Centre Number

Student Number

Mathematics Extension 2
TRIAL Examination

September 2019

General Instructions

- **Reading time** — **5 Minutes**
- **Working time** — **180 Minutes**
- Blue pen is preferred, but you may use any colour you want (who cares?)
- Computers, mobile phones, tablets, smart watches and other electronic devices that can transmit and receive data via the Internet may *not* be used during this examination
- Calculators not connected to the Internet may be used regardless of $\mathrm{N^ES_A}$ approval, but it probably won't help you much in this examination
- The Mathematics Extension 2 reference sheet is *not* provided for this paper; you may bring your own, but again it probably won't help you much in this examination
- For Questions 1–10, shade the correct bubble in the multiple choice section of the answer booklet (well duh?)
- For Questions 11–16, show all relevant mathematical reasoning and/or calculations; correct answers may receive little to no credit without an appropriate justification
- Failure to copy a geometry diagram onto your answer booklet when presenting a proof will result in the deduction of half a mark per such instance

There are 100 Marks in this Examination

- **Section 1** — **10 Marks**
  - Attempt Questions 1–10, as you will get 0 marks for questions not attempted
  - Allow about 18 minutes for this section
- **Section 2** — **90 Marks**
  - Attempt Questions 11–16, as you will get 0 marks for questions not attempted
  - Allow about 162 minutes for this section

Section 1 (10 Marks)

Attempt Questions 1–10
Allow about 18 minutes for this section

Use the multiple choice section of the answer booklet for Questions 1 – 10.

Question 1 Which of the following inequalities hold for all real numbers a, b, c?

(a) $a^3 + b^3 + c^3 \geqslant 3abc$

(b) $a^4 + b^4 + c^4 \geqslant abc(a + b + c)$

(c) $a^3b^3 + b^3c^3 + c^3a^3 \geqslant abc(a^3 + b^3 + c^3)$

(d) $a^2b^2 + b^2c^2 + c^2a^2 + \frac{1}{2} \geqslant ab + bc + ca$

Question 2 Let $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ be a primitive root of unity of $z^n - 1 = 0$, where $n \geqslant 1$ is odd. Ana computed the sum

$$A := \frac{1}{1-\omega} + \frac{1}{1-\omega^2} + \cdots + \frac{1}{1-\omega^{n-2}} + \frac{1}{1-\omega^{n-1}},$$

while Banana computed the similar sum

$$B := \frac{1}{1+1} + \frac{1}{1+\omega} + \frac{1}{1+\omega^2} + \cdots + \frac{1}{1+\omega^{n-1}}.$$

Which of the following is equal to $A - B$?

(a) 1 (b) 0 (c) -1 (d) n

Question 3 Find the value of $\displaystyle\int_0^{2\pi} \frac{x}{\phi - (\cos(x))^2} \cdot \mathrm{d}x$, where $\phi = \dfrac{1+\sqrt{5}}{2}$.

(a) π^2 (b) $2\pi^2$ (c) $3\pi^2$ (d) $4\pi^2$

Question 4 A circle on the Argand plane has equation $|z - \omega| = r$, where $\omega = a + bi$ is a complex number such that $a, b > r > 0$. Which of the following correctly represents the possible values of $\arg(z)$ and $|z|$ for each z on this circle?

(a) $\left|\arg(z) - \tan^{-1}\dfrac{b}{a}\right| \leqslant \sin^{-1}\dfrac{r}{\sqrt{a^2+b^2}}$, $\quad \left||z| - \sqrt{a^2+b^2}\right| \leqslant r$

(b) $\left|\arg(z) - \tan^{-1}\dfrac{b}{a}\right| \leqslant \sin^{-1}\dfrac{r}{\sqrt{a^2+b^2}}$, $\quad \left||z| - \sqrt{a^2+b^2}\right| \leqslant \sqrt{a^2+b^2-r^2}$

(c) $\left|\arg(z) - \tan^{-1}\dfrac{b}{a}\right| \leqslant \tan^{-1}\dfrac{r}{\sqrt{a^2+b^2}}$, $\quad \left||z| - \sqrt{a^2+b^2}\right| \leqslant r$

(d) $\left|\arg(z) - \tan^{-1}\dfrac{b}{a}\right| \leqslant \tan^{-1}\dfrac{r}{\sqrt{a^2+b^2}}$, $\quad \left||z| - \sqrt{a^2+b^2}\right| \leqslant \sqrt{a^2+b^2-r^2}$

Question 5 Consider the function $f(x) = \begin{cases} 1 - \frac{\sin x}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$ For a given real number α such that $f'(\alpha) \neq 0$, let $\beta = \alpha - \dfrac{f(\alpha)}{f'(\alpha)}$ be the number obtained by a single application of the Newton approximation formula. Which of the following is, to two decimal places, the set of all $\alpha \neq 0$ such that $|\alpha| < 2\pi$ and $|\beta| < |\alpha|$?

(a) $0 < |\alpha| < 3.14$

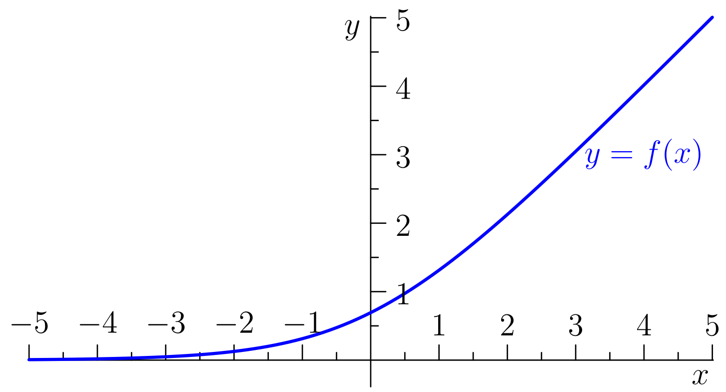
(b) $0 < |\alpha| < 3.61$

(c) $0 < |\alpha| < 3.86$

(d) $0 < |\alpha| < 4.49$

Question 6 Below shows the graph of the equation $y = f(x)$, where f is a function satisfying:

- $f(x) \to 0$ as $x \to -\infty$;
- $\frac{f(x)}{x} \to 1$ as $x \to \infty$.



