Year 3 — Partial Differential Equations

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine (especially the typos!).

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1 Introduction to PDEs

A differential equation that contains, in addition to the dependent variable and independent variables, one or more partial derivatives of the dependent variable is called a partial differential equation.

In general it may be written in the form,

$$f(x, y, \ldots, u, u_x, u_y, \ldots, u_{xx}, u_{yy}, \ldots) = 0 \tag{1}$$

involving several of the x, y, u_x, u_{xx} terms. Note that the notation $u_x = \frac{\partial u}{\partial x}$.

When we consider a PDE, we also consider it in a suitable domain. For us, the domain, D, will just be some domain of $\mathbb{R}^n$ in the variables $x, y, \ldots$. A solution of this equation will be a function $u = u(x, y, \ldots)$ which satisfy (1). We call the order of the PDE the highest order partial derivative appearing the equation.

Example. $u_{xxy} + xu_{yy} + 8u = 8y$ is a third order PDE.

Definition 1.1 (Linear)**.** We call a PDE linear if it is linear in the unknown function and all it's derivatives. For example,

$$yu_{xx} + 2xyu_{yy} + u = 1$$

and we have a further characterisation, called quasi-linear,

Definition 1.2 (Quasi-Linear)**.** A PDE is quasi linear if it linear in the highest-orderderivative of the unknown function,

$$u_x u_{xx} + xuu_y = \sin y$$

and furthermore, an equation that isn't linear is non-linear. IN this course we will consider mainly second order linear PDEs. The most general of these can be written as,

$$\sum_{i,j=1}^{n} A_{ij} u_{x_i x_j} + \sum_{i=1}^{n} B_i u_{x_i} + F_u = G$$

where we assume that $A_{ij} = A_{ji}$, we also assume that B_i, F and G are functions of the n independent variables x_i. If $G = 0$, then we have a homogenous PDE; otherwise it's non-homogenous.

If we consider an n^{th} order ODE, then what we end up with is a solution depending on n arbitrary constants. A similar thing applies to PDEs, but they are n arbitrary functions. To illustate, we solve $u_{xy} = 0$, where first we integrate wrt y, and we get $u_x = f(x)$ and then integrate wrt x and we get $u(x, y) = g(x) + h(y)$ where g and h are arbitrary functions.

1.1 Mathematical Problems

A mathematical problem is PDE along with some supplimentary conditions on that PDE. the conditions may be intial conditions that are of the form $u(x, 0) = f(x)$ or boundary conditions which depends on the boundary. Let us take the example of the following PDE,

$$u_t - u_{xx} = 0$$

Then an initial conditions for this PDE may be $u(x, 0) = \sin x$ and if we consider it on some boundary $0 \leq x \leq \ell$ some boundary conditions may be $u(0, t) = 0$ and $u(\ell, t) = 0$ for some $t \geq 0$ (This example is the heat equation for a rod of length ℓ). This problem is known as an initial boundary problem. Sometimes we have more conditions that specify the problem, for example some conditions on the derivative. If we have a boundary that is not bounded, then sometimes we won't have boundary conditions and then we have a initial-value problem or a Cauchy Problem.

Finally, we say that a problem is well posed if,

(i) Existence, there is at least one solution

(ii) Uniqueness, there is at most one solution

(iii) Continuity, the solutions depends continuously on the data. A small input in the input data must reach a small change in the output data.

1.2 Linear Operators

An operator is a mathematical rule which when applied to a function produces another function. For example where,

$$L[u] = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$
$$M[u] = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} + x\frac{\partial u}{\partial x}$$

then we say that $L = \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y}$ and $M = \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} + x\frac{\partial}{\partial x}$ are the differential linear operators. We note a few things before a formal definition, if we have linear operators L and M, then $(L+M)[u]$ is a linear operator and $(L+M)[u] = L[u] + M[u]$. Furthermore, we can do something similar with $LM[u] = L(M[u])$. In general, here is the definition,

Lemma 1.3. Let L, M and N be linear operators. In general, a linear operator satisfies the following,

- $L + M = M + L$ (commutativity of addition)
- $(L + M) + N = L + (M + N)$ (associativity of addition)
- $L(MN) = (LM)N$ (associativity of multiplication)
- $L(c_1 M + c_2 N) = c_1 LM + c_2 LN$ (distributivity)

and for Linear Differential operators with constant coefficients, we have that $LM = ML$.

Now consider a linear second order PDE,

$$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} + D(x,y)u_x + E(x,y)u_y + F(x,y)u = G(x,y)$$

then if we let,

$$L = A(x,y)\frac{\partial^2}{\partial x^2} + B(x,y)\frac{\partial u}{\partial x \partial y} + C(x,y)\frac{\partial^2}{\partial y^2} + D(x,y)\frac{\partial}{\partial x} + E(x,y)\frac{\partial}{\partial y} + F(x,y)$$

be a linear differential operator, then we can write $Lu = G$ and that is our PDE.

Let $v_1, v_2, \dots, v_n$ be n functions satisfying

$$L[v_j] = G_j$$

for j running from 1 to n. Let $w_1, w_2, \dots, w_n$ be n functions where $L[w_j] = 0$ for j running from 1 to n. If we let $u_j = v_j + w_j$ then $u = \sum_{j=1}^{n} u_j$ this is called the principle of linear superposition.

If we consider $u_{tt} - c^2 u_{xx} = G(x,t)$ if we solve this for $0 < x < L$ where $u(x,0) = g_1(x)$ and $u_t(x,0) = g_2(x)$ for $0 \leq x \leq L$ and $t \geq 0$. We also have boundary conditions $u(0,t) = g_3$ and $u(L,t) = g_4$. We can write this in the form, $l[u] = G$ and $m_1[u] = g_1$ and $M_2[u] = g_2$ and $M_3[u] = g_3$ and finally $M_4[u] = g_4$. We can now decomopse this into four different problems.

