Real Analysis (Incomplete)

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September 27, 2021

Contents

1 Overview

The reals ($\mathbb{R}$) have a few properties:

1. They are a field, i.e. a groupoid with two binary operations.
2. They are ordered
3. They are also complete.

We will also look at supremum and the infimum.

We are also going to look at the extended real numbers. We are going to add two more fictitious points. $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$.

2 Properties of the Reals

We will be taking the axiomatic view point of the real numbers. No construction with Dedekind cuts or Cauchy sequences. All of these are isomorphic.

2.1 Field Properties

The real numbers are a set, $\mathbb{R}$, with two binary operations, $+$ and $\times$. They must satisfy the following axioms. So take $a, b, c \in \mathbb{R}$:

1. $a + b = b + a$ and $ab = ba$ (commutativity)
2. $(a + b) + c = a + (b + c)$ and $a(bc) = (ab)c$ (associativity)
3. $a(b + c) = ab + ac$ (distributivity)
4. There are two distinctive identities 0 (additive identity) and 1 (multiplicative identity), such that $a + 0 = 0 + a = a$ and $a1 = 1a = a$
5. We also have inverses, $-a$ (additive inverse) such that $a + -a = 0$ and if $a \neq 0$, there is a real number $\frac{1}{a}$ such that: $a(\frac{1}{a}) = 1$

2.2 Order Relation

The real numbers are ordered, that means:

1. For each pair of reals a and b, exactly one of the following is true

$$a = b \qquad a < b \qquad b < a$$

2. It is also transitive, if $a < b$ and $b < c$, then $a < c$
3. If $a < b$ then $a + c < b + c$ for any c, and if $0 < c$, then $ac < bc$

2.3 Supremum

Let $S \subset \mathbb{R}$. If there exists $b \in \mathbb{R}$ such that $x \leq b \quad \forall x \in S$ then S is bounded above and b is an upper bound of S.

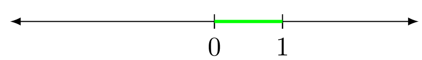
If β is an upper bound of S, but no number less than β is, then β is called the supremum of S, denoted:

$$\beta = \sup S$$

Figure 1: *Let S be the orange set and then b is an upper bound of S and β is $\sup S$*

We also call the supremum the least upper bound.

Example 1. *$S = [\,0, 1\,]$ and prove $\sup S = 1$*



Solution 1. *Take our diagram from above, We need to check that $x \leq 1 \quad \forall\, x \in S$, which is definitionally true.*
Secondly we need to prove that $\forall\, b < 1$, $\exists\, x \in S$, $b < x$, which is again trivially true. So $\sup S = 1$

Example 2. *Take $T = (\,0, 1\,)$ where $\sup T = 1$*

Solution 2. *Again every number is less than 1, but if you take any number less than one you can always find another element larger.*
NB: The supremum here isn't in the set

2.4 Infimum

Similarly, if there exists an $a \in \mathbb{R}$ such that $a \leq x \quad x \in S$, then S is bounded below and a is a lower bound of S.

If α is a lower bound of S, but no number is greater than α is, then α is called the infimum of S:

$$\alpha = \inf\ S$$

Figure 2: *Let S be the orange set and then a is a lower bound of S and α is* inf *S*

Another name for the infimum is the greatest lower bound.

2.5 Completeness Axiom

Do the supremum and the infimum actually exist? Well, not all subsets are bounded above, i.e. $\mathbb{R} \subset \mathbb{R}$ or what about the empty set? This is what the completeness axiom does:

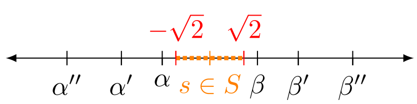
1. If a non-empty set of real numbers are bounded above, then it has a supremum.

So the reals are a **complete ordered field**

The completeness axiom is distinguishing of the reals. They are the only complete ordered field. The rationals possess everything but completeness in terms of our axioms.

Example 3. *We restrict to the $\mathbb{Q}$, $S = \{r \in \mathbb{Q} : r^2 < 2\}$. Find the supremum and infimum.*

Solution 3. *If we take the example below;*



we can say that we won't reach $\sqrt{2}$ in the supremum or $-\sqrt{2}$ in the infimum. This is because we are using rationals and $\sqrt{2}$ is an irrational. We can go either way and there is always a number closer to $\sqrt{2}$.

This proves that rationals are not complete.

3 Extended Real Numbers

It is convenient to attach ∞ and $-\infty$ to the reals. How do they fit in? Firstly lets look at orders. Take $x \in \mathbb{R}$, then:

$$-\infty < x < \infty$$

Now if a set S is unbounded above or below, we can write:

$$\sup\ S = \infty \quad \inf\ S = -\infty$$

Example 4. *Find the infimum of $S = \{x \in \mathbb{R} : x : 2\}$*

Solution 4. *As there is technically no lower bound, it is $-\infty$*

We usually denote the extended reals with the symbol, $\overline{\mathbb{R}}$ or $[-\infty, \infty]$ or $\mathbb{R} \cup \{-\infty, \infty\}$

3.1 Arithmetic

If $a \in \mathbb{R}$,

1. Then:

$$a + \infty = \infty + a = \infty$$
$$a - \infty = -\infty + a = -\infty$$
$$\frac{a}{\infty} = \frac{a}{-\infty} = 0$$

2. and $0 < a$, then:

$$a\infty = \infty a = \infty$$
$$a(-\infty) = (-\infty)a = -\infty$$

3. and $a < 0$, then:

$$a\infty = \infty a = -\infty$$
$$a(-\infty) = (-\infty)a = \infty$$

We also define:

1. $\infty + \infty = \infty\infty = (-\infty)(-\infty) = \infty$
2. and also $-\infty - \infty = \infty(-\infty) = (-\infty)\infty = -\infty$
3. and finally, $|\infty| = |-\infty| = \infty$

We say it isn't useful to define; $\infty - \infty$, $0 \cdot \infty$, $\frac{\infty}{\infty}$ and $\frac{0}{0}$. We call them indeterminate forms.

4 Triangle Inequality

As we can use the ordered relation of the reals we can produce something known as the triangle inequality.

Theorem 4.1: Triangle Inequality

It states for any $a, b \in \mathbb{R}$, we have:

$$|a+b| \leq |a| + |b|$$

Proof. There are four possibilities:

1. If $0 \leq a$ and $0 \leq b$, then $0 \leq a+b$, so $|a+b| = a+b = |a|+|b|$.
2. If $a \leq 0$ and $b \leq 0$, then $a+b \leq 0$, so $|a+b| = -a+(-b) = |a|+|b|$.
3. If $0 \leq a$ and $b \leq 0$, then $a+b = |a|-|b|$.
4. If $a \leq 0$ and $0 \leq b$, then $a+b = -|a|+|b|$.

