Year 4 — Topics in Fluid Mechanics

Based on lectures by Dr Graham Benham
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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine (especially the typos!).

Contents

The course is split up as follows, *Lecture 1*

- Thin Film and Lubrication Theory (5 Lectures)
- Flow in porous media (6 Lectures)
- Convection and Turbulence (5 Lectures)

4 Problem Sheets.
We will have all our work revolving around the Navier Stokes equations,

$$\rho(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}) = -\nabla p + \mu\nabla^2\mathbf{u}$$
$$\nabla\cdot\mathbf{u} = 0$$

Then we can nondimensionalise $\mathbf{x} = L\hat{\mathbf{x}}$, $t = \frac{L}{U}\hat{t}$, $\mathbf{u} = U\hat{\mathbf{u}}$, $p = \frac{pU}{L}\hat{p}$. We nondimensionalise,

$$\mathrm{Re}(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)u) = -\nabla p + \nabla^2\mathbf{u}$$

A low Reynolds number is slow, sticky flows like honey, while high reynolds numers are fast and sloshy. However, for high reynolds number something goes wrong, we need to non-dimensionalise using, $p = \rho U^2 p$. Then we get,

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}}\nabla^2\mathbf{u}$$

1 Thin Film and Lubrication Theory

1.1 Slider Bearing

Consider some object sitting above some impermiable surface. This is a problem about flow in narrow gaps, so we have $\varepsilon = H/L << 1$. So we need to re-nondimensionalise our system. We have $x \sim 1$ and $z \sim \varepsilon$, so our incompressibility condition goes to,

$$u_x + w_z = 0$$

The pressure scales like $p \sim 1/\varepsilon^2$. Therefore the rest of our Navier Stokes equations goes to,

$$\varepsilon^2 \mathrm{Re}[\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u}] = -p_x + \varepsilon^2 u_{xx} + u_{zz}$$

and the veritcal equation becomes,

$$\varepsilon^4 \mathrm{Re}[w_t + (\mathbf{u} \cdot \nabla)w] = -p_z + \varepsilon^4 w_{xx} + \varepsilon^2 w_{zz}$$

The key assumption is ε **is small, but also** $\varepsilon^2\mathrm{Re}$ **is small**.

Therefore, the governing equation becomes,

$$\begin{aligned} u_{zz} &= p_x = p'(x) \\ p_z &= 0 \end{aligned}$$

and the boundary conditions, we have the no slip condition, so $u = 0$ and the impermiablility, $w = 0$ on $z = 0$. On $z = h(x)$, we have the no slip says that $u = 1$ and $w = uh'(x)$. We also have atmospheric pressure, so $p = 0$ on $x = 0$ and $x = 1$.

These equations are solvable, $u = \frac{1}{2}p'z(z-h) + z/h$ and mass conservation says, $u_x + w_z = 0$,

$$\begin{aligned} \int_0^h u_x dz + [w]_0^h &= 0 \\ \frac{\partial}{\partial x}\int_0^h udz - uh'|_{z=h} + uh'|_{z=h} &= 0 \\ \frac{d}{dx}\int_0^h udz &= 0 \\ \frac{d}{dx}\left[\frac{1}{2}h + \frac{1}{2}p'(-\frac{1}{6}h^3)\right] &= 0 \end{aligned}$$

Then this is solvable for p.

Exercise. Do this for 3D. We have $\nabla_H = (\partial_x, \partial_y, 0)$ and then we get,

$$\nabla_H \cdot [h\mathbf{i} - \frac{1}{6}h^3\nabla_H p]$$

and suppose $\mathbf{u} = (1, 0, 0)$ on $z = h(x, y)$.

1.2 Free Surface

The kinematic condition on the free boundary is,

$$\frac{D}{Dt}[S - z] = 0 \quad z = S$$

what also about accumulation? Like rainfall or something being sprayed, then,

$$\frac{D}{Dt}[S-z] = a \quad z = S$$

and so we have,

$$w = S_t + uS_x + vS_y - a \quad z = S.$$

What if we apply similar to $z = b$ and define $h = S - b$, then we can integrate conservation of mass like before and get,

$$\frac{\partial h}{\partial t} + \nabla_H \cdot \int_0^h \mathbf{u}_H dz = 0$$

where $\mathbf{u}_H = (u, v, 0)$.

1.3 Free Surface Stress

We consider the tangent and normal to the surface,

$$\hat{\mathbf{n}} = \frac{(-S_x, 1)}{(1+S_x^2)^{\frac{1}{2}}} \quad \hat{\mathbf{t}} = \frac{(1, S_x)}{(1+S_x^2)^{\frac{1}{2}}}$$

Then we have continuous stress at $z = S(r)$ and we have for say, a droplet, as an example, $\sigma_{nn} = p_a$, $\sigma_{nt} = 0$. What are these defined as, $\sigma_{nn} = \sigma_{ij}n_in_j$, in suffix notation. We defined $\sigma_{ij} = -p\delta_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ and so,

$$\begin{aligned}\sigma_{nn} &= \sigma_{11}n_1n_1 + \sigma_{13}n_1n_3 + \sigma_{31}n_3n_1 + \sigma_{33}n_3n_3 \\ &= -p + [\tau_1(S_x^2 - 1) - 2\tau_3 S_x]/(1+S_x^2)\end{aligned}$$

where $\tau_1 = 2\mu u_x$ and $\tau_3 = \mu(u_z + w_x)$. Similarly,

$$\begin{aligned}\sigma_{nt} &= \sigma_{11}n_1t_1 + \sigma_{13}n_1t_3 + \sigma_{31}n_3n_1 + \sigma_{33}n_3t_3 \\ &= \frac{[\tau_3(1-S_x^2) - 2\tau_1 S_x]}{1+S_x^2}\end{aligned}$$

We now consider dimensional scalings, $x \sim L$, $z \sim \varepsilon L, S \sim \varepsilon L = K$. We will scale stress by $\tau^* = \frac{\mu U}{H}$, that is, $\tau_1 \sim \varepsilon\tau*^*, \tau_3 \sim \tau^*$ and $p - p_a \sim \frac{\mu U}{L\varepsilon^2} = \frac{\tau^*}{\varepsilon}$. That means,

$$\sigma_{nn} = -\frac{p}{\varepsilon} + [\varepsilon\tau_1(\varepsilon^2 S_x^2 - 1) - 2\varepsilon\tau_3 S_x]/(1+\varepsilon^2 S_x^2)$$

where $\tau_1 = 2u_x$ and $\tau_3 = u_z + \varepsilon^2 w_x$ and similarly,

$$\sigma_{nt} = \frac{[t_3(1-\varepsilon^2 S_x^2) - 2\varepsilon^2\tau_1 S_x]}{1+\varepsilon^2 S_x^2}$$

and so $\tau_3 = 0$ implies that $u_z = 0$ and $p = 0$ on $z = S$.