Algebraic Topology

Part III prepatory notes based on Algebraic Topology by Allen Hatcher

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Part III requires a load of complex and interesting Mathematics that I have not covered thus far in my course and will not. These collections of notes are my thoughts on topics and needed intuition to bits of Mathematics. They aren't here to teach or to be used a stand alone text.

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0 Underlying Notions

We are going to adopt the convention that every map is continuous unless otherwise stated.

We want to be able to talk about ways to transform some shapes into other shapes and think about them then being equal. We do this by considering an $I \in [0,1]$ that we can paramaterise a function by. So if we have two curves $s : X \to X$ and $t : X \to X$, we can define some $f : X \times I \to X$ such that,

$$\begin{cases} f_0(x) &= s(x) \\ f_1(x) &= t(x) \end{cases}$$

Definition 0.1 (Deformation Retraction)**.** A Deformation retraction of a space X onto an $A \subset X$ is a family of maps, $f_t : X \to X$ $t \in I$, such that, $f_0 = \mathbb{1}$, $f_t(X) = A$ and $f_t|A = \mathbb{1}$ $\forall t$. The family f_t should be continuous in the sense that the associated map $X \times I \to X$, $(x,t) \mapsto f_t(x)$ is continuous.

Definition 0.2 (Mapping Cylinder)**.** A mapping cylinder, M_f, for a $f : X \to Y$ is the quotient space of the disjoint union $(X \times I) \coprod Y$, obtained by identifying each $(x, 1) \in X \times I$ with $f(x) \in Y$.

A mapping cylinder deformation, of f, retracts to Y.

A deformation retraction $f_t : X \to X$ is just a special case of a homotopy, which is just simply any family of maps, $f_t : X \times I \to Y$, given by $F(x,t) = f_t(x)$ is continuous.

Definition 0.3 (Homotopic)**.** One says two maps are homotopic if there exists a homotopy f_t connecting them and one writes $f_0 \simeq f_1$

Retractions are the topological analogue of projection operators.

Definition 0.4 (Homotopy Relative)**.** A homotopy $f_t : X \to Y$ whose restriction to a subspace $A \subset X$ is independent of t is homotopy relative, written as homotopy rel.

Definition 0.5 (Homotopy Equivalence)**.** A map $f : X \to Y$ is called a homotopy equivalence if there exists a $g : Y \to X$ such that $fg \simeq \mathbb{1}$ and $gf \simeq \mathbb{1}$.

Moreover, we can now say that X and Y are homotopy equivalent or have the same homotopy type.

Lemma 0.6. Any two spaces X and Y are homotopy equivilent if there exists some other space Z containing both X and Y are deformation retracts.

Definition 0.7 (Contractible)**.** A space that has the homotopy type of a point is contractible.

This amounts to the identity map of this space to be nullhomotopic, that is homotopic to a constant map.

0.1 Cell Complexes

To construct a cell complex, let us follow these rules,

(i) Start with a discrete set X^0, whose points are regarded as 0-cells.

(ii) inductively, form the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via maps $\phi_\alpha : S^{n-1} \to X^{n-1}$ with a collection of n-disks D^n_α under the identifications $x \sim \phi_\alpha(x)$ for $x \in \partial D^n_\alpha$. Thus $X^n = X^{n-1} \coprod_\alpha e^n_\alpha$.

(iii) We can either stop at a finite amount or continue indefinitely. in the latter, X is given a weak topology, A set $A \subset X$ is open if and only if $A \cap X^n$ is open in X^n for each n. This similarly works for closed.

X is a cell complex or a CW complex. If $X = X^n$ then X is said to be finite dimensional and the smallest such n is the dimension of X, the maximum dimension of cells of X.

Notation. e^n denotes an n-cell.

Definition 0.8 (Characteristic Map)**.** Each cell e^n_α in a cell complex X has a characteristic map $\Phi_\alpha : D^n_\alpha \to X$ which extends ϕ_α and is a homeomorphism from the interior of D^n_α onto e^n_α. Namely, we can take ϕ_α to be the composition, $D^n_\alpha \hookrightarrow X^{n-1} \coprod_\alpha D^n_\alpha X^n \hookrightarrow X^n$, where the middle map is the quotient map defining X^n.

Definition 0.9 (Subcomplex)**.** A subcomplex of a cell complex X is a closed subspace $A \subset X$ this is a union of cells of X.

Definition 0.10 (CW Pair)**.** A complex X with a subcomplex A, (X, A), is called a CW pair.

0.2 Operations

- **Products:** We can take the product of two spaces, by just considering the product of the cells.
- **Quotients:** We can take quotients, of a CW pair (X, A), by just quotienting out $(X \setminus A)$ and adding the 0-cell, which acts as the image of A.

  Intuition. A quotient is just taking the A space and moving it to a point. For example, let $X = D^2$ and $A = \partial D^2$, then, $X \setminus A \cong S^2$

