Year 3 — Topology and Metric Spaces

Based on lectures by Prof. Nigel Byott
Notes taken by James Arthur

Spring Term 2022

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine (especially the typos!).

Contents

1 Introduction

1.1 Motivation

In this module we will look at ways to generalise Real Analysis.

1. Metric Spaces
2. Topological Spaces
3. Measure Spaces

A key idea in Real Analysis is continuity, a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if, given $a \in \mathbb{R}$ given $\varepsilon > 0$ there exists some $\delta > 0$ so that,

$$|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon$$

We have seen a version of this for $\mathbb{R}^n \to \mathbb{R}^m$ or $\mathbb{C} \to \mathbb{C}$. This can interpreted as a notion of a distance, we can ensure that the distance between $f(x)$ and $f(a)$ be less than ε. Here the distance between real numbers is $|x-y|$. This leads to metric spaces is a set where we have a distance function $d_X(a, b)$ for any points $a, b \in X$.

Another way to interpret the continuity of $f : \mathbb{R} \to \mathbb{R}$ is to day that for any U in $\mathbb{R}$, the set,

$$f^{-1}(U) := \{x \in \mathbb{R} : f(x) \in U\}$$

is also open.

We may ask what happens if we choose a U such that $f^{-1}(U) = \varnothing$, but we say that the empty set is open.

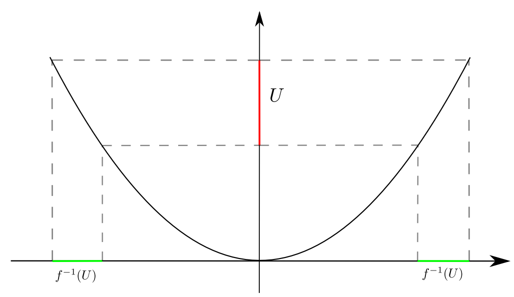

Figure 1: Image Convergence.

We can talk about continuity without talking about distances, provided that we know what we mean by the idea of open sets. Open sets may not be defined by distance. A space together with a collection of open subsets is a topological space. Metric spaces are topological spaces with a idea of distance.

Measure spaces are related to length of a subset, and also integration. These are linked since if A is a subset of $\mathbb{R}$ of length ℓ, then,

$$\ell = \int_{\mathbb{R}} 1_A(x)\, dx$$

where $1_A : \mathbb{R} \to \mathbb{R}$ is the indicator function,

$$1_A \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

This unproblematic if we have $A = [a, b]$, then we can integrate this nicely,

However, if $A = \mathbb{Q}$ it is not clear that we can make sense of this 'length' of $\mathbb{Q}$, and the integral is not defined (as a Riemann Integral). Measure Theory provides the theoretical framework for assigning a length to most (but not all, the measurable ones work) subsets of $\mathbb{R}$ and making corresponding integral as Lesbegue integrals. It turns out that $\mathbb{Q}$ has 'length' of 0, so there are way more irrational numbers, and $\mathbb{Q}$ is countable.

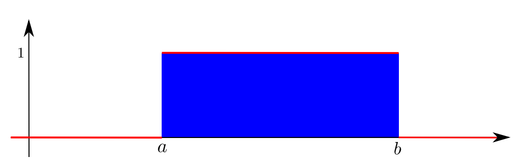

Figure 2: Image Convergence.

1.2 Review of Real Analysis

For real numbers $a \leq b$, we have the open interval,

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

and closed interval,

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$

We can also have the mixed intervals, $(a, b]$ or (a, ∞).

In general, a subset U is open, if for each $a \in U$ there is some $\varepsilon > 0$ such that $(a - \varepsilon, a + \varepsilon) \subset U$ (U does not contain it's boundary, every point is interior). A closed set is a set where it's complement is open. The empty set and $\mathbb{R}$ are clopen, open and closed.

Lemma 1.1 (Triangle Inequality)**.** For some $a, b \in R$,

$$|a + b| \leq |a| + |b|$$

and we can extend this to say $|a - b| \geq ||a| - |b||$.

Let $A \subset \mathbb{R}$. An upper bound is a number u such that $a \leq u$ for all $a \in A$. If u is an upper bound of a then it has many upper bounds, if at least one exists, the set is bounded. A least upper bound or supremum for A is a number u such that,

1. $a \leq u$ for all $a \in A$
2. if $u_* < u$ then there is some $a \in A$ with $a > u_*$

If A has a least upper bound u, then u might or might not be in A. There are similar definitions for greatest lower bound or infimum. A set is bounded, if it is bounded above and below, or there is some M such that $|a| \leq M$ for all $a \in A$. An important property of the real numbers is the completeness property: every non-empty set of real numbers which is bounded above has a least upper bound.

We say that a sequence converges to a, if given $\varepsilon > 0$ thiere is some $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for all $n > N$. Then a is the limit of a sequence. A sequence is a bounded if $|a_n| < M$ for all n. If a_n is bounded which is monotonically increasing, then it must converge, same for monotonically decreasing. In general a sequence that is bounded, doesn't have to converge. However, a bounded sequence always has a convergence subsequence.

A function is continuous at a point $a \in \mathbb{R}$, for all $\varepsilon > 0$ there is some $\delta > 0$ so that $|f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \delta$. We say that f is continuous if it holds for every a. If $f, g : \mathbb{R} \to \mathbb{R}$, then $f \pm g$, fg, $\frac{f}{g}$ $(g \neq 0)$ are all continuous. Suppose we have a continuous function on a closed and bounded interval

Theorem 1.2 (Intermediate Value Theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be continuous, for any v between $f(a)$ and $f(b)$, there is at least one $x \in [a, b]$ with $f(x) = v$.

Theorem 1.3. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then $f(x)$ is bounded and attains its bounds, i.e. f has a (finite) maximum M and minimum m in $[a, b]$. More precisely $x_{\min}$ and $x_{\max} \in [a, b]$ so that $m = f(x_{\min}) \leq f(x) \leq f(x_{\max})$ for all $x \in [a, b]$.

2 Metric Spaces

We firstly define a metric space,

Definition 2.1 (Metric Space)**.** A metric space, (X, d) consists of a non-empty set X and a function $d : X \times X \to \mathbb{R}$ satisfying,

1. $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$ for all $x, y \in X$ (symmetry)
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

Here are a load of examples,

Example. Take, $X = \mathbb{R}$ and $d_{\mathbb{R}}(x, y) = |x - y|$. Now, we can probably see normally that the three axioms hold. The first is how we define $|\cdot|$, then $|x - y| = |(-1)(y - x)| = |y - x|$ and the third is the triangle inequality.

and now for $\mathbb{R}^m$,

Example. If we let $\mathbb{R}^m$ and $d_{\mathbb{R}^m}(x, y) = \sqrt{\sum_{j=1}^{n} |x_j - y_j|^2}$ where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$. The axioms hold, as if $d_{\mathbb{R}^m}(\mathbf{x}, \mathbf{y}) = 0$, then we require that $x_j = y_j$ for all j and so $\mathbf{x} = \mathbf{y}$. For the second, we can use a similar argument to before as $|x_j - y_j| = |y_j - x_i|$. For the triangle inequality for this metric space, we need to use the Cauchy Schwartz inequality,

$$\left|\sum_{j=1}^{n} a_j \overline{b_j}\right| \leq \left(\sum_{j=1}^{n} |a_j|^2\right) \left(\sum_{j=1}^{n} |b_j|^2\right)$$

that is $|\mathbf{a} \cdot \overline{\mathbf{b}}| < |\mathbf{a}|^2 |\mathbf{b}|^2$.

We now can look at the taxicab metric,

Example. Take $X = \mathbb{R}^m$ and $d'_{\mathbb{R}^m}(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^{n} |x_i - y_j|$ for $x, y \in \mathbb{R}^m$. The first two are trivial for d', but the third easier than before,

$$\sum_{j=1}^{n} |x_j - z_j| = \sum_{j=1}^{n} |x_j - y_j - (y_j - z_j)| \leq \sum_{j=1}^{n} |x_j - y_j| + \sum_{j=1}^{n} |y_j - z_j| = d'(\mathbf{x}, \mathbf{y}) + d'(\mathbf{y}, \mathbf{z})$$

For an example not $\mathbb{R}^m$,

Example. Take any X that is non-empty, then

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

The first two axioms are clear, then for the third consider $x = z$,

$$d(x, z) = 0 \leq d(x, y) + d(y, z)$$

and this is always true. If $x \neq z$, then,

$$d(x, z) = 1 \leq d(x, y) + d(y, z)$$

if $x \neq z$, then either $x \neq y$ or $y \neq z$, so the above holds.

Now for something more abstract,

Example. Consider $\mathcal{C}[0,1]$ and let the metric be, $d(f,g) = \max\{f(t) - g(t) : t \in [0,1]\}$. Does this metric make sense? Are they bounded / why does this maximum make sense. This makes sense because of a Theorem in the last lecture. The first two of the conditions follow nicely, then the third,

$$\begin{aligned} |f(t) - h(t)| &= |(f(t) - g(t)) + (g(t) - h(t))| \\ &\leq |f(t) - g(t)| + |g(t) - h(t)| \\ &= d(f,g) + d(g,h) \end{aligned}$$

and so taking the maximum, we can get that $d(f,h) \leq d(f,g) + d(g,h)$.

We can remark, that this is not the only way to consider the distance between two functions, we could have integrated.

Definition 2.2 (Subspace)**.** A subspace of a metric space (X, d_X), is a non-empty subset Y together with the metric d_Y by restricting d_X to Y.

$$d_Y(y, y') = d_X(y, y') \qquad \forall y, y' \in Y$$

This is clearly a metric space as if the conditions hold for X, they will then hold for Y.

2.1 Continuity in Metric Spaces

We can talk nicely about continuity in metric space, in a rather obvious way once we realise it's all about distance,

Definition 2.3 (Limit)**.** Let (X,d) be a metric space, then let (a_n) be a sequence of points in X. For some $a \in X$ we say that (a_n) converges to a, written $a_n \to a$ as $n \to \infty$ if, for any real number $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ with $d(a_n, a) < \varepsilon$ for all $n > N$. We say that a is the limit of the sequence.

This is just a copy of the definition of a limit, just with our metric placed in. Here is an interesting quirk, if we take the discrete metric, then the sequence $\left(\frac{1}{n}\right)$ then this does not converge to zero. For, if we choose $\varepsilon > 0$ with $\varepsilon < 1$, then $d\left(\frac{1}{n}, 0\right) > \varepsilon$

Definition 2.4 (Continuity)**.** Let (X, d_X) and (Y, d_Y) be metric spaces, then $f : X \to Y$. For $a \in X$, we say that f is continuous at a if, given $\varepsilon > 0$, there is some $\delta > 0$ so that $d_Y(f(x), f(a)) < \varepsilon$ for all $x \in X$ with $d_X(x,a) < \delta$. We say f is continuous if it is continuous for every a.

We can prove that in the discrete metric then any function $f : X \to Y$ is convergent where X and Y have the discrete metric, just take $\delta = 1$.

2.2 Opens Sets

We can consider balls, as we have a distance metric we can move forwards to open sets and the required analytic tools.

Definition 2.5 (Open Ball)**.** Let (X,d) be a metric space, for any $a > 0$ and any $a \in X$, the set

$$B_\varepsilon(a) = \{x \in X : d(x,a) < \varepsilon\}$$

is called an open ball in X of radius ε and center a.

As a sanity check, when $X = \mathbb{R}$ we get an interval, $(a - \varepsilon, a + \varepsilon)$ and with $X = \mathbb{R}^2$ or $\mathbb{C}$, then we see we get an open disc

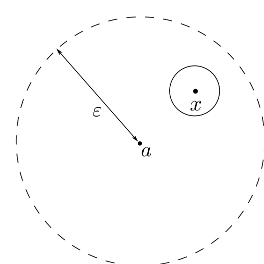

Figure 3: Open Ball.

Definition 2.6 (Open Set)**.** A subset U of a metric space X is open if, for every $x \in U$ there is some $\varepsilon > 0$ such that

$$B_\varepsilon(x) \subset U$$

A subset V is closed if $X \setminus V$ is open.

By convention, $\varnothing$ is open and now we prove that the epsilon ball is open.

Proposition 2.7. For any $a \in X$ and every $\varepsilon > 0$ the set $B_\varepsilon(a)$ is an open set in X.

Proof. Let $x \in B_\varepsilon(a)$, then we need to find a $\delta > 0$ such that $B_\delta(a) \subset B_\varepsilon(a)$. Take $\delta = \varepsilon - d(x,a)$. Then $\varepsilon > 0$ and if $y \in B_\delta(a)$ then $d(y,a) \leq d(y,x) + d(x,a) < \delta + d(x,a) = \varepsilon$. Thus $y \in B_\varepsilon(a)$. This holds for every $y \in B_\delta(a)$ and so $B_\delta(a) \subset B_\varepsilon(a)$. □

Here's a slight quirk, if we consider X and $Y \subset X$. If we consider a $U \subset Y$ which is open, this need not be open in X. Consider $Y = [0,1] \subset \mathbb{R}$, and $B_{\frac{1}{2}}(0)$ as our open set, which is just $\{x \in [0,1] : |x - 0| < \frac{1}{2}\}$. However, in $\mathbb{R}$ this subset is $\left[0, \frac{1}{2}\right)$.

Proposition 2.8. Let U and V be open sets in the metric space (X,d). Then $U \cap V$ is an open set.

Proof. If $x \in U \cap V$, then there are $\varepsilon_1, \varepsilon_2 > 0$ such that $B_{\varepsilon_1}(x) \subset U$ and $B_{\varepsilon_2}(x) \subset V$ and so we just choose $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. Then $B_\varepsilon(x) \subset U \cap V$. □

Then by induction we can generalise this,

Proposition 2.9. The intersection of any finite family of open sets is open, ie. if $n \geq 0$, then $U_1, \ldots, U_n$ are open sets then $U_1 \cap U_2 \cap \cdots \cap U_n$ is an open set.

We often write this to mean the above intersection,

$$\bigcap_{i=0}^{n} U_i$$

The same works for unions, but we can say more. Suppose we have a family of open sets, indexed by some set $\mathcal{I}$. This means for every $i \in \mathcal{I}$ we have an open set $U_i \subset X$. The indexing set doesn't need to be finite.

Proposition 2.10. If U_i, $i \in \mathcal{I}$ is a family of open sets $\bigcup_{i \in \mathcal{I}} U_i$ is open.

Proof. Let $U = \bigcup_{i \in \mathcal{I}} U_i$. We need to show that U is open. Let $x \in U$, then $x \in U_i$ for some $i \in \mathcal{I}$. As U_i is open, there is some $\varepsilon > 0$ with $B_\varepsilon(x) \subset U_i$. As $U_i \subseteq U$, we have $B_\varepsilon(x) \subseteq U$. Hence U is open. □

The intersection of infinitely many open sets, need not be open. Consider,

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n} \right) = \{0\}$$

which is then closed.

Now let us redefine the continuity and convergence in terms of these open sets,

Definition 2.11 (Limit)**.** Let (a_n) be a sequence in a metric space, (X,d) and let $a \in X$. Then $a_n \to a$ as $n \to \infty$ if and only if the following hold,

1. for every open set U containing a there is some $N \in \mathbb{N}$ such that $a_n \in U$ for all $n > N$

Proof. First suppose $a_n \to a$ as $n \to \infty$. We must show that the condition holds. Let $a \in U$, U is open. Then there is some ε with $B_\varepsilon(a) \subset U$. As $a_n \to a$ there exists $N \in \mathbb{N}$ with $d(a_n, a) < \varepsilon$ for all $n > N$. But then $a_n \in B_\varepsilon(a) \subseteq U$ for all $n > N$ as required.

Conversely, suppose the conditon holds, then we must show that $a_n \to a$. Let $\varepsilon > 0$. Then $B_\varepsilon(a)$ is an open set containing a, so by the condition there is some N with $a_n \in B_\varepsilon(a)$ is an open set containing a, so there is some N with $a_n \in B_e(a)$ for all $n > N$. Hence $d(a_n, a) < \varepsilon$ for all $n > N$. This shows $a_n \to a$. □

We can do a similar thing for continuity.

Proposition 2.12. Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$. Then f is continuous if and only if, for every open set U in Y, the set $\{x \in X : f(x) \in U\}$ is an open set in X.

We often use the notation $f^{-1}(U)$ for the set $\{x \in X : f(x) \in U\}$. This is the preimage of the set U. We use this notation even if there is no actual function f^{-1}.

