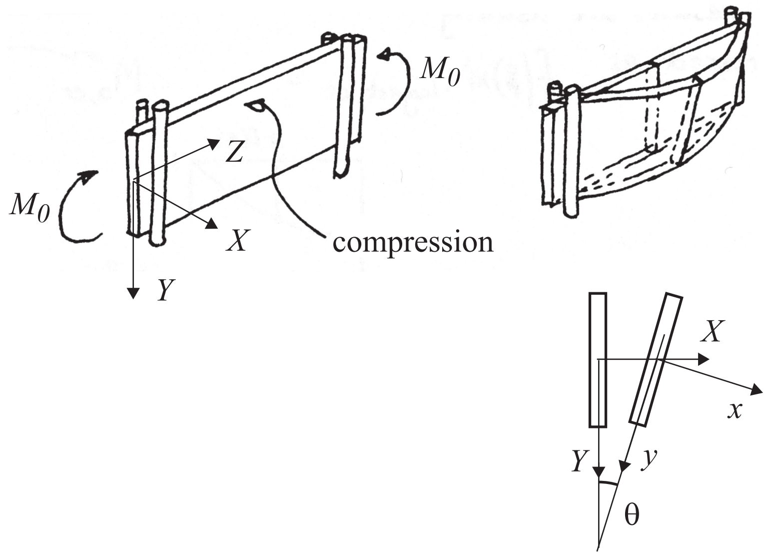
3.6.1 Rectangular section (no warping)

Consider a “simply supported” rectangular section (thin plank) beam loaded by equal and opposite in-plane bending couples, and supported in such a way that the ends are prevented from torsional rotation.

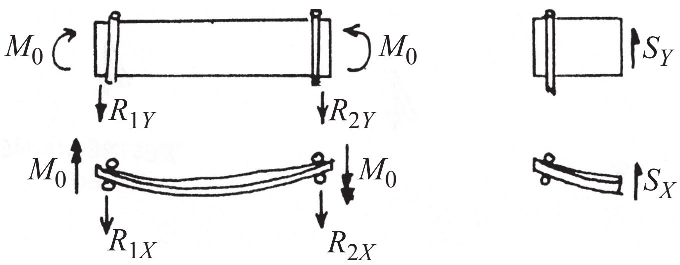


Note that X,Y,Z is a *global coordinate system* that is fixed; while x,y,z is a *local coordinate system* that moves with the cross-section.

Assume $I_{xx} >> I_{yy}$, and hence there are negligibly small vertical deflections prior to buckling.

Consider equilibrium *in the deformed shape*.

Force equilibrium:



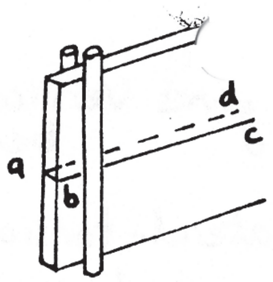
Reaction forces:

$$R_{1Y} = R_{2Y} = 0$$
$$R_{1X} = R_{2X} = 0$$

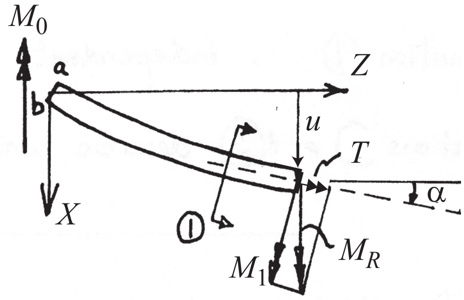
Shear forces:

$$S_X = S_Y = 0$$

Moment equilibrium:



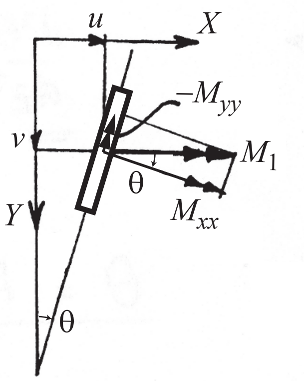
Top view (along Y-axis):



$$
\begin{aligned}
M_R &= M_0 \\
M_1 &= M_R \cos\alpha \approx M_0 \\
T &= M_R \sin\alpha \approx M_0 \alpha
\end{aligned}
$$

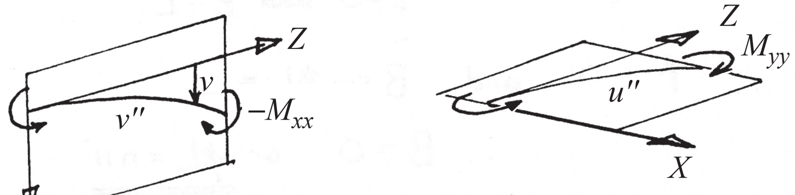
where $\alpha = \dfrac{du}{dz}$

View along direction 1:



$$
\begin{aligned}
M_{xx} &= M_1 \cos\theta \approx M_0 \\
M_{yy} &= -M_1 \sin\theta \approx -M_0 \theta
\end{aligned}
$$

