Year MAGIC — Algebraic Geometry

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine (especially the typos!).

Contents

In this course we will study some introductory algebraic geometry, we will study Classical Algebraic Geometry and Sheaves There are three chapters,

(i) Affine Varieties

(ii) Noetherian Rings

(iii) Algebraic Varieties in general

Literature: Karen Smith's Book, has lots of examples and is very readable. We will cover chapter one and the start of chapter two of Hardshawn.

Prerequistites: Commutative Algebra, Topology.

1 Chapter 1 - Affine Varieties

Algebraic Sets in n-space, we want to study zero sets of polynomials in several variables in affine spaces. The affine spaces are k-vector spaces. We will consider algebraically closed fields k.

Definition 1.1 (Affine n-space). Let k be a field. We write $\mathbb{A}^n(k)$ to be an affine n-space over k. This is the set, $\{a_1, a_2, \dots, a_n : a_i \in k\}$

Let $k[X_1, \dots, X_n]$ be the polynomial ring in n-variables over k where $n < \infty$.

Definition 1.2 (Vanishing Set). Let $f \in k[X_1, \dots, X_n]$ then the zero-set of f is,

$$\mathcal{V}(f) = \{(a_1, \dots, a_n) \in \mathbb{A}^n(k) : f(a_1, \dots, a_n) = 0\}$$

Example. Let $k = \mathbb{R}$ and $n = 1$, then $f(X) = X + 1$,

$$\mathcal{V}(f) = \{-1\} \in \mathbb{A}^1(\mathbb{R})$$

Example. Let $k = \mathbb{R}$, $n = 2$ and $f(X, Y) = X^2 + Y^2 - 1$, then,

$$\mathcal{V}(f) = \{X, Y \in \mathbb{R}^2 : X^2 + Y^2 = 1\}$$

Example. Let $k = \mathbb{R}$, $n = 3$ and $f(X, Y, Z) = Z^3 + Z^2Y^2 - X^2$, this is not as obvious. The vanishing set is just some curve, and if we intersect it with a sphere we get,

This is slightly odd, it intersects itself and so this isn't a manifold and so is slightly more complicated.

More generally: $f_1, \dots f_m \in k[X_1, \dots, X_n]$, we define,

$$\mathcal{V}(f_1, \dots, f_m) = \{a \in \mathbb{A}^n : f_1(a) = f_2(a) = \dots = f_m(a) = 0\}$$

Even more generally, we can take any $S \subseteq k[X_1, \dots, X_n]$, then

$$\mathcal{V}(S) = \{a \in \mathbb{A}^n : f(a) = 0 \, \forall f \in S\}$$

This allows us to have infinitely many functions. We call S an algebraic subset of $\mathbb{A}^n$.

Example.

$$\mathcal{V}(X^2 - Y, X^3 - Z) \subset \mathbb{A}^3(\mathbb{R})$$

This defines a smooth space curve.

Example. $M_{n\times n}(\mathbb{C})$ can be identified by $\mathbb{A}^{n^2}(\mathbb{C})$ and we can look at subsets of this space. Let $V = \{A \in M_{n\times n}(\mathbb{C}) : \det A = 1\}$. $V = \mathcal{V}(S)$ is an algebraic subset of $\mathbb{A}^{n^2}$. For $\mathbb{A}^{n^2}$ we associate $k[X_{ij}]$ where $1 \leq i, j \leq n$. Let $S = \Delta - 1$ where

$$\Delta(X_{ij}) = \det \begin{pmatrix} X_{11} & \dots & X_{1n} \\ & \ddots & \\ X_{n1} & \dots & X_{nn} \end{pmatrix}$$

We can say slightly more than this,

Remark. (i) $\mathbb{A}^n$ is a algebraic subset, 0 is a polynomial and we can see that $\mathcal{V}(0) = \mathbb{A}^n$.

(ii) $\varnothing$ is an algebraic set, $V(1) = \{a \in \mathbb{A}^n : 1(a) = 1 = 0\} = \varnothing$.

(iii) Algebraic sets are closed under intersection. Let $V(S_i)_{i\in\mathcal{I}}$ be a collection of algebraic sets in $\mathbb{A}^n$, then,

$$\bigcap_{i\in\mathcal{I}} V(S_i) = V\left(\bigcup_{i\in\mathcal{I}} S_i\right)$$

Proof. Exercise ☐

(iv) Algebraic sets are closed under **finite** unions. We want to show that the union of two algebraic sets is algebraic. Let $V(S), V(T)$ be algebraic sets in $\mathbb{A}^n$, let $S.T = \{fg : f \in S, g \in T\}$. Then we claim that $V(S) \cup V(T) = V(S.T)$. We aim to show both inclusions,

Proof. ($\subset$): Suppose $a \in V(S)$, then $f(a) = 0$ for all $f \in S$, but, $(f \cdot g)(a) = f(a) \cdot g(a) = 0$ for all $g \in T$. Therefore $a \in V(S.T)$.
($\supset$) Suppose $a \in V(S.T) \setminus V(S)$. Then there is some $f \in V(S)$ such that $f(a) \neq 0$, but then, for any $g \in T$ $fg(a) = f(a) \cdot g(a) = 0$ as $a \in V(S.T)$ and as we are in a field, and as $f(a) \neq 0$, then $g(a) = 0$ for all $g \in T$. Therefore $a \in V(T)$. ☐

Proposition 1.3. The collection of algebraic subsets of $\mathbb{A}^n(k)$ form the closed sets of a topology on $\mathbb{A}^n$. This topology is called the Zariski Topology on $\mathbb{A}^n$.

