The Real Number System

James Arthur

September 21, 2020

Contents

1 Overview

The reals ($\mathbb{R}$) have a few properties:

1. They are a field, i.e. a groupoid with two binary operations.
2. They are ordered
3. They are also complete.

We will also look at supremum and the infimum.

We are also going to look at the extended real numbers. We are going to add two more fictitious points. $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$.

2 Properties of the Reals

We will be taking the axiomatic view point of the real numbers. No construction with Dedekind cuts or Cauchy sequences. All of these are isomorphic.

2.1 Field Properties

The real numbers are a set, $\mathbb{R}$, with two binary operations, $+$ and $\times$. They must satisfy the following axioms. So take $a, b, c \in \mathbb{R}$:

1. $a + b = b + a$ and $ab = ba$ (commutativity)
2. $(a + b) + c = a + (b + c)$ and $a(bc) = (ab)c$ (associativity)
3. $a(b + c) = ab + ac$ (distributivity)
4. There are two distinctive identities 0 (additive identity) and 1 (multiplicative identity), such that $a + 0 = 0 + a = a$ and $a1 = 1a = a$
5. We also have inverses, $-a$ (additive inverse) such that $a + -a = 0$ and if $a \neq 0$, there is a real number $\frac{1}{a}$ such that: $a(\frac{1}{a}) = 1$

2.2 Order Relation

The real numbers are ordered, that means:

1. For each pair of reals a and b, exactly one of the following is true

$$a = b \qquad a < b \qquad b < a$$

2. It is also transitive, if $a < b$ and $b < c$, then $a < c$
3. If $a < b$ then $a + c < b + c$ for any c, and if $0 < c$, then $ac < bc$

2.3 Supremum

Let $S \subset \mathbb{R}$. If there exists $b \in \mathbb{R}$ such that $x \leq b \quad \forall x \in S$ then S is bounded above and b is an upper bound of S.

If β is an upper bound of S, but no number less than β is, then β is called the supremum of S, denoted:

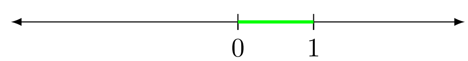
$$\beta = \sup S$$

Figure 1: *Let S be the orange set and then b is an upper bound of S and β is $\sup S$*

We also call the supremum the least upper bound.

Example 1 *$S = [\,0, 1\,]$ and prove $\sup S = 1$*

Solution 1 *Take our diagram from above:*



We need to check that $x \leq 1 \quad \forall x \in S$, which is definitionally true.
Secondly we need to prove that $\forall\, b < 1, \exists\, x \in S, b < x$, which is again trivially true. So $\sup S = 1$

Example 2 *Take $T = (\,0, 1\,)$ where $\sup T = 1$*

Solution 2 *Again every number is less than 1, but if you take any number less than one you can always find another element larger.*
NB: The supremum here isn't in the set

2.4 Infimum

Similarly, if there exists an $a \in \mathbb{R}$ such that $a \leq x \quad x \in S$, then S is bounded below and a is a lower bound of S.

If α is a lower bound of S, but no number is greater than α is, then α is called the infimum of S:

$$\alpha = \inf\ S$$

Figure 2: *Let S be the orange set and then a is a lower bound of S and α is $\inf S$*

Another name for the infimum is the greatest lower bound.

2.5 Completeness Axiom

Do the supremum and the infimum actually exist? Well, not all subsets are bounded above, i.e. $\mathbb{R} \subset \mathbb{R}$ or what about the empty set? This is what the completeness axiom does:

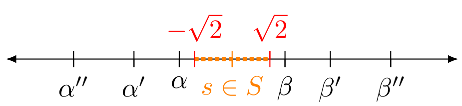
1. If a non-empty set of real numbers are bounded above, then it has a supremum.

So the reals are a **complete ordered field**

The completeness axiom is distinguishing of the reals. They are the only complete ordered field. The rationals possess everything but completeness in terms of our axioms.

Example 3 *We restrict to the $\mathbb{Q}$, $S = \{r \in \mathbb{Q} : r^2 < 2\}$. Find the supremum and infimum.*

Solution 3 *If we take the example below;*



we can say that we won't reach $\sqrt{2}$ in the supremum or $-\sqrt{2}$ in the infimum. This is because we are using rationals and $\sqrt{2}$ is an irrational. We can go either way and there is always a number closer to $\sqrt{2}$.

This proves that rationals are not complete.

3 Extended Real Numbers

It is convenient to attach ∞ and $-\infty$ to the reals. How do they fit in? Firstly lets look at orders. Take $x \in \mathbb{R}$, then:

$$-\infty < x < \infty$$

Now if a set S is unbounded above or below, we can write:

$$\sup S = \infty \quad \inf S = -\infty$$

Example 4 *Find the infimum of $S = \{x \in \mathbb{R} : x : 2\}$*

Solution 4 *As there is technically no lower bound, it is $-\infty$*

We usually denote the extended reals with the symbol, $\overline{\mathbb{R}}$ or $[-\infty, \infty]$ or $\mathbb{R} \cup \{-\infty, \infty\}$

3.1 Arithmetic

If $a \in \mathbb{R}$,

1. Then:

$$a + \infty = \infty + a = \infty$$
$$a - \infty = -\infty + a = -\infty$$
$$\frac{a}{\infty} = \frac{a}{-\infty} = 0$$

2. and $0 < a$, then:

$$a\infty = \infty a = \infty$$
$$a(-\infty) = (-\infty)a = -\infty$$

3. and $a < 0$, then:

$$a\infty = \infty a = -\infty$$
$$a(-\infty) = (-\infty)a = \infty$$

We also define:

1. $\infty + \infty = \infty\infty = (-\infty)(-\infty) = \infty$

2. and also $-\infty - \infty = \infty(-\infty) = (-\infty)\infty = -\infty$

3. and finally, $|\infty| = |-\infty| = \infty$

We say it isn't useful to define; $\infty - \infty$, $0 \cdot \infty$, $\frac{\infty}{\infty}$ and $\frac{0}{0}$. We call them indeterminate forms.