Year 2 — Vector Calculus

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine (especially the typos!).

Contents

1 Lecture 1: Basic Definitions

1.1 Suffix Notation

Let there be a vector $\underline{\mathbf{c}} = \underline{\mathbf{a}} + \underline{\mathbf{b}}$, where $\underline{\mathbf{a}} = a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}$ and $\underline{\mathbf{b}} = b_1\hat{\boldsymbol{\imath}} + b_2\hat{\boldsymbol{\jmath}} + b_3\hat{\boldsymbol{k}}$. Then $\underline{\mathbf{c}}$ is equivalent to:

$$c_i = a_i + b_i$$

In suffix notation:

$$c_j = a_j + b_j \qquad j = 1, 2, 3$$

The inner product of two vectors:

$$\begin{aligned} a \cdot b &= a_1b_1 + a_2b_2 + a_3b_3 \\ &= \sum_{j=1}^{3} a_j b_j \end{aligned}$$

For a vector $\underline{\mathbf{a}} = a_i$, i is a free index. For the dot product above: $\sum_{j=1}^{3} a_j b_j$, j is a dummy suffix.

For suffix notation, an index cannot be repeated more than two times in an equation.

Example 1. Write $(a \cdot b)(c \cdot d)$ in suffix notation

Solution. Here we take that:

$$a \cdot b = a_j b_j \quad j = 1, 2, 3$$

and that

$$c \cdot d = c_i d_i \quad i = 1, 2, 3$$

Now we can say that

$$(a \cdot b)(c \cdot d) = a_j b_j c_i d_i \quad i, j = 1, 2, 3$$

Example 2. Write $a_j b_i c_j$ in normal vector notation

Solution. We know that

$$a_j b_i c_j = a_j c_j b_i$$

Which is:

$$(a \cdot c)b$$

Example 3. Write the vector notation $\underline{\mathbf{u}} + (\underline{\mathbf{a}} \cdot \underline{\mathbf{b}})\underline{\mathbf{v}} = |\underline{\mathbf{a}}|^2(\underline{\mathbf{b}} \cdot v)\underline{\mathbf{a}}$ in suffix notation

Solution. We know that

$$a_j b_i c_j = a_j c_j b_i$$

Which is:

$$(a \cdot c)b$$

Example 4. Write the vector notation $\underline{\mathbf{u}} + (\underline{\mathbf{a}} \cdot \underline{\mathbf{b}})\underline{\mathbf{v}} = |\underline{\mathbf{a}}|^2(\underline{\mathbf{b}} \cdot v)\underline{\mathbf{a}}$ in suffix notation

Solution. Firstly:

$$[\underline{\mathbf{u}} + (\underline{\mathbf{a}} \cdot \underline{\mathbf{b}})\underline{\mathbf{v}}]_i = \left[|\underline{\mathbf{a}}|^2(\underline{\mathbf{b}} \cdot v)\underline{\mathbf{a}}\right]_i$$

Then,

$$u_i + (a_j b_j)v_i = a_j a_j b_l v_l a_i \qquad j, l = 1, 2, 3$$

1.2 The Kronecker Delta $\delta_{i,j}$

The function is defined:

$$\delta_{i,j} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

The suffixes i and j can each take the values $1, 2, 3$ so $\delta_{i,j}$ has nine elements.
We can write the function as the identity matrix:

$$\delta_{i,j} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\delta_{i,j}$ is called a substitution tensor, since it's effect when multiplied by a_j is to replace j with i.

$$\begin{aligned} \delta_{i,j} a_j &= \sum_{j=1}^{3} \delta_{i,j} a_j \\ &= \delta_{i1} a_1 + \delta_{i2} a_2 + \delta_{i3} a_3 \\ &= \delta_{11} a_1 + \delta_{12} a_2 + \delta_{13} a_3 \\ &\quad + \delta_{21} a_1 + \delta_{22} a_2 + \delta_{23} a_3 \\ &\quad + \delta_{31} a_1 + \delta_{32} a_2 + \delta_{33} a_3 \\ &= a_1 + a_2 + a_3 \end{aligned}$$

From this we can say: $\delta_{i,j} a_i = a_j$ and $\delta_{i,j} a_j = a_i$

Example 5. $\delta_{i,j}$ and dot product

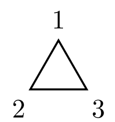
Solution.

$$\begin{aligned} a \cdot b &= a_i b_i \quad i = 1, 2, 3 \\ &= \delta_{i,j} a_j b_i \\ &= a_j \delta_{i,j} b_i \\ &= a_j b_j \end{aligned}$$

1.3 The Alternating Tensor, $\varepsilon_{i,j,k}$

$\varepsilon_{i,j,k}$ is useful for manipulating expressions involving the cross product of two vectors and curl of a vector.

$$\varepsilon_{i,j,k} = \begin{cases} +1 & \text{if } (i,j,k) = (1,2,3),\ (2,3,1) \text{ or } (3,1,2) \\ -1 & \text{if } (i,j,k) = (3,2,1),\ (2,1,3) \text{ or } (1,3,2) \\ 0 & \text{if any of } i,j,k \text{ are equal} \end{cases}$$



The +1 case can be also written as 1, 2 or 3 are in clockwise order. So if you take a triangle and then go clockwise around it from the first element, that the order they are in. The −1 are in anticlockwise order. Hence meaning the opposite of clockwise.

The six non-zero elements of ε_{ijk}:

$$\begin{aligned} \varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1 \\ \varepsilon_{321} = \varepsilon_{213} = \varepsilon_{132} = -1 \\ \varepsilon_{ijk} = 0, \text{ otherwise} \end{aligned}$$

We can take that; $\varepsilon_{ijk} = \varepsilon_{jki}$ as they are in clockwise order. This also imples $\varepsilon_{ijk} = -\varepsilon_{jik}$ because if ijk are in clockwise order then jik must be in counterclockwise order.

1.4 $\varepsilon_{i,j,k}$ and cross product

Let $\underline{\mathbf{a}} = a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}$ and $\underline{\mathbf{b}} = b_1\hat{\boldsymbol{\imath}} + b_2\hat{\boldsymbol{\jmath}} + b_3\hat{\boldsymbol{k}}$. Then their cross product is:

$$\underline{\mathbf{a}} \times \underline{\mathbf{b}} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

and in suffix notation, we can write the above as; $(\underline{\mathbf{a}} \times \underline{\mathbf{b}})_i = \varepsilon_{ijk}\, a_j b_k$ where j, k are dummy suffixes and must be summed over 1 to 3.

1.5 ε_{ijk} and the scalar triple product

We can take the scalar triple product, $\underline{\mathbf{a}} \cdot \underline{\mathbf{b}} \times \underline{\mathbf{c}}$, then we can do the following:

$$\begin{aligned} \underline{\mathbf{a}} \cdot \underline{\mathbf{b}} \times \underline{\mathbf{c}} &= a_i(\underline{\mathbf{b}} \times \underline{\mathbf{c}})_i \\ &= a_i\varepsilon_{ijk}b_jc_k \\ &= \varepsilon_{ijk}a_ib_jc_k \\ &= c_k\varepsilon_{ijk}a_ib_j \end{aligned}$$

from the above we show that $\underline{\mathbf{a}} \cdot \underline{\mathbf{b}} \times \underline{\mathbf{c}} = \underline{\mathbf{c}} \cdot \underline{\mathbf{a}} \times \underline{\mathbf{b}}$. We can expand $\varepsilon_{ijk}\, a_i\, b_j\, c_k$ to get:

$$\begin{aligned} &= \varepsilon_{123}a_1b_2c_3 + \varepsilon_{231}a_2b_3c_1 + \varepsilon_{312}a_3b_1c_2 \\ &\quad + \varepsilon_{321}a_3b_2c_1 + \varepsilon_{213}a_2b_1c_3 + \varepsilon_{132}a_1b_3c_2 \\ &= a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2 \end{aligned}$$

which is the expanded form of the triple scalar product.