Here are some examples of closed sets,

Example. If $a \in \mathbb{A}^n$ is a point then $\{a\} = V(X_1 - a_1, X_2 - a_2, \ldots, X_n - a_n)$ and so points are closed in the Zariski Topology.

Example. If $n = 1$ and $S = 0$, then $V(S) = \mathbb{A}^n$, but if $S \subseteq \mathbb{A}^n$ is algebraic, and if $\exists\, f \neq 0 \in S$ then since we have every polynomial in $k[X]$ has finitely many zeros. Then $\mathcal{V}(f)$ is finite. However $\mathcal{V}(S) \subseteq \mathcal{V}(f)$ and so $\mathcal{V}(S)$ must be finite. Therefore the Zariski Topology is cofinite, the sets are finite or the whole space.

We defined the most important algebraic variety last time and then we defined an algebraic set, $\mathcal{V}(S)$. We start with some remarks from last time. There is some issues about k, we assumed that we could take any k. If k is finite, then $p = p^e$, then for $a \in \mathbb{F}_p$ then $a^q = a$ by Euler Fermat Theorem. Then $f(X) = X^q - X \in \mathbb{F}_q[X]$, evaluates to 0 forall $a \in k$, but f is not the zero polynomial. So we have problems with finite fields.

The other issues, is more geometric. If we have $\mathbb{R}$, then $X^2 + 1 \in k[X]$ doesn't have any zeros. Hence, we need to work with algebraically closed sets (another example is the Whitney Umbrella).

From now on, we consider $k = \overline{k}$, here k is algebraically closed.
Also, all rings in the course are commutative, and contain 1 and all ring homomorphisms take 1 to 1.

2 Affine Varieties

Today, we give a definition of an affine variety that is dependant of the embedding in $\mathbb{A}^n$. Therefore, we need to define an algebra,

Definition 2.1 (Algebra)**.** Let k be a field and A be a ring, that is also a k-vector space. Then A is a k-algebra if $\lambda \cdot (ab) = (\lambda \cdot a) \cdot b$, for all $\lambda \in k$ and $a, b \in A$

The trivial example is k is a k-algebra. The second example is $A = k[X]$. The third is that k being any field and V a set, let $A := \text{Map}(V, K)$, this is a k-algebra as A is a ring with $(f + g)(v) = f(v) + g(v)$ and $(f \cdot g)(v) = f(v) \cdot g(v)$. A is a k-vector space as $(\lambda \cdot f)(v) = \lambda \cdot f(v)$ for all $\lambda \in k$, for all $v \in V$.

We now need morphisms,

Definition 2.2 (k-algebra homomorphism)**.** Let A, B be k-algebras. A map $\phi : A \to B$ is a morphism between k-algebras if it is a ring homomorphism and a k-linear map. We write,

$$\hom_{k-alg}(A, B) = \{k\text{-alg homom from} A \to B\}$$

Definition 2.3 (Subalgebra)**.** Let $C \subseteq A$, C is a subalgebra if C is a subring and a k-subspace.

If $A = k[X]$ and $B = k$ are k-algebras, then

$$\hom_{\text{k-alg}}(k[X], k) \ni \phi$$

Then ϕ is determined by $\phi(X) = a \in k$. Have a bijection $a \in k$, then we can associate a $\phi : k[X] \to K$ to it. We can associate $a \mapsto (\phi_a : X \mapsto a)$. We will see that in more generalaity that

$$\hom_{k-alg}(k[X], k) = \mathbb{A}^1(k)$$

In this course we will see that considering all algebras is too much, but there is one that is enough to describe what we want. We want to look at the right type of algebras, more specifically the finitely generated k-algebra

Definition 2.4 (Finitely generated k-algebra)**.** A is finitely generated if $A = k[a_1, a_2, \ldots a_n]$ for some finite set $S = \{a_1, a_2, \ldots, a_n\} \subseteq A$.

Then we define a morphism $\phi : k[X_1, \ldots, X_n] \to A = k[a_1, \ldots, a_n]$, we define $X_i \mapsto a_i$ for all i. Now we see that ϕ is surjective and so by the First Isomorphism Theorem for k-algebras we get $k[X_1, \ldots, X_n]/\operatorname{Ker}\phi \cong A$. We know $\operatorname{Ker}\phi$ is an ideal in $k[X_1, \ldots, X_n]$ and so finitely generated k-algebras are the same, in a bijection of rings $k[X_1, \ldots, X_n]/I$.

