Suffix Notation

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1 Basic Definitions

1.1 Suffix Notation

Let there be a vector $\underline{\mathbf{c}} = \underline{\mathbf{a}} + \underline{\mathbf{b}}$, where $\underline{\mathbf{a}} = a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}$ and $\underline{\mathbf{b}} = b_1\hat{\boldsymbol{\imath}} + b_2\hat{\boldsymbol{\jmath}} + b_3\hat{\boldsymbol{k}}$. Then $\underline{\mathbf{c}}$ is equivalent to:

$$c_i = a_i + b_i$$

In suffix notation:

$$c_j = a_j + b_j \qquad j = 1, 2, 3$$

The inner product of two vectors:

$$\begin{aligned} a \cdot b &= a_1 b_1 + a_2 b_2 + a_3 b_3 \\ &= \sum_{j=1}^{3} a_j b_j \end{aligned}$$

For a vector $\underline{\mathbf{a}} = a_i$, i is a free index. For the dot product above: $\sum_{j=1}^{3} a_j b_j$, j is a dummy suffix.

For suffix notation, an index cannot be repeated more than two times in an equation.

Example 1 *Write $(a \cdot b)(c \cdot d)$ in suffix notation*

Solution 1 *Here we take that:*

$$a \cdot b = a_j b_j \quad j = 1, 2, 3$$

and that

$$c \cdot d = c_i d_i \quad i = 1, 2, 3$$

Now we can say that

$$(a \cdot b)(c \cdot d) = a_j b_j c_i d_i \quad i, j = 1, 2, 3$$

Example 2 *Write $a_j b_i c_j$ in normal vector notation*

Solution 2 *We know that*

$$a_j b_i c_j = a_j c_j b_i$$

Which is:

$$(a \cdot c)b$$

Example 3 *Write the vector notation $\underline{\boldsymbol{u}} + (\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}})\underline{\boldsymbol{v}} = |\underline{\boldsymbol{a}}|^2(\underline{\boldsymbol{b}} \cdot v)\underline{\boldsymbol{a}}$ in suffix notation*

Solution 3 *We know that*

$$a_j b_i c_j = a_j c_j b_i$$

Which is:

$$(a \cdot c)b$$

Example 4 *Write the vector notation $\underline{\boldsymbol{u}} + (\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}})\underline{\boldsymbol{v}} = |\underline{\boldsymbol{a}}|^2(\underline{\boldsymbol{b}} \cdot v)\underline{\boldsymbol{a}}$ in suffix notation*

Solution 4 *Firstly:*

$$[\underline{\boldsymbol{u}} + (\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}})\underline{\boldsymbol{v}}]_i = \left[|\underline{\boldsymbol{a}}|^2(\underline{\boldsymbol{b}} \cdot v)\underline{\boldsymbol{a}}\right]_i$$

Then,

$$u_i + (a_j b_j) v_i = a_j a_j b_l v_l a_i \qquad j, l = 1, 2, 3$$

1.2 The Kronecker Delta $\delta_{i,j}$

The function is defined:

$$\delta_{i,j} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

The suffixes i and j can each take the values $1, 2, 3$ so $\delta_{i,j}$ has nine elements.

We can write the function as the identity matrix:

$$\delta_{i,j} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\delta_{i,j}$ is called a substitution tensor, since it's effect when multiplied by a_j is to replace j with i.

$$\begin{aligned} \delta_{i,j} a_j &= \sum_{j=1}^{3} \delta_{i,j} a_j \\ &= \delta_{i1} a_1 + \delta_{i2} a_2 + \delta_{i3} a_3 \\ &= \delta_{11} a_1 + \delta_{12} a_2 + \delta_{13} a_3 \\ &\quad + \delta_{21} a_1 + \delta_{22} a_2 + \delta_{23} a_3 \\ &\quad + \delta_{31} a_1 + \delta_{32} a_2 + \delta_{33} a_3 \\ &= a_1 + a_2 + a_3 \end{aligned}$$

