Year 2 — Colloquia

Based on lectures by Various

Notes taken by James Arthur

LMS Summer School 2021

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Contents

1 Markov Numbers and the free group on two generators - Caroline Series.

We are going to talk about three binary tree and the connections between them.

A Markov number is a solution to,

$$x^2 + y^2 + z^2 = 3xyz$$

If we set $x = y = z = 1$ and that's a solution. Let's not worry about negative solutions as here is another $(-x, -y, z)$.

Suppose x, y_1, z_1 is a solution you get,

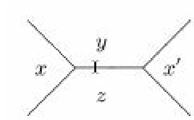
$$z^2 - 3xy_1z_1 + y_1^2 + z_1^2 = 0$$

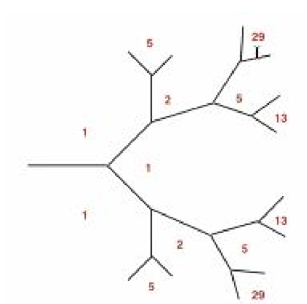
and so $x + x' = 3y_1z_1$. If we have (x, y_1, z_1) we can get $(3y_1z_1 - x, y_1, z_1)$. We could permute any of these.

Theorem 1.1. If we start with a solution, we can carry on permuting, we can get all the solutions,

$$(x, y, z) \to (3yz - x, y, z)$$

Proof. Start with (x, y, z), and let $(1, 1, 1)$ and thenget a load of solutions. We can now put these around the vertices of a binary tree. and we can do this again, to get a load more solutions, Let's now prove that this is





all of them,

Say it is special if two of x, y, z are equal. The only special solutions are $(1, 1, 1)$ and $(1, 1, 2)$. Say $x = y$ and then $2x^2 + z^2 = 3x^2z$. Hence $x^2 | z^2$ and so $z = kx$ and so $2 + k^2 = 3kx$ so $k|2$ and so $k = 1$ or 2.

Step 2: Show that if (x, y, z) is a solution with x ¡ y ¡ z if x' = 3yz - x and x ¿ y ¿ x'. **Step 3:** Take any non-special $x > y > z$ surrounding V and draw it's local tree with arrows. By step 2, there is an outgoing arrow from x to x'. **Claim:** The other two arrows at V point to V.

This follows from a change of variables, $\xi + \eta + \zeta = 1$ and so again, $\xi + \xi' = 1$ and for the other variables. Hence $\xi > \xi'$ and so $\xi > \frac{1}{2}$. But then, $\eta < \frac{1}{2}$ and $\zeta < \frac{1}{2}$, which means that $\eta < \eta'$ and $\zeta < \zeta'$, so the arrows point to V.

Step 4: From each non-special vertex there exists $\square$

There is a conjecture that says,

Conjecture. Suppose $x > y > z$ and $x > \hat{y} > \hat{z}$ are both solutions to Markov equations, then $y = \hat{y}$ and $z = \hat{z}$

The conjecture has been checked up to numbers 140 digits long.

Theorem. The uniqueness conjecture is truw if the largest number x is the triple of the form kp^n whiere p is prime and $k \leq 10^35$

A simpler proof is given in 2005 about x being a prime power.

1.1 Free Group on 2 generators

A free group of two generators, F_2, every element is a product of a, a^{-1}, b and b^{-1}. A string of these is called a word, there are no relations expect the identity relation.

A generator pair $(\alpha, \beta) \in F_2$ such that every $g \in F_2$ can be written in terms of α^{π} and $\beta^{\pm}$. You can get a new generator pair by inverting one generator or by conjugating.

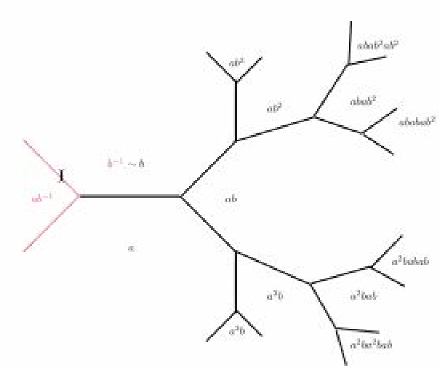
Theorem 1.2. Every generator pair can be obtained in this way starting from (a, b).

