An assortment of theory relating to the Euler Equations on Manifolds

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Abstract

ABSTRACT

Acknowledgements

Thanks to a lot of people

Contents

Chapter 1

Motivation, Introduction and Background

> Never reduce on the Hamiltonian and never code your own numerical methods.
>
> — Darryl D. Holm

The area of Geometric Mechanics is based around the idea of deriving equations from symmetries baked into physical systems. In most physical systems these symmetries are left unexploited and lead to a variety of theory and further information about our systems that is very useful. We can derive so called conserved quantities of these physical systems. Conserved quantities give insight into how the system behaves and further can help when we attempt to solve these systems numerically.

Historically the area can be traced back to least action principles applied to mechanical systems by Maurperius in 1744. Then optics used the ideas later, and the boom of the use of least action principle by Euler, Liebnitz and Lagrange was started. Lagrange deduced the Euler-Lagrange Equations in the 1750s,

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} = 0.$$

Hamilton deduced the 'Hamilton' least action principle in 1835,

$$\delta \int_{t_1}^{t_2} F[\mathbf{x}] \; \mathrm{d}\mathbf{x} = 0.$$

From the Hamilton's principle you can derive equations from a Lagrangian. Then in 1901, Poincaré wrote down a new set of equations, called the Euler-Poincaré equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta \ell}{\delta \xi} - \mathrm{ad}^*_\xi \frac{\delta \ell}{\delta \xi} = 0.$$

While what we call the Lagrangian framework was developing, there was a conncurent development doing on for the Hamiltonian framework. The history here is more convoluted as the path to understanding was rewritten several times. This framework takes a Lagrangian, then considers a Legendre transform of the Lagrangian. This involves using some Lagranage multipliers, which in turn can be written as Clebsch variables, which then leads to Hamilton's Equations. The canonical Hamilton's equations using the trivial Clebsch variables,

$$\frac{\partial \mathbf{q}}{\partial t} = \frac{\partial H}{\partial \mathbf{p}} \qquad \frac{\partial \mathbf{p}}{\partial t} = -\frac{\partial H}{\partial \mathbf{q}}.$$

If we consider these equations in higher generality. That is, written on some sort of manifold. Not only do Hamilton's equations conserve the Hamiltonian, they also conserve the canonical symplectic form.

$$\omega = \mathrm{d}p_i \wedge \mathrm{d}q_i$$

where p_i and q_i are local coordinates. By a quick calculation we can verify that,

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial q} := \{F,\, H\},$$

where H is the Hamiltonian. You can write an Hamiltonian system in this way where you write the Lie-Poisson bracket on the right hand side. This leads to the study of conserved quantities when the Lie bracket vanishes. These are Casimirs.
One of the current main figures in modern Geometric Mechanics is Professor Darryl Holm at Imperial College. Holm was inspired by work done before him by Poisson, Lie, Arnold, Marsden, Axton and Hamilton to name a few. His idea was to move forward and use the theory before him and to apply this, conglomerate it and present a new area of mathematics. Geometric Mechanics. Trained as a Physicist, Darryl presented ideas on Geometry in his work at Los Alamos labs. He then was then elected as the first director of the Nonlinear Systems Institute after presenting how Geometry can be used to create new solution criterion to equations of interest.

From there the idea spread further, with Darryl writing his infamous book, known as 'The Green Book' to all Geometric Mechanists. While Darryl continues to work at the coalface new ideas were developing and new mathematicians were starting work. The ideas of Sympelectic Geometry were starting to be developed into the area by Bridges. Further the applications of this work to Numerical Analysis were starting to form with mathematicians such as Cotter and Gay-Balmaz hearing of Geometric Mechanics and getting interested about the conservation of quantities in numerical schemes and how this relates to Geometric Mechanics.

Then the question of industrial applications was raised with the team of Professor Tom Bridges, Dr Matt Turner and Dr Hamid Alemi Ardakani. They were interested in the application of this work in wave energy generators. They studied new variational principles, similar to the Hamilton's Principle, such as the Luke-Bateman principle,

$$\delta \int_{t_1}^{t_2} \int_{\mathcal{Q}} -\rho \left(\frac{\partial \Phi}{\partial t} + \frac{1}{2}\nabla\Phi\cdot\nabla\Phi - \nabla\Phi\cdot(\mathbf{v}_0 + \boldsymbol{\omega}\times\mathbf{r}) + U \right)\, \mathrm{d}\mathcal{Q}\; \mathrm{d}t = 0.$$

These are more general and to consider more complicated fluid interactions, such as coupling and free surfaces. This is still an ongoing project, and one that the author will be joining next year under Dr Hamid Alemi Ardakani.

We now can move forward and consider this thesis. Alongside this work, Arnold wrote a lot of mathematics relating the Ideal Fluid Euler Equations and Euler-Poincaré reduction. In this thesis we will review that work, then use other sources and work forward to review and create new theory for the Euler Equations on general manifolds. One of our main examples of the thesis will be to recreate the paper of Vanneste [24] in terms of Geometric Mechanics. Then we will consider the derivation of the 2D Euler Equations from the 3D Euler Equations and then consider conserved quantities. The aim of this thesis is two fold; to understand and expose the current research being done in Geometric Mechanics and to fill holes in the theory by adding examples.

The motivation of using this type of mathematics is as follows. Many PDE systems can be described in a coordinate dependant environment. The derivation of these equations is usually found from considering conservation laws and then fitting terms to that, then integrating and pulling equations out. This assumes our system already has these conserved quantities. These derivations are made quite difficult when we consider when complex chemical behaviour. These conservation laws are no longer as obvious. Hence we propose to derive these 'standard' equations from just considering the energy. We consider the Lagrangian or the Hamiltonian,

$$L(q,\dot{q}) = K(q,\dot{q}) - V(q,\dot{q}) \qquad H(q,\dot{q}) = K(q,\dot{q}) + V(q,\dot{q}).$$

Then we can take a system, say a spherical pendulum, and derive the Hamiltonian and Lagrangian. This example is taken from Arthur's 'Lie Groups and Applications to Geometric Mechanics' [5]. Consider a spherical pendulum as in the figure below.

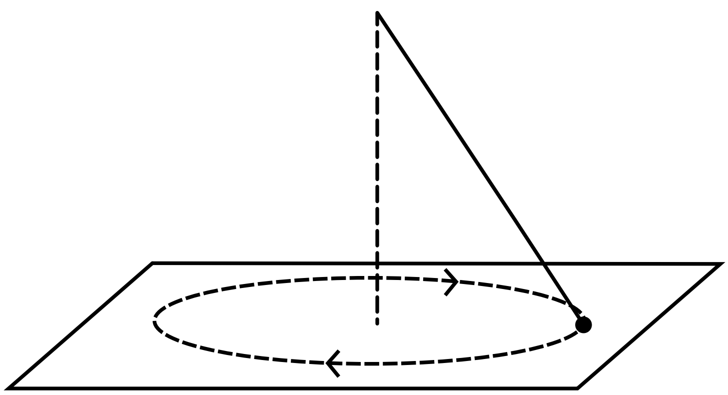

Figure 1.1: Spherical Pendulum.

Then we can define the Lagrangian of this system as,

$$L(\mathbf{x},\dot{\mathbf{x}}) = \frac{1}{2}m|\dot{\mathbf{x}}|^2 - mg\mathbf{x}\cdot\mathbf{e}_3.$$

We are interested in considering the motion from the spatial frame. We can use a body to space map through, $\mathbf{x} \mapsto \mathbf{R}(t)X$, where $\mathbf{x}, X \in \mathbb{R}^3$ and $\mathbf{R}(t) \in \mathrm{SO}(3)$. We can then move the Lagrangian from the body frame to the spatial frame,

$$L(\mathbf{R},\dot{\mathbf{R}}) = \frac{1}{2}m|\dot{\mathbf{R}}(t)X|^2 - mg\mathbf{R}(t)X\cdot\mathbf{e}_3. \tag{1.1}$$

Then by manipulating the Lagrangian (1.1), we can now rewrite it again letting $\Gamma(t) = \mathbf{R}(t)\mathbf{e}_3$,

$$L(\mathbf{R}, \dot{\mathbf{R}}, \boldsymbol{\Gamma}) = \frac{1}{2}m|\dot{\mathbf{R}}X|^2 - mg\boldsymbol{\Gamma} \cdot X.$$

Then we can notice that this Lagrangian with respect to symmetric part, the first term, is right symmetric and has a symmetry breaking parameter. Therefore we take $g \mapsto g\mathbf{R}^{-1}$. So,

$$L(\mathbf{R}\mathbf{R}^{-1}, \dot{\mathbf{R}}\mathbf{R}^{-1}, \tilde{\boldsymbol{\Gamma}}) := \ell(\hat{\boldsymbol{\Omega}}, \tilde{\boldsymbol{\Gamma}}) = \frac{1}{2}m\mathbb{I}|\hat{\boldsymbol{\Omega}}|^2 - mg\tilde{\boldsymbol{\Gamma}} \cdot X,$$

where $\mathbb{I} = |X|^2 I - XX^T$. Then you can plug this into the Euler-Poincaré equations for right invariant systems on SO(3) and get,

$$\mathbb{I}\hat{\boldsymbol{\Omega}}_t = \mathbb{I}\hat{\boldsymbol{\Omega}} \times \boldsymbol{\Omega} + mgX \times \tilde{\boldsymbol{\Gamma}}.$$

The thesis will be structured as follows,

- This first chapter will lay down the introductory material for the theory we will use later. This part of the thesis is critical for readers not well versed in Geometric Mechanics as it lays out the operators and functionals we use without definition in the rest of the thesis. The sections in this chapter cover; Fundamental Differential Geometry, Exterior Calculus, Symplectic Geometry and Geometric Fluid Dynamics.

- Then we will move on to study the mechanics and PDEs for different fluid systems on manifolds. We will write these PDEs in the language of exterior calculus. We will study both the Lagrangian framework and the Hamiltonian framework, through Euler-Poincaré reduction and Lie-Poisson reduction. We will derive the associated equations and introduce the canonical Lie-Poisson bracket. We will then move on to describe equations with advected quantities and give some motivation for the EP-Diff equations.

- Finally to complete the theory, we will then study the conservation laws of our systems with Noether and Casimir Theory. We will prove the generalised Noether Theorem and Kelvin-Noether Theorem for the Euler-Poincaré equations on SDiff(M) and then study the Casimirs and introduce the non-canonical Lie-Poisson bracket.

- To conclude the thesis we will then study several examples relating to the theory. Such as axisymmetric fluid flow on a cylinder, derivation of 2D Eulers Equations from 3D, Fluid flow on a sphere and then the Möbius strip.

1.1 Introduction to Geometric Fluid Dynamics

We will follow Chapter 1 of Arnold and Khesin's 'Topological Methods in Hydrodynamics' [26] and more generally Holm, Schmah and Stoica's 'Geometric Mechanics and Symmetry' [13].

We will start with some definitions relating to Group Theory and Lie Theory. We can define a group,

Definition 1.1.1 (Group on a Manifold)**.** *A set G of smooth transformations of a manifold M onto itself is a group if,*

1. *Given two transformations $g, h \in G$, the composition $g \circ h$ is in G,*
2. *Given some $g \in G$, the inverse, g^{-1}, is also in G.*

We define the group like this because then the definition of the Lie Group becomes obvious. If the functions induced by the two conditions in Definition 1.1.1 are smooth then we have a *Lie Group*. There are some pertinent examples of Lie Groups that are useful to mechanics. For rigid body dynamics, we tend to use the Lie Group of length preserving rotations, SO(3). For a detailed survey of the Geometric Mechanics of SO(3) see [5]. For hydrodynamics however we are interested in the volume preserving diffeomorphisms, $\mathrm{SDiff}(M)$.

Definition 1.1.2 (Diffeomorphism)**.** *Let M and N be manifolds, then a diffeomorphism is a bijective map $\phi : M \to N$ such that both ϕ and ϕ^{-1} are differentiable. The group of diffeomorphisms are denotes* $\mathrm{Diff}(M)$*.*

We can further extend this to volume preserving diffeomorphisms,

Definition 1.1.3 (Volume Preserving Diffeomorphism)**.** *Let $\phi : M \to N$ be a diffeomorphism and $\mathrm{d}\mu$ be the volume element of M. Then we say ϕ is volume preserving if, $\phi(\mathrm{d}\mu)$ is the volume element of N. The group of volume preserving diffeomorphisms are denoted* $\mathrm{SDiff}(M)$*.*

We say that the volume preserving diffeomorphism describe fluid. We mean that the flow of a particle can be described by a diffeomorphism. Given a terminal point we can describe all flows of the fluid particle by a diffeomorphism, they show the path the particle takes. We can see this in Figure 2.1.

As an example we can define the kinetic energy of a particle in a fluid as the following,

$$E = \frac{1}{2}\int_M \varphi_t^2 \,\mathrm{d}x.$$

This turns out to be nicely right invariant. Let $G = \mathrm{SDiff}(M)$. If we perform the operation, $R_h : G \to G$, defined by $R_h(g) = gh$, on the kinetic energy, we can write it as the following,

$$R_h E = \frac{1}{2}\int_M (\phi_t h)^2 \,\mathrm{d}x.$$

We note that h is volume preserving, then it is really just a relabelling of the fluid particles and so $R_h E = E$.

We also note the left multiplication by h can be written as, $L_h(g) = hg$.

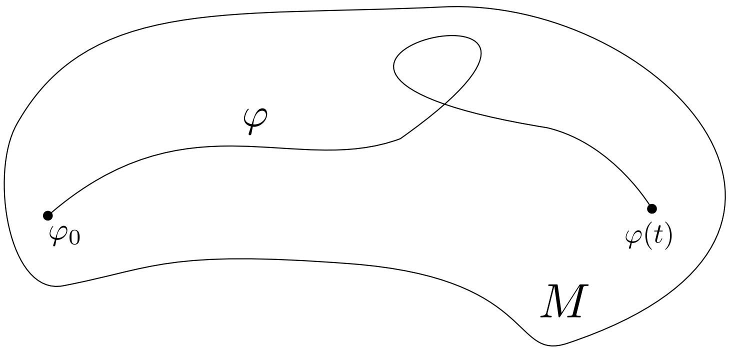

Figure 1.2: The path of a fluid particle.

1.1.1 Adjoint, Coadjoints and Group Actions

We spend the next subsection generalising and reviewing the content from [5] but for structure preserving diffeomorphisms. We shall define the inner automorphism that will quickly lead to the group adjoint the adjoint representation. We shall also discover the nature of the Lie Algebra of $\mathrm{SDiff}(M)$.

We can consider the L_h and the R_h operators we defined before and define the inner automorphism.

Definition 1.1.4 (Inner Automorphism)**.** *The inner automorphism, $A_g : G \to G$ is defined by, $A_h = L_h R_{h^{-1}}$. This can also be written, given some $g \in G$, $A_h g = hgh^{-1}$.*

This can then be used to talk about the adjoints of the group and algebra. For $\mathrm{SDiff}(M)$, our Lie algebra is going to be the space of divergence free vector fields in M. We also quickly need to note that to differentiate a map on a manifold we denote it as follows. Given some $F : M \to M$, then the differential at a point $x \in M$ is $F_*|_x : T_x M \to T_{F(x)} M$. We can now define our Ad_g and ad_g on $\mathrm{SDiff}(M)$ and $\mathfrak{g}$.

