Geometry of Nets of Conics

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Projective geometry is all geometry.

Arthur Cayley

Overview

The entire discussion of this mini-course shall be done in the language of projective geometry. As all conversations are done, an understanding of the language in which the conversation is carried must first be understood by all interlocutors. Thus we shall begin by introducing projective geometry in the first three sections before moving on to the purpose of this mini-course which is the discussion of the geometry of nets of conics. We shall work exclusively with the complex field $\mathbb{C}$.

Contents

1 The Complex Projective Plane $\mathbb{P}^2_{\mathbb{C}}$

Recall the notion of *equivalence relation* [14]. To define complex projective plane, we need to use an equivalence relation $\sim$ on the set $\mathbb{C}^3 \setminus (0,0,0)$ defined by

$$\big(x,y,z\big) \sim \big(x',y',z'\big) \iff \big(x,y,z\big) = \big(\lambda x', \lambda y', \lambda z'\big) \text{ for some complex number } \lambda \neq 0.$$

Exercise 1.1. Verify that $\sim$ is an equivalence relation.

For every $(x,y,z) \neq (0,0,0)$, we denote by $[x : y : z]$ the set consisting all points $(\lambda x, \lambda y, \lambda z)$ where λ runs through all non-zero complex numbers. Then $[x : y : z]$ consists of all points $(x',y',z') \in \mathbb{C}^3$ such that $(x',y',z') \neq (0,0,0)$ and $(x',y',z') \sim (x,y,z)$. The set $[x : y : z]$ is usually called the *equivalence class* of the point (x,y,z). Note that

$$\big[x : y : z\big] = \big[x' : y' : z'\big] \iff \big(x,y,z\big) \sim \big(x',y',z'\big).$$

For example, we have $[1 : 2 : 3] = [2 : 4 : 6] = [1+i : 2+2i : 3+3i]$.

Definition 1.2. The complex projective plane $\mathbb{P}^2_{\mathbb{C}}$ is the set consisting of all equivalent classes

$$\big[x : y : z\big].$$

Here (x,y,z) is a point in $\mathbb{C}^3$ such that $(x,y,z) \neq (0,0,0)$.

Similarly, we can also define real projective plane, denoted as $\mathbb{P}^2_{\mathbb{R}}$, or a projective plane over any other field. However, we shall restrict our discussions to the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$. We may sometimes drop the symbol $\mathbb{C}$ in $\mathbb{P}^2_{\mathbb{C}}$ and simply write $\mathbb{P}^2$ instead of $\mathbb{P}^2_{\mathbb{C}}$ for shorthand which should be unambiguous due to our restriction. For geometric reasons, we will refer to the elements of the set $\mathbb{P}^2_{\mathbb{C}}$ as points.

Let us repeat that, by definition, we consider points in $\mathbb{P}^2_{\mathbb{C}}$ as three-tuples $[x : y : z]$ such that

$$\big[x : y : z\big] = \big[x' : y' : z'\big] \iff \big(x,y,z\big) = \big(\lambda x', \lambda y', \lambda z'\big) \text{ for some non-zero } \lambda \in \mathbb{C}.$$

Here we exclude the point $[0 : 0 : 0]$, because this point does not exist!

Exercise 1.3. Define the complex projective line $\mathbb{P}^1_{\mathbb{C}}$.

The complex projective plane $\mathbb{P}^2_{\mathbb{C}}$ consists of $\mathbb{C}^2$ and an *infinite line.* To be precise, let U_z be the subset in $\mathbb{P}^2_{\mathbb{C}}$ consisting of points $[x : y : z]$ with $z \neq 0$. For each point $[x : y : z] \in U_z$, we have

$$\big[x : y : z\big] = \Big[\frac{x}{z} : \frac{y}{z} : 1\Big],$$

and the point $[x:y:z]$ is uniquely determined by two numbers $\frac{x}{z}$ and $\frac{y}{z}$. This means that we just constructed a map $U_z \to \mathbb{C}^2$ defined by

$$\big[x:y:z\big] \mapsto \Big(\frac{x}{z}, \frac{y}{z}\Big).$$

This map is bijection (it is one-to-one and onto). Thus, we can identify $U_z = \mathbb{C}^2$. To be pedantic, let $\overline{x} = \frac{x}{z}$ and $\overline{y} = \frac{y}{z}$. Then we can consider $\overline{x}$ and $\overline{y}$ as coordinated on $U_z = \mathbb{C}^2$.

What is the complement in $\mathbb{P}^2_{\mathbb{C}}$ to the subset U_z? The complement is the subset in $\mathbb{P}^2_{\mathbb{C}}$ consisting of all points $[x:y:0]$ where x and y are complex numbers such that either $x \neq 0$ or $y \neq 0$ (or both). Thus, we can identify the complement in $\mathbb{P}^2_{\mathbb{C}}$ to the subset U_z with the complex projective line $\mathbb{P}^1_{\mathbb{C}}$. This is the line at infinity. Hence, we split $\mathbb{P}^2_{\mathbb{C}}$ into two *disjoint* pieces:

- The subset U_z consisting of all points $[x:y:z]$ with $z \neq 0$. We identify $U_z = \mathbb{C}^2$ with coordinates $\overline{x} = \frac{x}{z}$ and $\overline{y} = \frac{y}{z}$ using $[x:y:z] = [\frac{x}{z} : \frac{x}{z} : 1]$.
- The subset consisting of all points $[x:y:0]$. We identify it with $\mathbb{P}^1_{\mathbb{C}}$.

Absolutely in the same way, we can split the complex projective line $\mathbb{P}^1_{\mathbb{C}}$ into two *disjoint* pieces:

- The subset consisting of all points $[x:y]$ with $y \neq 0$, which we can identify with $\mathbb{C}$ with coordinate $\frac{x}{y}$ using $[x:y] = [\frac{x}{y} : 1]$.
- The point $[1:0]$, which can be considered as a *point at infinity.*

We defined U_z as the subset in $\mathbb{P}^2_{\mathbb{C}}$ that consists of all points $[x:y:z]$ with $z \neq 0$. Then we identified $U_z = \mathbb{C}^2$ with coordinates $\overline{x} = \frac{x}{z}$ and $\overline{y} = \frac{y}{z}$. Why did we choose z here? There is no particular reason for this. Similar to U_z, we can define U_x and U_y. Namely, let U_x be the complement in $\mathbb{P}^2_{\mathbb{C}}$ to the line $x = 0$. If $[x:y:z] \in U_x$, then

$$\big[x:y:z\big] = \Big[1 : \frac{y}{x} : \frac{z}{x}\Big],$$

so that the couple $(\frac{y}{x}, \frac{z}{x})$ uniquely determines $[x:y:z]$. Thus, we can identify $U_x = \mathbb{C}^2$ with coordinates $\widetilde{y} = \frac{y}{x}$ and $\widetilde{z} = \frac{z}{x}$. Likewise, we define U_y to be the complement in $\mathbb{P}^2_{\mathbb{C}}$ to the line $y = 0$, and identify $U_y = \mathbb{C}^2$ with coordinates $\widehat{x} = \frac{x}{y}$ and $\widehat{z} = \frac{z}{y}$. Then

$$\mathbb{P}^2 = U_x \cup U_y \cup U_z,$$

since every point $[x:y:z]$ is contained in at least one of the subsets U_x, U_y and U_z. Informally speaking, the projective plane $\mathbb{P}^2_{\mathbb{C}}$ can be obtained from $U_x = \mathbb{C}^2$, $U_y = \mathbb{C}^2$, $U_z = \mathbb{C}^2$ by *patching* them together using the following formulas:

$$\boxed{\widetilde{y} = \frac{1}{\widehat{x}} = \frac{\overline{y}}{\overline{x}}, \widetilde{z} = \frac{\widehat{z}}{\widehat{x}} = \frac{1}{\overline{x}}}, \boxed{\widehat{x} = \frac{\overline{x}}{\overline{y}} = \frac{1}{\widetilde{y}}, \widehat{z} = \frac{1}{\overline{y}} = \frac{\widetilde{z}}{\widetilde{y}}}, \boxed{\overline{x} = \frac{1}{\widetilde{z}} = \frac{\widehat{x}}{\widehat{z}}, \overline{y} = \frac{\widetilde{y}}{\widetilde{z}} = \frac{1}{\widehat{z}}}.$$

1.1 Lines in $\mathbb{P}^2_{\mathbb{C}}$

To learn how to live in the complex projective plane, we have to consider simple geometric objects in $\mathbb{P}^2_{\mathbb{C}}$ and play with them. The simplest objects in $\mathbb{P}^2_{\mathbb{C}}$ are lines. How to define them? We already know one line: the line given by $z = 0$, which we informally called the *line at infinity*. Using this and our definition of a line in $\mathbb{C}^2$, let us define a line in $\mathbb{P}^2_{\mathbb{C}}$ as a subset that is given by

$$Ax + By + Cz = 0$$

for some (fixed) complex numbers A, B and C such that $(A, B, C) \neq (0, 0, 0)$. This subset does not change if we multiply the equation by a non-zero complex number λ, since the equation

$$\lambda Ax + \lambda By + \lambda Cz = 0$$

defines the same subset in $\mathbb{P}^2_{\mathbb{C}}$. Thus, the line given by $Ax + By + Cz = 0$ is determined by the point $[A : B : C] \in \mathbb{P}^2_{\mathbb{C}}$.

Our definition of a line in $\mathbb{P}^2_{\mathbb{C}}$ matches perfectly our definition of a line in $\mathbb{C}^2$. For instance, if L is the line in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$x + 2y + 3z = 0,$$

then $L \cap U_z$ consists of the points in U_z that is given by $\frac{x}{z} + 2\frac{y}{z} + 3$, so that $L \cap U_z$ is the line in $U_z = \mathbb{C}^2$ that is given by

$$\overline{x} + 2\overline{y} + 3 = 0,$$

where $\overline{x} = \frac{x}{z}$ and $\overline{y} = \frac{y}{z}$ are our coordinates on $U_z = \mathbb{C}^2$.

Remark 1.4. In $\mathbb{P}^2_{\mathbb{C}}$ we always use *homogeneous* equations. Non-homogeneous equations in $\mathbb{P}^2_{\mathbb{C}}$ do not make sense. For example, if we consider equation

$$2x - 3y + 5 = 0$$

in the projective plane, then point $[-1 : 1 : 0]$ satisfies this equation, while $[-2 : 2 : 0]$ does not. But $[-1 : 1 : 0] = [-2 : 2 : 0]$. Thus, the equation $2x - 3y + 5 = 0$ does not make any sense in $\mathbb{P}^2_{\mathbb{C}}$, because it is not homogeneous.

As in the complex plane $\mathbb{C}^2$, the line in $\mathbb{P}^2_{\mathbb{C}}$ is uniquely determined by two distinct points it contains. To be precise, if P and Q are points in $\mathbb{P}^2_{\mathbb{C}}$ such that $P \neq Q$, then there is a unique line in $\mathbb{P}^2_{\mathbb{C}}$ that contains both P and Q. The proof of this assertion is almost identical to the proof of its counterpart in $\mathbb{C}^2$. Namely, if $[a : b : c]$ and $[a' : b' : c']$ are distinct points in $\mathbb{P}^2_{\mathbb{C}}$, then the rank–nullity theorem implies that there is unique $[A : B : C] \in \mathbb{P}^2_{\mathbb{C}}$ such that

$$\begin{cases} Aa + Bb + Cc = 0, \\ Aa' + Bb' + Cc' = 0. \end{cases}$$

In this case, we can explicitly express $[A : B : C]$ in terms of a, b, c, a', b', c' as follows:

$$\begin{cases} A = bc' - cb', \\ B = ca' - ac', \\ C = ab' - ba'. \end{cases}$$

Thus, the unique line that contains $[a : b : c]$ and $[a' : b' : c']$ is given by the equation

$$\Big(bc' - cb'\Big)x + \Big(ca' - ac'\Big)y + \Big(ab' - ba'\Big)z = 0,$$

which can be rewritten in the following determinant form:

$$\det \begin{pmatrix} a & b & c \\ a' & b' & c' \\ x & y & z \end{pmatrix} = 0.$$

For instance, to find the line in $\mathbb{P}^2_{\mathbb{C}}$ that contains points $[11 : -7 : 1]$ and $[2 : 5 : 1]$, we have to solve the system of linear equations

$$\begin{cases} 11A - 7B + C = 0, \\ 2A + 5B + C = 0. \end{cases}$$

The solutions of this system form a one-dimensional vector space by the rank–nullity theorem. Using linear algebra, we see that $(A, B, C) = (4, 3, -23)$ is one solution, so that $(4, 3, -23)$ is a basis of this vector space. This means that the line $4x + 3y - 23z = 0$ is the unique line in $\mathbb{P}^2_{\mathbb{C}}$ that contains the points $[11 : -7 : 1]$ and $[2 : 5 : 1]$. Of course, we can find the same line using explicit determinant equation

$$\det \begin{pmatrix} 11 & -7 & 1 \\ 2 & 5 & 1 \\ x & y & z \end{pmatrix} = 0.$$

This gives us the line $-12x - 9y - 69z = 0$, which is the same line as $4x + 3y - 23z = 0$.

Exercise 1.5. Find the line in $\mathbb{P}^2_{\mathbb{C}}$ containing $[1 + 3i : 7 - 2i : 1]$ and $[3 : 8 - i : 1]$.

If three points $[a_1 : b_1 : c_1]$, $[a_2 : b_2 : c_2]$ and $[a_3 : b_3 : c_3]$ in $\mathbb{P}^2_{\mathbb{C}}$ are contained in one line, we say that they are collinear. Using determinant equation of a line that passes through two given points in $\mathbb{P}^2_{\mathbb{C}}$, we see that this happens if and only if

$$\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = 0.$$

Exercise 1.6. Find all lines in $\mathbb{P}^2_{\mathbb{C}}$ that contains exactly 2 points among $[0 : 0 : 1]$, $[0 : 1 : 1]$, $[1 : 1 : -1]$, $[1 : 3 : 1]$, $[2 : 5 : 1]$, $[1 : 1 : 1]$, $[1 : 4 : 2]$.

Why do we need projective plane? Is $\mathbb{C}^2$ not good enough? Yes. It is not good enough: the projective plane $\mathbb{P}^2_{\mathbb{C}}$ is *much better*. The first evidence for this is given by

Exercise 1.7. Let L and L' be two distinct lines in $\mathbb{P}^2_{\mathbb{C}}$. Show that the intersection $L \cap L'$ consists of exactly one point in $\mathbb{P}^2_{\mathbb{C}}$. Note that this implies that the intersection $L \cap L'$ is not empty.

Thus, there are no *parallel* lines in $\mathbb{P}^2_{\mathbb{C}}$. Every two lines in $\mathbb{P}^2_{\mathbb{C}}$ have a common point and this point is unique provided that the lines are distinct. For example, to find the unique intersection point of the lines $4x - 3y - 17z = 0$ and $7x + 5y - 11z = 0$ in $\mathbb{P}^2_{\mathbb{C}}$, we can use the determinant formula from the solution of Exercise 1.7. It gives us the point $[118 : -75 : 41]$, since

$$\det \begin{pmatrix} 4 & -3 & -17 \\ 7 & 5 & -11 \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \end{pmatrix} = 118\mathbf{i} - 75\mathbf{j} + 41\mathbf{k},$$

where $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$.

Alternatively, we can find the unique intersection point of the lines $4x - 3y - 17z = 0$ and $7x + 5y - 11z = 0$ by solving the following system of linear equations:

$$\begin{cases} 4x - 3y - 17z = 0, \\ 7x + 5y - 11z = 0. \end{cases}$$

This system has a unique solution $[x : y : z] \in \mathbb{P}^2_{\mathbb{C}}$ that is the same intersection point $[\frac{118}{41} : -\frac{75}{41} : 1] = [118 : -75 : 41]$.

Exercise 1.8. Find the intersection point in $\mathbb{P}^2_{\mathbb{C}}$ of the line given by

$$(-13+17i)x + (23+29i)y - (11-3i)z = 0$$

and the line given by $(2-5i)x + (15+19i)y + (37+41i)z = 0$.

Recall that we defined the subset U_z in the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$ by $z \neq 0$, and we identified $U_z = \mathbb{C}^2$ with coordinates $\overline{x} = \frac{x}{z}$ and $\overline{y} = \frac{y}{z}$. Every line in $U_z = \mathbb{C}^2$ is given by

$$A\overline{x} + B\overline{y} + C = 0,$$

where A, B and C are some fixed complex numbers such that either $A \neq 0$ or $B \neq 0$ (or both). Multiplying this equation by z, we obtain the line in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$Ax + By + Cz = 0.$$

This line contains the line $A\overline{x} + B\overline{y} + C = 0$ plus one extra point: the intersection point of the line $Ax + By + Cz = 0$ and the line $z = 0$, which is the point $[-B : A : 0]$. Therefore, every line in $U_z = \mathbb{C}^2$ can be naturally extended to a line in $\mathbb{P}^2_{\mathbb{C}}$ by adding one point at *infinity* (contained in the line $z = 0$). This process is usually called *projective completion.* Algebraically, this simply means the homogenization of the equation $A\overline{x} + B\overline{y} + C = 0$, which results in the homogeneous equation

$$Ax + By + Cz = 0.$$

We know that $U_z = \mathbb{C}^2$ contains lines that do not intersect. What happened with their projective completions? They intersect at some point in the line $z = 0$. For instance, the lines in $U_z = \mathbb{C}^2$ given by $2\overline{x} - 3\overline{y} + 5 = 0$ and $2\overline{x} - 3\overline{y} + 7 = 0$ do not intersect in U_z. But the lines $2x - 3y + 5z = 0$ and $2x - 3y + 7z = 0$ intersect at $[3 : 2 : 0]$.

1.2 Projective Transformations

There are no translations of the projective plane $\mathbb{P}^2_{\mathbb{C}}$, because $\mathbb{P}^2_{\mathbb{C}}$ does not have any special point like an origin $(0, 0)$ in $\mathbb{C}^2$. Instead, the projective plane allows a lot of *projective transformations* that can be used to simplify equations of curves. In particular, we can use them to simplify the equation of the conics in $\mathbb{P}^2_{\mathbb{C}}$. By definition, a projective transformation $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ is a function (map) that is given by

$$\big[x : y : z\big] \mapsto \big[a_{11}x + a_{12}y + a_{13}z : a_{21}x + a_{22}y + a_{23}z : a_{31}x + a_{32}y + a_{33}z\big]$$

for some (fixed) complex numbers a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32} and a_{33} such that

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \neq 0.$$

This transformation is induced by the linear transformations of $\mathbb{C}^3$ given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The 3×3 matrix in this expression is invertible, since its determinant is not zero by assumption. This means that there are complex numbers b_{11}, b_{12}, b_{13}, b_{21}, b_{22}, b_{23}, b_{31}, b_{32} and b_{33} such that

$$\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This gives us another projective transformation $\psi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ defined as

$$\psi\Big(\big[x : y : z\big]\Big) = \big[b_{11}x + b_{12}y + b_{13}z : b_{21}x + b_{22}y + b_{23}z : b_{31}x + b_{32}y + b_{33}z\big].$$

Then both $\psi \circ \phi$ and $\phi \circ \psi$ are identity maps, so that $\psi = \phi^{-1}$. Hence, the projective transformations of the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$ behave like linear transformations of the vector space $\mathbb{C}^3$. The only difference is that the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

gives an identity map of $\mathbb{P}^2_{\mathbb{C}}$ for every $\lambda \neq 0$. Alternatively, we can consider the projective transformations of $\mathbb{P}^2_{\mathbb{C}}$ as a way of choosing new *projective* coordinates on $\mathbb{P}^2_{\mathbb{C}}$. Namely, instead of x, y and z, we can introduce new projective coordinates:

$$\begin{cases} \mathbf{x} = a_{11}x + a_{12}y + a_{13}z, \\ \mathbf{y} = a_{21}x + a_{22}y + a_{23}z, \\ \mathbf{z} = a_{31}x + a_{32}y + a_{33}z, \end{cases}$$

where a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32} and a_{33} are complex numbers such that we can express x, y and z in terms of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ as follows:

$$\begin{cases} x = b_{11}\mathbf{x} + b_{12}\mathbf{y} + b_{13}\mathbf{z}, \\ y = b_{21}\mathbf{x} + b_{22}\mathbf{y} + b_{23}\mathbf{z}, \\ z = b_{31}\mathbf{x} + b_{32}\mathbf{y} + b_{33}\mathbf{z} \end{cases}$$

for some complex numbers b_{11}, b_{12}, b_{13}, b_{21}, b_{22}, b_{23}, b_{31}, b_{32} and b_{33}. Now substituting this expression for x, y and z into the defining equations of curve in $\mathbb{P}^2_{\mathbb{C}}$, we obtain its equation in new projective coordinates $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, which often can be simpler. For example, consider the line L given by

$$7x + 5y + 3z = 0,$$

then we let

$$\begin{cases} \mathbf{x} = 7x - 5y + 3z, \\ \mathbf{y} = y, \\ \mathbf{z} = z, \end{cases}$$

which implies that

$$\begin{cases} x = \frac{1}{7}\mathbf{x} + \frac{5}{7}\mathbf{y} - \frac{3}{7}\mathbf{z}, \\ y = \mathbf{y}, \\ z = \mathbf{z}, \end{cases}$$

so that substituting this into the defining equation of the line L, we get $\mathbf{x} = 0$, which is clearly simpler than the original equation. While this may seem trivial, the process will greatly ease more complicated problems that we shall encounter later on.

Projective transformations are bijections (one-to-one and onto). Furthermore, for every two lines L and L' in $\mathbb{P}^2_{\mathbb{C}}$, there exists a projective transformation that maps L into L'. Similarly, for every two points P and Q in $\mathbb{P}^2_{\mathbb{C}}$, there exists a projective transformation that maps P to Q. In fact, we can say more:

Exercise 1.9. Let P_1, P_2, P_3, P_4 be distinct points in $\mathbb{P}^2_{\mathbb{C}}$ such that no three among them are collinear. Show that there exists a projective transformation $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that

$$\begin{cases} \phi(P_1) = [1:0:0], \\ \phi(P_2) = [0:1:0], \\ \phi(P_3) = [0:0:1], \\ \phi(P_4) = [1:1:1]. \end{cases}$$

Let us show how to find the projective transformation ϕ such that

$$\begin{cases} \phi([1:2:3]) = [1:0:0], \\ \phi([1:0:-1]) = [0:1:0], \\ \phi([2:5:1]) = [0:0:1], \\ \phi([-1:7:1]) = [1:1:1]. \end{cases}$$

Observe that no three points among $[1:2:3]$, $[1:0:-1]$ $[2:5:1]$ and $[-1:7:1]$ are collinear, because

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \\ 2 & 5 & 1 \end{pmatrix} = 14 \neq 0, \det\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \\ -1 & 7 & 1 \end{pmatrix} = 28 \neq 0,$$
$$\det\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 1 \\ -1 & 7 & 1 \end{pmatrix} = 49 \neq 0, \det\begin{pmatrix} 1 & 0 & -1 \\ 2 & 5 & 1 \\ -1 & 7 & 1 \end{pmatrix} = -21 \neq 0.$$

By definition, the transformation ϕ is given by

$$\big[x:y:z\big] \mapsto \big[a_{11}x + a_{12}y + a_{13}z : a_{21}x + a_{22}y + a_{23}z : a_{31}x + a_{32}y + a_{33}z\big]$$

for some complex numbers a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32} and a_{33}. Let us find these numbers by *brute force*. By assumption, we have

$$\begin{cases} \phi([1:2:3]) = \big[a_{11} + 2a_{12}y + 3a_{13} : a_{21} + 2a_{22} + 3a_{23} : a_{31} + 2a_{32} + 3a_{33}\big] = [1:0:0], \\ \phi([1:0:-1]) = \big[a_{11} - a_{13} : a_{21} - a_{23} : a_{31} - a_{33}\big] = [0:1:0], \\ \phi([2:5:1]) = \big[2a_{11} + 5a_{12} + a_{13} : 2a_{21} + 5a_{22} + a_{23} : 2a_{31} + 5a_{32} + a_{33}\big] = [0:0:1], \\ \phi([-1:7:1]) = \big[-a_{11} + 7a_{12} + a_{13} : -a_{21} + 7a_{22} + a_{23} : -a_{31} + 7a_{32} + a_{33}\big] = [1:1:1]. \end{cases}$$

This gives us system of equations

$$\begin{cases} a_{11} + 2a_{12} + 3a_{13} = a, \\ a_{21} + 2a_{22} + 3a_{23} = 0, \\ a_{31} + 2a_{32} + 3a_{33} = 0, \\ a_{11} - a_{13} = 0, \\ a_{21} - a_{23} = b, \\ a_{31} - a_{33} = 0, \\ 2a_{11} + 5a_{12} + a_{13} = 0, \\ 2a_{21} + 5a_{22} + a_{23} = 0, \\ 2a_{31} + 5a_{32} + a_{33} = c, \\ -a_{11} + 7a_{12} + a_{13} = d, \\ -a_{21} + 7a_{22} + a_{23} = d, \\ -a_{31} + 7a_{32} + a_{33} = d, \end{cases}$$

where a, b, c and d are some numbers. Thus, we have 12 linear equations and 13 variables: a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}, a, b, c and d. By the rank–nullity theorem we know that solutions form at least one-dimensional vector space. However, we do not want solutions with $d = 0$, because there exists no such point in $\mathbb{P}^2_{\mathbb{C}}$ as $[0 : 0 : 0]$. Thus, we may add one extra equation $d = 1$. Solving the resulting system of equations, we get $a_{11} = -\frac{5}{21}$, $a_{12} = \frac{1}{7}$, $a_{13} = -\frac{5}{21}$, $a_{21} = -\frac{13}{49}$, $a_{22} = \frac{5}{49}$, $a_{23} = \frac{1}{49}$, $a_{31} = -\frac{1}{14}$, $a_{32} = \frac{1}{7}$, $a_{33} = -\frac{1}{14}$, $a = -\frac{2}{3}$, $b = -\frac{2}{7}$, $c = \frac{1}{2}$ and $d = 1$. Thus, the required projective transformation ϕ is given by

$$\big[x : y : z\big] \mapsto \Big[-\frac{5x}{21} + \frac{y}{7} - \frac{z}{21} : -\frac{13x}{49} + \frac{5y}{49} + \frac{z}{49} : -\frac{x}{14} + \frac{y}{7} - \frac{z}{14}\Big].$$

Multiplying all entries by $49 \cdot 2 \cdot 3 = 294$ or recomputing the system of equation with $d = 294$, we can rewrite the formula for ϕ as

$$\big[x : y : z\big] \mapsto \big[-70x + 42y - 70z : -78x + 30y + 6z : -21x + 42y - 21z\big].$$

Now let us find ϕ again using the idea described in the solution of Exercise 1.9. Let α be the projective transformation that is induced by the linear transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 5 \\ 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

and let β be the inverse of α. Then $\beta([1 : 2 : 3]) = [1 : 0 : 0]$, $\beta([1 : 0 : -1]) = [0 : 1 : 0]$ and $\beta([2 : 5 : 1]) = [0 : 0 : 1]$. Observe that

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \\ 2 & 5 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{5}{14} & \frac{-3}{14} & \frac{5}{14} \\ \frac{13}{14} & -\frac{5}{14} & -\frac{1}{14} \\ -\frac{2}{14} & \frac{4}{14} & -\frac{2}{14} \end{pmatrix} = \frac{1}{14} \begin{pmatrix} 5 & -3 & 5 \\ 13 & -5 & -1 \\ -2 & 4 & -2 \end{pmatrix}.$$

This shows that β is given by

$$\big[x : y : z\big] \mapsto \big[5x - 3y + 5z : 13x - 5y - z : -2x + 4y - 2z\big].$$

Then $\beta(P_4) = [-21 : -49 : 28]$. Let γ be the projective transformation that is given by

$$\big[x : y : z\big] \mapsto \Big[-\frac{x}{21} : -\frac{y}{49} : \frac{z}{28}\Big] = \big[28x : 12y : -21z\big].$$

Then the composition $\gamma \circ \beta$ is the projective transformation ϕ, which we already found by brute force. This can be verified by as follows:

$$\begin{pmatrix} 28 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & -21 \end{pmatrix} \begin{pmatrix} 5 & -3 & 5 \\ 13 & -5 & -1 \\ -2 & 4 & -2 \end{pmatrix} = \begin{pmatrix} 140 & -84 & 140 \\ 156 & -60 & -12 \\ -42 & 84 & 42 \end{pmatrix} = -2 \begin{pmatrix} -70 & 42 & -70 \\ -78 & 30 & 6 \\ -21 & 42 & -21 \end{pmatrix}.$$

Exercise 1.10. Find projective transformation $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that

$$\begin{cases} \phi\big([0:1:3]\big) = [1:0:0], \\ \phi\big([2:1:-1]\big) = [0:1:0], \\ \phi\big([1:1:5]\big) = [0:0:1], \\ \phi\big([7:1:-1]\big) = [1:1:1]. \end{cases}$$

The set of all projective transformations of $\mathbb{P}^2_{\mathbb{C}}$ is usually denoted by $\mathrm{PGL}_3(\mathbb{C})$. Similarly, we can define projective transformations of $\mathbb{P}^1_{\mathbb{C}}$, which form a set that is usually denoted by $\mathrm{PGL}_2(\mathbb{C})$. These are the maps $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ that are given by

$$\big[x:y\big] \mapsto \big[a_{11}x + a_{12}y : a_{21}x + a_{22}y\big]$$

for some (fixed) complex numbers a_{11}, a_{12}, a_{21} and a_{22} such that $a_{11}a_{22} - a_{12}a_{21} \neq 0$. These transformations are also known under different name. One can expands the set of all complex numbers $\mathbb{C}$ to *the Riemann sphere* by adding a *point at infinity*, which is usually denoted by ∞. Adding ∞ to the set $\mathbb{C}$ is useful since many natural transformations $\mathbb{C} \to \mathbb{C}$ behaves better if considered as transformations of the Riemann sphere. Among them are the so-called Möbius transformations:

$$t \mapsto \frac{at+b}{ct+d},$$

where a, b, c and d are fixed complex numbers such that $ad - bc \neq 0$. These transformations sends the point $t = -\frac{d}{c}$ to ∞. One can easily describe all of this using projective geometry. The Riemann sphere is just $\mathbb{P}^1_{\mathbb{C}}$ with projective coordinates $[x:y]$. It contains the subset U_y that is given by $y \neq 0$, which can be identified with $\mathbb{C}$ with coordinate $t = \frac{x}{y}$ using

$$\big[x:y\big] = \Big[\frac{x}{y}:1\Big] \mapsto \frac{x}{y} \in \mathbb{C}.$$

Here *the point at infinity* is just the point $[1:0]$. Moreover, for any complex numbers a, b, c and d such that $ad - bc \neq 0$, the projective transformation

$$\big[x:y\big] \mapsto \big[ax + by : cx + dy\big]$$

gives Möbius transformations $t \mapsto \frac{at+b}{ct+d}$ when restricted to $U_y = \mathbb{C}$. So Möbius transformations of the Riemann sphere are just projective transformations of $\mathbb{P}^1_{\mathbb{C}}$.

