Commutative Algebra: Lecture 1 05/10/2023.

Chapter 0:

Ring $\Leftrightarrow$ Commutative, Unital ring (with 1.)

(non-commutative exception: $End(M) = \{f: M \to M$ group hom$\}$
for $(M, +)$ abelian group. mult. = composition.)

DEF An R-module M is an abelian group M with a fixed group hom. $\rho: R \to End(M)$, such that:
$r \cdot m = \rho(r)(m)$ ($\forall r \in R$ & $m \in M$).

Properties: 1) $r(m_1 + m_2) = \rho(r)(m_1 + m_2)$
$= \rho(r)(m_1) + \rho(r)(m_2) = r(m_1) + r(m_2)$. ($\rho(r)$ is group hom $M \to M$)

2) $(r_1 + r_2)m = \rho(r_1 + r_2)m = \rho(r_1)(m) + \rho(r_2)(m)$
$= r_1(m) + r_2(m)$.
(ρ is ring hom.)

Examples (of modules). (K field)

1) K- module $\Leftrightarrow$ K-vector space

2) Every abelian group is a $\mathbb{Z}$- module in a unique way. $\mathbb{Z} \longrightarrow End(M)$
$$1_{\mathbb{Z}} \mapsto id$$

$\Rightarrow$ Abelian group $\Leftrightarrow$ $\mathbb{Z}$-module.

3) Every ring R is an R-module: $R \to End(R)$
$$r_0 \mapsto (r \mapsto r_0 r)$$
Similarly, $R^{\oplus N}$ & R^N are R-modules.

Chain Conditions.

DEF] An R-module M is <u>noetherian</u> iff:

① Any Ascending chain of submodules $M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots$ <u>stabilises</u> ($\exists N, M_{n+N} = M_N \ \forall n$)

2) Every nonempty set Σ of sub<u>modules</u> of M has maximal element.

M <u>artinian</u> $\Leftrightarrow$ same holds but ascending $\leftrightarrow$ descending & maximal $\leftrightarrow$ minimal.

Lemma An R-module M is <u>noetherian</u> $\Leftrightarrow$ every submodule of M is finitely generated.

$\Rightarrow$ Every Noetherian module is f.g.

<u>Note</u>: if $R = \mathbb{Z}[T_1, T_2, \dots]$ & $M = R$ (as R-mod) then M generated by 1_R. ($1_R \in M \Rightarrow r \cdot 1_R \in M = R$)

<u>But</u>: $M' = \langle T_1, \dots, T_n, \dots \rangle \subseteq M$, polynomials, constant term 0 generated by T_i, is <u>not</u> f.g. R-module.

DEF] A ring R is <u>Noetherian</u> (resp. artinian) if R (as R-module) is noetherian (resp. artinian).

<u>Examples]</u> Noeth & not artin: $\mathbb{Z}$

Artin & non-noeth: module: $\mathbb{Z}[\frac{1}{2}] / \mathbb{Z}$.

& A ring R is artinian $\Leftrightarrow$ noetherian and Krull-dimension 0.

DEF] $\cdots \longrightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow \cdots$ is exact $\Leftrightarrow$ $Im(f_i) = Ker(f_{i+1})$ $\forall i$.

Short exact sequence: $0 \longrightarrow A \xrightarrow{i} B \longrightarrow C \longrightarrow 0.$
$\quad\quad$ inj $\quad$ surj.

$(\Rightarrow C \cong B/i(A))$

Lemma $0 \to A \to B \to C \to 0.$ (SES)

Then: B noeth (artin) $\Leftrightarrow$ A, C noeth (artin).

Corollary If $M_1, \dots, M_n$ noeth (artin) modules, then so is $M_1 \oplus \cdots \oplus M_n$.

Reminder: module hom $\varphi : M_1 \oplus \cdots \oplus M_n \longrightarrow L$

$\Leftrightarrow$ module homs $\varphi_i : M_i \longrightarrow L$.

(all M_i & L are R-modules.)

PROP] For Noetherian (Artinian) R, every f.g. R-module is Noetherian (Artinian).

PROOF] M f.g. $\Leftrightarrow$ $(\exists n \geq 1)(\exists \varphi : R^n \to M)$ surjective.

& Know: R^n noetherian, & noetherian passes to quotients.
(same for Artinian.)

DEF] An R-algebra is ring A & fixed ring hom $\rho: R \to A$. Write: $ra \equiv \rho(r)(a)$.

Know: $\rho(r) = \rho(r) \cdot \mathbb{1}_A = r \cdot \mathbb{1}_A$

Example: $k \longrightarrow k[T_1, \dots, T_n]$. ring hom.

An R-algebra A is f.g. $\Longleftrightarrow$ ~~$(\exists n \geq 0)(\exists \varphi: k[T_1, \dots, T$~~
$(\exists n \geq 0)(\exists \varphi: k[T_1, \dots, T_n] \to A)(\varphi$ surj & R-algebra hom$)$.

DEF] $\varphi: A \to B$ is R-algebra hom. if: φ ring hom, and: $\varphi(r \cdot \mathbb{1}_A) = r \cdot \mathbb{1}_B$

Comm Alg: lecture 2.

Theorem (Hilbert Basis Theorem)

Every finitely gen algebra A over Noetherian ring R is Noetherian.

Proof Suffice to show for $R[T_1, -, T_n]$ [since any f.g. algebra is a quotient of this]. Suffice to show for $R[T]$ [since $R[T_1,-,T_n] \simeq R[T_1,-,T_{n-1}][T_n]$]

Let $I \leq R[T]$ ideal. & $\forall i \geq 0$: $I_i \equiv \{c_0 : c_0 T^i + \dots + c_i T^0 \in I\}$.

Then: $I_i \leq R$ is an ideal. & $I_i \subseteq I_{i+1}$.

$\Rightarrow$ This ACC stabilises (R noeth) $\exists N$, $I_n = I_N$ $\forall n \geq N$.

& each I_i is finitely generated.

Write: $I_i = (b_{i,1}, -, b_{i,n_i})$. $b_{ij} \in R$.

& $f_{ij} \equiv b_{ij} T^i + (\text{smaller order terms}) \in I$.

& $J \equiv (f_{ij})_{ij} \subseteq R[T]$ (ideal).

Then: $J \subseteq I$ (by construction) & $J_i = I_i \, \forall i$.

Claim: $J = I$. If not: find $f \in I \setminus J$, minimal degree i. Then: $J_i = I_i \Rightarrow \exists g \in J$, $\deg(\underbrace{f-g}_{\in J}) < i$.

$\Rightarrow f = \underbrace{(f-g)}_{\in J} + \underbrace{g}_{\in J} \in J$ #

If $S \subseteq R[T_1, -, T_n]/I$ then $\exists S_0 \subseteq S$ finite & $(S_0) = (S)$.

Tensor Products.

let: M, N R-modules.

$$M \otimes_R N \equiv \left\{ \sum_{i=1}^{\ell} m_i \otimes n_i : m_i \in M, n_i \in N \right\}.$$

Where:
$$\left.\begin{aligned} (m_1 + m_2) \otimes n &= (m_1 \otimes n) + (m_2 \otimes n) \\ m \otimes (n_1 + n_2) &= (m \otimes n_1) + (m \otimes n_2) \\ (rm) \otimes n &= r(m \otimes n) = m \otimes (rn). \end{aligned}\right] \quad (*)$$

Example 1) $(\mathbb{Z}/2) \otimes_{\mathbb{Z}} (\mathbb{Z}/3)$?

Know: $x \otimes y = (3x) \otimes y = x \otimes (3y) = x \otimes 0 = 0.$

$\Rightarrow$ Tensor product collapses.

2) $\mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{R}^m$? $\simeq \mathbb{R}^{m+n}$ (seen later).

Formal Def: recall: $f: \underbrace{M \times N}_{R\text{-mods}} \longrightarrow L$ is R-bilinear

if: $\left.\begin{aligned} n &\mapsto f(m_0, n) \\ m &\mapsto f(m, n_0) \end{aligned}\right]$ are R-bilinear $\forall m_0 \in M$, $n_0 \in N$.

For M, N R-modules: $\mathcal{F} \equiv R^{\oplus (M \times N)}$.

$= \operatorname{span}_R \left(e_{(m,n)} : (m,n) \in M \times N \right)$

& $K \subseteq \mathcal{F}$, generated by relations $(*)$.

& $M \otimes_R N \equiv \mathcal{F}/K$.

So, have: R-bilinear map $i_{M \otimes N} : M \times N \longrightarrow M \otimes_R N$

With: $i_{M\otimes N}(m,n) = m\otimes n$. (inclusion)

<u>Prop</u> $(M\underset{R}{\otimes} N, \ i_{M\otimes N})$ <u>satisfies</u>:

$\circledast$ $\forall L$ (R-module)

$\circledast$ $\forall f: M\times N \longrightarrow L$ (R-bilinear map)

$\exists!$ R-linear $h: M\underset{R}{\otimes} N \longrightarrow L$, with:

$$\boxed{\begin{tikzcd} M\times N \arrow[r, "i"] \arrow[d, "f"'] & M\underset{R}{\otimes} N \arrow[dl, dashed, "h"] \\ L & \end{tikzcd}}$$

commutes.

<u>Proof</u> <u>need</u>: h with: $(h\circ i_{M\otimes N} = f)$

$\Longleftrightarrow h(i_{M\otimes N}(m,n)) = f(m,n)$ $(\forall m,n\in M,N)$

$\Longleftrightarrow h(m\otimes n) = f(m,n)$ (hence: uniqueness)

Existence: know, $R^{\oplus(M\times N)} \longrightarrow L$ <u>vanishes</u>

$e_{m,n} \longmapsto f(m,n)$

on generators of K

$\Longrightarrow h(m\otimes n) = f(m,n)$ extends to R-linear map

$M\underset{R}{\otimes} N \longrightarrow L$.

<u>Prop</u> M,N R-modules & suppose (T,j) another pair (T R-module & $j: M\times N \longrightarrow T$ R-bilinear) also satisfying universal property. <u>Then</u>: $\exists!$ R-linear <u>isomorphism</u> $\varphi: M\underset{R}{\otimes} N \xrightarrow{\cong} T$ with $\varphi\circ i_{M\otimes N} = 1$.

Proof

$$\begin{tikzcd} M \otimes N \arrow[r, shift left, "\varphi"] & T \arrow[l, shift left, "\psi"] \\ M \times N \arrow[u, "i_{M\otimes N}"] \arrow[ur, "j"'] & \end{tikzcd}$$

Challenge $M\otimes N$ with j: get φ

Challenge ~~$M\otimes$~~ T with $i_{M\otimes N}$: get ψ

& $(\psi\circ\varphi)\circ i_{M\otimes N} = i_{M\otimes N}$.

$\Rightarrow$ $(\psi\circ\varphi)$ is solution for challenging $M\otimes N$ with $i_{M\otimes N}$,

but so is $i_{M\otimes N}$, so $\psi\circ\varphi = id_{M\otimes N}$.

Same for reverse.

Commutative Algebra: Lecture 3)

From last time: proved: $\forall M, N$ R-modules & L R-module,

$$\boxed{\begin{array}{ccc} \mathrm{Bilinear}_R(M\times N, L) & \xrightarrow{\sim} & \mathrm{Hom}(M \otimes_R N, L) \\ h \circ i_{M\otimes N} & \longleftarrow\!\!\!\shortmid & h \end{array}}$$

where: $i_{M\otimes N} : M \times N \hookrightarrow M \otimes_R N$ natural inclusion.

Prop M, N R-modules. Then:

$$\sum_i m_i \otimes n_i = 0 \text{ in } M \otimes_R N \iff \begin{array}{l} \forall R\text{-module } L \\ \forall R\text{-bilinear } f: M\times N \to L \\ \sum_i f(m_i, n_i) = 0. \end{array}$$

Proof $\Rightarrow$: Assume $\sum_i m_i \otimes n_i = 0$. Then, f factors through.

$$\begin{tikzcd} M\times N \arrow[r, "i"] \arrow[dr, bend right, "f"'] & M\otimes N \arrow[d, "h"] \\ & L \end{tikzcd}$$

$\Rightarrow \sum_i f(m_i, n_i) = \sum_i h(i_{M\otimes N}(m, n))$

$= h(\sum_i m_i \otimes n_i) = 0.$ ✓

$\Leftarrow$: If $\sum_i m_i \otimes n_i \neq 0$, then $\sum_i i_{M\otimes N}(m_i, n_i) \neq 0$. ✓

Example) $k^m \otimes k^n$ (k field).

Find basis: $\{e_i\}$ for k^m & $\{f_j\}$ for k^n.

$\Rightarrow k^m \otimes k^n = \mathrm{span}_k\{v\otimes w : v \in k^m, w \in k^n\}$.

$= \mathrm{span}_k\{e_i \otimes f_j : \begin{smallmatrix} 1 \le i \le m \\ 1 \le j \le n \end{smallmatrix}\}$.

Claim: $\{\underline{e}_i \otimes f_j\}$ is a basis.

Lin indep: If $\sum \alpha_{ij} \underline{e}_i \otimes \underline{f}_j = 0$:

$(\forall a \leq m)(\forall b \leq n)$ $T_{a,b}: k^m \times k^l \longrightarrow k$

$$((v_i), (w_j)) \longmapsto v_a \cdot w_b.$$

Then, by prop, $\sum a_{ij} T_{ab}(\underline{e}_i, f_j) = 0 \Rightarrow \alpha_{ab} = 0$ ✓

Example) $\mathbb{R}^2 \otimes \mathbb{R}^2$. Has: infinitely many pure tensors!

& has: basis $\underset{\mathbb{R}}{}$of size 4. $e_1 \otimes f_1$, $e_1 \otimes f_2$, $e_2 \otimes f_1$, $e_2 \otimes f_2$.

Some tensors not pure:

$$(\alpha e_1 + \beta e_2) \otimes (\gamma f_1 + \delta f_2) = (\alpha\gamma)\underline{e}_1 \otimes \underline{f}_1 + (\alpha\delta)\underline{e}_1 \otimes \underline{f}_2 + (\beta\gamma)\underline{e}_2 \otimes \underline{f}_1 + (\beta\delta)\underline{e}_2 \otimes \underline{f}_2.$$

$\Rightarrow$ know: $(\alpha\gamma, \alpha\delta)$ & $(\beta\gamma, \beta\delta)$ lin dep.

$\Rightarrow$ E.g. $1\underline{e}_1 \otimes \underline{f}_1 + 2\underline{e}_1 \otimes \underline{f}_2 + 3\underline{e}_2 \otimes \underline{f}_1 + 4\underline{e}_2 \otimes \underline{f}_2$ is not a pure tensor.

Example) • $\mathbb{Z} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}/2$. **Then**: $2 \otimes (1 + 2\mathbb{Z}) = 0$.

• $2\mathbb{Z} \underset{\mathbb{Z}}{\otimes} \frac{\mathbb{Z}}{2}$. Then: $2 \otimes (1 + 2\mathbb{Z}) \neq 0$!!

Indeed: $b: (2\mathbb{Z} \times \frac{\mathbb{Z}}{2}) \longrightarrow \mathbb{Z}/2$

$$(2n, x + 2\mathbb{Z}) \longmapsto nx + 2\mathbb{Z}$$

$$(2, 1 + 2\mathbb{Z}) \longmapsto 1 + 2\mathbb{Z} \neq 0.$$

So, have to be careful. (warning)

However: if $M' \subseteq M$, $N' \subseteq M$ R-modules & $\sum_i m_i \otimes n_i = 0$ in $M' \otimes N'$, then: $\sum_i m_i \otimes n_i = 0$ in $M \otimes N$.

Prop] If $\sum_i m_i \otimes n_i = 0$ in $M \otimes_R N$, then $\exists$ $\boxed{\text{f.g.}}$ R-submods $M' \subseteq M$ & $N' \subseteq N$, with: $\sum_i m_i \otimes n_i = 0$ in $M' \otimes_R N'$.

Proof $\sum_i m_i \otimes n_i = 0$ in $M \otimes_R N \equiv R^{\oplus(M \times N)} / K$.

$\Rightarrow \sum_i m_i \otimes n_i \in K$

But: $= \sum_i e_{m_i, n_i} = \sum_i \alpha_i k_i$ $(\alpha_i \in R, k_i \in K)$.

$\Rightarrow$ Take: $m_1', \dots, m_n'$ that appear on left sides of k_i, as generators of M'. Then, they are finitely generated.

Corollary A, B Torsion-free abelian groups $\Rightarrow A \otimes_{\mathbb{Z}} B$ also torsion free.

Proof By Prop: $\exists$ f.g. $A' \subseteq A$ & $B' \subseteq B$, such that: if $n \cdot \sum_i a_i \otimes b_i = 0$ in $A \otimes_{\mathbb{Z}} B$, then:

$$n \cdot \sum_i a_i \otimes b_i = 0 \text{ in } A' \otimes_{\mathbb{Z}} B'.$$

But: fin-gen & torsion free $\Rightarrow A' \cong \mathbb{Z}^n, B' \cong \mathbb{Z}^m$

$\Rightarrow A' \otimes_{\mathbb{Z}} B' \cong \mathbb{Z}^{nm}$ is torsion free.

$\Rightarrow \sum_i a_i \otimes b_i = 0$ in $A \otimes_{\mathbb{Z}} B$ ✓

Note $\mathbb{C}^2 \otimes_{\mathbb{C}} \mathbb{C}^3 \cong \mathbb{C}^6$. (as $\mathbb{C}$-modules)

$\cong \mathbb{R}^{12}$ (as $\mathbb{R}$-modules).

& $\mathbb{C}^2 \otimes_{\mathbb{R}} \mathbb{C}^3 \cong \mathbb{R}^4 \otimes_{\mathbb{R}} \mathbb{R}^6 \cong \mathbb{R}^{24}$ (as $\mathbb{R}$-modules).

$\Rightarrow$ Tensoring with larger field makes product smaller.

Prop) $M \otimes_R N \xrightarrow{\sim} N \otimes_R M$. (~~as~~ R-module isomorphism)

& $(M \otimes N) \otimes P \xrightarrow{\sim} M \otimes (N \otimes P) \cong M \otimes N \otimes P$.

& $(\bigoplus M_i) \otimes P \xrightarrow{\sim} \bigoplus (M_i \otimes P)$.

& $R \otimes_R M \xrightarrow{\sim} M$. $(r,m) \longmapsto rm$

So, $\mathbb{R}^m \otimes_R R^n = \left(\bigoplus_{i \le m} R\right) \otimes \left(\bigoplus_{i \le n} R\right)$

$\simeq \bigoplus_{i,j} R \simeq R^{mn}$.

Tensor Products.) (of R-linear maps).

Prop) For R-linear maps $f: M \longrightarrow M'$ & $g: N \longrightarrow N'$,

$\exists!$ R-linear map $f \otimes g: M \otimes N \longrightarrow M' \otimes N'$ such that:

$(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$.

Proof Define $T: M \times N \longrightarrow M' \otimes N'$ by: $T(m,n) \equiv f(m) \otimes g(n)$

By universal prop of $M \otimes N$: have $f \otimes g$ as in the statement ✓

Commutative Algebra: Lecture 4.

From last time $f: M \to M'$, $g: N \to N'$ R-linear

$\Rightarrow f \otimes g : M \otimes N \to M' \otimes N'$, $(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$.

Exercise: $(f \otimes g) \circ (h \otimes i) = (f \circ h) \otimes (g \circ i)$.

Example $\left.\begin{array}{l} T: k^a \to k^b \\ S: k^c \to k^d \end{array}\right]$ $T \otimes S : \underset{\simeq k^{ab}}{k^a \otimes_k k^b} \to \underset{\simeq k^{cd}}{k^c \otimes k^d}$.

Defined by: $(T \otimes S)(\underline{e}_i \otimes \underline{e}_j) \equiv (T\underline{e}_i) \otimes (S\underline{e}_j)$.

$$\equiv \sum_{l,t} [T]_{li} [S]_{tj} (f_l \otimes f_t).$$

Order basis of $k^a \otimes k^c$ by: $\underline{e}_1 \otimes \underline{e}_1, \ldots, \underline{e}_1 \otimes \underline{e}_c$, $\underline{e}_2 \otimes \underline{e}_1, \ldots, \underline{e}_2 \otimes \underline{e}_c, \ldots$

$$\Rightarrow [T \otimes S] = \begin{bmatrix} [T]_{11} \cdot [S] & \cdots & [T]_{1a} \cdot [S] \\ \vdots & & \vdots \\ [T]_{b1} \cdot [S] & & [T]_{ba} \cdot [S] \end{bmatrix}.$$

Prop $f: M \to M'$ & $g: N \to N'$ R-linear.

1) If f, g isomorphisms $\Rightarrow f \otimes g$ isomorphism

2) If f, g surjective $\Rightarrow f \otimes g$ surjective.

Proof 1) $f^{-1} \otimes g^{-1}$ is a 2-sided inverse, for $f \otimes g$. ✓

2) Note that ~~$f\otimes g$~~ $\mathrm{Im}(f \otimes g)$ contains all pure tensors of $M' \otimes N'$, so by linearity, is surjective. ✓

$\boxed{\text{Example}}$ Not true for "injective"!

$f: \mathbb{Z} \xrightarrow{\times p} \mathbb{Z}$ & $id: \mathbb{Z}/p \to \mathbb{Z}/p$.

$\Rightarrow (f \otimes id)(a \otimes b) = (pa) \otimes b = a \otimes (pb) = 0$.

$\Rightarrow f \otimes id : \mathbb{Z} \otimes \mathbb{Z}/p \to \mathbb{Z} \otimes \mathbb{Z}/p$ is 0-map, so not injective!

Tensor Products of Algebras.

Let: B, C algebras over R. Construct: $B \otimes_R C$, initially, as a R-module.

Want: $(b \otimes c)(b' \otimes c') = (bb') \otimes (cc')$.

⊛ Well-defined: Fix $(b, c) \in B \times C$ & consider R-bilinear

$$\left[\begin{array}{l} B \times C \longrightarrow B \otimes_R C \\ (b', c') \longmapsto (bb') \otimes (cc'). \end{array}\right] \Longrightarrow \left[\begin{array}{l} B \otimes_R C \longrightarrow B \otimes_R C \\ b' \otimes c' \longmapsto (bb') \otimes (cc'). \end{array}\right]$$

⊛ Ring axioms ✓

⊛ R-algebra structure: $R \longrightarrow B \otimes_R C$
$r \longmapsto r(1_B \otimes 1_C)$.

$\boxed{\text{Example}}$ $\varphi: R[X_1, \dots, X_n] \otimes_R R[Y_1, \dots, Y_m] \xrightarrow{\sim} R[X_i, Y_j]$

sending $\varphi(x \otimes y) = xy$.

R-basis for LHS: for $a \in R[X_i]$, $b \in R[Y_j]$ monomials, get a basis $\{ab\}$.

Is it R-algebra homomorphism? $\varphi(r\otimes 1) = r\cdot 1$ ✓

$\varphi(1\otimes 1) = 1$.

& $\varphi\left(\left(\sum_i p_i\otimes q_i\right)\left(\sum_j h_j\otimes g_j\right)\right)$

$= \sum_{i,j} (p_i h_j)(q_i g_j) = \sum_{i,j} \varphi(p_i\otimes q_i)\varphi(h_j\otimes g_j)$

$= \left(\sum_i \varphi(p_i\otimes q_i)\right)\left(\sum_j \varphi(h_j\otimes g_j)\right)$.

<u>More generally</u>: $\left(R[X_1,\dots,X_n]/I\right)\otimes\left(R[Y_1,\dots,Y_m]/J\right)$

$\cong R[X_i,Y_j]/(I^e+J^e)$

$\cong \left(R[X_i]\underset{R}{\otimes}R[Y_j]\right)/L$, $L \equiv$ submodule generated by

$\{p\otimes q : p\in I, q\in R[Y_i]\}\cup\{g\otimes h : g\in R[X_i], h\in J\}$.

