Linear Algebra - Coursework 1

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Problem 1. *Determine whether the following subsets of real vector spaces are subspaces. Justify your answers.*

1. $T = \{\mathbf{x} \in \mathbb{R}^3 : 3x_1 + 4x_2 + x_3 = 0\}$
2. $U = \{\mathbf{x} \in \mathbb{R}^2 : x_1 + x_2 = 1\}$
3. $V = \{f \in C(\mathbb{R}) : \int_0^1 f(x)\,dx = 3\}$
4. $W = \{A \in M_{n\times n}(\mathbb{R}) : Tr(A) = 0\}$

Solution 1. *i) We firstly have the space* $T = \{\mathbf{x} \in \mathbb{R}^3 : 3x_1 + 4x_2 + x_3 = 0\}$*, by Lemma 2.35 of the notes, we must have that* $\mathbf{0}_v \in T$*,* T *is closed under addition and scalar multiplication (smul).*
Firstly $\mathbf{0}_v = (0, 0, 0)$*, which is in* T *as,* $3(0) + 4(0 + 0 = 0)$*. For closure under addition, take arbitrary* $\mathbf{u}, \mathbf{v} \in T$ *and let* $\mathbf{w} = \mathbf{u} + \mathbf{v}$*. Now,*

$$\begin{aligned}
3w_1 + 4w_2 + w_3 &= 3(u_1 + v_1) + 4(u_2 + v_2) + (u_3 + v_3) \\
&= 3u_1 + 3v_1 + 4u_2 + 4v_2 + w_1 + w_2 \\
&= (3u_1 + 4u_2 + u_3) + (3v_1 + 4v_2 + v_3) \\
&= 0 + 0 && \text{as } \mathbf{u}, \mathbf{v} \in T \\
&= 0 && \text{Hence, } \mathbf{w} \in T
\end{aligned}$$

Hence, T *is closed under addition. For smul, we can do something similar, take an arbitrary* $u \in T$ *and* $a \in \mathbb{R}$*. Let there be a vector* $\mathbf{w} = a\mathbf{u}$*. Now,*

$$\begin{aligned}
3w_1 + 4w_2 + w_3 &= 3(au_1) + 4(au_2) + au_3 \\
&= a(3u_1 + 4u_2 + u_3) \\
&= a \cdot 0 && \text{as } \mathbf{u} \in T \\
&= 0 && \text{Hence, } a\mathbf{u} \in T
\end{aligned}$$

Hence we have that T *is closed under smul and* T *is a subspace of* $\mathbb{R}^3$*. A less formal way to see this would be to say that* T *is a plane passing through the origin and hence it has to have the required properties.*

ii) For $U = \{\mathbf{x} \in \mathbb{R}^2 : x_1 + x_2 = 1\}$*, it suffices to notice that there isn't a zero vector, for the zero vector in* $\mathbb{R}^2$ *doesn't satisfy* $x_1 + x_2 = 1$ *as by definition,* $x_1 + x_2 = 0$*. So* U *is not a subspace of* $\mathbb{R}^2$*.*

iii) $V = \{f \in C(\mathbb{R}) : \int_0^1 f(x)\,dx = 3\}$ *falls for a very similar problem. If there was a zero function in the space* $(f(x) = 0)$*, then we know that* $\displaystyle\int_0^1 0\,dx = 0$*. Hence it is excluded from the space.* V *is not a subspace of* $C(\mathbb{R})$

iv) $W = \{A \in M_{n\times n}(\mathbb{R}) : Tr(A) = 0\}$*, we can use the fact that* $M_{n\times n}(\mathbb{R})$ *is a vector space as we proved in the lecture notes. Here we have some extra constraints. So it suffices to prove that,* $\mathbf{0}_{n\times n} \in W$*, as* $Tr(\mathbf{0}_{n\times n}) = 0$ *this is quickly proved. Next we can take* $A, B \in M_{n\times n}(\mathbb{R})$ *and prove that,*