1.3 The Sylvester–Gallai Theorem

The Sylvester–Gallai theorem in geometry states that, given a finite number of points in $\mathbb{R}^2$, either all the points lie on a single line; or there is a line which contains exactly two of the points. It is named after James Sylvester, who posed it as a problem in [19], and Tibor Gallai, who proved it in [9]. Later, a simpler proof of this result was found by Leroy Kelly (see [6]). It is more natural to consider this problem for lines $\mathbb{P}^2_{\mathbb{C}}$ (cf. [10]). Surprisingly, the required assertion does not hold in the projective plane $\mathbb{P}^2_{\mathbb{C}}$ in general! For instance, the following 9 points

$$\begin{gathered}[1:-1:0],[0:1:-1],[1:0:-1],\\ [1:-\omega:0],[1:-\omega^2:0],[1:0:-\omega],\\ [1:0:-\omega^2],[0:1:-\omega],[0:1:-\omega^2],\end{gathered} \tag{1}$$

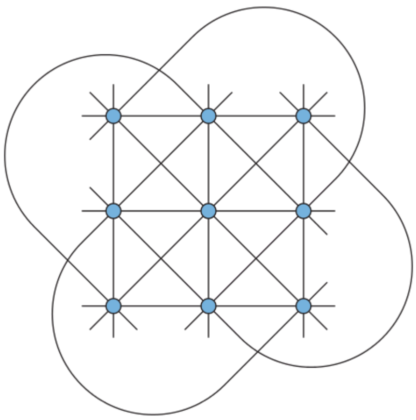
give a counter-example, where $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ (a cube root of unity).

Exercise 1.11. Show that every line in $\mathbb{P}^2_{\mathbb{C}}$ that passes through two points in (1) also contains a third point in (1). Show also that no four points among (1) are contained in one line.

On the other hand, the Sylvester–Gallai theorem holds every finite subset in $\mathbb{P}^2_{\mathbb{C}}$ that consists of at most 8 points. This follows from:

Exercise 1.12. Let Σ be a subset in $\mathbb{P}^2_{\mathbb{C}}$ such that $|\Sigma| \leqslant 8$. Suppose that Σ is not contained in one line in $\mathbb{P}^2_{\mathbb{C}}$. Prove that there exists a line L in $\mathbb{P}^2_{\mathbb{C}}$ that contains exactly two points of the set Σ.

One can then also ask: what condition(s) on $\Sigma \subset \mathbb{P}^2_{\mathbb{C}}$ which has 9 elements are then required so that the statement "either all the points lie on a single line; or there is a line which contains exactly two of the points" remains true? Indeed there are many cases for a subset of 9 elements of $\mathbb{P}^2_{\mathbb{C}}$ such that the statement above remains true and the collection of 9 points given in (1) is a rare case which gives a counterexample to the theorem. The collection is one of a special configuration of 9 points in $\mathbb{P}^2_{\mathbb{C}}$ called the Hesse configuration which has the automorphism group of order 216 called the Hessian group. The Hesse configuration can be visualised using the following figure of 9 points and 12 lines where there are always exactly 3 distinct points in each line and exactly 4 distinct lines passing through each point:



We will show later using the theory of cubic curves that such configuration is unique.

Exercise 1.13 (Challenge: no solution given). Let Σ be a subset of $\mathbb{P}^2_{\mathbb{C}}$ containing 9 points *not* projectively equivalent to (1) (i.e. not having the Hesse configuration). Find an elementary proof to show that there exists a line L in $\mathbb{P}^2_{\mathbb{C}}$ that contains exactly two points of the set Σ.

In fact, one can ask further: what are all the special configurations of a collection of finite points in $\mathbb{P}^2_{\mathbb{C}}$? The special configurations of such collections are intimately related to the G-orbit for some $G \subseteq \mathrm{PGL}_3(\mathbb{C})$ so first we need to consider *all* finite subgroups of $\mathrm{PGL}_3(\mathbb{C})$. This has in fact only been done (in 1916) by Miller, Blichfeldt, and Dickson [15] which has been found to contain the Hessian group. The reader is invited to make independent explorations on the points and questions raised here. Additionally, the reader is also invited to find the classifications of the finite groups in $\mathrm{PGL}_2(\mathbb{C})$ and note their relations to the platonic solids.

2 Conics in $\mathbb{P}^2_{\mathbb{C}}$

The only reasonable subsets we can define in the complex projective plane are given by homogeneous polynomial equations in x, y and z. Because of this, x, y and z are usually called *homogeneous coordinates* or *projective coordinates*. The simplest case is when we have one equation:

$$f_d(x, y, z) = 0,$$

where $f_d(x, y, z)$ is a homogeneous polynomial of degree $d \geqslant 1$. Such subsets are called *plane projective complex curves* of degree d or simply *plane curves*. Plane curves of degree 1 are just lines in $\mathbb{P}^2_{\mathbb{C}}$. Plane curves of degree 2 are called *conics*, and plane projective curves of degree 3 are called *cubic curves*. If the polynomial $f_d(x, y, z)$ is *irreducible*, we say that the curve it defines is *irreducible*. Otherwise, we have

$$f_d(x, y, z) = f_{d_1}(x, y, z) f_{d_2}(x, y, z)$$

for some homogeneous polynomial $f_{d_1}(x, y, z)$ of degree $d_1 \geqslant 1$ and some homogeneous polynomial $f_{d_2}(x, y, z)$ of degree $d_2 \geqslant 1$ such that $d_1 + d_2 = d$. In this case, the subset given by $f_d(x, y, z) = 0$ is a union of the subset $f_{d_1}(x, y, z) = 0$ and the subset $f_{d_2}(x, y, z) = 0$, so in some sense this case is simpler. Such *reducible* curves are often excluded from consideration. However, we will use them, since this is convenient sometimes. Note that lines are always irreducible. However, conics and cubic curves can be reducible.

One can show that the polynomial $f_d(x, y, z)$ can be uniquely written as a product of irreducible homogeneous polynomials (see [2]), where uniqueness is understood in a natural way: up to permutations of factors and up to scaling. Thus, if $f_d(x, y, z)$ is reducible, then

$$f_d(x, y, z) = f_{d_1}(x, y, z) f_{d_2}(x, y, z) \cdots f_{d_r}(x, y, z),$$

where each $f_{d_i}(x, y, z)$ is an irreducible homogeneous polynomials of degree $d_i \geqslant 1$, and

$$d_1 + d_2 + \cdots + d_r = d.$$

Then each polynomial equation $f_{d_i}(x, y, z) = 0$ defines an irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$. These curves are said to be *irreducible components* of the curve $f_d(x, y, z) = 0$.

Geometry of plane projective complex curves is a very rich subject that is way beyond the scope of this book (see [8, 12]). We will only work with the curves of degree 1 (lines), 2 (conics) and 3 (cubic curves). In Section 1.1, we played a bit with lines. Now it is our turn to play with conics. As we already mentioned, conics are not always irreducible. However, if a conic is reducible, then it is a union of two (not necessarily distinct) lines, so that we already know how to play with them.

Let $\mathcal{C}$ be a conic in $\mathbb{P}^2_{\mathbb{C}}$. Then it is given by

$$Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2 = 0$$

for some complex numbers A, B, C, D, E and F such that $(A,B,C,D,E,F) \neq (0,0,0,0,0,0)$. We can rewrite this equation in the matrix form as:

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} A & \frac{B}{2} & \frac{D}{2} \\ \frac{B}{2} & C & \frac{E}{2} \\ \frac{D}{2} & \frac{E}{2} & F \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

Denote the 3×3 matrix in this equation by M.

Exercise 2.1. Show that the conic $\mathcal{C}$ is irreducible if and only if $\det(M) \neq 0$.

Now observe the following matrix:

$$\begin{pmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 2F \end{pmatrix}. \tag{2}$$

Its determinant is $8 \times \det(M)$, so that we can use it to check the irreducibility of the conic $\mathcal{C}$. This is slightly better than matrix M, since we do not need to divide by 2 anywhere. Note also that the matrix (2) can be obtained as follows:

$$\begin{pmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 2F \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix},$$

where $f = Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2$. Thus, the matrix (2) is the Hessian matrix of this polynomial.

Let us consider one explicit example. Let $\mathcal{C}$ be the conic in $\mathbb{P}^2_{\mathbb{C}}$ that is given by the equation

$$(1+i)x^2 - 7xy + (3-2i)y^2 + (2+5i)xz + (-1+i)yz + (-11+13i)z^2 = 0.$$

Then $\mathcal{C}$ is irreducible conic, since

$$\det \begin{pmatrix} 2+2i & -7 & 2+5i \\ -7 & 6-4i & -1+i \\ 2+5i & -1+i & -22+26i \end{pmatrix} = 674 - 1000i \neq 0.$$

Exercise 2.2. Let $\mathcal{C}$ be the conic in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$x^2 + xy - 2y^2 + 3xz + 3yz + z^2 = 0.$$

Show that the conic $\mathcal{C}$ is *irreducible*. Verify that $\mathcal{C}$ contains the point $[-2:1:3]$.

2.1 Classification of Conics

Using projective transformations of the complex projective plane, we can obtain very simple classification of conics in $\mathbb{P}^2_{\mathbb{C}}$. Namely, for every conic $\mathcal{C}$ in $\mathbb{P}^2_{\mathbb{C}}$, there is a projective transformation $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that the conic $\phi(\mathcal{C})$ is given by one of the following equations:

1. $xy - z^2 = 0$ (irreducible conic);

2. $xy = 0$ (reducible conic that is a union of two distinct lines);

3. $x^2 = 0$ (a line taken with multiplicity two).

Indeed, let $\mathcal{C}$ be a conic in $\mathbb{P}^2_{\mathbb{C}}$. Then $\mathcal{C}$ is given by the equation

$$Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2 = 0,$$

where A, B, C, D, E and F are complex numbers such that $(A, B, C, D, E, F) \neq (0, 0, 0, 0, 0, 0)$. To find the projective transformation ϕ, we have to find an invertible change of coordinates

$$\begin{cases} x = b_{11}\mathbf{x} + b_{12}\mathbf{y} + b_{13}\mathbf{z}, \\ y = b_{21}\mathbf{x} + b_{22}\mathbf{y} + b_{23}\mathbf{z}, \\ z = b_{31}\mathbf{x} + b_{32}\mathbf{y} + b_{33}\mathbf{z}, \end{cases}$$

such that the equation $Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2 = 0$ becomes $\mathbf{xy} - \mathbf{z}^2 = 0$, or $\mathbf{xy} = 0$ or $\mathbf{x}^2 = 0$. Here b_{11}, b_{12}, b_{13}, b_{21}, b_{22}, b_{23}, b_{31}, b_{32} and b_{33} are some complex numbers. To find them is the same as to find the projective transformation ϕ. This can be done in the following six *algorithmic* steps:

Step 1. Pick a point P in $\mathcal{C}$. Let ϕ_1 be a projective transformation such that $\phi_1(P) = [0 : 0 : 1]$. There are plenty of points in $\mathcal{C}$ and there are a lot of choices for the map ϕ_1. Then the defining equation of $\phi_1(\mathcal{C})$ is given by

$$A_1x^2 + B_1xy + C_1y^2 + D_1xz + E_1yz = 0,$$

where A_1, B_1, C_1, D_1, E_1 are some complex numbers. This equation is simpler than the original one. Informally speaking, we just *killed* F.

Step 2. If $D_1 = E_1 = 0$, then $\phi_1(\mathcal{C})$ is a union of two lines that intersects at $[0 : 0 : 1]$. Indeed, in this case, we have

$$A_1x^2 + B_1xy + C_1y^2 = \big(\alpha_1x + \beta_1y\big)\big(\alpha_2x + \beta_2y\big)$$

for complex numbers α_1, β_1, α_2, β_2, so that $\phi_1(\mathcal{C})$ is a union of the lines $\alpha_1x + \beta_1y = 0$ and $\alpha_2x + \beta_2y = 0$. Vice versa, if $D_1 \neq 0$ or $E_1 \neq 0$, then the equation

$$D_1x + E_1y = 0$$

defines a line in $\mathbb{P}^2_{\mathbb{C}}$, which is called *the tangent line* to the conic $\phi_1(\mathcal{C})$ at the point $[0 : 0 : 1]$. The word *tangent* here can be explained as follows: in the subset U_z given by $z \neq 0$ the line $D_1\overline{x} + E_1\overline{y} = 0$ defines the tangent line to the conic

$$A_1\overline{x}^2 + B_1\overline{xy} + C_1\overline{y}^2 + D_1\overline{x} + E_1\overline{y} = 0,$$

where $\overline{x} = \frac{x}{z}$ and $\overline{y} = \frac{y}{z}$ are coordinates on $U_z = \mathbb{C}^2$. We will discuss this in Section 2.5.

Step 3. If $D_1 = E_1 = 0$ and $\phi_1(\mathcal{C})$ is a union of two distinct lines, we can find projective transformation ϕ_2 that maps these lines to the lines $x = 0$ and $y = 0$, so that $\phi_2 \circ \phi_1(\mathcal{C})$ is given by

$$xy = 0.$$

Similarly, if $\phi_1(\mathcal{C})$ is a double line, then there is a projective transformation ϕ_2 so that $\phi_2 \circ \phi_1(\mathcal{C})$ is given by $x^2 = 0$.

Step 4. We may assume that $D_1 \neq 0$ or $E_1 \neq 0$. Let ϕ_2 be any projective transformation that maps the line $D_1x + E_1y = 0$ to the line $x = 0$. This means that we change our projective coordinates as follows:

$$\begin{cases} \mathbf{x} = D_1x + E_1y, \\ \mathbf{y} = \alpha x + \beta y, \\ \mathbf{z} = z, \end{cases}$$

where α and β are any complex numbers such that $D_1\beta - \alpha E_1 \neq 0$. For example, if $D_1 \neq 0$, we let $\alpha = 0$ and $\beta = 1$. Similarly if $D_1 = 0$ and $E_1 \neq 0$, we let $\alpha = 1$ and $\beta = 0$. Then $\phi_2 \circ \phi_1(\mathcal{C})$ is the conic that is given by

$$A_2x^2 + B_2xy + C_2y^2 + D_2xz = 0,$$

where A_2, B_2, C_2 and D_2 are complex numbers such that $D_2 \neq 0$. This equation is much simpler than the original one: we *killed* E and F.

Step 5. Now, let us make the following coordinate change:

$$\begin{cases} \mathbf{x} = x, \\ \mathbf{y} = y, \\ \mathbf{z} = z + \lambda x + \delta y, \end{cases}$$

where λ and δ are complex numbers to be chosen later. Expressing x, y and z in terms of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{y}$, we get

$$\begin{cases} x = \mathbf{x}, \\ y = \mathbf{y}, \\ z = \mathbf{z} - \lambda\mathbf{x} - \delta\mathbf{y}. \end{cases}$$

Substituting them into the polynomial $A_2x^2 + B_2xy + C_2y^2 + D_2xz$, we obtain

$$\begin{aligned} A_2x^2 + B_2xy + C_2y^2 + D_2xz = \\ = A_2\mathbf{x}^2 + B_2\mathbf{xy} + C_2\mathbf{y}^2 + D_2\mathbf{x}\Big(\mathbf{z} - \lambda\mathbf{x} - \delta\mathbf{y}\Big) = \\ = A_2\mathbf{x}^2 + B_2\mathbf{xy} + C_2\mathbf{y}^2 + D_2\mathbf{xz} - D_2\lambda\mathbf{x}^2 - D_2\delta\mathbf{xy} = \\ = \big(A_2 - D_2\lambda\big)\mathbf{x}^2 + \big(B_2 - D_2\delta\big)\mathbf{xy} + C_2\mathbf{y}^2 + D_2\mathbf{xz}. \end{aligned}$$

Since $D_2 \neq 0$, we can put $\lambda = \frac{A_2}{D_2}$ and $\delta = \frac{B_2}{D_2}$. Then our new equation of $\phi_2 \circ \phi_1(\mathcal{C})$ simplifies as

$$C_2\mathbf{y}^2 + D_2\mathbf{xz} = 0.$$

This gives us the projective transformation ϕ_3 such that the composition $\phi_3 \circ \phi_2 \circ \phi_1$ maps our conic $\mathcal{C}$ into the conic that is given by the equation $C_2y^2 + D_2xz = 0$.

Step 6. If $C_2 = 0$, then $\phi_3 \circ \phi_2 \circ \phi_1(\mathcal{C})$ is given by $xz = 0$. In this case we are done by composing $\phi_3 \circ \phi_2 \circ \phi_1$ with a projective transformation $[x : y : z] \mapsto [x : z : y]$. If $C_2 \neq 0$, then changing coordinates as

$$\begin{cases} x = \mathbf{x}, \\ y = \dfrac{\mathbf{z}}{\sqrt{C_2}}, \\ z = -\dfrac{\mathbf{y}}{D_2}. \end{cases}$$

we obtain the equation

$$\boxed{\mathbf{xy} - \mathbf{z}^2 = 0.}$$

To illustrate the algorithm in this proof, let us consider one explicit example. Namely, let $\mathcal{C}$ be the conic in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$x^2 + y^2 - 2xy + xz - 3yz + 2z^2 = 0.$$

Let us show how to find the projective transformation $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that the conic $\phi(\mathcal{C})$ is given by the equation $4y^2 - xz = 0$. Observe first that $\mathcal{C}$ contains the point $[0:1:1]$. Let us introduce new projective coordinates x_1, x_2 and x_3 using the formula:

$$\begin{cases} x_1 = x, \\ y_1 = y - z, \\ z_1 = z. \end{cases}$$

Then the old coordinates x, y and z are expressed by

$$\begin{cases} x = x_1, \\ y = y_1 + z_1, \\ z = z_1. \end{cases}$$

Substituting this into $x^2 + y^2 - 2xy + xz - 3yz + 2z^2$, we see that $\mathcal{C}$ is given by the equation

$$x_1^2 + y_1^2 - 2x_1y_1 - x_1z_1 - y_1z_1 = 0.$$

Then the tangent line to $\mathcal{C}$ at the point $[0:1:1]$ is given by $x_1 + y_1 = 0$. Thus, we change the projective coordinates as follows:

$$\begin{cases} x_2 = x_1 + y_1, \\ y_2 = y_1, \\ z_2 = z_1. \end{cases}$$

Then $x_1 = x_2 - y_2$, $y_1 = y_2$, $z_1 = z_2$. Substituting this into $x_1^2 + y_1^2 - y_1z_1 - 2x_1y_1 - x_1z_1$, we see that $\mathcal{C}$ is given by $x_2^2 - 4x_2y_2 + 4y_2^2 - z_2x_2 = 0$. Finally, we let

$$\begin{cases} x_3 = x_2, \\ y_3 = y_2, \\ z_3 = z_2 - x_2 + 4y_2. \end{cases}$$

Then $x_2 = x_3$, $y_2 = y_3$, $z_2 = z_3 - x_3 + 4y_3$. Substituting this into $x_2^2 - 4x_2y_2 + 4y_2^2 - z_2x_2$, we see that $\mathcal{C}$ is given by the equation $4y_3^2 - x_3z_3 = 0$. Observe that

$$\begin{cases} x_3 = x_2 = x_1 + y_1 = x + y - z, \\ y_3 = y_2 = y_1 = y - z, \\ z_3 = z_2 - x_2 + 4y_2 = z_1 - (x_1 + y_1) + 4y_1 = z_1 - x_1 + 3y_1 = -x + 3y - 2z. \end{cases}$$

Substituting these expressions for x_3, y_3 and z_3 into to the polynomial $4y_3^2 - x_3z_3$, we indeed get $x^2 + y^2 - 2xy + xz - 3yz + 2z^2$. Let ϕ be the projective transformation of $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$\big[x : y : z\big] = \big[x + y - z : y - z : -x + 3y - 2z\big].$$

It corresponds to the linear transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ -1 & 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

We can compute the inverse of the 3×3 matrix in this expression:

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ -1 & 3 & -2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -3 & 1 \\ 1 & -4 & 1 \end{pmatrix}.$$

This gives the projective transformation $\psi \colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ that is given by

$$\psi\Big([x : y : z]\Big) = \big[x - y : x - 3y + z : x - 4y + z\big].$$

This is the inverse of the transformation ϕ. Of course, we can also recover ψ from our coordinate changes:

$$\begin{cases} x = x_1 = x_2 - y_2 = x_3 - y_3, \\ y = y_1 + z_1 = y_2 + z_2 = y_3 + (z_3 + x_3 - 4y_3) = x_3 - 3y_3 + z_3, \\ z = z_1 = z_2 = x_3 - 4y_3 + z_3. \end{cases}$$

One can double check that substituting the expression for x, y and z into our original polynomial $x^2 + y^2 - 2xy + xz - 3yz + 2z^2$, we obtain the polynomial $4y_3^2 - x_3 z_3$. This implies that the conic $\phi(\mathcal{C})$ is given by $4y^2 - xz = 0$. Indeed, we have

$$[x : y : z] \in \phi\big(\mathcal{C}\big) \iff \psi\big([x : y : z]\big) \in \mathcal{C},$$

since ψ and ϕ are inverses of each other. Thus, we have

$$[x : y : z] \in \phi\big(\mathcal{C}\big) \iff [x - y : x - 3y + z : x - 4y + z] \in \mathcal{C}.$$

Since the equation of the curve $\mathcal{C}$ is $x^2 + y^2 - 2xy + xz - 3yz + 2z^2 = 0$, the latter condition is equivalent to

$$\begin{aligned} (x-y)^2 + (x - 3y + z)^2 - 2(x-y)(x - 3y + z) + \\ + (x-y)(x - 4y + z) - 3(x - 3y + z)(x - 4y + z) + 2(x - 4y + z)^2 = 0. \end{aligned}$$

But we already know that the left hand side of this equation is the polynomial $4y^2 - xz$. Thus, we have

$$[x : y : z] \in \phi\big(\mathcal{C}\big) \iff 4y^2 - xz = 0,$$

so that the conic $\phi(\mathcal{C})$ is given by $4y^2 - xz = 0$.

Exercise 2.3. Let $\mathcal{C}$ be the conic in the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$4x^2 - 4xy + y^2 - 4xz - 13yz + 12z^2 = 0.$$

Then $\mathcal{C}$ contains $[0 : 1 : 1]$, $[-1 : 4 : 1]$, $[2 : 1 : 1]$, $[19 : 20 : 1]$, $[1 : 2 : 0]$. Show that $\mathcal{C}$ is irreducible, find a projective transformation $\phi \colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that $\phi(\mathcal{C})$ is given by

$$xz + y^2 = 0,$$

and compute $\phi([0 : 1 : 1])$, $\phi([-1 : 4 : 1])$, $\phi([2 : 1 : 1])$, $\phi([19 : 20 : 1])$, $\phi([1 : 2 : 0])$.

Let us consider another example when the conic is reducible. Namely, let $\mathcal{C}$ be the conic in $\mathbb{P}^2_{\mathbb{C}}$ that given by

$$5x^2 - 11xy + 14xz + 2y^2 - yz - 3z^2 = 0.$$

Now we can verify that

$$5x^2 - 11xy + 14xz + 2y^2 - yz - 3z^2 = \big(x - 2y + 3z\big)\big(5x - y - z\big).$$

This shows that $\mathcal{C}$ is a union of the line $x - 2y + 3z = 0$ and the line $5x - y - z = 0$. Let

$$\begin{cases} \mathbf{x} = x - 2y + 3z, \\ \mathbf{y} = 5x - y - z, \\ \mathbf{z} = z. \end{cases}$$

Using linear algebra, we can express x, y and z in terms of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$. To be precise, we get

$$\begin{cases} x = -\dfrac{\mathbf{x}}{9} + \dfrac{2\mathbf{y}}{9} + \dfrac{5\mathbf{z}}{9}, \\ y = -\dfrac{5\mathbf{x}}{9} + \dfrac{\mathbf{y}}{9} + \dfrac{16\mathbf{z}}{9}, \\ z = \mathbf{z}. \end{cases}$$

Substituting this into $5x^2 - 11xy + 14xz + 2y^2 - yz - 3z^2$, we see that

$$5x^2 - 11xy + 14xz + 2y^2 - yz - 3z^2 = \mathbf{x}\mathbf{y},$$

so that $\mathcal{C}$ is given by $\mathbf{x}\mathbf{y} = 0$. Let ϕ be the projective transformation

$$\big[x : y : z\big] \mapsto \big[x - 2y + 3z : 5x - y - z : z\big],$$

and let ψ be the projective transformation

$$\big[x : y : z\big] \mapsto \Big[-\frac{x}{9} + \frac{2y}{9} + \frac{5z}{9} : -\frac{5x}{9} + \frac{y}{9} + \frac{16z}{9} : z\Big] = \big[-x + 2y + 5z : -5x + y + 16z : 9z\big].$$

Then $\phi^{-1} = \psi$. This implies that the conic $\phi(\mathcal{C})$ is given by $xy = 0$, because the point $[x : y : z]$ is contained in the curve $\phi(\mathcal{C})$ if and only if the point $[-x+2y+5z : -5x+y+16z : 9z]$ is contained in the curve $\mathcal{C}$.

Exercise 2.4. Let $\mathcal{C}$ be the conic in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$-xy + 2y^2 - 3xz + 7yz + 3z^2 = 0.$$

Find a projective transformation $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that $\phi(\mathcal{C})$ is given by $xy = 0$.

Exercise 2.5. By observing the bijections

$$\{\text{homogeneous quadratic polynomials } Q : \mathbb{C}^2 \to \mathbb{C}\} \longleftrightarrow \{\text{symmetric bilinear forms on } \mathbb{C}^2\}$$

$$ax^2 + 2bxy + cy^2 + 2dxz + 2eyz + fz^2 \longleftrightarrow \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix},$$

use linear algebra methods to again deduce the classification of conics in $\mathbb{P}^2_{\mathbb{C}}$.

Exercise 2.6. Use the classification of conics in $\mathbb{P}^2_{\mathbb{C}}$ to give a second proof of Exercise 2.1.

Exercise 2.7. Let $\mathcal{C}_2 \subset \mathbb{P}^2_{\mathbb{C}}$ be a conic given by $xz = y^2$. Find all $\phi \in \mathrm{PGL}_3(\mathbb{C})$ such that $\phi(\mathcal{C}_2) = \mathcal{C}_2$. Show that these projective transformations form a group isomorphic to $\mathrm{PGL}_2(\mathbb{C})$.

2.2 How many points determine a conic?

For two points $[a_{11} : a_{12} : a_{13}] \neq [a_{21} : a_{22} : a_{23}]$ in the projective plane $\mathbb{P}^2_{\mathbb{C}}$, the equation

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ x & y & z \end{pmatrix} = 0$$

determines the unique line in $\mathbb{P}^2_{\mathbb{C}}$ containing them if $[a_{11} : a_{12} : a_{13}] \neq [a_{21} : a_{22} : a_{23}]$. This implies, in particular, that any three points $[a_{11} : a_{12} : a_{13}]$, $[a_{21} : a_{22} : a_{23}]$ and $[a_{31} : a_{32} : a_{33}]$ in $\mathbb{P}^2_{\mathbb{C}}$ are collinear (contained in one line) if and only if

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = 0.$$

Now let P_1, P_2, P_3, P_4 and P_5 be five distinct points in the plane $\mathbb{P}^2_{\mathbb{C}}$. If three points among P_1, P_2, P_3, P_4 and P_5 are contained in a line, then there are two lines in $\mathbb{P}^2_{\mathbb{C}}$ whose union contains five points P_1, P_2, P_3, P_4 and P_5. On the other hand, if no three points among P_1, P_2, P_3, P_4 and P_5 are collinear (contained in one line), then there exists unique conic in $\mathbb{P}^2_{\mathbb{C}}$ that contains these five points, and this conic is irreducible. This follows from

Exercise 2.8. Suppose that every line in $\mathbb{P}^2_{\mathbb{C}}$ contains at most 3 points among P_1, P_2, P_3, P_4, P_5. Show that there exists a unique conic $\mathcal{C}$ in $\mathbb{P}^2_{\mathbb{C}}$ that contains these five points. Moreover, also show that the conic $\mathcal{C}$ is irreducible if and only if every line in $\mathbb{P}^2_{\mathbb{C}}$ contains at most 2 points among P_1, P_2, P_3, P_4, P_5.

Exercise 2.9. Find a conic in $\mathbb{P}^2_{\mathbb{C}}$ that contains the points

$$[3 : 4 : 5], [5 : 12 : 13], [8 : 15 : 17], [7 : 24 : 25], [20 : 21 : 29].$$

Explain why this conic is unique.

How many conics in $\mathbb{P}^2_{\mathbb{C}}$ passes through four points? Infinitely many. In fact, we can say more. Fix four distinct points P_1, P_2, P_3 and P_4 in $\mathbb{P}^2_{\mathbb{C}}$. If three of them are contained in one line, say L, then every conic passing through P_1, P_2, P_3, P_4 must be a union of L and another line. Hence, this case is not very interesting. Because of this, we suppose that no three points among P_1, P_2, P_3 and P_4 are collinear. Then, by Exercise 1.9, there exists a projective transformation $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that

$$\begin{cases} \phi(P_1) = [1 : 0 : 0], \\ \phi(P_2) = [0 : 1 : 0], \\ \phi(P_2) = [0 : 0 : 1], \\ \phi(P_3) = [1 : 1 : 1]. \end{cases}$$

Thus, to describe all conics passing through P_1, P_2, P_3 and P_4, we may assume that $P_1 = [1 : 0 : 0]$, $P_2 = [0 : 1 : 0]$, $P_3 = [0 : 0 : 1]$ and $P_4 = [1 : 1 : 1]$. Let $\mathcal{C}$ be a conic in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2 = 0,$$

where A, B, C, D, E and F are complex numbers such that $(A, B, C, D, E, F) \neq (0, 0, 0, 0, 0, 0)$. If $\mathcal{C}$ contains all points P_1, P_2, P_3 and P_4, then $A = B = C = 0$ and $B + D + E = 0$, so that $\mathcal{C}$ is given by

$$Bxy + Dxz - (B + D)yz = 0.$$

Substituting various $(B, D) \neq (0, 0)$ in this equation, we obtain all conics passing through P_1, P_2, P_3 and P_4. For instance, if we let $(B, D) = (1, 1)$, we get the conic

$$x(y - z) = 0,$$

which is a union of the lines $x = 0$ and $y - z = 0$. Similarly, if we let $(B, D) = (1, 0)$, we get the conic

$$y(x - z) = 0,$$

which is a union of the lines $y = 0$ and $x - z = 0$. Likewise, if we let $(B, D) = (0, 1)$, we get the conic

$$z(x - y) = 0,$$

which is a union of the lines $z = 0$ and $x - y = 0$. These are all reducible conics that contains the points P_1, P_2, P_3 and P_4.

