Modular Form: Lecture 1. 06/10/2023.

DEF | $\mathbb{H} \equiv \{\tau \in \mathbb{C} : Im(\tau) > 0\}$.

$SL_2(\mathbb{Z}) \subseteq GL_2(\mathbb{Z})^+ \equiv \{g \in GL_2(\mathbb{Z}) : \det(g) > 0\}$.

$= \Gamma(1)$.

Lemma 1.2 | $GL_2(\mathbb{R})^+$ acts transitively on H; by Möbius transformations.

Proof let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})^+$ & $\tau \in H$.

$$Im(g\tau) = \frac{1}{2i}\left(\frac{a\tau+b}{c\tau+d} - \frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right) = \frac{1}{2i}\left(\frac{bc(\bar{\tau}-\tau)+ad(\tau-\bar{\tau})}{|c\tau+d|^2}\right)$$

$$= \det(g) \cdot \frac{Im(\tau)}{|c\tau+d|^2} > 0. \text{ So } g\tau \in H.$$

Transitivity: Know $\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} i = x+iy$ ✓

DEF 1.3 | $g \in GL_2(\mathbb{R})^+$ & $\tau \in H$. $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then: $j(g,\tau) \equiv c\tau + d$ ("modular cocycle")

If $k \in \mathbb{Z}$ & $f : H \longrightarrow \mathbb{C}$, then: $f|_k[g] : H \longrightarrow \mathbb{C}$:

$\tau \longmapsto (\det g)^{k-1} f(g\tau) j(g,\tau)^{-k}$.

Called: "weight k action of g on f".

Lemma 1.4 | Is a Right action of $GL_2(\mathbb{R})^+$. i.e.

$f|_k[gh] = (f|_k[g])|_k[h]$.

Proof $(f|_k[g])|_k[h](\tau) = \det(h)^{k-1} f|_k[g](h\tau) j(h,\tau)^{-k}$

$= \det(h)^{k-1} \det(g)^{k-1} f(gh\tau) j(g,h\tau)^{-k} j(h,\tau)^{-k}$.

Suffices to show: $j(gh,\tau) = j(g,h\tau)j(h,\tau)$.

<u>Have</u>: $g\begin{pmatrix}\tau\\1\end{pmatrix} = \begin{pmatrix}a\tau+b\\c\tau+d\end{pmatrix} = j(g,\tau)\begin{pmatrix}g\tau\\1\end{pmatrix}$.

$\circledast$ $j(gh,\tau)\begin{pmatrix}gh\tau\\1\end{pmatrix} = gh\begin{pmatrix}\tau\\1\end{pmatrix} = g\left(j(h,\tau)\begin{pmatrix}h\tau\\1\end{pmatrix}\right)$

$= j(h,\tau)j(g,h\tau)\begin{pmatrix}gh\tau\\1\end{pmatrix}$. ✓

<u>Formulae</u>: $\mathrm{Im}(g\tau) = \det(g)\dfrac{\mathrm{Im}(\tau)}{|j(g,\tau)|^2}$

& $j(g,\tau)\begin{pmatrix}g\tau\\1\end{pmatrix} = g\begin{pmatrix}\tau\\1\end{pmatrix}$.

<u>Have</u>: $H \curvearrowleft \underset{\underset{SL_2(\mathbb{Z})}{\vee\!/}}{GL_2(\mathbb{R})^+}$, and $\forall k\in\mathbb{Z}$, $\underset{\underset{SL_2(\mathbb{Z})}{\vee\!/}}{GL_2(\mathbb{R})^+} \curvearrowright \{f: H \to \mathbb{C}\}$.

<u>DEF 1.5</u>) let $k\in\mathbb{Z}$. $\Gamma \leq \Gamma(1)$. (finite index subgroup)

A <u>weakly modular function</u>, weight k, level Γ, is a meromorphic $f: H \to \mathbb{C}$, invariant under weight k action of Γ: $(\forall \tau\in H, \gamma\in\Gamma)$ $f|_k[\gamma] = f$.

Modular form: weakly modular function which is holomorphic both in H & "at ∞".

<u>Fact</u>: Modular forms of fixed weight & level, live in f.d. $\mathbb{C}$-VS $M_k(\Gamma)$. Main objects of study.

Why study Modular Forms?

<u>1)</u> Related to theory of Elliptic Functions.

<u>Let</u>: $E/\mathbb{C}$ elliptic curve & ω holomorphic 1-form.

Then: $\exists!\ \Lambda \subseteq \mathbb{C}$ lattice & isomorphism $\varphi: \mathbb{C}/\Lambda \to E$
s.t. $\phi^*(\omega) = dz$.
& E given by $Y^2 = 4X^3 - 60 \cdot G_4(\Lambda)X - 140\, G_6(\Lambda)$
where: if $k \in \mathbb{Z}$: $G_k(\Lambda) = \sum_{\lambda \in \Lambda \setminus 0} \lambda^{-k}$ (conv. absolutely if $k > 2$).

If $\tau \in H$: $\Lambda_\tau = \mathbb{Z}\tau \oplus \mathbb{Z} \subseteq \mathbb{C}$ is a lattice, and:
$G_k(\tau) = G_k(\Lambda_\tau)$ is modular form, weight k & level $\Gamma(1)$.
("Eisenstein series.").
& $H / SL_2(\mathbb{Z})$ can be identified with: set of isomorphism classes of elliptic curves over $\mathbb{C}$.

2) Modular forms f have Fourier expansions: $\sum_{n \in \mathbb{Z}} a_n \cdot q^n$ ($a_n \in \mathbb{C}$). Often same as generating functions for "arithmetically interesting" series.
E.g. $\theta(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau}$.
If $k \in 2\mathbb{N}$: then θ^k is modular form with q-expansion
$\theta^k = \sum_{n \in \mathbb{Z}} r_k(n) e^{\pi i n \tau}$ & $r_k(n) \equiv \#\{\underline{x} \in \mathbb{Z}^k : \sum x_i^2 = n\}$.

By expressing θ^k in terms of other modular functions, can prove stuff like: $r_4(n) = 8 \sum_{\substack{d \mid n \\ 4 \nmid n}} d$.

3) $\zeta(s)$: Riemann Zeta function.

- Euler Product $\zeta(s) = \prod_p (1-p^{-s})^{-1}$
- Meromorphic continuation to $\mathbb{C}$.
- Has Functional Equation relating $\zeta(s)$, $\zeta(1-s)$.

A Dirichlet series $\sum a_n \cdot n^{-s}$ is L-function if it has similar ~~series~~ properties.

Modular Forms can be used to construct interesting L-funcs. (Hecke operators & Hecke Eigenforms.)

4) Langlands programme. Predicts relations between modular forms & algebraic geometry.

E.g. Modularity Conjecture.

$$\{\text{Elliptic curves } E/\mathbb{Q}\}/_{\text{isogeny}} \longleftrightarrow \left\{\begin{matrix}\text{Hecke eigenforms} \\ \text{weight } 2\end{matrix}\right\}.$$

Bijection formulated in language of Hecke operators and L-functions.

Modular forms. Lecture 2 09/10/2023.

From last time: $f|_k[\gamma](\tau) \equiv f(\gamma\tau)\, j(\gamma,\tau)^{-k}$.

§2: Modular forms on $\Gamma(1)$.

Reminder: A meromorphic function on an open set $U \subseteq \mathbb{C}$ is a closed $A \subseteq U$ & $f : U \setminus A \to \mathbb{C}$ s.t. $(\forall a \in A)(\exists \delta > 0)\ D^*(a,\delta) \subset U \setminus A$. & $(\exists n)\ (z-a)^n f(z)$ extends to a holomorphic fn in $D(a,\delta)$.

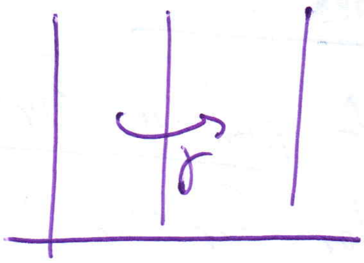
f has Laurent expansion: $\sum_{m \in \mathbb{Z}} a_m (z-a)^m$ in $D^*(a,\delta)$

Lemma 2.1 f weakly modular, weight k, level $\Gamma(1)$.
Then: $\exists! \tilde{f}$ meromorphic in $D^*(0,1)$ s.t. $f(\tau) = \tilde{f}(e^{2\pi i \tau})$.

Proof f meromorphic in H. & $\gamma \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma(1)$.

$$\Rightarrow f|_k[\gamma](\tau) = f(\gamma\tau) = f(\tau)$$
$$= f(\tau+1)$$

Define $\tilde{f}$ locally: let $a, \delta > 0$ s.t. $D(a,\delta) \subseteq D^*(0,1)$



$\to D^*(0,1)$

& $\tilde{f} : D(a,\delta) \to \mathbb{C}$

$$\tilde{f}(q) \equiv f\left(\frac{1}{2\pi i} \operatorname{Log} q\right)$$ for any branch of log defined on $D(a,\delta)$.

Clearly: indep. of choice of log. (since f periodic, period 1)
& defines $\tilde{f}$ uniquely on $D^*(0,1)$.
Know: $\tilde{f}$ unique, since $\tau \mapsto e^{2\pi i \tau}$ is surjective. ✓

If $\tilde{f}$ extends to meromorphic fn. on <u>whole</u> $D(0,1)$, then $\exists \delta > 0$, s.t. $\tilde{f}$ has Laurent expansion $\tilde{f}(q) = \sum_{n \in \mathbb{Z}} a_n q^n$ valid in $D^*(0, \delta)$.

$\Rightarrow$ In $\{\tau \in H : \mathrm{Im}(\tau) > \frac{\log \delta}{2\pi}\}$: have $f(\tau) = \sum_{n \in \mathbb{Z}} a_n q^n$ for $q = e^{2i\pi\tau}$.

This is called: <u>q-expansion</u> of weakly modular f.

<u>DEF 2.2</u> f weakly modular, weight k level $\Gamma(1)$.

Say f <u>meromorphic at ∞</u> if $\tilde{f}$ extends to meromorphic func. in $D(0,1)$.

Say f <u>holomorphic at ∞</u> if $\tilde{f}$ meromorphic at ∞, <u>and</u> has removable singularity at $q=0$.

In this case: define $\boxed{f(\infty) \equiv \tilde{f}(0) \equiv \lim_{\mathrm{Im}(\tau) \to \infty} f(\tau).}$

Say f <u>vanishes at ∞</u> if f holo. at ∞ & $f(\infty) = 0$

<u>DEF 2.3</u> A <u>modular function</u> (weight k level $\Gamma(1)$) is: weakly modular function, <u>meromorphic at ∞</u>.

A <u>modular form</u> is weakly modular function, holomorphic in H & holomorphic at ∞.

A <u>cuspidal modular form</u> is modular form, vanishing at ∞.

<u>Remark</u> $M_k(\Gamma(1)) \equiv$ set of modular forms, weight k, level $\Gamma(1)$. & $S_k(\Gamma(1)) \equiv$ same but cuspidal.

These are <u>vector spaces</u>.
If k odd: both are zero!
[Why: for $\gamma \equiv \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma(1)$, have:
$f(\tau) = f|_k[\gamma](\tau) = f(\tau)(-1)^k$. So, $f(\tau) = 0\ \forall \tau$.]

$\Rightarrow$ Consider even weights only.
If k <u>even</u>: let $G_k(\tau) \equiv \sum_{\lambda \in \Lambda_\tau - 0} \lambda^{-k} = \sum_{(m,n) \in \mathbb{Z}^2/(0,0)} (m\tau+n)^{-k}$;
where $\Lambda_\tau \equiv \mathbb{Z} \oplus \tau\mathbb{Z}$.
If $\gamma \in \Gamma(1)$: <u>formally</u> have: $G_k|_k[\gamma](\tau)$
$= G_k(\gamma\tau) j(\gamma,\tau)^{-k} = \sum_{\lambda \in \Lambda_{\gamma\tau} \setminus 0} \lambda^{-k} j(\gamma,\tau)^{-k}$.

[<u>But</u>: $\Lambda_{\gamma\tau} = \mathbb{Z}\frac{a\tau+b}{c\tau+d} \oplus \mathbb{Z}$ $\left(\gamma \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$
$= (c\tau+d)^{-1}\left(\mathbb{Z}(a\tau+b) \oplus \mathbb{Z}(c\tau+d)\right) = (c\tau+d)^{-1}\Lambda_\tau$.]

$\Rightarrow G_k|_k[\gamma](\tau) = \sum_{\lambda \in (c\tau+d)^{-1}\Lambda_\tau} \lambda^{-k}(c\tau+d)^{-k}$

$= \sum_{\lambda \in \Lambda_\tau - 0} \left((c\tau+d)^{-1}\lambda\right)^{-k}(c\tau+d)^{-k} = G_k(\tau)$.

Is justified ~~for~~ <u>when</u> series converges absolutely.

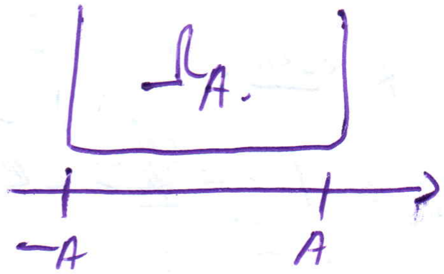
<u>Prop 2.4</u>] Let $k > 2$ even integer. Then: $G_k(\tau)$ defines a modular form, weight k & level $\Gamma(1)$.
& $G_k(\infty) = 2\zeta(k)$ (zeta function).

G_k is: weight k Eisenstein series.
(Later: $M_2(\Gamma(1)) = 0$.)

<u>Proof</u> Show: absolute & locally uniform convergence. (in H)
(shows that G_k is holomorphic.)

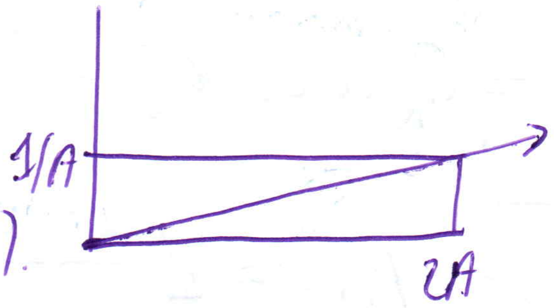
Let: $A \geq 2$ & $\Omega_A \equiv \{\tau \in H : \operatorname{Im}(\tau) \geq \frac{1}{A}$ & $\operatorname{Re}\tau \in (-A, A)\}$.

Will show: uniform conv. in Ω_A.



If $\tau \in \Omega_A$; $x \in \mathbb{R}$:

$$|\tau + x| \geq \begin{cases} 1/A & \text{if } |x| \leq 2A \\ |x|/2 & \text{if } |x| \geq 2A. \end{cases}$$



$$\Rightarrow |\tau + x| \geq \max\left(\frac{1}{A}, \frac{|x|}{2A^2}\right)$$

$$\geq \frac{1}{2A^2} \sup(1, |x|) \quad (A \text{ big}).$$

If $m, n \in \mathbb{Z}$: $|m\tau + n| = |m|\,|\tau + \frac{n}{m}| \geq \frac{1}{2A^2} \max(1, |\frac{n}{m}|)\,|m|$
$m \neq 0$.
$$= \frac{1}{2A^2} \max(|m|, |n|).$$

(also valid when $m = 0$.)

$$\Rightarrow \text{If } \tau \in \Omega_A: \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} |m\tau + n|^{-k} \geq \left(\frac{1}{2A^2}\right)^{-k} \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \max(|m|, |n|)^{-k}.$$

$$= (2A^2)^k \sum_{d \in \mathbb{Z}^+} d^{-k} \, \#\{(m,n) \in \mathbb{Z}^2 : \max(|m|, |n|) = d\}.$$

$$= (2A^2)^k \sum_d d^{-k} \cdot 8d = 8(2A^2)^k \sum_k d^{1-k} \text{ converges } \forall k > 2.$$

This shows <u>uniform convergence</u> on Ω_A (M-test)
Hence: G_k holomorphic & invariant under weight k action.

Remains to show: G_k holomorphic at ∞ & $G_k(\infty) = 2\zeta(k)$.

Need to show: $\lim_{\operatorname{Im}\tau \to \infty} G_k(\tau) = 2\zeta(k)$.

$$\lim_{\operatorname{Im}\tau \to \infty} G_k(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \lim_{\operatorname{Im}\tau\to\infty} (m\tau+n)^{-k}.$$

$$= \sum_{n\in\mathbb{Z}\setminus 0} n^{-k} = 2\zeta(k). \quad (k \text{ even}).$$

Modular Forms Lecture 3. 11/10/2023.

__Last time__: defined, $f: \mathbb{H} \to \mathbb{C}$ modular form (weight k, level $\Gamma(1)$) & $M_k(\Gamma(1)) \equiv$ such forms ($\mathbb{C}$-VS).

If f modular form $\Rightarrow \exists\, \tilde{f}: \mathbb{D} \to \mathbb{C}$ with $\forall \tau \in \mathbb{H}$:

$f(\tau) = \tilde{f}(e^{2i\pi\tau})$.

$\uparrow$ q-disc.

__Know__: $\tilde{f}$ has a Taylor expansion, which gives q-expansion of f:

$$\tilde{f}(q) = \sum_{n \geq 0} a_n \cdot q^n. \quad (q \equiv e^{2i\pi\tau}). \quad (f(\infty) \equiv \tilde{f}(0) = a_0.)$$

__Showed__: $\forall k > 2$ __even__ int, $G_k(\tau) \equiv \sum_{\lambda \in \Lambda_\tau \setminus 0} \lambda^{-k}$ converges absolutely ($\Lambda_\tau \equiv \mathbb{Z} \oplus \mathbb{Z}\tau$) & defines a modular form in $M_k(\Gamma(1))$ with $G_k(\infty) = 2\zeta(k)$.

__Define__: $E_k \equiv \dfrac{G_k(\tau)}{2\zeta(k)} = 1 + \sum_{n \geq 1} a_n q^n$.

__Will show__: $a_n \in \mathbb{Q}$ $\forall n \geq 1$. (later.)

check. holo. at ∞.

For $f \in M_k(\Gamma(1))$ & $g \in M_\ell(\Gamma(1))$: then $fg \in M_{k+\ell}(\Gamma(1))$.

$\Rightarrow$ Have: $E_4^3, E_6^2 \in M_{12}(\Gamma(1))$ & $E_4^3 - E_6^2 \in S_{12}(\Gamma(1))$.

This is: Ramanujan's Δ-function.

__Next__: want $M_k(\Gamma(1))$ finite-dimensional.

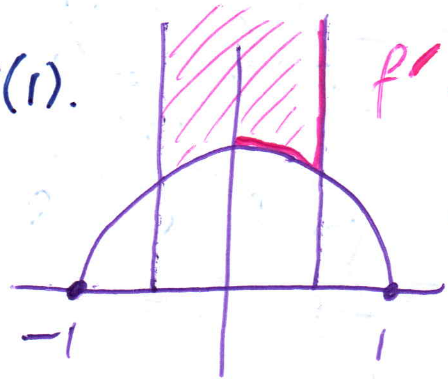
First study: $\Gamma(1) \backslash \mathbb{H}$.

Introduce: fundamental set $f' \subseteq H$ for $\Gamma(1)$-action: a subset intersecting each $\Gamma(1)$-orbit in exactly 1 elem ~~orbit~~

$$f \equiv \{\tau \in H : Re(\tau) \in (-\tfrac{1}{2}, \tfrac{1}{2}] \ \& \ |\tau| \geq 1\}$$

$$f' \equiv \{\tau \in H : Re(\tau) \in (-\tfrac{1}{2}, \tfrac{1}{2}], \ [|\tau| = 1 \Rightarrow Re(\tau) \in (-\tfrac{1}{2}, 0]]\}.$$

$T \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \Gamma(1)$.



$T(\tau) = \tau + 1$. & $S(\tau) = -1/\tau$.

Observation: any element of f is conjugate (under S or T^{-1}) to an element of f'.

Prop 2.5) let $G \equiv \Gamma(1)/\{\pm I\}$.

1) $\forall \tau \in H$: τ is $\Gamma(1)$-conj. to an element of f'

2) If τ, τ' are $\Gamma(1)$-conj then $\tau = \tau'$

3) If $\tau \in f'$ then $stab_G(\tau)$ is trivial except: $stab(i) = \langle S \rangle$ & $stab(\rho) = \langle ST \rangle$ $\rho \equiv e^{2i\pi/3}$.

4) $\Gamma(1)$ generated by S, T.

Proof. Let $K \leq G$, $K = \langle S, T \rangle$.

Claim: Every $\tau \in H$ is K-conj. to f'.

By observation: (since $S, T \in K$): suffices to show:

any $\tau \in H$ is K-conj. to f.

Let: $\tau \in H$ & recall if $\gamma \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ then

$$Im(\gamma\tau) = \frac{Im(\tau)}{|c\tau+d|^2}.$$

$\Rightarrow \forall R>0$: $H\tau \cap \{Im\,\tau > R\}$ finite

(because: $Im(\gamma\tau) > R \iff |c\tau+d|^2 < \frac{Im\,\tau}{R}$. But, $\Lambda\tau \equiv \mathbb{Z}\tau + \mathbb{Z}$ lattice $\Rightarrow \{(c,d) \in \mathbb{Z}^2 : |c\tau+d| < R'\}$ finite $\forall R' > 0$.]

$\Rightarrow \exists h \in K$: $Im(h\tau) \geq Im(h'\tau)$ $\forall h' \in K$.

Replace $\tau \mapsto h\tau$: assume, $Im(\tau) \geq Im(h\tau)$ $\forall h \in K$.

Acting by T does not change $Im(\tau)$ $\Rightarrow$ Can assume $Re(\tau) \in [-\frac{1}{2}, \frac{1}{2}]$

& $Im(\tau) \geq Im(S\tau) = \frac{Im(\tau)}{|\tau|^2} \Rightarrow |\tau| \geq 1$ ✓

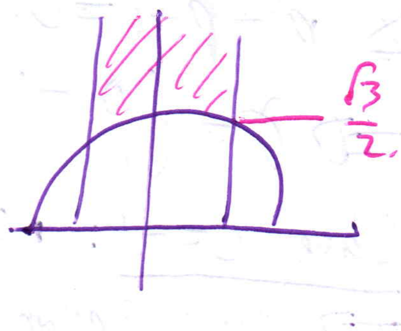
Take: $\tau, \tau' \in f'$ & $\gamma\tau = \tau'$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$.

Want: $\gamma = \pm I$, or: $\tau = i$ or ρ.

WLOG: $Im(\tau') = Im(\gamma\tau) \geq Im(\tau) \Rightarrow \frac{Im\,\tau}{|c\tau+d|^2} \geq Im\,\tau$.

$\Rightarrow |c\tau+d| \leq 1$.

& if $\tau \in f'$ $\Rightarrow Im(\tau) \geq \frac{\sqrt{3}}{2}$



$\Rightarrow |c\tau+d| \geq c\, Im(\tau) \geq c\frac{\sqrt{3}}{2}$

$\Rightarrow |c| \leq 2/\sqrt{3}$, so $|c| = 0, 1 \Rightarrow c = 0, \pm 1$.

If $c = 0 \Rightarrow \gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, $\gamma = \pm T^m \Rightarrow m = 0$, $\gamma = \pm I$,
so $\tau = \pm \tau'$.

If $c = 1 \Rightarrow \gamma = \begin{pmatrix} a & b \\ 1 & d \end{pmatrix}$ & $|\tau + d| \leq 1$.

Only circle centered at integers of radius 1 which intersect $\mathcal{F}'$ are $d = 0, -1$.

$\Rightarrow d = 0$, $|\tau| = 1$ or $d = 1$, $\tau = \rho$.

<u>Case $d=0$, $|\tau|=1$</u>: $\gamma = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow \gamma\tau = a - \frac{1}{\tau} = a - \bar{\tau}$.

$\Rightarrow \mathrm{Re}(\gamma\tau) = a - \mathrm{Re}(\tau) \in \mathrm{Re}(\mathcal{F}' \cap \{|\tau| = 1\}) = [-\frac{1}{2}, 0]$.

& $\mathrm{Re}(\gamma\tau) \in a - [-\frac{1}{2}, 0] = a + [0, \frac{1}{2}]$.

Intersection of $[-\frac{1}{2}, 0]$ & $a + [0, \frac{1}{2}]$ is nonempty $\Leftrightarrow$
$a = 0$ or $a = -1$.

If $a = 0$: $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -S$ ✓

If $a = -1$: $\gamma = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ & $ST = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$.

$\Rightarrow (ST)^2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$.

$(ST)^3 = I$.

<u>Case $d=1$, $\tau=\rho$</u>: $\gamma = \begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix} \Rightarrow \gamma\rho = \dfrac{a\rho + b}{\rho + 1} = \rho$

& $\rho^2 + \rho + 1 = 0$. $\Rightarrow a\rho + b = \rho^2 + \rho = -1$. $\Rightarrow a = 0, b = -1$.

$\Rightarrow \gamma = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = -(ST)$.

<u>Case $c = -1$</u>: reduce to $c = 1$ by replacing $\gamma \mapsto -\gamma$.

$\Rightarrow$ Have shown first 3 parts of prop.

Remains to show: $\Gamma(1) = \langle S, T \rangle$.

Since $S^2 = -I$: need, $H = G$.

For $\tau \in \operatorname{Int}(f)$: $\operatorname{stab}_G(\tau) = \{I\}$.

For $g \in G$: by claim 1, $\exists h \in H$, $hg\tau \in f'$.

$\Rightarrow$ Have: $hg\tau = \tau \Rightarrow hg \in \operatorname{stab}_G(\tau) = \{I\}$

$\Rightarrow g = h^{-1} \in H$ ✓

Modular Forms: Lecture 4. 13/10/2023.

Last time: introduced $\mathcal{F} \equiv \{\tau \in \mathbb{H} : \mathrm{Re}(\tau) \in [-\frac{1}{2}, \frac{1}{2}], |\tau| \geq 1\}$

& $\mathcal{F}' \equiv \{\tau \in \mathcal{F} : \mathrm{Re}(\tau) \in [-\frac{1}{2}, \frac{1}{2})$ & if $|\tau| = 1$ then $\mathrm{Re}(\tau) \in [-\frac{1}{2}, 0]\}$.

Proved: $\forall \tau \in \mathbb{H}$, is $\Gamma(1)$ - conjugate to a unique element of $\mathcal{F}'$.

Moreover: $G \equiv \Gamma(1)/\{\pm 1\} \Rightarrow \mathrm{stab}_G(\tau) = \{1\}$ except if

τ conj. to $\begin{cases} i, & \mathrm{stab}_G(\tau) \text{ cyclic order } 2 \\ \rho, & \mathrm{stab}_G(\tau) \text{ cyclic order } 3. \end{cases}$

Notation $\boxed{e_\tau = \# \mathrm{stab}_G(\tau).}$

Let: $f \not\equiv 0$ modular func (weight k, level $\Gamma(1)$)

If $\tau \in \mathbb{H}$: write $\boxed{v_\tau(f) \equiv \text{order of } f \text{ at } \tau.}$

($\Leftrightarrow$ unique $n \in \mathbb{Z}$ s.t. $f(z) = (z-\tau)^n g(z)$ for g holomorphic at τ).

& define $v_\infty(f) \equiv$ "order at ∞" $= v_0(\tilde{f})$. [$\tilde{f}$ mero. on disc $D(0,1)$ with $f(\tau) = \tilde{f}(e^{2i\pi\tau})$.]