It holds in cases (c) and (d), since

$$||a|-|b|| = \begin{cases} |a|-|b| & \text{if } |b| \leq |a|, \\ |b|-|a| & \text{if } |a| \leq |b|. \end{cases}$$

□

5 Open and Closed Sets

Definition 5.1: Open Interval

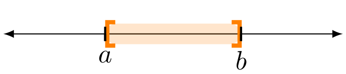
e define an open interval between a and b, $a, b \in \overline{\mathbb{R}}$, as such:

$$(a,\, b) = \{x : a < x < b\}$$

Definition 5.2: Closed Interval

e define a closed interval between a and b, $a, b \in \overline{\mathbb{R}}$, as such:

$$[a,\, b] = \{x : a \leq x \leq b\}$$

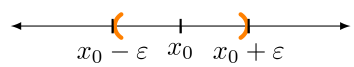


5.1 Neightbourhoods

A neighborhood is used to talk about closeness of points. We are now going to go through a load of set definitions!

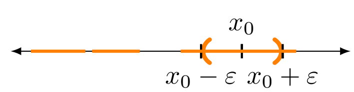
Definition 5.3: ε-neighborhood

f x_0 is a real number and $\varepsilon > 0$, then the open interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ is an ε-neighbourhood of x_0.



Definition 5.4: Neighborhood

f a set S contains an ε-neighbourhood of x_0, then S is a neighborhood of x_0. i.e. we need $(x_0 - \varepsilon,\, x_0 + \varepsilon) \subset S$



Definition 5.5: Interior Point

f S is a neighbourhood of x_0, then x_0 is an interior point of S.

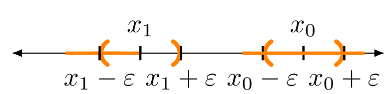


Figure 3: x_0 is an interior point, however x_1 is not

Definition 5.6: Interior

he set of interior points of S is the interior of S, denoted by S^0

Definition 5.7: Interior Point

If every point of S is an interior point, ($S^0 = S$), then S is open .

Definition 5.8: Closed

set is closed if S^c is open.

Example 5. *Any open interval $S = (a,\ b)$ is open.*

Solution 5. *Need to show that $\forall x_0 \in (a,\ b)$, $\exists\, \varepsilon > 0 : (x_0 - \varepsilon, x_0 + \varepsilon) \subset (a,\ b)$*
Assume that $a, b \in \mathbb{R}$. Let $x_0 \in (a,\ b)$ and let $\varepsilon = \min(x_0 - a, b - x_0)$. Then clearly $(x_0 - \varepsilon,\ x_0 + \varepsilon) \subset (a,\ b)$.
The rest of the proof is left as an exercise. (Where $a = -\infty$ or $b = \infty$).
Now we know that $\mathbb{R}$ is open and S^c, where $S^c = (-\infty,\ a) \cup [b, \infty)$, and $\varnothing$ is closed

We also note that because of a vacouity argument $\varnothing$ is also open, hence $\mathbb{R}$ is also closed. So $\mathbb{R}$ and $\varnothing$ are both open and closed.

5.2 Unions and Intersections

Theorem 5.1: Unions and Intersection of closed and open sets

1. The union of open sets is open
2. The intersection of closed sets is closed

These apply to abtritary collections (finite or infinite of open and closed sets).

Proof. First lets prove (1), so let $\mathcal{G}$ be a collection of open sets.
Let $S = \bigcup_{G \in \mathcal{G}} G$, If $x_0 \in S$, then $x_0 \in G_0$ for some $G_0 \in \mathcal{G}$. Since G_0 is open, it must contain an ε- neighborhood of x_0. The ε-neighborhood, $(x_0 - \varepsilon, x_0 + \varepsilon)$, is in S, hence S is a neighborhood of x_0 and x_0 is an interior point of S.
Since x_0 was arbitrary, then all points in S are interior points and hence, S is open.
Now for part (2) of the theorem. Let $\mathcal{F}$ be a collection of closed sets and let $T = \bigcap_{F \in \mathcal{F}} F$. Then $T^c = \bigcup_{F \in \mathcal{F}} F^c$.
Since each F^c is open, that means T^c is open by (1).
Therefore T is closed $\square$

Example 6. *For $a, b \in \mathbb{R}$, the sets $[a,\ b]$ is closed.*

Solution 6. *Since $[a,\ b]^c = (-\infty,\ a) \cup (b, \infty)$. Since its a union of open intervals, it is open. Hence making $[a,\ b]$ closed.*

Example 7. *What about $[a, b)$, or $(a, b]$ for $a, b \in \mathbb{R}$?*

Solution 7. *These are half-open or half-closed intervals. These are neither open nor closed. Take $[a,\ b)$, then a isn't an interior point of the set, hence it's not open. Now take the compliment of the set $[a,\ b)^c = (-\infty,\ a) \cup [b,\ \infty)$ and now b is no longer an interior point. Hence, not closed.*

Example 8. *What about: $(-\infty,\ a]$ or $[a,\ \infty)$*

Solution 8. *Exercise*

Now, what about the intersection of open sets and union of closed sets. Well, it can be proved that the intersection of finitely many open sets is open and union of finitely many closed sets is closed. However the infinite versions of these statements need not be the same.

The concept of open and closed sets, doesn't form a dichotomy (A set is partitioned into two. i.e. odd or even naturals). A set can be neither open or closed or both.

Definition 5.9: Deleted Neighborhood

deleted neighbourhood of a point x_0 is a set that contains every point of some neighborhood of x_0 except x_0. For example:

$$S = \{x : 0 < |x - x_0| < \varepsilon\}$$

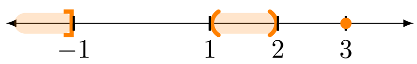
is a deleted neighborhood of x_0. We also say it is deleted ε-neighborhood of x_0.

Let S be a subset of $\mathbb{R}$.

Definition 5.10: Limit Points

x_0 is a limit point of S if every deleted neighbourhood of x_0 contains a point of S

Example 9. *Let $S = (-\infty, -1] \cup (1, 2) \cup \{3\}$. Find the limit points of S.*

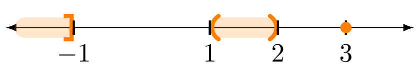


Solution 9. *Every point in S is limit point, apart from $\{3\}$*

Definition 5.11: Boundary Points

x_0 is a boundary point of S if every neighborhood of x_0 contains at least one point in S and one not in S. The set of boundary points of S is the boundary of S, denoted by ∂S. The closure of S, denoted by $\overline{S}$, is $\overline{S} = S \cup \partial S$

Example 10. *Let $S = (-\infty, -1] \cup (1, 2) \cup \{3\}$. Find the closure of S.*

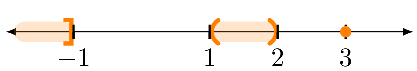


Solution 10. *The boundary points of S are: $\partial S = \{-1, 1, 2, 3\}$ and $\overline{S} = (-\infty, -1] \cup [1, 2] \cup \{3\}$*

Definition 5.12: Isolated Point

x_0 is an isolated point to S if $x_0 \in S$ and there is a neighborhood of x_0 that contains no other points of S.

Example 11. *Let $S = (-\infty, -1] \cup (1, 2) \cup \{3\}$. Find the isolated points of S.*

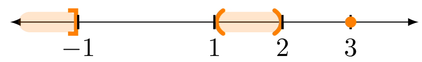


Solution 11. *Theres only one isolated point of S, $\{3\}$.*

Definition 5.13: Exterior

x_0 is interior to S if x_0 is in the interior of S^c. The collection of such points is the exterior of S.