If $A \subseteq \text{Map}(V, K)$ be a subalgebra and $x \in V$, then there is always a k-algebra homomorphism $\varepsilon_x : A \to k$ where $\varepsilon_x(f) \mapsto f(x)$. This ε_x is the evaluation homomorphism at the element x. Now assume $k = \overline{k}$ (algebraically closed), then,

Definition 2.5 (Affine k-variety)**.** An affine k-variety is a pair (V, A), where V is a set and $A \subseteq \text{Map}(V, K)$ is a finitely generated sub-algebra such that

$$V \to \hom_{k-alg}(A, k)$$

$$x \mapsto \varepsilon_x$$

is a bijection.

This means, the elements of V correspond one to one with k-algebra homomorphisms from $A \to k$. Here is an example, Consider the pair $(\mathbb{A}^n(k), k[X_1, \dots, X_n])$ this an affine variety. A is finitely generated by $X_1, \dots, X_n$. The X_i are defined coordinate function $X_i(x_1, x_2, \dots x_n) = x_i$, we now show this is a bijection. Assume we have $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n) \in \mathbb{A}^n$ and $\varepsilon_x = \varepsilon_y$. Then, $\varepsilon_x(X_i) = \varepsilon_y(Y_i)$ for all i. Then by Exercise 1, $(x_1, \dots, x_n) = (y_1, \dots, y_n)$. Hence, it is injective. We now show surjectivity, $\phi \in \hom_{k-alg}(A, k)$. Set $x_i := \phi(X_i)$ for all i and $X \in A^n$. Then $\phi(X_i) = x_i = \varepsilon_x(X_i)$ for all i. Since, X_i generate A, we must have $\phi(f) = \varepsilon_x(f)$ for all $f \in A$, and hence $\phi = \varepsilon_x$. Therefore we have surjectivity. In total, $\mathbb{A}^n \to \hom_{k-alg}(k[X_1, \dots, X_n], k)$ is a bijection and $(\mathbb{A}^n, k[X_1, \dots, X_n])$ is an affine variety.

We say V is an affine variety to mean, we consider the pair (V, A) for $A := k[V]$ is the coordinate algebra of A. We can now do the Zariski Topology of an affine variety.

2.1 Zariski Topology of an affine variety

Let (V, A) be an affine variety. Then $S \subseteq A$ define

$$\mathcal{V}(S) := \{x \in V : f(x) = 0 \, \forall f \in S\}$$

Exercise. Show that $\mathcal{V}(S)$ $S \subseteq A$ form the closed sets of a topology on V, the Zariski Topology.

Proposition 2.6. Let $W \subseteq V$, where (V, A) is an affine variety, and W is closed. Then W itself is an affine variety.

The issue is, what is $k[W]$? Then we have to show that $(W, k[W])$ satisfies an affine variety.

Proof. Denote, $B := \{f|_W : f \in A\}$ and let $\pi : A \to B : f \mapsto f|_W$ be the restriction map. Now we want to show that (W, B) is an affine variety. First, $B = \pi(A)$ is a finitely generated subalgebra of A, and so B is finitely generated. Since W is closed we have that $W = \mathcal{V}(S)$ $(S \subseteq A)$. Let $x \in W$ and $\varepsilon'_X B \to k$ be the evaluation at x. Note $\varepsilon'_X \circ \pi = \varepsilon_X : A \to k$ Now it remains to show that $W \to \mathrm{Hom}_{k-alg}(B, k)$ where $x \mapsto \varepsilon'_X$ is a bijection. This is injective as if $x, y \in W$ and $\varepsilon'_X = \varepsilon'_Y$, then $\varepsilon'_X \circ \pi = \varepsilon'_Y \circ \pi$ and so $\varepsilon_X = \varepsilon_Y$ hence $x = y$.

Surjective. Let $\theta \in \mathrm{Hom}_{k-alg}(B, k)$. Then $\theta \circ \pi = \mathrm{Hom}(A, k)$ and so $\theta \circ \pi = \varepsilon_x$ for some $x \in V$. Now we want to show this is just some evaluation map. ☐

Remark. This now gives us lots of examples.

Example. Last time we say that $(\mathbb{A}^n, k[X_1, X_2, \dots, X_n])$ is an affine variety, a closed subset $\mathcal{V}(S)$ where $S \subseteq k[X_1, \dots, X_n]$ and so these are just algebraic sets. Hence algebraic sets are affine k-varieties. Hence we have the varieties, $(\mathcal{V}, k[X_1, \dots, X_n]/I_{\mathcal{V}})$ where $I_{\mathcal{V}} = \{f \in k[X_1, \dots, X_n] : f|_{\mathcal{V}} = 0\}$.

Definition 2.7 (Morphism)**.** Let $(V, k[V])$ and $(W, k[W])$ be affine k-varieties. A map $\phi : V \to W$ is called a morphism of affine varieties if $g \circ \phi \in k[V]$ for all $g \in k[W]$.

Suprisingly we have another morphism, called a comorphism

Definition 2.8 (Co-morphism)**.** Let $\phi^\sharp : k[W] \to k[V]$ be defined by $g \mapsto g \circ \phi$, this is called the co-morphism of ϕ.

and now as expected, an isomorphism,

Definition 2.9 (Isomorphism)**.** ϕ is an isomorphism if and only if ϕ is morphism and there is a $\psi : W \to V$ is a morphism such that $\phi \circ \psi = \mathrm{id}_W$ and $\psi \circ \phi = \mathrm{id}_V$.

