

# Year 3 — Lie Groups and Applications in Geometric Dynamics

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine (especially the typos!).

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PART I:

LECTURES

# 1 Lie Groups and Algebras

To start, let us define some groups. Firstly, what is a group?

**Definition 1.1** (Group).  $G$  is a nonempty set and endowed with a binary operation such that,

- It's closed under  $(\cdot)$ ,  $\forall a, b \in G, a \cdot b \in G$
- It's associative, i.e.  $\forall a, b \in G, a \cdot (b \cdot c) = (a \cdot b) \cdot c$ .
- There is an identity element,  $\forall a \in G, a \cdot e = a = e \cdot a$ .
- Every element has an inverse,  $\forall a \in G, a \cdot a^{-1} = e = a^{-1} \cdot a$ .

The definition of a Lie group also uses something called a manifold, to define this let's go on a slight adventure into topology!

## 1.1 Manifolds

A manifold is a topological space that locally resembles Euclidean space near each point [5]. However, this doesn't sate me as, what does near mean? It's very informal and hand wavey. Let's define it properly!

**Definition 1.2** (Manifold). A manifold is a second countable Hausdorff space that is locally homeomorphic to Euclidean space.

Where we refer to a second countable space as being a topological equivalent to being finitely generated. There exists some countable base of this space  $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ , where any open subset of our space,  $T$ , can be written as a disjoint union of a finite subfamily of  $\mathcal{U}$ . This nicely restricts manifolds to be smaller spaces, by making them be the union of countably many open sets. On the point of Hausdorff, this relies on the following definition,

**Definition 1.3** (Pairwise Neighborhood-Separable). Two points are pairwise neighborhood-separable if there exists a neighborhood  $U$  of  $x$  and  $V$  of  $y$  such that  $U$  and  $V$  are disjoint.

Then we can say that a space is Hausdorff if all distinct points are pairwise neighborhood-separable. Finally, to say something is homeomorphic to another space, this means it can be stretched without creating holes or glueing. A homeomorphism is a bijective map between two spaces and local homeomorphisms relates to neighborhoods around points. Hence, saying that something is locally homeomorphic to Euclidean space directly means, that you can bijectively map the contents of the neighborhood around a point to an open ball in  $\mathbb{R}^n$ , i.e the Euclidean  $n$ -ball.<sup>1</sup>

## 1.2 Lie Groups

Now we know enough to define what a Lie group actually is,

**Definition 1.4** (Lie Group). A Lie group is a group that is also a smooth manifold, such that the binary product and inversion are smooth functions.

What we will be focusing our attention to is special Lie groups, the general linear group, special linear group and the special orthogonal group.

**Definition 1.5** (General Linear Group).  $GL(n, \mathbb{R})$  is the linear matrix group. The manifold of  $n \times n$  invertible square real matrices is a lie group denoted by  $GL(n, \mathbb{R})$

**Definition 1.6** (Special Linear Group). The  $SL(n, \mathbb{R})$  is the manifold of  $n \times n$  matrices with unit determinant.

**Definition 1.7** (Special Orthogonal Group).  $SO(n, \mathbb{R})$  is the manifold of rotation matrices in  $n$  dimensions. This may be denoted by  $SO(n)$

<sup>1</sup>This is long and a very non-succinct way to define this structure, I prefer to define manifolds via sheaves as I feel it is neater. A manifold is just a locally ringed space, whose sheaf structure is just locally isomorphic to continuous functions on Euclidean space.

### 1.3 Lie Algebras

To actually understand what Lie Algebras are, we need to generalise the notion of a vector and a tangent. We shall look at so called tangent spaces. To formally define them, we shall first define charts and atlas and along the way redefine what a manifold is. These definitions are adapted from [4].

**Definition 1.8** (Chart). Let  $X$  be a topological space. An  $\mathbb{R}^n$  chart on  $X$  is a homeomorphism  $\phi : U \rightarrow U'$  where  $U \subset X$  and  $U' \subset \mathbb{R}^n$ .

**Definition 1.9** (Atlas). A  $C^\infty$  atlas on a topological space  $X$  is a collection of charts  $\phi_\alpha : U_\alpha \rightarrow U'_\alpha$  where all the  $U'$ 's are open subsets of one fixed  $\mathbb{R}^n$  such that,

- (i) Each  $U_\alpha \subset X$  is open and  $\bigcup_\alpha U_\alpha = X$  ( $U_\alpha$  is an open subcover of  $X$ ) and,
- (ii) Changes of coordinates are smooth.

Two last definitions in this section are equivalence relation and equivalence class.

**Definition 1.10** (Equivalence Relation). An equivalence relation on a set  $X$  is a binary relation  $\sim$  satisfying,

- (i)  $\forall a \in X, a \sim a$
- (ii)  $\forall a, b \in X, a \sim b \implies b \sim a$
- (iii)  $\forall a, b, c \in X, a \sim b \text{ and } b \sim a \implies a \sim c$ .

Then the equivalence class is just all of the equivalent elements to a member of that set. For example, for the equivalence relation that  $x \sim y$  if and only if  $x - y$  is even, then the equivalence classes are all the even numbers and all the odd numbers.

**Remark.** It can be proven that equivalence classes are just a partition of a set. [2]

Here is another definition of a manifold, this definition explicitly output a smooth manifold,

**Definition 1.11** ( $C^\infty$  Manifold). An  $n$ -dimensional ( $C^\infty$ ) manifold is a topological space  $M$  together with an equivalence class of  $C^\infty$  atlases.

**Remark.** Our equivalence relation here is that two atlases are equivalent if their union is also an atlas.

Here are a few examples of manifolds,

- Let  $M = \mathbb{R}^n$ , this is a manifold covered by one open set and then if we take the identity map as our chart, we get the standard manifold on  $\mathbb{R}^n$ .
- Let  $M = \mathbb{C}^n$ , then we cover  $\mathbb{C}^n$  by just one open set and then chart the map,  $\phi : \mathbb{C}^n \rightarrow \mathbb{R}^{2n}$  which is just,

$$\phi(z_1, \dots, z_n) = (\operatorname{Re} z_1, \operatorname{Im} z_1, \dots, \operatorname{Re} z_n, \operatorname{Im} z_n)$$

- If  $M$  is a manifold, then any open  $V \subset M$  is also a manifold. This can be seen as the union of the atlases  $V$  and  $M$  is going to be  $M$  and so it has the same equivalence class and hence it must be a manifold.
- If we let  $M_n(\mathbb{R})$  be all real  $n \times n$  matrices, then this is a manifold as it's just  $\mathbb{R}^{n^2}$ . We also can say  $\operatorname{GL}(n, \mathbb{R}) \subset M_n(\mathbb{R})$  and so by the previous point,  $\operatorname{GL}(n, \mathbb{R})$  is a manifold.

Now we can formalise the idea of tangent vectors on a manifold[3]

**Definition 1.12** (Tangent Vectors). Let  $M$  be a  $C^\infty$  manifold, then we can say that  $x \in M$ . Let us take a chart of  $M$ ,  $\phi : U \rightarrow \mathbb{R}^n$  where  $x \in U$ . Now take two curves  $\gamma_1, \gamma_2 : (-1, 1) \rightarrow M$  with  $\gamma_1(0) = \gamma_2(0) = x$  such that we can form  $\phi \circ \gamma_1 \circ \phi \circ \gamma_2 : (-1, 1) \rightarrow \mathbb{R}^n$  are differentiable.

Now define an equivalence such that  $\gamma_1$  and  $\gamma_2$  are equivalent at 0 if and only if  $(\phi \circ \gamma_1)' = (\phi \circ \gamma_2)' = 0$ . Then take the equivalence class of all of these curves and these are the tangent vectors of  $M$ .

**Definition 1.13** (Tangent Space). The set of all of the tangent vectors at  $x$ . We denote it as  $T_x M$ .

Lie Algebras are tangent space to the lie group at the identity. Let  $G$  be a lie group, then the  $T_e G$  (tangent space at the identity) is an interesting vector space with a remarkable structure called the lie algebra structure.

**Lemma 1.14.** Let  $G$  be a matrix lie group, and  $g \in G$ , then,

$$\xi \in T_e G \implies g\xi g^{-1} \in T_e G$$

Note that,  $g\xi g^{-1}$  is a matrix expression.

*Proof.* Let  $c(t) \in G$  be a curve in  $G$ , such that  $c(0) = e$  and  $\dot{c}(0) = \xi$ . Define  $\gamma(t) = gc(t)g^{-1}$ . Then  $\gamma(0) = gc(0)g^{-1} = e$  and  $\dot{\gamma}(0) = g\dot{c}(0)g^{-1} = g\xi g^{-1} \in T_e G$ .  $\square$

**Proposition 1.15.** Let  $G$  be a matrix lie group and  $\xi, \eta \in T_e G$ . Then,  $\xi\eta - \eta\xi \in T_e G$

*Proof.* Let  $c(t) \in G$  be a curve such that  $c(0) = e$  and  $\dot{c}(0) = \xi$  also define  $b(t) = c(t)\eta c(t)^{-1} \in T_e G$  by Lemma 1.5. Then  $\dot{b}(t) \in T_e G$ .

$$\begin{aligned} \dot{b}(0) &= \dot{c}(0)\eta c(0)^{-1} + c(0)\eta + \frac{dc(t)}{dt}(0) \\ &= \dot{c}(0)\eta c(0)^{-1} - c(0)\eta c(0)^{-1}\dot{c}(0)c(0)^{-1} \\ &= \xi\eta - \eta\xi \end{aligned}$$

As  $\dot{b}(t) \in T_e G$ , then  $\xi\eta - \eta\xi \in T_e G$   $\square$

Now we have a Lie Algebra,

**Definition 1.16** (Lie Algebra). A lie algebra is a vector space endowed with a **commutator** (or Lie bracket), that is a bilinear map. If we have,

$$[\cdot, \cdot] : V \times V \rightarrow V$$

such that,

- $[B, A] = -[A, B]$  (skew-symmetry property)
- $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0 \quad \forall A, B, C \in V$  (Jacobi Identity)

**Theorem 1.17.** Let  $G$  be a matrix Lie group. Then  $T_e G$  is a lie algebra with bracket given by the matrix commutator. Denoted by  $\mathfrak{g}$ .

$$[A, B] = AB - BA$$

Assume we have a surface, of manifold  $M$ , the tangent space  $T_q M$ , then we can say that,

$$\bigcup T_q M = TM$$

Cotangent manifold is dual space of manifold

**Example.** - The lie algebra  $\mathfrak{GL}(n, \mathbb{R}) = T_e \text{GL}(n, \mathbb{R})$  which is vector space of real square  $n \times n$  matrices with commutator.