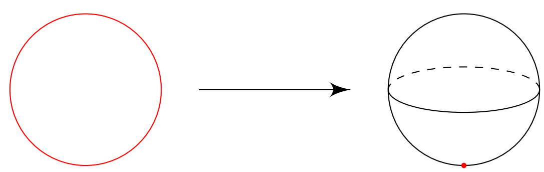
  

  Figure 1: $X \setminus A \cong S^2$

- **Suspension:** Suspension is obtained by turning a space X into a cylinder by considering $X \times I$ and then collapsing $X \times \{0\}$ and $X \times \{1\}$ to a point. We denote the suspension of a space by SX and a cone by CX, both of these are CW complexes if X is a CW complex.
- **Joins:** More generally we can talk about joins, this is where we consider the space $X \times Y \times I$, then instead of collapsing to a point, we collapse to a space. So we collapse $X \times Y \times \{0\}$ to X and $X \times Y \times \{1\}$ to Y. This creates line segments. We can a join as $X * Y$. Then we can write each line segment as a formal linear combination of $a_1 x + a_2 y$, where we say $a_1, a_2 \in [0, 1]$ and $a_1 + a_2 = 1$, subject to $0x + 1y = y$ and $1x + 0y = x$. This can be generalised to many different spaces. One interesting example is when every X_i is a single point. This leads to some thing called a simplex. If every point is a standard basis vector for $\mathbb{R}^n$, we get their join as the $(n-1)$-diensional simplex.
- **Wedge sum:** This is quite simple, this is just a single point union. We glue two spaces together at a single point. For example, taking $S^1 \vee S^1$ and then getting a figure of eight after a wedge sum. More formally $X \vee Y$ is the quotient of the disjoint union $X \coprod Y$ obtained by identitying a $x_0 \in X$ and $y_0 \in Y$ to a single point. Interestingly, for any cell complex X, the quotient $X^n \setminus X^{n-1}$ is just a wedge sum of n-spheres $\bigvee_\alpha S^n_\alpha$, with one sphere for each n-cell of X.
- **Smash Product:** This is slightly more complicated and will be used later. This is taking the points that we found with the wedge sum and just quotienting from the product space. So if we have X and Y, our smash product is, $X \wedge Y = X \times Y \setminus X \vee Y$. An amusing and satifying fact is that $S^m \wedge S^n = S^{m+n}$ as we know that S^n only has two cells, of dimension 0 and n. This then means that when we smash two together we get another n-sphere. An example of this is on the next page:

Figure 2: $S^1 \wedge S^1 \cong S^2$

0.3 Two Criteria for Homotopy Equivilence

0.3.1 Collapsing Subspaces

Claim 0.11. If (X, A) are a CW pair consisting of a CW complex X and a contractible subcomplex A, then the quotient map $X \to X \setminus A$ is a homotopy equivilence.

Then we can show that two spaces are homotopically equivilent by showing that through some quotienting and wedge sums the spaces are the same. For example, we can take a torus, then add four meridional discs, then transform it to three spheres, grab a line from a point between four of those discs and make 'beads on a string', and finally we can make a bouquet by just pulling on the spheres a bit.

0.4 Attaching Spaces

We can attach one space to another without changing the homotopy type. This can be done with some mappings and image magic.

Let us take two spaces X_0 and X_1 that we wish to attach. Now take $A \subset X_1$ and identify them with points in X_0. We now need the data of an map, $f : A \to X_0$, and form a quotient space of $X_0 \coprod X_1$ by identifying each $a \in A$ with its image $f(a) \in X_0$. Let us denote this space as $X_0 \sqcup_f X_1$, the space X_0 with X_1 attached along A via f.

The Mapping Cylinder M_f is an example of this construction and we can consider a mapping cone $C_f = M_f \setminus X$.

Claim 0.12. If (X, A) is a CW pair and the two attaching maps $f, g : A \to X_0$ are homotopic, then, $X_0 \sqcup_f X_1 \simeq X_0 \sqcup X_1$.

0.5 Homotopy Extension Properties

Consider the following, suppose we had a map $f_0 : X \to Y$ on a subspace $A \subset X$ and one is given a homotopy $f_t : A \to Y$ of $f_0|A$ that one would like to extend to $f_t : X \to Y$ of the given f_0. If a pair (X, A) is such a pair that this can always be solved we say that this pair has the homotopy extension property.

Claim 0.13. A pair, (X, A) has the homotopy extension property if and only if $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.

Proposition 0.14. If (X, A) is a CW pair, then $X \times \{0\} \cup A \times I$ is a defirmation retract of $X \times I$, hence (X, A) has the homotopy extention property.

Proof. Notice that $r : D^n \times I \to D^n \times \{0\} \times \partial D^n \times I$ is a retraction and then use that and the fact that you can just form X and A by attaching copies of $D^n \times I$ to $D^n \times \{0\} \times \partial D^n \times I$. Then preform a deformation retraction during $[\frac{1}{2^{n+1}}, \frac{1}{2^n}]$. $\square$

Proposition 0.15. If the pair (X, A) satisfies the homotopy extension property and A is constractible, then the quotient map $q : X \to X \setminus A$ is a homotopy equivilence.

$$\begin{tikzcd} X \arrow[r, "f_t"] \arrow[d, "q"'] & X \arrow[d, "q"] \\ X \setminus A \arrow[r, "\overline{f_t}"'] & X \setminus A \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "q"'] & X \arrow[d, "q"] \\ X \setminus A \arrow[r, "\overline{f_1}"'] \arrow[ur, "g"] & X \setminus A \end{tikzcd}$$

Proof. Let $f_t : X \to X$ be a homotopy extending contraction and then we can consider $f_0 = \mathbb{1}$ and then $qf_t : X \to X \setminus A$ sending A to a point and hence factors as a composition; $X \xrightarrow{q} X \setminus A \longrightarrow X \setminus A$. Denoting the later map by $\overline{f_t} : X \setminus A \to X \setminus A$, we have that $qf_t = \overline{f_t}q$ as in the first diagram above. From here we consider a g and a q and prove homotopy equivalence. $\square$

Proposition 0.16. If (X_1, A) is a CW pair and we have attaching maps $f, g : A \to X_0$ that are homotopic, then $X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1 \operatorname{rel} X_0$

Proof. Consider $F : A \times I \to X_0$ a homotopy from f to g. Now consider $X_0 \sqcup (X_1 \times I)$, then this must contain $X_0 \sqcup_f X_1$ and $X_0 \sqcup_g X_1$ hence using Prop 0.14 we can prove this with a bit of fiddling with homotopies. $\square$

Proposition 0.17. Suppose that (X, A) and (Y, A) satisfy the homotopy extention property, and $f : X \to Y$ is a homotopy equivilence with $f|A = \mathbb{1}$. Then f is a homotopy equivilence $\operatorname{rel} A$.

Corollary 0.18. If (X, A) satisfies the homotopy extension property and the inclusion $A \hookrightarrow X$ is a homotopy equivilence, then A is a deformation retract of A.

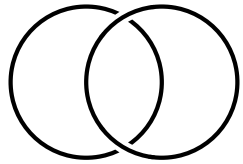
Proof. Apply the proposition to the inclusion $A \hookrightarrow X$. $\square$

Corollary 0.19. A map $f : X \to Y$ is a homotopy equivalence if and only if X is a deformation retract of the mapping cylinder M_f. Hence, two spaces X and Y are homotopy equivalence $\operatorname{rel} A$.

