Year 4 — Approximations of Functions

Based on lectures by Prof. Nick Trefethen

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Michaelmas 2022

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine (especially the typos!).

Contents

1 Introduction - Approximation Theory

This is the foundation of constructive analysis and the foundation of numerical analysis. The subject is 150 years old with Chebyshev and has 5 eras,

Chebyshev Era, 1800 - 1899

This is in the 19th Century. Some names include, Jacobi, Chebyshev, Zolotarev, Weirstrass and Runge. The flavour were expansions and series (Taylor and Fourier), Orthogonal Polynomials and best approximations (approximations that are optimal in ∞-norm).

Classical Era, 1900 - 1925

Some names include, Lebesgue, Bernstein, Jackson, De la Vallee Poussin, Faber, Fejer and Riesz. These are all names linked with analysis. These are the era of the foundation of analysis. We went from just formula and mapping from sets to sets. The approximation is how we bridge these ideas. This was all halted by the war.

Neoclassical Era, 1950-1975

This was the era of computers. This changed everything. Hence the field became into its own. Some names are, Davis, Cheney, Meinardes, Riblin, Lorentz, Rice, de Boor. These are people that have died very recently. Most of these people wrote great textbooks and created journals. They studied, splines, rational approximation.

Numerical Era, 1985 -

As time goes on, here we get proper computing. They studied, wavelets, radial basis functions, spectral methods, hp-finite element methods, chebfun.

High-Dimensional Era, 2010 -

Compressed Sensing, randomised algorithms, data science, deep learning, low rank approximation.

2 Chebyshev Points and Interpolants

Chebyshev is the same a fourier, but not for periodic functions. Let $n \geq 0$ and P_n is the set of polynomials of degree n ($\leq n$). Let $\{x_0, \ldots, x_n\}$ be $n+1$ distinct points in $[-1, 1]$. Suppose we have $\{f_0, \ldots, f_n\}$ a set of $\mathbb{R}$ or $\mathbb{C}$ numbers. We know,

Claim. There exists a unique interpolant $p \in P_n$ to $\{f_i\}$ in $\{x_i\}$.

This is true for arbitrary points. But we will ue Chebyshev points. That is,

$$x_j = \cos\left(\frac{j\pi}{n}\right) \qquad 0 \leq j \leq n$$

and so Chebyshev points are projections of the unit circle. They get denser towards the edge of the unit. That is important because interpolants on these points go well. In chebfun, these are `chebpts(n+1)`. The contrast to Chebyshev points are ewqually spaced points, which are awful for interpolation. When we speak of a Chebyshev interpolant, we mean a unique polynomial that interpolates some data on the amount of Chebyshev points.

2.1 Clustering

This is what makes these points so good. The clustering has a beautiful property. Think of the Chebyshev points as electrons, they will find the minimal energy configuration. This is what Chebyshev points are. Take a point, then the geometric mean distance from any point to the others is approximately a half.

3 Fourier, Laurent, Chebyshev

Fourier, Laurent and Chebshev are three equivalent ways of doing things. Each one is useful in their own area, they all have their different areas.

3.1 Fourier Analysis

- We have some $\theta \in [-\pi, pi]$.
- $F(\theta)$ with $F(\theta) = F(-\theta)$.
- Analytic in a strip.
- For interpolation, we need $2n$ equispaced points.
- Trigonometric (Fourier) Polynomial,
$$\frac{1}{2}\sum_{k=1}^{n} a_k(e^{i\theta k} + e^{-i\theta k})$$
- Forier Series,
$$\frac{1}{2}\sum_{k=1}^{\infty} a_k(e^{i\theta k} + e^{-i\theta k})$$

3.2 Laurent Analysis

- We have some $z \in D(0,1)$, where $z = e^{i\theta}$.
- $\mathbb{F}(z)$ with $\mathbb{F}(z) = \mathbb{F}(z^{-1})$.
- Analytic in some annulus
- For interpolation, we need $2n$ roots of unity.
- Laurent Polynomial,
$$\frac{1}{2}\sum_{k=0}^{n} a_k(z^k + z^{-k})$$
- Laurent Series,
$$\frac{1}{2}\sum_{k=0}^{\infty} a_k(z^k + z^{-k})$$

3.3 Chebyshev Analysis

- We have some $x \in [-1, 1]$ where $x = cos\theta = \frac{1}{2}(z + z^{-1})$.
- We have some $f(x)$ not restriction.
- Analytic in an ellipse (Bernstein Ellipse, which means focus at ± 1).
- For interpolation, we need $n+1$ Chebyshev points.
- Polynomial,
$$\sum_{k=0}^{n} a_k T_k(x)^1$$

[1] $T_k(x)$ is the degree k Chebyshev polynomial

- Chebyshev Series,

$$\sum_{k=0}^{\infty} a_k T_k(x)$$

3.4 Chebyshev Series

3.4.1 Chebychev Polynomials

It all comes from $z = e^{i\theta}$, which then says $z^k = e^{ik\theta}$ and $x = \frac{1}{2}(z + z^{-1}) = cos\theta$. Then we define, $T_k(x) = \frac{1}{2}(z^k + z^{-k}) = \cos k\theta$. Another way to spell that out is, $T_k(x) = \cos(k \arccos(x))$.