2.3 Intersections of Lines and Conics

Let L_z be the line in $\mathbb{P}^2_{\mathbb{C}}$ given by $z = 0$. As above, we denote by U_z its complement in $\mathbb{P}^2_{\mathbb{C}}$, so that U_z is given by $z \neq 0$. Then the intersection $\mathcal{C} \cap U_z$ is the subset given by

$$A\overline{x}^2 + B\overline{xy} + C\overline{y}^2 + D\overline{x} + E\overline{y} + F = 0,$$

where $\overline{x} = \frac{x}{z}$ and $\overline{y} = \frac{y}{z}$ are coordinates on U_z. If $(A, B, C) \neq (0, 0, 0)$, this equation defines a conic in $U_z = \mathbb{C}^2$, which we denote by $\overline{\mathcal{C}}$. If $A = B = C = 0$, then

$$Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2 = Dxz + Eyz + Fz^2 = z(Dx + Ey + Fz),$$

so that $\mathcal{C}$ is a union of the line L_z and the line L that is given by $Dx + Ey + Fz = 0$. A special case here is when $A = B = C = D = E = 0$, so that $L = L_z$, and $\mathcal{C}$ is given by $z^2 = 0$. Set-theoretically this is of course the line L_z. But we prefer to think of this conic as of the line L_z taken with multiplicity 2. Because of this, it is convenient to use the *additive* notation

$$L_z + L$$

instead of the set-theoretic union $L_z \cup L$. If $(A, B, C) \neq (0, 0, 0)$, then $\overline{\mathcal{C}}$ is a conic in $\mathbb{C}^2$. However, the projective conic $\mathcal{C}$ in $\mathbb{P}^2_{\mathbb{C}}$ always has more points than the conic $\overline{\mathcal{C}}$ in $\mathbb{C}^2$. To be precise, the intersection $\mathcal{C} \cap L_z$ consists of the points in $\mathbb{P}^2_{\mathbb{C}}$ given by

$$\begin{cases} Ax^2 + Bxy + Cy^2 = 0, \\ z = 0. \end{cases}$$

This system always has a solution $[x : y : 0]$, so that the intersection $\mathcal{C} \cap L_z$ is not empty. Indeed, if $A \neq 0$, then we can rewrite the above system as

$$\begin{cases} Ax^2 + Bx + C = 0, \\ y = 1, \\ z = 0, \end{cases}$$

which gives us the points

$$\Big[\frac{-B \pm \sqrt{B^2 - 4AC}}{2A} : 1 : 0\Big] = \Big[- B \pm \sqrt{B^2 - 4AC} : 2A : 0\Big].$$

Similarly, if $A = 0$ and $(C, B) \neq (0, 0)$, then the above system becomes

$$\begin{cases} y(Bx + Cy) = 0, \\ z = 0, \end{cases}$$

which gives us the points $[1 : 0 : 0]$ and $[-C : B : 0]$. Thus, if $(A, B, C) \neq (0, 0, 0)$, then the intersection $\mathcal{C} \cap L_z$ is not empty. Moreover, if $B^2 - 4AC \neq 0$, then $\mathcal{C} \cap L_z$ consists of exactly two points in $\mathbb{P}^2_{\mathbb{C}}$. If $B^2 - 4AC = 0$, then $\mathcal{C} \cap L_z = [-B : 2A : 0]$, which should be counted with multiplicity 2 for consistency.

Exercise 2.10. Let $\mathcal{C}$ be the conic in Exercise 2.2. Find its intersection with the line $z = 0$.

We see that if L_z is not an irreducible component of the conic $\mathcal{C}$, then the intersection $\mathcal{C} \cap L_z$ consists of two points (counted with multiplicities). The same result holds for every line in $\mathbb{P}^2_{\mathbb{C}}$ by:

Exercise 2.11. Let L be a line in $\mathbb{P}^2_{\mathbb{C}}$, and let $\mathcal{C}$ be an irreducible conic. Show that $\mathcal{C} \cap L$ consists of two points counted with multiplicities. Note that this implies that the intersection $\mathcal{C} \cap L$ is not empty.

To find the intersection points of a line and a conic in $\mathbb{P}^2_{\mathbb{C}}$, the only thing you need to know is how to solve quadratic equation in one variable. For example, let L be a line in $\mathbb{P}^2_{\mathbb{C}}$ that is given by $2x + 7y - 5z = 0$, and let $\mathcal{C}$ be a conic in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$2x^2 - 3xy + 7y^2 - 5xz + 11yz - 8z^2 = 0.$$

Note that the intersection $L_z \cap L \cap \mathcal{C}$ is empty, since the system

$$\begin{cases} 2x^2 - 3xy + 7y^2 - 5xz + 11yz - 8z^2 = 0, \\ 2x + 7y - 5z = 0, \\ z = 0, \end{cases}$$

does not have solutions in $\mathbb{P}^2_{\mathbb{C}}$. Thus, the intersection $L \cap \mathcal{C}$ is contained in the subset in $\mathbb{P}^2_{\mathbb{C}}$ that is given by $z \neq 0$. Hence, to find $L \cap \mathcal{C}$, we have to solve the following system:

$$\begin{cases} 2x^2 - 3xy + 7y^2 - 5xz + 11yz - 8z^2 = 0, \\ 2x + 7y - 5z = 0, \\ z = 1. \end{cases}$$

Using substitution and the quadratic equation, we immediately find that the intersection $L \cap \mathcal{C}$ consists of two points

$$\left[\frac{23 \pm \sqrt{385}}{12} : \frac{7 \mp \sqrt{385}}{42} : 1\right] = \left[161 \pm 7\sqrt{385} : 14 \mp 2\sqrt{385} : 84\right].$$

Exercise 2.12. Let $\mathcal{C}$ be the conic in Exercise 2.2. Find the intersection $L \cap \mathcal{C}$, where L is the line in $\mathbb{P}^2_{\mathbb{C}}$ that contains the points $[-2 : 1 : 3]$ and $[1 : 2 : 3]$.

2.4 Intersections of Two Conics

Let $\mathcal{C}$ and $\mathcal{C}'$ be two distinct conics in $\mathbb{P}^2_{\mathbb{C}}$. The conics $\mathcal{C}$ is given by

$$Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2 = 0$$

for some complex numbers A, B, C, D, E, F such that $(A, B, C, D, E, F) \neq (0, 0, 0, 0, 0, 0)$. Similarly, the conic $\mathcal{C}'$ is given by

$$A'x^2 + B'xy + C'y^2 + D'xz + E'yz + F'z^2 = 0,$$

where A', B', C', D', E', F' are complex numbers such that $(A', B', C', D', E', F') \neq (0, 0, 0, 0, 0, 0)$. To find the intersection $\mathcal{C} \cap \mathcal{C}'$, we have to solve the system of equations

$$\begin{cases} Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2 = 0, \\ A'x^2 + B'xy + C'y^2 + D'xz + E'yz + F'z^2 = 0. \end{cases} \tag{3}$$

Let us consider one example. Let $\mathcal{C}$ be the conic in $\mathbb{P}^2_{\mathbb{C}}$ that contains the points $P_1 = [0 : 3 : 1]$, $P_2 = [1 : 2 : 3]$, $P_3 = [-1 : 5 : 2]$, $P_4 = [2 : -5 : 2]$ and $P_5 = [1 : 2 : 3]$. Then $\mathcal{C}$ is given by

$$2717x^2 + 4411xy - 10186xz + 776y^2 - 3751yz + 4269z^2 = 0.$$

Let $\mathcal{C}'$ be the conic that contains P_1, P_2, P_3, $P_6 = [-11 : 7 : 5]$ and $P_7 = [8 : 13 : -3]$. Then it is given by

$$80586x^2 - 265815xy + 458579xz + 110301y^2 - 360404yz + 88503z^2 = 0.$$

Their intersection $\mathcal{C} \cap \mathcal{C}'$ consists of the points P_1, P_2, P_3 and the point

$$\left(\frac{169665714547021}{138235560925089}, \frac{203859180246131}{138235560925089}\right).$$

Here we can use the method of *resultant* to obtain the last point. Discussion on resultant can be found in Section A.1 of the Appendix of this document.

Exercise 2.13. Let $P_1 = [0 : 0 : 1]$, $P_2 = [1 : 1 : 1]$, $P_3 = [2 : 4 : 1]$, $P_4 = [3 : 9 : 1]$, $P_5 = [4 : 16 : 1]$, $P_6 = [2 : -5 : 2]$ and $P_7 = [1 : 2 : 3]$.

(a) Show that no six of these points are contained in one conic.

(b) Show that for every five of these points, there is unique conic that contains them.

(c) For every five of these points, find conic containing them and show that it is irreducible.

(d) For every two conics found in (c), find their intersection.

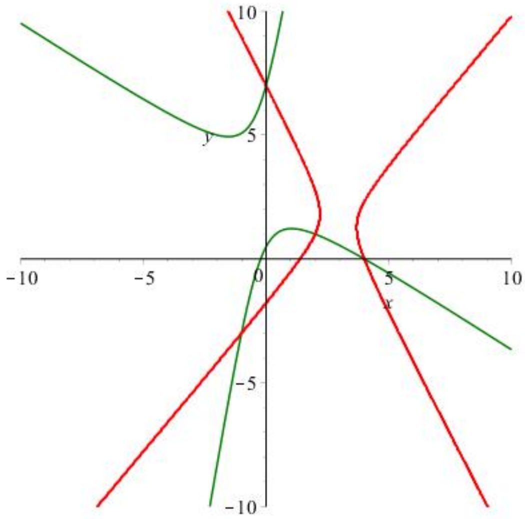
If one of the conics $\mathcal{C}$ and $\mathcal{C}'$ is reducible, then we know how to do solve (3) without resultant thanks to Exercise 2.11. If both conics $\mathcal{C}$ and $\mathcal{C}'$ are irreducible, then the intersection $\mathcal{C} \cap \mathcal{C}'$ cannot contain more than 4 points by Exercise 2.8. In fact, we can say more:

Exercise 2.14. Suppose that both conics $\mathcal{C}$ and $\mathcal{C}'$ are irreducible. Show that the intersection $\mathcal{C} \cap \mathcal{C}'$ is not empty, and it consists of at most 4 points.

By Exercise 2.14, the intersection $\mathcal{C} \cap \mathcal{C}'$ can consist of one, two, three or four points. All of these four cases are indeed possible. In fact, we know from the solution to Exercise 2.13 that the intersection of two irreducible conics in $\mathbb{P}_{\mathbb{C}}$ *usually* consists of exactly four points. Indeed, suppose that $\mathcal{C}$ is given by $f_2(x, y, z) = 0$, where

$$f_2(x, y, z) = 511x^2 + 709xy - 131y^2 - 1932xz + 981yz - 448z^2.$$

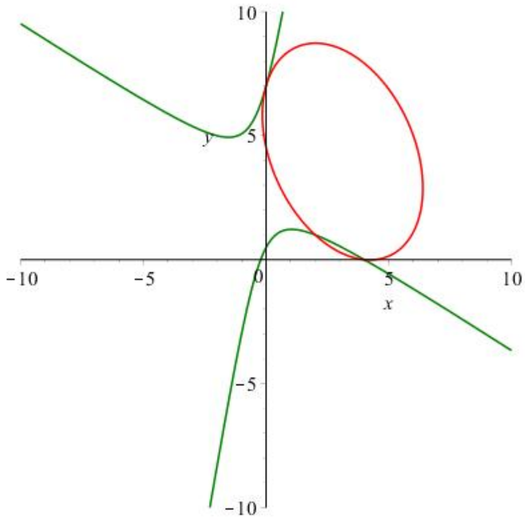
If the conic $\mathcal{C}'$ is given by $1217x^2 - 394xy - 541y^2 - 6555xz + 2823yz + 6748z^2 = 0$, their intersect by the points $[4 : 0 : 1]$, $[1 : 3 : -1]$, $[0 : 7 : 1]$, $[2 : 1 : 1]$. We can plot the real part of this picture:



On the other hand, if the second conic $\mathcal{C}'$ is given by

$$42049x^2 + 21271xy + 23536y^2 - 355005xz - 271500yz + 747236z^2 = 0,$$

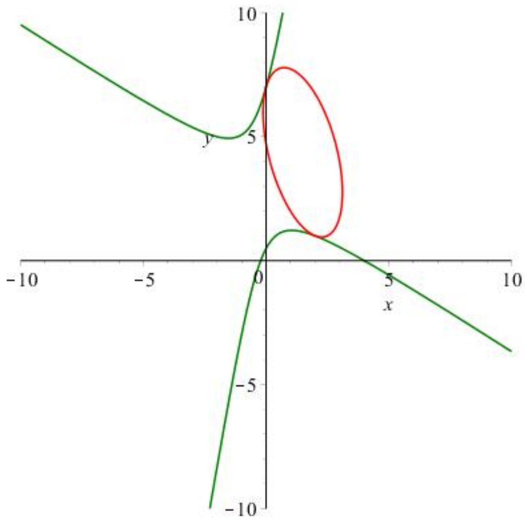
then their intersection $\mathcal{C} \cap \mathcal{C}'$ consists of just three points $[4 : 0 : 1]$, $[0 : 7 : 1]$, $[2 : 1 : 1]$. Plotting the real part of the plane $\mathbb{P}^2_{\mathbb{C}}$ away from the line $z = 0$, we obtain the following picture:



We see that these conics are tangent at the point $[0:7:1]$, so that we should count it with appropriate multiplicity. Likewise, if the second conic $\mathcal{C}'$ is given by the equation

$$(3031x - 853y + 5971z)(821x - 3779y + 2137z) - 9700f_2(x, y, z) = 0,$$

then both conics are irreducible and intersect by two points $[0:7:1]$ and $[2:1:1]$. Plotting their real finite part, we see that these conics tangent at both of these points:



This is easy to explain algebraically. In this case, the intersection $\mathcal{C} \cap \mathcal{C}'$ is given by the following system of polynomial equations:

$$\begin{cases} f_2(x, y, z) = 0, \\ (3031x - 853y + 5971z)(821x - 3779y + 2137z) - 9700f_2(x, y, z) = 0. \end{cases}$$

This system has the same solutions in $\mathbb{P}^2_{\mathbb{C}}$ as

$$\begin{cases} f_2(x, y, z) = 0, \\ (3031x - 853y + 5971z)(821x - 3779y + 2137z) = 0. \end{cases}$$

To solve this system, we can solve separately the system

$$\begin{cases} f_2(x, y, z) = 0, \\ 3031x - 853y + 5971z = 0, \end{cases}$$

and then solve the system

$$\begin{cases} f_2(x, y, z) = 0, \\ 821x - 3779y + 2137z = 0. \end{cases}$$

Geometrically, this means that we are looking for the intersection of the first conic $\mathcal{C}$ with the lines

$$3031x - 853y + 5971z = 0$$

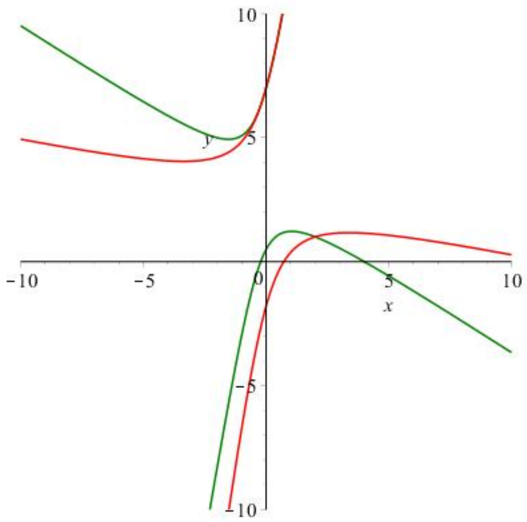
and

$$821x - 3779y + 2137z = 0.$$

These are tangent lines to the conic $\mathcal{C}$ at the points $[0:7:1]$ and $[2:1:1]$, respectively. We already know that we should count these points with multiplicity 2 in both these intersections. If the second conic $\mathcal{C}'$ is given by

$$(3031x - 853y + 5971z)(6x + 2y - 14z) - 50f_2(x, y, z) = 0,$$

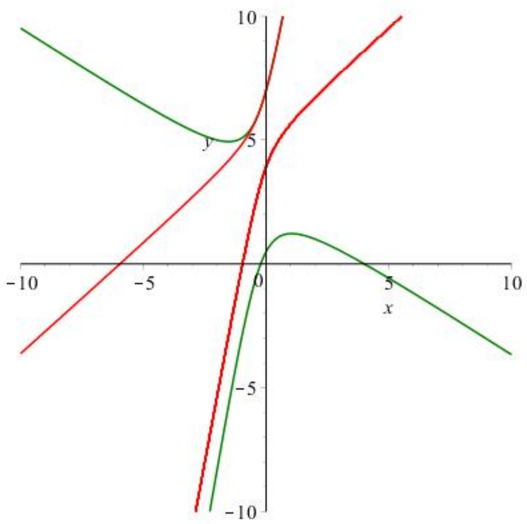
we get the following picture:



We see that these conics tangent at the point $[0:7:1]$, and they intersect *transversally* at the point $[2:1:1]$. We already know that we have to count $[0:7:1]$ in the intersection $\mathcal{C} \cap \mathcal{C}'$ with an appropriate multiplicity. What is this multiplicity? As in the previous case, algebra helps us to answer this question: the line $3031x - 853y + 5971z = 0$ is tangent to $\mathcal{C}$ at the point $[0:7:1]$, and the line $6x + 2y - 14z = 0$ intersects the conic $\mathcal{C}$ by $[0:7:1]$ and $[2:1:1]$ transversally. Thus, we should count $[0:7:1]$ in the intersection $\mathcal{C} \cap \mathcal{C}'$ with multiplicity 3. Finally, if the second conic $\mathcal{C}'$ is given by

$$(3031x - 853y + 5971z)^2 - 5000f_2(x, y, z) = 0,$$

then it is irreducible and the intersection $\mathcal{C} \cap \mathcal{C}'$ consists of the point $[0:7:1]$ and we get the following picture:



We already know that we should consider it with multiplicity 4.

Two distinct irreducible conics in $\mathbb{P}^2_{\mathbb{C}}$ always intersect by 4 points (if we count them with appropriate multiplicities), which is a special case of Bezout's theorem (see Theorem A.2.1 in Section A.2 of the Appendix). Moreover, the solution of Exercise 2.14 gives a algorithm how to find these 4 points. Unfortunately, to do this we have to solve a polynomial equation of degree 4 in one variable, which is usually not so easy to do. However, if we know 3 roots of this polynomial, then we can always find the fourth root by using Vieta's formulas (see [14]).

2.5 Smooth and Singular Curves

Let L be a line in $\mathbb{P}^2_{\mathbb{C}}$, and let $\mathcal{C}_2$ be an irreducible conic. Then L and $\mathcal{C}_2$ have a common point. Moreover, the intersection $L \cap \mathcal{C}_2$ often consists of two points. However, it may happen $L \cap \mathcal{C}_2$ consists of a single point. Then we say that the line L is *tangent* to the conic $\mathcal{C}_2$ at this point, so that we have to count it twice (with multiplicity 2).

Exercise 2.15. Let L be a line in $\mathbb{P}^2_{\mathbb{C}}$ given by $x + 2y + z = 0$. Find all irreducible conics in $\mathbb{P}^2_{\mathbb{C}}$ that contains the points $[1 : 0 : 2]$, $[3 : 1 : 2]$, $[1 : 2 : 1]$, $[1 : 1 : 1]$, and intersect L by one point.

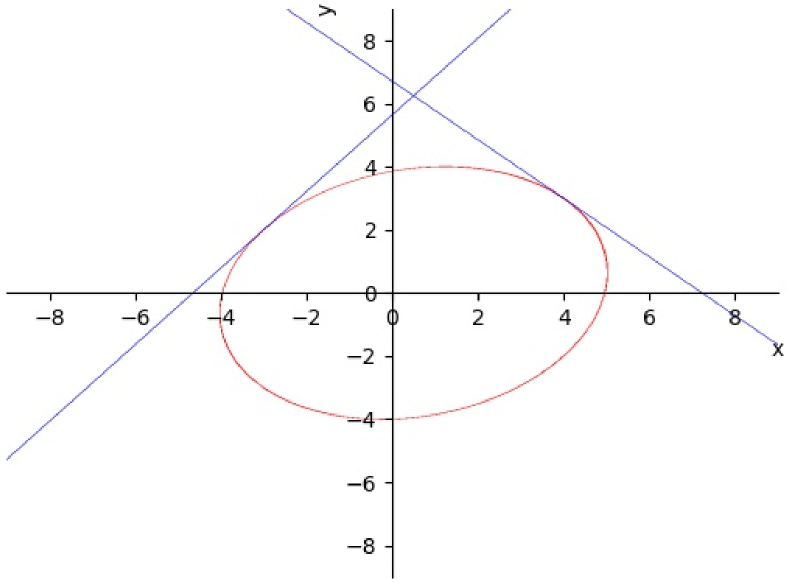
Actually, we did not yet define the tangency in the projective plane. This is easy to do. The tangent line to $\mathcal{C}_2$ at every point $[a : b : c] \in \mathcal{C}_2$ is given by

$$\frac{\partial f(x,y,z)}{\partial x}(a,b,c)x + \frac{\partial f(x,y,z)}{\partial y}(a,b,c)y + \frac{\partial f(x,y,z)}{\partial z}(a,b,c)z = 0. \tag{4}$$

Let us consider one example to illustrate how to use this formula in applications. Suppose now that the conic $\mathcal{C}_2$ is given by the following equation:

$$11x^2 - 4xy + 14y^2 - 11xz + 2yz - 216z^2 = 0.$$

Let P be the point $[2 : 25 : 4]$. Then P is not contained in the conic $\mathcal{C}_2$. We claim that there are two lines in $\mathbb{P}^2_{\mathbb{C}}$ containing P that are tangent to $\mathcal{C}_2$. In the real part of the projective plane given by $z \neq 1$, we can see our conic $\mathcal{C}_2$ together with these two lines on the following picture:



Now let us find these tangent lines algebraically. Let L be a line in $\mathbb{P}^2_{\mathbb{C}}$ such that L contains P, and L is tangent to $\mathcal{C}_2$. Then L is given by

$$Ax + By + Cz = 0$$

where A, B and C are some complex numbers. Since L contains P, we see that $2A + 25B + 4C = 0$. Let O be a point in $\mathcal{C}_2$ such that L is tangent to $\mathcal{C}_2$ at this point. Then $O = [a : b : c]$ for some numbers a, b and c. Taking partial derivatives and using (4), we get

$$\begin{cases} A = 22a - 4b - 11c, \\ B = -4a + 28b + 2c, \\ C = -11a + 2b - 432c. \end{cases}$$

Now, substituting these expressions for A, B and C into the equation $2A + 25B + 4C = 0$, we get the equation $a - 7b + 17c = 0$. Thus, since $\mathcal{C}_2$ contains O, we have

$$\begin{cases} 11a^2 - 4ab + 14b^2 - 11ac + 2bc - 216c^2 = 0, \\ a - 7b + 17c = 0. \end{cases}$$

This looks like an intersection of a line $x - 7y + 17z = 0$ and our conic $\mathcal{C}_2$. From Section 2.3, we know how to find this intersection. This gives us two points in the conic $\mathcal{C}_2$. One of them is $[-3 : 2 : 1]$, and another one is the point $[4 : 3 : 1]$. Now, using (4) again, we see that the tangent line to $\mathcal{C}_2$ at the point $[-3 : 2 : 1]$ is given by

$$-17x + 14y - 79z = 0.$$

Similarly, the tangent line to the conic $\mathcal{C}_2$ at the point $[4 : 3 : 1]$ is given by $13x + 14y - 94z = 0$. These are the lines we need! One can check that these two lines indeed intersect at the point P.

Remark 2.16. We see that there exists two lines in $\mathbb{P}^2_{\mathbb{C}}$ containing point $P = [2 : 25 : 4]$ that are tangent to the irreducible conic $11x^2 - 4xy + 14y^2 - 11xz + 2yz - 216z^2 = 0$ in two different points. These points are $[-3 : 2 : 1]$ or $[4 : 3 : 1]$. We know from Section 1.1 that there exists unique line passing through these two points. This line is given by

$$x - 7y + 17z = 0.$$

Denote it by L_P. We can repeat our construction in the case when P is any point in $\mathbb{P}^2_{\mathbb{C}}$ that is not contained in our conic. This gives us a map

$$P \mapsto L_P$$

that takes point P in $\mathbb{P}^2_{\mathbb{C}}$ and maps it to the line L_P that intersects our conic by two points. Moreover, we can define this map also for the points contained in the conic by taking L_P to be the tangent line to the conic at the point P. The resulting map is often called *polarity* with respect to our conic. Of course, we can define a similar map for every smooth conic in $\mathbb{P}^2_{\mathbb{C}}$.

Exercise 2.17. Find all lines in $\mathbb{P}^2_{\mathbb{C}}$ that are tangent to the conic

$$888x^2 - 131xy - 1396xz - 416y^2 - 907yz + 96z^2 = 0$$

and contains the point $[1723 : 1268 : 413]$.

Of course, the formula (4) works for all curves in $\mathbb{P}^2_{\mathbb{C}}$. It allows us to find tangent lines to curves of any degree. However, there is a hidden problem: what if all partial derivatives are zero at some point? Namely, if there is a point $[x : y : z] \in \mathcal{C}_2$ such that

$$\begin{cases} \dfrac{\partial f(x,y,z)}{\partial x} = 0, \\ \dfrac{\partial f(x,y,z)}{\partial y} = 0, \\ \dfrac{\partial f(x,y,z)}{\partial z} = 0. \end{cases}$$

In our example, this cannot happen, because the system of equations

$$\begin{cases} 26x - 10y - 44z = 0 \\ -10x + 4y + 16z = 0 \\ -44x + 16y + 30z = 0 \end{cases}$$

has only trivial solution $x = y = z = 0$, since

$$\det \begin{pmatrix} 26 & -10 & -44 \\ -10 & -4 & 16 \\ -44 & 16 & 30 \end{pmatrix} = -200 \neq 0.$$

This means that for every point $[a : b : c] \in \mathcal{C}_2$ there is a line in $\mathbb{P}^2_{\mathbb{C}}$ that is tangent to this conic at this point. Note that this also implies that $\mathcal{C}_2$ is irreducible. Indeed, if $f(x,y,z) = h(x,y,z)g(x,y,z)$ for some linear polynomials $h(x,y,z)$ and $g(x,y,z)$, then there is a point $P \in \mathbb{P}^2_{\mathbb{C}}$ such that $h(P) = 0$ and $g(P) = 0$, because two lines in $\mathbb{P}^2_{\mathbb{C}}$ always has non-empty intersection. Since

$$\frac{\partial f(z,y,x)}{\partial x} = \frac{\partial g(x,x,y)}{\partial x} h(x,y,z) + \frac{\partial h(x,y,z)}{\partial x} g(x,y,z) = 0,$$

we see that $\frac{\partial f(z,y,x)}{\partial x}$ vanishes at P. Similarly, we see that both $\frac{\partial f(z,y,x)}{\partial y}$ and $\frac{\partial f(z,y,x)}{\partial z}$ vanish at P. But we already proved that this could not happen. This shows that the conic $\mathcal{C}_2$ is irreducible. Of course, the irreducibility also follows from the fact that $\mathcal{C}_2$ is given by

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 26 & -10 & -44 \\ -10 & -4 & 16 \\ -44 & 16 & 30 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0,$$

and we just computed the determinant of this 3×3 matrix (cf. the solution of Exercise 2.1).

Exercise 2.18. Let $\mathcal{C}_2$ be the conic in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$x^2 + xy - 2y^2 + 3xz + 3yz + z^2 = 0.$$

Then $\mathcal{C}_2$ contains $[-2 : 1 : 3]$. Do the following:

(a) Verify that $\mathcal{C}_2$ does not have singular points.

(b) Find the equation of the line that is tangent to $\mathcal{C}_2$ at the point $[-2 : 1 : 3]$.

(c) Find the intersection of the conic $\mathcal{C}_2$ with this line.

(d) Find a projective transformation $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that $\phi(\mathcal{C})$ is given by $xz + y^2 = 0$.

Now we are ready to give a definition in general. Let $\mathcal{C}_d$ be a curve in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$f_d(x, y, z) = 0,$$

where $f_d(x, y, z)$ is a homogeneous polynomial of degree $d \geqslant 1$. Let $[a : b : c]$ be a point in $\mathcal{C}_d$.

Definition 2.19. We say that $[a : b : c]$ is a *singular point* of the curve $\mathcal{C}_d$ if $[a : b : c]$ is a solution of the following system of equations:

$$\begin{cases} \dfrac{\partial f_d(x, y, z)}{\partial x} = 0, \\ \dfrac{\partial f_d(x, y, z)}{\partial y} = 0, \\ \dfrac{\partial f_d(x, y, z)}{\partial z} = 0. \end{cases} \tag{5}$$

Otherwise, we say that $[a : b : c]$ is a *smooth point* or a *non-singular point* of the curve $\mathcal{C}_d$ or that the curve $\mathcal{C}_d$ is *smooth* or *non-singular* at the point $[a : b : c]$.

If $[a : b : c]$ is a smooth point of the curve $\mathcal{C}_d$, then the equation

$$\frac{\partial f_d(x, y, z)}{\partial x}(a, b, c)x + \frac{\partial f_d(x, y, z)}{\partial y}(a, b, c)y + \frac{\partial f_d(x, y, z)}{\partial z}(a, b, c)z = 0$$

defines a line in $\mathbb{P}^2_{\mathbb{C}}$. This is a *tangent* line to the curve $\mathcal{C}_d$ at the point $[a : b : c]$. If (5) does not have solutions in $\mathbb{P}^2_{\mathbb{C}}$, then all points of the curve $\mathcal{C}_d$ are smooth. In this case, we say that $\mathcal{C}_d$ is smooth. One can show that

$$df_d(x, y, z) = x\frac{\partial f_d(x, y, z)}{\partial x} + y\frac{\partial f_d(x, y, z)}{\partial y} + z\frac{\partial f_d(x, y, z)}{\partial z}.$$

This is known as Euler's formula. Thus, if all partial derivatives $\frac{\partial f_d(x,y,z)}{\partial x}$, $\frac{\partial f_d(x,y,z)}{\partial y}$ and $\frac{\partial f_d(x,y,z)}{\partial z}$ vanish at some point in $\mathbb{P}^2_{\mathbb{C}}$, then this point is contained in the curve $\mathcal{C}_d$.

Exercise 2.20. Find a smooth conic $\mathcal{C}_2$ in $\mathbb{P}^2_{\mathbb{C}}$ such that the following conditions are satisfies

- the conic $\mathcal{C}_2$ contains the points $[0:0:1]$, $[0:1:0]$, $[1:0:0]$;
- the line $y-z=0$ is tangent to $\mathcal{C}_2$ at the point $[1:0:0]$,
- the line $y+2x=0$ is tangent to $\mathcal{C}_2$ at the point $[0:0:1]$.

Arguing as in the solution of Exercise 2.1, we obtain

Corollary 2.21. A conic in $\mathbb{P}^2_{\mathbb{C}}$ is irreducible if and only if it is smooth.

Similarly, we will prove later that smooth cubic curves are irreducible (see Lemma A.2.2 below). However, there are irreducible singular cubic curves.

Exercise 2.22. Let $\mathcal{C}_3$ be the cubic curve in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$xyz + x^3 + y^3 = 0.$$

Find all singular points of the curve $\mathcal{C}_3$, and prove that the curve $\mathcal{C}_3$ is irreducible.

Singular points of plane curves can be very different in nature. To describe this poetically, Iain Gordon and Maurizio Martino rephrased Leo Tolstoy's famous quote as

Smooth points are all alike, every singular point is singular in its own way.

