Year 2 — Combinatorics of Young tableaux and symmetric groups

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Contents

1 Overview

- Basic reminders about groups

- Definitions of partitions, Young Diagrams and tableaux.

- How many standard tableaux are there?

- There are two answers; the hook formula and the determinant formula.

- Some ideas of the proof - prove the two formulas give the same answer. More detail about the hook formula, via a bijective proof.

- Sketch of the relevance to representations of the symmetric group.

Useful resource - Bruce E. Sagan, *The Symmetric Group: Representations, Combinatorial Algorithms and Symmetric Functions*

1.1 Symmetric Group and Permutations

Let $n \in \mathbb{N}$,

Definition 1.1. Symmetric Group The nth symmetric group is S_n, and it's the group of all bijections from $\{1, 2, \ldots, n\}$ to itself.

Permutations are multiplied from right to left as that's the convention for function composition.
There are several standard ways to write perms, The same permutations σ in S_7, two line notation,

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 2 & 5 & 7 & 3 & 4 & 1 \end{matrix}$$

and one line notation,

$$6 \quad 2 \quad 5 \quad 7 \quad 3 \quad 4 \quad 1$$

and in cycle notation, $(1647)(35)(2)$ or $(1647)(35)$ this means that $1 \to 6 \to 4 \to 7 \to 1 \ldots$ and $3 \to 5 \to 3 \to \ldots$ and 2 maps to itself. The two doesn't necessarily need to be written as it may be seen as redundant.

The composition of $\pi\sigma$ where $\sigma = (1647)(35)(2)$ and $\pi = (14)$, then,

$$\pi\sigma = (14)(1647)(35) = (16)(35)(47)$$

this isn't very efficient as there are things written more than once. Read right to left, and just work through it and write it down.

Definition 1.2. k-cycle A K-cycle is a cycle containing k elements

The cycle type of a permutation is an expression of the form $(1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$ where M_k is the number of k-cycles.
For example, σ has cycle type, $(1^1, 2^1, 3^0, 4^1, 5^0, 6^0, 7^0)$.

Definition 1.3. Conjugation In Group Theory, take a group G, the elements g and h are conjugate if the exists some $k \in G$, st, $g = khk^{-1}$.

Conjugacity is an equivalence relation, so the group splits up into non-intersecting conjugacy classes.

Definition 1.4. Conjugacy Class The set of all elements conjugate to $g \in G$ is called the conjugacy class of g.

Conjugacy Classes and Cycle Types

Remark. For the inverse, write down each of the elements in reverse order.

If,
$$\pi = (i_1, i_2, \ldots, i_l) \ldots (i_r, i_{r+1}, \ldots, i_n) \in S_n$$

then,
$$\sigma\pi\sigma^{-1} = (\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_l)) \ldots (\sigma(i_r), \sigma(i_{r+1}), \ldots, \sigma(i_n))$$

You have the same cycle type, you just 'apply' σ. Same cycle type and the number replaced with what you are conjugating by. For example, $\pi = (14)$ (2 cycle, transposition) and $\sigma = (1647)(35)$ as before, $\sigma\pi\sigma^{-1} = (67)$. It follows that,

Lemma 1.5. Two permutations are conjugate if and only if they have the same cycle type.

We haven't only done one side as if we take a σ, we can just take an appropriate π.

Conjugacy is very important in group theory, they are used in classification results and philoe theorems. A group acts on itself by conjugation, it acts by conjugation of that group action. You have orbits and stabilizers and you can deduce things about the structure of the group. There are irreducible representations and there is each one for each conjugacy class of the group.

2   Paritions and Young Tableaux

Definition 2.1. *Partition* A partition of n is a finite sequence of natural numbers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ where $\lambda_i \geq \lambda_{i+1}$ for all i and $\sum_{i=1}^{r} \lambda_i = n$. We write $(\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash n$ this to mean partition,

For example, $(5, 3, 1) \vdash 9$. We represent paritions by diagrams, in the form of left justified arrays of boxes. For the partition $(\lambda_1, \lambda_2, \ldots, \lambda_r)$, we draw λ_1 on the top, then λ_2, and so on. Let us take $(5, 3, 1)$.

We saw earlier that two elements of S_n are in the same conjugacy class if and only if they have the same cycle type.

