# Year 3 — Lie Groups and Applications in Geometric Mechanics

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine (especially the typos!).

## Contents

# 1 Hamilton's Variational Principle

This document started nearly a year ago in a zoom call with two words, 'plastic bag'. The challenge set was to get a mathematical model to describe the motions of pseudo rigid bodies, of which a plastic bag is an example of. The original plan, was for the derivations to be below as a first and motivating example. However, I underestimated the journey that the mathematics would take me on and this document now acts as the path that I took on this journey. I realised that energy is really important in mathematics and sometimes a problem cannot be solved by considering the particle position alone and so we need to use energy arguments. A simple example of an energy argument can be seen in the following example,

Consider a particle that is half way along a piece of massless string which is pulled past its natural extension. The question is, what is the velocity of the particle when it reaches a displacement of zero? Anybody trained in mathematics would start by trying to describe the position of the particle in this system and that is the wrong way to solve this problem.

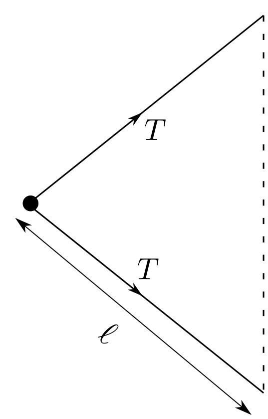



Figure 1: Motivating Problem.

To solve this problem, the best (and probably only) way is to describe the particle's energy. That is, we know that energy is conserved and so if at some point we know the total energy we can then describe the energy in the system at any point. At $t = 0$ there is zero kinetic energy, as the particle is held at rest. If we assume that the only energies to consider are potential and kinetic, the whole energy of the system can be described by just the elastic potential energy. Then we know that the kinetic energy of the particle at displacement zero is, $\frac{\lambda x^2}{2l_0}$ where $l_0$ is the natural extension of the spring, $x$ is the elastic extension and $\lambda$ is the spring constant. Hence, in our case, the energy in the string is $\frac{\lambda(\ell - l_0)^2}{2l_0}$ and so the velocity of the particle at displacement zero is going to be, $v = (\ell - l_0)\sqrt{\frac{\lambda}{ml_0}}$. In general, we can make these energy arguments about sets of particles in Euclidean space. To start to understand this, we must first go on a little detour to explain the role of the Lagrangian through the Euler-Lagrange equations and variations.

The rest of this chapter is in very close relation to first chapter of Holm, Schmah & Stoica (2009), Geometric Mechanics and Symmetry [2]. Consider a point mass, the position of this point is also called the **configuration** is a vector $\mathbf{q} \in \mathbb{R}^d$. If we consider $N$ points then the configuration becomes $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N) \in \mathbb{R}^{dN}$. This notation works with familiar mechanics concepts, that is we can write Newtons Second Law, $F_i = m_i \ddot{\mathbf{q}}_i$ for $i = 1, 2, \ldots, N$. We now define a special type of Newtonian system that we can use to illustrate a few important concepts.

**Definition 1.1** (Newtonian Potential System)**.** A **Newtonian potential system** is a system of equations,

$$m_i\ddot{\mathbf{q}}_i = -\frac{\partial V_i}{\partial \mathbf{q}_i}$$

for $i = 1, 2, \ldots, N$ where $V(\{\mathbf{q}_i\})$ is a real-valued function, called the potential energy.

This system has conserved energy and this is one of the most interesting thing for mathematicians. We define it to be kinetic energy plus potential energy, $E := K + V$.

**Theorem 1.2** (Conservation of Energy)**.** Energy is conserved in Newtonian Potential Systems.

*Proof.*

$$\begin{aligned}
\frac{dE}{dt} &= \frac{d}{dt}\left(\frac{1}{2}\sum_{i=1}^{N} m_i \left\|\ddot{\mathbf{q}}_i\right\|^2 + V(\mathbf{q})\right) \\
&= \sum_{i=1}^{N} m_i\dot{\mathbf{q}}_i \cdot \ddot{\mathbf{q}}_i + \sum_{i=1}^{N} \frac{\partial V}{\partial \mathbf{q}_i} \cdot \dot{\mathbf{q}}_i \\
&= \sum_{i=1}^{N} \dot{\mathbf{q}}_i \cdot \left(m_i\ddot{\mathbf{q}}_i + \frac{\partial V}{\partial \mathbf{q}_i}\right) = 0. && \text{these vectors are orthogonal.}
\end{aligned}$$

□

This is one of the first important ideas that we will use again and again throughout this dissertation. These are the main motivation behind Noethers Theorems; they tell us about different conserved quantities in the system we are studying. These different types of conserved quantities will change depending on different types of invariance. We can have different types of invariance in the systems we are studying, later we will mostly consider invariance in Lagrangians.

**Definition 1.3** (Rotational Invariance)**.** A function $V : \mathbb{R}^{dN} \to \mathbb{R}$ is rotationally invariant if

$$V(R\mathbf{q}_1, R\mathbf{q}_2, \ldots, R\mathbf{q}_N) = V(\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$$

for any $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$ and a rotation matrix $R \in M_{d\times d}(\mathbb{R})$.

As an interesting and motivating example, see Proposition 1.28 of [2], which says in any Newtonian potentially system with rotationally invariant $V$, angular momentum is conserved. This invariance can be used to reduce certain functions from $N$ variables to $N-1$ variables, this will be one the main focus' of study.

## 1.1 Lagrangian Mechanics

We firstly introduce the following theorem relating to a new set of equations called the Euler-Lagrange Equations,

**Theorem 1.4** (Euler-Lagrange Equations for Newtonian Potential System)**.** Every Newtonian potential system is equivalent to the Euler Lagrange Equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}_i} - \frac{\partial L}{\partial \mathbf{q}_i} = 0 \tag{1.1}$$

for the Lagrangian $L : \mathbb{R}^{2dN} \to \mathbb{R}$ defined by,

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \sum_{i=1}^{N} \frac{1}{2} m_i \left\|\dot{\mathbf{q}}_\mathbf{i}\right\|^2 - V(\mathbf{q}).$$

*Proof.*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}_i}\right) - \frac{\partial L}{\partial \mathbf{q}_i} = \frac{d}{dt}\left(m_i \dot{\mathbf{q}}_i\right) + \frac{\partial V}{\partial \mathbf{q}_i} = m_i \ddot{\mathbf{q}}_i + \frac{\partial V}{\partial \mathbf{q}_i} = 0.$$

□

Henceforth, we will work in Lagrangian systems, that is defined as,

**Definition 1.5** (Lagrangian System)**.** A **Lagrangian system** on a configuration space $\mathbb{R}^{dN}$ is the system of ODEs called the Euler-Lagrange equations (Equation 1.1), for some function $L : \mathbb{R}^{2dN} \to \mathbb{R}$ called the Lagrangian.

As we did before, we can talk about energy in terms of the Lagrangian, $E := \frac{\partial L}{\partial \dot{\mathbf{q}}_i} \cdot \dot{\mathbf{q}}_i - L$ and the energy is conserved. We now go the final ideas that I would like to introduce in the introduction, the variational derivative and Hamilton's Variational Principle. We will use these to derive the Euler-Lagrange equations again, in a more general case.

The Euler-Lagrange equations relate to a variational principle on the space of smooth paths with fixed end points. The main idea of this variational principle is that we can determine solutions of the Euler-Lagrange equations as stationary points of some action functional. In an example, consider a chain that is fixed at two ends. This will form a catenary, but a chain can form many more shapes or paths, but it turns out the minimal of the functional, which is a stationary point, is the catenary that the chain forms.

Consider some smooth path, $\mathbf{q} : [a, b] \to \mathbb{R}^{dN}$ with endpoints $\mathbf{q}(a) = \mathbf{q}_a$ and $\mathbf{q}(b) = \mathbf{q}_b$. We define a **deformation** of $\mathbf{q}$ as a smooth map $\mathbf{q}(s, t)$ where $s \in (-\varepsilon, \varepsilon)$ where $\varepsilon > 0$ such that $\mathbf{q}(0, t) = \mathbf{q}(t)$ for all $t \in [a, b]$.

**Definition 1.6** (Variation)**.** The **variation** of the curve $\mathbf{q}(t)$ corresponding to the following deformation $\mathbf{q}(s, t)$ is,

$$\delta\mathbf{q}(t) = \left.\frac{d}{ds}\right|_{s=0} \mathbf{q}(s, t).$$

Then the first variation is,

**Definition 1.7** (First Variation)**.** The **first variation** of a smooth $C^\infty$ functional $\mathcal{S} : [a, b] \to \mathbb{R}^{dN}$ is

$$\delta\mathcal{S} := \left.\frac{d}{ds}\right|_{s=0} \mathcal{S}[\mathbf{q}(s, t)].$$

Then we call $\mathbf{q}$ a **stationary point** of $\mathcal{S}$ if $\delta\mathcal{S} = 0$ for all deformations of $\mathbf{q}$. Furthermore, if $\mathbf{q}(s, t)$ has fixed endpoints, meaning that $\mathbf{q}(a, s) = \mathbf{q}_a$ and $\mathbf{q}(b, s) = \mathbf{q}_b$ for all $s = (-\varepsilon, \varepsilon)$ then $\delta\mathbf{q}(a) = \delta\mathbf{q}(b) = 0$. These are variations along paths with fixed endpoints. Finally, we prove that the Euler-Lagrange equations are equivalent to Hamilton's Principle,

**Theorem 1.8.** For any $L : \mathbb{R}^{2dN} \to \mathbb{R}$, the Euler-Lagrange equations (Equations 1.1) are equivalent to Hamilton's principle of stationary action $\delta\mathcal{S} = 0$ where $\mathcal{S}$ is defined as,

$$\mathcal{S}[\mathbf{q}(t)] = \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}})\, dt$$

with respect to variations along paths and fixed endpoints.

*Proof.* We will proceed using the fact that $\frac{d}{dt}\delta\mathbf{q} = \delta\dot{\mathbf{q}}$ and using integration by parts.

$$\begin{aligned}
\delta\mathcal{S} = \frac{d}{ds}\bigg|_{s=0} S[\mathbf{q}(s,t)] &= \frac{d}{ds}\bigg|_{s=0} \int_{t_1}^{t_2} L(\mathbf{q},\dot{\mathbf{q}})\,dt \\
&= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \mathbf{q}}\cdot\mathbf{q} + \frac{\partial L}{\partial \dot{\mathbf{q}}}\cdot\delta\dot{\mathbf{q}}\right) dt \\
&= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \mathbf{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)\cdot\delta\mathbf{q}\,dt + \left[\frac{\partial L}{\partial \dot{\mathbf{q}}}\cdot\delta\mathbf{q}\right]_{t_1}^{t_2} \\
&= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \mathbf{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)\cdot\delta\mathbf{q}\,dt. \qquad \text{applying end point conditions}
\end{aligned}$$

This then tells us that for any smooth $\delta\mathbf{q}(t)$ satisfying $\delta\mathbf{q}(a) = \delta\mathbf{q}(b) = 0$. If $\delta\mathcal{S} = 0$, then $\frac{\partial L}{\partial \mathbf{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} = 0$ and so Hamilton's Principle is equivalent to the Euler-Lagrange equations. □

We now have the foundation to use Lagrangian Mechanics throughout the rest of the document. We will again and again see the use of Hamilton's Variational Principle and different types of Euler-Legrange equations. The rest of this dissertation shall be used to explore this area further. We will focus on Euler-Poincaré Reduction, Noether Theory and several applications of this mathematics. The main focus will be on the pure background mathematics of Geometric Mechanics leading up to the last section of Pseudo-Rigid bodies. This is because the main aim of this project was to learn enough Geometric Mechanics to do a PhD in the area. Therefore heavy emphasis is put on Theorems and proofs rather than examples. Under this we will focus on three main activities; symmetry and reduction of Lagrangians, derivation of equations and finding conserved quantities of these systems.

# 2 Lie Groups, Algebras and their associated actions

Thus far, we have only considered mechanics, in this section we start to introduce what we mean by Geometric Mechanics, our toolbox for making arguments about Lagrangians. We will look briefly at the definitions of manifolds and Lie groups and then consider the connection between them. To start, let us define some groups. Firstly, what is a group?

**Definition 2.1** (Group)**.** $G$ is a nonempty set and endowed with a binary operation such that,

(i) It's closed under $(\cdot)$, $\forall\, a, b \in G$, $a \cdot b \in G$

(ii) It's associative, i.e. $\forall\, a, b \in G$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

(iii) There is an identity element, $\forall\, a \in G$, $a \cdot e = a = e \cdot a$.

(iv) Every element has an inverse, $\forall\, a \in G$, $a \cdot a^{-1} = e = a^{-1} \cdot a$.

Groups are a very useful and interesting structure. There is a rich area of research and study surrounding them. One of the things that will be the most useful to us is actions of groups. We can take groups and consider them acting on sets, with an identity and compatibility axiom. We will see this in the next chapter, where we consider the adjoint actions. However, before we get to that we need to define a Lie group. I feel that in order to define a Lie group it would help to define a manifold or get some idea of what structures we will be working with.

## 2.1 Manifolds

In the simplest definition, a manifold is a space where we can do geometry. We are studying Analytic and Differential Geometry so we need to consider differentiable, or smooth manifolds. These are manifolds that we can take paths along and we can consider the velocity (or tangent vectors) of these paths. We are particularly interested in one specific type of manifold, SO(3) and the associated tangent space. The tangent space is just a vector space where we can use all the rules of Linear Algebra to understand it.

For our purposes, we will use the fact that a Lie Group is just a manifold. For background, we will spend the rest of this section defining what a manifold is formally. A manifold is a topological space that locally resembles Euclidean space near each point [7]. However, this isn't entirely satisfactory as what does near mean? It's very informal. Let's define it properly!

**Definition 2.2** (Manifold)**.** A manifold is a second countable Hausdorff space that is locally homeomorphic to Euclidean space.

In the above definition, we refer to a second countable space as being a topological equivalent to having a countable basis. That is, there exists some countable base of this space $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$, where any open subset of our space, $T$, can be written as a disjoint union of a finite subfamily of $\mathcal{U}$. This nicely restricts manifolds to be smaller spaces, by making them be the union of countably many open sets. We define a Hausdorff space as,

**Definition 2.3** (Hausdorff)**.** A topological space is Hausdorff if, given any points $x, y \in X$ with $x \neq y$, there exists open sets $U, V \in X$ with $x \in U$, $y \in V$ and $U \cap V = \varnothing$.

Finally, to say something is homeomorphic to another space, means it can be stretched to the other space without creating holes or gluing. A homeomorphism is a bijective map between two spaces that has a bijective inverse. A local homeomorphism is a homeomorphism in a neighbourhood of a point. Hence, saying that something is locally homeomorphic to Euclidean space directly means, that you can bijectively map the contents of the neighbourhood around a point to an open ball in $\mathbb{R}^n$, i.e the Euclidean $n$-ball.[1]

[1] This is long and not very succinct way to define this structure, a slightly nicer definition of manifolds is: A manifold is just a locally ringed space, whose sheaf structure is just locally isomorphic to continuous functions on Euclidean space.

## 2.2 Lie Groups

Now we know enough to define what a Lie group actually is,

**Definition 2.4** (Lie Group)**.** A Lie group is a group that is also a smooth manifold, such that the binary product and inversion are smooth functions.

What we will be focusing our attention on is special Lie groups; the general linear group, special linear group and the special orthogonal group.

**Definition 2.5** (General Linear Group)**.** $\mathrm{GL}(n, \mathbb{R})$ is the linear matrix group. The manifold of $n \times n$ invertible square real matrices is a Lie group denoted by $\mathrm{GL}(n, \mathbb{R})$.

**Definition 2.6** (Special Linear Group)**.** The $\mathrm{SL}(n, \mathbb{R})$ is the manifold of $n \times n$ matrices with unit determinant.

**Definition 2.7** (Special Orthogonal Group)**.** $\mathrm{SO}(n, \mathbb{R})$ is the manifold of rotation matrices in $n$ dimensions. This may be denoted by $\mathrm{SO}(n)$

## 2.3 Lie Algebras

To actually understand what Lie Algebras are, we need to generalise the notion of a vector and a tangent. We shall look at so called tangent spaces. To formally define them, we will define manifolds a second way, one that leads to the definition of smooth (or $C^\infty$ manifolds). We shall first define charts and atlases. These definitions are adapted from [6].

**Definition 2.8** (Chart)**.** Let $X$ be a topological space. An $\mathbb{R}^n$ chart on $X$ is a homeomorphism $\phi : U \to U'$ where $U \subset X$ and $U' \subset \mathbb{R}^n$.

**Definition 2.9** (Atlas)**.** A $C^\infty$ atlas on a topological space $X$ is a collection of charts $\phi_\alpha : U_\alpha \to U'_\alpha$ where all the $U'$'s are open subsets of one fixed $\mathbb{R}^n$ such that,

1. Each $U_\alpha \in X$ is open and $\bigcup_\alpha U_\alpha = X$ ($U_\alpha$ is an open subcover of $X$) and,
2. Changes of coordinates are smooth[2].

Two last definitions in this section are equivalence relation and equivalence class.

**Definition 2.10** (Equivalence Relation)**.** An equivalence relation on a set $X$ is a binary relation $\sim$ satisfying,

1. $\forall\, a \in X, a \sim a$
2. $\forall\, a, b \in X, a \sim b \implies b \sim a$
3. $\forall\, a, b, c \in X, a \sim b$ and $b \sim a \implies a \sim c$.

Then the equivalence class is just all of the equivalent elements to a member of that set. For example, for the equivalence relation that $x \sim y$ if and only if $x - y$ is even, then the equivalence classes are all the even numbers and all the odd numbers.

**Remark.** It can be proven that equivalence classes are just a partition of a set. [4]

Here is another definition of a manifold, this definition explicitly output a smooth manifold,

[2]This is slightly more convoluted than what we hint to here. In the greatest formality, we should write that transition maps are smooth. Transitions maps are maps from two charts $(U_\alpha, \phi_\alpha)$ and $(U_\beta, \phi_\beta)$ where $U_\alpha \cap U_\beta$ is nonempty and

$$\tau_{\alpha,\beta} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

defined by $\tau_{\alpha,\beta} = \phi_\beta \circ \phi_\alpha^{-1}$ (this is a homeomorphism).

**Definition 2.11** ($C^\infty$ Manifold)**.** An $n$-dimensional ($C^\infty$) manifold a topological space $M$ together with an equivalence class of $C^\infty$ atlases.

**Remark.** Our equivalence relation here is that two atlases are equivalent if their union is also an atlas.

Here are a few examples of manifolds,

- Let $M = \mathbb{R}^n$, this is a manifold covered by one open set and then if we take the identity map as our chart, we get the standard manifold on $\mathbb{R}^n$.
- Let $M = \mathbb{C}^n$, then we cover $\mathbb{C}^n$ by just one open set and then chart the map, $\phi : \mathbb{C}^n \to \mathbb{R}^{2n}$ which is just,
$$\phi(z_1, \dots, z_n) = (\operatorname{Re} z_1, \operatorname{Im} z_1, \dots, \operatorname{Re} z_n, \operatorname{Im} z_n).$$
- If $M$ is a manifold, then any open $V \subset M$ is also a manifold. This can be seen as the union of the atlases $V$ and $M$ is going to be $M$ and so it has the same equivalence class and hence it must be a manifold.
- If we let $M_n(\mathbb{R})$ be all real $n \times n$ matrices, then this is a manifold as it's just $\mathbb{R}^{n^2}$. We also can say $\mathrm{GL}(n, \mathbb{R}) \subset M_n(\mathbb{R})$ and so by the previous point, $\mathrm{GL}(n, \mathbb{R})$ is a manifold.

This is very abstract, we should see that a manifold has similar behaviour to $\mathbb{R}^n$, but is more flexible and can have some sort of curvature inbuilt. The ideas of a manifold holding similarities to $\mathbb{R}^n$ can be seen in Whitney's Embedding Theorem, a theorem whose proof would not add to the work, so it is omitted. Its existence should suffice to prove to the reader that manifolds, however abstract, can be manipulated and argued with using similar ideas to $\mathbb{R}^n$.

**Theorem 2.12** (Whitney Embedding Theorem)**.** Every $m$-dimensional manifold can be embedded in $\mathbb{R}^{2m}$

Now we can formalise the idea of tangent vectors on a manifold [5]

**Definition 2.13** (Tangent Vectors)**.** Let $M$ be a $C^\infty$ manifold, then we can say that $x \in M$. Let us take a chart of $M$, $\phi : U \to \mathbb{R}^n$ where $x \in U$. Now take two curves $\gamma_1, \gamma_2 : (-1, 1) \to M$ with $\gamma_1(0) = \gamma_2(0) = x$ such that we can form $\phi \circ \gamma_1 \circ \phi \circ \gamma_2 : (-1, 1) \to \mathbb{R}^n$ are differentiable.
Now define an equivalence such that $\gamma_1$ and $\gamma_2$ are equivalent at 0 if and only if $(\phi \circ \gamma_1)' = (\phi \circ \gamma_2)' = 0$. Then take the equivalence class of all of these curves and these are the tangent vectors of $M$.

**Definition 2.14** (Tangent Space)**.** The set of all of the tangent vectors at $x$. We denote it as $T_x M$.

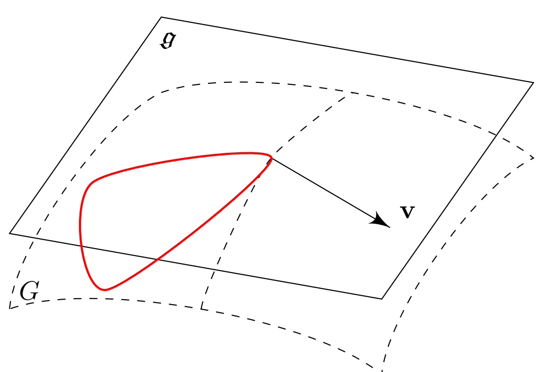



Figure 2: Lie Group and Associated Lie Algebra

These definitions aren't the most intuitive, so we provide Figure 2 to help the reader see the geometric interpretations. We let $G$ be a Lie group and $\mathfrak{g}$ be a Lie algebra, or a tangent space to the manifold, or Lie group. Lie algebras are tangent spaces to the Lie group at the identity. $T_eG$ (tangent space at the identity) is an interesting vector space with a remarkable structure called the Lie algebra structure. We now will prove a few results relating to this structure and then define the Lie algebra using the bilinear map called the Lie Bracket,

**Lemma 2.15.** Let $G$ be a matrix lie group, and $g \in G$, then,

$$\xi \in T_eG \implies g\xi g^{-1} \in T_eG.$$

**Note:** $g\xi g^{-1}$ is a matrix expression.

*Proof.* Let $c(t) \in G$ be a curve in $G$, such that $c(0) = e$ and $\dot{c}(0) = \xi$. Define $\gamma(t) = gc(t)g^{-1}$. Then $\gamma(0) = gc(0)g^{-1} = e$ and $\dot{\gamma}(0) = g\dot{c}(0)g^{-1} = g\xi g^{-1} \in T_eG$. $\square$

**Proposition 2.16.** Let $G$ be a matrix lie group and $\xi, \eta \in T_eG$. Then, $\xi\eta - \eta\xi \in T_eG$

*Proof.* Let $c(t) \in G$ be a curve such that $c(0) = e$ and $\dot{c}(0) = \xi$ also define $b(t) = c(t)\eta c(t)^{-1} \in T_eG$ by Lemma 1.5. Then $\dot{b}(t) \in T_eG$.

$$\begin{aligned}
\dot{b}(0) &= \dot{c}(0)\eta c(0)^{-1} + c(0)\eta + \frac{dc(t)}{dt}(0) \\
&= \dot{c}(0)\eta c(0)^{-1} - c(0)\eta c(0)^{-1}\dot{c}(0)c(0)^{-1} \\
&= \xi\eta - \eta\xi.
\end{aligned}$$

As $\dot{b}(t) \in T_eG$, then $\xi\eta - \eta\xi \in T_eG$ $\square$

Now we have a Lie Algebra,

**Definition 2.17** (Lie Algebra)**.** A lie algebra is a vector space endowed with a commutator (or Lie bracket), that is a bilinear map. If we have,

$$[\cdot, \cdot] : V \times V \to V$$

such that,

- $[B, A] = -[A, B]$ (skew-symmetry property)
- $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0 \quad \forall A, B, C \in V$ (Jacobi Identity).