From this we can say: $\delta_{i,j} a_i = a_j$ and $\delta_{i,j} a_j = a_i$

Example 5 *$\delta_{i,j}$ and dot product*

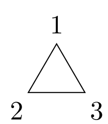
Solution 5

$$\begin{aligned} a \cdot b &= a_i b_i \quad i = 1,2,3 \\ &= \delta_{i,j} a_j b_i \\ &= a_j \delta_{i,j} b_i \\ &= a_j b_j \end{aligned}$$

1.3 The Alternating Tensor, $\varepsilon_{i,j,k}$

$\varepsilon_{i,j,k}$ is useful for manipulating expressions involving the cross product of two vectors and curl of a vector.

$$\varepsilon_{i,j,k} = \begin{cases} +1 & \text{if } (i,j,k) = (1,2,3),\ (2,3,1) \text{ or } (3,1,2) \\ -1 & \text{if } (i,j,k) = (3,2,1),\ (2,1,3) \text{ or } (1,3,2) \\ 0 & \text{if any of } i,j,k \text{ are equal} \end{cases}$$



The $+1$ case can be also written as 1, 2 or 3 are in clockwise order. So if you take a triangle and then go clockwise around it from the first element, that the order they are in. The -1 are in anticlockwise order. Hence meaning the opposite of clockwise.

The six non-zero elements of ε_{ijk}:

$$\begin{aligned} &\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1 \\ &\varepsilon_{321} = \varepsilon_{213} = \varepsilon_{132} = -1 \\ &\varepsilon_{ijk} = 0, \text{ otherwise} \end{aligned}$$

We can take that; $\varepsilon_{ijk} = \varepsilon_{jki}$ as they are in clockwise order. This also imples $\varepsilon_{ijk} = -\varepsilon_{jik}$ because if ijk are in clockwise order then jik must be in counterclockwise order.

1.4 $\varepsilon_{i,j,k}$ and cross product

Let $\underline{\mathbf{a}} = a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}$ and $\underline{\mathbf{b}} = b_1\hat{\boldsymbol{\imath}} + b_2\hat{\boldsymbol{\jmath}} + b_3\hat{\boldsymbol{k}}$. Then their cross product is:

$$\underline{\mathbf{a}} \times \underline{\mathbf{b}} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

and in suffix notation, we can write the above as; $(\underline{\mathbf{a}} \times \underline{\mathbf{b}})_i = \varepsilon_{ijk}\, a_j b_k$ where j,k are dummy suffixes and must be summed over 1 to 3.

1.5 ε_{ijk} and the scalar triple product

We can take the scalar triple product, $\underline{\mathbf{a}} \cdot \underline{\mathbf{b}} \times \underline{\mathbf{c}}$, then we can do the following:

$$\begin{aligned} \underline{\mathbf{a}} \cdot \underline{\mathbf{b}} \times \underline{\mathbf{c}} &= a_i(\underline{\mathbf{b}} \times \underline{\mathbf{c}})_i \\ &= a_i \varepsilon_{ijk} b_j c_k \\ &= \varepsilon_{ijk} a_i b_j c_k \\ &= c_k \varepsilon_{ijk} a_i b_j \end{aligned}$$

from the above we show that $\underline{\mathbf{a}} \cdot \underline{\mathbf{b}} \times \underline{\mathbf{c}} = \underline{\mathbf{c}} \cdot \underline{\mathbf{a}} \times \underline{\mathbf{b}}$. We can expand $\varepsilon_{ijk}\, a_i\, b_j\, c_k$ to get:

$$\begin{aligned} &= \varepsilon_{123} a_1 b_2 c_3 + \varepsilon_{231} a_2 b_3 c_1 + \varepsilon_{312} a_3 b_1 c_2 \\ &\quad + \varepsilon_{321} a_3 b_2 c_1 + \varepsilon_{213} a_2 b_1 c_3 + \varepsilon_{132} a_1 b_3 c_2 \\ &= a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_2 b_1 c_3 - a_1 b_3 c_2 \end{aligned}$$

which is the expanded form of the triple scalar product.

