Year 3 — Partial Differential Equations

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine (especially the typos!).

Contents

1 Introduction to PDEs

A differential equation that contains, in addition to the dependent variable and independent variables, one or more partial derivatives of the dependent variable is called a partial differential equation.

In general it may be written in the form,

$$f(x, y, \dots, u, u_x, u_y, \dots, u_{xx}, u_{yy}, \dots) = 0 \tag{1}$$

involving several of the x, y, u_x, u_{xx} terms. Note that the notation $u_x = \frac{\partial u}{\partial x}$.

When we consider a PDE, we also consider it in a suitable domain. For us, the domain, D, will just be some domain of $\mathbb{R}^n$ in the variables $x, y, \dots$. A solution of this equation will be a function $u = u(x, y, \dots)$ which satisfy (1). We call the order of the PDE the highest order partial derivative appearing the equation.

Example. $u_{xxy} + xu_{yy} + 8u = 8y$ is a third order PDE.

Definition 1.1 (Linear)**.** We call a PDE linear if it is linear in the unknown function and all it's derivatives. For example,

$$yu_{xx} + 2xyu_{yy} + u = 1$$

and we have a further characterisation, called quasi-linear,

Definition 1.2 (Quasi-Linear)**.** A PDE is quasi linear if it linear in the highest-orderderivative of the unknown function,

$$u_x u_{xx} + xuu_y = \sin y$$

and furthermore, an equation that isn't linear is non-linear. IN this course we will consider mainly second order linear PDEs. The most general of these can be written as,

$$\sum_{i,j=1}^{n} A_{ij} u_{x_i x_j} + \sum_{i=1}^{n} B_i u_{x_i} + F_u = G$$

where we assume that $A_{ij} = A_{ji}$, we also assume that B_i, F and G are functions of the n independent variables x_i. If $G = 0$, then we have a homogenous PDE; otherwise it's non-homogenous.

If we consider an n^{th} order ODE, then what we end up with is a solution depending on n arbitrary constants. A similar thing applies to PDEs, but they are n arbitrary functions. To illustate, we solve $u_{xy} = 0$, where first we integrate wrt y, and we get $u_x = f(x)$ and then integrate wrt x and we get $u(x, y) = g(x) + h(y)$ where g and h are arbitrary functions.

1.1 Mathematical Problems

A mathematical problem is PDE along with some supplimentary conditions on that PDE. the conditions may be intial conditions that are of the form $u(x, 0) = f(x)$ or boundary conditions which depends on the boundary. Let us take the example of the following PDE,

$$u_t - u_{xx} = 0$$

Then an initial conditions for this PDE may be $u(x, 0) = \sin x$ and if we consider it on some boundary $0 \leq x \leq \ell$ some boundary conditions may be $u(0, t) = 0$ and $u(\ell, t) = 0$ for some $t \geq 0$ (This example is the heat equation for a rod of length ℓ). This problem is known as an initial boundary problem. Sometimes we have more conditions that specify the problem, for example some conditions on the derivative. If we have a boundary that is not bounded, then sometimes we won't have boundary conditions and then we have a initial-value problem or a Cauchy Problem.

Finally, we say that a problem is well posed if,

1. Existence, there is at least one solution

2. Uniqueness, there is at most one solution

3. Continuity, the solutions depends continuously on the data. A small input in the input data must reach a small change in the output data.

1.2 Linear Operators

An operator is a mathematical rule which when applied to a function produces another function. For example where,

$$L[u] = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$
$$M[u] = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} + x\frac{\partial u}{\partial x}$$

then we say that $L = \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y}$ and $M = \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} + x\frac{\partial}{\partial x}$ are the differential linear operators. We note a few things before a formal definition, if we have linear operators L and M, then $(L+M)[u]$ is a linear operator and $(L+M)[u] = L[u] + M[u]$. Furthermore, we can do something similar with $LM[u] = L(M[u])$. In general, here is the definition,

Lemma 1.3. Let L, M and N be linear operators. In general, a linear operator satisfies the following,

- $L + M = M + L$ (commutativity of addition)
- $(L + M) + N = L + (M + N)$ (associativity of addition)
- $L(MN) = (LM)N$ (associativity of multiplication)
- $L(c_1M + c_2N) = c_1LM + c_2LN$ (distributivity)

and for Linear Differential operators with constant coefficients, we have that $LM = ML$.

Now consider a linear second order PDE,

$$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} + D(x,y)u_x + E(x,y)u_y + F(x,y)u = G(x,y)$$

then if we let,

$$L = A(x,y)\frac{\partial^2}{\partial x^2} + B(x,y)\frac{\partial u}{\partial x \partial y} + C(x,y)\frac{\partial^2}{\partial y^2} + D(x,y)\frac{\partial}{\partial x} + E(x,y)\frac{\partial}{\partial y} + F(x,y)$$

be a linear differential operator, then we can write $Lu = G$ and that is our PDE.

Let $v_1, v_2, \ldots, v_n$ be n functions satisfying

$$L[v_j] = G_j$$

for j running from 1 to n. Let $w_1, w_2, \ldots, w_n$ be n functions where $L[w_j] = 0$ for j running from 1 to n. If we let $u_j = v_j + w_j$ then $u = \sum_{j=1}^{n} u_j$ this is called the principle of linear superposition.

If we consider $u_{tt} - c^2u_{xx} = G(x,t)$ if we solve this for $0 < x < L$ where $u(x,0) = g_1(x)$ and $u_t(x,0) = g_2(x)$ for $0 \le x \le L$ and $t \ge 0$. We also have boundary conditions $u(0,t) = g_3$ and $u(L,t) = g_4$. We can write this in the form, $l[u] = G$ and $m_1[u] = g_1$ and $M_2[u] = g_2$ and $M_3[u] = g_3$ and finally $M_4[u] = g_4$. We can now decomopse this into four different problems.

- $L[u] = G$, $M_1[u] = 0$, $M_2[u_1] = 0$, $M_3[u_1] = 0$ and $M_4[u_1] = 0$

- $L[u_2] = 0$, $M_1[u_2] = g_1$, $M_2[u_2] = 0$, $M_3[u_1] = 0$ and $M_4[u_1] = 0$
- $L[u_2] = 0$, $M_1[u] = 0$, $M_2[u_1] = g_2$, $M_3[u_1] = 0$ and $M_4[u_1] = 0$
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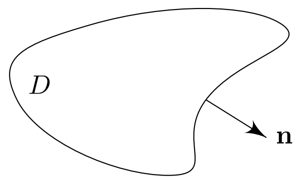
- $L[u_3] = 0$, $M_1[u] = 0$, $M_2[u_1] = 0$, $M_3[u_1] = 0$ and $M_4[u_1] = g_4$

and then solve the above and then add them together via the linear superposition.

1.3 Boundary Conditions

Assume we have $u_{xx} + u_{yy} = 0$ with a domain and boundary,



where $u(x, y) = f(x, y)$ along the boundary of D, then we have a Dirichlet Boundary condition. If $\frac{\partial u}{\partial x} = h(x, y) \to \partial D$ is a Neumann boundary condition. We can also have $\frac{\partial u}{\partial \mathbf{n}} = \nabla u \cdot \mathbf{n}$. If we can split the boundary into two then we can have a mixed type boundary condition; $u(x, y) + \frac{\partial u}{\partial n} = h(x, y)$ this is called a Robin Boundary condition.

Exercise. Prove that $\mathbb{R}^3$, gradient, curl and divergence are all linear differential operators, ie. prove that,

$$L(f + g) = L(f) + L(g)$$
$$L(cf) = cL(f)$$

where $c \in R^3$ and f, g are elements.

Exercise. Solve,

$$5u'' - 4u' + 4u = e^x \cos x$$

for a solution $u(x) = \frac{1}{6}e^x \sin x + c_1 e^{\frac{2}{5}x} \cos \frac{4x}{5} + c_2 e^{\frac{2x}{5}} \sin \frac{4x}{5}$

We now define classical solutions. Assume we have a PDE,

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} = D$$

with a solution, $u(x, y)$, for a classical solution we need this solution continuously second differentiable.

Definition 1.4 (Smooth)**.** A function is smooth if it can be differentiated sufficiently enough.

If the PDE has order n, then a solution has class $\mathcal{C}^n$. If we consider

$$\frac{\partial u}{\partial t} + k\frac{\partial^2 u}{\partial x^2} = 0$$

A solution is classical if $u(x, t)$ is differentiable by all the variables n times.

2 First Order Linear and Nonlinear waves

We want to solve,

$$\frac{\partial u}{\partial t} = 0$$

for $u(x,t)$. We can integrate both sides wrt time,

$$\int_0^t \frac{\partial u}{\partial s}\, ds = 0$$

and so we see $u(x,t) - u(x,0) = 0$ and so $u(x,t) = f(x)$ where $f(x)$ is defined by the IC. For this to be classical $f(x) \in \mathcal{C}^1$. If $f(x) = x$, then we get $u(x,t) = xt + f(x)$ where $f(x) \in \mathcal{C}^1$.

If we want to solve $u_t = x - t$, then $u(x,t) = xt - \frac{1}{2}t^2 + f(x)$, or $u_x + tu = 0$ then we can use an integrating factor and then get $\frac{\partial u}{\partial t}(e^{tx}u) = 0$ and so $u(x,t) = e^{-tx}f(t)$ where $f(t) \in \mathcal{C}^1$.

2.1 Transport Equations

Next let us add another term,

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

where c is a constant. This is a transport equation, and the solution is a travelling wave. This models a uniform fluid flow with speed c subject to the condition $u(x,t_0) = f(x)$. We aim to reduce this to an ODE. Let us introduce $\xi = x - ct$ (the characteristic variable), then $u(x,t) = v(\xi,t) = v(x-ct,t)$. Let us take partial derivatives,

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial \xi}\frac{\partial x}{\partial i}t = v_t - cv_\xi$$

and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} = v_\xi$$

and so we get, $v_t - cv_\xi + cv_\xi$ and so $v_t = 0$. Hence, $v = v(\xi) = v(x-ct)$. Now let us put this more formally,

Proposition 2.1. If $u(x,t)$ is a solution to the PDE

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

defined on all $\mathbb{R}^2$. Then, $u(x,t) = v(x-ct)$ where $v(\xi)$ is a $\mathcal{C}^1$ function of the characteristic variable $\xi = x - ct$.

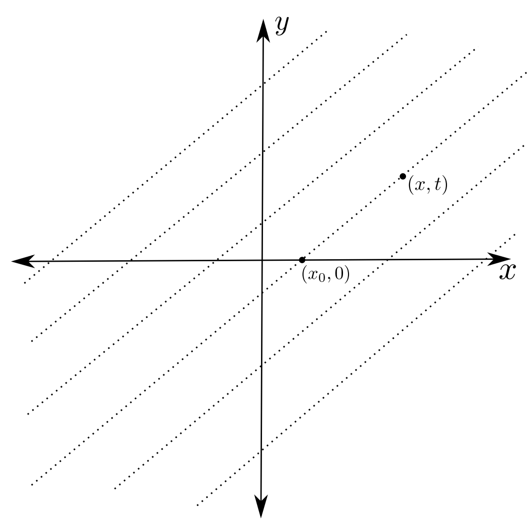


Figure 1: Characteristic Lines of a QLPDE

Now for an example,

Example. Solve,

$$\frac{\partial u}{\partial t} + 2\frac{\partial u}{\partial x} = 0$$

subject to $u(x,0) = \frac{1}{1+x^2}$. To solve this, consider the characteristic variable, $\xi = x - 2t$, then we can represent the solution in the form $v(x-ct)$. To see this we represent the PDE as,

$$\frac{\partial u}{\partial t} = -v_\xi + v_t$$

$$\frac{\partial u}{\partial x} = v_\xi \xi_x = v_\xi$$

and so we can plug these in and get,

$$v_t = 0$$

and so $v = v(x-2t)$. Now we plug in the IC and get that $v(x) = \frac{1}{1+x^2}$ and so $v = \frac{1}{1+(x-2t)^2}$ and hence, $u(x,t) = \frac{1}{1+(x-2t)^2}$.

Let's go one step further with the transport equation with decay.

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + au = 0 \qquad a > 0$$

Example. We want to again use characteritics, so let $\xi = x - ct$ and so $u(x,t) = v(\xi,t) = v(x-ct,t)$. Then we get $u_t = -cv_\xi + v_t$ and $u_x = v_\xi$. Hence again we get that $\frac{\partial v}{\partial t} + av = 0$, solve by an integrating factor of e^{at} and conclude that $\frac{\partial}{\partial t}(ve^{at}) = 0$ and so $v = e^{-at}f(\xi)$. We can hence conclude that $v(\xi,t) = e^{-at}f(\xi)$ and $u(x,t) = e^{-at}f(x-ct)$. $f \in \mathcal{C}^1$.

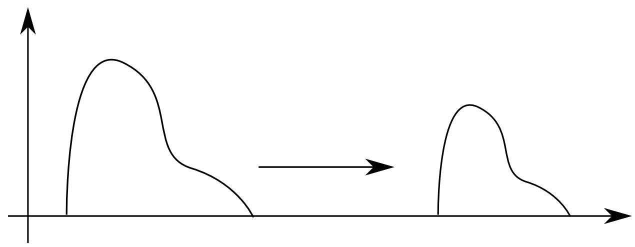


Figure 2: Transport Decay Equation

Exercise. Solve,

$$\begin{cases} \frac{\partial u}{\partial t} - 3\frac{\partial u}{\partial x} = 0 \\ u(x,0) = e^{-x^2} \end{cases}$$

$$\begin{cases} \frac{\partial u}{\partial t} + 2\frac{\partial u}{\partial x} = 1 \\ u(x,0) = e^{-x^2} \end{cases}$$

Now let us adapt this such that $c = f(x)$, a non-uniform transport equation. It is of the form,

$$\frac{\partial u}{\partial t} + c(x)\frac{\partial u}{\partial x} = 0$$

To use the method of characteristics, we would like to know how the solution varies along a curve in the (x,t) plane. We can parametrise any curve and so let us let $h(t) = u(x(t),t)$ and we want to measure the rate of change as the solutions moves along some curve in the plane. Now we take the derivative of $h(x)$ wrt time,

$$\frac{\partial h}{\partial t} = \frac{\partial u}{\partial t}(x(t),t) + \frac{\partial u}{\partial x}(x(t),t)\frac{dx}{dt}$$

Now we assume that $\frac{dx}{dt} = c(x)$, then we can conclude that,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial t}(x(t),t) + c(x)\frac{\partial u}{\partial x}(x(t),t)$$

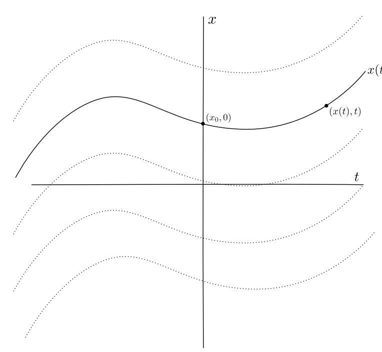


Figure 3: Non-uniform characteristic curves

and since we are assuming that the curve is a solution, then this is just our PDE. Hence, $\frac{\partial u}{\partial t} = 0$. The solution is constant along the characteristic.

Definition 2.2 (Characteristic Curve)**.** The graph of a solution $x(t)$ to the autonomous ODE $\frac{dx}{dt} = c(x)$ is called the characteristic curve. For the transport equation with wave speed $c(x)$.

Proposition 2.3. Solutions to the linear transport equation $u_t + c(x)u_x = 0$ are constant along characteristic curves.

Hence, from $\frac{dx}{dt} = c(x)$, we can find a characteristic curve for the PDE; if we integrate it then we can say that $\beta(x) = \int \frac{dx}{c(x)}$, then we can achieve that $\beta(x) = t + c$ and so we say that $\xi = \beta(x) - t$.

Example. Solve

$$\frac{\partial u}{\partial t} + \frac{1}{1+x^2}\frac{\partial u}{\partial x} = 0$$

using the method of characteristics.

2.2 Solutions to Quasi-Linear equations via methods of characteristics

We can write

$$F(x, y, u, u_x, u_y) = 0 \qquad (x, y) \in D \subset \mathbb{R}^2$$

Then if we write $u_x = p$ and $q = u_y$. Then this solution is quasi-linear if,

$$F(x, y, u, p, q) = 0$$

the PDE is linear in first partial derivatives of the unknown function $u(x, y)$. We can write the most general form as,

$$a(x, y, u)\frac{\partial u}{\partial x} + b(x, y, u)\frac{\partial u}{\partial y} = c(x, y, u)$$

Some examples are,

$$x(y^2 + u)u_x - y(x^2 + u)u_y = (x^2 - y^2)u$$

We call a PDE semi-linear if it further satisfies a and b being independent of u,

$$a(x, y)u_x + b(x, y)u_y = c(x, y, u)$$

For example,

$$xu_x + yu_y = u^2 + x^2$$

We call a PDE linear if it is linear in each of the variable,

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = d(x, y)$$

If $d(x, y) = 0$ we get a homogenous first order PDE and if $d(x, y) \neq 0$ then we have a non-homogenous first order PDE. For example, a homogenous PDE,

$$xu_x + yu_y - nu = 0$$

and a non-homogenous PDE,

$$nu_x + (x + y)u_y - u = e^x$$

More generally, these are geometric surfaces described by $f(x, y, z, a, b) = 0$ and if this exists, then the solution is complete. We can also reduce a and b out. A solution can be written as $f(\phi, \psi) = 0$ where $\phi, \psi \in \mathbb{R}^3 \to \mathbb{R}^3$.

Example. Show that a family of spheres $x^2 + y^2(z - c)^2 = r^2$ satisfies the first order linear PDE $yp - xq = 0$ where $p = z_x$ and $q = z_y$.

Exercise. Show that the family of spheres $(x - a)^2 + (y - b)^2 + z^2 = r^2$ satisfy $z^2(p^2 + q^2 + 1) = r^2$ where $p = z_x$ and $q = z_y$.

Theorem 2.4. If $\phi = \phi(x, y, z)$ and $\psi = \psi(x, y, z)$ are two given functions of x, y and z and if $f(\phi, \psi) = 0$ where f is an arbitrary function of ϕ and ψ. Then $z = z(x, y)$ satisfies a first order PDE,

$$p\frac{\partial(\phi, \psi)}{\partial(y, z)} + q\frac{\partial(\phi, \psi)}{\partial(z, x)} = \frac{\partial(\phi, \psi)}{\partial(x, y)}$$

where

$$\frac{\partial(\phi, \psi)}{\partial(x, y)} = \begin{vmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{vmatrix}$$

Proof. Let $f(\phi, \psi) = 0$ and now let us differentiate by x and y, then,

$$\frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial \psi}\frac{\partial \psi}{\partial x} + \frac{\partial f}{\partial \psi}\frac{\partial \psi}{\partial z}\frac{\partial z}{\partial x} = 0$$

and simplify,

$$\frac{\partial f}{\partial \phi}\left(\frac{\partial \phi}{\partial x} + p\frac{\partial \phi}{\partial z}\right) + \frac{\partial f}{\partial \psi}\left(\frac{\partial \psi}{\partial x} + p\frac{\partial \psi}{\partial z}\right) = 0$$

and now we do the same thing for y,

$$\frac{\partial f}{\partial \phi}\left(\frac{\partial \phi}{\partial y} + q\frac{\partial \phi}{\partial z}\right) + \frac{\partial f}{\partial \psi}\left(\frac{\partial \psi}{\partial y} + q\frac{\partial \psi}{\partial z}\right) = 0$$

and now let us write these in matrix form,

$$\begin{bmatrix} \frac{\partial \phi}{\partial x} + p\frac{\partial \phi}{\partial z} & \frac{\partial \psi}{\partial x} + p\frac{\partial \psi}{\partial z} \\ \frac{\partial \phi}{\partial y} + q\frac{\partial \phi}{\partial z} & \frac{\partial \phi}{\partial y} + q\frac{\partial \phi}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial \phi} \\ \frac{\partial f}{\partial \psi} \end{bmatrix} = \mathbf{0}'$$

There is a non-trivial solution is and only if the determinant matrix is zero. If we calculate this determinant we get the solution of the PDE. $\square$

If we consider a PDE of the form,

$$a(x, y, u)u_x + b(x, y, u)u_y - c(x, y, u) = 0$$

If we assume that $z = u$ is a solution surface, then we can define $f(x, y, u) = u(x, y) - u = 0$ Then we can write it as the following,

$$au_x + bu_y - c = (a, b, c) \cdot (u_x, u_y, -1) = 0$$

and so we can write it as $\nabla u \cdot (a, b, c)$ and so we know that ∇u is normal to the surface and so (a, b, c) must be tangent to the surface and we call the direction of the vector the characteristic direction. Now we seek to parametrise a curve such that (a, b, c) is tangent to the curve. If we paramerise the curve by $(x(t), y(t), u(t))$, then the tangent to the curve will be $(\frac{dx}{dt}, \frac{dy}{dt}, \frac{du}{dt}) = (a, b, c)$. Now we can find the chateristic curve as we see that

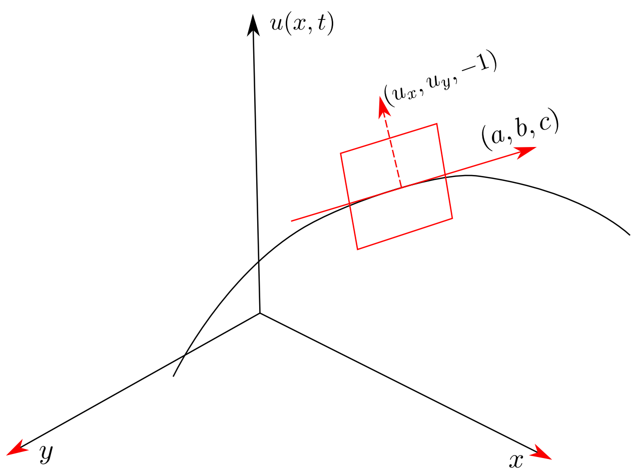


Figure 4: Geometric Interpretations.

$$\begin{cases} \frac{dx}{dt} = a(x, y, u) \\ \frac{dy}{dt} = b(x, y, u) \\ \frac{du}{dt} = b(x, y, u) \end{cases}$$

and we can write them as, $\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)} = dt$. Now we move to another theorem,

Theorem 2.5. The general solution of a first order quasi-linear PDE

$$a(x, y, u)\frac{\partial u}{\partial x} + b(x, y, u)\frac{\partial u}{\partial y} = c(x, y, u)$$

is $f(\phi, \psi) = 0$ where f is an arbitrary function of $\psi, \phi : \mathbb{R}^3 \to \mathbb{R}^3$, $\phi = c_1$ and $\psi = c_2$ are solution curves of the characteristic equations,

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}$$

and $\phi(x, y, u) = c_1$ and $\psi(x, y, u) = c_2$ are the family of characteristic curves.

