Year 3 — Lie Groups and Applications in Geometric Dynamics

Based on lectures by Dr Hamid Alemi Ardakani

Notes taken by James Arthur

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

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1 Lie Groups and Algebras

To start, let us define some groups. Firstly, what is a group?

Definition 1.1 (Group)**.** G is a nonempty set and endowed with a binary operation such that,

- It's closed under $(\cdot)$, $\forall\, a, b \in G$, $a \cdot b \in G$
- It's associative, i.e. $\forall\, a, b \in G$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
- There is an identity element, $\forall\, a \in G$, $a \cdot e = a = e \cdot a$.
- Every element has an inverse, $\forall\, a \in G$, $a \cdot a^{-1} = e = a^{-1} \cdot a$.

The definition of a Lie group also uses something called a manifold, to define this let's go on a slight adventure into topology!

1.1 Manifolds

A manifold is a topological space that locally resembles Euclidean space near each point [4]. However, this doesn't sate me as, what does near mean? It's very informal and hand wavey. Let's define it properly!

Definition 1.2 (Manifold)**.** A manifold is a second countable Hausdorff space that is locally homeomorphic to Euclidean space.

Where we refer to a second countable space as being a topological equivalent to being finitely generated. There exists some countable base of this space $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$, where any open subset of our space, T, can be written as a disjoint union of a finite subfamily of $\mathcal{U}$. This nicely restricts manifolds to be smaller spaces, by making them be the union of countably many open sets. On the point of Hausdorff, this relies on the following defintion,

Definition 1.3 (Pairwise Neighborhood-Separable)**.** Two points are pairwise neighborhood-separable if there exists a neighborhood U of x and V of y such that U and V are disjoint.

Then we can say that a space is Hausdorff if all distinct points are pairwise neighborhood-separable. Finally, to say something is homeomorphic to another space, this means it can be stretched without creating holes or glueing. A homeomorphism is a bijective map between two spaces and local homeomorphisms relates to neighborhoods around points. Hence, saying that something is locally homeomorphic to Euclidean space directly means, that you can bijectively map the contents of the neighborhood around a point to an open ball in $\mathbb{R}^n$, i.e the Euclidean n-ball.[1]

1.2 Lie Groups

Now we know enough to define what a Lie group actually is,

Definition 1.4 (Lie Group)**.** A Lie group is a group that is also a smooth manifold, such that the binary product and inversion are smooth functions.

What we will be focusing our attention to is special Lie groups, the general linear group, special linear group and the special orthogonal group.

Definition 1.5 (General Linear Group)**.** $\mathrm{GL}(n, \mathbb{R})$ is the linear matrix group. The manifold of $n \times n$ invertible square real matrices is a lie group denoted by $\mathrm{GL}(n, \mathbb{R})$

Definition 1.6 (Special Linear Group)**.** The $\mathrm{SL}(n, \mathbb{R})$ is the manifold of $n \times n$ matrices with unit determinant.

Definition 1.7 (Special Orthogonal Group)**.** $\mathrm{SO}(n, \mathbb{R})$ is the manifold of rotation matrices in n dimensions. This may be denoted by $SO(n)$

[1] This is long and a very non-succinct way to define this structure, I prefer to define manifolds via sheaves as I feel it is neater. A manifold is just a locally ringed space, whose sheaf structure is just locally isomorphic to continuous functions on Euclidean space.

1.3 Lie Algebras

To actually understand what Lie Algebras are, we need to generalise the notion of a vector and a tangent. We shall look at so called tangent spaces. To formally define them, we shall first define charts and atlas and along the way redefine what a manifold is. These definitions are adapted from [3].

Definition 1.8 (Chart)**.** Let X be a topological space. An $\mathbb{R}^n$ chart on X is is homeomorphism $\phi : U \to U'$ where $U \subset X$ and $U' \subset \mathbb{R}^n$.