- The lie algebra of  $\mathfrak{SL}(n, \mathbb{R}) := T_e \text{SL}(n, \mathbb{R})$  vector space of real traceless square  $n \times n$  matrices.

*Proof.* Take  $g(t) \in \mathrm{SL}(n, \mathbb{R})$ , and so  $\det g(t) = 1$  hence, take  $g(t)$  such that  $g(0) = e$ , and  $\dot{g}(0) = \xi$  and so  $\dot{g}(0) \in \mathfrak{SL}(n, \mathbb{R})$ . Now use the formula of the derivative of the determinant of a matrix to show,

$$\frac{d}{dt}(\det(g(t)))_{t=0} = \det g(0) \mathrm{Tr}(g^{-1}(0)\dot{g}(0))$$

and so,  $g^{-1}(0)\dot{g}(0) \in \mathfrak{SL}(n, \mathbb{R})$  and so we just have  $\mathrm{Tr}(\xi)$ .  $\square$

- The lie algebra of  $\mathfrak{SO}(3) = T_e\mathrm{SO}(3)$ , the vector space of skew-symmetric matrices.

**Lemma 1.18.** If  $v \in T_g G$ , then we can say,

- (i)  $g^{-1}v \in T_e G$
- (ii)  $vg^{-1} \in T_e G$

*Proof.* Suppose we have a  $c(t) \in G$  such that  $c(0) = g$  and  $\dot{c}(0) = v$ . Now we define a  $\gamma(t) = g^{-1}c(t)$  and we see that  $\gamma(0) = e$  and  $\dot{\gamma}(0) = g^{-1}\dot{c}(0) = g^{-1}v$ . Hence by Lemma 1.14 we can say if  $v \in T_e G$  then  $vg^{-1} \in T_e G$ , applying this to  $g^{-1}v$ , we can get that  $vg^{-1} \in T_e G$ , as required.  $\square$

## 2 Actions of a Lie Group and Lie Algebra

In this section we will focus on how our Lie Group will act on our Algebra. We will first use conjugation actions to define our adjoints which will become very useful once we see the Euler Poincare equations.

**Definition 2.1** (Conjugation Action). Let  $g \in G$ , then the operation  $I_g : G \rightarrow G$  (Inner Automorphism) and so you define it by  $h \mapsto ghg^{-1} \quad \forall h \in G$ .  $I_{gh} = AD_{gh}$ .

Take an arbitrary path  $h(t) \in G$  such that  $h(0) = e$  and now  $\xi = \dot{h}(0) \in T_e G$ . We now define  $Ad_g(\xi) = \frac{d}{dt} I_g h(t)_{t=0} = g\xi g^{-1} \in T_e G$  the adjoint action.

**Definition 2.2** (Adjoint and coadjoint actions of  $G$  on  $\mathfrak{g}$  and  $\mathfrak{g}^*$ ). The adjoint action of the matrix group  $G$  on it's lie algebra  $\mathfrak{g}$  is a map,

$$Ad : G \times \mathfrak{g} \rightarrow \mathfrak{g}$$

which is,

$$Ad_g \xi = g\xi g^{-1}$$

The dual map  $\langle Ad_g^* \mu, \xi \rangle = \langle \mu, Ad_g \xi \rangle$  where  $\mu \in \mathfrak{g}^*$  and  $\xi \in T_e G = \mathfrak{g}$ . is called the coadjoint map of  $G$  on the dual lie algebra  $\mathfrak{g}^*$ .

We will find that sometimes our classical ideas of vectorspaces doesn't work. Hence, we shall introduce functionals and use them to define dual vector spaces.

**Definition 2.3** (Dual Space for vectors). Let  $V$  be a finite dimensional vector space, of dimension  $n$ , over  $\mathbb{R}$ . The dual vector space is denoted by  $V^*$  is the space of all linear functionals from  $V \rightarrow \mathbb{R}$ ,  $f(v) = a$  where  $v \in V$  and  $a \in \mathbb{R}$ , then also  $f(\alpha v + \beta w) = \alpha f(v) + \beta f(w)$  and  $\alpha, \beta \in \mathbb{R}$  and  $v, w \in V$ . Hence,  $f(v) = Mv$  we call  $M$  the covector such that  $Mv \in \mathbb{R}$ . The vectorspace of all covectors is the dual space.

$$\langle m, v \rangle \in \mathbb{R} \quad m \in V^* \quad v \in V$$

Now we can see that the dual space is also a vector space so we can use the normal vector space ideas with it,

**Lemma 2.4.** Let  $V$  be a vector space of real  $n \times n$  real matrices. Then the dual vector space  $V^*$  is also a vector space of  $n \times n$  matrices and every linear functional  $f : V \rightarrow \mathbb{R}$  such that,

$$f(A) := Tr(B^T A), \quad B \in V^*, A \in V$$

We need to generalise the idea of a inner product to matrices and here is a particular inner product called trace pairing. From here on any inner product signs will indicate a trace pairing.

**Definition 2.5** (Trace Pairing). For every vector space  $V$  of real  $n \times n$  matrices with dual  $V^*$ , then the pairing is,

$$\langle B, A \rangle = Tr(B^T A) = Tr(BA^T)$$

**Proposition 2.6.** Suppose  $A^T = A$  and  $B^T = -B$ , then,  $Tr(B^T A) = 0$

*Proof.*

$$\begin{aligned} Tr(B^T A) &= -Tr(BA) \\ &= -Tr((BA)^T) \\ &= -Tr(B^T A^T) \\ &= -Tr(A^T B^T) \\ &= -Tr(B^T A) \end{aligned}$$

□

We say that,

$$\begin{aligned}
\langle \text{Ad}_g^* \mu, \xi \rangle &= \langle \mu, \text{Ad}_g \xi \rangle \\
&= \langle \mu, g \xi g^{-1} \rangle \\
&= \text{Tr}(\mu^T g \xi g^{-1}) \\
&= \text{Tr}(\xi g \mu^T g^{-1}) \\
&= \text{Tr}[(g^T \mu (g^{-1})^T)^T \xi] \\
&= \langle g^T \mu (g^{-1})^T, \xi \rangle \\
&= \langle g^T \mu (g^T)^{-1}, \xi \rangle
\end{aligned}$$

Let  $g(t) \in G$  such that  $g(0) = e$  where  $\eta = \dot{g}(0) \in T_e G$

$$\text{ad}_\eta \xi := \frac{d}{dt}_{t=0} \text{Ad}_{g(t)} \xi \quad \forall \xi \in \mathfrak{g}$$

which we can see to be

$$\dot{g}(0) \xi g(0)^{-1} + g(0) \xi \frac{d}{dt}_{t=0} g(t)^{-1} = \eta \xi - \xi \eta$$

Hence we can say that  $\text{ad}_\eta \xi = [\eta, \xi] = \eta \xi - \xi \eta$ .

Hence now we define the coadjoint action on  $\mu$ ,

**Definition 2.7** (Adjoint / Coadjoint action on  $\mathfrak{g}/\mathfrak{g}^*$ ). The adjoint action of the matrix lie algebra on itself is given by,

$$\text{ad} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$$

$$\text{ad}_\eta \xi = [\eta, \xi]$$

The dual map  $\langle \text{ad}_\eta^* \mu, \xi \rangle = \langle \mu, \text{ad}_\eta \xi \rangle$  is the coadjoint action of  $\mathfrak{g}$  on  $\mathfrak{g}^*$ .

**Exercise.** Find  $\text{ad}_\eta^* \mu$ .

**Solution.** Above we said that  $\langle \text{Ad}_g^* \mu, \xi \rangle = \langle g^T \mu (g^T)^{-1}, \xi \rangle$  and so  $\text{Ad}_g^* \mu = g^T \mu (g^T)^{-1}$ . As before we define,

$$\text{ad}_g^* \mu = \frac{d}{dt}_{t=0} \text{Ad}_g^* \mu$$

and now we can input our definitions and differentiate where we define  $g(0) = e$  and  $\dot{g}(0) = \eta \in T_e G$ ,

$$\begin{aligned}
\text{ad}_g^* \mu &= \frac{d}{dt}_{t=0} \text{Ad}_g^* \mu \\
&= \frac{d}{dt}_{t=0} g^T \mu (g^T)^{-1} \\
&= [\dot{g}^T \mu (g^T)^{-1} - g^T \mu (g^T)^{-1} \dot{g}^T (g^T)^{-1}]_{t=0} \\
&= \dot{g}(0)^T \mu (g(0)^T)^{-1} - g(0)^T \mu (g(0)^T)^{-1} \dot{g}(0)^T (g(0)^T)^{-1} \\
&= \eta^T \mu e - e \mu e^{-1} \eta^T e^{-1} \\
&= \eta^T \mu - \mu \eta^T \\
&= [\eta^T, \mu]
\end{aligned}$$

Hence,  $\text{ad}_g^* \mu = [\eta^T, \mu]$



### 3 Rotation

#### 3.1 Inertial Frame

A spatial coordinate system with origin at the centre of mass of the given rigid body. We denote it by,  $\mathbf{x}(t) \in \mathbb{R}^3$ , where  $\mathbf{x} = X$ . Assume we have a spatial coordinate system, We need a way to rotate things without constraints, so we denote a tensor  $R(t)$  and say  $\mathbf{x}(t) = R(t)\mathbf{X}$  where  $\mathbf{X}$  is in the body coordinate system. The configuration of the body particle at time  $t$  is given by a rotation matrix that takes the label  $\mathbf{X}$  to current position  $\mathbf{x}(t)$  where  $R \in \text{SO}(3)$  is a proper rotation matrix; this means,

$$R^T = R^{-1} \quad \det R = 1$$

The map  $\mathbf{X} \rightarrow R(t)\mathbf{X}$  is called the body-to-space map.

We can now talk about kinetic energy,

$$K = \frac{1}{2} \int_{\mathcal{B}} \rho \|\mathbf{x}\|^2 d^3\mathbf{X}$$

which we can change to,

$$\begin{aligned} \frac{1}{2} \int_{\mathcal{B}} \rho \|\mathbf{x}\|^2 d^3\mathbf{X} &= \frac{1}{2} \int_{\mathcal{B}} \rho(\mathbf{X}) \left\| \dot{R}(t)\mathbf{X} \right\|^2 d^3\mathbf{X} \\ &= \frac{1}{2} \int_{\mathcal{B}} \rho(\mathbf{X}) \dot{R}(t)\mathbf{X} \cdot \dot{R}(t)\mathbf{X} d^3\mathbf{X} \end{aligned}$$

Now we can say if  $V = 0$ . Hence,  $L = K$  and so,

$$\frac{d}{dt} \frac{\partial K}{\partial \dot{R}} - \frac{\partial K}{\partial R} = \mathbf{0}$$

This is difficult to deal with, so let's do something more cool!

