Year 4 — Supplementary Applied Maths

Based on lectures by Prof. Helen Byrne
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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine (especially the typos!).

Contents

1 Introduction

Study methods for solving inhomogenous BVP of the following form,

$$Lu = f(x) \qquad a < x < b$$

where L is a linear differential operator,

$$Lu = a_n \frac{d^n u}{dx^n} + a_{n-1} \frac{d^{n-1} u}{dx^{n-1}} + \cdots + a_1 \frac{du}{dx} + a_0 u$$

where $f(x)$ is a forcing function and some boundary conditions imposed at $x = a$ and b.

The theory for linear boundary value problem is much richer than the theory for initial value problems, and there are many applications of linear boundary value problems. Some examples are,

- Shooting an arrow
- Tracking the melting of a block of ice
- Propogation of vibrations on a suspension bridge

The sort of questions of interest when constructing solutions are,

- Can we construct a solution for a given boundary value problem for an arbitrary function $f(x)$?
- If there is a solution, is there always a solution? (Existence) Is it unique?
- How do the choice of the boundary conditions affect the solutions?
- Can we construct solutions when $a_k = a_k(x)$?

2 Eigenfunction Methods

Consider a differential operator L. If there exists functions $y_i(x)$ such that,

$$\begin{cases} Ly_i(x) = \lambda_i y_i(x) \\ y_i(a) = y_i(b) = 0 \end{cases}.$$

If we can do this, $y_i(x)$ is an **eigenfunction** and λ_i the associated **eigenvalue** of L.

Idea. We are talking about linear differential operator, we can exploit this by constructing a solution that is a superposition (or sum) of the eigenfunctions (y_i) of L. That is,

$$y(x) = \sum_i c_i y_i(x)$$

for some c_i.

2.1 Function Spaces

There is a natural question on whether we can approximate f as a sum of eigenfunctions of differential operator. Consider an infinite dimensional space of reasonably well-behaved functions on $[a, b]$. We can introduce a set of linearly independent basis functions y_n, such that, any reasonable f in the space can be written as a sum of the basis functions,

$$f(x) = \sum_{k=1}^{\infty} f_k y_k(x).$$

An example of this, are fourier series.

Definition 2.1 (Inner Product)**.**

$$\langle u\,, v \rangle = \int_a^b u(x)\overline{v}(x)\mathrm{d}x$$

2.1.1 Weighting Functions

Consider the following eigenvalue problem,

$$Ly_i = \lambda_i \rho(x) y_i(x).$$

Here $\rho(x)$ is a weighting function, $\rho(x) \in \mathbb{R}$ and one signed on $[a, b]$. Further,

$$\langle u\,, v \rangle = \int_a^b \rho(x) u(x)\overline{v}(x)\mathrm{d}x$$

2.2 Adjoint Operator

Definition 2.2. For an operator, L, with homogenous BC, then the **adjoint problem** (L^*, BC^*) is defined by,

$$\langle Ly\,, w \rangle = \langle y\,, L^* w \rangle\,.$$

The BC^* are just the boundary conditions needed such that the above equality holds.

Note. We use integration by parts to transfer derivatives from y to w. Further, the choice of boundary conditions for L^* will become clear.

Example. Let,

$$\begin{cases} Ly = \frac{d^2y}{dx^2} + \alpha\frac{dy}{dx} + \beta y & x \in (a,b) \\ y(a) = 0, \quad y_x(b) - y(b) = 0 \end{cases}$$

Then,

$$\begin{aligned} \langle Ly\,, w\rangle &= \int_a^b (y_{xx} + \alpha y_x + \beta y) w \mathrm{d}x \\ &= [wy_x - w_x y + \alpha wy]_{x=a}^b + \int_a^b (w_{xx} - \alpha w_x + \beta w) y \mathrm{d}x \end{aligned}$$

From this, we can write,

$$\begin{cases} L^* w = w_{xx} - \alpha w_x + \beta w \\ [wy_x - w_x y + \alpha wy]_{x=a}^b = 0 \end{cases}.$$

We now need to exploit the boundary conditions from above to find the adjoint boundary conditions. We get,

$$0 = [w(b) - w_x(b) + \alpha w(b)]y(b) - w(a)y_x(a)$$

This must be true for any $y(b)$ and any $y_x(a)$. Hence we must let, $w(a) = 0$ and $w_x(b) = (1+\alpha)w(b)$ as our adjoint boundary conditions.

Note. If in the above, $\alpha = 0$, then $L^*w = w_{xx} + \beta w$, which is just the same as L. Hence it is self-adjoint. Furthermore, the boundary conditions become, $w(a) = 0$ and $w_x(b) - w(b) = 0$, which are the same as the original problem.

Definition 2.3 (Self-adjoint)**.** If $L = L^*$ and $BC = BC^*$, then the problem is called **self-adjoint**. Also if $L = L^*$ but $BC \neq BC^*$ then we call the operator self-adjoint.

Here are some facts, which turn out to be useful,

- Eigenfunctions of adjoint problems have the same eigenvalues as the original problem,

$$Ly = \lambda y \implies \exists w, L^* w = \lambda w$$

- Eigenfunctions corresponding to distinct eigenvalues are orthogonal, that is, if $Ly_j = \lambda_j y_j$ and $Ly_k = \lambda_k y_k$ then $\langle y_j\,, w_k\rangle = 0$.