Proof. Suppose f is continuous, let $U \subseteq Y$ be open. We must show that $f^{-1}(U)$ is open. If $f^{-1}(U) = \varnothing$, then $f^{-1}(U)$ is open. Let $x \in f^{-1}(U)$, then $f(x) \in U$. Since U is open, there is some $\varepsilon > 0$ such $B_\varepsilon^Y(f(x)) \subseteq U$ (with metric Y). Since f is continuous, there is some $\delta > 0$ so that $d_Y(f(x'), f(x)) < \varepsilon$ for all x' such that $d(x, x') < \delta$. If $x' \in B_\delta^X(x)$ then $f(x') \in B_\varepsilon^Y(f(x)) \subseteq U$ and so $x' \in f^{-1}(U)$ and $B_\delta^X(x) \subseteq f^{-1}(U)$. So $f^{-1}(U)$ is open.

Conversely suppose $f^{-1}(U)$ is open for all open $U \subseteq Y$. Let $x \in X$ and $\varepsilon > 0$. Then $U = B_\varepsilon^Y(f(x))$ is an open set in Y, then $x \in f^{-1}(U)$, which is open in X. So there is some $\delta > 0$ with $B_\delta^Y(x) \subseteq f^{-1}(U)$. Therefore for all $x' \in B_\delta(x)$ where $x' \in f^{-1}(U)$ and so $f(x) \in B_\varepsilon^Y(f(x))$, that is forall x' with $d^X(x', x) < \delta$ and so we have

$$d_Y(f(x'), f(x)) < \varepsilon$$

Hence f is continuous. $\square$

2.3 Equivalent Metrics

Definition 2.13 (Equivalent Metrics)**.** Let d_1 and d_2 be two metrics on the same set X.

1. We say that d_1 and d_2 are topologically equivalent if the open sets with respect to d_1 are the same as the open sets with repsect to d_2
2. We say that d_1 and d_2 are Lipschitz equivalent if there are constants $A \geq B > 0$ such that,

$$Bd_1(x, y) \leq d_2(x, y) \leq Ad_1(x, y) \qquad \forall x, y \in X$$

Proposition 2.14. If d_1 and d_2 are Lipschitz equivalent metrics on X then they are topologically equivalent.

Proof. Let $B_\varepsilon^{d_1}(a)$ and $B_\varepsilon^{d_2}(a)$ be the open balls with respect to d_1 and d_2 respectively. By hypothesis, there are constants such that,

$$Bd_1(x, y) \leq d_2(x, y) \leq Ad_1(x, y) \qquad \forall x, y \in X$$

Let U be an open set with respect to d_1. Given an $a \in U$ there is some $\varepsilon > 0$ with $B_\varepsilon^{d_1}(a) \subseteq U$. Now if $d_2(x, a) < B\varepsilon$ then $Bd_1(x, a) \leq d_2(x, a) < B\varepsilon$ so $d_1(x, a) < \varepsilon$. Hence $B_{B\varepsilon}^{d_2}(a) \subset B_\varepsilon^{d_2}(a) \subseteq U$. This shows that U is an open set with respect to d_2. $\square$

Example. Let $X = \mathbb{R}$ with d_1 is the usual metric and d_2 is the taxi-cab metric. Then d_1 and d_2 are Lischitz equivalent. This is because, if $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ in $\mathbb{R}^2$. Then, for some $A \geq B > 0$,

$$Bd_1(\mathbf{x}, \mathbf{y}) \leq d_2(\mathbf{x}, \mathbf{y}) \leq Ad_1(\mathbf{x}, \mathbf{y})$$

that is,

$$\begin{aligned} B\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} &\leq |x_1 - y_1| + |x_2 - y_2| \\ &\leq A\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \end{aligned}$$

Let $u_1 = |x_1 - y_1|$ and $u_2 = |x_2 - y_2|$, and then squaring,

$$\begin{aligned} B^2(u_1^2 + u_2^2) &\leq (u_1 + u_2)^2 \\ &\leq A^2(u_1^2 + u_2^2) \end{aligned}$$

for all $u_1, u_2 \geq 0$. We now want to find such A and B. For B, we let $B = 1$ as $u_1^2 + u_2^2 \leq (u_1 + u_2)^2$. For A, $u_1^2 + u_2^2 - 2u_1u_2 \geq 0$ and so $u_1^2 + u_2^2 \geq 2u_1u_2$ and so $(u_1 + u_2)^2 \leq 2(u_1^2 + u_2^2)$, so $A = \sqrt{2}$.

Consider $X = \mathbb{R}_{>0}$ and d_1 be the usual metric and $d'(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$, it can be proved that this d' is a metric. Now let $x = \frac{1}{n}$ and $y = \frac{1}{n+1}$ and we can see that our normal distance, $d\left(\frac{1}{n}, \frac{1}{n+1}\right) = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)}$ and $d'\left(\frac{1}{n}, \frac{1}{n+1}\right) = 1$ and so we can pick points close together in d but not in d'. Now consider,

$$\frac{d'(x, y)}{d(x, y)} = n(n + 1)$$

and so we can make this whatever we want and so we cannot have these as Lipschitz equivalent. However, they are topologically equivalent because $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ where $x \mapsto \frac{1}{x}$ is continuous.

3 Topological Spaces

We are going to mainly start by focussing on defintions and examples.

Definition 3.1 (Topological Space)**.** A topological space $(X, \mathcal{T})$ is a non-empty set X along with a family $\mathcal{T}$ of subsets X satisfying,

1. $X \in \mathcal{T}$ and $\varnothing \in \mathcal{T}$
2. If $U_1, U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$
3. If $U_1 \in \mathcal{T}$ are any collection of sets in $\mathcal{T}$, indexed by $i \in \mathcal{I}$ for some set $\mathcal{I}$, then

$$\bigcup_{i \in \mathcal{I}} U_i \in \mathcal{T}$$

We call a collection $\mathcal{T}$ of subsets satisfying these axioms a topology on X and we call the elements of $\mathcal{T}$ the open sets of X in the topology $\mathcal{T}$.

It follows from (T2) by induction that the intersection of finitely many open sets is an open set. This (T1) - (T3) say that the open sets in a topology on X must satisfy,

- $\varnothing$ and X are open
- the intersection of finitely many open sets is open
- The union of an arbitrary collection of open sets is open.

Moreover, any collection of subsets of X with these properties form a topology on X. Note that the intersections and unions behave differently, the union of infinitely many open sets must be open but their intersection need not be. That's a definition, here are some examples,

Example. Let (X, d) be any metric space and let $\mathcal{T}$ be the collection of open sets defined with respect to d. We have seen these satisfy the axioms of a topological space. In particular, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{R}^n$ are topological spaces with the topology given by the usual metric. We call this the **usual topology**.

Here's (potentially) a different example,

Example. Let X be any non-empty set and $\mathcal{T}$ be the powerset of X. Clearly the axioms hold, so this is a topology on X, which we call the **discrete topology**. It is the topology with the most open sets, every subset of X is open. In fact, this is a special case of the previous example, with the discrete example.

Example. Let X be a set. Then $\mathcal{T} = \{\varnothing, X\}$ is a topology on X, called the **indescrete topology** on X.

Example. The **Sierpinski space** is the two-point set $\{0, 1\}$ with the open sets $\varnothing$, $\{0\}$, $\{0, 1\}$.

Example. Let X be a non-empty set and let $\mathcal{T}$ consist of all $U \subseteq X$ whose complement $(X \setminus U)$ is finite, together with the empty set, $\varnothing$. Then $\mathcal{T}$ is a topology on X, called the **cofinite topology**. WE check (T1) - (T3),

1. $\varnothing \in \mathcal{T}$ follows from the definition, also $X^c = \varnothing \in \mathcal{T}$.
2. Let $U, V \in \mathcal{T}$. We must show that $U \cap V \in \mathcal{T}$. If $U = \varnothing$ or $V = \varnothing$, then $U \cap V = \varnothing$. Otherwise $X \setminus U$ and $X \setminus V$ are finite. So $X \setminus (U \cap V) = (X \setminus U) \cap (X \setminus V)$ is finite, again $U \cap V \in \mathcal{T}$.
3. Let U_i for $i \in \mathcal{I}$ be a family of sets in $\mathcal{T}$. We must show $V := \bigcup_{i \in \mathcal{I}} U_i \in \mathcal{T}$. If $U_i = \varnothing$ for all i then $V = \varnothing$ and we are done. Otherwise. we can choose a $j \in \mathcal{I}$ such that $U_j \neq \varnothing$. As $U_j \in \mathcal{T}$, we have $X \setminus U_j$ is finite. As $U_j \subset V$ we have $X \setminus V \subseteq X \setminus U_j$ so $X \setminus V$ is also finite. Hence $V \in \mathcal{T}$.

3.1 Basis of a topology

Next we talk about how we start to adapt the definition such that we can define the sets in terms of building blocks, like in $\mathbb{R}$ where we talk about intervals and epsilon neighbourhoods. In fact, in a matric space, not every open set is from one open ball, but if we know of all the open balls we know of all the open sets. We can do something similar for topological spaces.

Definition 3.2 (Basis)**.** Given a topological space $(X, \mathcal{T})$, a basis of $\mathcal{T}$ is a subset $\mathcal{B}$ of $\mathcal{T}$ such that every open set is a union of sets from $\mathcal{B}$.

Remark. If $\mathcal{B}$ is a basis of $\mathcal{T}$, then every $B \in \mathcal{B}$ is open (since $\mathcal{B} \subseteq \mathcal{T}$) and hence every union of sets from $\mathcal{B}$ is open. So $\mathcal{T}$ consists exactly of the subsets of X which can be written as the unions of sets of $\mathcal{B}$.

Example. A basis for $\mathbb{R}$ is

$$\mathcal{B} = \{(a,b) : a, b \in \mathbb{R} \text{ with } a < b\}$$

the collection of all open intervals in $\mathbb{R}$. For id U is an open set, then for each $x \in U$ we can find $\varepsilon_x > 0$ so that the open interval $B_x = (x - \varepsilon_x, x + \varepsilon_x) \subset U$ and then,

$$U = \bigcup_{x \in U} B_x$$

and now a lemma,

Lemma 3.3. If $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on X, then,

1. For each $x \in X$, there is some $B \in \mathcal{B}$ with $x \in B$
2. If $x \in B_1$ and $x \in B_2$ with $B_1, B_2 \in \mathcal{B}$ then there exists a B_3 such that $B_3 \in \mathcal{B}$ with $x \in B_3 \subset B_1 \cap B_2$.

Conversely, let $\mathcal{B}$ be a collection of subsets of a non-empty set X. If $\mathcal{B}$ satisfies (B1), (B2) then there exists unique topology $\mathcal{T}$ on X such that $\mathcal{B}$ is a topology for $\mathcal{T}$.

Proof. **(B1):** $\mathcal{T}$ consists of all possible unions of sets in $\mathcal{B}$. $X \in \mathcal{T}$ so X is a union of sets in $\mathcal{B}$ therefore given a $x \in X$, so $x \in B$ for some $B \in \mathcal{B}$. Hence (B1) holds.

(B2): If $x \in B_1$ and $x \in B_2$ with $B_1, B_2 \in \mathcal{B}$, so B_1, B_2 are open sets as they are in $\mathcal{T}$, therefore $B_1 \cap B_2 \in \mathcal{T}$, so $x \in B_1 \cap B_2$ and $B_1 \cap B_2$ is a union of sets in $\mathcal{B}$. So there is a $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$. So (B2) holds.

Converse: Uniqueness is easy, if $\mathcal{B}$ is a basis, then the topology is just all the union of $B \in \mathcal{B}$. This is the only possible topology. We need to check that $\mathcal{T}$ satisfies (T1)-(T2):

(T1): To get the empty set, take no elements of $\mathcal{B}$ and take the union of them. For X, it is just $X = \bigcup_{B \in \mathcal{B}} B$.

(T2): If $U, V \in \mathcal{T}$ and $x \in U \cap V$ then there is a $B, C \in \mathcal{B}$ with $x \in B \subseteq U$ and $x \in C \subseteq V$, by (B2) there is some $W_x \in \mathcal{B}$ with $x \in W_x \subseteq B \cap C$. Then $U \cap V = \bigcup_{x \in U \cap V} W_x$. We have written $U \cap V$ as a union of sets in $\mathcal{B}$, hence $U \cap V \in \mathcal{T}$.

(T3): If $U_i \in \mathcal{T}$ for some $i \in \mathcal{I}$. Each U_i is a union of sets in $\mathcal{B}$, so $\bigcup_{i \in \mathcal{I}} U_i$ is a union of sets in $\mathcal{B}$. Therefore $\bigcup_{i \in \mathcal{I}} U_i \in \mathcal{T}$.

Hence we have a topology. $\square$

We can compare two topologies on the same set X.

Definition 3.4 (Coarse/Fine)**.** Let $\mathcal{T}_1, \mathcal{T}_2$ be topologies on X. We say $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$ (or weaker) if every open set of $\mathcal{T}_1$ is an open set in $\mathcal{T}_2$. We also say that $\mathcal{T}_2$ is finer than $\mathcal{T}_1$.

On any X, the coarsest topology is the indiscrete topology and the finest is the discrete topology.

Example. Let $X = \{1, 2\}$, we can ask what are the topologies on X? The subsets of X are $\varnothing$, $\{1\}$, $\{2\}$ and $\{1, 2\}$. Any topology of X contains $\varnothing$ and $\{1, 2\}$ so the possible topologies are $\mathcal{T}_1 = \{\varnothing, \{1, 2\}\}$ (indiscrete topology), $\mathcal{T}_2 = \{\varnothing, \{1\}, \{1, 2\}\}$, $\mathcal{T}_3 = \{\varnothing, \{2\}, \{1, 2\}\}$, $\mathcal{T}_2 = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$ (discrete topology).

We can say that $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$, $\mathcal{T}_3$ and $\mathcal{T}_4$. $\mathcal{T}_2$ is finer than $\mathcal{T}_1$ and coarser than $\mathcal{T}_4$, similarly for $\mathcal{T}_4$. We say $\mathcal{T}_4$ is finer than $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$. $\mathcal{T}_2$ and $\mathcal{T}_3$ are not comparable as neither is coarser than the other.

3.2 Closed Sets in a TS

Definition 3.5 (Closed)**.** A subset A of a topological space X is closed if its complement $X \setminus A$ is open.

Note that $\varnothing$ and X are closed. So a set can be both open and closed. It is also to have a set that is neither. USing demorgans laws for sets,

$$\bigcup_{i \in \mathcal{I}} X \setminus U_i = X \setminus \left(\bigcap_{i \in \mathcal{I}} U_i \right) \qquad \bigcap_{i \in \mathcal{I}} X \setminus U_i = X \setminus \left(\bigcup_{i \in \mathcal{I}} U_i \right)$$

and the properties of open sets, we can show

Proposition 3.6. 1. An arbitrary intersection of closed sets is closed

2. A finite union of closed sets is closed.

Proof. (i) Let C_i for $i \in \mathcal{I}$ be an arbitrary collection of closed sets in X. Then,

$$X \setminus \left(\bigcap_{i \in \mathcal{I}} U_i \right) = bigcup_{i \in \mathcal{I}} X \setminus U_i$$

Since the sets $X \setminus C_i$ are open, so is their union. Hence $\bigcap_{i \in \mathcal{I}} C_i$ is closed.

(ii) **Exercise** □

Again, the union of an infinite family of closed sets need not be closed.

3.3 Convergence and Continuity

Definition 3.7 (Limit of a sequence)**.** Let a_n, $n \geq 1$ be a sequence of points in a topological space X. We say that a_n converges to a point $a \in X$, written $a_n \to a$ as $n \to \infty$, if, for every open set U of X with $a \in U$, there is some $N \in \mathbb{N}$ such that $a_n \in U$ for all $n > N$.

Example. Let X be a topological space with the indiscrete topology (the only open sets are $\varnothing$ and X). Then every sequence (a_n) in X converges to every point $a \in X$. For, given an open set U containing a, we must have $U = X$, and then $a_n \in X$ for all n.

Remark. If X is a metric space, viewed as a topological space with topology given by it's metric, then the two definitions agree.

Definition 3.8 (Continuous)**.** A function $f : X \to Y$ between topological spaces is continuous if, for every open set U of Y, the subset $f^{-1}(U)$ is an open subset X.

