An Ode to Geometric Mechanics

Where Pure Maths meets Applications

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Recap - Lie Groups

1. $\circ$ is closed, so $\forall\, g_1, g_2 \in G$, $\circ g_2 \in G$.
2. Identity, $\exists\, e \in g\ \forall g \in G$, $e \circ g = g = g \circ e$.
3. Inverse, $\forall g \in G, \exists\, h \in G$, $g \circ h = e = h \circ g$
4. Associativity, $\forall\, g_1, g_2, g_3 \in G$, $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$

$$m : G \times G \to G \qquad m(g_1, g_2) = g_1 \circ g_2 \qquad g_1, g_2 \in G$$

$$i : G \times G \to G \qquad i(g_1, g_2) = g_1^{-1} g_2 \qquad g_1, g_2 \in G$$

$$[A, B] = AB - BA$$

Recap - Group Actions

$$\mathrm{Ad} : G \times \mathfrak{g} \to G$$

$$\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

$$\mathrm{Ad}_g \xi = g\xi g^{-1}$$

$$\mathrm{ad}_\eta \xi = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{g(t)} \xi = \eta\xi - \xi\eta = [\eta, \xi]$$

$$\mathrm{Ad}^*_g \mu = g^T \mu (g^T)^{-1}$$

$$\mathrm{ad}^*_\eta \xi = [\eta^T, \xi]$$

NB! More detailed proofs and definitions can be found in the thesis document.

What is Geometric Mechanics

Geometric Mechanics is the study of how we can use Geometric ideas to simplify, solve and explore real world problems.

What does this mean mathematically? We use manifolds and define Lagrangians on the tangent bundles on those manifolds to the real numbers. Then we can exploit certain symmetries in these manifolds in order to reduce a system in order. Let M be a manifold then this Lagrangian is,

$$L : TM \to \mathbb{R}$$

You may ask, what is this manifold? The project was interested in smooth manifolds and more specifically matrix Lie Groups and more specifically again $\mathrm{SO}(3)$.

What is Geometric Mechanics

Geometric Mechanics dates back to the 1800s when two different mathematical ideologies started fighting it out over which framework was better. The Eulerian or Lagrangian frameworks. These wars were fought by, surprisingly, Euler and Lagrange.

Figure: Euler

Figure: Lagrange

Figure: Poincaré

Figure: Noether

Eulerian or Lagrangian

As an example, here can found in fluids from Bennett 2006. The conservation of mass in Eulerian and Lagrangian is,

$$\frac{d}{dt}\int_V \rho dV = 0 \qquad \frac{\partial}{\partial t}(\rho J) = 0$$

where J is the determinant of the Jacobean of transformation. The conservation of momentum is,

$$\frac{d}{dt}\int_V \rho u dV = -\int_A p\hat{n} dA \qquad \rho \frac{\partial X_i}{\partial a_k}\frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial a_k}{}^{1}$$

and energy follows a similar pattern, but is messier:

$$\frac{d}{dt}\int_V \rho|\mathbf{u}|^2 dV = -\int_A p\mathbf{u}\cdot\mathbf{n} dA \qquad \frac{d}{dt}(\rho J(K+E)) = -J\frac{\partial A_j}{\partial x_k}\frac{\partial}{\partial a_j}(u_k p)\,.$$

The main point is this, **Lagrangian gives prettier expressions without integrals**.

[1] The derivation I used here is slightly cheating, but my point stands

Now

Why are we still working on this? That is especially where Poisson and Noether come in. They were able to move forward these mathematical ideas to introduce reduction and conserved quantities. Here are some of the current academics working on Geometric Mechanics,

Figure: Prof. Darryl Holm, Imperial College

Figure: Dr Hamid Alemi Ardakani, Exeter University

Figure: Prof. Tudor Ratiu, Shanghai Jiao Tong University

Figure: Prof. Cristina Stoica, Wilfred Laurier University

Euler-Poincaré and Holm

In Geometric Mechanics, we have Holm's books (Darryl D. Holm, Schmah, and Stoica 2009) that lays out the foundations and important things to know when embarking on a journey into GM. We will touch the surface here.
I defined the variational derivative in the last presentation, but as a reminder, it's this,

$$\delta \mathbf{q}(t) = \frac{d}{ds}\bigg|_{s=0} \mathbf{q}(s,t).$$

and Hamilton's Principle is based upon this,

$$\delta \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}) dt = 0$$

where we have endpoint conditons that say that the variational derivative disappears at t_1 and t_2.

