Year 3 — Partial Differential Equations

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine (especially the typos!).

Contents

1 Introduction to PDEs

A differential equation that contains, in addition to the dependent variable and independent variables, one or more partial derivatives of the dependent variable is called a partial differential equation.

In general it may be written in the form,

$$f(x, y, \ldots, u, u_x, u_y, \ldots, u_{xx}, u_{yy}, \ldots) = 0 \tag{1}$$

involving several of the x, y, u_x, u_{xx} terms. Note that the notation $u_x = \frac{\partial u}{\partial x}$.

When we consider a PDE, we also consider it in a suitable domain. For us, the domain, D, will just be some domain of $\mathbb{R}^n$ in the variables $x, y, \ldots$. A solution of this equation will be a function $u = u(x, y, \ldots)$ which satisfy (1). We call the order of the PDE the highest order partial derivative appearing the equation.

Example. $u_{xxy} + xu_{yy} + 8u = 8y$ is a third order PDE.

Definition 1.1 (Linear)**.** We call a PDE linear if it is linear in the unknown function and all it's derivatives. For example,

$$yu_{xx} + 2xyu_{yy} + u = 1$$

and we have a further characterisation, called quasi-linear,

Definition 1.2 (Quasi-Linear)**.** A PDE is quasi linear if it linear in the highest-orderderivative of the unknown function,

$$u_x u_{xx} + xuu_y = \sin y$$

and furthermore, an equation that isn't linear is non-linear. IN this course we will consider mainly second order linear PDEs. The most general of these can be written as,

$$\sum_{i,j=1}^{n} A_{ij} u_{x_i x_j} + \sum_{i=1}^{n} B_i u_{x_i} + F_u = G$$

where we assume that $A_{ij} = A_{ji}$, we also assume that B_i, F and G are functions of the n independent variables x_i. If $G = 0$, then we have a homogenous PDE; otherwise it's non-homogenous.

If we consider an n^{th} order ODE, then what we end up with is a solution depending on n arbitrary constants. A similar thing applies to PDEs, but they are n arbitrary functions. To illustate, we solve $u_{xy} = 0$, where first we integrate wrt y, and we get $u_x = f(x)$ and then integrate wrt x and we get $u(x, y) = g(x) + h(y)$ where g and h are arbitrary functions.

1.1 Mathematical Problems

A mathematical problem is PDE along with some supplimentary conditions on that PDE. the conditions may be intial conditions that are of the form $u(x, 0) = f(x)$ or boundary conditions which depends on the boundary. Let us take the example of the following PDE,

$$u_t - u_{xx} = 0$$

Then an initial conditions for this PDE may be $u(x, 0) = \sin x$ and if we consider it on some boundary $0 \leq x \leq \ell$ some boundary conditions may be $u(0, t) = 0$ and $u(\ell, t) = 0$ for some $t \geq 0$ (This example is the heat equation for a rod of length ℓ). This problem is known as an initial boundary problem. Sometimes we have more conditions that specify the problem, for example some conditions on the derivative. If we have a boundary that is not bounded, then sometimes we won't have boundary conditions and then we have a initial-value problem or a Cauchy Problem.

Finally, we say that a problem is well posed if,

(i) Existence, there is at least one solution

(ii) Uniqueness, there is at most one solution

(iii) Continuity, the solutions depends continuously on the data. A small input in the input data must reach a small change in the output data.

1.2 Linear Operators

An operator is a mathematical rule which when applied to a function produces another function. For example where,

$$L[u] = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$
$$M[u] = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} + x\frac{\partial u}{\partial x}$$

then we say that $L = \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y}$ and $M = \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} + x\frac{\partial}{\partial x}$ are the differential linear operators. We note a few things before a formal definition, if we have linear operators L and M, then $(L+M)[u]$ is a linear operator and $(L+M)[u] = L[u] + M[u]$. Furthermore, we can do something similar with $LM[u] = L(M[u])$. In general, here is the definition,

Lemma 1.3. Let L, M and N be linear operators. In general, a linear operator satisfies the following,

- $L + M = M + L$ (commutativity of addition)
- $(L + M) + N = L + (M + N)$ (associativity of addition)
- $L(MN) = (LM)N$ (associativity of multiplication)
- $L(c_1 M + c_2 N) = c_1 LM + c_2 LN$ (distributivity)

and for Linear Differential operators with constant coefficients, we have that $LM = ML$.

