Year 2 — Complex Analysis

Based on lectures by Ana Rodrgiues
Notes taken by James Arthur

Summer Term 2020

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Contents

1 Introduction to Complex Analysis

In this introduction we are going to prove some foundational things about complex numbers, which make them a field. Firstly we define the set,

$$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$$

as the complexes, so they are a set of '2D numbers'. This is most obvious in Lean where my convention is to write them as $\langle a, b\rangle = a + ib$.
Addition: Let $z = a + ib$ and $w = c + id$, then we can deduce,

$$z + w = (a + c) + (b + d)i$$

hence $\mathbb{C}$ is closed under addition (and by `sub_eq_neg_add` subtraction aswell).
Multiplication: Again let $z = a + ib$ and $w = c + id$, then we can deduce,

$$z \cdot w = (ac - bd) + (ad - bc)i$$

hence $\mathbb{C}$ is closed under multiplication.
Division: Again let $z = a + ib$ and $w = c + id$, then we can deduce,

$$\frac{z}{w} = \frac{z\overline{w}}{|w|}$$

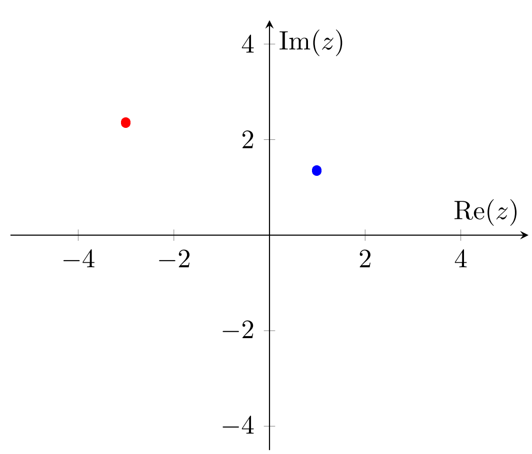
hence closed. In technicalities with a few more lemmas we have a field, but we don't bother too much about that, yet (hopefully).

Lemma 1.1. Let $z \in \mathbb{C}$, then we can say $z\overline{z} = |z|^2$

Proof.

$$z\overline{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$$

Argand Diagrams: An argand diagram is a way to visualise complex numbers. Let us plot $z = -3 + 2i$ and $w = 1 + i$.



Lemma 1.2. Let $z, w \in \mathbb{C}$, then,

(i) $(z \pm \overline{w}) = (\overline{z} \pm \overline{w})$

(ii) $\overline{(zw)} = \overline{z}\overline{w}$

(iii) $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}} \qquad \text{if } w \neq 0$

Corollary 1.3. If $z, w \in \mathbb{C}$, then $|zw| = |z||w|$

Proof. $|zw|^2 = (zw)(\overline{zw}) = (z\overline{z})(w\overline{w}) = |z|^2|w|^2$ 🎉

Corollary 1.4. Triangle Inequality If $z, w \in \mathbb{C}$ then, $|z+w| \leq |z| + |w|$.

Proof. If $z + w = 0$, then proof complete. If $z + w \neq 0$,

$$\frac{z}{z+w} + \frac{w}{z+w} = 1$$

and then,

$$\text{Re}\left(\frac{z}{z+w}\right) + \text{Re}\left(\frac{w}{z+w}\right) = 1$$

We know also that,

$$Re\left(\frac{z}{z+w}\right) \leq |\frac{z}{z+w}|$$

and similarly for the other. Hence,

$$\left|\frac{z}{z+w}\right| + \left|\frac{w}{z+w}\right| \geq 1$$
$$|z+w| \leq |z| + |w|$$

🎉

Polar Form: We can say

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

1.1 Roots of complex numbers and equations

Lemma 1.5. Every complex number has n-distinct nth roots

Theorem 1.6. (*De Mouvire's*) For all, $z \in \mathbb{C}$, then $r, \theta \in \mathbb{R}$,

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

Let $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$ and $\mu = \rho e^{i\alpha} = \rho(\cos\alpha + i\sin\alpha)$, then,

$$r(\cos\theta + i\sin\theta) = \rho^n(\cos n\alpha + i\sin n\alpha)$$

Which implies,

$$\rho^n = r \qquad n\alpha = \theta + 2k\pi \quad (k \in \mathbb{Z}^+)$$

Hence,

$$\rho = r^{\frac{1}{n}} \qquad \alpha = \frac{\theta + 2k\pi}{n}$$

1.2 Complex Functions

We shall consider functions of the form $f : D \to \mathbb{C}$, where $D \subset \mathbb{C}$.

Lemma 1.7. (*Remainder Theorem*) If g is a polynomial over $\mathbb{C}$ and $b \in \mathbb{C}$, then $\exists\, h(z)$ over $\mathbb{C}$ st, $g(z) = (z-b)h(z) + g(b)$.

Theorem 1.8. If $g(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ with $a_n \neq 0$ and $a_i \in \mathbb{C}(i \in \mathbb{N}_1)$, then $g(z)$ has at most n complex roots.

Proof. In general, every polynomial over $\mathbb{C}$ can be written as,

$$a(z-z_1)(z-z_2)\dots(z-z_n)$$

and the only polynomials $p(z)$ over $\mathbb{C}$ with no solutions are $p(z) = 0$ (by FTA). 🎉

1.2.1 Exponential and Logarithm

Definition 1.9. The complex Exponential is defined as,

$$e^z = e^x(\cos y + i \sin y)$$

Lemma 1.10. $\forall\, z \in \mathbb{C}$, we have,

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Lemma 1.11. $\forall\, z, w \in \mathbb{C}$,

(i) $e^{z+w} = e^z e^w$

(ii) $e^{z+2\pi i} = e^z$

(iii) $|e^z| = e^{\mathrm{Re}(z)}$

Proof.

$$\begin{aligned}
|e^z| &= |e^{x+iy}| \\
&= |e^x||e^{iy}| \\
&= |e^x||\cos y + i \sin y| \\
&= |e^x| \cdot 1 \\
&= e^x = e^{\mathrm{Re}(z)}
\end{aligned}$$

🎉

Definition 1.12. (*Complex Trigonometry*) $\forall\, z \in \mathbb{C}$,

$$\cos z = \frac{e^{iz} - e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} + e^{-iz}}{2i}$$

Definition 1.13. (*Complex Hyperbolic Trigonometry*) $\forall\, z \in \mathbb{C}$,

$$\cos ix = \frac{e^{-x} - e^{x}}{2} = \cosh x \qquad \sin ix = \frac{e^{-x} + e^{x}}{2i} = i \sinh x$$

Lemma 1.14. For $\theta, \phi \in \mathbb{R}$, we have $e^{i(\theta+\phi)} = e^{i\theta} e^{i\phi}$

Proof.
```
lemma comp_exp_add (θ φ : ℝ) : exp(θ * I) * exp(φ * I) = exp((θ + φ) * I) :=
begin
  repeat {rw exp_mul_I},
  simp only [add_mul, mul_add],
  rw [add_comm, mul_comm (sin ↑φ) I, mul_assoc _ I _,
      tactic.ring.mul_assoc_rev I I _, ← pow_two, I_sq],
  simp only [neg_mul_eq_neg_mul_symm, one_mul, mul_neg_eq_neg_mul_symm],
  rw ←[ mul_assoc (cos ↑θ), mul_comm (cos ↑θ), ← add_assoc, add_comm (I * cos ↑θ *
  sin ↑φ),
      add_right_comm (-(sin ↑θ * sin ↑φ)), add_comm (-(sin ↑θ * sin ↑φ)),
      tactic.ring.add_neg_eq_sub, ← cos_add, mul_comm _ I, add_assoc],
  have H1 : I * cos ↑θ * sin ↑φ + I * sin ↑θ * cos ↑φ = I * (cos ↑θ * sin ↑φ + sin ↑θ *
   cos ↑φ),
  { ring},
  rw [H1, add_comm (cos ↑θ * sin ↑φ), ← sin_add, mul_comm],
end
```

🎉

Corollary 1.15. For $r, s, \theta, \phi \in \mathbb{R}$, we have $re^{i\phi}(se^{i\theta}) = rse^{i(\theta+\phi)}$

Proof.
```
lemma exp_form_mul (φ θ : ℝ) (r s : ℂ) : (r*exp(φ * I)) * (s*exp(θ * I)) = r *
  s * exp((θ + φ) * I) := by rw [mul_mul_mul_comm, comp_exp_add, add_comm]
```

🎉

Definition 1.16. (*Complex Logarithm*) If we have $e^z = w$, then we can solve and get,

$$z = \log r + i(\theta + 2k\pi) \qquad k \in \mathbb{Z}^+$$

Definition 1.17. (*Principle Complex Logarithm*) We define the principle logarithm as,

$$Log(w) = log|w| + i\arg(w)$$

and we can deduce that,

Lemma 1.18. $\forall\, z, w \in \mathbb{C} \setminus 0$,

$$log(zw) = Log(z) + Log(w) + 2n\pi \qquad (n \in \mathbb{N})$$

2 Topology

Definition 2.1. (*Open Disc*) $D(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$

Definition 2.2. (*Closed Disc*) $\overline{D}(a, r) = \{z \in \mathbb{C} : |z - a| \leq r\}$

Definition 2.3. (*Punctured Disc*) $D'(a, r) = \{z \in \mathbb{C} : 0 < |z - a| < r\}$

Definition 2.4. *(Open Set)* A set $S \subset \mathbb{C}$ is open $\forall\, z \in \mathbb{C}$, $\exists\, r > 0$, $D(z; r) \subset S$.