1.6 A relation between ε_{ijk} and $\delta_{i,j}$

We are going to prove the following statement:

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Since all of the coordinate axis are the same, just consider $i = 1$:

If then $j = 1$, we get that $\varepsilon_{11k} = 0$ and so LHS $= 0$. Then considering the RHS, we get that $\delta_{1l}\delta_{1m} - \delta_{1m}\delta_{1l} = 0$, so equation holds.

If $j = 2$, then $\varepsilon_{ijk} = \varepsilon_{12k} = 0$, unless $k = 3$, so then only $k = 3$ contributes to the sum. So $\varepsilon_{klm} = \varepsilon_{3lm}$, so zero unless l and m are 1 and 2. So we can conclude that $\varepsilon_{ijk}\varepsilon_{klm} = \varepsilon_{123}\varepsilon_{312}$ or $\varepsilon_{123}\varepsilon_{321}$, so the LHS is either ± 1. Looking at RHS, we have either: $\delta_{11}\delta_{22} - \delta_{12}\delta_{21}$ or $\delta_{12}\delta_{21} - \delta_{11}\delta_{22}$. This gives ± 1 in the same perumtation as the LHS. So equation holds.

2 Gradient, Divergence and Curl

2.1 Gradient

Assume we have a $f = f(x, y, z)$ or $f = f(x_1, x_2, x_3)$, so a scalar calued function. Then we define grad f as:

$$\underline{\nabla} f = \left(\frac{\partial}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial}{\partial z}\hat{\boldsymbol{k}} \right) f$$

We say grad of f is a differential operator. So:

$$\underline{\nabla} f = \left(\frac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{k}} \right)$$

and we can write it in suffix notation aswell:

$$[\underline{\nabla} f]_i = \frac{\partial}{\partial x_i} \qquad i = 1, 2, 3$$

2.2 Divergence

Assume we have a vector field, $\underline{\mathbf{u}} = \underline{\mathbf{u}}(x, y, z, t)$. We define the divergence of this vector field as;

$$\underline{\nabla} \cdot \underline{\mathbf{u}} = \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right)$$

Placing this in suffix notation, we get that:

$$[\underline{\nabla} \cdot \underline{\mathbf{u}}]_j = \frac{\partial u_j}{\partial x_j}$$

2.3 Curl

the curl of a vector field can be written as:

$$\underline{\nabla} \times \underline{\mathbf{u}}$$

To write this in suffix notation, we can just use the cross produce formula:

$$[\underline{\nabla} \times \underline{\mathbf{u}}]_i = \varepsilon_{ijk} \underline{\nabla}_j u_k$$

which then can be manipulated into:

$$[\underline{\nabla} \times \underline{\mathbf{u}}]_i = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} \qquad j, k = 1, 2, 3$$

where i is a free index and j, k are dummy suffixes, so $j, k = 1, 2, 3$

3 Combinations of gradient, divergence and curl

3.1 Divergence of Gradient

If we take $\underline{\nabla} \cdot \underline{\nabla} f$ where $f = (x_1, x_2, x_3, t)$. We can write the div of grad as:

$$\begin{aligned}
\underline{\nabla} \cdot \underline{\nabla} f &= \left(\frac{\partial}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial}{\partial z}\hat{\boldsymbol{k}} \right) \cdot \left(\frac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{k}} \right) \\
&= \frac{\partial}{\partial x_1}\frac{\partial f}{\partial x_1} + \frac{\partial}{\partial x_2}\frac{\partial f}{\partial x_2} + \frac{\partial}{\partial x_3}\frac{\partial f}{\partial x_3} \\
&= \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} \\
&= \Delta f
\end{aligned}$$

Where the $\Delta = \underline{\nabla}^2$ is the laplacian. So how do we write this in suffix notation?