A word w which we can find a v such that they are generators are called primitive. We want all of the primitives of our group. We don't want conjuages so we assume that w isn't cyclically reduced.

$$w = e_1 e_2 \dots e_n$$
$$e_1^{-1} w e_1 = e_2 e_3 \dots e_n e_1$$

He shows that after cycliv permutations and cancelling, then if (α, β) is a generator we can just and shorten. For Example take, (ab^2abab^2, ab^2) can be reduced to (a, b).

We can now put these around our binary tree. Across an edge we have a generator pair. We have generator



pairs opposite eachother. We have some fascinating patterns called Sturmian sequences. There is every primite element is represented once and the otheres just have negative exponents.

Another proof uses albelianisation of $\mathbb{Z}^2$. If $w \in F_2$ and map it to $\psi(w) = (m, n) \in \mathbb{Z}^2$.We assume everything is commutative and so we can have $\hat{w} = a^m b^n$. We also note that, $\psi(w^{-1}) = -\psi(w)$ and $\psi(w) = \psi(w')$.

Theorem 1.3. Any primitive element of F_2 maps to a relatively prime pair. If w' is another primitive element with $\psi(w') = \psi(w)$, then they are conjuage.

What it's telling us that the rationals are an equivilence class around a tree. We shall now look at Farey tree.

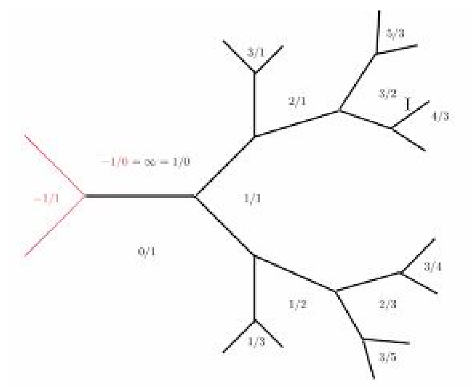
We say two rationals $\left(\frac{p}{q} \text{ and } \frac{r}{q}\right)$ are neighbours if $ps - rq = \pm 1$. This then means that,

$$\frac{p}{q} + \frac{r}{s} = \frac{p+r}{q+s}$$

this is called a farey sum and so,

$$\frac{p}{q} < \frac{p+r}{q+s} < \frac{r}{s}$$

Using the euclidean algorithm it is not hard to show that all positive rationals can be reached this way starting at $\frac{1}{0}$ and $\frac{0}{1}$. We can just add as we go around and now we have all of the rationals. We just go



around and multiply the trees.

Finally we make the connection. Consider elements of $SL(2,\mathbb{C})$ which all have determinant 1. Now we can consider the trace and it's invariant under conjigation. There are some other polynomial identities. They use the commutator and -2 and we can simplify things nicely,

$$TrATrBTrAB = (TrA)^2 + (TrB)^2 + (TrAB)^2$$

and so we divide by three and get the markov equation. We can also consider $TrAB^{-1}$ and get that $z + z' = 3xy$. Now it suffices to show that there exists these matrices. Let,

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

Now take the tree of generators of the free group and then put them into the Neilson tree and replace W bt $Tr\frac{W}{3}$.

Theorem. The tree of Markov number is found from the three of traces of the above matrices by dividing all entries by 3 and starting the triple $(1,1,1)$.

1.2 Tree of Traces

We used the tree of traces and got the special numbers. Why don't any old matrices work? We can sub in and do some generators and find it's trace.

