Year 3 — Dynamical Systems and Control

Based on lectures by Dr Tim Hughes

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine (especially the typos!).

Contents

1 Preliminaries

1.1 Continuous time Dynamical Systems

We are going to consider, *Lecture 1*

$$\begin{aligned}\frac{d\mathbf{x}}{dt}(t) &= f(x(t), u(t)) \\ y(t) &= g(x(t), u(t))\end{aligned}$$

We are going to call x the state and u the input and y the output. There may be a case where our variables are vector valued and hence have a system of differential equations. We call f and g time invariant as they do not vary with t and not explicitly dependent on t.

Example. The equations governing aerobic digestion are,

$$\begin{aligned}\frac{db}{dt} &= (e^{-s} - D)b \\ \frac{ds}{dt} &= ke^{-s}b + D(s_I - s)\end{aligned}$$

where b and s are biomass and substrate concentrations, which comprise the states. Then D and s_I are the dilution rate and input substrate concentration, these are the inputs.

We consider systems over some $0 \leq t \leq t_1$ and we consider where $x(t)$ is uniquely determined over our interval by the initial condition and the input on that same interval. This places a constrain on the functions f and g since, in general, $x(t)$ need not be uniquely determined by the initial condition and the input.

Our form may seem rather restrictive, however, it's less restrictive than it appears, let's consider a pendula *Lecture 2*

Example. The angle of a damped pendulum is defined by,

$$mL^2\frac{d^2\theta}{dt^2} = -\nu\frac{d\theta}{dt} - mgL\sin(\theta) + T$$

now, we let $x_1 = \theta$, $x_2 = \frac{d\theta}{dt}$, $u = T$ and $y = \theta$. Now we write this in the previous form,

$$\begin{aligned}\frac{dx_1}{dt} &= \frac{d\theta}{dt} = x_2 \\ \frac{dx_2}{dt} &= \frac{d^2\theta}{dt^2} = \frac{1}{mL^2}\left(-\nu x_2 - mgL\sin(x_1) + u\right) \\ y &= \theta = x_1\end{aligned}$$

Hence,

$$\begin{aligned}f_1 &= x_2 \\ f_2 &= \frac{1}{mL^2}\left(-\nu x_2 - mgL\sin(x_1) + u\right) \\ g_1 &= x_1\end{aligned}$$

Definition 1.1 (Autonomous)**.** If the input $u(t)$ is missing, then the system is said to be autonomous and the state and output depend only on the initial state.

particular attention is to be paid to linear time-invariant systems, the solutions to linear ODEs. Then we can write them as,

$$\begin{aligned}\frac{dx}{dt}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

1.2 Equilibria and Stability

Definition 1.2 (Equilibrium)**.** Consider a fixed input $u(t) = u_e \forall t \in \mathbb{R}$. Then the state and input pair (x_e, u_e) is called an equilibrium if $f(x_e, u_e) = 0$.

Example. We consider the anaerobic digester. Then, we need solutions to,

$$(e^{-s} - D)b = 0 \qquad ke^{-s}b + D(s_I - s) = 0$$

We can solve these equations nicely, and get the following equilibrium,

$$\begin{aligned}
(x_e, u_e) &= \left(\begin{bmatrix} 0 \\ c_1 \end{bmatrix}, \begin{bmatrix} 0 \\ c_2 \end{bmatrix} \right) \\
(x_e, u_e) &= \left(\begin{bmatrix} 0 \\ c_3 \end{bmatrix}, \begin{bmatrix} c_4 \\ c_3 \end{bmatrix} \right) \\
(x_e, u_e) &= \left(\begin{bmatrix} c_5 \\ c_6 \end{bmatrix}, \begin{bmatrix} e^{-c_6} \\ c_6 - kc_5 \end{bmatrix} \right)
\end{aligned}$$

If we let the input depend on the state, $u(t) = k(x(t)), \forall t \geq 0$ and some function k, then we can define new functions $F(x(t)) = f(x(t), u(x(t)))$ and $G(x(t)) = g(x(t), u(x(t)))$, whereupon we obtain an autonomous system,

$$\begin{aligned}
\frac{dx}{dt}(t) &= F(x(t)) \\
y(t) &= G(x(t))
\end{aligned}$$

For a autonomous system the state x_e is an equilibrium point if $F(x_e) = 0$. This is a lot simpler. A system may have many equilibria.

Definition 1.3 (Stability)**.** Informally we call an equilibria stable if whenever $x(0)$ is sufficiently close to x_e if $x(t)$ remains close to $x_e, \forall t \geq 0$

Definition 1.4 (Asymptotically Stable)**.** A system is asymptotically stable if it is stable ad in addition, if $x(0)$ is sufficiently close to x_e, then $x(t) \to x_e$ as $t \to \infty$.

If we want to see if a system is stable, we can do so by considering energy. In terms of our pendulum, the energy is,

$$V(x_1, x_2) = \frac{g}{L}(1 - \cos(x_1)) + \frac{1}{2}x_2^2$$

and if the system doesn't have an increasing change in energy, then we can say that it stays relatively close to an initial condition and hence can be asymptotically stable. This applies to our $V(x_1, x_2)$.

If we consider a torque input, $u = mL^2 x_1$ then will result in an unstable system with respect to it's equilibria point $(0, 0)$. *Lecture 3*

1.3 Linearisation

You can apply our linear tecniques to non-linear systems by linearising them. Firstly, consider an equilibrium point and let f and g be continuously differentiable. Then, we can linearise by using Taylor Series.

Let $u(t) = u_e + \delta u(t)$ and $x(t) = x_e + \delta x(t)$ and also $y_e = g(x_e, u_e)$. Then,

$$\begin{aligned}
\frac{d\delta x}{dt} &= A\delta x + B\delta u + O(x^2) \\
\delta y &= C\delta x + D\delta u + O(x^2)
\end{aligned}$$

where we define A, B, C, D as,

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_d} \\ \vdots & & \vdots \\ \frac{\partial f_d}{\partial x_1} & \cdots & \frac{\partial f_d}{\partial x_d} \end{bmatrix} (x_e, u_e) \qquad B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_d} \\ \vdots & & \vdots \\ \frac{\partial f_d}{\partial u_1} & \cdots & \frac{\partial f_d}{\partial u_d} \end{bmatrix} (x_e, u_e)$$

$$C = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_m} \end{bmatrix} (x_e, u_e) \qquad B = \begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \cdots & \frac{\partial g_1}{\partial u_m} \\ \vdots & & \vdots \\ \frac{\partial g_m}{\partial u_1} & \cdots & \frac{\partial g_m}{\partial u_m} \end{bmatrix} (x_e, u_e)$$

where $f \in (\mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R})^d$, $g \in (\mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R})^m$ and $u \in (\mathbb{R} \to \mathbb{R})^n$ and $x \in (\mathbb{R} \to \mathbb{R})^d$.

Example. We consider the Lotka-Volterra equations,

$$\frac{dx_1}{dt} = ax_1 - bx_1x_2 \qquad \frac{dx_2}{dt} = cx_1x_2 - dx_2$$

where x_1 and x_2 are the prey and predators and $a, b, c, d \in \mathbb{R}$. We have some equilibrium at $(0, 0)$ and also $\left(\frac{d}{c}, \frac{a}{b}\right)$.
We can calculate our matrix and get,

$$\frac{d\delta\mathbf{x}}{dt} = \begin{bmatrix} 0 & -\frac{bd}{c} \\ \frac{ac}{d} & 0 \end{bmatrix} \delta\mathbf{x} + O(x^2)$$

Example. Consider our friend, the damped pendulum,

$$\begin{aligned} \frac{dx_1}{dt} &= x_1 \\ \frac{dx_2}{dt} &= -\frac{\nu}{mL^2}x_2 - \frac{g}{L}\sin(x_1) + \frac{1}{mL^2}u \end{aligned}$$

where $x_1 = \frac{\pi}{6}$ and $x_2 = 0$ and $u = \frac{mgL}{2}$. Then we can form the linearisation matrix,

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \left(\frac{\pi}{6}, 0, \frac{mgL}{2}\right) = \begin{bmatrix} 0 & 1 \\ -\frac{\sqrt{3}g}{2L} & -\frac{\nu}{mL^2} \end{bmatrix}$$

and as we have an input, we need to form a second matrix,

$$\begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} \left(\frac{\pi}{6}, 0, \frac{mgL}{2}\right) = \begin{bmatrix} 0 \\ \frac{1}{mL^2} \end{bmatrix}$$

Thus,

$$\frac{d\delta\mathbf{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{\sqrt{3}g}{2L} & -\frac{\nu}{mL^2} \end{bmatrix} \delta\mathbf{x} + \begin{bmatrix} 0 \\ \frac{1}{mL^2} \end{bmatrix} \delta u + O(x^2)$$

1.4 Discrete Time Dynamical Systems

Here is a difference equation,

$$\begin{aligned} x(k+1) &= f(x(k), u(k)) \\ y(k) &= g(x(k), u(k)) \end{aligned}$$

Here x, y, u are sequences defined for all $k \geq 0 \in \mathbb{Z}$. We write $u, x, y \in \mathbb{Z}_+ \to \mathbb{R}$.