Definition 1.1.5 (Group Adjoint Operator)**.** *The differential of A_g at the unit is called the group adjoint operator,* Ad_g,

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g} \qquad \mathrm{Ad}_g \xi = (A_{g*}|_e)\xi \qquad\qquad \xi \in \mathfrak{g} = T_e G$$

We can further define the adjoint orbit,

Definition 1.1.6 (Adjoint Orbit)**.** *Fix $\xi \in \mathfrak{g}$. The set of images of $\mathrm{Ad}_g \xi$ of ξ under the action of Ad_g, $g \in G$, is called the adjoint orbit of ξ.*

If $g \in \mathrm{SDiff}(M)$, then $\mathrm{Ad}_g \xi = g\xi g^{-1}$ and is just the group of structure preserving diffeomorphisms acting on a vector field. Now if we let $g(0) = e$ and $\dot{g}(0) = \eta$. Then we can define the adjoint representation of the Lie algebra,

Definition 1.1.7 (Adjoint Representation of the Lie algebra)**.** *If we take the differential of* Ad *at the identity, we have the adjoint representation of the Lie algebra,*

$$\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \qquad \mathrm{ad}_\xi = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}_{g(t)} \, .$$

We can now take the derivative of the group adjoint on $\mathrm{SDiff}(M)$. We can then get that $\mathrm{ad}_\xi \eta = -[\xi, \eta]$ for $\xi, \eta \in \mathfrak{X}(M)$ such that $\mathrm{div}\, \xi = \mathrm{div}\, \eta = 0$.

We can further consider the coadjoint versions of the above. These are maps to the cotangent space and the dual to the Lie algebra. The dual to the Lie algebra can be described for arbitrary dimension. Let $G = \mathrm{SDiff}(M^n)$ the group of volume preserving diffeomorphisms on a manifold M with boundary ∂M. The commutator of divergence free vector fields on M is just $-\{v, w\}$. Then we have the following result,

Theorem 1.1.8 (Dual of $\mathrm{SVect}(M)$)**.** *The Lie Algebra, $\mathfrak{g}$, of the group G is naturally identified with the space of closed differential $(n-1)$-forms on M vanishing on ∂M. That is, a divergence free vector field v is identified with the $(n-1)$-form $v \lrcorner \mu$ where μ is the volume form of M. Therefore, the dual to the Lie algebra is, $\mathfrak{g}^* = \Omega^1(M)/\, \mathrm{d}\Omega^0(M)$.*

Proof. The proof here has two parts. Firstly we prove that the space of all $(n-1)$-forms $v \lrcorner \mu$ is naturally identified to $\Omega^1(M)/\Omega^0(M)$. Then we can prove that $\mathfrak{g} \cong \Omega^1(M)/\Omega^0(M)$.

Recall Cartans magic formula, $\pounds_X \alpha = X \lrcorner \, \mathrm{d}\alpha + \mathrm{d}(X \lrcorner \, \alpha)$. We can say that if $\alpha = \mu$ and $X = u$, some vector field, it reduces to just saying, $\pounds_u \alpha = d(u \lrcorner \, \mu)$. We are working over the space of volume preserving diffeomorphisms and so when we consider the Lie derivative of the volume form over an element of the algebra we want it to vanish. Hence we require $v = u \lrcorner \, \mu$ to be closed. Therefore, $v \in \Omega^1(M)/\Omega^0(M)$. As u was arbitrary, v is just any $(n-1)$-form and so we have our natural identification.

Now we seek an isomorphism, $\mathfrak{g} \cong \Omega^1(M)/\Omega^0(M)$. This is more complicated and can be found in Theorem 8.3 (pg 42) of 'Topological Methods of Hydrodynamics' [26]. ☐

As we are considering fluid flows on these manifolds. If M is simply connected we can nicely say that the dual algebra, $\mathfrak{g}^*$ is just the vorticities.

1.1.2 Lie-Poisson Brackets

As introduced in the introduction we can write a Hamiltonian system as,

$$\dot{F} = \{F, H\},$$

where $F : M \to \mathbb{R}$ and H is the Hamiltonian. The right hand side is just the Canonical Lie-Poisson bracket. We can write this as,

$$\{F, G\} := \sum_i \frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial F}{\partial q_i}.$$

This is the simplest Poisson bracket and so is called the canonical bracket. These brackets are much like the Lie Brackets reviewed in [5]. However, feed into the theory of conserved quantities.

Definition 1.1.9 (Poisson Structure)**.** *A Poisson structure on a smooth manifold with a bilinear form mapping,* $(f, g) \mapsto \{f, g\}$ *which satisfies,*

- *The Jacobi Identity,* $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{f, h\}, g\} = 0$*,*
- *The Liebnitz Identity,* $\{f, gh\} = \{f, g\}h + \{f, h\}g$*.*

One of the important notions surrounding these brackets alongside their stability detection, is Casimirs. We can define a Casimir,

Definition 1.1.10 (Casimir)**.** *Let* M *be a smooth manifold, then* $C : M \to \mathbb{R}$ *be a function. We call* C *a Casimir of this system if,*

$$\{F, C\} = 0,$$

for any $F : M \to \mathbb{R}$*.*

There are more complicated brackets for different groups and models. For example, when we consider rigid bodies over SO(3) we get,

$$\{f, h\}(\Pi) := -\left\langle \Pi \, , \left[\frac{\partial f}{\partial \Pi}, \frac{\partial g}{\partial \Pi} \right] \right\rangle,$$

where $\Pi = R^{-1}\dot{R} \in \mathfrak{so}(3)$, $R \in \mathrm{SO}(3)$. This is a specific form of Nambu's $\mathbb{R}^3$ Poisson bracket,

$$\{f, h\} = \nabla c \cdot \nabla f \times \nabla h.$$

In ideal fluids the bracket differs. Consider the dual of SVect(M), then the bracket of this space with respect to some $\alpha \in \Omega^1/\Omega^0$ is,

$$\{F, G\}(\omega) = -\int_M \alpha \left(\left[\frac{\delta F}{\delta \omega}, \frac{\delta G}{\delta \omega} \right] \right) \mathrm{d}V.$$

This is the Arnold bracket. In the section on conserved quantities we will show that this bracket doesn't conserve enstrophy and provide some alternate brackets.

1.2 A short interlude into Symplectic Geometry

Hamiltons equations present an insight to the time evolution of a system. They rely on the Hamiltonian in its entirety to present information about the future. We know that if we have M as a manifold, then Lagrangian dynamics happens on TM and Hamiltontian dynamics happens on T^*M. These present a very nice framework for rigid body dynamics. In fluid dynamics it may help to consider the flow and the flow space of our system. This requires some heavier duty tools and the generalisation of the idea of a Hamiltonian. We generalise this into the idea of a symplectic form. The symplectic form relates to the vector field for the flow, $\mathrm{d}H$.

Let M be a manifold, a specific manifold that we will define later, then TM be the tangent space, or phase space, and T^*M the cotangent space, or the flow space. Then let ω be a non-degenerate section of $T^*M \otimes T^*M$. The non-degeneracy of ω means that for every $\mathrm{d}H$ there is some V_H such that $\mathrm{d}H = \omega(V_H, \cdot)$. We want two properties of our system to line up with ω. Firstly, we want the Hamiltonian, H, to be constant along the flow lines. Hence,

$$dH(V_H) = \omega(V_H, V_H) = 0.$$

That is, ω is an alternating 2-form. Further we want the Lie derivative of ω to vanish,

$$\begin{aligned} \pounds_{V_H}\omega &= \mathrm{d}(V_H \lrcorner\, \omega) + V_H \lrcorner\, \mathrm{d}\omega \\ &= \mathrm{d}(\mathrm{d}H) + \mathrm{d}\omega(V_H) = \mathrm{d}\omega(V_H) = 0. \end{aligned}$$

As H is arbitrary, ω must be closed. Hence we can define this idea of a symplectic form.

Definition 1.2.1 (Sympelctic Form)**.** *A symplectic form on a smooth manifold, M, is a closed non-degenerate 2-form, ω.*

We further define a symplectic manifold as a pair (M, ω). Symplectic Manifolds are just Poisson manifolds and the Symplectic form is usually of more interest and use. These are very useful when we get to defining symplectic leaves. These are another gateway into conserved quantities.

Definition 1.2.2 (Symplectic Leaf)**.** *The symplectic leaf of a point on a Poisson manifold is the set of all points on the manifold that can be reached by paths starting at a given point, such that the velocity vectors of the paths are Hamiltonian at every moment.*

You can show that these leaves are just symplectic manifolds and have the same symplectic structure. That is, the flow from any point where the Hamiltonian behaves nicely is just a manifold with nice symplectic structure and you can use the theory we will develop on it!

Theorem 1.2.3. *The symplectic leaf of every point is a smooth even dimensional manifold. It has natural symplectic structure defined by, $\omega(\xi, \eta) = \{f, g\}$, where ξ and η are vectors at the point x of the Hamiltonian fields with Hamiltonian functions f and g.*

We finally define a functor between symplectic manifolds that will be useful later,

Definition 1.2.4 (Symplectomorphism)**.** *A diffeomorphism between two syplectic manifolds, $\phi : (M, \omega) \to (N, \omega')$ is a symplectomorphism if it preserves the symplectic form,*

$$\phi^*\omega' = \omega$$

Chapter 2

Reduction, Equations and Brackets

Groups will be known by their actions

Guillermo Moreno

In 1757 Euler wrote the following set of equation in paper 'Principes généraux du mouvement des fluides'.

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p$$
$$\nabla \cdot \mathbf{u} = 0.$$

These are known as Euler's Equation for perfect incompressible fluid flow. In the Chapter we will take these equations, pull them apart and put them back together again. We will study a more general theory of these equations, by opening the eyes of the reader to fluid dynamics on general manifolds. On $\mathbb{R}^n$ conserved quantity and theory can be obscured by the flatness or even the simplicity of the space. We will lift these equations to curved surfaces and aim to prepare for the next chapter where we will study conserved quantities. We first can manipulate these equations into something more amicable to generalise. We will then generalise them and then prove that these are the generalised Euler equations by deriving them from the Lie-Poisson reduction of a Hamiltonian. This whole chapter draws from and builds on a number of sources, [26, 13, 5, 12, 27, 10, 18, 8, 22, 19, 25]

2.1 Generalisation of Eulers Equations

We can replace the $\mathbf{u} \cdot \nabla$ with a covariant derivative. We know that,

$$[\mathbf{u} \cdot \nabla]_i = u^i \frac{\partial u^i}{\partial x^i}.$$

However this is very similar to the covariant derivative,

$$[\nabla_{\mathbf{v}} \mathbf{u}]_i = \left(v^j u^i \Gamma^k_{ij} + v^j \frac{\partial u^k}{\partial x^j} \right) \mathbf{e}_k.$$

We now that, Γ^k_{ij} is the Christoffel Symbol and represents some notion of twistyness of the space and so $\Gamma^k_{ij} = 0$ for $\mathbb{R}^n$. Hence for flat space,

$$\mathbf{u} \cdot \nabla = \nabla_{\mathbf{v}} \mathbf{u}.$$

Therefore we firstly propose the generalisation of Eulers Equations by writing,

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \nabla_{\mathbf{u}} \mathbf{u} &= -\nabla p \\ \nabla \cdot \mathbf{u} &= 0. \end{aligned} \tag{2.1}$$

We can rewrite the covariant derivative with something more geometric. We want to explicity refer to the manifold that we work on. Let M be a manifold with Riemannian metric, then we can prove the following theorem using the metric.

Theorem 2.1.1. *Let M be a manifold with Riemannian metric. The Lie derivative of the one-form corresponding to a vector field on a M differs from the covariant derivative along itself by a complete derivative.*

$$\pounds_v(v^\flat) = (\nabla_v v)^\flat + \frac{1}{2}\,\mathrm{d}\,\langle v\,, v\rangle\,.$$

Proof. Assume we have v, w such that they commute. That is, $\{v, w\} = 0$. Recall the translation property of the Lie derivative and the covariant derivative,

$$\pounds_a \langle b\,, c\rangle = \langle \nabla_a b\,, c\rangle + \langle b\,, \nabla_a c\rangle\,.$$

If we specialise this formula we can get,

$$\langle \nabla_w v\,, v\rangle = \frac{1}{2}\,\mathrm{d}\,\langle v\,, v\rangle\,(w). \tag{2.2}$$

$$\pounds_v \langle w\,, v\rangle = \langle \nabla_v w\,, v\rangle + \langle w\,, \nabla_v v\rangle\,. \tag{2.3}$$

By noting that $\pounds_v f = \mathrm{d}f(v)$. As $\{v, w\} = 0$, then,

$$\langle \nabla_v w\,, v\rangle = \langle \nabla_w v\,, v\rangle\,. \tag{2.4}$$

Therefore we can substitute Equations 2.2 and 2.4 into Equation 2.3. This yields,

$$\pounds_v \langle w\,, v\rangle = \langle \nabla_v v\,, w\rangle + \frac{1}{2}\,\mathrm{d}\,\langle v\,, v\rangle\,(w).$$

It suffices to show that, $\pounds_v \langle w\,, v\rangle = (\pounds_v v^\flat)(w)$. This follows from the naturality rule of the Lie derivative,

$$\pounds_\xi(v^\flat(w)) = (\pounds_\xi v^\flat)(w) + v^\flat(\pounds_\xi w).$$

Then let $\xi = v$,

$$\pounds_v(v^\flat(w)) = (\pounds_v v^\flat)(w).$$

Now given that $\flat$ is the musical ismorphism, we know $\pounds_v \langle v\,, w\rangle = \pounds_v(v^\flat(w))$. Therefore,

$$\pounds_v v^\flat = (\nabla_v v)^\flat + \frac{1}{2}\,\mathrm{d}\,\langle v\,, v\rangle\,.$$

□

That is, we can write Equation 2.1, as the following,

$$\begin{aligned} \frac{\partial \mathbf{u}^\flat}{\partial t} &= -\pounds_v \mathbf{u} - \mathrm{d}f \\ \mathrm{d}\mathbf{u}^\flat &= 0 \end{aligned} \tag{2.5}$$

2.2 Euler-Poincaré Reduction

The Euler-Poincaré Equations are a set of equations that relate to a reduced Lagrangian on the Lie algebra of the manifold. We recall that a Lagrangian is a function on the tangent bundle, $L : TG \to \mathbb{R}$, then the reduced Lagrangian is $\ell : \mathfrak{g} \to \mathbb{R}$, i.e. the restriction of L to $\mathfrak{g}$. The following result characterises the Euler-Poincaré Equations in our current case. We note this is the first of a few very similar theorems depending on the assumptions we have on our groups and our system. In this section, we will consider Euler-Poincaré reduction of manifolds without boundary for a basic Lagrangian, a Lagrangian with an advective quantity, a diffusion term and then for compressible fluids. We will then recall the theory without boundary and work towards a theory for manifolds with boundary.

2.2.1 Manifolds without boundary

We can introduce and prove the following theorem that relates to the basic reduction of Lagrangians on manifolds without boundary. We consider a Lagrangian of basic form, $L = K - V$.