Lemma 1.4. Given any triple of complex numbers (x, y, z) there are matrices $A, B \in SL(2,\mathbb{C})$ so that, $TrA = x$, $TrB = y$ and $TrAB = z$

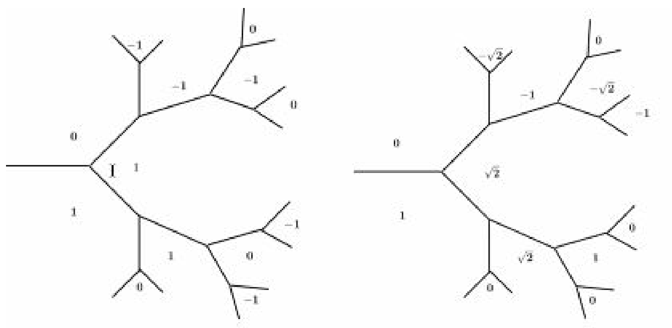
Proof. We take,

$$A = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & \zeta \\ \zeta^{-1} & y \end{pmatrix}$$

where $z = \zeta + \zeta^{-1}$ $\square$

and so now we just run through the tree and images of primitive elements. Some questions have been asked is, is the group generated by A and B free? If not, is it discrete?

Are the corresponding groups finite? If we look at the following,



You can see that you won't get past 0 and 1 for the first. Then in the second, taking $\sqrt{2}$, we can't more values than we have. If a group had generators 0, 1 would it be finite? So we now consider $SU(2)$,

$$M = \begin{pmatrix} a & b \\ \overline{a} & \overline{b} \end{pmatrix}$$

and they are unitary. These basically give us stereographic projections, it didn't preserve distance, but it does for angles. if we rotate our stereographic sphere, we get an angle preserving map. This then gives us $SU(2) \subset SL(2, \mathbb{C})$. This gives us a mobius transformation. Thn if we consider,

$$\begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix}$$

We get the trace as just $2\cos\frac{\theta}{2}$. Then out pops $0, 1, \sqrt{2}$.

Theorem 1.5. With one exception, every finite tree is associated to a regular solids, and correspeonds to finite representations to $F_2 \to S(U)$ with finite image. The exception is the dihedral group.

The other regular solid is the icosohedron, hence giving the icosohedral group. The sphere is coverd with twenty copies of the a equilateral triangle of angle $\frac{\pi}{5}$. This then moves forward with subgroups generated by rotations of orders 2, 3 and 5. So we expect to get a finite tree starting from the values, $2\cos\frac{\pi}{2} = 0$, $2\cos\frac{\pi}{3} = 1$ and $2\cos\frac{\pi}{5} = \omega$ and after some algebra we get that $\omega - \omega - 1 = 0$ and hence after some algebra we have finite values.

2 A Glimpse of Tropical Geometry - Felipe Ricon, QMUL

Resources - *Introduction to tropical Geometry (Book), First Steps in Tropical Geometry (Article)*

Definition 2.1 (Tropical Semiring)**.** The tropical semiring is,

$$\overline{\mathbb{R}} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$$

where,

$$\oplus := \min \qquad \odot := +$$

Example.

$$3 \oplus 5 = 3 \qquad 4 \odot 7 = 11$$

$$\infty \oplus a = a \qquad 0 \odot a = a$$

$\overline{\mathbb{R}}$ is an idemoptent remiring as there are no additive inverses.

$$a \odot (b \oplus c) = a \odot b \oplus a \odot c$$

but things like,

$$2 \oplus x = 5$$

doesn't have a solution in our semiring. But addition is idempotent,

$$a \oplus a = a$$

Example. $(x \oplus y)^3 = x^3 \oplus x^2 \odot y \oplus x \odot y^2 \oplus y^3 \equiv x^3 \oplus y^3$

Denote $\overline{\mathbb{R}}[x_1, \ldots, x_n]$ the semiring of the tropical polynomials on the variables $x_1, \ldots, x_n$.

Example. $f(x) = x^2 \oplus 1 \odot x \oplus 4$ which then we can plot,

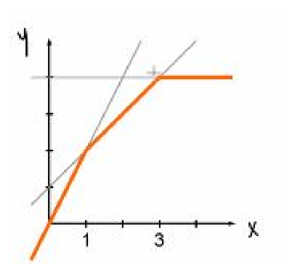


Figure 1: A graph of $f(x) = (x \oplus 1) \odot (x \oplus 3$

we note we can factor them as,

$$f(x) = (x \oplus 1) \odot (x \oplus 3)$$

and where we factor it is where the graph bends.

