1 — a) True, b) 2 Legendre transforms, c) all of the above

d) 4^{20}, e) i, ii, and iii

2a — Total length:

$$L = aN_a + bN_b, \qquad N = N_a + N_b.$$

Fixing L fixes N_a and N_b, but for a given L, there are

$$\Omega = \binom{N}{N_b} = \binom{N}{\frac{L-aN}{b-a}}$$

possible microstates.

The value of N_a that maximizes this is $N/2$, so $L = (a+b)N/2$.

2b — The entropy at a given L is

$$S = k_b \log \Omega(L) = k_B \log \binom{N}{\frac{L-aN}{b-a}}.$$

The Helmholtz free energy is just $-TS$, because there is no internal energy in this model, we consider only entropic effects.

$$A = -TS = -k_B T \log \binom{N}{\frac{L-aN}{b-a}}.$$

If you want, you can go to the canonical ensemble:

$$Q = \sum_\nu \omega_n e^{-\beta E_\nu}.$$

Here, $\omega_n = \Omega(L)$ and $E_\nu = 0$ for all states ν, so there's no sum over ν at all, because all the states have the same energy, and our degeneracy is $\Omega(L)$, so the canonical is the same as the microcanonical ensemble here:

$$A = -k_B T \log Q = -k_B T \log \Omega(L).$$

We use Stirling's approximation on $\binom{N}{N_b}$ and we get:

$$A = -k_B T \left(N \log N - N_b \log N_b - (N - N_b) \log(N - N_b) \right).$$

2c — From $dA = -SdT + JdL$, we see that

$$J = \left.\frac{\partial A}{\partial L}\right|_T.$$

and we have

$$\frac{\partial}{\partial L} = \frac{1}{b-a}\frac{\partial}{\partial N_a}.$$

So we get

$$J = -k_B T \frac{\partial}{\partial N_b}\left(-N_b \log N_b + (N - N_b)\log(N - N_b)\right).$$

$$J = \frac{k_B T}{b-a} \log \frac{N_b}{N - N_b}.$$

Notice that when $N_b > N/2$, when we have the polymer extended beyond its equilibrium length, $J > 0$, and when $N_b < N/2$, we have $J < 0$. This is the meaning of entropic elasticity.

3a — We're given

$$\left.\frac{\partial S}{\partial L}\right|_T < 0.$$

And from our fundamental relation

$$dE = TdS + JdL \rightarrow dA = dE - d(TS) = -SdT + JdL.$$

From this we can read off a Maxwell relation:

$$-\left.\frac{\partial S}{\partial L}\right|_T = \left.\frac{\partial J}{\partial T}\right|_L > 0.$$

3b — We can use the cyclic derivative rule here

$$\left.\frac{\partial T}{\partial L}\right|_S \left.\frac{\partial L}{\partial S}\right|_T \left.\frac{\partial S}{\partial T}\right|_L = -1.$$

$$\left.\frac{\partial T}{\partial L}\right|_S = -\left.\frac{\partial S}{\partial L}\right|_T \left(\frac{T}{C_L}\right) > 0.$$

4a — If we pin n particles to the surface, there are $\binom{N}{n}$ ways to arrange them on the surface. This also leave $N - n$ particles to roam free in the bulk. The number of ways to arrange *those* particles is $\binom{M}{N-n}$, so we have

$$\Omega = \binom{N}{n}\binom{M}{N-n}.$$

The entropy is

$$S = k_B \log \Omega.$$

4b — Here, it is *really hard* to work in the canonical ensemble. We stay in the microcanonical ensemble here. So we have

$$E = -\varepsilon n, \qquad \frac{1}{T} = \frac{\partial S}{\partial E}.$$

$$\frac{1}{T} = -\frac{1}{\varepsilon}\frac{\partial S}{\partial n}.$$

$$-\beta\varepsilon = \frac{\partial \log \Omega}{\partial n}.$$

So after doing all the Stirling expansion:

$$\frac{\partial \log \Omega}{\partial n} = \frac{\partial}{\partial n}\left(N\log N - n\log n - 2(N-n)\log(N-n) + M\log M - (M-N+n)\log(M-N+n)\right).$$

$$-\beta\varepsilon = \log\left(\frac{(N-n)^2}{n(M-N+n)}\right).$$

$$e^{-\beta\varepsilon} = \frac{(N-n)^2}{n(M-N+n)}.$$

It's really hard to invert this, I think, I haven't tried. But it's not very interesting if it is invertible.

4c — If $T \to 0$, then $\beta \to \infty$ and $e^{-\beta\varepsilon} \to 0$, so we have

$$(N-n)^2 = 0.$$

$$\frac{n}{N} = 1.$$

4d — Now we send $T \to \infty$, $\beta \to 0$, and $e^{-\beta\varepsilon} \to 1$,

$$(N-n)^2 = n(M-N+n).$$

$$\frac{n}{N} = \frac{N}{M+N}.$$

5a — The magnetic degree of freedom is just in the dipole, the orientation of the particle. The ideal gas partition function only cared about momentum and position, so this is a degree of freedom that isn't coupled at all to the ideal gas degrees of freedom, so I'll abbreviate those:

$$Q = Q_{\text{ideal}}Q_{\text{magnetic}} = Q_I Q_M.$$

$$Q_I = \frac{1}{N!}\left(\frac{V}{\Lambda^3}\right)^N.$$

The magnetic degree of freedom is quantized, and it only has six states:

$$Q_M = e^{\beta Bm} + 1 + 1 + 1 + 1 + e^{-\beta Bm} = e^{\beta Bm} + e^{-\beta Bm} + 4.$$

5b — Since the partition functions are multiplicatively decoupled, then the free energy separates into two additive terms:

$$A = A_I + A_M.$$

$$A = A_I - Nk_BT \log\left(e^{\beta Bm} + e^{-\beta Bm} + 4\right).$$

5c — Since the magnetic free energy doesn't depend on the volume, it won't contribute to the pressure:

$$P = -\left.\frac{\partial A}{\partial V}\right|_T = -\left.\frac{\partial A_I}{\partial V}\right|_T = \frac{Nk_BT}{V}.$$

5d — The internal energy also separates:

$$E = -\frac{\partial \log Q}{\partial \beta} = -\frac{\partial \log Q_I}{\partial \beta} - \frac{\partial \log Q_M}{\partial \beta}.$$

$$E = \frac{3}{2}\frac{N}{\beta} - NBm\frac{e^{\beta Bm} - e^{-\beta Bm}}{e^{\beta Bm} + e^{-\beta Bm} + 4}.$$