ECM1905 - Advanced Calculus

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Part I
Integration

1 Definite Integrals

1.1 Riemann Sums

Let $f(x)$ be continuous for $a \leq x \leq b$. Divide the interval $[a, b]$ into a partition $\mathcal{P}_n$ of subintervals of equal width $\Delta x = (b - a)/n$. So

$$\mathcal{P}_n : a = x_0, x_1 = a + \Delta, x_2 = a + 2\Delta, \dots, x_n = b$$

Then, the definite integral of f from a to b is defined by any one of the following limits:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j)\Delta x, \quad (\text{right-hand sum});$$

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{j=0}^{n-1} f(x_j)\Delta x, \quad (\text{left-hand sum});$$

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{j=1}^{n} f(c_j)\Delta x, \qquad x_{j-1} \leq c_j \leq x_j\,.$$

Note that the partition $\mathcal{P}_n$ gets finer as $n \to \infty$.

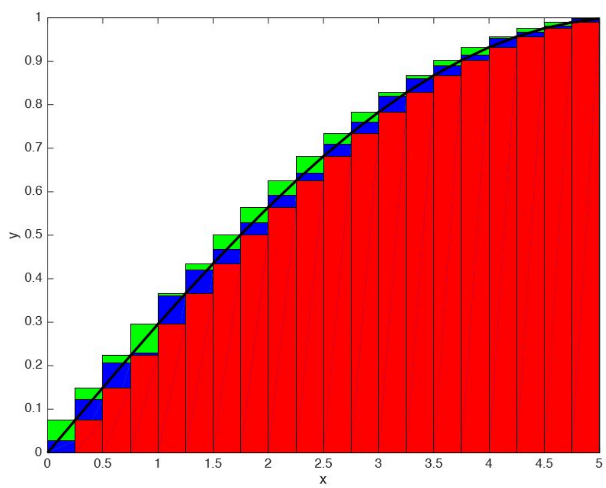


Figure 1: Riemann Sum Diagram on an arbitrary graph

- $f(x)$ is the integrand, a the lower limit, b the upper limit.
- Each finite sum is called a Riemann or partial sum.
- The definite integral can be interpreted as the "area under the curve" $y = f(x)$ from a to b (0 to 5 in the example above)

Claim:
$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{j=1}^{n} f(x_j)\Delta x = \lim_{n\to\infty}\sum_{j=0}^{n-1} f(x_j)\Delta x\,.$$

Proof. Recall from Calculus (ECM1901) that a sequence $\{s_n\}$ has a limit s, i.e.

$$s_n \to s \text{ as } n \to \infty\,,$$

if for each $\epsilon > 0$ we can find N so that if $n \geq N$, then $|s_n - s| < \epsilon$.

We will use notion of convergent sequence with $s = \int_a^b f(x)dx$ and

$$\text{either } s_n = \sum_{j=0}^{n-1} f(x_j)\Delta x \text{ or } s_n = \sum_{j=1}^{n} f(x_j)\Delta x\,.$$

Assume that

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{j=0}^{n-1} f(x_j)\Delta x\,.$$

Now

$$\int_a^b f(x)dx - \sum_{j=1}^{n} f(x_j)\Delta x = \int_a^b f(x)dx - \sum_{j=0}^{n-1} f(x_j)\Delta x - (\Delta x)\times(f(b)-f(a))\,.$$

Chose $\epsilon > 0$. By definition, there exists N_1 so that if $n \geq N_1$, then

$$\left|\int_a^b f(x)dx - \sum_{j=0}^{n-1} f(x_j)\Delta x\right| < \frac{\epsilon}{2}\,. \tag{1}$$

We can also choose N_2 (bigger than N_1 if needed) so that if $n \geq N_2$, then

$$(\Delta x)\times(f(b)-f(a)) = \frac{b-a}{n}\times(f(b)-f(a)) < \frac{\epsilon}{2}\,. \tag{2}$$

If now follows that for the given given $\epsilon > 0$, we can, using (1) and (2), find N so that if $n \geq N$ then

$$\left|\int_a^b f(x)dx - \sum_{j=1}^{n} f(x_j)\Delta x\right| < \epsilon \text{ and } \lim_{n\to\infty}\sum_{j=1}^{n} f(x_j)\Delta x = \int_a^b f(x)dx\,.$$

□

Example I1. Using the definition to evaluate a definite integral.
Let

$$f(x) = x^3 - 6x, a = 0, b = 3$$

Then we know (do we?) that

$$\begin{aligned} I := \int_0^3 (x^3 - 6x)dx &= \left[x^4/4 - 3x^2\right]_0^3 \\ &= 81/4 - 27 = (81-108)/4 = -27/4\,. \end{aligned}$$

Now let us compute I by Riemann sums. First we set up the partition of the interval $[0,3]$ so that

$$\Delta x = (b-a)/n = 3/n,\ x_0 = 0, x_1 = 3/n, x_2 = 6/n, ..., x_j = 3j/n,$$

Then from first principles:

$$I = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j)\Delta x = \lim_{n\to\infty} \sum_{n=1}^{n} f(\frac{3j}{n})\frac{3}{n}$$

$$\begin{aligned} I = & \lim_{n\to\infty} \sum_{j=1}^{n} \left[\left(\frac{3j}{n}\right)^3 - 6\left(\frac{3j}{n}\right)\right] \frac{3}{n} \\ = & \lim_{n\to\infty} \sum_{j=1}^{n} \left[\frac{27j^3}{n^3} - \frac{18j}{n}\right] \frac{3}{n} \\ = & \lim_{n\to\infty} \left[\frac{81}{n^4} \sum_{j=1}^{n} j^3 - \frac{54}{n^2} \sum_{j=1}^{n} j \right]. \end{aligned}$$

But

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}, \ \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}, \sum_{j=1}^{n} j^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

Hence

$$\begin{aligned} I &= \lim_{n\to\infty} \left[\frac{81}{n^4} \left(\frac{n(n+1)}{2}\right)^2 - \frac{54}{n^2}\frac{n(n+1)}{2} \right] \\ &= \lim_{n\to\infty} \left[\frac{81}{4}\left(1+\frac{1}{n}\right)^2 - 27\left(1+\frac{1}{n}\right)\right] \\ &= \frac{81}{4} - 27 = -\frac{27}{4} \end{aligned}$$

Note that we do not need to use a partition

$$\mathcal{P}_n : x_0 = a, x_1 = a + \Delta x, \ldots, x_{n-1}, x_n = a + n\Delta x = b$$

with uniform separation

$$\Delta x = \frac{b-a}{n},$$

and any partition $\mathcal{P}_n : x_0^n = a, x_1^n, \ldots, x_{n-1}^n, x_n^n = b$ of $[a, b]$ would be allowed.

What is crucial is that the partition $\mathcal{P}_n$ becomes finer as $n \to \infty$.

In this case, we can alternatively say that

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^n)(x_i^n - x_{i-1}^n)$$

where in the limit as $n \to \infty$ we assume a partition $\mathcal{P}_n$ that gets finer with increasing n.

- Note that in approximating the integral from a to b by a finite sum obtained from (increasingly finer) partitions of $[a, b]$, continuity of the function $f(x)$ plays a crucial role.
- If we drop continuity of $f(x)$ then the process collapses and bizarre things can happen.
- By way of illustration of how bizarre it gets — consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

This function is not continuous at any $x \in [0,1]$. Convince your self that for any given $A \in (0,1)$, there exists a sequence of partitions

$$\mathcal{P}_n : 0 = x_0^n, x_1^n, \ldots, x_{n-1}^n, x_n^n = 1, \qquad n = 1, 2, 3, \ldots$$

so that

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_{k-1}^n) \left(x_k^n - x_{k-1}^n\right) = A\,.$$

1.2 Properties of the definite integral

$$\begin{aligned}
&(1) \quad \int_a^b f(x)dx = -\int_b^a f(x)dx \\
&(2) \quad \int_a^a f(x)dx = 0 \\
&(3) \quad \int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx \\
&(4) \quad \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,\ a < c < b \\
&(5) \quad \int_a^b f(x)dx \geq 0, \quad \text{if}\ \ f(x) > 0,\ a \leq x \leq b \\
&(6) \quad \int_{-a}^a f(x)dx = 2\int_0^a f(x)dx, \quad \text{if}\ \ f(x) = f(-x) \\
&(7) \quad \int_{-a}^a f(x)dx = 0, \quad \text{if}\ \ f(x) = -f(x)
\end{aligned}$$

1.3 Some proofs

We can, without loss of generality, work with right-hand Riemann sum and partitions

$$a = x_0, \ldots, x_n = b,\ \text{and}\ \Delta = \frac{b-a}{n}$$

(1) **Claim:**

$$\int_a^b f(x)dx = -\int_b^a f(x)dx\,.$$

Proof.

$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_1^n f(x_i) = -\lim_{n\to\infty} \frac{a-b}{n} \sum_n^1 f(x_i) = -\int_b^a f(x)dx\,.$$

□

(3) **Claim:**

$$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

Proof. Using the algebra of limits we have that:

$$\begin{aligned}\int_a^b [f(x)+g(x)]dx &= \lim_{n\to\infty} \tfrac{b-a}{n} \textstyle\sum_1^n (f(x_i)+g(x_i)) \\ &= \lim_{n\to\infty} \frac{b-a}{n} \sum_1^n f(x_i) + \lim_{n\to\infty} \frac{b-a}{n} \sum_1^n g(x_i) \\ &= \int_a^b f(x)dx + \int_a^b g(x)dx\end{aligned}$$

□

(4) **Claim:**

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \ a < c < b$$

Proof. If

$$\mathcal{P}_n : x_0 = a, x_1, \ldots, x_n = c$$

is a partition of $[a, c]$ and

$$\mathcal{Q}_n : z_0 = c, z_1, \ldots, z_n = b$$

is a partition of $[c, b]$ then

$$\begin{aligned}\mathcal{R}_n &= \mathcal{P}_n \cup \mathcal{Q}_n \\ &: y_0 = x_0 = a, \ldots, y_n = x_n = c, y_{n+1} = z_1, \ldots, y_{2n} = b\end{aligned}$$

is a partition of $[a, b]$. Then

$$\begin{aligned}&\int_a^c f(x)dx + \int_c^b f(x)dx \\ &= \lim_{n\to\infty} \sum_1^n f(x_i)(x_i - x_{i-1}) + \lim_{n\to\infty} \sum_1^n f(z_i)(z_i - z_{i-1}) \\ &= \lim_{n\to\infty} \sum_1^{2n} f(y_i)(y_i - y_{i-1}) = \int_a^b f(x)dx\,.\end{aligned}$$

□

2 The Fundamental Theorem of Calculus

Theorem 2.1 (FTC)**.** *Suppose f is continuous on $[a, b]$*

$$\begin{aligned}(1) &\quad \textit{If } g(x) = \int_a^x f(t)\,dt, \quad \textit{then } \frac{dg}{dx} = f(x); \quad \textit{or} \\ &\quad \frac{d}{dx}\int_a^x f(t)dt = f(x) \\ (2) &\quad \int_a^a f(x)dx = F(b) - F(a),\end{aligned}$$

where F is any anti-derivative of f, i.e., $dF/dx = f$. Also

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)dt = \frac{d}{dx}\left(F(b(x)) - F(a(x))\right)$$
$$= F'(b(x))b'(x) - F'(a(x))a'(x) = f(b(x))b'(x) - f(a(x))a'(x).$$