$$Tr(A) + Tr(B) = Tr(A + B)$$

This is again pretty simple, let the traces of the matrices be,

$$Tr(A) = a_1 + a_2 + \dots a_n \qquad Tr(B) = b_1 + b_2 + \dots + b_n$$

and as $A, B \in W$*, by definition* $Tr(A) = Tr(B) = 0$*. Hence we need to show that,* $Tr(A + B) = 0$*. We can do this by just noticing that,*

$$\begin{aligned} Tr(A+B) &= a_1 + b_1 + \dots + a_n + b_n \\ &= (a_1 + \dots + a_n) + (b_1 + \dots + b_n) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

$$Tr(A + B) = a_1 + b_1 + \dots + a_n + b_n = (a_1 + \dots + a_n) + (b_1 + \dots + b_n) = 0 + 0 = 0$$

Hence W *is closed under addition. To prove closure under smul, it again suffices to show that, if* $a \in \mathbb{R}$ *and* $B \in M_{n\times n}(\mathbb{R})$

$$aTr(B) = Tr(aB)$$

This can be quickly shown by taking the scalar multiple of a matrix B*, we have that* $aB = \{ab_{i,j} : i, j \in 1, \dots, n\}$*, hence we can now just say that,*

$$\begin{aligned} Tr(aB) &= ab_1 + \dots + ab_n \\ &= a(b_1 + \dots + b_n) \\ &= aTr(B) \end{aligned}$$

Hence W *is a subspace of* $M_{n\times n}(\mathbb{R})$*.*[1]

[1] Interesting fact! We have now also proved that the $Tr(A)$ is a linear transformation!

Problem 2. *Determine whether the following lists of vectors are linearly independent, whether they are spanning sets and whether they are bases,*

1. *In* $\mathbb{R}^2$, *the vectors* $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ *where,*

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad \mathbf{v}_3 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

2. *In* $\mathbb{C}^3$ *(considered as a complex vector space), the vectors* $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ *where,*

$$\mathbf{v}_1 = \begin{pmatrix} 2+3i \\ 1 \\ 3 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \mathbf{v}_3 = \begin{pmatrix} i \\ 2 \\ 1+i \end{pmatrix}$$

3. *In* $M_{2\times 2}(\mathbb{R})$, *the matrices* $[A_1, A_2, A_3]$ *where,*

$$A_1 = \begin{pmatrix} 1 & 5 \\ -1 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1 & -4 \\ 4 & 1 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$$

4. *) In* $P_2(\mathbb{R})$, *the polynomials* $[p_1, p_2, p_3]$ *where,*

$$p_1(x) = 2 + x - 2x^2 \qquad p_2(x) = 3 - x + x^2 \qquad p_3(x) = -1 + 2x - 3x^2$$

Solution 2. *We are going to firstly take the vectors and place them into a matrix. By Lemma 5.4 in the notes, if we have a pivot in every row and column after gaussian elimination then they are linearly independent, spanning and a basis. If there is only pivots in the columns or rows then they are only linearly independent or spanning respectively. (Lemma 4.8 / Lemma 3.12)*

i) We form our matrix,

$$\begin{pmatrix} -1 & 2 & 4 \\ 3 & -1 & 1 \end{pmatrix}$$

which when we preform gaussian elimination, we get,

$$\begin{pmatrix} 1 & 0 & \frac{6}{15} \\ 0 & 1 & \frac{13}{5} \end{pmatrix}$$

which then tells us that they are not linearly indendent as we don't have pivots in every column and so hence not a basis but they are spanning as we have a pivot in every row.

ii) We form another matrix for our complex vectors,

$$\begin{pmatrix} 2+3i & 1 & i \\ 1 & 0 & 2 \\ 2 & 1 & 1+i \end{pmatrix}$$

which we can preform gaussian elimination on and get that,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Which we can look at quickly and tell that it's a basis, and hence linearly independent and spanning of $\mathbb{C}^3$.

iii) We first need to turn these matrices into co-ordinate vectors as the set of $M_{2\times 2} \cong \mathbb{R}^4$ (Example 8.2). Hence take the standard basis and rewite as coordinate vectors,

$$[m_0, m_1, m_2, m_3] = \left[\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right]$$

and hence we get the matrix,

$$\begin{pmatrix}1 & 1 & 2\\ 5 & -4 & 1\\ -1 & 4 & 3\\ 0 & 1 & 1\end{pmatrix}$$

which reduces to,

$$\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

Hence this set of vectors is not spanning or linearly independent as there are two pivots in the second row and third column.

