Alg geo: lecture 1 06/10/2023.

Plan: ⊛ Basics of Sheafs (on topological spaces).

⊛ Def. of Schemes & morphisms.

⊛ First properties of schemes (AG analogues of compactness)

⊛ Rapid introduction of Cohomology of ~~she~~ Sheaves.

Why Scheme Theory?

⊛ Moduli Theory: better to study families of varieties than one at a time. (or even: All varieties of a given 'type')

Playground: $\mathbb{P}^n_{\mathbb{C}} = \mathbb{P}^n$. & variety $\mathbb{V}(S)$.

Examples of moduli: set of all lines in $\mathbb{P}^2$.

(line $\equiv$ solution to linear poly, $\{aX_0 + bX_1 + cX_2 = 0\}$)

$$\{\text{lines in } \mathbb{P}^2\} = \left(\{(a,b,c) \in \mathbb{C}^3\} \setminus \underline{0}\right) / \sim \;\cong \mathbb{P}^2_{dual}.$$

Similarly: all deg $= d$ hypersurfaces in $\mathbb{P}^n$.

$$\Rightarrow \{\deg = d \text{ hypersurfaces}\} \cong \mathbb{P}^{\binom{n+d}{d}-1}.$$

Something wrong! Some polys: $f = f_1^2 \cdot f_2$ ($f_1 \neq$ const)

have: $\underset{\deg = d}{\mathbb{V}(f)} = \underset{\text{not } \deg d.}{\mathbb{V}(f_1 f_2)}$

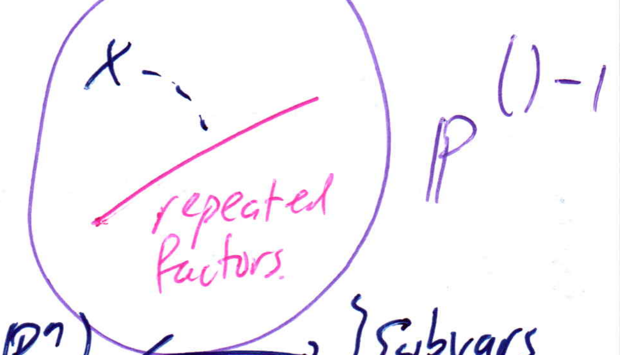
E.g. $\mathbb{V}((X_0 + X_1 + X_2)^2)$ is a line (not a conic).

Solution: take $U_d \subseteq \mathbb{P}^{\binom{n+d}{d}-1}$, $[f] \in U_d$ has no repeated factors.

<u>But</u>: U_d is no longer compact.

Output of Scheme Theory:

Fix a $\mathbb{P}^n$.



Product: "space" $\mathrm{Var}(\mathbb{P}^n) \subsetneq \mathrm{Hilb}(\mathbb{P}^n) \longleftrightarrow \{\text{Subvars of } \mathbb{P}^n\}$.

$\Updownarrow$ Sub-varieties of $\mathbb{P}^n$

(Hilbert Scheme) $\to$ (compact!) (in Eucl top.)

$\Rightarrow$ In scheme theory: $\mathbb{V}(X_0+X_1+X_2)$ & $\mathbb{V}((X_0+X_1+X_2)^2) \subseteq \mathbb{P}^2$

are <u>not</u> isomorphic as schemes!

<u>Weil Conjectures.</u> (1949).

<u>Fix</u>: $f \in \mathbb{Z}[X_0, \ldots, X_{n+1}]$ <u>homog</u>.

<u>2 worlds</u>: $X = \mathbb{V}(f) \subseteq \mathbb{P}^{n+1}_{\mathbb{C}}$. <u>Assume</u>: X smooth. (manifold).

X <u>compact</u> TS $\Rightarrow$ find numbers $b_0(X), \ldots, b_{2n}(X)$

Betti numbers $b_i(X) \equiv \mathrm{Rank}\, H^i(X; \mathbb{Z})$.

<u>World 2</u>: fix p prime & $N_m \equiv \# X(\mathbb{F}_{p^m})$.

& $\zeta(X;t) \equiv \exp\left(\sum_m \frac{N_m}{m} t^m\right)$.

<u>Theorem</u> (Grothendieck): <u>1)</u> $\zeta(X,t)$ <u>rational</u> func in t. $\frac{P_X(t)}{Q_X(t)}$

<u>2)</u> $\zeta(X,t) = \frac{P_0(t)\, P_2(t) \ldots P_{2n}(t)}{P_1(t)\, P_3(t) \ldots P_{2n-1}(t)}$ & $\deg P_i(t) = b_i(X)$.

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§1: Beyond Varieties

1.1: Classical AG.

For k alg closed field: set $\mathbb{A}^n \equiv k^n$ (as a set).

An affine variety $V \subseteq \mathbb{A}^n$ is a subset of the form $\mathbb{V}(S)$ for $S \subseteq k[X_1, \dots, X_n]$. ("vanishing locus").

Have: $\mathbb{V}(S) = \mathbb{V}(I(S)) = \mathbb{V}(S')$ by Hilbert basis theorem (for some finite S'), since $k[X]$ noetherian.

Also: $\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$ where $\sqrt{I} \equiv \{f \in k[\underline{X}] : f^n \in I, \text{ some } n\}$.

Morphisms: Given $V \subseteq \mathbb{A}^n$ & $W \subseteq \mathbb{A}^m$: morphism $\varphi: V \to W$ is a map s.t. $\varphi = (f_1, \dots, f_m)$ & f_i is a polynomial (or restriction thereof) in $\{X_1, \dots, X_n\}$.

Morphisms that have 2-sided inverses are isomorphisms.

Basic correspondence:

$$\left\{\begin{array}{c}\text{Affine varieties}\\ \text{over } k\end{array}\right\} \Big/ \text{isom.} \longleftrightarrow \left\{\begin{array}{c}\text{fin-gen } k\text{-algebra}\\ \text{with no nilpotent elements}\end{array}\right\}.$$

i.e. $\forall V$ variety, $V = \mathbb{V}(I)$ for $I \subseteq k[\underline{X}]$ radical.

Then, send: $V \longmapsto k[X]/I$

Reverse: if A is fin-gen k-algebra: $A \cong k[\underline{Y}]/J$ for some radical J (since nilpotent-free).

(*) Independence of choice: check!

The algebra associated with V is denoted $k[V]$, the <u>coordinate ring</u>. There is a bijection:

$$\{\text{Morphisms } V \to W\} \longleftrightarrow \{\text{Ring homs}_k \; k[W] \to k[V]\}.$$

<u>Topology</u>: (Zariski topology): closed sets are $\mathbb{V}(S)$ for $S \subseteq k[V]$ (any subset).

If $V \cong W$ then their Zariski spaces are homeomorphic.

<u>Nullstellensatz</u>: $\forall p \in V$, can find $ev_p : k[V] \longrightarrow k$
$f \longmapsto f(p)$.

Then: ev_p is <u>surjective</u>.

$\Rightarrow m_p \equiv \ker(ev_p)$ is <u>maximal ideal</u>.

$$\Rightarrow \{\text{Points of } V\} \longleftrightarrow \left\{\begin{matrix}\text{maximal ideals}\\ \text{in } k[V]\end{matrix}\right\}.$$

Nullstellensatz: this is a bijection.

So: given $m \leq k[V]$: $k[V] \twoheadrightarrow k[V]/m = k$.

Choose a presentation of $k[V]$.

<u>Hence</u>: $\{\text{Points of } V\} \longleftrightarrow \left\{\begin{matrix}\text{maximal ideals}\\ \text{of } k[V]\end{matrix}\right\}.$

<u>§1.2</u>: Limitations.

<u>Q1</u>: What is an "abstract" variety?

i.e. some space X, s.t. locally on a cover $\{U_i\}$ where each U_i is affine variety <u>&</u> compatible with overlaps.

E.g. $\mathbb{P}^n$, projective varieties.

<u>Example</u>: (non-alg closed fields). $I \equiv (X^2+Y^2+1) \subseteq \mathbb{R}[X,Y]$

& observe: $\mathbb{V}(I) = \phi$.

<u>But</u>: I prime $\Rightarrow$ radical $\Rightarrow$ Nullstellensatz fails.

<u>Q?</u>: From what top. space is $\mathbb{R}[X,Y]/(X^2+Y^2+1)$

"naturally" the set of functions?

(or, $\mathbb{Z}$ or $\mathbb{Z}[X]$ etc).

<u>Example</u> (Why Radical ideals? Nilpotent-free algebras?)

Take: $C = \mathbb{V}(Y-X^2) \subseteq \mathbb{A}_k^2$. & $D = \mathbb{V}(Y)$.

$\Rightarrow C \cap D = \mathbb{V}(Y, Y-X^2) = \mathbb{V}(X,Y) =$ origin.

If instead $D = D_\delta = \mathbb{V}(Y+\delta)$ then $C \cap D_\delta$ is 2 points.

$(\forall \delta \neq 0)$

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Here: A is comutative ring (with 1).

~~DEF~~ §1.3: Spectrum of a Ring.

DEF 1.3.1: Zariski Spectrum $\mathrm{Spec}(A) = \{\mathfrak{p} \subseteq A \text{ prime ideal}\}$.

⊛ Given ring hom $\varphi: A \to B$, have induced $\varphi^*: \mathrm{Spec}(B) \to \mathrm{spec}(A)$.
$q \mapsto \varphi^{-1}(q)$.

Note Would fail if only considered maximal ideals (pre-image of maximal ideal not necessarily a maximal ideal).

⊛ Given $f \in A$: & $\mathfrak{p} \in \mathrm{spec}\, A$: have $\bar{f} \in A/\mathfrak{p}$ obtained via a quotient:
(informally) evaluate $f \in A$ at "point" $\mathfrak{p} \in \mathrm{spec}(A)$, with caveat: codomain of evaluation depends on $\mathfrak{p}$.

Example 1.3.2 $A = \mathbb{Z}$, $\mathrm{Spec}(A) = \{(p) : p \text{ prime}\} \cup \{(0)\}$.

Pick value in $\mathbb{Z}$, say 42.
$\Rightarrow$ Given prime p: can look at $(42 \bmod p) \in \mathbb{Z}/(p)$.

Takeaway: $\mathrm{Spec}\,\mathbb{Z} \longrightarrow$ space
$42 \longrightarrow$ function
$42 \bmod p \longrightarrow$ value of function at (p).

Example 1.3.3 $A = \mathbb{R}[X]$. $\mathrm{Spec}(A) = (\mathbb{C}/(z \mapsto \bar{z})) \cup \{(0)\}$.

Spec $\mathbb{Z}(x)$ or Spec $k(x)$ k field?

Example 1.3.4) $A = \mathbb{C}[X]$ $\operatorname{Spec} A = \mathbb{C} \cup \{(0)\}$.
(send $a \in \mathbb{C}$ to $(X-a)$.)

§1.4: Topology.

For $f \in A$: $\mathbb{V}(f) \equiv \{p \in \operatorname{spec} A : f \equiv 0 \bmod p\} \subseteq \operatorname{Spec} A$.
$\Leftrightarrow f \in p$.

For $J \subseteq A$ ideal: $\mathbb{V}(J) \equiv \{p \in \operatorname{spec} A : f \in p \ \forall f \in J\}$.

Prop 1.4.1) $\{\mathbb{V}(J) : J \leq A\}$ forms closed sets of a topology on $\operatorname{spec}(A)$ (Zariski topology).

Proof ⊛ ϕ, $\operatorname{spec} A$ closed (easy)
⊛ $\mathbb{V}(\sum_\alpha I_\alpha) = \bigcap_\alpha \mathbb{V}(I_\alpha)$ (since $\sum_\alpha I_\alpha \equiv$ minimal ideal containing I_α)
⊛ $\mathbb{V}(I_1) \cup \mathbb{V}(I_2) = \mathbb{V}(I_1 \cap I_2)$

since: $\subseteq$ clear and: if $I_1 I_2 \subseteq I_1 \cap I_2 \subset p$ then since p prime, $I_1 \subseteq p$ or $I_2 \subseteq p$.

Example 1.4.2) $\operatorname{Spec} \mathbb{C}[X,Y]$?
⊛ $\overline{\{(0)\}} = \operatorname{spec} \mathbb{C}[X,Y]$ since any prime ideal contains (0) (Integral domain)
⊛ $I \equiv (Y^2 - X^3)$, $m_{a,b} \equiv (X-a, Y-b)$ maximal.

When is $m_{a,b} \in \overline{\{(y^2-x^3)\}}$?

$\Rightarrow$ When: $b^2 = a^3$. (e.g. $(1,1)$).

§1.5: Functions on Opens.

For $f \in A$: <u>distinguished open</u> $U_f \equiv (\operatorname{Spec} A) \setminus \mathbb{V}(f)$.

<u>Example</u> $A = \mathbb{C}[X]$, $\operatorname{Spec} A = \mathbb{C} \cup \{(0)\}$. $f = x$.

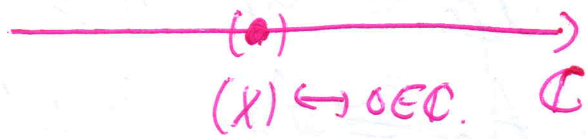
Had: bijection

$$\begin{aligned} \operatorname{Spec} A &\longrightarrow \mathbb{C} \cup \{(0)\} \\ (X-x) &\longleftarrow\!\shortmid\ x \\ (0) &\longleftarrow\!\shortmid\ (0) \end{aligned}$$

$\Rightarrow \mathbb{V}(X) = \{p \in \operatorname{spec} A : x \in p\} = \{(X)\}$.

$\Rightarrow U_f = \operatorname{Spec} A \setminus \{(X)\}$.

$\bigcirc$ (0).



More generally: $a_1, \dots, a_n \in \mathbb{C}$.

Then: $U_f = \operatorname{Spec} A \setminus \{(X-a_i)\}_{i \leq n}$, for $f = \prod (X - a_i)$.

<u>Lemma 1.5.1)</u> $\{U_f : f \in A\}$ form <u>basis</u> for Zariski Topology on $\operatorname{Spec}(A)$.

Given $f \in A$: <u>Localization</u> $A_f \cong A[X]/(Xf-1) = A[\frac{1}{f}]$.

<u>Lemma 1.5.2)</u> $U_f \overset{\text{homeo}}{\cong} \operatorname{Spec} A_f$ (naturally).

$$A \overset{j}{\longmapsto} A_f$$

inducing: $j^{-1} : \operatorname{Spec}(A_f) \longrightarrow \operatorname{Spec}(A)$.

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Proof primes in $A_f \iff$ primes of A missing f via j^{-1}.

$$q \longmapsto j^{-1}(q) \quad \leftarrow \text{(check: these}$$

$$j^{-1}(p) \cdot A_f \longleftarrow\!\mid p \quad \leftarrow \text{are inverse).}$$

Know: P_f prime $\iff f \notin P$.

(If $f \in P \Rightarrow f \in P_f$ unit, so $P_f = (1)$ not prime.

If $f \notin P \Rightarrow$ Observe: $A_f / P_f \cong (A/p)_{\bar{f}}$ $(\bar{f} = f + P)$.

$\Rightarrow A_f / P_f \subseteq FF(A/p)$ field, so is an integral domain.