Definition 2.5. *(Closed Set)* A set $S \subset \mathbb{C}$ is closed if $\mathbb{C} \setminus S$ is open.

Definition 2.6. *(Limit point)* A point $z \in \mathbb{C}$ is a limit point of S if $D'(z; r) \cap S \neq 0 \;\; \forall\, r > 0$. A point of S which isn't a limit point is an isolated point.

Definition 2.7. *(Closure)* the closure of S is the union of S and it's limit points.

Definition 2.8. *(Interior Point)* $\exists\, r > 0, D(z; r) \subset S$

Definition 2.9. *(Exterior Point)* $\exists\, r > 0, D(z; r) \cap S = \varnothing$

Definition 2.10. *(Boundary Point)* z is a boundary point if it's neither a interior or exterior point of S.

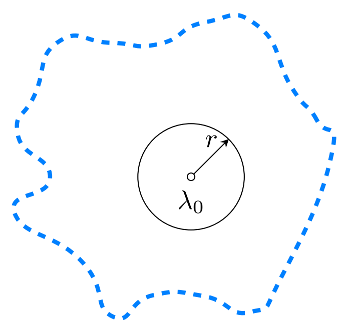
Lemma 2.11. Let $A \subset \mathbb{C}$, each point $a \in \mathbb{C}$ is either an interior of A, an exterior of A or a boundary point of A.

Proposition 2.12. (i) The following three statements are equivalent,

- S is closed.
- S contains all it's limit points.
- $\overline{S} = S$.

(ii) $z \in \overline{S} \iff V \cap S \neq \varnothing \; \forall\, z \in$ open set V

(iii) $\overline{S}$ is a closed set.



A disk, $D(\lambda_0; r)$ in an open set S.

Proof. TO DO

Definition 2.13. *(Bounded Set)* $S \subset \mathbb{C}$ is bounded if $\exists\, M \in \mathbb{R}$, $|z| \leq M \,\forall\, z \in S$.

Definition 2.14. *(Compactness)* A set is bounded and closed is compact.

Definition 2.15. An open set $U \subset \mathbb{C}$ is connected if any two points a and b in U, one can join a to b in a finite sequence of straight lines segments contained within U.

Definition 2.16. (*Domain*) If $A \subset \mathbb{C}$ is a domain if A is nonempty, open and connected.

3 Continuity

Definition 3.1. (*Limit*) Let $A \subset \mathbb{C}$, $f : A \to \mathbb{C}$ and $a \in \mathbb{C}$ be a limit point of A. $f(z) \to l$, as $z \to a$ if $\forall\, \varepsilon > 0\; \exists\, \delta > 0, z \in D(a, \delta) \cap A$ then $f(z) \in D(l, \varepsilon)$.

Theorem 3.2. Let $f : A \to \mathbb{C}$ and $a \in \mathbb{C}$ be a limit point of A. Then $f(z) \to l$ as $z \to a \iff f(a_n) \to l,\ \forall\, a_n : \mathbb{N} \to \mathbb{C}, a_n \to a$.

Question. Does a limit only exist when a is a limit point?

Definition 3.3. (*Continuous at*) Let $f : A \to \mathbb{C}$. If $a \in A$ is a limit point of A, and if $f(z) \to f(a)$ as $z \to a$, we say that f is continuous at a.

Definition 3.4. (*Continuous on*) Let $f : A \to \mathbb{C}$. Suppose that each point of A is a limit point of A. We say f is continuous on A if f is continuous at all $a \in A$.

Theorem 3.5. Continuity holds under addition, multiplication and their inverses, with the usual caveats.

Remark. Polynomials are continuous on all of $\mathbb{C}$, rational functions are continuous where defined.

4 Holomorphic Functions

Definition 4.1. (*Differentiable*) Let $A \subset \mathbb{C}$ be open, $f : A \to \mathbb{C}$. We say f is differentiable at $a \in A$ if,

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} \quad \text{exists}$$

This limit is called $f'(a)$.

Definition 4.2. (*Holomorphic*) Let $f : U \to \mathbb{C}$, which is differentiable at every point of U is Holomorphic.

Theorem 4.3. If $f : A \to \mathbb{C}$ and g are differentiable then,

$$f \pm g \qquad fg \qquad fg^{-1} \qquad f \circ g \qquad g \circ f$$

are differentiable

Theorem 4.4. (*Cauchy Riemann Equations*) Let $U \subset \mathbb{C}$ be open. Suppose that $f : U \to \mathbb{C}$ is a function,

$$f(x + iy) = u(x, y) + iv(x, y)$$

$x, y \in \mathbb{R}$, $u, v : \mathbb{R} \to \mathbb{R}$. If $z_0 \in U$ and if f is differentiable at z_0,

$$u_x = v_y \qquad u_y = -v_x$$

Lemma 4.5. (*Partial Converse of C-R*) Suppose $f(x + iy) = u(x, y) + iv(x, y)$ is a function on an open set U and suppose $z_0 \in U$. If f satisfies the CR (with u_x, v_y are continuous at z_0), then f is differentiable.

5 Integration

5.1 Path Integrals

Definition 5.1. (*Path*) A path is a continuous map $\gamma : [a,\, b] \to \mathbb{C}$. It is called smooth if γ is differentiable and γ' is continuous.

We write $\gamma(t) + x(t) + iy(t)$ where $x, y : [a,\, b] \to \mathbb{R}$.

- γ is continuous if $x(t)$ and $y(t)$ are continuous.
- γ is differentiable if $x(t)$ and $y(t)$ are differentiable.

Example. Let $z_1, z_2 \in \mathbb{C}$. The line segment from $z_1 \to z_2$ is the path $\gamma : [0,\, 1] \to \mathbb{C}$,

$$\gamma(t) = z_1 + (z_2 - z_1)t$$

Example. Let $z_0 \in \mathbb{C}$, $(r \in \mathbb{R}) > 0, \alpha, \beta \in \mathbb{R}, a < \beta$. We define $\gamma : [\alpha,\, \beta] \to \mathbb{C}$,

$$\gamma(t) = z_0 + re^{it}$$

Let $\gamma : [a,\, b] \to \mathbb{C}$,

- $\gamma(a)$ is the starting point
- $\gamma(b)$ is the end point
- If $\gamma(a) = \gamma(b)$, then the path is closed
- If $\forall\, t, s \in (a,\, b)$ and $\gamma(t) = \gamma(s) \iff t = s$ (i.e. γ is injective)

Notation. γ^- is a map $\gamma^- : [-b,\, -a] \to \mathbb{C}$, i.e. the reversal of γ

Definition 5.2. (*Path Integral*) Let f be continuous on an open set U, $\gamma : [a,\, b] \to \mathbb{C}$ be a smooth path contained within U. The path integral is defined as,

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

Example. Let $f(z) = z$ and γ be the line segment from 1 to $2 + 2i$,

$$\gamma(t) = 1 + (1 + 2i)t \qquad t \in [0, 1]$$

then,

$$\int_\gamma z\, dz = \int_0^1 (1 + (1 + 2i)t)(1 + 2i)dt = -\frac{1}{2} + 4i$$

5.1.1 Properties of the path integral

$$\int_\gamma f + g\, dz = \int_\gamma f\, dz + \int_\gamma g\, dz$$
$$\int_\gamma af\, dz = a \int_\gamma f\, dz \qquad \forall\, a \in \mathbb{C}$$
$$\int_{\gamma^-} f\, dz = -\int_\gamma f\, dz$$

Theorem. (*FTC for* $\mathbb{C}$) Assume $f : U \to \mathbb{C}$ is holomorphic and U is open. Assume also that f' is continuous. Then,

$$\int_\gamma f'(z)\,dz = f(\gamma(b)) - f(\gamma(a))$$

If γ is closed, then,

$$\int_\gamma f'(z)\,dz = 0$$

5.2 Contour Integral

Definition. (*Contour*) A contour, $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a sequence of smooth paths arranged end to end.

Definition. (*Closed Contour*) A contour is closed if $\gamma_1 = \gamma_n$.