Which of the following could be an expression for $f'(x)$?

(a) y (b) $x - y + \dfrac{y}{x}$ (c) e^{x-y} (d) e^{y-x}

Question 7 There are 10 people at a party. Unfortunately none of these people like each other, and so in a row of 2019 chairs every two people have at least two empty chairs in between. How many seating arrangements are possible?

(a) $10! \cdot \binom{1998}{10}$ (b) $10! \cdot \binom{1999}{10}$ (c) $10! \cdot \binom{2000}{10}$ (d) $10! \cdot \binom{2001}{10}$

Question 8 Let z_1, z_2, z_3 be three distinct complex numbers. It is given that the three points represented by the complex numbers lie on a line *not* passing through the origin. Which of the following is always true about the circle $\mathcal{C}$ passing through $1/z_1, 1/z_2, 1/z_3$?

(a) The origin O lies on $\mathcal{C}$

(b) The origin O lies strictly inside $\mathcal{C}$

(c) The origin O lies strictly outside $\mathcal{C}$

(d) A combination of (a), (b), (c) may occur depending on the location of z_1, z_2, z_3.

Question 9 How many integer solutions are there to the equation $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{2019}$?

(a) 1 (b) 4 (c) 8 (d) 17

Question 10 Let a, b, c be three distinct complex numbers on the unit circle, represented by points A, B, C. Suppose O is the origin, and point H is represented by the complex number $a+b+c$. Which of the following correctly represents the distance $|OH|^2$?

(a) $1-|AB|^2-|BC|^2-|CA|^2$

(b) $3-|AB|^2-|BC|^2-|CA|^2$

(c) $9-|AB|^2-|BC|^2-|CA|^2$

(d) $9+|AB|^2+|BC|^2+|CA|^2$

End of Section 1

Section 2 (90 Marks)

Attempt Questions 11–16
Allow about 162 minutes for this section

Answer each question in the appropriate writing booklet. In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (**15 Marks**) Use the Question 11 section of the writing booklet.

(a) [**2**] Show that $\displaystyle\int_0^{\pi/4} \frac{\ln(\cot\theta)}{(\cos\theta)^{2n}}\mathrm{d}\theta = \sum_{k=0}^{n-1} \frac{\binom{n-1}{k}}{(2k+1)^2}$ for all positive integers n.

(b) In this question, we show $\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(1+x^2)(\pi+\cos(x))} = \frac{\pi}{\sqrt{\pi^2-1}} \times \frac{\sqrt{\pi-1}+\sqrt{\pi+1}\tanh(\frac{1}{2})}{\sqrt{\pi+1}+\sqrt{\pi-1}\tanh(\frac{1}{2})}$. Denote this integral as I.

(i) [**1**] Let $r = \pi + \sqrt{\pi^2-1}$ be the larger root of the quadratic equation $z^2 - 2z\pi + 1 = 0$, where $z = \mathrm{cis}(x) = e^{ix}$. Show that $I = \displaystyle\frac{2}{r}\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(1+x^2)(1+\frac{z}{r})(1+\frac{1}{rz})}$.

(ii) [**1**] Use the Geometric Series formula ($1 + x + x^2 + \cdots = \frac{1}{1-x}$ for $|x| < 1$) to show that $I = \displaystyle\frac{2r}{r^2-1}\int_{-\infty}^{\infty} \frac{1}{x^2+1} \sum_{N=-\infty}^{\infty} \frac{(-z)^N}{r^{|N|}}$, where $|...|$ denotes the absolute value.

(iii) [**2**] Use the identity $\displaystyle\int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2}\,\mathrm{d}x = \pi \times e^{-|t|}$, which you may assume holds for all real arguments t, to prove the initial integral.

(c) Define the polynomial $P(x) = x^2 - x + \alpha$, where $0 < \alpha < 1$ is a real number.

(i) [**2**] Show that if $z = \cos\theta + i\sin\theta$ for real angle θ, then

$$|P(z)|^2 \geqslant \begin{cases} \dfrac{(\alpha-1)^2(4\alpha-1)}{4\alpha} & \text{if } \alpha \geqslant 1/3, \\ \alpha^2 & \text{if } \alpha \leqslant 1/3. \end{cases}$$

(ii) [**3**] Hence show that if ω is a complex number with $|\omega| \geqslant 1$, then there exists some complex z with $|z| = 1$ such that $|P(\omega)| \geqslant |P(z)|$.

(d) [**2**] Let x and y be two points on the unit circle such that $\frac{\pi}{3} \leqslant \arg(x) - \arg(y) \leqslant \frac{5\pi}{3}$. Show that any complex number z satisfies $|z| + |z-x| + |z-y| \geqslant \left|z - \dfrac{x}{y}\right|$.

(e) (i) [**1**] Show that $a^4 + b^4 + c^4 \geqslant a^3b + b^3c + c^3a$ for any positive real numbers a, b, c.