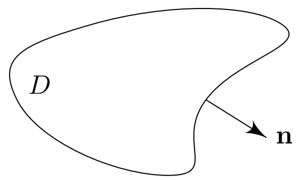
- $L[u] = G$, $M_1[u] = 0$, $M_2[u_1] = 0$, $M_3[u_1] = 0$ and $M_4[u_1] = 0$

- $L[u_2] = 0$, $M_1[u_2] = g_1$, $M_2[u_2] = 0$, $M_3[u_1] = 0$ and $M_4[u_1] = 0$
- $L[u_2] = 0$, $M_1[u] = 0$, $M_2[u_1] = g_2$, $M_3[u_1] = 0$ and $M_4[u_1] = 0$
- $L[u_3] = 0$, $M_1[u] = 0$, $M_2[u_1] = 0$, $M_3[u_1] = g_3$ and $M_4[u_1] = 0$
- $L[u_3] = 0$, $M_1[u] = 0$, $M_2[u_1] = 0$, $M_3[u_1] = 0$ and $M_4[u_1] = g_4$

and then solve the above and then add them together via the linear superposition.

1.3 Boundary Conditions

Assume we have $u_{xx} + u_{yy} = 0$ with a domain and boundary,



where $u(x, y) = f(x, y)$ along the boundary of D, then we have a Dirichlet Boundary condition. If $\frac{\partial u}{\partial x} = h(x, y) \to \partial D$ is a Neumann boundary condition. We can also have $\frac{\partial u}{\partial \mathbf{n}} = \nabla u \cdot \mathbf{n}$. If we can split the boundary into two then we can have a mixed type boundary condition; $u(x, y) + \frac{\partial u}{\partial n} = h(x, y)$ this is called a Robin Boundary condition.

Exercise. Prove that $\mathbb{R}^3$, gradient, curl and divergence are all linear differential operators, ie. prove that,

$$L(f + g) = L(f) + L(g)$$
$$L(cf) = cL(f)$$

where $c \in R^3$ and f, g are elements.

Exercise. Solve,

$$5u'' - 4u' + 4u = e^x \cos x$$

for a solution $u(x) = \frac{1}{6} e^x \sin x + c_1 e^{\frac{2}{5}x} \cos \frac{4x}{5} + c_2 e^{\frac{2x}{5}} \sin \frac{4x}{5}$

We now define classical solutions. Assume we have a PDE,

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} = D$$

with a solution, $u(x, y)$, for a classical solution we need this solution continuously second differentiable.

Definition 1.4 (Smooth)**.** A function is smooth if it can be differentiated sufficiently enough.

If the PDE has order n, then a solution has class $\mathcal{C}^n$. If we consider

$$\frac{\partial u}{\partial t} + k \frac{\partial^2 u}{\partial x^2} = 0$$

A solution is classical if $u(x, t)$ is differentiable by all the variables n times.

2 First Order Linear and Nonlinear waves

We want to solve,

$$\frac{\partial u}{\partial t} = 0$$

for $u(x, t)$. We can integrate both sides wrt time,

$$\int_0^t \frac{\partial u}{\partial s}\, ds = 0$$

and so we see $u(x, t) - u(x, 0) = 0$ and so $u(x, t) = f(x)$ where $f(x)$ is defined by the IC. For this to be classical $f(x) \in \mathcal{C}^1$. If $f(x) = x$, then we get $u(x, t) = xt + f(x)$ where $f(x) \in \mathcal{C}^1$.

If we want to solve $u_t = x - t$, then $u(x, t) = xt - \frac{1}{2}t^2 + f(x)$, or $u_x + tu = 0$ then we can use an integrating factor and then get $\frac{\partial u}{\partial t}(e^{tx}u) = 0$ and so $u(x, t) = e^{-tx}f(t)$ where $f(t) \in \mathcal{C}^1$.

2.1 Transport Equations

Next let us add another term,

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

where c is a constant. This is a transport equation, and the solution is a travelling wave. This models a uniform fluid flow with speed c subject to the condition $u(x, t_0) = f(x)$. We aim to reduce this to an ODE. Let us introduce $\xi = x - ct$ (the characteristic variable), then $u(x, t) = v(\xi, t) = v(x - ct, t)$. Let us take partial derivatives,

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial \xi}\frac{\partial x}{\partial i}t = v_t - cv_\xi$$

and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} = v_\xi$$

and so we get, $v_t - cv_\xi + cv_\xi$ and so $v_t = 0$. Hence, $v = v(\xi) = v(x - ct)$. Now let us put this more formally,

Proposition 2.1. If $u(x, t)$ is a solution to the PDE

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

defined on all $\mathbb{R}^2$. Then, $u(x, t) = v(x - ct)$ where $v(\xi)$ is a $\mathcal{C}^1$ function of the characteristic variable $\xi = x - ct$.

