An assortment of theories relating to the Euler Equations on Manifolds

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To all the mathematicians and my parents who stood before me and blocked me from being a Game Developer.

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Abstract

In this thesis, we discuss and present ideas surrounding fluid flow on manifolds. We model fluid flow as a Fréchet group of structure-preserving diffeomorphisms and use Euler-Poincaré and Lie-Poisson reduction to derive geodesic equations. We then study the conservation quantities of these equations using (Kelvin-)Noether Theorems and Casimir Theory.

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Chapter 1

Motivation, Introduction and Background

Never reduce on the Hamiltonian and never code your numerical methods.

Darryl D. Holm

Geometric Mechanics is based on deriving equations from symmetries baked into physical systems. In most physical systems, these symmetries are left unexploited and lead to a variety of theories and further information about our systems that is very useful. We can derive so-called conserved quantities of these physical systems. Conserved quantities give insight into how the system behaves and can help when we attempt to solve these systems numerically.

Historically, the area can be traced back to least action principles applied to mechanical systems by Maurperius in 1744. Then Optics used the ideas later, and Euler, Liebniz and Lagrange started the boom of the least action principle. Lagrange deduced the Euler-Lagrange Equations in the 1750s,

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} = 0.$$

Hamilton deduced the 'Hamilton' least action principle in 1835,

$$\delta \int_{t_1}^{t_2} F[\mathbf{x}]\ \mathrm{d}\mathbf{x} = 0.$$

From Hamilton's principle, you can derive equations from a Lagrangian. Then, in 1901, Poincaré wrote down a new set of equations called the Euler-Poincaré equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta \ell}{\delta \xi} - \mathrm{ad}^*_\xi \frac{\delta \ell}{\delta \xi} = 0.$$

While the Lagrangian framework was developing, there was a concurrent development for the Hamiltonian framework. The history here is more convoluted as the path to understanding was rewritten several times. This framework takes a Lagrangian and then considers a Legendre transform of the Lagrangian. This involves using some Lagrange multipliers, which can be written as Clebsch variables, leading to Hamilton's Equations. The canonical Hamilton's equations using the trivial Clebsch variables are,

$$\frac{\partial \mathbf{q}}{\partial t} = \frac{\partial H}{\partial \mathbf{p}} \qquad \frac{\partial \mathbf{p}}{\partial t} = -\frac{\partial H}{\partial \mathbf{q}}.$$

If we consider these equations in higher generality. That is, written on some manifold. Not only do Hamilton's equations conserve the Hamiltonian, but they also conserve the canonical symplectic form.

$$\omega = \mathrm{d}p_i \wedge \mathrm{d}q_i$$

where p_i and q_i are local coordinates and momenta. By a quick calculation, we can verify that,

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial q} := \{F,\, H\},$$

where H is the Hamiltonian. You can write a Hamiltonian system this way, writing the Lie-Poisson bracket on the right-hand side. This leads to the study of conserved quantities when the Lie bracket vanishes. These are Casimirs.

One of the current main figures in modern Geometric Mechanics is Professor Darryl Holm at Imperial College. Holm was inspired by work done before him by Poisson, Lie, Arnold, Marsden, Axton and Hamilton, to name a few. His idea was to move forward and use the theory before him and to apply this, conglomerate it and present a new area of mathematics— Geometric Mechanics. Trained as a Physicist, Darryl presented ideas on Geometry in his work at Los Alamos labs. He was then elected as the first director of the Nonlinear Systems Institute after presenting how Geometry can be used to create new solution criteria to equations of interest.

From there, the idea spread further, with Darryl writing his infamous book, The Green Book' to all Geometric Mechanists [14]. While Darryl continued to work at the coalface, new ideas were developing, and new mathematicians were starting work. The ideas of Symplectic Geometry were starting to be developed in the area by Bridges. Further, the applications of this work to Numerical Analysis were starting to form with mathematicians such as Cotter and Gay-Balmaz hearing of Geometric Mechanics and getting interested in the conservation of quantities in numerical schemes and how this relates to Geometric Mechanics.

Then, the question of industrial applications was raised with the team of Professor Tom Bridges, Dr Matt Turner, Dr Cesare Tronci and Dr Hamid Alemi Ardakani. They were interested in the application of this work in wave energy generators. They studied new variational principles similar to Hamilton's Principle, such as the Luke-Bateman principle,

$$\delta \int_{t_1}^{t_2} \int_{\mathcal{Q}} -\rho \left(\frac{\partial \Phi}{\partial t} + \frac{1}{2}\nabla\Phi \cdot \nabla\Phi - \nabla\Phi \cdot (\mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}) + U \right) \, \mathrm{d}\mathcal{Q} \; \mathrm{d}t = 0.$$

These are more general and consider more complicated fluid interactions, such as coupling and free surfaces. This is still an ongoing project and one that the author will be joining next year under Dr Hamid Alemi Ardakani.

We can now move forward and consider this thesis. Alongside this work, Arnold wrote a lot of mathematics relating to the Ideal Fluid Euler Equations and Euler-Poincaré reduction. In this thesis, we will review that work, then use other sources and work forward to review and create a new theory for the Euler Equations on general manifolds. One of our main examples of the thesis will be to recreate the paper of Vanneste [27] in the language of Geometric Mechanics. This thesis aims twofold: to understand and expose the current research in Geometric Mechanics and to fill holes in the theory by adding examples. This theory was written in very abstract ways so we aim to provide more intuitive descriptions and produce informative examples and intuitive proofs. The new material in this thesis relates to the examples, proofs and explanations.

The motivation for using this type of mathematics is as follows. Many PDE systems can be described in a coordinate-dependent environment. The derivation of these equations is usually found by considering conservation laws and then fitting terms to that, then integrating and pulling equations out. This assumes our system already has these conserved quantities. These derivations are made quite difficult when we consider complex chemical behaviour. These conservation laws are no longer as obvious. Hence, we propose to derive these 'standard' equations by considering the energy. We consider the Lagrangian or the Hamiltonian,

$$L(q, \dot{q}) = K(q, \dot{q}) - V(q, \dot{q}) \qquad H(q, \dot{q}) = K(q, \dot{q}) + V(q, \dot{q}).$$

Then, we can take a system, say a spherical pendulum, and derive the Hamiltonian and Lagrangian. This example is from Arthur's 'Lie Groups and Applications to Geometric Mechanics ' [5]. Consider a spherical pendulum as in the figure below.

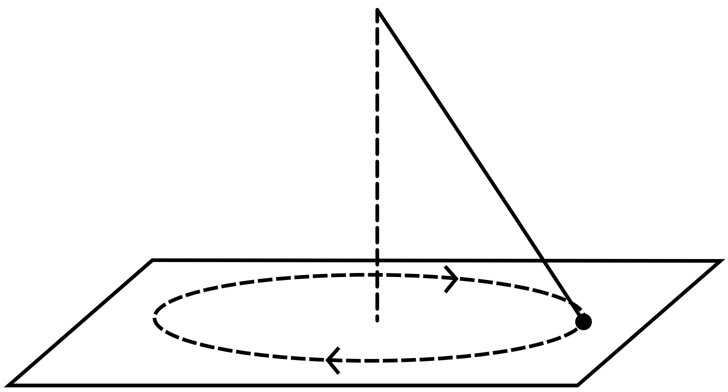


Figure 1.1: Spherical Pendulum.

Then, we can define the Lagrangian of this system as,

$$L(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2} m |\dot{\mathbf{x}}|^2 - mg\mathbf{x} \cdot \mathbf{e}_3.$$

We are interested in considering the motion from the spatial frame. We can use a body to space map through $\mathbf{x} \mapsto \mathbf{R}(t)X$, where $\mathbf{x}, X \in \mathbb{R}^3$ and $\mathbf{R}(t) \in \mathrm{SO}(3)$. We can then move the Lagrangian from the body frame to the spatial frame,

$$L(\mathbf{R},\dot{\mathbf{R}}) = \frac{1}{2}m|\dot{\mathbf{R}}(t)X|^2 - mg\mathbf{R}(t)X\cdot\mathbf{e}_3. \tag{1.1}$$

Then by manipulating the Lagrangian (1.1), we can now rewrite it again letting $\Gamma(t) = \mathbf{R}(t)\mathbf{e}_3$,

$$L(\mathbf{R},\dot{\mathbf{R}},\mathbf{\Gamma}) = \frac{1}{2}m|\dot{\mathbf{R}}X|^2 - mg\mathbf{\Gamma}\cdot X.$$

Then, we can notice that by considering the symmetric part of the Lagrangian, is right symmetric and the whole Lagrangian has a symmetry-breaking parameter. Therefore we take $g \mapsto g\mathbf{R}^{-1}$. So,

$$L(\mathbf{R}\mathbf{R}^{-1},\dot{\mathbf{R}}\mathbf{R}^{-1},\tilde{\mathbf{\Gamma}}) := \ell(\hat{\mathbf{\Omega}},\tilde{\mathbf{\Gamma}}) = \frac{1}{2}m\mathbb{I}|\hat{\mathbf{\Omega}}|^2 - mg\tilde{\mathbf{\Gamma}}\cdot X,$$

where $\mathbb{I} = |X|^2 I - XX^T$. Then you can plug this into the Euler-Poincaré equations for right invariant systems on SO(3) and get,

$$\mathbb{I}\hat{\mathbf{\Omega}}_t = \mathbb{I}\hat{\mathbf{\Omega}}\times\mathbf{\Omega} + mgX\times\tilde{\mathbf{\Gamma}}.$$

This motivates how we can go from Lagrangian and Hamiltonian descriptions of systems to very informative equations for systems in non-standard variables, such as the angular momentum, $\hat{\mathbf{\Omega}}$. We now aim for this thesis to derive similar equations in fluid dynamics situations. The thesis will be structured as follows,

- This first chapter will lay down the introductory material for the theory we will use later. This part of the thesis is critical for readers not well versed in Geometric Mechanics as it lays out the operators and functionals we use without definition in the rest of the thesis. This chapter covers Symplectic Geometry and Geometric Fluid Dynamics. For an introduction to Differential Geometry see Appendix A.

- Then, we will study the mechanics and PDEs for different fluid systems on manifolds. We will write these PDEs in the language of exterior calculus. We will learn the Lagrangian and Hamiltonian frameworks through Euler-Poincaré reduction and Lie-Poisson reduction. We will derive the associated equations and introduce the canonical Lie-Poisson bracket. We will then describe equations with advected quantities and give some motivation for the EP-Diff equations.

- Finally, to complete the theory, we will then study the conservation laws of our systems with Noether and Casimir Theory. We will prove the generalised Noether Theorem and Kelvin-Noether Theorem for the Euler-Poincaré equations on SDiff(M) and then study the Casimirs and introduce the non-canonical Lie-Poisson bracket.

- To conclude the thesis, we will then study several examples relating to the theory. Such as axisymmetric fluid flow on a cylinder, Rotating Stratified Fluid and then the Möbius strip.

1.1 Introduction to Geometric Fluid Dynamics

We will follow Chapter 1 of Arnold and Khesin's 'Topological Methods in Hydrodynamics' [29] and, more generally, Holm, Schmah and Stoica's 'Geometric Mechanics and Symmetry' [14].

We will start with some definitions relating to Group Theory and Lie Theory. We can define a group,

Definition 1.1.1 (Group on a Manifold)**.** *A set G of smooth transformations of a manifold M onto itself is a group if,*

1. *Given two transformations $g, h \in G$, the composition $g \circ h$ is in G,*
2. *Given some $g \in G$, the inverse, g^{-1}, is also in G.*

We define the group like this because then the definition of the Lie Group becomes obvious. If the functions induced by the two conditions in Definition 1.1.1 are smooth, then we have a *Lie group*. There are some pertinent examples of Lie groups that are useful to mechanics. We use the Lie group of length preserving rotations for rigid body dynamics, SO(3). For a detailed survey of the Geometric Mechanics of SO(3), see [5]. However, we are interested in the volume-preserving diffeomorphisms, SDiff(M), for hydrodynamics.

Definition 1.1.2 (Diffeomorphism)**.** *Let M and N be manifolds, then a diffeomorphism is a bijective map $\phi : M \to N$ such that both ϕ and ϕ^{-1} are differentiable. The group of diffeomorphisms are denoted* Diff(M)*.*

We can further extend this to volume-preserving diffeomorphisms,

Definition 1.1.3 (Volume Preserving Diffeomorphism)**.** *Let $\phi : M \to N$ be a diffeomorphism and μ be the volume element of M. Then we say ϕ is volume preserving if $\phi(\mu)$ is the volume element of N. The group of volume preserving diffeomorphisms are denoted* SDiff(M)*.*

We say that the volume-preserving diffeomorphism describes fluid. We mean that a diffeomorphism can describe the flow of a particle. Given a terminal point, we can describe all flows of the fluid particle by a diffeomorphism; they show the path the particle takes. We can see this in Figure 2.1. In SO(3) we have smooth left and right actions, this makes SO(3) a Lie group. However, diffeomorphism are not smooth by left action. Therefore they are not a Lie group. They are however, a topological group or a Fréchet group, that provides the structure needed to do Geometric Mechanics by right action. This marvelous result means although we don't have a Lie group, we have a Fréchet group and can do all of the mathematics required [22].

As an example, we can define the kinetic energy of a particle in a fluid as the following,

$$E = \frac{1}{2} \int_M \varphi_t^2 \, \mathrm{d}x.$$

This turns out to be nicely right invariant. Let $G = \mathrm{SDiff}(M)$. If we perform the operation, $R_h : G \to G$, defined by $R_h(g) = gh$, on the kinetic energy, we can write it as the following,

$$R_h E = \frac{1}{2} \int_M (\phi_t h)^2 \, \mathrm{d}x.$$

We note that h is volume preserving, then it is just a relabelling of the fluid particles, and so $R_h E = E$. This is known as the *relabelling symmetry* and is why the rest of this thesis holds. In the Lagrangian framework we consider blobs of particles, whilst in the Eularian framework we consider singular particles. The fact we consider the Lagrangian framework as equivalent to blobs of particles means that if we relabel the particles in the blob the dynamics don't change. This is the relabelling symmetry. It is equivalent to saying that for $\eta \in \mathrm{Diff}(M)$, then $\eta\mu = \mu$ where μ is the volume form.

We also note the left multiplication by h can be written as $L_h(g) = hg$.

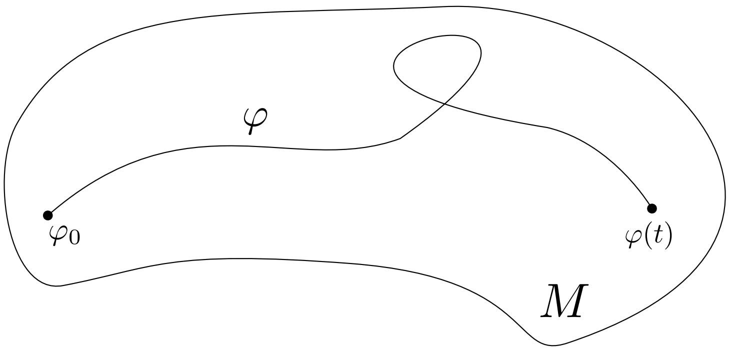


Figure 1.2: The path of a fluid particle.

1.1.1 Adjoint, Coadjoints and Group Actions

We spend the next subsection generalising and reviewing the content from [5] but for structure-preserving diffeomorphisms. We shall define the inner automorphism that quickly leads to the adjoint representation. We shall also discover the nature of the Lie algebra of $\mathrm{SDiff}(M)$.

We can consider the L_h and the R_h operators we defined before and define the inner automorphism.

Definition 1.1.4 (Inner Automorphism)**.** *The inner automorphism, $A_g : G \to G$ is defined by, $A_h = L_h R_{h^{-1}}$. Given some $g \in G$, $A_h g = hgh^{-1}$ can also be written.*

This can then be used to talk about the adjoints of the group and algebra. For $\mathrm{SDiff}(M)$, our Lie algebra will be the space of divergence-free vector fields in M. We also quickly need to note that to differentiate a map on a manifold, we denote it as follows. Given some $F : M \to M$, then the differential at a point $x \in M$ is $F_*|_x : T_x M \to T_{F(x)} M$. We can now define our Ad_g and ad_g on $\mathrm{SDiff}(M)$ and $\mathfrak{g}$.

Definition 1.1.5 (Group Adjoint Operator)**.** *The differential of A_g at the unit is called the group adjoint operator,* Ad_g*,*

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g} \qquad \mathrm{Ad}_g\,\xi = (A_{g*}|_e)\xi \qquad\qquad \xi \in \mathfrak{g} = T_eG$$

We can further define the adjoint orbit,

Definition 1.1.6 (Adjoint Orbit)**.** *Fix $\xi \in \mathfrak{g}$. The set of $\mathrm{Ad}_g\,\xi$ images of ξ under the action of Ad_g, $g \in G$, is called the adjoint orbit of ξ.*

If $g \in \mathrm{SDiff}(M)$, then $\mathrm{Ad}_g\,\xi = g\xi g^{-1}$ is the structure-preserving diffeomorphisms acting on a vector field. Now if we let $g(0) = e$ and $\dot{g}(0) = \eta$. Then we can define the adjoint representation of the Lie algebra,

Definition 1.1.7 (Adjoint Representation of the Lie algebra)**.** *If we take the differential of* Ad *at the identity, we have the adjoint representation of the Lie algebra,*

$$\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \qquad \mathrm{ad}_\xi = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}_{g(t)}\,.$$

We can now take the derivative of the group adjoint on $\mathrm{SDiff}(M)$. We can then get that $\mathrm{ad}_\xi\eta = -[\xi, \eta]$ for $\xi, \eta \in \mathfrak{X}(M)$ such that $\mathrm{div}\,\xi = \mathrm{div}\,\eta = 0$.

We can further consider the coadjoint versions of the above. These are maps to the cotangent space and the dual to the Lie algebra. The dual to the Lie algebra can be described for arbitrary dimension. Let $G = \mathrm{SDiff}(M)$ the group of volume-preserving diffeomorphisms on a manifold M with boundary ∂M. The commutator of divergence-free vector fields on M is just $-\{v, w\}$. Then we have the following result.

Theorem 1.1.8 (Dual of $\mathrm{SVect}(M)$)**.** *The Lie algebra, $\mathfrak{g}$, of the group G is naturally identified with the space of closed differential $(n-1)$-forms on M vanishing on ∂M. A divergence-free vector field v is identified with the $(n-1)$-form $v \lrcorner \mu$ where μ is the volume form of M. Therefore, the dual to the Lie algebra is $\mathfrak{g}^* = \Omega^1(M)/\,\mathrm{d}\Omega^0(M)$.*

Proof. The proof here has two parts. Firstly we prove that the space of all $(n-1)$-forms $v \lrcorner \mu$ is naturally identified to $\Omega^1(M)/\Omega^0(M)$. Then we can prove that $\mathfrak{g} \cong \Omega^1(M)/\Omega^0(M)$.

Recall Cartans magic formula, $\pounds_X\alpha = X \lrcorner \mathrm{d}\alpha + \mathrm{d}(X \lrcorner \alpha)$. If $\alpha = \mu$ and $X = \mathbf{u}$, some vector field, it reduces to just saying, $\pounds_{\mathbf{u}}\mu = \mathrm{d}(\mathbf{u} \lrcorner \mu)$. We are working over the space of volume preserving diffeomorphisms. So, when we consider the Lie derivative of the volume form over an algebra element, we want it to vanish. Hence, we require $v = \mathbf{u} \lrcorner \mu$ to be closed. Therefore, $v \in \Omega^1(M)/\Omega^0(M)$. As $\mathbf{u}$ was arbitrary, v is just any $(n-1)$-form, so we have our natural identification.

Now we seek an isomorphism, $\mathfrak{g} \cong \Omega^1(M)/\Omega^0(M)$. This is more complicated and can be found in Theorem 8.3 (pg 42) of 'Topological Methods of Hydrodynamics ' [29]. □

As we are considering fluid flows on these manifolds. If M is simply connected, we can say that the dual algebra, $\mathfrak{g}^*$, is just the vorticities.

1.1.2 Lie-Poisson Brackets

As introduced in the introduction, we can write a Hamiltonian system as,

$$\dot{F} = \{F, H\},$$

where $F : M \to \mathbb{R}$ and H is the Hamiltonian. The right-hand side is just the Canonical Lie-Poisson bracket. We can write this as,

$$\{F, G\} := \sum_i \frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial F}{\partial q_i}.$$

This is the simplest Poisson bracket, and so is called the canonical bracket. These brackets are much like the Lie Brackets reviewed in [14]. However, it feeds into the theory of conserved quantities.

Definition 1.1.9 (Poisson Structure)**.** *A Poisson structure on a smooth manifold with a bilinear form mapping, $(f, g) \mapsto \{f, g\}$ which satisfies,*

- *The Jacobi Identity, $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{f, h\}, g\} = 0$,*
- *The Liebniz Identity, $\{f, gh\} = \{f, g\}h + \{f, h\}g$.*

One of the important notions surrounding these brackets, alongside their stability detection, is Casimirs. We can define a Casimir,

Definition 1.1.10 (Casimir)**.** *Let M be a smooth manifold, then $C : M \to \mathbb{R}$ be a function. We call C a Casimir of this system if,*

$$\{F, C\} = 0,$$

for any $F : M \to \mathbb{R}$.

