Year 4 — Numerical Solutions of PDEs

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine (especially the typos!).

Contents

1 Introduction

The basic idea is very simple. Suppose that y is differentiable at $x \in \mathbb{R}$ then,

$$y' = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}$$

and so,

$$\frac{y(x+h) - y(x)}{h} = y'(x) + o(1) \quad \text{as } h \to 0$$

This then motivates,

$$y'(x) \approx \frac{y(x+h) - y(x)}{h}$$

and further if y' is differentiable,

$$y''(x) \approx \frac{y(x+h) - 2y(x) + y(x-h)}{h^2} \quad \text{as } h \to 0.$$

To check if this is a good approximation we use Taylor series and see what the other terms are (apart from y' and y'') and then check these terms go to zero.
Euler's Method Given $y'(x) = f(x, y(x))$ subject to $y(x_0) = y_0$. We write,

$$\frac{y(x_k + h) - y(x_k)}{h} \approx f(x_k, y(x_k)) \quad y(x_0) = y_0 \quad x_k = x_0 + kh \quad k \in \mathbb{Z}$$

2 Measuring Smoothness

2.1 Function Spaces

The accuracy of a numerical method for the aproximate solution of PDEs depends on its ability to capture the important feature of the analytic solution. One such feature is smoothness. To do this, we need some function spaces,

- $C(\Omega)$, Continuous functions
- $L_p(\Omega)$, Integrable Functions
- $H^k(\Omega)$, Sobolev spaces.

Notation. We let $\mathbb{N}$ be the nonnegative integers, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ is called a multi-index. The nonnegative integer, $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We let,

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$$

Definition 2.1 ($C^k(\Omega)$)**.** Let Ω be an open set in $\mathbb{R}^n$, and let $k \in \mathbb{N}$. WE denote $C^k(\Omega)$ the set of all continuous real-valued functions defined on Ω st, $D^\alpha u$ is continupus on Ω for all $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $|\alpha| \leq k$.

Definition 2.2 ($C^k(\Omega)$)**.** Assuming that Ω is a bounded open set, $C^k(\overline{\Omega})$ will denote the set of all u in $C^k(\overline{\Omega})$ st $D^\alpha u$ can be extended from Ω to a continuous function on $\overline{\Omega}$, the closure of the set Ω for all α with $|\alpha| \leq k$.

The linear space $C^k(\overline{\Omega})$ can then be equipped with the norm,

$$\|u\|_{C^k(\overline{\Omega})} := \sum_{|a| \leq k} \sup_{x \in \Omega} |D^\alpha u(x)|$$

Further, when $k = 0$ we just write $C(\Omega)$.
The support, $\operatorname{supp} u$ of a continuous function u on Ω is defined as the closure of the set,

$$\{x \in \Omega : u(x) \neq 0\}$$

In other words, $\operatorname{supp} u$ is the smallest closed subset of Ω such that $u = 0$ in $\Omega \setminus u$,
We denote $C_0^k(\Omega)$ the set of all $u \in C^k(\Omega)$ such that $\operatorname{supp} u \subset \Omega$ and $\operatorname{supp} u$ is bounded. Let,

$$C_0^\infty(\Omega) = \bigcap_{k \geq 0} C_0^k(\Omega)$$

2.2 Spaces of integrable functions

Let $p \in \mathbb{R}$, $p \geq 1$, we denote $L_p(\Omega)$ the set of all real-valued functions defined on Ω such that,

$$\int_\Omega (|u(x)|^p)^{\frac{1}{p}} \, \mathrm{d}x$$

Functions which equal almost everywhere on Ω are identified with eachother. [1] The norm is,

$$\|u\|_{L_p(\Omega)} = \left(\int_\Omega |u(x)|^2\right)^{\frac{1}{2}}$$

[1] This is equivalent to saying they are equal except on a set of measure zero. A subset of $\mathbb{R}^n$ is said to be a set of measure zero if it can be contained in the union of countably many open balls of arbitrarily small volume.

More specifically, we will focus on L_2,

$$(u,v) := \int_\Omega u(x)v(x)\mathrm{d}x$$

Lemma 2.3 (Cauchy-Schwartz)**.** Let $u, v \in L_2(\Omega)$, then

$$|(u,v)| \leq \|u\|_{L_2(\Omega)} \|v\|_{L_2(\Omega)}$$

Remark. The space $L_2(\Omega)$ equipped with the inner product $(\cdot,\cdot)$ is a Hilbert Space. This implies why the Sobolev spaces are denoted $H^k(\Omega)$

2.3 Sobolev Spaces

Consider,

$$\begin{aligned} -u''(x) &= f(x) \\ u(a) = A, \quad u(b) &= B \end{aligned}$$

If we say that $u \in \mathcal{C}^2([a,b])$, then in the classical sense then this solution doesn't have a solution. However consider some v, that vanishes at the endpoints. Then,

$$\int_a^b u'(x)v'(x)\mathrm{d}x = \int_a^b f(x)v(x)\mathrm{d}x$$

and then this is a new definition of a solution. This is a weak solution.

