

# Year 2 — Geometry of nets of conics

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

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**Suggested Reading** - Miles Reid, Undergrad Algebraic Geometry, contains other book recommended. Harris. Coxeter. Harsher, Schafrivich (?), Methods of Algebraic Geometry - Hodge, Ideals, Varieties and Algorithms.

# 1 Overview

This course is an introduction to Projective Geometry. In Lecture 1, we will consider and define the complex projective plane and lines in it, define projective transformations and their properties and finally Sylvester-Gallai Theorem.

## 1.1 What is a line?

In Algebraic Geometry we can use algebra, but it comes dry and we lose a geometric intuition.

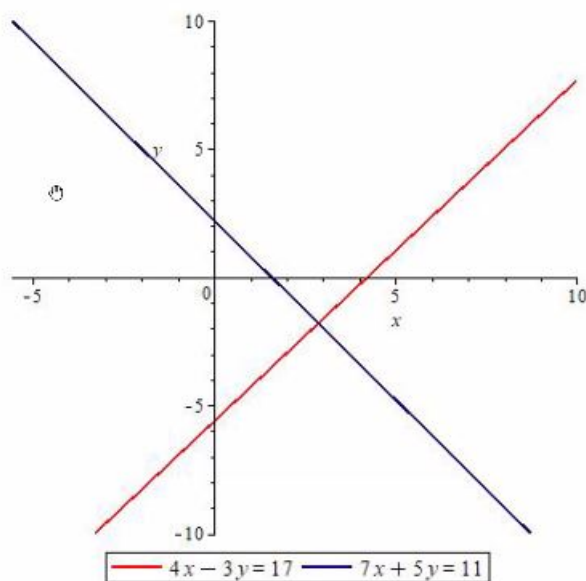


Figure 1: This shows two lines intersect at one point

This picture is real, which is no good for us. Reals aren't algebraically closed, which create lots of problems and holes. If we look at the Cremona Group, there's a theorem about generators of this group, if we consider the reals, this is a really new result. Reals makes the problem too difficult and so we consider the complexes to make everything so much simpler.

**Definition 1.1** (Line). A line in  $\mathbb{C}^2$  is a subset that is given by,

$$ax + by + c = 0$$

for some  $a, b, c \in \mathbb{C}$  such that  $(a, b) \neq (0, 0)$

and here is a lemma,

**Lemma 1.2.** There is a unique line in  $\mathbb{C}$  for each two distinct points.

*Proof.* Trivial by linear algebra □

**Lemma 1.3.** Suppose that  $L_1 \neq L_2$  then  $L_1 \cap L_2$  contains at least one point.

. If  $a_1b_2 - a_2b_1 \neq 0$ , then  $L_1 \cap L_2$  consists of a point, □

This is what Algebraic Geometry removes.

**What is a plane** For us a plane is just  $\mathbb{R}^2$ , but complex plane is also not perfect. If you don't travel over the plane that far, you are OK, but if you do, you get global problems. Take travelling the globe as an example. We can call a sphere a plane, if we wish, locally, we do, we have a plane, but global questions have a problem, the earth isn't a plane<sup>1</sup>.

When we talk about spaces, Algebraic Geometers mean a projective plane.

**Definition 1.4** (Projective Plane). Let  $(x, y, z) \in \mathbb{C}^3$  st  $(x, y, z) \neq (0, 0, 0)$  and let  $[x : y : z]$  be a subset of  $\mathbb{C}^3$  such that,

$$(a, b, c) \in [x : y : z] \iff \begin{cases} a = \lambda x \\ b = \lambda y \\ c = \lambda z \end{cases}$$

for some non-zero complex  $\lambda$

**Definition 1.5.** The projective plane  $\mathbb{P}_{\mathbb{C}}^2$  is the set of all possible  $[x : y : z]$

We refer to  $\mathbb{P}_{\mathbb{C}}^2$  as points and we can say that,

$$[1 : 2 : 3] = [7 : 14 : 21] = [2 - i : 4 - 4i : 3 - 3i]$$

and

$$[1 : 2 : 3] \neq [3 : 2 : 1]$$

and

$$[0 : 0 : 1] \neq [0 : 1 : 0]$$

**Remark.** There is no point  $[0 : 0 : 0]$

## 1.2 How to live in projective plane

Let  $U_z$  be the projective plane  $\mathbb{P}_{\mathbb{C}}^2$  consisting of points  $[x : y : z]$  with  $z \neq 0$ ,

**Lemma 1.6.** The map  $U_z \rightarrow \mathbb{C}$ ,

$$[x : y : z] = \left[ \frac{x}{z} : \frac{y}{z} : 1 \right] \mapsto \left( \frac{x}{z}, \frac{y}{z} \right)$$

is a bijection.