Theorem 2.2 (Fundemental Theorem of Algebra)**.** If $f(x) \in \overline{\mathbb{R}}[x]$ has degree d then,

$$f(x) \equiv c \odot (x \oplus a_1)^{m_1} \odot (x \oplus a_1)^{m_2} \odot \cdots \odot (x \oplus a_r)^{m_r}$$

where $a_1, \ldots, a_r \in \overline{\mathbb{R}}$ and $m_1 + \cdots + m_r = d$

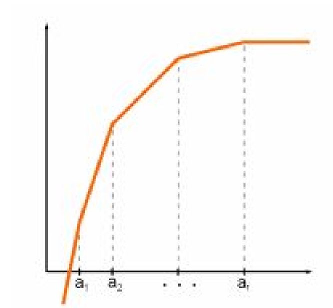
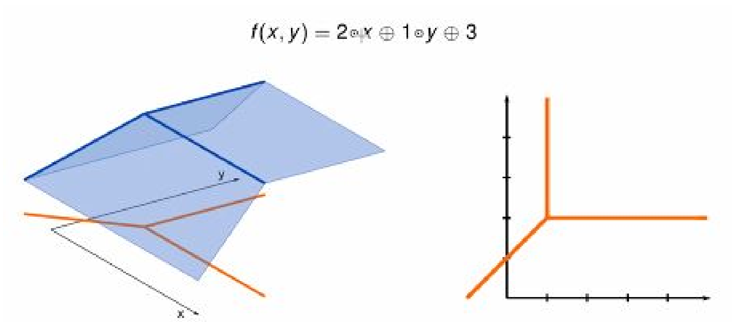


Figure 2

Any $f(\mathbf{x}) \in \overline{\mathbb{R}}[x_1, \ldots, x_n]$ is the minimum of a load of finite number of affine functions. Then we define,

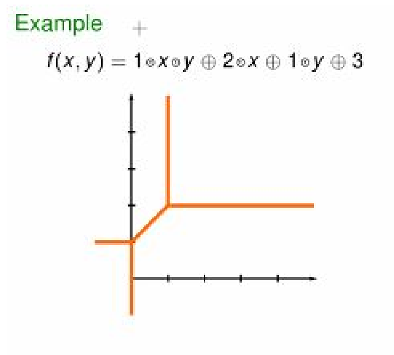
Definition 2.3 (Tropical Zero Set)**.**

$$\mathcal{V}(f) := \{\mathbf{a} \in \overline{\mathbb{R}}^n \mid \text{ the miniumu of } f(\mathbf{a}) \text{ is attained by at least two terms}\}$$



We notice here that instead of points, this time we get some lines where the graph breaks. The tropical zeros are the ‘bending sets.’

Now we can consider the following and find the smallest in each region,

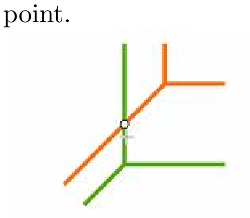


The points where they meet at a point is where they are equal. We have a tropical conic here. Changing the coefficient of the quadratic term from $1 \to -1$ the zero set changes the graph doesn't just flip.

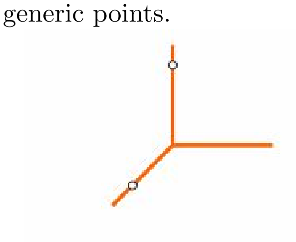
Tropical hypersurfaces are 'combinatorial' polyhedral complexes with an interesting structure.

2.1 The Tropical Plane

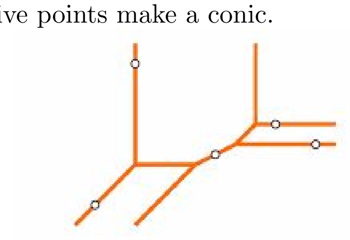
Any two generic tropical lines meets at exactly one point.



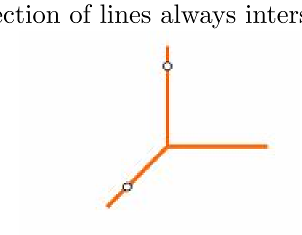
There is unique tropical line going through any two generic points.



