Differential Equations Week 2 - Second Order ODEs

James Arthur

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1 Linear Second Order

The DE is linear in y and of the form; $y'' + p(x)y' + q(x)y = r(x)$. we also know the superposition or linearity principle, which says that: if $y_1(x)$ and $y_2(x)$ are solutions of the DE, then $Ay_1(x) + By_2(x)$ are solutions. These are known as basis of the solutions.

Also theIVt requires two conditions. We also know that $y_1(x)$ and $y_2(x)$ must be linearly independent, hence a linearly dependent basis. So we know that $k_1y_1(x) + k_2y_2(x) = 0$.

1.1 Finding basis if one solution is known

Le us assume that we know $y_1(x)$ and we have an equation of the form, $y'' + y'p(x) + q(x)y = r(x)$. Then we can let $y = y_2$ and then $y = uy_1$ due to linear independence of solutions. Then:

$$\begin{aligned} y' &= y_2' = uy_1' + u'y_1 \\ y'' &= y_2'' = u'' + 2u'y_1' + uy_1'' \end{aligned}$$

Now we sub these into the ODE, we get that:

$$\begin{aligned} u''y_1 + 2u'y_1' + uy'' + p(u'y_1 + uy_1') + quy_1 = 0 \\ u''y_1 + u'(2y_1' + py_1) + u(y_1'' + py_1' + qy_1) = 0 \end{aligned}$$

The last coeeficient becomes 0 as we know that y_1 is a solution. We then transform it by letting $u' = v$ and then $u'' = v$. So:

$$\begin{aligned} v' + v\left(\frac{2y_1'}{y_1} + p\right) = 0 \\ \implies u = \int \frac{1}{y_1^2} e^{-\int p dx} \end{aligned}$$

So then, we know that:

$$\frac{y_2}{y_1} = \int \frac{1}{y_1^2} e^{-\int p dx}$$

1.2 Homogenous Linear ODEs

Let us take an ODE of the form: $y'' + ay' + by = 0$ and then let the solution be of the form: $y = e^{\lambda x}$. Then you get the characteristic equation, which is $\lambda^2 + a\lambda + b = 0$, and then $\lambda = \dfrac{-a \pm \sqrt{a^2 - 4b}}{2}$. Then we must see whether the solution is two real solutions, one real solution or a complex conjugate.

1.2.1 Case 1: Two distinct roots

If discriminant > 0, then as y_1, y_2 are defined for all x, then their quotient are not constant. Hence, we then know that: $y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$

1.2.2 Case 2: Repeated roots

If the discriminant $= 0$, then we must have $\lambda = -\dfrac{a}{2}$. So $y_1 = e^{-\frac{a}{2}x}$ and we have to work out what y_2 is. However, we know that:

$$y_2' = u'y_1 + y_1'u$$

and

$$y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

Then we can plug these into $y_2'' + ay_2' + by_2 = 0$ and we get that

$$u''y_1 + u'(2y_1' + ay_1) + u(y_1'' + ay_1' + by_1) = 0$$

and from differentiating y_1 we know that $2y_1' + ay_1 = 0$ and because we know y_1 is a root of the equation, $y_1'' + ay_1' + by_1 = 0$, so then:

$$u''y_1 = 0$$

However, we know that $y_1 \neq 0$, so then $u'' = 0$ and hence:

$$u(x) = c_1x + c_2$$

Given the structure of the solution, we can say again that:

$$y = Ay_1 + By_2 = Ay_1 + Buy_1$$

and hence,

$$y = e^{-\frac{a}{2}x}\left[\tilde{c_1} + \tilde{c_2}x\right]$$

1.2.3 Case 3: Complex Conjugates

If the discriminant is < 0, then $\lambda = -\frac{a}{2} \pm i\omega$, where $\omega = b - \frac{a^2}{4}$. Then we know that:

$$e^{\lambda_1 x} = e^{-\frac{a}{2}x}(\cos\omega x + i\sin\omega x) \tag{1}$$

$$e^{\lambda_2 x} = e^{-\frac{a}{2}x}(\cos\omega x - i\sin\omega x) \tag{2}$$

Then adding the two, we get that:

$$y_1 = e^{-\frac{a}{2}x}\cos\omega x \tag{3}$$

$$y_2 = e^{-\frac{a}{2}x}\sin\omega x \tag{4}$$

and hence that:

$$y(x) = e^{-\frac{a}{2}x}(\cos\omega x + \sin\omega x)$$

2 Differential Operators

You can write an ODE using much simpler expressions, so $y'' + ay' + by = 0$ is the same as $D^2 + aD + bI = L$, where $Dy = y' = \frac{d^2y}{dx^2}$.