Thus we can identify  $U_z = \mathbb{C}^2$ , we call these charts and affine charts. These are exactly manifolds are we have three affine charts.

- Put  $\bar{x} = \frac{x}{z}$  and  $\bar{y} = \frac{y}{z}$
- Then we can consider *overline* $x$  and  $\bar{y}$  coordinates on  $U_z = \mathbb{C}^2$

**Question.** What is  $\mathbb{P}_{\mathbb{C}}^2 \setminus U_z$

- The subset  $\mathbb{P}_{\mathbb{C}}^2$  consisting of points  $[x : y : 0]$
- We can identify this as  $\mathbb{P}_{\mathbb{C}}^1$

If a question isn't homogenous, it doesn't make sense to do it in  $\mathbb{P}_{\mathbb{C}}^2$ . This is the same as it is for Linear Algebra.

**Definition 1.7** (Line). A line in  $\mathbb{P}_{\mathbb{C}}^2$  is the subset given by,

$$Ax + By + Cz = 0$$

for some fixed point  $[A : B : C] \in \mathbb{P}_{\mathbb{C}}^2$

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<sup>1</sup>Sorry, flat eathers

We sometimes call  $\mathbb{P}_{\mathbb{C}}^2$  the dual plane here. We have polynomials of degree one.

**Example.** Let  $P = [5 : 0 : 2]$  and  $Q = [1 : -1 : 1]$ . Then,

$$2x - 3y + 5z = 0$$

is the unique line

**Example.** Let  $L$  be give by,

$$x + 2y + 3z = 0$$

Let  $L'$  be  $x - y = 0$ , we then have  $L \cap L' = [1 : -1 : 0]$

**Theorem 1.8.** There is a unique line in  $\mathbb{P}_{\mathbb{C}}^2$  that contains  $P$  and  $Q$

. Let  $L$  be a line in  $\mathbb{P}_{\mathbb{C}}^2$  that is given by  $Ax + By + Cz = 0$ , then take two different lines, solve the system and by rank-nullity implies that  $L$  exists and is unique  $\square$

**Theorem 1.9.** The intersections  $L \cap L'$  consist of one point in  $\mathbb{P}_{\mathbb{C}}^2$

*Proof.* The same as above.  $\square$

It's very easy to find intersections and lines through two points, but we can use python to do it. We can also use a determinant formula,

$$\det \begin{pmatrix} 11 & -7 & 1 \\ 2 & 5 & 1 \\ x & y & z \end{pmatrix}$$

### 1.3 Parallel Lines? Do they meet?

In classical geometry, this is at infinity. We can do this in algebra,

Let  $U_z$  be the complement of  $\mathbb{P}_{\mathbb{C}}^2$  to the line  $z = 0$ . Identity,

$$U_z = \mathbb{C}$$

with coordinates  $\bar{x} = \frac{x}{z}$  and  $\bar{y} = \frac{y}{z}$

$$- \text{ Let } \bar{L} = 2\bar{x} + 3\bar{y} + 5 = 0$$

$$- \text{ Let } \bar{L}' = 2\bar{x} + 3\bar{y} + 7 = 0$$

$$- L \cap L' = \emptyset$$

**Question.** Where do  $\bar{L}$  and  $\bar{L}'$

### 1.4 Projective Transformations

If we have an equation that can be simplified using some sort of change of variables. Let  $\mathbf{M} \in M_{3 \times 3}$  and  $\phi : \mathbb{P}_{\mathbb{C}}^2 \rightarrow \mathbb{P}_{\mathbb{C}}^2$  be given by,

$$[x : y : z] = M\mathbf{x}$$

where  $\mathbf{x}$  is just the vector formed of  $x$ . This is called the automorphism of  $\mathbb{P}_{\mathbb{C}}^2$ , but we don't have  $[0 : 0 : 0]$ , this is a problem as  $\phi$  may output it, if  $\mathbf{x}$  is in the kernel.

