

Year 2 — Colloquia

Based on lectures by Various

Notes taken by James Arthur

LMS Summer School 2021

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

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1 Markov Numbers and the free group on two generators - Caroline Series.

We are going to talk about three binary tree and the connections between them.

A Markov number is a solution to,

$$x^2 + y^2 + z^2 = 3xyz$$

If we set $x = y = z = 1$ and that's a solution. Let's not worry about negative solutions as here is another $(-x, -y, z)$.

Suppose x, y_1, z_1 is a solution you get,

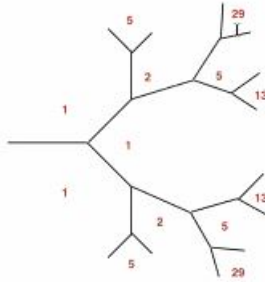
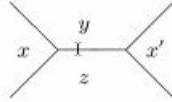
$$z^2 - 3xy_1z_1 + y_1^2 + z_1^2 = 0$$

and so $x + x' = 3y_1z_1$. If we have (x, y_1, z_1) we can get $(3y_1z_1 - x, y_1, z_1)$. We could permute any of these.

Theorem 1.1. If we start with a solution, we can carry on permuting, we can get all the solutions,

$$(x, y, z) \rightarrow (3yz - x, y, z)$$

Proof. Start with (x, y, z) , and let $(1, 1, 1)$ and then get a load of solutions. We can now put these around the vertices of a binary tree. and we can do this again, to get a load more solutions, Let's now prove that this is



all of them,

Say it is special if two of x, y, z are equal. The only special solutions are $(1, 1, 1)$ and $(1, 1, 2)$. Say $x = y$ and then $2x^2 + z^2 = 3x^2z$. Hence $x^2|z^2$ and so $z = kx$ and so $2 + k^2 = 3kx$ so $k|2$ and so $k = 1$ or 2 .

Step 2: Show that if (x, y, z) is a solution with $x \nmid y \nmid z$ if $x' = 3yz - x$ and $x \nmid y \nmid x'$. **Step 3:** Take any non-special $x > y > z$ surrounding V and draw it's local tree with arrows. By step 2, there is an outgoing arrow from x to x' . **Claim:** The other two arrows at V point to V .

This follows from a change of variables, $\xi + \eta + \zeta = 1$ and so again, $\xi + \xi' = 1$ and for the other variables. Hence $\xi > \xi'$ and so $\xi > \frac{1}{2}$. But then, $\eta < \frac{1}{2}$ and $\zeta < \frac{1}{2}$, which means that $\eta < \eta'$ and $\zeta < \zeta'$, so the arrows point to V .

Step 4: From each non-special vertex there exists

□

There is a conjecture that says,

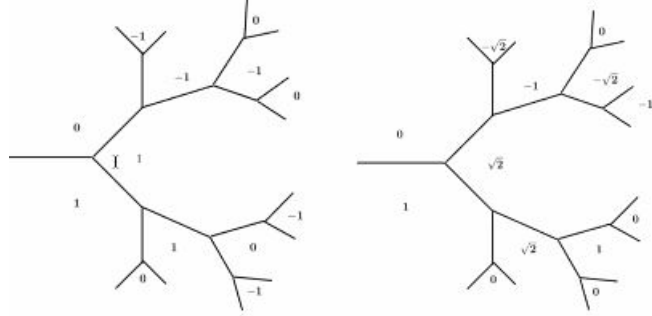
Proof. We take,

$$A = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & \zeta \\ \zeta^{-1} & y \end{pmatrix}$$

where $z = \zeta + \zeta^{-1}$ □

and so now we just run through the tree and images of primitive elements. Some questions have been asked is, is the group generated by A and B free? If not, is it discrete?

Are the corresponding groups finite? If we look at the following,



You can see that you won't get past 0 and 1 for the first. Then in the second, taking $\sqrt{2}$, we can't more values than we have. If a group had generators 0, 1 would it be finite? So we now consider $SU(2)$,

$$M = \begin{pmatrix} a & b \\ \bar{a} & \bar{b} \end{pmatrix}$$

and they are unitary. These basically give us stereographic projections, it didn't preserve distance, but it does for angles. if we rotate our stereographic sphere, we get an angle preserving map. This then gives us $SU(2) \subset SL(2, \mathbb{C})$. This gives us a mobius transformation. Thn if we consider,

$$\begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix}$$

We get the trace as just $2 \cos \frac{\theta}{2}$. Then out pops 0, 1, $\sqrt{2}$.

Theorem 1.5. With one exception, every finite tree is associated to a regular solids, and corresponds to finite representations to $F_2 \rightarrow S(U)$ with finite image. The exception is the dihedral group.

The other regular solid is the icosohedron, hence giving the icosohedral group. The sphere is covered with twenty copies of the a equilateral triangle of angle $\frac{\pi}{5}$. This then moves forward with subgroups generated by rotations of orders 2, 3 and 5. So we expect to get a finite tree starting from the values, $2 \cos \frac{\pi}{2} = 0$, $2 \cos \frac{\pi}{3} = 1$ and $2 \cos \frac{\pi}{5} = \omega$ and after some algebra we get that $\omega - \omega - 1 = 0$ and hence after some algebra we have finite values.

