NSPDEs

L_2 norm, continuous $\|U\|_{L_2} = \left(\int_\Omega |u(x)|^2\right)^{1/2}$, discrete, $\|U\|_{L_2} = \left(\sum_{i=1}^{N-1} h|U_i|^2\right)^{1/2}$.

Modified L_2 norm, $\|U]|_h = \left(\sum_{i=1}^{N-1} h|U_i|^2\right)^{1/2}$

Sobolev norm, continuous $\|U\|_{H^k} = \left(\sum_{|\alpha|\le k} \|D^\alpha u\|_{L_2}\right)^{1/2}$, discrete, $\|U\|_{1,h} = (\|U\|_h^2 + \|D_x^- U\|_h^2)^{1/2}$

Poincaré Friedrichs (Discrete) $\|V\|_h^2 \le c_*(\|D_x^+ V\|_x^2 \left\|D_y^+ V\right\|_x^2)$ where $c_* = 1/4$ for elliptic pde.

Summation by parts:

$$(-D_x^+ D_x^- U, V)_h + (-D_y^+ D_y^- U, V)_h = \sum_{i=1}^{N}\sum_{j=1}^{N-1} h^2 |D_x^- V_{ij}|^2 + \sum_{i=1}^{N}\sum_{j=1}^{N-1} h^2 |D_y^- V_{ij}|^2$$

$$(AV, V)_h = (-D_x^+ D_x^- V - D_x^+ D_x^- V + cV, V) \ge (-D_x^+ D_x^- V - D_x^+ D_x^- V, V)$$

Also we have, $(AV, V)_h \ge \|D_x^- V\|_x^2 + \left\|D_y^2\right\|_y^2$ (Notation for SBP) and so putting them together $(AV, V)_h \ge (1 + c_*)^{-1} \|V\|_{1,h}^2$.
For errors, $e_{ij} = u(x_i, y_j) - U_{ij}$ is global. Then consistency is $\varphi_{ij} = Ae_{ij}$. Then fiddle with it and use taylor series.
If we dont have continuity,

$$Tf_{ij} = \frac{1}{h^2}\int_{K_{ij}} f(x,y)\mathrm{d}x\mathrm{d}y \qquad K_{ij} = [x_i - h/2, x_i + h/2] \times [y_i - h/2, y_i + h/2]$$

For error here, note we can sum by parts, to move the differences.
Let $\overline{h}_i = \frac{1}{2}(h_{i+1} - h_i)$ such that,

$$D_x^+ U = \frac{U_{i+1} - U_i}{\overline{h}_i} \qquad D_x^- U_i = \frac{U_i - U_{i-1}}{\overline{h}_i} \quad D_x^+ D_x^- U = \frac{1}{\overline{h}_i}\left(\frac{U_{i+1} - U_i}{h_{i+1}} - \frac{U_i - U_{i-1}}{\overline{h}_i}\right)$$

Similarly for y, but with $\overline{k}$.
Max/Min Principle, the min/max of an elliptic equation will occur on the boundary. Less than max, more than min. You can use this to show stability.
CFL Number, $\mu := \frac{\Delta t}{(\Delta x)^2}$
Taylor Series in 2D at $X = (a, b)$,
$f(x,y) \approx f(X) + (x-a)f_x(X) + (y-b)f_y(X) + 1/2![(x-a)^2 f_{xx}(X) + 2(x-a)(y-b)f_{xy}(X) + (y-b)^2 f_{yy}(X)] + 1/3![(x-a)^3 f_{xxx}(X) + 3(x-a)^2(y-b)f_{xxy}(X) + 3(x-a)(y-b)^2 f_{xyy} + (y-b)^3 f_{yyy}(X)] + \ldots$
The discrete ℓ_2 norm, $\|U^m\|_{\ell_2} = \left(\Delta x \sum_{k=-\infty}^{\infty} |U_j^m|^2\right)^{1/2}$
Discrete Parseval's Identity $\|U\|_{\ell_2} = \frac{1}{\sqrt{2\pi}}\|U\|_{L_2}$.
Semidiscrete FT, $U_j^m = \int_{-\pi/\Delta x}^{\pi/\Delta x} e^{ijk\Delta x}\hat{U}^m \mathrm{d}x$

NLA

Choleskys, for sq symm pos def, $A = RR^T = LL^T$ where R, L are upper, lower.