Theorem 2.2.1 (Basic Euler-Poincaré)**.** *Let G be a topological group which admits a smooth manifold structure with smooth right translation, and let $: TG \to \mathbb{R}$ be a right invariant Lagrangian. Let $\mathfrak{g}$ denote the fiber T_eG, and let $\ell : \mathfrak{g} \to \mathbb{R}$, the restriction of L to $\mathfrak{g}$. For a curve $\eta(t) \in G$, let $\mathbf{u}(t) = TR_{\eta(t)^{-1}}\eta(t)$. Then TFAE,*

- *$\eta(t)$ satisfies the Euler-Lagrange Equations,*
- *$\eta(t)$ is an extremum of the action,*

$$S(\eta) = \int L(\eta(t), \dot{\eta}(t))\ \mathrm{d}t,$$

- *$u(t)$ solves the Basic Euler-Poincaré equations,*

$$\frac{\partial}{\partial t}\frac{\delta \ell}{\delta u} = -\mathrm{ad}^*_u \frac{\delta \ell}{\delta u},$$

  where the coadjoint action is defined by,

$$\langle \mathrm{ad}^*_u v \,, w \rangle = \langle v \,, [u, w] \rangle \,.$$

- *$u(t)$ is the extremum of the reduced action,*

$$s(u) = \int \ell(u(t))\, \mathrm{d}t,$$

  for variations,

$$\delta u = \dot{w} + [w, u], \qquad w = TR_{\eta^{-1}}\delta\eta.$$

Proof. This proof can be found in [13, 6, 12, 20]. We will derive the Euler-Poincaré equations from the variational principle to show where the equations come from. We denote, $\ell(\xi)$ as our reduced lagrangian then we write our variational principle and integrate by parts.

$$\begin{aligned}
\delta \int_{t_1}^{t_2} \ell(\xi)\, \mathrm{d}t &= \int_{t_1}^{t_2} \left\langle \frac{\delta \ell}{\delta \xi}\, , \delta\xi \right\rangle \mathrm{d}t \\
&= \int_{t_1}^{t_2} \left\langle \frac{\delta \ell}{\delta \xi}\, , \dot{\eta} - \mathrm{ad}_{\xi}\eta \right\rangle \mathrm{d}t \\
&= \int_{t_1}^{t_2} \left\langle -\frac{\partial}{\partial t}\frac{\delta \ell}{\delta \xi}\, , \eta \right\rangle - \left\langle \mathrm{ad}^*_{\xi}\frac{\delta \ell}{\delta \xi}\, , \eta \right\rangle = 0.
\end{aligned}$$

Therefore the Euler-Poincaré equations are,

$$\frac{\partial}{\partial t}\frac{\delta \ell}{\delta \xi} + \mathrm{ad}^*_{\xi}\frac{\delta \ell}{\delta \xi} = 0.$$

□

We note that for ideal fluids, the Lagrangian is just kinetic energy. Hence,

$$L(\eta, \dot{\eta}) = \frac{1}{2}\int_M (\dot{\eta}, \dot{\eta})\mu \qquad \eta \in \mathrm{SDiff}(M),\ \dot{\eta} \in T_e\mathrm{SDiff}(M)$$

Then we can show that this Lagrangian is right invariant. We remember we are reducing over some group of structure preserving diffeomorphisms, so,

$$\begin{aligned}
L(\eta g, \dot{\eta} g) &= \frac{1}{2}\int_M \langle \dot{\eta} g\, , \dot{\eta} g\rangle\, \mu \\
&= \frac{1}{2}\int_M \langle \dot{\eta}\, , \dot{\eta}\rangle\, g^2 \mu \\
&= \frac{1}{2}\int_M \langle \dot{\eta}\, , \dot{\eta}\rangle\, \mu = L(\eta, \dot{\eta}).
\end{aligned}$$

Hence our Lagrangian is right invariant and we can let $g = \eta^{-1}$ and hence rewrite our Lagrangian,

$$L(\eta, \dot{\eta}) = L(\eta\eta^{-1}, \dot{\eta}\eta^{-1}) = L(e, \dot{\eta}\eta^{-1}) := \ell(\xi).$$

We now note that,

$$\ell(\xi) = \frac{1}{2}\int_M \langle \xi\, , \xi\rangle\, \mu.$$

Then we can write the Euler-Poincaré equations for this system as,

$$\frac{\partial \xi}{\partial t} = \mathrm{ad}^*_{\xi}\xi = \pounds_{\xi}\xi.$$

We note that quite nicely, we can see that this is just the Lie-Poisson Equation.

Advective Terms

In fluid mechanics, some quantities are transported by the flow. That is, they are transported by the Lie Derivative along vector fields. These are called advected quantities. Advection is usually referring to two things; advected conserved quantities, as due to the advection term new and more interesting conserved quantities are present, or to model substrate or substance being moved through the fluid. When we consider a Hamiltonian of these systems we note that they have a parameter that turns from a parameter to a variable. We note that these variables form a vector space and our groups act on them linearly. Hence we have a representation space. To see this more clearly, consider the following Lagrangian[1] for a spherical pendulum,

$$H(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} m |\dot{\mathbf{q}}|^2 - mg\mathbf{q} \cdot \mathbf{e}_3.$$

We note how the only variable in the potential energy term is $\mathbf{q}$. Now considered the reduce Lagrangian,

$$\ell(\hat{\mathbf{\Omega}}, \mathbf{\Gamma}) = \frac{1}{2} m\mathbb{I}|\hat{\mathbf{\Omega}}|^2 - mg\mathbf{\Gamma} \cdot X.$$

We note that, in this case we have turned 'created' another variable out of the symmetry breaking parameter. Hence a parameter has been turned into a variable and $\mathbf{\Gamma} \in V^*$ to adjoint of the representation of SO(3).

We can follow Holm, Marsden and Ratiu's paper on Semidirect Product Theories [12], to consider advection in our system. Let us consider a representation space, V, for our Lie Group G and let L have invariance properties for both G and V.

Definition 2.2.2 (Representation)**.** *A representation of a Lie group is a tuple (V, ρ) where V is the representation space and $\rho : G \to \mathrm{GL}(V)$ the G-linear action on V.*

We can define a semidirect product on a Fréchet Group and some representation group as follows,

Definition 2.2.3 (Semidirect Product)**.** *Let V be a vectorspace and G be a Fréchet Group that acts on the right by linear maps on V. The semidirect product is the cartesian product, $G \times V$ where group multiplication is given by,*

$$(g_1, v_1)(g_2, v_2) = (g_1 g_2, v_2 + v_1 g_2).$$

We note that this group has identity, $(e, 0)$ and inverses, $(g, v)^{-1} = (g^{-1}, -g^{-1}v)$. Then we can consider the Lie algebra, $\mathfrak{s} = \mathfrak{g} \times V^*$, and it bestows a Lie bracket,

$$[(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2], v_1\xi_2 - v_2\xi_1).$$

Notation 2.2.4. *For notational purposes, we will consider the adjoint linear representation ρ^* of some vector v and $a \in V^*$ as,*

$$\rho_v^* a = v \diamond a \in \mathfrak{g}^*.$$

or,

$$\langle A \diamond a \,, w \rangle_{\mathfrak{g} \times \mathfrak{g}^*} = \langle A \,, -\pounds_w a \rangle_{V \times V^*} = -\int_{\mathfrak{D}} A \lrcorner \pounds_w a.$$

[1] This is equivalent to the Hamiltonian up to Legendre transform

We consider our advected parameter to be some $a \in V^*$. Then we can consider the semidirect product of G and V^*, $G \rtimes V^*$. We choose the semidirect product for the flexibility, the Lie algebra of $\mathfrak{g} \rtimes V^*$.

We can define the Lagrangian as $L : G \to V^* \to \mathbb{R}$. Then we assume there is right representation of the Lie Group on the vector space V and G acts on the right on $TG \times V^*$, $(\eta, \dot{\eta}, a)h = (\eta g, \dot{\eta} g, ag)$ for $g \in G$. Then $L : TG \times V^* \to \mathbb{R}$ is G-invariant. That is, if we define $L_{a_0}(v_g) = L(v_g, a_0)$, then L_{a_0} is right invariant under the lift and action. Finally we can now define for $\ell : \mathfrak{g} \times V^* \to \mathbb{R}$ by,

$$L(\eta\eta^{-1}, \dot{\eta}\eta^{-1}, a_0\eta^{-1}) := \ell(\xi, \alpha).$$

We now proceed with the Euler-Poincaré advection theorem.

Theorem 2.2.5 (Euler-Poincaré with advection). *Let M be a manifold and G be a Fréchet group. Then let (V, ρ) be the representation of G. Suppose we have an advective quantity $a \in V^*$, and a Lagrangian $L : G \rtimes V^* \to \mathbb{R}$ that is right invariant under tangent lift. Let $\eta \in G$ and $\dot{\eta} \in \mathfrak{g}$. Then the folloiwng are equivalent,*

- *Hamilton's variational principle holds, where variations vanish at end points,*

$$\int_{t_1}^{t_2} L_{a_0}(\eta(t), \dot{\eta}(t))\, dt = 0$$

- *$\eta(t)$ satisfies the Euler-Lagrange equations for L_{a_0},*
- *The constrained variational principle holds for $\mathfrak{g} \times V^*$,*

$$\delta \int_{t_1}^{t_2} \ell(\xi, a)\ \mathrm{d}t = 0 \qquad \delta\xi = \dot{\nu} - [\xi, \nu], \quad \delta a = -a\nu$$

- *The Euler-Poincaré equations hold on $\mathfrak{g} \times V^*$,*

$$\frac{\partial}{\partial t}\frac{\delta \ell}{\delta \xi} = -\mathrm{ad}^*_{\xi}\frac{\delta \ell}{\delta \xi} + \frac{\delta \ell}{\delta a} \diamond a.$$

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\xi}\right) a = 0$$

Proof. The proof here is very similar to 2.2.1. However there are a few results we need to prove that are different. We firstly consider the time derivative of a, and then the variation.

We start with a Lagrangian of the form $L(\eta, \dot{\eta}, a_0)$ and then assume that it's right invariant, that is $L(\eta, \dot{\eta}, a_0) = L(\eta g,\, \dot{\eta} g,\, a_0 g)$. Then we can form a reduced Lagrangian, $L(\eta\eta^{-1},\, \dot{\eta}\eta^{-1},\, a_0 g^{-1}) := \ell(\xi, a)$ where $a := a_0 g^{-1}$. From this we calculate,

$$\begin{aligned} \frac{\partial a}{\partial t} &= \frac{\partial}{\partial t}(a_0 g^{-1}) \\ &= -a_0 g^{-1}\frac{\partial g}{\partial t} g^{-1} = -a\xi. \end{aligned} \tag{2.6}$$

Then similarly, we can write,

$$\begin{aligned}\delta a &= \delta(a_0 g^{-1})\\ &= -a_0 g^{-1}\delta g g^{-1}\\ &= -a\nu,\end{aligned}$$

where $\nu := \delta g g^{-1}$. Now we can derive the equations,

$$\begin{aligned}\delta\int_{t_1}^{t_2}\ell(\xi,\, a)\;\mathrm{d}t &= \int_{t_1}^{t_2}\left\langle\frac{\delta\ell}{\delta\xi}\,,\delta\xi\right\rangle + \left\langle\frac{\delta\ell}{\delta a}\,,\delta a\right\rangle\;\mathrm{d}t\\ &= \int_{t_1}^{t_2}\left\langle\frac{\delta\ell}{\delta\xi}\,,\dot\nu-[\xi,\,\nu]\right\rangle + \left\langle\frac{\delta\ell}{\delta a}\,,-a\nu\right\rangle\;\mathrm{d}t\\ &= \int_{t_1}^{t_2}\left\langle\frac{\delta\ell}{\delta\xi}\,,\dot\nu\right\rangle - \left\langle\frac{\delta\ell}{\delta\xi}\,,\mathrm{ad}_\xi\nu\right\rangle + \left\langle\frac{\delta\ell}{\delta a}\,,-a\nu\right\rangle\;\mathrm{d}t\\ &= \int_{t_1}^{t_2}\left\langle-\frac{\partial}{\partial t}\frac{\delta\ell}{\delta\xi}\,,\nu\right\rangle - \left\langle\mathrm{ad}^*_\xi\frac{\delta\ell}{\delta\xi}\,,\nu\right\rangle + \left\langle\frac{\delta\ell}{\delta a}\diamond a\,,\nu\right\rangle\;\mathrm{d}t\\ &= \int_{t_1}^{t_2}\left\langle-\frac{\partial}{\partial t}\frac{\delta\ell}{\delta\xi} - \mathrm{ad}^*_\xi\frac{\delta\ell}{\delta\xi} + \frac{\delta\ell}{\delta a}\diamond a\,,\nu\right\rangle\;\mathrm{d}t.\end{aligned}$$

Therefore, using Hamilton's principle we have,

$$\frac{\partial}{\partial t}\frac{\delta\ell}{\delta\xi} = -\mathrm{ad}^*_\xi\frac{\delta\ell}{\delta\xi} + \frac{\delta\ell}{\delta a}\diamond a. \tag{2.7}$$

Then considering the right concatenation using 2.6 we get,

$$\left(\frac{\partial}{\partial t} + \pounds_\xi\right)a = 0.$$

as required. □

Using this theory we can derive the Euler Equations for fluids with variable densities. Consider a Lagrangian with a fluid density advective term,

$$L(\eta,\dot\eta,\rho) = \int_M \frac{\rho}{2}|\dot\eta|^2 + p(\rho-1)\mu.$$

This Lagrangian has a symmetry breaking parameter [5], hence we consider the symmetric part and it's easy to verify this is right invariant. Let $\eta\in\mathrm{SDiff}(M)$ and ρ be a 0-form. Then we can reduce this Lagrangian into,

$$L(\eta\eta^{-1},\dot\eta\eta^{-1},\rho\rho_0^{-1}) := \ell(\mathbf{u},D) = \int_M \frac{D}{2}|\mathbf{u}|^2 + p(D-1)\mu.$$

Then we take functional derivatives,

$$\begin{aligned}\left\langle\frac{\delta\ell}{\delta\mathbf{u}}\,,\phi\right\rangle &:= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\ell(u^\flat+\varepsilon\phi,D)\right]_{\varepsilon=0}\\ &= \left\langle[u^\flat]D\,,\phi\right\rangle.\\ \left\langle\frac{\delta\ell}{\delta D}\,,\phi\right\rangle &:= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\ell(u^\flat,u^\flat+\varepsilon\phi)\right]_{\varepsilon=0}\\ &= \left\langle\frac{1}{2}[u^\flat]^2+p\,,\phi\right\rangle.\end{aligned}$$

Then we can plug these into Equation (2.7) and get,

$$\frac{\partial [u^\flat]}{\partial t} + \pounds_{\mathbf{u}}[u^\flat] = \frac{(\frac{1}{2}[u^\flat]^2 + p) \diamond D}{D}.$$

Then considering cosets we have,

$$\frac{\partial u^\flat}{\partial t} + \pounds_{\mathbf{u}} u^\flat = F(u^\flat, D) := \frac{(\frac{1}{2}[u^\flat]^2 + p) \diamond D}{D} + \mathrm{d}f.$$

Then the advection equation being,

$$\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right) D = 0$$

2.2.2 Manifolds with Boundary

In order to have systems with boundary conditions we require a manifold with some sort of boundary. This is where the theory deviates from above. We now will have three objects, the group we reduce over, G, with its associated algebra, $\mathfrak{g}$, and its dual $\mathfrak{g}^*$, the principle we use to derive the equations, usually Hamilton's principle, and the manifold we consider the flow over, M. We note that the boundary conditions will come from M and will be imposed on G. Boundary conditions come in two different forms,

1. Finite conditions, so conditions along a point or a path. For example, axisymmetric flow in a cylinder.

2. Infinite conditions, limiting conditions as you approach an infinitum. For example, waves on a plane.

The first of these types of conditions requires manifolds with boundary and the second is more complicated and requires a generalisation called manifolds with corners [17]. We will primarily be interested in the first case, while diverting the second to subsequent work on this area. We will be using the ideas from papers by Marsden, Ratiu and Shkoller [19, 22] to develop a more general theory that we can apply to compressible Euler-Poincaré equations. We can derive; Neumann, Dirichlet and Mixed conditions. We will work on two new diffeomorphism groups and further mainly study the Dirichlet group that corresponds to the zero on the boundary as that is the boundary condition seen the most in this type of fluid dynamics. We introduce the manifold of diffeomorphisms with boundary as follows.