Example. Let $f : \mathbb{R} \to \mathbb{R}$,

$$\begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

is not continuous since, for the open set $U = \left(\frac{1}{2}, \frac{3}{2}\right)$ we have $f^{-1}(U) = [0, \infty)$

Here's a slightly more interesting example,

Example. Let $X = (\mathbb{R}, \mathcal{T}_d)$ and let $Y = (\mathbb{R}, \mathcal{T}_u)$ where $\mathcal{T}_d$ is the discrete topology and $\mathcal{T}_u$ is the usual topology on $\mathbb{R}$. Let $f : X \to Y$ and $f : Y \to X$ be the identity map on $\mathbb{R}$.

Then f is continuous, for if $U \subseteq Y$ is open then $f^{-1}(U) = U$ is certainly open in X. However g is not continuous, the set $V = \{0\}$ is open in Y (because every set is open in Y) but $f^{-1}(V)$ is not open in X (since $\{0\}$ is not an open set in the usual topology.)

Lemma 3.9. If $f : X \to Y$ and $g : Y \to Z$ are continuous maps between topological spaces, then $g \circ f : X \to Z$ is continuous

Proof. Let U be an open set in Z. Then $g^{-1}(U)$ is an open set in A since g is continuous, and therefore $f^{-1}(g^{-1}(U))$ is an open set in X since f is continuous. But,

$$\begin{aligned} f^{-1}(g^{-1}(U)) &= \{x \in X : f(x) \in g^{-1}(U)\} \\ &= \{x \in X : g(f(x)) \in U\} = (g \circ f)^{-1}(U) \end{aligned}$$

Hence $g \circ f$ is continuous. □

Continuous functions should be thought of as the structure-preserving functions between topological spaces, in the same as we have homomorphisms between groups, and linear maps between vector spaces. An isomorphism of topological spaces is called a homeomorphism.

Definition 3.10 (Homeomorphism)**.** A homeomorphism between topological dpaces X and Y as a continuous function $f : X \to Y$ which is bijective and whose inverse function $f^{-1} : Y \to X$ is also continuous. We say that X and Y are homeomorphic if there is a homeomorphism between them.

Example. The intervals $(0, 1)$ and $(0, \infty)$ in $\mathbb{R}$ (usual topology) are homeomorphic. Indeed, consider $f : (0, 1) \to (0, \infty)$ with

$$f(x) = \frac{1 - x}{x}$$

This is well defined and continuous, and is bijective with continuous inverse $g : (0, \infty) \to (0, 1)$ with,

$$g(y) = \frac{y}{1 + y}$$

The inverse of a homeomorphism is again, a homeomorphism, but a continuous bijection is not necessarily a homeomorphism.

Example. We have seen that $(\mathbb{R}, \mathcal{T}_d) \to (\mathbb{R}, \mathcal{T}_u)$ is a continuous bijection whose inverse is not continuous. So it is not a homeomorphism.

3.4 Interior and Closure

Here's a defintion

Definition 3.11 (Interior)**.** Let X be a topological space. For any $A \subseteq X$, the interior of A, written A°, is the union of all open subsets of X contained in A,

$$A^\circ = \bigcup_{U \text{ open}; U \subseteq A} U$$

Proposition 3.12. 1. A° is the (unique) largest open subset contained in A, that is A° is an open set, $A^\circ \subseteq A$ and if U is open and $U \subseteq A$ then $U \subseteq A^\circ$.

2. For $x \in X$ we have $x \in A^\circ \iff$ there exists an open set U with $x \in U \subseteq A$

3. $A^\circ = A \iff A$ is open.

Proof. 1. A° is a union of open sets, so it is open. If U is open and $U \subseteq A$ then U is one of the sets in the union, so $U \subseteq A^\circ$.

2. If $x \in A^\circ$ then $x \in U$ for some U, so $x \in U \subseteq A$. Conversely if $x \in U \subseteq A$ for some U, then U is one of the sets in the union and so $x \in U^\circ$.

3. If $A^\circ = A$, then A is open as A° by (i). Conversely, if A is open, then it is clearly the largest open set contained in A, so $A^\circ = A$ by (i).

□

Definition 3.13 (Closure)**.** Let X be a topological space. For any $A \subseteq X$, the closure of A, written $\overline{A}$ is the intersection of all closed subsets of X which contain A:

$$\overline{A} = \bigcap_{C \text{ closed; } A \subseteq C} C$$

Proposition 3.14. 1. $\overline{A}$ is the (unique) smallest closed subset containing A. That is, $\overline{A}$ is a closed set, $A \subseteq \overline{A}$, and if C is closed and $A \subseteq C$ then $\overline{A} \subseteq C$.

2. For $x \in X$ we have $x \in \overline{A} \iff$ there is no open set U with $x \in U$ and $U \cap A = \varnothing$

3. $\overline{A} = A \iff A$ is closed.

Proof. Exercise □

Here is an application,

Corollary 3.15. Let X be any topological space and let S be a subset X. Let (a_n) be a sequence in X with $a_n \in S$ for all n. If a_n converges to some point $a \in X$ then $a \in \overline{S}$.

Example. Let $X = \mathbb{R}$ and $S = \mathbb{R}^+$. Let $a_n = \frac{1}{n} \in S$ and $a_n \to 0$ but $0 \in S$, but $0 \in \overline{S}$.w

Proof. Recall that $x \in \overline{S}$ if and only if there no open $U \ni x$ with $U \cap S = \varnothing$. Suppose $U \subseteq X$ is open and suppose $a \in U$. Then there is some N so $a_n \in U$ for some $n > N$. Therefore, $a_n \in U \cap S$ for all $n > N$. So $U \cap S \neq \varnothing$. Hence $a \in \overline{S}$. □

3.5 Hausdorff Spaces

Definition 3.16 (Hausdorff)**.** A topological space is Hausdorff if, given any points $x, y \in X$ with $x \neq y$, there exists open sets $U, V \in X$ with $x \in U$, $y \in V$ and $U \cap V = \varnothing$.

We ask two questions, are metric spaces Hausdorff and are all topological spaces Hausdorff?

Example. Every metric space (X, d) is Hausdorff. If $x, y \in X$ and $x \neq y$. Let $\varepsilon = d(x, y)$, so $\varepsilon > 0$. Then $B_{\frac{\varepsilon}{2}}(x)$ and $B_{\frac{\varepsilon}{2}}(y)$ are disjoint open sets containing x, y respectively.

Example. Let X be a set with at least two elements and let X have the indiscrete topology. Then X is not hausdorff. Indeed, take $x, y \in X$ with $x \neq y$. The only open set that contains x is X, this also contains y. Therefore, it cannot be Hausdorff.

Remark. It follows that any topological space that is not Hausdorff, it cannot comes from a metric. For example, there is no metric on a set X with $|X| \geq 2$ which gives rise to the indiscrete topology.

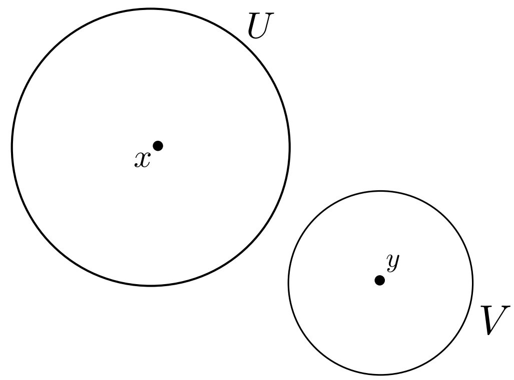

Figure 4: Hausdorff Spaces

Proposition 3.17. In a Hausdorff space, any sequence can converge to at most one point.

Proof. Suppose $a_n \to a$ and $a_n \to b$. We must prove that $a = b$. Suppose $a \neq b$, let U, V be open sets with $a \in U$, $b \in V$ and $U \cap V = \varnothing$. There exists an N_1 such that $a_n \in U$ for all $n > N_1$ and there is also an N_2 such that $a_n \in V$ for all $n > N_2$. For $n > \max(N_1, N_2)$, then $a_n \in U \cap V$ and so $U \cap V \neq \varnothing$ - Contradiction. Therefore, $a = b$. $\square$

Proposition 3.18. If $f : X \to Y$, X, Y are topological spaces, is injective and continuous and Y is Hausdorff. Then X is Hausdorff.

Proof. Let $x_1 \neq x_2$ where $x_1, x_2 \in X$. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then $y_1 \neq y_2$ as f is injective. As Y is hausdorff, then there are open sets $U, V \subseteq Y$ with $y_1 \in U$ and $y_2 \in V$ with $U \cap V \neq \varnothing$. Then $x_1 \in f^{-1}(U)$ and $x_2 \in f^{-1}(V)$, these preimages are open subsets in X as f is continuous. If $x \in f^{-1}(U) \cap f^{-1}(V)$, therefore $f(x) \in U \cap V$, with is $\varnothing$ - Contradiction. So $f^{-1}(U) \cap f^{-1}(V) = \varnothing$. Therefore X is Hausdorff. $\square$

Corollary 3.19. If X and Y are homeomorphic then X is Hausdorff if and only if Y is Hausdorff.

Proof. Let $f : X \to Y$ be a homeomorphism. Then we apply the previous proposition as we have two continuous injections, $f : X \to Y$ and $f^{-1} : Y \to X$ which gives us both directions. $\square$

3.6 Subspaces

If $(X, \mathcal{T})$ is a topological space and A is any non-empty subset of X, then there is a natural way to make A into a topological space,

Definition 3.20 (Subspace Topology)**.** With the above notation, we define

$$\mathcal{T}_A = \{U \cap A : U \in \mathcal{T}\}$$

Thus a subset $V \subseteq A$ is open in A if and only if there exists an open set U in X with $V = U \cap A$. We call $\mathcal{T}_A$ the subset topology on A induced by X.

We must check that $\mathcal{T}_A$ satisfies $(T1) - (T3)$.

1. $\varnothing = \varnothing \cap A$ and $A = A \cap X$ and so they are open sets in A.

2. If V_1 and V_2 are open in A, there are open sets $U_1, U_2 \in X$ where $V_1 = U_1 \cap A$ and $V_2 = U_2 \cap A$. Therefore, $V_1 \cap V_2 = (U_1 \cap A) \cap (U_2 \cap A) = (U_1 \cap U_2) \cap A$ where $U_1 \cap U_2$ is open and so $V_! \cap V_2$ is open.

3. If V_i where $i \in \mathcal{I}$ is a family of open sets in A, then for each i there is a $U_i \in X$ with $V_i = U_i \cap A$ and,

$$\bigcup_{i \in \mathcal{I}} V_i = \bigcup_{i \in \mathcal{I}} (U_i \cap A) = \left(\bigcup_{i \in \mathcal{I}} U_i \right) \cap A$$

   Then since $\left(\bigcup_{i\in\mathcal{I}} U_i\right)$ is open in X, then $\bigcup_{i\in\mathcal{I}} V_i$ is open in A.

Here is a remark,

Remark. If (X, d) is a metric space, then we have a metric topology on X, where a subset U of X is open if and only if for each $x \in U$ we have $B_\varepsilon(x) \subseteq U$ for some $\varepsilon > 0$. If A is a non-empty subset of X then the subspace topology on A induced by X is the sae as the topology of the restriction.

Lemma 3.21. Let A be a non-empty subset of a topological space X and let $i : A \to X$ be the inclusion map. Then,

1. i is continuous

2. For any topological space Z and any function $g : Z \to A$, g is continuous of and only if $i \circ g : Z \to X$ is continuous. This is the **universal property** for topological spaces.

$$\begin{tikzcd} Z \arrow[d, "g"'] \arrow[dr, "i\circ g"] & \\ A \arrow[r, hook, "i"'] & X \end{tikzcd}$$

3. The subspace topology on A is the only topology for which (ii) holds for all functions g.

Proof. (i), We first prove that $i : A \to X$ is continuous. That is we need to prove for all open sets $U \in X$, $i^{-1}(U)$ is open in A. Therefore, take $U \subseteq X$, then $i^{-1}(U) = \{a \in A : i(a) \in U\} = \{a \in A : a \in U\} = U \cap A$, which is an open set in A, by the definition of $\mathcal{T}_A$.

(ii), We just want to show that g is continuous iff $i \circ g$ is continuous.

$$\begin{tikzcd} Z \arrow[d, "g"'] \arrow[dr, "i\circ g"] & \\ A \arrow[r, hook, "i"'] & X \end{tikzcd}$$

For $U \subseteq X$, then

$$\begin{aligned} (i \circ g)^{-1}(U) &= \{z \in Z : i(g(z)) \in U\} \\ &= \{z \in Z : g(z) \in U\} \end{aligned}$$

Since $g(z) \in A$, this set can be written as $\{z \in Z : g(z) \in U \cap A\} = g^{-1}(U \cap A)$. If g is continuous, then we could also write $g^{-1}(V)$ is open for every open $V \subseteq A$, that is also the same as $g^{-1}(U\cap A)$ is open for every open $U \subseteq A$. We know that is then equivalent to $(i\circ g)^{-1}(U)$ is open for every $U \subseteq A$, which gives the required result.

(iii), Let $\mathcal{T}'$ be a topology on A such that (ii) holds. So A is a topological space in two ways, $(A, \mathcal{T}_A)$ and $(A, \mathcal{T}')$, we want to prove that $\mathcal{T}_A = \mathcal{T}'$. To do this we make an appropriate choice of Z and g. We can choose a function where even though we don't know what A is, we can still make this work. First, let $g = \mathrm{id}_A : (A, \mathcal{T}') \to (A, \mathcal{T}')$. g is continuous as $U \in \mathcal{T}'$, then $g^{-1}(U) = U \in \mathcal{T}'$ is open. By (ii) $\Longrightarrow$, $g \circ i$ is continuous. For open $U \subseteq X$, $(i \circ g)^{-1}(U)$ is open in $(A, \mathcal{T}')$,

$$
\begin{aligned}
(i \circ g)^{-1}(U) &= \{a \in A : i \circ g(a) \in U\} \\
&= \{a \in A : a \in U\} && = U \cap A
\end{aligned}
$$

Hence, for each $U \subseteq X$, $U \cap A \in \mathcal{T}'$. Therefore, $\mathcal{T}_A \subseteq \mathcal{T}'$. We now need the other inclusion. Secondly, let $g = \mathrm{id}_A : (A, \mathcal{T}_A) \to (A, \mathcal{T}')$. Now we consider $i \circ g = i : (A, \mathcal{T}') \to X$ and this is continuous by (i). Therefore, by $(ii) \Leftarrow$, g is continuous. If $V \subseteq A$ is open in $\mathcal{T}'$ then $g^{-1}(V) = V$ is again in $\mathcal{T}_A$. Therefore, $\mathcal{T}' \subseteq \mathcal{T}_A$. Hence, $\mathcal{T}' = \mathcal{T}_A$. $\square$

We've seen $\mathcal{T}_A = \{U \cap A : U \in \mathcal{T}\}$ is the only topology such that $\forall g : Z \to A$, g is continuous if and only if $i \circ g$ is continuous. If we try something else,

$$
\begin{tikzcd}
 & Z \arrow[d] \\
A \arrow[r] & X
\end{tikzcd}
$$

This doesn't make sense, but what about this,

$$
\begin{tikzcd}
A \arrow[r, hook, "i"] \arrow[dr, dashed, "h \circ i"'] & X \arrow[d, "h"] \\
 & Y
\end{tikzcd}
$$

But there wouldn't be a universal property like that. It is not true that $\forall h : X \to Y$, h is continuous if and only if $h \circ i$ is continuous.

3.7 Products of Topological Spaces

Definition 3.22 (Product Topology)**.** Let X, Y be topological spaces. The **product topology** on $X \times Y$ is the topology with basis,

$$
\mathcal{B} = \{U \times V : U \text{ open in } X, V \text{ open in } Y\}
$$

That means that a subset of A of $X \to Y$ is open if and only if it is the union of subsets of the form $U \times V$ with U and V open in X, Y respectively. Equivalently, A is open for each point $(x, y) \in A$, if there are open sets such that $x \in U \subseteq X$ and $y \in V \subseteq Y$.

Remark. If we have $\mathcal{B}_X$ and $\mathcal{B}_Y$ for the topologies on X, Y then we get the same open sets in $X \times Y$ if we restrict U, V to be sets in $\mathcal{B}_X$, $\mathcal{B}_Y$ respectively.

So a disc in $\mathbb{R}^2$ is an open set, since it can be filled with open rectangles. This isn't finitely many rectangles, but still it can be done. We now check that the product topology gives a usual topology on $\mathbb{R}^2$ formally,

Lemma 3.23. Let $X = Y = \mathbb{R}$. Then the product topology on $\mathbb{R}^2$ agrees with the usual topology on $\mathbb{R}^2$.

