Year 3 — Dynamical Systems and Control

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Contents

1 Preliminaries

1.1 Continuous time Dynamical Systems

We are going to consider, *Lecture 1*

$$\frac{d\mathbf{x}}{dt}(t) = f(x(t), u(t))$$
$$y(t) = g(x(t), u(t))$$

We are going to call x the state and u the input and y the output. There may be a case where our variables are vector valued and hence have a system of differential equations. We call f and g time invariant as they do not vary with t and not explicitly dependent on t.

Example. The equations governing aerobic digestion are,

$$\frac{db}{dt} = (e^{-s} - D)b$$
$$\frac{ds}{dt} = ke^{-s}b + D(s_I - s)$$

where b and s are biomass and substrate concentrations, which comprise the states. Then D and s_I are the dilution rate and input substrate concentration, these are the inputs.

We consider systems over some $0 \leq t \leq t_1$ and we consider where $x(t)$ is uniquely determined over our interval by the initial condition and the input on that same interval. This places a constrain on the functions f and g since, in general, $x(t)$ need not be uniquely determined by the initial condition and the input.

Our form may seem rather restrictive, however, it's less restrictive than it appears, let's consider a pendula *Lecture 2*

Example. The angle of a damped pendulum is defined by,

$$mL^2\frac{d^2\theta}{dt^2} = -\nu\frac{d\theta}{dt} - mgL\sin(\theta) + T$$

now, we let $x_1 = \theta$, $x_2 = \frac{d\theta}{dt}$, $u = T$ and $y = \theta$. Now we write this in the previous form,

$$\frac{dx_1}{dt} = \frac{d\theta}{dt} = x_2$$
$$\frac{dx_2}{dt} = \frac{d^2\theta}{dt^2} = \frac{1}{mL^2}\left(-\nu x_2 - mgL\sin(x_1) + u\right)$$
$$y = \theta = x_1$$

Hence,

$$f_1 = x_2$$
$$f_2 = \frac{1}{mL^2}\left(-\nu x_2 - mgL\sin(x_1) + u\right)$$
$$g_1 = x_1$$

Definition 1.1 (Autonomous)**.** If the input $u(t)$ is missing, then the system is said to be autonomous and the state and output depend only on the initial state.

particular attention is to be paid to linear time-invariant systems, the solutions to linear ODEs. Then we can write them as,

$$\frac{dx}{dt}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

1.2 Equilibria and Stability

Definition 1.2 (Equilibrium)**.** Consider a fixed input $u(t) = u_e \forall\, t \in \mathbb{R}$. Then the state and input pair (x_e, u_e) is called an equilibrium if $f(x_e, u_e) = 0$.

Example. We consider the anaerobic digester. Then, we need solutions to,

$$(e^{-s} - D)b = 0 \qquad ke^{-s}b + D(s_I - s) = 0$$

We can solve these equations nicely, and get the following equilibrium,

$$\begin{aligned}
(x_e, u_e) &= \left(\begin{bmatrix} 0 \\ c_1 \end{bmatrix}, \begin{bmatrix} 0 \\ c_2 \end{bmatrix}\right) \\
(x_e, u_e) &= \left(\begin{bmatrix} 0 \\ c_3 \end{bmatrix}, \begin{bmatrix} c_4 \\ c_3 \end{bmatrix}\right) \\
(x_e, u_e) &= \left(\begin{bmatrix} c_5 \\ c_6 \end{bmatrix}, \begin{bmatrix} e^{-c_6} \\ c_6 - kc_5 \end{bmatrix}\right)
\end{aligned}$$

If we let the input depend on the state, $u(t) = k(x(t)), \forall\, t \geq 0$ and some function k, then we can define new functions $F(x(t)) = f(x(t), u(x(t)))$ and $G(x(t)) = g(x(t), u(x(t)))$, whereupon we obtain an autonomous system,

$$\begin{aligned}
\frac{dx}{dt}(t) &= F(x(t)) \\
y(t) &= G(x(t))
\end{aligned}$$

For a autonomous system the state x_e is an equilibrium point if $F(x_e) = 0$. This is a lot simpler. A system may have many equilibria.

Definition 1.3 (Stability)**.** Informally we call an equilibria stable if whenever $x(0)$ is sufficiently close to x_e if $x(t)$ remains close to $x_e, \forall\, t \geq 0$

Definition 1.4 (Asymptotically Stable)**.** A system is asymptotically stable if it is stable ad in addition, if $x(0)$ is sufficiently close to x_e, then $x(t) \to x_e$ as $t \to \infty$.

