Group Cohomology lecture 1 18/01/2024.

§1: Def's & Resolutions.

Let: G group.

DEF 1.1 Integral group ring $\mathbb{Z}G = \{ n_g \cdot g : n_g \in \mathbb{Z}, g \in G \}$
(only finitely many n_g are nonzero).

Is: free Abelian group, under: $(\sum n_g \cdot g) + (\sum m_g \cdot g) = \sum (n_g + m_g) g$.

& Has: multiplication $(\sum m_g \cdot g)(\sum n_h \cdot h) = \sum_g \sum_h (m_g \cdot n_h) \cdot (gh)$.

$$= \sum_g \Big(\sum_{hk=g} m_h n_k \Big) g.$$

$\Rightarrow$ Is: associative ring, underlying integral rep. of G.

DEF 1.2 A (left)- $\mathbb{Z}G$ - module M is abelian group under $+$, and map $\mathbb{Z}G \times M \to M$, $(r, m) \mapsto rm$ satisfying:

$r(m_1 + m_2) = rm_1 + rm_2$, $(r_1 + r_2)m = r_1 m + r_2 m$, $r_1(r_2 m) = (r_1 r_2) m$ & $1 \cdot m = m$.

Trivial module: if $gm = m$ $\forall g \in G$, $m \in M$.

Call the trivial module $\mathbb{Z}$, with $gn = n$ $\forall n \in \mathbb{Z}$, $g \in \mathbb{Z}$.

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free $\mathbb{Z}G$-module (on X)

Formulation: $\sum r_x \cdot x$, for $r_x \in \mathbb{Z}G$ & $x \in X$.
(& finitely many r_x nonzero, etc). Denote this: $\mathbb{Z}G\{X\}$.

Have: submodules, quotients, etc as you would expect.

DEF 1.3 A $\mathbb{Z}G$-map (morphism) $\alpha : M_1 \to M_2$ of abelian groups has: $\alpha(rm_1) = r\cdot\alpha(m_1)$ $\forall r\in\mathbb{Z}G$, $m_1\in M_1$.

Example) Augmentation map $\varepsilon: \mathbb{Z}G \longrightarrow \mathbb{Z}$, $\sum_G n_g\cdot g \longmapsto \sum_G n_g$.

This is a (left) $\mathbb{Z}G$-map of rings.

Notation) $\mathrm{Hom}_G(M,N) = \{\alpha: M\longrightarrow N \ \ \mathbb{Z}G\text{-maps}\}$.

Example) Regard $\mathbb{Z}G$ as (left) $\mathbb{Z}G$-module.

$\Rightarrow \mathrm{Hom}_G(\mathbb{Z}G, M) \cong M$ $\forall M$ left $\mathbb{Z}G$-module.

$$\phi \longmapsto \phi(1).$$

This is because any $\mathbb{Z}G$-map ϕ is determined by $\phi(1)$, since $\phi(r) = r\cdot\phi(1)$ $\forall r\in \mathbb{Z}G$.

Note: $\mathrm{Hom}_G(\mathbb{Z}G, M)$ is itself a left $\mathbb{Z}G$-module, by:

$(s\phi)(r) \equiv \phi(rs)$ $\forall s\in\mathbb{Z}G$.

Also: $\mathrm{Hom}_G(\mathbb{Z}G, \mathbb{Z}G) \cong \mathbb{Z}G$ $(M=\mathbb{Z}G)$

$$\phi \longmapsto \phi(1)$$

$\Rightarrow \phi$ corresponds to multiplication on right by $\phi(1)$.

Note) G not necessarily Abelian, so need to be careful about left/right-ness.

DEF 1.4) If $f: M_1 \longrightarrow M_2$ is $\mathbb{Z}G$-map, have <u>dual map</u>

$f^*: \mathrm{Hom}_G(M_2, N) \longrightarrow \mathrm{Hom}_G(M_1, N)$ (N $\mathbb{Z}G$-module)

$$\phi \longmapsto \phi\circ f$$

& <u>induced maps</u>: If $f: N_1 \longrightarrow N_2$ $\mathbb{Z}G$-maps, then

$f_*: \mathrm{Hom}_G(M, N_1) \longrightarrow \mathrm{Hom}_G(M, N_2)$

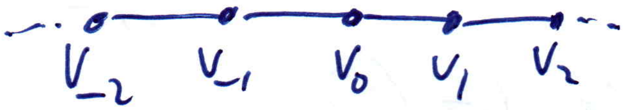
$$\phi \longmapsto f\circ\phi.$$

Prototype example. $G = \langle t \rangle$ infinite cyclic group.

$\Rightarrow G$ acts on $\mathbb{Z}$ by translations, where t sends $i \mapsto i+1$.

$\Rightarrow G$ acts on $V = \{v_i : i \in \mathbb{Z}\}$ & $E = \{\text{edges}\}$

& $\mathbb{Z}V$, $\mathbb{Z}E$ can be regarded as $\mathbb{Z}G$-modules.



& Are free: $\mathbb{Z}V = \mathbb{Z}G\{v_0\}$ & $\mathbb{Z}E = \mathbb{Z}G\{e\}$.

There are $\mathbb{Z}G$-maps
$$d : \mathbb{Z}E \longrightarrow \mathbb{Z}V$$
$$e \longmapsto v_1 - v_0$$

& since $\mathbb{Z}E = \mathbb{Z}G\{e\}$: have $\begin{matrix} \mathbb{Z}V \longrightarrow \mathbb{Z} \\ v_0 \longmapsto 1 \end{matrix}$, corresponding to augmentation $\varepsilon : \mathbb{Z}G \longrightarrow \mathbb{Z}$.

DEF 1.5 A *chain complex* of $\mathbb{Z}G$-modules is a sequence:
$$M_s \xrightarrow{d_s} M_{s-1} \xrightarrow{d_{s-1}} \cdots \xrightarrow{d_{t+1}} M_t \quad \text{s.t. } \forall\, t<n<s,\ d_n d_{n+1} = 0.$$
($\Rightarrow \operatorname{Ker}(d_n) \subseteq \operatorname{Im}(d_{n+1})$.)

Notation $M_* = (M_n, d_n)$, $t \leq n \leq s$.

Say: M_* *exact* at $n \iff \operatorname{im}(d_{n+1}) = \ker(d_n)$

& M_* *exact* $\iff$ exact $\forall n$.

$\Rightarrow$ Get Homology $H_s(M_*) = \ker(d_s)$

& $H_n(M_*) = \dfrac{\ker(d_n)}{\operatorname{im}(d_{n+1})}$ $\forall\, t<n<s$.

$H_t(M_*) = \dfrac{M_t}{\operatorname{im}(d_{t+1})} = \operatorname{coker}(d_{t+1})$.

Short exact seq $0 \to M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \to 0$

exact, i.e. α inj, β surj & $\operatorname{im}\alpha = \ker(\beta)$.

In proto example: $0 \to \mathbb{Z}E \to \mathbb{Z}V \to \mathbb{Z} \to 0$. SES.

$\iff$ $0 \to \mathbb{Z}G \to \mathbb{Z}G \to \mathbb{Z} \to 0$ SES

for $G = \langle t \rangle$.

mult on right

DEF 1.6) A $\mathbb{Z}G$-module P projective $\iff \forall\, \alpha : M_1 \to M_2$ surjective $\mathbb{Z}G$-map, & $\forall$ $\mathbb{Z}G$-map $\beta : P \to M_2$, $\exists\, \bar\beta : P \to M_1$ s.t. $\alpha \circ \bar\beta = \beta$.

$$\begin{tikzcd} & P \arrow[dl, dashed, "\bar\beta"'] \arrow[d, "\beta"] & \\ M_1 \arrow[r, "\alpha"'] & M_2 \arrow[r] & 0 \end{tikzcd}$$

If $0 \to N \xrightarrow{f} M_1 \xrightarrow{g} M_2 \to 0$ SES:

(*) $0 \to \operatorname{Hom}_G(P,N) \xrightarrow{f_*} \operatorname{Hom}_G(P,M_1) \xrightarrow{g_*} \operatorname{Hom}_G(P,M_2) \to 0$.

