Complex Analysis Coursework 2

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Problem 1. *By using the Cauchy-Riemann equations, or otherwise, find a function f, holomorphic on $\mathbb{C}$, such that*

$$\text{Re}(f(x+iy)) = 2x^3 - 6xy^2 + 2xy$$

Solution 1. *We can say that $u(x, y) = 2x^3 - 6xy^2 + 2xy$ and hence we can differentiate and solve the PDE produced,*

$$\frac{\partial u}{\partial x} = 6x^2 - 6y^2 + 2y = \frac{\partial v}{\partial y}$$

and so,

$$\begin{aligned} v &= \int 6x^2 - 6y^2 + 2y, dy \\ &= 6yx^2 - 2y^3 + y^2 + f(x) \end{aligned}$$

and we can differentiate with respect to x,

$$\begin{aligned} \frac{\partial v}{\partial x} = 12xy + f'(x) &= -\frac{\partial u}{\partial y} \\ &= -(-12xy + 2x) \end{aligned}$$

Hence, $f'(x) = -2x$ and so $f(x) = C - x^2$. Now we can write his together as,

$$f(x+iy) = 2x^3 - 6xy^2 + 2xy + i(6yx^2 - 2y^3 + y^2 - x^2) + C \quad C \in \mathbb{C}$$

and hence by partial converse of the Cauchy Riemann equations, this function is holomorphic.

Problem 2. *Suppose that f is a function holomorphic at every point of the open disc*

$$D = \{z \in \mathbb{C} : |z| < 1\}$$

such that

$$\mathrm{Re}(f(z)) + \mathrm{Im}(f(z)) = 10$$

for all $z \in D$. Show that f is constant in D.

Solution 2. *As f is holomorphic on the open disc D, we can use the Cauchy Riemann equations to prove the required result. We can rewrite the constraint as we know that $f(x+iy) = u(x,y) + iv(x,y)$ and so,*

$$u + v = 10$$

by differentiating $u = 10 - v$, we can find that,

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial x} \tag{$*$}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial y} \tag{$**$}$$

and we can hence rewrite the Cauchy Riemann equations using $(*)$ and $(**)$,

$$\begin{aligned} \frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x} \\ &= -\frac{\partial v}{\partial x} \end{aligned} \qquad\qquad \begin{aligned} \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \\ &= \frac{\partial u}{\partial x} \end{aligned}$$

Hence we can say that $v_y = -v_x$ which then leads us to say that $v_x = v_y = 0$ and with $u_y = u_x$ we can say $u_x = u_y = 0$ and hence f is constant.

□

Problem 3. *Show that*

$$\left|\int_\gamma \frac{dz}{2+z^2}\right| \le \pi$$

where γ is the upper half of the unit circle.

Solution 3. *Firstly we say,*

$$\left|\int_\gamma \frac{dz}{2+z^2}\right| \le \int_\gamma \left|\frac{dz}{2+z^2}\right|$$

and as γ is the upperhalf of the unit circle we can say that,

$$\left|\frac{1}{2+z^2}\right| \le 1$$

where the maximum is at $z = i$. Now applying the ML-bound we can say,

$$\left|\int_\gamma \frac{dz}{2+z^2}\right| \le \pi$$

as $\ell(\gamma) = \pi$.

$\square$

Problem 4. *If* $\sum_{n=0}^{\infty} a_n z^n$ *has radius of convergence* R*, show that,*

$$\sum_{n=0}^{\infty} \mathrm{Re}(a_n) z^n$$

has radius of convergence greater than or equal to R*.*

Solution 4. *We define the radius of convergence as,*

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

and so we can look at the expression inside the limit,

$$\begin{aligned}
\left| \frac{a_{n+1}}{a_n} \right| &\geq \frac{\mathrm{Re}(a_{n+1})}{|a_n|} \\
&= \frac{|\overline{a_n}|\,\mathrm{Re}(a_{n+1})}{|\overline{a_n} a_n|} \\
&\geq \frac{\mathrm{Re}(\overline{a_n})\,\mathrm{Re}(a_{n+1})}{|\overline{a_n} a_n|} \\
&= \frac{\mathrm{Re}(\overline{a_n})\,\mathrm{Re}(a_{n+1})}{\mathrm{Re}(\overline{a_n} a_n)} \\
&= \frac{\mathrm{Re}(\overline{a_n})\,\mathrm{Re}(a_{n+1})}{\mathrm{Re}(\overline{a_n})\,\mathrm{Re}(a_n)} \\
&= \frac{\mathrm{Re}(a_{n+1})}{\mathrm{Re}(a_n)}
\end{aligned}$$

and so

$$\left| \frac{a_{n+1}}{a_n} \right| \geq \frac{\mathrm{Re}(a_{n+1})}{\mathrm{Re}(a_n)}$$

and now taking limits,

$$\begin{aligned}
\left| \frac{a_{n+1}}{a_n} \right| &\geq \frac{\mathrm{Re}(a_{n+1})}{\mathrm{Re}(a_n)} \\
\left| \frac{a_n}{a_{n+1}} \right| &\leq \frac{\mathrm{Re}(a_n)}{\mathrm{Re}(a_{n+1})} \\
\lim_{n\to\infty} \left| \frac{a_n}{a_n + 1} \right| &\leq \lim_{n\to\infty} \frac{\mathrm{Re}(a_n)}{\mathrm{Re}(a_{n+1})} \\
R &\leq \lim_{n\to\infty} \frac{\mathrm{Re}(a_n)}{\mathrm{Re}(a_{n+1})}
\end{aligned}$$

and so the radius of convergence of the real part of a series is greater than the radius of convergence of the series.

$\square$