Year 3 — Lie Groups and Applications in Geometric Mechanics

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine (especially the typos!).

Contents

1 Hamilton's Variational Principle

This document started nearly a year ago in a zoom call with two words, 'plastic bag'. The challenge set was to get a mathematical model to describe the motions of pseudo rigid bodies, of which a plastic bag is an example of. The original plan, was for the derivations to be below as a first and motivating example. However, I underestimated the journey that the mathematics would take me on and this document now acts as the path that I took on this journey. I realised that energy is really important in mathematics and sometimes a problem cannot be solved by considering the particle position alone and so we need to use energy arguments. A simple example of an energy argument can be seen in the following example,

Consider a particle that is half way along a piece of massless string which is pulled past its natural extension. The question is, what is the velocity of the particle when it reaches a displacement of zero? Anybody trained in mathematics would start by trying to describe the position of the particle in this system and that is the wrong way to solve this problem.

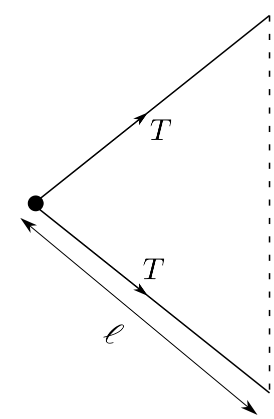


Figure 1: Motivating Problem.

To solve this problem, the best (and probably only) way is to describe the particle's energy. That is, we know that energy is conserved and so if at some point we know the total energy we can then describe the energy in the system at any point. At $t = 0$ there is zero kinetic energy, as the particle is held at rest. If we assume that the only energies to consider are potential and kinetic, the whole energy of the system can be described by just the elastic potential energy. Then we know that the kinetic energy of the particle at displacement zero is, $\frac{\lambda x^2}{2l_0}$ where l_0 is the natural extension of the spring, x is the elastic extension and λ is the spring constant. Hence, in our case, the energy in the string is $\frac{\lambda(\ell - l_0)^2}{2l_0}$ and so the velocity of the particle at displacement zero is going to be, $v = (\ell - l_0)\sqrt{\frac{\lambda}{ml_0}}$. In general, we can make these energy arguments about sets of particles in Euclidean space. To start to understand this, we must first go on a little detour to explain the role of the Lagrangian through the Euler-Lagrange equations and variations.

The rest of this chapter is in very close relation to first chapter of Holm, Schmah & Stoica (2009), Geometric Mechanics and Symmetry [2]. Consider a point mass, the position of this point is also called the **configuration** is a vector $\mathbf{q} \in \mathbb{R}^d$. If we consider N points then the configuration becomes $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N) \in \mathbb{R}^{dN}$. This notation works with familiar mechanics concepts, that is we can write Newtons Second Law, $F_i = m_i \ddot{\mathbf{q}}_i$ for $i = 1, 2, \ldots, N$. We now define a special type of Newtonian system that we can use to illustrate a few important concepts.

Definition 1.1 (Newtonian Potential System)**.** A **Newtonian potential system** is a system of equations,

$$m_i \ddot{\mathbf{q}}_i = -\frac{\partial V_i}{\partial \mathbf{q}_i}$$

for $i = 1, 2, \ldots, N$ where $V(\{\mathbf{q}_i\})$ is a real-valued function, called the potential energy.

This system has conserved energy and this is one of the most interesting thing for mathematicians. We define it to be kinetic energy plus potential energy, $E := K + V$.

Theorem 1.2 (Conservation of Energy)**.** Energy is conserved in Newtonian Potential Systems.

Proof.

$$\begin{aligned}
\frac{dE}{dt} &= \frac{d}{dt}\left(\frac{1}{2}\sum_{i=1}^{N} m_i \left\|\ddot{\mathbf{q}}_i\right\|^2 + V(\mathbf{q})\right) \\
&= \sum_{i=1}^{N} m_i \dot{\mathbf{q}}_i \cdot \ddot{\mathbf{q}}_i + \sum_{i=1}^{N} \frac{\partial V}{\partial \mathbf{q}_i} \cdot \dot{\mathbf{q}}_i \\
&= \sum_{i=1}^{N} \dot{\mathbf{q}}_i \cdot \left(m_i \ddot{\mathbf{q}}_i + \frac{\partial V}{\partial \mathbf{q}_i}\right) = 0 && \text{these vectors are orthogonal.}
\end{aligned}$$

□

This is one of the first important ideas that we will use again and again throughout this dissertation. These are the main motivation behind Noethers Theorems; they tell us about different conserved quantities in the system we are studying. These different types of conserved quantities will change depending on different types of invariance. We can have different types of invariance in the systems we are studying, later we will mostly consider invariance in Lagrangians.

Definition 1.3 (Rotational Invariance)**.** A function $V : \mathbb{R}^{dN} \to \mathbb{R}$ is rotationally invariant if

$$V(R\mathbf{q}_1, R\mathbf{q}_2, \ldots, R\mathbf{q}_N) = V(\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$$

for any $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$ and a rotation matrix $R \in M_{d\times d}(\mathbb{R})$.

As an interesting and motivating example, see Proposition 1.28 of [2], which says in any Newtonian potentially system with rotationally invariant V, angular momentum is conserved. This invariance can be used to reduce certain functions from N variables to $N - 1$ variables, this will be one the main focus' of study.

1.1 Lagrangian Mechanics

We firstly introduce the following theorem relating to a new set of equations called the Euler-Lagrange Equations,

Theorem 1.4 (Euler-Lagrange Equations for Newtonian Potential System)**.** Every Newtonian potential system is equivalent to the Euler Lagrange Equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}_i} - \frac{\partial L}{\partial \mathbf{q}_i} = 0 \tag{1.1}$$

for the Lagrangian $L : \mathbb{R}^{2dN} \to \mathbb{R}$ defined by,

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \sum_{i=1}^{N} \frac{1}{2} m_i \left\|\dot{\mathbf{q}}_\mathbf{i}\right\|^2 - V(\mathbf{q})$$

Proof.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}_i}\right) - \frac{\partial L}{\partial \mathbf{q}_i} = \frac{d}{dt}\left(m_i \dot{\mathbf{q}}_i\right) + \frac{\partial V}{\partial \mathbf{q}_i} = m_i \ddot{\mathbf{q}}_i + \frac{\partial V}{\partial \mathbf{q}_i} = 0$$

□

Henceforth, we will work in Lagrangian systems, that is defined as,

Definition 1.5 (Lagrangian System)**.** A **Lagrangian system** on a configuration space $\mathbb{R}^{dN}$ is the system of ODEs called the Euler-Lagrange equations (Equation 1.1), for some function $L : \mathbb{R}^{2dN} \to \mathbb{R}$ called the Lagrangian.

As we did before, we can talk about energy in terms of the Lagrangian, $E := \frac{\partial L}{\partial \dot{\mathbf{q}}_i} \cdot \dot{\mathbf{q}}_i - L$ and the energy is conserved. We now go the final ideas that I would like to introduce in the introduction, the variational derivative and Hamilton's Variational Principle. We will use these to derive the Euler-Lagrange equations again, in a more general case.

The Euler-Lagrange equations relate to a variational principle on the space of smooth paths with fixed end points. The main idea of this variational principle is that we can determine solutions of the Euler-Lagrange equations as stationary points of some action functional. In an example, consider a chain that is fixed at two ends. This will form a catenary, but a chain can form many more shapes or paths, but it turns out the minimal of the functional, which is a stationary point, is the catenary that the chain forms.

Consider some smooth path, $\mathbf{q} : [a, b] \to \mathbb{R}^{dN}$ with endpoints $\mathbf{q}(a) = \mathbf{q}_a$ and $\mathbf{q}(b) = \mathbf{q}_b$. We define a **deformation** of $\mathbf{q}$ as a smooth map $\mathbf{q}(s, t)$ where $s \in (-\varepsilon, \varepsilon)$ where $\varepsilon > 0$ such that $\mathbf{q}(0, t) = \mathbf{q}(t)$ for all $t \in [a, b]$.

Definition 1.6 (Variation)**.** The **variation** of the curve $\mathbf{q}(t)$ corresponding to the following deformation $\mathbf{q}(s, t)$ is,

$$\delta \mathbf{q}(t) = \left.\frac{d}{ds}\right|_{s=0} \mathbf{q}(s, t)$$

Then the first variation is,

Definition 1.7 (First Variation)**.** The **first variation** of a smooth C^∞ functional $\mathcal{S} : [a, b] \to \mathbb{R}^{dN}$ is

$$\delta \mathcal{S} := \left.\frac{d}{ds}\right|_{s=0} \mathcal{S}[\mathbf{q}(s, t)]$$

Then we call $\mathbf{q}$ a **stationary point** of $\mathcal{S}$ if $\delta \mathcal{S} = 0$ for all deformations of $\mathbf{q}$. Furthermore, if $\mathbf{q}(s, t)$ has fixed endpoints, meaning that $\mathbf{q}(a, s) = \mathbf{q}_a$ and $\mathbf{q}(b, s) = \mathbf{q}_b$ for all $s = (-\varepsilon, \varepsilon)$ then $\delta \mathbf{q}(a) = \delta \mathbf{q}(b) = 0$. These are variations along paths with fixed endpoints. Finally, we prove that the Euler-Lagrange equations are equivalent to Hamilton's Principle,

Theorem 1.8. For any $L : \mathbb{R}^{2dN} \to \mathbb{R}$, the Euler-Lagrange equations (Equations 1.1) are equivalent to Hamilton's principle of stationary action $\delta \mathcal{S} = 0$ where $\mathcal{S}$ is defined as,

$$\mathcal{S}[\mathbf{q}(t)] = \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}})\, dt$$

with respect to variations along paths and fixed endpoints.

Proof. We will proceed using the fact that $\frac{d}{dt}\delta\mathbf{q} = \delta\dot{\mathbf{q}}$ and using integration by parts.

$$\begin{aligned}
\delta\mathcal{S} = \frac{d}{ds}\bigg|_{s=0} S[\mathbf{q}(s,t)] &= \frac{d}{ds}\bigg|_{s=0} \int_{t_1}^{t_2} L(\mathbf{q},\dot{\mathbf{q}})\,dt \\
&= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \mathbf{q}}\cdot\mathbf{q} + \frac{\partial L}{\partial \dot{\mathbf{q}}}\cdot\delta\dot{\mathbf{q}}\right)dt \\
&= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \mathbf{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)\cdot\delta\mathbf{q}\,dt + \left[\frac{\partial L}{\partial \dot{\mathbf{q}}}\cdot\delta\mathbf{q}\right]_{t_1}^{t_2} \\
&= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \mathbf{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)\cdot\delta\mathbf{q}\,dt && \text{applying end point conditions}
\end{aligned}$$

This then tells us that for any smooth $\delta\mathbf{q}(t)$ satisfying $\delta\mathbf{q}(a) = \delta\mathbf{q}(b) = 0$. If $\delta\mathcal{S} = 0$, then $\frac{\partial L}{\partial \mathbf{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} = 0$ and so Hamilton's Principle is equivalent to the Euler-Lagrange equations. □

We now have the foundation to use Lagrangian Mechanics throughout the rest of the document. We will again and again see the use of Hamilton's Variational Principle and different types of Euler-Legrange equations. The rest of this dissertation shall be used to explore this area further. We will focus on Euler-Poincaré Reduction, Noether Theory and several applications of this mathematics. The main focus will be on the pure background mathematics of Geometric Mechanics leading up to the last chapter of pseudo-rigid bodies. We will focus on three main activities; symmetry and reduction of Lagrangians, derivation of equations and finding conserved quantities of these systems.

2 Lie Groups, Algebras and their associated actions

Thus far, we have only considered mechanics, in this section we start to introduce what we mean by Geometric Mechanics, our toolbox for making arguments about Lagrangians. We will look briefly at the definitions of manifolds and Lie groups and then consider the connection between them. To start, let us define some groups. Firstly, what is a group?

Definition 2.1 (Group)**.** G is a nonempty set and endowed with a binary operation such that,

(i) It's closed under $(\cdot)$, $\forall\, a, b \in G$, $a \cdot b \in G$

(ii) It's associative, i.e. $\forall\, a, b \in G$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

(iii) There is an identity element, $\forall\, a \in G$, $a \cdot e = a = e \cdot a$.

(iv) Every element has an inverse, $\forall\, a \in G$, $a \cdot a^{-1} = e = a^{-1} \cdot a$.

Groups are a very useful and interesting structure. There is a rich area of research and study surrounding them. One of the things that will be the most useful to us is actions of groups. We can take groups and consider them acting on sets, with an identity and compatibility axiom. We will see this in the next chapter, where we consider the adjoint actions. However, before we get to that we need to define a Lie group. I feel that in order to define a Lie group it would help to define a manifold or get some idea of what structures we will be working with.

2.1 Manifolds

In the simplest definition, a manifold is a space where we can do geometry. We are studying Analytic and Differential Geometry so we need to consider differentiable, or smooth manifolds. These are manifolds that we can take paths along and we can consider the velocity (or tangent vectors) of these paths. We are particularly interested in one specific type of manifold, SO(3) and the associated tangent space. The tangent space is just a vector space where we can use all the rules of Linear Algebra to understand it.

For our purposes, we will use the fact that a Lie Group is just a manifold. For background, we will spend the rest of this section defining what a manifold is formally. A manifold is a topological space that locally resembles Euclidean space near each point [7]. However, this isn't entirely satisfactory as what does near mean? It's very informal. Let's define it properly!

Definition 2.2 (Manifold)**.** A manifold is a second countable Hausdorff space that is locally homeomorphic to Euclidean space.

In the above definition, we refer to a second countable space as being a topological equivalent to having a countable basis. That is, there exists some countable base of this space $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$, where any open subset of our space, T, can be written as a disjoint union of a finite subfamily of $\mathcal{U}$. This nicely restricts manifolds to be smaller spaces, by making them be the union of countably many open sets. We define a Hausdorff space as,

Definition 2.3 (Hausdorff)**.** A topological space is Hausdorff if, given any points $x, y \in X$ with $x \neq y$, there exists open sets $U, V \in X$ with $x \in U$, $y \in V$ and $U \cap V = \varnothing$.

Finally, to say something is homeomorphic to another space, means it can be stretched to the other space without creating holes or gluing. A homeomorphism is a bijective map between two spaces that has a bijective inverse. A local homeomorphism is a homeomorphism in a neighbourhood of a point. Hence, saying that something is locally homeomorphic to Euclidean space directly means, that you can bijectively map the contents of the neighbourhood around a point to an open ball in $\mathbb{R}^n$, i.e the Euclidean n-ball.[1]

[1] This is long and not very succinct way to define this structure, a slightly nicer definition of manifolds is: A manifold is just a locally ringed space, whose sheaf structure is just locally isomorphic to continuous functions on Euclidean space.

2.2 Lie Groups

Now we know enough to define what a Lie group actually is,

Definition 2.4 (Lie Group)**.** A Lie group is a group that is also a smooth manifold, such that the binary product and inversion are smooth functions.

What we will be focusing our attention on is special Lie groups; the general linear group, special linear group and the special orthogonal group.

Definition 2.5 (General Linear Group)**.** $\mathrm{GL}(n, \mathbb{R})$ is the linear matrix group. The manifold of $n \times n$ invertible square real matrices is a Lie group denoted by $\mathrm{GL}(n, \mathbb{R})$.

Definition 2.6 (Special Linear Group)**.** The $\mathrm{SL}(n, \mathbb{R})$ is the manifold of $n \times n$ matrices with unit determinant.

Definition 2.7 (Special Orthogonal Group)**.** $\mathrm{SO}(n, \mathbb{R})$ is the manifold of rotation matrices in n dimensions. This may be denoted by $\mathrm{SO}(n)$

2.3 Lie Algebras

To actually understand what Lie Algebras are, we need to generalise the notion of a vector and a tangent. We shall look at so called tangent spaces. To formally define them, we will define manifolds a second way, one that leads to the definition of smooth (or C^∞ manifolds). We shall first define charts and atlases. These definitions are adapted from [6].

Definition 2.8 (Chart)**.** Let X be a topological space. An $\mathbb{R}^n$ chart on X is a homeomorphism $\phi : U \to U'$ where $U \subset X$ and $U' \subset \mathbb{R}^n$.

Definition 2.9 (Atlas)**.** A C^∞ atlas on a topological space X is a collection of charts $\phi_\alpha : U_\alpha \to U'_\alpha$ where all the U''s are open subsets of one fixed $\mathbb{R}^n$ such that,

1. Each $U_\alpha \in X$ is open and $\bigcup_\alpha U_\alpha = X$ (U_α is an open subcover of X) and,
2. Changes of coordinates are smooth[2].

Two last definitions in this section are equivalence relation and equivalence class.

Definition 2.10 (Equivalence Relation)**.** An equivalence relation on a set X is a binary relation $\sim$ satisfying,

1. $\forall\, a \in X, a \sim a$
2. $\forall\, a, b \in X, a \sim b \implies b \sim a$
3. $\forall\, a, b, c \in X, a \sim b$ and $b \sim a \implies a \sim c$.

Then the equivalence class is just all of the equivalent elements to a member of that set. For example, for the equivalence relation that $x \sim y$ if and only if $x - y$ is even, then the equivalence classes are all the even numbers and all the odd numbers.

Remark. It can be proven that equivalence classes are just a partition of a set. [4]

Here is another definition of a manifold, this definition explicitly output a smooth manifold,

[2]This is slightly more convoluted than what we hint to here. In the greatest formality, we should write that transition maps are smooth. Transitions maps are maps from two charts (U_α, ϕ_α) and (U_β, ϕ_β) where $U_\alpha \cap U_\beta$ is nonempty and

$$\tau_{\alpha,\beta} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

defined by $\tau_{\alpha,\beta} = \phi_\beta \circ \phi_\alpha^{-1}$ (this is a homeomorphism).

Definition 2.11 (C^∞ Manifold)**.** An n-dimensional (C^∞) manifold a topological space M together with an equivalence class of C^∞ atlases.

Remark. Our equivalence relation here is that two atlases are equivalent if their union is also an atlas.

Here are a few examples of manifolds,

- Let $M = \mathbb{R}^n$, this is a manifold covered by one open set and then if we take the identity map as our chart, we get the standard manifold on $\mathbb{R}^n$.
- Let $M = \mathbb{C}^n$, then we cover $\mathbb{C}^n$ by just one open set and then chart the map, $\phi : \mathbb{C}^n \to \mathbb{R}^{2n}$ which is just,
$$\phi(z_1, \dots, z_n) = (\operatorname{Re} z_1, \operatorname{Im} z_1, \dots, \operatorname{Re} z_n, \operatorname{Im} z_n)$$
- If M is a manifold, then any open $V \subset M$ is also a manifold. This can be seen as the union of the atlases V and M is going to be M and so it has the same equivalence class and hence it must be a manifold.
- If we let $M_n(\mathbb{R})$ be all real $n \times n$ matrices, then this is a manifold as it's just $\mathbb{R}^{n^2}$. We also can say $\mathrm{GL}(n, \mathbb{R}) \subset M_n(\mathbb{R})$ and so by the previous point, $\mathrm{GL}(n, \mathbb{R})$ is a manifold.