There are more complicated brackets for different groups and models. For example, when we consider rigid bodies over SO(3) we get,

$$\{f, h\}(\Pi) := -\left\langle \Pi\,, \left[\frac{\partial f}{\partial \Pi}, \frac{\partial g}{\partial \Pi}\right]\right\rangle,$$

where $\Pi = R^{-1}\dot{R} \in \mathfrak{so}(3)$, $R \in \mathrm{SO}(3)$. This is a specific form of Nambu's $\mathbb{R}^3$ Poisson bracket,

$$\{f, h\} = \nabla c \cdot \nabla f \times \nabla h.$$

In ideal fluids, the bracket differs. Consider the dual of SVect(M), then the bracket of this space with respect to some $\alpha \in \Omega^1/\Omega^0$ is,

$$\{F, G\}(\omega) = -\int_M \alpha\left(\left[\frac{\delta F}{\delta \omega}, \frac{\delta G}{\delta \omega}\right]\right)\,\mathrm{d}V.$$

This is the Arnold bracket. In the section on conserved quantities, we will show that this bracket doesn't conserve enstrophy and provide some alternate brackets.

1.2 A short interlude into Symplectic Geometry

Hamilton's equations present an insight into the time evolution of a system. They rely on the Hamiltonian in its entirety to give information about the future. If we have M as a manifold, then Lagrangian dynamics happens on TM and Hamiltonian dynamics on T^*M. These present a very nice framework for rigid body dynamics. In fluid dynamics, it may help to consider our system's flow and the flow space. This requires some heavier-duty tools and generalising the idea of a Hamiltonian. We generalise this into the notion of a symplectic form. The symplectic form relates to the vector field for the flow, $\mathrm{d}H$.

Let M be a manifold, a specific manifold we will define later, then TM be the tangent space, or phase space, and T^*M be the cotangent space, or the flow space. Then let ω be a non-degenerate section of $T^*M \otimes T^*M$. The non-degeneracy of ω means that for every $\mathrm{d}H$ there is some V_H such that $\mathrm{d}H = \omega(V_H, \cdot)$. We want two properties of our system to line up with ω. Firstly, we want the Hamiltonian, H, to be constant along the flow lines. Hence,

$$dH(V_H) = \omega(V_H, V_H) = 0.$$

That is, ω is an alternating 2-form. Further, we want the Lie derivative of ω to vanish,

$$\begin{aligned} \pounds_{V_H}\omega &= \mathrm{d}(V_H \lrcorner\, \omega) + V_H \lrcorner\, \mathrm{d}\omega \\ &= \mathrm{d}(\mathrm{d}H) + \mathrm{d}\omega(V_H) = \mathrm{d}\omega(V_H) = 0. \end{aligned}$$

As H is arbitrary, ω must be closed. Hence, we can define this idea of a symplectic form.

Definition 1.2.1 (Sympelctic Form)**.** *A symplectic form on a smooth manifold, M, is a closed non-degenerate 2-form, ω.*

We define a symplectic manifold as a pair (M, ω). Symplectic Manifolds are just Poisson manifolds, and the Symplectic form is usually of more interest and use. These are very useful when we get to defining symplectic leaves. These are another gateway into conserved quantities.

Definition 1.2.2 (Symplectic Leaf)**.** *The symplectic leaf of a point on a Poisson manifold is the set of all points on the manifold that can be reached by paths starting at a given point, such that the velocity vectors of the paths are Hamiltonian at every moment.*

You can show that these leaves are just symplectic manifolds and have the same symplectic structure. That is, the flow from any point where the Hamiltonian behaves nicely is just a manifold with a nice symplectic structure, and you can use the theory we will develop on it!

Theorem 1.2.3. *The symplectic leaf of every point is a smooth, even dimensional manifold. It has natural symplectic structure defined by, $\omega(\xi, \eta) = \{f, g\}$, where ξ and η are vectors at the point x of the Hamiltonian fields with Hamiltonian functions f and g.*

We finally define a morphism between symplectic manifolds that will be useful later,

Definition 1.2.4 (Symplectomorphism)**.** *A diffeomorphism between two syplectic manifolds, $\phi : (M, \omega) \to (N, \omega')$ is a symplectomorphism if it preserves the symplectic form,*

$$\phi^*\omega' = \omega$$

Chapter 2

Reduction, Equations and Brackets

Groups will be known by their actions

Guillermo Moreno

In 1757, Euler wrote the following equation in the paper 'Principes généraux du mouvement des fluides'.

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p$$
$$\nabla \cdot \mathbf{u} = 0.$$

These are known as Euler's equations for perfect incompressible fluid flow. In this Chapter, we will take these equations, pull them apart and put them back together. We will study a more general theory of these equations by opening the eyes of the reader to fluid dynamics on general manifolds. On $\mathbb{R}^n$, conserved quantities and theory can be obscured by the space's flatness or even simplicity. We will lift these equations to curved surfaces and aim to prepare for the next chapter, where we will study conserved quantities. We first can manipulate these equations into something more amicable to generalise. We will then generalise them and prove that these are the generalised Euler equations by deriving them from the Lie-Poisson reduction of a Hamiltonian. This whole chapter draws from and builds on several sources, [29, 14, 5, 13, 30, 11, 19, 9, 24, 20, 28].

2.1 Generalisation of Eulers Equations

We can replace the $\mathbf{u} \cdot \nabla$ with a covariant derivative. We know that,

$$[\mathbf{u} \cdot \nabla]_i = u^i \frac{\partial u^i}{\partial x^i}.$$

However, this is very similar to the covariant derivative,

$$[\nabla_{\mathbf{v}} \mathbf{u}]_i = \left(v^j u^i \Gamma^k_{ij} + v^j \frac{\partial u^k}{\partial x^j} \right) \mathbf{e}_k.$$

We know that Γ^k_{ij} is the Christoffel Symbol and represents some notion of twistyness of the space, so $\Gamma^k_{ij} = 0$ for $\mathbb{R}^n$. Hence, for flat space,

$$\mathbf{u} \cdot \nabla = \nabla_{\mathbf{v}} \mathbf{u}.$$

Therefore, we first propose the generalisation of Euler equations by writing,

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} + \nabla_{\mathbf{u}} \mathbf{u} = -\nabla p \\ &\nabla \cdot \mathbf{u} = 0. \end{aligned} \tag{2.1}$$

We can rewrite the covariant derivative with something more geometric. We want to refer to the manifold that we work on explicitly. Let M be a manifold with the Riemannian metric, and then we can prove the following theorem using the metric.

Theorem 2.1.1. *Let M be a manifold with Riemannian metric. The Lie derivative of the one-form corresponding to a vector field on a M differs from the covariant derivative along itself by a complete derivative.*

$$\pounds_v(v^\flat) = (\nabla_v v)^\flat + \frac{1}{2}\,\mathrm{d}\left\langle v\,, v\right\rangle.$$

Proof. Assume we have v, w such that they commute. That is, $\{v, w\} = 0$. Recall the translation property of the Lie derivative and the covariant derivative,

$$\pounds_a \left\langle b\,, c\right\rangle = \left\langle \nabla_a b\,, c\right\rangle + \left\langle b\,, \nabla_a c\right\rangle.$$

If we specialise this formula, we can get,

$$\left\langle \nabla_w v\,, v\right\rangle = \frac{1}{2}\,\mathrm{d}\left\langle v\,, v\right\rangle(w). \tag{2.2}$$

$$\pounds_v \left\langle w\,, v\right\rangle = \left\langle \nabla_v w\,, v\right\rangle + \left\langle w\,, \nabla_v v\right\rangle. \tag{2.3}$$

By noting that $\pounds_v f = \mathrm{d}f(v)$. As $\{v, w\} = 0$, then,

$$\left\langle \nabla_v w\,, v\right\rangle = \left\langle \nabla_w v\,, v\right\rangle. \tag{2.4}$$

Therefore, we can substitute Equations 2.2 and 2.4 into Equation 2.3. This yields,

$$\pounds_v \left\langle w\,, v\right\rangle = \left\langle \nabla_v v\,, w\right\rangle + \frac{1}{2}\,\mathrm{d}\left\langle v\,, v\right\rangle(w).$$

It suffices to show that, $\pounds_v \left\langle w\,, v\right\rangle = (\pounds_v v^\flat)(w)$. This follows from the naturality rule of the Lie derivative,

$$\pounds_\xi(v^\flat(w)) = (\pounds_\xi v^\flat)(w) + v^\flat(\pounds_\xi w).$$

Then let $\xi = v$,

$$\pounds_v(v^\flat(w)) = (\pounds_v v^\flat)(w).$$

Now given that $\flat$ is the musical isomorphism, we know $\pounds_v \left\langle v\,, w\right\rangle = \pounds_v(v^\flat(w))$. Therefore,

$$\pounds_v v^\flat = (\nabla_v v)^\flat + \frac{1}{2}\,\mathrm{d}\left\langle v\,, v\right\rangle.$$

□

That is, we can write Equation 2.1, as the following,

$$\begin{aligned} &\frac{\partial \mathbf{u}^\flat}{\partial t} = -\pounds_v \mathbf{u} - \mathrm{d}f \\ &\mathrm{d} \star \mathbf{u}^\flat = 0 \end{aligned} \tag{2.5}$$

2.2 Euler-Poincaré Reduction

The Euler-Poincaré Equations are a set of equations that relate to a reduced Lagrangian on the Lie algebra of the manifold. We recall that a Lagrangian is a function on the tangent bundle, $L : TG \to \mathbb{R}$, then the reduced Lagrangian is $\ell : \mathfrak{g} \to \mathbb{R}$, i.e. the restriction of L to $\mathfrak{g}$. The following result characterises the Euler-Poincaré Equations in our current case. We note this is the first of a few very similar theorems, depending on our assumptions about our groups and system. In this section, we will consider Euler-Poincaré reduction of manifolds without boundary for a basic Lagrangian, a Lagrangian with an advective quantity, a diffusion term and then for compressible fluids. We will then recall the theory without boundary and work towards a theory for manifolds with boundary.

2.2.1 Manifolds without boundary

We can introduce and prove the following theorem related to basic Lagrangian reduction on manifolds without boundary. We consider a Lagrangian of basic form, $L = K - V$.

Theorem 2.2.1 (Basic Euler-Poincaré)**.** *Let G be a topological group that admits a smooth manifold structure with smooth right translation, and let $L : TG \to \mathbb{R}$ be a right invariant Lagrangian. Let $\mathfrak{g}$ denote the fiber T_eG, and let $\ell : \mathfrak{g} \to \mathbb{R}$, the restriction of L to $\mathfrak{g}$. For a curve $\eta(t) \in G$, let $\mathbf{u}(t) = TR_{\eta(t)^{-1}}\eta(t)$. Then TFAE,*

- *$\eta(t)$ satisfies the Euler-Lagrange Equations,*
- *$\eta(t)$ is an extremum of the action,*

$$S(\eta, \dot{\eta}) = \int L(\eta(t), \dot{\eta}(t))\ \mathrm{d}t,$$

- *$\mathbf{u}(t)$ solves the Basic Euler-Poincaré equations,*

$$\frac{\partial}{\partial t}\frac{\delta \ell}{\delta \mathbf{u}} = -\mathrm{ad}^*_{\mathbf{u}}\frac{\delta \ell}{\delta \mathbf{u}},$$

  where the coadjoint action is defined by,

$$\langle \mathrm{ad}^*_{\mathbf{u}} v\,, \mathbf{w}\rangle = \langle v\,, [\mathbf{u}, \mathbf{w}]\rangle\,.$$

- *$\mathbf{u}(t)$ is the extremum of the reduced action,*

$$s(\mathbf{u}) = \int \ell(\mathbf{u}(t))\,\mathrm{d}t,$$

  for variations,

$$\delta\mathbf{u} = \dot{\mathbf{w}} + [\mathbf{w}, \mathbf{u}], \qquad \mathbf{w} = TR_{\eta^{-1}}\delta\eta.$$

Proof. This proof can be found in [14, 7, 13, 21]. We will derive the Euler-Poincaré equations from the variational principle to show where the equations come from. We denote $\ell(\mathbf{u})$ as our reduced Lagrangian, and then we write our variational principle and integrate it by parts.

$$\begin{aligned}\delta\int_{t_1}^{t_2}\ell(\mathbf{u})\,\mathrm{d}t &= \int_{t_1}^{t_2}\left\langle\frac{\delta\ell}{\delta\mathbf{u}}\,,\delta\mathbf{u}\right\rangle\,\mathrm{d}t\\ &= \int_{t_1}^{t_2}\left\langle\frac{\delta\ell}{\delta\mathbf{u}}\,,\dot{\mathbf{w}}-\mathrm{ad}_{\mathbf{u}}\mathbf{w}\right\rangle\,\mathrm{d}t\\ &= \int_{t_1}^{t_2}\left\langle-\frac{\partial}{\partial t}\frac{\delta\ell}{\delta\mathbf{u}}\,,\mathbf{w}\right\rangle-\left\langle\mathrm{ad}^*_{\mathbf{u}}\frac{\delta\ell}{\delta\mathbf{u}}\,,\mathbf{w}\right\rangle\,\mathrm{d}t=0.\end{aligned}$$

Therefore, the Euler-Poincaré equations are,

$$\frac{\partial}{\partial t}\frac{\delta\ell}{\delta\mathbf{u}}+\mathrm{ad}^*_{\mathbf{u}}\frac{\delta\ell}{\delta\mathbf{u}}=0.$$

□

We note that for ideal fluids, the Lagrangian is just kinetic energy. Hence,

$$L(\eta,\dot{\eta})=\frac{1}{2}\int_M(\dot{\eta},\dot{\eta})\mu\qquad\eta\in\mathrm{SDiff}(M),\ \dot{\eta}\in T_e\mathrm{SDiff}(M)$$

Then, we can show that this Lagrangian is right invariant. We remember we are reducing over some group of structure-preserving diffeomorphisms that have the relabelling symmetry, so,

$$\begin{aligned}L(\eta g,\dot{\eta}g) &= \frac{1}{2}\int_M\langle\dot{\eta}g\,,\dot{\eta}g\rangle\,\mu\\ &= \frac{1}{2}\int_M\langle\dot{\eta}\,,\dot{\eta}\rangle\,g^2\mu\\ &= \frac{1}{2}\int_M\langle\dot{\eta}\,,\dot{\eta}\rangle\,\mu=L(\eta,\dot{\eta}).\end{aligned}$$

Hence our Lagrangian is right invariant and we can let $g=\eta^{-1}$ and hence rewrite our Lagrangian,

$$L(\eta,\dot{\eta})=L(\eta\eta^{-1},\dot{\eta}\eta^{-1})=L(e,\dot{\eta}\eta^{-1}):=\ell(\mathbf{u}).$$

We now note that,

$$\ell(\mathbf{u})=\frac{1}{2}\int_M\langle\mathbf{u}\,,\mathbf{u}\rangle\,\mu.$$

Then we can write the Euler-Poincaré equations for this system as,

$$\frac{\partial\mathbf{u}}{\partial t}=\mathrm{ad}^*_{\mathbf{u}}[\mathbf{u}]=\pounds_{\mathbf{u}}[\mathbf{u}].$$

Note, that $[\mathbf{u}]\in\mathfrak{g}^*$ is the coset of $\mathbf{u}$ up to 1-forms. This is just the Lie-Poisson Equation, these are the variant of the Euler-Poincaré equations for the Hamiltonian, but in this case our Lagrangian is the total energy and hence the Hamiltonian.

Advective Terms

In fluid mechanics, some quantities are transported by the flow. That is, the Lie Derivative transports them along vector fields. These are called advected quantities. Advection usually refers to two things: advected conserved quantities, as due to the advection term, new and more interesting conserved quantities are present, or to model substrate or substance being moved through the fluid. When we consider a Hamiltonian of these systems, we note that they have a parameter that turns from a parameter to a variable. These variables form a vector space, and our groups act on them linearly. Hence, we have a representation space. To see this more clearly, consider the following Lagrangian[1] for a spherical pendulum,

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}m|\dot{\mathbf{q}}|^2 - mg\mathbf{q} \cdot \mathbf{e}_3.$$

Which is the same as the Hamiltonian up to Legendre transform. We note that the only potential energy term variable is $\mathbf{q}$. Now, considering the reduced Lagrangian,

$$\ell(\hat{\mathbf{\Omega}}, \mathbf{\Gamma}) = \frac{1}{2}m\mathbb{I}|\hat{\mathbf{\Omega}}|^2 - mg\mathbf{\Gamma} \cdot X.$$

In this case, we have turned 'created' another variable out of the symmetry-breaking parameter, $mg\mathbf{q} \cdot \mathbf{e}_3$. Hence a parameter has been turned into a variable and $\mathbf{\Gamma} \in V^*$ the adjoint of the representation of SO(3).

We can follow Holm, Marsden and Ratiu's paper on Semidirect Product Theories [13] to consider advection in our system. Let us consider a representation space, V, for our Lie Group G and let L have invariance properties for G and V.

Definition 2.2.2 (Representation)**.** *A representation of a Lie group is a tuple (V, ρ) where V is the representation space and $\rho : G \to \mathrm{GL}(V)$ the G-linear action on V.*

We can define a semidirect product on a Fréchet Group and some representation groups as follows,

Definition 2.2.3 (Semidirect Product)**.** *Let V be a vector space and G be a Fréchet group that acts on the right by linear maps on V. The semidirect product, $G \rtimes V$, is the cartesian product, $G \times V$, where group multiplication is given by,*

$$(g_1,\, v_1)(g_2,\, v_2) = (g_1 g_2, v_2 + v_1 g_2).$$

We note that this group has identity, $(e, 0)$ and inverses, $(g, v)^{-1} = (g^{-1}, -g^{-1}v)$. Then we can consider the Lie algebra, $\mathfrak{s} = \mathfrak{g} \times V^*$, and it bestows a Lie bracket,

$$[(\xi_1,\, v_1),\, (\xi_2,\, v_2)] = ([\xi_1,\, \xi_2],\, v_1\xi_2 - v_2\xi_1).$$

Notation 2.2.4. *For notational purposes, we will consider the adjoint linear representation ρ^* of some vector v and $a \in V^*$ as,*

$$\rho^*_v a = v \diamond a \in \mathfrak{g}^*.$$

or,

$$\langle A \diamond a\,, w \rangle_{\mathfrak{g}\times\mathfrak{g}^*} = \langle A\,, -\pounds_w a \rangle_{V \times V^*} = -\int_M A \lrcorner \pounds_w a.$$

[1]This is equivalent to the Hamiltonian up to Legendre transform

We consider our advected parameter to be some $a \in V^*$. Then, we can consider the semidirect product of G and V^*, $G \rtimes V^*$. We choose the semidirect product for flexibility, the Lie algebra of $\mathfrak{g} \rtimes V^*$.

We can define the Lagrangian as $L : G \rtimes V^* \to \mathbb{R}$. Then we assume there is the right representation of the Lie Group on the vector space V and G acts on the right on $TG \rtimes V^*$, $(\eta, \dot{\eta}, a)g = (\eta g, \dot{\eta} g, ag)$ for $g \in G$. Then $L : TG \rtimes V^* \to \mathbb{R}$ is G-invariant. That is, if we define $L_{a_0}(v_g) = L(v_g, a_0)$, then L_{a_0} is right invariant under the lift and action. Finally we can now define for $\ell : \mathfrak{g} \rtimes V^* \to \mathbb{R}$ by,

$$L(\eta\eta^{-1}, \dot{\eta}\eta^{-1}, a_0\eta^{-1}) := \ell(\xi, \alpha).$$

We now proceed with the Euler-Poincaré advection theorem.

Theorem 2.2.5 (Euler-Poincaré with advection)**.** *Let M be a manifold and G be a Fréchet group. Then let (V, ρ) represent G. Suppose we have an advective quantity $a \in V^*$, and a Lagrangian $L : G \rtimes V^* \to \mathbb{R}$ that is right invariant under the tangent lift. Let $\eta \in G$ and $\dot{\eta} \in \mathfrak{g}$. Then the following are equivalent,*

- *Hamilton's variational principle holds, where variations vanish at endpoints,*

$$\int_{t_1}^{t_2} L_{a_0}(\eta(t), \dot{\eta}(t))\, dt = 0$$

- *$\eta(t)$ satisfies the Euler-Lagrange equations for L_{a_0},*
- *The constrained variational principle holds for $\mathfrak{g} \rtimes V^*$,*

$$\delta \int_{t_1}^{t_2} \ell(\xi, a)\ \mathrm{d}t = 0 \qquad \delta\xi = \dot{\nu} - [\xi, \nu], \quad \delta a = -a\nu$$

- *The Euler-Poincaré equations hold on $\mathfrak{g} \rtimes V^*$,*

$$\frac{\partial}{\partial t}\frac{\delta \ell}{\delta \xi} = -\mathrm{ad}^*_\xi \frac{\delta \ell}{\delta \xi} + \frac{\delta \ell}{\delta a} \diamond a.$$

$$\left(\frac{\partial}{\partial t} + \pounds_\xi\right) a = 0$$

Proof. The proof here is similar to 2.2.1. However, there are a few results we need to prove that are different. We first consider the time derivative of a and then the variation.