We know that  $R^{-1} = R^T$  and so  $RR^T = RR^{-1} = I = e$ . If we have  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ , then  $\mathbf{v} \cdot \mathbf{w} = R\mathbf{v} \cdot R\mathbf{w}$ . Hence, consider  $\left\| \dot{R}\mathbf{X} \right\|^2$  and we know

$$\begin{aligned} \left\| \dot{R}\mathbf{X} \right\|^2 &= \dot{R}\mathbf{X} \cdot \dot{R}\mathbf{X} \\ &= R^{-1}(\dot{R}\mathbf{X}) \cdot R^{-1}(\dot{R}\mathbf{X}) \\ &= \left\| R^{-1}\dot{R}\mathbf{X} \right\|^2 \end{aligned}$$

and so,

$$K = \frac{1}{2} \int_{\mathcal{B}} \rho(\mathbf{X}) \left\| R^{-1}\dot{R}\mathbf{X} \right\|^2 d^3\mathbf{X}$$

Then  $K = K(R, \dot{R}) = K(R^{-1}R, R^{-1}\dot{R})$  this is called left symmetry. Hence, we can reduce this to  $K(e, R^{-1}\dot{R})$  and change notation let  $\kappa(R^{-1}\dot{R})$  and  $R^{-1}\dot{R}$  is angular velocity of the body. We can see this from the body and from an observation outside the system. Hence, we call this  $R^{-1}\dot{R} = \hat{\Omega}$ . Interestingly, we know  $RR^T = RR^{-1} = I$ . Hence,

$$\frac{d}{dt} I = \frac{d}{dt} (RR^{-1}) = \dot{R}R^{-1} + R \frac{d}{dt} R^{-1} = \mathbf{0}$$

and we can also write this as,

$$\begin{aligned} I &= RR^T \\ \mathbf{0} &= \frac{d}{dt}(RR^T) \\ \mathbf{0} &= \dot{R}^T R + R^T \dot{R} \\ R^T \dot{R} &= -(R^T \dot{R})^T \end{aligned}$$

and so  $R^{-1}\dot{R} = -(R^{-1}\dot{R})^T$  and so  $\hat{\Omega} = -\hat{\Omega}^T$ . This is the antisymmetric property we have noted about this vector.

Now we go back to kinetic energy to nicely write it as  $\hat{\Omega}$

$$K = \frac{1}{2} \int_{\mathcal{B}} \rho(\mathbf{X}) \|\hat{\Omega}\mathbf{X}\|^2 d^3\mathbf{X}$$

and now we can prove that  $\hat{\Omega}\mathbf{X} = \Omega \times \mathbf{X}$  where,

$$\hat{\Omega} = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}$$

where  $\Omega = \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix}$  where  $\Omega$  is the axel vector. and so,

$$\begin{aligned} K &= \frac{1}{2} \int_{\mathcal{B}} \rho(\mathbf{X}) \|\Omega \times \mathbf{X}\|^2 d^3\mathbf{X} \\ &= \frac{1}{2} \int_{\mathcal{B}} \rho(\mathbf{X}) (\Omega \times \mathbf{X}) \cdot (\Omega \times \mathbf{X}) d^3\mathbf{X} \\ &= \frac{1}{2} \int_{\mathcal{B}} \rho(\mathbf{X}) (\|\Omega\|^2 \|\mathbf{X}\|^2 - (\Omega \cdot \mathbf{X})^2) d^3\mathbf{X} \\ &= \frac{1}{2} \int_{\mathcal{B}} \rho(\mathbf{X}) (\Omega^T \Omega \|\mathbf{X}\|^2 - \Omega^T \mathbf{X} \mathbf{X}^T \Omega) d^3\mathbf{X} \\ &= \frac{1}{2} \int_{\mathcal{B}} \rho(\mathbf{X}) \Omega^T (\|\mathbf{X}\|^2 I - \mathbf{X} \mathbf{X}^T) \Omega d^3\mathbf{X} \\ &= \frac{1}{2} \int_{\mathcal{B}} \rho(\mathbf{X}) \Omega^T \Omega (\|\mathbf{X}\|^2 - \mathbf{X} \mathbf{X}^T) d^3\mathbf{X} \\ &= \frac{1}{2} \Omega \cdot \Omega \int_{\mathcal{B}} \rho(\mathbf{X}) (\|\mathbf{X}\|^2 - \mathbf{X} \mathbf{X}^T) d^3\mathbf{X} \\ &= \frac{1}{2} \mathbb{I} \Omega \cdot \Omega \end{aligned}$$

where  $\mathbb{I}$  is the moment of inertia tensor, which we define as,

$$\mathbb{I} = \int_{\mathcal{B}} \rho(\mathbf{X}) \|\mathbf{X}\|^2 I - \mathbf{X} \mathbf{X}^T d^3\mathbf{X}$$

where  $\mathbf{X} \mathbf{X}^T = \mathbf{X} \otimes \mathbf{X}$  and,

$$(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$$

for all  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ .

## 4 Calculus of variations

We are going to consider a continuous level, but you can use discrete level.

**Theorem 4.1** (The variation principle).

$$\mathcal{L} = \int_{t_1}^{t_2} \frac{1}{2} \mathbb{I} \boldsymbol{\Omega} \cdot \boldsymbol{\Omega} dt$$

and we find differential equations by letting  $\delta \mathcal{L} = 0$  but this is subject to  $\delta \boldsymbol{\Omega}(t_1) = \delta \boldsymbol{\Omega}(t_2) = \mathbf{0}$

and so,

$$\delta \int_{t_1}^{t_2} \frac{1}{2} \mathbb{I} \boldsymbol{\Omega} \cdot \boldsymbol{\Omega} dt = \int_{t_1}^{t_2} \frac{1}{2} \mathbb{I} \delta \boldsymbol{\Omega} \cdot \boldsymbol{\Omega} + \frac{1}{2} \mathbb{I} \boldsymbol{\Omega} \cdot \delta \boldsymbol{\Omega} dt = \int_{t_1}^{t_2} \mathbb{I} \boldsymbol{\Omega} \cdot \delta \boldsymbol{\Omega} dt$$

but what is  $\delta \boldsymbol{\Omega}$ , but remember we have  $\hat{\boldsymbol{\Omega}}$ , which is the lie algebra of  $\text{SO}(3)$ . We said,  $\hat{\boldsymbol{\Omega}} = R^T \dot{R} = R^{-1} \dot{R}$ . Now we take variations of  $\hat{\boldsymbol{\Omega}} = \boldsymbol{\Omega} \times \mathbf{X}$  and so,

$$(\delta \boldsymbol{\Omega}) \times \mathbf{X} = (\delta \hat{\boldsymbol{\Omega}}) \mathbf{X}$$

and so we see that,

$$\delta \hat{\boldsymbol{\Omega}} = \delta(R^{-1} \dot{R}) = \delta R^{-1} \dot{R} + R^{-1} \delta \dot{R} = 0$$

as  $\delta I = \delta R R^{-1} + R \delta R^{-1}$  and then we see that  $R^{-1} \delta R R^{-1} + R^{-1} R \delta R^{-1} = 0$  and so as  $R R^{-1} = I$ ,  $R^{-1} \delta R R^{-1} + \delta R^{-1} = \mathbf{0}$ . We have that  $\delta R^{-1} \dot{R} + R^{-1} \delta \dot{R} = \delta \hat{\boldsymbol{\Omega}}$  and  $\hat{\boldsymbol{\Omega}} = R^{-1} \dot{R}$  where  $\hat{\mathbf{A}} = R^{-1} \delta R$  and so we sub in,

$$\begin{aligned} \delta \hat{\boldsymbol{\Omega}} &= -R^{-1} \delta R R^{-1} \dot{R} + R^{-1} \frac{d}{dt} \delta R \\ &= R^{-1} \delta R \hat{\boldsymbol{\Omega}} + \frac{d}{dt} (R^{-1} \delta R) - \left( \frac{d}{dt} R^{-1} \right) \delta R \\ &= R^{-1} \delta R \hat{\boldsymbol{\Omega}} + \frac{d}{dt} (R^{-1} \delta R) + R^{-1} \dot{R} R^{-1} \delta R \\ &= -\hat{\mathbf{A}} \hat{\boldsymbol{\Omega}} + \frac{d}{dt} \hat{\mathbf{A}} + \hat{\boldsymbol{\Omega}} \hat{\mathbf{A}} \\ &= \dot{\hat{\mathbf{A}}} + [\hat{\boldsymbol{\Omega}}, \hat{\mathbf{A}}] \end{aligned}$$

**Exercise.** Prove,

$$\delta \boldsymbol{\Omega} = \dot{\mathbf{A}} + (\boldsymbol{\Omega} \times \mathbf{A})$$

**Solution.** We can use the fact that  $\widehat{[\boldsymbol{\Omega}, \mathbf{A}]} = [\hat{\boldsymbol{\Omega}}, \hat{\mathbf{A}}]$  and then we can get the required result.

$$\begin{aligned} \delta \hat{\boldsymbol{\Omega}} &= \dot{\hat{\mathbf{A}}} + [\hat{\boldsymbol{\Omega}}, \hat{\mathbf{A}}] \\ &= \dot{\hat{\mathbf{A}}} + \widehat{[\boldsymbol{\Omega}, \mathbf{A}]} \\ &= \dot{\hat{\mathbf{A}}} + \widehat{(\boldsymbol{\Omega} \times \mathbf{A})} \\ \widehat{\delta \boldsymbol{\Omega}} &= \widehat{\dot{\mathbf{A}} + (\boldsymbol{\Omega} \times \mathbf{A})} \end{aligned}$$

and so we can see that  $\delta \boldsymbol{\Omega} = \dot{\mathbf{A}} + (\boldsymbol{\Omega} \times \mathbf{A})$

Now, let us substitute this back into our variational principle.

$$\begin{aligned}
\delta \int_{t_1}^{t_2} \mathbb{I} \boldsymbol{\Omega} \cdot \boldsymbol{\Omega} dt &= 0 \\
\int_{t_1}^{t_2} \mathbb{I} \boldsymbol{\Omega} \cdot \delta \boldsymbol{\Omega} dt &= 0 \\
\int_{t_1}^{t_2} \mathbb{I} \boldsymbol{\Omega} \cdot (\dot{\boldsymbol{\Lambda}} + \boldsymbol{\Omega} \times \boldsymbol{\Lambda}) dt &= 0 \\
[\mathbb{I} \boldsymbol{\Omega} \cdot \boldsymbol{\Lambda}|_{t_2} - \mathbb{I} \boldsymbol{\Omega} \cdot \boldsymbol{\Lambda}|_{t_1}] - \int_{t_1}^{t_2} \frac{d}{dt} (\mathbb{I} \boldsymbol{\Omega}) \cdot \boldsymbol{\Lambda} dt + \int_{t_1}^{t_2} (\mathbb{I} \boldsymbol{\Omega} \times \boldsymbol{\Omega}) \cdot \boldsymbol{\Lambda} dt &= 0 \\
0 - 0 - \int_{t_1}^{t_2} (-\mathbb{I} \dot{\boldsymbol{\Omega}} + \mathbb{I} \boldsymbol{\Omega} \times \boldsymbol{\Omega}) \cdot \boldsymbol{\Lambda} dt &= 0
\end{aligned}$$

Hence,

$$\mathbb{I} \dot{\boldsymbol{\Lambda}} = \mathbb{I} \boldsymbol{\Omega} \times \boldsymbol{\Omega}$$

We can write the equations by considering the tangent space.