We can equivalent record a cycle by a partition of n. For our previous example, $\sigma = (1647)(35)(2)$ and record the cycle type as $(4, 2, 1)$, as per our pervious definition. This means that there are a bijection between partitions of n
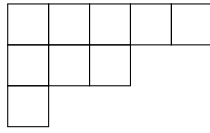

Figure 1: A Young Diagram

Definition 2.2. A Young Tableau of shape λ, where $\lambda \vdash n$, is a bijective assignment of the numbers $1, 2, \ldots, n$ to the boxes of the diagram of λ.
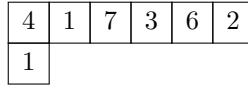

Figure 2: A Youngs Tableau

Clearly, the number of such tableaux is $n!$. Such a tableaux is called standard if each column and row is increasing.

First we fix n, then draw the diagram and then assign 1 up to n into the boxes with constraints of increasing in each row and column. We will write f^λ as the number of standard young tableaux of shape λ. Understanding the number of f^λ and our importance will be our focus.

2.1   Alfred Young

Reverend Alphred Young, came up with the tableaux. "The original man of the year, who would have ranked a higher place in the list had he directed his attention to the examination schedule". He attaned a 2:2 from Cambridge.

2.2   Tableaux

We are going to consider S_3. Here are the 6 young tableaux of shape $(2, 1)$.
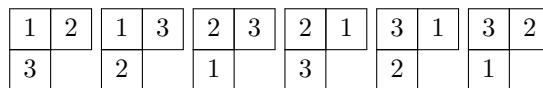

Figure 3: The Young Tableaux for $(2, 1)$.

The first two are standard tableaux and the rest are not. The conclusion is that $f^{(2,1)} = 2$.

3 Hook and Determinant Formula

3.1 Hook Formula

The box in row i and column j has coordinate (i, j). For example this is $(2, 3)$

Let $\lambda \vdash n$. For the box b with coordinates (i, j) in the diagram of the shape λ, its hook, H_b consists of all the boxes in the same row from b onwards and the boxes from b downwards.

$$H_n \{(i, j') \text{ a box of } \lambda \,|\, j' \geq j\} \cup \{(i', j) \text{ a box of } \lambda \,|\, i' \geq i\}$$
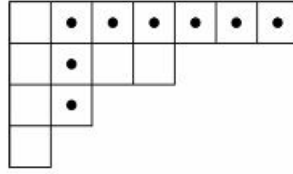
For example, the hook of $(1, 2)$ is shown,

Figure 4

The corresponding hooklength, $h_b = h_{i,j}$ is the number of boxes in the hook:

$$h_b = |H_b|$$

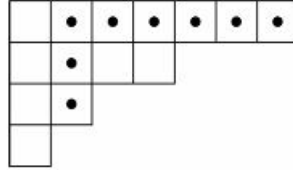each box has a hook length. In the example $h_{1,2} = 8$.

Figure 5

Theorem 3.1. For $\lambda \vdash n$,

$$f^\lambda = \frac{n!}{\prod_{(i,j) \in \lambda} h_{i,j}}$$

For $\lambda = (3, 2) \vdash 5$, the hook lengths are indicated for each box.

4	3	1
2	1	

and the hook formula gives us that,

$$f^{(3,2)} = \frac{5!}{4 \cdot 3 \cdot 2} = 5$$

Now consider this diagram, again same numbering convention,

9	7	6	4	2	1
6	4	3	1		
4	2	1			
1					

and the formula gives, $f^{(6,4,3,1)} = \frac{14!}{9 \cdot 7 \cdot 6 \cdot 4 \cdot 2 \cdot 6 \cdot 4 \cdot 3 \cdot 4 \cdot 2}$

3.2 Determinant Formula

Let's look at another formula for f^λ calculated via a determinant.

We will use the convention that $\frac{1}{r!} = 0$ if $r < 0$ and $0! = 1$.

Theorem 3.2. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash n$,

$$f^\lambda = n! \left| \frac{1}{(\lambda_i - i + j)!} \right|$$

where the determinant is of an $r \times r$ matrix (the thing in the mod symbols is what the matrix is defined as, i.e. $a_{ij} = \frac{1}{(\lambda_i - i + j)!}$).