The Total Change Theorem:

Theorem 2.2 (Total Change Theorem)**.**

$$\int_a^b F'(x)dx = F(b) - F(a).$$

Some examples of differentiating an integral:

$$\begin{aligned}\frac{d}{dx}\int_1^{x^3} y^2 dy \quad &= (x^3)^2(3x^2) - (1)^2(0) \\ &= 3x^8\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}\int_{-\cos x}^{\sin x} y^3 dy \quad &= (\sin x)^3(\cos x) - (-\cos x)^3(+\sin x) \\ &= (\sin x\cos x)(\sin^2 x + \cos^2 x) = \sin x\cos x\,.\end{aligned}$$

2.1 Gappy Proof of FTC

Proof. We assume that

$$f(x) = F'(x) \text{ for some function } F(x)$$

We know, from the Mean Value Theorem (for derivatives) that for any α and β, $\beta > \alpha$:

$$\frac{F(\beta) - F(\alpha)}{\beta - \alpha} = F'(c),\ \text{ some } c \in [\alpha, \beta]$$

Then, from the Riemann definition of the integral via partial sums:

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{j=1}^{n} f(c_j)\Delta x = \lim_{n\to\infty}\sum_{j=1}^{n} F'(c_j)\Delta x,\ \ x_{j-1} \le c_j \le x_j\,.$$

Choosing c_j according to the MVT with $\alpha = x_{j-1}$ and $\beta = x_j = x_{j-1} + \Delta x$ gives:

$$\int_a^b f(x)dx = \lim_{n\to\infty}\overbrace{\sum_{j=1}^{n}\frac{F(x_j) - F(x_{j-1})}{\Delta x}}^{\text{telescopic sum}}\Delta x = F(b) - F(a)$$

□

2.2 The Fundamental Theorem and Integration - an example

Example I2. Evaluate the definite integral

$$I = \int_0^3 f(x)dx = \int_0^3 (x^3 - 6x)dx,$$

First observe that:

$$\frac{d}{dx}\left(\frac{1}{4}x^4 - 3x^2\right) = x^3 - 6x = f(x).$$

So an anti-derivative of $f(x)$ is $F(x) = \frac{1}{4}x^4 - 3x^2$.

The fundamental Theorem of Calculus then gives

$$\begin{aligned} I &= \int_0^3 f(x)dx = F(3) - F(0) \\ &= \tfrac{1}{4}3^4 - 3\cdot 3^2 = \tfrac{3}{4}27 - 27 = -\tfrac{27}{4} \end{aligned}$$

3 Indefinite integrals

An anti-derivative of f, $\int f(x)dx = F(x)$ is called an indefinite integral, i.e.,

$$\int f(x)dx = F(x) \quad \text{means} \quad \frac{dF(x)}{dx} = f(x).$$

Integration is the inverse of differentiation.

- A definite integral is a number
- an indefinite integral is a function.

4 Methods of integration

4.1 Basic integrals obtained from differentiating well known functions

$$\begin{array}{ll}
(1) & \dfrac{d}{dx}(\sin x) = \cos x \;\rightarrow\; \displaystyle\int \cos x = \sin x + C \\
(2) & \dfrac{d}{dx}(\cos x) = -\sin x \;\rightarrow\; \displaystyle\int \sin x = -\cos x + C \\
(3) & \dfrac{d}{dx}(\tan x) = \dfrac{1}{\cos^2 x} \;\rightarrow\; \displaystyle\int \frac{1}{\cos^2 x}dx = \tan x + C \\
 & (\sec x = 1/\cos x) \\
(4) & \dfrac{d}{dx}(\cot x) = -\dfrac{1}{\sin^2 x} \;\rightarrow\; \displaystyle\int \frac{1}{\sin^2 x} = -\cot x + C \\
 & (\operatorname{cosec} x = 1/\sin x)
\end{array}$$

$$(5)\quad \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n \;\to\; \int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$
$$(6)\quad \frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \;\to\; \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$
$$(7)\quad \frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2} \;\to\; \int \frac{dx}{1+x^2} = \tan^{-1} x + C$$
$$(8)\quad \frac{d}{dx}(e^x) = e^x \;\to\; \int e^x = e^x + C$$
$$(9)\quad \frac{d}{dx}(\ln|x|) = \frac{1}{x} \;\to\; \int \left(\frac{1}{x}\right) dx = \ln|x| + C$$
$$(10)\quad \frac{d}{dx}(a^x) = a^x \ln a \;\to\; \int (a^x)\,dx = \frac{a^x}{\ln a} + C$$
$$(11)\quad \frac{d}{dx}(\sinh x) = \cosh x \;\to\; \int (\cosh x)\,dx = \sinh x + C,$$
$$\sinh x = (e^x - e^{-x})/2$$
$$(12)\quad \frac{d}{dx}(\cosh x) = \sinh x \;\to\; \int (\sinh x)\,dx = \cosh x + C,$$
$$\cosh x = (e^x + e^{-x})/2$$

4.2 Integration by substitution

If $u = u(x)$ is a differentiable function and f is continuous, then

$$\int f(u(x))\,(u'(x)dx) = \int f(u)du \quad \text{indefinite integral}$$

$$\int_a^b f(u(x))\,(u'(x)dx) = \int_{u(a)}^{u(b)} f(u)du \quad \text{definite integral.}$$

Here du **acts as if it is a differential.**

$$du = u'dx,\;\; d(c+x) = dx;\;\; d(x^2) = 2xdx,\;\; d(\sin x) = \cos x dx, ...,$$

Example I3. Evaluate the indefinite integral

$$I = \int 2x\sqrt{1+x^2}dx.$$

Introduce $u = 1 + x^2$, then $du = u'(x)dx = 2x\;dx$ and we can write

$$I = \int \sqrt{(1+x^2)}(2xdx) = \int u^{1/2}du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+x^2)^{3/2} + C.$$

Exercise I4. Evaluate the indefinite integral

$$I = \int x^3 \cos(2 + x^4) dx.$$

5 Methods of integration

5.1 Integrals of trigonometric and hyperbolic functions

Here we make extensive use of double-angle formulas, addition/subtraction formulas and trigonometric substitution:

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y, \quad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$
$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$
$$\sin 2x = 2 \sin x \cos x$$
$$\sin x \cos y = \frac{1}{2}\left[\sin(x-y) + \sin(x+y)\right]$$
$$\sin x \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$
$$\cos x \cos y = \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right]$$
$$\cosh^2 x - \sinh^2 x = 1.$$

Example I5. Evaluate the definite integral

$$\begin{aligned} I &= \int_0^\pi \sin^2 x dx \\ &= \int_0^\pi \frac{1}{2}\left(1 - \cos 2x\right) dx = \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^\pi = \frac{\pi}{2} \end{aligned}$$

Example I6. Evaluate the indefinite integral

$$\begin{aligned} I &= \int (\sin^2 x)^2 dx = \int \frac{1}{4}\left(1 - \cos 2x\right)^2 dx \\ &= \int \frac{1}{4}\left(1 - 2\cos 2x + \cos^2 2x\right) dx \\ &= \int \frac{1}{4}\left[1 - 2\cos 2x + \frac{1}{2}\left(1 - \cos 4x\right)\right] dx \\ &= \frac{1}{4}\left[\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right] + C \end{aligned}$$

Exercise I7. Evaluate the indefinite integral

$$I = \int (\sin 4x \cos 5x) dx$$

Example I8. Evaluate the indefinite integral

$$\begin{aligned} I &= \int \frac{\sqrt{9-x^2}}{x^2} dx \ \ [x = 3\sin\theta, \ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}] \\ &= \int \frac{\sqrt{9-(3\sin\theta)^2}}{(3\sin\theta)^2} d(3\sin\theta) \\ &= \int \frac{3\cos\theta}{9\sin^2\theta}(3\cos\theta) = \int \frac{\cos^2\theta}{\sin^2\theta} d\theta \\ &= \int \frac{1-\sin^2\theta}{\sin^2\theta} d\theta = \int \left(\frac{1}{\sin^2\theta} - 1\right) d\theta = -\cot\theta - \theta + C \end{aligned}$$

Exercise I7. Evaluate the indefinite integral

$$I = \int (\sin 4x \cos 5x) dx =$$

Exercise I8. Evaluate the indefinite integral

$$I = \int \frac{\sqrt{9-x^2}}{x^2} dx$$

Exercise I9. Evaluate the indefinite integral

$$I \quad = \int \frac{dx}{\sqrt{x^2 - a^2}} \quad [x = a \cosh t]$$

5.2 Integrals of rational functions using partial fractions

Let

$$d(x) = (x - a_1)(x - a_2)\dots(x - a_{n-1})(x - a_n), \qquad a_i \text{ distinct}$$

If $n(x)$ is a polynomial of degree n or less, then

$$\frac{n(x)}{d(x)} = \sum_{i=1}^{n} \left(\frac{n(a_i)}{\prod_{j \neq i}(x - a_j)} \right) \frac{1}{x - a_i}$$

Now we know that

$$\frac{d}{dx} \ln(x - a) = \frac{1}{x - a}$$

By using this fact repeatedly, we can then integrate any rational function of the form

$$\frac{n(x)}{d(x)}$$

above - but the answer depends on the details (because $\ln z$ is only defined for positive z).

Example I10. Evaluate the indefinite integral

$$
\begin{aligned}
I &= \int \frac{2x^2 - x + 4}{x(x^2+4)} dx \\
&\left[\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx + C}{(x^2+4)}, A = 1, B = 1, C = -1\right] \\
I &= \int \left[\frac{1}{x} + \frac{x-1}{x^2+4}\right] \\
&= \int \frac{dx}{x} + \int \frac{x}{x^2+4} dx - \int \frac{1}{x^2+4} dx \\
&= \ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\tan^{-1}(x/2) + C
\end{aligned}
$$

Exercise I11. Evaluate the indefinite integral (the degree of the numerator is not less than the degree of the denominator)

$$
I = \int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3} dx
$$

5.3 Integration by parts

According to the product rule from Calculus: If $u(x)$ and $v(x)$ are differentiable functions, then

$$
\frac{d(u(x)v(x))}{dx} = u(x)v'(x) + v(x)u'(x).
$$

From the Fundamental Theorem of Calculus it follows that

$$
\int \frac{d}{dx}\left(u(x)v(x)\right) dx = u(x)v(x) = \int \left[u(x)v'(x) + v(x)u'(x)\right] dx,
$$

or

$$
u(x)v(x) = \int u(x)v'(x)dx + \int v(x)u'(x)dx.
$$

Rearranging gives:

$$
\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx.
$$

So given

$$
\int f(x)g(x)dx
$$

we look to set $f(x) = u(x)$ and $g(x) = v'(x)$.

