Geometric Group Theory: lecture 1) 19/01/2023

I: Combinatorial Group Theory.

1) Free groups & permutation.

Let: $A = \{a_1, a_2, \dots\}$ an "alphabet". A group F is **free** on $A \iff$ [(i) $\exists$ map of sets $A \longrightarrow F$
(ii) $\forall G$ group & map $A \to G$, $\exists!$ hom $F \to G$ s.t.

$$\begin{tikzcd} & F \arrow[d, "(*)"] \\ A \arrow[ur] \arrow[r] & G \end{tikzcd}$$

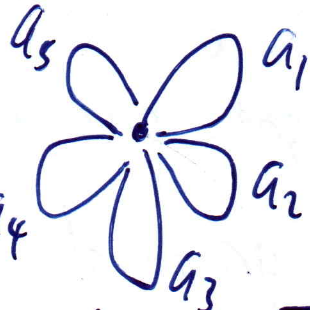
commutes. ("Universal Prop")

Exercise: F unique up to unique isom. (canonical.)

$\Rightarrow F$ determined by A, so write $F = F(A)$.

Existence: in 2 ways.

① Topologically: $X = \bigvee_{a \in A} S^1$



By Seifert-Van-Kampten Theorem: $\pi_1(X) \cong F(A)$.

② Combinatorial: let $A^* = \{\text{words in } A \sqcup A^{-1}\}$.

$(A^{-1} = \{a_1^{-1}, a_2^{-1}, \dots\})$ Formal inverses.

& say a word is reducible $\iff$ contains aa^{-1} or $a^{-1}a$ as sub-word. Else, called reduced.

Define: $F(A) = \{\omega \in A^* : \omega \text{ reduced}\}$.

Group operation is: concatenation then reduction.

(mult is well-defined, but won't prove lol)
identity e & obvious inverse.

<u>Presentations</u>: Consists of alphabet A ("generators") and a set $R \subseteq F(A)$ ("Relations"). <u>Write</u>: $\langle A \mid R \rangle$ for: group $F(A)/\langle\langle R \rangle\rangle$ ← Normal closure of R.

<u>Examples</u> (i) $\langle a \mid a^n \rangle \cong \mathbb{Z}/n\mathbb{Z}$. ($C_n$)

(ii) $\langle r, s \mid r^n, s^2, srsr \rangle \cong D_{2n}$.

(iii) $\langle A \mid \cdot \rangle \cong F(A)$

(iv) $\langle a, b \mid aba^{-1}b^{-1} \rangle \cong \mathbb{Z}^2$.

(v) $\langle a_1, \dots, a_g, b_1, \dots, b_g \mid \prod_{i \leq g} a_i b_i a_i^{-1} b_i^{-1} \rangle \cong \Sigma_g$

As we will see: presentations arise when we write down fundamental groups of spaces. In fact: All groups arise in this way.

<u>Corollary of SVK</u>: if $G = \langle a_1, a_2, \dots \mid r_1, r_2, \dots \rangle$, let X be space by taking $\bigvee_{a \in A} S^1$, and "gluing" discs whose boundary are r_i.

<u>Then</u>: $\pi_1(X) \cong G$. <u>Called</u>: Presentation Complex of G.

E.g. for $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$, get <u>Torus</u> T^2.

Group ~~Gr~~ Theory: lecture 2. 22/01/2024.

Remarks (i) CP is stronger than WP. ($w=1 \iff w$ conj to 1)
(ii) Dehn motivated by topology, but problems ask for algs.
Often will solve using geometry.
(iii) All 3 unsolved in 1950s.

- Novikov (1955) & Boone (1959) unsolved WP.
- Adyan (1959) & Rabin (1958) unsolved the IP.

Nevertheless: positive sols are known for many "reasonable" classes of groups. E.g. Free groups.

Let: $A = \{a_1, a_2, \dots\}$.

Example (in WP) For $w \in A^*$, if w reduced then $w = e$ $\iff w$ empty (length 0). Otherwise, w contains some aa^{-1} for $a \in A \sqcup A^{-1}$, and "cancelling" gives $w' \in A^*$ s.t. $w' =_{F(A)} w$ & $\text{length}(w') < \text{length}(w)$. So eventually terminates.

Can also solve CP in free groups!

DEF $\exists$ natural action ~~on~~ of $\mathbb{Z}$ on A^* that permutes cyclically:
$1(a_1 \dots a_n) = a_2 \dots a_n a_1$. $a_i \in A^*$.
$\Rightarrow$ Elements of $\mathbb{Z}$ are called: cyclic conjugation of w.
& $\mathbb{Z} \backslash A^*$ are called: cyclic words.
A word is cyclically reduced $\iff$ Every cyclic conjugate is reduced. So, e.g. aba^{-1} not cyclically reduced.

Note: reduced + not cyclically reduced $\Leftrightarrow$ $w = a w' a^{-1}$.

Where: w' reduced & length < length(w).

$\Rightarrow$ After finitely many steps/reps: Can assume w cyclically reduced.

Lemma (CP in Free groups).

If $u, v \in F(A)$ cyclically reduced, then u conj to v $\Leftrightarrow$ are cycles of each other.

Proof) $\Leftarrow$: u, v cyclically equal $\Rightarrow$ $u = a_1 a_2 \dots a_n$
$v = a_{k+1} - a_n a_1 - a_k$.

Then $u = g v g^{-1}$ for $g = a_1 \dots a_k$.

$\Rightarrow$: Say $u = g v g^{-1}$. By induction, assume g has 1 letter.

Say $g = a \in A \sqcup A^{-1}$. Then, if $u = a v a^{-1}$, either $v = a^{-1} v'$ or $v = v' a$.

In either case, $u = v' a^{-1}$ or $u = a v'$.

$\Rightarrow$ Are cyclic conj's as desired.

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§2: Historical Case Study.

Poincaré figured out: Homology classifies 2D compact surfaces.

Motivated: 3D sphere? $\Rightarrow$ Poincaré conjecture, V1.

Let M compact 3D manifold. If $H_*(M) = \begin{cases} \mathbb{Z}, & * = 0,3 \\ 0, & \text{else} \end{cases}$

then $M \underset{\text{homeo}}{\cong} S^3$.

Such a 3-manifold M is called: homology sphere.

Theorem (1904, Poincaré) There is a 3D homology sphere P s.t. $\pi_1(P) \twoheadrightarrow A_5$. $\Rightarrow$ Moral: Homology not enough! Need π_1.

Poincaré conjecture (V2) M compact, connected 3-manifold, $\pi_1(M)$ trivial. Then, $M \underset{\text{homeo.}}{\cong} S^3$

Proved in 2003 ~~using~~ by Perelman!

Are there any more homology spheres?

Theorem (1910, Dehn) There are infinitely many non-homeo 3D homology spheres.

Remark Isomorphism problem needed to make progress, i.e. to distinguish these manifolds.

In 1911, Max Dehn posed 3 problems.

1) <u>Word Problem</u>: Given $w \in A^*$, determine if $w=1$ in the group $G=\langle A \mid R\rangle$.

($\Leftrightarrow$ whether $w \in \langle\langle R\rangle\rangle$, in $F(A)$.)

2) <u>Conjugacy Problem</u>: Given $u, v \in A^*$, determine if u, v are conjugate in G.

3) <u>Isomorphism Problem</u>: Given $G=\langle A \mid R\rangle$ & $H=\langle A' \mid R'\rangle$, determine if $G \cong H$.

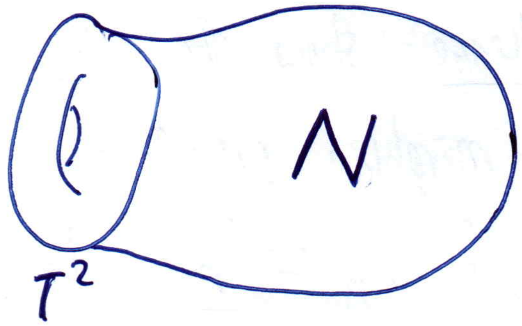
Geometric Groups: <u>Lecture 3.</u> 24/01/2024.

Dehn's Construction: let K trefoil knot:  $=K$.

<u>&</u> consider $K \subseteq \mathbb{R}^3 \subseteq \mathbb{S}^3$. Denote $\mathring{N}(K) =$ open, regular nbhood of K (i.e. "fatten up" K into a cylinder).

<u>&</u> define: $N = \mathbb{S}^3 \setminus \mathring{N}(K)$. Complement. $\Rightarrow$ Compact. <u>&</u> $\partial N \cong T^2$.

$$H_*(N) = \begin{cases} \mathbb{Z}, & * = 0,1 \\ 0, & \text{else.} \end{cases} \quad (MV)$$



<u>&</u> $\pi_1(N) = \langle x, y, z \mid x^2 = y^3 = z \rangle$.

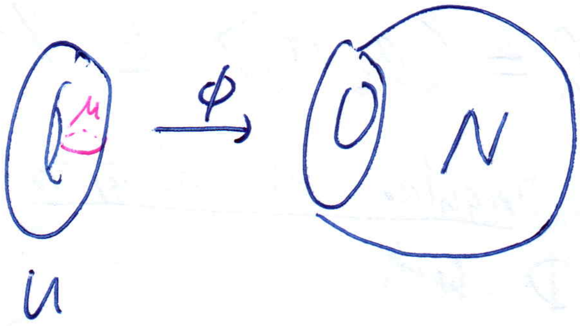
(redundant z, but want to name it.)

<u>Know</u>: $\pi_1(N) \xrightarrow{ab} H_1(N) \cong \mathbb{Z}$.

$$x \longmapsto 2$$
$$y \longmapsto 3$$
$$\Rightarrow z \longmapsto 6$$

<u>&</u> boundary T^2 is: $\pi_1(T^2) \cong \mathbb{Z}^2 = \langle xy, z \rangle$.

<u>Idea</u>: Glue $S^1 \times D^2$ to the border of N, to get a <u>closed</u> manifold.



$\Rightarrow M_\phi = N \underset{T}{\cup} U$.

$\Rightarrow M_\phi$ closed

<u>&</u> By SVK: $\pi_1(M_\phi) = \pi_1(N) / \langle\langle g \rangle\rangle$, $g = \phi_*(\mu)$.

<u>&</u> $H_1(M_\phi) = \mathbb{Z} / \langle [g] \rangle$.

To produce homology sphere: need to choose ϕ such that: $\sqrt{1}$

$g = \phi_*(\mu) \longmapsto 1$ in $H_1(N)$.

& $g = (xy)^a \cdot (z)^b$, then: $\overset{ab}{\longmapsto} 5a+6b$.

<u>Choose</u>: $a = 6m+5$, $b = -(5m+4)$ $\forall m \in \mathbb{Z}_{\geq 0}$

This constructs ϕ_n s.t. $\phi_n(\mu) = (xy)^a z^b = g_n$.

$\Rightarrow$ Gives family: $D_n = N \cup_{\phi_n} U$, with $\pi_1(D_n) = \langle x, y, z \mid x^2 = y^3 = z, (xy)^{6n+5} = z^{5n+4} \rangle = G_n$.

<u>Challenge</u>: show that D_n have different π_1.

(Isomorphism problem!)

<u>Note</u> If $G_n \cong G_m$, then g_n conj to g_m in $\pi_1 N$, so:

$g_n = g_m$ in $\pi_1 N$. So, need to solve CP for $g_n \in \pi_1(N)$.

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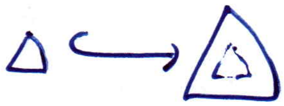
§3: Van Kampten Diagrams.

<u>DEF</u> A map of cell complexes $Y \xrightarrow{f} X$ is <u>combinatorial</u>, if:

$\forall k$ & $\forall k$-cell e^k of Y, f maps $\mathring{e}^k$ homeomorphically to interior of k-cell of X.



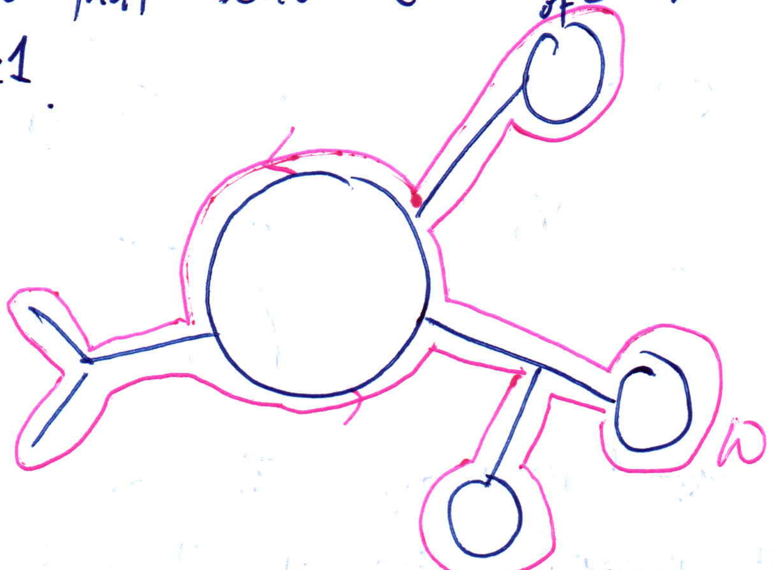
<u>Consider</u>: $G = \langle a_i \mid r_j \rangle$ & X = associated cell complex.

<u>DEF</u> A <u>singular disc diagram</u> is: compact & contractible 2-complex D with embedding $D \hookrightarrow \mathbb{R}^2$.

E.g.

Disc diagram D is <u>over</u> X $\Leftrightarrow$ equipped with a combinatorial map $D \longrightarrow X$.

$\Leftrightarrow$ Every oriented 1-cell of X is labelled with: some $a_i \in A$, so that: each 2-cell of D reads a cyclic conj of some $r_j^{\pm 1}$.

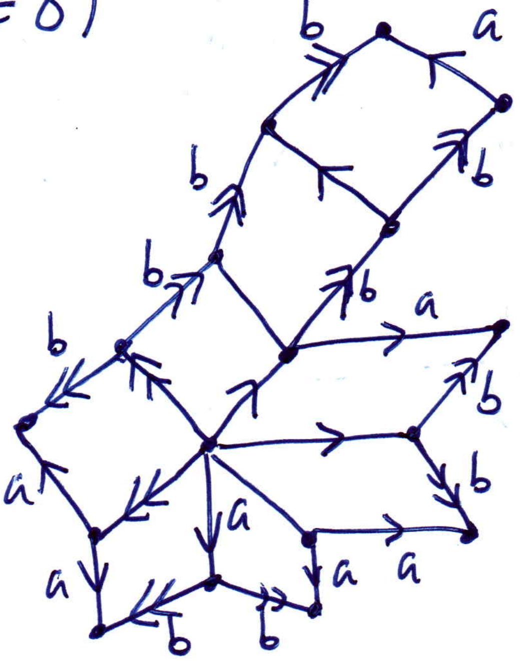


The boundary cycle <u>reads</u> a (cyclic) word $w \in A^*$, that reduces to an element $w' \in \langle\langle r_i \rangle\rangle \leq F(A)$.

D is said to be <u>Van-Kampten diagram</u> for w.

<u>Example</u>: $G = \langle a, b \mid aba^{-1}b^{-1} \rangle$, $w = b^{-1}b^3a^{-1}b^{-2}ab^{-1}ba^{-1} \cdots$
$\cdots ab^{-1}ba^{-1}a$.

$\Rightarrow$ Has the following VK-diagram.
(Area = 8)



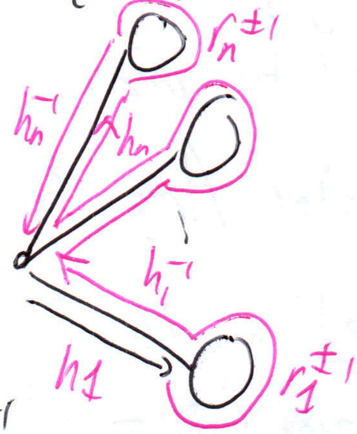
Geometric Group Theory: lecture 4. 26/01/2024

let: $G = \langle A | R \rangle = FA / \langle\langle R \rangle\rangle$. Group presentation.

Lemma (VK-lemma) $\forall w \in \langle\langle R \rangle\rangle$: $\exists$ VK-diagram for w.

Proof $w \in \langle\langle R \rangle\rangle \Rightarrow w \underset{FA}{=} \prod_{i=1}^{k} h_i r_i^{\pm 1} h_i^{-1}$ for $r_i \in R$ (normal closure)

$\Rightarrow$ Draw the "lollipop diagram":



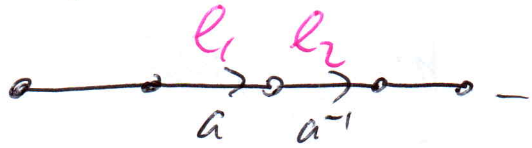
Call this word $w_0 \in F(A)$.

If w_0 is reduced: $\Rightarrow w = w_0$ in G,

so have the diagram.

Else: w_0 contains cancelling pair $a a^{-1}$.

(for some $a, a^{-1} \in A \sqcup A^{-1}$).

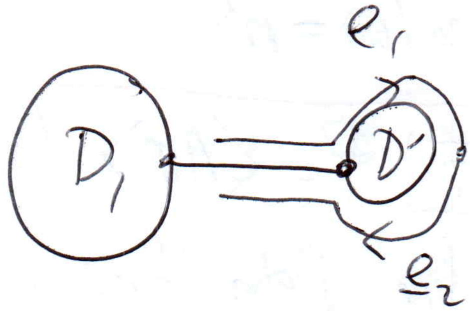


There are 2 cases to consider:

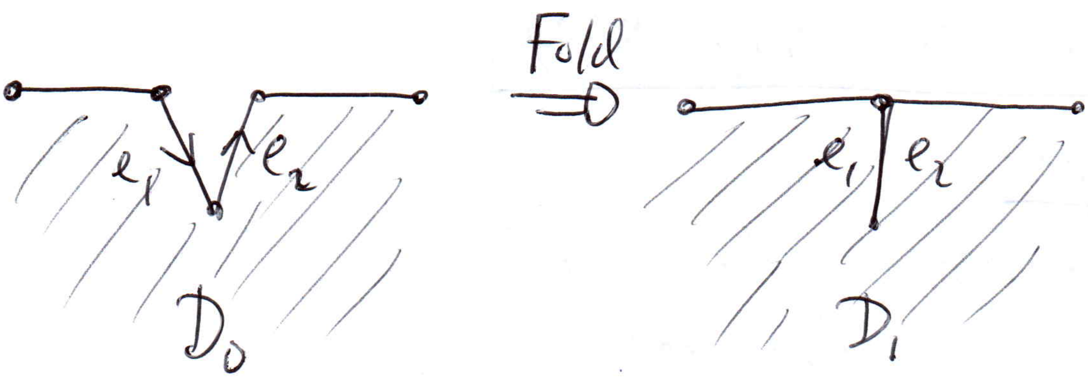
1) If origin of e_1 & terminates:

$\Rightarrow D_0 = D_1 \vee D'$.

$\Rightarrow D_1$ is VK-diagram for w.



2) Origin of e_1 is distinct from terminus of e_2.



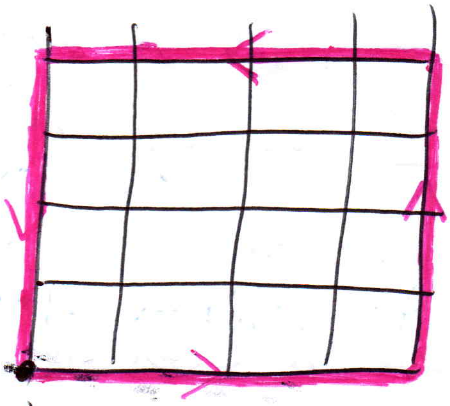
$\Rightarrow$ In either case: $w_1 = \partial D_1$ is obtained by cancelling (from w_0) a pair. So, can proceed finitely many times to get such VK-diagram.

$\Rightarrow$ Construct VK-diagram ~~for~~ D_n s.t. $\omega_n = \partial D_n$, and: $\omega_n \underset{F(A)}{=} \omega$. So, $\omega_n = \omega$ in G, so D_n is VK-diag for ω ✓

<u>Remark</u>) Minimal # of 2-cells in VK-diagram for ω is equivalent to: minimal k s.t. $\omega = \prod_{i=1}^{k} h_i r_i^{\pm 1} h_i^{-1}$.

k is called <u>Area</u> for ω.

<u>Example</u>) $G = \mathbb{Z}^2 = \langle a, b \mid [a,b] \rangle$ & $\omega = a^n b^n a^{-n} b^{-n}$.



<u>Has</u>: $\text{area}(D) = n^2 \sim C \cdot \ell(\omega)^2$.

In fact, this <u>is</u> minimal! so, $\text{area}(\omega) = n^2$.

<u>DEF</u>) $P = \langle A | R \rangle$ <u>finite</u> presentation for G.

Define Dehn function $\delta_P : \mathbb{N} \longrightarrow \mathbb{N}$, $\ell \longmapsto \max(\text{Area}(\omega))$ s.t. $\omega \in \langle\langle R \rangle\rangle$, $\text{length}(\omega) = \ell$.

<u>Remark</u>) Word problem solvable for P $\Leftrightarrow$ Dehn function for P is <u>computable</u>.

$\boxed{\text{II: Basics of Geometric Group Theory.}}$

§1: <u>Cayley Graphs.</u>

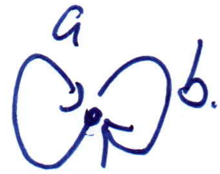
A <u>graph</u> is a 1-dimensional cell complex.

For G group: & $S \subset G$ finite generating set.

<u>DEF</u>) <u>Cayley Graph $\text{Cay}_S(G)$</u> defined by:

- Vertices of graph is G
- Edges of graph are: well, they correspond bijectively with $G \times S$ & $\underset{g}{\bullet} \longrightarrow \underset{gs}{\bullet}$ (act on RIGHT).

