Year 3 — Lie Groups and Applications in Geometric Dynamics

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

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Part I:

Lectures

1 Lie Groups and Algebras

To start, let us define some groups. Firstly, what is a group?

Definition 1.1 (Group)**.** G is a nonempty set and endowed with a binary operation such that,

- It's closed under $(\cdot)$, $\forall\, a, b \in G$, $a \cdot b \in G$
- It's associative, i.e. $\forall\, a, b \in G$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
- There is an identity element, $\forall\, a \in G$, $a \cdot e = a = e \cdot a$.
- Every element has an inverse, $\forall\, a \in G$, $a \cdot a^{-1} = e = a^{-1} \cdot a$.

The definition of a Lie group also uses something called a manifold, to define this let's go on a slight adventure into topology!

1.1 Manifolds

A manifold is a topological space that locally resembles Euclidean space near each point [5]. However, this doesn't sate me as, what does near mean? It's very informal and hand wavey. Let's define it properly!

Definition 1.2 (Manifold)**.** A manifold is a second countable Hausdorff space that is locally homeomorphic to Euclidean space.

Where we refer to a second countable space as being a topological equivalent to being finitely generated. There exists some countable base of this space $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$, where any open subset of our space, T, can be written as a disjoint union of a finite subfamily of $\mathcal{U}$. This nicely restricts manifolds to be smaller spaces, by making them be the union of countably many open sets. On the point of Hausdorff, this relies on the following defintion,

Definition 1.3 (Pairwise Neighborhood-Separable)**.** Two points are pairwise neighborhood-separable if there exists a neighborhood U of x and V of y such that U and V are disjoint.

Then we can say that a space is Hausdorff if all distinct points are pairwise neighborhood-separable. Finally, to say something is homeomorphic to another space, this means it can be stretched without creating holes or glueing. A homeomorphism is a bijective map between two spaces and local homeomorphisms relates to neighborhoods around points. Hence, saying that something is locally homeomorphic to Euclidean space directly means, that you can bijectively map the contents of the neighborhood around a point to an open ball in $\mathbb{R}^n$, i.e the Euclidean n-ball.[1]

1.2 Lie Groups

Now we know enough to define what a Lie group actually is,

Definition 1.4 (Lie Group)**.** A Lie group is a group that is also a smooth manifold, such that the binary product and inversion are smooth functions.

What we will be focusing our attention to is special Lie groups, the general linear group, special linear group and the special orthogonal group.

Definition 1.5 (General Linear Group)**.** $\mathrm{GL}(n, \mathbb{R})$ is the linear matrix group. The manifold of $n \times n$ invertible square real matrices is a lie group denoted by $\mathrm{GL}(n, \mathbb{R})$

Definition 1.6 (Special Linear Group)**.** The $\mathrm{SL}(n, \mathbb{R})$ is the manifold of $n \times n$ matrices with unit determinant.

Definition 1.7 (Special Orthogonal Group)**.** $\mathrm{SO}(n, \mathbb{R})$ is the manifold of rotation matrices in n dimensions. This may be denoted by $SO(n)$

[1] This is long and a very non-succinct way to define this structure, I prefer to define manifolds via sheaves as I feel it is neater. A manifold is just a locally ringed space, whose sheaf structure is just locally isomorphic to continuous functions on Euclidean space.

1.3 Lie Algebras

To actually understand what Lie Algebras are, we need to generalise the notion of a vector and a tangent. We shall look at so called tangent spaces. To formally define them, we shall first define charts and atlas and along the way redefine what a manifold is. These definitions are adapted from [4].

Definition 1.8 (Chart)**.** Let X be a topological space. An $\mathbb{R}^n$ chart on X is is homeomorphism $\phi : U \to U'$ where $U \subset X$ and $U' \subset \mathbb{R}^n$.

Definition 1.9 (Atlas)**.** A C^∞ atlas on a topological space X is a collection of charts $\phi_\alpha : U_\alpha \to U'_\alpha$ where all the U''s are open subsets of one fixed $\mathbb{R}^n$ such that,

(i) Each $U_\alpha \in X$ is open and $\bigcup_\alpha U_\alpha = X$ (U_α is an open subcover of X) and,

(ii) Changes of coordinates are smooth.

Two last definitions in this section are equivalence relation and equivalence class.

Definition 1.10 (Equivalence Relation)**.** An equivalence relation on a set X is a binary relation $\sim$ satisfying,

(i) $\forall\, a \in X, a \sim a$

(ii) $\forall\, a, b \in X, a \sim b \implies b \sim a$

(iii) $\forall\, a, b, c \in X, a \sim b$ and $b \sim a \implies a \sim c.$

Then the equivalence class is just all of the equivalent elements to a member of that set. For example, for the equivalence relation that $x \sim y$ if and only if $x - y$ is even, then the equivalence classes are all the even numbers and all the odd numbers.

Remark. It can be proven that equivalence classes are just a partition of a set. [2]

Here is another definition of a manifold, this definition explicitly output a smooth manifold,

Definition 1.11 (C^∞ Manifold)**.** An n-dimensional (C^∞) manifold a topological space M together with an equivalence class of C^∞ atlases.