1.6 A relation between ε_{ijk} and $\delta_{i,j}$

We are going to prove the following statement:

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Since all of the coordinate axis are the same, just consider $i = 1$:

If then $j = 1$, we get that $\varepsilon_{11k} = 0$ and so LHS $= 0$. Then considering the RHS, we get that $\delta_{1l}\delta_{1m} - \delta_{1m}\delta_{1l} = 0$, so equation holds.

If $j = 2$, then $\varepsilon_{ijk} = \varepsilon_{12k} = 0$, unless $k = 3$, so then only $k = 3$ contributes to the sum. So $\varepsilon_{klm} = \varepsilon_{3lm}$, so zero unless l and m are 1 and 2. So we can conclude that $\varepsilon_{ijk}\varepsilon_{klm} = \varepsilon_{123}\varepsilon_{312}$ or $\varepsilon_{123}\varepsilon_{321}$, so the LHS is either ± 1. Looking at RHS, we have either: $\delta_{11}\delta_{22} - \delta_{12}\delta_{21}$ or $\delta_{12}\delta_{21} - \delta_{11}\delta_{22}$. This gives ± 1 in the same perumtation as the LHS. So equation holds.

2 Gradient, Divergence and Curl

2.1 Gradient

Assume we have a $f = f(x, y, z)$ or $f = f(x_1, x_2, x_3)$, so a scalar calued function. Then we define grad f as:

$$\underline{\nabla}\, f = \left(\frac{\partial}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial}{\partial z}\hat{\boldsymbol{k}}\right) f$$

We say grad of f is a differential operator. So:

$$\underline{\nabla}\, f = \left(\frac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{k}}\right)$$

and we can write it in suffix notation aswell:

$$[\underline{\nabla}\, f]_i = \frac{\partial}{\partial x_i} \qquad i = 1, 2, 3$$

2.2 Divergence

Assume we have a vector field, $\underline{\mathbf{u}} = \underline{\mathbf{u}}(x, y, z, t)$. We define the divergence of this vector field as;

$$\underline{\nabla} \cdot \underline{\mathbf{u}} = \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)$$

Placing this in suffix notation, we get that:

$$[\underline{\nabla} \cdot \underline{\mathbf{u}}]_j = \frac{\partial u_j}{\partial x_j}$$

2.3 Curl

the curl of a vector field can be written as:

$$\underline{\nabla} \times \underline{\mathbf{u}}$$

To write this in suffix notation, we can just use the cross produce formula:

$$[\underline{\nabla} \times \underline{\mathbf{u}}]_i = \varepsilon_{ijk}\underline{\nabla}_j u_k$$

which then can be manipulated into:

$$[\underline{\nabla} \times \underline{\mathbf{u}}]_i = \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j} \qquad j, k = 1, 2, 3$$

where i is a free index and j, k are dummy suffixes, so $j, k = 1, 2, 3$

3 Combinations of gradient, divergence and curl

3.1 Divergence of Gradient

If we take $\underline{\nabla} \cdot \underline{\nabla} f$ where $f = (x_1, x_2, x_3, t)$. We can write the div of grad as:

$$\begin{aligned}
\underline{\nabla} \cdot \underline{\nabla} f &= \left(\frac{\partial}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial}{\partial z}\hat{\boldsymbol{k}}\right) \cdot \left(\frac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{k}}\right) \\
&= \frac{\partial}{\partial x_1}\frac{\partial f}{\partial x_1} + \frac{\partial}{\partial x_2}\frac{\partial f}{\partial x_2} + \frac{\partial}{\partial x_3}\frac{\partial f}{\partial x_3} \\
&= \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} \\
&= \Delta f
\end{aligned}$$

Where the $\Delta = \underline{\nabla}^2$ is the laplacian. So how do we write this in suffix notation?

$$\begin{aligned}
\underline{\nabla} \cdot \underline{\nabla} f &= \underline{\nabla}_j [\underline{\nabla} f]_j \\
&= \frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_j} \\
&= \frac{\partial^2 f}{\partial x_j}
\end{aligned}$$

3.2 Curl of Gradient

We can write the curl of gradient as:

$$\begin{aligned}
[\underline{\nabla} \times \underline{\nabla} f]_i &= \varepsilon_{ijk} \underline{\nabla}_j \underline{\nabla} f_k \\
&= \varepsilon_{ijk} \frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_k} \\
&= \varepsilon_{ikj} \frac{\partial}{\partial x_k}\frac{\partial f}{\partial x_j} \\
&= -\varepsilon_{ijk} \frac{\partial}{\partial x_k}\frac{\partial f}{\partial x_j} \\
&= -\varepsilon_{ijk} \frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_k} && \text{if } f \in c^2 \\
\implies \underline{\nabla} \times \underline{\nabla} f &= 0
\end{aligned}$$

3.3 Gradient of Divergence

Assume we have a $\underline{\mathbf{u}}$, vector field, and we want $\underline{\nabla} f \underline{\nabla} \cdot$.

$$\begin{aligned}
[\underline{\nabla} f \underline{\nabla} \cdot]_i &= \underline{\nabla}_i \frac{\partial u_j}{\partial x_j} \\
&= \frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j} \\
&= \frac{\partial^2 u_j}{\partial x_i \partial x_j}
\end{aligned}$$

3.4 Divergence of Curl

We can write divergence of curl as:

$$\begin{aligned}
[\underline{\nabla} \cdot \underline{\nabla} \times \underline{\mathbf{u}}]_i &= \frac{\partial}{\partial x_i} [\underline{\nabla} \times \underline{\mathbf{u}}]_i \\
&= \frac{\partial}{\partial x_i} \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} \\
i, j, k = 1, 2, 3, \text{ so } i \leftrightarrow j & \\
&= \frac{\partial}{\partial x_j} \varepsilon_{jik} \frac{\partial u_k}{\partial x_i} \\
&= -\varepsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial u_k}{\partial x_i} \\
&= -\varepsilon_{ijk} \frac{\partial}{\partial x_i} \frac{\partial u_k}{\partial x_j} && \text{as } \underline{\mathbf{u}} \in c^2
\end{aligned}$$

As $\underline{\nabla} \cdot (\underline{\nabla} \times \underline{\mathbf{u}}) = -\underline{\nabla} \cdot (\underline{\nabla} \times \underline{\mathbf{u}})$, then we know that $\underline{\nabla} \cdot (\underline{\nabla} \times \underline{\mathbf{u}}) = 0$

3.5 Curl of Curl

We can write curl of curl, $\underline{\nabla} \times (\underline{\nabla} \times \underline{\mathbf{u}})$, as:

$$\begin{aligned}
[\underline{\nabla} \times (\underline{\nabla} \times \underline{\mathbf{u}})]_i &= \varepsilon_{ijk} \frac{\partial}{\partial x_j} (\underline{\nabla} \times \underline{\mathbf{u}})_k \\
&= \varepsilon_{ijk} \frac{\partial}{\partial x_j} \varepsilon_{klm} \frac{\partial u_m}{\partial x_l} \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \frac{\partial^2 u_m}{\partial x_j \partial x_l} \\
&= \delta_{il}\delta_{jm} \frac{\partial^2 u_m}{\partial x_j \partial x_l} - \delta_{im}\delta_{jl} \frac{\partial^2 u_m}{\partial x_j \partial x_l} \\
&= \frac{\partial^2 u_j}{\partial x_j \partial x_i} - \frac{\partial^2 u_i}{\partial x_j \partial x_j} \\
&= \frac{\partial}{\partial x_i} \frac{\partial u_j}{\partial x_j} - \frac{\partial^2 u_i}{\partial x_j^2} \\
&= [\underline{\nabla}(\underline{\nabla} \cdot \underline{\mathbf{u}})]_i - [\Delta \underline{\mathbf{u}}]_i \\
&= [\underline{\nabla}(\underline{\nabla} \cdot \underline{\mathbf{u}}) - \underline{\nabla}^2 \underline{\mathbf{u}}]_i
\end{aligned}$$

4 Scalar Field / Vector Fields Defintions

A scalar or vector quantity is said to be a field if it is a function of position. Examples

(i) Temperature is a scalar field, $T = T(x, y, z) = T(\underline{\mathbf{r}})$

(ii) Pressure and Density are also scalr fields $P = P(\underline{\mathbf{r}})$ and $\rho = \rho(\underline{\mathbf{r}})$

(iii) if a physical quantity is a scalar we speak of a scalar field or function of position.