Example. Exercise 7 and 8 will be useful here.

Now we want to show a closed subsets of $\mathbb{A}^n$ correspond to affine varieties. We have seen the forward direction already.

Lemma 2.10. Let $\phi : V \to W$ be a morphism of affine varieties and assume $\phi^\sharp : k[W] \to k[V]$ is surjective. Then the image $\phi(V) \subseteq W$ is closed and $\phi|_V : V \to \phi(V)$ is an isomorphism.

Remark. We may ask why we need the comorphism to be surjective? Well then $\mathrm{im}(\phi)$ will not necessarily closed. Here is an example,

Example. Take $\phi : \mathbb{A}^2 \to \mathbb{A}^2$ where $(x, y) \mapsto (xy, y)$, then the comorphism, $\phi^\sharp : k[X, Y] \to k[Z, W]$ is $X \mapsto ZW$ and $Y \mapsto W$ and so we know $f(X, Y) \mapsto f(ZW, W)$. Why is the image not closed? Well the image is $\mathrm{im}(\phi) = \{(a, b) \in \mathbb{A}^2 : a = xy, b = y\}$. If $b \neq 0$ so $y = b$ and $x = \frac{a}{b}$, then we have a preimage. If $b = 0$ then $a = 0$ and the preimage is just the origin. Hence the image of ϕ is $\mathbb{A}^2 \setminus \{(x, 0) : x \neq 0\}$. Why is this not closed? Well $\mathbb{A}^2 \setminus \{(x, 0) : x \neq 0\} = (\mathbb{A}^2 \setminus \{(x, 0) : x \in \mathbb{R}\}) \cup \{(0, 0)\} = \mathbb{A}^2 \setminus \{(x, 0) : x \in \mathbb{R}\} \cap \mathbb{A}^2 \setminus \{(0, 0)\}$. *This is the union of an open and a closed set, this is still closed and is shown in the written notes. We assume that it's closed and then move to a contradiction.*

Proof. Let $Z := \phi(V) = \mathrm{im}(\phi)$ and $I := \mathrm{Ker}(\phi^\sharp)$. We use the first isomorphism theorem, we have

$$\begin{tikzcd} k[W] \arrow[r, "\phi^\sharp"] \arrow[d] & k[V] \\ k[W]/I \arrow[ur, "\overline{\phi^\sharp}"'] & \end{tikzcd}$$

and we have isomorphism. We denote the inverse of the isomorphism as $\theta : k[V] \to k[W]/I$. We claim $Z = V(I)$, so we show both inclusions.
($\subseteq$) Let $g \in I$, $x \in V$, then $g(\phi(X)) = \phi^\sharp(g)(x) = 0$ and so $\phi(x) \in V(I)$ for all $x \in V$. Therefore $Z \subseteq V(I)$.
($\supseteq$) Assume $y \in V(I) \subseteq W$. Then $\varepsilon'_y : k[W] \to k$ is zero for all $g \in I$. Therefore we get a homomorphism of algebra $\overline{\varepsilon'_y} : k[W]/I \to k$.

$$\begin{tikzcd} k[V] \arrow[r, "\theta"] \arrow[d, dashed] & k[W]/I \arrow[dl, "\overline{\varepsilon'_y}"] \\ k & \end{tikzcd}$$

Then we have a homomorphism $\overline{\varepsilon'_y} \circ \theta : k[V] \to k$. Since V is an affine variety we have $\overline{\varepsilon'_y} \circ \theta = \varepsilon_x$ for some $x \in V$. Let us write this out,

$$\overline{\varepsilon'_y} \circ \theta \circ \overline{\phi^\sharp} = \overline{\varepsilon'_y} = \varepsilon_x \circ \overline{\phi^\sharp}$$

For $g \in k[W]$ and so $\overline{\varepsilon'_y}(y + I) = \varepsilon_x \circ \overline{\phi^\sharp}(y + I)$ and so $\varepsilon'_y = \phi^\sharp(g)(x) = g(\phi(x)) = \varepsilon'_{\phi(x)}(g)$. Since W is affine $y = \phi(x) \subseteq \phi(V) = Z$. Therefore $V(I) \subseteq Z$. This shows that Z is closed.