The Lie Bracket, could be any bilinear map and the behaviour of this map is related to the space that we are considering. In particular, when we consider the matrix Lie groups, we are interested in the Lie Bracket defined in Theorem 2.18

**Theorem 2.18.** Let $G$ be a matrix Lie group. Then $T_eG$ is a Lie algebra with the Lie bracket given by the matrix commutator. Denoted by $\mathfrak{g}$.

$$[A, B] = AB - BA$$

Assume we have a surface, or manifold $M$, with tangent space $T_qM$, then we can say that,

$$\bigcup T_qM = TM$$

is the tangent bundle. We can further define a cotangent bundle, which leads to the cotangent manifold. The cotangent manifold is the dual space of the manifold, $M$.

We now present several examples of Lie algebras and their groups to help the reader understand the main space of study for the rest of the work.

**Example.**

- The Lie algebra $\mathfrak{gl}(n, \mathbb{R}) = T_e\mathrm{GL}(n, \mathbb{R})$ which is vector space of real square $n \times n$ matrices with commutator.
- The Lie algebra of $\mathfrak{sl}(n, \mathbb{R}) := T_e\mathrm{SL}(n, \mathbb{R})$ vector space of real traceless square $n \times n$ matrices.

  *Proof.* Take $g(t) \in \mathrm{SL}(n, \mathbb{R})$, and so $\det g(t) = 1$ hence, take $g(t)$ such that $g(0) = e$, and $\dot{g}(0) = \xi$ and so $\dot{g}(0) \in \mathfrak{sl}(n, \mathbb{R})$. Now use the formula of the derivative of the determinant of a matrix to show,

  $$\frac{d}{dt}(\det(g(t)))_{t=0} = \det g(0)\,\mathrm{Tr}(g^{-1}(0)\dot{g}(0))$$

  and so, $g^{-1}(0)\dot{g}(0) \in \mathfrak{sl}(n, \mathbb{R})$. Therefore, we just have $\mathrm{Tr}(\xi)$. □

- The Lie algebra of $\mathfrak{so}(3) = T_e\mathrm{SO}(3)$, the vector space of skew-symmetric matrices.

We finally present a lemma that we will use again and again throughout this work. In some sense this is the most important lemma in the whole document. If this were false, we would not be able to do many of the things we do in the next few chapters.

**Lemma 2.19.** If $v \in T_gG$, then we can say,

(i) $g^{-1}v \in T_eG$

(ii) $vg^{-1} \in T_eG$

*Proof.* Suppose we have a $c(t) \in G$ such that $c(0) = g$ and $\dot{c}(0) = v$. Now we define a $\gamma(t) = g^{-1}c(t)$ and we see that $\gamma(0) = e$ and $\dot{\gamma}(0) = g^{-1}\dot{c}(0) = g^{-1}v$. Hence by Lemma 2.15 we can say if $v \in T_eG$ then $gvg^{-1} \in T_eG$, applying this to $g^{-1}v$, we can get that $vg^{-1} \in T_eG$, as required. □

## 2.4 Actions of a Lie Group and Lie Algebra

In this section we will focus on how our Lie groups will act on our algebras. We will first use conjugation actions to define our adjoints which will become very useful once we see the Euler-Poincaré equations. As an introduction we will consider classical groups, then define a group action as how the structure can act on a set,

**Definition 2.20** (Group Action)**.** Let $(G, *)$ be a group and $A$ be a set. A group action is a map,

$$(\cdot) : G \times A \to A$$

$$(g, a) \mapsto g \cdot a$$

that satisfies the following axioms,

**(A1)** $(g_1 * g_2) \cdot a = g_1 \cdot (g_2 \cdot a)$ for $a \in A$

**(A2)** $e \cdot a = a$ for all $a \in A$

One of the most thoroughly studied actions in applied settings is conjugation. In our case we will also use the conjugation action to derive the adjoint and coadjoint actions for firstly the Lie group onto the Lie algebra and then the Lie algebra onto itself. The conjugation action is defined in the usual way,

**Definition 2.21** (Conjugation Action)**.** Let $g \in G$, then the operation $I_g : G \to G$ (Inner Automorphism) and so you define it by $h \mapsto ghg^{-1} \quad \forall\, h \in G$. $I_{gh} = AD_{gh}$.

Take an arbitrary path $h(t) \in G$ such that $h(0) = e$ and let $\xi = \dot{h}(0) \in T_eG$. We now define $Ad_g(\xi) = \frac{d}{dt} I_g h(t)_{t=0} = g\xi g^{-1} \in T_eG$, called the adjoint action. Here $I_g$ is the inner automorphism.

**Definition 2.22** (Adjoint and coadjoint actions of $G$ on $\mathfrak{g}$ and $\mathfrak{g}^*$)**.** The adjoint action of the matrix lie group $G$ on it's lie algebra $\mathfrak{g}$ is a map,

$$Ad : G \times \mathfrak{g} \to \mathfrak{g}$$

defined by,

$$\mathrm{Ad}_g \xi = g\xi g^{-1}.$$

The dual map $\left\langle Ad_g^* \mu \,, \xi \right\rangle = \langle \mu \,, Ad_g \xi \rangle$ where $\mu \in \mathfrak{g}^*$ and $\xi \in T_eG = \mathfrak{g}$ is called the coadjoint map of $G$ on the dual Lie algebra $\mathfrak{g}^*$.

We will find that sometimes our classical ideas of vector spaces don't work. Hence, we shall introduce functionals and use them to define dual vector spaces.

**Definition 2.23** (Dual Space for vectors)**.** Let $V$ be a finite dimensional vector space, of dimension $n$, over $\mathbb{R}$. The dual vector space, denoted by $V^*$, is the space of all linear functionals from $V \to \mathbb{R}$, $f(v) = a$ where $v \in V$ and $a \in \mathbb{R}$, then also $f(\alpha v + \beta w) = \alpha f(v) + \beta f(w)$ and $\alpha, \beta \in \mathbb{R}$ and $v, w \in V$. Hence $f(v) = Mv$, we call $M$ the covector such that $Mv \in \mathbb{R}$. The vector space of all covectors is the dual space.

$$\langle m \,, v \rangle \in \mathbb{R} \quad m \in V^* \quad v \in V$$

Now we can see that the dual space is also a vector space so we can use the normal vector space ideas with it,

**Lemma 2.24.** Let $V$ be a vector space of $n \times n$ real matrices. Then the dual vector space $V^*$ is also a vector space of $n \times n$ matrices and every linear functional $f : V \to \mathbb{R}$ such that,

$$f(A) := Tr(B^T A), \quad B \in V^*, A \in V.$$

We need to generalise the idea of an inner product to matrices and here is a particular inner product called the trace pairing. From here on any inner product signs will indicate a trace pairing.

**Definition 2.25** (Trace Pairing)**.** For every vector space $V$ of real $n \times n$ matrices with dual $V^*$, then the pairing is,

$$\langle B \,, A \rangle = Tr(B^T A) = Tr(BA^T).$$

**Proposition 2.26.** Suppose $A^T = A$ and $B^T = -B$, then, $\mathrm{Tr}(B^T A) = 0$

*Proof.*

$$\begin{aligned} \mathrm{Tr}(B^T A) &= -\mathrm{Tr}(BA) \\ &= -\mathrm{Tr}((BA)^T) \\ &= -\mathrm{Tr}(B^T A^T) \\ &= -\mathrm{Tr}(A^T B^T) \\ &= -\mathrm{Tr}(B^T A). \end{aligned}$$

□

We now seek a closed form for the adjoint action of $G$ onto $\mathfrak{g}$. This can be done through the following argument using trace pairings,

$$\begin{aligned}\left\langle Ad_g^*\mu\,,\xi\right\rangle &= \langle\mu\,,Ad_g\xi\rangle\\ &= \left\langle\mu\,,g\xi g^{-1}\right\rangle\\ &= \mathrm{Tr}(\mu^T g\xi g^{-1})\\ &= \mathrm{Tr}(\xi g\mu^T g^{-1})\\ &= \mathrm{Tr}[(g^T\mu(g^{-1})^T)^T\xi]\\ &= \left\langle g^T\mu(g^{-1})^T\,,\xi\right\rangle\\ &= \left\langle g^T\mu(g^T)^{-1}\,,\xi\right\rangle.\end{aligned}$$

We conclude this chapter with ideas about the adjoint action from the Lie algebra onto itself. This can be derived by taking what we will see to be the first variation. This is because we know to get from a Lie group to its tangent space (which is its Lie algebra) it suffices to just take the derivative and set the variable to zero. We do this with the adjoint from $G\to\mathfrak{g}$. Let $g(t)\in G$ such that $g(0)=e$ where $\eta=\dot g(0)\in T_eG$, then we define the adjoint from $\mathfrak{g}$ to $\mathfrak{g}$ as,

$$ad_\eta\xi := \frac{d}{dt}_{t=0}Ad_{g(t)}\xi \qquad \forall\,\xi\in\mathfrak{g}$$

which we can see to be

$$\dot g(0)\xi g(0)^{-1} + g(0)\xi\frac{d}{dt}_{t=0}g(t)^{-1} = \eta\xi-\xi\eta.$$

Hence we can say that $\mathrm{ad}_\eta\xi=[\eta,\xi]=\eta\xi-\xi\eta$. We define the coadjoint action on $\mu$,

**Definition 2.27** (Adjoint / Coadjoint action on $\mathfrak{g}/\mathfrak{g}^*$)**.** The adjoint action of the matrix Lie algebra on itself is given by,

$$\begin{aligned}\mathrm{ad}:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}\\ \mathrm{ad}_\eta\xi=[\eta,\,\xi].\end{aligned}$$

The dual map $\left\langle ad_\eta^*\mu\,,\xi\right\rangle=\langle\mu\,,\mathrm{ad}_\eta\xi\rangle$ is the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$.

We now seek again a closed form for the coadjoint action from a Lie algebra onto itself. Above we said that $\left\langle \mathrm{Ad}_g^*\,\mu\,,\xi\right\rangle=\left\langle g^T\mu(g^T)^{-1}\,,\xi\right\rangle$ and so $\mathrm{Ad}_g^*\,\mu=g^T\mu(g^T)^{-1}$. As before we define,

$$\mathrm{ad}_g^*\mu=\frac{d}{dt}_{t=0}Ad_g^*\mu$$

and now we can input our definitions and differentiate where we define $g(0)=0$ and $\dot g(0)=\eta\in T_eG$,

$$\begin{aligned}\mathrm{ad}_g^*\mu &= \frac{d}{dt}_{t=0}Ad_g^*\mu\\ &= \frac{d}{dt}_{t=0}g^T\mu(g^T)^{-1}\\ &= [\dot g^T\mu(g^T)^{-1}-g^T\mu(g^T)^{-1}\dot g^T(g^T)^{-1}]_{t=0}\\ &= \dot g(0)^T\mu(g(0)^T)^{-1}-g(0)^T\mu(g(0)^T)^{-1}\dot g(0)^T(g(0)^T)^{-1}\\ &= \eta^T\mu e-e\mu e^{-1}\eta^Te^{-1}\\ &= \eta^T\mu-\mu\eta^T\\ &= [\eta^T,\,\mu].\end{aligned}$$

Hence, $\mathrm{ad}_g^*\mu=[\eta^T,\,\mu]$. These results may seem arbitrary and slightly non-useful currently; however, after we have started to derive equations these results will be invaluable. Now we will move towards a more applied treatment of this area and consider the acts of rotation and different types of coordinate systems.

# 3 Integrating Mechanics and Geometry

In this final pure mathematics chapter we will use the Geometric ideas we met in Chapter 1 and Chapter 2 and integrate them to start to be able to discuss systems of equations. We will first talk about different coordinate systems and rotations that connect them. This section introduces specific Lagrangians that we will use several times in the next chapter. In the next section we present an introduction to calculus of variations. We will focus on the first variation and how that relates to Hamilton's variational principle, which we will use to derive our first equation for the Lagrangian in section one. Finally, we will formally introduce the hat map, we will prove it's a Lie algebra isomorphism and explain some of the theory surrounding it. This will then all be used in the next chapter which is the main topic of the thesis, Euler-Poincaré reduction.

## 3.1 Rotation

Consider a spatial coordinate system with its origin at the centre of mass of a given rigid body. We denote it by, $\mathbf{x}(t) \in \mathbb{R}^3$, where $\mathbf{x} = X$. We need a way to rotate the rigid body without constraints, so we denote a

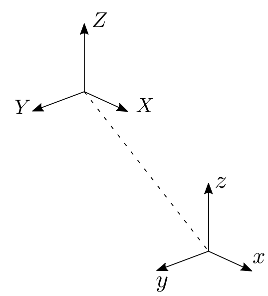



Figure 3: Spatial and Body Coordinates

rotation tensor $R(t)$ and say $\mathbf{x}(t) = R(t)\mathbf{X}$ where $\mathbf{X}$ is in the body coordinate system. The configuration of the body particle at time $t$ is given by a rotation matrix that takes the label $\mathbf{X}$ to current position $\mathbf{x}(t)$ where $R \in \mathrm{SO}(3)$ is a proper rotation matrix. This means,

$$R^T = R^{-1} \qquad \det R = 1.$$

The map $\mathbf{X} \to R(t)\mathbf{X}$ is called the body-to-space map. We can now talk about kinetic energy,

$$K = \frac{1}{2}\int_{\mathcal{B}} \rho \left\|\dot{\mathbf{x}}\right\|^2 \, d^3\mathbf{X}$$

which we can change to,

$$\begin{aligned} \frac{1}{2}\int_{\mathcal{B}} \rho \left\|\dot{\mathbf{x}}\right\|^2 \, d^3\mathbf{X} &= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X}) \left\|\dot{R}(t)\mathbf{X}\right\|^2 \, d^3\mathbf{X} \\ &= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X}) \dot{R}(t)\mathbf{X} \cdot \dot{R}(t)\mathbf{X} \, d^3\mathbf{X}. \end{aligned}$$

Now we can say if $V = 0$. Hence, $L = K$ and so,

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{R}} - \frac{\partial K}{\partial R} = \mathbf{0}.$$

This is difficult to deal with. We shall now derive kinetic energy that is more useful to Euler-Poincaré reduction.

We know that $R^{-1} = R^T$ and so $RR^T = RR^{-1} = I$. If we have $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^3$, then $\mathbf{v} \cdot \mathbf{w} = R\mathbf{v} \cdot R\mathbf{w}$. Hence, consider $\left\|\dot{R}\mathbf{X}\right\|^2$ and we know,

$$\begin{aligned}\left\|\dot{R}\mathbf{X}\right\|^2 &= \dot{R}\mathbf{X} \cdot \dot{R}\mathbf{X} \\ &= R^{-1}(\dot{R}\mathbf{X}) \cdot R^{-1}(\dot{R}\mathbf{X}) \\ &= \left\|R^{-1}\dot{R}\mathbf{X}\right\|^2\end{aligned}$$

and so,

$$K = \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X}) \left\|R^{-1}\dot{R}\mathbf{X}\right\|^2 d^3\mathbf{X}.$$

It can be seen that $K = K(R, \dot{R}) = K(R^{-1}R, R^{-1}\dot{R})$. This is called left symmetry. Hence, we can reduce this to $K(e, R^{-1}\dot{R})$ and change notation. Let $K(e, R^{-1}\dot{R}) := \kappa(R^{-1}\dot{R})$, where $R^{-1}\dot{R}$ is the angular velocity of the body. We can see this from the body and from an observation outside the system. Hence, we call $R^{-1}\dot{R} = \hat{\mathbf{\Omega}}$.
Interestingly, we know $RR^T = RR^{-1} = I$. Therefore,

$$\frac{d}{dt}I = \frac{d}{dt}(RR^{-1}) = \dot{R}R^{-1} + R\frac{d}{dt}R^{-1} = \mathbf{0}$$

and we can also write this as,

$$\begin{aligned}I &= RR^T \\ \mathbf{0} &= \frac{d}{dt}(RR^T) \\ \mathbf{0} &= \dot{R}^T R + R^T\dot{R} \\ R^T\dot{R} &= -(R^T\dot{R})^T\end{aligned}$$

and so $R^{-1}\dot{R} = -(R^{-1}\dot{R})^T$ that is, $\hat{\mathbf{\Omega}} = -\hat{\mathbf{\Omega}}^T$. This is the antisymmetric property we have noted about this vector. Now we go back to kinetic energy to nicely write it in terms of $\hat{\mathbf{\Omega}}$,

$$K = \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X}) \left\|\hat{\mathbf{\Omega}}\mathbf{X}\right\|^2 d^3\mathbf{X}.$$

We can now prove that $\hat{\mathbf{\Omega}}\mathbf{X} = \mathbf{\Omega} \times \mathbf{X}$ where,

$$\hat{\mathbf{\Omega}} = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix} \qquad \mathbf{\Omega} = \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix}.$$

This can be proven by just multiplying out the matrix,

$$\begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} \Omega_2 X_3 - X_2\Omega_3 \\ \Omega_3 X_1 - \Omega_1 X_3 \\ X_2\Omega_1 - X_1\Omega_2 \end{bmatrix} = \mathbf{X} \times \mathbf{\Omega}$$

where $\mathbf{\Omega}$ is the axel vector. We now write again,

$$\begin{aligned}
K &= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})\left\|\mathbf{\Omega}\times\mathbf{X}\right\|^2\, d^3\mathbf{X} \\
&= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})(\mathbf{\Omega}\times\mathbf{X})\cdot(\mathbf{\Omega}\times\mathbf{X})\, d^3\mathbf{X} \\
&= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})\left(\left\|\mathbf{\Omega}^2\right\|\left\|\mathbf{X}^2\right\| - (\mathbf{\Omega}\cdot\mathbf{X})^2\right)\, d^3\mathbf{X} \\
&= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})\left(\mathbf{\Omega}^T\mathbf{\Omega}\left\|\mathbf{X}^2\right\| - \mathbf{\Omega}^T\mathbf{X}\mathbf{X}^T\mathbf{\Omega}\right)\, d^3\mathbf{X} \\
&= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})\mathbf{\Omega}^T\left(\left\|\mathbf{X}^2\right\| I - \mathbf{X}\mathbf{X}^T\right)\mathbf{\Omega}\, d^3\mathbf{X} \\
&= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})\mathbf{\Omega}^T\mathbf{\Omega}\left(\left\|\mathbf{X}^2\right\| I - \mathbf{X}\mathbf{X}^T\right)\, d^3\mathbf{X} \\
&= \frac{1}{2}\mathbf{\Omega}\cdot\mathbf{\Omega}\int_{\mathcal{B}} \rho(\mathbf{X})\left(\left\|\mathbf{X}^2\right\| I - \mathbf{X}\mathbf{X}^T\right)\, d^3\mathbf{X} \\
&= \frac{1}{2}\mathbb{I}\mathbf{\Omega}\cdot\mathbf{\Omega}
\end{aligned}$$

where $\mathbb{I}$ is the moment of inertia tensor. This is defined as,

$$\mathbb{I} = \int_{\mathcal{B}} \rho(\mathbf{X})\left(\left\|\mathbf{X}\right\|^2 I - \mathbf{X}\mathbf{X}^T\right) d^3\mathbf{X}$$

where $\mathbf{X}\mathbf{X}^T = \mathbf{X}\otimes\mathbf{X}$ and,

$$(\mathbf{a}\otimes\mathbf{b})\mathbf{c} = (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$$

for all $\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbb{R}^3$.

## 3.2 Calculus of variations

Now we will continue our exploration of pure-applied mathematics by looking at calculus of variations. We are going to consider the mathematics on a continuous level, but you can consider a discrete level. This could be done via discrete differential geometry and is very useful to derive numerical methods subject to some conditions.

**Theorem 3.1** (Hamilton's Variation Principle)**.** Let $G$ be a Lie group and $\mathfrak{g}$ be its associated algebra. We consider a Lagrangian, $L : G\times\mathfrak{g}\to\mathfrak{g}$. Then Hamilton's Variational Principle holds, that is,

$$L = \int_{t_1}^{t_2} \frac{1}{2}\mathbb{I}\mathbf{\Omega}\cdot\mathbf{\Omega}\, dt.$$

We find differential equations by letting $\delta L = 0$ but this is subject endpoint conditions, $\delta\mathbf{\Omega}(t_1) = \delta\mathbf{\Omega}(t_2) = \mathbf{0}$.