Facts about distinguished opens:

⊛ $U_f \cap U_g = U_{fg}$

⊛ $U_{f^n} = U_f \ \forall n \geq 1$.

⊛ $A_f \cong A_{f^n}$ as rings. $A_f = A[x]/(xf-1)$, $A_{f^n} = A[x]/(xf^n-1)$

$x \longmapsto f^{n-1} \cdot y$

$y \longmapsto x^n$.

⊛ $U_f \subseteq U_g \iff \exists n, f^n \in (g)$.

[Why? have: $\sqrt{I} = \bigcap_{\substack{p \text{ prime} \\ p \supseteq I}} p$, so:

$U_f \subseteq U_g \iff \mathbb{V}(f) \supseteq \mathbb{V}(g) \iff$ Any prime ideal containing g also contains $f \iff \sqrt{(f)} \subseteq \sqrt{(g)}$.]

Foreshadow: for A ring: Distinguished Opens in Spec A $\longrightarrow$ Rings

$U_f \longmapsto A_f$.

$\sqrt{1}$

Is functorial: If $U_{f_1} \subseteq U_{f_2} \Rightarrow \exists n,\ f_1^n = f_2 \cdot f_3$.

$\Rightarrow U_{f_1} = U_{f_1^n} = U_{f_2 \cdot f_3} \subseteq U_{f_2}$.

$\Rightarrow \exists$ hom $A_{f_2} \to A_{f_1}$. & Is restriction map.

Can this be extended to all opens?

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§2: Sheafs.

§2.1: Pre-sheaves.

X topological space.

DEF 2.1.1) $\mathcal{F}$ pre-sheaf on X of abelian groups if:

$$\left.\begin{array}{rcl} \mathrm{Open}(X) & \longrightarrow & \mathrm{AbGps} \\ U & \longmapsto & \mathcal{F}(U). \end{array}\right] \quad \begin{array}{l} \forall U \subseteq V \text{ opens:} \\ \exists\, res_U^V : \mathcal{F}(V) \to \mathcal{F}(U). \end{array}$$

- $Res_U^U = id$.
- $Res_U^V\, Res_V^W = Res_U^W$ if $U \subseteq V \subseteq W$.

Key example: $\mathcal{F}(U) \equiv C^0(U, \mathbb{R})$.

Similar for ~~the~~ pre-sheaves of rings, sets, modules (over a fixed ring) etc.

DEF 2.1.3) Morphism $\varphi : \mathcal{F} \to \mathcal{G}$ on presheaves of X:

$\forall U \subseteq X$ open, $\exists$ homomorphism $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ compatible with restrictions: i.e. If $V \subseteq U$ then:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "Res"] & \mathcal{G}(U) \arrow[d, "Res"] \\ \mathcal{F}(V) \arrow[r, "\varphi(V)"'] & \mathcal{G}(V) \end{tikzcd}$$

commutes.

A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ <u>injective</u> (<u>surjective</u>) if:
$\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective (surjective) $\forall U$.

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§2.2: Sheaves.

DEF 2.2.1 Sheaf is pre-sheaf $\mathcal{F}$, s.t.:

1) If $U \subseteq X$ open & $\{U_i\}$ open cover of U, then: for $s \in \mathcal{F}(U)$ with $s|_{U_i} \equiv Res^{U}_{U_i}(s) = 0 \ \forall i \Rightarrow s = 0$.

2) If U, $\{U_i\}$ as in 1): given $s_i \in \mathcal{F}(U_i)$ with $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \ \forall i,j$, then: $\exists s \in \mathcal{F}(U)$, $s|_{U_i} = s_i \ \forall i$.

Remark Imply: $\mathcal{F}(\emptyset) = 0$.

A *morphism* of sheaves is morphism of underlying pre-sheaves.

[$f: \mathcal{F} \to \mathcal{G}$ means $f_U: \mathcal{F}(U) \to \mathcal{G}(U)$ such that compatible with restriction]

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f_U} & \mathcal{G}(U) \\ \downarrow{\scriptstyle Res} & & \downarrow{\scriptstyle Res} \\ \mathcal{F}(V) & \xrightarrow{f_V} & \mathcal{G}(V) \end{array}$$

Example 2.2.2 If X top space, $\mathcal{F}(U) \equiv \{f: U \to \mathbb{R} \text{ continuous}\}$ then $\mathcal{F}$ sheaf.

Non-example 2.2.3 $X = \mathbb{C}$ (Eucl topology).

$\mathcal{F}(U) \equiv \{f: U \to \mathbb{C} \text{ bounded \& holomorphic}\}$.

Bounded functions may give rise to unbounded functions; being bounded is not a local condition.

Non-example 2.2.4 Fix group G & set $\mathcal{F}(U) = G$.

If U_1, U_2 disjoint $\Rightarrow \mathcal{F}(U_1 \cup U_2) = G \times G$.

Example 2.2.5 ((constant sheaf)

Give: G discrete topology & $\mathcal{F}(U) \equiv \{f : U \to G$ continuous$\}$

$= \{ f : U \to G :$ locally constant $\}$.

Example 2.2.6 V irred variety. Set $\mathcal{O}_V(U) \equiv \{ f \in k(V) :$ f regular at p $\forall p \in U \}$.

(Regular at $p \Leftrightarrow f = g/h$, in nbhood of p, with g, h polynomials with $h(p) \neq 0$).

$\mathcal{O}_V \equiv$ structure sheaf of V. (Obviously a sheaf).

§2.3 : Basic Constructions.

Terminology: A section of $\mathcal{F}$ over U is an element $s \in \mathcal{F}(U)$.

DEF 2.3.1 (stalk): Fix $p \in X$ & $\mathcal{F}$ presheaf on X.

$\mathcal{F}_p \equiv$ "stalk at p" $\equiv \{ (U, s) : s \in \mathcal{F}(U)$ & $p \in U \} / \sim$

with: $(U, s) \sim (U', s')$ if $\exists W \subseteq U \cap U'$, $p \in W$, and: $s|_W = s'|_W$. Elements of $\mathcal{F}_p$ are germs.

Exercise 2.3.2 For $\mathbb{A}^1$ affine space:

$$\mathcal{O}_{\mathbb{A}^1, 0} = \left\{ \frac{f(t)}{g(t)} : g(0) \neq 0 \right\} \subseteq k(t)$$
$$= k[t]_{(t)}.$$

Prop 2.33 If $f: \mathcal{F} \to \mathcal{G}$ morphism of sheaves on X s.t. $\forall p \in X$, $f_p: \mathcal{F}_p \to \mathcal{G}_p$ is an <u>isomorphism</u>. Then, f itself is an isomorphism.

<u>Proof</u> Show: $f_V: \mathcal{F}(V) \to \mathcal{G}(V)$ is isomorphism, and then define f^{-1} by $(f^{-1})_U = (f_U)^{-1}$. <u>Check</u>: is morphism.

⊛ f_V <u>Injective</u>: if $s \in \mathcal{F}(U)$ & $f_V(s) = 0$. Since f_p inj:

means $(U, s) = 0$ in $\mathcal{F}_p$ $\forall p \in U$.

$\Rightarrow \forall p \in U$ $\exists U_p$ open nbhood of p, $s|_{U_p} = 0$.

<u>But</u>: $\{U_p : p \in U\}$ <u>cover</u> U, so by axiom 1, get: $s = 0$ in $\mathcal{F}(U)$.

⊛ f_U <u>surjective</u>: Fix $t \in \mathcal{G}(U)$.

$\forall p \in U$: find: $(U_p, s_p) \in \mathcal{F}_p$ & $f_p(U_p, s_p) = (U, t) \in \mathcal{G}_t$.

By shrinking (if necessary): assume: $f_{U_p}(s_p) = t|_{U_p}$.

For $p, p' \in U$: $f_{U_p \cap U_{p'}}(s_p|_{U_p \cap U_{p'}} - s_{p'}|_{U_p \cap U_{p'}})$

$= t|_{U_p \cap U_{p'}} - t|_{U_p \cap U_{p'}} = 0$.

$\Rightarrow$ By injectivity of $f_{U_p \cap U_{p'}}$: $\Rightarrow s_p|_{U_p \cap U_{p'}} - s_{p'}|_{U_p \cap U_{p'}} = 0$

$\Rightarrow$ By Axiom 2: $\exists s \in \mathcal{F}(U)$: $s|_{U_p} = s_p$ $\forall p$.

& $f_U(s)|_{U_p} = f_{U_p}(s|_{U_p}) = f_{U_p}(s_p) = t|_{U_p}$.

$\Rightarrow f_U(s) = t$ (by S1).

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Remark 2.3.4 i) $\mathcal{F}(U) \longrightarrow \prod_{p\in U} \mathcal{F}_p$ injective.

$$s \longmapsto ((U,s))_{p\in U}$$

ii) Given $\varphi, \psi : \mathcal{F} \longrightarrow \mathcal{G}$ with $\varphi_p = \psi_p$ $\forall p \in X$, have $\varphi = \psi$.

DEF 2.3.5 (Sheafification).

If $\mathcal{F}$ pre-sheaf on X, a morphism $sh : \mathcal{F} \longrightarrow \mathcal{F}^{sh}$ (for sheaf $\mathcal{F}^{sh}$) is a sheafification if: $\forall \varphi : \mathcal{F} \to \mathcal{G}$ map (sheaf $\mathcal{G}$), $\exists!$ commutative diagram:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "sh"] \arrow[dr, "\varphi"'] & \mathcal{F}^{sh} \arrow[d] \\ & \mathcal{G} \end{tikzcd}$$

i.e. ~~and~~ any $\varphi : \mathcal{F} \to \mathcal{G}$ extends uniquely to a map $\mathcal{F}^{sh} \to \mathcal{G}$.

Remark 2.3.6 i) Since def. by universal property: $\mathcal{F}^{sh}$ & $\mathcal{F} \to \mathcal{F}^{sh}$ are unique (up to unique isomorphism).

ii) A morphism of pre-sheaves $\mathcal{F} \to \mathcal{G}$ induces a morphism $\mathcal{F}^{sh} \longrightarrow \mathcal{G}^{sh}$.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "sh"] \arrow[d, "\varphi"'] \arrow[dr] & \mathcal{F}^{sh} \arrow[d, dotted] \\ \mathcal{G} \arrow[r, "sh"] & \mathcal{G}^{sh} \end{tikzcd}$$

Prop 2.3.7 Sheafification exists.

Proof Given pre-sheaf $\mathcal{F}$ on X: define:

$$\mathcal{F}^{sh}(U) \equiv \{ f : U \longrightarrow \prod_{p\in U} \mathcal{F}_p : f(p) \in \mathcal{F}_p \ \& \ \forall p \in U, \exists p \in V_p \subseteq_{open} U : \& \ s \in \mathcal{F}(V_p) : (V_p, s) = f(q) \in \mathcal{F}_q \ \forall q \in V_p \}$$

"Obviously": is a sheaf & has universal property (exercise).

<u>Remark</u>: by universal property: $\mathcal{F}^{sh} = \mathcal{F}$ if $\mathcal{F}$ sheaf!

<u>Corollary 2.3.8</u> Stalks of $\mathcal{F}$, $\mathcal{F}^{sh}$ <u>coincide</u>.

<u>Exercise 2.3.9</u> Find a nonzero pre-sheaf $\mathcal{F}$ with $\mathcal{F}^{sh} = 0$!

§2.4: Kernels & cokernels.

Let $\varphi : \mathcal{F} \to \mathcal{G}$ morphism of pre-sheaves. Define the presheaves $\ker\varphi$, $\operatorname{coker}\varphi$, $\operatorname{im}\varphi$, etc by:

- $(\ker\varphi)(U) \equiv (\ker \varphi_U) : \mathcal{F}(U) \to \mathcal{G}(U)$
- $(\operatorname{coker}\varphi)(U) \equiv \operatorname{coker}\varphi_U$
- $(\operatorname{im}\varphi)(U) \equiv \operatorname{im}\varphi_U$.

<u>Exercise 2.4.1</u> The presheaf kernel of a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is also a sheaf.

<u>Not</u> true for $\operatorname{coker}\varphi$ in general!

<u>Example 2.4.2</u> $X = \mathbb{C}$ & $\mathcal{O}_X \equiv$ sheaf of holomorphic functions on X. & $\mathcal{O}_X^* \equiv$ sheaf of nowhere-0 holomorphic funcs.

~~$\Rightarrow \mathcal{O}_X^* \subseteq \mathcal{O}_X$ sub-sheaf.~~

Have morphism of sheaves
$$\exp : \overset{+}{\mathcal{O}_X} \longrightarrow \overset{\times}{\mathcal{O}_X^*}$$
$$f \longmapsto e^f.$$

$\Rightarrow \ker(\exp) = 2\pi i\mathbb{Z}$

($\mathbb{Z}$ is constant sheaf).

& $\operatorname{Coker}(\exp)$ is <u>not</u> a sheaf.

E.g. take $U_1 = \mathbb{C} \setminus [0, \infty)$ & $U_2 \equiv \mathbb{C} \setminus (-\infty, 0]$.

$\Rightarrow U \equiv U_1 \cup U_2 = \mathbb{C}_*$.

Let $f(z) = z$, so $f \in \mathcal{O}_X^*(U)$ (nowhere 0 on U)

& $f \notin \text{image}(\exp : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X^*(U))$, as $\text{Log}(z)$ is not single-valued on U.

$\Rightarrow f$ defines nonzero section of $(\text{coker} \exp)(U)$.

<u>However</u>: $f|_{U_i}$ <u>is</u> in $\text{image}(\exp : \mathcal{O}_X(U_i) \longrightarrow \mathcal{O}_X^*(U_i))$.

~~Hence~~ (E.g. by choosing an appropriate branch).

Hence: $f|_{U_i} = 1$ in $\text{coker}(\exp)$. So, axiom 1 fails.

<u>Def 2.4.3</u>) $\varphi : \mathcal{F} \to \mathcal{G}$ morphism of sheaves. Define the <u>sheaf cokernel</u> & <u>sheaf image</u> by the sheafification of presheaf cokernel & presheaf image.

<u>Remark 2.4.4</u>) Crucial fact: $0 \longrightarrow 2\pi i \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 1$

is <u>exact</u> sequence of sheaves, and: $\ker(\exp) = 2\pi i \mathbb{Z}$
$\text{coker}(\exp) = 1$.

<u>Remark 2.4.5</u>) $\text{Ker}\,\varphi$, $\text{coker}\,\varphi$ satisfy category theory defs:

$$\begin{tikzcd}
\text{Ker}\,\varphi \arrow[r] \arrow[rr, bend left, "0"] & \mathcal{F} \arrow[r, "\varphi"] & \mathcal{G} \\
& L \arrow[u, "\psi"] \arrow[ur, "\varphi \circ \psi = 0"'] \arrow[ul, "\exists!"] &
\end{tikzcd}$$

<u>Coker</u>: reverse all arrows on the diagram.

<u>DEF 2.4.6)</u> 1) Sub-sheafs! $\mathcal{F} \subseteq \mathcal{G} \iff$ we have inclusions $\mathcal{F}(U) \subseteq \mathcal{G}(U)$ compatible with restrictions.

E.g. $\ker(\varphi : \mathcal{F} \to \mathcal{G}) \subseteq \mathcal{F}$.