If we want to see if a system is stable, we can do so by considering energy. In terms of our pendulum, the energy is,

$$V(x_1, x_2) = \frac{g}{L}(1 - \cos(x_1)) + \frac{1}{2}x_2^2$$

and if the system doesn't have an increasing change in energy, then we can say that it stays relatively close to an initial condition and hence can be asymptotically stable. This applies to our $V(x_1, x_2)$.

If we consider a torque input, $u = mL^2x_1$ then will result in an unstable system with respect to it's equilibria point $(0, 0)$. *Lecture 3*

1.3 Linearisation

You can apply our linear tecniques to non-linear systems by linearising them. Firstly, consider an equilibrium point and let f and g be continuously differentiable. Then, we can linearise by using Taylor Series.

Let $u(t) = u_e + \delta u(t)$ and $x(t) = x_e + \delta x(t)$ and also $y_e = g(x_e, u_e)$. Then,

$$\begin{aligned}
\frac{d\delta x}{dt} &= A\delta x + B\delta u + O(x^2) \\
\delta y &= C\delta x + D\delta u + O(x^2)
\end{aligned}$$

where we define A, B, C, D as,

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_d} \\ \vdots & & \vdots \\ \frac{\partial f_d}{\partial x_1} & \cdots & \frac{\partial f_d}{\partial x_d} \end{bmatrix} (x_e, u_e) \qquad B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_d} \\ \vdots & & \vdots \\ \frac{\partial f_d}{\partial u_1} & \cdots & \frac{\partial f_d}{\partial u_d} \end{bmatrix} (x_e, u_e)$$

$$C = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_m} \end{bmatrix} (x_e, u_e) \qquad B = \begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \cdots & \frac{\partial g_1}{\partial u_m} \\ \vdots & & \vdots \\ \frac{\partial g_m}{\partial u_1} & \cdots & \frac{\partial g_m}{\partial u_m} \end{bmatrix} (x_e, u_e)$$

where $f \in (\mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R})^d$, $g \in (\mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R})^m$ and $u \in (\mathbb{R} \to \mathbb{R})^n$ and $x \in (\mathbb{R} \to \mathbb{R})^d$.

Example. We consider the Lotka-Volterra equations,

$$\frac{dx_1}{dt} = ax_1 - bx_1x_2 \qquad \frac{dx_2}{dt} = cx_1x_2 - dx_2$$

where x_1 and x_2 are the prey and predators and $a, b, c, d \in \mathbb{R}$. We have some equilibrium at $(0, 0)$ and also $\left(\frac{d}{c}, \frac{a}{b}\right)$.
We can calculate our matrix and get,

$$\frac{d\delta\mathbf{x}}{dt} = \begin{bmatrix} 0 & -\frac{bd}{c} \\ \frac{ac}{d} & 0 \end{bmatrix} \delta\mathbf{x} + O(x^2)$$

Example. Consider our friend, the damped pendulum,

$$\begin{aligned} \frac{dx_1}{dt} &= x_1 \\ \frac{dx_2}{dt} &= -\frac{\nu}{mL^2}x_2 - \frac{g}{L}\sin(x_1) + \frac{1}{mL^2}u \end{aligned}$$

where $x_1 = \frac{\pi}{6}$ and $x_2 = 0$ and $u = \frac{mgL}{2}$. Then we can form the linearisation matrix,

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \left(\frac{\pi}{6}, 0, \frac{mgL}{2}\right) = \begin{bmatrix} 0 & 1 \\ -\frac{\sqrt{3}g}{2L} & -\frac{\nu}{mL^2} \end{bmatrix}$$

and as we have an input, we need to form a second matrix,

$$\begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} \left(\frac{\pi}{6}, 0, \frac{mgL}{2}\right) = \begin{bmatrix} 0 \\ \frac{1}{mL^2} \end{bmatrix}$$

Thus,

$$\frac{d\delta\mathbf{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{\sqrt{3}g}{2L} & -\frac{\nu}{mL^2} \end{bmatrix} \delta\mathbf{x} + \begin{bmatrix} 0 \\ \frac{1}{mL^2} \end{bmatrix} \delta u + O(x^2)$$

1.4 Discrete Time Dynamical Systems

Here is a difference equation,

$$\begin{aligned} x(k+1) &= f(x(k), u(k)) \\ y(k) &= g(x(k), u(k)) \end{aligned}$$

Here x, y, u are sequences defined for all $k \geq 0 \in \mathbb{Z}$. We write $u, x, y \in \mathbb{Z}_+ \to \mathbb{R}$.

We call (x_e, u_e) an equilibrium and input pair if x_e and u_e if there are constant vectors satisfying $f(x_e, u_e) = x_e$. There are also similar ideas to what we considered for continuous systems.