Now we can take Hamilton's Variational Principle with respect to the Lagrangian we derived in the last section,

$$\delta\int_{t_1}^{t_2}\frac{1}{2}\mathbb{I}\mathbf{\Omega}\cdot\mathbf{\Omega}\, dt = \int_{t_1}^{t_2}\frac{1}{2}\mathbb{I}\delta\mathbf{\Omega}\cdot\mathbf{\Omega} + \frac{1}{2}\mathbb{I}\mathbf{\Omega}\cdot\delta\mathbf{\Omega}\, dt = \int_{t_1}^{t_2}\mathbb{I}\mathbf{\Omega}\cdot\delta\mathbf{\Omega}$$

We have reached an impassable point, we currently don't know what $\delta\mathbf{\Omega}$ is. Remember $\hat{\mathbf{\Omega}}$ is in the Lie algebra of SO(3). We said that $\hat{\mathbf{\Omega}} = R^T\dot{R} = R^{-1}\dot{R}$. Now we take variations of $\hat{\mathbf{\Omega}} = \mathbf{\Omega}\times\mathbf{X}$ and so,

$$(\delta\mathbf{\Omega})\times\mathbf{X} = (\delta\hat{\mathbf{\Omega}})\mathbf{X}.$$

We can now rewrite this as,

$$\delta\hat{\mathbf{\Omega}} = \delta(R^{-1}\dot{R}) = \delta R^{-1}\dot{R} + R^{-1}\delta\dot{R} = 0$$

as $\delta I = \delta R R^{-1} + R\delta R^{-1}$. We see that $R^{-1}\delta R R^{-1} + R^{-1}R\delta R^{-1} = 0$ and so as $RR^{-1} = I$, $R^{-1}\delta R R^{-1} + \delta R^{-1} = \mathbf{0}$. Therefore, $\delta R^{-1}\dot{R} + R^{-1}\delta\dot{R} = \delta\hat{\mathbf{\Omega}}$ and $\hat{\mathbf{\Omega}} = R^{-1}\dot{R}$ where $\hat{\mathbf{\Lambda}} = R^{-1}\delta R$ and so we substitute for this quantity,

$$\begin{aligned}
\delta\hat{\mathbf{\Omega}} &= -R^{-1}\delta R R^{-1}\dot{R} + R^{-1}\frac{d}{dt}\delta R \\
&= R^{-1}\delta R\hat{\mathbf{\Omega}} + \frac{d}{dt}(R^{-1}\delta R) - (\frac{d}{dt}R^{-1})\delta R \\
&= R^{-1}\delta R\hat{\mathbf{\Omega}} + \frac{d}{dt}(R^{-1}\delta R) + R^{-1}\dot{R}R^{-1}\delta R \\
&= -\hat{\mathbf{\Lambda}}\hat{\mathbf{\Omega}} + \frac{d}{dt}\hat{\mathbf{\Lambda}} + \hat{\mathbf{\Omega}}\hat{\mathbf{\Lambda}} \\
&= \dot{\hat{\mathbf{\Lambda}}} + [\hat{\mathbf{\Omega}}, \hat{\mathbf{\Lambda}}].
\end{aligned}$$

We now aim to prove,

$$\delta\mathbf{\Omega} = \dot{\Lambda} + (\mathbf{\Omega} \times \mathbf{\Lambda}).$$

We can use the fact that $\widehat{[\mathbf{\Omega}, \mathbf{\Lambda}]} = [\hat{\mathbf{\Omega}}, \hat{\mathbf{\Lambda}}]$ to get the required result.

$$\begin{aligned}
\delta\hat{\mathbf{\Omega}} &= \dot{\hat{\mathbf{\Lambda}}} + [\hat{\mathbf{\Omega}}, \hat{\mathbf{\Lambda}}] \\
&= \dot{\hat{\mathbf{\Lambda}}} + \widehat{[\mathbf{\Omega}, \mathbf{\Lambda}]} \\
&= \dot{\hat{\mathbf{\Lambda}}} + \widehat{(\mathbf{\Omega} \times \mathbf{\Lambda})} \\
\widehat{\delta\mathbf{\Omega}} &= \widehat{\dot{\mathbf{\Lambda}} + (\mathbf{\Omega} \times \mathbf{\Lambda})}
\end{aligned}$$

Therefore, $\delta\mathbf{\Omega} = \dot{\mathbf{\Lambda}} + (\mathbf{\Omega} \times \mathbf{\Lambda})$. Now, let us substitute this back into our variational principle.

$$\begin{aligned}
0 &= \delta\int_{t_1}^{t_2} \mathbb{I}\mathbf{\Omega} \cdot \mathbf{\Omega}\, dt \\
0 &= \int_{t_1}^{t_2} \mathbb{I}\mathbf{\Omega} \cdot \delta\mathbf{\Omega}\, dt \\
0 &= \int_{t_1}^{t_2} \mathbb{I}\mathbf{\Omega} \cdot (\dot{\mathbf{\Lambda}} + \mathbf{\Omega} \times \mathbf{\Lambda})\, dt \\
0 &= [\mathbb{I}\mathbf{\Omega} \cdot \mathbf{\Lambda}|_{t_2} - \mathbb{I}\mathbf{\Omega} \cdot \mathbf{\Lambda}|_{t_1}] - \int_{t_1}^{t_2} \frac{d}{dt}(\mathbb{I}\mathbf{\Omega}) \cdot \mathbf{\Lambda}\, dt + \int_{t_1}^{t_2} (\mathbb{I}\mathbf{\Omega} \times \mathbf{\Omega}) \cdot \mathbf{\Lambda} \\
0 &= 0 - 0 - \int_{t_1}^{t_2} (-\mathbb{I}\dot{\mathbf{\Omega}} + \mathbb{I}\mathbf{\Omega} \times \mathbf{\Omega}) \cdot \mathbf{\Lambda}\, dt.
\end{aligned}$$

Hence,

$$\mathbb{I}\dot{\mathbf{\Lambda}} = \mathbb{I}\mathbf{\Omega} \times \mathbf{\Omega}.$$

We note that can write the equations by considering the tangent space.

## 3.3 The Hat Map as a Lie Algebra Isomorphism

In this section we will take a quick jaunt back into some purer topics, more specifically a Lie algebra isomorphism called the Hat Map. We can define a Lie algebra isomorphism in the usual way, as a bijective homomorphism. We define this homomorphism as,

**Definition 3.2** (Lie Algebra Homomorphism)**.** A Lie algebra homomorphism is a linear map, $\phi : \mathfrak{g} \to \mathfrak{g}'$ compatible with the respective Lie brackets,

$$\phi([x,y]_{\mathfrak{g}}) = [\phi(x), \phi(y)]_{\mathfrak{g}'} \qquad \forall x, y \in \mathfrak{g}.$$

We note that the Lie bracket for $\mathbb{R}^3$ is the cross product, and more specifically $(\mathbb{R}^3, \times)$ is a Lie algebra. Then we can talk about a Lie algebra homomorphism, $\phi : \mathbb{R}^3 \to \mathfrak{so}(3)$, which is a homomorphism that maps,

$$\phi(\mathbf{x} \times \mathbf{y}) = [\phi(x), \phi(y)]_{\mathfrak{so}(3)}.$$

Given $\boldsymbol{\omega} \in \mathbb{R}^3$, we let this $\phi$ be the hat map where we define,

$$\phi(\boldsymbol{\omega}) = \hat{\boldsymbol{\omega}} = \begin{pmatrix} 0 & -\omega_1 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

The Lie algebra of SO(3) is the space of skew-symmetric matrices, $\mathfrak{so}(3)$. For $\mathfrak{so}(3)$ we can derive that the Euler-Poincaré Equations are written as:

$$\frac{d}{dt}\frac{\partial \ell}{\partial \hat{\boldsymbol{\Omega}}} - \mathrm{ad}^*_{\hat{\boldsymbol{\Omega}}} \frac{\partial \ell}{\partial \hat{\boldsymbol{\Omega}}} = 0.$$

Let $\boldsymbol{\Pi}$ be any element in $\mathfrak{g}^*$, then the $\mathrm{ad}^*$ operator is defined by $\left\langle \mathrm{ad}^*_{\hat{\boldsymbol{\Omega}}}\boldsymbol{\Pi}\,, \hat{\boldsymbol{\omega}} \right\rangle = \left\langle \boldsymbol{\Pi}\,, \mathrm{ad}_{\hat{\boldsymbol{\Omega}}}\hat{\boldsymbol{\omega}} \right\rangle$ where $\hat{\boldsymbol{\omega}} \in \mathfrak{g}^*$.

$$\begin{aligned}
\left\langle \mathrm{ad}^*_{\hat{\boldsymbol{\Omega}}}\boldsymbol{\Pi}\,, \hat{\boldsymbol{\omega}} \right\rangle &= \left\langle \boldsymbol{\Pi}\,, \mathrm{ad}_{\hat{\boldsymbol{\Omega}}}\hat{\boldsymbol{\omega}} \right\rangle \\
&= \left\langle \boldsymbol{\Pi}\,, [\hat{\boldsymbol{\Omega}},\, \hat{\boldsymbol{\omega}}] \right\rangle \\
&= \mathrm{Tr}(\boldsymbol{\Pi}^\top [\hat{\boldsymbol{\Omega}},\, \hat{\boldsymbol{\omega}}]) \\
&= \mathrm{Tr}(\boldsymbol{\Pi}^\top \hat{\boldsymbol{\Omega}}\hat{\boldsymbol{\omega}} - \boldsymbol{\Pi}^\top \hat{\boldsymbol{\omega}}\hat{\boldsymbol{\Omega}}) \\
&= \mathrm{Tr}(\boldsymbol{\Pi}\hat{\boldsymbol{\Omega}}^\top \hat{\boldsymbol{\omega}} - \boldsymbol{\Pi}\hat{\boldsymbol{\Omega}}\hat{\boldsymbol{\omega}}^\top) \\
&= \mathrm{Tr}(\boldsymbol{\Pi}\hat{\boldsymbol{\Omega}}\hat{\boldsymbol{\omega}}^\top - \hat{\boldsymbol{\Omega}}\boldsymbol{\Pi}\hat{\boldsymbol{\omega}}^\top) \\
&= \mathrm{Tr}((\boldsymbol{\Pi}\hat{\boldsymbol{\Omega}} - \hat{\boldsymbol{\Omega}}\boldsymbol{\Pi})\hat{\boldsymbol{\omega}}^\top) \\
&= \mathrm{Tr}([\boldsymbol{\Pi},\, \hat{\boldsymbol{\Omega}}]\hat{\boldsymbol{\omega}}^\top) \\
&= \left\langle [\boldsymbol{\Pi},\, \hat{\boldsymbol{\Omega}}]\,, \hat{\boldsymbol{\omega}} \right\rangle.
\end{aligned}$$

Then, $\mathrm{ad}^*_{\hat{\boldsymbol{\Omega}}}\boldsymbol{\Pi} = [\boldsymbol{\Pi},\, \hat{\boldsymbol{\Omega}}]$. From here we can conclude that, the hat map is a Lie algebra isomorphism, i.e. $[\hat{\boldsymbol{\Omega}},\, \hat{\boldsymbol{\omega}}] = \widehat{\boldsymbol{\Omega} \times \boldsymbol{\omega}}$. We prove this from the definition of the Lie bracket, that is,

$$\begin{aligned}
[\hat{\boldsymbol{\Omega}}, \hat{\boldsymbol{\omega}}] &= \hat{\boldsymbol{\Omega}}\hat{\boldsymbol{\omega}} - \hat{\boldsymbol{\omega}}\hat{\boldsymbol{\Omega}} \\
&= \begin{pmatrix} 0 & -\Omega_1 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\omega_1 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -\omega_1 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\Omega_1 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -\Omega_3\omega_3 - \Omega_2\omega_2 & \Omega_2\omega_1 & \Omega_3\omega_1 \\ \Omega_1\omega_2 & -\Omega_3\omega_3 - \Omega_1\omega_1 & \Omega_3\omega_2 \\ \Omega_1\omega_3 & \Omega_2\omega_3 & -\Omega_2\omega_2 - \Omega_1\omega_1 \end{pmatrix} - \begin{pmatrix} -\omega_3\Omega_3 - \omega_2\Omega_2 & \omega_2\Omega_1 & \omega_3\Omega_1 \\ \omega_1\Omega_2 & -\omega_3\Omega_3 - \omega_1\Omega_1 & \omega_3\Omega_2 \\ \omega_1\Omega_3 & \omega_2\Omega_3 & -\omega_2\Omega_2 - \omega_1\Omega_1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \Omega_2\omega_1 - \Omega_1\omega_2 & \Omega_3\omega_1 - \Omega_1\omega_3 \\ \Omega_1\omega_2 - \Omega_2\omega_1 & 0 & \Omega_3\omega_2 - \Omega_2\omega_3 \\ \Omega_1\omega_3 - \Omega_3\omega_1 & \Omega_2\omega_3 - \Omega_3\omega_2 & 0 \end{pmatrix} \\
&= \widehat{\boldsymbol{\Omega} \times \boldsymbol{\omega}}.
\end{aligned}$$

We have proved that $[\phi(\mathbf{x}), \phi(\mathbf{y})] = \phi(\mathbf{x} \times \mathbf{y})$ where $\phi$ is the hat map. We now need to prove that the hat map is a bijective linear map. It is linear as it can be represented as a matrix in SO(3). Now we prove bijectivity by first proving injectivity, then surjectivity. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ and we know that $\hat{\mathbf{x}} = \hat{\mathbf{y}}$. In matrix form,

$$\begin{pmatrix} 0 & -x_1 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -y_1 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{pmatrix}.$$

Then we can see that $x_1 = y_1$, $x_2 = y_2$ and $x_3 = y_3$. Therefore, $\mathbf{x} = \mathbf{y}$. Hence the hat map is injective. Now we seek to prove that the hat map is surjective. Consider some $\hat{\mathbf{z}} \in \mathfrak{so}(3)$, then we can write it as,

$$\begin{pmatrix} 0 & -z_1 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{pmatrix}.$$

This will uniquely define some $\mathbf{z} = (z_1, z_2, z_3)^T \in \mathbb{R}^3$. Hence the hat map is surjective. Therefore, the hat map is a Lie algebra isomorphism.

# 4 Euler-Poincaré Reduction by Symmetry

We have seen several pieces of applied-adjacent mathematics over the past few sections. In this chapter we aim to bundle it all together and introduce Euler-Poincaré reduction by symmetry. The main idea underpinning this section is that Lagrangians can be classed as symmetric, that is if we take some Lagrangian we could have $L(R,\dot{R}) = L(gR, g\dot{R})$ for all $g \in \mathrm{SO}(3)$. If we now let $g = R^{-1}$, then we get that $L(R,\dot{R}) = L(I, R^{-1}\dot{R})$ and so we have reduced the Lagrangian from two parameters to one. In mathematical modelling and applied maths generally we try to reduce the number of parameters in a system, because this makes it easier to analyse. Here we use this reduction before we have defined our equations and so we end up with simpler equations. We also work with Lie Groups and Algebras, which come with a set of tools, such as the hat map and the diamond map (introduced in 4.2). These tools help us reduce the equations further and find nice closed forms for the conserved quantities (Noether Theorems) of the systems. We will firstly look at the simplest example of an Euler-Poincaré reduction. Then we will look at Noether Theory in more depth. Finally, we consider Euler-Poincaré reduction where we have symmetry-breaking parameters, leading to a few applications noted in the last chapter.

To gain a general idea of how the equations of motion appear for rotational dynamics with symmetry, we consider an arbitrary Lagrangian of this form,

$$L : T\mathrm{SO}(3) \to \mathbb{R}$$
$$L = L(R, \dot{R}).$$

This Lagrangian satisfies,

$$\delta \int_{t_1}^{t_2} L(R,\dot{R})\, dt = 0$$

which means,

$$\begin{aligned}\delta \int_{t_1}^{t_2} L(R,\dot{R})\, dt &= \int_{t_1}^{t_2} \left\langle \frac{\partial L}{\partial R}, \delta R \right\rangle + \left\langle \frac{\partial L}{\partial \dot{R}}, \delta R \right\rangle dt \\ &= \left\langle \frac{\partial L}{\partial \dot{R}}, \delta R \right\rangle \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left\langle \frac{\partial L}{\partial R} - \frac{d}{dt}\frac{\partial L}{\partial \dot{R}}, \delta R \right\rangle dt.\end{aligned}$$

Imposing endpoint conditions leads to the first term going to zero and so we can notice $\frac{d}{dt}\frac{\partial L}{\partial \dot{R}} - \frac{\partial L}{\partial R} = 0$. These are the Euler-Lagrange equations we have derived a few times from other means. We now look towards a Lagrangian that has what we call left-symmetry.

**Definition 4.1** (Left-Symmetric Lagrangian)**.** A Lagrangian is said to be left-symmetric or left-invariant under the action of the group of rotations if, $L(\chi R, \chi\dot{R}) = L(R,\dot{R})\ \forall\, \chi \in \mathrm{SO}(3)$.

We also know $\mathfrak{so}(3) = T_e\mathrm{SO}(3)$ and we have proven that $v \in T_eG$ implies $g^{-1}v \in \mathfrak{g}$ and $\mathfrak{g}$ is just the tangent space of SO(3). We know $\dot{R}(t) \in T_{R(t)}\mathrm{SO}(3)$ and so we can say $R^{-1}\dot{R} \in \mathfrak{so}(3)$. We say

$$\begin{aligned}L(R,\dot{R}) &= L(R^{-1}R, R^{-1}\dot{R}) \\ &= \widetilde{\ell}(R^{-1}\dot{R}) = \widetilde{\ell}(\hat{\mathbf{\Omega}}).\end{aligned}$$

We now seek to derive the Euler-Poincaré equations for this system. We shall start from Hamilton's Variational Principle,

$$\begin{aligned}\delta \int_{t_1}^{t_2} \ell(\hat{\mathbf{\Omega}})\, dt &= \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}}, \delta\hat{\mathbf{\Omega}} \right\rangle dt \\ &= \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \mathbf{\Omega}}, \delta\mathbf{\Omega} \right\rangle.\end{aligned}$$

Now we shall use a fact we proved in the last chapter, $\delta\mathbf{\Omega} = \dot{\mathbf{\Lambda}} + (\mathbf{\Omega}\times\mathbf{\Lambda})$, to derive the Euler-Poincaré equations.

$$\begin{aligned}
\int_{t_1}^{t_2}\left\langle \frac{\partial \ell}{\partial \mathbf{\Omega}}, \delta\mathbf{\Omega}\right\rangle &= \int_{t_1}^{t_2}\left\langle \frac{\partial \ell}{\partial \mathbf{\Omega}}, \dot{\mathbf{\Lambda}} + \mathbf{\Omega}\times\mathbf{\Lambda}\right\rangle \\
&= \int_{t_1}^{t_2}\left\langle \frac{\partial \ell}{\partial \mathbf{\Omega}}, \frac{d}{dt}\mathbf{\Lambda}\right\rangle + \left\langle \frac{\partial \ell}{\partial \mathbf{\Omega}}, \mathbf{\Omega}\times\mathbf{\Lambda}\right\rangle \\
&= \int_{t_1}^{t_2}\left\langle -\frac{d}{dt}\frac{\partial \ell}{\partial \mathbf{\Omega}}, \mathbf{\Lambda}\right\rangle + \left\langle -\mathbf{\Omega}\times\frac{\partial \ell}{\partial \mathbf{\Omega}}, \mathbf{\Lambda}\right\rangle \\
&= \int_{t_1}^{t_2}\left\langle -\frac{d}{dt}\frac{\partial \ell}{\partial \mathbf{\Omega}}, \mathbf{\Lambda}\right\rangle + \left\langle \frac{\partial \ell}{\partial \mathbf{\Omega}}\times\mathbf{\Omega}, \mathbf{\Lambda}\right\rangle \\
&= \int_{t_1}^{t_2}\left\langle -\frac{d}{dt}\frac{\partial \ell}{\partial \mathbf{\Omega}} + \frac{\partial \ell}{\partial \mathbf{\Omega}}\times\mathbf{\Omega}, \mathbf{\Lambda}\right\rangle = \mathbf{0}.
\end{aligned}$$

Hence we say that

$$\frac{d}{dt}\frac{\partial \ell}{\partial \mathbf{\Omega}} - \frac{\partial \ell}{\partial \mathbf{\Omega}}\times\mathbf{\Omega} = \mathbf{0}$$

is the Euler-Poincaré equation for rotational dynamics with symmetry under left multiplication.

**Theorem 4.2.** The spatial angular momentum is conserved along solutions of the Euler-Poincaré equations.

*Proof.* We know $\frac{d}{dt}\frac{\partial \ell}{\partial \mathbf{\Omega}} - \frac{\partial \ell}{\partial \mathbf{\Omega}}\times\mathbf{\Omega} = 0$ and we know that $\frac{d}{dt}R\frac{\partial \ell}{\partial \mathbf{\Omega}}$ is the spatial angular momentum. Hence we consider,

$$\begin{aligned}
\frac{d}{dt}R\frac{\partial \ell}{\partial \mathbf{\Omega}} &= \dot{R}\frac{\partial \ell}{\partial \mathbf{\Omega}} + R\frac{d}{dt}\frac{\partial \ell}{\partial \mathbf{\Omega}} \\
&= R\hat{\mathbf{\Omega}}\frac{\partial \ell}{\partial \mathbf{\Omega}} + R\left(\frac{\partial \ell}{\partial \mathbf{\Omega}}\times\mathbf{\Omega}\right) \\
&= R\left(\mathbf{\Omega}\times\frac{\partial \ell}{\partial \mathbf{\Omega}}\right) + R\left(\frac{\partial \ell}{\partial \mathbf{\Omega}}\times\mathbf{\Omega}\right) = 0.
\end{aligned}$$

□

Now we want to write a general form of the Euler-Poincaré Equations for left invariant systems. Let $L$ be a Lagrangian on the tangent bundle of a matrix Lie group $G$,

$$\begin{aligned}
&L : TG \to \mathbb{R} \\
&L = L(g, \dot{g}) \qquad \forall\, g \in G.
\end{aligned}$$

We assume that the Lagrangian is left-invariant,

$$L(g,\, \dot{g}) = L(hg,\, h\dot{g}) \quad \forall\, h \in G$$

and now let $h = g^{-1}$, and so $L(g,\, \dot{g}) = L(g^{-1}g\, g^{-1}\dot{g}) := \ell(\xi)$. We have moved from $g$ which is in the Lie group to, $\xi = g^{-1}\dot{g} \in T_eG = \mathfrak{g}$ which is the Lie algebra. We now aim to use the action functional and variational derivative,

$$\begin{aligned}
\delta\int_{t_1}^{t_2} L(g,\, \dot{g}) &= \mathbf{0} \\
\delta\int_{t_1}^{t_2} \ell(\xi)\, dt &= \mathbf{0} \\
\int_{t_1}^{t_2}\left\langle \frac{\partial \ell}{\partial \xi}, \delta\xi\right\rangle\, dt &= \mathbf{0}.
\end{aligned}$$

Now we want to consider $\delta\xi = \delta(g^{-1}\dot{g})$,

$$\begin{aligned}
\delta(g^{-1}\dot{g}) &= \delta g^{-1}\dot{g} + g^{-1}\delta\dot{g} \\
&= g^{-1}\delta g g^{-1}\dot{g} + g^{-1}\frac{d}{dt}\delta g \\
&= -(g^{-1}\delta g)g^{-1}\dot{g} + g^{-1}\frac{d}{dt}\delta g \\
&= -\eta\xi + \frac{d}{dt}\delta(g^{-1}\delta g) + (g^{-1}\dot{g})(g^{-1}\frac{d}{dt}\delta g) \\
&= -\eta\xi + \dot{\eta} + \xi\eta \\
&= \dot{\eta} + [\xi,\, \eta] \\
&= \dot{\eta} + \mathrm{ad}_\xi \eta.
\end{aligned}$$

This is useful to free up the $\eta$ terms which we use to move all of terms into one trace pairing,

$$\begin{aligned}
\int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \xi}\,, \delta\xi \right\rangle \, dt = 0 \\
\int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \xi}\,, \dot{\eta} + \mathrm{ad}_\xi \eta \right\rangle = 0 \\
\int_{t_1}^{t_2} \left\langle -\frac{d}{dt}\left(\frac{\partial \ell}{\partial \xi}\right)\,, \eta \right\rangle + \left\langle \mathrm{ad}_\xi^* \frac{\partial \ell}{\partial \xi}\,, \eta \right\rangle \, dt = 0.
\end{aligned}$$

Since $\eta$ is arbitrary our equation is of this form,

$$\frac{d}{dt}\frac{\partial \ell}{\partial \xi} - \mathrm{ad}_\xi^* \frac{\partial \ell}{\partial \xi} = 0$$

and this is our Euler-Poincaré equation for a left-invariant system.