Remark. Our equivalence relation here is that two atlases are equivalent if their union is also an atlas.

Here are a few examples of manifolds,

– Let $M = \mathbb{R}^n$, this is a manifold covered by one open set and then if we take the identity map as our chart, we get the standard manifold on $\mathbb{R}^n$.

– Let $M = \mathbb{C}^n$, then we cover $\mathbb{C}^n$ by just one open set and then chart the map, $\phi : \mathbb{C}^n \to \mathbb{R}^{2n}$ which is just,

$$\phi(z_1, \dots, z_n) = (\operatorname{Re} z_1, \operatorname{Im} z_1, \dots, \operatorname{Re} z_n, \operatorname{Im} z_n)$$

– If M is a manifold, then any open $V \subset M$ is also a manifold. This can be seen as the union of the atlases V and M is going to be M and so it has the same equivalence class and hence it must be a manifold.

– If we let $M_n(\mathbb{R})$ be all real $n \times n$ matrices, then this is a manifold as it's just $\mathbb{R}^{n^2}$. We also can say $\mathrm{GL}(n, \mathbb{R}) \subset M_n(\mathbb{R})$ and so by the previous point, $\mathrm{GL}(n, \mathbb{R})$ is a manifold.

Now we can formalise the idea of tangent vectors on a manifold[3]

Definition 1.12 (Tangent Vectors)**.** Let M be a C^∞ manifold, then we can say that $x \in M$. Let us take a chart of M, $\phi : U \to \mathbb{R}^n$ where $x \in U$. Now take two curves $\gamma_1, \gamma_2 : (-1, 1) \to M$ with $\gamma_1(0) = \gamma_2(0) = x$ such that we can form $\phi \circ \gamma_1 \circ \phi \circ \gamma_2 : (-1, 1) \to \mathbb{R}^n$ are differentiable.
Now define an equivalence such that γ_1 and γ_2 are equivalent at 0 if and only if $(\phi \circ \gamma_1)' = (\phi \circ \gamma_2)' = 0$. Then take the equivalence class of all of these curves and these are the tangent vectors of M.

Definition 1.13 (Tangent Space)**.** The set of all of the tangent vectors at x. We denote it as T_xM.

Lie Algebras are tangent space to the lie group at the identity. Let G be a lie group, then the T_eG (tangent space at the identity) is an interesting vector space with a remarkable structure called the lie algebra structure.

Lemma 1.14. Let G be a matrix lie group, and $g \in G$, then,

$$\xi \in T_eG \implies g\xi g^{-1} \in T_eG$$

Note that, $g\xi g^{-1}$ is a matrix expression.

Proof. Let $c(t) \in G$ be a curve in G, such that $c(0) = e$ and $\dot{c}(0) = \xi$. Define $\gamma(t) = gc(t)g^{-1}$. Then $\gamma(0) = gc(0)g^{-1} = e$ and $\dot{\gamma}(0) = g\dot{c}(0)g^{-1} = g\xi g^{-1} \in T_eG$. □

Proposition 1.15. Let G be a matrix lie group and $\xi, \eta \in T_eG$. Then, $\xi\eta - \eta\xi \in T_eG$

Proof. Let $c(t) \in G$ be a curve such that $c(0) = e$ and $\dot{c}(0) = \xi$ also define $b(t) = c(t)\eta c(t)^{-1} \in T_eG$ by Lemma 1.5. Then $\dot{b}(t) \in T_eG$.

$$\begin{aligned}
\dot{b}(0) &= \dot{c}(0)\eta c(0)^{-1} + c(0)\eta + \frac{dc(t)}{dt}(0) \\
&= \dot{c}(0)\eta c(0)^{-1} - c(0)\eta c(0)^{-1}\dot{c}(0)c(0)^{-1} \\
&= \xi\eta - \eta\xi
\end{aligned}$$

As $\dot{b}(t) \in T_eG$, then $\xi\eta - \eta\xi \in T_eG$ □

Now we have a Lie Algebra,

Definition 1.16 (Lie Algebra)**.** A lie algebra is a vector space endowed with a commutator (or Lie bracket), that is a bilinear map. If we have,

$$[\cdot, \cdot] : V \times V \to V$$

such that,

- $[B, A] = -[A, B]$ (skew-symmetry property)
- $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0 \quad \forall A, B, C \in V$ (Jacobi Identity)

Theorem 1.17. Let G be a matrix Lie group. Then T_eG is a lie algebra with bracket given by the matrix commutator. Denoted by $\mathfrak{g}$.

$$[A, B] = AB - BA$$

Assume we have a surface, of manifold M, the tangent space T_qM, then we can say that,

$$\bigcup T_qM = TM$$

Cotangent manifold is dual space of manifold

Example.

- The lie algebra $\mathfrak{gL}(n, \mathbb{R}) = T_e\mathrm{GL}(n, \mathbb{R})$ which is vector space of real square $n \times n$ matrices with commutator.
- The lie algebra of $\mathfrak{SL}(n, \mathbb{R}) := T_e\mathrm{SL}(n, \mathbb{R})$ vector space of real traceless square $n \times n$ matrices.