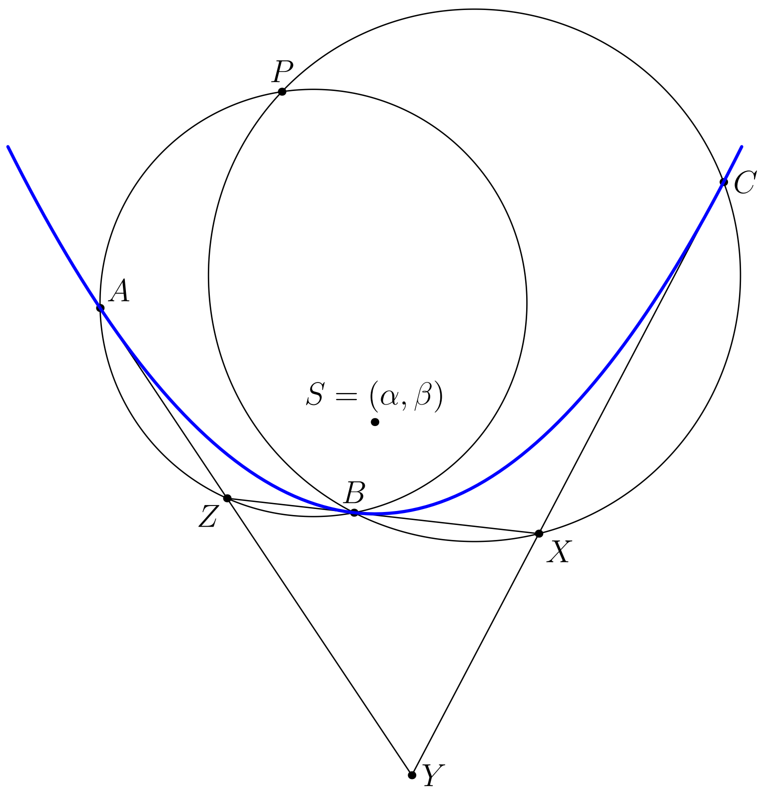
(ii) [**1**] By substituting $a = t^{x-1/4}$ for $t > 0$ and similar for b and c, and using an appropriate integration, show that any triple of positive real numbers x, y, z satisfies

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geqslant \frac{4}{3x+y} + \frac{4}{3y+z} + \frac{4}{3z+x}.$$

End of Question 11

Question 12 (**15 marks**) Use the Question 12 section of the writing booklet.

(a) The three points $A = (2ap, ap^2), B = (2aq, aq^2)$ and $C = (2ar, ar^2)$ lie on the parabola $x^2 = 4ay$ with focus S. The tangents at points A, B, C to the parabola intersect each other at distinct points X, Y, Z as shown below. The centre of the circle passing through points X, Y, Z has coordinates (α, β).



You may assume the formulas $X = (a(q + r), aqr)$ and similar for Y and Z.

(i) [**2**] By computing the equations of the perpendicular bisectors of XY and YZ, show that

$$(\alpha, \beta) = \left(\frac{a(p + q + r - pqr)}{2}, \frac{a(pq + qr + rp + 1)}{2}\right).$$

(ii) [**2**] Show that the radius R of the circle passing through points X, Y, Z satisfies

$$R = \frac{a}{2}\sqrt{(p^2 + 1)(q^2 + 1)(r^2 + 1)}.$$

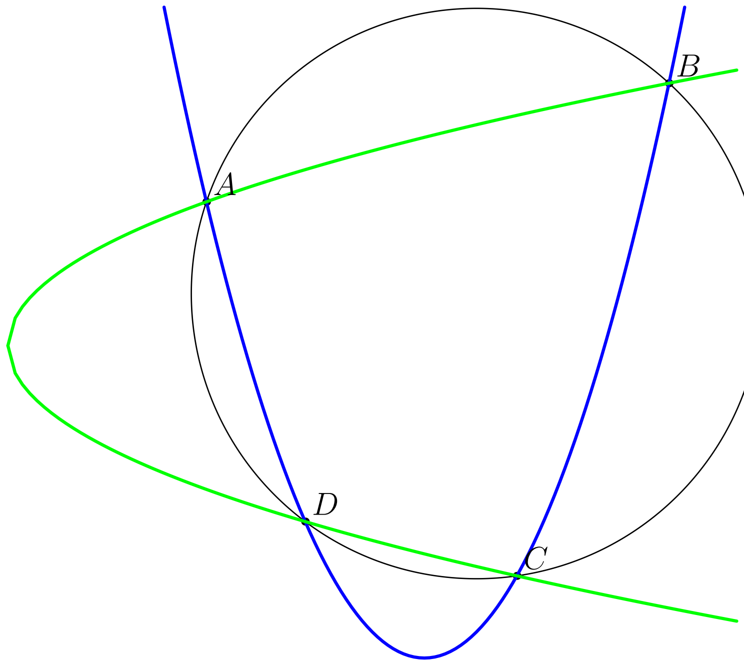
(iii) [**1**] Hence, show that X, Y, Z, S are concyclic.

(iv) [**1**] It is assumed that $p < q < r$ as in the above diagram. By using the distance formula, show that $|SX| = a\sqrt{(y^2 + 1)(z^2 + 1)}$ and $|YZ| = a(r - q)\sqrt{x^2 + 1}$, and hence prove that $|SX| \cdot |YZ| + |SZ| \cdot |XY| = |SY| \cdot |XZ|$.

(v) [**1**] Let P be the intersection of the circles through A, Z, B and B, X, C respectively, other than B. Show that $APCY$ is also cyclic.

