Chapter 3

Conservation

> As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection.
>
> Joseph-Louis Lagrange

In Noether's very influential paper in 1918, named 'Invariante Variationsprobleme' [13] she stated one of the most important theorems in the area of conservation. This was to be known as Noether's theorem. She stated, *'Every differentiable symmetry generated by local actions has a corresponds a conserved current.'* This innocuous looking theorem would lead to many years of work to find these conserved currents. The theorem said a little about how to find these quantities for the Euler-Lagrange, but there is an analogous result for Euler-Poincaré. These theorems then need to be derived for every new type of Euler-Poincaré equation. Noether's theorems apply to the Lagrangian framework. In order to calculate conserved quantites for the Hamiltonian framework you go to consider Casimir's and Poisson brackets.

In this penultimate chapter we will discuss the conservation properties of the Euler-Poincaré and Lie-Poisson equations. There are two different routes that we will take to find the quantities depending on what framework we work under. Firstly we will return to the Lagrangian framework and study the Noether Theorems for Euler-Poincaré theory and then consider a generalisation of these theorems called the Kelvin-Noether theorems which relate to first integrals over a path in our flow. We will then study the Hamiltonian point of view by returning to Lie-Poisson brackets, considering Casimirs and showing that certain functions that make the brackets vanish are conserved. We will also consider the relation between conserved quantities and Reeb graphs, another potential area for exploration in future work. Finally, we will attempt to show numerically that these quantities are conserved and conclude with ideas relating to how we can extend numerical methods with the preceeding theory to guarentee that the conserved quantites are conserved.

3.1 Noether Theorems

Noether Theorems can be derived from the variational principle and reduction. The process involves considering the terms we made vanish to produce the equations. We can state and prove the Noether Theorem for Euler-Poincaré equations with advected parameters.

Theorem 3.1.1 (Noether Theorem for Euler-Poincaré with advected parameters)**.** *Each symmetry vector field of the Euler-Poincaré reduced lagrangian for the infinitesimal variations,*

$$\delta u = \dot{\nu} - \mathrm{ad}_u \nu \qquad \delta a = -\nu a,$$

corresponds to an integral of the Euler-Poincaré motion and a conserved quantity,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_M \left\langle \frac{\delta \ell}{\delta u}, \nu \right\rangle \mu = 0$$

Proof. We will consider the derivation of the Euler-Poincaré equations for advection parameters. Then jump to the integration by parts of the $\frac{\partial u}{\partial t}$ term.

$$\begin{aligned} 0 &= \int_{t_1}^{t_2} \int_M \left\langle \frac{\delta \ell}{\delta u}, \frac{\partial \nu}{\partial t} \right\rangle - \left\langle \mathrm{ad}_u^* \frac{\partial \ell}{\partial u}, \nu \right\rangle + \left\langle a \diamond \frac{\delta \ell}{\delta a}, \nu \right\rangle \mathrm{d}t \\ &= \int_{t_1}^{t_2} \int_M \left\langle -\frac{\partial}{\partial t} \frac{\delta \ell}{\delta u} - \mathrm{ad}_u^* \frac{\partial \ell}{\partial u} + a \diamond \frac{\delta \ell}{\delta a}, \nu \right\rangle \mathrm{d}t + \int_M \left\langle \frac{\delta \ell}{\delta u}, \nu \right\rangle \Bigg|_{t_1}^{t_2} \mu. \end{aligned}$$

We now note that the first pairing is just the Euler-Poincaré equations. Hence we now can write,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_M \left\langle \frac{\delta \ell}{\delta u}, \nu \right\rangle \mu = 0,$$

as we use the fundemental theorem of calculus of variations. □

3.1.1 Vorticity and Helicity

We can now prove two lemmas for the specific case of the Euler Equations in barotropic fluids. The advective quantity is going to be $a = \rho \, \mathrm{d}V$ and then the infinitesimal symmetry for a becomes,

$$\pounds_\eta(\rho \, \mathrm{d}V) = \mathrm{d}(\eta \lrcorner \rho \, \mathrm{d}V) = 0.$$