Then: Projective $\iff$ (*) exact.

(Note: (*) not always exact, except at $\operatorname{Hom}_G(P, M_2)$.)

Group Cohomology: lecture 2. 23/01/2024.

lemma 1.7) Free modules are projective.

Proof) Given $\alpha: M_1 \to M_2$ surjective $(\mathbb{Z}G)$- map

& $\beta: \mathbb{Z}G\{X\} \longrightarrow M_2$ map.

For each $x \in X$: $\exists m_x \in M_1$, s.t. $\alpha(m_x) = \beta(x)$ (Surj α)

$\Rightarrow$ Define $\bar{\beta}: \mathbb{Z}G\{X\} \longrightarrow M_1$

$$\sum r_x \cdot x \longmapsto \sum r_x \cdot m_x. \quad \checkmark$$

DEF 1.8) A projective / free resolution of trivial module $\mathbb{Z}$ is exact sequence: $\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \mathbb{Z} \to 0$

with all P_i projective / free resp. (Chain complex).

Example) (1) $G = \langle t \rangle$. Infinite, cyclic.

$\Rightarrow$ Finite resolution, $0 \longrightarrow \mathbb{Z}G \xrightarrow[\cdot(t-1)]{} \mathbb{Z}G \xrightarrow[\text{aug.}]{} \mathbb{Z} \to 0$.

2) $G = \langle t \rangle$, cyclic, order n.

$$\cdots \xrightarrow{\beta} \mathbb{Z}G \xrightarrow{\alpha} \mathbb{Z}G \xrightarrow{\beta} \mathbb{Z}G \xrightarrow{\alpha} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

$\alpha(x) = x(t-1)$ & $\beta(x) = x(1+t+\cdots+t^{n-1})$.

From Alg Top: if X connected simplicial complex with fundamental group G, s.t. $\widehat{X}$ (universal cover) contractible, then: obtain: free resolution of $\mathbb{Z}$ from $\widehat{X}$.

[Point is that X contains lots of info about G.]

Trying to replicate this algebraically.

For calculation purposes: interested in small resolutions, e.g. when free module has small rank.

& for theory purposes: wanting general constructions & such tend to be "large".

DEF 1.9 G is of type FP_n if $\mathbb{Z}$ has projective resol.

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$ & $P_n, P_{n-1}, \ldots, P_0$ f.g. $\mathbb{Z}G$-modules.

& G of type FP_∞ if as above + All P_n f.g. $\mathbb{Z}G$-modules.

& G of type FP if $\mathbb{Z}$ has projective resol. with finite length.

Examples 1) $G = \langle t \rangle$ infinite $\Rightarrow$ Type FP

2) $G = \langle t \rangle$ order n $\Rightarrow$ Type FP_∞. (+ all P_n fin-gen.)

All of these regarded as "finiteness" conditions of G.

[The topological equiv. of FP_n is asking for X to have finite n-skeleton.]

General Constructions.

If $P_s \xrightarrow{d_s} P_{s-1} \xrightarrow{d_{s-1}} \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z}$ partial projective resolution (i.e. exact & finite length): can make it longer. Set $P_{s+1} = \mathbb{Z}G\{X_{s+1}\}$, $X_{s+1} = \ker(d_s)$.

& $d_{s+1} : P_{s+1} \longrightarrow P_s$, $\underbrace{\sum r_x \cdot x}_{\in P_{s+1}} \longmapsto \underbrace{\sum r_x x}_{\in P_s}$

$\Rightarrow$ Get $P_{s+1} \xrightarrow{d_{s+1}} P_s \xrightarrow{d_s} \cdots$ Longer partial resolution.

But: since free module, likely not fin-gen.

To make P_{s+1} smaller: instead take X_{s+1} as $\mathbb{Z}G$-generating set of $\ker(d_s)$. Is more useful if: $\ker(d_s)$ f.g.

Continue this $\Rightarrow$ Get infinite (formal) proj resolution.

<u>DEF 1.10</u>) [Standard / Bar resolution of $\mathbb{Z}$.] For group G:

$G^{(N)} := \{[g_1|\dots|g_n] : g_1, \dots, g_n \in G\}$ Set of symbols.

& $F_n = \mathbb{Z}G\{G^{(N)}\}$.

& $$\begin{aligned} d_n([g_1|\dots|g_n]) = {} & g_1[g_2|\dots|g_n] \\ & - [g_1g_2|g_3|\dots|g_n] \\ & + \\ & \vdots \\ & +(-1)^{n-1}[g_1|\dots|g_{n-1}g_n] \\ & +(-1)^n [g_1|\dots|g_{n-1}]. \quad \in F_{n-1}. \end{aligned}$$

$\Rightarrow$ Have: $d_{n-1}d_n = 0 \ \forall n$, so get sequence:

$$\to F_n \to F_{n-1} \to \dots \to F_0 \to \mathbb{Z} \to 0 \quad \text{Chain complex.}$$
$$[\,] \mapsto 1.$$

<u>Remark</u>) Bar resolution $\Leftrightarrow$ Standard resolution (in AT).

Consider: $(n+1)$-tuples G^{n+1} & Free abelian group $\mathbb{Z}G^{n+1}$. (under +)

G acts on G^n by $g(g_0, \dots, g_n) = (gg_0, \dots, gg_n)$.

$\Rightarrow \mathbb{Z}G^{n+1}$ is free $(\mathbb{Z}G)$-module, on basis of $(n+1)$-tuples that start with 1 (i.e. $g_0 = 1$)

$\Rightarrow (g_1|\dots|g_n)$ corresponds to $(1, g_1, g_1g_2, \dots, g_1\dots g_n)$.

& Note: Removing first entry gives $g_1(1, g_2, g_2g_3, \dots, g_2 \dots g_n)$

$\Rightarrow$ Removing 2nd entry in $(n+1)$-tuple gives $(1, g_1, g_1g_2, \dots, g_1 \dots g_n)$.

Lemma 1.11 Bar resolution exact. ($\Rightarrow$ Is a resolution)

Proof Consider d_n as maps of abelian groups.

$\Rightarrow F_n$ has basis $G \times G^{(n)}$ as free abelian group.

$G \times G^{(n)} = \{ g_0 [g_1 | \dots | g_n] : g_i \in G \}$.

Define: $s_n : F_n \longrightarrow F_{n+1}$ s.t. $id_{F_n} = d_{n+1} s_n + s_{n-1} d_n$. $(*)$

Given by: $s_n (g_0 [g_1 | \dots | g_n]) = [g_0 | \dots | g_n]$.

Then: $(*)$ holds on basis elements of $G \times G^{(n)}$ (check).

$\Rightarrow \forall x \in \ker(d_n)$: $x = id_{F_n}(x) = d_{n+1} s_n(x) + s_{n-1} \cancel{d_n(x)}$

$\Rightarrow$ Exact at F_n ✓ $\in Im(d_{n+1})$.

Corollary 1.12 Finite group G is of type FP_∞.

Proof Bar resolution.

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DEF 1.13 let $\to P_n \to P_{n-1} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \to \mathbb{Z} \to 0$

Projective resolution. ~~Then~~ (of $\mathbb{Z}$, by $\mathbb{Z}G$-modules).

& let: M (left)- $\mathbb{Z}G$ module. Apply $\mathrm{Hom}_G(\cdot, M)$:

$$\leftarrow \mathrm{Hom}_G(P_{n+1}, M) \leftarrow \mathrm{Hom}_G(P_n, M) \leftarrow \cdots \xleftarrow{d^1} \mathrm{Hom}_G(P_0, M). \quad (+)$$

(where: $d^n = d_n^*$)

The n-th Cohomology group $H^n(G,M)$ with coeffs in M:

$$H^n(G,M) = \mathrm{Ker}(d^{n+1})/\mathrm{Im}(d^n) \quad \forall n \geq 1.$$

& $H^0(G,M) = \mathrm{Ker}(d^1)$.

Remark 1) Have dropped $\mathbb{Z}$-term in $(+)$.

2) Those Cohomology groups are: Homology groups of the Chain complex $C_n \equiv \mathrm{Hom}_G(F_{-n}, M)$ $\forall n<0$.