Prop 26) $f \not\equiv 0$ modular func (weight k level $\Gamma(1)$)

Then
$$\boxed{\sum_{\tau \in \Gamma(1)\backslash\mathbb{H}} \frac{1}{e_\tau} v_\tau(f) + v_\infty(f) = \frac{k}{12}.}$$
$\Gamma(1)\backslash\mathbb{H} \leftarrow$ orbits of $\Gamma(1)$ mod $\mathbb{H}$

Proof 1) **need**: Sum well-defined.

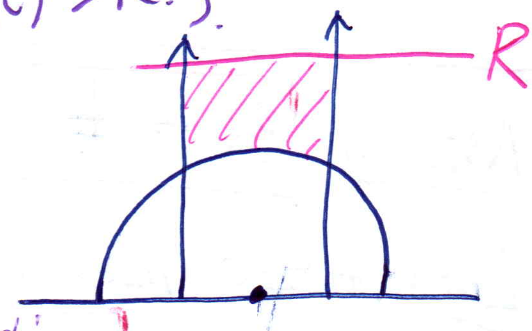
⊛ If $\tau \in \mathbb{H}$ then $e_\tau, v_\tau(f)$ depend only on $\Gamma(1)$ - orbit of τ.

If $\gamma \in \Gamma(1)$ & $\tau \in H$: $\mathrm{stab}_{\Gamma(1)}(\tau)$ & $\mathrm{stab}_{\Gamma(1)}(\gamma\tau)$ are conjugate subgroups of $\Gamma(1)$, so $e_\tau = e_{\gamma\tau}$.

& Know: $f(\gamma\tau) = f(\tau)\, j(\gamma,\tau)^k$. & $j(\gamma,\tau) \neq 0$ holomorphic on H. So, $V_{\gamma\tau}(f) = V_\tau(f)$ ✓

⊛ Need: Sum only has finite # of non-0 terms.

Since f modular $\Rightarrow \tilde{f}$ meromorphic on $D(0,1)$.
$\Rightarrow \exists \delta > 0$, $\tilde{f}$ holomorphic & non-vanishing at $D^*(0,\delta)$
$\Rightarrow \exists R > 0$, f holo & $\neq 0$ on $\{\mathrm{Im}(\tau) > R\}$.



Enough to show: f has finitely many zeros & poles in $\mathcal{F} \cap \{\mathrm{Im}(\tau) \leq R\}$.

True, since poles & zeros of a meromorphic function is discrete ✓

Identity: Contour Integration!

⊛ If $U \subseteq \mathbb{C}$ & $f: U \to \mathbb{C}$ holo. & $\gamma: [0,1] \to U$ path:

$$\int_\gamma f(z)\,dz = \int_0^1 f(\gamma(t))\,\gamma'(t)\,dt.$$

⊛ Pullback formula: If $u: U \to V$ holomorphic ($U, V \subseteq \mathbb{C}$) & $g: V \to \mathbb{C}$ holo. & γ path in U:

$$\int_{u\circ\gamma} g(z)\,dz = \int_\gamma u^*(g(z))\,dz \equiv \int_\gamma g(u(z))\,u'(z)\,dz.$$

In particular, for $g = h/h' \Rightarrow g(z)\,dz = d(\log h)$.

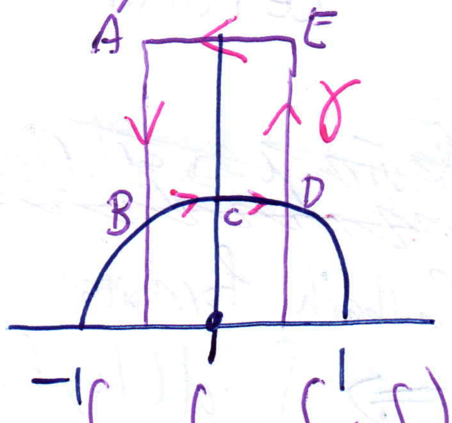
& $\int_{u\circ\gamma} d(\log h) = \int_\gamma u^*(d\log(h)) = \int_\gamma d(\log(h\circ u))$

$= \int_\gamma \frac{(h\circ u)'(z)}{(h\circ u)(z)}\,dz$.

(*) <u>Argument Principle</u>: $U \subseteq \mathbb{C}$ <u>simply connected</u> open
& $\gamma \subseteq U$ path (closed, simple, positively oriented)
<u>Then</u>: if g <u>mero</u> on U, no zeros & poles on γ, then

$$\frac{1}{2\pi i}\oint_\gamma d(\log g) \equiv \frac{1}{2\pi i}\oint_\gamma \frac{g'(z)}{g(z)}\,dz = \sum_{a\in \mathrm{Int}(\gamma)} v_a(g).$$

Apply to f: choose $R>0$ s.t. f has no zeros/poles in $\{\mathrm{Im}(\tau) \geq R\}$. & consider: $\frac{1}{2\pi i}\oint_\gamma d\log f$



On γ: no poles/zeros on $E \to A$.
First: consider case, no zeros or poles on <u>entire</u> contour.

<u>Arg principle</u> $\Rightarrow \frac{1}{2\pi i}\oint_\gamma d\log(f) = \frac{1}{2\pi i}\left(\int_{AB} + \int_{BC} + \int_{CD} + \int_{DE} + \int_{EA}\right)$

$= \sum_{\tau\in\Gamma(1)\backslash H} \frac{1}{e_\tau} v_\tau(f)$ (since: $v_\tau(f) \neq 0$ & $e_\tau = 1$ by assumptions).

Take: $v(\tau) \equiv S(\tau) \equiv -1/\tau$, sending $V(BC) = DC$

$\underline{\text{Know}}$: $f(\tau) = f|_k[S](\tau) = f(-\frac{1}{\tau})\tau^{-k}$. $\Rightarrow f\circ v = f(\tau)\tau^{-k}$

$$\Rightarrow \int_{DC} d\log(f) = \int_{v(BC)} d\log(f) = \int_{BC} d(\log(f(\tau)\tau^{-k}))$$

$$= \int_{BC} d\log(f) + k\, d\log(\tau) = \int_{BC} d\log(f) + k(\log C - \log B)$$

($\log$ is any branch of logarithm defined on BC)

$B = \tau_1$, $C = i \Rightarrow \log B = \frac{2\pi i}{3}$ & $\log C = \frac{2\pi}{2}$.

$$\Rightarrow \int_{CD} d\log(f) = -\int_{BC} d\log f + k\left(\frac{2\pi i}{3} - \frac{2\pi i}{4}\right)$$

$$\Rightarrow \left(\int_{BC} + \int_{CD}\right)(d\log f) = 2\pi i \frac{k}{12}.$$

$$\Rightarrow \sum_{\tau \in \Gamma(1)\backslash H} \frac{1}{e_\tau} v_\tau(f) = \frac{1}{2\pi i}\Big(\cancel{\int_{AB}} + \underbrace{\int_{BC} + \int_{CD}}_{(k/12)2\pi i} + \cancel{\int_{DE}} + \underbrace{\int_{EA}}_{-v_\infty(f)}\Big) d\log(f).$$

~~Rearrange to get result~~

Pullback formula: $u(\tau) = \tau + 1 \Rightarrow u(AB) = ED$ & $f\circ u = f$.

$$\Rightarrow \int_{u(AB)} d(\log f) = \int_{AB} d\log(f\circ u) = \int_{AB} d\log(f)$$

$$\Rightarrow \int_{AB} + \int_{DE} d\log(f) = 0.$$

Then, take $q = e^{2i\pi\tau} \Rightarrow f = \hat{f}\circ q$ & $q(AE)$ pos. oriented circle around 0

$$\Rightarrow v_\infty = \frac{1}{2\pi i}\int_{q(AE)} d(\log \hat{f}) = \frac{1}{2\pi i}\int_{AE} d\log(\hat{f}\circ q) = \frac{1}{2\pi i}\int_{AE} d\log f.$$

Modular Forms: Lecture 5
16/10/2023.

From last time If $f \not\equiv 0$ modular form, weight k level $\Gamma(1)$ then $\sum_{\tau \in \Gamma(1)\backslash \mathbb{H}^2} \frac{1}{e_\tau} v_\tau(f) + v_\infty(f) = \frac{k}{12}$

Example $(k=4)$ $f = E_4 \in M_4(\Gamma(1))$.

Since f holomorphic: $v_\tau(E_4) \geq 0 \ \forall \tau \in H$.

& $E_4(\tau) = 1 + \sum_{n\geq 1} a_n q^n \Rightarrow E_4(\infty) \neq 0$ & $v_\infty(E_4) = 0$.

$\Rightarrow \sum_{\tau \in \Gamma(1)\backslash H} \frac{1}{e_\tau} v_\tau(E_4) = \frac{1}{3} v_\rho(E_4) + \frac{1}{2} v_i(E_4) + \sum_{\tau \neq i,\rho} v_\tau(E_4) = \frac{1}{3}$.

$\Rightarrow \frac{a}{3} + \frac{b}{2} + c = \frac{1}{3}$ $(a,b,c \in \mathbb{Z}_{\geq 0})$.

Only solution: $a=1$ & $b=c=0$. $\Rightarrow E_4(\rho) = 0$ & $E_4(\tau) \neq 0 \ \forall \tau \notin \Gamma(1)\rho$.

Example $(k=6)$ $f = E_6$. $\frac{1}{3} v_\rho(E_6) + \frac{1}{2} v_i(E_6) + \sum_{\tau \neq i,\rho} v_\tau(E_6) = \frac{1}{2}$.

Forces: $E_6(i) = 0$ & $E_6(\tau) \neq 0$ $\forall \tau \neq i$.

Recall: $\Delta = \frac{E_4^3 - E_6^2}{1728}$. (important normalising term!)

Is: cuspidal modular form. $\in S_{12}(\Gamma(1))$.

Can check: $\Delta \neq 0$. Since: $\Delta(\rho) = \frac{E_4(\rho)^3 - E_6(\rho)^2}{1728} = \frac{-E_6(\rho)^2}{1728} \neq 0$.

& $v_\infty(\Delta) \geq 1$ (by construction).

$\Rightarrow \sum_\tau \frac{1}{e_\tau} v_\tau(\Delta) + \underbrace{v_\infty(\Delta)}_{\geq 1} = 1$. $\Rightarrow v_\infty(\Delta) = 1$, all other terms 0. $\Delta \neq 0$ on H.

Theorem 2.7 Let $k \in 2\mathbb{Z}$.

1) If $k<0$ or $k=2$: $M_k(\Gamma(1)) = 0$.
 & $M_0(\Gamma(1)) = \mathbb{C}\cdot 1$.

2) If $4 \leq k \leq 10 \Rightarrow M_k(\Gamma(1)) = \mathbb{C}\cdot E_k$

3) Mult. by Δ gives isomorphism $M_k(\Gamma(1)) \xrightarrow{\cdot\Delta} S_{k+12}(\Gamma(1))$.

Proof 1) For $f \in M_k(\Gamma(1))$: $\sum_\tau \frac{v_\tau(f)}{e_\tau} + v_\infty(f) = \frac{k}{12}$.

If $k<0 \Rightarrow$ RHS <0 if $f \neq 0$ #.

If $k=2 \Rightarrow \frac{a}{3} + \frac{b}{2} + c = \frac{1}{6} \Rightarrow 2a+3b+6c=1$ #

If $k=0$: & If f not constant ($f \in M_0(\Gamma(1)) - \mathbb{C}1$).

$\Rightarrow f - f(\infty)1 \in S_0(\Gamma(1))$ & nonzero.

$\Rightarrow \sum \frac{1}{e_\tau} v_\tau(f - f(\infty)1) + \underbrace{v_\infty(f - f(\infty)1)}_{\geq 1} = 0$ #

2) Let: $4 \leq k \leq 10$ & $f \in M_k(\Gamma(1))$.

& consider $g = f - f(\infty)E_k \in S_k(\Gamma(1))$.

By same argument: if nonzero: $\sum_\tau \frac{1}{e_\tau} v_\tau(g) + \underbrace{v_\infty(g)}_{\geq 1} = \frac{k}{12} < 1$. #

$\Rightarrow f = f(\infty)E_k$.

3) Note: $(\times\Delta): M_k(\Gamma(1)) \longrightarrow S_{k+12}(\Gamma(1))$ well-defined & $\mathbb{C}$-linear map. Need: isomorphism.

Injective: If $\Delta \cdot f = 0 \Rightarrow f=0$ (since Δ nonvanishing on $\mathbb{H}$)

Surjective: If $f \in S_{k+12}(\Gamma(1))$: $\frac{f}{\Delta}$ holomorphic & invariant under

under weight k action of ∞.

<u>Need to check</u>: f/Δ holomorphic at ∞.

$\Leftrightarrow v_\infty(f/\Delta) \geq 0. \Leftrightarrow \underbrace{v_\infty(f)}_{\geq 1} - \underbrace{v_\infty(\Delta)}_{=1} \geq 0$ ✓

<u>Corollary 2.8)</u> If $k \in 2\mathbb{Z}$ & $k \geq 0$ then $M_k(\Gamma(1))$ is finite-dimensional & $\dim_{\mathbb{C}} M_k(\Gamma(1)) = \begin{cases} \lfloor \frac{k}{12} \rfloor + 1 & \text{if } k \not\equiv 2 \bmod 12 \\ \lfloor k/12 \rfloor & \text{else.} \end{cases}$

<u>Proof)</u> Have proved this for $0 \leq k \leq 10$.

In general: use <u>induction</u> on k. Need: $\dim_{\mathbb{C}} M_{k+12}(\Gamma(1))$
$= 1 + \dim_{\mathbb{C}} M_k(\Gamma(1))$.

<u>Know</u>: $E_{k+12} \in M_k(\Gamma(1))$.

$$\& \; M_{k+12}(\Gamma(1)) = \mathbb{C}E_{k+12} \oplus S_{k+12}(\Gamma(1))$$
$$\cong \mathbb{C}E_{k+12} \oplus \Delta M_k(\Gamma(1)).$$

$\Rightarrow \dim_{\mathbb{C}} M_{k+12}(\Gamma(1)) = 1 + \dim_{\mathbb{C}} M_k(\Gamma(1))$ ✓

<u>Example)</u> $E_4^2 \in M_8(\Gamma(1)) = \mathbb{C}E_8$. & $E_8(\infty) = E_4(\infty)^2 = 1$
$\Rightarrow E_4^2 = E_8$

& $E_4 E_6 \in M_{10}(\Gamma(1)) = \mathbb{C}E_{10}$. $\Rightarrow E_4 E_6 = E_{10}$.

<u>Corollary 2.9)</u> If $k \in 2\mathbb{N} \Rightarrow M_k(\Gamma(1))$ spanned ~~by~~ as a $\mathbb{C}$-vector space by $\{E_4^a E_6^b : a, b \in \mathbb{Z}_{\geq 0} \;\&\; 4a + 6b = k\}$.

If $M = \bigoplus_{k \in \mathbb{Z}} M_k(\Gamma(1))$ then M is a graded ~~alg~~ $\mathbb{C}$-algebra generated by E_4 & E_6.

<u>Proof</u> Again, proved for $0 \leq k \leq 10$. (by example)

for $\underline{k \geq 12}$: $M_k(\Gamma(1)) = \mathbb{C} E_k \oplus \Delta M_{k-12}(\Gamma(1))$.

$= \mathbb{C} f \oplus \Delta M_{k-12}(\Gamma(1)) \quad \forall f \in M_k(\Gamma(1))$

$f(\infty) \neq 0$.

Always find $f = E_4^A E_6^B$ (i.e. $4A + 6B = k$) ✓

By induction: $M_{k-12}(\Gamma(1)) = \langle E_4^a E_6^b : 4a + 6b = k-12 \rangle$

$\Rightarrow \Delta M_{k-12}(\Gamma(1)) \leq \langle E_4^a E_6^b : 4a + 6b = k \rangle$ $(\Delta = E_4^3 E_6^2)$.

& $E_4^A E_6^B$ in here too.

So get the result ✓

<u>Theorem 2.10</u> (Proved next time).

Define: $j(\tau) \equiv \dfrac{E_4(\tau)^3}{\Delta(\tau)}$. Then: j modular function, of weight 0 & level $\Gamma(1)$, and defines a bijection:

$$\begin{array}{ccc} \Gamma(1) \backslash H & \xrightarrow{\sim} & \mathbb{C} \\ \tau & \longmapsto & j(\tau). \end{array}$$

<u>Moreover</u>: any such modular function (weight 0 level $\Gamma(1)$) is rational func. of j.

<u>Interpretation</u>: Possible to define Riemann Surface structure on $\Gamma(1) \backslash H \sqcup \{\infty\}$, s.t. <u>compact</u> RS whose meromorphic functions are exactly: modular funcs, weight 0.

$\Rightarrow$ This Riemann surface $(X(1))$ is $\cong \mathbb{C}_\infty$.

& Isomorphism given by j function.

2.6: If $\mathcal{L}$ invertible sheaf on compact RS & $s \neq 0$ mero $\Rightarrow \sum_a v_a(s) = \deg \mathcal{L}$.

Modular Forms: Lecture 6. 18/10/2023.

From last time: $\Delta \equiv \frac{E_4^3 - E_6^2}{1728}$ & $j(\tau) \equiv \frac{E_4^3}{\Delta}$.

Theorem 2.10 j modular func, weight 0, level $\Gamma(1)$.

Is holomorphic on $\mathbb{H}$, simple pole at ∞, & induces a map

$\Gamma(1)\backslash\mathbb{H} \longrightarrow \mathbb{C}$
$\tau \longmapsto j(\tau)$.

[& Every modular func, weight 0 level $\Gamma(1)$ is in $\mathbb{C}(j)$.]

Proof Δ nonzero on $\mathbb{H}$ & $V_\infty(\Delta) = 1$. So, j holomorphic in $\mathbb{H}$

& $V_\infty(j) = 3V_\infty(E_4) - V_\infty(\Delta) = 1$.

If $\gamma \in \Gamma(1)$: $j|_0[\gamma](\tau) = j(\gamma\tau) = j(\tau) \Rightarrow$ well defined map.

To show $j: \Gamma(1)\backslash\mathbb{H} \longrightarrow \mathbb{C}$ bijective: need: $\forall z \in \mathbb{C}: \exists! \Gamma(1)\tau$

with $j(\tau) = z$, i.e. $V_\tau(j-z) > 0$.

Know: $\sum_{\tau \in \Gamma(1)\backslash\mathbb{H}} \frac{1}{e_\tau} V_\tau(j-z) = 1$ & $V_\tau(j-z) \geq 0$ (Holomorphic).

$(V_\infty(j-z) = -1)$ (since $M_0(\Gamma(1)) = \mathbb{C}\cdot 1$)

$\Rightarrow \frac{a}{3} + \frac{b}{2} + c = 1 \quad (a,b,c \in \mathbb{Z})$

$a = V_\rho(j-z)$
$b = V_i(j-z)$
$c =$ sum of rest.

$\Rightarrow (a,b,c) = (3,0,0)$
or $(0,2,0)$
or $(0,0,1)$

$\Rightarrow$ $j-z$ vanishes at ρ, nowhere else

or $j-z$ vanishes at i, nowhere else

or $\exists! \Gamma(1)\tau$ $(\tau \neq \rho, i)$: $j-z$ vanishes at $\Gamma(1)\tau$ only ✓

Consider: a nonzero modular f, weight 0.

Consider: product $f \cdot \prod_{0\le i\le n} (j(\tau)-j(a_i))^{b_i}$: $a_i \in H$, $b_i \ge 0$

Where a_i's are ~~among~~ ~~from~~ among the poles of f in h.

$\Rightarrow$ Enough to show for case when f holomorphic in H.

Then: $\exists m \ge 0$, $\Delta^m f$ holo. at ∞.

$\Rightarrow \Delta^m f$ is modular form, in $M_{12m}(\Gamma(1)) = \langle E_4^a E_6^b \rangle$ $(4a+6b=12m)$

$\Rightarrow f$ linear comb. of functions $\dfrac{E_4^a E_6^b}{\Delta^m}$.

So, enough to show for these.

$4a+6b=12m \Rightarrow 2a+3b=6m$, so $\exists p,q \ge 0$: $a=3p$, $b=2q$, $p+q=m$.

$\Rightarrow \dfrac{E_4^a E_6^b}{\Delta^m} = \left(\dfrac{E_4^3}{\Delta}\right)^p \left(\dfrac{E_6^2}{\Delta}\right)^q = j^p \,(E_6^2/\Delta)^q$.

& $E_4^3 - E_6^2 = 1728\Delta \Rightarrow \dfrac{E_6^2}{\Delta} = j - 1728$.

So, indeed, rational function in j. ✓

Prop 2.11) Let $k \ge 4$ even. Then: ~~$G_k(\tau) = 2\zeta(k)$~~

$$G_k(\tau) = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n\ge 1} \sigma_{k-1}(n) q^n.$$

Where: $q \equiv e^{2i\pi\tau}$ & $\sigma_{k-1}(n) \equiv \sum_{d|n} d^{k-1}$.

Proof Know: $\pi \cot(\pi\tau) = \dfrac{1}{\tau} + \sum_{n\ge 1}\left(\dfrac{1}{\tau+n} + \dfrac{1}{\tau-n}\right)$.

& holds, converges locally uniformly in H.

& $\pi \cot(\pi\tau) = i\pi \dfrac{e^{\pi i\tau} + e^{-\pi i\tau}}{e^{\pi i\tau} - e^{-\pi i\tau}} = \pi i \dfrac{q+1}{q-1}$

$$= -i\pi(1+q)(1-q)^{-1} = -i\pi\Big(1+2\sum_{n\geq 1} q^n\Big).$$

Differentiate $k-1$ times:

$$\text{LHS} = -2\pi i \left(\frac{d}{d\tau}\right)^{k-1}\Big(\sum_{n\geq 1} q^n\Big) = -(2\pi i)^k \sum_{n\geq 1} n^{k-1} q^n.$$

$$\text{RHS} = (-1)^{k-1}(k-1)!\,\Big(\tau^{-k} + \sum_{n\geq 1} (\tau+n)^{-k} + (\tau-n)^{-k}\Big)$$

$$= (-1)^{k-1}(k-1)! \sum_{n\in\mathbb{Z}} (\tau+n)^{-k} \quad (\text{converges, } k\geq 4)$$

$$\Rightarrow \sum_{n\in\mathbb{Z}} (\tau+n)^{-k} \; \frac{(2\pi i)^k}{(k-1)!}\sum_{n\geq 1} n^{k-1} q^n. \quad (\tau\in H).$$

$$\& \; G_k(\tau) = \sum_{(m,n)\neq(0,0)} (m\tau+n)^{-k} = 2\zeta(k) + \sum_{\substack{(m,n)\in\mathbb{Z}^2\\ m\neq 0}} (m\tau+n)^{-k}$$

$$= 2\zeta(k) + 2\sum_{m\geq 1}\sum_{n\in\mathbb{Z}} (m\tau+n)^{-k}.$$

$$\Rightarrow G_k(\tau) = 2\zeta(k) + 2\sum_{m\geq 1} \frac{(2\pi i)^k}{(k-1)!}\sum_{n\geq 1} n^{k-1} q^{mn}.$$

$$= 2\zeta(k) + 2\,\frac{(2\pi i)^k}{(k-1)}\sum_{N\geq 1}\Big(\sum_{n\mid N} n^{k-1}\Big) q^N. \checkmark$$

<u>Corollary 2.12</u> $E_k \equiv \dfrac{G_k(\tau)}{2\zeta(k)} = 1 + \sum_{n\geq 1} a_n q^n \Rightarrow a_i \in \mathbb{Q}.$

<u>Moreover</u>: for $k = 4, 6$: all terms are <u>integers</u>.

<u>Proof</u> Know: $\pi^{-k}\zeta(k) \in \mathbb{Q}$ $\forall k \geq 4$ even.

$\& \; E_k(\tau) = 1 + \dfrac{(2\pi i)^k}{\zeta(k)(k-1)!}\sum_{n\geq 1} \sigma_{k-1}(n) q^n. \Rightarrow$ All coeffs $\in \mathbb{Q}$.

Also: $\zeta(4) = \frac{\pi^4}{90} \Rightarrow 1 + \frac{2^4 \pi^4 \cdot 90}{6\pi^4} \sum_{n\geq 1} \sigma_3(n) q^n$

$$= 1 + 240 \sum_{n\geq 1} \sigma_3(n) q^n \checkmark$$

& $\zeta(6) = \frac{\pi^6}{945} \Rightarrow E_6(\tau) = 1 \pm \frac{2^6 3^3 \cdot 5 \cdot 7}{2^3 \cdot 3 \cdot 5} \sum_{n\geq 1} \sigma_5(n) q^n$

$$= 1 - 504 \sum_{n\geq 1} \sigma_5(n) q^n.$$

Corollary 2.13 If $\Delta \equiv \sum_{n\geq 1} \tau(n) q^n$: then $\tau(n) \in \mathbb{Z}$, and $\tau(1) = 1$.

Proof $E_4 = 1 + 240U$

$E_6 = 1 - 504V$

$$\Rightarrow \Delta = \frac{E_4^3 - E_6^2}{1728} = \frac{(1+240U)^3 - (1-504V)^2}{1728}$$

$$= \frac{3 \cdot 240U + 2 \cdot 504V}{1728} + R \quad (R \in q^2 \mathbb{Z}[[q]])$$

$$= \frac{5}{12}(U - V) + V + R.$$

& know: $\sigma_3(n) - \sigma_5(n) \equiv 0 \bmod 12$ $(\forall n \geq 1)$, since: $d^5 \equiv d^3 \bmod 12$.

~~&~~ Need to compute: $\tau(1) = \frac{3 \cdot 240 + 2 \cdot 504}{1728} = 1 \checkmark$

Modular Forms: lecture 7

20/10/2023.

From last time Computed: q-expansion of $E_k(\tau)$.

$$E_4(\tau) = 1 + 240 \sum_{n\geq 1} \sigma_3(n)\cdot q^n.$$

$$E_6(\tau) = 1 - 504 \sum_{n\geq 1} \sigma_5(n)\cdot q^n$$

& $\Delta \equiv \dfrac{E_4^3 - E_6^2}{1728}$

$= \sum_{n\geq 1} \tau(n) q^n$, with $\tau(n)\in\mathbb{Z}$. $\tau(1)=1$.

Theorem 2.14 $k\geq 4$ (even). & $N \equiv \dim_{\mathbb{C}} S_k(\Gamma(1))$. Then:

$\exists!$ basis $\{f_0, \dots, f_N\}$ for $M_k(\Gamma(1))$ as $\mathbb{C}$-vector space, s.t.

a) $\forall 0\leq i\leq N$: $f_i = \sum_{n\geq 0} a_n(f_i) q^n$ & $(a_n(f_i)\in\mathbb{Z}\ \forall n\geq 0)$

b) If $0\leq i,n\leq N \Rightarrow a_n(f_i) = \delta_{in}$.

In particular: $f_i = q^i + O(q^{N+1})$.

Why is it Important: turns out $M_k(\Gamma(1))$ has $\mathbb{Z}$-structure:

$M_k(\Gamma(1)) \cong M_k(\Gamma(1),\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$. ~~$M_k(\Gamma(1),\mathbb{Z}) \equiv \{f\in M_k(\Gamma(1)):$ $a_n(f)\geq 0\ \forall n\geq 0\}$~~

& $M_k(\Gamma(1),\mathbb{Z}) \equiv \{f\in M_k(\Gamma(1)) : a_n(f) \geq 0\ \forall n\geq 0\}$.

