Year 3 — Mathematical Biology and Ecology

Based on lectures by Dr Ozgur Akman and Dr Marc Goodfellow
Notes taken by James Arthur

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Contents

1 Continuous Models for a single species

We are going to model simply how we model population dynamics for a single species. We are going to call $N(t)$ our population size at a certain time. We are going to say that $N(t)$ is continuous. We also say $N \in \mathbb{R}$ and so it's going to be a density measure. We are going to constrain this $N \geq 0, \forall t$. We can measure this and create a model, *Lecture 2*

$$\frac{dN}{dt} = f(N, t, \boldsymbol{\mu})$$

We call N the variable, then $\boldsymbol{\mu}$ the parameter. We let $t \in \mathbb{R}$ and $t > 0$.

Let's start off by thinking about an actual population of individuals, we have observed that there is some sort of growth dynamics. We can write a mechanistic model, by including mechanisms that affect the population.

$$\frac{dN}{dt} = +\text{births} + \text{resources} + \text{net migration} - \text{deaths}$$

If the positive things are greater, the population will grow, otherwise if they are smaller they decline, or they stay equal.

Now we make some assumptions, so we can write down a mathematical model.

(i) Births and deaths predominate:

$$\frac{dN}{dt} = \text{births} - \text{deaths}$$

(ii) Births and deaths are proportional to N:

$$\frac{dN}{dt} = \alpha N - \beta N \qquad \alpha, \beta \in \mathbb{R}^+$$

where we call α the birthrate and β the deathrate and $\boldsymbol{\mu} = (\alpha, \beta)$.

If we consider, as an aside,

$$\frac{dN}{dt} = \alpha N - \beta N + \gamma \qquad \gamma \in \mathbb{R}$$

this adds something like independent migration, where we assume migration is constant, this could be γN if it's proportional to N.

We can solve this equation nicely,

$$\begin{aligned}
\frac{dN}{dt} &= N(\alpha - \beta) \\
\int \frac{1}{N}\frac{dN}{dt}\,dt &= \int (\alpha - \beta) dt \\
\ln N &= (\alpha - \beta)t + C \\
N(t) &= Ae^{(\alpha-\beta)t}
\end{aligned}$$

Now assume this is an initial value problem, $N_0 = A$ and so,

$$N(t) = N_0 e^{(\alpha-\beta)t}$$

We are also interested in the long term dynamics of these models, the asymptotic dynamics. Hence, we consider $t \to \infty$.

$$\lim_{t\to\infty} N(t) = \begin{cases} \infty & \text{if } \alpha > \beta \\ 0 & \text{if } \alpha < \beta \\ N_0 & \text{if } \alpha = \beta \end{cases}$$

We call the case where $\alpha = \beta$ a steady state solution.

You don't usually see just a t in the equation, but we may want to use a forcing term, i.e. periodic migration in Cornwall.

$$\frac{dN}{dt} = \alpha N - \beta N + \cos(t)$$

We are not going to consider these non-autonomous systems. Hence we can write,

$$\frac{dN}{dt} = f(N, \boldsymbol{\mu})$$

We have a nice thing to make sure our growth doesn't go exponential. It's the logistic map,

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{k}\right) \qquad r, k > 0 \quad N \geq 0$$

where $f(N) = rN\left(1 - \frac{N}{k}\right)$ is called the logistic model. This model has a level of self regulation, so if N is high, then it will decrease later. Firstly, look at $f(N)$,

$$f(N) = rN\left(1 - \frac{N}{k}\right)$$

We start by graphing it,

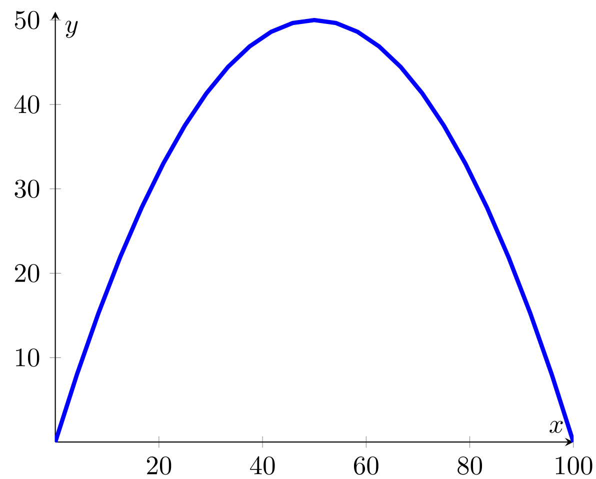

Figure 1: $y = 2x\left(1 - \frac{x}{100}\right)$

Exercise. Solve $\frac{dN}{dt} = rN(1 - \frac{N}{k})$

The solution to this equation is,

$$N(t) = \frac{N_0 k e^{rt}}{k - N_0 + N_0 e^{rt}}$$

As $t \to \infty$, we can consider it and see that it will depend on N_0. If $N_0 = 0$, then we get that $N(t) = 0, \forall t$. If we then take $N_0 > 0$, then we get,