Euler-Lagrange

Theorem

For any $L : \mathbb{R}^{2dN} \to \mathbb{R}$, the Euler-Lagrange equations are equivalent to Hamilton's principle of stationary action $\delta\mathcal{S} = 0$ where $\mathcal{S}$ is defined as,

$$\mathcal{S}[\mathbf{q}(t)] = \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}})\, dt$$

with respect to variations along paths and fixed endpoints.

Proof.

We will proceed using the fact that $\frac{d}{dt}\delta\mathbf{q} = \delta\dot{\mathbf{q}}$ and using integration by parts.

$$\begin{aligned}
\delta\mathcal{S} = \frac{d}{ds}\bigg|_{s=0} S[\mathbf{q}(s,t)] &= \frac{d}{ds}\bigg|_{s=0} \int_{t_1}^{t_2} L(\mathbf{q},\dot{\mathbf{q}})\,dt \\
&= \int_{t_1}^{t_2} \left(\left\langle \frac{\partial L}{\partial \mathbf{q}}, \mathbf{q} \right\rangle + \left\langle \frac{\partial L}{\partial \dot{\mathbf{q}}}, \delta\dot{\mathbf{q}} \right\rangle \right) dt \\
&= \int_{t_1}^{t_2} \left\langle \frac{\partial L}{\partial \mathbf{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}}, \delta\mathbf{q} \right\rangle dt + \left[\left\langle \frac{\partial L}{\partial \dot{\mathbf{q}}}, \delta\mathbf{q} \right\rangle \right]_{t_1}^{t_2} \\
&= \int_{t_1}^{t_2} \left\langle \frac{\partial L}{\partial \mathbf{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}}, \delta\mathbf{q} \right\rangle dt. \qquad \text{applying end point conditions}
\end{aligned}$$

This then tells us that for any smooth $\delta\mathbf{q}(t)$ satisfying $\delta\mathbf{q}(a) = \delta\mathbf{q}(b) = 0$. If $\delta\mathcal{S} = 0$, then $\frac{\partial L}{\partial \mathbf{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} = 0$ and so Hamilton's Principle is equivalent to the Euler-Lagrange equations. $\square$

Invariant Lagrangians

Consider a spatial coordinate system with it's origin at the centre of mass of a given ridid body, $\mathbf{x}(t) \in \mathbb{R}^3$. We define body coordinates related to the spatial coordinate system by $\mathbf{x}(t) = \mathbf{R}(t)\mathbf{X}$ where $\mathbf{R}(t) \in \mathrm{SO}(3)$ is the rotation tensor.

Further consider a Lagrangian $L : T\mathrm{SO}(3) \to \mathbb{R}$ where we define $L\left(\mathbf{R}, \dot{\mathbf{R}}\right) = \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X}) \left\|\dot{\mathbf{x}}\right\|^2 d^3\mathbf{X}$. We can write this as,

$$L\left(\mathbf{R}, \dot{\mathbf{R}}\right) = \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X}) \left\|\dot{\mathbf{R}}(t)\mathbf{X}\right\|^2 d^3\mathbf{X}$$

You can see that $L(\mathbf{R}, \dot{\mathbf{R}}) = L(\mathbf{R}^{-1}\mathbf{R}, \mathbf{R}^{-1}\dot{\mathbf{R}}) = L(\mathbf{I}, \mathbf{R}^{-1}\dot{\mathbf{R}}) := \ell(\hat{\mathbf{\Omega}})$.

This is the reduction.

$$\frac{d}{dt}\frac{\partial \ell}{\partial \xi} - \mathrm{ad}^*_{\xi}\frac{\partial \ell}{\partial \xi} = 0$$

Noether's Theorem

Theorem (Noether's Theorem for left-invariant systems)

The Euler-Poincaré equations associated with a left-invariant system preserve the generalised momentum along solutions of the Euler-Poincaré equations. That is,

$$\frac{d}{dt}\left(\mathrm{Ad}^*_{g^{-1}(t)}\,\frac{\partial \ell}{\partial \xi}(t)\right) = 0.$$

Noether's Theorem Proof

Proof.

