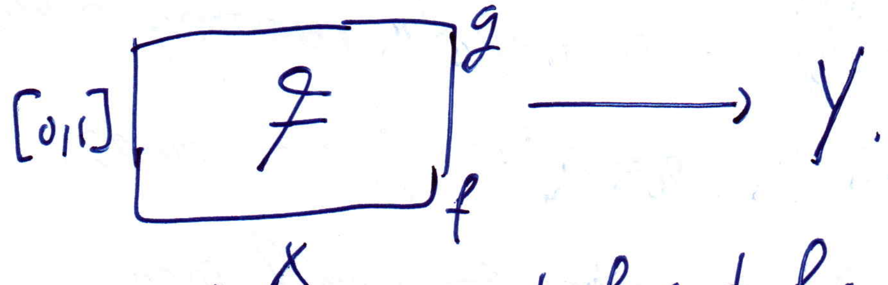
Alg Top: <u>lecture 1</u> 06/10/2023.

<u>Intro</u>. let X, Y top spaces & $f: X \to Y$, $g: X \to Y$ cont. ("a map"). Say f <u>homotopic</u> to g ($f \simeq g$) if: $\exists \mathcal{F}: X \times [0,1] \to Y$, $\mathcal{F}|_{t=0} = f$, $\mathcal{F}|_{t=1} = g$.
$\mathcal{F}$ <u>continuous</u>
(Here: $[0,1]$ has Eucl top.)



$\simeq$ is <u>equiv rel</u> on set of cont fns $X \to Y$.

<u>DEF</u> $f: X \to Y$ <u>homotopy equivalence</u> if $\exists g: Y \to X$ $f \circ g \simeq id_Y$ & $g \circ f \simeq id_X$. (g is homotopy <u>inverse</u>)

Then: Homotopy equiv. is <u>equiv rel.</u> on topological spaces X.

<u>Examples</u> <u>1)</u> If $f: X \to Y$ <u>homeo</u> $\Rightarrow g \equiv f^{-1}: Y \to X$ is exists & is continuous, with $f \circ g = id_Y$ & $g \circ f = id_X$.
$\Rightarrow$ Is homotopy equivalent.

<u>2)</u> $\{0\} \xhookrightarrow{i} \mathbb{R}^n$ is homotopy equivalence: let $p: \mathbb{R}^n \to \{0\}$.
$\Rightarrow p \circ i = id_{\{0\}}$ & $i \circ p = \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto 0$ is homotopy equiv. to $id_{\mathbb{R}^n}$ by $\mathcal{F}(x,t) \equiv tx$.

(At $t=0$: is $i \circ p$
At $t=1$: is identity)

(iii) Inclusion $S^{n-1} \underset{i}{\subseteq} \mathbb{R}^n \setminus 0$.

$\rho: \mathbb{R}^n \setminus 0 \longrightarrow S^{n-1}$, $x \longmapsto \dfrac{x}{\|x\|}$.

$\Rightarrow \rho \circ i = id_{S^{n-1}}$ & $i \circ \rho \simeq id_{\mathbb{R}^n \setminus 0}$ via linear homotopy.

$F(x,t) = tx + \dfrac{(1-t)x}{\|x\|}$.

Notation: X contractible $\Leftrightarrow$ homotopy equiv. to $*$.

Idea of subject: study objects up to homotopy equiv.

Interested in: "connectivity" properties of top. spaces.

Example: 1) X path-connected $\Leftrightarrow$ any 2 maps $* \longrightarrow X$ are homotopic.

Fact: $\mathbb{R}$ path-connected, $\mathbb{R} \setminus 0$ not.

Corollary: (IVT) If $f: \mathbb{R} \rightarrow \mathbb{R}$ cts, $f(x) < 0$, $f(y) > 0$ then $\exists z \in (x,y)$ $f(z) = 0$.

Proof If no such z then $f^{-1}((-\infty, 0))$ & $f^{-1}((0,\infty))$ disconnects (x,y) ↯

2) X (path-conn) is simply conn. if $\forall f: S^1 \longrightarrow X$ is homotopic to some constant map.

($\Leftrightarrow$ any 2 such maps $S^1 \rightarrow X$ are homotopic

$\Leftrightarrow$ any cont. $S^1 \xrightarrow{f} X$ extends to map $F: D^2 \rightarrow X$.)

Fact: $\mathbb{R}^2$ simply connected, $\mathbb{R}^2 \setminus 0$ not.

& If $\gamma: S^1 \longrightarrow \mathbb{R}^2 \setminus 0$ continuous, it has a winding number $\deg(\cdot\gamma) \in \mathbb{Z}$ with:

i) $\gamma_1 \simeq \gamma_2 \Rightarrow \deg(\gamma_1) = \deg(\gamma_2)$

ii) $\gamma_n(t) \equiv e^{2i\pi n t}$ has degree n.

For $n=0$, $\deg(\text{const}) = 0$.

[aside: can define $\deg(\gamma) \equiv \frac{1}{2\pi i}\int_\gamma \frac{dz}{z}$].

Corollary: (FTA). Any non-const poly has root.

Proof let f monic nonconst poly. (WLOG). Assume f has 0 roots. If $\gamma_R(t) \equiv f(Re^{2i\pi t}) : S^1 \to \mathbb{R}^2\setminus 0$ then γ_0 constant. $\Rightarrow \deg(\gamma_0) = 0. \Rightarrow \deg(\gamma_R) = 0 \ \forall R$.

If $R \gg \sum |a_i|$ for $f = z^n + a_{n-1}z^{n-1} + \dots + a_0$:

on $\{|z| = R\}$, $|z^n| \gg |a_{n-1}z^{n-1} + \dots + a_0|$.

$\Rightarrow$ If $f_s(z) \equiv z^n + s(a_{n-1}z^{n-1} + \dots + a_0)$, also have maps $t \mapsto f_s(Re^{2i\pi t})$. defines maps $S^1 \to \mathbb{R}^2\setminus 0$.

For $s=1$: get γ_R. degree $=0$

For $s=0$: get z^n. Has degree $=n$.

$\Rightarrow n = 0$ #

3) X (path-conn) is k-connected $\Leftrightarrow \forall i \le k$, any map $S^i \to X$ is homotopic to constant map.

Fact: $\mathbb{R}^n$ is $(n-1)$-connected & $\mathbb{R}^n\setminus 0$ is not.

Similarly, any $S^{n-1} \xrightarrow{f} \mathbb{R}^n\setminus 0$ has a degree $\deg(f) \in \mathbb{Z}$ such that $(\deg)$ is homotopy invariant & $\deg(\text{inclusion}) = 0$ & $\deg(\text{constant}) = 0$.

Corollary (Brouwer FPT): ~~any~~ any $B^n \to B^n$ cont. has a fixed point.

Proof: If not: $\exists f: B^n \to B^n$ no fixed pts.

$\forall v \in S^{n-1} = \partial B^n$, & $0 \leq R \leq 1$, define $\gamma_R(v) \equiv Rv - f(Rv)$.

$\Rightarrow$ View: $\gamma_R: S^{n-1} \to \mathbb{R}^n \setminus 0$. (since f no f.p.)

When $R=0$: γ_0 const. $\Rightarrow \deg(\gamma_0) = 0 \Rightarrow \deg(\gamma_1) = 0$.

Define: $\gamma_{1,s} \equiv v - s \cdot f(v)$. Then: $\gamma_{1,1} = \gamma_1$. & $\gamma_{1,0} =$ inc.

& If $s<1$ then $\|v\| > \|s f(v)\|$.

$\Rightarrow$ Get: $\gamma_{1,s}: S^{n-1} \to \mathbb{R}^n \setminus 0$. So, get $\deg(\text{inclusion}) = 0$

#.

Alg Top: <u>lecture 2</u> 09/10/2023

1.2: (Co)chain complexes.

Define invariants of spaces in 2 steps:

⊛ Associate to X a chain complex / cochain complex (topology)

⊛ Take (co)homology of said complex. (algebra)

<u>DEF</u>) <u>Chain complex</u> (C_*, ∂) has $(C_n)_{n\in\mathbb{Z}}$ of <u>abelian</u> groups & $\partial_n : C_n \longrightarrow C_{n-1}$ ("boundary homs") with $\underline{\partial^2 = 0}$ $(\Leftrightarrow \partial_n \circ \partial_{n+1} = 0 \ \forall n)$.

The i'th Homology group is: $H_i(C, \partial) \equiv \dfrac{\operatorname{Ker}(\partial_i : C_i \to C_{i-1})}{\operatorname{Im}(\partial_{i+1} : C_{i+1} \to C_i)}$.

$(\partial^2 = 0 \Rightarrow \operatorname{Im}(\partial_{i+1}) \subseteq \operatorname{Ker}(\partial_i))$

Write: $H_* \equiv \bigoplus_{i\in\mathbb{Z}} H_i$ (Graded abelian group)

<u>DEF</u>) <u>Cochain complex</u>: (C^*, ∂^*), $C_{i-1} \xrightarrow{\partial^{i-1}} C_i \xrightarrow{\partial^i} C_{i+1} \xrightarrow{\partial^{i+1}}$ with $\partial^2 = 0$.

Cohomology: $H^*(C^*, \partial)$ where $H^i(C^*) = \dfrac{\operatorname{Ker}(\partial^i)}{\operatorname{Im}(\partial^{i-1})}$.

<u>Terminology</u> Elements of $\operatorname{Ker}(\partial)$ are cycles / cocycles

Elements of $\operatorname{Im}(\partial)$ are boundaries / coboundaries.

Elements of H^*, H_* are homology / cohomology classes.

$\partial \equiv$ differential.

<u>DEF</u>) Given (C_*, ∂) & (D_*, ∂): a <u>chain map</u> $C_* \to D_*$ are group homs $f_i : C_i \to D_i \ \forall i$ s.t. the following diagram commute:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{i+1} \arrow[r] \arrow[d, "f_{i+1}"] & C_i \arrow[r] \arrow[d, "f_i"] & C_{i-1} \arrow[r] \arrow[d, "f_{i-1}"] & \cdots \\
\cdots \arrow[r] & D_{i+1} \arrow[r] & D_i \arrow[r] & D_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

$\Leftrightarrow \partial_i \circ f_i = f_{i+1} \circ \partial_i$. $\forall i$.

<u>Lemma</u> A chain map $f: C_* \to D_*$ induces homs
$f_* : H_i(C_*) \longrightarrow H_i(D_*)$.

<u>Proof</u> Let: $a \in H_i(C_*)$. Represented by $\alpha \in C_i(C_*)$, $\partial\alpha = 0$.

$\Rightarrow \partial(f_i(\alpha)) = f_{i-1}(\underbrace{\partial(\alpha)}_{=0}) = 0$. so $f_i(\alpha)$ is cycle in D_* — complex.

$\Rightarrow \cancel{f_i(\alpha) \in}$

Provisionally set: $f_*(a) \equiv [f_i(\alpha)] \in H_i(D_*)$.

<u>Need</u>: well-defined (indep. of α)

If $a = [\alpha] = [\alpha']$ $\Rightarrow \alpha - \alpha' \in im(\partial : C_{i+1} \to C_i)$.

$\Rightarrow \exists \tau$, $\alpha - \alpha' = \partial\tau$.

$\Rightarrow f_i(\alpha) - f_i(\alpha') = f_i(\partial\tau) = \partial(f_{i+1}(\tau)) \in Im\, \partial$.

$\Rightarrow [f_i(\alpha)] = [f_i(\alpha')]$ ✓

So, f_* is well-defined. Can check: f_* defines homs.

Correspondingly: cochain map $f: C^* \longrightarrow D^*$ has homs f^i, s.t.

$$\begin{tikzcd}
C_{i-1} \arrow[r] \arrow[d, "f^{i-1}"] & C_i \arrow[r] \arrow[d, "f^i"] & C_{i+1} \arrow[r] \arrow[d, "f^{i+1}"] & \cdots \\
D_{i-1} \arrow[r] & D_i \arrow[r] & D_{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

commute.

<u>Exercise</u>: Construction <u>natural</u>: $(id)_* = id_{H_i(C_*)}$.

& If $f: C_* \longrightarrow D_*$ & $g: D_* \longrightarrow E_*$, then:

$(g\circ f): C_* \to E_*$ also chain map. $(g\circ f)_* = g_* \circ f_*$.

<u>Goal</u>: associate, with X, (co)chain complexes C_*, C^*.

$\Rightarrow$ Singular (co)homology!

<u>DEF</u>] Standard simplex $\Delta^n \equiv \{(t_0,\dots,t_n) \subseteq \mathbb{R}^{n+1} : t_i \geq 0 \;\&\; \sum t_i = 1\}$.

<u>i'th face</u>: $\Delta^n_i \equiv \{t_i = 0\} \subseteq \Delta^n$.

<u>Have</u>: canonical homeo. $\Delta^{n-1} \to \Delta^n_i$, ~~$(t_0, t_1, t_2, \dots)$~~

$(t_0,\dots,t_{n-1}) \mapsto (t_0,\dots,t_{i-1},0,t_i,\dots,t_{n-1})$.

<u>DEF</u>] If X space: a singular n-simplex of X is a continuous map $\delta : \Delta^n \to X$.

<u>Singular chain complex</u>: $(C_*(X), \partial)$:

$$C_i(X) \equiv \Big\{ \sum_{j\leq N} n_j \sigma_j : N < \infty \;\&\; \sigma_j : \Delta^i \to X \text{ cts} \Big\}.$$

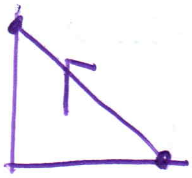
$\&$ $\partial : C_i(X) \to C_{i-1}(X)$

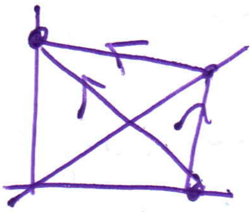
$$\sigma \mapsto \sum_{j=0}^{i} (-1)^j \sigma|_{\Delta^i_j} \equiv \sum (-1)^j \sigma \circ \delta_j.$$

(extended linearly).

<u>Note</u>: If $\{v_i\}_{0\leq i\leq n}$ ordered pts of $\mathbb{R}^{n+1}$ $\&$ $(v_i - v_0)_{1\leq i\leq n}$ are linearly indep, then $\{v_i\}$ determine n-simplex $[v_0,\dots,v_n]$ by taking convex hull. $\Delta^n \to \mathbb{R}^{n+1}$, $\underline{t} \mapsto \sum t_i v_i$.

Orient edges of standard simplex of Δ^n by saying $v_i < v_j \Leftrightarrow i < j$.





Lemma: $\partial^2 = 0$.

Proof: If σ is n-simplex: $\partial(\partial\sigma)$

denote: σ defined on some $[v_0, \ldots, v_n]$ $\Rightarrow \partial\sigma = \sum_i (-1)^j \sigma|_{[v_0, \ldots, \hat{v}_j, \ldots]}$

$$\Rightarrow \partial(\partial\sigma) = \sum_{i<j} (-1)^i (-1)^j \sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_n]}$$

$$+ \sum_{i>j} (-1)^{i+1} (-1)^j \sigma|_{[v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_n]}$$

$$= 0 \checkmark$$

DEF) Singular homology $H_i(X)$ are homology groups of singular chain complexes.

Note Obviously these are homeomorphism invariants of X.

If $f: X \to Y$ cts & $\sigma: \Delta^n \to X$ is in $C_n(X)$ then $f \circ \sigma: \Delta^n \to Y$. Also, $f \circ (\sigma \circ \delta_j) = (f \circ \sigma) \circ \delta_j$.

$\Rightarrow f_*: C_*(X) \to C_*(Y)$ is a chain map.

$\sigma \mapsto f \circ \sigma$.

Alg Top: Lecture 3. 11/10/2023

<u>Recall</u>: $C_i(X) \equiv$ free abelian group of i-simplices on X.
(cts maps $\Delta^i \xrightarrow{\sigma} X$).

Has: boundary operator $\partial\sigma \equiv \sum_j (-1)^j \sigma \circ \delta_i$, $\delta_i \equiv i$'th face.

& $H_*(X) = \bigoplus_{i \geq 0} H_i(C_*, \partial)$ (using $\partial^2 = 0$).

<u>DEF</u> i'th singular cochain complex $C^i(X) \equiv \mathrm{Hom}(C_i(X), \mathbb{Z})$.

& boundary operator $C^i(X) \xrightarrow[\partial^i]{} C^{i+1}(X)$ is <u>adjoint</u> to ∂.

<u>i.e.</u> $\forall \psi \in C^i(X)$: $\partial^* \psi \in C^{i+1}(X)$ defined so that:
$(\partial^*\psi)(\sigma) = \psi(\partial\sigma) \in \mathbb{Z}$ (σ is $(i+1)$-th simplex $\sigma: \Delta^{i+1} \to X$).

Since $\partial^2 = 0$, get $(\partial^*)^2 = 0$.

<u>Define</u>: singular Cohomology $H^*(X) = \bigoplus_{i \geq 0} H^i(C^*, \partial^*)$.

<u>Recall</u>: $f: X \to Y$ cts $\Rightarrow$ induces chain map $f_*: C_*(X) \to C_*(Y)$.
via: $\sigma \longmapsto f \circ \sigma$. $(\Delta^i \to X) \longrightarrow (\Delta^i \to Y)$.

<u>Similarly</u>: $f^*: C^*(Y) \to C^*(X)$ (direction change!)

$\forall \psi \in C^*(Y)$: $(f^*(\psi))(\tau) \equiv \psi(\underbrace{f_*(\tau)}_{\Delta^i \to Y})$.
$\tau: \Delta^i \to X$

Check: $(\partial^* f^* \psi)(\tau) = (f^*(\psi))(\partial \tau) = \psi(f_*(\partial \tau))$

$= (\partial^* \psi)(f_* \tau) = f^*(\partial^* \psi)(\tau).$

$\Rightarrow \partial^* f^* = f^* \partial^*$. So, f^* is a co-chain map.

$\Rightarrow f^*$ defines hom. $f^* : H^*(Y) \longrightarrow H^*(X)$

(i.e. $f^* : H^i(Y) \longrightarrow H^i(X)$. $\forall i$.)

<u>Warning</u>: $C^i(X) = \mathrm{Hom}(C_i(X), \mathbb{Z})$

$H^i(X) \neq \mathrm{Hom}(H_i(X), \mathbb{Z})$.

Basic Computations.

<u>Lemma</u> X point $\Rightarrow H_n(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{else.} \end{cases}$

<u>Proof</u> $C_i \equiv$ free abelian group, on cts maps $\Delta^i \longrightarrow X$.

Only 1 such map: <u>constant map</u>. to $\{pt\}$.

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & C_3 & \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 & \longrightarrow & C_{-1} & \longrightarrow \cdots \\ & \| & & \| & & \| & & \| & & \| & \\ \cdots \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow \cdots \end{array}$$

<u>Have</u>: $\partial(\Delta^1) = \partial(\bullet \longrightarrow \bullet) =$ Right endpt $-$ left endpt.

$\partial(\Delta^2) = \partial(\triangle) = e_0 - e_1 + e_2$.

$\Rightarrow \partial(\sigma_n) = \begin{cases} \sigma_{n-1} & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd.} \end{cases}$

($\sigma_n \equiv$ Generator of $C_n(X)$.)

$$\cdots \xrightarrow{0} \overset{2}{\mathbb{Z}} \xrightarrow{\cong} \overset{1}{\mathbb{Z}} \xrightarrow{0} \overset{0}{\mathbb{Z}} \longrightarrow 0 \longrightarrow \cdots$$

$\Rightarrow H_0(X) = \dfrac{\ker(\mathbb{Z} \longrightarrow 0)}{\operatorname{Im}(\mathbb{Z} \xrightarrow[0]{} \mathbb{Z})} = \mathbb{Z}$

& $H_i(X) = 0 \quad \forall i \geq 1$.

<u>Lemma</u>) If $X = \bigsqcup_{\alpha \in A} X_\alpha$ $\Leftarrow$ disjoint $\cup$ of path components:

then $H_i(X) = \bigoplus_{\alpha \in A} H_i(X_\alpha)$.

Moreover: if X path-connected: $H_0(X) \cong \mathbb{Z}$.

<u>Proof</u> Δ^i path-connected $\Rightarrow$ Any $\sigma: \Delta^i \longrightarrow X$ restricts to $\sigma: \Delta^i \to X_\alpha$ (some α). All $(\sigma \circ \delta_i)$ faces must also lie in X_α.

$\Rightarrow (C_*(X), \partial) = \bigoplus_{\alpha \in A} (C_*(X_\alpha), \partial)$.

[<u>Note</u>: any element of $C_i(X)$ is <u>finite</u> linear combination of i-simplices in X, meets only finitely many X_α.]

For X path-connected: define $\varepsilon: C_0(X) \longrightarrow \mathbb{Z}$

$$\sum n_i \sigma_i \longmapsto \sum n_i.$$

Since $X \neq \phi$: ε <u>surjective</u>.

& If $\tau: \underset{=[0,1]}{\Delta^1} \longrightarrow X$, then: $\varepsilon(\partial \tau) = \varepsilon(\tau(1) - \tau(0)) = 0$

$\Rightarrow \operatorname{Im}(\partial_1: C_1(X) \longrightarrow C_0(X)) \subseteq \operatorname{Ker}(\varepsilon)$

$$\cdots \longrightarrow C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z}$$

& $H_0(X) = \dfrac{C_0(X)}{\mathrm{im}(\partial_1)}$ (note $\ker(\partial_0) : C_0 \to \cancel{C_{-1}}$)

$\Rightarrow \varepsilon$ descends to $H_0(X) \to \mathbb{Z}$.

X path-connected.

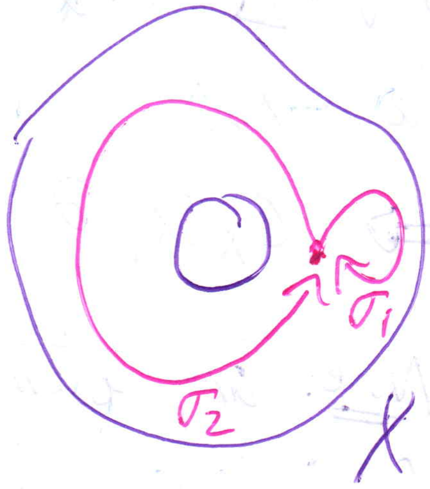
If $\sum_i n_i \sigma_i \in \ker(\varepsilon)$: fix p basepoint.

