Problem Set 10

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Instructions

- This problem set is based off the notes "*Polynomials*".
- They are in roughly difficulty order and get quite difficult, so you **not** expected to be able to solve every problem.
- However, please attempt as many questions as you can and submit your solutions to your mentor for marking and feedback.
- You may type your solutions or take a **clear** scan/photo of **legible** written solutions.
- Feel free to discuss these problems with your peers and on the forum but the solutions you submit must be written by yourself.

Problems

1. Let a, b, c be the roots of the polynomial $P(x) = 2x^3 - 3x^2 + 4x - 5$. Evaluate the following numbers in closed form (i.e. express them as a single rational number).
   (a) $a + b + c$;
   (b) $a^2 + b^2 + c^2$;
   (c) $(a + b)(b + c)(c + a)$.
2. Let $x \neq y$ be integers. Show, by factorisation or otherwise, that $x - y \mid x^n - y^n$. Hence show that $x - y \mid P(x) - P(y)$ whenever $P(x)$ is an integer polynomial.
3. Let $P(x)$ be a **cubic** polynomials with roots r_1, r_2, r_3. It is given that
   $$\frac{P(\frac{1}{2}) + P(-\frac{1}{2})}{P(0)} = 1000.$$
   Find the value of $\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1}$.
4. A nonzero real number a is given. Find all polynomials $P(x)$ with real coefficients for which $P(x+a) = P(x)+a$ for every real number x.

   Hint. What can you say about the polynomial $Q(x) = P(x) - x$?
5. Find all polynomials $P(x)$ with real coefficients such that
   $$(x - 27)P(3x) = 27(x - 1)P(x).$$