Example 12. *Let $S = (-\infty, -1] \cup (1, 2) \cup \{3\}$. Find the exterior of S.*



Solution 12. *We can write $S^c = (-1, 1] \cup [2, 3) \cup (3, \infty)$ and then we can find the exterior of S, $(S^c)^o = (-1, 1) \cup (2, 3) \cup (3, \infty)$.*

Theorem 5.2: Limit point closure

A set S is closed if and only if no point S^c is a limit point of S

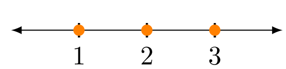
Proof. Begin by proving the forward direction. If S is closed, then S^c is open and then for any $x_0 \in S^c \exists$ ε-neighborhood contained in S^c, hence x_0 cannot be a limit point.

Next for the reverse direction. If no point of S^c is a limit point of S, every point $x_0 \in S^c$ has neighborhood contained in S^c, hence S^c is open. Therefore, S is closed. □

Corollary 1. *A set is closed if and only if it contains all its limit points*

What if it has no limit points? Well, it's just closed. Let us take an example:

Example 13. *Take $S = \{1, 2, 3\}$ and determine whether it closed or not.*



Solution 13. *You can take a deleted ε-neighborhood of all of these points and see that none of them are limit points. Hence the set is closed.*

5.3 Open Coverings

A collection $\mathcal{H}$ of open sets is an open covering of a set S if every point in S is contained in a set H belonging to $\mathcal{H}$; that is, if $S \subset \{H : H \in \mathcal{H}\}$

Theorem 5.3: Heine-Borel Theorem

If $\mathcal{H}$ is an open covering of a closed and bounded subset S of the real line, then S has an open covering $\widetilde{\mathcal{H}}$ consisting of finitely many open sets belonging to $\mathcal{H}$.

Proof. Since S is bounded, it has supremum, α, and infimum, β. As S is closed, we can say that α and β are in S. Now we shall define a couple of things:

$$S_t = S \cap [\alpha, t] \qquad \alpha \leq t$$

and let

$$F = \{t : \alpha \leq t \leq \beta \text{ and finitely many sets from } \mathcal{H} \text{ to cover } S_t\}$$

As we know that $S_\beta = S$, all we need to prove is $\beta \in F$. We shall use the completeness of the reals to prove this.

Starting with the things we know, $\alpha \in S$. We can then deduce that $S_\alpha = \{\alpha\}$, which is contained in some open set, H_α from $\mathcal{H}$ as we know that $\mathcal{H}$ covers S, therefore $\alpha \in F$. Since F is non-empty and bounded above by β we wish to show that the supremum, γ, can be seen the same as β. As we definitionally know that $\gamma \leq \beta$ from F, it suffices to know that $\gamma \nless \beta$. Let us consider two cases:

Case 1: Suppose that $\gamma < \beta$ and $\gamma \notin S$. Since S is closed, γ is not a limit point of S. This means that $\exists\, \varepsilon > 0$:

$$[\gamma - \varepsilon, \gamma + \varepsilon] \cap S = \varnothing$$

Then $S_{\gamma-\varepsilon} = S_{\gamma+\varepsilon}$, which can't happen, due to γ being a supremum $S_{\gamma-\varepsilon}$ has a subcovering from $\mathcal{H}$, but $S_{\gamma+\varepsilon}$ wouldn't. This is a contradiction and hence if $\gamma \notin S$, then $\gamma \nless \beta$.

Case 2: Suppose that $\gamma < \beta$ and $\gamma \in S$. Then there is an open set $H_\gamma \in \mathcal{H}$ along with an interval, for $\varepsilon > 0$, $[\gamma - \varepsilon, \gamma + \varepsilon]$. It then follows that since $S_{\gamma-\varepsilon}$ has a finite covering, so does $S_{\gamma+\varepsilon}$. This is a contradiction from the definition of γ. Hence, if $\gamma \in S$ then $\gamma \nless \beta$.

So we know that $\beta = \gamma$. Therefore H_β exists and is in $\mathcal{H}$. H_β contains β and an interval of the form, for some $\varepsilon > 0$: $[\beta - \varepsilon, \beta + \varepsilon]$. Since we know that $S_{\beta-\varepsilon}$ is covered finitely by some collection: $\{H_1, \ldots, H_k\}$, then S_β is covered by some collection: $\{H_1, \ldots, H_k, H_\beta\}$. Hence, $S_\beta = S$. □

Definition 5.14: Compactness

A set is compact if it is closed and bounded

Theorem 5.4: Bolzano-Weirstrass Theorem

Every bounded infinite set of real numbers has at least one limit point

Proof. It will suffice to show that a bounded nonempty set without a limit point can only contain a finite number of elements.

If S has no limit points, then S is closed and every point, $x \in S$ has a open neighborhood, N_x, that only contains itself. The collection:

$$\mathcal{H} = \{N_x : x \in S\}$$

is an open covering for S. Heine-Borel Theorem states that S can be covered by finitely many elements of $\mathcal{H}$, say $N_{x_1}, \ldots, N_{x_n}$. Since these sets only depend on a finite set of points, $x_1, \ldots x_n$. Then $S = \{x_1, \ldots x_n\}$ and hence, is finite. □

6 Limits

6.1 Defining Limits

We consider limits of real functions, that is $f : X \to \mathbb{R}$, with $X \subset \mathbb{R}$.

Definition 6.1: Limit

We say that $f(x)$ approaches the limit L as x approaches x_0, and write

$$\lim_{x \to x_0} f(x) = L$$

if f is defined on some deleted neighbourhood of x_0 and, for every $\varepsilon > 0$, there is a $\delta > 0$ such that:

$$|f(x) - L| < \varepsilon$$

if

$$0 < |x - x_0| < \delta$$

Theorem 6.1: Limit Uniqueness

If $\lim\limits_{x \to x_0} f(x)$ exists, then it is unique, that is, if:

$$\lim_{x \to x_0} f(x) = L_1 \qquad \text{and} \quad \lim_{x \to x_0} f(x) = L_2$$

then $L_1 = L_2$

Proof. Let $\exists \varepsilon > 0$, such that

$$|f(x) - L_i| < \varepsilon \text{ if } 0 < |x - x_0| < \delta_i$$

for $i = 1, 2$

Now, let us look at a $|L - 1 - L_2|$ and let $\delta = \min(\delta_1, \delta_2)$.

$$\begin{aligned} |L_1 - L_2| &= |L_1 - f(x) + f(x) - L_2| \\ &\le |L_1 - f(x)| + |L_2 - f(x)| < 2\varepsilon \end{aligned}$$

Given we know that ε is arbitrarily small, then $|L_1 - L_2|$ is arbitrarily small and hence, $L_1 = L_2$. □

Theorem 6.2: Algebra of Limits

If $\lim_{x \to x_0} f(x) = L_1$ and $\lim_{x \to x_0} g(x) = L_2$, then:

$$\begin{aligned} \lim_{x \to x_0} (f + g) &= L_1 + L_2 \\ \lim_{x \to x_0} (f - g) &= L_1 - L_2 \\ \lim_{x \to x_0} (fg) &= L_1 L_2 \\ \lim_{x \to x_0} \left(\frac{f}{g}\right) &= \frac{L_1}{L_2} \qquad \text{if } L_2 \neq 0 \end{aligned}$$