Proof Construction: ~~Assume $k\geq 2$.~~

Write: $k = 12a + d$: $a,d\geq 0$ & $d = 14$ if $k \equiv 2 \mod 12$
$0\leq d\leq 10$ if $k\not\equiv 2 \mod 12$.

$$\Rightarrow \left\lfloor \frac{k}{12} \right\rfloor = \begin{cases} a & k\not\equiv 2\ (12) \\ a+1 & k\equiv 2\ (12). \end{cases}$$

$\Rightarrow a = \begin{cases} \lfloor \frac{k}{12} \rfloor & \text{if } k \not\equiv 2 \bmod 12 \\ \lfloor \frac{k}{12} \rfloor - 1 & \text{if } k \equiv 2 \bmod 12. \end{cases}$

Also know: $\dim_{\mathbb{C}} M_k(\Gamma(1)) \equiv N+1 = \begin{cases} \lfloor \frac{k}{12} \rfloor + 1 & \text{if } k \not\equiv 2 \bmod 12 \\ \lfloor \frac{k}{12} \rfloor & \text{if } k \equiv 2 \bmod 12. \end{cases}$

<u>Hence</u>: $a = N$ & $k = 12N + d$.

<u>Choose</u>: $A, B \geq 0$ s.t. $d = 4A + 6B$. & $g_i \equiv \Delta^i E_4^A E_6^B \cdot E_6^{2(N-i)}$ $(0 \leq i \leq N)$

g_i has <u>weight</u>: $12i + 4A + 6B + 12(N-i) = k$. $\Rightarrow g_i \in M_k(\Gamma(1))$

As E_4, E_6, Δ have q-expansions in $\mathbb{Z}[[q]]$ (last lecture).

so does g_i, & g_i has leading term q^i.

$\Rightarrow$ Perform <u>row reduction</u> on $\{g_i\}$ to get the f_i. Satisfies <u>a), b)</u>.

<u>Next</u>: consider linear functionals $a_0, \ldots, a_N : M_k(\Gamma(1)) \longrightarrow \mathbb{C}$
$f \longmapsto a_i(f)$

where: $f = \sum_{n \geq 0} a_n(f) q^n$. <u>Then</u>: $a_i(f_j) = \delta_{ij}$.

$\Rightarrow \{a_i\}$ are <u>linearly indep</u>, and form basis of $M_k(\Gamma(1))^*$.

& $\{f_i\}$ is dual basis of $M_k(\Gamma(1))$

$\Rightarrow$ Form unique basis with properties <u>a)</u>, <u>b)</u>.

§3: Hecke Operators.

<u>What</u>: symmetries of <u>spaces</u> of modular forms.

Arise in <u>Rep Theory</u>: $\Gamma(1) \leq GL_2(\mathbb{Q})^+ \curvearrowright \{f : H \longrightarrow \mathbb{C}\}$ under weight k action.

Arise in Geometry: think of modular forms as functions on set $\mathcal{L}$ of lattices of $\mathbb{C}$. (Will use this point.)

Recall if V f.d. $\mathbb{R}$-VS, a lattice $\Lambda \subseteq V$ is a subgroup $\Lambda \leq V$ which is discrete & cocompact (V/Λ compact).

$\boxed{\text{Lemma 3.1}}$ $\Lambda \leq V$ lattice $\Leftrightarrow \exists$ basis $\{e_i\}$ for V (as $\mathbb{R}$-VS) s.t. $\Lambda = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$.

Study: $\mathcal{L} = \{\Lambda \leq \mathbb{C} \text{ lattice}\}$, with its action by $\mathbb{C}^\times$:
$z \cdot \Lambda = \{z\lambda : \lambda \in \Lambda\}$ ($\forall z \neq 0$ & $\Lambda \in \mathcal{L}$).

$\boxed{\text{Prop 3.2}}$ The map $\tau \longmapsto \Lambda_\tau = \mathbb{Z}\langle \tau, 1\rangle$ induces: bijection $\Gamma(1)\backslash H \xrightarrow{\sim} \mathbb{C}^\times \backslash \mathcal{L}$.

Proof. Well-defined: if $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma(1)$ & $\tau \in H$,

$$\Lambda_{\gamma\tau} = \mathbb{Z}\left\langle 1, \frac{a\tau+b}{c\tau+d}\right\rangle = (c\tau+d)^{-1}\mathbb{Z}\langle a\tau+b, c\tau+d\rangle$$
$$= (c\tau+d)^{-1}\Lambda_\tau.$$

Surjective: If Λ lattice: $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ & $\mathrm{Im}(e_1/e_2) \neq 0$.
So, by perhaps swapping e_1 & e_2, assume: > 0.

Then: $\Lambda = e_2\mathbb{Z}\langle 1, \frac{e_1}{e_2}\rangle = e_2\Lambda_\tau$ ($\tau = e_1/e_2$).

Injective: If τ, τ' have same image: $\exists z \in \mathbb{C}^\times$, $z(\Lambda_\tau) = \Lambda_{\tau'}$. So, $\exists \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in GL_2(\mathbb{Z})$: $\tau' = az\tau + bz$
$1 = cz\tau + dz$

Then: $\tau' = \cancel{\frac{az+b}{cz+}} \frac{az\tau+bz}{cz\tau+dz} = \frac{a\tau+b}{c\tau+d}$.

Also: $\mathrm{Im}(\tau') = \mathrm{Im}(\gamma\tau) = \det(\gamma)\frac{\mathrm{Im}(\tau)}{|c\tau+d|^2}$.

Since $\mathrm{Im}(\tau), \mathrm{Im}(\tau') > 0$: get $\det(\gamma) = 1 \Rightarrow \gamma \in \Gamma(1)$.

$\Rightarrow \tau, \tau'$ in same $\Gamma(1)$-orbit.

If $k \in \mathbb{Z}$: say $\mathcal{F}: \mathcal{L} \to \mathbb{C}$ is weight k if: $\forall z \in \mathbb{C}^\times, \Lambda \in \mathcal{L}$:

$\mathcal{F}(z\Lambda) = z^{-k}\mathcal{F}(\Lambda)$.

Prop 3.3 Let $V_k \equiv \{\mathcal{F}: \mathcal{L} \to \mathbb{C}, \text{ weight } k\}$

$W_k \equiv \{f: H \to \mathbb{C}, \forall \gamma \in \Gamma(1), f|_k[\gamma] = f\}$.

Then, the map $\mathcal{F} \mapsto f$ & $f(\tau) = \mathcal{F}(\Lambda\tau)$ induces a $\mathbb{C}$-vector space isomorphism $V_k \xrightarrow{\sim} W_k$.

Proof Well-defined: check, if $\mathcal{F} \in V_k$, ~~then~~ $f(\tau) = \mathcal{F}(\Lambda_\tau)$

then $f \in W_k$.

If $\gamma \in \Gamma(1)$: $f|_k[\gamma](\tau) = f(\gamma\tau)j(\gamma,\tau)^{-k} = F(\Lambda_{\gamma\tau})j(\gamma,\tau)^{-k}$

$= F(j(\gamma,\tau)\Lambda_{\gamma\tau}) = F(\Lambda_\tau) = f(\tau)$.

(since: $j(\gamma,\tau)\Lambda_{\gamma\tau} = \Lambda_\tau$)

To show isomorphism: write an inverse $\alpha: W_k \to V_k$ by:

$(\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2,\ \mathrm{Im}(e_1/e_2) > 0) \Rightarrow \alpha(f)(\Lambda) = e_2^{-k} f(e_1/e_2)$.

Well-def'ed: if $\begin{matrix} e_1' = ae_1 + be_2 \\ e_2' = ce_1 + de_2 \end{matrix}$ $\Rightarrow e_2'^{-k} f(e_1'/e_2') = (ce_1+de_2)^{-k} f\left(\frac{ae_1+b}{\cdots}\right)$

$= e_2^{-k} f(e_1/e_2)$.

Modular Forms: Lecture 8 23/10/2023.

From last time $\mathcal{L} \equiv \{\Lambda \leq \mathbb{C} \text{ lattice}\} \curvearrowleft \mathbb{C}^\times$.

Has: bijection $\Gamma(1)\backslash \mathbb{H} \xrightarrow{\sim} \mathbb{C}^\times \backslash \mathcal{L}$.

$$\tau \longmapsto \Lambda_\tau = \mathbb{Z} \oplus \mathbb{Z}\tau.$$

If $k \in \mathbb{Z}$: defined $V_k \equiv \{F : \mathcal{L} \to \mathbb{C} : \forall z \in \mathbb{C}^\times, \forall \Lambda \in \mathcal{L}, F(z\Lambda) = z^{-k} F(\Lambda)\}$

$W_k \equiv \{f : \mathbb{H} \to \mathbb{C}, \forall \gamma \in \Gamma(1) : f|_k[\gamma] = f\}$.

Showed: $V_k \xrightarrow{\sim} W_k$ by: $F \longmapsto (f : \tau \longmapsto F(\Lambda_\tau))$.

DEF 3.4 $n \in \mathbb{N}$. The n'th Hecke operator $T_n : V_k \to V_k$ is:

$$\boxed{(T_n F)(\Lambda) = n^{k-1} \sum_{\Lambda' \leq_n \Lambda} F(\Lambda')} \quad (\Lambda' \leq \Lambda \text{ \& index } n)$$

& $T_n : W_k \to W_k$ for endomorphism of W_k via identifying the isomorphism $V_k \xrightarrow{\sim} W_k$.

[Why is T_n well-def'ed endomorphism of V_k?

(1) Finite sum: $\exists$ bijection $\{\Lambda' \leq_n \Lambda\} \longleftrightarrow \{H \leq \Lambda/n\Lambda, \text{ index } n\}$

$$\Lambda' \longmapsto \Lambda'/n\Lambda$$
$$H + n\Lambda \longmapsfrom H$$

By Lagrange's theorem: $\Lambda' \leq_n \Lambda \Rightarrow n(\Lambda/\Lambda') = 0 \Rightarrow n\Lambda \leq \Lambda'$.

& $\Lambda/n\Lambda \cong (\mathbb{Z}/n\mathbb{Z}) \oplus (\mathbb{Z}/n\mathbb{Z})$ finite, so finitely many subgroups.

$\Rightarrow$ Sum finite ✓

& since $n\Lambda \leq \Lambda' \leq \Lambda$: Λ' is discrete & cocompact in $\mathbb{C}$.
$\Rightarrow$ Is lattice.

⊛ **Weight k:** need: $(T_nF)(z\Lambda) = z^{-k}(T_nF)(\Lambda)$

$$LHS = n^{k-1}\sum_{\Lambda' \underset{n}{\leq} z\Lambda} F(\Lambda') = n^{k-1}\sum_{\Lambda' \underset{n}{\leq} \Lambda} F(z\Lambda') = n^{k-1}\sum_{\Lambda' \underset{n}{\leq} \Lambda} z^{-k}F(\Lambda') = RHS.$$

Prop 3.5 1) m,n coprime. $\Rightarrow T_m T_n = T_{mn}$.

2) If p prime, $n \in \mathbb{Z}_{\geq 1}$ $\Rightarrow T_{p^n}T_p = T_{p^{n+1}} + p^{k-1}T_{p^{n-1}}$.

Proof let m,n be $\in \mathbb{Z}_{\geq 1}$, (not necessarily coprime).

$$(T_m(T_nF))(\Lambda) = m^{k-1}\sum_{\Lambda' \underset{m}{\leq} \Lambda} (T_nF)(\Lambda') = (mn)^{k-1}\sum_{\Lambda' \underset{m}{\leq} \Lambda}\sum_{\Lambda'' \underset{n}{\leq} \Lambda'} F(\Lambda'')$$

Know: $\Lambda'' \underset{mn}{\leq} \Lambda$.

$$= (mn)^{k-1}\sum_{\Lambda'' \underset{mn}{\leq} \Lambda} a(\Lambda,\Lambda'')F(\Lambda'') \text{ where } a(\Lambda,\Lambda'') \text{ is } \#\{\Lambda'' \underset{n}{\leq} \Lambda' \underset{m}{\leq} \Lambda\}$$

$a(\Lambda,\Lambda'') = \#\{H \leq \Lambda/\Lambda'' : |H| = n\}$.

⊛ If m,n coprime: $a(\Lambda,\Lambda'') = 1$ $\forall \Lambda'' \leq \Lambda$. [Since: any finite abelian group of order mn has unique subgroup, order n.]

$$\Rightarrow T_mT_nF(\Lambda) = (mn)^{k-1}\sum_{\Lambda'' \underset{mn}{\leq} \Lambda} F(\Lambda'') = (T_{mn}F)(\Lambda) \checkmark$$

⊛ If $m = p^n$ $n = p$: $(T_{p^n}T_pF)(\Lambda) = p^{(n+1)(k-1)}\sum_{\Lambda'' \underset{p^{n+1}}{\leq} \Lambda} a(\Lambda,\Lambda'')F(\Lambda'')$

where $a(\Lambda,\Lambda'') = \{\# H \leq \Lambda/\Lambda'' : |H| = p\}$.

(Not necessarily unique H in this case.)

Have: $\Lambda \simeq \mathbb{Z}^2$. $\Rightarrow \Lambda/\Lambda''$ finite abelian group, order p^{n+1}, that can be generated by 2 elements.

& Any such group is $\cong \mathbb{Z}/p^a\mathbb{Z} \oplus \mathbb{Z}/p^b\mathbb{Z}$ (for $a+b=n+1$)

Case 1: $b=0$, $a=n+1$. $\Lambda/\Lambda'' \cong \mathbb{Z}/p^{n+1}\mathbb{Z}$. Unique subgroup of order p. $\Rightarrow$ $a(\Lambda,\Lambda'')=1$

Case 2: $b>0$. Then: $\Lambda/\Lambda''[p] \equiv \{x \in \Lambda/\Lambda'' : px=0\}$ is a subgroup, and: $\{H \leq \Lambda/\Lambda'' : |H|=p\} = \{H \leq \Lambda/\Lambda''[p] : |H|=p\}$.

& $\Lambda/\Lambda''[p] \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$.

$$\Rightarrow a(\Lambda,\Lambda'') = \#\{H \leq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} : |H|=p\}$$
$$= \#\,\mathbb{P}^n(\mathbb{F}_p) = \#(\mathbb{A}^1(\mathbb{F}_p) \cup \{\infty\}) \boxed{= p+1.}$$

To distinguish the 2 cases: if $\Lambda'' \leq_{p^{n+1}} \Lambda$ then $\exists$ $\mathbb{Z}$-basis of Λ, say $\{e_1, e_2\}$, s.t. $\Lambda'' = \mathbb{Z}p^a e_1 \oplus \mathbb{Z}p^b e_2$.

$\Rightarrow$ (ii) $\Leftrightarrow$ $\Lambda'' \leq p\Lambda$.

Hence:
$$(T_{p^n} T_p F)(\Lambda) = p^{(n+1)(k-1)} \sum_{\Lambda'' \leq_{p^{n+1}} \Lambda} F(\Lambda'') + p^{(n+1)(k-1)} \sum_{\Lambda'' \leq_{p^{n-1}} p\Lambda} pF(\Lambda'')$$

[Each Λ'' in case 1 contribute to first sum, and case 2: once in first sum, $p\times$ in 2nd sum]

First term: $(T_{p^{n+1}} F)(\Lambda)$ ✓

2nd term:
$$p^{(n+1)(k-1)} \sum_{\Lambda'' \leq_{p^{n-1}} p\Lambda} pF(\Lambda'') = p^{(n-1)(k-1)} p^{2(k-1)} \sum_{\Lambda'' \leq_{p^{n-1}} \Lambda} pF(p\Lambda'')$$

$$= p^{(n-1)(k-1)} p^{2(k-1)} p^{1-k} \sum_{\Lambda'' \leq_{p^{n-1}} \Lambda} F(\Lambda'') = p^{k-1} (T_{p^{n-1}} F)(\Lambda) \checkmark$$

$$\Rightarrow T_{p^n} T_p = T_{p^{n+1}} + p^{k-1} T_{p^{n-1}}.$$

Corollary 3.6 $\forall m, n \in \mathbb{Z}$: $T_m T_n = T_n T_m$ as endomorphisms of V_k.

$\Rightarrow$ All operators <u>commute</u> with each other.

<u>Proof</u> let $n = \prod p_i^{\alpha_i}$, then $T_n = T_{p_1^{a_1}} \dots, T_{p_k^{a_k}}$.

& If p, q <u>distinct</u> primes then $T_{p^{a_1}} T_{q^b}$ commute.

So, suffices to show for equal primes. <u>Need</u>: $T_{p^a} T_{p^b} = T_{p^b} T_{p^a}$.

<u>Claim</u>: T_{p^a} is <u>polynomial</u> in T_p.

<u>Proof of claim</u>: induction on a. $a = 1$: ✓

& $T_{p^{a+1}} = T_{p^a} T_p \cancel{+} - p^{k-1} T_{p^{a-1}}$. So, done by induction.

So, obviously commute.

Modular Forms: lecture 9. 25/10/2023.

From last time $T_n \curvearrowright V_k = \{F : \mathcal{L} \longrightarrow \mathbb{C}\}$ (weight k)

$$\begin{array}{c} V_k \\ \cong \\ W_k \end{array} = \{F : \mathcal{H} \to \mathbb{C}, \text{ invariant under weight } k \text{ action}\}$$

& Proved: T_n commute.

& Can be expressed in terms of $T_p : p|n$.

We're interested in: $M_k(\Gamma(1)), S_k(\Gamma(1)) \leq W_K$.

By def: $(T_n F)(\Lambda) = n^{k-1} \sum_{\Lambda' \leq_n \Lambda} F(\Lambda')$.

Lemma 3.7 Let $n \in \mathbb{Z}_{\geq 0}$ & $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \leq \mathbb{C}$ lattice. Then:

$$\{\Lambda' \leq_n \Lambda\} = \{\mathbb{Z}(a\underline{e}_1 + b\underline{e}_2) \oplus \mathbb{Z}(d\underline{e}_2) : a, b, d \in \mathbb{Z}_{\geq 0},\ ad = n,\ 0 \leq b < d\}$$

Proof SES $0 \to \mathbb{Z}\underline{e}_2/\mathbb{Z}e_2 \cap \Lambda' \longrightarrow \Lambda/\Lambda' \longrightarrow \Lambda/\mathbb{Z}\underline{e}_2 + \Lambda' \longrightarrow 0$

$$\cong \mathbb{Z}\underline{e}_2/(\mathbb{Z}\underline{e}_1 \cap (\mathbb{Z}\underline{e}_2 + \Lambda'))$$

& $|\Lambda/\Lambda'| = n$.

Set: $d \equiv \# \left(\dfrac{\mathbb{Z}\underline{e}_2}{\mathbb{Z}\underline{e}_2 \cap \Lambda'}\right) = \inf\{d \geq 1 : d\underline{e}_2 \in \Lambda'\}$

$a = \# \left(\dfrac{\Lambda}{\mathbb{Z}\underline{e}_2 + \Lambda'}\right) = \inf\{a \geq 1 : \exists b \in \mathbb{Z},\ a\underline{e}_1 + b\underline{e}_2 \in \Lambda'\}$

$\Rightarrow ad = n$ & $\exists!\ 0 \leq b < d,\ a\underline{e}_1 + b\underline{e}_2 \in \Lambda'$.

Claim: $\Lambda' = \mathbb{Z}(a\underline{e}_1 + b\underline{e}_2) \oplus \mathbb{Z}(d\underline{e}_2)$.

Have: $\supseteq$ (trivially), and: if $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(\mathbb{Z})$, then

if $N \equiv \alpha\delta - \beta\gamma \neq 0$ then $[\Lambda : \mathbb{Z}(\alpha\underline{e}_1 + \beta\underline{e}_2) \oplus \mathbb{Z}(\gamma\underline{e}_1 + \delta\underline{e}_2)] = |N|$

$\Rightarrow [\Lambda : \mathbb{Z}(a\underline{e}_1 + b\underline{e}_2) \oplus \mathbb{Z}(d\underline{e}_2)] = \det\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = n$.

$\Rightarrow [\Lambda' : \mathbb{Z}(a\underline{e}_1 + b\underline{e}_2) \oplus \mathbb{Z}(d\underline{e}_2)] = 1$, so equal ✓

So, defined: $\{\Lambda' \underset{n}{\leq} \Lambda\} \xrightarrow{\varphi} \{(a,b,d) \in \mathbb{Z}_{\geq 0},\ ad = n \ \& \ 0 \leq b < d\}$

which has inverse $(a,b,d) \xmapsto{\varphi^{-1}} \mathbb{Z}(a\underline{e}_1 + b\underline{e}_2) \oplus \mathbb{Z}(d\underline{e}_2)$

$\Rightarrow$ Is a bijection.

Lemma 3.8 Let: $f \in W_k$. Then $(T_n f)(\tau) = n^{k-1} \sum_{\substack{ad=n \\ 0 \leq b < d}} f\left(\frac{a\tau+b}{d}\right) d^{-k}$

$= \sum_{\substack{ad=n \\ 0 \leq b < d}} f\big|_k \left[\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right].$

Proof $\left.\begin{matrix} f \longleftrightarrow F \in W_k \\ f(\tau) \equiv F(\Lambda_\tau) \end{matrix}\right]$ & by def. $(T_n f)(\tau) = (T_n F)(\Lambda_\tau)$

$= n^{k-1} \sum_{\Lambda' \underset{n}{\leq} \Lambda_\tau} F(\Lambda')$

$= n^{k-1} \sum_{\substack{ad=n \\ 0 \leq b < d}} F(\mathbb{Z}(a\tau+b) \oplus \mathbb{Z}d)$

$= n^{k-1} \sum_{a,b,d} F\left(d\left(\mathbb{Z}\left(\frac{a\tau+b}{d}\right) \oplus \mathbb{Z}\right)\right) = n^{k-1} \sum d^{-k} F\left(\Lambda_{\frac{a\tau+b}{d}}\right)$

$= n^{k-1} \sum d^{-k} f\left(\frac{a\tau+b}{d}\right)$ ✓

Second equality: recall, if $g \in GL_2(\mathbb{R})^+$ ~~$\Rightarrow f|_k[g] =$~~ then

$f|_k[g] = \det(g)^{k-1} f(g\tau) j(g,\tau)^{-k}$.

$\Rightarrow f\big|_k \left[\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right](\tau) = n^{k-1} f\left(\frac{a\tau+b}{d}\right) d^{-k}$ ✓

Corollary 3.9 If $f \in W_k$ & f holo. $\Rightarrow T_n f \in W_k$ holomorphic.

(since: finite sum of $f\left(\frac{a\tau+b}{d}\right)$'s.)

Prop 3.10 Let $f \in W_k$ holo. in $\mathbb{H}$. Then: $T_n f$ has a q-expansion $T_n f = \sum c_m \cdot q^m$ & $c_m \equiv \sum_{a \mid (m,n)} a^{k-1} b_{mn/a^2}$

$\left[\text{if } f(\tau)=\sum_{m\in\mathbb{Z}} b_m q^m \Rightarrow (T_n f)(\tau)=\sum_{m\in\mathbb{Z}} c_m q^m,\ c_m \equiv \sum_{a|(m,n)} a^{k-1} b_{\left(\frac{mn}{a^2}\right)}.\right]$

<u>Proof</u> $T_n f = n^{k-1} \sum_{\substack{ad=n \\ 0\le b<d}} d^{-k} f\left(\frac{a\tau+b}{d}\right)$

$$= n^{k-1}\sum_{\substack{ad=n\\0\le b<d}} d^{-k} \sum_{m\in\mathbb{Z}} b_m e^{2\pi i m a\tau/d} e^{2i\pi mb/d}.$$

$$= n^{k-1}\sum_{ad=n} d^{-k}\sum_{m\in\mathbb{Z}} b_m e^{2\pi i m a\tau/d} \underbrace{\left(\sum_{0\le b<d} e^{2i\pi mb/d}\right)}_{(=d \text{ if } d|m,\ 0 \text{ else})}.$$

$$= n^{k-1}\sum_{ad=n} d^{1-k}\sum_{m\in\mathbb{Z}} b_{dm} e^{2i\pi a m\tau}.$$

$$= \sum_{ad=n}\left(\frac{n}{d}\right)^{k-1}\sum_{m\in\mathbb{Z}} b_{dm} q^{am} = \sum_{m\in\mathbb{Z}}\sum_{a|n} a^{k-1} b_{mn/a} q^{am}.$$

$$= \sum_{N\in\mathbb{Z}} c_N q^N,\quad c_N \equiv \sum_{a|m,\,a|n} a^{k-1} b_{(nN/a^2)}.$$

<u>Theorem 3.11</u> $\forall n\in\mathbb{N}$: T_n preserve $S_k(\Gamma(1)) \subseteq M_k(\Gamma(1)) \subseteq W_k$.

Moreover: if $f\in M_k(\Gamma(1))$ then $a_0(T_n f) = \sigma_{k-1}(n)\, a_0(f)$

$a_1(T_n f) = a_n(f)$.

<u>Proof</u> To show T_n preserves $M_k(\Gamma(1))$: need to show that:

if $f\in M_k(\Gamma(1))$ then $\underbrace{T_n f \text{ holo. in } \mathbb{H}}_{\text{done by 3.9}}$, <u>and</u> at ∞.

$\Leftrightarrow a_N(T_n f) = 0\ \forall N<0$.

<u>But</u>: $a_N(T_n f) = \sum_{d|(N,n)} d^{k-1} a_{\left(\frac{Nn}{d^2}\right)}(f)$. & $\frac{Nn}{d^2}<0$, so, since f holo. at $\infty \Rightarrow$ <u>All</u> terms 0.

$\underline{\underline{\&}}$ $a_0(T_n f) = \sum_{d \mid (N,0)} d^{k-1} a_{\left(\frac{n \cdot 0}{d^2}\right)}(f) = \sum_{d \mid N} d^{k-1} a_0(f) = \sigma_{k-1}(N)\, a_0(f)$

$\underline{\underline{\&}}$ $a_1(T_n f) = \sum_{d \mid (n,1)} d^{k-1} a_{\frac{n \cdot 1}{d^2}}(f) = a_n(f)$.

If $f \in S_k(\Gamma(1))$: then $T_n f \in M_k(\Gamma(1))$ (as before).

$\underline{\underline{\&}}$ $a_0(T_n f) = \sigma_{k-1}(n)\, a_0(f) = 0 \Rightarrow T_n f \in S_k(\Gamma(1))$ ✓

<u>Next</u>: study Spectral Decomp of T_n.

$\Longleftrightarrow$ Decomposition of $M_k(\Gamma(1))$ as sum of simultaneous generalised eigenspaces (for the T_n)

<u>Simplest cases</u> When $M_k(\Gamma(1))$ or $S_k(\Gamma(1))$ is 1-dim, then: any nonzero element is an eigenvector.

E.g. $S_{12}(\Gamma(1))$ is 1-dimensional, $= \mathbb{C}\Delta$. $\Delta = \sum \tau(n) q^n$.

$\Rightarrow \Delta$ is (T_n)-eigenvector $\forall n \geq 1$. $T_n \Delta = \alpha_n \Delta$ $(\alpha_n \in \mathbb{C})$

So: $a_1(T_n \Delta) = a_1(\alpha_n \Delta) = \alpha_n \cdot a_1(\Delta) = \alpha_n$.

$\underline{\underline{\&}}$ $= a_n(\Delta) = \tau(n)$. So, $\underline{\alpha_n = \tau(n)}$ = coeff. of q^n of Δ.

<u>Ramanujan Conjecture</u> (Holds)

⊛ τ <u>multiplicative</u>: $\tau(mn) = \tau(m)\tau(n)$ $\forall (m,n) = 1$

⊛ $\tau(p^{n+1}) = \tau(p)\tau(p^n) - p^{11}\tau(p^{n-1})$.