If a physical quantity is a vector, such as force $\underline{\mathbf{F}} = \underline{\mathbf{F}}(x, y, z)$. We speak of a vector field or vector function.

A vector-valued function is an $f : A \subset \mathbb{R}^n \mapsto \mathbb{R}^m$. So, for each $\underline{\mathbf{x}} = (x_1, \ldots, x_n) \in A$, f assigns a value $f(\underline{\mathbf{x}})$, an m-tuple, in $\mathbb{R}^m$. These functions, f, are called vector-valued functions if $m > 1$ and scalar if $m = 1$.

Example 6. Take the function, $f : (x, y, z) \mapsto (x^2 + y^2 + z^2)^{\frac{3}{2}}$

Solution. It's a scalar function from $\mathbb{R}^3$ to $\mathbb{R}$.

Example 7. Take the function $g : (x_1, x_2, x_3) \mapsto (x_1x_2x_3, \sqrt{x_1x_3})$

Solution. This is a vector valued function from $\mathbb{R}^3$ to $\mathbb{R}^2$

To specify a temperature T in a region A of space requires a function T, $T : A \subset \mathbb{R}^m \mapsto \mathbb{R}$. $T = T(x, y, z)$.

To specify the velocity of a fluid moving in space requires a map, $\underline{\mathbf{v}} : \mathbb{R}^4 \mapsto \mathbb{R}^3$ where $\underline{\mathbf{v}}(x, y, z, t)$ is the velocity of the fluid at (x, y, z) at time t.

When $f : U \subset \mathbb{R}^n \mapsto \mathbb{R}$, we say that f is a real valued function of n-variables with domain U.

Let $f : U : \mathbb{R}^n \mapsto \mathbb{R}$, then graph $f = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^{n+1} : (x_1, \ldots, x^n)\}$ If $n = 1$, then we can conclude that graph f is curve in $\mathbb{R}^2$ and if $n = 2$, then graph f is a surface in $\mathbb{R}^3$.

4.1 Level Sets, Curves and Surfaces

A level set is a subset of $\mathbb{R}^3$ on which f is constant. For example, for $f(x, y, z) = x^2 + y^2 + z^2$, the set where $x^2 + y^2 + z^2 = 1$ is alevel set. A level set is a set of $(x, y, z) : f(x, y, z) = c$ where $c \in \mathbb{R}$.

For functions $f(x, y)$, we speak of level curves or contours. example, $f : \mathbb{R}^2 \mapsto \mathbb{R}$, $f(x, y) = x + y + 2$, has as it's graph the inclined plane $z = x + y + 2$. The plane intersects the xy plan where $z = 0$ in the line $y = -x - 2$ and the z-axis at $(0, 0, 2)$. For any $c \in \mathbb{R}$, the level curve of c is the straight line: $y = -x + (c - 2) : L_c\{(x, y) : y = -x + c - 2\} \subset \mathbb{R}^2$

5 Differentiating Scalar Fields

Definition 5.1: Partial Differentiation

Let $U \subset \mathbb{R}^n$ be an open set. The $\dfrac{\partial f}{\partial x_1}, \ldots, \dfrac{\partial^n f}{\partial x_n}$ partial derivatives of $f(x_1, \ldots, x_n)$ which at point $\underline{x}$ are defined by:

$$\frac{\partial f}{\partial x_j} =$$

$$\lim_{h\to 0} \frac{f(x_1, \ldots, x_j + h, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}$$

where the limit exists for j from 1 to n.

Example 8. If $f(x, y) = x^2y + y^3$, then find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$

Solution. We can simply work out that:

$$\frac{\partial f}{\partial x} = 2xy$$
$$\frac{\partial f}{\partial y} = x^2 + 3y^2$$

To say that a partial derivative shall be evaluated at a point (x_0, y_0), we write; $\left.\dfrac{\partial f}{\partial x}\right|_{(x_0,y_0)}$

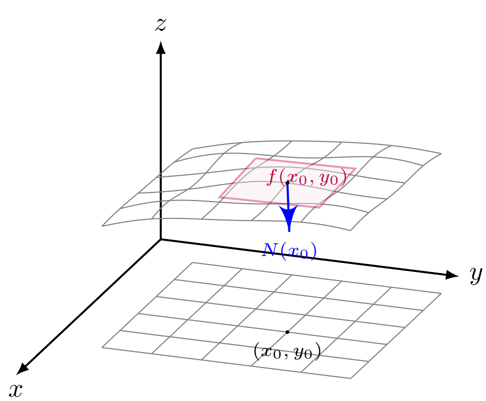
5.1 Equations of Tangent planes

Definition 5.2: Tangent Plane

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differeniable at (x_0, y_0), the plane described by:

$$z_p = f(x_0, y_0) + \left.\frac{\partial f}{\partial x_0}\right|(x - x_0) + \left.\frac{\partial f}{\partial y}\right|(y - y_0)$$

is called the tangent plane of f at (x_0, y_0).



Definition 5.3

Let f be a function $f : \mathbb{R}^2 \to \mathbb{R}$ we say that f is differentiable at (x_0, y_0), if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exists at (x_0, y_0) and if

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - z_p}{\|(x,y) - (x_0,y_0)\|}$$

then z_p is a good approximation of f.

5.2 Gradient of a scalar field

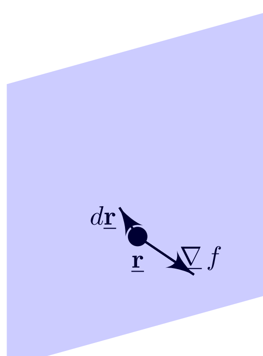
Definition 5.4

The gradient of a scalar field is a vector field with a direction that is perpendicular to the level surface and pointing in the direction of increasing f, with a magnitude equal to the rate of change of f in this direction.

$$\underline{\nabla} f = \frac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{k}}$$

Consider an infitesimal change in the position in space from $\underline{\mathbf{r}}$ to $d\underline{\mathbf{r}}$. This results in a small change in the value of f, from f to $f + df$.

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \\ &= \underline{\nabla} f \cdot d\underline{\mathbf{r}} \end{aligned}$$



Suppose that $d\underline{\mathbf{r}}$ lies in the level surface $f = C$, then $d\underline{\mathbf{f}} = \underline{\nabla} f \cdot d\underline{\mathbf{r}} = 0$ so $\underline{\nabla} f$ and $d\underline{\mathbf{r}}$ are perpendicular. To show that $\underline{\nabla} f$ has the required magnitude, let $d\underline{\mathbf{r}} = \underline{\hat{\mathbf{n}}}ds$, where $\underline{\hat{\mathbf{n}}}$ is normal to the surface and s is a distance measured along the normal.

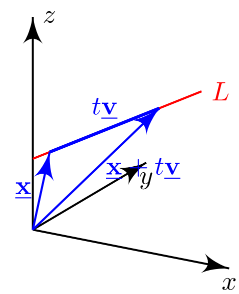
$$\begin{aligned} df &= \underline{\nabla} f \cdot d\underline{\mathbf{r}} \\ &= \underline{\nabla} f \cdot \underline{\hat{\mathbf{n}}}ds \\ &= |\underline{\nabla} f|ds \end{aligned}$$

So we know that $\underline{\nabla} f \parallel ds \implies \dfrac{df}{ds} = |\underline{\nabla} f|$.