Alg Geo: Lecture 7 20/10/2023.

§2.5: Moving between spaces.

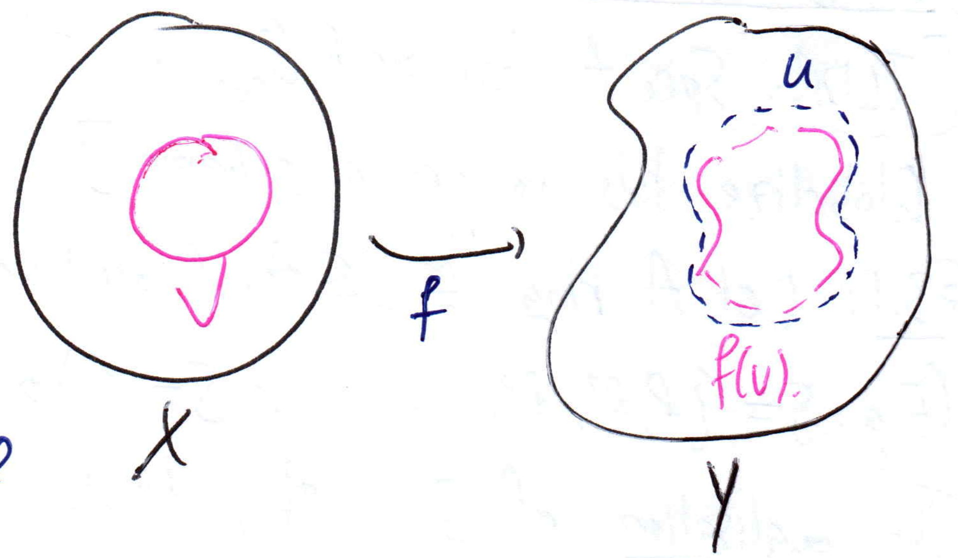
Given $f: X \longrightarrow Y$ cts: & $\mathcal{F}$ sheaf on X
$\mathcal{G}$ sheaf on Y.

DEF 2.5.1 (push-forward).
The presheaf pushforward for $\mathcal{F}$ by $\underset{\text{open}}{U} \mapsto \mathcal{F}(f^{-1}(U))$

Prop 2.5.2 This is also a sheaf.

DEF 2.5.3 (Inverse image).
The inverse image presheaf $(f^{-1}\mathcal{G})^{pre}$ is defined by:
$(f^{-1}\mathcal{G})^{pre}(V) \equiv \{(S_u, U) : U \text{ open}, U \supseteq f(V), S_u \in \mathcal{G}(U)\}/\sim$
where: $\sim$ identifies pairs that agree on a small open containing V.
The inverse image sheaf is: $f^{-1}\mathcal{G} \equiv ((f^{-1}\mathcal{G})^{pre})^{sh}$.

Example 2.5.4
(Why is sheafification necessary?)



Take: $X \equiv Y \sqcup Y$.
$\mathcal{G} \equiv \mathbb{Z}$ constant sheaf
$\mathcal{F} \equiv (f^{-1}\mathcal{G})^{pre}$.

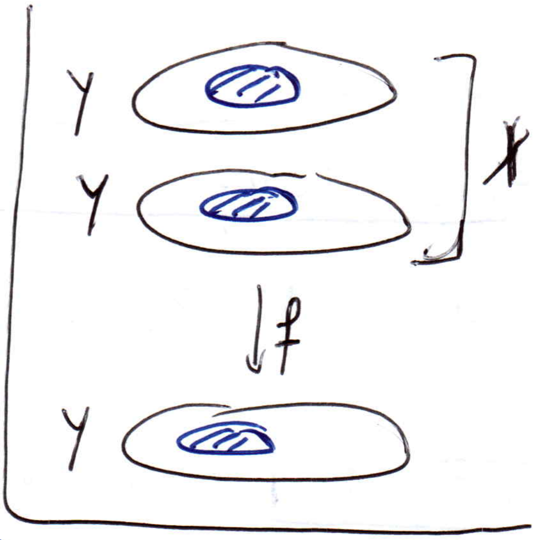
For $U \subseteq Y$ open & connected: & $V \equiv f^{-1}(U)$: ~~$\mathcal{F}(U) = \mathcal{G}(V)$~~
$\mathcal{F}(V) = \mathcal{G}(U) = \mathbb{Z}$. But, $V = U \sqcup U \Rightarrow \mathcal{F}^{sh}(V) = \mathcal{G}(U) \times \mathcal{G}(U)$
$= \mathbb{Z}^2$.

Example 2.5.5 $\mathcal{F}$ sheaf on X.

& $\pi : X \longrightarrow *$. (Unique map).

Then: $\pi_* \mathcal{F}$ is a sheaf on a point, so is an abelian group.

specifically: $\mathcal{F}(\pi^{-1}(*)) = \mathcal{F}(X)$.



Notation: $\mathcal{F}(X) \equiv \Gamma(X, \mathcal{F})$ "global sections"

$\equiv H^0(X, \mathcal{F})$ "0'th Cohomology* with coeffs in $\mathcal{F}$"

*For $p \in X$: & $i : \{p\} \hookrightarrow X$, $\mathcal{G}$ is a sheaf on p, i.e. an abelian group A.

Consider $i_* \mathcal{G}$: the sheaf on X s.t. $(i_*\mathcal{G})(U) \equiv \begin{cases} 0 & \text{if } p \notin U \\ A & \text{if } p \in U \end{cases}$ is the "skyscraper at p with value A".

§3: Back to Schemes.

[TLDR: Spec A has sheaf $\mathcal{O}_{\operatorname{Spec} A}$ s.t. values on U_f are A_f. Globalize this to get a scheme.]

§3.1: Let A ring & $S \subseteq A$. Closed under mult.

(E.g. $S \equiv \{f, f^2, f^3, \ldots\}$, or $S = A \setminus \mathfrak{p}$, $\mathfrak{p}$ prime)

The localisation of S at A, denoted $S^{-1}A$, are pairs ~~(a,s)~~ ~~a∈A s∈S~~ $S^{-1}A \equiv \{(a,s) : a \in A, s \in S\}/\sim$

where $(a,s) \sim (a',s') \iff \exists s'' \in S,\ s''(as' - a's) = 0$.

Warning $A \longrightarrow S^{-1}A$ (inclusion) may not be injective (if S has zero divisor). $\sqrt{2}$

Next Steps ⊛ Define: sheaf $\mathcal{O}_{\mathrm{Spec}\,A}$ on $\mathrm{Spec}\,A$, s.t.

1) stalks at a prime p is: $(A \setminus p)^{-1} A \equiv A_p$

2) If U_f distinguished open: $\mathcal{O}_{\mathrm{Spec}\,A}(U_f) = A_f$.

Sheaf on a basis. Fix X & basis $\mathcal{B}$ of topology of X.

A sheaf F on the basis $\mathcal{B}$ consists of assignments:

$B_i \longmapsto F(B_i)$ (abelian group / ring / etc)

& restriction maps: $F(B_i) \xrightarrow{\mathrm{Res}^{B_i}_{B_j}} F(B_j)$ whenever $B_i \supseteq B_j$

that satisfy usual commutativity & identity when $B_i \subseteq B_j \subseteq B_k$ or $B_i = B_j$.

And: $\boxed{SB1}$ If $B = \bigcup B_i$, $B_i \in \mathcal{B}$, and:

$f, g \in F(B)$ with $f|_{B_i} = g|_{B_i}$ $\forall i$, then: $f = g$

$\boxed{SB2}$ If $B = \bigcup B_i$, $B_i \in \mathcal{B}$ & $\forall i$, $\exists f_i \in F(B_i)$, s.t. $\forall i, j$ and $B' \subseteq B_i \cap B_j$ with $f_i|_{B'} = f_j|_{B'}$,

then $\exists f \in F(B)$: $f|_{B_i} = f_i$ $\forall i$.

Prop 3.1.2 F sheaf on base $\mathcal{B}$ of X. This uniquely determines a sheaf $\mathcal{F}$ by: $\mathcal{F}(B_i) \equiv F(B_i)$ $\forall i$, agreeing with restriction.

Proof Step 1: Define stalks of $\mathcal{F}$.

$\mathcal{F}_p \equiv \{(s_B, B) : B \text{ basic open containing } p\}/\sim$.

Step 2: Sheafification trick. Define $\mathcal{F}(U) \equiv \{(f_p \in \mathcal{F}_p)_{p \in U} : \forall p \in U, \exists \text{ basic open } B \text{ containing } p, \text{ \& } \exists s \in F(B) \text{ with } s_q = f_q \ \forall q \in B\}$

Clearly: is a sheaf, since conditions are local.

Step 3: Natural maps $F(B) \longrightarrow \mathcal{G}(B)$ are isomorphisms, by sheaf axioms.

Alg Geo: Lecture 8 | 23/10/2023.

Recall: (*) Spec(A) is topological space, with basis $\{U_f\}$ where $U_f \equiv \mathbb{V}(f)^c$ ($f \in A$).

(*) $U_f = U_g \Leftrightarrow \exists h \in A$, s.t. f, g are powers of h.

& If $U_f = U_g$ then $\cancel{A_f \cong A_g}$ $\begin{bmatrix} f^n = g \cdot a \\ g^m = f \cdot b \end{bmatrix}$ $a, b \in A \Rightarrow \sqrt{(f)} = \sqrt{(g)}$ $\Rightarrow$ well def'ed.

$\Rightarrow$ The assignment $U_f \longmapsto A_f$ is well-defined.

Prop 3.1.8 | This assignment defines a sheaf (of rings) on the base of topology of Spec(A) given by U_f's.

Hence: Spec A inherits a sheaf of rings, $\mathcal{O}_{\mathrm{spec}\, A}$. Called the structure sheaf.

Prelude. Suppose $\{U_{f_i} : i \in I\}$ ~~contains~~ covers Spec(A). Then: exist finite subcover. $\Rightarrow$ Is paracompact.

[Why? Since U_{f_i} covers: there is no prime ideal $p \subseteq A$ containing all (f_i). So, $\sum_{i \in I} (f_i) = (1) \Leftrightarrow 1 = \sum_i a_i f_i$.

Finite sum!

$\Rightarrow$ If $J \subseteq I$ finite & contain just all i s.t. $a_i \neq 0$, then get $\sum_{j \in J} (f_j) = (1) \Rightarrow \{U_{f_j}\}_{j \in J}$ covers.]

Proof of Prop | Need to check: SB1 & SB2.

Will check these for the basic open $B = \mathrm{spec}\, A$; general case is similar.

SB1) Suppose $\mathrm{Spec}(A) = \bigcup_{i \le n} U_{f_i}$ (by prelude, WLOG finite)

Given $s \in A$; s.t. $s|_{U_{f_i}} = 0 \ \forall i$: by def. of localisation cover:

$\exists m$, s.t. $f_i^m \cdot s = 0$.

Also: $(1) = \sum_i (f_i) = \sum_i (f_i^m) \ \forall m > 0$. (since: cover)

$\Rightarrow 1 = \sum_i r_i f_i^m$, so $s = \sum_i r_i f_i^m s = 0$ ✓

SB2) Say: $\mathrm{Spec}(A) = \bigcup_{i \in I} U_{f_i}$ & choose $s_i \in A_{f_i} \ \forall i \in I$

with $s_i = s_j$ on $A_{f_i} \cap A_{f_j} = A_{f_i f_j} \ \forall i,j \in I$.

For SB2, need to build: $s \in A$ with $s|_{A_{f_i}} = s_i$.

⊛ If I finite: $\forall i \in I$, choose $l_i \ge 0$ s.t. $\frac{a_i}{f_i^{l_i}} \in A_{f_i}$.

let $g_i \equiv f_i^{l_i} \Rightarrow U_{f_i} = U_{g_i} \ \forall i$.

Condition for SB2: $(g_i g_j)^{m_{ij}} (a_i g_j - a_j g_i) = 0$.

Since I finite (assumed): WLOG $m = m_{ij} \ \forall i,j$. (take maximum)

Write: $b_i \equiv a_i g_i^m$ & $h_i \equiv g_i^{m+1}$.

$\Rightarrow$ On each $U_{h_i} \equiv U_{f_i}$: have chosen $b_i|_{U_{h_i}}$.

Since $\{U_{h_i}\}$ cover $\mathrm{Spec}(A)$: $1 = \sum_i r_i h_i$ ($r_i \in A$).

& construct: $r \equiv \sum_i r_i b_i$ (as above). Then, this restricts

correctly to each U_{h_i} to b_i/h_i (in A_{h_i}).

⊛ Infinite case: Pick (by before) finite subcover $\{U_{f_i}\}$,

i.e. $(f_1, \dots, f_n) = A$. By finite case: can build r. But: given

$(f_1, \dots, f_n, f_\alpha) = A$: Same construction gives new element r' ↱

But: by SB1, $r = r'$ ✓

DEF 3.14 Structure sheaf on Spec(A) is the structure sheaf associated to the ~~presh~~ sheaf on the base $U_f \mapsto A_f$. It is denoted $\mathcal{O}_{\operatorname{Spec} A}$.

Observation: $\mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}} = A_{\mathfrak{p}}$ stalk

Terminology Ringed space $(X, \mathcal{O}_X)$ is top space X with a sheaf of rings $\mathcal{O}_X$.

An isomorphism of ringed spaces is: $\pi : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is: $\pi : X \longrightarrow Y$ homeomorphism + isomorphism of sheaves on Y:

$\mathcal{O}_Y \simeq \pi_* \mathcal{O}_X$.

An affine scheme is ringed space $(X, \mathcal{O}_X)$ that is isomorphic to $(\operatorname{spec} A, \mathcal{O}_{\operatorname{spec} A})$

DEF 3.16 A scheme is a ringed space $(X, \mathcal{O}_X)$ that is locally isomorphic to an affine scheme.

Note If $U \subseteq X$, then U is naturally a ringed space $\mathcal{O}_U(V) \equiv \mathcal{O}_X(V)$.

$\Rightarrow \forall p \in X, \exists p \in U_p \subseteq X$ open s.t. $(U_p, \mathcal{O}_{U_p})$ is isomorphic to some affine scheme (possibly depending on p).

Examples of schemes Spec(A), various rings A.

Open subschemes: let X scheme & $U \subseteq X$ open. $i : U \hookrightarrow X$.

Take $\mathcal{O}_U \equiv \mathcal{O}_X|_U$, structure sheaf of U. $\equiv i^{-1}\mathcal{O}_X$.
Is a scheme, since U_f's form a basis.

$\boxed{\text{Example 3.2.4}}$ $X = \mathrm{Spec}(\mathbb{C}[X,Y])$. & $U \equiv \{(X,Y)\}^c$.
$\Rightarrow$ Scheme U is not an affine scheme.

Is: $\boxed{\text{Prop } \cancel{\text{Example}}\ 3.2.3}$.