**Question.** When  $\phi$  well defined?

This is when  $M$  is invertible or  $\det M \neq 0$

**Definition 1.10** (Projective Transformation). If  $\det M \neq 0$  we say that  $\phi$  is a projective transformation.

## 1.5 Group of transformations

Projective transformations of  $\mathbb{P}_{\mathbb{C}}^2$  form a group.

- Let  $M$  be a matrix in  $GL_3(\mathbb{C})$
- Denote by  $\phi_M$  the corresponding projective transformation.

**Question.** When  $\phi_M$  is an identity map?

The map  $\phi_M$  is said to be the identity  $\iff M$  is a scalar.

**Definition 1.11.** We call  $M$  is said to be scalar if  $M$  is diagonal

**Corollary 1.12.** Let  $G$  be a subgroup in  $GL_3(\mathbb{C})$  consisting of scalar matrices. The group of projective transformations of  $\mathbb{P}_{\mathbb{C}}^2$  is isomorphic to,

$$PGL_3(\mathbb{C}) = GL_3(\mathbb{C})/G$$

We can look at mobius transformations, but for our plane. If you have four points in the plane, such that no three points are not collinear. Then there is a projective plane such that  $\mathbb{P}_{\mathbb{C}}^2 \rightarrow \mathbb{P}_{\mathbb{C}}^2$  such that,

$$[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1], [1 : 1 : 1]$$

so we let,  $P_1 = [a_{11} : a_{12} : a_{13}]$  and  $P_2 = [a_{21} : a_{22} : a_{23}]$  and  $P_3 = [a_{31} : a_{32} : a_{33}]$  and then do  $\phi$  on them and then it will map the basic points to ir points and so do the inverse. Let  $\psi$  be  $\phi$  and then apply it and then form a projective transformation and done.

## 1.6 Sylvester - Gallai Problem

**Problem.** Let  $\Sigma$  be the finite subset in  $\mathbb{R}^2$ . show that either  $\Sigma$  is contained in one line, or there is a line containing exactly two points in  $\Sigma$ .

- This problem was proposed by Sylvester in 1893
- It was solved by Gallai in 1944
- A simpler proof was given by Kelly in 1948

Sylvester was not just a Geometry, he was interested in Algebra aswell, he worked with Complexes and he knew the problem was wrong. He knew a very simpler counter example.

## 1.7 Hesse Configurations

- Nine points were the counterexample
- There are 12 lines passing through 2 points among them
- Each such line contain 3 points among nine.
- Let  $\mathbb{F}$  be a field
- Let  $\Sigma$  be a finite subset in  $\mathbb{P}_{\mathbb{F}}^2$

Suppose that the subset  $\Sigma$  is not contained in one line,

**Lemma 1.13.** There exists a line in  $\mathbb{P}_{\mathbb{F}}^2$  containing exactly two points in  $\Sigma$ ,

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a & \frac{b}{2} & \frac{d}{2} \\ \frac{b}{2} & c & \frac{e}{2} \\ \frac{d}{2} & \frac{e}{2} & f \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

- where  $|\Sigma| \leq 6$
- where  $|\Sigma| = 7$  and  $2 \neq 0$  in  $\mathbb{F}$
- where  $|\Sigma| = 8$

**Theorem 1.14.** Suppose  $|\Sigma| = 9$  and  $\mathbb{F} = \mathbb{C}$ . If there exists no line containing exactly two points of the set

## 2 Lecture 2

In this lecture we will cover,

- Conics in the complex projective Plane
- Classifications of conics up to projective Transformations
- Intersections of conics and lines in  $\mathbb{P}_{\mathbb{C}}^2$
- Intersections of two conics (Bezout Theorem in disguise)

### 2.1 Conics in the complex projective plane.

**Definition 2.1** (Conic). A conic in  $\mathbb{P}_{\mathbb{C}}^2$  is a subset that is given by,

$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$$

for  $a, b, c, d, e, f \in \mathbb{C}$  such that  $(a, b, c, d, e, f) \neq (0, 0, 0, 0, 0, 0)$

The right way to think about this, is a curve. This was traditionally defined as a section as a conic in the cartesian plane.