Now consider a linear second order PDE,

$$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} + D(x,y)u_x + E(x,y)u_y + F(x,y)u = G(x,y)$$

then if we let,

$$L = A(x,y)\frac{\partial^2}{\partial x^2} + B(x,y)\frac{\partial u}{\partial x \partial y} + C(x,y)\frac{\partial^2}{\partial y^2} + D(x,y)\frac{\partial}{\partial x} + E(x,y)\frac{\partial}{\partial y} + F(x,y)$$

be a linear differential operator, then we can write $Lu = G$ and that is our PDE.

Let $v_1, v_2, \ldots, v_n$ be n functions satisfying
$$L[v_j] = G_j$$
for j running from 1 to n. Let $w_1, w_2, \ldots, w_n$ be n functions where $L[w_j] = 0$ for j running from 1 to n. If we let $u_j = v_j + w_j$ then $u = \sum_{j=1}^{n} u_j$ this is called the principle of linear superposition.

If we consider $u_{tt} - c^2 u_{xx} = G(x,t)$ if we solve this for $0 < x < L$ where $u(x,0) = g_1(x)$ and $u_t(x,0) = g_2(x)$ for $0 \le x \le L$ and $t \ge 0$. We also have boundary conditions $u(0,t) = g_3$ and $u(L,t) = g_4$. We can write this in the form, $l[u] = G$ and $m_1[u] = g_1$ and $M_2[u] = g_2$ and $M_3[u] = g_3$ and finally $M_4[u] = g_4$. We can now decomopse this into four different problems.

- $L[u] = G$, $M_1[u] = 0$, $M_2[u_1] = 0$, $M_3[u_1] = 0$ and $M_4[u_1] = 0$

– $L[u_2] = 0$, $M_1[u_2] = g_1$, $M_2[u_2] = 0$, $M_3[u_1] = 0$ and $M_4[u_1] = 0$

– $L[u_2] = 0$, $M_1[u] = 0$, $M_2[u_1] = g_2$, $M_3[u_1] = 0$ and $M_4[u_1] = 0$

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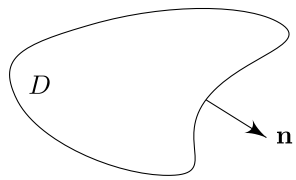
– $L[u_3] = 0$, $M_1[u] = 0$, $M_2[u_1] = 0$, $M_3[u_1] = 0$ and $M_4[u_1] = g_4$

and then solve the above and then add them together via the linear superposition.

1.3 Boundary Conditions

Assume we have $u_{xx} + u_{yy} = 0$ with a domain and boundary,



where $u(x, y) = f(x, y)$ along the boundary of D, then we have a Dirichlet Boundary condition. If $\frac{\partial u}{\partial x} = h(x, y) \to \partial D$ is a Neumann boundary condition. We can also have $\frac{\partial u}{\partial \mathbf{n}} = \underline{\nabla} u \cdot \mathbf{n}$. If we can split the boundary into two then we can have a mixed type boundary condition; $u(x, y) + \frac{\partial u}{\partial n} = h(x, y)$ this is called a Robin Boundary condition.

Exercise. Prove that $\mathbb{R}^3$, gradient, curl and divergence are all linear differential operators, ie. prove that,

$$L(f + g) = L(f) + L(g)$$
$$L(cf) = cL(f)$$

where $c \in R^3$ and f, g are elements.

Exercise. Solve,

$$5u'' - 4u' + 4u = e^x \cos x$$

for a solution $u(x) = \frac{1}{6} e^x \sin x + c_1 e^{\frac{2}{5}x} \cos \frac{4x}{5} + c_2 e^{\frac{2x}{5}} \sin \frac{4x}{5}$

We now define classical solutions. Assume we have a PDE,

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} = D$$

with a solution, $u(x, y)$, for a classical solution we need this solution continuously second differentiable.

Definition 1.4 (Smooth)**.** A function is smooth if it can be differentiated sufficiently enough.