This is very abstract, we should see that a manifold has similar behaviour to $\mathbb{R}^n$, but is more flexible and can have some sort of curvature inbuilt. The ideas of a manifold holding similarities to $\mathbb{R}^n$ can be seen in Whitney's Embedding Theorem, a theorem whose proof would not add to the work, so it is omitted. Its existence should suffice to prove to the reader that manifolds, however abstract, can be manipulated and argued with using similar ideas to $\mathbb{R}^n$.

Theorem 2.12 (Whitney Embedding Theorem)**.** Every m-dimensional manifold can be embedded in $\mathbb{R}^{2m}$

Now we can formalise the idea of tangent vectors on a manifold [5]

Definition 2.13 (Tangent Vectors)**.** Let M be a C^∞ manifold, then we can say that $x \in M$. Let us take a chart of M, $\phi : U \to \mathbb{R}^n$ where $x \in U$. Now take two curves $\gamma_1, \gamma_2 : (-1, 1) \to M$ with $\gamma_1(0) = \gamma_2(0) = x$ such that we can form $\phi \circ \gamma_1 \circ \phi \circ \gamma_2 : (-1, 1) \to \mathbb{R}^n$ are differentiable.
Now define an equivalence such that γ_1 and γ_2 are equivalent at 0 if and only if $(\phi \circ \gamma_1)' = (\phi \circ \gamma_2)' = 0$. Then take the equivalence class of all of these curves and these are the tangent vectors of M.

Definition 2.14 (Tangent Space)**.** The set of all of the tangent vectors at x. We denote it as $T_x M$.

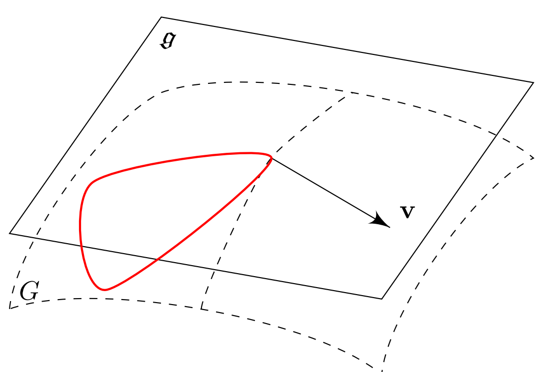


Figure 2: Lie Group and Associated Lie Algebra

These definitions aren't the most intuitive, so we provide Figure 2 to help the reader see the geometric interpretations. We let G be a Lie group and $\mathfrak{g}$ be a Lie algebra, or a tangent space to the manifold, or Lie group. Lie algebras are tangent spaces to the Lie group at the identity. T_eG (tangent space at the identity) is an interesting vector space with a remarkable structure called the Lie algebra structure. We now will prove a few results relating to this structure and then define the Lie algebra using the bilinear map called the Lie Bracket,

Lemma 2.15. Let G be a matrix lie group, and $g \in G$, then,

$$\xi \in T_eG \implies g\xi g^{-1} \in T_eG$$

Note: $g\xi g^{-1}$ is a matrix expression.

Proof. Let $c(t) \in G$ be a curve in G, such that $c(0) = e$ and $\dot{c}(0) = \xi$. Define $\gamma(t) = gc(t)g^{-1}$. Then $\gamma(0) = gc(0)g^{-1} = e$ and $\dot{\gamma}(0) = g\dot{c}(0)g^{-1} = g\xi g^{-1} \in T_eG$. □

Proposition 2.16. Let G be a matrix lie group and $\xi, \eta \in T_eG$. Then, $\xi\eta - \eta\xi \in T_eG$

Proof. Let $c(t) \in G$ be a curve such that $c(0) = e$ and $\dot{c}(0) = \xi$ also define $b(t) = c(t)\eta c(t)^{-1} \in T_eG$ by Lemma 1.5. Then $\dot{b}(t) \in T_eG$.

$$\begin{aligned}
\dot{b}(0) &= \dot{c}(0)\eta c(0)^{-1} + c(0)\eta + \frac{dc(t)}{dt}(0) \\
&= \dot{c}(0)\eta c(0)^{-1} - c(0)\eta c(0)^{-1}\dot{c}(0)c(0)^{-1} \\
&= \xi\eta - \eta\xi
\end{aligned}$$

As $\dot{b}(t) \in T_eG$, then $\xi\eta - \eta\xi \in T_eG$ □

Now we have a Lie Algebra,

Definition 2.17 (Lie Algebra)**.** A lie algebra is a vector space endowed with a commutator (or Lie bracket), that is a bilinear map. If we have,

$$[\cdot, \cdot] : V \times V \to V$$

such that,

- $[B, A] = -[A, B]$ (skew-symmetry property)
- $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0 \quad \forall\, A, B, C \in V$ (Jacobi Identity)

The Lie Bracket, could be any bilinear map and the behaviour of this map is related to the space that we are considering. In particular, when we consider the matrix Lie groups, we are interested in the Lie Bracket defined in Theorem 2.18

Theorem 2.18. Let G be a matrix Lie group. Then T_eG is a Lie algebra with the Lie bracket given by the matrix commutator. Denoted by $\mathfrak{g}$.

$$[A, B] = AB - BA$$

Assume we have a surface, or manifold M, with tangent space T_qM, then we can say that,

$$\bigcup T_qM = TM$$

is the tangent bundle. We can further define a cotangent bundle, which leads to the cotangent manifold. The cotangent manifold is the dual space of the manifold, M.

We now present several examples of Lie algebras and their groups to help the reader understand the main space of study for the rest of the work.

Example.

- The Lie algebra $\mathfrak{gl}(n,\mathbb{R}) = T_e\mathrm{GL}(n,\mathbb{R})$ which is vector space of real square $n \times n$ matrices with commutator.
- The Lie algebra of $\mathfrak{sl}(n,\mathbb{R}) := T_e\mathrm{SL}(n,\mathbb{R})$ vector space of real traceless square $n \times n$ matrices.

  Proof. Take $g(t) \in \mathrm{SL}(n,\mathbb{R})$, and so $\det g(t) = 1$ hence, take $g(t)$ such that $g(0) = e$, and $\dot{g}(0) = \xi$ and so $\dot{g}(0) \in \mathfrak{sl}(n,\mathbb{R})$. Now use the formula of the derivative of the determinant of a matrix to show,

  $$\frac{d}{dt}(\det(g(t)))_{t=0} = \det g(0)\,\mathrm{Tr}(g^{-1}(0)\dot{g}(0))$$

  and so, $g^{-1}(0)\dot{g}(0) \in \mathfrak{sl}(n,\mathbb{R})$. Therefore, we just have $\mathrm{Tr}(\xi)$. $\square$

- The Lie algebra of $\mathfrak{so}(3) = T_e\mathrm{SO}(3)$, the vector space of skew-symmetric matrices.

We finally present a lemma that we will use again and again throughout this work. In some sense this is the most important lemma in the whole document. If this were false, we would not be able to do many of the things we do in the next few chapters.

Lemma 2.19. If $v \in T_gG$, then we can say,

(i) $g^{-1}v \in T_eG$

(ii) $vg^{-1} \in T_eG$

Proof. Suppose we have a $c(t) \in G$ such that $c(0) = g$ and $\dot{c}(0) = v$. Now we define a $\gamma(t) = g^{-1}c(t)$ and we see that $\gamma(0) = e$ and $\dot{\gamma}(0) = g^{-1}\dot{c}(0) = g^{-1}v$. Hence by Lemma 2.15 we can say if $v \in T_eG$ then $gvg^{-1} \in T_eG$, applying this to $g^{-1}v$, we can get that $vg^{-1} \in T_eG$, as required. $\square$

2.4 Actions of a Lie Group and Lie Algebra

In this section we will focus on how our Lie Group will act on our Algebra. We will first use conjugation actions to define our adjoints which will become very useful once we see the Euler Poincare equations. As an introduction consider classical groups, then we define a group action on the group as how a group can act on a set,

Definition 2.20 (Group Action)**.** Let $(G,*)$ be a group and A be a set. A group action is a map,

$$(\cdot) : G \times A \to A$$

$$(g,a) \mapsto g \cdot a$$

that satisfies the following axioms,

(A1) $(g_1 * g_2) \cdot a = g_1 \cdot (g_2 \cdot a)$ for $a \in A$

(A2) $e \cdot a = a$ for all $a \in A$

One of the most thoroughly studied actions in applied settings is conjugation. In our case we will also use the conjugation action to derive the adjoint and coadjoint actions for firstly the Lie Group onto the Lie algebra and then the Lie algebra onto itself. The conjugation action is defined in the usual way,

Definition 2.21 (Conjugation Action)**.** Let $g \in G$, then the operation $I_g : G \to G$ (Inner Automorphism) and so you define it by $h \mapsto ghg^{-1} \quad \forall\, h \in G$. $I_{gh} = AD_{gh}$.

Take an arbitrary path $h(t) \in G$ such that $h(0) = e$ and let $\xi = \dot{h}(0) \in T_eG$. We now define $Ad_g(\xi) = \frac{d}{dt} I_g h(t)_{t=0} = g\xi g^{-1} \in T_eG$, called the adjoint action. Here I_g is the inner automorphism.

Definition 2.22 (Adjoint and coadjoint actions of G on $\mathfrak{g}$ and $\mathfrak{g}^*$)**.** The adjoint action of the matrix lie group G on it's lie algebra $\mathfrak{g}$ is a map,

$$Ad : G \times \mathfrak{g} \to \mathfrak{g}$$

defined by,

$$\mathrm{Ad}_g \xi = g\xi g^{-1}$$

The dual map $\left\langle Ad_g^* \mu\, , \xi \right\rangle = \langle \mu\, , Ad_g \xi \rangle$ where $\mu \in \mathfrak{g}^*$ and $\xi \in T_eG = \mathfrak{g}$ is called the coadjoint map of G on the dual lie algebra $\mathfrak{g}^*$.

We will find that sometimes our classical ideas of vector spaces doesn't work. Hence, we shall introduce functionals and use them to define dual vector spaces.

Definition 2.23 (Dual Space for vectors)**.** Let V be a finite dimensional vector space, of dimension n, over $\mathbb{R}$. The dual vector space is denoted by V^* is the space of all linear functionals from $V \to \mathbb{R}$, $f(v) = a$ where $v \in V$ and $a \in \mathbb{R}$, then also $f(\alpha v + \beta w) = \alpha f(v) + \beta f(w)$ and $\alpha, \beta \in \mathbb{R}$ and $v, w \in V$. Hence, $f(v) = Mv$ we call M the covector such that $Mv \in \mathbb{R}$. The vector space of all covectors is the dual space.

$$\langle m\, , v \rangle \in \mathbb{R} \quad m \in V^* \quad v \in V$$

Now we can see that the dual space is also a vector space so we can use the normal vector space ideas with it,

Lemma 2.24. Let V be a vector space of real $n \times n$ real matrices. Then the dual vector space V^* is also a vector space of $n \times n$ matrices and every linear functional $f : V \to \mathbb{R}$ such that,

$$f(A) := Tr(B^T A), \quad B \in V^*, A \in V$$

We need to generalise the idea of a inner product to matrices and here is a particular inner product called trace pairing. From here on any inner product signs will indicate a trace pairing.

Definition 2.25 (Trace Pairing)**.** For every vector space V of real $n \times n$ matrices with dual V^*, then the pairing is,

$$\langle B\, , A \rangle = Tr(B^T A) = Tr(BA^T)$$

Proposition 2.26. Suppose $A^T = A$ and $B^T = -B$, then, $\mathrm{Tr}(B^T A) = 0$

Proof.

$$\begin{aligned}
\mathrm{Tr}(B^T A) &= -\,\mathrm{Tr}(BA) \\
&= -\,\mathrm{Tr}((BA)^T) \\
&= -\,\mathrm{Tr}(B^T A^T) \\
&= -\,\mathrm{Tr}(A^T B^T) \\
&= -\,\mathrm{Tr}(B^T A)
\end{aligned}$$

□

We now seek a closed form for the adjoint action of G onto $\mathfrak{g}$. This can be done through the following argument using trace pairings,

$$\begin{aligned}\left\langle Ad_g^*\mu\,,\xi\right\rangle &= \langle\mu\,,Ad_g\xi\rangle\\ &= \left\langle\mu\,,g\xi g^{-1}\right\rangle\\ &= \mathrm{Tr}(\mu^T g\xi g^{-1})\\ &= \mathrm{Tr}(\xi g\mu^T g^{-1})\\ &= \mathrm{Tr}[(g^T\mu(g^{-1})^T)^T\xi]\\ &= \left\langle g^T\mu(g^{-1})^T\,,\xi\right\rangle\\ &= \left\langle g^T\mu(g^T)^{-1}\,,\xi\right\rangle\end{aligned}$$

We conclude this chapter with ideas about the adjoint action from the Lie Algebra to the Lie Algebra. This can be derived by taking what we will see to be the first variation. This is because we know to get from a Lie Group to it's tangent space (which is it's Lie Algebra) it suffices to just take the derivative and set the variable to zero. We do this with the adjoint from $G \to \}$. Let $g(t) \in G$ such that $g(0) = e$ where $\eta = \dot{g}(0) \in T_eG$, then we define the adjoint from $\mathfrak{g}$ to $\mathfrak{g}$ as,

$$ad_\eta\xi := \frac{d}{dt}_{t=0}Ad_{g(t)}\xi \qquad \forall\,\xi\in\mathfrak{g}$$

which we can see to be

$$\dot{g}(0)\xi g(0)^{-1} + g(0)\xi\frac{d}{dt}_{t=0}g(t)^{-1} = \eta\xi - \xi\eta$$

Hence we can say that $\mathrm{ad}_\eta\xi = [\eta, \xi] = \eta\xi - \xi\eta$. We define the coadjoint action on μ,

Definition 2.27 (Adjoint / Coadjoint action on $\mathfrak{g}/\mathfrak{g}^*$)**.** The adjoint action of the matrix lie algebra on itself is given by,

$$\begin{aligned}\mathrm{ad} : \mathfrak{g}\times\mathfrak{g} &\to \mathfrak{g}\\ \mathrm{ad}_\eta\xi &= [\eta,\,\xi]\end{aligned}$$

The dual map $\left\langle ad_\eta^*\mu\,,\xi\right\rangle = \langle\mu\,,\mathrm{ad}_\eta\xi\rangle$ is the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$.

We again, now seek a closed form for the coadjoint action from a lie algebra to itself. Above we said that $\left\langle \mathrm{Ad}_g^*\,\mu\,,\xi\right\rangle = \left\langle g^T\mu(g^T)^{-1}\,,\xi\right\rangle$ and so $\mathrm{Ad}_g^*\,\mu = g^T\mu(g^T)^{-1}$. As before we define,

$$\mathrm{ad}_g^*\mu = \frac{d}{dt}_{t=0}Ad_g^*\mu$$

and now we can input our definitions and differentiate where we define $g(0) = 0$ and $\dot{g}(0) = \eta \in T_eG$,

$$\begin{aligned}\mathrm{ad}_g^*\mu &= \frac{d}{dt}_{t=0}Ad_g^*\mu\\ &= \frac{d}{dt}_{t=0}g^T\mu(g^T)^{-1}\\ &= [\dot{g}^T\mu(g^T)^{-1} - g^T\mu(g^T)^{-1}\dot{g}^T(g^T)^{-1}]_{t=0}\\ &= \dot{g}(0)^T\mu(g(0)^T)^{-1} - g(0)^T\mu(g(0)^T)^{-1}\dot{g}(0)^T(g(0)^T)^{-1}\\ &= \eta^t\mu e - e\mu e^{-1}\eta^T e^{-1}\\ &= \eta^T\mu - \mu\eta^T\\ &= [\eta^T,\,\mu]\end{aligned}$$

Hence, $\mathrm{ad}_g^*\mu = [\eta^T,\,\mu]$. These results may seem arbitrary and slightly non-useful currently, however after we have started to derive equations these results will be invaluable. Now we will move towards a more applied treatment of this area and consider the acts of rotation and different types of coordinate systems.

3 Integrating Mechanics and Geometry

3.1 Rotation

A spatial coordinate system with origin at the centre of mass of the given rigid body. We denote it by, $\mathbf{x}(t) \in \mathbb{R}^3$, where $\mathbf{x} = X$. Assume we have a spatial coordinate system, We need a way to rotate things without constraints, so we denote a tensor $R(t)$ and say $\mathbf{x}(t) = R(t)\mathbf{X}$ where $\mathbf{X}$ is in the body coordinate system. The configuration of the body particle at time t is given by a rotation matrix that takes the label $\mathbf{X}$ to current position $\mathbf{x}(t)$ where $R \in \mathrm{SO}(3)$ is a proper rotation matrix; this means,

$$R^T = R^{-1} \qquad \det R = 1$$

The map $\mathbf{X} \to R(t)\mathbf{X}$ is called the body-to-space map.

We can now talk about kinetic energy,

$$K = \frac{1}{2}\int_{\mathcal{B}} \rho \left\|\mathbf{x}\right\|^2 \, d^3\mathbf{X}$$

which we can change to,

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{B}} \rho \left\|\mathbf{x}\right\|^2 \, d^3\mathbf{X} &= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X}) \left\|\dot{R}(t)\mathbf{X}\right\|^2 \, d^3\mathbf{X} \\ &= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})\dot{R}(t)\mathbf{X} \cdot \dot{R}(t)\mathbf{X} \, d^3\mathbf{X}\end{aligned}$$

Now we can say if $V = 0$. Hence, $L = K$ and so,

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{R}} - \frac{\partial K}{\partial R} = \mathbf{0}$$

This is difficult to deal with, so let's do something more cool!

