Vector Calculus Week 2 - More Suffix Notation

James Arthur

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Contents

1 Gradient, Divergence and Curl

1.1 Gradient

Assume we have a $f = f(x, y, z)$ or $f = f(x_1, x_2, x_3)$, so a scalar calued function. Then we define grad f as:

$$\underline{\nabla} f = \left(\frac{\partial}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial}{\partial z}\hat{\boldsymbol{k}} \right) f$$

We say grad of f is a differential operator. So:

$$\underline{\nabla} f = \left(\frac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{k}} \right)$$

and we can write it in suffix notation aswell:

$$[\underline{\nabla} f]_i = \frac{\partial}{\partial x_i} \qquad i = 1, 2, 3$$

1.2 Divergence

Assume we have a vector field, $\underline{\mathbf{u}} = \underline{\mathbf{u}}(x, y, z, t)$. We define the divergence of this vector field as;

$$\underline{\nabla} \cdot \underline{\mathbf{u}} = \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right)$$

Placing this in suffix notation, we get that:

$$[\underline{\nabla} \cdot \underline{\mathbf{u}}]_j = \frac{\partial u_j}{\partial x_j}$$

1.3 Curl

the curl of a vector field can be written as:

$$\underline{\nabla} \times \underline{\mathbf{u}}$$

To write this in suffix notation, we can just use the cross produce formula:

$$[\underline{\nabla} \times \underline{\mathbf{u}}]_i = \varepsilon_{ijk} \underline{\nabla}_j u_k$$

which then can be manipulated into:

$$[\underline{\nabla} \times \underline{\mathbf{u}}]_i = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} \qquad j, k = 1, 2, 3$$

where i is a free index and j, k are dummy suffixes, so $j, k = 1, 2, 3$

2 Combinations of gradient, divergence and curl

2.1 Divergence of Gradient

If we take $\underline{\nabla} \cdot \underline{\nabla} f$ where $f = (x_1, x_2, x_3, t)$. We can write the div of grad as:

$$\begin{aligned}
\underline{\nabla} \cdot \underline{\nabla} f &= \left(\frac{\partial}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial}{\partial z}\hat{\boldsymbol{k}} \right) \cdot \left(\frac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{k}} \right) \\
&= \frac{\partial}{\partial x_1}\frac{\partial f}{\partial x_1} + \frac{\partial}{\partial x_2}\frac{\partial f}{\partial x_2} + \frac{\partial}{\partial x_3}\frac{\partial f}{\partial x_3} \\
&= \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} \\
&= \Delta f
\end{aligned}$$

Where the $\Delta = \underline{\nabla}^2$ is the laplacian. So how do we write this in suffix notation?

$$\begin{aligned}
\underline{\nabla} \cdot \underline{\nabla} f &= \underline{\nabla}_j [\underline{\nabla} f]_j \\
&= \frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_j} \\
&= \frac{\partial^2 f}{\partial x_j}
\end{aligned}$$

2.2 Curl of Gradient

We can write the curl of gradient as:

$$\begin{aligned}
[\underline{\nabla} \times \underline{\nabla} f]_i &= \varepsilon_{ijk} \underline{\nabla}_j \underline{\nabla} f_k \\
&= \varepsilon_{ijk} \frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_k} \\
&= \varepsilon_{ikj} \frac{\partial}{\partial x_k}\frac{\partial f}{\partial x_j} \\
&= -\varepsilon_{ijk} \frac{\partial}{\partial x_k}\frac{\partial f}{\partial x_j} \\
&= -\varepsilon_{ijk} \frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_k} \qquad \text{if } f \in c^2 \\
\implies \underline{\nabla} \times \underline{\nabla} f &= 0
\end{aligned}$$

2.3 Gradient of Divergence

Assume we have a $\underline{\mathbf{u}}$, vector field, and we want $\underline{\nabla} f \underline{\nabla} \cdot$.

$$
\begin{aligned}
[\underline{\nabla}\, f \underline{\nabla}\cdot]_i &= \underline{\nabla}_i \frac{\partial u_j}{\partial x_j} \\
&= \frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j} \\
&= \frac{\partial^2 u_j}{\partial x_i \partial x_j}
\end{aligned}
$$

2.4 Divergence of Curl

We can write divergence of curl as:

$$
\begin{aligned}
[\underline{\nabla}\cdot\underline{\nabla}\times\underline{\mathbf{u}}]_i &= \frac{\partial}{\partial x_i}[\underline{\nabla}\times\underline{\mathbf{u}}]_i \\
&= \frac{\partial}{\partial x_i}\varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}
\end{aligned}
$$

$i, j, k = 1, 2, 3$, so $i \leftrightarrow j$

$$
\begin{aligned}
&= \frac{\partial}{\partial x_j}\varepsilon_{jik}\frac{\partial u_k}{\partial x_i} \\
&= -\varepsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial u_k}{\partial x_i} \\
&= -\varepsilon_{ijk}\frac{\partial}{\partial x_i}\frac{\partial u_k}{\partial x_j} \qquad \text{as } \underline{\mathbf{u}} \in c^2
\end{aligned}
$$