We call (x_e, u_e) an equilibrium and input pair if x_e and u_e if there are constant vectors satisfying $f(x_e, u_e) = x_e$. There are also similar ideas to what we considered for continuous systems.

2 LTI Systems

We are going to consider some linear ODES, *Lecture 4*

$$\frac{d^n y}{dt^n} + \frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \cdots + b_0 u$$

qhwew $a_i, b_i \in \mathbb{R}$ and $u, y : \mathbb{R} \to \mathbb{R}$ and we consider $m < n$ for the moment, so the order of differentiation of u doesnt exceed m. To solve these we are going to consider laplace transform.

2.1 Laplace Transforms

Definition 2.1 (exponentially bounded)**.** Here, f is called exponentially bounded, if $|f(t)| \leq Me^{\alpha t}$ for some $\alpha \in \mathbb{R}$

The laplace transforms for an exponentially bounded function $f(t)$ defined on $t \geq 0$ is defined by,

$$\mathcal{L}(f) = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

The above integral is defined for all $s \in \mathbb{C}$ where $\operatorname{Re} c > \alpha$.
This is a restriction for the existence of laplace transform, but not the DEs. When f is exponentially bounded, means that just that the integral converges nicely, it makes our life easier.

Lemma 2.2. If $f = g$, then $f = g$ for (almost[1]) all $t \geq 0$.

We will focus on piecewise continuous, in which $f = g$ for all $t \geq 0$.
This means that the laplace transform is invertible, then we can say $\mathcal{L}^{-1}(\mathcal{L}(f)) = f$ and we will use lookup tables for these transforms.

Remark. Note, we can say nothing about $\mathcal{L}^{-1}(\mathcal{L}(f))$ for $t < 0$

Let f and g be defined on $t \geq 0$

(i) $\mathcal{L}(af + bg) = a\mathcal{L}(f) + b\mathcal{L}(g)$

(ii) $\mathcal{L}(e^{at}) = \frac{1}{s-a}$

(iii) $\mathcal{L}(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}}$

(iv) $\mathcal{L}(\frac{d^k f}{dt^k}) = s^k \mathcal{L}(f) - s^{k-1} f(0) - \cdots - s\frac{d^{k-2}f}{dt^{k-2}}(0) - \frac{d^{k-1}f}{dt^{k-1}}(0)$

(v) $\mathcal{L}\left(\int_0^t f(\tau)\, d\tau\right) = \frac{1}{s}\mathcal{L}(f)$

(vi) $\mathcal{L}(\int_0^t g(t-\tau) f(\tau) d\tau) = \mathcal{L}(f(t))\mathcal{L}(g(t))$

Note, in 4, we assume that f is k differentiable. We will sometimes want to lift this assumption.
If we take the laplace transform of the ODE,

$$\frac{d^n y}{dt^n} + \frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \cdots + b_0 u$$

we can rearrage and get something of,

$$Y(s) = \frac{b(s)}{a(s)} U(s) + \frac{c(s) - d(s)}{a(s)}$$

[1]we just want the integrals to be equal, but at some t, $f(t) \neq g(t)$. If they are continuous, this doesn't matter

where,

$$\begin{aligned}
a(s) &= s^n + a_{n-1}s^{n-1} + \cdots + a_0 \\
b(s) &= b_m s^{n-1} + b_{m-1}s^{m-1} + \cdots + b_0 \\
c(s) &= y(0)s^{n-1} + \left(\frac{dy}{dt}(0) + a_{n-1}y(0)\right)s^{n-2} + \cdots + \left(\frac{d^{n-1}y}{dt^{n-1}}(0) + a_{n-1}\frac{d^{n-2}y}{dt^{n-2}}(0) + \cdots + a_1 y(0)\right) \\
d(s) &= b_m u(0)s^{m-1} + \left(b_m\frac{du}{dt}(0) + b_{m-1}u(0)\right)s^{m-2} + \cdots + \left(b_m\frac{d^{m-1}u}{dt^{m-1}}(0) + b_{m-1}\frac{d^{m-2}u}{dt^{m-2}} + b_1 u(0)\right)
\end{aligned}$$

Thus,

$$y(t) = \mathcal{L}^{-1}\left(\frac{b(s)}{a(s)}U(s) + \frac{c(s)-d(s)}{a(s)}\right)$$

We are going to use result 7, then we can continue from,

$$\begin{aligned}
&= \mathcal{L}^{-1}\left(\frac{b(s)}{a(s)}U(s) + \frac{c(s)-d(s)}{a(s)}\right) \\
&= \int_0^t \mathcal{L}^{-1}\left(\frac{b(s)}{a(s)}\right)(t-\tau)u(\tau)\,d\tau + \mathcal{L}^{-1}\left(\frac{c(s)-d(s)}{a(s)}\right)
\end{aligned}$$

We call the ratio $G(s) = \frac{b(s)}{a(s)}$ the transfer function, $W(t) = \mathcal{L}^{-1}\left(\frac{b(s)}{a(s)}\right)$ the impulse response, and $y_f(t) = \mathcal{L}^{-1}\left(\frac{c(s)-d(s)}{a(s)}\right)$ the free response, so,

$$y(t) = \int_0^t W(t-\tau)u(\tau)\,d\tau + y_f(t)$$

The inverse Laplace transforms can be obtained using a partial fraction decomposition. We will show how this works in general later, so consider the example,

Example. Consider, $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \frac{du}{dt} + 3u$ with $u(t) = \sin t$, $y(0) = 0$ and $\frac{dy}{dt}(0) = 1$. So,

– Find the free response

– Find the impulse response

– Find $y(t)$ $(t \geq 0)$

We note that both $W(t)$ and $y_f(t)$ both take forms of $\mathcal{L}^{-1}\left(\frac{p(s)}{a(s)}\right)$, so let's find the closed form for this, *Lecture 5*
We split it up into,

$$\frac{p(s)}{a(s)} = \sum_{i=1}^{N}\sum_{j=1}^{r_i}\frac{h_{i,j}}{(s-\lambda_i)^j}$$

where $\lambda_i, h_{i,j} \in \mathbb{C}$ and $a(s) = \prod_{j=1}^{N}(s-\lambda_i)^{r_i}$. Here with $f_k(s) = \frac{p(s)}{\prod_{i=1,i\neq j}^{N}(s-\lambda_i)^{r_i}}$ the coefficients $h_{k,j}$ can we obtained from some formula,

$$\begin{aligned}
h_{k,r_k} &= f_k(\lambda_k) \\
h_{k,r_k-1} &= \frac{df_k}{ds}(\lambda_k) \\
&\vdots \\
h_{k,1} &= \frac{1}{(r_k-1)!}\left(\frac{d^{r_k-1}f_k}{ds^{r_k-1}}(\lambda_k)\right)
\end{aligned}$$

thus,

$$\mathcal{L}^{-1}\left(\frac{p(s)}{a(s)}\right) = \sum_{i=1}^{N}\sum_{j=1}^{r_i}\frac{h_{i,j}t^{j-1}e^{\lambda_i t}}{(j-1)!}$$

Applying this we get some formula. We assumed that both $u(t)$ and $y(t)$ are exponentially bounded and sufficiently differentiable, in whih case $u(t)$ and $y(t)$ satisfy the differential equation even when they are not exponentially bounded. Now we onlu need to assume that they are integrable, then we can get a weak solution, i.e. when they are not differentiable. We can consider a step function. We need to have the correct initial conditions.

We only consider piecewise continuous functions, then we define $y(0) = y(0_-)$ and this is the left hand limit, and the same for everything else. To account for this, we modify the Laplace transform to say,

$$\mathcal{L}\left(\frac{d^n f}{dt^n}\right) = s^k\mathcal{L}(f) - s^{k-1}f(0_-) - \cdots - s\frac{d^{k-2}f}{dt^{k-2}}(0_-) - \frac{d^{k-1}f}{dt^{k-1}}(0_-)$$

Let's consider the degree of differentiation of the RHS is greater than the left, so we will do polynomial long division. So take the RHS and replace theoremstyle differentials with s^r, then we get,

$$g_r s^r + g_{t-1}s^{r-1} + \cdots + g_0 = (s^n + a_{n-1}s^{n-1} + \dots)(q_{r-n}s^{r-n} + q_{r-n-1}s^{r-n-1} + \dots) + b_m s^m + b_{m-1}s^{m-1} + \dots$$

Then we define $y = \hat{y} - (q_{r-n}\frac{d^{r-n}u}{dt^{r-n}} + q_{r-n-1}\frac{d^{r-n-1}u}{dt^{r-n-1}} + \dots)$ then we can transform back to our original equation.