Suppose that u is locally integrable[2] on Ω for each bounded open set ω, with $\overline{\omega} \subseteq \Omega$. Suppose also that there exists some w_α locally integrable on Ω such that,

$$\int_\Omega w_\alpha(x)v(x)\mathrm{d}x = (-1)^{|\alpha|}\int_\Omega u(x)D^\alpha v(x) \quad \forall v \in C_0^\infty(\Omega).$$

Then w_α is called the weak derivative of u (of order $|\alpha|$) and we write $w_\alpha = D^\alpha u(x)$.

Example. Let $\Omega = \mathbb{R}$ and let $u(x) = (1-|x|)_+$ $(u(x) > 0)$. Clearly u isn't differentiable at $0, \pm 1$. However, u is locally differentable and so will possibly have a weak derivative,

$$\begin{aligned} \int_{-\infty}^\infty uv'dx = \int_{-\infty}^\infty (1-|x|)_+v'dx &= \int_{-1}^1 (1-|x|)v'(x)dx \\ &= \int_{-1}^0 (1+x)v'dx + \int_0^1 (1-x)v'(x)dx \\ &= \int_{-1}^0 (-1)v(x)dx + \int_0^1 v(x)dx \\ &= -\infty_{-\infty}^\infty w(x)v(x)dx \end{aligned}$$

where,

$$w = \begin{cases} 0, & x < -1 \\ 1, & x \in (-1,0) \\ -1, & x \in (0,1) \\ 0, & x > 1 \end{cases}$$

[2]This means that a function is integrable on every subset of ω.

Let k be a nonnegative integer. We define,

$$H^k(\Omega) := \{u \in L_2(\Omega) : D^\alpha u \in L_2(\Omega), |\alpha| \leq k\}$$

$H^k(\Omega)$ is called a Sobolev space of order k; it is equipped with the Sobolev norm,

$$\|u\|_{H^k(\Omega)} = \left(\sum_{|a| \leq k} \|D^\alpha u\|^2_{L_2(\Omega)} \right)^{\frac{1}{2}}$$

and inner product,

$$(u, v)_{H^k(\Omega)} := \sum_{|a| \leq k} (D^\alpha u, D^\alpha v)$$

With this inner product, $H^k(\Omega)$ is a hilbert space. Letting,

$$|u|_{H^k(\Omega)} := \left(\sum_{|a|=k} \|D^\alpha u\|^2_{L_2(\Omega)} \right)$$

and we can write,

$$\|u\|_{H^K(\Omega)} = \left(\sum_{j=0}^{k} |u|^2_{H^j(\Omega)} \right)^{\frac{1}{2}}$$

and this is the Sobolev seminorm.
We define a special sobolev space,

$$H^1_0(\Omega) := \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega\}.$$

In Lesbegue you can change points on measure zero, the boundary is also measure zero. The trace theorems tell us what the implication of doing this happens.

Lemma 2.4 (Poincaré-Friedrichs Inequality)**.** Suppose that Ω is a bounded open set in $\mathbb{R}^n$ and let $u \in H^1_0(\Omega)$, then, there exists a positive constant $c_*(\Omega)$ independent of u, such that,

$$\int_\Omega u^2(x)dx \leq c_* \sum_{i=1}^{n} \int_\Omega |\partial_{x_i} u(x)|^2$$

. We shall the case for a square, that is, $\Omega = (a, b) \times (c, d)$. Then,

$$u(x, y) = u(a, y) + \int_a^x \partial_x u(\xi, y) d\xi = \int_a^x \partial_x u(\xi, y) d\xi \quad c < y < d$$

Then from the Cauchy Schwartz, we get,

$$\int_\Omega |u(x, y)|^2 \leq \frac{1}{2}(b - a)^2 \int_\Omega |\partial_x u(x, y)|^2 dxdy$$

and now we do it in the y direction,

$$\int_\Omega |u(x, y)|^2 \leq \frac{1}{2}(d - c)^2 \int_\Omega |\partial_y u(x, y)|^2 dxdy$$

Then we divide by the $(b - a)^2/2$ and $(d - c)^2/2$ and then just add them,

$$\int_\Omega u^2(x)dx \leq c_* \int_\Omega (|\partial_x u|^2 + \partial_y u|^2) dxdy$$

□

3 section name

REMEMBER: Stability + Consistency, implies Convergence. We will illustrate finite difference on a simple two point BVP,

$$
\begin{aligned}
-u'' + c(x)u = f(x) \quad x \in (0,1) \\
u(0) = 0, u(1) = 0
\end{aligned}
$$