2 A Glimpse of Tropical Geometry - Felipe Ricon, QMUL

Resources - *Introduction to tropical Geometry (Book)*, *First Steps in Tropical Geometry (Article)*

Definition 2.1 (Tropical Semiring). The tropical semiring is,

$$\overline{\mathbb{R}} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$$

where,

$$\oplus := \min \quad \odot := +$$

Example.

$$3 \oplus 5 = 3 \quad 4 \odot 7 = 11$$

$$\infty \oplus a = a \quad 0 \odot a = a$$

$\overline{\mathbb{R}}$ is an idempotent semiring as there are no additive inverses.

$$a \odot (b \oplus c) = a \odot b \oplus a \odot c$$

but things like,

$$2 \oplus x = 5$$

doesn't have a solution in our semiring. But addition is idempotent,

$$a \oplus a = a$$

Example. $(x \oplus y)^3 = x^3 \oplus x^2 \odot y \oplus x \odot y^2 \oplus y^3 \equiv x^3 \oplus y^3$

Denote $\overline{\mathbb{R}}[x_1, \dots, x_n]$ the semiring of the tropical polynomials on the variables x_1, \dots, x_n .

Example. $f(x) = x^2 \oplus 1 \odot x \oplus 4$ which then we can plot,

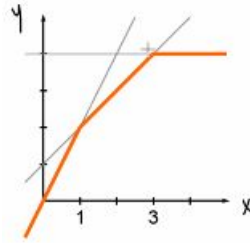


Figure 1: A graph of $f(x) = (x \oplus 1) \odot (x \oplus 3)$

we note we can factor them as,

$$f(x) = (x \oplus 1) \odot (x \oplus 3)$$

and where we factor it is where the graph bends.

Theorem 2.2 (Fundamental Theorem of Algebra). If $f(x) \in \overline{\mathbb{R}}[x]$ has degree d then,

$$f(x) \equiv c \odot (x \oplus a_1)^{m_1} \odot (x \oplus a_2)^{m_2} \odot \cdots \odot (x \oplus a_r)^{m_r}$$

where $a_1, \dots, a_r \in \overline{\mathbb{R}}$ and $m_1 + \cdots + m_r = d$

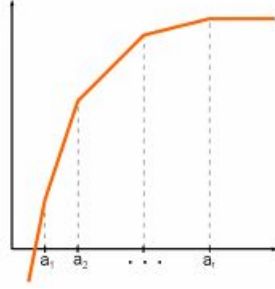
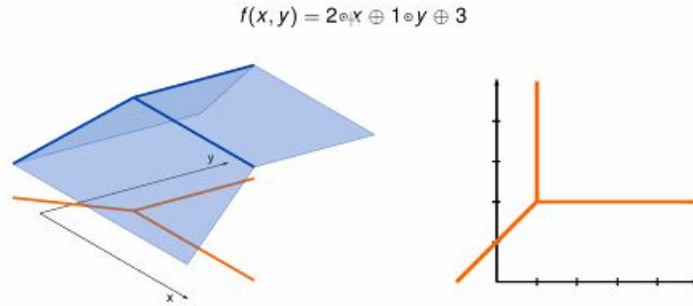


Figure 2

Any $f(\mathbf{x}) \in \overline{\mathbb{R}}[x_1, \dots, x_n]$ is the minimum of a load of finite number of affine functions. Then we define,

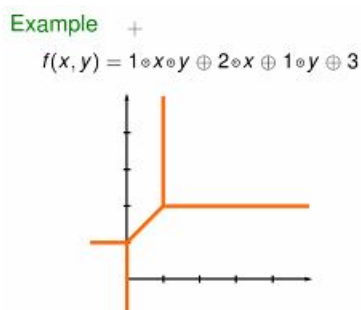
Definition 2.3 (Tropical Zero Set).

$$\mathcal{V}(f) := \{\mathbf{a} \in \overline{\mathbb{R}}^n \mid \text{the minimum of } f(\mathbf{a}) \text{ is attained by at least two terms}\}$$



We notice here that instead of points, this time we get some lines where the graph breaks. The tropical zeros are the ‘bending sets.’

Now we can consider the following and find the smallest in each region,



The points where they meet at a point is where they are equal. We have a tropical conic here. Changing the coefficient of the quadratic term from $1 \rightarrow -1$ the zero set changes the graph doesn't just flip.

Tropical hypersurfaces are 'combinatorial' polyhedral complexes with an interesting structure.

2.1 The Tropical Plane

Any two generic tropical lines meet at exactly one point. There is a unique tropical line going through any two generic points.



Five points make a conic.



There are no parallel lines in the tropical world. Every collection of lines always intersect once.