Definition. (*Contour Integral*) We define an integral over the contour γ as,

$$\int_\gamma f(z)\,dz = \sum_{i=1}^{n} \int_{\gamma_i} f(z)\,dz$$

Definition. (*Path Length*) If γ is a smooth path, $\gamma : [a,\, b] \to \mathbb{C}$, then we define it's length to be,

$$\ell\,(\gamma) = \int_a^b |\gamma'(t)|\,dt$$

Definition. (*Contour Length*) If we have a smooth contour we define it's lengh to be,

$$\ell\,(\gamma) = \sum_{i=1}^{n} \int_a^b |\gamma_i'(t)|\,dt$$

Example. If γ is a line segment from w_1 to w_2, $\gamma : [0,\, 1] \to \mathbb{C}$, then,

$$\ell\,(\gamma) = \int_0^1 |w_2 - w_1|\,dt = |w_2 - w_1|$$

Lemma. If f is complex valued then,

$$\left| \int_a^b f(t)\,dt \right| \le \int_a^b |f(t)|\,dt$$

Corollary 5.3. (*M-L Bounds*) Consider a contour γ and continuous function f on γ. Suppose $|f(z)| \le M \quad \forall\, z \in \gamma$. Then,

$$\left| \int_\gamma f(z)\,dz \right| \le ML \qquad \text{where } L = \ell\,(\gamma)$$

6 Sequences and Series of Complex Numbers

Let a_n be a sequence of complex numbers, let $a \in \mathbb{C}$, we say that a is the limit of a_n as $n \to \infty$ if $\forall\, \varepsilon > 0, \exists\, N \in \mathbb{R}, \forall\, n > N$,

$$|a_n - a| < \varepsilon$$

Theorem 6.1. Let z_n be a sequence of complex numbers, let $z \in \mathbb{C}$, then the following are equivalent,

(i) $z_n \to z$ as $n \to \infty$

(ii) $|z_n - z| \to 0$ as $n \to \infty$

(iii) $Re(z_n) \to Re(z)$, $Im(z_n) \to Im(z)$ as $n \to \infty$

Definition 6.2. (*Cauchy Sequences*) A sequence a_n of complex numbers is a cauchy sequence if $\forall\, \varepsilon > 0, \exists\, N \in \mathbb{R}$, if $n, m \geq N$, then

$$|a_n - a_m| < \varepsilon$$

Theorem 6.3. If a_n is a convergent sequence of complex numbers, then (a_n) is cauchy.

Proof. Since a_n is convergent $a_k \to l$ as $n \to \infty$. Take $\varepsilon > 0$, by def, $\exists\, N$, if $n > N$, then $|a_n - l| < \dfrac{\varepsilon}{2}$. Suppose $n, m > N$,

$$\begin{aligned} |a_n - a_m| &= |a_n - l + l - a_m| \\ &\leq |a_n - l| + |a_m - l| \\ &< \varepsilon \end{aligned}$$

🎉

Definition 6.4. (*Convergent Series*) Let $\sum_{n=0}^{\infty} z_n$ be an infinite series of complex numbers, we say the series converges if, $\sum_{n=0}^{N} z_n$ converges.

Definition 6.5. (*Absolutely convergent*) $\sum_{n=0}^{\infty} z_n$ converges absolutely if $\sum_{n=0}^{\infty} |z_n|$ converges.

Lemma 6.6. If $\sum_{n=0}^{\infty} |z_n|$ converges so does, $\sum_{n=0}^{\infty} z_n$.

Corollary 6.7. Let $(z_n)_{n=0}^{\infty}$ be a sequence of complex numbers. If $\forall\, \varepsilon > 0\, \exists\, N$, $\left| \sum_{m+1}^{n} z_k \right| < \varepsilon$ and $n, m > N$, then $\sum_{k=0}^{\infty} z_k$ is convergent.

6.1 Sequences of functions

Definition 6.8. (*Pointwise Limit of a sequence of functions*) Let (f_n) be a sequence of functions on U. Let $f : U \to \mathbb{C}$, we say that f is pointwise limit on f of f_n if $\forall\, x \in U$, we have $f_n(x) \to f(x)$ as $n \to \infty$

Definition 6.9. (*Uniform Convergence*) Let (f_n) be a sequence of functions in U. we say that f is the uniform limit of f_n if,

$$\sup_{x \in U} |f_n - f| \to 0 \text{ as } n \to \infty$$

we say that (f_n) is uniformly convergent to f.

Theorem 6.10. Let (f_n) be a sequence of continuous functions that converge uniformly to f on $U \subset \mathbb{C}$, st, $\forall\, z \in U$, z is a limit point. Then f is continuous.

Theorem 6.11. Let γ be a contour and f_n a sequence of functions integrable on γ. Assume that $f_n \to f$ uniformly on g. Then,

$$\int_\gamma f_n \, dz \to \int_\gamma f \, dz$$

Definition 6.12. (*Uniform Cauchy*) $(f_n)_{n=1}^\infty$ defined on U is uniformly cauchy on U if $\forall\, \varepsilon > 0 \,\exists\, N$, $\forall\, n, m > N, \forall\, z \in U$,

$$|f_n(z) - f_m(z)| < \varepsilon$$

Lemma 6.13. A sequence of functions defined on U is uniformly convergent on U if and only if it is uniformly cauchy on U.

Lemma 6.14. Let $(f_n)_{n=1}^\infty$ be a sequence of functions on U, the series $\displaystyle\sum_{n=0}^{i} nfty f_n(z)$ converges uniformly on U if,

$$S_N(z) = \sum_1^N f_n(z)$$

Theorem 6.15. (*Weirstrass M-Test*) Let $(f_n)_{n=1}^\infty$ be a sequence of functions defined on a subset of $U \subset \mathbb{C}$. The series, $\displaystyle\sum_1^\infty f_n$ converges uniformly and absolutely on U if, $\exists\ (M_n)_{n=1}^\infty \geq 0 \in \mathbb{R}$, st, $\forall\, n \in \mathbb{N}$, $\forall\, z \in U$ we have,

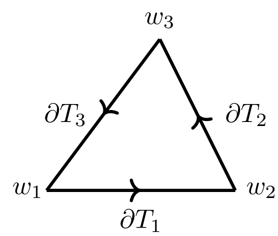
$$|f_n(z)| \leq M_n \text{ and } \sum_{n=0}^\infty M_n \text{ converges}$$

7 Cauchy Theorem(s) - Many of them

Theorem 7.1. (*Vauge Cauchy Theorem*) If f is holomorphic at every $z \in \gamma$ (a closed contour) then,

$$\int_\gamma f(z)\,dz = 0$$

Definition 7.2. (*Interior point γ in triangles*) Given any two edge points on distinct edges any point on the interval between these two points on the interior point.

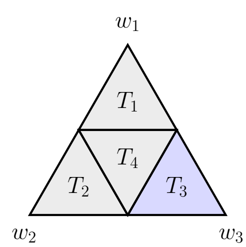


Theorem 7.3. If f is holomorphic on a domain U and $T \subset U$ is a triangle in U, then $\partial T = \partial T_1 + \partial T_2 + \partial T_3$ and,

$$\int_{\partial T} f(z)\,dz = 0$$

Lemma 7.4. Take a triangle T with vertices w_1, w_2, w_3. Subdivide T into subtraingles T_1, T_2, T_3, T_4, where each subtriangle has half the dimensions of the original triangle. Then,

$$\int_{\partial T} f(z)\,dz = \sum_{j=1}^{4} \int_{\partial T_j} f(z)\,dz$$



Lemma 7.5. (*Gorsats Lemma*) Let f be holomorphic in $U \subset \mathbb{C}$ and take $\alpha \in U$. Then, $\exists\ v(z)$ defined on U, st,

$$f(z) = f(\alpha) + (z-\alpha)f'(\alpha) + (z-\alpha)v(z)$$

and such that, $v(z) \to 0$ as $z \to \alpha$

7.1 Nested Sequence of compact sets

Lemma 7.6. Let U be a closed subset of $\mathbb{C}$ and let (a_n) be a convergent sequence of elements of U with limit a. Then $a \in U$.

Lemma 7.7. Let $U_1 \supseteq U_2 \supseteq \ldots \supseteq U_n \supseteq \ldots$ be a decreasing sequence of compact subsets of $\mathbb{C}$. Then $\exists\, \alpha \in \mathbb{C}$, st, $\alpha \in U_n \forall\, n \in \mathbb{N}$

Definition 7.8. (*Star Domain*) A domain in $\mathbb{C}$ is star if it has a star center.

Definition 7.9. (*Star Center*) We call $z_0 \in \mathbb{C}$ a star center of U if, $\forall\, z \in U$, the line segment between z_0 and z is contained in U.

Theorem 7.10. If f is holomorphic on a star domain U, then $f = g'$ for some g holomorphic on U.

