Year 2 — Geometry of nets of conics

Based on lectures by Ivan Cheltsov
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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Contents

Suggested Reading - Miles Reid, Undergrad Algebraic Geometry, contains other book recommended. Harris. Coxeter. Harsher, Shafarevich, Methods of Algebraic Geometry - Hodge, Ideals, Varieties and Algorithms.

1 Lecture 1 - Overview

This course is an introduction to Projective Geometry. In L3cture 1, we will consider and define the complex projective plane and lines in it, define projective transformations and their properties and finally Sylvester-Galllai Theorem.

1.1 What is a line?

In Algebraic Geometry we can use algebra, but it comes dry and we lose a geometric intuition.
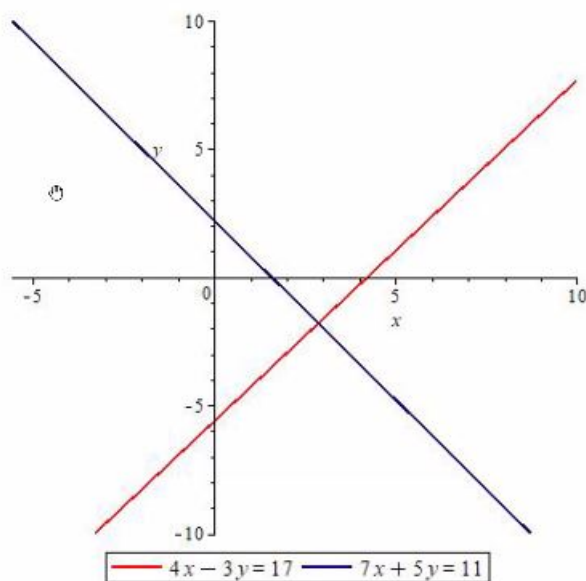

Figure 1: This shows two lines intersect at one point

This picture is real, which is no good for us. Reals aren't algebraically closed, which create lots of problems and holes. If we look at the Cremona Group, theres a theorem about generators of this group, if we consider the reals, this is a really new result. Reals makes the problem too difficult and so we consider the complexes to make everything so much simpler.

Definition 1.1 (Line)**.** A line in $\mathbb{C}^2$ is a subset that is given by,

$$ax + by + c = 0$$

for some $a, b, c \in \mathbb{C}$ such that $(a, b) \neq (0, 0)$

and here is a lemma,

Lemma 1.2. There is a unique line in $\mathbb{C}$ for rach two distinct points.

Proof. Trivial by linear algebra   □

Lemma 1.3. Suppose that $L_1 \neq L_2$ then $L_1 \cap L_2$ contains at least one point.

. If $a_1 b_2 - a_2 b_1 \neq 0$, then $L_1 \cap L_2$ consists of a point,   □

This is what Algebraic Geometry removes.

What is a plane   For us a plane is just $\mathbb{R}^2$, but complex plane is also not perfect. If you don't travel over the plane that far, you are OK, but if you do, you get global problems. Take travelling the globe as an example. We can call a sphere a plane, if we wish, locally, we do, we have a plane, but global questions have a problem, the earth isn't a plane[1].

When we talk about spaces, Algebraic Geometers mean a projective plane.

Definition 1.4 (Projective Plane)**.** Let $(x, y, z) \in \mathbb{C}^3$ st $(x, y, z) \neq (0, 0, 0)$ and let $[x : y : z]$ be a subset of $\mathbb{C}^3$ such that,

$$(a, b, c) \in [x : y : z] \iff \begin{cases} a = \lambda x \\ b = \lambda y \\ c = \lambda z \end{cases}$$

for some non-zero complex λ

Definition 1.5. The projective place $\mathbb{P}^2_{\mathbb{C}}$ is the set of all possible $[x : y : z]$

We refer to $\mathbb{P}^2_{\mathbb{C}}$ as points and we can say that,

$$[1 : 2 : 3] = [7 : 14 : 21] = [2 - i : 4 - 4i : 3 - 3i]$$

and

$$[1 : 2 : 3] \neq [3 : 2 : 1]$$

and

$$[0 : 0 : 1] \neq [0 : 1 : 0]$$

Remark. There is no point $[0 : 0 : 0]$

1.2   How to live in projective plane

Let U_z be the projective plane $\mathbb{P}^2_{\mathbb{C}}$ consisting of points $[x : y : z]$ with $z \neq 0$,

Lemma 1.6. The map $U_z \to \mathbb{C}$,

$$[x : y : z] = \left[\frac{x}{z} : \frac{y}{z} : 1 \right] \mapsto \left(\frac{x}{z}, \frac{y}{z} \right)$$

is a bijection.

Thus we can identify $U_z = \mathbb{C}^2$, we call these charts and affine charts. These are exactly manifolds are we have three affine charts.

  – Put $\overline{x} = \frac{x}{z}$ and $\overline{y} = \frac{y}{z}$

  – Then we can consider *overlinex* and $\overline{y}$ coordinates on $U_z = \mathbb{C}62$

Question. What is $\mathbb{P}^2_{\mathbb{C}} \setminus U_z$

  – The subset $\mathbb{P}^2_{\mathbb{C}}$ consisting of points $[x : y : 0]$

  – We can identify this as $\mathbb{P}^1_{\mathbb{C}}$

If a question isnt homogenous, it doesnt make sense to do it in $\mathbb{P}^2_{\mathbb{C}}$. This is the same as it is for Linear Algebra.

