Year 4 — Differential Manifolds

Based on lectures by Prof. Dominic Joyce

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Michaelmas 2022

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine (especially the typos!).

Contents

1 Manifolds

Reading: Hitchin Chapter 2
Manifolds are just geometric spaces that hold other geometric structures.

1.1 Topological Manifolds

Definition 1.1 (Topological Manifold)**.** A Topological space X, is a **topological manifold** of **dimension**, $n \in \mathbb{N}$ if,

1. X is **Hausdorff**,
2. X is **second countable**,
3. For all $x \in X$, there is an open neighbourhood $V \subseteq X$, and an open set $U \in \mathbb{R}^n$ and a homeomorphism, $\phi : U \to V$. That is, X is **locally homeomorphic** to $\mathbb{R}^n$.

Hausdorff and Second Countable are global topological conditions.

Definition 1.2 (Hausdorff)**.** X is Hausdorff if for all $x, y \in X$ where $x \neq y$, there is some open $U, V \subseteq X$ such that $x \in U$ and $y \in V$ then $U \cap V = \varnothing$.

Further,

Definition 1.3 (Second Countable)**.** X is second countable, if there exists a countable set $U_1, U_2, \ldots$ open sets in X such that every open set in X is the union of some of the U_i's.

What does this mean? Well, X being second countable means X is not 'too big'. For instance, we need X second countable to show that 'every manifold is a submanifold of $\mathbb{R}^n$ for n sufficiently large' (Whitney Embedding Theorem). Some authors assume X is **paracompact** instead.

We now show $\mathbb{R}^n$ is second countable. Take the U_i's to be all the $B_r(x_1, \ldots, x_n)$ for $x_1, \ldots, x_n > 0$, rational. Hence as $\mathbb{Q}$ is dense, then every real is in a ball, but also $\mathbb{Q}$ is countable, we get the second countable result. If it is compact, then it is second countable.

The only sensible notion of 'morhpisms' of topological manifolds are continuous maps. Here are some examples / non-examples,

Example.

- $\mathbb{R}^n$ and S^n with the induced topology are topological manifolds of dimension n.
- (Non-example) The line with two origins, $\mathbb{R} \cup \mathbb{R}$ glued on $\mathbb{R} \backslash \{0\}$. This has two open subsets homeomorphic to $\mathbb{R}$. This satisfies condition (2-3), but not (1) as it isn't countable. Limits in this set isn't unique.
- Let S be any set, make S into a topological set with the discrete topology. Then S is a topological manifold of dimension 0, if and only if S is countable (needed for S to be a Second Countable). As we need these TS's to be second countable, we need countably many connected components.

1.2 Smooth Manifolds

In some sense, a manifold is general place where you can do calculus. We are trying to avoid using coordinates (this is the interesting bit for applied maths and geometry). On topological manifolds there is no meaningful notion of differentiable function. A **smooth structure** is an additional structure on a topological manifold which functions are differentiable. We express this in terms of an **atlas of charts**. There is an alternative way to do this via sheaves of smooth functions.

Definition 1.4 (Chart)**.** Let X be a topological space. A **chart** of X, of dimension $n \in \mathbb{N}$, is a pair (U, ϕ) with $U \subseteq \mathbb{R}$ open and $\phi : U \to V$ is a continuous map, such that $\phi(U) \subseteq X$ is open, and $\phi : U \to \phi(U)$ is a homeomorphism.

That is relatively boring. This tells us that X is locally homeomorphic to $\mathbb{R}^n$. Here is a more interesting definition,

Definition 1.5 (Compatible)**.** Two charts (U, ϕ) and (V, ψ) are compatible if $\psi^{-1} \circ \phi : \phi(\psi(v)) \to \psi^{-1}(\phi(v))$ is a smooth map between open subsets of $\mathbb{R}^n$.

Definition 1.6 (Smooth)**.** All partial derivatives exist. We call them C^∞.

It is automatic that $\psi^{-1} \circ \phi$ is a **homeomorphism** between open subsets of $\mathbb{R}^n$. We want smooth aswell.

$$\begin{tikzcd}
\phi^{-1}(\psi(v)) \subseteq U \subseteq \mathbb{R}^n \arrow[dr, "\phi"] \arrow[dd, "\psi^{-1}\circ\phi"'] & \\
& X \arrow[dl, dashed, bend left, "\psi^{-1}"] \\
\phi^{-1}(\psi(v)) \subseteq U \subseteq \mathbb{R}^n \arrow[ur, "\psi"] &
\end{tikzcd}$$

Definition 1.7 (Atlas)**.** An **atlas** on X of dimension $n \in \mathbb{N}$ is a family $\{(U_i, \phi_i) : i \in \mathcal{I}\}$ is a family of charts of dimension n on X, such that,

1. (U_i, ϕ_i) and (U_j, ϕ_j) are compatable for all $i, j \in \mathcal{I}$
2. $X = \bigcup_{i \in \mathcal{I}} \phi_i(U_i)$

Definition 1.8 (Maximal Atlas)**.** An atlas is called **maximal** if it is not a proper subset of any other atlas.