Five points make a conic.



There are no parallel lines in the tropical world. Every collection of lines always intersect once.



2.2 Why Tropical Geometry

Let K be an algebraically closed field with a valuation map: $\mathrm{val} : K \to \overline{\mathbb{R}}$, that is,

$$\mathrm{val}(a \cdot b) = \mathrm{val}\, a \odot \mathrm{val}\, b \quad \mathrm{val}(a+b) \geq \mathrm{val}\, a \odot \mathrm{val}\, b \quad \mathrm{val}\, a = \infty \iff a = 0$$

Definition 2.4 (Trivial Valuation)**.**

$$\mathrm{val}\, x = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{otherwise} \end{cases}$$

Example. Here are some fields with evaluations,

- Let $K = \mathbb{C}$ and with the trivial evaluation.
- $K = \overline{\mathbb{Q}}$ with the p-adic evaluation.
- $K = \mathbb{C}\{\{t\}\}$ the field of Pulseux series; a formal power series of the form,

  $$a = c_0 t^{r_0} + c_1 t^{r_1} + \cdots + c_k t^{r_k} + \ldots$$

  with $c_i \in \mathbb{C}$ and $r_0 < r_1 < \ldots$ rational numbers with a common denominator and valuation $\mathrm{val}\, a = r_0$.

One can tropicalise any $F \in K[x_1, \ldots, x_n]$ to $\mathrm{trop}(F) \in \overline{\mathbb{R}}[x_1]$ by substituting,

$$+ \to \oplus \quad \cdot \to \times \quad f \to \mathrm{val}\, f$$

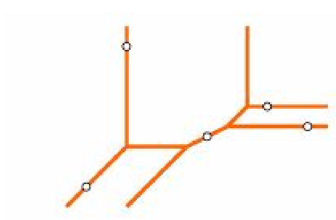
Suppose V be the zero locus of an ideal $J \subset K[x_1, x_2, \ldots, x_n]$. Consder an ideal,

$$\mathrm{trop}(J) := \langle \mathrm{trop}(F) := F \in J \rangle \subset \overline{\mathbb{R}}[x_1]$$

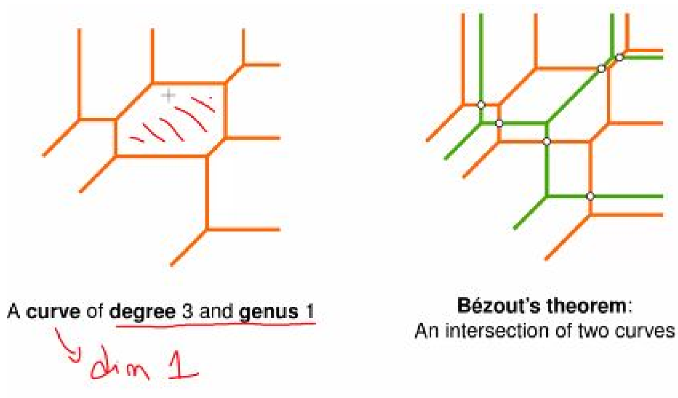
The tropicalisation of V is,

$$\mathrm{trop} := \bigcap_{f \in \mathrm{trop}\, J} \mathcal{V}(f)$$

Example. Let K= {{t}} and let $J = \langle x - t \cdot y + q \rangle \subset K[x, y]$ and $V = \{x - t \cdot y + 1 = 0\} \subset K^2$. Then we just take the functions and do the tropicalisation, and then we get a tropical line. Which is, $\mathrm{trop}\, V$.



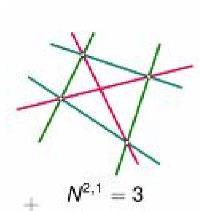
Tropical varieties preseves many invariants of their defining algebraic varieties. It preserves degrees and genus, which is really nice as you can get rid of singularities. We can use the tropicalisation to make sense of some thing yucky and complicated. The tropical degree is the number of rays doing in the imporant directions.



