Week 2: Limits and Functions

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Contents

1 Limits

1.1 Defining Limits

We consider limits of real functions, that is $f : X \to \mathbb{R}$, with $X \subset \mathbb{R}$.

Definition 1.1: Limit

We say that $f(x)$ approaches the limit L as x approaches x_0, and write

$$\lim_{x \to x_0} f(x) = L$$

if f is defined on some deleted neighbourhood of x_0 and, for every $\varepsilon > 0$, there is a $\delta > 0$ such that:

$$|f(x) - L| < \varepsilon$$

if

$$0 < |x - x_0| < \delta$$

Theorem 1.1: Limit Uniqueness

If $\lim\limits_{x \to x_0} f(x)$ exists, then it is unique, that is, if:

$$\lim_{x \to x_0} f(x) = L_1 \qquad \text{and} \quad \lim_{x \to x_0} f(x) = L_2$$

then $L_1 = L_2$

Proof. Let $\exists \varepsilon > 0$, such that

$$|f(x) - L_i| < \varepsilon \text{ if } 0 < |x - x_0| < \delta_i$$

for $i = 1, 2$

Now, let us look at a $|L - 1 - L_2|$ and let $\delta = \min(\delta_1, \delta_2)$.

$$\begin{aligned} |L_1 - L_2| &= |L_1 - f(x) + f(x) - L_2| \\ &\leq |L_1 - f(x)| + |L_2 - f(x)| < 2\varepsilon \end{aligned}$$

Given we know that ε is arbitrarily small, then $|L_1 - L_2|$ is arbitrarily small and hence, $L_1 = L_2$. $\square$

Theorem 1.2: Algebra of Limits

If $\lim_{x \to x_0} f(x) = L_1$ and $\lim_{x \to x_0} g(x) = L_2$, then:

$$\begin{aligned} \lim_{x \to x_0} (f + g) &= L_1 + L_2 \\ \lim_{x \to x_0} (f - g) &= L_1 - L_2 \\ \lim_{x \to x_0} (fg) &= L_1 L_2 \\ \lim_{x \to x_0} \left(\frac{f}{g}\right) &= \frac{L_1}{L_2} && \text{if } L_2 \neq 0 \end{aligned}$$

Proof. long and tedious $\square$

1.2 One Sided Limit

Definition 1.2: Left-hand limits

We say that $f(x)$ approaches the left-hand limit L as x approaches x_0 from the left and write:

$$\lim_{x \to x_0^-} f(x) = L$$

if f is defined on some open interval (a, x_0) and, for each $\varepsilon > 0, \exists\, \delta > 0$,

$$|f(x) - L| < \varepsilon \text{ if } x_0 - \delta < x < x_0$$

Definition 1.3: Right-hand limit

We say that $f(x)$ approaches the right-hand limit L as x approaches x_0 from the right and write:

$$\lim_{x \to x_0^+} f(x) = L$$

if f is defined on some open interval (x_0, b) and, for each $\varepsilon > 0, \exists\, \delta > 0$,

$$|f(x) - L| < \varepsilon \text{ if } x_0 < x < x_0 + \delta$$

Theorem 1.3

A function f has a limit at $x_0 \iff$ it has right and left handed limits and they are equal.

$$\lim_{x \to x_0} f(x) = L$$

if and only if

$$f(x_0-) = f(x_0+) = f(x_0)$$

Proof. coming soon □

1.3 Limits at $\pm\infty$

Definition 1.4: Limit at infinity

We say that $f(x)$ approaches the limit L as x approaches ∞, and write:

$$\lim_{x \to x_0} f(x) = L$$

if f is defined on an interval (a, ∞) and, for each $\varepsilon > 0$, there is a number β st,

$$|f(x) - L| < \varepsilon \qquad \text{if } x > \beta$$

Definition 1.5: Left infinite limit

We say $f(x)$ approaches ∞ as x approaches x_0 from the left, and write:

$$f(x_0-) = \infty$$

if f is defined on an interval (a, x_0) and, for each real number M, there is a $\delta > 0$ such that:

$$f(x) > M \text{ if } x_0 - \delta < x < x_0$$

NB! When we say a limit exists, we mean that it is finite, i.e. not $\pm\infty$. If it is, we can say it exists in the extended reals.