Using rules from the appendix this can now be written as for some vector function, $\boldsymbol{\Psi}$,

$$\eta \lrcorner \rho \, \mathrm{d}V = \mathrm{d}(\boldsymbol{\Psi} \cdot \mathrm{d}\mathbf{x}) = \mathrm{curl}\, \boldsymbol{\Psi} \cdot \mathrm{d}\mathbf{S}. \tag{3.1}$$

Lemma 3.1.2 (Conservation of Vorticity)**.** *In the Euler-Poincaré equations for advected quantities, the following holds and relates to vorticity being conserved,*

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \pounds_{\mathbf{u}} \right) \left(\mathrm{curl} \frac{1}{\rho} \frac{\delta \ell}{\delta \mathbf{u}} \cdot \mathrm{d}\mathbf{S} \right) = 0. \tag{3.2}$$

Proof. We start with Noether's theorem for these equations and then perform a calculation.

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left\langle \frac{\delta \ell}{\delta u}, \eta \right\rangle \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_M \frac{\delta \ell}{\delta u}\cdot \eta \,\mathrm{d}V \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_M \frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\cdot \mathrm{d}\mathbf{x}\wedge \eta \lrcorner (\rho \,\mathrm{d}V) \\
&= \int_M \frac{\partial}{\partial t}\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}} \wedge \mathrm{d}(\boldsymbol{\Psi}\cdot \mathrm{d}\mathbf{x}) + \frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\wedge \frac{\partial}{\partial t}\mathrm{d}(\boldsymbol{\Psi}\cdot \mathrm{d}\mathbf{x}) \\
&= -\int_M \left(\frac{\partial}{\partial t}\mathrm{d}\left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\cdot \mathrm{d}\mathbf{x}\right) + \pounds_{\mathbf{u}}\,\mathrm{d}\left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\cdot \mathrm{d}\mathbf{x}\right)\right)\wedge (\boldsymbol{\Psi}\cdot \mathrm{d}\mathbf{x}) \\
&= -\int_M \left(\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right)\mathrm{d}\left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\cdot \mathrm{d}\mathbf{x}\right)\right)\wedge (\boldsymbol{\Psi}\cdot \mathrm{d}\mathbf{x})
\end{aligned}
$$

Hence we can say that, as we know $\Psi\cdot \mathrm{d}x \neq 0$. Then the result appears after applying (3.1),

$$
\begin{aligned}
0 &= \left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right)\mathrm{d}\left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\cdot \mathrm{d}\mathbf{x}\right) \\
&= \left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right)\left(\mathrm{curl}\,\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\cdot \mathrm{d}\mathbf{S}\right).
\end{aligned}
$$

□

We now present two major ideas in this area. Firstly, iterated conserved quantities. Let us consider another important theorem, called Ertel's theorem.

Theorem 3.1.3 (Ertel's Theorem). *If a quantity a satisfied the advection equation, and η satisfy $\delta\eta = \dot{\nu} + \mathrm{ad}_\eta \nu$ for the labelling symmetry. Then $\pounds_\eta a$ is also advected.*

Proof. By a simple substitution we find,

$$
\pounds_\eta\left(\frac{\partial}{\partial t} + \pounds_u\right) a = \left(\frac{\partial}{\partial t} + \pounds_u\right)\pounds_\eta a = 0.
$$

This is the advection equation. Hence $\pounds_\eta a$ is advected by the flow. □

Now consider a conserved quantity, $c(t)$, this is carried by the flow. Hence we can say that $\pounds_u c(t)$ is also conserved by the Euler-Poincaré equations. Hence we can now consider a conserved quantity using the vorticity in (3.2). Hence we now can set the vorticity as our conserved 2-form in (3.1).

$$
\mathrm{d}(\boldsymbol{\Psi}\cdot \mathrm{d}\mathbf{x}) = \mathrm{d}\left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\cdot \mathrm{d}\mathbf{x}\right).
$$

This will lead us to a new conserved quantity called Helicity. Helicity is a topological invariant sometimes known as the Hopf invariant. It measures the knottedness of the vortex lines and can also be use to caclulate the number of linkages. As the vortex lines are baked in the Lagrangian and hence the flow this quantity is always conserved.