Tropical Geometry has another Bezout's Theorem (which is the same as normal geometry), we can use this with tropical varieties to see the points of intersections. In the diagram above, we have an orange degree 3 and green degree 2 curve. Then we have six intersections.

Definition 2.5 (Severi Degree)**.** The severi degree $N^{d,\delta}$ of $\mathbb{CP}^2$ is the number of plane curves of degree d and δ nodal singularities passing through $\frac{(d+3)d}{2} - \delta$ generic points.

Example.
- $N^{2,0}$ = number of smooth conics through five points. This is one.
- $N^{2,1}$ = number of 1-nodal conics through four points. This is three.



- $N^{3,1}$ = number of 1-nodal cubics through eight points. This is twelve.

Theorem 2.6 (Mikhalkin 2005)**.** Severi degree can be computer tropically

Theorem 2.7 (Formin-Mikhalkin 2010, conj. Francesco-Itzykson 1994)**.** For a fixed δ thre is a polynomial $N_\delta(d)$ such that for all $d \geq 2\delta$,

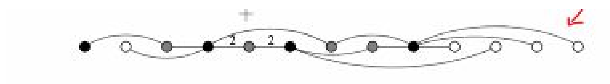
$$N^{d,\delta} = N_\delta(d)$$

Example. Here are some people that have solved certain values of this problem,

- Steiner 1848: $N^{d,1} = 3(d-1)^3$
- Cayley 1863: $N^{d,2} = \frac{3}{2}(d-1)(d-2)(3d^2 - 3d - 11)$
- Roberts 1867: $\delta = 3$
- Vainsencher 1995: $\delta = 4, 5, 6$

– Keiman-Piene 2001: $\delta = 7, 8$

– Block 2010: $\delta = 9, 10, 11, 12, 13, 14$



Similar approaches have succeeded in the study of Severi degree $N^{\Delta,d}$ of more general toric surfaces, double Hurwitz numbers $H_g(l, \mu)$, Welshinger invariants W_d, ...

3 Cluster Algebras, Quivers mutations and triangulated surfaces - Anna Felikson

3.1 Cluster Algebra

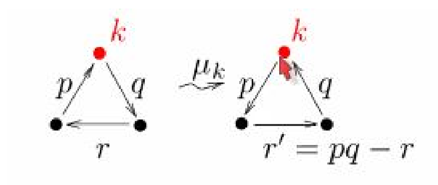
Very new, twenty years ago. It go connected to many many different areas of Maths.

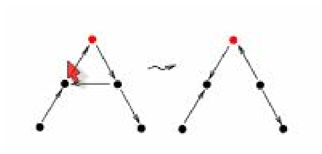
Theorem 3.1 (Ptolemy Theorem)**.** ef = ac + bd on a cyclic quadrilateral

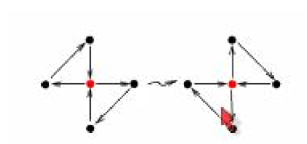
Definition 3.2 (Quiver)**.** A quiver is a direct graph without loops and 2-cycles.

Definition 3.3 (Mutation)**.** A mutation , μ_k of quiver:

- reverse all arrows incident to k
- for every path through k with and $p, q > 0$, do:







Example.

If we have a quiver with six arrows, we can mutate however we want and do it in all directions and then we get new quivers and we can mutate again.

$$\text{Iterated mutations} \longrightarrow \text{many other quivers}$$

$$Q \longrightarrow \text{It's mutation class}$$

Property. $\mu_k \circ \mu_k(Q) = Q$ for any quiver Q.

We get a regular graph.

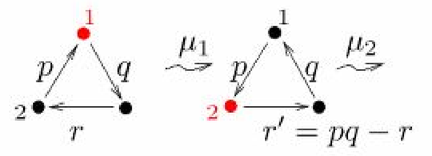
Definition 3.4. A quiver is of finite mutation type if its mutation class contains finitely many quivers

Question. Which quivers are of finite mutation type?

Quick answer, not many.

Lemma 3.5. If Q is connected $|Q| \geq 3$ and Q contains arrow $\longrightarrow$ with $p > 2$, then Q is mutation finite.

if $q > r > 0$ and $p > 2$, then you can make each step bigger and bigger at every step.