2 LTI Systems

We are going to consider some linear ODES, *Lecture 4*

$$\frac{d^n y}{dt^n} + \frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \cdots + b_0 u$$

qhwew $a_i, b_i \in \mathbb{R}$ and $u, y : \mathbb{R} \to \mathbb{R}$ and we consider $m < n$ for the moment, so the order of differentiation of u doesnt exceed m. To solve these we are going to consider laplace transform.

2.1 Laplace Transforms

Definition 2.1 (exponentially bounded)**.** Here, f is called exponentially bounded, if $|f(t)| \leq Me^{\alpha t}$ for some $\alpha \in \mathbb{R}$

The laplace transforms for an exponentially bounded function $f(t)$ defined on $t \geq 0$ is defined by,

$$\mathcal{L}(f) = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

The above integral is defined for all $s \in \mathbb{C}$ where $\operatorname{Re} c > \alpha$.
This is a restriction for the existence of laplace transform, but not the DEs. When f is exponentially bounded, means that just that the integral converges nicely, it makes our life easier.

Lemma 2.2. If $f = g$, then $f = g$ for (almost[1]) all $t \geq 0$.

We will focus on piecewise continuous, in which $f = g$ for all $t \geq 0$.
This means that the laplace transform is invertible, then we can say $\mathcal{L}^{-1}(\mathcal{L}(f)) = f$ and we will use lookup tables for these transforms.

Remark. Note, we can say nothing about $\mathcal{L}^{-1}(\mathcal{L}(f))$ for $t < 0$

Let f and g be defined on $t \geq 0$

(i) $\mathcal{L}(af + bg) = a\mathcal{L}(f) + b\mathcal{L}(g)$

(ii) $\mathcal{L}(e^{at}) = \frac{1}{s-a}$

(iii) $\mathcal{L}(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}}$

(iv) $\mathcal{L}(\frac{d^k f}{dt^k}) = s^k \mathcal{L}(f) - s^{k-1}f(0) - \cdots - s\frac{d^{k-2}f}{dt^{k-2}}(0) - \frac{d^{k-1}f}{dt^{k-1}}(0)$

(v) $\mathcal{L}\left(\int_0^t f(\tau)\, d\tau\right) = \frac{1}{s}\mathcal{L}(f)$

(vi) $\mathcal{L}(\int_0^t g(t-\tau)f(\tau)d\tau) = \mathcal{L}(f(t))\mathcal{L}(g(t))$

Note, in 4, we assume that f is k differentiable. We will sometimes want to lift this assumption.
If we take the laplace transform of the ODE,

$$\frac{d^n y}{dt^n} + \frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \cdots + b_0 u$$

we can rearrage and get something of,

$$Y(s) = \frac{b(s)}{a(s)}U(s) + \frac{c(s) - d(s)}{a(s)}$$

[1] we just want the integrals to be equal, but at some t, $f(t) \neq g(t)$. If they are continuous, this doesn't matter

where,

$$
\begin{aligned}
a(s) &= s^n + a_{n-1}s^{n-1} + \cdots + a_0 \\
b(s) &= b_m s^{n-1} + b_{m-1}s^{m-1} + \cdots + b_0 \\
c(s) &= y(0)s^{n-1} + \left(\frac{dy}{dt}(0) + a_{n-1}y(0)\right)s^{n-2} + \cdots + \left(\frac{d^{n-1}y}{dt^{n-1}}(0) + a_{n-1}\frac{d^{n-2}y}{dt^{n-2}}(0) + \cdots + a_1 y(0)\right) \\
d(s) &= b_m u(0)s^{m-1} + \left(b_m\frac{du}{dt}(0) + b_{m-1}u(0)\right)s^{m-2} + \cdots + \left(b_m\frac{d^{m-1}u}{dt^{m-1}}(0) + b_{m-1}\frac{d^{m-2}u}{dt^{m-2}} + b_1 u(0)\right)
\end{aligned}
$$

Thus,

$$
y(t) = \mathcal{L}^{-1}\left(\frac{b(s)}{a(s)}U(s) + \frac{c(s)-d(s)}{a(s)}\right)
$$

We are going to use result 7, then we can continue from,

$$
\begin{aligned}
&= \mathcal{L}^{-1}\left(\frac{b(s)}{a(s)}U(s) + \frac{c(s)-d(s)}{a(s)}\right) \\
&= \int_0^t \mathcal{L}^{-1}\left(\frac{b(s)}{a(s)}\right)(t-\tau)u(\tau)\,d\tau + \mathcal{L}^{-1}\left(\frac{c(s)-d(s)}{a(s)}\right)
\end{aligned}
$$