**Definition 2.2** (Irreducible). The conic is said to be irreducible if

$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$$

is irreducible, Otherwise the conic is reducible, i.e. can be factored as two linear terms or lines.

A conic is the union of two lines. The topology of an irreducible conic is just a sphere.

Let  $\mathcal{C}$  be a conic in  $\mathbb{P}_{\mathbb{C}}^2$ . Then  $\mathcal{C}$  that is given by,

$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$$

and we can rewrite the equation in matrix form,

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a & \frac{b}{2} & \frac{d}{2} \\ \frac{b}{2} & c & \frac{e}{2} \\ \frac{d}{2} & \frac{e}{2} & f \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

and this is just the Hessian of the conic.

**Lemma 2.3.** The conic  $\mathcal{C}$  is irreducible if and only if  $\det M \neq 0$ .

**Example.** The conic  $xy - z^2 = 0$  is irreducible.

### 2.2 Five Points make a conic

Let  $P_i \in \mathbb{P}_{\mathbb{C}}^2$  for  $1 \leq i \leq 5$ . Suppose no four points is collinear.

**Theorem 2.4.** There is a unique conic in  $\mathbb{P}_{\mathbb{C}}^2$  that contains these points

. Write down the coordinates of the points, every two points can be cooked up a line, then check three points. Find complex numbers such that, These are linear, apply rank-nullity and then it's dimension is not zero. Find null space, so there is at least one conic. Next proof that the rank is not larger than five, or is exactly five.  $\square$

$$\begin{cases} ax_1^2 + bx_1y_1 + cy_1^2 + dx_1z_1 + ey_1z_1 + fz_1^2 = 0, \\ ax_2^2 + bx_2y_2 + cy_2^2 + dx_2z_1 + ey_2z_1 + fz_1^2 = 0, \\ ax_3^2 + bx_3y_3 + cy_3^2 + dx_3z_1 + ey_3z_1 + fz_1^2 = 0, \\ ax_4^2 + bx_4y_4 + cy_4^2 + dx_4z_1 + ey_4z_1 + fz_1^2 = 0, \\ ax_5^2 + bx_5y_5 + cy_5^2 + dx_5z_1 + ey_5z_1 + fz_1^2 = 0. \end{cases}$$

Figure 2

### 2.2.1 How to find a conic?

It would be time consuming by hand, so do it with Python. We can plot the real part of the curve as just curves.

### 2.2.2 Intersection of line and a conic

Let  $L$  be a line in  $\mathbb{P}_{\mathbb{C}}^2$ . Let  $C$  be an irreducible conic in  $\mathbb{P}_{\mathbb{C}}^2$

**Lemma 2.5.** The intersection of  $L \cap C$  consists of 2 points.

. The  $L$  is given by,

$$\alpha x + \beta y + \gamma z = 0$$

for  $\alpha, \beta, \gamma \in \mathbb{C}$  such that they are non-zero. The conic  $C$  is given by,

$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$$

for  $a, b, c, d, e, f \in \mathbb{C}$  such that ... . Then  $L \cup C$  is,

$$\begin{cases} \alpha x + \beta y + \gamma z = 0 \\ ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0 \end{cases}$$

which then has solutions, which are 2 as we have a quadratic.  $\square$

If we consider, an example.

**Example.**  $L : 2x + 7y - 5z = 0$  and  $C : 2x^2 - 3xy + 7y^2 - 5xz + 11yz - 8z^2$ . Firstly we check at infinity. The intersection of  $L_z \cap L \cap C$  and see this is empty, then look at non-infinity, by letting  $z = 1$ , and substitute and get quadratic equations and get two numbers. If discriminant is negative, this isn't a problem as we are algebraically closed.

Let  $L$  be a line in  $\mathbb{P}_{\mathbb{C}}^2$ , and let  $C$  be irreducible in  $\mathbb{P}_{\mathbb{C}}^2$ .

**Question.** When  $|L \cap C| = 1$ ?

If we consider finite fields, our pictures are intuitive, but we can still study tangents. We may assume  $[0 : 0 : 1] \in L \cap C$ . Then  $C$  is given by,

$$ax^2 + bxy + cy^2 + dxz + eyz = 0$$

for some  $[a : b : c : d] \in \mathbb{P}_{\mathbb{C}}^4$ .

we may assume that  $L$  is given by  $x = 0$ . Then,

$$L \cap C = [0 : 0 : 1] \cup [0 : e : -c]$$

and so we have  $|L \cap C| = 0 \iff e = 0$ .