There are many ways how to measure the *singularity* of the plane curve $\mathcal{C}_d$ at a given point P. The simplest and the most naive measure is given by the multiplicity of the curve $\mathcal{C}_d$ at the point P, which is usually denoted as $\mathrm{mult}_P(\mathcal{C}_d)$. It can be defined as follows. First, just for simplicity, we apply an appropriate projective transformation of the plane $\mathbb{P}^2_{\mathbb{C}}$ to simplify the coordinates of the point P. This allows us to assume that $P=[0:0:1]$. Then the curve $\mathcal{C}_d$ is given by the equation

$$z^d h_0\big(x,y\big) + z^{d-1} h_1\big(x,y\big) + z^{d-2} h_2\big(x,y\big) + \cdots + h_d\big(x,y\big) = 0,$$

where each $h_i(x,y)$ is a homogenous polynomial in x and y of degree i. Thus, in the subset $U_z = \mathbb{C}^2$ with coordinates $\overline{x} = \frac{x}{z}$ and $\overline{y} = \frac{y}{z}$, the curve $\mathcal{C}_d \cap U_z$ is given by the equation

$$h_0\big(\overline{x},\overline{y}\big) + h_1\big(\overline{x},\overline{y}\big) + h_2\big(\overline{x},\overline{y}\big) + \cdots + h_d\big(\overline{x},\overline{y}\big) = 0.$$

The left hand side of this equality is just the Taylor expansion of the defining polynomial of the curve $\mathcal{C}_d \cap U_z$ in $U_z = \mathbb{C}^2$. Then we define the multiplicity of the curve $\mathcal{C}_d$ at the point P as

$$\mathrm{mult}_P\big(\mathcal{C}_d\big) = \min\Big\{ i \mid h_i(x,y) \text{ is not a zero polynomial} \Big\},$$

and we say that $\mathcal{C}_d$ has multiplicity $\mathrm{mult}_P(C)$ at the point P. Thus, in particular, we have

$$\mathrm{mult}_P\big(\mathcal{C}_d\big) \geqslant 1 \iff P \in \mathcal{C}_d.$$

Likewise, our definition implies that

$$\mathrm{mult}_P\big(\mathcal{C}_d\big) \geqslant 2 \iff P \in P \in \mathrm{Sing}\big(\mathcal{C}_d\big).$$

Finally, we observe that $\mathrm{mult}_P(\mathcal{C}_d) \leqslant d$. Moreover, if $\mathrm{mult}_P(\mathcal{C}_d) = d$, then the curve $\mathcal{C}_d$ is given by

$$h_d\big(x,y\big) = 0,$$

which means that $\mathcal{C}_d$ is a union of d lines passing through the point P. Why?

3 Cubic Curves

As described in Section 2, a cubic curve in $\mathbb{P}^2_{\mathbb{C}}$ is simply the solutions $[x : y : z] \in \mathbb{P}^2_{\mathbb{C}}$ of the equation

$$f_3(x, y, z) = 0$$

where $f_3(x, y, z)$ is a homogeneous polynomial of degree 3.

3.1 Inflection Points

Let $\mathcal{C}_3$ be an irreducible cubic curve in the projective plane, and let L be a line in $\mathbb{P}^2_{\mathbb{C}}$. By Theorem A.2.1, we know that the intersection $L \cap \mathcal{C}_3$ is not empty. Moreover, this intersection consists of three points if we count them with appropriate multiplicities.

Let P be a point in the intersection $L \cap \mathcal{C}_3$. In Section 2.5, we explained briefly how to determine the multiplicity of the point P in this intersection. Recall that the answer is not dependent on projective change of coordinates. Thus, we may assume that L is given by $x = 0$ and $P = [0 : 0 : 1]$. Then the cubic curve $\mathcal{C}_3$ is given by $f_3(x, y, z) = 0$, where

$$f_3(x, y, z) = A_1x^3 + A_2x^2y + A_3xy^2 + A_4y^3 + A_5x^2z + A_6xyz + A_7y^2z + A_8xz^2 + A_9yz^2$$

for some numbers A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8 and A_9 such that at least one of them is not zero. Then the intersection $L \cap \mathcal{C}_3$ is given by

$$\begin{cases} A_4y^3 + A_7y^2z + A_9yz^2 = 0, \\ x = 0, \end{cases}$$

where at least one number among A_4, A_7 and A_9 is not zero. We immediately get one solution:

- $x = 0$ and $y = 0$, which corresponds to the point $P = [0 : 0 : 1]$.

To find the remaining solutions, we factorize the quadratic polynomial $A_4y^2 + A_7yz + A_9z^2$ to get

$$A_4y^3 + A_7y^2z + A_9yz^2 = y\Big(\alpha_1 y - \beta_1 z\Big)\Big(\alpha_2 y - \beta_2 z\Big)$$

for some complex numbers α_1, β_1, α_2, β_2. This gives us the remaining two intersection points:

- the point $[0 : \beta_1 : \alpha_1]$,
- the point $[0 : \beta_2 : \alpha_2]$.

Observe that $\beta_1 = 0 \iff [0 : \beta_1 : \alpha_1] = P$. Similarly, we see that $\beta_2 = 0 \iff [0 : \beta_2 : \alpha_2] = P$. Thus, the multiplicity of the point P in $L \cap \mathcal{C}_3$ is the multiplicity of the root $y = 0$ in the equation

$$f_3(0, y, z) = 0.$$

If we have $\beta_1 \neq 0$ and $\beta_2 \neq 0$, then we should count P in the intersection $L \cap \mathcal{C}_3$ with multiplicity 1. In this case, the following two conditions hold:

1. the curve $\mathcal{C}_3$ is smooth at the point P,
2. the line L is not the tangent line to $\mathcal{C}_3$ at the point P.

Thus, we say that the line L intersects the curve $\mathcal{C}_3$ *transversally* at the point P.

If either $\beta_1 = 0$ or $\beta_2 = 0$ (but not both), then we should count P in the intersection $L \cap \mathcal{C}_3$ with multiplicity 2. Finally, if $\beta_1 = 0$ and $\beta_2 = 0$, then

$$f_3(0, y, z) = A_4 y^3$$

so that we should count P in the intersection $L \cap \mathcal{C}_3$ with multiplicity 3. In these two cases, one of the following two possibilities holds:

- either the curve $\mathcal{C}_3$ is singular at the point P,
- or $\mathcal{C}_3$ is smooth at the point P and L is the tangent line to $\mathcal{C}_3$ at this point.

In particular, if the intersection $L \cap \mathcal{C}_3$ consists of three distinct points, then the curve $\mathcal{C}_3$ must be smooth at these points, and L does not tangent $\mathcal{C}_3$ at any of these points.

Exercise 3.1. Let $\mathcal{C}_3$ be the cubic curve in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$\Gamma(x, y, z) = 6x^3 - 10x^2y - 3xy^2 + 13y^3 - x^2z + 4xz^2 + y^2z - 5yz^2 - 9z^3 = 0,$$

and let L be the line in $\mathbb{P}^2_{\mathbb{C}}$ that is given by $x + y - z = 0$. Find the intersection $L \cap \mathcal{C}_3$, and show that L is not tangent to $\mathcal{C}_3$ at any point of the intersection $L \cap \mathcal{C}_3$.

Let O be a smooth point of the curve $\mathcal{C}_3$, and let L_O be the line in $\mathbb{P}^2_{\mathbb{C}}$ that is tangent to $\mathcal{C}_3$ at this point. Then

- the intersection $L \cap \mathcal{C}_3$ contains less than 3 points,
- the point O should be counted in $\mathcal{C}_3 \cap L_O$ with multiplicity at least 2.

If the intersection $\mathcal{C}_3 \cap L_O$ contains another point, then the multiplicity of the point O in this intersection is 2. However, if we have

$$\mathcal{C}_3 \cap L_O = O,$$

then we should count O in the intersection $\mathcal{C}_3 \cap L_O$ with multiplicity 3. This is very special case. If this happens, we say that O is an *inflection* point of the curve $\mathcal{C}_3$.

Exercise 3.2. Every smooth cubic curve in $\mathbb{P}^2_{\mathbb{C}}$ has exactly 9 distinct inflection points.

Let us illustrate this result by an example. Let $\mathcal{C}_3$ be a cubic curve given by $f_3(x, y, z) = 0$, where

$$f_3(x, y, z) = x^3 + y^3 + z^3 + \lambda xyz \tag{6}$$

for some $\lambda \in \mathbb{C}$ such that $\lambda^3 \neq -27$.

Exercise 3.3. Show that $\mathcal{C}_3$ is smooth.

Let $P = [1 : -1 : 0]$. Then P is an inflection point of the curve $\mathcal{C}_3$. Indeed, the equation

$$3(x + y) - \lambda z = 0$$

defines the line in $\mathbb{P}^2_{\mathbb{C}}$ that is tangent to $\mathcal{C}_3$ at the point P. Solving the system

$$\begin{cases} 3(x + y) - \lambda z = 0, \\ x^3 + y^3 + z^3 + \lambda xyz = 0, \end{cases}$$

we see that this line intersects the curve $\mathcal{C}_3$ only at P. Indeed, we have $z = \frac{3}{\lambda}(x+y)$ and

$$x^3 + y^3 + 27\frac{(x+y)^3}{\lambda^3} + 3xy(x+y) = 0,$$

which can be rewritten as

$$(x+y)\left(x^2 - xy + y^2 + 27\frac{x^2+2xy+y^2}{\lambda^3} - 3xy\right) = 0,$$

which can be simplified further as

$$(x+y)^3\left(1 + \frac{27}{\lambda^3}\right) = 0.$$

But $\lambda^3 \neq -27$, so that $x+y=0$ and $z = x+y = 0$, which gives $[x : y : z] = [1 : -1 : 0] = P$. Therefore, the point P is an inflection point of the curve $\mathcal{C}_3$.

To find the remaining 8 inflection points of the curve $\mathcal{C}_3$, we should solve the system

$$\begin{cases} f_3(x,y,z) = 0, \\ g_3(x,y,z) = 0, \end{cases}$$

where $g_3(x,y,z)$ is the Hessian of the polynomial $f_3(x,y,z)$ defined in the solution of Exercise 3.2. Computing partial derivatives, we get

$$g_3(x,y,z) = \det\begin{pmatrix} 6x & \lambda z & \lambda y \\ \lambda z & 6y & \lambda x \\ \lambda y & \lambda x & 6z \end{pmatrix} = (6^3 + 2\lambda^3)xyz - 6\lambda^2(x^3+y^3+z^3).$$

Thus, if $g_3(x,y,z) = 0$ and $f_3(x,y,z) = 0$, then

$$\big(6^3 + 2\lambda^3\big)xyz - 6\lambda^2\big(x^3+y^3+z^3\big) = \big(6^3+2\lambda^3\big)xyz + 6\lambda^3 xyz = 8\big(27+\lambda^3\big)xyz = 0,$$

which implies that $xyz = 0$. Since $x^3 + y^3 + z^3 + \lambda xyz = 0$, this gives us 9 inflection points (1).

Exercise 3.4. Let $\mathcal{C}_3$ be the cubic curve in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

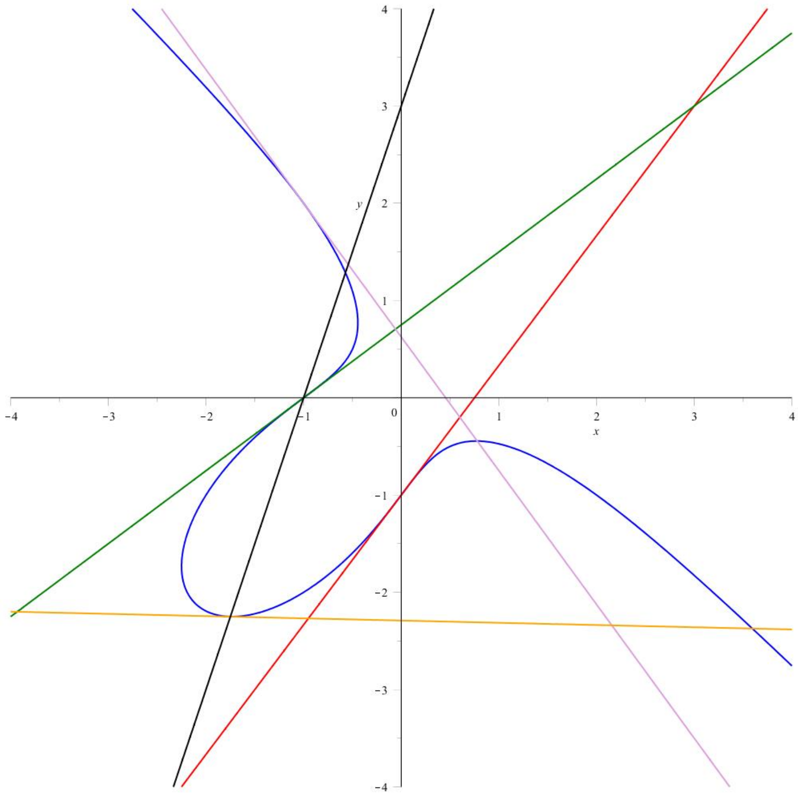
$$zy^2 = x^3 - 2z^3.$$

Show that $\mathcal{C}_3$ is smooth and find all its inflection points.

Observe that the points $[0 : 1 : -1]$, $[-1 : 0 : 1]$, $[-1 : 2 : 1]$ and $[-7 : -9 : 4]$ are contained in the smooth cubic curve

$$x^3 + y^3 + z^3 + 4xyz = 0,$$

where point $[0 : 1 : -1]$ and point $[-1 : 0 : 1]$ are two of its nine inflection points (1). The following picture shows the real part of this curve in the chart $z \neq 0$, the tangent lines to this curve at these points, and the line that passes through $[-1 : 0 : 1]$ and $[-7 : -9 : 4]$.



Exercise 3.5. Let $\mathcal{C}_3$ be a smooth cubic and Σ be the set of its inflection point. Verify that any line through two of the inflection points of $\mathcal{C}_3$ pass through a distinct third inflection point of $\mathcal{C}_3$. Then deduce that Σ forms a Hesse configuration.

Exercise 3.6. Suppose Σ is a set of 9 points in $\mathbb{P}^2_{\mathbb{C}}$ which has Hesse configuration. Show that there exists a smooth cubic $\mathcal{C}_3$ passing through the nine points. Verify that Σ is the set of inflection points of $\mathcal{C}_3$.

3.2 Classification of Cubic Curves

Up to projective transformations, there are just three conics in the plane:

1. an irreducible conic, e.g. $xy = z^2$;
2. a union of two distinct lines, e.g. $xy = 0$;
3. a single line taken with multiplicity 2, e.g. $z^2 = 0$.

A similar classification exists for cubic curves. Let us show how to classify *smooth* cubic curves.

Let $\mathcal{C}_3$ be a smooth cubic curve in the projective plane $\mathbb{P}^2_{\mathbb{C}}$. Then $\mathcal{C}_3$ is irreducible by Lemma A.2.2. Moreover, up to projective transformations, the curve $\mathcal{C}_3$ is given by

$$zy^2 = x(x - z)(x - \lambda z) \tag{7}$$

for some $\lambda \in \mathbb{C}$ such that $\lambda \neq 0$ and $\lambda \neq 1$. Indeed, it follows from Exercise 3.2 that there exists at least one inflection point of the curve $\mathcal{C}_3$. Let P be one such point. Let us choose new projective coordinates on $\mathbb{P}^2_{\mathbb{C}}$ such that $P = [0 : 1 : 0]$, and the tangent line to $\mathcal{C}_3$ at this point is $z = 0$. Then $\mathcal{C}_3$ is given by

$$\alpha y^2 z + y l_2(x, z) + l_3(x, z) = 0,$$

where α is a non-zero complex number, and each $l_i(x, y, z)$ is a homogeneous polynomial of degree i. Dividing this equation by α, we may assume that $\mathcal{C}_3$ is given by

$$y^2 z + y l_2(x, z) + l_3(x, z) = 0.$$

Since P is an inflection point, we see that $l_2(x, 0)$ is a zero polynomial. So, we have

$$l_2(x, z) = z g_1(x, z)$$

for some linear polynomial $g_1(x, z)$. Hence, the curve $\mathcal{C}_3$ is given by

$$y^2 z + y z g_1(x, z) + l_3(x, z) = 0.$$

Now let us replace y by $y + \mu g_1(x, z)$ for some complex number μ (to be chosen later). Then

$$\Big(y + \mu g_1(x, z)\Big)^2 z + \Big(y + \mu g_1(x, z)\Big) z g_1(x, z) + l_3(x, z) = 0$$

is the equation of the curve $\mathcal{C}_3$ in new projective coordinates. So, we have

$$\big(y^2 + 2\mu y g_1(x, z) + \mu^2 g_1^2(x, z)\big) z + \big(y + \mu g_1(x, z)\big) z g_1(x, z) + l_3(x, z) = 0.$$

Now we can put $\mu = -\frac{1}{2}$. We get the following equation:

$$y^2 z = -l_3(x, z) - \frac{z g_1^2(x, z)}{4} + \frac{g_1^2(x, z) z}{2}.$$

Expanding the polynomial in its right hand side, we see that $\mathcal{C}_3$ is given by

$$zy^2 = \alpha x^3 + \beta x^2 z + \gamma x z^2 + \delta z^3$$

for some complex numbers α, β, γ and δ. Moreover, since $\mathcal{C}_3$ is irreducible, we must have $\alpha \neq 0$. Now, replacing z by αz, the equation of the curve $\mathcal{C}_3$ becomes

$$zy^2 = x^3 + a x^2 z + b x z^2 + c z^3, \tag{8}$$

where a, b and c are complex numbers. Now using Fundamental Theorem of Algebra, we get

$$x^3 + a x^2 z + b x z^2 + c z^3 = (x + \delta_1 z)(x + \delta_2 z)(x + \delta_3 z)$$

for some complex numbers δ_1, δ_2 and δ_2. Replacing x by $x + \delta_1 z$, we obtain (7).

Exercise 3.7. Let $\mathcal{C}_3$ be the cubic curve in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$3x^3 - x^2 y + 6x^2 z - 3xy^2 + 8xyz - xz^2 + y^3 - 6y^2 z + 9yz^2 - 3z^3 = 0.$$

Observe that $\mathcal{C}_3$ contains $[1 : 1 : 0]$. Show that $[1 : 1 : 0]$ is an inflection point of the curve $\mathcal{C}_3$. Find a projective transformation $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that $\phi(\mathcal{C}_3)$ is given by (7).

The equation (7) is commonly known as the Legendre form of the smooth cubic curve $\mathcal{C}_3$. We also can find a projective transformation that maps the curve $\mathcal{C}_3$ to the curve given by (8). Since $\mathcal{C}_3$ is assumed to be smooth, we must have

$$-4a^3c + a^2b^2 + 18abc - 4b^3 - 27c^2 \neq 0,$$

where a, b and c are coefficients in (8). Using an appropriate change of homogeneous coordinates, we can simplify (8) a bit more to get $a = 0$ so that we are left with equation of the form

$$zy^2 = x^3 + pxz^2 + qz^3 \tag{9}$$

for some complex numbers p and q. Then (9) is called *Weierstrass normal form* of the smooth cubic curve.

Exercise 3.8. Let $\mathcal{C}_3$ be the curve in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$6x^3 - 7x^2y - 7x^2z + 3xy^2 + 4xyz + 2xz^2 - y^3 - y^2z = 0.$$

Observe that $\mathcal{C}_3$ contains $[0 : 1 : -1]$. Show that $[0 : 1 : -1]$ is an inflection point of the curve $\mathcal{C}_3$. Find a projective transformation $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that $\phi(\mathcal{C}_3)$ is given by (9).

Exercise 3.9. Prove that for every smooth cubic curve $\mathcal{C}_3$, there exists a projective transformation $\phi : \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that $\phi(\mathcal{C}_3)$ is given by (6) – which is called the Hesse normal form of the cubic curve.

Using (7), we can compute the number

$$\mathbf{j} = 256\frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2},$$

which is known as the **j**-invariant of the cubic curve $\mathcal{C}_3$.

Exercise 3.10. Let $\mathcal{C}_3$ be the cubic curve in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$7x^3 - 10x^2y + 10x^2z + 5xy^2 - 6xyz + 3xz^2 - y^3 + y^2z = 0.$$

Observe that $\mathcal{C}_3$ contains $[0 : 1 : 1]$. Do the following

(a) Show that $[0 : 1 : 1]$ is an inflection point of the curve $\mathcal{C}_3$.

(b) Find a projective transformation $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that $\phi(\mathcal{C}_3)$ is given by (7).

(c) Find a projective transformation $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that $\phi(\mathcal{C}_3)$ is given by (9).

(d) Check that

$$1728\frac{4p^3}{4p^3 + 27q^2} = 256\frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2},$$

where p and q are coefficients in (9), and λ is the number in (7).

The **j**-invariant is important, because smooth cubic curves with the same **j**-invariant are *isomorphic*. However, this does not mean that they can be obtained from each other by a projective transformation. Namely, we can use *quadratic birational* transformations

$$\mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}}$$

that simplify the equation of the curve $\mathcal{C}_3$. These transformations are given by

$$\big[x : y : z\big] \mapsto \big[f_2(x, y, z) : g_2(x, y, z) : h_2(x, y, z)\big]$$

for some homogeneous polynomials $f_2(x, y, z)$, $g_2(x, y, z)$ and $h_2(x, y, z)$ of degree 2 such that there exist inverse transformations of the same kind. For instance, the formula

$$\big[x : y : z\big] \mapsto \Big[\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\Big] = \big[yz : xz : xy\big].$$

gives us a quadratic birational transformation, which is also known as the Cremona involution. Observe that this involution is not defined at three points: $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$. In fact, all quadratic birational transformations of the projective plane are undefined in few points. Because of this, we use symbol $\dashrightarrow$ instead of $\to$ when working with them.

Exercise 3.11. Let $\mathcal{C}_3$ be a smooth cubic curve in $\mathbb{P}^2_{\mathbb{C}}$, and let P be a non-inflection point in it. Show that there is a quadratic birational transformation $\rho\colon \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}}$ such that

- the curve $\rho(\mathcal{C}_3)$ is a smooth cubic curve that is given by (9),
- the map ρ is well defined at P and $\rho(P) = [0 : 1 : 0]$.

Corollary 3.12. Since every smooth cubic curve are projectively equivalent to one another, we obtain the uniqueness of the Hesse configuration (up to projective transformations) by Exercises 3.5 and 3.6.

We classified smooth cubic curves. What about singular cubic curves? We can classify them too. For example, there are just two irreducible singular cubic curves up to projective transformations:

Exercise 3.13. Let $\mathcal{C}_3$ be an irreducible singular cubic curve. Show that there exists a projective transformation $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that $\phi(\mathcal{C}_3)$ is given by one of the following equations:

- $zxy + x^3 + y^3 = 0$ (irreducible nodal cubic curve);
- $zy^2 = x^3$ (irreducible cuspidal cubic curve).

Likewise, up to projective transformations of $\mathbb{P}^2_{\mathbb{C}}$, there are just six reducible cubic polynomials:

1. $x(yz + x^2) = 0$ (union of an irreducible conic and a line that does not tangent it);
2. $x(xz + y^2) = 0$ (union of an irreducible conic and a line that tangents it);
3. $xyz = 0$ (union of three lines that do not pass through one point);
4. $xy(x + y) = 0$ (union of three lines that all pass through one point);
5. $x^2y = 0$ (a union of two lines such that one is taken with multiplicity two);
6. $x^3 = 0$ (a triple line).

4 Pencils of Conics

Let $\mathcal{C}_2$ and $\mathcal{C}_2'$ be two conics in $\mathbb{P}^2_{\mathbb{C}}$ given by the homogeneous degree two polynomial equations $f_2(x,y,z) = 0$ and $g_2(x,y,z) = 0$ respectively. Then the pencil of conics generated by $\mathcal{C}_2$ and $\mathcal{C}_2'$ is the family of conics

$$\left\{\mathcal{C}_{[\lambda:\mu]}\right\}_{[\lambda:\mu]\in\mathbb{P}^1_{\mathbb{C}}} = \left\{\text{Conic in } \mathbb{P}^2_{\mathbb{C}} \text{ given by } \lambda f_2(x,y,z) + \mu g_2(x,y,z) = 0 \mid [\lambda:\mu] \in \mathbb{P}^1_{\mathbb{C}}\right\}.$$

But before we explore further into the topic of pencil of conic, let us first have a discussion on the vector space of homogeneous polynomials.

Let $\mathbb{V}_d$ be the vector space consisting of all homogeneous polynomials $f_d(x,y,z)$ with complex coefficients of degree d together with the zero-polynomial. Then $\mathbb{V}_d$ is a vector space over $\mathbb{C}$. What is its dimension? Every polynomial in $\mathbb{V}_1$ is given by

$$Ax + By + Cz$$

for some complex numbers A, B and C. This shows that $\mathbb{V}_1$ is a vector space of dimension 3. Similarly, every polynomial in $\mathbb{V}_2$ is given by

$$Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2$$

for some complex numbers A, B, C, D, E and F, so that $\mathbb{V}_2$ is a vector space of dimension 6. Likewise, every polynomial in $\mathbb{V}_3$ is given by

$$A_1x^3 + A_2x^2y + A_3xy^2 + A_4y^3 + A_5x^2z + A_6xyz + A_7y^2z + A_8xz^2 + A_9yz^2 + A_{10}z^3$$

for some A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10} in $\mathbb{C}$. Then $\mathbb{V}_3$ is a vector space of dimension 10.

Exercise 4.1 (cf. [14])**.** Show that $\mathbb{V}_d$ is a vector space of dimension $\frac{(d+2)(d+1)}{2}$.

Let $P_1, \ldots, P_k$ be distinct points in the plane $\mathbb{P}^2_{\mathbb{C}}$. Denote by $\mathbb{V}_d(P_1,\ldots,P_k)$ the vector subspace in $\mathbb{V}_d$ consisting of all homogeneous polynomials $f_d(x,y,z)$ in $\mathbb{V}_d$ such that

$$\begin{cases} f_d(P_1) = 0, \\ f_d(P_2) = 0, \\ \vdots \\ f_d(P_k) = 0. \end{cases}$$

By the rank–nullity theorem, the dimension of the space $\mathbb{V}_d(P_1,\ldots,P_k)$ is at least

$$\frac{(d+2)(d+1)}{2} - k.$$

If $k \leqslant \frac{(d+2)(d+1)}{2}$, this is the expected dimension of the vector space $\mathbb{V}_d(P_1,\ldots,P_k)$.

We say that the points $P_1,\ldots,P_k$ *impose independent linear conditions* on curves of degree d if

$$\dim_{\mathbb{C}}\Big(\mathbb{V}_d\big(P_1,\ldots,P_k\big)\Big) = \frac{(d+2)(d+1)}{2} - k.$$

Otherwise, we say that $P_1,\ldots,P_k$ *impose dependent linear conditions* on curves of degree d in $\mathbb{P}^2_{\mathbb{C}}$.

If $k > \frac{(d+2)(d+1)}{2}$, then the points $P_1, \ldots, P_k$ impose dependent linear conditions on curves of degree d, because $\dim_{\mathbb{C}}(\mathbb{V}_d(P_1, \ldots, P_k)) \geqslant 0$. If

$$k \leqslant \frac{(d+2)(d+1)}{2},$$

then $P_1, \ldots, P_k$ impose independent linear conditions on curves of degree d if they are *general*.

For instance, if $k = \frac{(d+2)(d+1)}{2}$, then the points $P_1, \ldots, P_k$ impose independent linear conditions on curves of degree d if and only if these points are not contained in any such curve. Similarly, if

$$k = \frac{(d+2)(d+1)}{2} - 1,$$

then $P_1, \ldots, P_k$ impose independent linear conditions on curves of degree d if and only if there exists a unique curve of degree d in $\mathbb{P}^2_{\mathbb{C}}$ that contains all points $P_1, \ldots, P_k$.

It is easy to understand when points in $\mathbb{P}^2_{\mathbb{C}}$ impose independent linear conditions on lines:

- two points always impose independent linear conditions on lines;
- three points impose dependent linear conditions on lines $\iff$ they are collinear;
- four or more points always impose dependent linear conditions on lines.

Similarly, four or five points in $\mathbb{P}^2_{\mathbb{C}}$ impose independent linear conditions on conics if and only if no four points among them are collinear. We proved this in Sections 2.2 and 2.4. The same proof implies that three or less points always impose independent linear conditions on conics.

If we know homogeneous coordinates of the points $P_1, \ldots, P_k$, then we can compute the dimension of the vector space $\mathbb{V}_d(P_1, \ldots, P_k)$, and check whether the points $P_1, \ldots, P_k$ impose independent linear conditions on curves of degree d or not.

For example, if $P_1 = [2:3:1]$, $P_2 = [-3:4:1]$, $P_3 = [-4:-5:1]$, $P_4 = [-6:2:1]$, $P_5 = [5:3:1]$, $P_6 = [3:2:1]$, $P_7 = [-2:-6:1]$, $P_8 = [4:8:1]$ and $P_9 = [1:2:0]$, then

$$\dim_{\mathbb{C}}\Big(\mathbb{V}_3\big(P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9\big)\Big) = 1,$$

so that the points P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8 and P_9 impose independent linear conditions on cubic curves. Indeed, if $f_3(x,y,z)$ is a polynomial in $\mathbb{V}_3(P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9)$, then

$$f_3(x,y,z) = A_1x^3+A_2x^2y+A_3xy^2+A_4y^3+A_5x^2z+A_6xyz+A_7y^2z+A_8xz^2+A_9yz^2+A_{10}z^3$$

for some complex numbers A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9 and A_{10} such that

$$\begin{cases} 8A_1 + 12A_2 + 18A_3 + 27A_4 + 4A_5 + 6A_6 + 9A_7 + 2A_8 + 3A_9 + A_{10} = 0, \\ 36A_2 - 27A_1 - 48A_3 + 64A_4 + 9A_5 - 12A_6 + 16A_7 - 3A_8 + 4A_9 + A_{10} = 0, \\ 16A_5 - 80A_2 - 100A_3 - 125A_4 - 64A_1 + 20A_6 + 25A_7 - 4A_8 - 5A_9 + A_{10} = 0, \\ 72A_2 - 216A_1 - 24A_3 + 8A_4 + 36A_5 - 12A_6 + 4A_7 - 6A_8 + 2A_9 + A_{10} = 0, \\ 125A_1 + 75A_2 + 45A_3 + 27A_4 + 25A_5 + 15A_6 + 9A_7 + 5A_8 + 3A_9 + A_{10} = 0, \\ 27A_1 + 18A_2 + 12A_3 + 8A_4 + 9A_5 + 6A_6 + 4A_7 + 3A_8 + 2A_9 + A_{10} = 0, \\ 4A_5 - 24A_2 - 72A_3 - 216A_4 - 8A_1 + 12A_6 + 36A_7 - 2A_8 - 6A_9 + A_{10} = 0, \\ 64A_1 + 128A_2 + 256A_3 + 512A_4 + 16A_5 + 32A_6 + 64A_7 + 4A_8 + 8A_9 + A_{10} = 0, \\ aA_1 + 2A_2 + 4A_3 + 8A_4 = 0. \end{cases}$$

Here we just substituted the coordinates of the points P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8 and P_9 into the polynomial equation $f_3(x, y, z) = 0$. The latter system can be rewritten in a matrix form:

$$\begin{pmatrix} 8 & 12 & 18 & 27 & 4 & 6 & 9 & 2 & 3 & 1 \\ -27 & 36 & -48 & 64 & 9 & -12 & 16 & -3 & 4 & 1 \\ -64 & -80 & -100 & -125 & 16 & 20 & 25 & -4 & -5 & 1 \\ -216 & 72 & -24 & 8 & 36 & -12 & 4 & -6 & 2 & 1 \\ 125 & 75 & 45 & 27 & 25 & 15 & 9 & 5 & 3 & 1 \\ 27 & 18 & 12 & 8 & 9 & 6 & 4 & 3 & 2 & 1 \\ -8 & -24 & -72 & -216 & 4 & 12 & 36 & -2 & -6 & 1 \\ 64 & 128 & 256 & 512 & 16 & 32 & 64 & 4 & 8 & 1 \\ 1 & 2 & 4 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \\ A_9 \\ A_{10} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We see that the rank of this 9×10 matrix is 9. Thus, its solutions form a one dimensional vector space by the rank–nullity theorem. This simply means that there exists unique non-zero solution $(A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10})$ up to scaling. Namely, we have

$$\begin{cases} A_1 = 4149128\lambda, \\ A_2 = -16611896\lambda, \\ A_3 = 13274044\lambda, \\ A_4 = -3002689\lambda, \\ A_5 = 38844860\lambda, \\ A_6 = -56087164\lambda, \\ A_7 = 16082273\lambda, \\ A_8 = -36085100\lambda, \\ A_9 = 34296718\lambda, \\ A_{10} = 13972672\lambda, \end{cases}$$

where $\lambda \in \mathbb{C}$. Hence, every polynomial in $\mathbb{V}_3(P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9)$ equals to $\lambda f_3(x, y, z)$, where λ is any complex number and $f_3(x, y, z)$ is the following polynomial:

$$\begin{aligned} 4149128x^3 - 16611896x^2y + 38844860x^2z + 13274044xy^2 - 56087164xyz - \\ - 36085100xz^2 - 3002689y^3 + +16082273y^2z + 34296718yz^2 + 13972672z^3. \end{aligned}$$

Hence, the points P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9 impose independent linear conditions on cubic curves in $\mathbb{P}^2_{\mathbb{C}}$.

