APDEs *Just go for the τ method, it will make your life better. Trust me.*
Uniqueness. $\mathcal{J}=0$, solve, plug into $x(t,s)$. chars $(x(t))$ at end pts s, for bounds, sketch the chars and envelope. Chars are tangent to envelope.
Grns Thm: $\int_\Gamma (P\mathrm{d}y - Q\mathrm{d}x) = \iint_\Gamma P\frac{\partial}{\partial x}(P\psi) + \frac{\partial}{\partial y}(Q\psi)$
Shock is causality when $\left[\frac{\mathrm{d}x}{dt}\right]_- > \frac{dX}{dt} > \left[\frac{\mathrm{d}x}{dt}\right]_+$ and $\frac{dX}{dt} = \frac{[Q]_-^+}{[P]_-^+}$
Charpits $\frac{dx}{d\tau} = \frac{\partial F}{\partial p}, \quad \frac{dy}{d\tau} = \frac{\partial F}{\partial q}, \quad \frac{dp}{d\tau} = -\frac{\partial F}{\partial x} - p\frac{\partial F}{\partial u}, \quad \frac{dq}{d\tau} = -\frac{\partial F}{\partial y} - q\frac{\partial F}{\partial u}, \quad \frac{du}{d\tau} = p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q}.$
For BCs, $u_0' = p_0x_0' + q_0y_0'$ and use PDE as second eqn.
Optics, $\psi(x,y,t) = \phi(x,y)e^{-i\omega t}$. WKBJ, $\phi(x,y) = A(x,y)e^{iku(x,y)}$.
Systems $\mathbf{A}(x,y,\mathbf{u})\frac{\partial \mathbf{u}}{\partial x} + \mathbf{B}(x,y,\mathbf{u})\frac{\partial \mathbf{u}}{\partial y} = \mathbf{c}(x,y,\mathbf{u})$. Note $\frac{dx}{dt} = \lambda$ where $\det(\mathbf{B} - \lambda\mathbf{A}) = 0$. For 2×2, if 2 λ then hyperbolic, if $\lambda_1 = \lambda_2$ then parabolic if just 1 λ then elliptic.
Solve, find left evec ℓ of $\mathbf{B} - \lambda\mathbf{A}$, then $\ell^T\mathbf{A}\frac{d\mathbf{u}}{dx} = \ell^T\mathbf{c}$. Try to intergrate the solns to produce Riemann invariants, e.g. $\frac{d}{dx}(u\mp v) = u+v$ on $\frac{dy}{dx} = \pm 1$ implies that $e^x(u\mp v) = c$ on $\frac{dy}{dx} = \pm 1$ and so we have $e^x(u\mp v) = f(y\mp x)$. Now intergate and solve for u and v.
Region of influence is where we solve u and v for the functions, plug in initial data and then replug those into u and v. Then the region is union of where funcs are defined.
Rankine–Hugoniot condition for systems of PDEs is, $(B - \frac{\partial u}{\partial x}A)[u]_-^+ = 0$ so need $\det(B - \frac{\partial u}{\partial x}A) = 0$
Second Order PDEs: $au_{xx} + 2bu_{xy} + cu_{yy} = f$. The Cauchy data we need for this is, $x = x_0(s)$, $y = y_0(s)$, $u = u_0(s)$ and $\frac{\partial u}{\partial n} = v_0(s)$. However,

$$u_0'(s) = \frac{\partial u}{\partial x}x_0'(s) + \frac{\partial u}{\partial y}y_0'(s)$$

$$v_0'(s) = \frac{\frac{\partial u}{\partial x}x_0'(s) + \frac{\partial u}{\partial y}y_0'(s)}{\sqrt{x_0'^2 + y_0'^2}}$$

So replace $\frac{\partial u}{\partial n} = v_0(s)$ with $\frac{\partial u}{\partial x} = p_0(s)$ and $\frac{\partial u}{\partial y} = q_0(s)$. Then diff p_0 and q_0 to get the uniqueness cond.
Further note $a\lambda^2 - 2b\lambda + c = 0$. **MINUS**.
Hyperbolic: $u_{\xi\eta} = \phi(\xi,\eta,u,u_\xi,u_\eta)$. Parabolic: $u_{\xi\xi} = \phi$. Elliptic, $\xi \pm i\eta = K$, then $u_{\xi\xi} + u_{\eta\eta} = \phi$.