Input output stability. We shall consider our general linear ODE, then with $m < n$. We have shown that this has a unique solution for any integrable u and $y(0_-)$ and all of it's derivatives up to $n-1$. We call a system input-output stable if, *Lecture 6*

- if $u(t) = 0$ for all $t \geq 0$ then $y(t) \to 0$ as $t \to \infty$.
- If $\sup_{t\geq 0}|u(t)| < \infty$, then $\sup_{t\geq 0}|y(t)| < \infty$.

We will now show that a system is stable if and only if all of the roots of $a(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$ are in the open left half plane. To prove this,

- 1 implies that the roots of $a(s)$ are all in the open left half plane
- If all the roots of $a(s)$ are in the open left half plane then 1 holds.
- If all of the roots of $a(s)$ are in the opel left half plane then 2 holds.

For (a) holds, let $u(t) = 0$, this implies that,

$$y(t) = \mathcal{L}^{-1}\left(\frac{c(s)}{a(s)}\right)$$

and the a and c are what we expect they are. If $\frac{d^{n-1}y}{dt^{n-1}}(0) = 1$ and $y(0) = \frac{dy}{dt}(0) = \cdots = 0$, then we can say that,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left(\frac{1}{a(s)}\right) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{r_i}\frac{h_{i,j}t^{j-1}e^{\lambda_i t}}{(j-1)!} \end{aligned}$$

where $\widetilde{h}_{i,r_i} \neq 0$. Thus, we can say that for these to converge, we must say $\operatorname{Re}\lambda_i < 0$ and λ_i are the roots. Hence, first is proved.

For (b) and (c), note that all the roots are in the open left half plane, then $W(t) = \sum_{i=1}^N \sum_{j=1}^{r_i} \frac{\hat{h}_{i,j} t^{j-1} e^{\lambda_i t}}{(j-1)!}$ and $y_f(t) = \sum_{i=1}^N \sum_{j=1}^{r_i} \frac{\widetilde{h}_{i,j} t^{j-1} e^{\lambda_i t}}{(j-1)!}$ for some $\widetilde{h_{i,j}}, \hat{h}_{i,j} \in \mathbb{C}$, where $a(s) = \prod_{i=1}^N (s - \lambda_i)^{r_i}$, so λ_i are all in the open left half plane.
Hence we can now choose a $0 > \lambda \in \mathbb{R}$ such that $\lambda_i - \lambda$ is in the open left half plane. We can then find $M, N \in \mathbb{C}$ such that $|W(t)| \leq Me^{\lambda t}$ and $|y_f(t)| \leq Ne^{\lambda t}$ forall $t \geq 0$. This then proves (b).

For (c),

$$\begin{aligned}
\sup_{t\geq 0} |y(t)| &= \sup_{t\geq 0} \left| \int_0^t W(t-\tau)u(\tau)\, d\tau + y_f(t) \right| \\
&\leq \sup_{t\geq 0} \left| \int_0^t W(t-\tau)u(\tau)\, d\tau \right| + \sup_{t\geq 0} |y_f(t)| \\
&\leq \sup_{t\geq 0} \int_0^t |W(t-\tau)u(\tau)|\, d\tau + \sup_{t\geq 0} |y_f(t)| \\
&\leq \sup_{t\geq 0} \int_0^t |W(t-\tau)|\, d\tau \times \sup_{t\geq 0} |u(\tau)| + \sup_{t\geq 0} |y_f(t)| \\
&\leq M \sup_{t\geq 0} \int_0^t e^{\lambda\tau}\, d\tau \times \sup_{t\geq 0} |u(t)| + \sup_{t\geq 0} |y_f(t)|
\end{aligned}$$

and so it follows that $\sup_{t\geq 0} |u(t)| < \infty$ implies $\sup_{t\geq 0} |y(t)| < \infty$ which proves (c).

If we have u differentiated the same amount of $\hat{y}$, then we can again use long division to prove the same result, by letting $\hat{y} = y + q_0 u$. Then we can show that this substitution still allows the lemma to be true.

2.2 Routh Hurwitz Stability Criterion

We have show that linear ODEs are stable if and only if $a(s)$'s roots are in the open left half plane. We could solve $a(s) = 0$, however this requires numerical techniques which could end up with rounding errors, for example,

Example. Consider $a(s) = (s - \alpha)^n$ and now consider a pertibation on those roots with a polynomial, $a_\varepsilon(s) = (s-\alpha)^n - \varepsilon$ with roots $\alpha + \varepsilon^{\frac{1}{n}} e^{\frac{2k\pi i}{n}}$ for $k \in \mathbb{Z}$ and $k < n-1$.

For a polynomial $a(s)$, the Routh Hurwitz criterion provides a computable condition based solely on coefficients. where $r_{0,0} = a_0$ and $r_{0,1} = a_2$, $r_{0,2} = a_4$ and $r_{1,0} = a_1$, $r_{1,1} = a_3$ and so on. Then onwards from

$$\begin{matrix}
r_{0,0} & r_{0,1} & r_{0,2} & \cdots \\
r_{1,0} & r_{1,1} & r_{1,2} & \cdots \\
\vdots & \vdots & \vdots & \\
r_{n,0} & r_{n,1} & r_{n,2} & \cdots
\end{matrix}$$

that we define $r_{i,j} = r_{i-1,0} \times r_{i-2,j+1} = r_{i-2,0} \times r_{i-1,j+1}$. Then we say that all of the roots of $a(s)$ are in the open left half plane if and only if $r_{i,0} > 0$ for $i = 0, 1, 2, \ldots, n$.

Proof. I am not LaTeXing that, I'm sorry, I've just looked at the slides. I'll give a brief idea though. *Lecture 7*

We firstly form our Routh array in the usual way, our criterion states that the roots are in the left open half plane when all the left entries are positive. We form some polynomials, and a series of polynomials where $a(s) = a_0(s)$.
Then, we can say that $r_{i+1,0} \neq 0$ and then we say that when the left entries are positive, then the degrees of the polynomials decrease. We aim to show the following are equivalent:

- The roots of $a_i(s)$ are all in the open left half place and $a_i, a_{i+1}, r_{i,0}, r_{i+1,0} > 0$
- The roots of $a_{i+1}(s)$ are all in the open half plane and (some other coefficients are ¿ 0).

We can note a_i and a_{i+1} differ by a degree of one. If our polynomials roots are in the left half plane, then we can say the coefficients must all have the same sign.
Hence, now consider $\hat{a}^{\gamma}_{i+1}$ with $0 \leq \gamma \leq 1$ and all the left hand entries are greater than 0. We can then increase γ, then we get something proportional to a_i and then at 1, we are proportional to sa_{i+1}. When $s = 1$, we can say that there is a zero at the origin of multiplicity one.
All of the polynomials of our new family have same same degree. Then if $a^{\gamma}_{i+1}(i\omega) = 0$ $0 \neq \omega \in \mathbb{R}$, then all of the polynomials have this root. We finally note that since all of the hat polynomials have the same degree, then the roots will vary continuously with γ. Then we can say as γ varies none of the roots of the polynomials can cross the imaginary axis.
If all of the roots of $a_i(s)$ then no roots will cross the imaginary axis and one approaches the origin, but that polynomial with that root is divisble by a_{i+1} and so then a_{i+1} must also be in the open left half plane. If we decrease γ, then again no roots cross the left open half plane, so the ones at the origin, move into the left or right. So we now show that $\alpha_{i+2}, r_{i+2,0} > 0$ moves into the left half plane. Hence, we now have the equivalence condition.
Now induct on the equivalence, then if all the original or initial polynomials have roots in the open left half plane, then so must the rest by assumption and induction. $\square$

3 Closed Loop Stability

We are now interested to use feedback control on our system to let u be influenced on y and pehaps an *Lecture 8*
addition external input signal w. We can capture what we are interested in a block diagram,

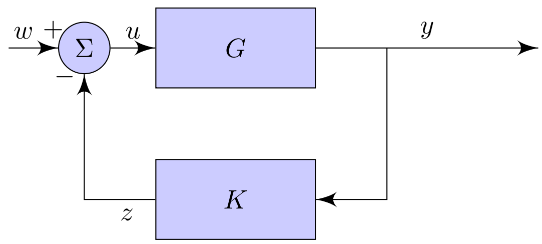

Figure 1: Closed Loop System

We are interested that G and K are ODEs of a similar form.

$$\frac{d^{n_1}y}{dt^{n_1}} + \frac{d^{n_1-1}y}{dt^{n_1-1}} + \cdots + a_0 y = b_m \frac{d^{m_1}u}{dt^{m_1}} + b_{m_2-1}\frac{d^{m_3-1}u}{dt^{m_3-1}} + \cdots + b_0 u$$

$$\frac{d^{n_2}y}{dt^{n_2}} + \frac{d^{n_2-1}y}{dt^{n_2-1}} + \cdots + a_0 y = b_m \frac{d^{m_2}u}{dt^{m_2}} + b_{m_2-1}\frac{d^{m_2-1}u}{dt^{m_2-1}} + \cdots + b_0 u$$

$n_1 \geq m_1$ and $n_2 \geq m_2$.
Let $a(s) = s^{n_1} + a_{n_1-1}s^{n_1-1} + \cdots + a_0$ etc. and then let $G(s) = \frac{b(s)}{a(s)}$ and $K(s) = \frac{q(s)}{p(s)}$, then we can look at the laplace transforms,

$$Y(s) = G(s)U(s) + \frac{c(s)}{a(s)} \qquad Z(s) = K(s)Y(s) + \frac{d(s)}{p(s)} \qquad U(s) = W(s) - Z(s)$$

and now we can eliminate u and z, to obtain,

Alternatively, breaking up $G(s)$ and $K(s)$

$$Y(s) = \frac{b(s)p(s)}{a(s)p(s) + b(s)q(s)}W(s) + \frac{c(s)p(s) - d(s)b(s)}{a(s)p(s) + b(s)q(s)}$$

we assume that $a(s)p(s)$ and $b(s)q(s)$ do not have any common roots in the closed right half plane. We will show that the closed root system is stable if and only if $1+G(s)K(s)$ has no zeros in the closed right half plane.