& path: $\tau_i : \Delta^1 \to X$, $\tau_i(1) = \sigma_i$ & $\tau_i(0) = p$.

$\Rightarrow \partial(\sum_i n_i \tau_i) = \sum_i n_i \sigma_i - \sum_i n_i p$ since $\sum_i n_i \sigma_i \in \ker \varepsilon$.

$\Rightarrow \ker(\varepsilon) = \mathrm{im}(\partial_1)$, so get $H_0(X) \underset{\varepsilon}{\cong} \mathbb{Z}$.

What are we doing? X = Annulus

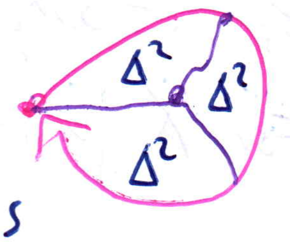


consider: 1-simplices $\sigma_1, \sigma_2 : \Delta^1 \to X$.

Note: $\partial(\sigma_1) = 0 = \partial(\sigma_2)$.

$\Rightarrow [\sigma_1], [\sigma_2] \in H_1(X)$.

"clearly": $[\sigma_1] = 0$: break up area bounded by σ_1 into regions.



$\partial(\tau_0 + \tau_1 + \tau_2)$ = union of 3 1-simplices, that I get by dividing σ_1 into 3 pieces

But: $[\sigma_2] \neq 0$. (?) To show: need structure thms.

Alg Top: Lecture 4. 13/10/2023.

Fundamental Properties.

Theorem (Homotopy invariance).

If $f: X \to Y$ & $g: X \to Y$ homotopic then $f_* = g_*$ & $f^* = g^*$

Corollary If $X \underset{f}{\simeq} Y$ then $H_*(X) \underset{f_*}{\cong} H_*(Y)$ & $H^*(Y) \underset{f^*}{\cong} H^*(X)$.

Proof: $X \underset{f}{\simeq} Y \Rightarrow \exists g: Y \to X$, $g \circ f \simeq id_X$ $f \circ g \simeq id_Y$

$\Rightarrow g_* \circ f_* = id$ & $f_* \circ g_* = id$. So, Isomorphism.

Saying: (co)homology insensitive to inessential deformations.

Example $\{0\} \underset{\simeq}{\hookrightarrow} \mathbb{R}^n \Rightarrow H_*(\mathbb{R}^n) = \begin{cases} \mathbb{Z} & *=0 \\ 0 & * \neq 0 \end{cases}$.

(doesn't detect dimension.)

Other key structural property: Mayer-V

DEF $\longrightarrow C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1} \longrightarrow -$ exact $\Leftrightarrow H_* = 0$.

$\Leftrightarrow im(\partial_{i+1}) = ker(\partial_i)\ \forall i$

Similarly: Cochain complex if $H^*(C^*, \partial^*) = 0$.

If A, B, C abelian groups: $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ exact at B if $im(\alpha) = ker(\beta)$.

Short exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$.

$\Leftrightarrow \alpha$ injective & β surjective & $im\,\alpha = ker\,\beta$.

Examples) $0 \to A \xrightarrow{\alpha} B \to 0 \Rightarrow \alpha$ isomorphism

$0 \to \mathbb{Z} \to G \to \mathbb{Z}/n \to 0$: $G = \mathbb{Z} \oplus \mathbb{Z}/n$ $\quad \alpha = (1,0)$
$\beta = (0,1)$

or: $G = \mathbb{Z}$, $\alpha =$ mult. by n.

Theorem (Mayer Vietoris).

$X = A \cup B$ (A, B open in X). There are MV boundary homs

$\partial_{MV} : H_{i+1}(X) \longrightarrow H_i(\cancel{X}_{A\cap B})$ $\forall i$, such that:

$$\to H_{i+1}(X) \xrightarrow[\partial_{MV}]{} H_i(A\cap B) \xrightarrow[(i_A, i_B)_*]{} H_i(A) \oplus H_i(B) \xrightarrow[(j_A)_* - (j_B)_*]{} H_i(X) \to H_{i-1}(A\cap B) \to \cdots$$

Similarly:
$$\begin{array}{ccc} A\cap B & \xrightarrow{i_A} & A \\ {\scriptstyle i_B}\downarrow & & \downarrow{\scriptstyle j_A} \\ B & \xrightarrow[j_B]{} & \cancel{A}X \end{array}$$

there are boundary homs $H^i(A\cap B) \xrightarrow[\partial^*_{MV}]{} H^{i+1}(X)$

$$H^{i-1}(A\cap B) \xrightarrow[\partial^*_{MV}]{} H^i(X) \xrightarrow[(j_A^*, -j_B^*)]{} H^i(A) \oplus H^i(B) \xrightarrow[(i_A)^* + (i_B)^*]{} H^i(A\cap B) \xrightarrow[\partial^*_{MV}]{} H^{i+1}(X) \to \cdots$$

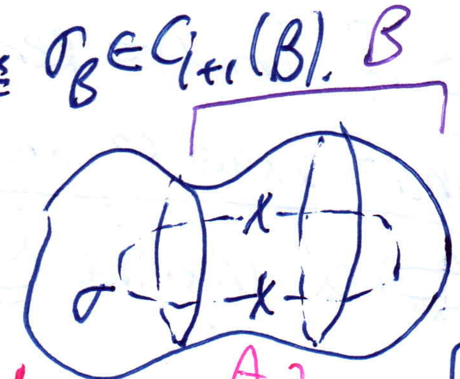
Notes 1) ∂_{MV} not induced by maps of spaces. (algebraic)

Construction sketch: take $\sigma \in C_{i+1}(X)$ cycle ($\partial\sigma = 0$).

& suppose $\sigma = \sigma_A + \sigma_B$, $\sigma_A \in C_{i+1}(A)$ & $\sigma_B \in C_{i+1}(B)$.

($\sigma_{A,B}$ chains, not cycles)

$\Rightarrow \partial\sigma_A = -\partial\sigma_B. \Rightarrow \partial\sigma_A \in A\cap B$.



Set: $\partial_{MV}[\sigma] \equiv [\partial\sigma_A]$.

Since $\partial^2=0$ & $\partial\sigma_A \subseteq A\cap B$: it represents a class in $H_i(A\cap B)$. Note: Correct intuition, not proof!

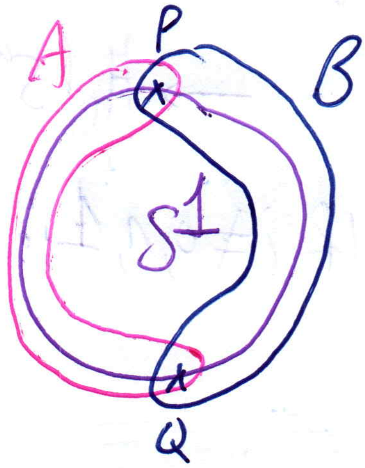
ii) MV sequence is natural! (for maps of pairs).

If $X=A\cup B$, $Y=C\cup D$ & $f: X\to Y$, $f(A)\subseteq C$, $f(B)\subseteq D$,

then:

$$\begin{tikzcd}
H_{i+1}(X) \arrow[r] \arrow[d, "f_*"] & H_i(A\cap B) \arrow[r] \arrow[d, "f_*"] & H_i(A)\oplus H_i(B) \arrow[r] \arrow[d, "f_*"] & H_i(X) \arrow[r] \arrow[d, "f_*"] & \cdots \\
H_{i+1}(Y) \arrow[r] & H_i(C\cap D) \arrow[r] & H_i(C)\oplus H_i(D) \arrow[r] & H_i(Y) \arrow[r] & \cdots
\end{tikzcd}$$

f_* is induced & all ~~par~~ squares commute.



Example: $H_*(S^1) = \begin{cases} \mathbb{Z} & *=0,1 \\ 0 & \text{else.} \end{cases}$

$S^1 = A\cup B$ & $A, B \cong (0,1) \simeq *$.

$A\cap B \cong (-1,0) \oplus (0,1) \simeq p \sqcup q$.

For $i\geq 2$: $\cancel{H_i(A)} \oplus \cancel{H_i(B)} \longrightarrow H_i(X) \longrightarrow \cancel{H_{i-1}(A\cap B)}$.

$\Rightarrow 0 \longrightarrow H_i(X) \longrightarrow 0$, so $H_i(X)=0 \ \forall i\geq 2$.

$i=1$: ~~$H_1(A)\oplus H_1(B) \longrightarrow H_1(X) \longrightarrow H_{0}(A\cap B)$~~

$\cancel{H_1(A\cap B)} \longrightarrow \cancel{H_1(A)} \oplus \cancel{H_1(B)} \longrightarrow H_1(X)$

$\hookrightarrow H_0(A\cap B) \longrightarrow H_0(A)\oplus H_0(B) \longrightarrow H_0(X) \longrightarrow 0$

$\mathbb{Z}^2 = \langle p, q\rangle \xrightarrow{\alpha} \mathbb{Z}^2 \qquad \mathbb{Z}$

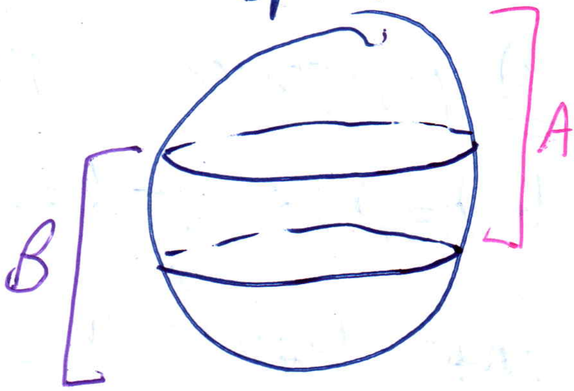
$\alpha(n,m) = (n+m, n+m)$.

$$0 \longrightarrow H_1(X) \xrightarrow{\varphi} \mathbb{Z}^2 \xrightarrow{\alpha} \mathbb{Z}^2 \longrightarrow 0$$

φ injective & $\operatorname{im}\varphi = \ker\alpha \cong \mathbb{Z}$.

$\Rightarrow \varphi : H_1(X) \xrightarrow{\cong} \ker\alpha \cong \mathbb{Z}\langle (1,-1)\rangle = \mathbb{Z}(p-q)$.

<u>Example</u> $H_*(S^n) = \begin{cases} \mathbb{Z} & * = 0, n \\ 0 & \text{else.} \end{cases}$



<u>Proof</u> $X = A \cup B$, $A, B \cong \mathbb{R}^n \simeq *$

& $A \cap B \cong S^{n-1} \times (-\varepsilon, \varepsilon) \simeq S^{n-1}$.

Work <u>Inductively</u> on n & use MV.

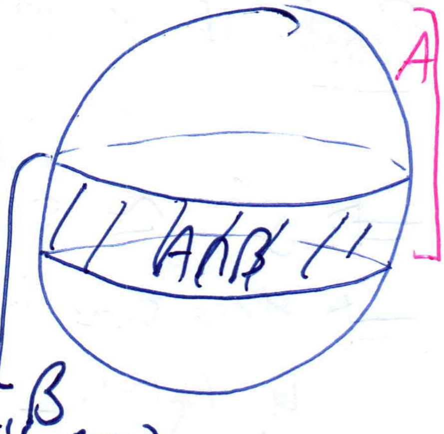
$$\cdots \longrightarrow H_i(S^n) \xrightarrow[\partial_{MV}]{} H_{i-1}(S^{n-1}) \longrightarrow H_{i-1}(*) \oplus H_{i-1}(*)$$

If $i \neq 0, n, 1$: $H_i(S^n) = 0$ $\quad (0 \longrightarrow H_i(S^n) \longrightarrow 0)$

Alg Top: Lecture 5 16/10/2023.

From last time: $H_j(S^n) = \begin{cases} \mathbb{Z} & j=0,n \\ 0 & \text{else.} \end{cases}$

$H^j(S^n) = \begin{cases} \mathbb{Z} & j=0,n \\ 0 & \text{else.} \end{cases}$



We compute: H^j using Mayer-Vietorus.

$S^n = A \cup B$: $A, B \simeq *$ & $A \cap B \simeq S^{n-1}$.

$$H^{i-1}(S^{n-1}) \to H^i(S^n) \to H^i(*) \oplus H^i(*) \to H^i(S^{n-1}) \to H^{i+1}(S^n) \to H^{i+1}(*) \oplus H^{i+1}(*) \to \dots$$

For $i \geq 1$: $0 \to H^i(S^{n-1}) \xrightarrow{\sim} H^{i+1}(S^n) \to 0$

$\Rightarrow H^i(S^{n-1}) \cong H^{i+1}(S^n)$.

For $\underline{i=0}$: $\underset{\mathbb{Z}}{H^0(S^n)} \to \underset{\mathbb{Z}}{H^0(\mathbb{R}^n)} \oplus \underset{\mathbb{Z}}{H^0(\mathbb{R}^n)} \xrightarrow{\alpha} \underset{\mathbb{Z}}{H^0(S^{n-1})} \xrightarrow{\delta} H^1(S^n) \to 0.$ $=0?$

$\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\delta} H^1(S^n) \to 0.$ Need: α surj.

For X path-conn: know: $H_0(X) \cong \mathbb{Z}$ & gen. by any $p \in X$. $p \in C_0(X)$.

Sheet 1 $\Rightarrow H^0(X) \simeq \mathbb{Z}$, gen. by:

$\varphi \in C^0(X) = \mathrm{Hom}(C_0(X), \mathbb{Z})$

$p \mapsto 1.$

$\Rightarrow \alpha(p, q) = p + q$, so indeed surjective ✓

Corollary $\mathbb{R}^n \cong \mathbb{R}^m \Leftrightarrow n=m$.

Proof If $f: \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^m \Rightarrow \mathbb{R}^n \setminus 0 \cong \mathbb{R}^m \setminus f(0)$.
$\cong \mathbb{R}^m \setminus 0$.

$\Rightarrow S^{n-1} \simeq S^{m-1}$ (homotopy equiv).
$\Rightarrow$ Have same homology & cohomology (homotopy invariance).
$H_*(S^{n-1}) = H_*(S^{m-1})$, so indeed $m=n$ ✓

(Contrast: space filling curves.)

DEF) If $f: S^n \to S^n$ cts $\Rightarrow$ induces $f_*: H_n(S^n) \to H_n(S^n)$.
Is: mult by unique integer, degree of f.

Remark $(n>0)$

$$\begin{tikzcd} H_n(S^n) \arrow[r, "f_*"] \arrow[d, phantom, "\cong" rotate=90] & H_n(S^n) \arrow[d, phantom, "\cong" rotate=90] \\ \mathbb{Z} \arrow[r, "f_*"'] & \mathbb{Z} \end{tikzcd}$$

If we choose same isom. $H_n(S^n) \cong \mathbb{Z}$ on both sides, then f_* well-def'ed (not just up to sgn).

Note If $f \simeq g \Rightarrow \deg(f) = \deg(g)$
If $f \simeq id \Rightarrow \deg(f) = 1$.

& Constant maps have degree 0.

[Why? if $S^n \xrightarrow{f} S^n$ const: $\begin{tikzcd} S^n \arrow[r] \arrow[rr, bend left, "f"] & * \arrow[r, hook] & S^n \end{tikzcd}$

$\Rightarrow$ By naturality: $\begin{tikzcd} H_n(S^n) \arrow[r] \arrow[rr, bend right, "f_*"'] & \underbrace{H_n(*)}_{=0} \arrow[r] & H_n(S^n). \end{tikzcd}$]

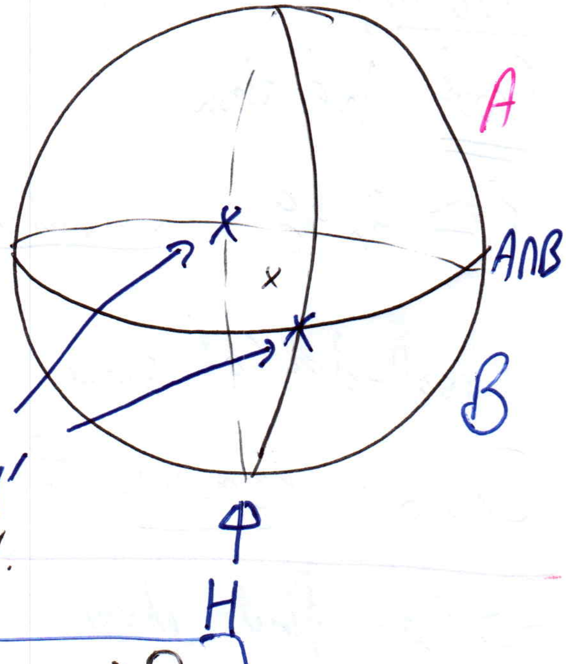
Lemma) If $A \in O(n+1)$ (orthogonal group), then A acts on $S^n \subseteq \mathbb{R}^{n+1}$ by a map of degree $\det(A) = \pm 1$.

Proof Note $O(n+1)$ has 2 path components: $\Rightarrow$ by homotopy invariance of degree, any A is homotopic to id_{S^n} or to $\mathrm{reflection}_H$ ($H \subseteq S^n$ hyperplane).

Divide S^n into A, B as before.

$\Rightarrow$ Notice: reflection in H preserves A, B & $A \cap B$

& H meets $A \cap B$ in a hyperplane H'. $H' \subseteq S^{n-1}$.



$\Rightarrow$ Get exact MV sequence:

$$\begin{tikzcd}
0 \arrow[r] & H_n(S^n) \arrow[r, "\partial_{MV}"] \arrow[d, "\mathrm{Refl}_H"] & H_{n-1}(S^{n-1}) \arrow[r] \arrow[d, "\mathrm{Refl}_{H'}"] & 0 \\
0 \arrow[r] & H_n(S^n) \arrow[r, "\partial_{MV}"] & H_{n-1}(S^{n-1}) \arrow[r] & 0
\end{tikzcd}$$

$(n \geq 2)$

This diagram commutes. & the 2 vertical maps agree.

$\Rightarrow$ Suffices to show result for reflection on $S^0 \subseteq S^1$.

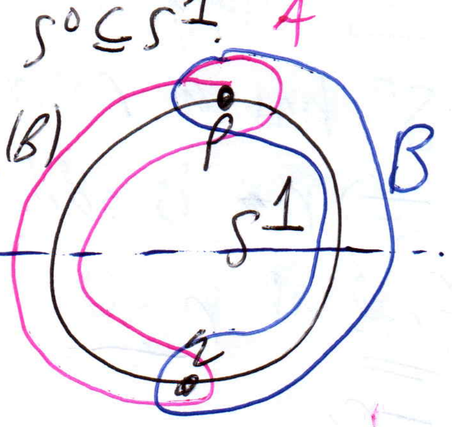
Have: $0 \to H_1(S^1) \to H_0(p \sqcup q) \to H_0(A) \oplus H_0(B)$

$\mathbb{Z} \oplus \mathbb{Z} \longmapsto \mathbb{Z} \oplus \mathbb{Z}$

$(u,v) \longmapsto (u+v, u+v)$.

$\Rightarrow H_1(S^1) \cong \mathbb{Z}$.

Generated by $p-q$.



Reflection in H swaps p, q, sends $p-q \longmapsto q-p$, so acts on $H_1(S^1)$ by -1 ✓

[Corollary] Antipodal map $a_n : S^n \to S^n$, $v \longmapsto -v$ has degree $(-1)^{n+1}$.

[Indeed: $v \longmapsto -v$ is a composition of $n+1$ reflections, and:

$\deg(f\circ g) = \deg(f)\cdot\deg(g)$ since $(f\circ g)_* = f_* \circ g_*$ ✓)

<u>Corollary</u> If $f: S^n \longrightarrow S^n$ <u>no</u> fixed pt, then $f \simeq a_n$.

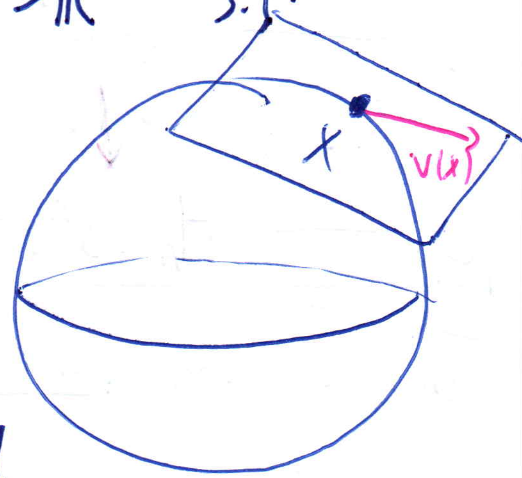
<u>Proof</u> We show: if $f(x) \neq g(x)$ $\forall x$ for $f, g: S^n \longrightarrow S^n$, then $f \simeq a_n \circ g$. <u>Consider</u>: $x \longmapsto \dfrac{t f(x) - (1-t) g(x)}{\|t f(x) - (1-t) g(x)\|}$, which is well-def'ed since denom $\neq 0$ ($f(x) \neq g(x)$).

Exhibits <u>homotopy</u> f to $a_n \circ g$. ✓

$\Rightarrow$ Can find degree of any fixed point free map.

<u>DEF</u> <u>Vector Field</u> on S^n is $v: S^n \longrightarrow \mathbb{R}^{n+1}$ s.t. $\forall x \in S^n$, $\langle x, v(x) \rangle = 0$.

$$\begin{array}{c} S^n \\ \cap \\ \mathbb{R}^{n+1} \end{array}$$



<u>Prop</u> (Hairy Ball Theorem).

S^n has a nowhere vanishing vector field $\Longleftrightarrow$ n is odd.

<u>Proof</u> If <u>n odd</u>: $v(x_1, y_1, \ldots, x_n, y_n) = (-y_1, x_1, -y_2, x_2, \ldots, -y_n, x_n)$.

TBC!

Alg Top: Lecture 6. Chain Homotopies. 18/10/2023.

From last time "Hairy Ball Theorem": S^n has nowhere-zero vector field $\iff$ n odd. (proved existence last time, n odd).