Exercise 4.2. Let Σ be the subset in $\mathbb{P}^2_{\mathbb{C}}$ that consists of the points

$$[2 : -1 : 2],\ [-6 : -5 : 2],\ [4 : 1 : 1],\ [15 : 25 : 3],\ [-3 : 13 : 3],$$
$$[-15 : 5 : 3],\ [102 : 1835 : 1086],\ [1015 : -405 : 277],\ [-447 : 485 : 51].$$

(a) Show that Σ imposes dependent linear conditions on cubic curves in $\mathbb{P}^2_{\mathbb{C}}$.

(b) Let Σ' be any subset of the set Σ consisting of 8 points. Show that $\mathbb{V}_3(\Sigma') = \mathbb{V}_3(\Sigma)$.

(c) Do the following:

- find a cubic polynomial $f_1(x, y, z) \in \mathbb{V}_3(\Sigma)$ such that $f_1(0, 0, 1) = 0$;

- find a cubic polynomial $f_2(x, y, z) \in \mathbb{V}_3(\Sigma)$ such that $f_2(0, 1, 0) = 0$;
- show that $f_1(x, y, z)$ and $f_2(x, y, z)$ form a basis of the vector space $\mathbb{V}_3(\Sigma)$.

(d) Let $g_1(x, y, z) = (29x + 12y - 23z)(22x - 3y - 85z)(25x + 12y + 105z)$ and let

$$g_2(x, y, z) = (13x + 36y + 5z)(10x + 15y - 55z)(65x - 48y + 75z).$$

Check that $g_1(P) = 0$ and $g_2(P) = 0$ for every point $P \in \Sigma$.

(e) Find numbers a, b, c and d such that

$$\begin{cases} g_1(x, y, z) = af_1(x, y, z) + bf_2(x, y, z), \\ g_2(x, y, z) = cf_1(x, y, z) + df_2(x, y, z). \end{cases}$$

Now consider the collection of points $P_1 = [1 : 0 : 0]$, $P_2 = [0 : 2 : 1]$, $P_3 = [0 : -2 : 1]$, and $P_4 = [-4 : 0 : 1]$ with respect to the vector space $\mathbb{V}_2$. Since we can write the polynomial in $\mathbb{V}_2(P_1, P_2, P_3, P_4)$ as

$$f_2(x, y, z) = A_1x^2 + A_2xy + A_3y^2 + A_4xz + A_5yz + A_6z^2,$$

for some complex numbers A_1, A_2, A_3, A_4, A_5, and A_6, we have the system

$$\begin{cases} A_1 = 0 \\ 4A_3 + 2A_5 + A_6 = 0 \\ 4A_3 - 2A_5 + A_6 = 0 \\ 16A_1 - 4A_4 + A_6 = 0. \end{cases}$$

Writing this in the matrix form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 2 & 1 \\ 0 & 0 & 4 & 0 & -2 & 1 \\ 16 & 0 & 0 & -4 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

we see that the 4×6 matrix above has rank 4 so its solutions form a two dimensional vector space by the rank–nullity theorem. Solving the equation tells us that the polynomial xy is a basis of the vector space $\mathbb{V}_2(P_1, P_2, P_3, P_4)$ and so is the polynomial $y^2 - xz - 4z^2$. This basis is not unique. Another basis is given by the polynomials $(y+2z)(x-2y+4z)$ and $(y-2z)(x + 2y + 4z)$. This means that every polynomial in $\mathbb{V}_2(P_1, P_2, P_3, P_4)$ is given by

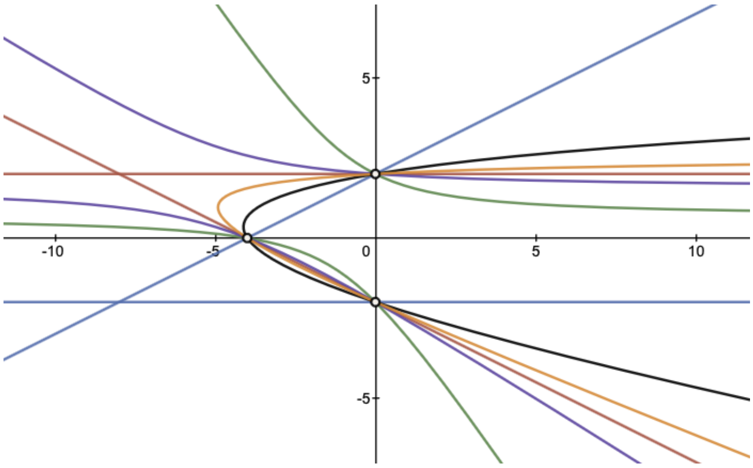
$$\lambda(y + 2z)(x - 2y + 4z) + \mu(y - 2z)(x + 2y + 4z). \tag{10}$$

Seems familiar? Indeed for any $\kappa \in \mathbb{C}$, the conics given by (10) and

$$\lambda\kappa(y + 2z)(x - 2y + 4z) + \mu\kappa(y - 2z)(x + 2y + 4z).$$

are identical so the variations of λ and μ that we need to consider to obtain unique conics are only those for which $[\lambda : \mu] \in \mathbb{P}^1_{\mathbb{C}}$.

Let $\mathcal{C}_2$ be the conic given by $f_2(x, y, z) = (y+2z)(x-2y+4z)$ and $\mathcal{C}_2'$ be the conic given by $g_2(x, y, z) = (y - 2z)(x + 2y + 4z)$. Thus the vector space $\mathbb{V}_2(P_1, P_2, P_3, P_4)$ contains the defining polynomials of the elements of the pencil of conics generated by $\mathcal{C}_2$ and $\mathcal{C}_2'$. On the other hand, note that P_1, P_2, P_3, and P_4 are distinct points in $\mathbb{P}^2_{\mathbb{C}}$ and $f_2(P_i) = g_2(P_j) = 0$ for all $i, j = 1, ..., 4$. So by Exercise 2.14, P_1, P_2, P_3, and P_4 are *all* the intersection points between the conic $\mathcal{C}_2$ defined by $f_2(x, y, z)$ and the conic $\mathcal{C}'_2$ defined by $g_2(x, y, z)$; and they intersect *transversally* at each of these points. Consequently, we can also say that the pencil of conics generated by $\mathcal{C}_2$ and $\mathcal{C}'_2$ contains the conics which defining polynomials are in the vector space $\mathbb{V}_2(P_1, P_2, P_3, P_4)$ where P_1, P_2, P_3, and P_4 are the intersection points of $\mathcal{C}_2$ and $\mathcal{C}'_2$. The pencil of conics generated by $\mathcal{C}_2$ and $\mathcal{C}'_2$ above produces the following picture in the subset $U_z \subset \mathbb{P}^2_{\mathbb{C}}$:

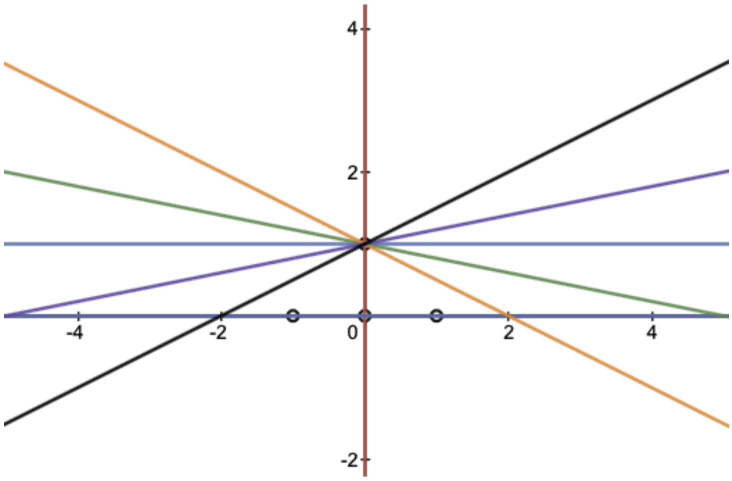


We call $\mathcal{C}_2 \cap \mathcal{C}'_2 = \{P_1, P_2, P_3, P_4\}$ the *base locus* of the pencil of conic while the points P_1, P_2, P_3, and P_4 themselves are called the *base points* of the pencil. We emphasise that every element in the pencil must pass through the base points of the pencil; and we note that the fourth intersection point in the picture above is at $[1 : 0 : 0]$ which lies on the line at infinity $z = 0$.

It should be clear from the exploration above that given four noncollinear points in $\mathbb{P}^2_{\mathbb{C}}$, we obtain a unique (up to projective transformation) pencil of conic. If three of the four points are collinear, we can use projective transformation so that these points, say P_1, P_2, and P_3, are placed at $[-1 : 0 : 1], [0 : 0 : 1]$, and $[1 : 0 : 1]$ respectively. Let us pick the coordinate of the last point, P_4, to be $[0 : 1 : 1]$; thus the three collinear points now lie on the line $y = 0$. Exercise 2.11 then tells us that any conic $\mathcal{C}$ that passes through P_1, P_2, P_3, and P_4 must be reducible with $g(x, y, z) = y$ as a factor of the defining polynomial of $\mathcal{C}$. That is, $\mathcal{C}$ must the union of the line $y = 0$ and one other line in $\mathbb{P}^2_{\mathbb{C}}$ that passes through P_4. We note that the line $x = 0$ and the line $y = 1$ serve this purpose and that the conics $\mathcal{C}_2$ and $\mathcal{C}'_2$ which are defined by the polynomials $f_2(x, y, z) = xy$ and $g_2(x, y, z) = (y - z)y$ form a basis for $\mathbb{V}_2(P_1, P_2, P_3, P_4)$. Then the elements of $\mathbb{V}_2(P_1, P_2, P_3, P_4)$ are given by

$$\lambda xy + \mu(y - z)y = (\lambda x + \mu y - \mu z)y = 0$$

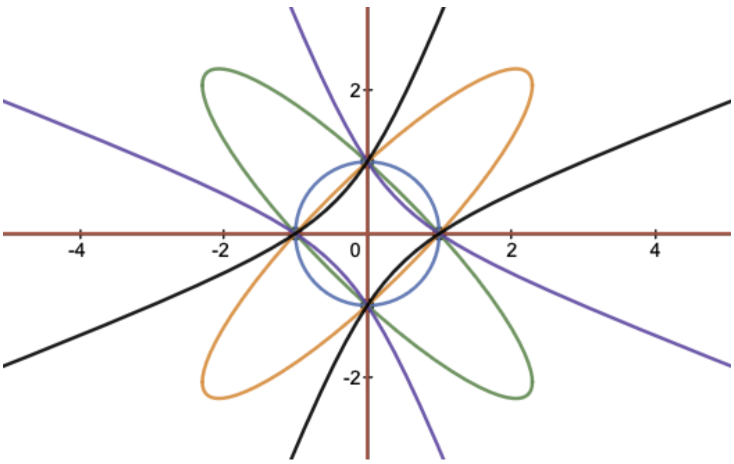
for $[\lambda : \mu] \in \mathbb{P}^1_{\mathbb{C}}$ – i.e. the union of the line $y = 0$ and a line that passes through $[0 : 1 : 1]$. In this case, every element of $\mathbb{V}_2(P_1, P_2, P_3, P_4)$, or equivalently the pencil of conic generated by $\mathcal{C}_2$ and $\mathcal{C}'_2$, is reducible – specifically, if $\mathcal{C}$ is a conic which defining polynomial is in $\mathbb{V}_2(P_1, P_2, P_3, P_4)$, it is simply that $\mathcal{C} = L_{123} + L_4$ where L_{123} is the line that passes through the collinear points P_1, P_2, and P_3 while L_4 is a line that passes through the last point P_4. This pencil of conic produces the following picture:



On the other hand, we obtain a more interesting problem when not even three of the four points P_1, P_2, P_3, an P_4 are collinear. Again, by using projective transformation we may assume that the points P_1, P_2, P_3, and P_4 are given by $[1:0:1], [0:1:1], [-1:0:1]$, and $[0:-1:1]$ respectively. We can immediately guess two (easy) elements of $\mathbb{V}_2(P_1, P_2, P_3, P_4)$. One is the polynomial $f_2(x,y,z) = x^2 + y^2 - z^2$ which corresponds to the unit circle and the other one is the polynomial $g_2(x,y,z) = xy$ which corresponds to the line pair $L_x + L_y$. Furthermore, since they are linearly independent, they form a basis for $\mathbb{V}_2(P_1, P_2, P_3, P_4)$ so that the elements of the vector space are given by

$$\lambda(x^2 + y^2 - z^2) + \mu xy = 0$$

where $[\lambda : \mu] \in \mathbb{P}^1_{\mathbb{C}}$. The picture produced by this pencil is as below:



Clearly $f_2(x,y,z) \in \mathbb{V}_2(P_1, P_2, P_3, P_4)$ is irreducible while $g_2(x,y,z) \in \mathbb{V}_2(P_1, P_2, P_3, P_4)$ is reducible. We can also see in the picture above that every other element of $\mathbb{V}_2(P_1, P_2, P_3, P_4)$ depicted are irreducible except for $L_x + L_y$ which corresponds to $g_2(x,y,z)$. It suggests that a reducible element of this pencil is *special*. We have found one such element (i.e. $L_x + L_y$) so it is only natural to ask whether there are other reducible elements in this pencil. The answer is: yes; but even more than this, we can say:

Exercise 4.3. Let P_1, P_2, P_3, and P_4 be four points in $\mathbb{P}^2_{\mathbb{C}}$ no three of which are collinear. Show that there are exactly three distinct reducible elements (up to scalar multiples) in $\mathbb{V}_2(P_1, P_2, P_3, P_4)$.

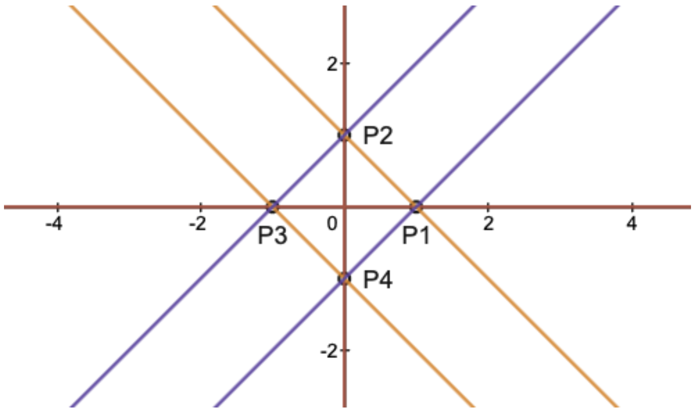
Following from the above, we note that $[\lambda : \mu] = [0 : 1]$ corresponds to the reducible element $xy = 0$ of $\mathbb{V}_2(P_1, P_2, P_3, P_4)$ which corresponds to the conic $L_x + L_y$ that we have found earlier. Now the values $[\lambda : \mu]$ equalling $[1 : 2]$ and $[-1 : 2]$ correspond to the conics given by

$$x^2 + y^2 - z^2 + 2xy = (x + y - z)(x + y + z) = 0$$

and

$$x^2 + y^2 - z^2 - 2xy = (x - y - z)(x - y + z) = 0$$

respectively. If we denote L_{ij} as the line that passes through the points P_i and P_j for $i, j = 1, ..., 4$, then the two newly found reducible conics above are $L_{12} + L_{34}$ and $L_{14} + L_{23}$ respectively; with $L_x + L_y = L_{13} + L_{24}$ in the new notation. These three are all the unique and distinct reducible elements of the pencil of conic above. Below is the figure that illustrates the three reducible elements of the pencil.



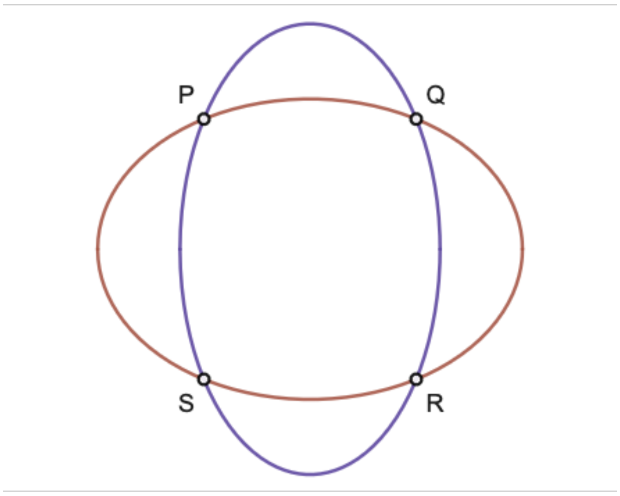
Finally, we also note that any two distinct elements of the pencil must intersect *transversally* at each of the base points of the pencil.

But two distinct conics do not need to intersect at four distinct points in $\mathbb{P}^2_{\mathbb{C}}$. Just as we can write:

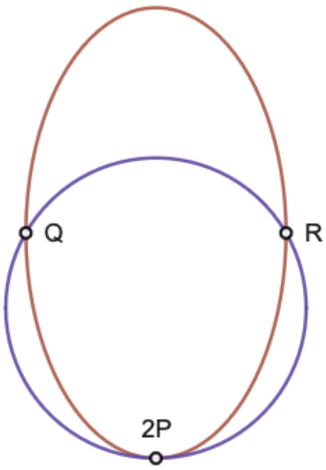
$$\begin{aligned} 4 &= 1 + 1 + 1 + 1 \\ &= 2 + 1 + 1 \\ &= 2 + 2 \\ &= 3 + 1 \\ &= 4, \end{aligned}$$

as we have seen in Section 2.5, there are five ways in which two distinct conics in $\mathbb{P}^2_{\mathbb{C}}$ can intersect:

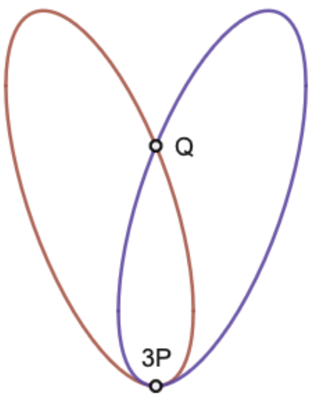
1. They can intersect transversally at 4 distinct points as the picture



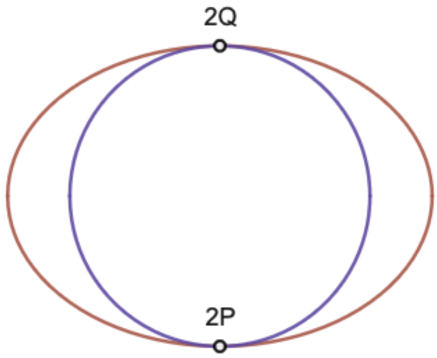
2. They can intersect at 3 distinct points, tangentially at one and transversally at the other two, as the picture



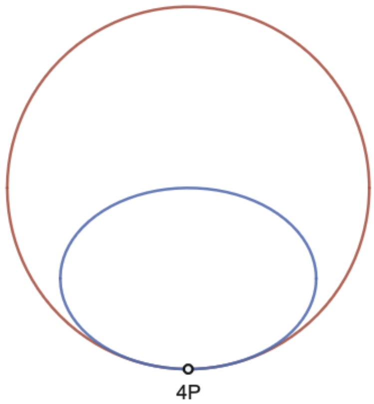
3. They can intersect at 2 distinct points, tangentially at one (with multiplicity 3) and transversally at the other one, as the picture



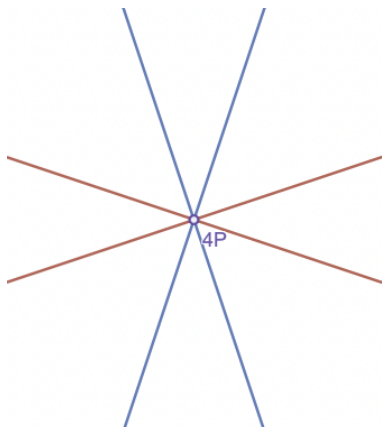
4. They can intersect at 2 distinct points, tangentially at both points, as the picture



5. They can intersect at a single point with multiplicity 4 as the picture



or alternatively as in the picture



For each of the point above, if $f_2(x,y,z)$ is the defining polynomial for the red conic $\mathcal{C}_2$ and $g_2(x,y,z)$ is the defining polynomial of the blue conic $\mathcal{C}_2'$, we can still define the pencil of conic which elements are defined by

$$\lambda f_2(x,y,z) + \mu g_2(x,y,z) = 0$$

for some $[\lambda : \mu] \in \mathbb{P}^1_{\mathbb{C}}$ regardless of how these two conics intersect. We still call the set $\mathcal{C}_2 \cap \mathcal{C}_2'$ the base locus of the pencil but we can classify them depending on *how* the conics intersect at the base points; by which means we actually classify conic pencils. Now let $\Lambda \subset \mathbb{P}^2_{\mathbb{C}}$ has $1 \leq |\Lambda| \leq 4$ with no three collinear points.

Remark 4.4. We have discussed what happens if we allow three points to become collinear for the case when $|\Lambda| = 4$ above. This was not particularly interesting. The reader is invited to think about the consequence when we allow three points to be collinear when $|\Lambda| = 3$.

Now we are ready to classify the conic pencils.

4.1 Type I

This class of conic pencils are exactly what we have discussed above – i.e. when the base locus is comprised of 4 distinct points in $\mathbb{P}^2_{\mathbb{C}}$ no three of which are collinear. We say that the configuration of the base locus is $(1, 1, 1, 1)$ and Exercise 4.3 tells us that there are exactly 3 distinct reducible elements of the pencil which are the line pairs that pass through the base points of the pencil. That is all there is to it.

4.2 Type II

The second class of conic pencils consists of conics which intersect at three distinct points in $\mathbb{P}^2_{\mathbb{C}}$, tangentially at the point P (with multiplicity 2 so we write it as $2P$) and transversally at the points Q and R. We say that the configuration of this base locus is $(2, 1, 1)$.

Let L_P be the tangent line of both $\mathcal{C}_2$ and $\mathcal{C}_2'$ at P and $S \in L_P \setminus \{P\}$. Exercise 2.11 tells us that no three points in $\{P, Q, R, S\}$ are collinear so using an appropriate projective transformation, $P, Q, R,$ and S may have the coordinate $[0 : 0 : 1], [-1 : 1 : 1], [1 : 1 : 1]$, and $[1 : 0 : 1]$ respectively. In this coordinate system, L_P is then simply given by $y = 0$. Elementary geometry tells us that three noncollinear points in $\mathbb{C}^2 \sim U_z \subset \mathbb{P}^2_{\mathbb{C}}$ defines a unique circle. Using this elementary method, we deduce that the circle defined by

$$x^2 + (y - z)^2 - z^2 = 0$$

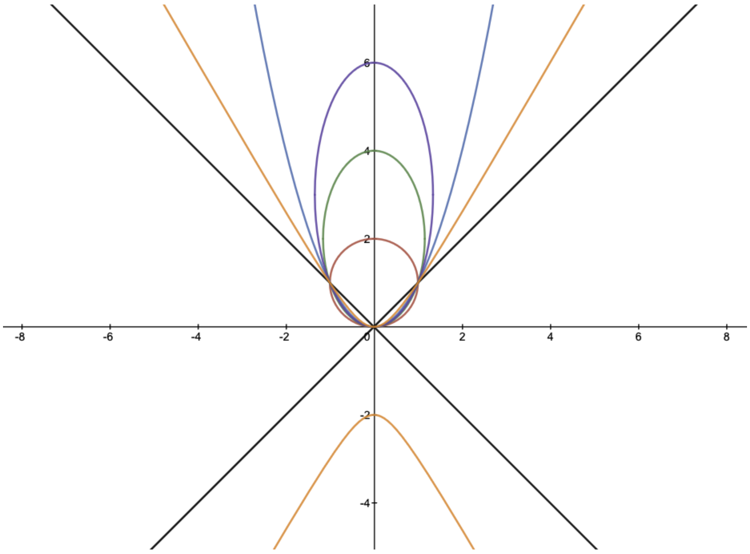
passes through the three points $P, Q,$ and R and that the tangent line of the circle at P is indeed given by $y = 0$. Now note that the configuration of the points $P, Q,$ and R are symmetric in the coordinate system $(\bar{x}, \bar{y})$ of U_z about $\bar{x} = 0$. What conic is symmetric about an axis? Yes! A parabola is symmetric about an axis. Using yet again elementary geometry, we deduce that the unique (why unique?) parabola that passes through the three points $P, Q,$ and R is defined by the polynomial

$$x^2 - yz = 0$$

and that the tangent of this parabola at P is given by $y = 0$. So we have found two linearly independent conics in the pencil and thus can write the defining polynomial of each element in the pencil as

$$\lambda(x^2 + y^2 - 2yz) + \mu(x^2 - yz) = 0.$$

This class of conic pencil exhibits the following picture.



We note that the black line pair in the picture is a reducible element in this pencil given by

$$(x^2 + y^2 - 2yz) - 2(x^2 - yz) = (y - x)(y + x) = 0.$$

It is natural to ask if there are any other reducible element in the pencil; and the answer is

Exercise 4.5. Show that Type II conic pencil has exactly two distinct reducible element.

The value $[\lambda : \mu] = [1 : -2]$ above corresponds to the reducible element $L_{PQ} + L_{PR}$ of the conic depicted in the picture above by the black line pair. For consistency, we say that this reducible element of the conic has the multiplicity 2. Now the other reducible polynomial in the pencil is one defined by

$$(x^2 + y^2 - 2yz) - (x^2 - yz) = y(y - z) = 0.$$

We note that this is the line pair $L_P + L_{QR}$ where we defined L_P to be the tangent line of the conics at P. Therefore we can deduce that every irreducible element in this pencil has a common tangent L_P at P and there are exactly two distinct reducible elements in this pencil, one is the line pair through Q and R which self-intersect at P and another is the line pair $L_P + L_{QR}$.

4.3 Type III

The third class of conic pencils consists of conics which intersect at two distinct points in $\mathbb{P}^2_{\mathbb{C}}$, tangentially at the point P (with multiplicity 3 so we write it as $3P$) and transversally at the point Q. We say that the configuration of this base locus is $(3, 1)$.

Again, let L_P be the the tangent line of both $\mathcal{C}_2$ and $\mathcal{C}'_2$ at P and $R \in L_P \setminus \{P\}$. Since the multiplicity of intersection at P is 3, Exercise 2.11 still applies here to tell us that the three points P, Q, and R are not collinear. Thus through an appropriate projective transformation, we may represent the points P, Q, and R as $[0 : 0 : 1], [0 : 1 : 1]$, and $[1 : 0 : 1]$ respectively so that L_{PQ} is given by $x = 0$ and L_P is given by $y = 0$. Here the reader is invited to produce a creative method to construct two linearly independent conics which intersect in the configuration considered in this classification.

Let $\mathcal{C}_2$ and $\mathcal{C}'_2$ be defined by the polynomial equations

$$(25x + 24y)^2 + (32y - 25z)^2 - 25^2z^2 = 0$$

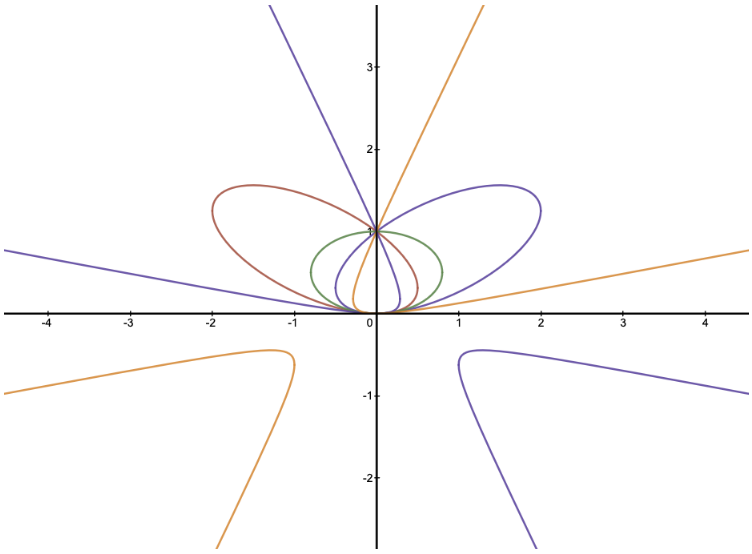
and

$$(25x - 24y)^2 + (32y - 25z)^2 - 25^2 z^2 = 0.$$

The suspicious reader is invited to verify that indeed $\{P, Q\} = \mathcal{C}_2 \cap \mathcal{C}_2'$ where P has multiplicity 3 and that $y = 0$ is the tangent of both $\mathcal{C}_2$ and $\mathcal{C}_2'$ at P. Then $\mathcal{C}_2$ and $\mathcal{C}_2'$ are two linearly independent elements of the pencil and every other element in the pencil is defined by

$$\lambda((25x + 24y)^2 + (32y - 25z)^2 - 25^2 z^2) + \mu((25x - 24y)^2 + (32y - 25z)^2 - 25^2 z^2) = 0.$$

This class of conic pencil exhibits the following picture.