These are also <u>algebra</u> homomorphisms.

<u>Example</u>) $\mathbb{C}[X,Y,Z]/_{(f,g)}\underset{\mathbb{C}}{\otimes}\mathbb{C}[V,U]/_{(h)}$.

$\simeq \mathbb{C}[X,Y,Z]/(f,g,h)$ (as $\mathbb{C}$-algebras)

<u>Prop</u>) (Universal Property of Tensor Product of Algebras)

A,B R-algebras. Then: $\forall C$ R-algebra & R-algebra maps $A\xrightarrow{f}C$ & $B\xrightarrow{g}C$, $\exists!$ R-algebra map $h: A\underset{R}{\otimes}B\longrightarrow C$ such that:

commutes:

$$\begin{tikzcd}
A \arrow[r, "i_A"] \arrow[dr, bend right, "f"'] & A\underset{R}{\otimes}B \arrow[d, "h"] & B \arrow[l, "i_B"'] \arrow[dl, bend left, "g"] \\
& C &
\end{tikzcd}$$

This characterises $(A \otimes_R B, i_A, i_B)$ uniquely.

<u>Proof</u>: note $A \otimes_R B$ is generated (as R-<u>algebra</u>) by:

$\{a \otimes 1 : a \in A\} \cup \{1 \otimes b : b \in B\}$.

$\Rightarrow$ Get uniqueness of h.

<u>Existence</u>: take $H : A \times B \longrightarrow C$ R-<u>bilinear</u> map.

$(a,b) \longmapsto f(a) \cdot g(b)$.

By universal property: produces $h : A \otimes_R B \longrightarrow C$

$a \otimes b \longmapsto f(a) g(b)$.

<u>Verify</u>: h is R-algebra hom.

Since h is R-<u>module</u> hom: suffices: $h(\underbrace{1_A \otimes 1_B}_{1_{A \otimes_R B}}) = 1_C$.

<u>and</u>: $h(xy) = h(x)h(y)$ $\forall x, y \in A \otimes_R B$.

* First condition ✓
* 2nd condition: Suffices to show on <u>pure</u> tensors ✓

$$\begin{tikzcd}
R[X_i] \arrow[r, hook] \arrow[dr, bend right, "f"'] & R[X_i, Y_j] \arrow[d, "\exists!"] & R[Y_j] \arrow[l, hook'] \arrow[dl, bend left, "g"] \\
& C &
\end{tikzcd}$$

$\Rightarrow R[X_i, Y_j] \cong R[X_i] \otimes_R R[Y_j]$

Commutative Algebra: Lecture 5

Tensor Products of algebras (continued):

1) If $f: A \to A'$, $g: B \to B'$ R-algebra homs, then:
$f \otimes g : A \otimes B \to A' \otimes B'$ is also R-algebra hom.

2) Have the following R-algebra homs:

- $R/I \otimes_R R/J \cong R/I+J$
- $A \otimes B \cong B \otimes A$
- $A \otimes (B \times C) \cong (A \otimes B) \times (A \otimes C)$ & $A \otimes B^n \cong (A \otimes B)^n$
- $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$.

Restriction & Extension of Scalars.

If $f: R \to S$ ring hom & M is S-module, it is also an R-module by $r.m = f(r)m$.

Get: $R \xrightarrow{f} S \longrightarrow \mathrm{End}(M)$, so indeed is R-module.

Example) $f: \mathbb{R} \hookrightarrow \mathbb{C}$ inclusion. Then: $\mathbb{C}^n$ is $\mathbb{C}$-module, but also an $\mathbb{R}$-module ($\mathbb{C}^n \simeq \mathbb{R}^{2n}$).

Extension of Scalars) $f: R \to S$ ring hom & M is S-module, N is R-module. Then: $M \otimes_R N$ is an R-module.

Then: $M \otimes_R N$ is also an S-module, by:

$\boxed{s(m \otimes n) = (sm) \otimes n.}$

⊛ <u>Well-def'ed</u>: have R-bilinear map $M \times N \longrightarrow M \otimes_R N$
$\Rightarrow$ By universal prop: have ~~[illegible]~~ $\quad (m,n) \longmapsto (sm) \otimes n.$

$h_s : M \otimes_R N \longrightarrow M \otimes_R N$, $h_s(m \otimes n) \equiv (sm) \otimes n.$

& h_s is R-linear.

<u>Then</u>: $\varphi : S \longrightarrow \mathrm{End}(M \otimes_R N)$, $\varphi(s) = h_s$ turns $M \otimes_R N$ into an S-module [<u>check</u>: $h_s \in \mathrm{End}(M \otimes_R N)$ & φ ring hom].

<u>Example</u> <u>1)</u> Know: $S \otimes_R R \simeq S$ (as R-modules).

$$(s \otimes r) \longmapsto s \cdot f(r)$$

This is also <u>S-linear</u>: $s'(s \otimes r) \longmapsto (s's) \cdot f(r) = s'(s \cdot f(r)) = s'(s \otimes r)$

In particular: $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R} \cong \mathbb{C}$ as $\mathbb{C}$-mods.

<u>2)</u> If M S-module & N_i R-modules, then:

$$M \otimes \left(\bigoplus_i N_i \right) \simeq \bigoplus_i (M \otimes N_i) \quad \text{as } S\text{-modules.}$$

In particular: $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^n \cong \mathbb{C}^n$ as $\mathbb{C}$-modules.

<u>3)</u> $\mathbb{C}^n$ (as $\mathbb{C}$-module).

<u>Restrict to $\mathbb{R}$</u>: $\mathbb{C}^n \cong \mathbb{R}^{2n}$ (as $\mathbb{R}$-module)

<u>Extend to $\mathbb{C}$</u>: $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^{2n} \cong \mathbb{C}^{2n}$ (as $\mathbb{C}$-module)

<u>4)</u> $\mathbb{R}^n$ (as $\mathbb{R}$-module)

<u>Extend to $\mathbb{C}$</u>: $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^n \cong \mathbb{C}^n$ (as $\mathbb{C}$-modules)

<u>Restrict to $\mathbb{R}$</u>: $\mathbb{C}^n \cong \mathbb{R}^{2n}$ (as $\mathbb{R}$-modules).

5] $\mathbb{Z}^n$ (as $\mathbb{Z}$-module) & $f : \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2$.
"Extend" to $\mathbb{Z}/2$: $(\mathbb{Z}/2) \otimes_{\mathbb{Z}} \mathbb{Z}^n \cong (\mathbb{Z}/2)^n$.

Example] What is $\mathbb{C}^n \otimes_{\mathbb{R}} \mathbb{R}^\ell$?

As $\mathbb{R}$-modules: $\mathbb{C}^n \otimes_{\mathbb{R}} \mathbb{R}^\ell \cong_{\mathbb{R}} \mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{R}^\ell \cong_{\mathbb{R}} \mathbb{R}^{2n\ell} \cong_{\mathbb{R}} \mathbb{C}^{n\ell}$.

As $\mathbb{C}$-modules: $\mathbb{C}^n \otimes_{\mathbb{R}} \mathbb{R}^\ell \cong \mathbb{C}^n \otimes_{\mathbb{C}} (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^\ell) \cong \mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}^\ell \cong \mathbb{C}^{n\ell}$.

$(v \otimes u) \longmapsto v \otimes (1 \otimes u) \longmapsto v \otimes u$.

Prop] M S-module, N R-module. $\Rightarrow M \otimes_R N \xrightarrow{\sim} M \otimes_S (S \otimes_R N)$

$m \otimes n \longmapsto m \otimes (1 \otimes n)$

$~~s \otimes \otimes~~$ $sm \otimes n \longleftarrow m \otimes (s \otimes n)$

Prop] M, M' S-modules, N, N' R-modules. $\underline{1)}$ $\Rightarrow M \otimes_R N \xrightarrow{\sim} N \otimes_R M$

$m \otimes n \longmapsto n \otimes m$.

$\underline{2)}$ $(M \otimes_R N) \otimes_R N' \xrightarrow{\sim} M \otimes_R (N \otimes_R N')$.

$\underline{3)}$ $(M \otimes_R N) \otimes_S M' \xrightarrow{\sim} M \otimes_S (N \otimes_R M')$

"Tensor over largest ring you can".

$\underline{4)}$ $M \otimes_R (\bigoplus_i N_i) \xrightarrow{\sim} \bigoplus_i (M \otimes N_i)$.

<u>Proof</u> $(M \otimes_R N) \otimes_S M' \simeq (M \otimes_{\cancel{R}S} (N \otimes_R S)) \otimes_S M'$

$\simeq M \otimes_S ((N \otimes_R S) \otimes_S M')$ (normal assoc.)

$\simeq M \otimes_S (N \otimes_R M')$. ✓ Similar for others.

<u>Example</u> $\mathbb{C} \otimes_{\mathbb{R}} (\mathbb{R}^l \otimes_{\mathbb{R}} \mathbb{R}^m)$

$\simeq (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^l) \otimes_{\mathbb{C}} (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^m) \simeq \mathbb{C}^l \otimes_{\mathbb{C}} \mathbb{C}^m \simeq \mathbb{C}^{lm}$.

<u>Corollary</u> N, N' R-modules. Then:

$S \otimes_R (N \otimes_R N') \simeq (S \otimes_R N) \otimes_S (S \otimes_R N')$.

<u>Proof</u> LHS $\cong (S \otimes_R N) \otimes_R N' \simeq (S \otimes_R N) \otimes_S (S \otimes_R N')$ ✓

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Corollary $f: R \to S$ ring hom. $S \otimes_R (N \otimes_R N) \simeq (S \otimes_R M) \otimes_S (S \otimes_R N)$.

Example $\mathbb{C} \otimes_{\mathbb{R}} (\mathbb{R}^m \otimes_{\mathbb{R}} \mathbb{R}^n) \simeq (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^m) \otimes_{\mathbb{C}} (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^n)$

$\simeq \mathbb{C}^m \otimes_{\mathbb{C}} \mathbb{C}^l \simeq \mathbb{C}^{ml}$.

By induction, $S \otimes_R (N_1 \otimes_R \cdots \otimes_R N_l) \simeq (S \otimes_R N_1) \otimes_S \cdots \otimes_S (S \otimes_R N_l)$.

Extension of Scalars on Morphisms.

Say: $f: N \longrightarrow N'$ R-linear (N, N' R-modules). & M S-mod.

& $(id_M \otimes f): M \otimes_R N \longrightarrow M \otimes_R N'$ (R-linear).

In fact: is S-linear map. $(id_M \otimes f)(s \cdot (m \otimes n))$

$= (sm \otimes f(n)) = s(m \otimes f(n)) = s \cdot (id_M \otimes f)(m \otimes n)$.

Say: $T: \mathbb{R}^n \to \mathbb{R}^m$ $\mathbb{R}$-linear. Bases $\{\underline{e}_i\}_{i \leq n}$, $\{\underline{f}_j\}_{j \leq m}$.

$\Rightarrow (id_{\mathbb{C}} \otimes T): \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^n \longrightarrow \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^m$

$1 \otimes \underline{e}_i \longmapsto 1 \otimes T\underline{e}_i$.

& $1 \otimes T\underline{e}_i = 1 \otimes \sum [T]_{ij} \underline{f}_j = \sum [T]_{ij} (1 \otimes \underline{f}_j)$.

$\Rightarrow [id_{\mathbb{C}} \otimes T]$ is same as $[T]$.

Extension of scalars over Algebras.

Let: A, B R-algebras. Then, $A \otimes_R B$ is R-algebra.
Is also: A-algebra: $A \to A \otimes_R B$
$a \mapsto a \otimes 1$.

& B-algebra: $B \to A \otimes_R B$
$b \mapsto 1 \otimes b$.

Example $\varphi: S \otimes_R (R[X_1, \dots, X_n]) \xrightarrow[S]{\sim} S[X_1, \dots, X_n]$

Proof Already have: S-module isomorphism:
$\varphi(s \otimes p) = sp$ & $\varphi(1 \otimes 1) = 1$
& Verify multiplication is preserved.

So: $S \otimes (R[X_1, \dots, X_n]/I) \simeq S[X_1, \dots, X_n]/I^e$
where $I^e \equiv f(I)$ ideal gen. by $\{f(i) : i \in I\}$.

Prop) A R-algebra & B S-algebra. Then:
$A \otimes_R B \cong (A \otimes_R S) \otimes_S B$ (as S-algebras).

Prop) A & B R-algebras. Then: $S \otimes (A \otimes_R B) \cong (S \otimes_R A) \otimes_S (S \otimes_R B)$
(as S-algebras).

Exactness Properties of Tensor Product.

Let: M R-module. Define: $T_M(N) \equiv M \otimes_R N$. ($N$ R-mod.)
For $f: N \to N'$ R-linear: $T_M(f) \equiv id_M \otimes f$.

First Goal: if $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ exact seq of R-modules, then: $M \otimes_R A \xrightarrow{id_M \otimes f} M \otimes_R B \xrightarrow{id_M \otimes g} M \otimes_R C \longrightarrow 0$ is also exact. ($\Rightarrow T_M$ is a Right-exact functor.)

DEF) Q, P R-modules. $\mathrm{Hom}_R(Q,P) \equiv \{f : Q \to P,\ R\text{-linear}\}$.

For $\varphi \in \mathrm{Hom}_R(Q,P)$: $(r \cdot \varphi)(q) \equiv r \cdot (\varphi(q))$
$r \in R$ ($\Rightarrow$ Turns into an R-module.)

Hom functions) $\mathrm{Hom}_R(Q, \cdot)$ & $\mathrm{Hom}_R(\cdot, P)$ (P, Q fixed R-modules).

For R-module hom $f: M \to N$:

$$\mathrm{Hom}_R(Q,M) \xrightarrow{\mathrm{Hom}_R(Q,f)} \mathrm{Hom}_R(Q,N)$$
$$\varphi \longmapsto f \circ \varphi \equiv f_*(\varphi)$$

&

$$\mathrm{Hom}_R(N,P) \xrightarrow{\mathrm{Hom}_R(f,P)} \mathrm{Hom}_R(M,P)$$
$$\varphi \longmapsto \varphi \circ f \equiv f^*(\varphi).$$

$\Rightarrow \mathrm{Hom}_R(\cdot, P)$ reverses direction of morphism.
Call: $\mathrm{Hom}_R(\cdot, P)$ contravariant functor
& $\mathrm{Hom}_R(Q, \cdot)$, T_M covariant functors.

Prop) 1) If $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$ exact R-modules:
so is $0 \longrightarrow \mathrm{Hom}_R(Q,A) \xrightarrow{f_*} \mathrm{Hom}_R(Q,B) \xrightarrow{g_*} \mathrm{Hom}_R(Q,C)$.

2) If $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ <u>exact</u>, so is:

$$0 \to \mathrm{Hom}_R(C,P) \xrightarrow{g^*} \mathrm{Hom}_R(B,P) \xrightarrow{f^*} \mathrm{Hom}_R(A,P).$$

("Hom-functors are left-exact").

Lemma $A \xrightarrow{f} B \xrightarrow{g} C$ R-modules (<u>not</u> necessarily exact). Such that: ~~The~~ $\forall P$ (R-module): $\mathrm{Hom}_R(C,P) \xrightarrow{g^*} \mathrm{Hom}_R(B,P) \xrightarrow{f^*} \mathrm{Hom}_R(A,P)$ is exact. <u>Then</u>, $A \to B \to C$ exact.

<u>Proof</u> <u>Step 1</u>: Take $P = C$. $\mathrm{Hom}_R(C,C) \to \mathrm{Hom}_R(B,C) \to \mathrm{Hom}_R(A,C)$.

$\Rightarrow g \circ f = 0 \Rightarrow \mathrm{im} f \subseteq \ker g.$ $\quad id_C \longmapsto \underbrace{id_C \circ g}_{=g} \longmapsto g \circ f.$

<u>Step 2</u>: Take ~~P~~ $P \equiv B/\mathrm{im} f$ (<u>cokernel</u>, $\mathrm{coker}(f)$).

$$\mathrm{Hom}_R\left(C, \frac{B}{\mathrm{im} f}\right) \longrightarrow \mathrm{Hom}_R\left(B, \frac{B}{\mathrm{im} f}\right) \longrightarrow \mathrm{Hom}_R\left(A, \frac{B}{\mathrm{im} f}\right).$$

$$(h : B \to B/\mathrm{im} f) \longmapsto h \circ f = 0$$

quotient map.

$\Rightarrow \exists e \in \mathrm{Hom}_R\left(C, \frac{B}{\mathrm{im} f}\right) : e \circ g = h$ (exactness).

$\Rightarrow \ker(g) \subseteq \ker(h) = \mathrm{im} f$, so <u>equal</u> ✓

For R-modules M, N, L: $\mathrm{Hom}_R(M \otimes_R N, L) \xrightarrow{\sim} \mathrm{Bilin}_R(M \times N, L)$

& $\mathrm{Bilin}_R(M \times N, L) \xrightarrow{\sim} \mathrm{Hom}_R(N, \mathrm{Hom}_R(M,L))$

$$b \longmapsto [n \mapsto (m \mapsto b(m,n))].$$

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From last time: $\mathrm{Hom}_R(M \otimes_R N) \xrightarrow{\cong} \mathrm{Hom}_R(N, \mathrm{Hom}_R(M, L))$

$\varphi \longmapsto n \mapsto (m \mapsto \varphi(m \otimes n))$.

Prop M R-module. Then: $T_M(\cdot) \equiv M \otimes_R (\cdot)$ right exact.

Proof Take: exact sequence $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$.

For any R-module P, apply $\mathrm{Hom}_R(\cdot, P)$ then $\mathrm{Hom}_R(M, \cdot)$.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(M, \mathrm{Hom}_R(C,P)) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Hom}_R(M, \mathrm{Hom}_R(B,P)) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Hom}_R(M, \mathrm{Hom}_R(A,P)) \arrow[d, phantom, "\cong" rotate=90] \\
0 \arrow[r] & \mathrm{Hom}_R(M \otimes_R C, P) \arrow[r] & \mathrm{Hom}_R(M \otimes_R B, P) \arrow[r] & \mathrm{Hom}_R(M \otimes_R A, P)
\end{tikzcd}$$

This diagram commutes since isom's are natural.

$\Rightarrow$ Bottom row exact

Fact: If $\mathrm{Hom}_R(C,P) \to \mathrm{Hom}_R(B,P) \to \mathrm{Hom}_R(A,P)$ exact $\forall P$, then $A \to B \to C$ exact.

$\Rightarrow$ By this fact: $M \otimes A \to M \otimes B \to M \otimes C \to 0$ exact ✓

Warning $A \to B \to C$ exact $\not\Rightarrow$ $M \otimes A \to M \otimes B \to M \otimes C$ exact.

Why: e.g. $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/2$ exact, but if we $\otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & (\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z} \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & (\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z} \arrow[d, phantom, "\cong" rotate=90] \\
0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "\times 2"] & \mathbb{Z}/2\mathbb{Z}
\end{tikzcd}$$

is not exact.

DEF R-module M <u>flat</u> $\iff$ $\forall f: N \to N'$ R-linear map, + injective,
then $(\mathrm{id}_M \otimes f): M \underset{R}{\otimes} N \to M \underset{R}{\otimes} N'$ injective.

Examples <u>1)</u> $\mathbb{Z}/2\mathbb{Z}$ <u>not</u> flat $\mathbb{Z}$-module. (previous example)

<u>2)</u> Free modules are exact. If $f: N \to N'$ injective + R-linear:

$$\begin{tikzcd} R^{\oplus I} \underset{R}{\otimes} N \arrow[r, "\mathrm{id}_{R^{\oplus I}} \otimes f"] \arrow[d, phantom, "\cong" rotate=90] & R^{\oplus I} \underset{R}{\otimes} N' \arrow[d, phantom, "\cong" rotate=90] \\ N^{\oplus I} \arrow[r] & (N')^{\oplus I} \\ (n_i)_{i \in I} \arrow[r, mapsto] & (f(n_i))_{i \in I}. \end{tikzcd}$$

Bottom arrow injective (applying injective map by coordinate)
+ diagram commutes
$\Rightarrow$ Top arrow injective ✓

<u>3)</u> <u>Base ring matters</u>: $\mathbb{Z}/2\mathbb{Z}$ is a ~~ba~~ flat $(\mathbb{Z}/2\mathbb{Z})$-module.
(free)

<u>4)</u> Call M <u>torsion-free</u> if $rm \neq 0$ $\forall r \in R$, $m \in M$.
$\Updownarrow$
r <u>not</u> zero-divisor

<u>Then: flat $\Rightarrow$ torsion-free.</u>

<u>Proof</u> If M <u>not</u> torsion-free: pick $r_0 \in R$ (<u>not</u> zero divisor) + $m_0 \in M$, with $r_0 \cdot m_0 = 0$. Then, $f: R \to R$, $r \mapsto r_0 \cdot r$ is injective. (r_0 <u>not</u> divisor)

$$\Rightarrow \begin{tikzcd} M \underset{R}{\otimes} R \arrow[r, "\mathrm{id}_M \otimes f"] \arrow[d, phantom, "\cong" rotate=90] & M \underset{R}{\otimes} R \arrow[d, phantom, "\cong" rotate=90] \\ M \arrow[r, mapsto, "m \mapsto r_0 \cdot m"] & M \end{tikzcd}$$

Diagram commutes.
& $m \mapsto r_0 \cdot m$ <u>not</u> injective.
$\Rightarrow$ Top <u>not</u> injective ✓

Prop (Flatness). TFAE:

1) T_M preserves exactness of all exact sequences

2) T_M preserves exactness of all short exact sequences

3) M flat

4) If $f: N \to N'$ R-linear injection: (for N, N' f.g. R-mods) then $id_M \otimes f$ injection.

Proof Clear: $1) \Rightarrow 2) \Rightarrow 3) \Rightarrow 4)$.

$2) \Rightarrow 1)$: Assume $A \xrightarrow{f} B \xrightarrow{g} C$ exact.

$\Rightarrow 0 \to A/\ker(f) \xrightarrow{\bar{f}} B \xrightarrow{g} im(g) \to 0$ exact.

$\overset{2)}{\Rightarrow} 0 \to M \otimes \frac{A}{\ker(f)} \xrightarrow{id_M \otimes \bar{f}} M \otimes B \xrightarrow{id_M \otimes g} M \otimes im(g) \to 0$ exact

But: $\ker(id_M \otimes g) = im(id_M \otimes \bar{f}) = im(id_M \otimes f)$

Hence, $M \otimes A \to M \otimes B \to M \otimes C$ is exact ✓

$4) \Rightarrow 3)$: Read notes.

[Let: $f: N \to N'$ injective R-module hom. & take:

$\sum_i m_i \otimes n_i \in \ker(id_M \otimes f: M \otimes N \to M \otimes N')$.

$\Rightarrow$ Have: $\sum_i m_i \otimes f(n_i) = 0$ in $M \otimes N'$.

Denote: $N_0 \subseteq N$ submodule gen. by $\{n_i\}$.

$\Rightarrow$ By Prop. 3.10: $\exists$ f.g. modules $M_0 \subseteq M$ & $N_0' \subseteq N'$, s.t. $\sum_i m_i \otimes f(n_i) = 0$ in $M_0 \otimes N_0'$.

$\Rightarrow$ This $\uparrow$ also holds in $M_0 \otimes (N_0' + f(N_0))$.

<u>Consider</u>: $id_{M_0} \otimes (f|_{N_0}) : M_0 \otimes N_0 \longrightarrow M_0 \otimes (N_0' + f(N_0))$.

$\Rightarrow \sum_i m_i \otimes n_i \in M_0 \otimes N_0$ is sent to 0 in this map.

& By injectivity of <u>4)</u> $\sum_i m_i \otimes n_i = 0$ in $M_0 \otimes N_0$, hence in $M \otimes N$. So, M is flat module.]