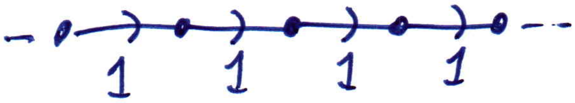
<u>Examples</u> • $1 \cong \langle a, b \mid a, b \rangle$ Graph =



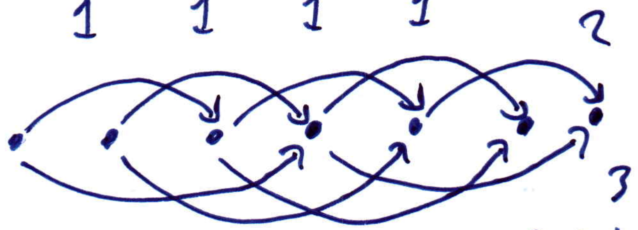
• $S_3 = \langle r, s \mid srsr, r^3, s^2 \rangle$

Graph =

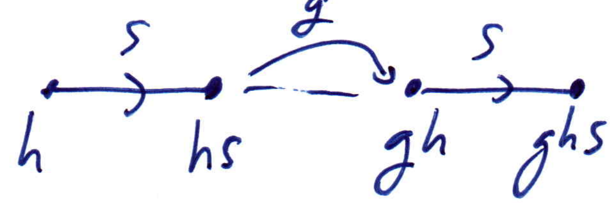
• $\mathbb{Z} = \langle 1 \rangle$



• $\mathbb{Z} = \langle 2, 3 \rangle$



<u>Note</u>: $G \curvearrowright G$ (acting by mult. on left) <u>extends</u> to action $G \curvearrowright Cay_S(G)$ by:



& $G \curvearrowright Cay_S(G)$ <u>Free</u>, i.e. $Stab_G(x) = 1 \;\; \forall x \in Cay_S(G)$.

<u>Prop</u> (<u>Cayley graphs & Pres cplxs</u>).
Let: $G = \langle S \mid R \rangle$, X presentation complex. Then, $\exists$ G-equiv isomorphism of graphs: $Cay_S(G) \cong \tilde{X}_{(1)}$.
[$\tilde{X}_{(1)}$ = 1-skeleton of universal cover $\tilde{X}$]

Geometric Group Theory : lecture 5

29/01/2024.

Proof (of Cayley graphs & Presentation Complexes).

Consider: Natural & Free action of G on $\widetilde{X}$ (by deck trans)
$(G = \pi_1(X))$.

Note: this action is by ~~combinatorial~~ combinatorial automorphs,
so: this restricts to an action on $\widetilde{X}_{(1)}$.

$\Rightarrow G \curvearrowright \widetilde{X}_{(1)}$, sending verts $\to$ verts & edges $\to$ edges.

The action of G on $\widetilde{X}_{(0)}$ is: Free & Transitive

$\Rightarrow \forall \widetilde{x}_0 \in \widetilde{X}$ vertex: $\exists$ G-equiv bijection $G \longrightarrow \widetilde{X}_{(0)}$, $g \longmapsto g\widetilde{x}_0$
(by Orbit-Stabiliser).

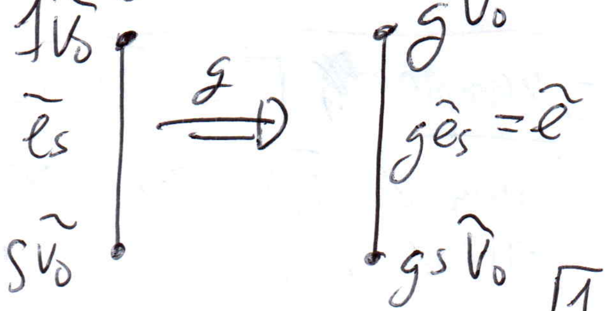
This matches up vertices (as claimed).

Edges: For each $s \in S$: denote e_s = edge corresponding to s, on X. & $\widetilde{e}_s$ = unique lift of e_s to $\widetilde{X}$, starting at $\widetilde{x}_0$.

$\Rightarrow$ By ~~...~~ def. of action $G \curvearrowright \widetilde{X}$:
$\widetilde{e}_s$ ends at $s\widetilde{v}_0$.

& An arbitrary edge $\widetilde{e}$ of $\widetilde{X}$ maps to some e_s
(under covering map), so: $\exists g \in G$, $\widetilde{e} = g \cdot \widetilde{e}_s$.

$\Rightarrow$ G-equivariant map $G \longrightarrow \widetilde{X}_{(0)}$
extends to G-equivariant isomorphism ✓



The next prop. deepens relation (generating $\longleftrightarrow$ Path connectedness).

<u>Prop</u>] let: $\tilde{X}$ path-connected space. & $G \curvearrowright \tilde{X}$ by homeomorphs.
If $U \subseteq \tilde{X}$ open & $G(U) = \tilde{X}$, then: $S = \{ g \in G : gU \cap U \neq \emptyset \}$
<u>generates</u> G.

<u>Proof</u>] Fix basepoint $\tilde{x}_0 \in U \subseteq \tilde{X}$. For $g \in G$, let
$\gamma : [0,1] \longrightarrow \tilde{X}$ path from $\tilde{x}_0$ to $g\tilde{x}_0$.

$\Rightarrow$ The set $\{ \gamma^{-1}(hU) : h \in G \}$ is: an open cover of the compact set $[0,1]$, hence has finite subcover.
Say $\{ \gamma^{-1}(U_1), \dots, \gamma^{-1}(U_n) \}$, $U_i = h_i U$ & choose indices so that $\tilde{x}_0 \in \gamma^{-1}(U_1)$ & $\gamma^{-1}(U_i) \cap \gamma^{-1}(U_{i+1}) \neq \emptyset \ \forall i$.
$g\tilde{x}_0 \in \gamma^{-1}(U_n)$

By <u>def</u>: $\tilde{x}_0 \in U \cap U_1 = U \cap g_1 U \Rightarrow g_1 \in S$.
Similarly: $\forall i$, $t_i \in \gamma^{-1}(U_i) \cap \gamma^{-1}(U_{i+1})$:

$$\Rightarrow x_i \equiv \gamma(t_i) \in U_i \cap U_{i+1} = g_i U \cap g_{i+1} U$$
$$= g_i (U \cap g_i^{-1} g_{i+1} U).$$

So, $g_i^{-1} g_{i+1} \in S$. Call it: s_i.

Then, $g = (g_1)(g_1^{-1} g_2) \dots (\underbrace{g_{n-1}^{-1} g_n}_{=s}) \in \langle S \rangle$ ✓

<u>Example</u> $\Gamma \leq \mathrm{Isom}(\mathbb{R}^2)$ Symmetry group of standard tiling of Equilateral triangles.
$\Rightarrow$ Using prop: can show Γ generated by reflections of sides of a single Δ.

DEF An action $G \curvearrowright \tilde{X}$ properly discontinuous, if:
$\forall K \subseteq \tilde{X}$ compact, $|\{g \in G : gK \cap K \neq \phi\}| < \infty$.

& $G \curvearrowright \tilde{X}$ co-compact $\Leftrightarrow$ $\exists K$ compact, s.t. $\{gK\}$ cover $\tilde{X}$

& $\tilde{X}$ locally compact $\Leftrightarrow$ $\forall x \in \tilde{X}$ & $\forall U \ni x$ nbhood,
$\exists$ open $x \ni V \supseteq U$ s.t. $\bar{V} \subseteq U$ & $\bar{V}$ compact.

Corollary If $G \curvearrowright \tilde{X}$ prop discts. & cocompact and $\tilde{X}$ is path-connected & locally compact, then: G is f.g.

Proof Let K compact, s.t. $GK = \tilde{X}$. By locally compact: may find U s.t. $K \subseteq U$ & $\bar{U}$ compact.
In particular: $GU = \tilde{X}$ & $S \equiv \{g : gU \cap U \neq \phi\}$
$\subseteq \{g : g\bar{U} \cap \bar{U} \neq \phi\}$ finite (by prop discts)
$\Rightarrow S$ is finite + Generating set for G. ✓

"Geometric" Group Theory: lecture 6 31/01/2024.

Corollary If X Locally compact & Compact, and has a universal cover (SEMI-LOCALLY SIMPLY CONNECTED) then $\pi_1(X)$ is finitely generated. [Sheet 1]

§2: Schwarz – Milner Lemma.

Cayley graphs not just Combinatorial $\Rightarrow$ Admit a natural metric (Word metric).

DEF (Word metric). Say $S \subseteq G$ generates G. Then $l_S(g) = \min\{n : \exists s_1, \dots, s_n \in S,\ g = s_1^{\pm 1} \cdots s_n^{\pm 1}\}$.

& Metric $d_S(g,h) = l_S(g^{-1}h)$. [Obviously a metric.]

The Word Metric is: Invariant under LEFT-action of G on itself.

Example $G = \mathbb{Z}^2 \Rightarrow$ Manhattan Distance.

Remark d_S extends naturally to ~~a~~ left G-invariant metric on $Cay_S(G)$, where: interior of each edge locally isometric to $[0,1]$.

Lemma Say: S, T generate G. Then, $\exists C, C' \geq 1$, with:
$\frac{1}{C} d_T \leq d_S \leq C' d_T$. ($S, T$ FINITE generating sets)

Proof Denote $C = \max_{s \in S} l_T(s)$. $\Rightarrow l_T(g) \leq C \cdot l_S(g)$ ✓

$\Rightarrow$ For f.g. groups G: word metric ~~equiv~~ well-defined up to Lipschitz equivalence.

DEF A map $f: X \to Y$ (of Metric spaces) Quasi-isometric embedding if: $\exists\, C \geq 1,\ D \geq 0$ with:

$$\frac{1}{C} d_X(x,x') - D \leq d_Y(f(x), f(x')) \leq C\, d_X(x,x') + D.$$

If also $\exists K$ s.t. $\forall y \in Y$, $\exists x \in X$, $d(y, f(x)) \leq K$, "Quasi ~~Surjective~~ isometry."

& Write: $X \underset{qi}{\sim} Y$. (Exercise: $\underset{qi}{\sim}$ is equiv rel!)

Example: All bounded metric spaces are $\underset{qi}{\sim}$ to a point.

DEF] Metric X proper $\Leftrightarrow$ Closed balls of X are compact.

A Geodesic in X is isometric embedding $[0,1] \hookrightarrow X$.

& X geodesic if: $\forall x,y \in X$, $\exists \gamma : [0,1]$ geodesic, $\gamma(0) = x$, $\gamma(1) = y$.

Theorem (Schwarz-Milner Lemma)

Suppose: X proper Geodesic metric space. & Let $G \curvearrowright X$ prop-disctsly & Co-compactly via isometries. Then: G f.g. & $X \underset{qi}{\sim} (G, d_S)$ $\forall$ word metric d_S. (S finite duh)

Proof] Fix base point $x_0 \in X$. & $B = \bar{B}(x_0, K) \subseteq X$ big enough so $GB = X$.

By Properness + Prop discontinuity: $\{g \in G : d(x_0, gx_0) \leq 3K\}$ finite

$\Rightarrow \exists\, \varepsilon > 0$ s.t. $d(x_0, gx_0) \leq 2K + \varepsilon \Leftrightarrow d(x_0, gx_0) \leq 2K$.

$\Rightarrow g^{-1}B \cap B \neq \phi$ (B closed)

Geometric Group Theory: lecture 7 02/02/2024.

Theorem (Schwarz–Milner).

X proper, geodesic MS. & $G \curvearrowright X$ prop discts & co-compactly (by isometries). Then: G f.g. & $\boxed{X \underset{qi}{\simeq} (G, d_S) \ \forall S \subseteq G}$
finite + generating.

Proof (continued).

Showed: $\exists \varepsilon > 0$, s.t. $\forall g \in G$: $d(x_0, gx_0) \leq 2K$
$\iff d(x_0, gx_0) < 2K + \varepsilon$.

$\Rightarrow$ For $U = B(x_0, K + \varepsilon/2)$: $\Rightarrow S = \{gU \cap U \neq \emptyset\} = \{gB \cap B \neq \emptyset\}$

Since B compact: $\Rightarrow S$ finite (since $G \curvearrowright X$ properly discts).

& By earlier prop. applied to U: $G = \langle S \rangle$

Since all finite generating sets are Bilipschitz: may prove result for this S.

Consider: map $f: G \to X$
$g \mapsto gx_0$.
Claim: f quasi-isometry.

Know: f quasi-surjective, because $GB = X$.

$\Rightarrow$ Remains to show: f is quasi-embedding.

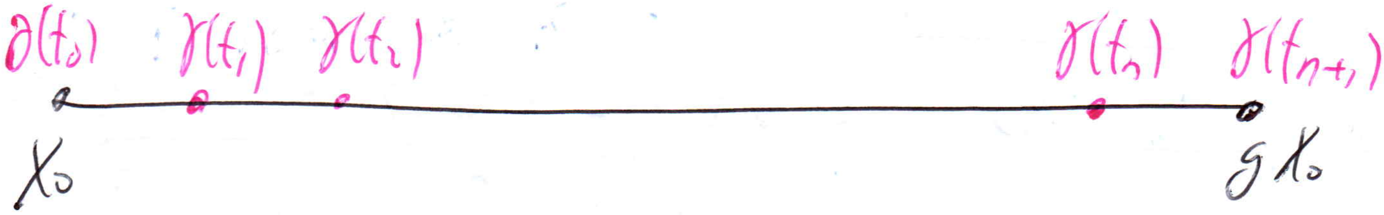
$\iff$ need: Upper + Lower bounds of $d(x_0, gx_0)$ WRT $l_S(g)$.

Upper Bound: $C = \max_{s \in S} d(x_0, gx_0)$

$\Rightarrow \forall g \in G$, $d(x_0, gx_0) \leq C\, l_S(g)$ (using Δ-ineq)

Lower Bound: Let γ geodesic between x_0, gx_0.

i.e. $\gamma : [0, d(x_0, gx_0)] \longrightarrow X$. & Dissect interval:

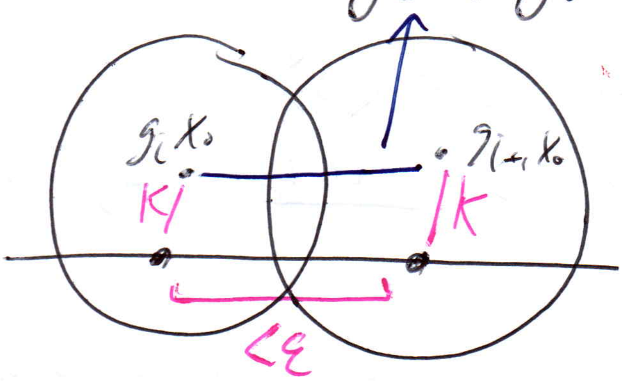


$0 = t_0 < t_1 < \dots < t_n < t_{n+1} = d(x_0, gx_0)$, s.t:

$\varepsilon > t_i - t_{i-1} \geq \frac{\varepsilon}{2}$ $\forall\, 1 \leq i \leq n$ (Note: <u>nothing</u> said about t_{n+1})

Since $GB = X$: $\forall\, 1 \leq i \leq n$: choose Balls $g_i B$, s.t. $\gamma(t_i) \in g_i B$ $\forall\, 1 \leq i \leq n$. (& $g_0 = 1$, $g_{n+1} = g$)

$\Rightarrow \forall\, 1 \leq i \leq n+1$: $d(g_i x_0, g_{i+1} x_0) < 2K + \varepsilon$.



$\Rightarrow g_{i+1}^{-1} g_i \in \{h \in G : U \cap hU \neq \emptyset\}$.

$\Rightarrow l_S(g) \leq n + 1$.

<u>Furthermore</u>: $t_i - t_{i-1} \geq \frac{\varepsilon}{2} \Rightarrow d(x_0, gx_0) \geq n\varepsilon/2$.

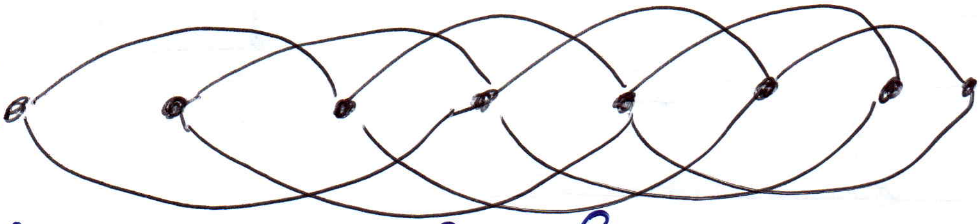
Combine $\Rightarrow l_S(g) \leq \frac{2}{\varepsilon} d(x_0, gx_0) + 1$.

Rearrange $\Rightarrow$ Get desired lower bound ✓

<u>Example</u> Any 2 Cayley graphs for G with sets S_1, S_2 are quasi-isometric.
In particular: $Cay_{\{1\}}(\mathbb{Z})$ & $Cay_{\{2,3\}}(\mathbb{Z})$.

$\mathbb{Z}$ 

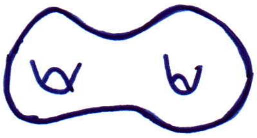
$\cong_{qi}$

$\mathbb{Z}$ 

"if you look them from far away they look the same"

Corollary] If G f.g. & $|G:H|<\infty$ then <u>H f.g.</u> and $G \underset{qi}{\sim} H$.

Proof] Note $H \curvearrowright Cay_S(G)$ <u>co-compact</u> (since $|G:H|<\infty$). So, it satisfies S-M, so $H \underset{qi}{\sim} Cay_S(G) \underset{qi}{\sim} G$. ✓

Example] $\Sigma_2 =$  $G = \pi_1(\Sigma_2)$.

$\Rightarrow$ <u>Choose</u>: Riemannian metric on Σ_2 with constant curvature -1.

This pulls back to: Riemannian metric on $\widetilde{\Sigma}_2$.

& By classic Diff geo: $\widetilde{\Sigma}_2 \underset{isom}{\cong} \mathbb{H}^2$. <u>Hence</u>, $\pi_2(\Sigma_2) \curvearrowright \mathbb{H}^2$ by isom.

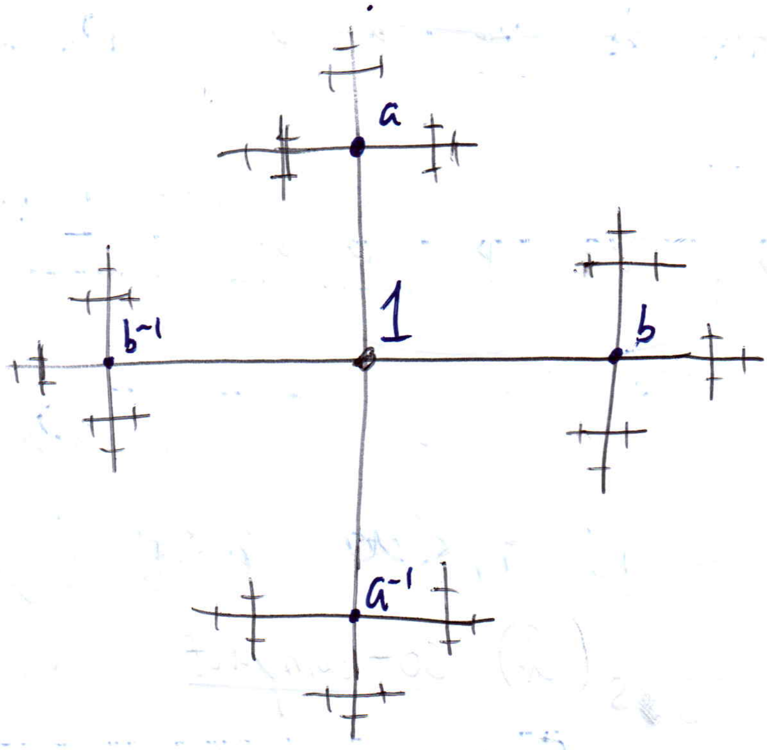
These actions are: properly discts & co-compact, because Σ_2 compact. So, by SM-lemma: $\pi_1(\Sigma_2)$ quasi-isometric to $\mathbb{H}^2$.

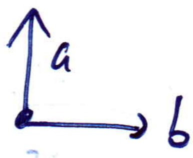
---

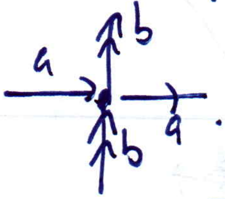
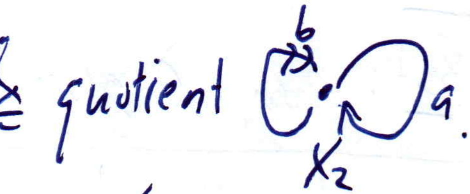
§3: Free Groups.] Let $A = \{a_1, \dots, a_n\}$ Finite alphabet, and: $F_n = F(A_n)$ free group.

The <u>Cayley-Tree</u> is: the $(2n)$-valent tree T_n:

$\underline{n=2}$:





In Free group F_2, $T_2 = Cay(F_2)$. Each vertex looks like . Also, know $F_n \curvearrowright T_n$ freely & quotient .

Recover F_n & T_n as universal cover of the X_n spaces.

Can: translate our combinatorial arguments about F_n with geometric properties about T_n.

1) Words of F_n: $\forall w \in A^*$, is equivalent to edge path $w: I \to X_n$ (I = some interval).

& This edge-path lifts to a unique path in T_n based at 1.

[Conversely, any such path projects to path in X_n.]

2) Reduced Words: $w \in A^*$ reduced $\Longleftrightarrow$ this path w is locally injective $\Leftrightarrow$ $w: I \to T_n$ locally injective.

(Since: edge path can only fail to be locally injective at a vertex.) Clearly: Shortest path in T_n from $1 \to g \in F_n$ is injective, so: every element is represented by a reduced word.

The fact that this representative is unique, follows from next lemma!

$\boxed{\text{Lemma}}$ T tree & $\gamma: I \to T$ locally injective (edge) path.

Then: γ injective (globally).

Proof Let $\gamma : [a,b] \longrightarrow T$ shortest counterexample.

Then: $\gamma(a) = \gamma(b)$ (else, shrink to make smaller)

But: then, γ gives a map $\tilde{\gamma} : S^1 \longrightarrow T$ $\not\exists$ ✓

Similarly, if g is shortest word that can be ~~reduced~~ represented by 2 distinct reduced words, then get $S^1 \hookrightarrow T$ $\not\exists$

$\Rightarrow$ Reduced maps are unique. Denote: $[1, g]$ unique path $1 \to g$ in T_n.

3) Cyclically Reduced WORDS

Each word $w \in A^*$ also defines: (based) edge loop, by gluing endpoints of interval.

