Lie algebras: lecture 1 06/10/2023

1] Introduction.

<u>Lie group</u> $\equiv$ group + smooth manifold, $(+, \times)$ are differentiable maps.

<u>Examples</u>: GL_n, SL_n, SO_n, SP_{2n}. & S^1. (prototypical example)

If G is Lie group: the <u>Lie algebra</u> of G is the <u>tangent space</u> of the identity (e) of G, denoted $T_e G \equiv g$.

Then: g is a vector space.

<u>Have</u>: conjugation map $\begin{pmatrix} G \longrightarrow Aut(G) \\ g \longmapsto g(\cdot)g^{-1} \end{pmatrix}$

"adjoint".

Take differential: turn into a linear map $ad: g \longmapsto End g$.

Gives: bilinear map $[\cdot,\cdot]: g^2 \longrightarrow g$, $[X,Y] \longmapsto ad(X)Y$.

<u>Example 1.2</u>] $G = GL_n(\mathbb{R})$. $\Rightarrow g = gl_n(\mathbb{R}) = M_n(\mathbb{R})$

& $[x,y] = xy - yx$. (commutator).

<u>Questions</u>: <u>a)</u> What does Lie algebras tell us about the structure of G?

- Define <u>root system</u> of g: tells commutator relations in G (Carter's book)
- Define <u>Weyl group</u> of g: e.g. Weyl group of $GL_n(\mathbb{C})$ is S_n. i.e. $\exists$ embedding $S_n \hookrightarrow GL_n(\mathbb{C})$ via permuting a basis element.

Let: $B \equiv$ upper triangular matrices of $GL_n(\mathbb{C})$. Then, exist Bruhat decomposition $G = \bigsqcup_{w \in S_n} BwB$

i.e. disjoint union of double cosets.

b) They tell us the rep. theory of G, and $\mathfrak{g}$.

Example: $\exists$ $\{$f.d. $\mathbb{C}$-reps $SL_n(\mathbb{C})\} \xleftrightarrow{\sim} \{$f.d. $\mathbb{C}$-reps of $Lie(SL_n(\mathbb{C}))\}$

where: $Lie(SL_n(\mathbb{C})) = \mathfrak{sl}_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : \operatorname{tr} A = 0\}$.

& RHS can be described completely.

c) They have applications in alg geo.

Can use Lie algebras to build families of surfaces, or algebraic curves. [Slodowy]

Define: Dynkin diagram of a semisimple Lie algebra.

e.g.

"type E_7".

Tells us: about singularities of surfaces.

d) Applications to number theory. Root systems & Weyl groups give structure of groups over $\mathbb{Q}_p$.

Local Langlands correspondence products relationship:

$\{$Galois Theory of Local fields$\} \longleftrightarrow \{$complete Lie theory$\}$.

Basic defs & Examples.

k field (usually $\mathbb{C}$).

DEF) Lie algebra over k is VS $\mathfrak{g}$ over k with a bilinear pairing $[\cdot,\cdot] : \mathfrak{g}^2 \to \mathfrak{g}$ with:

- $[x,x] = 0$ $\forall x \in G$.
- Jacobi identity: $[x[yz]] + [y[\cancel{xz}]] + [z[xy]] = 0$ (with zx written above)

Implies: anti-symmetry $[xy] = -[yx]$.

<u>DEF</u> A k-subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is (Lie)-subalgebra if it is closed under $[\cdot,\cdot]$.

<u>Examples</u>. V fin-dim $/k$.

<u>1)</u> <u>$\mathfrak{gl}(V)$</u> $\equiv \mathrm{End}_k(V)$, $[x,y] = xy - yx$.

If V has basis: identify V with $n \times n$ matrices over k.

Then, write: $\mathfrak{gl}(V) \equiv \mathfrak{gl}_n$ $(\equiv \mathfrak{gl}_n(k) \equiv M_n(k))$.

<u>2)</u> <u>$\mathfrak{sl}(V)$</u> $\equiv \{x \in \mathfrak{gl}(V) : tr(x) = 0\}$. $\equiv \mathfrak{sl}_n$.

Is: subspace & closed under $[\cdot,\cdot]$ $\Rightarrow$ Sub-algebra.

<u>Have</u>: $\dim(\mathfrak{sl}(V)) = n^2 - 1$. Basis: $e_{ii} - e_{i+1,i+1}$ and e_{ij} $(i \neq j)$.

Lie groups: Lecture 2. 09/10/2023.

Examples (continued).

3) k field ($\mathrm{char}\, k \neq 2$). & V is vector space with a symmetric bilinear form $\langle \cdot, \cdot \rangle : V^2 \to k$.

let: $\boxed{so(V) \equiv \{ x \in gl(V) : \langle xv, w \rangle = -\langle v, xw \rangle \ \forall v, w \}}$.

In coordinates: know, $\exists M \in GL(V)$ s.t. $\langle v, w \rangle = v^t M w$.

$\Rightarrow so(V) = \{ x : Mx + x^T M = 0 \}$.

Usually take: $M = \begin{bmatrix} 0 & I_\ell \\ I_\ell & 0 \end{bmatrix}$ if $n = 2\ell$, $\begin{bmatrix} 1 & & \\ & 0 & I_\ell \\ & I_\ell & 0 \end{bmatrix}$ if $n = 2\ell+1$.

Called: Orthogonal Lie algebras.

Remark For $n=2$, have: $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ & $g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, viewed as matrices in $sl_2(\mathbb{C})$. Forms standard basis of (3).

Note: $[ef] = h$, $[he] = \cancel{2f}\ 2e$, $[hf] = -2f$.

Will see: structure of semisimple Lie algebras (over $\mathbb{C}$) "comes from" $sl_2(\mathbb{C})$.

4) k as in 3), and $\langle \cdot, \cdot \rangle$ non-degen skew-symmetric bilinear form, i.e. $\langle v, w \rangle = -\langle w, v \rangle \ \forall v, w$.

Then: $\boxed{sp(V) \equiv \{ x \in gl(V) : \langle xv, w \rangle = -\langle v, xw \rangle \}}$

In coords: take ~~M=~~ $\langle \cdot, \cdot \rangle$ the skew-symmetric form associated with $M = \begin{bmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{bmatrix}$, $(n = 2\ell)$ (note that n must be even)

Then: this is called symplectic Lie algebras. (sp_n).

5) Any space V is a Lie algebra with $[xy]=0$ $\forall x,y$.
Defines an abelian Lie algebra.
(named so, since for linear Lie algebras, $[x,y]=0 \iff x,y$ commute. Linear $\iff$ subalgebra of $gl(V)$.)

6) $b_n \equiv$ upper triangular matrices. Borel subalgebra.
(associated with Borel subgroup of GL_n)

7) $n_n \equiv$ strictly upper triangular matrices: Nilpotent matrices.

Basic Rep Theory.

DEF 3.1 A linear transformation $\varphi: \mathfrak{g} \to \mathfrak{h}$ ($\mathfrak{g}, \mathfrak{h}$ Lie algs) is a homomorphism if $[\varphi x, \varphi y] = \varphi[x,y]$.
If φ is isomorphism of vector spaces, it's called an isomorphism of Lie algebras.

DEF 3.2 A representation of $\mathfrak{g}$ is a Lie algebra hom.
$\varphi: \mathfrak{g} \to gl(V)$, some V.

Notation V itself is also called the representation, or $\mathfrak{g}$-module. Write $\mathfrak{g} \curvearrowright V$ and "$\mathfrak{g}$ acts on V" (from the left).
Write: $x \cdot v$ for $\varphi(x)(v)$.

The dimension of the rep. is $\dim(V)$.

Examples) 1) Trivial rep : let $\dim(V)=1$ and $\mathfrak{g} \curvearrowright V$ by $x \cdot v = 0 \ \forall x \in \mathfrak{g}, v \in V$.

2) If $\mathfrak{g}$ subalg. of $\mathfrak{gl}(V)$, $\exists\, i : \mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$ natural inclusion (defining representation)

3) Adjoint rep : for $x \in \mathfrak{g}$, define $\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g}$, $y \mapsto [xy]$.

Then $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is Adjoint rep.

Remark Adjoint rep of $\mathfrak{sl}_2(\mathbb{C})$ with basis $\{e, h, f\}$ has:

$$\mathrm{ad}(h) = \begin{bmatrix} 2 & & \\ & 0 & \\ & & -2 \end{bmatrix}, \quad \mathrm{ad}(e) = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \ \& \ \mathrm{ad}(f) = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

4) If V, W reps, so is $V \oplus W$ by $x(v,w) = (xv, xw)$.

5) If V rep, so is V^*: by $(x \cdot f)(v) = -f(x \cdot v)$ ($\forall x \in \mathfrak{g}, v \in V, f \in V^*$).

6) If V, W reps, so is $\mathrm{Hom}(V,W)$ by: $(x \cdot f)(v) = x \cdot f(v) - f(xv)$

DEF 3.4) If V, W reps: a linear transformation $\varphi : V \to W$ is $\mathfrak{g}$-equivariant if $x \cdot \varphi(v) = \varphi(x \cdot v) \ \forall x \in \mathfrak{g}, v \in V$.

Say: V, W isomorphic (as reps) if $\exists \varphi$ as above which is an isomorphism.

DEF 3.5) A subrep $V' \subseteq V$ is subspace closed under the action of V: $(x \cdot v \in V' \ \forall v \in V')$.

DEF 3.6 $V \neq 0$ <u>irred</u> (simple) $\Leftrightarrow$ only subreps are V and 0.

<u>Notes</u> 1) Trivial rep is irreducible

2) For $sl_2(\mathbb{C})$: defining rep & adjoint rep are irreducible.

Lie Algebras: Lecture 3 11/10/2023.

DEF 3.7) V completely reducible $\iff$ decomposes as a direct sum of irred reps.

For V rep, completely reducible $\iff$ $\forall W \leq V$ subrep, $\exists W' \leq V$ s.t. $V = W \oplus W'$.

Example 7 V rep, $W \leq V$ subrep $\Rightarrow$ V/W is a $\mathfrak{g}$-rep by $x(v+W) = xv + W$.

Rep Theory of $\mathfrak{sl}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b, c \in \mathbb{C} \right\}$.

Example: $\mathfrak{b}_2 \equiv \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{C} \right\}$ (Borel subalgebra).

Let: $V \equiv \operatorname{span}\left\{ v_1 \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ defining rep. of $\mathfrak{b}_2$.

Then: V not completely reducible ($\langle v_1 \rangle$ is not).

DEF 4.1) A rep V of lie algebra $\mathfrak{g}$ is faithful if: $\mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ injective.

[Henceforth: All Lie algs are over $\mathbb{C}$.]

For V rep of $\mathfrak{sl}_2(\mathbb{C})$, recall $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Know 3 reps (already):

dim	name	h
1	trivial	0
2	defining	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
3	adjoint	$\begin{pmatrix} 2 & & \\ & 0 & \\ & & -2 \end{pmatrix}$.

DEF 4.2 | $\forall \lambda \in \mathbb{C}$, $V_\lambda \equiv \{v \in V : h(v) = \lambda v\}$.

"λ-weight space".

- ⊛ Trivial space V_0
- ⊛ Defining rep $V_{-1} \oplus V_1$
- ⊛ Adjoint rep $V_2 \oplus V_0 \oplus V_{-2}$ $(V_2 = \langle e \rangle, V_0 = \langle h \rangle, V_{-2} = \langle f \rangle)$.

Actions.

⊛ $\underline{e}$: $\forall v \in V_\lambda$: $hev = ([he] + eh)v = 2ev + \lambda ev$
$= (\lambda + 2)ev$.

⊛ $\underline{f}$: $\forall v \in V_\lambda$, $fv \in V_{\lambda-2}$.

$$\begin{tikzcd}
V_{\lambda-4} \arrow[r, bend left, "e"] & V_{\lambda-2} \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] & V_\lambda \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] & V_{\lambda+2} \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] & V_{\lambda+4} \arrow[l, bend left, "f"] \arrow[r] & \text{Higher weights}
\end{tikzcd}$$

DEF 4.3 | $v \in (V_\lambda \cap \ker(e))$, $v \neq 0$ is <u>highest weight vector</u> (<u>HWV</u>). (of weight λ).

In adjoint rep, e is a HWV. $(e^2 = 0)$

Lemma 4.4 | $v \in V_\lambda$ is HWV. $\Rightarrow \forall n \geq 1 : e f^n v = n(\lambda - n + 1) f^{n-1} v$.

Proof Induction on n.

Base case $n=1$: $efv = ([ef] + fe)v = (h + fe)v = \lambda v + 0$.

Inductive step: similar.

Lemma 4.5 | $v \in V_\lambda$ is HWV $\Rightarrow W \equiv \text{span}\{f^i v : i \geq 0\}$]2 is subrep of V.

Proof need: for $w = f^n v$: i) $ew \in W$ ii) $hw \in W$ iii) $fw \in W$

i) If $n \geq 1$, follows from lemma 4.4; if $n=0$, $ew=0$.

ii) $f^n \in V_{\lambda-2n} \Rightarrow hw = (\lambda - 2n) w \in W$

iii) Trivial from def. of W.

Prop 4.6) A f.-d. rep V has a HWV.

Proof choose any evec $v \neq 0$ of h and consider $W \equiv \{v, ev, e^2 v, \ldots\}$. Then, W lin-indep (as e changes the eigenvalue of v), but V fin-dim. So, $\exists n$, $e^n v \neq 0$ & $e^m v = 0 \ \forall m \geq n$. So, $e^n v$ is a HWV.

Lemma 4.7) V f.d. & $v \in V_\lambda$ HWV. $\Rightarrow \lambda \in \mathbb{Z}_{\geq 0}$.

Proof Any nonzero $f^n v$ is lin-indep, so $\exists n \geq 0$, with $f^n v \neq 0$ & $f^m v = 0 \ \forall m > n$.

By (4.4): $0 = e f^{n+1} v = (n+1)(\lambda - n) \underbrace{f^n v}_{\neq 0} \Rightarrow \lambda = n$ ✓

Conclusion: V irred $\dim = n+1$. Then, $\exists v$, HWV. $v \in V_\lambda$.

By (4.5): $\text{span}\{v, fv, f^2 v, \ldots\}$ is a sub-rep of V.

$\Rightarrow \{v, fv, \ldots, f^n v\}$ is a basis (since $f^i v$ linearly indep). Also, by (4.7): $\lambda = n \in \mathbb{Z}_{\geq 0}$.

Corollary 4.8 V irred rep of $\mathfrak{sl}_2\mathbb{C}$, dim $n+1$. Then: $\exists$ basis $v_0, \dots, v_n$ of V s.t.

$h \cdot v_i = (n-2i)v_i$ $\quad \begin{pmatrix} n & & & \\ & n-2 & & \\ & & \ddots & \\ & & & -n \end{pmatrix}$

& $f \cdot v_i = \begin{cases} v_{i+1} & \text{if } i<n \\ 0 & \text{if } i=n. \end{cases}$ $\quad \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$

& $e \cdot v_i = \begin{cases} i(n-i+1)v_{i-1}, & i>0 \\ 0, & i=0. \end{cases}$

In particular: $\exists!$ irred rep, dim $n+1$.

$$h \mapsto \begin{pmatrix} n & & & \\ & n-2 & & \\ & & \ddots & \\ & & & -n \end{pmatrix}$$

$$f \mapsto \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$$

$$e \mapsto \begin{pmatrix} 0 & n & & \\ & \ddots & 2(n-1) & \\ & & \ddots & n \\ & & & 0 \end{pmatrix}$$

$\Rightarrow$ Can view $V(n)$ as representation: $\mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{sl}_{n+1}(\mathbb{C})$.

Lie Groups : Lecture 4. 13/10/2023.

From last time: HWV $\Leftrightarrow v \in V_\lambda \cap \ker(e)$. $ev=0$. $hv=\lambda v$.

& If V fin-dim rep of sl_2, then: (4.2) $\exists$ HVW.

& $\forall n \geq 0$ $\exists!$ irred rep of sl_2, dim $n+1$ (up to $\cong$).

5] Irred Modules of sl_2.

Remark V is $(n+1)$-dim irred rep of sl_2 & $v \in V$ HWV.

Then: $\underbrace{(ef+fe+\frac{1}{2}h^2)}_{\in End(V)}(v) = (n^2/2+n)v$.

Write: $V(n) \equiv$ irred reps of sl_2, dim $= n+1$

DEF 5.1] Given rep V of sl_2: weights $\{\lambda \in \mathbb{C} : V_\lambda \neq 0\}$.

Theorem 5.2] (Weyl's Theorem)

Any f-d. rep of sl_2 is completely reducible.

This result, with (4.8), shows: action of h completely determines a fin-dim rep.

Example] V dim $=5$ rep of sl_2. & $\exists v \in V$, $hv = 3v$.

Means: weights contain $3, 1, -1, -3$. $\Rightarrow$ $\boxed{V = V(0) \oplus V(3)}$.

Facts: ⊛ If g any Lie algebra & $\varphi : g \to gl(V)$ rep, and $\exists \sigma$ commuting with $\varphi(x)$ $\forall x \in g$, then:

1) $\ker(\sigma - \lambda . id_V)$ is sub-rep of V ($\forall \lambda \in \mathbb{C}$)

2) If V irred $\Rightarrow \sigma$ is scalar mult. of id ("Schur's lemma").

DEF 5.3] V f.d. rep of $sl_2\mathbb{C}$. Then:

$\boxed{\Omega \equiv ef + fe + \frac{1}{2}h^2} \in gl(V)$ is: Casimir element of sl_2.

Lemma 5.4] If $\varphi: sl_2 \longrightarrow gl(V)$ then Ω commutes with $\varphi(x)$ $\forall x \in sl_2$ ("Ω is central"). (V fin-dim).

Proof To show Ω central: show $e\Omega = \Omega e$, $f\Omega = \Omega f$ and $h\Omega = \Omega h$. ✓

Corollary 5.5] If V fin-dim. rep of sl_2 then $\Omega \circlearrowright V$ by scalar mult. (follows from Schur's lemma)

By Remark: scalar is $\frac{n^2}{2} + n$.

Proof of 5.2] Let: $\varphi: sl_2 \longrightarrow gl(V)$ f.d. rep. (of sl_2) & $W \leq V$ sub-rep. Need: $\exists U \leq V: V \cong W \oplus U$.

Case 1: W has codim$=1$. $\Rightarrow V/W \cong V(0)$.

⊛ If W trivial $\Rightarrow \dim V = 2$ & $\exists$ basis $\{v_1, v_2\}$ of V with: sl_2 acts on V by $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$. ($W = \mathbb{C}v_1$)

Show: $V \cong V(0) \oplus V(0)$

$W = \mathbb{C}v_1$ $\quad V \overset{?}{\equiv} \mathbb{C}v_2$

Know: φ respects $[\,,\,]$ (since representation).

$\Rightarrow \varphi(h) = [\varphi(e)\varphi(f)] = 0$, similarly $\varphi(e) = \varphi(f) = 0$.

$\Rightarrow$ Action is trivial on the other factor.

⊛ If $W \cong V(n)$ irred ($n>0$):

<u>have</u>: $\Omega \in gl(V)$. <u>Show</u>: $V \cong V(n) \oplus \ker\Omega$.

By <u>Schur</u>: & W <u>irred</u> & Ω acts on V/W trivially:

$\exists$ basis for V, s.t. Ω acts via:

$$\Omega \longmapsto \begin{pmatrix} \lambda & & & * \\ & \ddots & & \vdots \\ & & \lambda & * \\ & & & 0 \end{pmatrix}$$

Since W nontrivial $\Rightarrow \ker(\Omega) \neq \phi$ & $W \cap \ker\Omega = 0$.

$\Rightarrow V = W \oplus \ker\Omega$.

⊛ If W <u>arbitrary</u>: induction (on $\dim V$).

Base case: $\dim V = 1$ ✓ Assume: $\dim(V) \geq 2$.

<u>Take</u>: $W' \leq W$ nonzero subrep. Have: $\dim(W/W') < \dim V$,

& $\mathrm{codim}(W/W'$ in $V/W') = 1$.

By <u>induction</u>: $(V/W') = (W/W') \oplus \underbrace{(W''/W')}_{\text{subrep of LHS}}$. ($W'' \subseteq V$)

<u>Know</u>: $W' \leq W''$ has codim 1. & $\dim W' < \dim V$.

& W'' is subrep of V (since: W''/W' subrep, so: $xw \in W''$ $\forall x \in sl_2$) ($xw \in W' \subseteq W''$.) (dim 1 subrep trivial)

$\Rightarrow$ By induction, $\exists U \leq W''$: $W'' = W' \oplus U$.

<u>Claim</u>: $V \cong W \oplus U$.

<u>Have</u>: $\dim(U) = 1$ & $W \cap U \subseteq W \cap W'' = W'$ (since $\underline{1}$ is a direct sum). ($W/W' \oplus U/W'$ direct)

By (2): sum direct $\Rightarrow W \cap U \leq W' \cap U = 0$
& since $\dim(U) = 1$ & $\operatorname{codim}(W) = 1$, done by dimension counting (rank-nullity).

$\boxed{\text{Case 2}}$ $\underline{W \text{ arbitrary}}$. $(W \leq V)$

$\underline{\text{Recall}}$: action on $\operatorname{Hom}(V, W)$: $(x\varphi)(v) = x(\varphi(v)) - \varphi(xv)$.

$\underline{\text{Define}}$: $\mathbb{V} \equiv \{\psi \in \operatorname{Hom}(V, W) : \psi|_W = \lambda \, id_W \text{ for } \underline{\text{some}} \ \lambda \in \mathbb{C}\}$

& $\mathbb{W} \equiv \{\psi \in \mathbb{V} : \psi|_W = 0\}$ (i.e. $\lambda = 0$)

$\Rightarrow$ Have: $\operatorname{codim}(\mathbb{W}, \mathbb{V}) = 1$. (lose 1 degree of freedom)

$\underline{\text{Suppose}}$: $\psi|_W = \lambda \cdot id_W$ & $x \in sl_2, w \in W$. Then:

$(x\psi)(w) = x\psi(w) - \psi(xw) = x(\lambda w) - \lambda(xw) = 0$.