Definition 1.7 (Line)**.** A line in $\mathbb{P}^2_{\mathbb{C}}$ is the subset given by,

$$Ax + By + Cz = 0$$

for some fixed point $[A : B : C] \in \mathbb{P}^2_{\mathbb{C}}$

---

[1]Sorry, flat eathers

We sometimes call $\mathbb{P}^2_{\mathbb{C}}$ the dual plane here. We have polynomials of degree one.

Example. Let $P = [5 : 0 : 2]$ and $Q = [1 : -1 : 1]$. Then,

$$2x - 3y + 5z = 0$$

is the unique line

Example. Let L be give by,

$$x + 2y + 3z = 0$$

Let L' be $x - y = 0$, we then have $L \cap L' = [1 : -1 : 0]$

Theorem 1.8. There is a unique line in $\mathbb{P}^2_{\mathbb{C}}$ that contains P and Q

. Let L be a line in $\mathbb{P}^2_{\mathbb{C}}$ that is given by $Ax + By + Cz = 0$, then take two different lines, solve the system and by rank-nullity implies that L exists and is unique   □

Theorem 1.9. The intersections $L \cap L'$ consirst of one point in $\mathbb{P}^2_{\mathbb{C}}$

Proof. The same as above.   □

It's very easy to find intersections and lines theough two points, but we can use python to do it. We can also use a determinant formula,

$$\det \begin{pmatrix} 11 & -7 & 1 \\ 2 & 5 & 1 \\ x & y & z \end{pmatrix}$$

1.3   Parallel Lines? Do they meet?

In classical geometry, this is at infinity. We can do this in agebra,

Let U_z be the complement of $\mathbb{P}^2_{\mathbb{C}}$ to the line$z = 0$. Idenity,

$$U_z = \mathbb{C}$$

with coordinates $\bar{x} = \frac{x}{z}$ and $y = \frac{y}{z}$

- Let $\overline{L} = 2\overline{x} + 3\overline{y} + 5 = 0$
- Let $\overline{L} = 2\overline{x} + 3\overline{y} + 7 = 0$
- $L \cap L' = \varnothing$

Question. Where do $\overline{L}$ and $\overline{L}'$

1.4   Projective Transformations

If we have an equation that can be simplified using some sort of change of variables. Let $\mathbf{M} \in M_{3\times 3}$ and $\phi : \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ be given by,

$$[x : y : z] = M\mathbf{x}$$

where $\mathbf{x}$ is just the vector formed of x. This is called the automorphism of $\mathbb{P}^2_{\mathbb{C}}$, but we dont have $[0 : 0 : 0]$, this is a problem as ϕ may output it, if $\mathbf{x}$ is in the kernel.

Question. When ϕ well defined?

This is when M is invertible or $\det M \neq 0$

Definition 1.10 (Projective Transformation)**.** If $\det M \neq 0$ we say that ϕ is a projective transformation.

1.5 Group of transformations

Projective transformations of $\mathbb{P}^2_{\mathbb{C}}$ form a group.

    – Let M be a atric in $GL_3(\mathbb{C})$

    – Denote by ϕ_M the corresponding projective transformation.

Question. When ϕ_M is an idenity map?

    The map ϕ_M is said to be the indentity $\iff$ M is a scalar.

Definition 1.11. We call M is said to be scalar if M is diagonal

Corollary 1.12. Let G be a subgroup in $GL_3(\mathbb{C})$ consistong of scalar matrices. The group of projective transformations of $\mathbb{P}^2_{\mathbb{C}}$ is isomorphic to,

$$PGL_3(\mathbb{C}) = GL_3(\mathbb{C})G$$

    We can look at mobius transformations, but for our plane. If you have four points in the plane, such that no three points are not collinear. Then there is a projective plane such that $\mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that,

$$[1:0:0], [0:1:0], [0:0:1], [1:1:1]$$

so we let, $P_1 = [a_{11} : a_{12} : a_{13}]$ and $P_2 = [a_{21} : a_{22} : a_{23}]$ and $P_3 = [a_{31} : a_{32} : a_{33}]$ and then do ϕ on them and then it will map the basic points to ir points and so do the inverse. Let ψ be ϕ and then apply it and then forma projective transformation and done.

1.6 Sylvester - Gallai Problem

Problem. Let Σ be the finite subset in $\mathbb{R}^2$. show that either Σ is contained in one line, or there is a line containing exactly two points in Σ.

    – This problem was proposed by Sylvester in 1893

    – It was solved by Gallai in 1944

    – A simpler proof was given by Kelly in 1948

    Sylvester was not just a Geometry, he was interested in Algebra aswell, he worked with Complexes and he knew the problem was wrong. He knew a very simpler counter example.

1.7 Hesse Configurations

    – Nine points were the counterexample

    – There are 12 lines passing through 2 points among them

    – Each such line contain 3 points among nine.

