1 Coupled Lattice Fields

I'm going to write a simple toy model to illustrate the approach to solving a system of equations to find the dispersion relation for some coupled fields on our lattice.

1.1 1D Chain

Let's start simple, with a one dimensional chain with some harmonic coupling between some arbitrary fields a_i and b_i that are defined on our one dimensional lattice of lattice spacing ℓ. The potential energy will be

$$V = \sum_i \frac{1}{2}A(a_i - a_{i+1})^2 + \frac{1}{2}B(b_i - b_{i+1})^2 + C(a_i - a_{i+1})(b_i - b_{i+1}) + \frac{1}{2}Ma_i^2.$$

The first two terms are the coupling between nearest neighboring lattice sites, the third term couples the a fields to the b fields, and the last term we'll see is what will give the a modes a mass, *i.e.* their frequencies don't go to zero at zero wavelength. Let's give the a particles a mass m and the b particles a mass μ. So the Lagrangian looks like

$$\mathcal{L} = \sum_i \frac{1}{2}m\dot{a}_i^2 + \frac{1}{2}\mu\dot{b}_i^2 - V.$$

Newton's equations then read:

$$m\ddot{a}_i = -2Aa_i + Aa_{i+1} + Aa_{i-1} - 2Cb_i + Cb_{i+1} + Cb_{i-1} - Ma_i.$$

$$m\ddot{b}_i = -2Ca_i + Ca_{i+1} + Ca_{i-1} - 2Bb_i + Bb_{i+1} + Bb_{i-1}.$$

Now we go to Fourier space to turn this difference equation into an algebraic equation. The transformation is:

$$a_j = \sum_j \tilde{a}(k)e^{-ik(j\ell)}, \qquad b_j = \sum_j \tilde{b}(k)e^{-ik(j\ell)}.$$

We put in these transforms in the above difference equations and exploit the orthonormality of the exponential to get equality term by term:

$$m\ddot{\tilde{a}} = -2A\tilde{a} + A\tilde{a}e^{-ik\ell} + A\tilde{a}e^{ik\ell} - 2C\tilde{b} + C\tilde{b}e^{-ik\ell} + C\tilde{b}e^{ik\ell} - M\tilde{a}.$$

$$\mu\ddot{\tilde{b}} = -2C\tilde{a} + C\tilde{a}e^{-ik\ell} + C\tilde{a}e^{ik\ell} - 2B\tilde{b} + B\tilde{b}e^{-ik\ell} + B\tilde{b}e^{ik\ell}.$$

We then look for time-harmonic solutions such that $\frac{d^2}{dt^2} \to -\omega^2$, and we finally have a matrix equation, setting $\sin(k\ell/2) = \gamma$:

$$\begin{pmatrix} m\omega^2 - 4\gamma^2 A - M & -4\gamma^2 C \\ -4\gamma^2 C & \mu\omega^2 - 4\gamma^2 B \end{pmatrix} \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = 0.$$

Now if we wanted to, we could put this in Mathematica and get the eigenmodes and corresponding dispersion relations. But let's just set $k = 0$ so $\gamma = 0$:

$$\begin{pmatrix} m\omega^2 - M & 0 \\ 0 & \mu\omega^2 \end{pmatrix} \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = 0.$$

The solution to this eigenvalue problem is $\omega^2 = 0$ and $\omega^2 = M/m$. The first case has eigenvector $\tilde{a} = 0$ and $\tilde{b} = b_0$, so this is a wave with zero frequency and zero wavelength with arbitrary amplitude, which corresponds to a global constant shift of the b field. This doesn't couple to the a field. However, for the other eigenvalue, $\omega = \sqrt{M/m}$ is a finite frequency with $k = 0$ and $\tilde{a} = a_0$ and $\tilde{b} = 0$. This is a global and spatially homogenous oscillation of the a field.

1.2 2D Spring Field

Now let's consider two coupled fields in two dimensions situated on a lattice. Suppose that we don't need a basis to characterize our unit cell (if we did, we'd expect some optical modes), and let the nearest neighbors to point α be given by the set of vectors $\{\mathbf{c}\}$.