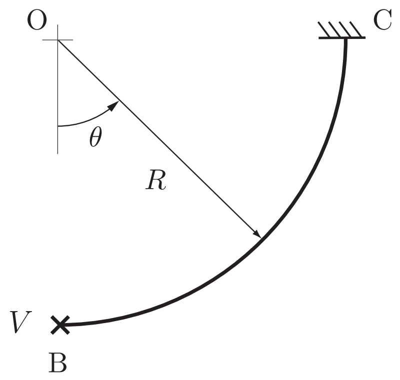
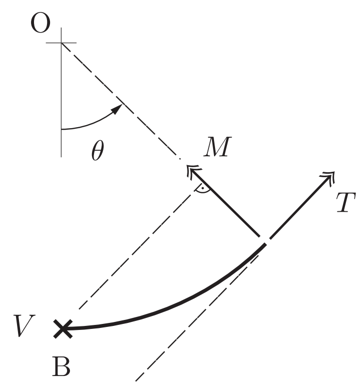
Module 3D4: Structural Analysis and Stability
— Crib for Examples Paper 2 —

1. Plan view of the cantilever.





Bending moment: $M = VR\sin\theta$

Torque: $T = VR(1-\cos\theta)$

Real curvature: $\kappa = \dfrac{M}{EI} = \dfrac{VR\sin\theta}{EI}$

Real rate of twist: $\theta' = \dfrac{T}{GJ} = \dfrac{VR(1-\cos\theta)}{GJ}$

(a) Apply unit load at tip.

Virtual moment: $M = R\sin\theta$ Virtual torque: $T = R(1-\cos\theta)$

Virtual work:

$$1 \cdot \delta = \int M\kappa \,\mathrm{d}s + \int T\theta' \,\mathrm{d}s$$

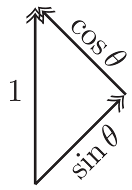
$$\delta = \int_0^{\pi/2} R\sin\theta \frac{VR\sin\theta}{EI} R\,\mathrm{d}\theta + \int_0^{\pi/2} \frac{VR(1-\cos\theta)}{GJ} R(1-\cos\theta) R\,\mathrm{d}\theta$$

$$\delta = VR^3\left(\frac{\pi}{4EI} + \frac{3\pi/4 - 2}{GJ}\right)$$

(b) Consider a unit couple applied about OB.

$$M = 1 \cdot \cos\theta$$

$$T = 1 \cdot \sin\theta$$



Rotation β

$$1 \cdot \beta = \int_0^{\pi/2} \cos\theta \frac{VR\sin\theta}{EI} R\,\mathrm{d}\theta + \int_0^{\pi/2} \sin\theta \frac{VR(1-\cos\theta)}{GJ} R\,\mathrm{d}\theta$$

$$\Longrightarrow \beta = \frac{VR^2}{2}\left(\frac{1}{EI} + \frac{1}{GJ}\right)$$

2. **Approach a)**

- The rotation at node B has to be zero.
- Assume that a real couple C acts at B.

$\Longrightarrow$ Real moment: $C\cos\theta$
Real torque: $C\sin\theta$

Use virtual couple as in Question 1 b).

$$\beta = CR\frac{\pi}{4}\left(\frac{1}{EI} + \frac{1}{GJ}\right)$$

For the system with the couple C plus the vertical force W.

$$\frac{WR^2}{4}\left(\frac{1}{EI} + \frac{1}{GJ}\right) + CR\frac{\pi}{4}\left(\frac{1}{EI} + \frac{1}{GJ}\right) = 0$$

$$\Longrightarrow \quad C = -\frac{WR}{\pi}$$

At support C:

$$M = \frac{WR}{2} \qquad T = WR\left(\frac{1}{2} - \frac{1}{\pi}\right)$$

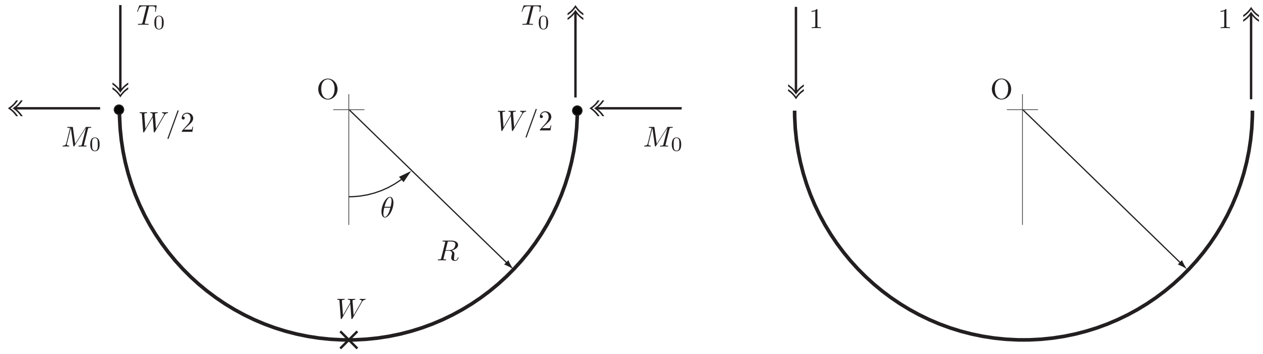
Approach b) The system has one redundancy T_0.

Moments about AC give $M_0 = WR/2$.

