Year 3 — Number Theory

Based on lectures by Professor Henri Johnston
Notes taken by James Arthur

Autumn Term 2021

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Contents

1 Divisibility

1.1 Division Algorithm

Definition 1.1 (Well Ordering Principle)**.** Every non-empty subset of $\mathbb{N}_0$ contains a least element

Theorem 1.2 (Division Algorithm)**.** Given a $a \in \mathbb{Z}$ and a $b \in \mathbb{N}_1$ there exists unique integers q and r satisfying $a = bq + r$ and $0 \leq r < b$.

The proof splits into uniqueness and existence.

Proof. We shall first prove existence, define $S := \{a - xb : x \in \mathbb{Z} \quad \text{and } a - xb \geq 0\}$. We know $S \neq 0$ since,

- if $a \geq 0$, then choose $m = 0$, them $a - mb = a \geq 0$
- if $a < 0$, then let $a = m$, so $a - mb = a - ab = (-a)(b-1) \geq 0$ since $-a > 0$ and $b > 0$[1]

Hence S is non-empty subset of $\mathbb{N}_0$ and so by the well ordering principle S must contain a least element $r \geq 0$. Since $r \in S$, then we have there exists a $q \in \mathbb{Z}$ such that $a - qb = r$ and so $a = qb + r$. Now it remains to check that $r < b$, so assume for a contradiction that $r \geq b$, then let there be a $r_1 = r - b \geq 0$. Then,

$$a = qb + r = qb + (r_1 + b) = (q+1)b + r_1$$

and so $a - (q+1)b = r_1 \in S$ and is smaller than r, a contradiction.

Now let us show uniqueness, assume that there exist another pair q', r' such that $a = q'b + r'$ where $0 \leq r' < b$. Then form $a = a + qb + r = q'b + r'$ we have that, $(q - q')b = r' - r$. If $q = q'$, then we must have $r = r'$, suppose for a contradiction that this isn't true, then,

$$b \leq |q - q'||b| = |r - r'|$$

However, since $0 \leq r, r' < b$ and so $|r - r'| < b$ which gives a contradiction. □

Here's a definition that I feel is useful that wasnt covered in the lectures,

Definition 1.3 (Divisible)**.** We say that some $a \in \mathbb{Z}$ is divisible by some $b \in \mathbb{Z}$ if and only is,

$$\exists\, n \in \mathbb{Z}, \text{ such that } b = na$$

and denote it, $a \mid b$

1.2 Greatest Common Divisor

Let us start with a theorem.

Theorem 1.4. Let $a, b \in \mathbb{Z}$, $\exists\, d \in \mathbb{N}_0$ and non-unique $x, y \in \mathbb{Z}$ such that,

(i) $d \mid a$ and $d \mid b$

(ii) and if $e \in \mathbb{Z}$, $e \mid a$ and $e \mid b$, then $e \mid d$

(iii) $d = ax + by$

[1] You absolute plank, there doesn't exist any numbers between 0 and 1 in $\mathbb{Z}$, so $b > 0$ is the same as $b \geq 1$

Proof. If $a = b = 0$, then $d = 0$
Suppose that $a \neq b \neq 0$, then let

$$S := \{am + bn : m, n \in \mathbb{Z} \text{ and } am + bn > 0\}$$

Now $a^2 + b^2 > 0$ so S is non-empty and a subset of $\mathbb{N}_1$. Hence, by the Well ordering principle then there must be some minimum element d. Then we can write $d = ax + by$ by definition of S.

By the division Algorithm, $a = qs + r$ for some $q, r \in \mathbb{Z}$ with $0 \leq q < d$. Suppose for a contradiction that $r \neq 0$. Then,

$$0 < r = a - qd = a - q(ax + by) = (1 - qx)a - qby$$

Hence, $r \in S$. But $r < d$, contradiciting the minimality of d in S. So we must have $r = 0$, i.e $d \mid a$. The same works for $d \mid b$.

Suppose that $e \in \mathbb{Z}$, $e \mid a$ and $e \mid b$. Then e divides any linear combination of a and b, so $e \mid d$. Suppose that $e \in \mathbb{N}_1$ also satisfies (i) and (ii). Then, $e \mid d$ and $d \mid e$ and so $d = \pm e$, but $d, e \geq 0$ and so $d = e$. Thus d is unique. □

Note that this is a standard trick to prove that integers divide, by just proving that $r = 0$ by contradiction.

Corollary 1.5. If $a, b \in \mathbb{Z}$ then there exists a unique $d \in \mathbb{N}_1$ such that.