If $\{(U_i, \phi_i) : i \in \mathcal{I}\}$ is an atlas on X, then the set of all charts (U, ϕ) on X that satisfy, they are compatible with (U_i, ϕ_i) for $i \in I$ is called a maximal atlas and is the unique maximal atlas containing the initial atlas.

Now for the punchline, the defintion of a smooth manifold

Definition 1.9 (Smooth Manifold)**.** A (smooth) manifold, (X, A) of dimension $n \in \mathbb{N}$, is a Hausdorff, second countable topological space X together with a maximal atlas A of dimension n. Then X is a topological manifold. Usually we just call X the manifold, leaving A implicit. [1]

A **chart** on X is an element of (U, ϕ) of A. Then $V = \phi(U)$ is open in X and $\phi^{-1} = (x_1, \ldots, x_n) : V \to \mathbb{R}^n$ is a **local coordinate system** on X.

Remark. We can use basically the same definition to define,

- C^k manifolds, modelled on $\mathbb{R}^n$ but the maps have k continuous derivatives. (C^0 manifolds are topological manifolds)
- Complex Manifolds, we just use $\mathbb{C}^n$ and holomorphic maps
- Banach Manifolds, we model of Banach spaces.

Example.
- The easiest example is $X = \mathbb{R}^n$, this has an atlas consisting of one chart $\{(\mathbb{R}^n, \mathrm{id})\}$, which isn't maximal but extends to a unique maximal atlas making $\mathbb{R}^n$ into an n-manifold.
- Let $X = S^n$. It has an atlas $\{(U_1, \phi_1), (U_2, \phi_2)\}$ where $U_1 = U_2 = \mathbb{R}^n$ and $\phi_1(U_1)$ is S^n minus the north pole and $\phi_2(U_2)$ is S^n minus the north pole.

[1] We are embarrassed about A as it is a really ugly piece of kit. Hence we leave it implicitly.

Another example of a manifold with an atlas,

Example. $X = T^n = \mathbb{R}^n \backslash \mathbb{Z}^n$ is an n-manifold, with an atlas $\{(U_{\mathbf{y}}, \phi_{\mathbf{y}}) : \mathbf{y} \in Y\}$ where $Y = \{\mathbf{y} = (y_1, \ldots, y_n) : y_i \in \{0, \frac{1}{2}\}\}$. Then $U_{\mathbf{y}} = (-1/3, 1/3)^n \subseteq \mathbb{R}^n$ for all $\mathbf{y}$ and $\phi_{\mathbf{y}} : (x_1, \ldots, x_n) \mapsto (x_1 + y_1 + \mathbb{Z}, \ldots, x_n + y_n + \mathbb{Z})$. The transition maps are,

$$\phi_{\mathbf{y}_2}^{-1} \circ \phi_{\mathbf{y}_1} = x_i \mapsto \begin{cases} x_i + 1/2 \\ x_i \\ x_i - 1/2 \end{cases}$$

locally smooth map with a smooth inverse.

1.3 Smooth maps between manifolds

Definition 1.10. Let (X, A) and (Y, B) be manifolds of dimension m, n respectively and $f : X \to Y$ be a continuous map. We say that f is smooth if whenever $(U, \phi) \in A$ and $(V, \psi) \in B$ then $\psi^{-1} \circ f \circ \phi : (f \circ \phi)^{-1}(\psi(v)) \to V$ is a smooth map between open subsets of $\mathbb{R}^m$, $\mathbb{R}^n$.

$$\begin{tikzcd}
U \arrow[drr] & \supseteq & (f \circ \psi)^{-1}\psi(v) \arrow[r, "\psi^{-1} \circ f \circ \phi"] \arrow[d] & V \arrow[d, "\psi"] \\
& & X \arrow[r] & Y
\end{tikzcd}$$

Remark.
- We note $\psi^{-1} \circ f \circ \phi$ is continuous, as f is continuous but we want it to be smooth.
- If $f = \mathrm{id}_X$, then this is the definition of compatibility of charts.
- You don't have to check this on all charts of X and Y. It is enough to check this for some subsets of charts convering X and Y, that is, for atlases not for maximal atlases.