<u>Real system</u>

$$EI\kappa = -\frac{WR}{2}\cos\theta + M_0\sin\theta + T_0\cos\theta$$

$$GJ\theta' = T = \frac{WR}{2}(1-\sin\theta) - M_0\cos\theta + T_0\sin\theta$$



Real system   Virtual system

<u>Virtual system</u>

$M = \cos\theta \qquad T = \sin\theta$

<u>Principle of virtual work</u>

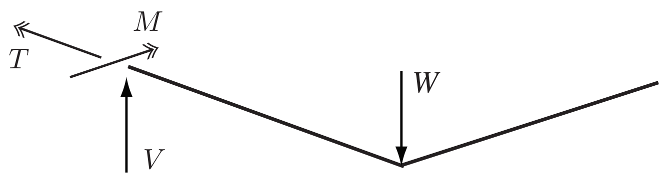
$$1 \cdot 0 = \int M\kappa \, \mathrm{d}s + \int T\theta' ds$$

$$0 = \frac{2}{EI}\int_0^{\pi/2} \cos\theta \left(\frac{WR}{2}(\sin\theta - \cos\theta) + T_0 \cos\theta \right) R \, \mathrm{d}\theta$$
$$+ \frac{2}{GJ}\int_0^{\pi/2} \sin\theta \left(\frac{WR}{2}(1 - \sin\theta - \cos\theta) + T_0 \sin\theta \right) R \, \mathrm{d}\theta$$

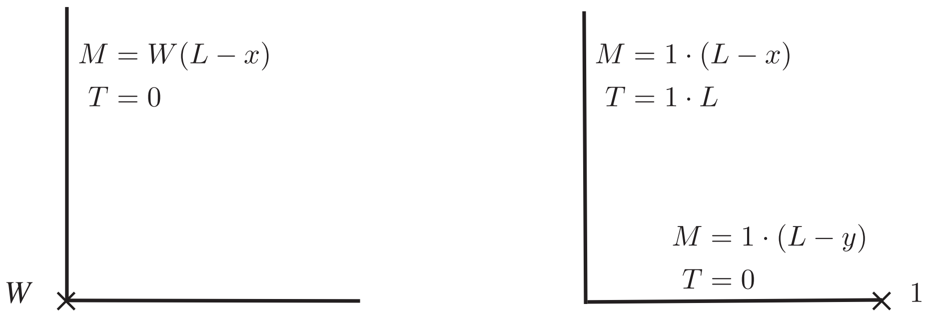
$$0 = \frac{1}{EI}\left(\frac{WR}{2}\left(\frac{1}{2} - \frac{\pi}{4}\right) + T_0\frac{\pi}{4} \right) + \frac{1}{GJ}\left(\frac{WR}{2}\left(1 - \frac{\pi}{4} - \frac{1}{2}\right) + T_0\frac{\pi}{4} \right)$$

$$\Longrightarrow T = WR\left(\frac{1}{2} - \frac{1}{\pi}\right)$$

3. The non-zero boundary reactions at the built-in support are V, M and T.



The degree of redundancy is one. We can, for instance, remove the roller support.



Tip deflection using the principle of virtual work.

$$\delta_0 = \frac{1}{EI}\int_0^L W(L-x)^2\,\mathrm{d}x = \frac{1}{3EI}WL^3$$

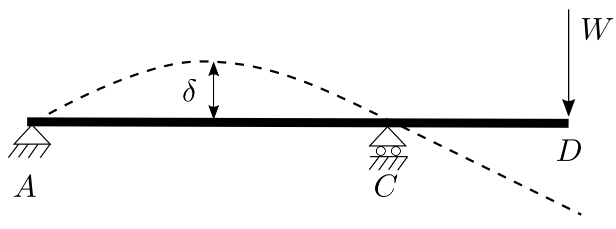
Tip deflection due to a unit load at the tip.

$$\begin{aligned}\delta_1 &= \frac{1}{EI}\int_0^L (L-x)^2\,\mathrm{d}x + \frac{1}{EI}\int_0^L (L-y)^2\,\mathrm{d}y + \frac{1}{GJ}\int_0^L L^2\,\mathrm{d}x \\ &= \frac{1}{EI}\frac{1}{3}L^3 + \frac{1}{EI}\frac{1}{3}L^3 + \frac{1}{GJ}L^3 = \frac{2}{3}\frac{L^3}{EI} + \frac{L^3}{GJ} = \frac{7}{6}\frac{L^3}{EI}\end{aligned}$$

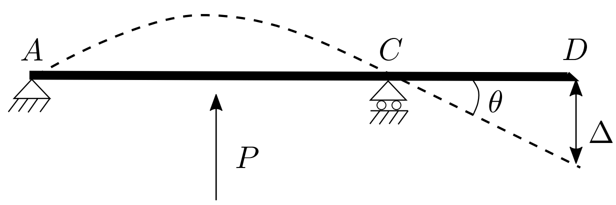
$$\Longrightarrow X = \frac{1}{3}WL^3 \cdot \frac{6}{7L^3} = \frac{2}{7}W$$

$$\Longrightarrow M = WL - \frac{2}{7}WL = \frac{\,}{5}7WL$$

4. Load case (a)



Load case (b)