**Theorem 4.3** (Noether's Theorem for left-invariant systems)**.** The Euler-Poincaré equations associated with a left-invariant system preserve the generalised momentum along solutions of the Euler-Poincaré equations. That is,

$$\frac{d}{dt}\left(\mathrm{Ad}^*_{g^{-1}(t)} \frac{\partial \ell}{\partial \xi}(t)\right) = 0.$$

*Proof.* Suppose we have a left invariant Lagrangian, i.e. $L(g, \dot{g}) = L(e,\, g^{-1}\dot{g}) = \ell(g^{-1}g) := \ell(\xi)$ where $\xi = g^{-1}\dot{g}$. Firstly, however, let us consider the following derivative where $\mu(t) \in \mathfrak{g}$,

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=t_0} \left(\mathrm{Ad}_{g^{-1}(t)}\, \mu(t)\right) &= \left.\frac{d}{dt}\right|_{t=t_0} \mathrm{Ad}_{g^{-1}(t)g(t_0)} \left(\mathrm{Ad}_{g^{-1}(t_0)}\, \mu\right) \\
&= -\mathrm{ad}_{g^{-1}(t_0)\dot{g}(t_0)} \left(\mathrm{Ad}_{g^{-1}(t_0)}\, \mu\right) \\
&= -\mathrm{ad}_{\xi(t_0)} \left(\mathrm{Ad}_{g^{-1}(t_0)}\, \mu\right).
\end{aligned}$$

Hence we can say,

$$\frac{d}{dt}\left(\mathrm{Ad}_{g^{-1}(t)}\, \mu(t)\right) = -\mathrm{ad}_{\xi(t)} \left(\mathrm{Ad}_{g^{-1}(t)}\, \mu(t)\right).$$

Now, we can move forward and consider the trace pairing of the conserved quantity and $\mu(t)$.

$$\begin{aligned}
\left\langle \frac{d}{dt}\left(\mathrm{Ad}^*_{g^{-1}(t)}\frac{\partial \ell}{\partial \xi}(t)\right), \mu(t)\right\rangle &= \frac{d}{dt}\left\langle \mathrm{Ad}^*_{g^{-1}(t)}\frac{\partial \ell}{\partial \xi}(t), \mu(t)\right\rangle \\
&= \frac{d}{dt}\left\langle \frac{\partial \ell}{\partial \xi}(t), \mathrm{Ad}_{g^{-1}(t)}\,\mu(t)\right\rangle \\
&= \left\langle \frac{d}{dt}\frac{\partial \ell}{\partial \xi}(t), \mathrm{Ad}_{g^{-1}(t)}\,\mu(t)\right\rangle + \left\langle \frac{\partial \ell}{\partial \xi}(t), \frac{d}{dt}\mathrm{Ad}_{g^{-1}(t)}\,\mu(t)\right\rangle \\
&= \left\langle \frac{d}{dt}\frac{\partial \ell}{\partial \xi}(t), \mathrm{Ad}_{g^{-1}(t)}\,\mu(t)\right\rangle + \left\langle \frac{\partial \ell}{\partial \xi}(t), -ad_{\xi(t)}(\mathrm{Ad}_{g^{-1}(t)}\,\mu(t))\right\rangle \\
&= \left\langle \frac{d}{dt}\frac{\partial \ell}{\partial \xi}(t), \mathrm{Ad}_{g^{-1}(t)}\,\mu(t)\right\rangle - \left\langle \frac{\partial \ell}{\partial \xi}(t), ad_{\xi(t)}(\mathrm{Ad}_{g^{-1}(t)}\,\mu(t))\right\rangle \\
&= \left\langle \frac{d}{dt}\frac{\partial \ell}{\partial \xi}(t), \mathrm{Ad}_{g^{-1}(t)}\,\mu(t)\right\rangle - \left\langle ad^*_{\xi(t)}\frac{\partial \ell}{\partial \xi}(t), \mathrm{Ad}_{g^{-1}(t)}\,\mu(t)\right\rangle \\
&= \left\langle \frac{d}{dt}\frac{\partial \ell}{\partial \xi}(t) - ad^*_{\xi(t)}\frac{\partial \ell}{\partial \xi}(t), \mathrm{Ad}_{g^{-1}(t)}\,\mu(t)\right\rangle \\
&= \left\langle \mathrm{Ad}^*_{g^{-1}(t)}\left[\frac{d}{dt}\frac{\partial \ell}{\partial \xi}(t) - ad^*_{\xi(t)}\frac{\partial \ell}{\partial \xi}(t)\right], \mu(t)\right\rangle.
\end{aligned}$$

Hence, we can say that,

$$\frac{d}{dt}\left(\mathrm{Ad}^*_{g^{-1}(t)}\frac{\partial \ell}{\partial \xi}(t)\right) = \mathrm{Ad}^*_{g^{-1}(t)}\underbrace{\left[\frac{d}{dt}\frac{\partial \ell}{\partial \xi}(t) - ad^*_{\xi(t)}\frac{\partial \ell}{\partial \xi}(t)\right]}_{\text{LHS of Euler-Poincaré Equations}}$$

and as we have a left-invariant system, we can use the left-invariant Euler-Poincaré equations to reduce the above derivative to zero, and hence Noether's Theorem for left-invariant systems follows from this. □

Now let us move forward with the derivation for right-invariant systems. A right-invariant Lagrangian can be defined as having the following property, $L(g, \dot{g}) = L(gh, \dot{g}h)$ for all $h \in G$. We then set $h = g^{-1}$ and get that $L(g, \dot{g}) = L(I, \dot{g}g^{-1})$ and so we let $\xi = \dot{g}g^{-1}$ and hence write our Lagrangian as $\ell(\xi)$. Considering Hamilton's Variational Principle,

$$\begin{aligned}
0 = \delta\int_{t_1}^{t_2} L(g, \dot{g})\,dt &= \delta\int_{t_1}^{t_2}\ell(\xi)\,dt \\
&= \int_{t_1}^{t_2}\left\langle \frac{\partial \ell}{\partial \xi}, \delta\xi\right\rangle dt.
\end{aligned}$$

Now we consider $\delta\xi = \delta(\dot{g}g^{-1})$,

$$\begin{aligned}
\delta(\dot{g}g^{-1}) &= \delta\dot{g}g^{-1} + \dot{g}\delta g^{-1} \\
&= \frac{d}{dt}(\delta g)g^{-1} - \dot{g}g^{-1}\delta g g^{-1} \\
&= \frac{d}{dt}(\delta g g^{-1}) - \delta g\frac{d}{dt}(g^{-1}) - \dot{g}g^{-1}\delta g g^{-1} \\
&= \frac{d}{dt}(\delta g g^{-1}) - \delta g g^{-1}\dot{g}g^{-1} - \dot{g}g^{-1}\delta g g^{-1} && \text{let } \nu = \delta g g^{-1} \\
&= \dot{\nu} + \nu\xi - \xi\nu \\
&= \dot{\nu} + [\nu, \xi] \\
&= \dot{\nu} + \mathrm{ad}_\nu\xi \\
&= \dot{\nu} - \mathrm{ad}_\xi\nu.
\end{aligned}$$

Hence, we now can move forward and complete the derivation of the right-invariant Euler-Poincaré equations.

$$\begin{aligned}\int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \xi}, \frac{d}{dt}\nu - \mathrm{ad}_\xi \nu \right\rangle dt &= \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \xi}, \frac{d}{dt}\nu \right\rangle - \left\langle \frac{\partial \ell}{\partial \xi}, \mathrm{ad}_\xi \nu \right\rangle dt \\ &= \int_{t_1}^{t_2} \left\langle -\frac{d}{dt}\frac{\partial \ell}{\partial \xi}, \nu \right\rangle - \left\langle \mathrm{ad}^*_\xi \frac{\partial \ell}{\partial \xi}, \nu \right\rangle dt \\ &= \int_{t_1}^{t_2} \left\langle -\frac{d}{dt}\frac{\partial \ell}{\partial \xi} - \mathrm{ad}^*_\xi \frac{\partial \ell}{\partial \xi}, \nu \right\rangle dt.\end{aligned}$$

So we can write down the Euler-Poincaré equations for the right-invariant system,

$$\frac{d}{dt}\frac{\partial \ell}{\partial \xi} + \mathrm{ad}^*_\xi \frac{\partial \ell}{\partial \xi} = 0.$$

We can restate Noether's Theorem as following,

**Theorem 4.4** (Noether's Theorem for right-invariant systems.)**.** The Euler-Poincaré equations associated to a right-invariant system preserve the generalised momentum along solutions of the Euler-Poincaré equations. That is,

$$\frac{d}{dt}\left(\mathrm{Ad}^*_{g(t)} \frac{\partial \ell}{\partial \xi}(t)\right) = 0.$$

*Proof.* This follows from a very similar argument to Theorem 4.3. By finding that $\frac{d}{dt}(\mathrm{Ad}_{g(t)}\,\mu) = \mathrm{ad}_\xi(\mathrm{Ad}_{g(t)}\,\mu)$ and applying this fact in an identical analysis of the trace pairings ending with $\frac{d}{dt}(\mathrm{Ad}^*_{g(t)}\frac{\partial \ell}{\partial \xi}) = \mathrm{Ad}^*_g(t)\left[\frac{d}{dt}\frac{\partial \ell}{\partial \xi} + \mathrm{ad}^*_\xi \frac{\partial \ell}{\partial \xi}\right]$ and then the result follows from the right-invariant version of the Euler-Poincaré equations. □

## 4.1 More Noether Theory

Now we have introduced the ideas of Euler-Poincaré reduction, we follow some advances made by Emmy Noether in her 1918 paper 'Invariante Variationsprobleme' [8]. This paper introduced the idea that whenever we have a symmetric Lagrangian there is some conserved current. That is, some value $f$, such that $\frac{d}{dt}f = 0$. If $\frac{d}{dt}f = 0$, then we can also say that $f$ is also a conserved quantity. We will favour the terminology of 'conserved quantity' to align more closely with modern literature and Holm, Schmah & Stoica (2009), Geometric Mechanics and Symmetry [2]. In this section we will take the main idea of Noether Theorems and introduce two main theorems, the first a conserved quantity (or Noether Theorem) for Euler-Lagrange equations and the second a conserved quantity for Euler-Poincaré Equations. These will be joined with a third theorem for Euler-Poincaré Reduction with Parameters in the next section.

### 4.1.1 Noethers Theorem for Euler-Lagrange equations

Consider a Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}})$ for $\mathbf{q} \in \mathbb{R}^3$ and $\dot{\mathbf{q}} \in T_{\mathbf{q}}\mathbb{R}^3$. Suppose that $L$ is left-invariant with respect to the tangent lift on SO(3), i.e. $R \in \mathrm{SO}(3)$ with $L(R\mathbf{q}\, R\dot{\mathbf{q}}) = L(\mathbf{q}, \dot{\mathbf{q}})$. Then we can prove that,

$$\mathcal{E} := \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} = 0.$$

These are the Euler-Lagrange equations we have met a few times in the past. We can use these to show a conserved quantity exists, that is, to prove the Noether Theorem for Euler-Lagrange equations.

**Theorem 4.5** (Noether Theorem for Euler-Lagrange Equations)**.** Corresponding to each one-parameter subgroup of SO(3) $R(s)$ where $R(0) = e$ and $R'(0) = \hat{\xi} \in \mathfrak{SO}(3)$. There is a conserved quantity,

$$A_\xi := \left\langle \mathbf{q} \times \frac{\partial L}{\partial \dot{\mathbf{q}}}, \xi \right\rangle$$

with $\frac{dA_\xi}{dt} = 0$ along solutions of the Euler-Lagrange equations, $\mathcal{E}(\mathbf{q}) = 0$.

*Proof.* Associated with the one-parameter subgroup $R(s)$ is the generator $\xi_{\mathcal{M}}(\mathbf{q}) := \frac{d}{ds}_{s=0} R(s)\mathbf{q} = \hat{\xi}\mathbf{q} = \xi \times \mathbf{q}$ where $\mathcal{M} = \mathbb{R}^3$. As the Lagrangian is symmetric we know that $L(R(s)\mathbf{q}, R(s)\dot{\mathbf{q}}) = L(\mathbf{q}, \dot{\mathbf{q}})$. Hence we can use Hamilton's Principle and consider,

$$\begin{aligned}
\delta \int_{t_1}^{t_2} L(R(s)\mathbf{q},\, R(s)q) &= \left[\int_{t_1}^{t_2} \left\langle \frac{\partial L}{\partial \mathbf{q}}(R(s)\mathbf{q}, R(s)\dot{\mathbf{q}})\,, R'(s)\mathbf{q} \right\rangle + \left\langle \frac{\partial L}{\partial \dot{\mathbf{q}}}(R(s)\mathbf{q}, R(s)\dot{\mathbf{q}})\,, R'(s)\dot{\mathbf{q}} \right\rangle dt\right]_{s=0} \\
&= \int_{t_1}^{t_2} \left\langle \frac{\partial L}{\partial \mathbf{q}}(R(0)\mathbf{q}, R(0)\dot{\mathbf{q}})\,, R'(0)\mathbf{q} \right\rangle + \left\langle \frac{\partial L}{\partial \dot{\mathbf{q}}}(R(0)\mathbf{q}, R(0)\dot{\mathbf{q}})\,, R'(0)\dot{\mathbf{q}} \right\rangle dt \\
&= \int_{t_1}^{t_2} \left\langle \frac{\partial L}{\partial \mathbf{q}}(\mathbf{q}, \dot{\mathbf{q}})\,, \xi_{\mathcal{M}}\mathbf{q} \right\rangle + \left\langle \frac{\partial L}{\partial \dot{\mathbf{q}}}(\mathbf{q}, \dot{\mathbf{q}})\,, \xi_{\mathcal{M}}\dot{\mathbf{q}} \right\rangle dt \\
&= \int_{t_1}^{t_2} \left\langle \frac{\partial L}{\partial \mathbf{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}}\,, \xi_{\mathcal{M}}\mathbf{q} \right\rangle + \left\langle \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}}\,, \xi_{\mathcal{M}}\mathbf{q} \right\rangle + \left\langle \frac{\partial L}{\partial \dot{\mathbf{q}}}\,, \xi_{\mathcal{M}}\dot{\mathbf{q}} \right\rangle dt \\
&= \int_{t_1}^{t_2} \langle \mathcal{E}(\mathbf{q})\,, \xi_{\mathcal{M}}\mathbf{q} \rangle + \frac{d}{dt}\left\langle \frac{\partial L}{\partial \dot{\mathbf{q}}}\,, \xi_{\mathcal{M}}\mathbf{q} \right\rangle dt \\
&= \int_{t_1}^{t_2} \frac{d}{dt}\left\langle \frac{\partial L}{\partial \dot{\mathbf{q}}}\,, \xi \times \mathbf{q} \right\rangle dt \\
&= \int_{t_1}^{t_2} \frac{d}{dt}\left\langle \frac{\partial L}{\partial \dot{\mathbf{q}}} \times \mathbf{q}\,, \xi \right\rangle dt = 0.
\end{aligned}$$

Then imposing end point conditions and considering this integral, we can say that $\frac{d}{dt}\left\langle \mathbf{q} \times \frac{\partial L}{\partial \dot{\mathbf{q}}}\,, \xi \right\rangle = 0$ and hence $\frac{dA_\xi}{dt} = 0$, as required. ☐

### 4.1.2 Noether Theory and Euler-Poincaré Reduction

In this subsection we will consider the conserved quantities for the Euler-Poincaré equations. We will first look at a half reduced version before formally defining what we mean by $\xi_{\mathcal{M}}$ and proving the final version of the Noether's Theorem for Euler-Poincaré reduction.

Suppose we have a Lagrangian $L$ that satisfies Hamilton's Variational Principle,

$$\int_{t_1}^{t_2} L(R,\, \dot{R})\, dt = 0$$

and $L$ is left-invariant and so, $L(SR,\, S\dot{R}) = L(R,\, \dot{R})$. Then we can state the half reduced Noether's Theorem.

**Theorem 4.6** (Half Reduced Noether's Theorem)**.** Corresponding to each one-parameter subgroup of SO(3), $S(s)$ with $S(0) = e$ and $S'(0) = \hat{\xi} \in \mathfrak{so}(3)$, then there is a conserved quantity,

$$A_\xi := \left\langle \mathrm{Ad}^*_{R^T} \frac{\partial \hat{\ell}}{\partial \hat{\boldsymbol{\Omega}}}\,, \hat{\xi} \right\rangle$$

with $\frac{dA_\xi}{dt} = 0$ along solutions of the Euler-Lagrange Equation.

$$\mathcal{E}(R) := \frac{d}{dt}\frac{\partial L}{\partial \dot{R}} - \frac{\partial L}{\partial R} = \mathbf{0}.$$

*Proof.* We will follow an almost identical argument to Theorem 4.5, until we reach the following step,

$$\int_{t_1}^{t_2} \frac{d}{dt}\left\langle \frac{\partial L}{\partial \dot{R}}\,, \hat{\xi}R \right\rangle dt = 0.$$

Then we note the following,

$$\int_{t_1}^{t_2} \frac{d}{dt}\left\langle \frac{\partial L}{\partial \dot{R}}, \hat{\xi} R \right\rangle dt = \int_{t_1}^{t_2} \frac{d}{dt}\left\langle R\frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}}R^{-1}, \hat{\xi} \right\rangle dt.$$

Further we note that $R\frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}}R^{-1} = \mathrm{Ad}_{R^T}\frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}}$, then we can say,

$$\int_{t_1}^{t_2} \frac{d}{dt}\left\langle R\frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}}R^{-1}, \hat{\xi} \right\rangle dt = \int_{t_1}^{t_2} \frac{d}{dt}\left\langle \mathrm{Ad}_{R^T}\frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}}, \hat{\xi} \right\rangle dt = 0.$$

Therefore,

$$\frac{d}{dt}\left\langle \mathrm{Ad}^*_{R^T}\frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}}, \hat{\xi} \right\rangle = 0,$$

as required. □

In order to consider the full reduction Noether's Theorem we will define what we mean by $\xi_{\mathcal{M}}$ more generally. This is what we will see to be the first variation of the infinitesimal generator,

**Definition 4.7** (Infinitesimal Generator)**.** Consider the left action of a Lie group $G$ on the manifold $\mathcal{M}$, $(g, \mathbf{x}) \to gx$ ($\mathbf{x} \in \mathcal{M}$). Let $\xi \in \mathfrak{g}$ be a vector in the Lie algebra of $G$ and consider a one-parameter subgroup

$$[\exp(t\xi) : t \in \mathbb{R}] \subseteq G.$$

Then the orbit of an element $\mathbf{x}$ with respect to this subgroup is a smooth map $t \to (\exp(t\xi))\mathbf{x}$ in $\mathcal{M}$. The infinitesimal generator associated to $\xi$ at $\mathbf{x} \in \mathcal{M}$ denoted by $\xi_{\mathcal{M}}(\mathbf{x})$ is the tangent vector (or velocity) to this curve at point $\mathbf{x}$,

$$\xi_{\mathcal{M}}(\mathbf{x}) = \left.\frac{d}{dt}\right|_{t=0} (\exp(t\xi)\mathbf{x}) \in T_{\mathbf{x}}\mathcal{M}.$$

This smooth vector field $\xi_{\mathcal{M}} : M \to TM$ and $x \mapsto \xi_{\mathcal{M}}(\mathbf{x})$ is called the infinitesimal generator vector field associated to $\xi$.

Let $G$ be an arbitrary matrix Lie group, and let $L$ be a left-invariant Lagrangian as defined above. The reduced system is $L(hg, h\dot{g})|_{h=g^{-1}} = L(e, g^{-1}\dot{g}) = \ell(\xi) = \ell(g^{-1}\dot{g})$. Now we can define the full reduction Noether's Theorem for the Euler-Poincaré equations.

**Theorem 4.8** (Full Reduction Noether's Theorem)**.** Corresponding to each one-parameter subgroup of $G$, $\chi(s)$ such that $\chi(0) = e$ and $\xi_s(0) = \eta \in \mathfrak{g}$. There is a conserved quantity,

$$\left\langle \mathrm{Ad}^*_{g^{-1}}\frac{\partial \ell}{\partial \xi}, \eta \right\rangle.$$

*Proof.* This follows a very similar, if not identical argument to Theorem 4.3. □

**Proposition 4.9.** The left-invariant Lagrangian $L(g, \dot{g})$ satisfies,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{g}} - \frac{\partial L}{\partial g} = \mathbf{0} \iff \frac{d}{dt}\frac{\partial \ell}{\partial \xi} - \mathrm{ad}^*\frac{\partial \ell}{\partial \xi} = 0.$$

*Proof.* The argument is as follows. In Chapter 1, we provided proof that Hamilton's Variational Principle is equivalent to the Euler-Lagrange equations, and at the start of Chapter 4 we provide a proof that Hamilton's Variational Principle with reduced Lagrangian is equivalent to the adjoint version of the Euler-Poincaré equations. The Hamilton's Variational Principle is equivalent to the version with reduced Lagrangian and so the two sets of equations are equivalent for a left-invariant Lagrangian. □

## 4.2 Diamond Map

We will quickly go back into pure mathematics, for the final time in this thesis, in order to define the diamond map. This new piece of mathematics is there in order to help us find a closed form for the Euler-Poincaré equations for systems with symmetry-breaking parameters.

Let $V$ be an $n$-dimensional vector space with dual $V^*$ and pairing $\langle \mathbf{w}\,, \mathbf{u} \rangle_V$ where $\mathbf{u} \in V$ and $\mathbf{w} \in V^*$. Let $\mathcal{M}(n,\, \mathbb{R})$ be a vector space of $n \times n$ matrices with dual $\mathcal{M}^*(n,\, \mathbb{R})$ and pairing $\langle B\,, A \rangle_{\mathcal{M}} := \mathrm{Tr}(B^T A)$ where $A \in \mathcal{M}(n,\, \mathbb{R})$ and $B \in \mathcal{M}^*(n,\, \mathbb{R})$.

The diamond map is a representation of the transformation of the pairing on $V$ to the pairing on $\mathcal{M}(n,\, \mathbb{R})$. Let $\mathbf{u} \in V$ and $\mathbf{w} \in V^*$ and consider the matrices $A \in \mathcal{M}(n,\, \mathbb{R})$ and $\mathbf{\Lambda} \in \mathcal{M}(n,\, \mathbb{R})$ where $A$ is a general matrix and $\mathbf{\Lambda}$ is a symmetric matrix ($\mathbf{\Lambda}^T = \mathbf{\Lambda}$). The diamond map is defined by $\langle \mathbf{w}\,, A\mathbf{\Lambda}\mathbf{u} \rangle_V = \langle \mathbf{u} \diamond \mathbf{w}\,, \mathbf{\Lambda} \rangle_{\mathcal{M}}$.

$$\begin{aligned}
\langle \mathbf{w}\,, A\mathbf{u} \rangle_V &= \mathrm{Tr}(\mathbf{u}\mathbf{w}^T A) \\
&= \mathrm{Tr}((\mathbf{w}\mathbf{u}^T)^T A) \\
&= \left\langle \mathbf{w}\mathbf{u}^T\,, A \right\rangle_{\mathcal{M}}.
\end{aligned}$$

This is for any matrix $A \in \mathcal{M}(n,\, \mathbb{R})$ and vectors $\mathbf{u} \in V$ and $\mathbf{w} \in V^*$. We now conclude that,

$$\begin{aligned}
\langle \mathbf{w}\,, A\mathbf{\Lambda}\mathbf{u} \rangle_V &= \mathrm{Tr}(\mathbf{u}\mathbf{w}^T A\mathbf{\Lambda}) \\
&= \left\langle (\mathbf{u}\mathbf{w}^T A)^T\,, \mathbf{\Lambda} \right\rangle_{\mathcal{M}} \\
&= \left\langle A^T\mathbf{w}\mathbf{u}^T\,, \mathbf{\Lambda} \right\rangle.
\end{aligned}$$

If $F = F^T$ and $G = -G^T$, then $\mathrm{Tr}(FG) = 0$. We see that $\mathbf{\Lambda}$ is symmetric so we consider the anti-symmetric part,

$$\begin{aligned}
&= \mathrm{Tr}\left(\mathbf{u}\mathbf{w}^T A\mathbf{\Lambda}\right) \\
&= \mathrm{Tr}\left(\frac{1}{2}(\mathbf{u}\mathbf{w}^T A + A^T\mathbf{w}\mathbf{u}^T)\mathbf{\Lambda}\right) && \text{we are spliitng this by it's symmetric part} \\
&= \mathrm{Tr}\left(\mathrm{Sym}\left(\mathbf{u}\mathbf{w}^T A\right)\mathbf{\Lambda}\right) \\
&= \left\langle \mathrm{Sym}\left(\mathbf{u}\mathbf{w}^T A\right)\,, \mathbf{\Lambda} \right\rangle.
\end{aligned}$$

We can say that $\mathbf{u} \diamond \mathbf{w} = \mathrm{Sym}\left(\mathbf{u}\mathbf{w}^T A\right) = \frac{1}{2}\left(\mathbf{u}\mathbf{w}^T A + A^T\mathbf{w}\mathbf{u}^T\right)$. This is going to appear in Euler-Poincaré theory in symmetry-breaking parameters.