2.2 Why Tropical Geometry

Let K be an algebraically closed field with a valuation map: $\text{val} : K \rightarrow \overline{\mathbb{R}}$, that is,

$$\text{val}(a \cdot b) = \text{val } a \odot \text{val } b \quad \text{val}(a + b) \geq \text{val } a \odot \text{val } b \quad \text{val } a = \infty \iff a = 0$$

Definition 2.4 (Trivial Valuation).

$$\text{val } x = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{otherwise} \end{cases}$$

Example. Here are some fields with evaluations,

- Let $K = \mathbb{C}$ and with the trivial evaluation.
- $K = \overline{\mathbb{Q}}$ with the p-adic evaluation.
- $K = \mathbb{C}\{\{t\}\}$ the field of Puiseux series; a formal power series of the form,

$$a = c_0 t^{r_0} + c_1 t^{r_1} + \dots + c_k t^{r_k} + \dots$$

with $c_i \in \mathbb{C}$ and $r_0 < r_1 < \dots$ rational numbers with a common denominator and valuation $\text{val } a = r_0$.

One can tropicalise any $F \in K[x_1, \dots, x_n]$ to $\text{trop}(F) \in \overline{\mathbb{R}}[x_1]$ by substituting,

$$+ \rightarrow \oplus \quad \cdot \rightarrow \times \quad f \rightarrow \text{val } f$$

Suppose V be the zero locus of an ideal $J \subset K[x_1, x_2, \dots, x_n]$. Consider an ideal,

$$\text{trop}(J) := \langle \text{trop}(F) := F \in J \rangle \subset \overline{\mathbb{R}}[x_1]$$

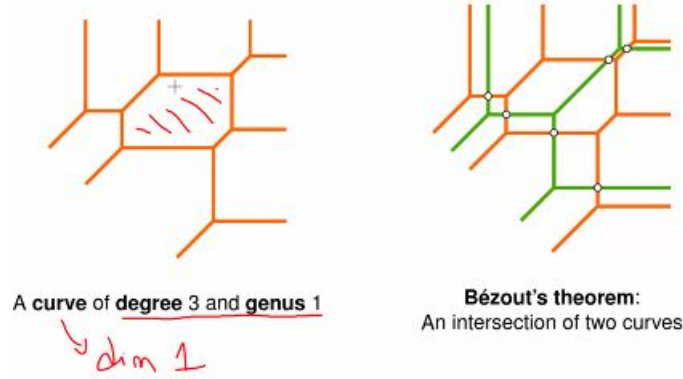
The tropicalisation of V is,

$$\text{trop} := \bigcap_{f \in \text{trop } J} \mathcal{V}(f)$$

Example. Let $K = \mathbb{C}\{\{t\}\}$ and let $J = \langle x - t \cdot y + q \rangle \subset K[x, y]$ and $V = \{x - t \cdot y + 1 = 0\} \subset K^2$. Then we just take the functions and do the tropicalisation, and then we get a tropical line. Which is, $\text{trop } V$.



Tropical varieties preserve many invariants of their defining algebraic varieties. It preserves degrees and genus, which is really nice as you can get rid of singularities. We can use the tropicalisation to make sense of some thing yucky and complicated. The tropical degree is the number of rays doing in the important directions.

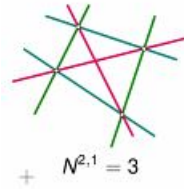


Tropical Geometry has another Bezout's Theorem (which is the same as normal geometry), we can use this with tropical varieties to see the points of intersections. In the diagram above, we have an orange degree 3 and green degree 2 curve. Then we have six intersections.

Definition 2.5 (Severi Degree). The severi degree $N^{d,\delta}$ of \mathbb{CP}^2 is the number of plane curves of degree d and δ nodal singularities passing through $\frac{(d+3)d}{2} - \delta$ generic points.

Example. – $N^{2,0}$ = number of smooth conics through five points. This is one.

– $N^{2,1}$ = number of 1-nodal conics through four points. This is three.



– $N^{3,1}$ = number of 1-nodal cubics through eight points. This is twelve.

Theorem 2.6 (Mikhalkin 2005). Severi degree can be computer tropically

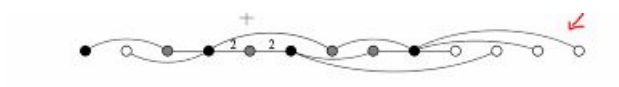
Theorem 2.7 (Formin-Mikhalkin 2010, conj. Francesco-Itzykson 1994). For a fixed δ there is a polynomial $N_\delta(d)$ such that for all $d \geq 2\delta$,

$$N^{d,\delta} = N_\delta(d)$$

Example. Here are some people that have solved certain values of this problem,

- Steiner 1848: $N^{d,1} = 3(d-1)^3$
- Cayley 1863: $N^{d,2} = \frac{3}{2}(d-1)(d-2)(3d^2 - 3d - 11)$
- Roberts 1867: $\delta = 3$
- Vainsencher 1995: $\delta = 4, 5, 6$

- Keiman-Piene 2001: $\delta = 7, 8$
- Block 2010: $\delta = 9, 10, 11, 12, 13, 14$



Similar approaches have succeeded in the study of Severi degree $N^{\Delta, d}$ of more general toric surfaces, double Hurwitz numbers $H_g(l, \mu)$, Welshinger invariants W_d, \dots