From the Databook

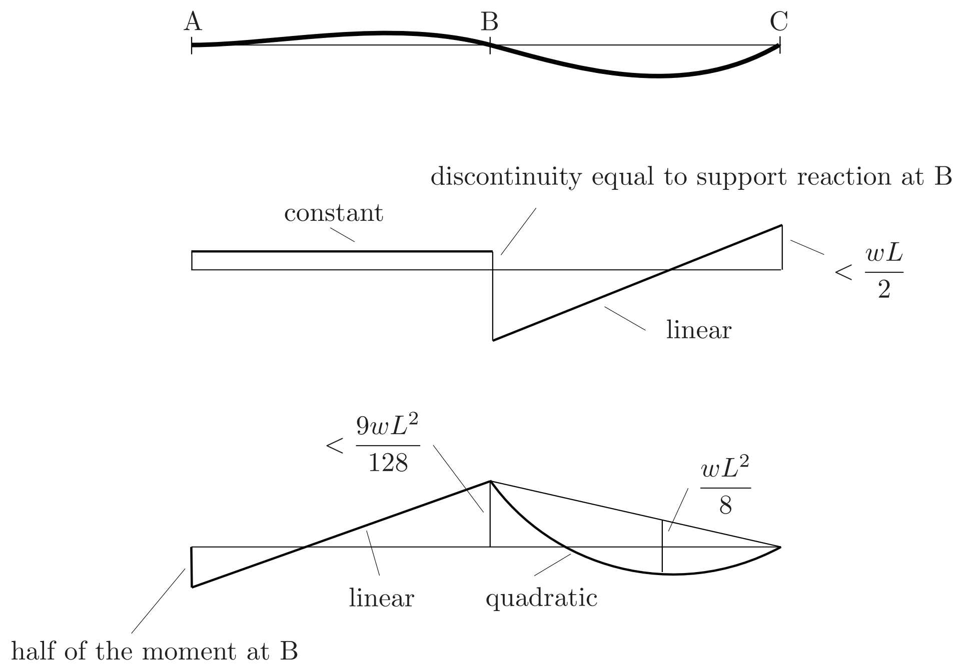
$$\Delta = \theta \cdot L = \frac{P(2L)^2}{16EI} \cdot L = \frac{PL^3}{4EI}$$

Reciprocal theorem

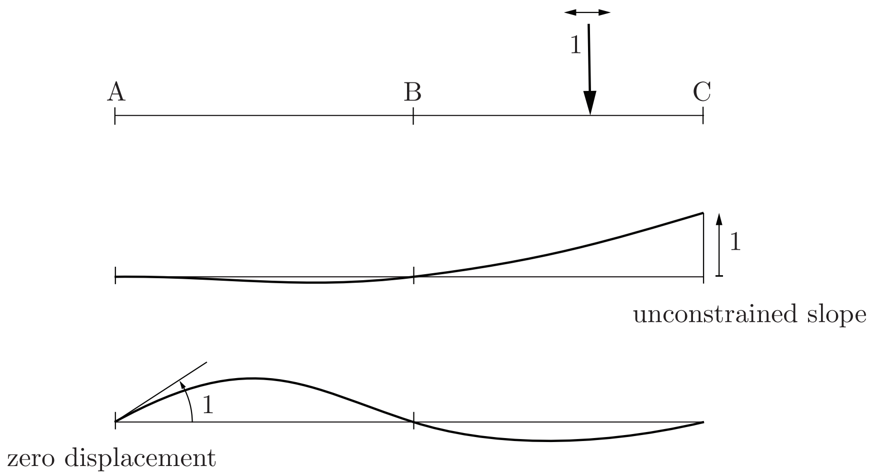
$$W \cdot \Delta = P \cdot \delta \quad \Longrightarrow \quad \delta = \frac{WL^3}{4EI}$$

5. For structural design it is important to be able to sketch the internal force distributions and the influence lines.

   a) Deflections, shear forces and bending moments.

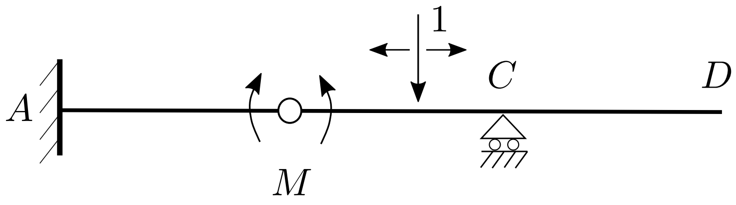


   b) Influence line for the support reaction at C and the bending moment at A.

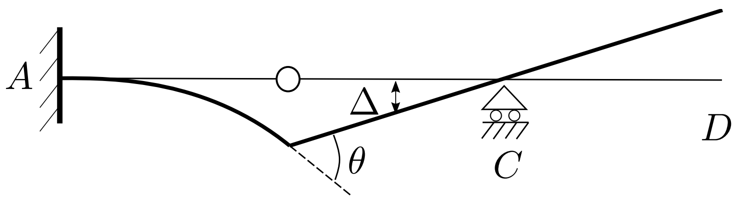


   To figure out the correct signs it is easiest to place the unit load at selected positions and to think what type of moments it will create. For instance, a unit load in the left span will generate a positive moment at A. From this we can conclude that the influence line in the left span will be positive and the right span negative.

6. Consider two load cases a) and b). The load case a) consists of a unit load and a moment M at point B.



The load case b) is the system with the rotation θ at point at B.

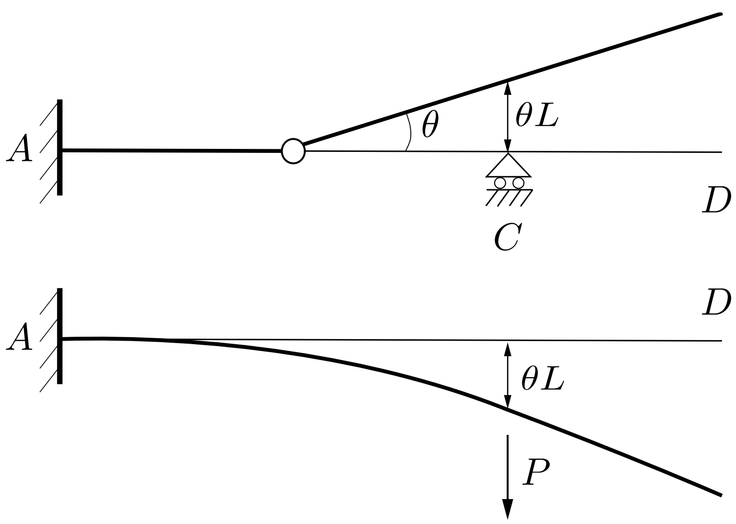


The deflection under the unit load is Δ. According to the reciprocal theorem we have

$$1 \cdot \Delta + M \cdot \theta = 0 \quad \Longrightarrow \quad M = -\frac{\Delta}{\theta}$$

This is the Müller-Breslau principle relating the moment in load case a) to the deflection in load case b).