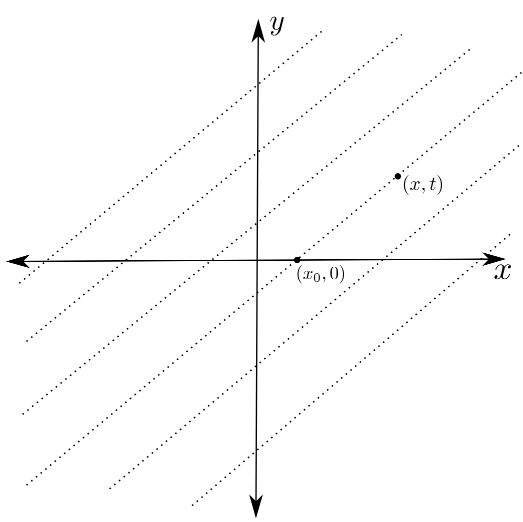


Figure 1: Characteristic Lines of a QLPDE

Now for an example,

Example. Solve,

$$\frac{\partial u}{\partial t} + 2\frac{\partial u}{\partial x} = 0$$

subject to $u(x, 0) = \frac{1}{1+x^2}$. To solve this, consider the characteristic variable, $\xi = x - 2t$, then we can represent the solution in the form $v(x - ct)$. To see this we represent the PDE as,

$$\frac{\partial u}{\partial t} = -v_\xi + v_t$$

$$\frac{\partial u}{\partial x} = v_\xi \xi_x = v_\xi$$

and so we can plug these in and get,

$$v_t = 0$$

and so $v = v(x-2t)$. Now we plug in the IC and get that $v(x) = \frac{1}{1+x^2}$ and so $v = \frac{1}{1+(x-2t)^2}$ and hence, $u(x,t) = \frac{1}{1+(x-2t)^2}$.

Let's go one step further with the transport equation with decay.

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + au = 0 \qquad a > 0$$

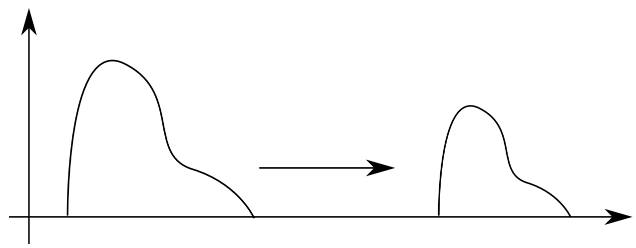


Figure 2: Transport Decay Equation

Example. We want to again use characteritics, so let $\xi = x - ct$ and so $u(x,t) = v(\xi,t) = v(x-ct,t)$. Then we get $u_t = -cv_\xi + v_t$ and $u_x = v_\xi$. Hence again we get that $\frac{\partial v}{\partial t} + av = 0$, solve by an integrating factor of e^{at} and conclude that $\frac{\partial}{\partial t}(ve^{at}) = 0$ and so $v = e^{-at}f(\xi)$. We can hence conclude that $v(\xi,t) = e^{-at}f(\xi)$ and $u(x,t) = e^{-at}f(x-ct)$. $f \in \mathcal{C}^1$.

Exercise. Solve,

$$\begin{cases} \frac{\partial u}{\partial t} - 3\frac{\partial u}{\partial x} = 0 \\ u(x,0) = e^{-x^2} \end{cases}$$

$$\begin{cases} \frac{\partial u}{\partial t} + 2\frac{\partial u}{\partial x} = 1 \\ u(x,0) = e^{-x^2} \end{cases}$$

Now let us adapt this such that $c = f(x)$, a non-uniform transport equation. It is of the form,

$$\frac{\partial u}{\partial t} + c(x)\frac{\partial u}{\partial x} = 0$$

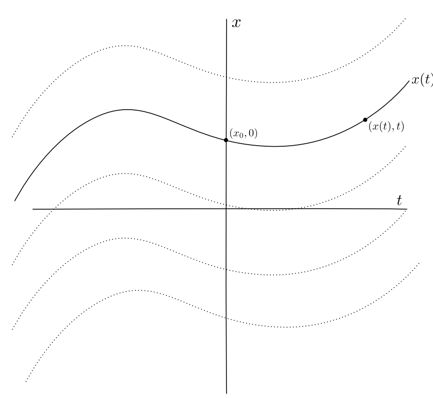
To use the method of characteristics, we would like to know how the solution varies along a curve in the (x,t) plane. We can parametrise any curve and so let us let $h(t) = u(x(t),t)$ and we want to measure the rate of change as the solutions moves along some curve in the plane. Now we take the derivative of $h(x)$ wrt time,

$$\frac{\partial h}{\partial t} = \frac{\partial u}{\partial t}(x(t),t) + \frac{\partial u}{\partial x}(x(t),t)\frac{dx}{dt}$$

Now we assume that $\frac{dx}{dt} = c(x)$, then we can conclude that,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial t}(x(t),t) + c(x)\frac{\partial u}{\partial x}(x(t),t)$$

and since we are assuming that the curve is a solution, then this is just our PDE. Hence, $\frac{\partial u}{\partial t} = 0$. The solution is constant along the characteristic.



Definition 2.2 (Characteristic Curve)**.** The graph of a solution $x(t)$ to the autonomous ODE $\frac{dx}{dt} = c(x)$ is called the characteristic curve. For the transport equation with wave speed $c(x)$.

Proposition 2.3. Solutions to the linear transport equation $u_t + c(x)u_x = 0$ are constant along characteristic curves.

Hence, from $\frac{dx}{dt} = c(x)$, we can find a characteristic curve for the PDE; if we integrate it then we can say that $\beta(x) = \int \frac{dx}{c(x)}$, then we can achieve that $\beta(x) = t + c$ and so we say that $\xi = \beta(x) - t$.

Example. Solve

$$\frac{\partial u}{\partial t} + \frac{1}{1+x^2}\frac{\partial u}{\partial x} = 0$$

using the method of characteristics.