$\Rightarrow \mathbb{V}$ is sub-rep of $\operatorname{Hom}(V, W)$

& $\mathbb{W}$ is sub-rep of $\operatorname{Hom}(V, W)$ (same logic).

$\underline{\text{Hence}}$: $\exists \, \mathbb{U}$ 1-dim rep of $\mathbb{V}$ s.t. $\boxed{\mathbb{V} = \mathbb{W} \oplus \mathbb{U}}$

$\Rightarrow$ write: $\mathbb{U} \equiv \langle \gamma \rangle$. $(\gamma \in \mathbb{V})$ & $\gamma|_W = \lambda \cdot id_W$.

Have: $\lambda \neq 0$ since not in $\mathbb{W}$.

$\underline{\text{Claim}}$: As vector spaces, $V = W \oplus \underbrace{\ker \gamma}_{U}$.

~~Commutative Algebra~~ Lie Groups: Lecture 5 16/10/2023.

From last time: $\mathbb{V} \equiv \{\varphi \in \mathrm{Hom}(V,W) : \varphi|_W = \lambda \cdot id_W\}$

[Proof Continued] $\mathbb{W} \equiv \{\varphi \in \mathbb{V} : \varphi|_W = 0\}$.

$\Rightarrow \mathbb{W} \subseteq \mathbb{V} \subseteq \mathrm{Hom}(V,W)$ (as subreps)

$\Rightarrow \exists\, \mathbb{U} : \mathbb{V} = \mathbb{W} \oplus \mathbb{U}$ & $\dim(\mathbb{U}) = 1$. $\Rightarrow \mathbb{U} = \langle \gamma \rangle$.

Claim: ~~$\mathbb{V} = \mathbb{W} \oplus \ker\gamma$~~ $V = W \oplus \ker\gamma$.

Proof: know, $W \cap \ker\gamma = 0$. & $\dim V = \dim W + \dim\ker\gamma$, (by dimension count)

since: $W = \mathrm{im}\,\gamma$.

Finally: need, $\ker\gamma \leq V$ as subreps.

For $v \in \ker\gamma$ & $x \in sl_2$. Since $\mathbb{U}$ is 1-dim rep. of sl_2, see: $\mathbb{U}$ is the trivial rep.

$\Rightarrow 0 = (x\gamma)(v) = x \cdot \cancel{\gamma(v)} - \gamma(xv)$. $\Rightarrow \gamma(xv) = 0$ ✓ $xv \in \ker\gamma$

Remark 1) The proof only needed:

- Existence of Ω
- Every 1-dim rep. of sl_2 is trivial rep.

2) Complete reducibility is rare! (for inf-dim Lie algebras and simple Lie algebras over char $p > 0$)

Example Adjoint rep. of $sl_n(\mathbb{F}_p)$ on $gl_n(\mathbb{F}_p)$ is not completely reducible if $p \mid n$.

6: Tensor Products.

Given: V, W f.d. vector spaces with bases $\{v_i\}$ & $\{w_j\}$.
Recall: $V \underset{\mathbb{C}}{\otimes} W$ has bases $\{v_i \otimes w_j : 1 \le i \le n, 1 \le j \le m\}$.

DEF 6.1) If V, W reps of lie algebra g, then: so is $V \otimes W$

$$\boxed{x \cdot (v \otimes w) \equiv (xv) \otimes w + v \otimes (xw).}$$

Example 6.2) If V, W reps of sl_2 & $v \in V_\lambda$, $w \in W_\mu$ then
$h(v \otimes w) = (\lambda + \mu) v \otimes w$.
In practice: weights of $V \otimes W$ are $\{\lambda + \mu : \lambda$ weight of V, μ weight of $W\}$.

⊛ $V(2)^{\otimes 2} \equiv V(2) \otimes V(2)$ decomposition:

	2	0	−2
2	4	2	0
0	2	0	−2
−2	0	−2	−4

$\Rightarrow V(2)^{\otimes 2} = V(4) \oplus V(2) \oplus V(0)$.

⊛ If v_n HVW of $V(n)$, then
$v_n \otimes v_m$ HVW of $V(n) \otimes V(m)$

⊛ Clebsch–Gordon Rule: $V(n) \otimes V(m) = \bigoplus_{\substack{r = |n-m| \\ r \equiv n-m \bmod 2}}^{n+m} V(r)$.

DEF 6.3) n-th symmetric power:

$$S^n V = \mathrm{Sym}^n(V) = \underbrace{V \otimes \dots \otimes V}_{n} / M_n$$ where M_n is

$\mathrm{span}\{u_1 \otimes \dots \otimes u_n - u_{\sigma(1)} \otimes \dots \otimes u_{\sigma(n)} : \sigma \in S_n\}$.

For $n=2$: $M_2 = \mathrm{span}\{u \otimes v - v \otimes u\}$.

Fact) M_n is subrep. of $V^{\otimes n}$ (whenever V rep of g).
$\Rightarrow \mathrm{Sym}^n(V)$ subrep.

Example 6.4 In S^2V, $v\otimes w \simeq w\otimes v$. S^2V has basis $\{v_i\otimes v_j : i\leq j\}$. Decomposing $S^2(V(2))$: $0\neq e\otimes e\in S^2(V(2))$ $\Rightarrow$ $V(4)$ is a subrep & $S^2(V(2)) = V(4)\oplus V(0)$.

DEF 6.5 The n'th exterior/alternating power: $\Lambda^n V \equiv V^{\otimes n}/N_n$, where $N_n = \langle u_1\otimes\cdots\otimes u_n : u_i = u_j, \text{ some } j\neq i\rangle$.

Example 6.6 For $n=2$, $N_2 = \langle v\otimes v : v\in V\rangle$. ~~Is subrep~~

Have: N_n is subrep of $V^{\otimes n}$.

Notation: write $u_1\otimes\cdots\otimes u_n \equiv u_1\wedge\cdots\wedge u_n$.

Example Decomposition of $\Lambda^2(V(2)) \cong V(2)$, with basis $\{e\wedge f, e\wedge h, f\wedge h\}$. Find dimensions!

Lie Groups: Lecture 6. 18/10/2023.

Let: g be a semisimple Lie algebra $/\mathbb{C}$.

$I \subseteq g$ is ideal of g, if: $[xy] \in I \ \forall x, \in g, y \in I$.

Remarks 1) Any ideal is subalgebra

2) I ideal $\Rightarrow g/I$ Lie algebra $([x+I, y+I] \equiv [xy]+I)$

3) I ideal $\Rightarrow$ Is subrep of adjoint rep. of g.

Examples 1) Centre of g: $Z(g) \equiv \{x \in g : [x,y] = 0 \ \forall y \in g\}$.

2) Derived subalgebra: $D[g] \equiv [g,g] \equiv \{[xy] : x, y \in g\}$.

~~3)~~ Recall: $sl_n \subseteq gl_n$ is a subalgebra. In fact, sl_n is the derived subalgebra of gl_n!

~~3)~~ Proof: $\forall x, y \in gl_n$, $tr[x,y] = tr(xy) - tr(yx) = 0$.

$\Rightarrow [gl_n, gl_n] \subseteq sl_n$. Conversely, $sl_n = [sl_n, sl_n] \subseteq [gl_n, gl_n]$.

3) If $\phi : g \to h$ hom, then $\ker\phi$ is ideal, since ϕ respects $[\cdot,\cdot]$. In fact, all ideals arise in this way!

DEF 7.2) Lie algebra g simple if $[g,g] \neq 0$, and only ideals of g are $0, g$. (non-abelian)

~~Remarks~~ Examples) sl_n $(n \geq 2)$, so_n $(n \geq 5)$, sp_{2l} $(l \geq 1)$ are all simple.

Notes 1) g simple $\Rightarrow [g,g] = g$.

2) g simple $\Rightarrow$ Every rep of g is either faithful, or

is a direct sum of trivial reps.

So, g simple $\Leftrightarrow$ Adjoint rep is irred.

DEF 7.3 Lie algebra g semi-simple if is direct sum of simple ideals, i.e. ideals which are simple when viewed as a Lie algebra.

Example $so_4 \simeq sl_2 \oplus sl_2$.

DEF 7.4 Central Series (lower central series): for g Lie algebra,

$$\boxed{g^0 \supseteq g^1 \supseteq g^2 \supseteq \ldots} = g \supseteq [g,g] \supseteq [g,[g,g]] \supseteq \ldots$$

$g^0 \equiv g$ & $g^n \equiv [g, g^{n-1}]$.

DEF 7.5 Derived Series (upper central series): for g Lie algebra,

$$\boxed{g^{(0)} \supseteq g^{(1)} \supseteq g^{(2)} \supseteq \ldots} = g \supseteq [g,g] \supseteq [[g,g],[g,g]] \supseteq \ldots$$

$g^{(0)} \equiv g$ & $g^{(n)} = [g^{(n-1)}, g^{(n-1)}]$.

Note ⊛ $g^{(n)} \subseteq g^n$, ⊛ $g^n, g^{(n)}$ ideals.

Proof (for 2nd bullet): Induction on n ($n=0$ is trivial)

Let: $x, y \in g, g^n$ resp. $\Rightarrow [x,y] \in g^{n-1}$ by induction (since g^{n-1} already an ideal & $g^n \subseteq g^{n-1}$)

& $g^n = \{[x,y] : x \in g \text{ \& } y \in g^{n-1}\} \supseteq \{[x,y] : x \in g, y \in g^n\}$.

& $g^{(n)} = \{[x,y] : x, y \in g^{(n-1)}\}$, so if $y \in g^{(n)}$, $y = [w,z]$ for $w, z \in g^{(n-1)}$.

$\Rightarrow \forall x \in \mathfrak{g}: [xy] = (x[wz]) = -[w(xz)] - [z(xw)]$

By <u>induction</u>: $[zx], \cancel{z}[xw] \in \mathfrak{g}^{(n-1)}$. So, $[xy] \in \mathfrak{g}^{(n)}$ by induction

Examples ⊛ If $\mathfrak{g}$ simple: both series are $\mathfrak{g} \supseteq \mathfrak{g} \supseteq \mathfrak{g} \supseteq \ldots$

⊛ If $\mathfrak{g}$ abelian, both series are $\mathfrak{g} \supseteq 0 \supseteq 0 \supseteq \ldots$

⊛ Let $\mathfrak{n}_n \equiv \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ & & 0 \end{pmatrix} \right\} \subseteq \mathfrak{gl}_n$. Then: central series is:

$$\left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ & & 0 \end{pmatrix} \right\} \supseteq \left\{ \begin{pmatrix} 0 & 0 & & * \\ & \ddots & \ddots & \\ & & \ddots & 0 \\ & & & 0 \end{pmatrix} \right\} \supseteq \left\{ \begin{pmatrix} 0 & 0 & 0 & * \\ & \ddots & \ddots & 0 \\ & & \ddots & 0 \\ & & & 0 \end{pmatrix} \right\} \supseteq \ldots$$

For $n=3$: <u>Heisenberg Lie-algebra.</u>

DEF 7.6 If $\mathfrak{g}^n = 0$ for some n, then $\mathfrak{g}$ <u>nilpotent</u>

If $\mathfrak{g}^{(n)} = 0$ for some n, $\mathfrak{g}$ <u>solvable</u>.

Example $\mathfrak{b}_n \equiv \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \subseteq \mathfrak{gl}_n$ solvable, <u>not</u> nilpotent!

(for $n \geq 2$). (By above, nilpotent $\Rightarrow$ solvable.)

<u>Proof</u> Write (abuse of notation): h, for <u>image</u> of h in $\mathfrak{gl}_n$ under irred rep of $\mathfrak{sl}_2$, of dim n. Analogous for e.

(Both h, e in basis for which they take standard form)

Since $h \in \mathfrak{b}$: & $[he] = 2e$: by <u>induction</u>, $e \in \mathfrak{b}^n$. $\forall n$.

$\Rightarrow$ <u>Not</u> nilpotent.

However: diagonal of <u>bracket</u> of any 2 elements of $\mathfrak{b}$ is 0.

$\Rightarrow$ Derived series of $\mathfrak{b}$ (after 0'th term) is contained in central series for $\mathfrak{n}$, hence is 0 after n steps.

Theorem 7.7 (Lie's Theorem). Let: $k=\mathbb{C}$. & $g \leq gl(V)$ is a subalgebra. Suppose: g solvable. Then: $\exists$ common eigenvector of ALL elements of g.

$(\exists\, 0 \neq v \in V : (\forall x \in g), (\exists \lambda_x \in \mathbb{C}), xv = \lambda v.)$

Corollary 7.8 There is a basis for V, s.t. every element is upper-triangular. [Follows from: (7.7) + induction on V]

Using Lie's theorem: can show, $\exists\, 0 = V_0 \subseteq V_1 \subseteq \dots \subseteq V_n = V$ chain of subspaces of V, s.t. $\dim(V_i) = i \ \forall i$, and $g(V_i) \subseteq V_i \ \forall i$. If $V_i = \langle e_1, \dots, e_i \rangle$, g can be rep'ed by upper-Δ matrices. Using such basis: $g \leq b_n$.

maximal flag

Prop 7.9 Let $I, J \subseteq g$ ideals.

1) If g solvable $\Rightarrow$ Any sub-algebra & quotient is solvable.
2) If I solvable & g/I solvable $\Rightarrow$ g solvable
3) If I, J solvable $\Rightarrow$ $I+J$ solvable.

Proof 1) Clear from definition

2) Choose n: $(g/I)^n = 0$, so $g^n \subseteq I$.

Also: $g^{(n+m)} \subseteq I^{(m)} \ \forall m \geq 0$. Since I solvable: done!

3) $\dfrac{(I+J)}{J} \cong \dfrac{I}{I \cap J}$ & RHS solvable by 1).

& J solvable by assumption.

& $(I+J)/J$ solvable. So, done by 2).

DEF 7.10 The radical of g, $Rad(g)$, is maximal solvable ideal of g. (Sum of all solvable ideals).

DEF 7.11 If $\varphi: g \to gl(V)$ is f.d. rep of g, then the trace form of V is: $(\cdot,\cdot)_V : g \times g \to \mathbb{C}$

composition!

$$(x,y) \mapsto tr(\varphi(x)\varphi(y)).$$

Example 1) If φ rep $\Rightarrow (\cdot,\cdot)_V$ symmetric & bilinear.

2) $([xy],z) = (x,[yz])$ "invariance"

DEF 7.12 The Killing form $K(\cdot,\cdot) \equiv (\cdot,\cdot)_{ad}$ is the trace form attached to adjoint rep, i.e.

$$K(x,y) \equiv Tr(ad(x) \circ ad(y) : g \to g) \equiv (x,y)_{ad}.$$

Remark $Rad(g/Rad(g)) = 0$.

Since: a solvable ideal of $g/Rad\,g$ lifts to an ideal J containing $Rad(g)$ s.t. $J/Rad\,g$ solvable, so: by (7.9): J solvable $\Rightarrow J \subseteq Rad(g)$. $\Rightarrow J = Rad(g)$.

Hence: $g/Rad(g)$ semisimple.

Theorem (Levi's Theorem) If $char(k) = 0$ & g fin-dim then: $\exists$ Lie algebra g' s.t. $\boxed{g' \cap Rad(g) = 0 \text{ \& } g = g' \oplus Rad\,g}$.

$\Rightarrow g' \cong g/Rad\,g$ is semisimple.

Is called: Levi's Decomposition & g' is: Levi subalgebra of g.

Fails if $char(k) = p > 0$, and $dim(g) = \infty$.

Lie groups: lecture 7)

From last time) $\varphi : g \to gl(V)$ $(\dim V < \infty)$.

Trace form: $(X,Y)_V \equiv tr(\varphi(X)\varphi(Y))$.

If $\varphi \equiv ad \Rightarrow$ killing form: invariant, symmetric bilinear form.

& $Rad(g) \equiv$ maximal solvable ideal of g $(\dim g < \infty)$.

Theorem 7.13) For FD Lie algebra g, TFAE:

1) g semisimple

2) $Rad(g) = 0$

3) Killing form on g is non-degenerate (Cartan-Killing criterion)

Remark) $Rad(g/Rad(g)) = 0$.

(since: any soluble ideal of $g/Rad\, g$ would lift to an ideal J of g containing $Rad(g)$ which is solvable (7.9) so $J = Rad\, g$.)

$\Rightarrow g/Rad(g)$ is semisimple.

Theorem* (Levi's Theorem)) If $char(k) = 0$ & g is f.d., then: $\exists\, g' \subseteq g$ Lie sub-algebra s.t. $g' \cap Rad\, g = 0$ & $g = g' \oplus Rad\, g$.

$\Rightarrow g' \cong g/Rad(g)$ semisimple.

[$0 \to Rad(g) \to g \to g/Rad(g) \to 0$. Splits.]

Decomp is: Levi-decomposition & g' is Levi-subalgebra of g.

Result fails (char $= p$ or g inf-dim.)

Lemma 7.14) Let g be a Lie algebra.

1) If $I \subseteq g$ ideal, so is $[I,I]$.

2) $Rad(g) = 0 \iff g$ has no non-trivial abelian ideals.

Proof 1) If $x,y \in I$ & $z \in g$: need: $[z(x,y)] \in [II]$.

Have: (by Jacobi) $[z(xy)] = -[x(yz)] - [y(zx)]$.

For both the summands on RHS: both components of $[\cdot,\cdot]$ are in I, so both terms are in $[II]$

$\Rightarrow [z(xy)] \in [II]$ ✓

2) Any abelian ideal is solvable if: I is solvable. & the last nonzero term in the derived series for I is abelian.

Let $g^\perp \equiv \{x \in g : K(x,y) = 0 \ \forall y \in g\}$. Is an ideal.

[Take: $x \in g^\perp$, $y,z \in g$. Need to show: $K([xy]z) = 0$.

& $K([xy]z) = K(x,[yz]) = 0$ since $x \in g^\perp$.]

Lemma 7.15] Let: I ideal of g, & $K_I \equiv$ Killing form of I. Then: $\boxed{K_I(x,y) = K(x,y) \quad \forall x,y \in I.}$

Proof Choose basis of I & extend to basis of g.

For $x,y \in I$: WRT this basis, $ad(x) = \begin{pmatrix} A & * \\ 0 & 0 \end{pmatrix}$, $ad(y) = \begin{pmatrix} B & * \\ 0 & 0 \end{pmatrix}$

$\Rightarrow K_I(xy) = tr(AB) = tr(ad(x)ad(y))$. ✓

Theorem 7.16] (Cartan's Criterion for solvability)

Suppose: $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ subalgebra (V fin-dim / $\mathbb{C}$). Then, if $K(x,y) \equiv 0$ on $\mathfrak{g} \times [\mathfrak{g}\mathfrak{g}]$, then $\mathfrak{g}$ is solvable. ($\mathfrak{g}^{(1)} \subseteq \mathfrak{g}^{\perp}$)

(no proof given)

Corollary 7.17) 1) If $\mathfrak{g} = \mathfrak{g}^{\perp} \Rightarrow \mathfrak{g}$ solvable

2) $\mathfrak{g}$ simple $\Rightarrow \mathfrak{g}^{\perp} = 0$

3) $\mathfrak{g}^{\perp}$ solvable $\forall \mathfrak{g}$ f.d. Lie algebra.

Proof) 1) Consider: $ad: \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$. ~~$\Rightarrow$ Image $ad(g) = g^\perp$~~

Has image $ad(\mathfrak{g}) = \mathfrak{g}/Z(\mathfrak{g})$ & $Z(\mathfrak{g})$ solvable (abelian).

Since $\mathfrak{g} = \mathfrak{g}^{\perp}$: (7.16) $\Rightarrow ad(\mathfrak{g})$ solvable, so $\mathfrak{g}$ solvable.

2) $\mathfrak{g}^{\perp}$ is an ideal, so either $\mathfrak{g}^{\perp} = 0$ (✓) or $\mathfrak{g}^{\perp} = \mathfrak{g}$, but then by 1) $\mathfrak{g}$ solvable ↯ (since $\mathfrak{g}$ simple)

3) $(\mathfrak{g}^{\perp})^{\perp} = \mathfrak{g}^{\perp}$ (by 7.15) so $\mathfrak{g}^{\perp}$ solvable by 1).

Proof of (7.13)) 2) $\Rightarrow$ 3): $Rad(\mathfrak{g}) = 0$ & $\supseteq \mathfrak{g}^{\perp}$ solvable $\Rightarrow$ have $\mathfrak{g}^{\perp} \subseteq Rad(\mathfrak{g}) = 0$. Hence, non-degenerate.

3) $\Rightarrow$ 2): let A abelian ideal of $\mathfrak{g}$. $\boxed{\text{Claim: } A \subseteq \mathfrak{g}^{\perp}}$.

For $x \in A$, $y \in \mathfrak{g}$: choose basis of A & extend to basis of $\mathfrak{g}$.

$$\Rightarrow Ad(x) = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \quad \& \quad Ad(y) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

$\Rightarrow tr(Ad(x)\, Ad(y)) = 0$, ~~so $A = 0$ ✓~~ Hence, if $Rad(\mathfrak{g}) \neq 0$ then $\mathfrak{g}^{\perp} \neq 0$. $\Rightarrow$ contains abelian ideal.

2) + 3) $\Rightarrow$ 1) If g simple $\Rightarrow$ ✓

Else: pick minimal (nontrivial) ideal ~~of~~ $I \subseteq g$. Then:

$g_I \equiv \{x \in g : K(x,y) = 0 \ \forall y \in I\}$ is an ideal of g.

Claim: $I \oplus g_I = g$.

Since I simple, $\Rightarrow$ by minimality, & non-abelian [redacted]

$\Rightarrow I \cap g_I \subseteq I^\perp = 0$.

Consider: map

$$\begin{bmatrix} g \xrightarrow{\cong} g^* \xrightarrow{Res} I^* \\ x \longmapsto K(x,\cdot) \longmapsto K|_{I^*}(x,\cdot) \end{bmatrix}$$

Kernel of entire map is g_I.

Proves: claim 1.