3.2 Cluster Algebra

Definition 3.6 (Seed)**.** A seed is a pair $(Q, \mathbf{u})$, where,

- Q is a quiver with $n := |Q|$ vertices.
- $\mathbf{u} = (u_1, \dots, u_n)$ is a set of rational functions.

Initial seed is $(Q_0, \mathbf{u}_0)$
Seed mutation: $\mu_k(Q, \mathbf{u}) = (\mu_k(Q), \mathbf{u'})$ where,

$$u'_k = \frac{1}{u_k}\left(\prod_{i \to k} u_i + \prod_{k \to j} u_j\right)$$

if $u'_i = u_i$ if $i = k$
Cluster Variable: A function of seeds **Cluster Algebra:** An alegbra with seeds and $+$ and $*$.

We can link this to Markov Equation using the Markov quiver.
By definition

$$u_i = \frac{P(x_1, \dots x_n)}{R(x_1, \dots, x_n)}$$

where P and R are polynomials.

Theorem 3.7 (Laurent Phenomenon)**.** R is a monomial, $R = x_1^{d_1} x_2^{d_2} \dots x_n^{d_n}$

Theorem 3.8 (Positivity)**.** P has positive coefficients.

Definition 3.9 (Finite)**.** A cluster algebra is of finite type, so contains finitely many cluster variables.

Theorem 3.10. A cluster algebra $\mathcal{A}$ is of finite type iff Q is mutation equivilent to an orientation of a Dynkin diagram, A_n, D_n, E_6, E_7, E_8.

Dynkin diagrams describe: finite reflection group, semisimple Lie Algebras, surface singularities...
A cluster algbra $\mathcal{A}(Q)$ is of finittype if Q is of finite mutation type.

Example. – $n = 2$

$$\bullet \longrightarrow \bullet$$

– Quivers arising from triangulated surfaces.

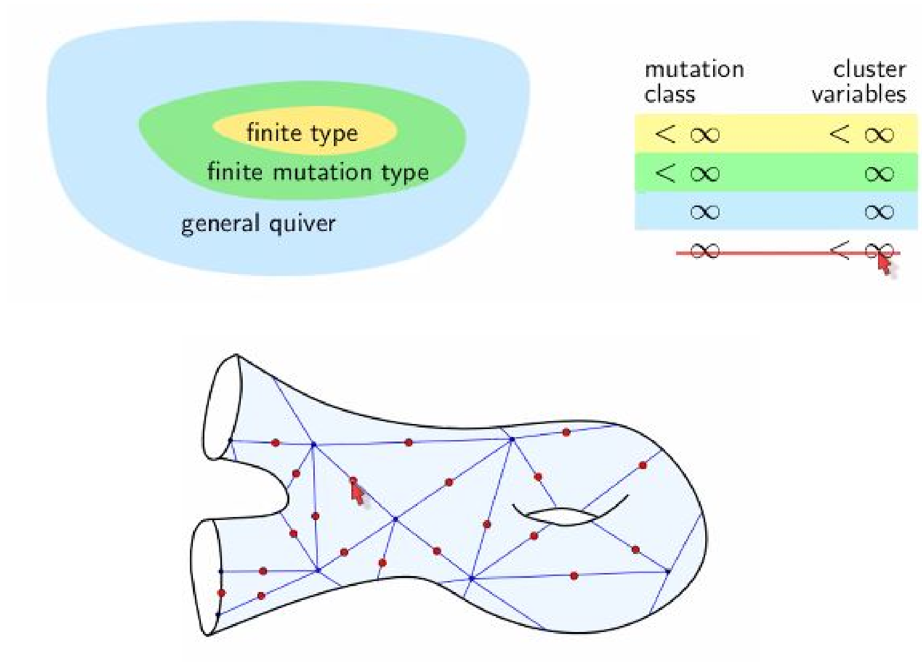


Figure 3

– Finitely many except that.