For $\lambda = (3, 2) \vdash 5$, we get,

$$f^{(3,2)} = 5! \begin{vmatrix} \frac{1}{3!} & \frac{1}{4!} \\ \frac{1}{1!} & \frac{1}{2!} \end{vmatrix}$$

and we can calculate this,

$$f^\lambda = 5!(\frac{1}{6} \cdot \frac{1}{2} - \frac{1}{24})$$
$$= 5! \left(\frac{1}{24} \right)$$
$$= 5$$

Remark. To write down the correct denominators in the determinant, note that the factorials of the parts of λ appear down the main diagonal. The other entries involve increasing the number in the factorial by 1 for each step to the right and decrease it by one to the left.

For $\lambda = (4, 2, 2, 1) \vdash 9$, we get,

$$f^{(4,2,2,1)} = 9! \begin{vmatrix} \frac{1}{4!} & \frac{1}{5!} & \frac{1}{6!} & \frac{1}{7!} \\ \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} \\ \frac{1}{0!} & \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} \\ 0 & 0 & \frac{1}{0!} & \frac{1}{1!} \end{vmatrix}$$

3.3 Proving Equality

Now we aim to assertain that for $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$,

$$\frac{n!}{\prod_{(i,j) \in \lambda} h_{i,j}} = n! \left| \frac{1}{(\lambda_i - i + j)!} \right|$$

Equivalently,

$$\frac{1}{\prod_{(i,j) \in \lambda} h_{i,j}} = \left| \frac{1}{(\lambda_i - i + j)!} \right|$$

We are going to write $\det_\lambda$ for the determinant on the right hand side. The first thing to do is to write $\det_\lambda$ in terms of hook lengths of boxes in the first column of the diagram of λ, using,

$$\lambda_i - i + r = h_{i,1}$$

Exercise. This is an exercise, draw a diagram. If a partition has r parts, it has r rows in the diagram.

So,

$$\det_\lambda = \left| \frac{1}{(h_{i,1} - r + j)!} \right|$$

Now we are going to do induction on n. The main step is removing the first column and comparing. So check some base case, for $n = 1$. Since $\lambda = (1)$ and,

$$\det_{(1)} = 1 = \frac{1}{h_{1,1}}$$

Lets write $\overline{\lambda}$ for the partition corresponding to removing the first column of the diagram λ. So,

$$\overline{\lambda} = (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_3 - 1)$$

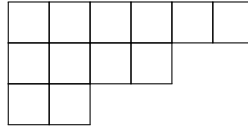($\overline{\lambda}$ could have less entries than λ).

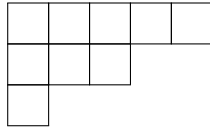Figure 6: $\lambda = (6, 4, 2)$

Figure 7: $\overline{\lambda} = (5, 3, 1)$

Notice that the hook length of $\overline{\lambda}$ are the hook lengths $h_{i,j}$ of λ, for $j \geq 2$, that is excluding those for boxes in the first column. Use row and column operations to show that,

$$\det_\lambda = \prod_{i=1}^r \frac{1}{h_{i,1}} \det_{\overline{\lambda}}$$

The result then follows by induction.
A bit of care is needed in the case where $\overline{\lambda}$ has fewer rows than λ, but it still works. This relates to the conventions we made, but it still works out OK.

3.3.1   Row and Column Operations

Lets write C_j for the jth column.

- Pull out $\frac{1}{h_{i,1}!}$ from row i for each i

- Replace C_1 by $C_2 - C_1$

– Replace C_2 by $C_3 - 2C_2$

– Continue in this way, replacing C_{j-1} by $C_j - (j-1)C_{j-1}$ for $4 \le j \le r$

– Put $\frac{1}{h_{i,1}!}$ back into row i.