We call the ratio $G(s) = \frac{b(s)}{a(s)}$ the transfer function, $W(t) = \mathcal{L}^{-1}\left(\frac{b(s)}{a(s)}\right)$ the impulse response, and $y_f(t) = \mathcal{L}^{-1}\left(\frac{c(s)-d(s)}{a(s)}\right)$ the free response, so,

$$
y(t) = \int_0^t W(t-\tau)u(\tau)\,d\tau + y_f(t)
$$

The inverse Laplace transforms can be obtained using a partial fraction decomposition. We will show how this works in general later, so consider the example,

Example. Consider, $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \frac{du}{dt} + 3u$ with $u(t) = \sin t$, $y(0) = 0$ and $\frac{dy}{dt}(0) = 1$. So,

- Find the free response
- Find the impulse response
- Find $y(t)$ $(t \geq 0)$

We note that both $W(t)$ and $y_f(t)$ both take forms of $\mathcal{L}^{-1}\left(\frac{p(s)}{a(s)}\right)$, so let's find the closed form for this, *Lecture 5*
We split it up into,

$$
\frac{p(s)}{a(s)} = \sum_{i=1}^{N}\sum_{j=1}^{r_i}\frac{h_{i,j}}{(s-\lambda_i)^j}
$$

where $\lambda_i, h_{i,j} \in \mathbb{C}$ and $a(s) = \prod_{j=1}^{N}(s-\lambda_i)^{r_i}$. Here with $f_k(s) = \frac{p(s)}{\prod_{i=1,i\neq j}^{N}(s-\lambda_i)^{r_i}}$ the coefficients $h_{k,j}$ can we obtained from some formula,

$$
\begin{aligned}
h_{k,r_k} &= f_k(\lambda_k) \\
h_{k,r_k-1} &= \frac{df_k}{ds}(\lambda_k) \\
&\vdots \\
h_{k,1} &= \frac{1}{(r_k-1)!}\left(\frac{d^{r_k-1}f_k}{ds^{r_k-1}}(\lambda_k)\right)
\end{aligned}
$$

thus,

$$\mathcal{L}^{-1}\left(\frac{p(s)}{a(s)}\right) = \sum_{i=1}^{N}\sum_{j=1}^{r_i} \frac{h_{i,j}t^{j-1}e^{\lambda_i t}}{(j-1)!}$$

Applying this we get some formula. We assumed that both $u(t)$ and $y(t)$ are exponentially bounded and sufficiently differentiable, in whih case $u(t)$ and $y(t)$ satisfy the differential equation even when they are not exponentially bounded. Now we onlu need to assume that they are integrable, then we can get a weak solution, i.e. when they are not differentiable. We can consider a step function. We need to have the correct initial conditions.

We only consider piecewise continuous functions, then we define $y(0) = y(0_-)$ and this is the left hand limit, and the same for everything else. To account for this, we modify the Laplace transform to say,

$$\mathcal{L}\left(\frac{d^n f}{dt^n}\right) = s^k \mathcal{L}(f) - s^{k-1}f(0_-) - \dots - s\frac{d^{k-2}f}{dt^{k-2}}(0_-) - \frac{d^{k-1}f}{dt^{k-1}}(0_-)$$

Let's consider the degree of differentiation of the RHS is greater than the left, so we will do polynomial long division. So take the RHS and replace theoremstyle differentials with s^r, then we get,

$$g_r s^r + g_{t-1}s^{r-1} + \dots + g_0 = (s^n + a_{n-1}s^{n-1} + \dots)(q_{r-n}s^{r-n} + q_{r-n-1}s^{r-n-1} + \dots) + b_m s^m + b_{m-1}s^{m-1} + \dots$$

Then we define $y = \hat{y} - (q_{r-n}\frac{d^{r-n}u}{dt^{r-n}} + q_{r-n-1}\frac{d^{r-n-1}u}{dt^{r-n-1}} + \dots)$ then we can transform back to our original equation.

Input output stability. We shall consider our general linear ODE, then with $m < n$. We have shown that this has a unique solution for any integrable u and $y(0_-)$ and all of it's derivatives up to $n-1$. We call a system input-output stable if, *Lecture 6*

– if $u(t) = 0$ for all $t \geq 0$ then $y(t) \to 0$ as $t \to \infty$.

– If $\sup_{t\geq 0}|u(t)| < \infty$, then $\sup_{t\geq 0}|y(t)| < \infty$.