(i) $d \mid a$ and $d \mid b$

(ii) if $e \in \mathbb{Z}$, then $e \mid a$ and $e \mid b$ then $e \mid d$

Proof. The existence of a d is given by the theorem. In the proof of uniqueness we only use (i) and (ii). □

Definition 1.6 (Greatest Common Divisor)**.** Let $a, b \in \mathbb{Z}$. Them d of the previous corollary is just the greatest common divisor of a and b, written $\gcd(a, b)$. Also sometimes seen as $\text{hcf}(a, b)$.

If $\gcd(a, b) = 1$, then a and b are coprime.

Identity (Bezouts Identity)**.** Given $a, b \in \mathbb{Z}$ there exist $x, y \in \mathbb{Z}$ such that $\gcd(a, b) = ax + by$.

Proposition 1.7. Let $a, b, c \in \mathbb{Z}$, then,

(i) $\gcd(a, b) = \gcd(b, a)$

(ii) $\gcd(a, \gcd(b, c)) = \gcd(\gcd(a, b), c)$

(iii) $\gcd(ac, bc) = |c| \gcd(a, b)$

(iv) $\gcd(1, a) = \gcd(a, 1) = a$

(v) $\gcd(0, a) = \gcd(a, 0) = |a|$

(vi) $c \mid \gcd(a, b)$ if and only if $c \mid a$ and $c \mid b$

(vii) $\gcd(a + cb, b) = \gcd(a, b)$

Then we can consider the following remark,

Remark. Note that $\gcd(a, b) = 0$ if and only if, $a = b = 0$. Otherwise, $\gcd(a, b) \geq 1$.

Proof. Checking these properties are pretty simple, for (vi) just use Bezouts.

We shall prove (iii), so let $d = \gcd(a, b)$ and $e = \gcd(ac, bc)$. By (vi), $cd \mid e = gcd(ac, bc)$ since $cd \mid ac$ and $cd \mid bc$. Then by Bezouts, there exists $x, y \in \mathbb{Z}$ such that $d = ax + by$. Then,

$$cd = acx + bcy$$

and as $e \mid ac$ and $e \mid bc$ and so by linearity we have $e \mid cd$. Therefore, $|e| = |cd|$ and so, $e = |c|d$.

Now, let's prove (vii), let $e = \gcd(a + bc, b)$ and $f = \gcd(a, b)$. Then $e \mid (a + bc)$ and $e \mid b$. Thus by linearity, we have $e \mid a$. Hence, $e \mid a$ and $e \mid b$ so by property (vi), we have $e \mid f$. Similarly we can get that $f \mid a + bc$ and $f \mid b$ and so again my (vi) we have $e = f$ as $f, e \geq 0$. $\square$

Lemma 1.8 (Euclids Lemma)**.** Let $a, b, c \in \mathbb{Z}$. If $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$.

Proof. Suppose that $a \mid bc$ and $\gcd(a, b) = 1$. By Bezouts, we get that for some $x, y \in \mathbb{Z}$ we get $1 + ax + by$. Hence, $c = acx + bcy$, but $a \mid acx$ and $a \mid bcy$, so $a \mid c$ by linearity. $\square$

Theorem 1.9 (Solubility of linear equations in $\mathbb{Z}$)**.** Let $a, b, c \in \mathbb{Z}$. The equation,

$$ax + by = c$$

is soluble with $x, y \in \mathbb{Z}$ if and only if $\gcd(a, b) \mid c$

Proof. Let $d = \gcd(a, b)$. Then $d \mid a$ and $d \mid b$ so if there exists $x, y \in \mathbb{Z}$ such that $c = ax + by$ then $d \mid c$ by linearity of divisibility. Now, suppose that $d \mid c$. Then we can write $c = qd$ for some $q \in \mathbb{Z}$. By Bezouts, there exists some $x', y' \in \mathbb{Z}$ such that $d = ax' + by'$. Hence, $c = qd = aqx' + bqy'$ and so $x = qx'$ and $y = qy'$ gives a suitable solution. $\square$

1.3 Euclids Algorithm

Theorem 1.10 (Euclids Algorithm)**.** Let $a, b \in \mathbb{N}_1$ with $a > b > 0$ and $b \nmid a$. Let $r_0 = a$, $r_1 = b$ and apply the division Algorithm repeatedly to obtain a sequence of remainders defined sucessively,

$$\begin{aligned}
r_0 &= r_1 q_1 + r_2 & 0 &< r_2 < r_1 \\
r_1 &= r_2 q_2 + r_3 & 0 &< r_3 < r_2 \\
&\vdots \\
r_{n-2} &= r_{n-1} q_{n-1} + r_n & 0 &< r_n < r_{n-1} \\
r_{n-1} &= r_n q_n + r_{n+1} & r_{n+1} &= 0
\end{aligned}$$

Then the last non-zero remainder, r_n is the $\gcd(a, b)$.