<u>Know</u>: these hold for T_n, and any ~~operato~~ identity on operators also hold for their eigenvalues ✓

Modular Forms: lecture 10 27/10/2023.

From last time: Hecke Operators $T_n : n \in \mathbb{N}$ act on $S_k(\Gamma(1))$ & $M_k(\Gamma(1))$ by $T_n(f) \equiv \sum_{m \geq 0} b_m \cdot q^m$, where:

$$b_m \equiv \sum_{d \mid (m,n)} d^{k-1} a_{\left(\frac{mn}{d^2}\right)}(f).$$

Goal: Study Spectral Decomp. of $M_k(\Gamma(1))$, and arithmetic properties of these eigenvalues.

DEF 3.1.2: If $f \in M_k(\Gamma(1))$: say f eigenform if is a T_n-eigenvector $\forall n \geq 1$. Say f normalised eigenform, if $a_1(f) = 1$.

Lemma 3.13: let $k \geq 1$. Then: Any eigenform $f \in M_k(\Gamma(1))$ is a scalar multiple of a unique normalised eigenform.

If f normalised: $T_n(f) = a_n(f) f \quad \forall n \geq 1$.

("n'th Hecke eigenvalue = n'th q-expansion coeff.")

Example: Δ is normalised eigenform, then: $T_n \Delta = \tau(n) \Delta$.

Proof: 1) Need: if f eigenform then $a_1(f) \neq 0$. (Then, take $\frac{f}{a_1(f)}$)

If $a_1(f) = 0$: & $\alpha_n \equiv$ eval of T_n on f: then

$a_n(f) = a_1(T_n f) = \alpha_n \, a_1(f) = 0 \quad \forall n \geq 1$.

$\Rightarrow f = \sum a_n(f) q^n = a_0(f)$ const # (constants are not modular forms weight $k > 0$)

2) If f normalised: $a_n(f) = \alpha_n$.

Prop 3.14: let: $k \geq 4$ even. Then: $G_k(\tau)$ eigenform.

<u>Proof</u> <u>Need</u>: G_k is T_n - e-vector $\forall n \geq 1$.

<u>Know</u>: T_n poly of T_p $(p|n)$ so enough to show for primes.

$$(T_p G_k)(\Lambda) = p^{k-1} \sum_{\Lambda' \underset{p}{\leq} \Lambda} G_k(\Lambda') = p^{k-1} \sum_{\Lambda' \underset{p}{\leq} \Lambda} \sum_{\lambda \in \Lambda' - 0} \lambda^{-k}$$

$$= p^{k-1} \sum_{\lambda \in \Lambda - 0} a(\Lambda, \lambda) \lambda^{-k} \text{ with: } a(\Lambda,\lambda) \equiv \# \{\Lambda' \underset{p}{\leq} \Lambda : \lambda \in \Lambda'\}$$

If $\Lambda' \underset{p}{\leq} \Lambda$ then: $p\Lambda \leq \Lambda' \leq \Lambda$, & have bijection:

$$\{\Lambda' \leq p\} \xrightarrow{\sim} \{H \leq \Lambda / p\Lambda \text{ \& } |H| = p\}.$$

& If $\lambda \in p\Lambda$ then: $\{\Lambda' \underset{p}{\leq} \Lambda : \lambda \in \Lambda'\} = \{\Lambda' \underset{p}{\leq} \Lambda\}$ so $\underline{a(\Lambda,\lambda) = p+1}$

If $\lambda \notin p\Lambda$ then $\lambda \bmod p\Lambda \neq 0$ so $\exists! H \leq \Lambda / p\Lambda$, order p, with $\lambda \in H$. So, $\{\Lambda' \underset{p}{\leq} \Lambda : \lambda \in \Lambda'\} = \{\mathbb{Z}\lambda + p\Lambda\}$, so $\underline{a(\Lambda,\lambda) = 1}$.

$$\Rightarrow (T_p G_k)(\Lambda) = p^{k-1} \sum_{\lambda \in \Lambda - 0} \lambda^{-k} + p^{k-1} \sum_{\lambda \in p\Lambda - 0} p \lambda^{-k}.$$

$$= p^{k-1} \sum_{\lambda \in \Lambda - 0} \lambda^{-k} + p^{k-1} \sum_{\lambda \in \Lambda - 0} p (p\lambda)^{-k} = (p^{k-1} + 1) G_k(\Lambda)$$

$$= \sigma_{k-1}(p) G_k(\Lambda).$$

$\Rightarrow$ Is a T_p - eigenvector ✓

Can even: compute T_n - eigenvalue $\forall n$: $a_0(T_n f) = \sigma_{k-1}(n) a_0(f)$.

$\Rightarrow$ If f eigenform & $a_0(f) \neq 0$, then eigenvalue <u>must</u> be $\sigma_{k-1}(n)$.

$\Rightarrow T_n G_k = \sigma_{k-1}(n) G_k \quad \forall n \geq 1$.

G_k has q-expansion: $2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) q^n$.

& $E_k(\tau) = 1 + \frac{(2\pi i)^k}{\zeta(k)(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) q^n$. (Not normalised!)

$\Rightarrow$ Normalised Eigenform $F_k(\tau) \equiv \frac{\zeta(k)(k-1)!}{(2\pi i)^k} + \sum_{n\geq 1} \sigma_{k-1}(n) q^n$.

$F_k(\tau) = \frac{-B_k}{2k} + \sum_{n \geq 1} \sigma_{k-1}(n) q^n$. (lead coeff $\frac{\zeta(1-k)}{2}$)

Have: decomposition $M_k(\Gamma(1)) = \mathbb{C} F_k \oplus S_k(\Gamma(1))$ $(k \geq 4)$,

and: both parts are preserved by T_n.

$\Rightarrow$ Enough to concern for just $S_k(\Gamma(1))$.

Remark T_n doesn't (usually) respect multiplication.

$\Rightarrow$ Product of 2 eigenforms not an eigenform. (usually)

$E_4^2 = E_8$ (both eforms) but: $E_4^3 \in M_{12}(\Gamma(1)) = \mathbb{C} E_{12} \oplus \mathbb{C}\Delta$.

& E_4^3 not an eigenform.

Prop 3.15) If f cuspidal eigenform ($f \in S_k(\Gamma(1))$) then all T_n-eigenvalues of f are algebraic integers.

& If f normalised: $\mathbb{Q}(\{a_n(f) : n \in \mathbb{N}_0\})$ is number field (fin. dim $/\mathbb{Q}$).

Proof Show: $\forall n \geq 1$, all evals of T_n on $S_k(\Gamma(1))$ are algebraic integers.

Consider: basis $f_1, \dots, f_N$ of $S_k(SL_2(\mathbb{Z}))$ characterised by: $\forall 1 \leq i \leq N$: $a_n(f_i) \in \mathbb{Z}$ (dual basis of $a_1, \dots, a_n$ of $S_k(\Gamma(1))^*$)

$\forall 1 \leq i, n \leq N$: $a_n(f_i) = \delta_{in}$.

$\Rightarrow \forall f \in S_k(\Gamma(1)) : f = \sum_{i=1}^N a_i(f)\, f_i$.

Claim: If A denote matrix of T_n in basis $\{f_i\}$, ~~in the basis~~

then A has integer coeffs.

[Since charpoly $T_n \equiv \det(X \cdot 1_N - A)$: char poly has monic & integer coeffs.]

By def: $T_n(f_j) = \sum_i A_{ij} f_i$. $\Rightarrow a_m(T_n f_j) = \sum_i A_{ij}\, a_m(f_i) = A_{mj}$.

& $a_m(T_n f_j) = \sum_{d \mid (m,n)} d^{k-1} a_{\frac{mn}{d^2}}(f_j) \in \mathbb{Z}$.

$\Rightarrow A_{mj} \in \mathbb{Z} \;\forall m,j \leq N$ ✓

If f normalised eigenform: $f = \sum_{i=1}^N a_i(f)\, f_i$.

$\Rightarrow \forall n \geq 1,\quad a_n(f) = \sum_i a_i(f) \underbrace{a_n(f_i)}_{\in \mathbb{Z}}$

$\Rightarrow \mathbb{Q}(\{a_n(f) : n \geq 1\})$

$= \mathbb{Q}(\{a_n(f) : n \leq N\})$ finite dim $/\mathbb{Q}$.

Modular Forms: Lecture 11 30/10/2023.

From last time showed: if $f \in S_k(\Gamma(1))$ normalised eigenform, then $\exists$ number field $K_f \equiv \mathbb{Q}(\{a_n(f)\}_{n\geq 1})$ s.t. $\forall n \geq 1$, $a_n(f) \in \mathcal{O}_{K_f} \equiv$ ring of alg ints of K_f.

To prove this: we showed: matrix of T_n WRT basis $f_1, \ldots, f_N$ dual to basis $a_1, \ldots, a_N$ of $S_k(\Gamma(1))^*$, has $\mathbb{Z}$-coeffs.

Can use same argument to compute Hecke eigenforms.

E.g. For $k=24$: $T_2 \curvearrowright S_{24}(\Gamma(1))$.

& $S_{24}(\Gamma(1))$ has unique basis $\{f_1, f_2\}$ with $f_1 = q + O(q^3)$
$f_2 = q^2 + O(q^3)$

& For any $f \in S_{24}(\Gamma(1))$: $f = a_1(f) f_1 + a_2(f) f_2$.

$$\Rightarrow T_2 f_1 = a_1(T_2 f_1) f_1 + a_2(T_2 f_1) f_2$$
$$= a_2(f_1) f_1 + (a_4(f_1) + 2^{23} a_1(f_1)) f_2$$
$$= (a_4(f_1) + 2^{23}) f_2.$$

$$\& \; T_2 f_2 = a_2(f_2) f_1 + (a_4(f_2) + 2^{23} a_1(f_2)) f_2$$
$$= f_1 + a_4(f_2) f_2.$$

In fact: $f_1 = \Delta E_6^2 + 1032\Delta^2 = q + 195660q^3 + 12080128q^4 + \ldots$

& $f_2 = \Delta^2 = q^2 - 48q^3 + 1080q^4 + \ldots$

$\Rightarrow$ Matrix for T_2: $\begin{pmatrix} 0 & 1 \\ 20468736 & 1080 \end{pmatrix}$

& Eigenvalues: $12\left(45 \pm \sqrt{144169}\right)$.

$\Rightarrow S_{24}(\Gamma(1))$ has basis of normalised eigenforms: g_1, g_2

$K_{g_2} \equiv \mathbb{Q}(\sqrt{144169})$. Prime!

DEF 3.16] Let $f: H \to \mathbb{C}$ cts fn & invariant under action weight 0 & level $\Gamma(1)$. ($\Rightarrow f(\gamma\tau) = f(\tau)\ \forall \gamma \in \Gamma(1)$)

Define: $\displaystyle\int_{\Gamma(1)\backslash H} f(\tau)\frac{dx\,dy}{y^2} \equiv \int_{H} f(\tau)\frac{dx\,dy}{y^2}$. $(\tau \equiv x+iy)$.

[Motivation: $\omega \equiv \dfrac{dx\,dy}{y^2}$ is invariant under $GL_2(\mathbb{R})^+$]

[Want to say: $\Gamma(1)\backslash H \rightsquigarrow \mathbb{C}$ is manifold, & ω descends to $\Gamma(1)/H$, so can use integration on manifolds.

Problem: 1) No diff geo $(M,\omega) \mapsto \int_M \omega$

2) ω does not descend (is ramified).

Solution: Choose finite index $\Gamma \subseteq \Gamma(1)$, with no nontrivial elements of finite order. Then, ω descends to ω_Γ on $\Gamma\backslash H$, and:

$\displaystyle\frac{1}{[\Gamma(1):\Gamma]}\int_{\Gamma\backslash H} f\cdot\omega_\Gamma$ is indep. of choice of Γ.]

Lemma 3.17] Let $f, g \in S_k(\Gamma(1))$. Then: the function $f(\tau)\overline{g(\tau)}\,\mathrm{Im}(\tau)^k$ is invariant under weight 0 action of $\Gamma(1)$.

& $\displaystyle\int_{\Gamma(1)\backslash H} f(\tau)\overline{g(\tau)}\,\mathrm{Im}(\tau)^k \frac{dx\,dy}{y^2}$ converges absolutely.

Proof If $\gamma\in\Gamma(1) \Rightarrow f(\gamma\tau) = f(\tau)\, j(\gamma,\tau)^k$.

$$\Rightarrow f(\gamma\tau)\overline{g(\gamma\tau)}\,\mathrm{Im}(\gamma\tau)^k = f(\tau)\overline{g(\tau)}\,\frac{j(\gamma,\tau)^k\,\overline{j(\gamma,\tau)}^k}{|j(\gamma,\tau)|^{2k}}\,\mathrm{Im}(\tau)^k$$

$$= f(\tau)\overline{g(\tau)}\,\mathrm{Im}(\tau)^k \;\checkmark$$

& Write: $f(\tau) = \tilde{f}(q)$, $\tilde{f}: D(0,1) \longrightarrow \mathbb{C}$. Holomorphic.

& $\tilde{f}$ vanishing at 0.

$\Rightarrow \tilde{f}(q) = q\cdot h(q)$ with $h: D(0,1) \longrightarrow \mathbb{C}$ holomorphic.

$\forall \delta \in (0,1): \exists C_\delta > 0$ s.t. $|h(q)| \leq C_\delta \; (\forall\, 0 \leq |q| \leq \delta.)$

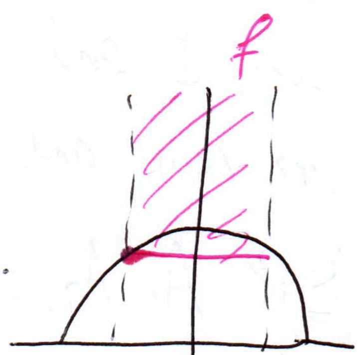
$\Rightarrow |\tilde{f}(q)| \leq |q|\, C_\delta. \; (\forall\, 0 \leq |q| \leq \delta.)$

$\Rightarrow \forall R > 0,\; \exists C_{f,R} > 0$ s.t. $\forall \tau \in H,\; \mathrm{Im}(\tau) > R,\; |f(\tau)| \leq |q|\, C_{f,R}.$

$\Rightarrow |f(\tau)| \leq e^{-2\pi\,\mathrm{Im}(\tau)}\, C_{f,R}.$

$$\Rightarrow \int_{\Gamma(1)\backslash H} \left| f(\tau)\overline{g(\tau)}\,\mathrm{Im}(\tau)^k \right| \frac{dx\,dy}{y^2}$$

$$\leq \int_{F} C_{f,\frac{\sqrt{3}}{2}}\, C_{g,\frac{\sqrt{3}}{2}}\, e^{-2\pi y}\, e^{-2\pi y}\, y^k\, \frac{dx\,dy}{y^2}.$$



$$\leq \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\sqrt{3}/2}^{\infty} e^{-4\pi y}\, y^{k-2}\, dx\,dy = \int_{\sqrt{3}/2}^{\infty} e^{-4\pi y}\, y^{k-2}\, dy < \infty.$$

Remark lemma fails if f, g not cuspidal.

DEF 3.18) Peterson inner product: on $S_k(\Gamma(1))$,

$$\langle f, g\rangle = \int_{\Gamma(1)\backslash H} f(\tau)\overline{g(\tau)}\, im(\tau)^k \frac{dx\,dy}{y^2}.$$

Why inner product? $\langle f, f\rangle = \int_{\Gamma(1)\backslash H} |f(\tau)|^2 \, Im(\tau)^k \frac{dx\,dy}{y^2}$

$\Rightarrow$ If $\langle f, f\rangle = 0$ then $|f|^2 y^k = 0 \Rightarrow \underline{f \equiv 0}$.

Theorem 3.19) $\forall n \geq 1$, T_n Hermitian under $\langle \cdot, \cdot \rangle$.

$\Leftrightarrow \forall f, g \in S_k(\Gamma(1)) : \langle T_n f, g\rangle = \langle f, T_n g\rangle$.

Theorem 3.20) $\forall k \geq 12$ even: $\exists$ basis $f_1, \ldots, f_N$ of normalised eigenforms for $S_k(\Gamma(1))$, unique (up to re-ordering) with:
$\forall i = 1, \ldots, N$: $K_{f_i} \equiv \mathbb{Q}(\{a_n(f_i) : n \geq 1\})$ is contained in $\mathbb{R}$
s.t. $\forall n \geq 1$, $a_n(f_i) \in \mathcal{O}_{K_{f_i}}$.

Proof From linear Algebra: if $(V, \langle \cdot, \cdot \rangle)$ inner product space over $\mathbb{C}$ and $T: V \longrightarrow V$ Hermitian, then: all evals of T are real and T diagonalisable.

& If $A_1, A_2, \ldots$ are an infinite family of Hermitian operators that commute, then simultaneously diagonalisable.

$\Rightarrow \exists$ Basis $f_1, \ldots, f_N$ of $S_k(\Gamma(1))$ of eigenforms.
(WLOG normalised).

So, need: unique up to re-ordering. $\Leftrightarrow$ Simultaneous eigenspaces are 1-dimensional. But, if $f, g \in S_k(\Gamma(1))$, same T_n-eval $\forall n$, then $\forall n$, $a_n(f) = a_n(g) \Rightarrow f = g$ ✓

Modular Forms. Lecture 12. 01/11/2023.

From last time: $\forall k$: $S_k(\Gamma(1))$ has unique basis of normalised eigen~~forms~~forms $f_1, \dots, f_N$. $f_i \equiv \sum_{n\geq 1} a_n(f_i) q^n$.

& $a_n(f_i) \equiv (T_n)$-eigenvalues $(\in \mathcal{O}_{K_{f_i}})$.

Sequences $(a_1(f), a_2(f), \dots)_{n\geq 1}$ of eigenvalues of Hecke operators on normalised eigenforms f are: ~~precisely~~ interesting objects in number theory.

[One reason: $(a_p(f) : p \text{ prime})$. Need to define a formulation of main conjectures of Langlands.

Another reason: Sequences related to concrete Qs in NT

E.g. Ramanujan Conjectures:

* $\tau(mn) = \tau(m)\tau(n)$ $\qquad \tau_n \equiv a_n(\Delta)$.
* $\tau(p^n)\tau(p) = \tau(p^{n+1}) + p^{11}\tau(p^{n-1})$
* One more conjecture.]

Lemma 3.21 p prime $\Rightarrow \displaystyle\sum_{n\geq 0} \tau(p^n) X^n = \frac{1}{1-\tau(p)X + p^{11}X^2}$.

Proof $(1-\tau(p)X + p^{11}X^2)\displaystyle\sum_{n\geq 0} \tau(p^n) X^n$

$$= 1 + \sum_{n\geq 2} \left(\tau(p^n) - \tau(p)\tau(p^{n-1}) + p^{11}\tau(p^{n-2})\right) X^n = 1 \;\checkmark$$

Factor quadratic: $1-\tau(p)X + p^{11}X^2 = (1-\alpha_p X)(1-\beta_p X)$.

2 possibilities: 1) $\tau(p)^2 - 4p^{11} > 0 \Rightarrow \alpha_p, \beta_p$ distinct & real.

(also, distinct absolute values)

2) $\tau(p)^2 - 4p^{11} \leq 0 \Rightarrow \alpha_p, \beta_p$ complex & conjugate. Same absolute value. $p^{11/2}$.

Ramanujan's 3rd conjecture: Only 2) occurs. $\Rightarrow |\tau(p)| \leq 2p^{11/2}$.

Conjecture 3.22 (Ramanujan–Peterson).
$f \in S_k(\Gamma(1))$ normalised e form. Then: $\forall p$ prime: $|a_p(f)| \leq 2p^{\frac{k-1}{2}}$.
[Proved by Deligne, 1973]

Ramanujan proved: $r_{24}(p) \equiv \{\#\{\underline{x} \in \mathbb{Z}^{24} : \|\underline{x}\|^2 = p\}\}$

$\Rightarrow r_{24}(p) = \frac{16}{691}(1+p^{11}) + \frac{33152}{691}\tau(p)$.

$\Rightarrow$ By conjecture: $r_{24}(p) = \frac{16}{691} p^{11} + O(p^{11/2})$.

Will prove Theorem 3.18: $\forall n$, $f, g \in S_k(\Gamma(1))$: $\langle T_n f, g\rangle = \langle f, T_n g\rangle$.

Proof* (sketch).

Recall: $\langle f, g\rangle = \int_{\Gamma(1)\backslash H} f(\tau)\overline{g(\tau)}\, \mathrm{im}(\tau)^k \frac{dx\,dy}{y^2}$.

Want: $\int_{\Gamma(1)\backslash H} (T_n f)\bar{g}\, \mathrm{Im}(\tau)^k \frac{dx\,dy}{y^2} = \int_{\Gamma(1)\backslash H} f\,(\overline{T_n g})\, \mathrm{Im}(\tau)^k \frac{dx\,dy}{y^2}$.

Reduction: Suffices to show for $n = p$ prime (since T_n is a poly of ~~coeffs~~ T_p's, $p \mid n$.)

Know: $f\bar{g}\,\mathrm{Im}^k$ invariant under weight 0 action of $\Gamma(1)$.

$\Rightarrow$ Corresponds to function $\mathcal{L} \longrightarrow \mathbb{C}$, invariant under $\mathbb{C}^\times$

<u>Claim</u>: this function is $\Lambda \longmapsto F(\Lambda)\overline{G(\Lambda)}\,\mathrm{covol}(\Lambda)^k$,
where $F(\Lambda\tau) \equiv f(\tau)$, $G(\Lambda\tau) \equiv g(\tau)$ & $\mathrm{covol}(\Lambda) = \int_{\mathbb{C}/\Lambda} dx\,dy$.

$\mathrm{covol}(\Lambda) = \left|\det\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}\right|$ where $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$
$(e_1 = x_1 + iy_1,\ e_2 = x_2 + iy_2)$

<u>Proof of claim</u> $\Lambda\tau \longmapsto F(\Lambda\tau)\overline{G(\Lambda\tau)}\,\mathrm{covol}(\Lambda\tau)^k$.

& $\mathrm{covol}(\Lambda\tau) = \left|\det\begin{pmatrix} x & y \\ 1 & 0 \end{pmatrix}\right| = y$. $\Rightarrow\ \rangle = f(\tau)\overline{g(\tau)}\,\mathrm{Im}(\tau)^k$.

If $A : \mathbb{C}^\times \backslash \mathcal{L} \longrightarrow \mathbb{C}$ cts fn, <u>define</u>: $\int_{\mathbb{C}^\times\backslash\mathcal{L}} A(\Lambda)\,d\Lambda$ by:

$$\int_{\Gamma(1)\backslash H} a(\tau)\,\frac{dx\,dy}{y^2} \qquad (a(\tau) \equiv A(\Lambda\tau)).$$

$$\Rightarrow \langle f, g\rangle = \int_{\mathbb{C}^\times\backslash\mathcal{L}} F(\Lambda)\overline{G(\Lambda)}\,\mathrm{covol}(\Lambda)^k\,d\Lambda$$

$$\Rightarrow \langle T_n f, g\rangle = p^{k-1}\int_{\mathbb{C}^\times\backslash\mathcal{L}} \sum_{\Lambda' \leq_p \Lambda} F(\Lambda')\overline{G(\Lambda)}\,\mathrm{covol}(\Lambda)^k\,d\Lambda.$$

Define: $\mathcal{L}_p \equiv \{(\Lambda', \Lambda) : \Lambda \in \mathcal{L}^p\ \&\ \Lambda' \leq_p \Lambda\} \longrightarrow \mathcal{L}$.
$(\Lambda', \Lambda) \longmapsto \Lambda$.

<u>Fact</u>: $\exists$ bijection $\Gamma_0(p)\backslash H \longrightarrow \mathbb{C}^\times\backslash\mathcal{L}_p$
$\tau \longmapsto (\mathbb{Z}p\tau \oplus \mathbb{Z} \leq_p \mathbb{Z}\tau \oplus \mathbb{Z})$

Where: $\Gamma_0(p) \equiv \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) : c \equiv 0 \mod p\right\}$.

If $A : \mathbb{C}^\times\backslash\mathcal{L}_p \longrightarrow \mathbb{C}$ cts: define:

$$\int_{\mathbb{C}^\times\backslash\mathcal{L}_p} A(\Lambda', \Lambda)\,d(\Lambda', \Lambda) \equiv \int_{\Gamma_0(p)\backslash H} a(\tau)\,\frac{dx\,dy}{y^2}.$$

$$\Rightarrow \langle T_p f, g\rangle = p^{k-1} \int_{\mathbb{C}^\times \backslash \mathcal{L}} \sum_{\Lambda' \underset{p}{\subseteq} \Lambda} F(\Lambda')\overline{G(\Lambda)}\, \mathrm{covol}(\Lambda)^k \, d\Lambda$$

$$= p^{k-1} \int_{\mathbb{C}^\times \backslash \mathcal{L}_p} F(\Lambda')\overline{G(\Lambda)}\, \mathrm{covol}(\Lambda)^k \, d(\Lambda', \Lambda)$$

<u>Observation</u>: If $\Lambda' \underset{p}{\subseteq} \Lambda \Rightarrow p\Lambda \underset{p}{\subseteq} \Lambda'$.

$\Rightarrow \exists\, i : \mathcal{L}_p \to \mathcal{L}_p$, $(\Lambda', \Lambda) \mapsto (p\Lambda, \Lambda')$.

& $i^2 = (p\Lambda', p\Lambda)$, so i descends to $\bar{i} : \mathbb{C}^\times \backslash \mathcal{L}_p \to \mathbb{C}^\times \backslash \mathcal{L}_p$.

<u>Key point</u>: $\bar{i}$ <u>preserves measures</u> & transforms $\langle T_p f, g\rangle$ to $\langle f, T_p g\rangle$.

<u>Why</u>? Under bijection $\mathbb{C}^\times \backslash \mathcal{L}_p \longrightarrow \Gamma_0(p)\backslash H$: i corresponds to the action of $\begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix} \in GL_2(\mathbb{Q})^+$.

& differential form $\omega = \frac{dx\,dy}{y^2}$ is invariant of $GL_2(\mathbb{Q})^+$.

Modular Forms: Lecture 13 03/11/2023.

From last time: $\langle T_n f, g\rangle = \langle f, T_n g\rangle$.

Prop 3.23 Let: $f: H \to \mathbb{C}$ continuous & Γ_∞ - invariant

$\left(\Gamma_\infty \equiv \left\{\begin{pmatrix} a & b \\ 0 & d\end{pmatrix} \in \Gamma(1)\right\}\right)$. $(\Leftarrow)\ f(\tau+1) = f(\tau))$

Suppose: $\forall \tau \in H, \sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} |f(\gamma\tau)| < \infty$. (1)

& $\int_{x=-\frac{1}{2}}^{1/2} \int_{y=0}^{\infty} |f(x+iy)| \frac{dx\,dy}{y^2} < \infty$ (2)

Then: $\sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} f(\gamma\tau)$ is measurable & $\Gamma(1)$-invariant

$$\boxed{\& \int_{\Gamma(1)\backslash H} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} f(\gamma\tau) \frac{dx\,dy}{y^2} = \int_{-1/2}^{1/2} \int_0^\infty f(x+iy) \frac{dx\,dy}{y^2}.}$$

Application: Unfolding.