Proof. Let $\phi(x, y, u) = c_1$ and $\psi(x, y, u) = c_2$. From the first, we can say that,

$$d\phi = \phi_x dx + \phi_y dy + \phi_u du = 0$$

and so,

$$\frac{d\phi}{dt} = \phi_x \frac{dx}{dt} + \phi_y \frac{dy}{dt} + \phi_u \frac{du}{dt} = 0$$

and so we get $a\phi_x + b\phi_y + c\phi_u = 0$ and similarly we can get $a\psi_x + b\psi_y + c\psi_u = 0$. If we now consider the following system of equations,

$$\begin{cases} a\phi_x + b\psi_y = -c\phi_u \\ a\psi_x + b\psi_y = -c\psi_u \end{cases}$$

and multiply the top equation by ψ_u and the bottom by ϕ_u we can conclude that,

$$a(\phi_x\psi_u - \psi_x\phi_u) + b(\psi_y\psi_u - \psi_y\phi_u) = 0$$

and we can write this as a Jacobean,

$$a\frac{\partial(\phi, \psi)}{\partial(x, u)} + b\frac{\partial(\psi, \phi)}{\partial(y, u)} = 0$$

and hence we can now find that,

$$\frac{a}{\frac{\partial(\phi,\psi)}{\partial(u,x)}} = \frac{b}{\frac{\partial(\phi,\psi)}{\partial(y,u)}}$$

Now we can do a very similar thing for the other systems we can form this way and get the desired result:

$$\frac{a}{\frac{\partial(\phi,\psi)}{\partial(y,u)}} = \frac{b}{\frac{\partial(\phi,\psi)}{\partial(u,x)}} = \frac{c}{\frac{\partial(\phi,\psi)}{\partial(x,y)}} \tag{$*$}$$

and so now from Theorem 2.4, and using the above result $(*)$ in the following way, consider $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$,

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial f}{\partial \phi}\left(\frac{\partial \phi}{\partial x} + p\frac{\partial \phi}{\partial u}\right) + \frac{\partial f}{\partial \psi}\left(\frac{\partial \psi}{\partial x} + p\frac{\partial \psi}{\partial u}\right) \\ \frac{\partial f}{\partial y} &= \frac{\partial f}{\partial \phi}\left(\frac{\partial \phi}{\partial y} + q\frac{\partial \phi}{\partial u}\right) + \frac{\partial f}{\partial \psi}\left(\frac{\partial \psi}{\partial y} + q\frac{\partial \psi}{\partial u}\right) \end{aligned}$$

and again write this as a matrix,

$$\begin{bmatrix} \frac{\partial \phi}{\partial x} + p\frac{\partial \phi}{\partial u} & \frac{\partial \psi}{\partial x} + p\frac{\partial \psi}{\partial u} \\ \frac{\partial \phi}{\partial y} + q\frac{\partial \phi}{\partial u} & \frac{\partial \psi}{\partial y} + q\frac{\partial \psi}{\partial u} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial \phi} \\ \frac{\partial f}{\partial \psi} \end{bmatrix} = \mathbf{0}$$

and so we can say, similarly to before that there is a unique solution if the determinant of the matrix is zero. Hence,

$$\left(\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial x}\frac{\partial \phi}{\partial y}\right) + p\left(\frac{\partial \phi}{\partial u}\frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial u}\frac{\partial \phi}{\partial y}\right) + q\left(\frac{\partial \psi}{\partial y}\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \phi}{\partial u}\right) = 0$$

and so we can then rewrite this with Jacobeans,

$$p\frac{\partial(\phi, \psi)}{\partial(y, u)} + q\frac{\partial(\phi, \psi)}{\partial(u, x)} = \frac{\partial(\phi, \psi)}{\partial(x, y)}$$

and so we can now divide through and get,

$$p\frac{\frac{\partial(\phi,\psi)}{\partial(y,u)}}{\frac{\partial(\phi,\psi)}{\partial(x,y)}} + q\frac{\frac{\partial(\phi,\psi)}{\partial(u,x)}}{\frac{\partial(\phi,\psi)}{\partial(x,y)}} = 1$$

and from the result above we can say that,

$$\frac{\frac{\partial(\phi,\psi)}{\partial(y,u)}}{\frac{\partial(\phi,\psi)}{\partial(x,y)}} = \frac{a}{c} \qquad \frac{\frac{\partial(\phi,\psi)}{\partial(u,x)}}{\frac{\partial(\phi,\psi)}{\partial(x,y)}} = \frac{b}{c}$$

and so,

$$p\frac{a}{c} + q\frac{b}{c} = 1$$

which yields,

$$ap + bq = c$$

□

Theorem 2.6 (Cauchy Problem for first order PDEs)**.** Suppose C is a given curve in the (x, y)-plane with it's parametric equation, $x = x_0(t)$ and $y = y_0(t)$ where $t \in I \subseteq \mathbb{R}$ and derivatives $x_0(t)$ and $y_0'(t)$ are piecewise continuous such that they satisfy $x_0'^2 + y_0'^2 \neq 0$. Suppose that $u = u_0(t)$ is a given function on the curve C. Then there exists a solution $u = u(x, y)$ of the equation,

$$F(x, y, u, u_x, u_y) = 0$$

in the domain $D \subseteq \mathbb{R}^2$ containing the curve C for all $t \in I$. $u(x, y)$ satisfies $u(x_0(t), y_0(t)) = u_0(t)$ for all values of $t \in I$.

Now for a lot of examples,

Example. Find the general solution of the PDE, $xu_x + yu_y = u$. We let $a = x$, $b = y$ and $c = u$, hence,

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u}$$

and now we split and solve to get $y = c_1 x$ and $u = c_2 x$. Hence, the solution is $f\left(\frac{y}{x}, \frac{u}{x}\right) = 0$. We could have written this as $\frac{u}{x} = F\left(\frac{y}{x}\right)$ or $u(x, y) = xF\left(\frac{y}{x}\right)$.

Example. Obtain the general solution of the linear equation $xu_x + yu_y = nu$ where n is a constant. Here we do the same thing as above,

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{nu}$$

and we get the solution $u(x, y) = x^n F\left(\frac{y}{x}\right)$

Example. Find the general solution of $x^2 u_x + y^2 u_y = (x + y)u$. Here the characteristic is,

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{du}{(x + y)u}$$

The first function is easy to construct, we find that $\frac{1}{y} - \frac{1}{x} = c_1$ and the second can be found from

$$(x + y)udx = x^2 du \qquad (2)$$

$$(x + y)udy = y^2 du \qquad (3)$$

and then solving. Hence $\frac{x-y}{u} = c_2$. Then we can say the solution is $f(\frac{y-x}{xy}, \frac{x-y}{u}) = 0$ or $u(x, y) = (x-y)h(\frac{y-x}{xy})$

Exercise. Verify the solution.

Example. Obtain the general solution of the linear equation $u_x - u_y = 1$ with the Cauchy data $u(x,0) = x^2$. We find the characteristics,

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{du}{1}$$

and so we find that $y + x = c_1$ and $u - y = c_2$. Therefore, $u(x,y) = -y - F(x+y)$ and using the Cauchy data we can get that $u(x,y) = (x+y)^2 - y$

Review: We have $a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u)$ and we write this as $(a,b,c)\cdot(u_x,u_y,u_z) = 0$ and wrote $u(x,y)$ as the third coordinate and then considered the level surface $f(x,y,u) = u(x,y) - u$ and got that $\nabla f = (u_x, u_y, -1)$ and hence concluded that $(a,b,c)\cdot\nabla f = 0$ recovers our PDE. ∇f is perpendicular to the solution surface, and (a,b,c) is tangent to the surface and some curve in the surface must have tangent vector (a,b,c) which we call the characteristic curve.

$$(a,b,c) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{du}{dt}\right)$$

and this yielded a way to solve the PDE. How do we now parmaterise this solution surface?

2.3 Characteristic Projections

We shall now introduce characteristic projections. Suppose that $u(x,y)$ is specified along some curve Λ in the (x,y)-plane we then have $u = u_0(s)$ when $x = x_0(s)$ and $y = y_0(s)$ where s parameterises Λ in 3D, $(x_0(s), y_0(s), u_0(s))$ is our initial curve.

Characteristics pass through this curve and they are tangent to (a,b,c), so

$$\frac{dx}{d\tau} = a, \quad \frac{dy}{d\tau} = b, \quad \frac{du}{d\tau} = c$$

with initial conditions $x = x_0(s)$, $y = y_0(s)$ and $u = u_0(s)$ at $\tau = 0$. Then we know that the parameterised surface will be $(x(s,\tau), y(s,\tau), u(s,\tau))$and these are the parametric equations of the solution surface.

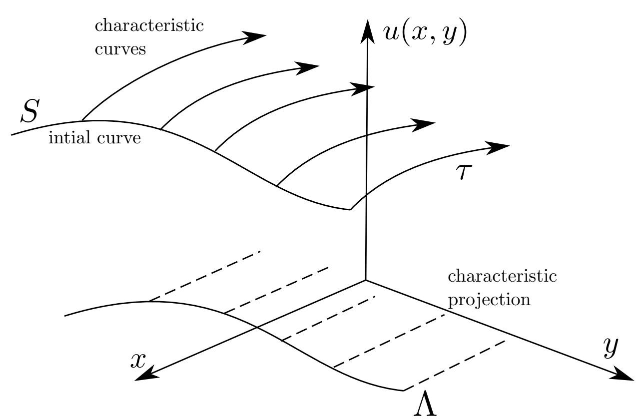


Figure 5: Geometric Interpretations.

Example. Solve $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 1$ subject to the boundary data $u = 0$ when $x + y = 0$. We can solve this by setting up characteristics,

$$\frac{dx}{d\tau} = 1 \quad \frac{dy}{d}\frac{dy}{d\tau} = 1 \quad \frac{du}{d\tau} = 1$$

Then we need initial conditions so we will find solutions depending on the parameter s, so our intial conditions are,

$$x = s \quad y = -s \quad u = 0 \quad \text{at } \tau = 0$$

and then we solve this system and get,

$$x(\tau) = \tau + s \quad y(\tau) = \tau - s \quad u(\tau) = \tau$$

Now we can eliminate τ and get the solution as $x + y = 2u$ or $u = \frac{x+y}{2}$

Example. Solve the PDE, $u_t + uu_x = 1$ for $u = u(x,t)$ in $t > 0$ subject to the initial condition $u = x$ at $t = 0$. We have the characteristics of,

$$\frac{dt}{d\tau} = 1 \quad \frac{dx}{d\tau} = u \quad \frac{du}{d\tau} = 1$$

We can solve the first and third very quickly, more specifically, $t(\tau) = \tau + c_1$ and $u(\tau) = \tau + c_3$, now given the data we can form some initial conditions, $x = s$, $u = s$ for $\tau = 0$. Hence, we have $t = \tau$ and $u = \tau + s$. Now we solve the second equation by plugging in u, $\frac{dx}{d\tau} = \tau + s$ and now we can solve them, $x(\tau) = \frac{1}{2}\tau^2 + s\tau + s$, or $s = \frac{x(\tau) - \frac{1}{2}\tau^2}{\tau + 1}$. Now we plug in the other solutions and find,

$$u(x,t) = \frac{\frac{1}{2}t^2 + t + x}{t+1}$$

We start with another example,

Example. Find the solution of $u(x+y)u_x + u(x-y)u_y = x^2 + y^2$ with the Cauchy data $u = 0$ on $y = 2x$. We start with the characteristics,

$$\frac{dx}{u(x+y)} = \frac{dy}{u(x-y)} = \frac{du}{x^2+y^2}$$

we can verify that $ydy + xdy - udu = 0$. Now we can turn this equation into an exact equation,

$$d\left(xy - \frac{1}{2}u^2\right) = xdy + ydx - udu = 0$$

and so, $xy - \frac{1}{2}u^2 = c_1 = \phi(x,y,u)$. We now need to find a ψ function, consider $xdx - ydy - udu = 0$ is satisfied by our characteristics and so we can form another exact equation,

$$d\left(\frac{1}{2}x^2 - \frac{1}{2}y - \frac{1}{2}u\right) = 0$$

and so, $\psi = x^2 - y^2 - u^2$. Therefore the general solution is,

$$u(x,y) = f\left(2xy - u^2, x^2 - y^2 - u^2\right) = 0$$

Now we consider the Cauchy data, $u(x,2x) = 0 = f(4x^2, -3x^2)$ and so $\frac{1}{2}c_1 = -\frac{1}{3}c_2$, hence, $\frac{1}{4}\left(2xy - u^2\right) = -\frac{1}{3}\left(x^2 - y^2 - u^2\right)$ and so $7u^2 = 6xy + 4(x^2 + y^2)$

Now we revisit the Cauchy Theorem in a slightly different form.

Theorem 2.7 (Cauchy Problem (revisited))**.** Suppose that $x_0(t)$, $y_0(t)$ and $u_0(t)$ are continuously differentiable functions of t in $t \in [0,1]$. Further suppose that $a(x,y,u)$, $b(x,y,u)$ and $c(x,y,u) \in \mathcal{C}^1$ with respect to their arguments t respect to some domain D of the (x,y,u)-space. $\Lambda : x = x_0(t) \quad y = y_0(t) \quad u = u_0(t)$ for $t \in [0,1]$ and $y_0'(t)a(x_0,y_0,u_0) - x_0'(t)b(x_0,y_0,z_0) \neq 0$ then there exists a unique solution $u = u(x,y)$ of the quasi-linear PDE $au_x + bu_y = c$.

This condition assures that the intial curve is not in the same direction as the solution as they should arise from the Cauchy data. Another interpretation of this is there is a, one to one mapping from x, y, z to t, τ, s. We can now state the Cauchy-Kowalevski Theorem,

Theorem 2.8 (Cauchy-Kowaleski)**.** A necessary condition for a unique solution $u(x,y)$ to exist in a neighbourhood Λ is for the first derivative of $u(x,y)$ to be determined on Λ

We say that along the curve if Λ is parameterised by s, then $u_0(x_0(s), y_0(s)) = u_0(s)$. Any point of this curve has a projection onto the (x, y)-plane. We want to find the first order derivatives of u,

$$\frac{du_0}{ds} = \frac{\partial u_0}{\partial x}\frac{dx_0}{ds} + \frac{\partial u_0}{\partial y}\frac{dy_0}{ds}$$

and the PDE says,

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c$$

Now, let us put these in a matrix and find a condition,

$$\begin{pmatrix} a & b \\ \frac{dx_0}{ds} & \frac{\partial y_0}{\partial s} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} c \\ \frac{du_0}{ds} \end{pmatrix}$$

Hence, we need the determinant to be non-zero and so $a\frac{dy_0}{ds} - b\frac{dx_0}{ds} \neq 0$. This is just what we said in the Cauchy Problem Theorem. Here is the formal statement,

Theorem 2.9. The PDE $a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c$ has a unique analytical solution in some neighbourhood of Λ, provided a, b and c are analytic and satisfy $a\frac{dy_0}{ds} - b\frac{dx_0}{ds} \neq 0$

Some more examples,

Example. Solve the following PDE, $\frac{\partial u}{\partial t} + xu\frac{\partial u}{\partial x} = u$ for $u(x, t)$ in $t > 0$ subject to the cauchy data $u = x$ at $t = 0$ for $0 < x < 1$. We can write the characteristics as,

$$\frac{dt}{1} = \frac{dx}{xu} = \frac{du}{u}$$

or,

$$\frac{dt}{d\tau} = 1 \qquad \frac{dx}{d\tau} = xu \qquad \frac{du}{d\tau} = u$$

We can solve the first of the second form of characteristics, with the initial curve of $t = 0, x = s, u = s$ and we can say that $t = \tau$. Now we aim to solve $dt = \frac{du}{u}$ which yields a solution of the form $u(s, t) = f(s, t)e^t$ which when we use our intial curve gives us a specific solution of $u(s, t) = se^t$. Now we can solve $\frac{dx}{d\tau} = xu$ by plugging in our solution for $u(s, t)$ and so we solve, $\frac{dx}{dt} = xse^t$ which we can separate and then solve. This yields a solution of $x(s, t) = se^{s(e^t-1)}$ for $0 < s < 1$. We can now talk about the domain of definition, which will be $0 < x < e^{e^t-1}$.

Now we aim to eliminate s, so find a solution in terms of physical variables, we can see quickly that $s = ue^{-t}$ and so after substituting it in we get that $x = ue^{u-t-ue^{-t}}$

Here is another example,

Example. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u^3$ subject to $u = y$ on $x = 0$ for $0 < y < 3$. We write the characteristics,

$$\frac{dx}{d\tau} = 1 \qquad \frac{dx}{d\tau} = 1 \qquad \frac{du}{d\tau} = u^3$$

We can also write the initial curve of $x = 0, y = s, u = s$. This leads to us being able to write $x(\tau) = \tau$ and $y(\tau) = \tau + s$. Now we aim to solve $\frac{du}{d\tau} = u^3$, we aim to solve it by seperation of variables so let $u(s, \tau) = A(s)B(\tau)$. Now, $\frac{du}{d\tau} = A(s)\frac{dB}{d\tau}$, as we are considering $\frac{du}{d\tau} = u^3$ we say that $u^3 = A(s)\frac{dB}{d\tau}$ and so, $\frac{dB}{B^3} = A^2 d\tau$ and integrating we reach that $B(\tau) = \frac{1}{\sqrt{c-2A^2(s)\tau}}$. Hence, $u(s, \tau) = \frac{A(s)}{\sqrt{c-2A^2(s)\tau}}$. Now consider the intial curve and we get that $A(s) = s\sqrt{c}$ and so,

$$u(s, t) = \frac{s}{\sqrt{1 - 2s^2\tau}}$$

Finally, by considering the fact that $x = \tau$ and $y = \tau + s$, we can say that $s = y - x$ and the the implicit solution is,

$$u = \frac{y - x}{\sqrt{1 - (y - x)^2 x}}$$

An interesting feature of this solution is that it blows up if $1-(y-x)^2 x = 0$, that is along the line $y = x + \frac{1}{\sqrt{2x}}$ the solution does not exist and it's a singularity. The domain of solution is $x < y < x + 3$. Hence we can plot the domain of solution,

2.4 Canonical Form for Linear First Order Equations

Assume that the PDE is linear,

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = d(x,y)$$

We aim to reduce this PDE to an ODE an integrate this ODE. This reduction is guided by the characteristics of this PDE. Assume that the characteristics are, $\xi = \xi(x,y)$, $\eta = \eta(x,y)$. This will be just a coordinate change, and this is a one-to-one mapping between $(x,y) \mapsto (\xi,\eta)$, that is $J = \xi_x\eta_y - \xi_y\eta_x \neq 0$. We want to substitute for these partial differentials.

$$u_x = u_\xi \xi_x + u_\eta \eta_x \qquad u_y = u_\xi \xi_y + u_\eta \eta_y$$

Now we substitute this into our linear PDE,

$$(a\xi_x + b\xi_y)u_\xi + (a\eta_x + b\eta_y)u_\eta + cu = d$$

Let $A = a\xi_x + b\xi_y$ and $B = a\eta_x + b\eta_y$, we get that $Au_\xi + Bu_\eta + cu = d$. If $B = 0$ we have an ODE. If $B = 0$, then $a\eta_x + b\eta_y = 0$. Our characteristics of our original equation is, $\frac{dx}{a} = \frac{dy}{b}$. The level curves of $B = 0$ are always characteristics of the original first order PDE. Hence, $\eta(x,y) = c$ and so $d\eta = 0 = \eta_x dx + \eta_y dy = 0$ or $\eta_x + \eta_y \frac{dy}{dx} = 0$. Now we substitute into the characteristcs, $a\eta + b\eta_y = 0$. This tells us that the general condition is just the characteristics. For the second characteristic we need to choose a one parameter family of curves such that the jacobean is zero. Hence we set $\xi = x$ (or $\xi = y$), then the jacobean is non-zero, or $\xi(x,y) = c$ and choose $\eta = y$ to satisfy $J \neq 0$. As $B = 0$ we can rewrite the PDE as,

$$u_\xi + \frac{c}{A}u = \frac{d}{A}$$

and we call this the cannonical form of this PDE and it is an ODE, hence we can just integrate it and get the required solution.