Simple case $X = \mathrm{spec}\, A$ & $U = U_f$ $(f \in A)$.
$\Rightarrow (U, \mathcal{O}_U) \cong (\mathrm{spec}\, A_f, \mathcal{O}_{\mathrm{Spec}\, A_f})$

Proof of prop Let: $p \in U \subseteq X$ open. Since X scheme:
find $(V_p, \mathcal{O}_X|_{V_p})$ inside X, with: $p \in V_p$. & $V_p \cong$ affine scheme.
$\Rightarrow$ Take: $(V_p \cap U) \subseteq U$, with structure sheaf restriction.
($V_p \cap U$ may not be affine!)
Say: $V_p \cong \mathrm{spec}(B)$ & distinguished opens in $\mathrm{spec}(B)$ form a basis for the topology $\Rightarrow$ Reduced to "simple case".

Define: $\mathbb{A}^n_k \equiv \mathrm{spec}\, k[X_1, \ldots, X_n]$.

$\boxed{\text{Example 3.2.4}}$ Take: $U = \mathbb{A}^n_k \setminus \{\det(X_{ij}) = 0\}$.
"$U = GL_n(k)$". (Eventually: $\underbrace{U \times U}_{?} \xrightarrow{\times} U$ is morphism of schemes)

$\boxed{\text{Example 3.2.5}}$ (Non-affine scheme.)
$X \equiv \mathbb{A}^2_k \equiv \mathrm{Spec}\, k[X,Y]$ & $U \equiv \mathbb{A}^2_k \setminus \{(X,Y)\}$.
"delete origin: $\mathbb{R}^2$ − origin".
Claim: U not affine scheme.

Alg Geo: <u>lecture 9</u> 25/10/2023.

<u>From last time</u> $U \equiv \mathbb{A}^2 \setminus \{(X,Y)\}$ <u>not</u> affine.

<u>Why</u>! Try to calculate: $\mathcal{O}_U(U)$. Write: $U_X \equiv \mathbb{V}(X)^c \subseteq \mathbb{A}^2$

$U_Y \equiv \mathbb{V}(Y)^c \subseteq \mathbb{A}^2$.

<u>Observe</u>: $U = U_X \cup U_Y$

& $U_X \cap U_Y = \mathbb{A}^2 \setminus \{(XY)\}$.

⊛ $\mathcal{O}_U(U_X) = k[X, X^{-1}, Y]$, $\mathcal{O}_U(U_Y) = k[X, Y, Y^{-1}]$

⊛ $\mathcal{O}_U(U_X \cap U_Y) = k[X, X^{-1}, Y, Y^{-1}]$.

& Restrictions $\mathcal{O}_U(U_X) \longrightarrow \mathcal{O}_U(U_{XY})$ etc are obvious ones.

By <u>Sheaf axioms</u>: $\mathcal{O}_U(U) = k[X, X^{-1}, Y] \cap k[X, Y, Y^{-1}]$.

$\Rightarrow \mathcal{O}_U(U) = k[X, Y]$.

<u>Know</u>: on $(U, \mathcal{O}_U)$: $\exists$ maximal ideal in global section ring with ϕ vanishing locus (e.g. $(X,Y) \subseteq k[X,Y]$)

<u>But</u>: no such maximal ideal in $k[X,Y]$ ↯

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§3.3: Gluing Sheaves.

let: X top. space & cover $\{U_\alpha\}_{\alpha \in I}$. & Sheaves $\{\mathcal{F}_\alpha\}$ on $\{U_\alpha\}$.

& Isom's $\phi_{\alpha\beta} : \mathcal{F}_\alpha|_{U_\alpha \cap U_\beta} \to \mathcal{F}_\beta|_{U_\alpha \cap U_\beta}$

With: $\phi_{\alpha\alpha} = id$, $\phi_{\alpha\beta} = \phi_{\beta\alpha}^{-1}$ & $\boxed{\phi_{\beta\gamma}\phi_{\alpha\beta} = \phi_{\alpha\gamma}.}$

(sort of like equivalence relation) <u>cocycle</u> condition.

Construction 3.3.1 Build sheaf $\mathcal{F}$ on X: for $V \subseteq X$ open:

$$\mathcal{F}(V) \equiv \left\{ (s_\alpha)_\alpha,\ s_\alpha \in \mathcal{F}_\alpha(U_\alpha \cap V) \,\middle|\, \phi_{\alpha\beta}(s_\alpha|_{V\cap U_\alpha \cap U_\beta}) = s_\beta|_{V\cap U_\alpha\cap U_\beta} \right\}.$$

$\Rightarrow \mathcal{F}(V)$ is presheaf.

[if $W \subseteq V$: take $(s_\alpha)_\alpha|_W \equiv \left(\mathrm{Res}^{V\cap U_\alpha}_{W\cap U_\alpha}(s_\alpha)\right)_\alpha \in \mathcal{F}(W)$ by sheaf axioms]

Prop 3.3.2 $\mathcal{F}$ is a sheaf and $\mathcal{F}|_{U_\alpha} \cong \mathcal{F}_\alpha$ on U_α $\forall \alpha$.

Proof: $\mathcal{F}$ is presheaf & sheaf axioms are clear.

Need to show: $\mathcal{F}|_{U_\gamma} \cong \mathcal{F}_\gamma$.

Given $V \subseteq U_\gamma$ & $s \in \mathcal{F}_\gamma(V)$: $\mathcal{F}|_{U_\gamma} \longmapsto \left(\phi_{\gamma\alpha}(s|_{V\cap U_\alpha})\right)_\alpha$

Need to check: lies in $\mathcal{F}|_{U_\gamma}(V) = \mathcal{F}(V)$. Follows from cocycle cond:

$$\phi_{\alpha\beta} \circ \phi_{\gamma\alpha}(s|_{V\cap U_\alpha \cap U_\beta}) = \phi_{\gamma\beta}(s|_{V\cap U_\alpha \cap U_\beta})$$

---

§3.4: More Schemes

Schemes $(X, \mathcal{O}_X)$ & $(Y, \mathcal{O}_Y)$. & opens $U \subseteq X$, $V \subseteq Y$ & an isomorphism $\varphi : (U, \mathcal{O}_X|_U) \xrightarrow{\sim} (V, \mathcal{O}_Y|_V)$.

Can glue both the spaces, and the schemes! $X \sqcup_\varphi Y$ & sheaf as in previous construction.

(No cocycle condition since gluing only 2 things together.)

Alg Geo: lecture 10 27/10/2023

From last time: $U \equiv \mathbb{A}^2_k \setminus \{(x,y)\}$ not affine scheme.

let: X scheme. & $f \in \Gamma(X, \mathcal{O}_X)$. & $p \in X$. Then: $\mathcal{O}_{X,p}$ is stalk of $\mathcal{O}_X$ at p. The stalk is of form $A_{\mathfrak{p}}$, where A is ring, and $\mathfrak{p} \subseteq A$ prime ~~pr~~ ideal.

In particular: $A_{\mathfrak{p}}$ has unique maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$.

Say: f vanishes on $\mathfrak{p}$, if $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ is 0 ($\Leftrightarrow f \in \mathfrak{p}A_{\mathfrak{p}}$).

There is an isomorphism: $p \in V_p$ (open) to $\mathrm{Spec}(A)$.

$\Rightarrow$ For $f \in \Gamma(X, \mathcal{O}_X)$: $\mathbb{V}(f) \equiv$ (vanishing locus of f) $\subseteq X$ is well-defined.

---

let: $(X, \mathcal{O}_X)$ & $(Y, \mathcal{O}_Y)$ ~~ringed spaces~~ sheaves, and: $U \subseteq X$ & $V \subseteq Y$ open, & $(U, \mathcal{O}_X|_U) \xrightarrow{\sim} (V, \mathcal{O}_Y|_V)$.

Can glue: $S \equiv (X \sqcup Y)/(u \sim v)$ (quotient topology).

$\Rightarrow$ The images of X, Y in S form open cover of S, and intersection is: image of U.

$\Rightarrow$ Glue structure sheaves of X, Y by prev lecture (noting: no cocycle condition to be checked).

Example 3.4.1 (Bug-eyed line)

let: $X = \mathrm{Spec}\, k[t]$ & $Y \equiv \mathrm{Spec}\, k[u]$

$U \equiv \mathrm{Spec}\, k[t, t^{-1}]$ & $V \equiv \mathrm{Spec}\, k[u, u^{-1}]$.

Have: isomorphism $U \longrightarrow V$ $\quad k[u,u^{-1}] \longrightarrow k[t,t^{-1}]$

$t \longmapsfrom u$ $\qquad u \longmapsto t.$

Topological level: $X = \mathbb{A}^1_k$ & $Y = \mathbb{A}^1_k$, $U = \mathbb{A}^1_k \setminus pt.$

$\Rightarrow$  (2 origins.)

Types of opens: 1) $W \subseteq (X \text{ or } Y) \subseteq S \Rightarrow$ nice opens.

2) $W = S \setminus \{p_1, \dots, p_n\}$, $p_i \in U$ or V? E.g. $W = S$

What is $\mathcal{O}_S(S)$? By sheafs (similar to $\mathbb{A}^2 \setminus pt$): get $\mathcal{O}_S(S) = k[t]$. ($\Rightarrow S$ not affine!)

Example 3.4.2) X, Y, U, V, as before.

& glue by isomorphism $U \longmapsto V$, $t^{-1} \longmapsfrom u$.

Define: $\mathbb{P}^1_k \equiv$ resulting (glued) scheme.

Prop 3.4.3) $\mathbb{P}^1_k$ is not S, i.e. $\mathcal{O}_{\mathbb{P}^1_k}(\mathbb{P}^1_k) \cong k$.

Proof Only elements of $k[t,t^{-1}]$ ~~&~~ that are polynomials of t and t^{-1} are constants (check).

Example 3.4.4) Can build "$\mathbb{A}^2_k$ with double origin". $\equiv S$

Has property: has $U_1, U_2 \subseteq S$ affine & open with $U_1 \cap U_2$ not affine ($U_i \equiv S -$ i'th origin).

Gluing Schemes.) (Sheet 1)

Given: schemes X_i ($i \in I$) with:

⊛ Open Subschemes $X_{ij} \subseteq X_i$ $(X_{ii} = X_i)$

⊛ Isomorphisms: $f_{ij} : X_{ij} \xrightarrow{\sim} X_{ji}$, $f_{ii} = id_{X_i}$

With: cocycle condition $f_{ik}\big|_{X_{ij} \cap X_{ik}} = f_{jk}\big|_{X_{ji} \cap X_{jk}} \circ f_{ij}\big|_{X_{ik} \cap X_{ij}}$.

Then: $\exists !$ X scheme, with open cover X_i, glued along $X_{ij} \xrightarrow{\sim} X_{ji}$.

Example 3.4.5) A any ring & $X_i \equiv \operatorname{Spec} A\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right]$ & $X_{ij} \equiv \mathbb{V}(x_j/x_i)^c \subseteq X_i$.

& Isomorphisms $X_{ij} \longrightarrow X_{ji}$, $\frac{x_k}{x_i} = \left(\frac{x_k}{x_j}\right) \cdot \left(\frac{x_i}{x_j}\right)^{-1}$.

Satisfies cocycle.

Gluing: is called $\mathbb{P}^n_A$, projective n-space.

Calculation: $\Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}) = A$.

---

§ Proj Construction.

DEF 3.5.1) A $\mathbb{Z}$-grading of ring A is decomposition $A = \bigoplus_{i \in \mathbb{Z}} A_i$, with: $A_i \cdot A_j \subseteq A_{i+j}$. ($A_i$ abelian group)

Example $A = k[x_0, \ldots, x_n]$ & $A_d \equiv$ degree d homog. polys.

& $I \subseteq k[x_0, \ldots, x_n]$, homogeneous ideal (generated by homog. elements of different degree). Then: A/I is also naturally a ~~gradient~~ graded ring.

<u>Assumption</u>: degree 1 elements generate A as an <u>algebra</u> over A_0.

Alg Geo: lecture 11) 30/10/2023

From last time: Graded ring $A = \bigoplus_{d \in \mathbb{Z}} A_d$ & $A_i \cdot A_j \subseteq A_{i+j}$.

& $A_d \equiv$ "degree d elements"

& $A_0 \subseteq A$ subring & assume: A_1 generates A over A_0.

& Assume: $A_i = 0 \ \forall i < 0$.

$\Rightarrow A_+ \equiv \bigoplus_{i>0} A_i \subseteq A$ positive degree subgroup & is ideal

("irrelevant ideal")

& Homogeneous element: something in A_d, some d.

An ideal ~~Ab~~ $I \subseteq A$ is homogeneous, if generated by homogeneous elements.

DEF 3.5.2) $\mathrm{Proj}(A) \equiv$ set of all homog ~~points~~ primes of A that don't contain A_+.

If $I \subseteq A$ homog: $\mathbb{V}(I) = \{ p \in \mathrm{Proj}(A) : p \text{ contains } I \}$

& Zariski Topology: has closed sets $\mathbb{V}(I)$, $I \subseteq A$ homog.

(motivation: $\mathbb{P}^n_k \equiv \mathbb{A}^{n+1}_k - 0 / \sim$).

let: $f \in A_1$ & $U_f \equiv \mathrm{Proj}(A) \setminus \mathbb{V}(f)$. Then: $\{U_f\}_{f \in A_1}$ covers $\mathrm{Proj}(A)$ (since: generate unit ideal)

& $A[1/f] \equiv A_f$ naturally $\mathbb{Z}$-graded. ($\deg(f^{-1}) \equiv \deg(f)^{-1}$)

Example $A = k[X_0, X_1]$. $\Rightarrow$ If $f = X_0$: $A[\frac{1}{f}] = k[X_0, X_1, X_0^{-1}]$.

$\Rightarrow$ Have deg $= 0$ elements λ ($\lambda \in k$), $\frac{x_0}{x_1}$, etc.)

$\boxed{\text{Prop 3.5.3}}$ $\exists$ natural bijection:

$$\left\{ \begin{array}{c} \text{Homog Primes of } A \\ \text{missing } f \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{Primes of } (A_f) \\ \text{of degree } 0 \end{array} \right\}$$

Proof (construction). Primes in A missing $f \Leftrightarrow$ hom. primes of A_f.

Suppose $q \subseteq (A[1/f])_0$ prime. Let $\psi(q)$ gen. by: $\bigcup_{d \geq 0} \{ a \in A_d : \frac{a}{f^d} \in q \} \subset A$.

$\Rightarrow$ Is a prime (check)

For $p \subseteq A$ prime: & homog, & missing f: take

$\varphi(p) \equiv (p \cdot A[1/f] \cap (A[1/f])_0)$.

$\Rightarrow \varphi \circ \psi = id$ (check)

but: $\psi \circ \varphi = id$ is trickier! Will show: $p \overset{?}{=} (\psi \circ \varphi)(p)$ by ~~showing containment~~ establishing both containment.

⊛ Suppose $p \in U_f \subseteq \mathrm{Proj}(A)$. If $a \in p \cap A_d$, then $\frac{a}{f^d} \in \varphi(p)$

$\Rightarrow a \in \psi(\varphi(p))$. ✓

⊛ If $a \in \psi(\varphi(p))$ then $\frac{a}{f^d} \in \varphi(p)$ (some d).

$\Rightarrow \exists b_p \in p$, $\frac{b}{f^e} = \frac{a}{f^d}$ (after inverting f).

$\Rightarrow$ for some $k \geq 0$, $f^k(bf^d - af^e) = 0$. & $f^{e+k} \notin p$.