## 4.3 Euler-Poincaré Reduction with Parameters

In this final section of Euler-Poincaré theory, we will now lift the assumption that all Lagrangians are symmetric and study what we should do if a Lagrangian isn't symmetric. It may seem that symmetry is everywhere in the world, but certain parameters may cause issues. The main one we consider in the applications section is gravity. Gravity only acts downwards, we model potential energy in a gravitational field using $-mg\mathbf{e}_3 \cdot R\mathbf{X}$. One can quickly verify that this doesn't hold any sort of symmetry or invariance. Therefore, we need to develop a theory to work around these restrictions. We start with an example that we have seen half of before in the previous chapter, but we now consider right symmetry, then we will consider gravity and end this section by considering the Noether Theorem for symmetry-breaking parameters.

We have described Lagrangians that have left or right invariance. We now look to Lagrangians that have symmetry-breaking parameters, like gravity. If we consider the spherical pendulum, we have defined $\hat{\mathbf{\Omega}} = R^T\dot{R}$ and $\hat{\boldsymbol{\omega}} = R\hat{\mathbf{\Omega}}$. Hence, $\hat{\boldsymbol{\omega}} = \dot{R}R^T = \dot{R}R^{-1}$ where $R \in \mathrm{SO}(3)$. This doesn't lead to a symmetric Lagrangian but we can still use our theory here.

We are going to study rigid body dynamics in the spatial frame. We look firstly to the Lagrangian. We have shown,

$$L(R,\, \dot{R}) = \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X}) \left\| \dot{R}\mathbf{X} \right\|^2 d^3\mathbf{X}$$

and we showed that $L(R,\, \dot{R}) = L(e,\, R^{-1}\dot{R})$. Then we used Euler-Poincaré Theory to show that $\ell = \frac{1}{2}\mathbb{I}\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}$. Now assume we would prefer to formulate rigid body dynamics in the spatial frame. We need to consider a $\boldsymbol{\omega}$ such that $\hat{\boldsymbol{\omega}} = \dot{R}R^{-1}$. We can now prove that, $L(R,\, \dot{R}) \neq L(R\chi,\, \dot{R}\chi)$ (right multiplication) hence we have broken symmetry,

$$\begin{aligned}
L(R\chi,\, \dot{R}\chi) &= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X}) \left\| \dot{R}\chi\mathbf{X} \right\|^2 d^3\mathbf{X} \\
&= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})(\dot{R}\chi\mathbf{X})\cdot(\dot{R}\chi\mathbf{X}) d^3\mathbf{X} \\
&= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})(\dot{R}\chi\mathbf{X})(\dot{R}\chi\mathbf{X})^T d^3\mathbf{X} \\
&= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})(\dot{R}\chi\mathbf{X})(\mathbf{X}^T\chi^T\dot{R}^T) d^3\mathbf{X} \\
&\neq \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X}) \left\| \dot{R}\mathbf{X} \right\|^2 d^3\mathbf{X} = L(R, \dot{R}).
\end{aligned}$$

Now we seek this Lagrangian,

$$\begin{aligned}
L(R,\, \dot{R}) &= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X}) \left\| \dot{R}\mathbf{X} \right\|^2 d^3\mathbf{X} \\
&= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X}) \left\| \dot{R}R^{-1}R\mathbf{X} \right\|^2 d^3\mathbf{X} \\
&= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X}) \left\| \hat{\boldsymbol{\omega}}R\mathbf{X} \right\|^2 d^3\mathbf{X} \\
&= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})(\hat{\boldsymbol{\omega}}\times R\mathbf{X})\cdot(\hat{\boldsymbol{\omega}}\times R\mathbf{X}) d^3\mathbf{X} \\
&= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})(\left\|\hat{\boldsymbol{\omega}}\right\|^2 \left\|R\mathbf{X}\right\|^2 - (\hat{\boldsymbol{\omega}}\cdot R\mathbf{X})^2) d^3\mathbf{X} \\
&= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})(\hat{\boldsymbol{\omega}}^T\hat{\boldsymbol{\omega}} \left\|R\mathbf{X}\right\|^2 - \hat{\boldsymbol{\omega}}^T(R\mathbf{X})(R\mathbf{X})^T\hat{\boldsymbol{\omega}}) d^3\mathbf{X} \\
&= \hat{\boldsymbol{\omega}}^T\frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})(R\left\|\mathbf{X}\right\|^2 R^T - R\mathbf{X}\mathbf{X}^T R^T d^3\mathbf{X}\,\hat{\boldsymbol{\omega}} \\
&= \hat{\boldsymbol{\omega}}^T R\frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})(\left\|\mathbf{X}\right\|^2 - \mathbf{X}\mathbf{X}^T d^3\mathbf{X}\, R^T\hat{\boldsymbol{\omega}} \\
&= \frac{1}{2}\hat{\boldsymbol{\omega}}^T(R\mathbb{I}R^T\hat{\boldsymbol{\omega}}) \\
&= \frac{1}{2}\hat{\boldsymbol{\omega}}\cdot(R\mathbb{I}R^T)\hat{\boldsymbol{\omega}} = L(\hat{\boldsymbol{\omega}},\, R).
\end{aligned}$$

We define a new parameter, $\mathbb{J} := R\mathbb{I}R^T$ and so $\ell = \ell(\mathbb{J},\, \hat{\boldsymbol{\omega}}) = \frac{1}{2}\hat{\boldsymbol{\omega}}(t)\cdot\mathbb{J}(t)\hat{\boldsymbol{\omega}}(t)$. Now we take variations as usual. We seek to find a closed form of $\delta\mathbb{J}(t)$,

$$\begin{aligned}\delta\mathbb{J}(t) &= \delta(R\mathbb{I}R^T)\\ &= \delta R\mathbb{I}R^T + R\mathbb{I}\delta R^T\\ &= \delta RR^{-1}R\mathbb{I}R^T - R\mathbb{I}R^{-1}\delta RR^{-1}\\ &= \hat{\mathbf{\Lambda}}\mathbb{J} - \mathbb{J}\hat{\mathbf{\Lambda}}\\ &= [\hat{\mathbf{\Lambda}},\, \mathbb{J}]\end{aligned}$$

and now we use this in the derivation,

$$\begin{aligned}0 &= \delta\int_{t_1}^{t_2}\ell(\mathbb{J},\, \hat{\boldsymbol{\omega}})\,dt\\ 0 &= \int_{t_1}^{t_2}\left\langle\frac{\partial\ell}{\partial\mathbb{J}}\,,\delta\mathbb{J}\right\rangle + \left\langle\frac{\partial\ell}{\partial\hat{\boldsymbol{\omega}}}\,,\delta\hat{\boldsymbol{\omega}}\right\rangle\,dt\\ 0 &= \int_{t_1}^{t_2}\left\langle\frac{\partial\ell}{\partial\mathbb{J}}\,,[\hat{\mathbf{\Lambda}},\mathbb{J}]\right\rangle + \left\langle\frac{\partial\ell}{\partial\hat{\boldsymbol{\omega}}}\,,\dot{\hat{\mathbf{\Lambda}}} + [\hat{\mathbf{\Lambda}},\hat{\boldsymbol{\omega}}]\right\rangle\,dt\\ 0 &= \int_{t_1}^{t_2}\left\langle\frac{\partial\ell}{\partial\mathbb{J}}\,,[\hat{\mathbf{\Lambda}},\mathbb{J}]\right\rangle + \left\langle\frac{\partial\ell}{\partial\hat{\boldsymbol{\omega}}}\,,\dot{\hat{\mathbf{\Lambda}}} + [\hat{\mathbf{\Lambda}},\hat{\boldsymbol{\omega}}]\right\rangle\,dt\\ 0 &= \int_{t_1}^{t_2}\left\langle-\mathrm{ad}^*_{\mathbb{J}}\frac{\partial\ell}{\partial\mathbb{J}}\,,\hat{\mathbf{\Lambda}}\right\rangle - \left\langle\frac{d}{dt}\frac{\partial\ell}{\partial\hat{\boldsymbol{\omega}}}\,,\hat{\mathbf{\Lambda}}\right\rangle - \left\langle\mathrm{ad}^*_{\hat{\boldsymbol{\omega}}}\frac{\partial\ell}{\partial\hat{\boldsymbol{\omega}}}\,,\hat{\mathbf{\Lambda}}\right\rangle\,dt\end{aligned}$$

where $\hat{\mathbf{\Lambda}} = \delta RR^{-1}$. This tells us that,

$$\frac{d}{dt}\frac{\partial\ell}{\partial\hat{\boldsymbol{\omega}}} = -\mathrm{ad}^*_{\mathbb{J}}\frac{\partial\ell}{\partial\mathbb{J}} - \mathrm{ad}^*_{\hat{\boldsymbol{\omega}}}\frac{\partial\ell}{\partial\hat{\boldsymbol{\omega}}}.$$

We proved before that $\mathrm{ad}^*_{\hat{\mathbf{\Omega}}}\mathbf{\Pi} = [\mathbf{\Pi}, \hat{\mathbf{\Omega}}]$ and so we can use this and find the equations linked to our Lagrangian. We can write the equation as this,

$$\frac{d}{dt}\frac{\partial\ell}{\partial\hat{\boldsymbol{\omega}}} = \left[\mathbb{J},\frac{\partial\ell}{\partial\mathbb{J}}\right] - \left[\frac{\partial\ell}{\partial\hat{\boldsymbol{\omega}}},\hat{\boldsymbol{\omega}}\right]. \tag{4.1}$$

We now seek to find the derivatives of a Lagrangian with respect to $\mathbb{J}$ and $\hat{\boldsymbol{\omega}}$. These can be found to be,

$$\frac{\partial\ell}{\partial\mathbb{J}} = \frac{1}{2}\hat{\boldsymbol{\omega}}\cdot\hat{\boldsymbol{\omega}} \qquad \frac{\partial\ell}{\partial\hat{\boldsymbol{\omega}}} = \hat{\boldsymbol{\omega}}\cdot\mathbb{J} + \frac{1}{2}\left(\frac{d\mathbb{J}}{d\hat{\boldsymbol{\omega}}}\hat{\boldsymbol{\omega}}\right)\cdot\hat{\boldsymbol{\omega}}.$$

Now we can consider the two terms of Equation 4.1 in turn. The details get slightly lengthy and so that's why we consider each term on its own. Firstly note that $\mathbb{J}^T = (R\mathbb{I}R^T)^T = R\mathbb{I}R^T = \mathbb{J}$ and then consider $\left[\mathbb{J}, \frac{\partial\ell}{\partial\mathbb{J}}\right]$,

$$\begin{aligned}\left[\mathbb{J},\frac{\partial\ell}{\partial\mathbb{J}}\right] &= \left[\mathbb{J},\frac{1}{2}\hat{\boldsymbol{\omega}}\cdot\hat{\boldsymbol{\omega}}\right]\\ &= \frac{1}{2}\left[\mathbb{J},\hat{\boldsymbol{\omega}}\cdot\hat{\boldsymbol{\omega}}\right]\\ &= \frac{1}{2}\left(\mathbb{J}\hat{\boldsymbol{\omega}}\cdot\hat{\boldsymbol{\omega}} - \mathbb{J}\hat{\boldsymbol{\omega}}\cdot\hat{\boldsymbol{\omega}}\mathbb{J}\right)\\ &= \frac{1}{2}\left(\mathbb{J}\hat{\boldsymbol{\omega}}^T\hat{\boldsymbol{\omega}} - \hat{\boldsymbol{\omega}}^T\hat{\boldsymbol{\omega}}\mathbb{J}\right)\\ &= \frac{1}{2}\left((\hat{\boldsymbol{\omega}}\mathbb{J})^T\hat{\boldsymbol{\omega}} - \hat{\boldsymbol{\omega}}^T\hat{\boldsymbol{\omega}}\mathbb{J}\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left((\hat{\boldsymbol{\omega}}\mathbb{J})\cdot\hat{\boldsymbol{\omega}} - \hat{\boldsymbol{\omega}}^T\hat{\boldsymbol{\omega}}\mathbb{J}\right) \\
&= \frac{1}{2}\left(\hat{\boldsymbol{\omega}}\cdot(\hat{\boldsymbol{\omega}}\mathbb{J}) - \hat{\boldsymbol{\omega}}^T\hat{\boldsymbol{\omega}}\mathbb{J}\right) \\
&= \frac{1}{2}\left(\hat{\boldsymbol{\omega}}^T\hat{\boldsymbol{\omega}}\mathbb{J} - \hat{\boldsymbol{\omega}}^T\hat{\boldsymbol{\omega}}\mathbb{J}\right) = 0.
\end{aligned}
$$

Therefore, we can remove the first term from Equation 4.1. Now we consider the second term,

$$
\left[\hat{\boldsymbol{\omega}}\cdot\mathbb{J} + \frac{1}{2}\left(\frac{\partial\mathbb{J}}{\partial\hat{\boldsymbol{\omega}}}\hat{\boldsymbol{\omega}}\right)\cdot\hat{\boldsymbol{\omega}}, \hat{\boldsymbol{\omega}}\right] = [\hat{\boldsymbol{\omega}}\cdot\mathbb{J}, \hat{\boldsymbol{\omega}}] + \frac{1}{2}\left[\left(\frac{\partial\mathbb{J}}{\partial\hat{\boldsymbol{\omega}}}\hat{\boldsymbol{\omega}}\right)\cdot\hat{\boldsymbol{\omega}}, \hat{\boldsymbol{\omega}}\right]. \tag{4.2}
$$

We can use a similar argument to the first term, along with the fact that $\hat{\boldsymbol{\omega}}$ is symmetric (that is, $\hat{\boldsymbol{\omega}}^T = -\hat{\boldsymbol{\omega}}$), to produce that $[\hat{\boldsymbol{\omega}}\cdot\mathbb{J}, \hat{\boldsymbol{\omega}}] = 2\mathbb{J}\hat{\boldsymbol{\omega}}\hat{\boldsymbol{\omega}}$. Then some further manipulation yields that $-2\mathbb{J}\hat{\boldsymbol{\omega}}\hat{\boldsymbol{\omega}} = 2\mathbb{J}\cdot(\hat{\boldsymbol{\omega}}\times\boldsymbol{\omega})$. For the second term of the right hand side of Equation 4.2, the major simplification we can make is to find $\frac{\partial\mathbb{J}}{\partial\hat{\boldsymbol{\omega}}}$. We can use the following, because $\mathbb{J}$ is at least twice differentiable,

$$
\frac{\partial}{\partial\hat{\boldsymbol{\omega}}}\frac{\partial\mathbb{J}}{\partial t} = \frac{\partial}{\partial t}\frac{\partial\mathbb{J}}{\partial\hat{\boldsymbol{\omega}}} \tag{4.3}
$$

and now we aim to find $\frac{d\mathbb{J}}{dt}$. This can be found from the following argument,

$$
\begin{aligned}
\frac{d\mathbb{J}}{dt} &= \frac{d}{dt}(R\mathbb{I}R^T) \\
&= \dot{R}\mathbb{I}R^{-1} + R\mathbb{I}R^{-1}\dot{R}R^{-1} \\
&= \dot{R}R^{-1}R\mathbb{I}R^{-1} + R\mathbb{I}R^{-1}\dot{R}R^{-1} \\
&= \hat{\boldsymbol{\omega}}R\mathbb{I}R^{-1} + R\mathbb{I}R^{-1}\hat{\boldsymbol{\omega}} \\
&= \hat{\boldsymbol{\omega}}\mathbb{J} - \mathbb{J}\hat{\boldsymbol{\omega}} = [\hat{\boldsymbol{\omega}}, \mathbb{J}].
\end{aligned}
$$

Now we can substitute this into Equation 4.3 and consider the derivative,

$$
\frac{\partial}{\partial\hat{\boldsymbol{\omega}}}[\hat{\boldsymbol{\omega}}, \mathbb{J}] = \frac{\partial}{\partial\hat{\boldsymbol{\omega}}}(\hat{\boldsymbol{\omega}}\mathbb{J} - \mathbb{J}\hat{\boldsymbol{\omega}}) = \left(\mathbb{J} + \hat{\boldsymbol{\omega}}\frac{\partial\mathbb{J}}{\partial\hat{\boldsymbol{\omega}}}\right) - \left(\frac{\partial\mathbb{J}}{\partial\hat{\boldsymbol{\omega}}}\hat{\boldsymbol{\omega}} + \mathbb{J}\right) = 0.
$$

Therefore we have the following PDE,

$$
\frac{\partial\mathbb{J}}{\partial t\partial\hat{\boldsymbol{\omega}}} = 0.
$$

This implies $\frac{\partial\mathbb{J}}{\partial\hat{\boldsymbol{\omega}}} = K$, where $K$ is a constant. This is true as both $\hat{\boldsymbol{\omega}}$ and $\mathbb{J}$ are functions of $t$ and so in essence we really have an ODE. Therefore, we can now consider the second term in Equation 4.2,

$$
\frac{1}{2}\left[\left(\frac{\partial\mathbb{J}}{\partial\hat{\boldsymbol{\omega}}}\hat{\boldsymbol{\omega}}\right)\cdot\hat{\boldsymbol{\omega}}, \hat{\boldsymbol{\omega}}\right] = \frac{K}{2}[\hat{\boldsymbol{\omega}}\cdot\hat{\boldsymbol{\omega}}, \hat{\boldsymbol{\omega}}] = 0.
$$

Therefore, we can write our equation as this,

$$
\frac{d}{dt}\left(\hat{\boldsymbol{\omega}}\cdot\mathbb{J} + \frac{1}{2}\left(\frac{d\mathbb{J}}{d\hat{\boldsymbol{\omega}}}\hat{\boldsymbol{\omega}}\right)\cdot\hat{\boldsymbol{\omega}}\right) = 2\mathbb{J}\cdot(\hat{\boldsymbol{\omega}}\times\boldsymbol{\omega}).
$$

Then expanding and neglecting the terms that go to zero, we get the following,

$$
\left(\mathbb{J} + \frac{\partial\mathbb{J}}{\partial\hat{\boldsymbol{\omega}}}\right)\cdot\dot{\hat{\boldsymbol{\omega}}} + (\dot{\mathbb{J}} + \mathbb{J}\times\boldsymbol{\omega})\cdot\hat{\boldsymbol{\omega}} = 0.
$$

### 4.3.1 Symmetry-Breaking Parameters and the Diamond Map

We have now derived a system of equations for a symmetry-broken Lagrangian. We will now write more formally what we have just done and then consider gravity. We then will have an equation that relates to the diamond map we introduced in the last section. Consider a Lie group, $G$, and a left action on a manifold, $\mathcal{M}$. Then for a given $a_0 \in \mathcal{M}$ (a parameter), let $L : TG \times \mathcal{M} \to \mathbb{R}$ be a Lagrangian with the symmetry-breaking parameter $a_0$. Suppose it is invariant under the left action: $G \times (TG \times \mathcal{M}) \to TG \times \mathcal{M}$ then $(h, (g, \dot{g}, a_0)) \to (hg, h\dot{g}, ha_0)$ for all $h \in G$. This means that $L(hg, h\dot{g}, a_0) = L(g, \dot{g}, a_0)$ for all $h \in G$. As usual let $h = g^{-1}$, then $L(g, \dot{g}, a_0) = L(g^{-1}g, g^{-1}\dot{g}, g^{-1}a_0) =: \ell(\xi, a)$ where $\xi := g^{-1}\dot{g}$ and $a = g^{-1}a_0$. From this the following Theorem arises.

**Theorem 4.10.** The following are equivalent,

(i) Hamilton's Variational Principle
$$\delta \int_{t_1}^{t_2} L(g, \dot{g}, a_0)\, dt = 0$$
with $\delta g(t_1) = \delta g(t_2) = 0$.

(ii) $g(t)$ satisfies the Euler-Lagrange equations associated with $L(g, \dot{g}, a_0)$.

(iii) The reduced variational principle (or Hamilton's principle),
$$\delta \int_{t_1}^{t_2} \ell(\xi, a)\, dt = 0$$
holds on $\mathfrak{g} \times \mathcal{M}$, using variations $\delta\xi = \dot{\eta} + \mathrm{ad}_\xi \eta$ and $\delta a = -\eta_{\mathcal{M}}(a)$ with free variations $\eta(t)$ satisfying end point conditions.

(iv) The Euler-Poincaré equations
$$\begin{aligned} \frac{d}{dt}\frac{\partial \ell}{\partial \xi} &= \mathrm{ad}^*_\xi \frac{\partial \ell}{\partial \xi} - a \diamond \frac{\partial \ell}{\partial a} \\ \dot{a} &= -\xi_{\mathcal{M}} a \end{aligned}$$
hold on $\mathfrak{g} \times \mathcal{M}$ where $\left\langle \frac{\partial \ell}{\partial a}, \alpha_{\mathcal{M}} a \right\rangle =: \left\langle a \diamond \frac{\partial \ell}{\partial a}, \alpha \right\rangle$ for all $\alpha \in \mathfrak{g}$ and for all $a \in \mathcal{M}$.

*Proof.* We already know that $\delta\xi = \dot{\eta} + [\xi, \eta]$ and $\eta = g^{-1}\delta g$ and then $\delta a = -g^{-1}\delta g g^{-1} a = -\eta_{\mathcal{M}} a = -\eta a$. Now we look at our variational principle,
$$\begin{aligned} 0 &= \delta \int_{t_1}^{t_2} \ell(\xi, a)\, dt \\ &= \int_{t_1}^{t_2} \left( \left\langle \frac{\partial \ell}{\partial \xi}, \delta\xi \right\rangle + \left\langle \frac{\partial \ell}{\partial a}, \delta a \right\rangle \right) dt \\ &= \int_{t_1}^{t_2} \left( \left\langle \frac{\partial \ell}{\partial \xi}, \dot{\eta} + \mathrm{ad}_\xi \eta \right\rangle - \left\langle \frac{\partial \ell}{\partial a}, \eta_{\mathcal{M}} a \right\rangle \right) dt \\ &= \int_{t_1}^{t_2} \left\langle -\frac{d}{dt}\frac{\partial \ell}{\partial \xi} + \mathrm{ad}^*_\xi \frac{\partial \ell}{\partial \xi}, \eta \right\rangle - \left\langle a \diamond \frac{\partial \ell}{\partial a}, \eta \right\rangle dt. \end{aligned}$$

Now we want the second equation,
$$\begin{aligned} \dot{a} &= \frac{d}{dt}(g^{-1}a_0) \\ &= \frac{d}{dt} g^{-1} a_0 \\ &= -g^{-1}\dot{g}g^{-1}a_0 \\ &= -\xi_{\mathcal{M}} a. \end{aligned}$$

Therefore, our set of equations are,

$$\frac{d}{dt}\frac{\partial \ell}{\partial \xi} = \mathrm{ad}_{\xi}^{*}\frac{\partial \ell}{\partial \xi} - a \diamond \frac{\partial \ell}{\partial a} \tag{4.4}$$

$$\frac{da}{dt} = -\xi_{\mathcal{M}} a \tag{4.5}$$

and the proof is complete. □

Now we look at Noether's Theorems.