#### 4.1 Euler-Poincare Reduction by Symmetry

To gain a general idea of how the equations of motion appear for rotational dynamics with symmetry, we consider an arbitrary Lagrangian of this form,

$$L : TSO(3) \rightarrow \mathbb{R}$$

$$L = L(R, \dot{R})$$

and satisfies,

$$\delta \int_{t_1}^{t_2} L(R, \dot{R}) dt = 0$$

this means,

$$\begin{aligned}
\delta \int_{t_1}^{t_2} L(R, \dot{R}) dt &= \int_{t_1}^{t_2} \left\langle \frac{\partial L}{\partial R}, \delta R \right\rangle + \left\langle \frac{\partial L}{\partial \dot{R}}, \delta \dot{R} \right\rangle dt \\
&= \left\langle \frac{\partial L}{\partial \dot{R}}, \delta R \right\rangle \Big|_{t_2} - \left\langle \frac{\partial L}{\partial \dot{R}}, \delta R \right\rangle \Big|_{t_1} + \int_{t_1}^{t_2} \left\langle \frac{\partial L}{\partial R} - \frac{d}{dt} \frac{\partial L}{\partial \dot{R}}, \delta R \right\rangle dt
\end{aligned}$$

and so we can notice  $\frac{d}{dt} \frac{\partial L}{\partial \dot{R}} - \frac{\partial L}{\partial R} = 0$

**Definition 4.2** (Left-Symmetric Lagrangian). A Lagrangian is said to be left-symmetric or left-invariant under the action of the group of the group of rotations if,  $L(\chi R, \chi \dot{R}) = L(R, \dot{R}) \forall \chi \in SO(3)$ .

We also know  $\mathfrak{SO}(3) = T_e SO(3)$  and we said that  $v \in T_e G \implies g^{-1}v \in \mathfrak{g} = T_e G$ . We know  $\dot{R}(t) \in T_{R(t)} SO(3)$  and so we can say  $R^{-1} \dot{R} \in \mathfrak{SO}(3)$ .

We say

$$\begin{aligned}
L(R, \dot{R}) &= L(R^{-1}R, R^{-1}\dot{R}) \\
&= \tilde{\ell}(R^{-1}\dot{R}) = \tilde{\ell}(\hat{\boldsymbol{\Omega}})
\end{aligned}$$

Now we write out Hamilton's principle,

$$\begin{aligned} 0 &= \delta \int_{t_1}^{t_2} \tilde{\ell}(\hat{\Omega}) dt \\ &= \delta \int_{t_1}^{t_2} \ell(\Omega) dt = 0 \end{aligned}$$

Euler Poincare equations,

$$\frac{d}{dt} \frac{\partial \ell}{\partial \Omega} - \frac{\partial \ell}{\partial \Omega} \times \Omega = 0$$

**Exercise.** Derive these

**Solution.** We shall start from Hamilton's Principle and move forward to derive our Euler-Poincare Equations.

$$\begin{aligned} \delta \int_{t_1}^{t_2} \ell(\hat{\Omega}) dt &= \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \hat{\Omega}}, \delta \hat{\Omega} \right\rangle dt \\ &= \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \Omega}, \delta \Omega \right\rangle dt \end{aligned}$$

Now, we shall use a fact we proved in the last exercise  $\delta \Omega = \dot{\Lambda} + (\Omega \times \Lambda)$  to derive the Euler-Poincare equations we wanted,

$$\begin{aligned} \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \Omega}, \delta \Omega \right\rangle &= \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \Omega}, \dot{\Lambda} + \Omega \times \Lambda \right\rangle dt \\ &= \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \Omega}, \frac{d}{dt} \Lambda \right\rangle dt + \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \Omega}, \Omega \times \Lambda \right\rangle dt \\ &= \int_{t_1}^{t_2} \left\langle -\frac{d}{dt} \frac{\partial \ell}{\partial \Omega}, \Lambda \right\rangle dt + \int_{t_1}^{t_2} \left\langle -\Omega \times \frac{\partial \ell}{\partial \Omega}, \Lambda \right\rangle dt \\ &= \int_{t_1}^{t_2} \left\langle -\frac{d}{dt} \frac{\partial \ell}{\partial \Omega}, \Lambda \right\rangle dt + \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \Omega} \times \Omega, \Lambda \right\rangle dt \\ &= \int_{t_1}^{t_2} \left\langle -\frac{d}{dt} \frac{\partial \ell}{\partial \Omega} + \frac{\partial \ell}{\partial \Omega} \times \Omega, \Lambda \right\rangle dt = 0 \end{aligned}$$

Hence we say that

$$\frac{d}{dt} \frac{\partial \ell}{\partial \Omega} - \frac{\partial \ell}{\partial \Omega} \times \Omega = 0$$

These are the Euler-Poincare equations for rotational dynamics with symmetry under left multiplication.

**Theorem 4.3.** The spatial angular momentum (in the spatial frame) is conserved along solutions of the Euler-Poincare equations.

*Proof.* We know  $\frac{d}{dt} \frac{\partial \ell}{\partial \Omega} - \frac{\partial \ell}{\partial \Omega} \times \Omega = 0$  and we know that  $R \frac{\partial \ell}{\partial \Omega}$  and the multiplication by  $R$  means spatial frame. Hence we prove,

$$\begin{aligned} \frac{d}{dt} R \frac{\partial \ell}{\partial \Omega} &= \mathbf{0} \\ \frac{d}{dt} R \frac{\partial \ell}{\partial \Omega} &= \dot{R} \frac{\partial \ell}{\partial \Omega} + R \frac{d}{dt} \frac{\partial \ell}{\partial \Omega} \\ &= R \hat{\Omega} \frac{\partial \ell}{\partial \Omega} + R \left( \frac{\partial \ell}{\partial \Omega} \times \Omega \right) \\ &= R \left( \Omega \times \frac{\partial \ell}{\partial \Omega} \right) + R \left( \frac{\partial \ell}{\partial \Omega} \times \Omega \right) \\ &= \mathbf{0} \end{aligned}$$

□

Now we want to write a general form of the Euler-Poincare Equations for left invariant systems. Let  $L$  be a lagrangian on the tangent bundle of a matrix lie group  $G$ ,

$$L : TG \rightarrow \mathbb{R}$$

$$L = L(g, \dot{g}) \quad \forall g \in G$$

Assume that the lagrangian is left-invariant,

$$L(g, \dot{g}) = L(hg, h\dot{g}) \quad \forall h \in G$$

and now let  $h = g^{-1}$ , and so  $L(g, \dot{g}) = L(g^{-1}g, g^{-1}\dot{g}) = \ell(\xi)$ . We have gone from a lie group to a lie algebra,  $\xi = g^{-1}\dot{g} \in T_e G = \mathfrak{g}$  which is a matrix lie algebra. We now aim to use the action functional and variational derivative,

$$\begin{aligned} \delta \int_{t_1}^{t_2} L(g, \dot{g}) &= \mathbf{0} \\ \delta \int_{t_1}^{t_2} \ell(\xi) dt &= \mathbf{0} \\ \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \xi}, \delta \xi \right\rangle dt &= \mathbf{0} \end{aligned}$$

Now we want to consider  $\delta \xi = \delta(g^{-1}\dot{g})$ ,

$$\begin{aligned} \delta(g^{-1}\dot{g}) &= \delta g^{-1}\dot{g} + g^{-1}\delta\dot{g} \\ &= g^{-1}\delta g g^{-1}\dot{g} + g^{-1}\frac{d}{dt}\delta g \\ &= -(g^{-1}\delta g)g^{-1}\dot{g} + g^{-1}\frac{d}{dt}\delta g \\ &= -\eta\xi + \frac{d}{dt}\delta(g^{-1}\delta g) + (g^{-1}\dot{g})(g^{-1}\frac{d}{dt}\delta g) \\ &= -\eta\xi + \dot{\eta} + \xi\eta \\ &= \dot{\eta} + [\xi, \eta] \\ &= \dot{\eta} + \text{ad}_\xi \eta \end{aligned}$$

and so back to the derivation,

$$\begin{aligned} \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \xi}, \delta \xi \right\rangle dt &= 0 \\ \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \xi}, \dot{\eta} + \text{ad}_\xi \eta \right\rangle &= 0 \\ \int_{t_1}^{t_2} \left\langle -\frac{d}{dt} \left( \frac{\partial \ell}{\partial \xi} \right), \eta \right\rangle + \left\langle \text{ad}_\xi^* \frac{\partial \ell}{\partial \xi}, \eta \right\rangle dt &= 0 \end{aligned}$$

Since  $\eta$  is arbitrary our equation is of this form,

$$\frac{d}{dt} \frac{\partial \ell}{\partial \xi} - \text{ad}_\xi^* \frac{\partial \ell}{\partial \xi} = 0$$

and these are our Euler-Poincare equations for a left invariant system.

