Year 4 — Numerical Linear Algebra

Based on lectures by
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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine (especially the typos!).

Contents

1 Introduction

The goal of this course is to solve and understand problems using (usually non-square) matrices. The course will cover,

1. SVD. Singular Value Decomposition, see Linear Algebra (Year 2). $A = U\Sigma V^T$.
2. Linear systems $Ax = b$ and eigenvalue problems $Ax = \lambda x$. Least square problems, $\min_x \|Ax - b\|_2$.

There are going to be three classes of approaches,

- Direct Methods, classical approach. Not useful for very big matrices.
- Iterative solvers, work even if matrix sizes are larger.
- Randomised Algorithms, use some sense of randomisation in order to get an algorithm that works with high probability better than direct for very large matrices.

1.1 Why do we care?

When we want to solve a non-linear equation a linear system pops out. Motivation: minimising a function, $\min_x f(x)$ where $f : \mathbb{R}^n \to \mathbb{R}$, where we let n be large. One powerful way to find a minimum is to find a stationary point. So we find,

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \dots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} = 0$$

if we have a nice (convex) function, then if we have a stationary point, we have a minimiser. This boils down to, letting $F = \nabla f : \mathbb{R}^n \to \mathbb{R}^n$. We need to find zeros of this non-linear problem. Hence we now have a root-finding problem, and so we can use Newton's Method.

$$x_{\text{new}} = x_{\text{old}} - \mathcal{J}^{-1} F(x_{\text{old}})$$

and we see,

$$\mathcal{J}_{ij} = \frac{\partial F_i}{\partial x_j}$$

and then this is the hessian of f. This is then a linear system,

$$\mathcal{J} \Delta x = F(x_{\text{old}}).$$

NB! Here $A^* = \overline{A}$ (instead of $A^* = \overline{A}^T$) We are going to stay under real matrices because there are only two cases where the difference matters. Further, $m \geq n$ for most matrices in this course. For orthonormal matrices,

$$A^T A = I_n \qquad AA^T \neq I_m$$

1.2 Norms

We need norms to quantify how big matrices are. These help us when approximating matrices. Given some $\mathbf{x} \in \mathbb{R}^n$, we have

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

and 1-norm,

$$\|x\|_1 = |x_1| + \cdots + |x_n|$$

and the p-norm,

$$\|x\|_p = (|x_1|^p + \cdots + |x_p|^p)^{\frac{1}{p}}$$

and finally the ∞-norm,

$$\|x\|_\infty = \max_i |x_i|$$

Definition 1.1 (Norm)**.** A norm satisfies three axioms,

- $\|\alpha x\| = |\alpha| \|x\|$
- $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$
- $\|x + y\| \leq \|x\| + \|y\|$

Lemma 1.2 (Holder's Inequality)**.** For $p > q$,

$$\|x\|_p \leq \|x\|_q$$

Definition 1.3 (Unitarily Invariant)**.** If A is orthonormal, then $\|Ax\|_2 = \|x\|_2$.

Lemma 1.4 (Cauchy-Schwartz)**.** For any $x, y \in \mathbb{R}^n$,

$$|x^T y| \leq \|x\|_2 \|x\|_2$$

Proof. For any scalar c, $\|x - cy\|_2^2 = \|x\|_2^2 - 2cc^T y + \|y\|_2^2$. Now we complete the square and we get,

$$\begin{aligned}
\|x - cy\|_2^2 &= \|x\|_2^2 - 2cc^T y + \|y\|_2^2 \\
&= \|y\|_2^2 \left(c - \frac{x^T y}{\|y\|_2} \right)^2 + \|x\|_2^2 - \frac{(x^T y)^2}{\|y\|_2^2}
\end{aligned}$$

and so we now minimise by letting $c = \frac{x^T y}{\|y\|_2}$ and so we get Cauchy Schwartz. $\square$

Now for matrix norms. We have the p-norm,

$$\|A\|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \max_{\|x\|_p = 1} \frac{\|Ax\|_p}{\|x\|_p}.$$

The most important case is when $p = 2$, this is the spectrum norm. This is,

$$\|A\|_2 = \max_{\|x\|_2 = 1} \|Ax\|_2$$

Exercise. Show,

$$\begin{aligned}
\|A\|_1 &= \text{maximum column sum} \\
\|A\|_\infty &= \text{maximum row sum}
\end{aligned}$$

Definition 1.5 (Frobenius Norm)**.**

$$\|A\|_F = \sqrt{\sum_i \sum_j |A_{ij}|^2} = \sqrt{\mathbf{A}^T \mathbf{A}} = \sqrt{\operatorname{tr}(A^T A)}$$

where

$$\mathbf{A} = \begin{pmatrix} A_{11} \\ \vdots \\ A_{1n} \\ A_{21} \ldots \end{pmatrix}$$

Definition 1.6 (Trace Norm)**.**

$$\|A\|_* = \sum_{i=1}^{\min(m,n)} \sigma_i(A)$$

For most p-norms,

$$\|AB\|_p \leq \|A\|_p \|B\|_p$$

and for the Frobenius norm,

$$\begin{aligned}\|AB\|_F &\leq \|A\|_F \|B\|_F \\ &\leq \|A\|_2 \|B\|_F\end{aligned}$$

this is the subordinate property. Where this goes wrong is,

$$\|A\|_\infty = \max_{i,j} |A_{ij}|$$