$$\begin{aligned} N(t) &= \frac{N_0 k}{\frac{k - N_0}{e^{kt}} + N_0} \\ &= k & t \to \infty \end{aligned}$$

lets try and formalise some of these ideas. So consider, *Lecture 3*

$$\frac{dN}{dt} = f(N)$$

where $N \in \mathbb{R}, t \in \mathbb{R}, f : \mathbb{R} \to \mathbb{R}$. Then we define steady states as, where $f(N^*) = 0$. These are also reffered to as fixed points or equilibrium. When a system in the first place depends on whether the state is attracting.

Definition 1.1 (Attrracting)**.** A steady state N^* is attracting if all trajectories that start close to N^* approach it as $t \to \infty$.

We consider $N = N^* + n$, recall the taylor series is an approximation to a function at a point,

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \dots$$

as we consider small values of n we can throw away higher terms. Hence, we can linearise it.

$$\begin{aligned}
\frac{dN}{dt} &= f(N) \\
\frac{dN^* + n}{dt} &= f(N^* + n) \\
\frac{dn}{dt} &= f(N^* + n) \\
\frac{dn}{dt} &\approx f(N^*) + f'(N^*)(N^* + n - N^*) + f''(a)\frac{(N^* + n - N^*)}{2!} + \dots \\
\frac{dn}{dt} &\approx 0 + f'(N^*)n + \frac{f''(N^*)}{2!}n^2 + \dots \\
\frac{dn}{dt} &\approx f'(N^*)n
\end{aligned}$$

and so we can model it by,

$$\frac{dn}{dt} = f'(N^*)n$$

which can be solved as,

$$n(t) \approx n_0 e^{f'(N^* t}$$

If $f'(N^*) > 0$, then we just have an exponential (unstable), but if $f'(N^*) < 0$ then we have a decaying exponential (stable).

Example. We shall consider,

$$\frac{dx}{dt} = rN\left(1 - \frac{N}{k}\right)$$

and so we want $f(N^*) = 0$ and hence we get that $N^* = 0$ and $N^{**} = k$. Then we find $f'(N) = r\left(1 - \frac{2N}{K}\right)$. Hence we can find $f'(0) = r$ and $r'(k) = -r$ and as $r > 0$, $f'(N^*) > 0$ hence N^* is unstable and as $f(N^{**}) < 0$ then N^{**} is stable.

We are going to recap last lectures, and study, *Lecture 4*

$$\frac{dN}{dt} = N(N-\alpha)(N-\beta)$$

where $N \in \mathbb{R}^+$ and $0 < \alpha < \beta$. We are going to focus on long term dynamics, i.e. $t \to \infty$. The steady state will give us information about that, once we know what they are and their stability, we know everything. Steady states are going to be where, $\frac{dN}{dt} = f(N) = 0$, then we can see that this is just $N^* = 0$, $N^{**}\alpha$, $N^{***} = \beta$.

Definition 1.2 (Trajectory)**.** From an initial N_0, a trajectory is how $N(t)$ evolves.

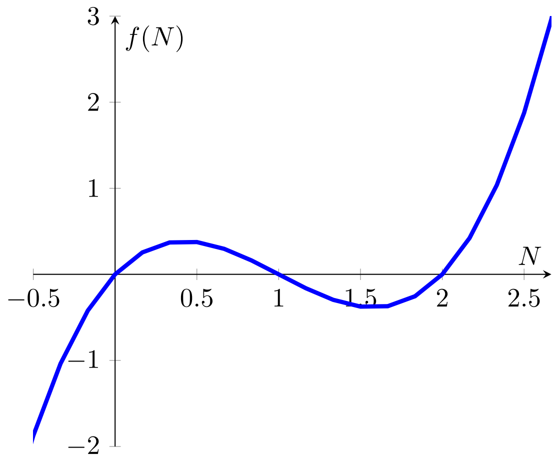

Figure 2: $y = N(N-\alpha)(N-\beta)$

Then we consider the small perturbations around the steady states and it was a linear ODE that resulted in exponential decay or just an exponential. We don't need to do this though, as we can look at the phase space as it has split up the graph into regions. These regions are $(-\infty, 0]$, $[0, \alpha]$, $[\alpha, \beta]$.