We will now show that a system is stable if and only if all of the roots of $a(s) = s^n + a_{n-1}s^{n-1} + \dots + a_0$ are in the open left half plane. To prove this,

– 1 implies that the roots of $a(s)$ are all in the open left half plane

– If all the roots of $a(s)$ are in the open left half plane then 1 holds.

– If all of the roots of $a(s)$ are in the opel left half plane then 2 holds.

For (a) holds, let $u(t) = 0$, this implies that,

$$y(t) = \mathcal{L}^{-1}\left(\frac{c(s)}{a(s)}\right)$$

and the a and c are what we expect they are. If $\frac{d^{n-1}y}{dt^{n-1}}(0) = 1$ and $y(0) = \frac{dy}{dt}(0) = \dots = 0$, then we can say that,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left(\frac{1}{a(s)}\right) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{r_i}\frac{h_{i,j}t^{j-1}e^{\lambda_i t}}{(j-1)!} \end{aligned}$$

where $\widetilde{h}_{i,r_i} \neq 0$. Thus, we can say that for these to converge, we must say $\operatorname{Re}\lambda_i < 0$ and λ_i are the roots. Hence, first is proved.

For (b) and (c), note that all the roots are in the open left half plane, then $W(t) = \sum_{i=1}^{N} \sum_{j=1}^{r_i} \frac{\hat{h}_{i,j} t^{j-1} e^{\lambda_i t}}{(j-1)!}$ and $y_f(t) = \sum_{i=1}^{N} \sum_{j=1}^{r_i} \frac{\widetilde{h}_{i,j} t^{j-1} e^{\lambda_i t}}{(j-1)!}$ for some $\widetilde{h_{i,j}}, \hat{h}_{i,j} \in \mathbb{C}$, where $a(s) = \prod_{i=1}^{N} (s - \lambda_i)^{r_i}$, so λ_i are all in the open left half plane.
Hence we can now choose a $0 > \lambda \in \mathbb{R}$ such that $\lambda_i - \lambda$ is in the open left half plane. We can then find $M, N \in \mathbb{C}$ such that $|W(t)| \leq M e^{\lambda t}$ and $|y_f(t)| \leq N e^{\lambda t}$ forall $t \geq 0$. This then proves (b).

For (c),

$$\begin{aligned}
\sup_{t \geq 0} |y(t)| &= \sup_{t \geq 0} \left| \int_0^t W(t-\tau) u(\tau)\, d\tau + y_f(t) \right| \\
&\leq \sup_{t \geq 0} \left| \int_0^t W(t-\tau) u(\tau)\, d\tau \right| + \sup_{t \geq 0} |y_f(t)| \\
&\leq \sup_{t \geq 0} \int_0^t |W(t-\tau) u(\tau)|\, d\tau + \sup_{t \geq 0} |y_f(t)| \\
&\leq \sup_{t \geq 0} \int_0^t |W(t-\tau)|\, d\tau \times \sup_{t \geq 0} |u(\tau)| + \sup_{t \geq 0} |y_f(t)| \\
&\leq M \sup_{t \geq 0} \int_0^t e^{\lambda \tau}\, d\tau \times \sup_{t \geq 0} |u(t)| + \sup_{t \geq 0} |y_f(t)|
\end{aligned}$$

and so it follows that $\sup_{t \geq 0} |u(t)| < \infty$ implies $\sup_{t \geq 0} |y(t)| < \infty$ which proves (c).
If we have u differentiated the same amount of $\hat{y}$, then we can again use long division to prove the same result, by letting $\hat{y} = y + q_0 u$. Then we can show that this substitution still allows the lemma to be true.

2.2 Routh Hurwitz Stability Criterion

We have show that linear ODEs are stable if and only if $a(s)$'s roots are in the open left half plane. We could solve $a(s) = 0$, however this requires numerical techniques which could end up with rounding errors, for example,

Example. Consider $a(s) = (s - \alpha)^n$ and now consider a pertibation on those roots with a polynomial, $a_\varepsilon(s) = (s - \alpha)^n - \varepsilon$ with roots $\alpha + \varepsilon^{\frac{1}{n}} e^{\frac{2k\pi i}{n}}$ for $k \in \mathbb{Z}$ and $k < n - 1$.