Also with infinite limits, we know that the 'Uniqueness of Limits' and the 'Algebra of Limits' are also valid when x_0 are replaced by $\pm\infty$.

The 'Alegbra of Limits' rules are also valid if $L_1, L_2 = \infty$ provided the RHS are not indeterminant forms.

1.4 Monotonics

Definition 1.6: Monotonicity

A function f is nondecreasing on an interval I if:

$$f(x_1) \leq f(x_2) \quad \text{if } x_1, x_2 \in I \text{ and } x_1 < x_2$$

or nondecreasing if,

$$f(x_1) \geq f(x_2) \quad \text{if } x_1, x_2 \in I \text{ and } x_1 < x_2$$

We further define that if the '$\leq$' can be replaced with a '$<$', then f is strictly monotonic on I

Theorem 1.4

Suppose that f is monotonic on $(a,\, b)$ and define

$$\alpha = \inf_{a<x<b} f(x) \text{ and } \sup_{a<x<b} f(x)$$

1. If f is nondecreasing, then $f(a+) = \alpha$ and $f(b-) = \beta$
2. If f is nonincreasing, then $f(a+) = \beta$ and $f(b-) = \alpha$.
3. If $a < x_0 < b$, then $f(x_0+)$ and $f(x_0-)$ exist and are finite; moreover;

   $$f(x_0-) \leq f(x_0) \leq f(x_0+)$$

   if f is nondecreasing, and

   $$f(x_0-) \geq f(x_0) \geq f(x_0+)$$

   if f is nonincreasing

Proof. Too long and tedious to typeset □

2 Continuity

Now we have defined limits, we can now define continuity.

Definition 2.1: Continuity at x_0

We say that f is continuous at x_0 if f is defined on an open interval (a, b) containing x_0 and that $\lim_{x\to x_0} f(x) = f(x_0)$.

Definition 2.2: Left continuity at x_0

We say f is continuous from the left at x_0 if f is defined on an open interval (a, x_0) and $f(x_0-) = f(x_0)$.

Definition 2.3: Right Continuity at x_0

we say f is continuous from the right at x_0 if f is defined on an open interval (x_0, b) and $f(x_0+) = f(x_0)$.

Theorem 2.1

A function f is continuous at x_0 if and only if f is defined on an open interval (a, b) containing x_0 and for each $\varepsilon > 0$ there is a $\delta > 0$ st,

$$|f(x) - f(x_0)| < \varepsilon \tag{1}$$

whenever $|x - x_0| < \delta$

Theorem 2.2

A function f is continuous from the right at x_0 if and only if f is defined on an interval $[x_0, b)$ and for each $\varepsilon > 0 \exists \delta > 0$ st (1) holds whenever: $x_0 \le x < x_0 + \delta$

Theorem 2.3

A function f is continuous from the left at x_0 if and only if f is defined on an interval $(a, x_0]$ and for each $\varepsilon > 0 \exists \delta > 0$ st (1) holds whenever: $x_0 - \delta < x \le x_0$

Note that f is continuous if and only if $f(x_0-) = f(x_0+) = f(x_0)$.

Definition 2.4: Continuous on a set

A function f is continuous on an open interval (a, b) if it is continuous at every point in (a, b). If, in addition,

$$f(b-) = f(b) \tag{2}$$

or

$$f(a+) = f(a) \tag{3}$$

then f is continuous on $(a, b]$ or $[a, b)$ respectively. If both are true then f is continuous on $[a, b]$.

More generally, if S is a subset of D_f consisting of finitely or infinitely many disjoint intervals, then f is continuous on S if f is continuous on every interval in S. (From here on, if we say "f is continuous on S" we mean S is a set of this kind.).

