1 Chapter 0 - Underlying Notions

We are going to adopt the convention that every map is continuous unless otherwise stated.

We want to be able to talk about ways to transform some shapes into other shapes and think about them then being equal. We do this by considering an $I \in [0,1]$ that we can paramaterise a function by. So if we have two curves $s : X \to X$ and $t : X \to X$, we can define some $f : X \times I \to X$ such that,

$$\begin{cases} f_0(x) & = s(x) \\ f_1(x) & = t(x) \end{cases}$$

Definition 1.1 (Deformation Retraction)**.** A Deformation retraction of a space X onto an $A \subset X$ is a family of maps, $f_t : X \to X$ $t \in I$, such that, $f_0 = \mathbb{1}$, $f_t(X) = A$ and $f_t|A = \mathbb{1}$ $\forall t$. The family f_t should be continuous in the sense that the associated map $X \times I \to X$, $(x,t) \mapsto f_t(x)$ is continuous.

Definition 1.2 (Mapping Cylinder)**.** A mapping cylinder, M_f, for a $f : X \to Y$ is the quotient space of the disjoint union $(X \times I) \coprod Y$, obtained by identifying each $(x,1) \in X \times I$ with $f(x) \in Y$.

A mapping cylinder deformation, of f, retracts to Y.

A deformation retraction $f_t : X \to X$ is just a special case of a homotopy, which is just simply any family of maps, $f_t : X \times I \to Y$, given by $F(x,t) = f_t(x)$ is continuous.

Definition 1.3 (Homotopic)**.** One says two maps are homotopic if there exists a homotopy f_t connecting them and one writes $f_0 \simeq f_1$

Retractions are the topological analogue of projection operators.

Definition 1.4 (Homotopy Relative)**.** A homotopy $f_t : X \to Y$ whose restriction to a subspace $A \subset X$ is independent of t is homotopy relative, written as homotopy rel.

Definition 1.5 (Homotopy Equivalence)**.** A map $f : X \to Y$ is called a homotopy equivalence if there exists a $g : Y \to X$ such that $fg \simeq \mathbb{1}$ and $gf \simeq \mathbb{1}$.

Moreover, we can now say that X and Y are homotopy equivalent or have the same homotopy type.

Lemma 1.6. *Any two spaces X and Y are homotopy equivilent if there exists some other space Z containing both X and Y are deformation retracts.*

Definition 1.7 (Contractible)**.** A space that has the homotopy type of a point is contractible.

This amounts to the identity map of this space to be nullhomotopic, that is homotopic to a constant map.

1.1 Cell Complexes

To construct a cell complex, let us follow these rules,

(i) Start with a discrete set X^0, whose points are regarded as 0-cells.

(ii) inductively, form the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via maps $\phi_\alpha : S^{n-1} \to X^{n-1}$ with a collection of n-disks D^n_α under the identifications $x \sim \phi_\alpha(x)$ for $x \in \partial D^n_\alpha$. Thus $X^n = X^{n-1} \coprod_\alpha e^n_\alpha$.

(iii) We can either stop at a finite amount or continue indefinitely. in the latter, X is given a weak topology, A set $A \subset X$ is open if and only if $A \cap X^n$ is open in X^n for each n. This similarly works for closed.

X is a cell complex or a CW complex.