Note that this is basically boiled down to showing the howing right half plane zeros of the equation coincide with the closed right half plane roots of $ap + bq$ and poles of $\frac{G(s)}{1+G(s)K(s)}$

3.1 Nyquist Stability Condition

We have considered closed loop systems characterised by the polynomials from the ODEs, and the relationship between their laplace transforms.

The Nyquist Stability Condition, proves a necessary and sufficient condition for all of the zeros fo $1 + G(s)K(s)$ to be in the open left half plane.

We will consider the Nyquist Diagram of $L(s) = G(s)K(s)$ is a plot of $L(s)$ in the complex plane as s traverses a path in the complex plane, ie. the Nyquist contour.

Example. For example, study $L(s) = \frac{s+1}{s^2+1}$ which has poles at $s = \pm i$, the Nyquist contour, so we start elsewhere.

Lemma 3.1 (Nyquist Stability Condition)**.** The Nyquist stability criterion states that the zeros of $1 + L(s)$ are all in the open left half plane if and only if the Nyquist diagram of $L(s)$ does not pass through the point $s = -1$ and encircles this point as many times anticlockwise as the number of poles of $L(s)$ is in the open right half plane.

We are now going to prove the Nyquist Stability Condition. *Lecture 9*

Proof. We note that the poles of $L(s)$ are the same as $1 + L(s)$. Hence, the contour will enclose all of the poles of $1 + L(s)$, and there is a zero on the imaginary axis if and only if it passes through $s = -1$.
Factor

$$1 + L(s) = \alpha \frac{(s - z_1)(s - z_2)\dots(z - p_M)}{(s - p_1)\dots(s - p_N)}$$

and now we say that for $s_0 \in \mathbb{C}$ the argument of $1 + L(s)$ is the sum of the arguments of $(s_0 - z_j)$ minus the sum of the argument of $(s_0 - p_j)$, $j = 1, \dots, N$.
Next as s traverses the contour, then the change of argument of $s - \beta$ is -2π if $\beta \in N_C$ otherwise zero. It follows that the change in argument of $1 + L(s)$ as s traverses the Nyquist contour is $2\pi(P - Z)$ where P is the number of poles and Z is the number of zeros. This $P - Z$ is just the number of anticlockwise rotations around -1 by $L(s)$.
Thus $1 + L(s)$ has no zeros in the ORHP if and only if the number of anticlockwise rotations encirclements of the point -1 by the Nyquist Diagram is equal to the number of poles of $L(s)$ in the ORHP. □

Useful sketching tips,

(i) Consider $L(i\omega)$ as $\omega \to \infty$ and $\omega \to 0$

(ii) Determine what happens at the imaginary axis poles of $L(s)$, you could use a Laurent Expansion, however here a few rules of thumb,

- When the Nyquist contour rotates through an angle, so does the diagram in the same direction and angle.
- As the Nyquist contour turns through a semicircle the diagram will span an arc of the circle through an angle of π times the multiplicity of the pole.

(iii) Sketch $L(i\omega)$ for $0 \leq \omega < \infty$ paying attention to the signs of it's real and imaginary parts and the points where the imaginary or real part is zero.

(iv) The plot of $L(i\omega)$ for $-\infty < \omega \leq 0$, just reflect through the real axis

(v) Determine what happens on the infinite circular arc.

Now we consider, *Lecture 10*

$$L(s) = \frac{1}{(s^2 + 1)^2}$$

we can then find the residue of this as,

$$A = \lim_{s \to i}((s - i)^2 L(s)) = -\frac{1}{4}$$

and now we can find the small semicircular arc and then the points here are $s = i + re^{i\theta}$ as $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ Now we put this into the Laurent series,

$$L(i + re^{i\theta}) = \frac{1}{4r^2} e^{i(\pi - 2\theta)} + \dots$$

For small r, we just have $\frac{1}{4r^2}$ and so we have a clockwise circle. Then the rest of the Nyquist diagram lies on the positive real axis. $L(i\omega) \to 0$ as $\omega \to \infty$ and $L(0) = 0$. There are two clockwise encirclements of -1 and

$L(s)$ has no poles in the ORHP, so the closed loop isn't stable.

We can use Matlab to sketch these Nyquist diagrams, however we can't use it when we have an imaginary axis pole.

Example. We have proved the Nyquist Stability condition for quotients of functions. However, it also holds for meromorphic functions in the OHRP. In particular, this allows us to consider time delays. We note that $\mathcal{L}(f(t-\tau)) = e^{-st}\mathcal{L}(f(t))$ and consider the system, *Lecture 11*

$$\begin{aligned}\frac{dz}{dt}(t) + z(t) &= 2y(t)\\ y(t) &= u(t-T)\\ u(t) &= w(t) - z(t))\end{aligned}$$

Now we are going to take Laplace transforms where $w = u = y = z$. Hence we get,

$$\begin{aligned}Y(s) &= e^{-sT}U(s)\\ (s+1)Z(s) &= 2Y(s)\end{aligned}$$

so we want to sketch $L(s) = \frac{2e^{-sT}}{s+1}$ to sketch this we can consider $\frac{2}{s+1}$, but now we consider the exponential, we know that $|e^{-i\omega T} = 1|$ and we know $\arg(e^{-i\omega T}) = -\omega T$. Thus we know that the Nyquist diagram can be obtained from the Nyquist diagram of $\frac{2}{s+1}$ by changing the argument by an amount $-\omega T$ and we know ω is increasing and so we will have a spiral.

We note that above a certain critial value, then it will be sufficient for it to encircle -1. We got our Nyquist diagram by changing the argument of each point. We note that $|\frac{2}{1+i\omega}| = 1$ we then require to find $\omega_c T$ is equal to the difference between $-\pi$ and $\arg(\frac{2}{1+i\omega_c})$ more specifically $\omega_c T = \pi + \arg(\frac{2}{1+i\omega_c})$.

Firstly, we can say that $|\frac{2}{1+\omega_c}|^2 = |\frac{4}{1+\omega_c^2}| = 1$, i.e. $\omega_c = \sqrt{3}$. Then, $\arg(\frac{2}{1+i\sqrt{3}}) = -\frac{\pi}{3}$. We therefore require $\omega_c T = \sqrt{T} = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$. Hence $T = \frac{2\pi}{3\sqrt{3}}$.

3.2 System Response

We have seen how laplace transforms are particularly suited to our uses and adding onto of the RH and NSC.