2.1 Discontinuities

Definition 2.5: Piecewise Continuity

f is piecewise continuous on $[a, b]$ if

1. $\exists f(x_0+) \forall x_0 \in [a, b)$
2. $\exists f(x_0-) \forall x_0 \in (a, b]$
3. $f(x_0+) = f(x_0-) = f(x_0)$ for all but finitely many points $x_0 \in (a, b)$

If (3) fails to hold at some x_0 in (a, b), f has a jump discontinuity.

Definition 2.6: Removable discontinuity

Let f be defined on a deleted neighborhood of x_0 and be discontinuous (perhaps even undefined) at x_0. We say that f has a removable discontinuity at x_0 if $\lim_{x\to x_0} f(x)$ exists. In this case, the function

$$g(x) = \begin{cases} f(x) & \text{if } x \in D_f \text{ and } x \neq x_0 \\ \lim_{x\to x_0} f(x) & \text{if } x = x_0 \end{cases}$$

is continuous at x_0.

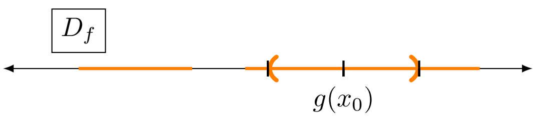
2.2 Continuity Arithmetic

Theorem 2.4

If f and g are continuous on a set S, then so are $f+g$, $f-g$ and fg. So is $\frac{f}{g}$ given $g \neq 0$ at x_0.

Theorem 2.5

Suppose that g is continous at x_0, $g(x_0)$ is an interior point of D_f and f is continuous at $g(x_0)$. Then $f \circ g$ is continuous at x_0.



So the above theorem is saying that we must have some $(g(x_0)-\varepsilon,\, g(x_0)+\varepsilon) \subset D_f$ or even that; $\lim_{x\to x_0} f(g(x)) = f(g(x_0))$.

Proof. Suppose $\varepsilon > 0$, since $g(x_0) \in D_f^o$ and f is continous at $g(x_0), \exists\, \delta_1 > 0$ st, $f(t)$ is defined and

$$|f(t) - f(g(x_0))| < \varepsilon \text{ if } |t - g(x_0)| < \delta_1 \tag{4}$$

Since g is continuous at x_0, $\exists\, \delta_2 > 0$ st, $g(x)$ is defined (why?) and

$$|g(x) - g(x_0)| < \delta_1 \text{ if } |x - x_0| < \delta_2 \tag{5}$$

Then (4) and (5) imply that,

$$|f(g(x)) - f(g(x_0))| < \varepsilon \text{ if } |x - x_0| < \delta_2$$

□

3 Boundedness

Definition 3.1: Bounded Below

A funtion f is bounded below on a set S if theres an $m \in \mathbb{R}$

$$f(x) \geq m \qquad \forall x \in S$$

In this case,

$$V = \{f(x) : x \in S\}$$

has an infimum, α, and we write,

$$\alpha = \inf_{x \in S} f(x)$$

If $\exists x_1 \in S$, such that $f(x_1) = \alpha$, then we say that α is the minimum of f on S and write:

$$\alpha = \min_{x \in S} f(x)$$

Definition 3.2: Bounded Above

f is bounded above on S, if $\exists M \in \mathbb{R}$, such that, $f(x) \leq M \qquad \forall x \in S$. Then we can write;

$$\beta = \sup_{x \in S} f(x)$$

If $\exists x_2 \in S$, such that $f(x_2) = \beta$, then we say that β is the minimum of f on S and write:

$$\beta = \max_{x \in S} f(x)$$

Definition 3.3: Bounded

If f is both bounded below and bounded above on a set S, then f is bounded on S.