Number of BCs needed on boundary equal to number of chars travelling out of boundary. Two chars out, boundary is space-like, two BCs. time-like boundary, one char in and one out, one condition.
Riemann Functions: $\mathcal{L}[u] = u_{xy} + au_x + bu_y + cu = f$ and adjoint is bring coeffs in and make odd derivs minus.

$$v\mathcal{L}[u] - u\mathcal{L}^*[v] = \frac{\partial}{\partial x}\left(v\frac{\partial u}{\partial y} + auv\right) + \frac{\partial}{\partial y}\left(-u\frac{\partial v}{\partial x} + buv\right)$$

Now take the double integral over D, $v = R$, , then Grns Thm, impose $\mathcal{L}^*[R] = 0$ and then $\mathrm{d}y = 0$ on AP and $\mathrm{d}x = 0$ on BP. Then $R_x = bR$, $R_y = aR$ and $R = 1$ at P. Then we have the properties,
$\mathcal{L}^*[R] = 0$, $R_x = bR$ $(y = \eta)$, $R_y = aR$ $(x = \xi)$ and $R = 1$ on $(x,y) = (\xi,\eta)$.
Uniqueness, Poisson. Take $u = u_1 - u_2$. Then consider,

$$\iint_D \nabla\cdot(\phi\nabla\phi)\mathrm{d}x\mathrm{d}y = \iint_D \frac{\partial}{\partial x}\left(\phi\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\phi\frac{\partial\phi}{\partial y}\right)\mathrm{d}x\mathrm{d}y$$

Then greens leads to $|\nabla\phi| = 0$
Let $\xi = \varepsilon^a x$, $\tau = \varepsilon^b t$ and $w = \varepsilon^c u(\varepsilon^a x, \varepsilon^b t)$. Then rewrite in terms of these variables and simplify. Then set the power of ε be zero. Now, $\varepsilon^a = \xi/x$, $\varepsilon^b = \tau/t$. Hence $(\tau/t)^{a/b} = \xi/x$ and $\xi/\tau^{a/b} = x/t^{a/b} = \eta$ and $w/\tau^{c/b} = u/t^{c/b} = \nu$. Then $u = f(\eta)$ and hey presto!

Inhomogenous Systems ($\mathcal{L}[u] = f(x)$ subject to $\mathcal{B}[u] = g(x)$), solve $\mathcal{L}[u_1] = f(x)$ with $\mathcal{B}[u_1] = 0$ and then $\mathcal{L}[u_2] = 0$ with $\mathcal{B}[u_2] = g(x)$ and then $u = u_1 + u_2$.
To go from $\mathcal{L}[u]$ to $\mathcal{L}^*[u]$, integrate $\mathcal{L}[u]$ by parts and collect the boundary terms, collect them and then force them to be zero.

- Eigenfunctions of the adjoint problem have the same eigenvalues as the original problem, $Ly = \lambda y \implies \exists\, w, L^* w = \lambda w$.
- Eigenfunctions corresponding to different eigenvalues are orthogonal (fiddle with $\lambda_j \langle y_j\,, w_k \rangle$ to get $\lambda_k \langle y_j\,, w_k \rangle$.)

To solve the BVP, $Lu = f(x)$ subject to $Bu = g(x)$, Solve the eigenvalue problem, then solve the adjoint eigenvalue problem. Then assume $u = \sum_i c_i y_i(x)$ where $\lambda_k c_k \langle y_k\,, w_k \rangle = \langle f\,, w_k \rangle$
If $\lambda_k = 0$, then we have two cases. If $\langle f\,, w_k \rangle = 0$, then infinitely many sols by adding y_0, if $\langle f\,, w_k \rangle \neq 0$ then no sols for f.
A SL operator is self adjoint and is the following,