Where f, c are real valued and continuous on the interval and $c(x) \geq 0$ for all $x \in [0,1]$. The first step is to define a mesh, we are choosing some N such that, $h = (1-0)/N = 1/N$ and $x_i = ih$ where $i = 0, \dots, N$. We define the set of mesh points,

$$\Omega_h = \{x_i : i = 1, \dots, N-1\}$$

the set of boundary mesh-points, $\Gamma_h := \{x_0, x_N\}$ and the set of all mesh-points, $\overline{\Omega_h} := \Omega_h \cup \Gamma_h$. Now we to fiddle with the equation. Suppose that u is sufficiently smooth ($u \in \mathcal{C}^4([0,1])$). Now we take the taylor series,

$$u(x_i \pm h) = u(x_i) \pm hu'(x_i) + \frac{h^2}{2}u''(x_i) \pm \frac{h^2}{6}u''(x_i) + \mathcal{O}(h^4)$$

so that,

$$
\begin{aligned}
D_x^+ u(x_i) &:= \frac{u(x_{i+1} - u(x_i))}{h} = u'(x_i) + \mathcal{O}(h) \\
D_x^- u(x_i) &:= \frac{u(x_i - u(x_{i-1}))}{h} = u'(x_i) + \mathcal{O}(h)
\end{aligned}
$$

and,

$$D_x^+ D_x^- u(x_i) = D_x^- D_x^+ = \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2} = u''(x_i) + \mathcal{O}(h^2)$$

We call these D_x^+ the forward difference and D_x^- the backward difference, and $D_x^+ D_x^-$ the symmetric difference. So we rewrite this equation,

$$-D_x^+ D_x^- + c_i u(x_i) \approx f(x), \quad i = 1, \dots, N-1$$

where $u(x_0) = 0$ and $u(x_N) = 0$. This now motivates the following equation,

$$-D_x^+ D_x^- U_i + c_i(x) U_i = f(x), \quad i = 1, \dots, N-1$$

where $U_0 = 0$ and $U_0 = N$. This is just a set of linear equations,

$$
\begin{pmatrix}
\frac{2}{h^2} + c(x_1) & -\frac{1}{h^2} & & & \\
-\frac{1}{h^2} & \frac{2}{h^2} + c(x_2) & -\frac{1}{h^2} & & \\
& \ddots & \ddots & \ddots & \\
& & -\frac{1}{h^2} & \frac{2}{h^2} + c(x_{N-2}) &
\end{pmatrix}
$$

3.1 Existence and Uniqueness of a solution

Q begin an analysis to help us how that the previous matrix is invertable. The iea is, take two functions V and W defined on the interior meshpoints, the inner product,

$$(V, W)_h = \sum_{i=1}^{N-1} hV_i W_i$$

which is very siimilar to the L_2 inner product. Consider our $-u'' + cu = f(x)$, the analogous idea to solving $AU = 0$ is $-u'' + cu = 0$ where $u(0) = 0$ and $u(1) = 0$, this is to show that $u = 0$. If we can do this, then we want to replicate this in finite difference. To replicate,

$$\begin{aligned}\int_0^1 -(-u'' + c(x)u(x))u(x)dx &= \int_0^1 |u'(x)|^2 + c(x)|u(x)|^2 dx \\ &\geq \int_0^1 |u'(x)|^2\end{aligned}$$

because $c(x) \geq 0$ for all $x \in [0,1]$. As we know $-u'' + cu = 0$, we can say that $u' = 0$ on $[0,1]$. We want to now do this for the finite difference scheme, hence we need a sum by parts,

Lemma 3.1. Suppose that V is a function defined at the mesh points $x_i = 0, \ldots, N$ and let $V_0 = V_N = 0$, then,

$$(-D_x^+ D_x^- V, V)_h = \sum_{i=1}^{N} h \left|D_x^- V_i\right|^2$$

We now consider, $(AV, V)_h$,

$$\begin{aligned}(AV, V)_h = (-D_x^+ D_x^- V + cV, V)_h \\ = (-D_x^+ D_x^- V, V)_h + (cV, V)_h \\ \geq \sum_{i=0}^{h} h \left|D_x^- V_i\right|^2\end{aligned}$$

and hence we have a non-singular matrix, and therefore we have a unique solution.

Theorem 3.2. Suppose c and f are continuous real-valued functions defined on $[0,1]$ and $c(x) \geq 0$ for all $x \in [0,1]$. Then the finite difference scheme posesses a unique solution.