Question 12 continues on the next page

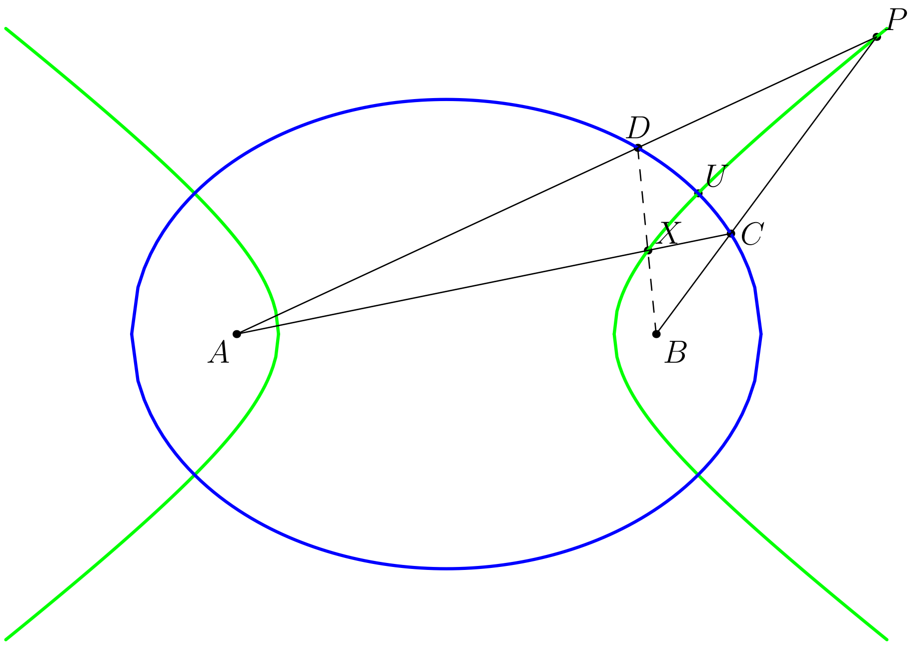
(b) The four points $A = (a, a^2), B = (b, b^2), C = (c, c^2)$ and $D = (d, d^2)$ lie on a parabola $\mathcal{P}_1$ with equation $y = x^2$. It is given that $ABCD$ is a cyclic quadrilateral.



(i) [**2**] By considering the angles $\angle ABC$ and $\angle CDA$, show that $a + b + c + d = 0$.

(ii) [**1**] Hence show that there exists another parabola $\mathcal{P}_2$ with its axis perpendicular to that of $\mathcal{P}_1$ passing through all four points A, B, C, D.

(c) An ellipse $\mathcal{E}$ and a hyperbola $\mathcal{H} : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ share the same foci $A \neq B$, and pass through a common point U as shown below. The point $P = (a \sec\theta, b \tan\theta)$ lies on $\mathcal{H}$, and the segments AP, BP intersect $\mathcal{E}$ again at points D, C respectively. The segment AC intersects $\mathcal{H}$ again at point X closer to C as shown.



(i) [**2**] Show that the three points B, X, D are collinear.

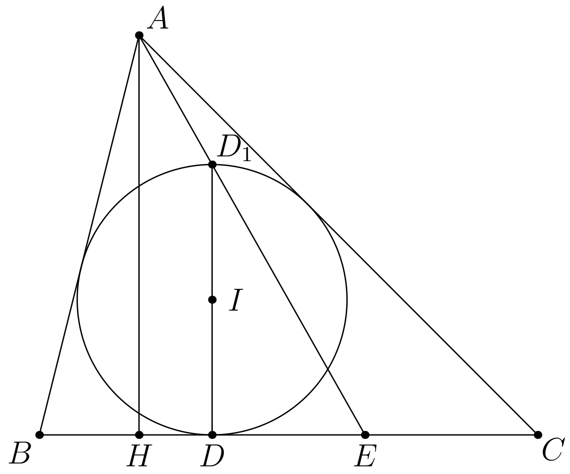
(ii) [**3**] It is further given that $\angle ABU = \frac{\pi}{2}$. Show that $\angle CBU = \angle UBD$.

End of Question 12

Question 13 (**15 Marks**) Use the Question 13 section of the writing booklet.

(a) Triangle ΔABC with $|AB| < |AC|$ has incentre I (intersection of internal angle bisectors). Let D be the incircle touchpoint with BC, and D_1 is the reflection of D across I. Also H is on BC such that $AH \perp BC$, and E is on BC with $|BD| = |CE|$. Let $|BC| = a$, $|CA| = b$ and $|AB| = c$.

Denote $|XY|$ as the distance between points X and Y for convenience.

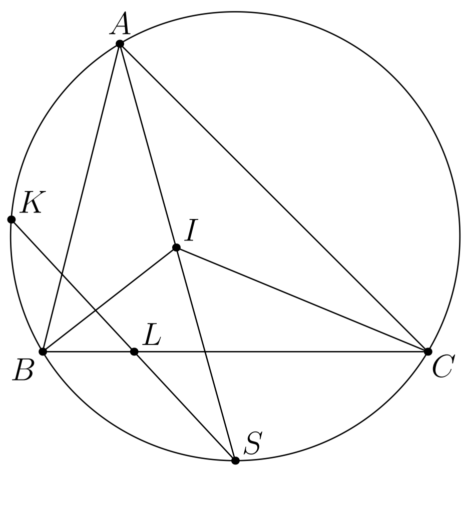


Copy the diagram onto your answer booklet.

(i) [**1**] By using the formula Area$(ABC) = r \cdot s$ where r is the inradius $|DI|$ and $s = (a+b+c)/2$ is the semi-perimeter, show that $\dfrac{|DD_1|}{|AH|} = \dfrac{2a}{a+b+c}$.

(ii) [**1**] By using $|BH| = c\cos B$, show that $|DE| = b - c$ and $|EH| = \dfrac{(b-c)(a+b+c)}{2a}$, and hence show that A, D_1, E are collinear.

(b) Let S be the point where the internal angle bisector from A intersects the circumcircle of ΔABC. Point K is any point on the major arc $\overbrace{BAC}$, and L is the intersection point of KS and BC.