Proof. long and tedious □

6.2 One Sided Limit

Definition 6.2: Left-hand limits

We say that $f(x)$ approaches the left-hand limit L as x approaches x_0 from the left and write:

$$\lim_{x \to x_0^-} f(x) = L$$

if f is defined on some open interval (a, x_0) and, for each $\varepsilon > 0, \exists\, \delta > 0$,

$$|f(x) - L| < \varepsilon \text{ if } x_0 - \delta < x < x_0$$

Definition 6.3: Right-hand limit

We say that $f(x)$ approaches the right-hand limit L as x approaches x_0 from the right and write:

$$\lim_{x \to x_0^+} f(x) = L$$

if f is defined on some open interval (x_0, b) and, for each $\varepsilon > 0, \exists\, \delta > 0$,

$$|f(x) - L| < \varepsilon \text{ if } x_0 < x < x_0 + \delta$$

Theorem 6.3

A function f has a limit at x_0 $\iff$ it has right and left handed limits and they are equal.

$$\lim_{x \to x_0} f(x) = L$$

if and only if

$$f(x_0-) = f(x_0+) = f(x_0)$$

Proof. coming soon □

6.3 Limits at $\pm\infty$

Definition 6.4: Limit at infinity

We say that $f(x)$ approaches the limit L as x approaches ∞, and write:

$$\lim_{x \to x_0} f(x) = L$$

if f is defined on an interval (a, ∞) and, for each $\varepsilon > 0$, there is a number β st,

$$|f(x) - L| < \varepsilon \qquad \text{if } x > \beta$$

Definition 6.5: Left infinite limit

We say $f(x)$ approaches ∞ as x approaches x_0 from the left, and write:

$$f(x_0-) = \infty$$

if f is defined on an interval (a, x_0) and, for each real number M, there is a $\delta > 0$ such that:

$$f(x) > M \text{ if } x_0 - \delta < x < x_0$$

NB! When we say a limit exists, we mean that it is finite, i.e. not $\pm\infty$. If it is, we can say it exists in the extended reals.

Also with infinite limits, we know that the 'Uniqueness of Limits' and the 'Algebra of Limits' are also valid when x_0 are replaced by $\pm\infty$.

The 'Alegbra of Limits' rules are also valid if $L_1, L_2 = \infty$ provided the RHS are not indeterminant forms.

6.4 Monotonics

Definition 6.6: Monotonicity

A function f is nondecreasing on an interval I if:

$$f(x_1) \leq f(x_2) \quad \text{if } x_1, x_2 \in I \text{ and } x_1 < x_2$$

or nondecreasing if,

$$f(x_1) \geq f(x_2) \quad \text{if } x_1, x_2 \in I \text{ and } x_1 < x_2$$

We further define that if the '$\leq$' can be replaced with a '$<$', then f is strictly monotonic on I

Theorem 6.4

Suppose that f is monotonic on (a, b) and define

$$\alpha = \inf_{a<x<b} f(x) \text{ and } \sup_{a<x<b} f(x)$$

1. If f is nondecreasing, then $f(a+) = \alpha$ and $f(b-) = \beta$
2. If f is nonincreasing, then $f(a+) = \beta$ and $f(b-) = \alpha$.
3. If $a < x_0 < b$, then $f(x_0+)$ and $f(x_0-)$ exist and are finite; moreover;

$$f(x_0-) \leq f(x_0) \leq f(x_0+)$$

if f is nondecreasing, and

$$f(x_0-) \geq f(x_0) \geq f(x_0+)$$

if f is nonincreasing

Proof. Too long and tedious to typeset □

7 Continuity

Now we have defined limits, we can now define continuity.

Definition 7.1: Continuity at x_0

We say that f is continuous at x_0 if f is defined on an open interval (a, b) containing x_0 and that $\lim_{x \to x_0} f(x) = f(x_0)$.

Definition 7.2: Left continuity at x_0

We say f is continuous from the left at x_0 if f is defined on an open interval (a, x_0) and $f(x_0-) = f(x_0)$.

Definition 7.3: Right Continuity at x_0

we say f is continuous from the right at x_0 if f is defined on an open interval (x_0, b) and $f(x_0+) = f(x_0)$.

Theorem 7.1

A function f is continuous at x_0 if and only if f is defined on an open interval (a, b) containing x_0 and for each $\varepsilon > 0$ there is a $\delta > 0$ st,

$$|f(x) - f(x_0)| < \varepsilon \tag{1}$$

whenever $|x - x_0| < \delta$

Theorem 7.2

A function f is continuous from the right at x_0 if and only if f is defined on an interval $[x_0, b)$ and for each $\varepsilon > 0 \exists \delta > 0$ st (1) holds whenever: $x_0 \leq x < x_0 + \delta$

Theorem 7.3

A function f is continuous from the left at x_0 if and only if f is defined on an interval $(a, x_0]$ and for each $\varepsilon > 0 \exists \delta > 0$ st (1) holds whenever: $x_0 - \delta < x \leq x_0$

Note that f is continuous if and only if $f(x_0-) = f(x_0+) = f(x_0)$.

Definition 7.4: Continuous on a set

A function f is continuous on an open interval (a, b) if it is continuous at every point in (a, b). If, in addition,

$$f(b-) = f(b) \tag{2}$$

or

$$f(a+) = f(a) \tag{3}$$

then f is continuous on $(a, b]$ or $[a, b)$ respectively. If both are true then f is continuous on $[a, b]$.

More generally, if S is a subset of D_f consisting of finitely or infinitely many disjoint intervals, then f is continuous on S if f is continuous on every interval in S. (From here on, if we say "f is continuous on S" we mean S is a set of this kind.).

7.1 Discontinuities

Definition 7.5: Piecewise Continuity

f is piecewise continuous on $[a, b]$ if

1. $\exists f(x_0+) \forall x_0 \in [a, b)$
2. $\exists f(x_0-) \forall x_0 \in (a, b]$
3. $f(x_0+) = f(x_0-) = f(x_0)$ for all but finitely many points $x_0 \in (a, b)$

If (3) fails to hold at some x_0 in (a, b), f has a jump discontinuity.

Definition 7.6: Removable discontinuity

Let f be defined on a deleted neighborhood of x_0 and be discontinuous (perhaps even undefined) at x_0. We say that f has a removable discontinuity at x_0 if $\lim_{x \to x_0} f(x)$ exists. In this case, the function

$$g(x) = \begin{cases} f(x) & \text{if } x \in D_f \text{ and } x \neq x_0 \\ \lim_{x \to x_0} f(x) & \text{if } x = x_0 \end{cases}$$

is continuous at x_0.

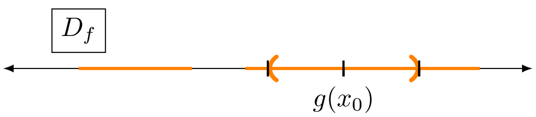
7.2 Continuity Arithmetic

Theorem 7.4

If f and g are continuous on a set S, then so are $f+g$, $f-g$ and fg. So is $\dfrac{f}{g}$ given $g \neq 0$ at x_0.