We know that $R^{-1} = R^T$ and so $RR^T = RR^{-1} = I = e$. If we have $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^3$, then $\mathbf{v} \cdot \mathbf{w} = R\mathbf{v} \cdot R\mathbf{w}$. Hence, consider $\left\|\dot{R}\mathbf{X}\right\|^2$ and we know

$$\begin{aligned}\left\|\dot{R}\mathbf{X}\right\|^2 &= \dot{R}\mathbf{X} \cdot \dot{R}\mathbf{X} \\ &= R^{-1}(\dot{R}\mathbf{X}) \cdot R^{-1}(\dot{R}\mathbf{X}) \\ &= \left\|R^{-1}\dot{R}\mathbf{X}\right\|\end{aligned}$$

and so,

$$K = \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X}) \left\|R^{-1}\dot{R}\mathbf{X}\right\| d^3\mathbf{X}$$

Then $K = K(R, \dot{R}) = K(R^{-1}R, R^{-1}\dot{R})$ this is called left symmetry. Hence, we can reduce this to $K(e, R^{-1}\dot{R})$ and change notation let $\kappa(R^{-1}\dot{R})$ and $R^{-1}\dot{R}$ is angular velocity of the body. We can see this from the body and from an observation outside the system. Hence, we call this $R^{-1}\dot{R} = \hat{\mathbf{\Omega}}$.
Interestingly, we know $RR^T = RR^{-1} = I$. Hence,

$$\frac{d}{dt}I = \frac{d}{dt}(RR^{-1}) = \dot{R}R^{-1} + R\frac{d}{dt}R^{-1} = \mathbf{0}$$

and we can also write this as,

$$\begin{aligned} I &= RR^T \\ \mathbf{0} &= \frac{d}{dt}(RR^T) \\ \mathbf{0} &= \dot{R}^T R + R^T \dot{R} \\ R^T \dot{R} &= -(R^T \dot{R})^T \end{aligned}$$

and so $R^{-1}\dot{R} = -(R^{-1}\dot{R})^T$ and so $\hat{\mathbf{\Omega}} = -\hat{\mathbf{\Omega}}^T$. This is the antisymmetric property we have noted about this vector.

Now we go back to kinetic energy to nicely write it as $\hat{\mathbf{\Omega}}$

$$K = \frac{1}{2} \int_{\mathcal{B}} \rho(\mathbf{X}) \left\| \hat{\mathbf{\Omega}} \mathbf{X} \right\|^2 d^3\mathbf{X}$$

and now we can prove that $\hat{\mathbf{\Omega}}\mathbf{X} = \mathbf{\Omega} \times \mathbf{X}$ where,

$$\hat{\mathbf{\Omega}} = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}$$

where $\mathbf{\Omega} = \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix}$ where Ω is the axel vector. and so,

$$\begin{aligned} K &= \frac{1}{2} \int_{\mathcal{B}} \rho(\mathbf{X}) \left\| \mathbf{\Omega} \times \mathbf{X} \right\|^2 d^3\mathbf{X} \\ &= \frac{1}{2} \int_{\mathcal{B}} \rho(\mathbf{X}) (\mathbf{\Omega} \times \mathbf{X}) \cdot (\mathbf{\Omega} \times \mathbf{X})\, d^3\mathbf{X} \\ &= \frac{1}{2} \int_{\mathcal{B}} \rho(\mathbf{X}) (\left\|\mathbf{\Omega}^2\right\| \left\|\mathbf{X}^2\right\| - (\mathbf{\Omega} \cdot \mathbf{X})^2)\, d^3\mathbf{X} \\ &= \frac{1}{2} \int_{\mathcal{B}} \rho(\mathbf{X}) (\mathbf{\Omega}^T \mathbf{\Omega} \left\|\mathbf{X}^2\right\| - \mathbf{\Omega}^T \mathbf{X}\mathbf{X}^T \mathbf{\Omega})\, d^3\mathbf{X} \\ &= \frac{1}{2} \int_{\mathcal{B}} \rho(\mathbf{X}) \mathbf{\Omega}^T (\left\|\mathbf{X}^2\right\| I - \mathbf{X}\mathbf{X}^T) \mathbf{\Omega}\, d^3\mathbf{X} \\ &= \frac{1}{2} \int_{\mathcal{B}} \rho(\mathbf{X}) \mathbf{\Omega}^T \mathbf{\Omega} (\left\|\mathbf{X}^2\right\| - \mathbf{X}\mathbf{X}^T)\, d^3\mathbf{X} \\ &= \frac{1}{2} \mathbf{\Omega} \cdot \mathbf{\Omega} \int_{\mathcal{B}} \rho(\mathbf{X}) (\left\|\mathbf{X}^2\right\| - \mathbf{X}\mathbf{X}^T)\, d^3\mathbf{X} \\ &= \frac{1}{2} \mathbb{I} \mathbf{\Omega} \cdot \mathbf{\Omega} \end{aligned}$$

where $\mathbb{I}$ is the moment of inertia tensor, which we define as,

$$\mathbb{I} = \int_{\mathcal{B}} \rho(\mathbf{X}) \left\|\mathbf{X}\right\|^2 I - \mathbf{X}\mathbf{X}^T d^3\mathbf{X}$$

where $\mathbf{X}\mathbf{X}^T = \mathbf{X} \otimes \mathbf{X}$ and,

$$(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$.

3.2 Calculus of variations

We are going to consider a continuous level, but you can use discrete level.

Theorem 3.1 (The variation principle)**.**

$$\mathcal{L} = \int_{t_1}^{t_2} \frac{1}{2}\mathbb{I}\mathbf{\Omega}\cdot\mathbf{\Omega}\,dt$$

and we find differential equations by letting $\delta\mathcal{L} = 0$ but this is subject to $\delta\mathbf{\Omega}(t_1) = \delta\mathbf{\Omega}(t_2) = \mathbf{0}$

and so,

$$\delta\int_{t_1}^{t_2}\frac{1}{2}\mathbb{I}\mathbf{\Omega}\cdot\mathbf{\Omega}\,dt = \int_{t_1}^{t_2}\frac{1}{2}\mathbb{I}\delta\mathbf{\Omega}\cdot\mathbf{\Omega} + \frac{1}{2}\mathbb{I}\mathbf{\Omega}\cdot\delta\mathbf{\Omega}\,dt = \int_{t_1}^{t_2}\mathbb{I}\mathbf{\Omega}\cdot\delta\mathbf{\Omega}$$

but what is $\delta\mathbf{\Omega}$, but remember we have $\hat{\mathbf{\Omega}}$, which is the lie algebra of SO(3). We said, $\hat{\mathbf{\Omega}} = R^T\dot{R} = R^{-1}\dot{R}$. Now we take variations of $\hat{\mathbf{\Omega}} = \mathbf{\Omega}\times\mathbf{X}$ and so,

$$(\delta\mathbf{\Omega})\times\mathbf{X} = (\delta\hat{\mathbf{\Omega}})\mathbf{X}$$

and so we see that,

$$\delta\hat{\mathbf{\Omega}} = \delta(R^{-1}\dot{R}) = \delta R^{-1}\dot{R} + R^{-1}\delta\dot{R} = 0$$

as $\delta I = \delta RR^{-1} + R\delta R^{-1}$ and then we see that $R^{-1}\delta RR^{-1} + R^{-1}R\delta R^{-1} = 0$ and so as $RR^{-1} = I$, $R^{-1}\delta RR^{-1} + \delta R^{-1} = \mathbf{0}$. We have that $\delta R^{-1}\dot{R} + R^{-1}\delta\dot{R} = \delta\hat{\mathbf{\Omega}}$ and $\hat{\mathbf{\Omega}} = R^{-1}\dot{R}$ where $\hat{\mathbf{\Lambda}} = R^{-1}\delta R$ and so we sub in,

$$\begin{aligned}
\delta\hat{\mathbf{\Omega}} &= -R^{-1}\delta RR^{-1}\dot{R} + R^{-1}\frac{d}{dt}\delta R \\
&= R^{-1}\delta R\hat{\mathbf{\Omega}} + \frac{d}{dt}(R^{-1}\delta R) - (\frac{d}{dt}R^{-1})\delta R \\
&= R^{-1}\delta R\hat{\mathbf{\Omega}} + \frac{d}{dt}(R^{-1}\delta R) + R^{-1}\dot{R}R^{-1}\delta R \\
&= -\hat{\mathbf{\Lambda}}\hat{\mathbf{\Omega}} + \frac{d}{dt}\hat{\mathbf{\Lambda}} + \hat{\mathbf{\Omega}}\hat{\mathbf{\Lambda}} \\
&= \dot{\hat{\mathbf{\Lambda}}} + [\hat{\mathbf{\Omega}}, \hat{\mathbf{\Lambda}}]
\end{aligned}$$

Exercise. Prove,

$$\delta\mathbf{\Omega} = \dot{\Lambda} + (\mathbf{\Omega}\times\mathbf{\Lambda})$$

Solution. We can use the fact that $\widehat{[\mathbf{\Omega},\, \mathbf{\Lambda}]} = [\hat{\mathbf{\Omega}},\, \hat{\mathbf{\Lambda}}]$ and then we can get the required result.

$$\begin{aligned}
\delta\hat{\mathbf{\Omega}} &= \dot{\hat{\mathbf{\Lambda}}} + [\hat{\mathbf{\Omega}},\, \hat{\mathbf{\Lambda}}] \\
&= \dot{\hat{\mathbf{\Lambda}}} + \widehat{[\mathbf{\Omega},\, \mathbf{\Lambda}]} \\
&= \dot{\hat{\mathbf{\Lambda}}} + \widehat{(\mathbf{\Omega}\times\mathbf{\Lambda})} \\
\widehat{\delta\mathbf{\Omega}} &= \widehat{\dot{\mathbf{\Lambda}} + (\mathbf{\Omega}\times\mathbf{\Lambda})}
\end{aligned}$$

and so we can see that $\delta\mathbf{\Omega} = \dot{\mathbf{\Lambda}} + (\mathbf{\Omega}\times\mathbf{\Lambda})$

Now, let us substitute this back into our variational principle.

$$\begin{aligned}
\delta \int_{t_1}^{t_2} \mathbb{I}\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}\, dt &= 0 \\
\int_{t_1}^{t_2} \mathbb{I}\boldsymbol{\Omega} \cdot \delta\boldsymbol{\Omega}\, dt &= 0 \\
\int_{t_1}^{t_2} \mathbb{I}\boldsymbol{\Omega} \cdot (\dot{\boldsymbol{\Lambda}} + \boldsymbol{\Omega} \times \boldsymbol{\Lambda})\, dt &= 0 \\
[\mathbb{I}\boldsymbol{\Omega} \cdot \boldsymbol{\Lambda}|_{t_2} - \mathbb{I}\boldsymbol{\Omega} \cdot \boldsymbol{\Lambda}|_{t_1}] - \int_{t_1}^{t_2} \frac{d}{dt}(\mathbb{I}\boldsymbol{\Omega}) \cdot \boldsymbol{\Lambda}\, dt + \int_{t_1}^{t_2} (\mathbb{I}\boldsymbol{\Omega} \times \boldsymbol{\Omega}) \cdot \boldsymbol{\Lambda} &= 0 \\
0 - 0 - \int_{t_1}^{t_2} (-\mathbb{I}\dot{\boldsymbol{\Omega}} + \mathbb{I}\boldsymbol{\Omega} \times \boldsymbol{\Omega}) \cdot \boldsymbol{\Lambda}\, dt &= 0
\end{aligned}$$

Hence,

$$\mathbb{I}\dot{\boldsymbol{\Lambda}} = \mathbb{I}\boldsymbol{\Omega} \times \boldsymbol{\Omega}$$

We can write the equations by considering the tangent space.

3.3 The Hat Map as a Lie Algebra Isomorphisms

In this Chapter we will take a quick jaunt back into some purer topics, more specifically a Lie Algebra Isomorphism called the Hat Map. We can define a Lie Algebra Isomorphism as a bijective Lie Algebra homomorphism, which we then define as,

Definition 3.2 (Lie Algebra Homomorphism)**.** A Lie Algebra homomorphism is a linear map, $\phi : \mathfrak{g} \to \mathfrak{g}'$ compatible with the respective lie brackets,

$$\phi([x, y]_{\mathfrak{g}}) = [\phi(x), \phi(y)]_{\mathfrak{g}'} \qquad \forall x, y \in \mathfrak{g}$$

We note that the Lie bracket for $\mathbb{R}^3$ is the cross product, and more specifically $(\mathbb{R}^3, \times)$ is a Lie Algebra. Then we can talk about a Lie Algebra homomorphism, $\phi : \mathbb{R}^3 \to \mathfrak{so}(3)$, which is a homomorphism that maps,

$$\phi(\mathbf{x} \times \mathbf{y}) = [\phi(x), \phi(y)]_{\mathfrak{so}(3)}.$$

We let this ϕ be the hat map where we define,

$$\hat{\boldsymbol{\omega}} = \begin{pmatrix} 0 & -\omega_1 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

The Lie algebra of SO(3) is the space of skew-symmetric matrices, SO(3). Then we can conclude that the Euler-Poincare Equations are written as:

$$\frac{d}{dt}\frac{\partial \ell}{\partial \hat{\boldsymbol{\Omega}}} - \mathrm{ad}^*_{\hat{\boldsymbol{\Omega}}} \frac{\partial \ell}{\partial \hat{\boldsymbol{\Omega}}} = 0$$

Let $\mathbf{\Pi}$ be any element in $\mathfrak{g}^*$, then ad^* operator is defined by $\left\langle \mathrm{ad}^*_{\hat{\mathbf{\Omega}}}\mathbf{\Pi}, \hat{\boldsymbol{\omega}} \right\rangle = \left\langle \mathbf{\Pi}, \mathrm{ad}_{\hat{\mathbf{\Omega}}}\hat{\boldsymbol{\omega}} \right\rangle$ where $\hat{\boldsymbol{\omega}} \in \mathfrak{g}^*$.

$$\begin{aligned}
\left\langle \mathrm{ad}^*_{\hat{\mathbf{\Omega}}}\mathbf{\Pi}, \hat{\boldsymbol{\omega}} \right\rangle &= \left\langle \mathbf{\Pi}, \mathrm{ad}_{\hat{\mathbf{\Omega}}}\hat{\boldsymbol{\omega}} \right\rangle \\
&= \left\langle \mathbf{\Pi}, [\hat{\mathbf{\Omega}}, \hat{\omega}] \right\rangle \\
&= \mathrm{Tr}(\mathbf{\Pi}^\top [\hat{\mathbf{\Omega}}, \hat{\omega}]) \\
&= \mathrm{Tr}(\mathbf{\Pi}^\top \hat{\mathbf{\Omega}}\hat{\omega} - \mathbf{\Pi}^\top \hat{\omega}\hat{\mathbf{\Omega}}) \\
&= \mathrm{Tr}(\mathbf{\Pi}\hat{\mathbf{\Omega}}^\top \hat{\omega} - \mathbf{\Pi}\hat{\mathbf{\Omega}}\hat{\omega}^\top) \\
&= \mathrm{Tr}(\mathbf{\Pi}\hat{\mathbf{\Omega}}\hat{\omega}^\top - \hat{\mathbf{\Omega}}\mathbf{\Pi}\hat{\omega}^\top) \\
&= \mathrm{Tr}((\mathbf{\Pi}\hat{\mathbf{\Omega}} - \hat{\mathbf{\Omega}}\mathbf{\Pi})\hat{\omega}^\top) \\
&= \mathrm{Tr}([\mathbf{\Pi}, \hat{\mathbf{\Omega}}]\hat{\omega}^\top) \\
&= \left\langle [\mathbf{\Pi}, \hat{\mathbf{\Omega}}], \hat{\omega} \right\rangle
\end{aligned}$$

Then, $\mathrm{ad}^*_{\hat{\mathbf{\Omega}}}\mathbf{\Pi} = [\mathbf{\Pi}, \hat{\mathbf{\Omega}}]$. From here we can conclude that, the hap map is a lie algebra isomorphism, i.e., $[\hat{\mathbf{\Omega}}, \hat{\omega}] = \widehat{\mathbf{\Omega} \times \boldsymbol{\omega}}$. We prove this from the definition of the Lie Bracket, that is,

$$\begin{aligned}
[\hat{\mathbf{\Omega}}, \hat{\omega}] &= \hat{\mathbf{\Omega}}\hat{\omega} - \hat{\omega}\hat{\mathbf{\Omega}} \\
&= \begin{pmatrix} 0 & -\Omega_1 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\omega_1 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -\omega_1 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\Omega_1 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
&= \begin{pmatrix} -\Omega_3\omega_3 - \Omega_2\omega_2 & \Omega_2\omega_1 & \Omega_3\omega_1 \\ \Omega_1\omega_2 & -\Omega_3\omega_3 - \Omega_1\omega_1 & \Omega_3\omega_2 \\ \Omega_1\omega_3 & \Omega_2\omega_3 & -\Omega_2\omega_2 - \Omega_1\omega_1 \end{pmatrix} - \begin{pmatrix} -\omega_3\Omega_3 - \omega_2\Omega_2 & \omega_2\Omega_1 & \omega_3\Omega_1 \\ \omega_1\Omega_2 & -\omega_3\Omega_3 - \omega_1\Omega_1 & \omega_3\Omega_2 \\ \omega_1\Omega_3 & \omega_2\Omega_3 & -\omega_2\Omega_2 - \omega_1\Omega_1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \Omega_2\omega_1 - \Omega_1\omega_2 & \Omega_3\omega_1 - \Omega_1\omega_3 \\ \Omega_1\omega_2 - \Omega_2\omega_1 & 0 & \Omega_3\omega_2 - \Omega_2\omega_3 \\ \Omega_1\omega_3 - \Omega_3\omega_1 & \Omega_2\omega_3 - \Omega_3\omega_2 & 0 \end{pmatrix} \\
&= \widehat{\mathbf{\Omega} \times \boldsymbol{\omega}}
\end{aligned}$$

We have proved that where ϕ is the hap map that $[\phi(\mathbf{x}), \phi(\mathbf{y})] = \phi(\mathbf{x} \times \mathbf{y})$. We now need to prove that the hap map is a bijective linear map. It is linear as it can be represented as a matrix in $\mathrm{SO}(3)$. Now we prove bijectivity by first proving injectivity, then surectivity. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}$ and we know that $\hat{\mathbf{x}} = \hat{\mathbf{y}}$, that is,

$$\begin{pmatrix} 0 & -x_1 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -y_1 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{pmatrix}$$

Then we can see that $x_1 = y_1$, $x_2 = y_2$ and $x_3 = y_3$. Therefore, $\mathbf{x} = \mathbf{y}$. Hence the hat map is injective. Now we seek to prove that the hap map is surjective. Consider some $\hat{\mathbf{z}} \in \mathfrak{so}(3)$, then we can write it as,

$$\begin{pmatrix} 0 & -z_1 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{pmatrix}$$

Then this will uniquely define some $\mathbf{z} = (z_1, z_2, z_3)^T \in \mathbb{R}^3$. Hence the hat map is surjective. Therefore the hat map is a Lie Algebra Isomorphism.