Example 9. Let $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, the euclidean norm.

Solution. Then we know that $\underline{\nabla} f = (\frac{x}{r}, \frac{y}{r} + \frac{z}{r}) = \frac{\underline{\mathbf{r}}}{r}$, where $r = \sqrt{x^2 + y^2 + z^2}$ and $\underline{\mathbf{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$

6 Directional Derivative



Suppose $f : \mathbb{R}^3 \to \mathbb{R}$, let $\underline{\mathbf{v}}, \underline{\mathbf{x}} \subset \mathbb{R}^3$ be fixed vectors. Consider the function from $\mathbb{R} \to \mathbb{R}$ defined as:

$$t \mapsto f(\underline{\mathbf{x}} + t\underline{\mathbf{v}}) \tag{†}$$

The set of points of the form $\underline{\mathbf{x}} + t\underline{\mathbf{v}}$, $t \in \mathbb{R}$ is the line L through which the point $\underline{\mathbf{x}}$ is parallel to $\underline{\mathbf{v}}$. (†) is a function, f, restricted to L.

Definition 6.1: Directional Derivative

If $f : \mathbb{R}^3 \to \mathbb{R}$, the directional derivative of f at $\underline{\mathbf{x}}$ along a vector $\underline{\mathbf{v}}$ is given by:

$$\frac{d}{dt}\bigg|_{t=0} f(\underline{\mathbf{x}} + t\underline{\mathbf{v}})$$

if it exists.

Note that we usually choose $\underline{\mathbf{v}}$ to be of length unity.

Theorem 6.1

If $f : \mathbb{R}^3 \to \mathbb{R}$ and differentiable, then all directional derivatives exist. The directional derivative at $\underline{\mathbf{x}}$ in direction $\underline{\mathbf{v}}$ is given by:

$$\frac{d}{dt}\bigg|_{t=0} f(\underline{\mathbf{x}} + t\underline{\mathbf{v}}) = \underline{\nabla} f(\underline{\mathbf{x}}) \cdot \underline{\mathbf{v}}$$

Proof. Let $\underline{\mathbf{c}}(t) = \underline{\mathbf{x}} + t\underline{\mathbf{v}}$, $f(\underline{\mathbf{x}} + t\underline{\mathbf{v}}) = f(\underline{\mathbf{c}}(t))$ and

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0} f(\underline{\mathbf{c}}(t)) &= \underline{\nabla} f(\underline{\mathbf{c}}(t)) \cdot \underline{\mathbf{c}}'(t) \\ &= \underline{\nabla} f \underline{\mathbf{c}}(0) \cdot \underline{\mathbf{c}}'(0) \\ &= \underline{\nabla} f(\underline{\mathbf{x}}) \cdot \underline{\mathbf{v}}\end{aligned}$$

□

Theorem 6.2

Assume that $\underline{\nabla} f \neq 0$. Then $\underline{\nabla} f(x)$ points in the direction along which f is increasing fastest

Proof. If $\underline{\hat{\mathbf{n}}}$ is a unit vector, the rate of change of f in the direction $\underline{\hat{\mathbf{n}}}$ is given by:

$$\underline{\nabla} f \cdot \underline{\hat{\mathbf{n}}} = |\underline{\nabla} f||\underline{\hat{\mathbf{n}}}| \cos\vartheta = |\underline{\nabla} f| \cos\vartheta$$

where ϑ is the angle between $\underline{\hat{\mathbf{n}}}$ and $\underline{\nabla} f$. This maximum is when $\vartheta = 0$, so $\underline{\hat{\mathbf{n}}}$ and $\underline{\nabla} f$ are parallel. If we wish to move in the direction in which f decreases the fastest, we should proceed in the direction, $-\underline{\nabla} f$. □

Example 10. Find the unique normal to $x^2 + y^2 - z = 0$ at $(1, 1, 2)$

Solution. We say that $f(x, y, z) = x^2 + y^2 - z = 0$, and that $\underline{\nabla} f$ is normal as f is a level surface. So:

$$\underline{\nabla} f = 2x\hat{\boldsymbol{\imath}} + 2y\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}$$

and we can work out $\underline{\hat{\mathbf{n}}}$ as:

$$\underline{\hat{\mathbf{n}}} = \frac{(2x, 2y, -1)}{\sqrt{1 + 4(x^2 + y^2)}}\Bigg|_{(1,1,2)}$$

and so $\underline{\hat{\mathbf{n}}} = \frac{1}{3}(2, 2, -1)$

6.1 Properties of Gradient

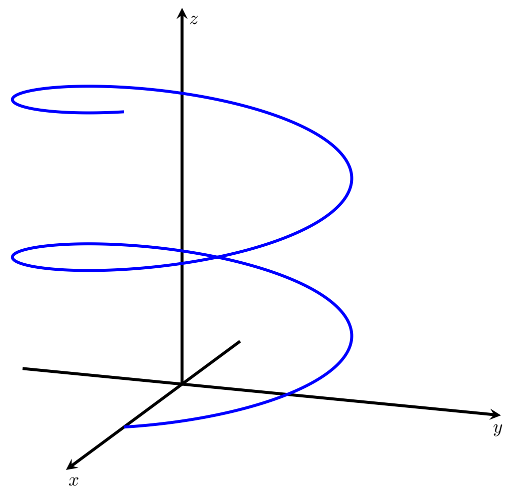
For any scalar functions of $f(x, y, z)$ and $g(x, y, z)$ and any $c \in \mathbb{R}$, we have:

$$\begin{aligned}
\underline{\nabla}(f + g) &= \underline{\nabla} f + \underline{\nabla} g \\
\underline{\nabla}(cf) &= c\underline{\nabla} f \\
\underline{\nabla}(fg) &= f\underline{\nabla} g + g\underline{\nabla} f \\
\underline{\nabla}(f \circ g) &= f'(g(x))\underline{\nabla} g
\end{aligned}$$

7 Parameterised Curves

We consider smooth curves in $\mathbb{R}^3$ specified in terms of rectangular cartesian coordinates (x, y, z). Such curves are generated by three smooth functions of a single parameter, t.

Example 11. A good example is a circular helix, $r(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}} + z(t)\hat{\boldsymbol{k}}$, where: $x(t) = a\cos t$, $y(t) = b\sin t$ and $z(t) = ct$



We can calculate the length of a path using an integral. Take a function that parameterised with three variables, $x(t), y(t), z(t)$ and between two points, $t_0 \leq t \leq t_1$, we can find the length, L:

$$L(\underline{\mathbf{r}}) = \int_{t_0}^{t_1} \sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)}$$

We could also parameterise a curve using an arc length parameter, s, where differential of arc-length satisfy the equation:

$$\begin{aligned} ds^2 &= dr \cdot dr \\ &= dx^2 + dy^2 + dz^2 \end{aligned}$$

We call ds the line element of the curve. We can also write this with respect to t:

$$\dot{s}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r} \cdot \dot{r}$$

Now we have a curve in a space $\underline{\mathbf{r}}(t)$. Then we can find a tangent, $\dot{\underline{\mathbf{r}}} = (\dot{x}, \dot{y}, \dot{z})$, which then we know that $|\dot{\underline{\mathbf{r}}}| = \dot{s}$ and $\hat{\underline{\mathbf{t}}} = \dfrac{\dot{\underline{\mathbf{r}}}}{|\dot{\underline{\mathbf{r}}}|}$. We have now swapped the parameter from t to s.

$$\hat{\underline{\mathbf{t}}} = \frac{dx}{ds}\hat{\boldsymbol{\imath}} + \frac{dy}{ds}\hat{\boldsymbol{\jmath}} + \frac{dy}{ds}\hat{\boldsymbol{k}}$$

As we then know that $\hat{\underline{\mathbf{t}}}$ is a unit vector, $\hat{\underline{\mathbf{t}}} \cdot \hat{\underline{\mathbf{t}}} = 1$, now differentiate and $\hat{\underline{\mathbf{t}}} \cdot \dfrac{d\hat{\underline{\mathbf{t}}}}{ds} = 0$, hence $\dfrac{d\hat{\underline{\mathbf{t}}}}{ds} \perp \hat{\underline{\mathbf{t}}}$. The $\frac{d\hat{\underline{\mathbf{t}}}}{ds}$ is in the direction of the principle normal, $\underline{\mathbf{n}}$, of the curve. So $\frac{d\hat{\underline{\mathbf{t}}}}{ds} = \kappa(s)\hat{\underline{\mathbf{n}}}$

The plane spanned by $\hat{\underline{\mathbf{t}}}(s)$ and $\hat{\underline{\mathbf{n}}}(s)$ is the osculating plane.