Example. Reduce, $u_x - u_y = u$. The characteristics are,

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{du}{u}$$

as $dx = -dy$ we let $x + y = \xi$, now set $\eta = y$. Hence we can say that $J \neq 0$. $u_x = u_\xi$ and $u_y = u_\xi + u_\eta$. We substitute this into the PDE, $u_x - u_y = -u_\eta = u$ or $u_\eta + u = 0$ and hence $u(\xi,\eta) = f(\xi)e^{-\eta}$ or $u(x,y) = f(x+y)e^{-y}$

3 Canonical Form of first and second order PDEs

3.1 Coordinate Method for the first order constant coefficient PDEs

Consider $au_x + bu_y = 0$ where a, b are constant. We can represent this as $(ab) \cdot \nabla fu$, if we let $\mathbf{v} = (uv)$ and so $\nabla fu \cdot v = 0$. This is the directional derivative of $u(x, y)$ in the direction of the vector of coefficients (ab), and it's zero. Now let us change the coordinates, such that the coordinates have their axis is parallel to (ab). Then $(\xi, \eta) = [(x, y) \cdot (ab),\ (x, y) \cdot (b - a)]$. Therefore, $\xi = ax + by$ and $\eta = bx - ay$.

Example. Solve $au_x + bu_y + cu = g(x, y)$ using the coordinate method. We let $\xi = ax + by$ and $\eta = bx - ay$. Then we can write out $u_x = u_\xi \xi_x + u_\eta \eta_x = au_\xi + bu_\eta$ and $u_y = u_\xi \xi_y + u_\eta \eta_y = bu_\xi - au_\eta$. Now we substitute these into our equation and get, $(a^2 + b^2)u_\xi + cu = g(\xi, \eta)$. We can now solve this with an integrating factor and get that,

$$u(\xi, \eta) = f(\eta)e^{-\frac{c}{a^2+b^2}\xi} + e^{-\frac{c}{a^2+b^2}} \int e^{-\frac{c}{a^2+b^2}\xi} g(\xi, \eta)\, d\xi$$

Exercise. Find the general solution of $-3u_x + 4u_y + 5u = e^{x+3y}$.

Solution. $u(x, y) = e^{\frac{1}{2}x-y}\left(f(4x + 2y) + \frac{1}{15}e^{\frac{1}{2}x+4y}\right)$

3.2 Classification of second order linear PDEs.

Let us write the general form of a second order PDE,

$$A(x, y)u_{xx} + B(x, y)u_{xy} + C(x, y)u_{yy} + D(x, y)u_x + E(x, y)u_y + F(x, y)u = G(x, y)$$

We want to classify the PDE, then we can find the right method to solve them. The classification is inspired by how we classify a quadratic curve in analytic geometry, we write the following,

$$Ax^2 + Bxy + Cy^2 + Dx + Ex + F = 0$$

$$\begin{cases} B^2 - 4AC > 0 & \text{hyperbola} \\ B^2 - 4AC = 0 & \text{parabola} \\ B^2 - 4AC < 0 & \text{ellipse} \end{cases}$$

A similar thing happens, but the signs of the coefficients may change in the domain of solution. However, we assume that this case will never happen, so it's classification holds for the whole domain of solution. The same definition as for quadratic curve works for PDEs.

This classification is on an invariant that is invariant under a change of coordinates. So we consider some change of variables that is guided by the characteristics. Let $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$. We assume that $(x, y) \mapsto (\xi, \eta)$ is one-to-one, that is $J \neq 0$. Let us input this change of coordinates, $u_x = u_\xi \xi_x + u_\eta \eta_x$ and $u_y = u_\xi \xi_y + u_\eta \eta_y$ and $u_{xx} = u_{\xi\xi}\xi_x^2 + u_{\xi\eta}\xi_x\eta_x + u_\xi \xi_{xx} + u_{\eta\eta}\eta_x^2 + u_{\xi\eta}\xi_x\eta_x + u_\xi \xi_{xx}$ and after a load of laborious maths we get,

$$A^* u_{\xi\xi} + B^* u_{\eta\xi} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^*$$

where $A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$, $B^* = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$, $C^* = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$, $D^* = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y$, $E^* = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y$, $F^* = F$ and $G^* = G$ we see that this PDE has the same form. If we consider the discriminant of the PDE, this is ${B^*}^2 - 4A^*C^*$, and if we substitute the definitions, we can prove that this is just $J^2(B^2 - 4AC)$ and so the sign of the discriminant is invariant under this change of coordinates. Hence, we can write the PDE in the form,

$$Au_{xx} + Bu_{xy} + C_{yy} = H(x, y, u_x, u_y)$$

Example. The wave equation $u_{xx} - u_{tt} = 0$ is hyperbolic.

Example. The Laplace equation $u_{xx} + u_{yy} = 0$ is elliptic.

Example. The diffusion / heat equation $u_t - u_{xx} = 0$ is parabolic.

Example. Classify,

- $u_{xx} - uxy = 0$, this is hyperbolic
- $4u_{xx} + 6u_{xy} + 9u_{yy} = 0$, this is elliptic.

3.3 Cannonical Forms of the semilinear second order PDE.

PDE Problem: We have a system or singular PDEs and we know of different types of boundary conditions; Dirichlet, Neumann and Robin boundary conditions, or it can be of mixed type.

Assume that the general form of our PDE is,

$$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} = H(x,y,u_x,u_y)$$

the cauchy data is to specify $\frac{\partial u}{\partial \mathbf{n}}$ and $u(x,y)$ on some curve Λ in the (x,y) - plane. Let $x = x_0(s)$, $y = y_0(s)$, $u = u_0(x_0(s), y_0(s)) = u_0(s)$ and $\frac{\partial u}{\partial \mathbf{n}} = v_0(s)$. However it's easier to write the last in other terms as $\frac{du_0}{ds} = \frac{\partial u}{\partial x}\frac{dx_0}{ds} + \frac{\partial u}{\partial y}\frac{dy_0}{ds}$ and $v_0(s) = \nabla fu \cdot \mathbf{n} = \frac{\partial u}{\partial x}$. What is $\mathbf{n}$? We know that $(\frac{dx_0}{ds}\frac{dy_0}{ds})$ is tangent to the curve Λ, hence we let $\mathbf{n} = (\frac{dy_0}{ds}, -\frac{dx_0}{ds})$. Hence we can write that $v_0(s) = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) \cdot (\frac{dy_0}{ds}, -\frac{dx_0}{ds}) = \frac{\partial u}{\partial x}\frac{dy_0}{ds} - \frac{\partial u}{\partial y}\frac{dx_0}{ds}$. Therefore we can write the cauchy data as, $x = x_0(s)$, $y = y_0(s)$, $u = u_0(s)$, $\frac{\partial u}{\partial x} = p_0(s)$ and $\frac{\partial u}{\partial y} = q_0(s)$.

We now want a condition for a unique solution in neighbourhood of the initial curve, and then to find the characteristics of the equation. Consider $\frac{dp}{ds} = \frac{\partial^2 u}{\partial x^2}\frac{dx_0}{ds} + \frac{\partial u}{\partial x \partial y}\frac{dy_0}{ds}$ and $\frac{dq}{ds} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x \partial y}\frac{dy}{ds}$. We now have our second order PDE, $Au_{xx} + Bu_{xy} + Cu_{yy} = H$. Now let us write these in matrix form,

$$\begin{pmatrix} A & B & C \\ \frac{dx_0}{ds} & \frac{dy_0}{ds} & 0 \\ 0 & \frac{dx_0}{ds} & \frac{dy_0}{ds} \end{pmatrix} \begin{pmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{pmatrix} = \begin{pmatrix} H \\ \frac{dp}{ds} \\ \frac{dq}{ds} \end{pmatrix}$$

Here we want $\hat{A} \neq 0$. Now we want the determinant of the coefficiet matrix,

$$|\hat{A}| = A\left(\frac{dy_0}{ds}\right)^2 + B\frac{dx_0}{ds}\frac{dy_0}{ds} + C\left(\frac{dx_0}{ds}\right)^2$$

if the characteristics are parameterised by τ then from $*$, we can say that

$$A\left(\frac{dy}{d\tau}\right) - B\frac{dx}{d\tau}\frac{dy}{d\tau} + C\left(\frac{dx}{d\tau}\right)^2 = 0$$

Hence, after some manipulation,

$$A\left(\frac{dy}{dx}\right)^2 - B\frac{dy}{dx} + C = 0$$

or $A\lambda^2 - B\lambda + C = 0$ and this is the characteristic equation of the second order semi-linear PDE.

We have already shown that a second order PDE is invariant under change of coordinates and achieved,

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} = H(\xi, \eta, y, u_\xi, u_\eta)$$

We want to choose characteristics such that $A^* = C^* = 0$, however we know that $A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$ and that $C^* = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$. Now we can write them in the form,

$$A\gamma_x^2 + \gamma_x\gamma_y + c\gamma_y^2 = 0$$

and so we get,

$$A\left(\frac{\gamma_x}{\gamma_y}\right)^2 + B\left(\frac{\gamma_x}{\gamma_y}\right) + C = 0$$

Now assume that one of them is a constant. Hence we differentiate, $d\gamma = \gamma_x dx + \gamma_y dy = 0$ and so $\frac{dy}{dx} = -\frac{\gamma_x}{\gamma_y}$ and so, $A(\frac{dy}{dx}) - B\frac{dy}{dx} + C = 0$ and we recover the characteristic equation. Hence we now want to find roots of this equation and get,

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

and so we have two equations for the characteristics and so we integrate to find the equations for characteristic curves. Here $\xi = \phi(x, y)$ and $\eta = \psi(x, y)$.

Let us consider three different cases,

Hyperbolic: $A^*u_{\xi\xi} + B^*u_{\xi\eta} + C^*u_{\eta\eta} = H^*$ and so we have $B^2 - 4AC > 0$ and so we get,

$$B^*u_{\xi\eta} = H^*$$

or,

$$\frac{\partial^2 u}{\partial\xi\partial\eta} = \frac{H^*}{B^*} = H$$

This is the first canonical form for hyperbolic PDEs. We can find the second order, let $\alpha = \xi+\eta$ and $\beta = \xi-\eta$. Therefore $u_\xi = u_\alpha\alpha_\xi + u_\beta\beta_\xi = u_\alpha + u_\beta$ and $u_{\xi\eta} = u_{\alpha\alpha} - u_{\alpha\beta} + u_{\beta\alpha} - u_{\beta\beta} = u_{\alpha\alpha} - u_{\beta\beta} = H_2(\alpha, u_\alpha, u_\beta)$. This is the second canonical form of hyperbolic PDEs.

Parabolic Equations: Here we have $B^2 - 4AC = 0$ and so we have a repeated root. Therefore, $\xi = c_1$ or $\eta = c_2$. Therefore, if $A^* = A\xi_x^2 + B\xi_x\xi_y + c\xi_y^2 = (\sqrt{A}\xi_x + \sqrt{c}\xi_y)^2 = 0$. Now consider $B^* = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y = 2(\sqrt{A}\xi_x + \sqrt{C}\xi_y)(\sqrt{A}\eta_x + \sqrt{C}\eta_y) = 0$ as $(\sqrt{A}\xi_x + \sqrt{C}\xi_y) = 0$. Hence, $A^* = B^* = 0$ and so we get the PDE of the form,

$$u_{\eta\eta} = \frac{H^*}{C^*} = H_3(\xi, \eta, u_\xi, u_\eta)$$

This is the canonical form for parabolic PDEs. If you set $C^* = 0$, you just get $u_{\xi\xi} = H_3(\xi, \eta, u_\xi, u_\eta)$. Similarly to before, if you solve $\frac{dy}{dx} = \frac{B}{2A}$ and call ξ the solution, then let $\eta = y$ you get a one-to-one mapping. Similarly, letting $\xi = x$ gives the same outcome.

Ellipic PDEs: The characteristic equation is,

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0$$

We know that $B^2 - 4AC < 0$, let the roots be $\hat{\alpha}(x, y) \pm i\hat{\beta}(x, y)$, now we integrate this and get $\xi = \alpha(x, y) + i\beta(x, y)$ and $\eta = \alpha(x, y) - i\beta(x, y)$. Now we let $\alpha = \frac{1}{2}(\xi + \eta)$ and $\beta = \frac{1}{2i}(\xi - \eta)$. Now we reduce using this change of transformation. The reduced PDE is,

$$A^{**}u_{\alpha\alpha} + B^{**}u_{\alpha\beta} + C^{**}u_{\beta\beta} = H^{**}(\alpha, \beta, u, u_\alpha, u_\beta)$$

What happens when we set $A^* = C^* = 0$. We substitute in for the derivatives into the equation for A^*,

$$\begin{aligned} A^* &= A(\alpha_x + i\beta_x) + B(\alpha_x + i\beta_x)(\alpha_y + i\beta_y) + C(\alpha_y + i\beta_y)^2 \\ &= A(\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2) - (A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2) + i\left[2A\alpha_x\beta_x + B[\alpha_x\beta_y + \beta_x\alpha_y] + 2C\alpha_y\beta_y\right] \end{aligned}$$

and the same thing for C^* and get that, $C^* = A(\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2) - (A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2) - i\left[2A\alpha_x\beta_x + B[\alpha_x\beta_y + \beta_x\alpha_y] + 2C\alpha_y\beta_y\right]$ and finally

$$\begin{aligned} B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\ &= 2A\alpha_x\beta_x + B[\alpha_x\beta_y + \beta_x\alpha_y] + 2C\alpha_y\beta_y \end{aligned}$$

Hence, $A^{**} - C^{**} + iB^{**} = 0$ and also $A^{**} - C^{**} - iB^{**} = 0$. Therefore, $A^{**} = C^{**}$ and $B^{**} = 0$. Hence, we can write,

$$u_{\alpha\alpha} + u_{\beta\beta} = \frac{H^{**}}{A^{**}} = H_4(\alpha, \beta, u, u_\alpha, u_\beta)$$

This is the canonical form of Elliptic PDEs.

3.4 Constant Coefficient Second Order PDEs

Last week we solved $A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} = H(x,y,u_x,u_y,u)$ and found cannonical form. Now we assume that we have a PDE of the form,

$$Au_{xx} + Bu_{xy} + Cu_{yy} = H(x,y,u_x,u_y,u)$$

where the coeffients are constant. We then get the characteristics of the form,

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

and so, $y = \lambda_{1,2}x + c_{1,2}$ hence we our characteristics care,

$$\xi = y - \lambda_1 x \qquad \eta = y - \lambda_2 x$$

If $H = 0$, then we call the PDE the Euler Equation. If $B^2 - 4AC$, then we have two roots, $\xi = y - \lambda_1 x$ and $\eta = y - \lambda_2 x$ and so $u_x = -\lambda_1 u_\xi - \lambda_2 u_\eta$ and $u_{xx} = \lambda_1^2 u_{\xi\xi} + \lambda_2^2 u_{\eta\eta} + 2\lambda_1\lambda_2 u_{\xi\eta}$ and $u_{xy} = -\lambda_1 u_{\xi\xi} - (\lambda_1 + \lambda_2)u_{\xi\eta} - \lambda_{2u_{\eta\eta}}$ and $u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$ and now we can substitute and it reduces to

$$(A\lambda_1^2 - \lambda_1 B + C)u_{\xi\xi} + (2\lambda_1\lambda_2 A - (\lambda_1 + \lambda_2)B + 2C)u_{\xi\eta} + (A\lambda_2^2 - \lambda_2 B + C)u_{\eta\eta} = H^*(\xi, \eta, u, u_\xi, u_\eta)$$

The coeffients are just the charactersitic equations and so,

$$u_{\xi\eta} = H_1^*$$

which is our expected canonical form. If $A = 0$, then $\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$, but our equation is $Bu_{xy} + B_{yy} = H(x,y,u,u_x,u_y)$ and then we get the geneeral transformation $(B\xi_x\xi_y + C\xi_\eta^2)u_{\xi\xi} + (B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y) + (B\eta_x\eta_y + C\eta_y^2)u_{\eta\eta} = H^*$ which is a changed version of what we expected from before, just with $A = 0$. Again let γ be ξ or η and then,

$$B\gamma_x\gamma_y + C\gamma_y^2 = 0$$

and this equation can now be written as, $B\frac{\gamma_x}{\gamma_y} + C = 0$ and look at it along the curves $\gamma = 0$. Then $d\gamma = \gamma_x dx + \gamma_y dy = 0$. Therefore, $\frac{dy}{dx} = -\frac{\gamma_x}{\gamma_x}$, and so substituting for this into $B\frac{\gamma_x}{\gamma_y} + C = 0$ gives us the usual characteristic equation.

Now consider $-B\frac{dy}{dx} + C = 0$, then we can say that $-Bx\frac{dx}{dy} + C\left(\frac{dx}{dy}\right)^2 = 0$ and so $\frac{dx}{dy}\left(-B + C\frac{dx}{dy}\right) = 0$ so either $\frac{dx}{dy} = 0$ or $-B + C\frac{dx}{dy} = 0$. Hence $\xi = x$ or $\eta = x - \frac{B}{C}y$. We can then reduce this to $u_{\xi\eta} = H^*(\xi, \eta, u, u_\xi, u_n)$. For the Euler Equation (with $A \neq 0$), $u_{\xi\eta} = 0$. We can simply integrate this equation,

$$u(\xi, \eta) = \int f(\xi)\, d\xi + G(\eta) = F(\xi) + G(\eta)$$

where F, G are arbitrary functions. Therefore in physical variables,

$$u(x,y) = F(y - \lambda_1 x) + G(y - \lambda_2 x) \qquad \textbf{where } A \neq 0$$

For **Parabolic Equations** we get a similar final form, which is $u_{\eta\eta} = H_1^*(\xi, \eta, u, u_\xi, u_\eta)$. For the Euler Equation, we just end up with $u_{\eta\eta} = 0$ and so we can just integrate these two equations to get the general solutions,

$$u(\xi, \eta) = \eta f(\xi) + g(\xi)$$

or in terms of physical variables,

$$u(x, y) = (y - \lambda_2 x) f(y - \lambda_1 x) + g(y - \lambda_1 x)$$

If $A = 0$, then,

$$u(x, y) = f(y - \lambda x) + y g(y - \lambda x)$$

For **Elliptical Equations** we again get a nice canonical form.

$$u_{\alpha\alpha} + u_{\beta\beta} = H^*(\alpha, \beta, u, u_\alpha, u_\beta)$$

Example. The Wave Equation. Solve ${}^{2}_{0}\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x, y)$ where a_0 is a constant.