Proof. There is a stage at which $r_{n+1} = 0$ because the r_i are strictly decreasing non-negative integers. We have,

$$\begin{aligned}
\gcd(r_i, r_{i+1}) &= \gcd(r_{i+1} q_{i+1} + r_{i+2} r_{i+1}) \\
&= \gcd(r_{i+2} r_{i+1}) \\
&= \gcd(r_{i+1}, r_{i+2})
\end{aligned}$$

Applying this result repeatedly,

$$\begin{aligned}
\gcd(a, b) &= \gcd(r_0, r_1) \\
&= \gcd(r_2, r_3) \\
&= \dots \\
&= \gcd(r_{n-1}, r_n) \\
&= r_n
\end{aligned}$$

Where the last equality is because $r_n \mid r_{n-1}$ □

Remark. One can also use Euclids Algorithm to find the $x, y \in \mathbb{Z}$ Bezouts Identity state to exist by working backwards. These aren't unique.

1.4 Extended Euclidean Algorithm

Instead of doing Euclids, and working backwards we can compute our bezouts x, y during euclids. This is the extended Euclids Algorithm. This time we are going to define sequnces of integers x_i and y_i, such that $r_i = ax_i + by_i$. Recall that r_n is the last non-zero remainder and that $r_n = \gcd(a, b)$. Therefore $\gcd(a, b) = r_n = ax_n + by_n$ and so $(x, y) := (x_n, y_n)$.

We have that $r_0 = a$ and $r_1 = b$. Hence, we see $r_0 = 1 \times a + 0 \times b$ and $r_1 = 0 \times a + 1 \times b$, and so we set $(x_0, y_0) := (1, 0)$ and $(x_1, y_1) := (0, 1)$. So, now we consider for $i \geq 2$ we have a pair (x_j, y_j) for $j < i$. Then $r_{i-2} = r_{i-1}q_{i-1} + r_i$ and so,

$$\begin{aligned} r_i &= r_{i-2} - r_{i-1}q_{i-1} \\ &= (ax_{i-2} + by_{i-2}) + (ax_{i-1} + by_{i-1})q_{i-1} \\ &= a(x_{i-2} - x_{i-1}q_{i-1}) + b(y_{i-2} - y_{i-1}q_{i-1}) \end{aligned}$$

Thus we set $x_i := x_{i-2} - x_{i-1}q_{i-1}$ and $y_i := y_{i-2} - y_{i-1}q_{i-1}$. These can be defined recursively this way.

$$(x_i, y_i) := (x_{i-2}, y_{i-2}) - q_{i-1}(x_{i-1}, y_{i-1})$$

Example. We compute $\gcd(841, 160)$ use Extended Euclidean Algorithm.

i	r_{i-2}		r_{i-1}		q_{i-1}		r_i	x_i	y_i
0							841	1	0
1							160	0	1
2	841	=	160	×	5	+	41	1	-5
3	160	=	41	×	3	+	37	-3	16
4	41	=	37	×	1	+	4	4	-21
5	37	=	4	×	9	+	1	-39	205
6	4	=	1	×	4	+	0		

Therefore, $\gcd(841, 160) = 1 = 841 \times (-39) + 160 \times 205$.

2 Primes and Congruences

We start by defining primes and composite numbers,

Definition 2.1 (Prime)**.** A number $p \in \mathbb{N}_1$ with $p > 1$ is prime if and only if it's only divisors are 1 and p, i.e.

$$n \mid p \implies n = 1 \text{ or } n = p$$

Definition 2.2 (Composite Numbers)**.** A number $n \in \mathbb{N}_1$ with $n > 1$ is composite if and only if it is not prime, i.e.

$$n = ab \qquad 1 < a, b \in \mathbb{N}$$

One is neither composite nor prime.

Proposition 2.3. If $n \in \mathbb{N}_1$ with $n > 1$, then n has a prime factor.

Proof. Use strong induction, so assume for $1 < m < n$ where $m \in \mathbb{N}_1$ that m has a prime factor.
Case (i): If n is prime, then n is a prime factor of n.
Case (ii): If n is composite, then $n = ab$ where $a, b > 1$ and so, $1 < a < n$. By the induction hypothesis, there is a prime p such that $p \mid a$. Hence, $p \mid a$ and $a \mid n$ so, by transitivity $p \mid n$. ☐

Proposition 2.4. If $1 < n \in \mathbb{N}_1$, then we can write $n = p_1 p_2 \ldots p_k$ where $k \in \mathbb{N}_1$ and p_i are primes.