Suppose we have a left invariant Lagrangian, i.e. $L(g, \dot{g}) = L(e, g^{-1}\dot{g}) = \ell(g^{-1}g) := \ell(\xi)$ where $\xi = g^{-1}\dot{g}$. Firstly, however, let us consider the following derivative where $\mu(t) \in \mathfrak{g}$,

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=t_0} \left(\mathrm{Ad}_{g^{-1}(t)}\, \mu(t)\right) &= \left.\frac{d}{dt}\right|_{t=t_0} \mathrm{Ad}_{g^{-1}(t)g(t_0)} \left(\mathrm{Ad}_{g^{-1}(t_0)}\, \mu\right) \\
&= -\,\mathrm{ad}_{g^{-1}(t_0)\dot{g}(t_0)} \left(\mathrm{Ad}_{g^{-1}(t_0)}\, \mu\right) \\
&= -\,\mathrm{ad}_{\xi(t_0)} \left(\mathrm{Ad}_{g^{-1}(t_0)}\, \mu\right).
\end{aligned}$$

Hence we can say,

$$\frac{d}{dt}\left(\mathrm{Ad}_{g^{-1}(t)}\, \mu(t)\right) = -\,\mathrm{ad}_{\xi(t)} \left(\mathrm{Ad}_{g^{-1}(t)}\, \mu(t)\right).$$

□

Noether's Theorem Proof Cont.

Proof Cont.

$$
\begin{aligned}
\left\langle \frac{d}{dt}\left(\mathrm{Ad}^*_{g^{-1}(t)}\frac{\partial \ell}{\partial \xi}(t)\right), \mu(t)\right\rangle &= \frac{d}{dt}\left\langle \frac{\partial \ell}{\partial \xi}(t), \mathrm{Ad}_{g^{-1}(t)}\,\mu(t)\right\rangle \\
&= \left\langle \frac{d}{dt}\frac{\partial \ell}{\partial \xi}(t), \mathrm{Ad}_{g^{-1}(t)}\,\mu(t)\right\rangle + \left\langle \frac{\partial \ell}{\partial \xi}(t), \frac{d}{dt}\mathrm{Ad}_{g^{-1}(t)}\,\mu(t)\right\rangle \\
&= \left\langle \frac{d}{dt}\frac{\partial \ell}{\partial \xi}(t), \mathrm{Ad}_{g^{-1}(t)}\,\mu(t)\right\rangle + \left\langle \frac{\partial \ell}{\partial \xi}(t), -ad_{\xi(t)}(\mathrm{Ad}_{g^{-1}(t)}\,\mu(t))\right\rangle \\
&= \left\langle \frac{d}{dt}\frac{\partial \ell}{\partial \xi}(t), \mathrm{Ad}_{g^{-1}(t)}\,\mu(t)\right\rangle - \left\langle \frac{\partial \ell}{\partial \xi}(t), ad_{\xi(t)}(\mathrm{Ad}_{g^{-1}(t)}\,\mu(t))\right\rangle \\
&= \left\langle \frac{d}{dt}\frac{\partial \ell}{\partial \xi}(t) - ad^*_{\xi(t)}\frac{\partial \ell}{\partial \xi}(t), \mathrm{Ad}_{g^{-1}(t)}\,\mu(t)\right\rangle \\
&= \left\langle \mathrm{Ad}^*_{g^{-1}(t)}\left[\frac{d}{dt}\frac{\partial \ell}{\partial \xi}(t) - ad^*_{\xi(t)}\frac{\partial \ell}{\partial \xi}(t)\right], \mu(t)\right\rangle.
\end{aligned}
$$

Psuedo-Rigid Bodies

We consider the configuration space of $\mathrm{GL}^+(3)$. We fix two coordinate systems, a body coordinate system $\mathbf{X}$ and a spatial coordinate system $\mathbf{x}$. These are related as follows,

$$\mathbf{x}(t, \mathbf{X}) = \mathbf{Q}(t)\mathbf{X}$$

We are primarily interested in matrices in $\mathrm{SO}(3)$, and so we consider a bipolar decomposition of $\mathbf{Q}$, which is as follows,

$$\mathbf{Q}(t) = \mathbf{R}(t)\mathbf{A}(t)\mathbf{S}(t)$$

where $R, S \in \mathrm{SO}(3)$ and $A \in \mathrm{diag}^+(3)$. We then define a Lagrangian for this system,

$$L_{\mathrm{ext}}\left(\mathbf{R}, \mathbf{A}, \mathbf{S}, \dot{\mathbf{R}}, \dot{\mathbf{A}}, \dot{\mathbf{S}}\right) = L\left(\mathbf{Q}, \dot{\mathbf{Q}}\right) = \frac{1}{2}\operatorname{Tr}\left(\dot{\mathbf{Q}}\dot{\mathbf{Q}}^T\right)$$