3.2 Stability, Consistency and Convergence

We define a norm,

$$\|U\|_h = \left(\sum_{i=1}^{N-1} h|U_i|^2\right)$$

and the discrete Sobolev norm,

$$|U|_{1,h} = (\|U\|_h^2 + \|D_x^- U\|_h^2)^{\frac{1}{2}}$$

Using thi we want to prove something like,

$$(AV, V)_h \geq c_0 \|V\|_{1,h}^2$$

and we can,

Lemma 3.3 (Discrete Poincaré Friedrichs inequality)**.** Let V be a function defined on the mesh $\{x_0 \ldots, x_N\}$, and such that $V_0 = V_N = 0$; then, there exists a positive constant c^*, independent of V and h, such that

$$\|V\|^2 \leq c_* \|D_x^- V\|_h^2$$

Further,

$$(AV, V)_h \geq c_0 \|V\|_{1,h}^2$$

where $c_0 = (1 + c_*)^{-1}$. Now we state,

Theorem 3.4. The scheme is stable in the sense that,

$$\|U\|_{1,h} \leq \frac{1}{c_0} \|f\|_h$$

Using the stability result we want to now go forward and look at convergence. Hence we consider the global error, $e_i := u(x_i) - U_i$. Now,

$$\begin{aligned} Ae_i &= Au(x_i) - AU_i = Au(x_i) - f(x_i) \\ &= -D_x^+ D_x^- u(x_i) + c(x_i)u(x_i) - f(x_i) \\ &= u''(x_i) - D_x^+ D_x^- u(x_i), \quad i = 1, \ldots, N-1 \end{aligned}$$

and hence we have derived the consistency error. Thus,

$$Ae_i = \phi_i \qquad i = 1, \ldots, N-1$$

where $e_0 = 0$ and $e_N = 0$. Now use the stability result,

$$\|u - U\|_{1,h} = \|e\|_{1,h} \leq \frac{1}{c_0} \|h\|$$

It remains to bound $\|\phi\|_h$, so we can then bound it by a constant times h^2. Hence, $\|\phi\|_h \leq Ch^2$. We deduce,

$$\|u - U\| \leq \frac{C}{c_0} h^2.$$

Hence we have quadratic convergence.

4 Finite Difference approximation for elliptic BVPs

We now want to start using the ideas from the PDE to the ODE, $-\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+c(x,y)u=f(x,y)$ on the rectangle $\Omega=(a,b)\times(c,d)$ with $u=0$ on $\partial\Omega$. We also remember $c(x)\geq 0$, we know how useful this now. Further assuming f is continuous leads to the case where the error analysis proceeds in the last lecture.

Let N be an integer and let $h=1/N$ the mesh points are (x_i,y_i) where we defined them as expected ($x_i=ih$ and $y_i=ih$). Then again we have points in the interior and on the exterior. The whole,

$$\overline{\Omega_h}=\{(x_i,y_i):i,j=0,\dots,N\}$$

To construct our finite difference method we note that there is now three points of finite difference in each direction. That is, if we take a point, then we have four points involved, North, South, East, West. Hence we have a five point finite difference scheme. This means our matrix looks different. Hence we get,

$$-(D_x^+D_x^-U_{i,j}+D_y^+D_y^-U_{i,j})+c(x_i,y_i)U_{i,j}=f(x_i,x_j).$$

Now we ask, how many unknowns we have, well it's $(n-1)^2$. We then have a pentadiagonal matrix,

4.1 Existence and Uniqueness

Now, is this unique and exists? We are going to follow similarly to yesterday,

$$(V,W)_h=\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}h^2V_{i,j}W_{i,j}$$

which is very similar to the L^2 inner product. So,

Lemma 4.1. Suppose V is a function defined on $\hat{\boldsymbol{\Omega}}_h$ such that $V=0$ on Γ_h, then,

$$(-D_x^+D_x^-V,V)_h+(-D_x^+D_x^-V,V)_h=\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}h^2|D_x^-V_{i,j}|^2+\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}h^2|D_y^-V_{i,j}|^2$$

Now we can go back and proceed to analysising this finite difference scheme,

$$\begin{aligned}(AV,V)_h&=(-D_x^+D_x^-V,V)_h+(-D_y^+D_y^-V,V)_h+(cV,V)_h\\&\geq\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}h^2|D_x^-V_{i,j}|^2+\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}h^2|D_y^-V_{i,j}|^2\end{aligned}$$

Hence, as $V=0$ on Γ_h we can now bound this by 0.

4.2 Stability and Convergence

Again, we look towards norms, just generalised,

$$\left\|D_x^-U\right\|_h=\left(\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}h^2|D_x^-V_{i,j}|^2\right),$$

and similarly for y. We have hence proved,

$$(AV,V)_h\geq\|D_x^-V]|_x^2+\|D_y^-V]|_y^2.$$

Now we need to prove some Poincaré-Friedrichs Inequality, so then we can bound by a full norm.

Lemma 4.2 (Discrete Poincaré-Friedrichs)**.**

$$\|V\|_h^2 \le c_* (\|D_x^- V]\|_h^2 + \|D_y^- V]\|_h^2)$$

and now we can bound the above inequality.