If the PDE has order n, then a solution has class $\mathcal{C}^n$. If we consider

$$\frac{\partial u}{\partial t} + k\frac{\partial^2 u}{\partial x^2} = 0$$

A solution is classical if $u(x, t)$ is differentiable by all th variables n times.

2 First Order Linear and Nonlinear waves

We want to solve,

$$\frac{\partial u}{\partial t} = 0$$

for $u(x,t)$. We can integrate both sides wrt time,

$$\int_0^t \frac{\partial u}{\partial s}\, ds = 0$$

and so we see $u(x,t) - u(x,0) = 0$ and so $u(x,t) = f(x)$ where $f(x)$ is defined by the IC. For this to be classical $f(x) \in \mathcal{C}^1$. If $f(x) = x$, then we get $u(x,t) = xt + f(x)$ where $f(x) \in \mathcal{C}^1$.

If we want to solve $u_t = x - t$, then $u(x,t) = xt - \frac{1}{2}t^2 + f(x)$, or $u_x + tu = 0$ then we can use an integrating factor and then get $\frac{\partial u}{\partial t}(e^{tx}u) = 0$ and so $u(x,t) = e^{-tx}f(t)$ where $f(t) \in \mathcal{C}^1$.

Next let us add another term,

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

where c is a constant. This is a transport equation, and the solution is a travelling wave. This models a uniform fluid flow with speed c subject to the condition $u(x,t_0) = f(x)$. We aim to reduce this to an ODE. Let us introduce $\xi = x - ct$ (the characteristic variable), then $u(x,t) = v(\xi,t) = v(x-ct,t)$. Let us take partial derivatives,

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial \xi}\frac{\partial x}{\partial i}t = v_t - cv_\xi$$

and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} = v_\xi$$

and so we get, $v_t - cv_\xi + cv_\xi$ and so $v_t = 0$. Hence, $v = v(\xi) = v(x - ct)$.
Now let us put this more formally,

Proposition 2.1. If $u(x,t)$ is a solution to the PDE

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

defined on all $\mathbb{R}^2$. Then, $u(x,t) = v(x-ct)$ where $v(\xi)$ is a $\mathcal{C}^1$ function of the characteristic variable $\xi = x - ct$.

Now for an example,

Example. Solve,

$$\frac{\partial u}{\partial t} + 2\frac{\partial u}{\partial x} = 0$$

subject to $u(x,0) = \frac{1}{1+x^2}$. To solve this, consider the characteristic variable, $\xi = x - 2t$, then we can represent the solution in the form $v(x-ct)$. To see this we represent the PDE as,

$$\frac{\partial u}{\partial t} = -v_\xi + v_t$$

$$\frac{\partial u}{\partial x} = v_\xi \xi_x = v_\xi$$

and so we can plug these in and get,

$$v_t = 0$$

and so $v = v(x - 2t)$. Now we plug in the IC and get that $v(x) = \frac{1}{1+x^2}$ and so $v = \frac{1}{1+(x-2t)^2}$ and hence, $u(x,t) = \frac{1}{1+(x-2t)^2}$.

Let's go one step further with the transport equation with decay.

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + au = 0 \qquad a > 0$$

Example. We want to again use characteritics, so let $\xi = x - ct$ and so $u(x,t) = v(\xi,t) = v(x-ct,t)$. Then we get $u_t = -cv_\xi + v_t$ and $u_x = v_\xi$. Hence again we get that $\frac{\partial v}{\partial t} + av = 0$, solve by an integrating factor of e^{at} and conclude that $\frac{\partial}{\partial t}(ve^{at}) = 0$ and so $v = e^{-at}f(\xi)$. We can hence conclude that $v(\xi,t) = e^{-at}f(\xi)$ and $u(x,t) = e^{-at}f(x-ct)$. $f \in \mathcal{C}^1$.