$$\begin{aligned}
\underline{\nabla} \cdot \underline{\nabla} f &= \underline{\nabla}_j [\underline{\nabla} f]_j \\
&= \frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_j} \\
&= \frac{\partial^2 f}{\partial x_j}
\end{aligned}$$

3.2 Curl of Gradient

We can write the curl of gradient as:

$$\begin{aligned}
[\underline{\nabla} \times \underline{\nabla} f]_i &= \varepsilon_{ijk} \underline{\nabla}_j \underline{\nabla} f_k \\
&= \varepsilon_{ijk} \frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_k} \\
&= \varepsilon_{ikj} \frac{\partial}{\partial x_k}\frac{\partial f}{\partial x_j} \\
&= -\varepsilon_{ijk} \frac{\partial}{\partial x_k}\frac{\partial f}{\partial x_j} \\
&= -\varepsilon_{ijk} \frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_k} \qquad \text{if } f \in c^2 \\
\implies & \underline{\nabla} \times \underline{\nabla} f = 0
\end{aligned}$$

3.3 Gradient of Divergence

Assume we have a $\underline{\mathbf{u}}$, vector field, and we want $\underline{\nabla}(\underline{\nabla} \cdot \underline{\mathbf{u}})$.

$$
\begin{aligned}
[\underline{\nabla}(\underline{\nabla} \cdot \underline{\mathbf{u}})]_i &= \underline{\nabla}_i \frac{\partial u_j}{\partial x_j} \\
&= \frac{\partial}{\partial x_i} \frac{\partial u_j}{\partial x_j} \\
&= \frac{\partial^2 u_j}{\partial x_i \partial x_j}
\end{aligned}
$$

3.4 Divergence of Curl

We can write divergence of curl as:

$$
\begin{aligned}
[\underline{\nabla} \cdot \underline{\nabla} \times \underline{\mathbf{u}}]_i &= \frac{\partial}{\partial x_i} [\underline{\nabla} \times \underline{\mathbf{u}}]_i \\
&= \frac{\partial}{\partial x_i} \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j}
\end{aligned}
$$

$i, j, k = 1, 2, 3$, so $i \leftrightarrow j$

$$
\begin{aligned}
&= \frac{\partial}{\partial x_j} \varepsilon_{jik} \frac{\partial u_k}{\partial x_i} \\
&= -\varepsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial u_k}{\partial x_i} \\
&= -\varepsilon_{ijk} \frac{\partial}{\partial x_i} \frac{\partial u_k}{\partial x_j} \quad \text{as } \underline{\mathbf{u}} \in c^2
\end{aligned}
$$

As $\underline{\nabla} \cdot (\underline{\nabla} \times \underline{\mathbf{u}}) = -\underline{\nabla} \cdot (\underline{\nabla} \times \underline{\mathbf{u}})$, then we know that $\underline{\nabla} \cdot (\underline{\nabla} \times \underline{\mathbf{u}}) = 0$

3.5 Curl of Curl

We can write curl of curl, $\underline{\nabla} \times (\underline{\nabla} \times \underline{\mathbf{u}})$, as:

$$
\begin{aligned}
[\underline{\nabla} \times (\underline{\nabla} \times \underline{\mathbf{u}})]_i &= \varepsilon_{ijk} \frac{\partial}{\partial x_j} (\underline{\nabla} \times \underline{\mathbf{u}})_k \\
&= \varepsilon_{ijk} \frac{\partial}{\partial x_j} \varepsilon_{klm} \frac{\partial u_m}{\partial x_l} \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \frac{\partial^2 u_m}{\partial x_j \partial x_l} \\
&= \delta_{il}\delta_{jm} \frac{\partial^2 u_m}{\partial x_j \partial x_l} - \delta_{im}\delta_{jl} \frac{\partial^2 u_m}{\partial x_j \partial x_l} \\
&= \frac{\partial^2 u_j}{\partial x_j \partial x_i} - \frac{\partial^2 u_i}{\partial x_j \partial x_j} \\
&= \frac{\partial}{\partial x_i} \frac{\partial u_j}{\partial x_j} - \frac{\partial^2 u_i}{\partial x_j^2} \\
&= [\underline{\nabla}(\underline{\nabla} \cdot \underline{\mathbf{u}})]_i - [\Delta \underline{\mathbf{u}}]_i \\
&= [\underline{\nabla}(\underline{\nabla} \cdot \underline{\mathbf{u}}) - \underline{\nabla}^2 \underline{\mathbf{u}}]_i
\end{aligned}
$$