Example I12. Evaluate the indefinite integral

$$I = \int \ln x \; dx = \int \ln x \; 1 \, dx,$$

where we take

$$u = \ln x, \quad v = x \implies v'(x) = 1$$

$$\begin{aligned} I \quad &= uv - \int vu' dx = x \ln x - \int x \frac{1}{x} dx \\ &= x \ln x - \int 1 \, dx \\ &= x \ln x - x + C \end{aligned}$$

Example I13. Evaluate the indefinite integral

$$I = \int x^2 e^x \; dx,$$

where we take

$$u = x^2, \quad v = e^x, \implies v'(x) = e^x.$$

$$I \quad = uv - \int vu' dx = x^2 e^x - \int e^x 2x \, dx = x^2 e^x - 2 \int e^x \, x \, dx$$

We need to use integration by parts again by taking

$$u = x, \quad v = e^x \implies v'(x) = e^x$$

$$\begin{aligned} I \quad &= x^2 e^x - \left(2 \int x e^x \; dx \right) = x^2 e^x - \left(2x e^x - 2 \int e^x \; dx \right) \\ &= x^2 e^x - (2x e^x - 2e^x + C) \end{aligned}$$

Class Example I14. Evaluate the indefinite integral

$$I = \int e^x \sin x \; dx,$$

where we take $u = e^x, \quad v = -\cos x, \quad v' = \sin x$

6 Improper Integrals:

The integral is called an improper integral when

- the interval is infinite or
- $f(x)$ has an infinite discontinuity in $[a, b]$.

6.1 Improper Integrals - Type I

If $\int_a^t f(x)dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx,$$

provided this limit exists (convergent).

Example I15. Evaluate the improper integral

$$I = \int_{-\infty}^{0} xe^x \, dx$$

i.e.,

$$\begin{aligned} I &= \int_{-\infty}^{0} xe^x \, dx = \lim_{t\to-\infty} \int_t^0 xe^x \, dx \\ &= \lim_{t\to-\infty} \left[\int_t^0 x de^x \right] = \lim_{t\to-\infty} \left[xe^x - \int_t^0 e^x \, dx \right] \\ &= \lim_{t\to-\infty} \left[xe^x - e^x \right]_t^0 = \lim_{t\to-\infty} \left[te^t - 1 + e^t \right] = -1 \end{aligned}$$

Example I16. Evaluate the improper integral

$$I = \int_0^\infty \frac{1}{1+x^2} \, dx$$

i.e.,

$$\begin{aligned} I &= \int_0^\infty \frac{1}{1+x^2} \, dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2} \, dx \\ &= \lim_{t\to\infty} \left[\tan^{-1} x \right]_0^t \\ &= \lim_{t\to\infty} \left[\tan^{-1} t - \tan^{-1} 0 \right] \\ &= \pi/2 \end{aligned}$$

6.2 Improper Integrals - Type II

If $f(x)$ is continuous on $(a, b]$, and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx,$$

provided this limit exists (convergent).

Example I17. Evaluate the improper integral

$$I = \int_2^5 \frac{1}{\sqrt{x-2}}\, dx$$

i.e.,

$$\begin{aligned} I &= \int_2^5 \frac{1}{\sqrt{x-2}}\, dx = \lim_{t\to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\ dx \\ &= \lim_{t\to 2^+} \int_t^5 \frac{d(x-2)}{\sqrt{x-2}} \\ &= \lim_{t\to 2^+} \left[2(x-2)^{1/2}\right]_t^5 = \lim_{t\to 2^+} \left[2\sqrt{3} - 2\sqrt{t-2}\right] = 2\sqrt{3} \end{aligned}$$

Exercise I18. Evaluate the improper integral

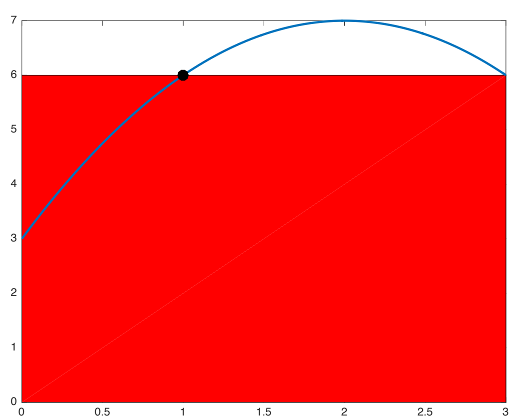
$$I = \int_0^1 \ln x\, dx$$

6.3 The Mean Value Theorem for integration

If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$\int_a^b f(x)dx = f(c)(b-a).$$

Figure 2: Mean Value Theorem with Integrals



7 Integration: General Examples

An example: Find dy/dx from

$$\int_0^y e^{t^2} dt + \int_0^{\sin x} \cos^2 t dt = 0$$

Note that

$$\frac{d}{dx} \int_0^{b(x)} f(t)dt = b'(x)f(b(x)).$$

Differentiate with x on the both side,

$$\frac{dy}{dx} e^{y^2} + \cos x \cos^2(\sin x) = 0,$$

i.e.

$$dy/dx = -e^{-y^2} \cos x \cos^2(\sin x).$$

An example: Evaluate the improper integral

$$I = \int_0^a x^2 \sqrt{a^2 - x^2}\, dx$$

i.e.,

$$\begin{aligned}
I &= \int_0^{\pi/2} (a \sin u)^2 (a \cos x)(a \cos u) du \quad (x = a \sin u) \\
&= \int_0^{\pi/2} a^4 \sin^2 u \cos^2 u \; du \\
&= \int_0^{\pi/2} \frac{a^4}{8} \sin^2(2u)\; d(2u) \quad (t = 2u) \\
&= \frac{a^4}{8} \int_0^{\pi} \sin^2 t \; dt \\
&= \frac{a^4}{8} \int_0^{\pi} (1 - \cos 2t)/2 \; dt = \frac{\pi a^4}{8 \times 2} = \frac{\pi a^4}{16}
\end{aligned}$$

Example: Evaluate the sum by associating it with a definite integral

$$I = \lim_{n \to \infty} \left[\frac{1^p + 2^p + 3^p + ... + n^p}{n^{p+1}} \right]$$

$$\begin{aligned}
I &= \lim_{n \to \infty} \frac{1}{n} \left[\frac{1^p}{n^p} + \frac{2^p}{n^p} + \frac{3^p}{n^p} + ... + \frac{n^p}{n^p} \right] \\
&= \lim_{n \to \infty} \left[\sum_{j=1}^{n} \left(\frac{j}{n} \right)^p \right] \frac{1}{n} \\
&= \int_0^1 x^p \; dx = \frac{1}{1+p}
\end{aligned}$$

Example: Evaluate the sum by associating it with a definite integral

$$I = \lim_{n\to\infty}\left[\frac{1}{\sqrt{n(n+1)}} + \frac{1}{\sqrt{n(n+2)}} + \frac{1}{\sqrt{n(n+3)}} + ... + \frac{1}{\sqrt{n(n+3n)}}\right]$$

$$\begin{aligned}
I &= \lim_{n\to\infty}\left[\frac{1}{\sqrt{1+1/n}} + \frac{1}{\sqrt{1+(2/n)}} + \frac{1}{\sqrt{1+(3/n)}} + ... + \frac{1}{\sqrt{1+(3n/n)}}\right]\frac{1}{n} \\
&= \lim_{n\to\infty}\sum_{j=1}^{3n}\left[\frac{1}{\sqrt{1+(j/n)}}\right]\frac{1}{n} \\
&= \lim_{n\to\infty}\sum_{j=1}^{3n} f\left(j/n\right)\frac{3}{3n}, \quad f(x) = \frac{1}{\sqrt{1+x}} \\
&= \int_0^3 \frac{1}{\sqrt{1+x}}dx, \quad \left[b = 3, a = 0, \Delta x = \frac{3}{3n}, x_j = \frac{j}{3n},\right] \\
&= \left[2\sqrt{1+x}\right]_0^3 dx = 2
\end{aligned}$$

Example: Show that

$$\int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\,dt = \frac{\pi}{4}, \quad 0 < x < \frac{\pi}{2}$$

Let

$$f(x) = \int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\,dt = \frac{\pi}{4}, \quad 0 < x < \frac{\pi}{2}$$

$$\begin{aligned}
\tfrac{df}{dx} &= \sin^{-1}\sqrt{\sin^2 x}\,(\sin^2 x)' - \cos^{-1}\sqrt{\cos^2 x}\,(\cos^2 x)' \\
&= x\sin 2x - x\sin 2x = 0
\end{aligned}$$

$$f(x) = C, \quad 0 < x < \frac{\pi}{2}.$$

$$\begin{aligned}
f(\pi/4) &= \int_0^{1/2} \sin^{-1}\sqrt{t}\,dt + \int_0^{1/2} \cos^{-1}\sqrt{t}\,dt \\
&= \left[t\sin^{-1}\sqrt{t}\,dt + \cos^{-1}\sqrt{t}\right]_0^{1/2} - \int_0^{1/2} t\left[\frac{1}{\sqrt{1-t}} - \frac{1}{\sqrt{1-t}}\right]dt \\
&= (1/2)\sin^{-1}\frac{1}{\sqrt{2}} + (1/2)\cos^{-1}\frac{1}{\sqrt{2}} = (1/2)\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \frac{\pi}{4}
\end{aligned}$$

Example: Evaluate the improper integral

$$I_n = \int_0^\infty x^n e^{-x}\ dx$$

$$\begin{aligned} I_n &= -\int_0^\infty x^n de^{-x} \\ &= -\left[x^n e^{-x}\right]_0^\infty + n\int_0^\infty x^{n-1}e^{-x}\ dx \\ &= 0 + nI_{n-1} \end{aligned}$$

i.e.,

$$I_n = nI_{n-1} = n(n-1)I_{n-2} = n(n-1)(n-2)I_{n-3} = ... = n!$$

Example: Evaluate the indefinite integral

$$\int \frac{xe^x}{\sqrt{e^x-1}}dx$$

$$\begin{aligned} I &= \int \frac{xe^x}{\sqrt{e^x-1}}\ dx \\ &= \int 2xd\sqrt{e^x-1} \\ &= 2x\sqrt{e^x-1} - 2\int \sqrt{e^x-1}\ dx \quad \left[t=\sqrt{e^x-1},\ x=\ln(t^2+1)\ dx=\frac{2t\ dt}{1+t^2}\right] \\ &\int \sqrt{e^x-1}\ dx = \int \frac{2t^2\ dt}{1+t^2} = \int \frac{2(t^2+1)-2\ dt}{1+t^2} = 2t - 2\tan^{-1}t + C \\ I &= 2x\sqrt{e^x-1} - 4\left(\sqrt{e^x-1} - \tan^{-1}\sqrt{e^x-1}\right) + C \end{aligned}$$

Example: Evaluate the indefinite integral

$$\int \frac{\sin x}{a\sin x + b\cos x}dx, \quad a \neq 0, b \neq 0;;$$

Let

$$T_s = \int \frac{\sin x}{a\sin x + b\cos x}dx, \quad T_c = \int \frac{\cos x}{a\sin x + b\cos x}dx.$$

Hence

$$\begin{aligned} aT_s + bT_c &= x + C_1 \\ aT_c - bT_s &= \int \frac{a\cos x - b\sin x}{a\sin x + b\cos x}dx \\ &= \int \frac{d(a\sin x + b\cos x)}{a\sin x + b\cos x}dx \\ &= \ln|a\sin x + b\cos x| + C_2 \end{aligned} \tag{3}$$