1 The Fundamental Group

1.1 Circles and Stuff

We are going to talk about circles and loops and use these to talk about the ideas of groups. Then we can



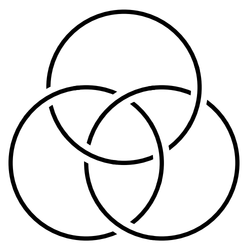
define something known as a loop,

Definition 1.1 (Loop)**.** An oriented curve that has the same start and end point.

Loops with the same start point can be glued together to create one loop and we can state the following,

Lemma 1.2. If we take two loops curling about a circle, then denote the number of turns as B_m and B_n and then as we connect these loops, we can say we have the curve B_{m+n}.

The proof of this is simple and obvious. We can do a similar thing with Borromean rings.



and pull the third ring into a U shape between the other two rings. We can then talk about a group, where going through the circle A, one way is a, the other being b^{-1}. Similarly, we can talk about b and b^{-1}. Hence, the Borromean rings have a loop of $aba^{-1}b^{-1}$. Moving onwards towards the fundamental group, we can say this,

Intuition. The fundamental group of a space X will be defined so that its elements are loops in X starting and ending at a point $x_0 \in X$.

Loops in the fundamental group are unique up to deformations.

1.2 Basic Constructions

We are going to define some stuff, firstly paths,

Definition 1.3 (Path)**.** A path in a space X we are talking about $f : I \to X$

and now we formalise a homotopy,

Definition 1.4 (Homotopy of paths)**.** A homotopy of paths in X is a family $f_t : I \to X$ for some $t \in [0, 1]$ and such that,

- $f_t(0) = x_0$ and $f_t(1) = x_1$, the endpoints of the curve are independent of t.
- The associated map $F : I \times I \to X$ is defined by $F(s, t) = f_t(s)$.

When we can say that f_0 and f_1 are connected by a homotopy f_t, they are said to be homotopic, notated by $f_0 \simeq f_1$.

Now we can define some sort of equivalence on them in the following proposition,

Proposition 1.5. The relation of homotopy on paths with fixed endpoints in any space is an equivalence relation.

Proof. We need to prove the following three things,

- **Reflexivity:** This is evident, where we just take $f = f_t$.
- **Symmetry:** This is also evident, we just take the inverse homotopy, i.e. f_{1-t}.
- **Transitivity:** For transitivity, we construct a homotopy from f_0 to f_1, then g_0 and g_1. We then define them as, $H(s,t) = F(s,2t)$ for $t \in \left[0,\, \frac{1}{2}\right]$ and $H(s,t) = G(s, 2t-1)$ for $t \in \left[\frac{1}{2},\, 1\right]$. It's continuous on the whole as it's continuous on it's parts.

$\square$

From this proof we can define a product paths,

Definition 1.6 (Product Paths)**.** If we have two paths, $f, g : I \to X$ such that $f(1) = g(0)$, there is a **composition** or **product path** $f \cdot g$ that traverses first f, then g, defined by,

$$\begin{cases} f(s) & s \in \left[0,\, \frac{1}{2}\right] \\ g(2s-1) & s \in \left[\frac{1}{2},\, 1\right] \end{cases}$$

Definition 1.7 (Fundemental Group)**.** We define the Fundemental group as the set of all homotopy classes $[f]$ for loops, $f : I \to X$ at the basepoint x_0. Denoted $\pi_1(X, x_0)$

Now, we don't really know if this a group yet, so let's prove it.

Proposition 1.8. $\pi_1(X, x_0)$ is a group with respect to the product, $[f][g] = [f \cdot g]$

Proof. We just have to prove the three axioms of the group, as we know that $[f][g] = [f \cdot g]$ is well defined. We are going to consider a reparameterisation of our maps, through a $\phi : I \to I$ where $\phi(0) = 0$ and $\phi(1) = 1$, this preserves the homotopy class as $f\phi \simeq f$ and finally define the homotopy of ϕ, just to be the linear homotopy.

- **Inverse:** We let the inverse loop, just be the loop where we replace $t \mapsto 1 - t$
- **Identity:** We just let the identity be the constant map.
- **Associativity:** This is basically taking three loops and then just joining them, so you go across every loop, just at different speeds.

$\square$

If we take some $h : I \to X$ and let it be a path, then we can consider the fact of whether the following fundamental groups are equivalent, $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$. Luckily, if we let h start at x_0 and end at x_1, then we can consider for each loop f based at x_1 to another loop $h \cdot f \cdot \overline{h}$ which starts at x_0, we note that $\overline{h}$ is just the reversal of h. Hence, we now define a change of basepoint map,

Definition 1.9 (Change Of Basepoint)**.** A map $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ by $\beta_h[f] = [h \cdot f \cdot \overline{h}]$.

This is well defined because if f_t is a homotopy of loops, then we have an inverse nicely.

Proposition 1.10. The map β_h is an isomorphism

Proof. Notice,

$$\begin{aligned}\beta_h[f \cdot g] &= [h \cdot f \cdot g \cdot \overline{h}] \\ &= [h \cdot f \cdot \overline{h} \cdot h \cdot g \cdot \overline{h}] \\ &= \beta_h[f]\beta_h[g]\end{aligned}$$

and it has an inverse as,

$$\begin{aligned}\beta_h\beta_{\overline{h}}[f] &= \beta_h[\overline{h} \cdot f \cdot h] \\ &= [h \cdot \overline{h} \cdot f \cdot h \cdot \overline{h}] \\ &= [f]\end{aligned}$$

and similarly for the other direction. □

Proposition 1.11. A space X is simply connected iff there is a unique homotopy class of paths connecting any two points in X.

Proof. This proof relies for the forward direction on proving uniqueness of paths, then if $f \simeq f \cdot \overline{g} \cdot g \simeq g$ and so it's unique. The other direction is if there is only one homotopy class, then all loops will be homotopic to the constant loop and so, $\pi_1(X, x_0) = 0$. □

1.3 Fundemental Group of the Circle

We are going to try and prove that $\pi_1(S^1) \simeq \mathbb{Z}$

Theorem 1.12. $\pi_1(S^1)$ is an infinite cyclic group generated by the homotopy class of the loop $\omega(s) = (\cos 2\pi s, \sin 2\pi s)$ based at $(1, 0)$.

