

# The Real Number System

James Arthur

October 2, 2020

## Contents

|          |                                    |          |
|----------|------------------------------------|----------|
| <b>1</b> | <b>Overview</b>                    | <b>2</b> |
| <b>2</b> | <b>Properties of the Reals</b>     | <b>2</b> |
| 2.1      | Field Properties . . . . .         | 2        |
| 2.2      | Order Relation . . . . .           | 2        |
| 2.3      | Supremum . . . . .                 | 2        |
| 2.4      | Infimum . . . . .                  | 2        |
| 2.5      | Completeness Axiom . . . . .       | 3        |
| <b>3</b> | <b>Extended Real Numbers</b>       | <b>3</b> |
| 3.1      | Arithmetic . . . . .               | 3        |
| <b>4</b> | <b>Triangle Inequality</b>         | <b>4</b> |
| <b>5</b> | <b>Open and Closed Sets</b>        | <b>4</b> |
| 5.1      | Neighbourhoods . . . . .           | 4        |
| 5.2      | Unions and Intersections . . . . . | 5        |
| 5.3      | Open Coverings . . . . .           | 6        |

# 1 Overview

The reals ( $\mathbb{R}$ ) have a few properties:

1. They are a field, i.e. a groupoid with **two binary operations**.
2. They are ordered
3. They are also **complete**.

We will also look at **supremum** and the **infimum**.

We are also going to look at the **extended real numbers**. We are going to add two more fictitious points.  $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ .

## 2 Properties of the Reals

We will be taking the axiomatic view point of the real numbers. No construction with **Dedekind cuts** or **Cauchy sequences**. All of these are isomorphic.

### 2.1 Field Properties

The real numbers are a set,  $\mathbb{R}$ , with two binary operations,  $+$  and  $\times$ . They must satisfy the following axioms. So take  $a, b, c \in \mathbb{R}$ :

1.  $a + b = b + a$  and  $ab = ba$  (**commutativity**)
2.  $(a + b) + c = a + (b + c)$  and  $a(bc) = (ab)c$  (**associativity**)
3.  $a(b + c) = ab + ac$  (**distributivity**)
4. There are two distinctive identities 0 (**additive identity**) and 1 (**multiplicative identity**), such that  $a + 0 = 0 + a = a$  and  $a1 = 1a = a$
5. We also have inverses,  $-a$  (**additive inverse**) such that  $a + -a = 0$  and if  $a \neq 0$ , there is a real number  $\frac{1}{a}$  such that:  $a(\frac{1}{a}) = 1$

### 2.2 Order Relation

The real numbers are **ordered**, that means:

1. For each pair of reals  $a$  and  $b$ , exactly one of the following is true

$$a = b \quad a < b \quad b < a$$

2. It is also **transitive**, if  $a < b$  and  $b < c$ , then  $a < c$
3. If  $a < b$  then  $a + c < b + c$  for any  $c$ , and if  $0 < c$ , then  $ac < bc$

### 2.3 Supremum

Let  $S \subset \mathbb{R}$ . If there exists  $b \in \mathbb{R}$  such that  $x \leq b \quad \forall x \in S$  then  $S$  is **bounded above** and  $b$  is an **upper bound of  $S$** .

If  $\beta$  is an upper bound of  $S$ , but no number less than  $\beta$  is, then  $\beta$  is called the **supremum** of  $S$ , denoted:

$$\beta = \sup S$$

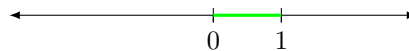


Figure 1: Let  $S$  be the orange set and then  $b$  is an upper bound of  $S$  and  $\beta$  is  $\sup S$

We also call the supremum the least upper bound.

**Example 1.**  $S = [0, 1]$  and prove  $\sup S = 1$

**Solution 1.** Take our diagram from above:



We need to check that  $x \leq 1 \quad \forall x \in S$ , which is definitionally true.

Secondly we need to prove that  $\forall b < 1, \exists x \in S, b < x$ , which is again trivially true. So  $\sup S = 1$

**Example 2.** Take  $T = (0, 1)$  where  $\sup T = 1$

**Solution 2.** Again every number is less than 1, but if you take any number less than one you can always find another element larger.

**NB: The supremum here isn't in the set**

### 2.4 Infimum

Similarly, if there exists an  $a \in \mathbb{R}$  such that  $a \leq x \quad x \in S$ , then  $S$  is **bounded below** and  $a$  is a **lower bound of  $S$** .