As $\underline{\nabla}\cdot(\underline{\nabla}\times\underline{\mathbf{u}}) = -\underline{\nabla}\cdot(\underline{\nabla}\times\underline{\mathbf{u}})$, then we know that $\underline{\nabla}\cdot(\underline{\nabla}\times\underline{\mathbf{u}}) = 0$

2.5 Curl of Curl

We can write curl of curl, $\underline{\nabla}\times(\underline{\nabla}\times\underline{\mathbf{u}})$, as:

$$
\begin{aligned}
[\underline{\nabla}\times(\underline{\nabla}\times\underline{\mathbf{u}})]_i &= \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\underline{\nabla}\times\underline{\mathbf{u}})_k \\
&= \varepsilon_{ijk}\frac{\partial}{\partial x_j}\varepsilon_{klm}\frac{\partial u_m}{\partial x_l} \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\frac{\partial^2 u_m}{\partial x_j \partial x_l} \\
&= \delta_{il}\delta_{jm}\frac{\partial^2 u_m}{\partial x_j \partial x_l} - \delta_{im}\delta_{jl}\frac{\partial^2 u_m}{\partial x_j \partial x_l} \\
&= \frac{\partial^2 u_j}{\partial x_j \partial x_i} - \frac{\partial^2 u_i}{\partial x_j \partial x_j} \\
&= \frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j} - \frac{\partial^2 u_i}{\partial x_j^2} \\
&= [\underline{\nabla}(\underline{\nabla}\cdot\underline{\mathbf{u}})]_i - [\Delta\underline{\mathbf{u}}]_i \\
&= [\underline{\nabla}(\underline{\nabla}\cdot\underline{\mathbf{u}}) - \underline{\nabla}^2\underline{\mathbf{u}}]_i
\end{aligned}
$$

3 Scalar Field / Vector Fields Defintions

A scalar or vector quantity is said to be a field if it is a function of position. Examples

1. Temperature is a scalar field, $T = T(x, y, z) = T(\underline{\mathbf{r}})$
2. Pressure and Density are also scalr fields $P = P(\underline{\mathbf{r}})$ and $\rho = \rho(\underline{\mathbf{r}})$
3. if a physical quantity is a scalar we speak of a scalar field or function of position.

If a physical quantity is a vector, such as force $\underline{\mathbf{F}} = \underline{\mathbf{F}}(x, y, z)$. We speak of a vector field or vector function.

A vector-valued function is an $f : A \subset \mathbb{R}^n \mapsto \mathbb{R}^m$. So, for each $\underline{\mathbf{x}} = (x_1, \ldots, x_n) \in A$, f assigns a value $f(\underline{\mathbf{x}})$, an m-tuple, in $\mathbb{R}^m$. These functions, f, are called vector-valued functions if $m > 1$ and scalar if $m = 1$.

Example 1 *Take the function,* $f : (x, y, z) \mapsto (x^2 + y^2 + z^2)^{\frac{3}{2}}$

Solution 1 *It's a scalar function from* $\mathbb{R}^3$ *to* $\mathbb{R}$.

Example 2 *Take the function* $g : (x_1, x_2, x_3) \mapsto (x_1 x_2 x_3, \sqrt{x_1 x_3})$

Solution 2 *This is a vector valued function from* $\mathbb{R}^3$ *to* $\mathbb{R}^2$

To specify a temperature T in a region A of space requires a function T, $T : A \subset \mathbb{R}^m \mapsto \mathbb{R}$. $T = T(x, y, z)$.

To specify the velocity of a fluid moving in space requires a map, $\underline{\mathbf{v}} : \mathbb{R}^4 \mapsto \mathbb{R}^3$ where $\underline{\mathbf{v}}(x, y, z, t)$ is the velocity of the fluid at (x, y, z) at time t.

When $f : U \subset \mathbb{R}^n \mapsto \mathbb{R}$, we say that f is a real valued function of n-variables with domain U.

Let $f : U : \mathbb{R}^n \mapsto \mathbb{R}$, then graph $f = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^{n+1} : (x_1, \ldots, x^n)\}$ If $n = 1$, then we can conclude that graph f is curve in $\mathbb{R}^2$ and if $n = 2$, then graph f is a surface in $\mathbb{R}^3$.

3.1 Level Sets, Curves and Surfaces

A level set is a subset of $\mathbb{R}^3$ on which f is constant. For example, for $f(x, y, z) = x^2 + y^2 + z^2$, the set where $x^2 + y^2 + z^2 = 1$ is alevel set. A level set is a set of $(x, y, z) : f(x, y, z) = c$ where $c \in \mathbb{R}$.

For functions $f(x, y)$, we speak of level curves or contours. example, $f : \mathbb{R}^2 \mapsto \mathbb{R}$, $f(x, y) = x + y + 2$, has as it's graph the inclined plane $z = x + y + 2$. The plane intersects the xy plan where $z = 0$ in the line $y = -x - 2$ and the z-axis at $(0, 0, 2)$. For any $c \in \mathbb{R}$, the level curve of c is the straight line: $y = -x + (c - 2) : L_c\{(x, y) : y = -x + c - 2\} \subset \mathbb{R}^2$