    – Let $\mathbb{F}$ be a field

    – Let Σ be a finite subset in $\mathbb{P}^2_{\mathbb{F}}$

    Suppose that the subset Σ is not contained in one line,

Lemma 1.13. There exists a line in $\mathbb{P}^2_{\mathbb{F}}$ contaning exactly two points in Σ,

$$\left(\begin{array}{ccc} x & y & z \end{array}\right) \left(\begin{array}{ccc} a & \frac{b}{2} & \frac{d}{2} \\ \frac{b}{2} & c & \frac{e}{2} \\ \frac{d}{2} & \frac{e}{2} & f \end{array}\right) \left(\begin{array}{c} x \\ y \\ z \end{array}\right) = 0.$$

- – where $|\Sigma| \leq 6$

- – where $|\Sigma| = 7$ and $2 \neq 0$ in $\mathbb{F}$

- – where $|\Sigma| = 8$

Theorem 1.14. Suppose $|\Sigma| = 9$ and $\mathbb{F} = \mathbb{C}$. If there exists no line containing exactly two points of the set

2   Lecture 2 - Conics

In this lecture we will cover,

- Conics in the complex projective Plane

- Classifications of conics up to projective Transformations

- Intersections of conics and lines in $\mathbb{P}^2_{\mathbb{C}}$

- Intersections of two conics (Bezout Theorem in disguize)

2.1   Conics in the complex projective plane.

Definition 2.1 (Conic). A conic in $\mathbb{P}^2_{\mathbb{C}}$ is a subset that is given by,

$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$$

for $a, b, c, d, e, f \in \mathbb{C}$ such that $(a, b, c, d, e, f) \neq (0, 0, 0, 0, 0, 0)$

The right way to think about this, is a curve. This was traditionally defined as a section as a conic in the cartesian plane.

Definition 2.2 (Irriducible). The conic is said to be irreducible if

$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$$

is irreducible, Otherwise the conic is reducible, i.e. can be factored as two linear terms or lines.

A conic is the union of two lines. The topology of an irreducible conic is just a sphere.

Let $\mathcal{C}$ be a conic in $\mathbb{P}^2_{\mathbb{C}}$. Then $\mathcal{C}$ that is given by,

$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$$

and we can rewrite the equation in matrix form,

$$
\begin{pmatrix} x & y & z \end{pmatrix}
\begin{pmatrix} a & \frac{b}{2} & \frac{d}{2} \\ \frac{b}{2} & c & \frac{e}{2} \\ \frac{d}{2} & \frac{e}{2} & f \end{pmatrix}
\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.
$$

and this is just the Hessian of the conic.

Lemma 2.3. The conic $\mathcal{C}$ is irreducible if and only if $\det M \neq 0$.

Example. The conic $xy - z^2 = 0$ is irreducible.

2.2   Five Points make a conic

Let $P_i \in \mathbb{P}^2_{\mathbb{C}}$ for $1 \leq i \leq 5$. Suppose no four points is collinear.

Theorem 2.4. There is a unique conic in $\mathbb{P}^2_{\mathbb{C}}$ that contains these points

. Write down the coordinates of the points, every two points can be cooked up a line, then check three points. Find complex numbers such that, These are linear, apply rank-nullity and then it's dimension is not zero. Find null space, so there is at least one conic. Next proof that the rank is not larger than five, or is exactly five. $\qquad\square$

$$\begin{cases} ax_1^2 + bx_1y_1 + cy_1^2 + dx_1z_1 + ey_1z_1 + fz_1^2 = 0, \\ ax_2^2 + bx_2y_2 + cy_2^2 + dx_2z_1 + ey_2z_1 + fz_1^2 = 0, \\ ax_3^2 + bx_3y_3 + cy_3^2 + dx_3z_1 + ey_3z_1 + fz_1^2 = 0, \\ ax_4^2 + bx_4y_4 + cy_4^2 + dx_4z_1 + ey_4z_1 + fz_1^2 = 0, \\ ax_5^2 + bx_5y_5 + cy_5^2 + dx_5z_1 + ey_5z_1 + fz_1^2 = 0. \end{cases}$$

Figure 2

2.2.1 How to find a conic?

It would be time consuming by hand, so do it with Python. We can plot the real part of the curve as just curves.

2.2.2 Intersection of line and a conic

Let L be a line in $\mathbb{P}^2_{\mathbb{C}}$. Let C be an irreducible conic in $\mathbb{P}^2_{\mathbb{C}}$

Lemma 2.5. The intersection of $L \cap C$ consists of 2 points.

. The L is given by,

$$\alpha x + \beta y + \gamma z = 0$$

for $\alpha, \beta, \gamma \in \mathbb{C}$ such that they are non-zero. The conic C is given by,

$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$$

for $a, b, c, d, e, f \in \mathbb{C}$ such that ... . Then $L \cup C$ is,

$$\begin{cases} \alpha x + \beta y + \gamma z = 0 \\ ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0 \end{cases}$$

which then has solutions, which are 2 as we have a quadratic. $\qquad\qquad\square$

If we consider, an example.

Example. $L : 2x + 7y - 5z = 0$ and $C : 2x^2 - 3xy + 7y^2 - 5xz + 11yz - 8z^2$. Firstly we check at infinity. The intersection of $L_z \cap L \cap C$ and see this is empty, then look at non-infinity, by letting $z = 1$, and substitute and get quadratic equations and get two numbers. If discriminant is negative, this isn't a problem as we are algebraically closed.

Let L be a line in $\mathbb{P}^2_{\mathbb{C}}$, and let C be irreducible in $\mathbb{P}^2_{\mathbb{C}}$.

Question. When $|L \cap C| = 1$?

If we consider finite fields, our pictures are intuitive, but we can still study tangents. We may assume $[0 : 0 : 1] \in L \cap C$. Then C is given by,

$$\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}xz + \mathbf{e}yz = 0$$

for some $[\mathbf{a} : \mathbf{b} : \mathbf{c} : \mathbf{d}] \in \mathbb{P}^4_{\mathbb{C}}$.
we may assume that L is given by $x = 0$. Then,

$$L \cap C = [0 : 0 : 1] \cup [0 : \mathbf{e} : -\mathbf{c}]$$

and so we have $|L \cap C| 0 \iff \mathbf{e} = 0$.