Definition 2.2.6 (Diffeomorphism group with Boundary)**.** *Let M be a manifold, $\tilde{M}$ be the double of M as $H^s(M, M)$ isn't smooth and* $\mathrm{Diff}^s(M)$ *be the group of C^s diffeomorphisms. Then we define the diffeomorphism group with boundary as,*

$$\mathrm{Diff}^s(M) = \{\eta \in H^s(M, \tilde{M}) \cap \mathrm{Diff}^s(M) : \eta(\partial M) = \partial M\}.$$

Then a calculation, similar to the ones done in a paper by Ebin and Marsden [8], with (E, π) being a vector bundle leads to,

$$T_\eta \mathrm{Diff}^s(M) = \{u \in H^s(M, TM) : \eta = \pi \circ u,\ g(u\eta^{-1}, n) = 0 \text{ on } \partial M,\ \mathrm{div}(u\eta^{-1}) = 0\}.$$

That is,

Lemma 2.2.7. *Let M be a manifold, (E, π) be a vector bundle on M and $\xi = u\eta^{-1}$. Then the tangent space of* $\mathrm{Diff}^s(M)$ *at η is,*

$$T_\eta \mathrm{Diff}^s(M) = \{u \in \mathfrak{X}^s(M) : g(\xi,\, n) = 0 \text{ on } \partial M,\, \mathrm{div}(\xi) = 0\},$$

with a map $\eta = \pi \circ u$ from the vector bundle to connection the group and tangent space.

Proof. See [8]. □

We now can define the structure preserving group,

Definition 2.2.8 (Structure Preserving Diffeomorphism Group with Boundary)**.** *Let M be a manifold, then* $\mathrm{Diff}^s(M)$ *be as defined above and μ be the volume form of M. Then the structure preserving diffeomorphism group with boundary is,*

$$\mathrm{SDiff}^s(M) = \{\eta \in \mathrm{Diff}^s(M) : \eta^*\mu = \mu\}.$$

For all our groups, we note that left action is class C^r, right action is C^∞ and inversion, $\eta \mapsto \eta^{-1}$, is C^0 and further not Lipschitz continuous. Hence we not work with topological group with smooth right action.

2.2.3 Three more diffeomorphism groups

We define the following, let a diffeomorphism have a smoothness class. This smoothness class is a Hilbert H^s-class. In practicality we will assume we have diffeomorphisms that are sufficiently smooth enough to do the construction, but it is useful to mention restrictions on this theory. We introduce a few bits of differential geometry,

Definition 2.2.9 (Weingarten map)**.** *Let $(M,\, g)$ be a Riemannian manifold, $p \in M$, n be the normal field, and v is a tangent vector. Then the Weingarten map (shape operator) is a map $S : T_pM \to T_pM$, defined by,*

$$S(n) = -\nabla_v n$$

We can introduce our three diffeomorphism groups,

Definition 2.2.10 (Neumann Group)**.** *Let M be a manifold and N be its normal bundle. Then the Neumann group is defined as,*

$$\mathrm{SDiff}^s_N(M) = \{\eta \in \mathrm{SDiff}^s(M) : T\eta \cdot n|_{\partial M} \in H^{s-3/2}(N) \text{ for all } n \in H^{s-1/2}(N)\}$$

Definition 2.2.11 (Dirichlet Group)**.** *Let M be a manifold. Then the Dirichlet group is defined as,*

$$\mathrm{SDiff}^s_D(M) = \{\eta \in \mathrm{SDiff}^s(M) : \eta|_{\partial M} = \mathrm{Id}\}$$

Definition 2.2.12 (Mixed Group)**.** *Let M be a manifold and N be its normal bundle. Let the boundary $\partial M = \Gamma_1 \cup \Gamma_2$ and $\overline{\Gamma_1} = M \setminus \Gamma_2$. Then the mixed group is,*

$$\begin{aligned}\mathrm{SDiff}^s_{\mathrm{Mi}}(M) = \{\eta \in \mathrm{SDiff}^s(M) : \quad & \eta(\Gamma_i) = \Gamma_i,\ T\eta \cdot n|_{\Gamma_2} \in H^{s-3/2}(N|_{\Gamma_1}) \\ & \text{for all } n \in H^{s-1/2}(N|_{\Gamma_1}),\ \eta|_{\Gamma_2} = \mathrm{Id}\}\end{aligned}$$

Then their associated lie algebras

Definition 2.2.13 (Neumann Algebra)**.** *Let M be a manifold and N be its normal bundle. Then the Neumann algebra is defined as,*

$$T_e\mathrm{SDiff}^s_N(M) = \{u \in T_e\mathrm{SDiff}^s(M) : 0 = (\nabla_n u|_{\partial M})^{tan} + S_n(u) \in H^{s-3/2}(T\partial M) \text{ for all } n \in H^{s-1/2}(N)\}$$

Definition 2.2.14 (Dirichlet Algebra)**.** *Let M be a manifold. Then the Dirichlet algebra is defined as,*

$$T_e\mathrm{SDiff}^s_D(M) = \{u \in T_e\mathrm{SDiff}^s(M) : u|_{\partial M} = 0\}$$

Definition 2.2.15 (Mixed Algebra)**.** *Let M be a manifold and N be its normal bundle. Let the boundary $\partial M = \Gamma_1 \cup \Gamma_2$ and $\overline{\Gamma_1} = M \setminus \Gamma_2$. Then the Mixed algebra is defined as,*

$$\begin{aligned} T_e\mathrm{SDiff}^s_{\mathrm{Mi}}(M) = \{u \in T_e\mathrm{SDiff}^s(M) : \quad & 0 = (\nabla_n u|_{\partial M})^{tan} + S_n(u) \in H^{s-3/2}(T\Gamma_2) \\ & \text{for all } n \in H^{s-1/2}(N|_{\Gamma_2}),\ u|_{\Gamma_1} = 0\} \end{aligned}$$

Now we can prove the required theorem about how this relates to Euler-Poincaré equations.

Theorem 2.2.16 (Advected Euler-Poincare with Boundary Conditions)**.** *Let G be a C^∞ topological group, either $\mathrm{SDiff}^s_N(M)$, $\mathrm{SDiff}^s_D(M)$ or $\mathrm{SDiff}^s_{\mathrm{Mi}}(M)$ and $\ell : \mathfrak{g} \times V^* \to \mathfrak{g}$ be a reduced Lagrangian of the form $\ell(\xi, a) = \tilde{\ell}(\xi, a) + b(\xi, a)$ where b is the tangent space boundary operator. Then the Euler-Poincaré equations for a system with advected quantities (similarly for no advected quantities) has the following form*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta\tilde{\ell}}{\delta\xi} - \mathrm{ad}^*_\xi \frac{\delta\tilde{\ell}}{\delta\xi} + a \diamond \frac{\delta\tilde{\ell}}{\delta a} &= 0 \\ b(\xi, a) &= 0. \end{aligned} \tag{2.8}$$

Proof. We seek to calculate the Euler-Poincaré equations for $\ell(\xi, a) = \tilde{\ell}(\xi, a) + b(\xi, a)$ via Hamilton's principle. Then show that no extra erroneous terms appear. Then finally, as b vanishes at the boundary, the terms on the boundary should vanish. We note that any term involving b is considered at the boundary. The calculation is standard, until we reach,

$$\delta \int_{t_1}^{t_2} -\frac{\partial}{\partial t}\left(\frac{\delta\tilde{\ell}}{\delta\xi} + \frac{\delta b}{\delta\xi}\right) - \mathrm{ad}_\xi \left(\frac{\delta\tilde{\ell}}{\delta\xi} + \frac{\delta b}{\delta\xi}\right) + a \diamond \left(\frac{\delta\tilde{\ell}}{\delta a} + \frac{\delta b}{\delta a}\right) \,\mathrm{d}t = 0$$

We note that we can decouple the boundary terms,

$$\frac{\partial}{\partial t}\frac{\delta b}{\delta\xi} = -\mathrm{ad}_\xi \frac{\delta b}{\delta\xi} + a \diamond \frac{\delta b}{\delta a}.$$

Then we know if $b(\xi, a) = 0$, this vanishes in the above. Hence we have,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta\tilde{\ell}}{\delta\xi} - \mathrm{ad}^*_\xi \frac{\delta\tilde{\ell}}{\delta\xi} + a \diamond \frac{\delta\tilde{\ell}}{\delta a} &= 0 \\ b(\xi, a) &= 0. \end{aligned}$$

□

2.3 Lie-Poisson Equation

There are two different formalisations of mechanics. Lagrangian, which we saw above, and Hamiltonian. These two formalisms come from the idea that,

$$L = K - V \qquad H = K + V,$$

and further, Lagrangian dynamics happen on the tangent space of the manifold and Hamiltonian dynamics happen on the cotagent space of the manifold. The Euler-Poincaré equations and the Lie-Poisson equations relate to dynamics on the cotangent space. This implies there is some way to get between them. This is called the Legendre transform.

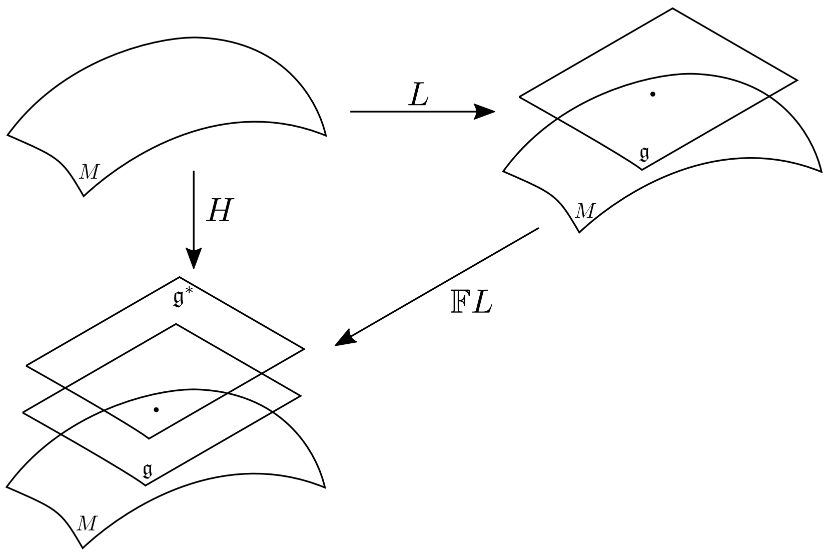

Figure 2.1: Dynamics and Legendre Transform.

In this section we will study how to reduce Hamiltonians via Lie-Poisson reduction and then show why it is usually better to reduce on the Lagrangian via Euler-Poincaré reduction and then use the Legendre transform. We will see that the reduction of the Lagrangian is a lot neater and further replicates most of modern theory. A lot of older Geometric Mechanics on fluids by Marsden and Arnold was mainly dedicated to Hamiltonian structures. This is because of the very close connection with conserved quantities and the intuitiveness of Hamiltonians being total energy.

There are several different ideas that arise in this area. We will look more closely at non-canonical Lie-Poisson brackets and how to calculate them and hint at the relation to Casimirs before moving to talk about them in greater detail in the next section.

We aim to study the way to get between Lagrangian and Hamiltonian through the Legendre transform. We include an exerpt from the author's special topic on Mechanical Mathematical Biology [4], that is relevant here.

2.3.1 Legendre Transform

Between Hamiltonians and Lagrangians, we are just working with a change of sign. A Hamiltonian is $K + V$ and a Lagrangian is $K - V$. However, if we have a Hamiltonian and want a Lagrangian, it is not necessarily obvious how to get between them. This is where the Legendre transform comes in. The Legendre transform is used in mechanics to go between a set of variables and their conjugate variables–for example, velocity and momentum. Regarding geometry, we know that Lagrangian mechanics happens on the tangent space of a manifold while Hamiltonian mechanics happens on the cotangent space of a manifold. Hence the Legendre transform is a map between the tangent space and the cotangent space that preserves a conserved quantity which transforms from the Lagrangian to the Hamiltonian or vice versa. We will briefly show the theory and derive the Legendre transform for this problem.

Let M be a manifold and (E, π) be a vector bundle. We quickly define a vector bundle [3].

Definition 2.3.1 (Vector Bundle)**.** *A $\mathbb{k}$-vector bundle over M of rank k consists of; a bundle $\pi : E \to M$ whose fibers are $\mathbb{k}$-vector spaces and around each point $p \in M$, there is some open $U \subset M$ and a diffeomorphism $\Phi : U \times \mathbb{k}^k \to E|U$ such that,*

- *$\pi \circ \Phi = \pi_1$ where $\pi_1 : U \times \mathbb{k}^k \to E|U$ is the projection of the first factor, and,*
- *for each $q \in U$, the map $\Phi_q : \mathbb{k}^k \to E_q$ such that $\Phi_q(\xi) := \Phi(q, \xi)$ is a $\mathbb{k}$-linear isomorphism.*

Let $L : E \to \mathbb{R}$ be a smooth function known as the Lagrangian. The Legendre transformation is a smooth map between E and E^*, the dual of E,

$$\mathbb{F}L : E \to E^*.$$

We define it by, $\mathbb{F}L(v) = \mathrm{d}(\,L|_{E_x})$, where E_x is the fiber of E over $x \in M$. This says we take a covector and map it to the directional derivative.

This localises via a trivialisation to just saying,

$$p_i = \frac{\partial L}{\partial q_i}.$$

This is exactly how we get between conjugate variables and generalised variables in the derivation of Hamilton's equations. Then further, in our case, $E = TM$ and $E^* = T^*M$. This leads to the Legendre transform being an isomorphism and further a diffeomorphism. Then we reach the Legendre transform being,

$$L(v) = H(p) - p \cdot v,$$

and further, $\mathbb{F}L = (\mathbb{F}H)^{-1}$, by using the natural isomorphism $TM \cong T^*M$ [13]. More specifically we can write,

$$\langle \mathbb{F}L(v)\,, w\rangle := \frac{\mathrm{d}}{\mathrm{d}s} L(v + sw)\bigg|_{s=0}.$$

Lie-Poisson reduction, in much the same way as Euler-Poincaré is a method to reduce equations into a more amenable form to consider conserved quantities and to use to write numerical methods via symmetries. We will study two main things, reduced legendre transforms and then the non-canonical brackets.