2.2 Solutions to Quasi-Linear equations via methods of characteristics

We can write

$$F(x, y, u, u_x, u_y) = 0 \qquad (x, y) \in D \subset \mathbb{R}^2$$

Then if we write $u_x = p$ and $q = u_y$. Then this solution is quasi-linear if,

$$F(x, y, u, p, q) = 0$$

the PDE is linear in first partial derivatives of the unknown function $u(x, y)$. We can write the most general form as,

$$a(x, y, u)\frac{\partial u}{\partial x} + b(x, y, u)\frac{\partial u}{\partial y} = c(x, y, u)$$

Some examples are,

$$x(y^2 + u)u_x - y(x^2 + u)u_y = (x^2 - y^2)u$$

We call a PDE semi-linear if it further satisfies a and b being independent of u,

$$a(x, y)u_x + b(x, y)u_y = c(x, y, u)$$

For example,

$$xu_x + yu_y = u^2 + x^2$$

We call a PDE linear if it is linear in each of the variable,

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = d(x, y)$$

If $d(x, y) = 0$ we get a homogenous first order PDE and if $d(x, y) \neq 0$ then we have a non-homogenous first order PDE. For example, a homogenous PDE,

$$xu_x + yu_y - nu = 0$$

and a non-homogenous PDE,

$$nu_x + (x + y)u_y - u = e^x$$

More generally, these are geometric surfaces described by $f(x, y, z, a, b) = 0$ and if this exists, then the solution is complete. We can also reduce a and b out. A solution can be written as $f(\phi, \psi) = 0$ where $\phi, \psi \in \mathbb{R}^3 \to \mathbb{R}^3$.

Example. Show that a family of spheres $x^2 + y^2(z-c)^2 = r^2$ satisfies the first order linear PDE $yp - xq = 0$ where $p = z_x$ and $q = z_y$.

Exercise. Show that the family of spheres $(x-a)^2 + (y-b)^2 + z^2 = r^2$ satisfy $z^2(p^2+q^2+1) = r^2$ where $p = z_x$ and $q = z_y$.

Theorem 2.4. If $\phi = \phi(x,y,z)$ and $\psi = \psi(x,y,z)$ are two given functions of x, y and z and if $f(\phi,\psi) = 0$ where f is an arbitrary function of ϕ and ψ. Then $z = z(x,y)$ satisfies a first order PDE,

$$p\frac{\partial(\phi,\psi)}{\partial(y,z)} + q\frac{\partial(\phi,\psi)}{\partial(z,x)} = \frac{\partial(\phi,\psi)}{\partial(x,y)}$$

where

$$\frac{\partial(\phi,\psi)}{\partial(x,y)} = \begin{vmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{vmatrix}$$

Proof. Let $f(\phi,\psi) = 0$ and now let us differentiate by x and y, then,

$$\frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial \psi}\frac{\partial \psi}{\partial x} + \frac{\partial f}{\partial \psi}\frac{\partial \psi}{\partial z}\frac{\partial z}{\partial x} = 0$$

and simplify,

$$\frac{\partial f}{\partial \phi}\left(\frac{\partial \phi}{\partial x} + p\frac{\partial \phi}{\partial z}\right) + \frac{\partial f}{\partial \psi}\left(\frac{\partial \psi}{\partial x} + p\frac{\partial \psi}{\partial z}\right) = 0$$

and now we do the same thing for y,

$$\frac{\partial f}{\partial \phi}\left(\frac{\partial \phi}{\partial y} + q\frac{\partial \phi}{\partial z}\right) + \frac{\partial f}{\partial \psi}\left(\frac{\partial \psi}{\partial y} + q\frac{\partial \psi}{\partial z}\right) = 0$$

and now let us write these in matrix form,

$$\begin{bmatrix} \frac{\partial \phi}{\partial x} + p\frac{\partial \phi}{\partial z} & \frac{\partial \psi}{\partial x} + p\frac{\partial \psi}{\partial z} \\ \frac{\partial \phi}{\partial y} + q\frac{\partial \phi}{\partial z} & \frac{\partial \phi}{\partial y} + q\frac{\partial \phi}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial \phi} \\ \frac{\partial f}{\partial \psi} \end{bmatrix} = \mathbf{0}'$$

There is a non-trivial solution is and only if the determinant matrix is zero. If we calculate this determinant we get the solution of the PDE. □

If we consider a PDE of the form,

$$a(x,y,u)u_x + b(x,y,u)u_y - c(x,y,u) = 0$$

If we assume that $z = u$ is a solution surface, then we can define $f(x,y,u) = u(x,y) - u = 0$ Then we can write it as the following,

$$au_x + bu_y - c = (a,b,c)\cdot(u_x,u_y,-1) = 0$$

and so we can write it as $\nabla u \cdot (a,b,c)$ and so we know that ∇u is normal to the surface and so (a,b,c) must be tangent to the surface and we call the direction of the vector the characteristic direction. Now we seek to parametrise a curve such that (a,b,c) is tangent to the curve. If we paramerise the curve by $(x(t),y(t),u(t))$, then the tangent to the curve

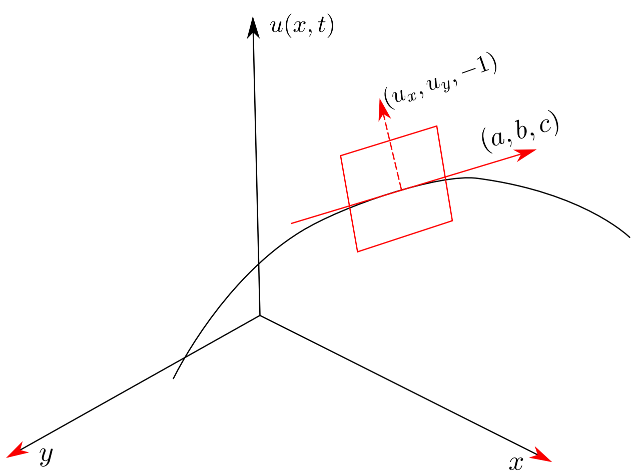


Figure 4: Geometric Interpretations.