We start with a Lagrangian of the form $L(\eta, \dot{\eta}, a_0)$ and then assume that it's right invariant, that is, $L(\eta, \dot{\eta}, a_0) = L(\eta g,\, \dot{\eta} g,\, a_0 g)$. Then we can form a reduced Lagrangian, $L(\eta\eta^{-1},\, \dot{\eta}\eta^{-1},\, a_0 g^{-1}) := \ell(\xi, a)$ where $a := a_0 g^{-1}$. From this, we calculate,

$$\begin{aligned} \frac{\partial a}{\partial t} &= \frac{\partial}{\partial t}(a_0 g^{-1}) \\ &= -a_0 g^{-1} \frac{\partial g}{\partial t} g^{-1} = -a\xi. \end{aligned} \tag{2.6}$$

Then, similarly, we can write,

$$\begin{aligned}\delta a &= \delta(a_0 g^{-1})\\ &= -a_0 g^{-1}\delta g g^{-1}\\ &= -a\nu,\end{aligned}$$

where $\nu := \delta g g^{-1}$. Now we can derive the equations,

$$\begin{aligned}\delta\int_{t_1}^{t_2}\ell(\xi,\,a)\;\mathrm{d}t &= \int_{t_1}^{t_2}\left\langle\frac{\delta\ell}{\delta\xi}\,,\delta\xi\right\rangle_{\mathfrak{g}\times\mathfrak{g}^*}+\left\langle\frac{\delta\ell}{\delta a}\,,\delta a\right\rangle_{V\times V^*}\;\mathrm{d}t\\ &= \int_{t_1}^{t_2}\left\langle\frac{\delta\ell}{\delta\xi}\,,\frac{\partial\nu}{\partial t}-[\xi,\,\nu]\right\rangle+\left\langle\frac{\delta\ell}{\delta a}\,,-a\nu\right\rangle\;\mathrm{d}t\\ &= \int_{t_1}^{t_2}\left\langle\frac{\delta\ell}{\delta\xi}\,,\frac{\partial\nu}{\partial t}\right\rangle-\left\langle\frac{\delta\ell}{\delta\xi}\,,\mathrm{ad}_\xi\nu\right\rangle+\left\langle\frac{\delta\ell}{\delta a}\,,-a\nu\right\rangle\;\mathrm{d}t\\ &= \int_{t_1}^{t_2}\left\langle-\frac{\partial}{\partial t}\frac{\delta\ell}{\delta\xi}\,,\nu\right\rangle-\left\langle\mathrm{ad}^*_\xi\frac{\delta\ell}{\delta\xi}\,,\nu\right\rangle+\left\langle\frac{\delta\ell}{\delta a}\diamond a\,,\nu\right\rangle\;\mathrm{d}t\\ &= \int_{t_1}^{t_2}\left\langle-\frac{\partial}{\partial t}\frac{\delta\ell}{\delta\xi}-\mathrm{ad}^*_\xi\frac{\delta\ell}{\delta\xi}+\frac{\delta\ell}{\delta a}\diamond a\,,\nu\right\rangle_{\mathfrak{g}\times\mathfrak{g}^*}\;\mathrm{d}t.\end{aligned}$$

Therefore, using Hamilton's principle, we have,

$$\frac{\partial}{\partial t}\frac{\delta\ell}{\delta\xi} = -\mathrm{ad}^*_\xi\frac{\delta\ell}{\delta\xi}+\frac{\delta\ell}{\delta a}\diamond a. \tag{2.7}$$

Then considering the right concatenation using (2.6) we get,

$$\left(\frac{\partial}{\partial t}+\pounds_\xi\right)a = 0.$$

as required. □

We can derive the Euler equations for fluids with variable densities using this theory. Consider a Lagrangian with a fluid density advective term,

$$L(\eta,\dot\eta,\rho) = \int_M \frac{\rho}{2}|\dot\eta|^2 + p(\rho-1)\mu.$$

This Lagrangian has a symmetry-breaking parameter [5]. Hence, we consider the symmetric part, and verifying this is right invariant is easy. Let $\eta\in\mathrm{SDiff}(M)$ and ρ be a 0-form. Then, we can reduce this Lagrangian into,

$$L(\eta\eta^{-1},\dot\eta\eta^{-1},\rho\rho_0^{-1}) := \ell(\mathbf{u},D) = \int_M \frac{D}{2}|\mathbf{u}|^2 + p(D-1)\mu.$$

Then, we take functional derivatives,

$$\begin{aligned}\left\langle\frac{\delta\ell}{\delta\mathbf{u}}\,,\phi\right\rangle &:= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\ell(\mathbf{u}+\varepsilon\phi,D)\right]_{\varepsilon=0}\\ &= \left\langle[\mathbf{u}^\flat]D\,,\phi\right\rangle. &(2.8)\\ \left\langle\frac{\delta\ell}{\delta D}\,,\phi\right\rangle &:= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\ell(\mathbf{u},D+\varepsilon\phi)\right]_{\varepsilon=0}\\ &= \left\langle\frac{1}{2}[\mathbf{u}^\flat]^2+p\,,\phi\right\rangle. &(2.9)\end{aligned}$$

We note the following argument relating to the Lie derivative of $\mathbf{u}^\flat D$ against some vector X,

$$\begin{aligned}\pounds_{\mathbf{u}}(\mathbf{u}^\flat D)X &= D\left\langle \mathbf{u}\,, \mathbf{u}^\flat \right\rangle X + D\pounds_{\mathbf{u}}\mathbf{u}^\flat X \\ &= D\pounds_{\mathbf{u}}\mathbf{u}^\flat X.\end{aligned} \tag{2.10}$$

Then we can plug the expressions (2.8 - 2.10) into Equation (2.7) and get,

$$\frac{\partial[\mathbf{u}^\flat]}{\partial t} + \pounds_{\mathbf{u}}[\mathbf{u}^\flat] = \frac{(\frac{1}{2}[\mathbf{u}^\flat]^2 + p) \diamond D}{D}.$$

Then, considering the cosets we have,

$$\frac{\partial \mathbf{u}^\flat}{\partial t} + \pounds_{\mathbf{u}}\mathbf{u}^\flat = F(\mathbf{u}^\flat,\, D,\, p) := \frac{(\frac{1}{2}[\mathbf{u}^\flat]^2 + p) \diamond D}{D} + \mathrm{d}f,$$

where $\mathrm{d}f = \nabla p/D$ and the remaining terms, $F(\mathbf{u}^\flat,\, D,\, p)$ act as the gravitational buoyancy term. Then the advection equation is,

$$\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right) D = 0$$

2.2.2 Manifolds with Boundary

We require a manifold with some boundary to have systems with boundary conditions. This is where the theory deviates from above. We now will have three objects: the group we reduce over, G, with its associated algebra, $\mathfrak{g}$, and its dual $\mathfrak{g}^*$, the principle we use to derive the equations, usually Hamilton's principle, and the manifold we consider the flow over, M. We note that the boundary conditions will come from M and be imposed on G. Boundary conditions come in two different forms,

1. Finite conditions, so conditions along a point or a path. For example, axisymmetric flow in a cylinder.

2. Infinite conditions, limiting conditions as you approach an infinitum. For example, waves on a plane.

The first of these conditions requires manifolds with boundaries; the second is more complicated and requires a generalisation called manifolds with corners [18]. We will primarily be interested in the first case while diverting the second to subsequent work on this area. We will be using the ideas from papers by Marsden, Ratiu and Shkoller [20, 24] to develop a more general theory that we can apply to compressible Euler-Poincaré equations. We can derive Neumann, Dirichlet and Mixed conditions. We will work on two new diffeomorphism groups and mainly study the Dirichlet group that corresponds to the zero on the boundary, as that is the boundary condition seen the most in this type of fluid dynamics. We introduce the manifold of diffeomorphisms with boundary as follows.

Definition 2.2.6 (Diffeomorphism group with Boundary)**.** *Let M be a manifold, $\tilde{M}$ be the double of M as $H^s(M, M)$ isn't smooth and* $\mathrm{Diff}^s(M)$ *be the group of C^s diffeomorphisms. Then, we define the diffeomorphism group with a boundary as,*

$$\mathrm{Diff}^s(M) = \{\eta \in H^s(M, \tilde{M}) \cap \mathrm{Diff}^s(M) : \eta(\partial M) = \partial M\}.$$

Then a calculation, similar to the ones done in a paper by Ebin and Marsden [9], with (E, π) being a vector bundle, leads to,

$$T_e\mathrm{Diff}^s(M) = \{\dot{\eta} \in H^s(M, TM) : \eta = \pi \circ \dot{\eta},\ g(\dot{\eta}\eta^{-1}, n) = 0 \text{ on } \partial M,\ \mathrm{div}(\dot{\eta}\eta^{-1}) = 0\}.$$

Lemma 2.2.7. *Let M be a manifold, (E, π) be a vector bundle on M and $\xi = u\eta^{-1}$. Then the tangent space of* $\mathrm{Diff}^s(M)$ *at η is,*

$$T_e\mathrm{Diff}^s(M) = \{u \in \mathfrak{X}^s(M) : g(\xi, n) = 0 \text{ on } \partial M,\ \mathrm{div}(\xi) = 0\},$$

with a map $\eta = \pi \circ u$ from the vector bundle to connect the group and tangent space.

Proof. See [9]. □

We now can define the structure-preserving group,

Definition 2.2.8 (Structure Preserving Diffeomorphism Group with Boundary)**.** *Let M be a manifold, then* $\mathrm{Diff}^s(M)$ *be as defined above and μ be the volume form of M. Then, the structure-preserving diffeomorphism group with boundary is,*

$$\mathrm{SDiff}^s(M) = \{\eta \in \mathrm{Diff}^s(M) : \eta^*\mu = \mu\}.$$

For all our groups, we note that the left action is class C^r, the right action is C^∞ and inversion, $\eta \mapsto \eta^{-1}$, is C^0 and further not Lipschitz continuous. Therefore we are now working with topological groups with smooth right action, Frénet groups.

2.2.3 Three more diffeomorphism groups

We define the following: let a diffeomorphism have a smoothness class. This smoothness class is a Hilbert H^s class. In practicality, we will assume we have diffeomorphisms that are sufficiently smooth enough to do the construction, but it is helpful to mention restrictions on this theory. We introduce a few bits of differential geometry,

Definition 2.2.9 (Weingarten map)**.** *Let (M, g) be a Riemannian manifold, $p \in M$, n be the normal field, and v be a tangent vector. Then the Weingarten map (shape operator) is a map $S : T_pM \to T_pM$, defined by,*

$$S(n) = -\nabla_v n.$$

Definition 2.2.10 (Normal Bundle)**.** *Let M be a manifold. If M has a metric, then given some $A \subseteq M$, $p \in M$ and $u \in T_pM$ we can say that the normal space of A is all vectors $v \in T_pM$ such that $g(u, v) = 0$. This denoted, N_pS. Then the normal bundle is,*

$$NS := \coprod_{p \in S} N_pS.$$

The disjoint union of the normal spaces.

We can introduce our three diffeomorphism groups,

Definition 2.2.11 (Neumann Group)**.** *Let M be a manifold and N be its normal bundle. Then, the Neumann group is defined as,*

$$\mathrm{SDiff}^s_N(M) = \{\eta \in \mathrm{SDiff}^s(M) : T\eta \cdot n|_{\partial M} \in H^{s-3/2}(N) \text{ for all } n \in H^{s-1/2}(N)\}.$$

Definition 2.2.12 (Dirichlet Group)**.** *Let M be a manifold. Then, the Dirichlet group is defined as,*

$$\mathrm{SDiff}^s_D(M) = \{\eta \in \mathrm{SDiff}^s(M) : \eta|_{\partial M} = \mathrm{Id}\}.$$

Definition 2.2.13 (Mixed Group)**.** *Let M be a manifold and N be its normal bundle. Let the boundary $\partial M = \Gamma_1 \cup \Gamma_2$ and $\overline{\Gamma_1} = M \setminus \Gamma_2$. Then, the mixed group is,*

$$\begin{aligned}\mathrm{SDiff}^s_{\mathrm{Mi}}(M) = \{\eta \in \mathrm{SDiff}^s(M) : \quad & \eta(\Gamma_i) = \Gamma_i,\ T\eta \cdot n|_{\Gamma_2} \in H^{s-3/2}(N|_{\Gamma_1}) \\ & \text{for all } n \in H^{s-1/2}(N|_{\Gamma_1}),\ \eta|_{\Gamma_2} = \mathrm{Id}\}.\end{aligned}$$

Then their associated Lie algebras,

Definition 2.2.14 (Neumann Algebra)**.** *Let M be a manifold and N be its normal bundle. Then, the Neumann algebra is defined as,*

$$T_e\mathrm{SDiff}^s_N(M) = \{u \in T_e\mathrm{SDiff}^s(M) : 0 = (\nabla_n u|_{\partial M})^{tan} + S_n(u) \in H^{s-3/2}(T\partial M) \text{ for all } n \in H^{s-1/2}(N)\}.$$

Definition 2.2.15 (Dirichlet Algebra)**.** *Let M be a manifold. Then, the Dirichlet algebra is defined as,*

$$T_e\mathrm{SDiff}^s_D(M) = \{u \in T_e\mathrm{SDiff}^s(M) : u|_{\partial M} = 0\}.$$

Definition 2.2.16 (Mixed Algebra)**.** *Let M be a manifold and N be its normal bundle. Let the boundary $\partial M = \Gamma_1 \cup \Gamma_2$ and $\overline{\Gamma_1} = M \setminus \Gamma_2$. Then, the Mixed algebra is defined as,*

$$\begin{aligned}T_e\mathrm{SDiff}^s_{\mathrm{Mi}}(M) = \{u \in T_e\mathrm{SDiff}^s(M) : \quad & 0 = (\nabla_n u|_{\partial M})^{tan} + S_n(u) \in H^{s-3/2}(T\Gamma_2) \\ & \text{for all } n \in H^{s-1/2}(N|_{\Gamma_2}),\ u|_{\Gamma_1} = 0\}.\end{aligned}$$

Now, we can prove the required theorem about how this relates to Euler-Poincaré equations.

Theorem 2.2.17 (Advected Euler-Poincare with Boundary Conditions)**.** *Let G be a C^∞ topological group, either $\mathrm{SDiff}^s_N(M)$, $\mathrm{SDiff}^s_D(M)$ or $\mathrm{SDiff}^s_{\mathrm{Mi}}(M)$ and $\ell : \mathfrak{g} \rtimes V^* \to \mathfrak{g}$ be a reduced Lagrangian of the form $\ell(\xi, a) = \tilde{\ell}(\xi, a) + b(\xi, a)$ where b is the tangent space boundary operator. Then the Euler-Poincaré equations for a system with advected quantities (similarly for no advected quantities) have the following form*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta\tilde{\ell}}{\delta\xi} - \mathrm{ad}^*_\xi \frac{\delta\tilde{\ell}}{\delta\xi} + a \diamond \frac{\delta\tilde{\ell}}{\delta a} &= 0, \\ b(\xi, a) &= 0.\end{aligned} \tag{2.11}$$

Proof. We seek to calculate the Euler-Poincaré equations for $\ell(\xi,\, a) = \tilde{\ell}(\xi,\, a) + b(\xi,\, a)$ via Hamilton's principle. Then, show that no extra erroneous terms appear. Then, finally, as b vanishes at the boundary, the terms on the boundary should vanish. We note that any term involving b is considered at the boundary. The calculation is standard until we reach,

$$\delta \int_{t_1}^{t_2} -\frac{\partial}{\partial t}\left(\frac{\delta \tilde{\ell}}{\delta \xi} + \frac{\delta b}{\delta \xi}\right) - \mathrm{ad}_\xi \left(\frac{\delta \tilde{\ell}}{\delta \xi} + \frac{\delta b}{\delta \xi}\right) + a \diamond \left(\frac{\delta \tilde{\ell}}{\delta a} + \frac{\delta b}{\delta a}\right)\, \mathrm{d}t = 0$$

We note that we can decouple the boundary terms,

$$\frac{\partial}{\partial t}\frac{\delta b}{\delta \xi} = -\mathrm{ad}_\xi \frac{\delta b}{\delta \xi} + a \diamond \frac{\delta b}{\delta a}.$$

Then we know if $b(\xi,\, a) = 0$, this vanishes in the above. Hence we have,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta \tilde{\ell}}{\delta \xi} - \mathrm{ad}^*_\xi \frac{\delta \tilde{\ell}}{\delta \xi} + a \diamond \frac{\delta \tilde{\ell}}{\delta a} &= 0 \\ b(\xi,\, a) &= 0. \end{aligned}$$

□

2.3 Lie-Poisson Equation

There are two different formalisations of mechanics. Lagrangian, which we saw above, and Hamiltonian. These two formalisms come from the idea that,

$$L = K - V \qquad H = K + V,$$

Furthermore, Lagrangian dynamics happen in the tangent space of the manifold and Hamiltonian dynamics happen in the cotangent space of the manifold. The Euler-Poincaré and Lie-Poisson equations relate to dynamics in the cotangent space. This implies there is some way to get between them. This is called the Legendre transform.

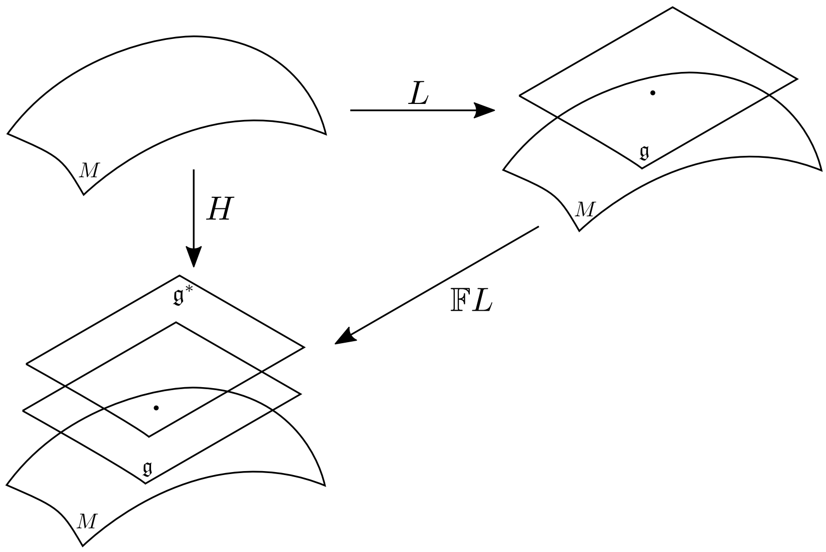


Figure 2.1: Dynamics and Legendre Transform.

In this section, we will study how to reduce Hamiltonians via Lie-Poisson reduction and then show why reducing on the Lagrangian via Euler-Poincaré reduction is usually better and then use the Legendre transform. We will see that the reduction of the Lagrangian is a lot neater and further replicates most of the modern theory. Many older Geometric Mechanics on fluids by Marsden and Arnold were mainly dedicated to Hamiltonian structures. This is because of the close connection with conserved quantities and the intuitiveness of Hamiltonians being total energy.

Several different ideas arise in this area. We will look more closely at non-canonical Lie-Poisson brackets and how to calculate them and hint at the relation to Casimirs before discussing them in greater detail in the next section.

We aim to study the way to get between Lagrangian and Hamiltonian through the Legendre transform. We include an excerpt from the author's special topic on Mechanical Mathematical Biology [4] that is relevant here.

2.3.1 Legendre Transform

Between Hamiltonians and Lagrangians, we are just working with a change of sign. A Hamiltonian is $K + V$ and a Lagrangian is $K - V$. However, if we have a Hamiltonian and want a Lagrangian, it is not necessarily obvious how to get between them. This is where the Legendre transform comes in. The Legendre transform is used in mechanics to go between a set of variables and their conjugate variables–for example, velocity and momentum. Regarding geometry, we know that Lagrangian mechanics happens on the tangent space of a manifold while Hamiltonian mechanics happens on the cotangent space of a manifold. Hence, the Legendre transform is a map between the tangent and cotangent spaces that preserves a conserved quantity, transforming from the Lagrangian to the Hamiltonian or vice versa. We will briefly show the theory and derive the Legendre transform for this problem.

Let M be a manifold and (E, π) be a vector bundle. We quickly define a vector bundle [3].

Definition 2.3.1 (Vector Bundle)**.** *A $\Bbbk$-vector bundle over M of rank k consists of; a bundle $\pi : E \to M$ whose fibers are $\Bbbk$-vector spaces and around each point $p \in M$, there is some open $U \subset M$ and a diffeomorphism $\Phi : U \times \Bbbk^k \to E|_U$ such that,*

- *$\pi \circ \Phi = \pi_1$ where $\pi_1 : U \times \Bbbk^k \to E|_U$ is the projection of the first factor, and,*
- *for each $q \in U$, the map $\Phi_q : \Bbbk^k \to E_q$ such that $\Phi_q(\xi) := \Phi(q, \xi)$ is a $\Bbbk$-linear isomorphism.*

Let $L : E \to \mathbb{R}$ be a smooth function known as the Lagrangian. The Legendre transformation is a smooth map between E and E^*, the dual of E,

$$\mathbb{F}L : E \to E^*.$$

We define it by, $\mathbb{F}L(v) = \mathrm{d}(L|_{E_x})$, where E_x is the fiber of E over $x \in M$. This says we take a covector and map it to the directional derivative.

This localises via a trivialisation to just saying,

$$p_i = \frac{\partial L}{\partial q_i}.$$

This is exactly how we get between conjugate and generalised variables in the derivation of Hamilton's equations. Then further, in our case, $E = TM$ and $E^* = T^*M$. This leads to the Legendre transform being an isomorphism and further a diffeomorphism. Then we reach the Legendre transform being,

$$L(v) = H(p) - p \cdot v,$$

and further, $\mathbb{F}L = (\mathbb{F}H)^{-1}$, by using the natural isomorphism $TM \cong T^*M$ [14]. More specifically, we can write,

$$\langle \mathbb{F}L(v)\,, w \rangle := \frac{\mathrm{d}}{\mathrm{d}s} L(v + sw) \bigg|_{s=0}.$$

In much the same way as Euler-Poincaré, Lie-Poisson reduction is a method to reduce equations into a more amenable form to consider conserved quantities and write numerical methods via symmetries. We will study two main things: reduced Legendre transforms and the non-canonical brackets.