Theorem 4.3. Our finite difference scheme is stable in the sense that,

$$\|U\|_{1,h} \le \frac{1}{c_0}\|f\|_h$$

We defined $e_i = u(x_i, y_i) - U_{i,j}$ global error, so we use it again. Hence we put this into the finite difference method.

$$\begin{aligned}
Ae_{i,j} &= \Delta u(x_i, x_j) - (D_x^+ D_x^- u(x_i, y_i) + D_x^+ D_x^- u(x_i, y_i)) \\
&= \left[\frac{\partial^2 u}{\partial x^2} - D_x^+ D_x^-\right] + \left[\frac{\partial^2 u}{\partial y^2} - D_y^+ D_x^-\right] \\
&= -\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i, y_i) - \frac{h^2}{12}\frac{\partial^4 u}{\partial y^4}(x_i, \eta_i)
\end{aligned}$$

and hence,

$$\|e\|_{1,h} \le \frac{1}{c_0}\|\phi\|_h = \mathcal{O}(h^2)$$

4.3 $f \in L_2(\Omega)$

We now want to consider $f \in L_2(\Omega)$, where we have a jump in the source term. Hence we modify our left hand side. The idea is to replace $f(x_i, y_i)$ by a cell average,

$$Tf_{i,j} := \frac{1}{h^2}\int_{K_{i,j}} f(x,y)\mathrm{xy}$$

where, $K_{i,j} = [x - hi/2, x + hi/2] \times [y - hi/2, y + hi/2]$. Hence we now have the same FD method but with $T_{i,j}$ on the right hand side. Does a solution exist and is it unique? Yes, because of the same arguments as before.

Theorem 4.4. The scheme is stable in the sense that,

$$\|U\|_{1,h} \le \frac{1}{c_0}\|Tf\|_h$$

Proof. Same as before. □

Now we consider $u - U$, to find the accuracy of this method.

$$\begin{aligned}
e &= u - U \\
Ae &= Au - AU \\
&= Au - Tf \\
&= Au - T(-\Delta u + cu) \\
&= (T\frac{\partial^2 u}{\partial x^2} - D_x^+ D_x^- u)_{ij} + (T\frac{\partial^2 u}{\partial y^2} - D_x^+ D_x^- u)_{ij} + (cu - T(cu))_{ij} \equiv \varphi
\end{aligned}$$

This has six terms, to make this better we realise that the integral of $\frac{\partial^2 u}{\partial x^2}$ turns into a difference operator, similarly for y. Hence,

$$T\left(\frac{\partial^2 i}{\partial x^2}\right)(x_i, y_i) = D_x^+\left[\frac{1}{h}\int_{y_j - h/2}^{y_j + h/2}\frac{\partial u}{\partial x}(x_i - h/2, y)\mathrm{dy}\right]$$

and,

$$T\left(\frac{\partial u}{\partial y}\right)(x_i, y_j) = D_y^+\left[\frac{1}{h}\int_{x_i-h/2}^{x_i+h/2}\frac{\partial u}{\partial y}(x, y_j - h/2)\mathrm{dx}\right]$$

and so we can write,

$$Ae = D_x^+\varphi_1 + D_y^+\varphi_2 + \psi.$$

Where,

$$\begin{aligned}
\varphi_1 &:= \frac{1}{h}\int_{x_i-h/2}^{x_i+h/2}\frac{\partial u}{\partial y}(x, y_j - h/2)\mathrm{dx} - \mathrm{D_x^- u}\\
\varphi &:= \frac{1}{h}\int_{x_i-h/2}^{x_i+h/2}\frac{\partial u}{\partial y}(x, y_j - h/2)\mathrm{dx} - \mathrm{D_y^- u}\\
\psi &:= (cu)(x_i, y_i) - T(cu)(x_i, y_i)
\end{aligned}$$

and we note that the usual stability bound can only give a crude bound. This makes no use of our new ϕ form. Hence,

$$\begin{aligned}
\frac{1}{c_0}\|e\|_{1,h}^2 &\leq (Ae, e)_h = (\phi, e)_h\\
&= (D_x^+\varphi_1, e)_h + (D_y^+\varphi_2)_h + (\psi, e)_h\\
&= (\varphi_1, D_x^- e)_h + (\varphi_2, D_y^- e) + (\psi, e)_h\\
&\leq \|\varphi_1\|_x \|D_x^- e]|_x + \|\varphi_2\|_y \|D_y^- e]|_y + \|\phi\|_h \|e\|_h\\
&= (\|\varphi_1]|_x^2 + \|\varphi_2]|_y^2 + \|\psi\|_h^2)^{\frac{1}{2}}(\|D_x^- e]|_x^2 + \|D_y^- e]|_y^2 + \|e\|_h^2)^{\frac{1}{2}}\\
&= (\|\varphi_1]|_x^2 + \|\varphi_2]|_y^2 + \|\psi\|_h^2)^{\frac{1}{2}}\|e\|_{1,h}
\end{aligned}$$

Lemma 4.5. The global error, e, of the finite difference scheme is,

$$\|e\|_{1,h} \leq \frac{1}{c_0}\left(\|\varphi_1]|_x^2 + \|\varphi_2]|_y^2 + \|\psi\|_h^2\right)$$

In fact we can see that we can bound ϕ by a third derivative, instead of a forth.