Let's consider $\lambda = (3,2) \vdash 5$,

4	3	1
2	1	

So we have $h_{1,1} = 4$, $h_{2,1} = 2$. Removing the first column gives $\overline{\lambda} = (2,1) \vdash 3$.

$$
\begin{aligned}
\det_{3,2} &= \begin{vmatrix} \frac{1}{3!} & \frac{1}{4!} \\ \frac{1}{1!} & \frac{1}{2!} \end{vmatrix} \\
&= \frac{1}{4!2!} \begin{vmatrix} 4 & 1 \\ 2 & 1 \end{vmatrix} \\
&= \frac{1}{4!2!} \begin{vmatrix} 4-1 & 1 \\ 2-1 & 1 \end{vmatrix} \\
&= \frac{1}{4!2!} \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} \\
&= \frac{1}{4!2!} \begin{vmatrix} \frac{3}{3!} & \frac{1}{3!} \\ 1 & 1 \end{vmatrix} \\
&= \frac{1}{4 \cdot 2} \begin{vmatrix} \frac{1}{2!} & \frac{1}{3!} \\ \frac{1}{0!} & \frac{1}{1!} \end{vmatrix} \\
&= \frac{1}{4 \cdot 2} \det_{(2,1)} \\
&= \frac{1}{h_{1,1} \cdot h_{2,1}} \det_{\overline{\lambda}}
\end{aligned}
$$

4 Bijective Proof of Hook Formula

We are going to sketch a bijective proof of this formula. All details won't be included, just some bits of intution.
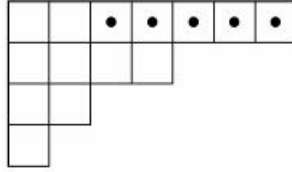
What to get from this section:

– Get across some idea / intuition about why this should work.

– The idea of doing a bijective proof. Where this comes from with the formula.

4.1 Arms and Legs

Definition 4.1 (Arms). Let $\lambda \vdash n$. For the box b with coordinates (i,j) in the diagram of shape λ, its arm, A_b, consists of all boxes on the same row to the right of b.

$$A_b = \{(i,j') \text{ a box of } \lambda \mid j' > j\}$$

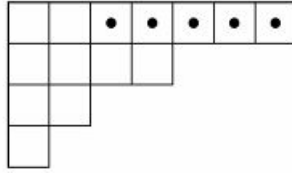It's arm length is $a_b = |A_b|$.

with $a_{1,2} = 5$

Definition 4.2 (Legs). Let $\lambda \vdash n$. For the box b with coordinates (i,j) in the diagram of shape λ, its leg, L_b, consists of all boxes on the same column below b.

$$L_b = \{(i',j) \text{ a box of } \lambda \mid i' > i\}$$

It's arm length is $l_b = |L_b|$.

with $l_{1,2} = 2$

Let $\lambda \vdash n$. It follows that, from the definitions,

$$h_{i,j} = a_{i,j} + l_{i,j} + 1$$

Definition 4.3 (Hook Tableux). A hoot tableaux of shape λ is a diagram J of shape λ with integer entries satisfying,

$$-l_{i,j} \leq J_{i,j} \leq a_{i,j}$$

Thus there are $h_{i,j}$ choices for the (i,j) entries in a hook tableau and so $\prod_{(i,j)\in\lambda} h_{i,j}$ hook tableaux of shape λ. We have produced a reasonable set with the size is an important player in the hook formula. This look vaugely relevant in the denominator of the hook formula.

4.2   The Bijective Proof

We can to rewrite the hook formula,

$$n! = f^\lambda \prod_{(i,j)\in\lambda} h_{i,j}$$

A bijective proof is forming a bijection between two finite sets, so look at the size and say they are the same size, so sets are basically the same. Hence we want natural number equal natural number. Recall that there are $n!$ tableaux of shape λ.

So we can prove the formula by establishing a bijection,

{Young tableux of shape λ} $\iff$ {$(P, J) \,|\, P$ a standard young tableau of shape λ and J a hook tableau of shape λ}

We are going to note that using pairs is useful in bijective proofs. We now have a plan. We shall show the forward direction, but not show it's a bijection.

4.2.1   The Forward Direction - Standardisation Algorithm

We want to standardise the tableau (P) and we can track what we are doing, i.e. tracking data J. We are going to describe an algorithm to a syandard youngs tableau and then track things using J. This gives a forward direction.

