Suffix Notation

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Contents

1 Lecture 1: Basic Definitions

1.1 Suffix Notation

Let there be a vector $\underline{\mathbf{c}} = \underline{\mathbf{a}} + \underline{\mathbf{b}}$, where $\underline{\mathbf{a}} = a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}$ and $\underline{\mathbf{b}} = b_1\hat{\boldsymbol{\imath}} + b_2\hat{\boldsymbol{\jmath}} + b_3\hat{\boldsymbol{k}}$. Then $\underline{\mathbf{c}}$ is equivalent to:

$$c_i = a_i + b_i$$

In suffix notation:

$$c_j = a_j + b_j \qquad j = 1, 2, 3$$

The inner product of two vectors:

$$\begin{aligned} a \cdot b &= a_1 b_1 + a_2 b_2 + a_3 b_3 \\ &= \sum_{j=1}^{3} a_j b_j \end{aligned}$$

For a vector $\underline{\mathbf{a}} = a_i$, i is a free index. For the dot product above: $\sum_{j=1}^{3} a_j b_j$, j is a dummy suffix.

For suffix notation, an index cannot be repeated more than two times in an equation.

Example 1 *Write $(a \cdot b)(c \cdot d)$ in suffix notation*

Solution 1 *Here we take that:*

$$a \cdot b = a_j b_j \quad j = 1, 2, 3$$

and that

$$c \cdot d = c_i d_i \quad i = 1, 2, 3$$

Now we can say that

$$(a \cdot b)(c \cdot d) = a_j b_j c_i d_i \quad i, j = 1, 2, 3$$

Example 2 *Write $a_j b_i c_j$ in normal vector notation*

Solution 2 *We know that*

$$a_j b_i c_j = a_j c_j b_i$$

Which is:

$$(a \cdot c)b$$

Example 3 *Write the vector notation $\underline{\boldsymbol{u}} + (\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}})\underline{\boldsymbol{v}} = |\underline{\boldsymbol{a}}|^2(\underline{\boldsymbol{b}} \cdot v)\underline{\boldsymbol{a}}$ in suffix notation*

Solution 3 *We know that*

$$a_j b_i c_j = a_j c_j b_i$$

Which is:

$$(a \cdot c)b$$

Example 4 *Write the vector notation $\underline{\boldsymbol{u}} + (\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}})\underline{\boldsymbol{v}} = |\underline{\boldsymbol{a}}|^2(\underline{\boldsymbol{b}} \cdot v)\underline{\boldsymbol{a}}$ in suffix notation*

Solution 4 *Firstly:*

$$[\underline{\boldsymbol{u}} + (\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}})\underline{\boldsymbol{v}}]_i = \left[|\underline{\boldsymbol{a}}|^2(\underline{\boldsymbol{b}} \cdot v)\underline{\boldsymbol{a}}\right]_i$$

Then,

$$u_i + (a_j b_j)v_i = a_j a_j b_l v_l a_i \qquad j, l = 1, 2, 3$$

1.2 The Kronecker Delta $\delta_{i,j}$

The function is defined:

$$\delta_{i,j} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

The suffixes i and j can each take the values $1, 2, 3$ so $\delta_{i,j}$ has nine elements.

We can write the function as the identity matrix:

$$\delta_{i,j} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\delta_{i,j}$ is called a substitution tensor, since it's effect when multiplied by a_j is to replace j with i.

$$\begin{aligned} \delta_{i,j} a_j &= \sum_{j=1}^{3} \delta_{i,j} a_j \\ &= \delta_{i1} a_1 + \delta_{i2} a_2 + \delta_{i3} a_3 \\ &= \delta_{11} a_1 + \delta_{12} a_2 + \delta_{13} a_3 \\ &\quad + \delta_{21} a_1 + \delta_{22} a_2 + \delta_{23} a_3 \\ &\quad + \delta_{31} a_1 + \delta_{32} a_2 + \delta_{33} a_3 \\ &= a_1 + a_2 + a_3 \end{aligned}$$

From this we can say: $\delta_{i,j} a_i = a_j$ and $\delta_{i,j} a_j = a_i$

Example 5 *$\delta_{i,j}$ and dot product*

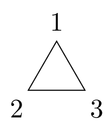
Solution 5

$$\begin{aligned} a \cdot b &= a_i b_i \quad i = 1,2,3 \\ &= \delta_{i,j} a_j b_i \\ &= a_j \delta_{i,j} b_i \\ &= a_j b_j \end{aligned}$$