We now restrict ϕ to V, we use Exercise 8 and so we show that the morphism is an isomorphism of k-algebras. We have subjectivity as of this diagram and then injectivity is we take $h \in k[Z]$ and then $\phi^\sharp|_V = 0$ and then we have that the $\mathrm{Ker}(\phi^\sharp|_V) = 0$ ☐

We use the algebra to show something geometrically. Algebra is more useful to prove things. Now we have a load of interesting examples. If we have some closed subset of $\mathbb{A}^n$ this is an affine variety. Now if we have an affine variety, then it is the vanishing set of some ideals. If we have some affine variety V and it's coordinatering $k[V]$, we assume it's fg by some $k[V] = k[f_1, f_2, \ldots, f_n]$ and so now we define a map $\phi : V \to \mathbb{A}^n$

$$\phi : x \mapsto (f_1(x), f_2(x), \ldots, f_n(x))$$

Now we want to prove this a morphism of varieties, so $X_i \in k[\mathbb{A}^n]$, the i^{th} coordinate function. We can look at $(X_i \circ \phi)(x) = X_i(\phi(x)) = f_i(x)$ and so $X_i \circ \phi = f_i \in k[V]$. Since the X_i generate $k[\mathbb{A}^n]$ and so $g \circ \phi \in k[V]$

for all $g \in k[\mathbb{A}^n]$ thus ϕ is a morphism. Moreover, $f_i = \phi^\sharp(X_i)$ and thus $k[V] = k[\phi^\sharp(X_1), \dots, \phi^\sharp(X_n)]$. Hence $\phi^\sharp : k[\mathbb{A}^n] \to k[V]$ is surjective and the lemma tells us that $\mathrm{im}(\phi)$ is closed and $\phi|_V : V \to \mathrm{im}\,\phi$ is an isomorphism. Therefore V is isomorphic to a closed set in $\mathbb{A}^n$.

Affine Varieties are precisely the closed sets in $\mathbb{A}^n$

Now we can immediately see that points are always closed in affine varieties. Points in $\mathbb{A}^n$ are closed. if $x \in \mathbb{A}^n$ then

$$x = (x_1, \dots, x_n) = \mathcal{V}(\{X_1 - x_1, \dots, X_n - x_n\})$$

since $V \cong$ a closed subsets of $\mathbb{A}^n$ for some n, it follows that points in V are closed.

3 Principal Open Sets and Products

We have seen closed sets in the Zariski topology, (V, A). The open sets, are the complements of closed sets, are in general not affine variety. But, there is one important example of open sets that have the structure of an affine variety: Let (V, A) be an affine variety, $0 \neq f \in A$

$$\mathcal{V}_f = \{x \in V : f(x) \neq 0\}$$

Example. Let $V = \mathbb{A}^2$ and $f = X^3 - Y^2$

Definition 3.1 (Principal Open Set)**.** $\mathcal{V}_f$ is called a principal open set of V, $V \setminus \mathcal{V}(f)$

Exercise. The sets $\mathcal{V}_f$ for a basis of the Zariski topology on V.

(i) $V = \bigcup \mathcal{V}_f$

(ii) if $x \in \mathcal{V}_f \cap \mathcal{V}_{f'}$ and there is some $g \in A$ and $x \in \mathcal{V}_g$ then $\mathcal{V}_g \in \mathcal{V}_f \cap V_{f'}$

Now we need a coordinate ring, so we localise, $A_f = \{\frac{a}{f^r} : a \in A, r \geq 0\} = A\left[\frac{1}{f}\right]$ this is not the localisation in a prime ideal, $A_{(f)} = \{\frac{a}{b} : a \in A, b \in A \setminus \langle f \rangle\}$. This is though, a finitely generated algebra. We now claim that

Claim. The pair $(\mathcal{V}_f, A_f)$ is an affine variety.

Proof. We need $A_f \subseteq \text{Map}(\mathcal{V}_f, k)$. We always have a map $\Phi : A_f \to \text{Map}(V_f, k)$, where $\Phi : \frac{a}{f^r} \to \frac{a}{f^r}(x) := \frac{a(x)}{f^r(x)}$ for any $f, a, x \in \mathcal{V}_f$. Φ is injective, because $\Phi(\frac{a}{f^r})$, then $a(x) = 0$ as $f(x) \neq 0$ and then $af = 0$ for all $x \in V$ and $\frac{a}{f^r} = \frac{af}{f^{r+1}} = 0$. Therefore Φ is injective. Therefore, $A_f \subseteq \text{Map}(\mathcal{V}_f, k)$ is a subalgebra.

Now we need to show that

(i) $A_f = A\left[\frac{1}{f}\right]$ is a finitely generated k-algebra.

(ii) $\mathcal{V}_f \to \text{Hom}_{k-alg}(A_f, k)$ is a bijection, $x \mapsto \varepsilon_x$ is a bijection. Show injective and surjective using properties of localisation.

□

Example. Consider $(V, A) = (\mathbb{A}^1, k[X])$. Let $f = X$ and $\mathcal{V}_f = \{x \in \mathbb{A}^1 : x \neq 0\}$. We have $\mathbb{A}^1 \setminus \{(0,0)\}$ now we claim this an affine variety, that is a zero set of some polynomial. By construction, $A_f = k[X]_X = k[X, \frac{1}{X}] = k[X, X^{-1}]$ and $(\mathbb{A}^1 \setminus \{0\}, k[X, X^{-1}])$ is an affine variety. We can look at a $\phi : Z = \mathcal{V}(XY - 1) \subset \mathbb{A}^1$ that is $t \mapsto (t, t^{-1})$ and this is an isomorphism. We use Exercise 7 and then show $\phi^\sharp$ is an algebra homomorphism, $\phi^\sharp : k[X, Y]/(XY - 1) \to k[X, X^{-1}]$. The nice picture you should get

3.1 Products

This is another construction of affine varieties: we take two affine varieties (V, A) and (W, B) then we can take cartesian products $(V \times W, A \otimes_k B)$ is an affine variety.