So, by primality, get: $a \in p$.

<u>Remark</u> Bijection constructed ~~by~~ is compatible with ideal containment: $\Rightarrow$ Homeo $U_f \longrightarrow \mathrm{Spec}(A_f)_0$.

Notice: ~~Spec(A)~~ $\mathrm{Proj}(A)$ is covered by opens, each isomorphic (homeo.) to $\mathrm{Spec}(A_f)_0$, some f.

(*) If $f, g \in A_1$: $U_f \cap U_g$ naturally homeo. to:

$$\mathrm{Spec}(A[\tfrac{1}{f}])_0[f/g] = \mathrm{Spec}(A[f^{-1}, g^{-1}])_0.$$

<u>Take</u>: open cover $\{U_f\}$, with structure sheaf: $\mathcal{O}_{\mathrm{spec}(A_f)_0}$ on each U_f, & isoms on $U_f \cap U_g$ by (*). Cocycle condition follows immediately from formal props of localisation (check!).

<u>Terminology</u>: for $A = k[X_0, \ldots, X_n]$ (standard grading), $\mathrm{Proj}(A)$ is denoted $\mathbb{P}^n_k$. ("Same as" gluing construction)

$\boxed{\text{§4: Morphisms.}}$

"Examples": ~~If~~ $U \subseteq X$ for an open subscheme.

& If $A \longrightarrow B$ ring hom, then $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ should be another example.

<u>§4.1</u>: Given ~~...~~ $(X, \mathcal{O}_X)$ scheme, the stalks $\mathcal{O}_{X,p}$ are <u>local rings</u> (unique maximal ideal). For $f \in \mathcal{O}_X(U)$, with

$p \in U$, can ask: does f <u>vanish</u> at p?

$\iff$ Is: "the image" of f in $\mathcal{O}_{X,p}$ contained in maximal ideal?

<u>DEF</u> A morphism of ringed spaces (i.e. top. space in a sheaf of rings) is $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ with:

* $f : X \longrightarrow Y$ ~~...~~ continuous
* $f^\# : \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X$ is morphism of ~~...~~ sheaves of rings on Y.

Alg Geo, lecture 12 01/11/2023.

From last time: morphism of ringed spaces $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ s.t. $f: X \longrightarrow Y$ & $f^\#: \mathcal{O}_Y \to f_* \mathcal{O}_X$.

Warning: Possible to find $(f, f^\#): (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ s.t. $\exists U \subseteq Y$ open, $\exists q \in U$, $h \in \mathcal{O}_Y(U)$ s.t. $h(q) = 0$, but $f^\#(h) \in \mathcal{O}_X(f^{-1}(U))$ does not vanish at some $p \in X$, $f(p) = q$.

Observation: Given $f: X \longrightarrow Y$ ringed space morphism: $\forall p \in X$, $\exists$ induced map $f^\#: \mathcal{O}_{Y, f(p)} \longrightarrow \mathcal{O}_{X,p}$.

[$\forall s \in \mathcal{O}_{Y, f(p)}$, represent by (s_U, U), U open, $f(p) \in U$ & $s_U \in \mathcal{O}_Y(U)$.

$\Rightarrow f^\#(s) = (f^\#(s_U), f^{-1}(U)) \in \mathcal{O}_{X,p}$ since $f^\#(s_U) \in \mathcal{O}_X(f^{-1}(U))$]

DEF 4.12) $(X, \mathcal{O}_X)$ ringed space is locally ringed if: $\forall p \in X$, $\mathcal{O}_{X,p}$ is local ring (has unique max ideal).

A morphism of locally ringed spaces $(f, f^\#): (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is: morphism of ringed spaces and:

(*) If m_p = max ideal of $\mathcal{O}_{X,p}$ then $f^\#(m_{f(p)}) \subseteq m_p$. (in the stalks)

DEF) A morphism of schemes is: morphism as locally ringed spaces.

Theorem 4.13

$$\left\{\begin{array}{c}\text{Scheme morphisms}\\ \mathrm{Spec}(B)\to\mathrm{Spec}\,A\end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{c}\text{Ring homs}\\ A\to B\end{array}\right\}.$$

Proof Recall: section of sheaf $\mathcal{F}$ on U is: $s(p)\in\mathcal{F}_p$ $\forall p\in U$.

~~Clearly~~ Given any $\varphi: A\to B$, take: $\varphi^{-1}:\mathrm{Spec}(B)\longrightarrow\mathrm{Spec}(A)$, as the topological map. (Is continuous).

& $\varphi^{\#}:\mathcal{O}_{\mathrm{Spec}\,A}\longrightarrow\varphi^{-1}_{*}\mathcal{O}_{\mathrm{Spec}\,B}$, defined by:

(*) $A_{\varphi^{-1}(p)}\longrightarrow B_p$ induced by φ (stalk level)

$\frac{a}{s}\longmapsto\frac{f(a)}{f(s)}$ Makes sense, since: if $s\notin\varphi^{-1}(p)\Rightarrow\varphi(s)\notin p$.

& local, since preimage of max ideal in max ideal.

(*) Open sets: recall, $s\in\mathcal{O}_{\mathrm{Spec}\,A}(U)$ is:

$\{[p\mapsto s(p)]: p\in U \;\&\; s(p)\in A_p\}$.

Define $\varphi^{\#}$ by: $[p\mapsto s(p)]_{p\in U}\longmapsto[q\mapsto\varphi_q(s(\varphi^{-1}(q)))]_{q\in(\varphi^{-1})^{-1}(U)}$.

[$\varphi_q\equiv$ map of stalks at q.]

Gives: coherent collection. $\varphi^{\#}:\mathcal{O}_{\mathrm{Spec}\,A}(U)\to\mathcal{O}_{\mathrm{Spec}\,B}((\varphi^{-1})^{-1}(U))$. ✓

Conversely: Suppose $(f,f^{\#}):\mathrm{Spec}\,B\longrightarrow\mathrm{Spec}\,A$.

Plug in global sections $\Rightarrow$ Get $\mathcal{O}_{\mathrm{Spec}\,A}(\mathrm{Spec}\,A)\longrightarrow\mathcal{O}_{\mathrm{Spec}\,B}(\mathrm{Spec}\,B)$.

$\Rightarrow$ Get ring hom $A\xrightarrow{g}B$.

Need to check:

1) g^{-1} gives right topological map $\operatorname{Spec} B \longrightarrow \operatorname{Spec} A$

2) Construction from before gives correct map (as sheaves)

Know: maps on stalks compatible with restriction:

$$\begin{tikzcd}
\Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \arrow[r] \arrow[d] & \Gamma(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \arrow[d] \\
\mathcal{O}_{\operatorname{Spec} A, f(p)} \arrow[r] & \mathcal{O}_{\operatorname{Spec} B, p}
\end{tikzcd} \quad (p \in \operatorname{Spec} B)$$

$$\Longrightarrow \begin{tikzcd}
A \arrow[r, "g"] \arrow[d] & B \arrow[d] \\
A_{f(p)} \arrow[r, "f^\#"] & B_p
\end{tikzcd} \quad \text{commutes } \forall p \in \operatorname{Spec} B.$$

Since $f^\#$ local: $(f^\#)^{-1} pB_p = f(p) A_{f(p)}$.

$\Longrightarrow$ By commutativity: $g^{-1} = f$ topologically.

Structure sheaf maps agree at stalk level by construction ✓

---

§4.2 Let: X, Y schemes.

DEF 4.2.1) A morphism $f: X \longrightarrow Y$ is open immersion if:
f induces isomorphism of X onto an open subscheme of Y.
i.e. $(U, \mathcal{O}_Y|_U)$ $(U \subseteq Y$ open$)$.

$g : X \longrightarrow Y$ closed immersion if: topological map is a homeo. to a closed subset of Y, $\underline{\underline{\&}}$ $g^{\#} : \mathcal{O}_Y \longrightarrow g_* \mathcal{O}_X$ is surjective.

<u>Exercise</u>: $k[t] \longrightarrow k[t]/t^2$ $\underline{\underline{\&}}$ take Spec. $\Rightarrow$ Closed immersion.

Alg Geo: <u>lecture 13</u> 03/11/2023.

<u>DEF 4.2.3</u> Y scheme. A <u>closed subscheme</u> of Y is an equivalence class of closed immersions ~~with~~ $\{X \to Y\}$, s.t.

$X \xrightarrow{f} Y$ & $X' \xrightarrow{f'} Y$ equivalent $\Leftrightarrow$

$$\begin{tikzcd} X' \arrow[r, "\cong"] \arrow[dr, "f'"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

commutes.

<u>Example</u> (Typical closed immersion):

if A ring & $I \subseteq A$ ideal, then

$\mathrm{Spec}(A/I) \longrightarrow \mathrm{Spec}(A)$ is a closed immersion.

---

§4.3: <u>Fibre Products.</u>

Simultaneously generalise:

- "Product" of schemes
- If $X_1, X_2 \subseteq Y$ <u>closed</u> subschemes then $X_1 \cap X_2 \subseteq Y$ is a fibre product
- Given morphism $X \xrightarrow{f} Y$ & subscheme $Z \subseteq Y$, the preimage $f^{-1}(Z)$ is subscheme of X, described by product.

<u>DEF 4.3.1</u> Consider: diagram

$$\begin{tikzcd} Z \arrow[dr, dashed, "\phi"] \arrow[drr, bend left, dashed, "q_X"] \arrow[ddr, bend right, "q_Y"'] & & \\ & X \times_S Y \arrow[r, "p_X"] \arrow[d, "p_Y"] & X \arrow[d] \\ & Y \arrow[r] & S \end{tikzcd}$$

Given X, Y, S: $X \times_S Y$ is the fibre ~~of~~ (scheme) s.t. for any other scheme Z with maps $Z \xrightarrow{q_X} X$ & $Z \xrightarrow{q_Y} Y$, then $\exists \phi : Z \longrightarrow X \times_S Y$ that makes the diagram commute.

Remark 1) Can define fibre products of sets.

If X, Y, S sets as before:

$$\begin{tikzcd} & X \arrow[d, "r_X"] \\ Y \arrow[r, "r_Y"'] & S \end{tikzcd}$$

$X \times_S Y = \{(x,y) \in X \times Y : r_X(x) = r_Y(y)\}$.

2) Fibre products also exist in topological spaces, with same def. as in Set (given subspace top. of product top.)

3)
$$\begin{tikzcd} & X \arrow[d, "r_X"] \\ \{*\} \arrow[r, "r_Y"'] & S \end{tikzcd}$$

If $r_Y(*) = s \in S$, then: $X \times_S Y = r_X^{-1}(s)$.

Theorem 4.3.2 Fibre products of schemes exist!

Proof (Affine case). Let X, Y, S affine schemes with rings A, B, R. Then, $X \times_S Y$ exists & $\cong \operatorname{Spec}(A \otimes_R B)$.

Check: universal property: given any other scheme Z with morphisms

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

: $\exists!\, Z \xrightarrow{\phi} \operatorname{Spec}(A \otimes_R B)$.

Fact: (sheet 2) Scheme theoretic map $Z \longrightarrow \operatorname{Spec}(A \otimes_R B)$

$\Longleftrightarrow A \otimes_R B \longrightarrow \Gamma(Z, \mathcal{O}_Z)$.

In general (non-affine) case: start with general $X \times_S Y$ & "cover by affines".

⊛ If $X \times_S Y$ & $U \subseteq X$ open subscheme: then $U \times_S Y$ also exists.
(because: take inverse image of U, under:
$X \times_S Y \xrightarrow{p_X} X$, with open subscheme structure.)

⊛ If X covered by opens $\{X_i\}$ and with $X_i \otimes_S Y$ exists $\forall i$, then $X \otimes_S Y$ exists. (by gluing: no cocycle condition)

$\Rightarrow$ For any X & S, Y affine, by above: $X \times_S Y$ exists
(by covering X with affines).
Since X, Y interchangeable $\Rightarrow$ $X \times_S Y$ exists $\forall S$ affine, $\forall X, Y$.

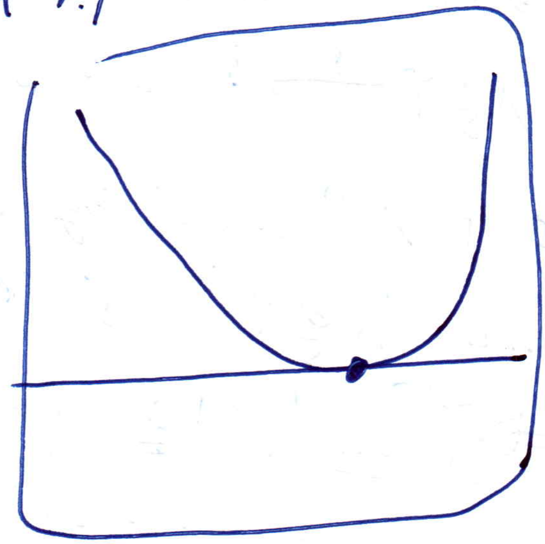
⊛ Cover S with affines $\{S_i\}$. & let: X_i, Y_i be the pre-images of S_i in X & Y resp.
$\Rightarrow X_i \otimes_{S_i} Y_i$ exists $\forall i$.
By Universal Property: $X_i \otimes_{S_i} Y_i = X_i \times_S Y_i$. Glue again ✓

Example 4.3.3) i) $\mathbb{P}^n_{\mathbb{C}} = \mathbb{P}^n_{\mathbb{Z}} \times_{\text{Spec}\,\mathbb{Z}} \text{Spec}(\mathbb{C})$.
Where: $\text{Spec}(\mathbb{C}) \longrightarrow \text{Spec}(\mathbb{Z})$ is induced by $\mathbb{Z} \hookrightarrow \mathbb{C}$.
& $\mathbb{P}^n_{\mathbb{Z}} \longrightarrow \text{Spec}(\mathbb{Z})$ induced by: $\mathbb{Z} \hookrightarrow \mathbb{Z}\left(\frac{X_1}{X_i}, \dots, \frac{X_n}{X_i}\right]$ (locally).
[Similar to: $\mathbb{Z}(X) \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}[X]$.]

ii) let $C \equiv \text{Spec}\left(\mathbb{C}(X, Y)/(Y - X^2)\right)$
$L \equiv \text{Spec}\left(\mathbb{C}(X, Y)/(Y)\right)$

& natural morphisms $C \longrightarrow \mathbb{A}^2_{\mathbb{C}}$ & $L \longrightarrow \mathbb{A}^2_{\mathbb{C}}$.

$\Rightarrow$ By algebra: $C \underset{\mathbb{A}^2}{\times} L = \operatorname{Spec}\left(\mathbb{C}[X]/(X^2)\right)$



Alg Geo: <u>lecture 14</u> 06/11/2023.

<u>Language</u>: In scheme theory, often fix scheme S, ~~then the~~ "base scheme", & consider all other schemes X with fixed morphism $X \longrightarrow S$. "Schemes over S"

$\Rightarrow$ Forms a category, morphisms being:
$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d] \\ & S \end{tikzcd}$$

Typical to keep in mind: $S = \operatorname{spec} k$, $\operatorname{spec} \mathbb{Z}$.

The product in Sch/S is <u>fibre product</u> $X \times_S Y$.