Consider our usual DE,

$$\frac{d^n y}{dt^n} + \frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_0 y = b_m\frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \cdots + b_0 u$$

Now, note that if $X \in \mathbb{C}$

$$\begin{aligned}X &= |X|e^{i\arg X}\\ |X|\cos(\omega_0 + \arg X) &= \mathrm{Re}(|X|e^{i(\omega_0 t + \arg X)}) = \mathrm{Re}(Xe^{i\omega_t t})\\ \mathcal{L}(|X|\cos(\omega_0 t + \arg(X))) &= \mathcal{L}(\mathrm{Re}(Xe^{i\omega_0 t}))\\ &= \mathcal{L}\left(\frac{1}{2}(Xe^{i\omega_0 t}) + \overline{X}e^{-\omega_0 t}\right)\\ &= \frac{1}{2}\left(X\frac{1}{s-i\omega_0} + \overline{X}\frac{1}{s+i\omega_0}\right)\end{aligned}$$

We let $u(t) = \cos(\omega_0 t)$. Using partial fraction expansions, it follows that,

$$\begin{aligned} Y(s) &= G(s)U(s) \\ &= \frac{1}{2}\left(G(i\omega_0)\frac{1}{s - i\omega_0} + G(-i\omega_0)\frac{1}{1 + i\omega_0}\right) + \frac{p(s)}{a(s)} \end{aligned}$$

for some polynomial $p(s)$. Since $G(-i\omega_0) = \overline{G(i\omega_0)}$, then

$$y(t) = |G(i\omega_0)|\cos(\omega_0 + \arg G(i\omega_0)) + \mathcal{L}^{-1}\left(\frac{p(s)}{a(s)}\right)$$

Moreover, if the roots of $a(s)$ are in the ORHP, then $\mathcal{L}^{-1}\left(\frac{p(s)}{a(s)}\right) \to 0$ as $t \to \infty$ so $y(t)$ tends towards to a soinasoidal system with the same frequence as $u(t)$ but with applitude multiplied by $|G(i\omega_0)|$ and phase increased by $\arg G(i\omega_0)$

Consider again, *Lecture 12*

$$\frac{d^n y}{dt^n} + \frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \cdots + b_0 u$$

with $m \leq n$ and suppose that $y(t) = u(t) = 0$ for all $t < 0$. We can get from Laplace transforms, $Y(s) = G(s)U(s)$. Now consider where the input is a step input, ie. $u(t) = A$ $(t \geq 0)$, $u(t) = 0$ for $t < 0$. Recall that $U(s) = \mathcal{L}(u(t)) = \frac{A}{s}$. Hence we can write,

$$\begin{aligned} Y(s) &= \frac{G(s)A}{s} \\ &= \frac{G(0)A}{s} + \frac{p(s)}{a(s)} \end{aligned}$$

where we use partial fraction decomposition. So, we then get $y(t) = G(0)A + \mathcal{L}\left(\frac{p(s)}{a(s)}\right)$. If all the roots of $a(s)$ are in the ORHP, then $\mathcal{L}\left(\frac{p(s)}{a(s)}\right) \to 0$ as $t \to \infty$and it follows that $y(t) \to G(0)A$. Hence, the steady state response to a step input of magnitude A is a steady state value of $G(0)A$.

Consider a closed loop system, and suppose $G(s)$ does not have a pole at the origin and the closed loop

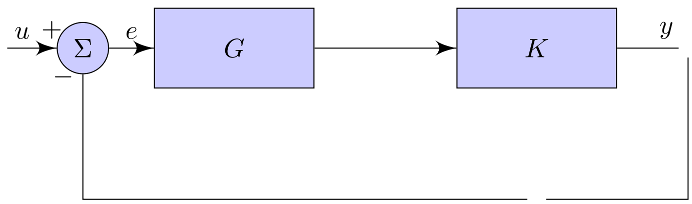

Figure 2: Closed Loop System

system is stable. We aim to find $\lim_{t\to\infty}(y(t))$ when (i) $K(s) = k_1$ and (ii) $K(s) = k_1 + \frac{k_2}{s}$.

Firstly we note that $Y(s) = \frac{G(s)K(s)}{1+G(s)K(s)}U(s)$ and $U(s) = \frac{A}{s}$. We can then consider the first case, $Y(s) = \frac{G(s)k_1}{1+G(s)k_1}\frac{A}{s}$ so $y(t) \to \frac{G(0)k_1 A}{1+G(0)k_1}$ as $t \to \infty$
For case (ii), then $K(s) = k_1 + \frac{k_2}{s}$, then, $Y(s) = \frac{G(s)(k_1 s + k_2)}{s + G(s)(k_1 s + k_2)}\frac{A}{s}$, so $y(t) \to \frac{G(0)k_2 A}{G(0)k_2} = A$ as $t \to \infty$.

Note that, with the second controller $K(s) = k_1 + \frac{k_2}{s}$, the response $y(t)$ of the system tends towards a constant value equal to the size A of the step input, and the error $e = u - y$ tends to zero.

This holds under some very general assumptions, mainly that $G(s)$ doesn't have a pole at the origin and the closed loop system is stable.

Definition 3.2 (Proportional Control)**.** This is referred to as proportional control as the controller $K(s) = k_1 + \frac{k_2}{s}$ is the transfer function $W(s)$ to $Y(s)$ of the system governed by the equation:

$$y(t) = k_1 w(t) + k_2 \int_0^t w(\tau)\, d\tau$$

4 Ball and Beam Experiment

The ball and beam is a common lab experiment for teaching control systems. We consider a beam and sat in the beam, which has a groove there is a ball. About the pivot, we have a motor that can pivot the beam.

If we don't add any torque we have an unstable system. The ball will cause a moment and hence induce some torque. Left to it's own devices the beam with rotate and the ball will fall off the beam. Hence, to stabilise the system we need some sort of control. This control is a motor around the pivot. We will need a sensor to see where the ball is; whether thats to see the angle of the beam, or a sensor to find the position of the ball. If we know the position of the ball, we can create a feedback control law to allow us to put the ball wherever we want.

We will design a controller in a simulation to move the ball $0.2m$ along the beam and hold it there. The model will be slightly out so we can consider slight sensitivities as we can't model the motor nicely. There will be some imperfections.

5 Linear Input-State-Output Systems

We will not consider systems of the form, *Lecture 13*

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx + Du$$

We can solve the DE by using an integrating factor,

$$\frac{dx}{dt} = Ae^{-At} + Bue^{-At}$$
$$\frac{dx}{dt}e^{-At} - Ae^{-At} = Bue^{-At}$$
$$\frac{d}{dt}(xe^{-At}) = Bue^{-At}$$
$$x(t)e^{-At} - x(0) = \int_0^t Bue^{a\tau}\, d\tau$$

5.1 Input-State Stability

Definition 5.1 (Input-State Stability)**.** Consider the system,

$$\frac{dx}{dt} = Ax + Bu \qquad y = Cx + Du$$

We call this stable if,

(i) if $u = 0$ for all $t \geq 0$ then $x(t) \to 0$ as $t \to \infty$

(ii) If $\sup_{t\geq 0} \max |u_i(t)| < \infty$ then $sup_{t\geq 0} \max |x_i(t)| < \infty$

If a system is input-state stable, then it is input -output stable. We aim to show that a system is input-state stable if and only if all the eigenvalues of A is in OLHP.

Proof. We first prove that condition (1) and forward holds and then we seek to prove (2) and backwards holds.

We recall,

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\, d\tau$$

We then consider each entry in $(sI - A)^{-1}$ and the obtain a partial fraction decomposition. Then we take inverse laplace transforms and the result follows.

For the second part, we take our decomposition and then consider the inverse laplace transform. Then proceed by contradiction, assume we have a eigenvalue in the ORHP, then by our inverse laplace, if follows that all matrices corresponding to our ORHP eigenvalue is zero, then we find that the ORHP eigenvalue has a non-zero eigenvector which can be written using our $(sI - A)^{-1}$ equation and so implies it's zero, but this causes a contradiction.

Now we prove a bounded input leads to a bounded state, we use variation of constants, then we proceed very similarly to input-output systems. □

We now consider choosing u to shape some sort of desired behaviour. We now consider controllability,

5.2 Controllability

Definition 5.2 (Controllability)**.** We call a system controllable if for z_0, $z_1 \in \mathbb{R}^d$ there exists some $t_1 > 0$ and an input u which steeps the system from the initial state $x(0) = z_0$ to the final state $x(t_1) = z_1$

Theorem 5.3. We will show the following are equivilent,

(i) The system is controllable

(ii) For any $T > 0$, $\int_0^T e^{-A\tau} BB^T e^{-A^T\tau}\, d\tau$ is non-singular.

(iii) If $z \in \mathbb{R}^d$ satisfies $z^T[B \quad AB \quad \dots \quad A^{d-1}B]$ then $z = 0$

The third is easiest to use, the second is more complicated, however, it does give a neat expression for u from $x(0) = z_0$ to $x(t_1) = z_1$ while minimising the integral. Finally the third leads to a linear control law $u = Kx$ where $K \in M_{m\times n}(\mathbb{R})$.

We will show that if (ii) does not hold, then (iii) does not hold, if (iii) does not hold, then (i) does not hold and if (ii) holds then so does (i).

Proof of Theorem 5.3. Case (i), Firstly, suppose there exists a $T > 0$ and $0 \neq z \in \mathbb{R}^d$ such that $z^T \int_0^T e^{-A\tau} BB^T e^{-A^T\tau}\, d\tau = 0$. This implies that $z^T e^{-A\tau} BB^T e^{-A^T\tau} = 0$ for all $0 \leq \tau \leq T$. If we define $w(t) = z^T e^{-At}B = 0$, hence we see $w(0) = 0$ and all the derivatives are zero, so then it follows that $z^T B = 0$ and $z^T AB = 0$ and so on. It hence follows that $z^T[B \quad AB \quad \dots \quad A^{d-1}B] = 0$.

The case (ii) requires Cayley Hamilton Theorem. It follows from the theorem that $z^T A^j B = 0$ for all $j \in \mathbb{Z}$ and moreover and considering the powerseries of matrix exponetial $z^T e^{At} B = 0$ and so the result follows from variation of constants formula. Thus the system is not controllable.