If $\exists v: S^n \longrightarrow \mathbb{R}^{n+1}$ vector field & n even, with $v(x) \neq 0 \ \forall x$.

By replacing v with $\frac{v}{\|v\|}$, WLOG: v takes values in S^n.

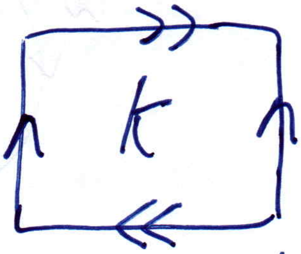
Consider: $v_t(x) \equiv (\cos t)x + (\sin t)v(x)$. Has: unit length $\forall t \ \forall x$.

$\Rightarrow$ Has: $id \simeq a_n \equiv (x \mapsto -x)$.

Comparing degrees: $deg(id) = deg(a_n) = (-1)^{n+1} = -1$ ↯.

Example | Klein Bottle. Gluing 2 Möbius bands around their boundaries. $K = A \cup B$.

$\Rightarrow K = [0,1]^2/\sim$ where $\sim$ is identifying:



$K = A \cup B$, A, B Mobius bands $(\Rightarrow A, B \simeq S^1)$ & $A \cap B \simeq S^1$.

~~$0 \longrightarrow H_2(K) \longrightarrow H_1(A) \oplus H_1(B) \longrightarrow H_1(A \cap B) \longrightarrow H_1(K) \longrightarrow H_0(A)$~~

~~$H_0(A) \oplus H_0(B) \longrightarrow$~~

$\mathbb{Z} \xrightarrow{\psi} \mathbb{Z} \oplus \mathbb{Z}$

$$0 \longrightarrow H_2(K) \longrightarrow H_1(A \cap B) \longrightarrow H_1(A) \oplus H_1(B) \longrightarrow H_1(K) \xrightarrow{0} H_0(A \cap B) \longrightarrow H_0(A) \oplus H_0(B).$$

$$\mathbb{Z} \xrightarrow{\varphi} \mathbb{Z} \oplus \mathbb{Z}$$
$$1 \longmapsto (1,1)$$

φ injective.

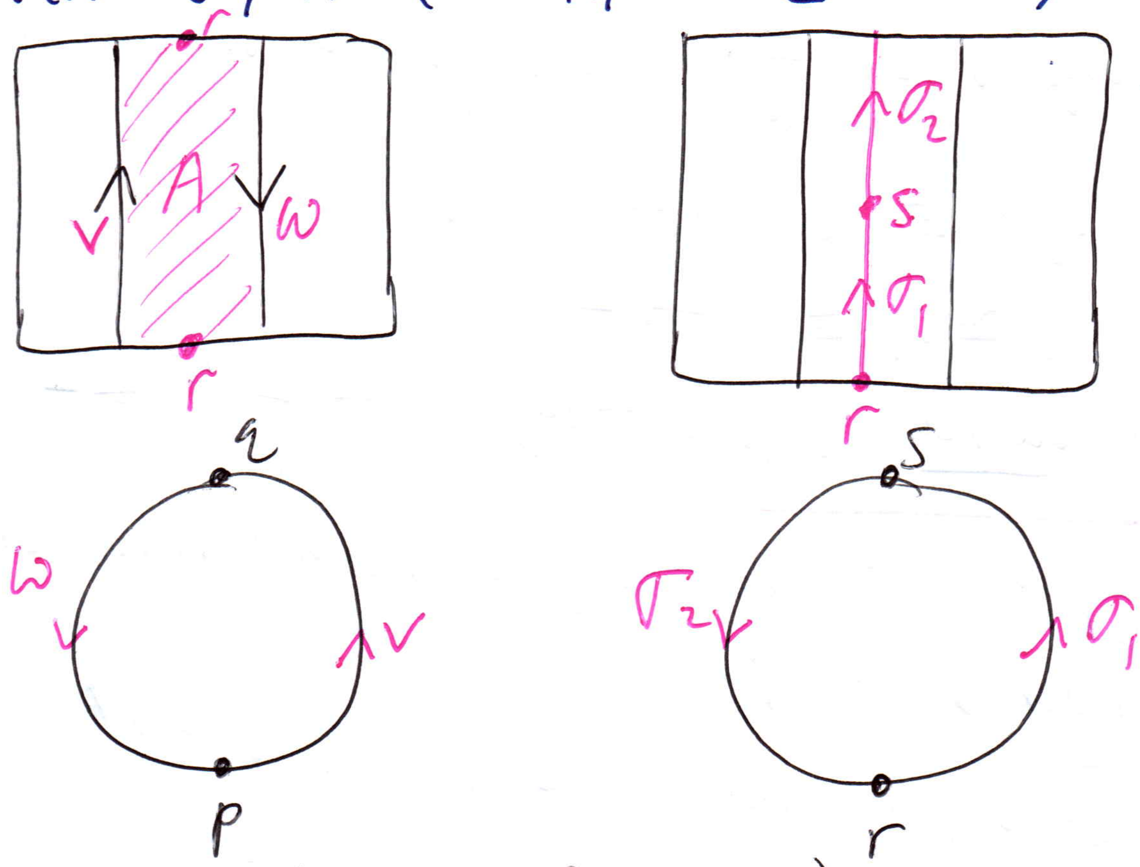
By exactness: $H_1(K) \cong (\mathbb{Z}\oplus\mathbb{Z})/\ker\psi$. (first isomorphism thms.)

& $H_2(K) \cong \ker\psi$.

$\Rightarrow$ Need: ψ.

Claim: $\psi(1) = (2,2)$. $(\Rightarrow H_1(K) \cong \mathbb{Z}\oplus(\mathbb{Z}/2))$

Proof:



Know: $A\cap B \cong S^1$ (boundary of A or B).

$\Rightarrow H_1(\cancel{A}\, A\cap B) = \mathbb{Z}[v_q + w]$ & $\Rightarrow H_1(A) = \mathbb{Z}\langle \sigma_1 + \sigma_2 \rangle$.

At chain level: $\exists$ collection of simplices (2-simplices) in A, with: boundary $v - (\sigma_1 + \sigma_2)$.

$\Rightarrow v \mapsto \sigma_1 + \sigma_2$

& $w \mapsto \sigma_1 + \sigma_2$

$\Rightarrow \psi(1) = (2,2)$ ✓

Intuitively: boundary of Mobius map winds 2x around its core.

Remarks 1) in Cohomology: $H^2(K) \cong \mathbb{Z}/2$ $\Rightarrow H^i \neq H_i$.
$H^1(K) \cong \mathbb{Z}$.

2) Defined: $C_i(X) \equiv \{ \sum_{\text{finite}} a_j \sigma_j : a_j \in \mathbb{Z} \;\&\; \sigma_j : \Delta^i \to X \}$.

Could have also taken: $C_i(X, G) \equiv$ same but $a_j \in G$.
& ∂ defined same before.
$\Rightarrow$ Defines chain complexes $C_*(X; G)$ & $H_*(X; G)$.

If we compute $H_*(K; \mathbb{Z}/2)$: $= \begin{cases} \mathbb{Z}/2 & *=0 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & *=1 \\ \mathbb{Z}/2 & *=2. \end{cases}$

(since: $\varphi \equiv 0$ if over $\mathbb{Z}/2$)

Interlude (Relative homology).

If $A \subseteq X$ subspace: any $\sigma : \Delta^i \to A$ is simplex of X.
& if image$(\sigma) \subseteq A$, then $\forall \delta_j$ boundary face, $\sigma \circ \delta_j$ contained in A.

$$\begin{tikzcd} C_i(A) \arrow[r, hook, "i_A"] \arrow[d, "\partial|_{C_i(A)}"] & C_i(X) \arrow[d, "\partial"] \\ C_{i-1}(A) \arrow[r, hook, "i_A"'] & C_{i-1}(X) \end{tikzcd}$$

Commutes.

Then: faces $(i_A \circ \sigma) \circ \delta_j$ are: $i_A \circ (\sigma \circ \delta_j)$, so: land in image i_A viewed as $C_{i-1}(A) \hookrightarrow C_{i-1}(X)$.

$\Rightarrow (C_*(A), \partial)$ is: sub-complex of $(C_*(X), \partial)$, i.e. is a subgroup $\forall i$ & preserved under ∂ operator.

In this setting, there is a quotient complex:

$C_i(X, A) \equiv C_i(X) / C_i(A)$ $\equiv$ abelian group of i-simplices in X not wholly contained in X.

Since ~~$\partial_i : C_i(X) \to C_i(A)$~~ $\partial : C_i(X) \to C_{i-1}(X)$ sends $C_i(A)$ to $C_{i-1}(A)$: induces $C_i(X,A) \to C_{i-1}(X,A)$.

Turns out: induced map also has $\partial^2 = 0$. So, we get another chain complex $(C_*(X,A), \partial)$.

Its homology is called Relative Homology.

Fact ~~1~~: a) $H_i(A) \overset{i_*}{\hookrightarrow} H_i(X) \underset{\text{quotient}}{\longrightarrow} H_i(X,A) \overset{\text{boundary map}}{\longrightarrow} H_{i-1}(A) \to \dots$

b) If X, A "nicely behaved"
(e.g. ~~X,~~ A is finite set of pts or circles on 2-dim surface,)
then: $H_i(X,A) \cong H_i(X/A)$ $\forall i > 0$.

Paying Debts. Recall: $f \simeq g : X \to Y \Rightarrow f_* = g_* : H_*(X) \to H_*(Y)$.
& $f^* = g^* : H^*(\cancel{X}\ Y) \to H^*(X)$

& $f_* : C_*(X) \to C_*(Y)$ chain map:

$f_* \circ \partial = \partial \circ f_*$.

DEF Let C_*, D_* chain complexes. & $f_*, g_* : C_* \to D_*$ chain maps. Say: chain-homotopic if:

$\exists P_n : C_n \to D_{n+1}$: $\boxed{P_{n-1} \circ \partial_n^C \pm \partial_{n+1}^D \circ P_n = f_n - g_n}$ $\forall n$.

$$\begin{tikzcd}
C_{n+1} \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] \arrow[dl, "P_n"'] & C_{n-1} \arrow[r] \arrow[d] \arrow[dl, "P_{n-1}"'] & \dots \\
D_{n+1} \arrow[r] & D_n \arrow[r] & D_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

Lemma: If $f_*, g_* : C_* \to D_*$ chain homotopic, their induced maps $H_i(C_*) \to H_i(D_*)$ are the same.

Proof $f_i(\alpha) - \cancel{f}\, g_i(\alpha) = (f_i - g_i)(\alpha) = (\partial P - P\partial)(\alpha) = \partial P\alpha$ (if α cycle).

Alg Top: Lecture 7. 20/10/2023.

Exercise Chain homotopy defines ~~map~~ equivalence relation on chain maps of chain complexes.

Theorem If $f \simeq g : X \to Y$ then $f_* = g_* : H_*(X) \to H_*(Y)$

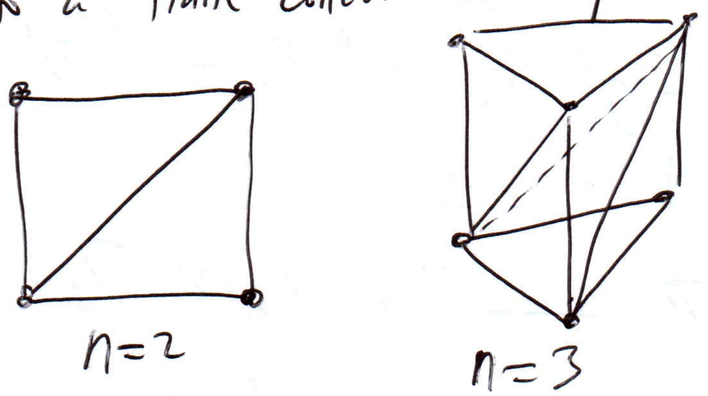
Proof $f \simeq g \Rightarrow \exists F : X \times [0,1] \to Y,\ F \circ i_0 = f,\ F \circ i_1 = g.$

$\Rightarrow f_* = F_* \circ (i_0)_*\ \&\ g_* = F_* \circ (i_1)_*.$

Suffices to show: $(i_0)_*$ & $(i_1)_*$ chain homotopic.

or: find $P : C_n(X) \to C_{n+1}(X \times [0,1])$, $\partial P + P\partial = (i_0)_* - (i_1)_*$.

P is: prism operator from usual way of decomposing $\Delta^n \times [0,1]$ into a finite collection of simplices.



Consider: ordered collections

$[v_0, \dots, v_i\, w_i, \dots, w_n]$:

$[v_0, \dots, v_n] \equiv$ standard Δ^n

$[w_0, \dots, w_n] \equiv$ upper vertices of $\Delta^n \times [0,1]$.

Define: $P : C_n(X) \to C_{n+1}(X \times [0,1])$

$$\sigma \mapsto \sum_{i=0}^{n} (-1)^i (\sigma \times 1)\Big|_{[v_0, \dots, v_i\, w_i \dots w_n]}$$

Key claim: $\partial P + P\partial = (i_1)_* - (i_0)_*$.

Proof: $$\partial P\sigma = \sum_{j \le i} (-1)^i (-1)^j (\sigma \times 1)\Big|_{[v_0 \dots \hat{v}_j \dots v_i\, w_i \dots w_n]}$$

$$+ \sum_{j > i} (-1)^i (-1)^{j+1} (\sigma \times 1)\Big|_{[v_0, \dots, v_i\, w_i, \dots, \hat{w}_j, \dots, w_n]}.$$

Terms $j=i$ on both sums give: $(\sigma\times 1)\Big|_{(\hat{v_i} w_i \ldots w_n)}$

$$- (\sigma \times 1)_{[v_0 \ldots v_n \hat{w_n}]}$$

<u>Check</u>: Other $j\neq i$ terms correspond to $P(\partial\sigma)$. ✓

<u>Remark</u>) If C^*, D^* cochain complexes, the cochain maps f^*, g^* are cochain-homotopic if:

$$\begin{tikzcd}
\cdots \arrow[r] & C^{i-1} \arrow[r] \arrow[d, "f-g"] & C^i \arrow[r] \arrow[d, "f-g"] \arrow[dl, "p^i"'] & C^{i+1} \arrow[r] \arrow[d, "f-g"] \arrow[dl, "p^{i+1}"'] & \cdots \\
\cdots \arrow[r] & D^{i-1} \arrow[r] & D^i \arrow[r] & D^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

$\exists\, p^i : C^i \longrightarrow D^{i+1}$, $\partial P + P\partial = f^* - g^*$.

Easy to check in this case: $f^* = g^* : H^*(C^*, \partial) \longrightarrow H^*(D^*, \partial)$.

<u>Know</u>: $P : C_n(X) \longrightarrow C_{n+1}(X\times[0,1])$ has <u>dual</u>:

$$p^* : \operatorname{Hom}(C_{n+1}(X\times[0,1]), \mathbb{Z}) \longrightarrow \operatorname{Hom}(C_n(X), \mathbb{Z})$$
$$= C^{n+1}(X\times[0,1]) \qquad\qquad = C^n(X)$$

So, $\partial P + P\partial = (i_1)_* - (i_0)_* \Longrightarrow \partial^* p^* + p^*\partial^* = (i_1)^* - (i_0)^*$

$\Longrightarrow f^* = g^* : H^*(Y) \longrightarrow H^*(X)$ if $f \simeq g : X \longrightarrow Y$.

<u>Recall</u>) Short exact sequences $0 \to A \to B \to C \to 0$.

(of abelian groups).

<u>DEF</u>) A short exact sequence of chain complexes $0 \to A_* \longrightarrow B_* \longrightarrow C_* \longrightarrow 0$ is a diagram:

Check: δ well-def'ed (then, easy to see is hom.)

Check: Sequence is exact.

Examples • If G abelian group: introduced: $C_*(X; G)$.

If $0 \to G_1 \to G_2 \to G_3 \to 0$ SES: have an associated SES.

$$0 \to C_*(X, G_1) \to C_*(X, G_2) \to C_*(X, G_3) \to 0.$$

The homs $H_n(X, G_3) \to H_{n-1}(X, G_1)$ are Bockstein homs.

(E.g. when $0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \xrightarrow{q} \mathbb{Z}/n \to 0$)

• $A \subseteq X$ subspace: $0 \to C_*(A) \to C_*(X) \to C_*(X, A) \to 0$.

Associated LES has boundary map $\delta: H_n(X, A) \to H_{n-1}(A)$.

& δ is: LES of the pair (X, A).

Theorem (Excision). X space, $A \subseteq X$ subspace, $Z \subseteq X$ with $\overline{Z} \subseteq \mathring{A}$.

Then: $(X \setminus Z, A \setminus Z) \xhookrightarrow[i]{} (X, A)$ (inclusion) induces isomorphism on relative homologies. $H_n(X \setminus Z, A \setminus Z) \xrightarrow[i_*]{\sim} H_n(X, A)$.

Lemma (5-lemma)

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r] \arrow[d, "\delta"'] & E \arrow[d, "\varepsilon"'] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

Have: commuting diagrams of abelian groups & exact rows.

Then: if $\alpha, \beta, \delta, \varepsilon$ isomorphisms, so is γ.

Corollary If $f: (X, A) \to (Y, B)$ map of pairs, i.e. $f(A) \subseteq B$

& induced $H_*(X) \to H_*(Y)$, $H_*(A) \to H_*(B)$ are isomorp $f: X \to Y$, then: $H_*(X, A) \to H_*(Y, B)$ isom.

$$\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \arrow[d] & \\
0 \arrow[r] & A_{n+1} \arrow[r, "\alpha"] \arrow[d] & B_{n+1} \arrow[r, "\beta"] \arrow[d] & C_{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_n \arrow[r, "\alpha"] \arrow[d] & B_n \arrow[r, "\beta"] \arrow[d] & C_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "\alpha"] \arrow[d] & B_{n-1} \arrow[r, "\beta"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
 & & & &
\end{tikzcd}$$

All squares commute

All rows exact

columns are chains.

Prop In this situation: there's a boundary map $H_n(C_*) \xrightarrow{\delta} H_{n-1}(A_*)$ s.t. get long exact seq of Homology:

$$\to H_{n+1}(C_*) \xrightarrow{\delta} H_n(A_*) \xrightarrow{\alpha} H_n(B_*) \xrightarrow{\beta} H_n(C_*) \xrightarrow{\delta} H_{n-1}(A_*) \xrightarrow{\alpha} \cdots$$

Proof sketch Diagram chasing.

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r, "\alpha"] \arrow[d, "\partial"'] & B_n \arrow[r, "\beta"] \arrow[d, "\partial"'] & C_n \arrow[r] \arrow[d, "\partial"'] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "\alpha"] & B_{n-1} \arrow[r, "\beta"] & C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

Suppose: $c_n \in C_n$ cycle representing $[c_n] \in H_{n+1}(C_*)$.

β surj $\Rightarrow \exists\, b_n \in B_n,\ \beta(b_n) = c_n$.

$\Rightarrow \beta(\partial b_n) = \partial(\beta b_n) = \partial c_n = 0$ (c_n cycle)

$\Rightarrow \partial b_n \in \ker(\beta) = \operatorname{im}(\alpha)$, so $\exists\, a_{n-1} \in A_{n-1},\ \alpha(a_{n-1}) = \partial b_n$.

$\Rightarrow \alpha(\partial a_{n-1}) = \partial(\alpha a_{n-1}) = \partial(\partial b_n) = 0 \Rightarrow \cancel{\alpha}(\partial a_{n-1}) = 0$.

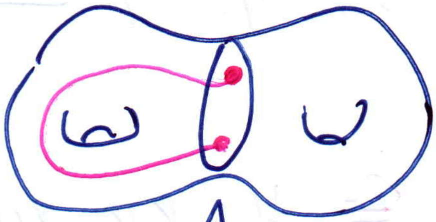
So, indeed: $\partial a_{n-1} = 0$ (α injective)

$\Rightarrow [a_{n-1}] \in H_{n-1}(A_*)$. Define: $\delta[c_n] \equiv [a_{n-1}]$.

Alg Top: Lecture 8

23/10/2023.

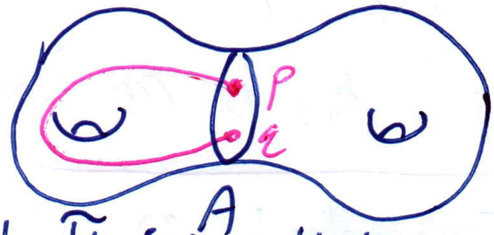
Key Picture:



Is: ~~a~~ cycle in $C_*(X,A)$ but <u>not</u> in $C_*(X)$.

(since: A is "modded out" to a point in X/A.)

It constructs a class in $H_*(X,A)$, where the image under $H_*(X,A) \longrightarrow H_0(A)$ is $p-q$.



DEF (Reduced Homology).

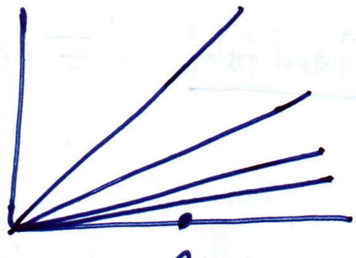
If X space & $x_0 \in X$ basepoint, set $\widetilde{H}_*(X) \equiv H_*(X, x_0)$.

Exercise: $\widetilde{H}_i(X) \equiv H_i(X)\ \forall i>0$ & $\widetilde{H}_0(X) \oplus \mathbb{Z} \simeq H_0(X)$

DEF (X,A) good if: $A \subseteq X$ closed & neighborhood deformation retract: i.e. $\exists A \subseteq U \underset{\text{open}}{\subseteq} X$ & $H: U \times [0,1] \longrightarrow U$ with $H(u,0)=u$, $H(u,1)\in A\ \forall u \in U$ & $H(a,t)=a\ \forall a \in A$.

Example

Non-example



Prop If (X,A) good: the natural map $(X,A) \longrightarrow (X/A, A/A)$ induces isomorphism $H_*(X,A) \xrightarrow{\sim} H_*(X/A, A/A) \simeq \widetilde{H}_*(X/A)$.

$A/A \leftrightarrow pt$

Proof let $A \subseteq U \subseteq X$ as in def. of good pair.