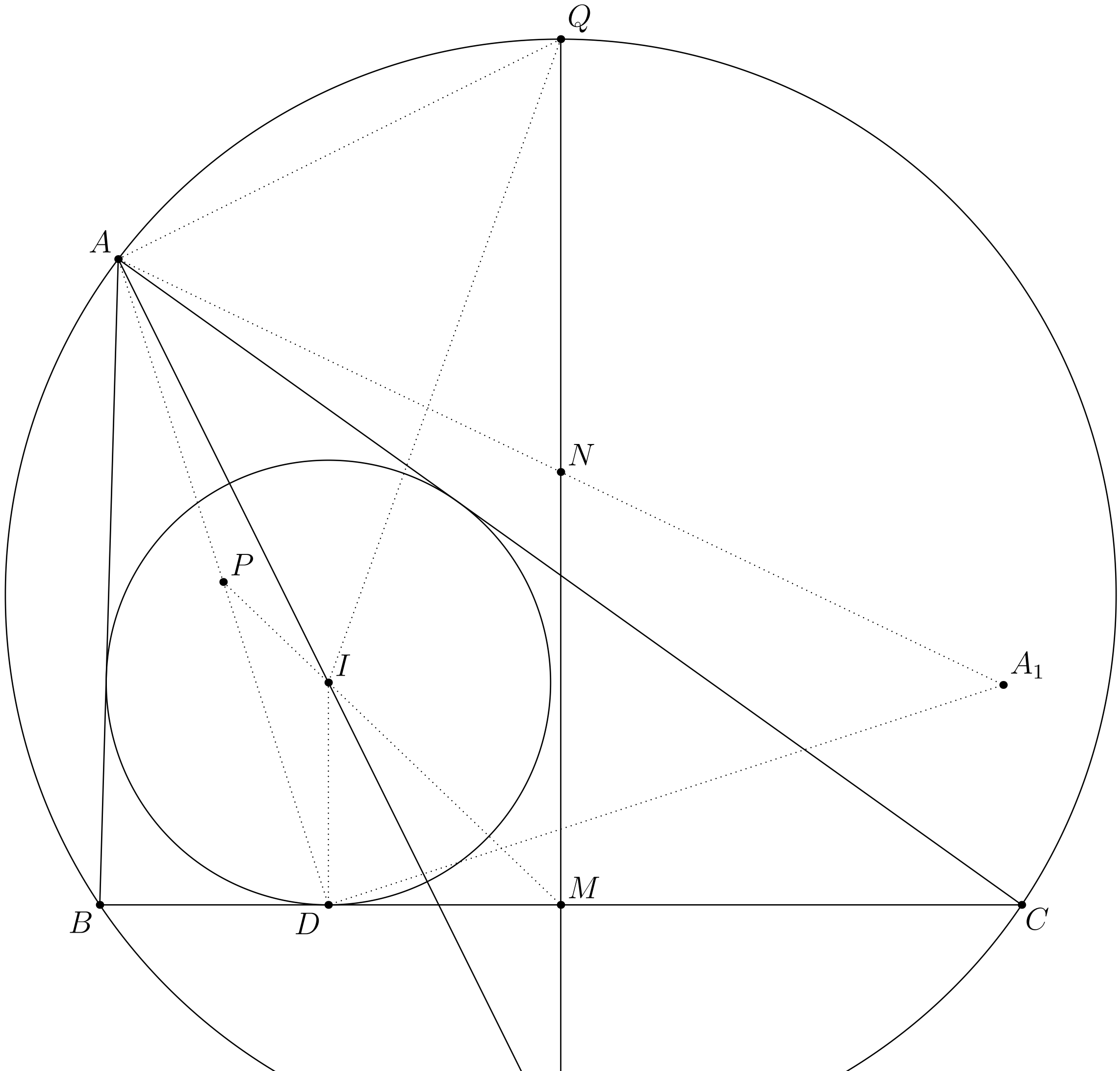
Copy the diagram onto your answer booklet.

Question 13 continues on the next page

(i) [**1**] Show that $|BS| = |IS| = |CS|$.

(ii) [**2**] Show that the circle (KIL) is tangent to line AS.

(c) In this figure below, I is the incentre, D is the incircle touchpoint, and S is the point where AI intersects the circumcircle of ΔABC again, as defined in (a) and (b). Furthermore, Q is the midpoint of arc $\overparen{BAC}$, and M, N, P are the midpoints of BC, MQ, AD respectively. Let A' be the reflection of A across N.



Copy the diagram onto your answer booklet.

(i) [**1**] Use the result from (a) to show that P, I, M are collinear.