3) (Proved next time): Cohomology groups indep of choice of resolution.

Prototype Example: $G = \langle t \rangle \cong \mathbb{Z}$

Recall: $0 \to \mathbb{Z}G \xrightarrow[\times(t-1)]{d_1} \mathbb{Z}G \to \mathbb{Z} \to 0$. SES.

Then: $\forall \phi \in \mathrm{Hom}_G(\mathbb{Z}G, M)$ & $x \in \mathbb{Z}G$:

$$d^1(\phi)(x) = \phi(d_1(x)) = \phi(x(t-1)).$$

Recall: $\mathrm{Hom}_G(\mathbb{Z}G, M) \cong M$. So, $d^1(\phi)$:

$$\theta \longmapsto \theta(1).$$

$$d^1(\phi) \longmapsto d^1(\phi)(1) = \phi(t-1) = (t-1)\mathbb{1}(\phi).$$

$\Rightarrow$ Get: Cohomology sequence $0 \leftarrow M \xleftarrow{(*)} M$

$(*)$ = left-mult by $(t-1)$.

$\Rightarrow H^0(G,M) = \{m \in M : tm = m\} = M^G$

$H^1(G,M) = M / \{(t-1)m : m \in M\} = M_G$

$H^n(G,M) = 0 \;\forall n \geq 2.$

Call: M^G = Group of invariants (largest Abelian subgroup with trivial action)

& M_G = Co-invariants. Largest quotient Abelian group with trivial G-action.

Note 1) $H^0(G,M) = M^G$ is true in general.

But: $H^1(G,M) \neq M_G$ in general!

2) for any group G of type FP, $H^n(G,M) = 0$ for all n large.

DEF 1.14) G has: Cohomological dimension m over $\mathbb{Z}$, if:

$\exists$ some $\mathbb{Z}G$-module M, with: $H^m(G,M) \neq 0$

& $\forall$ $\mathbb{Z}G$-module M, have $H^n(G,M) = 0 \;\forall n > m$.

Note: $\forall G$, $H^0(G,\mathbb{Z}) = \mathbb{Z} \neq 0$. $\Rightarrow$ All dimensions ≥ 0.

Remark If $G = \langle t \rangle \cong \mathbb{Z}$, G has Cohomological dimension 1 (over $\mathbb{Z}$). & If G free group of finite rank, also dim = 1.

Remark) Converse true! A finitely generated group of dim 1 is always free (Stallings, 19~~66~~ 68). True in general (1969).

Now: consider Bar Resolution in def. of Cohomology.

$\mathbb{Z}G\{G^{(n)}\}$ = Set of symbols $[g_1|\ldots|g_n]$.

& $\mathrm{Hom}_G(\mathbb{Z}G\{G^{(n)}\}, M) \cong C^n(G, M)$

$= \{\text{functions } G^{(n)} \longrightarrow M\} \ \forall n \geq 1$.

A $(\mathbb{Z}G)$-map is determined by: effect on basis.

$\Rightarrow = \{\text{Functions } G^n \longrightarrow M\}$.

& $C^0(G, M) = \{\text{Functions } [\cdot] \to M\} \cong M$.

DEF 1.15] The Abelian groups of n co-chains of G, with coeffs in M, is: $C^n(G, M)$.

& n'th Cohomology coboundary map: $d^n: C^{n-1}(G,M) \to C^n(G,M)$ dual to: d_n, in base resolution.

For $\phi \in C^{n-1}(G, M)$:
$$\begin{aligned}(d^n\phi)(g_1 \ldots g_n) = {} & g_1\phi(g_2, \ldots, g_n) \\ & - \phi(g_1g_2, g_3, \ldots, g_n) \\ & + \phi(g_1, g_2g_3, \ldots, g_n) \\ & + \ldots + (-1)^{n-1}(g_1, \ldots, g_{n-1}g_n) \\ & + (-1)^n\phi(g_1, \ldots, g_{n-1})\end{aligned}$$

& Group of n-Cocycles:

$Z^n(G, M) = \mathrm{Ker}\, d^{n+1} \leq C^n(G, M)$

& Group of n-Coboundaries:

$B^n(G, M) = \mathrm{Im}(d^n) \leq C^n(G, M)$.

$\Rightarrow H^n(G, M) = Z^n(G, M) / B^n(G, M)$.

Corollary 1.16] $H^0(G, M) = M^G \ \forall G$ group.

DEF 1.17 A *derivation* of G with coeffs in M is function $\phi : G \to M$, s.t. $\phi(gh) = g\phi(h) + \phi(g)$

Note: $Z^1(G,M)$ is: exactly set of derivations. "Crossed Homs".

An *inner derivation* is: ϕ of form $\phi(g) = gm - m$, for some *fixed* $m \in M$.

Corollary 1.18
$$H^1(G,M) = \frac{\{\text{Derivations } G \to M\}}{\{\text{Inner derivations } G \to M\}}.$$

In particular: if M *trivial* $\mathbb{Z}G$-module, then:
$H^1(G,M) = \mathrm{Hom}(G,M)$ (group homs).

Return to: Considering homology, arising from different resolutions.

DEF 1.20 *Let*: (A_n, α_n), (B_n, β_n) chain complexes, of $\mathbb{Z}G$-modules. A *chain map* (f_n) is: sequence of $\mathbb{Z}G$-maps $f_n : A_n \to B_n$, with:

$$\begin{tikzcd} A_n \arrow[r, "\alpha_n"] \arrow[d, "f_n"'] & A_{n-1} \arrow[d, "f_{n-1}"] \\ B_n \arrow[r, "\beta_n"] & B_{n-1} \end{tikzcd}$$

commutes.

Remark: Chain maps induce maps on Homology.

Lemma 1.20 Given chain map (f_n) as above: induces a map $f_* : H_n(A_*) \to H_n(B_*)$

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lemma 1.20 For chain map (f_n), induces $f_*: H_n(A_\bullet) \to H_n(B_\bullet)$

Proof let $x \in \mathrm{Ker}(\alpha_n)$, define $f_*([x]) = [f_n(x)]$

$\Rightarrow$ $f_n(x) \in \mathrm{Ker}(\beta_n)$, since: $\beta_n f_n(x) = f_{n-1}\alpha_n(x) = 0$.

& If $x' = x + \alpha_{n+1}(y)$ (some y), then:

$f_n(x') = f_n(x) + f_n \alpha_{n+1}(y) = f_n(x) + \beta_{n+1} f_{n+1}(y) \in f_n(x) + \mathrm{Im}\beta_{n+1}$

$\Rightarrow$ This is well-defined (and check: gives map of Abelian grps).

Theorem 1.21 Definition of $H^n(G, M)$ does not depend on choice of resolution.

Proof Choose: Projective resolutions (P_n, d_n) & (P_n', d_n').

Will: produce $(\mathbb{Z}G)$-maps:

$$\begin{bmatrix} f_n : P_n \to P_n' \\ g_n : P_1' \to P_n \\ s_n : P_n \to P_{n+1} \\ s_n' : P_1' \to P_{n+1}' \end{bmatrix} \quad \text{with:} \quad \begin{bmatrix} f_{n-1} d_n = d_n' f_n \\ g_{n-1} d_n' = d_n g_n \\ d_{n+1} s_n + s_{n-1} d_n = g_n f_n - id \\ d_{n+1}' s_n' + s_{n-1}' d_n' = f_n g_n - id. \end{bmatrix}$$

In this case, $(f_n), (g_n)$ give chain maps $\mathrm{Hom}_G(P_\bullet', M) \to \mathrm{Hom}_G(P_\bullet, M)$

& $\mathrm{Hom}_G(P_\bullet, M) \to \mathrm{Hom}_G(P_\bullet', M)$.

$\Rightarrow$ Induces: maps between respective Homology groups (by lemma 20), and: if $\phi \in \mathrm{Ker}(d^{n+1}) \in \mathrm{Hom}(P, M)$, then: $f_n^* g_n^*(\phi)(x) = \phi(g_n f_n(x)) = \phi(x) + \phi(d_{n+1} s_n(x)) + \phi(s_{n-1} d_n(x))$

$= \phi(x) + \cancel{S_n^* d^{n+1}(\phi)(x)} + d^n S^*_{n-1}(\phi)(x)$

$\Rightarrow f_n^* g_n^* = \phi + d^n(S^*_{n-1}(\phi))$.