<u>Prop 3.39</u>] (Ext. of scalars <u>preserve</u> flatness)

$f: R \to S$ ring hom & M <u>flat</u> R-module. Then, $S \otimes_R M$ is flat S-module.

<u>Proof</u>) Take $g: N \to N'$ S-linear + injective. hom.

$$\begin{tikzcd}
(S \otimes_R M) \otimes_S N \arrow[r, "id_{S \otimes_R M} \otimes g"] \arrow[d] & (S \otimes_R M) \otimes_S N' \arrow[d] \\
M \otimes_R N \arrow[r, "id_M \otimes g"] & M \otimes_R N'
\end{tikzcd}$$

$(s \otimes m) \otimes n \mapsto m \otimes (sn)$ $\qquad$ $(s \otimes m) \otimes n' \mapsto m \otimes (sn')$.

Diagram commutes (obvious from formulae)

& Bottom arrow <u>injective</u> (Def. of M flat)

$\Rightarrow$ Top arrow injective. Hence: $S \otimes_R M$ flat S-module. ✓

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§3.5: Further Examples of Tensor Products

Example 3.41) $\forall (x, \otimes y) \in \mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$: have:

$$(x \otimes y) = \left(n \cdot \frac{x}{n}\right) \otimes y = \frac{x}{n} \otimes (ny) = 0$$

$$\Rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0.$$

Used properties: 1) $\mathbb{Q}$ divisible : $\forall n \geq 1, a \in A, \exists a' \in A, na' = a$.

2) $\mathbb{Z}/n\mathbb{Z}$ torsion group: i.e. all elements finite order.

So: $(\text{divisible}) \otimes_{\mathbb{Z}} (\text{torsion}) = 0$.

Since $\mathbb{Q}/\mathbb{Z}$ torsion & divisible, get: $(\mathbb{Q}/\mathbb{Z})^{\otimes 2} = 0$.

But: $\forall M$ R-module, $(M \neq 0)$, if M f.g. $\Rightarrow M^{\otimes n} \neq 0$ $\forall n \geq 1$.

Example) V is $\mathbb{Q}$-VS. Then, $\mathbb{Q} \otimes_{\mathbb{Q}} V \xrightarrow{\cong} V$.

$$(x, v) \longmapsto xv.$$

What about $\mathbb{Q} \otimes_{\mathbb{Z}} V$? have: $\mathbb{Q} \otimes_{\mathbb{Z}} V \xrightarrow{\cong} V$

$$(x, v) \xmapsto{(*)} xv.$$

Proof Need: all tensors of $\mathbb{Q} \otimes_{\mathbb{Z}} V$ are pure. If $a_i, b_i \in \mathbb{Z}$:

$$\sum \frac{a_i}{b_i} \otimes v_i = \sum \frac{1}{b_i} \otimes (a_i v_i) = \sum \frac{1}{b_i} \otimes \left(b_i \cdot \frac{a_i}{b_i} v_i\right)$$

$$= \sum 1 \otimes \left(\frac{a_i}{b_i} v_i\right) = 1 \otimes \left(\sum \frac{a_i}{b_i} v_i\right). \checkmark$$

$(*)$ surjective: yes, since $1 \otimes v \mapsto v$.

$(*)$ injective: Suffices to show for pure tensors.

If $x \otimes v \longmapsto xv = 0$, then: $x=0$ or $v=0 \Rightarrow x \otimes v = 0$ ✓

Can generalise: $\mathrm{Frac}(R) \otimes_R V \cong V$, where V is $\mathrm{Frac}(R)$-module (as $\mathrm{Frac}(R)$-modules).

Prop 3.44) R integral dom. & V is $\mathrm{Frac}(R)$-module. & $M \neq 0$ R-submodule of $\mathrm{Frac}(R)$. Then: $M \otimes_R V \cong V$ as R-modules.

$(x, v) \longmapsto xv$.

More examples) $M \otimes_R \left(\bigoplus_{i \in I} M_i\right) \xrightarrow{\sim} \bigoplus_{i \in I} (M \otimes_R M_i)$

$m \otimes (m_i)_{i \in I} \longmapsto (m \otimes m_i)_{i \in I}$.

Is: R-module isomorphism.

Same formula $\Rightarrow$ $M \otimes_R \left(\prod_{i \in I} N_i\right) \longmapsto \prod_{i \in I} (M \otimes_R N_i)$.

Not generally an $\cong$!!

For example: consider $\mathbb{Q} \otimes_{\mathbb{Z}} \left(\prod_{n \geq 1} \mathbb{Z}/2^n\mathbb{Z}\right)$ vs $\prod_{n \geq 1} (\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/2^n\mathbb{Z})$.

Then: RHS $= 0$, since $\mathbb{Q}$ divisible & $\mathbb{Z}/2^n\mathbb{Z}$ torsion.

LHS: if $g = (1, 1, \dots)$ then g has infinite order. in $\prod_{n \geq 1} (\mathbb{Z}/2^n\mathbb{Z})$.

$$\Rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \underbrace{\langle g \rangle}_{\cong \mathbb{Z}} \cong \mathbb{Q}.$$

Since $\mathbb{Q}$ is flat $\mathbb{Z}$-module & $\langle g \rangle \hookrightarrow \prod_{n \geq 1} (\mathbb{Z}/2^n\mathbb{Z})$, get:

$\mathbb{Q} \cong \mathbb{Q} \otimes_{\mathbb{Z}} \langle g \rangle \hookrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \left(\prod_{n \geq 1} \mathbb{Z}/2^n\mathbb{Z}\right)$ non-empty. ✓

Example 3.48 $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ (as $\mathbb{C}$-algebra)?

aka: $\underset{\substack{\uparrow \\ \text{ext. of} \\ \text{scalars}}}{\mathbb{C}} \otimes_{\mathbb{R}} \underset{\substack{\uparrow \\ \mathbb{R}\text{-mod.} \\ \cong \mathbb{R}^2}}{\mathbb{C}}$ (as a $\mathbb{C}$-module).

$\Rightarrow \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^2$, with basis $\{1 \otimes 1$ & $1 \otimes i\}$.

As algebras: $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ ↖ $\mathbb{R}$-algebra (made choice)

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \otimes_{\mathbb{R}} \left(\mathbb{R}[T]/(T^2+1)\right) \cong \mathbb{C}[T]/(T^2+1).$$

$$\cong \mathbb{C}[T]/(T+i) \times \mathbb{C}[T]/(T-i) \quad (\text{CRT}).$$

$$\cong \mathbb{C} \times \mathbb{C}.$$

What is the isomorphism? $x \otimes y = (a+bi) \otimes (c+di)$

$\mapsto (a+bi) \otimes (c+dT + (T^2+1))$. (coset)

$\mapsto \underbrace{(a+bi)(c+dT)}_{=P} + (T^2+1)\mathbb{C}[T]$

$\mapsto (P + (T+i)\mathbb{C}[T],\ P + (T-i)\mathbb{C}[T])$

$\mapsto (x\bar{y},\ xy). = ((ac+bd + i(bc-ad)),\ (ac-bd) + i(bc+ad))$

So: $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} \mathbb{C} \times \mathbb{C}$ by $x \otimes y \mapsto (x\bar{y}, xy)$.

---

§4: Localisation.

DEF 4.1 $S \subseteq R$ multiplicative $\Leftrightarrow$ $1 \in S$ & $\forall a, b \in S$, $ab \in S$.

& If $U \subseteq R$ subset: multiplicative closure of U in R

is $\cap$ of all multiplicative sets containing U.

(Equivalently: all elements of form $\prod_{i \leq n} s_1 \dots s_n$, $n \geq 0$ & $s_i \in U$).

Examples] 1) R integral dom $\Rightarrow S = R \setminus 0$ mult.

2) p prime in $R \Rightarrow S = R \setminus p$ mult.

3) $x \in R \Rightarrow S = \{x^n : n \geq 0\}$ mult.

Note: $\mathbb{Q}$ obtained from $\mathbb{Z}$ by adding in inverses for elements in the set $\mathbb{Z} - 0$.

& $\exists$ ring hom $\mathbb{Z} \hookrightarrow \mathbb{Q}$. (injective)

Will: generalise this to general R (not just $\mathbb{Q}$) & multiplicative S (not just $\mathbb{Z}$). But, generally lose injectivity.

DEF 4.2] M R-module & $S \subseteq R$ mult. & consider $M \times S$.

Write: $(m_1, s_1) \sim (m_2, s_2) \Leftrightarrow \exists u \in S,\ u(s_2 m_1 - m_2 s_1) = 0$.

$\Rightarrow \sim$ is an equivalence relation, so write $\frac{m}{s}$ for the class (m, s).

& $S^{-1}M \equiv \{\frac{m}{s} : m \in M, s \in S\}$.

Then: $S^{-1}M$ is abelian group: $\frac{m_1}{s_1} + \frac{m_2}{s_2} = \frac{m_1 s_2 + m_2 s_1}{s_1 s_2}$

& is R-module by $r \cdot \frac{m}{s} = \frac{rm}{s}$.

& is Ring, by $\frac{m_1}{s_1} \cdot \frac{m_2}{s_2} = \frac{m_1 m_2}{s_1 s_2}$ (check well-def'ed).

& is $(S^{-1}R)$-module, by: $\frac{r}{s} \cdot \frac{m}{t} = \frac{rm}{st}$.

Commutative Algebra: lecture 9

From last time: defined $S^{-1}R$, $S^{-1}M$.

Localisation maps: $R \longrightarrow S^{-1}R$ & $M \longrightarrow S^{-1}M$

$r \longmapsto r/1$ $\quad m \longmapsto \frac{m}{1}$.

Both are R-linear.

Proof of equivalence relation of $\sim$

Reflexive: ✓ & symmetric: ✓ Transitive:

Say $(m_1, s_1) \sim (m_2, s_2) \sim (m_3, s_3)$.

$\Rightarrow \exists u, v \in S : u(s_2 m_1 - s_1 m_2) = 0 = v(s_3 m_2 - s_2 m_3)$.

$\Rightarrow uvs_3(s_2 m_1 - s_1 m_2) + uvs_1(s_3 m_2 - s_2 m_3) = 0$

$\Rightarrow uvs_2(s_3 m_1 - s_1 m_3) = 0$ so $(m_1, s_1) \sim (m_3, s_3)$ ✓.

Prop 4.4 $U \subseteq R$ subset & $S =$ mult. closure of U in R.

Then: $\forall B$ ring & ring hom $f : R \to B$ s.t. $f(u)$ unit $\forall u \in U$.

$\Rightarrow \exists!\ h : S^{-1}R \longrightarrow B$ ring hom s.t. commutes:

$$\begin{tikzcd} R \arrow[r, "r \mapsto r/1"] \arrow[dr, "f"'] & S^{-1}R \arrow[d, "!h"] \\ & B \end{tikzcd}$$

given by: $f(r) = h\left(\frac{r}{1}\right)$.

& $h\left(\frac{r}{s}\right) = f(s)^{-1} \cdot f(r)$.

$\Rightarrow \mathrm{Hom}_{ring}(S^{-1}R, B) \cong \{\varphi \in \mathrm{Hom}_{ring}(R, B) : \varphi(u) \in B^\times\}$

$f \longmapsto \left(r \mapsto f\left(\frac{r}{1}\right)\right)$.

Proof) Take $f: R \to B$, $f(U) \subseteq B^\times$. Then, $f(S) \subseteq B^\times$.

Want: $h: S^{-1}R \to B$, s.t. $f(r) = h(\frac{r}{1})$ $\forall r \in R$.

h must satisfy:

- $1 = h(1) = h(\frac{1}{s} \cdot \frac{s}{1}) = h(\frac{1}{s}) h(\frac{s}{1}) = h(\frac{1}{s}) f(s)$

$\Rightarrow h(s^{-1}) = f(s)^{-1}$. $\Rightarrow h(\frac{r}{s}) = f(r) \cdot f(s)^{-1}$.

So need to show h is well-defined with this formula.

$\Leftrightarrow$ if $\frac{r_1}{s_1} = \frac{r_2}{s_2}$, then $f(s_1)^{-1} f(r_1) = f(s_2)^{-1} f(r_2)$.

Know: $\exists t \in S$: $t(s_2 r_1 - s_1 r_2) = 0$.

$\Rightarrow f(t) f(s_2 r_1 - s_1 r_2) = 0$

$\Rightarrow f(t) f(s_2) f(r_1) = f(t) f(s_1) f(r_2)$.

$\Rightarrow f(s_1)^{-1} f(r_1) = f(s_2)^{-1} f(r_2)$ (since $f(t)$, $f(s_i)$ invertible) ✓

& Checking ring hom is obvious.

For uniqueness: need $(S^{-1}R, i_{S^{-1}R})$ is unique pair satisfying universal prop.

If (A, j) also satisfies:

- Challenge $S^{-1}R$ with j $\Rightarrow \exists \varphi: S^{-1}R \to A$, $j = \varphi \circ i_{S^{-1}R}$
- Challenge A with $i_{S^{-1}R}$ $\Rightarrow \exists \psi: A \to S^{-1}R$, $i_{S^{-1}R} = \psi \circ j$

$\Rightarrow \psi \circ \varphi \circ i_{S^{-1}R} = i_{S^{-1}R}$, so by universal prop, $\psi \circ \varphi = id_{S^{-1}R}$.

& $\varphi \circ \psi = id_A$.

$\Rightarrow \varphi, \psi$ isoms.

For formula: $j = \varphi \circ i_{S^{-1}A} \Rightarrow \varphi(\frac{r}{1}) = j(r) \Rightarrow \varphi(\frac{r}{s}) = j(s)^{-1} j(r)$ ✓

Facts of Localisation.

1) $\frac{r}{s} \in S^{-1}R. \Rightarrow$ have $\frac{r}{s} = \frac{0}{1} \Leftrightarrow \exists u \in S,\ ur = 0.$

2) In particular: $S^{-1}R = 0 \Leftrightarrow 0 \in S.$

3) $\mathrm{Ker}(i_{S^{-1}R}) = \{ r \in R : \exists u \in S,\ ur = 0 \}.$

4) $i_{S^{-1}R} : R \longrightarrow S^{-1}R$ injective $\Leftrightarrow$ S has no 0-divisors.

5) i always epimorphism, but not always surj.

[Recall: $f: A \to B$ epimorphism $\Leftrightarrow$ $(g \circ f = h \circ f \Rightarrow g = h)$ for any $g, h : B \to C$.)

Examples 1) $f \in R$, $S = \{ f^n : n \geq 0 \}$. $R_f \equiv S^{-1}R$.

For $R = \mathbb{Z}$, $f = 2 \Rightarrow \mathbb{Z}_2 = \{ \frac{a}{2^n} : a \in \mathbb{Z}, n \geq 0 \} \simeq \mathbb{Z}[\frac{1}{2}]$.

2) $p \subseteq \mathrm{Spec}(R) \equiv \{\text{Prime ideals of } R\}$, $S = R \setminus p$.

$\Rightarrow R_p \equiv (R \setminus p)^{-1} R.$

E.g. $\mathbb{Z}_{(3)} = \{ \frac{a}{b} : a, b \in \mathbb{Z}\ \&\ 3 \nmid b \}$.

Prop 4.6 M is R-module. Then: $(S^{-1}R) \otimes_R M \xrightarrow{\sim} S^{-1}M$ as $(S^{-1}R)$-modules,

$(\frac{r}{s}) \otimes m \longmapsto \frac{rm}{s}.$

Proof Define $\varphi : (S^{-1}R) \otimes_R M \longrightarrow S^{-1}M$, $\varphi((\frac{r}{s}) \otimes m) = \frac{rm}{s}$

(by first defining on $S^{-1}R \times M$, and using universal prop of tensor product).

Clearly: φ ~~is~~ surjective. & φ is also $(S^{-1}R)$-linear.

Injective: first show, all $\sum_i \frac{r_i}{s_i} \otimes m_i \in (S^{-1}R) \otimes_R M$ pure.

Let: $s = s_1 \cdots s_n$ & $t_i = \prod_{j \neq i} s_j$.

$$\Rightarrow \sum_i \left(\frac{r_i}{s_i}\right) \otimes m_i = \sum_i \underbrace{\left(\frac{1}{s_i}\right)}_{t_i/s} \otimes (r_i m_i) = \sum_i \frac{1}{s} \otimes (t_i r_i m_i)$$
$$= \frac{1}{s} \otimes \left(\sum_i t_i r_i m_i\right) \checkmark$$

So, to show inj: suffices to show for pure tensors.

If $\frac{1}{s} \otimes m \mapsto 0$ then $\exists u \in S$, $u(1 \cdot m - 0 \cdot s) = 0 \Rightarrow um = 0$.

$$\Rightarrow \frac{1}{s} \otimes m = \frac{u}{us} \otimes m = \frac{1}{us} \otimes um = 0 \quad (u \in R) \checkmark$$

So far: $S^{-1}(\cdot)$ acts on modules.

Next time: make it act on functions.

Commutative Algebra: <u>lecture 10.</u>

<u>From last time</u>: M R-module & $S \subseteq R$ mult.

$\Rightarrow S^{-1}R \otimes_R M \xrightarrow{\sim} S^{-1}M$ as $(S^{-1}R)$-modules.

If $f: N \to N'$ R-linear:

$$\begin{tikzcd}
(S^{-1}R) \otimes_R N \arrow[r, "\mathrm{id}_{S^{-1}R} \otimes f"] \arrow[d, "\frac{r}{s}\otimes n \mapsto \frac{rn}{s}"'] & (S^{-1}R) \otimes_R N' \arrow[d, "\frac{r}{s}\otimes n' \mapsto \frac{rn'}{s}"] \\
S^{-1}N \arrow[r, "\text{“}S^{-1}f\text{”}"] & S^{-1}N'
\end{tikzcd}$$

$\Rightarrow$ By chasing this diagram, $(S^{-1}f)\left(\frac{n}{s}\right) = \frac{f(n)}{s}$.

$\Rightarrow S^{-1}R \otimes_R (\cdot) \simeq S^{-1}(\cdot)$. "Naturally isomorphic".

<u>Remark</u> If A is R-algebra, then $S^{-1}R \otimes_R A \xrightarrow{\sim} S^{-1}A$ as a $(S^{-1}R)$-module $\cong$. But, also $\cong$ of $(S^{-1}R)$-<u>algebras</u>.

<u>Lemma 4.9</u> M is $(S^{-1}R)$-module.

<u>Write</u>: $S^{-1}M$ = module, resulting from: restricting scalars in M from $S^{-1}R$ to R, and localising at S.

<u>Then</u>: $S^{-1}M \simeq M$ as $(S^{-1}R)$-modules.

$$\frac{m}{1} \mapsto m$$

<u>Equivalently</u>: $M \xrightarrow{\sim} (S^{-1}R) \otimes_R M$ as $(S^{-1}R)$-modules.

$$m \mapsto 1 \otimes m.$$

Proof $M \mapsto S^{-1}M$ is $(S^{-1}R)$-linear.

$m \mapsto \frac{m}{1}$

Surjective: $\frac{1}{s} \cdot m \mapsto \frac{m}{s}$ ✓

Injective: $\frac{m}{1} = \frac{0}{1} \iff \exists u \in S,\ um = 0. \Rightarrow m = 0$ (mult by u^{-1}).

Recall: $\mathrm{Hom}_{ring}(S^{-1}R, B) \xrightarrow{\sim} \{f \in \mathrm{Hom}_{ring}(R, B) : f(S) \subseteq B^\times\}$.

Now: $\mathrm{Hom}_R(M, L) \xrightarrow{\sim} \mathrm{Hom}_{S^{-1}R}(S^{-1}M, L)$.

fixed R-module

$(S^{-1}R)$-module

Prop 4.10) M is R-mod & L is $(S^{-1}R)$-module. $f : M \to L$ R-linear.

$\Rightarrow \exists!\, h : S^{-1}M \to L$ $(S^{-1}R)$-linear

& $f = h \circ i_{S^{-1}M}$ (or: $f(m) = h(\frac{m}{1})$).

Proof Since $S^{-1}(\cdot)$ & $(S^{-1}R) \otimes_R (\cdot)$ naturally $\cong$: suffices to show, for $(S^{-1}R) \otimes_R (\cdot)$.

Then: $i : M \to (S^{-1}R) \otimes_R M$, $m \mapsto \frac{1}{1} \otimes m$.

Take: $f : M \to L$ R-linear. Want: $h : S^{-1}R \otimes_R M \to \underbrace{S^{-1}R \otimes_R L}_{\cong L}$.

$\Rightarrow h = id_{S^{-1}R} \otimes f$.

$\Rightarrow h(\frac{r}{s} \otimes m) = \frac{r}{s} f(m)$. In particular: $h(1 \otimes m) = f(m)$. ✓

Uniqueness: note $\{1 \otimes m\}_{m \in M}$ generate $S^{-1}R \otimes_R M$ as $(S^{-1}R)$-mod.

$\Rightarrow h$ determined by values here ✓

Prop] (Exactness of S^{-1}).

If $A \xrightarrow{f} B \xrightarrow{g} C$ exact (seq of R-mods), then:

$S^{-1}A \xrightarrow{S^{-1}f} S^{-1}B \xrightarrow{S^{-1}g} S^{-1}C$ exact (seq of $(S^{-1}R)$-mods).

Proof] $(S^{-1}g)\circ(S^{-1}f) = S^{-1}(g\circ f) = 0. \Rightarrow Im(S^{-1}f) \subseteq ker(S^{-1}g)$.

For $\frac{b}{s} \in ker(S^{-1}g)$: $\Rightarrow \frac{g(b)}{s} = 0 = \frac{0}{1} \Rightarrow \exists u \in S,\ g(b)\cdot u = 0.$

$\Rightarrow g(ub) = 0$ (g is R-linear)

$\Rightarrow ub \in ker(g) = im(f)$, so $\exists a \in A,\ f(a) = ub.$

Then: $\frac{b}{s} = \frac{ub}{us} = \frac{f(a)}{us} = (S^{-1}f)\left(\frac{a}{us}\right) \in im(S^{-1}f)$ ✓

Equivalently, $S^{-1}R$ is <u>flat</u> R-module.

In particular: if $N \subseteq M$ R-modules & $N \overset{i}{\hookrightarrow} M$ inclusion,

then $S^{-1}N \overset{S^{-1}i}{\hookrightarrow} S^{-1}M$ injective.

$$\frac{n}{s} \longmapsto \frac{n}{s}$$

Prop 4.13] $N, P \subseteq M$ submodules (M is R-module).

1) $S^{-1}(N+P) = S^{-1}N + S^{-1}P$

2) $S^{-1}(N \cap P) = S^{-1}N \cap S^{-1}P$

3) $S^{-1}M / S^{-1}N \xrightarrow{\sim} S^{-1}(M/N)$

$$\frac{m}{s} + S^{-1}N \longmapsto \frac{m+N}{s}.$$

Proof] 1) $\frac{n+p}{s} = \frac{n}{s} + \frac{p}{s} \in S^{-1}N + S^{-1}P$

& $\frac{n}{s_1} + \frac{p}{s_2} = \frac{ns_2 + ps_1}{s_1 s_2} \in S^{-1}(N+P).$

2) If $x \in S^{-1}N \cap S^{-1}P$ then $x = \frac{n}{s_1} = \frac{p}{s_2}$ $\Rightarrow u(s_2 n - s_1 p) = 0$

$\Rightarrow$ If $\omega = \underset{\in N}{us_2 n} = \underset{\in P}{us_1 p}$ then $\omega \in N \cap P$

$\Rightarrow \frac{n}{s_1} = \frac{\omega}{us_1 s_2} \in S^{-1}(N \cap P)$.