Example. $u_{tt} - c^2 u_{xx} = 0$,

Example. Find the general solution of $x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 0$

4 THE WAVE EQUATION

4.1 The Cauchy Problem for the wave equation

$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial u}{\partial x} = 0$ with $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ for $x \in R$, $t > 0$ and c being a constant. Let's find the solution, $A = -c^2$ and $B = 0$, then $C = -1$ and so we get $B^2 - 4AC = 4c^2 > 0$ which means it's hyperbolic. We then get characteristics of $1 - c^2 \left(\frac{dt}{dx}\right) = 0$ and so $(dx - cdt)(dx + cdt) = 0$ and so $\xi = x + ct$ and $\eta = x - ct$. This then reduces the PDE to $u_{\xi\eta} = 0$ and so $u(\xi, \eta) = \Phi(\xi) + \psi(\eta)$. Hence we get the general solution,

$$u(x,t) = \Phi(x + ct) + \psi(x - ct)$$

where $\phi, \psi \in \mathcal{C}^2$. The initial conditions can now be implemented, $u(x,0) = f(x) = \Phi(x) + \psi(x)$ and, $u_t(x,t) = c\Phi'(x + ct) - c\psi'(x - xt)$ and so $g(x) = \Phi'(x) - c\psi'(x)$. Therefore, $\Phi'(x) - \psi'(x) = \frac{1}{c}g(x)$ and so $\Phi(x) - \psi(x) = \frac{1}{c}\int_{x_0}^{x} g(\tau)\, d\tau + K$. We can now solve these two equations and get,

$$\Phi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(\tau)\, d\tau + \frac{K}{2}$$

$$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(\tau)\, d\tau - \frac{K}{2}$$

Now we can rewrite $u(x,t)$ as,

$$u(x,t) = \frac{1}{2}\left(f(x + ct) + f(x - ct)\right) + \frac{1}{2c}\int_{x_0}^{x+ct} g(\tau)\, d\tau - \frac{1}{2c}\int_{x_0}^{x-ct} g(\tau)\, d\tau$$

We then finally simplify this to the D'Alembent solution method,

$$u(x,t) = \frac{1}{2}\left(f(x + ct) + f(x - ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\, d\tau$$

Example. Find the solution of the initial value problem $u_{tt} - c^2 u_{xx} = 0$ where $x \in \mathbb{R}$ and $t > 0$ where $u(x,0) = \sin x$ and $u_t(x,0) = \cos x$

4.2 Nonhomogenous wave equation

Consider the Cauchy Problem $u_{tt} - c^2 u_{xx} = h^*(x,t)$ where $u(x,0) = f(x)$ and $u_t(x,0) = g^*(x)$. We are going to use Greens Theorem and so let us change coordinates, $y = ct$, so $\frac{\partial}{\partial t} = \frac{\partial}{\partial y}c$ and $\frac{\partial^2}{\partial t^2} = c^2\frac{\partial}{\partial y}$. Now we substitute in $u_{yy} - u_{xx} = \frac{1}{c^2}h^*(x,y) = h(x,y)$. Now we want to rewrite the Cauchy Data, $u_t(x,0) = cu_y(x,0) = g^*(x)$ or $u_y(x,0) = \frac{1}{c}g^*(x) = g(x)$. Hence our transformed equations are,

$$u_{xx} - u_{yy} = h(x,y)$$

with data $u(x,0) = f(x)$ and $u_y(x,0) = g(x)$. Now we seek the characteristics, which are just going to be $x \pm y = c_{1,2}$ and so $\xi = x + y$ and $\eta = x - y$. Let us integrate over the domain of dependance,

$$\iint_R (u_{xx} - u_{yy})\, dR = \iint_R h(x,y)\, dR$$

We can apply Greens Theorem, we know from Vector Calculus a bounded area the integrals of two functions can be related,

$$\oint_{\partial R} P(x,y) + Q(x,y)\, dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy$$

Now we can apply it,

$$\iint_R u_{xx} + u_{yy} dR = \oint_{\partial R} u_y dx + u_x dy$$

Now we can consider these along each side of R. Remember that we need to move counterclockwise around the surface, as stokes theorem needs the normal vector to be outward pointing. Therefore we get,

$$\begin{aligned} &= \int_{B_0} u_y dx + u_x dy + \int_{B_1} u_y dx + u_x dy + \int_{B_2} u_y dx + u_x dy \\ &= \int_{B_0} u_y dx + \int_{B_1} u_y dx + u_x dy + \int_{B_2} u_y dx + u_x dy && \text{as } x = 0 \text{ along } x \text{ axis.} \end{aligned}$$

Now we consider B_1 and B_2 which depend on the characteristics. On B_1 we know $x + y = c$ and so $dx = -dy$ and on B_2 we similarly know $dx = dy$. Hence we get,

$$\int_{B_0} u_y dx + \int_{B_1} -u_y dx - u_x dy + \int_{B_2} u_y dy + u_x dx$$

and hence we get that,

$$\begin{aligned} &= \int_{B_0} u_y dx + \int_{B_1} -du + \int_{B_2} du \\ &= \int_{x_0-y_0}^{x_0+y_0} u_y dx - u|_{p_2}^{p_0} + u|_{p_0}^{p_1} \\ &= \int_{x_0-y_0}^{x_0+y_0} u_y dx - u(x_0, y_0) + u(x_0 + y_0, 0) + u(x_0 - y_0, 0) - u(x_0, y_0) \\ &= \int_{x_0-y_0}^{x_0+y_0} u_y dx - 2u(x_0, y_0) + u(x_0 + y_0, 0) + u(x_0 - y_0, 0) \end{aligned}$$

Let us put this together, then

$$\int_{x_0-y_0}^{x_0+y_0} u_y dx - 2u(x_0, y_0) + u(x_0 + y_0, 0) + u(x_0 - y_0, 0) = \iint_R h(x, y) dR$$

and here from we can conclude that,

$$u(x_0, y_0) = \frac{1}{2}\left[u(x_0 - y_0, 0) + u(x_0 + y_0, 0)\right] + \frac{1}{2}\int_{x_0-y_0}^{x_0+y_0} u_y dx - \frac{1}{2}\iint_R h(x, y) dR$$

and further with the initial conditions,

$$u(x, y) = \frac{1}{2}\left(f(x + y) + f(x - y)\right) + \frac{1}{2}\int_{x-y}^{x+y} g(\tau)\, d\tau - \frac{1}{2}\iint h(x, y)\, dR$$

4.3 Solution of the Goursat Problem

Consider $u_{xy} = a_1(x, y)u_x + a_2(x, y)u_y + a_3(x, y)u + h(x, y)$. This PDE is prescribed along two lines $u(x, y) = f(x)$ along $y = 0$ and $u(x, y) = g(x)$ along $y = y(x)$ in the first quadrant. We will use the method of successive approximations.

Example. We consider $u_{tt} - c^2 u_{xx} = 0$ where $u(x,t) = f(x)$ on $x - ct = 0$ and $u(x,t) = g(x)$ on some $t = t(x)$ and $f(0) = g(0)$. We already know $u(x,t) = \Phi(x,t) + \psi(x,t)$. If we consider $x = ct$, then $f(x) = u(x,t) = \Phi(2x) + \psi(0)$, but on some $t = t(x)$ we get, $g(x) = \Phi(x + ct(x)) - \psi(x - ct(x))$. From these two we can verify that $f(0) = \Phi(0) + \psi(0) = g(0)$. To find the solution define s and let $s = x - ct(x)$, this means $x = \alpha(s)$. From $g(x)$ we can see that $g(\alpha(s)) = \Phi(x + ct(x)) + \psi(s)$ and from $f(x)$ we get $f(\alpha(s))$.

If we let $t = t(x) = kx$ for some $k > 0$. Then $s = x - ckx = (1 - ck)x$. Therefore, $x = \frac{s}{1-ck} = \alpha(s)$ and so $\alpha(x - ct) = \frac{x-ct}{1-ct}$ and so $t(\alpha(x - ct)) = \frac{k(x-ct)}{1-ck}$. Therefore, we can say that the solution is,

$$u(x,y) = f\left(\frac{x+ct}{2}\right) - f\left(\frac{(x-ct)(x+ct)}{x}\right)$$

Now consider $u(x,t)/u(x,y)$ is prescribed along both characteristics the linear hyperbolic problem is called a characteristic initial value problem.

Example. $u_{xy} = h(x,y)$ where $u(x,0) = f(x)$ and $u(0,y) = g(y)$. We say that f and g are continuously differentiable and $f(0) = g(0)$. We integrate this function,

$$u(x,y) = \int_0^x \int_0^y h(\xi,\eta) d\xi d\eta + \Phi(x) + \Psi(y)$$

We now need to find ϕ and Ψ in terms of the given functions. We substitute in the first initial condition,

$$u(x,0) = \Phi(x) + \Psi(0) = f(x)$$
$$u(0,y) = \Phi(0) + \Psi(y) = g(y)$$

Therefore, $\Phi(x) + \Psi(y) = f(x) + g(y) - \Phi(0) - \Psi(0)$. From the last condition, we know $f(0) = g(0)$ and so $\Phi(0) + \Psi(0) = f(0)$, which then gives us $\Phi(x) + \Psi(y) = f(x) + g(y) - f(0)$. Therefore,

$$u(x,y) = \int_0^x \int_0^y h(\xi,\eta) d\xi d\eta + f(x) + g(y) - f(0)$$

Example. Determine the solution of characteristic initial-value problem, $u_{tt} - c^2 u_{xx} = 0$ where $u(x,t) = f(x)$ on $x + ct = 0$ and $u(x,t) = g(x)$ on $x - ct = 0$ where $f(0) = g(0)$. We start with the solution of the wave equation,

$$u(x,t) = \Phi(x + ct) + \Psi(x - ct)$$

The first characteristic tells us $x = -ct$ and so $u(x,t) = \Phi(0) + \Psi(x - ct) = \Phi(0) + \Psi(2x) = f(x)$. Along the second characteristic $x = ct$ and so $u(x,t) = \Phi(2x) + \Psi(0) = g(x)$. From these we can see that $f(0) = g(0)$ is satisfied. We can now make a substitution $x \mapsto \frac{x-ct}{2}$ and see that $\Phi(0) + \Psi(x - ct) = f\left(\frac{x-ct}{2}\right)$ and make another substitution of $x \mapsto \frac{x+ct}{2}$ to get $\Phi(x + ct) + \Psi(0) = g\left(\frac{x+ct}{2}\right)$. Then we can then plug in to get $\Phi(x + ct) + \Psi(x - ct) = f\left(\frac{x-ct}{2}\right) + g\left(\frac{x+ct}{2}\right) - f(0)$. Therfore,

$$u(x,t) = f\left(\frac{x-ct}{2}\right) + g\left(\frac{x+ct}{2}\right) - f(0)$$

4.4 Semi-infinite string with fixed end point

We will consider the wave equation on the positive real line. We have $u_{tt} - c^2 u_{xx} = 0$ for $0 < x < \infty$ where $t > 0$ where $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ and $u(0,t) = 0$ for $t \geq 0$ (Dirichlet Boundary Condition).

Under $x = ct$ then we don't have a problem, it can be solved nicely. If we are above one of the characteristics hits the y-axis and then we have a problem, therefore we want to constrain this solution some how. We have the solution, $u(x,t) = \Phi(x + ct) + \Psi(x - ct)$ and the solution on the real line is,

$$\Phi(\xi) = \frac{1}{2} f(\xi) + \frac{1}{2c} \int_0^\xi g(\tau) d\tau + \frac{K}{2}$$
$$\Psi(\eta) = \frac{1}{2} f(\eta) - \frac{1}{2c} \int_0^\eta g(\tau) d\tau - \frac{K}{2}$$

Now we use the initial condition $u(0,t)=0$ and this gives $\Phi(ct)+\Psi(-ct)=0$ and so $\Psi(-ct)=-\Phi(ct)$ and so $\Psi(\alpha)=-\Phi(-a)$ where $\alpha=-ct$. Now we can say that $\Psi(x-ct)=-\Phi(-(x-ct))=-\Phi(ct-x)$. Then from the first equation above, $\Psi(x-ct)=\frac{1}{2}f(ct-x)-\frac{1}{2c}\int_0^{ct-x}g(\tau)d\tau-\frac{K}{2}$. From here we conclude that for $x<ct$,

$$u(x,t)=\Phi(x+ct)-\Phi(ct-x)=\frac{1}{2}\left(g(x+ct)-f(ct-x)\right)+\frac{1}{2c}\int_{ct-x}^{x+ct}g(\tau)d\tau$$

and for $x>ct$ we get the usual form of solution,

$$u(x,t)=\frac{1}{2}\left(f(x+ct)+f(x-ct)\right)+\frac{1}{2c}\int_{x-ct}^{x+ct}g(\tau)d\tau$$

Example. Determine the solution of the initial boundary value problem $u_{tt}-4u_{xx}=0$ for $x>0$ and $t>0$ with $u(x,0)=|\sin x|$ for $x>0$, $u_t(x,0)=0$ for $x\geq 0$ and the Dirichlet Boundary condition, $u(0,t)=0$.

Example. Determine the solution of the initial boundary value problem $u_{tt}-4u_{xx}=0$ for $x>0$ and $t>0$ with $u(x,0)=1$ for $x>0$, $u_t(x,0)=0$ for $x\geq 0$ and the Dirichlet Boundary condition, $u(0,t)=0$. This has solution of,

$$u(x,t)=\begin{cases}1 & x>2t\\ 0 & x<2t\end{cases}$$

4.5 Semi-infinite string with a free end point

The problem is set us as follows, we consider the PDE $u_{tt}-c^2u_{xx}=0$ on $0<x<\infty$ and $t>0$. We have the initial conditions $u(x,0)=f(x)$ for $0\leq x<\infty$, $u_t(0,t)=0$ for $0\leq x<\infty$ and the Neumann Boundary condition $u_x(0,t)=0$ for $0\leq t<\infty$.

We begin with the D'Alembert solution,

$$u(x,t)=\Phi(x+ct)+\Psi(x-ct)$$

and the usual definitions for Φ and Ψ. We consider the Neumann Boundary condition,

$$u_x(x,t)=\Phi'(x+ct)+\Psi'(x-ct)$$

and so,

$$u_x(0,t)=\Phi'(ct)+\Psi'(-ct)=0$$

Then we can integrate and get that $\Phi(ct)-\Psi(-ct)=K$. Now let $\alpha=-ct$ and so we get that $\Psi(\alpha)=\Phi(-\alpha)-K$ and so $\Psi(x-ct)=\Phi(ct-x)-K$. Therefore, we can now substitite this into the relation we have for Ψ,

$$\Psi(x-ct)=\frac{1}{2}f(ct-x)+\frac{1}{2c}\int_0^{ct-x}g(\tau)d\tau-\frac{K}{2}$$

Therefore placing this into the D'Alembert solution we get,

$$u(x,t)=\frac{1}{2}(f(x+ct)+f(ct-x))+\frac{1}{2c}\int_0^{x+ct}g(\tau)d\tau+\frac{1}{2c}\int_0^{ct-x}g(\tau)d\tau \qquad \text{for } x<ct$$

and the usual solution for $x>ct$. Furthermore we should have $f'(0)=g'(0)$ and $f\in\mathcal{C}^1$ and $g\in\mathcal{C}^m$ where $m>1$.

NB! In exams, we can start with D'Alemberts in exams.

Example. Find the solution of the initial value boundary problem for $u_{tt}-u_{xx}=0$ for $0<x<\infty$ and $t>0$, $u(x,0)=\cos\frac{\pi x}{2}$ for $0\leq x<\infty$, $u_t(x,0)=0$ for $0\leq x<\infty$ and $u_x(0,t)=0$ for $t\geq 0$.

4.6 Equations with nonhomogenous boundary conditions.

The problem is set us as follows, we consider the PDE $u_{tt} - c^2 u_{xx} = 0$ on $0 < x < \infty$ and $t > 0$. We have the initial conditions $u(x,0) = f(x)$ for $0 \leq x < \infty$, $u_t(0,t) = g(x)$ for $0 \leq x < \infty$ and the Boundary condition $u_t(0,t) = p(t)$ for $0 \leq t < \infty$.

We start with the D'Alembert solution, $u(x,t) = \Phi(x+ct) + \Psi(x-ct)$ with the initial conditions giving $u(x,0) = \Phi(x) + \Psi(x) = f(x)$ and $u_t(x,0) = c(\Phi'(x) + \Psi'(x)) = g(x)$. Now we consider the Boundary condition and $u(0,t) = \Phi(ct) + \Psi(-ct) = p(t)$. then letting $\alpha = -ct$ we get that $\Psi(\alpha) = -\Phi(-\alpha) + p\left(\frac{-\alpha}{c}\right)$ and now we let $\alpha = x - ct$ and as usual we arrive at $\Psi(x-ct) = \Phi(ct-x) + p\left(t - \frac{x}{c}\right)$. Therefore, for $0 \leq x < ct$ we have the solution,

$$u(x,t) = \frac{1}{2}(f(x+ct) - f(ct-x)) - \int_{ct-x}^{x+ct} \frac{1}{2c} g(\tau)d\tau + p\left(t - \frac{x}{c}\right)$$

where $p(0) = f(0)$, $p'(0) = g(0)$ and $p''(0) = c^2 f''(0)$.

Consider the where we have a Neumann Boundary Conditions, we consider the PDE $u_{tt} - c^2 u_{xx} = 0$ on $0 < x < \infty$ and $t > 0$. We have the initial conditions $u(x,0) = f(x)$ for $0 \leq x < \infty$, $u_t(0,t) = g(x)$ for $0 \leq x < \infty$ and the Neumann Boundary condition $u_x(0,t) = q(t)$ for $0 \leq t < \infty$.

Exercise. Verify that we reach the solution of,

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(ct-x)) + \frac{1}{2c}\int_0^{x+ct} g(\tau)d\tau + \frac{1}{2c} int_0^{ct-x} g(\tau)d\tau - c\int_0^{t-\frac{x}{c}} q(\tau)d\tau$$

for $x < ct$ where $f'(0) = q(0)$, $g'(0) = q'(0)$.

4.7 Vibration of finite string with fixed ends

Thus far we have considered the half string, now consider a finite string where both ends are fixed. Now let us consider both ends as fixed. The problem is set us as follows, we consider the PDE $u_{tt} - c^2 u_{xx} = 0$ on $0 < x < \ell$ and $t > 0$. We have the initial conditions $u(x,0) = f(x)$ for $0 \leq x < \ell$, $u_t(0,t) = g(x)$ for $0 \leq x < \ell$ for two boundary conditions, $u(0,t) = u(\ell,t) = 0$.

Using D'Alemberts solution we have $u(x,t) = \Phi(x+ct) + \Psi(x-ct)$ and the initial conditions give us that $u(x,0) = \Phi(x) + \Psi(x) = f(x)$ and $u_t(x,t) = c(\Phi'(x) - \Psi'(x))$ for $0 \leq x < \ell$. So we now integrate and get,

$$\Phi(\xi) = \frac{1}{2}f(\xi) + \frac{1}{2c}\int_0^{\xi} g(\tau)d\tau + \frac{K}{2} \qquad 0 \leq \xi \leq \ell$$

$$\Psi(\eta) = \frac{1}{2}f(\eta) + \frac{1}{2c}\int_0^{\eta} g(\tau)d\tau - \frac{K}{2} \qquad 0 \leq \eta \leq \ell$$

From here, we can conclude that $u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau$ where $0 \leq x + ct \leq \ell$ and $0 \leq x - ct \leq \ell$. This means the solution is unique determined by the initial data in $t \leq \frac{x}{c}$ and $t \leq \frac{\ell - x}{c}$.

We can now take the boundary condition and substitute this into the D'Alembert solution. We get

$$u(0,t) = \Phi(ct) + \Psi(-ct) = 0 \tag{4}$$

and

$$u(\ell,t) = \Phi(\ell+ct) + \Psi(\ell-ct) = 0 \tag{5}$$

for $t \geq 0$. Then from (4) we get that $\Psi(\alpha) = -\Phi(-\alpha)$ for $\alpha = -ct$. From (5) we get that $\Phi(\alpha) = -\Psi(2\ell - \alpha)$ where $\alpha = \ell + ct$. We can now use these two and the general solution to formulate the whole solution. Let us now replace $\xi = -\eta$ and we can get that,

$$\Phi(-\eta) = \frac{1}{2}f(-\eta) + \frac{1}{2c}\int_0^{-\eta} g(\tau)d\tau + K \qquad \text{for } 0 \leq -\eta \leq \ell$$

Therefore,

$$\Psi(\eta) - \Phi(-\eta) = -\frac{1}{2}f(-\eta) - \frac{1}{2c}\int_0^{-\eta} g(\tau)d\tau - K \qquad \text{for } -\ell \leq \eta \leq 0$$

Now use (2) and then let $\alpha = \xi$ and get a solution for up to $2\ell - \alpha$. Therefore,

$$\Phi(\xi) = -\psi(2\ell - \xi) = -\frac{1}{2}f(2\ell - \xi) + \frac{1}{2c}\int_0^{2\ell-\xi} g(\tau)d\tau - \frac{K}{2} \qquad \ell \leq \xi \leq 2\ell$$

Furthermore we can now extend the solution further to all the values of ℓ. The better way to solve this is through a series solution.

5 The Diffusion Equation

The diffusion (or heat) equation is parabolic equation denoted by $\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial t^2} = 0$ for $k > 0$.

Theorem 5.1 (Maximum Principle)**.** If $u(x,t)$ satisfies the diffusion equation in a rectangle for $0 \leq x \leq \ell$ and $0 \leq t < T$ in space-time, then the maximum value of $u(x,t)$ is assumed either initially at $t = 0$ or on lateral sides $x = 0, L$

Theorem 5.2 (Uniqueness)**.** Consider $u_t - ku_{xx} = f(x,t)$ for $0 \leq x \leq L$ and $t > 0$ subject to $u(x,0) = \Phi(x)$ as an initial condition and the boundary condition $u(0,t) = g(x)$ and $u(L,t) = h(t)$. Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions of this PDE problem. These are the same solution.

Proof. Then $w(x,t) = u_2(x,t) - u_1(x,t)$, satisfies $w_t - w_{xx} = 0$ and $w(x,0) = 0$ and $w(0,t) = w(L,t) = 0$. We know via the maximum principle, then if $T > 0$ we can say that $w(x,t) \leq 0$. We can make a similar argument for the minimal princliple then we can get that $w(x,t) \geq 0$. Therefore, $w(x,t) = 0$, that is $u_1(x,t) = u_2(x,t)$. ☐

Formal Proof. Consider $w_t - kw_{xx} = 0$ then mmultiply this by w, we get $ww_t - kww_{xx} = 0$ and so $\frac{\partial}{\partial t}(\frac{1}{2}w^2) - \frac{\partial}{\partial x}(kww_x) + kw_x^2 = 0$. This then gives us the PDE, $\frac{\partial}{\partial t}(\frac{1}{2}w^2) - \frac{\partial}{\partial x}(kww_x) = -kw_x^2$. Now from what we obtained, we can integrate this equation,

$$\int_0^L \left(\frac{1}{2}w^2\right)_x dx - [kww_x]_0^L + k\int_0^L w_x^2 = 0$$

$$\int_0^L \left(\frac{1}{2}w^2\right)_x dx = -k\int_0^L w_x^2$$

The RHS is always negative or zero. Therefore, $\int_0^L w_x^2 dx$ is decreasing and so

$$\int_0^L w_x^2 \leq \int_0^L w_x(x,0)^2 dx$$

and so $w(x,t) = 0$ and so $u_1(x,t) = u_2(x,t)$. ☐

We now consider stability of the equation. Consider $u_t - ku_{xx} = 0$ for $0 < x < \ell$ and $t > 0$ with initial and boundary conditions of $u(x,0) = \Phi(x)$ and $u(0,t) = u(L,t) = 0$. Let $u_1(x,0) = \Phi_1(x)$ and $u_2(x,t) = \Phi_2(x)$ we solutions of the equations. Call $w = u_1(x,t) - u_2(x,t)$ with the intial condition of $\Phi(x) = \Phi_1(x) + \Phi_2(x)$. Then from the energy method,

$$\int_0^L (u_1(x,t) - u_2(x,t))^2 dx \leq \int_0^L (\Phi_1(x) - \Phi_2(x))^2$$

The LHS is the nearness of the solutions at any later time and the RHS is the nearness of the initial data for two solutions. Stability means that if we stay near a point if we just change it a little bit. Therefore, the solution of the diffusion equation is stable.

