S1Q6 – December 2020

James Bang

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Abstract

This is the marker's document in the marking of S1Q6, December 2020.

Let $n \geq 2$ be a given integer. An integer x is called **funny** if the remainder after diving x by n is odd.

Prove that it is impossible to have more than $1 + \lfloor \sqrt{3n} \rfloor$ consecutive positive integers, all of which are funny.

1 Solution (James Bang)

The problem is obvious for n even, as any even number, when squared, will have an even remainder when divided by n and thus there are at most $1 < 1 + \lfloor \sqrt{3n} \rfloor$ consecutive funny numbers. Hence, we focus with n odd.

Suppose we have an interval $[x, x+k]$ which forms a funny consecutive sequence. Then by shifting the interval a distance a multiple of n, we can assume $0 \leq x < n$. Since obviously no integer in the interval can be a multiple of n (since $n^2 \equiv 0 \pmod{n}$), we can assume $x > 0$ and $x + k \leq n - 1$.

Notice that any pair of consecutive squares $(x+a)^2$ and $(x+a+1)^2$ differ by exactly $2(x+a)+1$, which is odd. However, by the question statement they must also have the same parity – this can only happen when there are an **odd** number of multiples of n between $(x+a)^2$ and $(x+a+1)^2$. Since they also differ by $2(x+a)+1 < 2n-1$, there can only be $0, 1$ or 2 multiples of n between the two. Hence, there must be exactly one multiple of n between any two consecutive squares from the interval. In particular, there exists $a \in \mathbb{N}$, for which

$$na < x^2 < n(a+1) < (x+1)^2 < \cdots < (x+k)^2 < n(a+k+1).$$

This is equivalent to saying, for each $0 \leq j \leq k$, we have $n(a+j) < (x+j)^2 < n(a+j+1)$, or

$$0 < (x+j)^2 - n(a+j) < n \iff 0 < \left(j + x - \frac{n}{2}\right)^2 + n\left(x - a - \frac{n}{4}\right) < n.$$

There are now two cases to consider:

- The function $f(j) \stackrel{\text{def}}{=} (x+j)^2 - n(a+j)$ is monotone on the interval $0 \leq j \leq k$. Since $f(j)$ is a monic quadratic equation, its range on the interval must have width at least k^2. Therefore $k^2 < n$ or $k < \sqrt{n}$, and so the funny sequence has at most $1 + \lfloor \sqrt{n} \rfloor$ numbers.
- The function $f(j)$ is decreasing, then increasing, on the interval $0 \leq j \leq k$. Then, the function $f(j)$ achieves minimum on some $j = j_0 = \frac{n+1}{2} - x$. However then $(j_0 + x - \frac{n}{2})^2 = \frac{1}{4} \notin \mathbb{Z}$ and $f(X) \in \mathbb{Z}[X]$,

and so $n(x - a - \frac{n}{4}) \neq 0$. It is easy to see that for $f(j_0)$ to be positive, we require $n(x - a - \frac{n}{4}) > 0$ and therefore $\geq \frac{n}{4}$. Hence we have

$$0 < \left(j + x - \frac{n}{2}\right)^2 < \frac{3n}{4}$$

and thus there are at most $\frac{1}{2}(\lfloor\sqrt{3n}\rfloor + 1)$ possible j's on either side of j_0. Hence we obtain the bound $k \leq 2 \times \frac{1}{2}(\lfloor\sqrt{3n}\rfloor + 1) = 1 + \lfloor\sqrt{3n}\rfloor$.

In either case, we obtain the desired bound. $\square$

2 Remarks

Overall, this question was poorly done amongst the students, with an average mark of 0.89/7. Many students did not submit any work for it at all, while the majority of those who did attempt the question got the trivial even case and proceeded to just do a lot of cases. I don't know whether this is because the students didn't have enough time to attempt them, or they simply had no idea.

However, at the heart of the question is the simple observation that, given appropriate shifting of the value x (the first funny number in the sequence), there must be exactly one multiple of n between any two consecutive squares. From there, the question isn't very hard – there are a few ways to finish from there, and all students who got the key observation managed to get very high marks on this question.

3 Marking Criteria

7− Criteria

- Any full solution will receive 7 marks.
- Minor errors, such as forgetting to check n eve,n will deduct 1 mark (down to a minimum of 5 marks).

0+ Criteria

- Solving the n even case: **0 Marks**
- Observing that if $x^2 \pmod n$ and $(x+1)^2 \pmod n$ have the same parity, then there must be some multiples of n in between: **0 Marks**
- Shows with proof that there must be **exactly** one multiple of n between any pair of squares: **1 Mark**
- Solves the case where the funny numbers are less/greater than $n/2$, or when the function $f(j)$ defined earlier is monotone: **3 Marks**
- Solves the other case where the funny numbers cross the $n/2$ mark: **3 Marks**