Before we do this, we shall talk about some stuff [1]

Definition 1.13 (Lift)**.** If we take a $p : \mathbb{R} \to S^1$ such that, $p(s) = (\cos 2\pi s, \sin 2\pi s)$ and another, $\widetilde{\omega} : I \to \mathbb{R}$, that we define as $\widetilde{\omega}_n(s) = ns$. Then, we call $\widetilde{\omega}_n$ the **lift** of ω_n.

and also,

Definition 1.14 (Covering Space)**.** A **covering space** of a space X consists of a space $\widetilde{X}$ and a map, $p : \widetilde{X} \to X$ satisfying the following,

- For each $x \in X$ there is an open neighbourhood U of x such that $p^{-1}(U)$ is a union of disjoint open sets each of which is mapped homeomorphically onto U by p.

Definition 1.15 (Evenly Covered)**.** We call such a U, as in the previous definition, **evenly covered**.

Here are three important things that we will need to prove the theorem,

(i) For each path starting at $x_0 \in X$ and each $\widetilde{x_0} \in p^{-1}(x_0)$ there exists a unique lift $\widetilde{f} : I \to \widetilde{X}$ starting at $\widetilde{x_0}$.

(ii) For each homotopy $f_t : I \to X$ of paths starting at x_0 and each $\widetilde{x_0} \in p^{-1}(x_0)$ there is a unique lifted homotopy $\widetilde{f} : I \to \widetilde{X}$ of paths starting at $\widetilde{x_0}$

(iii) Given a map $F : Y \times I \to X$ and a map $\widetilde{F} : Y \times \{0\} \to wtX$ lifting $F|Y \times \{0\}$, then there is a unique map $\widetilde{F} : Y \times I \to \widetilde{X}$ lifting F and restricting to the given $\widetilde{F}$ on $Y \times \{0\}$.

I shall not state the proofs, but they are true.

[1] A James note to James, I hate this section of the book, it's messy and not very readable.

Proof of Theorem 1.12. To prove this, take a loop with basepoint $(1, 0)$ and let that be an element of $\pi_1(S^1, x_0)$. Then apply (1), then note that $\widetilde{f}$ starts at 0 ends at some integer. Then we have another path from 0 to n, ω_n, such that $f \simeq \omega_n$ so that, $[f] = [\omega_n]$.
Next prove that n is unique, by stating two ω's and then let them be delimited by n and m, then prove that $n = m$ by using (b) and then the uniqueness part of (a). □

Then, finally both, (a) and (b) can be derived from (c). For (a), let Y be a point and for (b) we let $Y = I$.

Theorem 1.16 (FTA)**.** Every nonconstant polynomial with coefficients in $\mathbb{C}$ has roots in $\mathbb{C}$.

Proof. Take a polynomial $(p(z) = z^n + a_1 z^{n-1} + \cdots + a_n)$ that has no roots in $\mathbb{C}$ and then consider,

$$f_r(s) = \frac{p(re^{2\pi is})/p(r)}{|p(re^{2\pi is})/p(r)|}$$

which is a loop and as you vary r, you get a homotopy of loops based at 1. Hence, $[f_r] \in \pi_1(S^1)$. Now choose an $r > 1$ and $r > |a_1| + \cdots + |a_n|$. Then for $|z| = r$ we have,

$$|z^n| > (|a_1| + \cdots + |a_n|)|z^{n-1}| > |a_1 z^{n-1}| + \cdots + |a_n| \geq |a_1 z^{n-1} + \cdots + a_n|$$

Now define a $p_t = z^n + t(a_1 z^{n-1} + \cdots + a_n)$ and then define a lift and then use Theorem 1.12. This gives, $[\omega_n] = [f_r] = 0$ and hence $n = 0$ and so $p(z) = a_0$ and so that is the only polynomial with no roots and so FTA proved. □

Theorem 1.17 (Brouwers fixed points in dimension 2)**.** Every continuous map $h : D^2 \to D^2$ has a fixed point, that is, a point $x \in D^2$ with $h(x) = x$.

Proof. We are going to assume that $h(x) \neq x$ for all $x \in D^2$, then we can define a map $r : D^2 \to S^1$ as defined in the figure. It's obvious that r is continuous. To describe $r(x)$ we say, $r(x) = x$ if $x \in S^1$. Thus r is a retraction of D^2 onto S^1. Now show that this retraction can't exist.
Let f_0 be any loop in S^1. In D^2 there is a homotopy of f_0 to a constant loop. Since the retraction r is the identity on S^1, then rf_1 is a homotopy in S^1 from f_0 to x_0, however we know that $\pi_1(S^1) \neq 0$, hence contradiction is found. □

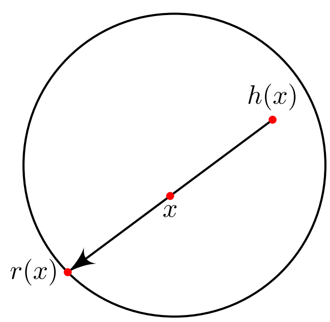


Figure 3: Brouwers in 2D

Theorem 1.18 (2D Borsuk-Ulam)**.** For every continuous map $f : S^2 \to \mathbb{R}^2$ there exists a pair of antipodal points x and $-x$ in S^2 with $f(x) = f(-x)$.

Corollary 1.19. Whenever S^2 van be represented as the union of three closed sets, then at least one of those sets contain a pair of antipodal points.

Proof. Consider a measure distance $d_i : S^2 \to \mathbb{R}$ and define it as, $d_i(x) = \inf_{y \in A_i} |x - y|$. Then apply Borsuk-Ulam and consider cases. $\square$

We choose three as this is the best case, for example consider a sphere inscribed in a tetrahedron. This object doesn't contain a pair of antipodal points. We can prove a similar result for S^n with $n+1$ different sets.

