Module 3D4: Structural Analysis and Stability
— Examples Paper 2 —

Plane elastic structures loaded out of plane

1. Figure 1 shows a curved *cantilever* in a horizontal plane, which is connected to a rigid support at C. It takes the form of a quadrant of a circle of radius R, and has a uniform flexural stiffness EI and uniform torsional stiffness GJ. A vertical force V is applied at the tip.

   Use virtual work to determine

   (a) the vertical deflection of the tip,

   (b) and the rotation θ of the tip about OB.

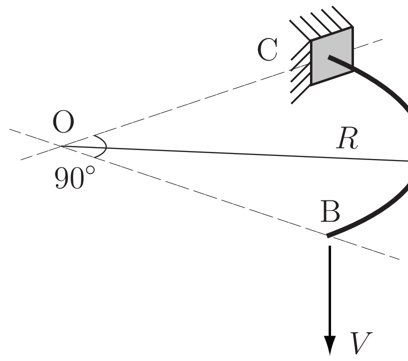

Figure 1:

2. Figure 2 shows a curved beam forming a semicircle in a horizontal plane and connected to rigid supports at A and C. Find the moment and torque reactions at A and C when a vertical load W is applied at B. Show that these reactions are independent of EI/GJ.

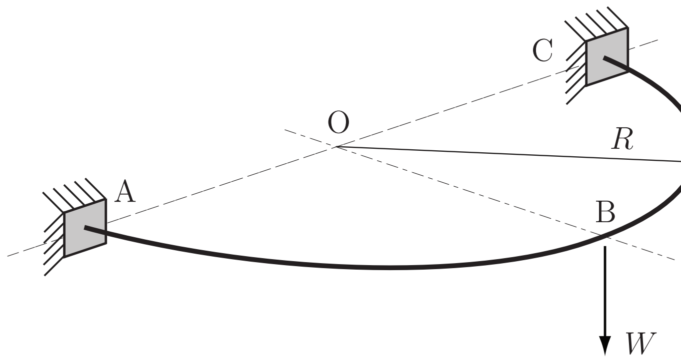

Figure 2:

   Note: This question can be done either (a) by extending the work of question 1 by considering a couple C about OB, and adapting the results to the semicircle problem; or (b), perhaps more interestingly, by considering the semicircle as a redundant structure and using virtual work.

3. Figure 3 shows a propped cranked cantilever in a horizontal plane. A vertical force of W is applied where the two beams rigidly connect. The flexural stiffness of both beams is EI and their torsional stiffness is $GJ = 2EI$. Determine the bending moment at the built-in support.

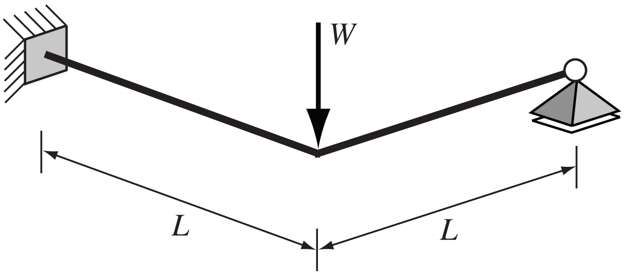

Figure 3:

Reciprocal theorem and influence lines

4. The uniform elastic beam shown in Figure 4 is simply supported at A and C. Find the vertical displacement at B due to the application of the load W at D by using the reciprocal theorem together with deflection coefficients taken from the Data Book.

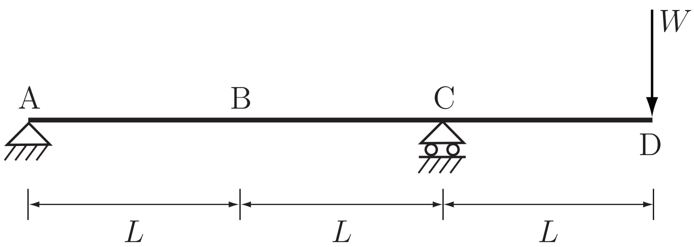

Figure 4:

5. The two-span beam shown in Figure 5 is clamped at A and simply supported at B and C. It is loaded on right span by a distributed load w. Without doing any numerical computations

   (a) sketch the bending moment and shear force diagrams while marking their salient features,

   (b) and sketch the influence lines for the support reaction at C and the bending moment at A due to a unit vertical load travelling across the beam.