**Theorem 4.4** (Noethers Theorem for left-invariant systems). The Euler Poincare equations associated a left-invariant system preserve the generalised momentum along solutions of the Euler-Poincare equations, that is,

$$\frac{d}{dt} \left( \text{Ad}_{g^{-1}(t)}^* \frac{\partial \ell}{\partial \xi}(t) \right) = 0$$

*Proof.* Suppose we have a left invariant lagrangian, i.e.  $L(g, \dot{g}) = L(e, g^{-1}\dot{g}) = \ell(g^{-1}g) := \ell(\xi)$  where  $\xi = g^{-1}\dot{g}$ . Firstly, however, let us consider the following derivative where  $\mu(t) \in \mathfrak{g}$ ,

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=t_0} (\text{Ad}_{g^{-1}(t)} \mu(t)) &= \left. \frac{d}{dt} \right|_{t=t_0} \text{Ad}_{g^{-1}(t)g(t_0)} (\text{Ad}_{g^{-1}(t_0)} \mu) \\ &= -\text{ad}_{g^{-1}(t_0)\dot{g}(t_0)} (\text{Ad}_{g^{-1}(t_0)} \mu) \\ &= -\text{ad}_{\xi(t_0)} (\text{Ad}_{g^{-1}(t_0)} \mu) \end{aligned}$$

and so we can say,

$$\frac{d}{dt} (\text{Ad}_{g^{-1}(t)} \mu(t)) = -\text{ad}_{\xi(t)} (\text{Ad}_{g^{-1}(t)} \mu(t))$$

Now, we can move forward and consider the trace pairing of our interested quantity and  $\mu(t)$ .

$$\begin{aligned} \left\langle \frac{d}{dt} \left( \text{Ad}_{g^{-1}(t)}^* \frac{\partial \ell}{\partial \xi}(t) \right), \mu(t) \right\rangle &= \frac{d}{dt} \left\langle \text{Ad}_{g^{-1}(t)}^* \frac{\partial \ell}{\partial \xi}(t), \mu(t) \right\rangle \\ &= \frac{d}{dt} \left\langle \frac{\partial \ell}{\partial \xi}(t), \text{Ad}_{g^{-1}(t)} \mu(t) \right\rangle \\ &= \left\langle \frac{d}{dt} \frac{\partial \ell}{\partial \xi}(t), \text{Ad}_{g^{-1}(t)} \mu(t) \right\rangle + \left\langle \frac{\partial \ell}{\partial \xi}(t), \frac{d}{dt} \text{Ad}_{g^{-1}(t)} \mu(t) \right\rangle \\ &= \left\langle \frac{d}{dt} \frac{\partial \ell}{\partial \xi}(t), \text{Ad}_{g^{-1}(t)} \mu(t) \right\rangle + \left\langle \frac{\partial \ell}{\partial \xi}(t), -\text{ad}_{\xi(t)} (\text{Ad}_{g^{-1}(t)} \mu(t)) \right\rangle \\ &= \left\langle \frac{d}{dt} \frac{\partial \ell}{\partial \xi}(t), \text{Ad}_{g^{-1}(t)} \mu(t) \right\rangle - \left\langle \frac{\partial \ell}{\partial \xi}(t), \text{ad}_{\xi(t)} (\text{Ad}_{g^{-1}(t)} \mu(t)) \right\rangle \\ &= \left\langle \frac{d}{dt} \frac{\partial \ell}{\partial \xi}(t), \text{Ad}_{g^{-1}(t)} \mu(t) \right\rangle - \left\langle \text{ad}_{\xi(t)}^* \frac{\partial \ell}{\partial \xi}(t), \text{Ad}_{g^{-1}(t)} \mu(t) \right\rangle \\ &= \left\langle \frac{d}{dt} \frac{\partial \ell}{\partial \xi}(t) - \text{ad}_{\xi(t)}^* \frac{\partial \ell}{\partial \xi}(t), \text{Ad}_{g^{-1}(t)} \mu(t) \right\rangle \\ &= \left\langle \text{Ad}_{g^{-1}(t)}^* \left[ \frac{d}{dt} \frac{\partial \ell}{\partial \xi}(t) - \text{ad}_{\xi(t)}^* \frac{\partial \ell}{\partial \xi}(t) \right], \mu(t) \right\rangle \end{aligned}$$

Hence, we can say that,

$$\frac{d}{dt} \left( \text{Ad}_{g^{-1}(t)}^* \frac{\partial \ell}{\partial \xi}(t) \right) = \text{Ad}_{g^{-1}(t)}^* \underbrace{\left[ \frac{d}{dt} \frac{\partial \ell}{\partial \xi}(t) - \text{ad}_{\xi(t)}^* \frac{\partial \ell}{\partial \xi}(t) \right]}_{\text{LHS of Euler-Poincare Equations}}$$

and as we have a left invariant system, we can use the left invariant Euler-Poincare equations to reduce the above derivative to zero, and hence Noethers Theorem for left invariant systems follows from this.  $\square$

**Exercise.** Repeat derivations for the Euler-Poincare Equations for right-invariant systems. What is Noether Theorem?<sup>2</sup>

**Solution.** Now let us carry forward with the derivation for right invariant systems. A right invariant lagrangian is one that the following is true,  $L(g, \dot{g}) = L(gh, \dot{g}h)$  for all  $h \in G$ . We then set  $h = g^{-1}$  and get that  $L(g, \dot{g}) = L(e, \dot{g}g^{-1})$  and so we let  $\xi = \dot{g}g^{-1}$  and hence write our lagrangian as  $\ell(\xi)$ . Now we again go back to Hamiltons Principle,

$$\begin{aligned} 0 &= \delta \int_{t_1}^{t_2} L(g, \dot{g}) dt = \delta \int_{t_1}^{t_2} \ell(\xi) dt \\ &= \int_{t_1}^{t_2} \delta \ell(\xi) dt \\ &= \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \xi}, \delta \xi \right\rangle dt \end{aligned}$$

Now we consider  $\delta \xi = \delta(\dot{g}g^{-1})$ ,

$$\begin{aligned} \delta(\dot{g}g^{-1}) &= \delta \dot{g}g^{-1} + \dot{g}\delta g^{-1} \\ &= \frac{d}{dt}(\delta g)g^{-1} - \dot{g}g^{-1}\delta g g^{-1} \\ &= \frac{d}{dt}(\delta g g^{-1}) - \delta g \frac{d}{dt}(g^{-1}) - \dot{g}g^{-1}\delta g g^{-1} \\ &= \frac{d}{dt}(\delta g g^{-1}) - \delta g g^{-1}\dot{g}g^{-1} - \dot{g}g^{-1}\delta g g^{-1} && \text{let } \nu = \delta g g^{-1} \\ &= \dot{\nu} + \nu \xi - \xi \nu \\ &= \dot{\nu} + [\nu, \xi] \\ &= \dot{\nu} + \text{ad}_{\nu} \xi \\ &= \dot{\nu} - \text{ad}_{\xi} \nu \end{aligned}$$

Hence, we now can move forward and complete the derivation of the right invariant Euler-Poincare Equations.

$$\begin{aligned} \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \xi}, \frac{d}{dt} \nu - \text{ad}_{\xi} \nu \right\rangle dt &= \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \xi}, \frac{d}{dt} \nu \right\rangle - \left\langle \frac{\partial \ell}{\partial \xi}, \text{ad}_{\xi} \nu \right\rangle dt \\ &= \int_{t_1}^{t_2} \left\langle -\frac{d}{dt} \frac{\partial \ell}{\partial \xi}, \nu \right\rangle - \left\langle \text{ad}_{\xi}^* \frac{\partial \ell}{\partial \xi}, \nu \right\rangle dt \\ &= \int_{t_1}^{t_2} \left\langle -\frac{d}{dt} \frac{\partial \ell}{\partial \xi} - \text{ad}_{\xi}^* \frac{\partial \ell}{\partial \xi}, \nu \right\rangle dt \end{aligned}$$

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<sup>2</sup>What about both left and right invariant?



and so we can write down the Euler-Poincare equations for the right invariant system,

$$\frac{d}{dt} \frac{\partial \ell}{\partial \xi} + \text{ad}_\xi^* \frac{\partial \ell}{\partial \xi} = 0$$

We can restate Noethers Theorem as following,

**Theorem 4.5** (Noethers Theorem for right invariant systems.). The Euler Poincare equations associated a right-invariant system preserve the generalised momentum along solutions of the Euler-Poincare equations, that is,

$$\frac{d}{dt} \left( \text{Ad}_{g(t)}^* \frac{\partial \ell}{\partial \xi}(t) \right) = 0$$

*Proof.* This follows from a very similar argument to before by finding that  $\frac{d}{dt}(\text{Ad}_{g^{-1}} \mu) = \text{ad}_\xi(\text{Ad}_g \mu)$  and applying this fact in an identical analysis of the trace pairings ending with  $\frac{d}{dt}(\text{Ad}_g^* \frac{\partial \ell}{\partial \xi}) = \text{Ad}_g^* \left[ \frac{d}{dt} \frac{\partial \ell}{\partial \xi} + \text{ad}_\xi^* \frac{\partial \ell}{\partial \xi} \right]$  and then the result follows from the right-invariant version of the Euler-Poincare equations.  $\square$

## 5 The Hat Map as a Lie Algebra Isomorphisms

The lie algebra of  $SO(3)$  is the space of skew-symmetric matrices,  $SO(3)$ . Then we can conclude that the Euler-Poincare Equations are written as:

$$\frac{d}{dt} \frac{\partial \ell}{\partial \hat{\Omega}} - \text{ad}_{\hat{\Omega}}^* \frac{\partial \ell}{\partial \hat{\Omega}} = 0$$

Let  $\Pi$  be any element in  $\mathfrak{g}^*$ , then  $\text{ad}^*$  operator is defined by  $\langle \text{ad}_{\hat{\Omega}}^* \Pi, \hat{\omega} \rangle = \langle \Pi, \text{ad}_{\hat{\Omega}} \hat{\omega} \rangle$  where  $\hat{\omega} \in \mathfrak{g}^*$ .

$$\begin{aligned} \langle \text{ad}_{\hat{\Omega}}^* \Pi, \hat{\omega} \rangle &= \langle \Pi, \text{ad}_{\hat{\Omega}} \hat{\omega} \rangle \\ &= \langle \Pi, [\hat{\Omega}, \hat{\omega}] \rangle \\ &= \text{Tr}(\Pi^T [\hat{\Omega}, \hat{\omega}]) \\ &= \text{Tr}(\Pi^T \hat{\Omega} \hat{\omega} - \Pi^T \hat{\omega} \hat{\Omega}) \\ &= \text{Tr}(\Pi \hat{\Omega}^T \hat{\omega} - \Pi \hat{\omega} \hat{\Omega}^T) \\ &= \text{Tr}(\Pi \hat{\Omega} \hat{\omega}^T - \hat{\Omega} \Pi \hat{\omega}^T) \\ &= \text{Tr}((\Pi \hat{\Omega} - \hat{\Omega} \Pi) \hat{\omega}^T) \\ &= \text{Tr}([\Pi, \hat{\Omega}] \hat{\omega}^T) \\ &= \langle [\Pi, \hat{\Omega}], \hat{\omega} \rangle \end{aligned}$$

Then,  $\text{ad}_{\hat{\Omega}}^* \Pi = [\Pi, \hat{\Omega}]$ . From here we can conclude that, the hat map is a lie algebra isomorphism, i.e.,  $[\hat{\Omega}, \hat{\omega}] = \widehat{\Omega \times \omega}$

*Proof.* Exercise □

We define,  $\hat{\cdot} : (\mathbb{R}^3, \times) \rightarrow (SO(3), [\cdot, \cdot])$ . This is the isomorphism.