2.3.2 The Reduced Legendre Transform

The next two subsections follow Holm, Schmah and Stoica [14], but we note that we consider a right-invariant version, while Holm considers a left-invariant system. We also loosely cite Vasylkevych and Masrden's paper [28]. We first can note some more theories about the regular Legendre transform. We define a class of Lagrangians called hyper-regular Lagrangians,

Definition 2.3.2 (Hyperregular)**.** *A hyperregular Lagrangian is a Lagrangian where the Legendre transform is a diffeomorphism, and the Hessian of the Lagrangian is invertible.*

Then, we can say that any system with a hyperregular Lagrangian has a guaranteed Hamiltonian related to it. This is because we have produced a differentiable bijection that is non-degenerate. For the rest of this thesis, we will assume that our Lagrangians are hyperregular. We can now define the reduced Legendre transform,

Definition 2.3.3 (Reduced Legendre transform)**.** *Let $L : TG \to \mathbb{R}$ be hyperregular and $\ell : \mathfrak{g} \to \mathbb{R}$ be the reduced Lagrangian defined by $\ell(\xi) = L(e, \xi)$. Then the reduced Legendre transform, $\mathrm{fl} : \mathfrak{g} \to \mathfrak{g}^*$, is defined by,*

$$\langle \mathrm{fl}(\xi) \, , \eta \rangle := \frac{\mathrm{d}}{\mathrm{d}s} \ell(\xi + s\eta) \bigg|_{s=0} = \left\langle \frac{\delta \ell}{\delta \xi} \, , \eta \right\rangle . \tag{2.12}$$

We want to show invariance on the Lagrangian leads to invariance in the Hamiltonian. We first define a special type of invariance,

Definition 2.3.4 (G-invariance)**.** *Let G be a group, then we say that a Lagrangian is G-invariant if for any $g_0 \in G$,*

$$L(qg_0, \, \dot{q}g_0) = L(q, \dot{q})$$

Lemma 2.3.5. *Let M be a manifold, $L : TM \to \mathbb{R}$ be G-invariant, then $\mathbb{F}L$ is G-invariant.*

Proof. Let $g_0 \in G$. We will consider the following calculation,

$$\begin{aligned} \langle \mathbb{F}L(\mathbf{q}g_0, \dot{\mathbf{q}}_1 g_0) \, , (\mathbf{q}g_0, \, \dot{\mathbf{q}}_2 g_0) \rangle &= \frac{\mathrm{d}}{\mathrm{d}t} L(\mathbf{q}g_0, \, \dot{\mathbf{q}}_1 g_0 + t\dot{\mathbf{q}}_2 g_0) \bigg|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} L((\mathbf{q}, \, \dot{\mathbf{q}}_1 + t\dot{\mathbf{q}}_2) g_0) \bigg|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} L(\mathbf{q}, \, \dot{\mathbf{q}}_1 + t\dot{\mathbf{q}}_2) \bigg|_{t=0} \\ &= \langle \mathbb{F}L(\mathbf{q}, \dot{\mathbf{q}}_1) \, , (\mathbf{q}, \, \dot{\mathbf{q}}_2) \rangle \end{aligned}$$

Therefore, $\mathbb{F}L(\mathbf{q}g_0, \, \dot{\mathbf{q}}g_0) = \mathbb{F}L(\mathbf{q}, \, \dot{\mathbf{q}})$ and so $\mathbb{F}L$ is G-invariant. ∎

This lemma tells us that if our Lagrangian is G-invariant, then the Legendre transform in G-invariant and as $H = E \circ (\mathbb{F}L)^{-1}$, where E is invariant from the Lagrangian, the Hamiltonian is invariant! We can now define these reduced quantities similarly to the full quantities, prove similar lemmas, and prove finally that the reduced Hamiltonian is reduced. We call the reduced energy, $\tilde{e} : \mathfrak{g} \to \mathbb{R}$, defined by

$$\tilde{e}(\xi) = \langle \mathrm{fl}(\xi)\,, \xi\rangle - \ell(\xi),$$

and the reduced Hamiltonian, $h : \mathfrak{g}^* \to \mathbb{R}$ defined by,

$$h(\mu) = \tilde{e} \circ \mathrm{fl}^{-1}.$$

We can name $\mu = \mathrm{fl}(\xi)$ and then we get,

$$h(\mu) = \tilde{e} \circ \mathrm{fl}^{-1}(\mu) = \langle \mu\,, \xi(\mu)\rangle - \ell(\xi(\mu)).$$

Lemma 2.3.6. *Let M be a Fréchet group, then $L : TG \to \mathbb{R}$ be a G invariant Lagrangian. If E is full energy, and $\tilde{e}$ is reduced energy corresponding to L and $\xi = gg^{-1}$, then,*

$$\tilde{e}(\xi) = E(e,\, \xi) = E(g\,\dot{g}).$$

Suppose L is hyperregular and H is the Hamiltonian corresponding to L. If $\mu = g^{-1}\alpha$, then,

$$h(\mu) = (e, \xi) = H(g, \alpha).$$

Proof. We consider the full energy, then reduce,

$$\begin{aligned}
\langle \mathbb{F}L(g, \dot{g})\,, (g, \dot{g})\rangle - L(g, \dot{g}) &= \left\langle \mathbb{F}L(gg^{-1}, \dot{g}g^{-1})\,, (gg^{-1}, \dot{g}g^{-1})\right\rangle - L(gg^{-1}, \dot{g}g^{-1}) \\
&= \langle \mathbb{F}L(e, \xi)\,, (e, \xi)\rangle - L(e, \xi) \\
&= \langle \mathrm{fl}(\xi)\,, (e, \xi)\rangle - \ell(\xi) \\
&= \langle (e, \mathrm{fl}(\xi))\,, (e, \xi)\rangle - \ell(\xi) \\
&= \langle \mathrm{fl}(\xi)\,, \xi\rangle - \ell(\xi) = \tilde{e}(\xi).
\end{aligned}$$

Then, the final part of the proof follows from energy and the Legendre transform being G-invariant. Then noting that $H = E \circ (\mathbb{F}L)^{-1}$ and running through a reduction procedure and noticing the reduction variable ends up being $g^{-1}\alpha$. □

2.3.3 Derivation of Lie-Poisson

We will now move from the Euler-Poincaré equations via the reduced Legendre transform to the Lie-Poisson equations. The Legendre transform moves invariances from the Lagrangian to the Hamiltonian side. The derivation is very similar to Hamilton's equations derivation. We consider the following application of product rule and Equation 2.12,

$$\begin{aligned}
\frac{\delta h}{\delta \mu} &= \left\langle \mu\,, \frac{\delta \xi}{\delta \mu}\right\rangle - \left\langle \frac{\delta \ell}{\delta \xi}\,, \frac{\delta \xi}{\delta \mu}\right\rangle + \xi(\mu) \\
&= \left\langle \mu\,, \frac{\delta \xi}{\delta \mu}\right\rangle - \left\langle \mu\,, \frac{\delta \ell}{\delta \mu}\right\rangle + \xi(\mu) \\
&= \xi(\mu).
\end{aligned}$$

Given this, we can now take the basic Euler-Poincaré equations and then rewrite them as follows,

$$\dot{\mu} = -\mathrm{ad}^*_{\frac{\delta h}{\delta \mu}} \mu.$$

This is the Lie-Poisson Equation! An equivalent derivation is from the right invariant Poisson bracket. For any two smooth functions on the manifold $F, G : M \to \mathbb{R}$, we can define the canonical Poisson bracket as,

$$\{F, G\}^{\text{right}}_{\mathfrak{g}^*} = \left\langle \mu\,, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right] \right\rangle.$$

The Lie-Poisson reduction theorem is redundant as we will reduce on the Lagrangian side. However for completeness we include it.

Theorem 2.3.7 (Lie-Poisson Reduction)**.** *Let G be a Fréchet group, then $\mathfrak{g}^*$ be the dual Lie algebra. If $\mathcal{F}_R(\mathfrak{g}^*)$ is the set of right G-invariant functions, then the right Lie-Poisson bracket is,*

$$\{F, G\}^{\textit{right}}_{\mathfrak{g}^*} = \left\langle \mu\,, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right] \right\rangle.$$

Further, this relates to the reduced bracket on T^*G/G via the isomorphism,

$$\psi : (T^*G/G) \mapsto \mathfrak{g}^*$$

$$[g,\, \alpha] \mapsto g^{-1}\alpha$$

Further, we have a reconstruction equation,

$$\dot{g} = g\frac{\delta h}{\delta \mu}.$$

This can be proven from momentum maps, from Appendix B

2.3.4 Lie-Poisson Reduction for advected quantities

We note that given a Lie algebra $\mathfrak{s} = \mathfrak{g} \rtimes V^*$. Then, we can start to derive the same equations as above for semidirect products. We consider the following two actions: the adjoint and the coadjoint. We can write these as,

Proposition 2.3.8 (Adjoint Action)**.** *The adjoint action on $\mathfrak{g} \rtimes V^*$ is,*

$$\mathrm{Ad}_{(g,v)}(\xi, u) = (\mathrm{Ad}_g \xi,\, ug^{-1} + vg^{-1}\,\mathrm{Ad}_g \xi)$$

Proof. We consider the following calculation,

$$\begin{aligned}
\mathrm{Ad}_{(g,v)}(\xi, u) &= (g,\, v) * (\xi,\, u) * (g,\, v)^{-1} \\
&= (g\xi,\, u + v\xi) * (g^{-1},\, -vg^{-1}) \\
&= (g\xi g^{-1},\, -vg^{-1} + (u + v\xi)g^{-1}) \\
&= (g\xi g^{-1},\, -vg^{-1} + ug^{-1} + v\xi g^{-1}) \\
&= (\mathrm{Ad}_g \xi,\, vg^{-1}\,\mathrm{Ad}_g \xi - vg^{-1}),
\end{aligned}$$

as required. ∎

We can do something similar for the coadjoint action,

Proposition 2.3.9. *The coadjoint action on* $\mathfrak{g} \rtimes V^*$ *is,*

$$\mathrm{Ad}^*_{(g,\,v)^{-1}}(\mu,\, a) = (\mathrm{Ad}_{g^{-1}}\,\mu - g \diamond (v^{-1}ga),\, g \diamond a)$$

Proof. We consider the trace pairing of the coadjoint action and then construct the required result.

$$\begin{aligned}
\left\langle \mathrm{Ad}^*_{(g,\,v)^{-1}}(\mu,\, a)\,,(\nu_1,\, \nu_2)\right\rangle &= \left\langle (\mu,\, a)\,, \mathrm{Ad}_{(g,\,v)^{-1}}(\nu_1,\, \nu_2)\right\rangle \\
&= \left\langle (\mu,\, a)\,, (g,\, v)^{-1}(\nu_1,\, \nu_2)(g,\, v)\right\rangle \\
&= \left\langle (\mu,\, a)\,, (g^{-1},\, -vg^{-1})(\nu_1,\, \nu_2)(g,\, v)\right\rangle \\
&= \left\langle (\mu,\, a)\,, (g^{-1}\nu_1,\, \nu_2 - vg^{-1}\nu_1)(g,\, v)\right\rangle \\
&= \left\langle (\mu,\, a)\,, (g^{-1}\nu_1 g,\, v + (\nu_2 - vg^{-1}\nu_1)g)\right\rangle \\
&= \left\langle (\mu,\, a)\,, (g^{-1}\nu_1 g,\, v + \nu_2 g - vg^{-1}\nu_1 g\right\rangle \\
&= \left(\left\langle \mu\,, \mathrm{Ad}_{g^{-1}}\,\nu_1\right\rangle,\, \left\langle (a\,, v + \nu_2 g - vg^{-1}\nu_1 g)\right\rangle\right) \\
&= \left(\left\langle \mathrm{Ad}^*_{g^{-1}}\,\mu\,, \nu_1\right\rangle,\, \langle a\,, v\rangle + \langle a\,, \nu_2 g\rangle - \left\langle a\,, vg^{-1}\nu_1 g\right\rangle\right) \\
&= \left(\left\langle \mathrm{Ad}^*_{g^{-1}}\,\mu\,, \nu_1\right\rangle,\, \langle g \diamond a\,, \nu_2\rangle - \left\langle g \diamond (v^{-1}ga)\,, \nu_1\right\rangle\right) \\
&= \left\langle (\mathrm{Ad}_{g^{-1}\mu} - g \diamond (v^{-1}ga),\, g \diamond a)\,, (\nu_1,\, \nu_2)\right\rangle .
\end{aligned}$$

Therefore,

$$\mathrm{Ad}^*_{(g,\,v)^{-1}}(\mu,\, a) = (\mathrm{Ad}_{g^{-1}}\,\mu - g \diamond (v^{-1}ga),\, g \diamond a)$$

□

We note the following facts about the semidirect products and Lie brackets,

$$[(\xi_1,\, v_1),\, (\xi_2,\, v_2)] = ([\xi_1,\, \xi_2],\, v_2\xi_1 - v_1\xi_2).$$

We now recall the definition of the Lie-Poisson bracket we gave for the basic Lie-Poisson equations and apply it to $\mathfrak{s}^* = \mathfrak{g}^* \rtimes V^*$.

$$\{F, G\}_{\mathfrak{s}} = \left\langle (\mu, a)\,, \left[\frac{\delta F}{\delta(\mu,\, a)},\, \frac{\delta G}{\delta(\mu,\, a)}\right]\right\rangle . \tag{2.13}$$

We can now prove,

Lemma 2.3.10. *The Lie-Poisson bracket for the Lie-Poisson equations with advected quantities is,*

$$\{F,\, G\}_{\mathfrak{s}^*} = \left\langle \mu\,, \left[\frac{\delta F}{\delta \mu},\, \frac{\delta G}{\delta \mu}\right]\right\rangle + \left\langle a\,, \frac{\delta F}{\delta \mu}\frac{\delta G}{\delta a} - \frac{\delta G}{\delta \mu}\frac{\delta F}{\delta a}\right\rangle$$

Proof. Consider Equation 2.13, and the following computation,

$$\begin{aligned}
\{F,\, G\}_{\mathfrak{s}^*} &= \left\langle (\mu, a)\,, \left[\frac{\delta F}{\delta(\mu,\, a)},\, \frac{\delta G}{\delta(\mu,\, a)}\right]\right\rangle \\
&= \left\langle (\mu, a)\,, \left[\left(\frac{\delta F}{\delta \mu},\, \frac{\delta F}{\delta a}\right),\, \left(\frac{\delta G}{\delta \mu},\, \frac{\delta G}{\delta a}\right)\right]\right\rangle \\
&= \left\langle (\mu,\, a)\,, -\left(\left[\frac{\delta F}{\delta \mu},\, \frac{\delta G}{\delta \mu}\right],\, \frac{\delta G}{\delta a}\frac{\delta F}{\delta \mu} - \frac{\delta F}{\delta a}\frac{\delta G}{\delta \mu}\right)\right\rangle \\
&= \left\langle \mu\,, \left[\frac{\delta F}{\delta \mu},\, \frac{\delta G}{\delta \mu}\right]\right\rangle + \left\langle a\,, \frac{\delta F}{\delta \mu}\frac{\delta G}{\delta a} - \frac{\delta G}{\delta \mu}\frac{\delta F}{\delta a}\right\rangle .
\end{aligned}$$

□

We can now further manipulate to prove the following,

Theorem 2.3.11 (Lie-Poisson Equations for advected parameters). *The Lie-Poisson equations for a system with advected parameters are,*

$$\begin{aligned} \frac{\mathrm{d}h}{\mathrm{d}t} &= -\mathrm{ad}^*_{\frac{\delta h}{\delta t}}\mu + \frac{\delta h}{\delta a} \diamond a \\ \frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{\delta h}{\delta \mu} a \end{aligned} \tag{2.14}$$

Proof. Consider the following calculation,

$$\begin{aligned} \{F, H\}_{\mathfrak{s}} &= \left\langle \mu \,, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu} \right] \right\rangle + \left\langle a \,, \frac{\delta F}{\delta \mu} \frac{\delta G}{\delta a} - \frac{\delta G}{\delta \mu} \frac{\delta F}{\delta a} \right\rangle \\ &= \left\langle \mu \,, - \left[\frac{\delta G}{\delta \mu}, \frac{\delta F}{\delta \mu} \right] \right\rangle + \left\langle a \,, \frac{\delta F}{\delta \mu} \frac{\delta G}{\delta a} \right\rangle - \left\langle a \,, \frac{\delta G}{\delta \mu} \frac{\delta F}{\delta a} \right\rangle \\ &= \left(\left\langle -\mathrm{ad}^*_{\frac{\delta G}{\delta \mu}} \mu \,, \frac{\delta F}{\delta \mu} \right\rangle, \left\langle \frac{\delta G}{\delta a} \diamond a \,, \frac{\delta F}{\delta \mu} \right\rangle - \left\langle a \frac{\delta G}{\delta \mu} \,, \frac{\delta F}{\delta a} \right\rangle \right) \\ &= \left\langle \left(-\mathrm{ad}^*_{\frac{\delta G}{\delta \mu}} \mu + \frac{\delta G}{\delta a} \diamond a, -\frac{\delta G}{\delta a} a \right) , \left(\frac{\delta F}{\delta \mu}, \frac{\delta F}{\delta a} \right) \right\rangle \end{aligned}$$

Then, we can decompose the inner products and pairings. Let G be the reduced Hamiltonian, and we can now write,

$$\begin{aligned} \frac{\mathrm{d}h}{\mathrm{d}t} &= -\mathrm{ad}^*_{\frac{\delta h}{\delta t}}\mu + \frac{\delta h}{\delta a} \diamond a \\ \frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{\delta h}{\delta \mu} a. \end{aligned} \tag{2.15}$$

□

Chapter 3

Conservation

> As long as algebra and geometry have been separated, their progress has been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces and have marched together towards perfection.
>
> Joseph-Louis Lagrange

In Noether's very influential paper in 1918, named 'Invariante Variationsprobleme ' [23], she stated one of the most important theorems in conservation. This was to be known as Noether's theorem. She stated, *'Every differentiable symmetry generated by local actions has a corresponding conserved current.'* This innocuous-looking theorem would lead to many years of work to find these conserved currents. The theorem said a little about how to find these quantities for the Euler-Lagrange, but there is an analogous result for Euler-Poincaré. These theorems then need to be derived for every new type of Euler-Poincaré equation. Noether's theorems apply to the Lagrangian framework. To calculate conserved quantities for the Hamiltonian framework, you go to consider Casimir's and Poisson brackets.

In this penultimate chapter, we will discuss the conservation properties of the Euler-Poincaré and Lie-Poisson equations. We will take two different routes to find the quantities depending on what framework we work under. Firstly, we will return to the Lagrangian framework and study the Noether Theorems for Euler-Poincaré theory and then consider a generalisation of these theorems called the Kelvin-Noether theorems which relate to first integrals over a path in our flow. We will then study the Hamiltonian point of view by returning to Lie-Poisson brackets, considering Casimirs and showing that certain functions that make the brackets vanish are conserved. We will also consider the relation between conserved quantities and Reeb graphs, another potential area for exploration in future work. Finally, we will attempt to show numerically that these quantities are conserved and conclude with ideas relating to extending numerical methods with the preceding theory to guarantee that the conserved quantities are conserved.

3.1 Noether Theorems

Noether Theorems can be derived from the variational principle and reduction. The process involves considering the terms we made vanish to produce the equations. We can state and prove the Noether Theorem for Euler-Poincaré equations with advected parameters.

Theorem 3.1.1 (Noether Theorem for Euler-Poincaré with advected parameters)**.** *Each symmetric vector field of the Euler-Poincaré reduced lagrangian for the infinitesimal variations,*

$$\delta \mathbf{u} = \dot{\nu} - \mathrm{ad}_{\mathbf{u}}\nu \qquad \delta a = -\nu a,$$

corresponds to an integral of the Euler-Poincaré motion and a conserved quantity,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M \left\langle \frac{\delta \ell}{\delta \mathbf{u}}, \nu \right\rangle \mu = 0.$$

Proof. We will consider the derivation of the Euler-Poincaré equations for advection parameters. Then jump to the integration by parts of the $\frac{\partial u}{\partial t}$ term.

$$\begin{aligned} 0 &= \int_{t_1}^{t_2}\int_M \left\langle \frac{\delta \ell}{\delta \mathbf{u}}, \frac{\partial \nu}{\partial t} \right\rangle - \left\langle \mathrm{ad}^*_{\mathbf{u}} \frac{\delta \ell}{\delta \mathbf{u}}, \nu \right\rangle + \left\langle a \diamond \frac{\delta \ell}{\delta a}, \nu \right\rangle \mathrm{d}t \\ &= \int_{t_1}^{t_2}\int_M \left\langle -\frac{\partial}{\partial t}\frac{\delta \ell}{\delta \mathbf{u}} - \mathrm{ad}^*_{\mathbf{u}} \frac{\delta \ell}{\delta \mathbf{u}} + a \diamond \frac{\delta \ell}{\delta a}, \nu \right\rangle \mathrm{d}t + \int_M \left\langle \frac{\delta \ell}{\delta \mathbf{u}}, \nu \right\rangle \Big|_{t_1}^{t_2} \mu. \end{aligned}$$

We now note that the first pairing is just the Euler-Poincaré equations. Hence, we now can write,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M \left\langle \frac{\delta \ell}{\delta u}, \nu \right\rangle \mu = 0,$$

as we use the fundamental theorem of calculus of variations. □

3.1.1 Vorticity and Helicity

We can now prove two lemmas for the specific case of the Euler Equations in barotropic fluids. The advective quantity is going to be $a = \rho\, \mathrm{d}V$ and then the infinitesimal symmetry for a becomes,

$$\pounds_\eta(\rho\, \mathrm{d}V) = \mathrm{d}(\eta \lrcorner \rho\, \mathrm{d}V) = 0.$$

Using rules from the Appendix A, this can now be written as for some vector function, $\mathbf{\Psi}$,

$$\eta \lrcorner \rho\, \mathrm{d}V = \mathrm{d}(\mathbf{\Psi} \cdot \mathrm{d}\mathbf{x}) = \mathrm{curl}\, \mathbf{\Psi} \cdot \mathrm{d}\mathbf{S}. \tag{3.1}$$

Lemma 3.1.2 (Conservation of Vorticity)**.** *In the Euler-Poincaré equations for advected quantities, the following holds and relates to vorticity being conserved,*

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \pounds_{\mathbf{u}}\right)\left(\mathrm{curl}\, \frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}} \cdot \mathrm{d}\mathbf{S}\right) = 0. \tag{3.2}$$

Proof. We start with Noether's theorem for these equations and then perform a calculation.