$\Rightarrow f_n^* g_n^*$ induces identity map on: $\mathrm{Ker}(\partial^{n+1})/\mathrm{Im}(d^n)$.

Similarly: for $g_n^* f_n^*$. Hence, these define $\cong$ between homology groups ✓

Remains to construct: the maps.

Consider: End of resolutions. $f_{-1}: \mathbb{Z} \to \mathbb{Z}$ is identity

$f_{-2}: 0 \to 0$ is zero map.

Suppose: defined f_{n-1} & f_n. Want: to construct f_{n+1}.

I.e. $f_n d_{n+1}: P_{n+1} \to P'_n$

& $d'_n \circ (f_n d_{n+1}) = f_{n-1} d_n d_{n+1} = 0$

$\Rightarrow f_n d_{n+1}$ has image in $\mathrm{Ker}(d'_n)$.

Define f_{n+1} by:

$$\begin{tikzcd}
 & P_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_n d_{n+1}"] \arrow[dl, dashed, "f_{n+1}"'] & P_n \arrow[r, "d_n"] \arrow[d, "f_n"] & P_{n-1} \arrow[d, "f_{n-1}"] \\
P'_{n+1} \arrow[r, two heads] & \mathrm{Ker}\, d'_n \arrow[r, hook] & P'_n \arrow[r, "d'_n"'] & P'_{n-1}
\end{tikzcd}$$

Observe: P_{n+1} projective, & $P'_{n+1} \to \mathrm{Ker}(d'_n)$ surj (by: def. of projective modules)

Similarly for ~~g_{n+1}~~ (g_n).

To define (S_n): first, set $h_n = g_n f_n - id : P_n \to P_n$.

$\Rightarrow h_n$ chain map (with $h_{-1} = 0$)

$\Rightarrow$ ~~&~~ $s_{-1} : \mathbb{Z} \to P_0$ is: zero map.

<u>Note</u>: $d_0 h_0 = h_{-1} d_0 = 0 \Rightarrow$ ~~$\mathrm{Im}(d_0) \subseteq$~~
$\mathrm{Im}(h_0) \subseteq \mathrm{Ker}(d_0)$.

As before: $d_1 : P_1 \longrightarrow \mathrm{Ker}(d_0)$ <u>surjective</u>.

$$\begin{tikzcd}
 & & P_0 \arrow[r] \arrow[dll, "s_0"'] \arrow[dl, "h_0"] \arrow[d, "h_0"] & \mathbb{Z} \arrow[d] \\
P_1 \arrow[r, two heads] & \mathrm{Ker}(d_0) \arrow[r, hook] & P_0 \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

(using def. of projective again, to get s_0.)

To define <u>rest</u>: suppose s_{n-1} & s_{n-2} are defined.

$\Rightarrow t_n = (h_n - s_{n-1} d_n) : P_n \longrightarrow P_n$

$\Rightarrow d_n t_n = d_n h_n - d_n s_{n-1} d_n = h_{n-1} d_n - (h_{n-1} - s_{n-2} d_{n-1}) d_n$

$$= s_{n-2} d_{n-1} d_n = 0.$$

$\Rightarrow \mathrm{Im}(t_n) \subseteq \mathrm{Ker}(d_n)$.

$$\begin{tikzcd}
 & & P_n \arrow[r, "d_n"] \arrow[dll, "s_n"'] \arrow[dl, "t_n"] \arrow[d, "h_n"] & P_{n-1} \arrow[d, "h_{n-1}"] \arrow[dl, dashed, "s_n"'] \\
P_{n+1} \arrow[r, two heads] & \mathrm{Ker}(d_n) \arrow[r, hook] & P_n \arrow[r, "d_n"'] & P_{n-1}
\end{tikzcd}$$

Similarly defined s_n.

<u>Remark</u> For any (left) $\mathbb{Z}G$-module N, can take ~~[illegible]~~ proj/free resolution of N, by $\mathbb{Z}G$-modules.

$\Rightarrow$ Repeating everything: applying $\mathrm{Hom}_G(-, M)$, gives:

Homology groups $\mathrm{Ext}^n_{\mathbb{Z}G}(N, M)$.

[So, $H^n(G, M) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M)$.]

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§2: Low-degree Cohomology (& Group extensions.)

<u>Recall</u>: $H^0(G, M) = M^G$. (invariants)

& Derivation / 1-cycle: $\phi : G \to M$, $\phi(g_1 g_2) = g_1 \phi(g_2) + \phi(g_1)$.

<u>Will get</u>: 2 interpretations of derivations. & inner derivations.

($\phi(g) = gm - m$ for <u>fixed</u> $m \in M$.)

<u>Interpretation 1)</u> Consider: possible $\mathbb{Z}G$-actions on Abelian group $M \oplus \mathbb{Z}$, of the form: $g(m, n) = (gm + n\phi(g), n)$.

$$\Rightarrow g_1(g_2(m,n)) = g_1(g_2 m + n\phi(g_2), n)$$
$$= (g_1 g_2 m + n g_1 \phi(g_2) + n\phi(g_1), n)$$

& $(g_1 g_2)(m, n) = (g_1 g_2 m + n \phi(g_1 g_2), n)$.

So, these are exactly same $\Leftrightarrow$ ϕ is derivation.

In particular: if M is <u>free</u> $\mathbb{Z}$-module of <u>finite</u> rank, then: $\exists$ map $g \mapsto \left(\begin{array}{c|c} \theta_1(g) & \phi(g) \\ \hline 0 & 1 \end{array}\right)$ $\theta_1(g)$ = Action of g on M.

which is: group homomorphism $\Leftrightarrow$ ϕ derivation.

<u>Check</u>: ϕ <u>inner</u> derivation $\Leftrightarrow$ $(-m, 1)$ generates a $\mathbb{Z}G$-module which is trivial inside $M \oplus \mathbb{Z}$.

Group Cohomology: Lecture 5

~~05~~ 02/02/2024

<u>From last time</u>: Representation theoretic interpretation of $H^1(G, M)$. We showed: we can construct G-action using derivation, and M was $\mathbb{Z}G$-module, quotient $\cong \mathbb{Z}$.
This construction is: direct sum of $\mathbb{Z}G$-modules $M \oplus \mathbb{Z}$, precisely if: derivation is an inner derivation.

<u>Second Interpretation:</u> Group theoretic. (Example sheet.)

<u>DEF 2.1</u>) G group & M $\mathbb{Z}G$-module. <u>Construct</u>: $M \rtimes G$ by defining group operation $(m_1, g_1) * (m_2, g_2)$

$$= (m_1 + g_1 m_2, \; g_1 g_2)$$

<u>Note</u>: $M \cong \{(m, 1) : m \in M\} \Rightarrow M$ is Abelian, normal subgrp.
& $G \cong \{(0, g) : g \in G\}$, & Conjugation by $(0, g)$ corresponds to: a given G-action, on M:

$$(M \rtimes G) / \{(m, 1) : m \in M\} \cong G.$$

There is a <u>hom</u>: $s : G \longrightarrow M \rtimes G$, $g \longmapsto (0, g)$

<u>Has</u>: $G \xrightarrow{s} M \rtimes G \xrightarrow{\pi} G$ is: identity map.

<u>Call</u>: s the <u>splitting map</u>.

If there is <u>another</u> splitting $G \xrightarrow{s_1} M \rtimes G$ with $s_1 \circ \pi = id$, then: define $\chi_{s_1} : G \to M$ such that $s_1(g) = (\chi_{s_1}(g), g)$.
<u>Then</u>: $\chi_{s_1} \in Z^1(G, M)$.

If S_1, S_2 splittings: then $\chi_{S_1} - \chi_{S_2} \in B^1(G,M)$ <u>iff</u> there is $m \in M$ with $(m,1)\, S_1(g)\, (m,1)^{-1} = S_2(g)$.