Definition 1.9 (Atlas)**.** A C^∞ atlas on a topological space X is a collection of charts $\phi_\alpha : U_\alpha \to U'_\alpha$ where all the U''s are open subsets of one fixed $\mathbb{R}^n$ such that,

(i) Each $U_\alpha \in X$ is open and $\bigcup_\alpha U_\alpha = X$ (U_α is an open subcover of X) and,

(ii) Changes of coordinates are smooth.

Two last definitions in this section are equivalence relation and equivalence class.

Definition 1.10 (Equivalence Relation)**.** An equivalence relation on a set X is a binary relation $\sim$ satisfying,

(i) $\forall\, a \in X, a \sim a$

(ii) $\forall\, a, b \in X, a \sim b \implies b \sim a$

(iii) $\forall\, a, b, c \in X, a \sim b$ and $b \sim a \implies a \sim c$.

Then the equivalence class is just all of the equivalent elements to a member of that set. For example, for the equivalence relation that $x \sim y$ if and only if $x - y$ is even, then the equivalence classes are all the even numbers and all the odd numbers.

Remark. It can be proven that equivalence classes are just a partition of a set. [1]

Here is another definition of a manifold, this definition explicitly output a smooth manifold,

Definition 1.11 (C^∞ Manifold)**.** An n-dimensional (C^∞) manifold a topological space M together with an equivalence class of C^∞ atlases.

Remark. Our equivalence relation here is that two atlases are equivalent if their union is also an atlas.

Here are a few examples of manifolds,

- Let $M = \mathbb{R}^n$, this is a manifold covered by one open set and then if we take the identity map as our chart, we get the standard manifold on $\mathbb{R}^n$.
- Let $M = \mathbb{C}^n$, then we cover $\mathbb{C}^n$ by just one open set and then chart the map, $\phi : \mathbb{C}^n \to \mathbb{R}^{2n}$ which is just,
$$\phi(z_1, \dots, z_n) = (\operatorname{Re} z_1, \operatorname{Im} z_1, \dots, \operatorname{Re} z_n, \operatorname{Im} z_n)$$
- If M is a manifold, then any open $V \subset M$ is also a manifold. This can be seen as the union of the atlases V and M is going to be M and so it has the same equivalence class and hence it must be a manifold.
- If we let $M_n(\mathbb{R})$ be all real $n \times n$ matrices, then this is a manifold as it's just $\mathbb{R}^{n^2}$. We also can say $\mathrm{GL}(n, \mathbb{R}) \subset M_n(\mathbb{R})$ and so by the previous point, $\mathrm{GL}(n, \mathbb{R})$ is a manifold.

Now we can formalise the idea of tangent vectors on a manifold[2]

Definition 1.12 (Tangent Vectors)**.** Let M be a C^∞ manifold, then we can say that $x \in M$. Let us take a chart of M, $\phi : U \to \mathbb{R}^n$ where $x \in U$. Now take two curves $\gamma_1, \gamma_2 : (-1, 1) \to M$ with $\gamma_1(0) = \gamma_2(0) = x$ such that we can form $\phi \circ \gamma_1 \circ \phi \circ \gamma_2 : (-1, 1) \to \mathbb{R}^n$ are differentiable.
Now define an equivalence such that γ_1 and γ_2 are equivalent at 0 if and only if $(\phi \circ \gamma_1)' = (\phi \circ \gamma_2)' = 0$. Then take the equivalence class of all of these curves and these are the tangent vectors of M.

Definition 1.13 (Tangent Space)**.** The set of all of the tangent vectors at x. We denote it as $T_x M$.

Lie Algebras are tangent space to the lie group at the identity. Let G be a lie group, then the $T_e G$ (tangent space at the identity) is an interesting vector space with a remarkable structure called the lie algebra structure.

Lemma 1.14. Let G be a matrix lie group, and $g \in G$, then,

$$\xi \in T_e G \implies g\xi g^{-1} \in T_e G$$

Note that, $g\xi g^{-1}$ is a matrix expression.