will be $(\frac{dx}{dt}, \frac{dy}{dt}, \frac{du}{dt}) = (a, b, c)$. Now we can find the chateristic curve as we see that

$$\begin{cases} \frac{dx}{dt} = a(x, y, u) \\ \frac{dy}{dt} = b(x, y, u) \\ \frac{du}{dt} = b(x, y, u) \end{cases}$$

and we can write them as, $\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)} = dt$. Now we move to another theorem,

Theorem 2.5. The general solution of a first order quasi-linear PDE

$$a(x, y, u)\frac{\partial u}{\partial x} + b(x, y, u)\frac{\partial u}{\partial y} = c(x, y, u)$$

is $f(\phi, \psi) = 0$ where f is an arbitrary function of $\psi, \phi : \mathbb{R}^3 \to \mathbb{R}^3$, $\phi = c_1$ and $\psi = c_2$ are solution curves of the characteristic equations,

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}$$

and $\phi(x, y, u) = c_1$ and $\psi(x, y, u) = c_2$ are the family of characteristic curves.

Proof. Let $\phi(x, y, u) = c_1$ and $\psi(x, y, u) = c_2$. From the first, we can say that,

$$d\phi = \phi_x dx + \phi_y dy + \phi_u du = 0$$

and so,

$$\frac{d\phi}{dt} = \phi_x \frac{dx}{dt} + \phi_y \frac{dy}{dt} + \phi_u \frac{du}{dt} = 0$$

and so we get $a\phi_x + b\phi_y + c\phi_u = 0$ and similarly we can get $a\psi_x + b\psi_y + c\psi_u = 0$. If we now consider the following system of equations,

$$\begin{cases} a\phi_x + b\psi_y = -c\phi_u \\ a\psi_x + b\psi_y = -c\psi_u \end{cases}$$

and multiply the top equation by ψ_u and the bottom by ϕ_u we can conclude that,

$$a(\phi_x\psi_u - \psi_x\phi_u) + b(\psi_y\psi_u - \psi_y\phi_u) = 0$$

and we can write this as a Jacobean,

$$a\frac{\partial(\phi, \psi)}{\partial(x, u)} + b\frac{\partial(\psi, \phi)}{\partial(y, u)} = 0$$

and hence we can now find that,

$$\frac{a}{\frac{\partial(\phi,\psi)}{\partial(u,x)}} = \frac{b}{\frac{\partial(\phi,\psi)}{\partial(y,u)}}$$

Now we can do a very similar thing for the other systems we can form this way and get the desired result:

$$\frac{a}{\frac{\partial(\phi,\psi)}{\partial(y,u)}} = \frac{b}{\frac{\partial(\phi,\psi)}{\partial(u,x)}} = \frac{c}{\frac{\partial(\phi,\psi)}{\partial(x,y)}} \qquad (*)$$

and so now from Theorem 2.4, and using the above result $(*)$ in the following way, consider $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$,

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial f}{\partial \phi}\left(\frac{\partial \phi}{\partial x} + p\frac{\partial \phi}{\partial u}\right) + \frac{\partial f}{\partial \psi}\left(\frac{\partial \psi}{\partial x} + p\frac{\partial \psi}{\partial u}\right) \\ \frac{\partial f}{\partial y} &= \frac{\partial f}{\partial \phi}\left(\frac{\partial \phi}{\partial y} + q\frac{\partial \phi}{\partial u}\right) + \frac{\partial f}{\partial \psi}\left(\frac{\partial \psi}{\partial y} + q\frac{\partial \psi}{\partial u}\right) \end{aligned}$$

and again write this as a matrix,

$$\begin{bmatrix} \frac{\partial\phi}{\partial x} + p\frac{\partial\phi}{\partial u} & \frac{\partial\psi}{\partial x} + p\frac{\partial\psi}{\partial u} \\ \frac{\partial\phi}{\partial y} + q\frac{\partial\phi}{\partial u} & \frac{\partial\psi}{\partial y} + q\frac{\partial\psi}{\partial u} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial\phi} \\ \frac{\partial f}{\partial\psi} \end{bmatrix} = \mathbf{0}$$

and so we can say, similarly to before that there is a unique solution if the determinant of the matrix is zero. Hence,

$$\left(\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial y}\right) + p\left(\frac{\partial\phi}{\partial u}\frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial u}\frac{\partial\phi}{\partial y}\right) + q\left(\frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial u}\right) = 0$$

and so we can then rewrite this with Jacobeans,

$$p\frac{\partial(\phi,\psi)}{\partial(y,u)} + q\frac{\partial(\phi,\psi)}{\partial(u,x)} = \frac{\partial(\phi,\psi)}{\partial(x,y)}$$

and so we can now divide through and get,

$$p\frac{\frac{\partial(\phi,\psi)}{\partial(y,u)}}{\frac{\partial(\phi,\psi)}{\partial(x,y)}} + q\frac{\frac{\partial(\phi,\psi)}{\partial(u,x)}}{\frac{\partial(\phi,\psi)}{\partial(x,y)}} = 1$$

and from the result above we can say that,

$$\frac{\frac{\partial(\phi,\psi)}{\partial(y,u)}}{\frac{\partial(\phi,\psi)}{\partial(x,y)}} = \frac{a}{c} \qquad \frac{\frac{\partial(\phi,\psi)}{\partial(u,x)}}{\frac{\partial(\phi,\psi)}{\partial(x,y)}} = \frac{b}{c}$$