We have described Lagrangians that have left or right invariance. We now look to Lagrangians that have Symmetry breaking parameters, like gravity. If we consider the spherical pendulum, we have defined $\hat{\mathbf{\Omega}} = R^T\dot{R}$ and we define $\hat{\boldsymbol{\omega}} = R\mathbf{\Omega}$ and then we can see that $\hat{\boldsymbol{\omega}} = \dot{R}R^T = \dot{R}R^{-1}$ where $R \in \mathrm{SO}(3)$. This doesn't lead to a symmetric Lagrangian but we can still use our theory here.

We are going to study rigid body dynamics in the spatial frame. We look firstly to the Lagrangian. We have showed,

$$L(R,\, \dot{R}) = \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X}) \left\| \dot{R}\mathbf{X} \right\|^2 d^3\mathbf{X}$$

and we showed that $L(R,\, \dot{R}) = L(e,\, R^{-1}\dot{R})$ and then we used Euler Poincare Theorem to show that $\ell = \frac{1}{2}\mathbb{I}\mathbf{\Omega}\cdot\mathbf{\Omega}$. Now assume we would prefer to formulate rigid body dynamics in the spatial frame. We need to consider a $\boldsymbol{\omega}$ such that $\hat{\boldsymbol{\omega}} = \dot{R}R^{-1}$. We can now prove that, $L(R,\, \dot{R}) \neq L(R\chi,\, \dot{R}\chi)$ (right multiplication) hence we have broken symmetry,

$$\begin{aligned}
L(R\chi,\, \dot{R}\chi) &= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X}) \left\| \dot{R}\chi\mathbf{X} \right\|^2 d^3\mathbf{X} \\
&= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})(\dot{R}\chi\mathbf{X})\cdot(\dot{R}\chi\mathbf{X}) d^3\mathbf{X} \\
&= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})(\dot{R}\chi\mathbf{X})(\dot{R}\chi\mathbf{X})^T d^3\mathbf{X} \\
&= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})(\dot{R}\chi\mathbf{X})(\mathbf{X}^T\chi^T\dot{R}^T) d^3\mathbf{X} \\
&\neq \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X}) \left\| \dot{R}\mathbf{X} \right\|^2 d^3\mathbf{X} = L(R, \dot{R})
\end{aligned}$$

Now we seek this Lagrangian,

$$\begin{aligned}
L(R,\, \dot{R}) &= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X}) \left\| \dot{R}\mathbf{X} \right\|^2 d^3\mathbf{X} \\
&= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X}) \left\| \dot{R}R^{-1}R\mathbf{X} \right\|^2 d^3\mathbf{X} \\
&= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X}) \left\| \hat{\boldsymbol{\omega}}R\mathbf{X} \right\|^2 d^3\mathbf{X} \\
&= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})(\boldsymbol{\omega}\times R\mathbf{X})\cdot(\boldsymbol{\omega}\times R\mathbf{X}) d^3\mathbf{X} \\
&= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})(\|\boldsymbol{\omega}\|^2 \|R\mathbf{X}\|^2 - (\boldsymbol{\omega}\cdot R\mathbf{X})^2) d^3\mathbf{X} \\
&= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})(\boldsymbol{\omega}^T\boldsymbol{\omega}\, \|R\mathbf{X}\|^2 - \boldsymbol{\omega}^T(R\mathbf{X})(R\mathbf{X})^T\boldsymbol{\omega}) d^3\mathbf{X} \\
&= \boldsymbol{\omega}^T\frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})(R\,\|\mathbf{X}\|^2 R^T - R\mathbf{X}\mathbf{X}^T R^T d^3\mathbf{X}\,\boldsymbol{\omega} \\
&= \boldsymbol{\omega}^T R\frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})(\|\mathbf{X}\|^2 - \mathbf{X}\mathbf{X}^T d^3\mathbf{X}\, R^T\boldsymbol{\omega} \\
&= \frac{1}{2}\boldsymbol{\omega}^T(R\mathbb{I}R^T\boldsymbol{\omega}) \\
&= \frac{1}{2}\boldsymbol{\omega}\cdot(R\mathbb{I}R^T)\boldsymbol{\omega} = L(\boldsymbol{\omega},\, R)
\end{aligned}$$

We define a new parameter, $\mathbb{J} := R\mathbb{I}R^T$ and so $\ell = \ell(\mathbb{J},\, \boldsymbol{\omega}) = \frac{1}{2}\boldsymbol{\omega}(t)\cdot\mathbb{J}(t)\boldsymbol{\omega}(t)$ Now we take variations as usual,

$$\delta\int_{t_1}^{t_2} \ell(\mathbb{J},\, \boldsymbol{\omega}) = 0$$

$$\delta \int_{t_1}^{t_2} \frac{1}{2}\boldsymbol{\omega}(t) \cdot \mathbb{J}(t)\boldsymbol{\omega}(t)dt = 0$$

and we ask what is $\delta\mathbb{J}(t)$,

$$\begin{aligned}
\delta\mathbb{J}(t) &= \delta(R\mathbb{I}R^T) \\
&= \delta R\mathbb{I}R^T + R\mathbb{I}\delta R^T \\
&= \delta RR^{-1}R\mathbb{I}R^T - R\mathbb{I}R^{-1}\delta RR^{-1} \\
&= \hat{\boldsymbol{\Lambda}}\mathbb{J} - \mathbb{J}\hat{\boldsymbol{\Lambda}} \\
&= [\hat{\boldsymbol{\Lambda}}, \mathbb{J}]
\end{aligned}$$

where $\hat{\boldsymbol{\Lambda}} = \delta RR^{-1}$

Exercise. Prove that $\delta\hat{\boldsymbol{\omega}} = \dot{\hat{\boldsymbol{\Lambda}}} + [\hat{\boldsymbol{\Lambda}}, \hat{\boldsymbol{\omega}}]$ and $\delta\boldsymbol{\omega} = \dot{\boldsymbol{\Lambda}} + \boldsymbol{\Lambda} \times \boldsymbol{\omega}$ and then take variations of $\frac{1}{2}\boldsymbol{\omega} \cdot \mathbb{J}\boldsymbol{\omega}$ and prove that $\frac{d}{dt}(\mathbb{J}\boldsymbol{\omega}) = \mathbf{0}$ and then $\frac{d\mathbb{J}}{dt}\hat{\boldsymbol{\omega}} + \mathbb{J}\dot{\hat{\boldsymbol{\omega}}} = \mathbf{0}$ and so $\frac{d\mathbb{J}}{dt} = [\hat{\boldsymbol{\omega}}, \mathbb{J}]$.

4 Euler-Poincaré Reduction by Symmetry

To gain a general idea of how the equations of motion appear for rotational dynamics with symmetry, we consider an arbitrary Lagrangian of this form,

$$L : T\mathrm{SO}(3) \to \mathbb{R}$$

$$L = L(R, \dot{R})$$

and satisfies,

$$\delta \int_{t_1}^{t_2} L(R, \dot{R})\, dt = 0$$

this means,

$$\begin{aligned}
\delta \int_{t_1}^{t_2} L(R, \dot{R})\, dt &= \int_{t_1}^{t_2} \left\langle \frac{\partial L}{\partial R}, \delta R \right\rangle + \left\langle \frac{\partial L}{\partial \dot{R}}, \delta R \right\rangle dt \\
&= \left\langle \frac{\partial L}{\partial \dot{R}}, \delta R \right\rangle \Bigg|_{t_2} - \left\langle \frac{\partial L}{\partial \dot{R}}, \delta R \right\rangle \Bigg|_{t_1} + \int_{t_1}^{t_2} \left\langle \frac{\partial L}{\partial R} - \frac{d}{dt}\frac{\partial L}{\partial \dot{R}}, \delta R \right\rangle dt
\end{aligned}$$

and so we can notice $\frac{d}{dt}\frac{\partial L}{\partial \dot{R}} - \frac{\partial L}{\partial R} = 0$

Definition 4.1 (Left-Symmetric Lagrangian)**.** A Lagrangian is said to be left-symmetric or left-invariant under the action of the group of the group of rotations if, $L(\chi R,\, \chi \dot{R}) = L(R, \dot{R})\ \forall\, \chi \in \mathrm{SO}(3)$.

We also know $\mathfrak{SO}(3) = T_e\mathrm{SO}(3)$ and we said that $v \in T_eG \implies g^{-1}v \in \mathfrak{g} = T_eG$. We know $\dot{R}(t) \in T_{R(t)}\mathrm{SO}(3)$ and so we can say $R^{-1}\dot{R} \in \mathfrak{SO}(3)$.

We say

$$\begin{aligned}
L(R, \dot{R}) &= L(R^{-1}R, R^{-1}\dot{R}) \\
&= \widetilde{\ell}(R^{-1}\dot{R}) = \widetilde{\ell}(\hat{\mathbf{\Omega}})
\end{aligned}$$

Now we write out Hamilton's principle,

$$\begin{aligned}
0 &= \delta \int_{t_1}^{t_2} \widetilde{\ell}(\hat{\mathbf{\Omega}})\, dt \\
&= \delta \int_{t_1}^{t_2} \ell(\mathbf{\Omega}) = \mathbf{0}
\end{aligned}$$

Euler Poincare equations,

$$\frac{d}{dt}\frac{\partial \ell}{\partial \mathbf{\Omega}} - \frac{\partial \ell}{\partial \mathbf{\Omega}} \times \Omega = \mathbf{0}$$

Exercise. Derive these

Solution. We shall start from Hamilton's Principle and move forward to derive our Euler-Poincare Equations.

$$\begin{aligned}
\delta \int_{t_1}^{t_2} \ell(\hat{\mathbf{\Omega}})\, dt &= \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}}, \delta \hat{\mathbf{\Omega}} \right\rangle dt \\
&= \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \mathbf{\Omega}}, \delta \mathbf{\Omega} \right\rangle
\end{aligned}$$

Now, we shall use a fact we proved in the last exercise $\delta\boldsymbol{\Omega} = \dot{\boldsymbol{\Lambda}} + (\boldsymbol{\Omega} \times \boldsymbol{\Lambda})$ to derive the Euler-Poincare equations we wanted,

$$\begin{aligned}
\int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \boldsymbol{\Omega}}, \delta\boldsymbol{\Omega} \right\rangle &= \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \boldsymbol{\Omega}}, \dot{\boldsymbol{\Lambda}} + \boldsymbol{\Omega} \times \boldsymbol{\Lambda} \right\rangle \\
&= \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \boldsymbol{\Omega}}, \frac{d}{dt}\boldsymbol{\Lambda} \right\rangle + \left\langle \frac{\partial \ell}{\partial \boldsymbol{\Omega}}, \boldsymbol{\Omega} \times \boldsymbol{\Lambda} \right\rangle \\
&= \int_{t_1}^{t_2} \left\langle -\frac{d}{dt}\frac{\partial \ell}{\partial \boldsymbol{\Omega}}, \boldsymbol{\Lambda} \right\rangle + \left\langle -\boldsymbol{\Omega} \times \frac{\partial \ell}{\partial \boldsymbol{\Omega}}, \boldsymbol{\Lambda} \right\rangle \\
&= \int_{t_1}^{t_2} \left\langle -\frac{d}{dt}\frac{\partial \ell}{\partial \boldsymbol{\Omega}}, \boldsymbol{\Lambda} \right\rangle + \left\langle \frac{\partial \ell}{\partial \boldsymbol{\Omega}} \times \boldsymbol{\Omega}, \boldsymbol{\Lambda} \right\rangle \\
&= \int_{t_1}^{t_2} \left\langle -\frac{d}{dt}\frac{\partial \ell}{\partial \boldsymbol{\Omega}} + \frac{\partial \ell}{\partial \boldsymbol{\Omega}} \times \boldsymbol{\Omega}, \boldsymbol{\Lambda} \right\rangle = \mathbf{0}
\end{aligned}$$

Hence we say that

$$\frac{d}{dt}\frac{\partial \ell}{\partial \boldsymbol{\Omega}} - \frac{\partial \ell}{\partial \boldsymbol{\Omega}} \times \boldsymbol{\Omega} = \mathbf{0}$$

These are the Euler-Poincare equations for rotational dynamics with symmetry under left multiplication.

Theorem 4.2. The spatial angular momentum (in the spatial frame) is conserved along solutions of the Euler-Poincare equations.

Proof. We know $\frac{d}{dt}\frac{\partial \ell}{\partial \mathbf{\Omega}} - \frac{\partial \ell}{\partial \mathbf{\Omega}} \times \Omega = 0$ and we know that $R\frac{\partial \ell}{\partial \mathbf{\Omega}}$ and the multiplcation by R means spatial frame. Hence we prove,

$$\frac{d}{dt}R\frac{\partial \ell}{\partial \mathbf{\Omega}} = \mathbf{0}$$

$$\begin{aligned}
\frac{d}{dt}R\frac{\partial \ell}{\partial \mathbf{\Omega}} &= \dot{R}\frac{\partial \ell}{\partial \mathbf{\Omega}} + R\frac{d}{dt}\frac{\partial \ell}{\partial \mathbf{\Omega}} \\
&= R\hat{\mathbf{\Omega}}\frac{\partial \ell}{\partial \mathbf{\Omega}} + R\left(\frac{\partial \ell}{\partial \mathbf{\Omega}} \times \mathbf{\Omega}\right) \\
&= R\left(\Omega \times \frac{\partial \ell}{\partial \mathbf{\Omega}}\right) + R\left(\frac{\partial \ell}{\partial \mathbf{\Omega}} \times \mathbf{\Omega}\right) \\
&= \mathbf{0}
\end{aligned}$$

□

Now we want to write a general form of the Euler-Poincare Equations for left invariant systems. Let L be a lagrangian on the tangent bundle of a matrix lie group G,

$$L : TG \to \mathbb{R}$$

$$L = L(g, \dot{g}) \qquad \forall\, g \in G$$

Assume that the Lagrangian is left-invariant,

$$L(g,\, \dot{g}) = L(hg,\, h\dot{g}) \quad \forall\, h \in G$$

and now let $h = g^{-1}$, and so $L(g,\, \dot{g}) = L(g^{-1}g\, g^{-1}\dot{g}) = \ell(\xi)$. We have gone from a lie group to a lie algebra, $\xi = g^{-1}\dot{g} \in T_eG = \mathfrak{g}$ which is a matrix lie algebra. We now aim to use the action functional and variational derivative,

$$\delta \int_{t_1}^{t_2} L(g,\, \dot{g}) = \mathbf{0}$$

$$\delta \int_{t_1}^{t_2} \ell(\xi)\, dt = \mathbf{0}$$

$$\int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \xi}, \delta\xi \right\rangle dt = \mathbf{0}$$

Now we want to consider $\delta\xi = \delta(g^{-1}\dot{g})$,

$$\begin{aligned}
\delta(g^{-1}\dot{g}) &= \delta g^{-1}\dot{g} + g^{-1}\delta\dot{g} \\
&= g^{-1}\delta g g^{-1}\dot{g} + g^{-1}\frac{d}{dt}\delta g \\
&= -(g^{-1}\delta g)g^{-1}\dot{g} + g^{-1}\frac{d}{dt}\delta g \\
&= -\eta\xi + \frac{d}{dt}\delta(g^{-1}\delta g) + (g^{-1}\dot{g})(g^{-1}\frac{d}{dt}\delta g) \\
&= -\eta\xi + \dot{\eta} + \xi\eta \\
&= \dot{\eta} + [\xi,\, \eta] \\
&= \dot{\eta} + \mathrm{ad}_\xi \eta
\end{aligned}$$

and so back to the derivation,

$$\int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \xi}, \delta\xi \right\rangle dt = 0$$
$$\int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \xi}, \dot{\eta} + \mathrm{ad}_\xi \eta \right\rangle = 0$$
$$\int_{t_1}^{t_2} \left\langle -\frac{d}{dt}\left(\frac{\partial \ell}{\partial \xi}\right), \eta \right\rangle + \left\langle \mathrm{ad}_\xi^* \frac{\partial \ell}{\partial \xi}, \eta \right\rangle dt = 0$$

Since η is arbitrary our equation is of this form,

$$\frac{d}{dt}\frac{\partial \ell}{\partial \xi} - \mathrm{ad}_\xi^* \frac{\partial \ell}{\partial \xi} = 0$$

and these are our Euler-Poincare equations for a left invariant system.