- Let U_z be the complement in $\mathbb{P}^2_{\mathbb{C}}$ to the line $z = 0$

- Identify $U_z = \mathbb{C}^2$ with coordinates $\overline{x} = \frac{x}{z}$ and $\overline{y} = \frac{y}{z}$

Then $U_z \cap \mathcal{C}$ is given by $\mathbf{a}\overline{x}^2$

Then $U_z \cap \mathcal{C}$ is given by $\mathbf{a}\overline{x}^2 + \mathbf{b}\overline{xy} + \mathbf{c}\overline{y}^2 + \mathbf{d}\overline{x} + \mathbf{e}\overline{y} = 0$.
- ▶ $\mathbf{d}\overline{x} + \mathbf{e}\overline{y} = 0$ is the tangent line to $U_z \cap \mathcal{C}$ at $(0,0)$.
- ▶ $\mathbf{d}x + \mathbf{e}y = 0$ is the tangent line to $\mathcal{C}$ at $[0:0:1]$.

Then $|L \cap \mathcal{C}| = 1 \iff L$ is tangent to $\mathcal{C}$ at point $L \cap \mathcal{C}$

2.3   Conics and projective transformations

Let $\mathcal{C}$ be a conic,

Theorem 2.6. There is projective transformations ϕ such that $\phi(\mathcal{C})$ is given by,

(i) either $xy = z^2$ (irreducible smooth conic)

(ii) $xy = 0$ (a union of two lines in $\mathbb{P}^2_{\mathbb{C}}$)

(iii) $x^2 = 0$ (a line in $\mathbb{P}^2_{\mathbb{C}}$ taken with multiplicity 2)

The proof could use Grahm Schmitz Orthogonalisation.[2] We can instead do group thory by studing an 8 dimensional group. We will then quotient this with three orbit (one normal, two special (unstable)).

- Pick a point in $\mathcal{C}$ and map it to $[0:0:1]$. This kills f.

- Map the tangent line $dx + ey = 0$ to $x = 0$. This kills e.

- Map the line $z = 0$ to the line,
$$z + \alpha y + \beta z$$
for appropriate α, β to kills a and b

- Scale x, y and z to get $b = 1$ and $c = -1$

This gives a projective Classification, i.e. the conic is just
$$xy = z^2$$

2.4   Intersecting Two Conics

Theorem 2.7. The intresection of $\mathcal{C} \cap \mathcal{C}'$ consists one one, two or three or four points.

. We can use Classification theorem, so just go to $xy = z^2$.

- Let L be the line $y = 0$. Then $L \cap \mathcal{C} \cap \mathcal{C}' \subset [1:0:0]$

- One has $L \cap \mathcal{C} \cap \mathcal{C}' = [1:0:0] \iff a = 0$

- Let U

$\square$

Here are some examples

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[2]This is course is made acessible ;)

▶ Let $U_y = \mathbb{P}^2_{\mathbb{C}} \setminus L$. Then $U_y \cap C \cap C'$ is given by

$$y - 1 = x - z^2 = az^4 + dz^3 + (b + f)z^2 + ez + c = 0.$$

If $a = 0$, then $L \cap C \cap C' = [1 : 0 : 0]$ and $0 \leqslant |U_y \cap C \cap C'| \leqslant 3$.
If $a \neq 0$, then $L \cap C \cap C' = \varnothing$ and $1 \leqslant |U_y \cap C \cap C'| \leqslant 4$.   □

Figure 3

3   Lecture 3 - Cubics

In this lecture,

- Cubics curves in the complex projective plane

- Each smooth cubic curves in $\mathbb{P}^2_{\mathbb{C}}$ has 9 infection points

- Classification of cubics up to projective transformations

- Legendre, Weirstrass and Hesse forms of cubics

Complex Irreducible Curves

Definition 3.1. An irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree $d \geq 1$ is a subset given by,

$$f(x, y, z) = 0$$

for an irreducible polynomial f of degree d

- $2x^2 - y^2 + 2z^2 = 0$ is an irreducible conic

- $zy^2 + x(x - z)(x + z) = 0$ is an irreducible cubib in $\mathbb{P}^2_{\mathbb{C}}$

To do the union of curves, just multiply them.

3.0.1   Smooth Curves

Let $\mathcal{C}$ be an irreducible curve and then,

Definition 3.2 (Singular). A point $[a : b : c] \in \mathbb{P}^2_{\mathbb{C}}$ is singlular if,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$$

This defines a smooth manifold, but for us this is a curve. We denote $\mathrm{Sing}(C)$ to be the set of singular points of C. Non-singular points are smooth and C is smooth if it has no singular points.

Example. If $f = zx^{d-1} - y^d$ and $d \geq 3$, then $\mathrm{Sing}\, C = [0 : 0 : 1]$.

If you take your point and change coordinates and take the chart, then the polynomial will become non-homogenous and then you get a taylor expansion. Then linearise by taking a linear term and this exactly tangent line, but it may vanish. This is a singular point.

Let P be a point then a tangent line is,

$$\frac{\partial f}{\partial x}x + \frac{\partial f}{\partial y}y + \frac{\partial f}{\partial z}z = 0$$

for $\mathcal{C}$ at P.

We can do projective to classify the different types of equations. We can do the same for cubics. This is what this lecture will mainly be about.

We can consider different equivalences of curves, i.e. projectively equivalent and isomorphic. If the degree is not three, they are the same. We won't define isomorphisms as it uses complicated algebraic geometry.

Exercise. Find all the matrices for a curve, using transformations and python. You can show that the corresponding respresentation is a symmetric square.