This process is carried out on the entries in the box of T, ordered from the bottom and the right. I.e,

$$(i,j) \le (i',j') \iff j > j' \text{ or } j = j' \text{ and } i \ge i'$$

For example, consider $\lambda = (3, 2, 1)$ and the order,

b_6	b_3	b_1
b_5	b_2	
b_4		

Notation. We write $T^{\le b}$ to be the part of T containing all the indicated boxes. Similarly $T^{<b}$

The algorithm will construct a sequence of pairs $(T_1, J_1) = (T, 0), (T_2, J_2), \ldots, (T_n, J_n) = (P, J)$ (the zero is a zero hook tableau and at the end P is standard), such that $T^{\le b_k}$ is standard and J_k is a hook tableau which is zero outside $J_k^{\le b_k}$. We are increasing the standardisation at each step.

We need to know what we do at each step, to each T's and the J's. For a box $b = (i, j)$ such that $T^{<b}$ is standard while $T^{\le b}$ is not standard. Do: Consider the entries next to b to the right and below, which ever is smaller (call it b'), swap them. Rename b' as b and repeat. When $T^{\le b}$ is standard, stop.

How do we know this terminates? $T^{\le b}$ will eventually become standard, in the worst case when there are no further boxes to the right or down from b. When the process terminates, we have our next T.

For the hook tableau J which tracks the changes, we start from J_1 with all entries of 0. Then if the entry in position (i, j) moves to position (i', j'), we change J as follows. It only changes in position in column j and i to i':

$$(J_{\text{new}})_{h,j} = \begin{cases} (J_{\text{old}})_{h+1,j-1} & \text{for } i \le h < i' \\ j' - j & \text{for } h = i' \end{cases}$$

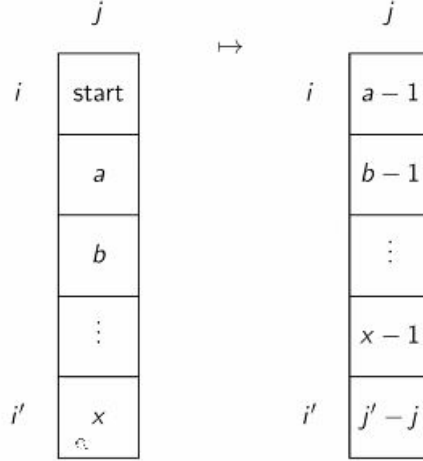So start position (i, j) and end position (i', j'), the change to J is:

Figure 8: The Algorithm on a few rows.

Now, let's look at an example:
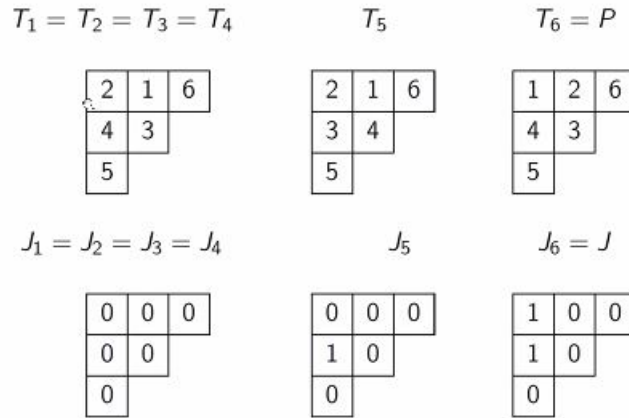
Figure 9: An example of the algorithm

We start with a non-standard tableaux T_1. Nothing happens for a while, we take the 6 and then that is standard. Next, we consider 6 and 3, then they are again standard. Now, $6, 3$ and 1, this is again standard. We have something to check, but these three are standard. Next, again they are disconnected and so standard. We now go to T_5 and we now have to do something, the 4 violates the standardness. We now look at the right and see a 3 and a 5 below. 3 is lower, so we now swap them. Now check if it's standard, but it is, so can output T_5. We add a just change the $(1, 2)$ zero to one in J_5. Now we go to the last box and look at the

2. Is it standard? No, so move through the standard steps, we see 1 and 3 and so swap 1 and 2. Then do the same previous case in the tracking data. [1]

4.2.2   Role of Hook Length

We must check that each J is a hook tableau. Essentially, J keeps track of how many places to the right and down each entry moves and this gives the constraints of a hook tableau. From (i, j) we can at most moce $a_{i,j}$ to the right, so the upper bound should be clear. And we can move at most $l_{i,j}$ places down. This gives the lower bound, which is attained in the case where everything below (i, j) has it's order reversed. It should be plausible that the tracking data in the J's provide enough data to go from T to P. This is proved by setting up a map in the other direction and prove it's an inverse of the previous process. It's an algorithm which reverses the provious steps. It's a bit more technical to describe and it involves some details that means its sometimes not well behaved.