Theorem 4.3 (Noether's Theorem for left-invariant systems)**.** The Euler Poincare equations associated a left-invariant system preserve the generalised momentum along solutions of the Euler-Poincare equations, that is,

$$\frac{d}{dt}\left(\mathrm{Ad}^*_{g^{-1}(t)} \frac{\partial \ell}{\partial \xi}(t)\right) = 0$$

Proof. Suppose we have a left invariant lagrangian, i.e. $L(g, \dot{g}) = L(e, g^{-1}\dot{g}) = \ell(g^{-1}g) := \ell(\xi)$ where $\xi = g^{-1}\dot{g}$. Firstly, however, let us consider the following derivative where $\mu(t) \in \mathfrak{g}$,

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=t_0} \left(\mathrm{Ad}_{g^{-1}(t)}\, \mu(t)\right) &= \left.\frac{d}{dt}\right|_{t=t_0} \mathrm{Ad}_{g^{-1}(t)g(t_0)} \left(\mathrm{Ad}_{g^{-1}(t_0)}\, \mu\right) \\
&= -\mathrm{ad}_{g^{-1}(t_0)\dot{g}(t_0)} \left(\mathrm{Ad}_{g^{-1}(t_0)}\, \mu\right) \\
&= -\mathrm{ad}_{\xi(t_0)} \left(\mathrm{Ad}_{g^{-1}(t_0)}\, \mu\right)
\end{aligned}$$

and so we can say,

$$\frac{d}{dt}\left(\mathrm{Ad}_{g^{-1}(t)}\, \mu(t)\right) = -\mathrm{ad}_{\xi(t)} \left(\mathrm{Ad}_{g^{-1}(t)}\, \mu(t)\right)$$

Now, we can move forward and consider the trace pairing of our interested quantity and $\mu(t)$.

$$\begin{aligned}
\left\langle \frac{d}{dt}\left(\mathrm{Ad}^*_{g^{-1}(t)} \frac{\partial \ell}{\partial \xi}(t)\right), \mu(t) \right\rangle &= \frac{d}{dt}\left\langle \mathrm{Ad}^*_{g^{-1}(t)} \frac{\partial \ell}{\partial \xi}(t), \mu(t) \right\rangle \\
&= \frac{d}{dt}\left\langle \frac{\partial \ell}{\partial \xi}(t), \mathrm{Ad}_{g^{-1}(t)}\, \mu(t) \right\rangle \\
&= \left\langle \frac{d}{dt}\frac{\partial \ell}{\partial \xi}(t), \mathrm{Ad}_{g^{-1}(t)}\, \mu(t) \right\rangle + \left\langle \frac{\partial \ell}{\partial \xi}(t), \frac{d}{dt}\mathrm{Ad}_{g^{-1}(t)}\, \mu(t) \right\rangle \\
&= \left\langle \frac{d}{dt}\frac{\partial \ell}{\partial \xi}(t), \mathrm{Ad}_{g^{-1}(t)}\, \mu(t) \right\rangle + \left\langle \frac{\partial \ell}{\partial \xi}(t), -ad_{\xi(t)}(\mathrm{Ad}_{g^{-1}(t)}\, \mu(t)) \right\rangle \\
&= \left\langle \frac{d}{dt}\frac{\partial \ell}{\partial \xi}(t), \mathrm{Ad}_{g^{-1}(t)}\, \mu(t) \right\rangle - \left\langle \frac{\partial \ell}{\partial \xi}(t), ad_{\xi(t)}(\mathrm{Ad}_{g^{-1}(t)}\, \mu(t)) \right\rangle \\
&= \left\langle \frac{d}{dt}\frac{\partial \ell}{\partial \xi}(t), \mathrm{Ad}_{g^{-1}(t)}\, \mu(t) \right\rangle - \left\langle ad^*_{\xi(t)}\frac{\partial \ell}{\partial \xi}(t), \mathrm{Ad}_{g^{-1}(t)}\, \mu(t) \right\rangle \\
&= \left\langle \frac{d}{dt}\frac{\partial \ell}{\partial \xi}(t) - ad^*_{\xi(t)}\frac{\partial \ell}{\partial \xi}(t), \mathrm{Ad}_{g^{-1}(t)}\, \mu(t) \right\rangle \\
&= \left\langle \mathrm{Ad}^*_{g^{-1}(t)}\left[\frac{d}{dt}\frac{\partial \ell}{\partial \xi}(t) - ad^*_{\xi(t)}\frac{\partial \ell}{\partial \xi}(t)\right], \mu(t) \right\rangle
\end{aligned}$$

Hence, we can say that,

$$\frac{d}{dt}\left(\mathrm{Ad}^*_{g^{-1}(t)}\frac{\partial \ell}{\partial \xi}(t)\right) = \mathrm{Ad}^*_{g^{-1}(t)}\underbrace{\left[\frac{d}{dt}\frac{\partial \ell}{\partial \xi}(t) - ad^*_{\xi(t)}\frac{\partial \ell}{\partial \xi}(t)\right]}_{\text{LHS of Euler-Poincare Equations}}$$

and as we have a left invariant system, we can use the left invariant Euler-Poincare equations to reduce the above derivative to zero, and hence Noethers Theorem for left invariant systems follows from this. □

Exercise. Repeat derivations for the Euler-Poincare Equations for right-invariant systems. What is Noether Theorem?[3]

Solution. Now let us carry forward with the derivation for right invariant systems. A right invariant lagrangian is one that the following is true, $L(g,\, \dot{g}) = L(gh,\, \dot{g}h)$ for all $h \in G$. We then set $h = g^{-1}$ and get that $L(g,\, \dot{g}) = L(e,\, \dot{g}g^{-1})$ and so we let $\xi = \dot{g}g^{-1}$ and hence write our lagrangian as $\ell(\xi)$. Now we again go back to Hamiltons Principle,

$$\begin{aligned}
0 = \delta\int_{t_1}^{t_2} L(g,\, \dot{g})\, dt = \delta\int_{t_1}^{t_2}\ell(\xi)\, dt \\
&= \int_{t_1}^{t_2}\delta\ell(\xi)\, dt \\
&= \int_{t_1}^{t_2}\left\langle \frac{\partial \ell}{\partial \xi}, \delta\xi\right\rangle dt
\end{aligned}$$

Now we consider $\delta\xi = \delta(\dot{g}g^{-1})$,

$$\begin{aligned}
\delta(\dot{g}g^{-1}) &= \delta\dot{g}g^{-1} + \dot{g}\delta g^{-1} \\
&= \frac{d}{dt}(\delta g)g^{-1} - \dot{g}g^{-1}\delta g g^{-1} \\
&= \frac{d}{dt}(\delta g g^{-1}) - \delta g\frac{d}{dt}(g^{-1}) - \dot{g}g^{-1}\delta g g^{-1} \\
&= \frac{d}{dt}(\delta g g^{-1}) - \delta g g^{-1}\dot{g}g^{-1} - \dot{g}g^{-1}\delta g g^{-1} && \text{let } \nu = \delta g g^{-1} \\
&= \dot{\nu} + \nu\xi - \xi\nu \\
&= \dot{\nu} + [\nu,\, \xi] \\
&= \dot{\nu} + \mathrm{ad}_\nu\xi \\
&= \dot{\nu} - \mathrm{ad}_\xi\nu
\end{aligned}$$

Hence, we now can move forward and complete the derivation of the right invariant Euler-Poincare Equations.

$$\begin{aligned}
\int_{t_1}^{t_2}\left\langle \frac{\partial \ell}{\partial \xi}, \frac{d}{dt}\nu - \mathrm{ad}_\xi\nu\right\rangle dt &= \int_{t_1}^{t_2}\left\langle \frac{\partial \ell}{\partial \xi}, \frac{d}{dt}\nu\right\rangle - \left\langle \frac{\partial \ell}{\partial \xi}, \mathrm{ad}_\xi\nu\right\rangle dt \\
&= \int_{t_1}^{t_2}\left\langle -\frac{d}{dt}\frac{\partial \ell}{\partial \xi}, \nu\right\rangle - \left\langle \mathrm{ad}^*_\xi\frac{\partial \ell}{\partial \xi}, \nu\right\rangle dt \\
&= \int_{t_1}^{t_2}\left\langle -\frac{d}{dt}\frac{\partial \ell}{\partial \xi} - \mathrm{ad}^*_\xi\frac{\partial \ell}{\partial \xi}, \nu\right\rangle dt
\end{aligned}$$

[3] What about both left and right invariant?

and so we can write down the Euler-Poincare equations for the right invariant system,

$$\frac{d}{dt}\frac{\partial \ell}{\partial \xi} + \mathrm{ad}^*_\xi \frac{\partial \ell}{\partial \xi} = 0$$

We can restate Noethers Theorem as following,

Theorem 4.4 (Noethers Theorem for right invariant systems.)**.** The Euler Poincare equations associated a right-invariant system preserve the generalised momentum along solutions of the Euler-Poincare equations, that is,

$$\frac{d}{dt}\left(\mathrm{Ad}^*_{g(t)}\frac{\partial \ell}{\partial \xi}(t)\right) = 0$$

Proof. This follows from a very similar argument to before by finding that $\frac{d}{dt}(\mathrm{Ad}_{g^{-1}}\,\mu) = \mathrm{ad}_\xi(\mathrm{Ad}_g\,\mu)$ and applying this fact in an identical analysis of the trace pairings ending with $\frac{d}{dt}(\mathrm{Ad}^*_g\,\frac{\partial \ell}{\partial \xi}) = \mathrm{Ad}^*_g\left[\frac{d}{dt}\frac{\partial \ell}{\partial \xi} + \mathrm{ad}^*_\xi\frac{\partial \ell}{\partial \xi}\right]$ and then the result follows from the right-invariant version of the Euler-Poincare equations. □

4.1 More Noether Theory

4.1.1 Noethers Theorem for EL equations

Consider $L(\mathbf{q},\, \dot{\mathbf{q}})$ for $\mathbf{q} \in \mathbb{R}^3$ and $\dot{\mathbf{q}} \in T_{\mathbf{q}}\mathbb{R}^3$. Suppose that L is left invariant with respect to the tangent lift on SO(3), ie. $R \in \mathrm{SO}(3)$ with $L(R\mathbf{q}\,R\dot{\mathbf{q}}) = L(\mathbf{q},\, \dot{\mathbf{q}})$. Then we can prove that,

$$\mathcal{E} := \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} = 0$$

Now we have theorem,

Theorem 4.5. Corresponding to each one-parameter subgroup of SO(3) $R(s)$ where $R(0) = e$ and $R'(0) = \hat{\xi} \in \mathfrak{SO}(3)$. There is a conserved quantity,

$$A_\xi := \left\langle \mathbf{q} \times \frac{\partial L}{\partial \dot{q}}\,, \xi \right\rangle$$

with $\frac{dA_\xi}{dt} = 0$ along solutions of the Euler Legrange equations $\mathcal{E}(\mathbf{q}) = 0$

Proof. Associated with the one parameter subgroup $R(s)$ is the generator $\xi_{\mathcal{M}}(\mathbf{q}) := \frac{d}{ds}_{s=0} R(s)\mathbf{q} = \hat{\xi}\mathbf{q} = \xi \times \mathbf{q}$ where $\mathcal{M} = \mathbb{R}^3$. Now we consider,

$$\int_{t_1}^{t_2} L(R(s)\mathbf{q},\, R(s)q) = \int_{t_1}^{t_2} L(q,\, \dot{q})\, dt$$

and now we differentiate this wrt s and then set $s = 0$ to obtain

$$\left\langle \frac{\partial L}{\partial \dot{q}}\,, \xi_{\mathcal{M}}(\mathbf{q}) \right\rangle = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\xi \times \mathbf{q} = \left\langle \mathbf{q} \times \frac{\partial L}{\partial \dot{q}}\,, \xi \right\rangle = A_\xi$$

□

4.1.2 Noether Theory and EP Reduction

We have,

$$\int_{t_1}^{t_2} L(R,\, \dot{R})\, dt = 0$$

and L is left-invariant and so, $L(SR,\, S\dot{R}) = L(R,\, \dot{R})$

Theorem 4.6 (Noethers Theorem)**.** Corresponding to each one parameter subgroup of SO(3), $S(s)$ with $S(0) = e$ and $S'(0) = \hat{\xi} \in \mathfrak{SO}(3)$, then there is a conserved quantity

$$A_\xi := \left\langle \mathrm{Ad}^*_{R^\top} \frac{\partial \hat{\ell}}{\partial \hat{\mathbf{\Omega}}} \,, \hat{\xi} \right\rangle$$

with $\frac{dA_\xi}{dt} = 0$ along solutions of the Euler Lagrange Equation.

$$\mathcal{E}(R) := \frac{d}{dt}\frac{\partial L}{\partial \dot{R}} - \frac{\partial L}{\partial R} = \mathbf{0}$$

Proof. Consider $S(s)$ and differentiate and let $s = 0$ much like before,

$$\int_{t_1}^{t_2} \frac{\partial}{\partial \xi_{\mathcal{M}(R)}} \frac{\partial L}{\partial R} + \left\langle \frac{\partial L}{\partial \dot{R}} \,, \xi_{\mathcal{M}(\dot{R})} \right\rangle$$

where here $\xi_{\mathcal{M}}(R) = \hat{\xi} R$ □

We now have a definition,

Definition 4.7 (Infinitesimal Generator)**.** Consider the left action of a Lie group G on the manifold $\mathcal{M}$, $(g, \mathbf{x}) \to gx$ ($\mathbf{x} \in \mathcal{M}$). Let $\xi \in \mathfrak{g}$ be a vector in the Lie algebra of G and consider one parameter subgroup

$$[exp(t\xi) : t \in \mathbb{R}] \subseteq G$$

Then the orbit of an element $\mathbf{x}$ with respect to this subgroup is a smooth map $t \to (exp(t\xi))\mathbf{x}$ in $\mathcal{M}$. The infinitesimal generator associated to ξ at $\mathbf{x} \in \mathcal{M}$ denoted by $\xi_{\mathcal{M}}(\mathbf{x})$ is the tangent vector (or velocity) to this curve at point $\mathbf{x}$,

$$\xi_{\mathcal{M}}(\mathbf{x}) = \left.\frac{d}{dt}\right|_{t=0} (exp(t\xi)\mathbf{x}) \in T_{\mathbf{x}}\mathcal{M}$$

this smooth vector field $\xi_{\mathcal{M}} : M \to TM$ and $x \mapsto \xi_{\mathcal{M}}(\mathbf{x})$ is called the infinitesimal generator vector field associated to ξ.

Let G be an arbitrary matrix lie group, and let L a left-invariant Lagrangian with variational principle $\int_{t_1}^{t_2} L(g,\, \dot{g}) = 0$ the reduced system is $L(hg,\, h\dot{g})|_{h=g^{-1}} = L(e,\, g^{-1}\dot{g}) = \ell(\xi) = \ell(g^{-1}\dot{g})$.

Theorem 4.8 (Noether's Theorem)**.** Corresponding to each one-parameter subgroup of G, $\chi(s)$ such that $\chi(0) = e$ and $\xi_s(0) = \eta \in \mathfrak{g}$. There is a conserved quantity

$$\left\langle \mathrm{Ad}^*_{g^{-1}} \frac{\partial \ell}{\partial \xi} \,, \eta \right\rangle$$

Proof. Done before □

Proposition 4.9. The left-invariant Lagrangian $L(g,\, \dot{g})$ satisfies,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{g}} - \frac{\partial L}{\partial g} = \mathbf{0} \iff \frac{d}{dt}\frac{\partial \ell}{\partial \xi} - \mathrm{ad}^* \frac{\partial \ell}{\partial \xi} = 0$$

Proof. I think I have done this

$$\int_{t_1}^{t_2} \langle \mathcal{E} \,, \delta g \rangle = \int_{t_1}^{t_2} \langle \mathcal{E}(\xi) \,, \nu \rangle$$

□

4.2 Diamond Map

Let V be an n-dimensional vector space with dual V^* and pairing $\langle \mathbf{w}\,, \mathbf{u}\rangle_V$ where $\mathbf{u} \in V$ and $\mathbf{w} \in V^*$. Let $\mathcal{M}(n, \mathbb{R})$ be a vector space of $n \times n$ matrices with dual $\mathcal{M}(n, \mathbb{R})^*$ and pairing $\langle B\,, A\rangle_{\mathcal{M}} := \mathrm{Tr}(B^\top A)$ where $A \in \mathcal{M}(n, \mathbb{R})$ and $B \in \mathcal{M}(n, \mathbb{R})$.

The diamond map is a representation of the transformation of the pairing on V to the pairing on $\mathcal{M}(n, \mathbb{R})$. Let $\mathbf{u} \in V$ and $\mathbf{w} \in V^*$ and consider the matrices $A \in \mathcal{M}(n, \mathbb{R})$ and $\mathbf{\Lambda} \in \mathcal{M}(n, \mathbb{R})$ where A is a general matrix and $\mathbf{\Lambda}$ is a symmetric matrix ($\mathbf{\Lambda}^\top = \mathbf{\Lambda}$). The diamond map is defined by $\langle \mathbf{w}\,, A\mathbf{\Lambda}\mathbf{u}\rangle_V = \langle \mathbf{u} \diamond \mathbf{w}\,, \mathbf{\Lambda}\rangle_{\mathcal{M}}$.

$$
\begin{aligned}
\langle \mathbf{w}\,, A\mathbf{u}\rangle_V &= \mathrm{Tr}(\mathbf{u}\mathbf{w}^\top A)\\
&= \mathrm{Tr}((\mathbf{w}\mathbf{u}^\top)^\top A)\\
&= \left\langle \mathbf{w}\mathbf{u}^\top\,, A\right\rangle_{\mathcal{M}}
\end{aligned}
$$

This is for any matrix $A \in \mathcal{M}(n, \mathbb{R})$ and vectors $\mathbf{u} \in V$ and $\mathbf{w} \in V^*$. We now conclude that,

$$
\begin{aligned}
\langle \mathbf{w}\,, A\mathbf{\Lambda}\mathbf{u}\rangle_V &= \mathrm{Tr}(\mathbf{u}\mathbf{w}^\top A\mathbf{\Lambda})\\
&= \left\langle (\mathbf{u}\mathbf{w}^\top A)^\top\,, \mathbf{\Lambda}\right\rangle_{\mathcal{M}}\\
&= \left\langle A^\top \mathbf{w}\mathbf{u}^\top\,, \mathbf{\Lambda}\right\rangle
\end{aligned}
$$

If $F = F^\top$ and $G = -G^\top$, then $\mathrm{Tr}(FG) = 0$. We see that $\mathbf{\Lambda}$ is symmetric so we consider the antisymmetric part,

$$
\begin{aligned}
&= \mathrm{Tr}(\mathbf{u}\mathbf{w}^\top A\mathbf{\Lambda})\\
&= \mathrm{Tr}(\frac{1}{2}(\mathbf{u}\mathbf{w}^\top A + A^\top \mathbf{w}\mathbf{u}^\top)\mathbf{\Lambda}) && \text{we are spliitng this by it's symmetric part}\\
&= \mathrm{Tr}(\mathrm{Sym}(\mathbf{u}\mathbf{w}^\top A)\mathbf{\Lambda})\\
&= \left\langle \mathrm{Sym}(\mathbf{u}\mathbf{w}^\top A)\,, \mathbf{\Lambda}\right\rangle
\end{aligned}
$$

We can say that $\mathbf{u} \diamond \mathbf{w} = \mathrm{Sym}(\mathbf{u}\mathbf{w}^\top A) = \frac{1}{2}(\mathbf{u}\mathbf{w}^T A + A^\top \mathbf{w}\mathbf{u}^\top)$. This is going to appear in EP theory in symmetry breaking paramaters.

4.3 EP Reduction with parameters

What are symmetry breaking parameters? We already know of $\mathbb{I}$ is a symmetry breaking parameter or $\mathbf{e}_3 = R\Gamma$ which is gravity.

Consider a Lie Group, G, and a left action on a manifold, $\mathcal{M}$. Then for a given $a_0 \in \mathcal{M}$ (a parameter), let $L : TG \times \mathcal{M} \to \mathbb{R}$ be a Lagrangian with symmetry breaking parameter a_0, and suppose it is invariant under the left action: $G \times (TG \times \mathcal{M}) \to TG \times \mathcal{M}$ then $(h, (g, \dot{g}, a_0)) \to (hg, h\dot{g}, ha_0)$ for all $h \in G$. This means that $L(hg, h\dot{g}, a_0) = L(g, \dot{g}, a_0)$ for all $h \in G$. As usual let $h = g^{-1}$, then $L(g, \dot{g}, a_0) = L(g^{-1}g, g^{-1}\dot{g}, g^{-1}a_0) =: \ell(\xi, a)$ where $\xi := g^{-1}\dot{g}$ and $a = g^{-1}a_0$.

Theorem 4.10. Then the following are equivalent,

(i) Hamiltons Principle

$$
\delta \int_{t_1}^{t_2} L(g, \dot{g}, a_0)\, dt = 0
$$

with $\delta g(t_1) = \delta g(t_2) = 0$.