3.0.2   Legendre Form

A cubic polynomial, can be listed by 10 different parameters (really 9 by scaling). We then have a large group, the group of projective transformations up to scaling, so eight dimensional. We can't expect a finite number of orbits, as not all planar cubics will be equivilent. So, they will be dependant of one parameter up to projections.

Let $\mathcal{C}$ be a curve in $\mathbb{P}^2_{\mathbb{C}}$,

Theorem 3.3 (Legendre Form)**.** There exists $\phi \in PGL_3(\mathbb{C})$ such that, $\phi(\mathcal{C})$ given by,

$$xz^2 = y(y - z)(y - \lambda x)$$

for some $\lambda \in \mathbb{C}$ such that $\lambda \neq 0$ and $\lambda \neq 1$.

The proof is similar, to the conic case, the algorithm is similar to conics. Here, however, when we look for the projective transformation we have a lot freedom in conics, but here we have very little. We have finite projective transformations. The group of projective transformations for cubics is finite.

Proof.

Pick a point in $\mathcal{C}$ and map it to $[0 : 0 : 1]$, this kills J.

Map a tangent to a line through $[0 : 0 : 1]$, i.e. $x = 0$. This kills I.

Suppose we want to know how to kill F.

Move the line $z = 0$ appropriately to kill G and E.

Move the line $y = 0$ to the line $y = \alpha x$ such that,

$$A + B\alpha^3 + C\alpha + D\alpha^2 = 0$$

This kills A. Now scale x, y and z appropriately. $\qquad\qquad\square$

3.1   How to kill F?

– Let P be a point of $\mathcal{C}$

– Let L be a tangent to the curve.

Then $L \cap \mathcal{C}$ contains P with multiplicity of at least two.

– Suppose $P = [0 : 0 : 1]$

– Suppose that L is $x = 0$

Question. When is $F = 0$?

After some cancelling we find that L is tangent to $\mathcal{C}$ with multiplicity 3. This means it's an infection point. So,

$$F = 0 \iff L \cap \mathcal{C} = P$$

3.1.1 Inflection Points

We want to measure how L tangents $\mathcal{C}$ at P.

– Suppose $P = [0 : 0 : 1]$

– and let L be $x = 0$

Then we take a taylor expansion, and sub in L,

$$z^{d-1}x + z^{d-2}h_2(x, y) + \cdots + zh_{d-1}(x, y) + h_d(x, y) = 0$$

Definition 3.4. Let $(L \cdot \mathcal{C})_P$ be the smallest such i such that $h_i(0, y) \neq 0$. Then $(L \cdot \mathcal{C})_P$ is the intersection multiplicity of L and $\mathcal{C}$ at P, and

$$(L \cdot \mathcal{C})$$

3.1.2 Hessian

The hessian is just a mess of partial derivatives, look at ECM1905 (Advanced Calculus),

Theorem 3.5. The polynomial $g(x, y, z)$ is not divisible by f. The set, of

$$\begin{cases} f(x, y, z) = 0 \\ g(x, y, z) = 0 \end{cases}$$

consists of all infection points

Theorem 3.6. If $d = 3$, then the curve has 9 inflection points.

3.2 Weirstrass (Normal) Form

Theorem 3.7 (Weirstrass Form)**.** There exists $\phi \in \mathrm{PGL}_3(\mathbb{C})$ such that $\phi(\mathcal{C})$ is given,

$$zy^3 = x^3 + axz^2 + bz^3$$

for some $a, b \in \mathbb{C}$

The same as Lengendre, but without final step Assume $[0 : 1 : 0]$ is an inflection point and there is a tangent of $z = 0$. Now obtain $G = I = 0$ by using porjective transformations.

$$[x : y : z] \mapsto \left[x : y - \frac{G}{2}x - \frac{I}{2}z : z\right]$$

using $[x : y : z] \mapsto \left[x + \frac{E}{3A}z : y : z\right]$ and get $E = 0$.

INSERT NUMBER THEORY STUFF

3.3 Singular Cubics

Singularity makes things so much easier, as you start with a singular point.

Theorem 3.8. There exists $\phi \in \mathrm{PGL}_3(\mathbb{C})$ such that $\phi(\mathcal{C})$ is given by,

– $zx^2 = y^3$ (Caspobel Curve)

– $zx^2 + zy^2 = y^3$ (Nodal Curve)

Take $[0 : 0 : 1]$ is a signular point of $\mathcal{C}$, then,

(i) FAST

3.4   Reducible Curves

Theorem 3.9. There exists $\phi \in \mathrm{PGL}_3(\mathbb{C})$ such that $\phi(\mathcal{C})$ is given by,

- $x(zy + x^2) = 0$

- $x(zx + y^2) = 0$

- $xyz = 0$

- $xy(x + y) = 0$

- $x^2 y = 0$

- $x^3 = 0$

. Must be trivial as he went at 100mph.   □

This was done by Newton

Definition 3.10 (Hasse Pencil). Let $\mathcal{P}$ be a family of cubic curves is given by,

$$\lambda(x^3 + y^3 + z^3) + \mu xyz = 0 \tag{1}$$

where $[\lambda : \mu] \in \mathbb{P}^1_{\mathbb{C}}$. We say $\mathcal{P}$ is a pencil.

If $\mu^3 \neq -27\lambda^3$, the curve is smooth. Otherwise it's a union of three lines. The inflection points are the counterexample for Gallai-Sylvester.