iv) We can do this one by a very similar method to (iii), so transform into coordinate vectors with the following mapping,

$$[l_0, l_1, l_2] = [1, x, x^2]$$

Hence we can rewrite,

$$p_1 \to \begin{pmatrix}2\\ 1\\ -2\end{pmatrix} \quad p_2 \to \begin{pmatrix}3\\ -1\\ 1\end{pmatrix} \quad p_2 \to \begin{pmatrix}-1\\ 2\\ -3\end{pmatrix}$$

and hence produce the matrix and reduce it,

$$\begin{pmatrix}2 & 3 & -1\\ 1 & -1 & 2\\ -2 & 1 & 3\end{pmatrix} \to \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{pmatrix}$$

which again isn't linearly independent or spanning as there is an empty row and two pivots in the second row and third column.

Problem 3. *Let V be a vector space. Prove that a list of vectors $[\mathbf{u}, \mathbf{v}]$ from V is linearly independent if and only if the list $[\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}]$ is linearly independent.*

Solution 3. *Suppose that $[\mathbf{u}, \mathbf{v}]$ is a list of linearly independent vectors. That means,*

$$a_1\mathbf{u} + a_2\mathbf{v} = \mathbf{0}_v \iff a_1 = a_2 = 0$$

Now as $a_1, a_2 \in \mathbb{R}$, we can write them as, $a_1 = b_1 + b_2$ and $a_2 = b_1 - b_2$ as the vectors, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are linearly independent and spanning, hence that system of equations is not constrained, and so a_1 and a_2 can still be any real numbers. So we plug in our change of variables,

$$(b_1 + b_2)\mathbf{u} + (b_1 - b_2)\mathbf{v} = 0 \iff b_1 = b_2 = 0$$

and hence we can rewrite the first part of that equation to get,

$$b_1(\mathbf{u}+\mathbf{v}) + b_2(\mathbf{u}-\mathbf{v}) = 0 \iff b_1 = b_2 = 0$$

Hence if $[\mathbf{u}, \mathbf{v}]$ are linearly independent so is $[\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}]$

Now suppose that $[\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}]$ is a list of linearly independent vectors. That means,

$$\alpha_1(\mathbf{u}+\mathbf{v}) + \alpha_2(\mathbf{u}-\mathbf{v}) = 0 \iff \alpha_1 = \alpha_2 = 0$$

Hence, we can rewite this proposition as,

$$(\alpha_1 + \alpha_2)\mathbf{u} + (\alpha_1 - \alpha_2)\mathbf{v} = 0 \iff \alpha_1 = \alpha_2 = 0$$

Now as the reals are closed under addition, and hence subtraction, we can let $\alpha_1 + \alpha_2 = \beta_1$ and $\alpha_1 - \alpha_2 = \beta_2$ and so,

$$\beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2 = 0 \iff \beta_1 = \beta_2 = 0$$

Hence if $[\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}]$ is linearly independent then so is $[\mathbf{u}, \mathbf{v}]$.

Pulling together both halfs of the proof, $[\mathbf{u}, \mathbf{v}]$ is linearly independent if and only if $[\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}]$ is also linearly independent.

$\square$

Problem 4. *Let $V = \mathbb{R}^3$. Let,*

$$\mathbf{u}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \mathbf{w}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad \mathbf{w}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

and,

$$U = \operatorname{span}(\{\mathbf{u}_1\}) \qquad W = \operatorname{span}(\{\mathbf{w}_1, \mathbf{w}_2\})$$

1. *Show that $V = U \oplus W$.*
2. *Determine the actions of the projections P_U onto U along W, and P_W onto W along U on an arbitrary vector $\mathbf{v} = (v_1, v_2, v_3) \in V$. Hence write down the matrices $[P_U]_E^E$ and $[P_W]_E^E$, where E is the standard basis in $\mathbb{R}^3$.*