Need to compute Δ for a given θ. In this example the following trick can be used.



Load case (b) is considered as the sum of these two cases.

- Compute P so that the displacement at support C is zero.

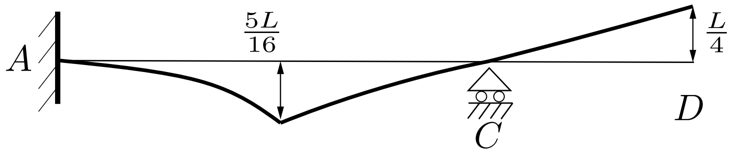
$$\theta L = \frac{P(2L)^3}{3EI} \quad \Longrightarrow \quad P = \frac{3}{8}\frac{EI}{L^2}\theta$$

- Deflection at B

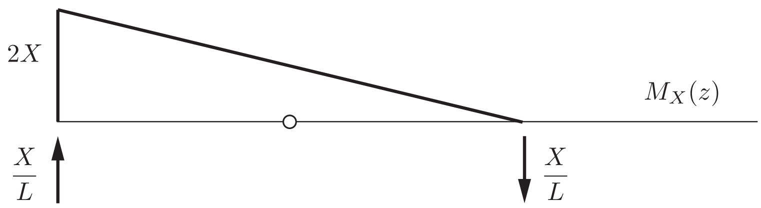
$$\frac{PL^3}{3EI} + \frac{PL \cdot L^2}{2EI} = \frac{5}{6}\frac{PL^3}{EI} = \frac{5}{16}\theta L$$

- Deflection at D

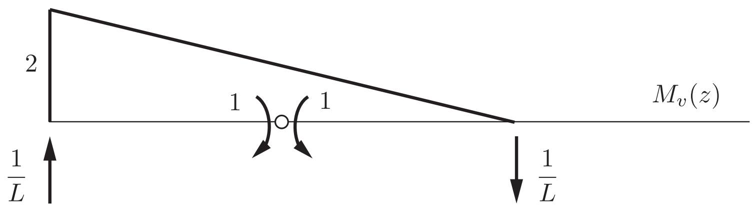
$$-2\theta L + \frac{P(2L)^2}{2EI} \cdot L + \frac{P(2L)^3}{3EI} = -\frac{\theta L}{4}$$



Alternative approach: To obtain the influence line in a more systematic manner consider the system with an unknown moment pair X and its moment diagram.



Determine using the principle of virtual work the moment pair X so that it yields a slope discontinuity of 1.

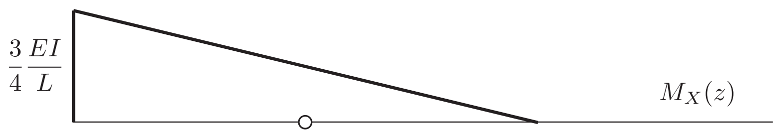


$$1 \times [\![\beta]\!] = \frac{1}{EI} \int M_X(z) M_v(z)\, dz = \frac{2L}{3} \times 2X \times 2 = \frac{1}{EI}\frac{8XL}{3}$$

Hence, choose

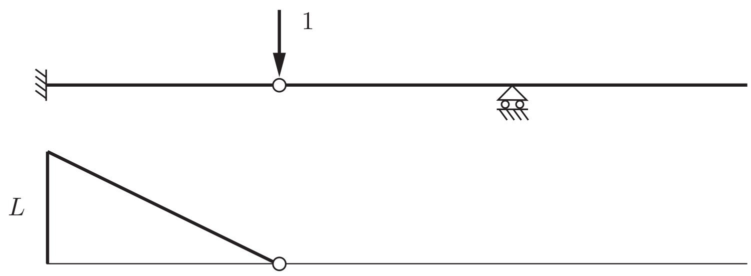
$$X = \frac{3}{8L} EI$$

so that $[\![\beta]\!] = 1$. The moment diagram for this pair looks as follows.



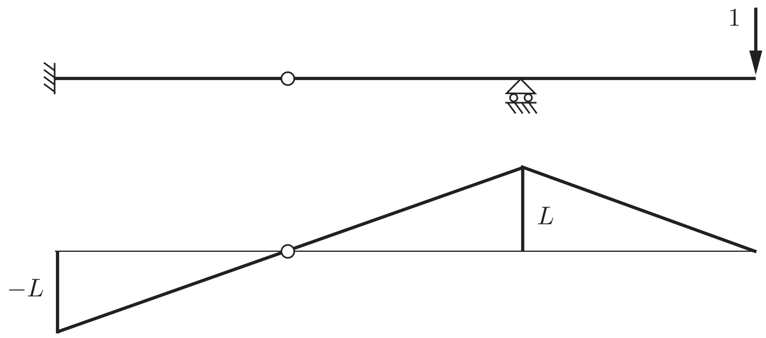
In the following this is the actual moment.

Use principle of virtual work to compute ordinates of the moment influence line. The deflection at the hinge is determined using virtual force and the corresponding moment.



