Year 4 — Numerical Solutions of PDEs

Based on lectures by Prof. Endre Süli
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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine (especially the typos!).

Contents

1 Introduction

The basic idea is very simple. Suppose that y is differentiable at $x \in \mathbb{R}$ then,

$$y' = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}$$

and so,

$$\frac{y(x+h) - y(x)}{h} = y'(x) + o(1) \quad \text{as } h \to 0$$

This then motivates,

$$y'(x) \approx \frac{y(x+h) - y(x)}{h}$$

and further if y' is differentiable,

$$y''(x) \approx \frac{y(x+h) - 2y(x) + y(x-h)}{h^2} \quad \text{as } h \to 0.$$

To check if this is a good approximation we use Taylor series and see what the other terms are (apart from y' and y'') and then check these terms go to zero.
Euler's Method Given $y'(x) = f(x, y(x))$ subject to $y(x_0) = y_0$. We write,

$$\frac{y(x_k + h) - y(x_k)}{h} \approx f(x_k, y(x_k)) \quad y(x_0) = y_0 \quad x_k = x_0 + kh \quad k \in \mathbb{Z}$$

2 Measuring Smoothness

2.1 Function Spaces

The accuracy of a numerical method for the aproximate solution of PDEs depends on its ability to capture the important feature of the analytic solution. One such feature is smoothness. To do this, we need some function spaces,

- $C(\Omega)$, Continuous functions
- $L_p(\Omega)$, Integrable Functions
- $H^k(\Omega)$, Sobolev spaces.

Notation. We let $\mathbb{N}$ be the nonnegative integers, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ is called a multi-index. The nonnegative integer, $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We let,

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$$

Definition 2.1 ($C^k(\Omega)$)**.** Let Ω be an open set in $\mathbb{R}^n$, and let $k \in \mathbb{N}$. WE denote $C^k(\Omega)$ the set of all continuous real-valued functions defined on Ω st, $D^\alpha u$ is continupus on Ω for all $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $|\alpha| \leq k$.

Definition 2.2 ($C^k(\Omega)$)**.** Assuming that Ω is a bounded open set, $C^k(\overline{\Omega})$ will denote the set of all u in $C^k(\overline{\Omega})$ st $D^\alpha u$ can be extended from Ω to a continuous function on $\overline{\Omega}$, the closure of the set Ω for all α with $|\alpha| \leq k$.

The linear space $C^k(\overline{\Omega})$ can then be equipped with the norm,

$$\|u\|_{C^k(\overline{\Omega})} := \sum_{|a| \leq k} \sup_{x \in \Omega} |D^\alpha u(x)|$$

Further, when $k = 0$ we just write $C(\Omega)$.
The support, $\operatorname{supp} u$ of a continuous function u on Ω is defined as the closure of the set,

$$\{x \in \Omega : u(x) \neq 0\}$$

In other words, $\operatorname{supp} u$ is the smallest closed subset of Ω such that $u = 0$ in $\Omega \setminus u$,
We denote $C_0^k(\Omega)$ the set of all $u \in C^k(\Omega)$ such that $\operatorname{supp} u \subset \Omega$ and $\operatorname{supp} u$ is bounded. Let,

$$C_0^\infty(\Omega) = \bigcap_{k \geq 0} C_0^k(\Omega)$$

2.2 Spaces of integrable functions

Let $p \in \mathbb{R}$, $p \geq 1$, we denote $L_p(\Omega)$ the set of all real-valued functions defined on Ω such that,

$$\int_\Omega (|u(x)|^p)^{\frac{1}{p}} \, \mathrm{d}x$$

Functions which equal almost everywhere on Ω are identified with eachother. [1] The norm is,

$$\|u\|_{L_p(\Omega)} = \left(\int_\Omega |u(x)|^2\right)^{\frac{1}{2}}$$

[1]This is equivalent to saying they are equal except on a set of measure zero. A subset of $\mathbb{R}^n$ is said to be a set of measure zero if it can be contained in the union of countably many open balls of arbitrarily small volume.

More specifically, we will focus on L_2,

$$(u, v) := \int_\Omega u(x)v(x)\mathrm{d}x$$

Lemma 2.3 (Cauchy-Schwartz)**.** Let $u, v \in L_2(\Omega)$, then

$$|(u, v)| \leq \|u\|_{L_2(\Omega)} \|v\|_{L_2(\Omega)}$$

Remark. The space $L_2(\Omega)$ equipped with the inner product $(\cdot, \cdot)$ is a Hilbert Space. This implies why the Sobolev spaces are denoted $H^k(\Omega)$

2.3 Sobolev Spaces

Consider,

$$-u''(x) = f(x)$$
$$u(a) = A, \quad u(b) = B$$

If we say that $u \in \mathcal{C}^2([a, b])$, then in the classical sense then this solution doesn't have a solution. However consider some v, that vanishes at the endpoints. Then,

$$\int_a^b u'(x)v'(x)\mathrm{d}x = \int_a^b f(x)v(x)\mathrm{d}x$$

and then this is a new definition of a solution. This is a weak solution.

