Week 2: Differentiation

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Contents

1 Definition of a Derivative

Definition 1.1: Derivative

A function f, is differentiable at an interior point, x_0, of it's domain if the difference qiotient:

$$\frac{f(x) - f(x_0)}{x - x_0}, \quad x \neq x_0$$

approaches a limit as x approaches x_0, in which case the limit is called the derivative of f at x_0 and is denoted: $f'(x_0)$, thus:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Definition 1.2

We say that $f(x_0)$ is a local extreme value of f, if there is a $\delta > 0$, such that the sign of $f(x) - f(x_0)$ diesnt change:

$$(x_0 - \delta,\, x_0 + \delta) \subset D_f$$

or a local minimum of f if:

$$f(x_0) \leq f(x)$$

if for all x in the set, these are true, then we have globals

Theorem 1.1

If f is differentiable at a local extreme point $x_0 \in D_f^0$, then $f(x_0) = 0$

Proof. We consider the case where x_0 is a local maximum. Then, $\exists\, \delta > 0$, $(x_0 - \delta,\, x_0 + \delta) \subset D_f$ and $f(x) \leq f(x_0)$ for all $x \in (x_0 - \delta,\, x_0 + \delta)$. We have:

$$0 \leq \lim_{x \to x_0 -} \frac{f(x) - f(x_0)}{x - x_0}$$

and

$$\lim_{x \to x_0 -} \frac{f(x) - f(x_0)}{x - x_0} \leq 0$$

So, as it exists,

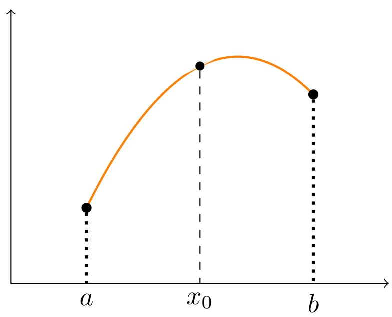
$$\lim_{x \to x_0 -} \frac{f(x) - f(x_0)}{x - x_0} = 0$$

The case of a local minimum at x_0 is obtained by applying the above to $-f$. □

2 Rolles Theorem

Theorem 2.1: Rolle's Theorem

Suppose that f is continuous on the closed interval $[a,\, b]$ and differentiable on the open interval $(a,\, b)$ and $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a,\, b)$



Proof. Since f is continuous on $[a,\, b]$, then, by EVT, f attains both a min and max values.

$$\alpha = \min_{x \in [a,\, b]} f(x) \qquad \beta = \max_{x \in [a,\, b]} f(x)$$

If $\alpha = \beta$, then f is constant on $(a,\, b)$, clearly $f'(x) = 0\, \forall x \in [a,\, b]$.
If $\alpha \neq \beta$, then at least one α or β is attained at a point $c \in (a,\, b)$ (Since $f(a) = f(b)$), and hence $f'(c) = 0$. □

3 Darboax's Theorem

Theorem 3.1: Darboax's Theorem

Suppose that f is differentiable on $[a,\, b]$, $f'(a) \neq f'(b)$, and μ is between $f'(a)$ and $f'(b)$. Then $f'(c) = \mu$ for some $c \in (a,\, b)$.

Proof. Suppose that $f'(a) < \mu < f'(b)$ aand then define:

$$g(x) = f(x) - \mu x$$

Then

$$g'(x) = f'(x) - \mu$$

and then:

$$g'(a) < 0 \qquad 0 < g'(b) \tag{$*$}$$

Since g is continuous on $[a,\, b]$, g attains a min, by EVT, at some point $c \in [a,\, b]$. Then, $(*)$, implies $\exists\, \delta > 0$,

$$g(x) < g(a), \quad \forall a < x < a + \delta$$

and

$$g(x) < g(b), \quad b - \delta < x < b$$

therefore $c \neq a$ and $c \neq b$. Hence $a < c < b$, and therefore $g'(c) = 0$ since c is a min in D_g^0, that is $f'(c) \neq \mu$
The proof when $f'(b) < \mu < f'(a)$ is obtained when you apply the above argument to $-f$. □

4 Mean Value Theorem

Theorem 4.1: Cauchy's Mean Value Theorem

If f and g are continuous on a closed interval $[a,\, b]$ nd differentiable on the open interval $(a,\, b)$, then:

$$[g(b) - g(a)]f'(c) = [f(b) - f(a)]g'(c)$$

for some $c \in (a,\, b)$

Proof. Let $h(x) = (g(b) - g(a))f(x) - (f(b) - f(a))g(x)$, then we can say h is continuous on $[a,\, b]$ and differentiable on $(a,\, b)$. Therefore Rolle's Theorem implies that $h'(c) = 0$ for some $c \in (a,\, b)$, that is:

$$h'(c) = (g(b) - g(a))f'(c) - (f(b) - f(a))g'(c) = 0$$

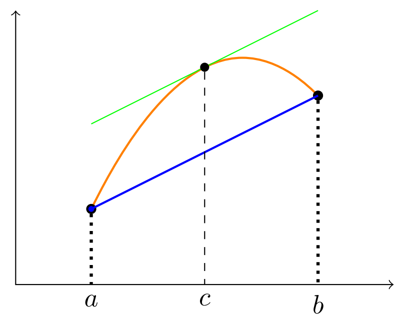
□

Theorem 4.2: Mean Value Theorem

If f is continuous on the closed interval $[a,\, b]$ and differentiable on $(a,\, b)$, then

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

for some $c \in (a,\, b)$



Proof. Apply Cauchy MVT and let $g(x) = x$ □

4.1 Consequences

Theorem 4.3

If $f'(x) = 0$ for all $x \in (a,\, b)$ then f is constant on $(a\, b)$.

Theorem 4.4

If f' exists and does not change sign on $(a,\, b)$, then f is monotonic on $(a,\, b)$

Theorem 4.5: Lipschitz Continuity

If $|f'(x)| \leq M \quad \forall x \in (a,\, b)$, then $|f(x) - f(x')| < M|x - x'|$ for all $x, x' \in (a,\, b)$.