There is a reducible element of the pencil in the picture above which is given by $xy = 0$. Here we propose

Exercise 4.6. Show that Type III conic pencil only has one distinct reducible element.

Indeed the solution $[1 : -1]$ above corresponds to the reducible conic in the picture which was illustrated by the black line pair. This is the line $L_P + L_{PQ}$. Again for consistency, we say that this reducible element of the conic has the multiplicity 3. Therefore we can conclude that every irreducible element in this pencil has a common tangent L_P at P which is the point with intersection multiplicity 3 for any two element in the pencil and the sole reducible element in this pencil is the line pair $L_P + L_{PQ}$.

4.4 Type IV

The fourth class of conic pencils consists of conics which intersect tangentially at two distinct points in $\mathbb{P}^2_{\mathbb{C}}$ (i.e. each with multiplicity 2 so we write $2P$ and $2Q$). We say that the configuration of this base locus is $(2, 2)$.

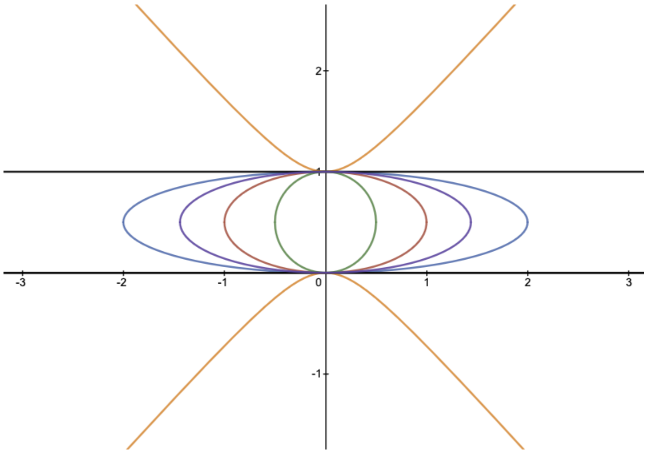
Using the same strategy as for the previous cases, let L_P and L_Q be the common tangents of $\mathcal{C}_2$ and $\mathcal{C}_2'$ at P and Q respectively. By choosing points $R \in L_P \backslash \{P\}$ and $S \in L_Q \backslash \{Q\}$, we may use the appropriate projective transformation so that the coordinates of P, Q, R, and S are given by $[0 : 0 : 1], [0 : 1 : 1], [1 : 0 : 1]$, and $[1 : 1 : 1]$ respectively. In this coordinate system, L_P is conveniently given by $y = 0$ whilst L_Q is conveniently given by $y - z = 0$. We note (real) ellipses centred at $[0 : 1 : 2]$ with its vertices (or co-vertices) at P and Q must be elements of this pencil since these ellipses pass through P and Q with tangent lines L_P and L_Q at these points respectively. Such ellipses are given by the equations

$$x^2 + a^2(2y - z)^2 = a^2 z^2$$

for some $a \in \mathbb{R}$. We pick two (simplest) elements in this family given by $a = 1$ and $a = 2$ from the equation above which corresponds an ellipse with vertices at $[-2 : 1 : 2]$ and $[2 : 1 : 2]$, and another with vertices at $[-4 : 1 : 2]$ and $[4 : 1 : 2]$ respectively. These are two linearly independent elements of the pencil so each element in the pencil is given by

$$\lambda(x^2 + (2y - z)^2 - z^2) + \mu(x^2 + 4(2y - z)^2 - 4z^2) = 0.$$

This class of conics exhibits the following picture.



Again we observe that there exists a reducible conic of the pencil in the picture above which is exactly the line pair $L_P + L_Q$. For Type IV conic pencil, we propose

Exercise 4.7. Show that Type IV conic pencil has two distinct reducible elements.

The value $[\lambda : \mu] = [1 : -1]$ corresponds to the line pair $L_P + L_Q$ depicted in the picture above. The other reducible element in the pencil is given by the polynomial equation $3x^2 = 0$ which means that this is the double line (i.e. one line counted with multiplicity 2) $2L_{PQ}$. Again, for consistency, we say that this reducible element of the conic has the multiplicity 2. Thus every irreducible element in this pencil are tangent to the lines L_P and L_Q at P and Q respectively; and there are exactly two reducible elements in the pencil which are the line pair $L_P + L_Q$ and the double line $2L_{PQ}$.

4.5 Type Va

The final class of conic pencils consists of conics which intersect at exactly one point P in $\mathbb{P}^2_{\mathbb{C}}$ with multiplicity 4 (and so we write it as $4P$). We say that the configuration of this base locus is (4).

As previously, let L_P be the common tangent line of $\mathcal{C}_2$ and $\mathcal{C}_2'$ at P and $Q \in L_P \setminus \{P\}$. Then using appropriate projective transformation, P can be represented by $[0 : 0 : 1]$ and Q by $[1 : 0 : 1]$ so that L_P is given by $y = 0$. It is easy to verify that two linearly independent elements of the pencil are the circle given by

$$x^2 + (y - z)^2 - z^2 = 0$$

and the parabola given by

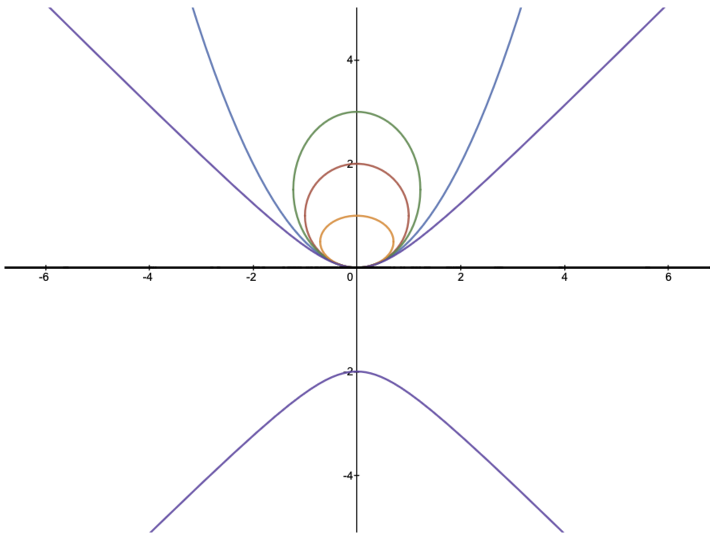
$$x^2 - 2yz = 0.$$

Warning: The pair of circles given by $x^2 + (y - z)^2 - z^2 = 0$ and $x^2 + (y - 2z)^2 - 4z^2$ are not distinct elements of the pencil. They are in fact related by the projective transformation $[x : y : z] \mapsto [2x : 2y : z]$. The same applies to the pair of parabolas given by $x^2 - 2yz = 0$ and $x^2 - 4yz$.

Thus every element in the pencil can be represented by the polynomial equation

$$\lambda(x^2 + (y - z)^2 - z^2) + \mu(x^2 - 2yz) = 0$$

and this pencil of conic exhibits the following picture.



For this class of conic,

Exercise 4.8. Show that Type V conic pencil only has one reducible element.

The value $[\lambda : \mu] = [1 : -1]$ above gives rise to the equation $y^2 = 0$ for the reducible conic in the pencil. This is the double line $2L_P$. For consistency, we say that this reducible element of the conic has the multiplicity 3. Thus every irreducible element in this pencil is tangent to the line L_P at P and every two distinct conics in the pencil intersect at P with multiplicity 4. Moreover, there is only one reducible element in this pencil and it is the double line $2L_P$.

4.6 Type Vb

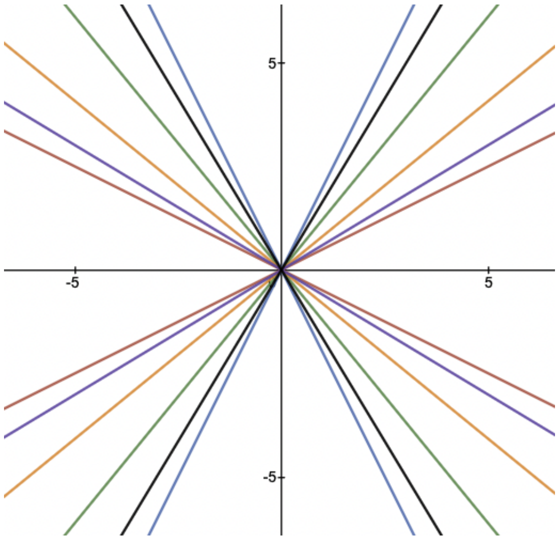
This is also a class of pencil of conics which intersect at exactly one point P in $\mathbb{P}^2_{\mathbb{C}}$ with multiplicity 4. However, this class is rather special because every element in this pencil is degenerate (unlike all the other types discussed above). So let $f_1(x, y, z), g_2(x, y, z), p_1(x, y, z)$, and $q_1(x, y, z)$ be four linear polynomials such that the conics defined by the quadratic polynomials $f_1(x, y, z)g_1(x, y, z)$ and $p_1(x, y, z)q_1(x, y, z)$ have no common reducible component. Then we find a projective transformation so that $P = [0 : 0 : 1]$, $f_1(x, y, z) = x$, and $p_1(x, y, z) = y$ which means that in this new coordinate system,

$$g_1(x, y, z) = ax + by \text{ and } q_1(x, y, z) = cx + dy$$

But then every element in the pencil can be represented by the polynomial equation

$$\lambda x(ax + by) + \mu y(cx + dy) = 0$$

and this pencil exhibits the following picture.



For this final class of conic, the question is not whether there exists a degenerate element but rather whether or not the exists elements that are double lines.

Exercise 4.9. Show that there exists exactly two distinct elements of the pencil which are double lines.

We summarise our classification of pencil of conics with base locus Λ which contains between one to four points in $\mathbb{P}^2_{\mathbb{C}}$ and no three of which are collinear in Table 1.

Type	Base Points	Reducible Elements	Generic Equation
I	P, Q, R, S	$(L_{PQ} + L_{RS}), (L_{PR} + L_{QS}), (L_{PS} + L_{QR})$	$\lambda(x^2 + y^2 + z^2) + \mu xy = 0$
II	$2P, Q, R$	$2(L_{PQ} + L_{PR}), (L_P + L_{QR})$	$\lambda(xz - y^2) + \mu x(x - z) = 0$
III	$3P, Q$	$3(L_P + L_{PQ})$	$\lambda(xz - y^2) + \mu xy = 0$
IV	$2P, 2Q$	$2(2L_{PQ}), (L_P + L_Q)$	$\lambda x^2 + \mu y(y - z) = 0$
Va	$4P$	$3(2L_P)$	$\lambda x^2 + \mu(xz - y^2) = 0$
Vb	$4P$	all elements	$\lambda x^2 + \mu y^2 = 0$

Table 1: Classification of Pencils of Conics

We shall now briefly discuss pencil of conics as surfaces. Observe that for linearly independent $f_2(x, y, z), g_2(x, y, z) \in \mathbb{V}_2$, the equation

$$\lambda f_2(x, y, z) + \mu g_2(x, y, z) = 0$$

which defines a conic is also actually an equation for a surface in $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$. That is, we can find coordinates $([\lambda : \mu], [x, y, z]) \in \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$ such that the conic equation is satisfied.

The problem that we are now presented is that of the classification of such surfaces. But indeed this classification is just the classification of the pencils of conics that we have done in Table 1. Since we have $\mathbb{C}^1 \subset \mathbb{P}^1_{\mathbb{C}}$ and $\mathbb{C}^2 \subset \mathbb{P}^2_{\mathbb{C}}$ as discussed in Section 1, one can plot (a part of) such surfaces by taking the equations in Table 1 and consider the part of the surface that lies in the $\mathbb{C}^3 \cong \mathbb{C}^1 \times \mathbb{C}^2 \subset \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$. It then can be observed that only the Type I pencil of conics produces a surfaces amongst all the surfaces considered and the singular points of the surfaces related to the other pencils of conics are points on the surface where the tangent plane is undefined.

5 Nets of Conics

Consider $\mathbb{P}^2_{\mathbb{C}}$ with homogeneous coordinates $[u : v : w]$. A net of conics, almost similar to a pencil of conic, is a family of conics in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$xf_2(u,v,w) + yg_2(u,v,w) + zh_2(u,v,w) = 0 \tag{11}$$

where $f_2(u,v,w), g_2(u,v,w)$, and $h_2(u,v,w)$ are homogeneous quadratic polynomials, and x, y, z are parameters such that $(x,y,z) \neq (0,0,0)$. For a fixed (x,y,z), the equation (11) defines a (possibly reducible) conic in $\mathbb{P}^2_{\mathbb{C}}$. Clearly, if we scale (x,y,z) like $(\lambda x, \lambda y, \lambda z)$ for some non-zero $\lambda \in \mathbb{C}$, then (11) would define the same conic, so that the conic given by (11) depends on the point $[x : y : z] \in \mathbb{P}^2_{\mathbb{C}}$. Thus, we may assume that x, y, z are homogeneous coordinates on $\mathbb{P}^2_{\mathbb{C}}$.

Observe that the equation (11) defines a three-dimensional algebraic variety (threefold) in $\mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$, where $([x : y : z], [u : v : w])$ are bi-homogeneous coordinates on the product of projective planes $\mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$. Let us denote this variety by X. Then X can be considered as a net of conics in $\mathbb{P}^2_{\mathbb{C}}$. Note that, the threefold X can be very singular. However, if the polynomials $f_2(u,v,w), g_2(u,v,w),, h_2(u,v,w)$ are general, then X is smooth. In this case, the threefold X is known as the smooth Fano threefold in the family №2.24 (see [4]).

Similar to what we did with pencil of conics, let us try to classify net of conics. This problem is much more difficult than the classification of pencils of conics. So, we shall only consider the classification of *smooth* nets of conics, i.e. the case when (11) defines a smooth threefold $X \subset \mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$. The claim is that for any smooth net of conic X given by (11), there exists projective transformations on the coordinates $([x : y : z], [u : v : w])$ on $\mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$ so that the net of conic is given by on of the following three equations:

1. $(\mu vw + u^2)x + (\mu uw + v^2)y + (\mu uv + w^2)z = 0$ for some $\mu \in \mathbb{C}$ such that $\mu^3 \neq -1$,
2. $(vw + u^2)x + (uw + v^2)y + w^2 z = 0$.
3. $(vw + u^2)x + v^2 y + w^2 z = 0$.

This is proved in [1]. To prove this assertion, it is enough to show that X can be given by

$$\big(a_1 vw + a_2 u^2\big)x + \big(b_1 uw + b_2 v^2\big)y + \big(c_1 uv + c_2 w^2\big)z = 0 \tag{12}$$

for some numbers a_1, a_2, b_1, b_2, c_1 and c_2. Indeed, suppose that X is given by (12). Then $a_2 b_2 c_2 \neq 0$, because X is smooth. Thus, scaling u, v and w appropriately, we may assume that $a_2 = b_2 = c_2 = 1$. Choose a, b and c such that $a^3 = a_1$, $b^3 = b_1$ and $c^3 = c_1$. If $abc \neq 0$, we scale our coordinates as $x \mapsto x, y \mapsto yt^2, z \mapsto zs^2, u \mapsto u, v \mapsto \frac{v}{s}, w \mapsto \frac{w}{t}$ for $s = \frac{a}{c}$ and $t = \frac{a}{b}$. Then we are in case (1) with $\mu = abc$, and X is singular if and only if $\mu^3 = -1$, so that the remaining assertions follows from [7]. Similarly, if $abc = 0$, then we can scale and permute the coordinates accordingly to get either case (2) or case (1).

Now, let us prove that we can choose u, v, w, x, y, z such that X is given by (12). We assume that X is smooth.

Let $\mathrm{pr}_1 \colon X \to \mathbb{P}^2$ be the projections to the first factor. For a point $[x : y : z] \in \mathbb{P}^2_{\mathbb{C}}$, the preimage $\mathrm{pr}_1^{-1}([x : y : z])$ is a conic in $\mathbb{P}^2_{\mathbb{C}}$ given by (11), so that we say that pr_1 is a conic bundle. If $[x : y : z]$ is a sufficiently general point in $\mathbb{P}^2_{\mathbb{C}}$, then the conic $\mathrm{pr}_1^{-1}([x : y : z])$ is smooth. However, for some $[x : y : z] \in \mathbb{P}^2_{\mathbb{C}}$, the conic $\mathrm{pr}_1^{-1}([x : y : z])$ is singular. Moreover, using Exercise 2.1, we can explicitly find all points $[x : y : z] \in \mathbb{P}^2_{\mathbb{C}}$ such that the conic $\mathrm{pr}_1^{-1}([x : y : z])$ is singular. In particular, we see that they form a cubic curve $\mathscr{C} \subset \mathbb{P}^2_{\mathbb{C}}$, which is usually called the discriminant curve of the conic bundle pr_1. Since X is smooth, the cubic curve $\mathscr{C}$ is either smooth or nodal (has simplest singularities).

Observe that the curve $\mathscr{C}$ maybe reducible. However, in this case, the required assertion is well-known (see [20] or [5, § 10]). Thus, we will assume that the curve $\mathscr{C}$ is irreducible. Then it follows from Section 3.2 that we can choose coordinates x, y and z such that $\mathscr{C}$ is given by

$$\alpha x^3 + \beta y^3 + \gamma z^3 + \delta xyz = 0 \tag{13}$$

for some α, β, γ and δ such that $\alpha \neq 0$ and $\beta \neq 0$. To prove the required assertion, it is enough to choose the coordinates u, v, w such that X is given by the equation (12). In the following, we will not change the coordinates x, y and z except for scaling (once).

Let C_x, C_y, C_z be the fibers of the conic bundle pr_1 over $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1]$, respectively. Since $\mathscr{C}$ contains neither $[1 : 0 : 0]$ nor $[0 : 1 : 0]$, both C_x and C_y are smooth. In particular, we can choose u, v and w such that C_x is given by $vw + u^2 = y = z = 0$. Then X is given by

$$\big(vw + u^2\big)x + f_2(u, v, w)y + f_3(u, v, w)z = 0,$$

where $f_2(u, v, w)$ and $f_3(u, v, w)$ are some quadratic polynomials such that C_y is given by the equation $f_2(u, v, w) = x = z = 0$, and the curve C_z is given by $f_3(u, v, w) = x = y = 0$. Abusing notations, we consider all three curves C_x, C_y and C_z as conics in one plane $\mathbb{P}^2$, which are given by the equations $vw + u^2 = 0$, $f_2(u, v, w) = 0$, $f_3(u, v, w) = 0$, respectively. If $\mathscr{C}$ is singular, then $[0 : 0 : 1] = \mathrm{Sing}(\mathscr{C})$, so that C_z is a double line.

Observe that $C_x \cap C_y \cap C_z = \varnothing$, since X is smooth. But $C_x \cap C_y \neq \varnothing$ and $C_x \cap C_z \neq \varnothing$. Therefore, since $\mathrm{Aut}(\mathbb{P}^2; C_x) \cong \mathrm{PGL}_2(\mathbb{C})$ and this groups acts faithfully on $C_x \cong \mathbb{P}^1$, we can choose u, v and w such that $[0 : 0 : 1] \in C_y$ and $[0 : 1 : 0] \in C_z$. Then

$$\begin{aligned} f_2(u, v, w) &= a_1 v^2 + a_2 u^2 + a_3 vu + a_4 vw + a_5 uw, \\ f_3(u, v, w) &= b_1 w^2 + b_2 u^2 + b_3 vu + b_4 vw + b_5 uw, \end{aligned}$$

where a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5 are some numbers. Note that we still have some freedom in changing the coordinates u, v and w. Namely, the subgroup in $\mathrm{Aut}(\mathbb{P}^2; C_x)$ that preserves the subset $\{[0 : 0 : 1], [0 : 1 : 0]\}$ is $\mathbb{G}_m \rtimes \boldsymbol{\mu}_2$, where the $\mathbb{G}_m$-action is just the scaling $u \mapsto u, v \mapsto sv, w \mapsto \frac{w}{s}$ for $s \in \mathbb{C}^*$. Using this scaling, we could get the following new equation for our threefold:

$$\big(u^2+vw\big)x+\Big(s^2a_1v^2+a_2u^2+a_3svu+a_4vw+\frac{a_5}{s}uw\Big)y+\Big(\frac{b_1}{s^2}w^2+b_2u^2+sb_3vu+b_4vw+\frac{b_5}{s}uw\Big)z = 0,$$

where $a_1b_1 \neq 0$, since $C_x \cap C_y \cap C_z = \varnothing$. Thus, if $a_5 \neq 0$, we can scale v, w, x and z such that $a_1 = a_5 = b_1 = 1$. Similarly, if $a_5 \neq 0$, we can scale y and z to get $a_1 = b_1 = 1$. Therefore, we can assume that $a_1 = b_1 = 1$, and either $a_5 = 0$ or $a_5 = 1$. Note also that

$$2a_3a_4a_5 - 2a_5^2 - 2a_2a_4^2 \neq 0, \tag{14}$$

because the conic C_y is smooth.

Now, we compute the equation of the curve $\mathscr{C}$ using the equation of the threefold X. Namely, the curve $\mathscr{C}$ is given by

$$\begin{gathered}x^3-\big(a_3a_4a_5-a_2a_4^2-a_5^2\big)y^3-\big(b_3b_4b_5-b_2b_4^2-b_3^2\big)z^3-\\ -\big(4-2a_2b_4+a_3b_5-2a_4b_2-2a_4b_4+a_5b_3\big)xyz+\\ +\big(a_2+2a_4\big)x^2y-\big(b_2+2b_4\big)x^2z-\big(a_3a_5-2a_2a_4-a_4^2\big)xy^2-\big(b_3b_5-2b_2b_4-b_4^2\big)xz^2-\\ -\big(4a_2-2a_2a_4b_4+a_3a_4b_5+a_3a_5b_4-a_4^2b_2+a_4a_5b_3-a_3^2-2a_5b_5\big)y^2z-\\ -\big(4b_2-a_2b_4^2+a_3b_4b_5-2a_4b_2b_4+a_4b_3b_5+a_5b_3b_4-2a_3b_3-b_5^2\big)yz^2=0.\end{gathered}$$

Thus, since $\mathscr{C}$ is given by (13), we obtain the following system of equations:

$$\begin{cases}a_2+2a_4=0,\\ b_2+2b_4=0,\\ a_3a_5-2a_2a_4-a_4^2=0,\\ b_3b_5-2b_2b_4-b_4^2=0,\\ 4a_2-2a_2a_4b_4+a_3a_4b_5+a_3a_5b_4-a_4^2b_2+a_4a_5b_3-a_3^2-2a_5b_5=0,\\ 4b_2-a_2b_4^2+a_3b_4b_5-2a_4b_2b_4+a_4b_3b_5+a_5b_3b_4-2a_3b_3-b_5^2=0.\end{cases}$$

Substituting $a_2=-2a_4$ and $b_2=-2b_4$ into the third equation, we get $2a_3a_5+6a_4^2=0$. Hence, if $a_5=0$, then $a_4=0$, which contradicts (14). Therefore, we see that $a_5=1$. Then our system of equations simplifies as

$$\begin{cases}a_2=-2a_4,\\ b_2=-2b_4,\\ 3a_4^2+a_3=0,\\ b_3b_5+3b_4^2=0,\\ a_3a_4b_5+6a_4^2b_4-a_3^2+a_3b_4+a_4b_3-8a_4-2b_5=0,\\ a_3b_4b_5+a_4b_3b_5+6a_4b_4^2-2a_3b_3+b_3b_4-b_5^2-8b_4=0,\end{cases}$$

so that $a_3=-3a_4^2$. In particular, the threefold X is given my

$$\big(u^2+vw\big)x+\big(v^2+uw-3a_4^2uv-2a_4u^2+a_4vw\big)y+\big(b_3uv-2b_4u^2+b_4vw+b_5uw+w^2\big)z=0.$$

Now, we change our u, v and w as follows: $u\mapsto w-a_4v$, $v\mapsto v$, $w\mapsto 2a_4w-a_4^2v-u$. Then, in new coordinates, the threefold X is given by the equation:

$$\big(u^2+vw\big)x+\big(uw+av^2\big)y+\big(u^2+c_1w^2+c_2v^2+c_3vu+c_4vw+c_5uw\big)z=0,$$

where $a=a_4^3+1$ and

$$\begin{cases}c_1=4a_4^2+2a_4b_5-2b_4,\\ c_2=a_4^4+a_4^3b_5-3a_4^2b_4-a_4b_3,\\ c_3=-2a_4^2-a_4b_5+b_4,\\ c_4=-4a_4^3-3a_4^2b_5+6a_4b_4+b_3,\\ c_5=4a_4+b_5.\end{cases}$$

Now, recomputing again the equation of the cubic curve $\mathscr{C}$ in terms of a, c_1, c_2, c_3, c_4, c_5, we see that $\mathscr{C}$ is given by

$$x^3 + ay^3 + \big(c_1c_3^2 + c_2c_5^2 - c_3c_4c_5 - 4c_1c_2 + c_4^2\big)z^3 - \big(4ac_1 + c_3\big)xyz + \big(2c_4 + 1\big)x^2z+ \\ + \big(2ac_5 + c_2\big)y^2z - \big(4c_1c_2 + c_3c_5 - c_4^2 - 2c_4\big)xz^2 + \big(ac_5^2 - 4ac_1 + 2c_2c_5 - c_3c_4\big)yz^2 = 0.$$

As above, this gives us the following system of equations:

$$\begin{cases} 2c_4 + 1 = 0, \\ 2ac_5 + c_2, \\ 4c_1c_2 + c_3c_5 - c_4^2 - 2c_4 = 0, \\ ac_5^2 - 4ac_1 + 2c_2c_5 - c_3c_4 = 0, \end{cases}$$

so that $c_4 = -\frac{1}{2}$ and $c_2 = -2ac_5$. Substituting these equalities back to the system, we get

$$\begin{cases} c_4 = -\dfrac{1}{2}, \\ c_2 = -2ac_5, \\ -8ac_1c_5 + c_3c_5 + \dfrac{3}{4} = 0, \\ 3ac_5^2 + 4ac_1 - \dfrac{c_3}{2} = 0. \end{cases}$$

Then $c_3 = 6ac_5^2 + 8ac_1$. Substituting this into $-8ac_1c_5 + c_3c_5 + \frac{3}{4} = 0$, we get $6ac_5^3 + \frac{3}{4} = 0$. In particular, we see that $c_5 \neq 0$. Summarizing, we get

$$\begin{cases} c_5 \neq 0, \\ c_4 = -\dfrac{1}{2}, \\ c_2 = -2ac_5, \\ c_3 = 6ac_5^2 + 8ac_1, \\ a = -\dfrac{1}{8c_5^3}. \end{cases}$$

Therefore, our threefold X is given by the following equation:

$$\big(u^2 + vw\big)x + \Big(uw - \frac{v^2}{8c_5^3}\Big)y + \Big(c_1w^2 + \frac{v^2}{4c_5^2} - \frac{3uv}{4c_5} - \frac{uvc_1}{c_5^3} - \frac{vw}{2} + c_5uw + u^2\Big)z = 0.$$

Now, if we change u, v and w as $u \mapsto c_5(u+v+2w)$, $v \mapsto 4c5^2u + c_5^2v - 4c_5^2w$, $w \mapsto 2u - v + w$, then X would be given by

$$\big(u^2c_5^2 + c_5^2vw\big)x + \Big(c_5uw - \frac{v^2c_5}{8}\Big)y + \Big(\big(2c_5^2 + c_1\big)w^2 + \frac{(c_5^2 - 4c_1)vu}{4}\Big)z,$$

which is a special case of (12). This completes the proof of the claim. We encourage the reader to find a simpler proof of this claim.