<u>Conversely</u>: given 1-cocycle $\phi \in Z^1(G,M)$, there is a splitting $S_1 : G \longrightarrow M \rtimes G$, with $\phi = \chi_{S_1}$.

<u>Theorem 2.2)</u> $H^1(G,M) \longleftrightarrow M$-conj classes of splittings.

<u>Next</u>: What about H^2? <u>$H^2(G,M)$</u>.

<u>DEF 2.3)</u> An <u>extension</u> E <u>of</u> G <u>by</u> M ($\mathbb{Z}G$-module M) is group s.t. $0 \longrightarrow M \longrightarrow E \longrightarrow G \longrightarrow 1$

s.t. $M \hookrightarrow E$ & $M \subseteq E$ image is Abelian <u>normal</u> subgroup of E; which is acted on by <u>conjugation</u> by E.

($\Rightarrow$ Since M Abelian, have induced map $E/M \cong G$.)

This action must: match given G-action on M.

<u>Example</u> Semidirect product $0 \longrightarrow M \longrightarrow M \rtimes G \longrightarrow G \longrightarrow 1$

<u>DEF 2.4)</u> 2 exts are <u>equivalent</u> if $\exists$ Commuting diag:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\phi"] & G \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & M \arrow[r] & E' \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

<u>Exercise</u>: Equivalent extensions yield E, E' that are isomorphic (as groups). <u>Converse is false!</u> Can find E, E' isomorphic as groups, but inequivalent exts.

DEF 2.5] (Central Extension)
Is: one where the given $\mathbb{Z}G$-module M is a trivial $\mathbb{Z}G$-module.

Prop 2.6] Let: E extension of G by M. If there is a splitting $s_1 : G \to E$, ~~st.~~ then: E is equivalent to $0 \to M \to M \rtimes G \to G \to 1$. (So, $E \cong M \rtimes G$.)

Theorem 2.7] G group, M $\mathbb{Z}G$-module. Then, $\exists$ bijection
$$H^2(G, M) \longleftrightarrow \left\{ \begin{matrix} \text{Equivalence classes of} \\ \text{extensions of } G \text{ by } M \end{matrix} \right\}.$$

Given an ext. $0 \to M \to E \to G \to 1$, there is a set-theoretic section $s : G \to E$ s.t. $G \xrightarrow{s} E \xrightarrow{\pi} G$ $(=id)$

But, s does not need to be a group hom!

WLOG: $s(1) = 1$. & $\phi(g_1, g_2) = s(g_1) s(g_2) s(g_1 g_2)^{-1}$.
(suppose:)

Then: $\pi(\phi(g_1, g_2)) = 1$, hence: $\phi(g_1, g_2) \in M$.

$\Rightarrow$ Have: $\phi : G^2 \to M$ is a 2-cochain.

In fact, it is a 2-cocycle!

$$\begin{aligned} s(g_1) s(g_2) s(g_3) &= \phi(g_1, g_2) s(g_1 g_2) s(g_3). \\ &= \phi(g_1, g_2) \phi(g_1 g_2, g_3) s(g_1 g_2 g_3) \end{aligned}$$

similarly

$$\begin{aligned} &= \cancel{\phi(g_2, g_3)} \; s(g_1) \phi(g_2, g_3) s(g_2 g_3) \\ &= s(g_1) \phi(g_2, g_3) s(g_1)^{-1} \cdot s(g_1) s(g_2 g_3) \end{aligned}$$

$= [s(g_1)\phi(g_2,g_3)s(g_1)^{-1}]\phi(g_1,g_2g_3)s(g_1g_2g_3)$.

Equate them & Convert to additive notation:

$\Rightarrow \phi(g_1g_2) + \phi(g_1g_2g_3) = g_1\phi(g_2,g_3) + \phi(g_1, g_2g_3)$

$\Rightarrow (\partial^2\phi)(g_1, g_2, g_3) = 0$. So, indeed, 2-cocycle.

Note ϕ normalised cocycle: $\phi(1,g) = \phi(g,1) = 0$.

$\Rightarrow$ An extension of G by M, with a choice of set-theoretic section, yields: normalised 2-cocycle $\phi \in Z^2(G,M)$.

Now: take another choice of section s', with $s'(1) = 1$.

$\Rightarrow$ The normalised cocycles ϕ, ϕ' differ by a coboundary, and: so, have map defined, from extensions $\longrightarrow H^2(G,M)$.

Group Cohomology: Lecture 6 (Example class Tues) 08/02/2024

Theorem 2.7 $\exists$ bijection:

$$\left\{\begin{array}{l}\text{Equiv classes of}\\ \text{exts } M \hookrightarrow E \twoheadrightarrow G\end{array}\right\} \longleftrightarrow H^2(G,M).$$

Last time: took set-theoretic section $s: G \to E$ & constructed a 2-cocycle ϕ, with: $\phi(g_1, g_2) = s(g_1)s(g_2)s(g_1g_2)^{-1}$.

& ϕ normalised, i.e. $\phi(1,g) = \phi(g,1) = 0 \ \forall g \in G$.

If take another (set-theoretic) section s' with $s'(1)=1$,

then: $\pi(s(g)s'(g)^{-1}) = 1 \Rightarrow \underbrace{s(g)s'(g)^{-1}}_{=\chi(g)} \in \mathrm{Ker}(\pi) = M$.

So, $\chi: G \to M$.

Then:
$$\begin{aligned}
s'(g_1)s'(g_2) &= \chi(g_1)s(g_1)\chi(g_2)s(g_2)\\
&= \chi(g_1)s(g_1)\chi(g_2)s(g_1)^{-1}s(g_1)s(g_2)\\
&= \chi(g_1)s(g_1)\chi(g_2)s(g_1)^{-1}\phi(g_1,g_2)s(g_1g_2)\\
&= \chi(g_1)[s(g_1)\chi(g_2)s(g_1)^{-1}] \phi(g_1,g_2)\chi(g_1g_2)^{-1}s'(g_1g_2)
\end{aligned}$$

Switch to additive notation:

$$\begin{aligned}
\phi'(g_1,g_2) &= \chi(g_1) + g_1\chi(g_2) + \phi(g_1,g_2) - \chi(g_1g_2)\\
&= \phi(g_1,g_2) + (d^2\chi)(g_1,g_2)
\end{aligned}$$

where: ϕ' is 2-cocycle, arising from the section s'.

$\Rightarrow \phi, \phi'$ differ by a co-boundary $d^2\chi$.

$\Rightarrow$ Have: map ~~A~~ from extensions, to: $H^2(G,M)$.

For rest of proof, need: ~~Equivalent exts yield same Cohomology class~~ ~~& Construct ~~ inverse map from Cohomology class, to an extension~~

a) Show: Equivalent exts $\Rightarrow$ Same Cohomology class

b) Construct inverse map from Cohomology classes to exts

c) Show that the maps are indeed inverses.

Lemma 2.8 $\phi \in Z^2(G,M)$. $\Rightarrow \exists \chi \in C^1(G,M)$, with: $\phi + d^2\chi$ normalised cocycle.

$\Rightarrow$ Every Cohomology class can by rep'ed by normalised 2-cycle.

Proof Denote $\chi(g) = -\phi(1,g)$.

Then: $(\phi + d^2\chi)(1,g) = \phi(1,g) - (\phi(1,g) - \phi(1,g) + \phi(1,1))$

$= \phi(1,g) - \phi(1,1)$ (*)

& $(\phi + d^2\chi)(g,1) = \cancel{g} \phi(g,1) - \phi(1,1)$. (**)

But, know: ~~$d^3\phi$~~ $d^3\phi(1,1,g) = 0 = d^3\phi(g,1,1)$ since cocycle.

Hence, a computation $\Rightarrow$ (*), (**) are both 0. ✓

Now: take normalised cocycle rep. ϕ, of the Cohomology class.

Then, construct an extension:

$0 \to M \to E_\phi \to G \to 1$, by:

$(m_1, g_1) * (m_2, g_2) = (m_1 + g_1 m_2 + \phi(g_1, g_2),\ g_1 g_2)$

For this to be a group op, use: ϕ normalised.

& Check: indeed, have an ext, $0 \to M \to E_\phi \xrightarrow{\pi} G$.

