Year MAGIC — Algebraic Geometry

Based on lectures by Prof. Eleonore Faber
Notes taken by James Arthur

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine (especially the typos!).

Contents

In this course we will study some introductory algebraic geometry, we will study Classical Algebraic Geometry and Sheaves There are three chapters,

(i) Affine Varieties

(ii) Noetherian Rings

(iii) Algebraic Varieties in general

Literature: Karen Smith's Book, has lots of examples and is very readable. We will cover chapter one and the start of chapter two of Hardshawn.

Prerequistites: Commutative Algebra, Topology.

1 Affine Varieties

Algebraic Sets in n-space, we want to study zero sets of polynomials in several variables in affine spaces. The affine spaces are k-vector spaces. We will consider algebraically closed fields k.

Definition 1.1 (Affine n-space)**.** Let k be a field. We write $\mathbb{A}^n(k)$ to be an affine n-space over k. This is the set, $\{a_1, a_2, \ldots, a_n : a_i \in k\}$

Let $k[X_1, \ldots, X_n]$ be the polynomial ring in n-variables over k where $n < \infty$.

Definition 1.2 (Vanishing Set)**.** Let $f \in k[X_1, \ldots, X_n]$ then the zero-set of f is,

$$\mathcal{V}(f) = \{(a_1, \ldots, a_n) \in \mathbb{A}^n(k) : f(a_1, \ldots, a_n) = 0\}$$

Example. Let $k = \mathbb{R}$ and $n = 1$, then $f(X) = X + 1$,

$$\mathcal{V}(f) = \{-1\} \in \mathbb{A}^1(\mathbb{R})$$

Example. Let $k = \mathbb{R}$, $n = 2$ and $f(X, Y) = X^2 + Y^2 - 1$, then,

$$\mathcal{V}(f) = \{X, Y \in \mathbb{R}^2 : X^2 + Y^2 = 1\}$$

Example. Let $k = \mathbb{R}$, $n = 3$ and $f(X, Y, Z) = Z^3 + Z^2Y^2 - X^2$, this is not as obvious. The vanishing set is just some curve, and if we intersect it with a sphere we get,

This is slightly odd, it intersects itself and so this isn't a manifold and so is slightly more complicated.

More generally: $f_1, \ldots f_m \in k[X_1, \ldots, X_n]$, we define,

$$\mathcal{V}(f_1, \ldots, f_m) = \{a \in \mathbb{A}^n : f_1(a) = f_2(a) = \cdots = f_m(a) = 0\}$$

Even more generally, we can take any $S \subset k[X_1, \ldots, X_n]$, then

$$\mathcal{V}(S) = \{a \in \mathbb{A}^n : f(a) = 0 \,\forall f \in S\}$$

This allows us to have infinitely many functions. We call S an algebraic subset of $\mathbb{A}^n$.

Example.

$$\mathcal{V}(X^2 - Y, X^3 - Z) \subset \mathbb{A}^3(\mathbb{R})$$

This defines a smooth space curve.

Example. $M_{n\times n}(\mathbb{C})$ can be identified by $\mathbb{A}^{n^2}(\mathbb{C})$ and we can look at subsets of this space. Let $V = \{A \in M_{n\times n}(\mathbb{C}) : \det A = 1\}$. $V = \mathcal{V}(S)$ is an algebraic subset of $\mathbb{A}^{n^2}$. For $\mathbb{A}^{n^2}$ we associate $k[X_{ij}]$ where $1 \le i, j \le n$. Let $S = \Delta - 1$ where

$$\Delta(X_{ij}) = \det \begin{pmatrix} X_{11} & \ldots & X_{1n} \\ & \ddots & \\ X_{n1} & \ldots & X_{nn} \end{pmatrix}$$

We can say slightly more than this,

Remark. (i) $\mathbb{A}^n$ is a algebraic subset, 0 is a polynomial and we can see that $\mathcal{V}(0) = \mathbb{A}^n$.

(ii) $\varnothing$ is an algebraic set, $V(1) = \{a \in \mathbb{A}^n : 1(a) = 1 = 0\} = \varnothing$.

(iii) Algebraic sets are closed under intersection. Let $V(S_i)_{i\in\mathcal{I}}$ be a collection of algebraic sets in $\mathbb{A}^n$, then,

$$\bigcap_{i\in\mathcal{I}} V(S_i) = V\left(\bigcup_{i\in\mathcal{I}} S_i\right)$$

Proof. Exercise □

(iv) Algebraic sets are closed under **finite** unions. We want to show that the union of two algebraic sets is algebraic. Let $V(S), V(T)$ be algebraic sets in $\mathbb{A}^n$, let $S.T = \{fg : f \in S, g \in T\}$. Then we claim that $V(S) \cup V(T) = V(S.T)$. We aim to show both inclusions,

Proof. ($\subset$): Suppose $a \in V(S)$, then $f(a) = 0$ for all $f \in S$, but, $(f \cdot g)(a) = f(a) \cdot g(a) = 0$ for all $g \in T$. Therefore $a \in V(S.T)$.
($\supset$) Suppose $a \in V(S.T) \setminus V(S)$. Then there is some $f \in V(S)$ such that $f(a) \neq 0$, but then, for any $g \in T$ $fg(a) = f(a) \cdot g(a) = 0$ as $a \in V(S.T)$ and as we are in a field, and as $f(a) \neq 0$, then $g(a) = 0$ for all $g \in T$. Therefore $a \in V(T)$. □

Proposition 1.3. The collection of algebraic subsets of $\mathbb{A}^n(k)$ form the closed sets of a topology on $\mathbb{A}^n$. This topology is called the Zariski Topology on $\mathbb{A}^n$.

Here are some examples of closed sets,

Example. If $a \in \mathbb{A}^n$ is a point then $\{a\} = V(X_1 - a_1, X_2 - a_2, \ldots, X_n - a_n)$ and so points are closed in the Zariski Topology.

Example. If $n = 1$ and $S = 0$, then $V(S) = \mathbb{A}^n$, but if $S \subset \mathbb{A}^n$ is algebraic, and if $\exists\, f \neq 0 \in S$ then since we have every polynomial in $k[X]$ has finitely many zeros. Then $\mathcal{V}(f)$ is finite. However $\mathcal{V}(S) \subset \mathcal{V}(f)$ and so $\mathcal{V}(S)$ must be finite. Therefore the Zariski Topology is cofinite, the sets are finite or the whole space.