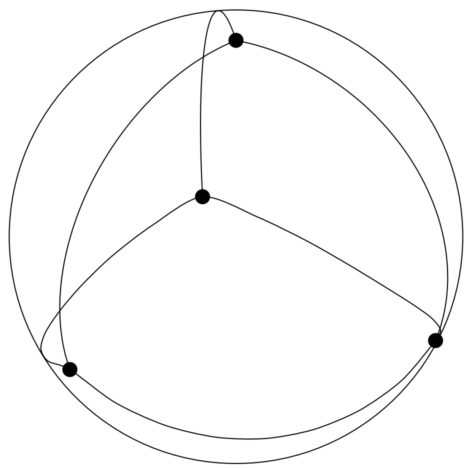


Figure 4: A sphere split into four

We can now say some more about our fundemental group,

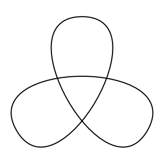
Proposition 1.20. $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$ if X and Y are both path connected

Proof. Use the fact that $f : Z \to X \times Y$ is path connected if and only if $g : Z \to X$ and $h : Z \to Y$ are continuous when f is defined as $f(z) = (g(z), h(z))$. We now say that a loop f in $X \times Y$ is equivalent to a pair of loops g and h in X and Y respectively, similarly with homotopies. Thus we get a bijection. Then we see $[f] \mapsto ([g], [h])$. $\square$

Example (Torus)**.** We can define a torus as $S^1 \times S^1$ and now using our new proposition, we can talk about,

$$\pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}$$

Hence, we can nicely talk about a point $(p, q) \in \mathbb{Z}^2$ as p times around one S^1 factor and q times around the other S^1 factor. If we take $(3, 2)$, we get the trefoil knot.



1.4 Induced Homomorpisms

We can consider our $\phi : X \to Y$ where $\phi : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and it takes the basepoint x_0 to y_0. Then we can say that $\phi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ defined by composing loops $f : I \to X$ based at x_0 with ϕ, so

$\phi_*[f] = [\phi f]$. The induced homomorphism ϕ_* is well-defined since a homotopy f_t of loops based at x_0 yields a homotopy of loops ϕf_t based at y_0. So,

$$\phi_*[f_0] = [\phi f_0] = [\phi f_1] = \phi_*[f_1]$$

We can also say ϕ_* is a homomorphism as,

$$\phi(f \cdot g) = (\phi f) \cdot (\phi g)$$

as we can consider $\phi f(2s)$ for $0 \leq s \leq \frac{1}{2}$ and $\phi g(2s-1)$ for $\frac{1}{2} \leq s \leq 1$.

Definition 1.21 (Induced homomorphisms)**.** Two basic properties of induced homomorphisms are,

- $(\psi\phi)_* = \psi_*\phi_*$ for a composition $(X, x_0) \xrightarrow{\psi} (Y, y_0) \xrightarrow{\phi} (Z, z_0)$.
- $\mathbb{1}_* = \mathbb{1}$ i.e. the identity map, $\mathbb{1} : X \to X$ induces the identity map $\mathbb{1} : \pi_1(X, x_0) \to \pi_1(X, x_0)$

Note that these two things make the fundemental group a functor[2]

Proposition 1.22. $\pi_1(S^n) = 0$ for $n \geq 2$

To prove this, we will need the following,

Proposition 1.23. If a space X is the union of a collection of path-connected open sets A_α each containing the basepoint $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is a path-connected, then every loop in X at x_0 is homotopic to a product of loops each of which is contained in a single A_α.

Proof. Take a loop at a basepoint x_0 and then a partition $0 = s_0 < s_1 < s_2 < \cdots < s_m = 1$ of I such that each subinterval $[s_{i-1}, s_i]$ is mapped to a single A_α.
Now denote A_i as $f([s_{i-1}, s_i])$ and f_i by restricting f to $[s_{i-1}, s_i]$. Then, as we have path connected sets, we can take a path from x_0 to $f(s_i) \in A_i \cap A_{i+1}$ and hence consider,

$$(f_1 \cdot \overline{g_1}) \cdot (g_1 \cdot f_2 \cdot \overline{g_2}) \cdot (g_2 \cdot f_3 \cdot \overline{g_3}) \cdot \ldots \cdot (g_{m-1} \cdot f_m)$$

and it's homotopic to f! Finally this loop is just the composition of a load of loops in each of the subsets. ☐

Proof of Prop 1.22. We can take S^n and express it as two open sets A_1 and A_2 (each homeomorphic to $\mathbb{R}^n$), such that $A_1 \cap A_2$ is homeomorphic to $S^{n-1} \times \mathbb{R}$. Choose a basepoint $x_0 \in A_1 \cap A_2$. If $n \geq 2$ then $A_1 \cap A_2$ is path connected. Hence, $\pi_1 A_1 = \pi_1 A_2 = 0$. Hence, every loop in S^n is nullhomotopic. ☐

Corollary 1.24. $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$

Proof. Suppose we have a homeomorphism and then for $n = 1$, it's trivial. Then, for $n > 2$ we consider the fundemental groups and then it is trivial for everything but $n = 2$. ☐

Proposition 1.25. If a space X retracts onto a space A, then the homeomorphism induced by the inclusion map, $i : A \hookrightarrow X$ is injective. If A is a deformation retract, then i_* is a isomorphism.

Proof. If $r : X \to A$ is a retraction, then $ri = r_*i_* = \mathbb{1}$, which implies that i_* is injective. If $r_t : X \to X$ is a deformation retraction, then we can say that there is a loop such that $r_t f$ gives a homotopy, so i_* is also surjective. ☐

Definition 1.26 (Basepoint-preserving homotopy)**.** We have a $\phi_t : (X, x_0) \to (Y, y_0)$ such that it is the case that $\phi_t(x_0) = y_0$ for all t.

Property (Induced homomorphism)**.** We can now talk about another basic property of induced homomorphisms,

[2]CATEGORY THEORY TIME

- If $\phi_t : (X, x_0) \to (Y, y_0)$ is a basepoint-preserving homotopy, then $\phi_{0*} = \phi_{1*}$

Now, at this point, we are really rather annoyed with all this basepoint nonsense, hence we look to drop this condition,

Proposition 1.27. If $\phi : X \to Y$ is a homotopy equivilence, then the induced homeomorphism $\phi_* : \pi_1(X, x_0) \to \pi_1(Y, \phi(x_0))$ is an isomorphism for all $x_0 \in X$.

