Differential Equations Week 4 - Systems of ODEs

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1 Preliminaries

Qualitive methods are about general behavior of a family of solutions. We aren't aiming to solve it, we are analysing behavior.

We are focusing on stability of the ODEs. If you make a small change an early time, you have a small change later.

2 Systems of ODEs.

We can take the characteristic equations and calculate the eigenvectors. with a general solution of: $y = c_1\underline{\mathbf{x}}_1 e^{\lambda_1 t} + \cdots + c_n \underline{\mathbf{x}}_n e^{\lambda_n t}$. We can also we can write any system as first order ODES. Let $y_n = y^(n-1)$ and then any equation is a system of ODEs. Then the solution is $y = \underline{\mathbf{h}}(t)$ and the IVP is $K = \begin{bmatrix} K_1 & \dots & K_n \end{bmatrix}^T$

Uniqueness and Existence: This is very similar to before. Let $f_1 \dots f_n$ be continuous with continuous partial derivative and there be some initial conditions in some domain, R. Then, the ODE has a solution on an interval $(t_0 - \alpha, t_0 + \alpha)$ satisfying the IC and this solution is unique.

We can write the system as: $\underline{\mathbf{y}}' = A\underline{\mathbf{y}} + \underline{\mathbf{g}}$ and if it is homogenous, then $\underline{\mathbf{g}} = \underline{\mathbf{0}}$ and then $\underline{\mathbf{y}}' = A\underline{\mathbf{y}}$.

If A and g are both continuous on $t_0 \in (\alpha, \beta)$, then the ODE has a solution $y(t)$ and it is unique.

We can apply superpoistion of linearity property and use the similar linear independence of the basis we learnt before. A wronskian can also be written. We shall again, be writing a lot of exponentials.

2.1 Systems of ODEs with constant Coefficients

For a system: $\underline{\mathbf{y}}' = A\underline{\mathbf{y}}$, the solution can be written as;

$$\underline{\mathbf{y}} = \underline{\mathbf{x}} e^{\lambda t} = A\underline{\mathbf{x}} e^{\lambda t}$$

This then yields the eigenvalue problem $Ax = \lambda x$ and the corresponding solutions are:

$$y_1 = \underline{\mathbf{x}}_1 e^{\lambda_1 t}, \dots, y_n = \underline{\mathbf{x}}_n e^{\lambda_n t}$$

and we can write a wronskian:

$$\begin{aligned} W &= (\underline{\mathbf{y}}_1, \dots, \underline{\mathbf{y}}_n) \\ &= e^{\lambda_1 + \cdots + \lambda_n} \begin{vmatrix} \underline{\mathbf{x}}_1 & \underline{\mathbf{x}}_2 & \cdots & \underline{\mathbf{x}}_n \\ \underline{\mathbf{x}}_1^{(1)} & \underline{\mathbf{x}}_2^{(1)} & \cdots & \underline{\mathbf{x}}_n^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{\mathbf{x}}_1^{(n-1)} & \underline{\mathbf{x}}_2^{(n-1)} & \cdots & \underline{\mathbf{x}}_n^{(n-1)} \end{vmatrix} \end{aligned}$$

W is only zero when the determinant is zero as an exponential is always positive. We also have a general solution:

$$\underline{\mathbf{y}} = c_1 \underline{\mathbf{x}}_1 e^{\lambda_1 t} + \cdots + c_n \underline{\mathbf{x}}_n e^{\lambda_n t}$$

2.2 Phase Portraits

A paramentric curve with parameter t is called a trajectory or orbit / path and the $y_1 - y_2$ plane is called a phase plane with trajectories gives phase portraits. Dividing we get:

$$\frac{dy_2}{dy_1} = \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2}$$

This associated every point P in the (y_1, y_2) plane with a unique tangent diection, $\dfrac{dy_2}{dy_1}$ of the trajectory passing through P except the point, $P = P_0$ where $\dfrac{dy_2}{dy_1} = \dfrac{0}{0}$ becomes indeterminant and is hence called a critical point.