Proof Sketch. The bounded open intervals $B_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$ for all $x \in \mathcal{T}$ and $\varepsilon > 0$ for a basis for the usual topology on $\mathbb{R}$. So $U \subseteq \mathbb{R}^2$ is open in the product topology if, for all $(x, y) \in U$ there is $\varepsilon, \delta > 0$ such that,

$$
(x, y) \in B_\varepsilon(x) \times B_\delta(y) \subseteq U
$$

Replacing ε and δ with $\min(\varepsilon, \delta)$ we can assume that $\delta = \varepsilon$. This agrees with the topology on $\mathbb{R}^2$ given by the metric d where

$$
d((x, y), (x', y')) = \max(|x - x'|, |y - y'|).
$$

It is not difficult to check that d is a metric on $\mathbb{R}^2$ and that it is Lipschitz equivalent to the Euclidean metric. So it gives the usual topology. $\square$

We really should check the product topology gives us a topology.

1. $\varnothing = \varnothing \times \varnothing$ and so it is open in the product topology as it is open in X and Y. Similarly for $X \times Y$ as X is open in X and Y is open in Y.

2. Let $W_1, W_2 \subseteq X \times Y$ be open sets in the product topology. We must show that $W_1 \cap W_2$ is an open set in the product topology. Let $(x, y) \in W_1 \cap W_2$, then as $(x, y) \in W_1$ we can find some set $U_1 \times V_1 \in \mathcal{B}$ with $(x, y) \in U_1 \times V_1 \subseteq W_1$. Similarly $(x, y) \in U_2 \cap V_2 \subseteq W_2$. Let $U = U_1 \cap U_2$ and $V = V_1 \cap V_2$, which are open. Then, $(x, y) \in U \times V \subseteq W_1 \cap W_2$. With $U \times V \in \mathcal{B}$. So $W_1 \cap W_2$ is the union of sets in $\mathcal{B}$, therefore it is open in the product topology.

3. Let W_i for $i \in \mathcal{I}$ be a family of open sets in $X \times Y$. We must show $\bigcup_{i \in \mathcal{I}} W_i =: W$ is open. Let $(x, y) \in W$, then $(x, y) \in W_j$ for some $j \in \mathcal{I}$ where W_j s open. Therefore $(x, y) \in U \times V \subseteq W_j$ for some open $U \subseteq X$ and $V \subseteq Y$. Then $U \times V \subseteq W$, so W is open in the product topology.

Remark. We can define the product topology of finitely many topological spaces by just iterating the construction. (The product of infinitely many topological spaces).

Lemma 3.24. Let X, Y be topological space and let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be the projection functions:

$$p_X((x, y)) = x \qquad p_Y((x, y)) = y$$

For any topological space Z and any function $f : Z \to X \times Y$, f is continuous if and only if $p_X \circ f$ and $p_Y \circ f$ are continuous.

$$\begin{tikzcd}
Z \arrow[dr, "f"] \arrow[drr, bend left, dotted] \arrow[ddr, bend right, dotted] & & \\
& X \times Y \arrow[r, "p_X"] \arrow[d, "p_Y"] & X \\
& Y &
\end{tikzcd}$$

In particular p_X and p_Y are continuous.

Proof. If $U \subseteq X$, then,

$$\begin{aligned}
(p_X \circ f)^{-1}(U) &= \{z \in Z : p_X \circ f(z) \in U\} \\
&= \{z \in Z : f(z) \in U \times Y\} = f^{-1}(U \times Y)
\end{aligned}$$

($\Longrightarrow$) If f is continuous, let $U \subseteq X$ be open. We consider $(p_X \circ f)^{-1}(U) = f^{-1}(U \times Y)$, this is open in Z, as f is continuous and $U \times Y$ is open in $X \times Y$. Hence $p_X \circ f$ is continuous. Similarly, $p_Y \circ f$ is continuous.

($\Longleftarrow$) Suppose that $p_X \circ f$ and $p_Y \circ f$ are continuous. We must show that if $W \subseteq X \times Y$ is open, then $f^{-1}(W)$ is open in Z. Since any union of open sets in Z is open it's enough to take $W = U \times V$ with $U \subseteq X$ and $V \subseteq Y$ open. Now we consider $f^{-1}(U \times V)$,

$$\begin{aligned}
f^{-1}(U \times V) &= f^{-1}((U \times Y) \cap (X \times V)) \\
&= f^{-1}((U \times Y)) \cap f^{-1}((X \times V)) \\
&= (p_X \circ f)^{-1}(U) \cap (p_Y \circ f)^{-1}(V)
\end{aligned}$$

This is a union of open sets, and so $f^{-1}(U \times V)$ is open in Z.

Finally, we can see that p_X and p_Y are continuous. Let $Z = X \times Y$ and $f = \mathrm{id}_{X \times Y}$, then $(p_X \circ f)(x, y) = (p_X \circ \mathrm{id}_{X \times Y})(x, y) = p_X(x, y) = x$. Hence, as f is continuous, then p_X is continuous. Similarly for p_Y. $\square$

Corollary 3.25. Let $f : X \to X'$ and $g : Y \to Y'$ be continuous functions and define $f \times g : X \times Y \to X' \times Y'$ by $(f \times g)(x, y) = (f(x), g(y))$. Then $f \times g$ is continuous.

Proof. We have $p_{X'} \circ (f \times g) = f \circ p_x : X \times Y \to X$, since both functions take (x, y) to $f(x)$.

$$\begin{array}{ccc} X \times Y & \xrightarrow{f \times g} & X' \times Y' \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle p_{X'}} \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

Since p_X is continuous (Lemma 3.24) and f is continuous, it follows that their composite $f \circ p_X$ is continuous. Hence $p_{X'} \circ (f \times g)$ is continuous. Similarly, so is $fp_{Y'} \circ (f \times g)$. so by Lemma 3.24, $f \times g$ is continuous. ☐

Corollary 3.26. For any topological space X, the diagonal map $\Delta : X \to X \times X$, $\Delta(x) = (x, x)$, is continuous.

Proof. Let p_1, p_2 be the projections from $X \times X$ to the first and second factors. Then $p_1 \circ \Delta$ and $p_2 \circ \Delta$ coincide with the identity function $id_X : X \to X$ (which is most certainly continuous). So by Lemma 3.24 Δ is continuous. ☐

Corollary 3.27. For continuous functions $f, g : X \to \mathbb{R}$, the functions $f \pm g$, fg etc. are continuous.

Proof. Let $m : \mathbb{R} \to \mathbb{R} \to \mathbb{R}$ be the multiplication function $m(x, y) = xy$. We know that this is continuous. Now $fg : X \times X \to \mathbb{R}$ is the composite of the continuous maps $\Delta : X \times X \to X$. $f \times g : X \times X \to \mathbb{R} \times \mathbb{R}$ and $m : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, so it is continuous. The cases for $f + g$, and $f - g$ are similar. ☐

3.8 Compact Spaces

We look back to the function on the closed and bounded interval. We need both 'closed' and 'bounded'. We can find a continuous function $(0, 1) \to \mathbb{R}$ defined by $\frac{1}{x}$ and $[0, \infty)$ defined by x that have no maximum.

It is also important that we work with $\mathbb{R}$ not $\mathbb{Q}$. On the closed bounded subset $S = [1, 2] \cap \mathbb{Q}$, the function,

$$f(x) = \frac{1}{x^2 - 2}$$

is well defined as $x^2 \neq 2$ but has no maximum or minimum. The essential property of $[a, b] \subseteq \mathbb{R}$ which makes this work is **compactness**.

Definition 3.28 (Open Cover, Compact)**.** Let X be a topological space and let A be any subset of X,

1. An **open cover** of A in X is a family of open sets U_i, $i \in \mathcal{I}$ such that,
$$A \subseteq \bigcup_{i \in \mathcal{I}} U_i$$
2. A is **compact** if every open cover U_i, $i \in \mathcal{I}$ has a finite subcover, that is there are $i_1, \ldots i_m \in \mathcal{I}$ with,
$$A \subseteq U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_n}$$

Remark. Taking A to be X itself, X is compact if, for any family of open sets U_i, $i \in \mathcal{I}$ with,

$$X = \bigcup_{i \in \mathcal{I}} U_i$$

we have,

$$X = \bigcup_{j=1}^{n} U_{i_j}$$

for some finite subcover $\{i_1, i_2, \ldots, i_n\}$ of $\mathcal{I}$.

Example. The intervals $U_n = (n-1, n+1)$ in $\mathbb{R}$ for $n \in \mathbb{Z}$ form an open cover of $\mathbb{R}$,

$$\bigcup_{n\in\mathbb{Z}} U_n = \mathbb{R}$$

but we cannot write $\mathbb{R}$ as a finite union of these intervals. Hence $\mathbb{R}$ is not compact.

Example. Any finite topological space $X = \{x_1, \dots, x_n\}$ is compact. If we have the open cover,

$$X = \bigcup_{i\in\mathcal{I}} U_i$$

then, $1 \le j \le n$ we can pick $i_j \in \mathcal{I}$ with $x_j \in U_{i_j}$. This means that $U_{i_1} \cup \dots \cup U_{i_n}$.

3.8.1 Compact subsets of $\mathbb{R}$

Our aim in this section is to show that a subset of $\mathbb{R}$ is compact if and only if it is closed and bounded. So for example $[0,1] \cup [2,3]$ is compact. The hardest part of this is to show that a closed interval $[a,b]$ is compact. This is the Heine-Borel Theorem. We will prove some easier results in greater results first,

Lemma 3.29. Let (X,d) be a metric space. Then any compact subset A of X is bounded.

Proof. For $n \ge 1$, let $U_n = \{y \in X : d(y,x) < n\}$, the open ball with center x and radius n. For every $y \in X$ we can find $n > d(y,x)$, so

$$X = \bigcup_{n=1}^{\infty} U_n,$$

and we have,

$$A \subseteq \bigcup_{n=1}^{\infty} U_n$$

that is, it's an open cover. Since A is compact, A is contained in the union of finitely many of these sets $U_{n_1}, \dots, U_{n_k}$. Taking $R = \max\{n_1, \dots, n_k\}$ we have $A \subseteq U_R$, so $d(a,x) < R$ for all $a \in A$. Hence A is bounded. $\square$

A compact subset of a metric space is also closed. In fact this holds in any Hausdorff space.

Lemma 3.30. A compact subset C of a Hausdorff space X is closed.

Proof. We have to show that $D = X \setminus C$ is open. We will show that, for each $x \in D$, there is an open set U_x with $x \in U_x \subseteq D$. Then $D = \bigcup_{x\in D} U_x$ is open.

Let $x \in D$. Since X is Hausdorff, for each $y \in C$, we can find open sets A_y and B_y with $x \in A_y$ and $y \in B_y$ and $A_y \cap B_y = \varnothing$. Then $C \subseteq \bigcup_{y\in C} B_y$. Since C is compact, there are $y_1, y_2, \dots, y_n \in C$ with $C \subseteq \bigcup_{k=1}^n B_{y_k}$. However,

$$\left(\bigcup_{k=1}^{n} B_{y_k}\right) \cap \left(\bigcap_{k=1}^{n} A_{y_k}\right) = \varnothing$$

so $\bigcap_{k=1}^n A_{y_k}$ is a subset of D containing x. Moreover, U is open since is it is the intersection of finitely many open sets. $\square$

Combining the two previous results, we get that,

Corollary 3.31. A compact subset of a metric space is closed and bounded. In particular, any compact subset of $\mathbb{R}$ is closed and bounded.

So far we have proven only that finite sets are compact,

Theorem 3.32 (Heine-Borel Theorem)**.** Let $a, b \in \mathbb{R}$ with $a < b$. Then the closed, bounded interval $[a, b]$ is compact.

The proof uses the completeness property of $\mathbb{R}$, every non-empty subset of $\mathbb{R}$ which is bounded above has a least upper bound.

Proof. Let U_i, $i \in \mathcal{I}$ be an open cover of $[a, b]$. We must show it has a finite subcover. Consider the set,

$$S = \{x \in [a, b] : [a, x] \subseteq U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_n} \text{ for some } i_1, \ldots, i_n \in \mathcal{I}\}$$

We will now show the following,

(i) S is bounded above by b,

(ii) $S \neq \varnothing$,

(iii) the least upper bound of S, is b,

(iv) $b \in S$.

This then means that $[a, b] \subseteq U_{i_1} \cup \cdots \cup U_{i_n}$ for some $i_1, \ldots, i_n$ as required.

(i), This is obvious, then if $x \in S$, then $x \leq b$.

(ii), We seek an element that is in S. It makes sense to let this be a. Since $[a, b] \subseteq \bigcup_{i \in \mathcal{I}} U_i$, there is some i_1 with $a \in U_{i_1}$. Then $[a, a] \subseteq U_{i_1}$, so $a \in S$.

(iii), Let c be the least upper bound of S. Then $a \leq c \leq b$. We have seen that $a \in U_{i_1}$ for some i_1. As U_{i_1} is open, we have

$$[a, a + \varepsilon) \in U_{i_1} \text{ for some } \varepsilon > 0$$

so $a + \frac{1}{2}\varepsilon \in S$. Thus $c > a$.

Suppose $a < c < b$. We have $c \in U_i$ for some i. As U_i is open, then there is some $\varepsilon > 0$ with $(c - \varepsilon, c + \varepsilon) \subseteq U_i$. As $c - \varepsilon$ is not an upper bound of S, we can find some $x \in S$ with $x > c - \varepsilon$. Then,

$$[a, x] \subseteq U_{i_1} \cup \cdots \cup U_{i_n}$$

for some $i_1, \ldots, i_n$. Hence,

$$[a, c - \varepsilon] \subseteq U_{i_1} \cup \cdots \cup U_{i_n}$$

and so,

$$[a, c + \varepsilon) \subseteq U_{i_1} \cup \cdots \cup U_{i_n} \cup U_i$$

This shows that $c + \frac{1}{2}\varepsilon \in S$, contradicting the choice of c as the least upper bound for S. Hence $c = b$.

(iv), $b \in U_i$, for some i, so $(b - \varepsilon, b] \subset U_i$ for some $\varepsilon > 0$. As b is the least upper bound for S, we can find some $x > b - \varepsilon$ with $x \in S$ with $x \in S$, so

$$[a, x] \subseteq U_{i_1} \cup \cdots \cup U_{i_n}$$

for some $i_1, \ldots, i_n$. Then,

$$[a, b] = [a, x] \cup (b - \varepsilon, b] \subseteq U_{i_1} \cup \cdots \cup U_{i_n} \cup U_i$$

Hence $b \in S$. These give us that $[a, b]$ is compact. □

Note: An open bounded interval isn't compact. Take $(-1, 1) = \bigcup_{n \geq 1} \left(-1 + \frac{1}{n}, 1 - \frac{1}{n}\right)$. You can't get away with finitely many of these, it's an open cover where no finite subcover will work.

Lemma 3.33. Let C be a compact subset in a topological space X and let A be a closed and bounded subset of X with $A \subseteq C$. Then A is compact.

Proof. Let U_i for $i \in \mathcal{I}$ be open sets such that $A \subseteq \bigcup_{i\in\mathcal{I}} U_i$. We must show that A is contained in the union of finitely many of the U_i. Let $B = X \setminus A$. Then B is open in X, since A is closed. We have,

$$C \subseteq X = B \cup \bigcup_{i\in\mathcal{I}} U_i$$

Since C is compact, there are $i_1, \dots, i_n \in \mathcal{I}$ so that,

$$C \subseteq B \cup \bigcup_{k=1}^{n} U_{i_k}$$

As we know that as $B = X \setminus A$ and $C \subseteq X$, then $A \subseteq \bigcup_{k=1}^{n} U_{i_k}$, as required. □

Corollary 3.34. A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.

Proof. We saw that a compact subset must be closed and bounded. Conversely, let A be a closed, bounded subset of R. Since A is bounded, there is some $R > 0$ so that $|a| < R$ for all $a \in A$. Then $A \subseteq [-R, R]$. The closed, bounded interval $[-R, R]$ is compact by Theorem 3.32. So A is compact by Lemma 3.33. □

3.9 Middle Cantor Set

Compact sets can be a lot more complicated than we give them credit for. Consider the following construction of the Middle Third Cantor Set. Start with $A_0 = [0, 1]$, then divide it into three pieces and remove the middle third, that is remove $(\frac{1}{3}, \frac{2}{3})$. Hence, $A_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Then do it again, $A_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}] \cup [\frac{8}{9}, 1]$. Hence we will have A_n as a union of 2^n closed intervals of length 3^{-n}. We further define,

$$A = \bigcap_{n=0}^{\infty} A_n$$

Then we can look at the length, then find that the length is zero, but it still an infinite set. We find A is closed and bounded. Hence A is compact.