Proof. If n is prime, then the result is clear. So suppose that n is composite. Then n must have a prime factor, so $n = p_1 n_1$ where $1 < n_1 \in \mathbb{N}_1$. If n_1 is prime, we are done. If n_1 is composite, then we can write $n_1 = p_2 n_2$ and so on... This process terminates as $n > n_1 > n_2 > \cdots > 1$. Hence after at least n steps we obtain a prime factorisation of n. ☐

Example.

$$666 = 3 \times 222 = 3 \times 2 \times 111 = 3 \times 2 \times 3 \times 37$$

Theorem 2.5. There are infinitely many primes

Euclid's Proof. For a contradiction, assume there are finitely many primes, $\{p_1, p_2, p_3, \ldots, p_n\}$ and that is a complete list. Consider $N := p_1 p_2 \ldots p_n + 1 \in \mathbb{N}$. Then $N > 1$ so by the first proposition, N has a prime factor p. However, every prime is one of the elements of the list, so $p = p_i$. Hence, $p_i \mid (p_1 p_2 \ldots p_n)$ so $p \mid (N - 1)$. However, $p \mid N$ and we can write $1 = N - (N - 1)$, so $p \mid 1$, which is a contradiction. ☐

2.1 Fundemental Theorem of Arithmetic

Lemma 2.6. Let $n \in Z$, then if $p \nmid n$ then $\gcd(p, n) = 1$

Proof. Let $d = \gcd(p, n)$. Then $d \mid p$ so by definition of prime either $d = 1$ or $d = p$. But $d \mid n$ so $d \neq p$ because $p \nmid n$. Hence, $d = 1$. ☐

Theorem 2.7 (Euclid's Lemma for Primes)**.** Let $a, b \in \mathbb{Z}$ and p be a prime. If $p \mid ab$, then $p \mid a$ or $p \mid b$.

Proof. Assume $p \mid ab$ and that $p \nmid a$. We shall prove $p \mid b$. By Lemma, $\gcd(p, a) = 1$, so by Euclid's lemma, $p \mid b$. ☐

Remark. Euclid's Lemma for primes immediately generalises to several factors.

Definition 2.8. Let $n \in \mathbb{N}_1$ and p be a prime. Then,

$$v_p(n) := \max\{k \in \mathbb{N} \cup \{0\} : p^k \mid n\}$$

In other words, k is the unique non-negative integer such that $p^k \mid n$ but $p^{k+1} \mid n$. Equivalently, $v_p(n) = k$ if and only if $n = p^k n'$ where $n' \in \mathbb{N}$ and $p \nmid n'$.

Example. We can see that,

- $v_2(720) = 4$ as $2^4 \mid 720$ but $2^5 \nmid 720$
- $v_3(720) = 2$ as $3^2 \mid 720$ but $3^3 \nmid 720$
- $v_5(720) = 1$ as $5^1 \mid 720$ but $5^2 \nmid 720$
- if $p \geq 7$, then $v_p(720) = 0$ as $p \nmid 720$.

Lemma 2.9. Let $n, m \in \mathbb{N}_1$ and p be a prime. Then $v_p(mn) = v_p(m) + v_p(n)$

Proof. Let $k = v_p(m)$ and $\ell = v_p(n)$. Then we write $m = p^k m'$ where $p \nmid m'$ and $n = p^\ell n'$ where $p \nmid n'$. Then $nm = p^{k+\ell} m'n'$ and so by Euclid's lemma $p \nmid m'n'$ as if it did then $p \mid n'$ or $p \mid m'$ but it doesn't. So $v_p(mn) = v_p(m) + v_p(n)$. □

Theorem 2.10 (Fundamental Theorem of Arithmetic)**.** Let $1 < n \in \mathbb{N}_1$. Then,

(i) (Existence) The number n can be written as a product of primes.

(ii) (Uniqueness) Suppose that,
$$n = p_1 \dots p_r = q_1 \dots q_s$$
where each p_i and q_j are prime. Assume further that,
$$p_1 \leq p_2 \leq \dots \leq p_r \qquad \text{and} \qquad q_1 \leq q_2 \leq \dots \leq q_s$$
Then $r = s$ and $p_i = q_i$ for all i

Remark. If 1 is a prime, then the Uniqueness here is broken, as,
$$6 = 3 \times 2 = 3 \times 2 \times 1 = \dots$$

Remark. A consequence of the FTA is that the integral domain $\mathbb{Z}$ is in fact a UFD.