Lemma 3.1.4 (Conservation of Helicity). *In the Euler-Poincaré equations for advected quantities, the following holds and relates to the Helicity being conserved.*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M \frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}} \cdot \operatorname{curl}\left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\right)\, \mathrm{d}V$$

Proof. We start at the third step of the previous argument and then move forward in a different way,

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\int_M \left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}} \cdot \mathrm{d}\mathbf{x}\right) \wedge \eta \lrcorner (\rho\, \mathrm{d}V) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_M \left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}} \cdot \mathrm{d}\mathbf{x}\right) \wedge \mathrm{d}\left(\mathbf{\Psi} \cdot \mathrm{d}\mathbf{x}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_M \left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}} \cdot \mathrm{d}\mathbf{x}\right) \wedge \mathrm{d}\left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}} \cdot \mathrm{d}\mathbf{x}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_M \frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}} \cdot \operatorname{curl}\left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\right)\, \mathrm{d}V
\end{aligned}$$

Therefore we can write Helicity as,

$$\mathcal{H} := \int_M \frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}} \cdot \operatorname{curl}\left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\right)\, \mathrm{d}V$$

□

3.1.2 Kelvin-Noether Theorem

We can now find other conservation quantities using similar ideas to above. Consider the following change of variables,

$$\oint_{\gamma_t} \frac{1}{\rho_t}\frac{\delta \ell}{\delta \mathbf{u}} = \oint_{\gamma_0} \eta^*\left[\frac{1}{\rho_0}\frac{\delta \ell}{\delta \mathbf{u}}\right] = \oint_{\gamma_0} \frac{1}{\rho_0}\eta^*\left[\frac{\delta \ell}{\delta \mathbf{u}}\right].$$

Now we can use the definition of the Lie derivative to write,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\eta^*\alpha) = \eta^*\left(\frac{\partial}{\partial t}\alpha + \pounds_{\mathbf{u}}\alpha\right).$$

Hence we can write,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\gamma_t} \frac{1}{\rho_t}\frac{\delta \ell}{\delta \mathbf{u}} &= \oint_{\gamma_0} \frac{1}{\rho_0}\eta^*\left(\frac{\partial}{\partial t}\alpha + \pounds_{\mathbf{u}}\alpha\right) \\
&= \oint_{\gamma_t} \frac{1}{\rho_t}\left(\frac{\partial}{\partial t}\alpha + \pounds_{\mathbf{u}}\alpha\right) \\
&= \oint_{\gamma_t} \frac{1}{\rho_t}\frac{\delta \ell}{\delta a} \diamond a.
\end{aligned}$$

The last step is from just considering the Euler-Poincaré equatios for advected parameters. From this we present the Kelvin-Noether theorem for advected Euler-Poincaré,

Theorem 3.1.5 (Kelvin-Noether Theorem for Euler Poincaré Equations)**.** *For the Euler-Poincaré equation with advected quantities, we can write the following,*

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_{\gamma_t} \frac{1}{\rho_t} \frac{\delta \ell}{\delta \mathbf{u}} = \oint_{\gamma_t} \frac{1}{\rho_t} \frac{\delta \ell}{\delta a} \diamond a.$$

We have two remarks,

Remark 3.1.6. *For basic Euler-Poincaré equations, the following is a conserved quantity,*

$$\oint_{\gamma} \frac{1}{\rho} \frac{\delta \ell}{\delta \mathbf{u}}.$$

Remark 3.1.7. *We can write the Euler-Poincaré equations in Kelvin-Noether form,*

$$\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}} \right) \frac{\delta \ell}{\delta \mathbf{u}} = \frac{\delta \ell}{\delta a} \diamond a.$$