$$\mathcal{L}[u] = -\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u$$

where $\mathcal{L}[u] = \lambda r(x)y(x)$, r is weighting function. Further, $\mathcal{L} = \mathcal{L}^*$. We can convert from a BVP ($\mathcal{L} = a_2 u'' + a_1 u' + a_0 u = f$) to SL form. Apply integrating factor then, $\mu a_2 u'' + \mu a_1 u' = -pu'' - p'u'$. Then find, $p = e^{\int \frac{a_2}{a_1}\mathrm{d}x}$, $\mu = -\frac{1}{a_2}e^{\int \frac{a_2}{a_1}\mathrm{d}x}$, $q = -\frac{a_0}{a_2}ue^{\int \frac{a_2}{a_1}\mathrm{d}x}$ and $g(x) = -\frac{f}{a_2}e^{\int \frac{a_2}{a_1}\mathrm{d}x}$.
Orthogonality, $\int_a^b y_k(x)y_j(x)r(x)\mathrm{d}x = 0$ for $j \neq k$. All the evals are real (fiddle with inner prods). If $a < x < b$ is finite, all λ_ks are discrete and countable.
We can decompose any function by a complete set of $\{y_k\}$, that is, $h(x) = \sum c_i y_i(x)$ where $c_j = \frac{\int_a^b h(x)y_j(x)r(x)\mathrm{d}x}{\int_a^b y_j^2(x)r(x)\mathrm{d}x}$
If we have $p(x) > 0$, $r(x) > 0$ and $q(x) \geq 0$ on $a \leq x \leq b$ and BCs have $\alpha_1\alpha_2 \leq 0$ and $\alpha_3\alpha_4 \geq 0$. Then $\lambda_k \geq 0$ for $k = 1, 2, 3, \ldots$. Proof, just consider $\langle y_k\,, Ly_k - \lambda_k r y_k \rangle = 0$, unfold and integrate by parts.
The k^{th} eigenfunction will have k zeroes on $a < x < b$.
Two SL problems, $\tilde{\lambda}_k > \lambda_k$, if $\tilde{p}(x) \geq p(x)$, $\tilde{q}(x) \geq q(x)$, $\tilde{r}(x) \leq r(x)$ plus $(\tilde{a}, \tilde{b}) \subseteq (a, b)$
Greens Funcs: We can consider c_k and the def of $\langle \cdot\,, \cdot \rangle$ then say,

$$y = \int_a^b \sum_k \frac{w_k(t)y_k(x)}{\lambda_k \langle y_k\,, w_k \rangle} f(t)\mathrm{d}t = \int_a^b g(x,t)f(t)\mathrm{d}t$$

and we call $g(x,t)$ the greens function. We can find it by defining,

$$\mathcal{L}g = \begin{cases} \mathcal{L}g_- = 0 & a < x < \xi \\ \mathcal{L}_+ g = 0 & \xi < x < b \end{cases}$$

and then we need two more BCs, so we impose,

$$\int_{\xi_-}^{\xi_+} \mathcal{L}g\mathrm{d}x = \int_{\xi_-}^{\xi_+} \delta(x-\xi)\mathrm{d}x = 1$$

Plug in $\mathcal{L}g = a_n g^{(n)} + a_{n-1}g^{(n-1)} + \ldots$ and let $a_n g^{(n-1)}\big|_{\xi_-}^{\xi^+} = 1$ and $g^{(j)}\big|_{\xi_-}^{\xi^+} = 0$ for $0 < j < n-2$.
Distributions: T is a distribution if it is linear and continuous, i.e.
(i) $\langle T, \alpha\phi_1 + \beta\phi_2 \rangle = \alpha\langle T, \phi_1 \rangle + \beta\langle T, \phi_2 \rangle$, $\forall \alpha, \beta \in \mathbb{R}$ and $\forall \phi_1, \phi_2 \in C_0^\infty(\mathbb{R})$.
(ii) If $\phi_n(x)$ is a sequence of test functions s.t. $\phi_n(x) \to 0$ as $n \to \infty$ then $\langle T, \phi_n \rangle \to 0$ as a sequence of real numbers. Then $\lim_{n\to\infty}\langle T, \phi_n \rangle = \langle T, \lim_{n\to\infty}\phi_n \rangle$.
Equivalent continuity condition (for checking): $\forall L > 0$, $\exists C > 0$ and $N \geq 0$ s.t.
$|\langle T, \phi \rangle| \leq C \sum_{m \leq N} \max_{x\in\mathbb{R}} |\phi^{(m)}(x)|$, $\forall \phi$ s.t. supp $\phi \subset [-L, L]$.
Translation property: $\langle T(x+\alpha), \phi(x) \rangle = \langle T(x), \phi(x-\alpha) \rangle$, $\forall \phi \in C_0^\infty(\mathbb{R})$. **Distributional derivative:** $\langle T', \phi \rangle = -\langle T, \phi' \rangle$, $\forall \phi \in C_0^\infty(\mathbb{R})$.
Convergence of T_j to T as $j \to \infty$ means $\lim_{j\to\infty}\langle T_j, \phi \rangle = \langle T, \phi \rangle$, $\forall \phi \in C_0^\infty(\mathbb{R})$.
If $T(\alpha)$ is a family of distributions with continuous parameter α then $T(\alpha) \to T(\alpha_0)$ for $\alpha \to \alpha_0$ means $\lim_{\alpha\to\alpha_0}\langle T(\alpha), \phi \rangle = \langle T(\alpha_0), \phi \rangle$, $\forall \phi \in C_0^\infty(\mathbb{R})$.