Vector Calculus Week 4 - Vector Operators

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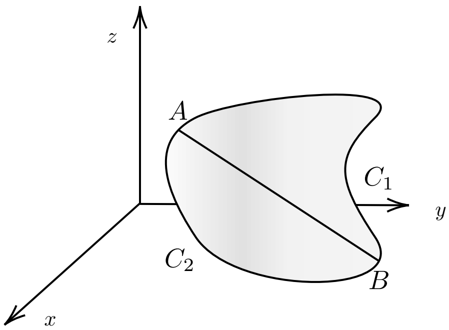
1 Conservative Fields

1.1 Gradients and Conservative Field



Definition 1.1: Conservative Vector Field

A conservative vector field is one which the line integral along a curve connecting two points does not depend on the path taken.

What this says, is that we can write:

$$\int_C \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} = \int_{C_1} \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} = \int_{C_2} \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}}$$

Theorem 1.1

Suppose that a vector field $\underline{\mathbf{F}}$ is related to a scalar field $\Phi(\underline{\mathbf{x}})$ by $\underline{\mathbf{F}} = \underline{\nabla}\Phi$ and $\underline{\nabla}\Phi$ exists everywhere in some region D. Conversely, if $\underline{\mathbf{F}}$ is conservative, then $\underline{\mathbf{F}}$ can be written as the gradient of a scalar field, $\underline{\mathbf{F}} = \underline{\nabla}\Phi$

Proof. Suppose that $\underline{\mathbf{F}} = \underline{\nabla}\Phi$, then F is conservative on D. So we can write;

$$\begin{aligned}
\int_C \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} &= \int_C \underline{\nabla}\Phi \cdot d\underline{\mathbf{r}} \\
&= \int_C \left(\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}, \frac{\partial\Phi}{\partial z}\right) \cdot (dx, dy, dz) \\
&= \int_C \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz \\
&= \int_C d\Phi \\
&= \Phi\Big|_A^B \\
&= \Phi(B) - \Phi(A)
\end{aligned}$$

So as this result only matters about the end points, $\underline{\mathbf{F}}$ is conservative. Now assume that $\underline{\mathbf{F}}$ is conservative, then a scalar field $\Phi(\underline{\mathbf{x}})$ can be defined as the line integral of $\underline{\mathbf{F}}$ from the origin to the point $\underline{\mathbf{x}}$:

$$\begin{aligned}
\Phi(\underline{\mathbf{x}}) &= \int_{\underline{\mathbf{0}}}^{\underline{\mathbf{x}}} \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} \\
d\Phi &= \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} \\
&= \underline{\nabla}\Phi \cdot \underline{\mathbf{r}} \\
&= \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz
\end{aligned}$$

and we can now say that $\underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} = \underline{\nabla}\Phi \cdot d\underline{\mathbf{r}}$ and hence, $F = \underline{\nabla}\Phi$ □

If a vector field is conservative, $\Phi(\underline{\mathbf{x}})$ which satisfies $\underline{\mathbf{F}} = \underline{\nabla}\Phi$ is called the potential of the vector field.

1.2 Curl and conservative vector fields

Suppose that $\underline{\mathbf{u}} = \underline{\nabla}\Phi$, then,

$$\begin{aligned}
\underline{\nabla} \times \underline{\mathbf{u}} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (u_1, u_2, u_3) \\
&= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_1 & u_2 & u_3 \end{vmatrix} \\
&= \left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right)\hat{\boldsymbol{\imath}} + \left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial z}\right)\hat{\boldsymbol{\jmath}} + \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)\hat{\boldsymbol{k}} \\
&= \left(\frac{\partial^2\Phi}{\partial y\partial z} - \frac{\partial^2\Phi}{\partial z\partial y}\right)\hat{\boldsymbol{\imath}} \left(\frac{\partial^2\Phi}{\partial z\partial x} - \frac{\partial^2\Phi}{\partial x\partial z}\right)\hat{\boldsymbol{\jmath}} \\
&\quad + \left(\frac{\partial^2\Phi}{\partial x\partial y} - \frac{\partial^2\Phi}{\partial y\partial x}\right)\hat{\boldsymbol{k}} \\
&= \underline{\mathbf{0}} \qquad \text{As } \Phi \in C^2
\end{aligned}$$

So for any vector $\underline{\mathbf{u}}$ that can be written as the gradient of a vector field is irrotational. Conversely, any irrotational vector field is conservative.