Euler-Poincaré Reduction

We will use the ideas of Euler-Poincaré reduction on this Lagrangian to produce what we will call the Euler-Poincaré equations. The ideal follows from this, $L(\mathbf{ARB}, \mathbf{A\dot{R}B}) = L(\mathbf{R}, \dot{\mathbf{R}})$ where $\mathbf{A}, \mathbf{B} \in SO(3)$. Therefore, we consider the extended Lagrangian and let $A = R^{-1}$ and $B = S^{-1}$,

$$L_{\text{ext}}\left(\mathbf{R}^{-1}\mathbf{R}, \mathbf{A}, \mathbf{S}\mathbf{S}^{-1}, \mathbf{R}^{-1}\dot{\mathbf{R}}, \dot{\mathbf{A}}, \dot{\mathbf{S}}\mathbf{S}^{-1}\right) = L_{\text{ext}}\left(\mathbf{R}, \mathbf{A}, \mathbf{S}, \dot{\mathbf{R}}, \dot{\mathbf{A}}, \dot{\mathbf{S}}\right)$$

We then can write the reduced Lagrangian, ℓ_{ext},

$$\begin{aligned} L_{\text{ext}}\left(\mathbf{R}^{-1}\mathbf{R}, \mathbf{A}, \mathbf{S}\mathbf{S}^{-1}, \mathbf{R}^{-1}\dot{\mathbf{R}}, \dot{\mathbf{A}}, \dot{\mathbf{S}}\mathbf{S}^{-1}\right) &= L_{\text{ext}}\left(\mathbf{I}, \mathbf{A}, \mathbf{I}, \hat{\boldsymbol{\Omega}}, \dot{\mathbf{A}}, \hat{\boldsymbol{\Lambda}}\right) \\ &= \ell_{\text{ext}}\left(\mathbf{A}, \dot{\mathbf{A}}, \hat{\boldsymbol{\Omega}}, \hat{\boldsymbol{\Lambda}}\right) \end{aligned}$$

We note that we have been able to reduce the number of variables in this equation from 6 down to 4.

Euler-Poincaré Reduction

We can now use Hamilton's Variational Principle to derive the equations for the Pseudo Rigid Body.

$$
\begin{aligned}
0 &= \delta \int_{t_1}^{t_2} \ell_{\text{ext}}(\mathbf{A}, \dot{\mathbf{A}}, \hat{\mathbf{\Omega}}, \hat{\mathbf{\Lambda}})\, dt \\
0 &= \delta \int_{t_1}^{t_2} -\left\langle \frac{d}{dt}\frac{\partial \ell}{\partial \dot{\mathbf{A}}} - \frac{\partial \ell}{\partial \mathbf{A}}, \delta \mathbf{A} \right\rangle - \left\langle \frac{d}{dt}\frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}} - \mathrm{ad}^*_{\hat{\mathbf{\Omega}}} \frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}}, \delta \hat{\mathbf{\Omega}} \right\rangle dt \\
&\qquad - \left\langle \frac{d}{dt}\frac{\partial \ell}{\partial \hat{\mathbf{\Lambda}}} + \mathrm{ad}^*_{\hat{\mathbf{\Lambda}}} \frac{\partial \ell}{\partial \hat{\mathbf{\Lambda}}}, \delta \hat{\mathbf{\Lambda}} \right\rangle dt
\end{aligned}
$$

We can then say our system for left-right-invariant Lagrangians is,

$$
\frac{d}{dt}\frac{\partial \ell}{\partial \dot{\mathbf{A}}} = \frac{\partial \ell}{\partial \mathbf{A}} \qquad \frac{d}{dt}\frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}} = \mathrm{ad}^*_{\hat{\mathbf{\Omega}}} \frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}} \qquad \frac{d}{dt}\frac{\partial \ell}{\partial \hat{\mathbf{\Lambda}}} = -\,\mathrm{ad}^*_{\hat{\mathbf{\Lambda}}} \frac{\partial \ell}{\partial \hat{\mathbf{\Lambda}}}
$$

