Complex Analysis Coursework 1

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Problem 1. *Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth path. Let f be a continuous function defined on an open set containing the contour γ. Prove that*

$$\left|\int_\gamma f(z)\,dz\right| \leq \ell(\gamma) \sup_{t\in[a,\,b]} |f(\gamma(t))|$$

Solution 1. *Let us start with the definition of a path integral (Definition 4.3),*

$$\begin{aligned}
\left|\int_\gamma f(z)\,dz\right| &= \left|\int_a^b f(\gamma(t))\gamma'(t)\,dt\right| \\
&\leq \int_a^b |f(\gamma(t))\gamma'(t)|\,dt && \textit{by Lemma 4.10}
\end{aligned}$$

We now note that,

$$|f(\gamma(t))| \leq \sup_{t\in[a,\,b]} |f(\gamma(t))|$$

as we define the supremum to be the least upper bound of a function over a domain. Hence we can write that,

$$\begin{aligned}
\int_a^b |f(\gamma(t))\gamma'(t)|\,dt &\leq \int_a^b |\gamma'(t)| \sup_{t\in[a,\,b]} |f(\gamma(t))|\,dt \\
&= \sup_{t\in[a,\,b]} |f(\gamma(t))| \int_a^b |\gamma'(t)|\,dt \\
&= \ell(\gamma) \sup_{t\in[a,\,b]} |f(\gamma(t))| && \textit{by Definition 4.7}
\end{aligned}$$

$\square$

Problem 2. *Verify the Cauchy-Riemann equations for the function $f : \mathbb{C} \to \mathbb{C}$ defined by*

$$f(z) = 2iz^2 + \sqrt{2}\pi z + 4\sqrt{2}$$

Solution 2. *First we have to let $z = x + iy$, then find $f(x + iy)$ in terms of a real and imaginary function. This leads to the following,*

$$f(x+iy) = (\pi\sqrt{2}x - 2xy + 4\sqrt{2}) + i(2x^2 - 2y^2 + \pi\sqrt{2}y)$$

We can now rename these two functions such that $f(x + iy) = u(x) + iv(x)$.

$$u(x) = \pi\sqrt{2}x - 4xy + 4\sqrt{2}$$
$$v(x) = 2x^2 - 2y^2 + \pi\sqrt{2}y$$

As $u(x)$ and $v(x)$ are compositions of differentiable functions we can differentiate them,

$$\begin{aligned}\frac{\partial u}{\partial x} &= \pi\sqrt{2} - 4y \\ &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -4x \\ &= -\frac{\partial v}{\partial x}\end{aligned}$$

Hence, we get that,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Which are exactly the Cauchy-Riemann Equations.

Problem 3. *Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth curve. Write down the formula for the length of γ. Using your formula, compute the length of the smooth curve*

$$\gamma(t) = e^{it} + t(\sin t - i\cos t)$$

from $t = 0$ to $t = \frac{\pi}{2}$.

Solution 3. *The formula for the arc length of a path γ is,*

$$\ell(\gamma) = \int_a^b |\gamma'(t)|\, dt$$

We can write $\gamma = (\cos t + i\sin t) + t(\sin t - i\cos t) = (\cos t + t\sin t) + i(\sin t - t\cos t)$ and hence we can write the differential as,

$$\gamma'(t) = t\cos t + it\sin t$$

and so $|\gamma'(t)| = t$. Now we plug into the formula,

$$\int_0^{\frac{\pi}{2}} t\, dt = \left[\frac{t^2}{2}\right]_0^{\frac{\pi}{2}} = \frac{\pi^2}{8}$$

Problem 4. *Let γ be the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ traversed once anti-clockwise. Prove that*

$$\left| \int_\gamma \frac{3z^3 \sin 2z}{3z^5 + 1}\, dz \right| \leq 3\pi e^2$$

Solution 4. *We are going to use the ML-Bound to prove an upper bound for this integral,*

$$\left| \int_\gamma f(z)\, dz \right| \leq ML$$

such that, $|f(x)| \leq M$ and $L = \ell(\gamma)$. We can quickly say that $\ell(\gamma) = 2\pi$ as γ is the unit circle traversed once. Now for M, this is a slight bit more work,

$$\begin{aligned}
\left| \frac{3z^3 \sin 2z}{3z^5 + 1} \right| &\leq \left| \frac{3z^3}{3z^5} \right| |\sin 2z| \\
&\leq |\sin 2z| && \textit{as } \left| \frac{1}{z^2} \right| \leq 1 \\
&= \left| \frac{e^{2iz} + e^{-2iz}}{2i} \right| && \textit{by definition of complex sine} \\
&\leq \left| e^{2iz} + e^{-2iz} \right| \\
&\leq |e^{2iz}| + |e^{-2iz}| && \textit{by Triangle Inequality} \\
&= e^{2y} + e^{-2y} \\
&\leq e^2 + 1 && \textit{as we know that } |z| = 1 \textit{ and so } x, y \leq 1 \\
&\leq \frac{3}{2} e^2
\end{aligned}$$

So as we now know that we can let $L = 2\pi$ and $M = \frac{3}{2}e^2$, we can now let $ML = 3\pi e^2$ and write,

$$\left| \int_\gamma \frac{3z^3 \sin 2z}{3z^5 + 1}\, dz \right| \leq 3\pi e^2$$

$\square$