Theorem 7.5

Suppose that g is continous at x_0, $g(x_0)$ is an interior point of D_f and f is continuous at $g(x_0)$. Then $f \circ g$ is continuous at x_0.



So the above theorem is saying that we must have some $(g(x_0)-\varepsilon,\, g(x_0)+\varepsilon) \subset D_f$ or even that; $\lim_{x \to x_0} f(g(x)) = f(g(x_0))$.

Proof. Suppose $\varepsilon > 0$, since $g(x_0) \in D_f^o$ and f is continous at $g(x_0), \exists\, \delta_1 > 0$ st, $f(t)$ is defined and

$$|f(t) - f(g(x_0))| < \varepsilon \text{ if } |t - g(x_0)| < \delta_1 \tag{4}$$

Since g is continuous at x_0, $\exists\, \delta_2 > 0$ st, $g(x)$ is defined (why?) and

$$|g(x) - g(x_0)| < \delta_1 \text{ if } |x - x_0| < \delta_2 \tag{5}$$

Then (4) and (5) imply that,

$$|f(g(x)) - f(g(x_0))| < \varepsilon \text{ if } |x - x_0| < \delta_2$$

□

8 Boundedness

Definition 8.1: Bounded Below

A funtion f is bounded below on a set S if theres an $m \in \mathbb{R}$

$$f(x) \geq m \qquad \forall x \in S$$

In this case,

$$V = \{f(x) : x \in S\}$$

has an infimum, α, and we write,

$$\alpha = \inf_{x \in S} f(x)$$

If $\exists\, x_1 \in S$, such that $f(x_1) = \alpha$, then we say that α is the minimum of f on S and write:

$$\alpha = \min_{x \in S} f(x)$$

Definition 8.2: Bounded Above

f is bounded above on S, if $\exists\, M \in \mathbb{R}$, such that, $f(x) \leq M \quad \forall x \in S$. Then we can write;

$$\beta = \sup_{x \in S} f(x)$$

If $\exists\, x_2 \in S$, such that $f(x_2) = \beta$, then we say that β is the minimum of f on S and write:

$$\beta = \max_{x \in S} f(x)$$

Definition 8.3: Bounded

If f is both bounded below and bounded above on a set S, then f is bounded on S.

Theorem 8.1: Boundedness Theorem

If f is continuous on a finite closed interval $[a, b]$, then f is bounded on $[a, b]$

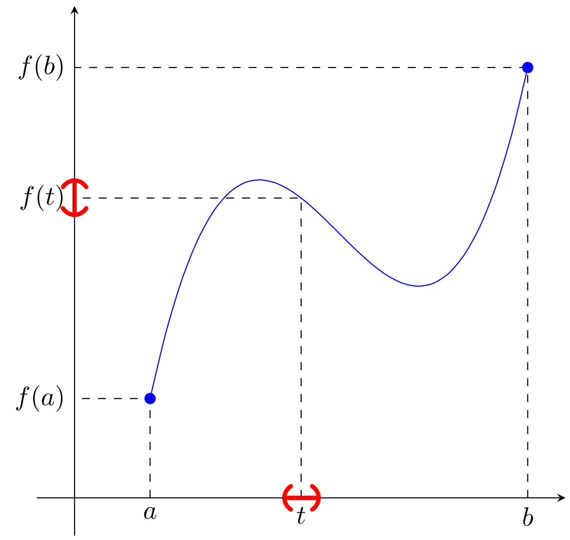


Figure 4: Assume f is bounded, it curves again, I promise...

Proof. Suppose we take a $t \in [a, b]$. Since f is continuous at t $\exists$ an open interval, $t \in I_t$, st,

$$|f(x) - f(t)| < 1 \qquad \text{if } x \in I_t \cap [a, b] \qquad (*)$$

The collection $\mathcal{H} = \{I - t : a \leq t \leq b\}$ is an open cover of $[a, b]$. Since, $[a, b]$ is compact, then by the Heine-Borel theorem, there exists a finite sub-cover made up of intervals $I_{t_1}, \ldots, I_{t_n}$. By $(*)$, taking $t = t_i$, then,

$$|f(x) - f(t_i)| < 1 \qquad \text{if } x \in I_{t_i} \cap [a, b]$$

Therefore,

$$\begin{aligned} |f(x)| &= |f(x) - f(t_i) + f(t_i)| \\ &\leq |f(x) - f(t_i)| + |f(t_i)| \\ &\leq 1 + |f(t_i)| \qquad \text{if } x \in I_{t_i} \cap [a, b] \qquad (**) \end{aligned}$$

Let $M = 1 + \max_{1 \leq i \leq n} |f(t_i)|$ and since,

$[a, b] \subset \bigcup_{i=1}^{n} I_{t_i} \cup [a, b]$, then apply $(**)$ and then

$$|f(x)| \leq M \qquad \forall x \in [a, b]$$

$\square$

Theorem 8.2: Extreme value Theorem

Suppose that f is continuous on a finite closed interval, $[a, b]$. Let,

$$\alpha = \inf_{a \leq x \leq b} f(x) \text{ and } \beta = \sup_{a \leq x \leq b} f(x)$$

Then α and β are respectively the minimum and maximum of f on $[a, b]$; that is there are points x_1 and x_2 in $[a, b]$ such that;

$$f(x_1) = \alpha \qquad f(x_2) = \beta$$

Proof. We'll show that x_1 exists first. Suppose for a contradiction, that there is no point $x_1 \in [a, b], f(x_1) = \alpha$. Then for $f(t) > \alpha \quad \forall t \in [a, b]$

$$f(t) > \frac{f(t) + \alpha}{2} > \alpha$$

Since, f is continuous at t, there is an open interval I_t about the point t, st,

$$f(x) > \frac{f(t) + \alpha}{2} \qquad x \in I_t \cap [a, b]$$

Then, the collection of $\mathcal{H} = \{I_t : a \leq x \leq b\}$ is an open covering of $[a, b]$. Since $[a, b]$ is compact, the Heine-Borel theorem implies that there is a finite sub-covering using some open intervals $I_{t_1}, \ldots, I_{t_n}$ around $t_1, \ldots, t_n$. Now we define:

$$\alpha_1 = \min_{1 \leq i \leq n} \frac{f(t_i) + \alpha}{2}$$

Then $f(t) > \alpha \, \forall \, t \in \bigcup_{i=1}^{n} I_{t_i} \cap [a, b] = [a, b]$, so we now have $a_1 > \alpha$ and hence a contradiction. So $f(x_1) = \alpha$ for some $x_1 \in [a, b]$.