Repeat this argument, with g_I. (choosing some minimal ideal of g_I.)

Can do this, since:

any ideal of g_I is ideal of g. $\Rightarrow Rad(g_I) = 0$.

Claim 2: $(g_I)^\perp = 0$, since: if $x \in (g_I)^\perp \Rightarrow x \in g^\perp$.

1 $\Rightarrow$ 2: Write: $g \equiv \bigoplus I_j$ (simple ideal sum)

& $\pi_j : g \to I_j$ projection onto j-th term.

Exercise: If J ideal of g then $\pi_j(J)$ ideal of I_j.

So if $A \subseteq g$ abelian ideal of g, then $\pi_j(A)$ abelian ideal of I_j, so $\pi_j(A) = 0 \ \forall j \Rightarrow \underline{A = 0}$. ✓

Lie Groups. Lecture 8.

Theorem 7.18 (Weyl). Any FD rep. of semisimple Lie alg is completely reducible.

Proof Is almost same as $sl_2(\mathbb{C})$. Main ingredient is a 2nd version of Casimir element.

Exercise: Any ideal or quotient of semisimple Lie alg. is semisimple.

Consider: $\varphi: g \to gl(V)$, fin-dim irred rep. of simple Lie alg. g.

WLOG: φ faithful. (otherwise, work with ~~$g/\ker\varphi$~~ $g/\ker\varphi$.)

By Cartan-Killing: $(\cdot,\cdot)_V$ non-degenerate.

Pick: $X_1, \ldots, X_n$ basis of g : since $(\cdot,\cdot)_V$ non-degen, can pick a dual basis $Y_1, \ldots, Y_n$.

DEF. 7.19 Let: $\Omega_\varphi \equiv \sum_i \varphi(X_i)\varphi(Y_i)$. "the Casimir element" of φ

⊛ Ω_φ endomorphism of V

⊛ Ω_φ commutes with $\varphi(X)$ $\forall X \in g$.

By Schur's lemma: Ω_φ scalar mult. of id_V, and has trace $\sum_i tr(\varphi(X_i)\varphi(Y_i)) = \dim(g)$ (check)

& clear that Ω_φ indep of choice of basis $\{X_i\}$ we chose.

Example $g \equiv sl_2 \subseteq gl_2$. & $V \equiv \mathbb{C}^2$. $\varphi \equiv id: g \to gl(V)$.

$\Rightarrow$ Basis $\left\{ e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$ of g

& Dual basis WRT trace form: $\{f, \frac{1}{2}h, e\}$.

$\Rightarrow \Omega_\varphi = ef + \frac{1}{2}h^2 + fe = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix} = \frac{3}{2} I. = \frac{\dim g}{\dim V} I.$

§8: Jordan Decomposition.

2 observations:

1) If g simple Lie alg & $\varphi : g \to gl(V)$ f.d. rep, then: $\varphi(g) \subseteq sl(V)$. [Because: simple $\Rightarrow [gg] = g$, so:

$$\varphi(g) = \varphi([gg]) = [\varphi(g)\varphi(g)] \in [gl(V), gl(V)] = sl(V).]$$

2) Recall: if $X \in gl(V)$: $\exists$ basis of V, s.t. X is block diagonal with "Jordan blocks" of form: $\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$

DEF 8.1) Say x nilpotent if: $\exists n$, $X^n = 0$.

& X semisimple if ~~the~~ roots of its min poly are distinct (i.e. diagonalisable)

~~DEF~~ PROP 8.2) If $X \in gl(V)$ & V f.d. then

1) $\exists$ unique $x_s, x_n \in gl(V)$, x_s semisimple, x_n nilpotent, $x = x_s + x_n$ & $[X_s, X_n] = 0$

2) $\exists$ polys $p_s, p_n \in \mathbb{C}[T]$ (no const terms) with: $X_s = p_s(X)$, $X_n = p_n(X)$. In particular: X_s, X_n commute with any endo. commuting with X.

3) If $A \subseteq B \subseteq V$ subspaces: & $x : B \to A$, then x_s, x_n are also $B \to A$.

The decomp. $x = x_s + x_n$ is additive Jordan decomp. of x. & X_s, X_n are semisimple & nilpotent parts of X. (resp)

Proof Routine linear algebra.

Example If x is rep'ed by a single Jordan block $\begin{pmatrix}\lambda & 1 & & 0\\ & \ddots & \ddots & \\ & & \ddots & 1\\ 0 & & & \lambda\end{pmatrix}$

then: x_s is $\begin{pmatrix}\lambda & & \\ & \ddots & \\ & & \lambda\end{pmatrix}$ & $x_n = \begin{pmatrix}0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1\\ & & & 0\end{pmatrix}$.

Why is this valuable / of interest?

Suppose: $\mathfrak{g}$ f.d. Lie-alg. Then: $\exists$ adjoint rep. $ad: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$.

If $x \in \mathfrak{gl}(V)$ nilpotent, so is $ad(x)$.
If $x \in \mathfrak{gl}(V)$ semisimple, so is $ad(x)$.

Lemma 8.3 Let $x \in \mathfrak{g} \leq \mathfrak{gl}(V)$ (V fin-dim). & take Jordan decomp $x = x_s + x_n$. Then: $ad(x) = ad(x_s) + ad(x_n)$ is Jordan decomp. of $ad(x)$. (in $\mathfrak{gl}(V)$)

Proof $ad(x_s)$ & $ad(x_n)$ are semisimple & nilpotent resp. & commute, since: $[ad(x_s), ad(x_n)] = ad([x_s, x_n]) = 0$.

$\Rightarrow$ By uniqueness (in prop) this is the Jordan decomp.

Theorem 8.4 $\mathfrak{g}$ semisimple & subalg of $\mathfrak{gl}(V)$.

Take $x \in \mathfrak{g}$. Then: $x_s, x_n \in \mathfrak{g}$.

Proof ~~Let~~ $\cancel{N(\mathfrak{g}) = \{y \in \mathfrak{gl}(V) \mid [yz] \in \mathfrak{g} \ \forall z \in \mathfrak{g}\}}$. ~~"Normaliser of $\mathfrak{g}$".~~

~~Claim 1) $N(\mathfrak{g})$ is subalgebra of $\mathfrak{gl}(V)$, and moreover contains $\mathfrak{g}$ as an ideal.~~

~~& 2) $x_s, x_n \in N(\mathfrak{g})$.~~

$N(g) \equiv \{ y \in gl(V) : (yz) \in g \ \forall z \in g \}$.

Claim: 1) $N(g)$ subalg. of $gl(V)$ & has g as an ideal

2) $x_s, x_n \in N(g)$.

poly of ad(x), no const term. (8.2) 2)

1) clear (def. of $N(g)$)

2) $\forall z \in g$, $[x_s, z] = ad(x_s)(z) = (ad.x)_s(z). \in g$

let $W \leq V$ irred subrep & define:

$g_W \equiv \{ y \in gl(V) : yw \in W \ \forall w \in W \ \& \ tr(y|_W) = 0 \}$.

Claim: ~~$g \leq g_W$~~ $\boxed{g \leq g_W.}$ subalgebra.

Know: W subrep ~~&~~ $\Rightarrow$ stabilised by g.

& image of g in $gl(W)$, say $\bar{g}$, is also semisimple.

$\Rightarrow [\bar{g}\,\bar{g}] = \bar{g}$. So, any element of $\bar{g}$ is sum of commutators,

all whose traces are 0. Hence, $g \leq g_W$.

$\Rightarrow \forall y \in g : tr(y|_W) = 0$

Note: x_s, x_n polys in x, so stabilise everything ~~..~~ that x does.

& ~~..~~ $tr(x_n|_W) = 0$, since: $x_n|_W$ nilpotent.

$\Rightarrow tr(x_s|_W) = tr(x|_W) - tr(x_n|_W) = 0$, so: $x_s, x_n \in g_W \ \forall W$.

To finish: define $g' \equiv N(g) \cap \left(\bigcap_{W \leq V \text{ irred}} g_W \right)$.

Claim: $g' = g$.

Lie Groups: Lecture 9 $\quad \mathfrak{g} \subseteq \mathfrak{g}' \subseteq N(\mathfrak{g}) \subseteq \mathfrak{gl}(V)$.

Claim: $\mathfrak{g}' = \mathfrak{g}$.

Know: $\mathfrak{g} \subseteq \mathfrak{g}' \subseteq N(\mathfrak{g})$ (by defn), so: $\mathfrak{g}$ is an ideal of $\mathfrak{g}'$.

$\Rightarrow \mathfrak{g}$ subrep of $\mathfrak{g}'$ under adjoint rep. of $\mathfrak{g}$.

By Weyl (7.18): $\mathfrak{g}' = \mathfrak{g} \oplus U$ (as reps. of $\mathfrak{g}$), need $U = 0$.

Take: $u \in U$. Have: $[u\mathfrak{g}] \subseteq \mathfrak{g}$ as $\mathfrak{g}$ ideal of $\mathfrak{g}'$.

But: $ad(\mathfrak{g})U \subseteq U$. Hence: $[u\mathfrak{g}] \in U$. Since sum direct:

$[u\mathfrak{g}] \in \mathfrak{g} \cap U = 0$, Thus: u commutes with all elements of $\mathfrak{g}$.

$\Rightarrow u$ is $\mathfrak{g}$-homomorphism of V

$\Rightarrow u$ stabilises all irred subreps W.

By Schur: $u|_W = \lambda \cdot id_W$ some λ. But, $tr(u|_W) = 0$ for any irred $W \leq V$, since $u \in \mathfrak{g}_W$ $\forall W$.

$\Rightarrow tr(\lambda \cdot id_W) = 0$, so $\lambda = 0$. So, $u|_W = 0$ $\forall W$ irred.

Finally: $V = \oplus V_i$ for irred V_i, and so $u = 0 \Rightarrow U = 0$ ✓

For $\mathfrak{g}$ as in statement: can define "abstract" JD by:

$ad(X) = (ad\, X)_s + (ad\, X)_n$. Since ad faithful: for $\mathfrak{g}$ semisimple, get: $\mathfrak{g} \cong ad(\mathfrak{g}) \subseteq \mathfrak{gl}(\mathfrak{g})$.

By (8.4): $(ad\, X)_s, (ad\, X)_n \in \mathfrak{g}$, hence are of form $ad(X_s)$, $ad(X_n)$ for some elements $X_s, X_n \in \mathfrak{g}$.

$\Rightarrow ad(X) = ad(X_s + X_n) \underset{\text{faithful}}{\Rightarrow} X = X_s + X_n$.

Suppose $\mathfrak{g} \subseteq \mathfrak{gl}(V)$. Write: $X = X_s + X_n$. Then by (8.3),
$ad(X_s) = (ad\,X)_s$ & $ad(X_n) = (ad\,X)_n$.
$\Rightarrow$ "abstract" decomp is as just given & agrees with initial def.

$\boxed{\text{Corollary 8.5}}$ Let $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ rep. of semisimple Lie alg. $\mathfrak{g}$. & $X \in \mathfrak{g}$. have Jordan decomp $X = X_s + X_n$.
Then: $\varphi(X)_s = \varphi(X_s)$ & $\varphi(X)_n = \varphi(X_n)$.
Defines JD of $\varphi(X)$.

Proof (David Stuart)

---

§10: Cartan Subalgebras & Root ~~systems~~ space decomposition.

Here: $\mathfrak{g}$ f.d. & semisimple over $\mathbb{C}$.

$\boxed{\text{DEF 10.1}}$ A subalgebra $\underline{\mathfrak{t}}$ of $\mathfrak{g}$ is toral if:

- $\underline{\mathfrak{t}}$ abelian
- $ad(X)$ semisimple $\forall X \in \underline{\mathfrak{t}}$.

A maximal, toral subalgebra is "maximal torus". / [Cartan Subalgebra]

$\boxed{\text{Remark}}$ Many authors define CSA's as nilpotent subalgebra that equals its normaliser in $\mathfrak{g}$.
$\{x : [x\mathfrak{t}] \in \mathfrak{t}\} = \mathfrak{t}$. [Equivalent to Def 10.1]

$\boxed{\text{Example}}$ 1) $\mathfrak{g} \subseteq \mathfrak{sl}_n$, $\mathfrak{t} \equiv$ diagonal matrices. $\Rightarrow$ Is maximal torus.
2) Similar for $\mathfrak{so}_n$ & $\mathfrak{sp}_{2n}$.

$\boxed{\text{Lemma 10.2}}$ Let: $t_1, \dots, t_n : V \to V$ commuting $(t_i t_j = t_j t_i)$ semisimple endomorphisms of V (f.d.). & $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$

Set: $V_\lambda \equiv \{v \in V : t_i(v) = \lambda_i(v) \quad \forall 1 \leq i \leq n\}$.

Then: $V = \bigoplus_{\lambda \in \mathbb{C}^n} V_\lambda$. (simultaneous eigenspaces)

Proof Induction on n.

$n=1$: Follows, since t_1 semisimple $\Rightarrow$ by diagonalisability, V has a t_1-eigenbasis.

$n>1$: by induction, $V = \bigoplus_{\lambda' \in \mathbb{C}^{n-1}} V_{\lambda'}$ (for action $t_1, \dots, t_{n-1}$).

Since t_i commute with t_n: $\Rightarrow t_n(V_{\lambda'}) \subseteq V_{\lambda'}$.

$\Rightarrow$ By decomposing each $V_{\lambda'}$ for t_n (as in $n=1$ case), get inductive step ✓

Lemma 10.3 Any $\mathfrak{g}$ contains a CSA.

Proof* Need Engel's Theorem & Zorn's Lemma.

Recasting (10.2): suppose $\underline{h} \leq \mathfrak{gl}(V)$, with basis of commuting semisimple $t_1, \dots, t_n$. & $\lambda \in \mathbb{C}^n$ corresponds to the element of h^* by: $[t_i \longmapsto \lambda_i]$.

Then: $V_\lambda = \{v \in V : hv = \lambda(h)(v) \quad \forall h \in \underline{h}\}$

In our situation: fix $\underline{t} \leq \mathfrak{g}$ CSA. Then: $\mathfrak{g} = \bigoplus_{\lambda \in \underline{t}^*} \mathfrak{g}_\lambda$

Where: $\mathfrak{g}_\lambda = \{x \in \mathfrak{g} : [tx] = \lambda(t)x \quad \forall t \in \underline{t}\}$.

Called: Root Spaces.

Lie Groups: Lecture 10

27/10/2023

From last time: V is rep. of semisimple g. Decompose V into simultaneous e-spaces for t: $V = \bigoplus_{\mathbb{C}^n} V_\lambda$ (Weight Spaces)

For $v \in V_\lambda$: $hv = \lambda(h)(v)$, some $\lambda : t \to \mathbb{C}$. $(\lambda \in t^*)$

If $t \leq g$ is CSA: $g = \bigoplus_{t^*} g_\lambda$, $g_\lambda \equiv \{x \in g : [tx] = \lambda(t)x \ \forall t \in t\}$

DEF 10.4 Let $\Phi \equiv \{\alpha \in t^* \setminus 0 : g_\alpha \neq 0\}$.

$\Rightarrow$ Elements of Φ are roots of g WRT t.

If $\alpha \in \Phi$: g_α is root space, and: $g = g_0 \oplus \sum_{\alpha \in \Phi} g_\alpha$. Is: root space decomp (Cartan decomp) of g.

Prop 10.5 1) $\forall \alpha, \beta \in t^*$: $[g_\alpha g_\beta] \subseteq g_{\alpha+\beta}$

2) $\forall \alpha \in \Phi$: $\forall x \in g_\alpha$: $ad(x)$ nilpotent

3) If $\alpha + \beta \neq 0$ then $K(g_\alpha, g_\beta) = 0 \ \forall \alpha, \beta \in t^*$. "Orthogonality"

Proof 1) Take $x \in g_\alpha$, $y \in g_\beta$. $t \in t$. By Jacobi identity:

$$[t[xy]] = -[x[yt]] - [y[tx]]$$
$$= [xy]\alpha(t) + [xy]\beta(t) = (\alpha+\beta)(t)([xy]) \checkmark$$

("Fundamental Calculation") (don't quite know why)

2) Follows from 1) & fin-dim of g

3) If $\alpha + \beta \neq 0 \Rightarrow \exists t \in t : (\alpha+\beta)(t) \neq 0$. Fix such t.

Fix $x \in g_\alpha$ & $y \in g_\beta$. Then:

$\alpha(t)\, K(x,y) = K([tx], y) = -K([xt], y) = -K(x, [ty])$
$= -\beta(t) K(x,y)$. So, $(\alpha+\beta)(t)\cdot K(x,y) = 0 \Rightarrow \underline{K(x,y) = 0}$

Corollary 10.6 1) $K|_{g_0 \times g_0}$ non-degenerate.

2) $\alpha \in \Phi \Rightarrow -\alpha \in \Phi$

Proof 1) If $z \in g_0$ & $K(z,x) = 0\ \forall x \in g_0$, then by (10.5), g_0 is orthogonal to $g_\alpha\ \forall \alpha \neq 0$. If $x \in g$, then: can write $x = x_0 + \sum_{\alpha \in \Phi} x_\alpha$, $x_\alpha \in g_\alpha$.

$\Rightarrow$ By linearity, $K(z,x) = 0\ \forall x \in g_\alpha$, and so, since K non-degenerate, get $\underline{z = 0}$ ✓ Similar argument for 2)

Prop 10.7 $g_0 = t$. [Humphreys, 8.2]

Corollary 10.8 $K|_{t \times t}$ is non-degenerate. In particular,
$$\left[\begin{array}{l} t \longrightarrow t^* \\ x \longmapsto K(x, \cdot) \end{array}\right]$$
is isomorphism with inverse $\lambda \mapsto t_\lambda$.
Call: $t_\lambda \in t^*$ Co-root. Defined by:
$K(t_\lambda, x) = \lambda(x)\ \forall x \in t$.

Exercises 1) $g = sl_2$. $\Rightarrow t = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle = \langle h \rangle$.
Define: $\alpha \in t^*$ by: $\alpha(h) = 2$. Then, $g_\alpha = \langle e \rangle$, and $g_{-\alpha} = \langle f \rangle$, & $sl_2 = t \oplus (g_\alpha \oplus g_{-\alpha})$.
Is: Root space decomp, for sl_2.

2) $\mathfrak{g} = sl_3$. Take: $h_1 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$ & $h_2 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}$.

$\Rightarrow$ Gives basis for diagonals, $\underline{t} = \langle h_1, h_2 \rangle$.

Let: $\alpha_i \in \underline{t}^*$ be, s.t. $\alpha_i \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} = a_i$.

Then: $sl_3 = \underline{t} \oplus \left[\mathfrak{g}_{\alpha_1 - \alpha_2} \oplus \mathfrak{g}_{\alpha_1 - \alpha_3} \oplus \mathfrak{g}_{\alpha_2 - \alpha_3} \right]$
$\oplus \mathfrak{g}_{\alpha_2 - \alpha_1} \oplus \mathfrak{g}_{\alpha_3 - \alpha_1} \oplus \mathfrak{g}_{\alpha_3 - \alpha_2}$

where: $\mathfrak{g}_{\alpha_i - \alpha_j} \equiv \langle e_{ij} \rangle$. Gives: Dimension 8 ✓

Try decomposing adjoint this way.

3) Try sl_n (similar), $\underline{t}$ = diagonals in $\mathfrak{g}$.

& Try: $\mathfrak{g} = so_{2n}$ & so_{2n+1} & sp_{2n}.

Prop 10.9 If $\alpha \in \Phi$ & $e_\alpha \in \mathfrak{g}_\alpha$. Then: $\exists f_\alpha \in \mathfrak{g}_{-\alpha}$ s.t. $\langle e_\alpha, f_\alpha, h_\alpha \equiv [e_\alpha f_\alpha] \rangle \cong sl_2$. Call it $\underline{m}_\alpha$.

$\Rightarrow$ Every semisimple Lie alg. is: "made up" of sl_2's.

First: note that if $t \in \underline{t}$ satisfies $\alpha(t) = 0 \ \forall \alpha \in \Phi$, then $t=0$. This is because $\forall x \in \mathfrak{g}_\alpha$, $0 = \alpha(t)x = [t, x]$, & since toral subalgebra Abelian, get: $[t, x] = 0 \ \forall x \in \mathfrak{g}$
$\Rightarrow t \in Z(\mathfrak{g}) = 0$.

Lemma 10.10 Φ spans $\underline{t}^*$.

Proof If not, then $\exists t \neq 0$ s.t. $\alpha(t) = 0 \ \forall \alpha \in \Phi$ $\#$.

Claim 1] $[g_\alpha \, g_{-\alpha}]$ 1-dim.

Proof] Take $x \in g_\alpha$, $y \in g_{-\alpha}$. Then, $[xy] \in \underline{t}$. Then, let $t \in \underline{t}$. Then $K([xy], t) = K(x, [yt])$

$= -K(x, [ty]) = \alpha(t) K(x, y)$

$\Rightarrow [xy] = K(x, y) t_\alpha \in \langle t_\alpha \rangle$, so $\dim [g_\alpha, g_{-\alpha}] \leq 1$.

But then K non-degen $\Rightarrow \exists x, y$ s.t. $K(x, y) \neq 0$, so $\dim = 1$.

Claim 2] $\alpha(t_\alpha) \neq 0$.

Proof Since K non-degen: by rescaling, can assume:

$x \in g_\alpha$ & $y \in g_{-\alpha}$, with $K(x, y) = 1$. Then:

$[xy] = t_\alpha$, $[t_\alpha x] = \alpha(t_\alpha) x$ & $[t_\alpha y] = -\alpha(t_\alpha) y$.

$\Rightarrow \langle x, y, t_\alpha \rangle$ is subalgebra $\underline{h} \subseteq g$.

If $\alpha(t_\alpha) \neq 0$, then: $[\underline{h}\,\underline{h}] = \langle t_\alpha \rangle \Rightarrow \underline{h}$ solvable.