Theorem 3.11 (Artebani, Dolgachev). Let $\mathcal{C}$ be an arbitrary smooth curve in $\mathbb{P}^2_{\mathbb{C}}$. There exists $\phi \in \mathrm{PGL}_3(\mathbb{C})$, such that $\phi(\mathcal{C}) \in \mathcal{P}$

4   Lecture 4 - Pencils of Conics and Surfaces

We are going to look at,

  – Pencils of conics

  – Surfaces of bi-degree $(1, 2)$ in $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^2$

  – Del Pezzo Surfaces.

It's Pure Linear Algebra and we can cook up examples. We are going to really study pencil of conics. How do we cook up examples? Take two points and then take two tangents and then create conics.

or we could start with a conic and then take a line passing through two points on the conic, and then a degenerate conic that also intersects the point there. Take the union and then we have a second basis point for the pencil, the multiply by constant until you get irreducibles.

4.1   Pencil of conics

Take two homogenous quadric polynomials and that they are linearly independent. Then if $\mathcal{P}$ is a family of conics given by,

$$\lambda f_2 + \mu g_2 = 0$$

then $\mathcal{P}$ is a pencil.

Definition 4.1 (Base Locus). Let $Bs(\mathcal{P})$ be the subset in $\mathbb{P}_{\mathbb{C}}^2$ given by,

$$\begin{cases} f_2 = 0 \\ g_2 = 0 \end{cases}$$

We say that $Bs(\mathcal{P})$ is the base locus of $\mathcal{P}$[3]

  – Either $Bs(\mathcal{P})$ contains a line

  – or $Bs(\mathcal{P})$ consists of at least most four point.

Why do we want to do this? Why do we want to classify them? If we can show there are finite cases, then to prove things all we need to do is just check these pencils, assuming that the generators are simple. However, there is another reason, the equation in the definition, λ and μ can be thought as coordinates and so then we can consider the pencil as a surface and it's going to be bidegree in $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^2$. Now we can consider this using surface mathematics.

The surfaces are irreducible and they are called a Quinticentric Del Pezzo Surface.

  – These are surfaces of degree n in $(n+1)$-dimensional space

  – They had already been studied by Serge and Veronese

  – Del Pezzo found all of them.

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[3]We can say that this is a zero dimensional sub scheme.

4.2   Classification

Up to projective transformations, one of the following cases hold:

Label	$\mathcal{P}$	$\det(M_{\lambda,\mu})$	$\mathrm{Bs}(\mathcal{P})$
$1+1+1+1$	$\lambda xy + \mu z(x+y+z) = 0$	$\lambda\mu(\lambda-\mu)$	4 points
$2+1+1$	$\lambda xy + \mu z(x+y) = 0$	$\mu^2\lambda$	3 points
$2+2$	$\lambda x^2 + \mu yz = 0$	$\mu^2\lambda$	2 points
$3+1$	$\lambda xy + \mu(xz + y^2) = 0$	μ^3	2 points
4	$\lambda x^2 + \mu(xy + z^2) = 0$	μ^3	1 point
$4'$	$\lambda x^2 + \mu y^2 = 0$	0	1 point
∞	$\lambda xy + \mu xz = 0$	0	line+point
∞'	$\lambda x^2 + \mu xy = 0$	0	line

. **Step 1:** Write them down in matrix form, then just add the conics. Then the determinant is going to polynomial in λ, it may be a zero polynomial or its going to be a homogenous polynomial of degree 3. These three points are three reducible conics. Now choose a basis of the pencil, such that our polynomial changes.
Step 2:   the polynomial has three, two dor one root. Then we can assume,

– it's a zero polynomial

– It's $\lambda\mu(\lambda - \mu)$

– $\lambda\mu^2$

– μ^3

Now to change x, y and z. Thus we assume $f_2 = x^2$ or $f_2 = 2xy$.

– If the polynomial is not μ^3, then g_2 is reducible

– If the polynomial is μ^3, then g_2 is irreducible.

Step 3: Take,
$$g_2 = 2ax^2 + bxy + 2cy^2 + dxz + eyz + eyz + 2fz^2$$

▶ If $f_2 = x^2$, then we may assume that $a = 0$ and

$$M_{\lambda,\mu} = \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mu \begin{pmatrix} 0 & b & d \\ b & 2c & e \\ d & e & e \end{pmatrix}.$$

▶ If $f_2 = 2xy$, then we may assume that $b = 0$ and

$$M_{\lambda,\mu} = \lambda \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mu \begin{pmatrix} 2a & 0 & d \\ 0 & 2c & e \\ d & e & e \end{pmatrix}.$$

Step 4: We have $f_2 = x^2$ and

$$h_3 = (2bde - 2b^2f - 2cd^2)\mu^3 + (4cf - e^2)\mu^2\lambda$$

If $h_3 = 0$, pencil is reducible,

$$\begin{cases} 2bde - 2b^2f - 2cd^2 = 0 \\ 4cf - e^2 = 0 \end{cases}$$

either $c = e = b = 0$, so g_2 doesn't depend on y, or $c \neq 0$, then we can assume $c = \frac{1}{4}$, which gives $f = e^2$ and $d = 2be$, so that,

$$g_2 = (y + 2ez)(4bx + y + 2ez)$$

In both cases, we can change $g_2 \in \mathcal{P}$ and coordinates so that $\mathcal{P}$ is given by,

$$\lambda x^2 + \mu y^2 = 0$$

similarly for $f_2 = 2xy$ and $h_3 = 0$.[4]

Remark. A pencil is 2D as nets is to 3D.