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left\langle \frac{\delta \ell}{\delta \mathbf{u}}, \eta \right\rangle \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_M \frac{\delta \ell}{\delta \mathbf{u}} \cdot \eta \, \mathrm{d}V \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_M \frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}} \cdot \mathrm{d}\mathbf{x} \wedge \eta \lrcorner (\rho \, \mathrm{d}V) \\
&= \int_M \frac{\partial}{\partial t}\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}} \wedge \mathrm{d}(\mathbf{\Psi} \cdot \mathrm{d}\mathbf{x}) + \frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}} \wedge \frac{\partial}{\partial t}\mathrm{d}(\mathbf{\Psi} \cdot \mathrm{d}\mathbf{x}) \\
&= -\int_M \left(\frac{\partial}{\partial t}\mathrm{d}\left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}} \cdot \mathrm{d}\mathbf{x}\right) + \pounds_{\mathbf{u}}\,\mathrm{d}\left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}} \cdot \mathrm{d}\mathbf{x}\right)\right) \wedge (\mathbf{\Psi} \cdot \mathrm{d}\mathbf{x}) \\
&= -\int_M \left(\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right)\mathrm{d}\left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}} \cdot \mathrm{d}\mathbf{x}\right)\right) \wedge (\mathbf{\Psi} \cdot \mathrm{d}\mathbf{x}).
\end{aligned}$$

Hence, we can say that, as we know, $\mathbf{\Psi} \cdot \mathrm{d}\mathbf{x} \neq 0$. Then the result appears after applying (3.1),

$$\begin{aligned}
0 &= \left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right)\mathrm{d}\left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}} \cdot \mathrm{d}\mathbf{x}\right) \\
&= \left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right)\left(\operatorname{curl}\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}} \cdot \mathrm{d}\mathbf{S}\right).
\end{aligned}$$

□

If we take the example of the variable density Euler equations, we can say that,

$$\frac{\delta \ell}{\delta \mathbf{u}} = D\mathbf{u}^\flat.$$

Then we can rewrite the above as,

$$\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right)\left(\operatorname{curl}\frac{1}{D}\frac{\delta \ell}{\delta \mathbf{u}} \cdot \mathrm{d}\mathbf{S}\right) = \left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right)(\operatorname{curl}\mathbf{u} \cdot \mathrm{d}\mathbf{S}).$$

This makes it clearer that this geometric term is directly referencing vorticity across each. infinitesimal element.

We now present two significant ideas in this area. We first iterated conserved quantities. Let us consider another vital theorem, called Ertel's theorem.

Theorem 3.1.3 (Ertel's Theorem)**.** *If a quantity a satisfied the advection equation, and η satisfy $\delta\eta = \dot{\nu} + \mathrm{ad}_\eta \nu$ for the labelling symmetry. Then $\pounds_\eta a$ is also advected.*

Proof. By a substitution of the advection equation for a, we find,

$$\pounds_\eta\left(\frac{\partial}{\partial t} + \pounds_u\right)a = \left(\frac{\partial}{\partial t} + \pounds_u\right)\pounds_\eta a = 0.$$

This is the advection equation. Hence, $\pounds_\eta a$ is advected by the flow. □

Consider a conserved quantity, $c(t)$; the flow carries this. Hence, we can say that $\pounds_u c(t)$ is also conserved by the Euler-Poincaré equations. Hence, we can now consider a conserved quantity using the vorticity in (3.2). Hence, we can now set the vorticity as our conserved 2-form in (3.1).

$$\mathrm{d}(\boldsymbol{\Psi}\cdot \mathrm{d}\mathbf{x}) = \mathrm{d}\left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\cdot \mathrm{d}\mathbf{x}\right).$$

This will lead us to a new conserved quantity called Helicity. Helicity is a topological invariant sometimes known as the Hopf invariant. It measures the knottedness of the vortex lines and can also be used to calculate the number of linkages. As the vortex lines are baked in the Lagrangian and hence the flow, this quantity is always conserved.

Lemma 3.1.4 (Conservation of Helicity). *The Euler-Poincaré equations for advected quantities, the following holds and relates to the Helicity being conserved.*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M \frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\cdot \operatorname{curl}\left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\right)\,\mathrm{d}V.$$

Proof. We start at the third step of the previous argument and then move forward differently,

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\int_M \left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\cdot \mathrm{d}\mathbf{x}\right)\wedge \eta \,\lrcorner\, (\rho\,\mathrm{d}V) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_M \left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\cdot \mathrm{d}\mathbf{x}\right)\wedge \mathrm{d}\left(\boldsymbol{\Psi}\cdot \mathrm{d}\mathbf{x}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_M \left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\cdot \mathrm{d}\mathbf{x}\right)\wedge \mathrm{d}\left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\cdot \mathrm{d}\mathbf{x}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_M \frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\cdot \operatorname{curl}\left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\right)\,\mathrm{d}V.
\end{aligned}$$

Therefore, we can write Helicity as,

$$\mathcal{H} := \int_M \frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\cdot \operatorname{curl}\left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}}\right)\,\mathrm{d}V.$$

□

Similarly to above consider the Euler equations for variable density, this time where the dimension of the space is greater equal to three. Then we can write Helicity in the more recognisable form,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M \mathbf{u}\cdot \operatorname{curl}\mathbf{u}\,\mathrm{d}V.$$

3.1.2 Kelvin-Noether Theorem

We can now find other conservation quantities using similar ideas to the above. Consider the following change of variables,

$$\oint_{\gamma_t} \frac{1}{\rho_t}\frac{\delta \ell}{\delta \mathbf{u}} = \oint_{\gamma_0} \eta^*\left[\frac{1}{\rho_0}\frac{\delta \ell}{\delta \mathbf{u}}\right] = \oint_{\gamma_0} \frac{1}{\rho_0}\eta^*\left[\frac{\delta \ell}{\delta \mathbf{u}}\right].$$

Now, we can use the definition of the Lie derivative to write,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\eta^* \alpha) = \eta^* \left(\frac{\partial}{\partial t}\alpha + \pounds_{\mathbf{u}}\alpha \right).$$

Hence, we can write,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \oint_{\gamma_t} \frac{1}{\rho_t} \frac{\delta \ell}{\delta \mathbf{u}} &= \oint_{\gamma_0} \frac{1}{\rho_0} \eta^* \left(\frac{\partial}{\partial t}\alpha + \pounds_{\mathbf{u}}\alpha \right) \\
&= \oint_{\gamma_t} \frac{1}{\rho_t} \left(\frac{\partial}{\partial t}\alpha + \pounds_{\mathbf{u}}\alpha \right) \\
&= \oint_{\gamma_t} \frac{1}{\rho_t} \frac{\delta \ell}{\delta a} \diamond a.
\end{aligned}$$

The last step is from just considering the Euler-Poincaré equations for advected parameters. From this, we present the Kelvin-Noether theorem for advected Euler-Poincaré,

Theorem 3.1.5 (Kelvin-Noether Theorem for Euler Poincaré Equations)**.** *For the Euler-Poincaré equation with advected quantities, we can write the following,*

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_{\gamma_t} \frac{1}{\rho_t} \frac{\delta \ell}{\delta \mathbf{u}} = \oint_{\gamma_t} \frac{1}{\rho_t} \frac{\delta \ell}{\delta a} \diamond a.$$

We have two remarks,

Remark 3.1.6. *For basic Euler-Poincaré equations, the following is a conserved quantity,*

$$\oint_{\gamma} \frac{1}{\rho} \frac{\delta \ell}{\delta \mathbf{u}}.$$

Remark 3.1.7. *We can write the Euler-Poincaré equations in Kelvin-Noether form,*

$$\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}} \right) \frac{\delta \ell}{\delta \mathbf{u}} = \frac{\delta \ell}{\delta a} \diamond a.$$

3.2 Casimirs

First, we will consider the Hamiltonian formalism to find conserved quantities with the idea of Casimir's. Then, we will briefly examine the ideas surrounding Reeb graphs and their application to this area. We cite Kolev's paper [19] as a good overview of this area.

3.2.1 The Arnold Bracket

As stated by Arnold in Topological Methods in Hydrodynamics [29], we can find a Hamiltonian system description through a Poisson bracket. That is,

$$\dot{F} = \{F, H\}.$$

To be able to use these brackets, we have to show that on a space of smooth functions, $\mathcal{F}$, they create a Hamiltonian structure,

Definition 3.2.1 (Hamiltonian Structure)**.** *A Hamiltonian structure is a bilinear operation, $\{\cdot,\, \cdot\}$ on the space of smooth functionals, $\mathcal{F}$, that satisfies the following, for any $F, G \in \mathcal{F}$,*

1. $\{F,\, G\} \in \mathcal{F}$,
2. $\{F,\, G\} = -\{G,\, F\}$,
3. $\{\{F,\, G\},\, H\} + \{\{G,\, H\},\, F\} + \{\{H,\, F\},\, G\} = 0.$

Remark 3.2.2. *The well-trained eye will notice that $\{\cdot,\, \cdot\}$ isn't technically a Poisson structure as it doesn't have a Liebniz rule. This is because local functionals don't necessarily have this property.*

Our first bracket is the Arnold bracket. For functionals with L^2 gradients on the Lie algebra, it is defined as,

$$\{F,\, G\}(\omega) = \int_M \omega \left(\frac{\delta F}{\delta \omega},\, \frac{\delta G}{\delta \omega} \right).$$

Proving this provides a Hamiltonian structure is a quick calculation of inner products. Further, this bracket is the Hamiltonian structure that relates to the incompressible Euler Equations. Recall we can write curl as follows,

$$\mu \lrcorner \operatorname{curl} \mathbf{u} = \, d\mathbf{u}^\flat,$$

and we know that the curl of a vector field relates to an exact 2-form. Hence, we can write an inertia operator $A : \mathrm{SVect}(M) \to \mathrm{SVect}^*(M)$ defined by $\mathbf{u} \mapsto \operatorname{curl} \mathbf{u}$. Then we can say that $\mathbf{u} = A^{-1}\omega$ is the unique solution of the Euler Equations subject to,

$$\operatorname{curl} \mathbf{u} = \omega \qquad \operatorname{div} \mathbf{u} = 0 \qquad \mathbf{u} \cdot n = 0 \text{ on } \partial M.$$

Our Hamiltonian is going to be,

$$H(\omega) = \int_M |\mathbf{u}|^2 \mu,$$

and $\mathrm{d}H = \mathbf{u} = A^{-1}\omega$. Now we consider the L^2 gradient of $\mathbf{u}$, and we can derive,

$$\begin{aligned}\int_M \partial_t \mathbf{u} \cdot \delta F &= \int_M \partial_t (A^{-1}\omega) \cdot \delta F \\ &= \int_M \omega \cdot (\delta F \times \mathbf{u}) \\ &= \int_M \delta F \cdot (\mathbf{u} \times \omega).\end{aligned}$$

Therefore, our bracket provides the Hamiltonian system modulo a 1-form,

$$\partial_t \mathbf{u} = \mathbf{u} \times \omega.$$

This is the vorticity form of the Euler Equations. We can consider the pullback of this functional to get something more amicable to work with,

$$\{F,\, G\}(\omega) = \int_M \operatorname{curl} \mathbf{u} \cdot \left(\frac{\delta F}{\delta \mathbf{u}} \times \frac{\delta G}{\delta \mathbf{u}} \right). \tag{3.3}$$

We can show that Helicity is conserved by finding the Casimir. We define a Casimir,

Definition 3.2.3 (Casimir)**.** *We define the Casimir as a smooth function $C : M \to \mathbb{R}$,*

$$\{C, H\} = 0.$$

Now we find the Casimirs of the bracket (3.3). We consider the following argument,

$$\begin{aligned}0 &= \int_M \operatorname{curl} \mathbf{u} \cdot \left(\frac{\delta C}{\delta \mathbf{u}} \times \frac{\delta H}{\delta \mathbf{u}} \right) \mu \\ &= \int_M \operatorname{curl} \mathbf{u} \cdot \left(\frac{\delta C}{\delta \mathbf{u}} \times \mathbf{u} \right) \mu \\ 0 &= -\int_M \frac{\delta C}{\delta \mathbf{u}} \cdot (\operatorname{curl} \mathbf{u} \times \mathbf{u})\, \mu \\ &= -\int_M \frac{\delta C}{\delta \mathbf{u}} \cdot \left((\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{2}\nabla(\mathbf{u}^2) \right) \mu \\ &= \int_M -\frac{\delta C}{\delta \mathbf{u}} \cdot (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{2}\frac{\delta C}{\delta \mathbf{u}} \cdot \nabla(\mathbf{u}^2)\, \mu \\ &= \int_M \left(\frac{\delta C}{\delta \mathbf{u}} \cdot \partial_t \mathbf{u} + \frac{1}{2}\nabla \frac{\delta C}{\delta \mathbf{u}} \cdot \mathbf{u}^2 \right) \mu + \int_{\partial M} \frac{1}{2}\frac{\delta C}{\delta \mathbf{u}} \cdot (\mathbf{u}^2 \cdot n)\mu_\partial \\ &= \int_M \left(\partial_t \frac{\delta C}{\delta \mathbf{u}} \cdot \mathbf{u} + \frac{1}{2}\nabla \frac{\delta C}{\delta \mathbf{u}} \cdot \mathbf{u}^2 \right) \mu + \left[\int_M \frac{1}{2}\frac{\delta C}{\delta \mathbf{u}} \cdot \mathbf{u}\, \mu \right]_{t_1}^{t_2} \\ &= \int_M \left(\partial_t \frac{\delta C}{\delta \mathbf{u}} + \frac{1}{2}\nabla \frac{\delta C}{\delta \mathbf{u}} \cdot \mathbf{u} \right) \cdot \mathbf{u}\, \mu + \left[\int_M \frac{1}{2}\frac{\delta C}{\delta \mathbf{u}} \cdot \mathbf{u}\, \mu \right]_{t_1}^{t_2}.\end{aligned}$$

Hence, we require,

$$\frac{\partial}{\partial t}\frac{\delta C}{\delta \mathbf{u}} + \frac{1}{2}\nabla \frac{\delta C}{\delta \mathbf{u}} \cdot \mathbf{u} = 0 \qquad \frac{\delta C}{\delta \mathbf{u}} = 0 \text{ on } \partial M. \tag{3.4}$$

We note that when we let the Casimir be Helicity, the equation (3.4) turns into,

$$\int_M (\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u})\cdot(1+\mathbf{u}\times\omega) + ((\omega_t + \mathbf{u}\cdot\nabla\omega)\times\mathbf{u})\cdot\mathbf{u} = 0.$$

Hence, we have proven that Helicity is a Casimir. This bracket seems to be conserving what we want to conserve. Let us now look to enstrophy for a 2D system. We consider generalised enstrophy,

$$C(\omega) = \int_M \phi(\omega)\,\mathrm{d}x \wedge \,\mathrm{d}y.$$

We can take the Fréchet derivative,

$$DC(\omega) = \int_M \operatorname{curl}(\phi'(\omega)\hat{\mathbf{k}})\,\mathrm{d}\alpha + \oint_{\partial M} \phi'(\omega)\,\mathrm{d}\alpha.$$

Using the Arnold bracket requires the function's derivative to be an L^2 gradient. This isn't as it has boundary terms. Hence, the Arnold bracket doesn't conserve enstrophy. Further, if we set $\phi'(\omega) = 0$, we would fall back to enstrophy, which is covered by the conservation of Helicity. Hence, we need a stronger bracket.

In Zackharov's 1968 paper [31], he presented the following Hamiltonian system for irrotational waves,

$$H = \frac{1}{2}\iiint (\operatorname{grad}\phi)^2\,\mathrm{d}V + \frac{1}{2}\lambda\iint \zeta^2\,\mathrm{d}S.$$

This was then generalised to include vorticity, and the bracket was found for a free boundary Σ and a velocity field v such that,

1. δv is a divergence-free vector field on some domain D_Σ. Further, we have a divergence-free vector field $\frac{\delta F}{\delta v}$ defined one some domain D_Σ such that,

$$D_v F(v,\,\Sigma) = \int_{D_\Sigma} \frac{\delta F}{\delta v}\cdot\delta v,$$

where D_v is the derivative with respect to v with Σ fixed.

2. Similarly, there's a $\delta\Sigma$ that's normal to Σ and some smooth function $\frac{\delta F}{\delta \Sigma}$ such that,

$$D_\Sigma F(v,\,\Sigma) = \int_{D_\Sigma} \frac{\delta F}{\delta \Sigma}\cdot\delta\Sigma.$$

Then the bracket is,

$$\{F,G\} = \int_{D_\Sigma} \omega\cdot\left(\frac{\delta F}{\delta v}\times\frac{\delta G}{\delta v}\right) + \int_\Sigma \left(\frac{\delta F}{\delta\Sigma}\frac{\delta G}{\delta\phi} - \frac{\delta G}{\delta\Sigma}\frac{\delta F}{\delta\phi}\right),$$

where $\omega = \operatorname{curl}\mathbf{u}$ and,

$$\frac{\delta F}{\delta\phi} = \left.\frac{\delta F}{\delta v}\right|_\Sigma \cdot n.$$

This proof of conservation here is more involved and will be covered in my PhD thesis.

3.2.2 Reeb Graphs

Another interesting way to derive the conserved quantities of a Hamiltonian system and derive all of the Casimir's of the system is to consider Reeb graphs of the manifold you work over. In this section, we will review and expose the work of Izosimov and Khesin [29, 17]. We will return to the problem we left in the previous section of proving in 2D that Enstrophy is conserved in the Hamiltonian form of the Euler Equations. There are two main theorems we are interested in, which give the following correspondences,

$$\text{Reeb Graphs} \longleftrightarrow \text{Morse Functions} \longleftarrow \text{Casimirs}$$

There are two main theorems relating to this diagram.

Theorem 3.2.4 (Reeb Graphs give Morse Functions)**.** *The mapping assigning the measured Reeb graph Γ_F to a simple Morse function F provides a one-to-one correspondence between simple Morse functions on M up to symplectomorphism and measured Reeb graphs compatible with M.*

Theorem 3.2.5 (Casimirs are moments of the Morse functions)**.** *A complete set of Casimirs to the 2D Euler Equation in a neighbourhood of a Morse-type coadjoint orbit is given by the momenta,*

$$I_{i,e} = \int_{M_e} F^i \omega, \qquad i \in \mathbb{N},$$

for each $e \in \Gamma$ and all circulations of the velocity v over cycles in the singular levels of the vorticity function F on M.

We will now define Morse functions and measures Reeb graphs and then give proof of Enstrophy being conserved in the 2D Euler Equations.

Definition 3.2.6 (Morse Function)**.** *Let M be a closed connected space, then a morse function $F : M \to \mathbb{R}$ is called simple if any level set of F contains at most one critical point.*

Now, we can associate a graph with a morse function. To do this, we take each critical point of F and place a critical point of the graph there, as shown in Figure 3.1. We can

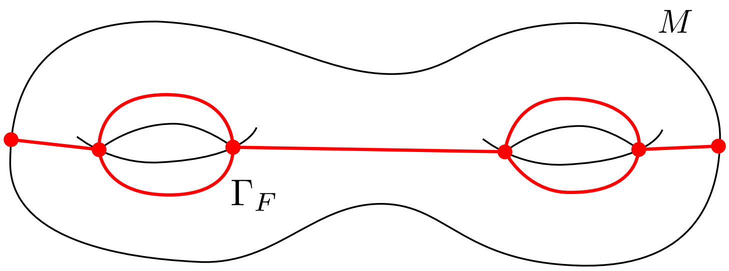


Figure 3.1: The Reeb Graph of a manifold.

now define a Reeb graph as follows,

Definition 3.2.7 (Reeb Graph)**.** *A Reeb graph, (Γ, f) is an oriented smooth connected finite graph Γ with a continuous function $f : \Gamma \to \mathbb{R}$ which satisfy the following,*

1. *All vertices of* Γ *are either* 1*-valent or* 3*-valent,*

2. *For each* 3*-valent vertex, there are either two incoming and two outgoing or vice versa,*

3. *The function* f *is strictly monotonous on each edge, and the edges of* Γ *are oriented towards the direction of increasing* f*.*

It is a standard result in Morse theory that a graph Γ_F associated with a simple Morse function $F : M \to \mathbb{R}$ on an orientable connected surface is a Reeb graph. Further, if this surface is endowed with an area or symplectic form ω, then we can take the pullback of that form and define a measured Reeb graph.

Definition 3.2.8 (Measured Reeb Graph)**.** *Let* $(\Gamma,\, f)$ *be a Reeb graph associated with a surface with symplectic form* ω*. Then we can define a measure* $\mu = f^*\omega$*. Then, a measured Reeb graph is a Reeb graph with a measure,* $(\Gamma,\, f,\, \mu)$*.*

We can say that the simple Morse functions are the same as the measured Reeb graphs up to symplectomorphism under the following compatibility conditions.

Definition 3.2.9 (Compatible)**.** *We say that a closed compact surface endowed with a symplectic form,* $(M,\, \omega)$*, is compatible with a measured Reeb graph,* $(\Gamma,\, f,\, \mu)$ *if the following are satisfied,*

1. *The dimension of the first homology group [12],* $H_1(\Gamma,\, \mathbb{R})$ *is equal to the genus of* M*,*

2. *The volume of* Γ *with respect to* μ *is the same as the volume of* M*,*

$$\int_\Gamma \mathrm{d}\mu = \int_M \omega.$$

Then from this Theorem 3.2.4 follows.