To prove this, we will prove the following,

Lemma 1.28. If $\phi_t : X \to Y$ is a homotopy and h is the path $\phi_t(x_0)$ formed by the images of a basepoint $x_0 \in X$, then the three maps in the diagram below satisfy $\phi_{0*} = \beta_h \phi_{1*}$

$$\begin{tikzcd}
 & \pi_1(Y, \phi_1(x_0)) \arrow[d, "\beta_h"] \\
\pi_1(X, x_0) \arrow[ur, "\phi_{1*}"] \arrow[dr, "\phi_{0*}"'] & \\
 & \pi_1(Y, \phi_0(x_0))
\end{tikzcd}$$

2 Van Kampen's Theorem

This theorem will allow us to calculate more fundemental groups by just decomposing spaces into smaller spaces and reconstituting those fundemental group.

If we take the example of two circles at a point we will get something in terms of a free product, namely $\mathbb{Z} * \mathbb{Z}$.

2.1 Free Products on Groups

To work out why on earth we want anything like this, let us consider a collection of groups. We want to get one singular group where each of these groups are subgroups. There are obviously two ways we know how to do this,

- The product group, $\prod_\alpha G_\alpha$, whose elements are regarded as functions $\alpha \mapsto G_\alpha$.
- The direct sum, $\bigoplus_\alpha G_\alpha$, by restricting these functions, taking the non-identities finitely often.

The problem comes as then all the elements will commute, we don't want this to happen and so we will construct a nonalbelian version of $\bigoplus_\alpha G_\alpha$ called the $*_\alpha G_\alpha$.

Definition 2.1 (Free Product)**.** As a set, three product $*_\alpha G_\alpha$ consists of all words $g_1 g_2 g_3 \dots g_m$ of arbitrary finite length $m \geq 0$, where each letter g_i belongs to a group G_{α_i}, and is not identity, and adjacent letters g_i and g_{i+1} belong to different groups G_α, that is $\alpha_i \neq \alpha_{i+1}$. Words satisfying this are called *reduced*. The empty word is the identity.
The group identity is juxtaposition $(g_1 \dots g_m)(h_1 \dots h_n) = g_1 \dots g_m h_1 \dots h_n$.

Remark 1. You can reduce a word by rewriting adjacent letters that lie in the same G_{α_i} to a single letter and cancel trivial letters.

We can nicely find that this group is associative.

Proof. Let W be thr set of all the reduced words $g_1 \dots g_m$ including the empty word. To each $g \in G_\alpha$ we associate a function $L_g(g_1 \dots g_m) = g g_1 \dots g_m$ and we combine with $g_1 \in G_\alpha$ such that it's a reduced word. We can say, $L_{gg'} = L_g L_{g'}$ and this is a special case of associativity, $g(g'(g_1 \dots g_m)) = (gg')(g_1 \dots g_m)$. We can also say that L_g is invertible with inverse $L_{g^{-1}}$. Hence it's a group homomorphism from G_α to $P(W)$, the permutations of W. More generally we can define $L : W \to P(W)$ by $L(g_1, \dots, g_m) = L_{g_1} \dots L_{g_m}$ for each reduced words $g_1 \dots g_m$. The product operation in W corresponds under L to composition in $P(W)$, because of the relation $L_{gg'} = L_g L_{g'}$. Since the composition in $P(W)$ is associative, we conclude that the product in W is associative. □

We can now say something nice about $\mathbb{Z} * \mathbb{Z}$, it's a free product but also a free group! We have some lingo to do with this, we can say that each element is uniquely representable as as reduced words in powers of generators for various copies of $\mathbb{Z}$, one generator for each $\mathbb{Z}$.

Definition 2.2 (Basis)**.** The basis of a free group is all of the generators

Definition 2.3 (Rank)**.** The rank of the free group is just the number of generators.

The abelianization of the free group is a free abelian group with basis of the same set of generators, and so the rank of the abelianization of a group is well defined, independent of choice of basis, the same is true for the rank of the free group.

Lemma 2.4. Any two different ways to reduce the word will then produce the same reduced word.

Proof. Associativity □

Any collection of homomorphisms $\phi_\alpha : G_\alpha \to H$ extends uniquely to $\phi : *_\alpha G_\alpha \to H$. Namely, the value of ϕ on a word $g_1 \dots g_m$ with $g_i \in G_{\alpha_i}$ must be $\phi_{\alpha_1}(g_1) \dots \phi_{\alpha_n}(g_n)$, and using this formula to derive ϕ gives a well defined homomorphism since the process of reducing an unreduced product doesn't change it's image.

2.2 The van Kampen Theorem

We will take a space X that can be decomposed as the union of a collection of path-connected open subsets A_α. Then, we define $i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ as an induced homomorphism by the inclusion $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$. Then basically we can get that the kernel of some map $\phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$ contains all the elements of the form, $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$.

Theorem 2.5 (Van Kampen Theorem)**.** If X is the union of path-connected open sets A_α each containing the basepoint $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path-connected, then the kernel of ϕ is the normal subgroup N generated by all elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$, and hence ϕ induces an isomorphism $\pi_1(X) \approx *_\alpha \pi_1(A_\alpha) \setminus N$.

Example. We shall consider some $\bigvee_\alpha S^1_\alpha$ and van Kampens Theorem states that $\pi_1(\bigvee_\alpha S^1_\alpha)$ is free and so, free product of copies of $\mathbb{Z}$. Hence, $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$. More generally, the fundemental group of any connected graph is free.

Definition 2.6 (Normal Subgroup)**.** A subgroup N of the group G is normal in G if and only if $gng^{-1} \in N$ for all $g \in G$ and $n \in N$ We denote this as, $N \lhd G$.

Proof of Van Kampens Theorem. We have already proved surjectivity of ϕ in Proposition 1.23. Now we prove that $\ker \phi = N$. We shall define the following,

Definition 2.7 (Factorisation)**.** A factorisation of an element $[f] \in \pi_1(X)$ we mean a formal product $[f_1] \ldots [f_k]$ where,

- Each f_i is a loop in some A_α at the basepoint x_0, and $[f_i] \in \pi_1(A_\alpha)$ is the homotopy class of f_i.
- The loop is homotopic to $f_1, \ldots, f_k$ in X.

A factorisation of $[f]$ is thus a word in $*_\alpha \pi_1(A_\alpha)$, possibly unreduced, that is mapped to $[f]$ by ϕ. So surjectivity of ϕ is equivalent to every word of $[f] \in \pi_1(X)$ has a factorisation.