If  $\alpha$  is a lower bound of  $S$ , but no number is greater than  $\alpha$  is, then  $\alpha$  is called the **infimum** of  $S$ :

$$\alpha = \inf S$$



Figure 2: Let  $S$  be the orange set and then  $a$  is a lower bound of  $S$  and  $\alpha$  is  $\inf S$

Another name for the infimum is the greatest lower bound.

## 2.5 Completeness Axiom

Do the supremum and the infimum actually exist? Well, not all subsets are bounded above, i.e.  $\mathbb{R} \subset \mathbb{R}$  or what about the empty set? This is what the completeness axiom does:

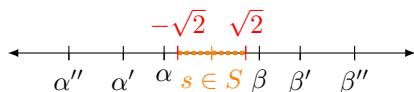
1. If a non-empty set of real numbers are bounded above, then it has a supremum.

So the reals are a **complete ordered field**

The completeness axiom is distinguishing of the reals. They are the only complete ordered field. The rationals possess everything but completeness in terms of our axioms.

**Example 3.** We restrict to the  $\mathbb{Q}$ ,  $S = \{r \in \mathbb{Q} : r^2 < 2\}$ . Find the supremum and infimum.

**Solution 3.** If we take the example below;



we can say that we won't reach  $\sqrt{2}$  in the supremum or  $-\sqrt{2}$  in the infimum. This is because we are using rationals and  $\sqrt{2}$  is an irrational. We can go either way and there is always a number closer to  $\sqrt{2}$ .

This proves that rationals are not complete.

## 3 Extended Real Numbers

It is convenient to attach  $\infty$  and  $-\infty$  to the reals. How do they fit in? Firstly let's look at orders. Take  $x \in \mathbb{R}$ , then:

$$-\infty < x < \infty$$

Now if a set  $S$  is unbounded above or below, we can write:

$$\sup S = \infty \quad \inf S = -\infty$$

**Example 4.** Find the infimum of  $S = \{x \in \mathbb{R} : x : 2\}$



**Solution 4.** As there is technically no lower bound, it is  $-\infty$

We usually denote the extended reals with the symbol,  $\bar{\mathbb{R}}$  or  $[-\infty, \infty]$  or  $\mathbb{R} \cup \{-\infty, \infty\}$

### 3.1 Arithmetic

If  $a \in \mathbb{R}$ ,

1. Then:

$$\begin{aligned} a + \infty &= \infty + a = \infty \\ a - \infty &= -\infty + a = -\infty \\ \frac{a}{\infty} &= \frac{a}{-\infty} = 0 \end{aligned}$$

2. and  $0 < a$ , then:

$$\begin{aligned} a\infty &= \infty a = \infty \\ a(-\infty) &= (-\infty)a = -\infty \end{aligned}$$

3. and  $a < 0$ , then:

$$\begin{aligned} a\infty &= \infty a = -\infty \\ a(-\infty) &= (-\infty)a = \infty \end{aligned}$$

We also define:

$$1. \infty + \infty = \infty \quad \infty(-\infty) = (-\infty)\infty = \infty$$

$$2. \text{ and also } -\infty - \infty = \infty(-\infty) = (-\infty)\infty = -\infty$$

$$3. \text{ and finally, } |\infty| = |-\infty| = \infty$$

We say it isn't useful to define;  $\infty - \infty$ ,  $0 \cdot \infty$ ,  $\frac{\infty}{\infty}$  and  $\frac{0}{0}$ . We call them indeterminate forms.

## 4 Triangle Inequality

As we can use the ordered relation of the reals we can produce something known as the **triangle inequality**.

**Theorem 1.** It states for any  $a, b \in \mathbb{R}$ , we have:

$$|a + b| \leq |a| + |b|$$

*Proof.* There are four possibilities:

1. If  $0 \leq a$  and  $0 \leq b$ , then  $0 \leq a + b$ , so  $|a + b| = a + b = |a| + |b|$ .
2. If  $a \leq 0$  and  $b \leq 0$ , then  $a + b \leq 0$ , so  $|a + b| = -a + (-b) = |a| + |b|$ .
3. If  $0 \leq a$  and  $b \leq 0$ , then  $a + b = |a| - |b|$ .
4. If  $a \leq 0$  and  $0 \leq b$ , then  $a + b = -|a| + |b|$ .