Suppose that u is locally integrable[2] on Ω for each bounded open set ω, with $\overline{\omega} \subseteq \Omega$. Suppose also that there exists some w_α locally integrable on Ω such that,

$$\int_\Omega w_\alpha(x)v(x)\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega u(x)D^\alpha v(x) \quad \forall v \in C_0^\infty(\Omega).$$

Then w_α is called the weak derivative of u (of order $|\alpha|$) and we write $w_\alpha = D^\alpha u(x)$.

Example. Let $\Omega = \mathbb{R}$ and let $u(x) = (1 - |x|)_+$ $(u(x) > 0)$. Clearly u isn't differentiable at $0, \pm 1$. However, u is locally differentable and so will possibly have a weak derivative,

$$\begin{aligned}
\int_{-\infty}^{\infty} uv'dx = \int_{-\infty}^{\infty} (1 - |x|)_+ v'dx &= \int_{-1}^{1} (1 - |x|)v'(x)dx \\
&= \int_{-1}^{0} (1 + x)v'dx + \int_0^1 (1 - x)v'(x)dx \\
&= \int_{-1}^{0} (-1)v(x)dx + \int_0^1 v(x)dx \\
&= -\infty_{-\infty}^{\infty} w(x)v(x)dx
\end{aligned}$$

where,

$$w = \begin{cases} 0, & x < -1 \\ 1, & x \in (-1, 0) \\ -1, & x \in (0, 1) \\ 0, & x > 1 \end{cases}$$

[2] This means that a function is integrable on every subset of ω.

Let k be a nonnegative integer. We define,

$$H^k(\Omega) := \{u \in L_2(\Omega) : D^\alpha u \in L_2(\Omega), |\alpha| \leq k\}$$

$H^k(\Omega)$ is called a Sobolev space of order k; it is equipped with the Sobolev norm,

$$\|u\|_{H^k(\Omega)} = \left(\sum_{|a|\leq k} \|D^\alpha u\|^2_{L_2(\Omega)} \right)^{\frac{1}{2}}$$

and inner product,

$$(u, v)_{H^k(\Omega)} := \sum_{|a|\leq k} (D^\alpha u, D^\alpha v)$$

With this inner product, $H^k(\Omega)$ is a hilbert space. Letting,

$$|u|_{H^k(\Omega)} := \left(\sum_{|a|=k} \|D^\alpha u\|^2_{L_2(\Omega)} \right)$$

and we can write,

$$\|u\|_{H^K(\Omega)} = \left(\sum_{j=0}^{k} |u|^2_{H^j(\Omega)} \right)^{\frac{1}{2}}$$

and this is the Sobolev seminorm.
We define a special sobolev space,

$$H^1_0(\Omega) := \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega\}.$$

In Lesbegue you can change points on measure zero, the boundary is also measure zero. The trace theorems tell us what the implication of doing this happens.

Lemma 2.4 (Poincaré-Friedrichs Inequality)**.** Suppose that Ω is a bounded open set in $\mathbb{R}^n$ and let $u \in H^1_0(\Omega)$, then, there exists a positive constant $c_*(\Omega)$ independent of u, such that,

$$\int_\Omega u^2(x)dx \leq c_* \sum_{i=1}^{n} \int_\Omega |\partial_{x_i} u(x)|^2$$

. We shall the case for a square, that is, $\Omega = (a, b) \times (c, d)$. Then,

$$u(x, y) = u(a, y) + \int_a^x \partial_x u(\xi, y)d\xi = \int_a^x \partial_x u(\xi, y)d\xi \quad c < y < d$$

Then from the Cauchy Schwartz, we get,

$$\int_\Omega |u(x, y)|^2 \leq \frac{1}{2}(b - a)^2 \int_\Omega |\partial_x u(x, y)|^2 dxdy$$

and now we do it in the y direction,

$$\int_\Omega |u(x, y)|^2 \leq \frac{1}{2}(d - c)^2 \int_\Omega |\partial_y u(x, y)|^2 dxdy$$

Then we divide by the $(b - a)^2/2$ and $(d - c)^2/2$ and then just add them,

$$\int_\Omega u^2(x)dx \leq c_* \int_\Omega (|\partial_x u|^2 + \partial_y u|^2)dxdy$$

$\square$