Classifying coadjoint orbits

We note that the coadjoint action of $\mathrm{SDiff}(M)$ on $\mathfrak{svect}(M)$ is just a symplectomorphism, a volume preserving map,

$$\mathrm{Ad}^*_\Phi[\alpha] = [\Phi^*\alpha].$$

The orbits of this action can be described by $\mathfrak{curl} : \Omega^1(M)/\,\mathrm{d}\Omega^0(M) \to C^\infty(M)$ given by vorticity,

$$\mathfrak{curl}\,[\alpha] := \frac{\mathrm{d}\alpha}{\omega}.$$

This mapping is equivariant, which means that if two maps differ by a symplectomorphism, they are in the same coadjoint orbit as each other. Hence, all the simple Morse functions are in one coadjoint orbit.

Definition 3.2.10 (Morse-Type forms)**.** *We say that a coset of* 1*-forms,* $[\alpha] \in \mathfrak{svect}^*(M)$*, is morse type if* $\mathfrak{curl}\,[\alpha]$ *is a simple morse function. A coadjoint orbit is Morse-type if any coset is Morse-type.*

For the remainder, let $[\alpha] \in \mathfrak{svect}^*(M)$ be morse-type and $F := \mathfrak{curl}\,[\alpha]$. Then Γ_F is invariant under the coadjoint action! However, if M is not simply connected, our theory fails as the graph branches and causes issues. Hence, we use a circulation function to deal with branches. Let $\pi : M \to \Gamma_F$ be the natural projection. Take any point x in the interior of some edge $e \in \Gamma_F$. Then $\pi^{-1}(x)$ is a circle. It is naturally oriented, and the integral does not depend on α. Hence we have a function $\mathfrak{c} : \Gamma_F \setminus V(\Gamma_F) \to \mathbb{R}$ defined by,

$$\mathfrak{c}(x) := \int_{\pi^{-1}(x)} \alpha.$$

We now define yet another graph.

Definition 3.2.11 (Circulation Measured Reeb Graph)**.** *A circulation measured Reeb graph,* $(\Gamma,\, f,\, \mu,\, \mathfrak{c})$ *, is a measured Reeb graph endowed with a circulation function* $\mathfrak{c} : \Gamma_F \setminus V(\Gamma_F) \to \mathbb{R}$ *defined by,*

$$\mathfrak{c}(x) := \int_{\pi^{-1}(x)} \alpha.$$

We note that non-circulation graphs work for us because Fluid Dynamics problems only have 'pants decompositions' and 'Dehn half twists'. These more complicated ideas don't arise in this example but should be considered for more complex phenomena. We can now prove that the following momenta,

$$m_{i,e}(F) = \int_{M_e} F^i \omega$$

form a complete set of invariants for the 2D Euler Equations.

Theorem 3.2.12. *Let $(M,\, \omega)$ be a closed connected symplectic surface, and let F and G be simple morse functions on M. Then let $\phi : \Gamma_F \to \Gamma_G$ be an isomorphism of abstract directed graphs which preserve moments on all edges. Then Γ_F and Γ_G are isomorphic as measured in Reeb graphs, and there is a symplectomorphism $\Phi : M \to M$ such that $\Phi_* F = G$*

Proof. We first construct two intervals of the two Reeb graphs. Let $[v,\, w] \in \Gamma_F$. Then we push forward by the measure using a homomorphism $f : e \to [v,\, w]$ we get a measure μ_f on an interval, $I_f = [f(v),\, f(w)]$ and similarly $I_g = [g(\phi(v)),\, g(\phi(w))]$ for $g : e \to [g(v),\, g(w)]$, we get a measure μ_g. These intervals have the same moments. We aim to prove that $\mu_f = \mu_g$. This follows from the Hausdorff moment problem. Consider an interval I that contains both I_f and I_g; then the measures are measures of I supported on their respective intervals. Then, by the Hausdorff moment problem, $\mu_f = \mu_g$. □

The above proves that if we take F as a Morse vorticity function for an ideal flow v on a closed surface M associated with a Reeb graph Γ. Then the moments of that graph $m_{i,e}(F)$ are just the generalised enstrophies,

$$m_{i,e}(F) = \int_{M_i} F^i \omega = \int_{M_i} \phi(\omega)\, \mathrm{d}x \wedge\, \mathrm{d}y,$$

in two dimensions. Then, you can single each conserved quantity out by Theorem 3.2.5, and we also have these conserved quantities of the 2D Euler Equations.

3.3 Numerics

Proving quantities are conserved is a great way of theoretically verifying our results are correct. However, in practicality, numerical analysts are interested in this area of mathematics to create more precise numerical schemes. We shall now show that normal numerical schemes sometimes provide non-realistic solutions and how we can produce more realistic results.

We shall take the standard Euler equations as a simplistic example to show our point. Consider the following Euler equation.

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = 0.$$

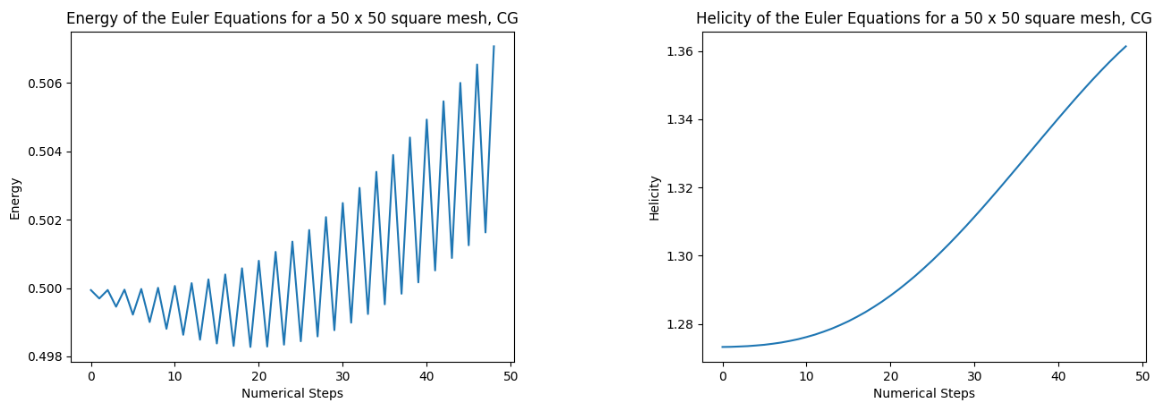
We can run a Crank-Nicholson scheme on this. It is well known that Crank-Nicholson is symplectic; that is, it preserves the symplectic form. Further, this means that it conserves energy. These schemes can be adapted to conserve other quantities. We consider a finite element scheme with a cubed 50×50 grid with order 2 Lagrange elements to motivate these. We set an initial condition of,

$$u_0(x, y) = \left(-\cos\left(\frac{\pi x}{2}\right)\sin\left(\frac{\pi x}{2}\right), \sin\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi y}{2}\right)\right),$$

and Neumann boundary conditions. The finite element scheme solves the spatial steps, and we use a Crank-Nicholson for the temporal steps. Hence, our scheme is,

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} = \frac{1}{2}\left(\left(\mathbf{u}^{n+1} \cdot \nabla\right)\mathbf{u}^{n+1} + \left(\mathbf{u}^n \cdot \nabla\right)\mathbf{u}^n\right).$$

We notice that we haven't produced a weak form of the equation. This is due to the Euler equations being unstable with weak forms. This produces the following simulation in Figure 3.3.



(a) t = 0.25 (b) t = 0.5

Figure 3.2: Conservation of Helicity and Energy in the FEM Scheme

We can also consider the conserved quantities. In this system, we will consider the total energy and enstrophy,

$$E = \int_M |\mathbf{u}|^2 \mu \qquad H = \int_M \mathbf{u} \cdot \mathrm{rot}\,\mathbf{u}\mu.$$

Unfortunately, the conserved quantities in Figure 3.2 present non-conservation. Due to us using a Crank-Nicholson timestep, which is symplectic, we can say that the error in our scheme comes from our spatial approximation. This is expected as, in general, finite element schemes are not structure-preserving or symplectic. This leads us to search for a new scheme to conserve these quantities. We will overview the theory of structure-preserving numerical methods.

The idea is as follows: We can discretise the theory presented in this document. Let $F : T^*Q \times \mathbb{R} \to T^*Q$ be an integrator, where T^*Q is just the tangent space of phase space, manifold, Q and $\mathbb{R}$ is the space where the time steps h live in. Then we can derive a discrete Euler-Lagrange equation from a discrete Lagrangian $L_d : TQ \times \mathbb{R} \to \mathbb{R}$ and discrete Hamilton's equations from $H_d : T^*Q \to \mathbb{R}$. Then we can define the so called discrete Euler-Lagrange equations. From this we ca discretise into methods in several ways.

One of the simplest examples of symplectic method is midpoint rule. We take a Hamiltonian, $H : T^*Q \to \mathbb{R}$ and define the method for $(p_0, q_0) \mapsto (p_1, q_1)$ where $z_0 = (p_0, q_0)$ and $z_1 = (p_1, q_1)$. Then we define,

$$\frac{z_1 - z_0}{h} = X_H\left(\frac{z_0 + z_1}{2}\right).$$

Then in the Lagrangian setup we define the discrete midpoint Lagrangian as,

$$L_d^{1/2}(q_1, q_0, h) = hL\left(\frac{q_1 + q_0}{2}, \frac{q_1 - q_0}{h}\right).$$

Then we can write the scheme as,

$$\begin{aligned}\frac{p_1 - p_0}{h} &= \frac{\partial L}{\partial q}\left(\frac{q_1 + q_0}{2}, \frac{q_1 - q_0}{h}\right)\\ \frac{p_1 + p_0}{2} &= \frac{\partial L}{\partial \dot{q}}\left(\frac{q_1 + q_0}{2}, \frac{q_1 - q_0}{h}\right).\end{aligned}$$

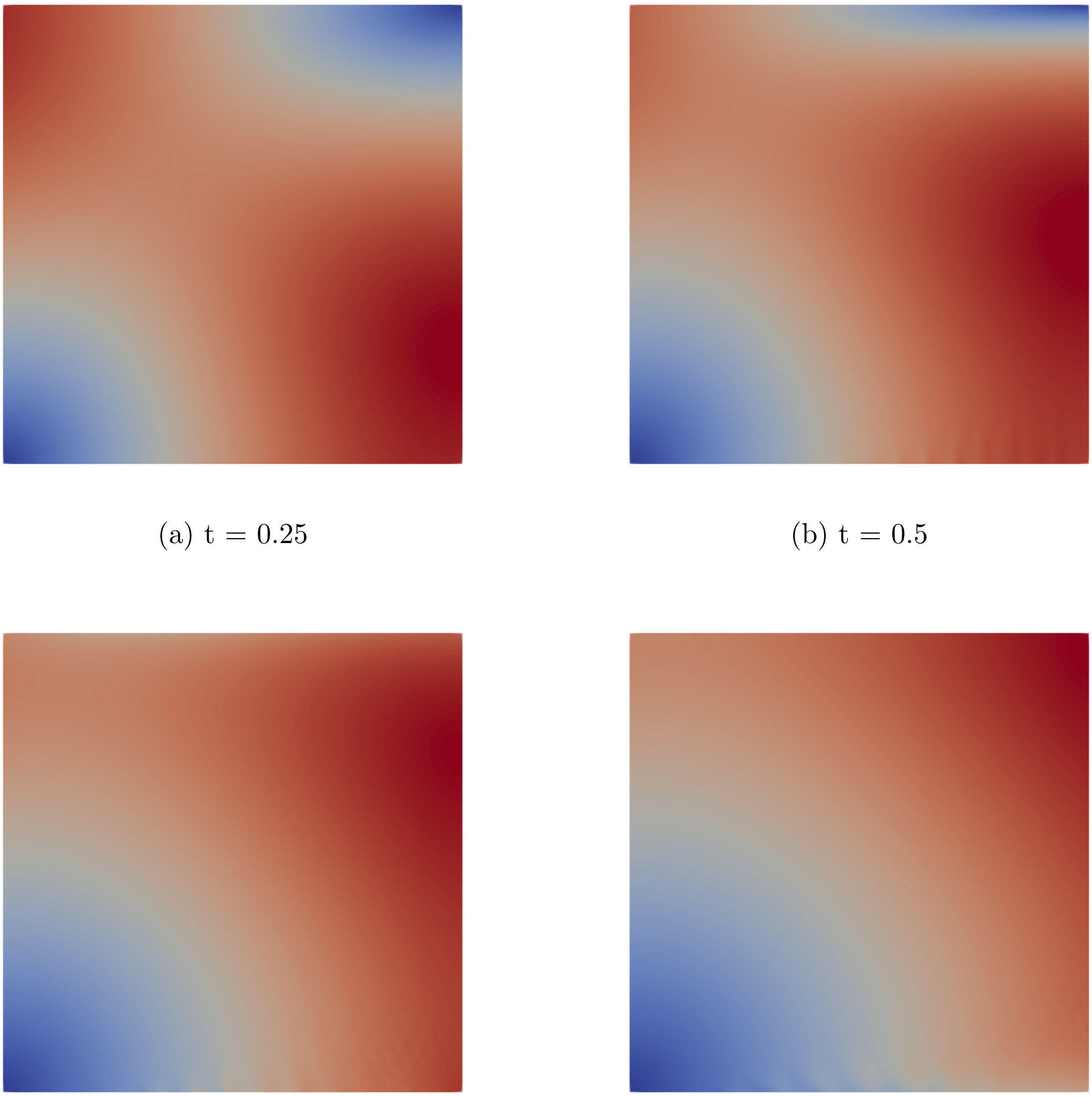
This can be used for our system,

$$L = H = \frac{1}{2}\int_M |\dot{\eta}|^2 \mu.$$

This unfortunately leaves us with a rather trivial scheme.

$$\begin{aligned}p_{k+1} &= p_k\\ q_{k+1} &= q_k + hp_k.\end{aligned}$$

where q_i increases by a fixed step. A similar scheme is derived from a Störmer-Verlet scheme. Hence we need something more complicated. A multisymplectic scheme. This again is an area that branches further than this thesis. They are methods that provide discrete conservation laws that agree with what we have seen in this conservation section. For details see Bridges' paper [25].



(a) t = 0.25 (b) t = 0.5

(c) t = 0.75 (d) t = 1

Figure 3.3: Solution to the Euler Equation using a FEM with Lagrange elements. The colours represent normalised velocity, where blue is -1 and red is 1.

Chapter 4

Examples

> All mathematics is divided into three parts: cryptography (paid for by CIA, KGB and the like), hydrodynamics (supported by manufacturers of atomic submarines) and celestial mechanics (financed by military and by other institutions dealing with missiles, such as NASA.).
>
> V. I. Arnold

In this final chapter, we will spend some time producing some examples to give the reader some idea of the scope of this theory. It has many applications, from Geophysical Fluid Dynamics [16] to computer graphics and image processing [14] to liquid crystals [15]. In this section, we will focus on Geophysical Fluid Dynamics and study some examples that will culminate in the Euler equations for a rotating stratified fluid and a recreation of the very popular paper by Vannaste on fluid flow on a Mobius strip [27] from a classical Geometric Mechanics point of view.

4.1 Axisymmetric Flow on a Cylinder

We start this examples section with a classical and simple example relating to axisymmetric flow on a cylinder. We will use much of the theory in subsection 2.2.2. Recall that we need three objects: a group to reduce over, a manifold to act as a space for dynamics to happen and a Lagrangian. Here, we let the manifold be the cylinder,

$$M = \overline{D}(0,\, 1) \times [0,\, 1].$$

The group is $\mathrm{SDiff}(M)$ with Dirichlet boundary conditions. We also make a modelling assumption that all the particles can be traced back to a plane, as seen in Figure 4.1. This doesn't impede the dynamics as it imposes the axisymmetric assumption but makes it easier to model. These assumptions give,

$$G = \{\eta \in \mathrm{SDiff}(M) : \eta(0) \in [0,\, 1] \times [0,\, 1] \text{ and } \eta = \mathrm{Id} \text{ on } \partial M = D(0,\, 1) \times [0,\, 1]\}.$$

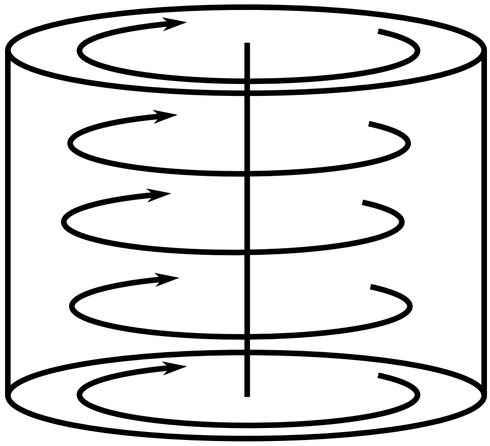


Figure 4.1: Axisymmetric Flow on a cylinder.

The Lagrangian is defined as,

$$L(\eta, \dot{\eta}) = \int_M (\dot{\eta}, \dot{\eta})\mu.$$

We can show that the Lagrangian is right invariant by letting $g \in \mathrm{SDiff}(M)$ then considering,

$$\begin{aligned} L(\eta g, \dot{\eta} g) &= \int_M (\dot{\eta} g, \dot{\eta} g)\mu \\ &= \int_M (\dot{\eta}, \dot{\eta}) g^2 \mu \\ &= \int_M (\dot{\eta}, \dot{\eta})\mu = L(\eta, \dot{\eta}). \end{aligned}$$

The second step comes from noting that g is volume preserving. Hence we let $g = \eta^{-1}$ and $\mathbf{u} = \dot{\eta}\eta^{-1}$ and we get that,

$$\ell(\mathbf{u}) = \int_M |\mathbf{u}|^2 \mu.$$

Then, we find the variational derivative of the Lagrangian and plug it into the Euler-Poincaré equation. Then we get,

$$\frac{\partial [\mathbf{u}^\flat]}{\partial t} = \mathrm{ad}_{\mathbf{u}}[\mathbf{u}^\flat].$$

We note that we have transferred from the algebra to the dual when we took the functional derivative. This means we mapped $[\mathbf{u}] \mapsto \mathbf{u} + \mathrm{d}f$ where f is a 0-form. We now need to impose boundary conditions. We get Dirichlet boundary conditions by using the group and considering the tangent space. Hence, our equations are,

$$\begin{aligned} \frac{\partial \mathbf{u}^\flat}{\partial t} = \mathrm{ad}_{\mathbf{u}} \mathbf{u}^\flat + \mathrm{d}f &= \pounds_{\mathbf{u}} \mathbf{u}^\flat + \mathrm{d}f \\ \mathbf{u}^\flat &= 0 \text{ on } \partial M \end{aligned} \tag{4.1}$$

where f is a 0-form.

4.2 Möbius Strip

The Möbius strip brings a new level of complexity to this theory. The Möbius strip is a non-orientable surface. This means that if you send a vortex around the strip spinning clockwise, it returns anticlockwise rotation. This is also equivalent to a manifold having a volume form [26]. This obviously can cause issues for the Euler Equations and the theory surrounding them. In a paper by Vanneste on this area, he produces some numerical simulations and equations for the Möbius strip [27]. We will first show the group that relates to this problem and then argue why this group is Fréchet Group. Then, we will discuss conserved quantities of this system.

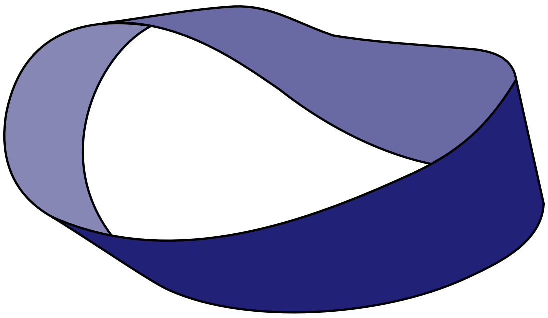


Figure 4.2: Mobius Strip.

We first note the Möbius strip diagram above and consider the following question.

Question 4.2.1. *How do we define the Möbius strip amicably?*

The answer to this is quite simple. Consider the following diagram.

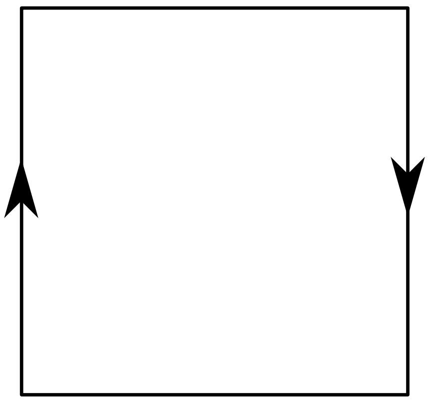


Figure 4.3: Plane representation of a Mobius Strip.

This is a Möbius strip. We note that we can connect the two arrowed sides by performing a half twist and glueing. This gives rise to the following definition of a Mobius Strip. Without loss of generality, imagine we are working on the square, $[-1, 1] \times [-1, 1]$. This holds as we can nondimensionalise in the x and y directions and get a unit square. Then, we can define our manifold as,

$$M = \{(x, y) \in [-1, 1] \times [-1, 1] : -x \sim x \text{ when } y = 1\}.$$

We can turn our manifold into boundary conditions on our diffeomorphism group. There has been much contention about what happens on the boundary of the Möbius strip when we consider the tangent space. There are a few ways to deal with this; the easiest is to set the vector fields to vanish at the boundary. Hence, our diffeomorphism group is,

$$G = \{\eta \in \mathrm{SDiff}(M) \colon \eta(-1, y) = \eta(1, -y), \eta(x, -1) = \eta(x, 1) = \mathrm{Id}\}.$$

Now we can find the tangent space and say the Lie algebra is isomorphic to,

$$\mathfrak{g} \cong \{\xi \in \mathfrak{X}(M) \colon \ \xi(0, y) = \xi(1, -y), \xi(x, -1) = \xi(x, 1) = 0\}.$$

Then, this is a Lie algebra because it doesn't introduce discontinuities into the system and performs a smooth genus-changing deformation between the square and the strip.