To complete the proof, show that x_2 exists. Suppose for a contradiction, that there is no point $x_2 \in [a, b], f(x_2) = \beta$. Then for $f(t) < \beta \quad \forall t \in [a, b]$

$$f(t) < \frac{f(t) + \beta}{2} < \beta$$

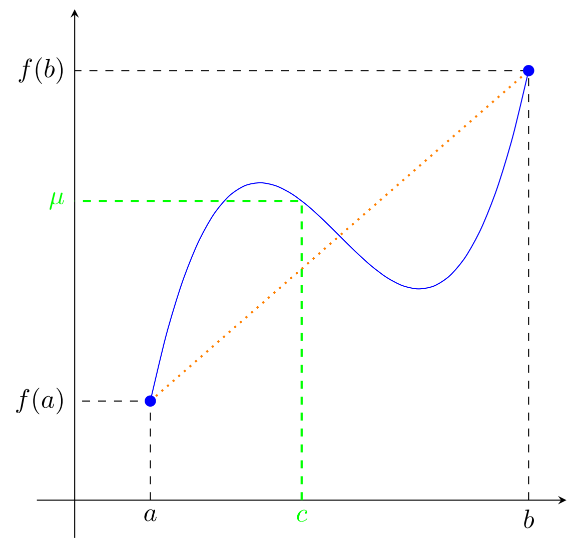
Since, f is continuous at t, there is an open interval I_t about the point t, st,

$$f(x) < \frac{f(t) + \beta}{2} \qquad x \in I_t \cap [a, b]$$

Then, the collection of $\mathcal{H} = \{I_t : a \leq x \leq b\}$ is an open covering of $[a, b]$. Since $[a, b]$ is compact, the Heine-Borel theorem implies that there is a finite sub-covering using some open intervals $I_{t_1}, \ldots, I_{t_n}$ around $t_1, \ldots, t_n$. Now we define:

$$\beta_1 = \max_{1 \leq i \leq n} \frac{f(t_i) + \beta}{2}$$

Then $f(t) < \beta \, \forall \, t \in \bigcup_{i=1}^{n} I_{t_i} \cap [a, b] = [a, b]$, so we now have $\beta < \beta_1$ and hence a contradiction. So $f(x_2) = \beta$ for some $x_2 \in [a, b]$. $\square$



Theorem 8.3: Intermediate Value Theorem

Suppose that f is continuous on $[a, b]$, $f(a) \neq f(b)$, and μ is between $f(a)$ and $f(b)$. Then $f(c) = \mu$, for some $c \in [a, b]$

Proof. Suppose that $f(a) < \mu < f(b)$. The set,

$$S = \{x : a \leq x \leq b \text{ and } f(x) \leq \mu\}$$

is bounded and is non-empty. Let $c = \sup S$. We will show that $f(c) = \mu$. If $f(c) > \mu$, then $c > a$ and since f is continuous at c, $\exists \, \varepsilon > 0$,st,

$$f(x) > \mu \qquad \text{if } c - \varepsilon < x \leq c$$

Therefore, $c - \varepsilon$ is an upper bound for S, contradicting the definition of c.
If $f(c) < \mu$, then $c < b$ and $\exists \, \varepsilon > 0$, st,

$$f(x) < \mu \text{ for } c \leq x < c + \varepsilon$$

so c is not an upper bound for S, which again contradicts the definition of c.
Therefore $f(c) = \mu$. The proof for $f(b) < \mu < f(a)$ is simply obtained by applying the above to the function $-f$. $\square$

8.1 Monotonics 2: God what a mess

9 Definition of a Derivative

Definition 9.1: Derivative

A function f, is differentiable at an interior point, x_0, of it's domain if the difference qiotient:

$$\frac{f(x) - f(x_0)}{x - x_0}, \quad x \neq x_0$$

approaches a limit as x approaches x_0, in which case the limit is called the derivative of f at x_0 and is denoted: $f'(x_0)$, thus:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Definition 9.2

We say that $f(x_0)$ is a local extreme value of f, if there is a $\delta > 0$, such that the sign of $f(x) - f(x_0)$ diesnt change:

$$(x_0 - \delta,\, x_0 + \delta) \subset D_f$$

or a local minimum of f if:

$$f(x_0) \leq f(x)$$

if for all x in the set, these are true, then we have globals

Theorem 9.1

If f is differentiable at a local extreme point $x_0 \in D^0_f$, then $f(x_0) = 0$

Proof. We consider the case where x_0 is a local maximum. Then, $\exists \delta > 0$, $(x_0 - \delta,\, x_0 + \delta) \subset D_f$ and $f(x) \leq f(x_0)$ for all $x \in (x_0 - \delta,\, x_0 + \delta)$. We have:

$$0 \leq \lim_{x \to x_0-} \frac{f(x) - f(x_0)}{x - x_0}$$

and

$$\lim_{x \to x_0-} \frac{f(x) - f(x_0)}{x - x_0} \leq 0$$

So, as it exists,

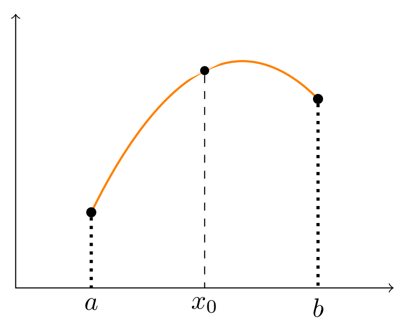
$$\lim_{x \to x_0-} \frac{f(x) - f(x_0)}{x - x_0} = 0$$

The case of a local minimum at x_0 is obtained by applying the above to $-f$. □

10 Rolles Theorem

Theorem 10.1: Rolle's Theorem

Suppose that f is continuous on the closed interval $[a,\, b]$ and differentiable on the open interval $(a,\, b)$ and $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a,\, b)$



Proof. Since f is continuous on $[a,\, b]$, then, by EVT, f attains both a min and max values.

$$\alpha = \min_{x \in [a,\, b]} f(x) \qquad \beta = \max_{x \in [a,\, b]} f(x)$$

If $\alpha = \beta$, then f is constant on $(a,\, b)$, clearly $f'(x) = 0 \, \forall x \in [a,\, b]$.
If $\alpha \neq \beta$, then at least one α or β is attained at a point $c \in (a,\, b)$ (Since $f(a) = f(b)$), and hence $f'(c) = 0$. □

11 Darboax's Theorem

Theorem 11.1: Darboax's Theorem

Suppose that f is differentiable on $[a,\, b]$, $f'(a) \neq f'(b)$, and μ is between $f'(a)$ and $f'(b)$. Then $f'(c) = \mu$ for some $c \in (a,\, b)$.

Proof. Suppose that $f'(a) < \mu < f'(b)$ aand then define:

$$g(x) = f(x) - \mu x$$

Then

$$g'(x) = f'(x) - \mu$$

and then:

$$g'(a) < 0 \qquad 0 < g'(b) \tag{$*$}$$

Since g is continuous on $[a, b]$, g attains a min, by EVT, at some point $c \in [a, b]$. Then, $(*)$, implies $\exists\, \delta > 0$,

$$g(x) < g(a), \quad \forall a < x < a + \delta$$

and

$$g(x) < g(b), \quad b - \delta < x < b$$

therefore $c \neq a$ and $c \neq b$. Hence $a < c < b$, and therefore $g'(c) = 0$ since c is a min in D_g^0, that is $f'(c) \neq \mu$
The proof when $f'(b) < \mu < f'(a)$ is obtained when you apply the above argument to $-f$. □