Corollary 7.11. (*Cauchy Theorem on Star Domains*) If U is a star domain with f as holomorphic on U, and γ is closed contour on U, then,

$$\int_\gamma f(z)\, dz = 0$$

Theorem 7.12. (*Cauchy Theorem*) Let U be a domain. Let γ be a closed contour, st, U contains $\gamma*$ and the interior of γ. Let F be holomorphic on U, then,

$$\int_\gamma f(z)\, dz = 0$$

7.2 Jordan Closed Curve

Theorem 7.13. Let γ be a simple closed curve, then $\mathbb{C} \setminus \gamma*$ is the disjoint union of a bounded region called the interior of γ and an unbounded region called the exterior of γ.

Theorem 7.14. (*Deformation Theorem*) Let f be a function, holomorphic on a domain U. Let γ_1, γ_2 be contours with the the same start and end points, st, U contains γ_1* and γ_2* and the region between them. Then,

$$\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz$$

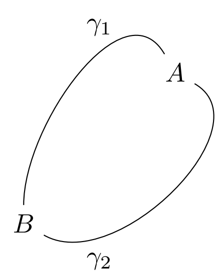


Figure 1: Diagram for Deformation Theorem

Definition 7.15. (*Positively Oriented Curve*) A simple closed curve is said to be positively oriented if the interior is to the left of the curve when travelling in the direction of the contour.

Theorem 7.16. Let γ_1 and γ_2 be positively oriented simple contours with γ_2* lying inside γ_1. If f is holomorphic on some domain that contains γ_1* and γ_2* and the region between the contours,

$$\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz$$

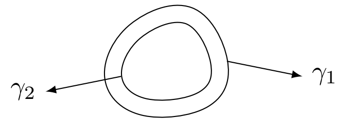


Figure 2: Diagram for Thm7.16

Theorem 7.17. (*Cauchy Integral Formula*) Let U be a domain, γ be a positively oriented simple contour with it's image and interior lying entirely inside U. Suppose that $a \in \gamma$. If f is holomorphic on U, then,

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz$$

Example. Let γ be the circle with center $(0,0)$ and radius 2. Then,

$$\int_\gamma \frac{e^{z^2}}{z+1}\, dz = 2\pi i e$$

We can then get from CIF,

$$\int_\gamma \frac{f(z)}{z-a}\, dz = 2\pi i f(a)$$

Let $a = -1$ and $f(z) = e^{z^2}$,

$$\int_\gamma \frac{e^{z^2}}{z+1}\, dz = 2\pi i f(-1) = 2\pi i e$$

7.3 Cauchys Integral Formula for the nth derivative

Theorem 7.18. (*Cauchys Integral Formula for the nth derivative*) Let U be a domain, γ a positively oriented simple contour with it's image and interior lying entirely in U. Suppose a is a path in the interior of γ. If f is holomorphic on U, then,

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}}\, dz$$

Example. Let γ be the unit circle, compute $\displaystyle\int_\gamma \frac{\sin z}{z^4}$. First take $a = 0$ and $n = 3$,

$$\int_\gamma \frac{f(z)}{z^4} = \frac{2\pi i}{3!} f^{(3)}(0) = -\frac{\pi i}{3}$$

7.4 Morera's Theorem

Theorem 7.19. Let U be a domain, f continuous on U, st, for all positively oriented simple contours γ,

$$\int_\gamma f(z)\, dz = 0$$

st, $\gamma *$ and it's interior are contained in U, then $\exists\, g : U \to \mathbb{C}$ st,

$$g' = f' \qquad \forall\, z \in U$$

Theorem 7.20. (*Morera's Theorem*) Let U be a domain, let f be continuous on U. If

$$\int_\gamma f(z)\, dz = 0 \qquad \forall \text{positively oriented simple closed contours}$$

st, $\gamma *$ and it's interior is contained in U, then f is holomorphic on U.

7.5 Cauchy Estimates

Let f be holomorphic on a domain containing $\overline{D}(a,r)$. If M is an upper bound for $|f(z)|$ on the boundary of the disc, st,

$$|f(z)| \leq M, \, \forall \, z \in D(a,r)$$

then,

$$f^{(n)}(a) \leq \frac{n!M}{r^n} \qquad \forall \, n \in Z^+$$

Proposition 7.21. Let U be a compact subset of $\mathbb{C}$ and let $f : U \to \mathbb{C}$ be continuous. Then, f is bounded.

Definition 7.22. (*Entire Function*) An entire function is holomorphic on $\mathbb{C}$

Theorem 7.23. (*Louiville's Theorem*) Let f be entire, if f is bounded then, f is constant.

Theorem 7.24. (*Generalised Louville*) Let f be entire, if $\exists \, n, C, R$, st,

$$|f(z)| \leq C|z|^n$$

whenever $|z| > R$. Then f is a polynomial of degree at most n.

Example. Suppose f is an entire function, satisfying

$$|f(z)| \leq |z| + 1 \qquad \forall \, z \in \mathbb{C}$$

Prove that $f(z)$ is a polynomial of degree 1, where $|A| \leq 1$ and $B \leq 1$.
So let $n = 1$, and so we want to show things when $|z| \geq 1$, then,

$$|z| + 1 \leq 2|z| \qquad \text{so let } C = 2 \text{ and } R = 1$$

We now know that $f(z) = Az + B$. We can differentiate and plug in zero to get the required inequalities,

$$\begin{aligned} |f(0)| &= |A(0) + B| \\ &\leq |0| + 1 \\ |B| &\leq 1 \end{aligned}$$

and now use Cauchy's estimate,

$$\begin{aligned} |f^{(n)}(a)| &\leq \frac{n!M}{r^n} \\ &= \frac{M}{r} && \text{as } n = 1 \\ &\leq \frac{1}{r} && \text{as } z = 0 \\ &\leq 1 && \text{as we are in } D(1,1) \end{aligned}$$

8 Power Series

Let $a \in \mathbb{C}$, (a_n) is a sequence of complex numbers, $\forall\, n \geq 0$ we define $f_n : \mathbb{C} \to \mathbb{C}$ by $f_n(z) = a_n(z-a)^n$,

$$\sum_{n=0}^{\infty} f_n \qquad \text{is the power series about } a$$

Lemma 8.1. Any differentiable complex function has a local power series expansion.

Theorem 8.2. (*Taylors Theorem*) Let f be holomorphic on a domain U, suppose that the $D(a,r) \subset U$, where $a \in \mathbb{C}$, $r > 0$. Then $\exists\, (a_n)_{n=0}^{\infty}$ of complex numbers st, $\forall\, z \in D(a,r)$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

γ is a circular contour, $D(a,r)$, where,

$$a_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}}$$

Proof. Assume $a = 0$, let f be holomorphic on a domain U. Suppose that $D(0,R) \subset U$. Let $z \in D(0,R)$, st, $|z| < R$. Let $\mu = D(0,S)$ with $|z| < S < R$.

By Cauchy's Integral Formula we have,

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}}\, dw \tag{$*$}$$

Take $n = 0$ in $(*)$,

$$\begin{aligned}
f^{(0)}(z) &= \frac{1}{2\pi i} \int_{\mu} \frac{f(w)}{(w-z)}\, dw \\
&= \frac{1}{2\pi i} \int_{\mu} \frac{f(w)}{w(1-\frac{z}{w})}\, dw \\
&= \frac{1}{2\pi i} \int_{\mu} \frac{f(w)}{w} \cdot \frac{1}{(1-\frac{z}{w})}\, dw \\
&= \frac{1}{2\pi i} \int_{\mu} \frac{f(w)}{w} \sum_{n=0}^{\infty} \left(\frac{z}{w}\right)^n dw \\
&= \frac{1}{2\pi i} \int_{\mu} \sum_{n=0}^{\infty} \left(\frac{f(w)}{w^{n+1}} z^n\right) dw \\
&= \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{\mu} \left(\frac{f(w)}{w^{n+1}}\right) dw\, z^n \\
&= \sum_{n=0}^{\infty} f^{(n)}(0)\, z^n
\end{aligned}$$

Let f be holomorphic on a domain U and suppose $\mathrm{id}\, aR \subset U$, where $a \in \mathbb{C}$ and $R > 0$. Then, $\exists\, (a_n)_{n=0}^{\infty}$ of complex numbers, $\forall\, z \in D(a,R)$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \qquad \text{where } a_n = \frac{f^{(n)}(a)}{n!}$$

γ is any circular contour, $D(a,r), (r < R)$.

8.1 Radius of Convergence

The sum $\sum_{n=0}^{\infty} z^n$ converges if $|z| < 1$ and diverges if $|z| > 1$. This is the series converges inside the unit circle at $(0,0)$ and diverges outside. So we can ask,

$$\sum_{n=0}^{\infty} a_n(z-a)^n \quad \text{for what values does it converge?}$$

There are three possibilities,

(i) The series converges only when $z = a$

(ii) The series converges everywhere $\forall\, z \in \mathbb{C}$

(iii) The series converges where $\exists\, R$, st, the series converges in $D(a,R)$ only.