2.3.2 The Reduced Legendre Transform

The next two subsections follow Holm, Schmah and Stoica [13] but we note that we consider a right invariant version, while Holm considers a left invariant system. We also loosely cite Vasylkevych and Masrden's paper [25]. We firstly can note some more theory about the regular Legendre transform. We define a class of Lagrangians called hyperregular Lagrangians,

Definition 2.3.2 (Hyperregular)**.** *A hyperregular Lagrangian is a Lagrangian where the Legendre transform is a diffeomorphism and the Hessian of the Lagrangian is invertible.*

Then we can say that any system with a hyperregular Lagrangian has a guaranteed Hamiltonian related to it. This is because we have produced a differentiable bijection that is non-degenerate. For the rest of this thesis we will assume that our Lagrangian's are hyperregular. We can now define the reduced Legendre transform,

Definition 2.3.3 (Reduced Legendre transform)**.** *Let $L : TG \to \mathbb{R}$ be hyperregular and $\ell : \mathfrak{g} \to \mathbb{R}$ be the reduced Lagrangian defined by $\ell(\xi) = L(e, \xi)$. Then the reduced Legendre transform, $\mathrm{fl} : \mathfrak{g} \to \mathfrak{g}^*$, is defined by,*

$$\langle \mathrm{fl}(\xi)\,, \eta\rangle := \frac{\mathrm{d}}{\mathrm{d}s}\ell(\xi + s\eta)\bigg|_{s=0} = \left\langle \frac{\delta\ell}{\delta\xi}\,, \eta\right\rangle. \tag{2.9}$$

We want to show invariance on the Lagrangian leads to invariance in the Hamiltonian. We firstly define a special type of invariance,

Definition 2.3.4 (G-invariance)**.** *Let G be a Group, then we say that a Lagrangian, is G-invariant if for any $g_0 \in G$,*

$$L(qg_0,\, \dot{q}g_0) = L(q, \dot{q})$$

Lemma 2.3.5. *Let M be a manifold, $L : TM \to \mathbb{R}$ be G-invariant, then $\mathbb{F}L$ is G-invariant.*

Proof. Let $g_0 \in G$. We will consider the following calculation,

$$\begin{aligned}
\langle \mathbb{F}L(\mathbf{q}g_0, \dot{\mathbf{q}}_1 g_0)\,, (\mathbf{q}g_0,\, \dot{\mathbf{q}}_2 g_0)\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}L(\mathbf{q}g_0,\, \dot{\mathbf{q}}_1 g_0 + t\dot{\mathbf{q}}_2 g_0)\bigg|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}L((\mathbf{q},\, \dot{\mathbf{q}}_1 + t\dot{\mathbf{q}}_2)g_0)\bigg|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}L(\mathbf{q},\, \dot{\mathbf{q}}_1 + t\dot{\mathbf{q}}_2)\bigg|_{t=0} \\
&= \langle \mathbb{F}L(\mathbf{q}, \dot{\mathbf{q}}_1)\,, (\mathbf{q},\, \dot{\mathbf{q}}_2)\rangle
\end{aligned}$$

Therefore, $\mathbb{F}L(\mathbf{q}g_0,\, \dot{\mathbf{q}}g_0) = \mathbb{F}L(\mathbf{q},\, \dot{\mathbf{q}})$ and so $\mathbb{F}L$ is G-invariant. $\square$

This lemma tells us that if our Lagrangian is G-invariant, then the Legendre transform in G-invariant and as $H = E \circ (\mathbb{F}L)^{-1}$, where E is invariant from the Lagrangian, the Hamiltonian is invariant! We can now define these reduced quantities in a similar way to the full quantities and prove similar lemma's and prove finally that the reduced Hamiltonian is reduced. We call the reduced energy, $\tilde{e} : \mathfrak{g} \to \mathbb{R}$, defined by

$$\tilde{e}(\xi) = \langle \mathrm{fl}(\xi) \,, \xi \rangle - \ell(\xi),$$

and the reduced Hamiltonian, $h : \mathfrak{g}^* \to \mathbb{R}$ defined by,

$$h(\mu) = \tilde{e} \circ \mathrm{fl}^{-1}.$$

We can name $\mu = \mathrm{fl}(\xi)$ and then we get,

$$h(\mu) = \tilde{e} \circ \mathrm{fl}^{-1}(\mu) = \langle \mu \,, \xi(\mu) \rangle - \ell(\xi(\mu)).$$

Lemma 2.3.6. *Let M be a Fréchet Manifold, then $L : TM \to \mathbb{R}$ be a G invariant Lagrangian. If E is full energy, and $\tilde{e}$ is reduced energy corresponding to L and $\xi = gg^{-1}$, then,*

$$\tilde{e}(\xi) = E(e, \, \xi) = E(g \, \dot{g}).$$

Now suppose L is hyperregular and H is the Hamiltonian corresponding to L. If $\mu = g^{-1}\alpha$, then,

$$h(\mu) = (e, \xi) = H(g, \alpha)$$

Proof. We consider the full energy, then reduce,

$$\begin{aligned}
\langle \mathbb{F}L(g, \dot{g}) \,, (g, \dot{g}) \rangle - L(g, \dot{g}) &= \left\langle \mathbb{F}L(gg^{-1}, \dot{g}g^{-1}) \,, (gg^{-1}, \dot{g}g^{-1}) \right\rangle - L(gg^{-1}, \dot{g}g^{-1}) \\
&= \langle \mathbb{F}L(e, \xi) \,, (e, \xi) \rangle - L(e, \xi) \\
&= \langle \mathrm{fl}(\xi) \,, (e, \xi) \rangle - \ell(\xi) \\
&= \langle (e, \mathrm{fl}(\xi)) \,, (e, \xi) \rangle - \ell(xi) \\
&= \langle \mathrm{fl}(\xi) \,, \xi \rangle - \ell(\xi) = \tilde{e}(\xi).
\end{aligned}$$

Then the final part of the proof following from energy and the Legendre transform being G-invariant. Then noting that $H = E \circ (\mathbb{F}L)^{-1}$ and running through a reduction procedure and noticing the reduction variable ends up being $g^{-1}\alpha$. □

2.3.3 Derivation of Lie-Poisson

We will now move from the Euler-Poincaré equations via the reduced Legendre transform to the Lie-Poisson equations. We know that the legendre transform moves invariances from the Lagrangian side to the Hamiltonian side. The derivation is very similar to the Hamilton's equations derivation. We consider the following application of product rule and Equation 2.9,

$$\begin{aligned}
\frac{\delta h}{\delta \mu} &= \left\langle \mu \,, \frac{\delta \xi}{\delta \mu} \right\rangle - \left\langle \frac{\delta \ell}{\delta \xi} \,, \frac{\delta \xi}{\delta \mu} \right\rangle + \xi(\mu) \\
&= \left\langle \mu \,, \frac{\delta \xi}{\delta \mu} \right\rangle - \left\langle \mu \,, \frac{\delta \ell}{\delta \mu} \right\rangle + \xi(\mu) \\
&= \xi(\mu).
\end{aligned}$$

Given this we can now take the basic Euler-Poincaré eqations and then rewrite it as follows,

$$\dot{\mu} = -\mathrm{ad}^*_{\frac{\delta h}{\delta \mu}} \mu.$$

This is the Lie-Poisson Equation! An equivalent derivation is from the right invariant Poisson bracket. For any two smooth functions on the manifold $F, G : M \to \mathbb{R}$ we can define the canonical Poisson bracket as,

$$\{F, G\}^{\text{right}}_{\mathfrak{g}^*} = \left\langle \mu \,, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu} \right] \right\rangle.$$

The Lie-Poisson reduction theorem is redundant to us as we will reduce on the Lagrangian side. However for completeness we include it,

Theorem 2.3.7 (Lie-Poisson Reduction)**.** *Let G be a Lie Group, then $\mathfrak{g}^*$ be the dual Lie algebra. If $\mathcal{F}_R(\mathfrak{g}^*)$ is the set of right G-invariant functions, then, the right Lie-Poisson bracket is,*

$$\{F, G\}^{right}_{\mathfrak{g}^*} = \left\langle \mu \,, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu} \right] \right\rangle.$$

Further this relates to the reduced bracket on T^*G/G via the isomorphism,

$$\psi : (T^* * G/G) \mapsto \mathfrak{g}^*$$
$$[g,\, \alpha] \mapsto g^{-1}\alpha$$

Further we have a reconstruction equation,

$$\dot{g} = g\frac{\delta h}{\delta \mu}.$$

This can be proven from momentum maps, from Appendix INSERT APPENDIX

2.3.4 Lie-Poisson Reduction for advected quantities

We note that given a Lie algebra $\mathfrak{s} = \mathfrak{g} \rtimes V^*$. Then we can start to derive the same equations as above for semidirect products. We consider the following two actions, the adjoint and the coadjoint. We can write these as,

Proposition 2.3.8 (Adjoint Action)**.** *The adjoint action on $\mathfrak{g} \ltimes V^*$ is,*

$$\mathrm{Ad}_{(g,v)}(\xi, u) = (\mathrm{Ad}_g \xi,\, ug^{-1} + vg^{-1}\,\mathrm{Ad}_g \xi)$$

Proof. We consider the following calculation,

$$\begin{aligned}
\mathrm{Ad}_{(g,v)}(\xi, u) &= (g,\, v) * (\xi,\, u) * (g,\, v)^{-1} \\
&= (g\xi,\, u + v\xi) * (g^{-1},\, -vg^{-1}) \\
&= (g\xi g^{-1},\, -vg^{-1} + (u + v\xi)g^{-1}) \\
&= (g\xi g^{-1},\, -vg^{-1} + ug^{-1} + v\xi g^{-1}) \\
&= (\mathrm{Ad}_g \xi,\, vg^{-1}\,\mathrm{Ad}_g \xi - vg^{-1}),
\end{aligned}$$

as required. ☐

We can do something similar for the coadjoint action,

Proposition 2.3.9. *The coadjoint action on $\mathfrak{g} \ltimes V^*$ is,*

$$\mathrm{Ad}^*_{(g,\,v)^{-1}}(\mu,\,a) = (\mathrm{Ad}_{g^{-1}}\,\mu - g \diamond (v^{-1}ga),\,g \diamond a)$$

Proof. We consider the trace pairing of the coadjoint action, then construct the required result.

$$\begin{aligned}
\left\langle \mathrm{Ad}^*_{(g,\,v)^{-1}}(\mu,\,a)\,,(\nu_1,\,\nu_2)\right\rangle &= \left\langle (\mu,\,a)\,,\mathrm{Ad}_{(g,\,v)^{-1}}(\nu_1,\,\nu_2)\right\rangle \\
&= \left\langle (\mu,\,a)\,,(g,\,v)^{-1}(\nu_1,\,\nu_2)(g,\,v)\right\rangle \\
&= \left\langle (\mu,\,a)\,,(g^{-1},\,-vg^{-1})(\nu_1,\,\nu_2)(g,\,v)\right\rangle \\
&= \left\langle (\mu,\,a)\,,(g^{-1}\nu_1,\,\nu_2 - vg^{-1}\nu_1)(g,\,v)\right\rangle \\
&= \left\langle (\mu,\,a)\,,(g^{-1}\nu_1 g,\,v + (\nu_2 - vg^{-1}\nu_1)g)\right\rangle \\
&= \left\langle (\mu,\,a)\,,(g^{-1}\nu_1 g,\,v + \nu_2 g - vg^{-1}\nu_1 g\right\rangle \\
&= \left(\langle \mu\,,\mathrm{Ad}_{g^{-1}}\,\nu_1\rangle\,,\,\left\langle (a\,,v + \nu_2 g - vg^{-1}\nu_1 g)\right\rangle\right) \\
&= \left(\left\langle \mathrm{Ad}^*_{g^{-1}}\,\mu\,,\nu_1\right\rangle\,,\,\langle a\,,v\rangle + \langle a\,,\nu_2 g\rangle - \left\langle a\,,vg^{-1}\nu_1 g\right\rangle\right) \\
&= \left(\left\langle \mathrm{Ad}^*_{g^{-1}}\,\mu\,,\nu_1\right\rangle\,,\,\langle g \diamond a\,,\nu_2\rangle - \left\langle g \diamond (v^{-1}ga)\,,\nu_1\right\rangle\right) \\
&= \left\langle (\mathrm{Ad}_{g^{-1}\mu} - g \diamond (v^{-1}ga),\,g \diamond a)\,,(\nu_1,\,\nu_2)\right\rangle .
\end{aligned}$$

Therefore,

$$\mathrm{Ad}^*_{(g,\,v)^{-1}}(\mu,\,a) = (\mathrm{Ad}_{g^{-1}}\,\mu - g \diamond (v^{-1}ga),\,g \diamond a)$$

□

We note the following fact about the semidirect products and Lie brackets,

We now recall the definition of the Lie-Poisson bracket that we gave for the basic Lie-Poisson equations and apply it to $\mathfrak{s}^* = \mathfrak{g}^* \times V^*$.

$$\{F, G\}_{\mathfrak{s}} = \left\langle (\mu, a)\,,\left[\frac{\delta F}{\delta(\mu,\,a)},\,\frac{\delta G}{\delta(\mu,\,a)}\right]\right\rangle . \tag{2.10}$$

We can now prove,

Lemma 2.3.10. *The Lie-Poisson bracket for the Lie-Poisson equations with advected quantities is,*

$$\{F,\,G\}_{\mathfrak{s}^*} = \left\langle \mu\,,\left[\frac{\delta F}{\delta \mu},\,\frac{\delta G}{\delta \mu}\right]\right\rangle + \left\langle a\,,\frac{\delta F}{\delta \mu}\frac{\delta G}{\delta a} - \frac{\delta G}{\delta \mu}\frac{\delta F}{\delta a}\right\rangle$$

Proof. Consider Equation 2.10, and the following computation,

$$\begin{aligned}
\{F,\,G\}_{\mathfrak{s}^*} &= \left\langle (\mu, a)\,,\left[\frac{\delta F}{\delta(\mu,\,a)},\,\frac{\delta G}{\delta(\mu,\,a)}\right]\right\rangle \\
&= \left\langle (\mu, a)\,,\left[\left(\frac{\delta F}{\delta \mu},\,\frac{\delta F}{\delta a}\right),\,\left(\frac{\delta G}{\delta \mu},\,\frac{\delta G}{\delta a}\right)\right]\right\rangle \\
&= \left\langle (\mu,\,a)\,,-\left(\left[\frac{\delta F}{\delta \mu},\,\frac{\delta G}{\delta \mu}\right],\,\frac{\delta G}{\delta a}\frac{\delta F}{\delta \mu} - \frac{\delta F}{\delta a}\frac{\delta G}{\delta \mu}\right)\right\rangle \\
&= \left\langle \mu\,,\left[\frac{\delta F}{\delta \mu},\,\frac{\delta G}{\delta \mu}\right]\right\rangle + \left\langle a\,,\frac{\delta F}{\delta \mu}\frac{\delta G}{\delta a} - \frac{\delta G}{\delta \mu}\frac{\delta F}{\delta a}\right\rangle .
\end{aligned}$$

□

We can now further manipulate to prove the following,

Theorem 2.3.11 (Lie-Poisson Equations for advected parameters)**.** *The Lie-Poisson equations for a system with advected parameters are,*

$$
\begin{aligned}
\frac{\mathrm{d}h}{\mathrm{d}t} &= -\mathrm{ad}^*_{\frac{\delta h}{\delta t}}\mu + \frac{\delta h}{\delta a} \diamond a \\
\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{\delta h}{\delta \mu} a
\end{aligned}
\tag{2.11}
$$

Proof. Consider the following calculation,

$$
\begin{aligned}
\{F, H\}_{\mathfrak{s}} &= \left\langle \mu\,, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right] \right\rangle + \left\langle a\,, \frac{\delta F}{\delta \mu}\frac{\delta G}{\delta a} - \frac{\delta G}{\delta \mu}\frac{\delta F}{\delta a} \right\rangle \\
&= \left\langle \mu\,, -\left[\frac{\delta G}{\delta \mu}, \frac{\delta F}{\delta \mu}\right] \right\rangle + \left\langle a\,, \frac{\delta F}{\delta \mu}\frac{\delta G}{\delta a} \right\rangle - \left\langle a\,, \frac{\delta G}{\delta \mu}\frac{\delta F}{\delta a} \right\rangle \\
&= \left(\left\langle -\mathrm{ad}^*_{\frac{\delta G}{\delta \mu}}\mu\,, \frac{\delta F}{\delta \mu} \right\rangle, \left\langle \frac{\delta G}{\delta a} \diamond a\,, \frac{\delta F}{\delta \mu} \right\rangle - \left\langle a\frac{\delta G}{\delta \mu}\,, \frac{\delta F}{\delta a} \right\rangle \right) \\
&= \left\langle \left(-\mathrm{ad}^*_{\frac{\delta G}{\delta \mu}}\mu + \frac{\delta G}{\delta a} \diamond a, \; -\frac{\delta G}{\delta a} a \right), \left(\frac{\delta F}{\delta \mu}, \frac{\delta F}{\delta a} \right) \right\rangle
\end{aligned}
$$

Then we can decompose the inner products and pairings. Let $G = h$ and we can now write,

$$
\begin{aligned}
\frac{\mathrm{d}h}{\mathrm{d}t} &= -\mathrm{ad}^*_{\frac{\delta h}{\delta t}}\mu + \frac{\delta h}{\delta a} \diamond a \\
\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{\delta h}{\delta \mu} a
\end{aligned}
\tag{2.12}
$$

□

Chapter 3

Conservation

> As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection.
>
> Joseph-Louis Lagrange

In Noether's very influential paper in 1918, named 'Invariante Variationsprobleme' [21] she stated one of the most important theorems in the area of conservation. This was to be known as Noether's theorem. She stated, *'Every differentiable symmetry generated by local actions has a corresponds a conserved current.'* This innocuous looking theorem would lead to many years of work to find these conserved currents. The theorem said a little about how to find these quantities for the Euler-Lagrange, but there is an analogous result for Euler-Poincaré. These theorems then need to be derived for every new type of Euler-Poincaré equation. Noether's theorems apply to the Lagrangian framework. In order to calculate conserved quantites for the Hamiltonian framework you go to consider Casimir's and Poisson brackets.