Definition 1.11 (Diffeomorphism)**.** A **diffeomorphism** $f : X \to Y$ is a smooth map with smooth inverse. This is the natural notion of isomprohism of manifolds.

Lemma 1.12. If $f : X \to Y$ and $g : Y \to Z$ are smooth maps of manifolds, then $g \circ f : X \to Z$ is also a smooth map. Further, identities $id_X : X \to X$ are smooth. Therefore, manifolds and smooth maps form a category. [2]

Proof. To show $g \circ f$ is smooth, let (U, ϕ), (V, ψ) and (W, χ) be charts on X, Y and Z. Then we have,

$$\begin{tikzcd}
(f \circ \phi)^{-1}(\psi(v)) \cap (g \circ f \circ \phi)^{-1}(\chi(w)) \arrow[r, "\chi^{-1} \circ (g \circ f) \circ \phi"] & (g \circ \psi)^{-1}(\chi(w)) \arrow[r, "\chi^{-1} \circ g \circ \psi"] \arrow[d, "\psi"] & W \arrow[d, "\chi"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

Then we have shown that, $\chi^{-1} \circ (g \circ f) \circ \phi$ is smooth on the open set $(f \circ \phi)^{-1}(\psi(v)) \cap (g \circ f \circ \phi)^{-1}(\chi(w))$. Of course[3] this is not what we want. We want $\chi^{-1} \circ (g \circ f) \circ \phi$ to be smooth on $(g \circ f \circ \phi)^{-1}(\chi(w))$. Luckily, Y is covered by $\phi(v)$ for charts (V, ψ). So $(g \circ f \circ \phi)^{-1})(\chi(W))$ is covered by subsets $(f \circ \phi)^{-1}(\psi(v)) \cap (g \circ f \circ \phi)^{-1}(\chi(w))$ over all charts (U, ψ) on Y. Therefore $\chi^{-1} \circ (g \circ f) \circ \phi$ is smooth on the whole set. So $g \circ f$ is smooth. The rest is easy [4] $\square$

[2] I got two lectures in before I met a category. RIP Applied Mathematician.
[3] this is not obvious
[4] I bet a tenner it isn't.

Another cool fact is, manifolds and smooth maps behave nicely under **products**. If X and Y are smooth manifolds of dimensions m and n, then there is a unique manifold structure on $X \times Y$ with dimension $m+n$, such that if (U, ϕ) and (V, ψ) are charts of X and Y. Then $(U \times V, \phi \times \psi)$ is a chart of $X \times Y$.

If $f : X \to Y$, $g : Y \to Z$ are smooth manifolds, then the direct product $(f, g) : X \to Y \times Z$ defined by $(f, g) : x \mapsto (f(x), g(x))$ is smooth. Further, if $f : W \to Y$ and $g : X \to Z$ are smooth, then the **product** $f \times g : W \times X \to Y \times Z$, defined by $(f \times g)(w, x) = (f(w), g(x))$ is also smooth.

2 Tangent Bundles and Cotangent Bundles

2.1 The Algebra $C^\infty(X)$ of a manifold X

Definition 2.1. Let X be a manifold. We write $C^\infty(X)$ for the set of smooth functions $f : X \to \mathbb{R}$. Then f is an $\mathbb{R}$-algebra under pointwise addition, multiplication and scalar multiplication.

If $\dim X > 0$ then $C^\infty(X)$ is infinitely dimensional. We can recover X completely, up canonical diffeomorphism from the $\mathbb{R}$-algebra $C^\infty(X)$. The points $x \in X$ are in a one-to-one correspondance with the $\mathbb{R}$-algevra morphisms $C^\infty(X) \to \mathbb{R}$ defined by $x \mapsto (x_* : f \mapsto f(x))$. This determines, X as a set.

The topology on X is the weakest such that $f : X \to \mathbb{R}$ is continuous for all $f \in C^\infty(X)$. There is then a unique manifold structure on X such that $f : X \to \mathbb{R}$ is smooth for all $f \in C^\infty(X)$[5].

Let $g : X \to Y$ be a smooth map an $g^* : C^\infty(Y) \to C^\infty(X)$ be an $\mathbb{R}$-algebra morphism. Coversely, any $\mathbb{R}$-algebra morphism $\gamma : C^\infty(Y) \to C^\infty(X)$ is g^* for some unique smooth $g : X \to Y$.

Moral: The $\mathbb{R}$-algebra knows everything about the manifold X.