So if we have a curve $\underline{\mathbf{r}}(t) \in \mathbb{R}^3$, then $\dfrac{d\underline{\mathbf{r}}}{dt}$, so we can now say that $\dfrac{\dot{\underline{\mathbf{r}}(t)}}{|\dot{\underline{\mathbf{r}}(t)}|} = \dfrac{d\underline{\mathbf{r}}}{ds} = \hat{\underline{\mathbf{t}}}$. Now we can take derivatives and hence:

$$\frac{d\hat{\underline{\mathbf{t}}}}{ds} = \kappa\hat{\underline{\mathbf{n}}}$$

Moving forward now, we can take $\hat{\underline{\mathbf{t}}} = \underline{\mathbf{r}}'(s)$ and then differentiating: $\hat{\underline{\mathbf{t}}} = \underline{\mathbf{r}}''(s)$, which then implies:

$$\kappa = |\underline{\mathbf{r}}''(s)|$$

and then we know that $\dot{\underline{\mathbf{r}}}(t) = \underline{\mathbf{r}}'(s)\dot{s}$ and then $\ddot{\underline{\mathbf{r}}(t)} = r''\dot{s}^2 + \dfrac{\ddot{s}}{\dot{s}}\dot{\underline{\mathbf{r}}}$ and hence we can say that: $\underline{\mathbf{r}}''(s) = \dfrac{1}{\dot{s}^2}\ddot{\underline{\mathbf{r}}} - \dfrac{\ddot{s}}{\dot{s}^3}\dot{\underline{\mathbf{r}}}$. So now,

$$\kappa^2(s) = \frac{1}{(\dot{\underline{\mathbf{r}}} \cdot \dot{\underline{\mathbf{r}}})^3}\left((\ddot{\underline{\mathbf{r}}} \cdot \ddot{\underline{\mathbf{r}}})(\dot{\underline{\mathbf{r}}} \cdot \dot{\underline{\mathbf{r}}}) - (\dot{\underline{\mathbf{r}}} \cdot \ddot{\underline{\mathbf{r}}})^2\right)$$

Given a unit tangent vector, $\hat{\underline{\mathbf{t}}}$ and a unit normal vector, $\hat{\underline{\mathbf{n}}}$ at a point on a curve in $\mathbb{R}^3$, we can define a third unit vector $\hat{\underline{\mathbf{b}}}$ which is the unit binormal vector.

$$\hat{\underline{\mathbf{b}}} = \hat{\underline{\mathbf{t}}} \times \hat{\underline{\mathbf{n}}}$$

They form a right handed system of unit vectors, this forms the moving trihedron as s varies.

$$\hat{\underline{\mathbf{b}}} = \hat{\underline{\mathbf{t}}} \times \hat{\underline{\mathbf{n}}}, \quad \hat{\underline{\mathbf{n}}} = \hat{\underline{\mathbf{b}}} \times \hat{\underline{\mathbf{t}}}, \quad \hat{\underline{\mathbf{t}}} = \hat{\underline{\mathbf{n}}} \times \hat{\underline{\mathbf{b}}}$$

7.1 Deriving Frenet-Serret Equations

We can now differentiate the other two equations, and get; $\dfrac{d\hat{\underline{\mathbf{b}}}}{ds} \perp \hat{\underline{\mathbf{b}}}$ and

$$\begin{aligned}
\frac{d\hat{\underline{\mathbf{b}}}}{ds} &= \frac{d\hat{\underline{\mathbf{t}}}}{ds} \times \hat{\underline{\mathbf{n}}} + \hat{\underline{\mathbf{t}}} \times \frac{d\hat{\underline{\mathbf{n}}}}{ds} \\
&= \kappa\hat{\underline{\mathbf{n}}} \times \hat{\underline{\mathbf{n}}} + \hat{\underline{\mathbf{t}}} \times \frac{d\hat{\underline{\mathbf{n}}}}{ds} \\
&= \hat{\underline{\mathbf{t}}} \times \frac{d\hat{\underline{\mathbf{n}}}}{ds}
\end{aligned}$$

which also tells us that:

$$\frac{d\hat{\underline{\mathbf{b}}}}{ds} \perp \hat{\underline{\mathbf{t}}}, \frac{d\hat{\underline{\mathbf{n}}}}{ds}$$

and hence $\dfrac{d\hat{\underline{\mathbf{n}}}}{ds} \parallel \hat{\underline{\mathbf{n}}}$ and so,

$$\frac{d\hat{\underline{\mathbf{b}}}}{ds} = -\tau\hat{\underline{\mathbf{n}}}$$

we call, τ the torsion of the curve.

Example 12. We shall take the helix again,

$$\begin{aligned}
d\underline{\mathbf{r}} &= -a\sin t dt\hat{\boldsymbol{\imath}} + a\cos t dt\hat{\boldsymbol{\jmath}} + cdt\hat{\boldsymbol{k}} \\
ds^2 &= (a^2(\sin^2 t + \cos^2 t) + c^2)dt^2 \\
ds &= (a^2 + c^2)^{\frac{1}{2}}dt \\
&\implies t = (a^2 + c^2)^{-\frac{1}{2}}s
\end{aligned}$$

Now we can find the tangent to any point.

$$\underline{\mathbf{r}}'(s) = \frac{1}{\sqrt{a^2+c^2}}\left(-a\sin\frac{s}{\sqrt{a^2+c^2}}\hat{\boldsymbol{\imath}} + a\cos\frac{s}{\sqrt{a^2+c^2}}\hat{\boldsymbol{\jmath}} + c\hat{\boldsymbol{k}}\right)$$

and now for $\hat{\underline{\mathbf{t}}}'(s)$

$$\hat{\underline{\mathbf{t}}}' = \underline{\mathbf{r}}''(s) = \frac{a}{a^2+c^2}\left(-\cos\frac{s}{\sqrt{a^2+c^2}}\hat{\boldsymbol{\imath}} - \sin\frac{s}{\sqrt{a^2+c^2}}\hat{\boldsymbol{\jmath}}\right)$$

comapring both sides, we can say that: $\kappa(s) = \dfrac{a}{a^2+c^2}$. Finally, we find $\hat{\underline{\mathbf{b}}}(s)$ as:

$$\hat{\underline{\mathbf{b}}} = \hat{\underline{\mathbf{t}}} \times \hat{\underline{\mathbf{n}}} = \frac{1}{\sqrt{a^2+c^2}}\left(-c\sin\frac{s}{\sqrt{a^2+c^2}}\hat{\boldsymbol{\imath}} - c\cos\frac{s}{\sqrt{a^2+c^2}}\hat{\boldsymbol{\jmath}} + a\hat{\boldsymbol{k}}\right)$$

and to find torsion:

$$\hat{\underline{\mathbf{b}}}' = \frac{c}{a^2+c^2}\left(\cos\frac{s}{\sqrt{a^2+c^2}}\hat{\boldsymbol{\imath}} + \sin\frac{s}{\sqrt{a^2+c^2}}\hat{\boldsymbol{\jmath}} + a\hat{\boldsymbol{k}}\right)$$

and so torsion:

$$\tau(s) = \frac{c}{a^2+c^2}$$

Now for $\hat{\underline{\mathbf{n}}}$, we can differentiate once and get:

$$\begin{aligned}\frac{d\hat{\underline{\mathbf{n}}}}{ds} &= \frac{d\hat{\underline{\mathbf{b}}}}{ds} \times \hat{\underline{\mathbf{t}}} + \hat{\underline{\mathbf{b}}} \times \frac{d\hat{\underline{\mathbf{t}}}}{ds} \\ &= -\tau(s)\hat{\underline{\mathbf{n}}} \times \hat{\underline{\mathbf{t}}} + \hat{\underline{\mathbf{b}}} \times \kappa\hat{\underline{\mathbf{n}}} \\ &= \tau(s)\hat{\underline{\mathbf{b}}} - \kappa(s)\hat{\underline{\mathbf{t}}}\end{aligned}$$