(π = proj, to 2nd coord).

If we have another normalised 2 cocycle ϕ' representing the Cohomology class, then: $\phi - \phi' = d^2 \chi$, some χ.

& Define: $E_\phi \to E_{\phi'}$ by: $(m, g) \mapsto (m + \chi(g), g)$.

This gives equivalence of Extensions ✓

Example of Central Extensions. $\mathbb{Z}^2$ by $\mathbb{Z}$.

Already know: $0 \to \mathbb{Z} \to \mathbb{Z}^3 \to \mathbb{Z}^2 \to 0$

$m \mapsto (m, 0, 0)$

$(m, r, s) \mapsto (r, s)$.

Can also set: Heisenberg group $H = \left\{ \begin{pmatrix} 1 & r & m \\ & 1 & s \\ & & 1 \end{pmatrix} : r, s, m \in \mathbb{Z} \right\}$.

& $0 \to \mathbb{Z} \to H \to \mathbb{Z}^2 \to 0$.

$m \mapsto \begin{pmatrix} 1 & & m \\ & 1 & \\ & & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & r & m \\ & 1 & s \\ & & 1 \end{pmatrix} \mapsto (r, s)$.

Now, write things ~~~~ multiplicatively: $\Rightarrow T \cong \mathbb{Z}^2$ gen. by $\{a, b\}$.

& Have Free resolution of trivial $(\mathbb{Z}T)$-module $\mathbb{Z}$:

$0 \to \mathbb{Z}T \xrightarrow{\beta} \mathbb{Z}T^2 \xrightarrow{\alpha} \mathbb{Z}T \xrightarrow{\varepsilon} \mathbb{Z} \to 0$

<u>Where</u>: $\beta(z) = \big(z(1-b),\ z(a-1)\big)$

$\alpha(x,y) = \big(x(a-1) + y(b-1)\big)$

& ε is Augmentation.

Apply $\mathrm{Hom}_T(-,\mathbb{Z})$ gives Chain complex:

$$0 \leftarrow \mathrm{Hom}_T(\mathbb{Z}T,\mathbb{Z}) \xleftarrow{\beta^*} \mathrm{Hom}_T(\mathbb{Z}T^2,\mathbb{Z}) \xleftarrow{\alpha^*} \mathrm{Hom}_T(\mathbb{Z}T,\mathbb{Z}).$$

<u>Claim</u>: α^*, β^* are both <u>Zero</u> maps.

$\Rightarrow H^2(T,\mathbb{Z}) = \mathrm{Hom}_T(\mathbb{Z}T,\mathbb{Z}) \cong \mathbb{Z}$, with generator rep'ed by $\varepsilon: \mathbb{Z}T \to \mathbb{Z}$.

<u>Check $\beta^*=0$</u>: take: $f: \mathbb{Z}T^2 \to \mathbb{Z}$ $\mathbb{Z}T$-map, & $z \in \mathbb{Z}T$.

$\Rightarrow (\beta^* f)(z) = f(\beta)(z) = f\big(z(1-b), z(a-1)\big)$

$= f(z-bz, 0) + f(0, za-z)$ [T acts trivially]

$= (1-b)f(z,0) + (a-1)f(0,z) = 0.$ [Again T acts trivially]

Similarly, $\alpha^* = 0$.

<u>Next</u>: Interpret $H^2(T,\mathbb{Z})$ in terms of 2-cocycles. (Bar resol.)

$\Rightarrow$ Construct chain map.

$$\begin{tikzcd}
\mathbb{Z}T\{T^{(2)}\} \arrow[r, "d_2"] \arrow[d, "f_2"] & \mathbb{Z}T\{T^{(1)}\} \arrow[r, "d_1"] \arrow[d, "f_1"] & \mathbb{Z}T\{T^{(0)}\} \arrow[r, "\epsilon"] \arrow[d, "id"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
\mathbb{Z}T \arrow[r, "\beta"] & \mathbb{Z}T^2 \arrow[r, "\alpha"] & \mathbb{Z}T \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

(In degree $0,-1$: have identity map, might as well).

Construct: $\underline{f_1}$. $\mathbb{Z}T\{T^{(1)}\} \longrightarrow \mathbb{Z}T^2$. $\alpha f_1 = d_1$.

So, need to give images for: symbols $[a^r b^s]$: $r, s \in \mathbb{Z}$.

i.e. $[a^r b^s] \longmapsto (x_{rs}, y_{rs}) \in \mathbb{Z}T^2$.

With: $\alpha(x_{r,s}, y_{r,s}) = d_1[a^r b^s] = a^r b^s - 1$

$$= (a^r - 1)b^s + (b^s - 1).$$

Define: $S(a,r) \equiv 1 + a + \cdots + a^{r-1} \quad (r > 0)$

$$= -a^{-1} - \cdots - a^{-r} \quad (r \leq 0)$$

$\Rightarrow$ Note: $(a-1)S(a,r) = a^r - 1$. (in all cases)

$\Rightarrow \alpha(S(a,r)b^s, S(b,s)) = S(a,r)b^s(a-1) + S(b,s)(b-1)$

$$= d_1[a^r b^s].$$

So, **define**: $f_1[a^r b^s] = (S(a,r)b^s, S(b,s))$.

Next time: define f_2.

Group Cohomology: lecture 7 | 13/02/2024

From last time:
$$\begin{tikzcd} \mathbb{Z}T\{T^{(2)}\} \arrow[r] \arrow[d, "f_2"] & \mathbb{Z}T\{T^{(1)}\} \arrow[r] \arrow[d, "f_1"] & \mathbb{Z}T\{T^{0}\} \arrow[r] \arrow[d, "id"] & \mathbb{Z} \arrow[d, equal] \\ \mathbb{Z}T \arrow[r] & \mathbb{Z}T^2 \arrow[r] & \mathbb{Z}T \arrow[r] & \mathbb{Z} \end{tikzcd}$$

& Defined $S(a,r) = \begin{cases} 1 + a + \dots + a^{r-1} & \text{if } r > 0 \\ -a^{-1} - \dots - a^{-r} & \text{if } r \leq 0 \end{cases}$

& Defined $f_1[a^r b^s] \equiv (S(a,r)b^s, S(b,s))$.

Want to define: f_2.

For each $[a^r b^s | a^t b^u]$: find $z_{r,s,t,u} \in \mathbb{Z}T$, with:

$f_1 d_2 [a^r b^s | a^t b^u] = \beta(z_{r,s,t,u})$

LHS $= f_1(a^r b^s [a^t b^u] - [a^{r+t} b^{s+u}] - [a^r b^s])$

$= (a^r b^s S(a,t) b^u - S(a, r+t) b^{s+u} + S(a,r) b^s,$
$a^r b^s S(b,u) - S(b, s+u) + S(b,s))$

& Notice that $z_{r,s,t,u} = S(a,r) b^s S(b,u)$ works.

$\Rightarrow$ Define $f_2[a^r b^s | a^t b^u]$ as $\uparrow$.

Next: Find cochain $\phi : T^2 \to \mathbb{Z}$ representing the cohomology class $p \in \mathbb{Z} = \mathrm{Hom}_T(\mathbb{Z}T, \mathbb{Z}) = \langle \text{Augmentation map } \varepsilon \rangle$
$= H^2(T, \mathbb{Z})$.

Such cochain is given by composition $\phi : T^2 \xrightarrow{f_2} \mathbb{Z}T \xrightarrow{p\varepsilon} \mathbb{Z}$.

Since $\varepsilon(S(a,r)) = r$: find:

$\phi(a^r b^s, a^t b^u) = p\varepsilon(z_{r,s,t,u}) = pru.$

The group structure on $\mathbb{Z} \times T$ corresponding to this is:

$(m, a^r b^s) * (n, a^t b^u) = (m+n+pru, a^{r+t} b^{s+u}).$

Corresponds to: $\left\{ \begin{pmatrix} 1 & pr & m \\ & 1 & s \\ & & 1 \end{pmatrix} : m, r, s \in \mathbb{Z} \right\}.$

Another approach to considering <u>extensions</u> (in particular, central extensions): Use <u>Partial</u> resolution arising from Generators and Relations.