The case (iii), follows from definiting the integral as $W(T)$ and if it's nonsingular, then we can define an input $u(t)$ using $W^{-1}(T)$. Then from variations of constant the result follows. Moreover we can show that the input minimises $u(t)^T u(t)$. □

5.3 State Space Isomorphism

Consider the system,

$$\frac{dx}{dt} = Ax + Bu \qquad y = Cx + Du$$

let $T \in R^{d\times d}$ be non-singular, and let

$$\hat{A} = TAT^{-1},\ \hat{B} = TB,\ \text{ and } \hat{C} = CT^{-1}$$

Then, defining $z = Tx$, we find that

$$\frac{dx}{dt} = \hat{A}z + \hat{B}u,\ y = \hat{C}z + Du$$

Thus the trajectories are in an one-to-one correspondence with the first.

Consider a controllable system, we have our V non-singular and there is a $w^T V = [0 \quad \dots \quad 1]$ and construct $w^T A^{k-1}$ and then,

$$TAT^{-1} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -a_0 & -a_1 & -a_2 & \dots & -a_{d-1} \end{bmatrix} \qquad TB = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

We are now going to prove the controller canonical form, *Lecture 16*

Proof. We note that by the definitions of w and V.

$$w^T V = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

and so $w^T A^k B = 0$ and $w^T A^{d-1} B = 1$. Now suppose $a_i \in \mathbb{R}$ satisfy $\begin{pmatrix} \alpha_0 & \alpha_1 & \dots & \alpha_{d-1} \end{pmatrix} T = 0$. To show that T is non-singular, it suffices to show that $a_i = 0$. Since the k^{th} row of T is $w^T A^{k-1}$, then $0 = w^T(\alpha_0 I + \alpha_1 A + \dots + \alpha_n A^{d-1})B = \alpha_{d-1}$, but $0 = w^T(\alpha_0 I + \alpha_1 A + \dots + \alpha_n A^{d-1})AB = \alpha_{d-2}$. Hence, by induction then $\alpha_0 = \alpha_1 = \dots = 0$

Now we note that,

$$TB = \begin{bmatrix} w^T B \\ w^T AB \\ \vdots \\ w^T A^{d-1} B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

and so TB has the requisite form. It remains to show that TAT^{-1} has the requistite form. We note that the (i,j) of TV and TAV are $wA^{i+j-2}B$ and $wA^{i+j-1}B$ respectively. Then by the Cayley-Hamilton Theorem,

$$TAV = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -a_0 & -a_1 & -a_2 & \dots & -a_{d-1} \end{bmatrix} TV$$

and the result should be clear. □

5.4 Pole Placement

The idea of that its possible to arbitrarily assign the poles of a system using a transformation.

if we let $\hat{K}$ and $u = -\hat{K}z$, then we will have that $\frac{dz}{dt} = (\hat{A} = \hat{B}\hat{K})z$ where,

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -a_0 - k_0 & -a_1 - k_1 & -a_2 - k_2 & \dots & -a_{d-1} - k_{d-1} \end{bmatrix}$$

It can be verified that the eigenvales can be placed arbitrarily by choice of $\hat{K}$. If we let $K = \hat{K}T$ and $u = -Kx$. Then, $\frac{dx}{dt} = (A - BK)x$ and the characteristic polynomial of $\hat{A} - \hat{B}\hat{K}$ is the same as $A - BK$ then we can arbitrarily place the eigenvales of $A - BK$ by choice of K, up to the laws of FTA.

We note that pole placement is also true for multi-input-output systems, ie. this is true for every controllable system. The proof for this is similar but with addition that there necessarily exists matrices $N \in \mathbb{R}^{n \times d}$ and $M \in \mathbb{R}^{n \times 1}$ such that $(A - BN, BM)$ is controllable.

We can now update our theorem,

Theorem 5.4. We will show the following are equivalent,

(i) The system is controllable

(ii) For any $T > 0$, $\int_0^T e^{-A\tau} BB^T e^{-A^T \tau} \, d\tau$ is non-singular.

(iii) If $z \in \mathbb{R}^d$ satisfies $z^T[B \quad AB \quad \dots \quad A^{d-1}B]$ then $z = 0$

(iv) The eigenvales of $A - BK$ can be arbitrarily assigned by a suitable choice of $K \in \mathbb{R}^{n \times d}$.

We can do all this tedious calculation in matlab with `charpoly`. We can use `place` to place roots wherever we want. *Lecture 17*

5.5 Stabilisability

We are now interested to pick the input to drive the system to the origin,

Definition 5.5 (Stabilisability)**.** If we consider $\frac{dx}{dt} = Ax + Bu$ this is stabilisible if, for any $z_0 \in \mathbb{R}^d$, there exists an input u such that $x(0) = z_0$ and $\lim_{t\to\infty} x(t) = 0$.

To determine whether a system is stabilisible, we will demonstatrate the existence of a non-singular matrix T which transforms the system into the form,

$$TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \text{ and } TB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

where $\frac{d\hat{x}}{dt} = A_{11}\hat{x} + B_1 u$ is controllable.

Proof. We consider the contrabililty matrix as before and $TV = \begin{bmatrix} \hat{V} \\ 0 \end{bmatrix}$ and this can be constructed via gaussian elimination. We let the number of rows of $\hat{V}$ is r. Now we can get that,

$$\begin{bmatrix} \hat{B} & \hat{A}\hat{B} & \dots \hat{A}^{d-1}\hat{B} \end{bmatrix} = V$$

and so the final $d - r$ rows of $\hat{B}$ is zero. By CHT then the final $d - r$ tows of $\hat{A}^d\hat{B}$ are zero. We can now show if,

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}$$

then $\hat{A}_{21} = 0$ by multiplying by V and the equating with V and considering the $d - r$ rows. □

Theorem 5.6. A system

$$\frac{dx}{dt} = Ax + Bu$$

is stabilisible if and only if the eigenvalues of A_{22} are in the OLHR.

Proof. Now we will partition $z = Tx$ compatibly with $TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ and $z \in \mathbb{R}^2$. We now note that $\frac{dz_2}{dt} = A_{22}z_2$ and then $z_2(t) = e^{A_{22}t}z_2(0)$, if A_{22} has an eigenvalue in the CRHP, then $z_2(t)$ doesnt tend to zero as $t \to \infty$, so the system is not stabilisible.

As $\frac{d\hat{x}}{dt} = A_{11}\hat{x} + B_1 u$ is controllable, $\exists\, \hat{K} \in \mathbb{R}^{r\times n}$ such that the eigenvalues of $A_{11} - B_1\hat{K}$ are all in the open left half plane. □

Lecture 18

5.6 Realisation Theory

In the section we aim to show that any system,

$$\frac{d^d y}{dt^d} + a_{d-1}\frac{d^{d-1}y}{dt^{d-1}} + \cdots + a_0 y = b_d\frac{d^d u}{dt^d} + \frac{d^{d-1}u}{dt^{d-1}} + \cdots + b_0 u$$

to a system of the form,

$$\frac{dx}{dt} = Ax + Bu \qquad y = Cx + Du$$

Firstly, we define a and b be the polynomials formed from the coefficients and degree of differentiation. We define $\hat{b}_i$ by the polynomial long division of $b(s)$ by $a(s)$:

$$b(s) - b_d a(s) = \hat{b}_{d-1}s^{d-1} + \hat{b}_{d-2}s^{d-2} + \cdots + \hat{b}_0$$

and so $\hat{b}_k = b_k - b_d a_k$. We now define,

$$x_d = y - b_d u$$
$$x_k = \frac{dx_{k+1}}{dt} + a_k x_d - \hat{b}_k u$$

Then we can prove that $\frac{dx_1}{dt} = -a_0 x_d + \hat{b}_0 u$. So then we can put into a matrix form and so,

$$\frac{dx}{dt} = Ax + Bu \quad y = Cx + Du$$

with,

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & 0 & \dots & 0 & -a_2 \\ 0 & 0 & 1 & \dots & 0 & -a_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -a_{d-1} \end{bmatrix} \qquad B = \begin{bmatrix} \hat{b}_0 \\ \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \\ \vdots \\ \hat{b}_{d-1} \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \quad D = b_d$$

5.7 Observability

We now seek to infer the initial state from measurement of the input and output into the future,

Definition 5.7 (Observable)**.** The following system

$$\frac{dx}{dt} = Ax + Bu \qquad y = Cx + Du$$

is observable if the initial state $x(0)$ can be deduced from knowledge of $u(t)$ and $y(t)$ for $t \geq 0$.

and we have the following theorem,

Theorem 5.8. TFAE,

(i) The system $\frac{dx}{dt} = Ax + Bu$, $y = Cx + Du$

(ii) For any given $T > 0$, $\int e^{A^T t} C^T C e^{At} \, dt$ is non-singular.