<u>§4.4</u>: <u>Separated morphisms.</u>

[Motivation: X space <u>Hausdorff</u> $\Leftrightarrow$ $\Delta_X \subseteq X \times X$ closed]

<u>DEF 4.4.1</u>] Let $X \longrightarrow S$ morphism of schemes. Diagonal is: morphism $\Delta_{X/S} : X \longrightarrow X \times_S X$, induced by universal prop:

$$\begin{tikzcd} X \arrow[dr, "\exists!"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "id"'] & & \\ & X \times_S X \arrow[r] \arrow[d] & X \arrow[d] \\ & X \arrow[r] & S \end{tikzcd}$$

[Write: Δ for $\Delta_{X/S}$ if $X \to S$ is clear from context.]

<u>Observation</u>: If $U, V \subseteq X$ open $\Rightarrow$ $\Delta^{-1}(U \times_S V) = U \cap V$.
(for $S = \operatorname{spec}(k)$, k field)

<u>DEF 4.4.2</u>] Morphism $X \longrightarrow S$ <u>separated</u> $\Leftrightarrow$ $\Delta_{X/S}$ is closed immersion.

<u>Example</u>] $X = \operatorname{spec} \mathbb{C}[t]$, $S = \operatorname{spec} \mathbb{C}$. $X \longrightarrow S$ induced by $\mathbb{C} \longrightarrow \mathbb{C}[t]$. <u>Then</u>: $X \times_S X = \operatorname{Spec}(\mathbb{C}[t] \otimes_{\mathbb{C}} \mathbb{C}[t])$

& Δ is: $\mathrm{Spec}(\mathbb{C}[t] \otimes_{\mathbb{C}} \mathbb{C}[t] \longrightarrow \mathbb{C}[t])$ multiplication

($\mathrm{Spec}(A \to B)$ is morphism in opposite directions)

Δ immersion since $\mathbb{C}[t] \otimes_{\mathbb{C}} \mathbb{C}[t] \xrightarrow{\times} \mathbb{C}[t]$ surjective.

<u>Prop 4.4.3</u> Let $X \longrightarrow S$ scheme morphism. Then $\Delta_{X/S}$ factors:

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend right, "\Delta_{X/S}"'] & U \arrow[r] & X \times_S X \end{tikzcd}$$

first map closed immersion

2nd map open immersion.

"locally closed morphism"

<u>Proof</u>: let $g: X \longrightarrow S$ morphism.

Say: S covered by <u>open</u> <u>affine</u> $\{V_i\}$ & X covered by $\{U_{ij}\}$ where $\{U_{ij}\}$ <u>open</u> <u>affine</u> & covers $g^{-1}(V_i)$. $\forall i$.

Have morphisms $U_{ij} \longrightarrow V_i$ induced by:

$$\begin{tikzcd} U_{ij} \arrow[r] & g^{-1}(V_i) \arrow[r] \arrow[d] & V_i \arrow[d] \\ & X \arrow[r] & S \end{tikzcd}$$

& $U_{ij} \times_{V_i} U_{ij}$ is <u>open</u> <u>affine</u> in $X \times_S X$, and their union contain image of $\Delta_{X/S}$.

& $\Delta^{-1}(U_{ij} \times_{V_i} U_{ij}) = U_{ij} \subseteq X$.

$\Rightarrow$ <u>Take</u>: $U \equiv$ Union of $U_{ij} \times_{V_i} U_{ij}$ (over all i, j)

& observe: "$T \to T'$ is closed immersion" can be proven locally on the codomain.

If U_{ij} affine $\Rightarrow$ diagonal $U_{ij} \longrightarrow U_{ij} \times_{V_i} U_{ij}$, and is "clearly" a closed immersion (similar to example 4.4.2)

Prop 4.4.3): If $X \to S$ morphism of <u>affine</u> schemes, then $\Delta_{X/S}$ is closed immersion.

[<u>Indeed</u>: $X = \operatorname{spec} A$, $S = \operatorname{spec} B$ & $X \to S$ is $\operatorname{Spec}(B \to A)$.
Then, $A \otimes_B A \to A$ is surjective ✓]

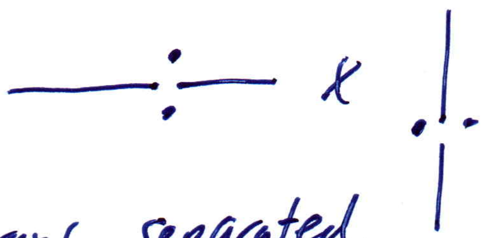
Example 4.4.5) ⊛ Bug-eyed line.

$X = (\mathbb{A}^1_k \sqcup \mathbb{A}^1_k)/\sim$, with: $U = \mathbb{A}^1_k - 0 \subset \mathbb{A}^1$, similar for V.
& $V \xrightarrow{\sim} U$ by $k[u, u^{-1}] \to k[t, t^{-1}]$
$u \mapsto t$.

<u>Claim</u>: Not separated. (over spec k)

What is $X \times_S X$? $\Rightarrow$ Compute via: gluing construction of fibre ~~products~~ product, ~~plane~~ $\Rightarrow$ Get plane with 4 origins.

& Diagonal only contains 2 of 4 origins.
$\Rightarrow$ Not a closed subset.



⊛ Open & closed immersions are <u>always</u> separated.

<u>Conseq of Prop 4.4.3</u>: $X \to S$ morphism of schemes. If $\operatorname{im}(\Delta_{X/S})$ <u>closed</u> (as topological space) then $X \to S$ separated.

Prop 4.4.6) A ring. $\Rightarrow$ Morphism $\mathbb{P}^n_A \to \operatorname{Spec}(A)$ is separated. (proof next time)

Prop 4.4.7) k field, $X \longrightarrow \mathrm{spec}(k)$ scheme morphism. & $U, V \subseteq X$ affine. If $X \longrightarrow \mathrm{Spec}\, k$ separated, then $U \cap V$ is also affine.

Alg Geo: lecture 15 Separated Morphisms. 08/11/2023.

From last time: $X \to S$ separated $\iff \Delta_{X/S} : X \to X \times_S X$ closed.

Example ⊛ $\mathbb{A}^n_k \to \operatorname{Spec} k$ separated.

Since: $\mathbb{A}^n_k \times_{\operatorname{Spec} k} \mathbb{A}^n_k = \operatorname{Spec}(k(\underline{x}) \otimes_k k(\underline{y}))$, and Δ is map induced by mult. $\left(k(\underline{x}) \otimes_k k(\underline{y}) \to \underline{k}[\underline{x}], \quad f(x) \otimes g(y) \mapsto f(x)g(x) \right)$.

Non-example ⊛ line with 2 origins.

Properties ⊛ Open & closed immersions are separated.

[For closed immersions: since $(A/I) \otimes_A (A/J) \cong A/(I+J)$, have, Δ is identity locally, hence closed.]

⊛ Compositions of separated morphisms is separated.

⊛ "Base extensions": Suppose $X \to S$ separated & $S' \to S$ arbitrary. Then: $X \times_S S' \to S'$ from

$$\begin{tikzcd} X \times_S S' \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

is also separated.

Prop 4.4.6 R any ring. Then:

$\mathbb{P}^n_R \to \operatorname{Spec}(R)$ separated.

$$\begin{tikzcd} \mathbb{P}^n_R \arrow[r, "\Delta"] & \mathbb{P}^n_R \otimes_R \mathbb{P}^n_R \arrow[r] \arrow[d] & \mathbb{P}^n_R \arrow[d] \\ & \mathbb{P}^n_R \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$

Proof Want: Δ closed.

Suffices to check on: open cover of $\mathbb{P}^n_R \otimes_R \mathbb{P}^n_R$.

Let: $A = R[X_0, \dots, X_n]$ (usual grading)

& $U_i \equiv \operatorname{Spec}(A[1/X_i]) \Rightarrow \{U_i\}_{i=0}^{n}$ cover $\mathbb{P}^n_R$.

& $U_i \otimes_R U_j = \operatorname{Spec} R\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}, \frac{Y_0}{Y_j}, \dots, \frac{Y_n}{Y_j}\right]$

Observe: restriction of Δ to $\Delta^{-1}(U_i \times_R U_j)$ is:

$$U_i \cap U_j \longrightarrow U_i \times_R U_j$$

$$R\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right]\left[\frac{X_i}{X_j}\right] \longleftarrow R\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}, \frac{Y_0}{Y_j}, \dots, \frac{Y_n}{Y_j}\right].$$

Map is: "Change y's to x's".

This map is clearly surjective, & $U_i \times_R U_j$ cover $\mathbb{P}^n_R \times_R \mathbb{P}^n$.

$\Rightarrow \Delta$ is closed ✓

Let: k algebraically closed field ($k = \bar{k}$) & $X \longrightarrow \operatorname{Spec} k$ scheme. (over spec k).

X is of finite type (over Spec k) if $\exists$ cover $\{U_i\}$ of X by affines, s.t. $\mathcal{O}_X(U_i)$ f.g. k-algebra $\forall i$.

& X reduced if: $\forall U \subseteq X$ open, $\mathcal{O}_X(U)$ has no nilpotence.

DEF] $X \longrightarrow \operatorname{Spec}(k)$ variety if: reduced & finite type & separated.

---

§4.6: Properness.

Let: $X \xrightarrow{f} S$ morphism. Then, f finite type if $\exists \{V_\alpha\}$ affine cover of S, $V_\alpha \equiv \operatorname{Spec}(U_\alpha)$, and covers $\{U_{\alpha\beta}\}_\beta$

of $f^{-1}(V_\alpha)$, by open affines, $U_{\alpha\beta} = \mathrm{Spec}(B_{\alpha\beta})$, s.t. $B_{\alpha\beta}$ f.g. A_α-algebra and: $\{U_{\alpha\beta}\}_\beta$ can be chosen to be finite.

DEF 4.6.1) $X \xrightarrow{f} S$ closed $\Leftrightarrow$ closed topological map

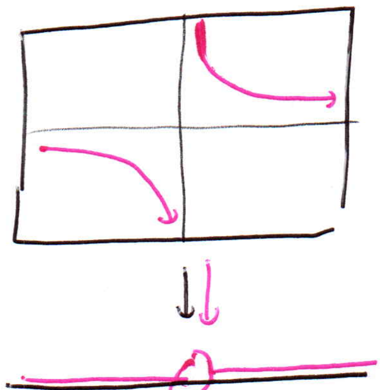
Universally closed $\Leftrightarrow$ f closed & $\forall S' \to S$, induced $X \times_S S' \to S'$ is closed.

& proper $\Leftrightarrow$ separated + finite type + Universally closed.

Examples ⊛ Closed immersions are proper.

Non-example: Usual map $\mathbb{A}^1_k \to \mathrm{Spec}(k)$ not proper.

Proof:
$$\begin{array}{ccc} \mathbb{A}^2 & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathrm{Spec}\, k \end{array}$$



Consider: $Z \equiv \mathbb{V}[xy-1] \subseteq \mathbb{A}^2 = \mathrm{Spec}\; k[x,y]$.

$\Rightarrow f^{-1}(Z)$ not Zariski closed.

Observation If $X \to S$ proper: any base extension $X \times_S S' \to S'$ also proper.

Prop 4.6.2) R any ring. Then: $\mathbb{P}^n_R \to \mathrm{Spec}(R)$ proper.

Alg Geo: lecture 16 10/11/2023.

Terminology: If $X \longrightarrow \mathrm{Spec}(k)$ morphism, say "X is _".

Examples) $\mathbb{A}^1_k$ separated & not proper.

& Line with 2 origins neither separated/universally closed.

Prop 4.6.2) R commutative ring $\Rightarrow \mathbb{P}^n_R \longrightarrow \mathrm{Spec}(R)$ proper.

Observation: Universal closedness of $X \longrightarrow S$ stable under base extension [i.e. $S' \longrightarrow S$, $X \times_S S' \longrightarrow S'$]

& Since already know $\mathbb{P}^n_R \longrightarrow \mathrm{Spec}(R)$ separated, finite type is immediate.

$\Rightarrow$ Suffices to show: when $R = \mathbb{Z}$ & just need closedness.

[since: $\mathrm{Spec}\, R \times_{\mathrm{Spec}\mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}} = \mathbb{P}^n_R$].

Proof (of $R = \mathbb{Z}$). Must show: for any $Y \longrightarrow \mathrm{Spec}\,\mathbb{Z}$, base extension $\mathbb{P}^n_{\mathbb{Z}} \times_{\mathrm{Spec}\mathbb{Z}} Y \longrightarrow Y$ is closed.

But: Y covered by affine schemes of form $\mathrm{Spec}(R)$, & closedness is local on target. So, suffices to show $\mathbb{P}^n_R \to \mathrm{Spec}\, R$ is closed

Let: $Z \subseteq \mathbb{P}^n_R$ Zariski closed. (i.e. $\mathbb{V}$ of hom polys). say: $g_1, g_2, \ldots,$. Goal: if $\pi: \mathbb{P}^n_R \longrightarrow \mathrm{Spec}(R)$, then need $\pi(Z)$ closed too.

Key: Need equations for $\pi(Z) \Longleftrightarrow$ Need to characterise those $p \in R$, s.t. $Z \cap \pi^{-1}(p) \neq \phi$.

<u>Let</u>: $k(p) \equiv FF(R/p)$. Have: morphism $\operatorname{Spec} k(p) \to \operatorname{Spec} R$.

Want to know for which p is $Z_p \equiv Z \times_{\operatorname{Spec} R} \operatorname{Spec} k(p) \neq \phi$?

<u>What is Z_p?</u> Take eqns $g_1, \ldots$ hom polys, with R-coeffs.

Reduce (mod p) to get $\bar{g}_1, \ldots$, hom polys with coeffs in $k(p)$.

$\Rightarrow Z_p \neq \phi \Leftrightarrow \bar{g}_1, \ldots$ cut out only <u>origin</u> in $\mathbb{A}^{n+1}_{k[p]}$

$\Leftrightarrow \sqrt{(\bar{g}_1, \bar{g}_2, \ldots)} \not\ni (X_0, \ldots, X_n)$ $\quad (\mathbb{P}^n_R \equiv \operatorname{Proj} R(X_0, \ldots, X_n))$

$\Leftrightarrow \forall d \geq 1,\ (X_0, \ldots, X_n)^d \nsubseteq (\bar{g}_1, \bar{g}_2, \ldots)$

<u>Write</u>: $A = R(\underline{X})$, with usual grading. Then, non-containment

$$\Leftrightarrow \bigoplus_i A_{d-\deg(g_i)} \longrightarrow A_d \quad (*)$$
$$f_i \longmapsto f_i \cdot g_i \quad \text{(for } i\text{'th factor)}$$

is <u>non-surjective mod p</u>. (or in $k(p)$) $\underline{\underline{\forall d}}$.

This condition is given by: Vanishing of maximal matrices ($\dim_R A_d \times \dim_R A_d$) of matrix associated to this $(*)$ map.

$\Rightarrow$ Infinitely many polys, coeffs of the g_i ✓

Hence forward: assume all schemes <u>Noetherian</u>.

<u>Valuative Criteria</u> (for separatedness + properness):

A discrete valuation ring is a <u>local PID</u>.