Given group G: take gen set X. $\Rightarrow \exists$ map $F_X \to G.$
(F_X = <u>free</u> group on X) $\qquad x \mapsto x$

Denote: R = <u>Kernel</u> of this map. $1 \to R \to F_X \to G \to 1.$

<u>Can</u>: think of R as set of <u>Relations</u>.

Often, take a generating set of R (as normal subgroup in F), <u>and</u>: take Normal closure.

<u>Let</u>: $R_{ab} = R/R'$, where R' = Derived subgroup.

It inherits action of F. <u>But</u>, R acts trivially, so: have an <u>induced</u> action by ~~from~~ $G = F/R$.

Since <u>Abelian</u>: R_{ab} is $\mathbb{Z}G$-module. "<u>Relation Module</u>"

<u>Have</u>: extension $1 \to R_{ab} \to F/R' \to G \to 1$

To get <u>Central</u> ext: $1 \to R/[R,F] \to F/[R,F] \to G \to 1$ $\qquad$ $\boxed{R}$

Where: $[R, F] = \{[r, f] : r \in R,\ f \in F\}$.

Unfortunately: no such "largest"/Universal Central ~~action~~ extension (since: can form direct prod with Abelian group).

But: Central Ext (as above) does have good properties.

Theorem 2.9] Given Presentation $1 \to R \to F \to G \to 1$ & M is (Left) $\mathbb{Z}G$-module. Then: $\exists$ Exact Sequence

$$H^1(F, M) \longrightarrow \mathrm{Hom}_G(R_{ab}, M) \longrightarrow H^2(G, M) \longrightarrow 0.$$

Thus: Any equiv class of exts on G by M, corresponding to a Cohomology Class, arises from taking some $\mathbb{Z}G$-map $R_{ab} \to M$.

Note: M is a $(\mathbb{Z}F)$-module, via map $F \to G$.

Corollary 2.10] In the above, if M is trivial $\mathbb{Z}G$-module, then: $\mathrm{Hom}(F, M) \longrightarrow \mathrm{Hom}_G(R/[R,F], M) \to H^2(G, M) \to 0$.

Proof] Know: M is trivial $\mathbb{Z}F$-module.

$\Rightarrow H^1(F, M) = \mathrm{Hom}(F, M) = \mathrm{Hom}(F_{ab}, M)$.

& $\mathrm{Hom}_G(R_{ab}, M) = \mathrm{Hom}_G(R/[R,F], M)$.

Trivial module

There is also Homology group connection.

Given projective resolution of Trivial $\mathbb{Z}G$-module $\mathbb{Z}$, can apply $\mathbb{Z} \otimes_{\mathbb{Z}G} (-)$ to resolution, to get Homology groups.

Group Cohomology: lecture 8 15/02/2024.

Take presentation $1 \to R \to F \to G \to 1$, and let:

$I_F = \ker(\mathbb{Z}F \longrightarrow \mathbb{Z})$, $\overline{I}_R = \ker(\mathbb{Z}F \to \mathbb{Z}G)$
$f \longmapsto 1$ (induced by $F \to G$).

$\boxed{\text{Prop 2.15}}$ $\exists$ Exact sequence:

$$\overline{I}_R / \overline{I}_R^2 \xrightarrow{d_2} I_F / \overline{I}_R I_F \xrightarrow{d_1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

Where: d_1 induced by $\mathbb{Z}F \to \mathbb{Z}G$

& d_2 induced by inclusion $\overline{I}_R \hookrightarrow I_F$.

Furthermore: $I_F / \overline{I}_R I_F$ Free $\mathbb{Z}G$-module.
& $\overline{I}_R / \overline{I}_R^2$ free $\mathbb{Z}G$-module.
] left modules.

and: $\mathrm{Im}(d_2) \subseteq I_F / \overline{I}_R I_F$ is: $\overline{I}_R / \overline{I}_R I_F$

and this is $\cong R_{ab}$ as $(\mathbb{Z}G)$-modules.

$\boxed{\text{Remark}}$ 1) Recall R_{ab} is $(\mathbb{Z}G)$-module, via action induced by Conjugation of R by F.

2) From Geometric Group Theory: Subgroups of Free groups are Free. So, deduce: R free group & R_{ab} free Abelian.

3) The partial resolution can be used to give full resolution (Gruenberg Resolution).

4) In practice: when wanting to deduce info about 2nd (co)homology, it is enough to know Image(d_2).

Lemma 2.16) G group, M left $\mathbb{Z}G$-module.

a) $I_G \equiv \mathrm{Ker}(\mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z})$ is free Abelian group, under $+$, on basis $\{g-1 : g \in G-1\}$.

b) $I_G / I_G^2 \cong G_{ab}$.
(under $+$) (under $\times$)

c) $\mathrm{Der}(G, M) \cong \mathrm{Hom}_G(I_G, M)$.

Proof) a) $\mathbb{Z}G$ free Abelian on basis $\{g : g \in G\}$.

$\Rightarrow \mathrm{Ker}(\varepsilon) = \{\sum n_g \cdot g : \sum n_g = 0\}$.

& Notice $\sum n_g \cdot g = \sum n_g \cdot (g-1)$, and anything of this form is in $\mathrm{Ker}(\varepsilon)$.

& Clearly, $\{g-1 : g \in G-1\}$ basis, since if $\sum n_g (g-1) = 0$ then $\sum n_g \cdot g = 0 \Rightarrow n_g = 0 \ \forall g$ ✓ (Freeness of $\mathbb{Z}G$)

b) Define $\theta : I_G \longrightarrow G_{ab}$
$g-1 \longmapsto gG'$.

Then: $(g_1 - 1)(g_2 - 1) = g_1 g_2 - g_1 - g_2 + 1 = (g_1 g_2 - 1) - (g_1 - 1) - (g_2 - 1)$

$\Rightarrow I_G^2 \subseteq \mathrm{Ker}(\theta)$. So, find map $I_G / I_G^2 \longrightarrow G_{ab}$.

Conversely: say $\phi : G \longrightarrow I_G / I_G^2$. Is group hom. onto
$g \longmapsto [g-1]$. additive group.

$\Rightarrow$ Factors to $\bar{\phi} : G_{ab} \longrightarrow I_G / I_G^2$.

Check: Inverses ✓ So $(G_{ab}, \times) \cong (I_G / I_G^2, +)$.

3) Need: $\mathrm{Der}(G, M) \cong \mathrm{Hom}_G(I_G, M)$.

Send: $\phi \longmapsto \theta$ $\quad (\theta = g \longmapsto \phi(g))$.

Check: θ is $\mathbb{Z}G$-map (since ϕ derivation)

Conversely, given $\theta \in \mathrm{Hom}_G(I_G, M)$: define ϕ by $\phi(g) = \theta(g-1)$.

Then check these are inverses. ✓

Lemma 2.17 a) Let: F free group on X. Then:
I_F is free $\mathbb{Z}G$-module, on $\tilde{X} \equiv \{x-1 : x \in X\}$.

b) Let $R \trianglelefteq F$. Then: R free on set (Y)
and: $\overline{I}_R$ is a free (left) $\mathbb{Z}F$-module on $\hat{Y} = \{y-1 : y \in Y\}$.

Proof a) Let $\alpha : \tilde{X} \longrightarrow M$ map, for $\mathbb{Z}F$-module M.

By def: need to show α extends to $\mathbb{Z}F$-map $I_F \longrightarrow M$.

First, let: $\alpha' : F \longrightarrow M \rtimes F$ semidirect prod (M is F-mod)

$$x \longmapsto (\alpha(x-1), x).$$

Thus: for each $f \in F$, $f \longmapsto (a, f)$, some $a \in M$.

(using: freeness of F)

$\Rightarrow$ ~~For each~~ There is a function $\overline{\alpha} : \underset{f \longmapsto a}{F \longrightarrow M}$, with:

$\alpha'(f) = (\overline{\alpha}(f), f)$.