5.1 Diffusion Equation on the real line

The problem of interest is $u_t - ku_{xx} = 0$ for $-\infty < x < \infty$ with initial condition of $u(x,0) = \Phi$. We consider the invariance properties,

1. The translate $u(x-y,t)$ for any solution is another solution for any fixed y.

2. Any derivative, u_x, u_t, u_{xx}, etc. of a solution is again a solution.
3. A linear combination of solutions of the diffusion equation is a solution of the diffusion equation.
4. An integral of a solution is again a solution. Thus if $S(x,t)$ is a solution to the diffusion equation, then so is $S(x-y,t)$ and so is,

$$v(x,t)=\int_{-\infty}^{\infty}S(x-y,t)g(y)dy$$

for any $g(y)$, as long as this convolution converges properly.

5. If $u(x,t)$ is a solution of the diffusion equation, then the dilated function $u(\sqrt{a}x,at)$ for any $a>0$ is also a solution.

Proof. Let $v(x,t)=u(\sqrt{a},at)$, then $v_t=au_t$ and $v_x=\sqrt{a}u_x$ and so $v_{xx}=au_{xx}$. Therefore, $v_t-kv_{xx}=a(u_t-ku_{xx})=0$ □

Consider $Q_t-kQ_{xx}=0$ with special initial conditions, such that $Q(x,0)=1$ for $x>0$ and $Q(x,0)=0$ for $x<0$. We let $Q(x,t)=g(p)$ where $p=\frac{x}{\sqrt{4kt}}$. We can see that $\frac{x}{\sqrt{t}}\to\frac{\sqrt{a}x}{\sqrt{at}}$ via dilation. This is now just $\frac{\sqrt{a}}{\sqrt{a}}\frac{x}{\sqrt{t}}$. We will now use this form to transform the PDE to an ODE. We see that $Q_t=\frac{dg}{dp}\frac{\partial p}{\partial t}=-\frac{1}{2t}\frac{x}{\sqrt{4kt}}\frac{dg}{dp}$ and also $Q_x=\frac{dg}{dp}\frac{\partial p}{\partial x}=\frac{1}{\sqrt{4kt}}g'(p)$ and $Q_{xx}=\frac{1}{4kt}g''(p)$. Then we get the following ODE,

$$\frac{1}{t}\left(-\frac{1}{2}pg'(p)-\frac{1}{4}g''(p)\right)=0$$

which is just,

$$g''(p)+2pg'(p)=0$$

and then we can solve this and get a solution of,

$$g'(p)=c_1e^{-p^2}$$

and then we integrate again to get an solution in terms of g

$$g(p)=c_1\int e^{-p^2}dp+c_2$$

But $Q(x,t)=g(p)$ and so

$$Q(x,t)=c_1\int_0^{\frac{x}{\sqrt{4kt}}}e^{-p^2}+c_2$$

We now want to impose initial conditions to get the constants. If $x>0$, then,

$$1=\lim_{t\to 0}Q(x,t)=c_1\int_0^{\infty}e^{-p^2}dp+c_2 \tag{6}$$

and impose the second condition, if $x<0$, then

$$0=\lim_{t\to 0}Q(x,t)=c_1\int_0^{-\infty}e^{-p^2}dp+c_2 \tag{7}$$

We now will find the closed form of the gaussian integral. Let $I = \int_0^\infty e^{-x^2}dx$ and so,

$$\begin{aligned}
I^2 &= \int_0^\infty e^{-x^2}dx \int_0^\infty e^{-y^2}dy \\
&= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy \\
&= \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r dr d\theta \\
&= \int_0^{\frac{\pi}{2}} -\frac{1}{2}e^{-r^2}\Big|_0^\infty d\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{2} d\theta = \frac{\pi}{4}
\end{aligned}$$

Therefore, we can say that $I = \frac{\sqrt{\pi}}{2}$. From here we can say that (6) is, $c_1\frac{\sqrt{\pi}}{2} + c_2$ and (7) becomes $-c_1\frac{\sqrt{\pi}}{2} + c_2$. We solve these and see that $c_1 = \frac{1}{\sqrt{\pi}}$ and $c_2 = \frac{1}{2}$. We can now substitute these into the solution for the equation,

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}}\int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2}dp \qquad t > 0$$

If $Q(x,t)$ is a solution, then $S(x,t) = \frac{\partial Q}{\partial x}$ is a solution. Then using the convolution integral, the following is also a solution,

$$u(x,t) = \int_{-\infty}^\infty S(x-y,t)\Phi(y)dy \qquad t > 0 \tag{$*$}$$

Furthermore, we claim this is unique. It suffices to show it satisfies the original initial condition.

$$\begin{aligned}
u(x,t) &= \int_{-\infty}^\infty S(x-y,t)\Phi(y)dy \\
&= \int_{-\infty}^\infty \frac{\partial Q}{\partial x}(x-y,t)\Phi(y)dy \\
&= -\int_{-\infty}^\infty \frac{\partial Q}{\partial y}(x-y,t)\Phi(y)dy \\
&= -Q(x-y,t)\Phi(y)|_{-\infty}^\infty + \int_{-\infty}^\infty Q(x-y,t)\Phi'(y)dy
\end{aligned}$$

We now impose the initial condition, $\Phi(y) \to 0$ as $|y|$ becomes large. Therefore,

$$u(x,t) = \int_{-\infty}^\infty Q(x-y,t)\Phi'(y)dy$$

Now let $t = 0$, then,

$$u(x,0) = \int_{-\infty}^\infty Q(x-y,0)\Phi'(y)dy$$

We also have initial conditions for Q, then,

$$u(x,0) = \int_{-\infty}^x \Phi'(y)dy = \Phi(x)$$

We said that $S(x,t) = \frac{\partial Q}{\partial x} = \frac{1}{2\sqrt{\pi kt}}e^{-\frac{x^2}{4kt}}$ for $t > 0$ (this is also known as Greens function). Now substituting in for the source function into $(*)$, we can conclude,

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^\infty e^{\frac{-(x-y)^2}{4kt}}\Phi(y)dy$$

We also note that,

$$\int_{-\infty}^{\infty} S(x,t)dt = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-q^2} dq = 1$$

We can't always talk about the solution in terms of elementary functions, but we are able to write it in terms of $\operatorname{erf} x$, which is defined as,

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2} dp$$

Example. $Q(x,t)$ can be written in terms of $\operatorname{erf} x$ as,

$$Q(x,t) = \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)$$

Example. Solve,

$$\begin{cases} u_t - ku_{xx} = 0 \\ u(x,0) = e^x \end{cases}$$

We already have a solution, we just have to replace Φ by the initial condition.

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} e^{-y} dy = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(y-x-2kt)^2}{4kt} + kt - x}$$

and so we now let $p = \frac{y+2kt-x}{\sqrt{4kt}}$, then,

$$u(x,t) = \frac{1}{\pi} e^{kt-x} \int_{-\infty}^{\infty} e^{-p^2} dp = e^{kt-x}$$

5.2 Diffusion on the half line

We consider $v_t - kv_{xx} = 0$ on $0 < x < \infty$ for $t > 0$ subject to initial condition $u(x,0) = \Phi(x)$ for $t = 0$ and $x > 0$ and Dirichlet Boundary condition, $v(0,t) = 0$. We need to restrict the equation, so we will consider the method of reflection or odd expansion. We will use odd expansion, consider

$$\Phi_{\text{odd}} = \begin{cases} \Phi(x) & x > 0 \\ 0 & x = 0 \\ -\Phi(-x) & x < 0 \end{cases}$$

Our problem of interest is $u_t - ku_{xx} = 0$ for $-\infty < x < \infty$ and $t > 0$ subject to $u(x,0) = \Phi_{\text{odd}}(x)$. Then the solution to this is,

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\Phi_{\text{odd}}(y)dy$$

where $v(x,t) = u(x,t)$ for $x > 0$. We can write,

$$\begin{aligned} u(x,t) &= \int_0^{\infty} S(x-y,t)\Phi(y)dy - \int_{-\infty}^{0} S(x-y,t)\Phi(-y)dy \\ &= \int_0^{\infty} S(x-y,t)\Phi(y)dy - \int_0^{\infty} S(x+y,t)\Phi(y)dy \\ &= \int_0^{\infty} \left(S(x-y,t) - S(x+y,t)\right)\Phi(y)dy \end{aligned}$$

Therefore,

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left(e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}}\right)\Phi(y)dy$$

Example. Solve the diffusion equation, $v_t - kv_{xx} = 0$ for $0 < x < \infty$ and $t > 0$ subject to $v(x, 0) = 1$ at $t = 0$ and $v(0, t) = 0$ at $x = 0$.

We get the solution of,

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left(e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}} \right) dy$$

We then change variables, let $p = \frac{x-y}{\sqrt{4kt}}$ and $q = \frac{x+y}{\sqrt{4kt}}$, then we get,

$$v(x,t) = -\frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{-\infty} e^{-p^2} dp - \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-q^2} dq$$

Then we get,

$$v(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} e^{-p^2} dp + \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2} dp + \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{4kt}} e^{-q^2} dq - \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-q^2} dq$$

and the first and last terms cancel,

$$v(x,t) = \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2} dp + \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{4kt}} e^{-q^2} dq$$

which is just a erf functions,

$$\operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)$$

Now consider a Neumnann Boundary condition,

Example. Consider $w_t - kw_{xx} = 0$ on the half line where $w(x, 0) = \Phi(x)$ and subject to $w_x(0, t) = 0$. For the Neumann problem we need to use the even extension as then we will get it's derivative as an odd function. Consider,

$$\Phi_{\text{even}}(x) = \begin{cases} \Phi(x) & x \geq 0 \\ \Phi(-x) & x < 0 \end{cases}$$

We consider $u_t - ku_{xx} = 0$ where $u(x, 0) = \Phi_{\text{even}}(x)$. We restrict this to $x > 0$ this will be the unique solution. We start with the solution on the entire real line, then,

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\Phi_{\text{even}} dy$$

Then we break it into two parts,

$$\begin{aligned} u(x,t) &= \int_0^\infty S(x-y,t)\Phi(y)dy + \int_{-\infty}^0 S(x-y,t)\Phi(-y)dy \\ &= \int_0^\infty S(x-y,t)\Phi(y)dy + \int_0^\infty S(x+y,t)\Phi(y)dy \\ &= \int_0^\infty \left(S(x-y,t) + S(x+y,t)\right)\Phi(y)dy \end{aligned}$$

Then we have that,

$$w(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left(e^{-\frac{(x-y)^2}{4kt}} + e^{-\frac{(x+y)^2}{4kt}} \right) \Phi(y)dy$$

Example. Solve $w_t - kw_{xx} = 0$ for the half line where $w(x,0) = \Phi(x) = 1$ and $w_x(0,t) = 0$. Then even extension of $\Phi_{\text{even}}(x) = \Phi(x)$. So we know $w(x,t) = u(x,t)$ for $x > 0$. Therefore, we have the solution,

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left(e^{-\frac{(x-y)^2}{4kt}} + e^{-\frac{(x+y)^2}{4kt}} \right) dy$$

Let $p = \frac{x-y}{\sqrt{4kt}}$ and $q = \frac{x+y}{4kt}$,

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{-\infty} e^{-p^2} dp + \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-q^2} dq$$

Then we can further simplify as before, and we get,

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-p^2} = 1$$

We consider the diffusion equation with a source,

5.3 Diffusion Equation with a source equation

$u_t - ku_{xx} = f(x,t)$ where $-\infty < x < \infty$, $t > 0$ and $k > 0$ where $u(x,0) = \Phi(x)$. When $f(x,t) = 0$, then we get,

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\Phi(y)dy$$

Then with $f(x,t) \neq 0$, we get that,

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\Phi(y)dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds$$

For simplicity, we let $\Phi(x) = 0$. Then we need to find that,

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t} \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds \\
&= \int_0^t \int_{-\infty}^{\infty} \frac{\partial}{\partial t} S(x-y,t-s)f(y,s)dyds + \lim_{s\to t} \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dy \\
&= \int_0^t \int_{-\infty}^{\infty} k\frac{\partial^2}{\partial x^2} S(x-y,t-s)f(y,s)dyds + \lim_{s\to t} \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dy
\end{aligned}$$

$$\begin{aligned}
&= \int_0^t \int_{-\infty}^{\infty} k\frac{\partial^2}{\partial x^2} S(x-y,t-s)f(y,s)dyds + \int_{-\infty}^{\infty} S(x-y,\varepsilon)f(y,t)dy \\
&= \int_0^t \int_{-\infty}^{\infty} k\frac{\partial^2}{\partial x^2} S(x-y,t-s)f(y,s)dyds + \int_{-\infty}^{\infty} \frac{\partial Q}{\partial x}(x-y,\varepsilon)f(y,t)dy \\
&= \int_0^t \int_{-\infty}^{\infty} k\frac{\partial^2}{\partial x^2} S(x-y,t-s)f(y,s)dyds - \int_{-\infty}^{\infty} \frac{\partial Q}{\partial y}(x-y,\varepsilon)f(y,t)dy \\
&= \int_0^t \int_{-\infty}^{\infty} k\frac{\partial^2}{\partial x^2} S(x-y,t-s)f(y,s)dyds - [-Q(x-y,\varepsilon)f(y,t)]_{y=-\infty}^{y=\infty} + \int_{-\infty}^{\infty} Q(x-y,\varepsilon)\frac{\partial f}{\partial y}dy \\
&= \int_0^t \int_{-\infty}^{\infty} k\frac{\partial^2}{\partial x^2} S(x-y,t-s)f(y,s)dyds + \int_{-\infty}^{\infty} Q(x-y,\varepsilon)\frac{\partial f}{\partial y}dy
\end{aligned}$$

Consider as $\varepsilon \to 0$ and we consider the initial conditions for Q,

$$= \int_{-\infty}^{x} \frac{\partial f}{\partial y} dy = f(x)$$

and so we can write that,

$$\begin{aligned}\frac{\partial u}{\partial t} &= k\frac{\partial^2}{\partial x^2}\int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds + f(x,t)\\ \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + f(x,t)\end{aligned}$$

Therefore it satisfies the given PDE and so now we need to show it satisfies the initial condition. Therefore,

$$\begin{aligned}u(x,0) &= \int_{-\infty}^{\infty} S(x-y,0)\Phi(y)dy + 0\\ &= \Phi(x)\end{aligned}$$

Then substituting into the source function $S(x,t)$, we get that,

$$\begin{aligned}u(x,t) &= \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds\\ &= \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{\frac{-(x-y)^2}{4k(t-s)}} f(y,s)dyds\end{aligned}$$

If $\Phi(x) \neq 0$, then we get,

$$\int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{\frac{-(x-y)^2}{4k(t-s)}} f(y,s)dyds + \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} \Phi(y)dy$$

Example. Use the method of reflection to solve the inhomogeneous diffusion equation on the half-line with Dirichlet boundary condition

$$\begin{cases} u_t - ku_{xx} = f(x,t)\\ u(0,t) = 0, u(x,0) = \Phi(x)\end{cases}$$

We consider the odd extensions of Φ and f. We denote them as f_{odd} and Φ_{odd}. Then we use the laws of integration to simplify the answer to,

$$v(x,t) = \int_0^{\infty} \left(S(x-y,t-s) - S(x+y,t-s)\right)\Phi(y)dy + \int_0^t \int_{-\infty}^{\infty} \left(S(x-y,t-s) - S(x+y,t-s)\right)dyds$$

6 Fourier Series

The Fourier Series of a function on an interval $[-\pi, \pi]$ is,

$$f(x) = \frac{a_0}{2} + \int_{k=1}^{\infty} a_k \cos kx + b \sin kx$$

where we define,

$$a_k = \langle f(x) \, , \cos kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx$$

$$b_k = \langle f(x) \, , \sin kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx$$

Where we use the $L-2$ inner product on function spaces. This is defined as,

$$\langle f \, , g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x) dx$$

We can now see that the trigonometric functions are orthogonal,

$$\int_{-\pi}^{\pi} \cos(kx) \cos(\ell x) = \begin{cases} 0 & k \neq \ell \\ 2\pi & k = \ell = 0 \\ \pi & k = \ell \neq 0 \end{cases}$$

and,

$$\int_{-\pi}^{\pi} \cos(kx) \sin(\ell x) dx = 0 \qquad \forall k, \ell \geq 0$$

and

$$\int_{-\pi}^{\pi} \sin(kx) \sin(\ell x) = \begin{cases} 0 & k \neq \ell \\ \pi & k = \ell \neq 0 \end{cases}$$

Therefore we that sin is orthogonal to sin and cos for $k \neq \ell$ and cos is orthogonal to cos for $k \neq \ell$. Furthermore, $\|1\| = \sqrt{2}$, $\|\cos kx\| = 1$ and $\|\sin kx\| = 1$ where $k \neq 0$.

Consider $\langle f(x) \, , \cos \ell x \rangle$,

$$\begin{aligned} \langle f(x) \, , \cos \ell x \rangle &= \left\langle \frac{a_0}{2} \, , \cos \ell x \right\rangle + \sum_{k=1}^{\infty} a_k \langle \cos kx \, , \cos \ell x \rangle + b_k \langle \sin kx \, , \cos \ell x \rangle \\ &= \left\langle \frac{a_0}{2} \, , \cos \ell x \right\rangle + \sum_{k=1}^{\infty} a_k \langle \cos kx \, , \cos \ell x \rangle \\ &= \sum_{k=1}^{\infty} a_k \langle \cos kx \, , \cos \ell x \rangle = a_\ell \end{aligned}$$

Therefore, $a_k = \langle f(x) \, , \cos kx \rangle$. We can now prove a similar thing for b_k.

Now assume f is an even function, then we have the following,

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

as $b_k = \langle f(x)\,, \sin kx \rangle = 0$. This is called the **Fourier Cosine Series**. If f is an odd function, then we have a **Fourier Sine series**,

$$f(x) = \sum_{k=1}^{\infty} b_k \sin kx$$

We notice that in the sine and cosine series the coefficients double. That is in the Fourier Cosine series we have,

$$a_k = \frac{2}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx$$

We know that $e^{ix} = \cos kx + i \sin kx$ and we also know that $\cos kx = \frac{1}{2}\left(e^{ikx} + e^{-ikx}\right)$ and $\sin kx = \frac{1}{2i}\left(e^{ikx} - e^{-ikx}\right)$. We cal also extend the $L-2$ norm to complex-valued functions,

$$\langle f\,, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g}(x)$$

Therefore we consider,

$$\left\langle e^{ikx}\,, e^{-i\ell x} \right\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-\ell)x} dx = \begin{cases} 1 & k = \ell \\ 0 & k \neq \ell \end{cases}$$

Therefore, the Fourier Series of the complex valued function is,

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \qquad c_k = \left\langle f(x)\,, e^{ikx} \right\rangle$$

Finally if we have a domain of definition of $[-\ell, \ell]$ instead of $[-\pi, \pi]$ we can use a change of variables of $x = \frac{\ell}{\pi} y$ for $-\pi \leq y \leq \pi$. Then our Fourier Series becomes,

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \frac{k\pi}{\ell} x + b_k \sin \frac{k\pi}{\ell} x$$

where,

$$a_k \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{k\pi}{\ell} x dx \qquad b_k = \frac{1}{\ell} \int_{-\pi}^{\pi} f(x) \sin \frac{k\pi}{\ell} x$$

7 Seperation of Vairables

We consider the diffusion equation, $u_t - u_{xx} = 0$ for $0 < x < \ell$ where $t > 0$ and $k > 0$. With boundary and initial conditions $u(0,t) = u(\ell,t) = 0$ and $u(x,0) = f(x)$. We consider a solution of the form $u(x,t) = X(x)T(t) \neq 0$. Then we get that, the PDE reduces to $XT' - kX''T = 0$ then we can write that,

$$\frac{T'}{T} = k\frac{X''}{X} = -\alpha^2$$

As they are equal, they must equal a constant. Therefore,

$$\frac{T'}{kT} = -\alpha^2 \qquad \frac{X''}{X} = -\alpha^2$$

We then have the two ODEs, $T' + \alpha^2 kT = 0$ and $X'' + \alpha^2 X = 0$, therefore we have the solutions of $T(t) = ce^{-\alpha^2 kt}$ (We have $-\alpha^2$ as we want a solution that decays), which means $u(x,t) = e^{-\lambda t}X(t)$. We consider this as a solution, then $u_t = -\lambda e^{-\lambda t}X(x)$ and $u_{xx} = e^{-\lambda t}X''(x)$. We then have that $-e^{-\lambda t}(\lambda X + kX'') = 0$ and so we then have that $X'' + \frac{\lambda}{k}X = 0$. We have the initial conditions of $u(0,t) = u(\ell,t) = 0$, that is $e^{-\lambda t}X(0) = e^{-\lambda t}X(\ell) = 0$ and so $X(0) = X(\ell) = 0$. We now solve the second order ODE with these boundary conditions.