Now, $D(e^{\lambda x}) = \lambda e^{\lambda x} = \lambda^2 e^{\lambda x}$.

$$\therefore L(e^{\lambda x} = e^{\lambda x}) = e^{\lambda x}(\lambda^2 + a\lambda + b) = P(\lambda)e^{\lambda x} = 0$$

So $e^{\lambda x}$ is a solution if and only if λ is a solution of the characteristic equation.

3 Euler Cauchy Equations

It has a general form: $x^2y'' + axy' + by = 0$. To solve follow the procedure:

Let one solution be: $y = x^m$, then $y' = xm^{m-1}$ and $y'' = m(m-1)x^{m-2}$. Then sub in and solve the following equation:

$$m^2 + m(a-1) + b = 0$$

Then again there are three cases.

3.1 Case 1: Two different roots

This one is pretty simple, plug in your m_1, m_2 from the quadratic above and use the general rule and you get that: $y(x) = c_1x^{m_1} + c_2x^{m_2}$

3.2 Case 2: Repeated Roots

We get that $b = \frac{(a-1)^2}{4}$, then you get that $m_1 = \frac{(1-a)}{2}$. Then we get that we need we need a $y_2 = uy_1$ and use the order reduction approach from above. You end up with $u = \ln x$. So the general solution is:

$$(c_1 + c_2 \ln x)x^{\frac{(1-a)}{2}}$$

3.3 Case 3: Complex Conjugates

We say that we have a $m_1, m_2 = c \pm id$. Then plugging it in, we get:

$$\begin{aligned} y &= x^{c \pm id} \\ &= x^c \exp(ln(x^{\pm id})) \\ &= x^c e^{\pm idlnx} \\ &= x^c(\cos(d \ln x) \pm i \sin(d \ln x)) \end{aligned}$$

Then adding and substracting, we get the general solution:

$$y(x) = x^c[A\cos(d \ln x) + B \sin(d \ln x)]$$

4 Nonhomegenous ODEs

The general form is $y'' + p(x)y' + q(x)y = r(x)$, where $r(x) \neq 0$. We have a general solution, that is of the form; $y(x) = y_{PI}(x) + y_{CF}(x) = y_p + C_1y_1 + C_2y_2$.

The method for solving this is:

1. If $r(x)$ is one of the entries on the table, use the corresponding y_p.
2. If y_p is the solution of the ODE, then multiply y_p by x, or x^2 if this solution corresponds to a doble root of the characteristic equation.
3. If $r(x)$ is the sum of functions, choose for sum of functions in the corresponding y_p.

4.1 Existence and Uniqueness of solutions

Let us take an ODE, $y'' + py' + qy = r(x)$, where $r(x) \neq 0$. Then we take the IVP and initial conditions of: $y(x_0) = k_0$ and $y'(x_0) = k_1$. If p and q are continuous in an interval I and x_0 is in I, then the IVP has a unique solution on the interval I, $y_h = c_1y_1 + c_2y_2$.

Linear independence is the same, so if $k_1y_1 + k_2y_2 = 0 \implies k_1 = 0, k_2 = 0$, then they are linearly independent. Two solutions are also linearly dependent, if and only if their Wronksian is zero at x_0.

$$W(y_1, y_2) = y_1y_2' - y_1'y_2 = ky_2y_2' - ky_2y_2' = 0$$

and we write the wronskian as:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

A general solution method, goes along like this. Find a general solution for $r(x) = 0$ and then, find the particular integral as:

$$y_p(x) = -y_1 \int \frac{y_r}{W} + y_2 \int \frac{y_1 r}{W}$$

Where y_1, y_2 are a basis of solutions for the ODE and W is the wronskian.