Example. Let $V = W = \mathbb{A}^1$ then $V \times W = \mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$ and so we have the affine variety $(A^2, k[X] \otimes_k k[Y] = k[X, Y])$ and there is something interesting about the topology.

If we have $(x, y) \in V \times W$ where $f \in A$ and $g \in B$ then we need to build up a function from the cartesian product to k. We write $f \otimes g : V \times W \to k$ and $(x, y) \mapsto f(x)g(y)$. We recall $A \otimes_k B$ is the k-span of the elements $f \otimes g$ where $f \in A$ and $g \in B$.

Remark. If $\{f_i : i \in I\}$ and $\{g_j : j \in J\}$ are bases of A and B respectively, then $\{f_i \otimes g_j\}$ are a k-basis of $A \otimes_k B$.

We now need the ring structure, but this can again be constructed, $A \otimes B$, $f_1, f_2 \in A$ and $g_1, g_2 \in B$ what happens to $(f_1 \otimes g_1) \cdot (f_2 \otimes g_2) := (f_1 \cdot f_2) \otimes (g_1 g_2)$ this means that $A \otimes B$ is a finitely generated k-algebra. We cal also show that $A \otimes B \subseteq \text{Map}(V \times W, k)$ and $(V \times W, A \otimes B)$ is an affine variety.

4 Chapter 2 - Noetherian Spaces

We recall the definition of a Noetherian Ring. Let A be a (comm) ring with elements $f_1, \dots, f_n \in A$ then the ideal generated by $f_1, \dots, f_n$ is $I = \langle f_1, f_2, \dots, f_n \rangle = \{a_1 f_1 + \dots a_n f_n\} \subseteq A$. An ideal $A \subseteq A$ is finitely generated if and only if we can fine $f_1, \dots f_n \in A$ such that $I = \langle f_1, \dots, f_n \rangle$.

Definition 4.1 (Noetherian Ring)**.** A is Noetherian if every ideal in A is finitely generated.

This is equivalent to A having the asending chain condition. That is, every chain of ascending ideals $I_1 \subseteq I_2 \subseteq \dots \subseteq I_n \subseteq A$ becomes stationary, i.e. $\exists\, N$ such that $I_n = I_N$ for all $n \geq N$.

Proof. Exercise ☐

Example. Consider $A = k[X]$, we use the Euclidean Algorithm for polynomials and show that every ideal in $k[X]$ can be generated by one element $f(X)$, $I = \langle f(X) \rangle$. Therefore, $k[X]$ is Noetherian.

A non-example,

Example. Consider $A = k[X_1, \dots]$ is not Noetherian. Consider the ideal generated by all of the X_i's.

Exercise. If $\phi : A \to B$ is a surjective ring homomorphism and A is Noetherian, then B is Noetherian.

Theorem 4.2 (Hilbert Basis Theorem)**.** If A is Noetherian, then $A[X]$ is Noetherian.

This tells us that if k is a field, then $k[X]$ is Noetherian, and so therefore $k[X_1, \dots, X_n]$ is Noetherian. If $A = k[a_1, \dots, a_n]$ is a finitely generated k-algebra, then we have a surjective algebra homomorphism $\phi : k[X_1, \dots, X_n] \to A$ where it's defined by $X_i \mapsto a_i$. Then we use Exercise 2, to see that A is Noetherian. This current isn't geometric, we will discuss the geometric interpretations now.

Definition 4.3 (Topological Noetherian Space)**.** *A topological space X is Noetherian if every descending chain of closed subsets $Z_1 \supseteq Z_2 \supseteq Z_3 \supseteq \dots \supseteq Z_n$ becomes stationary.*

This can be seen to be equivalent to saying that every chain of open subsets becomes stationary or every non-zero collection of closed subsets of X has a minimal element.

Now let $(V, k[V])$ be an affine variety, and $Z \subseteq V$ a subset. We define $I(Z) = \{f \in k[V] : f(x) = 0\, \forall x \in Z\}$. If $Z \subseteq V$, then $Z = \mathcal{V}(S)$ for some $S \subseteq k[V]$, since $S \subseteq I(Z) \implies \mathcal{V}(I(Z)) \subseteq \mathcal{V}(S) = Z$, clearly $\mathcal{V}(S) \subseteq \mathcal{V}(I(Z))$. Therefore $\mathcal{V}(S) = \mathcal{V}(I(Z))$. Therefore the vanishing set of a subset of $k[V]$ is an ideal.

If $Z_1 \supseteq Z_2 \supseteq \dots$ is a descending chain of closed subsets in V, then obtain thr ascending chain of ideals in $k[V]$:

$$I(Z_1) \subseteq I(Z_2) \subseteq \dots \subseteq k[V]$$

Since $k[V]$ is Noetherian, then there must be some N such that for all $n \geq N$, $I(Z_n) = I(Z_N)$ and so $\mathcal{V}(I(Z_n)) = \mathcal{V}(I(Z_N))$ and so $\mathcal{V}$ is a Noetherian space.