$\Rightarrow ad : g \longrightarrow gl(g)$

& $\underline{h}$ embeds to solvable subalgebra in $gl(g)$, WRT which $ad(\underline{h}) \subseteq \{\text{Upper } \Delta \text{ matrices}\}$, and $ad(t_\alpha) = [ad\, x, ad\, y]$
(Lie's Theorem) strictly upper Δ.

$\Rightarrow ad(t_\alpha)$ is Nilpotent. But by semisimplicity: $ad(t_\alpha) = 0$

$\Rightarrow t_\alpha \in Z(g) = 0$ #

Claim 3] $[[g_\alpha \, g_{-\alpha}], g_\alpha] \neq 0$. Because: $\forall x \in g_\alpha$, $y \in g_{-\alpha}$, $K(x, y) \neq 0$ then $\forall z \in g_\alpha$, $[[xy] z] = [K(x, y) t_\alpha, z] = K(x, y) \alpha(t_\alpha) z \neq 0$ $\forall z \neq 0$.

Lie Algebras: Lecture 11 30/10/2013.

From last time: g semisimple Lie alg. Try: to find a ~~set~~ set of subalgebras, each $\cong sl_2$. Pick: $\alpha \in \Phi$.

1) $\exists t_\alpha \in T$, s.t. $K(t_\alpha, t) = \alpha(t)$ $(\forall t \in T)$
& $[xy] = K(x,y)\, t_\alpha$ $(\forall x \in g_\alpha)(y \in g_{-\alpha})$

2) $\alpha(t_\alpha) \neq 0$ $\quad [K([xy]t) = \alpha(t) K(x,y)]$

3) $[[g_\alpha\ g_{-\alpha}]\ g_\alpha] \neq 0$.

4) $\alpha \in \Phi$, $e_\alpha \in g_\alpha \Rightarrow \exists f_\alpha \in g_{-\alpha}$, with:
$m_\alpha \equiv \langle e_\alpha, f_\alpha, h_\alpha \equiv [e_\alpha f_\alpha] \rangle \cong sl_2$.

Finally: take $e_\alpha \in g_\alpha$ & find $f_\alpha \in g_{-\alpha}$ with: $K(e_\alpha, f_\alpha) = \dfrac{2}{\alpha(t_\alpha)}$.

Define: $h_\alpha \equiv \dfrac{2}{K(t_\alpha, t_\alpha)} \cdot t_\alpha$

$\Rightarrow$ Check sl_2-relations: $[e_\alpha f_\alpha] = K(e_\alpha f_\alpha) t_\alpha = h(t_\alpha)$.
$[h_\alpha e_\alpha] = \dfrac{2}{\alpha(t_\alpha)} [t_\alpha e_\alpha] = 2e_\alpha$.
$[h_\alpha f_\alpha] = -2f_\alpha$.

$\Rightarrow$ Indeed $\cong sl_2$. Call: $\{e_\alpha, f_\alpha, h_\alpha\}$ a sl_2-tuple.

Exercise Weights "add": if g semisimple, & have root space decomp $g = t \oplus \bigoplus_\alpha g_\alpha$ & V, W reps with corresponding weight spaces V_α, W_α resp:

1) $g_\alpha V_\beta \subseteq V_{\alpha+\beta}$ 2) $V_\alpha \otimes W_\beta \subseteq (V \otimes W)_{\alpha+\beta}$.

Lemma 10.11 1) If V fin-dim rep, then $V|_{m_\alpha}$ is f.-d. rep.
2) If $\beta \in \Phi$ or $\beta = 0$, then let $V \equiv \bigoplus_{c \in \mathbb{C}} g_{\beta + c\alpha}$. (Sum ranges over $c \in \mathbb{C}$, with $\beta + c\alpha \in \Phi$.)
This is a rep of m_α, by restriction of adjoint.

Proof 1) clear by generic facts about restrictions
2) clear from (10.5).

Call: V the α-root string through β.

Prop 10.12 let $\alpha \in \Phi$. The root spaces $g_{\pm\alpha}$ are 1-dim.
Moreover: the multiples of α, which lie in Φ, are $\pm\alpha$.

Proof If $c\alpha \in \Phi$ then h_α takes: $c \cdot \alpha(h_\alpha) \equiv 2c$ as an eigenvalue. So, since eigenvalues of h_α integral: $2c \in \mathbb{Z}$.
Let: $V \equiv t \oplus \bigoplus_{c\alpha \in \Phi} g_{c\alpha}$. & $K \equiv \ker(\alpha) \subseteq t$. Check:
$K \oplus m_\alpha$ is a m_α-subrep. of V.
$\Rightarrow$ By Weyl's theorem: as a m_α-rep, $V = K \oplus m_\alpha \oplus W$.
(W is complementary subrep). Assume: either of conditions in statement fail $\Rightarrow W \neq 0$.
Let: $W_0 \equiv V(s)$ some irred rep of W.
$\Rightarrow W_0$ has HWV (w_0) & $w_0 \in g_{c\alpha}$, ~~since~~ some c.
& $[h_\alpha, w_0] = s w_0$, some s.

$$\begin{pmatrix} s & & \\ & s-2 & \\ & & \ddots \\ & & & -s \end{pmatrix}$$

Case 1) s even $\Rightarrow$ 0 eigenvalue.

12

Call e corresponding eigenvec (for 0).

The 0-eigenspace of h_α on V is t, contained in $K \oplus m_\alpha$.

$\Rightarrow e \in (K \oplus m_\alpha) \cap W_0 = 0$ ↯

Aside: If $2\alpha \in \Phi$ then h_α has $2\alpha(h_\alpha) = 4$ as eigenvalue. The eigenvals of h_α on $K \oplus m_\alpha$ are $0, \pm 2$, so this happens only if: W contains $V(s)$, s even. ↯

$\Rightarrow$ "2x root never a root"

Case 2: s odd. $\Rightarrow h_\alpha$ has eigenvalue 1. Since $\alpha(h_\alpha) = 2$: means $\frac{1}{2}\alpha$ root of g $\Rightarrow \alpha$ not root of g ↯

Exercise: $m_\alpha \cong m_{-\alpha}$ & $h_\alpha = -h_{-\alpha}$.

Prop 10.13 For $\alpha, \beta \in \Phi$ $(\beta \neq \pm\alpha)$:

1) $\beta(h_\alpha) \in \mathbb{Z}$ (Cartan integers)

2) $\exists$ integers $p, q \geq 0$ s.t. $\forall r \in \mathbb{Z}$: $\beta + r\alpha \in \Phi \Leftrightarrow -p \leq r \leq q$.
& $p - q = \beta(h_\alpha) = \langle \beta, \alpha^\vee \rangle$.

3) $[g_\alpha\, g_\beta] = g_{\alpha+\beta}$.

Proof 1) Consider $V = \bigoplus_{r \in \mathbb{Z}} g_{\beta + r\alpha}$ & let m_α act on V (by restriction). & $q \equiv \max\{r : \beta + r\alpha \in \Phi\}$.

For $v \in g_{\beta + q\alpha}$ $(v \neq 0)$: $(e_\alpha \cdot v) \in g_{\beta + (q+1)\alpha} = 0$
& $[h_\alpha \cdot v] = (\beta + q\alpha)(h_\alpha)(v) \in \langle v \rangle$,

$\Rightarrow v$ is HWV, weight $(\beta+q\alpha)(h_\alpha)$.

By known theory of sl_2: $\beta(h_\alpha)+q\alpha(h_\alpha)\in\mathbb{Z}_{\geq 0}$.

$\Rightarrow \beta(h_\alpha)+2q\in\mathbb{Z}_{\geq 0}$, hence $\underline{\beta(h_\alpha)\in\mathbb{Z}}$.

$\underline{\underline{2)}}$ By (4.8): $W\equiv\langle v, f_\alpha v, f_\alpha^2 v, \ldots\rangle$ is: $\underline{\text{irred}}$ rep of V

& h_α acts on it by: $\begin{pmatrix} (\beta+q\alpha)(h_\alpha) & & & \\ & (\beta+(q-1)\alpha)(h_\alpha) & & \\ & & \ddots & \\ & & & -(\beta+q\alpha)(h_\alpha) \end{pmatrix}$

$\Rightarrow W=\sum_{r=-p}^{q} g_{\beta+r\alpha}$ for some p $(=\max\{r : \beta-r\alpha\in\Phi\})$.

Suppose: $W'\subseteq V$ subrep $(\neq W)$ $\Rightarrow W'$ contains a HWV $w\in g_\gamma$ (for some γ). $\Rightarrow \gamma(h_\alpha)<-(\beta+q\alpha)(h_\alpha)\leq 0$ ↯.

Finally: $(\beta-p\alpha)(h_\alpha)=-(\beta+q\alpha)(h_\alpha)\Rightarrow p-q=\beta(h_\alpha)$.

$\underline{\underline{3)}}$ next time!

Lie Algebra: Lecture 12

From last time: $V = \bigoplus_{r\in\mathbb{Z}} \mathfrak{g}_{\beta+r\alpha}$ "α-string through β".

This: is irred rep, of $\mathfrak{m}_\alpha$

& $\{\beta + r\alpha : r\in\mathbb{Z}\} \cap \Phi$ of form: $\{\beta - p\alpha, \to \beta + q\alpha\}$

for $p, q \in \mathbb{Z}_{\geq 0}$ & $p - q = \beta(h_\alpha). = \langle \beta, \alpha^\vee\rangle$

Proof Continued (of $[\mathfrak{g}_\alpha \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$)

Know from before: $[\mathfrak{g}_\alpha \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$. $\Rightarrow$ If $\mathfrak{g}_{\alpha+\beta} = 0$: done ✓

If $\alpha+\beta\in\Phi$, then: take $0\neq v\in\mathfrak{g}_\beta$. If $[e_\alpha, v] = 0$,

then v is a HWV for V #

$\Rightarrow [e_\alpha, v] \neq 0$, and $\mathfrak{g}_{\alpha+\beta}$ is spanned by this ✓

DEF 10.14) For $\alpha\in\Phi$, define Reflection at α, by:

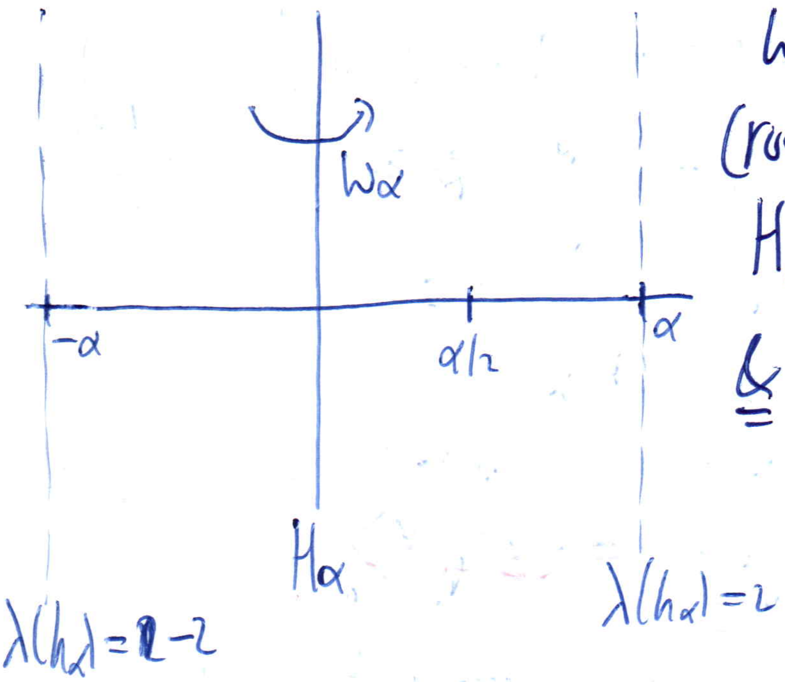
$$\left[\begin{array}{l} W_\alpha : \mathfrak{t}^* \longrightarrow \mathfrak{t}^*, \\ \qquad \beta \longmapsto \beta - \beta(h_\alpha)\alpha. \end{array}\right]$$

Corollary 10.15) $W_\alpha(\Phi) = \Phi$

Proof Let $\beta\in\Phi$ & p, q be as in 10.13. Need: $\beta - \beta(h_\alpha)\alpha \in \Phi$

Know: $\beta - \beta(h_\alpha)\alpha = \beta - (p-q)\alpha$

& $-p \leq -(p-q) \leq q$. So, this lies in the Root string ✓



w_α is: Reflection over the (root) hyperplane:

$H_\alpha = \{ \lambda \in \underline{t}^* : \lambda(h_\alpha) = 0 \}$.

& This preserves Φ.

Next: Define Root system as something having all the nice properties of Φ, and then show:

Root Systems $\Longleftrightarrow$ Semisimple Lie Algebras.

§11: Root Systems.

a) Roots in Euclidean Space.

Recall: Φ spans t^*. (10.10)

Prop 11.1 Bilinear form on t^* by $(\lambda, \mu) = K(t_\lambda, t_\mu)$ ← $\lambda, \mu \in t^*$.

Then: 1) If $\alpha, \beta \in \Phi$ then $(\alpha, \beta) \in \mathbb{Q}$.

2) If have basis $\{\alpha_1, \dots, \alpha_\ell\}$ of $\underline{t}^*$, and $\beta \in \Phi$,

then: $\beta = \sum c_i \alpha_i$ & $c_i \in \mathbb{Q}$.

[I.e. $\dim_{\mathbb{Q}}(\Phi) = \dim_{\mathbb{C}}(t)$.]

3) $(\cdot, \cdot)$ Positive Definite on $\mathbb{Q}\{\Phi\}$.

$$\beta(h_\alpha) = \frac{2(\alpha, \beta)}{(\alpha, \alpha)}.$$

b) Abstract Root Systems.

Have: Euclidean space $(E, (\cdot,\cdot))$ for Pos-def bilinear form.

If $\alpha \in E - \{0\}$, define $\alpha^\vee : E \to \mathbb{R}$ has:

$$\boxed{\alpha^\vee(\lambda) = \frac{2(\alpha, \lambda)}{(\alpha,\alpha)}.}$$

Also, define Reflection $w_\alpha : E \to E$ by $w_\alpha(\lambda) = \lambda - \alpha^\vee(\lambda)\alpha$.

DEF 11.2 A finite subset $\Phi \subseteq E$ is Abstract Root System if:

1) $0 \notin \Phi$

1.5) Φ Spans E

2) $\forall \alpha, \beta \in \Phi$: $\beta^\vee(\alpha) \in \mathbb{Z}$

3) $w_\alpha(\Phi) = \Phi$ $\forall \alpha \in \Phi$.

4) If $\alpha, c\alpha \in \Phi$ then $c = \pm 1$

Removing 4) gives "Non-reduced" system. (Not covered here)

Notation $\forall \mu \in E, \lambda \in E^*$: $\langle \mu, \lambda \rangle \equiv \lambda(\mu)$.

So, e.g. $\langle \beta, \alpha^\vee \rangle = \alpha^\vee(\beta)$, etc.

Each $\alpha \in \Phi$ is called a Root, and $\alpha^\vee$ co-root.

Example If $\mathfrak{g}$ Semisimple Lie-Algebra & $\mathfrak{t} \subseteq \mathfrak{g}$ CSA & Φ = Roots associated with $\mathfrak{t}$, then Φ Root system. (In the space $\mathbb{R}\Phi$).

DEF 11.3 1) Rank of root system $\dim_{\mathbb{R}}(E)$

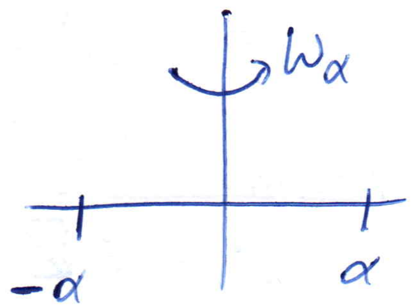
2) (Φ, E), (Φ', E') roots systems. Then, an Isomorphism is linear isom of spaces $\rho : E \to E'$, $\rho(\Phi) = \Phi'$ and

$\langle \rho(\alpha), \rho(\beta)^\vee \rangle = \langle \alpha, \beta^\vee \rangle \quad \forall \alpha, \beta \in \Phi$.

Examples (Rank ≤ 2)

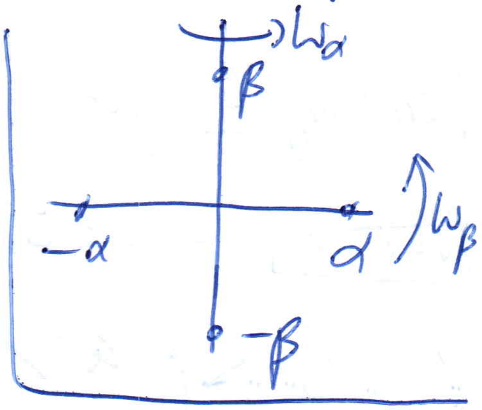
Rank = 1: (A_1) $E = \mathbb{R}$ & $(x,y) = xy$ inner product.

$\Phi = \{\pm\alpha\}$ ($\alpha \neq 0$) & $\langle \alpha, \alpha^\vee \rangle = +2$



Rank 2: 4 cases.

1) $A_1 \times A_1$: $E = \mathbb{R}^2$, usual inner product, $\Phi = \{\pm\alpha, \pm\beta\}$



2) A_2: $\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$. ($sl_3$)

3) B_2: $\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta)\}$.
(so_5), (sp_4)

4) G_2: Different points in Perimeter of star. D.

Returning to Lie-Algebras.

1) $sl_2 = \langle h \rangle \oplus g_\alpha \oplus g_{-\alpha}$ ($\alpha(h) = 2$). Taking h as generator of CSA, have: root system corresponding to sl_2.

2) $sl_3 = t \oplus (\text{lots})$, $t = \left\langle \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix} \right\rangle$.

$\Rightarrow \Phi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$, $\alpha(h_1) = 2$, $\alpha(h_2) = -1$, $\Phi \cong A_2$.
$\beta(h_1) = -1$, $\beta(h_2) = 2$.

Lie Algebras: Lecture 13

DEF 11.4 Weyl group of (Φ, E) is subgroup of $GL(E)$ generated by: $\{w_\alpha : \alpha \in \Phi\}$ $\quad w_\alpha(\lambda) = \lambda - \frac{2(\alpha,\lambda)}{(\alpha,\alpha)}\alpha$.

Know: W finite, because w_α invertible & $w_\alpha \circ \Phi = \Phi$, so w_α permutes Φ (finite). $\Rightarrow \exists \omega : W \hookrightarrow Sym(\Phi)$.

[This is injection, since $\mathbb{R}\Phi = E$, so if w_α, w_β agree on Φ then agree on all of E.]

Examples $A_1 \cong C_2$, $A_2 \cong D_6 \cong S_3$, ~~A_3~~ $B_2 \cong D_8$, $G_2 \cong D_{12}$.

Lemma 11.5 If (Φ_1, E_1) & (Φ_2, E_2) Root systems, then $(\Phi_1 \cup \Phi_2, E_1 \oplus E_2)$ also root system.

A root system is Reducible if of this form with $\Phi_i \neq \phi$, and Irreducible otherwise.

Examples $A_1 \times A_1$ reducible & A_1, A_2, B_2, G_2 irreducible.

& If Φ corresponds to CSA in Semisimple Lie Algebra, then: Φ irred $\Leftrightarrow$ g indecomposable.

Lemma 11.6 Φ Root System. & $\alpha, \beta \in \Phi$, $\alpha \neq \pm\beta$. Then:

$\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle \in \{0,1,2,3\}$. "Finiteness Lemma".

Proof $(\alpha, \beta) = \sqrt{(\alpha,\alpha)}\sqrt{(\beta,\beta)}\cos\theta$. $\Rightarrow \langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle = 4\cos^2\theta$.

But, $0 \leq \cos^2(\theta) \leq 1 \Rightarrow 4\cos^2\theta \in \{0,1,2,3,\cancel{3},4\}$.

Since $\alpha \neq \pm\beta \Rightarrow \cos^2(\theta) \neq 1 \Rightarrow < 4$ ✓

<u>Corollary 11.7</u>) If Φ Root system, & α, β <u>roots</u> then $\langle \alpha, \beta^\vee \rangle \in \{0, \pm 1, \pm 2, \pm 3\}$.

<u>Exercise</u>) The only Rank=2 systems are (up to $\cong$) those found in prev.

<u>Corollary 11.8</u>) If Φ <u>irred</u> root system, then (α, α) takes ≤ 2 values $\forall \alpha \in \Phi$.

<u>DEF 11.9</u>) An <u>irred</u> Φ <u>simply laced</u> if (α, α) <u>same</u> $\forall \alpha$.

<u>Examples</u>) A_1, $A_1 \times A_1$, A_2 <u>are</u>, but <u>not</u> B_2, G_2.

If Φ simply laced, then (Φ, E) is $\cong (\Phi', E')$, where $\langle \alpha, \beta^\vee \rangle$ takes only 3 values $\{0, \pm 1\}$ $\forall \alpha, \beta \in \Phi'$, $(\alpha \neq \pm\beta.)$

<u>§12: Weyl Chambers & Root Bases.</u>

Throughout chapter: (Φ, E) Root system.

For $\alpha \in \Phi$, denote $H_\alpha \equiv \{\lambda \in E : \langle \lambda, \alpha^\vee \rangle = 0\}$.

The connected components of $E \setminus \bigcup_{\alpha \in \Phi} H_\alpha$ is: <u>Weyl chambers</u>.

A subset $\Delta = \{\alpha_1, \ldots, \alpha_\ell\} \subseteq \Phi$ is <u>Root Basis</u> if:

<u>1)</u> Δ forms basis for E

<u>2)</u> If $\alpha \in \Phi$, $\alpha = \sum_i c_i \alpha_i$, <u>then</u> either $c_i \in \mathbb{Z}_{\geq 0}$ $\forall i$ <u>or</u> $c_i \in \mathbb{Z}_{\leq 0}$ $\forall i$.

DEF 12.1 If $\Delta = \{\alpha_i\}$ base, then α_j Simple roots.
Say $\alpha \in \Phi$ Positive roots (Φ^+) if $c_i \geq 0 \ \forall i$
& Negative roots (Φ^-) if $c_i \leq 0 \ \forall i$.