Here is some examples,

$$T_0(x) = 1 \quad T_1(x) = x \quad T_2(x) = 2x^2 - 1 \quad T_3(x) = 4x^3 - 3x \quad T_4(x) = 8x^4 - 8x^2 + 1$$

and from this we can write the three term recurrance,

$$T_{k+1} = 2xT_k(x) - T_{k-1}(x) \quad k \geq 1$$

To derive this, we note that

$$\begin{aligned} T_{k+1}(x) &= \frac{1}{2}(z^{k+1} + z^{-k-1}) \\ &= \frac{1}{2}(z^k + z^{-k})(z + z^{-1}) - \frac{1}{2}(z^{k-1} + z^{1-k}) \\ &= 2xT_k(x) - T_{k-1}(x) \end{aligned}$$

One last note is that these are Orthogonal polynomials.

Theorem 3.1. If f is Lipschitz continuous on $[-1, 1]$, it has a unique representation as a Chebyshev series,

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x)$$

and this sum is absolutely and uniformly convergent. The coefficients a_k are given by,

$$a_k = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1 - x^2}} \mathrm{d}x \quad (k \geq 1)$$

and for $k = 0$,

$$a_0 = \frac{1}{\pi} \int_{-1}^{1} \frac{f(x)}{\sqrt{1 - x^2}} \mathrm{d}x$$

Proof. Transplant to z or θ and use integrals. This is in the text. □

Example (Exercise 3.6)**.** It happens,

$$|x| = \sum_{k=0}^{\infty} a_k T_k(x)$$

where,

$$a_k = \begin{cases} a_k = 0 & k = 2n + 1 \\ a_k = \frac{4(-1)^n}{(2^n - 1)\pi} & \end{cases}$$

Example (Exercise 3.15)**.** What about e^x? We find, $a_0 = I_0(1)$ and then, $a_k = 2I_k(1)$ for $k \geq 1$.

How chebfun resolves a function

- Sample on grids of size, 17, 33, 65, 129,
- On each grid, find coefficients c_k of chebshev, interpolants (via FFT),
- Stop when coefficients reach machine precision,
- Trim the series to some degree n.

4 Interpolants, Projections and Aliasing

Our setting is we are given some f that is Lipschitz continuous on $[-1,1]$ and given some $n \geq 0$. There are two main ways to approximate f by a polynomial.

$$p_n(x) = \sum_{k=0}^{n} x_k T_k(x),$$

the Chebyshev interpolant, (we have already talked this), and,

$$f_n(x) = \sum_{k=0}^{n} a_k x_k T_k(x)$$

here we take the infinite series and then truncate it at n. This is the Chebyshev projection or truncation. The interpolant is natural for computation because it's a finite problem, while the truncation results in integrals and hence is infinite. We also call the interpolant 'by values' and the projection 'by projection'. We are going to see how c_k relate to a_k and they are basically the same.

4.1 Aliasing

Theorem 4.1. For any $n \geq 1$ and any $0 \leq m \leq n$ the following Chebyshev polynomials take the same values on the $(n+1)$ point Chebyshev grid,

$$T_m, \quad T_{2n-m}, \quad T_{2n+m}, \quad T_{4n-m}, \quad T_{4n+m}, \quad T_{6n-m}, \dots$$

Proof. Transplant to z. These polynomials are just $\frac{1}{2}$ times,

$$z^m + z^{-m}, \quad z^{2n-m} + z^{m-2n}, \quad z^{2n+m} + z^{-2n-m}, \dots$$

and further, $z^{2n} = 1$ on the $2n^{th}$ roots of unity and these are just the Chebyshev points. So all these are the same at the roots of unity of 1 and that implies all the Chebyshev polynomials are the same at the Chebyshev points. □

Theorem 4.1 implies,

Theorem 4.2. If f is Lipschitz continuous on $[-1,1]$, then $c_0 = a_0 + a_{2n} + a_{4n} + \dots$ and $c_n = a_n + a_{3n} + a_{5n} + \dots$ and for $0 < k < n$,

$$\begin{aligned} c_k = a_k + a_{k+2n} + a_{k+2n} + \dots \\ + a_{-k+2n} + a_{-k+4n} + \dots \end{aligned}$$

