Problem Set 7 Solutions

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1 Problem Set 7

1.1 Question 1

Problem 1.1

For all real x, show that $\dfrac{x^2+2}{\sqrt{x^2+1}} \geqslant 2$.

Solution (1). Squaring both sides and multiplying by x^2+1, this inequality is equivalent to

$$(x^2+2)^2 \geq 4(x^2+1).$$

However, after cancellation this is $x^4 \geq 0$, which is trivially true.

Solution (2). Notice by the AM-GM inequality that $(x^2+1)+1 \geq 2\sqrt{x^2+1}$. Then divide both sides by $\sqrt{x^2+1}$ to get the desired inequality.

1.2 Question 2

Problem 1.2

Prove that for all real numbers x, we have $x^4+6x^2+1 \geqslant 4x(x^2+1)$. When does inequality hold?

Solution. Notice that $LHS - RHS = x^4 - 4x^3 + 6x^2 - 4x + 1 = (x-1)^4 \geq 0$.

1.3 Question 3

Problem 1.3

Prove that for all positive reals a, b, c, we have $\dfrac{a^2}{b} + \dfrac{b^2}{c} + \dfrac{c^2}{a} \geqslant a+b+c$.

Solution (1). By the Cauchy-Schwarz Inequality, we have

$$(b+c+a)\left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}\right) \geq (a+b+c)^2.$$

Then, divide both sides by $a+b+c$ to get the desired inequality.

Solution (2). By the AM-GM Inequality, we have

$$\frac{a^2}{b}+a+b \geq 3a.$$

Then add the similar inequalities for $\frac{b^2}{c}$ and $\frac{c^2}{a}$ to get the desired inequality.

1.4 Question 4

Problem 1.4

What is the maximum value of $a^5(1-a)$ for $0<a<1$?

Solution. We have by the AM-GM Inequality

$$a^5(1-a)=5^5 \times \frac{a}{5}\cdot\frac{a}{5}\cdot\frac{a}{5}\cdot\frac{a}{5}\cdot\frac{a}{5}\cdot(1-a) \leq 5^5\cdot\left(\frac{\frac{a}{5}+\frac{a}{5}+\frac{a}{5}+\frac{a}{5}+\frac{a}{5}+(1-a)}{6}\right)^6 = 5^5\cdot\frac{1}{6^6}=\frac{5^5}{6^6}.$$

Equality occurs when $\frac{a}{5}=1-a$, or $a=5/6$.

1.5 Question 5

Problem 1.5

Let x, y be nonnegative reals with sum 2. Prove that $x^2y^2(x^2+y^2)\leqslant 2$.

Solution (1). Since we have $x+y=2$, the inequality is equivalent to $32x^2y^2(x^2+y^2)\leq(x+y)^6$, or after expansion

$$x^6+6x^5y-17x^4y^2+20x^3y^3-17x^2y^4+6xy^5+y^6\geq 0.$$

However, this is also equivalent to

$$(x-y)^2(x^4+8x^3y-2x^2y^2+8xy^3+y^4)\geq 0,$$

which is trivially true since $x^4+8x^3y-2x^2y^2+8xy^3+y^4=8xy(x^2+y^2)+(x^2-y^2)^2>0$.

Solution (2). Let $A=xy$. Then, since we have $4xy\leq(x+y)^2=4$, we have $A\leq 1$. We also have

$$x^2y^2(x^2+y^2)=(xy)^2((x+y)^2-2xy)=A^2(4-2A)\leq 2,$$

which is equivalent to

$$A^3-2A^2+1\geq 0 \iff (1-A)(1+A-A^2)\geq 0.$$

However, this is true since $1-A\geq 0$ (or $A\leq 1$) and thus $1\geq A^2$ meaning the second bracket is nonnegative as well.