For a polynomial $a(s)$, the Routh Hurwitz criterion provides a computable condition based solely on coefficients. where $r_{0,0} = a_0$ and $r_{0,1} = a_2$, $r_{0,2} = a_4$ and $r_{1,0} = a_1$, $r_{1,1} = a_3$ and so on. Then onwards from

$$\begin{matrix}
r_{0,0} & r_{0,1} & r_{0,2} & \cdots \\
r_{1,0} & r_{1,1} & r_{1,2} & \cdots \\
\vdots & \vdots & \vdots & \\
r_{n,0} & r_{n,1} & r_{n,2} & \cdots
\end{matrix}$$

that we define $r_{i,j} = r_{i-1,0} \times r_{i-2,j+1} = r_{i-2,0} \times r_{i-1,j+1}$. Then we say that all of the roots of $a(s)$ are in the open left half plane if and only if $r_{i,0} > 0$ for $i = 0, 1, 2, \ldots, n$.

Proof. I am not LaTeXing that, I'm sorry, I've just looked at the slides. I'll give a brief idea though. *Lecture 7*

We firstly form our Routh array in the usual way, our criterion states that the roots are in the left open half plane when all the left entries are positive. We form some polynomials, and a series of polynomials where $a(s) = a_0(s)$.
Then, we can say that $r_{i+1,0} \neq 0$ and then we say that when the left entries are positive, then the degrees of the polynomials decrease. We aim to show the following are equivalent:

- The roots of $a_i(s)$ are all in the open left half place and $a_i, a_{i+1}, r_{i,0}, r_{i+1,0} > 0$
- The roots of $a_{i+1}(s)$ are all in the open half plane and (some other coefficients are ¿ 0).

We can note a_i and a_{i+1} differ by a degree of one. If our polynomials roots are in the left half plane, then we can say the coefficients must all have the same sign.
Hence, now consider $\hat{a}_{i+1}^{\gamma}$ with $0 \leq \gamma \leq 1$ and all the left hand entries are greater than 0. We can then increase γ, then we get something proportional to a_i and then at 1, we are proportional to sa_{i+1}. When $s = 1$, we can say that there is a zero at the origin of multiplicity one.
All of the polynomials of our new family have same same degree. Then if $a_{i+1}^{\gamma}(i\omega) = 0$ $0 \neq \omega \in \mathbb{R}$, then all of the polynomials have this root. We finally note that since all of the hat polynomials have the same degree, then the roots will vary continuously with γ. Then we can say as γ varies none of the roots of the polynomials can cross the imaginary axis.
If all of the roots of $a_i(s)$ then no roots will cross the imaginary axis and one approaches the origin, but that polynomial with that root is divisble by a_{i+1} and so then a_{i+1} must also be in the open left half plane. If we decrease γ, then again no roots cross the left open half plane, so the ones at the origin, move into the left or right. So we now show that $\alpha_{i+2}, r_{i+2,0} > 0$ moves into the left half plane. Hence, we now have the equivalence condition.
Now induct on the equivalence, then if all the original or initial polynomials have roots in the open left half plane, then so must the rest by assumption and induction. $\square$

3 Closed Loop Stability

We are now interested to use feedback control on our system to let u be influenced on y and pehaps an addition external input signal w. We can capture what we are interested in a block diagram, *Lecture 8*

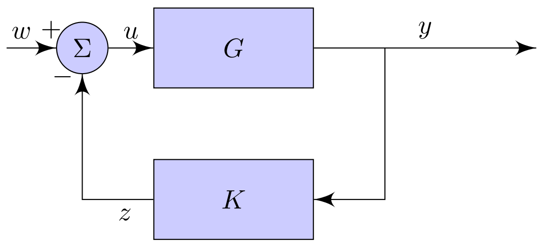

Figure 1: Closed Loop System

We are interested that G and K are ODEs of a similar form.

$$\frac{d^{n_1}y}{dt^{n_1}} + \frac{d^{n_1-1}y}{dt^{n_1-1}} + \cdots + a_0 y = b_m \frac{d^{m_1}u}{dt^{m_1}} + b_{m_2-1}\frac{d^{m_3-1}u}{dt^{m_3-1}} + \cdots + b_0 u$$

$$\frac{d^{n_2}y}{dt^{n_2}} + \frac{d^{n_2-1}y}{dt^{n_2-1}} + \cdots + a_0 y = b_m \frac{d^{m_2}u}{dt^{m_2}} + b_{m_2-1}\frac{d^{m_2-1}u}{dt^{m_2-1}} + \cdots + b_0 u$$

$n_1 \geq m_1$ and $n_2 \geq m_2$.
Let $a(s) = s^{n_1} + a_{n_1-1}s^{n_1-1} + \cdots + a_0$ etc. and then let $G(s) = \frac{b(s)}{a(s)}$ and $K(s) = \frac{q(s)}{p(s)}$, then we can look at the laplace transforms,

$$Y(s) = G(s)U(s) + \frac{c(s)}{a(s)} \qquad Z(s) = K(s)Y(s) + \frac{d(s)}{p(s)} \qquad U(s) = W(s) - Z(s)$$

and now we can eliminate u and z, to obtain,

Alternatively, breaking up $G(s)$ and $K(s)$

$$Y(s) = \frac{b(s)p(s)}{a(s)p(s) + b(s)q(s)}W(s) + \frac{c(s)p(s) - d(s)b(s)}{a(s)p(s) + b(s)q(s)}$$

we assume that $a(s)p(s)$ and $b(s)q(s)$ do not have any common roots in the closed right half plane. We will show that the closed root system is stable if and only if $1+G(s)K(s)$ has no zeros in the closed right half plane.