Proof. If f is Lipschitz, then the Chebyshev series converges absolutely. Therefore, all of the above series converge, hence they define some degree n polynomial, $q \in \mathcal{P}_n$. At a gridpoint x_j we write,

$$f(x_j) = \sum_{k=0}^{\infty} a_k T_k(x_j) \quad q(x_j) = \sum_{k=0}^{n} c_k T_k(x_j)$$

At x_j these are the same numbers in different order! (Note Thm 4.1). Therefore $f(x_j) = q(x_j)$ for all x_j. Therefore q is indeed the Chebyshev interpolant p. □

Corollary 4.3. The difference between f and f_n is,

$$f(x) - f_n(x) = \sum_{k=n+1}^{\infty} a_k T_k(x),$$

and the difference between p_n and f is,

$$f(x) - p_n(x) = \sum_{k=n+1}^{\infty} a_k(T_k(x) - T_m(x)),$$

where $m = |(k+n-1)(\bmod 2n) - (n-1)|$

5 Barycentric Interpolation Formula

Theorem 5.1 (Theorem 5.2 (Salzer 1972))**.** The degree n Chebyshev interpolant to data $f_0, \ldots, f_n$ is given by,

$$p(x) = \frac{\sum_{j=0}^{n}{}' (-1)^j f(j)/(x - x_j)}{\sum_{j=0}^{n}{}' (-1)^j / x - x_j}$$

we note $\sum'$ means we multiply the $j = 0, n$ by a half. Further if $x = x_j$, then $p(x) = f_j$.[2]

What's the point? Well theres two good ways to compute polynomial interpolants,

- By points, this is via the Barycentric Interpolation Formula. If we want to evaluate a interpolant at a point, we have $o(n)$.
- By coeffs, for this we calculate $\{c_k\}$ via FFT, then we use the series. If we want to evaluate a interpolant at a point, we have $o(n \log(n))$.

We now look to a better formula,

Theorem 5.2 (Theorem 5.1 (Dupuy 1948))**.** The degree n interpolant $x_0, \ldots, x_n$ is,

$$p(x) = \frac{\sum_{j=0}^{n} \frac{\lambda_j f_j}{x - x_j}}{\sum_{j=0}^{n} \frac{\lambda_j}{x - x_j}} \tag{$*$}$$

where, $\lambda_j = \frac{1}{\prod_{k \neq j}(x_j - x_k)}$ is the barycentric weight.

Note. We regard $|\lambda_j| = \frac{1}{(\text{geometric mean distance of } x_j \text{ to the other points})^n}$

Proof. We note that $*$ is in Lagrange form (the sum of $n + 1$ functions),

$$p(x) = \sum_{j=0}^{n} f_j \ell_j(x)$$

where

$$\ell_j(x) = \frac{\frac{\lambda_j}{x - x_j}}{\sum_{k=0}^{n} \frac{\lambda_k}{x - x_k}}$$

This is a sum of cardinal functions[3]. We have to show this does what we expect, that is again, that $\ell_j(x) \in \mathcal{P}_n$ and

$$\ell_j(x_k) = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}.$$

Now for the juicy derivation. We know (we probably don't), the lagrange interpolant,

$$\ell_j(x) = \frac{\prod_{k \neq j}(x - x_k)}{\prod_{k \neq j} x_j - x_k}.$$

We define the node polynomial,

$$\ell(x) = \prod_{k=0}^{n}(x - x_k).$$

[2] Nick Higham proved this was numerically stable (Special Topic?)
[3] a cardinal (or Lagrange) function is a function that is zero at all the grid points except one.

Then,

$$\begin{aligned}\ell_j(x) &= \frac{\ell(x)}{(x-x_j)\prod_{j\neq k}(x_j-x_k)}\\ &= \frac{\ell(x)\lambda_i}{x-x_j}.\end{aligned}$$

Why is this the same as what we wrote before? Well just divide by one! Isn't this so trivial! The reason we do this is because,

$$1 = \sum_{k=0}^{n} \ell_k(x).$$

We find,

$$\ell_j(x) = \frac{\ell(x)\frac{\lambda_j}{x-x_j}}{\ell(x)\sum_{k=0}^{n}\frac{\lambda_k}{x-x_k}} = \frac{\frac{\lambda_j}{x-x_j}}{\sum_{k=0}^{n}\frac{\lambda_k}{x-x_k}}.$$

□

6 Convergence for differentiable functions

We look at the contral dogma of approximation theory. We talk about the smoothness of f and that this corresponds to the rate of approximation of f. In this section we are going to consider f has several derivatives and we will see that this relates to algebraic convergence, and in the next section we are going to see that if f is analytic then we have exponential convergence.