It holds in cases (c) and (d), since

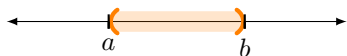
$$||a| - |b|| = \begin{cases} |a| - |b| & \text{if } |b| \leq |a|, \\ |b| - |a| & \text{if } |a| \leq |b|. \end{cases}$$

□

## 5 Open and Closed Sets

**Definition 1.** We define an **open interval** between  $a$  and  $b$ ,  $a, b \in \mathbb{R}$ , as such:

$$(a, b) = \{x : a < x < b\}$$



**Definition 2.** We define a **closed interval** between  $a$  and  $b$ ,  $a, b \in \mathbb{R}$ , as such:

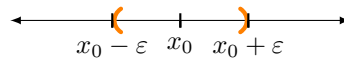
$$[a, b] = \{x : a \leq x \leq b\}$$



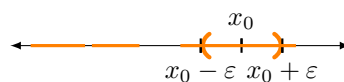
### 5.1 Neighbourhoods

A neighborhood is used to talk about closeness of points. We are now going to go through a load of set definitions!

**Definition 3.** If  $x_0$  is a real number and  $\varepsilon > 0$ , then the open interval  $(x_0 - \varepsilon, x_0 + \varepsilon)$  is an  **$\varepsilon$ -neighbourhood** of  $x_0$ .



**Definition 4.** If a set  $S$  contains an  $\varepsilon$ -neighbourhood of  $x_0$ , then  $S$  is a **neighborhood** of  $x_0$ . i.e. we need  $(x_0 - \varepsilon, x_0 + \varepsilon) \subset S$



**Definition 5.** If  $S$  is a neighbourhood of  $x_0$ , then  $x_0$  is an **interior point** of  $S$ .

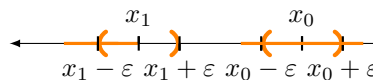


Figure 3:  $x_0$  is an interior point, however  $x_1$  is not

**Definition 6.** The set of interior points of  $S$  is the **interior** of  $S$ , denoted by  $S^0$

**Definition 7.** If every point of  $S$  is an interior point, ( $S^0 = S$ ), then  $S$  is **open**.

**Definition 8.** A set is **closed** if  $S^c$  is open.

**Example 5.** Any open interval  $S = (a, b)$  is open.

**Solution 5.** Need to show that  $\forall x_0 \in (a, b), \exists \varepsilon > 0 : (x_0 - \varepsilon, x_0 + \varepsilon) \subset (a, b)$ . Assume that  $a, b \in \mathbb{R}$ . Let  $x_0 \in (a, b)$  and let  $\varepsilon = \min(x_0 - a, b - x_0)$ . Then clearly  $(x_0 - \varepsilon, x_0 + \varepsilon) \subset (a, b)$ .

The rest of the proof is left as an exercise. (Where  $a = -\infty$  or  $b = \infty$ ).

Now we know that  $\mathbb{R}$  is open and  $S^c$ , where  $S^c = (-\infty, a) \cup [b, \infty)$ , and  $\emptyset$  is closed

We also note that because of a vacuity argument  $\emptyset$  is also open, hence  $\mathbb{R}$  is also closed. So  $\mathbb{R}$  and  $\emptyset$  are both open and closed.

## 5.2 Unions and Intersections

**Theorem 2.** 1. The union of open sets is open

2. The intersection of closed sets is closed

These apply to arbitrary collections (finite or infinite of open and closed sets).

*Proof.* First let's prove (1), so let  $\mathcal{G}$  be a collection of open sets.

Let  $S = \bigcup_{G \in \mathcal{G}} G$ . If  $x_0 \in S$ , then  $x_0 \in G_0$  for some  $G_0 \in \mathcal{G}$ .

Since  $G_0$  is open, it must contain an  $\varepsilon$ -neighborhood of  $x_0$ . The  $\varepsilon$ -neighborhood,  $(x_0 - \varepsilon, x_0 + \varepsilon)$ , is in  $S$ , hence  $S$  is a neighborhood of  $x_0$  and  $x_0$  is an interior point of  $S$ .

Since  $x_0$  was arbitrary, then all points in  $S$  are interior points and hence,  $S$  is open.

Now for part (2) of the theorem. Let  $\mathcal{F}$  be a collection of closed sets and let  $T = \bigcap_{F \in \mathcal{F}} F$ . Then

$T^c = \bigcup_{F \in \mathcal{F}} F^c$ . Since each  $F^c$  is open, that means

$T^c$  is open by (1). Therefore  $T$  is closed  $\square$

**Example 6.** For  $a, b \in \mathbb{R}$ , the sets  $[a, b]$  is closed.

**Solution 6.** Since  $[a, b]^c = (-\infty, a) \cup (b, \infty)$ . Since it's a union of open intervals, it is open. Hence making  $[a, b]$  closed.

**Example 7.** What about  $[a, b)$ , or  $(a, b]$  for  $a, b \in \mathbb{R}$ ?

**Solution 7.** These are half-open or half-closed intervals. These are neither open nor closed. Take  $[a, b)$ , then  $a$  isn't an interior point of the set, hence it's not open. Now take the complement of the set  $[a, b)^c = (-\infty, a) \cup [b, \infty)$  and now  $b$  is no longer an interior point. Hence, not closed.