4.1 Irriducible Components

Here is some motivation. Consider $W = V[X^2YZ + Y^2Z - YZ^2] \subseteq \mathbb{A}^3$, this is two planes intersecting with a curve as you can factor this to $YZ(X^2 + Y - Z)$. This has three components, $I(W) = \langle Y \rangle \cap \langle Z \rangle \cap X^2 + Y - Z$

Definition 4.4 (Irreducible)**.** A topological space X is called irreducible if it cannot be written as a union $X = Y \cup Z$ for some proper subsets $Y, Z \subsetneq Z$

Exercise. This for open subsets.

Exercise. An affine variety V is irreducible if and only if $k[V]$ is a integral domain.

A closed subset $Z \subseteq V$ is closed iff $I(Z)$ is a prime ideal in $k[V]$

Remark. If X is a Noetherian space, then it may be written as a finite union of closed subsets.

Definition 4.5 (irredundant union)**.** Let X be a set and we have some subsets $X_1, \dots, X_n \subseteq X$ such that $X = \bigcup_{i=1}^n X_i$. This is an irredundant union of $\forall i \neq j$, $X_i \not\subseteq X_j$.

Example. Let $X = \mathcal{V}(\langle X \rangle) \cup \mathcal{V}(\langle Y \rangle) \cup \mathcal{V}(\langle X, Y \rangle) \subseteq \mathbb{A}^2$. This is irredundant as $\mathcal{V}(\langle X, Y \rangle) \subseteq \mathcal{V}(\langle X \rangle)$.

Theorem 4.6. Let X be Noetherian, then X can be written as a finite irredundant union of $X_1, \dots, X_n$ of closed irreducible subsets $X = \bigcup_{i=1}^n X_i$

Proof. Ommited, too similar to primary decomposition. □

Definition 4.7 (Irreducible Compontents)**.** These X_i's are called the irreducible components of X

Example. Thinking back to the motivating example, the irreducible components are $\mathcal{V}(Y)$, $\mathcal{V}(Z)$ and $\mathcal{V}(X^2 - Y + Z)$.

Example. Let $V = \mathcal{V}(XZ, YZ) \subseteq \mathbb{A}^3$, this is just $\mathcal{V}(Z) \cup \mathcal{V}(X, Y)$. This shows us that it's not necessarily the same dimension!

We now want to get 'algebra-geometry dictionary', which are the Hilbert's Nullstellensatz,

4.2 Integral Extension + Nullstellensatz

Definition 4.8 (Integral element)**.** Let A be a subring of B, then an element $g \in B$ is **integral** over A if there is some $n > 0$ and soe $f_1, \dots, f_n \subseteq A$ such that,

$$g^n + fg^{n-1} + \cdots + f_n = 0$$

That is, g satisfies monic equations over A. That is, $\exists\, F \in A[X]$ such that $F(g) = 0$ where F is monic.

Example. Let $A = k[X^2, X^3]$, where $B = k[X]$. Then $X \in B$ is integral over A as $F(T) = T^2 - X^2$.

Lemma 4.9. Let A be a subring of B and $g \in B$,

(i) g is integral over A

(ii) $A[g] \subseteq B$ is a finitely generated A-module

(iii) there exists a subring C of B containing $A[g]$ which is a finitely generated A-module,

Proof. Skipped □

We say that B is integral over a subring A if every element of B is integral over A. A non-example is that $Z \subseteq \mathbb{Q}$ is not integral, or if we take $k[X^2, X^3] \subseteq k[X]$ this is integral.

Remark. Integral dependce is also important in AlgNumber Theory.

Here is a really integersting result about prime ideals,

Proposition 4.10 (Lying Over)**.** Lt B be an integral extension of A. Then for each prime ideal $P \subseteq A$ then there exists a prime Q 'lying over P', that is $Q \subseteq A = P$.

and we can improve this to be about chains of prime ideals,

Proposition 4.11. Let B be integral over a subring A. If $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ then there is an ascending sequence of prime ideals B such that $Q_i \cap A_i = P_i$

and now what is needed for the big german lemma,

Corollary 4.12. Let A be a subring of B and suppose that B is integral over A,

(i) if B is a field, so is A

(ii) if B is an integral domain and A is a field so is B

This is the most important result about integral dependence in algebraic geometry,

Proposition 4.13 (Noether Normalisation (NN))**.** LEt B be a finitely generated k-algebra. Then there exists a polynomial subring $A = k[X_1, \dots, X_n]$ of B with $X_1, \dots, X_n$ **something** independent, such that B is integral over A.

This tells us that B is a finitely generated A-module.

Corollary 4.14. If L is an extension of a field k, that is finitely generated as a k-algebra, then L is algebraic over k.

Now for the big theorem, Nullstellensatz means the 'theorem of zeroes'. We need that k is algebraically closed.