1.3 Laplacian of a scalar field

Suppose that a scalar field Φ, is twice dofferenctiable. Then $\underline{\nabla}\Phi$ is a differentiable vector field, so we can tak divergence of $\underline{\nabla}\Phi$ and obtain another scalar field

Definition 1.2: Laplacian

The scalar field $\underline{\nabla} \cdot \underline{\nabla}\Phi$ is called the Laplacian of Φ and is denoted, ∇^2 or Δ

The Laplacian can also act on a vector field, which results in another vector field.

$$\nabla^2 \underline{\mathbf{u}} = \nabla^2 u_1 \hat{\boldsymbol{\imath}} + \nabla^2 u_2 \hat{\boldsymbol{\jmath}} + \nabla^2 u_3 \hat{\boldsymbol{k}}$$

If we have $\Delta\Phi = 0$, this is a known PDE known as the laplace equation.

Theorem 1.2: Divergence of curl

For any $\mathcal{C}^2$ vector field, $\underline{\mathbf{F}}$,

$$\underline{\nabla} \cdot \underline{\nabla} \times \underline{\mathbf{F}} = 0$$

Proof.

$$\begin{aligned}
\underline{\nabla} \times \underline{\mathbf{F}} &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\
&= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\boldsymbol{\imath}} + \\
&\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial z}\right)\hat{\boldsymbol{\jmath}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{\boldsymbol{k}} \\
\underline{\nabla} \cdot \underline{\nabla} \times \underline{\mathbf{F}} &= \frac{\partial F_3}{\partial x \partial y} - \frac{\partial F_2}{\partial x \partial z} + \frac{\partial F_1}{\partial y \partial z} \\
&- \frac{\partial F_3}{\partial x \partial y} + \frac{\partial F_2}{\partial x \partial z} - \frac{\partial F_1}{\partial y \partial z} \\
&= \underline{\mathbf{0}}
\end{aligned}$$

□

1.4 Vector Operators Identities

Let Φ, f, g be scalar fields and $\underline{\mathbf{F}}, \underline{\mathbf{G}}$ be vector fields, then:

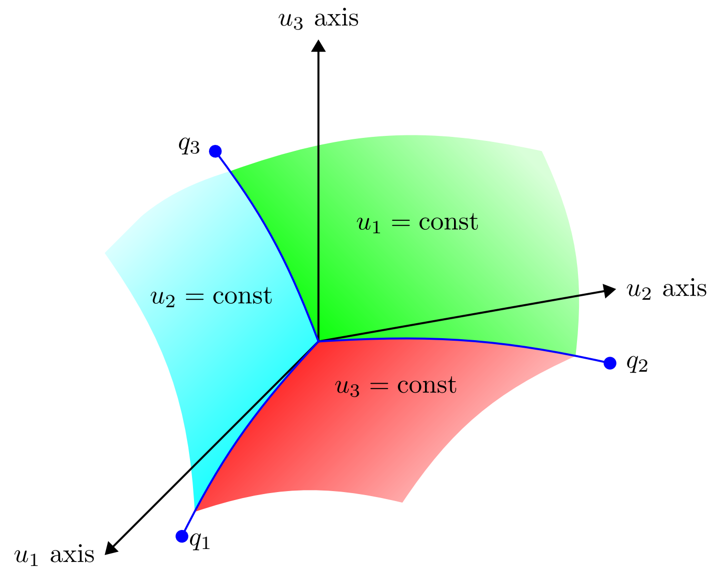
$$\begin{aligned}
\underline{\nabla} \cdot (\underline{\nabla} \times \underline{\mathbf{F}}) &= 0 && (1)\\
\underline{\nabla} \times \underline{\nabla}\Phi &= \underline{\mathbf{0}} && (2)\\
\underline{\nabla}(f+g) &= \underline{\nabla} f + \underline{\nabla} g && (3)\\
\underline{\nabla} \cdot (\underline{\mathbf{F}} + \underline{\mathbf{G}}) &= \underline{\nabla} \cdot \underline{\mathbf{F}} + \underline{\nabla} \cdot \underline{\mathbf{G}} && (4)\\
\underline{\nabla} \times (\underline{\mathbf{F}} + \underline{\mathbf{G}}) &= \underline{\nabla} \times \underline{\mathbf{F}} + \underline{\nabla} \times \underline{\mathbf{G}} && (5)\\
\underline{\nabla}(fg) &= f\underline{\nabla} g + g\underline{\nabla} f && (6)\\
\underline{\nabla} \cdot (\Phi\underline{\mathbf{F}}) &= \Phi\underline{\nabla} \cdot \underline{\mathbf{F}} + \underline{\mathbf{F}} \cdot \underline{\nabla}\Phi && (7)\\
\underline{\nabla} \times (\Phi\underline{\mathbf{F}}) &= \Phi\underline{\nabla} \times \underline{\mathbf{F}} - \underline{\mathbf{F}} \times \underline{\nabla}\Phi && (8)\\
\underline{\nabla}(\underline{\mathbf{F}} \cdot \underline{\mathbf{G}}) &= \underline{\mathbf{F}} \times (\underline{\nabla} \times \underline{\mathbf{G}}) + \underline{\mathbf{G}} \times (\underline{\nabla} \times \underline{\mathbf{F}}) && (9)\\
&+ (\underline{\mathbf{F}} \cdot \underline{\nabla})\underline{\mathbf{G}} + (\underline{\mathbf{G}} \cdot \underline{\nabla})\underline{\mathbf{F}} && (10)\\
& && (11)
\end{aligned}$$