Note that this is basically boiled down to showing the howing right half plane zeros of the equation coincide with the closed right half plane roots of $ap + bq$ and poles of $\frac{G(s)}{1+G(s)K(s)}$

3.1 Nyquist Stability Condition

We have considered closed loop systems characterised by the polynomials from the ODEs, and the relationship between their laplace transforms.

The Nyquist Stability Condition, proves a necessary and sufficient condition for all of the zeros fo $1 + G(s)K(s)$ to be in the open left half plane.

We will consider the Nyquist Diagram of $L(s) = G(s)K(s)$ is a plot of $L(s)$ in the complex plane as s traverses a path in the complex plane, ie. the Nyquist contour.

Example. For example, study $L(s) = \frac{s+1}{s^2+1}$ which has poles at $s = \pm i$, the Nyquist contour, so we start elsewhere.

Lemma 3.1 (Nyquist Stability Condition)**.** The Nyquist stability criterion states that the zeros of $1 + L(s)$ are all in the open left half plane if and only if the Nyquist diagram of $L(s)$ does not pass through the point $s = -1$ and encircles this point as many times anticlockwise as the number of poles of $L(s)$ is in the open right half plane.

We are now going to prove the Nyquist Stability Condition. *Lecture 9*

Proof. We note that the poles of $L(s)$ are the same as $1 + L(s)$. Hence, the contour will enclose all of the poles of $1 + L(s)$, and there is a zero on the imaginary axis if and only if it passes through $s = -1$.
Factor

$$1 + L(s) = \alpha \frac{(s - z_1)(s - z_2) \dots (z - p_M)}{(s - p_1) \dots (s - p_N)}$$

and now we say that for $s_0 \in \mathbb{C}$ the argument of $1 + L(s)$ is the sum of the arguments of $(s_0 - z_j)$ minus the sum of the argument of $(s_0 - p_j)$, $j = 1, \dots, N$.
Next as s traverses the contour, then the change of argument of $s - \beta$ is -2π if $\beta \in N_C$ otherwise zero. It follows that the change in argument of $1 + L(s)$ as s traverses the Nyquist contour is $2\pi(P - Z)$ where P is the number of poles and Z is the number of zeros. This $P - Z$ is just the number of anticlockwise rotations around -1 by $L(s)$.
Thus $1 + L(s)$ has no zeros in the ORHP if and only if the number of anticlockwise rotations encirclements of the point -1 by the Nyquist Diagram is equal to the number of poles of $L(s)$ in the ORHP. □

Useful sketching tips,

(i) Consider $L(i\omega)$ as $\omega \to \infty$ and $\omega \to 0$

(ii) Determine what happens at the imaginary axis poles of $L(s)$, you could use a Laurent Expansion, however here a few rules of thumb,

- When the Nyquist contour rotates through an angle, so does the diagram in the same direction and angle.
- As the Nyquist contour turns through a semicircle the diagram will span an arc of the circle through an angle of π times the multiplicity of the pole.

(iii) Sketch $L(i\omega)$ for $0 \leq \omega < \infty$ paying attention to the signs of it's real and imaginary parts and the points where the imaginary or real part is zero.

(iv) The plot of $L(i\omega)$ for $-\infty < \omega \leq 0$, just reflect through the real axis

(v) Determine what happens on the infinite circular arc.

Now we consider, *Lecture 10*

$$L(s) = \frac{1}{(s^2 + 1)^2}$$

we can then find the residue of this as,

$$A = \lim_{s \to i} ((s - i)^2 L(s)) = -\frac{1}{4}$$

and now we can find the small semicircular arc and then the points here are $s = i + re^{i\theta}$ as $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ Now we put this into the Laurent series,

$$L(i + re^{i\theta}) = \frac{1}{4r^2} e^{i(\pi - 2\theta)} + \dots$$

For small r, we just have $\frac{1}{4r^2}$ and so we have a clockwise circle. Then the rest of the Nyquist diagram lies on the positive real axis. $L(i\omega) \to 0$ as $\omega \to \infty$ and $L(0) = 0$. There are two clockwise encirclements of -1 and

$L(s)$ has no poles in the ORHP, so the closed loop isn't stable.