Definition 7.1: Frenet-Serret Equations in $\mathbb{R}^3$

$$\frac{d\hat{\underline{\mathbf{t}}}(s)}{ds} = \kappa(s)\hat{\underline{\mathbf{n}}}(s)$$

$$\frac{d\hat{\underline{\mathbf{b}}}(s)}{ds} = -\tau(s)\hat{\underline{\mathbf{n}}}(s)$$

$$\frac{d\hat{\underline{\mathbf{n}}}(s)}{ds} = \tau(s)\hat{\underline{\mathbf{b}}} - \kappa(s)\hat{\underline{\mathbf{t}}}$$

If you are given $\hat{\underline{\mathbf{t}}}$, $\hat{\underline{\mathbf{n}}}$, κ and τ, you can use the Frenet Serret equations to determine $\hat{\underline{\mathbf{t}}}$, $\hat{\underline{\mathbf{n}}}$ and $\hat{\underline{\mathbf{b}}}$ and thus determine the curve in its entirity.

8 Differentiation and Vector Fields

If $\underline{\mathbf{A}}(t) = A(t)_1\hat{\boldsymbol{\imath}} + A(t)_2\hat{\boldsymbol{\jmath}} + A(t)_3\hat{\boldsymbol{k}}$, then:

$$\frac{d\underline{\mathbf{A}}(t)}{dt} = \frac{d\underline{\mathbf{A}}(t)}{dt}_1\hat{\boldsymbol{\imath}} + \frac{d\underline{\mathbf{A}}(t)}{dt}_2\hat{\boldsymbol{\jmath}} + \frac{d\underline{\mathbf{A}}(t)}{dt}_3\hat{\boldsymbol{k}}$$

and let $\Phi = \Phi(x,y,z,t)$, $\underline{\mathbf{A}}(\underline{\mathbf{x}},t)$, $B(\underline{\mathbf{x}},t)$, then:

$$\frac{\partial}{\partial t}(\Phi\underline{\mathbf{A}}) = \frac{\partial\Phi}{\partial t}\underline{\mathbf{A}} + \Phi\frac{\partial\underline{\mathbf{A}}}{\partial t} \tag{$*$}$$

$$\frac{\partial}{\partial t}(\underline{\mathbf{A}} + \underline{\mathbf{B}}) = \frac{\partial\underline{\mathbf{A}}}{\partial t} + \frac{\partial\underline{\mathbf{B}}}{\partial t} \tag{$*^2$}$$

$$\frac{\partial}{\partial t}(\underline{\mathbf{A}} \cdot \underline{\mathbf{B}}) = \frac{\partial\underline{\mathbf{A}}}{\partial t} \cdot \underline{\mathbf{B}} + \underline{\mathbf{A}} \cdot \frac{\partial\underline{\mathbf{B}}}{\partial t} \tag{$*^3$}$$

$$\frac{\partial}{\partial t}(\underline{\mathbf{A}} \times \underline{\mathbf{B}}) = \frac{\partial\underline{\mathbf{A}}}{\partial t} \times \underline{\mathbf{B}} + \underline{\mathbf{A}} \times \frac{\partial\underline{\mathbf{B}}}{\partial t} \tag{$*^4$}$$

Now for the second derivatives

$$\begin{aligned}\frac{\partial^2\underline{\mathbf{A}}}{\partial x\partial y} &= \frac{\partial^2\underline{\mathbf{A}}}{\partial y\partial x}\\ &= \frac{\partial}{\partial x}\left(\frac{\partial\underline{\mathbf{A}}}{\partial y}\right)\\ &= \frac{\partial^2\underline{\mathbf{A}}_1}{\partial x\partial y}\hat{\boldsymbol{\imath}} + \frac{\partial^2\underline{\mathbf{A}}_2}{\partial x\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial^2\underline{\mathbf{A}}_3}{\partial x\partial y}\hat{\boldsymbol{k}}\end{aligned}$$

8.1 Divergence of a vector field

The divergence of a vector field $u(\underline{\mathbf{x}},t)$ is a scalar field. It's value at a point P is defined:

$$\underline{\nabla} \cdot u = \lim_{\delta\underline{\mathbf{V}}\to 0} \oiint \underline{\mathbf{u}} \cdot \underline{\mathbf{n}} ds$$

where $\underline{\mathbf{V}}$ is a small volume enclosing P. Physically this is the amount of flux in vector field, $\underline{\mathbf{U}}$ out of $\delta\underline{\mathbf{V}}$ divided by the volume.

$$\underline{\nabla} \cdot \underline{\mathbf{u}} = \frac{\partial\underline{\mathbf{u}}_1}{\partial x} + \frac{\partial\underline{\mathbf{u}}_2}{\partial y} + \frac{\partial\underline{\mathbf{u}}_3}{\partial z}$$

Assume $P(x,y,z)$ is enclosed by a cube of side length, $\delta x, \delta y, \delta z$. Assume P is at the centre of the cube. Then:

$$\begin{aligned}\oiint_S \underline{\mathbf{u}} \cdot \underline{\mathbf{n}} ds &= \iint_{S_1} \underline{\mathbf{u}} \cdot \underline{\mathbf{n}} ds + \iiint_{S_2} \underline{\mathbf{u}} \cdot \underline{\mathbf{n}} ds + \iint_{S_3} \underline{\mathbf{u}} \cdot \underline{\mathbf{n}} ds\\ &+ \iint_{S_4} \underline{\mathbf{u}} \cdot \underline{\mathbf{n}} ds + \iint_{S_5} \underline{\mathbf{u}} \cdot \underline{\mathbf{n}} ds + \iint_{S_6} \underline{\mathbf{u}} \cdot \underline{\mathbf{n}} ds\\ &= u_1(x + \frac{\delta x}{2}, y, z)\delta y\delta z - u_1(x - \frac{\delta x}{2}, y, z)\delta y\delta z\\ &+ u_2(x, y + \frac{\delta y}{2}, z)\delta x\delta z - u_2(x, y - \frac{\delta y}{2}, z)\delta x\delta z\\ &+ u_3(x, y, z + \frac{\delta z}{2})\delta x\delta y - u_3(x, y, z - \frac{\delta z}{2})\delta x\delta y\\ &= \frac{\partial u_1}{\partial x}\delta\underline{\mathbf{V}} + \frac{\partial u_2}{\partial y}\delta\underline{\mathbf{V}} + \frac{\partial u_3}{\partial z}\delta\underline{\mathbf{V}}\end{aligned}$$

So we can conclude that:

$$\lim_{\delta \underline{\mathbf{V}} \to 0} \oiint \underline{\mathbf{u}} \cdot \underline{\mathbf{n}} ds = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} = \underline{\nabla} \cdot \underline{\mathbf{u}}$$

Example 13. Compute divergence of $F = x^2 y \hat{\boldsymbol{\imath}} + z \hat{\boldsymbol{\jmath}} + xyz \hat{\boldsymbol{k}}$

Solution.

$$\begin{aligned} \underline{\nabla} \cdot F &= \frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(xyz) \\ &= 3xy \end{aligned}$$