Example (Example 2.1)**.** Define $a : \mathbb{R} \to \mathbb{R}$ by,

$$a(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

This function is smooth. Now we define $b : \mathbb{R} \to \mathbb{R}$ by,

$$b(t) = \frac{a(t)}{a(t) + a(1-t)}$$

This function is smooth with $b(t) = 0$ for $t \leq 0$ and $b(t) = 1$ for $t \geq 1$. Now let X be an n-manifold $x \in X$ and we choose a chart (U, ϕ) on X with $0 \in U \subseteq \mathbb{R}^n$ where $\phi(0) = x$. Now choose $\varepsilon > 0$ with $\overline{B_{\sqrt{2}\varepsilon}(0)} \subset U$. Now define $c : X \to \mathbb{R}$ by,

$$c(x) = \begin{cases} b(2 - \frac{x_1^2 + \cdots + x_n^2}{\varepsilon^2}) \text{ if} x' = \phi(x_1, \ldots, x_n) \in U \\ 0 & \text{otherwise} \end{cases}$$

and $\phi_{\mathbf{y}} : (x_1, \ldots, x_n) \mapsto (x_1 + y_1 + \mathbb{Z}, \ldots, x_n + y_n + \mathbb{Z})$. WE can say that c is a globally smooth function on X. It it 1 near x and 0 away from x. Further, the d_i are smooth on all of x and $(d_1, \ldots, d_n)$ are local coordinates on X near x.

2.2 Tangent Vectors and Tangent Space

Let X be a manifold and $x \in X$. We define a vector space T_xX called the **tangent space** to X at x. Elements $v \in T_xX$ are the **tangent vectors**. Heuristically they point in some direction. we think of them as some velocity of a point moving in X.

Definition 2.2 (Tangent Vector)**.** Let X be a manifold and $x \in X$. A **tangent vector** at x is a linear map $v : C^\infty(X) \to \mathbb{R}$ satisfying some Leibnitz rule, $v(ab) = a(x)v(b) + b(x)v(a)$ for all $a, b \in C^\infty(X)$.

We notice that this is a linear map and to we have a vector space of tangent vectors. This is a vector subspace of $C^\infty(X)^*$ (the vector space dual).

[5] This is a lie, there is apparently not a unique manifold structure

Proposition 2.3. Let X be an n-manifold (U, ϕ) be a chart on X, and $(u_1, \dots, u_n) \in U$ with $\phi(u_1, \dots, u_n) = x \in X$. Then $v : C^\infty(X) \to \mathbb{R}$ is a tangent vector if and only if it is od the form,

$$v(a) = \sum_{i=1}^{n} v_i \, \frac{\partial}{\partial x_i}(a \circ \phi)\bigg|_{(u_1, \dots u_n)}$$

for some unique $v_1, \dots, v_n \in \mathbb{R}$. Hence $T_x X \cong \mathbb{R}$, where $(x_1, \dots, x_n)$ are local coordinates of X near x.

Proof. For the 'if' part, take $v_1, \dots, v_n \in \mathbb{R}$ and set, $v(a) = \sum_{i=1}^{n} v_i \, \frac{\partial}{\partial x_i}(a \circ \phi)\Big|_{(u_1, \dots u_n)}$ for $a \in C^\infty(X)$. Then $v(ab) = a(x)b(v) + v(a)b(x)$ follows from product rule of differentiation, so v is a tangent vector. For the 'only if' part, we can define some smooth $d_1, \dots, d_n : X \to \mathbb{R}$ with $d_i \circ \phi(x_1, \dots, x_n) = x_i - u_i$ in an open neighbourhood of x in X. Let $v \in T_x X$, and set $v_i = v(d_i)$ for $i = 1, \dots, n$. Using Taylor's Theorem, for $a \circ \phi : U \to \mathbb{R}$ at $(u_1, \dots u_n)$ we can write,

$$a = a(x) \cdot 1 + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(a \circ \phi)\bigg|_{(u_1, \dots, u_n) \cdot d_i} + \sum_{i,j=1}^{n} F_{ij\,ij} + g,$$

where $F_{ij} : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are smooth with $g = 0$ in an neighbourhood of X. We write $g = g \cdot (1 - c)$ where $c = 1$ at $x \in X$. So,

$$\begin{aligned} v(a) &= a(x)v(1) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(a \circ \phi)\bigg|_{(u_1, \dots, u_n)} \cdot v_i + \sum_{i,j=1}^{n} v((F_{ij} d_i) d_j) + v(g(1 - c)) \\ &= \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(a \circ \phi)\bigg|_{(u_1, \dots, u_n)} v_i \end{aligned}$$

□

Example. Let $g : (-\varepsilon, \varepsilon) \to X$ be smooth with $\gamma(0) = x$. Define $v : C^\infty(X) \to \mathbb{R}$ by $v(a) = \frac{d}{dt}(a \circ \gamma(t)) = 0$. Then using the product rule we see $v \in T_x X$. So the velocity of a moving point $\gamma(t) \in X$ is a tangent vector at $\gamma(0)$.