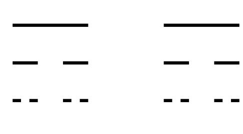

Figure 5: Middle Third Cantor Set

An alternative way to describe the Middle Third Cantor set A is in terms of infinite ternary expansions. For $x \in [0, 1]$, write x in base 3 as,

$$x = (0.c_1c_2c_3\dots)_3, \qquad c_j \in \{0, 1, 2\}$$

or, more formally,

$$x = \sum_{j=1}^{\infty} c_j 3^{-j}, \qquad c_j \in \{0, 1, 2\}$$

where the c_j are the base-3 digits of x.

Analogously to the 'recurring 9s problem' with infinite decimal expansions, a given x may have more than one ternary expansion. For example if $x = \frac{1}{3}$, we may take $c_1 = 0$ and $c_j = 0$ for all $j \geq 2$, or we may take $c_1 = 0$,

$c_j = 2$ for all $j \geq 2$. Since $2\sum_{j=2}^{\infty} 3^{-j} = \frac{1}{3}$. In general, if $x = 3^{-n}k$ for some $k \in \mathbb{Z}$ with $0 < l < 3^n$, then x has two ternary expansions, one terminating and one ending in recurring 2. Since our ternary expansions have 0 'before the ternary point', we have just one way of writing $x = 0$ $c_j = 0$ for all j and one way to write $x = 1$, $c_j = 2$ for all j. We then have,

$$A_n = \{x \in [0,1] : x \text{ has a ternary expansion with } c_j \neq 1 \text{ for all } j \leq n\}$$

For example $x \in A_1$ if x has a ternary expansion with $c_1 = 0$ or 2. In particular, $\frac{1}{3} \in A_1$ since it has the ternary expansion $c_1 = 0$ and $c_j = 2$ for all $j \geq 2$ (even though it has another expansion), and $\frac{2}{3}$ since it has a ternary expansion with $c_1 = 2$, $c_j = 0$ for all $j \geq 2$. The unique ternary expansions of 0 and 1 show that these also belong to A_1. Therefore, the ternary expansions do indeed give,

$$A_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

Similarly, the descriptions via ternary expansions correctly give A_n as the disjoint union of 2^n closed intervals, correspond to the 2^n choices $c_1, \ldots, c_n \in \{0, 2\}$ for the first n ternary digits of x. Then,

$$\begin{aligned} A &= \bigcap_{n=0}^{\infty} \\ &= \{x \in [0,1] : x \text{ has a ternary expansion with } c_j \neq 1 \text{ for all } j\} \end{aligned}$$

Proposition 3.35.

(i) Each point x in the Middle Third Cantor Set A has a unique ternary expansion such that $c_j \neq 1$ for all j,

(ii) A is uncountable infinite,

(iii) The interior of A° of A is the empty set.

Proof. (i), If x has two ternary expansions, then one will terminate, so, for some m, we have $c_m \neq 0$ but $c_j = 0$ for all $j > m$. The other will then have digits $c'_m = c_m - 1$ with $c'_j = 2$ for all $j > m$. Then either $c_m = 1$ or $c'_m = 1$

Cantor's Diagonal Argument: We are interested in finding some $x_1, x_2, x_3, \ldots$ containing all real numbers. If suffices to show this for $[0, 1]$. Suppose there is such a list for x_i as an infinite decimal. $x_1 = 0.x_{11}x_{21}x_{31}\ldots$ and $x_2 = 0.1x_{12}x_{22}x_{32}\ldots$. We avoid recurring 9s. We shall now write down $y = 0.y_1y_2y_3\ldots$ not in the list, then we have a contradiction. Let's say if $x_{jj} \neq 5$, let $y_j = 5$ and if $x_{55} = 5$, then $y_j \neq 5$. y differs from x_j in the jth place and y has just one decimal expansion. Therefore as y differs from each x_j in the list, then $\mathbb{R}$ is uncountable.

(ii), We use a variation of Cantor's famous diagonal argument, which uses decimal expansions to show $\mathbb{R}$ is uncountable. We must show that the elements of A cannot all be arranged into an infinite list $x_1, x_2, x_3, \ldots$. Suppose for a contradiction that such a list exists, and write,

$$x_n = \sum_{j=1}^{\infty} c_j^{(n)} 3^{-j}, \qquad c_j^{(n)} \in \{0, 2\}$$

We define a new number,

$$y = \sum_{j=1}^{\infty} d_j 3^{-j}$$

by setting $d_j = 2 - c_j^{(j)}$ for each j. Thus $d_j \in \{0, 2\}$ but $d_j \neq c_j^{(j)}$. Then $y \in A$ but y does not appear in our list since y differs from x_j in the jth ternary digit if its unique ternary expansion avoiding the digit 1.

(iii), Suppose for a contradiction that A is a nonempty open set U. Then U contains a closed interval of the form $[3^{-m}k, 3^{-m}(k+1)]$ for some $m \geq 1$ and some k with $0 < k < 3^m$. This consists of all points with a ternary expansion in which the c_j for $j \leq m$ are determined by k, but the c_j for $j > m$ can be chosen arbitrarily from $\{0, 1, 2\}$. In particular, this interval contains a point x with $c_j = 1$ for all $j > m$. Then x has a unique ternary expansion and $x \notin A$. As $x \in U \subseteq A$, this gives the required contradiction. Hence $A^\circ = \varnothing$. ☐

3.10 Compactness and continuous functions

Theorem 3.36. The continuous image of a compact space is compact, i.e. if $f : X \to Y$ is a continuous function between topological spaces, and X is compact, then the subset $f(X)$ of Y is also compact.

Proof. Let,

$$f(X) \subseteq \bigcup_{i \in \mathcal{I}} U_i$$

where U_i are open in Y. For each $x \in X$ we have $f(x) \in U_i$ for some $i \in \mathcal{I}$. Then $f^{-1}(U_i)$ is open in X and $x \in f^{-1}(U_i)$. So we have,

$$X = \bigcup_{i \in \mathcal{I}} f^{-1}(U_i)$$

Since X is compact, there are $i_1, \ldots, i_n \in \mathcal{I}$ such that,

$$X = \bigcup_{k=1}^{n} f^{-1}(U_{i_k})$$

Then,

$$f(X) \subseteq \bigcup_{k=1}^{n} U_{i_k}$$

Hence $f(X)$ is compact. ☐

Corollary 3.37. If X is any compact topological space and $f : X \to \mathbb{R}$ is any continuous function, then f is bounded and attains it's bounds.

Proof. By Theorem 3.36, $f(X)$ is a compact subset of $\mathbb{R}$, so, by an earlier Corollary it is closed and bounded. Since $f(X)$ is bounded and non-empty, it has supremum M and infimum m. Since $f(X)$ is closed, $M, m \in f(X)$, so these are the bounds. ☐

Theorem 3.38. If $f : X \to Y$ is a continuous bijection with X compact and Y Hausdorff, then f is a homeomorphism.

Proof. Since f is a bijection, it has an inverse function $g = f^{-1} : Y \to X$. All we need to show it that g is continuous. Now $f(X)$ is compact by Theorem 3.36 and $f(X) = Y$, since f is surjective and Y is compact.

Let U be an open set in X and let $C = X \setminus U$ be it's complement. We need to show that $g^{-1}(U)$ is open in Y, which is equivalent to showing that $g^{-1}(C)$ is closed in Y. Now C is a closed subset of the compact space X, so C is compact by an earlier lemma, and $g^{-1}(C) = f(C)$ is the continuous image of a compact set, so is compact by Theorem 3.36. Since $g^{-1}(C)$ is a compact set in the Hausdorff space Y, it is closed. ☐

3.11 Compact subsets of $\mathbb{R}^n$

Our characterisation of compact subgroups of $\mathbb{R}$ extends to $\mathbb{R}^n$, but there is one further tricky fact we need to prove this,

Theorem 3.39. Let X and Y be compact topological spaces. Then their product is compact.

We postpone the proof until after the next result,

Theorem 3.40. For any $n \geq 1$, a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof. Since $\mathbb{R}^n$ is a metric space with the usual Euclidean metric d, any compact subset is closed and bounded by a Lemma. Now let A be closed, bounded in $\mathbb{R}^n$. Since it is bounded we can find $R > 0$ so that,

$$\mathbf{A} \subseteq \{\mathbf{x} \in \mathbb{R}^n : d(\mathbf{x}, \mathbf{0}) \leq R\} \subseteq [-R, R]^n$$

By Heine-Borel Theorem, $[-R, R]$ is compact, and hence using Theorem 3.39 (and induction), so is $[-R, R]^n$. Since A is a closed subset of this compact set, it is compact. $\square$

Now we unpostpone that proof,

Proof of Theorem 3.39. By the definition of the product topology, it is enough to show that any open cover by sets of the form $U_i \times V_i$ for $i \in cI$ has a finite subcover, where U_i and V_i are open in X, Y respectively.

Given such an open cover, for each point $(x, y) \in X \times Y$, there is some $i \in \mathcal{I}$ with $(x, y) \in U_i \times V_i$. We write $U(x, y) = U_i$ and $V(x, y) = V_i$. So it suffices to find a finite subcover of the open cover,

$$\bigcup_{(x,y)\in X\times Y} (U(x, y) \times V(x, y))$$

For a fixed $y \in Y$ we have,

$$X \times \{y\} \subseteq \bigcup_{x\in X} (U(x, y) \times V(x, y))$$

So $\bigcup_{x\in X} U(x, y)$ is an open cover of X. Since X is compact, there exists $n(y) \in \mathbb{N}$ and $x_1(y), \ldots, x_{n(y)}(y) \in X$ such that,

$$X = \bigcup_{j=1}^{n(y)} U(x_j(y), y)$$

Now let $V_y = \bigcap_{j=1}^{n(y)} V(x_j(y), y)$. This is an open subset of Y since it is a finite intersection of open sets. Moreover, it contains y, and,

$$X \times \{y\} \subseteq \bigcup_{j=1}^{n(y)} (U(x_j(y), y) \times V_y) \subseteq \bigcup_{j=1}^{n(y)} (U(x_j(y), y) \times V(x_j(y), y))$$

Now let $y \in Y$ vary. The sets V_y form an open cover of Y, and since Y is compact, we can write Y as a finite union:

$$Y = \bigcup_{k=1}^{m} V_{y_k}$$

Then for each k, we have $X = \bigcup_{j=1}^{n(y_k)} U(x_j(y_k), y_k)$ so that,

$$\begin{aligned} X \times Y &= \bigcup_{k=1}^{m} \bigcup_{j=1}^{n(y_k)} (U(x_j(y_k), y_k) \times V_{y_k}) \\ &\subseteq \bigcup_{k=1}^{m} \bigcup_{j=1}^{n(y_k)} (U(x_j(y_k), y_k) \times V(x_j(y_k), y_k)) \end{aligned}$$

$\square$

Remark. The product of infinitely many compact spaces X_i, $\prod_{i\in\mathcal{I}} X_i$ is compact with the right definition of comapct for infinitely many spaces. The basic open sets are $\prod_{i\in\mathcal{I}} U_i$ with U_i open in X_i, and $U_i = X_i$ for all but finitely many i's.

3.12 Connected Spaces

Intuitively, a connected topology space doesn't fall apart into two or more pieces. Therefore if we would expect $(0,1)$ to be connected, but the union $(0,1)\cup(2,3)$ is disconnected. However if we take the topologists sine curve,

$$\{(x,\sin(1/x)) : x>0\}\cup\{(0,y) : -1\le y\le 1\}$$

it is less obvious if this should be connected. It turns out that this is connected, but not path-connected.

Definition 3.41 (Connected)**.** A topological space X is connected if there is no surjective continuous function $f : X\to\{0,1\}$ (where $\{0,1\}$ has the discrete topology). Otherwise it is disconnected.

We say a non-empty subspace Y of X is connected if it is connected as a topological space with its subspace topology induced by X.

Example. $X=(0,1)\cup(2,3)$ is disconnected, since the $f : X\to\{0,1\}$ can be defined as,

$$f(x)=\begin{cases}0 & x\in(0,1)\\ 1 & x\in(2,3)\end{cases}$$

This is surjective and continuous, the preimages $f^{-1}(\{0\})=(0,1)$ and $f^{-1}(\{1\})=(2,3)$ are both open in X, in addition to the basic open set.

Theorem 3.42. Any interval (a,b) with $a<b$ is connected.

Proof. We will use the Intermediate Value Theorem, whose proof ultimately depends on the completeness of $\mathbb{R}$. Suppose for a contradiction there is a surjective continuous function $f : (a,b)\to\{0,1\}$ and let $A=f^{-1}(0)$ and $B=f^{-1}(1)$. Then $A,B\ne\varnothing$ as f is surjective, and $A\cup B=(a,b)$. Also A,B are open since f is continuous. Pick $a_0\in A$ and $b_0\in B$. We may assume $a_0<b_0$.

Then f restricts to a continuous function on the closed interval $[a_0,b_0]$ with $f(a_0)=0$ and $f(b_0)=1$. By IVT there is some $x\in[a_0,b_0]$ with $f(x)=\frac{1}{2}$. This is a contradiction as f only takes the values of 0 or 1. □

Example. $\mathbb{Q}$ is disconnected. Indeed we can defined $f : \mathbb{Q}\to\{0,1\}$ by,

$$f(x)=\begin{cases}0 & x^2<2\\ 1 & x^2>2\end{cases}$$

This is well defined, (there is no $x\in\mathbb{Q}$ such that $x^2=2$) and it's surjective. The sets,

$$f^{-1}(\{0\})=(-\sqrt{2},\sqrt{2})\cap\mathbb{Q}$$

$$f^{-1}(\{1\})=((-\infty,-\sqrt{2})\cup(\sqrt{2},\infty))\cap\mathbb{Q}$$

are open in $\mathbb{Q}$, so f is continuous.

Definition 3.43 (Partition)**.** A partition of a topological space X is a pair of non-empty open subsets A,B such that $A\cup B=X$ and $A\cap B=\varnothing$.

Remark. If A,B is a partition of X then $A=X\setminus B$ and $B=X\setminus A$, so A and B are closed aswell as open.

Remark. In Theorem **??** we showed (a, b) isn't connected by showing that there is no partition of (a, b). We now show this in general.

Lemma 3.44. For a topological space X, the following are equivalent:

(i) X is connected,

(ii) there is no partition of X,

(iii) the only subsets of X which are both open and closed are $\varnothing$ and X.

Proof. We firstly show $(i) \implies (ii)$. We will prove the contrapositive statement, Suppose X has a partition, then $X = A \cup B$ and A, B are open, non-empty and $A \cap B = \varnothing$. We want to prove that if X is disconnected. We define $f : X \to \{0, 1\}$ where,

$$f(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B \end{cases}$$

We have a well defined function as $A \cap B = \varnothing$ and $A \cup B = X$. f is surjective, as $A, B \neq \varnothing$. f is continuous because $f^{-1}(\{0\}) = A$ and $f^{-1}(\{1\}) = B$, which are open, hence f is continuous. Therefore f is disconnected.

Now we prove $(ii) \implies (iii)$. Let $V \subseteq X$ be both open and closed and $W = X \setminus V$. So W is also both open and closed. We see $V \cup W = X$ and $V \cap W = \varnothing$. Hence we appear to have produced a partition. Since X has no partition, V and W cannot both be non-empty, so either $V = \varnothing$ or $W = \varnothing$ and so the other is just X.

(iii) $\implies$ (i). Assume X has no sets which are both open and closed expect $\varnothing$ and X. Let $f : X \to \{0, 1\}$ be continuous. We will show f is not surjective. This means X is connected. Let $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$, both open as f is continuous. So A is closed as $A = X \setminus B$ and B is open. Hence A is open and closed, so either $A = \varnothing$ or $A = X$. If $A = \varnothing$ then $X = B$ and if $A = X$ then $B = \varnothing$. Hence f is not surjective. □

Lemma 3.45. Let $f : X \to Y$ be a continuous function between topological spaces. If X is connected, so is $f(X)$.