In this penultimate chapter we will discuss the conservation properties of the Euler-Poincaré and Lie-Poisson equations. There are two different routes that we will take to find the quantities depending on what framework we work under. Firstly we will return to the Lagrangian framework and study the Noether Theorems for Euler-Poincaré theory and then consider a generalisation of these theorems called the Kelvin-Noether theorems which relate to first integrals over a path in our flow. We will then study the Hamiltonian point of view by returning to Lie-Poisson brackets, considering Casimirs and showing that certain functions that make the brackets vanish are conserved. We will also consider the relation between conserved quantities and Reeb graphs, another potential area for exploration in future work. Finally, we will attempt to show numerically that these quantities are conserved and conclude with ideas relating to how we can extend numerical methods with the preceeding theory to guarentee that the conserved quantites are conserved.

3.1 Noether Theorems

Noether Theorems can be derived from the variational principle and reduction. The process involves considering the terms we made vanish to produce the equations. We can state and prove the Noether Theorem for Euler-Poincaré equations with advected parameters.

Theorem 3.1.1 (Noether Theorem for Euler-Poincaré with advected parameters)**.** *Each symmetry vector field of the Euler-Poincaré reduced lagrangian for the infinitesimal variations,*

$$\delta u = \dot{\nu} - \mathrm{ad}_u \nu \qquad \delta a = -\nu a,$$

corresponds to an integral of the Euler-Poincaré motion and a conserved quantity,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_M \left\langle \frac{\delta \ell}{\delta u}, \nu \right\rangle \mu = 0$$

Proof. We will consider the derivation of the Euler-Poincaré equations for advection parameters. Then jump to the integration by parts of the $\frac{\partial u}{\partial t}$ term.

$$\begin{aligned} 0 &= \int_{t_1}^{t_2} \int_M \left\langle \frac{\delta \ell}{\delta u}, \frac{\partial \nu}{\partial t} \right\rangle - \left\langle \mathrm{ad}^*_u \frac{\partial \ell}{\partial u}, \nu \right\rangle + \left\langle a \diamond \frac{\delta \ell}{\delta a}, \nu \right\rangle \mathrm{d}t \\ &= \int_{t_1}^{t_2} \int_M \left\langle -\frac{\partial}{\partial t} \frac{\delta \ell}{\delta u} - \mathrm{ad}^*_u \frac{\partial \ell}{\partial u} + a \diamond \frac{\delta \ell}{\delta a}, \nu \right\rangle \mathrm{d}t + \int_M \left\langle \frac{\delta \ell}{\delta u}, \nu \right\rangle \Bigg|_{t_1}^{t_2} \mu. \end{aligned}$$

We now note that the first pairing is just the Euler-Poincaré equations. Hence we now can write,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_M \left\langle \frac{\delta \ell}{\delta u}, \nu \right\rangle \mu = 0,$$

as we use the fundemental theorem of calculus of variations. □

3.1.1 Vorticity and Helicity

We can now prove two lemmas for the specific case of the Euler Equations in barotropic fluids. The advective quantity is going to be $a = \rho \, \mathrm{d}V$ and then the infinitesimal symmetry for a becomes,

$$\pounds_\eta (\rho \, \mathrm{d}V) = \mathrm{d}(\eta \lrcorner \rho \, \mathrm{d}V) = 0.$$

Using rules from the appendix this can now be written as for some vector function, $\mathbf{\Psi}$,

$$\eta \lrcorner \rho \, \mathrm{d}V = \mathrm{d}(\mathbf{\Psi} \cdot \mathrm{d}\mathbf{x}) = \mathrm{curl}\, \mathbf{\Psi} \cdot \mathrm{d}\mathbf{S}. \tag{3.1}$$

Lemma 3.1.2 (Conservation of Vorticity)**.** *In the Euler-Poincaré equations for advected quantities, the following holds and relates to vorticity being conserved,*

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \pounds_{\mathbf{u}} \right) \left(\mathrm{curl} \frac{1}{\rho} \frac{\delta \ell}{\delta \mathbf{u}} \cdot \mathrm{d}\mathbf{S} \right) = 0. \tag{3.2}$$

Proof. We start with Noether's theorem for these equations and then perform a calculation.

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left\langle \frac{\delta \ell}{\delta u}, \eta \right\rangle \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_M \frac{\delta \ell}{\delta u}\cdot \eta \,\mathrm{d}V \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_M \frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\cdot \mathrm{d}\mathbf{x}\wedge \eta \lrcorner (\rho \,\mathrm{d}V) \\
&= \int_M \frac{\partial}{\partial t}\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}} \wedge \mathrm{d}(\mathbf{\Psi}\cdot \mathrm{d}\mathbf{x}) + \frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}} \wedge \frac{\partial}{\partial t}\mathrm{d}(\mathbf{\Psi}\cdot \mathrm{d}\mathbf{x}) \\
&= -\int_M \left(\frac{\partial}{\partial t}\mathrm{d}\left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\cdot \mathrm{d}\mathbf{x}\right) + \pounds_{\mathbf{u}}\,\mathrm{d}\left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\cdot \mathrm{d}\mathbf{x}\right)\right)\wedge (\mathbf{\Psi}\cdot \mathrm{d}\mathbf{x}) \\
&= -\int_M \left(\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right)\,\mathrm{d}\left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\cdot \mathrm{d}\mathbf{x}\right)\right)\wedge (\mathbf{\Psi}\cdot \mathrm{d}\mathbf{x})
\end{aligned}
$$

Hence we can say that, as we know $\Psi \cdot \mathrm{d}x \neq 0$. Then the result appears after applying (3.1),

$$
\begin{aligned}
0 &= \left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right)\,\mathrm{d}\left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\cdot \mathrm{d}\mathbf{x}\right) \\
&= \left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right)\left(\mathrm{curl}\,\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\cdot \mathrm{d}\mathbf{S}\right).
\end{aligned}
$$

□

We now present two major ideas in this area. Firstly, iterated conserved quantities. Let us consider another important theorem, called Ertel's theorem.

Theorem 3.1.3 (Ertel's Theorem)**.** *If a quantity a satisfied the advection equation, and η satisfy $\delta\eta = \dot{\nu} + \mathrm{ad}_\eta \nu$ for the labelling symmetry. Then $\pounds_\eta a$ is also advected.*

Proof. By a simple substitution we find,

$$
\pounds_\eta \left(\frac{\partial}{\partial t} + \pounds_u\right) a = \left(\frac{\partial}{\partial t} + \pounds_u\right)\pounds_\eta a = 0.
$$

This is the advection equation. Hence $\pounds_\eta a$ is advected by the flow. □

Now consider a conserved quantity, $c(t)$, this is carried by the flow. Hence we can say that $\pounds_u c(t)$ is also conserved by the Euler-Poincaré equations. Hence we can now consider a conserved quantity using the vorticity in (3.2). Hence we now can set the vorticity as our conserved 2-form in (3.1).

$$
\mathrm{d}(\mathbf{\Psi}\cdot \mathrm{d}\mathbf{x}) = \mathrm{d}\left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\cdot \mathrm{d}\mathbf{x}\right).
$$

This will lead us to a new conserved quantity called Helicity. Helicity is a topological invariant sometimes known as the Hopf invariant. It measures the knottedness of the vortex lines and can also be use to caclulate the number of linkages. As the vortex lines are baked in the Lagrangian and hence the flow this quantity is always conserved.

Lemma 3.1.4 (Conservation of Helicity)**.** *In the Euler-Poincaré equations for advected quantities, the following holds and relates to the Helicity being conserved.*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M \frac{1}{\rho}\frac{\delta\ell}{\delta\mathbf{u}}\cdot \operatorname{curl}\left(\frac{1}{\rho}\frac{\delta\ell}{\delta\mathbf{u}}\right)\,\mathrm{d}V$$

Proof. We start at the third step of the previous argument and then move forward in a different way,

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\int_M \left(\frac{1}{\rho}\frac{\delta\ell}{\delta\mathbf{u}}\cdot\,\mathrm{d}\mathbf{x}\right)\wedge \eta\,\lrcorner\,(\rho\,\mathrm{d}V)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_M \left(\frac{1}{\rho}\frac{\delta\ell}{\delta\mathbf{u}}\cdot\,\mathrm{d}\mathbf{x}\right)\wedge\,\mathrm{d}\,(\boldsymbol{\Psi}\cdot\,\mathrm{d}\mathbf{x})\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_M \left(\frac{1}{\rho}\frac{\delta\ell}{\delta\mathbf{u}}\cdot\,\mathrm{d}\mathbf{x}\right)\wedge\,\mathrm{d}\left(\frac{1}{\rho}\frac{\delta\ell}{\delta\mathbf{u}}\cdot\,\mathrm{d}\mathbf{x}\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_M \frac{1}{\rho}\frac{\delta\ell}{\delta\mathbf{u}}\cdot \operatorname{curl}\left(\frac{1}{\rho}\frac{\delta\ell}{\delta\mathbf{u}}\right)\,\mathrm{d}V
\end{aligned}$$

Therefore we can write Helicity as,

$$\mathcal{H} := \int_M \frac{1}{\rho}\frac{\delta\ell}{\delta\mathbf{u}}\cdot \operatorname{curl}\left(\frac{1}{\rho}\frac{\delta\ell}{\delta\mathbf{u}}\right)\,\mathrm{d}V$$

$\square$

3.1.2 Kelvin-Noether Theorem

We can now find other conservation quantities using similar ideas to above. Consider the following change of variables,

$$\oint_{\gamma_t}\frac{1}{\rho_t}\frac{\delta\ell}{\delta\mathbf{u}} = \oint_{\gamma_0}\eta^*\left[\frac{1}{\rho_0}\frac{\delta\ell}{\delta\mathbf{u}}\right] = \oint_{\gamma_0}\frac{1}{\rho_0}\eta^*\left[\frac{\delta\ell}{\delta\mathbf{u}}\right].$$

Now we can use the definition of the Lie derivative to write,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\eta^*\alpha) = \eta^*\left(\frac{\partial}{\partial t}\alpha + \pounds_{\mathbf{u}}\alpha\right).$$

Hence we can write,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\gamma_t}\frac{1}{\rho_t}\frac{\delta\ell}{\delta\mathbf{u}} &= \oint_{\gamma_0}\frac{1}{\rho_0}\eta^*\left(\frac{\partial}{\partial t}\alpha + \pounds_{\mathbf{u}}\alpha\right)\\
&= \oint_{\gamma_t}\frac{1}{\rho_t}\left(\frac{\partial}{\partial t}\alpha + \pounds_{\mathbf{u}}\alpha\right)\\
&= \oint_{\gamma_t}\frac{1}{\rho_t}\frac{\delta\ell}{\delta a}\diamond a.
\end{aligned}$$

The last step is from just considering the Euler-Poincaré equatios for advected parameters. From this we present the Kelvin-Noether theorem for advected Euler-Poincaré,

Theorem 3.1.5 (Kelvin-Noether Theorem for Euler Poincaré Equations)**.** *For the Euler-Poincaré equation with advected quantities, we can write the following,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\gamma_t}\frac{1}{\rho_t}\frac{\delta \ell}{\delta \mathbf{u}} = \oint_{\gamma_t}\frac{1}{\rho_t}\frac{\delta \ell}{\delta a}\diamond a.$$

We have two remarks,

Remark 3.1.6. *For basic Euler-Poincaré equations, the following is a conserved quantity,*

$$\oint_{\gamma}\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}.$$

Remark 3.1.7. *We can write the Euler-Poincaré equations in Kelvin-Noether form,*

$$\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right)\frac{\delta \ell}{\delta \mathbf{u}} = \frac{\delta \ell}{\delta a}\diamond a.$$

3.2 Casimirs

We are going to now consider the Hamiltonian formalism to find conserved quantities, firstly with the idea of Casimir's and then we will briefly look at the ideas surrounding Reeb graphs and their application to this area. We cite Kolev's paper [18] as a good overview of this area.

3.2.1 The Arnold Bracket

As stated by Arnold in Topological Methods in Hydrodynamics [26], we can find a Hamiltonian description of a system through a Poisson bracket. That is,

$$\dot{F} = \{F, H\}.$$

To actually be able to use these brackets we have to show that on a space of smooth functions, $\mathcal{F}$, they create a Hamiltonian structure,

Definition 3.2.1 (Hamiltonian Structure)**.** *A Hamiltonian structure is a bilinear operation, $\{\cdot, \cdot\}$ on the space of smooth functionals, $\mathcal{F}$, that satisfies the following, for any $F, G \in \mathcal{F}$,*

1. $\{F,\, G\} \in \mathcal{F}$,
2. $\{F,\, G\} = -\{G,\, F\}$,
3. $\{\{F,\, G\},\, H\} + \{\{G,\, H\},\, F\} + \{\{H,\, F\},\, G\} = 0.$

Remark 3.2.2. *The well trained eye will notice that $\{\cdot, \cdot\}$ isn't technically a Poisson bracket as it doesn't have a Liebnitz rule. This is because local functionals don't necessarily have this property.*

Our first bracket is the Arnold bracket. For functionals with L^2 gradients on the Lie algebra it is defined as,

$$\{F,\, G\}(\omega) = \int_M \omega\left(\frac{\delta F}{\delta \omega}, \frac{\delta G}{\delta \omega}\right)$$

Proving this provides a Hamiltonian structure is a quick calculation of inner products. Further will now show that this bracket is the Hamiltonian structure that relates to the incompressible Euler Equations. Recall we can write curl as follows,

$$\mu \lrcorner \operatorname{curl} u = \,\mathrm{d}u^\flat,$$

and we know that the curl of a vector field relates to an exact 2-form. Hence we can write an intertia operator $A : \mathrm{SVect}(M) \to \mathrm{SVect}^*(M)$ defined by $u \mapsto \operatorname{curl} u$. Then we can say that $u = A^{-1}\omega$ is the unique solution of the Euler Equations subject to,

$$\operatorname{curl} u = \omega \qquad \operatorname{div} u = 0 \qquad u \cdot n = 0 \text{ on } \partial M.$$

Our Hamiltonian is going to be,

$$H(\omega) = \int_M |\mathbf{u}|^2 \mu,$$

and $\mathrm{d}H = u = A^{-1}\omega$. Now we consider the L^2 gradient of $\mathbf{u}$, and we can derive,

$$\begin{aligned}\int_M \partial_t \mathbf{u} \cdot \delta F &= \int_M \partial_t (A^{-1}\omega) \cdot \delta F \\ &= \int_M \omega \cdot (\delta F \times \mathbf{u}) \\ &= \int_M \delta F \cdot (\mathbf{u} \times \omega).\end{aligned}$$