Definition 2.8 (Equivalent Factorisation)**.** We shall call two factorisations equivalent if they are related by a sequence of the following moves,

(i) Combine adjacent terms $[f_i][f_{i+1}]$ into a single term $[f_i \cdot f_{i+1}]$ if $[f_i]$ and $[f_{i+1}]$ are both in the same group $\pi_1(A_\alpha)$.

(ii) Regard the term $[f_i] \in \pi_1(A_\alpha)$ as lying in the group $\pi_1(A_\beta)$ rather than $\pi_1(A_\alpha)$ if f_i is a loop in $A_\alpha \cap A_\beta$.

The first move doesn't change the element, and the second doesnt change the image of the element in the quotient group $Q = *_\alpha \pi_1(A_\alpha) \setminus N$. We now aim to show that any two factorisations are equivalent, which means ϕ is injective, so $\ker \phi = N$.
If we take two factorisations and form a homotopy ($F : I \times I \to X$), we can then form a partition and now what we have is a covering of $I \times I$ with finitely many rectangles (R_r's). Now we do something odd and shift the middle row around such that each point is in at most three rectangles. We can assume that there are at least three rows of these rectangles. We shall index them from the bottom from left to right on each row.
If we take a γ on $I \times I$, from the left side to the right, we can consider $F|\gamma$, which is then a loop at the basepoint x_0 as F maps both the left edge and the right edge to x_0. We now consider a path γ_r as the path that separates off the first r rectangles, so γ_0 is the bottom edge and γ_{mn} the top.
Now let vertices, v, be any corner such that $F(v) \neq 0$. Then, we can choose paths g_v from x_0 to $F(v)$ that lie in the intersection of two or three A_{ij}'s (image of R_r in F). Then we obtain a factorisation of $[F|\gamma_r]$, different choices of which A_{ij} we go into can change our factorisation, however these factorisations are equivalent. Furthermore γ_r and γ_{r+1} are equivalent as we are just pushing through another rectangle.
Finally, we make our γ_0 equivalent to our first factorisation by choosing vertices that are on the bottom

edge and also not lying in the A_α such that v is in the domain of f_i. We then do something similar for γ_{mn} by letting it be the second factorisation and then we also know that all γ_r's are equivalent and so the factorisations are equivalent. □

Here is an interesting example that I have stewed over since it's been a while since I've picked this up.

Example. Torus Knots: Consider two positive integers that are relatively prime, m and n. The torus knot $K = K_{m,\,n} \subset \mathbb{R}^3$ is the image of the embedding $f : S^1 \to S^1 \times S^1 \subset \mathbb{R}^3$, $f(z) = (z^m,\, z^n)$. This is the interesting bit to me, the knot K winds arounds the torus a total of m times in the longitudinal direction and n in the meridional direction. We need to assume that m and n are relatively prime in order for f to be injective otherwise f would be gcd$(m,\, n)$-to-one, the image of this f would be $K_{\frac{m}{d},\,\frac{n}{d}}$ where $d = \gcd(m,\, n)$. We could also allow negative m and n and that would just change K to be a mirror-image knot.
Let us compute $\pi_1(\mathbb{R}^3 - K)$, it is easier to compute this fundamental group with $\mathbb{R}^3$ replaced with it's compactification S^3. Van Kampens Theorem tells us that this doesn't affect anything and this makes sense. Namely, we are taking the union of $\mathbb{R}^3 - K$ and an open ball B formed by the compactification point together with the complement a large closed ball in $\mathbb{R}^3$ containing K. As we used Van Kampens Theorem we require B and $B \cap (\mathbb{R}^3 - K)$ to be simply connected and we find that they are. Hence, $\mathbb{R}^3 - K \hookrightarrow S^3 - K$ induces an isomorphism on π_1.
To compute the fundamental group we show that it deformation retracts to a 2D complex $X = X_{m,\,n}$ homeomorphic to the quotient space of a cylinder $S \times I$ under the identifications $(z,\, 0) \sim (e^{\frac{2\pi i}{m}} z,\, 0)$ and $(z,\, 1) \sim (e^{\frac{2\pi i}{n}} z,\, 1)$. We define X_m and X_n as the left and right halves of X.
If we consider $\pi_1(X)$, then we can decompose this into the union of the fundemental groups of X_m and X_n. Both X_m and X_n deformation retract onto circles and so they fundemental group of (open balls around) X_m and X_n are just $\mathbb{Z}$, we also note that $X_m \cap X_n$ is also a circle and so the $\pi_1(()X_m \cap X_n) = \mathbb{Z}$. If we consider a loop in $X_m \cap X_n$ that is a generator is homotopic in X_m to m times a generator and in X_n it's homotopic to n times a generator. VKT then tells that $\pi_1(()X)$ is now just the quotient of the free group on generators a and b obtained by factoring out the normal subgroup generated by $a^m b^{-n}$.

We denote $G_{m,\,n}$ this group generated by a and b and with one relation $a^m = b^n$. If $m = 1$ or $n = 1$ then G is just infinite cyclic then one generator is a power of another. To find the behaviour, let us consider the center of this group. We can see that $a^m = b^n$ commutes with a and b and so the cyclic group C is generated by $a^m = b^n$. Moreover, C is normal, so we can consider $G \setminus C$, which is just the free product $\mathbb{Z}_m * \mathbb{Z}_n$. The free product of non-trivial groups has a trvial center, hence C is the center of G. Finally we can show that m and n are uniquely determined by $\mathbb{Z}_m * \mathbb{Z}_n$.

2.3 Application of Cell Complexes

Take some 2-cells e^2_α and attach them to a path-connected space X via some map, $\varphi : S^1 \to X$ (as we defined in the introduction) to produce a new space, Y. If s_0 is a basepoint of S^1, we shall call $\varphi_\alpha(s_0)$ a loop even though it isn't of the usual form. Now we get a loop from $x_0 \in X$, a basepoint, and a path γ_α from x_0 to $\varphi_\alpha(s_0)$ and now we have the basic setup for the next proposition.