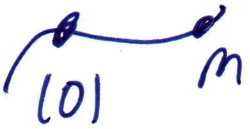
<u>Examples</u>: $\mathbb{C}[[t]]$, $\mathcal{O}_{\mathbb{A}^1,0} \equiv \left\{ \frac{f(t)}{g(t)} : g(0) \neq 0 \right\}$

$\mathbb{Z}_{(p)}$ & $\mathbb{Z}_p$ p-adic integers.

<u>Terminology / Observations</u>:

⊛ A DVR. $\Rightarrow$ $\mathrm{Spec}(A)$ consist of 2 : $(0) \subseteq A$ & $m \subseteq A$.

⊛ Topology in $\mathrm{Spec}(A) = \{(0), m\}$. & (0) <u>dense</u>, so closure of $\{(0)\}$ is $\mathrm{spec}(A)$. & <u>m closed</u>.



"Germ of a curve".

⊛ Any generator π for m is called: Uniformiser / uniformising parameter.

⊛ Any $a \in A$ is $u \cdot \pi^k$, u unit, k unique.

& $k \equiv$ valuation of a.

⊛ Gives a map: $A \setminus \{0\} \longrightarrow \mathbb{N}$ (indep of choice π).

$$a \longmapsto k$$

Alg Geo: lecture 17 13/10/2023.

Let: A valuation ring (local PID). Then, spec(A) is 2 points (0), m. & $m = (\pi)$ (π uniformizer).

& Valuation $A \setminus 0 \longrightarrow \mathbb{N}$ where $a = u \cdot \pi^k$ (u unit).
$a \longmapsto k$

& Valuation Field: $K = FF(A) \Rightarrow$ Valuation extends to
$K - 0 \longrightarrow \mathbb{Z}$, $\frac{a}{b} \longmapsto v(a) - v(b)$. ($\forall a, b \in A$).

Example) $A = k[[t]] \Rightarrow K = k((t))$. Valuation = "degree".

Picture $\operatorname{Spec} A \xleftarrow[\text{immersion}]{\text{open}} \operatorname{Spec} K$

Disk Disk $- *$.

Theorem 4.6.2) $X \xrightarrow{f} Y$ morphism of schemes. Then, f separated $\Leftrightarrow \forall A$ DVR, with fraction field K,

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} A \arrow[r] \arrow[ur] & Y
\end{tikzcd}$$

given blue arrows, there is ≤ 1 choice of red arrow to make it commute.

& f universally closed $\Leftrightarrow \exists$ red arrow as above.

Corollary 4.6.3)

i) $\mathbb{P}^n_R \longrightarrow \operatorname{Spec} R$ proper

ii) $\mathbb{A}^n_R \longrightarrow \operatorname{Spec} R$ separated, not proper

iii) Closed immersions are proper, so in particular: if $Z \to \mathbb{P}^n_R$ closed then $Z \longrightarrow \operatorname{Spec} R$ proper.

iv) Compositions of proper (or separated) morphisms remains so
v) (Base extension) If $X \xrightarrow{f} Y$ proper & $Y' \longrightarrow Y$ arbitrary then $X \times_Y Y' \longrightarrow Y'$ proper.

Sample Verifications.

i) $\mathbb{A}^1_k \longrightarrow \mathrm{Spec}(k)$ not proper (not universally closed).

Write: $\mathbb{A}^1_k = \mathrm{Spec}\, k[X]$, & consider: $A = k[[t]]$ & $K = k((t))$.

$$\begin{tikzcd}
\mathrm{Spec}\, k((t)) \arrow[r, "\varphi"] \arrow[d] & \mathbb{A}^1_k \arrow[d] \\
\mathrm{Spec}\, k[[t]] \arrow[r] & \mathrm{Spec}\, k
\end{tikzcd}$$

Let φ induced by:
$$k[X] \longrightarrow k((t))$$
$$X \longmapsto 1/t.$$

Then: does not extend.

Extension: if $\mathrm{Spec}\, A \xrightarrow{} \mathrm{Spec}\, k$ proper $\Rightarrow$ $\mathrm{Spec}\, A$ finite.

Observation: If we replace $\mathbb{A}^1_k$ with $\mathbb{P}^1_k$ then there is an affine patch s.t. map locally looks like $X \longmapsto t$.

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§5: Modules over $\mathcal{O}_X$.

Example 5.1.1 Let: $\mathbb{C}P^n \equiv (\mathbb{C}^{n+1} - 0)/\mathbb{C}^*$.

& $\mathcal{O}_{\mathbb{C}P^n}$ structure sheaf: if $U \subseteq \mathbb{C}P^n$ ~~open~~ Z-open, then

$$\mathcal{O}_{\mathbb{C}P^n}(U) = \left\{ \frac{P(\underline{X})}{Q(\underline{X})} : P, Q \text{ homog, same deg, regular } \forall p \in U \right\}.$$

& $\forall d \in \mathbb{Z}$: consider sheaf $\mathcal{O}_{\mathbb{C}P^n}(d)(U) \equiv \left\{ \frac{P(\underline{X})}{Q(\underline{X})} : P, Q \text{ homog } \& \deg P - \deg Q = d \right\}.$

note: $\mathcal{O}_{\mathbb{C}P^n}(d)(U)$ is module over ring $\mathcal{O}_{\mathbb{C}P^n}(U)$.

Example 5.1.2 A ring, M A-module. Define sheaf $\mathcal{F}_M$ on $\mathrm{spec}(A)$ by: if $U \subseteq \mathrm{Spec}\, A$ distinguished open ($U = U_f$) then $\mathcal{F}_M(U) \equiv M_f$. (& on general opens, use sheaf on a base construction)

§5.2 $\mathcal{O}_X$-modules.

Fix: $(X, \mathcal{O}_X)$ ringed space.

DEF 5.2.1 A sheaf of $\mathcal{O}_X$-modules on X is sheaf $\mathcal{F}$ of groups with mult. $\mathcal{F}(U) \times \mathcal{O}_X(U) \longrightarrow \mathcal{F}(U)$ giving a module, s.t. mult compatible with restriction.
(Similarly, can define a sheaf of $\mathcal{O}_X$-algebras.)

Morphisms between sheaves of modules $\varphi: \mathcal{F} \to \mathcal{G}$ on X is homomorphism of sheaves of abelian groups, compatible with this multiplication.

Alg Geo: <u>lecture 18</u> $\mathcal{O}_X$-modules. 15/11/2023.

<u>Motivation</u>: Analogous to morphisms of vector bundles.

From last time: $X = \operatorname{spec} A$, $\Rightarrow$ If M A-module, then M^{sh} is sheaf on X by $M^{sh}(U_f) = M_f$. (Sheaf associated to M.)

<u>Basic ops</u>: Given $\mathcal{O}_X$-module sheaf morphism $f: \mathcal{F} \to \mathcal{G}$:
Can take: kernel, cokernel*, image*, direct sum, tensor product.
$\Rightarrow$ All extend to sheaves of modules. [(*) require sheafification]

<u>E.g.</u> The sheaf tensor product $\mathcal{Y} \equiv \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ associates: to $U \underset{\text{open}}{\subseteq} X$, $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$ (& sheafify).

<u>Let</u>: $f: X \longrightarrow Y$ morphism of ringed spaces/schemes.
Given $\mathcal{O}_X$-module $\mathcal{F}$, pushforward $f_*\mathcal{F}$ is sheaf of abelian groups. But: $f^\#: \mathcal{O}_Y \to f_*\mathcal{O}_X$, so this gives $\mathcal{O}_Y$-module structure on $f_*\mathcal{F}$: <u>Given</u> $U \underset{\text{open}}{\subseteq} Y$, $a \in \mathcal{O}_Y(U)$, $m \in f_*\mathcal{F}(U)$:
define $a \cdot m \equiv f^\#(a) \cdot m$ $\quad (f^\#(a) \in \mathcal{O}_X(f^{-1}(U)))$.

<u>Conversely</u>: if $\mathcal{G}$ is sheaf of $\mathcal{O}_Y$-modules: <u>define</u>
$f^*\mathcal{G} \equiv f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ & $(f^{-1}\mathcal{O}_Y)$-module structure on $\mathcal{O}_X$ defined via adjoint to $f^\#$.

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§5.3: $\mathcal{O}_X$-modules on schemes & Quasi-coherence.

<u>DEF 5.3.1</u> A quasi-coherent sheaf $\mathcal{F}$ (on scheme X) is:

(of $\mathcal{O}_X$-modules) is sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, s.t.

$\exists$ cover of X $\{U_i\}$ of affines, s.t. $\mathcal{F}|_{U_i}$ is sheaf associated to a module over ring $\mathcal{O}_X(U_i)$.

If modules over these $\mathcal{O}_X(U_i)$ can be taken to be fin-gen: say $\mathcal{F}$ is <u>coherent</u>.

<u>Examples</u>: * If $\mathcal{O}_X$ (X scheme) q-coherent (or coherent) then $\mathcal{O}_X^{\oplus n}$ is also ($\forall n$).

* $\bigoplus_I \mathcal{O}_X$, $|I| = \infty$, quasi-coherent but <u>not</u> coherent

* If $i: X \hookrightarrow Y$ <u>closed</u> immersion: $i_* \mathcal{O}_X$ is q-coh. $\mathcal{O}_Y$-mod.
Say: $U \subseteq Y$ affine, $U = \operatorname{spec} A$. Then $X \cap U \hookrightarrow U$ gives ideal $I \subseteq A$, i.e. kernel of $\mathcal{O}_Y(U) \to \mathcal{O}_X(X \cap U)$.
$\Rightarrow$ On U, $(i_* \mathcal{O}_X)|_U$ is sheaf associated to A-module A/I.

<u>Prop 5.3.2)</u> An $\mathcal{O}_X$-module $\mathcal{F}$ is q-coh. <u>iff</u>: $\forall U \subseteq X$ open, $U = \operatorname{spec} A$, $\mathcal{F}|_U$ is the sheaf associated with a module over A.
(coherent: similar, + f.g. module.)

<u>lemma 5.3.3)</u> $X = \operatorname{spec} A$, $f \in A$, $\mathcal{F}$ q-coh ~~over~~ $\mathcal{O}_X$-mod. & $s \in \mathcal{F}(X) \equiv \Gamma(X, \mathcal{F})$.

<u>i)</u> If s restricts to 0 on U_f, then $\exists n$, $f^n s = 0$.

<u>ii)</u> If $t \in \mathcal{F}(U_f)$: then $\exists m$, $f^m \cdot t$ is restriction of a global section of $\mathcal{F}$.

Proof $\exists$ cover $\{V = \mathrm{spec}(B)\}$ of X by schemes, s.t.

$\mathcal{F}|_V = M^{sh}$ for some M B-module.

& Cover V, by distinguished ~~opens~~ affines U_g, $g \in A$.

With: $\mathcal{F}|_{U_g} = (M \otimes_B A_g)^{sh}$ (as: $\mathcal{F}|_V$ already q-coh.)

Recall: Spec A is "quasi-compact" i.e. every open cover has a finite subcover), so finitely many g_i, s.t. U_{g_i} & M_i will suffice to cover X by opens, s.t. $\mathcal{F}$ restricts to M^{sh} on U_{g_i}.

& lemma follows from formal properties of localisation

Proof of 5.3.2 Let: $\mathcal{F}$ q-coh. on X.

* Given $U \subseteq X$: $\mathcal{F}|_U$ also q-coh. $\Rightarrow$ Reduce 5.3.2 to the case $X = \mathrm{spec}\, A$.

* Take: $M = \Gamma(X, \mathcal{F})$ & M^{sh} associated sheaf.

Claim: $M^{sh} \cong M$ (isomorphic).

Let: $\alpha : M^{sh} \to \mathcal{F}$ given by restriction. (e.g. via stalks).

$\Rightarrow \alpha$ is isomorphism at stalk level. (Content of key lemma) ✓

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From last time: If $X = \mathrm{spec}\, A$, then q-coh sheaf on X is $\Leftrightarrow$ modules over A, & coh. sheaf on $X \Leftrightarrow$ f.g. modules.

Facts: (proofs omitted & non-examinable*)

⊛ Images, kernels, cokernels of maps of q-coh sheaves are also q-coh. (Same, when q-coh $\Leftrightarrow$ coh.)

⊛ If $X \xrightarrow{f} S$ scheme map & $\mathcal{F}$ quasi-coherent on S, then $f^*\mathcal{F}$ is also quasi-coherent

⊛ If $X \xrightarrow{f} S$ as above & $\mathcal{G}$ q-coh. on S, then: $f_*\mathcal{G}$ is also q-coh.

In general: if $\mathcal{G}$ coherent, $f_*\mathcal{G}$ not necessarily coherent.

Example: $f: \mathbb{A}_k \to \mathrm{Spec}\, k$ scheme map. Then, $f_* \mathcal{O}_{\mathbb{A}^1}$ is quasi-coherent sheaf on $\mathrm{spec}\, k$, so is k-vector space, i.e. $k[t]$. (not coherent.)

Observe: if instead $f: \mathbb{P}^1_k \to \mathrm{Spec}\, k$, then $f_* \mathcal{O}_{\mathbb{P}^1_k}$ is sheaf associated with k.

General fact: If $\mathcal{G}$ coherent sheaf on X & $f: X \to S$ proper, then: $f_*\mathcal{G}$ is coherent.

Source of examples. Let A graded ring (by $\mathbb{N}$).

Built: $\mathrm{Proj}(A)$, covered by $\mathrm{Spec}(A[f^{-1}]_0)$, $f \in A$.

<u>Construction</u>: M is A-module (graded), $M = \bigoplus_{d \in \mathbb{Z}} M_d$
for abelian M_d, s.t. $A_i \cdot M_j \subseteq M_{i+j}$.

Consider: sheaf determined by association:

$$\begin{array}{l} \mathrm{Proj}(A) \supseteq U_f \longmapsto (M[1/f]_\bullet)_0 \\ \qquad\qquad\; \shortparallel \\ \qquad\qquad \mathbb{V}(f)^c. \end{array}$$

($\Leftarrow$ deg 0 of localisation of f)

<u>Notation</u>: X scheme & $\mathcal{F}$ quasi-coherent $\mathcal{O}_X$-module.

Say: $\mathcal{F}$ <u>free</u> $\Leftrightarrow$ $\exists I$ set, $\mathcal{F} \cong \mathcal{O}_X^{\oplus I}$.

& $\mathcal{F}$ <u>vector bundle</u> if: $\exists \{U_i\}$ open cover of X, s.t. $\mathcal{F}|_{U_i}$ is (locally) free $\forall i$. (& note: coherent $\Leftrightarrow I$ finite)

& Line bundle $\Leftrightarrow$ Vector bundle + locally isomorphic to $\mathcal{O}_X$.
(module sheaf)

---

§5.4: Coherent Sheaves on Projective Schemes.

<u>DEF 5.4.1)</u> A graded ring & M graded A-module.
For $d \in \mathbb{Z}$, define $M(d) \equiv$ module s.t. $(M(d))_k = M_{k+d}$.

<u>DEF)</u> For $X = \mathrm{Proj}(A)$, the sheaf $\mathcal{O}_X(d)$ defined by: sheaf associated with graded module $A(d)$. ($d \in \mathbb{Z}$)

In particular: $\mathcal{O}_X(1) \equiv$ Twisting Sheaf
$\mathcal{O}_{\mathbb{P}^1}(1)$ important example.