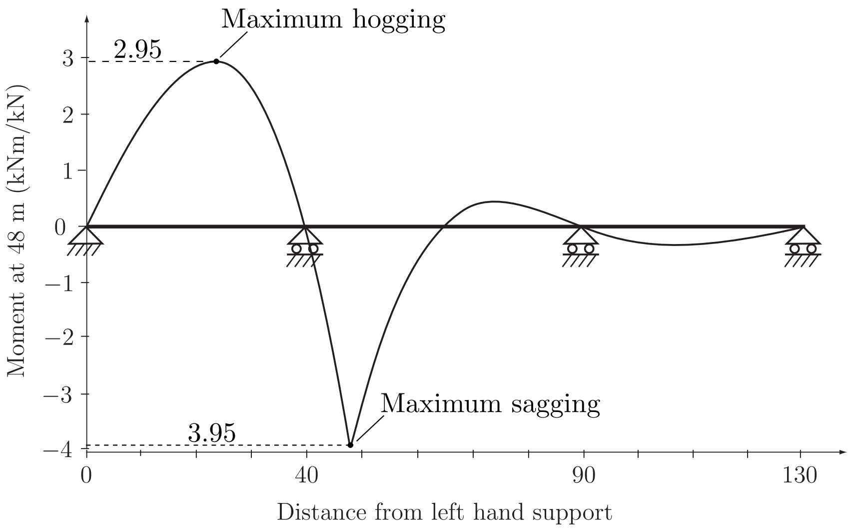
$$1 \times \delta = \frac{1}{EI}\left(\frac{L}{3} \times \frac{3EI}{4L} \times L + \frac{L}{6} \times \frac{3EI}{8L} \times L\right) = \frac{5L}{16}$$

Tip displacement of the cantilever is determined using the virtual force and the corresponding moment.



$$1 \times \delta = \frac{1}{EI}\left(\frac{2L}{2} \times \frac{3EI}{4L} \times (-L) + \frac{2L}{6} \times \frac{3EI}{4L} \times 2L\right) = -\frac{L}{4}$$

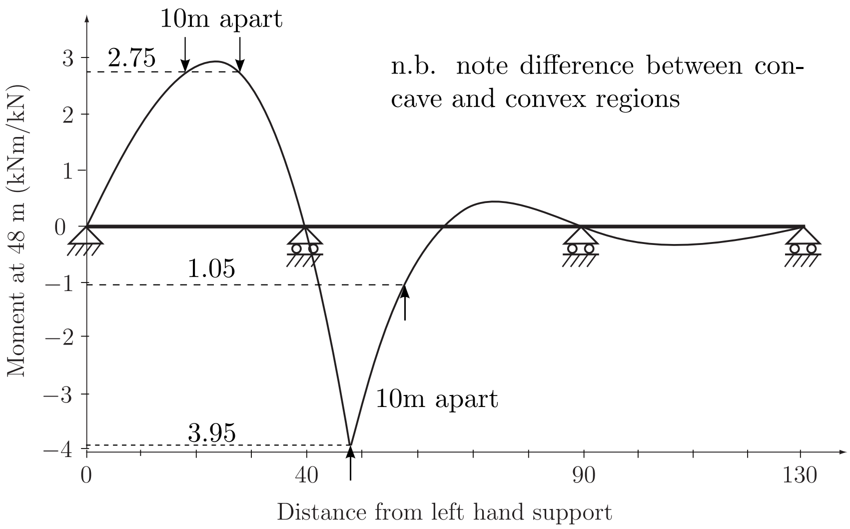
7. (a)



Maximum sagging: $800 \times 3.95 = 3160\,kNm$ (exact: $3145\,kNm$)

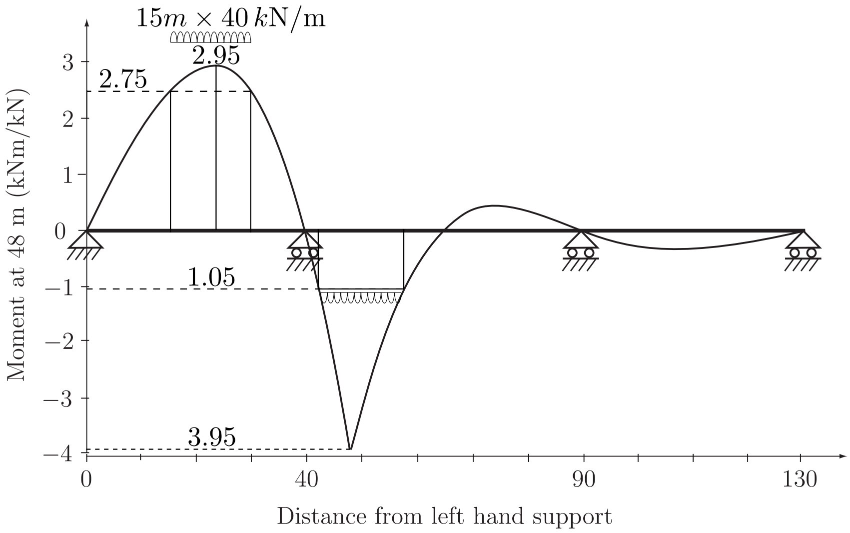
Maximum hogging: $800 \times 2.95 = 2360\,kNm$ (exact: $2360\,kNm$)

(b)



Maximum sagging: $400 \times (3.95 + 1.05) = 2000\,k$ Nm (exact: $1950\,kNm$)
Maximum hogging: $400 \times (2 \times 2.75) = 2200\,k$ Nm (exact: $2194\,kNm$)

(c)

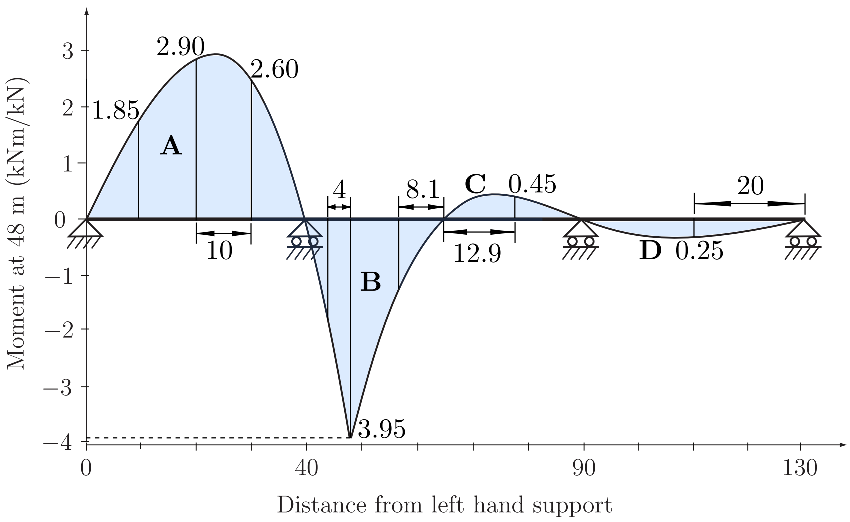


- Hogging
  Integrate using Simpson's rule.