Proof. Let $c(t) \in G$ be a curve in G, such that $c(0) = e$ and $\dot{c}(0) = \xi$. Define $\gamma(t) = gc(t)g^{-1}$. Then $\gamma(0) = gc(0)g^{-1} = e$ and $\dot{\gamma}(0) = g\dot{c}(0)g^{-1} = g\xi g^{-1} \in T_e G$. □

Proposition 1.15. Let G be a matrix lie group and $\xi, \eta \in T_e G$. Then, $\xi\eta - \eta\xi \in T_e G$

Proof. Let $c(t) \in G$ be a curve such that $c(0) = e$ and $\dot{c}(0) = \xi$ also define $b(t) = c(t)\eta c(t)^{-1} \in T_e G$ by Lemma 1.5. Then $\dot{b}(t) \in T_e G$.

$$\begin{aligned}
\dot{b}(0) &= \dot{c}(0)\eta c(0)^{-1} + c(0)\eta + \frac{dc(t)}{dt}(0) \\
&= \dot{c}(0)\eta c(0)^{-1} - c(0)\eta c(0)^{-1}\dot{c}(0)c(0)^{-1} \\
&= \xi\eta - \eta\xi
\end{aligned}$$

As $\dot{b}(t) \in T_e G$, then $\xi\eta - \eta\xi \in T_e G$ □

Now we have a Lie Algebra,

Definition 1.16 (Lie Algebra)**.** A lie algebra is a vector space endowed with a commutator (or Lie bracket), that is a bilinear map. If we have,

$$[\cdot, \cdot] : V \times V \to V$$

such that,

- $[B, A] = -[A, B]$ (skew-symmetry property)
- $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0 \quad \forall\, A, B, C \in V$ (Jacobi Identity)

Theorem 1.17. Let G be a matrix Lie group. Then $T_e G$ is a lie algebra with bracket given by the matrix commutator. Denoted by $\mathfrak{g}$.

$$[A, B] = AB - BA$$

Assume we have a surface, of manifold M, the tangent space $T_q M$, then we can say that,

$$\bigcup T_q M = TM$$

Cotangent manifold is dual space of manifold

Example.
- The lie algebra $\mathfrak{gL}(n, \mathbb{R}) = T_e \mathrm{GL}(n, \mathbb{R})$ which is vector space of real square $n \times n$ matrices with commutator.
- The lie algebra of $\mathfrak{SL}(n, \mathbb{R}) := T_e \mathrm{SL}(n, \mathbb{R})$ vector space of real traceless square $n \times n$ matrices.

Proof. Take $g(t) \in \mathrm{SL}(n, \mathbb{R})$, and so $\det g(t) = 1$ hence, take $g(t)$ such that $g(0) = e$, and $\dot{g}(0) = \xi$ and so $\dot{g}(0) \in \mathfrak{SL}(n, \mathbb{R})$. Now use the formula of the derivative of the determinant of a matrix to show,

$$\frac{d}{dt}(\det(g(t)))_{t=0} = \det g(0)\, \mathrm{Tr}(g^{-1}(0)\dot{g}(0))$$

and so, $g^{-1}(0)\dot{g}(0) \in \mathfrak{SL}(n, \mathbb{R})$ and so we just have $\mathrm{Tr}(\xi)$. □

- The lie algebra of $\mathfrak{SO}(3) = T_e\mathrm{SO}(3)$, the vector space of skew-symmetric matrices.

Lemma 1.18. If $v \in T_gG$, then we can say,

(i) $g^{-1}v \in T_eG$

(ii) $vg^{-1} \in T_eG$

2 Actions of a Lie Group and Lie Algebra

Definition 2.1 (Conjugation Action)**.** Let $g \in \mathfrak{G}$, then the operation $I_g : \mathfrak{G} \to \mathfrak{G}$ (Inner Automorphism) and so you define it by $h \mapsto ghg^{-1} \quad \forall\, h \in \mathfrak{G}$. $I_{gh} = AD_{gh}$.