12 Mean Value Theorem

Theorem 12.1: Cauchy's Mean Value Theorem

If f and g are continuous on a closed interval $[a, b]$ nd differentiable on the open interval (a, b), then:

$$[g(b) - g(a)]f'(c) = [f(b) - f(a)]g'(c)$$

for some $c \in (a, b)$

Proof. Let $h(x) = (g(b) - g(a))f(x) - (f(b) - f(a))g(x)$, then we can say h is continuous on $[a, b]$ and differentiable on (a, b). Therefore Rolle's Theorem implies that $h'(c) = 0$ for some $c \in (a, b)$, that is:

$$h'(c) = (g(b) - g(a))f'(c) - (f(b) - f(a))g'(c) = 0$$

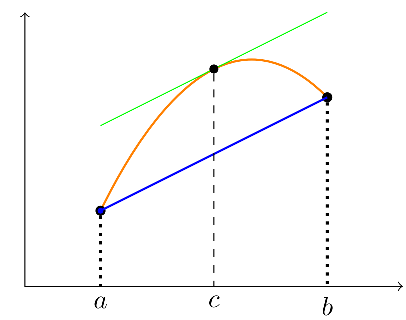
□

Theorem 12.2: Mean Value Theorem

If f is continuous on the closed interval $[a, b]$ and differentiable on (a, b), then

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

for some $c \in (a, b)$



Proof. Apply Cauchy MVT and let $g(x) = x$ □

12.1 Consequences

Theorem 12.3

If $f'(x) = 0$ for all $x \in (a, b)$ then f is constant on $(a\, b)$.

Theorem 12.4

If f' exists and does not change sign on (a, b), then f is monotonic on (a, b)

Theorem 12.5: Lipschitz Continuity

If $|f'(x)| \leq M \quad \forall x \in (a, b)$, then $|f(x) - f(x')| < M|x - x'|$ for all $x, x' \in (a, b)$.

13 Sequences

Definition 13.1: Limit of a Sequence

A sequence, $\{s_n\}$ converges to a limit s, if for every $\varepsilon > 0 \, \exists \, N \in \mathbb{Z}$,

$$|S_n - S| < \varepsilon \quad n \geq N$$

Definition 13.2: Divergence

We say that $\lim_{n\to\infty} a_n = \infty$ if $\forall a \in \mathbb{R}$, $s_n > a$ for $n > a$. Similarly for $-\infty$.

Theorem 13.1

Let $\lim_{x\to\infty} f(x) = L$, where $L \in \overline{\mathbb{R}}$ and suppose that $s_n = f(n)$ for large n, then:

$$\lim_{n\to\infty} s_n = L$$

Definition 13.3: Subsequence

A subsequence $\{t_k\}$ if $t_k = s_{n_k}$, where $\{n_k\}$ is an increasing subsequence of integers.

Theorem 13.2: Uniqueness of subsequence limit

If $\lim_{n\to\infty} s_n = s$, then $\lim_{n\to\infty} s_{n_k} = s \quad \forall \{s_{n_k}\}$ of $\{s_k\}$

Proof. Consider the finite case, $\forall \varepsilon > 0 \, \exists \, N$,

$$|S_n - S| < \varepsilon \quad k \geq K$$

Since, $\{n_k\}$ is increasing $\exists \, K$, $n_k \geq N$ if $k > K$

$$|S_{n_k} - S| < \varepsilon \qquad k \geq K$$

For infinite limits, $\forall \varepsilon > 0$, $\exists \, N$,

$$S_n > n \qquad n \geq N$$

as we know $\{n_k\}$ is increasing, then $n_k > n$ for $n \geq N$ but $S_{n_k} > n$ for some $n \geq N$ and so the limit is infinite. For limit to $-\infty$, use the sequence $-S_n$ $\square$

Theorem 13.3: Limit Points of Sequences

A point $\bar{x}$ is a limit point of a set S, iff there is a sequence $\{x_n\}$ of points in S, $x_n \neq \bar{x}$ for $n \geq 1$, and $\lim_{n\to\infty} x_n = \bar{x}$

Proof. Suppose such a $\{x_n\}$ exists. Then $\forall \varepsilon > 0$, $\exists \, N$,

$$0 < |x_n - \bar{x}| < \varepsilon \qquad \forall n \geq N$$

Therefore every ε-neigh. contains ∞ many points of S hence $\bar{x}$ is a limit point of S.

Now let $\bar{x}$ be a limit point of S. $\forall N \geq 1$, $(\bar{x} - \frac{1}{n}, \bar{x} + \frac{1}{n})$ has to contain some point $x_n \in S$, $x_n \neq \bar{x}$. Since,

$$|x_n - \bar{x}| \leq \frac{1}{n} \qquad m \geq n$$

and $\lim_{n\to\infty} x_n = \bar{x}$ $\square$

Theorem 13.4: Bounded and Subsequence Theorems

1. If $\{x_n\}$ is bounded then it has a convergent subsequence
2. If $\{x_n\}$ is unbounded above, then it has a subsequence $\{x_{n_k}\}$ st,
$$\lim_{x\to\infty} x_{n_k} = \infty$$
3. If $\{x_n\}$ is unbounded above, then $\{x_n\}$ has a subsequence st,
$$\lim_{x\to\infty} x_{n_k} = -\infty$$

Proof. **Proof of 1:** Let S be a set of distinct numbers of $\{s_n\}$, if s is finite, then $\exists$, $\bar{x} \in s$, which occurs infinitely often. Then,

$$\lim_{n\to\infty} x_{n_k} = \bar{x}$$

If s is infinte, then since s is bounded BWT applies, now s has a limit point, $\bar{x}$. Then by previous thm, $\exists \{y_j\} \in s$ with $y_j \neq s$,

$$\lim_{j\to\infty} y_j = \bar{x}$$

However, $\{y_j\}$ may not be a subsequence of $\{x_n\}$, so $y_j = x_{n_j}$ may not be true, where n_j is increasing. So now take an increasing subsequence of n_j, $\{n_j\}$, then $\{y_{j_k}\} = \{s_{n_{j_k}}\}$ is a subsequence. So it has the same limit as; $\{y_j\}$

$$\lim_{k\to\infty} \{s_{n_{j_k}}\} = \bar{x}$$

$\square$

13.1 Cauchy Sequences

Definition 13.4: Cauchy Sequences

A sequence $\{s_n\}$ of real numbers is said to be cauchy if $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$, $n \geq N$ and $m \geq N$, then:

$$|S_n - S_m| < \varepsilon$$

Lemma 13.1

Let $\{s_n\}$ be a convergent, then it's cauchy

Proof. Suppose that $s_n \to s$ as $n \to \infty$. Let $\varepsilon > 0$, $\exists\, N$, $n \geq N$

$$|s_n - s| < \frac{\varepsilon}{2}$$

Now take $m, n \geq N$, then

$$\begin{aligned} |s_n - s_m| &= |s_n - s - (s_m - s)| \\ &\leq |s_n - s| + |s_m - s| \\ &< \varepsilon \end{aligned}$$

$\square$

Lemma 13.2

Let $\{s_n\}$ be cauchy, then it's convergent

Proof. Let $\{s_n\}$ be cauchy, and hence it's bounded. By thm 3.14(a), there is a convergent subsequence $\{s_{n_k}\}$ for some $s \in \mathbb{R}$. Now claim, $s_k \in s$ as $k \to \infty$.