Now, we use the theory of subsection 2.2.2. This produces similar equations to the above, just with different boundary conditions. This produces,

$$\begin{aligned} \frac{\partial \mathbf{u}^\flat}{\partial t} &= \pounds_{\mathbf{u}} \mathbf{u}^\flat + \mathrm{d}f \qquad \mathbf{u} \in \mathfrak{X}([-1, 1]^2) \\ \mathbf{u}(-1, y) &= \mathbf{u}(1, -y) \\ \mathbf{u}(x, -1) &= \mathbf{u}(x, 1) = 0 \end{aligned} \tag{4.2}$$

where f is a 0-form.

As we have a continuous manifold with just a boundary condition that vanishes. Our Casimir functions will be equivalent to Vannaste's, the generalised enstrophies. However, the non-orientability of the Möbius strip produces the new constraint only for even functions in the enstrophy. This is because of the well-known fact that we can't integrate over non-orientable surfaces. Hence, we turn to the theory of pseudo-forms. The theory can be found in Frankel's Geometry of Physics [10]. This leads to the idea that we need even functions, f, to assert the following are conserved,

$$C_f(\omega) = \int_M f(\omega)\mu.$$

The theories reliance on pseudo-forms makes more sense when we define them.

Definition 4.2.2 (Pseudo-form). *Let M be a manifold, then a pseudo-k-form, α_o, is a k-form that for each orientation o of the manifold,*

$$\alpha_o = -\alpha_o,$$

when the orientation of the manifold is flipped.

We also note that even on non-orientable manifolds the tangent bundle is always orientable.

Proposition 4.2.3. *For any manifold, M, TM is orientable*

Proof. For any manifold T^*M has a symplectic form, ω. This symplectic form is a volume form for any $2n$-dimensional manifold. Hence the cotangent bundle, a $2n$-dimensional manifold, is orientable. Further as T^*M and TM are diffeomorphic, via some isomorphism ϕ, then TM also has volume form $\phi(\omega)$. Therefore TM is orientable. $\square$

We now have that TM is always orientable and that the idea of pseudo-forms. Hence let us rewrite Noether's Theorem in this context. Let any form we mention for the rest of the example be a pseudo-form. We need to ask the following: In order to perform integration on manifolds we require a volume form, the volume form comes from orientability, so how do we integrate non-orientable manifolds? We can't just integrate these manifolds as a non-orientable manifold doesn't have a volume form. However we know that integrals are invariant up to quotienting by zero measure sets. Therefore we aim to quotient our manifold by a zero measure set and make it orientable. This is done by considering an involution that relates to the orientation. Hence if we consider the involution,

$$\sigma : x \mapsto -x.$$

Then we can look at $M \setminus \sigma$ where M is just the Möbius strip. We see that $M \setminus \sigma = [-1,\, 1] \times [-1,\, 1]$, which is just the square. This explains why the above equations are on the square rather than the Möbius strip and the boundary conditions are the only part that enforces the Möbius strip. We are working with a group as diffeomorphisms, hence our group is,

$$G \setminus \sigma = \mathrm{SDiff}(M)([-1,\, 1]^2).$$

This is differentiable and has right inverse hence we write Noether's Theorem.

Theorem 4.2.4 (Noether Theorem for the Möbius Strip)**.** *Each symmetric vector field of the Euler-Poincaré reduced Lagrangian for the infinitesimal variation, $\delta\mathbf{u} = \dot{\nu} - \mathrm{ad}_{\mathbf{u}}\nu$, corresponds to an integral of the Euler-Poincaré motion and a conserved quantity,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1}\int_{-1}^{1} \left\langle \mathbf{u}^\flat \,, \nu \right\rangle \, \mathrm{d}x \, \mathrm{d}y = 0$$

Proof. The proof here involves specialising Noethers Theorem. Firstly let our domain of integration M be $N \setminus \sigma$ where N is the Möbius strip and $\sigma : x \mapsto -x$ across one of the vertical boundaries. Then $M \setminus \sigma = [-1,\, 1]^2$, which is orientable with volume for μ. Therefore Noethers Theorem becomes,

$$\int_{M\setminus\sigma} \left\langle \frac{\delta \ell}{\delta \mathbf{u}} \,, \nu \right\rangle \mu = \int_{-1}^{1}\int_{-1}^{1} \left\langle \frac{\delta \ell}{\delta \mathbf{u}} \,, \nu \right\rangle \, \mathrm{d}x \, \mathrm{d}y.$$

Now we note that $\frac{\delta \ell}{\delta \mathbf{u}} = \mathbf{u}^\flat$ and we are done. $\square$

Now we can notice that the above calculations for vorticity and helicity follows as they do above on $[-1,\, 1]^2$. We see the integration above when we have boundary is,

$$-\int_M \left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right) \mathrm{d}\left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}} \cdot \mathrm{d}\mathbf{x}\right) \wedge (\mathbf{\Psi} \cdot \mathrm{d}\mathbf{x}) - \int_{\partial M} \left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right) \left(\frac{1}{\rho}\frac{\delta \ell}{\delta \mathbf{u}} \cdot \mathrm{d}\mathbf{x}\right) \wedge (\mathbf{\Psi} \cdot \mathrm{d}\mathbf{x}) = 0.$$

We now set $\rho = 1$ for an incompressible flow and note $\frac{\delta \ell}{\delta \mathbf{u}} = \mathbf{u}^\flat$. Therefore the conserved quantities from this calculation is,

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right) \mathrm{d}\left(\mathbf{u}^\flat \cdot \mathrm{d}\mathbf{x}\right) &= 0 \\ \left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right) \left(\mathbf{u}^\flat \cdot \mathrm{d}\mathbf{x}\right) &= 0 \qquad\qquad \text{on } \partial M. \end{aligned}$$

This first of these quantities is the vorticity in the domain and the second allows for continuity and reconstruction of this conservation along the boundary. The second condition restricts the first and if these contradict, for example on more complex geometries, then the conservation quantity fails to be conserved. However, when we consider Helicity, we note that we never perform an integration by parts in the derivation. Therefore, there are no extra boundary terms. Further, Ertel's theorem doesn't require an integration by parts. Hence the following is conserved,

$$\mathcal{H} = \int_{-1}^{1} \int_{-1}^{1} \mathbf{u}^\flat \cdot \operatorname{rot} \mathbf{u} \, \mathrm{d}x \, \mathrm{d}y.$$

We note that $\operatorname{rot} \mathbf{u}$ is the 2D scalar curl, and further its equivalent to Helicity on the Möbius strip as the integral over the Möbius strip is equivalent to the integral over the square.

4.3 Rotating Stratified Fluid

We finally aim to derive the Euler equations for a rotating stratified fluid. This example opens up the possibilities of using this mathematics in geophysical fluid dynamics. What we will derive will be a simple model for storms, clouds and other objects in the atmosphere that have variable density, are buoyant in fluid and are affected by the Coriolis force. We will follow a similar procedure to the example after the advected quantity theorem. We can take the Lagrangian,

$$L = \int_M \left[\rho(1+b)\left(\frac{1}{2}|\dot{\eta}^2| + \dot{\eta}\cdot\mathbf{R} - gz\right) - p(\rho - 1)\right].$$

We can now reduce this to the following reduced Lagrangian, much like the compressible fluid example.

$$\ell(\mathbf{u},\, D,\, b) = \int_M \left[\rho_0(1+b)D\left(\frac{1}{2}|\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{R} - gz\right) - p(D-1)\right].$$

To deal with more conserved quantities, the Euler-Poincaré equations become,

$$\frac{\partial}{\partial t}\frac{\delta \ell}{\delta \xi} = -\mathrm{ad}^*_\xi \frac{\partial \ell}{\partial \xi} + \frac{\partial \ell}{\partial a} \diamond a + \frac{\partial \ell}{\partial b} \diamond b, \tag{4.3}$$

where a and b are advected quantities. These advected quantities satisfy,

$$\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right) a = \left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right) b = 0.$$

We can now find the variational derivatives. By performing the calculations, we arrive at the following.

$$\begin{aligned}
\left\langle \frac{\delta \ell}{\delta \mathbf{u}}\,, \phi \right\rangle &= \left\langle \rho_0 D(1+b)(\mathbf{u}^\flat + \mathbf{R})\,, \phi \right\rangle \\
\left\langle \frac{\delta \ell}{\delta b}\,, \phi \right\rangle &= \left\langle \rho_0 D\left(\frac{1}{2}|\mathbf{u}^\flat|^2 + \mathbf{u}\cdot\mathbf{R} - gz\right)\,, \phi \right\rangle \\
\left\langle \frac{\delta \ell}{\delta D}\,, \phi \right\rangle &= \left\langle \rho_0(1+b)\left(\frac{1}{2}|\mathbf{u}^\flat|^2 + \mathbf{u}\cdot\mathbf{R} - gz\right) - p\,, \phi \right\rangle.
\end{aligned}$$

These can now be plugged into Equation (4.3). We then get,

$$\begin{aligned}
\rho_0 D(1+b)\frac{\partial}{\partial t}(\mathbf{u}^\flat + \mathbf{R}) + \rho_0 D(1+b)\pounds_{\mathbf{u}}(\mathbf{u}^\flat + \mathbf{R}) + D\left(\frac{1}{2}|\mathbf{u}^\flat|^2 + \mathbf{u}^\flat\cdot\mathbf{R} - gz\right) \diamond b \\
+ (1+b)\left(\frac{1}{2}|\mathbf{u}^\flat|^2 + \mathbf{u}^\flat\cdot\mathbf{R} - gz - p\right) \diamond D = 0.
\end{aligned}$$

This isn't very manageable, so we can simplify and put this equation back into coordinates. This leaves us with,

$$\frac{\partial[\mathbf{u}^\flat]}{\partial t} + \pounds_{\mathbf{u}}[\mathbf{u}^\flat] + \pounds_{\mathbf{u}}\mathbf{R} - g\hat{\mathbf{z}} + B([\mathbf{u}^\flat],\, b,\, D,\, p) = 0. \tag{4.4}$$

We note that B is our buoyancy and density gradient term and is defined by,

$$B([\mathbf{u}^\flat],\, b,\, D,\, p) = \frac{D}{\rho_0}\left[\frac{\left(\frac{1}{2}|[\mathbf{u}^\flat]|^2 + [\mathbf{u}^\flat]\cdot\mathbf{R} - gz\right)\diamond b}{D(1+b)} + \frac{\left(\frac{1}{2}|[\mathbf{u}^\flat]|^2 + [\mathbf{u}^\flat]\cdot\mathbf{R} - gz - p\right)\diamond D}{D^2}\right].$$

Now we take Equation (4.4), then we can place this into coordinates and remove the cosets,

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{u}\cdot\operatorname{curl}\mathbf{u} - g\hat{\mathbf{z}} + \frac{\nabla p}{\rho_0} + B(\mathbf{u},\, b,\, D,\, p) = 0.$$

This is the coordinated rotating stratified Euler equations with buoyancy term, where we advect density and buoyancy,

$$\left(\frac{\partial}{\partial t} + D\cdot\nabla\right)D = \left(\frac{\partial}{\partial t} + b\cdot\nabla\right)b = 0.$$

For these equations, we have the usual conserved quantities, and Noether Theorem tells us that,

$$\oint_\gamma \frac{1}{D}\frac{\delta\ell}{\delta\mathbf{u}} = \oint_\gamma \rho_0(1+b)(\mathbf{u}+\mathbf{R}) = \oint_\gamma \nabla p\cdot\,\mathrm{d}\mathbf{x}.$$

Further, the advection equation for b gives us potential vorticity conservation given $q = \nabla b\cdot\operatorname{curl}(\mathbf{u}+\mathbf{R})$. This is another well known consequence of Ertel's theorem. This time vorticity is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M \left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right)(\mathrm{curl}\ ((1+b)(\mathbf{u}+R))\cdot\,\mathrm{d}S) = 0.$$

Further we have an idea of Helicity,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M (1+b)(\mathbf{u}+R)\cdot\mathrm{curl}\ ((1+b)(\mathbf{u}+R)) = 0.$$

Chapter 5

Conclusion

> Are you sitting comfortably? Good, then I'll begin
>
> Julia Lang, Daphne Oxenford
> Listen with mother

This chapter draws the dissertation to a close. In this thesis, we have seen several ideas surrounding the ideas of geometric and topological fluid dynamics and further Geometric Mechanics. We have presented and exposed many results in the area and produced simple examples towards the applications in this area. We have studied boundary conditions on manifolds with boundary and then studied conserved quantities of these systems. We saw examples of axisymmetric flow, the Möbius strip and rotating stratified fluid. Geometric Mechanics is still a huge area, and my two theses have only touched the surface in this area. My PhD thesis will be based upon this work and bring together many more areas.

In Chapter 1, we explored the area's history and presented some relevant pure background of the thesis. We studied a small amount of Symplectic Geometry, an area with vast branches and one I wish we could have explored more. This will be the basis of further work into multisymplectic reduction and the related numerical methods. In the second chapter, we studied reduction theorems on the diffeomorphism group and in both Lagrangian and Hamiltonian frameworks. These presented some extremely nice groundwork for the third chapter and were something that very nicely led from the author's undergraduate thesis to the Ph.D. thesis. This chapter went into great detail to consider the third chapter. In Chapter 3, we studied conserved quantities in three different ways. Firstly, via Noether Theorems, a usual method to derive conserved quantities. The second Casimirs, a slightly classical way to derive the conserved quantities and then a more interesting way via Reeb Graphs. The Reeb graph method provides an interesting approach that hasn't been generalised but provides an idea of how we can find all the Casimirs of more complicated systems. In the final chapter, we saw applications of all the preceding chapters. We studied a simplistic example, axisymmetric flow in a cylinder, an exotic example, flow on the Möbius strip and an industrial example, Rotating Stratified fluid.

Although there are now nearly a hundred pages of Geometric Mechanics written by the author, we have again only taken select areas to study. You only have to glance at Darryl Holm's publication list to tell us that. The author undertook some extra work in the area of numerics while writing this thesis; they implemented papers by Cotter and Bridges [8, 25] to show that the conserved quantities that were derived produced the expected numerical observations in a similar fashion to section 3.3. The methods were based on a finite element approach. The author's PhD thesis will give a broader idea of the area and develop new multisymplectic numerical methods for this problem.

There is also work towards ideas further than Euler's equations. The author spent some time proving theorems and results about a set of equations called the Lagrangian averaged Navier Stokes, which gives a Geometric Mechanics insight into the Navier Stokes problem. These provide interesting theories about different solutions between the Euler Equations and Navier Stokes. This area was discussed in detail with Darryl Holm, and there's a great wealth of knowledge toward ocean modelling using Stochastic Averaging Lagrangian Transport and Lagrangian averaging.

In addition, another whole area towards using variational principles differs from the usual Hamilton's principle. The work by Dr Hamid Alemi Ardakani [2, 1] in this area applied mathematics to wave energy generators and uses Geometric Mechanics in general to study waves.

We conclude this thesis and the author's time at Oxford similarly to my first thesis. Although one year on, a lot more knowledgeable about mathematics and a better mathematician, the author notes that they have equally enjoyed writing this thesis. We have studied some classical results, produced new examples and laid a perfect basis for a Ph.D. This thesis aimed to bring together the skills and knowledge the author attained during their year in Oxford. Further, Geometric Mechanics is still a handy tool to the world and one the author will dedicate their life to—solving the more pertinent problems with the most abstract objects. Solving problems need not be easy, but Geometric Mechanics is a toolbox to make it easier.

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Appendix A

An Introduction to Differential Geometry and Exterior Calculus

In this section we will aim to give a brief overview of exterior calculus and the tools that we use in the rest of the thesis. Specifically this will entail a lot of notation and some basic ideas around Hamiltonian mechanics written using different language.

The main idea we will show is that the so called symplectic two form is conserved across flows and hence Hamiltonian flows are symmetric. This leads to a lot of nice conservation results and opens up to levels of general theory. This general theory will then require some more heavy duty tools from differential geometry that we will introduce along the way. We first introduce the idea of a *manifold*.

Definition A.0.1 (Smooth Manifold)**.** *A smooth manifold, M, is a set with a countable set of subets $U_a \subset M$ such that their union covers the manifold, $\bigcup_\alpha U_\alpha = M$, and bijective functions $\phi_\alpha : U_\alpha \to \mathbb{R}$ such that the following condition is satisfied. For any nonempty intersection, $U_\alpha \cap U_\beta$, the set $\phi_\alpha(U_\alpha \cap U_\beta)$ is an open subset of $\mathbb{R}^n$ and the bijection $\phi_\beta \circ \phi_\alpha^{-1}$ is a smooth function is smooth on $\phi_\alpha(U_\alpha \cap U_\beta)$.*

This definition is very formal but is full of jargon. Simply speaking a manifold is a space that for each point it locally can be treated as $\mathbb{R}^n$. That is for every point on the manifold, $x \in M$, you can construct a continuous bijection with continuous inverse, a homeomorphism, from the epsilon ball around x, $B_\varepsilon^M(x)$ to an delta ball around some $y \in \mathbb{R}^n$, $B_\delta^{\mathbb{R}^n}(y)$. To hammer home the point, **manifolds are spaces that are locally flat**. Henceforth when say *manifold*, we mean *smooth manifold*.

The whole idea of using the manifold is to work in absense of coordinates. Hence we can talk about *generalised coordinates*. This is because for any arbitrary choices of coordinates, q and Q, we can a map, $q \to Q$ such that it's nondegenerate, $\det\left(\frac{\partial Q}{\partial q}\right) \neq 0$. Hence we call $\{q\}$ the *generalised coordinates*.

Now we have an idea of generalised coordinates, we can quickly define tangent spaces, bundles and the duals.

Definition A.0.2 (Tangent Spaces and Bundles)**.** *Take some $q(t) \in M$ at some fixed* *time t. The tangent space is all of the tangent vectors, $\dot{q}(t)$, to $q(t)$ at time t. This is denoted T_qM. The union for all $q \in M$ of the tangent spaces is the tangent bundle.*

As an example if we find the tangent bundle of a circle, we will see that it $\mathbb{R}^2$ minus the circle and its interior. The tangent space of one point on the circle is just a tangent vector.

Definition A.0.3 (Cotangent Spaces and bundles)**.** *The cotangent space is the dual to the tangent space*[1] *filled with covectors. Covectors are linear functionals $v : T_qM \to \mathbb{R}$ that undoes the action of vectors, in a loose sense. The cotangent bundle is the union of the cotangent spaces.*

In many places we will need to move between the tangent and cotangent spaces of a manifold. This can be done using the musical isomorphisms. These are the canonical maps between the tangent and cotangent spaces. We define $\flat : TM \to T^*M$ by a map $X \mapsto X_j\mathbf{e}^j$ where X is a vector field. We define sharp, $\sharp : T^*M \to TM$ by a map $\omega \mapsto \omega^j\mathbf{e}_j$ where ω is a covector.
An interesting mechanical fact of the cotangent spaces is that they are spanned by the *conjugate momenta* that come out of Hamilton's equations. This will be useful later when we start to talk about Hamilton's equations in more depth. Before moving onto some heavier machinery on these spaces we need to introduce one of the most important tools in this thesis, forms.

In order to talk about differentiation and integration we need to have some sort of way of measuring what our manifold looks like. In normal settings this would be down to the coordinate system we are working in. However, in differential geometry there is no coordinate system. This provides a slight problem when the whole of calculus is based on small coordinated changes. Hence we introduce the idea of differential forms in differential geometry.

A.1 Forms

Forms are a way of talking about these infinitesimal areas in a coordinate free way. A 1-form is an infinitesimal oriented length, a 2-form is an infinitesimal orientable area, and so on. We also have 0-forms which represent infinitesimal points, the scalar functions. More formally we can define *1-forms*, ω, which assigns a covector $v \in T_q^*M$ to each point $q \in M$. Hence we have mapping from $\mathbf{u}$ to scalar functions $\omega(\mathbf{u})$, by composing $\mathbf{u}$ with 1-forms. Further we can write 1-forms with basis $(\,\mathrm{d}x^1,\,\mathrm{d}x^2,\dots)$, and we can decompose them,

$$\omega = \sum_i \omega^i\,\mathrm{d}x^i.$$

Again we will formally define d later. Differential Geometry has an intolerable amount of overarching definitions. Now we need to go on a slight detour to look at the wedge product,

[1]The dual is associated to the differential map we will introduce later

A.2 Wedge Product

We define the **wedge product** of a k-form ($k = 0, 1, 2, \dots$) and an ℓ-form ($\ell = 0, 1, 2, \dots$) as a $(k + \ell)$-form. It satisfies the following,

1. Bilinearity, $(a\alpha+b\beta)\wedge\omega = a(\alpha\wedge\omega)+b(\beta\wedge\omega)$ and $\alpha\wedge(a\omega+b\varepsilon) = a(\alpha\wedge\omega)+b(\alpha\wedge\varepsilon)$
2. Anticommutativity, $\omega \wedge \gamma = (-1)^{k\ell}\gamma \wedge \omega$ where ω is a k-form and γ is a ℓ-form.
3. Associativity, $(\omega \wedge \gamma) \wedge \kappa = \omega \wedge (\gamma \wedge \kappa)$

Here are two other useful properties of the wedge product,

- Given two 1-forms, α, β on M, then $\alpha \wedge \beta$ is a two form defined by,
$$\alpha \wedge \beta(v_1, v_2) = \alpha(v_1)\beta(v_2) - \alpha(v_2)\beta(v_1).$$
- Given a scalar function (0-form), f, and a k-form, ω,
$$f \wedge \omega = f\omega.$$

Now we state a final result on the wedge product,

Lemma A.2.1. *Given, ω, γ are 1-forms. Then $\alpha \wedge \beta = \alpha \mathrm{d}x^1 \wedge \mathrm{d}x^2$, where $\alpha = w^1\gamma^2 - \omega^2\gamma^1$.*

Proof. Expand the forms in terms of a basis and then use bilinearity properties and note that $dx^i \wedge dx^i = 0$. $\square$

A *2- form* is a function that assigns a skew-symmetric bilinear map $T_xM \times T_xM \to \mathbb{R}$ on the tangent space T_xM to each point $x \in M$. This is used to define surface integrals and the surface form. In coordinates we see that,

$$\int_M \omega = \int_{\phi(M)} \alpha \mathrm{d}x^1 \wedge \mathrm{d}x^2$$

where $\phi_U(M)$ just places you in local coordinates (remember that a manifold is locally just Euclidian). Then we can show that there is a certain α, called α_S, which allows the above to make sense and further gives you the surface area of every submanifold, U M. This then gives us the *surface form* $\mathrm{dS} = \alpha \mathrm{d}x^1 \wedge \mathrm{d}x^2$. This then tells us that every 2-form can be written as, $\omega = \beta\mathrm{dS}$. Note again, all of this is still coordinate independent! At no point have we chosen a set of coordinates, if at any point we did, it cancelled out under a change of variable formula.