From last time: $H_*(A) \xrightarrow{\sim} H_*(U)$ & $H_*(X,A) \xrightarrow{\sim} H_*(X,U)$.

& since $H: U \times [0,1] \to U$ <u>fixed</u> on A: induces ~~homotopy~~ homotopy on the quotient of A. So, $A/A = pt \hookrightarrow U/A$ is <u>also</u> a nbhood detraction.

$$\begin{array}{ccccc}
H_*(X,A) & \xrightarrow{\cong} & H_*(X,U) & \xleftarrow{\cong} & H_*(X/A, U/A) \\
 & \uparrow & & \uparrow & \\
\downarrow & \text{Homotopy} & & \text{Excision} & \cong\downarrow \ \text{(Homeo. of pairs)} \\
 & \downarrow & & \downarrow & \\
H_*(X/A, pt) & \xrightarrow{\cong} & H_*\left(\frac{X}{A}, \frac{U}{A}\right) & \xleftarrow{\cong} & H_*\left((X/A)/pt, \frac{U}{A}/pt\right)
\end{array}$$

so, this is an isomorphism.

<u>Examples</u> i) $H_j(D^n, \partial D^n) \cong \tilde{H}_j(D^n/\partial D^n) = \tilde{H}_j(S^n)$ $= \begin{cases} \mathbb{Z} & j=n \\ 0 & \text{else} \end{cases}$

ii) $H_j(S^2, S^1_{eq}) = \tilde{H}_j(\text{◎◎})$

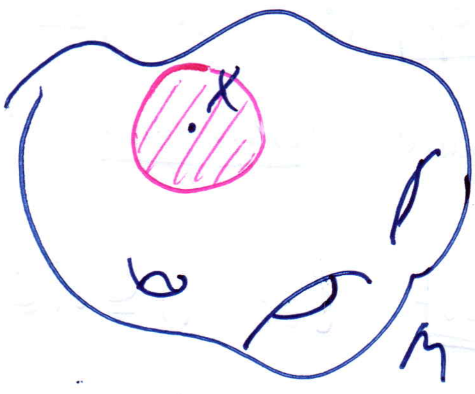
$= \tilde{H}_j(S^2 \vee S^2)$ (wedge sum of spaces)

$= \begin{cases} \mathbb{Z}^2 & \text{if } j=2 \\ 0 & \text{else.} \end{cases}$

iii) <u>Manifold</u> ($\equiv$ dim $= n$ Hausdorff top space, locally homeo to $\mathbb{R}^n$, i.e. $\forall x \in M \ \exists x \in \underset{\text{open}}{U \subseteq} X$, $\underset{\text{homeo}}{U \cong} \mathbb{R}^n$)

If M manifold, $x \in M$: $H_j(M, M-x) \cong H_j(\mathbb{R}^n, \mathbb{R}^n - 0)$ (by <u>excision</u>, removing ~~a disc~~ everything outside a disc.)

& $\mathbb{R}^n$ homotopic to ball.



$\Rightarrow = H_j(D^n, \partial D^n) = \begin{cases} \mathbb{Z}, & j=n \\ 0 & \text{else.} \end{cases}$

Back to MV & excision. (Both follow from "Small simplices them!")

Let: X space & $\mathcal{U} \equiv \{U_\alpha : \alpha \in I\}$ collection of subsets of X, s.t. interiors of U_α cover X.

Define: $C_j^{\mathcal{U}}(X) \equiv \left\{ \sum_{\text{finite}} a_i \sigma_i \;:\; a_i \in \mathbb{Z} \;\&\; \sigma_i : \Delta^j \to X, \; im(\sigma_i) \subseteq U_{\alpha(\sigma_i)}. \right\}$

$\alpha(\sigma_i) \in I$.

Is: subcomplex of $C_j(X)$.
(for usual boundary map.)

Theorem ("Small simplices")

The inclusion $C_*^{\mathcal{U}}(X) \hookrightarrow C_*(X)$ induces isomorphism of homology.

Remark If X, Y spaces, with covers $\mathcal{U}$ & $\mathcal{V}$, & if $f: X \to Y$ sends $U_\alpha \in \mathcal{U}$ to some $V_{f(\alpha)} \in \mathcal{V}$, then:

$C_*^{\mathcal{U}}(X) \xrightarrow[f_*]{} C_*^{\mathcal{V}}(X)$.

But: $C_*^{\mathcal{U}}(X)$ not functorial for arbitrary maps.

Corollary 1 (MV theorem) Let $\mathcal{U} = \{A, B\}$, $A, B \subseteq X$ open.

SES: $0 \to C_*(A \cap B) \to C_*(A) \oplus C_*(B) \to C_*^{\mathcal{U}}(X) \to 0$.

surj, since $C_*^{\mathcal{U}}(X)$.

By general theory: get LES in homology, which gives MV theorem.

Remarks ⊛ MV boundary map

⊛ Naturality of MV under maps of pairs is: the naturality

$C_*^u(X) \longrightarrow C_*^u(Y)$ if $X = A\cup B = \mathcal{U}$ & $A \xrightarrow{f} C$

$\qquad\qquad\qquad\qquad Y = C\cup D = \mathcal{U} \qquad B \xrightarrow{f} D$

Corollary 2 (Excision)

If $Z \subseteq A \subseteq X$ & $\overline{Z} \subseteq \mathring{A}$ then $(X\setminus Z, A\setminus Z) \hookrightarrow (X,A)$ is isom. on Homology.

Let: $B \equiv X\setminus Z$ & $\mathcal{U} \equiv \{A, B\}$.

$\Rightarrow \mathring{A} \cup \mathring{B} = X$, so allowable cover for small simplices.

Consider: $C_n^u(X)/C_n(X). \equiv C_n(B)/C_n(A\cap B)$

$\equiv$ free abelian group on simplices in B, not wholly in A.

$$\begin{tikzcd}
0 \arrow[r] & C_*(A) \arrow[r] \arrow[d, hook] & C_*^u(X) \arrow[r] \arrow[d, hook] & C_*^u(X)/C_*(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_*(A) \arrow[r] & C_*(X) \arrow[r] & C_*(X)/C_*(A) \arrow[r] & 0.
\end{tikzcd}$$

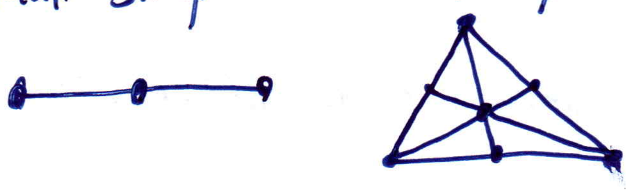
Get induced maps on Homology:

$$\begin{tikzcd}
H_i(A) \arrow[r] \arrow[d, "\|"] & H_i(C_*^u(X)) \arrow[r] \arrow[d, "\cong\ (\text{small simp.})"] & H_i(C_*^u(X)/C_*(A)) \arrow[r] \arrow[d, "\cong\ (5\text{ lemma})"] & H_{i-1}(A) \arrow[r] \arrow[d, "\cong"] & \cdots \\
H_i(A) \arrow[r] & H_i(X) \arrow[r] & H_i(X,A) \arrow[r] & H_{i-1}(A) \arrow[r] & \cdots
\end{tikzcd}$$

$\Rightarrow H_i\left(\dfrac{C_*^u(X)}{C_*(A)}\right) \cong H_i(X,A).$ $\quad\cong$

& $H_i \nearrow = H_*\left(\dfrac{C_*(B)}{C_*(A\cap B)}\right) = H_*(B, A\cap B) = H_*(X\setminus Z, A\setminus Z)$ ✓

Small simplices use Barycentric Subdivision.



Alg Top: lecture 9 $\qquad$ 25/10/2023.

Recall: If $\mathcal{U} = \{U_\alpha : \alpha \in I\} \subseteq \mathcal{P}(X)$ covers X (interiors cover X) then $C_*^{\mathcal{U}}(X) \hookrightarrow C_*(X)$ induces an isomorphism of homology.

Small simplices If Δ^n standard simplex: $b_n = \frac{1}{n+1}(1,\dots,1)$ "barycentre".

If $\sigma : \Delta^{i} \to \Delta^n$ i-simplex:

$$\mathrm{Cone}_i^{\Delta^n}(\sigma) : \Delta^{i+1} \longrightarrow \Delta^n, \quad (t_0,\dots,t_{i+1}) \longmapsto t_0 b_n + (1-t_0)\sigma\left(\frac{(t_1,\dots,t_n)}{1-t_0}\right).$$

Can view $\mathrm{Cone}_i^{\Delta^n} : C_i(\Delta^n) \to C_{i+1}(\Delta^n)$.

Exercise: $\partial(\mathrm{Cone}_i^{\Delta^n}(\sigma)) = \begin{cases} \sigma - \mathrm{Cone}_{i-1}^{\Delta^n}(\partial\sigma), & i>0 \\ \sigma - \varepsilon(\sigma) b_n, & i=0 \end{cases}$

$\varepsilon(\sigma) =$ "sum of coeffs of σ"

$\varepsilon(\sum_i n_i \sigma_i) = \sum_i n_i$.

If **define** $c_* : C_*(\Delta^n) \longrightarrow C_*(\Delta^n)$, $\sigma \longmapsto \begin{cases} \varepsilon(\sigma) b_n & * = 0 \\ 0 & * > 0 \end{cases}$

then $\partial(\mathrm{Cone}^{\Delta^n}) - \mathrm{Cone}^{\Delta^n}(\partial) = id - c_*$.

Aim: Introduce **subdivisions** $\varphi : C_*(X) \longrightarrow C_*(X)$ which does what our algorithm says (for each simplex, **divide** boundary & cone off to barycentre)

DEF A collection of chain maps $\varphi^X : C_*(X) \longrightarrow C_*(X)$ **natural** if: $f : X \to Y \Rightarrow f_* \circ \varphi^X = \varphi^Y \circ f_*$.

If $\sigma : \Delta^n \to X$ & $i_n : \Delta^n \xrightarrow{id} \Delta^n$: then σ is $\sigma \circ i_n$

$\Rightarrow \varphi_n^*(\sigma) = \varphi_n^*(\sigma_* \circ i_n) = \sigma_*(\varphi_n^{\Delta^n}(i_n))$

$\Rightarrow$ If subdivide n-simplex: can (naturally) extend & define φ^* $\forall *$: $\sigma \overset{\varphi_n^*}{\longmapsto} \sigma_*(\mathrm{Cone}_{n-1}^{\Delta^n}(\varphi_{n-1}^{\Delta^n}(\partial i_n)))$

as inductive def. in n

Lemma i) If σ simplex $[v_0, \dots, v_n] \subseteq \mathbb{R}^{n+1}$ (e.g. any n-simplex in a subdivision of Δ^N, some $N \geq n$) then: for any simplex in its barycentric subdivision,

$$\mathrm{diam}_{\mathrm{eucl}} \leq \frac{n}{n+1} \mathrm{diam}_{\mathrm{eucl}}(\sigma).$$

ii) If $\sigma \in C_n^{\mathcal{U}}(X) \Rightarrow \varphi_n^*(\sigma) \in C_n^{\mathcal{U}}(X)$.

iii) If $\sigma \in C_n(X) \Rightarrow \exists k > 0$, $(\varphi_n^*)^k(\sigma) \in C_n^{\mathcal{U}}(X)$.

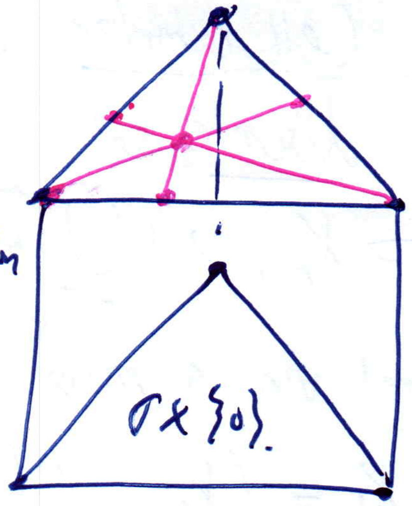
Proof i) Euclidean geo! ii) Obvious.

iii) Note: σ finite sum of simplices in X, so: suffices to show for a single $\sigma : \Delta^n \to X$.

Since $\{\sigma^{-1}(\mathrm{int}\, U_\alpha)\}_{\alpha \in A}$ open cover of Δ^n, then has Lebesgue number $\varepsilon > 0$, i.e. any open ε-ball in Δ^n lies in some subset of this cover. So, pick k s.t. $\left(\frac{n}{n+1}\right)^k < \varepsilon$ ✓

Fact there is a natural (WRT maps of spaces) chain bdry: $P_*^* : C_*(X) \to C_{*+1}(X)$ s.t. $\partial P_n^* + P_{n+1}^* \partial = \varphi_n^* - id_{C_n(X)}$

[Idea of P_n^*: form a <u>union</u> of $(n+1)$-simplices in $\Delta^n \times [0,1]$ by joining $\Delta^n \times \{0\}$ & barycentric subdivision of top to barycentre of top.]



<u>Conclusion of proof</u> [redacted]

let: $u: H_n(C_*^u(X)) \longrightarrow H_n(C_*(X))$.

let: $[c] \in H_n(X)$ & pick k, s.t. $(\varphi_n^*)^k(c) \in C_n^u(X)$.

<u>Know</u>: $\varphi^* \simeq id$ (chain homotopic) $\Rightarrow (\varphi^*)^k \simeq id$.

$\Rightarrow \exists F,\ \partial F + F\partial = (\varphi^*)^k - id$.

<u>Then</u>: $(\varphi^*)^k(c) = c +$ (image of ∂), so u is <u>surj</u> on $H_n(X)$.

& If $u([c]) = 0$: $([c] \in H_n(C_*^u(X)))$:

$\exists\, z \in C_{n+1}(X)$ s.t. $\partial z = c$,

$\exists\, k \geq 1$ s.t. $(\varphi_{n+1}^*)^k(z) \in C_{n+1}^u(X)$

& $(\varphi_{n+1}^*)^k(z) - z = (\partial F + F\partial)(\cancel{z}\ c)$

Can <u>check</u>: $\underbrace{\partial(\varphi^*)^k(z) - \partial F(\partial z)}_{\in C_{n+1}^u(X).} = \partial z = c$

$\Rightarrow$ Get $[c] = 0 \in H_n(C_*^u(X))$ ✓

Singular homology is <u>not</u> effective on "nice" spaces.

One such class is <u>cell complexes</u> (CW complexes).

Will introduce a more manageable chain complex for computing H_*.

DEF Cell complex obtained by:

- X_0 discrete set
- $X_n = X_{n-1} \cup \left(\bigcup_{i \in I_k} D_i^k \right)$, $D_i^k \equiv \{ x \in \mathbb{R}^k : \|x\| \leq 1 \}$

attached via a map $\partial D_i^k : S^{k-1} \longrightarrow X_{k-1}$

$\left(\text{i.e. } X_k = \left(X_{k-1} \sqcup \left(\bigsqcup_{i \in I_k} D_i^k \right) \right) \big/ \sim, \quad v \sim \varphi_i(v) \ \forall v \in \partial D_i^k \right)$

- $X \equiv \bigcup_{k \geq 0} X_k$. Equipped with Weak topology.

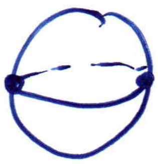
(i.e. $U \subseteq X$ open $\Longleftrightarrow$ $U \cap X_k$ open $\forall k$).

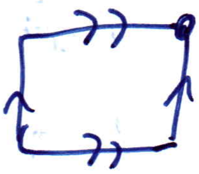
Notation D_i^k "k-cells" of X $\qquad X_k \equiv$ "k-skeleton" of X

φ_i^k "attaching maps"

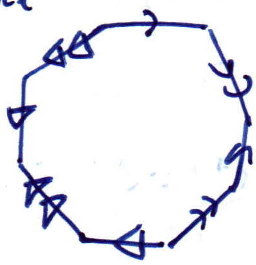
Examples i) $S^n = \{pt\} \cup \{\text{open disc}\} =$ 

x

ii) $S^2 =$  2 0-cells, 2 1-cells, 2 2-cells

iii) $T^2 =$  1 0-cell, 2 1-cells, 1 2-cell.

iv) $\Sigma_g \equiv$ surface with g holes,

1 0-cell, 1 2-cell, $2g$ 1-cells. 

v) Wedge Product: if X, Y cell complexes & $x_0 \in X$, $y_0 \in Y$ then $X \vee Y \equiv (X \sqcup Y) \big/ (x_0 \sim y_0)$ naturally a cell complex.

Note If X cell complex: X disjoint union of its open cells. $\mathrm{int}(D_i^k)$.

DEF If $\exists n$, $X = X_n$ then X finite-dimensional.

If $X = X_n$ & $\forall k \leq n$, $|I_k| < \infty$, then X finite cell complex.

[Such an X is compact.]

Alg Top: lecture 10: Homology of cell cplxs. 27/10/2023.

Recall: Cell Complex $X = \bigcup_{n\geq 0} X_n$, union of its n-skeletons.

DEF] A subcomplex of X is closed subspace of X, which is a union of cells of X.

Lemma] (Point-Set topology):

1) $A \subseteq X$ open (closed) $\Longleftrightarrow$ $\varphi_\alpha^{-1}(A) \subseteq D_\alpha$ open (closed) $\forall \alpha$, where: $D_\alpha \equiv \{\|x\| \leq 1\} \subseteq \mathbb{R}^n$, $D_\alpha \hookrightarrow X_{n-1} \sqcup (\bigsqcup_\alpha D_\alpha^n) \twoheadrightarrow X_n \hookrightarrow X$ (φ_α)

($\varphi_\alpha \equiv$ Characteristic map of this cell.)

2) Cell complexes are Hausdorff & locally contractible. (connected $\Longleftrightarrow$ path connected)

3) If $Z \subseteq X$ compact then $\exists N$, $Z \subseteq X_N$.

4) If $A \subseteq X$ subcomplex then (X, A) is good.

Corollary] If $A \subseteq X$ cell complex then $H_*(X;A) \cong \tilde{H}_*(X/A)$.

Corollary] $X = \bigcup_{n\geq 0} X_n \Rightarrow H_i(X_k, X_{k-1}) = \begin{cases} \bigoplus_{\alpha\in I_k} \mathbb{Z} & \text{if } i=k \\ 0 & \text{else.} \end{cases}$

(I_k indexes the k-discs.)

Proof] $\dfrac{X_k}{X_{k-1}} \cong \bigvee_{\alpha\in I_k} S_\alpha^k$ & follows from MV. To see: have:

$\left(\bigsqcup S_\alpha^k / \bigsqcup x_\alpha^k\right) \cong \bigvee_{\alpha\in I_k} S_\alpha^k$. $\Rightarrow \tilde{H}_*(\bigvee_\alpha S_\alpha^k) = H_*(\bigsqcup S_\alpha^k, \bigsqcup x_\alpha^k)$

$= \bigoplus_\alpha H_*(S_\alpha^k, x_\alpha)$. ✓

Prop] X is a cell complex.

$X = \bigcup_{n\geq 0} X_n$.

1) $H_k(X_n) = 0 \ \forall k > n$

2) $X_n \hookrightarrow X$ induces $H_j(X_n) \cong H_j(X) \ \forall j < n$.

Proof] LES of (X_n, X_{n-1}):

$$H_{k+1}(X_n, X_{n-1}) \longrightarrow H_k(X_{n-1}) \longrightarrow H_k(X_n) \longrightarrow H_k(X_n, X_{n-1}) \longrightarrow \dots$$

If $k > n$ then $H_{k+1}(X_n, X_{n-1}) = H_k(X_n, X_{n-1}) = 0$. $(\vee S^n_\alpha)$

$\Rightarrow H_k(X_{n-1}) \cong H_k(X_n)$.

$\Rightarrow H_k(X_n) \cong H_k(X_{n-1}) \cong \dots \cong H_k(X_0) = 0$ (X_0 discrete)

2) look at same LES, but for $k < n-1$.

$\Rightarrow H_k(X_{n-1}) \cong H_k(X_n) \cong \dots \cong H_k(X_N) \ \forall N \geq n+1$.

If X finite dimensional $\Rightarrow \exists N,\ X = X_N \Rightarrow$ done!

Else: for $\alpha \in H_k(X)$ then represented by finite set of k-simplices. But, this is compact & hence lies in some X_N.

$\Rightarrow \alpha \in \mathrm{Im}(H_k(X_N) \longrightarrow H_k(X))$.

similarly: if $\alpha = \sum_i \alpha_i \sigma_i \in H_k(X)$ bounds $(k+1)$-chain of X then this chain lies in $X_{N'}$ (some N'), so $H_k(X_{N'}) \ni [\alpha] = 0$

$\Rightarrow [\alpha] = 0$ in $H_k(X_N)$ so $= 0$ in $H_k(X)$.

Corollary] If X finite cell cplx of dimension N, then:

$H_j(X) = 0 \ \forall j > N$.

DEF] Cellular chain complex $C_*^{cell}(X)$ of a cell complex X (with its given cell structure) via:

$C_k^{cell}(X) \equiv H_*(X_k, X_{k-1}) \equiv$ Free abelian group on k-cells

$\left(\cong \bigoplus_{\alpha \in I_k} \mathbb{Z}(\alpha)\right)$

$\underline{k}$

$$\begin{tikzcd}
& H_k(X_k) \arrow[dr] & & & \\
H_{k+1}(X_{k+1}, X_k) \arrow[ur] \arrow[rr, "\partial_{k+1}^{cell}"'] \arrow[rr, bend left] & & H_k(X_k, X_{k-1}) \arrow[rr, "\partial_k^{cell}"] \arrow[dr] \arrow[rr, bend right] & & H_{k-1}(X_{k-1}, X_{k-2}) \\
& & & H_{k-1}(X_{k-1}) \arrow[ur] &
\end{tikzcd}$$

(Red from LES.)

Claim: $\partial_k^{cell} \circ \partial_{k+1}^{cell} = 0$.

From claim, write: $H_*^{cell}(X) \equiv H_*(C_*^{cell}(X), \partial_*^{cell})$.

Proof (of claim). The composition $\partial_k^{cell} \circ \partial_{k+1}^{cell}$ contains 2 consecutive terms of a LES, so is obviously 0.

Note $C_*^{cell}(X)$ depends on choice of ~~X~~. cell structure on X.