Lemma 8.3. Let $\sum_{n=0}^{\infty} a_n(z-a)^n$ be a power series. If the series converges for $z_0 \in \mathbb{C}$ with $z_0 \neq a$, $\forall\, r$, st, $0 < z < |z_0 - a|$ the series converges uniformly and absolutely on $\overline{D}(a,r)$

Theorem 8.4. (*Radius of Convergence*) Let $\sum_{n=0}^{\infty} a_n(z-a)^n$ be a power series. Suppose $\exists\, z_0 \neq a$, st, the power series converges when $z = z_0$. If the series doesn't converge $\forall\, z \in \mathbb{C}$, then $\exists\, R > 0$, $R \in \mathbb{R}$, st, the series converges absolutely when $|z-a| < R$ and diverges when $|z-a| > R$.

The number R is called a radius of convergence of the power series,

– If a power series converges $\forall\, z \in \mathbb{C}$, we say it has infinite radius of convergence.

– If the series converges only at a, the radius of convergence must be zero.

Theorem 8.5. Let f be a function of $z \in \mathbb{C}$ defined by,

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n \quad \text{with Radius of convergence, } R$$

Then, f is holomorphic on $D(a,R)$ and,

$$f'(z) = \sum_{n=0}^{\infty} n a_n (z-a)^{n-1} \quad \forall\, z \in D(a,R)$$

Theorem 8.6. If, $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ is a power series that converges in a domain containing a $D(a,R)$ where $a \in \mathbb{C}$ and $R > 0$ ($R \in \mathbb{R}$) then, $f(z)$ is holomorphic on $D(a,R)$ and $a_n = \dfrac{f^{(n)}(a)}{n!} \quad \forall\, n \in \mathbb{N}$

Example. What is the power series of $f(z) = z \sin z$ around π?
We know that,

$$\sin w = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} w^{2n+1}$$

Let $w = z - \pi$,

$$\sin(z-\pi) = \sum_{n=0}^{\infty} \frac{(-1)^n (z-\pi)^{2n+1}}{(2n+1)!}$$

and so,

$$\begin{aligned} z\sin z &= (w+\pi)\sin(w+\pi) \\ &= w\sin(w+\pi) + \pi\sin(w+\pi) \\ &= -w\sin w - \pi\sin w \\ &= -(w+\pi)\sum_{n=0}^{\infty}\frac{(-1)^n w^{2n+1}}{(2n+1)!} \end{aligned}$$

Example. Find the taylor series for $f(z) = \cos(3z^2)$ around $z = 0$ and state the radius of convergence, Let $w = 3z^3$ and let us use the taylor series for $\cos w$,

$$\begin{aligned} \cos w &= \sum_{n=0}^{\infty}\frac{(-1)^n w^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n (3z^3)^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n 9^n z^{6n}}{(2n)!} \end{aligned}$$

This series converges $\forall\, w$ and so it converges $\forall\, z$.

9 Zeros of holomorphic functions

Let f be a holomorphic function of a complex variable. A zero of f is a complex number z_0, st, $f(z_0) = 0$.

Suppose f is holomorphic in a domain containing a point $a \in \mathbb{C}$. Then, $\exists\, r > 0$, st, f has a power series,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{in } D(a,r)$$

and now suppose that a is a zero of f. Then,

- Either all of $a_n = 0, \quad \forall\, n > 0 \implies f(x) = 0$ on $D(a,r)$.
- $\exists\, N \in \mathbb{N}$, st, $a_0 = a_1 = \cdots = a_{N-1}$ and $a_N \neq 0$.

For the second point there, we say that f has a zero of order N at a. By Taylors Theorem, f has a zero of order N at $a \in \mathbb{C}$ if $f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0$ and $f^{(n)}(a) \neq 0$.

Definition. (*Simple Zero*) A zero of order one is a simple zero, i.e. $f(a) = 0$, but $f'(a) \neq 0$.

Definition. (*Double Zero*) A zero of order two is a double zero.

Example. Let $f(z) = z$, then we have a simple zero at $z = 0$ as,

- $f(0) = 0$
- $f'(z) = 1 \implies f'(0) = 1$

and now let $f(z) = z^2$, then we have a double zero at $z = 0$ as,

- $f(0) = 0^2 = 0$
- $f'(z) = 2z, \implies f'(0) = 0$
- $f''(z) = 2, \implies f''(0) = 2$

Lemma 9.1. Suppose that f and g have zeros of order n and m respectively at $a \in \mathbb{C}$, then fg has a zero of order $n + m$ at a.

Lemma 9.2. (*Isolated Zeros*) Let f be holomorphic on a domain U containing a point a. If $\exists\, m \in \mathbb{N}$, st, f has a zero of order m at a, then the zero is isolated.

More intuitively, a zero is isolated, if $\exists\, r > 0$, st, $f(z) \neq 0$ if $z \in D'(a,r)$

Theorem 9.3. Let f be holomorphic on a domain U, if $\exists\, a \in U$ and $r > 0$, st, $D(a,r) \subset U$ and st, $f(z) = 0\, \forall\, z \in D(a,r)$, then $f(z) = 0\, \forall\, z \in U$.

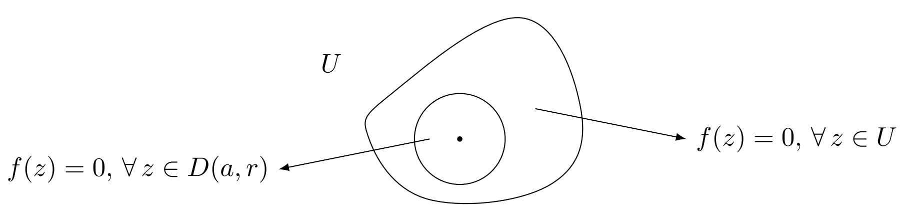


Figure 3: Diagram for locally zero $\implies$ globally zero

Let S be an open subset of $\mathbb{C}$, consider all $A \subset S$, st,

- A is open

– $S \setminus A$ is open

If the only set A that satisfy (1) and (2) are $\varnothing$ and S itself, then S is topologically connected.

Lemma 9.4. S is topologically connected if and only if it is connected.

Theorem 9.5. (*Identity Theorem*) Let U be a domain and let $f : U \to \mathbb{C}$ be holomorphic. The following are equivalent:

(i) $f(z) = 0 \quad \forall\, z \in U$

(ii) $\exists\, a \in U$, $r > 0$, st, $f(z) = 0$, $\forall\, z \in D(a, r)$

(iii) The set S of zeros of f has a limit point $z_0 \in U$.

9.1 Laurent Series

Let,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n = a_0 + a_1(z-a) + \dots$$

Theorem 9.6. (*Laurent Theorem*) If f is holomorphic on an annulus,

$$A = \{z \in \mathbb{C} : R < |z-a| < S\}$$

for $0 < R \leq S$, then, $\exists\, (b_n)_{n \in \mathbb{Z}^+} \in \mathbb{C}$, st,

$$f(z) = \sum_{n=-\infty}^{\infty} b_n (z-a)^n \qquad \text{the laurent series } \forall\, z \in A$$

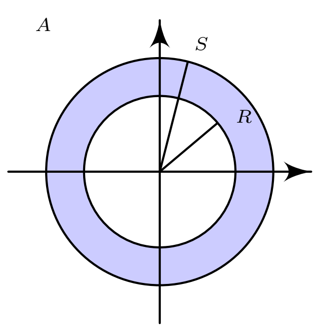


Figure 4: Then annulus A from R to S

$f(z)$ moreover $\forall\, r$, st, $R < r < S$ and $\forall\, n \in \mathbb{Z}^+$, if $\gamma$is the circular contour with center a and center r, then,

$$b_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}}$$

Suppose $\displaystyle\sum_{n=-\infty}^{\infty} a_n (z-a)^n$ is a laurent series convergent on the annulus. Then,

$$\sum_{n=-\infty}^{-1} a_n (z-a)^n \quad \text{is the principle part of the Laurent Series.}$$

Theorem 9.7. (*Uniqueness*) Let $A = \{z \in \mathbb{C} : R < |z-a| < S\}$, $0 < R < S < \infty$. If the series $\sum_{n=-\infty}^{\infty} b_n(z-a)^n$ converges $\forall\, z \in A$, then,

$$f(z) = \sum_{n=-\infty}^{\infty} b_n(z-a)^n \quad \text{is holomorphic on } A \text{ and } \forall\, n \in \mathbb{Z}^+ \text{ with the usual } b_n \text{ defined above.}$$

where γ is the circular contour, $D(a,r)$, $R < r < S$.

Example. $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$. Let $f : \mathbb{C} \setminus \{1,2\} \to \mathbb{C}$, st, $f(z) = \dfrac{1}{(z-1)(z-2)}$ and find the Laurent series about 0.

$$\begin{aligned} f(z) &= \frac{1}{z-2} - \frac{1}{z-1} \\ &= \frac{1}{-2\left(1-\frac{z}{2}\right)} + \frac{1}{z\left(1-\frac{1}{z}\right)} \\ &= -\frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} + \sum_{n=-\infty}^{-1} z^n \end{aligned}$$

which is the Laurent expansion around 0.