(ii) $g(t)$ satisfies the Euler-Lagrange equations associated with $L(g, \dot{g}, a_0)$

(iii) The reduced variational principle (or Hamiltons principle),

$$\delta \int_{t_1}^{t_2} \ell(\xi, a)\, dt = 0$$

holds on $\mathfrak{g} \times \mathcal{M}$, using variations $\delta\xi = \dot{\eta} + \mathrm{ad}_\xi \eta$ and $\delta a = -\eta_{\mathcal{M}}(a)$ with free variations $\eta(t)$ satisfying end point conditions.

(iv) The Euler-Poincare equations

$$\begin{aligned} \frac{d}{dt}\frac{\partial \ell}{\partial \xi} &= \mathrm{ad}^*_\xi \frac{\partial \ell}{\partial \xi} - a \diamond \frac{\partial \ell}{\partial a} \\ \dot{a} &= -\xi_{\mathcal{M}} a \end{aligned}$$

hold on $\mathfrak{g} \times \mathcal{M}$ where $\left\langle \frac{\partial \ell}{\partial a}\,, \alpha_{\mathcal{M}} a \right\rangle =: \left\langle a \diamond \frac{\partial \ell}{\partial a}\,, \alpha \right\rangle$ for all $\alpha \in \mathfrak{g}$ and for all $a \in \mathcal{M}$.

Proof. We already know that $\delta\xi = \dot{\eta} + [\xi, \eta]$ and $\eta = g^{-1}\delta g$ and then $\delta a = -g^{-1}\delta g a = -\eta_{\mathcal{M}} a = -\eta a$. Now we look at our variational principle,

$$\begin{aligned} 0 &= \delta \int_{t_1}^{t_2} \ell(\xi, a)\, dt \\ &= \int_{t_1}^{t_2} (\left\langle \frac{\partial \ell}{\partial \xi}\,, \delta\xi \right\rangle + \left\langle \frac{\partial \ell}{\partial a}\,, \delta a \right\rangle)\, dt \\ &= \int_{t_1}^{t_2} (\left\langle \frac{\partial \ell}{\partial \xi}\,, \dot{\eta} + \mathrm{ad}_\xi \eta \right\rangle - \left\langle \frac{\partial \ell}{\partial a}\,, \eta_{\mathcal{M}} \right\rangle\, dt \\ &= \int_{t_1}^{t_2} \left\langle -\frac{d}{dt}\frac{\partial \ell}{\partial \xi} + \mathrm{ad}^*_\xi \frac{\partial \ell}{\partial \xi}\,, \eta \right\rangle - \left\langle a \diamond \frac{\partial \ell}{\partial a}\,, \right\rangle dt \qquad \text{we used integration by parts} \end{aligned}$$

Now we want the second equation,

$$\begin{aligned} \dot{a} &= \frac{d}{dt}(g^{-1} a_0) \\ &= \frac{d}{dt} g^{-1} a_0 \\ &= -g^{-1}\dot{g}g^{-1}a_0 \\ &= -\xi a \\ &= -\xi_{\mathcal{M}} a \end{aligned}$$

and we are done. □

Now we look at Noethers Theorems.

Theorem 4.11 (Noether's Theorem for Symmetry Breaking Parameters)**.** Let $\xi = g^{-1}\dot{g}$ be a solution of the Euler Poincare Equations with parameters $a = g^{-1}a_0$. Then,

$$\frac{d}{dt}\,\mathrm{Ad}^*_{g^{-1}(t)}\,\mu = -\,\mathrm{Ad}^*_{g^{-1}(t)}(a \diamond \frac{\partial \ell}{\partial a})$$

and $\mu(t) = \frac{\partial \ell}{\partial \xi} \in \mathfrak{g}^*$.

Proof. Exercise □

Exercise. Do the same thing for right invariant actions. If the unreduced Lagrangian $L : TG \times \mathcal{M} \to \mathbb{R}$ is invariant under a right action

$$G \times (TG \times \mathcal{M}) \to TG \times \mathcal{M}$$

$$(h, (g, \dot{g}, a_0)) \mapsto (gh, \dot{g}h, a_0 h)$$

and

$$\frac{d}{dt}\frac{\partial \ell}{\partial \xi} = -\mathrm{ad}^*_{\xi}\frac{\partial \ell}{\partial \xi} - a \diamond \frac{\partial \ell}{\partial a}$$

$$\doteq -\,\xi_{\mathcal{M}} a$$

The Noether Theorem is,

Theorem 4.12.

$$\frac{d}{dt} Ad^*_{g(t)}\mu = -\,\mathrm{Ad}^*_{g(t)}\left(a \diamond \frac{\partial \ell}{\partial a}\right)$$

5 Applications of Geometric Mechanics

5.1 Spherical Pendulum

5.1.1 Euler-Legrange Equations

We want to consider a pendulum in 3D space. We will think about this through the definition of spherical coordinates, as in actuality our motion of the bob will just be on S^2. However, we need to define what are Euler Lagrange equations?[1]

Firstly, here is what the Euler Lagrange equations are,

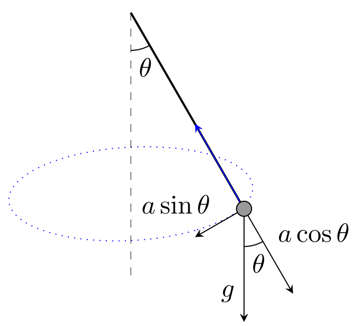
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} - \frac{\partial L}{\partial q^a}$$

where we range through the different basis vectors q^a and their associated derivatives $\dot{q}^a$.

We also define L as the Lagrangian. We define this simply as,

$$L(q,\dot{q}) = T(q,\dot{q}) - V(\mathbf{r}(q))$$

where further we define $T(q,\dot{q})$ as the kinetic energy of the system and $V(\mathbf{r}(q))$ the potential energy of the system.



We are going to use polar coordinates to derive our system of equations.

$$\begin{aligned} x &= R\sin\theta\cos\phi \\ y &= R\sin\theta\sin\phi \\ z &= R(1-\cos\theta) \end{aligned}$$

Our first focus is $T(q,\dot{q})$, which will just be $\frac{1}{2}mv^2$. We can see that $v = |\dot{\mathbf{r}}(t)|$ and so $v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$. Hence, we now find what v is and then find the lagrangian. Firstly, we note that,

$$\frac{d}{dt}(x(t)) = \frac{\partial x}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial x}{\partial \phi}\frac{d\phi}{dt}$$

and similarly for $y(t)$ and $z(t)$. Hence,

$$\begin{aligned} \frac{dx(t)}{dt} &= R\cos\theta\cos\phi\,\dot{\theta} - R\sin\theta\sin\phi\,\dot{\phi} \\ \frac{dy(t)}{dt} &= R\cos\theta\sin\phi\dot{\theta} + R\sin\theta\cos\phi\dot{\phi} \\ \frac{dz(t)}{dt} &= R\sin\theta\dot{\theta} \end{aligned}$$

and now we derive our $T(q,\dot{q})$,

$$\begin{aligned} T(q,\dot{q}) = \frac{1}{2}m\Bigg(&\Big(R\cos\theta\cos\phi\,\dot{\theta} - R\sin\theta\sin\phi\,\dot{\phi}\Big)^2 + \\ &\Big(R\cos\theta\sin\phi\dot{\theta} + R\sin\theta\cos\phi\dot{\phi}\Big)^2 + \\ &\Big(R\sin\theta\dot{\theta}\Big)^2\Bigg) \end{aligned}$$

which then can be simplified down to,

$$T(\theta, \dot{\theta}, \phi, \dot{\phi}) = \frac{mR^2}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)$$

and we note that our system has only one potential energy, gravitation potential! Hence,

$$V(\theta, \dot{\theta}, \phi, \dot{\phi}) = -mgz = -mgR(1 - \cos\theta)$$

Hence, we can now talk about Lagrangian explicitly,

$$L(\theta, \dot{\theta}, \phi, \dot{\phi}) = \frac{mR^2}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + mgR\,(1 - \cos\theta)$$

Finally, we can now take derivatives of this function and produce the Euler Lagrange equations. We need to find the basis vectors, θ and ϕ, as we have two basis vectors, we will have two equations. Firstly, θ

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \frac{d}{dt}mR^2\dot{\theta} - \left(\dot{\phi}^2 mR^2 \sin\theta\cos\theta - mgR\sin\theta\right) &= 0\\ mR^2\ddot{\theta} - mR^2\sin\theta\cos\theta\,\dot{\phi}^2 + mgR\sin\theta &= 0\\ R\ddot{\theta} - R\sin\theta\cos\theta\,\dot{\phi}^2 + g\sin\theta &= 0\end{aligned}$$

and secondly, ϕ

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = \frac{d}{dt}(mR^2\dot{\phi}\sin^2\theta) &= 0\\ mR^2\ddot{\phi}\sin^2\theta + 2mR^2\dot{\phi}\,\dot{\theta}\sin\theta\cos\theta &= 0\\ \ddot{\phi}\sin\theta + 2\dot{\phi}\,\dot{\theta}\cos\theta &= 0\end{aligned}$$

We have derived the Euler-Lagrange equations for the spherical pendulum, which are

$$\begin{cases} R\ddot{\theta} - R\sin\theta\cos\theta\,\dot{\phi}^2 + g\sin\theta = 0\\ \ddot{\phi}\sin\theta + 2\dot{\phi}\,\dot{\theta}\cos\theta = 0\end{cases}$$

5.1.2 Euler-Legrange Equations II

We will now consider an Euler-Poincaré treatment of the Spherical Pendulum. The spherical pendulum is just a particle that moves along the surface of a sphere. We will consider two coordinates, the body coordinates $\mathbf{x}(t)$ and the spatial coordinates $\mathbf{X}$, where $\mathbf{x}(t) = R(t)\mathbf{X}$ and we let $R \in \mathrm{SO}(3)$. In terms of the body coordinates, we can write the Lagrangian as,

$$L(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2}m\left|\dot{\mathbf{x}}\right|^2 - mg\mathbf{e}_3 \cdot \mathbf{x}$$

and then using the body to spatial coordinate transformation,

$$L\left(R, \dot{R}\right) = \frac{1}{2}m\left|\dot{R}\mathbf{X}\right|^2 - mg\mathbf{e}_3 \cdot R\mathbf{X}$$

This is similar to the Lagrangian we will see in the next example, but we aren't integrating over the body. We need to check whether this Lagrangian is invariant under symmetry. We can notice that the second term won't be. As $R\mathbf{X} \neq gR\mathbf{X}$. This is in contrast to the first part of the Lagrangian, which is symmetric. Let $g \in \mathrm{SO}(3)$,

$$\begin{aligned}
\frac{1}{2}m\left|g\dot{R}\mathbf{X}\right|^2 &= \frac{1}{2}m\left(g\dot{R}\mathbf{X}\cdot g\dot{R}\mathbf{X}\right) \\
&= \frac{1}{2}m\left(g\dot{R}\mathbf{X}\left(g\dot{R}\mathbf{X}\right)^T\right) \\
&= \frac{1}{2}m\left(gg^T\dot{R}\mathbf{X}\left(\dot{R}\mathbf{X}\right)^T\right) \\
&= \frac{1}{2}m\left(\dot{R}\mathbf{X}\left(\dot{R}\mathbf{X}\right)^T\right) = \frac{1}{2}m\left|\dot{R}\mathbf{X}\right|^2
\end{aligned}$$

We call $-mg\mathbf{e}_3\cdot R\mathbf{X}$ a symmetry breaking parameter and we proceed by introducing a new parameter $\mathbf{\Gamma} = \mathbf{e}_3 R^{-1}$. Then the Lagrangian becomes,

$$L(\dot{R},\mathbf{\Gamma}) = \frac{1}{2}m\left|\dot{R}\mathbf{X}\right|^2 - mg\mathbf{\Gamma}(t)\cdot\mathbf{X}$$

Now we can start using Hamilton's Variational Principle in order to try and derive the Euler Poincaré equations for our system. We firstly consider $\delta\mathbf{\Gamma}$, which we can see is,

$$\delta\mathbf{\Gamma} = \delta R^{-1}\mathbf{e}_3 = R^{-1}\delta R R^{-1}\mathbf{e}_3 = \hat{\mathbf{\Lambda}}\mathbf{\Gamma} = \mathbf{\Lambda}\times\mathbf{\Gamma}$$

where we define $\mathbf{\Lambda} = R^{-1}\delta R$. Now we can carry on with the derivation,

$$\begin{aligned}
0 &= \delta\int_{t_1}^{t_2} L(\dot{R},\mathbf{\Gamma})dt \\
&= \int_{t_1}^{t_2}\left\langle\frac{\partial L}{\partial\dot{R}},\delta\dot{R}\right\rangle + \left\langle\frac{\partial L}{\partial\mathbf{\Gamma}},\delta\mathbf{\Gamma}\right\rangle dt \\
&= \int_{t_1}^{t_2}\left\langle-\frac{d}{dt}\frac{\partial L}{\partial\dot{R}},\delta R\right\rangle + \left\langle\frac{\partial L}{\partial\mathbf{\Gamma}},\mathbf{\Lambda}\times\mathbf{\Gamma}\right\rangle dt \\
&= \int_{t_1}^{t_2}\left\langle-\frac{d}{dt}\frac{\partial L}{\partial\dot{R}},+\right\rangle\left\langle-\frac{\partial L}{\partial\mathbf{\Gamma}}\times\mathbf{\Gamma},\mathbf{\Lambda}\right\rangle dt \\
&= \int_{t_1}^{t_2}\left\langle-\frac{d}{dt}\frac{\partial L}{\partial\dot{R}},\delta R\right\rangle + \left\langle\mathbf{\Gamma}\times\frac{\partial L}{\partial\mathbf{\Gamma}},R^{-1}\delta R\right\rangle dt \\
&= \int_{t_1}^{t_2}\left\langle-\frac{d}{dt}\frac{\partial L}{\partial\dot{R}},\delta R\right\rangle + \left\langle R\mathbf{\Gamma}\times\frac{\partial L}{\partial\mathbf{\Gamma}},RR^{-1}\delta R\right\rangle dt \\
&= \int_{t_1}^{t_2}\left\langle R\mathbf{\Gamma}\times\frac{\partial L}{\partial\mathbf{\Gamma}} - \frac{d}{dt}\frac{\partial L}{\partial\dot{R}},\delta R\right\rangle dt
\end{aligned}$$

Therefore, one of our Euler-Poincaré equation becomes,

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{R}} = R\mathbf{\Gamma}\times\frac{\partial L}{\partial\mathbf{\Gamma}}$$

and the other comes from just differentiating $\mathbf{\Gamma} = R^{-1}\mathbf{e}_3$ and letting $\mathbf{\Omega} = R^{-1}\dot{R}$, then we get,

$$\dot{\mathbf{\Gamma}} = \mathbf{\Lambda}\times\mathbf{\Omega}$$

Finally, we can differentiate our specific Lagrangian giving the system,

$$\begin{cases} \ddot{R}\mathbf{X} = mgR\mathbf{X} \times \mathbf{\Gamma} \\ \dot{\mathbf{\Gamma}} = \mathbf{\Gamma} \times \mathbf{\Omega} \end{cases}$$

We could further rewrite the first equation using the hat map isomorphism

$$\begin{aligned} \ddot{R}\mathbf{X} &= -mgR\mathbf{\Gamma} \times \mathbf{X} \\ \ddot{R}\mathbf{X} &= -mgR\hat{\mathbf{\Gamma}}\mathbf{X} \\ \ddot{R} &= -mgR\hat{\mathbf{\Gamma}} \end{aligned}$$

as $\ddot{R} = -mgR\hat{\mathbf{\Gamma}}$, which is then invariant of the coordinate systems and only depends on the rotation tensors. Therefore, we have the final system of,

$$\begin{cases} \ddot{R} = -mgR\hat{\mathbf{\Gamma}} \\ \dot{\mathbf{\Gamma}} = \mathbf{\Gamma} \times \mathbf{\Omega} \end{cases}$$

5.2 Heavy Top

The heavy top problem is a rigid body with a fixed point. We are going to study it's dynamics in the body frame. $x(t) = R(t)\mathbf{X}$ and $R(t) \in \mathrm{SO}(3)$. We know the potential energy of a point mass in a gravitation field is $V = mg\mathbf{e}_3 \cdot x(t)$. The lagrangian,

$$L(R, \dot{R}) = \int_{\mathcal{B}} \rho(\mathbf{X}) \left(\frac{1}{2} \left\| \dot{R}\mathbf{X} \right\| - g\mathbf{e}_3 \cdot R\mathbf{X} \right) d^3\mathbf{X}$$

Now we can verify this neither left nor right invariant.

Exercise. Prove that $L(R,\, \dot{R}) \neq L(RR^{-1},\, \dot{R}R^{-1})$ and $L(R,\, \dot{R}) \neq L(R^{-1}R,\, \dot{R}^{-1}\dot{R})$

We know $\hat{\mathbf{\Omega}} = R^{-1}\dot{R}$, then we know $L = \frac{1}{2}\mathbf{\Omega} \cdot \mathbb{I}\mathbf{\Omega} - g\mathbf{e}_3 \cdot R\boldsymbol{\beta}$ where $\beta = \int_B \rho(\mathbf{X})\mathbf{X}\, d^3\mathbf{X}$ and $\boldsymbol{\beta}$ is the centre of mass in the body frame. (We could also use $\beta = \mathbf{X}_b$). We define $\mathbf{\Gamma} = R^{-1}\mathbf{e}_3$ and so

$$\ell(\hat{\mathbf{\Omega}},\, \mathbf{\Gamma}) = \frac{1}{2}\mathbf{\Omega}(t) \cdot \mathbb{I}\mathbf{\Omega}(t) - g\mathbf{\Gamma}(t) \cdot \mathbf{X}_b$$

and we see that $\mathbf{\Gamma}(t)$, the gravitational force, is the symmetry breaking parameter. Now we use Hamiltons principle,

$$\begin{aligned} \delta \int_{t_1}^{t_2} \ell(\hat{\mathbf{\Omega}},\, \mathbf{\Gamma})\, dt &= 0 \\ \int_{t_1}^{t_2} \left(-\frac{d}{dt}\mathbb{I}\hat{\mathbf{\Omega}} + \mathbb{I}\mathbf{\Omega} \times \mathbf{\Omega} \right) \cdot \Lambda\, dt - \int_{t_1}^{t_2} g\mathbf{X}_b \cdot \delta\mathbf{\Gamma} &= 0 \end{aligned}$$

The first integral is the rigid body terms and the second is the symmetry breaking term. We also define $\hat{\mathbf{\Lambda}} = R^{-1}\delta R$. We now look to $\delta\mathbf{\Gamma}$,

$$\begin{aligned} \delta\mathbf{\Gamma} &= \delta R^{-1}\mathbf{e}_3 \\ &= -R^{-1}\delta R^{-1}\mathbf{e}_3 \\ &= -\hat{\mathbf{\Lambda}}\mathbf{\Gamma} \\ &= -\mathbf{\Lambda} \times \mathbf{\Gamma} \end{aligned}$$

We can now write the following,

$$\int_{t_1}^{t_2}\left(-\frac{d}{dt}\mathbb{I}\hat{\mathbf{\Omega}}+\mathbb{I}\mathbf{\Omega}\times\mathbf{\Omega}\right)\cdot\Lambda\,dt+\int_{t_1}^{t_2}-g\left\langle\mathbf{X}_b\,,\mathbf{\Lambda}\times\mathbf{\Gamma}\right\rangle=0$$

and so we can write,

$$\int_{t_1}^{t_2}\left[-\frac{d}{dt}\mathbb{I}\hat{\mathbf{\Omega}}+\mathbb{I}\hat{\mathbf{\Omega}}\times\hat{\mathbf{\Omega}}\cdot\mathbf{\Lambda}-g(\mathbf{X}_b\times\mathbf{\Gamma})\cdot\mathbf{\Lambda}\right]=0$$

Now we get,

$$\mathbb{I}\dot{\mathbf{\Omega}}=\mathbb{I}\hat{\mathbf{\Omega}}\times\hat{\mathbf{\Omega}}+g\mathbf{\Gamma}\times\mathbf{X}_b$$
$$\dot{\mathbf{\Gamma}}=\mathbf{\Gamma}\times\hat{\mathbf{\Omega}}$$

Exercise. Find $\frac{d}{dt}\left(R\mathbb{I}\hat{\mathbf{\Omega}}\right)=gR(\mathbf{\Gamma}\times\mathbf{X}_b)\neq 0$. Find what kind of angular momentum is conserved. Find something that is conserved.