Proof. The existence is something we have done before. The harder part is uniqueness. Let ℓ be any prime. Then we have,
$$\begin{aligned} v_e ll(n) &= v_\ell(p_1 \dots p_r) \\ &= v_\ell(p_1) + \dots + v_\ell(p_r) \end{aligned}$$
However,
$$v_\ell(p_i) = \begin{cases} 1 & \text{if } \ell = p_i \\ 0 & \text{if } \ell \neq p_i \end{cases}$$
Therefore,
$$\begin{aligned} v_\ell(n) &= \#\text{ of } i \text{ for which } \ell = p_i \\ &= \#\text{ of times } \ell \text{ appears in the factorisation } n = p_1 \dots p_r \end{aligned}$$
Similarly,
$$v_\ell(n) = \#\text{ of times } \ell \text{ appears in the factorisation } n = q_1 \dots q_s$$
Thus every prime ℓ appears the same number of times in each factorisation, giving the desired result. □

Remark. Another way of interpreting this result is to say that for $n \in \mathbb{N}_1$,
$$n = p_1^{v_{p_1}(n)} p_2^{v_{p_2}(n)} \dots p_r^{v_{p_r}(n)}$$
where $p_1, \dots, p_r$ are the distinct prime factors of n. Note that we take the empty product to be 1, which covers the case for $n = 1$.

Lemma 2.11. Let $n = \prod_{i=1}^r p_i^{a_i}$ where each $a_i \in \mathbb{N}_0$ and the p_i's are distinct primes. The set of positive divisors of n is the set of numbers of the form $\prod_{i=1}^r p_i^{c_i}$ where $0 \leq c_i \leq a_i$ for $i = 1, \dots, r$.

Proof. Exercise □

2.2 Congruences

Definition 2.12. Suppose $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}_1$. We write $a \equiv b \mod n$, and say 'a is congruent to b mod n', if and only if $n \mid (a-b)$. If $n \nmid (a-b)$ we say that a and b are incongruent mod n.

Remark. In particular, $a \equiv 0 \mod n$ if and only if $m \mid a$

Example. Here are some examples:

- $4 \equiv 30 \mod 13$ since $13 \mid (4-30) = -26$
- $17 \not\equiv -17 \mod 4$ since $17 - (-17) = 34$ but $4 \nmid 34$.
- n is even if and only if $n \equiv 0 \mod 2$
- n is odd if and only if $n \equiv 1 \mod 2$
- $a \equiv b \mod 1$ for all $a, b \in \mathbb{Z}$

Proposition 2.13. Let $n \in \mathbb{N}_1$ being congruent mod n is an equivalence relation, so,

(i) Reflexive: $\forall\, a \in \mathbb{Z}, a \equiv a \mod n$

(ii) Symmetric: $\forall\, a, b \in \mathbb{Z}$, $a \equiv b \mod n \implies b \equiv a \mod n$

(iii) Transitive: $\forall\, a, b \in \mathbb{Z}$, $a \equiv b \mod n$ and $b \equiv c \mod n \implies a \equiv c \mod n$.

Proof. The proof follows from,

(i) $n \mid 0$.

(ii) If $n \mid (a-b)$ then $n \mid (b-a)$

(iii) If $n \mid (a-b) + (b-c) = (a-c)$

□

Proposition 2.14. Congruences respect addition, subtraction and multiplication. Then let $a, b, \alpha, \beta \in \mathbb{Z}$. Suppose that $a \equiv \alpha \mod n$ and $b \equiv \beta \mod n$. Then,

(i) $a + b \equiv \alpha + \beta \mod n$

(ii) $a - b \equiv \alpha - \beta \mod n$

(iii) $ab \equiv \alpha\beta \mod n$

Moreover, if $f(x) \in \mathbb{Z}[x]$ then $f(a) \equiv f(\alpha) \mod n$

Proof. Check that $ab \equiv \alpha\beta \mod n$. Since, $a \equiv \alpha \mod n$ and so, $n \mid (a - \alpha)$ and so $a = \alpha + ns$ for some $s \in \mathbb{Z}$, Similarly $b = \beta + nt$. Hence,

$$ab = (\alpha + ns)(\beta + nt) = \alpha\beta + n(s\beta + t\alpha + nst)$$

and so $n \mid (ab - \alpha\beta)$. Therefore, $ab \equiv \alpha\beta \mod n$, as required. □

