MTH2001

UNIVERSITY OF EXETER

COLLEGE OF ENGINEERING, MATHEMATICS AND PHYSICAL SCIENCES

MATHEMATICS

May 2020

Analysis

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Other examiner: Dr Ana Rodrigues

You have 24 hours to complete this paper from the time of its release
Intended duration 2 hours

Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).

Marks shown in questions are merely a guideline. Candidates are permitted to use approved portable electronic calculators in this examination.

This is an **OPEN NOTE** examination

SECTION A

1. (a) (i) Let A be a domain and $\gamma : [a,b] \to \mathbb{C}$ be a **smooth curve** such that $\gamma([a,b]) \subset A$. Let $f : A \to \mathbb{C}$ be a continuous function. State the formula for the integral of f along γ.

(ii) Let $z = x + iy$ and γ be the line segment starting at $1+i$ and ending at 0. Evaluate, **using the formula in** (i),

$$\int_\gamma (x - y + ix^2)\, \mathrm{d}z.$$

(12)

(b) (i) State the *Cauchy-Riemann equations*.

(ii) Verify them for the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = (2z-5)^2.$$

(12)

(c) (i) State the definition of an open set in $\mathbb{C}$.

(ii) State the definition of a closed set in $\mathbb{C}$. (2)

(d) Determine the radius of convergence of the following series:

$$\sum_{n=0}^{\infty} \frac{(3n)!}{n!(2n)!} z^n.$$

(12)

(e) How many zeros, counted with multiplicity, has the function

$$f(z) = z^5 - 6z^2 + 3z - 1,$$

in the annulus $\{z \in \mathbb{C} : 1 \le |z| < 2\}$? (12)

[**50**]

SECTION B

2. (a) State the *Fundamental Theorem of Contour Integration.* (1)

(b) Is the set

$$D = \{z \in \mathbb{C} : \ \mathrm{Re}(z) \geq 1 \text{ } \mathit{or} \text{ } \mathrm{Re}(z) \leq -1\}$$

closed? Justify your answer. (12)

(c) For any domain U, for any function f holomorphic on U, for any closed contour $\gamma : [a, b] \to \mathbb{C}$ such that the image $\gamma([a, b])$ is completely contained in U, the integral

$$\int_\gamma f(z) \, \mathrm{d}z = 0.$$

Is the above statement true? Justify your answer. (12)

[**25**]

3. (a) State the formula for the length of a smooth curve and use it to compute the length of the smooth curve

$$\gamma(t) = 8e^{i2\pi t} + 6 - 4i$$

from $t = 0$ to $t = 1/4$. (7)

(b) Find the Laurent series expansion of the function

$$f(z) = \frac{1}{(z+i)(z-5i)}$$

which is valid in the annulus where $1 < |z| < 5$. (9)

(c) Compute the integral

$$\int_0^\infty \frac{x^{10}}{1+x^{38}} \, \mathrm{d}x$$

expressing your answer as an evidently real number. (9)

[**25**]

4. Let

$$f(z) = \frac{1}{(e^z + 1)(z - i\pi)^2}.$$

(a) Find the order of each pole of f and calculate the residue of f at each simple pole. (8)

(b) Consider the Laurent expansion of $f(z)$ around $i\pi$:

$$f(z) = \sum_{n=-N}^{\infty} b_n (z - i\pi)^n$$

where N is the order of the pole of f at $i\pi$. Calculate b_{-2} and b_{-1}. (9)

(c) Evaluate $\int_\gamma f(z)\ \mathrm{d}z$ where γ is the circular contour, transversed anticlockwise, with centre 0 and radius 2π. (8)

[**25**]