$$\begin{aligned}
\underline{\nabla} \cdot (\underline{\mathbf{F}} \times \underline{\mathbf{G}}) &= \underline{\mathbf{G}} \cdot (\underline{\nabla} \times \underline{\mathbf{F}}) - \underline{\mathbf{F}}(\underline{\nabla} \times \underline{\mathbf{G}}) && (12)\\
\underline{\nabla} \times (\underline{\mathbf{F}} \times \underline{\mathbf{G}}) &= \underline{\mathbf{F}}(\underline{\nabla} \cdot \underline{\mathbf{G}}) - \underline{\mathbf{G}}(\underline{\nabla} \cdot \underline{\mathbf{F}}) && (13)\\
&+ (\underline{\mathbf{G}} \cdot \underline{\nabla})\underline{\mathbf{F}} - (\underline{\mathbf{F}} \cdot \underline{\nabla})\underline{\mathbf{G}} && (14)\\
\underline{\nabla} \times (\underline{\nabla} \times \underline{\mathbf{F}}) &= \underline{\nabla}(\underline{\nabla} \cdot \underline{\mathbf{F}}) - \underline{\nabla}^2\underline{\mathbf{F}} && (15)
\end{aligned}$$

2 Orthoginal Curvilinear Co-ordinate Systems

Assume a one to one map from x_i to u_i, the surfaces $u_i = k$ are defined as a co-ordinate surface and the intersection of the co-ordinate curves.

$$d\underline{\mathbf{r}} = (dx_1, dx_2, dx_3) = \frac{\partial \mathbf{r}}{\partial u_1} du_1 + \frac{\partial \mathbf{r}}{\partial u_2} du_2 + \frac{\partial \mathbf{r}}{\partial u_3} du_3$$



2.1 Scale Factors

If we let $\underline{\mathbf{e}}_1$ be an arbitrary unit vector in the direction of u_1, and similarly for $\underline{\mathbf{e}}_2$ and $\underline{\mathbf{e}}_3$, then:

$$e_1 = \frac{\partial \underline{\mathbf{r}}}{\partial u_1}\frac{1}{h_1} \qquad h_1 = \left|\frac{\partial \underline{\mathbf{r}}}{\partial u_1}\right|$$

and similarly for $\underline{\mathbf{e}}_2$ and $\underline{\mathbf{e}}_3$. Now we can rewrite $d\underline{\mathbf{r}}$:

$$d\underline{\mathbf{r}} = h_1\underline{\mathbf{e}}_1 du_1 + h_2\underline{\mathbf{e}}_2 du_2 + h_3\underline{\mathbf{e}}_3 du_3$$

We want, $\underline{\mathbf{e}}_i \cdot \underline{\mathbf{e}}_j = \delta_{ij}$ and $(\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \underline{\mathbf{e}}_3)$ to be right handed.

2.2 Differential of arc length

Let $d\underline{\mathbf{r}} = h_1 du_1 \underline{\mathbf{e}}_1 + h_2 du_2 \underline{\mathbf{e}}_2 + h_3 du_3 \underline{\mathbf{e}}_3$, then, $ds^2 = h_1^2 du_1^2 + h_2 du_2^2 + h_3^2 du_3^2$. Now we find dS, by taking the pross product between $\frac{\partial \underline{\mathbf{r}}}{\partial u_1} u_1$ and $\frac{\partial \underline{\mathbf{r}}}{\partial u_3} du_3$. Hence for u_1 surface, $dS = h_2 h_3 du_2 du_3$

2.3 Grad, Curl and Div in Curvilinear Co-ordinates

2.4 Cylindrical and Spherical Co-ordinate Systems