$$\begin{cases} kX'' + \lambda X = 0 \\ X(0) = X(\ell) = 0 \end{cases}$$

If $\lambda = 0$, then $\frac{dX}{dt} = c$ and so $X(x) = Cx + D$, therefore $X(0) = D = 0$ and $X(\ell) = c\ell = 0$. Therefore, $X(x) = 0$, which is the trivial solution. Now consider $\ell < 0$, let $\lambda = -\beta$ where $\beta > 0$. Therefore we get the ODE, $X'' - \frac{\beta}{k}X = 0$, and so we have the solution $X(x) = c_1 e^{\sqrt{\frac{\beta}{k}}x} + c_2 e^{-\sqrt{\frac{\beta}{k}}x}$. Now impose boundary conditions, $X(0) = c_1 + c_2 = 0 \implies c_1 = -c_2$ and so $X(\ell) = c_1 e^{\sqrt{\frac{\beta}{k}}\ell} + c_2 e^{-\sqrt{\frac{\beta}{k}}\ell} = 0$. Therefore, we get that $c_1 = 0$ and so $c_2 = 0$ as well so $X(x) = 0$. Now consider if $\lambda > 0$, then $X'' + \frac{\lambda}{k}X = 0$ and so we get, $X(x) = c_1\cos\omega x + c_2\sin\omega x$ where $\omega^2 = \frac{\lambda}{k}$. We impose the conditions and get that $X(0) = c_1 = 0$ and the second we get that $X(\ell) = c_2\sin\omega\ell = 0$, so we let $\sin\omega\ell = 0$ and so $\omega_n\ell = n\pi$ where $n \in \mathbb{Z}$. We then have $\omega_n = \frac{n\pi}{\ell}$, this is the eigenfrequency. Therefore $\lambda_n = k\omega_n^2 = k\left(\frac{n\pi}{\ell}\right)^2$. Now we can find the eigenfunctions. Therefore, $X_n(x) = \sin\frac{n\pi}{\ell}x$ for $n \in \mathbb{Z}$ are the eigenfunctions. The eigensolutions are,

$$u_n(x,t) = X_n(x)T_n(t) = e^{-\lambda_n t}X_n(x) = e^{-\lambda_n t}\sin\frac{n\pi}{\ell}x$$

Therefore we have,

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-k\left(\frac{n\pi}{\ell}\right)^2 t}\sin\frac{n\pi}{\ell}x$$

Now we find c_n by imposing the initial condition.

$$u(x,0) = \sum_{n=1}^{\infty} c_n\sin\frac{n\pi}{\ell}x = f(x) \qquad n \in \mathbb{Z}$$

This is a Fourier Sine series, so

$$c_n = \frac{2}{\ell}\int_0^\ell f(x)\sin\frac{n\pi}{\ell}dx$$

Example. Consider $u_t - ku_{xx} = 0$ for $0 < x < \ell$ and $t > 0$ where $u(0,t) = 0$ and $u(\ell,t) = u_0$ subject to $u(x,0) = f(x)$ for $0 < x < \ell$. We introduce that,

$$u(x,t) = v(x,t) + \frac{u_0 x}{\ell}$$

and then $u(0,t) = v(0,t) = 0$ and $u(\ell,t) = v(\ell,t) + u_0 = u_0$ and so $v(\ell,t) = 0$. We solve another PDE for $v(x,t)$, then we can find $u(x,t)$. We can quickly check that $u_t = v_t$ and $u_{xx} = v_{xx}$. Therefore the PDE problem is $v_t - kv_{xx} = 0$ for $0 < x < \ell$ and $t > 0$ where $v(0,t) = v(\ell,t) = 0$ subject to $v(x,0) = u(x,0) - \frac{u_0 x}{\ell} = f(x) - \frac{u_0 x}{\ell}$ for $0 < x < \ell$. We can now solve this.

$$v(x,t) = \sum_{n=1}^{\infty} v_n(x,t) = \sum_{n=1}^{\infty} c_n e^{-k\left(\frac{n\pi}{\ell}\right)^2 t} \sin \frac{n\pi}{\ell} x$$

where

$$c_n = \frac{2}{\ell} \int_0^{\ell} \left(f(x) - \frac{u_0 x}{\ell} \right) \sin \frac{n\pi}{\ell} x dx.$$

Therefore we get,

$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{2}{\ell} \int_0^{\ell} \left(f(\tau) - \frac{u_0 \tau}{\ell} \right) \sin \frac{n\pi}{\ell} \tau d\tau \right] e^{-k\left(\frac{n\pi}{\ell}\right)^2 t} \sin \frac{n\pi}{\ell} x + \frac{u_0 x}{\ell}$$

7.1 Diffusion Equation with Neumann Boundary Condition

Consider $u_t - ku_{xx} = 0$ for $0 < x < \ell$ where $t, k > 0$ with $u_x(0,t) = u_x(\ell,t) = 0$ for $u(x,0) = \Phi(x)$. We have a solution of the form $u(x,t) = e^{-\lambda t} X(x)$, then we substitute this into the PDE, we get that $X'' + \frac{\lambda}{k} X = 0$ where $\lambda > 0$. This is just $X'' + \omega^2 X = 0$ where $\omega^2 = \frac{\lambda}{k}$. We have that $X(x) = c_1 \cos \omega x + c_2 \sin \omega x$. Now we consider the boundary conditions, $u_x(0,t) = u_x(\ell,t) = 0$. That is $X'(0) = X'(\ell) = 0$. We consult our solution for $X(x)$ and get, $X'(x) = -c_1 \omega \sin \omega x + c_2 \omega \cos \omega x$ and so $X'(0) = c_2 \omega = 0$ and so $c_2 = 0$. Now $X'(\ell) = -c_1 \omega \sin \omega \ell = 0$. For non-trivial solutions $\sin \omega \ell = 0$ and so $\omega_n = \frac{n\pi}{\ell}$. Therefore, $\lambda_n = k\omega_n^2 = k\left(\frac{n\pi}{\ell}\right)^2$. Therefore, $X(x) = c_n \cos \frac{n\pi}{\ell} x$. Therefore,

$$u_n(x,t) = e^{-\lambda_n t} X_n(x) = e^{-k\left(\frac{n\pi}{\ell}\right)^2 t} \cos \frac{n\pi}{\ell} x$$

We also consider $\lambda = 0$, we then get another solution, $X(t) = D$. Therefore we have a solution (letting $D = \frac{1}{2} c_0$),

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-k\left(\frac{n\pi}{\ell}\right)^2 t} \cos \frac{n\pi}{\ell} x$$

Now we impose the initial conditions. Therefore,

$$u(x,0) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos \frac{n\pi}{\ell} x$$

and so we have that,

$$c_n = \frac{2}{\ell} \int_0^{\ell} \cos \frac{n\pi}{\ell} x \Phi(x) dx$$

7.2 Seperation of variables for wave equation

Consider $u_{tt} - c^2 u_{xx} = 0$ subject to $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ and the boundary conditions $u(0,t) = u(\ell,t) = 0$. We know the D'Alembert solution for the wave equation, but we consider seperation of variables. Let $u(x,t) = X(x)T(t)$, then we get the ODE, $XT'' - c^2 X'' T = 0$. Then we seperate and see,

$$\frac{T''}{T} = c^2 \frac{X''}{X} = \lambda$$

Then we have to solve,

$$\begin{cases} T'' - \mathcal{L}T = 0 \\ X'' - \frac{\lambda}{c^2}X = 0 \end{cases}$$

Then we consider different cases of λ. Firstly, let $\lambda = -\omega^2$, then we get two ODEs that solve to $T(t) = c_1 \cos \omega t + c_2 \sin \omega t$ and $X(x) = d_1 \cos \frac{\omega}{c}x + d_2 \sin \frac{\omega}{c}x$. Now we consider the boundary conditions, these yield that $d_1 = 0$ and $d_2 \sin \frac{\omega \ell}{x} = 0$ and so $\omega_n = \frac{n\pi c}{\ell}$ for $n = 1, 2, 3 \ldots$ and further, $\lambda_n = -\left(\frac{n\pi c}{\ell}\right)^2$. Therefore, $X_n = d_n \sin \frac{n\pi}{\ell}x$ for $n = 1, 2, 3 \ldots$. Therefore the eigensolution is,

$$u_n(x,t) = X_n T_n(t) = d_n \sin \frac{n\pi}{\ell}x \left(c_{1n} \cos \frac{n\pi c}{\ell}t + c_{2n} \sin \frac{n\pi c}{\ell}t\right)$$

This can be written as,

$$u_n(x,t) = C_n \sin \frac{n\pi}{\ell}x \cos \frac{n\pi c}{\ell}t + d_n \sin \frac{n\pi}{\ell}x \sin \frac{n\pi c}{\ell}t$$

Therefore, we can finally write,

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{\ell}x \cos \frac{n\pi c}{\ell}t + d_n \sin \frac{n\pi}{\ell}x \sin \frac{n\pi c}{\ell}t$$

Now consider the initial conditions,

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{\ell}x = f(x)$$

and so,

$$C_n = \frac{2}{\ell} \int_0^L f(x) \sin \frac{n\pi}{\ell} x dx$$

Now we differentiate,

$$u_t(x,0) = \sum_{n=1}^{\infty} d_n \frac{n\pi c}{\ell} \sin \frac{n\pi}{\ell}x = g(x)$$

and so,

$$d_n = \frac{2}{n\pi c} \int_0^L g(x) \sin \frac{n\pi}{\ell} x dx$$

We can verify that this is the only place where we don't have trivial solutions, that is $\lambda = 0$ and $\lambda > 0$ is trivial.

Example. Consider the wave equation where $u(x,0) = f(x)$, $u(x,0) = g(x)$ and $u_x(0,t) = u_x(\ell,t) = 0$. The boundary conditions tell us that $X'(0) = 0$ and $X'(\ell) = 0$. So we consider the separated ODEs and continue from there. We consider the ODE in terms of X, this has solution of $X(x) = d_1 \cos \frac{\omega}{c}x + d_2 \sin \frac{\omega}{c}x$, then we consider the boundary conditions. $X'(x) = -\frac{d_1\omega}{c} \sin \frac{\omega}{c}x + \frac{d_2\omega}{c} \cos \frac{\omega}{c}x$. Then imposing boundary conditions, we get that $d_2 = 0$ and secondly $X'(\ell) = -\frac{d_1\omega}{c} \sin \frac{\omega}{c}\ell = 0$, that is $\omega_n = \frac{n\pi c}{\ell}$ for $n = 1, 2, \ldots$. Therefore the eigenfunctions are,

$$X_n(x) = d_n \cos \frac{n\pi}{\ell}x$$

for $n = 1, 2, \ldots$. Then the other eigenfunction is,

$$T_n = c_{1n} \cos \frac{n\pi c}{\ell}t + c_{2n} \sin \frac{n\pi c}{t}$$

Putting them together,

$$u_n(x,t) = c_n \cos\frac{n\pi}{\ell}x \cos\frac{n\pi c}{\ell}t + d_n \cos\frac{n\pi}{\ell}x \sin\frac{n\pi c}{\ell}t$$

We consider $\lambda = 0$, then we get that, $X'' = 0$, that gives $X(x) = Cx + D$, then imposing the boundary conditions, we get that $X'(0) = C = 0$ and $X'(\ell) = C = 0$. Therefore, $X_0(x) = D$ and for T, we get $T(t) = c_0 + d_0 t$. This gives,

$$u(x,t) = c_0 + d_0 t + \sum_{n=1}^{\infty} c_n \cos\frac{n\pi}{\ell}x \cos\frac{n\pi c}{\ell}t + d_n \cos\frac{n\pi}{\ell}x \sin\frac{n\pi c}{\ell}t$$

Now we impose the intial conditions,

$$u(x,0) = c_0 + \sum_{n=1}^{\infty} c_n \cos\frac{n\pi}{\ell}x = f(x)$$

and differentiate to get,

$$u_t(x,0) = d_0 + \sum_{n=1}^{\infty} \frac{d_n n\pi c}{\ell} \cos\frac{n\pi}{\ell}x = g(x)$$

and these are Fourier series. This gives us,

$$c_n = \frac{2}{\ell}\int_0^L f(x)\cos\frac{n\pi}{\ell}x dx$$

and,

$$d_n = \frac{2}{n\pi c}\int_0^L g(x)\cos\frac{n\pi}{\ell}x dx$$

8 Laplace Equation

We have $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ for a Laplace equation and $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$ for the Poisson equations. We have a simply connected region that is the domain of definition, which we call Ω. We have a selection of boundary conditions on this problem,

1. Dirichlet BC, $u(x,y) = h(x,y)$ for $(x,y) \in \partial\Omega$
2. Neumann BC, $\frac{\partial u}{\partial \mathbf{n}} = \nabla u \cdot \mathbf{n} = k(x,y)$ on $\partial\Omega$
3. Robin BC, $\frac{\partial u}{\partial \mathbf{n}} + \beta(x,y)u = k(x,y)$ on $\partial\Omega$.

or even a mixture of these. We can separate variables, let $u = X(x)Y(y)$, then we can reduce this by substituting this into the PDE, we get that

$$\frac{X''}{X} = \frac{Y''}{Y} = \lambda$$

We look at different possible values for λ, if $\lambda = -\omega^2 < 0$, then we get solutions. We can run through each case and get the following,

$$\begin{cases} c_1 \cos \omega + c_2 \sin \omega x & \lambda = -\omega^2 \\ c_1 x + c_2 & \lambda = 0 \\ c_1 e^{\omega x} + c_2 e^{-\omega x} & \lambda = \omega^2 \end{cases}$$

If we consider the Laplace equation on $u(x,0) = f(x)$, then $u(x,b) = 0$, $u(0,y) = 0$ and $u(a,y) = 0$ (This is a rectangle). We want to induce the BC to reduce the number of solutions.

$$\begin{cases} c_1 \sin \omega x \\ cx \\ c \sinh \omega x \end{cases}$$

The last two of these provide trivial solutions. We now impose the BC,

$$X(a) = c \sin \omega a = 0$$

and so $\sin \omega a = 0$ and so $\omega_n a = n\pi$ for $n \in \mathbb{Z}$. Then we get that $\omega_n = \frac{n\pi}{a}$. Hence $\lambda_n = -\omega_n^2 = -\left(\frac{n\pi}{a}\right)$. Therefore our eigenfuctions are,

$$X_n(x) = c_n \sin \frac{n\pi}{a} x$$

and now we consider Y, we get that,

$$Y'' + \lambda Y = 0$$

and so we can say that $Y(y) = d_1 e^{\omega y} + d_2 e^{-\omega y}$. Now we impose boundary conditions, $Y(b) = d_1 e^{\omega b} + d_2 e^{-\omega b} = 0$. Hence, $d_2 = d_1 e^{-2\omega b}$. Hence we get the general solution,

$$Y(y) = d_1 e^{\omega b} \left(e^{\omega(y-b)} - r^{\omega(b-y)} \right)$$

and so we can conclude,

$$Y_n(y) = d_n \sinh \omega_n (b-y)$$

where $n \in \mathbb{Z}$. We can substitute for ω_n,

$$Y_n(y) = d_n \sinh \frac{n\pi}{a}(b-y)$$

We can now find $u(x,y) = X(x)Y(y)$,

$$u_n(x,y) = C_n \sin\frac{n\pi}{a}x \sinh\frac{n\pi}{a}(b-y)$$

where $C_n = c_n d_n$. Therefore,

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{a}x \sinh\left(\frac{n\pi(b-y)}{a}\right)$$

Now we consider the bottom edge, where $y = 0$, we get,

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{a} \sinh\frac{n\pi b}{a} = f(x)$$

This is fourier sine series and so,

$$C_n \sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi}{a}x dx$$

8.1 Laplace Equation in Polar Coordinates

Assume we want to solve $\Delta u = 0$ in Ω where $\Omega = x^2 + y^2 \leq 1$, where $\partial\Omega$ is just the unit circle. Along the boundary $u(x,y) = h(x,y)$. We know that in polar coordinates we have that $x = r\cos\theta$, $y = r\sin\theta$ where $x^2 + y^2 = r^2$ and $\theta = \tan^{-1}\frac{y}{x}$.

We know $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. We know that $\frac{\partial}{\partial r} = \frac{\partial}{\partial x}x_r + \frac{\partial}{\partial y}y_r = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}$, similarly, $\frac{\partial}{\partial \theta} = \frac{\partial}{\partial x}x_\theta + \frac{\partial}{\partial y}y_\theta = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}$. Then from these two equations we ge that,

$$\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$

and so we can see that,

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix}$$

Therefore we get that,

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial \theta} \qquad \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial \theta}$$

We can now transform the laplacian to,

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \to \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

If $r = 1$, then $u(1,\theta) = h(\theta)$. Hence we have transformed the problem to polar coordinates. We will now solve this. We also need periodic boundary conditions, that is $u(r,\theta+2\pi) = u(r,\theta)$ and $h(\theta+2\pi) = h(\theta)$. We will use seperation of variables, let $u(r,\theta) = R(r)\Theta(\theta)$. We substitute this into the PDE to get,

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

We will multiply by $\frac{r^2}{R\Theta}$,

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = 0$$

or,

$$\frac{r^2R''+rR'}{R}=-\frac{\Theta''}{\Theta}=\lambda$$

This is a set of ODEs,

$$\begin{cases} r^2R''+rR'-\lambda R &= 0 \\ \Theta''+\lambda\Theta &= 0 \end{cases}$$

We let $\lambda=\omega^2$. Then the second equation becomes $\Theta(\theta)=c_1\cos\omega\theta+c_2\sin\omega\theta$. Then we assume that the solution is 2π-periodic. That is, $\Theta(\theta+2\pi)=\Theta(\theta)$. We conclude,

$$\Theta(-\pi)=c_1\cos\omega\pi-c_2\sin\omega\pi$$

$$\Theta(\pi)=c_1\cos\omega\pi+c_2\sin\omega\pi$$

Therefore $\Theta(-\pi)=\Theta(\pi)$ implies that $c_2\sin\omega\pi=0$, that is $\omega_n=n$ for $n\in\mathbb{Z}$. If $\lambda=0$, then $\Theta''=0$, that is $\Theta(\theta)=C\theta+D$. Considering this under 2π-periodic, we get that $\Theta(\theta)=D$, that is it's a constant. We now conclude that $\lambda=n^2$ where $n\in\mathbb{Z}$. Therefore $\Theta_n=a_n\cos n\theta+b_n\sin n\theta$ for $0\neq n\in Z$ and $\Theta_0(\theta)=a_n$ for $n=0$.

Now we consider the radial direction,

$$r^2R''+rR'-\lambda R=0$$

We let $z=\ln r$, that is $\frac{d}{dr}=\frac{d}{dz}\frac{dz}{dr}=\frac{1}{r}\frac{d}{dz}$ and we also get that $\frac{d^2}{dr^2}=\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dz}\right)=-\frac{1}{r^2}\frac{d}{dz}+\frac{1}{r^2}\frac{d^2}{dz^2}$. We substitute this back in and get,

$$\frac{d^2R}{dz^2}-\lambda R=0$$

and so we get that $R(z)=c_1e^{\sqrt{\lambda}z}+c_2e^{-\sqrt{\lambda}z}$, now we substitute in for λ_n,

$$R_n(z)=c_{1n}e^{nz}+c_{2n}e^{-nz}\rightarrow R_n(r)=c_{1n}e^{n\ln r}+c_{2n}e^{-nlnr}=c_{1n}r^n+c_{2n}r^{-n}$$

Therefore,

$$R_n(r)=c_{1n}r^n+c_{2n}r^{-n}\quad n\in\mathbb{Z}^+\setminus\{0\}$$

Now we consider $\lambda=0$, then we have the equation

$$r^2R''+rR'=0$$

and so we get that,

$$\frac{d^2R}{dz^2}=0$$

and this has solution $R(z)=\tilde{a_0}z+\tilde{b_0}$, which is then just $R_0(r)=\tilde{a_0}\ln r+\tilde{b_0}$. Then we get the following solution,

$$u_n(r,\theta)=R_n(r)\Theta_n(\theta)=(\tilde{a_0}\ln r+\tilde{b_0})+(c_{1n}r^n+c_{2n}r^{-n})(a_n\cos n\theta+b_n\sin n\theta)$$

Then $\tilde{a_0}\ln r$ and $c_{2n}r^{-n}$ are singular as $r\to 0$ and so we remove them.

$$u(r,\theta)=\sum_{n=1}^{\infty}u_n(r,\theta)=\frac{a_0}{2}+\sum_{n=0}^{\infty}a_nr^n\cos n\theta+b_nr^n\sin n\theta$$

As a series solution for this problem. To find the coefficients we consider the boundary condition along $\partial\Omega$. We consider $u(1,\theta)$,

$$u(1,\theta)=\frac{a_0}{2}+\sum_{n=0}^{\infty}a_n\cos n\theta+b_n\sin n\theta=h(\theta)$$

This is now a fourier series and so,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} h(\theta)\cos n\theta d\theta \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} h(\theta)\sin n\theta d\theta$$

We want to find an explicit expression, so we will substitute the coefficients into the solution,

$$u(r,\theta) = \sum_{n=1}^{\infty} u_n(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\varphi)d\varphi + \sum_{n=0}^{\infty}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} h(\varphi)\cos n\varphi d\varphi\right) r^n \cos n\theta + \left(\frac{1}{\pi}\int_{-\pi}^{\pi} h(\varphi)\sin n\varphi d\varphi\right) r^n \sin n\theta$$

This is then just,

$$u(r,\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} h(\varphi)\left(\frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos(n-\phi)\right) d\varphi$$

Then considering complex numbers,

$$u(r,\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} h(\varphi)\left(\frac{1}{2} + \sum_{n=1}^{\infty} \mathrm{Re}\left(\frac{1+z}{2(1-z)}\right)\right) d\varphi$$

$$u(r,\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} h(\varphi)\left(\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{\mathrm{Re}(1+z-\overline{z}-|z|^2)}{2|1-z|^2}\right)\right) d\varphi$$

$$u(r,\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} h(\varphi)\left(\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{1-|z|^2}{2|1-z|^2}\right)\right) d\varphi$$

$$u(r,\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} h(\varphi)\left(\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{1-r^2}{2(1+r^2-2r\cos(\theta-\phi))}\right)\right) d\varphi$$

Theorem 8.1 (Poisson Integral Formula)**.** The solution to the Laplace equation in the unit disc subject to Dirichlet boundary condition $u(1,\theta) = h(\theta)$ is,

$$u(r,\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} h(\varphi)\frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)} d\varphi$$

This is the Poisson Formula.