and so,

$$p\frac{a}{c} + q\frac{b}{c} = 1$$

which yields,

$$ap + bq = c$$

□

w

Theorem 2.6 (Cauchy Problem for first order PDEs)**.** Suppose C is a given curve in the (x, y)-plane with it's parametric equation, $x = x_0(t)$ and $y = y_0(t)$ where $t \in I \subset \mathbb{R}$ and derivatives $x_0(t)$ and $y_0'(t)$ are piecewise continuous such that they satisfy $x_0'^2 + y_0'^2 \neq 0$. Suppose that $u = u_0(t)$ is a given function on the curve C. Then there exists a solution $u = u(x, y)$ of the equation,

$$F(x, y, u, u_x, u_y) = 0$$

in the domain $D \subset \mathbb{R}^2$ containing the curve C for all $t \in I$. $u(x, y)$ satisfies $u(x_0(t), y_0(t)) = u_0(t)$ for all values of $t \in I$.

Now for a lot of examples,

Example. Find the general solution of the PDE, $xu_x + yu_y = u$. We let $a = x$, $b = y$ and $c = u$, hence,

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u}$$

and now we split and solve to get $y = c_1x$ and $u = c_2x$. Hence, the solution is $f\left(\frac{y}{x}, \frac{u}{x}\right) = 0$. We could have written this as $\frac{u}{x} = F\left(\frac{y}{x}\right)$ or $u(x, y) = xF\left(\frac{y}{x}\right)$.

Example. Obtain the general solution of the linear equation $xu_x + yu_y = nu$ where n is a constant. Here we do the same thing as above,

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{nu}$$

and we get the solution $u(x,y) = x^n F\left(\frac{y}{x}\right)$

Example. Find the general solution of $x^2u_x + y^2u_y = (x+y)u$. Here the characteristic is,

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{du}{(x+y)u}$$

The first function is easy to construct, we find that $\frac{1}{y} - \frac{1}{x} = c_1$ and the second can be found from

$$(x+y)udx = x^2du \tag{2}$$
$$(x+y)udy = y^2du \tag{3}$$

and then solving. Hence $\frac{x-y}{u} = c_2$. Then we can say the solution is $f(\frac{y-x}{xy}, \frac{x-y}{u}) = 0$ or $u(x,y) = (x-y)h(\frac{y-x}{xy})$

Exercise. Verify the solution.

Example. Obtain the general solution of the linear equation $u_x - u_y = 1$ with the Cauchy data $u(x,0) = x^2$. We find the characteristics,

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{du}{1}$$

and so we find that $y + x = c_1$ and $u - y = c_2$. Therefore, $u(x,y) = -y - F(x+y)$ and using the Cauchy data we can get that $u(x,y) = (x+y)^2 - y$

Review: We have $a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u)$ and we write this as $(a,b,c)\cdot(u_x,u_y,u_z) = 0$ and wrote $u(x,y)$ as the third coordinate and then considered the level surface $f(x,y,u) = u(x,y) - u$ and got that $\nabla f = (u_x, u_y, -1)$ and hence concluded that $(a,b,c)\cdot\nabla f = 0$ recovers our PDE. ∇f is perpendicular to the solution surface, and (a,b,c) is tangent to the surface and some curve in the surface must have tangent vector (a,b,c) which we call the characteristic curve.

$$(a,b,c) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{du}{dt}\right)$$

and this yielded a way to solve the PDE. How do we now parmaterise this solution surface?

2.3 Characteristic Projections

We shall now introduce characteristic projections. Suppose that $u(x,y)$ is specified along some curve Λ in the (x,y)-plane we then have $u = u_0(s)$ when $x = x_0(s)$ and $y = y_0(s)$ where s parameterises Λ in 3D, $(x_0(s), y_0(s), u_0(s))$ is our initial curve.

Characteristics pass through this curve and they are tangent to (a,b,c), so

$$\frac{dx}{d\tau} = a, \quad \frac{dy}{d\tau} = b, \quad \frac{du}{d\tau} = c$$

with initial conditions $x = x_0(s)$, $y = y_0(s)$ and $u = u_0(s)$ at $\tau = 0$. Then we know that the parameterised surface will be $(x(s,\tau), y(s,\tau), u(s,\tau))$and these are the parametric equations of the solution surface.

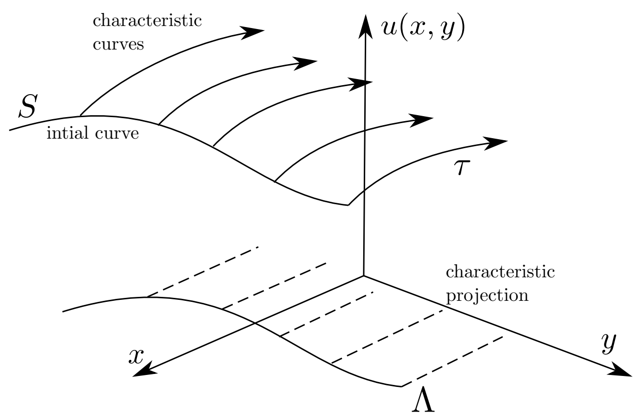


Figure 5: Geometric Interpretations.

