+c^2}}\hat{\boldsymbol{\imath}} - c\cos\frac{s}{\sqrt{a^2+c^2}}\hat{\boldsymbol{\jmath}} + a\hat{\boldsymbol{k}}\right)$$

and to find torsion:

$$\hat{\underline{\boldsymbol{b}}}' = \frac{c}{a^2+c^2}\left(\cos\frac{s}{\sqrt{a^2+c^2}}\hat{\boldsymbol{\imath}} + \sin\frac{s}{\sqrt{a^2+c^2}}\hat{\boldsymbol{\jmath}} + a\hat{\boldsymbol{k}}\right)$$

and so torsion:

$$\tau(s) = \frac{c}{a^2+c^2}$$

Now for $\hat{\underline{\mathbf{n}}}$, we can differentiate once and get:

$$\begin{aligned}
\frac{d\hat{\underline{\mathbf{n}}}}{ds} &= \frac{d\hat{\underline{\mathbf{b}}}}{ds} \times \hat{\underline{\mathbf{t}}} + \hat{\underline{\mathbf{b}}} \times \frac{d\hat{\underline{\mathbf{t}}}}{ds} \\
&= -\tau(s)\hat{\underline{\mathbf{n}}} \times \hat{\underline{\mathbf{t}}} + \hat{\underline{\mathbf{b}}} \times \kappa\hat{\underline{\mathbf{n}}} \\
&= \tau(s)\hat{\underline{\mathbf{b}}} - \kappa(s)\hat{\underline{\mathbf{t}}}
\end{aligned}$$

Definition 3.1: Frenet-Serret Equations in $\mathbb{R}^3$

$$\frac{d\hat{\underline{\mathbf{t}}}(s)}{ds} = \kappa(s)\hat{\underline{\mathbf{n}}}(s)$$

$$\frac{d\hat{\underline{\mathbf{b}}}(s)}{ds} = -\tau(s)\hat{\underline{\mathbf{n}}}(s)$$

$$\frac{d\hat{\underline{\mathbf{n}}}(s)}{ds} = \tau(s)\hat{\underline{\mathbf{b}}} - \kappa(s)\hat{\underline{\mathbf{t}}}$$

If you are given $\hat{\underline{\mathbf{t}}}$, $\hat{\underline{\mathbf{n}}}$, κ and τ, you can use the Frenet Serret equations to determine $\hat{\underline{\mathbf{t}}}$, $\hat{\underline{\mathbf{n}}}$ and $\hat{\underline{\mathbf{b}}}$ and thus determine the curve in its entirity.

4 Differentiation and Vector Fields

If $\underline{\mathbf{A}}(t) = A(t)_1\hat{\boldsymbol{\imath}} + A(t)_2\hat{\boldsymbol{\jmath}} + A(t)_3\hat{\boldsymbol{k}}$, then:

$$\frac{d\underline{\mathbf{A}}(t)}{dt} = \frac{d\underline{\mathbf{A}}(t)}{dt}_1\hat{\boldsymbol{\imath}} + \frac{d\underline{\mathbf{A}}(t)}{dt}_2\hat{\boldsymbol{\jmath}} + \frac{d\underline{\mathbf{A}}(t)}{dt}_3\hat{\boldsymbol{k}}$$

and let $\Phi = \Phi(x,y,z,t)$, $\underline{\mathbf{A}}(\underline{\mathbf{x}},t)$, $B(\underline{\mathbf{x}},t)$, then:

$$\frac{\partial}{\partial t}(\Phi\underline{\mathbf{A}}) = \frac{\partial\Phi}{\partial t}\underline{\mathbf{A}} + \Phi\frac{\partial\underline{\mathbf{A}}}{\partial t} \tag{$*$}$$
$$\frac{\partial}{\partial t}(\underline{\mathbf{A}}+\underline{\mathbf{B}}) = \frac{\partial\underline{\mathbf{A}}}{\partial t} + \frac{\partial\underline{\mathbf{B}}}{\partial t} \tag{$*^2$}$$
$$\frac{\partial}{\partial t}(\underline{\mathbf{A}}\cdot\underline{\mathbf{B}}) = \frac{\partial\underline{\mathbf{A}}}{\partial t}\cdot\underline{\mathbf{B}} + \underline{\mathbf{A}}\cdot\frac{\partial\underline{\mathbf{B}}}{\partial t} \tag{$*^3$}$$
$$\frac{\partial}{\partial t}(\underline{\mathbf{A}}\times\underline{\mathbf{B}}) = \frac{\partial\underline{\mathbf{A}}}{\partial t}\times\underline{\mathbf{B}} + \underline{\mathbf{A}}\times\frac{\partial\underline{\mathbf{B}}}{\partial t} \tag{$*^4$}$$

Now for the second derivatives

$$\begin{aligned}\frac{\partial^2\underline{\mathbf{A}}}{\partial x\partial y} &= \frac{\partial^2\underline{\mathbf{A}}}{\partial y\partial x}\\ &= \frac{\partial}{\partial x}\left(\frac{\partial\underline{\mathbf{A}}}{\partial y}\right)\\ &= \frac{\partial^2\underline{\mathbf{A}}_1}{\partial x\partial y}\hat{\boldsymbol{\imath}} + \frac{\partial^2\underline{\mathbf{A}}_2}{\partial x\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial^2\underline{\mathbf{A}}_3}{\partial x\partial y}\hat{\boldsymbol{k}}\end{aligned}$$