Solve the above two equations, we obtain

$$\begin{aligned} T_s &= \frac{1}{a^2+b^2}\left(ax - b\ln|a\sin x + b\cos x| + C\right) \\ T_c &= \frac{1}{a^2+b^2}\left(bx + a\ln|a\sin x + b\cos x| + C\right) \end{aligned}$$

Example: Evaluate the improper integral

$$I = \int_0^2 \frac{dx}{\sqrt{x(2-x)}}.$$

$$\begin{aligned}
I &= \int_0^1 \frac{dx}{\sqrt{x(2-x)}} + \int_1^2 \frac{dx}{\sqrt{x(2-x)}} \\
&= \lim_{\epsilon_1 \to 0} \int_{\epsilon_1}^1 \frac{dx}{\sqrt{x(2-x)}} + \lim_{\epsilon_2 \to 0} \int_1^{2-\epsilon_2} \frac{dx}{\sqrt{x(2-x)}} \\
&= \lim_{\epsilon_1 \to 0} 2 \int_{\epsilon_1}^1 \frac{d\sqrt{x}}{\sqrt{(2-x)}} + \lim_{\epsilon_2 \to 0} 2 \int_1^{2-\epsilon_2} \frac{d\sqrt{x}}{\sqrt{(2-x)}} \\
&= \lim_{\epsilon_1 \to 0} 2 \int_{\epsilon_1}^1 \frac{d\sqrt{x/2}}{\sqrt{(1-x/2)}} + \lim_{\epsilon_2 \to 0} 2 \int_1^{2-\epsilon_2} \frac{d\sqrt{x/2}}{\sqrt{(1-x/2)}} \\
&= \lim_{\epsilon_1 \to 0} 2 \sin^{-1} \left[\frac{\sqrt{x}}{\sqrt{2}}\right]_{\epsilon_1}^1 + \lim_{\epsilon_2 \to 0} 2 \sin^{-1} \left[\frac{\sqrt{x}}{\sqrt{2}}\right]_1^{2-\epsilon_2} \\
&= \lim_{\epsilon_1 \to 0} 2 \left[\frac{\pi}{4} - \sin^{-1} \sqrt{\epsilon_1/2}\right] + \lim_{\epsilon_2 \to 0} 2 \left[\sin^{-1} \sqrt{(2-\epsilon_2)/2} - \frac{\pi}{4}\right] \\
&= \frac{\pi}{2} + 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \pi
\end{aligned}$$

Example: Evaluate the improper integral

$$I = \int_{1/2}^{3/2} \frac{dx}{\sqrt{|x-x^2|}}.$$

$$\begin{aligned}
I &= \int_{1/2}^1 \frac{dx}{\sqrt{x-x^2}} + \int_1^{3/2} \frac{dx}{\sqrt{x^2-x}} = I_1 + I_2 \\
I_1 &= \int_{1/2}^1 \frac{d(2x)}{\sqrt{1-(4x^2-4x+1)}} = \int_{1/2}^1 \frac{d(2x-1)}{\sqrt{1-(2x-1)^2}} = \sin^{-1}[2x-1]_{1/2}^1 = \sin^{-1} 1 = \pi/2 \\
I_2 &= \int_1^{3/2} \frac{dx}{x^2-x} = \int_1^{3/2} \frac{d(2x-1)}{\sqrt{(4x^2-4x+1)-1}} \\
&= \cosh^{-1} [2x-1)]_1^{3/2}; \quad [\cosh^{-1} x = y, e^{2y} - 2xe^y + 1 = 0, y = \ln(x + \sqrt{x^2-1})] \\
&= \left[\ln((2x-1) + \sqrt{(2x-1)^2-1})\right]_1^{3/2} = \ln(2+\sqrt{3}) - \ln 1 = \ln(2+\sqrt{3}).
\end{aligned}$$

Hence

$$I_1 + I_2 = \pi/2 + \ln(2+\sqrt{3}).$$

8 Integral Test for Infinite Series

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}} \qquad \sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

are all examples of infinite sums

$$\sum_{n=1}^{\infty} a_n$$

where the ratio test fails because

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$$

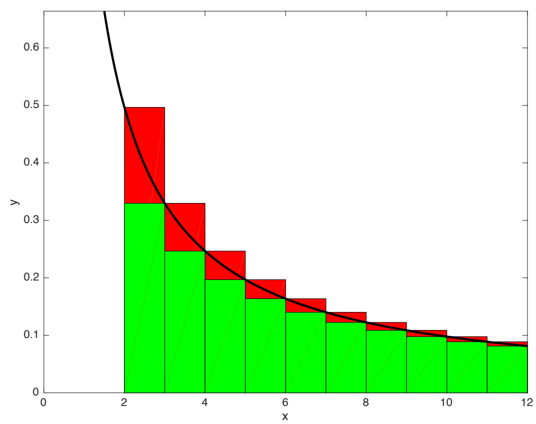


Figure 3: Use "green rectangles" for convergent sums from finite integrals; Use "red rectangles" for divergent sums from infinite integrals

Let $f(x)$ be non-increasing and non-negative on an unbounded interval $[N, \infty)$.

$$\text{(i)} \int_N^{\infty} f(x)dx < \infty \qquad \Longrightarrow \qquad \sum_N^{\infty} f(n) < \infty$$

$$\text{(ii)} \int_N^{\infty} f(x)dx = \infty \qquad \Longrightarrow \qquad \sum_N^{\infty} f(n) = \infty$$

Proof of (i):

Proof. Use the right Riemann (partial) sum.

$$\sum_N^{\infty} f(n) = f(N) + \sum_{N+1}^{\infty} f(n) \leq \int_N^{\infty} f(x)dx < \infty$$

□

Proof of (ii):

Proof. Use the left Riemann (partial) sum

$$\sum_{N}^{\infty} f(n) \geq \int_{N}^{\infty} f(x)dx = \infty$$

$\square$

NB. Here, the width of the "rectangles" $\Delta x = 1$.

8.1 Examples

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \quad \text{is finite if, and only if,} \quad p > 1\,.$$

Proof

$$\int_1^L x^{-p} dx = \begin{cases} \frac{1}{1-p}\left(L^{1-p} - 1\right) \text{ if } p \neq 1 \\ \\ \ln(L) \text{ if } p = 1 \end{cases}$$

If $p > 1$, then

$$\lim_{L \to \infty} \int_1^L x^{-p} dx = \frac{1}{p-1} < \infty\,.$$

If $p \leq 1$, then

$$\lim_{L \to \infty} \int_1^L x^{-p} dx = \infty\,.$$

Result follows by applying the integral test.

8.2 Estimating infinite sums

Using the idea that underpins the "Integral Test" we can obtain numerical approximations to "tricky" infinite sums.

For example, using some fancy mathematics (see the module *Data Signals and Systems*) we can show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

For the moment, we can estimate this sum: Let

$$S := \sum_{n=1}^{\infty} \frac{1}{n^4} \text{ and } S_N = \sum_{n=1}^{N} \frac{1}{n^4}$$

Then

$$S - S_N = \sum_{n=N+1}^{\infty} \frac{1}{n^4}$$

Clearly, we can link

$$\sum \frac{1}{n^4} \text{ and } \int \frac{1}{x^4} dx\,.$$

Indeed, using Riemann partial sums we have

$$\left[\frac{-1}{3x^3}\right]_{N+1}^{\infty} = \int_{N+1}^{\infty} \frac{1}{x^4} dx \leq \sum_{n=N+1}^{\infty} \frac{1}{n^4} \leq \int_{N}^{\infty} \frac{1}{x^4} dx = \left[\frac{-1}{3x^3}\right]_{N}^{\infty}$$

It follows that

$$\frac{1}{3(N+1)^3} \leq \sum_{n=N+1}^{\infty} \frac{1}{n^4} \leq \frac{1}{3N^3}$$

so that

$$\frac{1}{3(N+1)^3} \leq S - S_N \leq \frac{1}{3N^3}$$

Using some simple Matlab code we have that:

Table 1: Values of $1/3N^3$ and S_N for various N

N	1	2	3	4	5	6
$1/3N^3$	0.3333	0.0417	0.0123	0.0052	0.0027	0.0015
S_N	1.0000	1.0625	1.0748	1.0788	1.0804	1.0811

So, for example,

$$\underbrace{1.0804 + 0.0015}_{1.0819} = \overbrace{S_5}^{1.0804} + \overbrace{\frac{1}{3 \times 6^3}}^{0.0015} \leq \sum_{n=1}^{\infty} \frac{1}{n^4} \leq \overbrace{S_5}^{1.0804} + \overbrace{\frac{1}{3 \times 5^3}}^{0.0027} = \underbrace{1.0804 + 0.0027}_{1.0831}$$

In fact

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} = 1.082323233711138\ldots$$

9 Estimating $N!$

$2! = 2$, $3! = 3 \times 2 \times 1 = 6$, $N! = N \times (N-1)!$.

So how big is $N!$. Well

$$\begin{aligned} \ln N! &= \ln N + \ln(N-1) + \ldots + \ln 2 + \ln 1 \\ &\leq \int_1^{N+1} \ln x dx = (x \ln x - x)_1^{N+1} \end{aligned}$$

whilst

$$\begin{aligned} \ln N! &= \ln N + \ln(N-1) + \ldots + \ln 2 + \ln 1 \\ &\geq \int_1^{N} \ln x dx = (x \ln x - x)_1^{N} \end{aligned}$$

It follows that

$$N \ln N - N + 1 \leq \ln N! \leq (N+1)\ln(N+1) - N$$

or, exponentiating:

$$e\left(\frac{N}{e}\right)^N \leq n! \leq e\left(\frac{N+1}{e}\right)^{N+1}$$

Also we can estimate the number of permutations of a standard deck of 52 playing cards:

$$1.2110 \times 10^{67} < 52! < 6.3578 \times 10^{68}$$

Part II
Differential Equations

10 Basic Definitions

10.1 What is a differential equation?

- **A differential equation**: An equation containing an unknown function and one or more of its derivatives.
- The **order of a differential equation**: the highest order derivative that occurs in the equation.
- **A linear ordinary differential equation of order** n may be written as:

$$p_0(x)\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + p_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \ldots + p_n(x)y(x) = r(x);$$

  y is the dependent variable, x is the independent variable.
- **A nonlinear differential equation**: An equation that cannot be put in the above form.
- **Homogeneous differential equation**: $r(x) = 0$.
- **Inhomogeneous differential equation**: $r(x) \neq 0$.

10.2 Motivating Examples

- **Falling under constant gravity**

$$m\frac{d^2h}{dt^2}(t) = -mg \quad \text{(Newton's falling apple)}$$

- **Newton's Law of Cooling**

$$\frac{dT_{in}}{dt} = c(T_{out} - T_{in}) \quad \text{(recall climate change energy balance)}$$

- **Cash in the bank**

$$\frac{dc}{dt} = rc \quad \text{interest rate } r$$

- **Mass on the end of a spring**

$$m\frac{d^2x}{dt^2} = -kx \quad \text{(Newton + Hooke)}$$

10.3 What is the solution of a differential equation

- **General solution**: General solution of an nth order ODE contains n arbitrary constants. If n boundary conditions are specified, these n arbitrary constants can be determined.
- **Particular solution**: A particular solution is obtained from a general solution by assigning particular values of the arbitrary constants.