Material law and compatibility:



u'' and v'' are +ve as shown

Consider the beam centroid plotted in the YZ and ZX planes. We have bending relationships (with particular +ve and -ve signs because of our right-handed coordinate set)

$$
\begin{aligned}
M_{xx} &= -EI_{xx} v'' \\
M_{yy} &= +EI_{yy} u''
\end{aligned}
$$

We also have a torsional relationship

$$T = GJ\frac{d\theta}{dz}$$

Combining equilibrium, compatibility and the material law: we obtain *one independent* equation relating to vertical bending:

$$M_0 = M_{xx} = -EI_{xx}v''$$

and *two coupled equations* relating to the combined lateral-torsional behaviour of the beam

$$-M_0\theta = M_{yy} = EI_{yy}u'' \qquad (*)$$

$$M_0u' = T = GJ\frac{d\theta}{dz} \qquad (\dagger)$$

Governing differential equation for lateral-torsional behaviour:
Differentiate $(\dagger)$ by Z (equivalently, z) and subsitute u'' from $(*)$ gives

$$\frac{d^2\theta}{dz^2} + \left(\frac{M_0^2}{GJEI_{yy}}\right)\theta = 0$$

General solution:

$$\theta = A\cos\beta Z + B\sin\beta Z$$

where $\beta = \dfrac{M_0}{\sqrt{GJEI_{yy}}}$.
The boundary conditions are:

$$\theta = 0 \quad \text{at } Z = 0 \text{ and } Z = L$$

hence

$$A = 0 \quad \text{and} \quad B\sin\beta L = 0$$

and so

$$B = 0 \quad \text{or} \quad \beta L = n\pi$$

The lowest critical moment has $n = 1$, and hence

$$M_{0,cr} = \frac{\pi}{L}\sqrt{GJEI_{yy}}$$

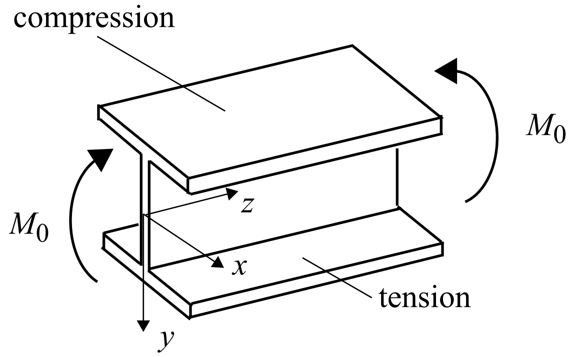
Note that:

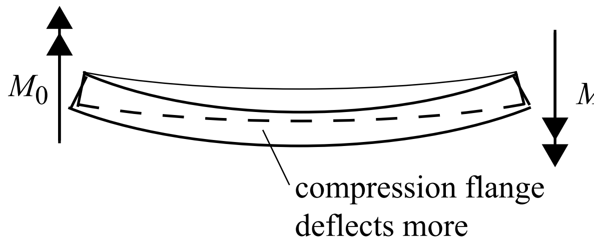
- $M_{0,cr}$ depends on both the torsional rigidity GJ *and* the minor axis flexural rigidity EI_{yy}
- $M_{0,cr} \propto \dfrac{1}{L}$ whereas $P_E \propto \dfrac{1}{L^2}$

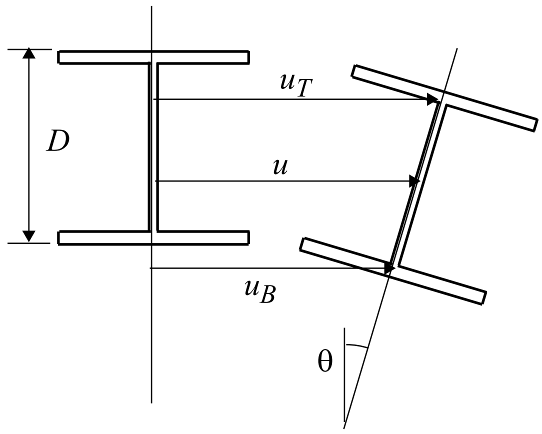
Try Question 6, Examples Paper 4

3.6.2 I-beam (including warping)

A pure bending moment M_0 about the major axis can cause lateral-torsional buckling about the minor axis of bending of the beam.







It was shown in the first half of 3D4 that the torque carried by the beam, T, is related to the section rotation θ by

$$T = GJ\theta' - E\Gamma\theta'''$$

where Γ is the *restrained warping torsion constant*.