Proof. Take $g(t) \in \mathrm{SL}(n, \mathbb{R})$, and so $\det g(t) = 1$ hence, take $g(t)$ such that $g(0) = e$, and $\dot{g}(0) = \xi$ and so $\dot{g}(0) \in \mathfrak{SL}(n, \mathbb{R})$. Now use the formula of the derivative of the determinant of a matrix to show,

$$\frac{d}{dt}(\det(g(t)))_{t=0} = \det g(0)\,\mathrm{Tr}(g^{-1}(0)\dot{g}(0))$$

and so, $g^{-1}(0)\dot{g}(0) \in \mathfrak{SL}(n, \mathbb{R})$ and so we just have $\mathrm{Tr}(\xi)$. □

– The lie algebra of $\mathfrak{SO}(3) = T_e\mathrm{SO}(3)$, the vector space of skew-symmetric matrices.

Lemma 1.18. If $v \in T_gG$, then we can say,

(i) $g^{-1}v \in T_eG$

(ii) $vg^{-1} \in T_eG$

2 Actions of a Lie Group and Lie Algebra

Definition 2.1 (Conjugation Action)**.** Let $g \in \mathfrak{G}$, then the operation $I_g : \mathfrak{G} \to \mathfrak{G}$ (Inner Automorphism) and so you define it by $h \mapsto ghg^{-1} \quad \forall\, h \in \mathfrak{G}$. $I_{gh} = AD_{gh}$.

Take an arbitrary path $h(t) \in \mathfrak{G}$ such that $h(0) = e$ and now $\xi = \dot{h}(0) \in T_eG$. We now define $Ad_g(\xi) = \frac{d}{dt} I_g h(t)_{t=0} = g\xi g^{-1} \in T_eG$ the adjoint action.

Definition 2.2 (Adjoint and coadjoint actions of $\mathfrak{G}$ on $\mathfrak{g}$ and $\mathfrak{g}^*$)**.** The adjoint action of the matrix group G on it's lie algebra $\mathfrak{g}$ is a map,

$$Ad : G \times \mathfrak{g} \to \mathfrak{g}$$

which is,

$$\mathrm{Ad}_g\, \xi = g\xi g^{-1}$$

The dual map $\left\langle Ad^*_g \mu\,, \xi \right\rangle = \langle \mu\,, Ad_g \xi \rangle$ where $\mu \in \mathfrak{g}^*$ and $\xi \in T_eG = \mathfrak{g}$. is called the coadjoint map of G on the dual lie algebra $\mathfrak{g}^*$.

Definition 2.3 (Dual Space for vectors)**.** Let V be a finite dimensional vector space, of dimension n, over $\mathbb{R}$. The dual vector space is denoted by V^* is the space of all linear functionals from $V \to \mathbb{R}$, $f(v) = a$ where $v \in V$ and $a \in \mathbb{R}$, then also $f(\alpha v + \beta w) = \alpha f(v) + \beta f(w)$ and $\alpha, \beta \in \mathbb{R}$ and $v, w \in V$. Hence, $f(v) = Mv$ we call M the covector such that $Mv \in \mathbb{R}$. The vectorspace of all covectors is the dual space.

$$\langle m\,, v \rangle \in \mathbb{R} \quad m \in V^* \quad v \in V$$

Lemma 2.4. Let V be a vector space of real $n \times n$ real matrices. Then the dual vector space V^* is also a vector space of $n \times n$ matrices and every linear functional $f : V \to \mathbb{R}$ such that,

$$f(A) := Tr(B^T A), \quad B \in V^*, A \in V$$

Definition 2.5 (Trace Pairing)**.** For every vector space V of real $n \times n$ matrices with dual V^*, then the pairing is,

$$\langle B\,, A \rangle = Tr(B^T A) = Tr(BA^T)$$

Proposition 2.6. Suppose $A^T = A$ and $B^T = -B$, then, $\mathrm{Tr}(B^T A) = 0$

Proof.

$$\begin{aligned}
\mathrm{Tr}(B^T A) = -\,\mathrm{Tr}(BA) \\
&= -\,\mathrm{Tr}((BA)^T) \\
&= -\,\mathrm{Tr}(B^T A^T) \\
&= -\,\mathrm{Tr}(A^T B^T) \\
&= -\,\mathrm{Tr}(B^T A)
\end{aligned}$$

□

We say that,

$$\begin{aligned}
\left\langle Ad^*_g \mu\,, \xi \right\rangle &= \langle \mu\,, Ad_g \xi \rangle \\
&= \left\langle \mu\,, g\xi g^{-1} \right\rangle \\
&= \mathrm{Tr}(\mu^T g\xi g^{-1}) \\
&= \mathrm{Tr}(\xi g \mu^T g^{-1}) \\
&= \mathrm{Tr}[(g^T \mu (g^{-1})^T)^T \xi] \\
&= \left\langle g^T \mu (g^{-1})^T\,, \xi \right\rangle \\
&= \left\langle g^T \mu (g^T)^{-1}\,, \xi \right\rangle
\end{aligned}$$