Proposition 2.9. The following three hold,

(i) If Y is obtained from X by attaching 2-cells as described above, then the inclusion $X \hookrightarrow Y$ induces a surjection $\pi_1(X, x_0) \to \pi_1(Y, x_0)$ whose kernel is the normal subgroup generated by all the loops $\gamma_\alpha \varphi_\alpha \overline{\gamma_\alpha}$, N. Thus, $\pi_1(Y) \cong \pi_1(X)/N$

(ii) If Y is obtained from X by attaching n-cells for a fixed $n > 2$, then the inclusion $X \hookrightarrow Y$ induces an isomorphism $\pi_1(X, x_0) \cong \pi_1(Y, x_0)$

(iii) For a path connected cell complex X the inclusion of the 2-skeleton $X^2 \hookrightarrow X$ induces an isomorphism $\pi_1(X^2, x_0) \cong \pi_1(Y, x_0)$

Proof. The proof of (i) relies on considering a slightly larger space. Consider the path between x_0 and each of the e^2_α cells, then we extrude these upwards and call them S_α. Now we call this space Z. Z deformation retracts onto Y. Now we consider two different situations, take Z and puncture each of the 2-cells anywhere by where they intersect S_α, then we can say this space deformation retracts to X, call it A. Now consider $Z - X$, then this will contract to a point and so it is contractible, call this space B. Since $\pi_1(B) = 0$, then an application of Van Kampens Theorem tells us that $\pi_1(Z) \cong \pi_1(A)/N$ where N is the image of $\pi_1(A \cap B) \to \pi_1(A)$

For (ii), we can use a similar if not identical argument but with e^n_α cells instead of e^2_α and so we get the $\pi_1(S^{n-1}) = 0, (n > 2)$ as we proved earlier.

For (iii), we prove by induction for the finite case. For the non-finite case we argue as such, we aim to take a loop $f : I \to X$ that has a basepoint $x_0 \in X^2$. Then as aim to prove that $\pi_1(X^2, x_0) \to \pi_1(X, x_0)$ is bijective. The surjection comes from the fact f has a compact image in X^n and so f is homotopic to some loop in X^2. For the injection suppose we have some g that is nullhomotopic in X^2, then again it's compact and we apply (b) to get the injection and so we can conclude that f is also nullhomotopic in X^2. □

Now we consider an orientable surface, which is 0-cell, $2g$ 1-cells and a 2-cell. We can then talk about the fundamental group of this object we know a 1-skeleton is just the wedge sum of $2g$ circles and it has a fundamental group that's free on two generators. Then we know the 2- cell is just the product of the commutators of the generators and so,

$$\pi_1(M_g) = \langle\, a_1, b_1, \dots, a_g, b_g \,|\, [a_1, b_1] \dots [a_g, b_g] \,\rangle$$

This is some new notation,

Notation. We write $\langle\, g_\alpha \,|\, r_\beta \,\rangle$ to mean a group with generators g_α and relators r_β. In other words, $\pi_1(M_g) = \langle\, g_\alpha \,\rangle \,/\, \langle\, r_\beta \,\rangle$.

Now here a fun little result,

Corollary 2.10. The surface M_g is not homeomorphic, or even homotopy equivalent to M_h if $g \neq h$.

2.3.1 Commutator

Here's also some more slightly interesting background mathematics that we need for the proof. We call the commutator of two elements g and h, $[g, h] = g^{-1}h^{-1}gh$; however we also may say that $[g, h] = ghg^{-1}h^{-1}$. This commutator is equal to e if and only if $gh = hg$. Here are a couple of facts,

(i) $[g, h]^{-1} = [h, g]$

(ii) $[g, h]^s = [g^s, h^s]$ where $g^s = sgs^{-1}$, the conjugate of g by s.

(iii) For any homomorphism, $f : G \to H$, we can say $f([g, h]) = [f(g), f(h)]$.

Note that the third implies the second as it's a generalisation, it tells us that the set of conjugators is stable under some $f : G \to G$ (where we take $G = H$). These also show us that the set of commutators is closed under inversion and conjugation. Which takes us to a new definition,

Definition 2.11 (Commutator Subgroup)**.** Take the set of commutators, G', of a group, by the identities above, then G' is a subgroup of G.

Furthermore, $([g_1, h_1][g_2, h_2] \dots [g_n, h_n])^s = [g^s_1, h^s_1][g^s_2, h^s_2] \dots [g^s_n, h^s_n]$ and so we can say G' is a normal subgroup.

Definition 2.12 (Abelianisation)**.** Consider a group G and it commutator subgroup G', then the abelianisation of G is G/G' and is denoted G^{ab}.

We can do some category theory and see that the map $\varphi : G \to G^{\text{ab}}$ is universal.

Proof. The abelianisation of $\pi_1(M_g)$ is the direct sum of $2g$ copies of $\mathbb{Z}$. Hence if, $M_g \simeq M_h$, then $\pi_1(M_g) \cong \pi_1(M_h)$. Hence the abelianisations are isomorphic and so $g = h$. $\square$

For non-orientable surfaces we can do something similar. Consider g circles and we attach a 2-cell by the word $a_1^2 a_2^2 a_3^2 \dots a_n^2$ we then get a non-orientable surface, N_g. For some examples N_1 is just $\mathbb{R}\mathrm{P}^2$ (for babies) and N_2 just a klein bottle. We can say that $\pi_1(N_g) = \left\langle a_1, \dots, a_n \,\middle|\, a_1^2 \dots a_n^2 \right\rangle$ and so the abelianisation of this is the direct sum of $\mathbb{Z}_2$ with g copies of $\mathbb{Z}$ as we can choose the generators such that $2(a_1 + \dots + a_n) = 0$. Hence $N_g \not\simeq N_h$ if $g \neq h$ or even $N_g \not\simeq M_h$ for any surfaces.

Here is something else,

Corollary 2.13. For every group G there is a 2-dimensional cell complex X_G with $\pi_1(X_G) \cong G$.

Proof. Choose a representation of $G = \langle g_\alpha \,|\, r_\beta \rangle$ as every group is the quotient of some free group. Then let g_α be the generators of the free group and r_β be the generators of the kernel of the map from the free group to the group. Now let X_G be constructed from $\bigvee_\alpha S^1_\alpha$ by attaching the loops of r_β (which are 2-cells) . $\square$