4.4 Non-uniform meshes on square domains

When Ω has a curved boundary, a non-uniform mesh has to be used. To be more precise, let us introduce, $h_{i+1} := x_{i+1} - x_i$, $h_i := x_i - x_{i-1}$ and let,

$$\hbar_i := \frac{1}{2}(h_{i+1} + h_i).$$

We define,

$$D_x^+ U_i := \frac{U_{i+1} - U_i}{\hbar}, \qquad D_x^- U_i := \frac{U_i - U_{i-1}}{h_i}$$

and,

$$D_x^+ D_x^- \frac{1}{\hbar_i}\left(\frac{U_{i+1} - U_i}{h_{i+1}} - \frac{U_i - U_{i-1}}{h_i}\right)$$

Similarly for y, but we let,

$$\overline{k}_j := \frac{1}{2}(k_{j+1} + k_j).$$

So we have,

$$\overline{\Omega}_h := \{(x_i, y_i) : x_{i+1} - x_i = h_i, y_{j+1} - y_j = k_i\}.$$

The finite difference approximation looks the same.

4.5 The Discrete Maximum Principle

The maximum Principle is a key property (See PDEs in Y3). Under suitable sign-conditions imposed on the source terms and the coefficients of a differential operator. It ensures that the maximum value of the solution is attained at the boundary rather than the interior, and if the maximum isn't on the boundary, then it must be constant.

We will now aim to prove the maximum principle for the finite difference scheme. We prove in two parts, '$<$' by contradition, then the '$=$' case.

Supppose that $f < 0$ for all $(x_i, y_j) \in \Omega_h$ and that the maximum value U is attained at some $(x_{i0}, y_{j0}) \in \Omega_h$. Then, we rearrange,

$$\left(\frac{1}{\hbar_i}\left(\frac{1}{h_{i+1}} - \frac{1}{h_i}\right) + \frac{1}{\bar{k}_j}\left(\frac{1}{k_{j+1}} + \frac{1}{k_j}\right)\right) = \frac{U_{i+1,j}}{\hbar_i h_{i+1}} + \frac{U_{i-1,j}}{\hbar_i h_i} + \frac{U_{i,j+1}}{\bar{k}_j k_{j+1}} + \frac{U_{i,j-1}}{\bar{k}_j k_j} + f$$

and so we bound this by the biggest $U_{i_0 j_0}$ and we get a contradition of $0 < 0$. Hence we see why we need to split this into two cases. Now if suffices to prove for the '$=$' case. We want to cook up some problem where we can push down f so we can use the previous result. Now suppose $f \leq 0$. Let $\varepsilon > 0$ and define,

$$V_{i,j} := U_{i,j} + \frac{\varepsilon}{4}(x_i^2 + y_j^2) \text{ for } (x_i, y_j) \in \overline{\Omega}_h$$

Hence,

$$\begin{aligned} -(D_x^+ D_x^- V_{i,j} + D_y^+ D_y^- V_{i,j}) &= -(D_x^+ D_x^- U_{i,j} + D_y^+ D_y^- U_{i,j}) - \varepsilon \\ &= f(x_i, y_j) - \varepsilon < 0 \end{aligned}$$

which implies the maximum of $V_{i,j}$ is on the boundary. Thus,

$$\begin{aligned} \max_{(x_i,y_j)\in\Gamma_h} U_{i,j} &= \max\left[V_{i,j} - \frac{\varepsilon}{4}(x_i^2 + y_j^2)\right] \\ &\geq \max_{(x,y)\in\Gamma_h} V_{i,j} - \frac{\varepsilon}{4}\max_{(x_i,y_j)\in\Gamma_h}(x_i^2 + y_j^2) \\ &= \max_{(x_i,y_j)\in\overline{\Omega}_h} V_{i,j} - \frac{\varepsilon}{4}\max_{(x_i,y_j)\in\Gamma_h}(x_i^2 + y_j^2) \end{aligned}$$

As by definition, $V_{i,j}$ is bigger than $U_{i,j}$. Hence,

$$\max U_{i,j} \geq \max_{(x_i,y_j)\in\Omega_h} U_{i,j} - \frac{\varepsilon}{4}\max(x_i^2 + y_j^2)$$

and in the limit,

$$\max_{(x_i,y_j)\in\Gamma_h} U_{i,j} \geq \max_{(x_i,y_j)\in\Omega_h} U_{i,j}.$$