Step 5: Now we assume that a cubic polynomial is not zero. So h_3 is $\lambda\mu(\lambda - \mu)$ or $\lambda\mu^2$ or μ^3 and play the same game.
If $f_2 = x^2$ and $\mathcal{C}$, then,

$$\mathcal{C} = L + L'$$

If $L = L'$ and then $h_3 = 0$, which is impossible. Then,

$$L \neq L'$$

so we can't have the lines being $x = 0$,

– L is then given by $y = 0$

– L' is given by $z = 0$

so that $\mathcal{P}$ is given by $\lambda x^2 + \mu yz = 0$
Step 6: If $f_2 = x^2$ and $\mathcal{C}$ is irreducible then $h_3 = \mu^3$ and,

– $x = 0$ intersects with teo points

– $x = 0$ only intersects at some points

Changing coordinates we can say,

– $x = 0$ instrsects $\mathcal{C}$ by $[0:0:1]$ and $[0:1:0]$.

– $x = 0$ is the tangent $\mathcal{C}$ at $[0:0:1]$

Step 7: Assume $f_2 = 2xy$ and h_3 is a polynomial and g_2 is some other polynomial. So we compute the determinant of the matrix and compare cases. [5] Then,

$$\lambda xy + \mu(xz + y^2) = 0$$

Corollary 4.2. We may assume that either $h_3 = \lambda\mu(\lambda - \mu)$ or $h_3 = \lambda\mu^2$

We conclude that $\mathcal{C}$ is reducible

Step 8: We do some stuff relating to the fact $g_2^2 = z^{26}$ and then say that $\lambda xy + \mu z^2 = 0$.
Now we assume that the lines are distinct. So assume one of the lines don't pass through $[0:0:1]$, so map to $z = 0$ and then pencil simplifies and the fiddle for a bit.
Step 9: Do some more cases. $\square$

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[4]This is a special case of Bertini's Theorem
[5]I'm done, I'm tired.
[6]Typo here, I don't quite understand

In the first case, we have $c = f = 0$, so that

$$h_3(\lambda, \mu) = 2bde\mu^3 - e^2\mu^2\lambda,$$

so that $e = 0$, which implies that $\mathcal{C}$ is reducible.
In the second case, we have $c = e = 0$, so that

$$g_2 = bxy + dxz + 2fz^2,$$

so that, after a coordinate change, $\mathcal{P}$ is given by

$$\lambda x^2 + \mu(xy + z^2) = 0.$$

5   Lecture 5

– Fano Varieties

– Classification of smooth Fano threefolds

– Which smooth Fano manifolds admit Kahler-Einstein metric

– Smooth Fano threefolds (No 2.24)

5.1   Fano Varieties

They are named after Gino Fano. He wrote a lot of varieties and were named after him with a new definition from a Russian Mathematician. Unfortunataly, the italian school proofs were very incorrect, so they had to be rewritten. Fano cut corners and streamlined things.

Let X be a projective smooth complex variety of dimension n.

Definition 5.1 (Fano Variety). We say that X is a Fano variety if $-K_X$ is ample.

– The number $(-K)X)^n is the degree$

$\mathbb{P}^n$ is a fano variety. The grassmannian $\mathrm{Gr}(r, m)$ is a fano varierty. Let X be a normal hyperspace in $\mathbb{P}^m$ of degree d. Then,

$$X \text{ is a fano variety} \iff d \leq m$$

Definition 5.2 (Del Pezzo). Del Pezzo surfaces are two dimensional Fano varieties

We can characterise all of the smooth Pezzo surfaces, called the S_d surfaces of degree d.

– $S_9 = \mathbb{P}^2$

– S_8 is $\mathbb{P}^1 \times \mathbb{P}^1$ or a blow up od $\mathbb{P}^2$ in one point.

– S_7 is a blow up of $\mathbb{P}^2$ in two points.

– and so on

A blow up is a lovely Algebraic Geometry operation.[7]
There exists a webpage called Fanography, that holds all of the characterisations. Lot's of conference talks, lots of nice results and a new classification was unveiled. The end of the list is **simple**.

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[7]Insert story about Italian Coffee Shop here and Ivan nearly getting arrested.

5.2   Kahler-Einstein smooth Fano threefold

Which smooth Fano threefold admit a Kahler-Einstein metric? There a lot of metrics on a specific manifold, so people looked for something nicer. If you give a matric an Einstein condition, you get some nice properties and this metric must be unique. So this only makes sense for Fano varieties. People searched for conditions to make sure the metric exists.

Ivan and eight friends studied this for a bit of fun, they met in California and looked at it.

Theorem 5.3. Let X be a general smooth Fano threefolds in the family $No\mathcal{N}$. Then X admits a Kahler-Einstein metric if and only if,

$$\mathcal{N} \notin \begin{Bmatrix} 2.23, 2.26, 2.28, 2.30, 2.31, 2.33, 2.35, 2.36, 3.14, \\ 3.16, 3.18, 3.21, 3.22, 3.23, 3.24, 3.26, 3.28, 3.29, \\ 3.30, 3.31, 4.5, 4.8, 4.9, 4.10, 4.11, 4.12, 5.2 \end{Bmatrix}$$

It took so long as it was proved case by case.

We shall look at $No2.24$ we take X being a smooth threefold in $\mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$ of degree $(1, 2)$. This is a net of conics, so a family of conics. For some homogenous quatratic polynomials,

$$x \cdot f_2(u, v, w) + y \cdot g_2(u, v, w) + z \cdot h_2(u, v, w) = 0$$

(i) f_2

(ii) g_2

(iii) h_2

Ivan studied this and if the threefold also had symmetries, it helps.

