geometry: angles

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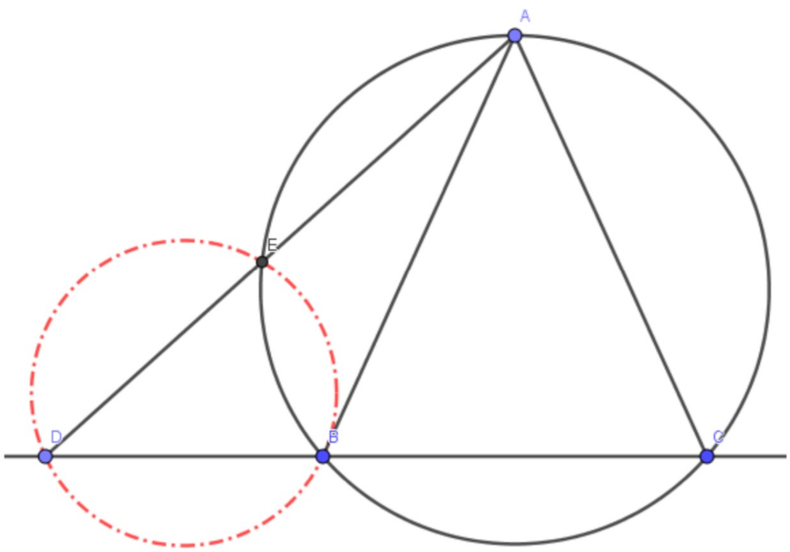
In Olympiad mathematics, geometry is one of the 4 core subjects in which the problems come out. For many, it is also one of the hardest topics since it requires one to recognise angles/similar triangles/lengths in an unfamiliar, convoluted diagram. However, most of the concepts that are required to solve even the hardest geometry problems are elementary, and we shall discuss these in the lessons.

Also, if you haven't been to Andy's maths club, then it might be worthy looking into the "/Triangle Centers" PDF which is available on Google Classroom with code **frboex** (?). This knowledge may come handy in these classes and possibly contests such as the AIMO.

1 Angles

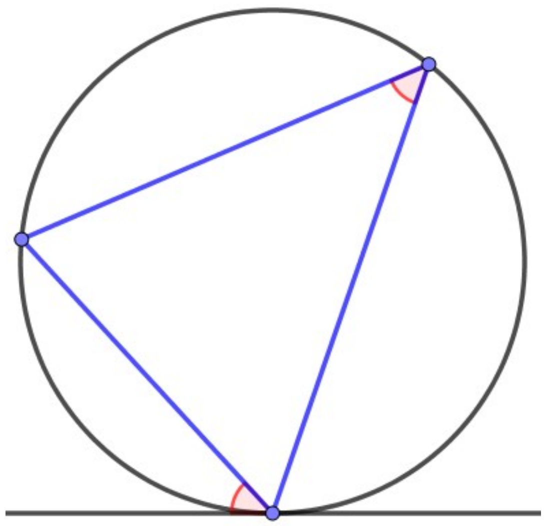
Problem 1.1. Given an isosceles triangle ABC with $AB = AC$, pick a point D on the line BC that does not lie on segment BC. Suppose $\overline{AD}$ intersects the circumcircle of ABC again at E. Show that the circle BDE is tangent to $\overline{AB}$.

This is probably a slightly hard question to start with but whatever. First step is **always to draw a large, clear diagram!**



With a diagram drawn, how could we possibly solve this problem? There's a tangent circle thingo which is bothering us a bit, so let's think about how this tangent circle could come into play.

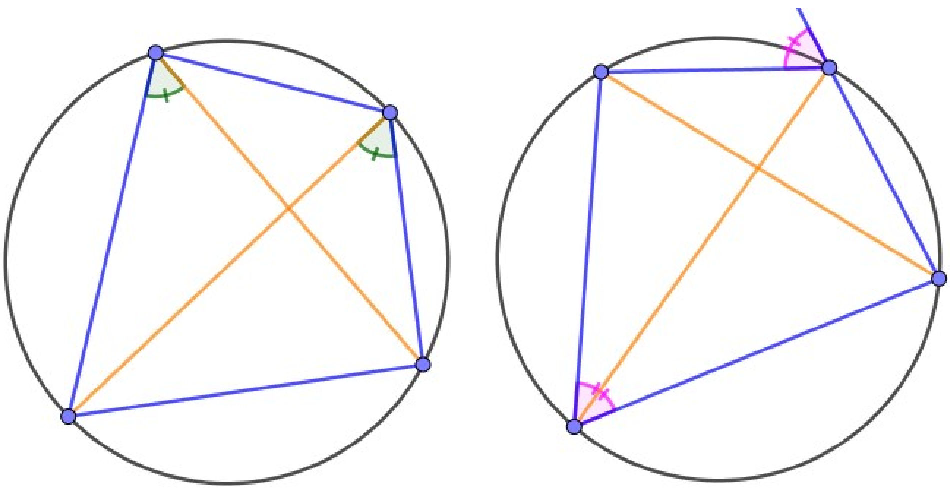
Theorem 1.1 (AST). In the figure below, if the circle and line are tangent then the two marked angles are equal.



In fact, we are more likely interested in the converse of this statement, which states that if the two angles as in the above figure are equal, then the circle is tangent to the line. Looking familiar, anyone?