Example. Let $n \in \mathbb{N}_1$ and write n in decimal notation,

$$n = \sum_{i=0}^{k} a_i \times 10^i \qquad 0 \leq a_i \leq 9$$

Then, define $f(x)$ by,

$$f(x) = \sum_{i=0}^{k} a_i x^i$$

Then, since $10 \equiv -1 \mod 11$, we see that $n = f(10) \equiv f(-1) \mod 11$, whence,

$$11 \mid n \iff 11 \mid f(-1) \iff 11 \mid (a_0 - a_1 + a_2 - a_3 + \cdots + (-1)^k a_k)$$

This is an easy way to test for divisibility by 11.

Example. Does $x^2 - 3y^2 = 2$ have a solution with $x, y \in \mathbb{Z}$. Let $x, y \in \mathbb{Z}$. Note that $x^2 - 3y^2 \equiv x^2 \mod 3$. Now, $x \equiv 0, 1, 2 \mod 3$, so $x^2 \equiv 0, 1, 4 \mod 3 \equiv 0, 1 \mod 3$. Hence, $x^2 - 3y^2 \equiv x^2 \not\equiv 2 \mod 3$ and so $x^2 - 3y^2 \neq 2$.

Remark. Suppose we have $f \in \mathbb{Z}[x_1, \ldots, x_m]$ if we have $a_1, \ldots, a_m \in \mathbb{Z}$ such that $f(a_1, \ldots, a_m) = 0$ then $f(a_1, \ldots, a_m) \equiv 0 \mod n$ for every $n \in \mathbb{N}$. Therefore if there exist an $n \in \mathbb{N}_1$ such that $f(x_1, \ldots, x_m) \equiv 0 \mod n$ has no solution, there cannot exist $a_1, \ldots, a_m \in \mathbb{Z}$ such that $f(a_1, \ldots, a_n) = 0$.

We are going to prove the following theorem,

Theorem 2.15. There are infinitely many primes p with $p \equiv 3 \mod 4$

Proof. Suppose that p is a prime. Then $p \equiv 0, 1, 2, 3 \mod 4$, but $p \not\equiv 0 \mod 4$ because $4 \nmid p$. If $p \equiv 2 \mod 4$ then $p = 4k + 2$ for some $k \in \mathbb{Z}$, so $2 \mid p$ so in fact $p = 2$. Therefore there are three types of primes,

(i) $p = 2$

(ii) $p \equiv 1 \mod 4$

(iii) $p \equiv 3 \mod 4$

Let $N \in \mathbb{N}$ it suffices to show that there exist a type (iii) prime with $p > N$. Let $4(N!) - 1$ and so $M \geq 3$ and so by the existence of FTA we can write $M = p_1 \ldots p_k$. If $p \leq N$, then $M \equiv -1 \mod p$ so $p \nmid M$. Hence, $p_j > N$ for all j. Moreover $p_j \neq 2$ for all j because M is odd. Therefore for each j we have $p_j \equiv 1, 3 \mod 4$. If $p_j \equiv 3 \mod 4$ for any j then we are done. If this is not the case, then $p_j \equiv 1 \mod 4$ for all j, and so, $M \equiv 1 \times 1 \times \cdots \times 1 \mod 4 \equiv 1 \mod 4$; but by definition of M we have $M \equiv -1 \equiv 3 \mod 4$ - contradiction! □

Remark. Congruences do not respect division, $4 \equiv 14 \mod 10$ but $2 \not\equiv 7 \mod 10$

Proposition 2.16. Let $a, b, s \in \mathbb{Z}$ and $d, n \in \mathbb{N}_1$.

(i) If $a \mid b \mod n$ and $d \mid n$ them $a \mid b \mod d$

(ii) Suppose $s \neq 0$. Then $a \equiv b \mod n$ if and only if $as \equiv bs \mod ns$

Proof. (i) follows from transitivity of divisibility;
(ii) follows from multiplication and cancellation properties. □

Theorem 2.17 (Cancellation law for Congruences)**.** Let $a, b, c \in \mathbb{Z}$ and $n \in \mathbb{N}_1$. Let $d = \gcd(c, n)$. Then $ac \mid bc \mod n \iff a \equiv b \mod \frac{n}{d}$. In particular, if n and c are coprime, then $ac \equiv bc \mod n \iff a \equiv b \mod n$.