Heres how it works, $\lambda = (2, 1) \vdash 3$. Considering the lengths we see that J has $-1 \leq J_{1,1} \leq 1$ and $H_{2,1} = J_{2,1} = 0$. There are 6 young tableux of shape $(2, 1)$, as seen above. The first two are the only standard ones. Then we can see corresponding pairs to them under the bijection,

we can see the J's below with tracking data and tracking data of -1, as we are moving things down.

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[1]Note a typo on the second row, swap on T_6.

5   Links to Symmetric Group

This lecture is an attempt to answer the question, who cares?

One answer is,

$f^\lambda = $ the dimension of the irreducible representation S^λ of the symmetric group S_n to the partition $\lambda \vdash n$.

Very loosely, a representation is an object on which the group acts, and an irreducible one that doesn't contain any smaller non-trivial ones. We will be looking specifically at vector spaces and we can build our actions from smaller building blocks, which are our irreducible representations.

5.1   Group Representation

Definition 5.1 (Group Representations)**.** Let G be a finite group with idenity e. A (finite dimensional complex vector space) representation of G is a finite dimensional complex vectorspace V, with an action of G.

Definition 5.2 (Group Action)**.** For $g \in G$ and $v \in V$, we have $gv \in V$ satisfying,

- $ev = v$ (identity acts as identity)

- $(gh)v = g(hv)$ (compatibility of the action with group multiplication)

- $g(\alpha v + \beta w) = \alpha(gv) + \beta(gw)$ (compatibility of the action with the vector space structure)

for all $g, h \in G$, $v, w \in V$ and $\alpha, \beta \in \mathbb{C}$.

Let's look at some examples:

- The *trivial* representation. Let $V = \mathbb{C}$ and let $\pi v = v$ for all $\pi \in S_n$ and $v \in \mathbb{C}$

- The sign representation: let $V = \mathbb{C}$ and let $\pi v = \text{sgn}(\pi)v$ for all $\pi \in S_n$ and all $v \in \mathbb{C}$.

Remark. You can always write a permutation as a composite of transposition (2-cycles), not uniquely, but if you see an even number of transpositions you will always get an even number, same for odd. It's not invariant, but the sign defined as $+1$ if it's even, but -1 if it's odd and it's invariant.
Also the sign of a product is the product of the signs.

These are the only one dimensional representations.

- The *natural* representation: let $V = \mathbb{C}$ and let S_n act by permuting the n standard basis vectors. So,

$$\pi \left(\sum_{i=1}^{n} \alpha_i e_i \right) = \sum_{i=1}^{n} \alpha_i e_{\pi(i)}$$

- The regular representation: let V be an $n!$ dimensional complex vector space with basis $\{e_\sigma = \sigma \in S_n\}$. The action is determined by the action od the group on itself, $\pi(e_\sigma) = e_{\pi\sigma}$.

The first two of these examples are irreducible, the trivial is always irreducible.

The natural representation is not irreducible, its a sum of two irreducible pieces. The regular representation is also very much not irreducible. Every single irreducible representation lives in the regular representation.

5.2   Tabloids

We want to start with a partition and construct an irreducible representation of the symmetric group.

Definition 5.3 (Row Equivilent)**.** We say that two tableaux of shape λ are row equivalent if the corresponding rows contain the same elements.

$$
\begin{array}{|c|c|c|}
\hline
1 & 2 & 3 \\
\hline
4 \\
\cline{1-1}
\end{array}
\qquad
\begin{array}{|c|c|c|}
\hline
3 & 4 & 1 \\
\hline
1 \\
\cline{1-1}
\end{array}
$$

Definition 5.4 (Tabloid)**.** The tabloid of shape λ is an equivilence class of shale λ under row equivilence. We denote it as **t**

We shall write a tabloid as,

$$
\begin{array}{ccc}
\hline
1 & 2 & 3 \\
\hline
4 \\
\hline
\end{array}
$$

5.3   Constructing S^λ

Firstly start with a young tableau t of shape λ.
Define an associated column group C_t.
Define κ_t as a signed sum over the column group, C_t.
Define $e_t = \kappa_t \mathbf{t}$ a linear combination of tabloids of shape λ.
Then S^λ is thr set of complex linear combination of e_t where t is the tableau of shape λ.