Theorem 3.1: Boundedness Theorem

If f is continuous on a finite closed interval $[a, b]$, then f is bounded on $[a, b]$

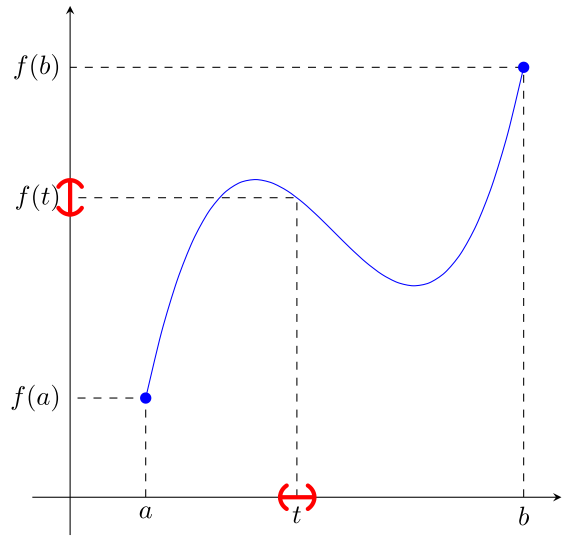


Figure 1: Assume f is bounded, it curves again, I promise...

Proof. Suppose we take a $t \in [a, b]$. Since f is continuous at t $\exists$ an open interval, $t \in I_t$, st,

$$|f(x) - f(t)| < 1 \qquad \text{if } x \in I_t \cap [a, b] \qquad (*)$$

The collection $\mathcal{H} = \{I - t : a \leq t \leq b\}$ is an open cover of $[a, b]$. Since, $[a, b]$ is compact, then by the Heine-Borel theorem, there exists a finite sub-cover made up of intervals $I_{t_1}, \ldots, I_{t_n}$. By $(*)$, taking $t = t_i$, then,

$$|f(x) - f(t_i)| < 1 \qquad \text{if } x \in I_{t_i} \cap [a, b]$$

Therefore,

$$\begin{aligned} |f(x)| &= |f(x) - f(t_i) + f(t_i)| \\ &\leq |f(x) - f(t_i)| + |f(t_i)| \\ &\leq 1 + |f(t_i)| \qquad \text{if } x \in I_{t_i} \cap [a, b] \qquad (**) \end{aligned}$$

Let $M = 1 + \max_{1 \leq i \leq n} |f(t_i)|$ and since,

$[a, b] \subset \bigcup_{i=1}^{n} I_{t_i} \cup [a, b]$, then apply $(**)$ and then

$$|f(x)| \leq M \qquad \forall x \in [a, b]$$

$\square$

Theorem 3.2: Extreme value Theorem

Suppose that f is continuous on a finite closed interval, $[a,b]$. Let,

$$\alpha = \inf_{a\le x\le b} f(x) \text{ and } \beta = \sup_{a\le x\le b} f(x)$$

Then α and β are respectively the minimum and maximum of f on $[a,\,b]$; that is there are points x_1 and x_2 in $[a,\,b]$ such that;

$$f(x_1) = \alpha \qquad f(x_2) = \beta$$

Proof. We'll show that x_1 exists first. Suppose for a contradiction, that there is no point $x_1 \in [a,\,b], f(x_1) = \alpha$. Then for $f(t) > \alpha \quad \forall t \in [a,\,b]$

$$f(t) > \frac{f(t)+\alpha}{2} > \alpha$$

Since, f is continuous at t, there is an open interval I_t about the point t, st,

$$f(x) > \frac{f(t)+\alpha}{2} \qquad x \in I_t \cap [a,\,b]$$

Then, the collection of $\mathcal{H} = \{I_t : a \le x \le b\}$ is an open covering of $[a,\,b]$. Since $[a,\,b]$ is compact, the Heine-Borel theorem implies that there is a finite sub-covering using some open intervals $I_{t_1}, \ldots, I_{t_n}$ around $t_1, \ldots, t_n$. Now we define:

$$\alpha_1 = \min_{1\le i\le n} \frac{f(t_i)+\alpha}{2}$$

Then $f(t) > \alpha\, \forall\, t \in \bigcup_{i=1}^{n} I_{t_i} \cap [a,\,b] = [a,\,b]$, so we now have $a_1 > \alpha$ and hence a contradiction. So $f(x_1) = \alpha$ for some $x_1 \in [a,\,b]$.