Let $g(t) \in G$ such that $g(0) = e$ where $\eta = \dot{g}(0) \in T_eG$

$$ad_\eta \xi := \frac{d}{dt}_{t=0} Ad_{g(t)}\xi \qquad \forall \xi \in \mathfrak{g}$$

qhich we can see to be

$$\dot{g}(0)\xi g(0)^{-1} + g(0)\xi \frac{d}{dt}_{t=0} g(t)^{-1} = \eta\xi - \xi\eta$$

Hence we can say that $\mathrm{ad}_\eta^\xi = [\eta, \xi] = \eta\xi - \xi\eta$.
Hence now we define the coadjoint action on μ,

Definition 2.7 (Adjoint / Coadjoint action on $\mathfrak{g}/\mathfrak{g}^*$)**.** The adjoint action of the matrix lie algebra on itself is given by,

$$\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

$$\mathrm{ad}_\eta \xi = [\eta, \, \xi]$$

The dual map $\left\langle ad_\eta^* \mu \, , \xi \right\rangle = \langle \mu \, , \mathrm{ad}_\eta \xi \rangle$ is the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$.

Exercise. Find $\mathrm{ad}_\eta^* \mu$.

3 Rotation

3.1 Inertial Frame

A spatial coordinate system with origin at the centre of mass of the given rigid body. We denote it by, $\mathbf{x}(t) \in \mathbb{R}^3$, where $\mathbf{x} = X$. Assume we have a spacial coordinate system, We need a way to rotate things without constraints, so we denote a tensor $R(t)$ and say $\mathbf{x}(t) = R(t)\mathbf{X}$ where $\mathbf{X}$ is in the body coordinate system. The configuration of the body particle at time t is given by a rotation matrix that takes the label $\mathbf{X}$ to current position $\mathbf{x}(t)$ where $R \in \mathrm{SO}(3)$ is a proper rotation matrix; this means,

$$R^T = R^{-1} \qquad \det R = 1$$

The map $\mathbf{X} \to R(t)\mathbf{X}$ is called the body-to-space map.

We can now talk about kinetic energy,

$$K = \frac{1}{2}\int_\beta \rho \left\|\mathbf{x}\right\|^2 \, d^3\mathbf{X}$$

which we can change to,

$$\begin{aligned}
\frac{1}{2}\int_\beta \rho \left\|\mathbf{x}\right\|^2 \, d^3\mathbf{X} &= \frac{1}{2}\int_\beta \rho(\mathbf{X}) \left\|\dot{R}(t)\mathbf{X}\right\|^2 \, d^3\mathbf{X} \\
&= \frac{1}{2}\int_\beta \rho(\mathbf{X})\dot{R}(t)\mathbf{X} \circ \dot{R}(t)\mathbf{X} \, d^3\mathbf{X}
\end{aligned}$$

Now we can say if $V = 0$. Hence, $L = K$ and so,

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{R}} - \frac{\partial K}{\partial R} = \mathbf{0}$$

This is difficult to deal with, so let's do something more cool!

We know that $R^{-1} = R^T$ and so $RR^T = RR^{-1} = I = e$. If we have $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^3$, then $\mathbf{v} \cdot \mathbf{w} = R\mathbf{v} \cdot R\mathbf{w}$. Hence, consider $\left\|\dot{R}\mathbf{X}\right\|^2$ and we know

$$\begin{aligned}
\left\|\dot{R}\mathbf{X}\right\|^2 &= \dot{R}\mathbf{X} \cdot \dot{R}\mathbf{X} \\
&= R^{-1}(\dot{R}\mathbf{X}) \cdot R^{-1}(\dot{R}\mathbf{X}) \\
&= \left\|R^{-1}\dot{R}\mathbf{X}\right\|
\end{aligned}$$

and so,

$$K = \frac{1}{2}\int_\beta \rho(\mathbf{X}) \left\|R^{-1}\dot{R}\mathbf{X}\right\| d^3\mathbf{X}$$

Then $K = K(R, \dot{R}) = K(R^{-1}R, R^{-1}\dot{R})$ this is called left symmetry. Hence, we can reduce this to $K(e, R^{-1}\dot{R})$ and change notation let $\kappa(R^{-1}\dot{R})$ and $R^{-1}\dot{R}$ is angular velocity of the body. We can see this from the body and from an observation outside the system. Hence, we call this $R^{-1}\dot{R} = \hat{\mathbf{\Omega}}$.
Interestingly, we know $RR^T = RR^{-1} = I$. Hence,

$$\frac{d}{dt}I = \frac{d}{dt}(RR^{-1}) = \dot{R}R^{-1} + R\frac{d}{dt}R^{-1} = \mathbf{0}$$

and we can also write this as,

$$\begin{aligned} I &= RR^T \\ \mathbf{0} &= \frac{d}{dt}(RR^T) \\ \mathbf{0} &= \dot{R}^T R + R^T \dot{R} \\ R^T \dot{R} &= -(R^T \dot{R})^T \end{aligned}$$

and so $R^{-1}\dot{R} = -(R^{-1}\dot{R})^T$ and so $\hat{\mathbf{\Omega}} = -\hat{\mathbf{\Omega}}^T$. This is the antisymmetric property we have noted about this vector.