We have described Lagrangians that have left or right invariance. We now look to Lagrangians that have Symmetry breaking parameters, like gravity. If we consider the spherical pendulum, we have defined  $\hat{\Omega} = R^T \dot{R}$  and we define  $\hat{\omega} = R \Omega$  and then we can see that  $\hat{\omega} = \dot{R} R^T = \dot{R} R^{-1}$  where  $R \in SO(3)$ . This doesn't lead to a symmetric Lagrangian but we can still use our theory here.

We are going to study rigid body dynamics in the spatial frame. We look firstly to the Lagrangian. We have showed,

$$L(R, \dot{R}) = \frac{1}{2} \int_{\mathcal{B}} \rho(\mathbf{X}) \left\| \dot{R} \mathbf{X} \right\|^2 d^3 \mathbf{X}$$

and we showed that  $L(R, \dot{R}) = L(e, R^{-1} \dot{R})$  and then we used Euler Poincare Theorem to show that  $\ell = \frac{1}{2} \mathbb{I} \Omega \cdot \Omega$ . Now assume we would prefer to formulate rigid body dynamics in the spatial frame. We need to consider a  $\omega$  such that  $\hat{\omega} = \dot{R} R^{-1}$ . We can now prove that,  $L(R, \dot{R}) \neq L(R\chi, \dot{R}\chi)$  (right multiplication) hence we have broken symmetry,

*Proof.* Exercise □

Now we seek this Lagrangian,

$$\begin{aligned}
 L(R, \dot{R}) &= \frac{1}{2} \int_{\beta} \rho(\mathbf{X}) \left\| \dot{R}\mathbf{X} \right\|^2 d^3\mathbf{X} \\
 &= \frac{1}{2} \int_{\beta} \rho(\mathbf{X}) \left\| \dot{R}R^{-1}R\mathbf{X} \right\|^2 d^3\mathbf{X} \\
 &= \frac{1}{2} \int_{\beta} \rho(\mathbf{X}) \left\| \hat{\omega}R\mathbf{X} \right\|^2 d^3\mathbf{X} \\
 &= \frac{1}{2} \int_{\beta} \rho(\mathbf{X}) (\omega \times R\mathbf{X}) \cdot (\omega \times R\mathbf{X}) d^3\mathbf{X} \\
 &= \vdots \\
 &= \frac{1}{2} \omega \cdot (R\mathbb{I}R^T)\omega = L(\omega, R)
 \end{aligned}$$

We define a new parameter,  $\mathbb{J} := R\mathbb{I}R^T$  and so  $\ell = \ell(\mathbb{J}, \omega) = \frac{1}{2}\omega(t) \cdot \mathbb{J}(t)\omega(t)$

Now we take variations as usual,

$$\delta \int_{t_1}^{t_2} \ell(\mathbb{J}, \omega) dt = 0$$

$$\delta \int_{t_1}^{t_2} \frac{1}{2} \omega(t) \cdot \mathbb{J}(t)\omega(t) dt = 0$$

and we ask what is  $\delta\mathbb{J}(t)$ ,

$$\begin{aligned}
 \delta\mathbb{J}(t) &= \delta(R\mathbb{I}R^T) \\
 &= \delta R\mathbb{I}R^T + R\mathbb{I}\delta R^T \\
 &= \delta R R^{-1} R\mathbb{I}R^T - R\mathbb{I}R^{-1} \delta R R^{-1} \\
 &= \hat{\mathbf{A}}\mathbb{J} - \mathbb{J}\hat{\mathbf{A}} \\
 &= [\hat{\mathbf{A}}, \mathbb{J}]
 \end{aligned}$$

where  $\hat{\mathbf{A}} = \delta R R^{-1}$

**Exercise.** Prove that  $\delta\hat{\omega} = \dot{\hat{\mathbf{A}}} + [\hat{\mathbf{A}}, \hat{\omega}]$  and  $\delta\omega = \dot{\mathbf{A}} + \mathbf{A} \times \omega$  and then take variations of  $\frac{1}{2}\omega \cdot \mathbb{J}\omega$  and prove that  $\frac{d}{dt}(\mathbb{J}\omega) = \mathbf{0}$  and then  $\frac{d\mathbb{J}}{dt}\hat{\omega} + \mathbb{J}\dot{\hat{\omega}} = \mathbf{0}$  and so  $\frac{d\mathbb{J}}{dt} = [\hat{\omega}, \mathbb{J}]$ .

## 6 More Noether Theory

### 6.1 Noethers Theorem for EL equations

Consider  $L(\mathbf{q}, \dot{\mathbf{q}})$  for  $\mathbf{q} \in \mathbb{R}^3$  and  $\dot{\mathbf{q}} \in T_{\mathbf{q}}\mathbb{R}^3$ . Suppose that  $L$  is left invariant with respect to the tangent lift on  $\text{SO}(3)$ , ie.  $R \in \text{SO}(3)$  with  $L(R\mathbf{q} R\dot{\mathbf{q}}) = L(\mathbf{q}, \dot{\mathbf{q}})$ . Then we can prove that,

$$\mathcal{E} := \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} = 0$$

Now we have theorem,

**Theorem 6.1.** Corresponding to each one-parameter subgroup of  $\text{SO}(3)$   $R(s)$  where  $R(0) = e$  and  $R'(0) = \hat{\xi} \in \mathfrak{SO}(3)$ . There is a conserved quantity,

$$A_{\xi} := \left\langle \mathbf{q} \times \frac{\partial L}{\partial \dot{\mathbf{q}}}, \xi \right\rangle$$

with  $\frac{dA_{\xi}}{dt} = 0$  along solutions of the Euler Lagrange equations  $\mathcal{E}(\mathbf{q}) = 0$

*Proof.* Associated with the one parameter subgroup  $R(s)$  is the generator  $\xi_{\mathcal{M}}(\mathbf{q}) := \frac{d}{ds} R(s)\mathbf{q} = \hat{\xi}\mathbf{q} = \xi \times \mathbf{q}$  where  $\mathcal{M} = \mathbb{R}^3$ . Now we consider,

$$\int_{t_1}^{t_2} L(R(s)\mathbf{q}, R(s)\dot{\mathbf{q}}) dt = \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}) dt$$

and now we differentiate this wrt  $s$  and then set  $s = 0$  to obtain

$$\left\langle \frac{\partial L}{\partial \dot{\mathbf{q}}}, \xi_{\mathcal{M}}(\mathbf{q}) \right\rangle = \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{q}}} \xi \times \mathbf{q} = \left\langle \mathbf{q} \times \frac{\partial L}{\partial \dot{\mathbf{q}}}, \xi \right\rangle = A_{\xi}$$

□

### 6.2 Noether Theory and EP Reduction

We have,

$$\int_{t_1}^{t_2} L(R, \dot{R}) dt = 0$$

and  $L$  is left-invariant and so,  $L(SR, S\dot{R}) = L(R, \dot{R})$

**Theorem 6.2** (Noethers Theorem). Corresponding to each one parameter subgroup of  $\text{SO}(3)$ ,  $S(s)$  with  $S(0) = e$  and  $S'(0) = \hat{\xi} \in \mathfrak{SO}(3)$ , then there is a conserved quantity

$$A_{\xi} := \left\langle \text{Ad}_{R^{\top}}^* \frac{\partial \hat{\ell}}{\partial \hat{\Omega}}, \hat{\xi} \right\rangle$$

with  $\frac{dA_{\xi}}{dt} = 0$  along solutions of the Euler Lagrange Equation.

$$\mathcal{E}(R) := \frac{d}{dt} \frac{\partial L}{\partial \dot{R}} - \frac{\partial L}{\partial R} = \mathbf{0}$$

*Proof.* Consider  $S(s)$  and differentiate and let  $s = 0$  much like before,

$$\int_{t_1}^{t_2} \frac{\partial}{\partial \xi_{\mathcal{M}(R)}} \frac{\partial L}{\partial R} + \left\langle \frac{\partial L}{\partial \dot{R}}, \xi_{\mathcal{M}(\dot{R})} \right\rangle$$

where here  $\xi_{\mathcal{M}}(R) = \hat{\xi}R$  □

We now have a definition,

**Definition 6.3** (Infinitesimal Generator). Consider the left action of a Lie group  $G$  on the manifold  $\mathcal{M}$ ,  $(g, \mathbf{x}) \rightarrow gx$  ( $\mathbf{x} \in \mathcal{M}$ ). Let  $\xi \in \mathfrak{g}$  be a vector in the Lie algebra of  $G$  and consider one parameter subgroup

$$[\exp(t\xi) : t \in \mathbb{R}] \subseteq G$$

Then the orbit of an element  $\mathbf{x}$  with respect to this subgroup is a smooth map  $t \rightarrow (\exp(t\xi))\mathbf{x}$  in  $\mathcal{M}$ . The infinitesimal generator associated to  $\xi$  at  $\mathbf{x} \in \mathcal{M}$  denoted by  $\xi_{\mathcal{M}}(\mathbf{x})$  is the tangent vector (or velocity) to this curve at point  $\mathbf{x}$ ,

$$\xi_{\mathcal{M}}(\mathbf{x}) = \left. \frac{d}{dt} \right|_{t=0} (\exp(t\xi)\mathbf{x}) \in T_{\mathbf{x}}\mathcal{M}$$

this smooth vector field  $\xi_{\mathcal{M}} : M \rightarrow TM$  and  $x \mapsto \xi_{\mathcal{M}}(\mathbf{x})$  is called the infinitesimal generator vector field associated to  $\xi$ .

Let  $G$  be an arbitrary matrix lie group, and let  $L$  a left-invariant Lagrangian with variational principle  $\int_{t_1}^{t_2} L(g, \dot{g}) = 0$  the reduced system is  $L(hg, h\dot{g})|_{h=g^{-1}} = L(e, g^{-1}\dot{g}) = \ell(\xi) = \ell(g^{-1}\dot{g})$ .