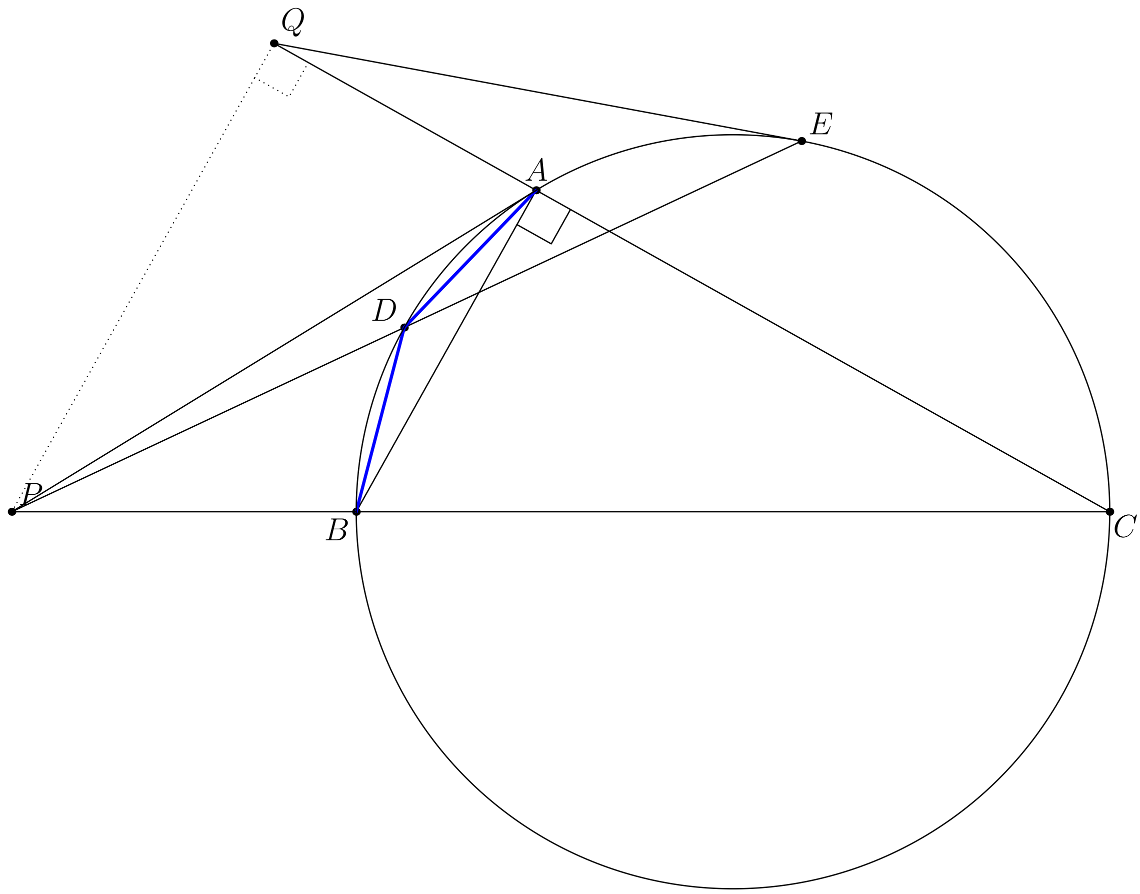
(ii) [**2**] By using the result from (b), show that $\Delta AQI \sim \Delta DMI$.

(iii) [**3**] By showing $\dfrac{|DM|}{|MA'|} = \dfrac{|DI|}{|IA|}$, show that $\Delta AA'D \sim \Delta IMD$.

(iv) [**1**] Hence, show that the perpendicular bisector of AD passes through N.

Question 13 continues on the next page

(d) [**3**] Below shows a diagram of a right-angled scalene triangle ΔABC, with $\angle A = 90°$. Let D be the midpoint of the smaller arc $\overgroup{AB}$, and P the point where the tangent at A intersects BC. Also PD intersects the circumcircle again at E, and the tangent at E to the circumcircle intersects AC at point Q.

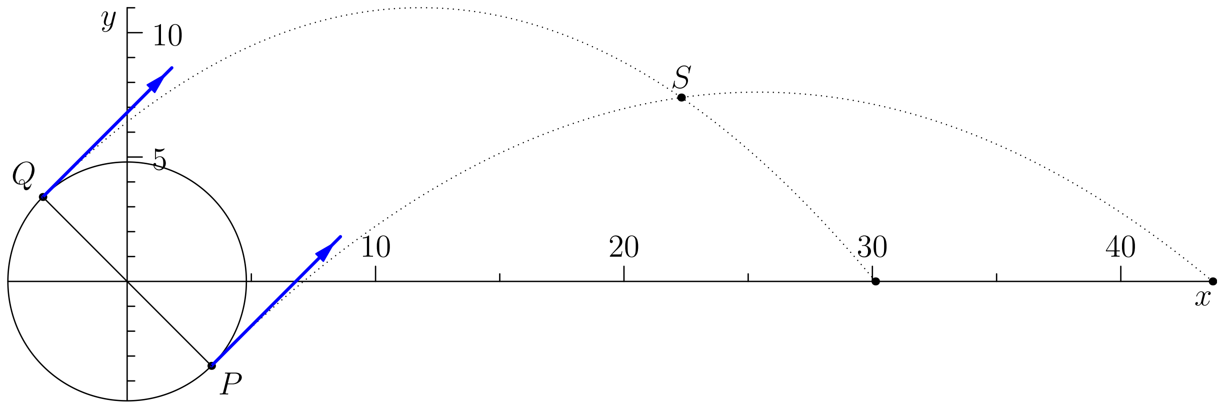


Copy the diagram onto your answer booklet and show that $\angle PQC = 90°$.

End of Question 13

Question 14 (**15 Marks**) Use the Question 14 section of the writing booklet.

(a) In the following figure, two points P and Q are given on a circle radius r centered at the origin, such that OP makes angle $-\pi/4$ with the positive x-axis and Q is diametrically opposite P. Two weightless strings, each of length r carrying a particle mass m, are swung in a vertical circle in opposite directions centered at O. At time $t = 0$, both strings are simultaneously cut in such a way that one projectile leaves from P and the other from Q, both with positive upwards velocity.



It is given that both strings are swung in such a way that the particles have velocity U at the points $(\pm r, 0)$, and $U^2 \gg 2gr$.

(i) [**1**] Show that the projectiles from P and Q satisfies the pair of equations

$$P : \begin{cases} x = \dfrac{V_P \cdot t + r}{\sqrt{2}} \\ y = \dfrac{V_P \cdot t - r}{\sqrt{2}} - \dfrac{1}{2}gt^2 \end{cases} \quad \text{and} \quad Q : \begin{cases} x = \dfrac{V_Q \cdot t - r}{\sqrt{2}} \\ y = \dfrac{V_Q \cdot t + r}{\sqrt{2}} - \dfrac{1}{2}gt^2 \end{cases}$$

where $V_P = \sqrt{U^2 + \sqrt{2}gr}$ and $V_Q = \sqrt{U^2 - \sqrt{2}gr}$.

(ii) [**1**] Prove that both trajectories share the same directrix, and find its equation.