As $\Gamma_h \subseteq \overline{\Omega}_h$, and so the oppsite is trivial and so, we have show that if $f(x_i, y_j) \leq 0$ and so,

$$\max_{(x_i,y_j)\in\Gamma_h} U_{i,j} = \max_{(x_i,y_j)\in\Omega_h} U_{i,j}$$

We have a lemma,

Lemma 4.6. The finite difference scheme has a unique solution

Proof. The existence is obvious. We have zero solution. We know $0 \leq 0$ and $0 \geq 0$. Hence, we for the maximum principle we have,

$$0 = \max U_{i,j}$$

and by the minimum principle,

$$0 = \min U_{i,j}$$

Hence we have uniqueness. □

Now for stability. We have seen this argument before, we cook up two problems of $U_{i,j}^{(1)}$ and $U_{i,j}^{(2)}$. We want to show that if $g^{(1)}$ and $g^{(2)}$ are close so are the U's. Let $U := U^{(1)} - U^{(2)}$ and $g := g^{(1)} - g^{(2)}$. Then we use the same trick with the principles. Hence,

$$\max_{\overline{\Omega}_h} U = \max_{\Gamma_h} U = \max_{\Gamma_h} g \leq \max_{\Gamma_h} |g|$$

Now consider $-U$, we get a similar argument

$$-U_{i,j} \leq \max_{\Gamma_h} |g|$$

and so,

$$\max_{\overline{\Omega}_h} |U_{i,j}| \leq \max_{\Gamma_h} |g|.$$

Therefore we have,

$$\max_{\overline{\Omega}_h} |U^{(1)} - U^{(2)}| \leq \max_{\Gamma_h} |g^{(1)} - g^{(2)}|.$$

5 Time dependant PDEs

We will consider time dependant PDEs, for example,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

where $x \in (-\infty, \infty)$ with $u(x,0) = x_0$ for $x \in (-\infty, \infty)$. We can find an exact solution using Fourier transform, (See PDEs Y3). We have been considering the supremum norm, we can prove something similar to what we saw with this equation in the least square sense. That is, we are going to do something similar to this in a finite case, we can do this in the L^2-norm. We show Parseval's identity, that is the L^2-norm of the solution at any time $t > 0$ is bounded by the L^2 norm.

Lemma 5.1 (Parseval's Indentity)**.** Suppose that $u \in L_2((-\infty, \infty))$. Then $\hat{u} \in L_2((\infty, \infty))$ and the following equality holds,

$$\|u\|_{L_2} = \frac{1}{\sqrt{2\pi}} \|\hat{u}\|_{L_2}$$

Proof. We observe,

$$\begin{aligned}
\int_{-\infty}^{\infty} \hat{u}(\xi) v(\xi)\, d\xi &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} u(x) e^{-ix\xi} \mathrm{dx} \right) v(\xi) \mathrm{d}\xi \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} v(\xi) e^{-ix\xi} \mathrm{dx} \right) u(x) \mathrm{dx} \\
&= \int_{-\infty}^{\infty} u(x) \hat{v}(x) \mathrm{dx}
\end{aligned}$$

Now let $v(\xi) = \overline{\hat{u}(\xi)} = 2\pi F^{-1}[\hat{u}](\xi)$ and so this goes into the above and provides the result. □

Now we can do the following, we can bound the solution using the initial condition. So we consider,

$$\begin{aligned}
\|u(\cdot, t)\|_{L_2} &= \frac{1}{\sqrt{2\pi}} \left\| e^{-t \cdot^2} \hat{u}_0(\cdot) \right\|_{L_2} \\
&\leq \frac{1}{\sqrt{2\pi}} \|\hat{u}_0\|_{L_2} \\
&= \|u_0\|_{L_2}
\end{aligned}$$

We now use the usual trick by applying this to $u - \tilde{u}$ with initial conditions $u_0 - \tilde{u}_0$. Hence we get a stability condition.

5.1 Model Problem : Heat Equation

Our computational domain is $\{(x,t) \in (-\infty, \infty) \times [0,T]\}$. We then create a mesh where $\Delta t = T/M$. Then we follow the usual story, where $x_j = j\Delta x$ and $t_m = m\Delta t$. Hence,

$$\frac{\partial u}{\partial t}(x_j, t_m) = \frac{u(x_j, t_{m+1}) - u(x_j, t_m)}{\Delta t} \qquad \frac{\partial^2 u}{\partial x^2}(x_j, t_m) = \frac{u(x_{j+1}, t_m) - 2u(x_j, t_m) + u(x_{j-1}, t_m)}{(\Delta x)^2}.$$

We can create an explicit euler method,

$$U_j^{m+1} = U_j^m + \mu(U_{j+1}^m - 2U_j^m + U_{j-1}^m) \qquad U_j^0 = u_0$$

where $\mu = \frac{\Delta x}{\Delta t}$. We could also do an implit or a θ-method.

5.2 Stability of finite difference schemes

To replicate this property of the heat equation in the L_2 norm at the discrete level, we need a suitable notion of stability. We shall say that a finite difference scheme for the unsteadt heat equation is **(practically) stable in ℓ_2 norm** if,

$$\|U^m\|_{\ell_2} \leq \left\|U^0\right\|_{\ell_2}$$

where,

$$\|U^m\|_{\ell_2} = \left(\Delta x \sum_{j=-\infty}^{\infty} |U_j^m|^2\right)^{\frac{1}{2}}.$$

Thinking about before we need a Parsevals and a fourier transform, so we consider a semidiscrete fourier transform.