Remark Δ defines Partial Order on E, by saying
$\lambda \leq \mu \Leftrightarrow \mu - \lambda =$ Sum of positive roots, or $\mu = \lambda$.

Lemma 12.2 Let W Weyl group of (Φ, E). Then, if Δ base & $\omega \in W$, then $\omega(\Delta)$ is also base.

Proof Know $\omega \in GL(E)$, so $\omega(\Delta)$ basis of E. & $(\subseteq \Phi)$
& If $\alpha \in \Phi$ has property that (if $\alpha = \sum c_i \alpha_i$ then $c_i \geq 0 \ \forall i$ or $c_i \leq 0 \ \forall i$,) then this property is preserved when we apply ω. ✓

Conclusion: W acts on set of root bases.

Question: How to construct one?? [Do they exist]

⊛ Choose $\gamma \in E \setminus \bigcup_\alpha H_\alpha$, and define Φ_γ^+ as:
$$\Phi_\gamma^+ = \{\alpha \in \Phi : \langle \gamma, \alpha^\vee \rangle > 0\} \quad \& \quad \Phi_\gamma^- = -\Phi_\gamma^+.$$

⊛ Define $\Delta_\gamma = \{\alpha \in \Phi_\gamma^+ : \alpha \neq \beta_1 + \beta_2 \ \forall \beta_1, \beta_2 \in \Phi_\gamma^+\}$.

[**Note**: Def of $\Phi_\gamma^\pm$ depends only on the Weyl chamber that γ is in, not necessarily γ.]

Theorem 12.3 1) Δ_γ is Root Basis

2*) All Root Bases are of the form Δ_γ, some γ.

Proof Claim 1: $\forall \alpha, \beta \in \Delta_\gamma : \alpha - \beta \notin \Delta_\gamma$.

If not: WLOG $\alpha - \beta \in \Phi_\gamma^+$. Then, $\alpha = (\alpha - \beta) + \beta \Rightarrow \alpha \notin \Delta_\gamma$.

Proof (Continued). Claim 2: $\forall \alpha, \beta \in \Delta_\gamma : (\alpha \neq \beta)$ $\langle \alpha, \beta^\vee \rangle \leq 0$

Recall: (11.6) $\langle \alpha, \beta^\vee \rangle \cdot \langle \beta, \alpha^\vee \rangle \in \{0,1,2,3\}$.

So, WLOG $\langle \alpha, \beta^\vee \rangle = 1$ (else consider $\langle \beta, \alpha^\vee \rangle$)

$\Rightarrow w_\beta(\alpha) = \alpha - \langle \alpha, \beta^\vee \rangle \beta = \alpha - \beta \in \Delta_\gamma$, since W preserves Δ_γ, contradicting Claim 1. So, $\langle \alpha, \beta^\vee \rangle = 0$.

Claim 3: If $\Delta_\gamma = \{\alpha_1, \dots, \alpha_\ell\}$, & $\forall \alpha \in \Phi_\gamma^+$, if $\alpha = \sum c_i \alpha_i$ then $c_i \geq 0$ $\forall i$.

Indeed: Suppose $\exists \alpha$, that can't be written in such way. Then, pick such α, with (γ, α) minimal.

So, $\alpha \notin \Delta_\gamma \Rightarrow \alpha = \beta_1 + \beta_2$, $\beta_i \in \Phi_\gamma^+$.

$\Rightarrow (\alpha, \gamma) = (\beta_1, \gamma) + (\beta_2, \gamma) \Rightarrow (\beta_i, \gamma) < (\alpha, \gamma)$ #

[since, then, β_i can be written as $\mathbb{Z}_{\geq 0}$-linear combination of α_i's by minimality, hence, sum to get one for α.]

Remark: This means Any element of Φ_γ^- can be written as a $\mathbb{Z}_{\leq 0}$-linear span of the α_i.

& Also, Δ_γ spans E, since Φ spans E.

Claim 4: Δ_γ is linearly independent set.

If not, then for some $c_i \in \mathbb{R}$, $\sum c_i \cdot \alpha_i = 0$.

Write as: $v = \sum_{i=1}^{m} c_i \alpha_i = -\sum_{m+1 \leq j \leq n} c_j \alpha_j$, where: $c_i \geq 0 \ \forall i \leq m$ & $c_j \leq 0 \ \forall i > m$.

$\Rightarrow 0 \leq (v, v) \leq -\sum_{i,j} c_i c_j (\alpha_i \alpha_j) \leq 0$

(because: $c_i c_j \leq 0$ & $(\alpha_i, \alpha_j) \leq 0$ by claim 2).

$\Rightarrow$ $v = 0$.

Hence: $0 = (\gamma, v) = \sum_{1 \leq i \leq m} c_i (\gamma, \alpha_i) \Rightarrow c_i = 0 \ \forall i \leq m$

& Similarly for $i > m$.

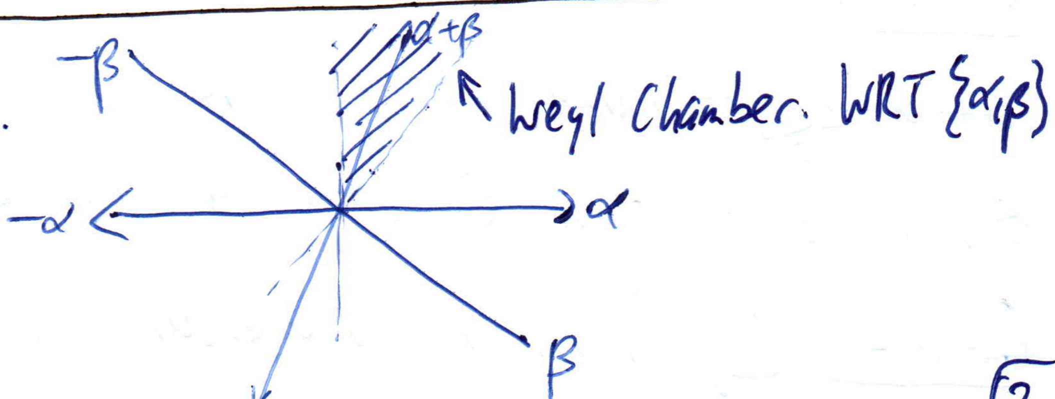
2*): Non-examinable Proof.

Corollary 12.4] $\exists$ Bijection {Weyl Chambers}

$\longleftrightarrow$ {Root ~~Systems~~ bases}.

Proof] Let C Weyl chamber, & choose $\gamma \in C$ and a Root basis Δ_γ. Then: given $\Delta = \Delta_\gamma$, write $C_\Delta = C_\gamma$ and call it fundamental Chamber relative to Δ.

Example] Type A_2.



DEF 12.5 If $\Delta = \{\alpha_1, \dots, \alpha_n\}$ Root basis, and $\alpha \in \Phi$.
The Height of α is $\sum c_i$, where $\alpha = \sum c_i \alpha_i$.

Lemma 12.6 If $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ Root basis & $\beta \in \Phi^+ - \Delta$.
Then: $\exists i$, $\beta - \alpha_i \in \Phi$.

Proof Given β, if $(\beta, \alpha_i) \leq 0$ $\forall i$ then $\Delta \cup \{\beta\}$ is linearly indep, by Claim 4. $\not\!\!\!\#$

So, $\exists i$, $\langle \beta, \alpha_i^\vee \rangle > 0$.

Since $\langle \beta, \alpha^\vee \rangle \langle \alpha, \beta^\vee \rangle \in \{0, 1, 2, 3\}$, either $\langle \beta, \alpha_i^\vee \rangle = 1$
or $\langle \alpha_i, \beta^\vee \rangle = 1$.

$\Longleftrightarrow$ Either $w_{\alpha_i}(\beta) = \beta - \alpha_i \in \Phi$
or $w_\beta(\alpha_i) = \alpha_i - \beta \in \Phi$ $\Rightarrow \beta - \alpha_i \in \Phi$ ✓

Corollary 12.7 If $\beta \in \Phi^+$ then $\beta = \sum \alpha_{ij}$, α_{ij} simple, for each j, & $\forall k$, $\sum_{j \leq k} \alpha_{ij}$ is a root.

Proof Follows from Lemma 12.6 & Induction on Height.

§13: Facts about Weyl Group.

Recall: W acts on set of Root Basis, hence, preserves the Weyl Chambers. (or the set thereof)

Lemma 13.1 If $w \in W$: $\lambda, \mu \in E$ then $\langle \lambda, \mu^\vee \rangle = \langle w(\lambda), w(\mu)^\vee \rangle$

Prop 13.2 If Δ Root Basis, then: if $w \in W$ then

$$C_{w(\Delta)} = w(C_\Delta).$$

Lemma 13.3 Φ Root System. & Δ Root basis. W weyl group, $\alpha \in \Delta$. Then, w_α <u>permutes</u> $\Phi^+ - \Delta$.

Lie Algebras: Lecture 15

08/11/2023

Let: Δ Root Basis & $w \in W$. $C_{w(\Delta)} = w(C_\Delta)$.

Lemma 13.3 For Φ Root System; Δ root basis; W weyl group and $\alpha \in \Delta$, then w_α permutes $\Phi^+ \setminus \Delta$.

Proof Take $\alpha = \alpha_1$, $\Delta = \{\alpha_1, \dots, \alpha_n\}$ & $\beta \in \Delta^+ \setminus \{\alpha\}$.

$\Rightarrow \beta = \sum c_i \alpha_i$, $c_i \in \mathbb{Z}_{\geq 0}$

& $w_{\alpha_1}(\beta) = \beta - \langle \beta, \alpha_1^\vee \rangle \alpha_1 = (c_1 - \langle \beta, \alpha_1^\vee \rangle)\alpha_1 + \sum_{i=2}^{\ell} c_i \alpha_i$

Since β positive root: & $\beta \neq \alpha_1$, have ~~...~~ $w_\alpha(\beta) \neq -\alpha_1$.

$\Rightarrow \exists i \geq 2$, $c_i > 0$, so $w_{\alpha_1}(\beta)$ is positive root different from α (since $\beta \neq -\alpha$) ✓

Theorem 13.4

1) W acts simply transitively (transitive + stabiliser trivial) on set of Root Bases (& on ~~the~~ set of Weyl chambers).

2) Given Root Basis Δ & $\alpha \in \Phi$, $\exists w \in W$, s.t $w(\alpha) \in \Delta$. [Not necessarily unique w!]

3) If $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ Root Basis, then ~~...~~ W is generated by $\{w_{\alpha_i} : 1 \leq i \leq \ell\}$.

[Proof Omitted]

§14: Classification of Root Systems.

<u>Throughout</u>: (Φ, E) Root System & Root basis $\Delta := \{\alpha_1, \ldots, \alpha_\ell\}$

<u>Have</u>: W Weyl Group, of Φ.

Define <u>Cartan Matrix</u> of Φ as $C = (a_{ij})_{1 \le i,j \le \ell}$

with $a_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$ (Cartan Integers).

This matrix is <u>indep</u> of choice of root basis Δ, because if Δ' another such, then $w(\Delta) = \Delta'$ exists & preserves $\langle \cdot, \cdot \rangle$.

<u>Note</u>: $\det(C) \neq 0$ (since Gram matrix).

<u>Example</u> (G_2):

$\alpha_1 = \alpha$ & $\alpha_2 = \beta$.

$\Rightarrow \langle \alpha_1, \alpha_2^\vee \rangle = -1$, $\langle \alpha_2, \alpha_1^\vee \rangle = -3$. & $\langle \alpha_i, \alpha_i^\vee \rangle = 2$

$\Rightarrow C = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$. [Uniqueness up to reordering base].

<u>Recall</u>: W has D_{12}.

<u>Other examples</u> $A_1 \times A_1 : \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $A_2 : \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $B_2 : \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$

<u>Prop 14.1</u> Suppose (Φ', E') Root System, with: base $\{\alpha_1', \ldots, \alpha_\ell'\}$ s.t. $\langle \alpha_i, \alpha_j^\vee \rangle = \langle \alpha_i', \alpha_j'^\vee \rangle$. $\forall i,j$.

<u>Then</u>: the linear map $\alpha_i \longrightarrow \alpha_i'$ induces Isomorphism ϕ: of Root Systems, s.t. $\langle \phi(\alpha), \phi(\beta)^\vee \rangle = \langle \alpha, \beta^\vee \rangle$.

In particular, C <u>determines</u> Φ up to $\cong$.

<u>Proof</u> Since Δ (resp. Δ') <u>Basis</u> of E (resp. E'),

$\exists!$ Vector Space $\cong$: $\phi : E \to E'$ sending $\alpha_i \to \alpha_i'$.

If $\alpha, \beta \in \Delta$:

$$w_{\phi(\alpha)}(\phi(\beta)) = w_{\alpha'}(\beta') = \beta' - \langle \beta', \alpha'^{\vee} \rangle \alpha'$$

$$= \phi(\beta) - \langle \beta, \alpha^{\vee} \rangle \phi(\alpha). \text{ (By assumption)}$$

$$= \phi(\beta - \langle \beta, \alpha^{\vee} \rangle \alpha) = \underline{\phi(w_\alpha(\beta))}.$$

$\Rightarrow \exists$ Commutative Diag:

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "w_\alpha"'] & E' \arrow[d, "w_{\phi(\alpha)}"] \\ E \arrow[r, "\phi"'] & E' \end{tikzcd}$$

By <u>(13.3)</u>: the Weyl groups W, W' are generated by <u>Simple</u> reflections, hence, the map $w \mapsto \phi \circ w \circ \phi^{-1}$ is an Isomorphism $W \to W'$, with $w_\alpha \mapsto w_{\phi(\alpha)}$.

Now: $\forall \beta \in \Phi$, it is conjugate (under W) to some simple root α (i.e. $\exists w$, $\beta = w(\alpha)$).

$\Rightarrow \phi(\beta) = (\phi \circ w \circ \phi^{-1})(\phi(\alpha)) \in \Phi'$

$\Rightarrow \phi$ maps Φ to Φ'.

By formula for reflections, <u>follows</u>: ϕ preserves Cartan integers, i.e. $\langle \alpha, \beta^{\vee} \rangle = \langle \phi(\alpha), \phi(\beta)^{\vee} \rangle$.

Remark Proposition suggests: it is possible to recover Φ from knowledge of Cartan Integers.

DEF 14.2 Recall: If $\alpha \neq \pm\beta \Rightarrow \langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle \in \{0,1,2,3\}$. The Coxeter Graph of Φ is: graph of l vertices $\{1, \dots, i\}$ s.t. (i,j) joined by $\langle \alpha_i, \alpha_j^\vee \rangle \langle \alpha_j, \alpha_i^\vee \rangle$ edges. $(\forall i \neq j)$

Note Coxeter graph determines $\langle \alpha_i, \alpha_j^\vee \rangle$ in the case when All roots have Equal lengths.
If more than 1 root length occurs, e.g. B_2 & G_2, graph fails to tell us: which, of a pair of edges, should correspond to short simple root, or long simple root, in the case when i,j joined by 2 or 3 edges.

Remark Coxeter Graph determines Weyl group completely, since determines the orders of products of generators of W.

DEF The Dynkin Diagram of Φ has:

- Vertices $\Leftrightarrow \Delta$
- Edges: (i,j) joined by $\langle \alpha_i, \alpha_j^\vee \rangle \langle \alpha_j, \alpha_i^\vee \rangle$ times, & If multiple edge occurs, then point arrow to shorter root.

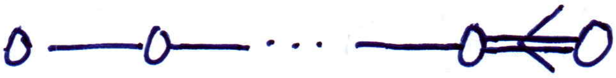
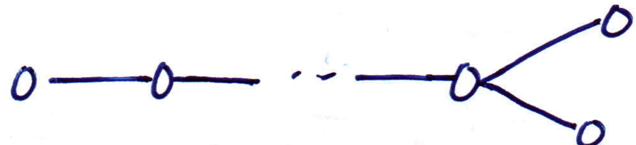
Examples [A_1 : •] [A_2 : •—•] [$A_1 \times A_1$: • •] [B_2 : •⇒•] [G_2 : •⇚•]

Lie Algebras: Lecture 16 | 10/11/2023

Remark Maximal # edges between any 2 vertices is 3, and Φ simply laced $\Leftrightarrow$ Dynkin Diagram has no mult edges.

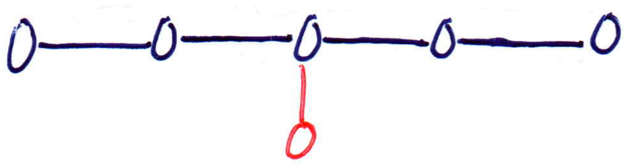
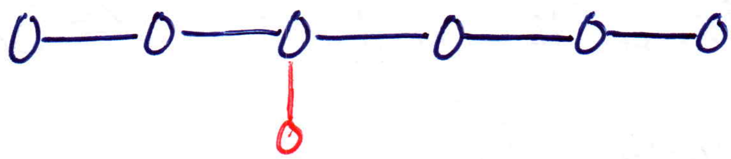
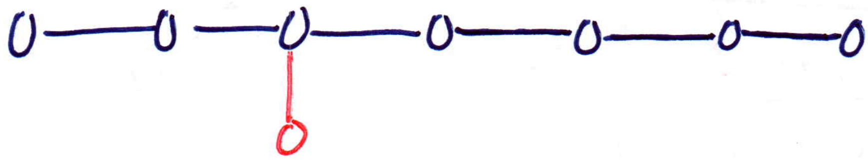
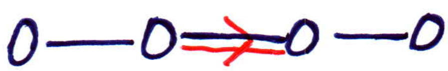
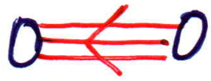
Exercise Φ Irred $\Leftrightarrow$ Dynkin Diagram simply connected.

Theorem 14.3 If Φ Irred root system, of rank ℓ "in the first batch", then: Dynkin diagram is one of 2 types:

- A_ℓ $(\ell \geq 1)$:  (ℓ nodes) $\cong \mathfrak{sl}_{n+1}$
- B_ℓ $(\ell \geq 2)$:  $\cong \mathfrak{so}_{2n+1}$
- C_ℓ $(\ell \geq 3)$:  $\cong \mathfrak{sp}_{2n}$
- D_ℓ $(\ell \geq 4)$:  $\cong \mathfrak{so}_{2n}$.

[The restrictions on ℓ are to avoid duplication]

OR, is one of Exceptional Root Systems:

- E_6: 
- E_7: 
- E_8: 
- F_4: 
- G_2: 

Theorem 11.4 For Any Dynkin Diagram $\mathcal{D}$, listed, $\exists$ simple Lie Algebra $\mathfrak{g}$ & CSA $\underline{t}$, corresponding to the roots of $\underline{t}$ ($\hat{\Phi}$), s.t. Dynkin Diagram of Φ is $\mathcal{D}$.

Proof (Sketch).

For A_ℓ: Let e_i Standard Vector of $\mathbb{R}^{\ell+1}$.

& $\Phi = \{e_i - e_j : i \neq j\} \subset \mathbb{R}^{\ell+1}$.

Then: Φ spans an ℓ-dim Subspace of $\mathbb{R}^{\ell+1}$, called E.

& Φ is a Root System in E, with root basis given by:

$\alpha_i = e_i - e_{i+1}$, $1 \leq i \leq \ell$.

[Note: $e_i - e_j = (e_i - e_{i+1}) + (e_{i+1} - e_{i+2}) + \dots + (e_{j-1} - e_j)$]

Cartan Integers: $\langle \alpha_i, \alpha_j^\vee \rangle = \begin{cases} -1 & \text{if } |i-j| \leq 1 \\ 2 & \text{if } i = j \\ 0 & \text{else} \end{cases}$

$\Rightarrow$ Dynkin Diagram $\underset{\alpha_1}{o} - \underset{\alpha_2}{o} - \cdots - \underset{\alpha_\ell}{o} = A_\ell$

$\Rightarrow$ Φ is of type A_ℓ.

Now: w_{α_i} flips i'th & $(i+1)$'th coords, so: $W \cong S_{\ell+1}$.

& Corresponding Lie Algebra: $\mathfrak{sl}_{\ell+1}(\mathbb{C})$

& Cartan Subalgebra: Diagonals $\begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}$

and $\alpha_i \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_{\ell+1} \end{pmatrix} = t_i - t_{i+1}$.

& Cartan Matrix: $\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & \ddots & \end{pmatrix}$

In summary: for other classical ones:

Type	$\Phi \subseteq \mathbb{R}^l$	$\Delta \subseteq \Phi$	W	$\mathfrak{g}$	dim
B_l	$\{\pm e_i, \pm e_i \pm e_j \ (i \neq j)$	$\{e_i - e_{i+1} : 1 \leq i \leq l\} \cup \{e_l\}$.	$S_l \ltimes C_2^l$	$\mathfrak{so}_{2l+1}$	$2l^2+l$
C_l					
D_l					

(Where: $S_l \curvearrowright C_2$ by sign change & coords by permutation)

Exceptional Cases:

G_2: 12 roots, $E = \{v = \sum_{1 \leq i \leq 3} c_i e_i : \sum c_i = 0\}$, $W = D_{12}$

F_4: 1152 roots, $E = \mathbb{R}^4$, $|\Phi| = 48$

E_6, E_7, E_8: First do E_8. $|\Phi| = 240$, $|W| = 2^{14} 3^5 5^2 7$, $|\Delta| = 8$.

Let: $W_c = \prod_{1 \leq i \leq 8} w_{\alpha_i} \in W$. Coxeter element. Order 30.

There is a plane of $\mathbb{R}^8$ that is acted on by Rotation.

Remarks 1) To look up Root Systems, Moodle page.

2) When doing Computations: Helpful to compute in terms of Simple roots.

3) The Exceptional Lie Algebras are: $\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$.

& $\mathfrak{e}_2$ = Algebra of "Derivations of the Octonions". $\mathbb{O}$.

[Derivation = linear map δ, s.t. $\delta(ab) = a\delta(b) + \delta(a)b$]

$\mathbb{O}$ is a dim $= 8$ Division Algebra over $\mathbb{R}$, having 1D Centre on which $\mathfrak{g}_2$ acts Trivially.
There is a Representation $\mathfrak{g}_2 \longrightarrow SO_7$ (lowest dim nontrivial rep) & Others can be constructed.