**Theorem 4.11** (Noether's Theorem for Symmetry-Breaking Parameters (Left-Symmetry))**.** Let $\xi = g^{-1}\dot{g}$ be a solution of the Euler-Poincaré Equations with parameters $a = g^{-1}a_0$. Then,

$$\frac{d}{dt}\,\mathrm{Ad}^{*}_{g^{-1}(t)}\,\mu = -\,\mathrm{Ad}^{*}_{g^{-1}(t)}\left(a \diamond \frac{\partial \ell}{\partial a}\right)$$

and $\mu(t) = \frac{\partial \ell}{\partial \xi} \in \mathfrak{g}^*$.

*Proof.* We will prove this with a argument that relies heavily on the trace pairing. We will consider the following trace pairing and work from there, where $\nu(t) \in \mathfrak{g}$

$$\begin{aligned}
\left\langle \mathrm{Ad}^{*}_{g^{-1}}\frac{\partial \ell}{\partial \xi}\,,\nu(t)\right\rangle &= \frac{d}{dt}\left\langle \mathrm{Ad}^{*}_{g^{-1}}\frac{\partial \ell}{\partial \xi}\,,\nu(t)\right\rangle \\
&= \frac{d}{dt}\left\langle \frac{\partial \ell}{\partial \xi}\,,\mathrm{Ad}_{g^{-1}}\,\nu(t)\right\rangle \\
&= \left\langle \frac{d}{dt}\frac{\partial \ell}{\partial \xi}\,,\mathrm{Ad}_{g^{-1}}\,\nu(t)\right\rangle + \left\langle \frac{\partial \ell}{\partial \xi}\,,\frac{d}{dt}\,\mathrm{Ad}_{g^{-1}}\,\nu(t)\right\rangle \\
&= \left\langle \frac{d}{dt}\frac{\partial \ell}{\partial \xi}\,,\mathrm{Ad}_{g^{-1}}\,\nu(t)\right\rangle + \left\langle \frac{\partial \ell}{\partial \xi}\,,-\mathrm{ad}^{*}_{\xi}\,\mathrm{Ad}_{g^{-1}}\,\nu(t)\right\rangle \\
&= \left\langle \frac{d}{dt}\frac{\partial \ell}{\partial \xi}\,,\mathrm{Ad}_{g^{-1}}\,\nu(t)\right\rangle - \left\langle \mathrm{ad}_{\xi}\frac{\partial \ell}{\partial \xi}\,,\mathrm{Ad}_{g^{-1}}\,\nu(t)\right\rangle \\
&= \left\langle \frac{d}{dt}\frac{\partial \ell}{\partial \xi} - \mathrm{ad}_{\xi}\frac{\partial \ell}{\partial \xi}\,,\mathrm{Ad}_{g^{-1}}\,\nu(t)\right\rangle - \left\langle a \diamond \frac{\partial \ell}{\partial a} - a \diamond \frac{\partial \ell}{\partial a}\,,\mathrm{Ad}_{g^{-1}}\,\nu(t)\right\rangle \\
&= \left\langle \frac{d}{dt}\frac{\partial \ell}{\partial \xi} - \mathrm{ad}_{\xi}\frac{\partial \ell}{\partial \xi} + a \diamond \frac{\partial \ell}{\partial a}\,,\mathrm{Ad}_{g^{-1}}\,\nu(t)\right\rangle - \left\langle a \diamond \frac{\partial \ell}{\partial a}\,,\mathrm{Ad}_{g^{-1}}\,\nu(t)\right\rangle \\
&= -\left\langle a \diamond \frac{\partial \ell}{\partial a}\,,\mathrm{Ad}_{g^{-1}}\,\nu(t)\right\rangle \\
&= \left\langle -\,\mathrm{Ad}^{*}_{g^{-1}}\left(a \diamond \frac{\partial \ell}{\partial a}\right)\,,\nu(t)\right\rangle
\end{aligned}$$

The result follows from this and so,

$$\mathrm{Ad}^{*}_{g^{-1}(t)}\frac{\partial \ell}{\partial \xi} = -\,\mathrm{Ad}^{*}_{g^{-1}(t)}\left(a \diamond \frac{\partial \ell}{\partial a}\right)$$

is the conserved quantity, as required. □

We can prove an equivalent theorem to Theorem 4.10 for right-invariant systems,

**Theorem 4.12.** The following are equivalent,

(i) Hamilton's Variational Principle

$$\delta \int_{t_1}^{t_2} L(g, \dot{g}, a_0)\, dt = 0$$

with $\delta g(t_1) = \delta g(t_2) = 0$.

(ii) $g(t)$ satisfies the Euler-Lagrange equations associated with $L(g, \dot{g}, a_0)$

(iii) The reduced variational principle (or Hamiltons principle),

$$\delta \int_{t_1}^{t_2} \ell(\xi, a)\, dt = 0$$

holds on $\mathfrak{g} \times \mathcal{M}$, using variations $\delta\xi = \dot{\eta} - \mathrm{ad}_\xi \eta$ and $\delta a = -\eta_{\mathcal{M}}(a)$ with free variations $\eta(t)$ satisfying end point conditions.

(iv) The Euler-Poincare equations

$$\begin{aligned} \frac{d}{dt}\frac{\partial \ell}{\partial \xi} &= -\mathrm{ad}^*_\xi \frac{\partial \ell}{\partial \xi} - a \diamond \frac{\partial \ell}{\partial a} \\ \dot{a} &= -\xi_{\mathcal{M}} a \end{aligned}$$

hold on $\mathfrak{g} \times \mathcal{M}$ where $\left\langle \frac{\partial \ell}{\partial a}\, , \alpha_{\mathcal{M}} a \right\rangle =: \left\langle a \diamond \frac{\partial \ell}{\partial a}\, , \alpha \right\rangle$ for all $\alpha \in \mathfrak{g}$ and for all $a \in \mathcal{M}$.

*Proof.* We consider a reduced Lagrangian and work from there, we will see that the proof is almost identical to Theorem 4.10, apart from a minus sign,

$$\begin{aligned} 0 &= \delta \int_{t_1}^{t_2} \ell(\xi, a)\, dt \\ &= \int_{t_1}^{t_2} \left( \left\langle \frac{\partial \ell}{\partial \xi}\, , \delta\xi \right\rangle + \left\langle \frac{\partial \ell}{\partial a}\, , \delta a \right\rangle \right) dt \\ &= \int_{t_1}^{t_2} \left( \left\langle \frac{\partial \ell}{\partial \xi}\, , \dot{\eta} - \mathrm{ad}_\xi \eta \right\rangle - \left\langle \frac{\partial \ell}{\partial a}\, , \eta_{\mathcal{M}} a \right\rangle \right) dt \\ &= \int_{t_1}^{t_2} \left\langle -\frac{d}{dt}\frac{\partial \ell}{\partial \xi} - \mathrm{ad}^*_\xi \frac{\partial \ell}{\partial \xi}\, , \eta \right\rangle - \left\langle a \diamond \frac{\partial \ell}{\partial a}\, , \eta \right\rangle dt. \end{aligned}$$

Now we want the second equation,

$$\begin{aligned} \dot{a} &= \frac{d}{dt}(g^{-1} a_0) \\ &= \frac{d}{dt} g^{-1} a_0 \\ &= -g^{-1}\dot{g}g^{-1} a_0 \\ &= -\xi_{\mathcal{M}} a \end{aligned}$$

Therefore our set of equations are,

$$\frac{d}{dt}\frac{\partial \ell}{\partial \xi} = -\mathrm{ad}^*_\xi \frac{\partial \ell}{\partial \xi} - a \diamond \frac{\partial \ell}{\partial a} \tag{4.6}$$

$$\frac{da}{dt} = -\xi_{\mathcal{M}} a \tag{4.7}$$

and the proof is complete. □

We now prove the right-invariant Noether's Theorem, which is very similar, but differs by an inverse sign,

**Theorem 4.13** (Noether's Theorem for Symmetry Breaking Parameters (Right-Symmetry))**.** Let $\xi = g^{-1}\dot{g}$ be a solution of the Euler-Poincaré Equations with parameters $a = g^{-1}a_0$. Then,

$$\frac{d}{dt}Ad^*_{g(t)}\mu = -\operatorname{Ad}^*_{g(t)}\left(a \diamond \frac{\partial \ell}{\partial a}\right)$$

*Proof.* As will all left and right Noether's Theorems we have seen, this theorem uses a similar argument to the left symmetric version but we use the fact that $\frac{d}{dt}\operatorname{Ad}_g \mu = \operatorname{ad}^*_\xi \operatorname{Ad}_g \mu$ and then run through the steps to get the equation above. $\square$

# 5 Applications of Geometric Mechanics

In this final section of the thesis, we will work towards using the mathematics we have defined in the previous chapters to describe real world situations. We will focus on three examples, the Spherical Pendulum, the Heavy Top and Pseudo Rigid Bodies. These will follow the mathematics in the order it was introduced in the text. For example, the Spherical Pendulum is the simplest of the examples as we can describe it with just the content from Chapter 1, although we will then describe it using more sophisticated means. The Heavy Top is a classic example of how to use Euler-Poincaré reduction with symmetry breaking parameters and Pseudo Rigid Bodies expands on the work we have thus far by considering the implications of bipolar decomposition on the body coordinate.

## 5.1 Spherical Pendulum

The Spherical Pendulum is a canonical example of where Geometric Mechanics is useful. The description of a pendulum in a plane is a usual exercise in undergraduate dynamics courses. However, the Spherical Pendulum is a lot harder to describe and to describe it simply you need either the Euler-Lagrange or Euler-Poincaré equations. The simplicity of this example does not mean it is uninteresting as, with a slight modification of putting the bob on a spring, this example can be used to model climate dynamics. We will first derive two sets of Euler-Lagrange Equations before deriving some Euler-Poincaré Equations and using the associated Noether's Theorem to derive a conserved quantity, which we will see is very nice.

### 5.1.1 Euler-Lagrange Equations

We want to consider a pendulum in 3D space. We will think about this through the definition of spherical coordinates, as in actuality the motion of the bob will just be on $S^2$. However, to describe the motion more precisely, we need to derive the Euler-Lagrange equations. [1]

Firstly, here is what the Euler-Lagrange equations are,

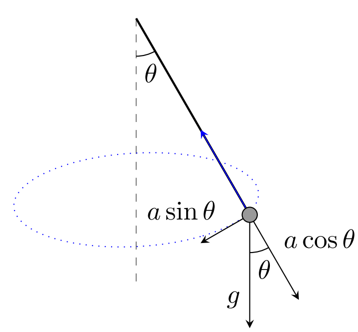
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} - \frac{\partial L}{\partial q^a}$$

where we range through the different basis vectors $q^a$ and their associated derivatives $\dot{q}^a$.

We also define $L$ as the Lagrangian. We define this simply as,

$$L(q,\dot{q}) = T(q,\dot{q}) - V(\mathbf{r}(q))$$

where we further define $T(q,\dot{q})$ as the kinetic energy of the system and $V(\mathbf{r}(q))$ as the potential energy of the system.




We are going to use polar coordinates to derive our system of equations.

$$\begin{aligned} x &= R\sin\theta\cos\phi \\ y &= R\sin\theta\sin\phi \\ z &= R(1-\cos\theta). \end{aligned}$$

Our first focus is to derive $T(q,\dot{q})$, which will be $\frac{1}{2}mv^2$. We can see that $v = |\dot{\mathbf{r}}(t)|$ and so $v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$. Hence, we now find what $v$ is and then find the Lagrangian. Firstly, we note that,

$$\frac{d}{dt}(x(t)) = \frac{\partial x}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial x}{\partial \phi}\frac{d\phi}{dt}$$

and similarly for $y(t)$ and $z(t)$. Hence,

$$\begin{aligned}\frac{dx(t)}{dt} &= R\cos\theta\cos\phi\,\dot\theta - R\sin\theta\sin\phi\,\dot\phi\\ \frac{dy(t)}{dt} &= R\cos\theta\sin\phi\dot\theta + R\sin\theta\cos\phi\dot\phi\\ \frac{dz(t)}{dt} &= R\sin\theta\dot\theta.\end{aligned}$$

Now we derive our $T(q,\dot q)$,

$$\begin{aligned}T(q,\dot q) = \frac{1}{2}m\Bigg(&\Big(R\cos\theta\cos\phi\,\dot\theta - R\sin\theta\sin\phi\,\dot\phi\Big)^2 +\\ &\Big(R\cos\theta\sin\phi\dot\theta + R\sin\theta\cos\phi\dot\phi\Big)^2 +\\ &\Big(R\sin\theta\dot\theta\Big)^2\Bigg).\end{aligned}$$

This can be nicely simplified,

$$T(\theta,\dot\theta,\phi,\dot\phi) = \frac{mR^2}{2}\left(\dot\theta^2 + \dot\phi^2\sin^2\theta\right).$$

We note that our system has only one potential energy, gravitation potential. Therefore,

$$V(\theta,\dot\theta,\phi,\dot\phi) = -mgz = -mgR(1-\cos\theta).$$

Hence, we can now talk about Lagrangian explicitly,

$$L(\theta,\dot\theta,\phi,\dot\phi) = \frac{mR^2}{2}\left(\dot\theta^2 + \dot\phi^2\sin^2\theta\right) + mgR\left(1-\cos\theta\right).$$

Finally, we can now take derivatives of this function and produce the Euler-Lagrange equations. We have two basis vectors, $\theta$ and $\phi$. We need to differentiate the Lagrangian with respect to each of these. Firstly, $\theta$

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} - \frac{\partial L}{\partial\theta} = \frac{d}{dt}mR^2\dot\theta - \left(\dot\phi^2 mR^2\sin\theta\cos\theta - mgR\sin\theta\right) &= 0\\ mR^2\ddot\theta - mR^2\sin\theta\cos\theta\,\dot\phi^2 + mgR\sin\theta &= 0\\ R\ddot\theta - R\sin\theta\cos\theta\,\dot\phi^2 + g\sin\theta &= 0\end{aligned}$$

and secondly, $\phi$,

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial\dot\phi} - \frac{\partial L}{\partial\phi} = \frac{d}{dt}(mR^2\dot\phi\sin^2\theta) &= 0\\ mR^2\ddot\phi\sin^2\theta + 2mR^2\dot\phi\,\dot\theta\sin\theta\cos\theta &= 0\\ \ddot\phi\sin\theta + 2\dot\phi\,\dot\theta\cos\theta &= 0.\end{aligned}$$

We have now derived the Euler-Lagrange equations for the Spherical Pendulum, which are

$$\begin{cases} R\ddot\theta - R\sin\theta\cos\theta\,\dot\phi^2 + g\sin\theta = 0\\ \ddot\phi\sin\theta + 2\dot\phi\,\dot\theta\cos\theta = 0.\end{cases}$$

### 5.1.2 Euler-Lagrange Equations II

We will now consider a second Euler-Lagrange treatment of the Spherical Pendulum. The Spherical Pendulum is just a particle that moves along the surface of a sphere. We will consider two coordinates, the body coordinates $\mathbf{x}(t)$ and the spatial coordinates $\mathbf{X}$, where $\mathbf{x}(t) = R(t)\mathbf{X}$ and we let $R \in \mathrm{SO}(3)$. In terms of the body coordinates, we can write the Lagrangian as,

$$L(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2}m\left|\dot{\mathbf{x}}\right|^2 - mg\mathbf{e}_3 \cdot \mathbf{x}$$

and then using the body-to-space map we can rewrite it as,

$$L\left(R, \dot{R}\right) = \frac{1}{2}m\left|\dot{R}\mathbf{X}\right|^2 - mg\mathbf{e}_3 \cdot R\mathbf{X}.$$

This is similar to the Lagrangian we will see in the next example, but we aren't integrating over the body. We need to check whether this Lagrangian is invariant under symmetry. We can notice that the second term won't be, as $R\mathbf{X} \neq gR\mathbf{X}$. This is in contrast to the first part of the Lagrangian, which is symmetric. Let $g \in \mathrm{SO}(3)$,

$$\begin{aligned}
\frac{1}{2}m\left|g\dot{R}\mathbf{X}\right|^2 &= \frac{1}{2}m\left(g\dot{R}\mathbf{X} \cdot g\dot{R}\mathbf{X}\right) \\
&= \frac{1}{2}m\left(g\dot{R}\mathbf{X}\left(g\dot{R}\mathbf{X}\right)^T\right) \\
&= \frac{1}{2}m\left(gg^T\dot{R}\mathbf{X}\left(\dot{R}\mathbf{X}\right)^T\right) \\
&= \frac{1}{2}m\left(\dot{R}\mathbf{X}\left(\dot{R}\mathbf{X}\right)^T\right) = \frac{1}{2}m\left|\dot{R}\mathbf{X}\right|^2.
\end{aligned}$$

We call $-mg\mathbf{e}_3 \cdot R\mathbf{X}$ a symmetry breaking parameter and we proceed by introducing a new parameter $\mathbf{\Gamma} = \mathbf{e}_3 R^{-1}$. Then the Lagrangian becomes,

$$L(\dot{R}, \mathbf{\Gamma}) = \frac{1}{2}m\left|\dot{R}\mathbf{X}\right|^2 - mg\mathbf{\Gamma}(t) \cdot \mathbf{X}.$$

Now we can start using Hamilton's Variational Principle to derive the Euler-Lagrange equations for our system. We firstly consider $\delta\mathbf{\Gamma}$, which we can see is,

$$\delta\mathbf{\Gamma} = \delta R^{-1}\mathbf{e}_3 = R^{-1}\delta R R^{-1}\mathbf{e}_3 = \hat{\mathbf{\Lambda}}\mathbf{\Gamma} = \mathbf{\Lambda} \times \mathbf{\Gamma}$$

where we define $\mathbf{\Lambda} = R^{-1}\delta R$. Now we can carry on with the derivation,

$$\begin{aligned}
0 &= \delta\int_{t_1}^{t_2} L(\dot{R}, \mathbf{\Gamma})dt \\
&= \int_{t_1}^{t_2} \left\langle \frac{\partial L}{\partial \dot{R}}, \delta\dot{R} \right\rangle + \left\langle \frac{\partial L}{\partial \mathbf{\Gamma}}, \delta\mathbf{\Gamma} \right\rangle dt \\
&= \int_{t_1}^{t_2} \left\langle -\frac{d}{dt}\frac{\partial L}{\partial \dot{R}}, \delta R \right\rangle + \left\langle \frac{\partial L}{\partial \mathbf{\Gamma}}, \mathbf{\Lambda} \times \mathbf{\Gamma} \right\rangle dt \\
&= \int_{t_1}^{t_2} \left\langle -\frac{d}{dt}\frac{\partial L}{\partial \dot{R}}, + \right\rangle \left\langle -\frac{\partial L}{\partial \mathbf{\Gamma}} \times \mathbf{\Gamma}, \mathbf{\Lambda} \right\rangle dt \\
&= \int_{t_1}^{t_2} \left\langle -\frac{d}{dt}\frac{\partial L}{\partial \dot{R}}, \delta R \right\rangle + \left\langle \mathbf{\Gamma} \times \frac{\partial L}{\partial \mathbf{\Gamma}}, R^{-1}\delta R \right\rangle dt \\
&= \int_{t_1}^{t_2} \left\langle -\frac{d}{dt}\frac{\partial L}{\partial \dot{R}}, \delta R \right\rangle + \left\langle R\mathbf{\Gamma} \times \frac{\partial L}{\partial \mathbf{\Gamma}}, RR^{-1}\delta R \right\rangle dt \\
&= \int_{t_1}^{t_2} \left\langle R\mathbf{\Gamma} \times \frac{\partial L}{\partial \mathbf{\Gamma}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{R}}, \delta R \right\rangle dt.
\end{aligned}$$

Therefore, one of our Euler-Lagrange equations becomes,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{R}} = R\mathbf{\Gamma} \times \frac{\partial L}{\partial \mathbf{\Gamma}}$$

and the other comes from just differentiating $\mathbf{\Gamma} = R^{-1}\mathbf{e}_3$ and letting $\mathbf{\Omega} = R^{-1}\dot{R}$, then we get,

$$\dot{\mathbf{\Gamma}} = \mathbf{\Lambda} \times \mathbf{\Omega}.$$

We call this last equation the reconstruction equation.
Finally, we can differentiate our specific Lagrangian to give the system,

$$\begin{cases} \ddot{R}\mathbf{X} = mgR\mathbf{X} \times \mathbf{\Gamma} \\ \dot{\mathbf{\Gamma}} = \mathbf{\Gamma} \times \mathbf{\Omega}. \end{cases}$$

We could further rewrite the first equation using the hat map isomorphism

$$\begin{aligned} \ddot{R}\mathbf{X} &= -mgR\mathbf{\Gamma} \times \mathbf{X} \\ \ddot{R}\mathbf{X} &= -mgR\hat{\mathbf{\Gamma}}\mathbf{X} \\ \ddot{R} &= -mgR\hat{\mathbf{\Gamma}} \end{aligned}$$

as $\ddot{R} = -mgR\hat{\mathbf{\Gamma}}$, which is then invariant of the coordinate systems and only depends on the rotation tensors. Therefore, we have the final system of,

$$\begin{cases} \ddot{R} = -mgR\hat{\mathbf{\Gamma}} \\ \dot{\mathbf{\Gamma}} = \mathbf{\Gamma} \times \mathbf{\Omega}. \end{cases}$$

### 5.1.3 Euler-Poincaré Equations

We now take the problem of interest and fully reduce it. To do so we will use a similar method to how we reduced the Lagrangian in Chapter 3.1 using vector identities. We start with the Lagrangian we introduced in the last section,

$$L(R, \dot{R}, \mathbf{\Gamma}) = \frac{1}{2}m\left|\dot{R}\mathbf{X}\right|^2 - mg\mathbf{\Gamma}(t) \cdot \mathbf{X}.$$

This can then be reduced to something of the form,

$$L(\hat{\mathbf{\Omega}}, \mathbf{\Gamma}) = \frac{1}{2}\widetilde{\mathbb{I}}\hat{\mathbf{\Omega}} \cdot \hat{\mathbf{\Omega}} - mg\mathbf{\Gamma} \cdot \mathbf{X}$$

where $\widetilde{\mathbb{I}} := |\mathbf{X}|^2 - \mathbf{X}\mathbf{X}^T$. We can now consider Hamilton's Variational Principle to derive some general equations for reduced Lagrangians of this form,

$$\begin{aligned} \delta \int_{t_1}^{t_2} \ell(\hat{\mathbf{\Omega}}, \mathbf{\Gamma})dt &= \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}}, \delta\hat{\mathbf{\Omega}} \right\rangle + \left\langle \frac{\partial \ell}{\partial \mathbf{\Gamma}}, \delta\mathbf{\Gamma} \right\rangle dt \\ &= \int_{t_1}^{t_2} \left\langle \frac{\partial L}{\partial \hat{\mathbf{\Omega}}}, \dot{\hat{\mathbf{\Lambda}}} + \mathrm{ad}_{\hat{\mathbf{\Omega}}}\hat{\mathbf{\Lambda}} \right\rangle + \left\langle \frac{\partial \ell}{\partial \mathbf{\Gamma}}, -\hat{\mathbf{\Lambda}} \times \mathbf{\Gamma} \right\rangle dt \\ &= \int_{t_1}^{t_2} \left\langle -\frac{d}{dt}\frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}} + \mathrm{ad}^*_{\hat{\mathbf{\Omega}}}\frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}} + \frac{\partial \ell}{\partial \mathbf{\Gamma}} \times \mathbf{\Gamma}, \hat{\mathbf{\Lambda}} \right\rangle dt = 0 \end{aligned}$$

Therefore, we reach the equation,

$$\frac{d}{dt}\frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}} = \mathrm{ad}^*_{\hat{\mathbf{\Omega}}}\frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}} + \frac{\partial \ell}{\partial \mathbf{\Gamma}} \times \mathbf{\Gamma}.$$

Now we can differentiate the Lagrangian to complete the derivation for this example,

$$\frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}} = \widetilde{\mathbb{I}}\hat{\mathbf{\Omega}} \qquad \frac{\partial \ell}{\partial \mathbf{\Gamma}} = mg\mathbf{X}$$

Then by considering the definition of the adjoint and then running through the algebra we reach the system of equations of,

$$\widetilde{\mathbb{I}}\hat{\mathbf{\Omega}}_t = \widetilde{\mathbb{I}}\hat{\mathbf{\Omega}} \times \mathbf{\Omega} + mg\mathbf{X} \times \mathbf{\Gamma}$$

with the reconstruction equation $\dot{\mathbf{\Gamma}} = \mathbf{\Gamma} \times \mathbf{\Omega}$. We can now consider the conserved quantity for this set of equations. To do this, we can consider the Noether Theorem we saw in section 4.1, which gives is that,

$$A_\xi = \left\langle \mathrm{Ad}_{R^{-1}}(\widetilde{\mathbb{I}}\hat{\mathbf{\Omega}})\, , \eta \right\rangle$$

This is a slightly unrevealing form, so we can consider it as this,

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} m \left\| \dot{\mathbf{q}} \right\|^2 - V(\mathbf{q})$$

Then we get a conserved quantity of this form,

$$A_\xi = \langle m\mathbf{q} \times \dot{\mathbf{q}}\, , \xi \rangle$$

and this is the classical conservation of angular momentum, the nice conserved quantity I alluded to at the start of this chapter.