Therefore our bracket provides the Hamiltonian system, modulo a 1-form,

$$\partial_t \mathbf{u} = \mathbf{u} \times \omega.$$

This is the vorticity form of the Euler Equations. We can consider the pullback of this functional to get something more amicable to work with,

$$\{F,\, G\}(\omega) = \int_M \operatorname{curl} \mathbf{u} \cdot \left(\frac{\delta F}{\delta \mathbf{u}} \times \frac{\delta G}{\delta \mathbf{u}} \right). \tag{3.3}$$

We can show that Helicity is conserved by finding the Casimir. We define a Casimir as,

Definition 3.2.3 (Casimir)**.** *We define the Casimir as a smooth function $C : M \to \mathbb{R}$,*

$$\{C, H\} = 0.$$

Now we find the Casimirs of the bracket (3.3). We consider the following argument,

$$\begin{aligned}0 &= \int_M \operatorname{curl} \mathbf{u} \cdot \left(\frac{\delta C}{\delta \mathbf{u}} \times \frac{\delta H}{\delta \mathbf{u}} \right) \mu \\ &= \int_M \operatorname{curl} \mathbf{u} \cdot \left(\frac{\delta C}{\delta \mathbf{u}} \times \mathbf{u} \right) \mu \\ 0 &= -\int_M \frac{\delta C}{\delta \mathbf{u}} \cdot (\operatorname{curl} \mathbf{u} \times \mathbf{u})\, \mu \\ &= -\int_M \frac{\delta C}{\delta \mathbf{u}} \cdot \left((u \cdot \nabla) u - \frac{1}{2} \nabla (u^2) \right) \mu \\ &= \int_M -\frac{\delta C}{\delta \mathbf{u}} \cdot (u \cdot \nabla) u + \frac{1}{2} \frac{\delta C}{\delta \mathbf{u}} \cdot \nabla (u^2)\, \mu \\ &= \int_M \left(\frac{\delta C}{\delta \mathbf{u}} \cdot \partial_t \mathbf{u} + \frac{1}{2} \nabla \frac{\delta C}{\delta \mathbf{u}} \cdot \mathbf{u}^2 \right) \mu + \int_{\partial M} \frac{1}{2} \frac{\delta C}{\delta \mathbf{u}} \cdot (u^2 \cdot n) \mu_\partial \\ &= \int_M \left(\partial_t \frac{\delta C}{\delta \mathbf{u}} \cdot \mathbf{u} + \frac{1}{2} \nabla \frac{\delta C}{\delta \mathbf{u}} \cdot \mathbf{u}^2 \right) \mu + \left[\int_M \frac{1}{2} \frac{\delta C}{\delta \mathbf{u}} \cdot \mathbf{u}\, \mu \right]_{t_1}^{t_2} \\ &= \int_M \left(\partial_t \frac{\delta C}{\delta \mathbf{u}} + \frac{1}{2} \nabla \frac{\delta C}{\delta \mathbf{u}} \cdot \mathbf{u} \right) \cdot \mathbf{u}\, \mu + \left[\int_M \frac{1}{2} \frac{\delta C}{\delta \mathbf{u}} \cdot \mathbf{u}\, \mu \right]_{t_1}^{t_2}.\end{aligned}$$

Hence we require,

$$\frac{\partial}{\partial t} \frac{\delta C}{\delta \mathbf{u}} + \frac{1}{2} \nabla \frac{\delta C}{\delta \mathbf{u}} \cdot \mathbf{u} = 0 \qquad \frac{\delta C}{\delta \mathbf{u}} = 0 \text{ on } \partial M. \tag{3.4}$$

We note that when we let the Casimir be Helicity, the equation (3.4) turns into,

$$\int_M (\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u})\cdot(1+\mathbf{u}\times\omega) + ((\omega_t + \mathbf{u}\cdot\nabla\omega)\times\mathbf{u})\cdot\mathbf{u} = 0.$$

Hence we have proven that Helicity is a Casimir. This bracket seems to be conserving what we want to conserve. Let us now look to enstrophy for a 2D system. We consider generalised enstrophy,

$$C(\omega) = \int_M \phi(\omega)\,\mathrm{d}x \wedge \,\mathrm{d}y.$$

We can take the Fréchet derivative,

$$DC(\omega) = \int_M \mathrm{curl}\,(\phi'(\omega)\hat{\mathbf{k}})\,\mathrm{d}\alpha + \oint_{\partial M} \phi'(\omega)\,\mathrm{d}\alpha.$$

For us to use the Arnold bracket, we require the derivative of the functional to be an L^2 gradient. This isn't as it has boundary terms. Hence the Arnold bracket doesn't conserve enstropy. Further if we set $\phi'(\omega) = 0$ then we would fall back to enstrophy and then this is covered by the conservation of Helicity. Hence we need a stronger bracket.

In Zackharov's 1968 paper [28] he presented the following Hamiltonian system for irrotational waves,

$$H = \frac{1}{2}\iiint (\mathrm{grad}\,\phi)^2\,\mathrm{d}V + \frac{1}{2}\lambda \iint \zeta^2\,\mathrm{d}S.$$

This was then generalised to include vorticity and the bracket was found for a free boundary Σ and a velocity field v such that,

1. δv is a divergence free vector field on some domain D_Σ. Further we have a divergence free vector field $\frac{\delta F}{\delta v}$ defined one some domain D_Σ such that,

   $$D_v F(v,\,\Sigma) = \int_{D_\Sigma} \frac{\delta F}{\delta v}\cdot\delta v,$$

   where D_v is the derivative with respect to v with Σ fixed.

2. Similarly, there's a $\delta\Sigma$ thats normal to Σ and some smooth function $\frac{\delta F}{\delta \Sigma}$ such that,

   $$D_\Sigma F(v,\,\Sigma) = \int_{D_\Sigma} \frac{\delta F}{\delta \Sigma}\cdot\delta\Sigma.$$

Then the bracket is,

$$\{F, G\} = \int_{D_\Sigma} \omega\cdot\left(\frac{\delta F}{\delta v}\times\frac{\delta G}{\delta v}\right) + \int_\Sigma \left(\frac{\delta F}{\delta\Sigma}\frac{\delta G}{\delta\phi} - \frac{\delta G}{\delta\Sigma}\frac{\delta F}{\delta\phi}\right),$$

where $\omega = \mathrm{curl}\,u$ and,

$$\frac{\delta F}{\delta\phi} = \left.\frac{\delta F}{\delta v}\right|_\Sigma \cdot n.$$

This proof of conservation here is more involved and will be covered in my PhD thesis.

3.2.2 Reeb Graphs

Another interesting way to derive the conserved quantities of a Hamiltonian system, and further, derive all of the Casimir's of the system is to consider Reeb graphs of the manifold you work over. In this section we will review and expose the work of Izosimov and Khesin [26, 16]. We will return to the problem we left in the previous section of proving in 2D that enstrophy is conserved in the Hamiltonian form of the Euler Equations. There are two main theorems we are interested in, which give the following correspondences,

$$\text{Reeb Graphs} \longleftrightarrow \text{Morse Functions} \longleftarrow \text{Casimirs}$$

The two theorems that relate to this diagram are,

Theorem 3.2.4 (Reeb Graphs give Morse Functions)**.** *The mapping assigning the measured Reeb graph Γ_F to a simple Morse function F provides a one-to-one correspondance between simple morse functions on M up to symplectomorphism and measured Reeb graphs compatible with M.*

Theorem 3.2.5 (Casimirs are moments of the Morse functions)**.** *A complete set of Casimirs to the 2D Euler Equation in a neighbourhood of a Morse-type coadjoint orbit is given by the momenta,*

$$I_{i,e} = \int_{M_e} F^i \omega, \qquad i \in \mathbb{N},$$

for each $e \in \Gamma$ and all circulations of the velocity v over cycles in the singular levels of the vorticity function F on M.

We will now define morse functions and measures reeb graphs and then give proof of Enstrophy being conserved in the 2D Euler Equations.

Definition 3.2.6 (Morse Function)**.** *Let M be a closed connected space, then a morse function $F : M \to \mathbb{R}$ is called simple if any level set of F contains at most one critical point.*

Now we can associate a graph with a morse function. To do this we take each critical points of F and then place a critical point of the graph there, as we see in Figure 3.1. We can now define a Reeb graph as follows,

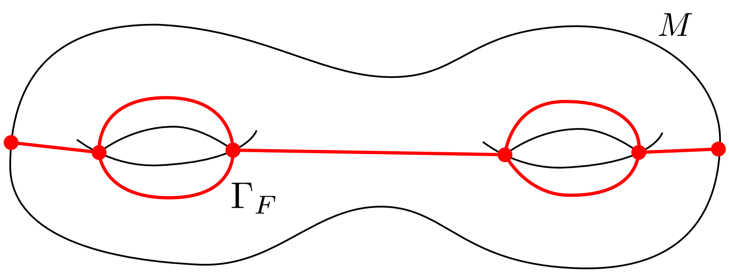

Figure 3.1: The Reeb Graph of a manifold.

Definition 3.2.7 (Reeb Graph). *A Reeb graph,* $(\Gamma,\, f)$ *is an oriented smooth connected finite graph* Γ *with a continuous function* $f : \Gamma \to \mathbb{R}$ *which satisfy the following,*

1. *All vertices of* Γ *are either* 1*-valent or* 3*-valent,*
2. *For each* 3*-valent vertex there are either two incoming and two outgoing or vice versa,*
3. *The function* f *is strictly monotonous on each edge, and the edges of* Γ *are oriented towards the direction of increasing* f*.*

It is a standard result in morse theory that a graph Γ_F associated with a simple morse function $F : M \to \mathbb{R}$ on an orientable connected surface is a Reeb graph. Further if this surface is endowed with an area or symplectic form ω, then we can take the pullback of that form and defined a measured reeb graph.

Definition 3.2.8 (Measured Reeb Graph). *Let* $(\Gamma,\, f)$ *be a Reeb graph associated with a surface with symplectic form* ω*. Then we can define a measure* $\mu = f^*\omega$*. Then a measured Reeb graph is a Reeb graph with a measure,* $(\Gamma,\, f,\, \mu)$*.*

Now we can say that the simple morse functions are the same as the measured Reeb graphs up to symplectomorphism under the following compatibility conditions.

Definition 3.2.9 (Compatible). *We say that a closed compact surface endowed with a symplectic form,* $(M,\, \omega)$*, is compatible with a measured reeb graph,* $(\Gamma,\, f,\, \mu)$ *if the following are satisfied,*

1. *The dimension of the first homology group [11],* $H_1(\Gamma,\, \mathbb{R})$ *is equal to the genus of* M*,*
2. *The volume of* Γ *with respect to* μ *is the same as the volume of* M*,*
$$\int_\Gamma \mathrm{d}\mu = \int_M \omega.$$

Then from this Theorem 3.2.4 follows.

Classifying coadjoint orbits

We note that the coadjoint action of $\mathrm{SDiff}(M)$ on $\mathfrak{svect}(M)$ is just a symplectomorphism, a volume preserving map,
$$\mathrm{Ad}^*_\Phi[\alpha] = [\Phi^*\alpha].$$
The orbits of this action can be described by $\mathfrak{curl} : \Omega^1(M)/\,\mathrm{d}\Omega^0(M) \to C^\infty(M)$ given by vorticity,
$$\mathfrak{curl}\,[\alpha] := \frac{\mathrm{d}\alpha}{\omega}.$$
This mapping is equivariant and so that means that if two maps differ by a symplectomorphism, then they are in the same coadjoint orbit as each other. Hence all the simple morse functions are in one coadjoint orbit.

Definition 3.2.10 (Morse-Type forms)**.** *We say that a coset of* 1*-forms,* $[\alpha] \in \mathfrak{svect}^*(M)$*, is morse type if* $\mathfrak{curl}\,[\alpha]$ *is a simple morse function. A coadjoint orbit is morse-type if any coset is morse type.*

For the remainder, let $[\alpha] \in \mathfrak{svect}^*(M)$ be morse-type and $F := \mathfrak{curl}\,[\alpha]$. Then Γ_F is invariant under the coadjoint action! However if M is not simply connected our theory fails as the graph branches and causes issues. Hence we use a circulation function to deal with branches. Let $\pi : M \to \Gamma_F$ be the natural projection. Take any point x in the interior of some edge $e \in \Gamma_F$. Then $\pi^{-1}(x)$ is a circle. It is naturally oriented and the integral does not depend on α. Hence we have a function $\mathfrak{c} : \Gamma_F \setminus V(\Gamma_F) \to \mathbb{R}$ defined by,

$$\mathfrak{c}(x) := \int_{\pi^{-1}(x)} \alpha.$$

We now define yet another graph,

Definition 3.2.11 (Circulation Measured Reeb Graph)**.** *A circulation measured Reeb graph,* $(\Gamma,\, f,\, \mu,\, \mathfrak{c})$ *, is a measured Reeb graph endowed with a circulation function* $\mathfrak{c} : \Gamma_F \setminus V(\Gamma_F) \to \mathbb{R}$ *defined by,*

$$\mathfrak{c}(x) := \int_{\pi^{-1}(x)} \alpha.$$

We note that non-circulation graphs work for us due to Fluid Dynamics problems only having 'pants decompositions' and 'Dehn half twists'. These more complicated idea's don't arise in this example but should be considered for more complex phenomena. We can now prove that the following momenta,

$$m_{i,e}(F) = \int_{M_e} F^i \omega$$

form a complete set of invariants for the 2D Euler Equations.

Theorem 3.2.12. *Let* $(M,\, \omega)$ *be a closed connected symplectic surface, and let* F *and* G *be simple morse functions on* M*. Then let* $\phi : \Gamma_F \to \Gamma_G$ *be an isomorphism of abstract directed graphs which preserve moments on all edges. Then* Γ_F *and* Γ_G *are isomorphic as measured Reeb graphs, and there is a symplectomorphism* $\Phi : M \to M$ *such that* $\Phi_* F = G$

Proof. We first construct two intervals of the two Reeb graphs. Let $[v,\, w] \in \Gamma_F$. Then we push forward by the measure using a homomorphism $f : e \to [v,\, w]$ we get a measure μ_f on an interval, $I_f = [f(v),\, f(w)]$ and similarly $I_g = [g(\phi(v)),\, g(\phi(w))]$ for $g : e \to [g(v),\, g(w)]$, we get a measure μ_g. These intervals have the same moments. We aim to prove that $\mu_f = \mu_g$. This follows from the Hausdorff moment problem. Consider an interval I that contains both I_f and I_g, then the measures are are measures of I supported on their respetive intervals. Then by the Hausdorff moment problem, $\mu_f = \mu_g$. □

The above then proves that if we take F to be a morse vorticity function for an ideal flow v on a closed surface M associated with a Reeb graph Γ. Then the moments of that graph $m_{i,e}(F)$ are just the generalised enstrophies,

$$m_{i,e}(F) = \int_{M_i} F^i \omega = \int_{M_i} \phi(\omega)\, \mathrm{d}x \wedge \mathrm{d}y,$$

in two dimensions. Then you can single each conserved quantity out by Theorem 3.2.5 and we have additionally these are all the conserved quantities of the 2D Euler Equations.

Chapter 4

Examples

> All mathematics is divided into three parts: cryptography (paid for by CIA, KGB and the like), hydrodynamics (supported by manufacturers of atomic submarines) and celestial mechanics (financed by military and by other institutions dealing with missiles, such as NASA.).
>
> V. I. Arnold

In this final chapter will will spend some time producing some examples to give the reader some idea of the scope of this theory. It has many applications from Geophysical Fluid Dynamics [15], to computer graphics and image processing [13] to liquid crystals [14]. In this section we will focus on Geophysical Fluid Dynamics and study some examples that will culminate in the Euler-Boussinesq equations for a rotating stratified fluid and a recreation of the very popular paper by Vannaste on fluid flow on a mobius strip [24] from a classical Geometric Mechanics point of view..