**Theorem 6.4** (Noether's Theorem). Corresponding to each one-parameter subgroup of  $G$ ,  $\chi(s)$  such that  $\chi(0) = e$  and  $\xi_s(0) = \eta \in \mathfrak{g}$ . There is a conserved quantity

$$\left\langle \text{Ad}_{g^{-1}}^* \frac{\partial \ell}{\partial \xi}, \eta \right\rangle$$

*Proof.* Done before □

**Proposition 6.5.** The left-invariant Lagrangian  $L(g, \dot{g})$  satisfies,

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{g}} - \frac{\partial L}{\partial g} = \mathbf{0} \iff \frac{d}{dt} \frac{\partial \ell}{\partial \xi} - \text{ad}^* \frac{\partial \ell}{\partial \xi} = 0$$

*Proof.* I think I have done this

$$\int_{t_1}^{t_2} \langle \mathcal{E}, \delta g \rangle = \int_{t_1}^{t_2} \langle \mathcal{E}(\xi), \nu \rangle$$

□

## 7 Diamond Map

Let  $V$  be an  $n$ -dimensional vector space with dual  $V^*$  and pairing  $\langle \mathbf{w}, \mathbf{u} \rangle_V$  where  $\mathbf{u} \in V$  and  $\mathbf{w} \in V^*$ . Let  $\mathcal{M}(n, \mathbb{R})$  be a vector space of  $n \times n$  matrices with dual  $\mathcal{M}(n, \mathbb{R})^*$  and pairing  $\langle B, A \rangle_{\mathcal{M}} := \text{Tr}(B^\top A)$  where  $A \in \mathcal{M}(n, \mathbb{R})$  and  $B \in \mathcal{M}(n, \mathbb{R})$ .

The diamond map is a representation of the transformation of the pairing on  $V$  to the pairing on  $\mathcal{M}(n, \mathbb{R})$ . Let  $\mathbf{u} \in V$  and  $\mathbf{w} \in V^*$  and consider the matrices  $A \in \mathcal{M}(n, \mathbb{R})$  and  $\Lambda \in \mathcal{M}(n, \mathbb{R})$  where  $A$  is a general matrix and  $\Lambda$  is a symmetric matrix ( $\Lambda^\top = \Lambda$ ). The diamond map is defined by  $\langle \mathbf{w}, A\Lambda\mathbf{u} \rangle_V = \langle \mathbf{u} \diamond \mathbf{w}, \Lambda \rangle_{\mathcal{M}}$ .

$$\begin{aligned} \langle \mathbf{w}, A\mathbf{u} \rangle_V &= \text{Tr}(\mathbf{u}\mathbf{w}^\top A) \\ &= \text{Tr}((\mathbf{w}\mathbf{u}^\top)^\top A) \\ &= \langle \mathbf{w}\mathbf{u}^\top, A \rangle_{\mathcal{M}} \end{aligned}$$

This is for any matrix  $A \in \mathcal{M}(n, \mathbb{R})$  and vectors  $\mathbf{u} \in V$  and  $\mathbf{w} \in V^*$ . We now conclude that,

$$\begin{aligned} \langle \mathbf{w}, A\Lambda\mathbf{u} \rangle_V &= \text{Tr}(\mathbf{u}\mathbf{w}^\top A\Lambda) \\ &= \langle (\mathbf{u}\mathbf{w}^\top A)^\top, \Lambda \rangle_{\mathcal{M}} \\ &= \langle A^\top \mathbf{w}\mathbf{u}^\top, \Lambda \rangle \end{aligned}$$

If  $F = F^\top$  and  $G = -G^\top$ , then  $\text{Tr}(FG) = 0$ . We see that  $\Lambda$  is symmetric so we consider the antisymmetric part,

$$\begin{aligned} &= \text{Tr}(\mathbf{u}\mathbf{w}^\top A\Lambda) \\ &= \text{Tr}\left(\frac{1}{2}(\mathbf{u}\mathbf{w}^\top A + A^\top \mathbf{w}\mathbf{u}^\top)\Lambda\right) && \text{we are splitting this by its symmetric part} \\ &= \text{Tr}(\text{Sym}(\mathbf{u}\mathbf{w}^\top A)\Lambda) \\ &= \langle \text{Sym}(\mathbf{u}\mathbf{w}^\top A), \Lambda \rangle \end{aligned}$$

We can say that  $\mathbf{u} \diamond \mathbf{w} = \text{Sym}(\mathbf{u}\mathbf{w}^\top A) = \frac{1}{2}(\mathbf{u}\mathbf{w}^\top A + A^\top \mathbf{w}\mathbf{u}^\top)$ . This is going to appear in EP theory in symmetry breaking parameters.

## 8 EP Reduction with parameters

What are symmetry breaking parameters? We already know of  $\mathbb{I}$  is a symmetry breaking parameter or  $\mathbf{e}_3 = R\Gamma$  which is gravity.

Consider a Lie Group,  $G$ , and a left action on a manifold,  $\mathcal{M}$ . Then for a given  $a_0 \in \mathcal{M}$  (a parameter), let  $L : TG \times \mathcal{M} \rightarrow \mathbb{R}$  be a Lagrangian with symmetry breaking parameter  $a_0$ , and suppose it is invariant under the left action:  $G \times (TG \times \mathcal{M}) \rightarrow TG \times \mathcal{M}$  then  $(h, (g, \dot{g}, a_0)) \rightarrow (hg, h\dot{g}, ha_0)$  for all  $h \in G$ . This means that  $L(hg, h\dot{g}, a_0) = L(g, \dot{g}, a_0)$  for all  $h \in G$ . As usual let  $h = g^{-1}$ , then  $L(g, \dot{g}, a_0) = L(g^{-1}g, g^{-1}\dot{g}, g^{-1}a_0) =: \ell(\xi, a)$  where  $\xi := g^{-1}\dot{g}$  and  $a = g^{-1}a_0$ .

**Theorem 8.1.** Then the following are equivalent,

(i) Hamiltons Principle

$$\delta \int_{t_1}^{t_2} L(g, \dot{g}, a_0) dt = 0$$

with  $\delta g(t_1) = \delta g(t_2) = 0$ .

(ii)  $g(t)$  satisfies the Euler-Lagrange equations associated with  $L(g, \dot{g}, a_0)$

(iii) The reduced variational principle (or Hamiltons principle),

$$\delta \int_{t_1}^{t_2} \ell(\xi, a) dt = 0$$

holds on  $\mathfrak{g} \times \mathcal{M}$ , using variations  $\delta \xi = \dot{\eta} + \text{ad}_\xi \eta$  and  $\delta a = -\eta_{\mathcal{M}}(a)$  with free variations  $\eta(t)$  satisfying end point conditions.

(iv) The Euler-Poincare equations

$$\begin{aligned} \frac{d}{dt} \frac{\partial \ell}{\partial \xi} &= \text{ad}_\xi^* \frac{\partial \ell}{\partial \xi} - a \diamond \frac{\partial \ell}{\partial a} \\ \dot{a} &= -\xi_{\mathcal{M}} a \end{aligned}$$

hold on  $\mathfrak{g} \times \mathcal{M}$  where  $\langle \frac{\partial \ell}{\partial a}, \alpha_{\mathcal{M}} a \rangle =: \langle a \diamond \frac{\partial \ell}{\partial a}, \alpha \rangle$  for all  $\alpha \in \mathfrak{g}$  and for all  $a \in \mathcal{M}$ .

*Proof.* We already know that  $\delta \xi = \dot{\eta} + [\xi, \eta]$  and  $\eta = g^{-1} \delta g$  and then  $\delta a = -g^{-1} \delta g a = -\eta_{\mathcal{M}} a = -\eta a$ . Now we look at our variational principle,

$$\begin{aligned} 0 &= \delta \int_{t_1}^{t_2} \ell(\xi, a) dt \\ &= \int_{t_1}^{t_2} \left( \left\langle \frac{\partial \ell}{\partial \xi}, \delta \xi \right\rangle + \left\langle \frac{\partial \ell}{\partial a}, \delta a \right\rangle \right) dt \\ &= \int_{t_1}^{t_2} \left( \left\langle \frac{\partial \ell}{\partial \xi}, \dot{\eta} + \text{ad}_\xi \eta \right\rangle - \left\langle \frac{\partial \ell}{\partial a}, \eta_{\mathcal{M}} \right\rangle \right) dt \\ &= \int_{t_1}^{t_2} \left\langle -\frac{d}{dt} \frac{\partial \ell}{\partial \xi} + \text{ad}_\xi^* \frac{\partial \ell}{\partial \xi}, \eta \right\rangle - \left\langle a \diamond \frac{\partial \ell}{\partial a}, \eta \right\rangle dt \end{aligned} \quad \text{we used integration by parts}$$

Now we want the second equation,

$$\begin{aligned}
 \dot{a} &= \frac{d}{dt}(g^{-1}a_0) \\
 &= \frac{d}{dt}g^{-1}a_0 \\
 &= -g^{-1}\dot{g}g^{-1}a_0 \\
 &= -\xi a \\
 &= -\xi_{\mathcal{M}}a
 \end{aligned}$$

and we are done.  $\square$

Now we look at Noethers Theorems.

**Theorem 8.2** (Noether's Theorem for Symmetry Breaking Parameters). Let  $\xi = g^{-1}\dot{g}$  be a solution of the Euler Poincare Equations with parameters  $a = g^{-1}a_0$ . Then,

$$\frac{d}{dt} \text{Ad}_{g^{-1}(t)}^* \mu = -\text{Ad}_{g^{-1}(t)}^* (a \diamond \frac{\partial \ell}{\partial a})$$

and  $\mu(t) = \frac{\partial \ell}{\partial \xi} \in \mathfrak{g}^*$ .