Following through all of the previous lateral-torsional buckling analysis gives

$$E\Gamma\frac{d^4\theta}{dz^4} - GJ\frac{d^2\theta}{dz^2} - \frac{M_0^2}{EI_{yy}}\theta = 0$$

giving a **Governing Differential Equation:**

$$\theta'''' - 2\alpha\theta'' - \beta\theta = 0$$

where α is a structural parameter, and β is a measure of the load,

$$\alpha = \frac{GJ}{2E\Gamma} \quad \text{and} \quad \beta = \frac{M_0^2}{E^2 I_{yy}\Gamma}$$

The **General Solution** is

$$\theta = A\cos k_1 z + B\sin k_1 z + C\cosh k_2 z + D\sinh k_2 z$$

where

$$k_1^2 = -\alpha + \sqrt{\alpha^2+\beta}$$
$$k_2^2 = +\alpha + \sqrt{\alpha^2+\beta}$$

Imposing the boundary conditions ($\theta = 0$, $\theta'' = 0$ at $x = 0$, $x = L$), we find that all terms vanish except $B\sin k_1 z$. Thus, either $B = 0$ (trivial solution) or $k_1 L = n\pi$, which gives the critical buckling condition for the case $n = 1$. Hence,

$$\sqrt{\alpha^2+\beta} - \alpha = \frac{\pi^2}{L^2} (= k_1^2)$$

from which we obtain

$$\beta = 2\alpha\frac{\pi^2}{L^2} + \frac{\pi^4}{L^4}$$

or

$$M_{0,cr}^2 = \left(\frac{\pi}{L}\sqrt{GJEI_{yy}}\right)^2 + \left(\frac{\pi^2}{L^2}E\sqrt{I_{yy}\Gamma}\right)^2$$
$$M_{0,cr} = \frac{\pi}{L}\sqrt{GJEI_{yy}}\sqrt{1+\frac{\pi^2}{L^2}\frac{E\Gamma}{GJ}}$$

The last term is a factor that allows for warping, and typically has a value of about 1.25 for $L/D \approx 20$ for an I-section. For long spans, the additional warping term is of less importance.

For an I-beam the restrained warping term comes from the requirement to bend the flanges about their major axis, and is given by

$$\Gamma = \frac{I_f D^2}{2} \approx \frac{I_y D^2}{4}$$

where I_f is the second moment of area of a single flange about its major axis. Because the web is thin and at the neutral axis for lateral bending, we approximate $I_f \approx I_y/2$.

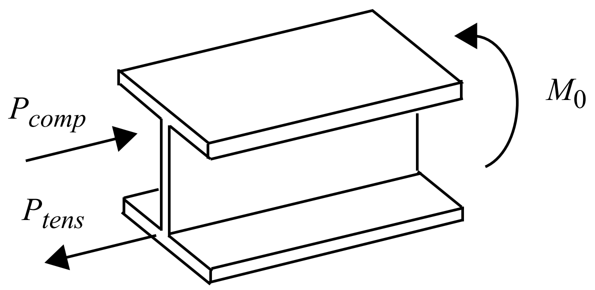
For short spans, the warping correction can dominate. In that case,

$$M_{0,cr} = \frac{\pi}{L}\sqrt{GJ}\sqrt{EI_{yy}}\left(\frac{\pi}{L}\frac{\sqrt{E\Gamma}}{\sqrt{GJ}}\right) = \frac{\pi^2}{L^2}E\sqrt{I_{yy}\Gamma}$$

Substituting $\Gamma = I_{yy}D^2/4 = I_f D^2/2$

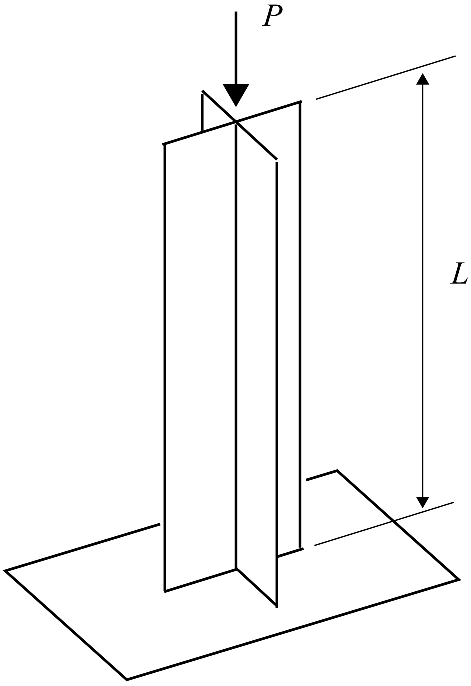
$$\frac{M_{0,cr}}{D} = \frac{\pi^2}{L^2}EI_f$$

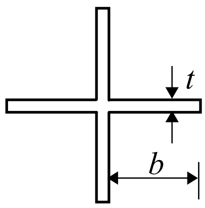
Note: Force in compression flange is $P_{comp} = M_0/D$, and in this case the equation reduces to the Euler buckling of the compression flange.



3.6.3 Axial-torsional buckling

Consider a cruciform column, fixed at the base and free at the top.





This has an open section and hence low torsional stiffness

$$J = 4 \times \frac{1}{3} b t^3$$

Consider a mode of buckling in which the top end rotates through an angle θ relative to the bottom end, with the entire column in a state of uniform shear. An analysis of this buckling mode leads to the following expression for the *critical stress for torsional buckling*:

$$\sigma_{cr,T} = G \left(\frac{t}{b} \right)^2$$

Note that this does not depend on the length L.

Try Question 7, Examples Paper 4