Solution 4. *i) To prove that $V = U \oplus W$, we must show that $V = U + W$ and then $U \cup W = \{\mathbf{0}_v\}$. To show that $V = U + W$, it suffices to show that $[\mathbf{u}_1, \mathbf{w}_1, \mathbf{w}_2]$ is a spanning set for $\mathbb{R}^3$. So we create a matrix and do gaussian elimiation,*

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence the vectors are a spanning set, in fact they are a basis for $\mathbb{R}^3$. Hence, $V = U + W$. Now given that these set of vectors are linearly independent, there is no linear combination of $[\mathbf{w}_1, \mathbf{w}_2]$ that is equal to $\mathbf{u}_1$. Hence we can say that $U \cup W = \{0_v\}$ and that $V = U \oplus W$.

ii) We can rewrite $[\mathbf{u}_1, \mathbf{w}_1, \mathbf{w}_2]$ in an augmented matrix with repsect to a general vector and row reduce it,

$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & v_1 \\ 0 & 1 & -1 & v_2 \\ 1 & 1 & 0 & v_3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & v_3 - v_2 - v_1 \\ 0 & 1 & 0 & v_1 + v_2 \\ 0 & 0 & 1 & v_1 \end{array}\right)$$

We can now go and find the projections from this matrix,

$$P_U = (v_3 - v_2 - v_1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ v_3 - v_2 - v_1 \end{pmatrix}$$

$$P_W = (v_2 + v_1) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + v_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_2 + v_1 \end{pmatrix}$$

and we can now write out the matrices for this transformations.

$$[P_U]_E^E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 1 \end{pmatrix} \qquad [P_W]_E^E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

Problem 5. *Let V and W be 3-dimensional real vector spaces with ordered bases $P = [v_1, v_2, v_3]$ and $Q = [w_1, w_2, w_3]$ respectively. Let $T : V \to W$ be a linear transformation. The matrix of T with respect to P and Q is,*

$$[T]_P^Q = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 5 & 6 \\ 4 & 9 & 5 \end{pmatrix}$$

Find the matrix $[T]_{Q'}^{P'}$ of T with respect to the bases $P' = [v_1', v_2', v_3']$ and $Q' = [w_1', w_2', w_3']$ where,

$$\begin{aligned} \mathbf{v}_1' &= \mathbf{v}_1 + \mathbf{v}_2 \\ \mathbf{v}_2' &= \mathbf{v}_2 + \mathbf{v}_3 \\ \mathbf{v}_3' &= -2\mathbf{v}_1 + \mathbf{v}_3 \end{aligned}$$

and

$$\begin{aligned} \mathbf{w}_1' &= \mathbf{w}_1 + 2\mathbf{w}_2 - 3\mathbf{w}_3 \\ \mathbf{w}_2' &= \mathbf{w}_1 \\ \mathbf{w}_3' &= \mathbf{w}_3 + \mathbf{w}_2 \end{aligned}$$

Solution 5. *We can use Corrolary 7.14 and so we can write $[T]_{P'}^{Q'}$ as,*

$$[T]_{P'}^{Q'} = [id_W]_Q^{Q'} [T]_P^Q [id_v]_{P'}^P$$

and so we need to calculate $[id_W]_Q^{Q'}$ and $[id_V]_{P'}^P$. We can get the following quickly,

$$[id_V]_{P'}^P = \begin{pmatrix} 1 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad [id_W]_{Q'}^Q = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ -3 & 0 & 1 \end{pmatrix}$$

and by Corollary 7.13, we now have $[id_W]_Q^{Q'} = ([id_W]_{Q'}^Q)^{-1}$ and so we can write,

$$\begin{aligned} [T]_{P'}^{Q'} &= \begin{pmatrix} 1 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 2 \\ 1 & 5 & 6 \\ 4 & 9 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ -3 & 0 & 1 \end{pmatrix}^{-1} \\ &= \frac{1}{5} \begin{pmatrix} -7 & -3 & 7 \\ 27 & 18 & -27 \\ 44 & 61 & 6 \end{pmatrix} \end{aligned}$$