Prop $H_*(C_k^{cell}(X)) \cong H_*(X)$.

Proof

$$\begin{tikzcd}
& & & 0 & \\
0 = H_k(X_{k-1}) \arrow[dr] & & H_k(X_{k+1}) = H_k(X) \arrow[ur] & & \\
& H_k(X_k) \arrow[ur] \arrow[dr, "i_k"] & & & \\
H_{k+1}(X_{k+1}, X_k) \arrow[ur, "\partial_{k+1}"] \arrow[rr, "\partial_{k+1}^{cell}"'] & & H_k(X_k, X_{k-1}) \arrow[rr, "\partial_k^{cell}"] \arrow[dr, "\partial_k"'] & & H_{k-1}(X_{k-1}, X_{k-2}) \\
& & & H_{k-1}(X_{k-1}) \arrow[ur, "i_{k-1}"'] &
\end{tikzcd}$$

$$H_k(X) = H_k(X_{k+1}) = \frac{H_k(X_k)}{\operatorname{im}(\partial_{k+1})}$$

$$= \frac{i_k(H_k(X_k))}{\operatorname{im}(i_k \circ \partial_{k+1})} = \frac{\ker(\partial_k)}{\operatorname{im}(\partial_{k+1}^{cell})} = \frac{\ker(i_{k-1} \circ \partial_k)}{\operatorname{im}(\partial_{k+1}^{cell})}$$

$\uparrow$ i_k injective.

$$= \frac{\ker(\partial_k^{cell})}{\operatorname{Im}(\partial_{k+1}^{cell})} \checkmark$$

Corollary X finite cell complex.

i) $H_k(X)$ is f.g. abelian group, rank $\leq n_k \equiv |I_k| \equiv \#$ k-cells

ii) If $H_k(X) \neq 0 \Rightarrow$ Every cell structure of X has k-cells

iii) If X has cell structure with only even dim cells, then:

$H_*(X) \cong C_*^{cell}(X)$.

iv) If $\mathbb{F}$ field then $H_*(X, \mathbb{F})$ is a f.d. $\mathbb{F}$-vector space.

Note $\mathbb{C}P^1$ satisfies iii)

This is true for the Grassmanian $Gr(k, \mathbb{C}^n)$ of k-dim linear spaces in $\mathbb{C}^n$. (E.g. $Gr(1, \mathbb{C}^n) = \mathbb{C}P^{n-1}$)

Remark $\mathbb{R}P^n$ has cell structure with 1 cell in each degree $0 \leq i \leq n$: $C_*^{cell}(\mathbb{R}P^n)$ $\underset{n}{\mathbb{Z}} \to \underset{n-1}{\mathbb{Z}} \to \cdots \to \underset{0}{\mathbb{Z}} \to 0$

$\Rightarrow$ New tools required, for computing ∂^{cell}!

Alg Top: Lecture 11: Homology of Cell cplxs. 30/10/2023.

Want: tool to compute $\partial_k^{cell} : \underset{\bigoplus_{\alpha\in I_k}\mathbb{Z}}{C_k^{cell}(X)} \longrightarrow \underset{\bigoplus_{\alpha\in I_{k-1}}\mathbb{Z}}{C_{k-1}^{cell}(X)}$.

i.e. If e_α k-cell:

$$\partial_k^{cell}(e_\alpha) = \sum_{\beta\in I_{k-1}} d_{\alpha\beta}\, e_\beta. \quad \underline{\&} \text{ want } d_{\alpha\beta}. \quad \cancel{(\beta\in I_{k-2})}$$

Lemma $d_{\alpha\beta}$ = degree of map of spheres:

$$\partial D^k_\alpha \cong S^{k-1}_\alpha \xrightarrow{\varphi_\alpha} X_{k-1} \longrightarrow X_{k-1}/X_{k-2} = \bigvee_{\beta\in I_{k-1}} S^{k-1}_\beta \xrightarrow{\pi} S^{k-1}_\beta$$

Remark To be well-def'ed (not just up to sign), recall, need to fix isoms $H_{k-1}(S^{k-1}) \cong \mathbb{Z}$.

$1 \mapsto$ deg of claimed map.

Proof

$$\begin{tikzcd}
1\in H_k(D^k_\alpha, \partial D^k_\alpha) \arrow[r, "\partial_{LES}"] \arrow[d, "\varphi_\alpha"] & H_{k-1}(\partial D^k_\alpha) \arrow[r] \arrow[d, "\varphi_\alpha|_{\partial D_\alpha}"] & H_{k-1}(S^{k-1}_\beta) \\
e_\alpha \in H_k(X_k, X_{k-1}) \arrow[r, "\partial_{LES}"'] \arrow[dr, "\partial_k^{cell}"] \arrow[ddr, bend right, mapsto] & H_{k-1}(X_{k-1}) \arrow[d] & \\
 & H_{k-1}(X_{k-1}, X_{k-2}) \cong & \tilde{H}_{k-1}(X_{k-1}/X_{k-2}) \arrow[uu] \\
 & \sum d_{\alpha\beta}\, e_\beta &
\end{tikzcd}$$

This is useful if we can compute degrees of maps of spheres.

Let: $f: S^n \longrightarrow S^n$ & suppose: $y\in S^n$ has fin-many preimages.

$f^{-1}(y) = \{x_1, \dots, x_m\}$. $\Rightarrow$ Can pick: open nbhoods $U_i \ni x_i$ pairwise disjoint discs in domain.

& some nbhood $V \ni y$, disc in target, s.t. $f|_{U_i}$ has image in $V \subseteq S^n$.

Then: f defines map $(U_i, x_i) \longrightarrow (V, y)$

& map $H_n(U_i, U_i - x_i) \longrightarrow H_n(V, V-y)$ (excision)

$\cong \mathbb{Z} \xmapsto{\circledast} \cong \mathbb{Z}$.

(via: $H_n(S^n, S^n - y) \cong H_n(S^n)$)

$\circledast$ "local degree" of f, at x_i, is this hom. defined WRT a fixed identification $H_n(S^n) \longleftrightarrow \mathbb{Z}$.

Write: $\deg_{x_i}(f)$ (or $\deg_f(x_i)$) $\in \mathbb{Z}$.

Lemma) Under this hypothesis: $f^{-1}(y) = \{x_1, \ldots, x_m\}$

$\Rightarrow \deg(f) = \sum_{i=1}^m \deg_{x_i}(f)$.

Proof

$$\begin{array}{ccc}
H_n(S^n) & \xrightarrow{\deg f} & H_n(S^n) \\
\downarrow & & \downarrow \\
H_n(S^n, S^n - \{x_i\}) & & H_n(S^n, S^n - y) \\
\downarrow \wr\wr & & \downarrow\wr\wr \\
H_n(\sqcup U_i, \sqcup U_i - x_i) & & H_n(V, V-y) \\
\wr\wr & \nearrow \oplus \deg_{x_i}(f) & \\
\bigoplus_{1 \le i \le m} H_n(U_i, U_i - x_i) & &
\end{array}$$

Follows by: commutativity of diagram.

Example) $\mathbb{RP}^n \equiv \{\text{lines of } \mathbb{R}^{n+1}, \text{ through } 0\} = e^n \cup \mathbb{RP}^{n-1}$.

($e^n \equiv n$'th cell).

$\Rightarrow \mathbb{RP}^n$ has 1 cell for each dimension $0 \leq i \leq n$.

& Attaching map: $\partial e^n : S^{n-1} \longrightarrow \mathbb{RP}^{n-1}$ is canonical 2-1 map

$\Rightarrow$ View: $\mathbb{RP}^n$ as $S^n / \pm I$.

<u>Cellular complex</u>: $\underset{n}{\mathbb{Z}} \longrightarrow \underset{n-1}{\mathbb{Z}} \longrightarrow \cdots \longrightarrow \underset{1}{\mathbb{Z}} \longrightarrow \underset{0}{\mathbb{Z}} \longrightarrow 0$.

<u>Consider</u>: $\partial_k^{cell} : C_k^{cell} \longrightarrow C_{k-1}^{cell}$.

& $\underset{= \partial e^k}{\partial D^k} = S^{k-1} \longrightarrow \mathbb{RP}^{k-1} \longrightarrow \mathbb{RP}^{k-1} / \mathbb{RP}^{k-2} = S^{k-1}$

η

For general pt of <u>target</u> S^{n-1}, have: η has 2 preimages.

Near each: η is <u>homeo</u>.

$\Rightarrow$ The map $S^{k-1} \longrightarrow \mathbb{RP}^{k-1}$ is s.t. $\eta|_{u_2} = \eta|_{u_1} \circ$ antipodal.

$u_1 \sqcup u_2 \longrightarrow V$

$\Rightarrow \deg_{x_1}(f) = (-1)^k \deg_{x_2}(f)$ (deg of antipodal is $(-1)^k$)

$\Rightarrow \partial_k^{cell}$ is: mult. by $1 + (-1)^k$. (possibly up to sign)

$$\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \cdots \xrightarrow[2]{} \mathbb{Z} \xrightarrow[0]{} \mathbb{Z} \xrightarrow[2]{} \mathbb{Z} \xrightarrow[0]{} \mathbb{Z} \longrightarrow 0$$

$$\Rightarrow H_*(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2 & 0 < * < n,\ * \text{ odd} \\ \mathbb{Z} & * = n,\ n \text{ odd} \\ 0 & \text{else} \end{cases}$$

Exercise (Example) If $p(z)$ polynomial then $p(z)$ extends to cts map $\hat{p}$ on $S^2 = \mathbb{C} \cup \{\infty\}$ of degree p.

Moreover: if $x \in \mathbb{C}$ root of p: local degree of $\hat{p}$ at $x \in p^{-1}(0)$ is multiplicity of x as root of p.

Remark If $f: S^n \to S^n$ smooth (viewing $S^n \subseteq \mathbb{R}^{n+1}$) then:

a) $f^{-1}(y)$ finite $\forall y$ regular value of f

b) Almost all $y \in S^n$ is finite (Sard's theorem)

c) Any continuous $f: S^n \to S^n$ homotopic to smooth map.

Digression of Cohomology.

$C^*_{cell} \equiv \mathrm{Hom}(C^{cell}_*(X), \mathbb{Z})$ & $\partial^*_{cell} \equiv \mathrm{adjoint}(\partial^{cell}_*)$.

$$\begin{tikzcd}
H^i(X_i, X_{i-1}) \arrow[r] \arrow[d, "\cong"] & H^i(X_i) \arrow[r] \arrow[d] & H^{i+1}(X_{i+1}, X_i) \arrow[d, "\cong"] \\
\mathrm{Hom}(H_i(X_i, X_{i-1}), \mathbb{Z}) \arrow[r] & \mathrm{Hom}(H_i(X_i), \mathbb{Z}) \arrow[r] & \mathrm{Hom}(H_{i+1}(X_{i+1}, X_i), \mathbb{Z})
\end{tikzcd}$$

$\Rightarrow$ Diagram shows that: Can define $C^i_{cell}(X) \equiv H^i(X_i, X_{i-1})$ or as: $\mathrm{Hom}(C_i^{cell}(X), \mathbb{Z})$ & natural candidate bdry ops yield the same chain complex.

Prop X finite cell complex. Then: $H^i(X) \cong \dfrac{H_i(X)}{\text{Torsion}} \oplus \mathrm{Tors}(H_{i-1}(X))$

($\mathrm{Tors}(G) \equiv$ elements of finite order)

Alg Top: lecture 12. 01/11/2023.

Prop X finite cell cplx $\Rightarrow H^j(X) \cong H_j(X)/_{Tors} \oplus Tors(H_{j-1}(X))$.

Proof: Pure algebra!

Let: $C_* \equiv$ chain complex, s.t. C_i fin-gen & free.

& $C^* \equiv Hom(C_*, \mathbb{Z})$. Suffices to prove: same relation.

(Applies, since H_* computed using C_*^{cell}.)

Break C_* into collection of SES: $0 \to Z_n \to C_n \to B_{n-1} \to 0$

$0 \to B_n \to Z_n \to H_n(C_*) \to 0$

$$\left.\begin{array}{l} Z_n \equiv Ker(\partial_n : C_n \to C_{n-1}) \\ B_n \equiv Im(\partial_{n+1} : C_{n+1} \to C_n). \end{array}\right]$$

In top line: all groups free $\Rightarrow$ can split top sequence:

$\exists \alpha_n : B_{n-1} \to C_n$ s.t. $\partial_n \circ \alpha_n = id$.

$\Rightarrow C_n \underset{\uparrow \text{non-canonical}}{\cong} Z_n \oplus B_{n-1}$ $(B_{n-1} \cong im(\alpha_n))$

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, equal] & C_n \arrow[r, "\partial_n"] \arrow[d, equal] & C_{n-1} \arrow[r] \arrow[d, equal] & \cdots \\
& Z_{n+1} \oplus B_n \arrow[r] & Z_n \oplus B_{n-1} \arrow[r] & Z_{n-1} \oplus B_{n-2} &
\end{tikzcd}$$

$\Rightarrow C_*$ breaks up into direct sum of length 2 sequences.

Can further simplify by choosing bases for B_n & Z_n:

by Smith normal form: can write ∂_n as the matrix:

$\begin{pmatrix} d_1 & & & & & \\ & d_2 & & & & \\ & & \ddots & & & \\ & & & d_k & & \\ & & & & 0 & \\ & & & & & \ddots 0 \end{pmatrix}$ where: $d_1 | d_2 | d_3 | \ldots$

$\Rightarrow C_*$ splits into direct sum of complexes of form:

$0 \longrightarrow \mathbb{Z} \xrightarrow{d} \mathbb{Z} \longrightarrow 0 \quad (d \neq 0)$

$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0 \quad (d = 0)$

For these 2, the prop is obvious, so done.

Remark For abelian G, H: let $\mathrm{Ext}^1(H, G) \equiv \{0 \to G \to J \to H \to 0\} / \sim$
("extensions of H by G") modulo equiv rel:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \longrightarrow & J_1 & \longrightarrow & H & \longrightarrow & 0 \\ & & \| & & \downarrow \phi & & \| & & \\ 0 & \longrightarrow & G & \longrightarrow & J_2 & \longrightarrow & H & \longrightarrow & 0 \end{array}$$

equiv $\Leftrightarrow \exists \phi : J_1 \to J_2$ s.t. squares commute ($\Rightarrow \phi$ is $\cong$ by 5-lemma)

Universal Coeff Theorem: $\exists$ exact sequences

$$0 \to \cancel{\mathrm{Hom}}\ \mathrm{Ext}^1(H_{n-1}(X), G) \longrightarrow H^n(\cancel{Y}X, G) \to \mathrm{Hom}(H_n(X), G) \to 0$$

& (non-canonically) split. No need for X cell complex!

Recall: if X finite cell complex $\Rightarrow H_i(X)$ f.g. abelian grp.

& $H_i(X, \mathbb{F}) \equiv$ f.d. $\mathbb{F}$-vector space ($\mathbb{F}$ field).

DEF Euler Characteristic $\chi(X) \equiv \sum_{i \geq 0} (-1)^i \mathrm{Rank}_{\mathbb{Z}}(H_i(X))$

& $\chi(X, \mathbb{F}) \equiv \sum_{i \geq 0} (-1)^i \dim_{\mathbb{F}} (H_i(X))$.

Lemma $\chi(X) = \sum_i (-1)^i N_i \quad (N_i \equiv \#\, i\text{-cells of } X)$.

Hence: RHS indep. of choice of cell structure.

<u>Proof</u> SES $0 \to Z_n \to C_n \to B_{n-1} \to 0$

$0 \to B_n \to Z_n \to H_n(C_*) \to 0.$

$(C_* \equiv C_*^{cell}(X).)$

$\Rightarrow \text{Rank}(C_n) = \text{Rank}(Z_n) + \text{Rank}(B_n)$ & $\text{Rank}(H_n) = z_n - b_n.$

$$\Rightarrow \sum (-1)^k \underset{N_n}{}\text{rank}_{\mathbb{Z}} H_k(X) = \sum (-1)^k (\overset{z_n}{}z_k - \overset{b_n}{}b_k)$$

$$= \sum (-1)^k (z_k - (N_{k+1} - z_{k+1})) = \underset{\uparrow N_0}{z_0} + \sum (-1)^{k+1} N_{k+1} = \sum (-1)^k N_k.$$

By same arg: can do for $\mathbb{F}$.

<u>Example</u> $\chi(S^4) = 2$, $\chi(\mathbb{CP}^2) = 3 \Rightarrow S^4 \not\simeq \mathbb{CP}^2$.

<u>a)</u> If $X = A \cup B \Rightarrow \chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$.

<u>b)</u> If X, Y cell complexes: $X \times Y$ admits a cell complex structure, and: open cells in $X \times Y$ is (open cell of X) $\times$ (open cell of Y).

$\Rightarrow \chi(X \times Y) = \chi(X)\chi(Y).$

<u>DEF</u> Generalised Homology Theory: is an assignment:

$$(X, A) \to h_*(X, A) = \bigoplus_{i \in \mathbb{Z}} h_i(X, A)$$

of graded abelian group to pair (X, A) of (space, subspace) s.t.

1) <u>Functorial</u>: a map $f: (X, A) \to (Y, B)$ induces

$$f_*: h_*(X, A) \to h_*(Y, B)$$

s.t. $(id)_* = id$, $(f\circ g)_* = f_* \circ g_*$

b) Homotopy Invariance: if $f \simeq g$ (map of pairs) $\Rightarrow f_* = g_*$

c) LES: If $h_i(X) \equiv h_i(X, \phi)$

$\Rightarrow h_i(A) \xrightarrow[(incl)_*]{} h_i(X) \xrightarrow[(incl)_*]{} h_i(X, A) \longrightarrow h_{i+1}(A) \longrightarrow \cdots$ exact

& is natural.

d) Excision: If $cl(Z) \subseteq int(A) \Rightarrow h_*(X\setminus Z, A\setminus Z) \underset{(incl)_*}{\cong} h_*(X,A)$

e) Unions: $\bigoplus_\alpha h_*(X_\alpha) \xrightarrow[\bigoplus_\alpha (incl)_\alpha]{\cong} h_*(\bigsqcup_\alpha X_\alpha)$

(if: $\bigsqcup_\alpha X_\alpha \equiv X$ disjoint union & index set discrete top.)

Remark $h_*(pt)$ "coefficients" of theory.

Can build examples of these, of form: $(X,A) \longmapsto H_*(X,A) \otimes_{\mathbb{Z}} R$.

(R some abelian group)

But: such things aside: is meta-theorem saying interesting GHTs (other than H_*) don't come from chain complexes.

Prop If h_* & k_* GHTs on set of pairs (X,A) of a (cell complex, subcomplex), & $\Phi: h_* \to k_*$ natural trans.,

then: (Φ is isomorphism $h_*(pt) \to k_*(pt)$)

$\Rightarrow$ (Φ isomorphism on such pairs).

Alg Top: Lecture 13 Cup products. 03/11/2023.

Prop If h_* & k_* GHTs & $\Phi: h_* \longrightarrow k_*$ is a natural transformation, then $\Phi: h_*(pt) \longrightarrow k_*(pt)$ implies Φ is $\cong$ for ANY cellular pairs (X, A)

$\to$ (cplx, subcplx).

Will give: outline of proof in the case $X = X_n$ (fin-dim).

Sketch Proof Induction on $\dim(X)$. For $\dim(X) = 0$, X discrete $\Rightarrow$ result follows from Union axiom of GHT.

Assume: $\Phi: h_*(X, A) \longrightarrow k_*(X, A)$ is $\cong$ $\forall \dim(X) < n$.

& consider $X = X_n$.

$$\Rightarrow \begin{tikzcd} h_{i+1}(X, X_{n-1}) \arrow[r] \arrow[d] & h_i(X_{n-1}) \arrow[r] \arrow[d, "\cong"] & h_i(X) \arrow[r] \arrow[d] & h_i(X, X_{n-1}) \arrow[r] \arrow[d] & \cdots \arrow[d, "\cong"] \\ k_{i+1}(X, X_{n-1}) \arrow[r] & k_i(X_{n-1}) \arrow[r] & k_i(X) \arrow[r] & k_i(X, X_{n-1}) \arrow[r] & \cdots \end{tikzcd}$$

5-lemma: if Φ is $\cong$ for $h_i(X, X_{n-1})$ $\forall i$, then result follows.

& $h_*(X, X_{n-1}) \cong h_*(\bigsqcup_\alpha D^n_\alpha, \bigsqcup_\alpha \partial D^n_\alpha)$ (excision)

(use fact: X_{n-1} has neighborhood $N_\varepsilon(X_{n-1})$ $\forall \varepsilon$ small, constructed inductively cell-to-cell, & retract to boundary on each cell.)

& $h_*(\bigsqcup_\alpha D^n_\alpha, \bigsqcup_\alpha \partial D^n_\alpha) \cong \bigoplus_\alpha h_*(D^n_\alpha, \partial D^n_\alpha)$ (unions)

(Similar for k_*.)

$$\begin{tikzcd}
h_i(\partial D^n) \arrow[r] \arrow[d, "\cong"] & h_i(D^n) \arrow[r] \arrow[d, "\cong"] & h_i(D^n, \partial D^n) \arrow[r] \arrow[d] & h_{i-1}(\partial D^n) \arrow[r] \arrow[d, "\cong"] & h_{i-1}(D^n) \arrow[d, "\cong"] \\
k_i(\partial D^n) \arrow[r] & k_i(D^n) \arrow[r] & k_i(D^n, \partial D^n) \arrow[r] & k_{i-1}(\partial D^n) \arrow[r] & k_{i-1}(D^n)
\end{tikzcd}$$

(homotopy)

$\Rightarrow$ by 5-lemma: Φ is $\cong$ $h_*(D^n, \partial D^n) \longrightarrow k_*(D^n, \partial D^n)$.

By <u>induction</u>: $\Phi_{(X,\phi)} : h_*(X) \longrightarrow k_*(X)$ is $\cong$ $\forall X$, $\dim = n$.

Same arg. with LES of pairs & 5-lemma shows: $\Phi_{(X,\phi)}$ is an isomorphism for <u>any</u> cellular pairs (X,A), $\dim(X) = n$. ✓

<u>Remark</u>] Result is true for inf-dim (X,A) cellular (uses "telescope" construction).