Now we go back to kinetic energy to nicely write it as $\hat{\mathbf{\Omega}}$

$$K = \frac{1}{2}\int_\beta \rho(\mathbf{X}) \left\|\hat{\mathbf{\Omega}}\mathbf{X}\right\|^2 d^3\mathbf{X}$$

and now we can prove that $\hat{\mathbf{\Omega}}\mathbf{X} = \mathbf{\Omega} \times \mathbf{X}$ where,

$$\hat{\mathbf{\Omega}} = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}$$

where $\mathbf{\Omega} = \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix}$ where Ω is the axel vector. and so,

$$\begin{aligned} K &= \frac{1}{2}\int_\beta \rho(\mathbf{X}) \left\|\mathbf{\Omega} \times \mathbf{X}\right\|^2 d^3\mathbf{X} \\ &= \frac{1}{2}\int_\beta \rho(\mathbf{X})(\mathbf{\Omega} \times \mathbf{X}) \cdot (\mathbf{\Omega} \times \mathbf{X}) d^3\mathbf{X} \\ &= \frac{1}{2}\int_\beta \rho(\mathbf{X})(\left\|\mathbf{\Omega}^2\right\| \left\|\mathbf{X}^2\right\| - (\mathbf{\Omega} \cdot \mathbf{X})^2)^2 \cdot (\mathbf{\Omega} \times \mathbf{X}) d^3\mathbf{X} \\ &= \frac{1}{2}\mathbb{I}\mathbf{\Omega} \cdot \mathbf{\Omega} \end{aligned}$$

where I is the moment of inertia tensor, which we define as,

$$\mathbb{I} = \int_\beta \rho(\mathbf{X}) \left\|\mathbf{X}\right\|^2 I - \mathbf{X}\mathbf{X}^T d^3\mathbf{X}$$

where $\mathbf{X}\mathbf{X}^T = \mathbf{X} \otimes \mathbf{X}$ and,

$$(a \otimes b)c = (b \cdot c)a$$

for all $a, b, c \in \mathbb{R}^3$.

4 Calculus of variations

We are going to consider a continuous level, but you can use discrete level.

Theorem 4.1 (The variation principle)**.**

$$\mathcal{L} = \int_{t_1}^{t_2} \frac{1}{2}\mathbb{I}\mathbf{\Omega} \cdot \mathbf{\Omega}\, dt$$

and we find differential equations by letting $\delta\mathcal{L} = 0$ but this is subject to $\delta\mathbf{\Omega}(t_1) = \delta\Omega(t_2) = \mathbf{0}$

and so,

$$\delta \int_{t_1}^{t_2} \frac{1}{2}\mathbb{I}\mathbf{\Omega} \cdot \mathbf{\Omega}\, dt = \int_{t_1}^{t_2} \frac{1}{2}\mathbb{I}\delta\mathbf{\Omega} \cdot \mathbf{\Omega} + \frac{1}{2}\mathbb{I}\mathbf{\Omega} \cdot \delta\mathbf{\Omega}\, dt = \int_{t_1}^{t_2} \mathbb{I}\mathbf{\Omega} \cdot \delta\mathbf{\Omega}$$

but what is $\delta\mathbf{\Omega}$, but remember we have $\hat{\Omega}$, which is the lie algebra of SO(3). We said, $\hat{\mathbf{\Omega}} = R^T\dot{R} = R^{-1}\dot{R}$. Now we take variations of $\hat{\mathbf{\Omega}} = \mathbf{\Omega} \times \mathbf{X}$ and so,

$$(\delta\mathbf{\Omega}) \times \mathbf{X} = (\delta\hat{\mathbf{\Omega}})\mathbf{X}$$

and so we see that,

$$\delta\hat{\mathbf{\Omega}} = \delta(R^{-1}\dot{R}) = \delta R^{-1}\dot{R} + R^{-1}\delta\dot{R} = 0$$

as $\delta I = \delta R R^{-1} + R\delta R^{-1}$ and then we see that $R^{-1}\delta R R^{-1} + R^{-1}R\delta R^{-1} = 0$ and so as $RR^{-1} = I$, $R^{-1}\delta R R^{-1} + \delta R^{-1} = \mathbf{0}$. We have that $\delta R^{-1}\dot{R} + R^{-1}\delta\dot{R} = \delta\hat{\mathbf{\Omega}}$ and $\hat{\mathbf{\Omega}} = R^{-1}\dot{R}$ where $\hat{\mathbf{\Lambda}} = R^{-1}\delta R$ and so we sub in,

$$\begin{aligned}
\delta\hat{\mathbf{\Omega}} &= -R^{-1}\delta R R^{-1}\dot{R} + R^{-1}\frac{d}{dt}\delta R \\
&= R^{-1}\delta R\hat{\mathbf{\Omega}} + \frac{d}{dt}(R^{-1}\delta R) - (\frac{d}{dt}R^{-1})\delta R \\
&= R^{-1}\delta R\hat{\mathbf{\Omega}} + \frac{d}{dt}(R^{-1}\delta R) + R^{-1}\dot{R}R^{-1}\delta R \\
&= -\hat{\mathbf{\Lambda}}\hat{\mathbf{\Omega}} + \frac{d}{dt}\hat{\mathbf{\Lambda}} + \hat{\mathbf{\Omega}}\hat{\mathbf{\Lambda}} \\
&= \dot{\hat{\Lambda}} + [\hat{\mathbf{\Omega}}, \hat{\mathbf{\Lambda}}]
\end{aligned}$$