10.4 Linear or Nonlinear DEs

Example D1: The differential equation

$$\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = \cos x$$

is (i) linear, (ii) second order and (iii) inhomogeneous. Linearity is determined solely by the way that the dependent variable y and its derivative occur in the equation.

Example D2: The differential equation

$$\frac{d^2y}{dx^2} + \sin y = 0$$

is second order and nonlinear because of the term $\sin y$.

11 First Order Differential Equations

11.1 Separable equations

Separation of variables is a key approach to solving differential equations. A separable differential equations takes the form:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}, \quad \text{or} \quad g(y)\,dy = f(x)\,dx.$$

The general solution of such separable ODEs are obtained by integration:

$$\int g(y)\,dy = \int f(x)\,dx.$$

When/if these integrals have been done, the constant of integration gives the one arbitrary constant required for the general solution. If a condition is specified, use it to determine the value of the constant to get the Particular Solution.

Example D3.: Solve the differential equation

$$\frac{dy}{dx} = \frac{1-y^2}{y(1-x)}$$

Separating the terms x, dx and y, dy gives

$$\frac{y}{1-y^2}dy = \frac{1}{1-x}dx$$

so that

$$\int \frac{y\,dy}{(1-y^2)} = \int \frac{dx}{(1-x)} \quad \text{or} \quad \int -\frac{1}{2}\frac{d(1-y^2)}{1-y^2} = -\int \frac{d(1-x)}{(1-x)}$$

Integrating the above equation, we obtain an equation defining y as an implicit function of x:

$$\ln|1-y^2| = 2\ln|1-x| - C,$$

i.e.

$$\ln\frac{|1-x|^2}{|1-y^2|} = C, \quad \frac{|1-x|^2}{|1-y^2|} = e^C = k^2, \quad k^2 \neq 0$$

Clearing the fractions and eliminating the absolute values gives:

$$(1-x)^2 = \lambda(1-y^2), \quad \lambda = \pm k^2 \neq 0; \quad \frac{(x-1)^2}{\lambda} + y^2 = 1$$

where λ takes on any real value, except for 0. If $\lambda > 0$, the solution curves are all ellipses; If $\lambda < 0$, the solution curves are all hyperbolas.

For example, the particular solution passing through the point $(x = 0, y = 0)$, we then have $\lambda = 1$ giving

$$(x-1)^2 + y^2 = 1.$$

11.2 First order homogeneous differential equations

The first order differential equation

$$\frac{dy}{dx} = F(x, y)$$

is homogeneous in x and y when

$$F(\alpha x, \alpha y)$$

is independent of α. In this case,

$$F(x, y) = f\left(\frac{y}{x}\right)$$

To see this put $\alpha = \frac{1}{x}$. Then

$$F(x, y) = F(\alpha x, \alpha y) = F\left(1, \frac{y}{x}\right) = f\left(\frac{y}{x}\right)$$

So DEs with homogeneous $F(x, y)$ have a "standard form"

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Using the substitution $u = \frac{y}{x}$ gives

$$y = ux$$

and

$$\frac{dy}{dx} = x\frac{du}{dx} + u = f(u)$$

Hence the differential equation

$$\frac{dy}{dx} = f(\frac{y}{x})$$

in x and y simplifies to a separated equation

$$x\frac{du}{dx} = f(u) - u \text{ or } \frac{du}{f(u) - u} = \frac{dx}{x}$$

in x and u.

Example D4. Solve the differential equation

$$\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}.$$

We write the DE in standard form by dividing top/bottom by x^2:

$$\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy} = \frac{1 + 3y^2/x^2}{2(y/x)} = \frac{1 + 3u^2}{2u}.$$

With

$$y = x\,u(x), \quad \frac{dy}{dx} = x\frac{du}{dx} + u$$

$$x\frac{du}{dx} + u = \frac{dy}{dx} = \frac{1+3u^2}{2u} \implies x\frac{du}{dx} = \frac{1+3u^2}{2u} - u = \frac{1+u^2}{2u}$$

Separating variables, we obtain

$$\frac{dx}{x} = \frac{2u\,du}{1+u^2}$$

then, by integrating, we find

$$\ln|x| - \ln|1+u^2| = C.$$

This can be written as

$$\ln\left|\frac{x}{1+u^2}\right| = \ln e^C,$$

Clearing of fractions and eliminating the absolute value

$$\left|\frac{x}{1+(y/x)^2}\right| = k^2,\ \ k^2 = e^C,\ \ \frac{x}{1+(y/x)^2} = \pm k^2.$$

Note that $k = 0, (x = 0)$ is also a solution of the given equation.

11.3 Linear first order differential equations

Linear first order inhomogeneous DEs can be written in the form

$$\frac{dy}{dx} + p(x)y = r(x).$$

To solve these equations we use the integrating factor method. We multiply the DE by the "integrating" factor

$$u(x) = \exp\int p\,dx.$$

Then

$$\begin{aligned}\tfrac{d}{dx}(u(x)y(x)) &= \tfrac{du}{dx}y(x) + u(x)\tfrac{dy}{dx}\\ &= p(x)u(x)y(x) + u(x)\left(r(x) - p(x)y(x)\right) = u(x)r(x)\,.\end{aligned}$$

It follows that

$$u(x)y(x) = \int u(x)r(x)\,dx + C\,.$$

This means that

$$y(x) = \int \left[\left(e^{\int p(x)\,dx}\right) r(x)\,dx + C\,\right] e^{-\int p(x)\,dx}.$$

is the general solution of the linear first order inhomogeneous DE

$$\frac{dy}{dx} + p(x)y = r(x).$$

Example D5. Find the solution of the differential equation

$$\frac{dy}{dx} - \frac{2x}{1+x^2}y = 1, \quad y(0) = 1$$

The integrating factor $u(x)$ is

$$\exp\left(-\int\frac{2x}{1+x^2}\,dx\right) = \exp\left(-\ln(1+x^2)\right) = \frac{1}{\exp(\ln(1+x^2))} = \frac{1}{1+x^2}.$$

Multiplying the equation by this factor

$$\frac{1}{1+x^2}\frac{dy}{dx} - \frac{2x}{(1+x^2)^2}y = \frac{d}{dx}\left(\frac{y}{1+x^2}\right) = \frac{1}{1+x^2}.$$

Integrating

$$\left(\frac{y}{1+x^2}\right) = \int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C.$$

When $x = 0, y = 1$ so that

$$C = 1, \text{ and } y(x) = (1+x^2)(\tan^{-1}x + 1).$$

12 Linear Second Order Differential Equations

12.1 Formulation

The general second order linear differential equation has the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x).$$

where $p(x)$ and $q(x)$ are coefficient functions.

When $r(x) = 0$, the equation is homogeneous

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0.$$

If $y_1(x)$ and $y_2(x)$ are linearly independent solutions of the equation, then the general solution is given by

$$y(x) = C_1y_1(x) + C_2y_2(x)$$

where C_1 and C_2 are arbitrary constants.

Independence of functions: A set of functions $\{y_1(x), \ldots, y_k(x)\}$ are linearly independent[1] if

$$c_1y_1(x) + \ldots + c_ky_k(x) = 0, \text{ for all } x$$

means

$$c_1 = \ldots = c_k = 0\,.$$

For example $\sin x$ and $\cos x$ are linearly independent. Indeed, if

$$c_1 \sin x + c_2 \cos x \equiv 0, \quad x = 0 \implies c_2 = 0, \quad x = \frac{\pi}{2} \implies c_1 = 0\,.$$

The monomials $y_1(x) = 1, y_2(x) = x, \ldots, y_k = x^{k-1}$ are linearly independent also.

There is no general, closed form solution for linear differential equations with non-constant coefficients and approaches to sol. However, there is "analytical" mileage in rewriting the second order DE in first order form by setting

$$z(x) = \begin{pmatrix} y(x) \\ y'(x) \end{pmatrix} \qquad \text{so that}$$

$$z'(x) = \begin{pmatrix} y'(x) \\ y''(x) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q(x) & -p(x) \end{pmatrix}\begin{pmatrix} y(x) \\ y'(x) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r(x)$$

[1] See extra notes on linear independence of functions for more info

12.2 Using one solution to find another

Let $y_1(x)$ be a solution of the homogeneous DE:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0.$$

Using $y_1(x)$ we seek a second solution in the form

$$y(x) = v(x)y_1(x).$$

If we can find $v(x)$, then we have $y(x)$ and the problem is solved. First we have that

$$\frac{dy}{dx} = v\frac{dy_1}{dx} + \frac{dv}{dx}y_1; \quad \frac{d^2y}{dx^2} = v\frac{d^2y_1}{dx^2} + 2\frac{dv}{dx}\frac{dy_1}{dx} + \frac{d^2v}{dx^2}y_1$$

So that substituting $y = v\,y_1$ into the equation gives

$$v\left(\frac{d^2y_1}{dx^2} + p(x)\frac{dy_1}{dx} + q(x)y_1\right) + y_1\frac{d^2v}{dx} + \left(2\frac{dy_1}{dx} + py_1\right)\frac{dv}{dx} = 0$$

But $y_1(x)$ is a solution, i.e.

$$\frac{d^2y_1}{dx^2} + p(x)\frac{dy_1}{dx} + q(x)y_1 = 0\,.$$

It follows that

$$y_1\frac{d^2v}{dx} + \left(2\frac{dy_1}{dx} + py_1\right)\frac{dv}{dx} = 0$$

or

$$\frac{v''}{v'} = -2\frac{y_1'}{y_1} - p(x), \quad \frac{dv'}{v'} = -2\frac{dy_1}{y_1} - p(x)dx.$$

An integration gives

$$\ln v' = -2\ln y_1 - \int p(x)dx, \quad \Longrightarrow \quad v'y_1^2 = e^{-\int p(x)dx}$$

and hence

$$v = \int \left(y_1^{-2}e^{-\int p(x)dx}\right)dx$$

Example D6.: Find the general solution of the differential equation

$$x^2y''(x) + xy'(x) - y(x) = 0 \quad \text{i.e.} \quad y''(x) + \frac{1}{x}y'(x) - \frac{1}{x^2}y(x) = 0$$

By inspection, $y_1 = x$ is a solution of the equation. In this case

$$p(x) = \frac{1}{x} \text{ and } q(x) = -\frac{1}{x^2}$$

So a second linearly independent solution is then given by $y_2 = v(x)y_1$ with

$$\begin{aligned} v \quad &= \int\left(y_1^{-2}e^{-\int p(x)dx}\right)dx = \int\left(x^{-2}e^{-\int(1/x)dx}\right)dx \\ \Longrightarrow v \quad &= \int\left(x^{-2}e^{-\ln x}\right)dx = \int\left(x^{-2}x^{-1}\right)dx = -x^{-2}/2 \end{aligned}$$

This yields

$$y_2 = x\,v = -x^{-\frac{1}{2}} \qquad \text{and a general solution} \quad y(x) = C_1x + C_2x^{-1}.$$

Homogeneous equation with constant coefficients Standard form is

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0,$$

where p and q are constants. Consider $y = e^{mx}$ as a possible solution. Substitution yields

$$\left(m^2 + pm + q\right) e^{mx} = 0, \ \ \rightarrow \ \ \left(m^2 + pm + q\right) = 0,$$

which is called the auxiliary equation (or characteristic equation) (since e^{mx} is never zero). The two roots m_1, m_2 are are given by the quadratic formula

$$m_{1,2} = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

Distinct real roots when $(p^2 - 4q) > 0$**.** In this case, we get the two linearly independent solutions

$$y_1 = e^{m_1 x}, \ \ y_2 = e^{m_2 x}$$

and the general solution is

$$y(x) = C_1 e^{m_1 x} + C_2 e^{m_2 x}.$$

Distinct complex roots when $(p^2 - 4q) < 0$**.** In this case,

$$m_1 = a + ib = \frac{-p + i\sqrt{|p^2 - 4q|}}{2}; \ \ m_2 = a - ib = \frac{-p - i\sqrt{|p^2 - 4q|}}{2}$$

we get the two linearly independent solutions

$$y_1 = e^{(a+ib)x} = e^{ax}(\cos bx + i \sin bx), \ \ y_2 = e^{(a-ib)x} = e^{ax}(\cos bx - i \sin bx),$$

and the general solution is

$$y(x) = e^{ax}\left[C_1(\cos bx + i \sin bx) + C_2(\cos bx - i \sin bx)\right] = e^{ax}\left(c_1 \cos bx + c_2 \sin bx\right)$$

where $c_1 = C_1 + C_2; \ \ c_2 = i(C_1 - C_2)$.