<u>Remark</u>] There is a corresponding notion of generalised cohomology theory. In this:

- Functoriality <u>contravariant</u>: if $f : (X,A) \longrightarrow (Y,B)$ induces $f^* : h^*(Y,B) \longrightarrow h^*(X,A)$
- LES $h^i(X,A) \longrightarrow h^i(X) \longrightarrow h^i(A) \longrightarrow h^{i+1}(X,A)$
- Union axiom: $h^*(\bigsqcup_\alpha X_\alpha) = \prod_{\alpha \in A} h^*(X_\alpha)$.

& Analogue of previous proposition holds (same proof).

<u>DEF</u>] X any space & $\phi \in C^k(X)$, $\psi \in C^l(X)$. <u>Cup product</u>: $\phi \smile \psi \in C^{k+l}(X)$ s.t. for $\sigma : [v_0 \ldots v_{k+l}] \longrightarrow X \in C_{k+l}$,

$$(\phi \smile \psi)(\sigma[v_0 \ldots v_{k+l}]) = \phi(\sigma|_{[v_0 \ldots v_k]}) \, \psi(\sigma|_{[v_k \ldots v_{k+l}]}).$$

$\sqrt{2}$

Lemma If $\partial^*: C^i(X) \to C^{i+1}(X)$ bdry op, then:

$$\partial^*(\phi \cdot \psi) = (\partial^* \phi) \cdot \psi + (-1)^k \phi \cdot (\partial^* \psi).$$

Proof Let $[v_0 - v_{k+l+1}]$ $(k+l+1)$-simplex in X.

$$(\partial^* \phi) \cdot \psi(\sigma) = \sum_{0 \le i \le k+1} (-1)^i \phi|_{[v_0 - \hat{v_i} - v_{k+1}]} \psi|_{[v_{k+1} - v_{k+l+1}]}.$$

$$(-1)^k \phi \cdot (\partial^* \psi)(\sigma) = \sum_{k \le i \le k+l+1} (-1)^i \phi|_{[v_0 - v_{k+1}]} \psi|_{[v_{k+1} - \hat{v_i} - v_{k+l+1}]}.$$

The $\phi|_{[v_0 - v_k]} \psi|_{[v_{k+1} - v_{k+l+1}]}$ appears in both sums, but cancel, since coeff is $(-1)^k$ in first, $(-1)^{k+1}$ in second.

Remaining terms: $(\phi \cdot \psi)(\sum_{0 \le i \le k+l+1} (-1)^i [v_0 - \hat{v_i} - v_{k+l+1}])$.

$= \partial^*(\phi \cdot \psi)(\sigma)$.

Corollary Cup product descends to cohomology.

i.e. induces $H^k(X) \times H^l(X) \to H^{k+l}(X)$, making $H^*(X)$ into <u>graded</u>, <u>unital</u> ring.

Proof If $\phi \in C^k(X)$, $\psi \in C^l(X)$ <u>cocycles</u> then: closed WRT ∂^*, representing $[\phi] \in H^k(X)$, $[\psi] \in H^l(X)$, and since $\partial^*(\phi \cdot \psi) = 0$, $(\phi \cdot \psi)$ defines <u>class</u> in $H^{k+l}(X)$.

If we ~~choose~~ change $\phi \mapsto \phi + \partial^*(\alpha)$ (in same $[\phi]$)

then: $\phi \cdot \psi$ becomes $(\phi + \partial^* \alpha) \cdot \psi = \phi \cdot \psi + \partial^*(\alpha \psi)$ (as: $\partial^* \psi = 0$)

$\Rightarrow [(\phi + \partial^*\alpha)\cdot\psi] = [\phi\cdot\psi] \in H^{k+l}(X)$ ✓

$\Rightarrow$ Element $[\phi]\cdot[\psi]$ is well-defined.

Finally: let $1 \in C^0(X)$ by $1(p) = 1$ $\forall p \in X$.

$\Rightarrow \partial^*(1)(\sigma : [x_0, x_1] \longrightarrow X) = 1(x_0) - 1(x_1) = 0$

$\Rightarrow \partial^*(1) = 0$ & $[1] \in H^0(X)$.

Can check: $1\cdot\phi = \phi\cdot 1 = \phi$ $\forall \phi \Rightarrow$ Is a unit ✓

Remark) Recall: $\forall G$ abelian group, $C_j(X, G)$ was:

$\{\sum_{\text{finite}} a_i\sigma_i : a_i \in G$ & $\sigma_i : \Delta^j \longrightarrow X\}$.

& $C^j(X, G) \equiv \mathrm{Hom}_{\mathbb{Z}}(C_j(X, \mathbb{Z}), G)$.

If G is ring: then can define cup-product on $C^*(X, G)$ & $H^*(X, G)$ is then graded ring.

If G ring & unital, then: $H^*(X, G)$ unital ring.

Properties of Cup products.

if $\phi \in C^k$, $\psi \in C^l$, $\tau \in C^r$ then

i) Assoc (at chain level): $(\phi\cdot\psi)\cdot\tau = \phi\cdot(\psi\cdot\tau)$.

ii) If $f : X \longrightarrow Y$ then $f^* : H^*(Y) \longrightarrow H^*(X)$ is a ring hom.

Indeed: $f^* : C^*(Y) \longrightarrow C^*(X)$ has $f^*(\phi\psi) = (f^*(\phi))(f^*\psi)$.

& f^* is also unital $(1 \longrightarrow 1)$.

iii) Cross Product: $H^i(Y) \times H^j(Z) \longrightarrow H^{i+j}(Y \times Z)$

$(\phi, \psi) \longmapsto (Pr_Y)^*\phi \cdot (Pr_Z)^*(\psi)$.

Alg Top: Lecture 14.

06/11/2023.

From last time: $\phi \cdot \psi \in C^{k+l}(X)$ for $\phi \in C^k(X)$, $\psi \in C^l(X)$.

$\Rightarrow$ Descends to $H^k(X) \times H^l(X) \longrightarrow H^{k+l}(X)$.

Example i) $X = pt. \Rightarrow H^*(X) = \begin{cases} \mathbb{Z}, & * = 0 \\ 0 & * \neq 0 \end{cases}$

(acquires its usual ring structure)

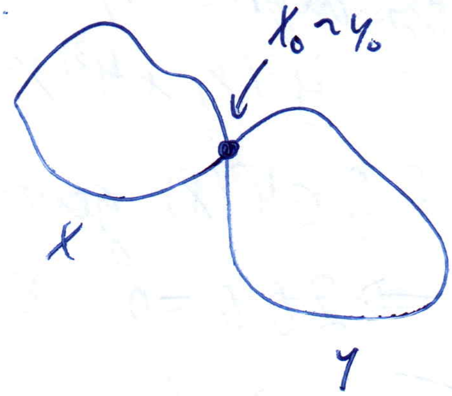
ii) $X = S^n$ $(n>0)$. $\Rightarrow H^*(X) = \begin{cases} \mathbb{Z} & * = 0, n \\ 0 & \text{else.} \end{cases}$

$\Rightarrow$ If $x \in H^n(S^n)$ generator, then $x \cdot x = 0$ (lies in $H^{2n}(X)$)

$\Rightarrow H^*(S^n) = \mathbb{Z}[x]/(x^2)$. & $|x| \equiv \deg(x) = n$.

(denote: $|\cdot| \equiv$ cohomology degree).

iii) X, Y cell complexes, $x_0 \in X$ & $y_0 \in Y$.



Recall: $X \vee Y \equiv (X \sqcup Y)/x_0 \sim y_0$.

There are maps:

(projections & inclusions)

$$\begin{tikzcd} & X \vee Y \arrow[dl] \arrow[dr] & \\ X \arrow[ur, bend left, dashed, "i_X"] & & Y \arrow[ul, bend right, dashed, "i_Y"'] \end{tikzcd}$$

$\Rightarrow$ Ring homs: $H^*(X) \xrightarrow[p_X^*]{} H^*(X \vee Y) \xleftarrow[p_Y^*]{} H^*(Y)$.

Since x_0, y_0 nbhood retracts (deformation retracts): there is an obvious open cover of $X \cup Y$.

$\Rightarrow$ By MV: will show, $\tilde{H}^*(X) \oplus \tilde{H}^*(Y) \xrightarrow[p_X^* \oplus p_Y^*]{} \tilde{H}^*(X \vee Y)$

is an isomorphism (additively).

$\Rightarrow$ Know: $H^*(X \vee Y)$ in terms of $H^*(X)$ & $H^*(Y)$.

(If $\alpha \in H^i(X)$, $\beta \in H^j(Y)$ with $i>0$ & $j>0$, then $\alpha \cdot \beta = 0$)

More simply: $H^*(X \sqcup Y) \cong H^*(X) \oplus H^*(Y)$ ($\cong$ of rings).

To make cup-products useful, need 2 deeper properties.

Prop) $H^*(X)$ graded commutative, i.e. $\phi \in H^k(X)$, $\psi \in H^l(X)$
$\Rightarrow \phi \cdot \psi = (-1)^{kl} \psi \circ \phi. \quad \in H^{k+l}(X)$.

Remarks) i) If R commutative $\Rightarrow H^*(X;R)$ commutative.

ii) Unlike assoc: this is not at chain level. Only on homology.

Examples) Suppose: $H^*(X) = \begin{cases} \mathbb{Z}, & * = 0, 3, 6 \\ 0 & \text{else.} \end{cases}$

By degree argument: only possible (nontrivial) cup-product is: $H^3(X) \times H^3(X) \longrightarrow H^6(X)$.

If $\theta \in H^3(X)$ generator: $\theta \cdot \theta = -\theta \cdot \theta$ (graded comm)
$\Rightarrow 2\theta \cdot \theta = 0$. But, $H^6(X)$ torsion-free, so $\underline{\theta \cdot \theta = 0}$.

Theorem) (Kunneth).

Let: Y space s.t. $H^i(Y)$ free & fin-gen $\forall i$. Then, cross product $H^k(X) \times H^l(Y) \longrightarrow H^{k+l}(X \times Y)$ induces $\cong$:

$$\bigoplus_{k+l=n} H^k(X) \times H^l(Y) \xrightarrow{\cong} H^n(X \times Y).$$, whenever:

(X is finite cell complex.)

Remark) In general, cross prod. induces hom: (of groups)

$$H^*(X, R) \otimes_R H^*(Y, R) \longrightarrow H^*(X \times Y, R).$$

If we <u>declare</u> that LHS is a ring via $(a\otimes b)\cdot(c\otimes d)$

then the map from cross prod. is a ring hom $\quad \equiv (-1)^{|b|\cdot|c|}(ac\otimes bd)$

<u>&</u> if Kunneth applies, then this is an isomorphism of rings:

$$H^*(X,R)\otimes_R H^*(Y,R) \xrightarrow{\cong} H^*(X\times Y, R).$$

<u>Example</u> $H^*(S^1) = \mathbb{Z}(x)/(x^2)$, $|x|=1$. $= \Lambda(x)$, exterior algebra, 1 generator.

$$\Rightarrow H^*(T^2) \cong \begin{cases} \mathbb{Z}, & * = 0,2 \\ \mathbb{Z}\oplus\mathbb{Z}, & * = 1 \\ 0, & \text{else} \end{cases}$$

<u>&</u> ~~$H^0(T^2) = H^0(S^1)\otimes H^0(S^1)$~~
~~$H^1(T^2) = H^0(S$~~

<u>&</u> $H^0(T^2) = H^0(S^1)\otimes H^0(S^1)$

$H^1(T^2) = H^0(S^1)\otimes H^1(S^1) \oplus H^1(S^1)\otimes H^0(S^1)$

$H^2(T^2) = H^1(S^1)\otimes H^1(S^1)$.

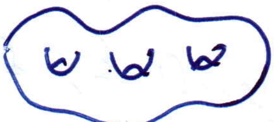
If $x_i \in H^1(S^1)$ generator: (for i'th factor) then:

$H^1(T^2)\otimes H^1(T^2) \longrightarrow H^2(T^2)$ $\quad\Rightarrow$ Abstractly $\cong$ to

$(1\otimes x_2)\otimes(x_1\otimes 1) \longmapsto -x_1x_2$ $\quad \Lambda(x_1, x_2)$.

$\Rightarrow H^2(T^2)$ generated by $x_1x_2 = -x_2x_1$.

[Iteratively: $H^*(T^n) \cong \Lambda(x_1, \dots, x_n)$ <u>&</u> $H^1(T^n)\cong\mathbb{Z}$ has generators $x_1, \dots, x_n$.]

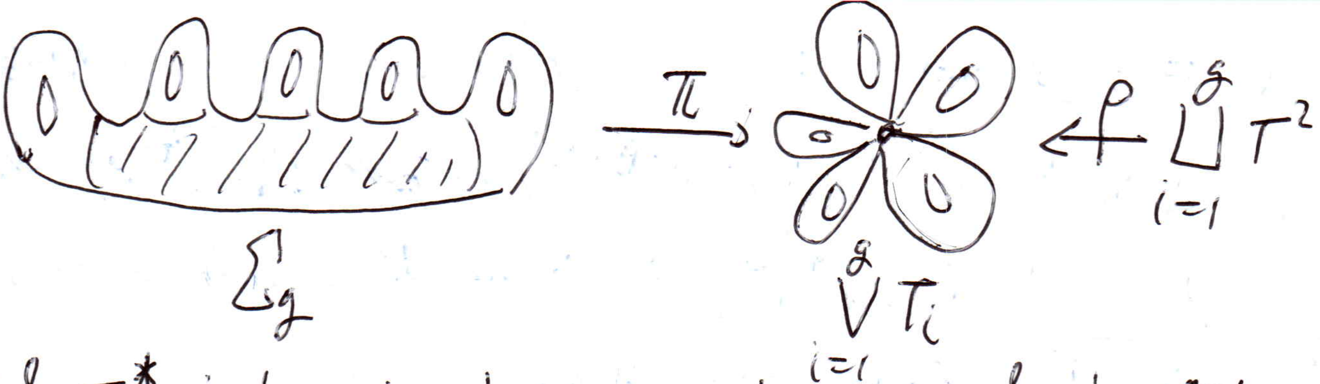
<u>Example</u> Σ_g  $H^*(\Sigma_g) = \begin{cases} \mathbb{Z}, & * = 0,2 \\ \mathbb{Z}^{2g}, & * = 1. \end{cases}$

let: $1 \in H^0(\Sigma_g)$ <u>&</u> $u \in H^2(\Sigma_g)$ generators.

<u>Claim</u>: $H^*(\Sigma_g) = \mathbb{Z}\left\langle x_1, y_1, \dots, x_g, y_g \;\middle|\; \begin{matrix} x_ix_j = 0 \\ y_iy_j = 0 \end{matrix} \quad x_iy_j = \delta_{ij}u \right\rangle$

(<u>&</u> other relations from skew-symmetry).

Proof



Know: p^* & π^* induce ring homs on cohomology, & by MV, define isomorphisms on H^1 additively.

& know: $H^*(\sqcup T^2) \underset{ring}{\cong} \oplus_{i=1}^{g} \Lambda(x_i, y_i)$ & $|x_i| = |y_i| = 1$.

& x_i, y_i generate $H^1(T^2)$ for i'th summand.

Since p^* & π^* isomorphisms: (H^1 – iso's): can view x_i, y_i as defining classes in $H^1(\Sigma_g)$.

On H^2:

$$\underset{\cong \mathbb{Z}}{H^2(\Sigma_g)} \xleftarrow{\pi^*} \underset{\cong \mathbb{Z}^g}{H^2(\bigvee_{i=1}^{g} T^2)} \xrightarrow{p^*} \underset{\cong \mathbb{Z}^g}{H^2(\sqcup T^2)}$$

If $u_i \equiv x_i y_i \in H^2(T^2)$ generator: the map $\mathbb{Z} \leftarrow \mathbb{Z}^g$ sends each $u_i \mapsto u$.

[Why? Interested in map $H^2(T^2) \to H^2(\Sigma_g)$, when $\Sigma_g \to T^2$ is the map to one factor.

Recall: had degrees of maps of spheres $\deg(f) \equiv \sum_i \deg_{x_i}(f)$, if $f^{-1}(y) = \{x_1, \dots, x_n\}$ is finite. & $\deg_{x_i} \equiv$ local degree.

By exact same argument: computes $\deg(f)$ for $f: \Sigma_g \to \Sigma_h$, since we know $H^2(\Sigma_g) \cong \mathbb{Z}$ $\Rightarrow$ Degree well-defined.

$\Rightarrow (x_i \cdot y_i)$ in our case is fixed generator of $H^2(\Sigma_g)$, changing signs of y_i if necessary ✓

$(f^*(x_1 \dots x_n) = f^*(x_1) \dots f^*(x_n) = 0.)$

Corollary: $f: S^n \to T^n$ continuous $(n > 1) \Rightarrow f$ degree $= 0$.

Alg Top: lecture 15 : Künneth's Theorem. 08/11/2023.

Theorem Y space, s.t. $\forall i$, $H^i(Y)$ f.g. & free. Then: cross product $\bigoplus_{i+j=n} H^i(X) \otimes H^j(Y) \longrightarrow H^n(X \times Y)$ is an isomorphism $\forall X$ cell complex. (finite)

Proof Recall: $C^*(X,A) = \{ \phi \in C^k(X) : \phi(\sigma) = 0$ whenever $\sigma : \Delta^k \to X$, $\mathrm{im}(\sigma) \subseteq A \}$

& If $\phi \in C^k(X,A)$, $\psi \in C^l(X)$:

$\phi \cdot \psi(\sigma) = \phi(\text{front face}) \cdot \psi(\text{back face}) = 0$

$(\sigma : \Delta^{k+l} \to X)$. $\Rightarrow$ $\underline{\phi \cdot \psi \in C^{k+l}(X)}$

$\Rightarrow$ Relative cup product $H^k(X,A) \otimes H^l(X) \longrightarrow H^{k+l}(X,A)$.

(In particular, $H^*(X,A)$ graded ring, though not unital in general)

Relative cross product: $C^k(X,A) \otimes C^l(Y) \longrightarrow C^{k+l}(X \times Y, A \times Y)$

$$\Rightarrow H^k(X,A) \otimes H^l(Y) \longrightarrow H^{k+l}(X \times Y, A \times Y).$$

Consider: associations: $(X,A) \longmapsto h^*(X,A) \equiv H^*(X,A) \otimes H^*(Y)$

$\longmapsto k^*(X,A) \equiv H^*(X \times Y, A \times Y)$.

(X,A) cellular, finite.

Then: relative cross product ~~...~~ defines: $\Phi : h^*(X,A) \to k^*(X,A)$.

& If $(X,A) = (pt, \phi)$ then $\Phi : h^*(pt) \longrightarrow k^*(pt)$

$= \mathbb{Z} \otimes H^*(Y) \quad = H^*(pt \times Y)$.

is: an <u>isomorphism</u>.

$\Rightarrow$ By discussion of GCT: if Φ <u>natural</u> transformation ~~then~~ & if h^*, k^* are GCT, then: Φ is $\cong$ $\forall$ cellular pairs (X,A).

i) h^*, k^* are GCT:

- For k^*, all axioms inherited by known properties of singular cohomology.
- For h^*, naturality, homotopy invariance, excision immediate.

& LES, union axioms hold since by assumption, $H^l(Y)$ f.g. $\forall i$ & free.

a) $\otimes$(free f.g.) preserves exact seq's

b) $(\prod M_\alpha) \otimes N = \prod(M_\alpha \otimes N)$ if N f.g. & free.

ii) Need: Φ natural transformation.

Know: cup & cross products natural. (for maps of spaces)

$\Rightarrow$ Naturality, homotopy invariance & excision axioms fine.

Consider:

$$\begin{array}{ccc} H^k(A) \otimes H^l(Y) & \xrightarrow{\delta \otimes id} & H^{k+1}(X,A) \otimes H^l(Y) \\ \downarrow \Phi & & \downarrow \Phi \\ H^{k+l}(A \times Y) & \xrightarrow[\delta]{} & H^{k+l+1}(X \times Y, A \times Y) \end{array}$$

Need square commute for Φ natural. (In that case, Φ is compatible with LES for h^*, k^*.)

Recall: δ defined by:

take $\phi \in C^k(A)$ cocycle $(\partial^* \phi = 0)$ & extend it to $\hat{\phi} \in C^k(X)$ cochain, and set: $\delta\phi \equiv \partial^* \hat{\phi}$.

(Represents relative cohomology class.)

If $\psi \in C^l(Y)$ any cocycle, then $\hat{\phi} \times \psi$ is an extension of $\phi \times \psi$, from $A \times Y \longrightarrow X \times Y$.

$\Rightarrow$ Can use this to define $\hat{\delta}$ downstairs.

$\Rightarrow$ Gives commutativity of Φ ✓

<u>Remark</u> Exact seq $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$ & $\otimes \frac{\mathbb{Z}}{2}$

gets: $0 \to \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \to \mathbb{Z}/2 \to 0$ <u>not</u> exact.

$\Rightarrow$ need free.

Other cup-product debt is: graded commutativity:

$\phi \cdot \psi = (-1)^{kl} \psi \cdot \phi$ on cohomology $(\phi \in H^k(X), \psi \in H^l(X))$.

<u>Sketch-proof</u> (of graded commutativity).

<u>Let</u>: $\varepsilon_n \equiv (-1)^{n(n+1)/2}$ & $\rho : C_n(X) \to C_n(X)$

$[v_0 \cdots v_n] \mapsto [v_n - v_0]\,\varepsilon_n$.

<u>Claim</u>: ρ <u>chain map</u> & chain homotopic to id.

<u>Given this</u>: $(\rho^* \phi)(\rho^* \psi)([v_0 - v_n])$

$= \phi(\varepsilon_k [v_0 - v_k]) \, \psi(\varepsilon_l [v_k \cdots v_{k+l}])$

& $\rho^*(\psi \cdot \phi)([v_0 - v_{k+l}]) = \varepsilon_{k+l} \, \psi([v_{k+l} - v_k]) \, \phi([v_k \cdots v_0])$

& $\varepsilon_k \cdot \varepsilon_l = (-1)^{kl} \varepsilon_{k+l}$. $\Rightarrow \rho^*\phi \cdot \rho^*\psi = (-1)^{kl} \rho^*(\psi \cdot \phi)$.