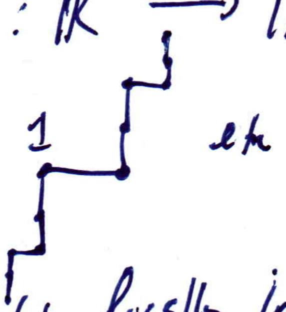
$\Rightarrow$ If we forget basepoint of S, then 2 elements $u, v \in A^*$ determine same cyclic word $\Leftrightarrow$ Represent same edge loop $S^1 \to X_n$

$\Rightarrow$ $w \in A^*$ cyclically reduced $\Leftrightarrow$ Corresponding map $S^1 \xrightarrow{w} X_n$ is locally injective.

$$\begin{tikzcd} \mathbb{R} \arrow[r] \arrow[d, "p"'] \arrow[dr, dashed, "w\circ p"] & T_n \arrow[d, "\pi"] \\ S^1 \arrow[r, "w"'] & X_n \end{tikzcd}$$

The composition map $w \circ p : \mathbb{R} \to X_n$ lifts to T_n. $\tilde{w}_* : \mathbb{R} \longrightarrow T_n$.

Example $w = ab$?



By lemma: since $\tilde{w}$ locally inj: $\Rightarrow$ Globally injective $\tilde{w}$.

Call: image$(\tilde{w})$ axis of w. & w acts by translating axis.

By def of Action of F_n on T_n: Can think w as Deck transf. of T_n, and preserves axis of w. & Translates axis of w by $l(w)$.

This is called: Translation Length of w, $\tau(w)$.

Geometric Group Theory: <u>lecture 9</u> 07/02/2024.

A <u>Geometric</u> soln. to Conjugacy problem follows from next lemma:

<u>Lemma</u> let $u, v \in F_n$ Cyclically reduced. If u, v are conjugate then $\exists g \in F_n$, with $\ell(g) \leq \frac{1}{2}(\tau(u) + \tau(v))$, <u>with</u>: $u = gvg^{-1}$.

The CP follows, because by lemma, need to check $u = gvg^{-1}$ for finitely many g, <u>and</u>: each can be checked by WP.

<u>Remark</u> Not all g that satisfy $u = gvg^{-1}$ has $\ell(g) \leq$ bound.

E.g. $u = (gv^k)v(gv^k)^{-1}$ $\forall k \in \mathbb{Z}_{\geq 0}$.

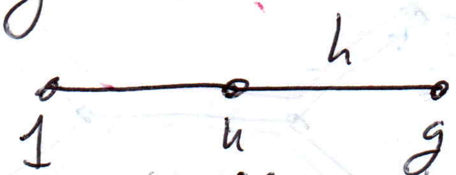
& $u = (u^\ell g)v(u^\ell g)^{-1}$ $\forall \ell \in \mathbb{Z}_{\geq 0}$.

[In fact, such conjugates form a Double coset $\langle u \rangle g \langle v \rangle$.]

<u>Proof of Lemma</u> Suppose $\exists g$, $u = gvg^{-1}$, and pick g such that $\ell(g)$ <u>minimal</u>. First, make 2 remarks:

<u>1)</u> If $u \in [1, g]$ then: $g = uh$.

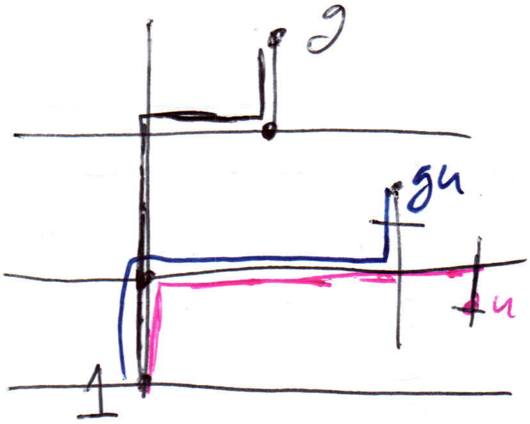
$\Rightarrow u = hvh^{-1}$ & $\ell(h) < \ell(g)$ $\times$



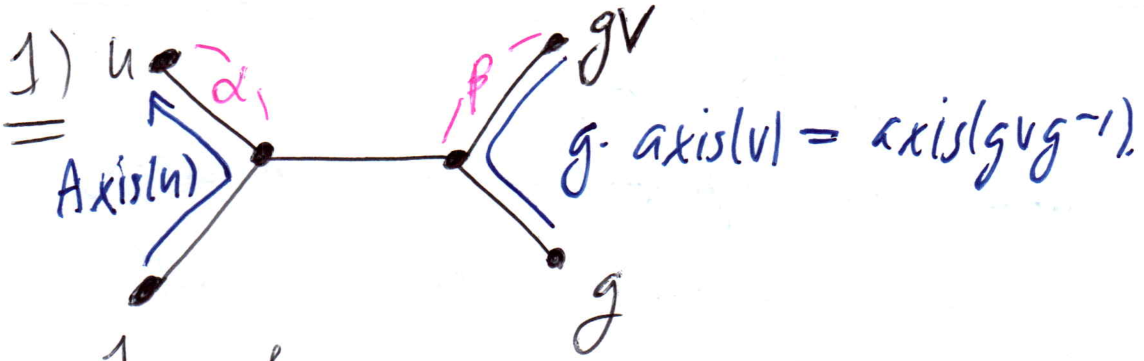
<u>2)</u> Similarly, if $v \in [1, g^{-1}]$, then get $\ell(g)$ <u>not minimal</u>.

~~Hence~~ <u>Idea</u>: Consider <u>Convex hull</u> of $\{1, g, u, gv\}$.

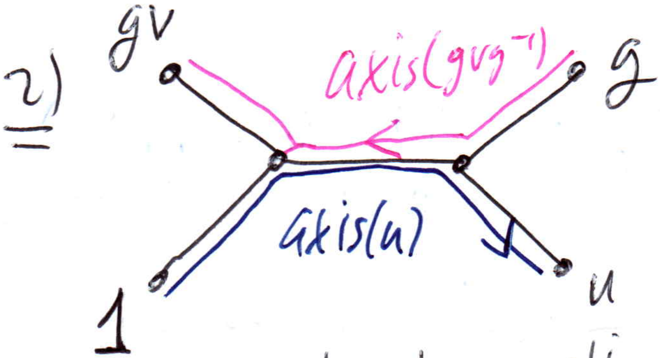
[Convex hull = Union of all shortest paths between the pairs of points in the set]



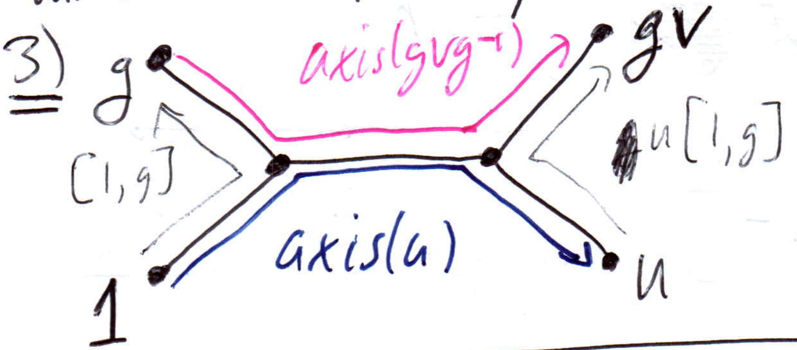
There are only 3 different combinatorial types for the convex hull: (including degenerate cases):

1) 

Normality of g $\Rightarrow$ $\ell(\alpha), \ell(\beta) > 0$. Hence: $\mathrm{Axis}(u) \cap \mathrm{Axis}(gvg^{-1}) = \emptyset$
$\Rightarrow u \neq gvg^{-1}$ ⨳.

2) 

If central interval positive length $\Rightarrow$ Translates in different directions $\Rightarrow$ Not equal. (Since, axes should be same!)

3) 

$\Rightarrow \tau(u) + \tau(v) \geq 2\,\ell(g)$.

§4 : Subgroups of Free Groups.

Prop X undirected Graph, connected. Then $\pi_1(X)$ Free.

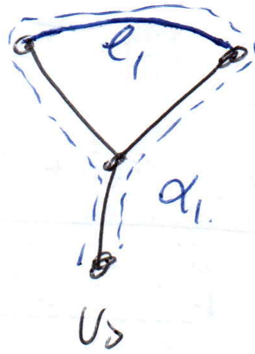
Proof (X countable): $T \subseteq X$ maximal tree & $\{e_i\}_{i \geq 1}$ edges not in T.
Let $X = \bigcup_{n \geq 1} X_n$, $X_n = T \cup \{e_1, \dots, e_n\}$.

Geometric Group Theory: lecture 10

09/02/2024

From last time: **Prop** If X ctable connected graph, then $\pi_1(X)$ is free. [True if X not countable.]

Proof Constructed: $T \subseteq X$ maximal tree, $X \setminus T = \{e_1, e_2, \dots\}$.
& $X_n = X \cup \{e_1, \dots, e_n\}$ has ~~$\pi_1(X_n)$~~ $X = \bigcup_{n \geq 1} X_n$.



Denote: $v_0 \in T$ basepoint.

Note: $X_{n+1} = X_n \cup e_{n+1} = X_n \underset{v_n}{\cup} (S^1 \cup I)$.

& By Seifert VK: $\pi_1(X_{n+1}) = \pi_1(X_n) * \pi_1(\alpha_{n+1})$.

$\Rightarrow$ By induction, $\pi_1(X_n) \cong \langle \alpha_1, \dots, \alpha_n \rangle$. Finite case ✓

For infinite case, any loop $\gamma \subseteq X$ compact $\Rightarrow$ Contained in some X_n. & $\pi_1(X)$ generated by $\langle \alpha_1, \alpha_2, \alpha_3, \dots \rangle$.

$\Rightarrow \exists$ Surjection $F_\infty = \langle \alpha_1, \alpha_2, \dots \rangle \xrightarrow{\eta} \pi_1(X)$
$\alpha_i \longmapsto \alpha_i$.

Claim: η injective.

If $\gamma \in \ker(\eta)$ (or a loop corresponding to such an element): then, $\exists n$, $\gamma \subseteq X_n$. $\Rightarrow \gamma$ lies in kernel of $\pi_1(X_n) \to \pi_1(X)$.

Since X_n retract of X: (i.e. inclusion $X_n \hookrightarrow X$ has a left-inverse $r: X \to X_n$): every loop $\gamma \subseteq X_n$ null-homotopic in X is also null-homotopic in X_n

$\Rightarrow \gamma = 1$ in $\pi_1(X_n) \subseteq F_\infty$ ✓

Corollary If $G \curvearrowright T$ (T tree) is free action then G free.

Proof The action $G \curvearrowright T$ is: covering space action, so:
$X = G \backslash T$ is a graph. $\Rightarrow$ $G = \pi_1(X)$, so G free ✓

Corollary (Nielsen–Schreier thm). If $H \leq F_n$ then H free.

Proof Let T = Cayley tree of F_n, so $F_n \curvearrowright T$ freely.
Restrict action to H: $H \curvearrowright T_n$ Freely, so H is free.

III: Bass–Serre Theory

Study: Groups acting on Trees (not necessarily freely)
Will see: how to glue groups together / cut them into pieces.

DEF A comm diag of groups

$$\begin{tikzcd} C \arrow[r, "i"] \arrow[d, "j"'] & A \arrow[d, "k"] \arrow[ddr, bend left, "f"] & \\ B \arrow[r, "l"'] \arrow[drr, bend right, "g"'] & \Gamma \arrow[dr, "h"] & \\ & & G \end{tikzcd}$$

is Pushout if: $f \circ i = g \circ j$
$\Rightarrow \exists! h : \Gamma \to G$, $g = h \circ l$, $f = h \circ k$.

Exercise: Γ is Unique map, up to unique isomorphism. Write: $\Gamma = A \underset{C}{\sqcup} B$.

Theorem (SVK). Suppose: K, L sub-complexes of $X = K \cup L$, s.t. $K, L, K \cap L$ path-connected. Then: $\pi_1(X) = \pi_1(K) \underset{\pi_1(K\cap L)}{\sqcup} \pi_1(L)$.

Exercise $A = \langle S_A \mid R_A \rangle$, $B = \langle S_B \mid R_B \rangle$
& $C = \langle \Sigma \mid \cdot \rangle$.
Let: $\hat{i} : \Sigma \to F(S_A)$ & $\hat{j} : \Sigma \to F(S_B)$ represent i, j.
Then: $A \underset{C}{\sqcup} B = \langle S_A, S_B \mid R_A, R_B, \{\hat{i}(\sigma)\hat{j}(\sigma)^{-1} : \sigma \in \Sigma\} \rangle$.

Example If $B \cong 1$, then: $A \bigsqcup_C 1 \cong A / \langle\langle C \rangle\rangle$.

DEF If i, j injective, then: $\Gamma = A *_C B$ Amalgamated Free Product of A, B. In particular, if $C \cong 1$ then $\Gamma = A * B$.
(Free product)

Theorem (Britton's Lemma):
The Vertex Group A (or B) inject into $A *_C B$.

Remark Not true (in general) for Pushouts!

E.g.
$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}/2 \\ \mathbb{Z}/3 & \end{tikzcd} \implies \mathbb{Z}/2 \bigsqcup_{\mathbb{Z}} \mathbb{Z}/3 \cong 1.$$

Next time: Prove it.

Geometric Group Theory: lecture 11 | 12/02/2024

From last time:

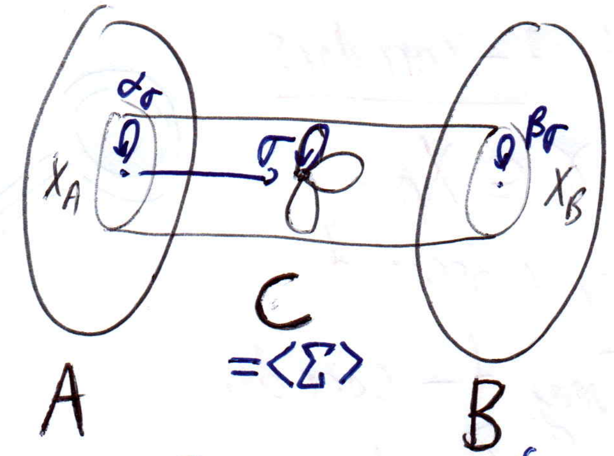
Theorem (Britton's Lemma). The Vertex groups A, B inject into $A \underset{C}{*} B = G$.

To prove it: Construct "Graphs of Spaces" X, s.t. $G \cong \pi_1(X)$.

Let: X_A presentation cplx for A

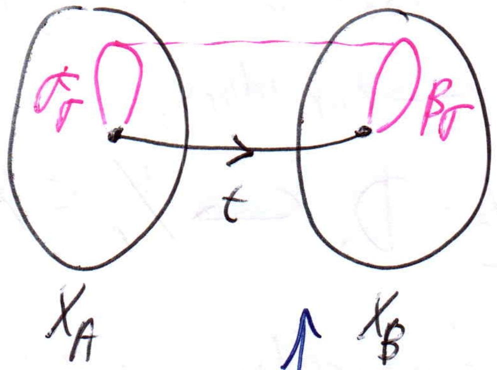
X_B presentation cplx for B

& Σ = Generating set, for C.

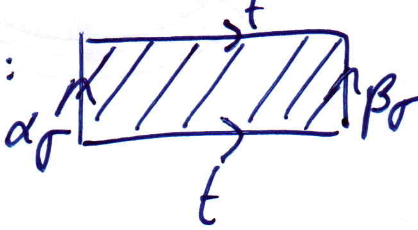


For $\sigma \in \Sigma$, let α_σ be: based edge loop in X_A, representing $i(\sigma)$. Similarly, β_σ. (rep $j(\sigma)$.)

Glue: vertices of X_A, X_B to either end of an edge, t.



For each $\sigma \in \Sigma$: consider Rectangular 2-cell:



& Attach them as in ↑

Do this for ALL such rectangular 2-cells in this way, get X.

$\Rightarrow$ By SVK: $\pi_1(X) \cong G \cong A \underset{C}{\sqcup} B$.

Proof of Theorem Suppose $g \in A$ maps to $1 \in A \underset{C}{\sqcup} B$

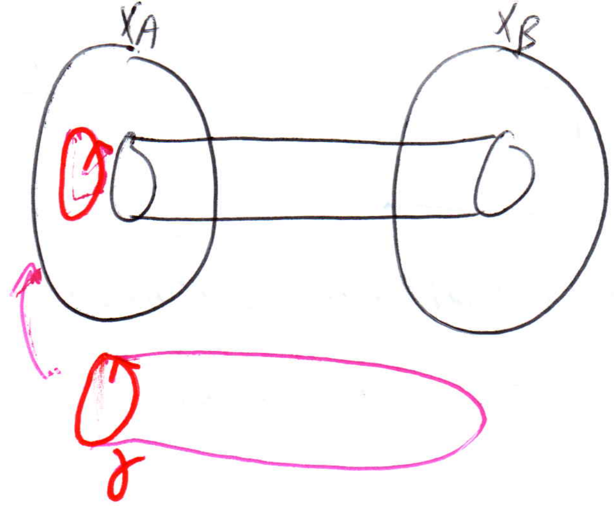
$\Rightarrow g$ represented by Based loop γ in X_A.

& γ is Null-homotopic in X.

By Van-Kampten Theorem:

γ bounds singular disc $\mathbb{D} \to X$.

Because t (edge) appears $2x$ in each rectangle & Nowhere else:

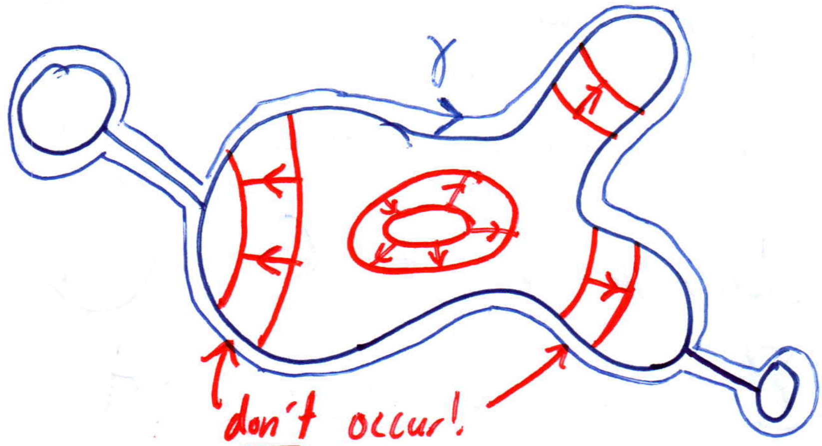


the rectangular 2-cells in $\mathbb{D}$ are arranged in strips, called t-corridors.

Since $\gamma \subseteq X_A$:

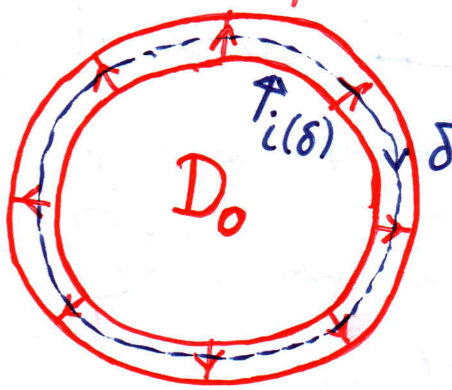
γ never crosses t

$\Rightarrow$ Every t-corridor is an Annulus.



Next: look at an Innermost disc D_0, bounded by some t-corridors.

WLOG: $D_0 \hookrightarrow X_A \subseteq X$. (Since no t's on boundary)

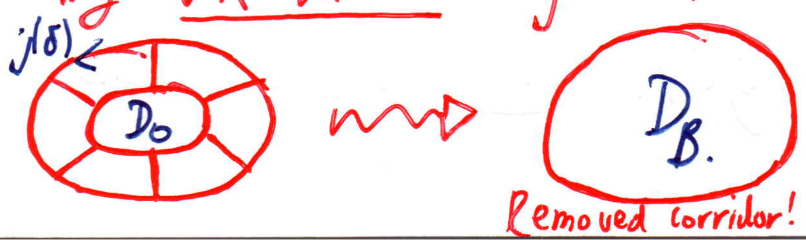


The t-corridor corresponds to: some Cyclic word δ in $\Sigma \sqcup \Sigma^{-1}$

& D_0 is Proof that: $i(\delta) = 1$, in A.

By hypothesis: since i injective, get $\delta = 1$ in C. $\Rightarrow j(\delta) \overset{B}{=} 1$.

$\Rightarrow$ By VK lemma: $j(\delta)$ has VK-diagram $D_B \to X_B$.



After fin. many replacements:

Removed ALL t-corridors. ✓

$\Rightarrow \exists \Delta$ VK diag for γ, in X_A. $\boxed{2}$

Geometric Group Theory: Lecture 12 | 14/02/2024

§2: Higman–Neumann–Neumann (HNN) extension.

DEF (HNN Pushout): Suppose $i, j : H \to G$ maps of groups. Then, the ~~HNN Product~~ HNN Pushout is quotient:

$$G \underset{H}{\sqcup} \equiv (G * \langle t \rangle) \Big/ \langle\langle t \cdot i(h) \cdot t^{-1} \cdot j(h)^{-1} : h \in H \rangle\rangle.$$

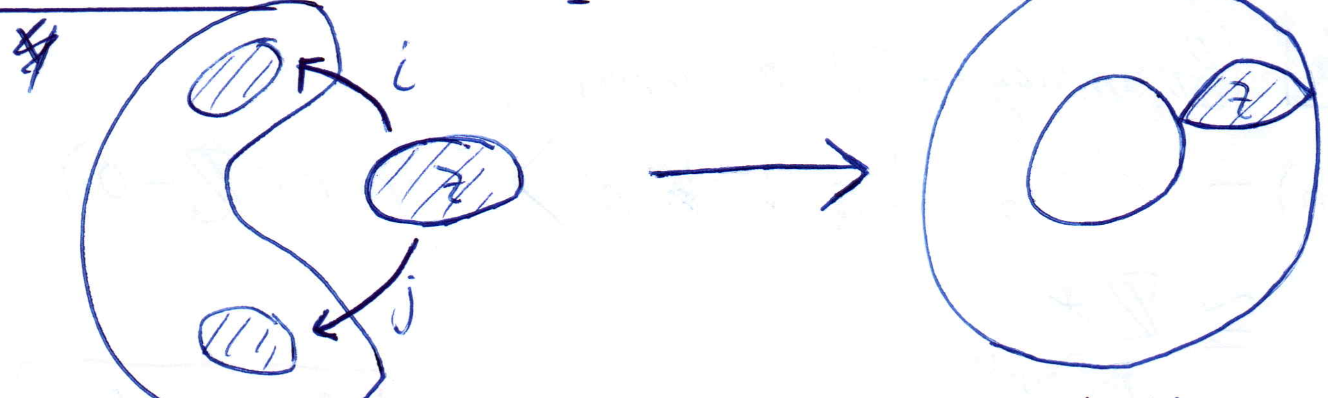
[That is: force $i(h)$ & $j(h)$ conjugate $\forall h \in H$.]