LU. $PA = LU$ where L is lower, U is upper, P is permutation. The way we find this is we take the first entry, make it non-zero and then write the matrix as the sum of two vectors (the first column divided by a) times by (the first row) add the remainder. Then repeat. Then we get lower and triangular matrices.

Schurs, $A = UTU^*$, where U is unitary and T upper triangular. To construct, find an eigenvalue, λ of A and evec v_1. Let $V_\perp$ be the orthogonal complement of v_1 and let $U_1 = [v_1 V_\perp]$ and now we see,

$$AU_1 = U_1 \begin{pmatrix} * & 0 & \dots & 0 \\ 0 & * & *** & * \\ \vdots & * & *** & * \\ 0 & * & *** & * \end{pmatrix}$$

and then repeat on the $*$ matrix.

SVD For any matrix, $A = U\Sigma V^T$ where U, V are orthog and Σ is diag with σ_i. To find SVD, σ_i are the root of the evals of A^TA. Find an orthonormal set of evecs of A^TA and then order then with σ_i to produce U and Σ, then $U = \frac{1}{\sigma_i}Av_i$. Full SVD, $A = \begin{pmatrix} V & V_\perp \end{pmatrix} \begin{pmatrix} \Sigma \\ 0 \end{pmatrix} V^T$

QR, $A = QR$ where Q is orthog and R is upper triangular. To construct,

Gram Schmidtt, take $A = [a_1\, a_2 \dots a_N]$. Let $q_1 = \frac{a_1}{\|a_1\|}$ and then, $\hat{q}_j = a_j - \sum_{i=1}^{j-1}(q_i cdq_i)a_j$ and then, $q_j = \frac{\hat{q}_j}{\|\hat{q}_j\|}$. To see QR, we have $r_{ij} = q_i^T a_j$ and $\|r_{jj}\| = \left\| a_j - \sum_{i=1}^{j-1} a_{ij} q_i \right\|$. So $a_1 = r_{11}q_1$ and $a_j = \sum_{i=1}^{j} r_{ij} q_i$.

Householder You can use householder reflectors. They just zero the first column apart from $a_{11} = \|a_{11}\|$. Consider smaller inner matrix and repeat.

CR Theorem For sq symm matrix,

$$\lambda_i(A) = \max_{\dim S = i} \min_{x \in S} \frac{x^T A x}{x^T x}$$

For any other matrix,

$$\max_{\dim S = i} \min_{x \in S} \frac{\|Ax\|_2}{\|x\|_2} = \max_{\dim S = i} \min_{\|y\|=1} \|AQy\|$$

$\frac{\|Ax\|_2}{\|x\|_2}$ is how much a matrix stretches or shrinks a vector. The $\min_{x \in S}$ over these values for a given subspace S is equivalent to $\|AQy\| = \left\|U\Sigma V^T y\right\| = \left\|\Sigma V^T y\right\|$ where $V = S$ since $U\Sigma V^T$ is an SVD of AQ. Then, the smallest value attained for $\left\|\Sigma V^T y\right\|$ with $\|y\| = 1$ is $\sigma_{\min}$ of AQ. Then we have $\max_{\dim S = i} \sigma_{\min}(AQ)$. Since we know that Q has a rank of i (from $\dim S = i$), it must therefore have exactly i nonzero singular values, so $\sigma_{\min}(AQ) = \sigma_i(AQ)$. Multiplication by an orthogonal matrix Q does not change the singular values, so $\sigma_i(AQ) = \sigma_i(A)$. Because we are restricting the subspace S by the use of Q, we have effectively removed the need for the max and obtained the result.