Week 5: Series of Functions

James Arthur

October 28, 2020

Contents

1 Limits of function series

Definition 1.1: Limit of a functional series

Suppose $\{f_n\}$, $n \in \mathbb{N}_1$ is a sequence of functions defined on a set E, then suppose the limit exists,

$$f(x) = \lim_{n\to\infty} f_n(x)$$

Now we say that $f_n(x)$ converges to $f(x)$ or $\{f_n\}$ converges to f pointwise on E. Similarly:

Definition 1.2: Sum of a series

If $\sum f_n(x)$ converges $\forall\, x \in E$, we say:

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

Theorem 1.1: Contunity of a series of continuous functions

To say that a series of continuous functions is continuous, it suffices to show:

$$\lim_{t\to x} \lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} \lim_{t\to x} f_n(t)$$

1.1 Convergence

Definition 1.3: Uniform Convergence (Sequence)

A sequence of functions $\{f_n\}, n \in \mathbb{N}_1$, converges uniformly on E to a function f if $\forall \varepsilon > 0$, $\exists\, N$, $n \geq N \implies$

$$|f_n(x) - f(x)| \leq \varepsilon \qquad \forall x \in E$$

Definition 1.4: Uniform Convergence (Series)

We say that the sequence (series) $\sum f_n(x)$ converges uniformly on E if the sequence $\{s_n\}$ of the partial sums is:

$$s_n = \sum_{i=1}^{n} f_i(x)$$

1.2 Cauchy Time

Theorem 1.2

The sequence $\{f_n\}$ defined on E, converges uniformly on E, $\iff$ $\forall \varepsilon > 0$, $\exists\, N$, $m \geq N$ and $n \geq N$, $x \in E$, $\implies$

$$|f_n(x) - f(x)| \leq \varepsilon$$

Proof. Suppose that $\{f_n\}$ converges uniformly on E, and let f be the limit of the sequence. Then $\exists\, N$, $n \geq N$, $x \in E$, $\implies$

$$|f_n(x) - f(x)| \leq \frac{1}{2}\varepsilon$$

$$\begin{aligned} |f_n - f_m| &\leq |f_n - f| + |f_m - f| \\ &\leq \varepsilon \end{aligned}$$

Suppose that cauchy holds, then we know every cauchy sequence converges on the real line. So we have to prove that the convergence is uniform; Let $\varepsilon > 0$, $\exists\, N$, st the theorem holds. Now fix n, and take $m \to \infty$, this gives:

$$|f_m - f_n| \leq \varepsilon \forall n \geq N, x \in E$$

$\square$

Theorem 1.3

Suppose $\lim\limits_{n\to\infty} f_n = f$, $x \in E$. Let $M_n = \sup\limits_{x \in E} |f_n - f|$. Then $f_n \to f$ uniformly on E $\iff$ $M_n \to 0$ as $n \to \infty$

Theorem 1.4: Wierstrass

Suppose $\{f_n\}$ is a sequence of functions defined on E, and

$$|f_n| \leq M \qquad (x \in E, n \in \mathbb{N}_1)$$

Proof. If $\sum M_n$ converges, then for $\varepsilon > 0$,

$$\left|\sum_{i=n}^{M} f_i\right| \leq \sum_{i=n}^{m} M_i \leq \varepsilon$$

if m and n are large enough. $\square$

2 Continuity

Let's prove Thm 1.1

Proof. Let $\varepsilon > 0$ by uniform convergence of $\{f_n\}$, then $\exists N$, $n, m \geq N$, $t \in E$, $\implies$

$$|f_n - f_m| \leq \varepsilon$$

Letting $t \to x$, we obtain: $|A_m - A_n| \leq \varepsilon$ for $n, m \geq N$, st. $\{A_n\}$ is a cauchy sequence and so converges to A.

$$|f - A| \leq |f - f_n| + |f_n - A_n| + |A_n - A|$$

Now let them all be less than a third by the usual limit nonsence and hence,

$$|f - A| \leq \varepsilon$$

□

Theorem 2.1

If $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \to f$ uniformly on E, then f is continuous on E (from above)

Theorem 2.2

Suppose K is compact and

1. $\{f_n\}$ is continuous on K
2. $\{f_n\}$ converges pointwise on K
3. $f_n \geq f_{n+1}$ $\forall n \in \mathbb{N}$

Proof. Let $g_n = f_n - f$, then g_n is continuous, $g_n \to 0$ pointwise and $g_n \geq g_{n+1}$. So prove that $g_n \to 0$ uniformly om K.

Let $\varepsilon > 0$, $K_n = \{x \in K : g_n(x) \geq \varepsilon\}$ as g_n is continuous, K is closed and hence compact. Since $g_n \geq g_{n+1}$, we have $K_n \geq K_{n+1}$. Fix an $x \in K$. Since $g_n \to 0$, then $x \notin K_n$ if n is large, thus $x \notin \bigcup K_n$. Hence K_N is empty for $n \geq N$, then:

$$0 \leq g_n < \varepsilon \qquad \forall x \in K \; n \geq N$$

□