4) Given Φ, $\exists$ Natural Construction of Lie Algebra, with that Root System.

To summarise:

$$\{\mathfrak{g} \text{ Simple } \& \ \underline{t} \text{ CSA}\} \longrightarrow\!\!\!\!\!\rightarrow \{\text{Irred Root systems } \Phi\}$$

$$\Updownarrow$$

$$\{\text{Connected Dynkin diagrams}\}$$

Next:

⊛ Show: Root systems corresponding to $\mathfrak{g}$ is Indp of choice of CSA

⊛ Show: Any 2 Lie Algebras with same root systems are Isomorphic.

Lie Algebras: Lecture 17 | 13/11/2023

§: Brief Intro to Inner Automorphisms.

An **Automorphism** of $\mathfrak{g}$ is: isom. $\mathfrak{g} \to \mathfrak{g}$. & The group of all Automorphisms $\mathrm{Aut}(\mathfrak{g})$.

Example $\mathfrak{g} = \mathfrak{gl}(V)$ or $\mathfrak{sl}(V)$ & $A \in GL(V)$ invertible endomorphism $V \to V$, then $x \mapsto AxA^{-1}$ auto of $\mathfrak{g}$.

Take: V F.D./$\mathbb{C}$ & $x \in \mathfrak{g}$, with ad(x) nilpotent. $(\mathrm{ad}(x))^m = 0$.

Then: $\exp(\mathrm{ad}\, x) = 1 + \mathrm{ad}(x) + \frac{\mathrm{ad}(x)^2}{2!} + \dots + \frac{\mathrm{ad}(x)^{m-1}}{(m-1)!}$

[makes sense, since only finitely many terms]

Easy to see: $\exp(\mathrm{ad}\, x) \in \mathrm{Aut}(\mathfrak{g})$.

Call: any such automorphism Inner.

The subgroup of $\mathrm{Aut}\, \mathfrak{g}$, generated by these inner Autos ~~are~~ is: denoted $\mathrm{Inn}(\mathfrak{g})$.

[Normal subgroup: $\forall \phi \in \mathrm{Aut}(\mathfrak{g})$ & $x \in \mathfrak{g}$, have

$\phi(\mathrm{ad}\, x)\phi^{-1} = \mathrm{ad}(\phi(x))$

$\Rightarrow \phi \exp(\mathrm{ad}\, x)\phi^{-1} = \exp(\mathrm{ad}\, \phi(x))$]

Lemma 9.1 If $\mathfrak{g} \leq \mathfrak{gl}(V)$ (over $\mathbb{C}$) & x nilpotent, then ad(x) nilpotent & $\boxed{\exp(x) \cdot y \cdot (\exp x)^{-1} = \exp(\mathrm{ad}\, x)(y)}$ $(\forall y \in \mathfrak{g})$.

Exercises 1) $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) \Rightarrow \mathrm{Inn}(\mathfrak{g}) = PGL_n(\mathbb{C})$.

$\underline{\underline{1)}}$ $\mathfrak{g} = su_n(\mathbb{C}) \Rightarrow Inn(\mathfrak{g}) = SU_n(\mathbb{C})/Z$

$\underline{3)}$ $\mathfrak{g} = sp_{2n}(\mathbb{C}) \Rightarrow Inn(\mathfrak{g}) = Sp_{2n}(\mathbb{C})/Z$.

15: Conjugacy Results.

<u>Here</u>: $\mathfrak{g}$ semisimple Lie Algebra & $\mathfrak{t}$ CSA of $\mathfrak{g}$ and Φ Root system WRT $\mathfrak{t}$. $\left(\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha\right)$ & $\Delta \subseteq \Phi$ Root Basis.

<u>Lemma 15.1</u> If $\mathfrak{t}'$ another CSA of $\mathfrak{g}$, then $\exists \psi \in Inn(\mathfrak{g})$ with $\psi(\mathfrak{t}) = \mathfrak{t}'$.

[Proof Omitted].

<u>Define</u>: Rank of $\mathfrak{g}$, as dimension of CSA (indep of choice of $\mathfrak{t}$ by 15.1). If $\mathfrak{g}$ Semisimple, then: $\text{Rank}(\mathfrak{g}) = \text{Rank}(\Phi)$, for Any Φ Root system of $\mathfrak{g}$ WRT any $\mathfrak{t}$.

<u>Lemma 15.2</u> If $\mathfrak{t}'$ CSA, root system Φ', then Φ, Φ' are Isomorphic.

<u>Proof</u> Take ψ as in (15.1). & $t \in \mathfrak{t}$ & $\alpha \in \Phi$ and $e_\alpha \in \mathfrak{g}_\alpha$. Then, $[\psi(t), \psi(e_\alpha)] = \psi[t, e_\alpha] = \psi(\alpha(t) e_\alpha)$

$$= \alpha(t)\psi(e_\alpha).$$

As $\psi(e_\alpha)$ spans the root space of $\mathfrak{t}'$, $\Phi' = \{\alpha \circ \psi^{-1} : \alpha \in \Phi\}$

<u>Theorem 15.3</u> If $\mathfrak{g}'$ Semisimple Lie Algebra, with Root system Φ, then $\mathfrak{g}' \cong \mathfrak{g}$.

Proof*] Apperson [Carter: §7] using theory of finite structure constants.

__Choose__: Basis h_{α_i} of $\underline{t}$ & basis e_α in each root space, such that: $\forall\alpha$, $[e_\alpha, e_{-\alpha}] = h_\alpha$. __Gives__: basis of Lie Alg.

& $[h_{\alpha_i}, h_{\alpha_j}] = 0$, $[h_{\alpha_i} e_\alpha] = \alpha(h_{\alpha_i}) e_\alpha$

& $[e_\alpha e_\beta] = \begin{cases} N_{\alpha\beta}\, e_{\alpha+\beta}, & \alpha+\beta \in \Phi \\ h_\alpha, & \alpha+\beta = 0 \\ 0 & \text{else.} \end{cases}$

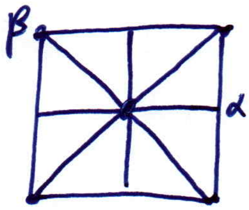
__§16: Weights.__

__Example__ $g = so_5(\mathbb{C})$ & α, β simple roots, for root system of g, (call it Φ). __Recall__: $\underline{m}_\alpha = g_\alpha \oplus \langle [g_\alpha g_{-\alpha}] \rangle \oplus g_{-\alpha}$
$\cong sl_2(\mathbb{C})$.

& Decompose: adjoint rep of g, under the action of $\underline{m}_\alpha, \underline{m}_\beta$.

If $e_\alpha \in g_\alpha$, then: $\forall \gamma \in \Phi$, $e_\alpha g_\gamma = g_{\alpha+\gamma}$.

$\Rightarrow$ Each α-root string ~~through~~ β corresponds to: an irreducible subrep m_α of g. (Theorem 10.13)



In fact: $g|_{m_\alpha} = V(2) \oplus V(2) \oplus V(2) \oplus V(0)$

$g|_{m_\beta} = V(2) \oplus V(1) \oplus V(1) \oplus V(0) \oplus V(0) \oplus V(0)$.

__Now__: (Φ, E) root system & for base $\Delta = \{\alpha_1, \dots, \alpha_l\}$.

DEF 16.1] The __Root ~~Base~~ Lattice__ $\mathbb{Z}\Phi = \left\{ \sum_{\alpha\in\Phi} c_\alpha \cdot \alpha : c_\alpha \in \mathbb{Z} \right\}$.

$\Rightarrow \mathbb{Z}\Phi \subseteq E$.

The weight line X is: $\{\lambda \in E : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \ \forall \alpha \in \Phi\}$.

or: $\{\beta \in \underline{t}^* : \beta(h_\alpha) \in \mathbb{Z} \ \forall \alpha \text{ root}\}$.

Note: ⊛ $\mathbb{Z}\Phi \subseteq X$

⊛ If $\lambda \in X$, so is $w(\lambda)$ $\forall w \in W$
(since: $\langle \lambda, \alpha^\vee \rangle = \langle w(\lambda), w(\alpha)^\vee \rangle$)

⊛ Root lattice is a lattice in E, in technical sense.

Lie Algebras: Lecture 18 | 15/11/2024

From last time (Φ, E) Root system & Δ base. $\{\alpha_1, \dots, \alpha_\ell\}$.

A Root Lattice $\mathbb{Z}\Phi = \{\sum_{\alpha \in \Phi} c_\alpha \cdot \alpha : c_\alpha \in \mathbb{Z}\}$.

& Weight Lattice $X = \{\lambda \in E : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \; \forall \alpha \in \Phi\}$.

Know: $W \curvearrowright X$ & $\mathbb{Z}\Phi \subseteq X$.

The Root Lattice is a lattice in E, in technical sense.
(L lattice $\Leftrightarrow$ Fh free Abelian group s.t. exist form $(\cdot,\cdot) : L^2 \longrightarrow \mathbb{Z}$ with $(L \otimes_{\mathbb{Z}} \mathbb{R}, (\cdot,\cdot))$ Inner prod space)

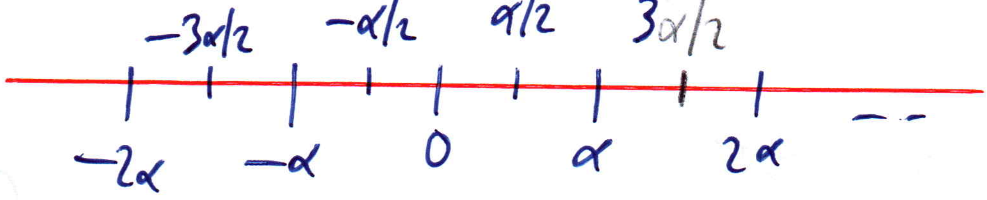
Lemma 16.2 $\lambda \in X \Leftrightarrow \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \; \forall \alpha \in \Delta$.

DEF 16.3 For $1 \leq i \leq \ell$: define $\boxed{w_i \in E, \text{ by } : \langle w_i, \alpha_j^\vee \rangle = \delta_{ij}}$.
& Call w_i fundamental weights WRT Δ.

By 16.2: $X = \{\sum_i c_i w_i : c_i \in \mathbb{Z}\}$

Exercise $X / \mathbb{Z}\Phi$ is Finite group. Called: Fundamental Group, and: $|X/\mathbb{Z}\Phi| = \det(C)$ for $C =$ Cartan matrix.

Example $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ & $\Phi = \{\pm\alpha\}$. $\Rightarrow \mathbb{Z}\Phi = \mathbb{Z}\alpha$.
& $\langle \alpha, \alpha^\vee \rangle = 2$, and $X = \mathbb{Z}\frac{\alpha}{2}$. So, $|X/\mathbb{Z}\Phi| = 2$



$X/\mathbb{Z}\Phi$. $= \det C$.

Root Lattice

For type A_ℓ, $|X : \mathbb{Z}\Phi| = \ell+1$.

DEF 16.4 $\lambda \in X$ Dominant if $\langle \lambda, \alpha^\vee \rangle \geq 0 \ \forall \alpha \in \Phi^+$, and Strongly Dominant if >0.

The set of Dominant weights is X^+.

This is equivalent to:

⊛ λ in ~~Fun~~ Closure of Fundamental Weyl Chamber (WRT Δ)

⊛ λ of form $\sum_{i=1}^{\ell} c_i w_i$ where $c_i \geq 0 \ \forall i$, $c_i \in \mathbb{Z}$.

$\left(X^+ = \bigoplus_{i \geq 1} \mathbb{Z}_{\geq 0} w_i \right)$.

Assume: g Semisimple, Root system Φ, CSA $\underline{t}$.

Choose: $e_\alpha \in g_\alpha \ \forall \alpha \in \Phi$, s.t. $[e_\alpha e_{-\alpha}] = h_\alpha$

& $\varphi : g \to gl(V)$ F.D. rep $/\mathbb{C}$.

Lemma 16.5 $V = \bigoplus_{\lambda \in \underline{t}^*} V_\lambda$, $V_\lambda \equiv \{ v \in V : tv = \lambda(t) v \ \forall t \in \underline{t} \}$.

Proof Clear, from Lemma 10.2, where the commuting semisimple endomorphisms are Basis elements of $\underline{t}$.

Some Terminology:

Recall, for $\lambda, \mu \in \underline{t}^*$, wrote $\mu \leq \lambda \iff \lambda - \mu = \sum k_i \alpha_i$ where $k_i \in \mathbb{Z}_{\geq 0}$.

& If V rep. of g, then say:

⊛ Weight of vector $0 \neq v \in V$ is: λ, if $v \in V_\lambda$

⊛ $\lambda \in \mathfrak{t}^*$ Highest weight of V $\Leftrightarrow$ $V_\lambda \neq 0$, and if $V_\mu \neq 0$, then $\mu \leq \lambda$.

Prop 16.6) 1) If $v \in V_\lambda$ then $e_\alpha V_\lambda \subset V_{\alpha+\lambda}$

2) If $V_\lambda \neq 0$ then $\lambda \in X$ (i.e. $\lambda(h_\alpha) \in \mathbb{Z}$ $\forall \alpha$)

3) $\dim(V_\lambda) = \dim(V_{w(\lambda)})$ $\forall w \in W$.

Proof) 1) Fix $t \in \mathfrak{t}$. Then, $t e_\alpha v = ([t e_\alpha] + e_\alpha t) v$
$= \alpha(t) e_\alpha(v) + e_\alpha \lambda(t) v = (\alpha+\lambda)(t) e_\alpha(v)$ ✓

2) Consider: $V|_{m_\alpha}$. Know: h_α acts by Integer weights.
$\Rightarrow \lambda(h_\alpha) \in \mathbb{Z}$. [~~$\Rightarrow \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \Rightarrow \lambda \in X$~~]

3) Enough to assume: $w = w_\alpha$, some $\alpha \in \Phi$ (Reflection)
$\Rightarrow$ $V_\lambda|_{m_\alpha} = \bigoplus_j V^{(j)}$ for $V^{(j)}$ irred m_α-reps.

Since h_α-weight spaces of $V^{(j)}$ are 1-dim: Can choose basis $v_1, \dots, v_n$ for V_λ, s.t. v_i in distinct $V^{(j)}$.

Enough to prove: given $v_i \in V^{(j)}$, $\exists x \in m_\alpha$, s.t. $x(v_i) \in V_{w_\alpha(\lambda)}$.

Know: $\{e_{-\alpha}^k v_i, e_\alpha^k v_i : k \in \mathbb{Z}_{\geq 0}\}$ span $V^{(j)}$.

$$\begin{tikzcd}
\underset{\lambda-3\alpha}{\bullet} & \underset{\lambda-2\alpha}{\bullet} \arrow[l, bend right, "e_{-\alpha}"'] & \underset{\lambda-\alpha}{\bullet} \arrow[l, bend right, "e_{-\alpha}"'] & \underset{\lambda}{\bullet} \arrow[l, bend right, "e_{-\alpha}"'] \arrow[r, bend left, "e_\alpha"] & \underset{\lambda+\alpha}{\bullet} \arrow[r, bend left, "e_\alpha"] & \underset{\lambda+2\alpha}{\bullet} \arrow[r, bend left, "e_\alpha"] & \underset{\lambda+3\alpha}{\bullet}
\end{tikzcd}$$

Define: $M = \max\{k : e_\alpha^k v_i \neq 0\}$

$m = \min\{k : e_{-\alpha}^k v_i \neq 0\}$.

Need to show: $-m \leq -\langle \lambda, \alpha^\vee \rangle \leq M$. (as in 10.13, 2))

Have: $(\lambda + M\alpha)(h_\alpha) = -(\lambda - m\alpha)(h_\alpha)$

$\Rightarrow \lambda(h_\alpha) = m - M$, so indeed ✓ $(\lambda(h_\alpha) = \langle \lambda, \alpha^\vee \rangle)$

(So, $\lambda - \langle \lambda, \alpha^\vee \rangle \alpha \in \{\lambda - m\alpha, \ldots, \lambda + M\alpha\}$.)

DEF 16.7 $v \in V$ Highest Weight Vector (HWV) if:

$v \neq 0$, $v \in V_\lambda$ (some λ), and $\forall \alpha \in \Phi^+$, $e_\alpha(v) = 0$.

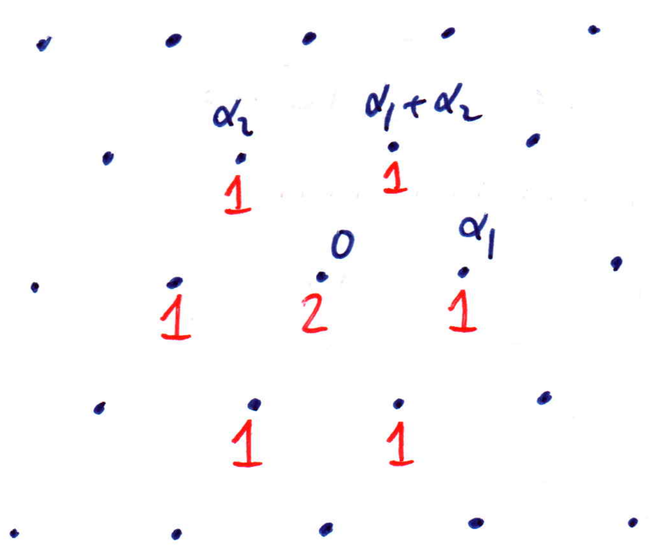
On example sheet: Showed, $\exists$ root α_0 of maximal height (12.5) WRT (Δ), the highest root.

Any nonzero $v \in \mathfrak{g}_{\alpha_0}$ is a HWV, WRT adjoint action.

---

§17: The PBW Theorem

Example $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$. Root lattice



• Dimension.

Lie Algebras: Lecture 19

17/11/2023

From last time: $v \in V$ HWV, if: $v \neq 0$, $v \in V_\lambda$ (some λ) and $e_\alpha(v) = 0$ $\forall \alpha \in \Phi^+$.

Example $g = sl_3$, $\underline{t}$ CSA, with basis $h_{\alpha_1} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$ and $h_{\alpha_2} = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}$.

Let: V Defining rep. of g, with standard basis $\{e_1, -, e_3\}$.

Look for: $\lambda_i \in \underline{t}^*$ s.t. $V = \bigoplus V_{\lambda_i}$ & for each V_{λ_i},

$h_{\alpha_1} e_1 = e_1 \qquad h_{\alpha_1} e_2 = -e_2 \qquad h_{\alpha_1} e_3 = 0$

$h_{\alpha_2}(e_1) = 0 \qquad h_{\alpha_2} e_2 = e_2 \qquad h_{\alpha_2} e_3 = -e_3.$

$\Rightarrow$ Let: $\lambda_1(h_{\alpha_1}) = 1$, $\lambda_1(h_{\alpha_2}) = 0 \Rightarrow V_{\lambda_1} = \langle e_1 \rangle$

$\lambda_2(h_{\alpha_1}) = -1$, $\lambda_2(h_{\alpha_2}) = 1 \Rightarrow V_{\lambda_2} = \langle e_2 \rangle$

$\lambda_3(h_{\alpha_1}) = 0$, $\lambda_3(h_{\alpha_3}) = -1 \Rightarrow V_{\lambda_3} = \langle e_3 \rangle$

& Note: $\lambda_1 = \omega_1$, $\lambda_2 = -\omega_1 + \omega_2$, $\lambda_3 = -\omega_2$ (p81, notes)

Check: e_1 is HWV, [since $e_\alpha \cdot e_1 \in V_{\lambda+\alpha}$ & etc.]

Lemma 17.1 1) V has a HWV

2) If $v \in V_\lambda$ HWV, then λ is Dominant.

Proof 1) Choose any $0 \neq v_0 \in V_\lambda$, some λ.

If v_0 HWV $\Rightarrow$ Done.

Else, choose $\alpha \in \Phi^+$, s.t. $e_\alpha v_0 \neq 0$. Then, define:

$k' = \max\{k : e_\alpha^k v_0 \neq 0\}$ & $v_1 \equiv e_\alpha^{k'} v_0 . \in V_{\lambda + k'\alpha}$.

Repeat this, replacing $v_0 \to v_1 \to v_2 \to \cdots$

& Process must end, since V FD & each v_i in different V_λ (since α positive root). ✓

2) Take $\alpha \in \Phi^+$, need: $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$.

Consider: $m_\alpha = \langle e_\alpha, h_\alpha, e_{-\alpha} \rangle$, acting on V.

$\Rightarrow e_\alpha v = 0$ & $h_\alpha v = \lambda(h_\alpha) v$ (sl_2-theory).

$\Rightarrow v$ is HWV, for any ~~...~~ $m_\alpha \cong sl_2$, ~~...~~ acting on V.

Hence, $\lambda(h_\alpha) \in \mathbb{Z}^+$.

Next: Show $\exists$ Correspondence $\{$FD irred reps of $\mathfrak{g}\}$

$\Updownarrow$

$\{$Dominant weights$\}$.

Universal Enveloping Algebra.

Let: k any field. Associate, to each Lie Algebra over k, an Associative Algebra, with 1 (inf-dim in general), which is generated "as freely as possible" by $\mathfrak{g}$, subject to the Commutation Relations of $\mathfrak{g}$.

DEF 17.2 Let V Vector space $/k$. The Tensor Algebra of V: $T(V) \equiv k \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$

$$= \bigoplus_{n \geq 0} V^{\otimes n} = \bigoplus_{n \geq 0} T^j V.$$

Has: Associative product, defined on Homogeneous generators of $T(V)$, by: $(v_1 \otimes \cdots \otimes v_n) \times (u_1 \otimes \cdots \otimes u_m)$

$$\equiv v_1 \otimes \cdots \otimes v_n \otimes u_1 \otimes \cdots \otimes u_m. \in T^{(m+n)} V$$

The Symmetric Algebra on V is: $Sym(V) \equiv S(V) = T(V)/I$ where I is (2-sided) Ideal, generated by:

$$I = span\{x \otimes y - y \otimes x : x, y \in V\}.$$

Notice: ⊛ $S(V) = \bigoplus_{n \geq 0} S^n V$.