(iii) [**2**] Show that the ranges R_P and R_Q of the projectiles from P and Q respectively are given by

$$R_P = \frac{U^2 + 2\sqrt{2}gr + \sqrt{U^4 - 2(gr)^2}}{2g}, \text{ and } R_Q = \frac{U^2 - 2\sqrt{2}gr + \sqrt{U^4 - 2(gr)^2}}{2g}.$$

(iv) [**1**] Explain why there exists a point S above the x-axis where the trajectories of the projectiles from P and Q intersect.

(v) [**2**] By using the equations from (i) or otherwise, show that

$$S = \left(\frac{-U^2 + \sqrt{5(U^4 - 2(gr)^2)}}{2g}, \ \frac{-2U^2 + \sqrt{5(U^4 - 2(gr)^2)}}{g} \right).$$

Question 14 continues on the next page

(vi) [**3**] Hence, or otherwise, show that the acute angle α formed between the two tangents at S satisfies $\tan\alpha = \left|\dfrac{r}{S_y \cos(\frac{\pi}{4})}\right|$, where $S = (S_x, S_y)$.

(vii) [**1**] Consider a third projectile, identical to the first two, being fired from O at the same angle $\pi/4$ and at speed U. Will this projectile pass through point S, and if not, does it pass above or below S? Justify your answer.

(b) Let $f(x) = 1 + x + x^2 + a_3x^3 + a_4x^4 + \cdots + a_{2020}x^{2020}$ be a polynomial with $a_{2020} \neq 0$. It is given that f has 2020 nonzero roots $r_1, r_2, \ldots, r_{2020}$.

(i) [**1**] Use the sums and products of polynomial roots to deduce that

$$\frac{1}{r_1} + \frac{1}{r_2} \cdots + \frac{1}{r_{2020}} = -1, \quad \text{and} \quad \sum_{i \neq j} \frac{1}{r_i r_j} = 1$$

where $\sum_{i \neq j}$ refers to the sum of all $1/r_i r_j$ for each pair of integers $1 \leqslant i \neq j \leqslant 2020$.

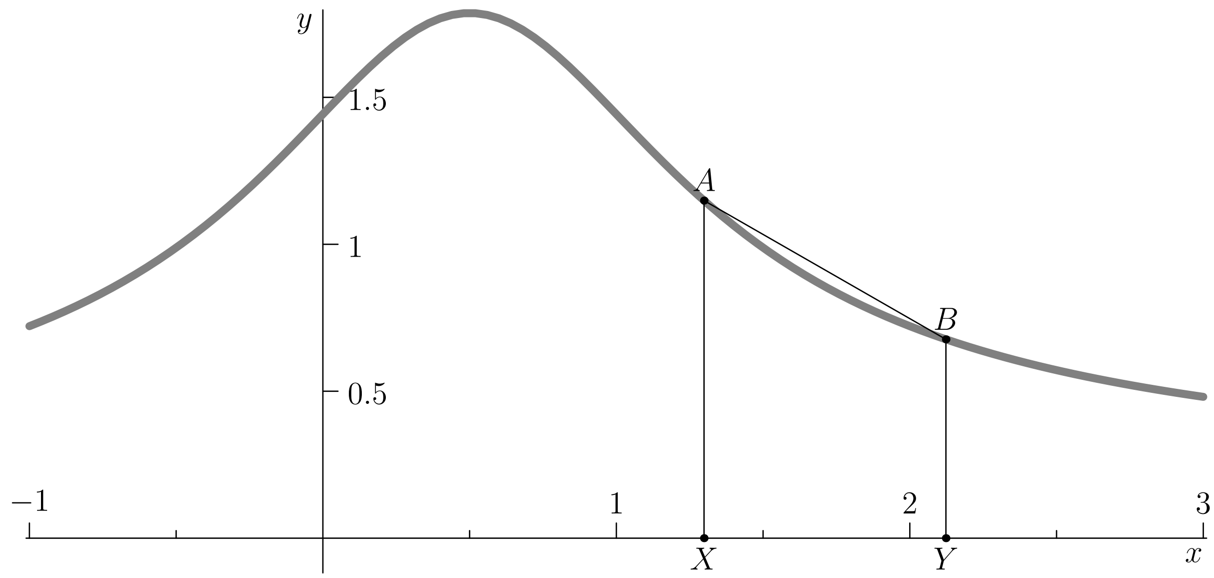
(ii) [**2**] Hence, prove that f has at most 2018 real roots.

(iii) [**1**] Is it possible to construct such a polynomial f in the above form, which has exactly 2018 nonzero real roots? Justify your answer. (If "Yes", then construct one such f, and if "No" then prove this is not possible.)

End of Question 14

Question 15 (**15 Marks**) Use the Question 15 section of the writing booklet.

(a) Given is a continuous, differentiable function f with domain $1 \leqslant x < +\infty$ and range $0 < f(x) < +\infty$. Let V be the volume of the solid of revolution formed when f is rotated about the x-axis across its domain, and S is the surface area of this solid (not including the circle at $x = 1$). It is given that S is a finite real number.



Consider two points A, B on the graph $y = f(x)$, both with x-coordinates at least 1. It is given that $A = (a, f(a))$ and $B = (a + \Delta x, f(a + \Delta x))$ where $\Delta x > 0$.

(i) [**1**] Show using Pythagoras' theorem that $|AB| = \sqrt{1 + \left(\dfrac{f(a+\Delta x) - f(a)}{\Delta x}\right)^2}\,\Delta x$.

(ii) [**1**] It is given that the curved surface area of a truncated cone is $\pi \cdot \ell(r_1 + r_2)$ where ℓ is the slant length and r_1, r_2 are the inner and outer radii of the cone. Show that

$$S = 2\pi \int_1^{+\infty} f(x)\sqrt{1 + (f'(x))^2}\,\mathrm{d}x.$$

(iii) [**2**] Let $u > 1$ be a real number. By using the equality $f(u)^2 - f(1)^2 = \displaystyle\int_1^u (f(x)^2)'\,\mathrm{d}x$, show that $f(u)^2 \leqslant \dfrac{S}{\pi} + f(1)^2$.