Example. Solve $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 1$ subject to the boundary data $u = 0$ when $x + y = 0$. We can solve this by setting up characteristics,

$$\frac{dx}{d\tau} = 1 \quad \frac{dy}{d}\frac{dy}{d\tau} = 1 \quad \frac{du}{d\tau} = 1$$

Then we need initial conditions so we will find solutions depending on the parameter s, so our intial conditions are,

$$x = s \quad y = -s \quad u = 0 \quad \text{at } \tau = 0$$

and then we solve this system and get,

$$x(\tau) = \tau + s \quad y(\tau) = \tau - s \quad u(\tau) = \tau$$

Now we can eliminate τ and get the solution as $x + y = 2u$ or $u = \frac{x+y}{2}$

Example. Solve the PDE, $u_t + uu_x = 1$ for $u = u(x,t)$ in $t > 0$ subject to the initial condition $u = x$ at $t = 0$. We have the characteristics of,

$$\frac{dt}{d\tau} = 1 \quad \frac{dx}{d\tau} = u \quad \frac{du}{d\tau} = 1$$

We can solve the first and third very quickly, more specifically, $t(\tau) = \tau + c_1$ and $u(\tau) = \tau + c_3$, now given the data we can form some initial conditions, $x = s$, $u = s$ for $\tau = 0$. Hence, we have $t = \tau$ and $u = \tau + s$. Now we solve the second equation by plugging in u, $\frac{dx}{d\tau} = \tau + s$ and now we can solve them, $x(\tau) = \frac{1}{2}\tau^2 + s\tau + s$, or $s = \frac{x(\tau) - \frac{1}{2}\tau^2}{\tau + 1}$. Now we plug in the other solutions and find,

$$u(x,t) = \frac{\frac{1}{2}t^2 + t + x}{t + 1}$$

We start with another example,

Example. Find the solution of $u(x+y)u_x + u(x-y)u_y = x^2 + y^2$ with the Cauchy data $u = 0$ on $y = 2x$. We start with the characteristics,

$$\frac{dx}{u(x+y)} = \frac{dy}{u(x-y)} = \frac{du}{x^2 + y^2}$$

we can verify that $ydy + xdy - udu = 0$. Now we can turn this equation into an exact equation,

$$d\left(xy - \frac{1}{2}u^2\right) = xdy + ydx - udu = 0$$

and so, $xy - \frac{1}{2}u^2 = c_1 = \phi(x, y, u)$. We now need to find a ψ function, consider $xdx - ydy - udu = 0$ is satisfied by our characteristics and so we can form another exact equation,

$$d\left(\frac{1}{2}x^2 - \frac{1}{2}y - \frac{1}{2}u\right) = 0$$

and so, $\psi = x^2 - y^2 - u^2$. Therefore the general solution is,

$$u(x, y) = f\left(2xy - u^2, x^2 - y^2 - u^2\right) = 0$$

Now we consider the Cauchy data, $u(x, 2x) = 0 = f(4x^2, -3x^2)$ and so $\frac{1}{2}c_1 = -\frac{1}{3}c_2$, hence, $\frac{1}{4}\left(2xy - u^2\right) = -\frac{1}{3}\left(x^2 - y^2 - u^2\right)$ and so $7u^2 = 6xy + 4(x^2 + y^2)$

Now we revisit the Cauchy Theorem in a slightly different form.

Theorem 2.7 (Cauchy Problem (revisited))**.** Suppose that $x_0(t)$, $y_0(t)$ and $u_0(t)$ are continuously differentiable functions of t in $t \in [0, 1]$. Further suppose that $a(x, y, u)$, $b(x, y, u)$ and $c(x, y, u) \in \mathcal{C}^1$ with respect to their arguments t respect to some domain D of the (x, y, u)-space. $\Lambda : x = x_0(t) \quad y = y_0(t) \quad u = u_0(t)$ for $t \in [0, 1]$ and $y_0'(t)a(x_0, y_0, u_0) - x_0'(t)b(x_0, y_0, z_0) \neq 0$ then there exists a unique solution $u = u(x, y)$ of the quasi-linear PDE $au_x + bu_y = c$.

This condition assures that the intial curve is not in the same direction as the solution as they should arise from the Cauchy data. Another interpretation of this is there is a, one to one mapping from x, y, z to t, τ, s. We can now state the Cauchy-Kowalevski Theorem,

Theorem 2.8 (Cauchy-Kowaleski)**.** A necessary condition for a unique solution $u(x, y)$ to exist in a neighbourhood Λ is for the first derivative of $u(x, y)$ to be determined on Λ

We say that along the curve if Λ is parameterised by s, then $u_0(x_0(s), y_0(s)) = u_0(s)$. Any point of this curve has a projection onto the (x, y)-plane. We want to find the first order derivatives of u,

$$\frac{du_0}{ds} = \frac{\partial u_0}{\partial x}\frac{dx_0}{ds} + \frac{\partial u_0}{\partial y}\frac{dy_0}{ds}$$

and the PDE says,

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c$$

Now, let us put these in a matrix and find a condition,

$$\begin{pmatrix} a & b \\ \frac{dx_0}{ds} & \frac{\partial y_0}{\partial s} \end{pmatrix}\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} c \\ \frac{du_0}{ds} \end{pmatrix}$$

Hence, we need the determinant to be non-zero and so $a\frac{dy_0}{ds} - b\frac{dx_0}{ds} \neq 0$. This is just what we said in the Cauchy Problem Theorem. Here is the formal statement,

Theorem 2.9. The PDE $a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c$ has a unique analytical solution in some neighbourhood of Λ, provided a, b and c are analytic and satisfy $a\frac{dy_0}{ds} - b\frac{dx_0}{ds} \neq 0$

