First Order Linear ODEs

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Contents

1 Recap of previous modules

A differentiable equation is an mathematical relation involving a derivative of a dependant variable w.r.t. single/many independant variables

1.1 Notation:

$$\frac{dx}{dt} = \dot{x}, \qquad \frac{dy}{dx} = y', \qquad f\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}\right) = 0$$

$$f_{xy} = \frac{\partial^f}{\partial x \partial y}$$

1.2 Prerequisites:

Integration, ODEs and PDEs from Advanced Calculus.

2 Basic Defintions and Concepts

2.1 Classifications

1. Use of full or partial derivative
2. Coeeficients are functions of independant variables only / contant, otherwise non Linear
3. The highest derivative is the order of the description DE
4. Degree is the highest derivative in rationalised form
5. An explicit solution is; $f = F(x, y, z, t)$ and implicit solution; $F(f, x, y, z, t) = 0$
6. Initial Value Problem (time) or boundary value problem (space).

2.2 Review of integration methods

1. List of commonly used integrals (link)
2. Polynomials, logarithms, trigonometric, inverse, hyperbolic and inverse hyperbolic trig.

2.3 Concepts

1. Given a DE, we want a solution.
2. A solution is a derived relation between the dependant and independant variables without any derivative term and defined in the iterval / domain / region.
3. replacing the solution within the domain satisfies the description DE
4. Needs integration on one or two variables
5. Not always analytical and closed forms possible. (could use numerical integration, iterative solution schemes.)

1. Linear higher order ODEs (Laplace transforms)
   (a) transforms linear ODEs in algebraic forms
   (b) needs table of laplace, inverse laplace formula
2. the Geometric meaning is the slope of $y(x)$: $y'(x) = f(x_0, y_0)$ implies at a point (x_0, y_0) is the slope at $\frac{dy}{dx}$
3. There is something called a direction field that we can use to visualise the DE without solving it.
4. Curves of equal inclination $f(x, y) = c$ along which derivative is constant. Lots of paralell lines.
5. Limitation of direction field give an overall idea about the solution but have limited accuracy
6. Orthoganal trajectories are a family of courves that intersect another family of curves at right angles. For a curve $G(x, y, c)$ firstly final out $\frac{dy}{dx} = f(x, y)$. General solution of the orthoganal trajector $\frac{dy}{dx} = \frac{-1}{f(x,y)}$.
7. existence is under what condition there is at least one Solution
8. uniqueness is what condition it has at most one Solution item the general solution contains the constants of integration
9. particular solution are when you ise the initial / boundary conditions

2.4 Famous models

- Van der Pol oscillator

$$\dot{x} = y$$
$$\dot{y} = \mu(1 - x^2)y - x$$

- Lorentz Attractor

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(\rho - z) - y$$
$$\dot{z} = xy - \beta z$$

Have to be careful between phase portrait and direction fields. Phase portraits are almost a guess and direction field as you have a solution at every point with a direction field.

3 Analytical Solutions

3.1 Seperation of variables

$$g(y)y' = f(x)$$
$$\int g(y)dy = \int f(x)dx + c$$

3.2 Reduction to seperable form

$$y' = f(\frac{y}{x})$$
$$v + x\frac{dv}{dx} = f(v) \qquad \text{by letting } y = vx$$
$$\int \frac{dv}{f(v) - v} = \int \frac{dx}{x} + c$$
$$= \ln|x| + c$$

3.3 Exact ODEs and integrating factors

Let us have an ODE: $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ which can be written as $M(x,y)dx + N(x,y)dy$. We can then write a total differential as partial derivatives:

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y} = 0 \implies u(x,y) = c \qquad (1)$$

Which then we can compare the two and we get:

$$M = \frac{\partial u}{\partial x}, \qquad N = \frac{\partial u}{\partial y}$$

$$\implies \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}, \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

So then we can say for an ODE to be exact:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

To solve the DE, take (1) and integrate with respect to u:

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y} = 0 \qquad (1)$$
$$u = \int M dx + K(y) = c$$
$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\int M dx\right] + \frac{dK(y)}{dy} = N$$
$$\implies \frac{dK}{dy} = N - \frac{\partial}{\partial y}\left[\int M dx\right]$$
$$\implies k = \int \left[N - \frac{\partial}{\partial y}\left[\int M dx\right]\right] dy$$