Pseudo-Rigid Body Equations

We can now use our specific Lagrangian to produce a system of equations for the Psudo-Rigid body. The equations on the last slide hold no resemblance to the structure of the Lagrangian, only the variables we write it as. Hence we now use the Lagrangian in order to write our actual equations. Consider,

$$\ell_{\text{ext}}(\mathbf{A}, \dot{\mathbf{A}}, \hat{\mathbf{\Omega}}, \hat{\mathbf{\Lambda}}) = \frac{1}{2}\,\mathrm{Tr}\Big((\hat{\mathbf{\Omega}}\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\hat{\mathbf{\Lambda}})(\hat{\mathbf{\Omega}}\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\hat{\mathbf{\Lambda}})^T\Big)$$

Then we consider $\delta\ell$ and see when we only consider $\hat{\mathbf{\Omega}}$, we have $\delta\ell = \left\langle \frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}}\,, \delta\hat{\mathbf{\Omega}} \right\rangle$. After come calculations we reach,

$$\left.\frac{d}{dt}\right|_{t=s=0} \ell(\hat{\mathbf{\Omega}} + \varepsilon\delta\hat{\mathbf{\Omega}}) = \left\langle \mathbf{A}(\hat{\mathbf{\Omega}}\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\hat{\mathbf{\Lambda}})\,, \delta\hat{\mathbf{\Omega}} \right\rangle$$

Pseudo-Rigid Body Equations

We then work further and get the following,

$$
\begin{aligned}
\frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}} &= \hat{\mathbf{\Omega}}\mathbf{A}^2 + \mathbf{A}^2\hat{\mathbf{\Lambda}} \\
\frac{\partial \ell}{\partial \mathbf{A}} &= 2\hat{\mathbf{\Lambda}}\mathbf{A}\hat{\mathbf{\Lambda}} \\
\frac{\partial \ell}{\partial \dot{\mathbf{A}}} &= \dot{\mathbf{A}} \\
\frac{\partial \ell}{\partial \hat{\mathbf{\Lambda}}} &= \hat{\mathbf{\Lambda}}\mathbf{A}^2 + \mathbf{A}^2\hat{\mathbf{\Omega}}
\end{aligned}
$$

Then we can plug these into our equations above and get the explicit matrix form of these equations.

Noether Theory

We are often interested in conserved quantities of systems and Noether Theory presents us with these. We can find conserved quantities and Noether Theory for Euler-Poincaré equations. Here, interestingly we have two conserved quantities as we reduced two of the parameters.

$$K = \left\langle \mathrm{Ad}^*_{R^{-1}} \frac{\partial \ell}{\partial \hat{\boldsymbol{\Omega}}}, \eta \right\rangle \qquad K = \left\langle \mathrm{Ad}^*_{S^{-1}} \frac{\partial \ell}{\partial \hat{\boldsymbol{\Lambda}}}, \eta \right\rangle$$

We can manipulate these and now write them as follows,

$$\left\langle \mathrm{Ad}^*_{\mathbf{R}^{-1}} \frac{\partial \ell}{\partial \hat{\boldsymbol{\Omega}}}, \eta \right\rangle = \mathbf{A}^2 \left\langle \mathrm{Ad}^*_{\mathbf{R}^{-1}} \hat{\boldsymbol{\Omega}}, \eta \right\rangle + \mathbf{A}^2 \left\langle \mathrm{Ad}^*_{R^{-1}} \hat{\boldsymbol{\Lambda}}, \eta \right\rangle$$

and,

$$\left\langle \mathrm{Ad}^*_{\mathbf{S}^{-1}} \frac{\partial \ell}{\partial \hat{\boldsymbol{\Lambda}}}, \eta \right\rangle = \mathbf{A}^2 \left\langle \mathrm{Ad}^*_{\mathbf{S}^{-1}} \hat{\boldsymbol{\Lambda}}, \eta \right\rangle + \mathbf{A}^2 \left\langle \mathrm{Ad}^*_{S^{-1}} \hat{\boldsymbol{\Omega}}, \eta \right\rangle$$

Further Work

This just scratches the surface of the area of Geometric Mechanics. Here a few of the places you can take the area,