Some more examples,

Example. Solve the following PDE, $\frac{\partial u}{\partial t} + xu\frac{\partial u}{\partial x} = u$ for $u(x,t)$ in $t > 0$ subject to the cauchy data $u = x$ at $t = 0$ for $0 < x < 1$. We can write the characteristics as,

$$\frac{dt}{1} = \frac{dx}{xu} = \frac{du}{u}$$

or,

$$\frac{dt}{d\tau} = 1 \qquad \frac{dx}{d\tau} = xu \qquad \frac{du}{d\tau} = u$$

We can solve the first of the second form of characteristics, with the initial curve of $t = 0, x = s, u = s$ and we can say that $t = \tau$. Now we aim to solve $dt = \frac{du}{u}$ which yields a solution of the form $u(s,t) = f(s,t)e^t$ which when we use our intial curve gives us a specific solution of $u(s,t) = se^t$. Now we can solve $\frac{dx}{d\tau} = xu$ by plugging in our solution for $u(s,t)$ and so we solve, $\frac{dx}{dt} = xse^t$ which we can separate and then solve. This yields a solution of $x(s,t) = se^{s(e^t-1)}$ for $0 < s < 1$. We can now talk about the domain of definition, which will be $0 < x < e^{e^t-1}$.

Now we aim to eliminate s, so find a solution in terms of physical variables, we can see quickly that $s = ue^{-t}$ and so after substituting it in we get that $x = ue^{u-t-ue^{-t}}$

Here is another example,

Example. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u^3$ subject to $u = y$ on $x = 0$ for $0 < y < 3$. We write the characteristics,

$$\frac{dx}{d\tau} = 1 \qquad \frac{dx}{d\tau} = 1 \qquad \frac{du}{d\tau} = u^3$$

We can also write the initial curve of $x = 0, y = s, u = s$. This leads to us being able to write $x(\tau) = \tau$ and $y(\tau) = \tau + s$. Now we aim to solve $\frac{du}{d\tau} = u^3$, we aim to solve it by seperation of variables so let $u(s,\tau) = A(s)B(\tau)$. Now, $\frac{du}{d\tau} = A(s)\frac{dB}{d\tau}$, as we are considering $\frac{du}{d\tau} = u^3$ we say that $u^3 = A(s)\frac{dB}{d\tau}$ and so, $\frac{dB}{B^3} = A^2 d\tau$ and integrating we reach that $B(\tau) = \frac{1}{\sqrt{c-2A^2(s)\tau}}$. Hence, $u(s,\tau) = \frac{A(s)}{\sqrt{c-2A^2(s)\tau}}$. Now consider the intial curve and we get that $A(s) = s\sqrt{c}$ and so,

$$u(s,t) = \frac{s}{\sqrt{1-2s^2\tau}}$$

Finally, by considering the fact that $x = \tau$ and $y = \tau + s$, we can say that $s = y - x$ and the the implicit solution is,

$$u = \frac{y-x}{\sqrt{1-(y-x)^2 x}}$$

An interesting feature of this solution is that it blows up if $1-(y-x)^2x = 0$, that is along the line $y = x + \frac{1}{\sqrt{2x}}$ the solution does not exist and it's a singularity. The domain of solution is $x < y < x+3$. Hence we can plot the domain of solution,

2.4 Canonical Form for Linear First Order Equations

Assume that the PDE is linear,

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = d(x,y)$$

We aim to reduce this PDE to an ODE an integrate this ODE. This reduction is guided by the characteristics of this PDE. Assume that the characteristics are, $\xi = \xi(x,y)$, $\eta = \eta(x,y)$. This will be just a coordinate change, and this is a one-to-one mapping between $(x,y) \mapsto (\xi,\eta)$, that is $J = \xi_x\eta_y - \xi_y\eta_x \neq 0$. We want to substitute for these partial differentials.

$$u_x = u_\xi\xi_x + u_\eta\eta_x \qquad u_y = u_\xi\xi_y + u_\eta\eta_y$$

Now we substitute this into our linear PDE,

$$(a\xi_x + b\xi_y)u_\xi + (a\eta_x + b\eta_y)u_\eta + cu = d$$

Let $A = a\xi_x + b\xi_y$ and $B = a\eta_x + b\eta_y$, we get that $Au_\xi + Bu_\eta + cu = d$. If $B = 0$ we have an ODE. If $B = 0$, then $a\eta_x + b\eta_y = 0$. Our characteristics of our original equation is, $\frac{dx}{a} = \frac{dy}{b}$. The level curves of $B = 0$ are always characteristics of the original first order PDE. Hence, $\eta(x, y) = c$ and so $d\eta = 0 = \eta_x dx + \eta_y dy = 0$ or $\eta_x + \eta_y \frac{dy}{dx} = 0$. Now we substitute into the characteristcs, $a\eta + b\eta_y = 0$. This tells us that the general condition is just the characteristics. For the second characteristic we need to choose a one parameter family of curves such that the jacobean is zero. Hence we set $\xi = x$ (or $\xi = y$), then the jacobean is non-zero, or $\xi(x, y) = c$ and choose $\eta = y$ to satisfy $J \neq 0$. As $B = 0$ we can rewrite the PDE as,

$$u_\xi + \frac{c}{A}u = \frac{d}{A}$$

and we call this the cannonical form of this PDE and it is an ODE, hence we can just integrate it and get the required solution.

Example. Reduce, $u_x - u_y = u$. The characteristics are,

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{du}{u}$$

as $dx = -dy$ we let $x + y = \xi$, now set $\eta = y$. Hence we can say that $J \neq 0$. $u_x = u_\xi$ and $u_y = u_\xi + u_\eta$. We substitute this into the PDE, $u_x - u_y = -u_\eta = u$ or $u_\eta + u = 0$ and hence $u(\xi, \eta) = f(\xi)e^{-\eta}$ or $u(x, y) = f(x + y)e^{-y}$