*Proof.* Exercise  $\square$

**Exercise.** Do the same thing for right invariant actions. If the unreduced Lagrangian  $L : TG \times \mathcal{M} \rightarrow \mathbb{R}$  is invariant under a right action

$$\begin{aligned}
 G \times (TG \times \mathcal{M}) &\rightarrow TG \times \mathcal{M} \\
 (h, (g, \dot{g}, a_0)) &\mapsto (gh, \dot{g}h, a_0h)
 \end{aligned}$$

and

$$\begin{aligned}
 \frac{d}{dt} \frac{\partial \ell}{\partial \xi} &= -\text{ad}_{\xi}^* \frac{\partial \ell}{\partial \xi} - a \diamond \frac{\partial \ell}{\partial a} \\
 &\doteq -\xi_{\mathcal{M}}a
 \end{aligned}$$

The Noether Theorem is,

**Theorem 8.3.**

$$\frac{d}{dt} \text{Ad}_{g(t)}^* \mu = -\text{Ad}_{g(t)}^* \left( a \diamond \frac{\partial \ell}{\partial a} \right)$$



PART II:

EXAMPLES

## 9 Spherical Pendulum

We want to consider a pendulum in 3D space. We will think about this through the definition of spherical coordinates, as in actuality our motion of the bob will just be on  $S^2$ . However, we need to define what are Euler Lagrange equations?[1]

Firstly, here is what the Euler Lagrange equations are,

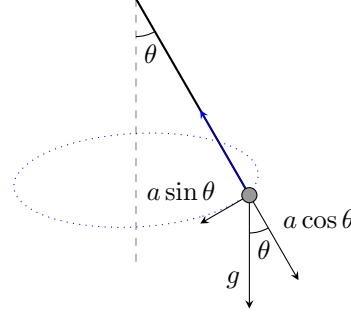
$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^a} - \frac{\partial L}{\partial q^a}$$

where we range through the different basis vectors  $q^a$  and their associated derivatives  $\dot{q}^a$ .

We also define  $L$  as the Lagrangian. We define this simply as,

$$L(q, \dot{q}) = T(q, \dot{q}) - V(\mathbf{r}(q))$$

where further we define  $T(q, \dot{q})$  as the kinetic energy of the system and  $V(\mathbf{r}(q))$  the potential energy of the system.



We are going to use polar coordinates to derive our system of equations.

$$\begin{aligned} x &= R \sin \theta \cos \phi \\ y &= R \sin \theta \sin \phi \\ z &= R(1 - \cos \theta) \end{aligned}$$

Our first focus is  $T(q, \dot{q})$ , which will just be  $\frac{1}{2}mv^2$ . We can see that  $v = |\dot{\mathbf{r}}(t)|$  and so  $v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$ . Hence, we now find what  $v$  is and then find the lagrangian. Firstly, we note that,

$$\frac{d}{dt}(x(t)) = \frac{\partial x}{\partial \theta} \frac{d\theta}{dt} + \frac{\partial x}{\partial \phi} \frac{d\phi}{dt}$$

and similarly for  $y(t)$  and  $z(t)$ . Hence,

$$\begin{aligned} \frac{dx(t)}{dt} &= R \cos \theta \cos \phi \dot{\theta} - R \sin \theta \sin \phi \dot{\phi} \\ \frac{dy(t)}{dt} &= R \cos \theta \sin \phi \dot{\theta} + R \sin \theta \cos \phi \dot{\phi} \\ \frac{dz(t)}{dt} &= R \sin \theta \dot{\theta} \end{aligned}$$

and now we derive our  $T(q, \dot{q})$ ,

$$\begin{aligned} T(q, \dot{q}) &= \frac{1}{2}m \left( \left( R \cos \theta \cos \phi \dot{\theta} - R \sin \theta \sin \phi \dot{\phi} \right)^2 + \right. \\ &\quad \left( R \cos \theta \sin \phi \dot{\theta} + R \sin \theta \cos \phi \dot{\phi} \right)^2 + \\ &\quad \left. \left( R \sin \theta \dot{\theta} \right)^2 \right) \end{aligned}$$

which then can be simplified down to,

$$T(\theta, \dot{\theta}, \phi, \dot{\phi}) = \frac{mR^2}{2} \left( \dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \right)$$

and we note that our system has only one potential energy, gravitation potential! Hence,

$$V(\theta, \dot{\theta}, \phi, \dot{\phi}) = -mgz = -mgR(1 - \cos \theta)$$

Hence, we can now talk about Lagrangian explicitly,

$$L(\theta, \dot{\theta}, \phi, \dot{\phi}) = \frac{mR^2}{2} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + mgR(1 - \cos \theta)$$

Finally, we can now take derivatives of this function and produce the Euler Lagrange equations. We need to find the basis vectors,  $\theta$  and  $\phi$ , as we have two basis vectors, we will have two equations. Firstly,  $\theta$

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} &= \frac{d}{dt} mR^2 \dot{\theta} - (\dot{\phi}^2 mR^2 \sin \theta \cos \theta - mgR \sin \theta) = 0 \\ mR^2 \ddot{\theta} - mR^2 \sin \theta \cos \theta \dot{\phi}^2 + mgR \sin \theta &= 0 \\ R\ddot{\theta} - R \sin \theta \cos \theta \dot{\phi}^2 + g \sin \theta &= 0 \end{aligned}$$

and secondly,  $\phi$

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} &= \frac{d}{dt} (mR^2 \dot{\phi} \sin^2 \theta) = 0 \\ mR^2 \ddot{\phi} \sin^2 \theta + 2mR^2 \dot{\phi} \dot{\theta} \sin \theta \cos \theta &= 0 \\ \ddot{\phi} \sin \theta + 2\dot{\phi} \dot{\theta} \cos \theta &= 0 \end{aligned}$$

We have derived the Euler-Lagrange equations for the spherical pendulum, which are

$$\begin{cases} R\ddot{\theta} - R \sin \theta \cos \theta \dot{\phi}^2 + g \sin \theta = 0 \\ \ddot{\phi} \sin \theta + 2\dot{\phi} \dot{\theta} \cos \theta = 0 \end{cases}$$

## 10 Heavy Top

The heavy top problem is a rigid body with a fixed point. We are going to study its dynamics in the body frame.  $x(t) = R(t)\mathbf{X}$  and  $R(t) \in \text{SO}(3)$ . We know the potential energy of a point mass in a gravitation field is  $V = mg\mathbf{e}_3 \cdot x(t)$ . The lagrangian,

$$L(R, \dot{R}) = \int_B \rho(\mathbf{X}) \left( \frac{1}{2} \|\dot{R}\mathbf{X}\|^2 - g\mathbf{e}_3 \cdot R\mathbf{X} \right) d^3\mathbf{X}$$

Now we can verify this neither left nor right invariant.

**Exercise.** Prove that  $L(R, \dot{R}) \neq L(RR^{-1}, \dot{R}R^{-1})$  and  $L(R, \dot{R}) \neq L(R^{-1}R, \dot{R}^{-1}\dot{R})$

We know  $\hat{\Omega} = R^{-1}\dot{R}$ , then we know  $L = \frac{1}{2}\Omega \cdot \mathbb{I}\Omega - g\mathbf{e}_3 \cdot R\beta$  where  $\beta = \int_B \rho(\mathbf{X})\mathbf{X} d^3\mathbf{X}$  and  $\beta$  is the centre of mass in the body frame. (We could also use  $\beta = \mathbf{X}_b$ ). We define  $\Gamma = R^{-1}\mathbf{e}_3$  and so

$$\ell(\hat{\Omega}, \Gamma) = \frac{1}{2}\Omega(t) \cdot \mathbb{I}\Omega(t) - g\Gamma(t) \cdot \mathbf{X}_b$$

and we see that  $\Gamma(t)$ , the gravitational force, is the symmetry breaking parameter. Now we use Hamilton's principle,

$$\begin{aligned} \delta \int_{t_1}^{t_2} \ell(\hat{\Omega}, \Gamma) dt &= 0 \\ \int_{t_1}^{t_2} \left( -\frac{d}{dt} \mathbb{I}\hat{\Omega} + \mathbb{I}\Omega \times \Omega \right) \cdot \Lambda dt - \int_{t_1}^{t_2} g\mathbf{X}_b \cdot \delta\Gamma &= 0 \end{aligned}$$

The first integral is the rigid body terms and the second is the symmetry breaking term. We also define  $\hat{\Lambda} = R^{-1}\delta R$ . We now look to  $\delta\Gamma$ ,

$$\begin{aligned} \delta\Gamma &= \delta R^{-1}\mathbf{e}_3 \\ &= -R^{-1}\delta R^{-1}\mathbf{e}_3 \\ &= -\hat{\Lambda}\Gamma \\ &= -\Lambda \times \Gamma \end{aligned}$$

We can now write the following,

$$\int_{t_1}^{t_2} \left( -\frac{d}{dt} \mathbb{I}\hat{\Omega} + \mathbb{I}\Omega \times \Omega \right) \cdot \Lambda dt + \int_{t_1}^{t_2} -g \langle \mathbf{X}_b, \Lambda \times \Gamma \rangle dt = 0$$

and so we can write,

$$\int_{t_1}^{t_2} \left[ -\frac{d}{dt} \mathbb{I}\hat{\Omega} + \mathbb{I}\hat{\Omega} \times \hat{\Omega} + \mathbb{I}\Omega \times \Omega - g(\mathbf{X}_b \times \Gamma) \cdot \Lambda \right] dt = 0$$

Now we get,

$$\begin{aligned} \mathbb{I}\dot{\Omega} &= \mathbb{I}\hat{\Omega} \times \hat{\Omega} + g\Gamma \times \mathbf{X}_b \\ \dot{\Gamma} &= \Gamma \times \hat{\Omega} \end{aligned}$$

**Exercise.** Find  $\frac{d}{dt} (R\mathbb{I}\hat{\Omega}) = gR(\Gamma \times \mathbf{X}_b) \neq 0$ . Find what kind of angular momentum is conserved. Find something that is conserved.

**Exercise.** Have a look at the spherical pendulum.

# PART III:

# APPENDICES

## 11 Automorphisms

This is an appendices for Chapter II of the thesis, where we mention briefly Inner Automorphisms. Here I would like to define what they are and give some general background. We have seen before what an isomorphism is, here is the definition of an Automorphism,

**Definition 11.1** (Automorphism). Let  $X$  be some mathematical object or structure, then an Automorphism is a bijection  $f : X \rightarrow X$

Here are a few examples of what automorphisms are,

- (i) Consider  $(\mathbb{Z}, +)$  as a group, then negation is an automorphism. Moreover, if we consider  $(\mathbb{Z}, +, \times)$  as a group, then that automorphism is the only automorphism.
- (ii) Of any abelian group there will always be the automorphism of negation.

As we saw in the main text, the inner automorphism is just the action of conjugation on the group itself. A further interesting fact is that the group of inner automorphisms form a normal subgroup of  $\text{Aut}(G)$  that we denote  $\text{Inn}(G)$ .

*Proof.* by Goursat's Lemma<sup>3</sup>

□

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<sup>3</sup>This lemma leads to my favourite lemma in the whole of mathematics, Snake Lemma.

## References

- [1] Darryl D. Holm. *Geometric mechanics: Part I Dynamics and Symmetry*. Imperial College Press, 2011.
- [2] Xena Project Kevin Buzzard. *Lean summer lectures 2/18: partitions*. (YouTube). July 2020. URL: <https://www.youtube.com/watch?v=FEKsZj3WkTY>.
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