5.3 Pseudo-Rigid Bodies

In the following section we will firstly follow Chapter 10 of Holm, Schmah & Stoica, Geometric Mechanics and Symmetry [2], which I summarise then derive the left, right and left and right invariant Euler Poincaré equations for Pseudo Rigid bodies. After, we will look at introducing potential energy into the Lagrangian following a 2010 paper from Toshihiro Iwai [3].

Let us assume that our body can stretch and sheer, this will be called a psudo rigid body. I have done the following derivations with the assumption that the configuration space we are working in is $\mathrm{GL}^+(3)$, ie. the set of matrices with postive determinant. We make a few assumptions, firstly the moment of inertia tensor is rotationally invariant, it is sufficient that the density function $\rho(\mathbf{X})$ is spherically symmetric. We will also assume that the Lagrangian only depends on the kinetic energy and so we study free ellipsoid motion.

We fix a reference configuration via a fixed spatial coordinate system and a moving body coordinate system, both with origin of the fixed point of the body. We will assume that the configuration of the system is a matrix $\mathbf{Q}(t)\in\mathrm{GL}^+(3)$ which takes the label $\mathbf{X}$ to the spacial position $\mathbf{x}(t)$, that is,

$$\mathbf{x}(t,\mathbf{X})=\mathbf{Q}(t)\mathbf{X}\qquad\dot{\mathbf{x}}(t,\mathbf{X})=\dot{\mathbf{Q}}\mathbf{X}=\dot{\mathbf{Q}}(t)\mathbf{Q}^{-1}(t)\mathbf{x}(t,\mathbf{X})$$

as before let $\rho(\mathbf{X})$ be the density function and $\mathcal{B}$ be the region occupied by the body in it's configuration space. The moment of inertia tensor is assumed to be spherically symmetric, that is,

$$\mathbb{J}=\int_{\mathcal{B}}\rho(\mathbf{X})\mathbf{X}\mathbf{X}^T d^3\mathbf{X}=kI\qquad k\in\mathbb{R}$$

and I is the identity matrix. We assume without loss of generality that $k=1$ and so,

$$\mathbb{J}=\int_{\mathcal{B}}\rho(\mathbf{X})\mathbf{X}\mathbf{X}^T d^3\mathbf{X}=I$$

We now consider the kinetic energy,

$$\begin{aligned}
K &= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})\,\|\dot{\mathbf{x}}\|\ d^3\mathbf{X} \\
&= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})\left\|\dot{\mathbf{Q}}\dot{\mathbf{X}}\right\|\ d^3\mathbf{X} \\
&= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{X})\,\mathrm{Tr}\left((\dot{\mathbf{Q}}\mathbf{X})(\dot{\mathbf{Q}}\mathbf{X})^T\right)\ d^3\mathbf{X} \\
&= \frac{1}{2}\,\mathrm{Tr}\left(\dot{\mathbf{Q}}\int_{\mathcal{B}} \rho(\mathbf{X})\mathbf{X}\mathbf{X}^T\, d^3\mathbf{X}\,\dot{\mathbf{Q}}^T\right) \\
&= \frac{1}{2}\,\mathrm{Tr}\left(\dot{\mathbf{Q}}\mathbb{J}\dot{\mathbf{Q}}^T\right) \\
&= \frac{1}{2}\,\mathrm{Tr}\left(\dot{\mathbf{Q}}\dot{\mathbf{Q}}^T\right)
\end{aligned}$$

We can notice that this Lagrangian is symmetric and invariant left and right actions, that is, if $L = \frac{1}{2}\,\mathrm{Tr}\left(\dot{\mathbf{Q}}\dot{\mathbf{Q}}^T\right)$, then

$$\begin{aligned}
L(g\mathbf{Q}h, g\dot{\mathbf{Q}}h) &= \frac{1}{2}\,\mathrm{Tr}\left(g\dot{\mathbf{Q}}h(g\dot{\mathbf{Q}}h)^T\right) \\
&= \frac{1}{2}\,\mathrm{Tr}\left(g\dot{\mathbf{Q}}hh^T\dot{\mathbf{Q}}g^T\right) \\
&= \frac{1}{2}\,\mathrm{Tr}\left(\dot{\mathbf{Q}}\dot{\mathbf{Q}}\right)
\end{aligned}$$

as $g, h \in \mathrm{SO}(3)$. From Linear Algebra last year we saw that we can decompose a matrix using single value decomposition. That is, take a matrix $\mathbf{A}$ and we can represent this as $\mathbf{U\Sigma V}$ where $\mathbf{U}, \mathbf{V} \in O(3)$ and $\mathbf{\Sigma} \in \mathrm{diag}^+(3)$. We want $\mathbf{U}, \mathbf{V}$ to be in SO(3) and so we now do the following. Take a decomposition of $\mathbf{Q} = \mathbf{RAS}$ and we know that $\det\mathbf{R} = \pm 1$, if $\det\mathbf{R} = 1$ leave it as it is, if $\det\mathbf{R} = -1$, then we tag on an additional matrix,

$$\mathbf{M} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

to $\mathbf{R}$ creating $\mathbf{R}' = \mathbf{RM}$ and similarly for $\mathbf{S}' = \mathbf{RS}$ if $\det\mathbf{S} = -1$. Now we have the following decomposition, $\mathbf{Q} = \mathbf{R}'\mathbf{MAMS}'$, noting that $\mathbf{R}', \mathbf{S}' \in \mathrm{SO}(3)$, $\mathbf{MAM} \in \mathrm{diag}^+(3)$ and $\mathbf{M}^2 = \mathbf{I}$ and so this makes sense. The decomposition $\mathbf{Q} = \mathbf{RAS}$ can be thought of as a body rotation, a stretch and a spatial rotation. That is, if we consider $\mathbf{S}$, this rotates the $\mathbf{X}$ (body) coordinates in the reference congifuration, the $\mathbf{A}$ stretches the axis and $\mathbf{R}$ rotates the $\mathbf{x}$ (spatial) coordinates. Now I present a nice example,

Example. Consider a bipolar decomposition of $\mathbf{Q}$ acting on a pseudo-rigid sphere. If $\mathbf{R}(t) = \mathbf{I}$, where $\mathbf{A}(t) = \mathrm{diag}(a_1, a_2, a_3)$. This is called the Jacobi ellipsoid motion, this is where only the outside of the sphere moves. However, if $\mathbf{S}(t) = \mathbf{I}$, then we have Dedekind ellipsoid motion, where only the inside moves and the outside stays still.

After that interlude, we continue. We will work in an extended configuration space, that is, instead of just working in $\mathbf{Q} : \mathrm{SO}(3)$, we work in $\mathbf{Q}_{\text{ext}} : \mathrm{SO}(3) \times \mathrm{diag}^+(3) \times \mathrm{SO}(3)$. We call this the extended configuration space for psuedo rigid bodies. We also define a submersion[4], $\phi : \mathbf{Q} \to \mathrm{GL}^+(3)$, defined by $\phi(\mathbf{R}, \mathbf{A}, \mathbf{S}) = \mathbf{RAS}$, which allows us to define the extended Lagrangian, $L_{\text{ext}} : T\mathbf{Q}_{\text{ext}} \to \mathbb{R}$ defined by $L_{\text{ext}} = L \circ T\phi$. Furthermore, we assume that this Lagrangian is invariant under left and right symmetry, that is

$$L_{\text{ext}}(g\mathbf{R}, \mathbf{A}, \mathbf{S}h, g\dot{\mathbf{R}}, \dot{\mathbf{A}}, \dot{\mathbf{S}}h) = L_{\text{ext}}(\mathbf{R}, \mathbf{A}, \mathbf{S}, \dot{\mathbf{R}}, \dot{\mathbf{A}}, \dot{\mathbf{S}})$$

[4] This is an everywhere surjective map between the tangent spaces of differentiable manifolds

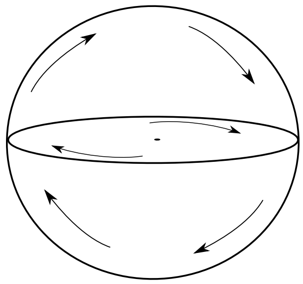


Figure 3: Dedekind Motion

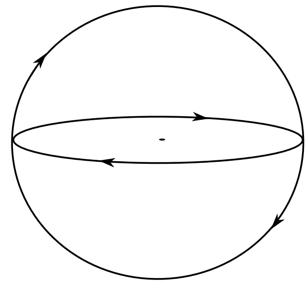


Figure 4: Jacobi Motion

To do the usual reduction, we now need some sort of Hamilton's variational principle. We will find in the following poroposition that it indeed satisfies the expected Hamilton's variational principle,

Proposition 5.1. $(\mathbf{R}, \mathbf{A}, \mathbf{S})$ satisfies Hamilton's Principle for L_{ext} if and only if $\mathbf{Q}$ satisfies Hamilton's Principle for L.

Proof. All deformations of $\mathbf{Q}(t,s)$ are of the form $\phi(\mathbf{R}(t,s), \mathbf{A}(t,s), \mathbf{S}(t,s))$. for some $(\mathbf{R}(t,s), \mathbf{A}(t,s), \mathbf{S}(t,s)) \in \mathbf{Q}_{\text{ext}}$. Hence by chain rule, $(\mathbf{Q}, \dot{\mathbf{Q}}) = T\phi(\mathbf{R}(t,s), \mathbf{A}(t,s), \mathbf{S}(t,s), \dot{\mathbf{R}}(t,s), \dot{\mathbf{A}}(t,s), \dot{\mathbf{S}}(t,s))$ and so $L(\mathbf{Q}(t,s), \dot{\mathbf{Q}}(t,s)) = L_{\text{ext}}(\mathbf{R}(t,s), \mathbf{A}(t,s), \mathbf{S}(t,s), \dot{\mathbf{R}}(t,s), \dot{\mathbf{A}}(t,s), \dot{\mathbf{S}}(t,s))$. Now we can write the following,

$$
\begin{aligned}
\delta \int_{t_1}^{t_2} L_{\text{ext}}(\mathbf{R}(t,s), \mathbf{A}(t,s), &\mathbf{S}(t,s), \dot{\mathbf{R}}(t,s), \dot{\mathbf{A}}(t,s), \dot{\mathbf{S}}(t,s))dt = 0 \\
&\iff \frac{d}{ds}\bigg|_{s=0} \int_{t_1}^{t_2} L_{\text{ext}}(\mathbf{R}(t,s), \mathbf{A}(t,s), \mathbf{S}(t,s), \dot{\mathbf{R}}(t,s), \dot{\mathbf{A}}(t,s), \dot{\mathbf{S}}(t,s))dt = 0 \\
&\iff \frac{d}{ds}\bigg|_{s=0} \int_{t_1}^{t_2} L(\mathbf{Q}(t,s), \dot{\mathbf{Q}}(t,s))dt = 0 \\
&\iff \frac{d}{ds}\bigg|_{s=0} \int_{t_1}^{t_2} L(\mathbf{Q}(t), \dot{\mathbf{Q}}(t))dt = 0 \\
&\iff \delta \int_{t_1}^{t_2} L(\mathbf{Q}(t), \dot{\mathbf{Q}}(t))dt = 0
\end{aligned}
$$

which gives the Hamilton's Principle for L. □

For the rest of the chapter we will refer to L_{ext} as L as the previous proposition proves that the analysis we do on L_{ext} suffices. We will use the following Lagrangian,

$$L(\mathbf{Q}, \dot{\mathbf{Q}}) = \frac{1}{2}\operatorname{Tr}(\dot{\mathbf{Q}}\dot{\mathbf{Q}}^T)$$

We note that $\frac{d}{dt}(\mathbf{RAS}) = \dot{\mathbf{R}}\mathbf{AS} + \mathbf{R}\dot{\mathbf{A}}\mathbf{S} + \mathbf{RA}\dot{\mathbf{S}}$, which we denote as $(RAS)^\bullet$, therefore we can write the Lagrangian as, $L(\mathbf{Q}, \dot{\mathbf{Q}}) = \frac{1}{2}\operatorname{tr}((\mathbf{RAS})^\bullet(\mathbf{S}^T\mathbf{A}\mathbf{R}^T)^\bullet)$. We will use this later to derive one of the sets of equations. We will now move to the reduction of the Lagrangian and then the Euler-Poincare equations.

5.3.1 Euler-Poincaré Reduction

We now seek to reduce the Lagrangian, we recall the symmetric invariance on the Lagrangian we stated above,

$$L(g\mathbf{R}, \mathbf{A}, \mathbf{S}h, g\dot{\mathbf{R}}, \dot{\mathbf{A}}, \dot{\mathbf{S}}h) = L(\mathbf{R}, \mathbf{A}, \mathbf{S}, \dot{\mathbf{R}}, \dot{\mathbf{A}}, \dot{\mathbf{S}})$$

and we let $g = \mathbf{R}^{-1}$, $h = \mathbf{S}^{-1}$ and the define $\hat{\mathbf{\Omega}} := \mathbf{R}^{-1}\dot{\mathbf{R}}$ and $\hat{\mathbf{\Lambda}} := \dot{\mathbf{S}}\mathbf{S}^{-1}$ and we can now write,

$$L(e, \mathbf{A}, e, \hat{\mathbf{\Omega}}, \dot{A}, \hat{\mathbf{\Lambda}}) = L(\mathbf{R}, \mathbf{A}, \mathbf{S}, \dot{\mathbf{R}}, \dot{\mathbf{A}}, \dot{\mathbf{S}})$$

We define $\ell(\mathbf{A}, \hat{\mathbf{\Omega}}, \dot{\mathbf{A}}, \hat{\mathbf{\Lambda}}) := L(e, \mathbf{A}, e, \hat{\mathbf{\Omega}}, \dot{\mathbf{A}}, \hat{\mathbf{\Lambda}}) = L(\mathbf{R}, \mathbf{A}, \mathbf{S}, \dot{\mathbf{R}}, \dot{\mathbf{A}}, \dot{\mathbf{S}})$ and now start to work towards the Euler-Poincare equations. We firstly consider Hamilton's Principle for this Lagrangian,

$$\begin{aligned}
0 &= \delta \int_{t_1}^{t_2} \ell(\mathbf{A}, \hat{\mathbf{\Omega}}, \dot{\mathbf{A}}, \hat{\mathbf{\Lambda}}) dt \\
0 &= \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \mathbf{A}}, \delta \mathbf{A} \right\rangle + \left\langle \frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}}, \delta \hat{\mathbf{\Omega}} \right\rangle + \left\langle \frac{\partial \ell}{\partial \dot{\mathbf{A}}}, \delta \dot{\mathbf{A}} \right\rangle + \left\langle \frac{\partial \ell}{\partial \hat{\mathbf{\Lambda}}}, \delta \hat{\mathbf{\Lambda}} \right\rangle dt \\
0 &= \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \mathbf{A}} - \frac{d}{dt}\frac{\partial \ell}{\partial \dot{\mathbf{A}}}, \delta \mathbf{A} \right\rangle + \left\langle \frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}}, \delta \hat{\mathbf{\Omega}} \right\rangle + \left\langle \frac{\partial \ell}{\partial \hat{\mathbf{\Lambda}}}, \delta \hat{\mathbf{\Lambda}} \right\rangle dt \\
0 &= \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \mathbf{A}} - \frac{d}{dt}\frac{\partial \ell}{\partial \dot{\mathbf{A}}}, \delta \mathbf{A} \right\rangle + \left\langle \frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}}, \dot{\xi} + [\hat{\mathbf{\Omega}}, \hat{\xi}] \right\rangle + \left\langle \frac{\partial \ell}{\partial \hat{\mathbf{\Lambda}}}, \dot{\eta} - [\hat{\mathbf{\Lambda}}, \hat{\eta}] \right\rangle dt
\end{aligned}$$

$$\begin{aligned}
0 &= \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \mathbf{A}} - \frac{d}{dt}\frac{\partial \ell}{\partial \dot{\mathbf{A}}}, \delta \mathbf{A} \right\rangle + \left\langle \frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}}, \dot{\xi} \right\rangle + \left\langle \frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}}, \mathrm{ad}_{\hat{\mathbf{\Omega}}}\xi \right\rangle + \left\langle \frac{\partial \ell}{\partial \hat{\mathbf{\Lambda}}}, \dot{\eta} \right\rangle - \left\langle \frac{\partial \ell}{\partial \hat{\mathbf{\Lambda}}}, \mathrm{ad}_{\hat{\mathbf{\Lambda}}}\eta \right\rangle dt \\
0 &= \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \mathbf{A}} - \frac{d}{dt}\frac{\partial \ell}{\partial \dot{\mathbf{A}}}, \delta \mathbf{A} \right\rangle + \left\langle -\frac{d}{dt}\frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}}, \xi \right\rangle + \left\langle \mathrm{ad}^*_{\hat{\mathbf{\Omega}}}\frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}}, \xi \right\rangle + \left\langle -\frac{d}{dt}\frac{\partial \ell}{\partial \hat{\mathbf{\Lambda}}}, \eta \right\rangle - \left\langle \mathrm{ad}^*_{\hat{\mathbf{\Lambda}}}\frac{\partial \ell}{\partial \hat{\mathbf{\Lambda}}}, \eta \right\rangle dt \\
0 &= \int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \mathbf{A}} - \frac{d}{dt}\frac{\partial \ell}{\partial \dot{\mathbf{A}}}, \delta \mathbf{A} \right\rangle + \left\langle \mathrm{ad}^*_{\hat{\mathbf{\Omega}}}\frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}} - \frac{d}{dt}\frac{\partial \ell}{\partial \hat{\mathbf{\Omega}}}, \xi \right\rangle - \left\langle \mathrm{ad}^*_{\hat{\mathbf{\Lambda}}}\frac{\partial \ell}{\partial \hat{\mathbf{\Lambda}}} + \frac{d}{dt}\frac{\partial \ell}{\partial \hat{\mathbf{\Lambda}}}, \eta \right\rangle dt
\end{aligned}$$