Exercise. Solve,

$$\begin{cases} \frac{\partial u}{\partial t} - 3\frac{\partial u}{\partial x} = 0 \\ u(x,0) = e^{-x^2} \end{cases}$$

$$\begin{cases} \frac{\partial u}{\partial t} + 2\frac{\partial u}{\partial x} = 1 \\ u(x,0) = e^{-x^2} \end{cases}$$

Now let us adapt this such that $c = f(x)$, a non-uniform transport equation. It is of the form,

$$\frac{\partial u}{\partial t} + c(x)\frac{\partial u}{\partial x} = 0$$

To use the method of characteristics, we would like to know how the solution varies along a curve in the (x,t) plane. We can parametrise any curve and so let us let $h(t) = u(x(t),t)$ and we want to measure the rate of change as the solutions moves along some curve in the plane. Now we take the derivative of $h(x)$ wrt time,

$$\frac{\partial h}{\partial t} = \frac{\partial u}{\partial t}(x(t),t) + \frac{\partial u}{\partial x}(x(t),t)\frac{dx}{dt}$$

Now we assume that $\frac{dx}{dt} = c(x)$, then we can conclude that,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial t}(x(t),t) + c(x)\frac{\partial u}{\partial x}(x(t),t)$$

and since we are assuming that the curve is a solution, then this is just our PDE. Hence, $\frac{\partial u}{\partial t} = 0$. The solution is constant along the characteristic.

Definition 2.2 (Characteristic Curve)**.** The graph of a solution $x(t)$ to the autonomous ODE $\frac{dx}{dt} = c(x)$ is called the characteristic curve. For the transport equation with wave speed $c(x)$.

Proposition 2.3. Solutions to the linear transport equation $u_t + c(x)u_x = 0$ are constant along characteristic curves.

Hence, from $\frac{dx}{dt} = c(x)$, we can find a characteristic curve for the PDE; if we integrate it then we can say that $\beta(x) = \int \frac{dx}{c(x)}$, then we can achieve that $\beta(x) = t + c$ and so we say that $\xi = \beta(x) - t$.

Example. Solve

$$\frac{\partial u}{\partial t} + \frac{1}{1+x^2}\frac{\partial u}{\partial x} = 0$$

using the method of characteristics.

2.1 Quasi-Linear equations and methods of MoC

We can write

$$F(x, y, u, u_x, u_y) = 0 \qquad (x, y) \in D \subset \mathbb{R}$$

Then if we write $u_x = p$ and $q = u_y$. Then this solution is quasi-linear if,

$$F(x, y, u, p, q) = 0$$

is quasi-linear if the PDE is linear in first partial derivatives of the unknown function $u(x, y)$. We can write the most general form as,

$$a(x, y, u)\frac{\partial u}{\partial x} + b(x, y, u)\frac{\partial u}{\partial y} = c(x, y, u)$$

Some examples are,

$$x(y^2 + u)u_x - y(x^2 + u)u_y = (x^2 - y^2)u$$

We call a PDE semi-linear if it further satisfies a and b being independent of u,

$$a(x, y)u_x + b(x, y)u_y = c(x, y, u)$$

For example,

$$xu_x + yu_y = u^2 + x^2$$

We call a PDE linear if it is linear in each of the variable,

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = d(x, y)$$

If $d(x, y) = 0$ we get a homogenous first order PDE and if $d(x, y) \neq 0$ then we have a non-homogenous first order PDE. For example, a homogenous PDE,

$$xu_x + yu_y - nu = 0$$

and a non-homogenous PDE,

$$nu_x + (x + y)u_y - u = e^x$$

More generally, these are geometric surfaces described by $f(x, y, z, a, b) = 0$ and if this exists, then the solution is complete. We can also reduce a and b out. A solution can be written as $f(\phi, \psi) = 0$ where $\phi, \psi \in \mathbb{R}^3 \to \mathbb{R}^3$.

Example. Show that a family of spheres $x^2 + y^2(z - c)^2 = r^2$ satisfies the first order linear PDE $yp - xq = 0$ where $p = z_x$ and $q = z_y$.

Exercise. Show that the family of spheres $(x - a)^2 + (y - b)^2 + z^2 = r^2$ satisfy $z^2(p^2 + q^2 + 1) = r^2$ where $p = z_x$ and $q = z_y$.