Example. Let μ be the circular contour, $D(0,\dfrac{3}{2})$. Compute,

$$I = \mu_\mu f(z)\, dz \quad \text{with } f(z) = \frac{1}{(z-1)(z-2)}$$

and we know that,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \qquad a_n = \int_\mu \frac{f(w)}{w^{n+1}}$$

and so let $n = -1$,

$$\begin{aligned} a_{-1} &= \frac{1}{2\pi i}\int_\mu f(w)\, dw \\ 2\pi i a_{-1} &= \int_\mu f(w)\, dw \end{aligned}$$

and so as we know that $f(z) = -\dfrac{1}{2}\sum_{n=0}^{\infty}\dfrac{z^n}{2^n} + \sum_{n=-\infty}^{-1} z^n$. So now compute a_{-1}. WE can get that $a_{-1} = -1$, and so,

$$\int_\mu f(w)\, dw = -2\pi i$$

9.2 Singularities

Definition 9.8. (*Isolated Singularity*) Let U be a domain on which f is holomorphic. If $a \notin U$, st, $D'(a,r)$ is a subset of U for some $r > 0$. Then, f has an isolated singularity.

If f has an isolated singularity at a, then by Laurents Theorem,

$$f(z) = \sum_{n=-\infty}^{\infty} b_n(z-a)^n \quad \text{above } D'(a,r) \text{ for some } r > 0$$

Example. Find the Laurent Series of $f(z) = \dfrac{\sin z}{z}$,

$$\begin{aligned} f(z) &= \frac{1}{z}\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!} \end{aligned}$$

Definition 9.9. (*Removable Singularity*) f has a removable singularity if it's Laurent Series has zero principle part, $b_n = 0, \forall\, n < 0$

Example. $f(z) = e^{\frac{1}{z}}$ has a laurent series about $z = 0$,

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} (\frac{1}{z})^n \frac{1}{n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{z^n n!} \\ &= \frac{1}{n!} + \sum_{n=-\infty}^{-1} \frac{1}{(-n)!} z^n \end{aligned}$$

Definition 9.10. (*Essential Singularity*) There are infinitely many terms in the principle part, hence, it is a essential singularity. $\nexists\, m$, st, $b_n = 0, \forall\, n < -m$.

To find singulaties look at the Laurent Series,

$$\dots + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \dots$$

– principle part all zeros, f has a removable singularity

– principle part not all zero, f has an essential singularity

Theorem 9.11. (*Picards Great Theorem*) If f is defined on a punctured disc and has an essential singularity at a, then f takes all complex values with at most one exception on $D'(a,r)$.

10 Residues and Cauchy (again...)

If f is a function holomorphic on a punctured disc, $D'(a,r)$ for some $a \in \mathbb{C}$ and $r > 0$, with Laurent Series,

$$\sum_{n=-\infty}^{\infty} b_n(z-a)^n \qquad \text{for } z \in D'(a,r)\text{, then,}$$

the residue of f at a is,

$$\operatorname{Res}(f,a) = b_{-1}$$

The residue of f at a is, $\operatorname{Res}(f,a) = b_{-1}$ and if f has a removable singulairty at a, then, $\operatorname{Res}(f,a) = 0$ and if a is a simple pole, then, $\operatorname{Res}(f,a) \neq 0$

Theorem 10.1. (*Cauchy Residue Theorem*) If γ is a closed simple contour, traversed anticlockwise, if f is a holomorphic function on a domain containing the image and the interior of γ except for a finite number of isolated singularities in the interior of the whole curve $(a_1, a_2, \dots, a_n)$, then,

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{j=0}^{n} \operatorname{Res}(f,a_j)$$

10.1 Computing Residues

If f has laurent series,

$$\sum_{\infty}^{\infty} b_n(z-a)^n \qquad \text{for } t \in D'(a,r)$$

Then,

$$\operatorname{res}(f,a) = b_{-1}$$

Given this, we have Cauchy Residue Theorem, suppose γ is a closed, simple contour traversed anticlockwise. f is holomorphic except at a finite number of isolated singulairties, say $(a_1, \dots, a_k)$, then,

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{j=0}^{k} \operatorname{res}(f,a_j)$$

If f has a simple pole at a, then f has a laurent series,

$$f(z) = \frac{b_{-1}}{z-a} + b_0 + b_1(z-a) + b_2(z-a)^2 + \dots$$

and we have,

$$\operatorname{res}(f,a) = b_{-1} = \lim_{z\to a}(z-a)f(z)$$

Example. Let γ be $D(0,3)$ and $f(z) = \dfrac{1}{(z-1)(z-2)}$. f has two singularities.

$$\operatorname{res}(f,1) = \lim_{z\to 1}(z-1)\frac{1}{(z-1)(z-2)} = \lim_{z\to 1}\frac{1}{z-2} = -1$$

$$\operatorname{res}(f,2) = \lim_{z\to 2}(z-2)\frac{1}{(z-1)(z-2)} = \lim_{z\to 2}\frac{1}{z-1} = 1$$

and so,

$$\int_\gamma f(z)\,dz = 2\pi i(-1+1) = 0$$

Lemma 10.2. If $f(z) = \dfrac{h(z)}{k(z)}$ and it has an isolated singularity at a, h and k are holomorphic on $D(a,r)$ if $h(a) \neq 0$ and k has a simple zero at a. Then,

$$\operatorname{res}(f, a) = \frac{h(a)}{k'(a)}$$

Example. Consider $f(z) = \dfrac{\sin z}{\cos z}$ and

$$\operatorname{res}\left(f, \frac{3\pi}{2}\right) = \frac{\sin \frac{3\pi}{2}}{-\sin \frac{3\pi}{2}} = -1$$

For poles of higher order, we look towards the laurent series.

Example. Compute the residue of $f(z) = \dfrac{\sin z}{(z-\pi)^6}$.
Firstly, let $w = z - \pi$.

$$\begin{aligned} f(z+\pi) &= \frac{\sin(w+\pi)}{w^6} \\ &= -\frac{\sin w}{w^6} \\ &= -\frac{1}{w^6}\left(w - \frac{w^3}{3!} + \frac{w^5}{5!} + \dots\right) \\ &= -\frac{1}{w^5} + \frac{1}{3!w^3} - \frac{1}{5!w} + \dots \end{aligned}$$

Hence, $b_{-1} = -\dfrac{1}{5!}$ an hence $\operatorname{res}(f, \pi) = \dfrac{1}{5!}$

Notation. Note that,

$$1 + w + w^2 + w^3 + \dots$$
$$1 + w + w^2 + w^3 + \mathcal{O}(w^4)$$

Proposition 10.3. Suppose f has a pole of order n at $a \in \mathbb{C}$, then,

$$\operatorname{res}(f, a) = \lim_{z \to a} \frac{g^{(n-1)(z)}}{(n-1)!}$$

where $g(z) = (z-a)^n f(z)$.

Example. Let $f(z) = \dfrac{\sin z}{(z-1)^3}$ and f has a triple pole at $z = 1$. Let us write,

$$g(z) = (z-1)^3 f(z) = \sin z$$

and so,

$$\operatorname{res}(f, 1) = \lim_{z \to 1} \frac{(\sin z)''}{2!} = -\frac{1}{2}\sin 1$$

When we consider residues at essential singularities, it suffices to just compute and consider the laurent series,

Example. Find $\operatorname{res}(f, 0)$ where $f(z) = e^{\frac{2}{z}}$. Let $w = \dfrac{2}{z}$.

$$\begin{aligned} f\left(\frac{w}{2}\right) &= e^w \\ &= 1 + w + w^2 + w^3 + \dots \\ &= 1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \dots \end{aligned}$$

and so, $\operatorname{res}(f, 0) = 2$

11 How to integrate 101

Complex Analysis is very useful for solving a wide amount of integrals.

11.1 Integrating Trigonometric Functions

If $z = e^{it}$, then we can write,

$$\cos t = \frac{1}{2}(z + z^{-1}) \qquad \text{and} \qquad \sin t = \frac{1}{2i}(z - z^{-1})$$

We want to write,

$$\int_0^{2\pi} F(\cos t, \sin t)\, dt$$

as a function of z alone.