Theorem (SVK, for non-Separating Decomps.)

Suppose: Y connected cell complex & $i, j : Z \hookrightarrow Y$ inclusions to a connected cell complex ~~Z~~ Z, with disjoint images.

Then: if $X = Y \underset{Z}{\sqcup} \equiv Y \big/ \{ i(z) \sim j(z) \ \forall z \in Z \}$:

$$\Rightarrow \boxed{\pi_1(X) \cong \pi_1(Y) \underset{\pi_1(Z)}{\sqcup}}$$



Y

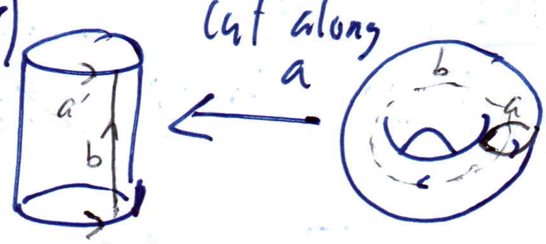
$X = Y \underset{Z}{\sqcup}$

Remark Let $G = \langle a_1, \dots, a_m, t \mid r_1, \dots, r_n, p_1 t q_1^{-1} t^{-1}, \dots, p_\ell t q_\ell^{-1} t^{-1} \rangle$.

Then: for $A = \langle a_1, \dots, a_m \mid r_1, \dots, r_n \rangle$, have map $F_\ell \to A$ given by $i : x_i \mapsto p_i$ & $j : x_i \mapsto q_i$.

& Follows that $G = A \sqcup_{F_e}$.

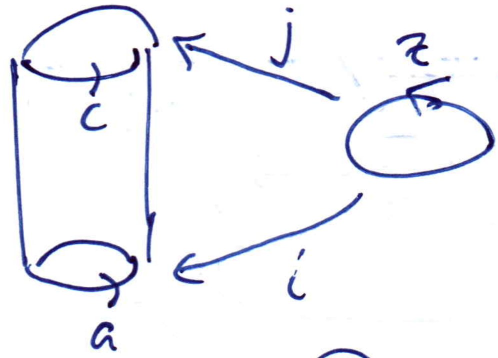
DEF] (HNN Extension). If $G = A \sqcup_B$ & $i, j : B \hookrightarrow A$, then call G a <u>HNN-Extension</u>. ($G = A *_B$)



Example] i) $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$.

$\Rightarrow \pi_1(LHS) = \mathbb{Z} = \langle a, c \mid ac^{-1} \rangle$, $c = ba'b^{-1}$.

So, <u>consider</u>: $\mathbb{Z} = \langle z \rangle \underset{i,j}{\hookrightarrow} \mathbb{Z}$, with: $i(z) = a$, $j(z) = c$.



The resulting HNN extension $\mathbb{Z} *_{\mathbb{Z}}$ has presentation: $\langle a, c, t \mid ac^{-1}, tat^{-1}c^{-1} \rangle$
$\cong \mathbb{Z}^2$.

Example]

$\pi_1 = \langle a_1, b_1, a_2, b_2 \mid a_1 b_1 a_1^{-1} b_1^{-1}, a_2 b_2 a_2^{-1} b_2^{-1} \rangle$

Example] (Baumslag – Solitar Groups)

$BS(m,n) = \langle a, b \mid ba^m b^{-1} a^n \rangle$. $(m, n \in \mathbb{Z} - 0)$

$\cong \mathbb{Z} *_{m\mathbb{Z} \sim n\mathbb{Z}}$.

Theorem] (<u>Britton's Lemma, HNN-Exts</u>): $\boxed{A \hookrightarrow A *_C}$

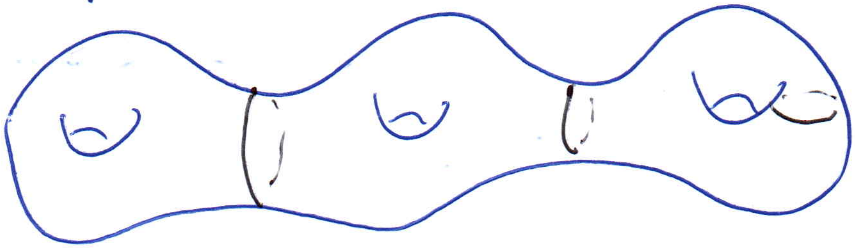
Proof] Same as for AFP proof: Build graph of Spaces & Apply method of t-Corridors.

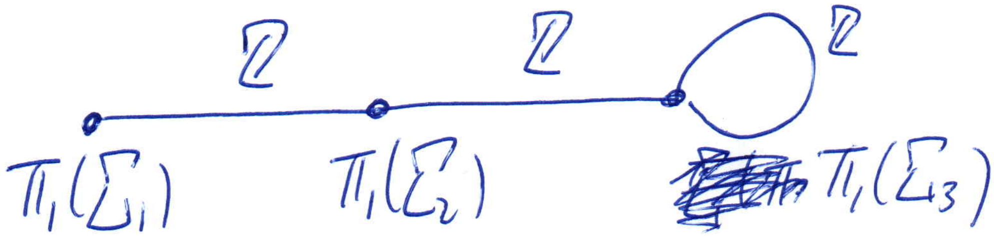
Simple applications of HNN's., AFP's etc.

- ⊛ $\exists$ Infinite group with exactly 2 conj classes
- ⊛ $\exists$ FG (indeed, finitely presented) group, $f : G \to G$ with $\mathrm{Ker}(f) \neq 1$. ("non-Hopfian")
- ⊛ $\exists$ Infinite + F.g. Simple group.
- ⊛ Every Countable group embeds into a 2-generator group.
- ⊛ $\exists$ Group with Unsolvable word problem.

What about: Cutting Surface along Multicurve?
(i.e. Systems of disjoint curves).

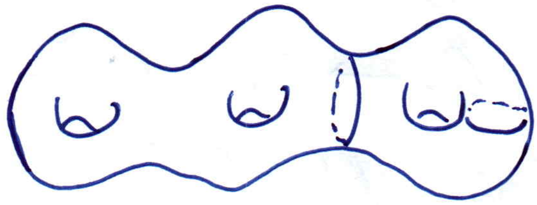
§3: Graphs of Groups.





Geometric Group Theory: lecture 13 | 16/02/2024

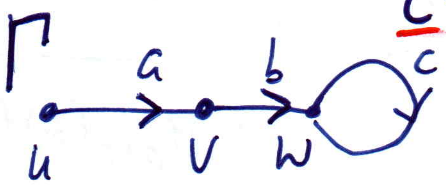
§3: Graphs of Groups.



First: should carefully define directed / undirected graphs.

DEF An Oriented / Directed graph Γ is: Pair of sets $V = V_\Gamma$, $E = E_\Gamma$, with: V = Vertices, E = edges

& Equipped with 2 maps $\iota = \iota_\Gamma : E \to V$ "Origin" map"
$\tau = \tau_\Gamma : E \to V$ "Terminus map".

Example 

$V = \{u, v, w\}$, $E = \{a, b, c\}$

$\tau : a \mapsto v,\ b \mapsto w,\ c \mapsto w.$
$\iota : a \mapsto u,\ b \mapsto v,\ c \mapsto w.$

The Realisation $|\Gamma|$ of Γ is natural 1D cell complex, defined by this data. "Draw it!".

DEF "Graph of Groups": $\mathcal{G}$, consists of:

- Underlying graph Γ.
- Assignments: $V \to$ Groups, $v \mapsto$ some group G_v
- Assignments: $E \to$ Groups, $e \mapsto G_e$
- Injective homs $\iota_e : G_e \hookrightarrow G_{\iota(e)}$

$$\tau_e : G_\tau \hookrightarrow G_{\tau(e)}.$$

Example 

$G_u = \pi_1(\Sigma_1)$
$G_v = \pi_1(\Sigma_2)$
$G_w = \pi_1(\Sigma_3)$

* $i_a : G_a \to G_u$
$\tau_a : G_a \to G_v$, 4 others. Induced by loops included in Σ_i.

(over here, All G_e's are $\mathbb{Z}$, induced by inclusion.)

DEF] (π_1 of graph of groups)

Given $\mathcal{G}$, underlying Γ, let $T \subseteq \Gamma$ spanning tree.
The Fundamental Group of $\mathcal{G}$, WRT T, denoted $\pi_1(\mathcal{G}, T)$, is defined as:

(*) Take Free Product of vertex groups:

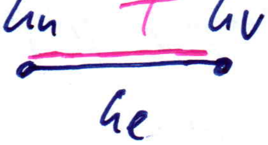
$$\boxed{\dfrac{\left(\ast_{v\in V} G_v\right) \ast F(E_\Gamma)}{\left\langle\!\left\langle \bigcup_{e\in E_\Gamma} \{t_e i_e(h) t_e^{-1} i_e(h)^{-1} : h \in G_e\} \cup \{t_e = 1,\ e \in T\} \right\rangle\!\right\rangle}}$$

[$F(E_\Gamma)$ gen set $\{t_e : e \in E\}$.]

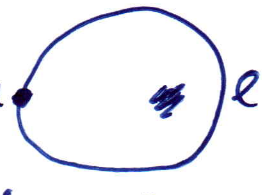
Visualise as:

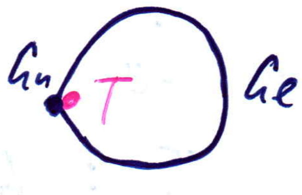
$$\pi_1(\mathcal{G}) = \frac{\left(\ast_{v\in V} G_v\right) \ast F(E_\Gamma).}{\left\langle\!\left\langle \bigcup_{e\in E_\Gamma} \{t_e i_e(h) t_e^{-1} i_e(h)^{-1} : h \in G_e\} \cup \{t_e = 1, e \in T\} \right\rangle\!\right\rangle}$$

Examples] 1) $\Gamma =$ 

$\Rightarrow \mathcal{G} =$ 

$\Rightarrow \pi_1(\mathcal{G}, T) \cong G_u \ast_{G_e} G_v$

2) $\Gamma =$ 

$\Rightarrow \mathcal{G} =$ 

$\Rightarrow \pi_1(\mathcal{G}, T) \cong G_u \ast_{G_e}.$

Theorem (SVK, for Graphs of groups)

Let: Γ graph. For each $v \in V$, $e \in E$: denote X_v, X_e cell cplx's (connected), & $i_e : X_e \longrightarrow X_{i(e)}$

$$\tau_e : X_e \longrightarrow X_{\tau(e)}$$

are: π_1-injective inclusions of sub-complexes.

Let: $X = \left(\bigsqcup_{v \in V} X_v \right) \Big/ \{ i_e(x) = \tau_e(x) \ \forall x \in X_e,\ e \in E \}$.

Setting $G_v = \pi_1(X_v)$ & $G_e = \pi_1(X_e)$, etc defines: graph of groups $\mathcal{G}$.

Then: $\boxed{\pi_1(\mathcal{G}, T) \cong \pi_1(X)}$ $\forall T$ spanning of Γ.

Proof Use induction on $\# E_\Gamma$ + two SVK theorems we already have (for finite E_Γ).

Remark Follows: $\pi_1(\mathcal{G}, T)$ does not depend (up to isom) on T, so write $\pi_1(\mathcal{G})$.

Quotients

Let: G group & $G \circlearrowright T$ tree (or, any graph).

So, G acts on vertices of T, and also edges of T, and:

$g(i(e)) = i(ge)$ & $g(\tau(e)) = \tau(ge)$ $\forall e \in E$.

$\Longrightarrow \exists$ Natural quotient graph $\Gamma = G \backslash T$. (Left action)

Defined by: $V_\Gamma = G \backslash V_T$, $E_\Gamma = G \backslash E_T$

& $i_\Gamma(G\tilde{e}) = G \cdot i_\Gamma(\tilde{e})$, $\tau_\Gamma(G\tilde{e}) = G \cdot \tau_\Gamma(\tilde{e})$ $\forall \tilde{e} \in E_T$.

This is well-defined by ~~strict G-action on ẽ~~. G-linearity on τ, i as before.

Furthermore: $G \backslash T = \Gamma$ is naturally: a graph of groups.

Let: $V = G\tilde{v} \in V_\Gamma$. $\Rightarrow$ Set: $G_V = \mathrm{Stab}_G(\tilde{v})$.

[Note: G_V well-defined up to conjugation in G, hence, up to $\cong$]

Similarly: if $e = G\tilde{e}$ then $G_e \equiv \mathrm{Stab}_G(\tilde{e})$.

Maps: Suppose $i(e) = V$, so: $G\, i(\tilde{e}) = G\tilde{v}$.

$\Rightarrow$ May choose $\tilde{e}$ s.t. $i(\tilde{e}) = \tilde{v}$.

$\Rightarrow G_e \equiv \mathrm{Stab}_G(\tilde{e}) \leq \mathrm{Stab}_G(\tilde{v}) = G_V$.

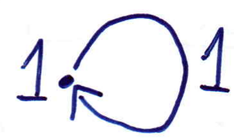
So, this is my map, $G_e \hookrightarrow G_V$ (so, injective).

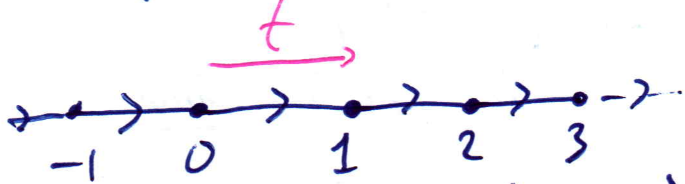
Remark] i_e is determined up to conjugation in G

& τ_e = defined similarly.

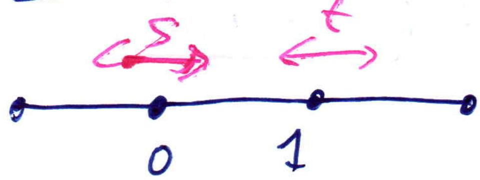
Examples] 1) $\mathbb{Z} = \langle t \rangle \curvearrowright \mathbb{R}$

$\Rightarrow \mathbb{Z} \backslash \mathbb{R} = \{\bullet\}$.

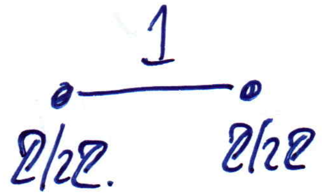




2) $D_\infty = \langle s, t \mid s^2, t^2 \rangle \curvearrowright \mathbb{R}$. s, t reflections on $0, 1$.



$D_\infty \backslash \mathbb{R}$:



$= (\mathbb{Z}/2\mathbb{Z}) *_1 (\mathbb{Z}/2\mathbb{Z})$

Geometric GT: lecture 14

19/02/2024

From last time: Graphs of Groups $\leftarrow$ Graphs of Spaces

$\leftarrow$ Quotients $G\backslash T$

§4: Bass-Serre Theorem.

[Main theorem of Subject, due to Serre. But: adopt topological approach, due to Scott-Wall]

Theorem (Fundamental Theorem of BS Theory)

Let: $\mathcal{G}$ connected graph of groups & $G = \pi_1(\mathcal{G})$.

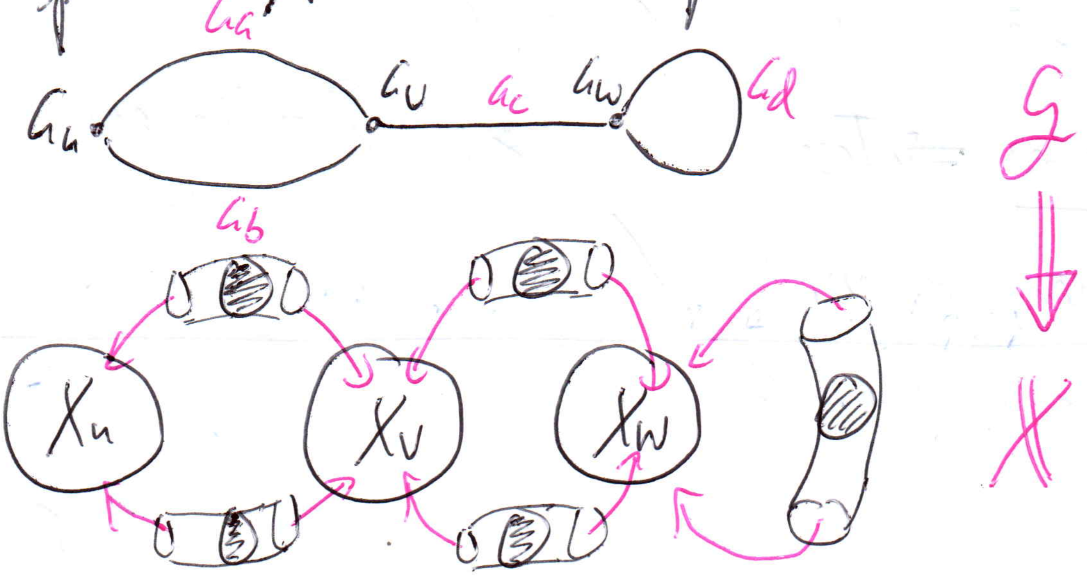
Then: $G \curvearrowright T$ (tree) s.t. $\mathcal{G} \cong G\backslash T$. $\boxed{T = \text{BS-tree}}$

Remark • Hence: letting G act on tree T is equivalent to "cutting" G into pieces.

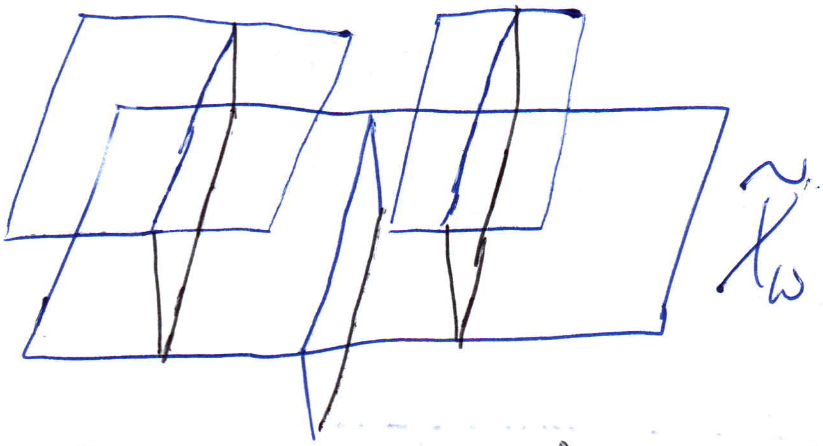
• The theorem says: $\mathcal{G}$ has "Universal Cover" T, for which $G = \pi_1(\mathcal{G})$ acts, and recover $\mathcal{G}$ as quotient $G\backslash T$.

Proof (Sketch)

Step 1 Using Presentation Complexes, build "Graph of spaces" $\mathbb{X}$ with: correspondence to G:



Let: $\widehat{\mathbb{X}} \equiv$ Universal Cover for $\mathbb{X}$.



$\Rightarrow$ Result is graph of spaces $\mathbb{X}$, with: each vertex space $\widetilde{X}_{\tilde{v}}$ is universal cover of some X_v

& each edge space is $[-1,1] \times \widetilde{X}_{\tilde{e}}$, for $\widetilde{X}_{\tilde{e}}$ some universal cover of X_e.

Let: $\widehat{\Gamma}$ underlying graph of $\widetilde{\mathbb{X}}$.

$\Rightarrow \widetilde{X}$ Retracts onto $\widetilde{\Gamma}$ ("Crush all X_u, X_e into pt")

ie $\exists$ maps $\widetilde{\Gamma} \underset{i}{\overset{r}{\leftrightarrows}} \widetilde{X}$ s.t. $r \circ i \simeq id$.

Apply π_1: $\Rightarrow i_*: \pi_1(\widetilde{\Gamma}) \longrightarrow \pi_1(\widetilde{X})$ Injective.

$$\pi_1(\widetilde{X}) \cong * \quad (\widetilde{X} = \text{Universal Cover})$$

$\Rightarrow \widetilde{\Gamma}$ simply connected $\Rightarrow$ Tree.

Set $T = \widetilde{\Gamma}$. Then, it works ✓

Prop) Let $G \circlearrowright T$, such that $\mathcal{G} = G \backslash T$. **Then:**

(i) $V_T \longleftrightarrow \bigsqcup_{v \in V_\Gamma} G/G_v$

(ii) $E_T \longleftrightarrow \bigsqcup_{e \in E_\Gamma} G/G_e$

G-equivariant $\longleftrightarrow$'s.

$\{g(f)(a)\} = f(ga)$.

$A \xrightarrow{f} B$.

iii) $\forall \tilde{v} \in V_T$ mapping $v \in V_\Gamma$, the set of edges of T incident at $\tilde{v}$ is: G-equivariantly bijective with:

$$\left(\bigsqcup_{\substack{i(e)=v \\ e \in E_\Gamma}} G_v / i_e(G_e) \right) \sqcup \left(\bigsqcup_{\substack{\tau(e)=v \\ e \in V_\Gamma}} G_v / \tau_e(G_e) \right).$$

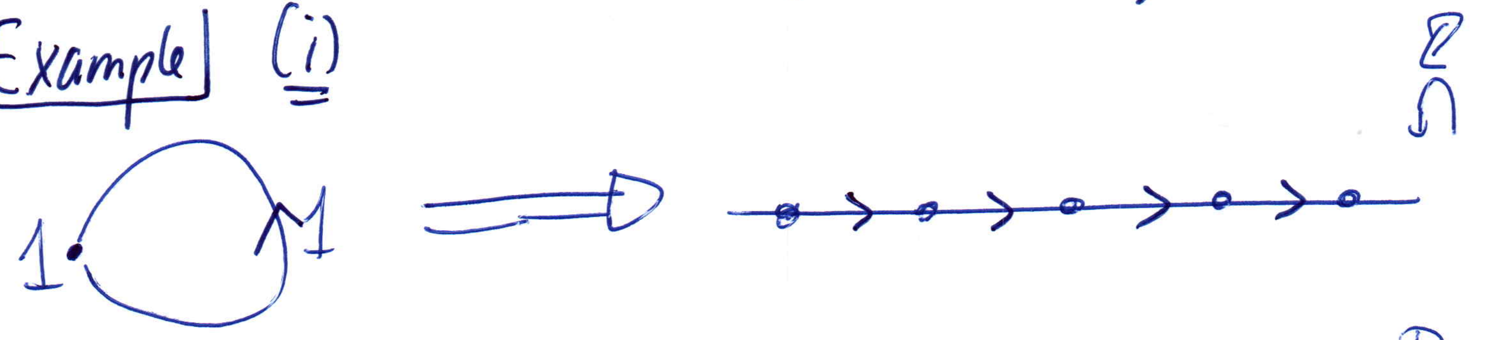
Proof (i) Choose orbit reps: $\tilde{v} \in G\tilde{v} = v \in V_\Gamma$.

