Week 4: Sequences

James Arthur

October 16, 2020

Contents

1 Sequences

Definition 1.1: Limit of a Sequence

A sequence, $\{s_n\}$ converges to a limit s, if for every $\varepsilon > 0 \,\exists\, N \in \mathbb{Z}$,

$$|S_n - S| < \varepsilon \quad n \geq N$$

Definition 1.2: Divergence

We say that $\lim_{n\to\infty} a_n = \infty$ if $\forall a \in \mathbb{R}$, $s_n > a$ for $n > a$. Similarly for $-\infty$.

Theorem 1.1

Let $\lim_{x\to\infty} f(x) = L$, where $L \in \overline{\mathbb{R}}$ and suppose that $s_n = f(n)$ for large n, then:

$$\lim_{n\to\infty} s_n = L$$

Definition 1.3: Subsequence

A subsequence $\{t_k\}$ if $t_k = s_{n_k}$, where $\{n_k\}$ is an increasing subsequence of integers.

Theorem 1.2: Uniqueness of subsequence limit

If $\lim_{n\to\infty} s_n = s$, then $\lim_{n\to\infty} s_{n_k} = s \quad \forall\{s_{n_k}\}$ of $\{s_k\}$

Proof. Consider the finite case, $\forall\varepsilon > 0 \,\exists\, N$,

$$|S_n - S| < \varepsilon \quad k \geq K$$

Since, $\{n_k\}$ is increasing $\exists\, K$, $n_k \geq N$ if $k > K$

$$|S_{n_k} - S| < \varepsilon \qquad k \geq K$$

For infinite limits, $\forall\varepsilon > 0$, $\exists\, N$,

$$S_n > n \qquad n \geq N$$

as we know $\{n_k\}$ is increasing, then $n_k > n$ for $n \geq N$ but $S_{n_k} > n$ for some $n \geq N$ and so the limit is infinite. For limit to $-\infty$, use the sequence $-S_n$ □

Theorem 1.3: Limit Points of Sequences

A point $\bar{x}$ is a limit point of a set S, iff there is a sequence $\{x_n\}$ of points in S, $x_n \neq \bar{x}$ for $n \geq 1$, and $\lim_{n\to\infty} x_n = \bar{x}$

Proof. Suppose such a $\{x_n\}$ exists. Then $\forall\varepsilon > 0$, $\exists\, N$,

$$0 < |x_n - \bar{x}| < \varepsilon \qquad \forall n \geq N$$

Therefore every ε-neigh. contains ∞ many points of S hence $\bar{x}$ is a limit point of S.

Now let $\bar{x}$ be a limit point of S. $\forall N \geq 1$, $(\bar{x} - \frac{1}{n}, \bar{x} + \frac{1}{n})$ has to contain some point $x_n \in S$, $x_n \neq \bar{x}$. Since,

$$|x_n - \bar{x}| \leq \frac{1}{n} \qquad m \geq n$$

and $\lim_{n\to\infty} x_n = \bar{x}$ □

Theorem 1.4: Bounded and Subsequence Theorems

1. If $\{x_n\}$ is bounded then it has a convergent subsequence
2. If $\{x_n\}$ is unbounded above, then it has a subsequence $\{x_{n_k}\}$ st,
$$\lim_{x\to\infty} x_{n_k} = \infty$$
3. If $\{x_n\}$ is unbounded above, then $\{x_n\}$ has a subsequence st,
$$\lim_{x\to\infty} x_{n_k} = -\infty$$