4 Scalar Field / Vector Fields Defintions

A scalar or vector quantity is said to be a field if it is a function of position. Examples

1. Temperature is a scalar field, $T = T(x, y, z) = T(\underline{\mathbf{r}})$
2. Pressure and Density are also scalr fields $P = P(\underline{\mathbf{r}})$ and $\rho = \rho(\underline{\mathbf{r}})$
3. if a physical quantity is a scalar we speak of a scalar field or function of position.

If a physical quantity is a vector, such as force $\underline{\mathbf{F}} = \underline{\mathbf{F}}(x, y, z)$. We speak of a vector field or vector function.

A vector-valued function is an $f : A \subset \mathbb{R}^n \mapsto \mathbb{R}^m$. So, for each $\underline{\mathbf{x}} = (x_1, \dots, x_n) \in A$, f assigns a value $f(\underline{\mathbf{x}})$, an m-tuple, in $\mathbb{R}^m$. These functions, f, are called vector-valued functions if $m > 1$ and scalar if $m = 1$.

Example 6 *Take the function,* $f : (x, y, z) \mapsto (x^2 + y^2 + z^2)^{\frac{3}{2}}$

Solution 6 *It's a scalar function from* $\mathbb{R}^3$ *to* $\mathbb{R}$.

Example 7 *Take the function* $g : (x_1, x_2, x_3) \mapsto (x_1 x_2 x_3, \sqrt{x_1 x_3})$

Solution 7 *This is a vector valued function from* $\mathbb{R}^3$ *to* $\mathbb{R}^2$

To specify a temperature T in a region A of space requires a function T, $T : A \subset \mathbb{R}^m \mapsto \mathbb{R}$. $T = T(x, y, z)$.

To specify the velocity of a fluid moving in space requires a map, $\underline{\mathbf{v}} : \mathbb{R}^4 \mapsto \mathbb{R}^3$ where $\underline{\mathbf{v}}(x, y, z, t)$ is the velocity of the fluid at (x, y, z) at time t.

When $f : U \subset \mathbb{R}^n \mapsto \mathbb{R}$, we say that f is a real valued function of n-variables with domain U.

Let $f : U : \mathbb{R}^n \mapsto \mathbb{R}$, then graph $f = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^{n+1} : (x_1, \dots, x^n)\}$ If $n = 1$, then we can conclude that graph f is curve in $\mathbb{R}^2$ and if $n = 2$, then graph f is a surface in $\mathbb{R}^3$.

4.1 Level Sets, Curves and Surfaces

A level set is a subset of $\mathbb{R}^3$ on which f is constant. For example, for $f(x, y, z) = x^2 + y^2 + z^2$, the set where $x^2 + y^2 + z^2 = 1$ is alevel set. A level set is a set of $(x, y, z) : f(x, y, z) = c$ where $c \in \mathbb{R}$.

For functions $f(x, y)$, we speak of level curves or contours. example, $f : \mathbb{R}^2 \mapsto \mathbb{R}$, $f(x, y) = x + y + 2$, has as it's graph the inclined plane $z = x + y + 2$. The plane intersects the xy plan where $z = 0$ in the line $y = -x - 2$ and the z-axis at $(0, 0, 2)$. For any $c \in \mathbb{R}$, the level curve of c is the straight line: $y = -x + (c - 2) : L_c\{(x, y) : y = -x + c - 2\} \subset \mathbb{R}^2$