9 Curl of a Vector Field

The curl of a vector field $\underline{\mathbf{u}}(\underline{\mathbf{x}}, t)$ is a vector field. The component in the direction of the $\underline{\hat{\mathbf{n}}}$,

$$\underline{\hat{\mathbf{n}}} \cdot \underline{\nabla} \times \underline{\mathbf{u}} = \lim_{\delta S \to 0} \frac{1}{\delta S} \oint_{\delta C} \underline{\mathbf{u}} \cdot d\underline{\mathbf{r}}$$

$\underline{\nabla} \times \underline{\mathbf{u}}$ is related to the rotatio or tisting of the vector field.

$$\underline{\nabla} \times \underline{\mathbf{u}} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_1 & u_2 & u_3 \end{vmatrix} =$$

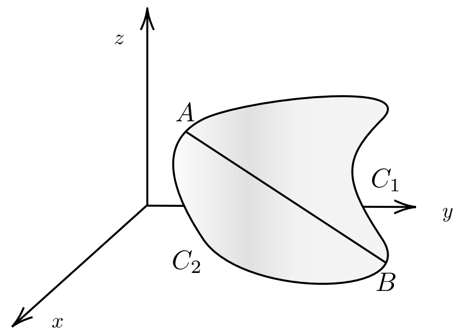
To prove this:

$$\begin{aligned}
\underline{\hat{\mathbf{n}}} \cdot \underline{\nabla} \times \underline{\mathbf{u}} &= \lim_{\delta S \to 0} \frac{1}{\delta S} \oint_{\delta C} \underline{\mathbf{u}} \cdot d\underline{\mathbf{r}} \\
&= \oint_{C_1} \underline{\mathbf{u}} \cdot d\underline{\mathbf{r}} + \oint_{C_2} \underline{\mathbf{u}} \cdot d\underline{\mathbf{r}} \\
&\quad + \oint_{C_3} \underline{\mathbf{u}} \cdot d\underline{\mathbf{r}} + \oint_{C_4} \underline{\mathbf{u}} \cdot d\underline{\mathbf{r}} \\
&\approx u_1(x, y - \frac{\delta y}{2}, z)\delta x + -u_1(x, y + \frac{\delta y}{2}, z)\delta x \\
&\quad + u_2(x + \frac{\delta x}{2}, y, z)\delta y - u_2(x - \frac{\delta x}{2}, y, z)\delta y \\
&= -\frac{\partial u_1}{\partial y}\delta y \delta x + \frac{\partial u_2}{\partial x}\delta x \delta y \\
&= \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}
\end{aligned}$$

The other components of $\underline{\nabla} \times \underline{\mathbf{u}}$ can be found with similar arguments.

10 Conservative Fields

10.1 Gradients and Conserivative Field



Definition 10.1: Conservative Vector Field

A conservative vector field is one which the line integral along a curve connecting two points does not depend on the path taken.

What this says, is that we can write:

$$\int_C \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} = \int_{C_1} \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} = \int_{C_2} \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}}$$

Theorem 10.1

Suppose that a vector field $\underline{\mathbf{F}}$ is related to a scalar field $\Phi(\underline{\mathbf{x}})$ by $\underline{\mathbf{F}} = \underline{\nabla}\Phi$ and $\underline{\nabla}\Phi$ exists everywhere in some region D. Conversely, if $\underline{\mathbf{F}}$ is conservative, then $\underline{\mathbf{F}}$ can be written as the gradient of a scalar field, $\underline{\mathbf{F}} = \underline{\nabla}\Phi$

Proof. Suppose that $\underline{\mathbf{F}} = \underline{\nabla}\Phi$, then F is conservative on D. So we can write;

$$\begin{aligned}
\int_C \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} &= \int_C \underline{\nabla}\Phi \cdot d\underline{\mathbf{r}} \\
&= \int_C \left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z}\right) \cdot (dx, dy, dz) \\
&= \int_C \frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy + \frac{\partial \Phi}{\partial z} dz \\
&= \int_C d\Phi
\end{aligned}$$

$$\begin{aligned}
&= \Phi\Big|_A^B \\
&= \Phi(B) - \Phi(A)
\end{aligned}$$

So as this result only matters about the end points, $\underline{\mathbf{F}}$ is conservative. Now assume that $\underline{\mathbf{F}}$ is conservative, then a scalar field $\Phi(\underline{\mathbf{x}})$ can be defined as the line integral of $\underline{\mathbf{F}}$ from the origin to the point $\underline{\mathbf{x}}$:

$$\begin{aligned}\Phi(\underline{\mathbf{x}}) &= \int_{\underline{\mathbf{0}}}^{\underline{\mathbf{x}}} \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} \\ d\Phi &= \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} \\ &= \underline{\nabla}\Phi \cdot \underline{\mathbf{r}} \\ &= \frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy + \frac{\partial \Phi}{\partial z} dz\end{aligned}$$

and we can now say that $\underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} = \underline{\nabla}\Phi \cdot d\underline{\mathbf{r}}$ and hence, $F = \underline{\nabla}\Phi$ □

If a vector field is conservative, $\Phi(\underline{\mathbf{x}})$ which satisfies $\underline{\mathbf{F}} = \underline{\nabla}\Phi$ is called the potential of the vector field.

10.2 Curl and conservative vector fields

Suppose that $\underline{\mathbf{u}} = \underline{\nabla}\Phi$, then,

$$\begin{aligned}\underline{\nabla} \times \underline{\mathbf{u}} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (u_1, u_2, u_3) \\ &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_1 & u_2 & u_3 \end{vmatrix} \\ &= \left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right)\hat{\boldsymbol{\imath}} + \left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial z}\right)\hat{\boldsymbol{\jmath}} + \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)\hat{\boldsymbol{k}} \\ &= \left(\frac{\partial^2 \Phi}{\partial y \partial z} - \frac{\partial^2 \Phi}{\partial z \partial y}\right)\hat{\boldsymbol{\imath}} \left(\frac{\partial^2 \Phi}{\partial z \partial x} - \frac{\partial^2 \Phi}{\partial x \partial z}\right)\hat{\boldsymbol{\jmath}} \\ &\quad + \left(\frac{\partial^2 \Phi}{\partial x \partial y} - \frac{\partial^2 \Phi}{\partial y \partial x}\right)\hat{\boldsymbol{k}} \\ &= \underline{\mathbf{0}} \qquad \text{As } \Phi \in C^2\end{aligned}$$

So for any vector $\underline{\mathbf{u}}$ that can be written as the gradient of a vector field is irrotational. Conversely, any irrotational vector field is conservative.

10.3 Laplacian of a scalar field

Suppose that a scalar field Φ, is twice dofferenctiable. Then $\underline{\nabla}\Phi$ is a differentiable vector field, so we can tak divergence of $\underline{\nabla}\Phi$ and obtain another scalar field

Definition 10.2: Laplacian

The scalar field $\underline{\nabla} \cdot \underline{\nabla}\Phi$ is called the Laplacian of Φ and is denoted, ∇^2 or Δ

The Laplacian can also act on a vector field, which results in another vector field.

$$\nabla^2 \underline{\mathbf{u}} = \nabla^2 u_1 \hat{\boldsymbol{\imath}} + \nabla^2 u_2 \hat{\boldsymbol{\jmath}} + \nabla^2 u_3 \hat{\boldsymbol{k}}$$

If we have $\Delta\Phi = 0$, this is a known PDE known as the laplace equation.