Example. Compute,

$$\int_0^{2\pi} \frac{\cos 2t}{5 - 3\cos t}\, dt$$

Firstly we can let $z = e^{it}$ and preform a substitution.

$$\begin{aligned}
\int_0^{2\pi} \frac{\cos 2t}{5 - 3\cos t}\, dt &= \int_0^{2\pi} \frac{\frac{1}{2}z^2 + z^{-2}}{5 - \frac{3}{2}(z + z^{-1})}\, dt \\
&= \int_0^{2\pi} \frac{\frac{1}{2}z^2 + z^{-2}}{5 - \frac{3}{2}(z + z^{-1})} \frac{dz}{iz} \\
&= \int_0^{2\pi} \frac{z^4 + 1}{z^2(3z^2 - 10z + 3)}\, dt \qquad \text{This step took half an hour of the lecture}
\end{aligned}$$

Looking at the integrand we can say, it has a double pole at $z = 0$ and simple poles at $z = 3$ and $z = \frac{1}{3}$. We shall take the disc of center 0 and radius 1. Now we can say that we want the residue of $z = 0$ and $z = \frac{1}{3}$.

We can calculate the residue at $z = 0$, by doing the following,

$$\begin{aligned}
\operatorname{res}(f, 0) &= \lim_{z \to 0} g'(z) \\
&= \lim_{z \to 0} (z^2 f(z))' \\
&= \lim_{z \to 0} \left(\frac{z^4 + 1}{3z^2 - 10z + 3}\right) \\
&= \frac{10}{9} i
\end{aligned}$$

and now the other residue as $z = \frac{1}{3}$ is a simple pole,

$$\begin{aligned}
\operatorname{res}\left(f, \frac{1}{3}\right) &= \lim_{z \to \frac{1}{3}} \left(z - \frac{1}{3}\right) f(z) \\
&= \lim_{z \to \frac{1}{3}} \frac{i(z^4 + 1)}{z^2} \\
&= -\frac{41}{36} i
\end{aligned}$$

and then by Cauchy Residue Theorem,

$$\begin{aligned}\int_0^{2\pi} \frac{\cos 2t}{5-3\cos t}\,dt &= 2\pi i\left[\operatorname{res}(f,0)+\operatorname{res}\left(f,\frac{1}{3}\right)\right]\\ &= 2\pi i\left[\frac{10}{9}i+\frac{41}{36}i\right] = \frac{\pi}{18}\end{aligned}$$

11.2 Semi-circle Method

We want to compute $\int_{-\infty}^{\infty} f(x)\,dx$ using a semicircle contour with radius R and letting $R \to \infty$.

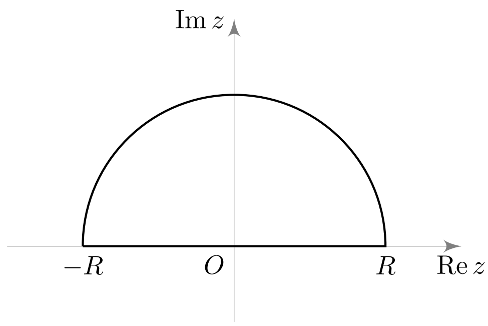


Figure 5: Semicircle Method Diagram

11.2.1 Odd function using Semi-circle

Consider,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$

We can consider the following integral over $\gamma = \gamma_1 + \gamma_2$, where we define $\gamma_1 = [-R, R]$ and $\gamma_2 = Re^{it}$ where $t \in [0, \pi]$. The integrand has two singularities, $z = \pm i$, only one of which is in γ, $z = i$. Consider the residue of the single pole $z = i$,

$$\begin{aligned}\operatorname{res}(f,i) &= \lim_{z\to i}(z-i)f(z)\\ &= \lim_{z\to i}\frac{1}{z+i}\\ &= \frac{1}{2i} = -\frac{i}{2}\end{aligned}$$

Hence we can say that,

$$\int_\gamma \frac{1}{1+z^2}\,dz = 2\pi i(-\frac{i}{2}) = \pi$$

and we can also split up,

$$\begin{aligned}\int_\gamma \frac{1}{1+z^2}\,dz &= \int_{\gamma_1}\frac{1}{1+z^2}\,dz + \int_{\gamma_2}\frac{1}{1+z^2}\,dz\\ &= \int_{-R}^{R}\frac{1}{1+z^2}\,dz + \int_{\gamma_2}\frac{1}{1+z^2}\,dz\end{aligned}$$

So we now consider I_2 as $R \to \infty$, so do the ML-inequality,

$$\begin{aligned}|f(z)| &= \left|\frac{1}{1+z^2}\right| \\ &\leq \frac{1}{R^2-1}\end{aligned}$$

and we know that $\ell(\gamma) = \pi R$, hence,

$$\left|\int_{\gamma_1} \frac{1}{1+z^2}\, dz\right| \leq \frac{\pi R}{R^2-1}$$

and so as $R \to \infty$,

$$\int_{\gamma_1} \frac{1}{1+z^2}\, dz \to 0$$

and now,

$$\begin{aligned}\int_{-R}^{R} f(z)\, dz &= \pi - \int_{\gamma_1} \frac{1}{1+z^2}\, dz \\ &= \pi && \text{as } R \to \infty\end{aligned}$$

11.2.2 Trigonometric using Jordan's Inequality and Semi-circle

Lemma 11.1. (*Jordan's Inequality*) If $0 < t < \frac{\pi}{2}$, then, $\sin t \geq \frac{2t}{\pi}$

Example. Compute,

$$I = \int_0^\infty \frac{x \sin x}{x^2+1}\, dx$$

Take $f(z) = \dfrac{ze^{iz}}{z^2+1}$ and so,

$$I = \operatorname{Im}\left(\int_0^\infty \frac{ze^{iz}}{z^2+1}\, dz\right)$$

and so let γ be defined the same as before, then we consider I_2,

$$\begin{aligned}I_2 &= \int_{\gamma_2} \frac{ze^{iz}}{z^2+1}\, dz \\ &\leq \int_0^\pi \left|\frac{Re^{it}e^{Re^{it}}}{(Re^{it})^2+1}\, iRe^{it}\right|\, dt \\ &\leq \frac{R^2}{R^2-1}\int_0^\pi |e^{Re^{it}}|\, dt \\ &= \frac{R^2}{R^2-1}\int_0^\pi e^{-R\sin t}\, dt \\ &= \frac{2R^2}{R^2-1}\int_0^{\frac{\pi}{2}} e^{-R\sin t}\, dt \\ &\leq \frac{2R^2}{R^2-1}\int_0^{\frac{\pi}{2}} e^{-\frac{2Rt}{\pi}}\, dt \\ &= \frac{2R^2}{R^2-1}K && K \in \mathbb{C}\end{aligned}$$

Hence as $R \to \infty$, $I_2 \to 0$ and so we can just apply CRT to the integral and achieve the solution. There are two simple poles in the integrand at $z = \pm i$ and as before only $i \in \gamma$ and so we only need to calculate one residue.

$$\begin{aligned}\operatorname{res}(f, i) &= \lim_{z\to i} (z-i)\frac{ze^{iz}}{z^2+1} \\ &= \lim_{z\to i} \frac{ze^{iz}}{z+i} = \frac{1}{2e}\end{aligned}$$

and so, by CRT,

$$\begin{aligned}\int_\gamma f(z)\,dz &= 2\pi i \operatorname{res}(f, i) \\ &= \frac{\pi i}{e}\end{aligned}$$

and so,

$$\int_{\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{e}$$

11.2.3 Even integrand using Semi-Circle

Let us consider the following integral,

$$I = \int_0^\infty \frac{dx}{(1+x^2)^2}\,dx$$

and, as the integrand is even,

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}\,dx \tag{$*$}$$

Hence, we consider $f(z) = \dfrac{1}{(1+z^2)^2}$ over the contour $\gamma = \gamma_1 + \gamma_2$, where $\gamma_1 = [-R, R]$ and $\gamma_2 = Re^{it}$ with $t \in [0, \pi]$.

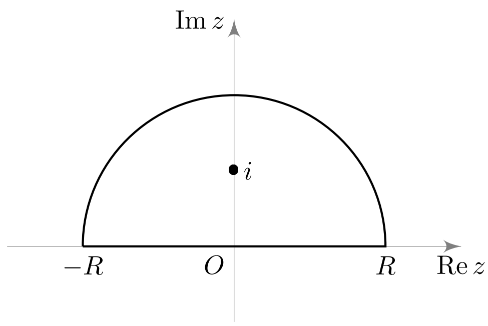


Figure 6: Diagram of γ

The singularity of $f(z)$ is $z = i$, if $R > i$. As i is a double pole, then let $g(z) = (z-i)^2 f(z)$ and now,

$$g'(z) = \frac{-2}{(z+i)^3}$$

and hence,

$$\operatorname{res}(f, i) = \lim_{z\to i} \frac{-2}{(z+i)^3} = -\frac{i}{4}$$

and so, by CRT,

$$\begin{aligned}\int_\gamma f(z)\,dz &= 2\pi i(\text{res}(f,i))\\ &= -2\pi i\cdot\frac{i}{4}=\frac{\pi}{2}\end{aligned}$$