More generally we are interested in k-forms, these can be written in the following form,

$$\omega = \sum_{i_1,\dots,i_k=1}^{n} \omega^{i_1,\dots,i_k}\, \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \dots \wedge \mathrm{d}x^{i_k}.$$

We are saying that the wedge of any k dx^j's is a k-form. We define the space of k-forms as $\Lambda^k(M)$ ($k = 0, 1, 2, \dots$).

A.3 Contractions and vector fields

We now look to study the interplay between vector fields and differential forms. These are two very important objects and we want to be able to use them together in the following theory. The contraction, $\lrcorner$, is an operation that defined a pairing between vector fields and forms. Contraction defined the following duality relations,

$$\partial_q \lrcorner \, \mathrm{d}q = \partial_p \lrcorner \, \mathrm{d}p = 1 \qquad \partial_q \lrcorner \, \mathrm{d}p = \partial_p \lrcorner \, \mathrm{d}q = 0.$$

In order to define this formally we define the exterior derivative and define vector fields,

A.4 Exterior Derivatives

We want some way to get between different for spaces. We also want to relate vector calculus operations to geometric operations. This is where the exterior derivative comes in. We define $d : \Lambda^k(M) \to \Lambda^{k+1}(M)$ for some $\alpha = \alpha_{i_1,\ldots,i_k}\, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$ as,

$$d\alpha = \mathrm{d}\alpha_{i_1,\ldots,i_k}\, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k},$$

where, $\mathrm{d}\alpha = \left(\frac{\partial \alpha_{i_1,\ldots,i_k}}{\partial x^j}\right) \mathrm{d}x^j$.

Hence we can show the following properties,

Proposition A.4.1. *The following properties hold for the exterior derivative,*

1. *If α is a 0-form then* $\mathrm{d}f$ *is 1-form given by the differential of f,*
2. $\mathrm{d}\alpha$ *is linear in α,*

$$\mathrm{d}(c_1\alpha_1 + c_2\alpha_2) = c_1\, \mathrm{d}\alpha_1 + c_2\, \mathrm{d}\alpha_2, \qquad c_1, c_2 \in \mathbb{R}$$

3. $\mathrm{d}\alpha$ *has a product rule. Let α be a k-form and β is a ℓ-form,*

$$\mathrm{d}(\alpha \wedge \beta) = (\mathrm{d}\alpha) \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta$$

4. $\mathrm{d}^2 = 0$ *for any k-form*
5. d *is a local operator, hence it is determined by its local properties in a neighbourhood of any $x \in M$.*

We can now define a vector field, and then the contraction,

Definition A.4.2 (Vector Field)**.** *A vector field, X, on a manifold M, is a map $X : M \to TM$ that assigns a vector, $X(m)$, to a point on the manifold $m \in M$. The real vector space of vector fields is denoted $\mathfrak{X}(M)$.*

As we know the basis of TM is going to be given by $\nabla := (\partial_1, \ldots, \partial_n)$ and X has components X^j we can say the basis of the vector field is,

$$X = X^j \partial_j.$$

We can finally define a contraction.

Definition A.4.3 (Contraction). *Let $\alpha \in \Lambda^k$ lie on M, where,*

$$\alpha = \alpha_{i_1,\ldots,i_n} \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k},$$

and $X = X^j \partial_j$ be a vector field. The contraction $X \lrcorner \alpha$ is defined by,

$$X \lrcorner \alpha = X^j \alpha_{j i_2 \ldots i_k} \, \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_x}.$$

We note quite nicely that contraction can split wedge products,

Proposition A.4.4. *Let α be a k-form and β be a 1-form, X be a vector field on a manifold M. Then the contraction of X through $\alpha \wedge \beta$ is,*

$$X \lrcorner (\alpha \wedge \beta) = (X \lrcorner \alpha) \wedge \beta + (-1)^k \alpha \wedge (X \lrcorner \beta).$$

Proof. This follows from unfolding definitions, then splitting into two sums and carefully swapping the forms around to achieve the $(-1)^k$ from moving the one form to the front. $\square$

A.5 Pullbacks and pushforwards

We discussed briefly about change of coordinates on manifolds and they coordinates don't matter because you can just construct a map between any two coordinates to create a change of coordinates. For changes of basis, these mappings are usually pullbacks or pushforwards. We can define them as smooth invertible maps that act on vector fields and functions.

Let $\phi : M \to N$ be a smooth invertible functions from a manifold M to a manifold N. The pull-back of some function f is just right composition by ϕ, $\phi^* f = f \circ \phi$. The push-forward is just right composition of ϕ^{-1}, $\phi_* g = g \circ \phi^{-1}$. We can do similar things to a vector field, but its slightly more involved to see whats actually happening. We can push forward a vector field X by ϕ, that is,

$$(\phi_* X)(\phi(z)) = T_z \phi \cdot X(z).$$

We can impose local coordinates and see that the components are,

$$(T_z \phi \cdot X(z))^l := \frac{\partial \phi^l}{\partial z^J} X^J(z) = (\phi_* X)^l.$$

We can then write the pull-back of a vector field as, $\phi^* Y = (\phi^{-1})_* Y$, then use the same argument as above.

A.6 Lie Derivatives

The final tool in our arsenal is called the Lie Derivative. This gives a generalisation of the idea of directional derivative. We want to be able to differentiate along a vector field. We want to define this derivative and then we can prove **Cartan's Magic Formula**!

Definition A.6.1 (Lie Derivative)**.** *Let α be a k-form and let X be a vector field with flow ϕ_t on a manifold M. The Lie derivative of α along X is,*

$$\pounds_X \alpha = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\phi_t^* \alpha).$$

We are in need of two results to actually prove Cartans magic formula. One to guarantee its well behaved and one to help us in the proof. The following proofs come from Tu [26].

Proposition A.6.2. *Let X be a vector field on a manifold M. Then $\pounds_X$ is $\mathbb{R}$-linear and if $\omega \in \Lambda^k$ and $\tau \in \Lambda^l$,*

$$\pounds_X(\omega \wedge \tau) = (\pounds_X \omega) \wedge \tau + \omega \wedge (\pounds_X \tau) \tag{A.1}$$

Proof. The $\mathbb{R}$-linearity of $\pounds_X$ is trivial by the definition. We focus on the second part. It follows from some chasing around of definitions,

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\omega \wedge \tau) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\phi_t^*(\omega \wedge \tau)) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi_t^*\omega \wedge \phi_t^*\tau \\
&= \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi_t^*\omega \right) \wedge \tau + \omega \wedge \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi_t^*\tau \right) \\
&= (\pounds_X \omega) \wedge \tau + \omega \wedge (\pounds_X \tau).
\end{aligned}$$

□

Proposition A.6.3. *Let X be a vector field on a manifold M. Then d commutes with $\pounds_X$.*

Proof. This follows again from chasing around definitions,

$$\begin{aligned}
\pounds_X \,\mathrm{d}\omega &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi_t^* \,\mathrm{d}\omega \\
&= \mathrm{d} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi_t^*\omega = d\pounds_X \omega
\end{aligned}$$

□

Lemma A.6.4 (Cartans Magic Formula)**.** *Let α be a k-form and X be a vector field on a manifold M. Then,*

$$\pounds_X \alpha = X \lrcorner \,\mathrm{d}\alpha + \mathrm{d}(X \lrcorner\, \alpha)$$

Proof. We shall use the fact that d is a local operator. This implies that we can prove this formula for an arbitrary point in an arbitrary neighbourhood on our manifold. That is we can write,

$$\alpha = \alpha_{i_1 \dots i_k} \, \mathrm{d}x^1 \wedge \cdots \wedge \, \mathrm{d}x^k.$$

If we consider the left hand side and the right hand side, we know that they are both derivations ($\mathbb{R}$-linear and satisfies a property like Equation A.1). Hence we know that if the formula works for ω and τ it will also work for $\omega \wedge \tau$ and further if it works for ω it will work for $\mathrm{d}\omega$. Hence we need to prove the above formula for a 0-form, f. That is check,

$$\pounds_X f = \mathrm{d}(X \lrcorner f) + X \lrcorner \, \mathrm{d}f.$$

This follows from the following. Note that $X \lrcorner f = 0$ as f is a 0-form. Then,

$$\pounds_X f = X \lrcorner \, \mathrm{d}f.$$

However, we also note,

$$\begin{aligned} \pounds_X f &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi_t^* f \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f \circ \phi_t \\ &= Xf = X \lrcorner \, \mathrm{d}f. \end{aligned}$$

That then completes the proof. □

A.7 Covariant Derivatives

One of the main types of terms in fluid equations are the covariant terms. These relate to derivatives of the solution along vectors. These terms in the Euler Equations look like $(\mathbf{u} \cdot \nabla)\mathbf{u}$ and are also known as directional derivatives. In addition to Lie derivatives we can state these in terms of covariant derivatives. They are derivatives of the solution along tangent vectors. You can take covariant derivatives of functions, $f : M \to \mathbb{R}$ or of vector fields.

Definition A.7.1 (Covariant Derivative of a function)**.** *Let M be a manifold, $p \in M$ and $f : M \to \mathbb{R}$. Then define some curve $\gamma : [-1, 1] \to M$ such that $\phi(0) = p$ and $\phi'(0) = \mathbf{v}$. Then the covariant derivative of f along $\mathbf{v}$ at p is,*

$$(\nabla_v f)_p = \lim_{t \to 0} \frac{f(\phi(t)) - \phi(p)}{t}$$

We note for scalar f, the covariant derivative is the same as the Lie derivative and the exterior derivative.

Definition A.7.2 (Covariant Derivative of a vector field)**.** *Let M be a manifold, $p \in M$, a, b be scalars, $f : M \to \mathbb{R}$, $\mathbf{u} : M \to T_pM$. Then the covariant derivative is defined as follows,*

- *It's linear in v,* $\nabla_{a\mathbf{x}+b\mathbf{y}}\mathbf{u} = a\nabla_x\mathbf{u} + b\nabla_y\mathbf{u}$*,*
- *It's additive in u,* $\nabla_\mathbf{v}(\mathbf{u}+\mathbf{w}) = \nabla_\mathbf{v}\mathbf{u} + \nabla_\mathbf{v}\mathbf{w}$*,*
- *It obeys a product rule,* $\nabla_\mathbf{v}(f\mathbf{u}) = (\nabla_\mathbf{v}f)\mathbf{u} + f(\nabla_\mathbf{v}\mathbf{u})$.

In coordinates we can write the covariant derivative as,

$$\nabla_\mathbf{v}\mathbf{u} = \left(v^i u^j \Gamma^k_{ij} + v^j \frac{\partial u^k}{\partial x^j}\right)\mathbf{e}_k.$$

We note Γ^k_{ij} is a Christoffel symbol and can be written in Euclidian space as, $\Gamma^k_{ij} = \frac{\partial \mathbf{e}_i}{\partial x^j} \cdot \mathbf{e}^k$.

A.8 De Rham Complex

We take the following from the authors special topic on finite element methods [6].

Consider a sequence of Hilbert spaces, each mapped into eachother by maps d^k.

$$\cdots \longrightarrow W^{k-1} \xrightarrow{\mathrm{d}^{k-1}} W^k \xrightarrow{\mathrm{d}^k} W^{k+1} \longrightarrow \cdots$$

Further, consider that $\mathrm{d}^{k+1} \circ \mathrm{d}^k = 0$, where d also satisfies Range d^k dom d^{k+1}. This is a Hilbert Complex. This is nice, but we can do better. Consider the spaces of smooth functions, 1-forms and 2-forms. These are Hilbert spaces. Further consider the exterior derivative as our d.

$$0 \longrightarrow \Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \Omega^2(M) \longrightarrow 0.$$

This is the de Rham complex. Further we can write it in the following way. This will help us when we convert from exterior derivatives to vector calculus.

$$0 \longrightarrow \Omega^0(M) \xrightarrow{\text{grad}} \Omega^1(M) \xrightarrow{\text{curl}} \Omega^2(M) \xrightarrow{\text{div}} 0.$$

Appendix B

Momentum Maps

We have nice symmetries through what we call Noether Theorems and these are familiar to us. They say, symmetries omit conserved quantities. We can that given a Hamiltonian H on a phase space P, that is invariant under the action of a Lie Group G there is usually a conserved map called the *momentum map*, $\mathbf{J} : P \to \mathfrak{g}^*$. The intuition here is that vector fields are induced on Lie algebras, $\mathfrak{g}$, that then have infinitesimal generators, ξ_P. Then most of these are Hamiltonian, which then emits a Hamiltonian vector field J_ξ and the Hamiltonian functions are unique up to a constant ξ, creating a map. Then as this map is of Hamiltonians it is conserved under the current!

Definition B.0.1. *A momentum map for a canonical action G on a poisson manifold P is a map $\mathbf{J} : P \to \mathfrak{g}^*$ such that, for every $\xi \in \mathfrak{g}$, the Hamiltonian vector field of the map $J_\xi : P \to \mathbb{R}$ defined by,*

$$J_\xi = \langle \mathbf{J}(z) \, , \xi \rangle$$

satisfies,

$$X_{J_\xi} = \xi_P$$

Now we take an example from rotations on $\mathbb{R}^3$. Consider $T^*\mathbb{R}^3$, this has coordinates $(\mathbf{q}, \mathbf{p}) \in T^*\mathbb{R}^3$, and is equipt with a Poisson bracket. Then we note that $\mathbf{R} \in \mathrm{SO}(3)$ acts on $T^*\mathbb{R}^3$ in the following way,

$$\mathbf{R}(\mathbf{q}, \mathbf{p}) = (\mathbf{R}\mathbf{q}, \mathbf{R}\mathbf{p}).$$

Let $\hat{\boldsymbol{\xi}} \in \mathfrak{so}(3)$ be an infinitesimally small rotation. We can calculate the value of the infinitesimal generator. Consider some $\mathbf{R}(t) \in \mathrm{SO}(3)$ such that $\mathbf{R}(0) = \mathbf{I}$ and $\dot{\mathbf{R}(t)} = \hat{\boldsymbol{\xi}} \in \mathfrak{so}(3)$. Now we can write,

$$\hat{\boldsymbol{\xi}}_{T^*\mathbb{R}^3}(\mathbf{q}, \mathbf{p}) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathbf{R}(t)(\mathbf{q}, \mathbf{p}) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathbf{R}(t)\mathbf{q}, \mathbf{R}(t)\mathbf{p}) = (\hat{\boldsymbol{\xi}}\mathbf{q}, \hat{\boldsymbol{\xi}}\mathbf{p}) = (\boldsymbol{\xi} \times \mathbf{q}, \boldsymbol{\xi} \times \mathbf{p}).$$

We know that $\hat{\boldsymbol{\xi}}_{T^*\mathbb{R}^3}$ is associated with a Hamiltonian vector field, J_ξ, and so we can write,

$$\left(\frac{\partial J_{\hat{\boldsymbol{\xi}}}}{\partial \mathbf{p}}, \frac{\partial J_{\hat{\boldsymbol{\xi}}}}{\partial \mathbf{q}} \right) = (\boldsymbol{\xi} \times \mathbf{p}, \boldsymbol{\xi} \times \mathbf{q}).$$

Therefore, after solving this system we can say J_ξ is,

$$J_\xi = (\boldsymbol{\xi} \times \mathbf{q}) \cdot \mathbf{p} = (\mathbf{p} \times \mathbf{q}) \cdot \boldsymbol{\xi}.$$

Therefore we can say, $\left\langle \mathbf{J}(\mathbf{q},\mathbf{p})\,,\hat{\boldsymbol{\xi}}\right\rangle = (\mathbf{q}\times\mathbf{p})\cdot\boldsymbol{\xi}$. Hence we can evaluate $J(\mathbf{q},\mathbf{p}) = \mathbf{q}\times\mathbf{p}$.

The above example shows nicely how we can study momentum maps and how they can produce conserved quantities of the systems. As with many conservation theorems Noether got her hands dirty in this area. We now present Noether's formula for cotangent bundles. Firstly though we note that if we let $P = T^*Q$ with the canonical poisson bracket we can write the Hamiltonian vector field in any tangent lifted coordinates as,

$$X_{J_\xi} = \left(\frac{\partial J_\xi}{\partial \mathbf{p}}, -\frac{\partial J_\xi}{\partial \mathbf{q}}\right).$$

Therefore the momentum map condition becomes,

$$\left(\frac{\partial J_\xi}{\partial \mathbf{p}}, \frac{\partial J_\xi}{\partial \mathbf{q}}\right) = \xi_{T^*Q}(\mathbf{q},\mathbf{p})$$

Theorem B.0.2 (Noether formula for cotangent bundles)**.** *Let G act on Q and by cotangent lifts on T^*Q. Then the momentum map $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ is,*

$$J_\xi(\mathbf{q},\mathbf{p}) = \langle \mathbf{p}\,, \xi_Q(\mathbf{q})\rangle\,,$$

where for all $\xi \in \mathfrak{g}$ there is a map $J_\xi : T^*Q \to \mathbb{R}$ such that $J_\xi(\mathbf{q},\mathbf{p}) = \langle \mathbf{J}(\mathbf{q},\mathbf{p})\,,\xi\rangle$.

Proof. We can think about this in coordinates. We get that, $J_\xi(\mathbf{q},\mathbf{p}) = \mathbf{p}^T\xi_Q(\mathbf{q})$ and so,

$$\frac{\partial J_\xi}{\partial \mathbf{p}} = \xi_Q(\mathbf{q}),$$

and further,

$$\frac{\partial J_\xi}{\partial \mathbf{q}}\cdot\dot{\mathbf{q}} = \left((D\xi_Q(\mathbf{q}))^T\mathbf{p}\right)^T\dot{\mathbf{q}}.$$

Therefore the infinitesimal generator is,

$$\xi_{T^*Q} = \left(\xi_Q(\mathbf{q}), (D\xi_Q(\mathbf{q}))^T\mathbf{p}\right),$$

and this is just the definition of infinitesimal generators in cotangent bundles in coordinates. □

If we return to the example of rotations on $\mathbb{R}^3$ we can use Noethers formula,

$$\mathbf{J}_{\hat{\boldsymbol{\xi}}} = \left\langle \mathbf{p}\,,\hat{\boldsymbol{\xi}}_{\mathbb{R}^3}(\mathbf{q})\right\rangle = \left\langle \mathbf{p}\,,\hat{\boldsymbol{\xi}}\mathbf{q}\right\rangle = \langle \mathbf{p}\,,\boldsymbol{\xi}\times\mathbf{q}\rangle = \mathbf{p}\cdot(\boldsymbol{\xi}\times\mathbf{q}) = (\mathbf{q}\times\mathbf{p})\cdot\boldsymbol{\xi}.$$

We can now go and define some properties of momentum maps. We first define the Lie Derivative,

Definition B.0.3 (Lie Derivative)**.** *Let X be a field and f be a smooth function. Then the Lie Derivative of f along X is the scalar field $\mathcal{L}_X f$ defined by,*

$$(\mathcal{L}_X f)(z) := \mathrm{d}f(z)\cdot X(z)$$

On a note of notation, we note that if we take a Lie derivative with respect to a vector field we denote it $\pounds_X f$.

Definition B.0.4 (Infinitesimal invariance)**.** *Let $H : P \to \mathbb{R}$. Then H is infinitesimally invariant if,*

$$\pounds_{\xi_P} H = 0,$$

for all $\xi \in \mathfrak{g}$.

Proposition B.0.5. *Let $H : P \to \mathbb{R}$ be G-invariant, then H is infinitesimally invariant.*

Proof. Let $g(t)$ be a path in G, such that $g(0) = 0$ and $g'(0) = \xi$. Then for all $x \in P$,

$$(\pounds_{\xi_P} H)(x) = \mathrm{d}H(\xi_P(x)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} H(g(t)x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} H(x) = 0.$$

□

We can now use this to prove the Noether Theorem in terms of momentum maps,

Theorem B.0.6 (Noether Theorem)**.** *Let G act canonically on $(P, \{\cdot, \cdot\})$ with momentum map $\mathbf{J}$. If H is G-invariant, then $\mathbf{J}$ is conserved by the flow of X_H.*

Proof. The momentum map $\mathbf{J}$ is conserved along the flow, if for every $\xi \in \}$ the map J_ξ is conserved. That is for every $\xi \in \mathfrak{g}$, the Lie derivative of J_ξ along X_H is zero. In this case,

$$\pounds_{X_H} J_\xi = \{J_\xi, H\} = -\{H, J_\xi\} = -\pounds_{\xi_P} H = 0$$

□