Proof. Replacing f by the continuous map $f_1 : X \to f(X)$ with $f_1(x) = x$ for all x, we may assume f is surjective. We prove the contrapositive, if $f(X)$ is disconnected then X is disconnected. We use use (ii) from the previous lemma. Suppose $f(X)$ is disconnected, so there is a partition A, B of $f(X)$. Then $f^{-1}(A)$, $f^{-1}(B)$ is a partition of X, showing X is also disconnected. □

3.12.1 Connected Components

Lemma 3.46. Let X be a topological space, let $x \in X$, and let V_i $i \in \mathcal{I} \neq \varnothing$, be a family of connected sets with $x \in V_i$ for each i. Then $\bigcup_{i \in \mathcal{I}} V_i$ is connected.

Proof. Suppose $V = \bigcup_{i \in \mathcal{I}} V_i$ has a partition, A, B. Without loss of generality $x \in A$. Pick some $y \in B$. Then $y \in V_j$ for some j. So the sets $A \cap V_j$ and $B \cap V_j$ contain x, y respectively and are disjoint and open in V_j. Hence they form a partition of V_j. This is impossible as V_j is connected. □

Definition 3.47 (Connected Component)**.** Let X be a topological space and let $x \in X$. Then the connected components C_x of $x \in X$ is the union of all connected subsets of X containing x:

$$C_x = \bigcup_{x \in V \subseteq X, V \text{ connected}}$$

Then C_x is connected by Lemma 3.46, so C_x is the unique largest connected subset of X containing x.

Clearly, X is connected if and only if $C_x = X$ for every $x \in X$.

Example. The subspace $X = (0,1) \cup (2,3)$ of $\mathbb{R}$ has two connected components, if $x \in (0,1)$ then $C_x = (0,1)$ and if $x \in (2,3)$ then $C_x = (2,3)$.

Proposition 3.48. For any $x, y \in X$, either $C_x = C_y$ or $C_x \cap C_y = \varnothing$.

Proof. Suppose $C_x \cap C_y \neq \varnothing$, so there is some $z \in C_x \cap C_y$. As C_x is a connected subset containing z, we have $C_x \subseteq C_z$. Hence $x \in C_z$. Since $z \in C_z$ is connected, this means that $C_z \subseteq C_x$. So $C_x = C_z$. Similarly $C_y = C_z$. Thus $C_x = C_y$. ☐

Remark. Proposition 3.48 means that $x \in C_y \iff C_x = C_y$. Moreover the relation $x \in C_y$ is an equivalence relation on X: it is,

1. reflexive, $x \in C_x$ for all x
2. symmetric, $x \in C_y \implies C_x = C_y \implies C_y = C_x \implies y \in C_x$
3. transitive, $x \in C_y$ and $y \in C_z$, then $C_x = C_y = C_z$ and so $x \in C_z$.

Proposition 3.49. If A is a connected subset of X, then its closure $\overline{A}$ is also connected.

Proof. Let $f \to \overline{A} \to \{0,1\}$ be a continuous function and let f_A be the restriction to A. Then f_A cannot be surjective, as A is connected, so, without loss of generality, $f(a) = 0$ for all $a \in A$.

Then $f^{-1}(1)$ is an open subset of $\overline{A}$, so $f^{-1}(1) = \overline{A} \cap U$ for some open subset U of X for which $A \cap U = \varnothing$. Then $\overline{A} \cap U = \varnothing$ aswell ($X \setminus U$ is a closed set containing A, so by the definition of $\overline{A}$ we have $\overline{A} \subseteq X \setminus U$). Thus $f(x) = 0$ for all $x \in \overline{A}$, so f is not surjective. Hence $\overline{A}$ is connected. ☐

Corollary 3.50. Connected components are closed. If there are only finitely many of them they are also open.

Proof. If C is a connected component in X then $\overline{C}$ is connected, so $\overline{C} \subseteq C$. Hence $C = \overline{C}$ is closed.

If there are only finitely mant of them $C_1, \ldots, C_n$ then the complement of each component C_i is the union $\bigcup_{i \neq j} C_j$ of finitely many closed sets, so this complement is closed and C_i is open. ☐

Can we find examples where there are infinitely many closed connected components?

Example. The connected compontents of $\mathbb{Q}$ are singletons. They are closed but not open. To see that a subset S of $\mathbb{Q}$ containing at least two points cannot be connected, let $x, y \in S$ with $x < y$. Choose an irrational number α with $x < \alpha < y$. and define $f : S \to \{0,1\}$ by,

$$f(s) = \begin{cases} 0 & \text{if } s < \alpha \\ 1 & \text{if } s > \alpha \end{cases}$$

Then f is continuous and surjective.

Example. Now for the topologists sine curve. We will show it's connected,

$$S = \{(x, \sin 1/x) : x > 0\} \cup \{(0, y) : -1 \leq y \leq 1\}.$$

This set is the union of two pieces,

$$S_1 = \{(x, \sin 1/x) : x > 0\} \qquad S_2 = \{(0, y) : -1 \leq y \leq 1\}$$

Since $(0, \infty]$ and $[-1, 1]$ are connected, then S_1 and S_2 are connected. Thus either S has two connected components or one. However, if S_1 and S_2 were compontents, they would be open subsets of S. But S_2 is not open because any open neighbourhood of $(0,0) \in S_2$ contains points from S_1, namely $((n\pi)^{-1}, 0)$ for large enough $n \in \mathbb{N}$. This shows that S is connected. (In fact S_1 is open but not closed and S_2 is closed but not open).

3.13 Path Connected Spaces

Definition 3.51 (Path Connected)**.** A topological space X is path connected if, for any $x, y \in X$, there is a continuous function $p : [0,1] \to X$ with $p(0) = x$ and $p(1) = y$. We call p a path from x to y.

Example. Any open ball $B_\varepsilon(\mathbf{a}) \subset \mathbb{R}^n$ for $\varepsilon > 0$ is path connected. Indeed, given $\mathbf{x}, \mathbf{y} \in B_\varepsilon(\mathbf{a})$, we can define a path $\mathbf{p}$ from $\mathbf{x}$ to $\mathbf{y}$ by,

$$\mathbf{p}(t) - (1-t)\mathbf{x} + t\mathbf{y} \qquad \text{for } t \in [0,1]$$

We must check that $\mathbf{p}(t) \in B_\varepsilon(\mathbf{a})$ for all t. But, writing $\|\mathbf{v}\|$ for the euclidean norm, we have,

$$\begin{aligned}\|\mathbf{p}(t) - \mathbf{a}\| &= \|(1-t)(\mathbf{x} - \mathbf{a}) + t(\mathbf{y} - \mathbf{a})\| \\ &\le \|(1-t)(\mathbf{x} - \mathbf{a})\| + \|t(\mathbf{y} - \mathbf{a})\| \\ &= (1-t)\,\|\mathbf{x} - \mathbf{a}\| + t\,\|\mathbf{y} - \mathbf{a}\| \\ &< (1-t)\varepsilon + t\varepsilon \\ &= \varepsilon\end{aligned}$$

Lemma 3.52. A connected open subset U of $\mathbb{R}^n$ is path connected.

Proof. Fix some $\mathbf{x} \in U$ and let,

$$A = \{\mathbf{y} \in U : \text{ there is a path from } \mathbf{x} \text{ to } \mathbf{y}\}$$

We will show that A and it's complement $B = X \setminus A$ are both open. Since U is the disjoint union of A and B, and U is connected, this will show that $A = U$.

Let $\mathbf{y} \in A$. As U is open, there is $\varepsilon > 0$ with $B_\varepsilon(y) \subseteq U$. We will show that $B_\varepsilon(y) \subseteq A$. Let $\mathbf{z} \in B_\varepsilon(y)$. We will find a path in U connecting $\mathbf{x}$ to $\mathbf{z}$. Since $y \in A$, there is a path $\mathbf{p} : [0,1] \to U$ connecting $\mathbf{x}$ to $\mathbf{y}$. By a previous example, there is a path $\mathbf{q} : [0,1] \to B_\varepsilon(\mathbf{y})$ joining $\mathbf{y}$ to $\mathbf{z}$. We will concatinate these to form a new path from $\mathbf{x}$ to $\mathbf{z}$.

$$\mathbf{r}(t) = \begin{cases} p(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ q(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

This is well defined since $\mathbf{p}(1) = \mathbf{q}(0) = \mathbf{y}$, and it is continuous since $\mathbf{p}$ and $\mathbf{q}$ are. Moreover, $\mathbf{r}(t) \in U$ for all $t \in [0,1]$. Hence $\mathbf{z} \in A$. Since this works for all $\mathbf{z} \in B_\varepsilon(\mathbf{y})$, this shows that A is open.

We now show that B is also open by a similar argument. Let $\mathbf{y} \in B$. There is some $\varepsilon > 0$ with $B_\varepsilon(y) \subseteq U$. Let $\mathbf{z} \in B_\varepsilon(y)$. We must show that there is no path in U from $\mathbf{x}$ to $\mathbf{z}$. If we had such a path $\mathbf{p}$, we could find a path $\mathbf{q}$ from $\mathbf{y}$ to $\mathbf{z}$ as above and then define,

$$\mathbf{r}(t) = \begin{cases} p(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ q(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

This gives a path in U from $\mathbf{x}$ to $\mathbf{y}$, which is impossible since $\mathbf{y} \notin A$. $\square$

Lemma 3.53. Any path connected topological space is connected.

Proof. Let X be a path connected topological space, and let $x, y \in X$. We will show that y is in the connected component C_x of x. Then $C_x = X$ so X is connected.

Since X is path connected, there is a path $[0,1] \to X$ with $p(0) = x$ and $p(1) = y$. Now $[0,1]$ is connected so, so by a lemma above, $p([0,1]) = \{p(t) : t \in [0,1]\}$ is a connected subset of X containing x and y.

Since C_x is the unique largest connected subset of X containing x, it follows that $y \in C_x$ as required. $\square$

Now the question is, is the converse true? It isn't true in general. A counter example is the topologists since curve,

Example. The topologists sine curve isn't connected.

Proof. Let $A = (1/\pi, 0)$, i.e. the last point on the curve with $y = 0$ and let $B = (0, 0)$. Suppose for a contradiction that there is a path $\mathbf{p} \in S$ with $\mathbf{p}(0) = A$ and $\mathbf{p}(1) = B$. We will write $\mathbf{p}(t) = (p_1(t), p_2(t))$, so p_1 and p_2 are the projections of $\mathbf{p}$ to the x-axis of $\mathbf{p}$ onto the x and y axis. Thus p_1 and p_2 are continuous. It is possible that the path reaches the y-axis before $t = 1$, so let t_0 be the first time it happens. More precisely, let,

$$T = \{t \in [0, 1] : p_1(t) = 0\}.$$

Then $1 \in T$ and T is bounded below, so it has infinimum t_0. As p_1 is continuous, $p_1(t_0) = 0$. As $p_1(0) > 0$, we have $0 < t_0 \leq 1$ and $p_1(t) > 0$ if $t \in [0, t_0)$.
Since $\mathbf{p}$ is continuous, there is some $\varepsilon > 0$ so that if $t_0 - \varepsilon < t < t_0$ then $\|\mathbf{p}(t) - \mathbf{p}(t_0)\| < 1$. In particular, if $t_0 - \varepsilon < t < t_0$ then $|p_2(t) - p_2(t_0)| < 1$. Now p_1 is continuous on the interval $[t_0 - \frac{1}{2}\varepsilon, t_0]$, so by IVT, p_1 takes all the values between $p_1(t_0) = 0$ ad $p_1(t_0 - \frac{1}{2}\varepsilon) > 0$.
Thus for large enough $n \in \mathbb{Z}$ we can find $t', t'' \in [t_0 - \frac{1}{2}\varepsilon, t_0]$ and $p_1(t') = [(2n + \frac{1}{2})\pi]^{-1}$ and $p_1(t'') = [(2n + \frac{3}{2})\pi]^{-1}$. Then $p_2(t') = \sin(1/p_1(t')) = +1$ and $p_2(t'') = -1$. This can't happen as $|p_2(t) - p_2(t_0)| < 1$. $\square$

4 Metric Spaces II

4.1 Complete Metric Spaces

We will now return to metric spaces and talk about Cauchy Sequences,

Definition 4.1 (Cauchy Sequence)**.** A sequence $(a_n)_{n\geq 1}$ in a metric space (X, d) is a Cauchy Sequence if for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ with $d(a_m, a_n) < \varepsilon$ for all $m, n > N$.

An important result is that,

Theorem 4.2. A sequence in $\mathbb{R}$ converges if and only if it is a Cauchy Sequence.

It is easy to show the forward direction, but is hard to show the backward direction. It requires the completeness property of $\mathbb{R}$. Can we prove this without the completeness property? Well, no, on $\mathbb{Q}$ this theorem doesn't hold. We will turn things on their head and define completeness from Cauchy Sequences,

Definition 4.3 (Complete Metric Space)**.** A metric space (X, d) if every Cauchy sequence in X converges to an element of X.

Example. $\mathbb{R}$ is a complete metric space.

Example. $\mathbb{Q}$ isn't complete, we can find a Cauchy sequence that converges to $\sqrt{2}$, but not a rational number.

Example. Recall that $\mathcal{C}[0,1] = \{f : [0,1] \to \mathbb{R} : f \text{ continuous}\}$. Then $\mathcal{C}[0,1]$ is complete in the sup metric,

$$d(f, g) = \sup_{t\in[0,1]} |f(t) - g(t)|.$$

Proof. Let $f_1, f_2, \ldots$ be a Cauchy sequence in $\mathcal{C}[0,1]$. Given some $\varepsilon > 0$ there is an N such that for all $m, n > N$, $|f_m(t) - f_n(t)| < \varepsilon$ for all $t \in [0,1]$.

Let $s \in [0,1]$. Then, given a $\varepsilon > 0$ there is N so that, for all $m, n > N$,

$$|f_n(s) - f_m(s)| \leq d(f_n, f_m) < \varepsilon$$

so $(f_n(s))_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. Therefore, it has a unique limit $f(s)$. We have now defined a function $f : [0,1] \to \mathbb{R}$ which is our candidate for the limit of the given sequence. We need to show,

1. $f \in \mathcal{C}[0,1]$,
2. $f_n \to f$ with respect to d.

To show that f is continuous. We fix a $s \in [0,1]$ and show f is continuous at s. We choose $\varepsilon > 0$ and N' such that,

$$\max_{t\in[0,1]} |f_n(t) - f_m(t)| < \frac{1}{4}\varepsilon \quad \forall m, n > N'$$

Choose some $\delta > 0$ so,

$$|f_{N'+1}(t) - f_{N'+1}(s)| < \frac{1}{4}\varepsilon \quad \forall t \in [0,1] \text{ with } |t - s| < \delta$$

For $m > N'$ we have and $|t - s| < \delta$,

$$\begin{aligned}
|f_m(t) - f_m(s)| &\leq |f_m(t) - f_{N'+1}(t)| + |f_{N'+1} - f_{N'+1}(s)| + |f_{N'+1}(t) - f_m(s)| \\
&< \frac{1}{4}\varepsilon + \frac{1}{4}\varepsilon + \frac{1}{4}\varepsilon \\
&= \frac{3}{4}\varepsilon.
\end{aligned}$$

Since $f(t) = \lim_{m\to\infty} f_m(t)$. Taking limits as $m \to \infty$. We get,

$$|f(t) - f(s)| \leq \frac{3}{4}\varepsilon < \varepsilon$$

for all $t \in [0,1]$ with $|t-s| < \delta$. So f is continuous at s. This holds for all $s \in [0,1]$. Therefore, $f \in \mathcal{C}[0,1]$. We next show $f_n \to f$ in the metric d. i.e,

$$\sup_{t\in[0,1]} |f_n(t) - f(t)| \to 0 \text{ as } n \to \infty.$$

Given $\varepsilon > 0$, pick N such that $\max_{t\in[0,1]} |f_n(t) - f_m(t)| < \frac{1}{2}\varepsilon$ for all $n, m > N$. Fix $n > N$ and let $m \to \infty$. Therefore,

$$\max_{t\in[0,1]} |f_n(t) - f(t)| \leq \frac{1}{2}\varepsilon < \varepsilon$$

That is, $d(f_n, f) < \varepsilon$ for all $n > N$, and so $f_n \to f$. Our Cauchy sequence $f_1, f_2, \dots$ converges to f. □

We now show that the same space of functions, with a different metric, is not complete.