Exercise. Prove,

$$\delta\mathbf{\Omega} = \dot{\Lambda} + (\mathbf{\Omega} \times \mathbf{\Lambda})$$

Now, let us substitute this back into our variational principle.

$$\begin{aligned}
\int_{t_1}^{t_2} \mathbb{I}\mathbf{\Omega} \cdot \mathbf{\Omega}\, dt &= 0 \\
\int_{t_1}^{t_2} \mathbb{I}\mathbf{\Omega} \cdot \delta\mathbf{\Omega}\, dt &= 0 \\
\int_{t_1}^{t_2} \mathbb{I}\mathbf{\Omega} \cdot (\dot{\Lambda} + \mathbf{\Omega} \times \mathbf{\Lambda})\, dt &= 0 \\
[\mathbb{I}\mathbf{\Omega} \cdot \Lambda|_{t_2} - \mathbb{I}\mathbf{\Omega} \cdot \Lambda|_{t_1}] - \int_{t_1}^{t_2} \frac{d}{dt}(\mathbb{I}\mathbf{\Omega}) \cdot \Lambda\, dt + \int_{t_1}^{t_2} (\mathbb{I}\mathbf{\Omega} \times \mathbf{\Omega}) \cdot \mathbf{\Lambda} &= 0 \\
0 - 0 - \int_{t_1}^{t_2} (-\mathbb{I}\dot{\mathbf{\Omega}} + \mathbb{I}\mathbf{\Omega} \times \mathbf{\Omega}) \cdot \Lambda\, dt &= 0
\end{aligned}$$

Hence,

$$\mathbb{I}\dot{\mathbf{\Lambda}} = \mathbb{I}\mathbf{\Omega} \times \mathbf{\Omega}$$

We can write the equations by considering the tangent space.

4.1 Euler-Poincare Reduction by Symmetry

To gain a general idea of how the equations of motion appear for rotational dynamics with symmetry, we consider an arbitrary Lagrangian of this form,

$$L : T\mathrm{SO}(3) \to \mathbb{R}$$

$$L = L(R, \dot{R})$$

and satisfies,

$$\delta \int_{t_1}^{t_2} L(R, \dot{R})\, dt = 0$$

this means,

$$\begin{aligned}\int_{t_1}^{t_2} L(R, \dot{R})\, dt &= \int_{t_1}^{t_2} \left\langle \frac{\partial L}{\partial R}, \delta R \right\rangle + \left\langle \frac{\partial L}{\partial \dot{R}}, \delta R \right\rangle dt \\ &= \left\langle \frac{\partial L}{\partial \dot{R}}, \delta R \right\rangle \Bigg|_{t_2} - \left\langle \frac{\partial L}{\partial \dot{R}}, \delta R \right\rangle \Bigg|_{t_1} + \int_{t_1}^{t_2} \left\langle \frac{\partial L}{\partial R} - \frac{d}{dt}\frac{\partial L}{\partial \dot{R}}, \delta R \right\rangle dt\end{aligned}$$

and so we can notice $\frac{d}{dt}\frac{\partial L}{\partial \dot{R}} - \frac{\partial L}{\partial R} = 0$

Definition 4.2 (Left-Symmetric Lagrangian)**.** A Lagrangian is said to be left-symmetric or left-invariant under the action of the group of the group of rotations if, $L(\chi R, \chi \dot{R}) = L(R, \dot{R})\ \forall \chi \in \mathrm{SO}(3)$.

We also know $\mathfrak{SO}(3) = T_e\mathrm{SO}(3)$ and we said that $v \in T_eG \implies g^{-1}v \in \mathfrak{g} = T_eG$. We know $\dot{R}(t) \in T_{R(t)}\mathrm{SO}(3)$ and so we can say $R^{-1}\dot{R} \in \mathfrak{SO}(3)$.

We say

$$\begin{aligned}L(R, \dot{R}) &= L(R^{-1}R, R^{-1}\dot{R}) \\ &= \widetilde{\ell}(R^{-1}\dot{R}) = \widetilde{\ell}(\hat{\mathbf{\Omega}})\end{aligned}$$

Now we write out Hamilton's principle,

$$\begin{aligned}0 &= \delta \int_{t_1}^{t_2} \widetilde{\ell}(\hat{\mathbf{\Omega}})\, dt \\ &= \delta \int_{t_1}^{t_2} \ell(\mathbf{\Omega}) = \mathbf{0}\end{aligned}$$

Euler Poincare equtions,

$$\frac{d}{dt}\frac{\partial \ell}{\partial \mathbf{\Omega}} - \frac{\partial \ell}{\partial \mathbf{\Omega}} = \mathbf{0}$$

Exercise. Derive these

These are the euler-poincare equations for rotational dynamics with symmetry under left multiplication.

Theorem 4.3. The spatial angular momentum (in the spatial frame) is conserved along solutions of the Euler-Poincare equations.