**Example 8.** What about:  $(-\infty, a]$  or  $[a, \infty)$

**Solution 8.** Exercise

Now, what about the intersection of open sets and union of closed sets. Well, it can be proved that the intersection of finitely many open sets is open and union of finitely many closed sets is closed. However the infinite versions of these statements need not be the same.

**The concept of open and closed sets, doesn't form a dichotomy (A set is partitioned into two. i.e. odd or even naturals).** A set can be neither open or closed or both.

**Definition 9.** A *deleted neighbourhood* of a point  $x_0$  is a set that contains every point of some neighborhood of  $x_0$  except  $x_0$ . For example:

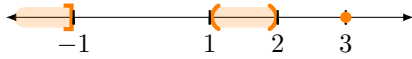
$$S = \{x : 0 < |x - x_0| < \varepsilon\}$$

is a deleted neighborhood of  $x_0$ . We also say it is deleted  $\varepsilon$ -neighborhood of  $x_0$ .

Let  $S$  be a subset of  $\mathbb{R}$ .

**Definition 10.**  $x_0$  is a *limit point* of  $S$  if every deleted neighbourhood of  $x_0$  contains a point of  $S$

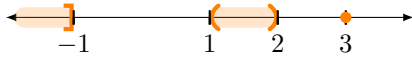
**Example 9.** Let  $S = (-\infty, -1] \cup (1, 2) \cup \{3\}$ . Find the limit points of  $S$ .



**Solution 9.** Every point in  $S$  is limit point, apart from  $\{3\}$

**Definition 11.**  $x_0$  is a *boundary point* of  $S$  if every neighborhood of  $x_0$  contains at least one point in  $S$  and one not in  $S$ . The set of boundary points of  $S$  is the *boundary* of  $S$ , denoted by  $\partial S$ . The *closure* of  $S$ , denoted by  $\bar{S}$ , is  $\bar{S} = S \cup \partial S$

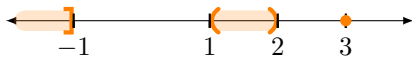
**Example 10.** Let  $S = (-\infty, -1] \cup (1, 2) \cup \{3\}$ . Find the closure of  $S$ .



**Solution 10.** The boundary points of  $S$  are:  $\partial S = \{-1, 1, 2, 3\}$  and  $\bar{S} = (-\infty, -1] \cup [1, 2] \cup \{3\}$

**Definition 12.**  $x_0$  is an *isolated point* to  $S$  if  $x_0 \in S$  and there is a neighborhood of  $x_0$  that contains no other points of  $S$ .

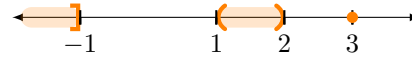
**Example 11.** Let  $S = (-\infty, -1] \cup (1, 2) \cup \{3\}$ . Find the isolated points of  $S$ .



**Solution 11.** There's only one isolated point of  $S$ ,  $\{3\}$ .

**Definition 13.**  $x_0$  is *interior* to  $S$  if  $x_0$  is in the interior of  $S^c$ . The collection of such points is the *exterior* of  $S$ .

**Example 12.** Let  $S = (-\infty, -1] \cup (1, 2) \cup \{3\}$ . Find the exterior of  $S$ .



**Solution 12.** We can write  $S^c = (-1, 1] \cup [2, 3) \cup (3, \infty)$  and then we can find the exterior of  $S$ ,  $(S^c)^o = (-1, 1) \cup (2, 3) \cup (3, \infty)$ .

**Theorem 3.** A set  $S$  is closed if and only if no point  $S^c$  is a limit point of  $S$

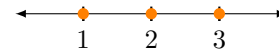
*Proof.* Begin by proving the forward direction. If  $S$  is closed, then  $S^c$  is open and then for any  $x_0 \in S^c \exists \varepsilon$ -neighborhood contained in  $S^c$ , hence  $x_0$  cannot be a limit point.

Next for the reverse direction. If no point of  $S^c$  is a limit point of  $S$ , every point  $x_0 \in S^c$  has neighborhood contained in  $S^c$ , hence  $S^c$  is open. Therefore,  $S$  is closed.  $\square$

**Corollary 1.** A set is closed if and only if it contains all its limit points

What if it has no limit points? Well, it's just closed. Let us take an example:

**Example 13.** Take  $S = \{1, 2, 3\}$  and determine whether it closed or not.



**Solution 13.** You can take a deleted  $\varepsilon$ -neighborhood of all of these points and see that none of them are limit points. Hence the set is closed.