Proof. Since, $d = \gcd(c, n)$, we may write $n = dn'$ and $c = dc'$ where $n', c' \in \mathbb{Z}$. Suppose $ac \equiv bc \mod n$. Then $n \mid c(a - b)$ and so $n' \mid c'(a - b)$. However, $\gcd(n', c') = 1$ and so $n' \mid (a - b)$ by Euclid's Lemma. Thus, $a \equiv b \mod n'$.
Suppose conversely $a \equiv b \mod n'$ and so, $n' \mid (a - b)$ and so $n \mid d(a - b)$. But $d \mid c$ and so $d(a - b) \mid c(a - b)$ and thus $n \mid c(a - b)$ by the transitivity of divisibility. Thus $ac \equiv bc \mod n$. □

Proposition 2.18. Let $a, m, n \in \mathbb{Z}$. If m and n are coprime and if $m \mid a$ and $n \mid a$ then $nm \mid a$.

Proof. Since $m \mid a$ we can write $a = mc$ for some $c \in \mathbb{Z}$. Now $n \mid a = mc$ and $\gcd(m, n) = 1$ and so by Euclid's Lemma, $n \mid c$. Hence, $mn \mid mc = a$. $\square$

Corollary 2.19. Let $m, n \in \mathbb{N}$ be coprime and let $a, b \in \mathbb{Z}$. If $a \equiv b \mod m$ and $a \mid b \mod n$ then $a \equiv b \mod mn$.

Proof. We have $n \mid (a - b)$ and $m \mid (a - b)$. Since m and n are coprime we therefore have $mn \mid (a - b)$. $\square$

3 Residue Classes

Proposition 3.1. Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}_1$. If $a \equiv b \mod n$ and $|b-a| < n$ then $a = b$.

Proof. Since $n \mid (a-b)$, by the comparison property of divisibility we have $n \leq |a-b|$ unless $a-b=0$. ☐

As $\mod n$ is an equivalence relation,

Definition 3.2 (Residue Class)**.** Consider $n \in \mathbb{N}$, then $a \in \mathbb{Z}$ we write $[a]_n$ for an equivalence class $a \mod n$. Thus,

$$[a]_n = \{x \in \mathbb{Z} : x \equiv a \mod n\} = \{a + qn : q \in \mathbb{Z}\}$$

This is called the residue class of a modulo n

$[a]_n$ is the coset, $\mathbb{Z}/n\mathbb{Z}$.

Example. Consider $n = 2$, then,

$$[0]_2 = \{x \in \mathbb{Z} : x \equiv 0 \mod 2\}$$
$$[1]_2 = \{x \in \mathbb{Z} : x \equiv 1 \mod 2\}$$

Proposition 3.3. Let $n \in \mathbb{Z}$. The n residue classes are disjoint and thier union is the set of all integers. Or $\forall\, x \in \mathbb{Z}$, $x \equiv y \mod n$ such that y is precisely one of $\{0, 1, \dots, n-1\}$.

Proof. The integers $0, 1, \dots, n-1$ are incongruent $\mod n$ by the Proposition 3.1. Hence, the residue classes are distinct and thus disjoint. Every integer must be in one of these classes by the division algorithm, as we can write $x = nq + r$. The result then follows from taking $x \equiv r \mod n$ and hence, $x \in [r]_n$. ☐

Distinct left cosets of $\mathbb{Z}/n\mathbb{Z}$ are always disjoint and partition $\mathbb{Z}$.

3.1 Complete Residue Systems

Definition 3.4 (Complete Residue System)**.** Let $n \in \mathbb{N}_1$. If S is a subset of $\mathbb{Z}$ containing exctly one element of each residue class modulo n we say that S is a complete residue system modulo n.

Proposition 3.5. The last proposition says $S = \{0, 1, \dots, n-1\}$ is a complete residue system. Note, that if S is any complete residue system, then $|S| = n$. Any set of integers that are incongruent $\mod n$ are a complete residue system $\mod n$.

Example. The following are complete residue systems,

$$\{1, 2, \dots, n\}$$
$$\{1, n+2, 2n+3, 3n+4, \dots, n^2\}$$
$$\{x \in \mathbb{Z} : -\frac{n}{2} < x \leq \frac{n}{2}\}$$

Proposition 3.6. Let $n \in \mathbb{N}_1$ an $k \in \mathbb{Z}$. Assume n and k are coprime. If $\{a_1, \dots, a_n\}$ is a complete residue system modulo n then so is $\{ka_1, \dots, ka_n\}$.