3) $0 \longrightarrow N \xrightarrow{i} M \xrightarrow{\pi} M/N \longrightarrow 0$

$\Rightarrow 0 \longrightarrow S^{-1}N \xrightarrow{S^{-1}i} S^{-1}M \xrightarrow{S^{-1}\pi} S^{-1}(M/N) \longrightarrow 0$

But: $(S^{-1}i)(S^{-1}N) = S^{-1}N \subseteq S^{-1}M$.

& $S^{-1}\pi$ sends $\frac{m}{s} \longmapsto \frac{m+N}{s}$

$\Rightarrow \ker(S^{-1}M \longrightarrow S^{-1}(M/N)) = S^{-1}N$.

$\Rightarrow S^{-1}(M/N) \cong S^{-1}M / S^{-1}N$. ✓

Prop 4.14 $S^{-1}M \underset{S^{-1}R}{\otimes} S^{-1}N \xrightarrow{\sim} S^{-1}(M \underset{R}{\otimes} N)$ as $(S^{-1}R)$-modules.

$$\frac{m}{s_1} \otimes \frac{n}{s_2} \longmapsto \cancel{\frac{mn}{s_1 s_2}}\ \frac{m \otimes n}{s_1 s_2}.$$

Proof $(S^{-1}R \underset{R}{\otimes} N) \underset{S^{-1}R}{\otimes} (S^{-1}R \underset{R}{\otimes} M) \cong S^{-1}R \underset{R}{\otimes} (M \underset{R}{\otimes} N)$.

So, done by natural isomorphism.

If $p \subseteq R$ prime ideal, then in particular: $M_p \underset{R_p}{\otimes} N_p \cong (M \underset{R}{\otimes} N)_p$.

Extension & Contraction under $R \mapsto S^{-1}R$.

For ring hom $A \xrightarrow{f} B$, recall:

⊛ Contraction map: $\underline{b} \mapsto \underbrace{f^{-1}(\underline{b})}_{=\underline{b}^c}$, ($\underline{b} \subseteq B$ ideal), and:
$\underline{b}^c \subseteq A$ ideal (call $\underline{b}^c$ <u>contracted</u> ideals)

⊛ Extension map: $\underline{a} \mapsto f(\underline{a}) \equiv \underline{a}^e$, ($\underline{a} \subseteq A$ ideal),
and $\underline{a}^e \subseteq B$ ideal. (call $\underline{a}^e$ <u>extended</u> ideals).

⊛ $\underline{b}^c$ prime if $\underline{b}$ prime

⊛ $\underline{a} \subseteq A$ contracted $\iff \underline{a} = \underline{a}^{ec}$
& $\underline{b} \subseteq B$ contracted $\iff \underline{b} = \underline{b}^{ce}$.

If $S \subseteq R$ multiplicative: have, $R \xrightarrow{i} S^{-1}R$, $r \mapsto \frac{r}{1}$.
$\Rightarrow \underline{a}^e = \left\{ \frac{a}{s} \in S^{-1}R : a \in \underline{a},\ s \in S \right\}$
& $\underline{b}^{c} = \left\{ r \in R : \frac{r}{1} \in \underline{b} \right\}$.

Commutative Algebra: lecture 11.

In general: say $f: A \to B$ ring hom. Then:

$$\{\text{Contracted ideals of } A\} \longleftrightarrow \{\text{Extended ideals of } B\}$$
$$\underline{a} \longmapsto \underline{a}^e$$
$$\underline{b}^c \longleftarrow\!\!\shortmid\ \underline{b}$$

& want to study: this for $R \to S^{-1}R$, $r \mapsto \frac{r}{1}$.

⊛ Extension: $\underline{a} \subseteq R$ ideal $\Rightarrow \underline{a}^e = \left\{\frac{a}{s} : a \in \underline{a}, s \in S\right\} = S^{-1}\underline{a}$

Claim: $a^{ec} = \bigcup_{s \in S} (\underline{a} : s)$ [where $(\underline{a} : s) = \{r \in R : rs \in \underline{a}\}$]

Proof. $\forall r \in \bigcup_{s \in S} (\underline{a} : s)$: $\Rightarrow rs = a$ (some $s \in S$)($a \in \underline{a}$)

$\Rightarrow \frac{rs}{1} = \frac{a}{1}$ in $S^{-1}R$ $\Rightarrow \frac{r}{1} = \frac{a}{s} \in \underline{a}^e$. $\Rightarrow r \in \underline{a}^{ec}$.

$\Leftarrow$: If $r \in \underline{a}^{ec} \Rightarrow \frac{r}{1} = \frac{a}{s} \Rightarrow \exists u \in S$, $u(rs - a) = 0$.

So, $rus = ua \in \underline{a} \Rightarrow r \in (\underline{a} : \underbrace{us}_{\in S})$ ✓

⊛ Contraction: let $\underline{b} \subseteq S^{-1}R$ ideal, $\underline{b}^c = \{r \in R : \frac{r}{1} \in \underline{b}\}$

Claim: $\underline{b}^{ce} = \underline{b}$.

Proof: in example sheet, $\subseteq$ always holds.

$\supseteq$: $\forall \frac{r}{s} \in \underline{b}$: $\Rightarrow \frac{r}{1} \in b$, so $r \in b^c \Rightarrow \frac{r}{1} \in b^{ce} \Rightarrow \frac{r}{s} \in b^{ce}$ ✓

Prop 4.15] Consider: $R \to S^{-1}R$, $r \mapsto \frac{r}{1}$.

1) Every ideal of $S^{-1}R$ is extended.

2) $\underline{a} \subseteq R$ ideal is contracted $\Leftrightarrow$ Image of S in $R/\underline{a}$ has no zero divisors.

3) $\underline{a}^e = R \iff \underline{a} \cap S \neq \phi$.

4) $\{p \in Spec(R) : p \cap S = \phi\} \longleftrightarrow Spec(S^{-1}R)$

$$p \longmapsto p^e$$
$$q^c \longleftarrow\!\shortmid\; q$$

Proof) 1) True, since $\underline{b}^{ce} = \underline{b}$ $\forall\, \underline{b} \subseteq S^{-1}R$ ideal.

2) a contracted $\iff \underline{a}^{ec} \subseteq \underline{a} \iff \forall s \in S, (\underline{a} : s) \subseteq \underline{a}$

$\iff \forall r \in R : (Sr \cap \underline{a} \neq \phi \Rightarrow r \in \underline{a})$

$\iff$ The image $\overline{S}$ of S in $R/\underline{a}$ contains no zero-divisors.

[since: $Sr \cap \underline{a} \neq \phi \iff 0 + \underline{a} \in \overline{S} \cdot (r + \underline{a})$

& $r \in \underline{a} \iff r + \underline{a} = 0 + \underline{a}$]

3) If $\underline{a} \cap S \neq \phi$, take $x \in \underline{a} \cap S$. Then, $1 = \frac{x^{\in R}}{x_{\in S}} \in a^e$.

So $a^e = S^{-1}R$ ✓

& If $\frac{1}{1} \in a^e$, then: $\exists a \in \underline{a}, s \in S, \frac{a}{s} = \frac{1}{1}$.

$\Rightarrow \exists u \in S, u(a-s) = 0$, so $us = ua \in S \cap \underline{a} \neq \phi$ ✓

4) For general ring hom, contraction of prime ideal is prime.

$\Rightarrow$ By 2): $p \subseteq R$ contracted $\iff$ Image of S in R/p has no zero divisors. $\iff p \cap S = \phi$ (as p prime)

$\Rightarrow$ Have contraction map: $\{p \in Spec(R) : p \cap S = \phi\} \leftarrow Spec(S^{-1}R)$.

~~Explanation~~ This map injective ($q^{ce} = q$ $\forall q \in Spec(S^{-1}R)$)

[since: every ideal of $S^{-1}R$ is extended)

$\leq$ $\forall \mathfrak{p} \in$ LHS: $\mathfrak{p}$ contracted $\Rightarrow \mathfrak{p}^{ec} = \mathfrak{p}$.

$\Rightarrow$ Remains to show: $\mathfrak{p}^e \subseteq S^{-1}R$ prime. $\Leftrightarrow (S^{-1}R)/\mathfrak{p}^e$ int dom.

$\Leftrightarrow (S^{-1}R)/\mathfrak{p}^e$ has no zero-divisors.

Strategy: Embed $\frac{S^{-1}R}{\mathfrak{p}^e} \hookrightarrow FF(R/\mathfrak{p})$. (Fields are int doms)

Indeed: the composite map $R \longrightarrow R/\mathfrak{p} \longrightarrow FF(R/\mathfrak{p})$ sends any elements of S to invertible element of $FF(R/\mathfrak{p})$.

$\Rightarrow$ By Universal Prop of $S^{-1}R$, $\exists \varphi: S^{-1}R \longrightarrow FF(R/\mathfrak{p})$ ring hom, given by: $\frac{r}{s} \longmapsto \frac{r+\mathfrak{p}}{s+\mathfrak{p}}$.

Suffices to show: $\ker(\varphi) = \mathfrak{p}^e$ $\left(\Rightarrow \frac{S^{-1}R}{\mathfrak{p}^e} \xhookrightarrow{\varphi} FF(R/\mathfrak{p})\right)$

For $\frac{r}{s} \in \ker(\varphi)$: $\Rightarrow \frac{r+\mathfrak{p}}{s+\mathfrak{p}} = \frac{0}{1}$ (in $FF(R/\mathfrak{p})$)

$\Rightarrow \operatorname{Im}(\varphi) \subseteq \bar{S}^{-1}(R/\mathfrak{p})$, where $\bar{S}$ = image of S in $R/\mathfrak{p}$

(noting: image of mult. set is mult.)

So, restrict range: $\bar{\varphi}: S^{-1}R \longrightarrow \bar{S}^{-1}(R/\mathfrak{p})$.

So, $\varphi(\frac{r}{s}) = \frac{0}{1} \Leftrightarrow \exists\, u+\mathfrak{p} \in \bar{S}$, with $u \in S$ and:

$(u+\mathfrak{p})(r+\mathfrak{p}) = 0 \Leftrightarrow ur \in \mathfrak{p}$.

$\Rightarrow \frac{r}{s} = \frac{ur}{us} \in \mathfrak{p}^e$ ✓

For reverse direction: $\forall x \in \mathfrak{p}^e$, $x = \frac{p}{s}$, $p \in \mathfrak{p}$, $s \in S$.

$\Rightarrow \varphi(x) = \frac{p+\mathfrak{p}}{s+\mathfrak{p}} = 0 \Rightarrow x \in \ker(\varphi)$ ✓

Application I ideal of R. $\Rightarrow$ Radical of I:

$\sqrt{I} = \{x \in R : \exists n,\ x^n \in I\}$.

Prop 4.17 $\sqrt{I} = \bigcap_{I \subseteq \mathfrak{p} \in \operatorname{Spec} R} \mathfrak{p}$.

Proof $\boxed{\subseteq}$: $\forall x \in \sqrt{I}$, $\exists n$, $x^n \in I$.

$\Rightarrow \forall \mathfrak{p} \in \operatorname{Spec}(R)$ with $I \subseteq \mathfrak{p}$, have $x^n \in \mathfrak{p} \Rightarrow x \in \mathfrak{p}$ ✓

$\boxed{\supseteq}$: Say $x \in R$, $x \notin \sqrt{I}$. Then: $I \neq R \Rightarrow R/I \neq 0$.

So write $\bar{x} \equiv$ image of x, in R/I.

& Consider: $(R/I)_{\bar{x}} \equiv \{\bar{x}^n : n \geq 0\}^{-1}(R/I)$ localisation

$\neq 0$ (Since $\bar{x}$ not nilpotent, $x \notin \sqrt{I}$).

$\Rightarrow (R/I)_{\bar{x}}$ has prime ideal $\mathfrak{p}$. (Since, it has maximal ideal)

Then, $\mathfrak{p}$ corresponds to prime ideal of R/I, disjoint from $\{\bar{x}^n : n \geq 0\}$. In particular, $x \notin \mathfrak{p}$. ✓

Commutative Algebra: lecture 12.

Local Properties.

DEF 4.18 R ring is *local* $\Leftrightarrow$ has unique maximal ideal.
Call it $\underline{m}$ & write $(R, \underline{m})$ to denote local ring.

Example $p \in \mathrm{Spec}(R)$ & $R_p = (R-p)^{-1}R$.

Write: pR_p for p^e. Then: prime ideals of R_p are in bijection with extensions of primes in R contained in p

$\Rightarrow$ All prime ideals of R_p are in pR_p.
So, (R_p, pR_p) is a local ring.

Example: $\mathbb{Z}_{(2)} = \left\{\frac{a}{b} : a, b \in \mathbb{Z}, 2 \nmid b\right\}$.

$(2)\mathbb{Z}_{(2)} = \left\{\frac{2a}{b} : a, b \in \mathbb{Z}, 2 \nmid b\right\}$

For any R-module M, M_p is module over R_p.

Prop 4.20 For R-module M, TFAE:

1) $M = 0$
2) $M_p = 0 \ \forall p$ prime ideal of R
3) $M_{\underline{m}} = 0 \ \forall \underline{m}$ max ideal of R

Proof 1) $\Rightarrow$ 2) $\Rightarrow$ 3) are obvious.

3) $\Rightarrow$ 1): Assume $M \neq 0 \Rightarrow$ pick $0 \neq m_0 \in M$, and consider

$\mathrm{Ann}_R(m_0) \equiv \{r \in R : rm_0 = 0\} \subseteq R$ *ideal* (of R)

& take: $\underline{m}$ ideal, s.t. $\mathrm{Ann}_R(m_0) \subseteq m$.

(can do, since $1 \notin \mathrm{Ann}_R(m)$.)

Then: $\frac{m_0}{1} \in M_{\underline{m}} = 0$ (by 3)), so: $\frac{m_0}{1} = \frac{0}{1}$ in $M_{\underline{m}}$.

$\Rightarrow \underbrace{\exists u \in R \setminus \underline{m}}$, $u \cdot m_0 = 0$.

$\Rightarrow u \notin \mathrm{Ann}_R(m_0)$ $\lightning$. ✓

Prop 4.21 Let $f: M \to N$ R-linear. TFAE.

1) f injective

2) $f_{\underline{p}} : M_{\underline{p}} \to N_{\underline{p}}$ injective $\forall \underline{p} \in \mathrm{Spec}(R)$.

3) $f_{\underline{m}} : M_{\underline{m}} \to N_{\underline{m}}$ injective $\forall \underline{m} \in \mathrm{mspec}(R)$.

Same, for "surjective".

Proof 1) $\Rightarrow$ 2): true since $(R \setminus \underline{p})^{-1}(\cdot)$ exact functor, hence preserves inj & surj.

2) $\Rightarrow$ 3): yes, since primes $\Leftarrow$ maximal.

3) $\Rightarrow$ 1): Assume $f_{\underline{m}}$ injective $\forall \underline{m} \in \mathrm{mspec}(R)$.

Consider: exact seq $0 \to \ker(f) \to M \xrightarrow{f} N$

$\Rightarrow$ Exact seq $0 \to (\ker f)_{\underline{m}} \to M_{\underline{m}} \xrightarrow{f_{\underline{m}}} N_{\underline{m}}$

$\Rightarrow (\ker(f))_{\underline{m}} = \ker(f_{\underline{m}}) = 0$ (assumption)

$\Rightarrow \ker(f) = 0$, (since being 0 is local prop.)

For surj: just invert all the arrows.

Prop 4.22 (Flatness is local prop.) TFAE:

1) M flat R-module

2) $M_{\mathfrak{p}}$ flat $R_{\mathfrak{p}}$-module $\forall \mathfrak{p} \in \operatorname{Spec}(R)$

3) $M_{\mathfrak{m}}$ flat $R_{\mathfrak{m}}$-module $\forall \mathfrak{m} \in \operatorname{mSpec}(R)$.

Proof | 1) $\Rightarrow$ 2): $M_{\mathfrak{p}} \cong R_{\mathfrak{p}} \otimes_R M$ (since: extension of scalars R to $R_{\mathfrak{p}}$), and: by Prop 3.39, Ext of scalars preserve flatness.

2) $\Rightarrow$ 3): True since maximals are prime

3) $\Rightarrow$ 1): Take R-linear + injective $f : N \to P$ & $\mathfrak{m} \in \operatorname{mspec}$

$\Rightarrow$ By Prop 4.21: $f_{\mathfrak{m}} : N_{\mathfrak{m}} \longrightarrow P_{\mathfrak{m}}$ injective.

$\Rightarrow (N \otimes M)_{\mathfrak{m}} \xrightarrow{(f \otimes id)_{\mathfrak{m}}} (P \otimes M)_{\mathfrak{m}}$ is injective (right diagram)

& this is true $\forall \mathfrak{m} \in \operatorname{mSpec}(R)$.

$\Rightarrow f \otimes id_M$ is injective (by injectivity local to global) ✓

$$\begin{tikzcd}
N_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}} \arrow[r, "f_{\mathfrak{m}} \otimes id_{\mathfrak{m}}"] \arrow[d, equal] & P_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}} \arrow[d, equal] \\
(N \otimes_R M)_{\mathfrak{m}} \arrow[r, "(f \otimes id)_{\mathfrak{m}}"] & (P \otimes_R M)_{\mathfrak{m}}
\end{tikzcd}$$

Example | Say R-module M locally-free if $M_{\mathfrak{p}}$ free $R_{\mathfrak{p}}$-module $\forall \mathfrak{p} \in \operatorname{Spec}(R)$.

Take: $R = \mathbb{C}^2 \Rightarrow \operatorname{Spec}(R) = \{\underbrace{\mathbb{C} \times (0)}_{=\mathfrak{p}_1}, \underbrace{(0) \times \mathbb{C}}_{=\mathfrak{p}_2}\}$

Then, what is $R_{\mathfrak{p}_1}$?

There is a surj map $\mathbb{C}^2 \longrightarrow \mathbb{C}$, $(x,y) \longmapsto y$.

This map sends: $\cancel{\mathbb{C}\times\{0\}\setminus\mathbb{C}^*}$ $\mathbb{C}^2\setminus\mathbb{C}\times\{0\}$ to units.

$\Rightarrow$ Descends to: $\varphi: (\mathbb{C}^2)_{\mathbb{C}\times\{0\}} \longrightarrow \mathbb{C}$ clearly surjective.

$\Rightarrow R_{p_1} \cong \mathbb{C}$. $\quad \dfrac{(a,b)}{(c,d)} \longmapsto \dfrac{b}{d}$ + injective.

So: $R_{p_1} \xrightarrow{\sim} \mathbb{C}$ & $R_{p_2} \xrightarrow{\sim} \mathbb{C}$, so R locally a field.

$\Rightarrow \forall M$ $\underset{=R}{\mathbb{C}^2}$-module, & $p \in \mathrm{Spec}(R)$, M_p is R_p-module,

so vector space $\Rightarrow$ free.

If $M = \mathbb{C}\times\{0\}$, M is $\mathbb{C}^2$-module.

<u>Claim</u>: $\left.\begin{array}{l} M \text{ \underline{not} free of rank } 0 \\ \& \ M \text{ \underline{not} free of rank} \geq 1. \end{array}\right] \Rightarrow$ M locally free & <u>not</u> free.

<u>Later</u>: $\exists R$ int dom & M R-module, not-free & locally free!

<u>Localisation of Ring as Quotient.</u>

<u>Let</u>: $S \subseteq R$ mult & $U \subseteq R$ subset. [mult. closure of U is S.]

<u>Consider</u>: $R_U = R[\{T_u\}_{u\in U}] / (\{u\cdot T_u - 1\}_{u\in U})$

<u>Claim</u> $R_U \cong S^{-1}R$ (as R-algebras).

<u>Proof</u> <u>Idea</u>: R_U has universal prop. of $S^{-1}R$

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[dr, "f"'] & R_U \arrow[d, "\exists! h"] \\ & A \end{tikzcd}$$

<u>Let</u>: A R-algebra via f. <u>Want</u>: to find h that makes this diagram commute.

~~Consider: $\mathrm{Hom}_{R\text{-alg}}(R_U, A) \xrightarrow{\sim} \{\varphi: U \to A : f(u)\varphi(u) = 1\}$.~~

So, f chosen, s.t. $f(u)$ unit $\forall u \in U$.

Can think of: R_U, ~~as~~ and A as R-algebras, via i, f.

$\Rightarrow$ Diagram commutes $\Leftrightarrow$ h is R-algebra hom.

Any R-algebra hom $R[\{T_u\}_{u\in U}] \longrightarrow A$ is determined uniquely by images of T_u, $u \in U$, ~~[illegible]~~

~~[illegible]~~

& R-algebra homs $R_U \longrightarrow A$ correspond to R-algebra homs $R[\{T_u\}_{u\in U}] \xrightarrow{\Phi} A$ s.t. $\Phi(T_u) = f(u)^{-1}$ $\forall u \in U$.

There is exactly one such Φ (since all its values are determined), so $h: R_U \to A$ is given by $h(p + I_U) = \left(p\,\big|_{T_u \mapsto f(u)^{-1}}\right)$ for $p \in R[\{T_u\}_{u\in U}]$.

$\Rightarrow (R_U, i)$ satisfy universal prop of $(S^{-1}R, i_{S^{-1}R})$

$\Rightarrow R_U \cong S^{-1}R$ by $p + I_U \longmapsto p\,\big|_{T_u \mapsto f(u)^{-1}}$.

$$r \prod_i T_{u_i} + I_U \longleftarrow \frac{r}{u_1 \cdots u_\ell}$$

This is clearly a R-algebra hom. (by viewing R_U, $S^{-1}R$ as R-algebras via i, $i_{S^{-1}R}$ resp.) ✓

<u>lemma 4.28</u>) (Special case) $R_u = \{u^n : n \geq 0\}^{-1}R$.

$\Rightarrow R_u \xrightarrow{\sim} R[T]/(uT-1)$.

By: $\frac{r}{u^n} \longmapsto r \cdot T^n + (uT-1)$

$p\left(\frac{1}{u}\right) \longleftarrow\!\mapsto p + (uT-1) \quad \forall p \in R[T].$

Commutative Algebra: Lecture 13.

Nakayama's lemma

Prop 5.1 (Cayley–Hamilton). Let M fin-gen R-module and $f: M \to M$ R-linear, and $\underline{a} \subseteq M$ ideal. with $f(M) \subseteq \underline{a}M$.

Then: $\exists n \geq 1$, $a_1, \ldots, a_n \in \underline{a}$ s.t. $f^{(n)} + a_1 f^{(n-1)} + \cdots + a_n = 0$.

$(f^{(i)} = \underbrace{f \circ \cdots \circ f}_{i})$.

Proof Since M is fin-gen: $M = \mathrm{Span}_R(m_1, \ldots, m_n)$.

$\Rightarrow \underline{a}M = \mathrm{span}_{\underline{a}}\{m_1, \ldots, m_n\}$.

$\Rightarrow \begin{pmatrix} f(m_1) \\ \vdots \\ f(m_n) \end{pmatrix} = P \cdot \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}$ for some $P \in \mathrm{Mat}_{n \times n}(\underline{a})$.

Note: the R-module M is defined by a ring ~~structure~~ hom:

$\rho: R \to \mathrm{End}(M)$ (by def). $\Rightarrow$ Makes $\mathrm{End}(M)$ into R-algebra

& An R-algebra hom $R[T] \to \mathrm{End}(M)$ is determined
$T \mapsto f$
by image of T. (& Any choice of f works)

$\Rightarrow M$ becomes $R[T]$-module, where R acts on M as before, and: $Tm = f(m)$.

So, $T \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = P \circ \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}$, $T \in R[T]$

$\Rightarrow Q \cdot \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0$, for $Q = T \cdot I_n - P \in \mathrm{Mat}_{n \times n}(R[T])$

$\Rightarrow (\det(Q))m = 0 \ \forall m \in M$. (since M gen. by m_i)

So, $(m \mapsto (\det Q)m)$ is 0-end. of $\mathrm{End}_R(M)$

But: $\det(Q)$ is a monic polynomial of T, say

$\det(Q) = T^n + a_1 T^{n-1} + \dots + a_n$ $\quad (a_i \in \underline{a})$

$\Rightarrow$ By replacing $T \longrightarrow f$, get $f^n + a_1 f^{n-1} + \dots = 0$ in $\mathrm{End}_R(M)$.