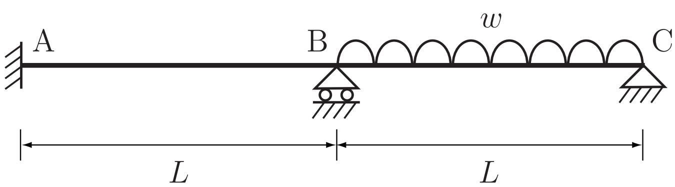

Figure 5:

6. The end A of the beam in question 4 is now clamped. Use the reciprocal theorem to sketch the influence line for bending moment at B due to a unit vertical load travelling across the beam. Determine the influence line ordinates at B and D.

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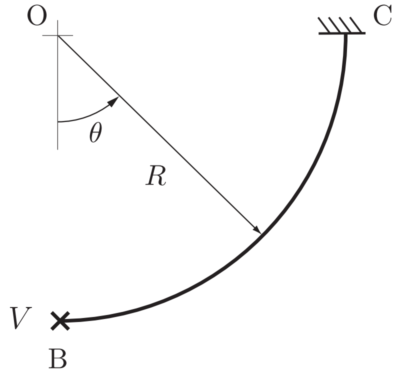
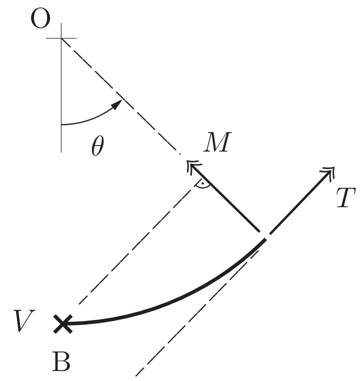
— Crib for Examples Paper 2 —

1. Plan view of the cantilever.





Bending moment: $M = VR\sin\theta$

Torque: $T = VR(1-\cos\theta)$

Real curvature: $\kappa = \dfrac{M}{EI} = \dfrac{VR\sin\theta}{EI}$

Real rate of twist: $\theta' = \dfrac{T}{GJ} = \dfrac{VR(1-\cos\theta)}{GJ}$

(a) Apply unit load at tip.

Virtual moment: $M = R\sin\theta$ Virtual torque: $T = R(1-\cos\theta)$

Virtual work:

$$1\cdot\delta = \int M\kappa\,\mathrm{d}s + \int T\theta'\,\mathrm{d}s$$

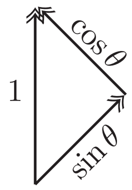
$$\delta = \int_0^{\pi/2} R\sin\theta\frac{VR\sin\theta}{EI}R\,\mathrm{d}\theta + \int_0^{\pi/2}\frac{VR(1-\cos\theta)}{GJ}R(1-\cos\theta)R\,\mathrm{d}\theta$$

$$\delta = VR^3\left(\frac{\pi}{4EI} + \frac{3\pi/4-2}{GJ}\right)$$

(b) Consider a unit couple applied about OB.

$$M = 1 \cdot \cos\theta$$

$$T = 1 \cdot \sin\theta$$



Rotation β

$$1 \cdot \beta = \int_0^{\pi/2} \cos\theta \frac{VR\sin\theta}{EI} R\,\mathrm{d}\theta + \int_0^{\pi/2} \sin\theta \frac{VR(1-\cos\theta)}{GJ} R\,\mathrm{d}\theta$$

$$\Longrightarrow \beta = \frac{VR^2}{2}\left(\frac{1}{EI} + \frac{1}{GJ}\right)$$

2. **Approach a)**

- The rotation at node B has to be zero.
- Assume that a real couple C acts at B.