Proof. If $ka_i \equiv ka_j \mod n$ then by the cancellation law for congruences we have $a_i \equiv a_j \mod n$ since $\gcd(k, n) = 1$. Therefore no two distinct elements in this set, $\{ka_1, \dots, ka_n\}$, are congruent modulo n. ☐

Example. The set $\{0, 1, 2, 3, 4\}$ is a complete residue system $\mod 5$ and so $\{0, 2, 4, 6, 8\}$ is also a complete residue system $\mod 5$.

3.2 Linear Congruences

The most basic congruences are linear congruence, for example,

$$ax \equiv b \mod n$$

When n is small, we can brute force it, however, it becomes impractical quickly.

Theorem 3.7 (Linear Congruences with exactly one solution)**.** Let $a, b \in \mathbb{Z}$ and let $n \in \mathbb{N}$. Suppose that a and n are coprime. Then the linear congruence,

$$ax \equiv b \mod n$$

has exactly one solution.

Proof. We need only to test $1, 2, \dots, n$ since they constitute a complete residue system. Therefore, we consider the products, $a, 2a, \dots, na$. Since a and n are coprime, these numbers are also a complete residue system. Hence, exactly one of the elements of this sets is congruent to $b \mod n$. □

Theorem 3.8 (Solubility of a Linear Congruence)**.** Let $a, b \in \mathbb{Z}$ and let $n \in \mathbb{N}$. Then the linear congruence,

$$ax \equiv b \mod n \tag{1}$$

has one or more solutions if and only if $\gcd(a, b) \mid b$.

Proof. By definition, the congruence (1) is soluble if and only if $n \mid (b - ax)$ for some $x \in \mathbb{Z}$, and this is true if and only if $b - ax = ny$ for some $x, y \in \mathbb{Z}$. Hence (1) is soluble if and only if,

$$ax + ny = b$$

for some $x, y \in \mathbb{Z}$. Therefore this result follows from the solubility of linear equations theorem □

Theorem 3.9. Let $a, b \in \mathbb{Z}$ and let $n \in \mathbb{N}$. Let $d = \gcd(a, n)$. Suppose $d \mid b$ and write $a = da'$, $b = db'$ and $n = dn'$. Then the linear congruence

$$ax \equiv b \mod n \tag{2}$$

has exactly d solutions modulo n. These are,

$$t, t + n' + t + 2n', \dots, t + (d-1)n' \tag{3}$$

where t is the unique solution $\mod n'$ to,

$$a'x \equiv b' \mod n' \tag{4}$$

Proof. Every solution of (2) is a solution of (4) and vice versa. Since a' and n' are coprime, (4) has exactly one solution, t, $\mod n'$ by the Theorem 3.7. Thus the d numbers in (3) are solutions of (4) and hence (2).

No two items in the list are congruent $\mod n$ since the relationships

$$\begin{aligned} t + rn' &\equiv t + sn' \mod n && \text{with } 0 \le r < d,\ 0 \le s < d \\ rn' &\equiv sn' \mod n && \text{and hence } r \equiv s \mod d \end{aligned}$$

But $0 \le |r - s| < d$ so $r = s$. It remains to show that (2) has no solutions other than (3). If y is a solution of (2), then $ay \equiv b \mod n$. But we also have $at \equiv b \mod n$. Thus $y \equiv t \mod n'$ by the cancellation law for congruences. Hence, $y = t + kn'$ for some $k \in \mathbb{Z}$. But $r \equiv k \mod d$ for some $r \in \mathbb{Z}$ such that $0 \le r < d$. Therefore we have,

$$kn' \equiv rn' \mod n \qquad \text{and so } y \equiv t + rn' \mod n$$

Therefore y is congruent $\mod n$ to one of these numbers in (3). □

Algorithm. Let $a, b \in \mathbb{Z}$ and let $n \in \mathbb{N}$. Suppose we want to solve,

$$ax \equiv b \mod n \tag{5}$$

Firstly apply Extended Euclidian algorithm to compute $d := \gcd(a, n)$ to find $x', y' \in Z$ such that,

$$ax' + ny' = d \tag{6}$$

if $d \nmid b$ then there are no solutions. Otherwise, these are exactly d solutions $\mod n$, which we find as follows. Write $a = da'$, $b = db'$ and $n = dn'$. Dividing (6) through by d gives,

$$a'x' + n'y' = 1 \tag{7}$$

Thus reducing this $\mod n'$ gives $a'x' \equiv 1 \mod n'$ and multiplying through by b' gives $a'(b'x') \equiv b' \mod n$. Therefore $t := b'x'$ is the unique solution to $a'x' \equiv b' \mod n'$. Now the solutions to (5) are,

$$t, t + n', t + 2n', \ldots, t + (d-1)n'$$