Appendix

A.1 Resultant

Consider two cubic curves C_1 and C_2 which intersections consist of the points: $(2,3)$, $(-3,4)$, $(-4,-5)$, $(-6,2)$, $(5,3)$, $(3,2)$, $(-2,-6)$,*and* $(4,8)$. Theorem A.2.1 will tell us that either there exists a ninth intersection point or one of the eight points from before should be counted with multiplicity 2. In fact, we have a ninth intersection point and it is the point

$$\Big(\frac{1439767504290697562}{409942054104759719}, \frac{4853460637572644276}{409942054104759719}\Big). \tag{15}$$

How did we find the ninth intersection point? To construct this example, we chose the eight points $(2,3)$, $(-3,4)$, $(-4,-5)$, $(-6,2)$, $(5,3)$, $(3,2)$, $(-2,-6)$ and $(4,8)$. Then we found two cubic curves C_1 and C_2 that contain all of them. Namely, let C be a cubic curve that is given by

$$a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3 + a_5x^2 + a_6xy + a_7y^2 + a_8x + a_9y + a_{10} = 0,$$

where a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9 and a_{10} are some numbers. Suppose that C contains the points $(2,3)$, $(-3,4)$, $(-4,-5)$, $(-6,2)$, $(5,3)$, $(3,2)$, $(-2,-6)$ and $(4,8)$. This gives us 8 equations:

$$\begin{cases} 8a_1 + 12a_2 + 18a_3 + 27a_4 + 4a_5 + 6a_6 + 9a_7 + 2a_8 + 3a_9 + a_{10} = 0, \\ 36a_2 - 27a_1 - 48a_3 + 64a_4 + 9a_5 - 12a_6 + 16a_7 - 3a_8 + 4a_9 + a_{10} = 0, \\ 16a_5 - 80a_2 - 100a_3 - 125a_4 - 64a_1 + 20a_6 + 25a_7 - 4a_8 - 5a_9 + a_{10} = 0, \\ 72a_2 - 216a_1 - 24a_3 + 8a_4 + 36a_5 - 12a_6 + 4a_7 - 6a_8 + 2a_9 + a_{10} = 0, \\ 125a_1 + 75a_2 + 45a_3 + 27a_4 + 25a_5 + 15a_6 + 9a_7 + 5a_8 + 3a_9 + a_{10} = 0, \\ 27a_1 + 18a_2 + 12a_3 + 8a_4 + 9a_5 + 6a_6 + 4a_7 + 3a_8 + 2a_9 + a_{10} = 0, \\ 4a_5 - 24a_2 - 72a_3 - 216a_4 - 8a_1 + 12a_6 + 36a_7 - 2a_8 - 6a_9 + a_{10} = 0, \\ 64a_1 + 128a_2 + 256a_3 + 512a_4 + 16a_5 + 32a_6 + 64a_7 + 4a_8 + 8a_9 + a_{10} = 0. \end{cases}$$

Here, we consider a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9 and a_{10} as unknowns. Now observe the corresponding 8×10 matrix:

$$\begin{pmatrix} 8 & 12 & 18 & 27 & 4 & 6 & 9 & 2 & 3 & 1 \\ -27 & 36 & -48 & 64 & 9 & -12 & 16 & -3 & 4 & 1 \\ -64 & -80 & -100 & -125 & 16 & 20 & 25 & -4 & -5 & 1 \\ -216 & 72 & -24 & 8 & 36 & -12 & 4 & -6 & 2 & 1 \\ 125 & 75 & 45 & 27 & 25 & 15 & 9 & 5 & 3 & 1 \\ 27 & 18 & 12 & 8 & 9 & 6 & 4 & 3 & 2 & 1 \\ -8 & -24 & -72 & -216 & 4 & 12 & 36 & -2 & -6 & 1 \\ 64 & 128 & 256 & 512 & 16 & 32 & 64 & 4 & 8 & 1 \end{pmatrix}.$$

Its rank is 8, so that, by the rank-nullity theorem, the solutions form two-dimensional vector space. Therefore, if we find two linearly independent solutions, they must form a basis of this vector space. To get one solution, we use $a_8 = 11$ and $a_9 = -700$. Multiplying the resulting numbers a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9 and a_{10} by their common denominators, we obtain the polynomial:

$$\begin{aligned} &-5913252577x^3 + 30222000280x^2y - 21634931915xy^2 + 5556266591y^3 - 73906985473x^2 + \\ &+ 102209537669xy - 37300172365y^2 + 1389517162x - 88423819400y + 204616284808. \end{aligned}$$

Let us denote it by $f(x,y)$. Then C_1 is the curve given by $f(x,y) = 0$. Likewise, using $a_8 = 2$ and $a_9 = 1$, we get the second solution. It gives the polynomial

$$\begin{aligned} &-4844332x^3 - 8147864x^2y - 4067744xy^2 - 1866029y^3 + 32668904x^2 - \\ &\qquad - 28226008xy + 41719157y^2 + 252639484x + 126319742y - 960898976. \end{aligned}$$

Denote this polynomial by $g(x,y)$. Then C_2 is the curve given by $g(x,y) = 0$. Two obtained solutions are linearly independent. Thus, every cubic curve C that contains the points $(2,3)$, $(-3,4)$, $(-4,-5)$, $(-6,2)$, $(5,3)$, $(3,2)$, $(-2,-6)$ and $(4,8)$ is given by

$$\lambda f(x,y) + \mu g(x,y) = 0$$

for some numbers λ and μ. In particular, it contains all points of the intersection $C_1 \cap C_2$. It is now our task to find $C_1 \cap C_2$. How can we do this? Basically, we need only to solve the following system of equations:

$$\begin{cases} f(x,y) = 0, \\ g(x,y) = 0. \end{cases}$$

While this may seem daunting since it is, after all, a system of polynomials in two variables, it is exactly in situations such as this when the *resultant* method is extremely useful. In order to illustrate this method, let us consider both polynomials $f(x,y)$ and $g(x,y)$ as polynomials in y whose coefficients are polynomials in x. To be precise, we have $f(x,y) = a_3y^3 + a_2y^2 + a_1y + a_0$, where

$$\begin{cases} a_3 = 5556266591, \\ a_2 = -21634931915x - 37300172365, \\ a_1 = 30222000280x^2 + 102209537669x - 88423819400, \\ a_0 = 5913252577x^3 - 73906985473x^2 + 1389517162x + 204616284808, \end{cases}$$

and we have $g(x,y) = b_3y^3 + b_2y^2 + b_1y + b_0$, where

$$\begin{cases} b_3 = -1866029, \\ b_2 = -4067744x + 41719157, \\ b_1 = -8147864x^2 - 28226008x + 126319742, \\ b_0 = -4844332x^3 + 32668904x^2 + 252639484x - 960898976. \end{cases}$$

Then the resultant of the polynomials $f(x,y)$ and $g(x,y)$ (considered as polynomials in y) is the polynomial

$$R(f,g,y) = \det \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 \end{pmatrix} = \det \begin{pmatrix} f(x,y) & a_1 & a_2 & a_3 & 0 & 0 \\ yf(x,y) & a_0 & a_1 & a_2 & a_3 & 0 \\ y^2f(x,y) & 0 & a_0 & a_1 & a_2 & a_3 \\ g(x,y) & b_1 & b_2 & b_3 & 0 & 0 \\ yg(x,y) & b_0 & b_1 & b_2 & b_3 & 0 \\ y^2g(x,y) & 0 & b_0 & b_1 & b_2 & b_3 \end{pmatrix}.$$

Note that $R(f,g,y)$ depends only on x. Moreover, the last formula shows that

$$R(f,g,y) = a(x,y)f(x,y) + b(x,y)g(x,y)$$

for some polynomials $a(x,y)$ and $b(x,y)$. Thus, if $f(\alpha,\beta)=0$ and $g(\alpha,\beta)=0$ for some $(\alpha,\beta)\in\mathbb{C}^2$, then α must be a root of the resultant $R(f,g,y)$. On the other hand, we can expand the above expression for $R(f,g,y)$ and get

$$
\begin{aligned}
R(f,g,y) = {} & 319168411614335505141855887784472124841956719232716 9x^9 - \\
& - 80179076502326448020959208485535781077792914885854 93x^8 - \\
& - 19951895461883394788720945351923685301295332302821563 3x^7 + \\
& + 56880707484802669486621609640000274581156521359635915 7x^6 + \\
& + 38806142666086015230321945019845701520691647539989334 64x^5 - \\
& - 11708714303403885204269002049013593498191154175608876 232x^4 - \\
& - 27936678172063675450258473952703104020433424068758015 952x^3 + \\
& + 86672526536406322333733242006002412277456517441705929 808x^2 + \\
& + 61609026384389751204137037731562203601860663683619173 632x - \\
& - 19370174572297727746873020967216261287511627800617079 9360.
\end{aligned}
$$

It has 9 roots (counted with appropriate multiplicities) by Fundamental Theorem of Algebra. But we already know 8 of its roots: 2, 3, 4, 5, -6, -4, -3 and -2. These are x-coordinates of our first 8 points of the intersection $C_1\cap C_3$. Now, using Vieta's formulas (see [14]), we see that the sum of all 9 roots is equal to

$$\frac{8017907650232644802095920848553578107779291488585493}{3191684116143355051418558877844721248419567192327169},$$

so that the ninth root is $\frac{1439767504290697562}{409942054104759719}$. This is the x-coordinate of the ninth intersection point! Substituting it into $f(x,y)=0$ and $g(x,y)=0$, we can find the y-coordinate of this point.

Let us consider a slightly simpler example. Let

$$f(x,y)=x^2+xy+y^2-6x-2y+7$$

and let $g(x,y)=2x^2-xy-y^2-6x+5y+2$. Suppose now that C_1 is the conic $f(x,y)=0$, and C_2 is the conic $g(x,y)=0$. Let us find $C_1\cap C_2$ using resultants. First, we compute the resultant of the polynomials $f(x,y)$ and $g(x,y)$ considered as polynomials in x. Denote it by $R(f,g,x)$. Observe that

$$f(x,y)=x^2+(y-6)x+y^2-2y+7$$

and

$$g(x,y)=2x^2-(y+6)x-y^2+5y+2.$$

Thus, the polynomial $R(f,g,x)$ is the determinant of the following matrix:

$$\begin{pmatrix} y^2-2y+7 & y-6 & 1 & 0\\ 0 & y^2-2y+7 & y-6 & 1\\ -y^2+5y+2 & -y-6 & 2 & 0\\ 0 & -y^2+5y+2 & -y-6 & 2 \end{pmatrix}.$$

Computing it we get $R(f,g,x)=9y^4-9y^3-36y^2+72y-36$. Now we multiply the second column of the above matrix by x and add the result to the first column, then we multiply the third column of the resulted matrix by x^2 and add the result to the first column, and finally we multiply the fourth column of the resulted matrix by x^3 and add the result to the first column again. This gives the following matrix

$$\begin{pmatrix} f(x,y) & y-6 & 1 & 0 \\ xf(x,y) & y^2-2y+7 & y-6 & 1 \\ g(x,y) & -y-6 & 2 & 0 \\ xg(x,y) & -y^2+5y+2 & -y-6 & 2 \end{pmatrix}.$$

Its determinant is also equal to the resultant $R(f,g,x)$ (see [16]). Moreover, we can compute this determinant by expanding it along the first column:

$$f(x,y)\det\begin{pmatrix} y^2-2y+7 & y-6 & 1 \\ -y-6 & 2 & 0 \\ -y^2+5y+2 & -y-6 & 2 \end{pmatrix} - xf(x,y)\det\begin{pmatrix} y-6 & 1 & 0 \\ -y-6 & 2 & 0 \\ -y^2+5y+2 & -y-6 & 2 \end{pmatrix} +$$

$$+\, g(x,y)\det\begin{pmatrix} y-6 & 1 & 0 \\ y^2-2y+7 & y-6 & 1 \\ -y^2+5y+2 & -y-6 & 2 \end{pmatrix} + xg(x,y)\det\begin{pmatrix} y-6 & 1 & 0 \\ y^2-2y+7 & y-6 & 1 \\ -y-6 & 2 & 0 \end{pmatrix} =$$

$$= f(x,y)\Big(9y^2-6y-12\Big) - xf(x,y)\Big(6y-12\Big) + g(x,y)\Big(-15y+24\Big) + xh(x,y)\Big(-3y+6\Big) =$$

$$= f(x,y)\Big(9y^2-6xy+12x-6y-12\Big) + g(x,y)\Big(3xy-6x-15y+24\Big).$$

Let $a(x,y) = 9y^2-6xy+12x-6y-12$ and $b(x,y) = 3xy-6x-15y+24$. Then

$$a(x,y)f(x,y) + b(x,y)g(x,y) = R(f,g,x) = 9y^4-9y^3-36y^2+72y-36.$$

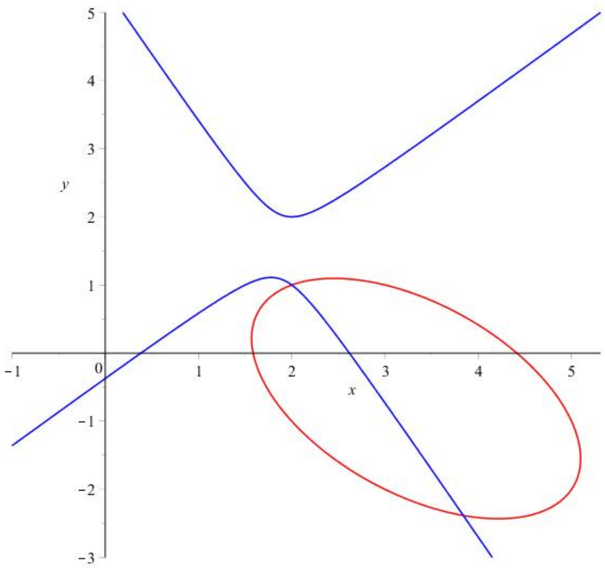
This shows that y-coordinates of all intersection points in $C_1 \cap C_2$ are roots of the polynomial equation $9y^4-9y^3-36y^2+72y-36=0$. By Fundamental Theorem of Algebra, it has 4 complex roots. Actually, we can guess one of its roots: $y=1$. This gives

$$9y^4-9y^3-36y^2+72y-36 = (y-1)(y^3-4y+4).$$

However, the polynomial y^3-4y+4 does not have nice looking (rational) roots. Using Computer Algebra System, one can show that it has only one real root $y \approx -2.3829$. Similarly, we can compute the resultant, which we denote by $R(f,g,y)$, of the polynomials $f(x,y)$ and $g(x,y)$ considered as polynomials in y. Namely, one has

$$R(f,g,y) = 9x^4-81x^3+261x^2-369x+198.$$

The roots of this polynomial are x-coordinates of the intersection points of our conics C_1 and C_2. One of them is $x=2$. Besides it, the resultant $R(f,g,x)$ has another real root $x \approx 3.8392$. This means that the intersection $C_1 \cap C_2$ contains only two real points. One of them is $(2,1)$, and another one is approximately $(3.8392,-2.3829)$. The remaining (two) points in $C_1 \cap C_2$ are complex (conjugate). We can plot the real part of the conics C_1 and C_2 to confirm this:



Exercise A.1.1. Find polynomials $a(x,y)$ and $b(x,y)$ such that the polynomial

$$a(x,y)f(x,y) + b(x,y)g(x,y)$$

does not depend on x, where $f(x,y)$ are $g(x,y)$ are the following polynomials:

(a) $f(x,y) = x^2 + xy + y^2 - 6x - 2y + 7$ and $g(x,y) = x^2 - 2xy - 2y^2 - 2x + 9y - 3$;

(b) $f(x,y) = x^2 + xy + y^2 - 3x - y - 5$ and $g(x,y) = x^2 + 2y^2 - 8$.

Resultants can be also applied to polynomials in one variable. For example, since x^3 and $(x-1)^4$ do not have common roots, we can find polynomials $a(x)$ and $b(x)$ such that

$$a(x)x^3 + b(x)\big(x-1\big)^2 = 1.$$

This follow from Hilbert's Nullstellensatz (see [17]). How to find $a(x)$ and $b(x)$? We can mimic (extended) Euclidean algorithm for polynomials. Indeed, dividing x^3 by $x^2 - 2x + 1$, we get

$$x^3 = \big(x^2 - 2x + 1\big)\big(x + 2\big) + 3x - 2.$$

Then dividing $x^2 - 2x + 1$ by $3x - 2$, we get

$$x^2 - 2x + 1 = \big(3x - 2\big)\Big(\frac{x}{3} - \frac{4}{9}\Big) + \frac{1}{9}.$$

This gives

$$\frac{1}{9} = \big(x^2 - 2x + 1\big) - \Big(x^3 - \big(x^2 - 2x + 1\big)\big(x + 2\big)\Big)\Big(\frac{x}{3} - \frac{4}{9}\Big).$$

Simplifying and multiplying by 9, we obtain

$$1 = \big(3x^2 + 2x + 1\big)\big(x^2 - 2x + 1\big) + x^3\big(-3x + 4\big),$$

so that we have $a(x) = -3x + 4$ and $b(x) = 3x^2 + 2x + 1$. We can find the same polynomials also by brute force. Indeed, let $a(x) = Ax + B$ and $b(x) = Cx^2 + Dx + E$ for some numbers A, B, C, D and E. If $a(x)x^3 + b(x)(x-1)^2 = 1$, then

$$\big(A + C\big)x^4 + \big(B - 2C + D\big)x^3 + \big(C + E - 2D\big)x^2 + \big(D - 2E\big)x + E = 1,$$

so that $A + C = 0$, $B - 2C + D = 0$, $C + E - 2D = 0$, $D - 2E = 0$ and $E = 1$. This gives $A = -3$, $B = 4$, $C = 3$, $D = 2$ and $E = 1$, so that $a(x) = -3x + 4$ and $b(x) = 3x^2 + 2x + 1$ as before. Moreover, we can find the same polynomials $a(x)$ and $b(x)$ using resultants. Namely, the resultant of the polynomials $x^2 - 2x + 1$ and x^3 is

$$\det\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = 1.$$

Multiplying ith column by x^{i-1} and adding it to the first one, we get

$$\det\begin{pmatrix} x^2 - 2x + 1 & -2 & 1 & 0 & 0 \\ x^3 - 2x^2 + x & 1 & -2 & 1 & 0 \\ x^4 - 2x^3 + x^2 & 0 & 1 & -2 & 1 \\ x^3 & 0 & 0 & 1 & 0 \\ x^4 & 0 & 0 & 0 & 1 \end{pmatrix} = 1.$$

Expanding this determinant along the first column, we obtain

$$1 = \Big(x^2 - 2x + 1\Big) + 2\Big(x^3 - 2x^2 + x\Big) + 3\Big(x^4 - 2x^3 + x^2\Big) + 4x^3 - 3x^4,$$

so that $(3x^2 + 2x + 1)(x^2 - 2x + 1) + (-3x + 4)x^3 = 1$ again. Which method is better?

A.2 Bezout's Theorem

Let L and L' be two distinct lines in $\mathbb{P}^2_{\mathbb{C}}$. Then the intersection $L \cap L'$ consists of a single point by Exercise 1.7. Similarly, if $\mathcal{C}$ is an irreducible conic in $\mathbb{P}^2_{\mathbb{C}}$, then the intersection $L \cap \mathcal{C}$ is not empty and consists of at most two points (see Section 2). Moreover, the intersection $L \cap \mathcal{C}$ consists of a single point if and only if the line L is tangent to the curve $\mathcal{C}$ at this point, so that this point should be counted in $L \cap \mathcal{C}$ with multiplicity 2. In Section 2.4, we proved that two irreducible conics in $\mathbb{P}^2_{\mathbb{C}}$ always intersect each other by 4 points if we count them with appropriate multiplicities, and we provided many examples that illustrate this fact.

Let $\mathcal{C}$ and $\mathcal{C}'$ be two irreducible conics in $\mathbb{P}^2_{\mathbb{C}}$. Then both of them are smooth (see Definition 2.19). Moreover, arguing as in the proposed solution of Exercise 2.14, one can show that the intersection $\mathcal{C} \cap \mathcal{C}'$ consists of exactly 4 points if and only if the conics $\mathcal{C}$ and $\mathcal{C}'$ intersect each other *transversally* at each point of the intersection $\mathcal{C} \cap \mathcal{C}'$. This means the following: for every point P in $\mathcal{C} \cap \mathcal{C}'$, the tangent lines to the conics $\mathcal{C}$ and $\mathcal{C}'$ at the point P are different. We can use this fact to define the same notion for any two curves that are both smooth at their intersection point.

The examples we described are part of a much more general assertion known as Bezout's theorem. To state its algebraic form, we fix non-constant homogeneous polynomials $f(x, y, z)$ and $g(x, y, z)$. Let d be the degree of the polynomial $f(x, y, z)$, and let $\widehat{d}$ be the degree of the polynomial $g(x, y, z)$. Consider the following system of polynomial equations:

$$\begin{cases} f(x, y, z) = 0 \\ g(x, y, z) = 0 \end{cases} \tag{16}$$

How many solutions in $\mathbb{P}^2_{\mathbb{C}}$ does it have? Infinite if $f(x, y, z)$ and $g(x, y, z)$ have a common factor. However, if $f(x, y, z)$ and $g(x, y, z)$ do not have common factors, then (16) has finitely many solutions in $\mathbb{P}^2_{\mathbb{C}}$. In fact, we can say more:

Theorem A.2.1 (Bezout)**.** Suppose that $f(x, y, z)$ and $g(x, y, z)$ do not have common factors. Then the following assertions hold:

- (16) has at least one solution in $\mathbb{P}^2_{\mathbb{C}}$;
- (16) has at most $d\widehat{d}$ solutions in $\mathbb{P}^2_{\mathbb{C}}$;
- (16) has $d\widehat{d}$ solutions in $\mathbb{P}^2_{\mathbb{C}}$ if we count them with appropriate multiplicities.

This is an algebraic form of Bezout's theorem. To make it more explicit, we should explain what we mean by *counted with appropriate multiplicities*. We have done this throughout the document but to do the same in general case, we have to introduce the quite technical notion of *intersection multiplicities*. Before doing this, let us consider one very handy application of Theorem A.2.1, which we implicitly used in the proposed solution of Exercise 2.1.

Lemma A.2.2. Suppose that the system of polynomial equations

$$\begin{cases} \dfrac{\partial f(x,y,z)}{\partial x} = 0 \\ \dfrac{\partial f(x,y,z)}{\partial y} = 0 \\ \dfrac{\partial f(x,y,z)}{\partial z} = 0 \end{cases}$$

does not have solutions in $\mathbb{P}^2_{\mathbb{C}}$. Then $f(x, y, z)$ is irreducible.

Proof. Suppose that $f(x, y, z)$ is not irreducible. Then

$$f\big(x, y, z\big) = g(x, y, z)h(x, y, z),$$

where $g(x, y, z)$ and $h(x, y, z)$ are some homogeneous polynomials of positive degrees. By Theorem A.2.1, there is a point $[a : b : c] \in \mathbb{P}^2_{\mathbb{C}}$ such that

$$\begin{cases} g(a, b, c) = 0, \\ h(a, b, c) = 0. \end{cases}$$

Since

$$\frac{\partial f(x,y,z)}{\partial x} = \frac{\partial g(x,y,z)}{\partial x} h(x, y, z) + g(x, y, z)\frac{\partial h(x,y,z)}{\partial x} = 0,$$

we see that $[a : b : c]$ is a solution to the polynomial equation

$$\frac{\partial f(x,y,z)}{\partial x} = 0.$$

In this case, we say that the polynomial $\frac{\partial f(x,y,z)}{\partial x}$ *vanishes* at the point $[a : b : c]$. Similarly, we see that $\frac{\partial f(x,y,z)}{\partial y}$ and $\frac{\partial f(x,y,z)}{\partial z}$ also vanish at the point $[a : b : c]$, which contradicts our assumption. $\square$

Now we are about to formally introduce *intersection multiplicities*. To do this, we suppose that the polynomials $f(x, y, z)$ and $g(x, y, z)$ do not have common factors. Then the system (16) defines a non-empty finite subset in $\mathbb{P}^2_{\mathbb{C}}$ that consists of at most $d\widehat{d}$ points. To describe it more geometrically, let C be the subset in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$f(x, y, z) = 0,$$

and let Z be the subset in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$g(x,y,z)=0.$$

Then $C\cap Z$ is the solution to (16). We call C and Z *subsets*, because $f(x,y,z)$ and $g(x,y,z)$ can be pretty bad in general. Say, we may have $f(x,y,z)=x^d$ and $g(x,y,z)=y^{\widehat{d}}$.

For every $P\in C\cap Z$, we can define a positive integer $(f,g)_P$, known as intersection multiplicity, as follows. First, just for simplicity, we apply an appropriate projective transformation to simplify the coordinates of the point P. Thus, without loss of generality, we may assume that $P=[0:0:1]$. Now we let $\overline{C}=C\cap U_z$ and $\overline{Z}=Z\cap U_z$, where we have $U_z=\mathbb{C}^2$ with coordinates $\overline{x}=\frac{x}{z}$ and $\overline{y}=\frac{y}{z}$. Then $P=(0,0)$ in U_z, the subset $\overline{C}$ is given by the equation

$$f(\overline{x},\overline{y},1)=0,$$

and the subset $\overline{Z}$ is given by the equation

$$g(\overline{x},\overline{y},1)=0.$$

Taking Taylor decomposition of the polynomial $f(\overline{x},\overline{y},1)$ at $(\overline{x},\overline{y})=(0,0)$, we can rewrite it as

$$f(\overline{x},\overline{y},1)=h_1\big(\overline{x},\overline{y}\big)+h_2\big(\overline{x},\overline{y}\big)+\cdots+h_d\big(\overline{x},\overline{y}\big)=0,\tag{17}$$

where each $h_i(\overline{x},\overline{y})$ is a homogeneous polynomial of degree i. Likewise, we have

$$g(\overline{x},\overline{y},1)=\widehat{h}_1\big(\overline{x},\overline{y}\big)+\widehat{h}_2\big(\overline{x},\overline{y}\big)+\cdots+\widehat{h}_{\widehat{d}}\big(\overline{x},\overline{y}\big)=0,\tag{18}$$

where each $\widehat{h}_i(\overline{x},\overline{y})$ is a homogeneous polynomial of degree i. If $h_1(\overline{x},\overline{y})$ and $\widehat{h}_1(\overline{x},\overline{y})$ are non zero-polynomials that define different lines in $U_z=\mathbb{C}^2$, then we let

$$\big(f,g\big)_P=1.$$

The geometrical meaning of this is the following: if the curves C and Z intersect each other transversally at the point P, then the point P should be counted with multiplicity 1 in the intersection $C\cap Z$. Intuitively, this makes perfect sense.

In general, to define the intersection multiplicity $(f,g)_P$, we have to consider localization of the ring $\mathbb{C}[\overline{x},\overline{y}]$ at the maximal ideal $\langle\overline{x},\overline{y}\rangle$. This ring, let us call it $\mathbf{R}$, can be defined as a subring in in the field of all rational functions $\mathbb{C}(\overline{x},\overline{y})$ that consists of all fractions

$$\frac{a(\overline{x},\overline{y})}{b(\overline{x},\overline{y})}$$

with $a(\overline{x},\overline{y})$ and $b(\overline{x},\overline{y})$ in $\mathbb{C}[\overline{x},\overline{y}]$ such that $b(P)\neq 0$. Informally speaking, the ring $\mathbf{R}$ consists of all fractions in $\mathbb{C}(\overline{x},\overline{y})$ that are well-defined at P. For instance, we have

$$\frac{\overline{x}}{\overline{y}+\overline{x}+1}\in\mathbf{R},$$

while $\frac{\overline{x}}{\overline{y}+\overline{x}}$ and $\frac{1}{\overline{x}}$ are not contained in $\mathbf{R}$. Recall that we assume that $P=(0,0)$.

The ring $\mathbf{R}$ contains exactly three prime ideals: the zero ideal, the whole ring $\mathbf{R}$, and the maximal ideal $\langle\overline{x},\overline{y}\rangle$. However, the ring $\mathbf{R}$ contains many other ideals. In particular, it contains the ideal

$$\big\langle f(\overline{x},\overline{y},1),g(\overline{x},\overline{y},1)\big\rangle,$$

where we consider the polynomials $f(\overline{x},\overline{y},1)$ and $g(\overline{x},\overline{y},1)$ as elements of the ring $\mathbf{R}$. Denote the later ideal by $\mathbf{I}$. Then the quotient ring $\mathbf{R}/\mathbf{I}$ is a finite dimensional vector space over $\mathbb{C}$. This follows from the fact that the polynomials $f(x,y,z)$ and $g(x,y,z)$ do not have common factors. Now we let

$$\big(f,g\big)_P=\dim_{\mathbb{C}}\Big(\mathbf{R}/\mathbf{I}\Big). \tag{19}$$

This definition implies that

$$\big(f,g\big)_P=\big(g,f\big)_P.$$

Moreover, if $h_1(\overline{x},\overline{y})$ in (17) and $\widehat{h}_1(\overline{x},\overline{y})$ in (18) are non-zero linear polynomials that define two different lines in $U_z=\mathbb{C}^2$, then our definition also gives $(f,g)_P=1$. This follows from

Exercise A.2.3. Suppose that $h_1(\overline{x},\overline{y})$ and $\widehat{h}_1(\overline{x},\overline{y})$ are linearly independent. Show that $(f,g)_P=1$.

The solution to Exercise A.2.3 shows that even the simplest properties of intersection multiplicities are not easy to prove. Because of this, we will not prove them. Let us just present some of them in the following proposition, whose proof can be found in Bill Fulton's book [8].

Proposition A.2.4. Let $h(x,y,z)$ be a homogeneous polynomial such that $f(x,y,z)$ and $h(x,y,z)$ do not have common factors. If $h(P)=0$, then

$$\big(f,gh\big)_P=\big(f,g\big)_P+\big(f,h\big)_P.$$

Likewise, if $h(P)\neq 0$, then

$$\big(f,gh\big)_P=\big(f,g\big)_P.$$

Let $m_P(f)$ be the smallest i in (17) such that $h_i(\overline{x},\overline{y})$ is not a zero-polynomial, and let $m_P(g)$ be the smallest i in (18) such that $\widehat{h}_i(\overline{x},\overline{y})$ is not a zero-polynomial. Then

$$(f,g)_P\geqslant m_P(f)m_P(g).$$

Now we can restate Theorem A.2.1 as

$$\sum_{P\in C\cap Z}\big(f,g\big)_P=d\widehat{d}, \tag{20}$$

where $(f,g)_P\geqslant 1$, since we assumed earlier that $f(P)=0$ and $g(P)=0$.

Remark A.2.5. In our definition of the intersection multiplicity $(f,g)_P$, we never used the assumption that $f(P)=0$ and $g(P)=0$. Indeed, if $f(P)\neq 0$ or $g(P)\neq 0$, then (19) gives $(f,g)_P=0$. Thus, we can rewrite (20) as

$$\sum_{P\in\mathbb{P}^2_{\mathbb{C}}}\big(f,g\big)_P=d\widehat{d}.$$

This formula implies all three assertions of Theorem A.2.1.

One can prove Bezout's theorem using *resultants*, which were briefly explained in Section A.1. Moreover, the examples we considered in Section A.1 illustrate this proof and the algorithm of finding all solutions to (16). However, the full proof of Bezout's theorem is way beyond the scope of this book, so we will not give it here (see [8, 12]).

Exercise A.2.6. Let $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ be a map given by

$$\big[x : y : z\big] \mapsto \big[u(x,y,z) : v(x,y,z) : w(x,y,z)\big]$$

for some homogeneous polynomials u, v, w of degree d. Suppose that the system the system

$$\begin{cases} u(x,y,z) = 0 \\ v(x,y,z) = 0 \\ w(x,y,z) = 0 \end{cases}$$

does not have solutions in $\mathbb{P}^2_{\mathbb{C}}$, so that ϕ is *well defined*. Suppose that the map ϕ is a bijection. Use Theorem A.2.1 to give an indication that $d = 1$, which simply means that ϕ is a projective transformation (see Section 1.2).

One special case of Bezout's theorem follows easily from Fundamental Theorem of Algebra. Namely, if $f(x,y,z)$ is a homogeneous polynomial of degree d, and $g(x,y,z)$ is a homogeneous polynomial of degree 1 that does not divide $f(x,y,z)$, then (16) has d solutions counted with multiplicities. Indeed, applying appropriate projective transformation, we may assume that $g(x,y,z) = z$. Then (16) simplifies as

$$\begin{cases} z = 0, \\ f(x,y,0) = 0. \end{cases}$$

On the other hand, it follows from Fundamental Theorem of Algebra that

$$f(x,y,0) = \prod_{i=1}^{d} h_i(x,y)$$

for some polynomials $h_1(x,y), \ldots, h_d(x,y)$ of degree 1. We have

$$h_i(x,y) = \alpha_i x + \beta_i y$$

for some complex numbers α_i and β_i such that $(\alpha_i, \beta_i) \neq (0,0)$. This gives us d solutions

$$\big[\beta_1 : -\alpha_1 : 0\big], \big[\beta_2 : -\alpha_2 : 0\big], \ldots, \big[\beta_d : -\alpha_d : 0\big],$$

which should be counted with multiplicities.

Exercise A.2.7. Let $f(x,y,z)$ be a homogeneous polynomial of degree d, and let $g(x,y,z)$ is an irreducible homogeneous polynomial of degree 2 that does not divide the polynomial $f(x,y,z)$. Prove that (16) has at least one solution in $\mathbb{P}^2_{\mathbb{C}}$. Show that it has $2d$ solutions in $\mathbb{P}^2_{\mathbb{C}}$ if we count them with appropriate multiplicities.