⊛ Can identify: $S(V)$ with $k[V]$, algebra of Polys on V.

⊛ $T(V)$ & $S(V)$ are Graded.

DEF 17.3 Given arbitrary Lie algebra $\mathfrak{g}$: (possibly ∞-dim): the Universal Enveloping Algebra:

$$\boxed{U(\mathfrak{g}) = T(\mathfrak{g})/J.} \quad J = span\{x \otimes y - y \otimes x - [x,y]\}_{x,y \in \mathfrak{g}}.$$

Is: associative k-algebra.

Some facts & exercises:

⊛ Often write $xy \equiv x \otimes y$

⊛ If V rep. of g, then V is $U(g)$-module by:
$(x_1 \otimes \cdots \otimes x_n)v \equiv x_1 \cdots x_n v. \in V$.
This is well-defined, since $(x \otimes y - y \otimes x)v = [xy]v$.
⊛ If V FD rep of $g = sl_2(\mathbb{C})$: defined $\Omega = ef + fe + \frac{1}{2}h^2$.
Then, Ω Naturally element of $U(g)$, indep of V.
<u>In general</u>: if g Semisimple Lie alg $/\mathbb{C}$, basis $\{x_i\}$, then let $\{y_i\}$ Dual basis WRT K. Then,
$\Omega = \sum x_i y_i \in U(g)$ is Casimir element, $\Omega \in Z(U(g))$.
⊛ $U(g)$ <u>Not graded</u>! E.g. $g \otimes g$ <u>not</u> closed under $+$.
($x \otimes y - y \otimes x - [xy]$ Not homogeneous.)
<u>But</u>: It has a <u>filtration</u>.
<u>Let</u>: $U_n \equiv$ Image, of $\bigoplus_{0 \le i \le n} g^{\otimes i}$, in $U(g)$.
<u>Then</u>: $U_n U_m \subset U_{m+n}$.
Exercise $\forall x \in U_n,\ y \in U_m \Rightarrow xy - yx \in U_{m+n-1}$
(<u>Write</u> $x = \sum_{1 \le i \le k} \lambda_i \cdot x_1^{i_1} \cdots x_n^{i}$ & $y = \sum_{1 \le i \le q} \mu_j \cdot y_1^{j} \cdots y_n^{j}$)
DEF 17.4 Given any filtration $F_0 \subset F_1 \subset \cdots$, call:
$\boxed{gr(F) \equiv \bigoplus_{i \ge 0} F_i / F_{i-1}}$, <u>associated Graded algebra</u>.
In our case: $gr(U(g)) = U_0 \oplus \left(\bigoplus_{n \ge 1} U_n / U_{n-1}\right)$.

Lie Algebras: Lecture 20 | 20/11/2023

From last time: Universal Enveloping Algebra $\mathcal{U}(\mathfrak{g}) \equiv \mathfrak{T}(\mathfrak{g})/J$

& Filtration $\{U_n\}$, $U_n \equiv \left(\text{Image of } \bigoplus_{0 \leq i \leq n} \mathfrak{g}^{\otimes i}\right) \subset \mathcal{U}(\mathfrak{g})$.

& Associated Graded Algebra $gr(U) = U_0 \oplus \left(\bigoplus_{n \geq 1} U_n / U_{n-1}\right)$.

Theorem 17.5 (PBW Theorem)

$\exists$ Algebra hom. $S(\mathfrak{g}) \xrightarrow{\cong} gr(\mathcal{U}(\mathfrak{g}))$.

Equivalently, if $\{x_1, \dots, x_n\}$ basis of $\mathfrak{g}$, then $\{x_1^{k_1} \cdots x_n^{k_n}\}$ is basis for $\mathcal{U}(\mathfrak{g})$, as $k_i \in \mathbb{Z}_{\geq 0}$.

In particular: $\mathfrak{g} \hookrightarrow \mathcal{U}(\mathfrak{g})$.

Proof* | Done in [Hu, 17.4].

Sketch: For isomorphism, defining $\mathfrak{g} \longrightarrow U_n \longrightarrow U_n / U_{n-1}$ gives a map from Tensor Algebra to the associated graded ring.

It is (by def): factors through $S(\mathfrak{g})$.

$$\Rightarrow \begin{tikzcd} \mathfrak{T}(\mathfrak{g}) \arrow[r] \arrow[dr] & gr(\mathcal{U}) \arrow[d] \\ & S(\mathfrak{g}) \end{tikzcd}$$

This map is Surjective (easy) & Injective (Hard).

For the Basis part: a basis of $S(\mathfrak{g})$ is a Basis of $gr(\mathcal{U})$, which is a basis of $\mathcal{U}(\mathfrak{g})$ [Hu, 17.3, Cor C]. ✓

Lemma 17.6 Suppose V rep. of $\mathfrak{g}$ & $v \in V$. Then: the minimal subrep of V, containing v, is:

$$\mathcal{U}(\mathfrak{g})v \equiv \{uv : u \in \mathcal{U}(\mathfrak{g})\}.$$

Proof $\mathcal{U}(\mathfrak{g})v$ Contains:

- Elements $X_1 \dots X_n v$ $\forall X_i \in \mathfrak{g}$
- Elements which are Scalar multiples & Sums of above. ✓

Example [For next Chapter].

Let: V be $\mathbb{C}$-VS, basis $\{X_0, \dots, X_n\}$. Define sl_2-action on V, by: $eX_0 = 0$, $hX_0 = 0$ & $fv_i = v_{i+1}$.

Claim: X_0, X_1 are HWV's for this $sl_2(\mathbb{C})$-action.

Indeed: need, $ev_j = 0$ $(j = 0, 1)$.

- $j = 0$: ✓
- $j = 1$: $ev_1 = efv_0 = ([ef] + fe)v_0 = [ef]v_0 = hv_0 = 0$.

Also, require ~~$\{v_0\}, \{v$~~ $\langle X_0 \rangle, \langle X_1 \rangle$ to contain image under h.

- X_0: ✓
- X_1: $hX_1 = hfX_0 = ([hf] + fh)X_0 = [hf]X_0 = -2fX_0 = -2X_1$.

$\Rightarrow X_1 \in V_{-2}$ is HWV.

Also, note: ⊛ $\langle X_1, X_2, \dots \rangle \equiv W$ is subrep of V, with: $V/W \cong V(0)$.

- More generally: if $V^{(n)} \equiv$ Space, with basis $\{v_0, v_1, \dots\}$ with $sl_2(\mathbb{C})$-action $ev_0 = 0$, $hv_0 = nv_0$ & $fv_i = v_{i+1}$

then v_{n+1} is HWV & $W_n \equiv \langle v_{n+1}, v_{n+2}, \dots \rangle \leq V^{(n)}$ has $V^{(n)}/W_n \cong V(n)$.

§18: Heighest Weight & Verma Modules.

Assumptions: $\mathfrak{g}$ Semisimple, $\underline{t}$ CSA, Φ Roots of $\mathfrak{g}$, $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ Root basis.

If V is $\mathfrak{g}$-rep, $V_\lambda \equiv \{v \in V : tv = \lambda(t)v \;\; \forall t \in \underline{t}\}$.

Remarks) 1) V_λ makes sense, even if V inf-dim.

2) Def. of HWV also makes sense if V inf-dim

3) If $e_\alpha \in \mathfrak{g}_\alpha$, $e_\alpha \neq 0 \Rightarrow e_\alpha V_\lambda \subset V_{\lambda+\alpha}$.

DEF 18.1) V $\mathfrak{g}$-rep. Is: Heighest Weight Module, if V contains HWV v, with: $V = \mathcal{U}(\mathfrak{g})v$. HWM.

Example 18.2) ⊛ Any F.D. irred rep V of $\mathfrak{g}$ is HWM.
(Since: 17.1 $\Rightarrow$ V has HWV v, & 17.6 $\Rightarrow$ $\mathcal{U}(\mathfrak{g})v$ Subrep, so $\mathcal{U}(\mathfrak{g})v = V$ by irred of V)

⊛ Example at end of §17. $\Rightarrow$ x_0 HWV, $x_i = f^i x_0$, $\Rightarrow V \equiv \mathcal{U}(\mathfrak{g})x_0$ is a HWM.

Remarks) 1) Not necessarily HWM $\Rightarrow$ irred.

2) If V inf-dim HWM & $v \in V_\lambda$ HWV, then λ is not necessarily dominant. (17.1, 2)).

Notation $\eta^+ \equiv \sum_{\alpha \in \Phi^+} g_\alpha$, $\eta^- = \sum_{\alpha \in \Phi^-} g_\alpha$. $(\Rightarrow g = \eta^- \oplus \underline{t} \oplus \eta^+)$

Lemma 18.3 V HWM, & $v \in V$ s.t. $V = \mathcal{U}(g)v$.
Then, $V = \mathcal{U}(\eta^-)v$.

Proof Choose basis $\{x_1, \dots, x_n\} \subset \eta^-$ & $\{t_1, \dots, t_\ell\} \subset \underline{t}$ & $\{y_1, \dots, y_n\} \subset \eta^+$. Then: $\mathcal{U}(g) = \mathcal{U}(\eta^-) \otimes \mathcal{U}(\underline{t}) \otimes \mathcal{U}(\eta^+)$
& $\mathcal{U}(g)v = \langle x_1^{k_1} \cdots x_n^{k_n} t_1^{m_1} \cdots t_\ell^{m_\ell} \cdots y_1^{r_1} \cdots y_n^{r_n} v \rangle$.
But: $y_i v = 0 \; \forall i$ & $t_i v \in \langle v \rangle$. So, $\underline{\mathcal{U}(g)v = \mathcal{U}(\eta^-)v}$

Prop 18.4 V HVM, HWV $v_\lambda \in V_\lambda$, s.t. $V = \mathcal{U}(g)v_\lambda$. Then:

1) $V = \bigoplus_{\mu \in D(\lambda)} V_\mu$, $D(\lambda) \equiv \{\lambda - \sum_{1 \le i \le \ell} k_i \alpha_i,\; k_i \in \mathbb{Z}_{\geq 0}\}$.
"Descent of λ".

2) Any sub-module of V is discrete sum of weight spaces V_μ.

3) $\dim(V_\lambda) = 1$ & Any other V_μ is finite-dim.

4) V irred $\Leftrightarrow$ every HVW lies in V_λ.

5) V contains: Maximal (proper) Subrepresentation.

Lie Algebras: Lecture 21 | 22/11/2023

Proof of Prop 18.4

1) $V = \mathcal{U}(\eta^-)v_\lambda$. $\Rightarrow$ Contains expressions of form:

$e_{-\beta_1} \cdots e_{-\beta_k} v_\lambda$, for $\beta_i \in \Phi^+$, $k \geq 0$ & $e_{-\beta_i} \in \mathfrak{g}_{-\beta_i}$.

The weight of this expression is: $\lambda - \sum \beta_i$.

$\Rightarrow$ Generators lie in: $V_{\lambda - \sum \beta_i}$ $(\beta_i \in \Phi^+)$.

So, 1) & 3) hold, since: $\forall \mu$, only finitely many ways $\mu = \lambda - \sum \beta_i$, for $\beta_i \in \Phi^+$.

2) Let: $W \leq V$ submodule. & $\forall w \in W$, write w as sum of $v_i \in V_{\mu_i}$, belonging to distinct weight spaces.

Need: all $v_i \in W$.

If not: Pick $w = \sum_{1 \leq k \leq n} v_i$ (n minimal) s.t. $n > 1$. Then, in particular, $v_i \notin W$ $\forall i$.

Find: $t \in \mathfrak{t}$ s.t. $\mu_1(t) \neq \mu_2(t)$. $\Rightarrow tw = \sum \mu_i(t) v_i \in W$.

As does: $(t - \mu_1(t) \cdot 1) w = (\mu_2(t) - \mu_1(t)) v_2 + \cdots + (\mu_n(t) - \mu_1(t)) v_n$

$\neq 0$.

$\Rightarrow$ By minimality of w: get $v_2 \in W$ ↯. ✓

4) Suppose V has HWV $v_\mu \in V_{\mu_i}$ for some $\mu \neq \lambda$.

$\Rightarrow \mathcal{U}(\mathfrak{g})v_\mu$ is subrep & $v_\lambda \notin \mathcal{U}(\mathfrak{g})v_\mu$.

(Because: weights of $\mathcal{U}(g)v_\mu$ are of form $\mu - \sum k_i \alpha_i$.)

$\Rightarrow \mathcal{U}(g)v_\mu$ is a nontrivial proper subrep.

Conversely: say $U \subsetneq V$ proper subrep. Then, by 2), U is direct sum of weight spaces V_μ.

Choose: μ, s.t. $\mu = \lambda - \sum_{1 \le i \le n} k_i \alpha_i$ & $\sum k_i$ minimal, and $v_\mu \in U$.

Then, if $v_\mu \neq 0$ (in V_μ), then $\forall e_\alpha \in g_\alpha$ ($\alpha \in \Phi$):

$e_\alpha v_\mu \in V_{\alpha+\mu} \cap U = \{\phi\}$ (By choice of U)

$\Rightarrow v_\mu$ is HWV, for V ✓

5) Easy

Exercises 1) V indecomposable (as g-module)

2) Any nonzero homomorphic image of V is also HWM.

Towards Verma Modules

If Φ roots, $\alpha \in \Delta$ & λ dominant, then: $\lambda - \alpha$ also dominant. In particular, $(\lambda-\alpha, \lambda-\alpha) \le (\lambda, \lambda)$.

(Because: LHS $= (\lambda,\lambda) - (\alpha,\lambda) - (\lambda-\alpha,\alpha)$).

DEF 18.5 If V HWM, $v \in V_\lambda$ & $V = \mathcal{U}(g)v$ for some HWV v, say: V is of Highest weight λ.

Let: $\mathfrak{t}$ CSA for g, Φ root system & Δ basis.

$\Phi^+ =$ Positive roots WRT Δ; & choose basis $\{h_1, \ldots, h_\ell\} \subset \mathfrak{t}$ with $h_i = h_{\alpha_i}$ $(\alpha_i \in \Delta)$.

For $\lambda \in \mathfrak{t}^*$: let $J(\lambda)$ be: left ideal of $\mathcal{U}(\mathfrak{g})$, generated by e_α $(\alpha \in \Phi)$ & $h_i - \lambda(h_i)1$ $(1 \le i \le \ell)$.

$\Rightarrow J(\lambda)$ has elements $\{\sum u_\alpha e_\alpha + \sum y_i (h_i - \lambda(h_i)1)\}$ for $u_\alpha, y_i \in \mathcal{U}(\mathfrak{g})$. This is left-module of $\mathcal{U}(\mathfrak{g})$.

DEF 18.6) $M(\lambda) \equiv \mathcal{U}(\mathfrak{g}) / J(\lambda)$

Is: $\mathcal{U}(\mathfrak{g})$ - module, with Action: $u \cdot (v + J(\lambda)) = uv + J(\lambda)$.

We say: $M(\lambda)$ is the Verma module associated with $\lambda \in \mathfrak{t}^*$.

Prop 18.7) 1) $M(\lambda)$ is HWM, heighest weight λ

2) $M(\lambda)$ Universal: $\forall m_\lambda \in M(\lambda)_\lambda$ HWV & V HWM of heighest weight λ: $v_\lambda \in V_\lambda$: $\uparrow$ λ-weight space

$\Rightarrow \exists!$ $\mathfrak{g}$-equivariant linear map $M(\lambda) \to V$

$$m_\lambda \mapsto v_\lambda.$$

Proof) 1) Let $\boxed{m_\lambda = 1 + J(\lambda)} \in M(\lambda)$. (Generator)

$\Rightarrow h_i m_\lambda = h_i + J(\lambda) = \lambda(h_i)1 + J(\lambda)$

(Since: $h_i - \lambda(h_i)1 \in J(\lambda)$)

& If $\alpha \in \Phi^+$, $e_\alpha \in \mathfrak{g}_\alpha$ then $e_\alpha m_\lambda = e_\alpha + J(\lambda) = J(\lambda)$.

Hence: $M(\lambda)$ module & m_λ HWV of ~~height~~ highest weight λ.

(and clearly: $M(\lambda) = \mathcal{U}(\mathfrak{g}) m_\lambda$)

So, any other HWV is a scalar multiple of this one.

2) By PBW theorem: If $\Phi^+ = \{\beta_1, \dots, \beta_r\}$, then:

$e_{-\beta_1} \dots e_{-\beta_r} m_\lambda$ $(r \in \mathbb{Z}_{\geq 0})$ basis for m_λ.

So, define: $\varphi : M(\lambda) \longrightarrow V$, $e_{-\beta_1} \dots e_{-\beta_r} m_\lambda \mapsto$ same.

Then, this is equivariant: $\varphi(e_\alpha w) = e_\alpha \cdot \varphi(w)$.

Lie Algebras: lecture 22

24/11/2023

From last time: Verma Module is HWM, heighest wt λ,

(*) Universal: If $m_\lambda \in M(\lambda)_\lambda$ HVW & V HWM, heighest wt λ & HWV $v_\lambda \in V_\lambda$, then: $\exists!$ g-equiv map

$$\begin{aligned} M(\lambda) &\longrightarrow V \\ m_\lambda &\longmapsto v_\lambda . \end{aligned}$$

Lemma 18.8 Given $\lambda \in \mathfrak{t}^*$: $\exists!$ irred HWM, of heighest weight λ (called $V(\lambda)$).

Proof Know: by 18.4: ~~∃~~ $M(\lambda)$ has Unique, Maximal (proper) ~~subspa~~ submodule J. $\Rightarrow M(\lambda)/J$ irred & Uniqueness comes from Universal Property. ✓

Examples 1) Example, at end of §17. $V = M(0)$.
$J = \langle v_1, v_2, \ldots \rangle$ & $M(0)/J \cong V(0)$ (trivial sl_2-rep)

2) [EW, Ex 15.12]: $\exists$ Irred Verma module for $sl_2(\mathbb{C})$.
$\Rightarrow$ Shows $g = sl_2(\mathbb{C})$ has inf-dim irred reps.

Remarks 1) Verma modules are building blocks for so-called "Category $\mathcal{O}$". Although each $M(\lambda)$ is inf-dim, when viewed as $\mathcal{U}(g)$-modules it has finite Length: i.e. $\exists$ Submodules $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_r = M(\lambda)$

s.t. M_{i+1}/M_i irred $\forall i$.

2) In 1985: Drinfeld & Jimbo defined Quantum Groups by "deforming" Universal enveloping algebras of Lie Algebras. They have many applications in Theoretical Physics, knot theory & Rep theory of Algebraic Groups.

Theorem 18.9 $V = V(\lambda)$ is f.d. (irred) $\mathfrak{g}$-module iff λ dominant.

Proof $\Rightarrow$: If V f.d. $\Rightarrow \forall \alpha_i$ simple root α_i, let m_{α_i} be corresponding $sl_2(\mathbb{C})$-copy in $\mathfrak{g}$.
Then, V is also f.d. module, for m_{α_i}.
& A HWV for $\mathfrak{g}$ is also HWV for m_{α_i}.
In particular, Since $\exists$ HWV of weight λ, ~~the weight for the CSA~~ $\cancel{\mathfrak{t}_i \subset m_{\alpha_i}}$ ~~is determined by:~~ $\cancel{\lambda(h_i)}$ $\cancel{(h_i = h_{\alpha_i})}$. ~~Since~~ $h_i(v) = \lambda(h_i)v = \langle \lambda, \alpha_i^\vee \rangle v$, it follows $\underline{\lambda(h_i) \in \mathbb{Z}_{\geq 0}}$ ✓

$\Leftarrow$: $V(\lambda)$ is direct sum of f.d. weight spaces (since $M(\lambda)$ is). Idea: Show that $\Pi(\lambda) \equiv \{\mu : V(\lambda)_\mu \neq 0\}$ (Set of weights) is Finite.

Know: $\Delta = \{\alpha_1, \dots, \alpha_\ell)$ Simple roots, & $\forall \alpha_i \in \Delta$, have:

12

Know: $\{x_i, y_i, h_i\}$ is $(\mathfrak{m}_{\alpha_i})$-tuple.

We need: $\forall k \geq 0$, (in $\mathcal{U}(\mathfrak{g})$):

i) $[x_j, y_i^{k+1}] = 0 \quad \forall i \neq j$, ii) $[x_i, y_i^{k+1}] = -(k+1) y_i^k (k1 - h_i)$

For proof, [Hu, 21.2].

$\boxed{\text{1) } V(\lambda) \text{ contains nonzero FD } \mathfrak{m}_{\alpha_i}\text{-module } \forall i.}$

Let: v HWV in $V(\lambda)$. Since λ dominant:

$n_i \equiv \langle \lambda, \alpha_i^\vee \rangle = \lambda(h_i) \in \mathbb{Z}_{\geq 0} \ \forall i.$

Claim: $u \equiv y_i^{n_i+1} v = 0$.

i) $\Rightarrow x_j u = y_i^{n_i+1}(x_j v) + [x_j\, y_i^{n_i+1}] v = 0 \quad (i \neq j)$

ii) $\Rightarrow x_i u = y_i^{n_i+1}(x_i v) + [x_i, y_i^{n_i+1}] v$

$= -(n_i+1) y_i^{n_i}(n_i - h_i) v = 0$ Since $h_i v = n_i v$.

So, if $u \neq 0$, then it would be a HWV of weight

$\lambda - (n_i+1)\alpha_i < \lambda$ ⨳ since HWV is unique.

Hence: $W \equiv \langle v, y_i v, \ldots, y_i^{n_i} v \rangle$ is nonzero FD $\mathfrak{m}_{\alpha_i}$-subrep of $V(\lambda)$ ✓

[Check: $h_i W \subseteq W$, $y_i W \subseteq W$ & $x_i W \subseteq W$.]

$\boxed{\text{2) } \forall i: V(\lambda) \text{ is Sum of all FD } \mathfrak{m}_{\alpha_i}\text{-~~modules~~ subreps contained in it.}}$

Let: W be this sum. & Show, W is $\mathfrak{g}$-submodule of $V(\lambda)$. By 1): this forces: $W = V(\lambda)$.