### 5.3 Open Coverings

A collection  $\mathcal{H}$  of open sets is an open covering of a set  $S$  if every point in  $S$  is contained in a set  $H$  belonging to  $\mathcal{H}$ ; that is, if  $S \subset \{H : H \in \mathcal{H}\}$

**Theorem 4** (Heine-Borel Theorem). If  $\mathcal{H}$  is an open covering of a closed and bounded subset  $S$  of the real line, then  $S$  has an open covering  $\tilde{\mathcal{H}}$  consisting of finitely many open sets belonging to  $\mathcal{H}$ .

*Proof.* Since  $S$  is bounded, it has supremum,  $\alpha$ , and infimum,  $\beta$ . As  $S$  is closed, we can say that  $\alpha$  and  $\beta$  are in  $S$ . Now we shall define a couple of things:

$$S_t = S \cap [\alpha, t] \quad \alpha \leq t$$

and let

$$F = \{t : \alpha \leq t \leq \beta\}$$

and finitely many sets from  $\mathcal{H}$  to cover  $S_t$

As we know that  $S_\beta = S$ , all we need to prove is  $\beta \in F$ . We shall use the completeness of the reals to prove this.

Starting with the things we know,  $\alpha \in S$ . We can then deduce that  $S_\alpha = \{\alpha\}$ , which is contained in some open set,  $H_\alpha$  from  $\mathcal{H}$  as we know that  $\mathcal{H}$  covers  $S$ , therefore  $\alpha \in F$ . Since  $F$  is non-empty and bounded above by  $\beta$  we wish to show that the supremum,  $\gamma$ , can be seen the same as  $\beta$ . As we definitionally know that  $\gamma \leq \beta$  from  $F$ , it suffices to know that  $\gamma \not< \beta$ . Let us consider two cases:

*Case 1:* Suppose that  $\gamma < \beta$  and  $\gamma \notin S$ . Since  $S$  is closed,  $\gamma$  is not a limit point of  $S$ . This means that  $\exists \varepsilon > 0$  :

$$[\gamma - \varepsilon, \gamma + \varepsilon] \cap S = \emptyset$$

Then  $S_{\gamma-\varepsilon} = S_{\gamma+\varepsilon}$ , which can't happen, due to  $\gamma$  being a supremum  $S_{\gamma-\varepsilon}$  has a subcovering from  $\mathcal{H}$ , but  $S_{\gamma+\varepsilon}$  wouldn't. This is a contradiction and hence if  $\gamma \notin S$ , then  $\gamma \not< \beta$ .

*Case 2:* Suppose that  $\gamma < \beta$  and  $\gamma \in S$ . Then there is an open set  $H_\gamma \in \mathcal{H}$  along with an interval, for  $\varepsilon > 0$ ,  $[\gamma - \varepsilon, \gamma + \varepsilon]$ . It then follows that since  $S_{\gamma-\varepsilon}$  has a finite covering, so does  $S_{\gamma+\varepsilon}$ . This is a contradiction from the definition of  $\gamma$ . Hence, if  $\gamma \in S$  then  $\gamma \not< \beta$ .

So we know that  $\beta = \gamma$ . Therefore  $H_\beta$  exists and is in  $\mathcal{H}$ .  $H_\beta$  contains  $\beta$  and an interval of the form, for some  $\varepsilon > 0$ :  $[\beta - \varepsilon, \beta + \varepsilon]$ . Since we know that  $S_{\beta-\varepsilon}$  is covered finitely by some collection:  $\{H_1, \dots, H_k\}$ , then  $S_\beta$  is covered by some collection:  $\{H_1, \dots, H_k, H_\beta\}$ . Hence,  $S_\beta = S$ .  $\square$

**Definition 14** (Compactness). A set is *compact* if it is closed and bounded

**Theorem 5** (Bolzano-Weirstrass Theorem). *Every bounded infinite set of real numbers has at least one limit point*

*Proof.* It will suffice to show that a bounded nonempty set without a limit point can only contain a finite number of elements.

If  $S$  has no limit points, then  $S$  is closed and every point,  $x \in S$  has an open neighborhood,  $N_x$ , that only contains itself. The collection:

$$\mathcal{H} = \{N_x : x \in S\}$$

is an open covering for  $S$ . Heine-Borel Theorem states that  $S$  can be covered by finitely many elements of  $\mathcal{H}$ , say  $N_{x_1}, \dots, N_{x_n}$ . Since these sets only depend on a finite set of points,  $x_1, \dots, x_n$ . Then  $S = \{x_1, \dots, x_n\}$  and hence, is finite.  $\square$