Proof. **Proof of 1:** Let S be a set of distinct numbers of $\{s_n\}$, if s is finite, then $\exists$, $\bar{x} \in s$, which occurs infinitely often. Then,

$$\lim_{n\to\infty} x_{n_k} = \bar{x}$$

If s is infinte, then since s is bounded BWT applies, now s has a limit point, $\bar{x}$. Then by previous thm, $\exists\{y_j\} \in s$ with $y_j \neq s$,

$$\lim_{j\to\infty} y_j = \bar{x}$$

However, $\{y_j\}$ may not be a subsequence of $\{x_n\}$, so $y_j = x_{n_j}$ may not be true, where n_j is increasing. So now take an increasing subsequence of n_j, $\{n_j\}$, then $\{y_{j_k}\} = \{s_{n_{j_k}}\}$ is a subsequence. So it has the same limit as; $\{y_j\}$

$$\lim_{k\to\infty} \{s_{n_{j_k}}\} = \bar{x}$$

□

1.1 Cauchy Sequences

Definition 1.4: Cauchy Sequences

A sequence $\{s_n\}$ of real numbers is said to be cauchy if $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$, $n \geq N$ and $m \geq N$, then:

$$|S_n - S_m| < \varepsilon$$

Lemma 1.1

Let $\{s_n\}$ be a convergent, then it's cauchy

Proof. Suppose that $s_n \to s$ as $n \to \infty$. Let $\varepsilon > 0$, $\exists\, N$, $n \geq N$

$$|s_n - s| < \frac{\varepsilon}{2}$$

Now take $m, n \geq N$, then

$$\begin{aligned} |s_n - s_m| &= |s_n - s - (s_m - s)| \\ &\leq |s_n - s| + |s_m - s| \\ &< \varepsilon \end{aligned}$$

□

Lemma 1.2

Let $\{s_n\}$ be cauchy, then it's convergent

Proof. Let $\{s_n\}$ be cauchy, and hence it's bounded. By thm 3.14(a), there is a convergent subsequence $\{s_{n_k}\}$ for some $s \in \mathbb{R}$. Now claim, $s_k \in s$ as $k \to \infty$.

Let $\varepsilon > 0$, $\exists\, N_2$, $k \geq N_1$, then

$$|s_{n_k} s| < \frac{\varepsilon}{2}$$

Then $\exists\, N_2$, $m, n \geq N_2$, then $|s_m - s_n| < \frac{\varepsilon}{2}$. If $K \geq \max(m, n)$,

$$\begin{aligned} |s_k - s| &= |(s_k - s_{n_k}) - (s - s_{n_k})| \\ &\leq |s_k - s_{n_k}| + |s - s_{n_k}| \\ &< \varepsilon \end{aligned}$$

□

2 Series

Definition 2.1: Series

If $\{a_k\}_k^\infty = \sum_{n=k}^{\infty} a_k$ is infinite and a_n is the n^th term. If $\sum_{k=1}^{\infty} = A$, then it converges. Also we say $A_n = a_k + \cdots + a_n \quad n \geq k$ is the n^th partial sum of the sum. We can also say that,

$$\lim_{k\to\infty} A_n = A$$

Theorem 2.1: Cauchy Criterion for Series

A series $\sum a_n$ converges iff $\forall \varepsilon > 0$, $\exists\, N$,

$$|a_n + a_{n-1} + \cdots + a_m| < \varepsilon \qquad m \geq n \geq N \quad (*)$$

Proof. Let $\{A_n\}$ be the series of partial sums of our series. Then

$$A_m - A_{n-1} = a_n + \cdots + a_m$$

If $(*)$ holds, then

$$|A_m - A_{n-1}| < \varepsilon \text{ if } m \geq n \geq N \quad (**)$$

To say $\sum a_n$ is convergent, then $\{A_n\}$ is convergent. This is equiv to $\{A_n\}$ being cauchy, which is what $(**)$ says. □

Corollary 1. *If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$*

Proof. Taking $m = n$ in the previous thm, then $\forall \varepsilon > 0$, $\exists\, N > 0$,

$$|a_n| < \varepsilon \text{ if } n \geq N$$

which is $\lim_{n\to\infty} a_n = 0$ □

Corollary 2. *(Divergence Test) If* $\lim_{n\to\infty} a_n \neq 0$*, then* $\sum a_n$ *divergent*