Corollary 5.2) M f.g. R-module & $\underline{a} \subseteq R$ ideal, with $\underline{a}M = M$. Then, $\exists a \in \underline{a}$ s.t. $am = m \ \forall m \in M$.

Proof) Apply CH to $f = id_M$. $\Rightarrow f(M) = M = \underline{a}M \subseteq \underline{a}M$.

$\Rightarrow (1 + \underbrace{a_1 + \dots + a_n}_{-a}) \, id_M = 0$ in $\mathrm{End}_R(M)$ ✓

DEF 5.3) Jacobson Radical $J(R)$ is $\bigcap_{m \in \mathrm{mspec}(R)} m$.

Examples 1) $(R, \underline{m})$ local ring $\Rightarrow J(R) = \underline{m}$

2) $J(\mathbb{Z}) = \{0\}$ (only number divisible by all primes).

Prop 5.5) $\forall x \in R$: $x \in J(R) \Leftrightarrow (1 - xy$ unit of $R \ \forall y \in R.)$

Proof) $\Rightarrow$: Assume $x \in J(R)$ & assume $\exists y$, $1 - xy$ not unit.

$\Rightarrow (1 - xy) \neq R$, so find max ideal $1 - xy \in \underline{m}$.

So, $x \in J(R) \subseteq \underline{m} \Rightarrow 1 \in \underline{m}$ ↯

$\Leftarrow$: Say $x \notin J(R)$. $\Rightarrow \exists \underline{m}$, $x \notin \underline{m}$. So, $\underline{m} + (x) = R$.

$\Rightarrow \exists t \in \underline{m}$, $y \in R$ s.t. $t + xy = 1$. $\Rightarrow 1 - xy = t \in \underline{m}$ not unit ✓ ↯

Prop 5.6 (Nakayama) M f.g. R-module & $\underline{a} \subseteq J(R)$ ideal of R s.t. $\underline{a}M = M$. Then, $M = 0$.

Proof By Corollary 5.2: $\exists a \in \underline{a}$ s.t. $am = m$ $\forall m \in M$.

$\Rightarrow \underbrace{(1-a)}_{\text{unit of } R} m = 0. \Rightarrow m = 0.$ ✓

Corollary 5.7 M f.g. R-module & $N \subseteq M$ R-submod. & $\underline{a} \subseteq J(R)$ ideal of R, s.t. $\underline{a}M + N = M$. Then: $N = M$.

Proof $\underline{a}(M/N) = \dfrac{\underline{a}M + N}{N} = \dfrac{M}{N} \overset{5.6}{\Rightarrow} M/N = 0.$ ✓

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§6: Integral & Finite Exts (Part I)

DEF 6.1 A is R-algebra. Say $x \in A$ integral over R if $\exists$ monic poly $f \in R[T]$ s.t. $f(x) = 0$.

Example 6.2 1) K field, A is K-algebra, $x \in A$. Then x R-algebraic $\Leftrightarrow$ x R-integral.

2) a) $\mathbb{Z}$ = integral elements of $\mathbb{Q}$

b) $\mathbb{Z}[\sqrt{2}] = \mathbb{Z}$-integral elts of $\mathbb{Q}(\sqrt{2})$

c) $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right] = \mathbb{Z}$-integral elts of $\mathbb{Q}(\sqrt{5})$.

DEF An R-module M faithful if: $R \xrightarrow{f} \mathrm{End}(M)$ injective ($\Leftrightarrow$ $\forall r \in R$, $r \neq 0 \Rightarrow \exists m \in M$, $rm \neq 0$).

Example $R \subseteq A$ ring. $\Rightarrow$ A is R-module. Must be faithful, since $r \cdot 1_A = r \neq 0$ $\forall r \neq 0$.

Lemma 6.4 $R \subseteq A$ ring & $x \in A$. Can consider: $R[x] \subseteq A$, so, A is $R[x]$-module. Then: x is R-integral iff:
$\exists M \subseteq A$ sub-R module of A, s.t.

1) M faithful $R[x]$-module.
2) M f.g. R-module. (not $R[x]$!!)

[M faithful $R[x]$-module $\Leftrightarrow$:
  i) M is R-submodule of A
  ii) $xM \subseteq M$
  iii) $\forall p \in R[x]$ $(p \neq 0)$, $\exists m \in M$, s.t. $pm \neq 0$.]

Proof Assume 1) & 2) $\Rightarrow$ Since $xM \subseteq M$, there is R-linear map $f: M \longrightarrow M$, $f(m) = x \cdot m$.
$\Rightarrow$ By CH, $f^n + r_1 f^{n-1} + \dots + r_n = 0$ $(r_i \in R)$ in $\mathrm{End}_R(M)$
Evaluate at $m \in M$: $(x^n + r_1 x^{n-1} + \dots + r_n) m = 0$ $\forall m \in M$.
& By faithfulness: $x^n + r_1 x^{n-1} + \dots + r_n = 0$, so x integral $/R$.
$\Leftarrow$: Say x integral $/R$ $\Rightarrow$ $x^n + r_1 x^{n-1} + \dots + r_n = 0$.
Define: $M = \mathrm{span}_R \{x^0, x^1, \dots, x^n\}$. Then, $xM \subseteq M$.
& M faithful over $R[x]$, since contains $x^0 = 1$.
& M is $R[x]$-submodule of A ✓

Commutative Algebra, lecture 14.

DEF 6.5) A is R-algebra.

1) A integral /R $\Leftrightarrow$ $\forall x \in A$, x integral /R

2) A finite /R $\Leftrightarrow$ A f.g. as R-module. (WRT R-mod structure on A)

Prop 6.6) TFAE: (A R-algebra)

i) A f.g. integral R-algebra

ii) A generated as R-algebra by finite set of integral elts

iii) A finite /R

Proof) 1) $\Rightarrow$ 2) trivial (All integral $\Rightarrow$ generators integral)

2) $\Rightarrow$ 3): Say A gen. by $\alpha_1, \dots, \alpha_n$ as R-algebra.

& α_i R-integral. $\Rightarrow \alpha_i^{n_i} + r_{n_i-1}\alpha_i^{n_i-1} + \dots + r_0\alpha_i^0 = 0$

$\Rightarrow \alpha_i^{n_i} \in \operatorname{span}(\alpha_i^0, \dots, \alpha_i^{n_i-1})$.

$\Rightarrow \forall e_1, \dots, e_n \geq 0$, s.t. $\alpha_1^{e_1} \dots \alpha_n^{e_n} \in \underbrace{\boxed{\operatorname{Span}\{\alpha_1^{f_1} \dots \alpha_n^{f_n} : 0 \leq f_i \leq n_i - 1\}}}_{\text{finite set.}}$

So, finite set generates A as R-module ✓

3) $\Rightarrow$ 1): A f.g. R-mod $\Rightarrow$ A f.g. R-alg.

Remains to show: $\forall \alpha \in R$ integral. (over R)

let: $\rho: R \to A$ structure hom of A, as R-algebra.

Then, $\rho(R) \subseteq A \Rightarrow \rho(R)[\alpha] \subseteq A$. So, A is $\rho(R)[\alpha]$-mod.

~~Since A is~~ But: A is $\rho(R)[\alpha]$ - faithful (since has 1)

& ~~f.g.~~ A is f.g. $\rho(R)[\alpha]$-module (f.g. /R).

$\Rightarrow$ By lemma 6.4, α is $\rho(R)$-integral. $\Rightarrow$ α is R-integral.

Prop 6.7] A R-algebra & $\mathcal{O} \equiv \{R\text{-integral elements of } A\}$.
Then, $\mathcal{O}$ is R-subalgebra of A.

Proof) For $x, y \in \mathcal{O}$, $\{x, y\}$ generate an integral R-subalgebra of A (by 6.6). So, $x+y$, xy, $x-y \in \mathcal{O}$. ✓ $\{r \cdot 1_A\} \subseteq \mathcal{O}$.

Prop 6.8] $A \subseteq B \subseteq C$ rings.

i) C finite $/B$ & B finite $/A$ $\Rightarrow$ C finite $/A$

ii) C integral $/B$ & B integral $/A$ $\Rightarrow$ C integral $/A$

Proof) i) $\forall c \in C$: $c = \sum_i b_i \gamma_i$ $\left(C = \mathrm{span}_B\{\gamma_1, \dots, \gamma_n\}\right)$
& $\forall i$, $b_i = \sum_j a_{ij} \beta_j$ $\left(B = \mathrm{span}_A\{\beta_1, \dots, \beta_\ell\}\right)$
$\Rightarrow$ $c = \sum_i \sum_j a_{ij} \gamma_i \beta_j$ for $a_{ij} \in A$ ✓

ii) $\forall c \in C$: c is B-integral, so $\exists f = T^n + b_1 T^{n-1} + \dots + b_n \in B[T]$ with $f(c) = 0$. So, f in $A' = A[b_1, \dots, b_n]$.
$\Rightarrow$ c integral over A' which is ~~f.g.~~ finite over A.
So, $A'[c]$ ~~f.g.~~ finite over A', which is ~~f.g.~~ finite over A, so by i), $A'[c]$ finite over A.
$\Rightarrow$ By Prop 6.6, c integral over A. ✓

DEF 6.9] A ring.

i) If $A \subseteq B$: Integral closure of A in B is:
$\overline{A} = \{b \in B : b \text{ integral } /A\}$.

& A integrally closed in $B \iff \bar{A} = A$.

2) If A int-dom: Integral closure of A is integral closure of A in $\mathrm{Frac}(A)$, and is integrally closed if so in $FF(A)$.

<u>Example 6.10</u> $\mathbb{Z}[\sqrt{5}]$ <u>not</u> integrally closed, since $\mathbb{Q}(\sqrt{5})$ has an element $\alpha = \frac{1+\sqrt{5}}{2}$ root of $T^2 - T - 1$.

& $\mathbb{Z}$, $k[T_1, \dots, T_n]$ <u>are</u> integrally closed.

<u>Prop 6.11</u> A UFD $\Rightarrow$ A integrally closed

<u>Proof</u> let A UFD, and take: $x \in \mathrm{Frac}(A) - A$. $x = \frac{a}{b}$. $a, b \in A$.

Since UFD: find $p \in A$ <u>prime</u>, $p \mid b$ & $p \nmid a$.

Integrality equation: $\left(\frac{a}{b}\right)^n + a_1 \left(\frac{a}{b}\right)^{n-1} + \dots + a_n = 0$.

$\Rightarrow a^n = -\left[a^{n-1} b \cdot a_1 + \dots + b^n \cdot a_n\right] \equiv 0 \mod p$

$\Rightarrow p \mid a^n$, so $p \mid a$ ↯.

<u>Lemma 6.12</u> $A \subseteq B$ rings. Then, $\bar{A}$ integraly closed in B.

<u>Proof</u> For $x \in B$ integral over $\bar{A}$, notice that:

$A \subseteq \bar{A} \subseteq \bar{A}[x]$, and both are integral extensions.

(Know $\bar{A} \subseteq \bar{A}[x]$ integral, by Prop 6.6, x is $\bar{A}$-integral)

$\Rightarrow$ By <u>Prop 6.8</u>: $A \subseteq \bar{A}[x]$ integral ext.

$\Rightarrow x$ is A-integral, hence $x \in \bar{A}$ ✓

<u>Prop 6.13</u> $A \subseteq B$ rings.

1) If B integral $/A$, then:

a) $B/\underline{b}$ integral / $\frac{A}{\underline{b}^c}$ ($\forall \underline{b} \subseteq B$ ideal)

b) $S^{-1}B$ integral / $S^{-1}A$ $\forall S \subseteq A$ multiplicative.

ii) ~~$S^{-1}A$~~ $\overline{S^{-1}A} = S^{-1}\bar{A}$ in $S^{-1}B$.

[Integral closure of $S^{-1}A$ in $S^{-1}B$ is $S^{-1}\bar{A}$.]

lemma 6.14) $A \subseteq B$ Integral ext of Rings.

i) $A \cap B^\times = A^\times$

ii) If A, B int-doms then B field $\Leftrightarrow A$ field.

Proof) i) $\supseteq$ clear

$\subseteq$: Take $a \in A \cap B^\times$. $\Rightarrow \exists b \in B,\ ab = 1$ in B.

Know: b integral $/A \Rightarrow b^n + a_1 b^{n-1} + \dots + a_n = 0$. ($a_i \in A$)

$\Rightarrow b + \underbrace{a_1 + a \cdot a_2 + \dots + a^{n-1} a_n}_{\in A} = 0$ so $\underline{b \in A}$ ✓

ii) If B field: $A^\times = A \cap B^\times = A \cap (B - \{0\}) = A - \{0\}$. ✓

If A field: take $b \in B$ nonzero. Need to invert it.

b A-integral $\Rightarrow b^n + a_1 b^{n-1} + \dots + a_n = 0$. ($n \geq 1$ minimal)

$\Rightarrow b(\underbrace{b^{n-1} + \dots + a_{n-1}}_{=\Delta}) = -a_n$

By minimality of n, have $\Delta \neq 0$, and $b \neq 0$ (assumed),

So: $a_n \neq 0$ (int dom) $\Rightarrow \exists a_n^{-1} \in A,\ a_n \cdot a_n^{-1} = 1$.

$\Rightarrow b \cdot (-a_n^{-1}\Delta) = 1$ ✓

Commutative Algebra: lecture 15.

Corollary 6.15) $A \subseteq B$ $(A \neq 0)$ Integral ext. of rings, and $\mathfrak{q} \subseteq B$ prime. Then: $\mathfrak{q} \cap A$ maximal in $A \Leftrightarrow \mathfrak{q}$ maximal in B.

Proof) $\mathrm{Ker}(A \hookrightarrow B \to B/\mathfrak{q}) = \mathfrak{q} \cap A$.

$\Rightarrow$ Induces: $\dfrac{A}{\mathfrak{q} \cap A} \hookrightarrow B/\mathfrak{q}$. & since $\mathfrak{q} \cap A$ prime in A, both $A/\mathfrak{q}\cap A$ & $B/\mathfrak{q}$ are integral domains.

By Prop 6.14: $\dfrac{A}{\mathfrak{q}\cap A}$ field $\Leftrightarrow$ $\dfrac{B}{\mathfrak{q}}$ field.

So, $\mathfrak{q} \cap A$ maximal in $A \Leftrightarrow \mathfrak{q}$ maximal in B.

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§7: Noether Normalisation & Hilbert Nullstellensatz.

§7.1: Noether Normalisation Theorem.

DEF 7.1) A is k-algebra (k field). Say: $x_1, \dots, x_n \in A$ are k-algebraically indep $\Leftrightarrow p(x_1, \dots, x_n) \neq 0$ $\forall\, 0 \neq p \in k[T_i]$

$\Leftrightarrow$ The k-algebra hom. $k[T_1, \dots, T_n] \to A$ injective.
$T_i \longmapsto x_i$

Theorem 7.2) (Noether):
$A \neq 0$ f.g. k-algebra over k. Then: $\exists\, x_1, \dots, x_n \in A$ k-alg indep, s.t. A <u>finite</u> $/ A' = k[x_1, \dots, x_n]$.

Example of proof method.

Consider: $A = k[T, T^{-1}]$. Then: $k[T] \subseteq A$ is <u>not</u> finite extension, because T^{-1} is not integral $/ k[T]$.

Indeed: if it were, $(T^{-1})^n \in \mathrm{Span}_{k[T]}\{1, T^{-1}, \dots, T^{-(n-1)}\}$

$\Rightarrow 1 \in \mathrm{Span}_{k[T]}\{T, T^2, \dots, T^n\}$ ↯.

<u>But</u>: $\forall c \neq 0$, note: $\{T, T^{-1} - cT\}$ generate A as k-algebra. <u>Show</u>: $k[T^{-1}-cT] \subseteq A$ <u>is</u> finite extension.

Indeed: $T[(T^{-1}-cT)+cT]-1=0$

$$\Rightarrow \underbrace{c}_{\in k}T^2 + \underbrace{(T^{-1}-cT)}_{\in k[T^{-1}-cT]}T - \underbrace{1}_{\in k[T^{-1}-cT]} = 0$$

$\Rightarrow$ If $c \neq 0$, then T integral over $k[T^{-1}-cT]$.

Hence, $k[T^{-1}-cT] \subseteq k[T,T^{-1}] = k[T^{-1}-cT][T]$ finite.

<u>Proof of 7.2</u> (Assuming k infinite, but works for all fields.)

<u>Strategy</u>: Induction on minimal # of generators, of A, as a k-algebra.

<u>Base case</u> $(n=0)$: $A=k$ & $A'=A$ ✓

<u>Inductive Step</u>: Say $\{x_1, \dots, x_m\}$ generate A as k-algebra & assume theorem holds when A is k-gen. by $<m$ elts.

If $x_1, \dots, x_m$ alg-indep $/k \Rightarrow$ done.

<u>Else</u>: <u>Want</u> to find $c_1, \dots, c_{m-1} \in k$, s.t. x_m integral over $B = k[x_1 - c_1 x_m, \dots, x_{m-1} - c_{m-1} x_m] \subseteq A$ subalgebra.

<u>If so</u>: then by induction, B <u>finite</u> $/ A' = k[z_1, \dots, z_{m-1}]$ for some $z_1, \dots, z_{m-1} \in B$ <u>alg-indep</u> $/k$. So, A finite$/A'$. ☐

Remains to show: $\exists c_1, \dots, c_{m-1} \in k$, s.t. X_m integral $/B$.

Proof: Take poly $0 \neq f \in k[T_1, \dots, T_m]$ with degree r, s.t. $f(X_1, \dots, X_m) = 0$. (Exists, since $\{X_i\}$ not alg-indep)

Write: f as sum of homog. parts & F = part with degree r.

For $c_1, \dots, c_{m-1} \in k$, have:

$$f(T_1 + c_1 T_m, \dots, T_{m-1} + c_{m-1} T_m, T_m) = F(c_1, \dots, c_{m-1}, 1)\, T_m^r + (\text{terms, deg of } T_m < r)$$

& The coefficient $F(c_1, \dots, c_{m-1}, 1)$ of T_m^r is in k, while those with lower powers of T_m are in ~~[illegible]~~ $k[T_1, \dots, T_{m-1}]$

So, define $g \in k[T_1, \dots, T_m]$ such that: (e.g. by shifting)

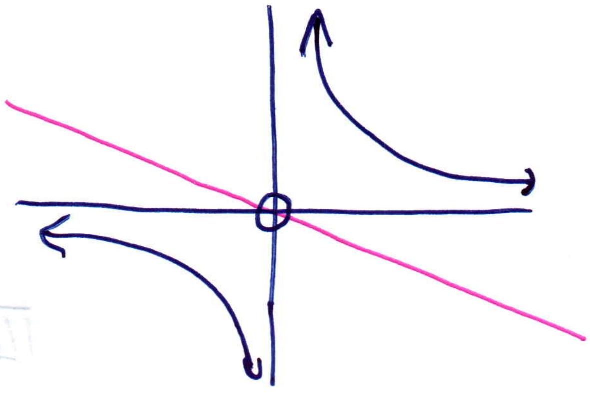
$$g(X_1 - c_1 X_m, \dots, X_{m-1} - c_{m-1} X_m, X_m) = f(X_1, \dots, X_m) = 0$$

Treat: g as polynomial in T_m, over $k[T_1, \dots, T_{m-1}]$.

$\Longrightarrow X_m$ is integral over B, if: $\underline{F(c_1, \dots, c_{m-1}, 1) \neq 0}$.

But, by example sheet 1, such c_i exist ✓

Geometric Interpretation:



$$k[T, T^{-1}] \cong k[X, Y]/(XY - 1).$$

Projection to $\{y = 0\}$ is not surj

But: proj to any other line is!

§7.2: Hilbert's Nullstellensatz.

Prop 7.6 $k \subseteq K$ (fields) where K fin-gen k-algebra.
Then, K is finite k-algebra ($\Rightarrow \dim_k(K) < \infty$).

Proof By Noether: $k[X_1, \ldots, X_n] \subseteq K$ finite ($\Rightarrow$ integral).
$\Rightarrow$ Get integral ext of integral domain.
$\Rightarrow k[X_1, \ldots, X_n]$ is a field
$\Rightarrow$ $n=0$ (since X_1 has No inverse), so K/k finite ✓

Commutative Algebra: lecture 16.

Let: $k \subseteq \Omega$, where Ω algebraically closed fields.

DEF 7.7 1) For $S \subseteq k[X_1, \dots, X_n]$: denote $\mathbb{V}(S) = \{ x \in \Omega^n : f(x) = 0 \ \forall f \in S \}$.

Sets of form $\mathbb{V}(S)$ are called k-algebraic subsets of Ω^n.

2) For $X \subseteq \Omega^n$, $I(X) = \{ f \in k[X_1, \dots, X_n] : f(x) = 0 \ \forall x \in X \}$.

Note: $\mathbb{V}(S) = \mathbb{V}((S))$.

Recall: (field theory): if $k \subseteq L$ finite ext, then:

$\exists L \longrightarrow \Omega$ injective k-algebra hom.

$\Rightarrow \Omega$ "contains all finite extensions" of k.

Theorem 7.9 $\underline{a} \subseteq k[T_1, \dots, T_n]$ ideal.

1) Weak Nullstellensatz: $\mathbb{V}(\underline{a}) \neq \emptyset \Leftrightarrow 1 \notin \underline{a}$.

2) Strong Nullstellensatz: $\sqrt{\underline{a}} = I(\mathbb{V}(\underline{a}))$.

Proof 1) $\Leftarrow$: clear. For $\Rightarrow$: assume $1 \notin \underline{a}$.

~~$\Rightarrow \exists \underline{m} \in \operatorname{mspec}(\underline{a})$ s.t.~~

$\Rightarrow \exists \underline{m} \in \operatorname{mspec}(k[T_1, \dots, T_n])$ s.t. $\underline{a} \subseteq \underline{m}$.

$\Rightarrow k[T_1, \dots, T_n]/\underline{m}$ field & f.g. as k-algebra over k

$\Rightarrow \dim_k (k[T_1, \dots, T_n]/\underline{m}) < \infty$.

$\Rightarrow \exists$ injective hom $k[T_1, \dots, T_n]/\underline{m} \hookrightarrow \Omega$.

$$\begin{tikzcd} k[T_1,\dots,T_n]/\underline{m} \arrow[r, hook] & \Omega \\ k[T_1,\dots,T_n] \arrow[u, two heads] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

$(\ker(\varphi) = \underline{m})$.

Write: $x = (\varphi(T_1), \dots, \varphi(T_n))$. Then, $\forall f \in k[T_1,\dots,T_n]$, have $\varphi(f) = f(x)$, so: $\forall f \in \underline{a} \subseteq \underline{m}$, $f(x) = \varphi(f) = 0$.

$\Rightarrow x \in \mathbb{V}(\underline{a})$, so $\mathbb{V}(\underline{a}) \neq \phi$.

ii) Say $f \in \sqrt{\underline{a}}$. $\Rightarrow \exists\, l \geq 1$, $f^l \in \underline{a}$ $\Rightarrow f^l(\underline{x}) = 0$ $\forall \underline{x} \in \mathbb{V}(\underline{a})$

$\Rightarrow f(\underline{x}) = 0 \ \forall x \in \mathbb{V}(\underline{a})$ [Ω int dom]

$\Rightarrow f \in I(\mathbb{V}(\underline{a}))$.

Conversely: let $f \in I(\mathbb{V}(\underline{a})) \subseteq k[T_1,\dots,T_n]$. Want $f \in \sqrt{\underline{a}}$

$\Leftrightarrow$ Want: $\bar{f}$ nilpotent in $k[T_1,\dots,T_n]/\underline{a}$.