Note: $\alpha(f_1 f_2) = \alpha(f_1)\alpha(f_2) = (\overline{\alpha}(f_1), f_1)(\overline{\alpha}(f_2), f_2)$

$$= (\overline{\alpha}(f_1) + f_1 \overline{\alpha}(f_2), f_1 f_2).$$

$\Rightarrow \underline{\overline{\alpha}(f_1 f_2) = \overline{\alpha}(f_1) + f_1 \overline{\alpha}(f_2)}$, hence, $\overline{\alpha}$ derivation,

$F \to M$. Then, the corresponding $\mathbb{Z}F$-map $I_F \longrightarrow M$
using 2.16 c).

b) Suppose: $\sum r_y \cdot (y-1) = 0$ for $r_y \in F$. Then:

Choose traversal T of cosets of $R \subseteq F$.

$\Rightarrow$ Write: $r_y = \sum_{t \in T} t s_{t,y}$, $s_{t,y} \in \mathbb{Z}$.

$\Rightarrow \sum_{\substack{t \in T \\ y \in Y}} t \cdot s_{t,y} (y-1) = 0$. But: since $\mathbb{Z}F$ Free Abelian, get: $t \sum s_{t,y} (y-1) = 0$.

$\Rightarrow \forall t, \sum s_{t,y} (y-1) = 0$.

But: since I_R free $\mathbb{Z}R$-module on $\tilde{Y}$, get $s_{t,y} = 0 \ \forall y, t$ ✓

Proof of Proposition: I_F is Free Left $\mathbb{Z}F$-module on $\tilde{X}$ (2.17)(a)

$\Rightarrow I_F / \overline{I_R} I_F$ free Left $(\mathbb{Z}G)$-module on $\tilde{X}$

Also: $\overline{I_R}$ free (left) $\mathbb{Z}G$-module by (2.17)(b), on $\tilde{Y}$

$\Rightarrow \overline{I_R} / \overline{I_R}^2$ free (left) $\mathbb{Z}G$-module on $\tilde{Y}$.

The image of d_2 is: $\overline{I_R} / \overline{I_R} I_F$ (d_2 = inclusion map)

So, consider: $\overline{I_R}$ as Right-module.

By Right-version of (2.17)(b): is free Right $\mathbb{Z}F$-module (on $\tilde{Y}$)

$\Rightarrow \overline{I_R} / \overline{I_R} I_F$ Free Abelian group on $\tilde{Y}$ (kill off F-action)

$\Rightarrow \cong R_{ab}$ as Abelian groups.

For G-action: notice that: $g(y-1) \equiv (gyg^{-1} - 1) g \mod \overline{I_R} I_F$

$\equiv (gyg^{-1} - 1) \mod$

(since we killed: Right action)

$\Rightarrow$ Corresponds to: Action of G on R_{ab} obtained by conj ✓

Exactness: $im(d_1) = I_G = \ker(\varepsilon) \Rightarrow \ker(d_1) = \overline{I_R} / \overline{I_R} I_F$ ✓ $= im\, d_2$

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Lemma 2.8 Given Projective resol: $\cdots \to P_1 \xrightarrow{d_1} P_0 \to \mathbb{Z}$.

Denote: $J_n \equiv \mathrm{im}(d_n) \subseteq P_n$ & induced $\psi : P_n \to J_n$.

a) For (left) $\mathbb{Z}G$-module M, $\exists$ exact sequence:

$$\mathrm{Hom}_G(P_{n-1}, M) \xrightarrow{\text{res}} \mathrm{Hom}_G(J_n, M) \to H^n(G,M) \to 0.$$

b) $\exists$ Exact seq: $0 \to H_n(G,\mathbb{Z}) \to \mathbb{Z} \otimes_{\mathbb{Z}G} J_n \to \mathbb{Z} \otimes_{\mathbb{Z}G} P_{n-1}$.

Proof a)

$$\begin{tikzcd} P_{n+1} \arrow[r, "d_{n+1}"] & P_n \arrow[r, "\psi"] \arrow[dr] & J_n \arrow[r] & 0 \\ & & P_{n-1} \arrow[u] & \end{tikzcd}$$

exact.

$$\Rightarrow \begin{tikzcd} \mathrm{Hom}_G(P_{n+1}, M) \arrow[r, "d^{n+1}"] & \mathrm{Hom}_G(P_n, M) \arrow[r, "\psi^*"] \arrow[dr, "d^n"] & \mathrm{Hom}_G(J_n, M) \arrow[r] & 0 \\ & & \mathrm{Hom}_G(P_{n-1}, M) \arrow[u, "i^*"] & \end{tikzcd}$$

$i^* =$ restriction.

By row exactness: $\mathrm{im}(\psi^*) = \ker(d^{n+1})$ & ψ^* injective.

$\Rightarrow \mathrm{im}(\psi^*) \cong \mathrm{Hom}_G(J_n, M)$.

& $\mathrm{im}(d^n) \cong \mathrm{im}(\psi^* \circ i^*) \cong \mathrm{im}(i^*)$ Since ψ^* injective.

$$\Rightarrow H^n(G,M) = \frac{\ker(d^{n+1})}{\mathrm{Im}(d^n)} \cong \frac{\mathrm{im}(\psi^*)}{\mathrm{im}(i^*)} = \frac{\mathrm{Hom}_G(J_n, M)}{\mathrm{im}(i^*)} \checkmark$$

b): Follows similarly.

Proof of Theorem 2.9 (Maclane's Theorem).

Recall: $1 \to R \to F \to G \to 1$. Presentation of G.

& M some $\mathbb{Z}G$-module.

Want: Exact seq $H^1(F,M) \longrightarrow \mathrm{Hom}_G(R_{ab}, M) \longrightarrow H^2(G,M) \longrightarrow 0$

Apply (2.18) a) to Partial Resol (2.15):

$$\mathrm{Hom}_G(I_F/\overline{I_R}\,\overline{I_F}) \xrightarrow{res} \mathrm{Hom}_G(R_{ab}, M) \longrightarrow H^2(G,M) \longrightarrow 0$$

But: $\mathrm{Hom}_G(I_F/\overline{I_R}\,\overline{I_F}) = \mathrm{Hom}_F(I_F/\overline{I_R}\,I_F, M)$

$= \mathrm{Hom}_F(I_F, M)$.

$= \mathrm{Der}(F, M)$ [2.16 c)]

~~But Any~~ Since: M trivial, as any $\mathbb{Z}F$-map $I_F \to M$ factors through $I_F/\overline{I_R}\,I_F$.

& Any inner derivation $F \to M$ (of form $f \mapsto (f-1)m$, some $m \in M$ fixed) has effect $r \mapsto 0$ $\forall r \in R$.

$\Rightarrow$ The map $\mathrm{Hom}_G(I_F/\overline{I_R}\,I_F, M) \xrightarrow{res} \mathrm{Hom}_G(R_{ab}, M)$ maps inner derivations to 0.

$\Rightarrow H^1(F,M) \longrightarrow \mathrm{Hom}_G(R_{ab}, M)$ induced map ✓

Proof of Hopf's Formula (2.14)

Apply: (2.18)(b) to partial resolution (2.15):

$$0 \longrightarrow H_2(G,\mathbb{Z}) \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}G} R_{ab} \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}G} I_F/\overline{I_R}I_F.$$

& Tensor, with $(\mathbb{Z} \otimes_{\mathbb{Z}G} -)$ is same as taking the co-invariants $\mathbb{Z} \otimes_{\mathbb{Z}G} R_{ab} = R/[R,F]$

$\boxed{R}$

$\cong \mathbb{Z} \otimes_{\mathbb{Z}G} I_F / I_R I_F = I_F / I_F^2 = F/[F,F]$.

$\Rightarrow$ Kernel of $R/[R,F] \longrightarrow F/[F,F]$ is: $F' \cap R / [R,F]$.

$\Rightarrow H_2(G,\mathbb{Z}) \cong F' \cap R / [R,F]$.

<u>Example</u> $G =$ Klein 4-group V $(\cong (\mathbb{Z}/2\mathbb{Z})^2)$

$\cong$ Generators x, y. $F =$ Free group on x, y

R gen. ~~by~~ as normal subgroup of F.

(e.g. $x^2, y^2, [x,y]$ or $x^2, y^2, (xy)^2$)