$$M = \left(\frac{1}{3} \times 7.5\right)(2.5 + 4 \times 2.95 + 2.5) \times 40 = 1680\,kNm \qquad \text{(exact:}1676\,kNm)$$

- Sagging
  Simpson's rule is inaccurate for functions with discontinuity. Therefore, use trapezoidal rule for either side of discontinuity.

$$\text{Area } = 15(1.05) + \frac{1}{2}(15)(3.95 - 1.05) = 37.5$$
$$\text{Moment } = 40 \times 37.5 = 1500k \text{ Nm} \qquad \text{(exact: } 1416\,kNm)$$



(d) Estimate area of the 4 regions.

$$
\begin{aligned}
\text{Region A:}\quad \text{Area} &= \frac{1}{3}(10)(0 + 4 \times 1.85 + 2 \times 2.9 + 4 \times 2.6 + 0) = 78.7 \quad \text{(exact: 76.6)}\\
\text{Region B:}\quad \text{Area} &= \text{left of } 48 \text{ m} + \text{right of } 48 \text{ m}\\
&= \frac{1}{3}(4)(0 + 4 \times 1.6 + 3.95) + \frac{1}{3}(8.1)(3.95 + 4 \times 1.45 + 0)\\
&= 13.8 + 26.3\\
&= 40.1 \quad \text{(exact: 39.9)}\\
\text{Region C:}\quad \text{Area} &= \frac{1}{3}(12.9)(0 + 4 \times 0.45 + 0) = 7.7 \quad \text{(exact: 7.7)}\\
\text{Region D:}\quad \text{Area} &= \frac{1}{3}(20)(0 + 4 \times 0.25 + 0) = 6.7 \quad \text{(exact: 7.1)}
\end{aligned}
$$

For the worst sagging, load regions B and D:

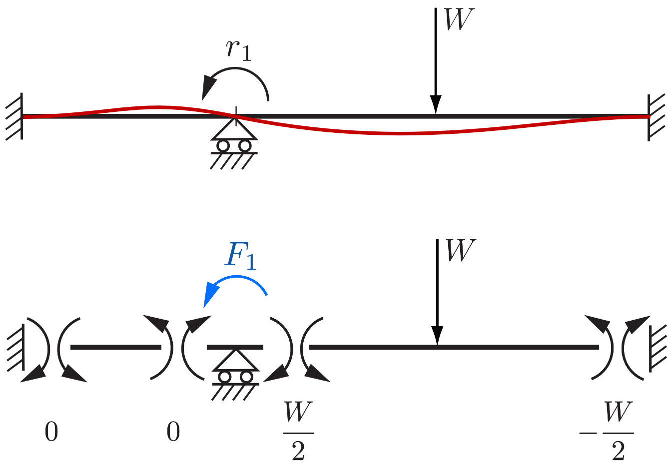
Moment $= 30(40.1 + 6.7) = 1404\, kNm$ (exact: $1410\, kNm$)

For the worst hogging, load regions A and C:

Moment $= 30(78.7 + 7.7) = 2592\, kNm$ (exact: $2592 kNm$)

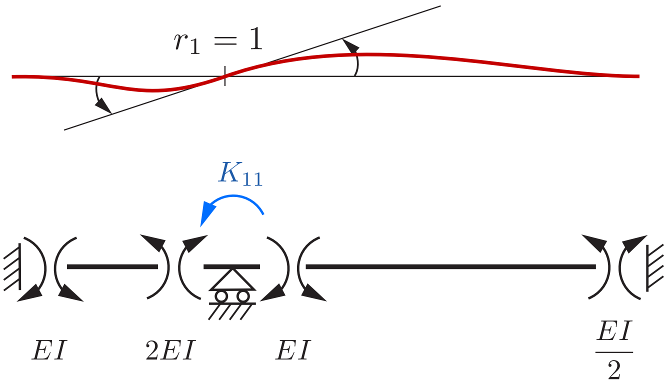
8. This example is very similar to the frame example in the Handout 5. Rotation of joint B is the only kinematic unknown.

   The restraining moment at B for the with W loaded system is



$$F_1 = \frac{W}{2}$$

The restraining moment at B for the with $r_1 = 1$ rotated system is



$$K_{11} = 3EI$$

Hence,

$$\frac{W}{2} + 3EIr_1 = 0 \qquad \Longrightarrow \qquad r_1 = -\frac{W}{6EI}$$

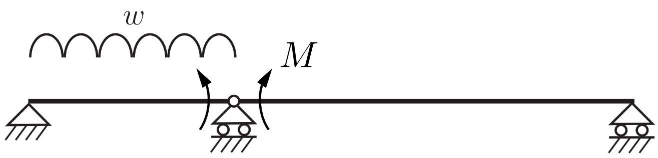
The moment at B can be obtained either from

$$M_B = 0 - 2EI\frac{W}{6EI} = -\frac{W}{3}$$

or

$$M_B = \frac{W}{2} - EI\frac{W}{6EI} = \frac{W}{3}$$

9. (a) Degree of statical indeterminacy, or redundancy, is 1. We can, for instance, introduce a pin at B.