Well, we still need to deal with the main circle in the question (the one around the isosceles triangle). Since the quadrilateral $AEBC$ is cyclic, we do need to recall our basic cyclic quad theorems:

Theorem 1.2 (Basic cyclic quad theorems). In the figure below, the marked angles are equal (they are two separate diagrams!).

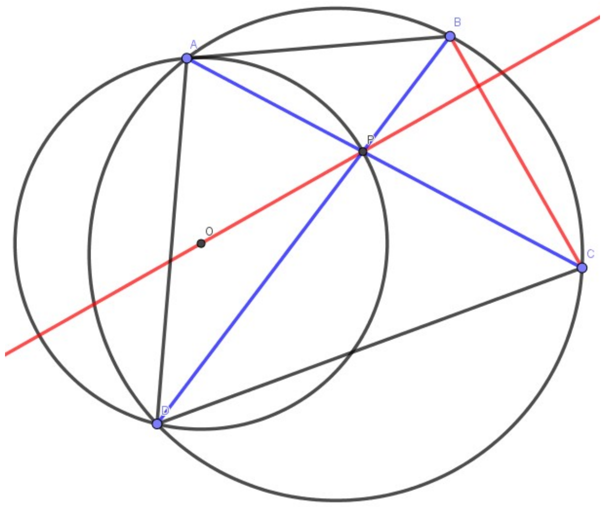


With those two theorems out of the way, it is now left to find the necessary angles and apply the theorems to solve the problem.

Exercise 1.1. Finish off **Problem 1.1** using the hints supplied above (**this is actually the hard part of the question!**).

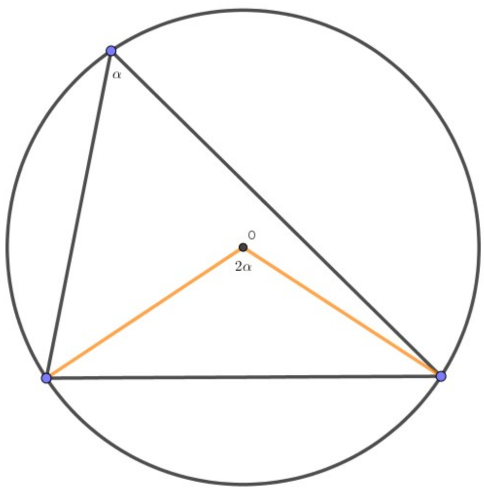
Well yay we managed to get this problem out. Let's try another one!

Problem 1.2. A cyclic quad $ABCD$ is given, where the intersection of its diagonals is P. Suppose O is the center of the circle (APD). Show that $\overline{OP}$ is perpendicular to $\overline{BC}$.



Well, this one looks even harder than the previous one! Fortunately, the problem can still be solved with just the ideas we have discussed so far. We want to show something is a right angle, so maybe try showing two of the angles add up to 90 degrees? We need to know one more theorem first:

Theorem 1.3. If ABC is a triangle with circumcenter O, then $\angle BOC = 2\angle BAC$.



It is then up to you to show that $\angle OPD + \angle PBC = 90$, which will finish the problem.

Exercise 1.2. Finish off **Problem 1.2** using the hints, and explain why showing that the two angles add to 90 degrees finishes the problem.

From this point are problems which you will have to draw your own diagram. Beware, some of these problems get rather difficult! (The end few problems are around the difficulty of Q1,2 of the Australian Mathematical Olympiad.)

Problem 1.3. In a triangle ΔABC, points X and Y lie on sides $\overline{AB}$ and $\overline{AC}$ such that the circle (AXY) is tangent to the circle (ABC) (at A, duh) and the line $\overline{BC}$ at a point Z. Show that ΔXYZ is isosceles.

Problem 1.4. Two circles are **internally**[1] tangent at a point T. Let $\overline{AB}$ be any chord of the larger circle which is tangent to the smaller circle at point P. Show that $\angle ATP = \angle BTP$.

Problem 1.5. Triangle ΔABC has circumcenter O. The lines $\overline{AC}$ and $\overline{BC}$ intersect the circle (AOB) again at points P and Q respectively. Show that $\overline{PQ} \perp \overline{CO}$.

Problem 1.6. In a cyclic quadrilateral $ABCD$, the lines $\overline{AB}$ and $\overline{CD}$ intersect at X, while lines $\overline{BC}$ and $\overline{AD}$ intersect at Y. Show that the angle bisectors from the angles X and Y are perpendicular.

Problem 1.7. Two circles Γ_1 and Γ_2 intersect at points P and Q. A line through P intersects Γ_1, Γ_2 again at points A, B respectively. The line through Q and the midpoint of AB intersects Γ_1, Γ_2 again at points X, Y respectively. Show that $AXBY$ is a parallelogram.

Problem 1.8. In an acute triangle ABC, let P, Q be points on sides $\overline{AC}$ and $\overline{BC}$ such that $APBQ$ is cyclic. Suppose there exists a point R such that $\overline{PR} \perp \overline{AC}$ and $\overline{QR} \perp \overline{BC}$. Show that $\overline{CR} \perp \overline{AB}$.

Problem 1.9. In triangle ΔABC, let AB be the shortest side. X, Y are the midpoints of $\overline{BC}, \overline{AC}$ respectively. Suppose there is a point P on $\overline{AC}$ such that $\overline{PX} \perp \overline{BC}$. The circle (ABP) intersects $\overline{BC}$ again at Q. Show that $\overline{QY} \perp \overline{AC}$.

[1] This means that one circle is inside another and tangent from the inside.