  Proof. The proof goes as follows,

$$\begin{aligned} \lambda_k \langle y_j\,, w_k\rangle &= \langle y_j\,, L^* w_k\rangle \\ &= \langle Ly_j\,, w_k\rangle = \lambda_j \langle y_j\,, w_k\rangle\,. \end{aligned}$$

  That is, we have,

$$(\lambda_j - \lambda_k)\langle y\,, w_k\rangle = 0$$

  □

2.3 Solution Process

Consider the BVP,

$$\begin{cases} Lu = f(x) \\ BC_1(a) = 0, \quad BC_2(b) = 0 \end{cases}$$

What is the method?

1. Solve the eigenvalue problem,
$$Ly = \lambda y \qquad BC_1(a) = 0 = BC_2(b),$$
which will give $\{\lambda_j, y_j(x)\}$

2. Solve the adjoint eigenvalue problem,
$$L^* w = \lambda w \qquad BC_1^*(a) = 0 = BC_2^*(b),$$
which will give $\{\lambda_j, w_j(x)\}$

3. Seek a solution to the boundary value problem of the form,
$$y(x) = \sum_{j=1}^{\infty} c_i y_i(x),$$
where the coeffients c_i are determined as follows:
$$\begin{aligned} Lu = f \implies \langle f\,, w_k\rangle &= \langle Ly\,, w_k\rangle \\ &= \langle y\,, L^* w_k\rangle \\ &= \lambda_k \langle y\,, w_k\rangle \\ &= \lambda_k \left\langle \sum_i c_i y_i\,, w_k \right\rangle \\ &= \lambda_k c_k \langle y_k\,, w_k\rangle \end{aligned}$$
and so we have, $c_k = \frac{\langle f\,, w_k\rangle}{\lambda_k \langle y_k\,, w_k\rangle}$

Example. Consider,
$$\begin{cases} Ly = y'' + 4y = f(x) \\ y(0) = y(1) = 0 \end{cases}$$
To solve,

1. Solve the eigenvalue problem,
$$Ly = y'' + 4y = \lambda y, \text{ with } y(0) = 0 = y(1)$$
and this solves to, $y_n(x) = \sin(n\pi x)$, with $\lambda_n = n^2\pi^2 + 4$.

2. Solve the adjoint problem,
$$L^* w = w'' + 4w = \lambda w, \text{ with } w(0) = 0 = w(1)$$
and this solves to, $w_n(x) = \sin(n\pi x)$, with $\lambda_n = n^2\pi^2 + 4$.

3. Seek a solution $y(x) = \sum_n c_n y_n(x)$ where, $c_n = \frac{\langle f(x)\,, w_n\rangle}{\lambda_n \langle y_n\,, w_n\rangle}$.

Exercise. Follow the same procedure for the BVP,
$$\begin{cases} Ly = x^2 y'' - 2xy' = f(x) \\ y(1) = 0 = y(1) \end{cases}$$

2.4 Solution Process for inhomogenous BC

The problem in question is,

$$\begin{cases} Lu = f(x) \\ B_i u = \gamma_i \end{cases} \tag{$*$}$$

There are two solution methods, one is decomposition and the other non-decomposition,

1. Split the solution into two its, $u = u_1 + u_2$ where,

$$Lu_1 = f(x) \quad B_i u_1 = 0$$

$$Lu_2 = 0 \quad B_i u_2 = \gamma_i$$

   Then by linearity $u = u_1 + u_2$ solves our problem $(*)$

2. We seek solutions of the form, $u = \sum c_j u_j$ where $\{\lambda_j, u_j\}$ are the eigensolutions of the linear operator with homogenous boundary conditions, i.e. $Lu_i = \lambda_i u_i$ and $B_1 u_i = 0 = B_2 u_i$. In this case, care is needed iwth BCs when using the inner product to determine the coefficients c_j.

Example.

$$\begin{cases} y'' = f(x) & x \in (0,1) \\ y(0) = \alpha, y(1) = \beta \end{cases}.$$

With option 2,

1. Solve the eigenvalue problem,

$$Ly = y'' = \lambda y \quad y(0) = 0 = y(1),$$

   which has solution $y_k(x) = \sin(k\pi x)$, where $\lambda_k = -k^2\pi^2$.

2. Solve the adjoint eigenvalue problem. The problem is self-adjoint,

$$w_k = y_k = \sin(k\pi x)$$

3. Form an inner product of y with w_k:

$$\begin{aligned} y'' = f(x) \implies \int_0^1 y'' w_k \mathrm{d}x &= \int_0^1 f(x) w_k \mathrm{d}x \\ (y'w_k - yw_k')_{x=0}^1 + \int_0^1 w_k'' y \mathrm{d}x &= \int_0^1 w_k f \mathrm{d}x \\ [y'w_k - yw_k']_{x=0}^1 + \lambda_k \int_0^1 y w_k \mathrm{d}x &= \int_0^1 w_k f \mathrm{d}x \\ [y'w_k - yw_k']_{x=0}^1 + \lambda_k c_k \int_0^1 y_k w_k \mathrm{d}x &= \int_0^1 w_k f \mathrm{d}x \\ [y'w_k - yw_k']_{x=0}^1 - k^2\pi^2 c_k \int_0^1 \sin^2(k\pi x) \mathrm{d}x &= \int_0^1 w_k f \mathrm{d}x \end{aligned}$$

   Given $w_k(x) = \sin(k\pi x)$, we get,

$$[y'w_k - yw_k']_{x=0}^1 = -k\pi(-1)^k \beta + k\pi\alpha$$

   and so,

$$c_k = -\frac{2}{k^2\pi^2} \int_0^1 \int_0^1 f(x) \sin(k\pi x) + \frac{2}{k\pi}(\alpha - (-1)^k \beta)$$