Proof. We know $\frac{d}{d}\frac{\partial \ell}{\partial \mathbf{\Omega}} - \frac{\partial \ell}{\partial \mathbf{\Omega}} \times \Omega = 0$ and we know that $R\frac{\partial \ell}{\partial \mathbf{\Omega}}$ and the multiplcation by R means spatial frame. Hence we prove,

$$\frac{d}{dt}R\frac{\partial \ell}{\partial \mathbf{\Omega}} = \mathbf{0}$$

$$\begin{aligned}
\frac{d}{dt}R\frac{\partial \ell}{\partial \mathbf{\Omega}} &= \dot{R}\frac{\partial \ell}{\partial \mathbf{\Omega}} + R\frac{d}{dt}\frac{\partial \ell}{\partial \mathbf{\Omega}} \\
&= R\hat{\mathbf{\Omega}}\frac{\partial \ell}{\partial \mathbf{\Omega}} + R(\frac{\partial \ell}{\partial \mathbf{\Omega}} \times \mathbf{\Omega}) \\
&= R(\Omega \times \frac{\partial \ell}{\partial \mathbf{\Omega}}) + R(\frac{\partial \ell}{\partial \mathbf{\Omega}} \times \mathbf{\Omega}) \\
&= \mathbf{0}
\end{aligned}$$

□

Now we want to write a general form of the Euler-Poincare Equations for left invariant systems. Let L be a lagrangian on the tangent bundle of a matrix lie group G,

$$L : TG \to \mathbb{R}$$

$$L = L(g, \dot{G}) \qquad g \in G$$

Assume that the lagrangian is left-invariant,

$$L(g,\, \dot{g}) = L(hg,\, h\dot{g}) \quad \forall\, h \in G$$

and now let $h = g^{-1}$, and so $L(g,\, \dot{g}) = L(g^{-1}g\, g^{-1}\dot{g}) = \ell(\xi)$. We have gone from a lie group to a lie algebra, $\xi = g^{-1}\dot{g} \in T_eG = \mathfrak{g}$ which is a matrix lie algebra. We now aim to use the action functional and variational derivative,

$$\begin{aligned}
\delta \int_{t_1}^{t_2} L(g,\, \dot{g}) &= \mathbf{0} \\
\delta \int_{t_1}^{t_2} \ell(\xi)\, dt &= \mathbf{0} \\
\int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial xi}\, , \delta\xi \right\rangle\, dt &= \mathbf{0}
\end{aligned}$$

Now we want to consider $\delta\xi = \delta(g^{-1}\dot{g})$,

$$\begin{aligned}
\delta(g^{-1}\dot{g}) &= \delta g^{-1}\dot{g} + g^{-1}\delta\dot{g} \\
&= g^{-1}\delta g g^{-1}\dot{g} + g^{-1}\frac{d}{dt}\delta g \\
&= -(g^{-1}\delta g)g^{-1}\dot{g} + g^{-1}\frac{d}{dt}\delta g \\
&= -\eta\xi + \frac{d}{dt}\delta(g^{-1}\delta g) + (g^{-1}\dot{g})(g^{-1}\frac{d}{dt}\delta g) \\
&= -\eta\xi\dot{\eta} + \xi\eta \\
&= \dot{\eta} + [\xi,\, \eta] \\
&= \dot{\eta} + \mathrm{ad}_{\xi}\eta
\end{aligned}$$

and so back to the derivation,

$$\int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial xi}, \delta\xi \right\rangle dt = \mathbf{0}$$

$$\int_{t_1}^{t_2} \left\langle \frac{\partial \ell}{\partial \xi}, \dot{\eta} + \mathrm{ad}_{\xi}\eta \right\rangle$$

$$\int_{t_1}^{t_2} \left\langle -\frac{d}{dt}\left(\frac{\partial \ell}{\partial \xi}\right), \eta \right\rangle + \left\langle \mathrm{ad}_{\xi}^{*}\frac{\partial \ell}{\partial xi}, \eta \right\rangle dt$$

Since η is arbitrary our equation is of this form,

$$\frac{d}{dt}\frac{\partial \ell}{\partial \xi} - \mathrm{ad}_{\xi}^{*}\frac{\partial \ell}{\partial \xi} = 0$$

and these are our Euler-Poincare equations for a left invariant system.

Theorem 4.4 (Noether Theorem)**.** The Euler Poincare equations associated a left-invariant system preserve the generalised momentum along solutions of the Euler-Poincare equations, that is,

$$\frac{d}{dt}\left(\mathrm{Ad}_{g^{-1}(t)}^{*}\frac{\partial \ell}{\partial \xi}(t)\right) = 0$$

Proof. Exercises □

Exercise. Repeat derivations for the Euler-Poincare Equations for right-invariant systems. What is Noether Theorem?[2]

[2] What about both left and right invariant?