Since ρ chain homotopic to id: ρ acts on cohomology of chain complex (its dual) by identity, so result follows.

To prove ρ chain-map:

$$\begin{array}{ccc} C_n(X) & \longrightarrow & C_{n-1}(X) \\ \downarrow \rho & & \downarrow \rho \\ C_n(X) & \xrightarrow[\partial]{} & C_{n-1}(X) \end{array}$$

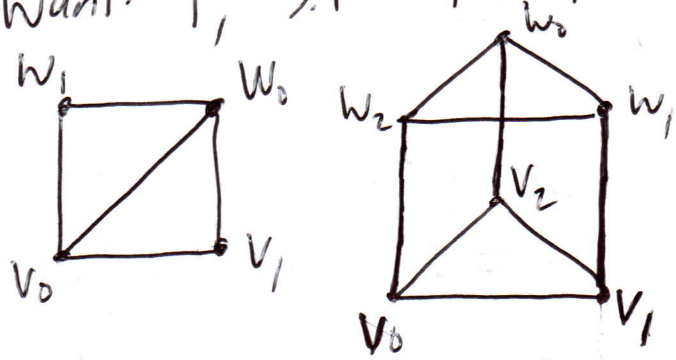
$\partial(\rho\sigma) = \varepsilon_n \sum_i (-1)^i \sigma|_{[v_n \cdots \hat{v}_{n-i} \cdots v_0]}$

$\rho(\partial\sigma) = \rho\left(\sum_i (-1)^i \sigma|_{[v_0 - \hat{v}_i \cdots v_n]}\right)$

$= \varepsilon_{n-1} \sum_i (-1)^{n-i} \sigma|_{(v_n \ldots \hat{v}_{n-i} \ldots v_0)}$. ✓

To show ρ chain-homotopic to id: "twisted prism".

Want: P, s.t. $\partial P + P\partial = \rho - \mathrm{id}$.



$\pi : \Delta^n \times [0,1] \longrightarrow \Delta^n$

$P_\sigma = \sum_i (-1)^i \varepsilon_{n-i} (\sigma \circ \pi)$?

$\Rightarrow$ Works. (check) $([v_0 \ldots v_i w_n \ldots v_i])$.

<u>Recap</u> Cup-product is $\phi \cdot \psi ([v_0 \ldots v_{k+l}])$
$= \phi([v_0 \ldots v_k]) \psi([v_k \ldots v_{k+l}])$.

* Descends to H^*
* Graded-commutative & assoc
* Kunneth Theorem.

& If X, Y surfaces / nice spaces, then: if $f : X \longrightarrow Y$ has $f^{-1}(y)$ finite, can <u>hope</u> to prove something about $\deg(f)$.

Alg Top: Lecture 16 10/11/2023

Recall: $\mathbb{CP}^n$ has cell structure, with 1 cell in each dimension $0, 2, 4, \ldots, 2n$. So, $C^*_{cell}(\mathbb{CP}^n)$ is:
$$\underset{0}{\mathbb{Z}} \to \underset{1}{0} \to \underset{2}{\mathbb{Z}} \to \underset{3}{0} \to \cdots 0 \to \underset{2n}{\mathbb{Z}}$$

$\Rightarrow \partial^*_{cell} = 0$, so: $H^*(\mathbb{CP}^n) \cong C^*_{cell}(\mathbb{CP}^n)$.

Prop $H^*(\mathbb{CP}^n) \cong \dfrac{\mathbb{Z}[x]}{(x^{n+1})}$ & $|x| = 2$.

$\Rightarrow x^i$ generates $H^{2i}(X)$ $\forall\, 0 \leq i \leq n$.

Lemma $\exists\, \underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{n} \xrightarrow{\pi} \mathbb{CP}^n$ (natural map), inducing: homeo. $(\mathbb{P}^1)^n / \mathrm{Sym}_n \xrightarrow[\pi]{\cong} \mathbb{CP}^n$.

Proof: If $[a,b] \in \mathbb{P}^1$: associate homog poly $(bx - ay)$ that vanishes at $[a,b]$

& If $[a_1, b_1], \ldots, [a_n, b_n] \in \mathbb{P}^1$: consider $\prod_{i=1}^{n} (b_i x - a_i y)$.

$\equiv \alpha_0 x^n + \alpha_1 y x^{n-1} + \cdots + \alpha_n y^n$. (homog poly).

& Define: $\pi: ([a_1, b_1], \ldots, [a_n, b_n]) \mapsto [\alpha_0 : \cdots : \alpha_n] \in \mathbb{P}^n$.

Clearly: well-defined &

i) π continuous

ii) Descends to map $(\mathbb{P}^1)^n / \mathrm{Sym}(n) \xrightarrow[\overline{\pi}]{\cong} \mathbb{CP}^n$.

iii) Surjective (Fundamental theo. of algebra)

iv) Induced $\overline{\pi}$ is bijection $\Rightarrow$ Homeomorphism ✓

So, $H^{2n}(\mathbb{CP}^n) \cong \mathbb{Z}$. & $H^*(\mathbb{CP}^1) = H^*(S^2) = \mathbb{Z}[u]/u^2$, $|u| = 2$

By Kunneth theorem: since $H^*(S^2)$ f.g. & free, in each degree ~~as~~ $*$, ~~associate~~ compute: $H^*(\mathbb{P}^1 \times \dots \times \mathbb{P}^1) \underset{\text{ring}}{\cong} \bigotimes_{i=1}^{n} H^*(\mathbb{P}^1)$

$$= \frac{\mathbb{Z}[u_1, \dots, u_n]}{(u_1^2, u_2^2, \dots, u_n^2)} \text{ \& } |u_i| = 2$$

$\Rightarrow H^{2n}((\mathbb{P}^1)^n) \cong \mathbb{Z}$, gen. by $u_1 u_2 \dots u_n$.

Since $\pi: (\mathbb{P}^1)^n \longrightarrow \mathbb{P}^n$: makes sense to ask for $\deg(\pi)$, i.e. induced map $H^{2n}(\mathbb{P}^n) \longrightarrow H^{2n}((\mathbb{P}^1)^n)$.

$$\mathbb{Z} \dashrightarrow \mathbb{Z}$$

If $[\alpha_0 \dots \alpha_n] \equiv q \in \mathbb{P}^n$ corresponds to poly. of distinct roots, then $\pi^{-1}(q) \equiv \{p_\sigma : \sigma \in \mathrm{Sym}(n)\}$ finite & $n!$ elements.

Local degree: fix small $q \in V \cong \mathbb{R}^{2n}$ in $\mathbb{CP}^n$, s.t.

$$\pi^{-1}(V) = \bigsqcup_{\sigma \in S_n} U_\sigma \quad \& \quad \pi: U_\sigma \xrightarrow[\text{homeo}]{\cong} V.$$

[Away from polys with repeated roots: action of S_n is free]

$\Rightarrow \deg(\pi) = \sum_{\sigma \in S_n} \deg_{p_\sigma}(\pi)$. & since homeo: $\deg_{p_\sigma}(\pi) = \pm 1$.

& $\forall \sigma, \tau \in S_n$: $\left.\begin{array}{l} p_\sigma : U_\sigma \longrightarrow V \\ p_\tau : U_\tau \longrightarrow V \end{array}\right]$ differ by homeo of ~~$\mathbb{P}$~~ $(\mathbb{P}^1)^n$, associated with $\sigma\tau^{-1}$.

$\Rightarrow$ If I fix $H^2(\mathbb{P}^1) \xrightarrow{\cong} \mathbb{Z}$ & let u_i correspond to 1 under this isomorphism: then, $\mathrm{Sym}(n)$-action on $(\mathbb{P}^1)^n$

acts on $H^2((\mathbb{P}^1)^n) \cong \mathbb{Z}u_1 \oplus \dots \oplus \mathbb{Z}u_n$ by permuting factors.

$\Rightarrow$ this preserves $u_1 u_2 \dots u_n \in H^{2n}((\mathbb{P}^1)^n)$.

$\Rightarrow$ Local degrees all agree, so $\deg(\pi) = n!$.

<u>Consider</u>: $H^*(\mathbb{P}^n) \longrightarrow H^*((\mathbb{P}^1)^n) = \mathbb{Z}[u_1, \dots, u_n] / (u_1^2, \dots, u_n^2)$.

<u>&</u> let x generate $H^2(\mathbb{P}^n)$.

Since cell complex of $\mathbb{P}^n$: the dim $= 0, 2$ cells of $\mathbb{P}^n$ determine a copy of $\mathbb{P}^1$ in $\mathbb{P}^n$.

<u>&</u> (incl): $\mathbb{P}^1 \hookrightarrow \mathbb{P}^n$ (cellular map) induces isomorphism

$$H^2(\mathbb{P}^n) \xrightarrow{\cong} H^2(\mathbb{P}^1).$$

<u>Consider</u>:

$$\begin{tikzcd}
\mathbb{P}^1 \times \dots \times \mathbb{P}^1 \arrow[r, "\pi"] & \mathbb{P}^n \\
\mathbb{P}^1 \times pt \times \dots \times pt \arrow[u, hook] \arrow[ur, dashed] &
\end{tikzcd}$$

$\leftarrow$ This is a <u>line</u> in $\mathbb{P}^n$, so can <u>choose</u> x s.t. it reduces to u_1 in $H^2(\mathbb{P}^1)$.

By <u>symmetry</u>: (Sym_n - invariance of π): $\pi^*(x) = u_1 + \dots + u_n$.

$\Rightarrow \pi^*(x^n) = \deg(\pi) \cdot (u_1 \dots u_n) \neq 0$.

<u>&</u> $(u_1 + \dots + u_n)^n = n! \, u_1 \dots u_n$. $\Rightarrow$ <u>$x^n \neq 0$</u> ✓

<u>Corollary</u>: $f : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ cannot have degree -1.

(<u>note</u>: since self-map, $\deg(f)$ defined, not just up to sign)

<u>Proof</u> $f^*(x^2) = \deg(f) \cdot x^2$ <u>&</u> $\exists \lambda \in \mathbb{Z}$, $f^*(x) = \lambda x$.

$\Rightarrow f^*(x^2) = \lambda^2 x^2$ <u>&</u> $\lambda^2 \neq -1$ ✓

<u>Consequence</u>: for $M = (\mathbb{P}^1)^n$ or $\mathbb{P}^n$: $\forall \alpha \in H^*(M)$, $\exists \beta \in H^*(M)$, $\alpha \cdot \beta$ have top deg.

Is: general feature of cohomology rings of "oriented" manifolds (contrasted with: suspensions).

Like to understand: H^*(Manifold). Locally, $M \cong \mathbb{R}^n$, and $H^*(\text{disk}) = \mathbb{Z}$ only in deg 0. (not interesting)

Let: X space & $K_1, K_2 \subseteq X$ compact. If $K_1 \subseteq K_2$, then $X \setminus K_1 \supseteq X \setminus K_2$, so $\exists$ inclusion of pairs $(X, X-K_2) \to (X, X-K_1)$.

$\Rightarrow$ Induces: $H^*(X, X-K_1) \longrightarrow H^*(X, X-K_2)$.

DEF) Cohomology of X, with compact support $H^*_{ct}(X) \equiv \varinjlim_{\substack{K \subseteq X \\ \text{compact}}} H^*(X, X-K)$.

Recap of direct limits.

Let: A poset s.t. $\forall a, b \in A$, $\exists c \in A$, $a \leq c$ & $b \leq c$.

A direct system of groups over A comprises: $\{G_a\}_{a \in A}$ (abelian groups) s.t. if $a \leq b$, $\exists \rho_{ab} : G_a \to G_b$ hom. s.t.

i) $\rho_{aa} = id$, ii) $\rho_{ab}\rho_{bc} = \rho_{ac}$ $(a \leq b \leq c)$

Then: direct limit $\varinjlim_a G_a \equiv \left(\bigoplus_{a \in A} G_a\right) / \sim$, where:

$\sim \equiv \langle x - \rho_{ab}(x) : x \in G_a$ & $a \leq b \rangle$.

For $x \in G_a$, $y \in G_b$: pick c, s.t. $a \leq c$ & $b \leq c$.

$\Rightarrow$ Choose: $[x] + [y] \equiv [\rho_{ac}(x) + \rho_{bc}(y)]$.

Then: $\varinjlim_a G_a$ is well-defined abelian group.

Key: If $\Gamma \subseteq A$ "cofinal", i.e. $\forall a \in A$, $\exists \gamma \in \Gamma$, $a \leq \gamma$, then: $\varinjlim_A G_a \cong \varinjlim_{\gamma \in \Gamma} G_\gamma$.

Alg Top: <u>lecture 17</u> : Compact Supports. 13/10/2023.

From last time: if $K_1 \subseteq K_2 \subseteq X$ compact then

$\exists\, H^*(X, X-K_1) \longrightarrow H^*(X, X-K_2)$. (From inclusion).

$\Rightarrow H^*_{ct}(X) \equiv \varinjlim_{K \subseteq X} H^*(X, X-K)$.

<u>Examples</u> (of direct limits).

<u>1)</u> $A = \mathbb{N}$, $G_n \equiv \mathbb{Z}/p^n$. ($p$ fixed prime).

<u>&</u> $\rho_{a,a+1} : \mathbb{Z}/p^n \longrightarrow \mathbb{Z}/p^{n+1}$, <u>mult</u>. by p

$\Rightarrow \varinjlim_n G_n = \mathbb{Z}(p^\infty) = \{z \in S^1 : p^n$'th root of 1 for some $n\}$

<u>2)</u> $A = \mathbb{N}$, $m \leq_\lambda n \Leftrightarrow m | n$. "divisibility".

Take: $G_a = \mathbb{Z}$ <u>&</u> ρ_{ab} mult. by b/a. Then, $\varinjlim G_a = \underline{\mathbb{Q}}$.

[The elements $\{n!\}$ in $\mathbb{N}$ form a cofinal family.]

<u>&</u> $\varinjlim G_n = \varinjlim G_{n!} \cong (\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} -)$

$\cong (\mathbb{Z} \xrightarrow{id} \mathbb{Z}/_{3!} \xrightarrow{id} \mathbb{Z}/_{5!} \rightarrow \dots)$

$= \bigcup_{n \geq 1} \frac{\mathbb{Z}}{n!} = \mathbb{Q}$.]

<u>Example</u> <u>i)</u> If X compact then poset K of subsets of X ordered by inclusion has final element X.

$\Rightarrow \varinjlim_K H^*(X, X-K) = H^*(X, X-X) = H^*(X)$.

ii) $H^*_{ct}(\mathbb{R}^n) = \begin{cases} \mathbb{Z}, & * = n \\ 0 & \text{else.} \end{cases}$

Why?. Any $K \subseteq \mathbb{R}^n$ lies in some $\overline{B}(0,N)$.

$\Rightarrow \varinjlim_K H^*(\mathbb{R}^n, \mathbb{R}^n - K) = \varinjlim_N H^*(\mathbb{R}^n, \mathbb{R}^n - \overline{B}(0,N))$.

But: $H^i(\mathbb{R}^n, \mathbb{R}^n - \overline{B}(0,N)) = \tilde{H}^{i-1}(S^{n-1})$ (homotopy invariance)

$$\Rightarrow H^*_{ct}(\mathbb{R}^n) = (\underset{\substack{\| \\ H^{n-1}(S^{n-1})}}{\mathbb{Z}} \xrightarrow{id} \mathbb{Z} \xrightarrow{id} \mathbb{Z} \longrightarrow \cdots) \cong \mathbb{Z}.$$

Because: $H^*(\mathbb{R}^n, \mathbb{R}^n - \overline{B}(0,N)) \hookrightarrow H^*(\mathbb{R}^n, \mathbb{R}^n - \overline{B}(0,N+1))$.

$$\begin{array}{ccc} H^*(S^{n-1}(N+2)) & & H^*(S^{n-1}(N+2)) \\ \cong & & \cong \\ H^*(S^{n-1}(1)) & \xrightarrow[id.]{\qquad} & H^*(S^{n-1}(1)). \end{array}$$

Note i) $H^*_{ct}(pt) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else.} \end{cases} \neq H^*_{ct}(\mathbb{R}^n)$. $\Rightarrow$ No homotopy invariance.

ii) H^*_{ct} not functorial. (under general continuous maps).

But: if $f: X \longrightarrow Y$ proper (f closed & f^{-1}(compact) compact) then f induces map $H^*_{ct}(Y) \longrightarrow H^*_{ct}(X)$.

iii) If $i: U \hookrightarrow X$ inclusion (X Hausdorff), there is an "extension by zero" map ~~$H^*_{ct}(U)$~~ $i_*: H^*_{ct}(X) \to H^*_{ct}(U)$.

(If $K \subseteq U$ compact $\Rightarrow K \subseteq X$ compact $\Rightarrow$ closed.

Gives map: $\mathcal{K}_U \longrightarrow \mathcal{K}_X$

Recall: n-manifold is Hausdorff space, locally homeo. to $\mathbb{R}^n$.
[$\forall p \in M$, $\exists p \in U \subseteq X$ open s.t. $\phi: U \to \mathbb{R}^n$ homeo.]

DEF] R commutative ring (with 1). A local R-orientation for manifold M at $x \in M$ is choice of generator:

$$\varepsilon_x \in \underbrace{H^n(M, M-x, R)}_{\cong R} \left(\underset{ex.}{\cong} H^n(U, U-x) \cong H^n(\mathbb{R}^n, \mathbb{R}^n - \phi x)\right]$$

Say M R-oriented if: we chose local orientations $\forall x \in M$ s.t. if $\varphi: U \xrightarrow{\cong} \mathbb{R}^n$ chart of M, & $p, q \in M$,

$$\begin{tikzcd}
\varepsilon_p \arrow[d] & H_n(M, M-p) \arrow[d] \arrow[r, phantom, "\cong"] & H_n(U, U-p) \arrow[r, "\cong"] & H_n(\mathbb{R}^n, \mathbb{R}^n - \phi p) \arrow[d, "\text{[Isom. from translation in } \mathbb{R}^n\text{.]}", "\wr" '] \\
\varepsilon_q & H_n(M, M-q) \arrow[r, phantom, "\cong"] & H_n(U, U-q) \arrow[r, "\cong"] & H_n(\mathbb{R}^n, \mathbb{R}^n - \phi q)
\end{tikzcd}$$

Remark] If $U, V \subseteq \mathbb{R}^n$ open & $f: U \to V$ homeo, then:
f orientation-preserving $\iff$ $\forall x \in U$, $f(x) \in V$,

$$\begin{tikzcd}
H_n(\mathbb{R}^n, \mathbb{R}^n - 0) \arrow[r] \arrow[d, dashed, "id"] & H_n(\mathbb{R}^n, \mathbb{R}^n - x) \arrow[r] & H_n(U, U-x) \arrow[d] \\
H_n(\mathbb{R}^n, \mathbb{R}^n - 0) & H_n(\mathbb{R}^n, \mathbb{R}^n - f(x)) \arrow[l] & H_n(V, V - f(x)) \arrow[l]
\end{tikzcd}$$

Exercise: M orientable $\iff$ admits an atlas $\{(U_\alpha, \phi_\alpha)\}$ of charts whose ~~dia~~ transition maps are orientation-preserving.

Note If $R = \mathbb{Z}/2 \Rightarrow H_n(\mathbb{R}^n, \mathbb{R}^n - 0, R) \cong \mathbb{Z}/2$.

Has only 1 generator, so all manifolds are $\mathbb{Z}/2$ – orientable.

Theorem (Poincare Duality). M is R–oriented manifold of dimension n. Then, there is a (preferred) isomorphism

$$H^i_{ct}(M,R) \xrightarrow{D} H_{n-i}(M,R).$$

In particular, if M compact, $H^i(M,R) \cong H_{n-i}(M,R)$.

This isom. is attained from cap-product.

DEF X space. $C_k(X) \otimes C^\ell(X) \xrightarrow{\cap} C_{k-\ell}(X)$

$$([v_0 - v_k], \psi) \longmapsto \psi([v_0 - v_\ell])[v_\ell - v_k].$$

(& is 0 by def, if $\ell > k$.)

Lemma For any space:

i) $\partial(\sigma \cap \phi) = (-1)^\ell (\partial\sigma \cap \phi - \sigma \cap \partial^* \phi)$. $\sigma \in C_k(X)$, $\phi \in C^\ell(X)$.

Indeed: cap prod. induces pairing $H_k(X) \otimes H^\ell(X) \to H_{k-\ell}(X)$.

ii) If $f: X \to Y$: $f_*(\alpha) \cap \psi = f_*(\alpha \cap f^*\psi)$.

(for: $\alpha \in H_*(X)$, $\psi \in H^*(X)$)

iii) $\psi(\sigma \cap \phi) = (\phi\psi)(\sigma)$ for $\sigma \in C_{k+\ell}(X)$, $\phi \in C^k(X)$, $\psi \in C^\ell(X)$

iv) For pair (X,A): $\exists$ relative cap prod.

$C_k(X,A) \otimes C^\ell(X,A) \longrightarrow C_{k-\ell}(X,A)$, which descends to cohomology.

Alg Top: lecture 18. Poincaré Duality. 15/11/2023.

Recall: i) $\partial(\sigma \frown \phi) = (-1)^{l}(\partial\sigma \frown \phi - \sigma \frown \partial^{*}\phi)$.

ii) If $f: X \to Y$: $f_{*}\alpha \frown \psi = f_{*}(\alpha \frown f^{*}\psi)$

iii) $\psi(\sigma \frown \phi) = (\phi \cdot \psi)(\sigma) \in \mathbb{Z}$.

Proof. i): $\partial\sigma \frown \phi = \sum_{i=0}^{l} (-1)^{i} \phi(\sigma|_{[v_0 .. \hat{v}_i .. v_{l+1}]}) \sigma|_{[v_{l+1} .. v_k]}$

$+ \sum_{i=l+1}^{k} (-1)^{i} \phi(\sigma|_{[v_0 - v_l]}) \sigma|_{[v_l - \hat{v}_i - v_k]}$.