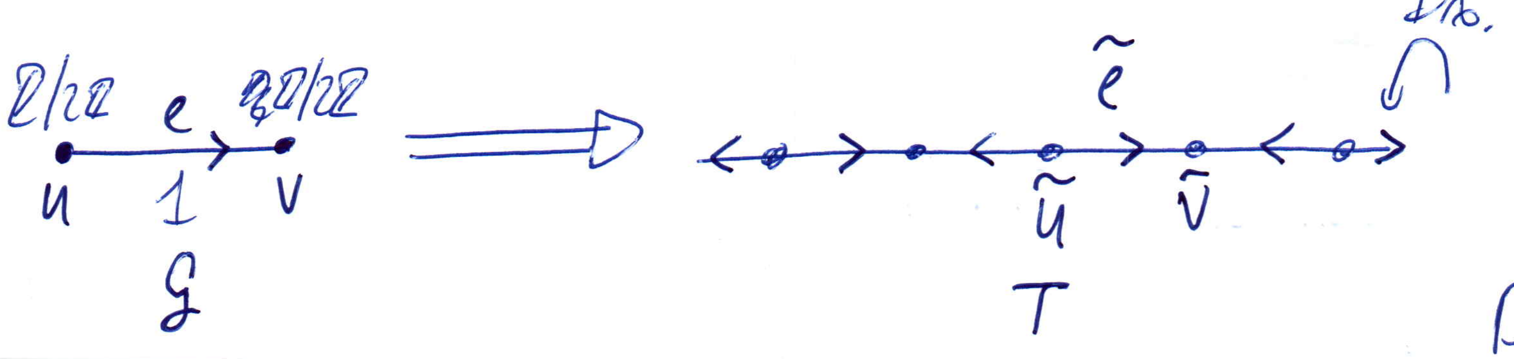
By OS: $G \longrightarrow G\tilde{v}$, $g \longmapsto g\tilde{v}$ defines: G-equiv bijections between: $G/G_v \longrightarrow G\tilde{v}$. ✓

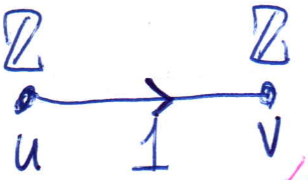
(ii) Same proof, and (iii) same as well. $\mathrm{Stab}_G(\tilde{v}) \curvearrowright \{\text{incident edges}\}$

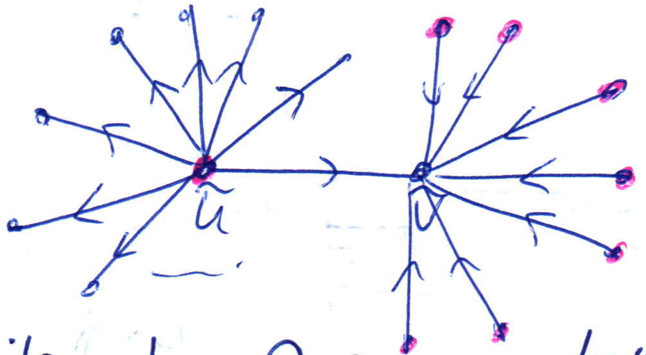
Remark T determined by Algebra of $\mathcal{G}$, hence Unique.

Example (i)



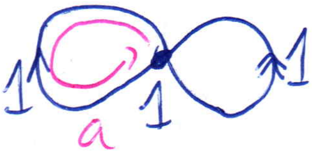


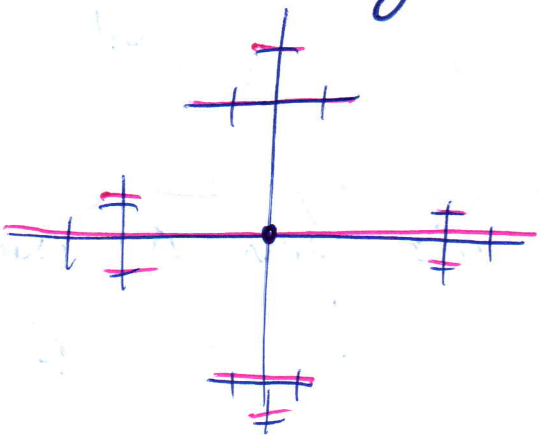




$F_2 = \pi_1(g) = \overset{\langle a\rangle}{\mathbb{Z}} \underset{1}{*} \mathbb{Z}$

$\Rightarrow$ Tree, with countably infinite valence @ every vertex.

iv) $F_2 =$ 



(F_2)

Geometric Group Theory: lecture 15.

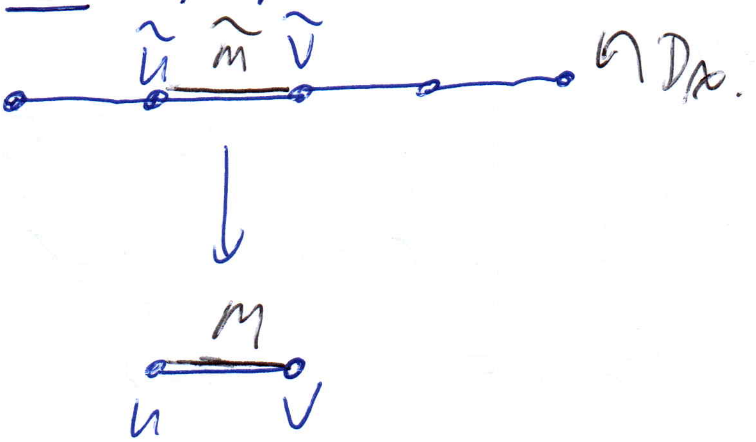
21/02/2024

From last time proved: $\mathcal{G} \cong G \backslash T$, $T =$ BS tree.

How do "stable letters" $t_e \in \pi_1 \mathcal{G}$ act on T?

$\Rightarrow$ Choose: maximal tree $M \subseteq \Gamma$. The action of $G \curvearrowright T$ depends on choice of $\tilde{M} \subseteq T$.

Example)

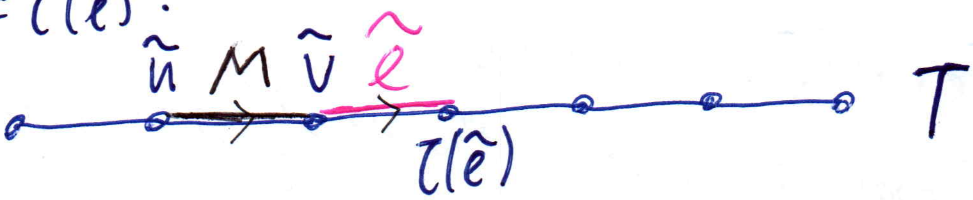


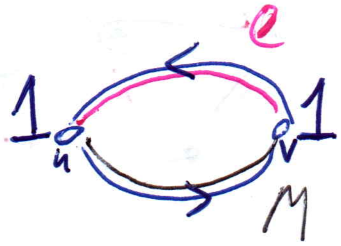
$$\begin{tikzcd} \tilde{M} \arrow[r, "incl"] \arrow[d] & \tilde{T} \arrow[d, "g"] \\ M \arrow[r, "incl"] & T \end{tikzcd}$$

The choice of $\tilde{M}$ determines choices of lifts of vertices $\tilde{u}$ and $\tilde{v}$. $(\tilde{v} \in T) \longmapsto (v \in \Gamma)$.

For each $e \in E_\Gamma$: not contained in M, choose lift $\tilde{e} \in E_T$ s.t. $i(\tilde{e}) = i(e)$.

Example)





$S^1 = \Gamma = M$.

$\Rightarrow$ The Action of t_e on T is determined, by the fact that: $t_e \tau(e) = \tau(\tilde{e})$.

[Proof: Think about action of π

Most importantly: Can understand elements of $G = \pi_1(\mathcal{G}, M)$ via Reduced Words.

DEF Fix base vertex $v_0^{(1)} \in V_\Gamma$. Consider element:

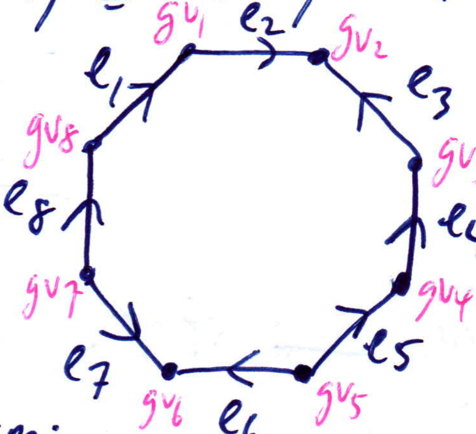
$$w \equiv (g_0 t_1^{\pm 1} g_1 \cdots g_{k-1} t_k^{\pm 1} g_k) \in \left(\mathop{\ast}_{v \in V_\Gamma} G_v\right) * F(E_\Gamma).$$

Where: $g_i \in G_{v_i^{(2)}}$ & $t_i \equiv t_{e_i}$. Then, is $\exists$ based loop if: (i) $v_0^{(1)} = v_0^{(2)} = v_k$

(ii) $e_1^{\pm 1} \cdots e_k^{\pm 1}$ loop in Γ starting & Ending at Γ

(iii) If "$t_i g_i$" then $v_i = \tau(e_i)$

If "$t_i^{-1} g_i$" then $v_i = i(e_i)$.



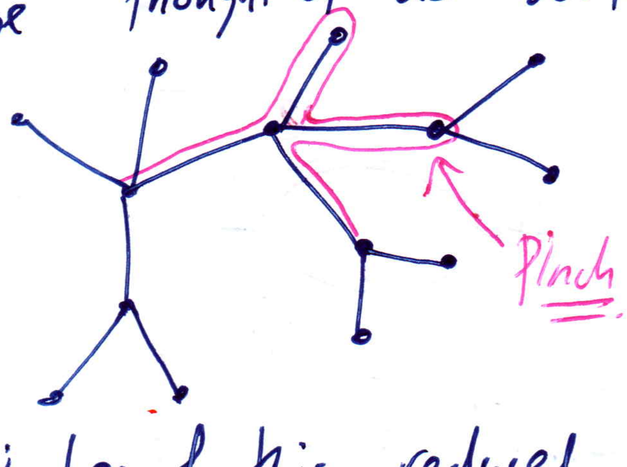
Recall: Relations $t_e i_e(G_e) t_e^{-1} = \tau_e(G_e)$.

A sub-path of loop is *pinch* if of form:

(i) $t_e i_e(h) t_e^{-1}$, some $h \in G_e$

(ii) $t_e^{-1} \tau_e(h) t_e$, some $h \in G_e$.

Remark Loops should be "thought of" as: defining paths along the BS-tree.



Call: based with No pinches of this, reduced.

Theorem [Normal Form, for Graphs of Groups.]

Let: $\mathcal{G}$ graph of groups.

(i) Any $g \in \pi_1(\mathcal{G})$ is represented by some based γ.

(ii) If γ reduced, then g nontrivial.

Remark (about the proof)

(i) The unique path $[\tilde{v}_0, g\tilde{v}_0]$ defines loop representing g.

(ii) Reduced loops $\gamma \Leftrightarrow$ Locally Injective paths $\tilde{\gamma}: I \to T$

& Locally injective $\Rightarrow$ Globally injective $\Rightarrow g\tilde{v}_0 \neq \tilde{v}_0$.

$\Rightarrow g \neq 1$.

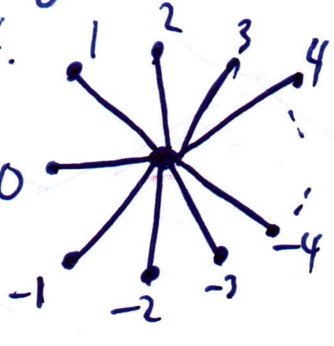
Property FA

Suppose: $G \curvearrowright T$ & A globally fixed pt for G is $x_0 \in T$, s.t.

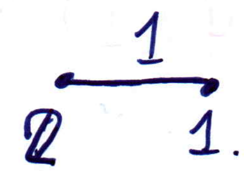
$\mathrm{Stab}(x_0) = G$. & **Say:** G acts **Trivially** on T

if $\exists$ globally fixed pt.

Example $\mathbb{Z} \curvearrowright T =$



$\Rightarrow \mathbb{Z} \backslash T =$



If $G \curvearrowright$ (some tree) Non-trivially, say: G splits.

Otherwise: say G has Property FA. "Fixe Arbres".

Lemma If T tree & ϕ isometry of T, then: either:

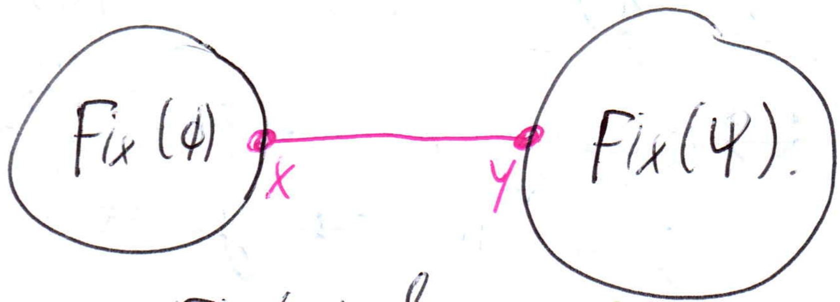
1) T fixes some pt (elliptic)

2) T translates some line by some positive distance. (hyperbolic)

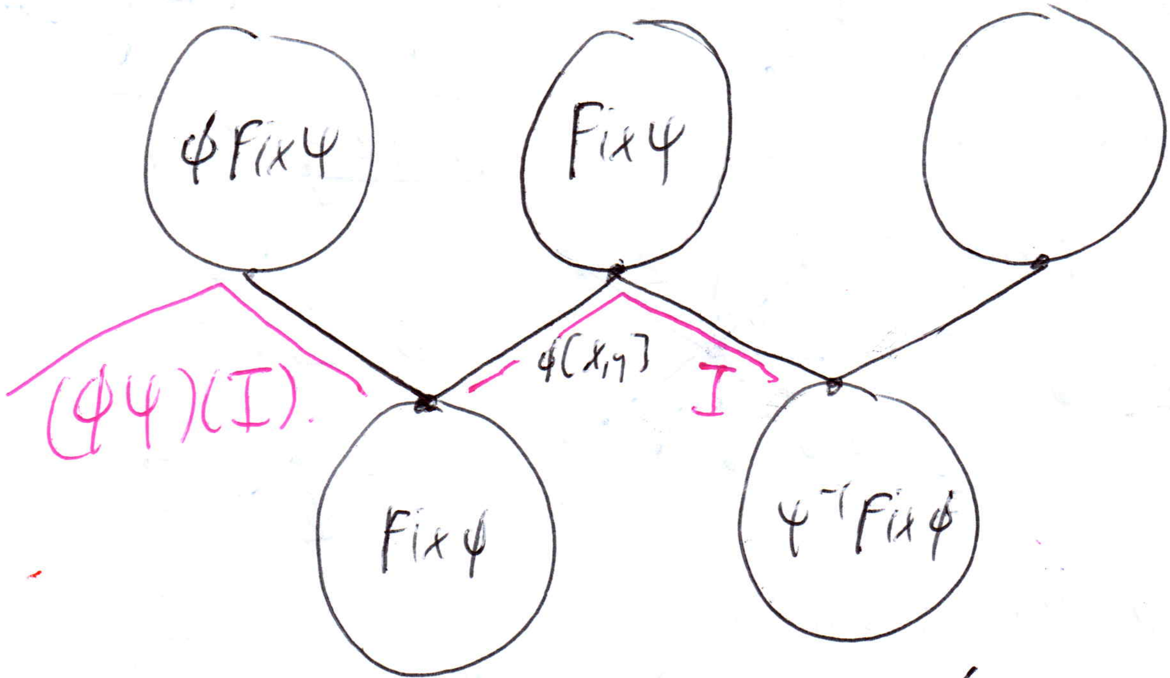
Remark order$(\phi) < \infty \Rightarrow \phi$ elliptic.

Lemma If ϕ, ψ both isometries of T, and: (both elliptic) $\mathrm{Fix}(\phi) \cap \mathrm{Fix}(\psi) = \emptyset$, then: $\phi \circ \psi$ hyperbolic.

Proof First notice $\mathrm{Fix}(\phi)$ connected subtrees of T, because if ϕ fixes x, y then ϕ fixes all points between x and y (since unique path $x - y$).



Let: $x \in \mathrm{Fix}(\phi)$ & $y \in \mathrm{Fix}(\psi)$. & $[x, y]$ unique path $x - y$.



So: I interval & $(\phi\psi)(I) \cap I = \{x =$

$\Rightarrow \bigcup_{n \in \mathbb{Z}} (\phi\psi)^n(I)$ is: invariant line preserved by $\phi\psi$.

$\Rightarrow$ Translation ✓

Geometric Group Theory: lecture 16

23/02/2024

<u>From last time</u> <u>Seek</u>: Criterion for proving that G has property FA. <u>We proved</u>:

<u>Lemma</u> $\phi, \psi \in \mathrm{Isom}(T)$ elliptic. If $\mathrm{Fix}(\phi) \cap \mathrm{Fix}(\psi) = \emptyset$ then $\phi \circ \psi$ is Hyperbolic.

<u>Next</u>: "Helly property" for Trees.

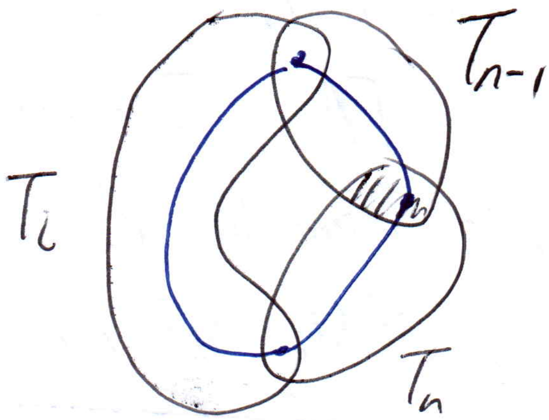
Lemma [Helly Property]. Suppose T tree & $T_1, \dots, T_n \subseteq T$ sub-trees with $T_i \cap T_j \neq \emptyset$ $\forall i, j$. <u>Then</u>: $\bigcap_{i \leq n} T_i \neq \emptyset$.

<u>Proof</u>: By <u>induction</u> on n. Trivial for $n = 1, 2$.

<u>Let</u>: $T_n' = T_{n-1} \cap T_n$. <u>Claim</u>: $T' \cap T_i \neq \emptyset$ $\forall i < n-1$.

If not, then:

$\Longrightarrow$ Contradiction.



Theorem [Criterion for FA].

<u>Let</u>: G group & $S \subseteq G$, $S = \{g_1, \dots, g_n\}$ <u>Finite</u> gen set.

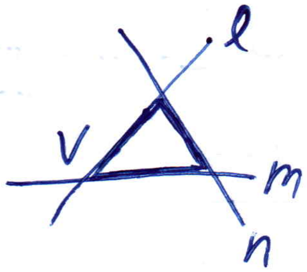
If <u>i)</u> $\mathrm{order}(s_i) < \infty$ $\forall i$

<u>ii)</u> $\mathrm{order}(s_i s_j)$ or $\mathrm{order}(s_j s_i)$ is $< \infty$, $\forall i \neq j$.

Then, G has FA.

Proof] Let $T_i = \mathrm{Fix}(s_i)$. $\Rightarrow$ Nonempty tree of G.
(<u>Assuming</u>: $G \curvearrowright T$).
& By <u>ii)</u>: $\mathrm{order}(s_i s_j) < \infty \Rightarrow T_i \cap T_j \neq \phi \ \forall i,j$.
$\Rightarrow$ By Helly Property, have $\bigcap_{i \leq n} T_i \neq \phi$.
$\Rightarrow \exists$ element Fixed by <u>all</u> of G. ✓ FA.

Example] $\Gamma \equiv$ Group generated by: <u>reflection of sides</u> of an equilateral Δ.



$= \langle r_m, r_n, r_l \rangle$ as on right
& $r_m^2 = r_n^2 = r_l^2 = id$.
& $r_m r_l$ generate Dihedral group around v. So, have finite order (Similarly, $r_m r_n$ & $r_n r_l$ have finite order).
$\Rightarrow \Gamma$ has property FA. (Infinite set with FA)
By <u>Sheet 3</u>: Dehn's examples also have FA.
[Corresponding 3-manifolds are non-Haker]

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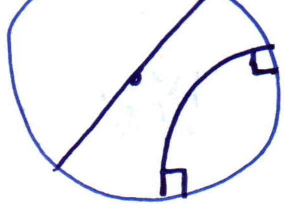
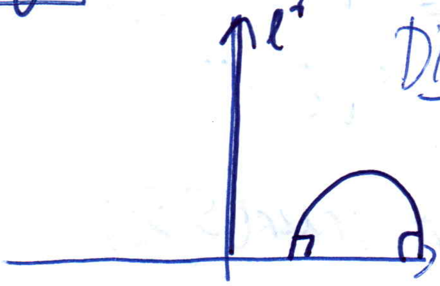
§4: FUCHSIAN GROUPS.

1. Hyperbolic Geometry.] Denote $\mathbb{H}_2 \equiv$ Hyperbolic plane:



$\mathbb{D}^2$



<u>Disc</u>: $ds^2 = \dfrac{4|dz|^2}{(1-|z|^2)^2}$

$\mathbb{H}^2$ $ds^2 = \dfrac{|dz|^2}{|\mathrm{Im}(z)|^2}$

<u>Geodesics on $\mathbb{H}^2$</u>: Are lines/circles, intersecting boundary of $\mathbb{H}^2$ orthogonally.