Part II:

Examples

5 Spherical Pendulum

We want to consider a pendulum in 3D space. We will think about this through the definition of spherical coordinates, as in actuality our motion of the bob will just be on S^2. However, we need to define what are Euler Lagrange equations?[1]

Firstly, here is what the Euler Lagrange equations are,

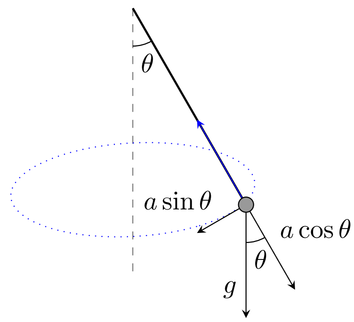
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} - \frac{\partial L}{\partial q^a}$$

where we range through the different basis vectors q^a and their associated derivatives $\dot{q}^a$.

We also define L as the Lagrangian. We define this simply as,

$$L(q, \dot{q}) = T(q, \dot{q}) - V(\mathbf{r}(q))$$

where further we define $T(q, \dot{q})$ as the kinetic energy of the system and $V(\mathbf{r}(q))$ the potential energy of the system.



We are going to use polar coordinates to derive our system of equations.

$$x = R\sin\theta\cos\phi$$
$$y = R\sin\theta\sin\phi$$
$$z = R(1-\cos\theta)$$

Our first focus is $T(q, \dot{q})$, which will just be $\frac{1}{2}mv^2$. We can see that $v = |\dot{\mathbf{r}}(t)|$ and so $v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$. Hence, we now find what v is and then find the lagrangian. Firstly, we note that,

$$\frac{d}{dt}(x(t)) = \frac{\partial x}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial x}{\partial \phi}\frac{d\phi}{dt}$$

and similarly for $y(t)$ and $z(t)$. Hence,

$$\frac{dx(t)}{dt} = R\cos\theta\cos\phi\,\dot{\theta} - R\sin\theta\sin\phi\,\dot{\phi}$$
$$\frac{dy(t)}{dt} = R\cos\theta\sin\phi\dot{\theta} + R\sin\theta\cos\phi\dot{\phi}$$
$$\frac{dz(t)}{dt} = R\sin\theta\dot{\theta}$$

and now we derive our $T(q, \dot{q})$,

$$T(q, \dot{q}) = \frac{1}{2}m\Bigg(\Big(R\cos\theta\cos\phi\,\dot{\theta} - R\sin\theta\sin\phi\,\dot{\phi}\Big)^2 + \Big(R\cos\theta\sin\phi\dot{\theta} + R\sin\theta\cos\phi\dot{\phi}\Big)^2 + \Big(R\sin\theta\dot{\theta}\Big)^2\Bigg)$$

which then can be simplified down to,

$$T(\theta, \dot{\theta}, \phi, \dot{\phi}) = \frac{mR^2}{2}\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right)$$

and we note that our system has only one potential energy, gravitation potential! Hence,

$$V(\theta,\dot{\theta},\phi,\dot{\phi}) = -mgz = -mgR(1-\cos\theta)$$

Hence, we can now talk about Lagrangian explicitly,

$$L(\theta,\dot{\theta},\phi,\dot{\phi}) = \frac{mR^2}{2}\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + mgR\,(1-\cos\theta)$$

Finally, we can now take derivatives of this function and produce the Euler Lagrange equations. We need to find the basis vectors, θ and ϕ, as we have two basis vectors, we will have two equations. Firstly, θ

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = \frac{d}{dt}mR^2\dot{\theta} - \left(\dot{\phi}^2 mR^2\sin\theta\cos\theta - mgR\sin\theta\right) &= 0\\ mR^2\ddot{\theta} - mR^2\sin\theta\cos\theta\,\dot{\phi}^2 + mgR\sin\theta &= 0\\ R\ddot{\theta} - R\sin\theta\cos\theta\,\dot{\phi}^2 + g\sin\theta &= 0\end{aligned}$$

and secondly, ϕ

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} - \frac{\partial L}{\partial\phi} = \frac{d}{dt}(mR^2\dot{\phi}\sin^2\theta) &= 0\\ mR^2\ddot{\phi}\sin^2\theta + 2mR^2\dot{\phi}\,\dot{\theta}\sin\theta\cos\theta &= 0\\ \ddot{\phi}\sin\theta + 2\dot{\phi}\,\dot{\theta}\cos\theta &= 0\end{aligned}$$

We have derived the Euler-Lagrange equations for the spherical pendulum, which are

$$\begin{cases} R\ddot{\theta} - R\sin\theta\cos\theta\,\dot{\phi}^2 + g\sin\theta = 0\\ \ddot{\phi}\sin\theta + 2\dot{\phi}\,\dot{\theta}\cos\theta = 0\end{cases}$$

References

[1] Darryl D. Holm. *Geometric mechanics: Part I Dynamics and Symmetry.* Imperial College Press, 2011.

[2] Xena Project Kevin Buzzard. *Lean summer lectures 2/18: partitions.* (YouTube). July 2020. URL: `https://www.youtube.com/watch?v=FEKsZj3WkTY`.

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[4] Eugene Lerman. *An Introduction to Differential Geometry.* (PDF). Date Accessed: 29/09/2021. URL: `https://faculty.math.illinois.edu/~lerman/518/f11/8-19-11.pdf`.

[5] *Manifold.* Date Accessed: 28/09/2021. URL: `https://mathworld.wolfram.com/Manifold.html`.