& Note: $\mathcal{O}_X(d) = \mathcal{O}_X(1)^{\otimes d}$.

<u>Orient</u>: let $\mathrm{Proj}\ k[X_0, \ldots, X_n] = \mathbb{P}^n_k$. Then, global sections of $\mathcal{O}_{\mathbb{P}^n}(d)$ are homogeneous, degree d polys in X_i.

$\Rightarrow$ In particular: ~~the~~ if $d<0$, then $\Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}^n}(d)) = 0$.

<u>DEF 5.4.5)</u> An $\mathcal{O}_X$-module $\mathcal{F}$ is <u>globally generated</u> if: is quotient of $\mathcal{O}_X^{\oplus r}$, some r.

$\Leftrightarrow \exists$ surj map $\mathcal{O}_X^{\oplus r} \longrightarrow \mathcal{F}$.

$\Leftrightarrow \exists\, s_1, \ldots, s_r \in \Gamma(X, \mathcal{F})$ s.t. $\{s_i\}$ generate <u>stalks</u> $\mathcal{F}_p$ over $\mathcal{O}_{X,p}$ $\forall p \in X$.

Let: $i : X \hookrightarrow \mathbb{P}^n_{R_0}$ closed immersion. & $\mathcal{O}_X(1) \equiv i^* \mathcal{O}_{\mathbb{P}^n}(1)$ be restriction of $\mathcal{O}_{\mathbb{P}^n}(1)$.

Theorem 5.4.6) let: $\mathcal{F}$ coherent sheaf on X. Then, $\exists\, d_0$ s.t. $\forall d \geq d_0$, $\mathcal{F}(d) \equiv \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d)$ is globally generated.

Proof By formal properties, ~~if~~ equivalent to show: statement for $i_* \mathcal{F}$ ($\Longleftrightarrow$ $i_* \mathcal{F}(d)$ globally generated on $\mathbb{P}^n_R$).

Strategy: 1) Cover by affines $U_i \equiv \operatorname{Spec} R_0 \left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right]$.

2) $i_* \mathcal{F}|_{U_i}$ is sheaf associated to module M_i. Pick a generator $\{S_{ij}\}_j$ for M

3) "Clear denoms" by multiplying S_{ij} by X_i^d (large d)

4) Extend them to global sections of $\mathcal{F}(d)$.

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From last time: $X \overset{i}{\hookrightarrow} \mathbb{P}^n_R$ closed immersions, such that: $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^1}(1)$.

Theorem 5.4.6: If $\mathcal{F}$ coherent on X: $\exists d_0$, s.t. $\forall d \geq d_0$, $\mathcal{F}(d) \equiv \mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X(1)$ is globally generated.

Proof From last time: Suffices to show for $X = \mathbb{P}^n$.

Have: $\mathbb{P}^n_R = \mathrm{Proj}\, R[X_0, \ldots, X_n]$ cover $\mathbb{P}^n$ by $U_i \equiv \mathrm{Spec}\, \underbrace{R[X_s/X_i]}_{B_i}$

& know: $\mathcal{F}|_{U_i} = M_i^{sh}$, where: M_i is f.g. B_i-module.

$\Rightarrow$ Can take generators $\{s_{ij}\}$ for M_i.

Claim: Sections $\{X_i^d \cdot s_{ij}\}_j$, which are sections of $\mathcal{F}(d)|_{U_i}(U_i)$ $\equiv (\mathcal{F} \otimes \mathcal{O}_X(d))|_{U_i}(U_i)$, are restrictions of global sections $L_{ij} \in \Gamma(\mathbb{P}^n, \mathcal{F}(d))$. (for all sufficiently large d, indep of i,j)

Why generate? On U_i, the s_{ij} generate M_i^{sh}. But, have morphism $X_i^d : \mathcal{F} \to \mathcal{F}(d)$

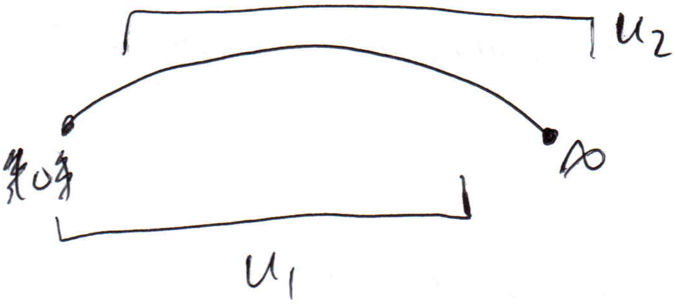
$$s \mapsto (s \otimes X_i^d) \quad (\equiv X_i^d s)$$

& On U_i, this restricts to ~~isomorphism~~ isomorphism: $\mathcal{F}|_{U_i} \xrightarrow{\sim} \mathcal{F}(d)|_{U_i}$ (since X_i invertible on U_i).

$\Rightarrow$ Since $\{s_{ij}\}$ generate $\mathcal{F}|_{U_i}$, the $X_i^d \cdot s_{ij}$ generate $\mathcal{F}(d)|_{U_i}$ hence L_{ij} globally generate.

Back to claim. Say: s_{ij} element of $M_i \equiv \mathcal{F}(U_i)$, and: let $X_i \in \mathcal{O}_{\mathbb{P}^n}(1)$.

Claim: $X_i^{d} \cdot s_{ij} \in (\mathcal{F} \otimes \mathcal{O}(d))(U_i)$, restriction of global section.



Have: $s_{ij} \in \mathcal{F}(U_i)$ restricts to $s_{ij}|_{U_i \cap U_j}$ (lemma 5.3.3)

$\Rightarrow$ Is: ~~extension~~ restriction, after multiplying high power of X_i ✓

Corollary $\mathcal{F}$ quotient of $\mathcal{O}(-d)^{\oplus N}$, some $N \geq 0$, $d \in \mathbb{Z}$.

(Theorem $\Rightarrow \mathcal{O}_X^{\oplus N} \twoheadrightarrow \mathcal{F}(d)$, & Tensor by $\mathcal{O}(-d)$.)

§6: Divisors.

In Rings: 2 types of ideals special:

- Principal ideals / principal prime ideals
- Height 1 prime ideals.

Recall: $p \subseteq R$: (prime) height $\equiv$ largest n, s.t.

$\exists\, p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_n = p$.

$\Rightarrow$ If R integral domain: $p \subseteq R$ height 1, if p doesn't contain any nonzero prime ideal.

Example (0) has height 0

(X) in $\mathbb{C}[X,Y]$ has height 1 & (X,Y) height 2

($\Rightarrow$ In UFD: height 1 = principal)

Globalize height 1 primes $\Rightarrow$ Weil divisors.

Weil Divisors. (Hypotheses: Noeth, integral, Separated &
(*) Regular in codim 1).

If X integral & $U = \operatorname{spec} A$ open affine: ideal $(0) \subseteq A$ is generic point of X. (for: any U open, affine) Often denoted: η_X.

§6.1: Topological Facts.

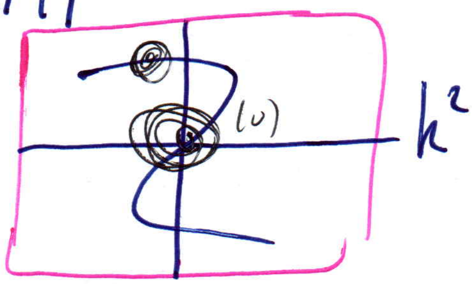
i) dimension of X $\equiv$ length of longest chain $Z_0 \subsetneq \cdots \subsetneq Z_n$ of nonempty; closed, irred subsets of X.

ii) For $Z \subseteq X$: (closed & irred) codim of Z in X is: largest such chain starting at Z ($Z = Z_0 \subsetneq \cdots \subsetneq Z_n$)

iii) If X noeth, any closed $Z \subseteq X$ has: decomp into finitely many irred closed sets.

Regularity in codim 1: X scheme, usual hypotheses. Then, X is (*) if: $\forall Y \subseteq X$ closed, irred, codim $= 1$ subspace: then $\mathcal{O}_{X, \eta_Y}$ is a DVR. ($\Rightarrow$ Local PID)

Picture:



§6.2: Weil Divisors.

DEF 6.2.1 A prime divisor of X is: integral, closed subscheme, of codim 1.
& A Weil Divisor is element of free abelian group $\operatorname{Div}(X)$, generated by prime divisors.

Write: $D \in Div(X)$ as: $\sum_i n_{Y_i} [Y_i]$ where Y_i are prime.
& D effective if $n_{Y_i} \geq 0 \ \forall i$.

DEF 6.2.2 Let $f \in \mathcal{O}_{X,\eta_X} = k[X]$.
Since $\forall Y \subseteq X$ prime divisor, $\mathcal{O}_{X,\eta_Y}$ is DVR, can calculate valuation $v_Y(f)$ of f in DVR.

$$\Rightarrow div(f) \equiv \sum_{Y \subseteq X \text{ codim } 1} v_Y(f) \cdot [Y].$$

Claim: this sum is finite.
Know: since X integral, for $U \subseteq X$, $U = \text{spec } A$, have:
$\mathcal{O}_{X,\eta} = FF(A)$. (Since: η contained in every open affine so $\mathcal{O}_{X,\eta}$ allows any denom)

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Again: X Noetherian, integral & separated.

⊛ Generic pt of X: take, $\mathrm{spec}(A) \subseteq X$ affine, A domain, and $(0) \subseteq A$ prime. & Get a point of $\mathrm{spec}(A) \subseteq X$.

If $\mathrm{Spec}\, A$, $\mathrm{spec}\, B$ affine opens in X: they intersect, since both are dense (irreducibility).

$\Rightarrow$ Common pt is identified in X (localisation).

$\Rightarrow$ Unique pt. Called: generic pt of X.

So, η_X generic pt, then: $\mathcal{O}_{X,\eta_X} = FF(A) = FF(B)$, and is denoted $k[X]$.

DEF 6.2.2) For $f \in k[X]^\times$. Define:

$$\mathrm{div}(f) \equiv \sum_{\substack{Y \subseteq X \\ \text{prime ideal divisors.}}} v_Y(f) \cdot [Y]. \quad \& \quad v_Y : k(X)^\times \longrightarrow \mathbb{Z}$$

induced by DVR $\mathcal{O}_{X,\eta_Y}$.

PROP 6.2.3) The sum is finite.

Proof Choose A, s.t. $\mathrm{Spec}(A)$ affine open with $f \in A$. ("f regular on $\mathrm{spec}\, A$"). $(FFA = k(X))$ & denote $U \equiv \mathrm{spec}\, A$.

$\Rightarrow X - U$ closed, codim ≥ 1. So, only finitely many prime Weil divisors of X contained in $X-U$.

& On the rest (i.e. on U), f regular, but: $v_Y(f) > 0 \Leftrightarrow Y$ contained in $V(f) \subseteq U$.

$\Rightarrow$ By same argument: only finitely many Y's contained in $\mathbb{V}(f)$.
(closed)

DEF 6.2.4) Weil divisors of form $\mathrm{div}(f)$ is principal.
& the set of principal divisors form subgroup of $\mathrm{Div}(X)$, denoted $\mathrm{Prin}(X)$. The class group (Weil divisor class group) is: $\mathrm{Div}(X)/\mathrm{Prin}(X) \equiv \mathrm{Cl}(X)$.

6.2.5 Basic facts:

1) A Noetherian domain. Then: A is UFD $\Leftrightarrow$ (A integrally closed & $\mathrm{Cl}(\mathrm{Spec}\, A)$ trivial). In particular: $\mathrm{Cl}(\mathbb{A}^n_k) = 0$.
(& $\exists A$, $\mathrm{Spec}(A)$ nontrivial Cl.)

2) $\mathrm{Cl}(\mathbb{P}^n_k) \cong \mathbb{Z}$

3) If $Z \subseteq X$ closed and $U = X - Z$, then: $\exists$ surjection

$$\mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \qquad (\& = 0 \text{ if empty}).$$
$$[Y] \longmapsto [Y \cap U] \qquad (Y \text{ prime ideal})$$

4) If Z has codim ≥ 2 then the above surj is an $\cong$

5) If $Z \subseteq X$ integral, closed & codim 1, then: $\exists$ exact seq

$$\mathbb{Z} \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \longrightarrow 0$$
$$1 \longmapsto [Z]$$
$$[Y] \longmapsto [Y \cap U].$$

"Excision exact seq"

Class group of $\mathbb{P}^n_k$. (k field).

$\cancel{X)}$ For $X \subseteq \mathbb{P}^n_k$: (closed, codim 1) let $X = \mathbb{V}(f)$, where f homog, degree d. Define: $\deg(X) \equiv d$.

Extend linearly to get: $\deg : \mathrm{Div}(\mathbb{P}^n_k) \longrightarrow \mathbb{Z}$.

Claim: This gives an isomorphism $Cl(\mathbb{P}^n) \longrightarrow \mathbb{Z}$.

⊛ Well-def'ed: if $f = g/h$, g, h homog & same degree, then $\deg(\operatorname{div} f) = 0$.

⊛ Surjective: Take $H \equiv \mathbb{V}(X_0)$ for X_0 homog & linear

⊛ Injective: If $D = \sum n_{Y_i}[Y_i]$ & $\sum_i n_{Y_i} \deg(Y_i) = 0$ then write $Y_i \equiv \mathbb{V}(g_i)$ for g_i homog. & $f \equiv \prod g_i^{n_i}$.

Then, f is homog & rational func, degree 0. ✓

Proof of excision

⊛ $Cl(X) \longrightarrow Cl(U)$ well-defined.

Note: $k(X)$ & $k(U)$ naturally isomorphic, so principal divisors are sent to principal divisors (in the map of class groups).

⊛ Surjective: If $D \subseteq U$ prime ideal: can choose $\overline{D} \subseteq X$ prime ideal, s.t. it restricts to D on U.

⊛ $Cl(X) = Cl(X - Z)$ if $\mathrm{codim}(Z) \geq 2$, since Z doesn't enter the definition of $Cl(X)$.

⊛ The kernel of $Cl(X) \longrightarrow Cl(U)$, when $Z = X - U$, is: integral, codim $= 1$, are exactly the divisors in X, contained in Z.

§6.3: Cartier Divisors. "Things locally looking like a principal ideal".

DEF 6.3.1 A Cartier divisor is: section of the sheaf $k_X^*/\mathcal{O}_X^*$.

⊛ For X scheme: take presheaf $U = \mathrm{Spec}(A) \longmapsto S^{-1}A$, where S = set of all nonzero divisors (& sheafify).

Call this: k_X. (Is: sheaf of rings).

& Take: $k_X^* \subseteq k_X$ the invertible elements. (Is: sheaf of abelian groups.)

⊛ $\mathcal{O}_X^* \subseteq \mathcal{O}_X$ subsheaf of invertible elements.

Practically: Every section of $k_X^*/\mathcal{O}_X^*$ can be (pre)-described by $\{(U_i, f_i)\}$ where U_i cover X, f_i section of $k_X^*(U_i)$ and: on $U_i \cap U_j$, ratio f_i/f_j lies in $\mathcal{O}_X^*(U_i \cap U_j)$.