If we consider $x^2 + y^2 = a$ for the boundary condition, then we get that from this boundary condition,

$$h(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n a^n \cos n\theta + b_n a^n \sin n\theta)$$

Then we get that,

$$a_n = \frac{1}{a\pi}\int_{-\pi}^{\pi} h(\theta)\cos n\theta d\theta \qquad b_n = \frac{1}{a^n}\int_{-\pi}^{\pi} h(\theta)\sin n\theta$$

Exercise. Finish the derivation,

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\varphi)\frac{a^2-r^2}{a^2-2ar\cos(\theta-\varphi)}$$

using $z = \frac{r}{a}e^{i\theta}$

8.2 Laplace Equation for wedge

Consider a wedge along $u = 0$, where the angle θ is $\theta = 0$ and $\theta = \beta$. That is, $u(r, 0) = 0$ and $u(r, \beta) = 0$. We also have $\frac{\partial u}{\partial r}(a, \theta) = h(\theta)$. As usual assume that $u(r, \theta) = R(r)\Theta(\theta)$, then we have $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$. We have the two ODEs,

$$\begin{cases} \Theta'' + \lambda\Theta = 0 \\ r^2 R'' + rR' - \lambda R = 0 \end{cases}$$

Then we have $\Theta'' + \lambda\Theta = 0$ for $\Theta(0) = \Theta(\beta) = 0$. We let $\lambda = \omega^2$, then we solve $\Theta(\theta) = c_1 \cos\omega\theta + c_2 \sin\omega\theta$, then using the boundary condition, we get that $c_1 = 0$, and the second tells us that $c_2 \sin\omega\beta = 0$. Then for a non-trivial solution we have $\sin\omega\beta = 0$ and so $\omega_n = \frac{n\pi}{\beta}$, therefore $\lambda_n = \left(\frac{n\pi}{\beta}\right)^2$. Now we consider R, we get that $r^2R'' + rR' - \lambda R = 0$, we let $p = \ln r$. We then have $\frac{d^2R}{dp^2} - \lambda R = 0$, therefore, $R(p) = c_1 e^{\sqrt{\lambda}p} + c_e^{-\sqrt{\lambda}p}$, then substituting for λ and p, we get,

$$R_n(r) = d_n r^{\frac{n\pi}{\beta}}$$

and we impose the BC at $r = 0$ by removing terms of form r^{-n} and $n \in \mathbb{Z}^+ \setminus \{0\}$. Therefore we get the solution,

$$u(r, \theta) = \sum_{n=1}^{\infty} a_n r^{\frac{n\pi}{\beta}} \sin\frac{n\pi\theta}{\pi}$$

We now impose the other BC,

$$\frac{\partial u}{\partial r} = \sum_{n=1}^{\infty} a_n \frac{n\pi}{\beta} r^{\frac{n\pi}{\beta}-1} \sin\frac{n\pi\theta}{\beta}$$

and now let $r = a$ to get,

$$h(\theta) = \sum_{n=1}^{\infty} a_n \frac{n\pi}{\beta} a^{\frac{n\pi}{\beta}-1} sin\frac{n\pi\theta}{\beta}$$

This is a fourier series in the interval $[0, \beta]$,

$$\frac{n\pi}{\beta} a^{\frac{n\pi}{\beta}-1} a_n = \frac{2}{\beta}\int_0^{\beta} h(\theta)\sin\frac{n\pi\theta}{\beta} d\theta$$

and so,

$$a_n = \frac{2}{n\pi} a^{1-\frac{n\pi}{\beta}} \int_0^{\beta} h(\theta)\sin\frac{n\pi\theta}{\beta} d\theta$$

8.3 Laplace Equation for an annulus

We now want to solve the poisson equation in an annulus. We say $u(b, \theta) = h(\theta)$ and also $u(a, \theta) = g(\theta)$. We consider $u_{xx} + u_{yy} = 0$ for $a^2 < x^2 + y^2 < b^2$ subject to $u = g(\theta)$ for $x^2 + y^2 = a^2$ and $u = h(\theta)$ for $x^2 + y^2 = b^2$. We follow the usual method for solution,

$$u(r, \theta) = \frac{1}{2}(a_0 + b_0 \ln r) + \sum_{n=1}^{\infty}(a_n r^n + b_n r^{-n})\cos n\theta + (c_n r^n + d_n r^{-n})\sin n\theta$$

then we have the terms that have a singularity at $r = 0$ as $r = 0$ is not in the domain of solution.

Exercise. Find the coefficients by letting $r = a$ and $r = b$.

8.4 Laplace Equation on an exterior of circle

Here we have boundary conditions on the boundary of the circle, $u(a,\theta) = h(\theta)$. We consider $\Delta u = 0$ on $x^2 + y^2 > a$. We also impose that the solution is bounded. We know that,

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n} \left(a_n \cos n\theta + b_n \sin n\theta\right)$$

Then,

$$h(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a^{-n} \left(a_n \cos n\theta + b_n \sin n\theta\right)$$

Then we can write this as,

$$a_n = \frac{a^n}{\pi} \int_{-\pi}^{\pi} h(\theta) \cos n\theta d\theta \qquad a_n = \frac{a^n}{\pi} \int_{-\pi}^{\pi} h(\theta) \sin n\theta d\theta$$

9 Eigenfunction Expansion

We recall from vector calculus,

Definition 9.1 (Gauss Divergence Theorem)**.**

$$\iiint \nabla \cdot \mathbf{u} dV = \iint_{\partial V} \mathbf{u} \cdot \mathbf{n} = \iint_{\partial V} \mathbf{u} \cdot d\mathbf{S}$$

Now we let $\mathbf{u} = f\nabla g$, where f, g are scalar differentiable functions. Then we can substitute this in,

$$\iiint_V \nabla \cdot (f\nabla g) dV = \iint_{\partial V} f\nabla g \cdot \mathbf{n} dS$$

but we know that $\nabla \cdot (f\nabla g) = \nabla f \cdot \nabla g + f\Delta g$. We substitute this in and get, greens first identity.

$$\iiint_V$$

Let $\mathbf{u} = g\nabla ff$, then,

$$\nabla f \cdot \nabla g + g\nabla^2 f dV = \iint_{\partial V} g\nabla f \cdot n dS$$

Then we subtract the two and get that,

$$\iiint_V (f\nabla^2 g = g\nabla^2 f) dV = \iint_{\partial V} (f\nabla g - g\nabla f) \cdot \mathbf{n} cS$$

This is Greens second identity.

10 Eigenfunction Expansion

For inhomogeneous PDEs, it's hard to solve the PDE using separation of variables.

We now consider the $\frac{\partial^2 u}{\partial t^2}$ and then consider the BC with Greens second identity,

$$W_n(t) = \frac{2}{L}\int_0^L \frac{\partial^2 u}{\partial x^2}\sin\frac{n\pi x}{L}dx$$

but we know that,

$$\iiint_V f\nabla g - g\nabla f dV = \iint_{\partial V}(f\nabla g - g\nabla f)\cdot\mathbf{n} = \iint_{\partial V} f\frac{\partial g}{\partial \mathbf{n}} - g\frac{\partial f}{\partial \mathbf{n}}.$$

Then using the one dimensional variant,

$$\begin{aligned} W_n(t) &= \frac{2}{L}\int_0^L \frac{\partial^2 u}{\partial x^2}\sin\frac{n\pi x}{L}dx \\ &= \frac{2}{L}\int_0^L g\Delta f dx + \frac{2}{L}\left[f\nabla g - g\nabla f\right]_0^L \\ &= \frac{2}{L}\int_0^L -u(x,t)\left(\frac{n\pi}{L}\right)^2\sin\frac{n\pi x}{L}dx + \left[\frac{2}{L}\sin\frac{n\pi x}{L}\frac{\partial u}{\partial x} - \frac{2}{L}u(x,t)\frac{n\pi}{L}\cos\frac{n\pi x}{L}\right]_0^L. \end{aligned}$$

We now impose the boundary conditions,

$$\begin{aligned} W_n(t) &= \frac{2}{L}\int_0^L -u(x,t)\left(\frac{n\pi}{L}\right)^2\sin\frac{n\pi x}{L}dx + \left[\frac{2}{L}\sin\frac{n\pi x}{L}\frac{\partial u}{\partial x} - \frac{2}{L}u(x,t)\frac{n\pi}{L}\cos\frac{n\pi x}{L}\right]_0^L \\ &= -\frac{2}{L}\int_0^L u(x,t)\left(\frac{n\pi}{L}\right)^2\sin\frac{n\pi x}{L}dx - \frac{2}{L}\left((-1)^n\frac{n\pi}{L}g(t) + \frac{n\pi}{L}h(t)\right). \end{aligned}$$

Let $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, then we can see that,

$$W_n = -\lambda_n u_n(t) + \frac{2}{L}\frac{n\pi}{L}\left(-(-1)^n g(t) + h(t)\right).$$

Then we substitute this into the PDE, then we get that,

$$V_n(t) - kW_n(t) = 0$$

and we now substitute for V_n and W_n, we get,

$$\frac{2}{L}\int_0^L (u_t - ku_{xx})\sin\frac{n\pi x}{L} = 0$$

We can substitute now using $V_n = kW_n$, $\frac{du_n}{dt} = V_n$, then get,

$$\frac{du_n}{dt} = k(-\lambda_n u_n - \frac{2n\pi}{L^2}\left((-1)^n g(t) - h(t)\right))$$

and now we seek to solve this ODE with respect to $u_n(0) = 0$. We use an integrating factor, $e^{\int k\lambda_n dt} = e^{k\lambda_n t}$. This gives,

$$\frac{d}{dt}\left(e^{k\lambda_n t}u_n(t)\right) = \frac{2n\pi k}{L^2}e^{k\lambda_n t}\left((-1)^n g(t) - h(t)\right)$$

Then we get the solution,

$$u_n(t) = Ce^{-k\lambda_n t} - \frac{2n\pi}{L^2}\int_0^t e^{-k\lambda_n(t-s)}((-1)^n g(s) - h(s))ds$$

and considering initial conditions we see that $u_n(0) = 0 = C$, and so,

$$u_n(t) = \frac{2n\pi}{L^2}\int_0^t e^{-k\lambda_n(t-s)}(h(s) - (-1)^n g(s))ds$$

10.1 Inhomogeneous wave problem

We consider $u_{tt} - c^2 u_{xx} = f(x,t)$, where $u(0,t) = g(t)$, $u(L,t) = h(t)$ and the initial conditions $u(x,0) = \Phi(x)$ and $u_t(x,0) = \Psi(x)$. If we have a homogenous wave problem, then using seperation of variables we get the following solution,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi x}{L}$$

Then we will get that,

$$u_n(t) = \frac{2}{L} \int_0^L u(x,t) \sin \frac{n\pi x}{L}$$

If we now expand each of the terms in the PDE we get,

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= \sum_{n=1}^{\infty} V_n(t) \sin \frac{n\pi x}{L} \\
\frac{\partial^2 u}{\partial x^2} &= \sum_{n=1}^{\infty} W_n(t) \sin \frac{n\pi x}{L} \\
f(x,t) &= \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi x}{L} \\
\Phi(x) &= \sum_{n=1}^{\infty} \Phi_n \sin \frac{n\pi x}{L} \\
\Psi(x) &= \sum_{n=1}^{\infty} \Psi_n \sin \frac{n\pi x}{L}
\end{aligned}$$

We now seek an ODE, so we consider V_n first,

$$\begin{aligned}
V_n(t) &= \frac{2}{L} \int_0^L \frac{\partial^2 u}{\partial t^2} \sin \frac{n\pi x}{L} dx = \frac{d^2 u_n}{dt^2} \\
W_n(t) &= \frac{2}{L} \int_0^L \frac{\partial^2 u}{\partial x^2} \sin \frac{n\pi x}{L} \\
&= \frac{2}{L} \int_0^L -u(x,t) \left(\frac{n\pi}{L}\right)^2 \sin \frac{n\pi x}{L} dx + \frac{2}{L} \left[u_x(x,t) \sin \frac{n\pi x}{L} - \frac{n\pi}{L} u(x,t) \cos \frac{n\pi x}{L}\right]_0^L \\
&= -\lambda_n u_n(t) + \frac{2}{L} \left(g(t) - (-1)^n h(t)\right)
\end{aligned}$$

Now we seek an ODE, from the PDE we get that $V_n - c^2 W_n = f_n(t)$, from substituting the expansions into the PDE. Hence we substitute for $V_n(t)$ and $W_n(t)$ in terms of u_n. We then get,

$$\frac{d^2 u_n}{dt^2} + c^2 \lambda_n u_n(t) = \frac{2n\pi}{L^2} c^2 \left(g(t) - (-1)^n h(t)\right) + f_n(t)$$

We also have two initial conditions, $u(0) = \Phi_n$ and $u_t(0) = \Psi_n$. We call the RHS, $S(t) = \frac{2n\pi}{L^2} c^2 \left((-1)^n h(t) - g(t)\right) + f_n(t)$. Therefore,

$$\frac{d^2 u_n}{dt^2} + c^2 \lambda_n u_n(t) = S(t)$$

and so we solve this equation. We solve it using variation of parameters. We assume $u_p = b_1(t) \cos \frac{n\pi ct}{L} + b_2(t) \sin \frac{n\pi ct}{L}$. We can see that $u_h(t) = A \cos \frac{n\pi ct}{L} + B \sin \frac{n\pi ct}{L}$, we now consider $W(t)$,

$$W(t) = \begin{vmatrix} \cos \frac{n\pi ct}{L} & \sin \frac{n\pi ct}{L} \\ -\frac{n\pi ct}{L} \sin \frac{n\pi ct}{L} & \frac{n\pi ct}{L} \cos \frac{n\pi ct}{L} \end{vmatrix} = \frac{n\pi c}{L}$$

Therefore, we get,

$$u_p(t) = \cos\frac{n\pi ct}{L}\int \frac{-L}{n\pi c}\sin\frac{n\pi ct}{L}S(t)dt + \sin\frac{n\pi ct}{L}\int\frac{L}{n\pi c}\cos\frac{n\pi ct}{L}S(t)dt$$

We want to make initial conditions explicit and so we now write,

$$\cos\frac{n\pi ct}{L}\int_0^t -\frac{L}{n\pi c}\sin\frac{n\pi c\tau}{L}S(\tau)d\tau + \sin\frac{n\pi ct}{L}\int_0^t\frac{L}{n\pi c}\cos\frac{n\pi c\tau}{L}S(\tau)d\tau$$

Then we can write it as the following,

$$u_p(t) = \int_0^t \frac{L}{n\pi c}\sin\left(\frac{n\pi c}{L}(t-\tau)\right)S(\tau)dt$$

Now we can write down $S(0)$, and the find the complementary solution to the problem. Hence we add them and get the solution,

$$u_n(t) = A\cos\frac{n\pi ct}{L} + B\sin\frac{n\pi ct}{L} + \frac{L}{n\pi c}\int_0^t \sin\left(\frac{n\pi c}{L}(t-\tau)\right)S(\tau)dt$$

To find A and B, consider the auxillary conditions, $u_n(0) = \Phi_n = A$ and $u_n'(0) = \Psi_n = B\frac{n\pi c}{L}$ and so $B = \frac{2}{n\pi L}\int_0^L \Psi(x)\sin\frac{n\pi x}{L}dx$. Then similarly for $A = \frac{2}{L}\int_0^L\Phi(x)\sin\frac{n\pi x}{L}dx$. Then,

$$\begin{aligned}u_n(t) = \frac{2}{L}\int_0^L \Phi(q)\sin\frac{n\pi q}{L}dq\cos\frac{n\pi ct}{L} + \frac{2}{n\pi c}\int_0^L\Psi(q)\sin\sin\frac{n\pi q}{L}dq\sin\frac{n\pi ct}{L}\\ + \frac{L}{n\pi c}\int_0^t\left[\sin\left(\frac{n\pi c}{L}(t-\tau)\right)\frac{-2n\pi}{L^2}c^2\left((-1)^n h(\tau) - g(\tau)\right) + \frac{2}{L}\int_0^L f(q,\tau)\sin\frac{n\pi q}{L}dq\right]d\tau\end{aligned}$$

10.2 Method of Subtraction - Nonexaminable

We can move inhomogeneous boundary conditions to homogeonous conditions by the method of subtraction. Consider $u_{tt} - c^2u_{xx} = f(x,t)$ where $u(0,t) = g(t)$, $u(L,t) = h(t)$ with initial conditions, $u(x,0) = \Phi(x)$ and $u_t(x,0) = \Psi(x)$. Let $\hat{u}(x,t) = \left(1 - \frac{x}{L}\right)g(t) + \frac{x}{L}h(t)$, where we see that $\hat{u}(0,t) = g(t)$ and $\hat{u}(L,t) = h(t)$. Then let $v(x,t) = u(x,t) - \hat{u}(x,t)$. Then we see that,

$$u_{tt} - c^2u_{xx} = v_{tt} + \hat{u}_{tt} - c^2(v_{xx} + \hat{u}_{xx})$$

This gives us,

$$v_{tt} - c^2v_{xx} + \hat{u}_{tt} = f(x,t)$$

and so,

$$v_{tt} - c^2v_{xx} = f - \hat{u}_{tt}$$

This then gives the boundary condition $v(0,t) = v(L,t) = 0$. We also need the initial conditions, $v(x,0) = u(x,0) - \hat{u}(x,0) = \Phi(x) - \hat{u}(x,0)$ and $v_t(x,0) = \Psi(x) - \hat{u}_t(x,0)$. If the source function on the RHS of the PDE is just a function of x, we would be able to make the PDE homogenous. If we have $u_{tt} - c^2u_{xx} = f(x)$, where $u(0,t) = g(t)$ and $u(L,t) = h$ subject to $u(x,0) = \Phi(x)$ and $u_t(x,0) = \Psi(x)$. We want to make this problem homogenous. We find a solution to the problem where $u(x,t) = \hat{u}(x)$. Then we get that $-c^2u_{xx} = f(x)$ where $\hat{u}(0) = g$ and $\hat{u}(L) = h$. Now let $v(x,t) = u(x,t) - \hat{u}(x)$. Then $v(t)$ solves the PDE problem with zero RHS and zero BC. From the new variable we conclude that $u(x,t) = v(x,t) - \hat{u}$, we get that,

$$u_{tt} - c^2u_{xx} = v_{tt} - c^2(v_{xx} + \hat{u}_{xx}) = f(x)$$

which we can write as,

$$v_{tt} = c^2 v_{tt} = c^2 \hat{u}_{xx} + f(x) = 0$$

Now we can show that $u(0,t) = v(0,t) + \hat{u}(0) = g$ and so $v(0,t) = 0$. That same thing is true for $u(L,t) = v(L,t) + \hat{u}(L) = h$ and so $v(L,t) = 0$. Then $u(x,0) = v(x,0) + \hat{u}(x) = \Phi(x)$ and so $v(x,0) = \Phi(x) - \hat{u}(x)$ and we look to $u_t(x,0) = v_t(x,0) = \Psi(x)$, then we get the problem,

$$v_{tt} - c^2 u_{tt} = 0$$

where $v(0,t) = 0$, $v(L,t) = 0$ subject to $v(x,0) = \Phi(x) - \hat{u}(x)$ and $v_t(x,0) = \Psi(x)$.