There are five types of critical point:

1. Improper Nodes: Where all trajectories except two, have the same limiting direction of the tangent
2. Proper Nodes: Where all the trajectories have a definite limiting direction
3. Saddle Points: where there re two incoming, two outgoing, rest in the neighbourhood.
4. Centers: enclosed by infinitely many closed trajectories
5. Spiral Points: about which the trajectories spiral approaching P_0 as $t \to \infty$
6. Degenerate Node: Does not happen if A is symmetric ($a_{kj} = a_{jk}$) or skew symmetric ($a_{kj} = -a_{jk}$)

2.3 Critical Points and Stability

The family of solution curves can be obtained through:

$$y(t) = \begin{bmatrix} y_1(t) & y_2(t) \end{bmatrix}^T$$

Then we say the solutions are of the form: $\underline{\mathbf{y}}(t) = \underline{\mathbf{x}}e^{\lambda t}$ where $\{\lambda, \underline{\mathbf{x}}, \underline{\mathbf{y}}\}$ are the eigenvalues and eigenvectors. Then we can derive the $\underline{\mathbf{A}}\underline{\mathbf{x}} = \lambda\underline{\mathbf{x}}$ equation back from $\underline{\mathbf{y}}'(t)$.

Now, let us say the two eigenvalues of the C.Eq are: λ_1, λ_2. Then we can write the C.Eq as:

$$\lambda^2 - (a_{11} - a_{22}) + \det(\underline{\mathbf{A}}) = 0$$

and now we can compare coefficients with a quadratic equation and let: $p = a_{11} + a_{22}$, $q = \det(\underline{\mathbf{A}})$ and $\Delta = p^2 - 4q$ and then:

$$\lambda_1 = \frac{1}{2}(p + \sqrt{\Delta}) \qquad \frac{1}{2}(p - \sqrt{\Delta})$$

and we can also say: $\lambda_1 - \lambda_2 = \sqrt{\Delta}$

From here we can now look at the critical points and stability, now we can say: If $p = 0$, then you have a center
If $p \neq 0$, then you have a spiral
If $q > 0$, then you have a node or a center
If $q < 0$, then you have a saddle
If $\Delta \geq 0$, then you have a node
If $\Delta < 0$, then you have a spiral

Real, same sign $\implies$ node
Real, opposite signs $\implies$ saddle
Pure imaginary $\implies$ center
Complex $\implies$ spiral

1. Stable: Trajector initiating at point P_1 stays within the disk of radius ε

2. Unstable: Trajectory intiating at P_1 diverges outside the disk of radius ε

3. Stable, attrative: Every trajectory apporoaches P_0 as $t \to \infty$.

Stable and attractive $\implies$ $p < 0$and $q > 0$
Stable $\implies$ $p \leq 0$ and $q > 0$
Unstable $\implies$ $P > 0$ and $q < 0$

3 Nonlinear Systems

Definition 3.1: Autonomous

We shall call a system autonomous if the independent variable doesn't appear on the RHS.

We can linearise a nonlinear system, so we can linearise near a P_0 and yield,

$$\underline{\mathbf{y}}' = f(\underline{\mathbf{y}}) = \underline{\mathbf{A}}\underline{\mathbf{y}} = h(\underline{\mathbf{y}})$$

and since P_0 is critical, then $f_1(0,0) = 0$, $f_2(0,0) = 0$ and hence,

$$y_1' = a_{11}y_1 + a_{12}y_2 + h_1(y_1, y_2)$$
$$y_2' = a_{21}y_1 + a_{22}y_2 + h_2(y_1, y_2)$$

Theorem 3.1

If f_1 and f_2 are continuous and have continuous partial derivatives in a neighbourhood of the critical point $P_0(0,0)$ and if $\det(\underline{\mathbf{A}}) \neq 0$, then the kind and stability of the critical point is the same as the linearised system.

Exception: If $\underline{\mathbf{A}}$ has equal or pure imaginary eigenvalues, then the nonlinear and the linearised system may have the ame kind of critical points or spiral points.

If the equation is non-homogenous, we can follow similar processes to the previos weeks and what we saw earlier. Solve for $\underline{\mathbf{y}}_h$ and then we can apply the two methods:

Method of undetermined Coeffieicients can only be applied when $\underline{\mathbf{A}}$ has constants or $\underline{\mathbf{g}}$ is positive integers of t, exponential, cosine and sine. Then y_p is assumed to be similar to $\underline{\mathbf{g}}$