Let us illustrate this exercise by solving (16) in the case when $f(x,y,z)$ is a polynomial of degree 3, and $g(x,y,z)$ is a polynomial of degree 2 that does not divide $f(x,y,z)$. In this case, we may assume that $g(x,y,z) = xy - z^2$ (see Section 2.1). One the other hand, we have

$$f(x,y,z) = A_1x^3 + A_2x^2y + A_3xy^2 + A_4y^3 + A_5x^2z + A_6xyz + A_7y^2z + A_8xz^2 + A_9yz^2 + A_{10}z^3$$

where A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9 and A_{10} are some complex numbers such that not all of them are zero. Then (16) can be rewritten as

$$\begin{cases} A_1x^3 + A_2x^2y + A_3xy^2 + A_4y^3 + A_5x^2z + A_6xyz + A_7y^2z + A_8xz^2 + A_9yz^2 + A_{10}z^3 = 0, \\ xy - z^2 = 0, \end{cases}$$

If $y = 0$, then this system has a solution in $\mathbb{P}^2_{\mathbb{C}}$ if and only if $A_1 = 0$. Moreover, if $A_1 = 0$, then the only solution with $y = 0$ is the point $[1 : 0 : 0]$. Vice versa, if $y \neq 0$, then we can rewrite (16) as

$$\begin{cases} A_1x^3 + A_2x^2 + A_3x + A_4 + A_5x^2z + A_6xz + A_7z + A_8xz^2 + A_9z^2 + A_{10}z^3 = 0, \\ x - z^2 = 0, \\ y = 1. \end{cases}$$

This system can be simplified as

$$\begin{cases} A_1z^6 + A_5z^5 + (A_2 + A_8)z^4 + (A_6 + A_{10})z^3 + (A_3 + A_9)z^2 + A_7z + A_4 = 0, \\ x - z^2 = 0, \\ y = 1, \end{cases}$$

Note that at least one number among A_1, A_5, $A_2 + A_8$, $A_6 + A_{10}$, $A_3 + A_9$, A_7, A_4 is not zero. Indeed, if

$$\begin{cases} A_1 = 0, \\ A_5 = 0, \\ A_2 + A_8 = 0, \\ A_6 + A_{10} = 0, \\ A_3 + A_9 = 0, \\ A_7 = 0, \\ A_4 = 0, \end{cases}$$

then

$$A_1x^3+A_2x^2+A_3x+A_4+A_5x^2z+A_6xz+A_7z+A_8xz^2+A_9z^2+A_{10}z^3 = \Big(A_2x+A_3y+A_6z\Big)\Big(xy-z^2\Big),$$

which is impossible by assumption: the polynomial $g(x, y, z)$ does not divide $f(x, y, z)$. If $A_1 \neq 0$, then the polynomial equation

$$A_1z^6 + A_5z^5 + (A_2 + A_8)z^4 + (A_6 + A_{10})z^3 + (A_3 + A_9)z^2 + A_7z + A_4 = 0 \tag{21}$$

has exactly 6 roots (counted with multiplicities) by Fundamental Theorem of Algebra, so that the system (16) has $2d = 6$ solutions:

$$\big[\zeta_1^2 : 1 : \zeta_1\big], \big[\zeta_2^2 : 1 : \zeta_2\big], \big[\zeta_3^2 : 1 : \zeta_3\big], \big[\zeta_4^2 : 1 : \zeta_4\big], \big[\zeta_5^2 : 1 : \zeta_5\big], \big[\zeta_5^2 : 1 : \zeta_5\big],$$

where ζ_1, ζ_2, ζ_3, ζ_4, ζ_5 and ζ_6 are (possibly repeated) roots of the polynomial equation (21).

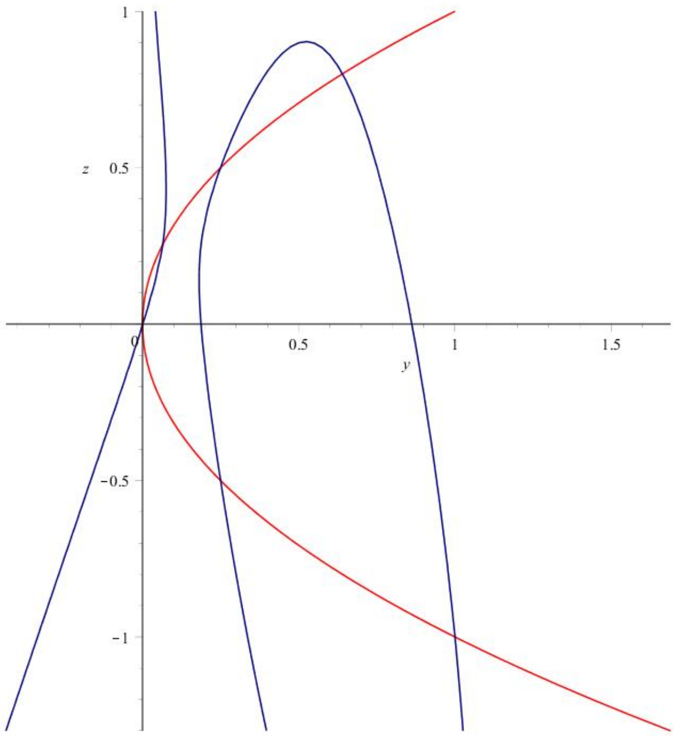
If $A_1 = 0$ and $A_5 \neq 0$, then (21) gives us 5 possibilities for z, which gives 5 solutions to (16). In this case the point $[1 : 0 : 0]$ is also a solution to (16), so that we have 6 solutions in total provided that we count then with multiplicities. For instance, if

$$f(x, y, z) = -1516x^2y+468x^2z+9828xy^2-2106xyz-473xz^2-9360y^3+468y^2z+468yz^2+117z^3,$$

then (16) has exactly 6 solutions in $\mathbb{P}^2_{\mathbb{C}}$, which are the points

$$[1 : 0 : 0],\ [4 : 1 : 2],\ [3 : 1 : 9],\ [16 : 1 : 4],\ [1 : 1 : -1] \text{ and } [4 : 1 : -2].$$

Each of these points is counted with multiplicity 1 in (16), so that we say that the corresponding curves intersect *transversally* at them. This can be seen on the following real picture:



This picture displays the corresponding curves and their six intersection points in the part of the projective plane $\mathbb{P}^2_{\mathbb{C}}$ which is given by $x \neq 1$ with coordinates $\mathbf{y} = \frac{y}{x}$ and $\mathbf{z} = \frac{z}{x}$. The red curve is the conic $\mathbf{y} = \mathbf{z}^2$, and the blue curve is the smooth cubic curve

$$-1516\mathbf{y} + 468\mathbf{z} + 9828\mathbf{y}^2 - 2106\mathbf{yz} - 473\mathbf{z}^2 - 9360\mathbf{y}^3 + 468\mathbf{y}^2\mathbf{z} + 468\mathbf{yz}^2 + 117\mathbf{z}^3 = 0.$$

If $A_1 = 0$, $A_5 = 0$ and $A_2 + A_8 \neq 0$, then (21) has 4 solutions (counted with multiplicities). In this case, the curves $xy = z^2$ and $f(x, y, z) = 0$ tangent at the point $[1 : 0 : 0]$, so that it should be counted with multiplicity 2. For instance, if

$$f(x,y,z) = 275x^2y+1925xy^2+1925xyz+1925xz^2+325920y^3-46280y^2z-92205yz^2+9645z^3,$$

then we have exactly 6 solutions to (16) in $\mathbb{P}^2_{\mathbb{C}}$, which are the points

$$[1 : 0 : 0] \text{ (counted twice)}, [4 : 1 : 2], [3 : 1 : 9], [16 : 1 : 4] \text{ and } [4 : 1 : -2].$$

If $A_1 = 0$, $A_5 = 0$ and $A_2 + A_8 = 0$, then (21) has at most 3 solutions, and the intersection multiplicity of the polynomials $xy - z^2$ and $f(x, y, z)$ at the point $[1 : 0 : 0]$ is at least 3. In the remaining cases, one can show that that (16) also has 6 solutions in $\mathbb{P}^2_{\mathbb{C}}$ if we count them with appropriate multiplicities.

Bezout's Theorem has many applications. For example, one can use it to solve the following exercise, which is the Steiner–Lehmus theorem in elementary geometry.

Exercise A.2.8. Prove that every triangle with two angle bisectors of equal lengths is isosceles.

Let us conclude this section by translating Theorem A.2.1 into language of projective geometry. Thanks to Proposition A.2.4, it is enough to do this for irreducible curves in $\mathbb{P}^2_{\mathbb{C}}$.

Because of this, we assume that both polynomials $f(x,y,z)$ and $g(x,y,z)$ are irreducible. Then C and Z are irreducible curves in $\mathbb{P}^2_{\mathbb{C}}$. We let

$$\big(C \cdot Z\big)_P = \big(f, g\big)_P$$

for every point $P \in C \cap Z$. Then Theorem A.2.1 gives

Theorem A.2.9 (Bezout)**.** Suppose that $f(x,y,z) \neq \lambda g(x,y,z)$ for any $\lambda \in \mathbb{C}$. Then

$$1 \leqslant \big|C \cap Z\big| \leqslant \sum_{P \in C \cap Z} \big(C \cdot Z\big)_P = d\widehat{d}.$$

Corollary A.2.10. $C = Z \iff f(x,y,z) = \lambda g(x,y,z)$ for some $\lambda \in \mathbb{C}$.

We say that C intersects Z transversally at $P \in C \cap Z$ if the following two conditions are satisfied:

1. both curves C and Z are smooth at the point P,
2. the lines in $\mathbb{P}^2_{\mathbb{C}}$ that are tangent to C and Z at the point P are different.

By Exercise A.2.3, we have $(C \cdot Z)_P = 1$ if and only if the curve C intersects the curve Z transversally at the point P. Thus, it follows from Theorem A.2.9 that

$$\big|C \cap Z\big| = d\widehat{d} \iff C \text{ intersects } Z \text{ transversally at every point of the intersection } C \cap Z.$$

In particular, if $|C \cap Z| = d\widehat{d}$, then C does not contain singular points of the curve Z, and the curve Z does not contain singular points of the curve C.

Exercise A.2.11. Suppose that $f(x,y,z) = xy^3 + yz^3 + zx^3$ and

$$g(x,y,z) = \det \begin{pmatrix} \frac{\partial^2 f(x,y,z)}{\partial x \partial x} & \frac{\partial^2 f(x,y,z)}{\partial x \partial y} & \frac{\partial^2 f(x,y,z)}{\partial x \partial z} \\ \frac{\partial^2 f(x,y,z)}{\partial y \partial x} & \frac{\partial^2 f(x,y,z)}{\partial y \partial y} & \frac{\partial^2 f(x,y,z)}{\partial y \partial z} \\ \frac{\partial^2 f(x,y,z)}{\partial z \partial x} & \frac{\partial^2 f(x,y,z)}{\partial z \partial y} & \frac{\partial^2 f(x,y,z)}{\partial z \partial z} \end{pmatrix}.$$

Prove that $f(x,y,z)$ and $g(x,y,z)$ are irreducible. Show that C and Z are smooth curves, and

$$3 \leqslant |C \cap Z| \leqslant 24.$$

Check that $[0:0:1] \in C \cap Z$. Show that C and Z intersect each other transversally at $[0:0:1]$. Find $L \cap C$, where L is the line in $\mathbb{P}^2_{\mathbb{C}}$ that is tangent to C at $[0:0:1]$.

Note that Proposition A.2.4 also implies that

$$(C \cdot Z)_P \geqslant \mathrm{mult}_P\big(C\big)\mathrm{mult}_P\big(Z\big).$$

Thus, if P and Q are two different points of the curve C and $d \geqslant 2$, then

$$\mathrm{mult}_P\big(C\big) + \mathrm{mult}_Q\big(C\big) \leqslant d.$$

Indeed, let L be a line in $\mathbb{P}^2_{\mathbb{C}}$ that passes through P and Q. Then $L \neq C$, so that

$$d = \sum_{O \in L \cap C} \big(L \cdot C\big)_O \geqslant \big(L \cdot C\big)_P + \big(L \cdot C\big)_Q \geqslant \mathrm{mult}_P\big(C\big) + \mathrm{mult}_Q\big(C\big).$$

Here, d is the degree of the curve C, which we assume to be irreducible.

Corollary A.2.12. An irreducible conic in $\mathbb{P}^2_{\mathbb{C}}$ is smooth.

Corollary A.2.13. An irreducible cubic curve in $\mathbb{P}^2_{\mathbb{C}}$ has at most one singular point.

Corollary A.2.14. If an irreducible quartic curve in $\mathbb{P}^2_{\mathbb{C}}$ has a singular point of multiplicity three, then it does not have other singular points.

Exercise A.2.15. Prove that an irreducible quartic curve in $\mathbb{P}^2_{\mathbb{C}}$ has at most three singular points.

A.3 Cubic Curves as Groups

Let $\mathcal{C}_3$ be a smooth cubic curve in $\mathbb{P}^2_{\mathbb{C}}$, and let O be a point in the curve $\mathcal{C}_3$. Then we can equip $\mathcal{C}_3$ with an addition operation $+$ such that for every points A and B in $\mathcal{C}_3$, there exists unique point

$$A + B \in \mathcal{C}_3$$

and the following four properties holds.

(♣) For every points A, B and C in the curve $\mathcal{C}_3$, one has

$$\boxed{(A+B)+C = A+(B+C).}$$

This property is known as *associativity*.

(♢) For every point A in the curve $\mathcal{C}_3$, one has

$$\boxed{A+O = O+A = A.}$$

The point O is usually called *identity element* or *identity* or *zero*.

(♡) For every point $A \in \mathcal{C}_3$ there is a point $B \in \mathcal{C}_3$ such that

$$\boxed{A+B = B+A = O.}$$

This point B is usually denoted by $-A$. It is often called the *inverse* of the point A.

(♠) For every points A and B in the curve $\mathcal{C}_3$, one has

$$\boxed{A+B = B+A.}$$

This property is known as *commutativity*.

Thus, the pair $(\mathcal{C}_3, +)$ is an abelian (commutative) group.

Given two points A and B in the cubic curve $\mathcal{C}_3$, there is an explicit algorithm for constructing the point $A+B$ in $\mathcal{C}_3$. Namely, for two points A and B in $\mathcal{C}_3$, we define $A+B$ as follows:

- If $A \neq B$, let L be the unique line in the plane $\mathbb{P}^2_{\mathbb{C}}$ that passes through the points A and B. If $A = B$, let L be the line in $\mathbb{P}^2_{\mathbb{C}}$ that is tangent to $\mathcal{C}_3$ at the point $A = B$.
- Then $L \cap \mathcal{C}_3$ consists of the points A, B and some *third* point P. Then

  $$P = A \iff L \text{ is tangent to } \mathcal{C}_3 \text{ at } A.$$

  Similarly, we have $P = B$ if and only if the line L is tangent to the curve $\mathcal{C}_3$ at the point B. If $P = A = B$, then L is tangent to $\mathcal{C}_3$ at P, which is an inflection point of the curve $\mathcal{C}_3$. In all possible cases, the *third* point P is uniquely determined by A and B.

- If $P \neq O$, let L' be the unique line in the plane $\mathbb{P}^2_{\mathbb{C}}$ that passes through the points P and O. If $P = O$, let L' be the line in $\mathbb{P}^2_{\mathbb{C}}$ that is tangent to the curve $\mathcal{C}_3$ at the point O.
- The intersection $L' \cap \mathcal{C}_3$ consists of the points P, O and *third* point Q. Then

  $$Q = P \iff \text{if and only if } L' \text{ is tangent to } \mathcal{C}_3 \text{ at the point } P.$$

  Similarly, we have $Q = O$ if and only if the line L' is tangent to the curve $\mathcal{C}_3$ at the point O. If $Q = P = O$, then L' tangents $\mathcal{C}_3$ at the point O that is an inflection point of the curve $\mathcal{C}_3$.
- The point Q is uniquely determined by the points A and B. We let $A + B = Q$.

Let us illustrate this algorithm by one explicit example. Let $\mathcal{C}_3$ be the curve given by

$$x^3 + y^3 + z^3 + 4xyz = 0.$$

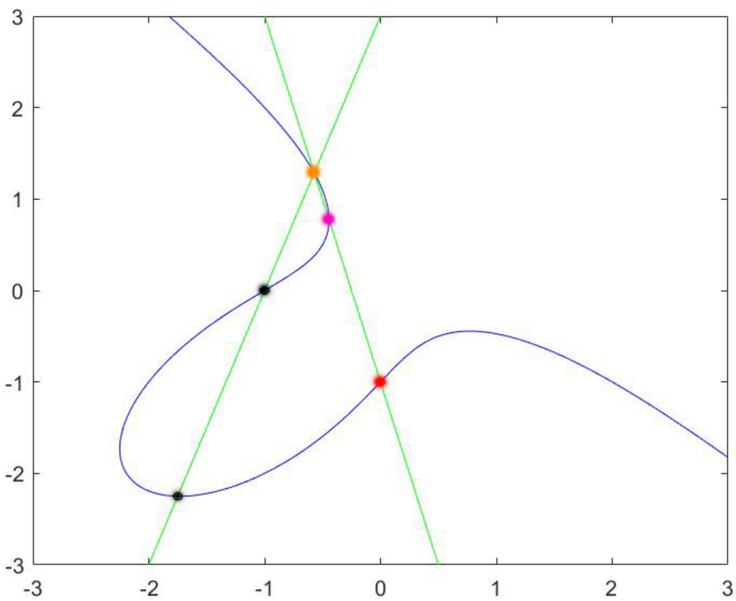
Then $\mathcal{C}_3$ is smooth by Exercise 3.4, and it contains $[0 : 1 : -1]$, $[-1 : 0 : 1]$ and $[-7 : -9 : 4]$. Let $O = [0 : 1 : -1]$, $A = [-1 : 0 : 1]$, $B = [-7 : -9 : 4]$. The line L containing A and B is given by

$$3x - y + 3z = 0,$$

and the intersection $L \cap \mathcal{C}_3$ consists of three distinct points: the points A, B and $P = [-4 : 9 : 7]$. Similarly, the line L' containing P and O is given by

$$4x + y + z = 0,$$

and the intersection $L' \cap \mathcal{C}_3$ consists of the points P, O, $Q = [7 : -4 : 9]$. Then $A + B = [-4 : 7 : 9]$. In the chart $z \neq 1$, this can be illustrated by the following real picture:



Here, the blue curve is our cubic curve, the red point is the point O, the black points are A and B, the green lines are the lines L and L', the orange point is P, and the pink point is the point $A + B$.

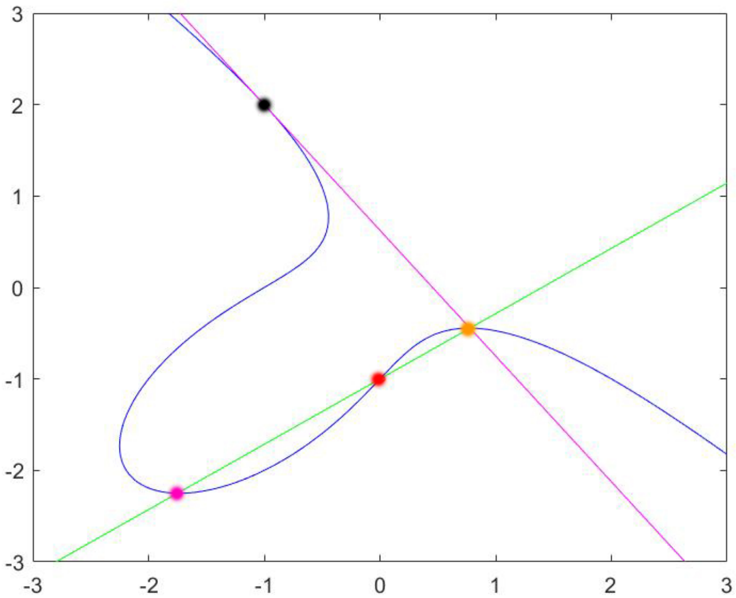
Let us show how to compute $2A = A + A$. As above, we let $O = [0 : 1 : -1]$ and $A = [-1 : 2 : 1]$. Then the tangent line in $\mathbb{P}^2_{\mathbb{C}}$ to the curve $\mathcal{C}_3$ at the point A is given by

$$11x + 8y - 5z = 0.$$

This line intersects $\mathcal{C}_3$ by the point A (counted with multiplicity 2) and the point $P = [7 : -4 : 9]$. The line L' containing $[7 : -4 : 9]$ and O is given by

$$5x - 7y - 7z = 0,$$

and $L' \cap \mathcal{C}_3$ consists of the points $[7 : -4 : 9]$, O and $[7 : 9 : -4]$. This shows that $2A = [7 : 9 : -4]$. In the chart $z \neq 0$, our computations can be seen on the following picture:



Here the blue curve is our curve $\mathcal{C}_3$, the red point is the point O, the black point is the point A, the purple line is the tangent line to the curve $\mathcal{C}_3$ at the point A, the green line is the line L', the orange point is P, and the pink point is $2A$.

Exercise A.3.1. Let $\mathcal{C}_3$ be the smooth cubic curve in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$zy^2 = x^3 + 17z^3.$$

Let $A = [0 : 1 : 0]$, $B = [-1 : 4 : 1]$ and $C = [2 : 5 : 1]$. Then $\mathcal{C}_3$ contains the points A, B and C. Let O be one of these points. Equip $\mathcal{C}_3$ with an operation $+$ such that O is the identity element.

(a) Compute $B + C$, $B + B$ and $C + C$ in the case when $O = A$.

(b) Compute $A + C$, $A + A$ and $C + C$ in the when $O = B$.

Let us prove that $\mathcal{C}_3$ equipped with the addition operation $+$ has 4 properties described above. Let L_O be the line tangent to $\mathcal{C}_3$ at the point O. Then $L_O \cap \mathcal{C}_3$ consists of the following points:

- the point O (counted with multiplicity at least two),
- the *third* point $\widehat{O}$, which may coincide with the point O.

Remember the following:

$$\boxed{O = \widehat{O} \iff O \text{ is an inflection point of the curve } \mathcal{C}_3.}$$

The addition $+$ is commutative by construction, so that (♠) holds. Let us prove (♢).

Exercise A.3.2. Let A be a point in $\mathcal{C}_3$. Show that $A + O = O + A = A$.

Now let us prove that the addition $+$ has property (♡).

Exercise A.3.3. Let A be a point in $\mathcal{C}_3$. Show that there exists a point $B \in \mathcal{C}_3$ such that $A + B = O$.

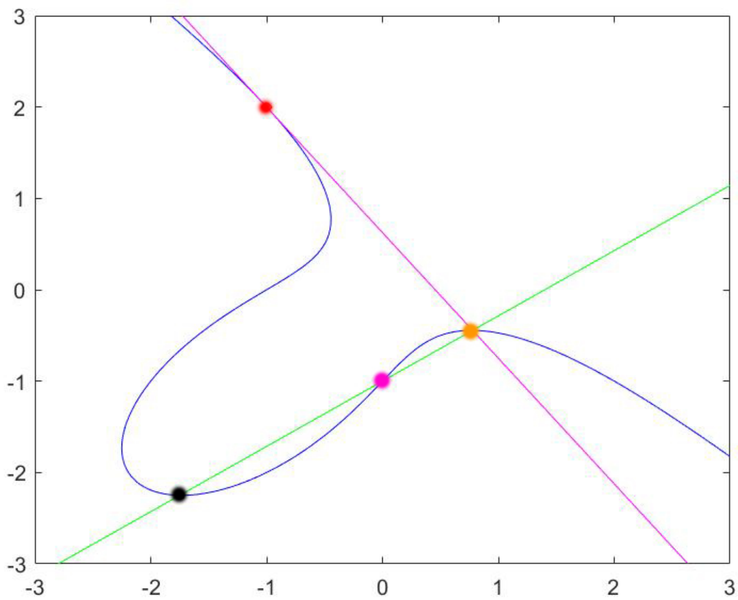
Let us illustrate Lemma A.3.3. As above, let $\mathcal{C}_3$ be the curve given by $x^3+y^3+x^3+4xyz = 0$. This curve contains the points $[-1 : 2 : 1]$ and $[7 : 9 : -4]$. Let $O = [-1 : 2 : 1]$ and $A = [7 : 9 : -4]$. The tangent line to $\mathcal{C}_3$ at the point O is given by

$$11x + 8y - 5z = 0.$$

It intersects the curve $\mathcal{C}_3$ at the point O (counted with multiplicity 2) and the point $P = [7 : -4 : 9]$. Then the line L' containing $[7 : -4 : 9]$ and A is given by the equation

$$5x - 7y - 7z = 0.$$

Then $L' \cap \mathcal{C}_3$ consists of the points $[7 : -4 : 9]$, A and $[0 : -1 : 1]$. This gives $-A = [0 : -1 : 1]$. This can be illustrated by the following picture:



Here the blue curve is the curve $\mathcal{C}_3$, the red point is the point O, the black point is the point A, the purple line is the line that is tangent to the curve $\mathcal{C}_3$ at the point O, the green line is L', the orange point is the point P, and the pink point is $-A$.

Finally, let us show that the addition $+$ is associative. This means that $+$ has property (♣).

Exercise A.3.4. Let A, B and C be three points in $\mathcal{C}_3$. Show that $(A+B)+C = A+(B+C)$.

Exercise A.3.5. Let $O = [0 : 1 : 1]$, $A = [2 : 9 : 8]$, $B = [72 : 611 : 1]$, $C = [-10332 : 40879 : 46656]$, and let $\mathcal{C}_3$ be the smooth cubic curve in $\mathbb{P}^2_{\mathbb{C}}$ given by

$$zy^2 - x^3 - xz^2 - z^3 = 0.$$

Then $\mathcal{C}_3$ contains O, A, B and C. Equip it with an operation $+$ such that O is the identity element. Compute $A + B$, $A + C$, $B + C$, $2A$, $2B$, $2C$, $-B$, $-A$, $-C$.

If O is an inflection point of the curve $\mathcal{C}_3$, then the addition $+$ is simpler than usual. In this case, for every three points A, B and C on the curve $\mathcal{C}_3$ we have

$$\boxed{A + B + C = O \iff \text{the points } A,\ B \text{ and } C \text{ are collinear.}}$$

Being collinear means that there exists a line L such that $L \cap \mathcal{C}_3$ consists of the points A, B and C. Here, we do not assume that the points A, B and C are distinct. In particular, this gives

Corollary A.3.6. Equip the cubic curve $\mathcal{C}_3$ with an addition $+$ such that O is the identity element. Suppose that O is an inflection point of this curve. Then $3A = O \iff A$ is an inflection points.

Remark A.3.7. Any point P in the cubic curve $\mathcal{C}_3$ equipped with an addition $+$ that satisfies $nP = O$ for some positive integer n is called a *torsion point* of $\mathcal{C}_3$. Thus an inflection point of a cubic curve is always a torsion point of the cubic curve.

Exercise A.3.8. Let $\mathcal{C}_3$ be a smooth cubic curve in $\mathbb{P}^2_{\mathbb{C}}$ that is given by (9). Let $O = [0 : 1 : 0]$. Equip $\mathcal{C}_3$ with an operation $+$ such that O is the identity element. Prove that

$$-[\alpha : \beta : \gamma] = [\alpha : -\beta : \gamma]$$

for every point $[\alpha : \beta : \gamma]$ in the curve $\mathcal{C}_3$.

Let us consider one example. Let $\mathcal{C}_3$ be the cubic curve in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$zy^2 = x^3 - zx^2 - 4xz^2 + 4z^3.$$

Let $O = [0 : 1 : 0]$, $A = [1 : 0 : 1]$ and $B = [0 : 2 : 1]$. Then we can find $A + B$ as follows.

- The line containing A and B is $2x+y-2z = 0$. It intersects $\mathcal{C}_3$ by A, B and $[4 : -6 : 1]$.
- Then $A + B + [4 : -6 : 1] = O$, so that $A + B = -[4 : -6 : 1] = [4 : 6 : 1]$.

Now let us find $n[4 : 6 : 1]$ for small n. This can be done as follows.

- The tangent line to $\mathcal{C}_3$ at $[4 : 6 : 1]$ is given by

  $$3x - y - 6z = 0.$$

  It intersects $\mathcal{C}_3$ by $[4 : 6 : 1]$ and $[2 : 0 : 1]$, so that $2[4 : 6 : 1] = [2 : 0 : 1]$.

- The line $3x - y - 6z = 0$ contains $[4 : 6 : 1]$ and $[2 : 0 : 1]$, and it tangents $\mathcal{C}_3$ at $[4 : 6 : 1]$. Then

  $$O = 2\big[4 : 6 : 1\big] + 2\big[4 : 6 : 1\big] = 4\big[4 : 6 : 1\big],$$

In this case, we say that $[4 : 6 : 1]$ is a point of *order* 4.

Exercise A.3.9. Let $\mathcal{C}_3$ be the smooth cubic curve in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$zy^2 = yz^2 + x^3 - x^2 z.$$

Let $O = [1 : 1 : 1]$ and $P = [0 : 1 : 0]$. Observe that the cubic curve $\mathcal{C}_3$ contains both these points. Equip $\mathcal{C}_3$ with an addition $+$ such that O is an identity element. Show that $5P = O$.

For every point P in the curve $\mathcal{C}_3$, we say that P is a point of order n if

$$nP = \underbrace{P + P + \cdots + P + P}_{n \text{ times}} = O,$$

where n is the smallest natural number such that this equality holds.

Exercise A.3.10. Show that every smooth cubic curve in $\mathbb{P}^2_{\mathbb{C}}$ equipped with an addition operation contains at most 3 points of order two. Try to prove that there are exactly 3 such points.

In fact, every smooth cubic curve equipped with an addition operation has 3 points of order two. This is proved in the solution to Exercise A.3.10, where we also show how to find all such points.

Exercise A.3.11. Let $\mathcal{C}_3$ be the cubic curve in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$zy^2 = x(x - z)(x - 2z).$$

Observe that $[0 : 1 : 0] \in \mathcal{C}_3$. Equip $\mathcal{C}_3$ with an addition $+$ such that $[0 : 1 : 0]$ is an identity element. Find all points of order 2 in the curve $\mathcal{C}_3$.

We equipped smooth cubic curves with additions $+$, so that we can "add" points on them. Can we do the same with irreducible singular cubic curves? Yes. But we must avoid singular points. From Exercise 3.13, we know that there are just two such curves (up to projective transformations). One of them (nodal cubic) is given by

$$zxy + x^3 + y^3 = 0.$$

This curve is singular at $[0 : 0 : 1]$. The second (cuspidal cubic) is given by

$$z^2 y = x^3.$$

It is singular at $[0 : 1 : 0]$. We can equip the sets of smooth points of these curves with the addition operations absolutely in the same way as we did for smooth cubic curves.

Exercise A.3.12. Let $\mathcal{C}_3$ be the irreducible cubic curve in $\mathbb{P}^2_{\mathbb{C}}$ that is given by the equation $z^2 y = x^3$. Let $O = [0 : 0 : 1]$. Equip $\mathcal{C}_3 \setminus [0 : 1 : 0]$ with an addition $+$ such that O is an identity element. Construct a function $\phi\colon \mathcal{C}_3 \setminus [0 : 1 : 0] \to \mathbb{C}$ such that

$$\phi\big(A + B\big) = \phi\big(A\big) + \phi\big(B\big)$$

for all points A and B in $\mathcal{C}_3 \setminus [0 : 1 : 0]$, where $\phi(A) + \phi(B)$ is an addition of numbers.

Therefore, as a group, irreducible cuspidal cubic curve looks like $\mathbb{C}$ with usual addition operation. A similar result holds for irreducible nodal cubic curves:

Exercise A.3.13. Let $\mathcal{C}_3$ be the irreducible cubic curve in $\mathbb{P}^2_{\mathbb{C}}$ that is given by $zxy+x^3+y^3 = 0$. Let $O = [1 : -1 : 0]$. Equip $\mathcal{C}_3 \setminus [0 : 0 : 1]$ with an addition $+$ such that O is an identity element. Construct a function $\phi\colon \mathcal{C}_3 \setminus [0 : 0 : 1] \to \mathbb{C}^*$ such that

$$\phi\big(A + B\big) = \phi\big(A\big)\phi\big(B\big)$$

for all points A and B in $\mathcal{C}_3 \setminus [0 : 0 : 1]$, where $\phi(A)\phi(B)$ is a multiplication of numbers.

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