Take an arbitrary path $h(t) \in \mathfrak{G}$ such that $h(0) = e$ and now $\xi = \dot{h}(0) \in T_eG$. We now define $Ad_g(\xi) = \frac{d}{dt} I_g h(t)_{t=0} = g\xi g^{-1} \in T_eG$ the adjoint action.

Definition 2.2 (Adjoint and coadjoint actions of $\mathfrak{G}$ on $\mathfrak{g}$ and $\mathfrak{g}^*$)**.** The adjoint action of the matrix group G on it's lie algebra $\mathfrak{g}$ is a map,

$$Ad : G \times \mathfrak{g} \to \mathfrak{g}$$

which is,

$$_g\xi = g\xi g^{-1}$$

The dual map $\left\langle Ad_g^* \mu \,, \xi \right\rangle = \langle \mu \,, Ad_g \xi \rangle$ where $\mu \in \mathfrak{g}^*$ and $\xi \in T_eG = \mathfrak{g}$. is called the coadjoint map of G on the dual lie algebra $\mathfrak{g}^*$.

Definition 2.3 (Dual Space for vectors)**.** Let V be a finite dimensional vector space, of dimension n, over $\mathbb{R}$. The dual vector space is denoted by V^* is the space of all linear functionals from $V \to \mathbb{R}$, $f(v) = a$ where $v \in V$ and $a \in \mathbb{R}$, then also $f(\alpha v + \beta w) = \alpha f(v) + \beta f(w)$ and $\alpha, \beta \in \mathbb{R}$ and $v, w \in V$. Hence, $f(v) = Mv$ we call M the covector such that $Mv \in \mathbb{R}$. The vectorspace of all covectors is the dual space.

$$\langle m \,, v \rangle \in \mathbb{R} \quad m \in V^* \quad v \in V$$

Lemma 2.4. Let V be a vector space of real $n \times n$ real matrices. Then the dual vector space V^* is also a vector space of $n \times n$ matrices and every linear functional $f : V \to \mathbb{R}$ such that,

$$f(A) := Tr(B^T A), \quad B \in V^*, A \in V$$

Definition 2.5 (Trace Pairing)**.** For every vector space V of real $n \times n$ matrices with dual V^*, then the pairing is,

$$\langle B \,, A \rangle = Tr(B^T A) = Tr(BA^T)$$

Proposition 2.6. Suppose $A^T = A$ and $B^T = -B$, then, $\mathrm{Tr}(B^T A) = 0$

Proof.

$$\begin{aligned}
\mathrm{Tr}(B^T A) = -\mathrm{Tr}(BA) \\
&= -\mathrm{Tr}((BA)^T) \\
&= -\mathrm{Tr}(B^T A^T) \\
&= -\mathrm{Tr}(A^T B^T) \\
&= -\mathrm{Tr}(B^T A)
\end{aligned}$$

□

We say that,

$$\begin{aligned}
\left\langle Ad_g^* \mu \,, \xi \right\rangle &= \langle \mu \,, Ad_g \xi \rangle \\
&= \left\langle \mu \,, g^{-1} \right\rangle \\
&= \mathrm{Tr}(\mu^T g^{-1}) \\
&= \mathrm{Tr}(\mu^T g^{-1}) \\
&= \mathrm{Tr}[(g^T \mu (g^{-1})^T)^T \xi] \\
&= \left\langle g^T \mu (g^{-1})^T \,, \xi \right\rangle \\
&= \left\langle g^T \mu (g^T)^{-1} \,, \xi \right\rangle
\end{aligned}$$

References

[1] Xena Project Kevin Buzzard. *Lean summer lectures 2/18: partitions.* (YouTube). July 2020. URL: https://www.youtube.com/watch?v=FEKsZj3WkTY.

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