4.1 Divergence of a vector field

The divergence of a vector field $u(\underline{\mathbf{x}},t)$ is a scalar field. It's value at a point P is defined:

$$\underline{\nabla}\cdot u = \lim_{\delta\underline{\mathbf{V}}\to 0}\oiint \underline{\mathbf{u}}\cdot\underline{\mathbf{n}}ds$$

where $\underline{\mathbf{V}}$ is a small volume enclosing P. Physically this is the amount of flux in vector field, $\underline{\mathbf{U}}$ out of $\delta\underline{\mathbf{V}}$ divided by the volume.

$$\underline{\nabla}\cdot\underline{\mathbf{u}} = \frac{\partial\underline{\mathbf{u}}_1}{\partial x} + \frac{\partial\underline{\mathbf{u}}_2}{\partial y} + \frac{\partial\underline{\mathbf{u}}_3}{\partial z}$$

Assume $P(x,y,z)$ is enclosed by a cube of side length, $\delta x, \delta y, \delta z$. Assume P is at the centre of the cube. Then:

$$\begin{aligned}\oiint_S \underline{\mathbf{u}}\cdot\underline{\mathbf{n}}ds &= \iint_{S_1}\underline{\mathbf{u}}\cdot\underline{\mathbf{n}}ds + \iiint_{S_2}\underline{\mathbf{u}}\cdot\underline{\mathbf{n}}ds + \iint_{S_3}\underline{\mathbf{u}}\cdot\underline{\mathbf{n}}ds\\ &\quad+ \iint_{S_4}\underline{\mathbf{u}}\cdot\underline{\mathbf{n}}ds + \iint_{S_5}\underline{\mathbf{u}}\cdot\underline{\mathbf{n}}ds + \iint_{S_6}\underline{\mathbf{u}}\cdot\underline{\mathbf{n}}ds\\ &= u_1(x+\frac{\delta x}{2},y,z)\delta y\delta z - u_1(x-\frac{\delta x}{2},y,z)\delta y\delta z\\ &\quad+ u_2(x,y+\frac{\delta y}{2},z)\delta x\delta z - u_2(x,y-\frac{\delta y}{2},z)\delta x\delta z\\ &\quad+ u_3(x,y,z+\frac{\delta z}{2})\delta x\delta y - u_3(x,y,z-\frac{\delta z}{2})\delta x\delta y\\ &= \frac{\partial u_1}{\partial x}\delta\underline{\mathbf{V}} + \frac{\partial u_2}{\partial y}\delta\underline{\mathbf{V}} + \frac{\partial u_3}{\partial z}\delta\underline{\mathbf{V}}\end{aligned}$$

So we can conclude that:

$$\lim_{\delta\underline{\mathbf{V}}\to 0}\oiint \underline{\mathbf{u}}\cdot\underline{\mathbf{n}}ds = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} = \underline{\nabla}\cdot\underline{\mathbf{u}}$$

Example 6. *Compute divergence of* $F = x^2y\hat{\boldsymbol{\imath}} + z\hat{\boldsymbol{\jmath}} + xyz\hat{\boldsymbol{k}}$

Solution 4.

$$\begin{aligned}\underline{\nabla}\cdot F &= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(xyz)\\ &= 3xy\end{aligned}$$

5 Curl of a Vector Field

The curl of a vector field $\underline{\mathbf{u}}(\underline{\mathbf{x}},t)$ is a vector field. The component in the direction of the $\underline{\hat{\mathbf{n}}}$,

$$\underline{\hat{\mathbf{n}}}\cdot\underline{\nabla}\times\underline{\mathbf{u}} = \lim_{\delta S\to 0}\frac{1}{\delta S}\oint_{\delta C}\underline{\mathbf{u}}\cdot d\underline{\mathbf{r}}$$

$\underline{\nabla}\times\underline{\mathbf{u}}$ is related to the rotatio or tisting of the vector field.

$$\underline{\nabla}\times\underline{\mathbf{u}} = \begin{vmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ u_1 & u_2 & u_3\end{vmatrix} =$$

To prove this:

$$
\begin{aligned}
\underline{\hat{\mathbf{n}}} \cdot \underline{\nabla} \times \underline{\mathbf{u}} &= \lim_{\delta S \to 0} \frac{1}{\delta S} \oint_{\delta C} \underline{\mathbf{u}} \cdot d\underline{\mathbf{r}} \\
&= \oint_{C_1} \underline{\mathbf{u}} \cdot d\underline{\mathbf{r}} + \oint_{C_2} \underline{\mathbf{u}} \cdot d\underline{\mathbf{r}} \\
&\quad + \oint_{C_3} \underline{\mathbf{u}} \cdot d\underline{\mathbf{r}} + \oint_{C_4} \underline{\mathbf{u}} \cdot d\underline{\mathbf{r}} \\
&\approx u_1(x, y - \frac{\delta y}{2}, z)\delta x + -u_1(x, y + \frac{\delta y}{2}, z)\delta x \\
&\quad + u_2(x + \frac{\delta x}{2}, y, z)\delta y - u_2(x - \frac{\delta x}{2}, y, z)\delta y \\
&= -\frac{\partial u_1}{\partial y}\delta y \delta x + \frac{\partial u_2}{\partial x}\delta x \delta y \\
&= \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}
\end{aligned}
$$

The other components of $\underline{\nabla} \times \underline{\mathbf{u}}$ can be found with similar arguments.