

Year 3 — Mathematical Biology and Ecology

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

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1 Introduction

In this module we are going to consider continuous time models under the pretense of biological systems,

$$\frac{dx}{dt} = f(x, t, \mu)$$

Example. Here's an example we will use a couple of times,

$$\frac{dN}{dt} = f(N, t, \mu) = rN \left(1 - \frac{N}{K}\right)$$

where $\mu = \begin{pmatrix} r \\ K \end{pmatrix}$

We are also going to look at discrete systems, these are going to be through difference equations.

$$\mathbf{x}(t+1) = A\mathbf{x}(t)$$

and also we shall study PDEs,

$$\frac{\partial u}{\partial t} = f(x, t, x, \mu, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2})$$

Example.

$$\frac{\partial u}{\partial t} = Ku(1-u) + D\frac{\partial^2 u}{\partial x^2}$$

2 Continuous Models for a single species

We are going to model simply how we model population dynamics for a single species. We are going to call $N(t)$ our population size at a certain time. We are going to say that $N(t)$ is continuous. We also say $N \in \mathbb{R}$ and so it's going to be a density measure. We are going to constrain this $N \geq 0, \forall t$. We can measure this and create a model,

$$\frac{dN}{dt} = f(N, t, \mu)$$

We call N the variable, then μ the parameter. We let $t \in \mathbb{R}$ and $t > 0$.

Let's start off by thinking about an actual population of individuals, we have observed that there is some sort of growth dynamics. We can write a mechanistic model, by including mechanisms that affect the population.

$$\frac{dN}{dt} = +\text{births} + \text{resources} + \text{net migration} - \text{deaths}$$

If the positive things are greater, the population will grow, otherwise if they are smaller they decline, or they stay equal.

Now we make some assumptions, so we can write down a mathematical model.

(i) Births and deaths predominate:

$$\frac{dN}{dt} = \text{births} - \text{deaths}$$

(ii) Births and deaths are proportional to N :

$$\frac{dN}{dt} = \alpha N - \beta N \quad \alpha, \beta \in \mathbb{R}^+$$

where we call α the birthrate and β the deathrate and $\mu = (\alpha, \beta)$.

If we consider, as an aside,

$$\frac{dN}{dt} = \alpha N - \beta N + \gamma \quad \gamma \in \mathbb{R}$$

this adds something like independent migration, where we assume migration is constant, this could be γN if it's proportional to N .

We can solve this equation nicely,

$$\begin{aligned} \frac{dN}{dt} &= N(\alpha - \beta) \\ \int \frac{1}{N} \frac{dN}{dt} dt &= \int (\alpha - \beta) dt \\ \ln N &= (\alpha - \beta)t + C \\ N(t) &= Ae^{(\alpha - \beta)t} \end{aligned}$$

Now assume this is an initial value problem, $N_0 = A$ and so,

$$N(t) = N_0 e^{(\alpha - \beta)t}$$

We are also interested in the long term dynamics of these models, the asymptotic dynamics. Hence, we consider $t \rightarrow \infty$.

$$\lim_{t \rightarrow \infty} N(t) = \begin{cases} \infty & \text{if } \alpha > \beta \\ 0 & \text{if } \alpha < \beta \\ N_0 & \text{if } \alpha = \beta \end{cases}$$

We call the case where $\alpha = \beta$ a steady state solution.

You don't usually see just a t in the equation, but we may want to use a forcing term, i.e. periodic migration in Cornwall.

$$\frac{dN}{dt} = \alpha N - \beta N + \cos(t)$$

We are not going to consider these non-autonomous systems. Hence we can write,

$$\frac{dN}{dt} = f(N, \mu)$$

We have a nice thing to make sure our growth doesn't go exponential. It's the logistic map,

$$\frac{dN}{dt} = rN \left(1 - \frac{N}{k}\right) \quad r, k > 0 \quad N \geq 0$$

where $f(N) = rN \left(1 - \frac{N}{k}\right)$ is called the logistic model. This model has a level of self regulation, so if N is high, then it will decrease later. Firstly, look at $f(N)$,

$$f(N) = rN \left(1 - \frac{N}{k}\right)$$

We start by graphing it,

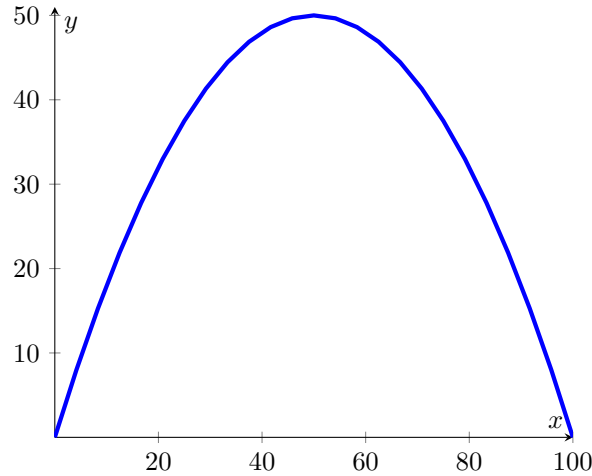


Figure 1: $y = 2x \left(1 - \frac{x}{100}\right)$

Exercise. Solve $\frac{dN}{dt} = rN \left(1 - \frac{N}{k}\right)$

The solution to this equation is,

$$N(t) = \frac{N_0 k e^{rt}}{k - N_0 + N_0 e^{rt}}$$

As $t \rightarrow \infty$, we can consider it and see that it will depend on N_0 . If $N_0 = 0$, then we get that $N(t) = 0, \forall t$. If we then take $N_0 > 0$, then we get,

$$\begin{aligned} N(t) &= \frac{N_0 k}{\frac{k - N_0}{e^{kt}} + N_0} \\ &\rightarrow k \end{aligned} \quad t \rightarrow \infty$$