$\Longrightarrow$ Real moment: $C\cos\theta$
Real torque: $C\sin\theta$

Use virtual couple as in Question 1 b).

$$\beta = CR\frac{\pi}{4}\left(\frac{1}{EI} + \frac{1}{GJ}\right)$$

For the system with the couple C plus the vertical force W.

$$\frac{WR^2}{4}\left(\frac{1}{EI} + \frac{1}{GJ}\right) + CR\frac{\pi}{4}\left(\frac{1}{EI} + \frac{1}{GJ}\right) = 0$$

$$\Longrightarrow \quad C = -\frac{WR}{\pi}$$

At support C:

$$M = \frac{WR}{2} \qquad T = WR\left(\frac{1}{2} - \frac{1}{\pi}\right)$$

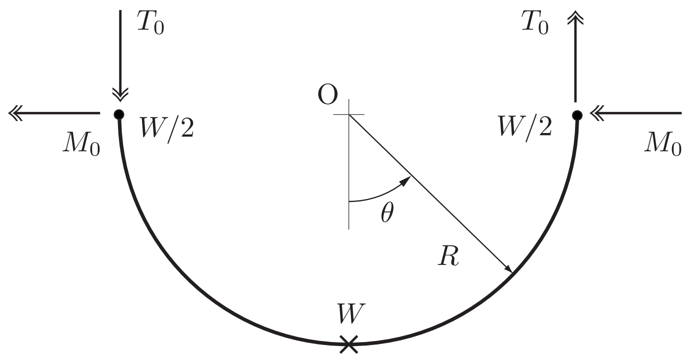
Approach b) The system has one redundancy T_0.

Moments about AC give $M_0 = WR/2$.

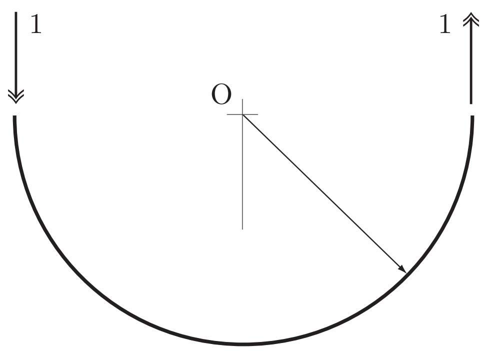
Real system

$$EI\kappa = -\frac{WR}{2}\cos\theta + M_0\sin\theta + T_0\cos\theta$$

$$GJ\theta' = T = \frac{WR}{2}(1-\sin\theta) - M_0\cos\theta + T_0\sin\theta$$



Real system



Virtual system

Virtual system

$M = \cos\theta \qquad T = \sin\theta$

Principle of virtual work

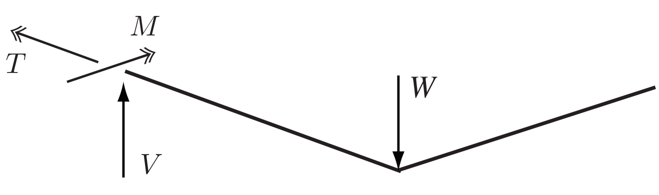
$$1 \cdot 0 = \int M\kappa \, \mathrm{d}s + \int T\theta' ds$$

$$0 = \frac{2}{EI}\int_0^{\pi/2} \cos\theta \left(\frac{WR}{2}(\sin\theta - \cos\theta) + T_0\cos\theta\right) R\,\mathrm{d}\theta$$
$$+ \frac{2}{GJ}\int_0^{\pi/2} \sin\theta \left(\frac{WR}{2}(1 - \sin\theta - \cos\theta) + T_0\sin\theta\right) R\,\mathrm{d}\theta$$