Slope discontinuity due to w using principal of virtual work.

$$EI[\![\beta_0]\!] = \frac{1}{3}2\frac{w}{2}M = \frac{wM}{3}$$

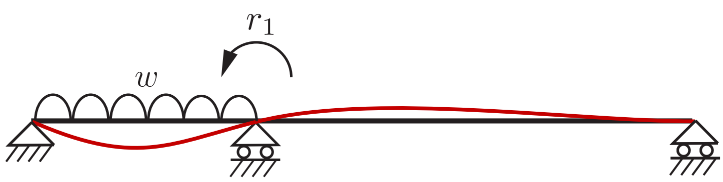
Slope discontinuity due to unknown moment M using principal of virtual work.

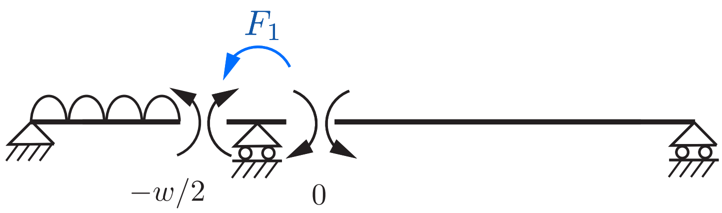
$$EI[\![\beta_1]\!] = \frac{1}{3}2M^2 + \frac{1}{3}4M^2 = 2M^2$$

Note there are tables for most common integrals. Google, e.g., 'virtual work integration table'.
Compatibility of the rotations, i.e. continuity of the slope, gives

$$[\![\beta_0]\!] + [\![\beta_1]\!] = 0 \quad \Longrightarrow \quad M = -\frac{w}{6}$$

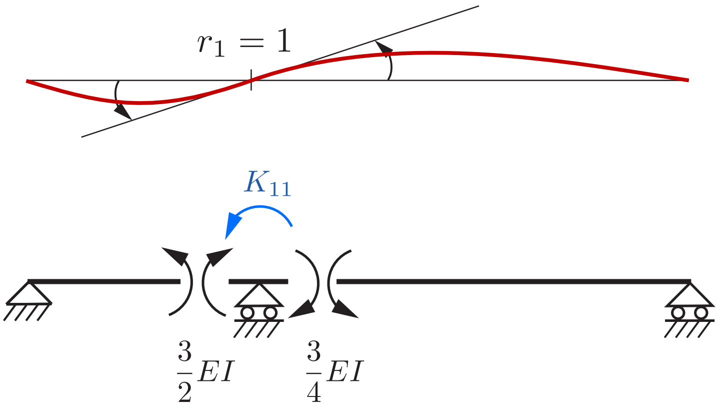
(b) Rotation of joint B can be considered as the only kinematic unknown (using the element stiffness matrix of the one-sided clamped beam). The restraining moment at B for the with W loaded is obtained from the Databook.





$$F_1 = -\frac{w}{2}$$

The restraining moment at B for the with $r_1 = 1$ rotated system from the element stiffness matrix.



$$K_{11} = \frac{9}{4}EI$$

Hence,

$$F_1 + K_{11}r_1 = 0 \Longrightarrow -\frac{w}{2} + r_1\frac{9}{4}EI = 0 \Longrightarrow r_1 = \frac{2w}{9EI}$$

Moment at support B

$$M = -\frac{w}{2} + \frac{2w}{9EI}\frac{3EI}{2} = -\frac{w}{6}$$

(c) To obtain the support reaction A use the superposition principle. For the force method, the sum of the support reactions due to w and M are

$$A = \frac{2w}{2} + \frac{M}{2} = w - \frac{w}{12} = \frac{11w}{12}$$

For the displacement method, r_1 has according to the stiffness matrix of the one-sided clamped beam the corresponding support reaction

$$A_1 = \frac{3EI}{L^2}r_1 = \frac{w}{6}$$

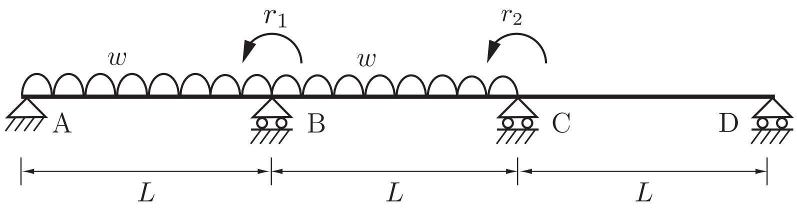
In addition, a one-sided clamped beam loaded with w has the support reaction

$$A_0 = \frac{3w}{4}$$

so that

$$A = A_0 + A_1 = \frac{w}{6} + \frac{3w}{4} = \frac{11w}{12}$$

10. Rotations of joints B and C can be considered as the only kinematic unknowns.



Restraining moments at joints B and C for the with w loaded system are obtained from the Databook.

$$F_1 = -\frac{wL^2}{8} + \frac{wL^2}{12} = -\frac{wL^2}{24}$$

$$F_2 = -\frac{wL^2}{12}$$

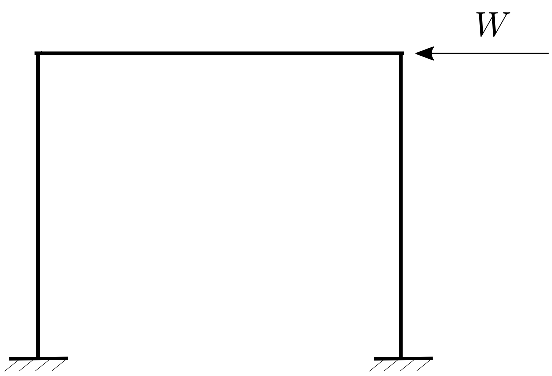
The stiffness matrix encoding the restraining moments for $r_1 = 1$ or $r_2 = 1$ has the following form.