$\Leftrightarrow$ Want: $(k[T_1,\dots,T_n]/\underline{a})_{\bar{f}} = 0$ (localisation)

This: $\cong \dfrac{k[T_1,\dots,T_{n+1}]}{(\underline{a}^e) + (T_{n+1}f - 1)}$ ($\underline{a}^e$ = extension of $\underline{a}$ from $k[T_1,\dots,T_n]$ to $k[T_1,\dots,T_{n+1}]$)

denote: $\underline{b} = (\underline{a}^e) + (T_{n+1}f - 1)$. Want: $1 \in \underline{b}$.

i.e. $\mathbb{V}(\underline{b}) = \phi$.

If $\overline{x} = (x_1,\dots,x_{n+1}) \in \mathbb{V}(\underline{b}) \subseteq \Omega^{n+1}$ ~~then: $f(\underline{x}) = 0 \ \forall f \in \underline{a}$~~

& $\overline{x}_0 = (x_1,\dots,x_n) \in \mathbb{V}(\underline{a}) \subseteq \Omega^n$

$\Rightarrow f(\bar{X}_0) = 0$ so $f(\bar{X}) = 0$

$\Rightarrow (\underbrace{T_{n+1} f - 1}_{\in \underline{b}})(\bar{X}) = -1 \neq 0$ $\#$ ✓

Remarks $\sqrt{\sqrt{I}} = \sqrt{I}$.

Fact: 1) If $X \subseteq Y \subseteq \Omega^n$ then $I(X) \supseteq I(Y)$.
& If $S \subseteq T \subseteq k[T_1, \ldots, T_n]$, $\mathbb{V}(S) \supseteq \mathbb{V}(T)$.

2) If $S \subseteq k[T_1, \ldots, T_n]$, have $S \subseteq I(\mathbb{V}(S))$.

3) If $X \subseteq \Omega^n$ then $X \subseteq \mathbb{V}(I(X))$

4) If $X \subseteq \Omega^n$ algebraic set, then $X = \mathbb{V}(I(X))$.

Proof) If $X = \mathbb{V}(\underline{a})$ then $\mathbb{V}(I(X)) = \mathbb{V}(I(\mathbb{V}(\underline{a})))$
$\subseteq \mathbb{V}(\underline{a}) = X$ ✓

5) If $X \subseteq \Omega^n$, $I(X)$ radical ($\sqrt{I(X)} = I(X)$.)

Prop 7.13) let $k \subseteq \Omega$, with $\Omega = \bar{\Omega}$, and: $n \geq 0$. Then:

$$\{k\text{-algebraic subsets of } \Omega^n\} \longleftrightarrow \{\text{Radical Ideals of } k[T_i]\}.$$

$$X \longmapsto I(X)$$

$$\mathbb{V}(\underline{a}) \longleftarrow\!\!| \ \underline{a}$$

Proof) Know, $I(X)$ radical $\forall X$ k-algebraic subset of Ω^n.
& $X = \mathbb{V}(I(X))$ for such sets.
For $\underline{a}$ radical ideal of $k[T_1, \ldots, T_n]$: $I(\mathbb{V}(\underline{a})) = \sqrt{\underline{a}} = \underline{a}$ ✓
(by strong NSZ).

Corollary 7.15] Ω alg closed. Then $\exists$ bijection:

$$\Omega^n \longleftrightarrow \operatorname{mspec}(\Omega[T_1, \dots, T_n])$$

$$\underbrace{(x_1, \dots, x_n)}_{=\underline{x}} \longmapsto \underbrace{(T_1 - x_1, \dots, T_n - x_n)}_{=\underline{m}_{\underline{x}}}.$$

Proof] $\forall \underline{x} \in \Omega^n$, $\{\underline{x}\} = \mathbb{V}(\underline{m}_{\underline{x}})$. Is: $\Omega-$ Algebraic set.

$\Rightarrow \{\{\underline{x}\} : \underline{x} \in \Omega^n\}$ = Collection of minimal & nonempty $\Omega-$ algebraic subsets of Ω^n.

& By prev: Such subsets of Ω^n correspond to elements of $\operatorname{mspec} \Omega[T_1, \dots, T_n]$ via $I(\cdot)$.

Since $\underline{m}_{\underline{x}} = I(\{\underline{x}\})$: claim follows.

Fact: Every prime ideal $\mathfrak{p}$ is radical: if $x^n \in \mathfrak{p}$, then $x \in \mathfrak{p}$ (by primality).

DEF 7.17] $X \subseteq \Omega^n$ irreducible if: $X \neq$ union of 2 proper algebraic sets of X.

Prop 7.18] $X \subseteq \Omega^n$ algebraic set. Then, X irreducible iff $I(X)$ prime ideal.

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§8: Integral & Finite exts (Part 2)

DEF 8.1] $A \subseteq B$ rings & $\underline{a} \subseteq A$ ideal. & $x \in B$. $\begin{matrix} A[T] \\ \cup \end{matrix}$

Say: x integral over $\underline{a}$ $\Leftrightarrow$ $\exists$ monic poly $f = T^n + a_1 T^{n-1} + \dots + a_n$ s.t. $a_i \in \underline{a}$ & $f(x) = 0$.

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DEF 8.1 let: $A \subset B$ rings, $\underline{a} \subseteq A$ ideal.

1) $x \in B$ $\underline{a}$-integral $\iff \exists f = T^n + a_1 T^{n-1} + \dots + a_n \in A[T]$, with $a_i \in \underline{a}$ $\forall i$ & $f(x) = 0$.

2) Integral Closure of $\underline{a}$ in B is: $\{x \in B : x \text{ } \underline{a}\text{-integral}\}$.

Prop 8.2 $A \subseteq B$ rings & $\overline{A}$ = integral closure of A in B.
If $\underline{a} \in A$ ideal: then $\boxed{\text{integral closure of } \underline{a} \text{ in } B \text{ is: } \sqrt{\underline{a}\overline{A}}.}$

Proof If $b \in B$ $\underline{a}$-integral: Then: $b^n + a_{n-1} b^{n-1} + \dots + a_n = 0$.
So, $b \in \overline{A} \Rightarrow b^0, \dots, b^{n-1} \in \overline{A}$.
$\Rightarrow b^n = -(a_1 b^{n-1} + \dots + a_n) \in \underline{a}\overline{A}$, so $b \in \sqrt{\underline{a}\overline{A}}$.
Conversely: say $b \in \sqrt{\underline{a}\overline{A}}$. $\Rightarrow \exists n \geq 1$, $b^n \in \underline{a}\overline{A}$.
$\Rightarrow b^n = \sum_{i=1}^{m} a_i x_i$, (2) for $a_i \in \underline{a}$ & $x_i \in \overline{A}$. $(m \geq 0)$
Then: $M = A[x_1, \dots, x_m]$ is finite A-algebra (Prop 6.6).
$\Rightarrow$ Since $b^n \in M$: have $b^n M \subseteq \underline{a} M$. (by 2))
$\Rightarrow$ By Prop 5.1 to A-linear map $f : M \to M$, $m \mapsto b^n m$, since $f(M) \subseteq \underline{a}M$,
(Cayley Hamilton) $\Rightarrow f^l + \alpha_1 f^{l-1} + \dots + \alpha_l = 0$ (in $\mathrm{End}_R M$). $(\alpha_i \in \underline{a})$
Evaluate at $m = 1_A$: $\Rightarrow b^{nl} + \alpha_1 b^{n(l-1)} + \dots + \alpha_l = 0$. (in B)
$\Rightarrow b$ is $\underline{a}$-integral ✓

Corollary 8.3 $A \subseteq B$ rings, $\underline{a} \subseteq A$ ideal, $b \in B$. Then:
~~b is B-integral ⟺~~

b is $\underline{a}$-integral $\iff$ b is $\sqrt{\underline{a}}$-integral.

Proof] Want: $\sqrt{\underline{a}\overline{A}} = \sqrt{\sqrt{\underline{a}}\,\overline{A}}$. (By prev prop).

$\subseteq$: clear. $\supseteq$: Use fact $\sqrt{I}^e \subseteq \sqrt{I^e}$, so:

$(\sqrt{\underline{a}})\overline{A} \subseteq \sqrt{\underline{a}\overline{A}} \Rightarrow \sqrt{\sqrt{\underline{a}}\,\overline{A}} \subseteq \sqrt{\sqrt{\underline{a}\overline{A}}} = \sqrt{\underline{a}\overline{A}}$. ✓

Prop 8.4] let: A integrally closed ~~domain~~ integral domain, $A \subseteq B$ (B int-dom), $\underline{a} \subseteq A$ ideal, $b \in B$.

Consider field ext $\mathrm{Frac}(A) \subseteq \mathrm{Frac}(B)$. Then, TFAE:

1) b integral / $\underline{a}$

2) b algebraic over $\mathrm{Frac}(A)$ with min poly over $\mathrm{Frac}(A)$ of the form: $T^n + \frac{a_1}{1}T^{n-1} + \dots + \frac{a_n}{1}T^0$ ($n \geq 1$ & $a_i \in \sqrt{\underline{a}}$).

Proof] $2) \Rightarrow 1)$: Have $b^n + \frac{a_1}{1}b^{n-1} + \dots + \frac{a_n}{1} = \frac{0}{1}$ (in Frac B)

$\Rightarrow b^n + a_1 b^{n-1} + \dots + a_n = 0$ (in B)

$\Rightarrow b$ is $\sqrt{\underline{a}}$-integral, hence $\underline{a}$-integral (Cor 8.3).

$1) \Rightarrow 2)$: have $b^n + a_1 b^{n-1} + \dots + a_n = 0$ ($a_i \in \underline{a}$)

$\Rightarrow$ If $h = T^n + \frac{a_1}{1}T^{n-1} + \dots + \frac{a_n}{1}T^0 \in (\mathrm{Frac}\, A)[T]$, then have $h\left(\frac{b}{1}\right) = 0$.

Denote: $f \in (\mathrm{Frac}\, A)[T]$ min poly of $\frac{b}{1}$ over $\mathrm{Frac}(A)$.

& let Ω alg-closed field, with $\mathrm{Frac}(A) \subseteq \Omega$.

Want: Coeffs of f in $\sqrt{\underline{a}}/1 = \left\{\frac{a}{1} : a \in \sqrt{\underline{a}}\right\}$.

By Prop 8.2: Integral closure of $\underline{a}$ in $\mathrm{Frac}(A)$ is: $\frac{\sqrt{\underline{a}}}{1}$. $\boxed{2}$

(Since: A integrally closed, so $\overline{A} = A$.)

<u>Denote</u>: $f = \prod_{i \leq \ell} (T - \alpha_i)$, ($\alpha_i \in \Omega$ & $\alpha_1 = \frac{b}{1}$)

$\Rightarrow$ Coeffs of f are: sums & products of α_i.

& By <u>Prop 8.2</u>: set of $\mathfrak{a}$-integral elements of Ω is an <u>ideal</u> of a subring of Ω, hence <u>closed</u> under $+, \times$.

So, suffices to show each α_i are $\mathfrak{a}$-integral.

<u>Know</u>: α_i & $\frac{b}{1}$ have same min poly (f) over $\mathrm{Frac}(A)$.

$$\Rightarrow \exists\, \varphi_i : (\mathrm{Frac}(A))\left[\tfrac{b}{1}\right] \longrightarrow (\mathrm{Frac}(A))[\alpha_i]$$

$$\tfrac{b}{1} \longmapsto \alpha_i$$

Where: φ_i is $(\mathrm{Frac}\, A)$-algebra isomorphism.

$\Rightarrow h(\alpha_i) = h(\varphi_i(\frac{b}{1})) = \varphi_i(h(\frac{b}{1})) = 0$

(<u>Since</u>: h is $(\mathrm{Frac}\, A)$-algebra hom & coeffs of h in $\mathrm{Frac}\, A$)

$\Rightarrow$ Each α_i integral $/\mathfrak{a}$ ✓

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§9: Cohen-Seidenberg Theorem ("going up/down").

Given integral ext $A \subset B$, & inclusion $i : A \hookrightarrow B$, $\exists$ contraction $i^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ where $i^*(\mathfrak{q}) = \mathfrak{q} \cap A$ $\forall \mathfrak{q} \in \mathrm{Spec}\, B$.

<u>Prop 9.1</u> (Incomparability)

Let A, B as above. (integral ext of rings) & $\mathfrak{q}, \mathfrak{q}' \in \mathrm{Spec}\, B$ with $\mathfrak{q} \cap A = \mathfrak{q}' \cap A$ & $\mathfrak{q} \subseteq \mathfrak{q}'$. Then, $\mathfrak{q} = \mathfrak{q}'$.

Proof Denote $\mathfrak{p} = \mathfrak{q} \cap A = \mathfrak{q}' \cap A$. & $S = A \setminus \mathfrak{p}$.

Then: S multiplicative & $\mathfrak{q}, \mathfrak{q}'$ primes of B that do not intersect S. $\Rightarrow$ By Prop 4.15: $\mathfrak{q} = (S^{-1}\mathfrak{q})^c$, $\mathfrak{q}' = (S^{-1}\mathfrak{q}')^c$.

[Recall: $\left\{\begin{array}{l}\text{Prime ideals } \mathfrak{p} \subseteq R \\ \mathfrak{p} \cap S = \emptyset\end{array}\right\} \Leftrightarrow \operatorname{Spec}(S^{-1}R)$]

Denote: $B_\mathfrak{p} = S^{-1}B = (A \setminus \mathfrak{p})^{-1}B$.

$\Rightarrow$ Suffices to show: $\mathfrak{q}B_\mathfrak{p} = \mathfrak{q}'B_\mathfrak{p}$ (Since: $S^{-1}\mathfrak{q} = \mathfrak{q}B_\mathfrak{p}$)

Have: $\mathfrak{q}B_\mathfrak{p} \cap A_\mathfrak{p} = S^{-1}\mathfrak{q} \cap S^{-1}A = S^{-1}(\mathfrak{q} \cap A) = S^{-1}\mathfrak{p}$.
$= \mathfrak{p}A_\mathfrak{p}$.

Similarly, $\mathfrak{q}'B_\mathfrak{p} \cap A_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$.

Since $A \hookrightarrow B$ integral ext: $A_\mathfrak{p} \hookrightarrow B_\mathfrak{p}$ integral ext. (Prop 6.13)

But: $\mathfrak{p}A_\mathfrak{p}$ maximal ideal of $A_\mathfrak{p}$

$\Rightarrow \mathfrak{q}B_\mathfrak{p}$ maximal ideal of $B_\mathfrak{p}$ (Corollary 6.15)

Similarly for $\mathfrak{q}'B_\mathfrak{p}$.

But: $\mathfrak{q} \subseteq \mathfrak{q}' \Rightarrow \mathfrak{q}B_\mathfrak{p} \subseteq \mathfrak{q}'B_\mathfrak{p} \Rightarrow \mathfrak{q}B_\mathfrak{p} = \mathfrak{q}'B_\mathfrak{p}$ as both are maximal ideals ✓

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Prop 9.2 (Lying over).

Let: $A \subseteq B$ integral ext (of rings), $\mathfrak{p} \in \operatorname{Spec}(A)$. Then, $\exists \mathfrak{q} \in \operatorname{Spec}(B)$ s.t. $\mathfrak{q} \cap A = \mathfrak{p}$.

Proof Have following commutative diag:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "\beta"] \\ A_{\mathfrak{p}} \arrow[r] & B_{\mathfrak{p}} = (A \setminus \mathfrak{p})^{-1} B \end{tikzcd}$$

Take: $\mathfrak{m} \in \operatorname{mspec}(B_{\mathfrak{p}})$.

$\Rightarrow \mathfrak{m} \cap A_{\mathfrak{p}} \in \operatorname{mspec}(A_{\mathfrak{p}})$.

But: $A_{\mathfrak{p}}$ localisation $\Rightarrow$ Local ring, maximal ideal $\mathfrak{p} A_{\mathfrak{p}}$.

So, $\mathfrak{m} \cap A_{\mathfrak{p}} = \mathfrak{p} A_{\mathfrak{p}}$.

Know: $\mathfrak{p} A_{\mathfrak{p}}$ contracts to $\mathfrak{p}$ under $A \to A_{\mathfrak{p}}$.

$\Rightarrow \underline{\mathfrak{m}}$ contracts to $\mathfrak{p}$ under $A \to A_{\mathfrak{p}} \to B_{\mathfrak{p}}$.

$= A \to B \to A_{\mathfrak{p}}$.

Hence: $\beta^{-1}(\mathfrak{m}) \cap A = \mathfrak{p}$ & $\beta^{-1}(\mathfrak{m}) \in \operatorname{spec}(B)$ ✓

Prop 9.3 (Going up).

$A \subset B$ integral ext of rings, & $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Spec}(A)$, $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$.

& $\mathfrak{q}_1 \in \operatorname{Spec}(B)$ with $\mathfrak{q}_1 \cap A = \mathfrak{p}_1$.

Then: $\exists\, \mathfrak{q}_2 \in \operatorname{Spec}(B)$, $\mathfrak{q}_2 \cap A = \mathfrak{p}_2$.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A/\mathfrak{p}_1 \arrow[r] & B/\mathfrak{q}_1 \end{tikzcd}$$

Proof Have: $\mathfrak{p}_1 = \mathfrak{q}_1 \cap A$.

$\Rightarrow \exists$ injective map: $A/\mathfrak{p}_1 \hookrightarrow B/\mathfrak{q}_1$

$a + \mathfrak{p}_1 \to a + \mathfrak{q}_1$.

$\&$ $B/\mathfrak{q}_1$ integral over $A/\mathfrak{p}_1$ (Prop 6.13).

By lying over (Prop 9.2): $\exists\, \mathfrak{q}_2/\mathfrak{q}_1 \in \operatorname{Spec}\left(\frac{B}{\mathfrak{q}_1}\right)$,

with: $\mathfrak{q}_2 \in \operatorname{Spec}(B)$ $\&$ $\frac{\mathfrak{q}_2}{\mathfrak{q}_1}$ contracts to $\frac{\mathfrak{p}_2}{\mathfrak{p}_1}$ in $A/\mathfrak{p}$.

Claim: $\mathfrak{q}_2 \cap A = \mathfrak{p}_2$.

Look at: commutative diagram.

$$\begin{tikzcd} \mathfrak{p}_2 \arrow[dd, mapsto] & A \arrow[r] \arrow[d] & B \arrow[d] & \mathfrak{q}_2 \arrow[d, mapsto] \\ & A/\mathfrak{p}_1 \arrow[r] & B/\mathfrak{q}_1 & \\ \mathfrak{p}_2/\mathfrak{p}_1 \arrow[rrr] & & & \mathfrak{q}_2/\mathfrak{q}_1 \end{tikzcd}$$

$\Rightarrow$ Get $\mathfrak{q}_2 \cap A = \mathfrak{p}_2$ ✓

Prop 9.4 (Going down).

Let: $A \subset B$ integral ext (of integral domains).

$\&$ Assume: A integrally closed. (in Frac(A), recall)

Let: $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Spec}(A)$, $\mathfrak{p}_1 \supseteq \mathfrak{p}_2$ $\&$ $\mathfrak{q}_1 \in \operatorname{Spec}(B)$

with $\mathfrak{q}_1 \cap A = \mathfrak{p}_1$. Then: $\exists\, \mathfrak{q}_2 \in \operatorname{Spec}(B)$, $\mathfrak{q}_2 \cap A = \mathfrak{p}_2$.

Proof Consider map $A \hookrightarrow B \hookrightarrow B_{\mathfrak{q}_1}$ (Injective)

Want: $\exists\, \mathfrak{n} \in \operatorname{Spec}(B_{\mathfrak{q}_1})$, s.t. $\mathfrak{p}_2$ contracted from $\mathfrak{n}$.

[Why? ~~In this case, $\mathfrak{q}_2 \cap A \subseteq \mathfrak{q}_1$ & contracts to $\mathfrak{p}_2$ in A. So $(\mathfrak{n} \cap B) \cap A = \mathfrak{p}_2$~~

If we let $\mathfrak{q}_2 = \mathfrak{n} \cap B$, then: have, $(\mathfrak{n} \cap B) \cap A = \mathfrak{p}_2$

and $\mathfrak{n} \cap B \in \operatorname{Spec}(B)$, and contains $\mathfrak{q}_1$. ✓]

Want: $(\mathfrak{p}_2 B_{\mathfrak{q}_1}) \cap A \subseteq \mathfrak{p}_2$ (reverse always holds)

Think of $\mathfrak{p}_2 B_{\mathfrak{q}_1}$ as extension in 2 steps: $\mathfrak{p}_2 \mapsto \mathfrak{p}_2 B \mapsto \mathfrak{p}_2 B_{\mathfrak{q}_1}$

$\Rightarrow \mathfrak{p}_2 B_{\mathfrak{q}_1} = (\mathfrak{p}_2 B) B_{\mathfrak{q}_1}$.

<u>Take</u>: $\frac{y}{s} \in (\mathfrak{p}_2 B_{\mathfrak{q}_1}) \cap A$, for: $y \in \mathfrak{p}_2 B$, $s \in B \setminus \mathfrak{q}_1$.

Since $B \supseteq A$ <u>integral ext</u>: integral closure of $\mathfrak{p}_2$ in B is: $\sqrt{\mathfrak{p}_2 \overline{A}} = \sqrt{\mathfrak{p}_2 B}$.

$\Rightarrow y$ is integral over $\mathfrak{p}_2$.

$\Rightarrow$ Min poly of y over $\mathrm{Frac}(A)$ is of form:

$T^r + u_1 T^{r-1} + \dots + u_r = 0 \quad (u_i \in \sqrt{\mathfrak{p}_2} = \mathfrak{p}_2$ since prime$)$.

<u>Know</u>: $y = \underbrace{\left(\frac{y}{s}\right)}_{\in A} \cdot \underbrace{s}_{\in B}$, $\Rightarrow y, s \in \mathrm{Frac}(B)$ & $\frac{y}{s} \in \mathrm{Frac}(A)$.

$\Rightarrow \left(\frac{y}{s} \cdot s\right)^r + u_1 \left(\frac{y}{s} \cdot s\right)^{r-1} + \dots + u_r = 0. \quad / \left(\frac{y}{s}\right)^r$:

$\Rightarrow s^r + \left(\frac{s}{y}\right)^1 u_1 s^{r-1} + \dots + \left(\frac{s}{y}\right)^r u_r = 0$ (*)

<u>But</u>: $s \in B \Rightarrow s$ integral $/A$, hence by <u>Prop 8.4</u>: get all coeffs $\left(\frac{s}{y}\right)^1 u_1, \dots, \left(\frac{s}{y}\right)^r u_r \in A$.

<u>Assume</u>: $\frac{s}{y} \notin \mathfrak{p}_2$. <u>Write</u>: $\underbrace{u_i}_{\in \mathfrak{p}_2} = \underbrace{\left(\frac{y}{s}\right)^i}_{\notin \mathfrak{p}_2} \underbrace{\left(\frac{s}{y}\right)^i u_i}_{\in A}$

$\Rightarrow \left(\frac{s}{y}\right)^i u_i \in \mathfrak{p}_2$.

$\Rightarrow$ By (*): $s^r \in \mathfrak{p}_2 \cdot B \subseteq \mathfrak{p}_1 B = (\mathfrak{q}_1 \cap A) B \subseteq \mathfrak{q}_1$.

$\Rightarrow s \in \mathfrak{q}_1$.

But, assumed before $s \in B \setminus \mathfrak{q}_1$ ↯. ✓

§10: Primary Decomp.

DEF 10.1) $I \subseteq R$ ideal.

1) I prime $\iff$ $R/I \neq 0$ & Only zero divisor of R/I is 0

2) I radical $\iff$ only nilpotent element in R/I is 0

3) I primary $\iff$ $R/I \neq 0$ & All zero divisors of R/I nilpotent

Examples.) $(6) \subset \mathbb{Z}$ radical, not primary

$(9) \subset \mathbb{Z}$ primary, not radical.