- Hamiltonian Formalisations as in Darryl D Holm, Marsden, and Ratiu 1998.

$$p = \frac{\partial L}{\partial \dot{q}} \qquad \frac{\partial q}{\partial t} = \frac{\partial H}{\partial p} q \frac{\partial p}{\partial t} = -\frac{\partial H}{\partial q}$$

- Different body to space maps. For example the swinging spring has a body to space map of $\mathbf{x}(t) = \mathbf{XR}(t)f(t)$ where $\mathbf{R}(t) \in \mathrm{SO}(3)$ and $f : \mathbb{R} \to \mathbb{R}$ where $f(0) = 1$. We have two groups here $\mathrm{SO}(3)$ and the dilation group.
- Discretisation. What if Hamilton's principle wasn't an integral, what if it was a sum? As in Brecht et al. 2021
- Geometric Mechanics applied to fluids. We can work with a Lagrangian dependent on time $L(\mathbf{x}, t)$ and develop variational principles (Ardakani et al. 2019; Alemi Ardakani 2019) and Noether Theorems for fluid problems, like the shallow water equations.

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Lie Groups and Algebras

Consider a group, $(G, \circ)$. We call G a Lie group if the following two functions are smooth,

$$m : G \times G \to G \qquad m(g_1, g_2) = g_1 \circ g_2 \qquad g_1, g_2 \in G$$

and

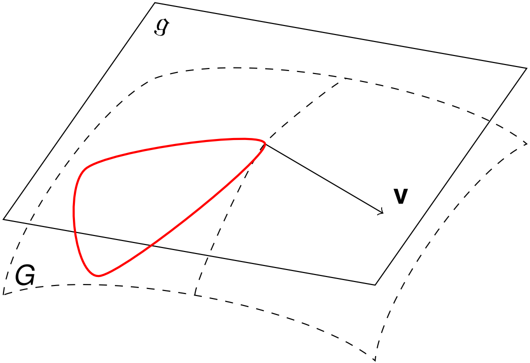
$$i : G \times G \to G \qquad i(g_1, g_2) = g_1^{-1} g_2 \qquad g_1, g_2 \in G$$

The mathematically interesting part is that Lie groups are manifolds. From this arises the ideal of a Lie Algebra. We can consider tangent spaces of Lie Groups and then we see that Lie Algebras are just vector space like objects.

Lie Groups and Algebras

For our purposes we consider matrix Lie Groups as many of our problems involve rotations (hence Geometric Mechanics). Lie Algebras are equipped with a Lie Bracket, which for us is just the matrix commutator,

$$[A, B] = AB - BA$$



Lagrangians, Hamilton's Variational Principle and Trace Pairings

We mostly work in Lagrangian systems, that is systems on a configuration space $\mathbb{R}^{dN}$ with ODEs of the form of the Euler-Legrange equations for some function $L : \mathbb{R}^{2dN} \to \mathbb{R}$ called the **Lagrangian**.

The most simplest example of a Lagrangian that we will use later is $L(R, \dot{R}) = K - V$ where K is kinetic energy and V is potential energy.

Let $S : [a, b] \to \mathbb{R}^{dN}$ and it to be a C^{∞} functional, then the **first variation** of $\mathcal{S}$ is,

$$\delta\mathcal{S} = \frac{d}{dt}\bigg|_{s=0} \mathcal{S}[\mathbf{q}(s, t)]$$

Lagrangians, Hamilton's Variational Principle and Trace Pairings

We consider a special matrix Lie group, the special orthogonal group, $\mathrm{SO}(3)$. This is where most of the nice mathematics that we do happens.

Consider $R \in \mathrm{SO}(3)$ and $\dot{R} \in \mathfrak{so}(3)$ (the Lie algebra of $\mathrm{SO}(3)$). We can extend the ideas of Lagrangians and first variations to Hamilton's Variational Principle,

$$\delta \int_{t_1}^{t_2} L(R, \dot{R})dt = 0$$

We will see that there are so called endpoint conditions, that take terms nicely to zero.

Before we get to the meaty part of the presentation, I introduce pairing, an inner product on our space. Let V be a vector space and $A \in V$ and $B \in V^*$. Then we define the **trace pairing**,

$$\langle B, A \rangle = \mathrm{Tr}(BA^T)$$