We can use Matlab to sketch these Nyquist diagrams, however we can't use it when we have an imaginary axis pole.

Example. We have proved the Nyquist Stability condition for quotients of functions. However, it also holds for meromorphic functions in the OHRP. In particular, this allows us to consider time delays. We note that $\mathcal{L}(f(t-\tau)) = e^{-st}\mathcal{L}(f(t))$ and consider the system, *Lecture 11*

$$\begin{aligned}\frac{dz}{dt}(t) + z(t) &= 2y(t)\\ y(t) &= u(t-T)\\ u(t) &= w(t) - z(t))\end{aligned}$$

Now we are going to take Laplace transforms where $w = u = y = z$. Hence we get,

$$\begin{aligned}Y(s) &= e^{-sT}U(s)\\ (s+1)Z(s) &= 2Y(s)\end{aligned}$$

so we want to sketch $L(s) = \frac{2e^{-sT}}{s+1}$ to sketch this we can consider $\frac{2}{s+1}$, but now we consider the exponential, we know that $|e^{-i\omega T} = 1|$ and we know $\arg(e^{-i\omega T}) = -\omega T$. Thus we know that the Nyquist diagram can be obtained from the Nyquist diagram of $\frac{2}{s+1}$ by changing the argument by an amount $-\omega T$ and we know ω is increasing and so we will have a spiral.

We note that above a certain critial value, then it will be sufficient for it to encircle -1. We got our Nyquist diagram by changing the argument of each point. We note that $|\frac{2}{1+i\omega}| = 1$ we then require to find $\omega_c T$ is equal to the difference between $-\pi$ and $\arg(\frac{2}{1+i\omega_c})$ more specifically $\omega_c T = \pi + \arg(\frac{2}{1+i\omega_c})$.

Firstly, we can say that $|\frac{2}{1+\omega_c}|^2 = |\frac{4}{1+\omega_c^2}| = 1$, i.e. $\omega_c = \sqrt{3}$. Then, $\arg(\frac{2}{1+i\sqrt{3}}) = -\frac{\pi}{3}$. We therefore require $\omega_c T = \sqrt{T} = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$. Hence $T = \frac{2\pi}{3\sqrt{3}}$.

3.2 System Response

We have seen how laplace transforms are particularly suited to our uses and adding onto of the RH and NSC.

Consider our usual DE,

$$\frac{d^n y}{dt^n} + \frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \cdots + b_0 u$$

Now, note that if $X \in \mathbb{C}$

$$\begin{aligned}X &= |X|e^{i\arg X}\\ |X|\cos(\omega_0 + \arg X) &= \mathrm{Re}(|X|e^{i(\omega_0 t + \arg X)}) = \mathrm{Re}(Xe^{i\omega_t t})\\ \mathcal{L}(|X|\cos(\omega_0 t + \arg(X))) &= \mathcal{L}(\mathrm{Re}(Xe^{i\omega_0 t}))\\ &= \mathcal{L}\left(\frac{1}{2}(Xe^{i\omega_0 t}) + \overline{X}e^{-\omega_0 t}\right)\\ &= \frac{1}{2}\left(X\frac{1}{s-i\omega_0} + \overline{X}\frac{1}{s+i\omega_0}\right)\end{aligned}$$

We let $u(t) = \cos(\omega_0 t)$. Using partial fraction expansions, it follows that,

$$\begin{aligned} Y(s) &= G(s)U(s) \\ &= \frac{1}{2}\left(G(i\omega_0)\frac{1}{s - i\omega_0} + G(-i\omega_0)\frac{1}{1 + i\omega_0}\right) + \frac{p(s)}{a(s)} \end{aligned}$$

for some polynomial $p(s)$. Since $G(-i\omega_0) = \overline{G(i\omega_0)}$, then

$$y(t) = |G(i\omega_0)| \cos(\omega_0 + \arg G(i\omega_0)) + \mathcal{L}^{-1}\left(\frac{p(s)}{a(s)}\right)$$

Moreover, if the roots of $a(s)$ are in the ORHP, then $\mathcal{L}^{-1}\left(\frac{p(s)}{a(s)}\right) \to 0$ as $t \to \infty$ so $y(t)$ tends towards to a soinasoidal system with the same frequence as $u(t)$ but with applitude multiplied by $|G(i\omega_0)|$ and phase increased by $\arg G(i\omega_0)$