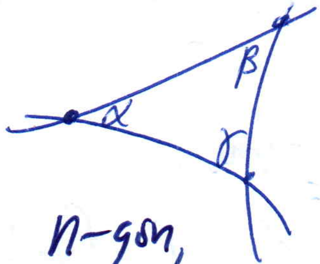
(<u>Examples</u>: $\ell^+ \subseteq \mathbb{H}^2$, positive imaginary numbers)

& $d(is, it) = \int_t^s \frac{dy}{y} = \log(s/t)$. $(s > t)$

Gauss Bonnet:

Prop If $\Delta \subseteq \mathbb{H}^2$ is Geodesic triangle, with interior angles α, β, γ then $\text{Area}(\Delta) = \pi - (\alpha+\beta+\gamma)$.

In particular, $\alpha+\beta+\gamma < \pi$.



Corollary If $P \subseteq \mathbb{H}^2$ Geodesic n-gon, then: $\text{Area}(P) = (n-2)\pi - \sum_\alpha \text{int}(P)$.

<u>Recall</u>: $\text{Isom}^+(\mathbb{H}^2) \cong PSL_2(\mathbb{R})$. ($\curvearrowright \mathbb{H}^2$ by Möbius Trans)

DEF If $\Gamma \leq PSL_2(\mathbb{R})$ acts <u>Properly Disctsly on $\mathbb{H}^2$</u>, then say Γ is <u>Fuchsian</u>.

Basic Props of $PSL_2(\mathbb{R})$.

Prop <u>i)</u> $PSL_2(\mathbb{R}) \curvearrowright \mathbb{H}^2$ extends <u>naturally</u> (<u>continuously</u>) to $\overline{\mathbb{H}^2} = \mathbb{H}^2 \cup \partial\mathbb{H}^2$. (<u>Proof</u>: Because action by Mobius maps)

<u>ii)</u> $PSL_2(\mathbb{R})$ <u>Transitive</u> on triples of pts in $\mathbb{R} \cup \{\infty\}$.

(iii) If $\phi \in PSL_2(\mathbb{R})$ fixes 3 distinct pts of $\overline{\mathbb{H}^2}$ then it is identity.

Corollary [Classification of Isometries of $\mathbb{H}^2$]

Suppose $\phi \in \mathrm{Isom}^+(\mathbb{H}^2)$. Then, one of the following holds:

1) ϕ Fixes pt of $\mathbb{H}^2$ (Unique unless $\phi = id$) [elliptic]

2) ϕ Fixes unique pt of $\partial \mathbb{H}^2$ [Parabolic]

3) ϕ Preserves unique Geodesic of $\mathbb{H}^2$. Translates it a positive distance. [Hyperbolic]

Prove it next time!

Remark If $\phi \in \Gamma$ fuchsian & elliptic, then: ~~Stab~~ Order of ϕ is finite.

Geometric Group Theory: lecture 17 | 26/02/2024

Proof of Corollary

Recall: $\phi: \mathbb{H}^2 \longrightarrow \mathbb{H}^2$ extends continuously to $\phi: \overline{\mathbb{H}^2} \longrightarrow \overline{\mathbb{H}^2}$ (e.g. for $\mathbb{H}$ = Disc model). So, by Brouwer FPT, $\mathrm{Fix}(\phi) \neq \phi$.

Also, know if $\mathrm{Fix}(\phi) \geq 3$, then ϕ = identity.

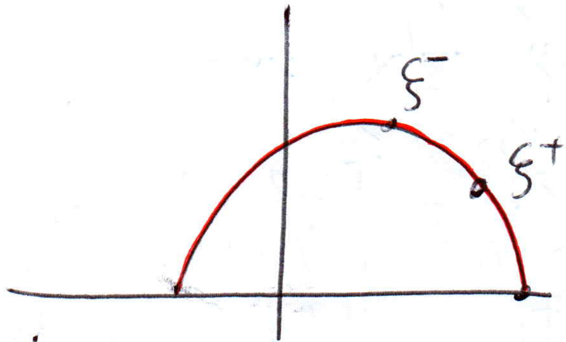
$\Rightarrow$ If $\phi \neq id$, then $\mathrm{fix}(\phi) \in \{1, 2\}$.

1) $\mathrm{Fix}(\phi) = \{\xi\} \subseteq \overline{\mathbb{H}^2}$

$\Rightarrow$ $\xi \in \mathbb{H}^2$ means ϕ elliptic, and $\xi \in \partial\mathbb{H}^2$ means ϕ Parabolic.

2) $\mathrm{Fix}(\phi) = \{\xi^+, \xi^-\}$

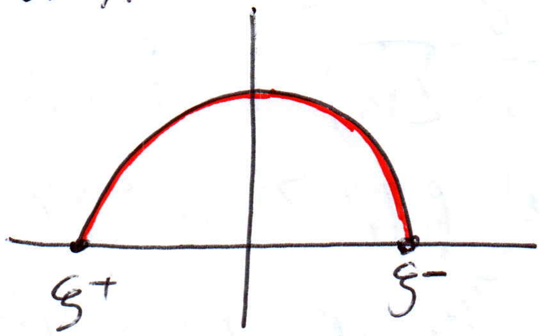
If $\xi^+ \notin \partial\mathbb{H}^2$, then: consider geodesic $[\xi^-, \xi^+]$.



$\Rightarrow$ Since ϕ isometry: must fix this entire geodesic

$\Rightarrow$ # fixed pts ≥ 3 $\Rightarrow$ $\phi = id$ ⚡

So, get: $\xi^\pm \subseteq \partial\mathbb{H}^2$



$\Rightarrow$ ϕ fixes unique geodesic $[\xi^+, \xi^-] \underset{isom}{\cong} \mathbb{R}$

& Since ϕ fixes endpoints pointwise, ϕ must be a translation (on the line)

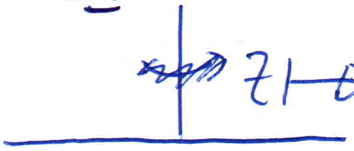
2) Examples of Fuchsian Groups.

Recall: $\Gamma \subseteq \mathrm{Isom}^+(\mathbb{H}^2)$ Fuchsian $\Leftrightarrow \Gamma \curvearrowright \mathbb{H}^2$ prop disc.

[In particular, $\forall x \in \mathbb{H}^2$, $\mathrm{Stab}_\Gamma(x) < \infty$]

Examples 1) $\mathbb{Z}/n\mathbb{Z}$

2a) $\mathbb{Z}$.



$z \mapsto \lambda z$, some fixed $\lambda > 1$. Call it ϕ.

$\Rightarrow \mathbb{Z} = \langle \phi \rangle$. (Hyperbolic)

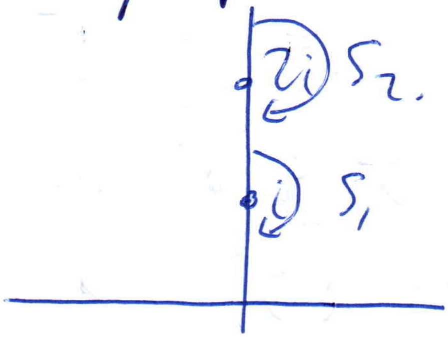
2b) $\mathbb{Z}$. $\phi : z \mapsto z+1$. (Parabolic)

These examples are "Elementary" Fuchsian groups.

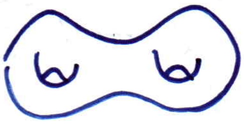
3) $\langle S_1, S_2 \rangle$.

$= \langle S_1, S_2 \mid S_1^2, S_2^2, \text{Composition} \rangle$

$\cong D_\infty$. "Also elementary."



4) Σ_g.



$(g \geq 2) \Rightarrow \widetilde{\Sigma}_g \underset{\text{isom}}{\cong} \mathbb{H}^2$.

$\Rightarrow \pi_1(\Sigma_g)$ is Fuchsian.

5) Let $p, q, r \geq 1$ integers.

DEF] The (p,q,r) – Triangle group is defined by Pres.

$$\Gamma(p,q,r) = \langle a, b, c \mid a^p, b^q, c^r, abc \rangle.$$

Notice this does not split. since it has property FA.

So, is $\Gamma(p,q,r)$ nontrivial? Infinite?

$\Rightarrow$ Not always!

E.g. $p=2$, $q=3$, $r=1$ $\Rightarrow$ Still get trivial group.

However, many interesting examples arise by. Poincare

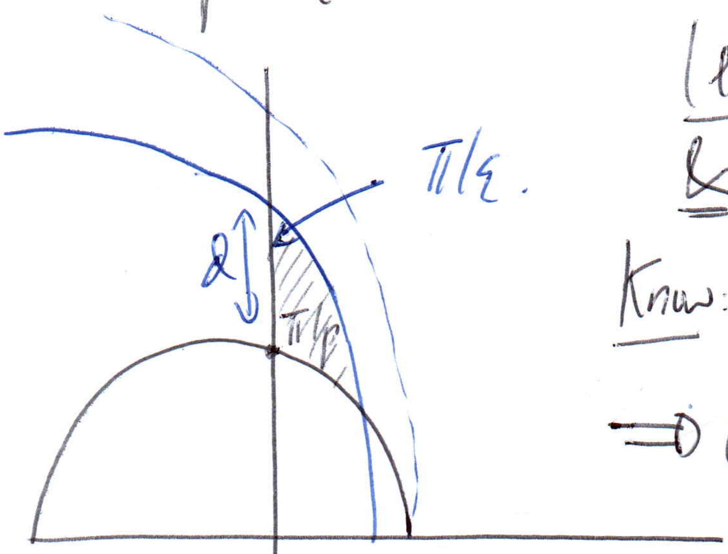
<u>Theorem</u> [Poincare Polygon Theorem.]

If $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$, then: $\Gamma(p,q,r)$ Infinite Fuchsian.

[E.g. (2, 3, 7).]

Converse is also true! (Sort of)

<u>Proof</u> Start off by taking $\Delta \subseteq \mathbb{H}^2$, with Interior angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$. To construct it, do following:



<u>Let</u>: d vary around.

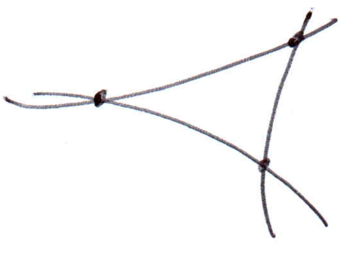
<u>&</u> Let Δ_d = triangle formed.

<u>Know</u>: $\text{Area}(\Delta_d) = \pi - \text{Sum of angles}$.

$\Rightarrow 0 \leq \text{Area}(\Delta_d) < \pi - \left(\frac{\pi}{p}+\frac{\pi}{q}\right)$

$\Rightarrow$ By continuity: must have some d s.t. $\theta_d = \pi/r$.

$\Rightarrow$ Successfully constructed Δ.



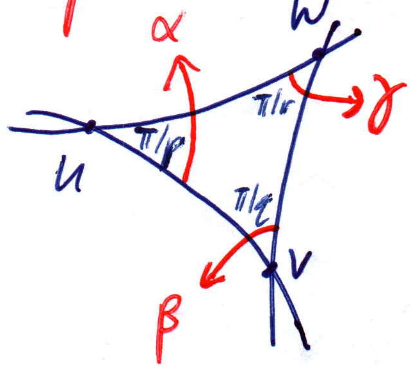
<u>Consider</u>: Group generated by reflections across the <u>sides</u> <u>&</u> <u>Get</u> the Orientation Preserving isometries.

Geometric Group Theory: Lecture 18 | 28/02/2024

From last time $\Gamma(p,q,r) = \langle a,b,c \mid a^p, b^q, c^r, abc \rangle$.

Theorem If $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$ then $\Gamma(p,q,r)$ Infinite & Fuchsian.

Proof From last lecture: construct Δ with interior angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$ on Hyperbolic plane.



Let: $\alpha = \mathrm{Rot}(u, \frac{2\pi}{p})$

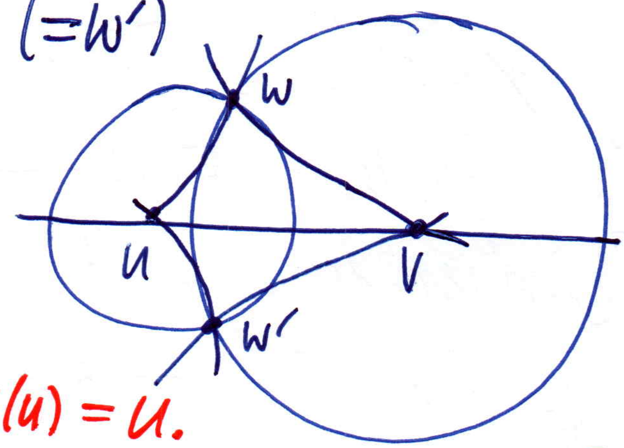
$\beta = \mathrm{Rot}(v, \frac{2\pi}{q})$

$\gamma = \mathrm{Rot}(w, \frac{2\pi}{r})$.

& $G = \langle \alpha, \beta, \gamma \rangle \leq \mathrm{Isom}^+(\mathbb{H}^2)$ & clearly $\alpha^p = \beta^q = c^r$.

By diagram: $\beta(w) = \alpha^{-1}(w)$. $(=w')$

$\Rightarrow \alpha\beta\gamma(w) = \alpha\beta(w) = w$.



Similarly: $\gamma(u) = \beta^{-1}(u)$

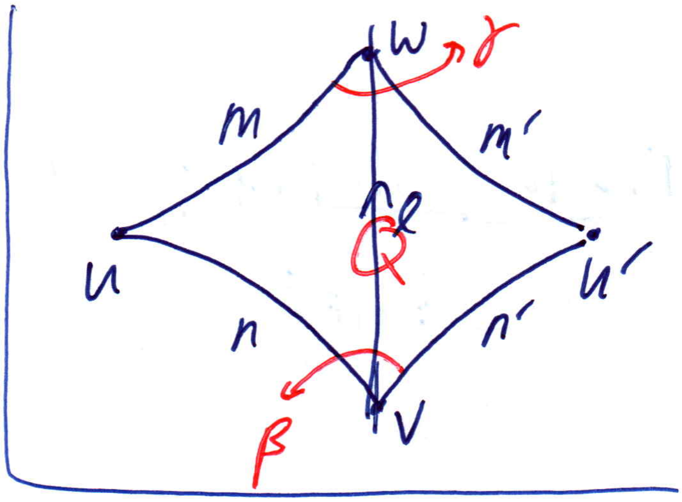
$\Rightarrow \alpha\beta\gamma(u) = \alpha\beta\beta^{-1}(u) = \alpha(u) = u$.

$\Rightarrow \alpha\beta\gamma$ is: orientation-preserving isometry of $\mathbb{H}^2$ that fixes 2 distinct pts on $\mathbb{H}^2$, hence $\alpha\beta\gamma = 1$.

$\Rightarrow \exists$ Surjective hom $\Gamma \xrightarrow{f} G$, $(a,b,c) \mapsto (\alpha,\beta,\gamma)$.

Aim: f is an isomorphism. By: Ingenious construction!

Denote: $r_\ell \equiv$ Reflection in ℓ & let $Q = \Delta \cup r_\ell(\Delta)$.



Define: $\tilde{Q} = (\Gamma \times Q)/\sim$

$(g\gamma, x) \sim (g, \gamma(x)) \quad \forall x \in m$

$(g\beta, x') \sim (g, \beta(x')) \quad \forall x' \in n'$.

(This is for: $g \in \Gamma$)

& Define: $F : \tilde{Q} \longrightarrow \mathbb{H}$

$(g, x) \longmapsto f(g)x$.

Know: $\tilde{Q}$ is complete, geodesic metric space. (Hopf-Rienow)

& F is a Local isometry, sending small balls in $\tilde{Q}$ to small open balls in $\mathbb{H}$ isometrically.

[In fact: F is isometric embedding].

Indeed: $x, y \in \tilde{Q}$ & $[x, y]$ geodesic. Then: $F([x, y])$ is local geodesic, $F(x) \longrightarrow F(y)$.

But: Local geodesics in $\mathbb{H}^2$ are Global geodesics.

$\Rightarrow d_{\mathbb{H}^2}(F(x), F(y)) = d_{\tilde{Q}}(x, y)$.

Remains to show: F surjective.

Know: F sends open balls to open balls. $\Rightarrow$ Open map.

But: $F(\tilde{Q}) \underset{\text{isom}}{\cong} \tilde{Q}$ complete $\Rightarrow$ Closed in $\mathbb{H}^2$.

$\Rightarrow F$ surjective, hence, isometry ✓

So, $\tilde{Q} \underset{\text{isom}}{\cong} \mathbb{H}^2$. But: $\Gamma \curvearrowright \tilde{Q}$ properly disctsly (by const.)

$\Rightarrow \Gamma$ is Fuchsian, and $|\Gamma| = \infty$ since we need

Γ-many compact tiles to cover $\mathbb{H}^2$.

Remark Follows from construction that only $\Gamma u, \Gamma v, \Gamma w$ have Stab $\neq 1$.

Call Q a Fundamental Domain for $\Gamma \curvearrowright \mathbb{H}^2$.

§3: Centres & Dehn Examples.

Lemma If $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$, $g \in \Gamma(p,q,r)$ & order$(g)<\infty$.

Then: g conjugate ~~to~~ into $\langle a\rangle, \langle b\rangle, \langle c\rangle$.

Proof Saw: Finite order elements of $\mathbb{H}^2$ fix some point z of $\mathbb{H}^2$. If $g \neq 1$, then WLOG $z = h\cdot u$, some $h \in \Gamma$.

$ghu = hu \Rightarrow (h^{-1}gh)u = u. \Rightarrow h^{-1}gh \in \mathrm{Stab}(u) = \langle a\rangle$.

Prop If Γ non-Elementary Fuchsian, then $Z(\Gamma) = 1$.

Proof Suppose $z \in Z(\Gamma)$, $z \neq 1$. & Consider $\mathrm{Fix}(z) \subseteq \overline{\mathbb{H}^2}$.

Note $\forall g \in \Gamma$, $\forall x \in \mathrm{Fix}(z)$, $gx = gzx = zgx$.

$\Rightarrow gx \in \mathrm{Fix}(z)$

Finish next time!

Geometric Group Theory: Lecture 19 | 01/03/2024

Prop | If Γ Fuchsian & Non-elementary, then $Z(\Gamma) \cong 1$.

Proof | If $\gamma \in Z(\Gamma) - 1$, then: g preserves $\mathrm{Fix}(\gamma) \subset \overline{\mathbb{H}^2}$ for any $g \in \Gamma$.

⊛ If γ elliptic $\Rightarrow \mathrm{Fix}(\gamma) = \{x\} \subset \overline{\mathbb{H}^2}$, WLOG $x = 0 \in \mathbb{D}^2$.

$\Rightarrow \mathrm{Stab}(x) \cong \{z \mapsto e^{i\theta} z\}$.

$\Rightarrow \Gamma \cong \mathbb{Z}/n\mathbb{Z}$, some n.

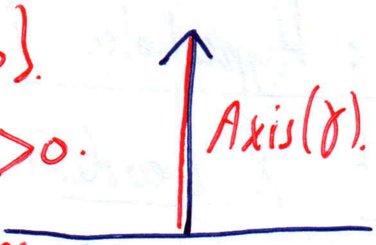
⊛ If γ parabolic $\Rightarrow$ WLOG $\mathrm{Fix}(\gamma) = \{\infty\}$ (upper half model)

$\Rightarrow$ By direct computation, $g(z) = az + b$.

& $\mathrm{Fix}(\gamma) = \infty \Rightarrow \gamma(z) = z + c$, some c.

& g commutes with $\gamma \Leftrightarrow a = 1$, i.e. $\Gamma \leq \{z \mapsto z + b\} \cong \mathbb{Z}$.

⊛ If γ Hyperbolic, WLOG: $\mathrm{Fix}(\gamma) = \{0, \infty\}$.



$\Rightarrow$ By direct computation: $g(z) = \lambda z$, $\lambda > 0$.

$\Rightarrow g$ preserves ℓ^+. So, Γ acts by isometries on some line, so $\Gamma \leq \mathbb{Z}$ or D_∞. (By Prop discts).

Next: Analyse Dehn examples.

$$G_n \equiv \langle x, y, z \mid x^2 = y^3 = z \text{ \& } (xy)^{6n+5} = z^{5n+4} \rangle$$

$$\Rightarrow z \in Z(G_n), \text{ so } \underbrace{G_n/\langle z \rangle}_{\Gamma_n \equiv} = \langle x, y \mid x^2, y^3, (xy)^{6n+5} \rangle$$

$$= \Gamma(2, 3, 6n+5).$$

$\Rightarrow \Gamma_n$ is fuchsian triangular group, as long as $n \geq 1$.

$\Rightarrow Z(\Gamma_n) = 1 \quad \forall n \geq 1$.

$\Rightarrow Z(G_n) = \langle z \rangle$.

So, if $G_n \cong G_m$, then $Z(G_n) \cong Z(G_m)$

& $\Gamma_n = G_n / Z(G_n) \cong G_m / Z(G_m) \cong \Gamma_m$.

But: Look at Torsion. The orders of torsion in Γ_n were exactly the divisors of $2, 3, 6n+5$.

Hence: $\{\text{divisors of } 2, 3, 6n+5\} = \{\text{divisors of } \cancel{6}, 2, 3, 6m+5\}$

$\Rightarrow m = n$ ✓

$\Rightarrow$ There are infinitely many non-homotopic 3D homology spheres.

§5: Hyperbolic Groups.

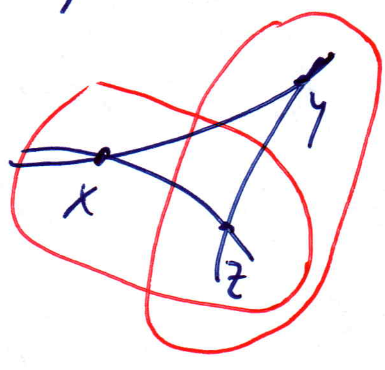
Goal: Define "Coarse" Hyperbolic ~~plane~~ geometry, that is invariant under $\widetilde{q_i}$.

1: Hyperbolic Metric Spaces.

Let: X Geodesic metric space. (Any 2 pts joined by geodesic)

A Geodesic Triangle is a triple $[x,y] \cup [y,z] \cup [z,x]$.

$\boxed{\text{DEF}}$ for $\delta > 0$, denote $N_\delta[x,y]$ as δ-neighborhood of (x,y).