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Example 10.2 For $R=\mathbb{Z}$, (0) is prime ($\Rightarrow$ radical + primary).

1) (x) prime $\Leftrightarrow$ $|x|$ prime number
2) (x) radical $\Leftrightarrow$ x squarefree
3) (x) primary $\Leftrightarrow$ $x=p^n$, p prime, & $n\geq 1$.

Prop 10.3 Let $I \subsetneq R$ ideal.

1) I primary $\Rightarrow$ $\underbrace{\sqrt{I}}_{\mathfrak{p}} \in \mathrm{Spec}(R)$. [Say I $\mathfrak{p}$-primary.]
2) $\sqrt{I} \in \mathrm{mspec}(R) \Rightarrow I$ is primary ($\sqrt{I}$-primary)
3) If $\mathfrak{q}_1,\dots,\mathfrak{q}_n$ $\mathfrak{p}$-primary then $\mathfrak{q}_1\cap\dots\cap\mathfrak{q}_n$ $\mathfrak{p}$-primary
4) If I has <u>primary decomp</u> $I=\mathfrak{q}_1\cap\dots\cap\mathfrak{q}_n$ where $\mathfrak{q}_i$ primary $\forall i$, then: I has <u>minimal</u> primary decomp
[i.e. $I=\mathfrak{q}_1\cap\dots\cap\mathfrak{q}_n$ where $\sqrt{\mathfrak{q}_1},\dots,\sqrt{\mathfrak{q}_n}$ distinct, and $I \subsetneq \bigcap_{j\neq i} \mathfrak{q}_j$ $\forall j$.]
5) If R Noetherian then any I has primary decomp.

Examples 10.4 In $\mathbb{Z}$, $(90)=(2)\cap(3^2)\cap(5)$ primary decomp.

1) (<u>Primary $\not\Rightarrow$ Power of prime ideal</u>)

$R=k[X,Y]$, $\mathfrak{q}=(X,Y^2)$. $\Rightarrow$ $R/\mathfrak{q} \cong k[Y]/(Y^2)$.

Any zero divisor of $R/\mathfrak{q}$ is factor of Y, so $\mathfrak{q}$ primary.

<u>Also</u>: $\sqrt{\mathfrak{q}}=(X,Y)$. $\in \mathrm{mspec}(R)$.

So, if $q = p^n$ for some n, then necessarily: $p = \underline{m} = (X, Y)$.
But: $(X,Y)^2 \subsetneq q \subsetneq (X,Y)$ ~~$\#$~~. $\Rightarrow$ q not power of prime.

2) (p^n not necessarily primary for p prime):
$R = k[X,Y,Z]/(XY - Z^2)$. & $\bar{X}, \bar{Y}, \bar{Z}$ images of X, Y, Z.
$\Rightarrow p = (\bar{X}, \bar{Z})$ prime (since: $R/\bar{p} \cong k[Y]$ int dom).
& $p^2 = (\bar{X}^2, \bar{X}\bar{Z}, \bar{Z}^2)$. Note $\bar{X}\bar{Y} = \bar{Z}^2 \in p^2$, but
$\bar{X} \notin p^2$ $\Rightarrow$ $\bar{Y}$ is zero divisor of R/p^2.
& $R/p^2 = k[X,Y,Z]/(XY - Z^2, X^2, XZ, Z^2)$
If $\bar{Y}$ nilpotent in R/p^2 then $\bar{Y} \in \sqrt{p^2} = p$, but
clearly $\bar{Y} \notin (\bar{X}, \bar{Z})$ since $R/p \cong k[Y]$.

$\boxed{\text{Theorem 10.5}}$ $I = q_1 \cap \dots \cap q_n$ minimal primary decomp
& $p_i = \sqrt{q_i}$ $\forall i$. Then,

1) [The Associated prime ideals of I]: p_i depend only on I, and in fact: $\{p_i\} = \{(I : x) : x \in R\} \cap \operatorname{spec} R$.

2) [Isolated prime ideals of I]:
The set of minimal elements in $\{p_i\}$ are exactly the prime ideals of R corresponding to minimal prime ideals of R/I.

[Embedded prime ideals of I]:
An associated prime ideal of I is embedded if not isolated.

3) If $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ ($t \leq n$) are the isolated prime ideals of I, then $\mathfrak{q}_1, \dots, \mathfrak{q}_t$ depend only on I.

Example 10.6) $R = k[X,Y]$ (k field), $I = (X^2, XY)$.

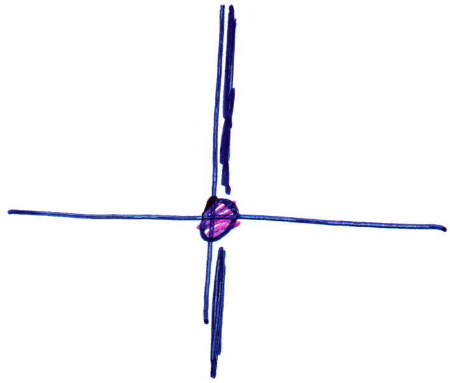
Then: $I = (X) \cap (X,Y)^2 = (X) \cap (X^2, Y)$.

& $\sqrt{(X)} = (X)$, $\sqrt{(X^2, Y^2)} = \sqrt{(X,Y)^2} = (X,Y)$.

What are I's associated primes?

$(X) \subseteq (X,Y)$. $\Rightarrow$ Only (X).

$\uparrow$ isolated prime $\uparrow$ embedded prime.



Remark 10.7) Let: $I = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_n$ minimal primary decomp, & $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$. & $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ isolated primes.

Then: $\sqrt{I} = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_t$

& this is the min primary decomp of $\sqrt{I}$.

& All associated primes of $\sqrt{I}$ are isolated.

$\Rightarrow$ * Taking $I \longrightarrow \sqrt{I}$ "only remembers isolated primes" of I.

For $k \subseteq \mathbb{C}$ & $R = k[T_1, \dots, T_n]$: $\forall I \subset R$ ideal,

$\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$ & $I(\mathbb{V}(I)) = \sqrt{I}$.

$\Rightarrow \mathbb{V}(I)$ "remembers" $\sqrt{I}$, and nothing else.

§11: Direct + Inverse Limits.

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Direct & Inverse limits + Completions ($\mathcal{C}$ = Category)

DEF 11.1) i) $(I, \leq)$ **Directed set** is a poset, s.t. $\forall a, b \in I: \exists c \in I$, $a \leq c$ & $b \leq c$.

ii) **Direct system over I**: is pair $\left((X_i)_{i \in I}, (f_{ij})_{i,j \in I, i \leq j}\right)$ where: X_i object, & $f_{ij}: X_i \to X_j$ morphism, s.t.

1) $f_{ii} = id_{X_i}$ $\forall i$
2) $f_{jk} \circ f_{ij} = f_{ik}$ $\forall i \leq j \leq k$ in I.

iii) **Inverse system over I**: is pair $\left((Y_i)_{i \in I}, (h_{ij})_{i,j \in I, i \leq j}\right)$ where: Y_i object, $h_{ij}: Y_j \to Y_i$ morphism, s.t.

1) $h_{ii} = id_{Y_i}$ $\forall i$
2) $h_{ij} \circ h_{jk} = h_{ik}$ $\forall i \leq j \leq k$.

Example 11.2) $I = (\mathbb{N}, \leq)$.

1) Direct system on I: $X_i = \mathbb{F}_{p^{i!}}$ & $f_{ij}: \mathbb{F}_{p^{i!}} \hookrightarrow \mathbb{F}_{p^{j!}}$.

[If $a | b$, then: $\exists$ ring embedding $\mathbb{F}_{p^a} \hookrightarrow \mathbb{F}_{p^b}$]

$\Rightarrow f_{ij} = f_{j-1, j} \circ f_{j-2, j-1} \circ \dots \circ f_{i, i+1}$.

$$\mathbb{F}_p \to \mathbb{F}_{p^2} \to \mathbb{F}_{p^6} \to \mathbb{F}_{p^{24}} \to$$

2) If $Y_i = \mathbb{Z}/p^i\mathbb{Z}$ & $f_{ij}: \mathbb{Z}/p^j\mathbb{Z} \to \mathbb{Z}/p^i\mathbb{Z}$ Is: natural projection. Then, $((Y_i)$ & $(f_{ij}))$ is inverse system.

DEF 11.3) Let $(I, \leq)$ Directed set.

i) Let $D = \left((X_i)_{i \in I}, (f_{ij})_{i \leq j}\right)$ direct system on I. Then:

$$\varinjlim X_i = \Big(\bigsqcup_{i\in I} X_i\Big)/\sim$$

where: $\forall x_i \in X_i,\ x_j \in X_j : x_i \sim x_j \iff \exists k \in I$ with $k \geq i,\ k \geq j$ & $f_{ik}(x_i) = f_{jk}(x_j)$.

($\iff$ "$x_j \sim f_{ij}(x_i)$")

2) let $E = \big((Y_i)_{i\in I}, (h_{ij})_{i\leq j}\big)$ inverse system on I. Then

$$\varprojlim Y_i = \Big\{ y \in \prod_{i\in I} Y_i : y_i = h_{ij}(y_j)\ \forall i \leq j \Big\}.$$

Examples 11.4) 1) $\varinjlim \mathbb{F}_{p^{i}}$ = Algebraic closure of $\mathbb{F}_p$.

Proof) Note: $\varinjlim \mathbb{F}_{p^{i}}$ algebraic over $\mathbb{F}_p$, because:

$\forall [x] \in \varinjlim \mathbb{F}_{p^{i}}$, $\exists n$, $x \in \mathbb{F}_{p^{n!}} \Rightarrow x^{p^{n!}} - x = 0$.

$\Rightarrow [x]^{p^{n!}} - [x] = 0$, so $[x]$ algebraic over $\mathbb{F}_p$ ✓

& $\varinjlim \mathbb{F}_{p^{i}}$ algebraically closed, because:

$\forall [h] \in (\varinjlim \mathbb{F}_{p^{i}})[T]$, $\exists n$, $[h] \in \mathbb{F}_{p^{n!}}[T]$.

So, h splits under some embedding $\mathbb{F}_{p^{n!}} \longrightarrow \mathbb{F}_{p^{j!}}$

$\Rightarrow h$ splits in $\varinjlim \mathbb{F}_{p^{i!}}$. ✓

2) $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$. p-adic integers.

$\Rightarrow$ Every element of $\mathbb{Z}_p$ of form: $\Big(\sum_{0\leq k<i} d_k p^k + p^i\mathbb{Z}\Big)_{i\geq 1}$.

DEF 11.7) R ring, $\underline{a} \subseteq R$ ideal. The $\underline{a}$-adic completion of R is: $\hat{R} = \varprojlim R/\underline{a}^n$.

Examples 11.8) i) $R = \mathbb{Z}$, $\underline{a} = (p) \Rightarrow \hat{R} = \mathbb{Z}_p$.

ii) $R = k[T]$, $\underline{a} = (T) \Rightarrow \hat{R} = k[[T]]$. Formal power series.

DEF 11.9) M is R-module & $\underline{a} \in R$ ideal. The $\underline{a}$-adic completion of M is: $\hat{M} = \varprojlim M/\underline{a}^i M$.

DEF 11.10) M R-module.

i) A Filtration of M is: sequence of submodules $(M_i)_{i \geq 0}$ such that: $M_n \supseteq M_{n-1}$ $\forall n$ & $M_0 = M$.

ii) Completion of M WRT filtration $(M_n)_{n \geq 0}$ is: $\varprojlim M/M_n$. (Objects $= M/M_n$ & morphisms projections)

Theorem 11.11) R Noetherian, $\hat{R} = \underline{a}$-completion of R.

1) $\hat{R}$ Noetherian

2) $\hat{R} \otimes_R (\cdot)$ exact functor

3) If M f.g. R-module, then: map $\hat{R} \otimes_R M \to \hat{M}$, $(x \otimes m) \mapsto xm$ is $\hat{R}$-linear isomorphism.

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Filtrations & Graded Rings.

§12.1: Graded rings & Modules.

DEF 12.1 A graded ring $\Leftrightarrow A = \bigoplus_{n \geq 0} A_n$ s.t. $A_m A_n \subseteq A_{m+n}$

Claim: $A_0 \subseteq A$ is subring

Proof Clearly, A_0 abelian subgroup of A & closed under mult ($A_0 A_0 \subseteq A_0$).

Need: $\mathbb{1}_A \subseteq A_0$.

Write: $\mathbb{1}_A = \sum_{i \leq m} y_i$ ($y_i \in A_i$) & let $z_n \in A_n$ ($n \geq 0$).

Then, $\underbrace{z_n}_{\in A_n} = z_n \cdot \mathbb{1}_A = \sum_{i \leq m} \underbrace{y_i z_n}_{\in A_{n+i}} \Rightarrow z_n = y_0 z_n \ \forall z \in A$.

Hence: $\mathbb{1}_A = y_0 \in A_0$ ✓

Each A_m is A_0-module (Since: $A_0 A_m \subseteq A_m$).

Example 12.2 $k[T_1, \ldots, T_n] = \bigoplus_{m \geq 0} A_m$, where A_m are homogeneous polys of degree m.

DEF 12.3 let $A = \bigoplus A_n$ graded ring.

1) a graded A-module is: module M (over A), s.t. $M = \bigoplus_{n \geq 0} M_n$, s.t. M_n subgroup of M & $A_m M_n \subseteq M_{n+m}$.

($\Rightarrow$ Each M_n is A_0-module).

2) $A_+ \equiv \bigoplus_{n\geq 1} A_n = \ker(A \to A_0)$. Then $A_+ \subseteq A$ is an ideal & $A/A_+ \cong A_0$.

Prop 12.4 Let $A = \bigoplus_{n\geq 0} A_n$ Graded ring. TFAE:

1) A Noetherian

2) A_0 Noetherian & A f.g. (A_0)-algebra.

Proof (ii) $\Rightarrow$ (i): True by Hilbert basis theorem.

(1) $\Rightarrow$ 2): So, A Noeth $\Rightarrow A/A_+ \cong A_0$ Noetherian.

& Notice: A_+ is generated by set of all Homogeneous ~~polys~~ elements, of positive degree.

Since A Noetherian: $\Rightarrow A_+$ generated by finite set $\{x_1, \dots, x_n\}$, $x_i \in A_{k_i}$.

If $A' = A_0[x_1, \dots, x_n]$: (Subalgebra generated by x_i)

then: want: $A' = A$.

Suffices to show: $A_n \subseteq A'$ $\forall n$ (since, clearly, $A' \subseteq A$)

Base case $(n=0)$: $A_0 \subseteq A'$ ✓ (by def)

Inductive step: $\forall y \in A_n$: $(n>0) \Rightarrow \underbrace{y}_{\in A_n} = \sum_{0\leq i\leq s} \underbrace{r_i}_{\in A} \underbrace{x_i}_{\in A_{k_i}}$

Apply projection $A \to A_n$:

$\Rightarrow y = \sum_i a_i x_i$ $(a_i \in A_{n-k_i})$.

$\Rightarrow$ By induction: each a_i is polynomial in $x_1, \dots, x_s$ over A_0.

Hence, $y \in A'$ ✓

The Associated Graded Ring.

DEF 12.5 let: $\underline{a} \subseteq R$ ideal & M R-module.

1) Filtration of M is: $(M_n)_{n\geq 0}$ s.t. $M_n \supseteq M_{n+1}$ & $M_0 = M$.
(M_n is: R-submodule of M.)

2) $(M_n)_{n\geq 0}$ $\underline{a}$-filtration if: $\underline{a} M_n \subseteq M_{n+1}$ $\forall n$.

3) an $\underline{a}$-filtration $(M_n)_{n\geq 0}$ is: $\underline{a}$-stable if: for all n large enough, $\underline{a} M_n = M_{n+1}$.

Example 12.6 $(\underline{a}^n M)_{n\geq 0}$ is a stable $\underline{a}$-filtration.

DEF 12.7 $\underline{a} \subseteq R$ ideal. The Associated Graded Ring is:

$$G_{\underline{a}}(R) = \bigoplus_{n\geq 0} \underline{a}^n / \underline{a}^{n+1}. \quad (\& \ \underline{a}^0 \equiv A)$$

Note: $\forall x \in \underline{a}^n$ & $y \in \underline{a}^m$: $(x + \underline{a}^{n+1})(y + \underline{a}^{m+1}) = xy + \underline{a}^{n+m+1}$

$\Rightarrow$ If $\bar{x}, \bar{y}$ images of x, y in $\underline{a}^n/\underline{a}^{n+1}$ & $\underline{a}^m/\underline{a}^{m+1}$, then $\bar{x}\bar{y} \in \underline{a}^{n+m}/\underline{a}^{n+m+1}$. Hence: indeed graded ring.

For R-module M: & $\underline{a}$-filtration $(M_n)_{n\geq 0}$: define:

$$G(M) = \bigoplus_{n\geq 0} M_n / M_{n+1}.$$

$\Rightarrow$ $G(M)$ is a $G_{\underline{a}}(M)$-module (naturally), since:

$\forall x \in \underline{a}^n$, $m \in M_\ell \Rightarrow (x + \underline{a}^{n+1})(m + M_{\ell+1}) = xm + M_{n+\ell+1}$.

Prop 12.8 let: R Noetherian ring. & $\underline{a} \subseteq R$ ideal.

1) $G_{\underline{a}}(R)$ Noetherian

2) M f.g. R-module & $(M_n)_{n\geq 0}$ stable $\underline{a}$-filtration of M.

Then: $G(M)$ is f.g. $G_{\underline{a}}(M)$-module.

Proof) R Noetherian $\Rightarrow$ $\underline{a} = (X_1, \dots, X_s)$. Denote: $\overline{X}_i$ image of X_i in $\underline{a}/\underline{a}^2$. Then:

$G_{\underline{a}}(R) = (R/\underline{a}) \oplus (\underline{a}/\underline{a}^2) \oplus (\underline{a}^2/\underline{a}^3) \oplus \cdots$

& $G_{\underline{a}}(R)$ is generated (as $R/\underline{a}$-algebra) by $\overline{X}_1, \dots, \overline{X}_s$.

(E.g. anything in $\underline{a}^2/\underline{a}^3$ can be received by sum of product of 2 things in $\underline{a}/\underline{a}^2$, which are generated by $\overline{X}_i$.)

& Since $R/\underline{a}$ Noetherian (R Noeth): by HBT, $G_a(R)$ is Noetherian.

2) $(M_n)_{n\geq 0}$ stable $\Rightarrow$ $\exists n_0$, $M_{n_0+r} = \underline{a}^r M_{n_0}$ $\forall r\geq 0$.

$\Rightarrow$ $G(M)$ is generated by: $\bigoplus_{n\leq n_0} M_n/M_{n+1}$, as $G_{\underline{a}}(R)$-module,

~~by~~ $(M_0/M_1) \oplus (M_1/M_2) \oplus \cdots \oplus (M_{n_0}/M_{n_0+1})$.

(Next one is multiplying previous by $\underline{a}$)

Note: each M_i/M_{i+1} is Noetherian $(R/\underline{a})$-module, & is annhilated by $\underline{a}$. (Since: M Noetherian since f.g. $/R$, so M_n is)

$\Rightarrow$ M_i/M_{i+1} f.g. $(R/\underline{a})$-module, so $\bigoplus_{n\leq n_0} M_n/M_{n+1}$ f.g. $(R/\underline{a})$-mod

$\Rightarrow$ $G_a(R)$ f.g. as $G_{\underline{a}}(R)$-module ✓

DEF 12.9) M is R-module, & (M_n) & (M'_n) filtrations of M.

Say: they are equivalent $\Leftrightarrow$ $\exists n_0\geq 0$, s.t. $M_{n+n_0} \subseteq M'_n$ & $M'_{n+n_0} \subseteq M_n$.

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Lemma 12.10) $\underline{a} \subseteq R$ ideal. Then, $\forall (M_n)_{n \geq 0}$ stable $\underline{a}$-filtration, it is equivalent to $(\underline{a}^n M)_{n \geq 0}$.

($\Rightarrow$ All stable $\underline{a}$-filtrations of M are equivalent.)

Proof) $(M_n)_{n \geq 0}$ $\underline{a}$-filtration $\Rightarrow M_n \supseteq \underline{a} M_{n-1} \supseteq \underline{a}^2 M_{n-2} \supseteq \dots$
$$\supseteq \underline{a}^n M \supseteq \underline{a}^{n+n_0} M \quad \forall n_0.$$

Also, $\exists n_0 \geq 0$, s.t. $\underline{a} M_n = M_{n+1}$ $\forall n \geq n_0$ (stable)

$\Rightarrow M_{n+n_0} = \underline{a}^n M_{n_0} \subseteq \underline{a}^n M$. ✓

The Artin-Rees Lemma.) Let: $\underline{a} \subseteq R$ ideal, M R-module and: $(M_n)_{n \geq 0}$ $\underline{a}$-filtration of M.

Define: $R^* = \bigoplus_{n \geq 0} \underline{a}^n$ $(\underline{a}^0 = R)$ $(x \in \underline{a}^n, y \in \underline{a}^m \Rightarrow xy \in \underline{a}^{n+m})$

& $M^* = \bigoplus_{n \geq 0} M_n$ $(x \in \underline{a}^n, m \in M_\ell \Rightarrow xm \in M_{n+\ell})$

$\Rightarrow M^*$ is: graded R^*-module.

If R Noeth, say $\underline{a} = (x_1, \dots, x_r)$. Then, R^* is generated (as R-module) by: $R^* = R \oplus \underline{a} \oplus \underline{a}^2 \oplus \dots$, $\underline{a} = (x_1, \dots, x_r)$

$\Rightarrow R^*$ Noetherian (HBT).

Lemma 12.11) Let: R Noetherian, M f.g. R-module. & $(M_n)_{n \geq 0}$ $\underline{a}$-filtration of M. Then, TFAE:

1) M^* is f.g. R^*-module

2) $(M_n)_{n\geq 0}$ is stable $\underline{a}$-filtration.

Proof) Facts: 1) Each M_n is f.g. R-module (since R Noeth, M f.g. $\Rightarrow$ M Noeth $\Rightarrow$ M_n ~~...~~ f.g.)

2) Denote: $M_n^* = M_0 \oplus \cdots \oplus M_n \oplus \bigoplus_{i\geq 1} \underline{a}^i M_n$.

Then: (M_n^*) is clearly ascending, and: it stabilises iff $(M_n)_{n\geq 0}$ is stable.

[Why: $M_n^* = M_0 \oplus - \oplus M_n \oplus (\underline{a} M_n \oplus \underline{a}^2 M_n \oplus \cdots)$

$M_{n+1}^* = M_0 \oplus - \oplus M_n \oplus M_{n+1} \oplus (\underline{a} M_{n+1} \oplus \underline{a}^2 M_{n+1} \oplus \cdots)$

So, (M_n^*) stabilises at $n \Leftrightarrow \underline{a} M_n = M_{n+1}$.]

Assume 1). Know: R Noetherian $\Rightarrow R^*$ Noetherian.

$\Rightarrow M^*$ is Noetherian R^*-module (by 1))

$\Rightarrow (M_n^*)$ stabilises

$\Rightarrow (M_n)_{n\geq 0}$ stable (Fact 2).

Assume 2). $\Rightarrow (M_n)_{n\geq 0}$ stable, so: (M_n^*) stabilises at $n_0 \geq 0$. (by fact 2)

~~$\Rightarrow$ ...~~ But: $M^* = \bigcup_{n\geq 0} M_n^*$

$\Rightarrow M^* = M_{n_0}^*$.

now: $M_{n_0}^* = \underbrace{M_0 \oplus - \oplus M_{n_0}}_{\text{generates: } M_{n_0}^* \text{ as } R^*\text{-module.}} \oplus (\underline{a} M_{n_0} \oplus \underline{a}^2 M_{n_0} \oplus \cdots)$

2

& Each M_i is a finitely generated R-module (Fact 1)

$\Rightarrow$ Have: $M_0 \oplus \ldots \oplus M_{n_0} = \mathrm{Span}_R\{x_1, \ldots, x_r\}$.