To complete the proof, show that x_2 exists. Suppose for a contradiction, that there is no point $x_2 \in [a,\,b], f(x_2) = \beta$. Then for $f(t) < \beta \quad \forall t \in [a,\,b]$

$$f(t) < \frac{f(t)+\beta}{2} < \beta$$

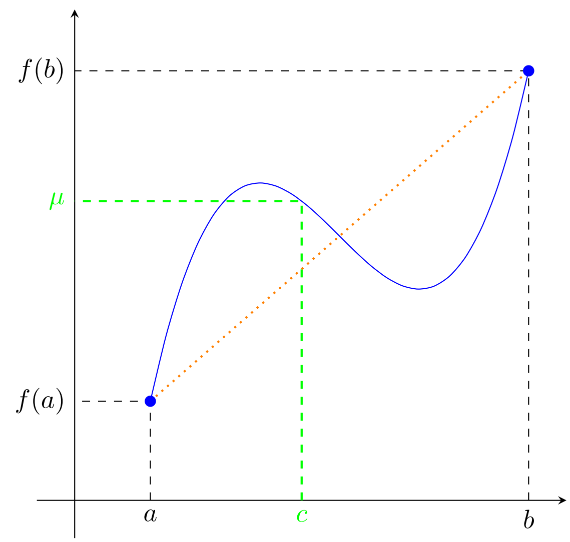
Since, f is continuous at t, there is an open interval I_t about the point t, st,

$$f(x) < \frac{f(t)+\beta}{2} \qquad x \in I_t \cap [a,\,b]$$

Then, the collection of $\mathcal{H} = \{I_t : a \le x \le b\}$ is an open covering of $[a,\,b]$. Since $[a,\,b]$ is compact, the Heine-Borel theorem implies that there is a finite sub-covering using some open intervals $I_{t_1}, \ldots, I_{t_n}$ around $t_1, \ldots, t_n$. Now we define:

$$\beta_1 = \max_{1\le i\le n} \frac{f(t_i)+\beta}{2}$$

Then $f(t) < \beta\, \forall\, t \in \bigcup_{i=1}^{n} I_{t_i} \cap [a,\,b] = [a,\,b]$, so we now have $\beta < \beta_1$ and hence a contradiction. So $f(x_2) = \beta$ for some $x_2 \in [a,\,b]$. ☐



Theorem 3.3: Intermediate Value Theorem

Suppose that f is continuous on $[a,\,b]$, $f(a) \ne f(b)$, and μ is between $f(a)$ and $f(b)$. Then $f(c) = \mu$, for some $c \in [a,\,b]$

Proof. Suppose that $f(a) < \mu < f(b)$. The set,

$$S = \{x : a \le x \le b \text{ and } f(x) \le \mu\}$$

is bounded and is non-empty. Let $c = \sup S$. We will show that $f(c) = \mu$. If $f(c) > \mu$, then $c > a$ and since f is continuous at c, $\exists\, \varepsilon > 0$,st,

$$f(x) > \mu \qquad \text{if } c - \varepsilon < x \le c$$

Therefore, $c - \varepsilon$ is an upper bound for S, contradicting the definition of c.
If $f(c) < \mu$, then $c < b$ and $\exists\, \varepsilon > 0$, st,

$$f(x) < \mu \text{ for } c \le x < c + \varepsilon$$

so c is not an upper bound for S, which again contradicts the definition of c.
Therefore $f(c) = \mu$. The proof for $f(b) < \mu < f(a)$ is simply obtained by applying the above to the function $-f$. $\square$

3.1 Monotonics 2: God what a mess