& $\sigma \frown \partial^{*}\phi = \sum_{i=0}^{l+1} (-1)^{i} \phi(\sigma|_{[v_0 - \hat{v}_i - v_{l+1}]}) \sigma|_{[v_{l+1} - v_k]}$

& $\partial(\sigma \frown \phi) = \sum_{i=l}^{k} (-1)^{i-l} \phi(\sigma|_{[v_0 - v_h]}) \sigma|_{[v_l - \hat{v}_i - v_k]}$.

Rearrange to get i) ✓

Identities 2), 3) hold at chain level, directly from def.

At level of cohomology: 2), 3) say that: Commutes:

$$\begin{tikzcd} H_k(X) \otimes H^{l}(Y) \arrow[r, "f_* \otimes id"'] \arrow[d, equal] & H_k(Y) \otimes H^{l}(Y) \arrow[r, "\frown"'] & H_{k-l}(Y) \\ H_k(X) \otimes H^{l}(Y) \arrow[r, "id \otimes f^*"'] & H_k(X) \otimes H^{l}(X) \arrow[r, "\frown"'] & H_{k-l}(X) \arrow[u, "f_*"'] \end{tikzcd}$$

Similarly: for $\phi \in C^{k}(X)$, $\psi \in C^{l}(X)$:

$$\begin{tikzcd} H^{l}(X) \arrow[r] \arrow[d, "(\phi\cdot)"'] & \mathrm{Hom}(H_l(X), \mathbb{Z}) \arrow[d, "(\cdot \frown \phi)^*"] \\ H^{k+l}(X) \arrow[r] & \mathrm{Hom}(H_{k+l}(X), \mathbb{Z}) \end{tikzcd}$$

($* \equiv$ adjoint) Commutes.

The horizontal maps <u>aren't</u> always isomorphisms, but if we worked over <u>fields</u> they would be.

We want: to find map D occurring in Poincaire Duality.

<u>PROP</u>) M manifold (oriented, n-fold). $H^*_{ct}(M) \longrightarrow H_{n-k}(M)$.

$\Rightarrow$ Know: $\forall x \in M, \exists \omega_x \in H_n(M, M-x)$. (coherent)

For each $K \subseteq M$ compact: $\exists! \, \omega_K \in H_n(M, M-K)$, s.t.

$$\left.\begin{array}{c} (M, M-K) \hookrightarrow (M, M-x) \\ \omega_K \longmapsto \omega_x \quad \forall x \in K. \end{array}\right] \quad \& \; H_i(M, M-K) = 0 \quad \forall i > n.$$

<u>GIVEN</u> this: let M oriented manifold & $K \subseteq L \subseteq M$ compact.

$$\begin{array}{ccccc} H_i(M, M-L) & \xrightarrow{\times} & H^k(M, M-L) & \xrightarrow{\frown} & H_{i-k}(M) \\ \downarrow (incl)_* & & \uparrow (incl)^* & & \\ H_i(M, M-K) & \times & H^k(M, M-K) & \xrightarrow{\frown} & H_{i-k}(M). \end{array}$$

If $i = n = \dim M$: $\omega_L \frown \phi = i_K \omega_L \frown \phi = \omega_K \frown i^* \phi$, using fact: $(incl)_* \omega_L = \omega_K$ by prop.

The map $\phi \longmapsto \omega_K \frown \phi$ <u>compatible</u> with maps in direct system defining H^*_{ct} by: inclusions $K \hookrightarrow L$.

$\Rightarrow \exists$ induced map $D: H^*_{ct}(M) \longrightarrow H^*_{n-k}(M)$.

<u>Note</u> If M compact $\Rightarrow$ prop states: $\omega_M \in H_n(M)$. Then, $D(\phi) = \omega_M \frown \phi$. We call: $[\phi]$ the fundamental class.

Need to i) prove prop; ii) prove P.D.

Proof of prop Construct ω_K in successive stages of generality.

Say: $K \subseteq M$ compact good if: satisfies conditions of prop.

i) If $A, B, A\cap B$ good: so is $A \cup B$.

$$\cancel{H_{n+1}(M, M - A\cup B)} \to H_n(M, M - A\cup B) \xrightarrow{j} H_n(M, M-A) \oplus H_n(M, M-B) \to H_n(M, M - A\cap B) \to \dots$$

This seq is exact (relative MV)

Exactness of $A\cap B$: $\omega_A \longmapsto \omega_{A\cap B}$ under relevant inclusions.
$\omega_B \longmapsto \omega_{A\cap B}$

$\Rightarrow (\omega_A, \omega_B) \longmapsto 0$, so: $\exists$ class $\omega_{A\cup B} \in H_n(M, M - A\cup B)$ mapping to it. & Unique, since j injective.

& $\forall x \in A\cup B$, $\omega_{A\cup B} \longmapsto \omega_x$ under inclusion

& $H_i(M, M - A\cup B) = 0$ $\forall i > n$ by exactness of MV seq.

ii) If $K \subseteq \mathbb{R}^n$ convex $\Rightarrow K$ good.

Know: $H_*(\mathbb{R}^n, \mathbb{R}^n - K) \cong H_*(\mathbb{R}^n, \mathbb{R}^n - 0)$ (by deformation retracting K to a point). & Trivially follows.

iii) Every $K \subseteq \mathbb{R}^n$ compact is good.

Know: $\exists R > 0$, $K \subseteq \overline{B}(0,R)$. Define ω_K by: $(\omega_{\overline{B}(0,R)})|_K$

Via: $(\mathbb{R}^n, \mathbb{R}^n - \overline{B}(0,R)) \longrightarrow (\mathbb{R}^n, \mathbb{R}^n - K)$.

With this def. of ω_K: know, $\forall x \in K$, $\omega_K \longmapsto \omega_x$ under inclusion.
(Since: this is clearly true for $\bar{B}(0,R)$).

<u>Uniqueness</u>: need, no other element of $H_n(\mathbb{R}^n, \mathbb{R}^n - K)$ also satisfies: push forward to ω_x $\forall x \in K$.

<u>Want</u>: If $\lambda \in H_n(\mathbb{R}^n, \mathbb{R}^n - K)$ has $\lambda|_x = 0$ $\forall x \in K$, then $\lambda = 0$

Suppose we have such λ.

$\Rightarrow \partial\lambda$ finite union of simplices of $\mathbb{R}^n - K$. So, there is a finite union of balls B_j, s.t. $K \subseteq \tilde{K} \equiv \bigcup B_j$ & $\partial\lambda \cap \tilde{K} = \phi$.

$\Rightarrow \lambda \in \text{image}\left(H_n(\mathbb{R}^n, \mathbb{R}^n - \tilde{K}) \longrightarrow H_n(\mathbb{R}^n, \mathbb{R}^n - K)\right)$.

Since $\tilde{K}$ union of convex sets (balls): is good $\Rightarrow \lambda = 0$. ✓
(by exactness of $\tilde{K}$)

<u>iv)</u> <u>If $K \subseteq M$ compact $\Rightarrow$ Is good.</u>

<u>Since</u>: $K = \bigcup_{\text{finite}} K_i$, $K_i \subseteq \underset{\cong \mathbb{R}^n}{D^n} \subseteq M$.

Since K_i good $\Rightarrow$ union good, by <u>i)</u>. ✓

Alg Top: Lecture 19 (Poincaire duality 2) 17/11/2023.

Constructed: $D: H^{*k}_{ct}(M) \longrightarrow H_{n-k}(M)$ for M oriented manifold, degree n. ($\phi \longmapsto \omega_k \frown \phi$, ~~for~~ $\omega_k \in H_n(M, M-K)$)

Poincaire Duality: D is isomorphism.

Consequences 1) Take coeffs in field $\mathbb{F}$. Since $\psi(\sigma \frown \phi) = (\phi \cdot \psi)(\sigma)$: & $H^k(M, \mathbb{F}) \xrightarrow{\cong} \mathrm{Hom}(H_k(M, \mathbb{F}), \mathbb{F})$

$\cong H_k(M, \mathbb{F})^*$:

$\Rightarrow$ Get pairing

$$H^k_{ct}(M, \mathbb{F}) \otimes H^{n-k}(M, \mathbb{F}) \longrightarrow \mathbb{F}. \quad (\phi, \psi) \longmapsto \psi(D\phi).$$

& by Poincaire Duality, this pairing is non-degenerate.

If M compact $\Rightarrow H^k(M, \mathbb{F}) \otimes H^{n-k}(M, \mathbb{F}) \to \mathbb{F}$
(is non-degen.) $\quad (\phi, \psi) \longmapsto \langle \phi \cdot \psi, [M] \rangle$.

fundamental class.

Corollary M, N oriented, compact, connected n-manifolds.
($\Rightarrow H^n(M), H^n(N) \cong \mathbb{Z}$ & $\exists$ preferred isomorphisms via orientations)

Then: $f: M \to N$ has a degree $\deg(f) \in \mathbb{Z}$.
& If $\deg(f) \neq 0$ then $\forall \mathbb{F}$: $f^*: H^*(N, \mathbb{F}) \to H^*(M, \mathbb{F})$ injective $\forall$ degree $*$.

Proof If $\alpha \in H^i(N, \mathbb{F})$ nonzero: by non-degeneracy of cup prod, get $\exists \beta_1 \in H^{n-i}(N, \mathbb{F})$ s.t. $\alpha \cdot \beta \neq 0$, so: $\alpha \cdot \beta \in H^n(N, \mathbb{F}) \xrightarrow{\cong} \mathbb{F}$.

Since $\deg(f) \neq 0$: $f^*: H^n(N, \mathbb{F}) \longrightarrow H^n(M, \mathbb{F})$ nonzero $\Rightarrow$ Isomorphism.

i.e. $f^*(\alpha \cdot \beta) \neq 0 \Rightarrow f^*(\alpha) \neq 0$.

Prop M compact $(n = 2k-1)$-manifold. Then, $\chi(M) = 0$.

Proof By ex. sheet 3: $H^i(M, \mathbb{Z})$ f.g. $\forall i$ and non-zero only if $i \leq n$.

$$\Rightarrow \chi(M) = \sum_{i \geq 0} (-1)^i \operatorname{rank}(H^i(M, \mathbb{Z})). \quad \text{(well-defined)}$$

We can compute it by working over $\mathbb{F}$.

If M oriented: $\chi(M) = \sum_{0 \leq i \leq 2k-1} (-1)^i \operatorname{Rank}_{\mathbb{F}}(H^i(M, \mathbb{F}))$

$= b_0 - b_1 + b_2 - \cdots + b_{2k-2} - b_{2k-1}$. By Duality: $b_i = b_{n-i}$, so

since n odd $\Rightarrow$ All cancel ✓ $\chi(M) = 0$.

If we take $\mathbb{F} = \mathbb{Z}/2$ then M is $\mathbb{F}$-oriented, so above holds.

(Can also deduce existence of oriented double cover, sheet 3)

DEF A manifold with boundary is Hausdorff space, locally homeo. to $\mathbb{R}^n_{\geq 0} \equiv \{ \underline{x} \in \mathbb{R}^n : x_1 \geq 0 \}$.

(i.e. $\forall p \in M \ \exists p \in U \subseteq M$ open & $V \subseteq \mathbb{R}^n_{\geq 0}$ open s.t. $U \underset{\text{homeo}}{\cong} V$.)

The boundary ∂M are all points in M s.t. under such homeo, $\phi(pt) \in \{x_1 = 0\}$. (well-defined).

Question: Given compact manifold M: is $\exists$ compact manifold with boundary W, with $\partial W \underset{\text{homeo}}{\cong} M$?

If so: say M null-cobordant

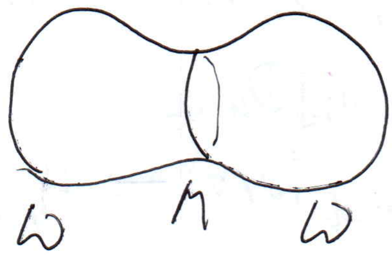
(Variation: M oriented, & W compatibly oriented)

Lemma) If $M = \partial W$ as above $\Rightarrow \chi(M)$ even.

(In particular: $\nexists W, \partial W = \mathbb{CP}^2$)

Proof WLOG $\dim(M) = n = 2k$ (else, follows by previous).

Suppose $M = \partial W$. Form: double Z of W:



$Z = W_{left} \bigcup_{M} W_{right}$

Fact: Double of W is compact manifold with No boundary.

& $\dim(Z) = \text{odd}$. So, $\chi(Z) = 0$. (by previous).

Can compute $\chi(Z)$ using MV ($Z = U \cup V$, $U \cong V \cong W$, $U \cap V \cong M$)

(using fact: neighborhood of ∂W in W is homeo. to $\partial W \times [0, \varepsilon)$)

$$\cdots \longrightarrow H^i(Z) \longrightarrow H^i(W) \oplus H^i(W) \longrightarrow H^i(M) \longrightarrow H^{i+1}(Z) \longrightarrow \cdots$$

Take χ of chain complex itself: $\chi(M) = 2\chi(W)$. $\Rightarrow$ Even ✓

Back to: proving Poincaire Duality.

Ingredients: a) MV sequence for H^*_{ct}.

PROP) X locally compact Hausdorff. If $X = U \cup V$ union of open sets: $\exists$ MV sequence:

$$\cdots \longrightarrow H^{i+1}_{ct}(X) \longrightarrow H^i_{ct}(U \cap V) \longrightarrow H^i_{ct}(U) \oplus H^i_{ct}(V) \longrightarrow H^i_{ct}(X) \longrightarrow \cdots$$

b): Inductive Argument.

Say: M oriented n-fold. Say: open subset $U \subseteq M$ "good" if: PD holds for U. (I.e. $D_U : H^k_{ct}(U) \xrightarrow{\cong} H_{n-k}(U)$)

(note: U also oriented).

<u>Hope</u>: if $U, V, U\cap V$ good $\Rightarrow U\cup V$ good.

$$\begin{tikzcd}
H^k_{ct}(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}", "\cong"'] & H^k_{ct}(U)\oplus H^k_{ct}(V) \arrow[r] \arrow[d, "D_U\oplus D_V", "\cong"'] & H^k_{ct}(U\cup V) \arrow[r] \arrow[d, "D_{U\cup V}"] & H^{k+1}_{ct}(U\cap V) \arrow[d, "D_{U\cap V}", "\cong"'] \\
H_{n-k}(U\cap V) \arrow[r] & H_{n-k}(U)\oplus H_{n-k}(V) \arrow[r] & H_{n-k}(U\cup V) \arrow[r] & H_{n-k-1}(U\cap V)
\end{tikzcd}$$

If so, then by <u>5-lemma</u>: $D_{U\cup V}$ is $\cong$ $\Rightarrow U\cup V$ good.

$\Rightarrow$ Need: squares commute.

For squares <u>not</u> involving boundary maps of LES: this is elementary.

For square

$$\begin{tikzcd}
H^k_{ct}(U\cup V) \arrow[r] \arrow[d] & H^{k+1}_{ct}(U\cap V) \arrow[d] \\
H_{n-k}(U\cup V) \arrow[r] & H_{n-k-1}(U\cap V).
\end{tikzcd}$$

It <u>doesn't</u> commute! However: it <u>does</u> commute up to sign depending on k (genuinely annoying)

& 5-lemma still applies (easy).

Alg Top: lecture 20. (Vector Bundles.) 20/11/2023.

From last time: $U \underset{\text{open}}{\subseteq} M$ good $\Leftrightarrow$ U satisfies: duality map of U is $\cong$.

Showed: $U, V, U \cap V$ good $\Rightarrow$ $U \cup V$ good.

Next: If $M = \bigcup_{i \geq 1} U_i$, $U_1 \subseteq U_2 \subseteq \cdots$ all good, then: want: M good.

For any K compact: $K \subseteq U_N$, some N.

$\Rightarrow \varinjlim H^*_{ct}(U_i) \longrightarrow H^*_{ct}(M)$ is isomorphism.

& On other hand: know: H_* is represented by: finite union of simplices, hence: $\varinjlim H_*(U_i) \xrightarrow{\cong} H_*(M)$.

Since any open subset of $\mathbb{R}^n$ is ctable union of balls: can check, any open subset of $\mathbb{R}^n$ good.

$\Rightarrow$ for M second ctable: covered by ctably many discs, so done. If not: Zorn's lemma on set of all good open subsets of M.

Still need: MV for H^*_{ct}.

Prop) If X locally compact & Hausdorff, and $X = U \cup V$ union of open sets: then exact sequence:

$$\cdots \longrightarrow H^{i-1}_{ct}(X) \longrightarrow H^i_{ct}(U \cap V) \longrightarrow H^i_{ct}(U) \oplus H^i_{ct}(V) \longrightarrow H^i(X) \longrightarrow \cdots$$

Proof: For $(X, Y) = (A \cup B, C \cup D)$, there is relative MV sequence:

$$\cdots \longrightarrow H^{i-1}(X,Y) \longrightarrow H^i(A,C) \oplus H^i(B,D) \longrightarrow H^i(A\cap B, C\cap D) \longrightarrow H^{i+1}(X,Y) \longrightarrow \cdots$$

& If $X = U \cup V$ & $K \subseteq U$, $L \subseteq V$ compact: set:

$A = B = X$, $C = X \setminus K$, $D = X \setminus L$.

$\Rightarrow Y = C \cup D = X \setminus (K \cap L)$ & $C \cap D = X \setminus (K \cup L)$.

$$\cdots \longrightarrow H^i(X, X \setminus (K\cap L)) \longrightarrow H^i(X, X-K) \oplus H^i(X, X-L) \longrightarrow H^i(X, X-(K\cup L))$$
$$\longrightarrow H^{i+1}(X, X-(K\cap L))$$

& excise: $X \setminus U\cap V$, $X\setminus U$, $X \setminus V$:

$$\cdots \longrightarrow H^i(U\cap V, U\cap V - K\cap L) \longrightarrow H^i(U, U-K) \oplus H^i(V, V-L) \longrightarrow H^i(X, X-(K\cup L))$$

& Use facts: i) Any compact ~~set~~ $Q \subseteq U \cap V$ has form $K \cap L$ for $K \subseteq U$ & $L \subseteq V$ compact (e.g. $K = L = Q$)

ii) Every compact set in X is contained in $K \cup L$ for some compact $K \subseteq U$ & $L \subseteq V$ (Use: X locally compact)

$\Rightarrow$ Compact sets of form $K \cup L$ are cofinal (amongst all compact sets in X)

Take: $\varinjlim_{\substack{K\subseteq U \\ L \subseteq V}}$ in above sequence & use: direct limit of exact sequences is exact.

Vector Bundles.

DEF) B space. A vector bundle, rank d, $(E \to B)$ is a family $\{E_b\}_{b\in B}$ s.t. $E_b \cong \mathbb{R}^d$, and has topology on $E \equiv \bigsqcup E_b$ such that:

i) Projection $\rho: E \to B$ is continuous

ii) $\forall b \in B$, $\exists\, b \in U \underset{\text{open}}{\subseteq} B$ & Local trivialisation: Ψ, s.t.

$$\begin{tikzcd}
\rho^{-1}(U) \equiv E|_U \arrow[r, "\Psi"] \arrow[d, "\rho"'] & U \times \mathbb{R}^d \arrow[d, "\pi_1"] \\
U \arrow[r, "="'] & U
\end{tikzcd}$$

this diagram commutes.

& $\Psi: E|_y \to \{y\} \times \mathbb{R}^d$ is linear isomorphism $\forall y \in U$.

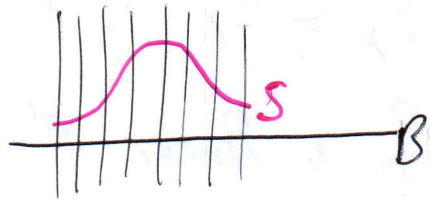
Remark: by def: topology on E extends/induces linear Euclidean topology, on each E_b.

Terminology: E "total space", B base space & E_b fibres.

& A map $s: B \to E$ is section $\Leftrightarrow \rho \circ s = id_B$.

& Inclusion $B \hookrightarrow E$ is zero-section.

$b \longmapsto 0 \in E_b$



Example Trivial vector bundle: $B \times \mathbb{R}^d$ (product topology).

Operations of vector Bundles.

a) Pullback: If $\rho: E \to X$ vector bundle & $f: Y \to X$ map:

$f^*E \to Y$, $f^*E = \{(e, y) \in E \times Y : \rho(e) = f(y)\} \xrightarrow{\text{proj}} Y$.

Then: $f^*E|_y = E|_{f(y)}$.

b) Whitney Sum: If $\rho: E \to X$, $q: F \to X$ vector bundles,

Sum $E \oplus F \to X$, $E \oplus F \equiv \{(e, f) \in E \times F : p(e) = q(e)\}$.

(So: under natural map $E \times F \to X \times X$, $E \oplus F$ maps to Δ_X.)

Remark Both pullback & sum:

i) take trivial bundles to trivial bundles

ii) Commute with restricting to open subsets of base.

(i.e. $f^*(E|_U) = (f^*E)|_{f^{-1}U}$ & $(E \oplus F)_U = E_U \oplus F_U$.)

$\Rightarrow$ Automatically: pullbacks & sums satisfy local triviality.)

More generally: $\exists$ dual E^* of bundle E, and tensor product $E \otimes F \to X$ of bundles $E \to X$ & $F \to X$.

If $E \to X$ bundle: say $F \subseteq E$ sub-bundle if: $\forall x \in X$, $F_x \subseteq E_x$ is linear subspace, and:

$$\begin{array}{ccc} E|_U & \xrightarrow{\Psi} & U \times \mathbb{R}^d \\ \cup & & \cup \\ F|_U & \longrightarrow & U \times \mathbb{R}^c \end{array}$$

for a linear $\mathbb{R}^c$: diag $\longrightarrow$.

& If $F \subseteq E$ sub-bundle:

$\exists$ quotient $E/F \to X$, fibre E_x/F_x.

& Say E, F isomorphic if:

$$\begin{array}{ccc} E & \xrightarrow{\alpha} & F \\ \downarrow p_E & & \downarrow p_F \\ X & \xrightarrow[g]{} & X \end{array}$$

commutes,

for some $\alpha : E \to F$ homeo

$g : X \to X$ homeo

with: $E_x \xrightarrow{\alpha} F_{g(x)}$ is a linear isomorphism $\forall x$.

(Key case: $\alpha = id$.)