Say: Δ is δ-slim, if:

$[x,y] \subset N_\delta[x,z] \cup N_\delta[y,z]$

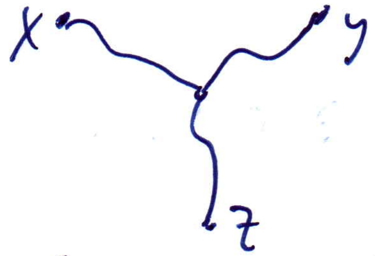
& similar with other sides.

Call: X δ-hyperbolic if: $\forall \Delta \subseteq X$ Geodesic triangle, Δ is δ-slim.

If δ-hyperbolic for some δ $\Rightarrow$ Gromov-Hyperbolic (or just Hyperbolic)

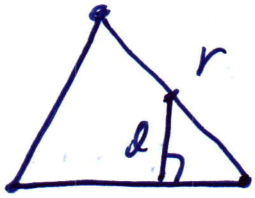
Examples] 1) If X has bounded (diameter δ) then clearly X is δ-hyperbolic.

2) X tree

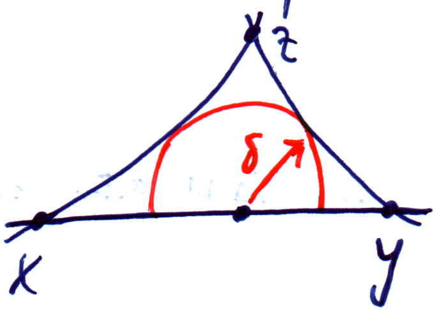


$\Rightarrow$ 0-hyperbolic.

3) $\mathbb{R}^2$ Not Gromov-hyperbolic.



4) $X = \mathbb{H}^2$.



Have: Δ is δ-slim, where: δ = ~~Area of~~ Radius of largest semicircle that fits.

Have: $\frac{1}{2}\operatorname{Area}(\delta) \leq \operatorname{Area}(\Delta) = \pi - (\alpha+\beta+\gamma) < \pi$

$\Rightarrow \operatorname{Area}(\delta) < 2\pi$, so $\delta < A^{-1}(2\pi)$.

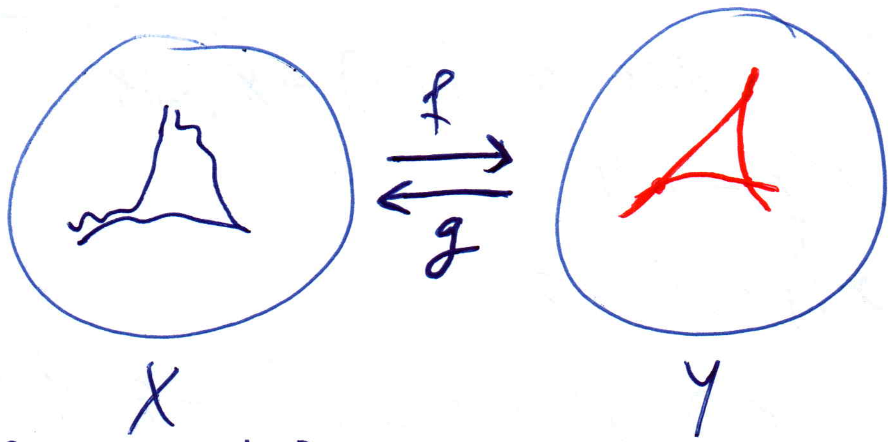
$\Rightarrow \mathbb{H}^2$ is Gromov-Hyperbolic.

2]: Mostow-Morse Lemma.

Goal: Prove, Gromov-Hyperbolic property is $\widehat{qi}$-invariant (possibly for different δ).

Geometric Group Theory: Lecture 20 | 04/03/2024

Goal: Prove G-hyperbolicity is Quasi-isometry invariant.



DEF) $\gamma:[a,b]\to X$ Path (λ,ε)-Quasigeodesic,

if: $\forall s,t\in[a,b]$, $\frac{1}{\lambda}|s-t|-\varepsilon\le d(\gamma(s),\gamma(t))\le\lambda|s-t|+\varepsilon$.

DEF) [Hausdorff Distance].

Let: $A,B\in X$ (X Metric Space) & $N_c(A)=\bigcup_{a\in A}B_c(a)$

$=\{x\in X:\exists a\in A,\ d(x,a)<c\}$.

For $A,B\neq\phi$, define Hausdorff Distance:

$d_{Haus}(A,B)=\inf\{C>0:\ A\subseteq N_c(B)\ \&\ B\subseteq N_c(A)\}$.

DEF) [Length].

Let: $\gamma:[a,b]\to X$ Continuous path. Then, the length of γ is: $\ell(\gamma)=\sup_{\mathcal{D}}\left\{\sum_{1\le i\le n}d(\gamma(t_{i-1}),\gamma(t_i))\right\}$

where $\mathcal{D}$ is Dissection $a=t_0<t_1<\dots<t_n=b$

(& Sup ranges across all such $\mathcal{D}$).

Lemma $\forall \lambda \geq 1,\ \varepsilon \geq 0 : \exists\ \lambda' \geq 1,\ \varepsilon' \geq 0$ such that:
$\forall X$ Geodesic metric space & Any (λ, ε)-quasigeodesic, $(\alpha : [a,b] \to X)$
$\exists$ Continuous (λ', ε')- Geodesic $\alpha' : [a,b] \to X$, with:

1) $\alpha'(a) = \alpha(a)$ & $\alpha'(\beta) = \alpha(b)$.
2) $d_{Haus}(im(\alpha), im(\alpha')) \leq \lambda + \varepsilon$
3) ~~[illegible]~~ $\ell\left(\alpha'|_{[s,t]}\right) \leq \lambda' d\left(\alpha'(s), \alpha'(t)\right) + \varepsilon \quad \forall s,t \in [a,b]$

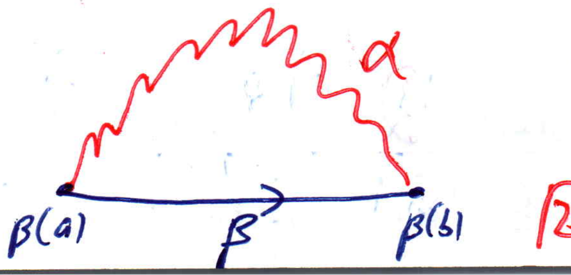
Proof

Let: $I = \{a,b\} \cup \left((a,b) \cap \mathbb{Z}\right)$.
Define α', by: setting $\alpha'(t) = \alpha(t)\ \forall t \in I$, and then linearly interpolate a geodesic between points of I.
Continuous: ✓
Ends & starts at correct points: ✓
& By easy & Tedious computations, get proof.

Lemma X δ-Hyperbolic MS. Suppose: $\beta : [a,b] \to X$ Geodesic, & $\alpha : [a,b]$ Continuous path s.t. $\alpha(a) = \beta(a)$, $\alpha(b) = \beta(b)$.

Then: $\forall t \in [a,b]$:

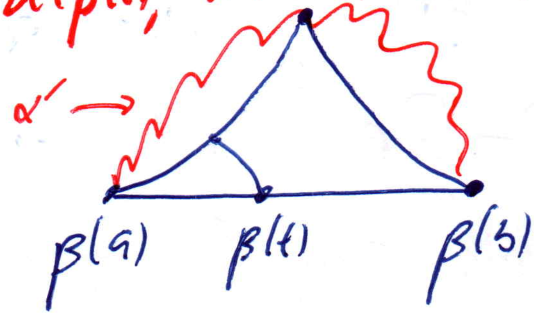
$d(\beta(t), im\,\alpha) \leq \delta \lfloor \log_2(\ell(\alpha)) \rfloor + 1.$



Proof Slim Triangles!

Let $N = \lfloor \log_2(\ell(\alpha)) \rfloor$. Proof by Induction on N.

If $N=0 \Rightarrow \ell(\alpha) \leq 1 \Rightarrow d(\beta(t), im(\alpha)) \leq 1$ ✓



Inductive Step:

There is a **Geodesic** β'

& some subpath $\alpha' \subseteq \alpha$

such that: has length $\ell(\alpha') \leq \ell(\alpha)/2$ & there's some point $\beta'(t')$ on ~~β~~ β' s.t. $d(\beta(t), d(\beta'(t'))) \leq \delta$.

$\Rightarrow$ By **Induction**: $d(\beta(t), im(\alpha)) \leq d(\beta(t), \beta'(t')) + d(\beta'(t'), im\,\alpha)$.

$\leq \delta + (\delta(N-1)+1) = \delta N + 1$ ✓

Theorem (Mostow-Morse Lemma)

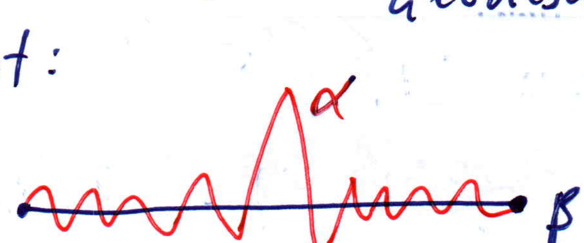
Let: X Geodesic δ-Hyperbolic space.

& $\alpha : [a', b'] \longrightarrow X$ (λ, ε)- ~~hyperbolic~~ quasi-geodesic

& $\beta(a) = \alpha(a')$, $\beta(b) = \alpha(a')$ for $\beta : [a,b] \to X$ Geodesic.

Then: $\exists\, C = C(\lambda, \varepsilon, \delta)$ such that:

$d_{Haus}(im(\alpha), im(\beta)) \leq C$



Proof As above, may assume: α "tame", i.e. in the sense of previous lemmas. (satisfies 3 properties)

i.e. α Continuous, $\ell(\alpha|_{[s,t]}) \leq \lambda|s-t| + \varepsilon$.

Need to bound: both $\inf\{C : im(\beta) \subset N_C(im\,\alpha)\}$ (1)

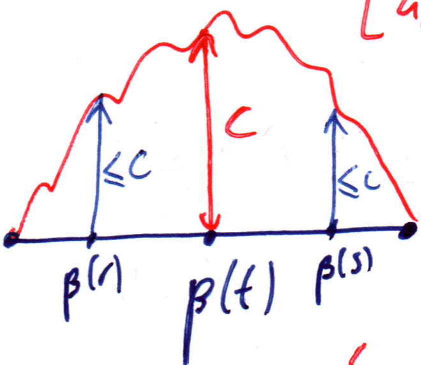
$\&\ \inf\{C : im(\alpha) \subseteq N_C(im\,\beta)\}$. (2)

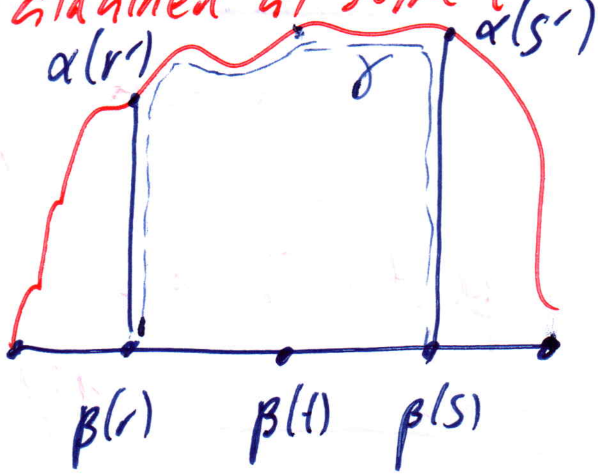
$\underline{\underline{(1)}}$: $d(\beta(t), im(\alpha))$. $\left[= \inf(d(\beta(t), d(\alpha(t')))\right]$.

$\forall a' \le t' \le t$.

$\underline{\text{Let}}$: $C = \sup\{d(\beta(t), im(\alpha))\}$.

$\Rightarrow$ Exists, since $[a,b]$ compact, & attained at some t.





$\underline{\text{Let}}$: $r = \max\{0, t-2c\}$.

$s = \min\{b, t+2c\}$.

Let γ be the path illustrated on Right.

$\Rightarrow \ell(\gamma) \le 2C + \ell(\alpha|_{[r',s']})$

$\le 2C + \lambda|r'-s'| + \varepsilon$

$\le 2C + \lambda(6C) + \varepsilon$. Linear func on C!

Combine with Log term from prev lemma.

Geometric Group Theory: Lecture 21) 06/03/2024

Mostow-Morse Lemma.

Let: $\alpha : [a', b'] \to X$ δ-hyperbolic & (λ, ε)-quasi geodesic.

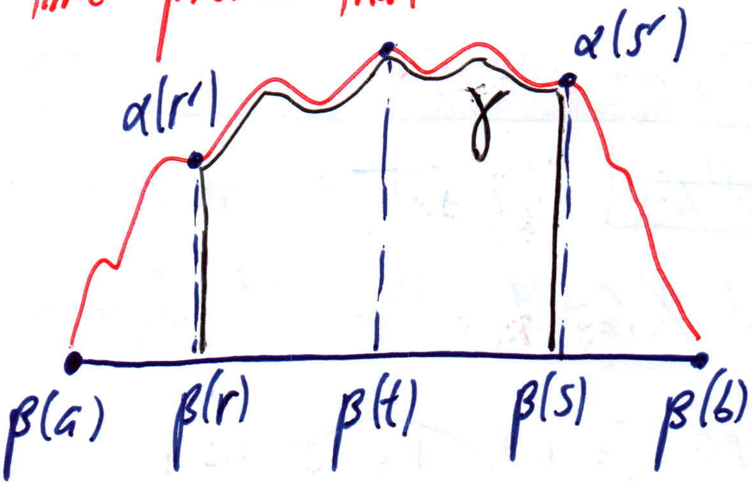
& $\beta : [a,b] \to X$ geodesic from $\alpha(a') \to \alpha(b')$.

Then: $d_{Haus}(im(\alpha), im(\beta)) \leq C. = C(\lambda, \varepsilon, \delta)$.

Proof) (Continued). From last time: proved that:

$\ell(\gamma) \leq 2C + 6C\lambda + \varepsilon$.

(γ illustrated to the right).



& From last time also, had the bound:

$$C = d(\beta(t), im(\gamma)) \leq \delta \lfloor \log_2 \ell(\gamma) \rfloor + 1.$$

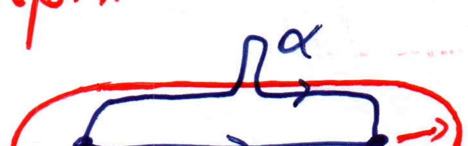
In conclusion: $C \leq \delta \lfloor \log_2 (2C + 6\lambda C + \varepsilon) \rfloor + 1$.

Because LHS linear & RHS logarithmic: $\exists$ Upper Bound for C. ✓

---

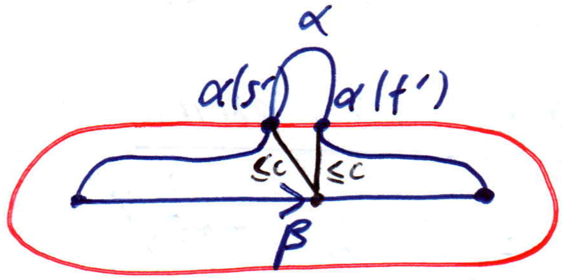
2): Remains to **bound**: $d(\alpha(t), im(\beta))$.

"Worried about" picture:



So, find $[s', t'] \subset [a', b']$ **maximal** s.t. $\alpha|_{(s', t')}$ lies **Outside** $N_c(im\, \beta)$.

By Continuity: $\exists t \in [a,b]$, $s \in [a', s']$ & $r \in [r', b']$



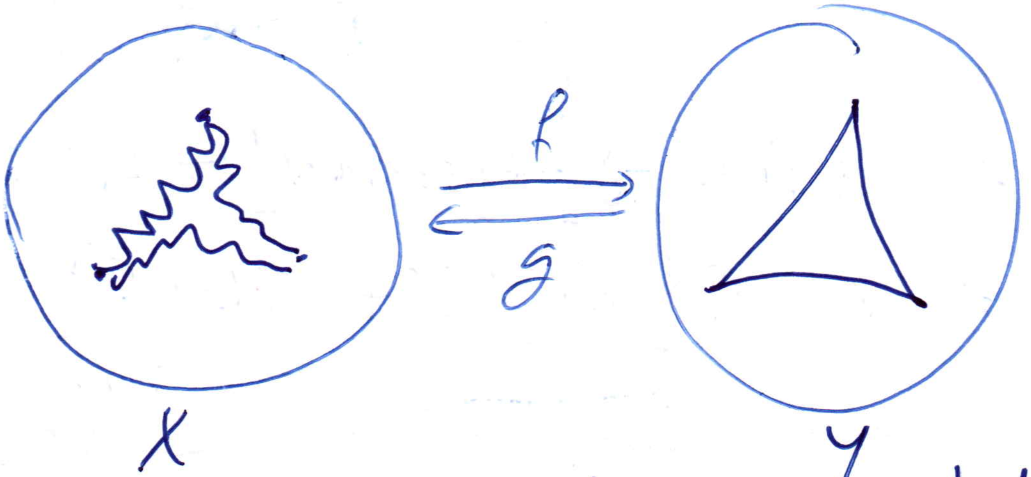
with: $d(\beta(t), \alpha(s)) \leq C$
& $d(\beta(t), \alpha(r)) \leq C$.

$\Rightarrow \ell(\alpha|_{[s',r']}) \leq \ell(\alpha|_{[s,r]}) \leq \lambda d(\alpha(s), \alpha(r)) + \varepsilon$
$< 2C\lambda + \varepsilon$.

$\Rightarrow$ Any point of α is distance $< C(2\lambda+1)+\varepsilon$ from im β ✓

Corollary Let X, Y Geodesic MS. If X δ-hyperbolic and $X \underset{qi}{\simeq} Y$, then Y Gromov-Hyperbolic.

Idea:

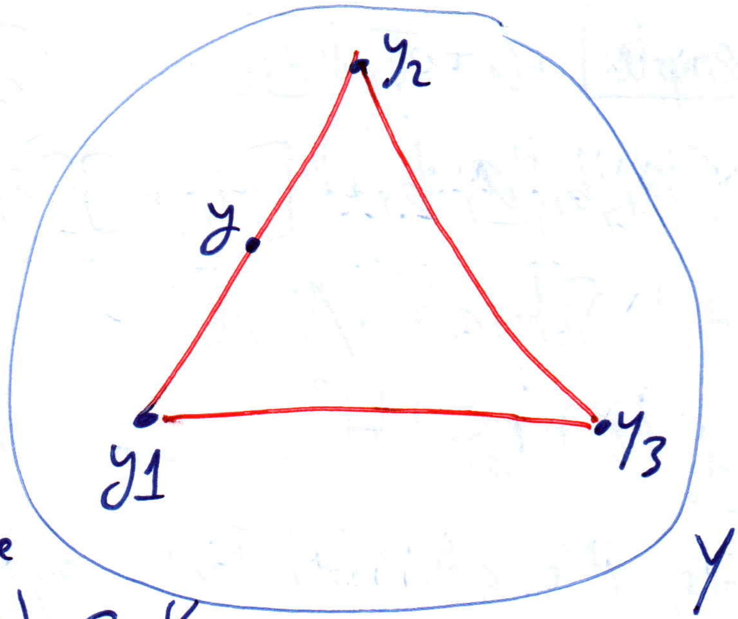


Can replace "image" with some close geodesic between them.

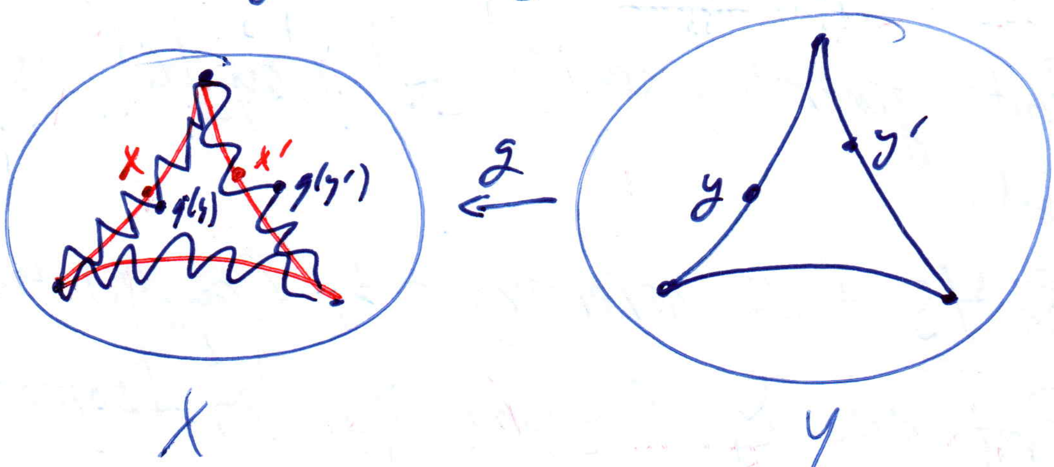
Proof Let: $f: X \to Y$, $g: Y \to X$ quasi-Isometries.
[(λ, ε)-quasi-isometries & $d(f\circ g(y), y) \leq \varepsilon$
& $d(g \circ f(x), x) \leq \varepsilon$.]

Consider: Geodesic triangle $(y_1, y_2, y_3) \subset Y$.

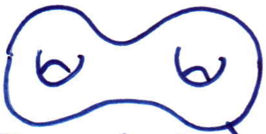
<u>Let</u>: $y \in [y_1, y_2]$.



$\Rightarrow$ By M-M lemma:
$\exists x \in [g(y_1), g(y_2)]$, with
$d(x, g(y)) \leq C$.
Apply δ-Hyperbolicity to the triangle: $(g(y_1), g(y_2), g(y_3)) \subset X$:
$\Rightarrow \exists x' \in [g(y_2), g(y_3)]$ (wlog) s.t. $d(x, x') \leq \delta$.



$\Rightarrow d(g(y), g(y')) \leq 2C + \delta$.
[<u>since</u>: by MM, $\exists y' \in [y_1, y_2]$, $d(x', g(y')) \leq C$]
$\Rightarrow d(f\circ g(y), f\circ g(y')) \leq \lambda(2C + \delta) + \varepsilon$.
($\cancel{\Rightarrow}$ f is a (λ, ε)-isometry)
because:
$\Rightarrow d(y, y') \leq 2\varepsilon + d(f\circ g(y), f\circ g(y'))$
$\leq \lambda(2C + \delta) + 3\varepsilon$. Function indep of Δ.
$\Rightarrow$ Y is indeed Gromov-Hyperbolic. ✓

Example $G = \pi_1(\Sigma_2)$ 

$= \langle a_1, b_1, a_2, b_2 \mid [a_1, b_1][a_2, b_2] \rangle.$

Apply: Schwarz-Milner

$Cay_S(G) \underset{qi}{\simeq} \mathbb{H}^2$. (Is: Gromov-Hyperbolic)

Hence, it's also true for $Cay_S(G)$.