Know: $\{\tau \in H, \mathrm{Re}(\tau) \in [-\frac{1}{2}, \frac{1}{2}]\}$ is fundamental set for $\Gamma_\infty \backslash \{\pm 1\}$. So, $\int_{\Gamma(1)\backslash H} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} f(\gamma\tau) \frac{dx\,dy}{y^2} = \int_{\Gamma_\infty \backslash H} f(\tau) \frac{dx\,dy}{y^2}$.

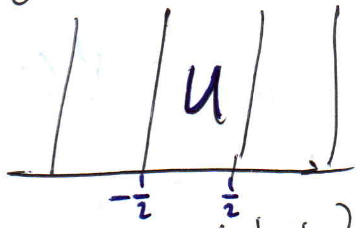
Proof Want to show $\sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} f(\gamma\tau)$ measurable on $\mathcal{F}$.

By Fubini: if $\sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} \int_{\mathcal{F}} |f(\gamma\tau)| \frac{dx\,dy}{y^2} < \infty$. Then, $\sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} f(\gamma\tau)$ measurable & absolutely ~~measurable~~ integrable on $\mathcal{F}$, and: can swap integral & sum.

So, need: $\displaystyle\sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} \int_{\mathcal{F}} f(\gamma\tau) \frac{dx\,dy}{y^2} = \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_0^\infty f(x+iy)\frac{dx\,dy}{y^2}$.

LHS $= \displaystyle\sum_{\gamma\in\Gamma_\infty\backslash\Gamma(1)} \int_{\gamma\mathcal{F}} f(\tau)\frac{dx\,dy}{y^2}$ (since: $\frac{dx\,dy}{y^2}$ invariant of $GL_2(\mathbb{R})^+$)

By sheet 1: if $\mathcal{F}^\circ = \mathrm{int}(\mathcal{F})$ then: $\forall \gamma\in\Gamma(1)$, have:



$\gamma\mathcal{F}^\circ \cap \{\tau\in H : \mathrm{Re}(\tau)\in \frac{1}{2}+\mathbb{Z}\} = \emptyset$.

$\gamma\mathcal{F}^\circ$ contained in vertical strip.

$\Rightarrow$ For $\gamma\in\Gamma(1)$: $\gamma\mathcal{F}^\circ$ contained in vertical strip.

$\Rightarrow$ Unique $\delta\in\Gamma_\infty\backslash\{\pm 1\}$ s.t. $(\delta\gamma)\mathcal{F}^\circ \subseteq \{\tau\in H : \mathrm{Re}(\tau)\in(-\frac{1}{2},\frac{1}{2})\} =: U$.

$\Rightarrow \{\gamma\in\Gamma(1)/\{\pm1\} \mid \gamma\mathcal{F}^\circ\subseteq U\}$ is set of coset reps of $\Gamma_\infty\backslash\Gamma(1)$.

So: $\displaystyle\sum_{\gamma\in\Gamma_\infty\backslash\Gamma(1)}\int_{\gamma\mathcal{F}} f(\tau)\frac{dx\,dy}{y^2} = \sum_{\substack{\gamma\in\Gamma(1)\backslash\{\pm1\} \\ \gamma\mathcal{F}^\circ\subseteq U}}\int_{\gamma\mathcal{F}^\circ} f(\tau)\frac{dx\,dy}{y^2}$

Know: $U = \Big(\displaystyle\bigsqcup_{\gamma\in\Gamma(1)\backslash\{\pm1\}} \gamma\mathcal{F}^\circ\Big)\sqcup W$, where W has measure 0.

$\Rightarrow U = \Big(\displaystyle\bigsqcup_{\gamma\in\Gamma(1)\backslash\{\pm1\}} \gamma\mathcal{F}^\circ\cap U\Big)\sqcup (W\cap U)$

$= \Big(\displaystyle\bigsqcup_{\substack{\gamma\in\Gamma(1)/\{\pm1\} \\ \gamma\mathcal{F}^\circ\subseteq U}} \gamma\mathcal{F}^\circ\Big)\sqcup (W\cap U)$

So, integrating gives $= \displaystyle\int_U f(\tau)\frac{dx\,dy}{y^2}$ ✓

§4: L-functions.

Normalised eigenforms can be used to construct L-funcs.

$\zeta(s) \equiv \sum_{n\geq 1} n^{-s}$ (Zeta function), converges absolutely in $\{s : Re(s) > 1\}$ (& holomorphic there).

Key properties:

- Euler product $\prod_{p\in\mathbb{P}} (1-p^{-s})^{-1}$, converges absolutely on $Re(s)>1$.
- Meromorphic continuation to $\mathbb{C}$. Simple pole at $s=1$, no other poles.
- Functional Equation. $\xi(s) \equiv \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$. ("$\xi$ func") $\Rightarrow \xi(s) = \xi(1-s)$.
- Special values of $\zeta(s)$ at $s\in\mathbb{Z}$. Has arithmetic meaning.

Other examples of such functions:

- Dirichlet L-function $L(\chi, s) \equiv \sum_{\substack{n\in\mathbb{N} \\ (n,N)=1}} \chi(n \bmod N)\, n^{-s}$ with $\chi : (\mathbb{Z}/N\mathbb{Z})^* \longrightarrow \mathbb{C}^\times$.
- $E/\mathbb{Q}$ elliptic curve $\Rightarrow L(E,s) = \sum_{n\geq 1} a_n \cdot n^{-s}$.

In general: L-function $\equiv$ Dirichlet series: $\sum_{n\geq 1} a_n \cdot n^{-s}$ $(a_n \in \mathbb{C})$ expecting to have properties similar to ~~the~~ zeta function.

$\boxed{\text{DEF 4.1}}$ Let $f(\tau) \equiv \sum_{n\geq 0} a_n q^n \in M_k(\Gamma(1))$. The associated Dirichlet series is: $L(f,s) \equiv \sum_{n\geq 1} a_n \cdot n^{-s}$.

Will consider separately: Eisenstein series, and cuspidal forms.

Let $F_k(\tau) \equiv$ normalised eigenform from G_k. $(k\geq 4$, even$)$

$$\Rightarrow L(F_k, s) = \sum_{n\geq 1} \sigma_{k-1}(n)\, n^{-s} = \sum_{n\geq 1} \sum_{d|n} d^{k-1} n^{-s}.$$

$$= \sum_{n\geq 1}\sum_{d|n} d^{k-1} d^{-s} \left(\frac{n}{d}\right)^{-s} = \sum_{a,d\geq 1} d^{k-1-s} a^{-s} = \underline{\zeta(s)\zeta(s+1-k)}.$$

<u>Lemma 4.2</u>) $f \in S_k(\Gamma(1))$. Then: $L(f,s)$ converges absolutely in $\{Re(s) > 1+\frac{k}{2}\}$, & is holomorphic there.

<u>Proof</u> From <u>Sheet 2</u>: $\exists C_f > 0$ s.t. $\forall n \geq 1$, $|a_n| \leq C_f \cdot n^{k/2}$.

<u>Claim</u>: $\forall \delta > 0$, $\sum_{n\geq 1} a_n \cdot n^{-s}$ converges absolutely & uniformly on $Re(s) > 1 + \frac{k}{2} + \delta$. (Enough, since uniform limit of holomorphic is holomorphic).

To prove: use Weierstrass M-test. Write: $s = \sigma + it$.

$\Rightarrow n^{-s} = \exp(-s\log(n)) \Rightarrow |n^{-s}| = n^{-\sigma}$.

If $\sigma > 1 + \frac{k}{2} + \delta$, then: $\sum_{n\geq 1} |a_n \cdot n^{-s}| \leq \sum_{n\geq 1} C_f \cdot n^{\frac{k}{2}} n^{-(1+\frac{k}{2}+\delta)}$

$= C_f \cdot \sum_{n\geq 1} n^{-(1+\delta)} < \infty$. ✓

Modular Form: <u>lecture 14.</u> 06/11/2023.

<u>From last time</u>: introduced L-functions $\sum_1 a_n \cdot n^{-s}$, with FE, AC & EP properties.

If ~~...~~ $f \equiv \sum_1 a_n \cdot q^n \in M_k(\Gamma(1))$, its Dirichlet series $L(f,s) \equiv \sum_1 a_n \cdot n^{-s}$.

We <u>showed</u>: if $f \in S_k(\Gamma(1))$ then $L(f,s)$ converges absolutely when $\mathrm{Re}(s) > 1 + k/2$.

[Under Ramanujan–Peterson: converge absolutely on $\mathrm{Re}(s) > \frac{1+k}{2}$.]

<u>Theorem 4.3</u> $f \in S_k(\Gamma(1))$.

1) $L(f,s)$ admits: analytic cont. to $\mathbb{C}$

2) If $\Lambda(f,s) \equiv (2\pi)^{-s}\Gamma(s)L(f,s) \Rightarrow \Lambda(f,s) = i^k \Lambda(f, k-s)$.

Warm-up: $\Gamma(s) = \int_0^\infty e^{-y} y^s \frac{dy}{y}$. (when integral converges absolutely)

<u>Prop 4.4</u> 1) $\Gamma(s)$ converges absolutely $\forall \mathrm{Re}(s) > 0$, & is holomorphic there.

2) $\Gamma(s)$ admits: meromorphic cont. to $\mathbb{C}$, simple poles $\mathbb{Z}_{\leq 0}$, & no ~~poles~~ other poles.

<u>Proof</u> $\Gamma(s)$ converges absolutely $\Leftrightarrow \int \left| e^{-y} y^s \frac{dy}{y} \right| < \infty$, which is true (elementary).

<u>Continuous</u>: for $N \geq 1$, let $\Gamma_N(s) \equiv \int_{1/N}^{N} e^{-y} y^s \frac{dy}{y}$. Then, <u>$\Gamma_N(s)$ continuous</u> on $\mathrm{Re}(s) > 0$.

[Why? know, $(y,s) \mapsto y^s : [\frac{1}{N}, N] \times \mathbb{C} \longrightarrow \mathbb{C}$ continuous.

$\Rightarrow \forall \varepsilon > 0$, $Re(s) > 0$, $\exists \delta > 0$ s.t. $\forall s' \in B_\delta(s)$, $|y^s - y^{s'}| < \varepsilon$.

$$\Rightarrow |\Gamma_n(s) - \Gamma_n(s')| \leq \int_{1/N}^{N} e^{-y} |y^s - y^{s'}| dy \leq \varepsilon \cdot C_N$$

$\Rightarrow$ Is continuous.]

To show $\Gamma_N(s)$ <u>holomorphic</u>: need, (Morera's theorem):

$\oint_\gamma f(z)dz = 0$ $\forall \gamma$ closed & continuous on $U \subseteq \mathbb{C}$.

& have:
$$\oint_\gamma \Gamma_N(s) = \oint_\gamma \int_{1/N}^{N} e^{-y} y^s \frac{dy}{y} = \underbrace{\int_{1/N}^{N} e^{-y} \underbrace{\oint_\gamma y^s \, ds}_{=0} \frac{dy}{y}}_{=0\ \checkmark}$$

$\Rightarrow \Gamma_N(s)$ holomorphic ✓

To show $\Gamma(s)$ holomorphic: show, $\Gamma_N(s) \longrightarrow \Gamma(s)$ <u>locally uniformly</u>.

<u>Will show</u>: Uniform convergence in $\{s \in \mathbb{C} : Re(s) \in [\sigma_0, \sigma_1]\}$ for $\forall\, 0 < \sigma_1 < \sigma_2$. ("vertical strips")

$$|\Gamma(s) - \Gamma_N(s)| \leq \int_0^{1/N} |y^s e^{-y}| \frac{dy}{y} + \int_N^\infty |y^s e^{-y}| \frac{dy}{y}$$

$$\leq \int_0^{1/N} y^{\sigma_0 - 1} e^{-y} dy + \int_N^\infty y^{\sigma_1 - 1} e^{-y} dy$$

$$\longrightarrow 0 \quad \text{Indep of } s\ \checkmark$$

For <u>2)</u>: use relation $\Gamma(s+1) = s\Gamma(s)$. (IBP)

$\Rightarrow$ use this (& induction) to <u>extend</u> Γ to meromorphic function on $\{Re(s) > -k\}$ $\forall k$. Description of poles also follow.

Proof of Theorem 4.3 (AC of $L(f,s)$ & FE of Λ)

Let: $F(s) \equiv \int_0^\infty f(iy) y^s \frac{dy}{y}$ ("Mellin Transform")

Claim: F converges absolutely $\forall s \in \mathbb{C}$ & is Holomorphic.

$$F(s) = \int_0^1 f(iy) y^{-s} \frac{dy}{y} + \int_1^\infty f(iy) y^{-s} \frac{dy}{y}.$$

Know: $|f(\tau)| \leq C_f |e^{2i\pi\tau}| \Rightarrow |f(iy)| \leqslant C_f e^{-2\pi y}$.

$\Rightarrow$ 2nd integral $< \infty$ $\forall s \in \mathbb{C}$.

& Since f modular form: $f(\tau) = f(-\frac{1}{\tau})\tau^{-k}$ ~~$\Rightarrow f(iy) = f(\frac{i}{y})y$~~

$$\Rightarrow f(iy) = f(\tfrac{i}{y})(iy)^{-k}$$

$\Rightarrow$ |1st integral| $= \left|\int_1^\infty f(\frac{i}{y}) y^{-s} \frac{dy}{y}\right| \leq \int_1^\infty |f(iy)| y^{k-s} \frac{dy}{y}$

$< \infty$ $\forall s \in \mathbb{C}$.

Holomorphic: trivial.

What is $F(s)$? $= \sum_{n=1}^\infty a_n \int_{y=0}^\infty e^{-2\pi i n y} y^{s} \frac{dy}{y}$ (Fubini)

(provided: $\sum |a_n| \int |e^{-2i\pi n y} y^{s}| \frac{dy}{y} < \infty$) (*)

$$= \sum_{n=0}^\infty a_n \int_0^\infty e^{-y} y^s (2\pi n)^{-s} \frac{dy}{y}$$

$$= \sum_{n=0}^\infty (2\pi)^{-s} a_n \cdot n^{-s} \int_{y=0}^\infty e^{-y} y^s \frac{dy}{y} = \underline{\Lambda(f,s)}.$$

(*) $= (2\pi)^\sigma \Gamma(\sigma) \sum_{n \geq 1} |a_n| n^{-\sigma} < \infty$. $\sigma \equiv \mathrm{Re}(s)$.

$|y^s| = y^\sigma$

$|n^{-s}| = n^{-\sigma}$

Conclusion: ~~F(s) holo ... then~~

Conclusion: $F(s)$ holomorphic & $= \Lambda(f,s)$ when $Re(s) > 1 + \frac{k}{2}$.

$\Rightarrow \Lambda(f,s)$ has analytic continuation to $\mathbb{C}$.

& $L(f,s) = \dfrac{\Lambda(f,s)}{(2\pi)^{-s}\Gamma(s)}$ analytic on $\mathbb{C}$ (since $\frac{1}{\Gamma}$ entire).

Have: $\Lambda(f,s) = \int_0^\infty f(iy)\, y^{-s} \frac{dy}{y}$

$= \int_1^\infty f(i/y)\, y^{-s} \frac{dy}{y} + \int_1^\infty f(iy) y^s \frac{dy}{y}$.

$= \int_1^\infty f(iy)\left(i^k y^{k-s} + y^s\right) \frac{dy}{y}$ (using: $f(\frac{i}{y}) = f(iy)\,(iy)^k$)

$\Rightarrow$ **Get**: $\underline{\Lambda(f,s)\, i^k = \Lambda(f, k-s)}$ ✓

Theorem 4.5) Let $f \in S_k(\Gamma(1))$. ~~Then~~ Normalised eigenform.

$\Rightarrow L(f,s) \equiv \sum_{n \geq 1} a_n n^{-s} = \prod_{p \text{ prime}} \left(1 - a_p p^{-s} + p^{k-1-2s}\right)^{-1}$.

Interpreted either: as formal Dirichlet series, or as complex numbers provided Dirichlet series converges absolutely.

Proof By ex. sheet 3: enough to consider formal identity.

Know: $a_{mn} = a_m a_n$ $\forall m,n$ coprime.

$\Rightarrow \sum_{n \geq 1} a_n \cdot n^{-s} = \prod_p \left(1 + a_p p^{-s} + a_{p^2} p^{-2s} + \dots\right)$

$= \prod_p \left(1 - a_p p^{-s} + p^{k+1-2s}\right)^{-1}$

Since: equivalent to $a_{p^{n+1}} = a_p a_{p^n} - p^{k-1} a_{p^{n-1}}$ ✓

Modular Forms: Lecture 15 08/11/2023.

From last time If $f \in S^k(\Gamma(1))$ normalised ~~eigenvalue~~ eigenform, then: $\mathcal{L}(f,s) \equiv \sum a_n(f)\cdot n^{-s}$ is L-function, and has analytic cont. to $\mathbb{C}$, has FE & EP.

Today: Application of L-funcs to distribution of primes. (and related quantities).

Theorem 4.6) (Weiner -

Consider: $f(s) \equiv \sum a_n \cdot n^{-s}$ converging absolutely for $\mathrm{Re}(s) > 1$.

($\Rightarrow$ Holomorphic in this region)

& Suppose: f has meromorphic continuation to open nbhood of $\{\mathrm{Re}(s) \geq 1\}$, and holomorphic on $\{\mathrm{Re}(s) = 1\}$ except possibly at $s=1$. (simple pole, residue α)

Then: $\sum_{1 \leq n \leq N} a_n = (\alpha + o(1)) N$ (as $N \to \infty$).

Prop 4.7) Suppose: $\zeta(s) \equiv \sum n^{-s}$ has meromorphic cont. to open nbhood of $\{\mathrm{Re}(s) \geq 1\}$, holomorphic & non-vanishing at $\{\mathrm{Re}(s) = 1\}$ except simple pole at $s=1$.

Then: PNT holds: $\pi(X) = \frac{X}{\log(X)} + o\left(\frac{X}{\log X}\right)$.

Proof Taylor exp. of $-\log(1-z)$ is: $\sum_{n \geq 1} \frac{z^n}{n}$ ($|z| < 1$).

A branch of $\mathrm{Log}\, \zeta(s) \equiv \mathrm{Log} \prod_p (1-p^{-s})^{-1}$ is given by:

$$\sum_p \log(1-p^{-s})^{-1} = \sum_p \sum_{n\geq 1} \frac{p^{-ns}}{n} \quad \text{Absolutely conv } \forall \operatorname{Re}(s)>1.$$

$$\Rightarrow \frac{-\zeta'(s)}{\zeta(s)} = -\sum_p \sum_{n\geq 1} \frac{d}{ds}\left(\frac{p^{-ns}}{n}\right) = \sum_p \sum_{k\geq 1} (\log p) p^{-ks}.$$

$$= \sum_p (\log p) p^{-s} + \underbrace{\sum_p \sum_{n\geq 2} (\log p) p^{-ns}}_{\text{Absolutely conv for } \operatorname{Re}(s) > \frac{1}{2}.}$$

also: $-\dfrac{\zeta'(s)}{\zeta(s)} = -\dfrac{d}{ds}\log(\zeta(s))$.

$\Rightarrow$ If $\zeta(s)$ has pole/zero (order k) at s_0 then $-\dfrac{\zeta'(s)}{\zeta(s)}$ has simple pole of order k.

By assumption on $\zeta(s)$: get: $\dfrac{-\zeta'(s)}{\zeta(s)}$ has ~~nonsingular~~ meromorphic cont. to $\{\operatorname{Re}(s)=1\}$, except for simple pole at $s=1$ (residue 1).

$\Rightarrow \sum_p (\log p) p^{-s}$ has meromorphic cont. to nbhood of $\operatorname{Re}(s)\geq 1$, holomorphic on $\{\operatorname{Re}(s)=1\}$ except for simple pole of residue 1 ($s=1$).

$\Rightarrow$ By Theorem 4.6: $\underbrace{\sum_{p\leq X} \log(p)}_{\theta(X).} = X + o(X) \quad (X\to\infty).$

Lemma 4.8 (Partial summation).

Let $(a_n)_{n\geq 0}$ sequence of complex numbers, & $0<X<Y$ real, $f:[X,Y]\longrightarrow\mathbb{R}$ is C^1.

Define $A(t)\equiv\sum_{0\leq n\leq t} a_n$. Then: $\sum_{X\leq n\leq Y} a_n f(n) = A(Y)f(Y) \overset{-A(X)f(X)}{\downarrow} - \int_{t=X}^{Y} A(t)f'(t)\,dt$.

Back to proof: $a_n \equiv \begin{cases} 0, & \text{not prime} \\ \log p, & n=p \text{ prime.}\end{cases}$ & $f(t)\equiv 1/\log(t)$.

$$\Rightarrow \pi(X) = 1+\sum_{e\leq n\leq X} \mathbb{1}_{n \text{ prime}} = 1+\sum_{e<n\leq X} a_n f(n).$$

$$= 1 + A(X)f(X) - A(e)f(e) + \int_{t=e}^{X} \frac{A(t)}{t(\log t)^2}\,dt.$$

$$= \frac{X}{\log X} + o\left(\frac{X}{\log X}\right) + \int_{t=e}^{X}\frac{\theta(t)}{t(\log t)^2}\,dt.$$

& $$\int_{t=e}^{X}\frac{\theta(t)}{t((\log(t))^2}\,dt \leq C\int_{t=e}^{X}\frac{1}{(\log(t))^2}\,dt \quad (\text{some } C)$$

$$= C\left[\int_{t=e}^{\sqrt{X}} + \int_{t=\sqrt{X}}^{X} \frac{1}{\log(t)^2}\,dt\right] \leq C\left(\sqrt{X} + \frac{X}{\log(\sqrt{X})^2}\right) = o\left(\frac{X}{\log X}\right)$$

✓

Will establish all required properties of $\zeta(s)$ (later in course).

Theorem 4.9 Fix $n\geq 1$ & suppose: given $\forall p$, matrix $\Phi_p \in M_{n\times n}$ which is either 0 or all evals $|\lambda|=1$.

Define: $\mathcal{L}(\{\Phi_p\}, s) \equiv \prod_p \left(\det(\mathbb{1}_n - p^{-s}\Phi_n)\right)^{-1}$.

Then: $\mathcal{L}(\{\Phi_p\}, s)$ absolutely convergent $\forall \operatorname{Re}(s)>1$.

If furthermore $\mathcal{L}(\{\Phi_p\} s)$ admits meromorphic cont. to open nbhood of $\{\operatorname{Re}(s)\geq 1\}$, holomorphic & nonzero on $\operatorname{Re}(s)=1$ (except possibly at $s=1$, ~~simple~~ pole, $\exists$ order δ)

Then: $\displaystyle\sum_{p\leq X} \operatorname{tr}(\Phi_p) = \delta \frac{X}{\log X} + o\left(\frac{X}{\log X}\right)$.

Proof: sheet 3 & generalisation of $\Phi_p = \mathbb{1}_n \ \forall p$ (Zeta func).

Example (Dirichlet's Theorem on AP's).

Fix: $N\geq 2$ & $\forall \chi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^\times$ hom, consider:

$$L(\chi, s) \equiv \sum_{(n,N)=1} \chi(n \bmod N) n^{-s} = \prod_{p\nmid N} (1-\chi(p \bmod N)p^{-s})^{-1}$$

"Dirichlet L-functions".

Fact: Theorem 4.9 apply to $L(\chi,s)$

$$\Rightarrow \sum_{p\leq X} \chi(p \bmod N) = \operatorname{ord}_{s=1} L(\chi,s) \frac{X}{\log X} + o\left(\frac{X}{\log X}\right).$$

Can show: $\operatorname{ord}_{s=1} L(\chi,s) = \begin{cases} -1 & \text{if } \chi \text{ trivial} \\ 0 & \chi \text{ non-trivial.} \end{cases}$

If $a\in (\mathbb{Z}/N\mathbb{Z})^*$ then $\mathbb{1}_{a \bmod N}(g) = \frac{1}{\phi(N)} \sum_{\chi} \overline{\chi}(n)\chi(g)$.

$$\Rightarrow \sum_{\substack{p\leq X\\ p\nmid N}} \mathbb{1}_{a \bmod N}(p) = \frac{1}{\phi(N)} \sum_{\chi} \overline{\chi(n)} \cdot \sum_{p\leq N} \chi(p \bmod N)$$

$$= \frac{1}{\phi(N)} \frac{X}{\log X} + o\left(\frac{X}{\log X}\right).$$

Modular Forms: lecture 16 10/11/2023.

From last time: for $n \geq 1$ & p prime, $\Phi_p \in M_n(\mathbb{C})$ with either $\Phi_p = 0$ or all eigenvalues λ. $|\lambda| = 1$. Then:

$L(\{\Phi_p\}, s) \equiv \prod_p \det(\mathbb{1}_n - \Phi_p \cdot p^{-s})^{-1}$ converges absolutely $\forall \mathrm{Re}(s) > 1$.

& If this has meromorphic cont. to nbhood of $\{\mathrm{Re}(s) \geq 1\}$ holo. & $\neq 0$ on $\{\mathrm{Re}(s) = 1\}$ except possibly simple pole at $s=1$, of order $\delta \geq 0$, then: $\sum_{p \leq X} tr(\Phi_p) \sim \delta \frac{X}{\log X}$.

Examples $n=1$, $\Phi_p = 1$ $\forall p$ $\Rightarrow L = \zeta$ (Zeta func) $\Rightarrow$ PNT.

$n=1$, $\Phi_p = \chi(p \bmod N)$ $\forall p \nmid N$. $\Rightarrow L = L(\chi, s)$. $\Rightarrow$ Dirichlet.

$f \in S_k(\Gamma(1))$ eigenform. (normalised). $\Rightarrow L(f,s) = \prod_p (1 - a_p p^{-s} + p^{k+1-2s})^{-1}$.

& $1 - a_p X + p^{k-1} X^2 = (1-\alpha_p X)(1 - \beta_p X)$

& $\Phi_p \equiv \begin{pmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{pmatrix}$ "Satake parameter of f at p".

$$\Rightarrow L(\{\Phi_p\}, s) = \prod_p \det \begin{pmatrix} 1-\alpha_p p^{-s} & 0 \\ 0 & 1-\beta_p p^{-s} \end{pmatrix}^{-1}$$

$$= \prod_p \left((1-\alpha_p p^{-s})(1-\beta_p p^{-s}) \right)^{-1}$$

$$= L(f,s).$$

If Ramanujan's 3rd conjecture holds: $|\alpha_p| = |\beta_p| = p^{\frac{k-1}{2}}$.

$\Rightarrow L(\{ p^{\frac{1-k}{2}} \Phi_p\}, s)$ fits into theorem.

$\underline{\underline{\&}} = \prod_p \left((1-\alpha_p p^{-s} p^{\frac{-(k-1)}{2}})(1-\beta_p p^{-s} p^{\frac{-(k-1)}{2}}) \right)^{-1} = L(f, s+\frac{k-1}{2})$.

<u>Corollary 4.10</u> Suppose $f \in S_k(\Gamma(1))$ normalised eform. Then, if R-P conj holds for f $\underline{\underline{\&}}$ $L(f,s)$ non-vanishing on $\{Re(s) = \frac{k+1}{2}\}$. <u>Then</u>: $\lim_{X \to \infty} \left(\sum_{p \leq X} a_p(f) / p^{\frac{k-1}{2}} \right) / \pi(X) = 0$.

$\Rightarrow$ Average val. of $\frac{a_p}{p^{\frac{k-1}{2}}} \in [-2,2]$ is 0.

<u>Recall</u>: p odd prime $\underline{\underline{\&}}$ $r_{24}(p) = \frac{16}{691}(1+p^{11}) + \frac{33152}{691}\tau(p)$.