Theorem 2.4. If $\phi = \phi(x, y, z)$ and $\psi = \psi(x, y, z)$ are two given functions of x, y and z and if $f(\phi, \psi) = 0$ where f is an arbitrary function of ϕ and ψ. Then $z = z(x, y)$ satisfies a first order PDE,

$$p\frac{\partial(\phi, \psi)}{\partial(y, z)} + q\frac{\partial(\phi, \psi)}{\partial(z, x)} = \frac{\partial(\phi, \psi)}{\partial(x, y)}$$

where

$$\frac{\partial(\phi, \psi)}{\partial(x, y)} = \begin{vmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{vmatrix}$$

Proof. Let $f(\phi, \psi) = 0$ and now let us differentiate by x and y, then,

$$\frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial \psi}\frac{\partial \psi}{\partial x} + \frac{\partial f}{\partial \psi}\frac{\partial \psi}{\partial z}\frac{\partial z}{\partial x} = 0$$

and simplify,

$$\frac{\partial f}{\partial \phi}\left(\frac{\partial \phi}{\partial x} + p\frac{\partial \phi}{\partial z}\right) + \frac{\partial f}{\partial \psi}\left(\frac{\partial \psi}{\partial x} + p\frac{\partial \psi}{\partial z}\right) = 0$$

and now we do the same thing for y,

$$\frac{\partial f}{\partial \phi}\left(\frac{\partial \phi}{\partial y} + q\frac{\partial \phi}{\partial z}\right) + \frac{\partial f}{\partial \psi}\left(\frac{\partial \psi}{\partial y} + q\frac{\partial \psi}{\partial z}\right)$$

and now let us write these in matrix form, There is a non-trivial solution is and only if the determinant matrix is zero. If we calculate this determinant we get the solution of the PDE. □

If we consider a PDE of the form,

$$a(x, y, u)u_x + b(x, y, u)u_y - c(x, y, u) = 0$$

If we assume that $z = u$ is a solution surface, then we can define $f(x, y, u) = u(x, y) - u = 0$ Then we can write it as the following,

$$au_x + bu_y - c = (a, b, c) \cdot (u_x, u_y, -1) = 0$$

and so we can write it as $\underline{\nabla} u \cdot (a, b, c)$ and so we know that $\underline{\nabla} u$ is normal to the surface and so (a, b, c) must be tangent to the surface and we call the direction of the vector the characteristic direction. Now we seek to parametrise a curve such that (a, b, c) is tangent to the curve. If we paramerise the curve by $(x(t), y(t), u(t))$, then the tangent to the curve will be $(\frac{dx}{dt}, \frac{dy}{dt}, \frac{du}{dt}) = (a, b, c)$. Now we can find the chateristic curve as we see that

$$\begin{cases} \frac{dx}{dt} = a(x, y, u) \\ \frac{dy}{dt} = b(x, y, u) \\ \frac{du}{dt} = b(x, y, u) \end{cases}$$

and we can write them as, $\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)} = dt$. Now we move to another theorem,

Theorem 2.5. The general solution of a first order quasi-linear PDE

$$a(x, y, u)\frac{\partial u}{\partial x} + b(x, y, u)\frac{\partial u}{\partial y} = c(x, y, u)$$

is $f(\phi, \psi) = 0$ where f is an arbitrary function of $\psi, \phi : \mathbb{R}^3 \to \mathbb{R}^3$, $\phi = c_1$ and $\psi = c_2$ are solution curves of the characteristic equations,

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}$$

and $\phi(x, y, u) = c_1$ and $\psi(x, y, u) = c_2$ are the family of characteristic curves.

Proof. Let $\phi(x, y, u) = c_1$ and $\psi(x, y, u) = c_2$. From the first, we can say that,

$$d\phi = \phi_x dx + \phi_y dy + \phi_u du = 0$$

and so,

$$\frac{d\phi}{dt} = \phi_x\frac{dx}{dt} + \phi_y\frac{dy}{dt} + \phi_u\frac{du}{dt} = 0$$

and so we get $a\phi_x + b\phi_y + c\phi_u = 0$ and similarly we can get $a\psi_x + b\psi_y + c\psi_u = 0$. (Exercise). We can now take each of the terms to the right hand side in term and then show that,

$$\frac{a}{\frac{\partial(\phi,\psi)}{\partial(y,u)}} = \frac{b}{\frac{\partial(\phi,\psi)}{\partial(u,x)}} = \frac{c}{\frac{\partial(\phi,\psi)}{\partial(x,y)}} \tag{$*$}$$

and so now from Theorem 2.4, and using the above result $(*)$ we get an expression of the form,

$$ap + bq = c$$

$\square$