Differential Equations Week 2 - Linear Higher Order ODEs

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1 Overview

A higher order ODE and its nth derivatives, form an ODE.

We know that the solution is nth times differentiable

Theorem 1.1: Fundemental Theorem for homogenous linear ODEs

The sum and constant multiples of solutions are again solutions of the ODE

The general solution is $y(x) = c_1y_1 + \cdots + c_ny_n$, where $\{c_1, \ldots, c_n\}$ are arbitrary constants and $\{y_1, \ldots, y_n\}$ is a basis or fundemental system of solution.

The n solutions are linearly independant when $k_1y_1 + \cdots + k_ny_n = 0$ implies that $\{k_1, \ldots, k_n\}$ are zero.

1.1 Initial Value Problem

An IVP consists of the ODE,

$$y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1y' + p_0y = 0$$

and n initial conditions with a given x_0 and several constants.

Theorem 1.2: Uniqueness of HODEs

If the coefficients are continuous on some open interval, I, and $x_0 \in I$, then the IVP has a unique solutions $y(x)$ on I.

The Wronskian of n solutions:

$$W = \begin{vmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{vmatrix}$$

has to be non-zero for linear independence.

Theorem 1.3: Existence

If the coefficients are continuous on some open interval, I, then the ODE has a general solution on I

Theorem 1.4: General Solution

If the ODE has continuous coefficients on some open interval I, then every solution $y = Y(x)$ is of the form $Y(x) = C_1y_1 + \cdots + C_ny_n$, where the '$y$'s are the basis and '$C$'s are the contants.

2 Homogenous ODEs with constant coefficients

The ODE can be solved rather easily by subbing in $y = e^{\lambda x}$ and getting a characteristic equation. Hence solve and follow usual procedure.

2.1 Real Roots

Case 1: Distinct Roots, then the solution is; $y_1 = e^{\lambda_1 x}, \ldots, y_n = e^{\lambda_n x}$. Then linearly combine and get general solution. Now plug into the Wronskian and pull out the exponential terms:

$$\begin{aligned} W &= \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} & \ldots & e^{\lambda_n x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} & \ldots & \lambda_n e^{\lambda_n x} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{(n-1)} e^{\lambda_1 x} & \lambda_2^{(n-1)} e^{\lambda_2 x} & \ldots & \lambda_n^{(n-1)} e^{\lambda_n x} \end{vmatrix} \\ &= E \begin{vmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{(n-1)} & \lambda_2^{(n-1)} & \ldots & \lambda_n^{(n-1)} \end{vmatrix} \end{aligned}$$

where $E = e^{\lambda_1 + \cdots + \lambda_n}$ and hence $W = 0$, iff the determinant on the right is zero. This is known as the vandermonde or cauchy determinant.

Theorem 2.1: Linearly Independence Theorem

Any number of solutions of the form $y = e^{\lambda x}$ are linearly independent on an open interval, I, iff the corresponding roots, λ, are all different.

Case III: Multiple Real Roots, For $\lambda_1 = \lambda_2$ the solutions are $y_1, y_2 = xy_1$ which are linearly independant. More generally, for real roots of order m, the linearly independant solutions are:

$$e^{\lambda x}, xe^{\lambda x}, x^2 e^{\lambda x}, \dots, x^{m-1} e^{\lambda x}$$

2.2 Complex Roots

Case 2: Simple Complex Roots For complex roots, they must be conjugate pairs, since the coefficients of the ODE are real. So for a $\lambda = \gamma \pm i\omega$, then the two linearly independant solutions are:

$$y_1 = e^{\gamma x} \cos \omega x \qquad y_2 = e^{\gamma x} \sin \omega x$$

Case 4: Multiple Complex Roots, let us have complex double roots, $\lambda = \gamma \pm i\omega$. The coresponding linearly independant solutions are: $e^{\gamma x} \sin \omega x, e^{\gamma x} \cos \omega x, xe^{\gamma x} \sin \omega x, xe^{\gamma x} \cos \omega x$. Therefore, we gain the general solution:

$$y = e^{\gamma x} \left[(A_1 + A_2 x) \cos \omega x + (B_1 + B_2 x) \sin \omega x \right]$$

For triple roots, just add x^2 terms.

3 Non-homogenous Linear ODEs

This is very similar to the second order, the proess is basically the same. We can solve via 'Method of Undetermined Coefficients' or 'Method of variation of paramaters'.

3.1 Method of Undetermined Coefficients

The basic rule is to know off, $r(x)$ and then solve. Next take a general particular integral and then follow usual solution. Then you can multiply by x^k where k is the number of y_p's in the solution plus one. Then if you have a summation of many particular integrals, just break it up.

3.2 Variation of Parameters

Particular solutions of the nonhomogenous equations:

$$\begin{aligned} y_p(x) &= \sum_{k=1}^{n} y_k(x) \int \frac{W_k}{W(x)} r(x) dx \\ &= y_1(x) \int \frac{W_1}{W(x)} r(x) dx + \dots \\ &\quad + y_n(x) \int \frac{W_n}{W(x)} r(x) dx \end{aligned}$$

The wronskians $W_j (j = 1, \dots, n)$ are obtained from W by replacing the j^{th} column by $\begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix}^T$, where $\{y_1, \dots, y_n\}$ are a basis.

Therefore, Wronskians $W_j (j = 1, \dots, n)$ for each component are:

$$W_1 = \begin{vmatrix} 0 & e^{\lambda_2 x} & \dots & e^{\lambda_n x} \\ 0 & \lambda_2 e^{\lambda_2 x} & \dots & \lambda_n e^{\lambda_n x} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_2^{(n-1)} e^{\lambda_2 x} & \dots & \lambda_n^{(n-1)} e^{\lambda_n x} \end{vmatrix}$$

all the way to

$$W_n = \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} & \dots & 0 \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{(n-1)} e^{\lambda_1 x} & \lambda_2^{(n-1)} e^{\lambda_2 x} & \dots & 1 \end{vmatrix}$$

This method should be used for ODEs greater than 2.