Need: $\forall x \in \mathfrak{g}$, $w \in W$, need: $xw \in W$.

Since $w \in W$, $\exists W' \subseteq$ finite $\mathfrak{m}_{\alpha_i}$-module with $w \in W'$.

Let: $x = \sum_{\beta \in \Phi \cup \{0\}} x_\beta$ & $x_\beta \in \mathfrak{g}_\beta$ $\forall \beta$.

$\Rightarrow xw \in \mathrm{Span}_\beta \{x_\beta W'\} = W''$, F.D. & $\mathfrak{m}_{\alpha_i}$-invariant.

[Since: $x_i W'' \subseteq \mathrm{Span}_\beta \{x_i x_\beta W'\}$,

& $x_i x_\beta W' = x_\beta x_i W' + [x_i, x_\beta] W' \subseteq x_\beta W' + \mathfrak{g}_{\alpha_i+\beta} W' \subseteq W''$]

(and similarly for y_i & h_i).

$\Rightarrow xw \in W'' \subseteq W$.

Then, show: Weyl group acts on $\Pi(\lambda)$ by Permutations.

If true: $\Pi(\lambda)$ decomposes into disjoint union of Weyl group orbits. So, it's enough to show only Finitely many orbits: use this with Finite Weyl group $\Rightarrow$ $\Pi(\lambda)$ Finite.

---

3) $\mu \in \Pi(\lambda) \Rightarrow w_{\alpha_i}\mu \in \Pi(\lambda)$ (w = Reflection).

& $\mathrm{Dim}\, V(\lambda)_\mu = \mathrm{Dim}\, V(\lambda)_{w_{\alpha_i}\mu}$.

Lie Algebras: Lecture 23 27/11/2023

Proof [Continued].

3) Need: $\forall \mu \in \Pi(\lambda)$: $w_{\alpha_i}(\mu) \in \Pi(\lambda)$
& $\mathrm{Dim}(V(\lambda)_\mu) = \mathrm{Dim}(V(\lambda)_{w_{\alpha_i}(\mu)})$.

Since $V(\lambda)_\mu$ f.d.: by 2), $\exists$ f.d. module m_{α_i}-module $U \supseteq V(\lambda)_\mu$. Then, for $0 \neq w \in V(\lambda)_\mu$, have $h_i w = \mu(h_i) w$.

But: $w \in U$ is then a weight vector for h_i.

$\Rightarrow$ By sl_2-theory: $\mu(h_i) = \langle \mu, \alpha_i^\vee \rangle \in \mathbb{Z}$. say m.

($\Rightarrow$ All weights $\in \mathbb{Z}$, as this holds $\forall i$).

As m occurs as a weight of U, so does $-m$, and:

$\dim(U_m) = \dim(U_{-m})$.

If $m \geq 0$, then: $y_i^m w \neq 0$ & lives in U_{-m}. But,

$y_i^m w \in V(\lambda)_{\mu - \langle \mu, \alpha_i^\vee \rangle \alpha_i}$ (since weights add)

$\Rightarrow w_{\alpha_i}(\mu)$ is also a weight.

Similarly, if $m < 0$, use $x_i^{-m} w$ instead. & Similar argument.

Hence: $w_{\alpha_i}(\mu) \in \Pi(\lambda)$ ✓

For equality of Dimensions: say $\{w_1, \dots, w_r\}$ Basis $\subset V(\lambda)_\mu$

then: $\{w_1, \dots, w_r\}$ linearly indep in U_m.

$\Rightarrow$ Apply y_i^m or x_i^{-m}, this keeps them linearly indep.

& sends w_i to $V(\lambda)_{\mu - \langle \mu, \alpha_i^\vee \rangle \alpha}$.

$\Rightarrow \mathrm{Dim}(V(\lambda)_\mu) \leq \mathrm{Dim}(V(\lambda)_{w_{\alpha_i}\mu}) \leq \mathrm{Dim}(V(\lambda)_{w_{\alpha_i}^2 \mu})$

Gives equality. $= \mathrm{Dim}(V(\lambda)_\mu)$ ✓

4) For $\mu \in \Pi(\lambda)$: its Weyl orbit W_μ contains a Dominant weight.

Know: W_μ is finite $\Rightarrow \exists\, \eta \in W_\mu$, maximal, WRT $\leq$.

Then: η is Dominant. [If not: then $\langle \eta, \alpha_i^\vee \rangle \in \mathbb{Z}_{<0}$, for some i. ~~Then, $w_i \mu \in W_\mu$ &~~

Then: $w_i \eta \in W_\mu$ & $w_i \eta = \eta - \langle \eta, \alpha_i^\vee \rangle \alpha_i > \eta$. ※]

5) The set $S \equiv \{\eta : \eta$ dominant & $\eta \leq \lambda\}$ Finite.

If $\eta \in S \Rightarrow \lambda - \eta =$ Sum of positive roots, with $\mathbb{Z}_{\geq 0}$-coefficients. Also, $\lambda + \eta$ dominant: since, $\langle \lambda + \eta, \alpha_i^\vee \rangle \geq 0 \ \forall i$.

$\Rightarrow$ In particular: $(\lambda + \eta, \lambda - \eta) \geq 0 \Rightarrow (\lambda, \lambda) \geq (\eta, \eta)$.

$\Rightarrow \eta$ lies in Compact set (Closed ball radius $\sqrt{(\lambda,\lambda)}$).

Hence: S Compact & Discrete $\Rightarrow$ Finite.

To finish: From (4): any W-orbit of $\Pi(\lambda)$ contains an element of S. By (5) S finite. ✓

Hence, just showed Bijection:

$\{\text{Dominant weights } \lambda\} \Longleftrightarrow \{\text{FD irred reps } V(\lambda) \text{ of } \mathfrak{g}\}$.

§19: Weyl Character Formula.

Let: $\mathfrak{g}$ Semisimple Lie Algebra & $\mathfrak{t}$ CSA for $\mathfrak{g}$.

$\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ & Φ Roots of $\mathfrak{g}$, X weight lattice, W Weyl grp.

Example) Define $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$.

Compute: $\langle \rho, \alpha^\vee \rangle$ $\forall \alpha \in \Delta$ in types: A_1, A_2, B_2.

For A_2, B_2: $\Delta = \{\alpha_1, \alpha_2\}$.

Claim: $\rho = \omega_1 + \omega_2$ (ω_i = fundamental dominant wts $\Longleftrightarrow$ Dual basis to coroots.)

* A_1: $\Phi = \{-\alpha, \alpha\}$, $\rho = \frac{\alpha}{2} \Rightarrow \langle \rho, \alpha^\vee \rangle = 1$.
* A_2: $\langle \rho, \alpha_i^\vee \rangle = \langle \omega_1 + \omega_2, \alpha_i^\vee \rangle = 1$
* B_2: Same computation.

Claim: $\rho = \sum_{1 \leq i \leq \ell} \omega_i$ (ω_i from 16.3)

Enough to check $\langle \rho, \alpha_i^\vee \rangle = 1$ $\forall \alpha_j \in \Delta$.

Because: $w_{\alpha_j}(\rho) = \rho - \langle \rho, \alpha_j^\vee \rangle \alpha_j$.

& also $w_{\alpha_j}(\rho) = w_{\alpha_j}\left(\frac{1}{2}\left(\sum_{\alpha \in \Phi^+ - \{\alpha_j\}} \alpha\right) + \frac{1}{2}\alpha_j\right)$

& and know: w_{α_j} permutes $\Phi^+ - \{\alpha_j\}$, so this is:

$$= \frac{1}{2} \sum_{\alpha \in \Phi^+ - \{\alpha_j\}} \alpha - \frac{1}{2}\alpha_j = \rho - \alpha_j.$$

By comparing the 2 expressions $\langle \rho, \alpha_i^\vee \rangle = 1$ ✓

<u>Recall</u>: from (18.9), that $\Pi(\lambda) \equiv \{\mu : V(\lambda)_\mu \neq 0\}$.

<u>Possible Questions</u>:

* What is $\Pi(\lambda)$?
* What is $\dim V(\lambda)$?

<u>DEF 19.1</u> Partial Ordering on X:

$$\mu \leq \lambda \iff \lambda - \mu = \sum_{1 \leq i \leq l} k_i \alpha_i \text{ for } k_i \in \mathbb{Z}_{\geq 0}. \; \forall i.$$

<u>Note</u> 1) $\Pi(\lambda) \subseteq \{\mu : \mu \leq \lambda\}$.

2) To determine $\Pi(\lambda)$, need to only dominant weights in it.

<u>Prop 19.2</u> Suppose λ, μ both dominant. Then:

$\mu \in \Pi(\lambda) \iff \mu \leq \lambda$.

<u>Proof</u> Suppose $\mu \leq \lambda \Rightarrow \mu = \lambda + \sum_{\alpha \in \Phi^+} k_\alpha \cdot \alpha \quad (k_\alpha \geq 0)$

So prove by <u>Induction</u> on $\sum k_\alpha$.

For $\sum k_\alpha = 0 \Rightarrow \mu = \lambda \Rightarrow \mu \in \Pi(\lambda)$

<u>Inductive step</u>: If $\mu = \lambda - \alpha, (\alpha \in \Phi^+) \Rightarrow \langle \mu, \alpha^\vee \rangle$
$= \langle \lambda, \alpha^\vee \rangle - 2 \geq 0. \Rightarrow \underline{\langle \lambda, \alpha^\vee \rangle \geq 2}$.

$\Rightarrow$ For $v_\lambda \in V_\lambda$ nonzero: Since $h_\alpha \cdot v_\lambda = n v_\lambda \quad (n \geq 2)$, know $e_{-\alpha} v_\lambda \neq 0$ (by usual sl_2-theory) in $V(\lambda)_{\lambda-\alpha} = V(\lambda)_\mu$)

$\boxed{\text{Continued next time!}}$

Lie Algebra: Lecture 24) 01/12/2023

Proof of 19.2) [Continued].

Suppose: know claim for $\sum k_\alpha = n-1$.

& Suppose: $\sum k_\alpha = n$, so that $\mu = \lambda - (\beta_1 + \dots + \beta_n)$.

Case 1) $\langle \beta_i, \beta_j^\vee \rangle < 0$, some (i,j). WLOG $i<j$.

Then, $\beta_i + \beta_j$ Positive root

$$\Rightarrow \sum_{1 \le k \le n} \beta_k = \sum_{1 \le k < i} \beta_k + \sum_{i<k<j} \beta_k + (\beta_i + \beta_j). \checkmark$$

Case 2) $\langle \beta_i, \beta_j^\vee \rangle \ge 0 \ \forall i,j \ (i \ne j)$.

Claim $\lambda - \sum_{1 \le i \le r} \beta_i \in \Pi(\lambda) \ \forall 1 \le r \le n$.

Proof of claim Induction on r. For $r=1$, notice:

$$0 \le \langle \lambda - \sum_{1 \le i \le n} \beta_i, \beta_1^\vee \rangle = \langle \lambda, \beta_1^\vee \rangle - 2 - \sum_{2 \le i \le n} \langle \beta_i, \beta_1^\vee \rangle$$

$\Rightarrow \langle \lambda, \beta_1^\vee \rangle \ge 2$

$\Rightarrow$ By considering action of m_β, get: $\lambda - \beta_1 \in \Pi(\lambda)$.

Induction: Same logic applies $\Rightarrow \langle \lambda - \sum_{1 \le i \le r} \beta_i, \beta_r^\vee \rangle \ge 0$.

$\Rightarrow$ By considering action of m_{β_r}: $\lambda - \sum_{1 \le i \le r} \beta_i \in \Pi(\lambda) \checkmark$

DEF 19.3) Let $\mathbb{Z}[X]$ Free $\mathbb{Z}$-module, with Basis $\{e^\mu : \mu \in X\}$, with mult. $e^\mu \cdot e^\lambda = e^{\mu+\lambda}$.

$\Rightarrow$ This is Commutative Ring with $1 = e^0$.

Let: V F.d. rep of $\mathfrak{g}$. The formal character of V is:

$$ch(V) \equiv \sum_{\mu \in X} (\dim V_\mu) e^\mu \in \mathbb{Z}[X].$$

Theorem 19.4 [Weyl Character Formula]

If λ Dominant weight & $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \sum_{1 \leq j \leq n} \omega_j$, then:

$$ch(V(\lambda)) = \frac{\sum_{w \in W} sgn(w) e^{w(\lambda+\rho)}}{e^\rho \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})}$$

Corollary 19.5 [Weyl Denominator Identity].

$$e^\rho \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}) = \sum_{w \in W} sgn(w) e^{w\rho}.$$

Corollary 19.6 If λ dominant, then

$$\dim(V(\lambda)) = \frac{\prod_{\alpha \in \Phi^+} \langle \lambda + \rho, \alpha^\vee \rangle}{\prod_{\alpha \in \Phi^+} \langle \rho, \alpha^\vee \rangle} = \prod_{\alpha \in \Phi^+} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)}.$$

Proof* By def: $(V = V(\lambda))$ $ch(V) = \sum_{\mu \in X^+} \dim(V_\mu) e^\mu$.

Would: like to "substitute" $e^\mu = 1$ into (19.4), but: could get $0/0$, so doesn't work.

Instead: for $\mu \in X$, $p \in \mathbb{Z}[X]$, define $F_\mu(p) : \mathbb{R}_{>0} \to \mathbb{R}$ by: $F_\mu(e^\lambda)(q) \equiv q^{-(\mu, \lambda)}$ & Extend linearly.

Note: $F_\mu(\rho)$ Multiplicative & Continuous, differentiable function on $\mathbb{R}_{>0}$.

Clearly: $F_0(e^\lambda) = 1 \Rightarrow F_0(ch(V(\lambda))) = \dim(V(\lambda))$.

First: apply F_μ to (19.5). $\Rightarrow \forall \mu \in X,\ q \in \mathbb{R}^+$:

$$q^{-(\rho,\mu)} \prod_{\alpha \in \Phi^+} (1 - q^{(\alpha,\mu)}) = \sum_{w \in W} (\operatorname{sgn} w) q^{-(w\rho,\mu)}$$

$$= \sum_{w \in W} (\operatorname{sgn} w) q^{-(\rho, w\mu)} \quad (*)$$

[since: $\operatorname{Sgn}(w) = \operatorname{Sgn}(w^{-1})$ & $(x, wy) = (w^{-1}x, y)$]

Now, apply F_ρ to (19.4):

$$F_\rho(ch(V(\lambda)))(q) = \frac{\sum_{w \in W} (\operatorname{sgn} w) q^{-(\rho, w(\lambda+\mu))}}{q^{-(\rho,\rho)} \prod_{\alpha \in \Phi^+} (1 - q^{(\rho,\alpha)})}.$$

[If $(\rho,\alpha) \neq 0\ \forall \alpha \in \Phi^+$. But, recall $(\rho, \alpha_i) = 1 > 0$ for any simple root α_i, so indeed $(\rho,\alpha) > 0\ \forall \alpha \in \Phi^+$]

Use (*) with $\mu = \lambda + \rho$:

$$F_\rho(ch(V(\lambda))) = \frac{q^{-(\rho,\lambda+\rho)} \prod_{\alpha \in \Phi^+} (1 - q^{(\alpha,\lambda+\rho)})}{q^{-(\rho,\rho)} \prod_{\alpha \in \Phi^+} (1 - q^{(\rho,\alpha)})}$$

Where: we used (*) for Weyl denominator identity, and applied to the numerator.

Finally: $F_\rho(ch(V(\lambda)))(q) = \sum (\dim V(\lambda)_\mu) q^{-(\rho,\mu)}$

$\Rightarrow$ Take: limit $q \to 1$ & use L'Hôpital's identity:

$$\dim V(\lambda) = \frac{\prod_{\alpha\in\Phi^+}(\lambda+\rho,\alpha)}{\prod_{\alpha\in\Phi^+}(\rho,\alpha)} = \frac{\prod_{\alpha\in\Phi^+}\langle\lambda+\rho,\alpha\rangle}{\prod_{\alpha\in\Phi^+}\langle\rho,\alpha\rangle}. \quad \checkmark$$

Examples | 1) $\mathfrak{g} = sl_2$, $w_1 = \frac{1}{2}\alpha = \rho$, $X^+ = \{\underbrace{mw_1}_{=\lambda}\}$. (some $m \geq 0$).

$\Rightarrow \operatorname{Dim} V(\lambda) = \dfrac{(m+1)(\alpha,\alpha)}{(\alpha,\alpha)} = m+1$ (as expected)

2) $\mathfrak{g} = sl_3$. $\Phi^+ = \{\alpha, \alpha+\beta, \beta\}$. Let $\lambda = m_1 w_1 + m_2 w_2$.

Here, $\rho = \alpha+\beta = w_1 + w_2$.

To compute dimension:

	α	β	$\alpha+\beta$
$(\lambda+\rho, \cdot)$	m_1+1	m_2+1	m_1+m_2+2
$(\rho, \cdot)$	1	1	2

$$\Rightarrow \prod\langle\lambda+\rho,\alpha^\vee\rangle = \langle m_1w_1+m_2w_2+\alpha_1+\alpha_2, \alpha_1^\vee\rangle \times \langle -, \alpha_2^\vee\rangle \times \langle -, (\alpha_1+\alpha_2)^\vee\rangle$$

$$\Rightarrow \dim V(\lambda) = \tfrac{1}{2}(m_1+1)(m_2+1)(m_1+m_2+2).$$

Exercise: Compute Dimensions for FD irred reps, of B_2 and G_2. [Hu, p140].

Example) [2018]. $\mathfrak{g} = sp_4(\mathbb{C})$. α_1 short. Type B_2.

1) Find dim $V(\lambda)$. $\lambda = aw_1 + bw_2 \in X^+$.

Calculate: $\langle w_1, \alpha_1^\vee\rangle = 1$. $\Rightarrow \prod_{\alpha\in\Phi^+}\langle\rho,\alpha^\vee\rangle = 6$.

$\lambda+\rho = (a+1)\omega_1 + (b+1)\omega_2$.

$\Rightarrow \prod_{\alpha\in\Phi^+} \langle \lambda+\rho, \alpha^\vee\rangle = (a+1)(b+1)(a+2b+3)(a+b+2)$.

Let: V defining rep, dim 4. $\underline{\underline{2)}}$ Find: Highest weight:

$\dim(V(\omega_1)) = 4$. (using Dimension Formula).

$\Rightarrow$ If W nontrivial rep of sp_4, then: by Dimension Formula, get $\dim(W) > 4$. So, $V \cong V(\omega_1)$.

$\underline{\underline{3)}}$ Decompose $V\otimes W$ into irred subreps:

Let: $v \in V(\omega_1)$ HWV. Then, $v\otimes v$ HWV for $V\otimes V$.

$\Rightarrow v\otimes v$ of weight $2\omega_1$. Hence, $V(2\omega_1) \subset V\otimes V$ subrep.

Now, $2\omega_1 = 2\alpha_1 + \alpha_2 \Rightarrow V(2\omega_1)$ Adjoint rep.

In particular, $V(2\omega_1)_{\omega_2}$ has dim $= 1$.

Take: $\{v_\gamma : \gamma \in \Pi(\omega_1)\}$ Basis of weight vectors of V.

$\Rightarrow \{v_\delta \otimes v_\gamma : \delta, \gamma \in \Pi(\omega_1)\}$ Basis of weight vectors for $V^{\otimes 2}$

& Know: $(V\otimes V)_{\omega_1}$ dim $=1$, $(V\otimes V)_{\omega_2}$ dim $=2$, $(V\otimes V)_0$ dim 4.

and all other weight spaces correspond to non-Dominant wts.

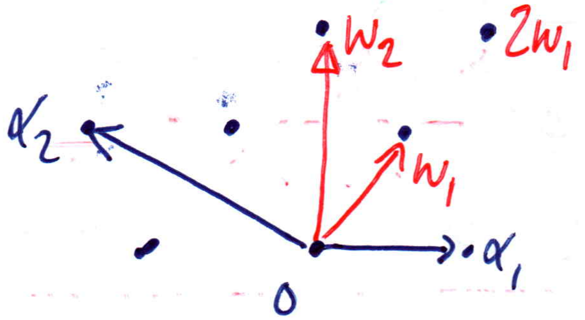
$\Rightarrow V(2\omega_1)_{\omega_2}$ dim 1

& $V(\omega_2)$ must be a Subrep of $V^{\otimes 2}$, of dim $=5$. (Weyl)

By counting dims: ~~dim~~ $V\otimes V = V(2\omega_1) \oplus V(\omega_2) \oplus V(0)$

$16 \quad 10 \quad 5 \quad 1$ ✓

Example [For 19.2]. Let G_2 have basis $\{\alpha_1, \alpha_2\}$ with α_1 Short.



Can you Compute $\Pi(2w_1)$?

Know: $\left.\begin{array}{l}\langle w_1, \alpha_1^\vee\rangle = 1 \\ \& \ \langle w_1, \alpha_2^\vee\rangle = 0.\end{array}\right]$

$\Rightarrow$ Dominant weights in $\Pi(2w_1)$ are: exactly the dominant μ, with: $\mu < 2w_1$ (by 19.2).

& $2w_1 = 2(\alpha_2 + 2\alpha_1) = 2\alpha_2 + 4\alpha_1$

& $w_2 = 3\alpha_1 + 2\alpha_2$. So, $w_2 < 2w_1$.

But, $w_1 + w_2 \not< 2w_1$. $\Rightarrow$ Dominant wts in $\Pi(2w_1)$ are: $\{w_1, 2w_1, w_2, 0\}$.

The Weyl Conjugates for w_1 are: Short roots.

The Weyl Conjugates for w_2 are: Long roots.

$\Rightarrow \Pi(2w_1) = \{$Short roots$\} \cup \{2\times$ Short roots$\} \cup \{$Long roots$\}$

$$= \Phi \cup \{\pm 2w_1, \pm 2\alpha_1, \pm 2(\alpha_1 + \alpha_2), 0\}.$$

[Typical exam question!]