From this we can substitute $k(y)$ back in and get the general solution of an exact ODE:

$$u(x,y) = \int M dx + \int \left[N + \frac{\partial}{\partial y}\left[\int M dx\right]\right] dy$$

3.4 Reduction to Integrating Factors

We have a $P(x,y)dx + Q(x,y)dy = 0$ can be moved into $F \cdot Pdx + F \cdot Qdx = 0$ and is exact. So we hope that:

$$\frac{\partial}{\partial y}(FP) = \frac{\partial}{\partial y}(FQ)$$
$$\implies \frac{\partial F}{\partial y}P + \frac{\partial P}{\partial y} = \frac{\partial F}{\partial x}Q + F\frac{\partial Q}{\partial x}$$

Now let, $F = F(x)$ only, then we can say that $\frac{\partial F}{\partial y} = 0$ and $\frac{\partial F}{\partial x} = \frac{dF}{dx}$

$$\therefore \frac{dF}{dx} = \frac{F}{Q}\left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right] = F \cdot R$$

where $R = \frac{1}{Q}\left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right]$ and we can solve nicely as:

$$F = Ke^{\int R dx}$$

R must be of x only, not x and y or both. We can do the reverse with y, the maths is the same, but the R this time, which denote as R^*:

$$R^* = \frac{1}{P}\left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]$$

4 Linear ODEs

First order linear ODEs:

$$\frac{dy}{dx} + p(x)y = r(x)$$

Linear Homogenous ODEs if $r(x) = 0$:

$$\frac{dy}{dx} + p(x)y = 0$$

which has a solution:

$$y(x) = ce^{-\int p(x)dx}$$

> **Example 1** *Solve $x^3dx + y^3dy$*
>
> **Solution 1** *Firstly let us test for exactness. Our $M = x^3$ and $N = y^3$. The test says that the DE is exact if we know that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. We know that from a differential:*
>
> $$\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x} = 0$$
>
> *This means that the equation is exact, so we can now plug this into the general solution to an exact equation:*
>
> $$u(x,y) = \int Mdx + \int\left[N + \frac{\partial}{\partial y}\left[\int Mdx\right]\right]dy$$
>
> *and we get the solution:*
>
> $$u(x,y) = \frac{1}{4}x^4 + \frac{1}{4}y^4 + C$$

4.1 Nonhomogeneous Linear ODEs

When $r(x) \neq 0$ in $\frac{d^2y}{dx^2} + p(x)y = r(x)$. We can then do the following:

$$\begin{aligned} \implies dy + pydx &= rdx \\ dy + (py - r)dx &= 0 \\ (py - r)dx = dy &= 0 \end{aligned}$$

Now we can look at the integrating factor, which we know is of form:

$$R = \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]$$

which we can sub in $M = py - r$ and $N = 1$ and we get:

$$R = 1 \cdot (p - 0) = p(x)$$

So for a non-homogenous linear ODE, the integrating factor is $p(x)$.

5 Non-linear ODE

We can take a bernoulli equation:

$$\frac{d^2y}{dx^2} + p(x)y = g(x) \cdot y^n$$

which we can write it as:

$$\frac{d^2y}{dx^2} = gy^n - py$$

The equation is linear for $n = 0, 1$, so we shall consider it without this condition. Firstly we are going to apply a substitution, $u(x) = y(x)^{1-n}$. Working it through we get:

$$\begin{aligned} \frac{du}{dx} &= (1-n)\left[y(x)\right]^{-n}\frac{d^2y}{dx^2} \\ &= (1-n)y^{-n}(gy^n - py) \\ &= (1-n)(g - py^{1-n}) \\ &= (1-n)(g - pu) \end{aligned}$$

Which is linear and solvable.

6 Existence and Uniqueness

An initial value problem $\frac{d^2y}{dx^2} = f(x,y), \quad y(x_0) = y_0$ has:

- no Solution
- precisely one solution
- many solutions

Defintion 1 *Existence: Under what condition the IVP has at least one solution.*

More formally: A function $f(x, y)$ is continuous in some bounded rectangle R, $R : |x - x_0| < a$, $|y - y_0| < b$ there exists some K, such that $|f(x, y)| \leq K \quad \forall (x, y) \in R$ then the IVP exists.

Defintion 2 *Uniqueness: Under what conditions the IVP has at most one solution.*

More formally: If the function and its partial derivatives f, f_y are continuous in R: $|f(x, y)| \leq K |f_y(x, y)| \leq M, \quad \forall (x, y) \in R$