$\Rightarrow$ If hypothesis of corollary hold: then avg val. of

$$\frac{r_{24}(p) - \frac{16}{691}(1+p^{11})}{p^{11/2}} \text{ is } 0.$$

<u>In fact</u>: Can go much further! Introduce family of L-funcs associated with normalised eigenform f.

For $n \geq 1$, define: $L(f, \mathrm{Sym}^n, s) \equiv L(\{\mathrm{Sym}^n \Phi_p\}, s)$

$$= \prod_p \prod_{i=0}^{n} (1-\alpha_p^i \beta_p^{n-i} p^{-s})^{-1}.$$

$\underline{\underline{\&}}$ $\mathrm{Sym}^n : GL_2 \to GL_{n+1}$ n'th symmetric power of standard representation.

A priori: these converge absolutely on some right half plane, and: for $n=1$, $L(f, \mathrm{Sym}^1, s) = L(f,s)$.

$\boxed{R}$

Prop 4.11 (Langlands 1967) If $\forall n \geq 1$, $L(f, \mathrm{Sym}^n, s)$ admits analytic cont. to $\mathbb{C}$, then: R-P holds for f.

Prop 4.12 (Serre) If R-P holds for f & $\forall n \geq 1$, $L(f, \mathrm{Sym}^n, s)$ admits analytic cont. which is nonvanishing on $\{\mathrm{Re}(s) = 1 + \frac{n(k-1)}{2}\}$, then: Sato-Tate conjecture hold for f:

the numbers $\frac{a_p(f)}{2p^{\frac{k-1}{2}}} \in [-1,1]$ equi-distributed WRT the sato-tate density. $\frac{2}{\pi}\sqrt{1-t^2}\,dt$.

& $\forall g \in C[-1,1]$: $\lim_{X \to \infty} \frac{1}{\pi(X)} \sum_{p \leq X} g\left(\frac{a_p}{2p^{\frac{k-1}{2}}}\right) = \int_{-1}^{1} g(t) \frac{2}{\pi}\sqrt{1-t^2}\,dt.$

$\Rightarrow \frac{691}{66304}\left(r_{24}(p) - \frac{16}{691}(1+p^{11})\right)\frac{1}{p^{11/2}} \sim$ distributed to the density $\frac{2}{\pi}\sqrt{1-t^2}\,dt$.

Now know: $L(f, \mathrm{Sym}^n, s)$ do have the required properties.

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§5: Modular Forms on Congruent Subgroups of $\Gamma(1)$.

DEF 5.1 $\Gamma \leq \Gamma(1)$ congruent subgroup $\Leftrightarrow$ Contains $\ker(SL_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{Z}/N\mathbb{Z}))$ for some $N \geq 1$.

Main examples $\Gamma(N) = \uparrow$

$\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \bmod N\right\}$

$\Gamma_1(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \bmod N \ \&\ a \equiv d \equiv 1 \bmod N\right\}$.

Remark If $\Gamma \leq \Gamma(1)$ congruence subgroup then $[\Gamma(1) : \Gamma]$ finite (since: $|\Gamma(1) : \Gamma(N)|$ finite $\forall N$).

Many of most interesting modular forms only exist at $\Gamma < \Gamma(1)$.

<u>Examples</u>: ⊛ θ-function of Λ.

⊛ Normalised eigenforms assoc. to $E/\mathbb{Q}$.

(defined on $\Gamma_0(N_E)$, $N_E \equiv$ "conductor" of E)

<u>DEF 5.2</u>] $k \in \mathbb{Z}$, $\Gamma \leq \Gamma(1)$ congruent subgroup. A <u>weakly modular function (weight k, level Γ)</u> is: meromorphic f on H such that $\forall \gamma \in \Gamma$, $f|_k[\gamma] = f$.

<u>FACT</u>: $\mathcal{F}_0(2) \equiv \{\tau \in H : Re(\tau) \in [0,1]$ & $|\tau - \frac{1}{2}| \geq \frac{1}{2}\}$ is (closure of) fundamental set of $\Gamma_0(2) \curvearrowright H$.

$\Rightarrow$ More than 1 way to "go to ∞".

<u>DEF 5.3</u>] $\Gamma \leq \Gamma(1)$ cong subgroup. A <u>cusp</u> of Γ is: a <u>Γ-orbit of $\mathbb{P}^1(\mathbb{Q})$</u>.

$$\begin{array}{ccccc} GL_2(\mathbb{C}) & \curvearrowright & \mathbb{P}^1(\mathbb{C}) = & \mathbb{C} & \cup \infty \\ \vee\!\! / & & \cup\!\! / & & \cup\!\! / \\ GL_2(\mathbb{Q}) & \curvearrowright & \mathbb{P}^1(\mathbb{Q}) = & \mathbb{Q} & \cup \infty \\ \vee\!\! / & & & & \\ \Gamma & & & & \end{array}$$

<u>Lemma 5.4</u> $\Gamma(1)$ has a unique cusp.

<u>Proof</u> <u>Need</u>: $\Gamma(1)$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$. [<u>STP</u>: $\forall \frac{a}{c} \in \mathbb{Q}$ conj to ∞.]

$a, c \in \mathbb{Z}$, $\gcd(a,c) = 1$, $c \in \mathbb{N}$. $\Rightarrow \exists r, s : ar + cs = 1$.

& $\gamma \equiv \begin{pmatrix} a & -s \\ c & r \end{pmatrix} \in \Gamma(1)$. Then $\gamma\infty = \frac{a\infty - s}{c\infty + r} = \frac{a}{c}$ ✓

$\Rightarrow$ Transitive.

Modular Forms: lecture 17. 13/10/2023.

From last time: $\Gamma \leq \Gamma(1)$ congruence subgroups $\iff \Gamma(N) \subseteq \Gamma$ for some $N \in \mathbb{N}$. Also defined cusp of Γ to be Γ-orbit in $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$.

& Proved: $\Gamma(1)$ has a unique cusp.

Corollary 5.5 If Γ congruence $\Rightarrow$ Finitely many Cusps.

Proof By Orbit-Stabiliser: $\exists$ $\Gamma(1)$-equivalent bijection

$$\Gamma(1)/\Gamma_\infty \xrightarrow{\sim} \mathbb{P}'(\mathbb{Q}) \qquad \text{where: } \Gamma_\infty \equiv \mathrm{stab}_{\Gamma(1)}(\infty)$$

$$\gamma\Gamma_\infty \longmapsto \gamma(\infty) \qquad = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Gamma(1) \right\}.$$

If $\Gamma \leq \Gamma(1)$ congruence: $\exists$ induced bijection:

$$\boxed{\Gamma \backslash \Gamma(1) / \Gamma_\infty \xrightarrow{\sim} \Gamma \backslash \mathbb{P}'(\mathbb{Q})} \quad (\text{double cosets})$$

& $\Gamma\backslash\Gamma(1)/\Gamma_\infty$ finite (since: set of right Γ_∞-orbits on $\Gamma\backslash\Gamma(1)$, and $\Gamma\backslash\Gamma(1)$ is finite. ✓

Idea: $Y(\Gamma) \equiv \Gamma\backslash H$ is non-compact RS, which we can compactify by adding finitely many pts corresponding to the cusps in $\Gamma\backslash\mathbb{P}'(\mathbb{Q})$.

We know: how to define charts around ∞, & deal with general case by transforming to this.

Let: f weakly modular func, weight k level Γ. The index $[\Gamma_\infty : \Gamma \cap \Gamma_\infty]$ finite, since: if $\Gamma(N) \subseteq \Gamma$, then $\boxed{1}$

$\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \Gamma_\infty \cap \Gamma.$

Define: width of ∞ (as a cusp of Γ) as $\min\left(h \geq 1 : \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma \cap \Gamma_\infty\right)$.

If h width: $f|_k\left[\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}\right] = f(\tau + h)$. (weight k)
$= f(\tau)$ (f has level Γ).

By same argument for $\Gamma = \Gamma(1)$: $\exists!$ meromorphic $\tilde{f}$ on $D_*(0,1)$ s.t. $f(\tau) = \tilde{f}(e^{2\pi i \tau / h})$. Define: f is:

$$\left.\begin{array}{l} \text{mero. at } \infty \\ \text{holo. at } \infty \\ \text{vanishes at } \infty \end{array}\right] \iff \left[\begin{array}{l} \tilde{f} \text{ extends to mero. fn. } \cancel{\text{on}} \text{ to } D_*(0,1) \\ f \text{ mero. at } \infty \ \underline{\&}\ 0 \text{ removable sing. of } \tilde{f}. \\ \tilde{f}(0) = 0 \ \underline{\&}\ f \text{ holo at } \infty. \end{array}\right.$$

If f mero. at ∞: has q-expansion $f(\tau) = \sum_{-\infty}^{\infty} a_n q_h^n$, where $q_h \equiv e^{2i\pi\tau/h}$ $\underline{\&}$ derived from Laurent exp. of $\tilde{f}$.

($\Rightarrow$ Absolutely convergent in $\{\tau : \operatorname{Im}(\tau) > R\}$ for some $R > 0$, and finitely many a_n, $n < 0$, nonzero.)

For general cusp $\Gamma \cdot z$ ($z \in \mathbb{P}^1(\mathbb{Q})$): choose: $\underline{\alpha \in \Gamma(1)}$ s.t. $\underline{\alpha \infty = z}$ (transitive). Say f is:

$$\left[\begin{array}{l} \text{mero.} \\ \text{holo.} \\ \text{vanishing} \end{array}\right] \text{ at } \Gamma \cdot z \text{ if } f|_k[\alpha] \text{ is } \left[\begin{array}{l} \text{mero.} \\ \text{holo.} \\ \text{vanishing} \end{array}\right] \text{ at } \infty,$$

when we consider $f|_k[\alpha]$ as weakly modular function of weight k, level $\alpha^{-1}\Gamma\alpha$.

[$\alpha^{-1}\Gamma\alpha$ is congruence subgroup since $\Gamma(N) \trianglelefteq \Gamma(1)$.]

$((f|_k[\alpha])|_k[\alpha^{-1}\gamma\alpha] = f|_k[\alpha\alpha^{-1}\gamma\alpha] = f|_k[\gamma\alpha] = f|_k[\alpha].)$

<u>Lemma 5.6</u>) The property of being holo/mero/vanishing at $\Gamma\cdot z$ is <u>indep</u> of choice α, $\alpha\cdot\infty = z$, <u>and</u> choice of z.

<u>Proof</u> First show: choice of α doesn't matter.

If $\alpha, \beta \in \Gamma(1)$ & $\alpha\infty = \beta\infty = z$ then $\beta = \alpha\delta$, for some $\delta \in \mathrm{Stab}_{\Gamma(1)}\infty. = \Gamma_\infty$.

$\Rightarrow \delta = \pm\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$, $m \in \mathbb{Z}$, $f|_k[\beta] = f|_k[\alpha]|_k[\delta]$.

$= f|_k[\alpha](\tau+n).(-1)^k$.

Want to show: $f|_k[\alpha]$ holomorphic $(\alpha^{-1}\Gamma\alpha)$ at $\infty \Leftrightarrow f|_k[\beta]$ holomorphic $(\beta^{-1}\Gamma\beta)$ at ∞.

<u>Claim</u> Width of cusp ∞ for $\alpha^{-1}\Gamma\alpha$
$=$ Width of cusp ∞ for $\beta^{-1}\Gamma\beta$.

LHS $= \min\left\{h \geq 1 : \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \alpha^{-1}\Gamma\alpha \cap \Gamma_\infty\right\}$.

& $\beta^{-1}\Gamma\beta \cap \Gamma_\infty = \beta^{-1}(\Gamma \cap \beta\Gamma_\infty\beta^{-1})\beta = \delta^{-1}\alpha^{-1}(\Gamma \cap \alpha\overset{\delta}{\vee}\Gamma_\infty\overset{\delta^{-1}}{\vee}\alpha^{-1})\alpha\delta$

$\underset{\substack{\Uparrow \\ \delta\in\Gamma_\infty}}{=} \delta^{-1}\alpha^{-1}(\Gamma \cap \alpha\Gamma_\infty\alpha^{-1})\alpha\delta = \delta^{-1}(\alpha^{-1}\Gamma\alpha \cap \Gamma_\infty)\delta \underset{\substack{\Uparrow \\ \Gamma_\infty \text{ abelian.}}}{=} \alpha^{-1}\Gamma\alpha \cap \Gamma_\infty$.

$\Rightarrow \widetilde{f|_k[\alpha]}(e^{2i\pi\tau/h}) = f|_k[\alpha](\tau)$

& $\widetilde{f|_k[\beta]}(e^{2i\pi\tau/h}) = f|_k(\beta)(\tau) = f|_k[\alpha](\tau+m)(-1)^k$
$= (-1)^k\widetilde{f|_k[\alpha]}(e^{2i\pi\tau/h}e^{2i\pi m/h})$.

$\Rightarrow f|_k[\alpha]$ holo. at $\cancel{\infty}$ 0 $\Leftrightarrow f|_k[\beta]$ holo at 0 etc ✓

<u>Next</u> need independence of z. If $\Gamma z = \Gamma z'$ $(z, z' \in \mathbb{P}^1(\mathbb{Q}))$

<u>Write</u>: $z' = \gamma z$, $\gamma \in T$. If $\alpha \in \Gamma(1)$: $\alpha\infty = z$, then: $(\gamma\alpha)\infty = z'$

Need to show: $f|_k[\alpha]$ holo. at $\infty \Leftrightarrow f|_k[\gamma\alpha]$ holo. at ∞.

Is true, since: $f|_k[\gamma\alpha] = f|_k[\alpha]$ & $\alpha^{-1}\Gamma\alpha =$ [redacted] $(\gamma\alpha)^{-1}\Gamma(\gamma\alpha)$
(since $\gamma \in \Gamma$.)

Can define: ~~cusp of~~ width of cusp $\Gamma \cdot z$ to be the width of ∞ as cusp of $\alpha^{-1}\Gamma\alpha$ (follows from proof).

DEF 5.7 Let: f weakly modular, weight k level Γ.
Say f modular function if meromorphic at EVERY cusp of Γ.
& f modular form if f holo on H & at every cusp of Γ.
& f cuspidal modular form if modular form & 0 at every cusp.

Notation $M_k(\Gamma) \equiv \mathbb{C}$-VS of modular forms, weight k level Γ.

$S_k(\Gamma) \equiv \mathbb{C}$-VS of cuspidal modular forms.

<u>Exercise</u>: If f weakly modular func, holo. in $\mathbb{H}^2$, then f is modular form $\Leftrightarrow \forall \alpha \in \Gamma(1)$, $\exists R > 0$, s.t. $f|_k[\alpha]$ bdd in $\{\operatorname{Im} \tau > R\}$.

Lemma 5.8 $k, \ell \in \mathbb{Z}$ & $\Gamma \leq \Gamma(1)$ congruent. Then:

1) $\forall f \in M_k(\Gamma)$, $g \in M_\ell(\Gamma) \Rightarrow fg \in M_{k+\ell}(\Gamma)$
2) If $\Gamma' \leq \Gamma$ congruent & $f \in M_k(\Gamma) \Rightarrow f \in M_k(\Gamma')$
3) If $\Gamma' \leq \Gamma(1)$ congruent & $\alpha \in GL_2(\mathbb{Q})^+$, $\Gamma' \leq \alpha^{-1}\Gamma\alpha$, and $f \in M_k(\Gamma)$, then: $f|_k[\alpha] \in M_k(\Gamma')$.

Modular Forms: lecture 18. 15/11/2023.

From last time: Defined: $M_k(\Gamma)$ & $S_k(\Gamma)$ for $\Gamma \leq \Gamma(1)$ congruence.

& Defined: f holo. on H, weakly modular of weight k & level Γ.

& $f \in M_k(\Gamma) \iff \forall \alpha \in \Gamma(1): f|_k[\alpha](\tau)$ bdd (as $\mathrm{Im}\,\tau \to \infty$).

~~Shown~~ lemma 5.8:

i) $f \in M_k(\Gamma)$, $g \in M_l(\Gamma) \Rightarrow f * g \in M_{k+l}(\Gamma)$

ii) If $\Gamma' \leq \Gamma$ congruent $\Rightarrow M_k(\Gamma') \subseteq M_k(\Gamma)$

iii) If $\Gamma' \leq \Gamma$ congruent $\Rightarrow \forall \alpha \in GL_2(\mathbb{Q})^+$: if $\alpha^{-1}\Gamma\alpha \geq \Gamma'$ then $\forall f \in M_k(\Gamma)$, $f|_k[\alpha] \in M_k(\Gamma')$.

Proof i) Follows from def, as in case $\Gamma = \Gamma(1)$.

ii) Follows from iii), for $\alpha = 1$.

iii) $f|_k(\alpha)$ holo. on H & weakly modular, level Γ'.

For $\gamma' \in \Gamma'$: $f|_k(\alpha)|_k(\gamma') = f|_k(\alpha\gamma'\alpha^{-1}\alpha]$

$= f|_k[\alpha\gamma'\alpha^{-1}]|_k(\alpha] = f|_k[\alpha]$ (since: $\alpha\gamma'\alpha^{-1} \in \Gamma$)

Need: $\forall \beta \in \Gamma(1)$: $f|_k[\alpha\beta](\tau)$ bounded as $\mathrm{Im}(\tau) \to \infty$.

Know: $\alpha\beta \in GL_2(\mathbb{Q})^+$, which acts on $\mathbb{P}^1(\mathbb{Q})$

& $\Gamma(1) \leq GL_2(\mathbb{Q})^+$ acts transitively.

$\Rightarrow \exists \gamma \in \Gamma(1)$ s.t. $\alpha\beta\infty = \gamma\infty$, so $\exists \delta \in \mathrm{Stab}_{GL_2(\mathbb{Q})^+}(\infty)$
$\alpha\beta = \gamma\delta$.

$\Rightarrow \delta = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ some $a, b, d \in \mathbb{Q}$. $ad > 0$.

$\Rightarrow f|_k[\alpha p](\tau) = f|_k[\gamma\delta](\tau) = f|_k[\gamma]|_k[\delta](\tau)$

$= f|_k[\gamma]\left(\frac{a\tau+b}{d}\right) \cdot d^{-k}(ad)^{k-1}$

<u>Know</u>: $f|_k[\gamma](\tau)$ <u>bdd</u> as $\mathrm{Im}(\tau) \to \infty$.

Say: bounded on $\{\tau \in \mathbb{C} : \mathrm{Im}(\tau) \geq R\}$.

$\Rightarrow f|_k[\gamma]\left(\frac{a\tau+b}{d}\right)$ <u>bdd</u> on $\{\tau \in \mathbb{C} : \mathrm{Im}(\tau) \geq \frac{d}{a} R\}$ ✓

<u>Corollary 5.9</u> Suppose: $M, d \in \mathbb{N}$ & $N = dM$. Then: If $f \in M_k(\Gamma_0(M))$ then $f(d\tau) \in M_k(\Gamma_0(N))$.

<u>Proof</u> Recall: $\Gamma_0(M) = \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} : C \equiv 0 \bmod M\right\}$.

By lemma: if $f \in M_k(\Gamma_0(M))$ then $f|_k\begin{bmatrix} d & 0 \\ 0 & 1 \end{bmatrix} \in M_k(\Gamma')$,

for any $\Gamma' \leq \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Gamma_0(M) \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow f|_k\left[\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}\right](\tau) = f(d\tau)\, d^{k-1} \in M_k(\Gamma') \Leftrightarrow f(d\tau) \in M_k(\Gamma')$

<u>Claim</u>: $\Gamma_0(N) \leq \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Gamma_0(M) \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$

$\Leftrightarrow \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0(N) \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}^{-1} \leq \Gamma_0(M)$.

Since: $\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} A & dB \\ d^{-1}C & D \end{pmatrix}$.

$\Rightarrow$ Have: $C \equiv 0 \bmod N \Leftrightarrow d^{-1}C \equiv 0 \bmod M$ ✓

<u>Example</u> If $k \geq 4$ even then: $M_k(\Gamma_0(N))$ contains: $G_k(d\tau)$ $\forall d | N$.

Now: construct more modular forms, using θ-functions.

e.g. $\theta(\tau) \equiv \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau} = \sum_{n\in\mathbb{Z}} q_2^{n^2}$ $\quad (q_2 \equiv e^{i\pi\tau})$

The power series definitely convergent for $|q_2|<1$, so θ is holomorphic in H.

Will show: certain powers of θ are modular forms.

For $k\in\mathbb{Z}$: $\theta^k = \sum_{n\in\mathbb{Z}} r_k(n)\, q_2^n$, where: $r_k(n) = \#\{\underline{x}\in\mathbb{Z}^k : \|\underline{x}\|^2 = n\}$

Prop 5.10 (Poisson Summation)

Consider: $f:\mathbb{R}\to\mathbb{C}$ cts, $\exists C,\delta>0$, s.t. $|f(t)| \le \frac{C}{(1+|t|)^{\delta+1}}$.

Let: $\hat{f}(s) \equiv \int_{-\infty}^{\infty} f(t)e^{-2i\pi s t}\,dt$. & suppose $\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty$

Then: $\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$.

Proof Define $F:\mathbb{R}\to\mathbb{C}$, $F(t) \equiv \sum_{n\in\mathbb{N}} f(n+t)$. Is uniformly conv. on any $[a,b]$.

$\Rightarrow$ F continuous & $\mathbb{Z}$-periodic.

& Define $\hat{F}:\mathbb{R}\to\mathbb{C}$, $\hat{F}(t) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi i n t}$. Also uniformly conv (since $\sum_{n\in\mathbb{Z}}|\hat{f}(n)|<\infty$), so $\hat{F}$ is continuous & periodic.

Claim: $F = \hat{F}$ (enough, since plug $t=0$).

Will show: $\forall n$, $\int_0^1 F(t)e^{-2i\pi n t}\,dt = \int_0^1 \hat{F}(t)e^{-2i\pi n t}\,dt$.

$$LHS = \int_0^1 \sum_{n\in\mathbb{Z}} f(n+t) e^{-2i\pi m t}\, dt \,\cdot = \sum_{n\in\mathbb{Z}} \int_0^1 f(n+t) e^{-2i\pi m t}\, dt.$$

$$= \int_{-\infty}^{\infty} f(t) e^{2i\pi m t}\, dt = \hat{f}(m).$$

$$RHS = \int_0^1 \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{2i\pi(n-m)t}\, dt = \sum_{n\in\mathbb{Z}} \int_0^1 \hat{f}(n) e^{2i\pi(n-m)t}\, dt$$

$$= \hat{f}(m) \checkmark$$

Apply to: $f_y(t) \equiv e^{-\pi t^2 y}$ ($y>0$ fixed).

$$\Rightarrow \theta(iy) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 y} = \sum_{n\in\mathbb{Z}} f_y(n).$$

$$\& \quad \hat{f_y}(s) = \int_{-\infty}^{\infty} e^{-\pi t^2 y} e^{-2\pi i s t}\, dt \,\cdot = \int_{-\infty}^{\infty} e^{-\pi\left(t\sqrt{y} + \frac{is}{\sqrt{y}}\right)^2} e^{-\frac{\pi s^2}{y}}\, dt$$

$$= e^{-\pi s^2/y} \frac{1}{\sqrt{y}} \int_{\mathbb{R}} e^{-\pi(x + is/\sqrt{y})^2}\, dx.$$

$$= \frac{1}{\sqrt{y}} e^{-\pi s^2/y} \int_{-\infty + is/\sqrt{y}}^{\infty + is/\sqrt{y}} e^{-\pi x^2}\, dx. \quad \text{\& Move the contour!}$$

$$= \frac{1}{\sqrt{y}} e^{-\pi s^2/y} = \frac{1}{\sqrt{y}} f_{y^{-1}}(s).$$

$$\Rightarrow \text{By Poisson: } \theta(iy) = \sum_{n\in\mathbb{Z}} f_y(n) = \sum_{n\in\mathbb{Z}} \hat{f_y}(n) = \frac{1}{\sqrt{y}} \sum_{n\in\mathbb{Z}} f_{y^{-1}}(n)$$

$$= \frac{1}{\sqrt{y}} \sum_{n\in\mathbb{Z}} e^{-\pi n^2/y} = \frac{\theta(1/y)}{\sqrt{y}}.$$

$\Rightarrow$ Functions $\theta(y)$ & $\frac{\theta(1/y)}{\sqrt{y}}$ holomorphic on H, & equal on line $\tau = iy$. So, by identity principle, $\theta(\tau) = \sqrt{\tau/i}^{\,-1}\, \theta(-\frac{1}{\tau})$.

Here: $\sqrt{\tau/i}$ is unique branch of $\sqrt{\cdot}$ defined on H, with $\sqrt{y}>0$ on $\tau = iy$.

Prop 5.11 If $k \in 8\mathbb{N}$ then $\theta^k \in M_{k/2}(\Gamma)$, where: $\Gamma \equiv \Gamma(2) \cup S\Gamma(2)$.

Proof Know: θ^k holo. on H, & function of q_2. So, $\theta(\tau+2) = \theta(\tau)$.

$\Rightarrow \theta^k(\tau+2) = \theta^k(\tau) \Rightarrow \underline{\theta^k|_{k/2}[T^2] = \theta^k}$. $T^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

Claim: $\theta^k|_{k/2}[S] = \theta^k$.

proof: $\theta^k(-1/\tau)\,(\tau)^{-k/2} = \left(\theta(\tau)\sqrt{\tau/i}\right)^k \tau^{-k/2} = \theta^k(\tau)\left(\frac{\tau}{i}\right)^{k/2}\tau^{-k/2}$

$= \theta^k(\tau)$ (since $\underline{8|k}$).

Fact: $\Gamma = \langle S, T^2 \rangle$.

$\Rightarrow \theta^k|_{k/2}(\gamma) = \theta^k \;\forall \gamma \in \Gamma$, so θ^k weakly modular, of weight k & level Γ.

Modular Forms: Lecture 19 — 17/11/2023.

From last time Introduced: $\theta(\tau) \equiv \sum_{n\in\mathbb{Z}} q_2^{n^2}$. $(q_2 \equiv e^{i\pi\tau})$

Prop 5.11: If $k \in 8\mathbb{N}$ then $\theta^k \in M_{k/2}(\Gamma)$ for $\Gamma = \Gamma(2) \cup S\Gamma(2)$.

Proof Already showed: θ^k is weakly modular, of weight $\frac{k}{2}$, level Γ. Proved using:

a) formula: $\theta(-\frac{1}{\tau}) = \sqrt{\tau/i}\,\theta(\tau)$ (Poisson formula)

b) $\Gamma = \langle S, T^2 \rangle$.

To complete proof: need to show: θ^k ~~[illegible]~~ holomorphic at the cusps of Γ.

& Cusps $\Longleftrightarrow \Gamma \backslash \mathbb{P}^1(\mathbb{Q}) \Longleftrightarrow \Gamma \backslash \Gamma(1) / \Gamma_\infty$.

First: describe $\Gamma\backslash\Gamma(1)$ as a right $\Gamma(1)$-set. Then compute $\Gamma\backslash\Gamma(1)/\Gamma_\infty$ as set of right Γ_∞-orbits.

If $\{g_i\}$ set of double coset reps. then, $\{\Gamma \cdot g_i\infty\}$ are cusps of Γ.

To deduce $\Gamma\backslash\Gamma(1)$: Write down right $\Gamma(1)$-set X & $x \in X$ with $\mathrm{stab}_{\Gamma(1)}(x) = \Gamma$.