and now consider I_2 under the ML-bound. We can say,

$$\begin{aligned}\left|\int_{\gamma_2} f(z)\,dz\right| &\le \frac{1}{(R^2-1)^2}\ell(\gamma)\\ &\le \frac{\pi R}{(R^2-1)^2}\end{aligned}$$

and so as $R\to\infty$, $I_2\to 0$. Hence as we take $R\to\infty$,

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int f(z)\,dz\\ &= \int_{-\infty}^{\infty}\frac{1}{(1+z^2)^2}\,dz\\ &= \frac{\pi}{2}\end{aligned}$$

and as $(*)$ we can say,

$$\int_0^\infty \frac{1}{(1+z^2)^2}\,dz=\frac{\pi}{4}$$

11.2.4 Large powered denomenator using Semicircle

Now, we shall evaluate the following integral,

$$\int_0^\infty \frac{dx}{x^{1000}+1}$$

and let $f(z)=\dfrac{1}{z^{1000}+1}$, which has a simple pole at $\alpha=e^{\frac{\pi}{1000}i}$. Now, we let $\gamma_1=[-R,R]$ and then $\gamma_2=Re^{it}$ where $t\in[0,\dfrac{\pi}{500}]$ and γ_3 is reversal for $[0,\alpha^2R]$ which we let,

$$\gamma_3^- = \alpha^2 t \qquad t\in[0,R]$$

Then α is inside γ. Now, consider I_3,

$$\begin{aligned}I_3 &= \int_{\gamma_3^-}\frac{dx}{z^{1000}+1}\\ &= -\int_\gamma \frac{\alpha^2}{(\alpha^2 t)^{1000}+1}\\ &= -\int_0^R \frac{\alpha^2}{(\alpha^2 t)^{1000}+1}\\ &= -\alpha^2 I_1\end{aligned}$$

We can now see that the integrand in γ_2 is $mathcalO(\dfrac{1}{R^{1000}})$, hence, the length of γ is $\dfrac{\pi R}{500}$, by the ML-inequality, $mathcalO(\dfrac{1}{R^{999}})\to 0$ as $R\to\infty$. This means,

$$\int_{\gamma_2} f(z)\,dz\to 0 \qquad \text{as } R\to\infty$$

Then the residue at f is simply,

$$\operatorname{res}(f,\alpha) = \frac{1}{1000\alpha^{999}}$$

and by Cauchy Residue Theorem,

$$\begin{aligned} I &= 2\pi i \operatorname{res}(f,\alpha) \\ &= \frac{2\pi i}{1000\alpha^{999}} \end{aligned}$$

and so,

$$\begin{aligned} I &= I_1 + I_2 + I_3 \\ &= (1-\alpha^2)I_2 + I_3 \\ \frac{2\pi i}{1000\alpha^{999}} &= (1-\alpha^2)I_2 && \text{as } R \to \infty \\ \frac{2\pi i}{1000(1-\alpha^2)\alpha^{999}} &= I_2 \\ \frac{2\pi i}{1000(\alpha^{-1}+\alpha)} &= I_2 \\ \frac{\pi}{1000}\frac{2i}{\alpha^{-1}+\alpha} &= I_2 \\ \frac{\pi}{1000}\csc\left(\frac{\pi}{1000}\right) &= I_2 \end{aligned}$$

then,

$$\int_0^\infty \frac{1}{1+x^{1000}}\,dx = \frac{\pi}{1000}\csc\left(\frac{\pi}{1000}\right)$$

11.2.5 Large powered denomenator and powered numerator using Semicircle

Compute,

$$\int_0^\infty \frac{x^{666}}{x^{1000}+1}\,dx$$

Consider, $\gamma = \gamma_1 + \gamma_2 + \gamma_3$, where $\gamma_1 = [0, R]$, $\gamma_2 = Re^{it}$ where $t \in [0, \frac{\pi}{500}]$ and γ_3 is the reversal of $[0, \alpha^2 R]$, which we parameterised as,

$$\gamma_3^- = \alpha^2 t \qquad t \in [0, R]$$

We can show $I_2 \to 0$ as $R \to \infty$. Then the integral over γ_3 is,

$$\begin{aligned} I_3 &= -\int_0^R \frac{(\alpha^2 t)^{666}}{(\alpha^2 t)^{1000}+1}\,\alpha^2 dt \\ &= -\alpha^{2\times 667} I_1 \end{aligned}$$

We see that the residue of the integrand at α is,

$$\operatorname{res}(f,\alpha) = \frac{\alpha^{666}}{1000\alpha^{999}}$$

and so by Cauchy Residue Theorem, as $R \to \infty$,

$$\begin{aligned} 2\pi i \frac{\alpha^{666}}{1000\alpha^{999}} &= I_1 + I_2 + I_3 \\ &= (1-\alpha^{2\times 667})I_1 \end{aligned}$$

Hence,

$$\begin{aligned}
I_1 &= \frac{2\pi i \alpha^{666}}{1000(1-\alpha^{2\times 667})a^{999}} \\
&= \frac{\pi}{1000}\frac{2i\alpha^{666}}{\alpha^{667}\alpha^{999}(\alpha^{-667}+\alpha^{667})} \\
&= \frac{\pi}{1000}\frac{2i}{(\alpha^{-667}+\alpha^{667})} \\
&= \frac{\pi}{1000}\csc\left(\frac{667\pi}{1000}\right)
\end{aligned}$$

12 Argument Principle and Rouche's Theorem

Definition 12.1. (*Meromorphic*) Let $U \subseteq \mathbb{C}$ be a domain and take $S \subseteq U$. A function $f : U \setminus S \to \mathbb{C}$ is meromorphic on U if f is differnetiable at every point of $U \setminus S$, and every point of S is a pole of f.

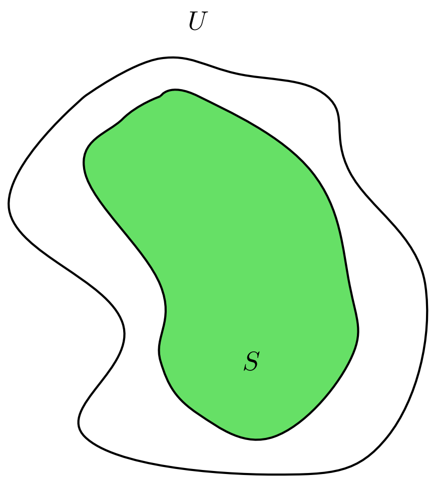


Figure 7: A Meromorhic function is holomorphic on $U \setminus S$.

Theorem 12.2. (*Argument Principle*) Let U be a domain and f be meromorphic on U. If γ is a simple positively oriented closed contour, such that, γ and it's interior is contained in U, and γ passes through no zero or poles of f, then,

$$\frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)}\,dz = Z + P$$

number of poles in theinterior of γ (counted w/ multiplicity)

number of zeros inthe interior of γ

and the final main result of the course,

Theorem 12.3. (*Rouche's Theorem*) Let γ be a simple closed contour. Let f and g be holomorphic in a domain that contains the image and the interior of γ. Suppose for all $z \in \gamma^*$ we have that,

$$|f(z) - g(z)| < |f(z)| + |g(z)|$$

then, f and g are non-zero on γ^* and $Z_f = Z_g$.

Example. Prove that all of the zeros of $f(z) = z^5 + 7z + 12$ lie in the annulus,

$$A = \{z \in \mathbb{C} : 1 \leq |z| < 2\}$$

Let γ be $D(0, 2)$ and we seek a function $g(z)$ that approximates f well on γ,

$$|f(z) - g(z)| < |f(z)| + |g(z)|$$

Take $g(z) = z^5$ and hence, in $D(0,2)$,

$$\begin{aligned}|f(z) - g(z)| &= |z^5 + 7z + 12 - z^5| \\ &= |7z + 12| \\ &\leq 7|z| + 12 \\ &= 26\end{aligned}$$

and so now we consider $|g(z)| = |z^5| = |z|^5 = 32$. Hence, we have shown that, $|f(z) - g(z)| < |f(z)| - |g(z)|$. Hence, by Rouche's theorem, we can say that $Z_f = Z_g$ on $D(0,2)$. So we find all the zeros of $g(z)$ on $D(0,2)$ and we can say that $f(z)$ has 5 zeros on $D(0,2)$. Now, let γ be the circular contour $D(0,1)$ and again $f(z) = z^5 + 7z + 12$. Now we want to select our $g(z)$. We take $g(z) = 12$. Then,

$$\begin{aligned}|f(z) - g(z)| &= |z^5 + 7z + 12 - 12| \\ &= |z^5 + 7z| \\ &\leq |z|^5 + 7|z| \\ &= 8\end{aligned}$$

and as we can say that $|g(z) = 12|$, then using Rouche's Theorem we can say $Z_f = Z_g$ and $g(z)$ has zero zeros on $f(z)$ on $D(0,1)$.
Hence,

- f has 5 zeros on $D(0,2)$
- f has 0 zeros on $D(0,1)$.

and so all of f's zeros are in A.