Equal real roots when $(p^2 - 4q) = 0$**.** In this case ($m_1 = m_2 = m = -p/2$), we get the two linearly independent solutions

$$y_1 = e^{mx}, \ \ y_2 = xe^{mx}$$

and the general solution is

$$y(x) = C_1 e^{mx} + C_2 x e^{mx}$$

From the first known solution, we can easily find a second linearly independent solution by

$$y_2 = vy_1 = v(x)e^{mx},$$

where

$$v = \int \left((y_1)^{-2} e^{-\int p dx}\right) dx = \int \left((e^{-px/2})^{-2} e^{-\int p dx}\right) dx = \int \left((e^{px} e^{-pdx}\right) dx = x.$$

Initial-value problems. It consists of finding a solution of y that satisfies initial conditions of the form

$$y(a) = A, \;\; y'(a) = B,$$

where A and B are given constants.
Boundary-value problems. It consists of finding a solution of y that satisfies boundary conditions of the form

$$y(a) = A, \;\; y(b) = B,$$

where A and B are given constants.

Example: Solve the initial-value problem

$$y'' + y = 0 \;\; y(0) = 2 \;\; y'(0) = 3$$

The auxiliary equation is ,

$$m^2 + 1 = 0$$

whose roots are $m_1 = i, m_2 = -i$. The general solution is

$$y(x) = c_1 \cos x + c_2 \sin x$$

$$y'(x) = -c_1 \sin x + c_2 \cos x$$

This yields

$$y(0) = c_1 = 2, \;\; y'(0) = c_2 = 3.$$

Therefore, the solution of the initial value problem is

$$y(x) = 2\cos x + 3\sin x.$$

Inhomogeneous equation with constant coefficients Standard form is

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = r(x),$$

where $r(x) \neq 0$ and p and q are constants.

The First step is to solve the equivalent homogeneous problem, the complementary equation,

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0,$$

the obtained by setting $r(x) = 0$. Since this equation is linear and homogeneous, If we find two independent solutions, $y_1(x)$ and $y_2(x)$, the complementary function is

$$y_c = C_1 y_1 + C_2 y_2.$$

The Second step is to find a Particular Integral, y_p. This is basically a matter of trial and error, but the following guidelines are helpful.

(i) If $r(x)$ is a polynomial of degree m, try

$$y = a_n x^m + a_{n-1} x^{m-1} + \cdots + a_0$$

and substitute in to the ODE. Equate the powers of x in the result with $r(x)$ to determine the coefficients. [This is sometimes called the method of undetermined coefficients].

(ii) If $r(x)$ contains $he^{\lambda x}$, try for the P.I.

$$y = ae^{\lambda x}.$$

Again substitute into the ODE and equate both sides to determine a.

(iii) If $r(x) = C\cos\lambda x + D\sin\lambda x$, try $y = a\cos\lambda x + b\sin\lambda x$, substitute in and equate coefficients of $\cos\lambda x$ and $\sin\lambda x$ to determine a and b.

(iv) If $r(x)$ is any combination of the above, find P.I.'s for each piece separately, and add to find the full P.I.

(v) If $r(x) = he^{\lambda x}$ and $e^{\lambda x}$ happens to be a part of the C.F., try $axe^{\lambda x}$ instead of $ae^{\lambda x}$.
If $r(x) = he^{\lambda x}$ is a double root of the homogeneous problem, try $ax^2e^{\lambda x}$.
Similarly, if $r(x) = \exp(px)[C\cos qx + D\sin qx]$ happens to be part of the C.F., try $y = xe^{px}(a\cos qx + b\sin qx)$ for the P.I.

The third step. The general solution of the nonhomogeneous equation is written as

$$y(x) = y_p + y_c$$

Since the complementary function has two arbitrary constants if the equation is of second order, this general solution has the correct number of arbitrary constants.

Example: Solve

$$y'' + 4y = e^{3x}.$$

The auxiliary equation is

$$m^2 + 4 = 0,$$

whose roots are $m_1 = 2i, m_2 = -2i$. The solution of the complementary equation is

$$y_c = c_1\cos 2x + c_2\sin 2x.$$

For a particular integral (solution), we try $y_p = ae^{3x}$ (the method of undetermined coefficients)

$$9ae^{3x} + 4ae^{3x} = e^{3x}, \;\; \rightarrow \;\; a = 1/13,$$

leading to the general solution

$$y = y_p + y_c = \frac{1}{13}e^{3x} + c_1\cos 2x + c_2\sin 2x.$$

Example: Solve

$$y'' + y = \sin x.$$

The auxiliary equation is

$$m^2 + 1 = 0,$$

whose roots are $m_1 = i, m_2 = -i$. The solution of the complementary equation is

$$y_c = c_1\cos x + c_2\sin x.$$

For a particular integral (solution), we try

$$y_p = x(a\cos x + b\sin x).$$

Then

$$y_p' = a\cos x + b\sin x + x(-a\sin x + b\cos x)$$
$$y_p'' = -2a\sin x + 2b\cos x - ax\cos x - bx\sin x$$

Substitution in the equation gives

$$(-2a\sin x + 2b\cos x - ax\cos x - bx\sin x) + (ax\cos x + bx\sin x) = \sin x,$$

$$-2a\sin x + 2b\cos x = \sin x; \quad \rightarrow \quad a = -1/2; b = 0; \quad y_p = -\frac{x}{2}\cos x.$$

The general solution is

$$y = y_p + y_c = c_1\cos x + c_2\sin x - \frac{x}{2}\cos x.$$

Higher-order homogeneous equation with constant coefficients Standard form is

$$\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + ... + a_{n-1}\frac{dy}{dx} + a_n y = r(x),$$

where $a_j, j = 1, ...n$ are constants.

When $r(x) = 0$ (homogeneous equation), consider $y = e^{mx}$ as a possible solution. Substitution in the equation leads to the characteristic equation

$$m^n + a_1 m^{n-1} + ... + a_{n-1}m + a_n = 0;$$

the degree of this algebraic equation will be the same as the order of the differential equation. The procedure is similar to that of the second-order differential equation.

Example Find the general solution of

$$y''' + 3y'' + 3y' + y = 0;$$

the characteristic equation is

$$m^3 + 3m^2 + 3m + 1 = (m+1)^3 = 0.$$

Hence, $m = -1$ is a triple root and three linearly independent solutions are

$$y_1 = e^{-x}, \;\; y_2 = xe^{-x}, \quad y_3 = x^2e^{-x}$$

and the general solution is

$$y(x) = C_1e^{-x} + C_2xe^{-x} + C_3x^2e^{-x}.$$

Example Find the general solution of

$$y^{(4)} + 32y'' + 256y = 0.$$

the characteristic equation is

$$m^4 + 32m^2 + 256 = 0, \;\; \rightarrow \;\; (m^2+16)^2 = 0$$

The double complex roots are $m_1 = i4, m_2 = i4, m_3 = -i4, m_4 = -i4$. The general solutions of the equation

$$y = C_1 x \cos 4x + C_2 x \sin 4x + C_3 \cos 4x + C_4 \sin 4x.$$

General Examples and Exercises

Example Assume that $x^3 e^{-x}$ is a solution of a fourth-order homogeneous differential equation with constant coefficients.
(i) Find the general solution and (ii) determine the corresponding differential equation.

If $x^3 e^{-x}$ is a solution, then $x^2 e^{-x}, xe^{-x} e^{-x}$ are also solutions of the equation, i.e.,

$$y = C_1 x^3 e^{-x} + C_2 x^2 e^{-x} + C_3 x e^{-x} + C_4 e^{-x}$$

The four equal real roots are -1, or the characteristic equation is

$$(m+1)^4 = 0$$

which can be expanded in the form

$$m^4 + 4m^3 + 6m^2 + 4m + 1 = 0.$$

Hence, the corresponding differential equation is

$$y^{(4)} + 4y''' + 6y'' + 4y + y = 0.$$

Exercise Solve $y^3 y'' + 1 = 0$.