To create a quiver from a triangulation
Every edge of the triangulation, is a vertex of the quiver. Two eges of one triangle is an arrow of the quiver. This allows us to mutate by just rotating the diagonal.

Remark. Q from a triangulation $\implies$ weights of arrow ≤ 2.

as every arc lies at most in two triangules.

Theorem 3.11 (Hatcher 1991)**.** Every two triangulations of the same surface are connected by a sequence of flips.

Corollary 3.12. (i) Quivers from triangulations of the same surface are mutation-equivalent (and form the whole mutation class).

(ii) Quivers from triangulations are mutation-finite.

Question. What else is mutation finite?

Any triangulated surface can be glued of,

Proposition. $\{Q$ is from triangulation$\} \iff \{Q$ is block-decomposable$\}$

Question. How to find all mutation-finite but not block-decomposable quivers

Interlude: How to classify hyperbolic space,

(i) They correspond to some polytopes.

(ii) Combinatorics of these polytopes are described by:

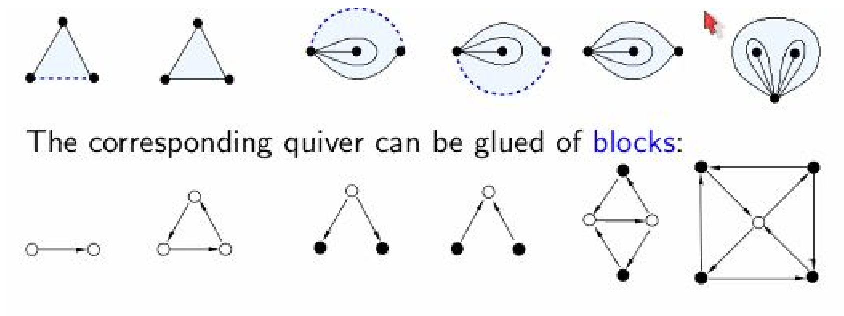


Figure 4

(a) subdiagrams corresponding to finite objects

(b) minimal subdiagrams corresponding to infinite objects.

Idea: Classify minimal non-decomposable quivers!

Lemma 3.13. If Q is minimal non-decomposable quiver then $|Q| \leq 7$

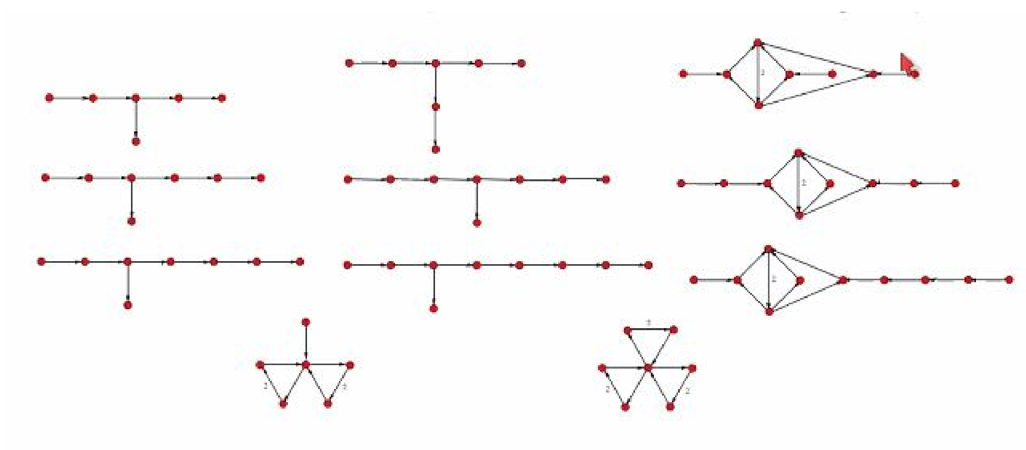
Lemma 3.14. If Q is a minimal non-decomposable mutation-finite quiver, then it is equivilent to one of,

Theorem 3.15. Let Q be a connected quiver of finite mutation type:

– $|Q| = 2$

– Q is obtained from a triangulated surface.

– Q is mutation-equivalent to one of the following,



Proof. Terrible and technical. It follows the same step as classifications of tesselations. □