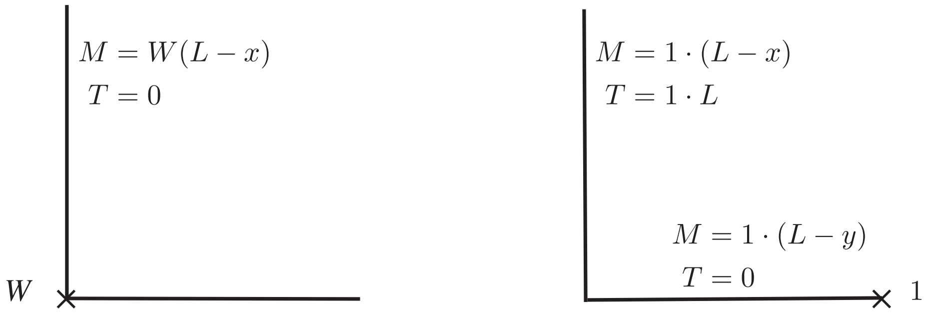
$$0 = \frac{1}{EI}\left(\frac{WR}{2}\left(\frac{1}{2} - \frac{\pi}{4}\right) + T_0\frac{\pi}{4}\right) + \frac{1}{GJ}\left(\frac{WR}{2}\left(1 - \frac{\pi}{4} - \frac{1}{2}\right) + T_0\frac{\pi}{4}\right)$$

$$\Longrightarrow T = WR\left(\frac{1}{2} - \frac{1}{\pi}\right)$$

3. The non-zero boundary reactions at the built-in support are V, M and T.



The degree of redundancy is one. We can, for instance, remove the roller support.



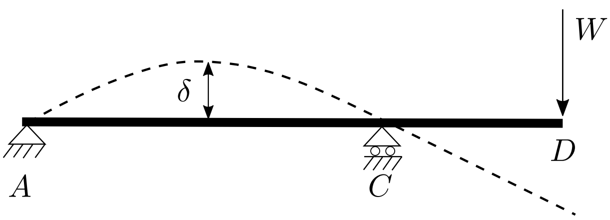
Tip deflection using the principle of virtual work.

$$\delta_0 = \frac{1}{EI}\int_0^L W(L-x)^2 \,\mathrm{d}x = \frac{1}{3EI}WL^3$$

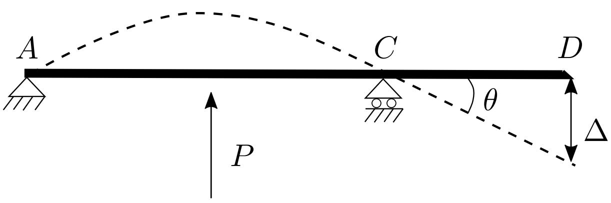
Tip deflection due to a unit load at the tip.

$$\delta_1 = \frac{1}{EI}\int_0^L (L-x)^2 \,\mathrm{d}x + \frac{1}{EI}\int_0^L (L-y)^2 \,\mathrm{d}y + \frac{1}{GJ}\int_0^L L^2 \,\mathrm{d}x$$
$$= \frac{1}{EI}\frac{1}{3}L^3 + \frac{1}{EI}\frac{1}{3}L^3 + \frac{1}{GJ}L^3 = \frac{2}{3}\frac{L^3}{EI} + \frac{L^3}{GJ} = \frac{7}{6}\frac{L^3}{EI}$$
$$\Longrightarrow X = \frac{1}{3}WL^3 \cdot \frac{6}{7L^3} = \frac{2}{7}W$$
$$\Longrightarrow M = WL - \frac{2}{7}WL = \frac{}{5}7WL$$

4. Load case (a)



Load case (b)



From the Databook

$$\Delta = \theta \cdot L = \frac{P(2L)^2}{16EI} \cdot L = \frac{PL^3}{4EI}$$

Reciprocal theorem

$$W \cdot \Delta = P \cdot \delta \quad \Longrightarrow \quad \delta = \frac{WL^3}{4EI}$$