Theorem 4.15 (Hilbert's Nullstellensatz (HNS))**.** Let k be an algebraically closed field,

(i) Let A be a finitely generated k-algebra then for every maximal ideal $M \subseteq A$ we have that the $\dim_k(A/M) = 1$ (that is $A/M = k$)

(ii) Let (V, A) be an affine variety and let $I \subseteq A$ be an ideal, $I \neq A$, then $\mathcal{V}(I) \neq \varnothing$.

Proof. (i), Let $M \subseteq A$ be maximal, and $B := A/M$. Then by NN, B is integral over $C = k[T_1, \dots, T_n]$. Since M is maximal, then B is a field. By Corrolary 4.12, then as B is a field, then C has to be a field. This can only happen if $n = 0$, and so $C = k$. B is then an algebraic extension of k, but since $k = \overline{k}$, $B = k$. Hence, $\dim_k(B) = \dim_k(k) = 1$.

(ii), Let $M \subseteq A$ be a maximal ideal containing I (by Zorn's Lemma). Let $\theta : A \to k$ be a k-algebra map such that we define $a \mapsto a + M$, $\operatorname{Ker}\theta = M$ and θ is surjective. Then $\theta = \varepsilon_x$ for some $x \in V$ [(V, A) is an affine variety]. For $f \in I \subseteq M$ we have $f(x) = \varepsilon_x(f) = \theta(f) = 0$. That is $x \in \mathcal{V}(I)$. $\square$

Definition 4.16 (Nilradical)**.** Let R be a ring. The Nilradical of R is,

$$\mathcal{N}(R) = \{f \in R : \exists\, n \geq 1 : f^n = 0\}$$

This is an ideal in R, and,

$$\mathcal{N}(R) = \bigcap_{r \subseteq R,\text{prime}} r$$

That is, it's the intersection of all ideals.

Definition 4.17 (Radical)**.** Let $I \subseteq R$ be an ideal,

$$\sqrt{I} = \{f \in R : \exists\, n \geq 1 : f^n \in I\}$$

We know $\sqrt{I}$ is an ideal in R, $I \subseteq \sqrt{I}$ and $\sqrt{I}$ is the intersection of all prime ideals contained in I. We can pass to the quotient, $R/I = \mathcal{N}(R/I) = \sqrt{I}/I$. Also note that $\sqrt{0} = \mathcal{N}(R)$.

Theorem 4.18 (Strong Nullstellensatz)**.** Let $k = \overline{k}$,

(i) Let A be a finitely generated k-algebra. Then

$$\bigcap_{m\subseteq A,\text{maximal}} m = \mathcal{N}(A)$$

(ii) Let (V, A) be an affine variety, $J \subseteq A$ an ideal. Then, $I(\mathcal{V}(J)) = \sqrt{J}$.

Proof. (i), this is quite messy. We skip it, but essentially need NN.

(ii), We show both inclusions, it is clear that $\sqrt{J} \subseteq I(\mathcal{V}(J))$, we take $f \in \sqrt{J}$, then we fine some $f^n \in J$ and so $f^n(x) = 0 \forall x \in J$. Then we have $f(x)$ times by itself n times. Then we have $f(x) = 0$, as we are in an integral domain. Therefore, $f \in I(\mathcal{V}(J))$. For the other inclusion, we assume that $\sqrt{J} \subsetneq I(\mathcal{V}(J))$, this means that there exists some $f \in I(\mathcal{V}(J)) \setminus \sqrt{J}$, that is $f^n \notin J$ for all $m \geq 1$. We apply (i) to the ring A/J, therefore $\mathcal{N}(A/J) = \bigcup_{m\in A/J,\text{maximal}} m$. We know that the maximal ideals in A/J are in bijection with the maximal ideals in A that contain J. Therefore, there exists some maximal ideal $M \subseteq A$ where $J \subseteq M$ such that $f \notin M$. By HNS, we have $A/M = k$ and so we consider the short exact sequence,

$$0 \to M \to A \to k \to A,$$

then we call $\theta : A \to k$, where $\operatorname{Ker}\theta = M$ and so (V, A) is an affine variety, $\theta = \varepsilon_x$ for some $x \in V$. For any $g \subseteq J \subseteq M$ we have, $\theta(g) = \varepsilon_x(g) = g(x) = 0$. Therefore $x \in \mathcal{V}(J)$. But $J \notin M$, therefore $0 \neq \theta(f)$, but $\theta(f) = \varepsilon_x(f) = f(x) = 0$. Therefore $f \in I(\mathcal{V}(J))$. ☐

Remark. Let A be a finitely generated k-algebra that is reduced (that is, $\mathcal{N}(A) = 0$, or A has no nilpotent elements). Set $\text{Var}(A) = \hom_{k-alg}(A, k)$. We can construct a natural map $q : A \to \text{Map}(\text{Var}(A), k)$ by $\theta(a)(\alpha) = \alpha(a)$ for $a \in \text{Var}(A)$. By HNS θ is injective and $A \subseteq \text{Map}(\text{Var}(A), k)$. Therefore, $(\text{Var}(A), A)$ is an affine variety. This is an equivalence of categories, affine varieties and finitely generated k-algebras.