Theorem 5.4. Suppose X is given by,

$$u^2 x + v^2 y + w^2 z = \mu(vwx + uwy + uvz) = 0$$

Then X admits a Kahler-Einstein metric.

Then Ivan polished the paper, he noticed if count paprameter on which x depends, it's 17, but $\mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$ has a lot of symmetries. There are 16 dimensional group on the three folds (7 dimensional). Hence, we should have one parameter like the cubic curves. The whole space should depend on one parameter, μ. Can you find the coordinate change to go between the two equations? The classifications of all is, possibly, open.Ivan played with the equations and got,

Theorem 5.5 (Classifications of net conics)**.** We can get,

(i) either X is given by,
$$u^2 x + v^2 y + w^2 z = \mu(vwx + uwy + uvz) = 0$$

(ii) or X is given by,
$$u^2 x + v^2 y + w^2 z + vwx + uwy = 0$$

(iii) or X is given by
$$u^2 + v^2 y + w^2 z + vwx = 0$$

Remark. We see that the last two does not admit a Kahler-Einstein metric.

This wasn't new, there was a preprint in the Edinburugh Library, written in French. Unpublished in 1972, but it was in French so we don't know what they actually proved. Ivan contacted them, they had forgot about the preprint. The authors aren't quite sure though.

5.3   Conic Bundle and discriminant cubic curve

Take a project $\mathrm{pr}_1 : X \to \mathbb{P}^2_{\mathbb{C}}$ be a projection of the first factor. For a point $[x:y:z] in \mathbb{P}^2_{\mathbb{C}}$, the preimage,

$$\mathrm{pr}_1^{-1}([x:y:z])$$

is a possibly reducible conic in $\mathbb{P}^2_{\mathbb{C}}$ with corrdinates $[u:v:w]$.

(i) Of $[x:y:z]$ is general, then $\mathrm{pr}_1^{-1}([x:y:z])$ is reducible.

(ii) For some $[x:y:z] in \mathbb{P}^2_{\mathbb{C}}$ is redicible.

(iii) Such points for a cubic curve. $C \subset \mathbb{P}^2_{\mathbb{C}}$

It is enough to show X can be given by,

$$(a_1 vw + a_2 u^2)x + (b_1 uw + b_2 v^2)y + (c_1 uv + c_2 w^2)z = 0 \tag{$*$}$$

If X is given by above,

$$a_2 b_2 c_2 \neq 0$$

as X is smoothm then, $a_2 = b_2 = c_2 = 1$ and then we can scale again,

$$x \mapsto x \quad y \mapsto yt^2 \quad z \mapsto zs^2 \quad u \mapsto u \quad v \mapsto \frac{v}{s} \quad w \mapsto \frac{w}{t}$$

for $s = \frac{a}{c}$ and $t = \frac{a}{b}$.

Now choose our parameters such hthat X is given by the equation above.

– C is a discriminant cubic curve

– Then C is either smooth or nodal, since X is smooth.

If C is reducible, then we may assume it is given by,

$$uvw = 0$$

This easily implies X is given by $(*)$. Choose coordinates such that,

$$\alpha x^3 + \beta y^3 + \gamma z^3 + \delta xyz = 0$$

such that $\alpha, \beta \neq 0$. Now change u, v and w such that we get a $(*)$.

Let $C_x = \mathrm{pr}_1^{-1}([1:0:0])$, $C_y = \mathrm{pr}_1^{-1}([0:1:0])$ and $C_z = \mathrm{pr}_1^{-1}([0:0:1])$. We do this to reduce from 18 unknowns, as 18 is too many. Consider C_x, C_y and C_z as conics in $\mathbb{P}^2$. Then they are smooth, as $\alpha, \beta \neq 0$ and so it maps nicely and hence smooth. Choose our u, v, w and so, $C_x = \{vw + u^2 = 0\}$,

$$(vw + u^2)x + f_2 y + f_3 z = 0$$

and so we now have 10 parameters, after scaling. If we take the hessian, we get the equation of C. If C is singular, then $[0:0:1] = \mathrm{Sing}\, C$ and C_z is a double line. Observe, that $C_x \cap C_y \cap C_z = \varnothing$, since X is smooth. But, $C_x \cap C_y \neq \varnothing$ and $C_x \cap C_z \neq \varnothing$. In lecture 2, we see that $(\mathbb{P}^2, C_x) \equiv \mathrm{PGL}_2(\mathbb{C})$. So we choose another u, v, w and somore scaling. So,

$$u \mapsto u \quad v \mapsto sv \quad w \mapsto \frac{w}{s}$$

for $s \neq 0$, then we get,

If $a_5 \neq 0$, we get $a_1 = a_5 = b_1 = 1$ and if $a_5 = 0$, $a_1 = b_1 = 1$. Now we fiddle and get what we want (this is ommited because it's honestly a mess, use Python).

$$\left(u^2 + vw\right)x + \left(s^2 a_1 v^2 + a_2 u^2 + a_3 svu + a_4 vw + \frac{a_5}{s} uw\right)y +$$
$$+ \left(\frac{b_1}{s^2} w^2 + b_2 u^2 + sb_3 vu + b_4 vw + \frac{b_5}{s} uw\right)z = 0,$$

6   Ivan Quotes

– What is blowup? Let me google.

– $\mathbb{P}^1_{\mathbb{C}}$ is for babies

– All my friends

– This is simple

– This is an exericse

– Me and my friends.

– Topology is just Algebraic Geometry