Let $\varepsilon > 0$, $\exists\, N_2$, $k \geq N_1$, then

$$|s_{n_k} s| < \frac{\varepsilon}{2}$$

Then $\exists\, N_2$, $m, n \geq N_2$, then $|s_m - s_n| < \frac{\varepsilon}{2}$. If $K \geq \max(m, n)$,

$$\begin{aligned} |s_k - s| &= |(s_k - s_{n_k}) - (s - s_{n_k})| \\ &\leq |s_k - s_{n_k}| + |s - s_{n_k}| \\ &< \varepsilon \end{aligned}$$

$\square$

14 Series

Definition 14.1: Series

If $\{a_k\}_k^\infty = \sum_{n=k}^{\infty} a_k$ is infinite and a_n is the $n^t h$ term. If $\sum_{k=1}^{\infty} = A$, then it converges. Also we say $A_n = a_k + \cdots + a_n \qquad n \geq k$ is the $n^t h$ partial sum of the sum. We can also say that,

$$\lim_{k\to\infty} A_n = A$$

Theorem 14.1: Cauchy Criterion for Series

A series $\sum a_n$ converges iff $\forall \varepsilon > 0$, $\exists\, N$,

$$|a_n + a_{n-1} + \cdots + a_m| < \varepsilon \qquad m \geq n \geq N \quad (*)$$

Proof. Let $\{A_n\}$ be the series of partial sums of our series. Then

$$A_m - A_{n-1} = a_n + \cdots + a_m$$

If $(*)$ holds, then

$$|A_m - A_{n-1}| < \varepsilon \text{ if } m \geq n \geq N \quad (**)$$

To say $\sum a_n$ is convergent, then $\{A_n\}$ is convergent. This is equiv to $\{A_n\}$ being cauchy, which is what $(**)$ says. $\square$

Corollary 2. *If $\sum a_n$ converges, then* $\lim_{n\to\infty} a_n = 0$

Proof. Taking $m = n$ in the previous thm, then $\forall \varepsilon > 0$, $\exists N > 0$,

$$|a_n| < \varepsilon \text{ if } n \geq N$$

which is $\lim_{n\to\infty} a_n = 0$ $\square$

Corollary 3. *(Divergence Test) If* $\lim_{n\to\infty} a_n \neq 0$*, then* $\sum a_n$ *divergent*

15 Limits of function series

Definition 15.1: Limit of a functional series

Suppose $\{f_n\}$, $n \in \mathbb{N}_1$ is a sequence of functions defined on a set E, then suppose the limit exists,

$$f(x) = \lim_{n\to\infty} f_n(x)$$

Now we say that $f_n(x)$ converges to $f(x)$ or $\{f_n\}$ converges to f pointwise on E. Similarly:

Definition 15.2: Sum of a series

If $\sum f_n(x)$ converges $\forall\, x \in E$, we say:

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

Theorem 15.1: Contunity of a series of continuous functions

To say that a series of continuous functions is continuous, it suffices to show:

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t)$$

15.1 Convergence

Definition 15.3: Uniform Convergence (Sequence)

A sequence of functions $\{f_n\}, n \in \mathbb{N}_1$, converges uniformly on E to a function f if $\forall\varepsilon > 0$, $\exists\ N$, $n \geq N \implies$

$$|f_n(x) - f(x)| \leq \varepsilon \qquad \forall x \in E$$

Definition 15.4: Uniform Convergence (Series)

We say that the sequence (series) $\sum f_n(x)$ converges uniformly on E if the sequence $\{s_n\}$ of the partial sums is:

$$s_n = \sum_{i=1}^{n} f_i(x)$$

15.2 Cauchy Time

Theorem 15.2

The sequence $\{f_n\}$ defined on E, converges uniformly on E, $\iff$ $\forall\varepsilon > 0$, $\exists\ N$, $m \geq N$ and $n \geq N$, $x \in E$, $\implies$

$$|f_n(x) - f(x)| \leq \varepsilon$$

Proof. Suppose that $\{f_n\}$ converges uniformly on E, and let f be the limit of the sequence. Then $\exists\ N$, $n \geq N$, $x \in E$, $\implies$

$$|f_n(x) - f(x)| \leq \frac{1}{2}\varepsilon$$

$$\begin{aligned}|f_n - f_m| &\leq |f_n - f| + |f_m - f| \\ &\leq \varepsilon\end{aligned}$$

Suppose that cauchy holds, then we know every cauchy sequence converges on the real line. So we have to prove that the convergence is uniform; Let $\varepsilon > 0$, $\exists\ N$, st the theorem holds. Now fix n, and take $m \to \infty$, this gives:

$$|f_m - f_n| \leq \varepsilon \forall n \geq N, x \in E$$

$\square$

Theorem 15.3

Suppose $\lim_{n\to\infty} f_n = f, \quad x \in E$. Let $M_n = \sup_{x\in E} |f_n - f|$. Then $f_n \to f$ uniformly on $E \iff M_n \to 0$ as $n \to \infty$

Theorem 15.4: Wierstrass

Suppose $\{f_n\}$ is a sequence of functions defined on E, and

$$|f_n| \leq M \qquad (x \in E, n \in \mathbb{N}_1)$$

Proof. If $\sum M_n$ converges, then for $\varepsilon > 0$,

$$\left|\sum_{i=n}^{M} f_i\right| \leq \sum_{i=n}^{m} M_i \leq \varepsilon$$

if m and n are large enough. $\square$

16 Continuity

Let's prove Thm 1.1

Proof. Let $\varepsilon > 0$ by uniform convergence of $\{f_n\}$, then $\exists N$, $n, m \geq N$, $t \in E$, $\implies$

$$|f_n - f_m| \leq \varepsilon$$

Letting $t \to x$, we obtain: $|A_m - A_n| \leq \varepsilon$ for $n, m \geq N$, st. $\{A_n\}$ is a cauchy sequence and so converges to A.

$$|f - A| \leq |f - f_n| + |f_n - A_n| + |A_n - A|$$

Now let them all be less than a third by the usual limit nonsence and hence,

$$|f - A| \leq \varepsilon$$

$\square$

Theorem 16.1

If $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \to f$ uniformly on E, then f is continuous on E (from above)

Theorem 16.2

Suppose K is compact and

1. $\{f_n\}$ is continuous on K
2. $\{f_n\}$ converges pointwise on K
3. $f_n \geq f_{n+1}$ $\forall n \in \mathbb{N}$

Proof. Let $g_n = f_n - f$, then g_n is continuous, $g_n \to 0$ pointwise and $g_n \geq g_{n+1}$. So prove that $g_n \to 0$ uniformly om K.

Let $\varepsilon > 0$, $K_n = \{x \in K : g_n(x) \geq \varepsilon\}$ as g_n is continuous, K is closed and hence compact. Since $g_n \geq g_{n+1}$, we have $K_n \geq K_{n+1}$. Fix an $x \in K$. Since $g_n \to 0$, then $x \notin K_n$ if n is large, thus $x \notin \bigcup K_n$. Hence K_N is empty for $n \geq N$, then:

$$0 \leq g_n < \varepsilon \qquad \forall x \in K\, n \geq N$$

$\square$