5.3.1   Details

Let t be a young tableau of hape $\lambda \vdash n$ and let C_i be the subgroup of S_n consisting of perms of the entries of the column i. Then C_t is the associated column group,

$$
C_t = C_1 \times C_2 \times \cdots \times C_k
$$

For example, if t is, [2] we can say,

$$
\begin{array}{ccc}
\hline
1 & 2 & 5 \\
\hline
3 & 4 \\
\hline
\end{array}
$$

$$
C_t = \{1, (13), (24), (13)(24)\}
$$

Then,

$$
\kappa_t = \sum_{\pi \in C_t} \mathrm{sgn}(\pi)\pi
$$

---

[2]Is this correct?

This is a linear combination of elements of th symmetric group. For example, its just,

$$\kappa_t = 1 - (13) - (24) + (13)(24)$$

and now we get to the last step,

$$e_t = \kappa_t \mathbf{t}$$

Remark. e_t is called the polytabloid associated to the tabloid $\mathbf{t}$.

For example,

$$
\begin{array}{|cc|c}\hline 1 & 2 & 5 \\\hline 3 & 4 \\\hline\end{array}
\ -\
\begin{array}{|cc|c}\hline 3 & 2 & 5 \\\hline 1 & 4 \\\hline\end{array}
\ -\
\begin{array}{|cc|c}\hline 1 & 4 & 5 \\\hline 3 & 2 \\\hline\end{array}
\ +\
\begin{array}{|cc|c}\hline 3 & 4 & 5 \\\hline 1 & 2 \\\hline\end{array}
$$

Exercise. Checking the details of the examples is one of the exercises

- $S^{(n)}$ is just the trivial representation.

- $S^{(1,1,\dots,1)}$ is the sign representation.

- $S^{(n-1,1)}$ is an $n-1$ dimensional representation and the natural representation is a direct sum of this and the trivial representation.

Theorem 5.5. The set,

$$\{e_t \mid t \text{ is a stanrd tableau of a shape } \lambda\}$$

is a basis for S^λ.

Proof. Ommited   □

Corollary 5.6.

$$\dim(S^\lambda) = f^\lambda$$

Proof. Trivial   □

Example of a basis, let $\lambda = (2,1) \vdash 3$. We have two standard tableaux, say t and t' of shape λ: Then e_t is, Then $e_{t'}$ is, and S^λ is the 2-dimensional representation of S_3 with basis, $\{e_t, e_{t'}\}$

$$
\begin{array}{|cc|}\hline 1 & 2 \\\hline 3 \\\hline\end{array}
\ -\
\begin{array}{|cc|}\hline 3 & 2 \\\hline 1 \\\hline\end{array}
$$

$$
\begin{array}{|cc|}\hline 1 & 3 \\\hline 2 \\\hline\end{array}
\ -\
\begin{array}{|cc|}\hline 2 & 3 \\\hline 1 \\\hline\end{array}
$$

Remark. Every finite dimensional representation is a direct sum of irriductible representations.

5.4   A Run around Representation Theory.

– The number of irreducible representation of a finite group G is equal to the number of conjuagcy classes of the group.

– So for S_n, the number of irriductible representations is equal to the number o partitions.

– For a partition of λ, the representation S^λ is constructed above and irriducible.

– And the S^λ are different so we have a complete list of irriducible representations.

– For a finite group G,
$$\sum_V (\dim V)^2 = |G|$$
where the sum is over all the irriducible representations.

– This follows from the fact that the regular representation contains each irriducible and the number of times each one apparears is equal to its dimension.

– So for S_n, this gives,
$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$$

There is a bijection,

$$\text{partition } \lambda \vdash n \longleftrightarrow S^\lambda \text{ irriducible representation of } S_n$$

S^λ has a basis described in terms of the standard young tableaux of shape λ and so dimension f^λ.