Definition 5.2. The semidiscrete Fourier transform of a function U defined on the infinite mesh $x_j = j\Delta x$, $j = 0, \pm 1, \pm 2, \ldots$ is,

$$\hat{U}(k) = \Delta x \sum_{j=-\infty}^{\infty} U_j e^{-ikx_j} \qquad k \in [-\pi/\Delta x, \pi/\Delta x]/$$

We also need an inverse for Parsevals identity that connects the transforms,

Definition 5.3. Let $\hat{U}$ be defined on $[-\pi/\Delta x, \pi/\Delta x]$. The inverse of the semidiscrete Fourier transform of $\hat{U}$ is defined by,

$$U_j := \frac{1}{2\pi} \int_{-pi/\Delta x}^{\pi/\Delta x} \hat{U}(k) e^{ik\Delta x} \mathrm{dk}$$

Lemma 5.4 (Discrete parseval's identity)**.** If we have as before our norms, then, if $\|U\|_{\ell_2}$ is finite, then also $\left\|\hat{U}\right\|_{L_2}$ is finite, and

$$\|U\|_{\ell_2} = \frac{1}{\sqrt{2\pi}} \left\|\hat{U}\right\|_{L_2}.$$

Now we want to consider finite difference schemes. We want to solve,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad x \in (-\infty, \infty), t > 0$$

with respect to $u(x, 0) = u_0(x)$. Then we want to consider some grid $x_j = j\Delta x$ where $j \in \mathbb{Z}$. We insert,

$$U_m = \frac{1}{2\pi} \int_{-\pi/\Delta x}^{\pi/\Delta x} e^{ikj\Delta x} \hat{U}^m(k) \mathrm{dk}$$

intro the euler scheme we can say,

$$\frac{1}{2\pi} \int_{-\pi/\Delta x}^{\pi/\Delta x} e^{ikj\Delta x} \frac{\hat{U}^{m+1} - \hat{U}^m}{\Delta t} \mathrm{dk} = \frac{1}{2\pi} \int_{-\pi/\Delta x}^{\pi/\Delta x} \frac{e^{ik(j+1)\Delta x} - 2e^{ikj\Delta x} + e^{ik(j-1)\Delta x}}{(\Delta x)^2} \hat{U}^m \mathrm{dk}$$

and so we have that,

$$\frac{1}{2\pi} \int_{-\pi/\Delta x}^{\pi/\Delta x} e^{ikj\Delta x} \frac{\hat{U}^{m+1} - \hat{U}^m}{\Delta t} \mathrm{dk} = \frac{1}{2\pi} \int_{-\pi/\Delta \mathrm{x}}^{\pi/\Delta \mathrm{x}} \mathrm{e}^{\mathrm{ikj}\Delta \mathrm{x}} \frac{\mathrm{e}^{\mathrm{ik}\Delta \mathrm{x}} - 2 + \mathrm{e}^{-\mathrm{ik}\Delta \mathrm{x}}}{(\Delta \mathrm{x})^2} \hat{\mathrm{U}}^{\mathrm{m}} \mathrm{dk}$$

We now we want to remove the integrals. We note that we have two inverse fourier transforms equalling eachother. Hence,

$$\hat{U}^{m+1}(k) = \hat{U}^m(k) + \mu(e^{ik\Delta x} - 2 + e^{-k\Delta x})\hat{U}^m(k)$$

Thus we have,

$$\hat{U}^{m+1}(k) = \lambda(k)\hat{U}^m(k),$$

where $\lambda(k) = 1 + \mu(e^{ik\Delta x} - 2 + e^{-k\Delta x})$. We want to bound $\lambda(k)$ by one. So consider,

$$\begin{aligned}
\left\|U^{m+1}\right\|_{\ell_2} &= \frac{1}{\sqrt{2\pi}} \left\|\hat{U}^{m+1}\right\|_{L_2} \\
&= \frac{1}{\sqrt{2\pi}} \left\|\lambda\hat{U}^{m}\right\|_{L_2} \\
&\leq \frac{1}{\sqrt{2\pi}} \max_k |\lambda(k)| \left\|\hat{U}^{m}\right\|_{L_2} \\
&= \max_k |\lambda(k)| \left\|U^{m}\right\|_{\ell_2}.
\end{aligned}$$

We want,

$$\left\|U^{m+1}\right\|_{\ell_2} \leq \left\|U^{m}\right\|_{\ell_2},$$

and to do this we demand, $\max_k |\lambda(k)| \leq 1$. Hence we have a real number,

$$\max_k \left|1 - 4\mu \sin^2\left(\frac{k\Delta x}{2}\right)\right| \leq 1.$$

So we want to say,

$$-1 \leq 1 - 4\mu \sin^2\left(\frac{k\Delta x}{2}\right) \leq 1$$

The left hand side is easy, but the right hand side, not as trivial. We note,

$$\mu = \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$$

for stability.