Then $\exists\, \Gamma(1)$-~~invariant~~ equivalent bijection
$$\Gamma\backslash\Gamma(1) \xrightarrow{\sim} X$$
$$\Gamma\gamma \longmapsto x\gamma.$$

Let: $\Gamma(1)$ act on $X \equiv \mathbb{F}_2^2 - 0$. (take pre-image under $\Gamma(1) \to SL_2(\mathbb{F}_2)$)

& acting ~~on~~ by right mult. on row vectors. $v\cdot\gamma \equiv v\gamma$.

$X = \{(0,1), (1,0), (1,1)\}$. $\quad (1,1)\begin{pmatrix} a & b \\ c & d\end{pmatrix} = (a+c, b+d)$.
$= (1,1)$ iff $a \neq c$, $b \neq d$.

<u>Possibilities</u>: $a=1 \Rightarrow \begin{pmatrix}1&0\\0&1\end{pmatrix}$
$a=0 \Rightarrow \begin{pmatrix}0&1\\1&0\end{pmatrix}$ $\Big] \Rightarrow \mathrm{Stab}_{\Gamma(1)}(1,1) = \left\{\begin{pmatrix}a&b\\c&d\end{pmatrix} \in \Gamma(1) : \equiv I_2 \text{ or } S \bmod 2\right\}$

$= \underline{\Gamma(2) \cup \Gamma(2)S = \Gamma}.$

& Compute: Γ_∞ – orbits: $(1\ 0)\begin{pmatrix}1&1\\0&1\end{pmatrix} = (1\ 1)$

$(0\ 1)\begin{pmatrix}1&1\\0&1\end{pmatrix} = (1\ 1).$

$\Rightarrow X\backslash\Gamma_\infty$ has 2 elements.

$\Rightarrow \Gamma\backslash\Gamma(1)/\Gamma_\infty$ has 2 elements, reps are: I_2 & any $\gamma \in \Gamma(1)$ with $x\gamma = (0\ 1)$ (on $X \equiv \mathbb{F}_2^2 - 0$). E.g. $\gamma = \begin{pmatrix}1&-1\\1&0\end{pmatrix}.$

$\Rightarrow \Gamma$ has 2 cusps: $\Gamma\cdot\infty$ & $\Gamma\cdot\gamma\infty = \Gamma\cdot 1$

<u>Need to show</u>: θ^k & $\theta^k\Big|_{k/2}\left[\begin{pmatrix}1&-1\\1&0\end{pmatrix}\right]$ holo. at ∞.

<u>Have</u>: $\theta^k = \left(\sum_{n\in\mathbb{Z}} q_2^{n^2}\right)^k.$ $\quad \theta(\tau) = \sum_{n\in\mathbb{Z}} q_2^{n^2}$

$\theta(\tau+1) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2} e^{\pi i n^2\tau}$

& $\gamma\tau = \frac{\tau-1}{\tau} = 1 - \frac{1}{\tau}.$

$= \sum_{n\in\mathbb{Z}} (-1)^n e^{\pi i n^2 \tau}.$

$\Rightarrow \theta(\tau) + \theta(\tau+1)$

$= 2\sum_{n\in\mathbb{Z}} e^{\pi i (2n)^2\tau} = 2\theta(4\tau).$

$\Rightarrow \theta(-1/\tau) + \theta(1-\frac{1}{\tau}) = 2\theta(-\frac{4}{\tau}).$

$\Rightarrow \theta(1-\frac{1}{\tau}) = 2\theta(-\frac{4}{\tau}) - \theta(-\frac{1}{\tau}) = 2\theta(\frac{\tau}{4})\sqrt{\frac{\tau}{4i}} - \theta(\tau)\sqrt{\frac{\tau}{i}}.$

$= \sqrt{\frac{\tau}{i}}\left(\theta(\frac{\tau}{4}) - \theta(\tau)\right).$

$\Rightarrow \theta^k|_{k/2}[\gamma](\tau) = \theta(1-\frac{1}{\tau})^k \tau^{-k/2} = \left(\sqrt{\frac{\tau}{i}}\right)^k \tau^{-k/2}(\theta(\frac{\tau}{4})-\theta(\tau))^k$

$= (\theta(\frac{\tau}{4})-\theta(\tau))^k.$

So, we see: $\theta^k|_{k/2}[\gamma]$ vanishes at ∞, hence is holomorphic there ✓

$\Rightarrow$ Is a modular form.

Theorem 5.12) Let $n\in\mathbb{N}$. Then

$$r_{24}(n) = \frac{65536}{691}\sigma_{11}(\tfrac{n}{2}) - (-1)^n\frac{16}{691}\sigma_{11}(n) - \frac{65536}{691}\tau(\tfrac{n}{2}) - (-1)^n\frac{33152}{691}\tau(n)$$

& Convention: $\sigma_{11}(\frac{n}{2}) = \sigma(\frac{n}{2}) = 0$ for n odd.

(If n odd $\Rightarrow r_{24}(n) = \frac{16}{691}\sigma_{11}(n) - \frac{33152}{691}\tau(n)$.)

Proof $\theta^{24}(\tau) = \sum_{n\geq 0} r_{24}(n) q_2^n \in M_{12}(\Gamma)$.

By example sheet: $\dim M_k(\Gamma) \leq 1 + \frac{k[\Gamma(1):\Gamma]}{12}$.

& $|\Gamma(1):\Gamma| = 3$. $(=\# X)$. So, $\dim M_{12}(\Gamma) \leq 1+3 = 4$.

Since $\Gamma \leq \Gamma(1)$: get $\langle F_{12}, \Delta\rangle = M_{12}(\Gamma(1)) \leq M_{12}(\Gamma)$.

& If $\alpha = \begin{pmatrix}1&1\\0&2\end{pmatrix}$: then $\Gamma \leq \alpha^{-1}\Gamma(1)\alpha$

$\Rightarrow$ If $f\in M_k(\Gamma(1))$ then $f|_k[\alpha]\in M_k(\Gamma)$.

[Need: $\Gamma \leq \alpha^{-1}\Gamma(1)\alpha \Leftrightarrow \alpha\Gamma\alpha^{-1} \leq \Gamma(1)$.

$\alpha\begin{pmatrix}a&b\\c&d\end{pmatrix}\alpha^{-1} = \begin{pmatrix}a+c & \frac{1}{2}(b+d-a-c)\\ 2c & d-c\end{pmatrix}$. Has integer entries,

since: a,d odd $\Rightarrow b,c$ even & a,d even $\Leftrightarrow b,c$ odd.)]

$\Rightarrow f|_k[\alpha](\tau) = f(\frac{\tau+1}{2})2^{k-1}2^{-k} = \frac{1}{2}f(\frac{\tau+1}{2})$.

So, have 4 elements: F_{12}, Δ, $F_{12}\left(\frac{1+\tau}{2}\right)$ & $\Delta\left(\frac{1+\tau}{2}\right)$

<u>Check</u>: modular forms are linearly independent.

$\Rightarrow M_{12}(\Gamma)$ has dim 4 & has these as basis.

Since $\theta^{24} \in M_{12}(\Gamma) \Rightarrow \theta^{24} = A F_{12}(\tau) + B\Delta(\tau) + C F\left(\frac{\tau+1}{2}\right) + D\Delta\left(\frac{\tau+1}{2}\right)$.

Solve for $A, B, C, D \Rightarrow$ Get result.

Another application of θ: Meromorphic cont. of $\xi(s)$.

<u>Theorem 5.13</u> Let $\xi(s) \equiv \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$. This has meromorphic cont. to $\mathbb{C}$, simple poles $s = 0, 1$, residues $-1, 1$, no other poles & $\xi(1-s) = \xi(s)$.

<u>Consider</u>: $$\int_{y=0}^{\infty} \theta(iy)\, y^{s/2} \frac{dy}{y}. \quad \& \quad \theta(iy) = 1 + O(e^{-\pi y}). \quad (y \to \infty).$$

$\Rightarrow$ Get poles of $\xi(s)$.

Modular Forms: lecture 20. 20/11/2023.

Theorem 5.13: $\xi(s) \equiv \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$. Has: meromorphic cont. to $\mathbb{C}$, simple pole $s=0,1$ residues $-1,1$, no other poles and $\xi(s)=\xi(1-s)$.

Proof let $\theta(\tau) \equiv \sum e^{\pi i n^2 \tau}$ & define: $F(s) \equiv \int_0^\infty (\theta(iy)-1)y^{\frac{s}{2}}\frac{dy}{y}$.

Know: $\theta(iy) = 1 + O(e^{-\pi y})$ as $y \to \infty$.

$$\Rightarrow F(s) = \int_{y=1}^\infty (\theta(iy)-1)y^{s/2}\frac{dy}{y} + \int_{y=0}^1 (\theta(iy)-1)y^{s/2}\frac{dy}{y}$$

$$= \int_{y=1}^\infty (\theta(iy)-1)y^{s/2}\frac{dy}{y} + \int_{y=1}^\infty (\theta(iy)\sqrt{y}-1)y^{-s/2}\frac{dy}{y}$$

(using: $y \mapsto 1/y$ & $\theta(i/y) = \theta(iy)\sqrt{y}$).

Know: $\theta(iy)\sqrt{y} - 1 = \sqrt{y} - 1 + o(e^{-\pi y/2})$ as $y \to \infty$.

$\Rightarrow$ Will converge when $\frac{1-\sigma}{2} < 0$, or $\sigma > 1$.

$\Rightarrow F(s)$ defined + holomorphic at: $\{\sigma > 1\}$.

& in this region: $$F(s) = \int_{y=1}^\infty (\theta(iy)-1)y^{s/2}\frac{dy}{y} + \int_{y=1}^\infty \left[(\theta(iy)-1)y^{\frac{1-s}{2}} + \underline{\left(y^{\frac{1-s}{2}} - y^{-\frac{s}{2}}\right)}\right]\frac{dy}{y}$$

$$= \underbrace{\int_{y=1}^\infty (\theta(iy)-1)y^{\frac{s}{2}}\frac{dy}{y}}_{\text{entire}} + \underbrace{\int_{y=1}^\infty (\theta(iy)-1)y^{\frac{1-s}{2}}\frac{dy}{y}}_{\text{entire}} + \left(\frac{2}{s-1} - \frac{2}{s}\right)$$

$\Rightarrow$ Conclusion: F has meromorphic cont. to $\mathbb{C}$, with simple poles $s=0,1$ of residue $2,-2$.

$\cancel{\circledast}$ & clearly: $F(s) = F(1-s)$ (by formula) ✓

Also: $F(s) = \int_{y=0}^{\infty} 2\sum_{n\geq 1} e^{-\pi n^2 y} y^{s/2} \frac{dy}{y}$.

$$\overset{(*)}{=} 2\sum_{n\geq 1} \int_{y=0}^{\infty} e^{-\pi n^2 y} y^{s/2} \frac{dy}{y}$$

$$= 2\sum_{n\geq 1} \int_{y=0}^{\infty} (\pi n^2)^{-s/2} y^{s/2} e^{-y} \frac{dy}{y} = 2\sum_{n\geq 1} \pi^{-s/2} n^{-s} \xi(s)$$

$$= 2\,\xi(s) ✓$$

& $(*)$ justified when: $\int_{y=0}^{\infty} \sum_{n\geq 1} e^{-\pi n^2 y} y^{\sigma/2} \frac{dy}{y} < \infty$.

But, we know this for $\sigma > 1$. So, holds for $\sigma > 1$ ✓

<u>Consider</u>: θ- function for a <u>lattice</u> $\Lambda \leq \mathbb{R}^n$.

<u>Define</u>: $\theta_\Lambda(\tau) \equiv \sum_{\lambda\in\Lambda} e^{\pi i \langle \lambda, \lambda \rangle \tau}$ ($\langle \cdot , \cdot \rangle$ standard in $\mathbb{R}^n$)

notice: if $\Lambda = \mathbb{Z} \leq \mathbb{R}$ then $\theta_\Lambda = \theta$.

<u>Know</u>: θ_Λ holomorphic in H $\forall \Lambda$. (sheet 4)

<u>Prop 5.14</u> (Poisson sum, $\mathbb{R}^n$). <u>Let</u>: $\Lambda \leq \mathbb{R}^n$ lattice, and

$f : \mathbb{R}^n \to \mathbb{C}$ <u>cts</u> s.t. $\exists c, \delta > 0$, $\forall x \in \mathbb{R}^n$, $|f(x)| \leq \frac{c}{(1+\|x\|)^{\delta+n}}$

& let $\hat{f}(y) \equiv \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, y\rangle} dx$.

& Assume: $\sum_{\mu\in\Lambda^\vee} |\hat{f}(\mu)| < \infty$, where $\Lambda^\vee \equiv \{\mu \in \mathbb{R}^n : \forall \lambda \in \Lambda, \langle \mu, \lambda \rangle \in \mathbb{Z}\}$.

Then: $\sum_{\lambda \in \Lambda} f(\lambda) = m(\Lambda)^{-1} \sum_{\mu \in \Lambda^\vee} \hat{f}(\mu)$.

Use this to get: transformation formula for Θ_Λ.

Let: $f: \mathbb{R}^n \to \mathbb{R}$, $f(x) \equiv e^{-\pi \|x\|^2} = \prod_{i \le n} e^{-\pi x_i^2}$.

$\Rightarrow \hat{f}(y) = f(y)$. (Separation of variables)

& $\Theta_\Lambda(iy) = \sum_{\lambda \in \Lambda} e^{-\pi \langle \lambda, \lambda \rangle y} = \sum_{\lambda \in \sqrt{y}\Lambda} e^{-\pi \langle \lambda, \lambda \rangle}$

$= \sum_{\mu \in (\sqrt{y}\Lambda)^\vee} e^{-\pi \langle \mu, \mu \rangle} \cdot m(y^{1/2}\Lambda)^{-1}$

$= m(\Lambda)^{-1} y^{-n/2} \sum_{\mu \in \Lambda^\vee} e^{-\pi \langle \mu, \mu \rangle y^{-1}} = m(\Lambda)^{-1} y^{-n/2} \Theta_{\Lambda^\vee}\left(\frac{i}{y}\right)$.

So, by identity principle: $\boxed{\Theta_\Lambda(\tau) = \frac{1}{m(\Lambda)} \sqrt{\tau/i}^{\,n}\, \Theta_{\Lambda^\vee}(-1/\tau).}$

Prop 5.15) Suppose: $n \in 8\mathbb{N}$, $\Lambda \subseteq \mathbb{R}^n$ lattice such that:

- Λ self-dual, $\Lambda = \Lambda^\vee$
- Λ even, $\forall \lambda \in \Lambda$, $\langle \lambda, \lambda \rangle \in 2\mathbb{Z}$.

Then: $\Theta_\Lambda \in M_{n/2}(\Gamma(1))$.

Non-example $\Lambda = \mathbb{Z}^n \le \mathbb{R}^n$. $\Rightarrow \Lambda = \Lambda^\vee$, **not** even.

In this case: $\Theta_\Lambda = \theta^n \in M_{n/2}(\Gamma)$.

Proof $\Theta_\Lambda(\tau)$ holomorphic in H & $\Theta_\Lambda(\tau) = \sum_{\lambda \in \Lambda} e^{\pi i \langle \lambda, \lambda \rangle \tau}$.

Even $\Rightarrow$ Is q-series. & invariant under $\tau \mapsto \tau+1$.

$(\theta_\Lambda|_{n/2}[T] = \theta_\Lambda.)$

& $\theta_\Lambda|_{n/2}[S](\tau)$

$= \theta_\Lambda(-1/\tau)\tau^{-n/2} = \theta_\Lambda(-1/\tau)\sqrt{\frac{\tau}{i}}^{-n}$ $(8|n)$

$= m(\Lambda^\vee)\,\theta_{\Lambda^\vee}(\tau)$.

For any lattice $\Lambda \subseteq \mathbb{R}^n$: $m(\Lambda)m(\Lambda^\vee) = 1$, and since assumed $\Lambda = \Lambda^\vee$, get $m(\Lambda^\vee) = 1$. So, $\theta_\Lambda|_{n/2}[S] = \theta_\Lambda$.

Since S, T generate Γ: get θ_Λ weakly modular, weight $\frac{n}{2}$, level $\Gamma(1)$. Since θ_Λ holomorphic at ∞: get modular ✓

Example $\Lambda = E_8$ root lattice in $\mathbb{R}^8$.

Then, Λ even (root lattice even), self-dual. (E_8 simply connected)

$\Rightarrow \theta_\Lambda \in M_4(\Gamma(1)) = \langle E_4 \rangle$.

Since $\theta_\Lambda = \sum_{\lambda \in \Lambda} e^{-\pi i \langle \lambda, \lambda \rangle \tau}$: get $= 1 + O(q)$.

$\Rightarrow$ Get: $\theta_\Lambda = E_4$. & $240 = \#\{\lambda \in \Lambda : \langle \lambda, \lambda \rangle = 2\}$

& $\dim E_8 = \dim \mathbb{R}^8 + \#\text{Roots} = 248$

Modular Forms: lecture 21

22/11/2023

From last time: If $\Lambda \le \mathbb{R}^n$ lattice, $\theta_\Lambda(\tau) \equiv \sum_{\lambda \in \Lambda} e^{\pi i \langle \lambda, \lambda \rangle \tau}$

& Transformation Formula: $\theta_\Lambda(\tau) = \frac{1}{m(\Lambda)} \sqrt{\frac{\tau}{i}}^{-n} \theta_{\Lambda^\vee}(-1/\tau)$.

DEF Epstein Zeta function $\zeta_\Lambda(s) \equiv \sum_{\lambda \in \Lambda - 0} \langle \lambda, \lambda \rangle^{-s}$.

Example $\zeta_{\mathbb{Z}}(s) = 2\zeta(2s)$.

In sheet 4: $\zeta_\Lambda(s)$ converges absolutely if $\mathrm{Re}(s) > \frac{n}{2}$, so it is holomorphic in that region.

Theorem 5.16 Define $\xi_\Lambda(s) \equiv \pi^{-s}\Gamma(s)\zeta_\Lambda(s)$. This admits meromorphic cont. to $\mathbb{C}$, with 2 poles $s = 0, n/2$ with residues $-1, 1/m(\Lambda)$ (& no other poles)

& satisfies: $\xi_\Lambda(s) = \frac{1}{m(\Lambda)} \xi_{\Lambda^\vee}\left(\frac{n}{2} - s\right)$.

Proof Define $F(s) \equiv \int_{t=0}^{\infty} (\theta_\Lambda(it) - 1) t^s \frac{dt}{t}$.

Then: $\theta_\Lambda(it) = 1 + O(e^{-ct})$ for some $c > 0$.

& converges absolutely, for $\mathrm{Re}(s) > \frac{n}{2}$, so well-def'ed & holo. here.

& $F(s) = \int_0^\infty \sum_{\lambda \in \Lambda - 0} e^{-\pi \langle \lambda, \lambda \rangle t} t^s \frac{dt}{t} = \sum_{\lambda \in \Lambda - 0} \pi^{-s} \langle \lambda, \lambda \rangle^{-s} \Gamma(s)$

$= \pi^{-s}\Gamma(s)\zeta_\Lambda(s) = \xi_\Lambda(s)$. ✓

& $F(s) = \int_1^\infty (\theta_\Lambda(it) - 1) t^s \frac{dt}{t} + \int_1^\infty (\theta_\Lambda(i/t) - 1) t^{-s} \frac{dt}{t}$.

$= \text{first term} + \int_1^\infty \left(\frac{1}{m(\Lambda)}\theta_{\Lambda^\vee}(it)t^{n/2} - 1\right)t^{-s}\frac{dt}{t}$

$= \text{first term} + \frac{1}{m(\Lambda)}\int_1^\infty (\theta_{\Lambda^\vee}(it)-1)t^{\frac{n}{2}-s}\frac{dt}{t} + \underbrace{\int_1^\infty \left(\frac{1}{m(\Lambda)}t^{\frac{n}{2}-s} - t^{-s}\right)\frac{dt}{t}}_{= \frac{1}{m(\Lambda)}\frac{1}{\frac{n}{2}-s} - \frac{1}{s}}$

Hence: results about poles + FE.

Remark $\xi_\Lambda(s)$ usually doesn't have Euler product, so isn't L-function.

§6: Non-holomorphic Eisenstein Series.

Modular forms are: beginning of automorphic forms.

General problem: decompose $L^2(\Gamma \backslash SL_2(\mathbb{R}))$, as a representation of $SL_2(\mathbb{R})$.

Modular forms arise from D_k-isotypic subspace, for a certain series of reps D_k $(k \geq 0)$ of $SL_2(\mathbb{R})$.

Will study simplest examples of non-holomorphic automorphic forms.

DEF 6.1 $G(\tau, s) \equiv \sum_{(m,n)\in\mathbb{Z}^2-0} \frac{\operatorname{Im}(\tau)^s}{|m\tau+n|^{2s}}$ $(\tau \in H,\ s \in \mathbb{C})$.

Can check: the series converges absolutely & locally uniformly on $H \times \{s \in \mathbb{C} : \operatorname{Re}(s) > 1\}$. So, is defined & continuous here. (Not holo. as function of τ, since $|m\tau+n|$ is not)

Think of these as family of automorphic forms on H, induced by s. $\sqrt{2}$

Can give more group-theoretic description:

$$G(\tau,s) \equiv \sum_{d\geq 1} d^{-2s} \sum_{(m,n)=1} \frac{\mathrm{Im}(\tau)^s}{|m\tau+n|^{2s}}.$$

& note: $\{(m,n)\in\mathbb{Z}^2 : \gcd(m,n)=1\}/\{\pm I\} \Longleftrightarrow \Gamma_\infty\backslash\Gamma(1)$, $\quad (0,1)\gamma \longleftarrow\!\shortmid \gamma$

$$=2\sum_{d\geq 1} d^{-2s} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma(1)} \frac{\mathrm{Im}(\tau)^s}{|j(\gamma,\tau)|^{2s}}$$

$$=2\zeta(2s)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma(1)} \mathrm{Im}(\gamma\tau)^s \equiv 2\zeta(2s)\underline{E(\tau,s)}$$

Consequence: If $\delta\in\Gamma(1)$ then $E(\delta\tau,s)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma(1)} \mathrm{Im}(\gamma\delta\tau)^s = E(\tau,s)$

$\Rightarrow G(\delta\tau,s)=G(\tau,s)$

Also see: $G(\tau,s)=\sum_{\lambda\in\Lambda-0} y^{-s}\langle\lambda,\lambda\rangle^{-2s} = \zeta_{y^{-\frac{1}{2}}\Lambda_\tau}(s)$

(where inner prod comes from $\mathbb{C}\cong\mathbb{R}^2$) (basis $1,i$)

Lemma 6.2 $m(y^{-\frac{1}{2}}\Lambda_\tau)=1$ & $(y^{-\frac{1}{2}}\Lambda_\tau)^\vee = iy^{-\frac{1}{2}}\Lambda_\tau$, where $\tau=x+iy$.

Proof $y^{-\frac{1}{2}}\Lambda_\tau$ has basis $\{y^{-\frac{1}{2}}, \tau y^{-\frac{1}{2}}\}$

& $\tau y^{-\frac{1}{2}} = xy^{-\frac{1}{2}}+iy^{\frac{1}{2}}$. $\Rightarrow$ Covolume: $\left|\det\begin{pmatrix} y^{-\frac{1}{2}} & 0 \\ xy^{-\frac{1}{2}} & y^{\frac{1}{2}}\end{pmatrix}\right|=1.$

Similar for the dual.

Prop 6.3 1) Define $G^*(\tau,s) \equiv \pi^{-s}\Gamma(s)G(\tau,s)$.

Then: $\forall \tau \in H$ fixed: $G^*(\tau,s)$ has mero cont to $\mathbb{C}$, with simple poles $s=0,1$, residue $-1,1$, no other poles.

2) $G^*(\tau,s) = G^*(\tau,1-s)$

3) $G^*(\tau,s) - \frac{1}{s(s-1)}$ extends to C^∞- function of $H \times \mathbb{C}$.

(Can think of as 4-dim manifold).

Proof 1) Follows from properties of $\xi_{y^{-\frac{1}{2}}\Lambda_\tau}(s)$.

2) $\xi_{y^{-\frac{1}{2}}\Lambda_\tau}(s) = m(y^{-\frac{1}{2}}\Lambda_\tau)^{-1}\,\xi_{i y^{-\frac{1}{2}}\Lambda_\tau}(1-s) = \xi_{y^{-\frac{1}{2}}\Lambda_\tau}(1-s)$.

3) Know: $\xi_{y^{-\frac{1}{2}}\Lambda_\tau}(s) = \int_{t=1}^{\infty} \left(\Theta_{y^{-\frac{1}{2}}\Lambda_\tau}(it) - 1\right)\left(t^s + t^{1-s}\right)\frac{dt}{t} + \frac{1}{s-1} - \frac{1}{s}$.

$\cancel{\Rightarrow G^*(\tau,s) - \frac{1}{s(s-1)} = \int_{t=1}^{\infty} \sum_{(m,n)\in\mathbb{Z}^2-0} e^{-\pi|m\tau+n|^2 \ldots}}$

$$\Rightarrow G^*(\tau,s) - \frac{1}{s(s-1)} = \int_{t=1}^{\infty}\left(\sum_{(m,n)\in\mathbb{Z}^2-0} e^{-\pi|m\tau+n|^2 t y^{-1}}\right)\left(t^s + t^{1-s}\right)\frac{dt}{t}.$$

& Can justify $\frac{\partial}{\partial s}$ under $\int$ in this case. ✓

Modular forms: Lecture 22

24/11/2023

<u>From last time</u>: Defined $G(\tau,s) \equiv \sum_{(m,n)\neq(0,0)} \frac{y^s}{|m\tau+n|^{2s}}$

<u>and</u>: $G^*(\tau,s) \equiv \pi^{-s}\Gamma(s)G(\tau,s)$.

<u>Know</u>: $G^*(\tau,s) - \frac{1}{s(s-1)}$ <u>extends</u> to C^∞-function on $\mathbb{H}\times\mathbb{C}$.

$\Rightarrow G^*(\tau,s) = \sum_{n\in\mathbb{Z}} A_n^*(y,s)e^{2\pi i n x}$ $\left[A_n^*(y,s) = \int_{x=0}^{1} \frac{G^*(\tau,s)}{e^{2i\pi n s}}\right]$

If $n\neq 0 \Rightarrow A_n^*(y,s)$ is C^∞ on $(0,\infty)\times\mathbb{C}$, and is <u>entire</u> as function of s (y fixed).

If $n=0 \Rightarrow A_0^*(y,s)$ is C^∞ on $(0,\infty)\times(\mathbb{C}-\{0,1\})$

and: $A_0^*(y,s) - \frac{1}{s(s-1)}$ extends to C^∞-function, on $(0,\infty)\times\mathbb{C}$, and entire as a function of s.

Theorem 6.4 $A_0^*(y,s) = 2\xi(2s)y^s + 2\xi(\cancel{1-2s}\ 2s-1)y^{1-s}$.

$$= 2\xi(2s)y^s + 2\xi(2(1-s))y^{1-s}.$$

Proof <u>Note</u>: Both LHS & RHS of this is Meromorphic on $\mathbb{C}$, so, it suffices to show agree on some open U.