Example Solve the second ode

$$(t-1)\frac{d^2x}{dt^2}+(t+1)\frac{dx}{dt}+x=2t.$$

Let

$$y=(t-1)x(t),\ \to\ \frac{dy}{dt}=x+(t-1)\frac{dx}{dt}=x+(t+1)\frac{dx}{dt}-2\frac{dx}{dt},$$

$$\frac{dy^2}{dt^2}=2\frac{dx}{dt}+(t-1)\frac{d^2x}{dt^2},$$

i.e.,

$$(t+1)\frac{dx}{dt}=\frac{dy}{dt}-x+2\frac{dx}{dt},$$

$$(t-1)\frac{d^2x}{dt^2}=\frac{dy^2}{dt^2}-2\frac{dx}{dt},$$

Substitution in the equation yields

$$\frac{dy^2}{dt^2}+\frac{dy}{dt}=2t.$$

It has the complementary solution

$$(m^2+m)=0,\ \ m_1=0,\ m_2=-1;\ \ y_c=C_1+C_2e^{-t}$$

For a particular solution, try

$$y_p=At^2+Bt$$

which leads to

$$2A+(2At+B)=2t,\ \to\ C=0,\ 2A+B=0,\ 2A=2,$$

$$A=1,\quad B=-1/2.$$

The general solution is

$$y=y_c+y_p=C_1+C_2e^{-t}+t^2-t/2,\ \ \to\ \ x=\frac{1}{t-1}\left(C_1+C_2e^{-t}+t^2-t/2\right).$$

Example Solve the third-order ode which has a known solution $x_1=1/t$

$$t\frac{d^3x}{dt^3}+3\frac{d^2x}{dt^2}-t\frac{dx}{dt}+x=0.$$

Let $x=y/t$, then

$$\frac{dx}{dt}=\frac{1}{t}\frac{dy}{dt}-\frac{y}{t^2}$$

$$\frac{d^2x}{dt^2}=\frac{1}{t}\frac{d^2y}{dt^2}-\frac{2}{t^2}\frac{dy}{dt}+\frac{2}{t^3}y$$

$$\frac{d^3x}{dt^3}=\frac{1}{t}\frac{d^3y}{dt^3}-\frac{3}{t^2}\frac{d^2y}{dt^2}+\frac{6}{t^3}\frac{dy}{dt}-\frac{6}{t^4}y$$

Substitution then in the equation

$$\left(\frac{d^3y}{dt^3}-\frac{3}{t}\frac{d^2y}{dt^2}+\frac{6}{t^2}\frac{dy}{dt}-\frac{6}{t^3}y\right)+\left(\frac{3}{t}\frac{d^2y}{dt^2}-\frac{6}{t^2}\frac{dy}{dt}+\frac{6}{t^3}y\right)-\left(\frac{dy}{dt}-\frac{y}{t}\right)+y/t=0$$

$$\frac{d^3y}{dt^3}-\frac{dy}{dt}=0.$$

Let $dy/dt = z$, then

$$\frac{d^2 z}{dt^2} - z = 0$$

which has the general solution

$$z = C_1 e^t + C_2 e^{-t}, \ \rightarrow \ y = C_1 e^t - C_2 e^{-t} + C_3$$

$$x = \frac{1}{t}\left(C_1 e^t - C_2 e^{-t} + C_3\right).$$

Example Solve the second-order ode

$$\frac{d^2 y}{dx^2} + (x + e^{2y})\left(\frac{dy}{dx}\right)^3 = 0.$$

Note that

$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1}$$

$$\frac{d^2 y}{dx^2} = \frac{dy'}{dy}\frac{dy}{dx} = \frac{d}{dy}\left(\frac{1}{dx/dy}\right)\frac{1}{dx/dy} = -\left(\frac{d^2x/dy^2}{(dx/dy)^2}\right)\frac{1}{dx/dy} = -\frac{d^2x/dy^2}{(dx/dy)^3}$$

Substitution in the equation

$$-\frac{d^2x/dy^2}{(dx/dy)^3} + (x + e^{2y})\left(\frac{dx}{dy}\right)^{-3} = 0, \ \rightarrow \ -\frac{d^2 x}{dy^2} + (x + e^{2y}) = 0$$

which is a linear equation for $x(y)$

$$\frac{d^2 x}{dy^2} - x = e^{2y}$$

It has the complementary solution

$$(m^2 - 1) = 0, \ \ m_1 = 1, \ m_2 = -1; \ \ x_c = C_1 e^y + C_2 e^{-y}$$

For a particular solution, try

$$x_p = Ae^{2y}$$

which leads to

$$4A - A = 1, \ \rightarrow \ A = 1/3.$$

The general solution is

$$x = x_c + x_p = C_1 e^y + C_2 e^{-y} + e^{3y}/3.$$

Exercise Solve the second-order ode

$$(x+1)\frac{d^2 y}{dx^2} + \frac{dy}{dx} = \ln(x+1).$$

13 Linear Differential Equations with Constant Coefficients

13.1 Second Order Recap

The standard form is

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0,$$

where p and q are constants.

We seek two linearly independent solutions.

Inspired guesswork suggests trying a solution $y(x) = e^{mx}$. Substitution yields

$$\left(m^2 + pm + q\right) e^{mx} = 0$$

Since $\exp(x) \not\equiv 0$, e^{mx} is a solution if, and only if, m satisfies:

$$m^2 + pm + q = 0\,. \qquad \text{Auxiliary Equation}$$

How we continue depends on the nature of the roots of the auxiliary equation

$$m_1, m_2 = \frac{1}{2}\left(-p \pm \sqrt{p^2 - 4q}\right).$$

- **Case 1**: Auxiliary equation has distinct real roots.

  If $p^2 - 4q > 0$ then m_1 and m_2 are real and distinct.

  In this case, we obtain the two linearly independent solutions

  $$y_1 = e^{m_1 x}, \quad y_2 = e^{m_2 x}$$

  and a general solution

  $$y(x) = C_1 e^{m_1 x} + C_2 e^{m_2 x}.$$

- **Case 2**: Auxiliary equation has distinct complex roots.

  If $p^2 - 4q < 0$ then $m_1 = a + ib$ and $m_2 = a - ib$ are distinct, non real complex conjugates.

  In this case we obtain the two linearly independent solutions

  $$\begin{aligned} y_1 &= e^{(a+ib)x} = e^{ax}(\cos bx + i \sin bx) \\ y_2 &= e^{(a-ib)x} = e^{ax}(\cos bx - i \sin bx) \end{aligned}$$

  and the general solution is

  $$\begin{aligned} y(x) &= e^{ax}\left[C_1(\cos bx + i \sin bx) + C_2(\cos bx - i \sin bx)\right] \\ &= c_1 e^{ax} \cos bx + c_2 e^{ax} \sin bx \end{aligned}$$

  where $c_1 = C_1 + C_2$; $c_2 = i(C_1 - C_2)$.

  Note that whilst the auxiliary equation has non-real roots, we do end up with two independent real valued solutions.

- **Case 3**: Auxiliary equation has equal real roots.

  If $p^2 - 4q = 0$, then $m_1 = m_2 = m = -p/2$

  In this case we get the two linearly independent solutions

  $$y_1 = e^{mx}, \quad y_2 = xe^{mx}$$

  and the general solution is

  $$y(x) = C_1 e^{mx} + C_2 x e^{mx}$$

  To see how we obtain the solution $y_2 = xe^{mx}$, recall that for general linear ODEs we can use a known solution $y_1 = e^{mx}$ to find a second solution $y_2 = v(x)y_1(x) = v(x)e^{mx}$ where:

  $$\begin{aligned} v(x) &= \int \left((y_1)^{-2} e^{-\int p\,dx}\right) dx = \int \left((e^{-px/2})^{-2} e^{-\int p\,dx}\right) dx \\ &= \int \left((e^{px} e^{-px}\right) dx = x. \end{aligned}$$

13.2 Initial-value Problems

In initial value problems we find a solution of $y(x)$ that satisfies initial conditions of the form

$$y(a) = A, \;\; y'(a) = B,$$

where A and B are given constants.

Example D7.: Solve the initial-value problem

$$y'' + y = 0 \qquad y(0) = 2 \;\; y'(0) = 3$$

The auxiliary equation is

$$m^2 + 1 = 0 \quad \text{so that} \quad m_1 = i, m_2 = -i\,.$$

Then the general solution is

$$y(x) = c_1 \cos x + c_2 \sin x \qquad \text{with} \quad y'(x) = -c_1 \sin x + c_2 \cos x$$

So

$$y(0) = c_1 = 2, \;\; y'(0) = c_2 = 3 \text{ and}$$

$$y(x) = 2\cos x + 3 \sin x.$$

13.3 Boundary-value Problems

In boundary-value problems we find a solution of $y(x)$ that satisfies boundary conditions of the form

$$y(a) = A, \;\; y(b) = B,$$

where A and B are given constants.

Example D8.: Solve the boundary-value problem

$$y'' - 3y' + 2y = 0 \qquad y(0) = 2 \quad y(1) = 1 + e$$

The auxiliary equation is

$$m^2 - 3m + 2 = (m-1)(m-2) = 0 \text{ so that} \quad m_1 = 1, m_2 = 2\,.$$

Then the general solution is

$$y(x) = c_1 e^x + c_2 e^{2x}$$

$$y(0) = c_1 + c_2 = 2, \quad y(1) = c_1 e + c_2 e^2 = 1 + e \implies c_1 = 2 + e^{-1}, \quad c_2 = -e^{-1}$$

Then

$$y(x) = (2 + e^{-1})e^x - e^{-1}e^{2x}\,.$$

13.4 2^{nd} order inhomogeneous ODEs with constant coefficients

Standard form is

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = r(x),$$

where $r(x) \neq 0$ and p and q are constants.

To solve such equations we proceed in steps:

Step 1: Solve the corresponding homogeneous problem, i.e complementary ODE

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0,$$

obtained by setting $r(x) = 0$.

This equation will have two linearly independent solutions, $y_1(x)$ and $y_2(x)$, giving the complementary function

$$y_C(x) = C_1 y_1(x) + C_2 y_2(x).$$

Step 2: Find a Particular Integral, $y_P(x)$ a (NOT the general) solution of the inhomogeneous. This is basically a matter of trial and error, but the following guidelines are helpful:

- If $r(x)$ is a polynomial of degree m, try

  $$y_P(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0$$

  and substitute in to the ODE. Equate the powers of x in the result with $r(x)$ to determine the coefficients a_k .

- If $r(x)$ contains $e^{\lambda x}$ try

  $$y_P(x) = a e^{\lambda x}.$$

  Again substitute into the ODE and equate both sides to determine a.

- If $r(x) = C \cos \lambda x + D \sin \lambda x$, try

  $$y_P(x) = a \cos \lambda x + b \sin \lambda x$$

  substitute in and equate coefficients of $\cos \lambda x$ and $\sin \lambda x$ to find a and b.

- If $r(x)$ is any combination of the above, find P.I.'s for each piece separately, and combine to find the full P.I.

- If $r(x) = h e^{\lambda x}$ and $e^{\lambda x}$ happens to be a part of the C.F., try $a x e^{\lambda x}$ instead of $a e^{\lambda x}$.
  If $r(x) = h e^{\lambda x}$ is a double root of the homogeneous problem, try $a x^2 e^{\lambda x}$.
  Similarly, if $r(x) = \exp(px)[C \cos qx + D \sin qx]$ happens to be part of the C.F., try $y = x e^{px}(a \cos qx + b \sin qx)$ for the P.I.

  Keep increasing powers until a PI is found.

Step 3: The general solution of the inhomogeneous ODE is then

$$y_{GS}(x) = y_P(x) + y_C(x)$$

The number of free constants in $y_C(x)$ will match the order of the ODE and so $y_{GS}(x)$ will be the general solution

We can understand the general solution as a solution of an ODE

$$\frac{d^{k+l}}{dx^{k+l}} y(x) + a_1 \frac{d^{k+l-1}}{dx^{k+l-1}} y(x) + \ldots + a_{k+l} y(x) = 0$$

where

$$m^{k+l} + a_1 m^{k+l-1} + \ldots + a_{k+l} = c(m) d(m) \,,$$

$c(m) = 0$ is the auxiliary equation for the complementary function and $r(x)$ is the solution of an ODE with an auxiliary equation $d(m) = 0$.