Theorem 10.2: Divergence of curl

For any $\mathcal{C}^2$ vector field, $\underline{\mathbf{F}}$,

$$\underline{\nabla} \cdot \underline{\nabla} \times \underline{\mathbf{F}} = 0$$

Proof.

$$\begin{aligned}
\underline{\nabla} \times \underline{\mathbf{F}} &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\
&= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\boldsymbol{\imath}} + \\
&\quad \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial z}\right)\hat{\boldsymbol{\jmath}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{\boldsymbol{k}} \\
\underline{\nabla} \cdot \underline{\nabla} \times \underline{\mathbf{F}} &= \frac{\partial F_3}{\partial x \partial y} - \frac{\partial F_2}{\partial x \partial z} + \frac{\partial F_1}{\partial y \partial z} \\
&\quad - \frac{\partial F_3}{\partial x \partial y} + \frac{\partial F_2}{\partial x \partial z} - \frac{\partial F_1}{\partial y \partial z} \\
&= \underline{\mathbf{0}}
\end{aligned}$$

□

10.4 Vector Operators Identities

Let Φ, f, g be scalar fields and $\underline{\mathbf{F}}, \underline{\mathbf{G}}$ be vector fields, then:

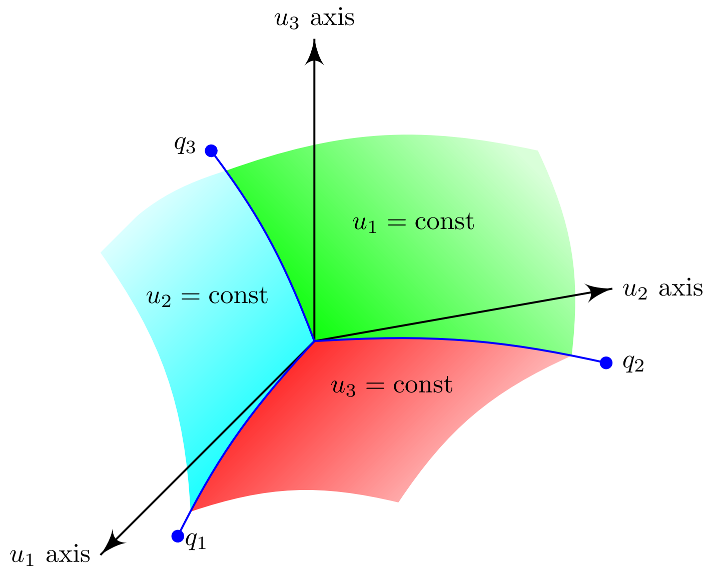
$$\begin{aligned}
\underline{\nabla} \cdot (\underline{\nabla} \times \underline{\mathbf{F}}) &= 0 && (1) \\
\underline{\nabla} \times \underline{\nabla}\Phi &= \underline{\mathbf{0}} && (2) \\
\underline{\nabla}(f+g) &= \underline{\nabla} f + \underline{\nabla} g && (3) \\
\underline{\nabla} \cdot (\underline{\mathbf{F}} + \underline{\mathbf{G}}) &= \underline{\nabla} \cdot \underline{\mathbf{F}} + \underline{\nabla} \cdot \underline{\mathbf{G}} && (4) \\
\underline{\nabla} \times (\underline{\mathbf{F}} + \underline{\mathbf{G}}) &= \underline{\nabla} \times \underline{\mathbf{F}} + \underline{\nabla} \times \underline{\mathbf{G}} && (5) \\
\underline{\nabla}(fg) &= f\underline{\nabla} g + g\underline{\nabla} f && (6) \\
\underline{\nabla} \cdot (\Phi\underline{\mathbf{F}}) &= \Phi\underline{\nabla} \cdot \underline{\mathbf{F}} + \underline{\mathbf{F}} \cdot \underline{\nabla}\Phi && (7) \\
\underline{\nabla} \times (\Phi\underline{\mathbf{F}}) &= \Phi\underline{\nabla} \times \underline{\mathbf{F}} - \underline{\mathbf{F}} \times \underline{\nabla}\Phi && (8) \\
\underline{\nabla}(\underline{\mathbf{F}} \cdot \underline{\mathbf{G}}) &= \underline{\mathbf{F}} \times (\underline{\nabla} \times \underline{\mathbf{G}}) + \underline{\mathbf{G}} \times (\underline{\nabla} \times \underline{\mathbf{F}}) && (9) \\
&\quad + (\underline{\mathbf{F}} \cdot \underline{\nabla})\underline{\mathbf{G}} + (\underline{\mathbf{G}} \cdot \underline{\nabla})\underline{\mathbf{F}} && (10) \\
& && (11)
\end{aligned}$$

$$\begin{aligned}
\underline{\nabla} \cdot (\underline{\mathbf{F}} \times \underline{\mathbf{G}}) &= \underline{\mathbf{G}} \cdot (\underline{\nabla} \times \underline{\mathbf{F}}) - \underline{\mathbf{F}}(\underline{\nabla} \times \underline{\mathbf{G}}) && (12) \\
\underline{\nabla} \times (\underline{\mathbf{F}} \times \underline{\mathbf{G}}) &= \underline{\mathbf{F}}(\underline{\nabla} \cdot \underline{\mathbf{G}}) - \underline{\mathbf{G}}(\underline{\nabla} \cdot \underline{\mathbf{F}}) && (13) \\
&\quad + (\underline{\mathbf{G}} \cdot \underline{\nabla})\underline{\mathbf{F}} - (\underline{\mathbf{F}} \cdot \underline{\nabla})\underline{\mathbf{G}} && (14) \\
\underline{\nabla} \times (\underline{\nabla} \times \underline{\mathbf{F}}) &= \underline{\nabla}(\underline{\nabla} \cdot \underline{\mathbf{F}}) - \underline{\nabla}^2\underline{\mathbf{F}} && (15)
\end{aligned}$$

11 Orthoginal Curvilinear Co-ordinate Systems

Assume a one to one map from x_i to u_i, the surfaces $u_i = k$ are defined as a co-ordinate surface and the intersection of the co-ordinate curves.

$$d\underline{\mathbf{r}} = (dx_1, dx_2, dx_3) = \frac{\partial \underline{\mathbf{r}}}{\partial u_1} du_1 + \frac{\partial \underline{\mathbf{r}}}{\partial u_2} du_2 + \frac{\partial \underline{\mathbf{r}}}{\partial u_3} du_3$$



11.1 Scale Factors

If we let $\underline{\mathbf{e}}_1$ be an arbitrary unit vector in the direction of u_1, and similarly for $\underline{\mathbf{e}}_2$ and $\underline{\mathbf{e}}_3$, then:

$$e_1 = \frac{\partial \underline{\mathbf{r}}}{\partial u_1} \frac{1}{h_1} \qquad h_1 = \left| \frac{\partial \underline{\mathbf{r}}}{\partial u_1} \right|$$

and similarly for $\underline{\mathbf{e}}_2$ and $\underline{\mathbf{e}}_3$. Now we can rewrite $d\underline{\mathbf{r}}$:

$$d\underline{\mathbf{r}} = h_1 \underline{\mathbf{e}}_1 du_1 + h_2 \underline{\mathbf{e}}_2 du_2 + h_3 \underline{\mathbf{e}}_3 du_3$$

We want, $\underline{\mathbf{e}}_i \cdot \underline{\mathbf{e}}_j = \delta_{ij}$ and $(\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \underline{\mathbf{e}}_3)$ to be right handed.

11.2 Differential of arc length

Let $d\underline{\mathbf{r}} = h_1 du_1 \underline{\mathbf{e}}_1 + h_2 du_2 \underline{\mathbf{e}}_2 + h_3 du_3 \underline{\mathbf{e}}_3$, then, $ds^2 = h_1^2 du_1^2 + h_2 du_2^2 + h_3^2 du_3^2$. Now we find dS, by taking the pross product between $\frac{\partial \underline{\mathbf{r}}}{\partial u_1} u_1$ and $\frac{\partial \underline{\mathbf{r}}}{\partial u_3} du_3$. Hence for u_1 surface, $dS = h_2 h_3 du_2 du_3$

11.3 Grad, Curl and Div in Curvilinear Co-ordinates

11.4 Cylindrical and Spherical Co-ordinate Systems