(iv) [**3**] Hence, by considering $V = \displaystyle\int_1^\infty \pi f(x)^2\,\mathrm{d}x$, deduce that if the surface area S is finite then the volume V is also finite.

(v) [**1**] Consider $f(x) = 1/x$ in the domain $1 \leqslant x < +\infty$. Use the result from (ii) to show that

$$V_t = \int_1^t \pi f(x)^2\,\mathrm{d}x = \pi\left(1 - \frac{1}{t}\right), \quad \text{and} \quad S_t = 2\pi \int_1^t f(x)\sqrt{1 + (f'(x))^2}\,\mathrm{d}x > 2\pi\ln(t).$$

Conclude that converse of (iv) is not necessarily true.

Question 15 continues on the next page

(b) Let $a > b$ be two positive integers and $p \geqslant 3$ an odd prime such that $p \mid a - b$, but $p \nmid a, p \nmid b$ and $p^2 \nmid a - b$. (We denote $k \mid n$ if k divides n, and $k \nmid n$ otherwise.)

(i) [**1**] Show that $p^2 \mid (a + p\ell)^n - (a^n + n \cdot p\ell a^{n-1})$ whenever $n, \ell \geqslant 1$ are integers.

(ii) [**1**] By using the result from (i), prove that the number

$$a^{p-1} + a^{p-2}(a + p\ell) + a^{p-3}(a + p\ell)^2 + \cdots + (a + p\ell)^{p-1}$$

leaves a remainder of $p \cdot a^{p-1}$ when divided by p^2.

(iii) [**2**] Hence, use mathematical induction to show that for each integer $n \geqslant 0$,

$$p^{n+1} \mid a^{p^n} - b^{p^n}, \text{ but } p^{n+2} \nmid a^{p^n} - b^{p^n}. \qquad (*)$$

(iv) [**1**] Now, consider the result where $p = 2$. Show, by factoring the expression $a^{2^n} - b^{2^n}$, or otherwise, that if a, b are **odd** positive integers satisfying $4 \mid a - b$, then the equation $(*)$ also holds.

Does $(*)$ still hold when the condition $4 \mid a - b$ is dropped?

(v) [**2**] Suppose p is a prime number and n, k are positive integers. Show that if the equation $20^p + 19^p = n^k$ holds, then $k = 1$.

End of Question 15

Question 16 (**15 Marks**) Use the Question 16 section of the writing booklet.

(a) A sequence $x_1, x_2, x_3, \ldots$ of positive real numbers is given, recursively defined by $x_1 = 1$ and $x_{n+1} = \dfrac{1}{x_1^2 + x_2^2 + \cdots + x_n^2}$ for each $n \geqslant 1$.

(i) [**2**] Show that $x_n \leqslant \dfrac{1}{\sqrt[3]{n-1}}$ for each $n \geqslant 2$.

(ii) [**1**] Prove that every integer $n \geqslant 2$ satisfies $\dfrac{1}{2^{2/3}} + \dfrac{1}{3^{2/3}} + \cdots + \dfrac{1}{n^{2/3}} \leqslant \displaystyle\int_1^n x^{-2/3}\,\mathrm{d}x$, and hence show that $x_{n+1} > \dfrac{1}{3\sqrt[3]{n-1}}$ for each integer $n \geqslant 2$.

(iii) [**1**] Prove that every $n \geqslant 2$ also satisfies $\dfrac{1}{\sqrt[3]{1}} + \dfrac{1}{\sqrt[3]{2}} + \cdots + \dfrac{1}{\sqrt[3]{n}} \geqslant \displaystyle\int_1^{n+1} \frac{1}{\sqrt[3]{x}}\,\mathrm{d}x$, and hence deduce that $x_1 + x_2 + \cdots + x_n > \dfrac{(n-1)^{2/3}}{2}$ for each $n \geqslant 2$.

(iv) [**1**] By considering a similar integral in (iii), show that for all sufficiently large positive integers n the inequality $\dfrac{1 - \frac{1}{2019}}{2} < \dfrac{x_1 + x_2 + \cdots + x_n}{n^{2/3}} < \dfrac{3 + \frac{1}{2019}}{2}$ holds.

(v) [**2**] It is given (**Do NOT prove**) that the limit $\displaystyle\lim_{n\to+\infty} \frac{x_1 + x_2 + \cdots + x_n}{n^{2/3}}$ exists, and is equal to a positive real number C.

Show that $C = \dfrac{\sqrt[3]{9}}{2}$.

(b) There are n balls, labelled $1, 2, \ldots, n$ and k buckets where $n \geqslant k$.

(i) [**1**] Assuming the buckets are distinguishable, show that there are k^n ways to place the n balls into the k buckets.

(ii) [**3**] Suppose there are n sets $A_1, A_2, \ldots, A_n$. It is given (**Do NOT prove**) that

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|+1} \left|\bigcap_{i \in I} A_i\right|.$$

Denote $S(n,k)$ as the number of ways to place all n balls into k *indistinguishable* buckets, such that no bucket is empty after the operation. Show, by selecting appropriate sets A_i in the above result, that $S(n,k) = \dfrac{1}{k!}\displaystyle\sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{\ell} (k-\ell)^n$.

(iii) [**1**] Show that $S(n,k) = kS(n-1,k) + S(n-1,k-1)$ for each $n \geqslant k \geqslant 1$.

(iv) [**3**] Hence, using mathematical induction or otherwise, show that each integer $n \geqslant 1$ satisfies $\displaystyle\sum_{k=0}^{n} S(n,k) \cdot n(n-1)\ldots(n-k+1) = n^n$.

——— *End of Examination* ———