Example D9. Find the general solution of

$$y'' + 4y = e^{3x}.$$

The auxiliary equation

$$m^2 + 4 = 0 \quad \text{has roots} \quad m_1 = 2i, m_2 = -2i \,.$$

The complementary function is therefore

$$y_C(x) = c_1 \cos 2x + c_2 \sin 2x.$$

$r(x) = e^{3x}$ so for a particular integral we try $y_P(x) = ae^{3x}$. Then

$$9ae^{3x} + 4ae^{3x} = e^{3x}, \quad \to \quad a = 1/13$$

giving the general solution

$$y_{GS}(x) = y_P(x) + y_C(x) = \frac{1}{13}e^{3x} + c_1 \cos 2x + c_2 \sin 2x.$$

Example D10. Solve

$$y'' + y = \sin x.$$

The auxiliary equation is

$$m^2 + 1 = 0 \quad \text{with roots} \quad m_1 = i, m_2 = -i.$$

The complementary function is

$$y_C = c_1 \cos x + c_2 \sin x.$$

For a particular integral we try $y_P = x(a \cos x + b \sin x)$. Then

$$\left.\begin{array}{rcl} y_P' & = & a \cos x + b \sin x + x(-a \sin x + b \cos x) \\ y_P'' & = & -2a \sin x + 2b \cos x - ax \cos x - bx \sin x \end{array}\right\} \quad \text{So}$$

$$y_P''(x) + y_P(x) = -2a \sin x + 2b \cos x = \sin x \implies a = -1/2; b = 0$$

So $y_P = -\frac{x}{2} \cos x$ and the general solution is

$$y = y_P + y_C = c_1 \cos x + c_2 \sin x - \frac{x}{2} \cos x.$$

13.5 Higher-order inhomogeneous ODEs with constant coefficients

Standard form is

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + ... + a_{n-1} \frac{dy}{dx} + a_n y = r(x),$$

where $a_j, j = 1, \ldots, n$ are constants.

As before we seek a complementary function $y_C(x)$ — a general solution of the corresponding homogeneous equation with $r(x) \equiv 0$ This leads to an auxiliary (or characteristic) equation

$$m^n + a_1 m^{n-1} + ... + a_{n-1} m + a_n = 0;$$

with degree equal to the order of the differential equation. Then we look for a particular solution $y_P(x)$ by trial and error.

Example D11. Find the general solution of

$$y''' + 3y'' + 3y' + y = e^{2x}.$$

The auxiliary/characteristic equation is

$$m^3 + 3m^2 + 3m + 1 = (m + 1)^3 = 0.$$

$m = -1$ is a triple root. Three linearly independent solutions are

$$y_1 = e^{-x}, \;\; y_2 = xe^{-x}, \quad y_3 = x^2 e^{-x}$$

giving the complementary function

$$y_C(x) = C_1e^{-x} + C_2xe^{-x} + C_3x^2e^{-x}.$$

Trying $y_P(x) = ae^{2x}$ gives

$$8ae^{2x} + (3)(4ae^{2x}) + (3)(2ae^{2x}) + ae^{2x} = e^{2x} \implies a = \frac{1}{27}$$

So the general solution is

$$y_{GS}(x) = C_1e^{-x} + C_2xe^{-x} + C_3x^2e^{-x} + \frac{1}{27}e^{2x}.$$

Example D12. Find the general solution of

$$y^{(4)} + 32y'' + 256y = e^x.$$

The auxiliary/characteristic equation is

$$m^4 + 32m^2 + 256 = 0, \quad \rightarrow \quad (m^2+16)^2 = 0$$

The double complex roots are $m_1 = m_2 = 4i, m_3 = m_4 = -4i$. The complementary function is

$$y_C(x) = C_1x\cos 4x + C_2x\sin 4x + C_3\cos 4x + C_4\sin 4x.$$

The particular solution is $y_P(x) = ae^x$ so that

$$(a + 32a + 256a)\, e^x = e^x \implies a = \frac{1}{289}$$

and the general solution is

$$y_{GS}(x) = C_1x\cos 4x + C_2x\sin 4x + C_3\cos 4x + C_4\sin 4x + \frac{1}{289}e^x.$$

A Miscellaneous Check. Let the auxiliary equation of

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$$

be such that

$$m^2 + pm + q = (m-m_1)(m-m_2) \implies p = -(m_1+m_2)$$

Then we know that the general solution is

$$y(x) = Ae^{m_1x} + Be^{m_2x}$$

But we also know that the general solution is

$$y(x) = y_1(x)(C + Dv(x)) \quad \text{where } y_1(x) = e^{m_1x} \quad \text{and}$$

$$v(x) = \int \frac{1}{y_1(x)^2}\exp\left(-\int p dx\right) = \int e^{-(2m_1+p)x}dx = \frac{-1}{2m_1+p}e^{-(2m_1+p)x}$$

Then

$$y(x) = y_1(x)(C + Dv(x)) = Ce^{m_1x} + D'e^{-(m_1+p)x} = Ce^{m_1x} + D'e^{m_2x}$$

The ODE Challenge. Find the fifth order homogeneous linear, constant coefficient ODE which has solution

$$y(x) = e^{2x} + x \sin 3x$$

Let its auxiliary equation be

$$p(m) = m^5 + a_1 m^4 + \ldots + a_4 m + a_5 = 0$$

Find first, second, third and fourth order inhomogeneous linear, constant coefficient ODEs which have $y(x)$ as a solution and are such that if their auxiliary equation is

$$q(m) = 0$$

then q dives p.

Part III
Appendix

14 Linear Independence of Functions

A set of functions $\{f_1, \ldots, f_n\}$, where each $f_i : D \mapsto \mathbb{R}$ for some common domain D (usually an interval), are linear independent if

$$\sum_{j=1}^{n} c_j f_j(x) = 0 \quad \text{for all} \quad x \in D$$

implies $c_1 = \ldots = c_n = 0$.

We use this concept to make sure that when we solve a differential equation, we are constructing a general solution.

That is we do not want one function in the solution to be another function in disguise.

15 Linear Independence of Common Functions

15.1 Polynomials

The set of polynomial functions $f_i : x \mapsto x^{i-1}$, $i = 1, \ldots, n$, defined on some interval $[0, b], b > 0$, are linearly independent.

To see this, suppose that

$$c_1 + c_2 x + \ldots + c_n x^{n-1} = 0 \text{ for all } x \in [0, b]\,. \tag{4}$$

- Putting $x = 0$ in (4) gives $c_1 = 0$.
- Differentiating (4) gives

$$c_2 + 2c_3 x + \ldots + (n-1)c_n x^{n-2} = 0 \text{ for all } x \in [0, b]\,. \tag{5}$$

- Putting $x = 0$ in (5) gives $c_2 = 0$.
- Repeating this "differentiate-evaluate at 0" process will give

$$c_3 = c_4 = \ldots = c_n = 0$$

  and the polynomials are linearly independent as claimed.

15.2 Exponentials

The set of exponential functions $f_i : x \mapsto e^{a_i x}$, $i = 1, \ldots, n$, defined on some interval $[0, b], b > 0$, are linearly independent when all the a_i are distinct.

To see this, suppose that

$$c_1 e^{a_1 x} + c_2 e^{a_2 x} + \ldots + c_n e^{a_n x} = 0 \text{ for all } x \in [0, b]\,. \tag{6}$$

- Putting $x = 0$ in (6) gives

$$c_1 + c_2 + \ldots + c_n = 0$$

- Differentiating (6), $n - 1$ times and evaluating at $x = 0$ after each successive differentiation will give

$$a_1^{j-1} c_1 + a_2^{j-1} c_2 + \ldots + a_n^{j-1} c_n = 0, \quad j = 2, 3, \ldots, n \tag{7}$$

  We can summarise (6) and (7) in matrix form as follows:

$$\begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ a_1 & a_2 & \dots & a_{n-1} & a_n \\ a_1^2 & \dots & \dots & \dots & a_n^2 \\ \vdots & \dots & \dots & \dots & \vdots \\ a_1^{n-1} & \dots & \dots & \dots & a_n^{n-1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \tag{8}$$

The above $n \times n$ matrix is called the "van der Monde" matrix and has determinant equal to

$$\prod_{1 \le i < j \le n} (a_j - a_i) \qquad\qquad \text{(Check this!)}$$

From this it follows that when all the a_i are distinct, then the determinant is non-zero. Hence the van der Monde matrix is invertible which then means that

$$c_1 = c_2 = \dots = c_n = 0$$

is the only solution of (8).

15.3 Trigonometric Functions

Fix $L > 0$. The trigonmetric functions $x \mapsto \sin(k\pi x/L)$, $x \mapsto \cos(k\pi x/L)$, $k = 1, 2, \dots, n$, and the constant function $x \mapsto 1$ are linearly independent on the interval $[-L, L]$ (actually also on any non-zero interval).

To see this we will use integration, specifically the following facts

$$\begin{aligned} \textstyle\int_{-L}^{L} \cos(\frac{k\pi t}{L}) \cos(\frac{l\pi t}{L}) dt = &\quad 0 \quad l \neq k \\ \textstyle\int_{-L}^{L} \sin(\frac{k\pi t}{L}) \sin(\frac{l\pi t}{L}) dt = &\quad 0 \quad l \neq k \\ \textstyle\int_{-L}^{L} \cos(\frac{k\pi t}{L}) \sin(\frac{l\pi t}{L}) dt = &\quad 0 \end{aligned}$$

$$\int_{-L}^{L} \cos(\frac{k\pi t}{L}) dt = 0 \qquad \int_{-L}^{L} \sin(\frac{k\pi t}{L}) dt = 0$$

Check this e.g.

$$\int_{-L}^{L} \cos(\frac{k\pi t}{L}) \cos(\frac{l\pi t}{L}) dt = \frac{1}{2} \int_{-L}^{L} \cos((k-l)\pi t/L) + \cos((k+l)\pi t/L) dt = 0$$

Now suppose that for some $a_0, a_1, \dots, a_n$ and $b_1, \dots, b_n$ we have

$$a_0 \times 1 + \sum_{j=1}^{n} a_j \cos(j\pi x/L) + b_j \sin(j\pi x/L) = 0 \tag{9}$$

Integrating (9) with respect to x from $-L$ to L gives

$$0 = \int_{-L}^{L} \left(a_0 \times 1 + \sum_{j=1}^{n} a_j \cos(j\pi x/L) + b_j \sin(j\pi x/L) \right) dx = 2a_0 L$$

It follows that $a_0 = 0$

Multiplying (9) by $\cos(k\pi x/L)$ and integrating

$$\begin{aligned}
0 &= \int_{-L}^{L} \cos(k\pi x/L) \left(a_0 \times 1 + \sum_{j=1}^{n} a_j \cos(j\pi x/L) + b_j \sin(j\pi x/L) \right) dx \\
&= \int_{-L}^{L} \cos(k\pi x/L) a_k \cos(k\pi x/L) dx \\
&= a_k \int_{-L}^{L} \cos^2(k\pi x/L) dx = \frac{1}{2} a_k \int_{-L}^{L} \left(\cos^2(k\pi x/L) + \sin^2(k\pi x/L) \right) dx \\
&= \frac{1}{2} a_k \int_{-L}^{L} 1\, dx = a_k L
\end{aligned}$$

So each a_k is zero.

Multiplying by (9) by $\sin(k\pi x/L)$ and integrating will also give $b_k = 0$.

16 Summary

The following functions are independent:

- exponential functions with distinct exponents
- trig functions with distinct periods
- monomials with distinct powers
- hyperbolic functions (as with trig functions)
- various combinations of the above