10.3 Seperation of variables in higher dimensions - Nonexaminable

We cosider the wave equations in the place and in space and the Helmholtz equation. We have a higher order wave equation,

$$u_{tt} = c^2 \Delta u$$

which just just the laplacian of u, then for diffusion we have,

$$u_t = k\Delta u$$

We aim to see what happens in higher variables. Assume we are in $\mathbb{R}^3$, that is we seek a solution of the form $u(x,y,z,t)$. We aim to find $v(x,y,z)T(t) = v(\mathbf{x})T(t)$. We substitute this back into the wave equation PDE we get,

$$v(\mathbf{x})T''(t) - c^2 T(t)\Delta v(\mathbf{x}) = 0$$

Now we divide through and get,

$$\frac{T''}{T} = c^2 \frac{\Delta v(\mathbf{x})}{v(\mathbf{x})} = \lambda$$

and for diffusions we get something similar,

$$\frac{T'}{kT} = \frac{\Delta v}{v} = -\lambda$$

We now seek to find the ODEs, and get the Helmholtz equation,

$$\Delta v + \lambda v = 0.$$

This satisfies one of the BC we have found on ∂D. We now seek to solve the Helmholtz equation. This is an eigenvalue problem with eigenvalues λ. If we call the eigenvalues of the problem λ_n and eigenfunctions $v_n(\mathbf{x}) = v(x,y,z)$. Then we interested in solving the second ODE to form our series solution. This equation solves to give us a series solution of,

$$u(\mathbf{x},t) = \sum_n \left(A_n \cos\sqrt{\lambda}_n ct + B_n \sin\sqrt{\lambda}_n ct \right) v_n(\mathbf{x})$$

Then for diffusions we get,

$$u(\mathbf{x},t) = \sum_n A_n e^{-\lambda_n kt} v_n(\mathbf{x})$$

Now we consider $\langle f\,, g\rangle$,

$$\begin{aligned}\langle f\,, g\rangle &= \iiint_D f(\mathbf{x})\overline{g(x)}d\mathbf{x} \\ &= \iiint_D f(\mathbf{x})g(x)d\mathbf{x}\end{aligned}$$

Now we use the Greens second identity,

$$\begin{aligned}\iiint_D u\Delta v - v\Delta u d\mathbf{x} &= \iint_{\partial D} (u\nabla v - v\nabla u)\cdot \mathbf{n} ds \\ &= \iint_{\partial D} u\frac{\partial v}{\partial \mathbf{n}} - v\frac{\partial u}{\partial \mathbf{n}} ds\end{aligned}$$

Then with homogenous boundary conditions, we then get that the RHS is zero,

$$\iiint_D u\Delta v - v\Delta u d\mathbf{x} = 0$$

Now using Helmholtz equation we have that $-\Delta u = \lambda_1 u$ and $-\Delta v = \lambda_2 v$ in D. Where u and v satisfy some boundary condition on ∂D. Therefore,

$$\iiint_V u\Delta v - v\Delta u d\mathbf{x} = \iiint_D -uv(\lambda_1 - \lambda_2) d\mathbf{x}$$

Therefore, $(\lambda_1 - \lambda_2)\langle u\,, v\rangle = 0$. Therefore, they are both orthogonal. Now we know that $\Phi(x) = \sum_n A_n v_n(\mathbf{x})$ where $A_n = \frac{\langle \Phi(x)\,, v_n(\mathbf{x})\rangle}{\langle v_n(\mathbf{x})\,, v_n(\mathbf{x})\rangle}$. this is then,

$$A_n = \frac{\iiint_D \Phi(x) v_n(\mathbf{x}) d\mathbf{x}}{\iiint_D v_n^2(\mathbf{x}) d\mathbf{x}}$$

Example. Consider the 3D heat equation, $u_t - k(u_{xx} + u_{yy} + u_{zz})$, then we can solve this with the solution,

$$u(\mathbf{x}, t) = \sum_\ell \sum_m \sum_n A_{mn\ell} \sin nx \sin my \sin \ell z$$

with coefficient,

$$A_{mn\ell} = \frac{\int_0^\pi \int_0^\pi \int_0^\pi \Phi(x) \sin nx \sin my \sin \ell z dx dy dz}{\int_0^\pi \int_0^\pi \int_0^\pi \sin^2 nx \sin^2 my \sin^2 \ell z dx dy dz}$$

which is just,

$$A_{mn\ell} \left(\frac{2}{\pi}\right)^3 \int_0^\pi \int_0^\pi \int_0^\pi \Phi(x) \sin nx \sin my \sin \ell z dx dy dz$$

11 Generalised functions and Green's functions

Assume we have $A\mathbf{u} = f$, where $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $f = (f_1, f_2, \ldots, f_n)$. We know $f \in \mathbb{R}^n$ and so we can write it as, $\mathbf{f} = f_1\mathbf{e}_1 + f_2\mathbf{e}_2 + \cdots + f_n\mathbf{e}_n$ and so we can write our original problem as $A\mathbf{u}_j = \mathbf{e}_j$, and get a solution $\mathbf{u} = f_1\mathbf{u}_1 + f_2\mathbf{u}_2 + \cdots + f_n\mathbf{u}_n$. This is the idea of Greens function at discrete level.

11.1 Delta Function

A unit impulse function at position $a < \xi < b$ will be described by the delta function, denoted by $\delta_\xi(x)$. We say $\delta_\xi(x) = 0$ if $x \neq \xi$. In addition,

$$\int_a^b \delta_\xi(x)dx = 0 \qquad a < x < b$$

We can define this function in two ways,

11.1.1 Limit Method

Regard, the delta function $\delta_\xi(x)$ as a limit of a sequence of ordinary function $g_n(x)$. This represents progressively more and more concentrated unit forces, in the limit , these converge to the desired limit impulse concentrated at $x = \xi$. That is,

$$\lim_{n\to\infty} g_n(x) = 0 \qquad x \neq \xi$$

and $\int_a^b g_n(x) = 1$ for all n. We could take the example of $g_n(x) = \frac{n}{\pi(1+n^2x^2)}$. This gives us $\delta_0(x)$.

11.1.2 Duality Method

The critical property is that if $u(x)$ is any continuous function, then

$$\int_a^b \delta_\xi(x)dx = u(\xi) \qquad a < \xi < b$$

This is the same as taking a linear functional, $L_\xi[u] = \langle \delta_\xi(x)\, , u(x)\rangle = u(\xi)$. This serves to define the linear functional $L_\xi : \mathcal{C}^0[a,b] \to \mathbb{R}$ that maps a continuous function to it's value at $x = \xi$.

We consider,

$$\begin{aligned} u(\xi) &= \lim_{n\to\infty} \langle g_n(x)\, , u\rangle \\ &= \lim_{n\to\infty} \int_a^n g_n(x)u(x)dx \\ &= \int_a^b \delta_\xi(x)u(x) \\ &= \langle \delta_\xi(x)\, , u(x)\rangle \end{aligned}$$

If we let $h(x) = 2\delta(x) - 3\delta(x-1)$, then we note that $L_h[u] = 2u(0) - 3u(1)$.

11.2 Integral of delta function

The integral of the delta function is the unit step function,

$$\int_a^x \delta_\xi(t)dt = \begin{cases} 0 & x < \xi \\ 1 & x > \xi \end{cases} = \sigma(x - \xi)$$

Furthere more the integral of the step function is a ramp function,

$$\int_a^x \sigma(t)dt = \sigma_\xi(x) = \begin{cases} 0 & x < \xi \\ x - \xi & x > \xi \end{cases}$$

11.3 Fourier Transforms

a fourier transform is the limiting case of a fourier series. Let $f(x)$ is a piecewise continuous function defined for all $-\infty < x < \infty$. Then,

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ik\frac{\pi x}{\ell}}$$

where,

$$c_k = \int_{-\ell}^{\ell} f(x) e^{-ik\frac{\pi x}{\ell}} dx$$

This function decays as $|x| \to \infty$. It's Fourier transform $\hat{f}(k)$ is defined by,

$$\mathcal{F}(f(k)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx = \hat{f}(k)$$

Then we can do the inverse,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dx$$

The Fourier transform and it's inverse define linear operators on function space.

Theorem 11.1. If the Fourier transform of the function $f(x)$ is $\hat{f}(x)$, then the Fourier transform of $\hat{f}(x)$ is $f(-k)$.

Proof. Something something proof □

Theorem 11.2. If $f(x) \to \hat{f}(k)$, then we want to show $f'(x) \mapsto (ik)f'(k)$

Proof. We know that,

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

and the inverse is,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk$$

Then we can differentiate,

$$f'(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ik\hat{f}(k) e^{ikx} dk = ik\hat{f}(k)$$

□

Similarly, we have the Fourier Transform of product function $xf(x)$, obtained by differentiating the Fourier transform of $f(x)$. Hence we want to show,

$$\mathcal{F}(xf(x)) = i\frac{d\hat{f}}{dk}$$

We start with the definition of $\hat{f}(k)$ and differentiate,

$$i\frac{d\hat{f}}{dk} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} xf(x) e^{-ikx} dx = \mathcal{F}(xf(x))$$

Corollary 11.3. The Fourier Transform of $f^{(n)}(x)$ is $(ik)^n \hat{f}(x)$.

11.4 Solution of boundary-value problem using Fourier Transform

We consider $-\frac{d^2u}{dx^2}+\omega^2u=h(x)$ for $-\infty<x<\infty$ and $\omega>0$. We then take Fourier transforms,

$$\mathcal{F}\left[-\frac{d^2u}{dx^2}+\omega^2u\right]=\mathcal{F}[h(x)]$$

Then this becomes,

$$-(ik)^3\hat{u}(k)+\omega^2\hat{u}(k)=\hat{h}(k)$$

That is,

$$\hat{u}(k)=\frac{\hat{h}(k)}{k^2+\omega^2}$$

Therefore, we have,

$$u(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\hat{h}(k)}{k^2+\omega^2}e^{ikx}dk$$

For an example consider $h(x)=e^{|x|}$, then we have $\hat{h}(k)=\frac{1}{\sqrt{2\pi}}\frac{1}{k^2+1}$. Therefore, we get,

$$u(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{ikx}}{(1+k^2)(k^2+\omega^2)}dk$$

We now use partial fractions,

$$u(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\frac{1}{\omega^2-1}\left(\frac{1}{k^2+1}-\frac{1}{k^2+\omega^2}\right)e^{ikx}dk$$

Then using inverse Fourier transforms, we can get that,

$$u(x)=\frac{e^{|x|}}{\omega^2-1}+\frac{e^{\omega|x|}}{\omega(\omega^2-1)}$$

11.5 Green's Function

We replace $h(x)=\delta_\xi(x)=\delta(x-\xi)$. We had that $\hat{u}(k)=\frac{\hat{h}(k)}{k^2+\omega^2}$. If we replace h with the delta function we can get Greens function. Hence we have $\hat{G}(k;\xi)=\frac{1}{\sqrt{2\pi}}\frac{e^{ik\xi}}{k^2+\omega^2}$. We now want to find $\hat{G}(x;\xi)$. We have alreadt see that $\mathcal{F}[f(x-\xi)]=e^{-ik\xi}\hat{f}(k)$. We also know that $\mathcal{F}[e^{a|x|}]=\sqrt{\frac{2}{\pi}}\frac{\omega}{\omega^2+k^2}$. Hence we get that,

$$\mathcal{F}[e^{-\omega|x-\xi|}]=\sqrt{\frac{2}{\pi}}\frac{\omega e^{-ik\xi}}{k^2+\omega^2}$$

Therefore, we can find that $\hat{G}(k;\xi)$ transforms to $\hat{G}(x;\xi)=\frac{1}{2\omega}e^{-\omega|x-\xi|}$. We also know that,

$$h(x)=\int_{-\infty}^{\infty}\delta(x-\xi)h(\xi)d\xi$$

Then the superposition principle based upon Green's function implies that the solution to the inhomogeneous boundary-value problem,

$$-u''+\omega^2u=h(x)\qquad -\infty<x<\infty\quad \omega>0$$

Under general forcing function $h(x)$ takes the following form,

$$u(x)=\int_{-\infty}^{\infty}G(x;\xi)h(\xi)d\xi=\int_{-\infty}^{\infty}\frac{1}{2\pi}e^{-\omega|x-\xi|}h(\xi)d\xi$$

Definition 11.4 (Convolution)**.** If we have a function $h(x) = f * g = \int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi$ is the convolution.

Here are the properties of convolutions,

1. Symmetry, $f * g = g * f$,
2. Blinearity, $f * (ag + bh) = af * g + bf * h$ and $(af + bg) * h = af * h + bg * h$,
3. Associativity, $f * (g * h) = (f * g) * h$,
4. Zero function, $f * 0 = 0$,
5. Delta function $f * \delta = \delta * f = f$.

We now want to show that $\mathcal{F}[f * g(x)] = \sqrt{2\pi}\hat{f}(k)\hat{g}(k)$. We start with the definition of Fourier transform,

$$\begin{aligned}\mathcal{F}[f * g(x)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f * g(x)e^{-ikx}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi)g(\xi)e^{-ikx}dxd\xi \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\eta)g(\xi)e^{-k(\xi+\eta)}d\xi d\eta \\ &= \sqrt{2\pi}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\eta)e^{-ik\eta}d\eta\right)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\xi)e^{-ik\xi}d\xi\right) = \sqrt{2\pi}\hat{f}(k)\hat{g}(k)\end{aligned}$$

We now look at the Fourier transform of $f(x)g(x)$, up to multiple, is the convolution of the Fourier transforms of f and g,

$$\mathcal{F}[f(x)g(x)] = \frac{1}{\sqrt{2\pi}}\hat{f} * \hat{g}(k)$$

We can use a similar argument to before, we know that,

$$\begin{aligned}fg(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbb{F}[f(k)g(k)]e^{ikx}dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(k-\xi)\hat{g}(\xi)d\xi e^{ikx}dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\eta)\hat{g}(\xi)e^{ik\eta}e^{ik\xi}d\eta d\xi \\ &= \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\eta)e^{ik\eta}d\eta\right)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{g}(\xi)e^{ik\xi}d\xi\right)\end{aligned}$$

11.6 The Fundamental Solution to the diffusion equation

The problem of interest is $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$ for $-\infty < x < \infty$ subject to $u(x,0) = f(x)$ for $-\infty < x < \infty$ and $u(x,t) \to 0$ in the limit as $x \to \pm\infty$. Apply Fourier transform in the x direction, $\mathcal{F}[u(x,t)] = \hat{u}(k,t)$,

$$\frac{\partial \hat{u}}{\partial t} + k^2\hat{u}(k,t) = 0$$

We can consider the IC, and see,

$$u(k,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,0)e^{-ikx}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx = \hat{f}(x)$$

and so now we can solve the first order ODE.

$$\hat{u}(k,t) = \hat{f}(k)e^{-k^2 t}$$

We now can write the following,

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx - k^2 t} dk$$

The idea now is to replace $f(x)$ with $\delta(\xi) = \delta(x - \xi)$ the use linear superposition theorem. We know,

$$\hat{\delta}_\xi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x-\xi)e^{-ikx}dx = \frac{1}{\sqrt{2\pi}} g(\xi) = \frac{1}{\sqrt{2\pi}} e^{-ik\xi}$$

We will call G by F to let it align with current literature. Therefore,

$$F(x;\xi,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-\xi) - k^2 t} dk$$

We aim now to find some function,

$$\mathcal{F}[\qquad] = \frac{1}{2\pi} e^{-ik\xi - k^2 t}$$

We know that

$$\mathcal{F}[e^{-ax^2}] = \frac{e^{-\frac{k^2}{4a}}}{\sqrt{2a}}$$

Now if we take $a = \frac{1}{4t}$, then we can use the above transform,

$$\mathbb{F}[e^{-\frac{x^2}{4t}}] = \sqrt{2t}e^{-k^2 t}$$

and so we consider a transformed function and find,

$$\mathcal{F}[e^{-\frac{(x-\xi)^2}{4t}}] = \sqrt{2t}e^{-ik\xi - k^2 t}$$

Therefore,

$$\mathbb{F}[\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2t}} e^{-\frac{(x-\xi)^2}{4t}}] = F(k;\xi,t)$$

Therefore,

$$F(x;\xi,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-\xi)^2}{4t}}$$

We now aim to use linear superposition theorem. We then get,

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4t}} f(\xi) d\xi$$

11.7 The forced heat /diffusion equation and Dyhamel's principle

Assume we have a forced diffusion equation,

$$\begin{cases} u_t - u_{xx} = h(x,t) \\ u(x,0) = 0 \end{cases}$$

Consider $h(x,t) = \delta(x-\xi)\delta(x-\tau)$. This will yield the general fundamental solution, which is actually Greens function. We use the critical property of the delta function,

$$h(x,t) = \int_0^\infty \int_{-\infty}^\infty \delta(x-\xi)\delta(t-\tau)h(\xi,\tau)d\xi d\tau$$

Then we can find the Greens function, $G(x,t;\xi,\tau)$ and so we get the solution,

$$u(x,t) = \int_0^t \int_\infty^\infty G(x,t;\xi,\tau)h(\xi,\tau)d\xi d\tau$$

Then if we have $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = h(x,t)$ where $u(x,0) = f(x)$. We can decompose this problem into two parts.

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = h(x,t) \\ u(x,0) = 0 \end{cases} \qquad + \qquad \begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \\ u(x,0) = f(x) \end{cases}$$

Then we can find the solution to the original problem, by summing them. We know already that we have,

$$u(x,t) = \int_a^b F(x,t;\xi)f(\xi)d\xi + \int_0^t \int_a^b G(x,t;\xi,\tau)h(\xi,\tau)d\xi d\tau$$

We shall now derive the second term. Our problem is,

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = h(x,t) \\ u(x,0) = 0 \end{cases}$$

We use Fourier transforms and get,

$$\frac{\partial \hat{u}}{\partial t} + k^2\hat{u} = \hat{h}(k,t)$$

Now we have an ODE, but we want to replace the right hand side with a delta function. Let $\hat{h}(k,t) = \hat{\delta}(x-\xi)\delta(t-\tau) = \frac{1}{\sqrt{2\pi}}e^{ik\xi}\delta(t-\tau)$. Therefore we have,

$$\frac{\partial \hat{u}}{\partial t} + k^2\hat{u} = \frac{1}{\sqrt{2\pi}}e^{ik\xi}\delta(t-\tau)$$

We want to solve this subject to $\hat{u}(k,0) = 0$. We can solve and see that,

$$\frac{\partial}{\partial t}\left(e^{k^2t}\hat{u}\right) = \frac{1}{\sqrt{2\pi}}e^{ik\xi+k^2t}\delta(t-\tau)$$

Now we integrate from 0 to t, we get a solution of this form,

$$\hat{u}(k,t) = \frac{1}{\sqrt{2\pi}}e^{-k^2(t-\tau)-ik\xi}\sigma(t-\tau)$$

Then we have,

$$u(x,t) = G(x,t;\xi,\tau) = \frac{\sigma(t-\tau)}{2\pi}\int_{-\infty}^\infty e^{-k^2(t-\tau)+ik(x-\xi)}dk$$

This is then actually,

$$u(x,t) = \frac{\sigma(t-\tau)}{2\sqrt{\pi(t-\tau)}}e^{-\frac{(x-\xi)^2}{4(t-\tau)}} = \sigma(t-\tau)F(x,t-\tau;\xi)$$

This is called Duhamels principle. That is, the general fundemental solution is obtained by translating the fundemental solution $F(x,t;\xi)$ for the initial-value problem to a starting time of $t=\tau$ instead of $t=0$. Applying linear superposition theorem,

$$u(x,t) = \int_0^t \int_{-\infty}^\infty \frac{h(\tau,\xi)}{2\sqrt{\pi(t-\tau)}}e^{-\frac{(x-\xi)^2}{4(t-\tau)}}$$

11.8 The source function for the wave equation

In the literature, Greens function for the wave equation is called the source function. The problem of interest is,

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \\ u(x,0) = 0 \\ u_t(x,0) = g(x) \end{cases}$$

We consider the source function S this satisfies,

$$\begin{cases} \frac{\partial^2 S}{\partial t^2} - c^2 \frac{\partial^2 S}{\partial x^2} = 0 \\ S(x,0) = 0 \\ S_t(x,0) = \delta(x) \end{cases}$$

We take fourier transforms of this problem and see,

$$\frac{\partial^2 S}{\partial t^2} + c^2 k^2 \hat{S} = 0$$

subject to $\hat{S}(k,0) = 0$ and $\hat{S}_t(k,0) = \frac{1}{\sqrt{2\pi}}$. The solution to this second order ODE is,

$$\hat{S}(k,t) = A\cos ckt + B\sin ckt$$

Then using the initial conditions, we see that $\hat{S}(k,0) = A = 0$ and $\hat{S}_t(k,0) = B = \frac{1}{\sqrt{2\pi}}$. Hence we have,

$$\hat{S}(k,t) = \frac{1}{\sqrt{2\pi}} \sin ckt = \frac{e^{ickt} - e^{-ickt}}{2\sqrt{2\pi} ick}$$

Hence we can use the inverse Fourier transform,

$$S(x,t) = \int_{-\infty}^{\infty} \frac{e^{ik(x+ct)} - e^{-ik(x-ct)}}{4\pi ick}$$

We know that $\mathcal{F}[\text{sgn}(x)] = \sqrt{\frac{2}{\pi}}\frac{1}{ik}$. Therefore we can say $\mathcal{F}[\text{sgn}(x+ct)] = \sqrt{\frac{2}{\pi}}\frac{1}{ik}e^{ikct}$ and $\mathcal{F}[\text{sgn}(x-ct)] = \sqrt{\frac{2}{\pi}}\frac{1}{ik}e^{-ikct}$. We can now look at the Fourier transforms,

$$\mathcal{F}\left[\frac{\text{sgn}(x+ct) - \text{sgn}(x-ct)}{4c}\right] = \frac{e^{ikct} - e^{-ikct}}{4\pi ick}$$

Therefore, we say that,

$$S(x,t) = \frac{1}{4c}\left(\text{sgn}(x+ct) - \text{sgn}(x-ct)\right)$$

If we let $\delta(x)$ now be $\delta(x-\xi)$, then we get that $\hat{S}(k,0) = \frac{1}{\sqrt{2\pi}}e^{-ik\xi}$. Hence we get,

$$\hat{S}(k,t) = \frac{e^{ik(ct-\xi) - e^{-ik(x+\xi)}}}{2\sqrt{2\pi} ick}$$

and so we get that,

$$S(x;\xi,t) = \frac{1}{4c}\left(\text{sgn}(x-ct-\xi) + \text{sgn}(x-ct-\xi)\right)$$

Now we apply linear superposition theorem and get,

$$u(x,t) = \int_{\infty}^{\infty} S(x;\xi,t) g(\xi) d\xi$$

Then we can rewrite this as,

$$u(x,t) = \int_{x-ct}^{x+ct} \frac{1}{2c} g(\xi) d\xi$$

Assume we have, $u_{tt} - c^2 u_{xx} = 0$ subject to $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$. We can now solve this similarly to before!