- Let  $U_z$  be the complement in  $\mathbb{P}_{\mathbb{C}}^2$  to the line  $z = 0$
- Identify  $U_z = \mathbb{C}^2$  with coordinates  $\bar{x} = \frac{x}{z}$  and  $\bar{y} = \frac{y}{z}$

Then  $U_z \cap \mathcal{C}$  is given by  $\mathbf{a}\bar{x}^2$

Then  $U_z \cap \mathcal{C}$  is given by  $\mathbf{a}\bar{x}^2 + \mathbf{b}\bar{x}\bar{y} + \mathbf{c}\bar{y}^2 + \mathbf{d}\bar{x} + \mathbf{e}\bar{y} = 0$ .

- ▶  $\mathbf{d}\bar{x} + \mathbf{e}\bar{y} = 0$  is the **tangent** line to  $U_z \cap \mathcal{C}$  at  $(0, 0)$ .
- ▶  $\mathbf{d}x + \mathbf{e}y = 0$  is the **tangent** line to  $\mathcal{C}$  at  $[0 : 0 : 1]$ .

Then  $|L \cap \mathcal{C}| = 1 \iff L$  is tangent to  $\mathcal{C}$  at point  $L \cap \mathcal{C}$

## 2.3 Conics and projective transformations

Let  $\mathcal{C}$  be a conic,

**Theorem 2.6.** There is projective transformations  $\phi$  such that  $\phi(\mathcal{C})$  is given by,

- (i) either  $xy = z^2$  (irreducible smooth conic)
- (ii)  $xy = 0$  (a union of two lines in  $\mathbb{P}_{\mathbb{C}}^2$ )
- (iii)  $x^2 = 0$  (a line in  $\mathbb{P}_{\mathbb{C}}^2$  taken with multiplicity 2)

The proof could use Gram-Schmidt Orthogonalisation.<sup>2</sup> We can instead do group theory by studying an 8 dimensional group. We will then quotient this with three orbits (one normal, two special (unstable)).

- Pick a point in  $\mathcal{C}$  and map it to  $[0 : 0 : 1]$ . This **kills**  $f$ .
- Map the tangent line  $dx + ey = 0$  to  $x = 0$ . This **kills**  $e$ .
- Map the line  $z = 0$  to the line,

$$z + \alpha y + \beta x$$

for appropriate  $\alpha, \beta$  to **kill**  $a$  and  $b$

- Scale  $x, y$  and  $z$  to get  $b = 1$  and  $c = -1$

This gives a projective Classification, i.e. the conic is just

$$xy = z^2$$

## 2.4 Intersecting Two Conics

**Theorem 2.7.** The intersection of  $\mathcal{C} \cap \mathcal{C}'$  consists one one, two or three or four points.

. We can use Classification theorem, so just go to  $xy = z^2$ .

- Let  $L$  be the line  $y = 0$ . Then  $L \cap \mathcal{C} \cap \mathcal{C}' \subset [1 : 0 : 0]$
- One has  $L \cap \mathcal{C} \cap \mathcal{C}' = [1 : 0 : 0] \iff a = 0$
- Let  $U$

□

Here are some examples of

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<sup>2</sup>This is course is made accessible ;)

► Let  $U_y = \mathbb{P}_{\mathbb{C}}^2 \setminus L$ . Then  $U_y \cap \mathcal{C} \cap \mathcal{C}'$  is given by

$$y - 1 = x - z^2 = az^4 + dz^3 + (b + f)z^2 + ez + c = 0.$$

If  $a = 0$ , then  $L \cap \mathcal{C} \cap \mathcal{C}' = [1 : 0 : 0]$  and  $0 \leq |U_y \cap \mathcal{C} \cap \mathcal{C}'| \leq 3$ .

If  $a \neq 0$ , then  $L \cap \mathcal{C} \cap \mathcal{C}' = \emptyset$  and  $1 \leq |U_y \cap \mathcal{C} \cap \mathcal{C}'| \leq 4$ . □

Figure 3