In classical approximation theory we usually take a conservative view on approximation, but we can do better. We will consider the variation of a function.

Definition 6.1 (Variation)**.** We define the variation of a function, f, on $[a, b]$ is,

$$\mathcal{V}(f) = \sup \sum |f(x_{i+1}) - f(x_i)|$$

for $a \leq x_1 < \cdots < x_n \leq b$.

If $\mathcal{V}(f) < \infty$, then we say that f has bounded variation, $f \in \text{BV}$. We also can think about f as the one-norm of the derivative. That is,

$$\mathcal{V}(f) = \int_a^b |f'(x)|\text{dx} = \|\text{f}\|_1$$

This holds if f has a continuous derivative. We can extend these ideas to non-continuous functions via Stieltjes Integration.

Example.

- Let $f(x) = |x|$, then $\mathcal{V}(f) = \|f\|_1 = 2$.
- Let $f(x) = \text{sgn}(x)$, then $\mathcal{V}(f) = 2$.

We shall assume that for some $\nu \in \mathbb{Z}_{\geq 0}$, $f, f', \ldots, f^{(\nu-1)}$ are continuous and $f^{(\nu)}$ has $\mathcal{V}(f^{(\nu)}) < \infty$. We can now derive some theorems from this,

Theorem 6.2. Assume $\mathcal{V}(f^{(\nu)}) < \infty$ for some $\nu \geq 0$. Then,

$$|a_k| \leq \frac{2\mathcal{V}(f^{(\nu)})}{\pi(k-\nu)^{\nu+1}} \qquad k \geq \nu + 1$$

This is saying we decrease and the approximations converge by $\mathcal{O}(k^{-\nu-1})$.

Idea (Too fiddly) Prof. Süli did it. Instead of doing the Chebyshev idea, we shall look at the Fourier analogue. Integration by parts. Given some 2π-periodic function $F(\theta)$ and suppose $\mathcal{V}(f^{(\nu)}) < \infty$. We now see,

$$F(\theta) = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}, \quad a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\theta) e^{-ik\theta} \text{d}\theta$$

and look at the coefficients,

$$\begin{aligned}
a_k &= \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\theta) e^{-ik\theta} \text{d}\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{F'(\theta) e^{-ik\theta}}{ik} \text{d}\theta \\
&= \vdots \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{F^{(\nu+1)}(\theta) e^{-ik\theta}}{(ik)^{\nu+1}}
\end{aligned}$$

□

From the work in previous chapters, we can now say,

Theorem 6.3. Assume that $\nu \geq 1$. Then,

$$\|f - f_n\| \leq \frac{2\mathcal{V}(f^{(\nu)})}{\pi\nu(n-\nu)^{\nu}} \qquad \mathcal{O}(n^{-\nu})$$

and,

$$\|f - p_n\| \leq \frac{4\mathcal{V}(f^{(\nu)})}{\pi\nu(n-\nu)^{\nu}} \qquad \mathcal{O}(n^{-\nu})$$

Proof. We know,

$$\begin{aligned} f - f_n &= a_{n+1}T_{n+1} + a_{n+2}T_{n+2} + \dots \qquad \|T_k\| = 1 \\ \|f - f_n\|_\infty &\leq \sum_{n+1}^{\infty} |a_k| \\ &\leq \frac{2\mathcal{V}(f^{(\nu)})}{\pi} \sum_{n+1}^{\infty} \frac{1}{(k-\nu)^{\nu+1}} \\ &\leq \frac{2\mathcal{V}}{\pi} \int_n^\infty \frac{\mathrm{ds}}{(s-\nu)^{\nu+1}} \\ &= \frac{2\mathcal{V}}{\pi\nu(n-\nu)^{\nu}} \end{aligned}$$

and similarly for $\|f - p_n\|$, there is just an extra constant. □

Example. Consider $f(x) = \operatorname{sgn}(x)$, then we have $a_k = \mathcal{O}(k^{-1})$ and so we have $\|f - p_n\| = \mathcal{O}(1)$. Hence we can't approximate this nicely.

If $f(x) = |x|$, then $a_k = \mathcal{O}(k^{-2})$ and so $\|f - p_n\| = \mathcal{O}(n^{-1})$.

If $f(x) = |x|^3$, then $a_k = \mathcal{O}(k^{-4})$ and so $\|f - p_n\| = \mathcal{O}(n^{-3})$