(iii) If $z \in \mathbb{R}^d$ satisfies $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{d-1} \end{bmatrix} z = 0$ then $z = 0$.

The proof of this is similar to the contrabililty theorem. Show that if 2 holds, so does 1, if 3 does not hold then one does not hold and if 2 does not hold then 3 does not hold.

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Proof. Firstly we are going to show if 2 holds then so does 1. To show this we use the variation of constants formula and say $W(T) = \int_0^T e^{A^T} C^T C e^{At} \, dt$ and then if $W(T)$ is non-singular, then,

$$x(0) = W(T)^{-1} \int_0^T e^{A^T t} C^T (y(t - Du(t) - \int_0^t C e^{A(t-\tau)} Bu(\tau)) \, d\tau) \, dt$$

so the system is observable.

Secondly we look to case (ii), if $z \in \mathbb{R}^d$ satisfies the condition, then $Cz = 0$ and $CA = 0$ and so $CA^k z = 0$ by the CH Theorem, $Ce^{At}\lambda z = 0$ for all $t \in \mathbb{R}$ and for any λ. Thus, if $u(t) = 0$ for all $t \geq 0$ then $y(t) = Ce^{At}x(0)$, so $y(t) = 0$ whenever $x(0) = \lambda z$ for some $\lambda \in \mathbb{R}$ and it follows that $x(0)$ cannot be uniquely determined by $u(t)$ and $y(t)$ in $t \geq 0$.

Finally we consider our case (iii), if $\exists T > 0$ and $z \neq 0$ such that $\int_0^T a^{A^T t} C^T C e^{At}\, dt z = 0$ then $z^T \int_0^T a^{A^T t} C^T C e^{At} z\, dt = 0$ and so again $h(t) = Ce^{At}z = 0$. But this implies all the derivatives of h are zero. Hence,

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{d-1} \end{bmatrix} z = 0$$

□

Remark. It can also be shown that, if the system is observable, then the eigenvalues of $A - LC$ can be placed arbitrarily by choice of $L \in \mathbb{R}^{d \times m}$.

5.8 Observer

An observer is a dynamical system that estimates the states of another dynamical system. These are very useful for control. They are basically a replica, but with additional inputs to correct for the difference.

Specifically, for the classical system, the observer takes the form

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu - L(y - \hat{y}), \quad \hat{y} = C\hat{x} - Du$$

We define the observer error $\hat{e} = \hat{x} - x$ and rearrange to obtain.

$$\frac{d}{dt}\begin{bmatrix} \hat{x} \\ \hat{e} \end{bmatrix} = \begin{bmatrix} A & -LC \\ 0 & A - LC \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{e} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u \quad y = \begin{bmatrix} C & -C \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{e} \end{bmatrix} + Du$$

We notice that $\frac{d\hat{e}}{dt} = (A - LC)\hat{e}$ and this can be solve nicely. If this system is observable, then we can place all the eigenvales where we want and so we can place them in the OLHP and so we have $\hat{e} \to 0$ as $t \to \infty$.

We can apply state feedback using the state of the observer as opposed to the real system. Specifivally we let $u = -K\hat{x}$ where $K \in \mathbb{R}^{n \times d}$ is to be assigned. This results in,

$$\frac{d}{dt}\begin{bmatrix} \hat{x} \\ \hat{e} \end{bmatrix} = \begin{bmatrix} A - BK & -LC \\ 0 & A - LC \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{e} \end{bmatrix}, \quad y = \begin{bmatrix} C & -C \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{e} \end{bmatrix} - DK\hat{x}$$

Finally, we can now say that the poles of the system are the eigenvalues of

$$\begin{bmatrix} A - BK & -LC \\ 0 & A - LC \end{bmatrix}$$

and it can be show that the eignvalues of that matrix can be seen to be $A - BK$ and $A - LC$. We can now describe a controller by choosing the gain matrix L and a state-feedback control law $u = -K\hat{x}$.

More matlab bits and bobs *Lecture 19*

5.9 Detectability

Detectability is just concerned whether the state is blowing up by considering the output. Consider our system,

$$\frac{dx}{dt} = Ax + Bu, \quad y = Cx + Du$$

there exists a non-singular $T \in \mathbb{R}^{d\times d}$ such that,

$$TAT^{-1} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \text{ and } CT^{-1} = \begin{bmatrix} C_1 & 0 \end{bmatrix}$$

Definition 5.9 (Detectability)**.** A system is detectable if $u(t) = 0$ and $y(t) = 0$ for $t \geq 0$ imply that $x(t) \to 0$ as $t \to \infty$.

And a test,

Theorem 5.10. We will show that the system is detectable if and only if the eigenvalues fo A_{22} are in the OLHP

Proof. Consider a non-singular T such that, with $Tx = z = \begin{bmatrix} z_1^T & z_2^T \end{bmatrix}$ then $\frac{dz}{dt} = TAT^{-1}z + TBu$ and by expanding this and using the variation of constants formula we can say that,

$$\begin{aligned} z_1(t) &= e^{A_{11}t}z_1(0) + \int_0^t e^{A_{11}(t-\tau)}B_1u(\tau)\,d\tau \\ z_2(t) &= e^{A_{22}t}z_2(0) + \int_0^t e^{A_{21}z_1(\tau)+B_2u(\tau)}\,d\tau \\ y(t) &= C_1e^{A_{11}t}z_1(0) + Du(t) + \int_0^t C_1e^{A_{11}(t-\tau)}B_1u(\tau)\,d\tau \end{aligned}$$

Thus, $u = y = 0$ for $t \geq 0$ if and only if $h(t) = C_1e^{A_{11}t}z_1(0) = 0$ for all $t \geq 0$. In particular, all the derivatives are zero. Since the system is observable, this holds if and only if $z_1(0) = 0$.
Thus if $u = y = 0$, then $z_1(0) = 0$ so,

$$\begin{aligned} z_1(t) &= 0 \\ z_2(t) &= e^{A_{22}t}z_2(0) \\ y(t) &= 0 \end{aligned}$$

But $Tx = \begin{bmatrix} z_1^T & z_2^T \end{bmatrix}$ and so it follows that $x(t) \to 0$ as $t \to \infty$ if and only if $z_2(t) \to 0$ as $t \to \infty$. This holds for all initial states if and only if all the eigenvalues of A_{22} are in the OLHP. □

5.10 Tracking

Sometimes we are not only interested in the stability of the system, but we also want to manipulate the systems state. We consider our system $\frac{dx}{dt} = Ax + Bu$ where our $u = -K(x - x_{\text{ref}})$ where x_{ref} is our reference for the systems state and K is the stabilising gain. We can apply Laplace transforms and get,

$$x(s) = (sI - A + Bk)^{-1}BKx_{\text{ref}}$$

Now, if $x_{\text{ref}}(s) = \frac{X}{s}$ we can use the final value theorem,

$$\lim_{t\to\infty} e(s) = -(A + BK)^{-1}AX$$

If the states are not all available for control, and it is instead necessary to use the output $y = Cx + Du$, then we can instead use the input $u = -K(\hat{x} - x_{\text{ref}})$ where $\hat{x}$ is the state estimate obtained from an observer.

5.11 LQR and Kalman Filter

We have considered design of (state feedback) controllers, observers and observer-based controllers using pole placement. This allows for the placement of the closed-loop eigenvalues of the system.

An alternative approach to design is to optimise some cost functional. We have seen an example already of this when we constructed an input to drive the system $\frac{dx}{dt} = Ax + Bu$ between an initial state $x(0) = z_0$ and a final state $x(t_1) = z_1$.

Definition 5.11 (LQR)**.** The LQR (linear quadratic regulator) provides an optimal control approach to designing a state feedback controller.

Definition 5.12 (Kalman Filter)**.** The Kalman filter (linear quadratic estimator) provides an analogous approach to designing optimal observers.

5.11.1 LQR Controllers

The LQR controller is a stabilising state feedback control law that minimises,

$$\int_0^\infty (x^T QX + u^T Ru(t))\, dt$$

where Q and R matrices are typically defined such that $u^T Ru > 0$ for all $0 \neq u \in \mathbb{R}^n$ and $x^T Qx \geq 0$ for all $x \in \mathbb{R}^d$. This is derived from the solution on,

$$A^T X + XA + Q - XBR^{-1}B^T X = 0$$

This solution is non-unique but there is one with the property of the eigenvales of $A - BR^{-1}B^T X$ being all in the OLHP. The cost functional is then reduced by $u = -R^{-1}B^T Xx$

5.11.2 Kalman Filters

The Kalman filter provides an optimal estimator for,

$$\frac{dx}{dt} = Ax + Bu + w \qquad y = Cx + Du + v$$

where w and v are stochatic variables, with power spectral densities W and V respectively. Typically $v^T V v > 0 \,\forall\, 0 \neq v \in \mathbb{R}^m$ and $w^T W w \geq 0 \,\forall\, w \in \mathbb{R}^d$. The optimal estimator is,

$$\frac{dx}{dt} = A\hat{x} + Bu - L(\hat{y} - y) \qquad \hat{y} = C\hat{x} + Du$$

and minimises the power spectral density of the observer error, $\lim_{T\to\infty} \frac{1}{T}\int_0^T (x - \hat{x})^T (x - \hat{x})\, dt$ and this is obtained by solving,

$$YA^T + AY + W - YC^T V^{-1} CY = 0$$

where $A - YC^T V^{-1} C$ has eigenvalues in the OLHP. The optimal gain is $L = YC^T V^{-1}$

6 Non-linear Systems

Thus far, we have considered linear systems, now we consider non-linear systems.