## 5.2 Heavy Top

Our second example is the heavy top problem. This is a second example of a body with the symmetry breaking parameter, gravity. The Heavy Top problem is a rigid body with a fixed point. We are going to study its dynamics in the body frame. That means we consider, $\mathbf{x}(t) = R(t)\mathbf{X}$ and $R(t) \in \mathrm{SO}(3)$. We know the potential energy of a point mass in a gravitational field is $V = mg\mathbf{e}_3 \cdot \mathbf{x}(t)$. We can write the following Lagrangian for our problem,

$$L(R, \dot{R}) = \int_{\mathcal{B}} \rho(\mathbf{X}) \left( \frac{1}{2} \left\| \dot{R}\mathbf{X} \right\|^2 - g\mathbf{e}_3 \cdot R\mathbf{X} \right) d^3\mathbf{X}.$$

Now we can verify this Lagrangian is neither left nor right invariant. We first verify the left non-invariance,

$$\begin{aligned}
L(RS, \dot{R}S) &= \int_{\mathcal{B}} \rho(\mathbf{X}) \left( \frac{1}{2} \left\| \dot{R}S\mathbf{X} \right\|^2 - g\mathbf{e}_3 \cdot RS\mathbf{X} \right) d^3\mathbf{X} \\
&= \int_{\mathcal{B}} \rho(\mathbf{X}) \left( \frac{1}{2} (\dot{R}S\mathbf{X})(\dot{R}S\mathbf{X})^T - g\mathbf{e}_3 \cdot RS\mathbf{X} \right) d^3\mathbf{X} \\
&= \int_{\mathcal{B}} \rho(\mathbf{X}) \left( \frac{1}{2} (\dot{R}\mathbf{X})(\dot{R}\mathbf{X})^T - g\mathbf{e}_3 \cdot RS\mathbf{X} \right) d^3\mathbf{X} \\
&= \int_{\mathcal{B}} \rho(\mathbf{X}) \left( \frac{1}{2} \left\| \dot{R}\mathbf{X} \right\|^2 - g\mathbf{e}_3 \cdot RS\mathbf{X} \right) d^3\mathbf{X} \neq L(R, \dot{R})
\end{aligned}$$

and now the right non-invariance,

$$L(SR, S\dot{R}) = \int_{\mathcal{B}} \rho(\mathbf{X}) \left( \frac{1}{2} \left\| S\dot{R}\mathbf{X} \right\|^2 - g\mathbf{e}_3 \cdot SR\mathbf{X} \right) d^3\mathbf{X}$$

**Exercise.** Prove that $L(R,\, \dot{R}) \neq L(RR^{-1},\, \dot{R}R^{-1})$ and $L(R,\, \dot{R}) \neq L(R^{-1}R,\, \dot{R}^{-1}\dot{R})$.

We know $\hat{\mathbf{\Omega}} = R^{-1}\dot{R}$, then we know $L = \frac{1}{2}\mathbf{\Omega} \cdot \mathbb{I}\mathbf{\Omega} - g\mathbf{e}_3 \cdot R\boldsymbol{\beta}$ where $\beta = \int_B \rho(\mathbf{X})\mathbf{X}\, d^3\mathbf{X}$ and $\boldsymbol{\beta}$ is the centre of mass in the body frame. (We could also use $\beta = \mathbf{X}_b$). We define $\mathbf{\Gamma} = R^{-1}\mathbf{e}_3$ and so

$$\ell(\hat{\mathbf{\Omega}}, \mathbf{\Gamma}) = \frac{1}{2}\mathbf{\Omega}(t) \cdot \mathbb{I}\mathbf{\Omega}(t) - g\mathbf{\Gamma}(t) \cdot \mathbf{X}_b$$

and we see that $\mathbf{\Gamma}(t)$, the gravitational force, is the symmetry breaking parameter. Now we use Hamilton's Variational Principle,

$$\delta \int_{t_1}^{t_2} \ell(\hat{\mathbf{\Omega}}, \mathbf{\Gamma})\, dt = 0$$

$$\int_{t_1}^{t_2} \left(-\frac{d}{dt}\mathbb{I}\hat{\mathbf{\Omega}} + \mathbb{I}\mathbf{\Omega} \times \mathbf{\Omega}\right) \cdot \Lambda\, dt - \int_{t_1}^{t_2} g\mathbf{X}_b \cdot \delta\mathbf{\Gamma} = 0.$$

The first integral contains the terms relating to the rigid body and the second relates to the symmetry breaking term. We also define $\hat{\mathbf{\Lambda}} = R^{-1}\delta R$. We now look to $\delta\mathbf{\Gamma}$,

$$\begin{aligned}
\delta\mathbf{\Gamma} &= \delta R^{-1}\mathbf{e}_3 \\
&= -R^{-1}\delta R^{-1}\mathbf{e}_3 \\
&= -\hat{\mathbf{\Lambda}}\mathbf{\Gamma} \\
&= -\mathbf{\Lambda} \times \mathbf{\Gamma}.
\end{aligned}$$

We can now write the following,

$$\int_{t_1}^{t_2} \left(-\frac{d}{dt}\mathbb{I}\hat{\mathbf{\Omega}} + \mathbb{I}\mathbf{\Omega} \times \mathbf{\Omega}\right) \cdot \Lambda\, dt + \int_{t_1}^{t_2} -g \left\langle \mathbf{X}_b \,, \mathbf{\Lambda} \times \mathbf{\Gamma} \right\rangle = 0$$

and so we can write,

$$\int_{t_1}^{t_2} \left[-\frac{d}{dt}\mathbb{I}\hat{\mathbf{\Omega}} + \mathbb{I}\hat{\mathbf{\Omega}} \times \hat{\mathbf{\Omega}} \cdot \mathbf{\Lambda} - g(\mathbf{X}_b \times \mathbf{\Gamma}) \cdot \mathbf{\Lambda}\right] = 0.$$

Now we get,

$$\begin{aligned}
\mathbb{I}\dot{\mathbf{\Omega}} &= \mathbb{I}\hat{\mathbf{\Omega}} \times \hat{\mathbf{\Omega}} + g\mathbf{\Gamma} \times \mathbf{X}_b \\
\dot{\mathbf{\Gamma}} &= \mathbf{\Gamma} \times \hat{\mathbf{\Omega}}.
\end{aligned}$$

**Exercise.** Find $\frac{d}{dt}\left(R\mathbb{I}\hat{\mathbf{\Omega}}\right) = gR(\mathbf{\Gamma} \times \mathbf{X}_b) \neq 0$. Find what kind of angular momentum is conserved. Find something that is conserved.

## 5.3 Pseudo-Rigid Bodies

In the following section we will firstly follow Chapter 10 of Holm, Schmah & Stoica, Geometric Mechanics and Symmetry [2], from which I summarise then derive the left, right and left and right invariant Euler-Poincaré equations for Pseudo-Rigid bodies.

Let us assume that our body can stretch and sheer, we will call this a Pseudo-Rigid body. The following derivations are done with the assumption that the configuration space we are working in is $\mathrm{GL}^+(3)$, i.e. the set of matrices with positive determinant. We make a few assumptions, firstly the moment of inertia tensor is rotationally invariant, for this to happen it is sufficient that the density function $\rho(\mathbf{X})$ is spherically symmetric. We will also assume that the Lagrangian only depends on kinetic energy, that means we study free ellipsoid motion.

We fix a reference configuration via a fixed spatial coordinate system and a moving body coordinate system, the origin of both of these systems is the fixed point of the body. We will assume that the configuration of the system is a matrix $\mathbf{Q}(t) \in \mathrm{GL}^+(3)$ which takes the label $\mathbf{X}$ to the spatial position $\mathbf{x}(t)$, that is,

$$\mathbf{x}(t, \mathbf{X}) = \mathbf{Q}(t)\mathbf{X} \qquad \dot{\mathbf{x}}(t, \mathbf{X}) = \dot{\mathbf{Q}}\mathbf{X} = \dot{\mathbf{Q}}(t)\mathbf{Q}^{-1}(t)\mathbf{x}(t, \mathbf{X}).$$

As before, let $\rho(\mathbf{X})$ be the density function and $\mathcal{B}$ be the region occupied by the body in its configuration space. The moment of inertia tensor is assumed to be spherically symmetric, that is,

$$\mathbb{J} = \int_{\mathcal{B}} \rho(\mathbf{X})\mathbf{X}\mathbf{X}^T d^3\mathbf{X} = kI \qquad k \in \mathbb{R}$$

with $I$ as the identity matrix. We assume without loss of generality that $k = 1$ and so,

$$\mathbb{J} = \int_{\mathcal{B}} \rho(\mathbf{X})\mathbf{X}\mathbf{X}^T d^3\mathbf{X} = I.$$

We now consider the kinetic energy,

$$\begin{aligned}
K &= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})\, \|\dot{\mathbf{x}}\| \; d^3\mathbf{X} \\
&= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X}) \left\|\dot{\mathbf{Q}}\dot{\mathbf{X}}\right\| \; d^3\mathbf{X} \\
&= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})\, \mathrm{Tr}\left((\dot{\mathbf{Q}}\mathbf{X})(\dot{\mathbf{Q}}\mathbf{X})^T\right) \; d^3\mathbf{X} \\
&= \frac{1}{2}\,\mathrm{Tr}\left(\dot{\mathbf{Q}}\int_{\mathcal{B}} \rho(\mathbf{X})\mathbf{X}\mathbf{X}^T \; d^3\mathbf{X}\, \dot{\mathbf{Q}}^T\right) \\
&= \frac{1}{2}\,\mathrm{Tr}\left(\dot{\mathbf{Q}}\mathbb{J}\dot{\mathbf{Q}}^T\right) \\
&= \frac{1}{2}\,\mathrm{Tr}\left(\dot{\mathbf{Q}}\dot{\mathbf{Q}}^T\right).
\end{aligned}$$

We can notice that this Lagrangian is symmetric and invariant under left and right actions, that is, if $L = \frac{1}{2}\,\mathrm{Tr}\left(\dot{\mathbf{Q}}\dot{\mathbf{Q}}^T\right)$, then

$$\begin{aligned}
L(g\mathbf{Q}h, g\dot{\mathbf{Q}}h) &= \frac{1}{2}\,\mathrm{Tr}\left(g\dot{\mathbf{Q}}h(g\dot{\mathbf{Q}}h)^T\right) \\
&= \frac{1}{2}\,\mathrm{Tr}\left(g\dot{\mathbf{Q}}hh^T\dot{\mathbf{Q}}g^T\right) \\
&= \frac{1}{2}\,\mathrm{Tr}\left(\dot{\mathbf{Q}}\dot{\mathbf{Q}}\right)
\end{aligned}$$

as $g, h \in \mathrm{SO}(3)$. We can decompose a matrix using single value decomposition. That is, take a matrix $\mathbf{A}$ and we can represent this as $\mathbf{U\Sigma V}$ where $\mathbf{U}, \mathbf{V} \in O(3)$ and $\mathbf{\Sigma} \in \mathrm{diag}^+(3)$. We want $\mathbf{U}, \mathbf{V}$ to be in $\mathrm{SO}(3)$ and so we now do the following derivation. Take a decomposition of $\mathbf{Q} = \mathbf{RAS}$. We know that $\det \mathbf{R} = \pm 1$, if $\det \mathbf{R} = 1$ leave it as it is, if $\det \mathbf{R} = -1$, then we tag on an additional matrix,

$$\mathbf{M} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

to $\mathbf{R}$. This creates $\mathbf{R}' = \mathbf{RM}$ and similarly for $\mathbf{S}' = \mathbf{RS}$ if $\det \mathbf{S} = -1$. Now we have the following decomposition, $\mathbf{Q} = \mathbf{R}'\mathbf{MAMS}'$, noting that $\mathbf{R}', \mathbf{S}' \in \mathrm{SO}(3)$, $\mathbf{MAM} \in \mathrm{diag}^+(3)$ and $\mathbf{M}^2 = \mathbf{I}$ and so this makes sense. This derivation is known as a bipolar decomposition. The decomposition $\mathbf{Q} = \mathbf{RAS}$ can be thought of as a body rotation, a stretch and a spatial rotation. That is, if we consider $\mathbf{S}$, this rotates the $\mathbf{X}$ (body) coordinates in the reference configuration, the $\mathbf{A}$ stretches the axis and $\mathbf{R}$ rotates the $\mathbf{x}$ (spatial) coordinates. Now I present a nice example.

**Example.** Consider a bipolar decomposition of $\mathbf{Q}$ acting on a pseudo-rigid sphere. If $\mathbf{R}(t) = \mathbf{I}$, where $\mathbf{A}(t) = \text{diag}(a_1, a_2, a_3)$. This is called the Jacobi ellipsoid motion, where only the outside of the sphere moves. However, if $\mathbf{S}(t) = \mathbf{I}$, then we have Dedekind ellipsoid motion, where only the inside moves and the outside stays still.

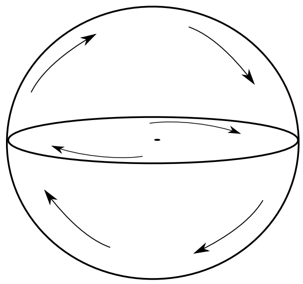


Figure 4: Dedekind Motion

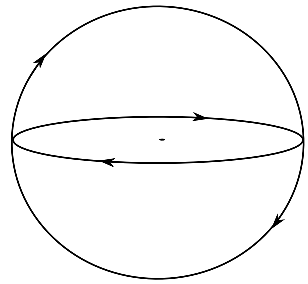


Figure 5: Jacobi Motion

After that interlude, we continue. We will work in an extended configuration space, that is, instead of just working in $\mathbf{Q} : \text{SO}(3)$, we work in $\mathbf{Q}_{\text{ext}} : \text{SO}(3) \times \text{diag}^+(3) \times \text{SO}(3)$. We call this the extended configuration space for Pseudo-Rigid bodies. We also define a submersion[3], $\phi : \mathbf{Q} \to \text{GL}^+(3)$, defined by $\phi(\mathbf{R}, \mathbf{A}, \mathbf{S}) = \mathbf{RAS}$, which allows us to define the extended Lagrangian, $L_{\text{ext}} : T\mathbf{Q}_{\text{ext}} \to \mathbb{R}$ defined by $L_{\text{ext}} = L \circ T\phi$. Furthermore, we assume that this Lagrangian is invariant under left and right symmetry, that is

$$L_{\text{ext}}(g\mathbf{R}, \mathbf{A}, \mathbf{S}h, g\dot{\mathbf{R}}, \dot{\mathbf{A}}, \dot{\mathbf{S}}h) = L_{\text{ext}}(\mathbf{R}, \mathbf{A}, \mathbf{S}, \dot{\mathbf{R}}, \dot{\mathbf{A}}, \dot{\mathbf{S}}).$$

To do the usual reduction, we now need some sort of variational principle. We will find in the following proposition that it indeed satisfies the expected Hamilton's Variational Principle,

**Proposition 5.1.** $(\mathbf{R}, \mathbf{A}, \mathbf{S})$ satisfies Hamilton's Principle for $L_{\text{ext}}$ if and only if $\mathbf{Q}$ satisfies Hamilton's Principle for $L$.

*Proof.* All deformations of $\mathbf{Q}(t,s)$ are of the form $\phi(\mathbf{R}(t,s), \mathbf{A}(t,s), \mathbf{S}(t,s))$. For some $(\mathbf{R}(t,s), \mathbf{A}(t,s), \mathbf{S}(t,s)) \in \mathbf{Q}_{\text{ext}}$. Hence by chain rule, $(\mathbf{Q}, \dot{\mathbf{Q}}) = T\phi(\mathbf{R}(t,s), \mathbf{A}(t,s), \mathbf{S}(t,s), \dot{\mathbf{R}}(t,s), \dot{\mathbf{A}}(t,s), \dot{\mathbf{S}}(t,s))$ and so $L(\mathbf{Q}(t,s), \dot{\mathbf{Q}}(t,s)) = L_{\text{ext}}(\mathbf{R}(t,s), \mathbf{A}(t,s), \mathbf{S}(t,s), \dot{\mathbf{R}}(t,s), \dot{\mathbf{A}}(t,s), \dot{\mathbf{S}}(t,s))$. Now we can write the following,

$$\begin{aligned}
&\delta \int_{t_1}^{t_2} L_{\text{ext}}(\mathbf{R}(t,s), \mathbf{A}(t,s), \mathbf{S}(t,s), \dot{\mathbf{R}}(t,s), \dot{\mathbf{A}}(t,s), \dot{\mathbf{S}}(t,s))dt = 0 \\
&\qquad \iff \frac{d}{ds}\bigg|_{s=0} \int_{t_1}^{t_2} L_{\text{ext}}(\mathbf{R}(t,s), \mathbf{A}(t,s), \mathbf{S}(t,s), \dot{\mathbf{R}}(t,s), \dot{\mathbf{A}}(t,s), \dot{\mathbf{S}}(t,s))dt = 0 \\
&\qquad \iff \frac{d}{ds}\bigg|_{s=0} \int_{t_1}^{t_2} L(\mathbf{Q}(t,s), \dot{\mathbf{Q}}(t,s))dt = 0 \\
&\qquad \iff \frac{d}{ds}\bigg|_{s=0} \int_{t_1}^{t_2} L(\mathbf{Q}(t), \dot{\mathbf{Q}}(t))dt = 0 \\
&\qquad \iff \delta \int_{t_1}^{t_2} L(\mathbf{Q}(t), \dot{\mathbf{Q}}(t))dt = 0
\end{aligned}$$

which gives the Hamilton's Principle for $L$. $\square$

For the rest of the chapter we will refer to $L_{\text{ext}}$ as $L$ as the previous proposition proves that the analysis we do on $L_{\text{ext}}$ suffices. We will use the following Lagrangian,

$$L(\mathbf{Q}, \dot{\mathbf{Q}}) = \frac{1}{2}\,\text{Tr}(\dot{\mathbf{Q}}\dot{\mathbf{Q}}^T).$$

[3] This is an everywhere surjective map between the tangent spaces of differentiable manifolds

We note that $\frac{d}{dt}(\mathbf{RAS}) = \dot{\mathbf{R}}\mathbf{AS} + \mathbf{R}\dot{\mathbf{A}}\mathbf{S} + \mathbf{RA}\dot{\mathbf{S}}$, which we denote as $(RAS)^\bullet$. Therefore, we can write the Lagrangian as, $L(\mathbf{Q}, \dot{\mathbf{Q}}) = \frac{1}{2}\operatorname{tr}((\mathbf{RAS})^\bullet(\mathbf{S}^T\mathbf{A}\mathbf{R}^T)^\bullet)$. We will use this later to derive one of the sets of equations. We will now move to the reduction of the Lagrangian and then the Euler-Poincaré equations.