4.1 Axisymmetric Flow on a Cylinder

We start this examples section with a classical and simple example relating to axisymmetric flow on a cylinder. We will be using a lot of the theory in subsection 2.2.2. Recall that we need three objects a group to reduce over, a manifold to act as a space for dynamics to happen on and a Lagrangian. Here we let the manifold be the cylinder,

$$M = \overline{D}(0,\, 1) \times [0,\, 1].$$

The group is $\mathrm{SDiff}(M)$ with Dirichlet boundary conditions. We also make a modelling assumption that all the particles can be traced back to a plane, as seen in Figure 4.1. This doesn't impeed the dynamics as it imposes the axisymmetric assumption, but instead makes it easier to model. These assumptions give,

$$G = \{\eta \in \mathrm{SDiff}(M) : \eta(0) \in [0,\, 1] \times [0,\, 1] \text{ and } \eta = \mathrm{Id} \text{ on } D(0,\, 1) \times [0,\, 1]\}.$$

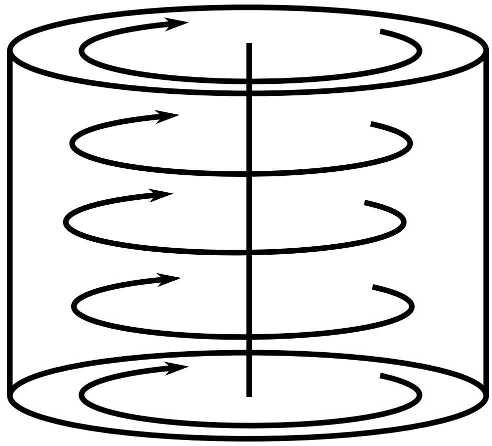

Figure 4.1: Axisymmetric Flow on a cylinder.

The Lagrangian is defined as,

$$L(\eta, \dot{\eta}) = \int_M (\dot{\eta}, \dot{\eta})\mu.$$

We can show that the Lagrangian is right invariant by letting $g \in \mathrm{SDiff}(M)$ then considering,

$$\begin{aligned} L(\eta g, \dot{\eta} g) &= \int_M (\dot{\eta} g, \dot{\eta} g)\mu \\ &= \int_M (\dot{\eta}, \dot{\eta}) g^2 \mu \\ &= \int_M (\dot{\eta}, \dot{\eta})\mu = L(\eta, \dot{\eta}). \end{aligned}$$

The second step comes from noting that g is volume preserving. Hence we let $g = \eta^{-1}$ and $\mathbf{u} = \dot{\eta}\eta^{-1}$ and we get that,

$$\ell(\mathbf{u}) = \int_M |\mathbf{u}|^2 \mu.$$

Then we find the variational derivative of the lagrangian and plug into the Euler-Poincaré equation. Then we get,

$$\frac{\partial[\mathbf{u}^\flat]}{\partial t} = \mathrm{ad}_{\mathbf{u}}[\mathbf{u}^\flat].$$

We note that we have transferred from the algebra to the dual when we took the functional derivative. We now need to impose boundary conditions. Using the group and considering the tangent space we get Dirichlet boundary conditions. Hence our equations are,

$$\begin{aligned} \frac{\partial \mathbf{u}^\flat}{\partial t} &= \mathrm{ad}_{\mathbf{u}} \mathbf{u}^\flat + \mathrm{d}f = \pounds_{\mathbf{u}} \mathbf{u}^\flat + \mathrm{d}f \\ \mathbf{u}^\flat &= 0 \text{ on } \partial M \end{aligned} \tag{4.1}$$

where f is a 0-form.

4.2 Möbius Strip

The Möbius strip brings a new level of complexity to this theory. The Möbius strip is a non-orientable surface. This means that if you send a vortex around the strip spinning clockwise, then it returns spinning anticlockwise. This obviously can cause some issues for the Euler Equations and the theory surrounding them. In a paper by Vanneste on this area he produces some numerical simulations and equations for the Möbius strip [24]. We will first show the group that relates to this problem and then give an argument of why this group is Fréchet Group. Then we will discuss conserved quantities of this system.

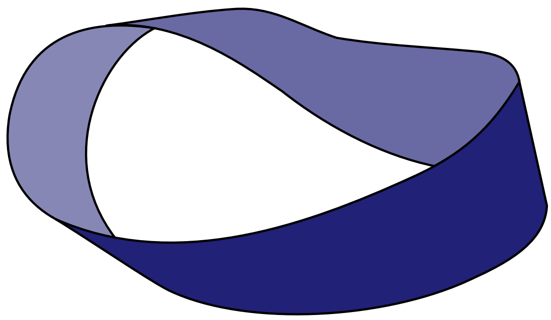

Figure 4.2: Mobius Strip.

We firstly note the diagram of the Möbius strip above and consider the following question.

Question 4.2.1. *How do we define the Möbius strip in an amicable way?*

The answer to this is quite simple. Consider the following diagram.

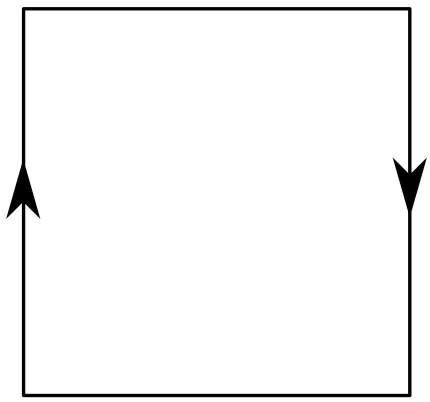

Figure 4.3: Plane representation of a Mobius Strip.

This is a Möbius strip. We note that we can connect the two arrowed sides by performing a half twist and glueing. This gives arise to the following definition of a Mobius Strip. Without loss of generality, imagine we are working on the square, $[-1,\,1]\times[-1,\,1]$. This holds as we can just nondimensionalise in both the x and y directions and get a unit square. Then we can define our manifold as,

$$M = \{(x,\,y) \in [-1,\,1]\times[-1,\,1] : -x \sim x \text{ when } y = 1\}.$$

We can use this to turn our manifold into boundary conditions on our diffeomorphism group. There has been much contention on the fact of what happens on the boundary of the Möbius strip when we consider the tangent space. There are a few ways to deal with this, the easiest, set the vector fields to vanish at the boundary. Hence our diffeomorphism group is,

$$G = \{\eta \in \mathrm{SDiff}(M) \colon \eta(-1,\, y) = \eta(1,\, -y),\, \eta(x,\, -1) = \eta(x,\, 1) = \mathrm{Id}\}.$$

Now we can find the tangent space and say the Lie algebra is isomorphic to,

$$\mathfrak{g} \cong \{\xi \in \mathfrak{X}(M) \colon\ \xi(0,\, y) = \xi(1,\, -y),\, \xi(x,\, -1) = \xi(x,\, 1) = 0\}.$$

Then this is a Lie algebra because it doesn't introduce discontinuities into the system and performs a smooth genus changing deformation between the square and the strip.

Now we use the theory of subsection 2.2.2. This produces similar equations to above, just with different boundary conditions. This produces,

$$\begin{aligned} \frac{\partial \mathbf{u}^\flat}{\partial t} &= \pounds_{\mathbf{u}} \mathbf{u}^\flat + \mathrm{d}f \qquad \mathbf{u} \in \mathfrak{X}([-1,\, 1]^2) \\ \mathbf{u}(-1,\, y) &= \mathbf{u}(1,\, -y) \\ \mathbf{u}(x, -1) &= \mathbf{u}(x,\, 1) = 0 \end{aligned} \tag{4.2}$$

where f is a 0-form.

As we have a continuous manifold with just a boundary condition that vanishes. Our Casimir functions will be equivalent to Vannaste's, which are the generalised enstrophies. However the non-orientability of the Möbius strip produces the new constraint that only for even functions in the enstrophy. This is because of the well known fact that we can't integrate over non-orientable surfaces. Hence we turn to the theory of pseudo-forms. The theory can be found in Frankel's Geometry of Physics [9]. This leads to the idea that we need even functions, f, to assert the following are conserved,

$$C_f(\omega) = \int_M f(\omega)\mu.$$

4.3 Rotating Stratified Fluid

We finally aim to derive the Euler equations for a rotating stratified fluid. We will follow a similar procedure to the example after the advected quantity theorem. We can take the Lagrangian,

$$L = \int_M \left[\rho(1+b)\left(\frac{1}{2}|\dot{\eta}^2| + \dot{\eta}\cdot\mathbf{R} - gz\right) - p(\rho-1)\right].$$

We can now reduce this down to the following reduced lagrangian, much like the compressible fluid example.

$$\ell(\mathbf{u},\, D,\, b) = \int_M \left[\rho_0(1+b)D\left(\frac{1}{2}|\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{R} - gz\right) - p(D-1)\right].$$

In order to deal with more conserved quantities the Euler-Poincaré equations become,

$$\frac{\partial}{\partial t}\frac{\delta \ell}{\delta \xi} = -\mathrm{ad}^*_\xi \frac{\partial \ell}{\partial \xi} + \frac{\partial \ell}{\partial a}\diamond a + \frac{\partial \ell}{\partial b}\diamond b, \tag{4.3}$$

where a and b are advected quantities. These advected quantities satisfy,

$$\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right) a = \left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right) b = 0.$$

We can now find the variational derivatives. By performing the calculations we arrive at,

$$\begin{aligned}
\left\langle \frac{\delta \ell}{\delta \mathbf{u}}\,,\phi\right\rangle &= \left\langle \rho_0 D(1+b)(\mathbf{u}^\flat + \mathbf{R})\,,\phi\right\rangle \\
\left\langle \frac{\delta \ell}{\delta b}\,,\phi\right\rangle &= \left\langle \rho_0 D\left(\frac{1}{2}|\mathbf{u}^\flat|^2 + \mathbf{u}\cdot\mathbf{R} - gz\right)\,,\phi\right\rangle \\
\left\langle \frac{\delta \ell}{\delta D}\,,\phi\right\rangle &= \left\langle \rho_0(1+b)\left(\frac{1}{2}|\mathbf{u}^\flat|^2 + \mathbf{u}\cdot\mathbf{R} - gz\right) - p\,,\phi\right\rangle.
\end{aligned}$$

These can now be plugged into Equation (4.3). We then get,

$$\begin{aligned}
D(1+b)\frac{\partial}{\partial t}(\mathbf{u}^\flat + \mathbf{R}) + D(1+b)\pounds_{\mathbf{u}}(\mathbf{u}^\flat + \mathbf{R}) + D\left(\frac{1}{2}|\mathbf{u}^\flat|^2 + \mathbf{u}^\flat\cdot\mathbf{R} - gz\right)\diamond b& \\
+ (1+b)\left(\frac{1}{2}|\mathbf{u}^\flat|^2 + \mathbf{u}^\flat\cdot\mathbf{R} - gz\right)\diamond D = 0.&
\end{aligned}$$

This is very complicated, so we can simplify and put this equations back into coordinates. This leaves us with,

$$\frac{\partial \mathbf{u}^\flat}{\partial t} + \mathrm{d}(\mathbf{u}^\flat\cdot\mathbf{R}) - g\hat{\mathbf{z}} + P(\mathbf{u}^\flat,\, b,\, D) = 0.$$

We note that P is our pressure gradient term and is defined by,

$$P(\mathbf{u}^\flat,\, b,\, D) = \frac{1}{\rho_0}\left[\frac{\left(\frac{1}{2}|\mathbf{u}^\flat|^2 + \mathbf{u}^\flat\cdot\mathbf{R} - gz\right)\diamond b}{1+b} + \frac{\left(\frac{1}{2}|\mathbf{u}^\flat|^2 + \mathbf{u}^\flat\cdot\mathbf{R} - gz\right)\diamond b}{D}\right].$$

For these equations we have the usual conserved quantities and Noether Theorem tells us that,

$$\oint_\gamma \frac{1}{D}\frac{\delta \ell}{\delta \mathbf{u}} = \oint_\gamma \rho_0(1+b)(\mathbf{u}+\mathbf{R}) = \oint_\gamma \nabla p \cdot \,\mathrm{d}\mathbf{x}.$$

Further the advection equation for b gives us potential vorticity conservation given $q = \nabla b \cdot \operatorname{curl}(\mathbf{u}+\mathbf{R})$.

Chapter 5

Conclusion

This chapter draws the dissertation to a close. In this thesis we have seen several ideas surrounding the ideas of geometric and topological fluid dynamics and further Geometric Mechanics. We have presented and exposed many results in the area and produced simple examples towards the applications in this area. We have studied boundary conditions on manifolds with boundary and then studied conserved quantities of these systems. We saw the examples of axisymmetric flow, the Möbius strip and rotating stratified fluid. Geometric Mechanics is still a very large area and my two theses have only touched the surface in this area. My PhD thesis will be based upon this work and bring together many more areas.

In the final chapter we saw applications of all the preceding chapters. In Chapter 1 we explored the history of the area and presented some relevant pure background of the thesis. We studied a small amount of Symplectic Geometry, an area that has vast branches and one I wish we could have explored more. This will be the basis of further work into multisymplectic reduction and the related numerical methods. In the second chapter we studied reduction theorems on the diffeomorphism group and in both Lagrangian and Hamiltonian frameworks. These presented some extremely nice groundwork for the third chapter and were something that very nicely lead from the author's undergraduate thesis onto the PhD thesis. This chapter went into great detail in order to consider the third chapter. In Chapter 3 we studied conserved quantities in three different ways. Firstly via Noether Theorems, a usual method to derive conserved quantities. The second Casimirs, a slightly classical way to derive the conserved quantities and then a more interesting way via Reeb Graphs. The Reeb graph method provides an interesting approach that hasn't yet been generalised, but provides an idea of how we can find all the Casimirs of more complicated systems.

Although there is now nearly a hundred pages of Geometric Mechanics written by me, we have again only taken select areas to study. You only have to glance at Darryl Holm's publication list to tell us that. The author undertook some extra work in the area of numerics while writing this thesis, they implemented papers by Cotter and Bridges [7, 23] to show that the conserved quantities that were derived produced the expected numerical observations. The methods were based off a finite element approach. The author's PhD thesis will give a broader idea of the area and develop new multisymplectic numerical methods for these problem.

There is also work in the area towards ideas further than Euler's equations. The author spent some time proving theorems and results about a set of equations called the Lagrangian averaged Navier Stokes which give a Geometric Mechanics insight into the Navier Stokes problem. These provide interesting theory relating to different solutions between the Euler Equations and Navier Stokes. This area was discussed in detail with Darryl Holm and there's a great welth of knowledge toward ocean modelling using Stochastic Averaging Lagrangian Transport.

In addition there is another whole area towards using variational principles that are different to the usual Hamilton's principle. Work by Dr Hamid Alemi Ardakani [2, 1] in this area applied the mathematics to wave energy generators and uses Geometric Mechanics in general to study waves.

We conclude this thesis and the authors time at Oxford in a very similar way to my first thesis. Although one year on, a lot more knowledgable about mathematics and a better mathematician for it the author notes that they have equally enjoyed writing this thesis. We have studied some classical results, produced new examples and laid a perfect basis for a PhD. This thesis aimed to bring together the skills and knowledge the author attained at their year in Oxford and it has succeeded. Further Geometric Mechanics is still a very useful tool to the world and one that the author will dedicate their life to. Solving the more pertinent problems with the most abstract of objects. Solving problems need not be easy, but Geometric Mechanics acts as a tool box to make it easy.

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