<u>Take</u>: $U = \{\mathrm{Re}(s)>1\}$.

$$\Rightarrow G^*(\tau,s) = \int_{t=0}^{\infty} (\theta_{y^{-\frac{1}{2}}\Lambda_\tau} - 1)(\tau)t^s\frac{dt}{t}.$$

$$\Rightarrow A_0^{*}(y,s) = \int_{x=0}^{1}\int_{t=0}^{\infty} \sum_{(m,n)\neq(0,0)} e^{-\pi|m\tau+n|^2 t/y}\, t^s \frac{dt}{t}.$$

This converges Absolutely (since Epstein zeta does here).

$$\Rightarrow A_0^{*}(y,s) = I_{m=0} + I_{m\neq 0}.$$

<u>Where</u>: $$I_{m=0} = 2\sum_{n\geq 1}\int_{x=0}^{1}\int_{t=0}^{\infty} e^{-\pi n^2 t/y}\, t^s \frac{dt}{t}$$

& $$I_{m\neq 0} = 2\sum_{m\geq 1}\sum_{n\in\mathbb{Z}}\int_{x=0}^{1}\int_{t=0}^{\infty} e^{-\pi(mx+n)^2 t/y}\, e^{-\pi m^2 t y}\, t^s \frac{dt}{t}\, dx.$$

So. <u>$I_{m=0}$</u>:

$$= 2\sum_{n\geq 1}(\pi n^2/y)^{-s}\,\Gamma(s) = 2\pi^{-s} y^s\, \Gamma(s)\, \zeta(2s)$$
$$= 2\xi(2s)\, y^s$$

[<u>where</u>: $\xi(s) \equiv \pi^{-s/2}\,\Gamma(\tfrac{s}{2})\,\zeta(s)$]

& <u>$I_{m\neq 0}$</u>:

First, consider $\displaystyle\sum_{n\in\mathbb{Z}}\int_{x=0}^{1} e^{-\pi(mx+n)^2 t/y}\, dx$ ($m\geq 1$ fixed).

$$= \frac{1}{m}\sum_{n\in\mathbb{Z}}\int_{x=0}^{m} e^{-\pi(x+n)^2 t/y}\, dx.$$

$$= \frac{1}{m}\sum_{n\in\mathbb{Z}}\int_{x=n}^{n+m} e^{-\pi x^2 t/y}\, dx.$$

$$= \frac{1}{m}\cdot m\cdot\int_{x=-\infty}^{\infty} e^{-\pi x^2 t/y}\, dx = \sqrt{\frac{y}{t}}.$$

$$\Rightarrow I_{m\neq 0} = 2\sum_{m\geq 1}\int_{t=0}^{\infty} \sqrt{y}\cdot e^{-\pi m^2 t \cancel{} y}\, t^{s-\frac{1}{2}}\,\frac{dt}{t}$$

$$= 2\sum_{m\geq 1} (\pi m^2 y)^{\frac{1}{2}-s}\, y^{\frac{1}{2}}\, \Gamma(s-\tfrac{1}{2})$$

$$= 2\pi^{\frac{1-2s}{2}}\,\Gamma\left(\frac{2s-1}{2}\right)\zeta(\cancel{}\,2s-1)\, y^{1-s} = 2\,\xi(2s-1)\,y^{1-s} \checkmark$$

The non-const Fourier coeffs are expressed in terms of the Bessel Function $K_s(c) = \int_{t=0}^{\infty} e^{-c(t+t^{-1})}\, t^s\,\frac{dt}{t}$

[$s\in\mathbb{C}$ & $c>0$]

Note: Integrand of K_s decays very quickly, as $t\to\infty$, $t\to 0$.

$\Rightarrow \forall c$ fixed: $K_s(c)$ Entire func in s.

Theorem 6.5) ~~Let~~ $k\in\mathbb{Z}_{\neq 0}$. Then, $A_k^*(y,s)$

$$= 2\sqrt{y}\,|k|^{s-\frac{1}{2}}\,\sigma_{1-2s}(|k|)\,K_{s-\frac{1}{2}}(\pi y|k|)$$

Where: $\sigma_s(n) \equiv \sum_{d|n} d^s$.

Proof) Again, both LHS & RHS entire as function of s, so enough to show, for $\mathrm{Re}(s)>1$.

In this case: $A_k^*(y,s)$

$$= \int_{x=0}^{1}\int_{t=0}^{\infty}\sum_{(m,n)\neq(0,0)} e^{-\pi(mx+n)^2 t/y}\, e^{-\pi m^2 t y}\, e^{-2\pi i k x}\, t^s\,\frac{dt}{t}\,dx.$$

for $m=0$: $\int_{x=0}^{1} 2\sum_{n\geq 1} e^{-\pi n^2 t/y}\, e^{-\pi m^2 t y}\, e^{-2\pi i k x}\,dx = 0$,

because $\int_{x=0}^{1} e^{-2\pi i k x}\,dx = 0 \;\forall k\neq 0$.

$$\Rightarrow A_k^+(y,s) = 2\sum_{m\geq 1}\int_{t=0}^{\infty}\sum_{n\in\mathbb{Z}}\int_{x=0}^{1} e^{-\pi(mx+n)^2 t/y - 2\pi i k x}\, dx. \quad e^{-\pi m^2 t y}\, t^s \frac{dt}{t}\, dx.$$

for *fixed* $m \geq 1$:

$$\sum_{n\in\mathbb{Z}}\int_{x=0}^{1} e^{-\pi(mx+n)^2 t/y - 2\pi i k x}\, dx.$$

$$= \frac{1}{m}\sum_{n\in\mathbb{Z}}\int_{x=0}^{m} e^{-\pi(x+n)^2 t/y}\, e^{-2\pi i k x/m}\, dx$$

$$= \frac{1}{m}\sum_{n\in\mathbb{Z}}\int_{x=n}^{n+m} e^{-\pi x^2 t/y}\, e^{-2\pi i k x/m}\, e^{2\pi i k n/m}\, dx$$

$$= \frac{1}{m}\sum_{a\in\mathbb{Z}/m\mathbb{Z}} e^{2i\pi k a/m}\sum_{\substack{n\in\mathbb{Z}\\ n\equiv a \bmod m}}\int_{x=n}^{n+m} e^{-\pi x^2 t/y - 2\pi i k x/m}\, dx.$$

$$= \frac{1}{m}\Big(\sum_{a\in\mathbb{Z}/m\mathbb{Z}} e^{2i\pi k a/m}\Big)\int_{x=-\infty}^{\infty} e^{-\pi x^2 t/y - 2\pi i k x/m}\, dx.$$

So, if $\underline{m|k}$, this is $= \int_{x=-\infty}^{\infty} e^{-\pi x^2 t/y - 2\pi i k x/m}\, dx$

$$= \int_{x=-\infty}^{\infty} e^{-\pi\left(x\sqrt{\frac{t}{y}} - \frac{ik}{m}\sqrt{\frac{y}{t}}\right)^2} e^{-\pi k^2 y/m^2 t}\, dx.$$

$$= \sqrt{\frac{y}{t}}\, e^{-\pi k^2 y/m^2 t}.$$

(& If $m \nmid k$, this is $= 0$.)

$$\Rightarrow A_k^+(y,s) = 2\sum_{m | |k|}\int_{t=0}^{\infty}\sqrt{\frac{y}{t}}\, e^{-\pi k^2 y/m^2 t}\, e^{-\pi m^2 t y}\, t^s \frac{dt}{t}$$

$$= 2\sqrt{y}\cdot\sum_{m | |k|}\int_{t=0}^{\infty} e^{-\pi |k| y\left(\frac{m^2 t}{|k|} + \frac{|k|}{m^2 t}\right)}\, t^{s-\frac{1}{2}}\frac{dt}{t}$$

$$= 2\sqrt{y} \sum_{m \mid |k|} \left(\frac{m^2}{|k|}\right)^{\frac{1}{2}-s} \int_{t=0}^{\infty} e^{-\pi |k| y (t + t^{-1})} t^{s-\frac{1}{2}} \frac{dt}{t}.$$

$$= 2\sqrt{y}\, K_{s-\frac{1}{2}}(\pi |k| y)\, |k|^{s-\frac{1}{2}} \Big(\sum_{m \mid |k|} m^{1-2s}\Big) \checkmark$$

<u>Observation</u> $A_0^*(y,s) = 2\xi(2s) y^s + 2\xi(2s-1) y^{1-s}.$

$= 2\xi(2s) y^s + 2\xi(2(1-s)) y^{1-s}.$

So, plug in $s = \frac{1+it}{2}$:

$= 2\xi(1+it) y^{\frac{1+it}{2}} + 2\xi(1-it) y^{\frac{1-it}{2}}.$

<u>Hope to show</u>: Non-vanishing of $\xi(1+it)$ $\forall t \in \mathbb{R}^\times$.

If $\xi(1+it) = 0$: observe $\xi(1+it) = \xi(1-it). = 0.$

$\Rightarrow A_0^*(y, s_0) = 0$, for $s = \frac{1+it}{2}.$

For $s = s_0$: $G^*(\tau, s_0)$ behaves "like a Cuspidal modular form".

<u>General Principle</u>: "Eisenstein series are Orthogonal to Cuspidal forms": $\langle f, E_k \rangle = 0.$

Lemma next lecture.

Modular forms: Lecture 23 | 27/11/2023

from last time $G(\tau,s) = \sum_{(m,n)\neq(0,0)} \frac{y^s}{|m\tau+n|^{2s}}$

& $G^*(\tau,s) = \pi^{-s}\Gamma(s)\,G(\tau,s) = 2\xi(2s)\,E(\tau,s)$

where $E(\tau,s) \equiv \sum_{\gamma\in\Gamma_\infty\backslash\Gamma(1)} \mathrm{Im}(\gamma\tau)^s$.

Computed: $G^*(\tau,s) = \sum_{k\in\mathbb{Z}} A_k^*(y,s)e^{2\pi i k x}$, where:

⊛ $A_0^*(y,s) = 2\xi(2s)y^s + 2\xi(2(1-s))y^{1-s}$

⊛ $A_k^*(y,s) = 2\sqrt{y}\,|k|^{s-\frac12}\sigma_{1-2s}(|k|)\,K_{s-\frac12}(\pi|k|y)$.

Where $K_s(c) \equiv \int_{t=0}^{\infty} e^{-c(t+\frac1t)}t^s\,\frac{dt}{t}$.

Lemma 6.6 1) If $\sigma_0<\sigma_1$ & $c_0>0$ then $\exists C = C(\sigma_0,\sigma_1,c_0)$ s.t. $\forall s\in\mathbb{C}$: $\mathrm{Re}(s)\in[\sigma_1,\sigma_2]$, $\forall c\geq c_0$, $|K_s(c)|\leq Ce^{-c}$.

~~Proof 1)~~ 2) If $s_0\in\mathbb{C}$, then $\exists c>0$, with $K_{s_0}(c)\neq 0$.

Proof 1) $K_s(c) = \int_{t=1}^{\infty} e^{-c(t+t^{-1})}(t^s+t^{-s})\frac{dt}{t}$.

$\Rightarrow$ Suffices to find a bound for $\int_{t=1}^{\infty} e^{-c(t+t^{-1})}t^{\sigma}\frac{dt}{t}$ for t in some compact interval.

This is: $= \int_{t=1}^{2}(-) + \int_{t=2}^{\infty}(-)$.

For $t \geq 1$: $t + t^{-1} \geq 1 \Rightarrow \int_{t=1}^{2} e^{-c(t+t^{-1})} t^{\sigma} \frac{dt}{t}$

$\leq e^{-c} \int_{t=1}^{2} t^{\sigma-1}\, dt = O(e^{-c})$.

& For $t \geq 2$: $\frac{t}{2} \geq 1 \Rightarrow t \geq 1 + \frac{t}{2}. \Rightarrow e^{-ct} \leq e^{-c} e^{-ct/2}$.

$\Rightarrow \int_{t=2}^{\infty} e^{-c(t+t^{-1})} t^{\sigma} \frac{dt}{t} \leq e^{-c} \int_{t=2}^{\infty} e^{-c(\frac{t}{2}+t^{-1})} t^{\sigma} \frac{dt}{t}$.

$\leq e^{-c} \int_{t=2}^{\infty} e^{-c_0(\frac{t}{2}+t^{-1})} t^{\sigma} \frac{dt}{t} = O(e^{-c})$ ✓

2) Want to show, $K_s(c)$ not 0 as function of $c > 0$.

Have: $\int_{c=0}^{\infty} K_{s_0}(c) \cdot c^{s} \frac{dc}{c} = \int_{c=0}^{\infty} \int_{t=0}^{\infty} e^{-ct} e^{-ct^{-1}} c^{s} t^{s_0} \frac{dc\, dt}{ct}$.

Let: $u = ct$ & $v = ct^{-1}$. $\Rightarrow du = c\, dt + t\, dc$

$dv = -\frac{c}{t^2} dt + t^{-1} dc$.

$\Rightarrow du \wedge dv = \frac{c}{t} dt \wedge dc - \frac{c}{t} dc \wedge dt = -2v\, dc \wedge dt$.

$\Rightarrow \frac{dc \wedge dt}{ct} = -\frac{1}{2} \frac{du \wedge dv}{uv}$.

& $\int_{c=0}^{\infty} K_{s_0}(c) c^{s} \frac{dc}{c} = \frac{1}{2} \int_{u=0}^{\infty} \int_{v=0}^{\infty} e^{-u} e^{-v} (uv)^{\frac{s}{2}} \left(\frac{u}{v}\right)^{\frac{s_0}{2}} \frac{du\, dv}{uv}$

$= \frac{1}{2} \int_{u=0}^{\infty} \int_{v=0}^{\infty} e^{-u} u^{\frac{s+s_0}{2}} e^{-v} v^{\frac{s-s_0}{2}} \frac{du\, dv}{uv}$.

$= \frac{1}{2} \Gamma\left(\frac{s+s_0}{2}\right) \Gamma\left(\frac{s-s_0}{2}\right)$.

This computation is valid whenever: $Re(\frac{s+s_0}{2}), Re(\frac{s-s_0}{2}) > 1$

e.g. when $Re(s)$ sufficiently large.

Know: Γ non-vanishing. $\Rightarrow$ for any such s, $\int_{c=0}^{\infty} K_{s_0}(c)c^s\frac{dc}{c} \neq 0$.

$\Rightarrow K_{s_0}(c)$ not 0-function ✓

Corollary 6.7) Let $s \in \mathbb{C} - \{0,1\}$. Then:

1) $G^*(\tau, s)$ not 0-function (as function of τ)

2) $|G^*(\tau,s) - A_0^*(\tau,s)| = O(e^{-\pi y/2})$, as $y \to \infty$

3) $|G^*(\tau,s)| = O(\max(y^\sigma, y^{1-\sigma}))$, as $y \to \infty$.

Notice that 2) is Analogue of $|G_k(\tau) - G_k(\infty)| = O(e^{-2\pi y})$ as $y \to \infty$ for Holomorphic Eisenstein series G_k

& 3) shows $G^*(\tau,s)$ has "Moderate growth".

Proof) 1) If $G^*(\tau,s) \equiv 0$, then $A_1^*(y,s) = 2\sqrt{y}\,K_{s-\frac{1}{2}}(\pi y) \equiv 0$.

But, then, lemma shows $\exists y > 0$, s.t. $A_1^*(y,s) \neq 0$ ✓

2) $|G^*(\tau,s) - A_0^*(y,s)| < 2\sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \sqrt{y}\,|k|^{\sigma-\frac{1}{2}}\sigma_{1-2\sigma}(|k|)\,|K_{s-\frac{1}{2}}(\pi|k|y)|$

& By lemma: $\forall y_0 > 0$, $\exists C > 0$, with $|K_{s-\frac{1}{2}}(\pi|k|y)| \leq Ce^{-\pi|k|y}$ for any $y \geq y_0$.

& $\forall N \geq 1$, $|k|^{\sigma-\frac{1}{2}}\sigma_{1-2\sigma}(|k|) \leq |k|^{N-1}$.

$\Rightarrow \forall y \geq y_0: |G^*(\tau,s) - A_0^*(\tau,s)| \leq 4\sqrt{y} \sum_{k\geq 1} k^N e^{-\pi k y}.$

$= O(e^{-\pi y/2}).$

$\underline{\underline{3)}}$ $|G^*(\tau,s)| \leq O(e^{-\pi y/2}) + |2\xi(2s)y^s + 2\xi(2(1-s))y^{1-s}|.$

$\leq O(e^{-\pi y/2}) + C\max\{y^\sigma, y^{1-\sigma}\}$ as $y\to\infty$.

Theorem 6.18) $\forall t \in \mathbb{R}^\times: \zeta(1+it) \neq 0.$ (Prime NT)

Proof) If $\exists t$, $\zeta(1+it) = 0$, then $\overline{\zeta(s)} = \zeta(\bar{s})$, so $\zeta(1-it) = 0$. So, if $s_0 = \frac{1+it}{2}$, then:

$A_0^*(y, s_0) = 2\xi(1+it)y^{s_0} + 2\xi(1-it)y^{s_0} = 0$

By $\underline{\underline{2)}}$ of Corollary: $|G^*(\tau,s_0)| = O(e^{-\pi y/2})$. $(y\to\infty)$

Consider: $F(s) \equiv \int_{\Gamma(1)\backslash\mathbb{H}} G^*(\tau,s)\overline{G^*(\tau,s_0)} \frac{dx\,dy}{y^2}$, $[s\neq 0,1]$

Is: Absolutely Convergent $\forall s \neq 0,1$. & Holomorphic as a function $s \in \mathbb{C} - \{0,1\}$.

Need to show: $\int |G^*(\tau,s)||G^*(\tau,s_0)| \frac{dx\,dy}{y^2} < \infty$.

But, integrand is $\overset{\text{f}}{:}$ $O(e^{-\pi y/2}\max(y^\sigma, y^{1-\sigma}))$.

$\Rightarrow$ Converges for $s\neq 0,1$ (exponential decay).

Next: Compute $F(s)$, by "Unfolding":

When $\mathrm{Re}(s)>1$: $G^*(\tau,s)=2\xi(2s)E(\tau,s)$

& $E(\tau,s)\equiv\sum_{\gamma\in\Gamma_\infty\backslash\Gamma(1)}\mathrm{Im}(\gamma\tau)^s$.

$\Rightarrow \forall \mathrm{Re}(s)>1$: $f(s)=2\xi(2s)\int_{\Gamma(1)\backslash\mathbb{H}}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma(1)}\mathrm{Im}(\gamma\tau)^s\,\overline{G^*(\gamma\tau,s_0)}\frac{dx\,dy}{y^2}$

$\overset{?}{=}2\xi(2s)\int_{x=-\frac{1}{2}}^{\frac{1}{2}}\int_{y=0}^{\infty}\overline{G^*(\tau,s_0)}\,y^s\frac{dy}{y}$

Provided that $\int_{x=-1/2}^{1/2}\int_{y=0}^{\infty}|G^*(\tau,s_0)|\,|y^\sigma|\frac{dx\,dy}{y^2}<\infty$

& note, this is finite, when $\underset{=\sigma}{\mathrm{Re}(s)}$ is large enough.

[Because: as $y\to\infty$, $|G^*(\tau,s_0)|=O(e^{-\pi y/2})$.
& as $y\to 0$, $|G^*(\tau,s_0)|$ Bounded in $\mathbb{H}^2$, since it is $\Gamma(1)$-invariant & Bounded in $\mathcal{F}$.]

So, when σ big: $f(s)=2\xi(2s)\int_{y=0}^{\infty}\underbrace{\int_{x=-\frac{1}{2}}\overline{G^*(x+iy,s_0)}\,dx}_{=A_0^*(y,s_0)=0.}\cdot y^s\frac{dy}{y^2}$

$=2\xi(2s)\int_{y=0}^{\infty}0\cdot y^{s}\frac{dy}{y^2}=0.$

Now, f is Holomorphic in $\mathbb{C}-\{0,1\}$.

$\Rightarrow f\equiv 0$ in $\mathbb{C}-\{0,1\}$ (Identity Principle)

& Plug in $\underline{s = s_0}$: $f(s_0) = \int_{\Gamma(1)\backslash\mathbb{H}^2} |G^*(\tau, s_0)|^2 \frac{dx\,dy}{y^2} = 0$

But: $G^*(\tau, s_0)$ is C^∞-function & not uniformly 0

(by 1) of Corollary) $\Rightarrow$ This $\neq 0$ $\lightning$.

So, indeed. ✓

Modular forms: Lecture 24* 29/11/2023

Today: Non-examinable!

Discussion of Relation between Modular forms, and Langlands Programme. [Galois Representations.]

Recall: $K/\mathbb{Q}$ Normal ~~sep~~ (e.g. $\overline{\mathbb{Q}}$)

& $\mathrm{Gal}(K/\mathbb{Q}) \equiv \mathrm{Aut}(K/\mathbb{Q})$.

Make into Topological group, by declaring each subgroup $\mathrm{Gal}(K/M)$ as Open ($\forall M/\mathbb{Q}$ finite).

Then: Galois Representation is: Continuous representation

$\rho: \mathrm{Gal}(K/\mathbb{Q}) \longrightarrow GL_n(L)$, for local field L (e.g. $\mathbb{C}$, or $L/\mathbb{Q}_p$)

Example: $E/\mathbb{Q}$ elliptic, $E: y^2z = x^3 + axz^2 + bz^3$. (for $a, b \in \mathbb{Q}$). Then E gives Complete system of Galois representations $\rho_{E,p}: \underbrace{G_{\mathbb{Q}}}_{=\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \longrightarrow GL_2(\mathbb{Q}_p)$

[E.g. $G_{\mathbb{Q}} \circlearrowright E[p^\infty](\overline{\mathbb{Q}})$, or $H^1_{\acute{e}t}(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)$]

Each $\rho_{E,p}$ is Unramified $\forall$ prime $\ell \nmid \Delta_E\, p$.

i.e. $\forall \ell$ as given, can choose embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$ extending $\mathbb{Q} \hookrightarrow \mathbb{Q}_\ell$.

$\Rightarrow$ Get map $G_{\mathbb{Q}_\ell} \hookrightarrow G_{\mathbb{Q}}$ (Decomposition Group)

$\sigma \longmapsto \sigma|_{\overline{\mathbb{Q}}}$

Say: $\rho_{E,p}$ unramified at ℓ $\Leftrightarrow$

$$\rho_{E,p}\big|_{G_{\mathbb{Q}_\ell}} : G_{\mathbb{Q}_\ell} \longrightarrow GL_2(\mathbb{Q}_p)$$

factors through $Gal(\mathbb{Q}_\ell^{ur}/\mathbb{Q}_\ell)$, where $\mathbb{Q}_\ell^{ur} \subset \overline{\mathbb{Q}}_\ell$ is maximal unramified extension of $\mathbb{Q}_\ell$.

If this is unramified at ℓ, then: $\rho_{E,p}(Frob_\ell)$ is defined.

$\Rightarrow$ Have Characteristic poly $\det(X - \rho_{E,p}(Frob_\ell))$.

This does not depend on ANY choices, and is $X^2 - a_\ell X + \ell$, where: $a_\ell \equiv \ell + 1 - \#\tilde{E}(\mathbb{F}_\ell)$.

Modular forms also give rise to Galois representations!

Theorem [Deligne]: let $f \in S_k(\Gamma(1))$ Normalised eigenform and λ be non-Archimedean place of $K_f = \mathbb{Q}(\{a_n(f)\}_{n\geq 1})$.

Then: $\exists$ Galois rep. $\rho_{f,\lambda} : G_{\mathbb{Q}} \longrightarrow GL_2(K_{f,\lambda})$ unramified at all $\ell \nmid \lambda$ & satisfies $\det(X - \rho_{f,\lambda}(Frob_\ell))$

$$= X^2 - a_\ell(f)X + \ell^{k-1}.$$

These reps can be applied to Number theoretic questions:

Theorem (Kummer). Let p odd prime. Then, p is Regular $\Leftrightarrow$ $\forall k$ even, $2 \leq k \leq p-3$, $p \nmid$ Numerator of B_k.

[Regular $\equiv p \nmid h_{\mathbb{Q}(\zeta_p)}$]

Kummer could prove FLT in exponent p, if p regular.

$\boxed{\text{Theorem}}$ [Harbrand – Ribet, 1975]

Consider: Action of $Gal(\mathbb{Q}(e^{2i\pi/p})/\mathbb{Q}) = (\mathbb{Z}/p\mathbb{Z})^{\times}$ on $Pic(\mathbb{Z}[e^{2i\pi/p}])[p] = \bigoplus H_k$.

Where: $H_k = \{ x \in Pic(\mathbb{Z}[e^{2i\pi/p}])[p] : \forall \sigma \in (\mathbb{Z}/p\mathbb{Z})^{\times}, \ \sigma(x) = \sigma^k x \}$.

Then: if k even & $2 \leq k \leq p-3$, then:

p divides Numerator of $B_k \iff H_{1-k} \neq 0$.

Hardest part to prove is that $p \mid B_k \Rightarrow H_{1-k} \neq 0$.

To show this: consider $f_k = \frac{-B_k}{2k} + \sum_{n \geq 1} \sigma_{k-1}(n) q^n$.

$\Rightarrow$ Constant term $= 0$ mod p means: there is a congruence mod p, between f_k and a cuspidal modular form.

(E.g. $f_{12} \equiv \Delta$ mod 691)

Let: f be this normalised Eigenform $\equiv f_k$ mod p.

Then: $\rho_{f,p} : G_{\mathbb{Q}} \longrightarrow GL_2(K_{f,p})$ has property that: associated residual rep $\bar{\rho}_{f,p} : G_{\mathbb{Q}} \longrightarrow GL_2(k/p)$ is isomorphic to $\mathbb{1} \oplus \chi_{cyc}^{k-1}$.

[Residual rep: is Unique Semisimple rep (up to $\cong$), such that for all unram primes ℓ, $\det(X - \rho_{f,p}(Frob_\ell))$ mod p is: $\det(X - \bar{\rho}_{f,p}(Frob_\ell))$.]

In this case: $\det(X - \overline{\rho}_{f,p}(\mathrm{Frob}_\ell)) = X^2 - (1+\ell^{k-1})X + \ell^{k-1}$
$= \det(X - (\mathbb{1} \oplus \chi_{cyc}^{k-1}))(\mathrm{Frob}_\ell)$.

& Know: $\rho_{f,p} : G_{\mathbb{Q}} \to GL_2(K_{f,p})$ factors through $\mathrm{Gal}(L_{f,p}/\mathbb{Q})$, where $L_{f,p} = \overline{\mathbb{Q}}^{\ker(\rho_{f,p})}$ "Field cut out by $\overline{\rho}_{f,p}$".

Observation of Ribet: Existence of Congruence forces $L_{f,p}$ to contain an everywhere unramified ext. of $\mathbb{Q}(e^{2i\pi/p})$. Then, by CFT, $H_{1-k} \neq 0$.

2014]: "Main Conjecture of Iwasawa Theory, GL_2".
$\Rightarrow$ Birch – Swinnerton-Dyer Conjecture holds, for "most" elliptic curves $/\mathbb{Q}$.

Big Picture $\left(L^2(GL_n(\mathbb{Q}))\backslash {}^{GL_n(\mathbb{A}_\mathbb{Q})}/\mathbb{R}_{>0}\right) \curvearrowleft GL_n(\mathbb{A}_\mathbb{Q})$

& An irred subrep π is an Automorphic rep of $GL_n(\mathbb{A}_\mathbb{Q})$.

Conjecture: $\exists$ Bijection $\left\{\begin{matrix}\text{Cuspidal Automorphic} \\ \text{reps of } GL_n(\mathbb{A}_\mathbb{Q})\end{matrix}\right\}$

$$\Updownarrow$$

$\left\{\begin{matrix}\text{Compatible Systems of} \\ \text{irreducible reps:} \\ (\rho_\lambda : G_\mathbb{Q} \to GL_n(\overline{\mathbb{Q}}_\lambda))_\lambda\end{matrix}\right\}$.

For $n=2$: $\pi \Leftrightarrow$ Holomorphic Modular forms