### 5.3.1 Euler-Poincaré Reduction

We now seek to reduce the Lagrangian, we recall the symmetric invariance on the Lagrangian we stated above,

$$L(g\mathbf{R}, \mathbf{A}, \mathbf{S}h, g\dot{\mathbf{R}}, \dot{\mathbf{A}}, \dot{\mathbf{S}}h) = L(\mathbf{R}, \mathbf{A}, \mathbf{S}, \dot{\mathbf{R}}, \dot{\mathbf{A}}, \dot{\mathbf{S}}).$$

We let $g = \mathbf{R}^{-1}$, $h = \mathbf{S}^{-1}$ and then define $\hat{\mathbf{\Omega}} := \mathbf{R}^{-1}\dot{\mathbf{R}}$ and $\hat{\mathbf{\Lambda}} := \dot{\mathbf{S}}\mathbf{S}^{-1}$ and we can now write,

$$L(e, \mathbf{A}, e, \hat{\mathbf{\Omega}}, \dot{A}, \hat{\mathbf{\Lambda}}) = L(\mathbf{R}, \mathbf{A}, \mathbf{S}, \dot{\mathbf{R}}, \dot{\mathbf{A}}, \dot{\mathbf{S}}).$$

We define $\ell(\mathbf{A}, \hat{\mathbf{\Omega}}, \dot{\mathbf{A}}, \hat{\mathbf{\Lambda}}) := L(e, \mathbf{A}, e, \hat{\mathbf{\Omega}}, \dot{\mathbf{A}}, \hat{\mathbf{\Lambda}}) = L(\mathbf{R}, \mathbf{A}, \mathbf{S}, \dot{\mathbf{R}}, \dot{\mathbf{A}}, \dot{\mathbf{S}})$ and now start to work towards the Euler-Poincaré equations. We firstly consider Hamilton's Principle for this Lagrangian,

$$\begin{aligned}
0 &= \delta\int_{t_1}^{t_2} \ell(\mathbf{A}, \hat{\mathbf{\Omega}}, \dot{\mathbf{A}}, \hat{\mathbf{\Lambda}})dt \\
0 &= \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \mathbf{A}}, \delta\mathbf{A} \right\rangle + \left\langle \frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}}, \delta\hat{\mathbf{\Omega}} \right\rangle + \left\langle \frac{\partial \ell}{\partial \dot{\mathbf{A}}}, \delta\dot{\mathbf{A}} \right\rangle + \left\langle \frac{\partial \ell}{\partial \hat{\mathbf{\Lambda}}}, \delta\hat{\mathbf{\Lambda}} \right\rangle dt \\
0 &= \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \mathbf{A}} - \frac{d}{dt}\frac{\partial \ell}{\partial \dot{\mathbf{A}}}, \delta\mathbf{A} \right\rangle + \left\langle \frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}}, \delta\hat{\mathbf{\Omega}} \right\rangle + \left\langle \frac{\partial \ell}{\partial \hat{\mathbf{\Lambda}}}, \delta\hat{\mathbf{\Lambda}} \right\rangle dt \\
0 &= \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \mathbf{A}} - \frac{d}{dt}\frac{\partial \ell}{\partial \dot{\mathbf{A}}}, \delta\mathbf{A} \right\rangle + \left\langle \frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}}, \dot{\xi} + [\hat{\mathbf{\Omega}}, \hat{\xi}] \right\rangle + \left\langle \frac{\partial \ell}{\partial \hat{\mathbf{\Lambda}}}, \dot{\eta} - [\hat{\mathbf{\Lambda}}, \hat{\eta}] \right\rangle dt
\end{aligned}$$

$$\begin{aligned}
0 &= \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \mathbf{A}} - \frac{d}{dt}\frac{\partial \ell}{\partial \dot{\mathbf{A}}}, \delta\mathbf{A} \right\rangle + \left\langle \frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}}, \dot{\xi} \right\rangle + \left\langle \frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}}, \operatorname{ad}_{\hat{\mathbf{\Omega}}}\xi \right\rangle + \left\langle \frac{\partial \ell}{\partial \hat{\mathbf{\Lambda}}}, \dot{\eta} \right\rangle - \left\langle \frac{\partial \ell}{\partial \hat{\mathbf{\Lambda}}}, \operatorname{ad}_{\hat{\mathbf{\Lambda}}}\eta \right\rangle dt \\
0 &= \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \mathbf{A}} - \frac{d}{dt}\frac{\partial \ell}{\partial \dot{\mathbf{A}}}, \delta\mathbf{A} \right\rangle + \left\langle -\frac{d}{dt}\frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}}, \xi \right\rangle + \left\langle \operatorname{ad}^*_{\hat{\mathbf{\Omega}}}\frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}}, \xi \right\rangle + \left\langle -\frac{d}{dt}\frac{\partial \ell}{\partial \hat{\mathbf{\Lambda}}}, \eta \right\rangle - \left\langle \operatorname{ad}^*_{\hat{\mathbf{\Lambda}}}\frac{\partial \ell}{\partial \hat{\mathbf{\Lambda}}}, \eta \right\rangle dt \\
0 &= \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \mathbf{A}} - \frac{d}{dt}\frac{\partial \ell}{\partial \dot{\mathbf{A}}}, \delta\mathbf{A} \right\rangle + \left\langle \operatorname{ad}^*_{\hat{\mathbf{\Omega}}}\frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}} - \frac{d}{dt}\frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}}, \xi \right\rangle - \left\langle \operatorname{ad}^*_{\hat{\mathbf{\Lambda}}}\frac{\partial \ell}{\partial \hat{\mathbf{\Lambda}}} + \frac{d}{dt}\frac{\partial \ell}{\partial \hat{\mathbf{\Lambda}}}, \eta \right\rangle dt.
\end{aligned}$$

Therefore, considering endpoint conditions we reach the following set of Euler-Poincaré equations,

$$\frac{d}{dt}\frac{\partial \ell}{\partial \mathbf{\Omega}} = \operatorname{ad}^*_{\mathbf{\Omega}}\frac{\partial \ell}{\partial \mathbf{\Omega}} \tag{5.1}$$

$$\frac{d}{dt}\frac{\partial \ell}{\partial \dot{\mathbf{A}}} = \frac{\partial \ell}{\partial \mathbf{A}} \tag{5.2}$$

$$\frac{d}{dt}\frac{\partial \ell}{\partial \mathbf{\Lambda}} = -\operatorname{ad}^*_{\mathbf{\Lambda}}\frac{\partial \ell}{\partial \mathbf{\Lambda}} \tag{5.3}$$

which then give arise to (10.12) - (10.14) in [2]. We can now work forward, in a very similar vein to Holm and replace the coadjoints and partial derivatives back with $\hat{\mathbf{\Lambda}}$, $\mathbf{A}$, $\hat{\mathbf{\Omega}}$ and $\dot{\mathbf{A}}$. We go straight to the rewritten derivatives,

$$\ell(Q, \dot{Q}) = \frac{1}{2}\operatorname{tr}\left((\hat{\mathbf{\Omega}}\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\hat{\mathbf{\Lambda}})(\hat{\mathbf{\Omega}}\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\hat{\mathbf{\Lambda}})^T\right).$$

We now need to take variations of this Lagrangian in order to get the direct equations. Consider the following,

$$\begin{aligned}\left.\frac{d}{dt}\right|_{\varepsilon=0}\ell(\hat{\mathbf{\Omega}}+\varepsilon\delta\hat{\mathbf{\Omega}}) &= \left.\frac{d}{dt}\right|_{\varepsilon=0}\frac{1}{2}\operatorname{tr}\left(((\hat{\mathbf{\Omega}}+\varepsilon\delta\hat{\mathbf{\Omega}})\mathbf{A}+\dot{\mathbf{A}}+\mathbf{A}\hat{\mathbf{\Lambda}})((\hat{\mathbf{\Omega}}+\varepsilon\delta\hat{\mathbf{\Omega}})\mathbf{A}+\dot{\mathbf{A}}+\mathbf{A}\hat{\mathbf{\Lambda}})^T\right)\\ &= \frac{1}{2}\operatorname{tr}\left((\delta\hat{\mathbf{\Omega}}\mathbf{A})^T(\hat{\mathbf{\Omega}}\mathbf{A}+\dot{\mathbf{A}}+\mathbf{A}\hat{\mathbf{\Lambda}})+(\hat{\mathbf{\Omega}}\mathbf{A}+\dot{\mathbf{A}}+\mathbf{A}\hat{\mathbf{\Lambda}})^T\delta\hat{\mathbf{\Omega}}\mathbf{A}\right)\\ &= \operatorname{tr}\left((\delta\hat{\mathbf{\Omega}}\mathbf{A})^T(\hat{\mathbf{\Omega}}\mathbf{A}+\dot{\mathbf{A}}+\mathbf{A}\hat{\mathbf{\Lambda}})\right).\end{aligned}$$

This step can be done as we know that $\operatorname{tr}(\mathbf{A}^T\mathbf{B})=\operatorname{tr}(\mathbf{A}\mathbf{B}^T)$ and so $\frac{1}{2}\operatorname{tr}(\mathbf{A}^T\mathbf{B}+\mathbf{A}\mathbf{B}^T)=\frac{1}{2}\operatorname{tr}(\mathbf{A}^T\mathbf{B})+\frac{1}{2}\operatorname{tr}(\mathbf{A}^T\mathbf{B})=\operatorname{tr}(\mathbf{A}^T\mathbf{B})$.

$$\begin{aligned}\left.\frac{d}{dt}\right|_{\varepsilon=0}\ell(\hat{\mathbf{\Omega}}+\varepsilon\delta\hat{\mathbf{\Omega}}) &= \operatorname{tr}\left((\delta\hat{\mathbf{\Omega}}\mathbf{A})(\hat{\mathbf{\Omega}}\mathbf{A}+\dot{\mathbf{A}}+\mathbf{A}\hat{\mathbf{\Lambda}})^T\right)\\ &= \operatorname{tr}\left((\delta\hat{\mathbf{\Omega}}\mathbf{A})(\hat{\mathbf{\Omega}}\mathbf{A}+\dot{\mathbf{A}}+\mathbf{A}\hat{\mathbf{\Lambda}})^T\right)\\ &= \operatorname{tr}\left(\mathbf{A}^T(\hat{\mathbf{\Omega}}\mathbf{A}+\dot{\mathbf{A}}+\mathbf{A}\hat{\mathbf{\Lambda}})^T\delta\hat{\mathbf{\Omega}}\right)\\ &= \left\langle\mathbf{A}(\hat{\mathbf{\Omega}}\mathbf{A}+\dot{\mathbf{A}}+\mathbf{A}\hat{\mathbf{\Lambda}}\,,\delta\hat{\mathbf{\Omega}}\right\rangle.\end{aligned}$$

Now we consider the fact that $\delta\ell=\left\langle\frac{\partial\ell}{\partial\hat{\mathbf{\Omega}}}\,,\delta\hat{\mathbf{\Omega}}\right\rangle$ and so $\frac{\partial\ell}{\partial\hat{\mathbf{\Omega}}}=\hat{\mathbf{\Omega}}\mathbf{A}^2+\dot{\mathbf{A}}\mathbf{A}+\mathbf{A}\hat{\mathbf{\Lambda}}\mathbf{A}$. We now need to consider,

$$\begin{aligned}\left\langle\dot{\mathbf{A}}\mathbf{A}\,,\delta\hat{\mathbf{\Omega}}\right\rangle &= \operatorname{tr}((\dot{\mathbf{A}}\mathbf{A})^T\delta\hat{\mathbf{\Omega}})\\ &= \operatorname{tr}(\mathbf{A}\dot{\mathbf{A}}\delta\hat{\mathbf{\Omega}})=0.\end{aligned}$$

This is true as $\mathbf{A}\dot{\mathbf{A}}$ is symmetric and $\delta\hat{\mathbf{\Omega}}$ is antisymmetric. We recall the result from the previous section that says that the trace must then be zero. Finally, we note that $\hat{\mathbf{\Omega}}\mathbf{A}^2=\frac{1}{2}\left(\hat{\mathbf{\Omega}}\mathbf{A}^2+\mathbf{A}^2\hat{\mathbf{\Omega}}\right)$, as $\mathbf{A}^2$ is diagonal and so commutative. Hence the first equation now must be,

$$\frac{\partial\ell}{\partial\hat{\mathbf{\Omega}}}=\frac{1}{2}\left(\hat{\mathbf{\Omega}}\mathbf{A}^2+\mathbf{A}^2\hat{\mathbf{\Omega}}\right)+\mathbf{A}\hat{\mathbf{\Lambda}}\mathbf{A}. \tag{5.4}$$

We continue with a similar argument for $\hat{\mathbf{\Lambda}}$, again taking variations,

$$\begin{aligned}\left.\frac{d}{dt}\right|_{\varepsilon=0}\ell(\hat{\mathbf{\Lambda}}+\varepsilon\delta\hat{\mathbf{\Lambda}}) &= \left.\frac{d}{dt}\right|_{\varepsilon=0}\frac{1}{2}\operatorname{tr}\left((\hat{\mathbf{\Omega}}\mathbf{A}+\dot{\mathbf{A}}+\mathbf{A}(\hat{\mathbf{\Lambda}}+\varepsilon\delta\hat{\mathbf{\Lambda}}))(\hat{\mathbf{\Omega}}\mathbf{A}+\dot{\mathbf{A}}+\mathbf{A}(\hat{\mathbf{\Lambda}}+\varepsilon\delta\hat{\mathbf{\Lambda}}))^T\right)\\ &= \frac{1}{2}\operatorname{tr}\left((\mathbf{A}\delta\hat{\mathbf{\Lambda}})^T(\hat{\mathbf{\Omega}}\mathbf{A}+\dot{\mathbf{A}}+\mathbf{A}\hat{\mathbf{\Lambda}})+(\hat{\mathbf{\Omega}}\mathbf{A}+\dot{\mathbf{A}}+\mathbf{A}\hat{\mathbf{\Lambda}})^T(A\delta\hat{\mathbf{\Lambda}})\right)\\ &= \operatorname{tr}\left((\mathbf{A}\delta\hat{\mathbf{\Lambda}})(\hat{\mathbf{\Omega}}\mathbf{A}+\dot{\mathbf{A}}+\mathbf{A}\hat{\mathbf{\Lambda}})^T\right)\\ &= \operatorname{tr}\left(\mathbf{A}^T(\hat{\mathbf{\Omega}}\mathbf{A}+\dot{\mathbf{A}}+\mathbf{A}\hat{\mathbf{\Lambda}})^T\delta\hat{\mathbf{\Lambda}}\right)\\ &= \left\langle\mathbf{A}(\hat{\mathbf{\Omega}}\mathbf{A}+\dot{\mathbf{A}}+\mathbf{A}\hat{\mathbf{\Lambda}})\,,\delta\hat{\mathbf{\Lambda}}\right\rangle\end{aligned}$$

and so we get $\frac{\partial\ell}{\partial\hat{\mathbf{\Lambda}}}=\mathbf{A}\hat{\mathbf{\Omega}}\mathbf{A}+\mathbf{A}\dot{\mathbf{A}}+\mathbf{A}^2\hat{\mathbf{\Lambda}}$. As before, we get that $\left\langle\mathbf{A}\dot{\mathbf{A}}\,,\delta\hat{\mathbf{\Lambda}}\right\rangle=0$. Therefore,

$$\frac{\partial\ell}{\partial\hat{\mathbf{\Lambda}}}=\mathbf{A}\hat{\mathbf{\Omega}}\mathbf{A}+\frac{1}{2}\left(\hat{\mathbf{\Lambda}}\mathbf{A}^2+\mathbf{A}^2\hat{\mathbf{\Lambda}}\right). \tag{5.5}$$

Finally, we derive the $\mathbf{A}$ version of the equations using a slightly different argument, where we split it half way through,

$$\begin{aligned}
\frac{d}{dt}\Big|_{\varepsilon=0} \ell(\mathbf{A}+\varepsilon\delta\mathbf{A}) &= \frac{d}{dt}\Big|_{\varepsilon=0} \frac{1}{2}\operatorname{tr}\left((\hat{\mathbf{\Omega}}(\mathbf{A}+\varepsilon\delta\mathbf{A})+\dot{\mathbf{A}}+(\mathbf{A}+\varepsilon\delta\mathbf{A})\hat{\mathbf{\Lambda}})(\hat{\mathbf{\Omega}}(\mathbf{A}+\varepsilon\delta\mathbf{A})+\dot{\mathbf{A}}+(\mathbf{A}+\varepsilon\delta\mathbf{A})\hat{\mathbf{\Lambda}})^T\right) \\
&= \frac{1}{2}\operatorname{tr}\left((\hat{\mathbf{\Omega}}\delta\mathbf{A}+\delta\mathbf{A}\hat{\mathbf{\Lambda}})^T(\hat{\mathbf{\Omega}}\mathbf{A}+\dot{\mathbf{A}}+\mathbf{A}\hat{\mathbf{\Lambda}})+(\hat{\mathbf{\Omega}}\mathbf{A}+\dot{\mathbf{A}}+\mathbf{A}\hat{\mathbf{\Lambda}})^T(\hat{\mathbf{\Omega}}\delta\mathbf{A}+\delta\mathbf{A}\hat{\mathbf{\Lambda}})\right) \\
&= \operatorname{tr}\left((\hat{\mathbf{\Omega}}\delta\mathbf{A}+\delta\mathbf{A}\hat{\mathbf{\Lambda}})(\hat{\mathbf{\Omega}}\mathbf{A}+\dot{\mathbf{A}}+\mathbf{A}\hat{\mathbf{\Lambda}})^T\right) \\
&= \operatorname{tr}\left(\hat{\mathbf{\Omega}}\delta\mathbf{A}(\hat{\mathbf{\Omega}}\mathbf{A}+\dot{\mathbf{A}}+\mathbf{A}\hat{\mathbf{\Lambda}})^T\right)+\operatorname{tr}\left(\delta\mathbf{A}\hat{\mathbf{\Lambda}}(\hat{\mathbf{\Omega}}\mathbf{A}+\dot{\mathbf{A}}+\mathbf{A}\hat{\mathbf{\Lambda}})^T\right) \\
&= \operatorname{tr}\left(\hat{\mathbf{\Omega}}^T(\hat{\mathbf{\Omega}}\mathbf{A}+\dot{\mathbf{A}}+\mathbf{A}\hat{\mathbf{\Lambda}})^T\delta\mathbf{A}\right)+\operatorname{tr}\left((\hat{\mathbf{\Omega}}\mathbf{A}+\dot{\mathbf{A}}+\mathbf{A}\hat{\mathbf{\Lambda}})^T\hat{\mathbf{\Lambda}}^T\delta\mathbf{A}\right) \\
&= \operatorname{tr}\left(\left(\hat{\mathbf{\Omega}}^2\mathbf{A}+\mathbf{A}\hat{\mathbf{\Lambda}}^2+\hat{\mathbf{\Omega}}\dot{\mathbf{A}}+\dot{\mathbf{A}}\hat{\mathbf{\Lambda}}+2\hat{\mathbf{\Omega}}\mathbf{A}\hat{\mathbf{\Lambda}}\right)^T\delta\mathbf{A}\right) \\
&= \left\langle \hat{\mathbf{\Omega}}^2\mathbf{A}+\mathbf{A}\hat{\mathbf{\Lambda}}^2+\hat{\mathbf{\Omega}}\dot{\mathbf{A}}+\dot{\mathbf{A}}\hat{\mathbf{\Lambda}}+2\hat{\mathbf{\Omega}}\mathbf{A}\hat{\mathbf{\Lambda}}\,,\delta\mathbf{A}\right\rangle.
\end{aligned}$$

Now we reduce this further as we know that $\hat{\mathbf{\Lambda}}^2\mathbf{A}+\mathbf{A}\hat{\mathbf{\Lambda}}^2=\mathbf{A}(\hat{\mathbf{\Omega}}^2+\hat{\mathbf{\Lambda}}^2)$. We consider the trace pairing and we get that this term disappears. We can make a similar argument for $\hat{\mathbf{\Omega}}\dot{\mathbf{A}}+\dot{\mathbf{A}}\hat{\mathbf{\Lambda}}$ and the equations reduce to,

$$\frac{\partial \ell}{\partial \mathbf{A}} = 2\hat{\mathbf{\Omega}}\mathbf{A}\hat{\mathbf{\Lambda}} = \hat{\mathbf{\Omega}}\mathbf{A}\hat{\mathbf{\Lambda}}+\hat{\mathbf{\Lambda}}\mathbf{A}\hat{\mathbf{\Omega}}. \tag{5.6}$$

In addition to,

$$\frac{\partial \ell}{\partial \dot{\mathbf{A}}} = \dot{\mathbf{A}} \tag{5.7}$$

these are then the equations (10.15)-(10.18) that are stated by Holm, Schmah & Stoica in [2].

#### 5.3.2 Noether Theory for general Euler-Poincaré reduction

Let $G$ be an arbitrary matrix Lie group and let $L$ be a left-invariant Lagrangian,

$$L(hg,h\dot{g}) = L(g,\dot{g}) \quad \forall g,h\in G$$

with variational principle

$$\delta\int_{t_1}^{t_2} L(g,\dot{g})dt = 0.$$

The reduced system is just $L(hg,h\dot{g})|_{h=g^{-1}} = L(g^{-1}g,g^{-1}\dot{g}) := \ell(\xi)$ where $\xi = g^{-1}g$ and $\xi\in\mathfrak{so}(3)$. This leads to the Noether Theorem for this Pseudo-Rigid bodies, which is stated in the following theorem.

**Theorem 5.2** (Pseudo-Rigid Body Noether Theorem)**.** Corresponding to each one-parameter subgroup of $G$, $\chi(s)$ with $\chi(0)=e$ and $\chi_s(s)=\eta\in\mathfrak{g}$ there is a conserved quantity,

$$\left\langle \operatorname{Ad}^*_{g^{-1}}\frac{\partial \ell}{\partial \xi}\,,\eta\right\rangle = K.$$

*Proof.* Take a one-parameter subgroup $\chi(s)$ and multiply this by the Lagrangian on the left,

$$\int_{t_1}^{t_2} L(\chi(s)g,\chi(s)\dot{g}) = \int_{t_1}^{t_2} L(g,\dot{g}).$$

Now differentiate with respect to $s$ and set $s=0$ (that is, take the first variation).

$$\int_{t_1}^{t_2}\left(\left\langle\frac{\partial L}{\partial g},\chi_s(0)g\right\rangle+\left\langle\frac{\partial L}{\partial \dot g},\chi_s(0)\dot g\right\rangle\right)dt=0.$$

Now we integrate by parts,

$$0=\int_{t_1}^{t_2}\left\langle\frac{\partial L}{\partial g}-\frac{d}{dt}\frac{\partial L}{\partial \dot g},\chi_s(0)g\right\rangle+\left\langle\frac{\partial L}{\partial \dot g},\chi_s(0)g\right\rangle\Bigg|_{t_1}^{t_2}.$$

The first part is zero as it is just the Euler-Lagrange equations. Therefore,

$$\left\langle\frac{\partial L}{\partial \dot g},\chi_s(0)g\right\rangle=K.$$

But we can write $\chi_s(0)g=\eta g=gg^{-1}\eta g=g(g^{-1}\eta g)=g\,\mathrm{Ad}_{g^{-1}}\eta$. Therefore,

$$\left\langle\frac{\partial L}{\partial \dot g},g\,\mathrm{Ad}_{g^{-1}}\eta\right\rangle=K.$$

Now we need to add $\xi$. We want to deform $g_t(t,s)$ but not $g(t)$. Assume that $L(g(t),g_t(t,s))=L(g^{-1}(t)g(t),g^{-1}(t)g_t(t,s))=\ell(g^{-1}(t)g_t(t,s))$. Now differentiate with respect to $s$. Now we conclude,

$$\left\langle\frac{\partial L}{\partial \dot g},g_{ts}(t,s)\right\rangle=\left\langle\frac{\partial \ell}{\partial \xi},g^{-1}g_{ts}(t,s)\right\rangle$$

and set $s=0$. We get,

$$\left\langle\frac{\partial L}{\partial \dot g},\delta g_t\right\rangle=\left\langle\frac{\partial \ell}{\partial \xi},g^{-1}\delta g_t(t)\right\rangle.$$

Therefore,

$$\left\langle\frac{\partial L}{\partial \dot g},\delta g_t\right\rangle=\left\langle g\frac{\partial \ell}{\partial \xi},\delta g_t(t)\right\rangle.$$

That is,

$$g^{-1}\frac{\partial L}{\partial \dot g}=\frac{\partial \ell}{\partial \xi}.$$

Therefore,

$$K=\left\langle g\frac{\partial \ell}{\partial \xi},g\,\mathrm{Ad}_{g^{-1}}\eta\right\rangle=\left\langle\frac{\partial \ell}{\partial \xi},\mathrm{Ad}_{g^{-1}}\eta\right\rangle=\left\langle\mathrm{Ad}^*_{g^{-1}}\frac{\partial \ell}{\partial \xi},\eta\right\rangle.$$

The result follows from this. □

We note that the conserved quantity is the constant that arises from integration by parts. I find this quite nice.

**Example.** Now apply this to Pseudo Rigid bodies.

# References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