$\boxed{3:}$ Hyperbolic Groups.

Using previous corollary, **TFAE**:

1) G has finite generating set S, s.t. $Cay_S(G)$ is Gromov-Hyperbolic

2) G f.g. & $\forall S$ finite generating set, $Cay_G(S)$ Hyp.

3) G acts Prop disc & Co-compactly by Isometries, on some Proper geodesic & Gromov-Hyperbolic MS X

4) Every (Proper & Geodesic) MS X, on which G acts Prop disc & Cocompactly, is Gromov-Hyperbolic.

DEF) If ANY of the above is true, then G is (word)-Hyperbolic.

Geometric Group Theory: Lecture 22

08/03/2024

From last time:

DEF If G f.g. & some (any) Cayley graph $Cay_S(G)$ is Hyperbolic (for some finite S), then G is called: Word-Hyperbolic.

Examples 1) Any finite G is Bounded $\Rightarrow$ Hyperbolic.

2) If $G = F_m$: Standard generating set S gives $Cay_G(G) =$ Tree (0-hyperbolic). So, F_m hyperbolic.

3) $\mathbb{Z}^2$ not hyperbolic: if it were, $\mathbb{Z}^2 \curvearrowright \mathbb{R}^2$ naturally so (by Schwarz-Milner) $\mathbb{R}^2$ Hyperbolic #

4) If $g \geq 2$, $G = \pi_1(\Sigma_g) \Rightarrow \pi_1(\Sigma_g) \curvearrowright \mathbb{H}^2$ geometrically (i.e. prop discts + Cocompactly & by isom.) $\Rightarrow G$ is Hyperbolic.

5) Holds for $\pi_1(M)$ for any M closed Riemannian manifold, with Sectional Curvature <0.

6) $G = SL_2(\mathbb{Z}) \cong \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$
$\Rightarrow$ BS tree is 3-valent tree T, so $G \curvearrowright T$ geometrically and so $SL_2(\mathbb{Z})$ is Hyperbolic.

7) Randomly finitely-Presented groups! $\langle a_1, \ldots, a_n \mid r_1, \ldots, r_n \rangle$ chosen "At random" (in suitable sense).

$\Rightarrow G$ is infinite & Hyperbolic. (almost always)

§4: Local Geodesics.

Goal: Solve WP in Hyperbolic Groups.

Key Ingredient is local-to-global statement about geodesics, in Hyperbolic metric Spaces.

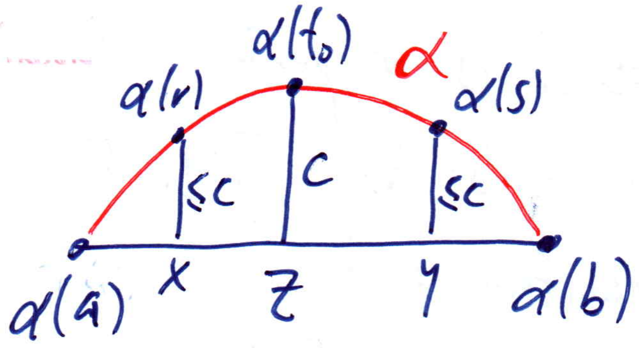
DEF) A path $\gamma : [a,b] \to X$ is C-local geodesic, if $d(\gamma(s), \gamma(t)) = |s-t|$ whenever $|s-t| \leq C$.

Lemma [Local Geodesics, in Hyperbolic Spaces]
Let X δ-Hyperbolic MS. Then, if $\alpha : [a,b] \to X$ is a (10δ)-Local Geodesic, then: $im_\alpha \subseteq N_{2\delta}([\alpha(a), \alpha(b)])$.
[Holds for ANY choice of geodesic α!]

Proof) $C = \sup_t \{ d(\alpha(t), [\alpha(a), \alpha(b)]) \}$ & say realised at some $t_0 \in [a,b]$.

& let $r = \max(a, t_0 - 5\delta)$

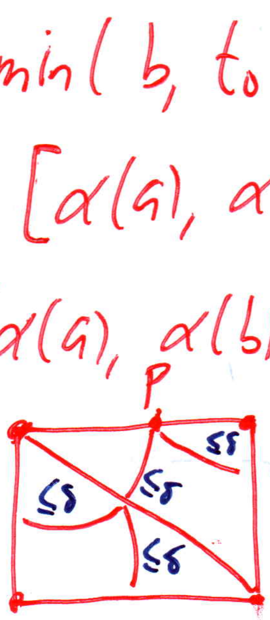
$s = \min(b, t_0 + 5\delta)$



& Let: $z \in [\alpha(a), \alpha(b)]$ minimize distance to $\alpha(t_0)$

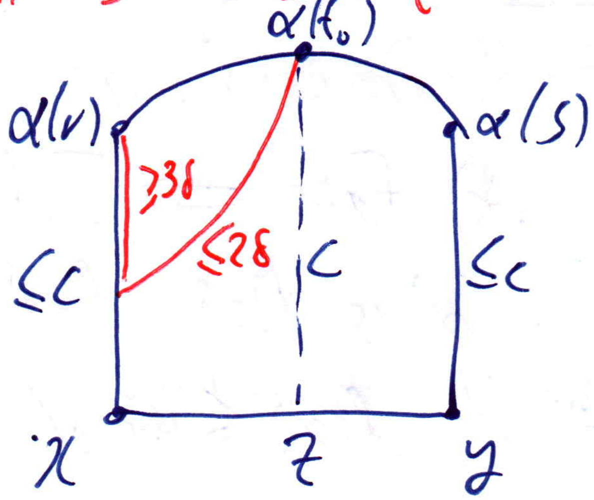
& $x, y \in [\alpha(a), \alpha(b)]$ defined similarly.

Observe:



$\exists$ Point on Quadrilateral, that is distance $\leq 2\delta$ from other sides.

Apply observation to $[\alpha(r), \alpha(s), y, x]$ on side $[\alpha(r), \alpha(s)]$



Suppose: $\exists\, w \in [\alpha(r), x]$, with $d(w, \alpha(t_0)) \leq 2\delta$.

$$\Rightarrow d(\alpha(r), w) \geq d(\alpha(r), \alpha(t)) - d(\alpha(t), w)$$

$$\geq (5\delta - 2\delta) = 3\delta.$$

$\Rightarrow d(\alpha(r), x)$ is $\leq C$ & $\geq 3\delta$

$$d(\alpha(t_0), x) \leq (2\delta) + d(w, x) < 3\delta + d(w, x) \leq d(\alpha(r), x) \leq C. \ \#$$

Hence, indeed, $C \leq 2\delta$.

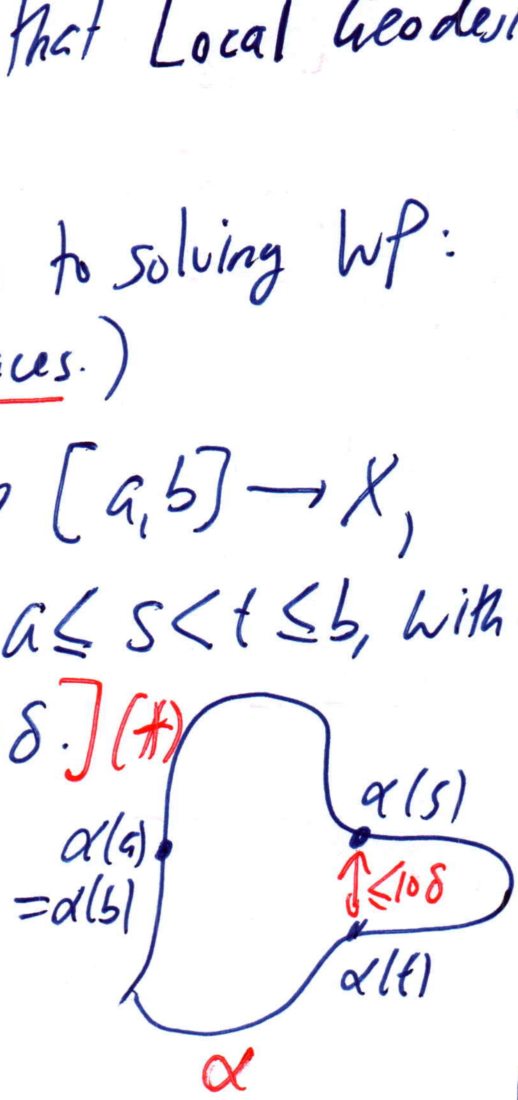
Remark Coarse analogue of fact that Local Geodesics in Trees are Global geodesics.

The next Consequence of this is key to solving WP:

Lemma (Shortcuts in Hyperbolic Spaces.)

Let: X δ-hyperbolic & α loop $[a,b] \to X$, with: $\ell(\alpha) \geq 4\delta$. [Then, α contains $a \leq s < t \leq b$, with:

$$d(\alpha(s), \alpha(t)) < \ell(\alpha|_{[s,t]}) \leq 10\delta.] \ (*)$$



($\Rightarrow \alpha$ not Geodesic!)

Proof) Unless (*) is satisfied: α is (10δ)-local Geodesic

$\Rightarrow$ By Previous Lemma: $Im(\alpha) \subset N_{2\delta}([\alpha(a), \alpha(b)]) = B_{2\delta}$.

So, (*) fails $\Rightarrow$ $Im(\alpha)$

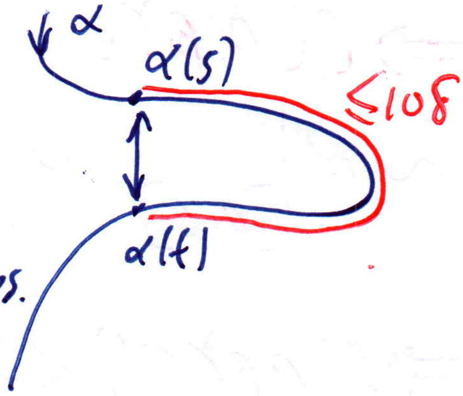
Since α is (10δ)-local Geodesic & $diam(B_{2\delta}) \leq 4\delta$,

follows: $l(\alpha) \leq 4\delta$ ✓

Geometric Group Theory: Lecture 23 | 11/03/2024

From last time:

<u>Lemma</u> X Hyperbolic. Any loop $\alpha : [a,b] \to X$ s.t. $\ell(\alpha) > 4\delta$ contains $[s,t] \subset [a,b]$ with:
$$d(\alpha(s), \alpha(t)) < \ell(\alpha|_{[s,t]}) \leq 10\delta.$$



§5: Dehn's Algorithm.

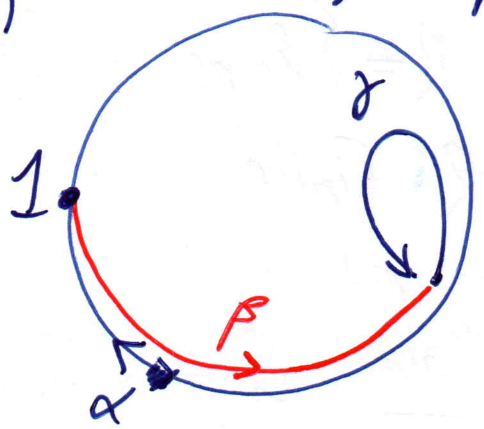
<u>Will</u>: solve WP in ALL hyperbolic groups.

Use: Algorithm exhibited by Dehn.

<u>Theorem</u> [Relations in Hyperbolic Groups.]

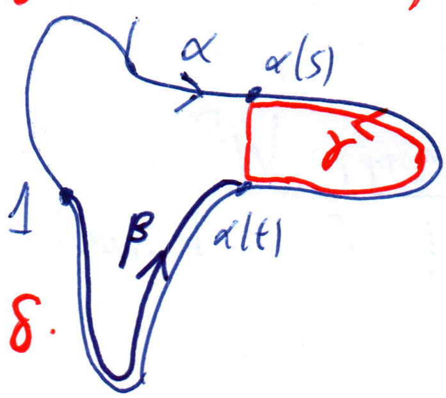
<u>Let</u>: Γ Hyperbolic group. & $S \subset \Gamma$ finite gen set.

For any nontrivial edge loop $\alpha \subset Cay_S(\Gamma)$, there is: edge loop γ of length $< 20\delta$, such that:
$$\ell(\alpha\beta\gamma\beta^{-1}) < \ell(\alpha)$$
for <u>some</u> choice of path β from 1 to some point on γ.



<u>Proof</u> If $\ell(\alpha) \leq 20\delta$ then take $\gamma = \alpha^{-1}$. Then,
$\alpha\gamma \sim * \Rightarrow \ell(\alpha\gamma) = 0 < \ell(\alpha)$.

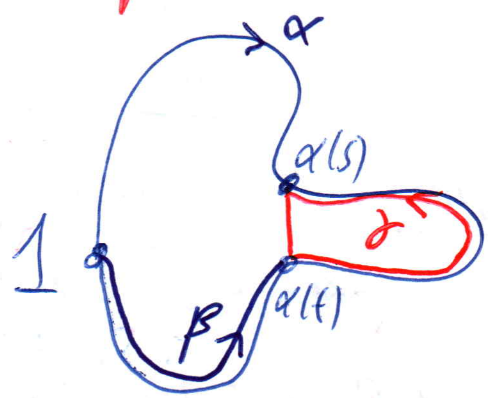
<u>Otherwise</u>: by <u>Lemma</u>: find s,t with $d(\alpha(s), \alpha(t)) < \ell(\alpha|_{[s,t]}) \leq 10\delta$.



So, define β, γ as in diagram (previous page)

$$\gamma = \left(\alpha|_{[s,t]}\right)^{-1} \cup [\alpha(s), \alpha(t)]$$

(Any geodesic)



$$\beta = \left(\alpha|_{[t,b]}\right)^{-1}$$

$\Rightarrow \ell(\alpha\beta\gamma\beta^{-1})$

$\sim \alpha|_{[a,s]} \cup \underbrace{[\alpha(s), \alpha(t)]}_{\ell < \ell(\alpha|_{[s,t]})} \cup \alpha|_{[t,b]}$

$\Rightarrow \ell(\alpha\beta\gamma\beta^{-1}) < \ell(\alpha)$ ✓

Corollary Hyperbolic groups are finitely Presented.

Proof Let: S fin-gen set, for hyperbolic group G.

Consider: $\mathrm{Cay}_S(G)$. Is: δ-hyperbolic (some δ)

Let: $R = \{$loops in $\mathrm{Cay}_S(G)$, Based at 1 & length $< 20\delta\}$

$\Rightarrow R$ finite. (Since: All such paths are contained in finite Ball, & all degrees finite degree)

Claim: $G \cong \langle S, | R \rangle$.

Indeed: By Theorem: & induction on length, any relation (loop) is Product of Conjugates, of elements of R. ✓

Corollary WP is solvable in Hyperbolic groups.

i.e. If G is Hyperbolic group, then the WP in G is solvable.

Proof Consider presentation $\langle S | R \rangle$, constructed in the previous Corollary.

& Let: $w \in F(S)$.

By Theorem: if $w \underset{G}{=} 1$ then $\exists$ Cyclic conjugate $w' \sim w$ & $r \in R$, with: $\ell(w' \cdot r) < \ell(w)$.

[Theorem says $\ell(\alpha \cdot \beta\gamma\beta^{-1})$, but you can "move" the β^{-1} to left, which amounts to change of base pt of loop]

Let: $\alpha = w$ in theorem, & $w' = \beta^{-1}\alpha\beta$, $r = \gamma$.

$\Rightarrow$ finitely many possibilities (w', r) to check.

If one found: replace $w \mapsto w' \cdot r$ & Repeat.

If not found: $w \underset{G}{\neq} 1$ $\Rightarrow$ Algorithm terminates with this information ✓

Because $\ell(w'r) < \ell(w)$: this can only happen finitely many times. So, if reached 1, then follows $w \underset{G}{=} 1$. ✓

Remark) A presentation (as in Corollary) is a Dehn presentation: $\langle S | R \rangle$, s.t. $\forall w \underset{G}{=} 1$, $\exists h \in G$ and $r^{\pm} \in R$, s.t. $\ell(w h r h^{-1}) < \ell(w)$.

Turns out: a group G has Dehn Presentation iff G is Hyperbolic!

Geometric Group Theory: Lecture 24* | 13/03/2024

§6: Outlook & Further Topics and Open Problems.

1) Random Groups.

Fix: generating set $S=\{a_1,\dots,a_m\}$, fix $n\geq 1$, and **choose** $\{r_1,\dots,r_n\}\subset F(S)$ uniformly at Random, with $\ell(r_i)=\ell$ (ℓ fixed).

Consider: $G=\langle a_1,\dots,a_m \mid r_1,\dots,r_n\rangle$. "Random Group"

For any property P of groups, can ask: $\mathbb{P}(G\in P)$.

Say random group **has** property P, if $\mathbb{P}(G\in P)\longrightarrow 1$ as $\ell\to\infty$.

Theorem [Gromov & Olshanski].
A random group is Infinite & Hyperbolic, for $m\geq 2$, $n\geq 1$.

2. Subgroups. Two open problems in area.

Conjecture [Surface Subgroup Conjecture]
Unless G is "virtually free", if G is word-hyperbolic then $\exists$ Closed surface Σ_g ($g\geq 2$) with $\pi_1(\Sigma_g)\subset G$.

Proved in case $G=\pi_1(M^3)$ for M Compact manifold (by Kahn & Markovic, 2009)

3. Representations & Residual finiteness.

A group G is *linear* if $G \leq GL_n(\mathbb{C})$, some n.
(i.e. Has Faithful $\mathbb{C}$-rep.)

Theorem [M. Kapovich]
There is a Hyperbolic group that is *not* linear.

But: consider a weaker notion of this.

DEF G *Residually finite* if: $\forall g \in G - e$, $\exists f : G \to Q$ (Q finite group) s.t. $f(g) \neq 1$.

Question: Is every Hyperbolic group Residually finite?

Recent progress includes:

Theorem [Olliner – Wise] Random groups are Residually finite.
(*In fact*: are linear!)

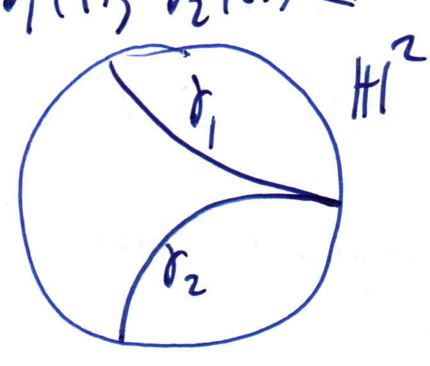
4. Boundaries.

Recall: $\partial \mathbb{H}^2 \cong S^1$.

DEF Let X Proper (i.e. closed balls are compact) Hyperbolic MS. A *geodesic ray* is Isometric embedding $\gamma : [0, \infty) \to X$.

Write: $\gamma_1 \sim \gamma_2 \iff \exists C > 0$ s.t. $d(\gamma_1(t), \gamma_2(t)) \leq C \ \forall t$.

The Gromov Boundary $\partial_\infty X$ is:
(*Set* of Geodesic rays of X)$/\sim$.



Rules 1) $\partial_\infty X$ admits natural metrisable topology so that $\partial_\infty X$ & $X \cup \partial_\infty X$ are compact.

2) A q.i. $f: X \to Y$ induces: homeo $\partial_\infty X \to \partial_\infty Y$.

$\Rightarrow$ If G Hyperbolic group, define $\partial_\infty G \equiv \partial_\infty \mathrm{Cay}_S(G)$.

Example) If G ~~comp~~ co-compact & Fuchsian (e.g. Σ_g & triangle groups), then $\partial_\infty G \cong \partial_\infty \mathbb{H}^2 = S^1$.

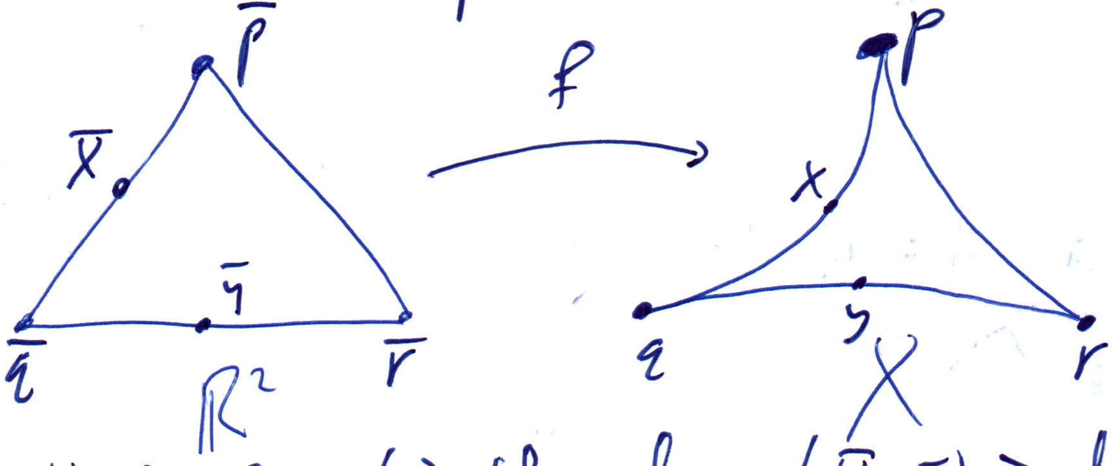
Theorem)

If G hyperbolic & $\partial_\infty G \cong S^1$, then G is virtually Fuchsian.

Cannon Conjecture) If G hyperbolic & $\partial_\infty G \cong S^2$, does it follow G is virtually $\pi_1(M^3)$?

5. Non-positive Curvature.

Def) X Geodesic MS. Each geodesic $\Delta \subset X$ has: a well-defined Companion triangle $\bar{\Delta} \subset \mathbb{R}^2$.



Say X is CAT(0) if: $d_{eucl}(\bar{x}, \bar{y}) \geq d_X(x, y)$.

<u>Question</u>) Does every Hyperbolic group act geometrically in a CAT(0) space?