$\Rightarrow M^* = \mathrm{Span}_{R^*}\{x_1, \ldots, x_r\}$ ✓

Prop 12.12 (Artin-Rees Theorem).

Let: R Noetherian ring, $\underline{a} \subseteq R$ ideal, M f.g. R-module.

& $(M_n)_{n\geq 0}$ stable $\underline{a}$-filtration of M.

& $N \subseteq M$ submodule.

Then: $(N \cap M_n)_{n\geq 0}$ is a stable $\underline{a}$-filtration of N.

Proof First, show this is a filtration. (& $\underline{a}$-filtration)

Have: $\underline{a}(N \cap M_n) \subseteq N \cap \underbrace{\underline{a}M_n}_{\subseteq M_{n+1}} \subseteq N \cap M_{n+1}$ ✓

Denote $N^* = \bigoplus_{n\geq 0} N_n$.

$\Rightarrow N^*$ is a (graded) R^*-submodule of $M^* = \bigoplus_{n\geq 0} M_n$.

Also: R Noetherian $\Rightarrow R^*$ Noetherian.

So, $(M_n)_{n\geq 0}$ stable $\Rightarrow M^*$ is f.g. R^*-module. (Lemma 12.11)

$\Rightarrow M^*$ is ~~f.g. R^*~~ Noetherian R^*-module.

$\Rightarrow N^*$ is f.g. R^*-module

$\Rightarrow (N \cap M_n)_{n\geq 0}$ is stable $\underline{a}$-filtration (Lemma 12.11). ✓

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§13: Dimension Theory.

DEF 13.1 1) The *length* of a chain $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_d$ is d.
2) Height of $\mathfrak{p} \in \operatorname{Spec}(R)$ is Supremum of all lengths of chains of prime ideals *contained* in $\mathfrak{p}$.
3) Krull dimension of R is: $\dim(R) = \sup\{\operatorname{height}(\mathfrak{p}) : \mathfrak{p} \in \operatorname{Spec} R\}$.
$[= \sup\{\operatorname{height}(\underline{m}) : \underline{m} \in \operatorname{mSpec}(R)\}]$.
(Since: chain of prime ideals *not* ending at maximal ideal can be extended.)
Also denote: $\operatorname{height}(\mathfrak{p}) = \dim R_{\mathfrak{p}}$. $\Rightarrow \dim R = \sup\limits_{\underline{m} \in \operatorname{mspec}(R)} \{\dim R_{\underline{m}}\}$.

DEF 13.2 $I \subseteq R$ ideal.
$\Rightarrow ht(I) = \inf\{ht(\mathfrak{p}) : I \subseteq \mathfrak{p} \in \operatorname{Spec} R\}$.

Prop 13.4 Let: $A \subseteq B$ integral ext of *rings*. Then:
i) $\dim(A) = \dim(B)$
ii) If A, B integral domains & are k-algebras, then:
$\operatorname{trdeg}_k(A) = \operatorname{trdeg}_k(B)$. (Transcendence degree).

Proof *$\dim A \leq \dim B$*: Take $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_d$ in $\operatorname{Spec}(A)$.
$\Rightarrow$ Find $\mathfrak{q}_0 \subseteq \cdots \subseteq \mathfrak{q}_d$ in $\operatorname{Spec}(B)$ s.t. $\mathfrak{q}_i \cap A = \mathfrak{p}_i$
(by: lying over & Going up theorems).
Notice that $\mathfrak{q}_i \neq \mathfrak{q}_{i+1}$ since $\mathfrak{p}_i \neq \mathfrak{p}_{i+1}$ & . So, $\dim B \geq d$.
$\dim B \leq \dim A$: Take $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_d$ in $\operatorname{Spec}(B)$.
$\Rightarrow \mathfrak{q}_0 \cap A \subseteq \cdots \subseteq \mathfrak{q}_d \cap A$.
& Since $\mathfrak{q}_i \subsetneq \mathfrak{q}_{i+1}$, by *incomparability*, $\mathfrak{q}_i \cap A \subsetneq \mathfrak{q}_{i+1} \cap A$. ✓ 14

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<u>Let</u>: A f.g. k-algebra. (k field).

By <u>Noether normalisation</u>: $\exists\ k[T_1, \dots, T_d] \hookrightarrow A$ injective ring hom ($d \geq 0$). By ex. sheet 3, $\dim k[T_1, \dots, T_d] = d$.

So, by Prop 13.4 (i), have $\dim(A) = d$.

§13.1 : Hilbert Polynomials & Functions

<u>Let</u>: $A = \bigoplus_{n \geq 0} A_n$ Noetherian graded ring. $\Leftrightarrow$ A_0 Noeth + A f.g. as A_0-algebra.

<u>Let</u>: $M = \bigoplus_{n \geq 0} M_n \neq 0$ f.g. graded A-module.

($\Rightarrow$ Each M_n is A_0-module.)

<u>Claim</u>: M_n is <u>f.g.</u> A_0-module.

<u>Proof</u> M f.g. $\Rightarrow M = \mathrm{Span}_A \{m_1, \dots, m_t\}$, $m_i \in M_{r_i}$.

Then, $M_n = \{a_1 m_1 + \dots + a_t m_t : a_i \in A_{n-r_i}\}$.

$$= \mathrm{Span}_{A_0} \{X_1^{e_1} \cdots X_t^{e_t} \cdot m_i : \sum_{j \leq s} r_j e_j = n - r_i \ \& \ 1 \leq i \leq t\}$$

<u>Where</u>: A generated as A_0 by $\{X_1, \dots, X_t\}$.

Note: finite generating set for M_n as A_0-module ✓

From now on: make extra assumption: $\boxed{A_0 \text{ Artinian}}$

$\Rightarrow M_n$ is both Noetherian & Artinian A_0-module.

$\Rightarrow M_n$ has <u>finite length</u>. $\ell(M_n) < \infty$.

<u>DEF 13.6</u> Let A, M as above. The Poincaré series for M is: power series $P(M, T) = \sum_{n \geq 0} \ell(M_n) T^n \in \mathbb{Z}[[T]]$.

Theorem 13.7 (Hilbert–Serre).

$P(M,T)$ is Rational function of form $\dfrac{f(T)}{\prod_{1\le i\le s}(1-T^{k_i})}$, $f(T)\in\mathbb{Z}[T]$

(Recall: A gen. (as A_0-algebra) by $x_1,\dots,x_s$, $x_i\in A_{k_i}$.)

Proof Induction on s. For $s=0$: $A=A_0 \Rightarrow M=\mathrm{Span}_{A_0}(S)$, for S finite. ($S\subseteq M_0\oplus\dots\oplus M_{n_0}$, some n_0)

$\Rightarrow M_n=0\ \forall n\ge n_0$, so indeed $P(M,T)$ is polynomial. ✓

Inductive step: Write $M=\bigoplus_{n\ge 0} M_n=\bigoplus_{n\in\mathbb{Z}} M_n$ ($M_n=0\ \forall n<0$)

Then $\forall n\in\mathbb{Z}$, the map $M_n\longrightarrow M_{n+k_s}$, $m\longmapsto x_s\cdot m$ is A_0-module hom.

$$\Rightarrow 0\longrightarrow K_n\longrightarrow M_n\xrightarrow{\times x_s} M_{n+k_s}\longrightarrow L_{n+k_s}\longrightarrow 0 \quad (*)$$

exact, where $K_n=\ker(m\longmapsto x_s m)$

$L_n=M_{n+k_s}/\mathrm{im}(m\longmapsto x_s m)$.

Check: $K=\bigoplus_{n\in\mathbb{Z}}K_n$ & $L=\bigoplus_{n\in\mathbb{Z}}L_n$ are graded A-modules

& K,L are annhilated by x_s. $\Rightarrow$ $\boxed{K,L \text{ f.g. } A_0[x_1,\dots,x_{s-1}]\text{-mods.}}$

(e.g. $\forall (y_n)_{n\in\mathbb{Z}}\in K$: $x_s(y_n)_{n\in\mathbb{Z}}=(x_s y_n)_{n\in\mathbb{Z}}=0$)

Apply $\ell(\cdot)$ to $(*)$: $\Rightarrow \ell(K_n)-\ell(M_n)+\ell(M_{n+k_s})-\ell(L_{n+k_s})=0$

$\Rightarrow \ell(M_{n+k_s})T^{n+k_s}-T^{k_s}(\ell(M_n)T^n)=\ell(L_{n+k_s})T^{n+k_s}-T^{k_s}(\ell(K_n)T^n)$

Sum over n: $\Rightarrow (1-T^{k_s})P(M,T)=P(L,T)-T^{k_s}P(K,T)$.

$\Rightarrow$ Since K,L have dim $<s$: $\Rightarrow$ follows by induction ✓

Note: $R(T) = \dfrac{f(T)}{\prod_{1\leq i\leq s}(1-T^{k_i})}$ holomorphic on $\mathbb{C}-\{1,-1\}$.

If $d(M)$ = order of pole at $T=1$, then: if $M\neq 0$, then $\underline{d(M)\geq 0}$.

Example 13.8) Denote $A = k[T_1, \dots, T_s] = \bigoplus_{n\geq 0} A_n$.

1) A is gen. as $A_0 = k$-module by $T_1, \dots, T_s \in A_1$. So, $k_1 = \dots = k_s = 1$. (All k_i in A_1).

2) $\ell(A_n) = \dim_k(A_n) = \binom{n+s-1}{n}$ (since: degree n monomials in T_i form basis of A_n)

3) $P(A,T) = \sum_{n\geq 0}\binom{n+s-1}{n}T^n = (1-T)^{-s}$.

Prop 13.9) If $k_1 = \dots = k_s = 1$ (All generators in A_1), then:

$\exists\, HP_M \in \mathbb{Q}(T)$ s.t. $\ell(M_n) = HP_M(n)$ for all n big enough.

& $\deg(HP_M) = d(M) - 1$.

Proof) Denote $d = d(M)$. By Theorem 13.7: $\sum_{n\geq 0}\ell(M_n)T^n = \dfrac{f(T)}{(1-T)^d}$

with $f(T)\in\mathbb{Z}[T]$ & $f(1)\neq 0$. (by def. of d)

Denote: $f = \sum_{0\leq i\leq N} a_i T^i$ $(N = \deg(f))$.

& $(1-T)^{-d} = \sum_{j\geq 0}\underbrace{\binom{j+d-1}{j}}_{=b_j} T^j \Rightarrow \ell(M_n) = \sum_{0\leq i\leq N} a_i b_{n-i}$

(for $n\geq N = \deg f$).

Notice: b_{n-i} is poly of n, of degree $d-1$.

& Coeff of n^{d-1} is: $\dfrac{1}{(d-1)!}$.

$\Rightarrow \forall n\geq N$, $\ell(M_n) = p(n)$ for $p\in\mathbb{Q}[T]$.

& degree $(p) \le d-1$. (Since: Sum of degree $d-1$ terms).

Have: Coeff of T^{d-1} is: $\frac{1}{(d-1)!}\sum_i a_i = \frac{f(1)}{(d-1)!} \neq 0$

$\Rightarrow$ Degree is exactly $d-1$. ✓

§13.2: Dimension Theory of Noetherian Local Rings.

Lemma 13.11 $(A, \underline{m})$ Noetherian & Local ring.

1) An ideal $q \subseteq A$ is $\underline{m}$-primary $\Leftrightarrow \underline{m}^t \subseteq q \subseteq \underline{m}$, for some $t \geq 1$.

2) If q is $\underline{m}$-primary, then A/q is Artinian.

Proof 1) If $\underline{m}^t \subseteq q \subseteq \underline{m}$ then $\sqrt{\underline{m}^t} \subseteq \sqrt{q} \subseteq \sqrt{\underline{m}}$, so $\sqrt{q} = \underline{m} \Rightarrow q$ is $\underline{m}$-primary.

Conversely: if q is $\underline{m}$-primary then $\sqrt{q} = \underline{m} \Rightarrow \exists t \geq 1$, with $\underline{m}^t \subseteq q$ (since ring Noetherian), and obviously $q \subseteq \underline{m}$.

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2) Note $(A/q, \underline{m}/q)$ Noetherian local ring.

If $q \subseteq p \subseteq \underline{m}$ & $p \in \mathrm{Spec}(A)$, then: $\sqrt{q} \subseteq p \Rightarrow p = \underline{m}$.

Thus: $\underline{m}/q$ is only prime ideal of $A/q \Rightarrow \dim(A/q) = 0$.

So, A/q Artinian. ($\dim = 0$ + Noetherian $\Rightarrow$ Artinian)

Commutative Algebra: lecture 24.

Theorem 13.12 Fix Noetherian ring $(A, \underline{m})$. Then:

$\delta(A) = d(G_{\underline{m}}(A)) = \dim(A)$, where:

- $\dim(A)$ = Krull dimension
- $\delta(A) = \min\{\delta(q) : q\ \underline{m}\text{-primary ideal of } A\}$
  where $\delta(q)$ = Smallest # of generators for q
- $d(G_{\underline{m}}(A))$ = Order of pole of $P(G_{\underline{m}}(A), T) = \sum_{n \geq 0} \ell\left(\frac{\underline{m}^n}{\underline{m}^{n+1}}\right) T^n$
  at $T = 1$.

Corollary 13.13 (Krull's Height Theorem).

If $\underline{a} = (x_1, \dots, x_r) \subseteq A$ ideal of Noetherian A, then:

$\forall \mathfrak{p} \supseteq \underline{a}$ minimal prime ideal, $ht(\mathfrak{p}) \leq r$.

Proof (of 13.13) Will prove: $\sqrt{\underline{a} \cdot A_{\mathfrak{p}}} = \mathfrak{p} \cdot A_{\mathfrak{p}}$.

Why: take $\underline{a} A_{\mathfrak{p}} \subseteq \underline{m} \in \text{mspec}(R)$.

Then: $\underline{a} \subseteq (\underline{a} \cdot A_{\mathfrak{p}})^c \subseteq \underline{m}^c \subseteq \mathfrak{p}$. $\Rightarrow \underline{m}^c = \mathfrak{p}$. ($\mathfrak{p}$ minimal)

$\Rightarrow \underline{m} = \underline{m}^{ce} = \mathfrak{p} A_{\mathfrak{p}}$ ($\Rightarrow$ Only maximal ideal containing $\underline{a} A_{\mathfrak{p}}$ is $\mathfrak{p} A_{\mathfrak{p}}$)

$\Rightarrow \sqrt{\underline{a} A_{\mathfrak{p}}} = \mathfrak{p} \cdot A_{\mathfrak{p}}$ (remember: Radical of ideal = $\bigcap$ of all prime ideals containing it.)

Hence: $\underline{a} A_{\mathfrak{p}}$ is $(\mathfrak{p} A_{\mathfrak{p}})$-primary. (Since $\mathfrak{p} A_{\mathfrak{p}} \in \text{mspec } A_{\mathfrak{p}}$)

But: $\underline{a} A_{\mathfrak{p}}$ is generated by $\frac{x_1}{1}, \dots, \frac{x_r}{1}$.

$\Rightarrow \dim(A_{\mathfrak{p}}) \underset{\text{dim thm}}{=} \delta(A_{\mathfrak{p}}) \leq \delta(\underline{a} A_{\mathfrak{p}}) \leq r$. ✓

From now on: let $(A, \underline{m})$ Noetherian local ring, and: $\underline{q} \subseteq A$ $\underline{m}$-primary ideal. Say $\underline{q} = (x_1, \dots, x_s)$, $s = \delta(\underline{q})$.

$$\Rightarrow G_{\underline{q}}(A) = \underbrace{A/\underline{q}}_{\text{Artinian}} \oplus \underbrace{\underline{q}/\underline{q}^2}_{\text{Generated by } x_i} \oplus \bigoplus_{n \geq 2} \underline{q}^n/\underline{q}^{n+1}.$$

& $\ell(\underline{q}^n/\underline{q}^{n+1}) < \infty$ $\forall n$.

By Hilbert's theorem: $\ell(\underline{q}^n/\underline{q}^{n+1})$ is eventually a polynomial, of degree $\leq s-1 = \delta(\underline{q}) - 1$.

<u>Lemma 13.14</u>) $\forall p \in \mathbb{Q}[T]$, $\sum_{0 \leq k < n} p(k) = q(n)$ for some $q \in \mathbb{Q}[T]$ where leading term of q depends only on leading poly of p & $\deg(q) = \deg(p) + 1$.

$\Rightarrow$ So, $\ell(A/\underline{q}^n)$ is eventually a poly, of degree $\leq s = \delta(\underline{q})$.

<u>Fix</u>: $\underline{q}_0 \subseteq A$ $\underline{m}$-primary ideal. $\delta(\underline{q}_0) = \delta(A)$.

$\Rightarrow \deg(\ell(\underline{q}_0^n/\underline{q}_0^{n+1})) \leq \delta(A) - 1$ [deg = deg as polynomial]

$\Rightarrow \deg(\ell(A/\underline{q}_0^n)) = \sum_{0 \leq i < n} \deg(\ell(\underline{q}_0^i/\underline{q}_0^{i+1})) \leq \delta(A)$.

& $\deg \ell(\underline{m}^n/\underline{m}^{n+1}) = d(G_{\underline{m}}(A)) - 1$

$\Rightarrow \deg \ell(A/\underline{m}^n) = d(G_{\underline{m}}(A))$.

By <u>lemma 13.11</u>: $\exists t \geq 1$, $m^t \subseteq \underline{q}_0 \subseteq m$.

$\Rightarrow \ell(A/\underline{m}^n) \leq \ell(A/\underline{q}_0^n) \leq \ell(A/\underline{m}^{tn})$.

$\Rightarrow \deg \ell(A/\underline{m}^n) = \deg \ell(A/\underline{q}_0^n)$.

Prop 13.15 $\delta(A) \geq d(G_{\underline{m}}(A))$.

Proof $\delta(A) = \delta(q_0) \geq \deg \ell(A/q_0^n) = \deg \ell(A/\underline{m}^n) = d(G_{\underline{m}}(A))$

Prop 13.16 If $x \in \underline{m}$ & NOT zero divisor, then: $[xA = (x)]$

$$d(G_{\underline{m}/xA}(A/xA)) \leq d(G_{\underline{m}}(A)) - 1.$$

Proof Consider local ring $(A/xA, \underline{m}/xA)$. Then:

$$d(G_{\underline{m}/xA}(A/xA)) = \deg \ell\left(\frac{A/xA}{(\underline{m}/xA)^n}\right) = \deg \ell\left(\frac{A}{\underline{m}^n + xA}\right)$$

$(\underline{m}^n + xA)/xA$

On the other hand: $d(G_{\underline{m}}(A)) = \deg \ell(A/\underline{m}^n)$.

Want: $\deg \ell\left(\frac{A}{\underline{m}^n + xA}\right) \leq \deg \ell\left(\frac{A}{\underline{m}^n}\right) - 1$.

Have: SES $0 \longrightarrow \frac{\underline{m}^n + xA}{\underline{m}^n} \longrightarrow A/\underline{m}^n \longrightarrow A/(\underline{m}^n + xA) \longrightarrow 0$

$\cong xA/(\underline{m}^n \cap xA)$

$\Rightarrow$ By additivity of length: $\ell\left(\frac{A}{\underline{m}^n + xA}\right) = \ell(A/\underline{m}^n) - \ell\left(\frac{xA}{\underline{m}^n \cap xA}\right)$.

Suffices to show: Leading term of $\ell(A/\underline{m}^n)$ & $\ell(xA/(\underline{m}^n \cap xA))$ are the same (so they cancel, giving smaller degree on LHS).

Know: $(\underline{m}^n)_{n \geq 0}$ is stable $\underline{m}$-filtration of A (Noetherian).

$\Rightarrow (\underline{m}^n \cap xA)_{n \geq 0}$ is stable $\underline{m}$-filtration of $xA \subseteq A$

$\Rightarrow (\underline{m}^n \cap xA)_{n \geq 0}$ & $(\underline{m}^n xA)_{n \geq 0}$ are equivalent (lemma 12.10)

$\Rightarrow \exists n_0 \geq 0$, s.t. $\ell\left(\frac{xA}{\underline{m}^n \cap xA}\right) \leq \ell\left(\frac{xA}{\underline{m}^{n+n_0} xA}\right)$ $\forall n$.

& $\ell\left(\frac{xA}{\underline{m}^n xA}\right) \leq \ell\left(\frac{xA}{\underline{m}^{n+n_0} xA}\right)$

$$\left.\begin{array}{l} \ell\left(xA/\underline{m}^n \cap xA\right) \leq \ell\left(xA/\underline{m}^{n+n_0}xA\right) \\ \underline{\&} \leq \ell\left(xA/\underline{m}^n xA\right) \leq \ell\left(xA/\underline{m}^{n+n_0} \cap xA\right) \end{array}\right\}$$

$\Rightarrow$ These 2 have same leading term.

[$\underline{\&}$ $\ell(xA/\underline{m}^n xA) = \ell(A/\underline{m}^n)$ since $A \xrightarrow{\cdot x} xA$ A-linear $\cong$]
as x not zero divisor.

Prop 13.17) $d(G_{\underline{m}}(A)) \geq \dim(A)$. [Proof in notes]

Prop 13.18) $\dim(A) \geq \delta(A)$.

[$\Leftrightarrow \exists\, q \subseteq A$ $\underline{m}$-primary ideal, generated by $\overbrace{\dim(A)}^{=d}$- elements.]

Proof) Know $ht(\underline{m}) = d$. $\underline{\&}$ $\forall\, p \in Spec(A)$, $p \neq \underline{m} \Rightarrow ht(p) < d$.

__Want:__ $q = (x_1, \dots, x_d) \subseteq R$ s.t. $ht(q) \geq d$.

[__Why?__ Since then, ~~$\forall q$~~ $\forall\, p \in Spec(A)$ with $q \subseteq p$, have: $ht(p) \geq d$, so $p = \underline{m} \Rightarrow \sqrt{q} = \underline{m} \Rightarrow q$ $\underline{m}$-primary.]

Construct x_i __inductively__ (so $ht((\underbrace{x_1, \dots, x_i}_{=q_i})) \geq i$).

__Base case__ $q_0 = (0)$ ✓

__Inductive step__: Say we constructed $q_{i-1} = (x_1, \dots, x_{i-1})$, with $ht(q_{i-1}) \geq i-1$. Then, there are finitely many primes $p_n \subseteq A$ of height $i-1$ that contain q_{i-1}, since:

1) Each such p_i must be __minimal__ prime ideal of q_{i-1} (since $ht(q_{i-1}) \geq i-1$)
2) q_{i-1} has finitely many minimal prime ideals, as A Noetherian.

Know: $i-1 < d = ht(\underline{m}) \Rightarrow \underline{m} \nsubseteq \mathfrak{p}_i$ for each such $\mathfrak{p}_i$.

$\Rightarrow \underline{m} \nsubseteq \bigcup_{j \leq t} \mathfrak{p}_j$ (Prime avoidance)

Take: $x_i \in \underline{m} \setminus \bigcup_j \mathfrak{p}_j$. & $\mathfrak{q}_i = (x_1, \ldots, x_i)$.

$\Rightarrow \forall \mathfrak{p} \in \mathrm{Spec}(A)$, if $\mathfrak{p} \supseteq \mathfrak{q}_i$, then $\mathfrak{p} \notin \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$

and $\mathfrak{p} \supseteq \mathfrak{q}_{i-1}$. Hence, $ht(\mathfrak{p}) \geq i \Rightarrow ht(\mathfrak{q}_i) \geq i$ ✓