$$\frac{EI}{L}\begin{pmatrix} 3+4 & 2 \\ 2 & 3+4 \end{pmatrix}$$

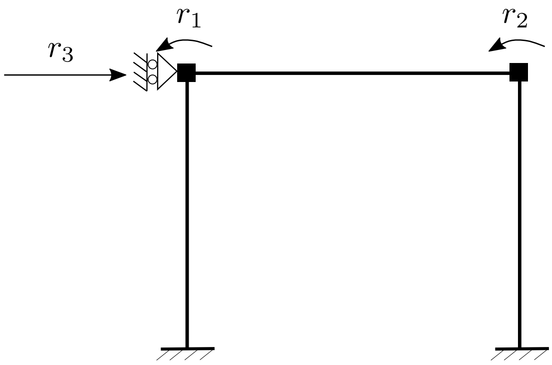
Hence,

$$\frac{EI}{L}\begin{pmatrix} 7 & 2 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = wL^2\begin{pmatrix} \frac{1}{24} \\ \frac{1}{12} \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \frac{L^3 w}{EI}\begin{pmatrix} \frac{1}{360} \\ \frac{1}{90} \end{pmatrix}$$

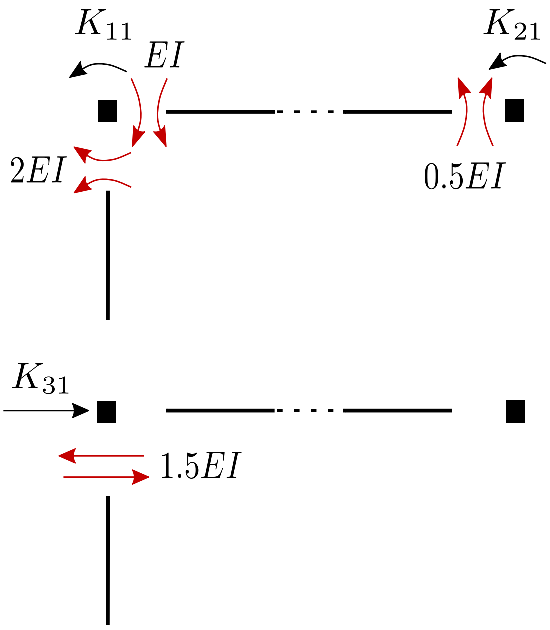
11. No axial displacements because for all bems $EA = \infty$. Still, the frame can sway to the side by bending of the vertical beams.



Rotations of the two joints and their horizontal displacement are the three kinematic degrees of freedom.

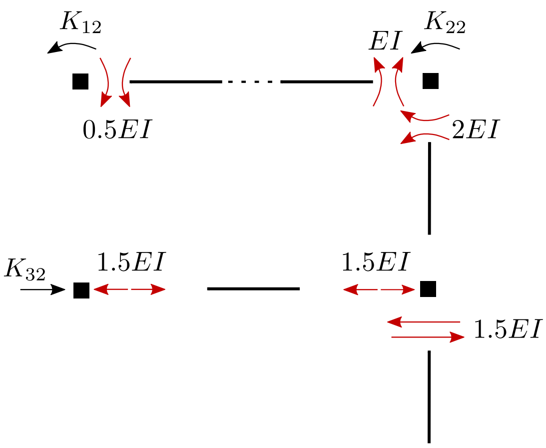


Restraining moments and forces for $r_1 = 1$ are determined with the element stiffness matrix of a doubly-clamped beam.



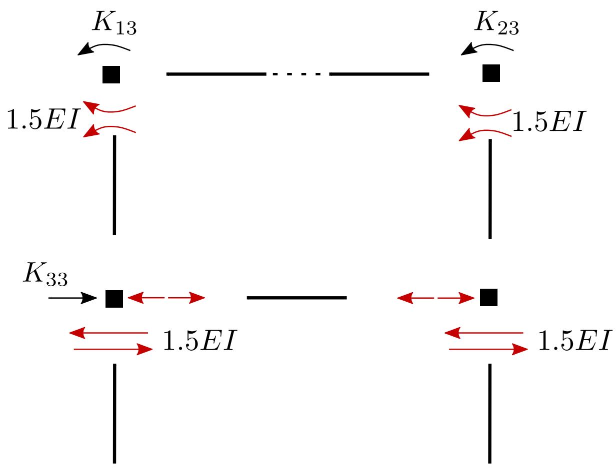
$K_{11} = 3EI \qquad K_{21} = 0.5EI \qquad K_{31} = 1.5EI$

Restraining moments and forces for $r_2 = 1$ are determined with the element stiffness matrix of a doubly-clamped beam.



$K_{12} = 0.5EI \qquad K_{22} = 3EI \qquad K_{32} = 1.5EI$

Restraining moments and forces for $r_3 = 1$ are determined with the element stiffness matrix of a doubly-clamped beam.



$K_{13} = 1.5EI \qquad K_{23} = 1.5EI \qquad K_{33} = 3EI$

Superposition of the four states yields the linear system of equations.

$$\begin{pmatrix} 3 & 0.5 & 1.5 \\ 0.5 & 3 & 1.5 \\ 1.5 & 1.5 & 3 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = wL^4 \begin{pmatrix} 0 \\ 0 \\ -W \end{pmatrix}$$

Hence,

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \frac{W}{EI} \begin{pmatrix} 0.25 \\ 0.25 \\ -0.5833 \end{pmatrix}$$

Moment at node B from the superposition of the four states with the obtined r_1, r_2 and r_3.

$$M_B = 2EI \cdot 0.25\frac{W}{EI} - 1.5EI \cdot 0.5833\frac{W}{EI} = -0.375W$$