Year 4 — Approximations of Functions

Based on lectures by Prof. Nick Trefethen
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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine (especially the typos!).

Contents

1 Introduction - Approximation Theory

This is the foundation of constructive analysis and the foundation of numerical analysis. The subject is 150 years old with Chebyshev and has 5 eras,

Chebyshev Era, 1800 - 1899

This is in the 19th Century. Some names include, Jacobi, Chebyshev, Zolotarev, Weirstrass and Runge. The flavour were expansions and series (Taylor and Fourier), Orthogonal Polynomials and best approximations (approximations that are optimal in ∞-norm).

Classical Era, 1900 - 1925

Some names include, Lebesgue, Bernstein, Jackson, De la Vallee Poussin, Faber, Fejer and Riesz. These are all names linked with analysis. These are the era of the foundation of analysis. We went from just formula and mapping from sets to sets. The approximation is how we bridge these ideas. This was all halted by the war.

Neoclassical Era, 1950-1975

This was the era of computers. This changed everything. Hence the field became into its own. Some names are, Davis, Cheney, Meinardes, Riblin, Lorentz, Rice, de Boor. These are people that have died very recently. Most of these people wrote great textbooks and created journals. They studied, splines, rational approximation.

Numerical Era, 1985 -

As time goes on, here we get proper computing. They studied, wavelets, radial basis functions, spectral methods, hp-finite element methods, chebfun.

High-Dimensional Era, 2010 -

Compressed Sensing, randomised algorithms, data science, deep learning, low rank approximation.

2 Chebyshev Points and Interpolants

Chebyshev is the same a fourier, but not for periodic functions. Let $n \geq 0$ and P_n is the set of polynomials of degree n ($\leq n$). Let $\{x_0, \ldots, x_n\}$ be $n+1$ distinct points in $[-1, 1]$. Suppose we have $\{f_0, \ldots, f_n\}$ a set of $\mathbb{R}$ or $\mathbb{C}$ numbers. We know,

Claim. There exists a unique interpolant $p \in P_n$ to $\{f_i\}$ in $\{x_i\}$.

This is true for arbitrary points. But we will ue Chebyshev points. That is,

$$x_j = \cos\left(\frac{j\pi}{n}\right) \qquad 0 \leq j \leq n$$

and so Chebyshev points are projections of the unit circle. They get denser towards the edge of the unit. That is important because interpolants on these points go well. In chebfun, these are `chebpts(n+1)`. The contrast to Chebyshev points are ewqually spaced points, which are awful for interpolation. When we speak of a Chebyshev interpolant, we mean a unique polynomial that interpolates some data on the amount of Chebyshev points.

2.1 Clustering

This is what makes these points so good. The clustering has a beautiful property. Think of the Chebyshev points as electrons, they will find the minimal energy configuration. This is what Chebyshev points are. Take a point, then the geometric mean distance from any point to the others is approximately a half.

3 Fourier, Laurent, Chebyshev

Fourier, Laurent and Chebshev are three equivalent ways of doing things. Each one is useful in their own area, they all have their different areas.

3.1 Fourier Analysis

- We have some $\theta \in [-\pi, pi]$.
- $F(\theta)$ with $F(\theta) = F(-\theta)$.
- Analytic in a strip.
- For interpolation, we need $2n$ equispaced points.
- Trigonometric (Fourier) Polynomial,

$$\frac{1}{2}\sum_{k=1}^{n} a_k(e^{i\theta k} + e^{-i\theta k})$$

- Forier Series,

$$\frac{1}{2}\sum_{k=1}^{\infty} a_k(e^{i\theta k} + e^{-i\theta k})$$

3.2 Laurent Analysis

- We have some $z \in D(0,1)$, where $z = e^{i\theta}$.
- $\mathbb{F}(z)$ with $\mathbb{F}(z) = \mathbb{F}(z^{-1})$.
- Analytic in some annulus
- For interpolation, we need $2n$ roots of unity.
- Laurent Polynomial,

$$\frac{1}{2}\sum_{k=0}^{n} a_k(z^k + z^{-k})$$

- Laurent Series,

$$\frac{1}{2}\sum_{k=0}^{\infty} a_k(z^k + z^{-k})$$

3.3 Chebyshev Analysis

- We have some $x \in [-1, 1]$ where $x = cos\theta = \frac{1}{2}(z + z^{-1})$.
- We have some $f(x)$ not restriction.
- Analytic in an ellipse (Bernstein Ellipse, which means focus at ± 1).
- For interpolation, we need $n + 1$ Chebyshev points.
- Polynomial,

$$\sum_{k=0}^{n} a_k T_k(x)^1$$

[1] $T_k(x)$ is the degree k Chebyshev polynomial

- Chebyshev Series,

$$\sum_{k=0}^{\infty} a_k T_k(x)$$

3.4 Chebyshev Series

3.4.1 Chebychev Polynomials

It all comes from $z = e^{i\theta}$, which then says $z^k = e^{ik\theta}$ and $x = \frac{1}{2}(z + z^{-1}) = cos\theta$. Then we define, $T_k(x) = \frac{1}{2}(z^k + z^{-k}) = \cos k\theta$. Another way to spell that out is, $T_k(x) = \cos(k \arccos(x))$.

Here is some examples,

$$T_0(x) = 1 \quad T_1(x) = x \quad T_2(x) = 2x^2 - 1 \quad T_3(x) = 4x^3 - 3x \quad T_4(x) = 8x^4 - 8x^2 + 1$$

and from this we can write the three term recurrance,

$$T_{k+1} = 2xT_k(x) - T_{k-1}(x) \quad k \geq 1$$

To derive this, we note that

$$\begin{aligned} T_{k+1}(x) &= \frac{1}{2}(z^{k+1} + z^{-k-1}) \\ &= \frac{1}{2}(z^k + z^{-k})(z + z^{-1}) - \frac{1}{2}(z^{k-1} + z^{1-k}) \\ &= 2xT_k(x) - T_{k-1}(x) \end{aligned}$$

One last note is that these are Orthogonal polynomials.

Theorem 3.1. If f is Lipschitz continuous on $[-1, 1]$, it has a unique representation as a Chebyshev series,

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x)$$

and this sum is absolutely and uniformly convergent. The coefficients a_k are given by,

$$a_k = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1 - x^2}} \mathrm{d}x \quad (k \geq 1)$$

and for $k = 0$,

$$a_0 = \frac{1}{\pi} \int_{-1}^{1} \frac{f(x)}{\sqrt{1 - x^2}} \mathrm{d}x$$

Proof. Transplant to z or θ and use integrals. This is in the text. □

Example (Exercise 3.6)**.** It happens,

$$|x| = \sum_{k=0}^{\infty} a_k T_k(x)$$

where,

$$a_k = \begin{cases} a_k = 0 & k = 2n + 1 \\ a_k = \frac{4(-1)^n}{(2^n - 1)\pi} & \end{cases}$$

Example (Exercise 3.15)**.** What about e^x? We find, $a_0 = I_0(1)$ and then, $a_k = 2I_k(1)$ for $k \geq 1$.

How chebfun resolves a function

- Sample on grids of size, 17, 33, 65, 129,
- On each grid, find coefficients c_k of chebshev, interpolants (via FFT),
- Stop when coefficients reach machine precision,
- Trim the series to some degree n.