Example. $\mathcal{C}[0,1]$ is not complete in the L^1 metric,

$$d_1(f,g) = \int_0^1 |f(t) - g(t)| dt$$

Proof. We give a Cauchy sequence of functions for d_1 which does not converge to any function in $\mathcal{C}[0,1]$. For $n \geq 1$, define,

$$f_n(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{1}{2} - \frac{1}{2n} \\ n(t - \frac{1}{2} - \frac{1}{2n}) & \text{if} \frac{1}{2} - \frac{1}{2n} \leq n \leq \frac{1}{2} + \frac{1}{2n} \\ 1 & \text{if } \frac{1}{2} + \frac{1}{2n} \leq t \leq 1 \end{cases}$$

Then $f_n \in \mathcal{C}[0,1]$. If $N \in \mathbb{N}$ and $m \geq n \geq N$ then $f_n(t) = f_m(t)$ unless $\frac{1}{2} - \frac{1}{2n} \leq t \leq \frac{1}{2} + \frac{1}{2n}$, and $|f_n(t) - f_m(t)| \leq 1$ for all t. Hence,

$$d_1(f_m, f_n) \leq \frac{1}{n} < \frac{1}{N} \text{ if } m, n > N$$

This shows that (f_n) is a Cauchy sequence with respect to d_1.

We will show that the sequence $(f_n) \in \mathcal{C}[0,1]$ does not converge to d_1 to any function in $\mathcal{C}[0,1]$. It is clear that f is 'trying to converge' to a function f with,

$$f(t) = \begin{cases} 0 & \text{if } t < \frac{1}{2} \\ 1 & \text{if } t > \frac{1}{2}. \end{cases}$$

This function can't be in $\mathcal{C}[0,1]$ whether we assign to $t = \frac{1}{2}$.

Since f is not in our metric space $(\mathcal{C}[0,1], d_1)$, this is not quite enough to show the sequence does not converge, perhaps maybe to some other limit $g \in \mathcal{C}[0,1]$. Suppose for a contradiction that such a continuous g does exist. We first show that if $0 < t < \frac{1}{2}$ then $g(t) = 0$. Suppose there is some $t_0 \in (0, \frac{1}{2})$ with $|g(t_0)| = \alpha > 0$. Since g is continuous, there is some $\delta > 0$ with $|g(t) - g(t_0)| < \frac{1}{2}\alpha$ for all t with $|t - t_0| < \delta$. Choosing δ even smaller if necessary, we may assume that $(t_0 - \delta, t_0 + \delta) \subseteq (0, \frac{1}{2})$.
If $|t - t_0| < \delta$ we then have $|g(t)| \geq \frac{1}{2}\alpha$. For any n large enough that $\frac{1}{2} - \frac{1}{2n} > t + \delta$ we have,

$$d_1(f_n, g) = \int_0^1 |f_n(t) - g(t)| dt \leq \int_{t_0-\delta}^{t_0+\delta} |0 - g(t)| dt \geq (2\delta)\frac{1}{2}\alpha = \delta\alpha$$

Then $d_1(f_n, g)$ doesn't tend to zero as $n \to \infty$. This shows that $g(t) = 0$ for all $t \in (0, \frac{1}{2})$. By continuity, $g(0) = 0$. As similar argument shows $g(t) = 1$ for all $t \in (\frac{1}{2}, 1]$. Thus we have our required contradiction. □

Lemma 4.4. In a complete metric space, a subspace is complete if and only if it is closed.

Proof. Let (X, d) be a complete metric space and Y a subspace.
Firstly suppose Y is complete. If a is any point in the closure $\overline{Y}$ of Y, we can find a sequence a_n in Y so that $a_n \to a$. Then (a_n) is Cauchy sequence, so by the completeness of Y, it's limit a must be an element of Y. Hence $\overline{Y} \subseteq Y$, so Y is closed.

Conversely, suppose that Y is closed. Any Cauchy sequence b_n in Y converges to a point $b \in X$ since X is complete. But then $\lim b_n = b$ must be in Y as Y is closed. Hence Y is complete. □

Lemma 4.5. The product of two complete metric spaces is complete.

Proof. Given metric space (X, d_X) and (Y, d_Y), the product topology on $X \times Y$ is given by the metric

$$d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1 + y_2)$$

(or any metric equivalent to this). If (x_n, y_n) is a Cauchy sequence in $X \times Y$ then (x_n) is a Cauchy sequence in X, so it converges to some $x \in X$ since X is complete. Similarly (y_n) converges to some $y \in Y$. Then $(x_n, y_n) \to (x, y)$. Hence $X \times Y$ is complete. □

Example. $\mathbb{R}^n$ is complete.

4.2 Banach's Fixed Point Theorem

Definition 4.6 (Fixed Point)**.** Let S be any set and $f : S \to S$ a function. A fixed point of f is a point $s \in S$ with $f(s) = s$.

It is useful to know when a function has a fixed point. This is closely related to finding a solution to an equation by successive approximation.

Definition 4.7 (Contraction)**.** Let (X, d) be a complete metric space. A function $f : X \to X$ is a contraction if there is some $K < 1$ such that,

$$d(f(x), f(y)) \leq K d(x, y) \quad \text{for all } x, y \in X$$

Theorem 4.8 (Banach's Fixed Point Theorem / Contraction Mapping Theorem)**.** Let (X, d) be a complete metric space and let $f : X \to X$ be a contraction. Then f has a unique fixed point.

Proof. Choose any $x_1 \in X$ and define a sequence (x_n) by $x_{n+1} = f(x_n)$ forall $n \geq 1$. Let $K < 1$ satisfy,

$$d(f(x), f(y)) \leq K d(x, y) \quad \text{for all } x, y \in X$$

Then for $n \geq 2$ we have,

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq K d(x_n, x_{n-1})$$

Iterating, we get,

$$d(x_{n+1}, x_n) = K^{n-1} d(x_2, x_1).$$

Hence if $m > n \geq 2$, using the triangle inequality in X repeatedly to get,

$$\begin{aligned} d(x_n, x_m) \leq d(x_m, x_{m-1}) + d(x_{m-1}, d_{m-2}) + \cdots + d(x_{n+1}, x_n) & \\ & \leq (K^{m-2} + K^{m-3} + \cdots + K^{n-1}) d(x_2, x_1) \\ & = \frac{1 - K^{m-n}}{1 - k} K^{n-1} d(x_2, x_1) \end{aligned}$$

As $K < 1$ this shows that (x_n) is a Cauchy sequence. Since X is complete, it converges to some $p \in X$. Since $x_{n+1} = f(x_n)$ for all n, it follows that $f(p) = p$, so p is a fixed point of f. If q is another fixed point, we have $d(q, p) = d(f(p), f(q)) \leq K d(q, p)$. As $K < 1$, this shows that $d(p, q) = 0$, so $q = p$. □

Remark. If $X = [a, b]$ and $f : X \to X$ is differentiable, and there is a $K < 1$ with $|f'(X) \leq K|$ for all $x \in (a, b)$, then f is a contraction on X.

This follows from the mean value theorem, if $a \leq y < x \leq b$, then,

$$\frac{f(x) - f(y)}{x - y} = f'(c) \text{ for some } c \in (y, x).$$

As $|f'(c)| \leq K$ it follows that $|f(x) - f(y)| \leq K|x - y|$ for all x and $y \in [a, b]$. So in the presvious example we could have shown that f is a contraction on $X = [1, 2]$ by noting that $f'(x) \leq \frac{1}{2}$ for all $x \in (1, 2)$

Example. We will show that $f(x) = x^5 - 7x + 1 = 0$ on $[0, 1]$ using BFP. To do so, we have to rewrite such that x is a fixed point of $[0, 1]$. Now we have,

$$f(x) = \frac{1}{7}(x^5 + 1).$$

Then for $x \in [0, 1]$ we have $\frac{1}{7} \leq f(x) \leq \frac{2}{7}$, so $f(x) \in [0, 1]$. Moreover, as $f'(x) = \frac{5}{7}x^4$, so we have $|f'(x)| \leq \frac{5}{7}$ on $[0, 1]$. Hence f is a contraction on $[0, 1]$ and so by BFP we have f has a unique fixed point in that interval. This shows that $x^5 - 7x + 1 = 0$ has a unique solution in $[0, 1]$.

5 Measure Theory

5.1 Motivation

This is basically trying to find some better way to do integration, but also we are interested in lengths of sets. If we have this length $m(A)$ of any $A \subseteq \mathbb{R}$. This is closely related to the idea of integration over A. If we define the indicator function of A,

$$1_A : \mathbb{R} \to \mathbb{R} \qquad 1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

then we should have,

$$\int_A dx = \int_R 1_A(x)dx = m(A)$$

If $A = [a, b]$ then this is straightforward. However, if $A = \mathbb{Q}$ it isn't straightforward. It isn't obvious what the length of $\mathbb{Q}$ should mean and the integral,

$$\int_{\mathbb{R}} 1_{\mathbb{Q}} dx$$

isn't defined (as a Riemann integral). It turns out we can make sense of these subsets of $\mathbb{R}$ and the ones that work are called the measurable sets, which includes $\mathbb{Q}$.

Let us begin realise what we want to happen with these subsets. Here are some properties,

1. The length of an interval should have the obvious meaning,
$$m((a, b)) = m([a, b]) = b - a \text{ if } b \geq a$$
In particular $m(\varnothing) = m(\{a\}) = 0$.
2. Unbounded intervals should have length ∞, but no set should have negative length. Hence the values for m should be in $\{\infty\} \cup [0, \infty)$. We write this set as $[0, \infty]$.
3. We expect $m(A \cup B) \leq m(A) + m(B)$ for all sets A, B and,
$$m(A \cup B) = m(A) + m(B) \text{ if } A \cup B = \varnothing.$$
By induction, this should be true for any union of finite collection of sets. Note that because we have ∞, we have $m + \infty = \infty$, $\infty + m = \infty$ and $\infty + \infty = \infty$ for $m \geq 0$.
4. More generally we should expect m to respect countably infinite unions,
$$m\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m(A_n)$$
and,
$$m\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m(A_n) \text{ if } A_i \cap A_j = \varnothing \text{ when } i \neq j.$$
Note that the sums may diverge to ∞, however if the sum is absolutely convergent, so the order of the terms in the sum doesn't matter. For example, $\mathbb{Q}$ is countable. So we should have,
$$\begin{aligned} m(\mathbb{Q}) &= m\left(\bigcup_{q \in \mathbb{Q}} \{q\}\right) \\ &\leq \sum_{q \in \mathbb{Q}} m(\{q\}) \\ &= \sum_{q \in \mathbb{Q}} 0 = 0. \end{aligned}$$

Note that is does not make sense to expect a similar property for uncountably infinite unions since, for example,

$$[0,1] = \bigcap_{x\in[0,1]} \{x\}$$

where $m([0,1]) = 1$ and $m(\{x\}) = 0$ for each of the uncountably many elements.

5. m should be translation invariant, i.e. if $c \in \mathbb{R}$ and we write,

$$A + c = \{a + c : a \in A\}$$

then $m(A + c) = m(A)$.

It turns out that we can't have all 1-5 for all $A \subseteq \mathbb{R}$.

5.1.1 A strange example

We now show that is not possible to define $m(A)$ of all subsets $A \subseteq \mathbb{R}$ so that $1 - 5$ are all satisfies. Roughly, the ideas is that 'real numbers modulo the rational numbers' gives a set too weird to have a length. However, to avoid the possibility of sets of infinite length, we work within $[0, 1)$ instead of $\mathbb{R}$ fully.

We first define an equivalence relation $\sim$ on $[0, 1)$ by $x \sim y \iff x - y \in \mathbb{Q}$. So $\mathbb{R}$ is disjoint union of it's equivalence classes, and for $x \in [0, 1)$, its equivalence class is the set,

$$\{x + q : q \in \mathbb{Q}, 0 \le x + q < 1\}.$$

The cooresponding values of q will be those in $[-x, 1 - x) \cap \mathbb{Q}$, so each equivalence class is a countably infinite set. As $[0, 1)$ is uncountable, it follows that there are uncountably many equivalence classes. Now let $S \subsetneq [0, 1)$ be a set containing exactly one representation of each equivalence class. (Another way to say this is that S is a system of coset representatives for the subgroup $\mathbb{Q}$ as a subgroup of $\mathbb{R}$, with these representatives in $[0, 1)$.) The axiom of choice guarantees that S exists, even though we cannot write it down explicity.

For each $q \in \mathbb{Q}$, let $S_q = S + q := \{s + q : s \in S\}$, so $S_q \subseteq [q, q + 1)$. In particular, if $q \in [0, 1) \cap \mathbb{Q}$, then S_q is the disjoint union of it's 'part below 1' and 'part above 1':

$$S_q^1 = S_q \cap [0, 1) = S_q \cap [q, 1)$$

$$S_q^2 = S_q \cap [1, 2) = S_q \cap [1, q + 1)$$

We will show that if $m(S_q^1)$ and $m(S_q^2)$ are defined for all q, then 1-5 will lead to a contradiction. This will show that S, and the closely related sets S_q^1, S_q^2 are too badly behaed to have a length satisfying our desired properties.

If $m(S_q^1)$ and $m(S_q^2)$ are defined we should have,

$$m(S_q) = m(S_q^1) + m(S_q^2)$$

by 3. By the translation property, $m(S_q) = m(S)$. USing the translation property again to move S_q^2 back by 1, we have $m(S_q^2) = m(T_q)$ for,

$$T_q = \{x - 1 : x \in S_q^2\} = S_{q-1} \cap [0, 1) \subseteq [0, q).$$

So writing $A_q = T_q \cup S_q^1 \subsetneq [0, 1)$, we have,

$$m(A_q) = m(T_q) + m(S_q^1) = m(S_q) = m(S).$$

Now $[0,1)$ is the disjoint union of sets A_q for $q \in [0,1)$. Indeed, given some $x \in [0,1)$, there is a unique $s \in S$ and a unique $q \in (-1,1) \cap \mathbb{Q}$ with $x = s + q$. If $q \geq 0$, then $x \in S_q^1 \subseteq A_q$. If $q < 0$, let $r = q + 1 \in [0,1)$ and $y = x + 1$. Then $y = s + r \geq 1$ so $y \in S_r^2$ and hence $x \in T_r$.

Now consider the countably infinite disjoint union,

$$[0,1) = \bigcup_{q \in [0,1) \cap \mathbb{Q}} A_q$$

with $m(A_q) = m(S)$ for each q. If property 4 holds and $s(S)$ is defined then either $m(S) = 0$ and $m([0,1)) = 0$, or $m(S) > 0$ and $m([0,1)) = \infty$. Both of these give a contradiction since $m([0,1)) = 1$.

5.2 Nulls Sets and Outer Measure

Null sets are sets that we can assign a length of 0. We can 'over-estimate' the length of a set by covering with a countable union of open sets. If we can make the over-estimate arbitrarily small, then our set should have length zero,

Definition 5.1 (Null Set)**.** A subset A of $\mathbb{R}$ is a null set if, given $\varepsilon > 0$, there is a countable family of open intervals I_n, $n \geq 1$ such that,

$$A \subseteq \bigcup_{n=1}^{\infty} I_n \text{ and } \sum_{n=1}^{\infty} m(I_n) < \varepsilon$$

where, for an open interval $I = (u,v)$ with $u \leq v$ we define $m(I) = v - u$.

Example. Given any finite set $A = \{a_1, a_2, \dots, a_k\}$ is a null set. For a given $\varepsilon > 0$ we pick $\delta > 0$ with $2k\delta < \varepsilon$. Define I_n by,

$$I_n = \begin{cases} (a_n - \delta, a_n + \delta) & \text{if } 1 \leq n \leq k \\ \varnothing & \text{if } n > k \end{cases}$$

Then $a_n \in I_n$ for $1 \leq n \leq m$, so,

$$A \subseteq \bigcup_{n=1}^{\infty} I_n$$

and,

$$\sum_{n=1}^{\infty} m(I_n) = \sum_{n=1}^{k} 2\delta < \varepsilon.$$

Example. The middle third cantor set C is a null set. Recall $C = \bigcap_{k=0}^{\infty} C_k$ where C_k is the disjoint union of 2^k closed intervals, each of length 3^{-k}. We choose some α with $\frac{1}{3} < \alpha < \frac{1}{2}$. Given $\varepsilon > 0$, let k be large enough that $(2\alpha)^k < \varepsilon$. For each of the 2^k closed intervals J of length 3^{-k} making up C_k, pick an open interval $I \supset J$ of length α^k, and label these intervals $I_1, I_2, \dots, I_r$ with $r = 2^k$. For $n > r$, let $I_n = \varnothing$. Then,

$$C \subset C_k \subseteq \bigcup_{n=1}^{\infty} I_n$$

and

$$\sum_{n=1}^{\infty} m(I_n) = \sum_{n=1}^{r} \alpha^k = r\alpha^k = (2\alpha)^k < \varepsilon$$

This shows that C is indeed a null set.