6.1 Linearisation

We consider the system,

$$\frac{dx}{dt} = f(x(t))$$

we assume that f is continuously differentiable in some $D \subset \mathbb{R}^d$. We say that D contains the origin and let the origin be an equilibrium point of this function, ie, $f(0, 0) = 0$. Then for some $x \in D$ then there exists some $0 \leq \lambda_i \leq 1$,

$$f_i(x) = f_i(0) + \sum_{j=1}^{d} \frac{\partial f_i}{\partial x_j}(\lambda_i x) x_j$$

and so it follows that $f(x) = Ax + h(x)$ such that,

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(0) & \dots & \frac{\partial f_1}{\partial x_d}(0) \\ \vdots & & \vdots \\ \frac{\partial f_d}{\partial x_1}(0) & \dots & \frac{\partial f_d}{\partial x_d}(0) \end{bmatrix} \quad \text{and} \quad h(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\lambda_1 x) - \frac{\partial f_1}{\partial x_1}(0) & \dots & \frac{\partial f_1}{\partial x_d}(\lambda_1 x) - \frac{\partial f_1}{\partial x_d}(0) \\ \vdots & & \vdots \\ \frac{\partial f_d}{\partial x_1}(\lambda_d x) - \frac{\partial f_d}{\partial x_1}(0) & \dots & \frac{\partial f_d}{\partial x_d}(\lambda_d x) - \frac{\partial f_d}{\partial x_d}(0) \end{bmatrix}$$

since f is continuously differentiable, then $\frac{\|h(x)\|}{\|x\|} \to 0$ as $\|x\| \to 0$.

Now we consider a system $\frac{d\hat{x}}{dt} = \hat{f}(\hat{x}(t))$, again $\hat{f}$ is continuously differentiable in some $\hat{D} \in \mathbb{R}^d$ which contains $\hat{x} = \widetilde{x}$. Now let $x = \hat{x} - \widetilde{x}$ and so $f(x) = f(\hat{x} + \widetilde{x})$. Then $\frac{d\hat{x}}{dt} = \hat{f}(\hat{x}(t))$ if and only if $\frac{dx}{dt} = f(x(t))$. Now without loss of generality we consider the equilibrium to be the origin. This will make the notation and proof less verbose[2].

Now consider our system and let $(\widetilde{x}, \widetilde{u})$ be an equilibrium and let f and g[3] be continuously differentiable in some region of $(\widetilde{x}, \widetilde{u})$ then,

$$f(x) = A(x - \widetilde{x}) + B(u - \widetilde{u}) + h(x, u)$$

where

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_d} \\ \vdots & & \vdots \\ \frac{\partial f_d}{\partial x_1} & \dots & \frac{\partial f_d}{\partial x_d} \end{bmatrix} (\widetilde{x}, \widetilde{u})$$

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \dots & \frac{\partial f_1}{\partial u_n} \\ \vdots & & \vdots \\ \frac{\partial f_d}{\partial u_1} & \dots & \frac{\partial u_d}{\partial x_d} \end{bmatrix} (\widetilde{x}, \widetilde{u})$$

and we note that $\frac{\|h(x,u)\|}{\sqrt{\|x-\widetilde{x}\|^2 + \|u-\widetilde{u}\|^2}} \to 0$ as $\|x - \widetilde{x}\| \to 0$ and $\|u - \widetilde{u}\| \to 0$.

6.2 Stability

We consider a standard form system where we have an equilibrium of $(\widetilde{x}, \widetilde{u})$ and it can be decomposed into $f(x) = A(x - \widetilde{x}) + h(x)$. We consider f being locally Lipschitz on $D \subset \mathbb{R}^d$ and let $x = \widetilde{x} \in \mathbb{R}^d$ be an equilibrium point. Let $B(\widetilde{x}, \alpha)$ denote a ball of radius α centred on $\widetilde{x}$.

[2] if this disturbs you, consider a change of coordinates

[3] g is something we use if $y = g(x, u)$

Definition 6.1 (Stability)**.** We call the equilibrium point $\widetilde{x}$ stable if there exists some $R_0 > 0$ and $t_0 \in \mathbb{R}$ then there is an r satisfying $0 < r < R$ such that $x(t_0) \in B(\widetilde{x}, r)$ then $x(t) \in B(\widetilde{x}, R)$ for all $t > t_0$

We now consider a different type of stability that is slightly stronger,

Definition 6.2 (Asymptotically Stable)**.** We call the equilibrium asymptotically stable if it is stable if and there is an $R_1 > 0$ such that if $x_0 \in B(\widetilde{x}, R_1)$ then $\lim_{t\to\infty}(x(t)) \to \widetilde{x}$

6.3 Lyapunov's Indirect Method

We consider our usual form and the decomposition more particularly, the A from the linearisation. Then we can state the following,

Theorem 6.3 (Lyapunov's Indirect Method)**.** There are two different bases,

(i) If all of the eigenvalues of A are in the open left half plane, then the equilibrium point $x = \widetilde{x}$ is stable.

(ii) If an eigenvalue of A is in the open right half plane, then the equilibrium point $x = \widetilde{x}$ is unstable

We note that if there is an imaginary axis eigenvalue, then we have that the test is inconclusive.[4]

6.4 Existence and Uniqueness of solutions

We consider our general system and we define what f to be locally Lipschitz,

Definition 6.4 (Locally Lipschitz)**.** A function f is locally Lipschitz in a domain $D \subset \mathbb{R}^d$ if, for any $x_0 \in D$, then there exists some $\varepsilon > 0$ and $L \geq 0$ such that $\|f(x) - f(y)\| \leq L\,\|x - y\|$ for all $x, y \in B(x_0, \varepsilon)$

and moreover we define globally Lipschitz,

Definition 6.5 (Globally Lipschitz)**.** We say f is globally Lipschitz if there exists $L \geq 0$ such that $\|f(x) - f(y)\| \leq L\,\|x - y\|$ for all $x, y \in \mathbb{R}^d$.

If f is continuously differentiable in D, then it is locally Lipschitz,

Theorem 6.6 (Local existence and uniqueness)**.** If f is locally Lipschitz for some $x_0 \in \mathbb{R}^d$ and $t_0 \in \mathbb{R}^d$ then $\frac{dx}{dt}(t) = f(x(t))$ with $x(t_0) = x_0$ has a unique solution over $t \geq t_0$.

and we can consider global existence and uniqueness,

Definition 6.7 (Global Existence and Uniqueness)**.** If f is globally Lipschitz, then $\frac{dx}{dt} = f(x(t))$ with $x(t_0) = x_0$ has a unique solution over $t \geq t_0$.

This is a very demanding condition on our system, so we can weaken this to another condition,

Definition 6.8 (Global Existence and Uniqueness (2))**.** Let f be locally Lipschitz in D and $t_0 \in \mathbb{R}$, let W be a closed and bounded subset of D, let $x_0 \in W$, and let every solution of $\frac{dx}{dt} = f(x(t))$ with $x(t_0) = x_0$ lie in W. Then $\frac{dx}{dt} = f(x)$ with $x(t_0) = x_0$ has a unique solution over the interval $t \geq t_0$.

[4] The centre manifold theorem allows us to remove this remark

7 Lyapunovs Direct Method

Theorem 7.1. The equilibrium is stable if there exists a function $V : D \mapsto \mathbb{R}$ such that,

(i) V is continuously differentiable

(ii) $V(\widetilde{x}) = 0$ and $V(x) > 0$ for all other $x \in D$

(iii) $\dot{V}(x) = \sum_{i=1}^{d} \frac{\partial V}{\partial x} \frac{dx_i}{dt} = \sum_{i=1}^{d} \frac{\partial V}{\partial x_i}$ satisfies $\dot{V}(x) \leq 0$ for all $x \in D$

and asymptotically stable,

Theorem 7.2. All above holds and,

(i) $\dot{V}(x) < 0$ for all $\widetilde{x} = x \in D$

Lyapunov's. Nope, not LaTeXing that. We consider the equilibrium to be at the origin and so then prove the conditions imply for all $\varepsilon > 0$ there exists $\delta > 0$ then $\|x(0)\| < \delta$ implies that $\|x(t)\| < \varepsilon$ for all $t \geq 0$. □

Asymptotic Lyapunov's. We show now that $V(x(t)) \to 0$ as $t \to \infty$. □