1.3 The Alternating Tensor, $\varepsilon_{i,j,k}$

$\varepsilon_{i,j,k}$ is useful for manipulating expressions involving the cross product of two vectors and curl of a vector.

$$\varepsilon_{i,j,k} = \begin{cases} +1 & \text{if } (i,j,k) = (1,2,3),\ (2,3,1) \text{ or } (3,1,2) \\ -1 & \text{if } (i,j,k) = (3,2,1),\ (2,1,3) \text{ or } (1,3,2) \\ 0 & \text{if any of } i,j,k \text{ are equal} \end{cases}$$



The $+1$ case can be also written as 1, 2 or 3 are in clockwise order. So if you take a triangle and then go clockwise around it from the first element, that the order they are in. The -1 are in anticlockwise order. Hence meaning the opposite of clockwise.

The six non-zero elements of ε_{ijk}:

$$\begin{aligned} &\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1 \\ &\varepsilon_{321} = \varepsilon_{213} = \varepsilon_{132} = -1 \\ &\varepsilon_{ijk} = 0, \text{ otherwise} \end{aligned}$$

We can take that; $\varepsilon_{ijk} = \varepsilon_{jki}$ as they are in clockwise order. This also imples $\varepsilon_{ijk} = -\varepsilon_{jik}$ because if ijk are in clockwise order then jik must be in counterclockwise order.

1.4 $\varepsilon_{i,j,k}$ and cross product

Let $\underline{\mathbf{a}} = a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}$ and $\underline{\mathbf{b}} = b_1\hat{\boldsymbol{\imath}} + b_2\hat{\boldsymbol{\jmath}} + b_3\hat{\boldsymbol{k}}$. Then their cross product is:

$$\underline{\mathbf{a}} \times \underline{\mathbf{b}} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

and in suffix notation, we can write the above as; $(\underline{\mathbf{a}} \times \underline{\mathbf{b}})_i = \varepsilon_{ijk}\, a_j b_k$ where j,k are dummy suffixes and must be summed over 1 to 3.

1.5 ε_{ijk} and the scalar triple product

We can take the scalar triple product, $\underline{\mathbf{a}} \cdot \underline{\mathbf{b}} \times \underline{\mathbf{c}}$, then we can do the following:

$$\begin{aligned} \underline{\mathbf{a}} \cdot \underline{\mathbf{b}} \times \underline{\mathbf{c}} &= a_i(\underline{\mathbf{b}} \times \underline{\mathbf{c}})_i \\ &= a_i \varepsilon_{ijk} b_j c_k \\ &= \varepsilon_{ijk} a_i b_j c_k \\ &= c_k \varepsilon_{ijk} a_i b_j \end{aligned}$$

from the above we show that $\underline{\mathbf{a}} \cdot \underline{\mathbf{b}} \times \underline{\mathbf{c}} = \underline{\mathbf{c}} \cdot \underline{\mathbf{a}} \times \underline{\mathbf{b}}$. We can expand $\varepsilon_{ijk}\, a_i\, b_j\, c_k$ to get:

$$\begin{aligned} &= \varepsilon_{123}a_1b_2c_3 + \varepsilon_{231}a_2b_3c_1 + \varepsilon_{312}a_3b_1c_2 \\ &\quad + \varepsilon_{321}a_3b_2c_1 + \varepsilon_{213}a_2b_1c_3 + \varepsilon_{132}a_1b_3c_2 \\ &= a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2 \end{aligned}$$

which is the expanded form of the triple scalar product.

1.6 A relation between ε_{ijk} and $\delta_{i,j}$

We are going to prove the following statement:

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Since all of the coordinate axis are the same, just consider $i = 1$:

If then $j = 1$, we get that $\varepsilon_{11k} = 0$ and so LHS $= 0$. Then considering the RHS, we get that $\delta_{1l}\delta_{1m} - \delta_{1m}\delta_{1l} = 0$, so equation holds.

If $j = 2$, then $\varepsilon_{ijk} = \varepsilon_{12k} = 0$, unless $k = 3$, so then only $k = 3$ contributes to the sum. So $\varepsilon_{klm} = \varepsilon_{3lm}$, so zero unless l and m are 1 and 2. So we can conclude that $\varepsilon_{ijk}\varepsilon_{klm} = \varepsilon_{123}\varepsilon_{312}$ or $\varepsilon_{123}\varepsilon_{321}$, so the LHS is either ± 1. Looking at RHS, we have either: $\delta_{11}\delta_{22} - \delta_{12}\delta_{21}$ or $\delta_{12}\delta_{21} - \delta_{11}\delta_{22}$. This gives ± 1 in the same perumtation as the LHS. So equation holds.