- For $(-\infty, 0]$ the system declines and so we draw arrows going away from 0 to the left.
- Then it's going to move towards α from 0 where it stops at the steady state α.
- From α to β, the particle is going to move from β to α.
- Anything larger than β, it shoots off to infinity.

From this, we can talk about the stability of the stable states. We can call 0 and β unstable and we can say α is a stable fixed point. So now we ask how do we describe $N(t)$ from this information?

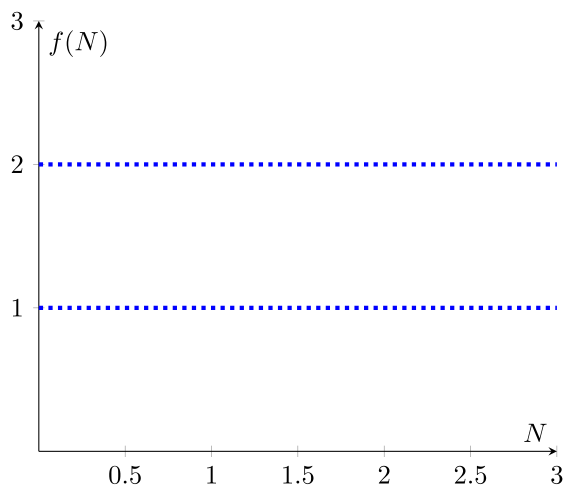

Figure 3: Steady States of the system

2 Modelling Spruce Budworm

Spruce Budworm eats spruce trees and populations have been watched and there exists some oscillatory behavior. Let's formulate some sort of model, firstly with a cartoon and then the assumptions to create a mathematical model. We consider $N =$ population density $\in \mathbb{R}$. We assume that;

- bounded growth, the logistic growth model (e.g. limited resources)
- predation (by birds)

and so we can write,

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{k}\right) - p(N)$$

and now a few assumptions for p,

- no predation if no prey, $p(0) = 0$
- predator population is not infinite, this means the rate of predation saturates as $n \to \infty$.
- If n is small, $p(N)$ is small.

We want a sigmoidal form of function so we can encode these assumptions.

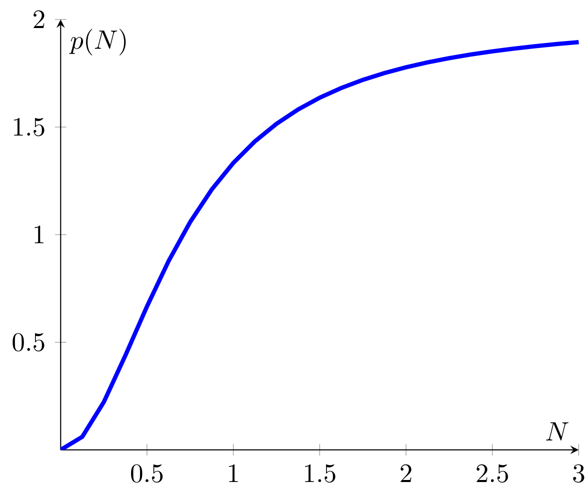

Figure 4: Sigmoidal Function, $p(N)$

We will let $p(N) = \frac{BN^2}{A^2+N^2}$, which will work quite nicely. This is a function seen all over Maths Biology. Hence, we write,

$$\frac{dN}{dt} = r_B N\left(1 - \frac{N}{k_B}\right) - \frac{BN^2}{A^2 + N^2}$$

where $r_B, k_B, B, A > 0$ and $N \geq 0$. In mathematical Biology we don't really have any big ideas about what our parameters are, we talk more generally about what our parameters actually mean. This makes it harder to know what's going on with multiple parameters.

2.1 Scaling and Non-dimensionalisation

The goal is to scale variables and time such that we get an equivalent system with fewer parameters, equivalent is in relation to the steady states and their stabilities.

Our stategy is to find constants α and β so that $u = \alpha$ and $\tau = \beta t$ yields a simpler system. The first thing we do is write down a new system, $\frac{du}{d\tau}$.

$$\frac{du}{d\tau} = \frac{dN}{dt}\frac{dt}{d\tau}\frac{du}{dN}$$

and so we can do some algebra (ommited), then we get to a point where we want to choose some values for our placeholders of α and β, given the form of the system, we choose that $\alpha = \frac{1}{A}$ and $\beta = B\alpha = \frac{B}{A}$ and hence the system simplifies to,

$$\delta u\tau = ru(1 - \frac{u}{q}) + \frac{u^2}{1+u^2}$$