Lemma 5.2. If A and B are null sets, so is $A \cup B$.

Proof. Given $\varepsilon > 0$, we can find open intervals I_n and J_n for $n \geq 1$ such that,

$$A \subseteq \bigcup_{n=1}^{\infty} I_n \text{ and } \sum_{n=1}^{\infty} m(I_n) < \frac{1}{2}\varepsilon,$$

and,

$$B \subseteq \bigcup_{n=1}^{\infty} J_n \text{ and } \sum_{n=1}^{\infty} m(J_n) < \frac{1}{2}\varepsilon.$$

Let $W_1, W_2, \ldots$ be the sequence of intervals $I_1, J_1, I_2, J_2, \ldots$. Then,

$$A \cup B \subseteq \bigcup_{n=1}^{\infty} W_n = \bigcup_{n=1}^{\infty} I_n \cup \bigcup_{n=1}^{\infty} J_n$$

and,

$$\sum_{n=1}^{\infty} W_n = \sum_{n=1}^{\infty} I_n + \sum_{n=1}^{\infty} J_n < \varepsilon$$

We can rearrange these sums since it is absolutely convergent. □

It follows by induction that the union of finitely many null sets is a null set. More generally,

Lemma 5.3. The union of countably many null sets is a null set.

Proof. Let $A_1, A_2, \ldots$ be null sets, Given $\varepsilon > 0$ for each $j \geq 1$, there is a countable collection of open sets $I_1^{(j)}, I_2^{(2)}, \ldots$ such that

$$A_j \subseteq \bigcup_{n=1}^{\infty} I_n^{(j)} \text{ and } \sum_{n=1}^{\infty} m(I_n^{(j)}) < 2^{-j}\varepsilon.$$

The collection of intervals $I_n^{(j)}$ for all j, $n \geq 1$ is countable. We can list it is $I_1^{(1)}, I_1^{(2)}, I_2^{(1)}, I_1^{(3)}, I_2^{(2)}, I_3^{(1)}, \ldots$. Label these intervals $W_1, W_2, \ldots$. Then we have,

$$\bigcup_{j=1}^{\infty} A_j \subseteq \bigcup_{j=1}^{\infty} \bigcup_{i=1}^{\infty} I_n^{(j)} = \bigcup_{n=1}^{\infty} W_n$$

and,

$$\begin{aligned} \sum_{j=1}^{\infty} m(W_n) &= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} m(I_n^{(j)}) \\ &< \sum_{j=1}^{\infty} 2^{-j}\varepsilon = \varepsilon. \end{aligned}$$

(We can rearrange the sum since it's absolutely convergent.) Hence $\bigcup_{j=1}^{\infty} A_j$ is a null set. □

Example. $\mathbb{Q}$ is a null set. In $\mathbb{Q}$ is countable, so we may list it's elements $q_1, q_2, \ldots$. Then $\mathbb{Q}$ is the union of countably many null sets $\{q_n\}$ for $n \geq 1$.

Now we seek to look at the Outer Measure. We wanted the following properties,

1. $m((a,b)) = m([a,b]) = b - a$ if $b \geq a$,
2. $m(A) \in [0, \infty]$,
3. $m(A \cup B) \leq m(A) + m(B)$ for all set A, B,

4. More generally we should expect m to respect countably infinite unions.

5. m should be translation invariant, $m(A+c) = m(A)$.

We will now consider an m^* which turns out to have all the properties above apart from the equality conditions (not listed). We will apply the same over-estimate strategy for null sets and define $m^*(A)$ for an arbitrary A as the best over estimate. Consider the set of all possible over-estimates,

$$Z(A) = \left\{ \sum_{n=1}^{\infty} m(I_n) : I_1, I_2 \ldots \text{ are open intervals with } A \subseteq \sum_{n=1}^{\infty} I_n \right\}$$

Clearly $\infty \in Z(A)$. Either $Z(A) = \{\infty\}$ or $Z(A) \setminus \{\infty\}$ is a non-empty subset of $\mathbb{R}$ bounded below by 0. Hence $Z(A)$ has an infinimum, which may be ∞.

Definition 5.4 (Outer Measure)**.** The outer measure of A is $m^*(A) = Z(A)$

This means that for any family $I_1, I_2, \ldots$ of open intervals covering A, we have,

$$m^*(A) \leq \sum_{n=1}^{\infty} m(I_n)$$

but, given $\varepsilon > 0$, there is such a family with,

$$\sum_{n=1}^{\infty} m(I_n) \leq m^*(A) + \varepsilon$$

In summary, we have defined $m^*(A) \in [0, \infty]$ for every subset $A \subseteq \mathbb{R}$, and shown that is has the following properties,

1. If A is an interval, then $m^*(A)$ is the length of A.

2. m^* is countably subadditive,
$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^*(A_n)$$

3. m^* is translation-invariant, $m^*(A+c) = m^*(A)$.

However, the example above shows that m^* cannot be additive on countable disjoint unions. That is,

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m^*(A_n) \text{ if } A_i \cap A_j = \varnothing \text{ when } i \neq j$$

cannot always hold. (Indeed, it doesn't always hold even for finite unions.)

5.3 Lebesgue Measurable Sets

Theorem 5.5. m^* satisfies the countable subadditity property,

$$m^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} m^*(A_j).$$

Proof. The argument is similar to the proof that a countable union of null sets is a null set. Given $\varepsilon > 0$ for each $j \geq 1$, there is a countable collection of open intervals $I_1^{(j)}, I_2^{(j)}, \dots$, such that

$$A_j \subseteq \bigcup_{n=1}^{\infty} I_n^{(j)} \text{ and } \sum_{m=1}^{\infty} m(I_n^{(j)}) \leq m^*(A_j) + 2^{-j-1}\varepsilon.$$

Then,

$$\bigcap_{j=1}^{\infty} A_j \subseteq \bigcap_{j=1}^{\infty} \bigcap_{n=1}^{\infty} I_n^{(j)}$$

and,

$$\begin{aligned}\sum_{j=1}^{\infty}\sum_{n=1}^{\infty} m(I_n^{(j)}) &\leq \sum_{j=1}^{\infty}(m^*(A_j) + 2^{-j-1}\varepsilon) \\ &= \left(\sum_{j=1}^{\infty} m^*(A_j)\right) + \varepsilon\end{aligned}$$

As this holds for all ε, then the result follows. □

Recall the for a bounded open interval $I = (a, b)$ we have defined $m(I) = b - a$, the length in the usual sense. The definition of m^* is comparable with this.

Lemma 5.6. Let $I \subseteq \mathbb{R}$ be an interval. Then $m^*(I) = m(I)$.

Proof. Lengthy but trivial, so omited. □

So the outer measure has,

1. the length of an interval,
2. sub-additive property,
3. Translation Invariance.

5.4 Lebesgue Measure

We now describe a large class $\mathcal{M}$ of subsets where m^* behaves nicely.

Definition 5.7 (Lesbegue Measurable)**.** We say $E \subseteq \mathbb{R}$ is (Lesbegue) Measurable if, for all $A \subseteq \mathbb{R}$,

$$m^*(A) = M^*(A \cap E) + M^*(A \cap E^c)$$

We write $\mathcal{M}$ for the collection of measurable sets of $\mathbb{R}$.

Remark. By sub-additivity, we always have,

$$\begin{aligned}m^*(A) &= m^*((A \cap E) \cup (A \cap E^c)) \\ &\leq m^*(A \cap E) + m^*(A \cap E^c),\end{aligned}$$

so to check that $E \in \mathcal{M}$ it is enough to show,

$$m^*(A \cap E) + m^*(A \cap E^c) \leq m^*(A).$$

Remark. Some properties of $\mathcal{M}$ follows easily from the properties of m^*,

1. If E is a null set, then $E \in \mathcal{M}$. Indeed $m^*(A \cap E) \leq m^*(E) = 0$, ad similarly for $m^*(A \cap E^c)$. In particular $\varnothing \in \mathcal{M}$.
2. If $E \in \mathcal{M}$, then $E^c \in \mathcal{M}$.
3. $\mathcal{M}$ is translation invariant. If $E \in \mathcal{M}$ then $E + t \in \mathcal{M}$.

Theorem 5.8. $\mathcal{M}$ admits countable unions and intersections. If $E_1, E_2, \dots \in \mathcal{M}$ then,

$$\bigcup_{n=1}^{\infty} E_n \in \mathcal{M} \text{ and } \bigcap_{n=1}^{\infty} E_n \in \mathcal{M}$$

Moreover, if $E_i \cap E_j = \varnothing$ for $i \neq j$, then

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m^*(E_n)$$

Proof. Omitted □

Theorem 5.9. Every interval is in $\mathcal{M}$

Proof. Omitted. □

Corollary 5.10. Every subset of $\mathbb{R}$ is in $\mathcal{M}$, and every closed subset of $\mathbb{R}$ is in $\mathcal{M}$.

Proof. As $E \in \mathcal{M} \iff E^c \in \mathcal{M}$, it suffices to consider the open subsets. By the theorems it suffices to show that any open set U is a countable union of open intervals.

Now U is the disjoint union of it's connected components. If $x \in U$ then the connected components C_x of U containing x is the open interval (a, b), where $a = \inf\{a' : (a', x] \subseteq U\}$ and $b = \sup\{b' : [x, b') \subseteq U\}$.

Thus each connected component contains rational numbers. As $\mathbb{Q}$ is countable and the components are disjoints, it follows that there are at most countable many components, so U is indeed a union of countably man open intervals. □

Definition 5.11 (Lebesgue)**.** For $E \in \mathcal{M}$ we define $m(E) = m^*(E)$. Then m is the Lebesgue measure on $\mathbb{R}$.

We now have achieved our aim as far as possible. We have assigned a 'length' for $m(A)$ for any reasonable subset $A \subseteq \mathbb{R}$ with some of the properties we wanted.

Remark. We have done this just for $\mathbb{R}$, but we can do this similarly for $\mathbb{R}^2$ and moreover for $\mathbb{R}^n$.

5.5 σ-algebras and measure spaces

We now set up some abstract ideas to talk about measures more generally.

Definition 5.12 (σ-algebra)**.** A σ-algebra $\mathcal{B}$ on a set X is a faimly of subsets such that,

1. $\varnothing \in \mathcal{B}$,
2. if $A \in \mathcal{B}$ then $X \setminus A \in \mathcal{B}$,
3. if $A_1, A_2, \dots$ is a countable sequence of sets in $\mathcal{B}$, then,

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{B}$$

Example. On any set X, the powerset $\mathcal{P}(X)$ of X is a σ-algebra.

Example. Let $X = \{a, b, c, d\}$ then one possible σ-algebra is,

$$\{\varnothing, \{a, b\}, \{c, d\}, X\}$$

Remark. If $\mathcal{B}$ is a σ-algebra then $X \in \mathcal{B}$ since $X = X \setminus \varnothing$.

Also for a countable sequence we have,

$$\bigcap_{n=1}^{\infty} A_n \in \mathcal{B}$$

as,

$$\bigcup_{n=1}^{\infty} (X \setminus A_n) = X \setminus \bigcap_{n=1}^{\infty} A_n \in \mathcal{B}$$

Of course, this also holds for finite unions and intersections. We just take $A_n = \varnothing$ for all but finitely many n.

Remark. A family of set $\mathcal{B}$ satisfying the definition for a σ-algebra, except that only finite unions are allowed is called an algebra on X.

Proposition 5.13. Let $\{\mathcal{B}_i : i \in \mathcal{I}\}$ be any set of σ-algebras on X. Then their intersection,

$$\mathcal{B} = \bigcap_{i \in \mathcal{I}} \mathcal{B}_i$$

is also a σ-algebra on X.

. We check the three conditions from the definition.

1. As $\varnothing \in \mathcal{B}_i$ for all i, then $\varnothing \in \mathcal{B}$,
2. If $A \in \mathcal{B}$,, then $A \in \mathcal{B}_i$ for each i, so $X \setminus A \in \mathcal{B}_i$. Thus $X \setminus A \in \mathcal{B}$.
3. If $A_1, A_2, \dots \in \mathcal{B}$, then $A_1, A_2, \dots \in \mathcal{B}_i$ for each i, so $\bigcup_{n=1}^{\infty} A_n \in \mathcal{B}_i$ for each i. Thus $\bigcup_{n=1}^{\infty} A_n \in \mathcal{B}$.

□

Why is this important? We can make the following defintion,

Definition 5.14 (σ-algebra generated by a set)**.** Let $\mathcal{E}$ be any family of subsets of a set X. Then the σ-algebra generated by $\mathcal{E}$ is the smallest σ-algebra containing $\mathcal{E}$.

Further we define the Borel Algebra,

Definition 5.15 (Borel Algebra)**.** Let $(X, \mathcal{T})$ be a topological space. The Borel algebra on X is the σ-algebra generated by $\mathcal{T}$. It's elements are called Borel subsets of X.

Example. On $\mathbb{R}$ the open sets are precisely unions of open intervals (a, b). So the Borel Algebra of $\mathbb{R}$ are any sets that can be built up from open intervals by taking complements and countable infinite unions and intersections.

Example. The set $\mathcal{M}$ of measurable functions in $\mathbb{R}$ is a σ-algebra. This follows from the properties given above. Since open intervals are in $\mathcal{M}$, it follows that $\mathcal{M}$ contains the Borel algebra on $\mathbb{R}$.

Remark. It turns out that not every null set is a Borel set. Since null sets are measurable, this means that there are measurable sets which are not Borel sets. However, every Borel set is contained in a measurable set with the same outer measure.

Example. Here is a sketch of a contruction of a null set which isn't Borel. We can write each $x \in [0,1]$ as a binary expansion $x = \sum_{n=1}^{\infty} a_n/2^n$, with $a_n \in \{0,1\}$. Choose the non-terminating expansion unless $x = 0$. Then the expansion is unique.

Define $b_n = 2a_n \in \{0,2\}$ and set $f(x) = \sum_{n=1}^{\infty} b_n/3^n$. This maps $[0,1]$ bijectively to the middle-third cantor set. Then $f(S)$ is a null set. In particular, it is Lebesgue measurable.

Now suppose $f(S)$ is a Borel set. Now $f : [0,1] \to [0,1]$ is injective and increasing, and the preimage of an interval under f is an interval. Since unions and complements behave nicely under preimages, the preimage of a Borel set is a Borel set. This is $f(S)$ is a Borel set, so is S, and this is a contradiction.

Definition 5.16 (Measure / Measure Space)**.** Let X be a set and $\mathcal{B}$ a σ-algebra on X. A measure is a function $\mu : \mathcal{B} \to [0,\infty]$ such that,

1. $\mu(\varnothing) = 0$,
2. for any countably infinite family of set $A_1, A_2, \cdots \in \mathcal{B}$ with $A_i \cap A_j = \varnothing$ for $i \neq j$, we have,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

We call the triple $(X, \mathcal{B}, \mu)$ a measure space. Further $\mu(X) < \infty$ and a probability measure if $\mu(X) = 1$.

Example. The triple $(\mathbb{R}, \mathcal{M}, m)$ is a measure space, where $\mathcal{M}$ is the set of measurable sets, and m the Lebesgue measure.

The tripe $(\mathbb{R}, \mathcal{B}, m)$, where $\mathcal{B}$ is the Borel algebra. These two measure spaces are different since $\mathcal{B} \neq \mathcal{M}$.

Example. For any set $X \neq \varnothing$ and any $p \in X$ we have the Dirac measure at p,

$$\mu_p(A) = \begin{cases} 1 & \text{if } p \in A \\ 0 & \text{if } p \notin A \end{cases}$$

on the σ-algebra $\mathcal{P}(X)$.

Definition 5.17 (Measurable Function)**.** Let $(X, \mathcal{B}, \mu)$ be a measure space, let $E \subseteq \mathcal{B}$ and let $f : E \to \mathbb{R}$ be a function. We stay f is a measurable function if, for each interval $I \subseteq \mathbb{R}$, the set $f^{-1}(I) \subseteq E$ is measurable.

Remark. There are functions $f : \mathbb{R} \to \mathbb{R}$ which are Lebesgue measurable but not Borel measurable. Take the indicator function of a set which is Lebesgue measurable but not Borel measurable.