Therefore, considering endpoint conditions we reach the following set of Euler-Poincare equations,

$$\frac{d}{dt}\frac{\partial \ell}{\partial \mathbf{\Omega}} = \mathrm{ad}^*_{\mathbf{\Omega}}\frac{\partial \ell}{\partial \mathbf{\Omega}} \tag{5.1}$$

$$\frac{d}{dt}\frac{\partial \ell}{\partial \dot{\mathbf{A}}} = \frac{\partial \ell}{\partial \mathbf{A}} \tag{5.2}$$

$$\frac{d}{dt}\frac{\partial \ell}{\partial \mathbf{\Lambda}} = -\mathrm{ad}^*_{\mathbf{\Lambda}}\frac{\partial \ell}{\partial \mathbf{\Lambda}} \tag{5.3}$$

which then give arise to (10.12) - (10.14) in [2]. We can now work forward, in a very similar vein to Holm and replace the coadjoints and partial derivatives back with $\hat{\mathbf{\Lambda}}$, $\mathbf{A}$, $\hat{\mathbf{\Omega}}$ and $\dot{\mathbf{A}}$. We go straight the rewritten derivatives,

$$\ell(Q, \dot{Q}) = \frac{1}{2}\operatorname{tr}\left((\hat{\mathbf{\Omega}}\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\hat{\mathbf{\Lambda}})(\hat{\mathbf{\Omega}}\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\hat{\mathbf{\Lambda}})^T \right)$$

We now need to take variations of this Lagrangian in order to get the direct equations. Consider the following,

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{\varepsilon=0} \ell(\hat{\mathbf{\Omega}} + \varepsilon\delta\hat{\mathbf{\Omega}}) &= \left.\frac{d}{dt}\right|_{\varepsilon=0} \frac{1}{2}\operatorname{tr}\left(((\hat{\mathbf{\Omega}} + \varepsilon\delta\hat{\mathbf{\Omega}})\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\hat{\mathbf{\Lambda}})((\hat{\mathbf{\Omega}} + \varepsilon\delta\hat{\mathbf{\Omega}})\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\hat{\mathbf{\Lambda}})^T \right) \\
&= \frac{1}{2}\operatorname{tr}\left((\delta\hat{\mathbf{\Omega}}\mathbf{A})^T(\hat{\mathbf{\Omega}}\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\hat{\mathbf{\Lambda}}) + (\hat{\mathbf{\Omega}}\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\hat{\mathbf{\Lambda}})^T\delta\hat{\mathbf{\Omega}}\mathbf{A} \right) \\
&= \operatorname{tr}\left((\delta\hat{\mathbf{\Omega}}\mathbf{A})^T(\hat{\mathbf{\Omega}}\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\hat{\mathbf{\Lambda}}) \right)
\end{aligned}$$

This step can be done as we know that $\operatorname{tr}(\mathbf{A}^T\mathbf{B}) = \operatorname{tr}(\mathbf{A}\mathbf{B}^T)$ and so $\frac{1}{2}\operatorname{tr}(\mathbf{A}^T\mathbf{B} + \mathbf{A}\mathbf{B}^T) = \frac{1}{2}\operatorname{tr}(\mathbf{A}^T\mathbf{B}) + \frac{1}{2}\operatorname{tr}(\mathbf{A}^T\mathbf{B}) = \operatorname{tr}(\mathbf{A}^T\mathbf{B})$

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{\varepsilon=0} \ell(\hat{\mathbf{\Omega}} + \varepsilon\delta\hat{\mathbf{\Omega}}) &= \operatorname{tr}\left((\delta\hat{\mathbf{\Omega}}\mathbf{A})(\hat{\mathbf{\Omega}}\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\hat{\mathbf{\Lambda}})^T\right) \\
&= \operatorname{tr}\left((\delta\hat{\mathbf{\Omega}}\mathbf{A})(\hat{\mathbf{\Omega}}\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\hat{\mathbf{\Lambda}})^T\right) \\
&= \operatorname{tr}\left(\mathbf{A}^T(\hat{\mathbf{\Omega}}\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\hat{\mathbf{\Lambda}})^T\delta\hat{\mathbf{\Omega}}\right) \\
&= \left\langle \mathbf{A}(\hat{\mathbf{\Omega}}\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\hat{\mathbf{\Lambda}}\,, \delta\hat{\mathbf{\Omega}} \right\rangle
\end{aligned}$$

Now we consider the fact that $\delta\ell = \left\langle \frac{\partial\ell}{\partial\hat{\mathbf{\Omega}}}\,, \delta\hat{\mathbf{\Omega}} \right\rangle$ and so $\frac{\partial\ell}{\partial\hat{\mathbf{\Omega}}} = \hat{\mathbf{\Omega}}\mathbf{A}^2 + \dot{\mathbf{A}}\mathbf{A} + \mathbf{A}\hat{\mathbf{\Lambda}}\mathbf{A}$. We now need to consider,

$$\begin{aligned}
\left\langle \dot{\mathbf{A}}\mathbf{A}\,, \delta\hat{\mathbf{\Omega}} \right\rangle &= \operatorname{tr}((\dot{\mathbf{A}}\mathbf{A})^T\delta\hat{\mathbf{\Omega}}) \\
&= \operatorname{tr}(\mathbf{A}\dot{\mathbf{A}}\delta\hat{\mathbf{\Omega}}) = 0
\end{aligned}$$

This is true as $\mathbf{A}\dot{\mathbf{A}}$ is symmetric and $\delta\hat{\mathbf{\Omega}}$ is asymmetric. We recall the result from the previous section that says that the trace must then be zero. Finally we note that $\hat{\mathbf{\Omega}}\mathbf{A}^2 = \frac{1}{2}\left(\hat{\mathbf{\Omega}}\mathbf{A}^2 + \mathbf{A}^2\hat{\mathbf{\Omega}}\right)$, as $\mathbf{A}^2$ is diagonal and so commutative. Hence the first equation now must be,

$$\frac{\partial\ell}{\partial\hat{\mathbf{\Omega}}} = \frac{1}{2}\left(\hat{\mathbf{\Omega}}\mathbf{A}^2 + \mathbf{A}^2\hat{\mathbf{\Omega}}\right) + \mathbf{A}\hat{\mathbf{\Lambda}}\mathbf{A} \tag{5.4}$$

We continue with a similar argument for $\hat{\mathbf{\Lambda}}$, again taking variations,

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{\varepsilon=0} \ell(\hat{\mathbf{\Lambda}} + \varepsilon\delta\hat{\mathbf{\Lambda}}) = & \left.\frac{d}{dt}\right|_{\varepsilon=0} \frac{1}{2}\operatorname{tr}\left((\hat{\mathbf{\Omega}}\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}(\hat{\mathbf{\Lambda}} + \varepsilon\delta\hat{\mathbf{\Lambda}}))(\hat{\mathbf{\Omega}}\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}(\hat{\mathbf{\Lambda}} + \varepsilon\delta\hat{\mathbf{\Lambda}}))^T\right) \\
&= \frac{1}{2}\operatorname{tr}\left((\mathbf{A}\delta\hat{\mathbf{\Lambda}})^T(\hat{\mathbf{\Omega}}\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\hat{\mathbf{\Lambda}}) + (\hat{\mathbf{\Omega}}\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\hat{\mathbf{\Lambda}})^T(A\delta\hat{\mathbf{\Lambda}})\right) \\
&= \operatorname{tr}\left((\mathbf{A}\delta\hat{\mathbf{\Lambda}})(\hat{\mathbf{\Omega}}\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\hat{\mathbf{\Lambda}})^T\right) \\
&= \operatorname{tr}\left(\mathbf{A}^T(\hat{\mathbf{\Omega}}\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\hat{\mathbf{\Lambda}})^T\delta\hat{\mathbf{\Lambda}}\right) \\
&= \left\langle \mathbf{A}(\hat{\mathbf{\Omega}}\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\hat{\mathbf{\Lambda}})\,, \delta\hat{\mathbf{\Lambda}} \right\rangle
\end{aligned}$$

and so we get $\frac{\partial\ell}{\partial\hat{\mathbf{\Lambda}}} = \mathbf{A}\hat{\mathbf{\Omega}}\mathbf{A} + \mathbf{A}\dot{\mathbf{A}} + \mathbf{A}^2\hat{\mathbf{\Lambda}}$, and as before we get that $\left\langle \mathbf{A}\dot{\mathbf{A}}\,, \delta\hat{\mathbf{\Lambda}} \right\rangle = 0$, therefore,

$$\frac{\partial\ell}{\partial\hat{\mathbf{\Lambda}}} = \mathbf{A}\hat{\mathbf{\Omega}}\mathbf{A} + \frac{1}{2}\left(\hat{\mathbf{\Lambda}}\mathbf{A}^2 + \mathbf{A}^2\hat{\mathbf{\Lambda}}\right) \tag{5.5}$$

Finally, we derive the $\mathbf{A}$ version of the equations using a slightly different argument, where we split it half way through,

$$\begin{aligned}
\frac{d}{dt}\bigg|_{\varepsilon=0} \ell(\mathbf{A}+\varepsilon\delta\mathbf{A}) = & \frac{d}{dt}\bigg|_{\varepsilon=0} \frac{1}{2}\operatorname{tr}\Big((\hat{\mathbf{\Omega}}(\mathbf{A}+\varepsilon\delta\mathbf{A}) + \dot{\mathbf{A}} + (\mathbf{A}+\varepsilon\delta\mathbf{A})\hat{\mathbf{\Lambda}})(\hat{\mathbf{\Omega}}(\mathbf{A}+\varepsilon\delta\mathbf{A}) + \dot{\mathbf{A}} + (\mathbf{A}+\varepsilon\delta\mathbf{A})\hat{\mathbf{\Lambda}})^T\Big) \\
&= \frac{1}{2}\operatorname{tr}\Big((\hat{\mathbf{\Omega}}\delta\mathbf{A} + \delta\mathbf{A}\hat{\mathbf{\Lambda}})^T(\hat{\mathbf{\Omega}}\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\hat{\mathbf{\Lambda}}) + (\hat{\mathbf{\Omega}}\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\hat{\mathbf{\Lambda}})^T(\hat{\mathbf{\Omega}}\delta\mathbf{A} + \delta\mathbf{A}\hat{\mathbf{\Lambda}})\Big) \\
&= \operatorname{tr}\Big((\hat{\mathbf{\Omega}}\delta\mathbf{A} + \delta\mathbf{A}\hat{\mathbf{\Lambda}})(\hat{\mathbf{\Omega}}\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\hat{\mathbf{\Lambda}})^T\Big) \\
&= \operatorname{tr}\Big(\hat{\mathbf{\Omega}}\delta\mathbf{A}(\hat{\mathbf{\Omega}}\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\hat{\mathbf{\Lambda}})^T\Big) + \operatorname{tr}\Big(\delta\mathbf{A}\hat{\mathbf{\Lambda}}(\hat{\mathbf{\Omega}}\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\hat{\mathbf{\Lambda}})^T\Big) \\
&= \operatorname{tr}\Big(\hat{\mathbf{\Omega}}^T(\hat{\mathbf{\Omega}}\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\hat{\mathbf{\Lambda}})^T\delta\mathbf{A}\Big) + \operatorname{tr}\Big((\hat{\mathbf{\Omega}}\mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\hat{\mathbf{\Lambda}})^T\hat{\mathbf{\Lambda}}^T\delta\mathbf{A}\Big) \\
&= \operatorname{tr}\Big(\Big(\hat{\mathbf{\Omega}}^2\mathbf{A} + \mathbf{A}\hat{\mathbf{\Lambda}}^2 + \hat{\mathbf{\Omega}}\dot{\mathbf{A}} + \dot{\mathbf{A}}\hat{\mathbf{\Lambda}} + 2\hat{\mathbf{\Omega}}\mathbf{A}\hat{\mathbf{\Lambda}}\Big)^T\delta\mathbf{A}\Big) \\
&= \Big\langle\hat{\mathbf{\Omega}}^2\mathbf{A} + \mathbf{A}\hat{\mathbf{\Lambda}}^2 + \hat{\mathbf{\Omega}}\dot{\mathbf{A}} + \dot{\mathbf{A}}\hat{\mathbf{\Lambda}} + 2\hat{\mathbf{\Omega}}\mathbf{A}\hat{\mathbf{\Lambda}}\,, \delta\mathbf{A}\Big\rangle
\end{aligned}$$

Now we reduce this further as we know that $\hat{\mathbf{\Lambda}}^2\mathbf{A} + \mathbf{A}\hat{\mathbf{\Lambda}}^2 = \mathbf{A}(\hat{\mathbf{\Omega}}^2 + \hat{\mathbf{\Lambda}}^2)$ and so considering the trace pairing, we get that this term disapears. We can make a similar argument for $\hat{\mathbf{\Omega}}\dot{\mathbf{A}} + \dot{\mathbf{A}}\hat{\mathbf{\Lambda}}$ and so we get the equations reduce to,

$$\frac{\partial\ell}{\partial\mathbf{A}} = 2\hat{\mathbf{\Omega}}\mathbf{A}\hat{\mathbf{\Lambda}} = \hat{\mathbf{\Omega}}\mathbf{A}\hat{\mathbf{\Lambda}} + \hat{\mathbf{\Lambda}}\mathbf{A}\hat{\mathbf{\Omega}} \tag{5.6}$$

In addition to,

$$\frac{\partial\ell}{\partial\dot{\mathbf{A}}} = \dot{\mathbf{A}} \tag{5.7}$$

these are then the Equations (10.15)-(10.18) that are stated by Holm, Schmah & Stoica in [2]

5.3.2 Noether Theory for general Euler-Poincaré reduction

Let G be an arbitrary matrix Lie group and let L be a left-invariant Lagrangian,

$$L(hg, h\dot{g}) = L(g,\dot{g}) \quad \forall g,h \in G$$

with variational principle

$$\delta\int_{t_1}^{t_2} L(g,\dot{g})dt = 0$$

The reduced system is just $L(hg,h\dot{g})|_{h=g^{-1}} = L(g^{-1}g, g^{-1}\dot{g}) := \ell(\xi)$ where $\xi = g^{-1}g$ and $\xi \in \mathfrak{so}(3)$.

Theorem 5.2 (Euler-Poincare Noether Theorem)**.** Corresponding to each one-parameter subgroup of G, $\chi(s)$ with $\chi(0) = e$ and $\chi_s(s) = \eta \in \mathfrak{g}$ there is a conserved quantity

$$\Big\langle \operatorname{Ad}^*_{g^{-1}}\frac{\partial\ell}{\partial\xi}\,, \eta\Big\rangle = K$$

Proof. Take a one parameter subgroup $\chi(s)$ and multiply this by the Lagrangian left,

$$\int_{t_1}^{t_2} L(\chi(s)g, \chi(s)\dot{g}) = \int_{t_1}^{t_2} L(g,\dot{g})$$

Now differentiate with respect to s and set $s = 0$ (take first variation).

$$\int_{t_1}^{t_2} \left(\left\langle \frac{\partial L}{\partial g}, \chi_s(0) g \right\rangle + \left\langle \frac{\partial L}{\partial \dot{g}}, \chi_s(0) \dot{g} \right\rangle \right) dt = 0$$

Integrate by parts,

$$0 = \int_{t_1}^{t_2} \left\langle \frac{\partial L}{\partial g} - \frac{d}{dt} \frac{\partial L}{\partial \dot{g}}, \chi_s(0) g \right\rangle + \left\langle \frac{\partial L}{\partial \dot{g}}, \chi_s(0) g \right\rangle \Bigg|_{t_1}^{t_2}$$

The first part is zero as they are just the Euler Legrange equations. Therefore,

$$\left\langle \frac{\partial L}{\partial \dot{g}}, \chi_s(0) g \right\rangle = K$$

But we can write $\chi_s(0) g = \eta g = g g^{-1} \eta g = g(g^{-1} \eta g) = g \operatorname{Ad}_{g^{-1}} \eta$. Therefore,

$$\left\langle \frac{\partial L}{\partial \dot{g}}, g \operatorname{Ad}_{g^{-1}} \eta \right\rangle = K$$

Now we need to add ξ. We want to deform $g_t(t, s)$ but not $g(t)$. Assume that $L(g(t), g_t(t, s)) = L(g^{-1}(t) g(t), g^{-1}(t) g_t(t, s)) = \ell(g^{-1}(t) g_t(t, s))$. Now differentiate with respect to s. Now we conclude,

$$\left\langle \frac{\partial L}{\partial \dot{g}}, g_{ts}(t, s) \right\rangle = \left\langle \frac{\partial \ell}{\partial \xi}, g^{-1} g_{ts}(t, s) \right\rangle$$

and set $s = 0$ we get,

$$\left\langle \frac{\partial L}{\partial \dot{g}}, \delta g_t \right\rangle = \left\langle \frac{\partial \ell}{\partial \xi}, g^{-1} \delta g_t(t) \right\rangle$$

Therefore,

$$\left\langle \frac{\partial L}{\partial \dot{g}}, \delta g_t \right\rangle = \left\langle g \frac{\partial \ell}{\partial \xi}, \delta g_t(t) \right\rangle$$

That is,

$$g^{-1} \frac{\partial L}{\partial \dot{g}} = \frac{\partial \ell}{\partial \xi}$$

Therefore,

$$K = \left\langle g \frac{\partial \ell}{\partial \xi}, g \operatorname{Ad}_{g^{-1}} \eta \right\rangle = \left\langle \frac{\partial \ell}{\partial \xi}, \operatorname{Ad}_{g^{-1}} \eta \right\rangle = \left\langle \operatorname{Ad}^*_{g^{-1}} \frac{\partial \ell}{\partial \xi}, \eta \right\rangle$$

□

We note that the conserved quantity is the constant that arises from integration by parts. I find this quite nice.

Example. Now apply this to Pseudo Rigid bodies.

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