Definition 2.4 (Covariantly Functorial)**.** Let $f : X \to Y$ be a smooth map of manifolds and $x \in X$ with $f(x) = y$. Define $T_x f : T_x X \to T_x Y$ by $(T_x f)(a) : a \mapsto v(a \circ f)$, for $v \in T_x X$ and $a \in C^\infty(Y)$. This is well defined. If $g : Y \to Z$ is smooth with $g(y) = z$ then $T_x(g \circ f) = T_y g \circ T_x f : T_x X \to T_z Z$. So tangent spaces are covariantly functorial.

Remark. Let $X = Y = \mathbb{R}$, then the tangent spaces are also naturally indentified with $\mathbb{R}$ by the basis of ∂_x and ∂_y. Hence it can be proved that, $T_x f = \frac{df}{dx}$.

2.3 Cotangent Spaces and 1-forms

Definition 2.5. Let X be a manifold and $x \in X$. Then the **cotangent space** $T_x^* X$ to be the dual vector space $(T_x X)^*$.

Elements of $T_x^* X$ are called 1-forms. If $(x_1, \dots, x_n)$ are local coordinates on X near x then $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ are a basis for $T_x X$. We write $dx_1, \dots, dx_n$ for the dual basis for $T_x^* X$. If $f : X \to Y$ is smooth and $x \in X$ with $f(x) = y$, we write $T_x^* X : T_y^* Y \to T_x^* X$ for the linear map dual to $T_x f : T_x X \to T_X Y$. For $g : Y \to Z$ smooth with $g(y) = z$ we have $T_x^*(g \circ f) : T_x^* X \circ T_y^* g$, so cotangent spaces are **contravariantly functorial**.

Proposition 2.6. Let X be a manifold and $x \in X$. We write $I_x = \{a \in C^\infty : a(x) = 0\}$, an ideal in $C^\infty(X)$. We write I_x^2 for the vector subspace of $C^\infty(X)$ generated by ab for $a, b \in I_x$, also an ieal in $C^\infty(X)$. Then there is a canonical isomorphism $T_x^* X \cong C^\infty(X)/(\langle 1 \rangle_R \oplus I_x^2)$. If $(x_1, \dots, x_n)$ are local coordinates of X near x, them $\langle 1 \rangle_R \oplus I_x^2$ is the kernel of some surjective linear map $C^\infty(X) \to \mathbb{R}$ mapping $a \mapsto \left(\frac{\partial a}{\partial x_1}\Big|_x, \dots, \frac{\partial a}{\partial x_n}\Big|_x \right.$

Proof. By definition, $T_xX \subset C^\infty(X)^*$. Thus there is a natural isomorphism, $T_x^* \cong C^\infty(X)/W$ where $W \subset C^\infty(X)$ is the vector subspace of $a \in C^\infty(X)$ with $v(a) = 0$ for all $v \in T_xX$, and the dual pairing $T_x^*X \times T_x^*X \to \mathbb{R}$ maps $(a + W, b) \mapsto v(a)$. If $(x_1, \dots, x_n)$ are local coordinates at x, then $\frac{\partial}{\partial x_1}, \dots \frac{\partial}{\partial x_n}$ are a basis for T_xX. So W is the kernel of $C^\infty(X) \to \mathbb{R}^n$ mapping the basis for $a \mapsto \left(\frac{\partial a}{\partial x_1}\Big|_x, \dots, \frac{\partial a}{\partial x_n}\Big|_x \right.$. Usig Taylors Theorem, we can also see that $W = \langle 1 \rangle_R \oplus I_x^2$. □

Definition 2.7 (Derivative)**.** Let X be a manifold, $x \in X$ and $a \in C^\infty(X)$. Define $d_xa \in T_x^*X$ to be the linear map $T_xX \to \mathbb{R}$ mapping $v \mapsto v(a)$. Equivalent under the isomorpihsm defined above, d_xa is $a + (\langle 1 \rangle_R I_x^2)$. We call d_xa the **derivative** of a.

If $(x_1, \dots, x_n)$ are local coordinates on X near x, and $dx_1, \dots, dx_n$ are the corresponding basis for T_x^*X, then $d_xa = \frac{\partial a}{\partial x_1}\Big|_x dx_1 + \cdots + \frac{\partial a}{\partial x_n}\Big|_x dx_n$. But dx_a makes sense without choosing coordinates.