Angle + Length Chasing

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As this is a Senior lecture, I will be assuming that all of you are familiar with the fundamentals of angle and length chasing. Hence this entire lecture will be spent solving questions, as opposed to learning any significant theory.

Rules for this lecture

- No $a + bi, (a, b), \overline{z}, (x, y, z), \overrightarrow{XY}$ or $re^{i\theta}$ are allowed without any exceptions
- Trigonometry is strongly discouraged, and may only be used under limited circumstances (i.e. only in the absence of long, explicit computations)
- Theorems from inversive and projective geometry are allowed, but not recommended
- Only the lecturer may use Geogebra for drawing diagrams; students must only use paper to attempt any of the questions for the duration of the lecture
- Common sense should be used when reading the following problems. In particular:
  - Given three points A, B, C, (ABC) denotes the circumcircle of ΔABC and (AB) denotes the circle with diamater AB;
  - For lines/circles λ, μ we denote $\lambda \cap \mu$ as their set of intersection points.

1. In ΔABC, ω_B is the circle through B and tangent to AC at A, and ω_C is symmetrically defined. Point M is the midpoint of AC, and $\omega_B \cap \omega_C = \{A, D\}$.

   Prove that the point $DM \cap BC$ lies on ω_B.

2. Triangle ΔABC has orthocentre H. Let $\omega \stackrel{\text{def}}{=} (BCH)$, $\Gamma \stackrel{\text{def}}{=} (AH)$ and $\omega \cap \Gamma = \{H, X\}$. Let Γ' be the reflection of Γ over AX. Suppose $\omega \cap \Gamma' = \{X, Y\}$ and $\omega \cap AH = \{H, Z\}$.

   Show that if M is the midpoint of BC, then $AMYZ$ is cyclic.

3. In triangle ΔABC, point M is the midpoint of BC, and $N \in \overline{AM}$ such that $|MA| \times |MN| = \frac{1}{4}|BC|^2$. Suppose $(ABC) \cap BN = \{B, Q\}$ and $(ABC) \cap CN = \{C, P\}$. Suppose points R, S lie on PQ such that the following **directed** angle conditions hold:

$$\angle ARN = \angle ANB, \quad \text{and} \quad \angle ASN = \angle ANC.$$

   Prove that $|NR| = |NS|$.

4. Triangle ΔABC has orthocentre H and altitudes AD, BE, CF. Points E', F' are the reflections of E, F over AD, respectively. Denote $X \stackrel{\text{def}}{=} BF' \cap CE'$ and $Y \stackrel{\text{def}}{=} BE' \cap CF'$.

   Prove that AX, YH, BC are concurrent.

5. Triangle ΔABC has circumcentre O and orthocentre H. Line AO intersects (ABC) again at D. Denote D_1, D_2, and H_1, H_2, the reflections of D and H across sides AB, AC respectively. Suppose $K \stackrel{\text{def}}{=} H_1H_2 \cap D_1D_2$, $L \stackrel{\text{def}}{=} UD_2 \cap DH_2$ and $AH \cap (AD_1D_2) = \{A, U\}$.

   Prove that one of the two intersection points of (D_1KH_1) and (DLU) lies on KL.

6. Triangle ΔABC satisfies $AB < AC$, and has circumcircle Γ. The internal angle bisector of $\angle BAC$ intersects the side BC at D. Suppose $\Gamma \cap (BD) = \{B, P\}$, $\Gamma \cap (CD) = \{C, Q\}$ and $X \stackrel{\text{def}}{=} PQ \cap BC$.

   Prove that AX is tangent to the circumcircle of $\triangle ABC$.

7. Triangle ΔABC has circumcircle Γ. Points D, E are chosen on $\overline{BC}$ such that $\angle BAD = \angle CAE$. The circle ω is tangent to AD at A and has its circumcentre lie on Γ. The reflection of A across BC is A'. Suppose that $\omega \cap A'E = \{K, L\}$.

   Prove that at least one of the points $BL \cap CK$ and $BK \cap CL$ lies on Γ.

8. Let points P, Q lie on side $\overline{BC}$ of triangle ΔABC such that $BP = CQ$ and $BP < BQ$. The circle (APQ) intersects AB, AC at E, F respectively. Let $T = EP \cap FQ$. The two lines passing through the midpoint of BC and parallel to AB and AC intersects EP and FQ at X, Y respectively.

   Prove that (APQ) and (TXY) are tangent to each other.

9. Triangle ΔABC has A-excentre I_A. Let ω be an arbitrary circle that passes through A and I_A, which intersects AB, AC at X, Y respectively. Let S, T be points on I_AB, I_AC such that $\angle AXI_A = \angle BTI_A$ and $\angle AYI_A = \angle CSI_A$. Denote $K \stackrel{\text{def}}{=} BT \cap CS$ and $Z \stackrel{\text{def}}{=} KI_A \cap TS$.

   Prove that X, Y, Z are collinear.

10. Triangle ΔABC has A-excentre I_A and A-excircle Ω that is tangent to sides AB and AC at F, E respectively. Point D is the reflection of A through I_AB, and $K \stackrel{\text{def}}{=} DI_A \cap EF$.

    Prove that the circumcentre of ΔDKE lies on the line through I_A and the midpoint M of BC.

11. Triangle ΔABC has circumcentre O and orthocentre H. The line through A perpendicular to OH intersects OH at K and (ABC) again at P. Suppose that P, A are **not** on the same side of BC as A. Points E, F are chosen on sides $\overline{AB}, \overline{AC}$ respectively such that $|BE| = |PC|$ and $|CF| = |PB|$.

    Prove that $\angle EKF = 90$.

12. Triangle ΔABC has altitudes AD, BE, CF, circumcircle Γ and circumcentre O. It is given that $\Gamma \cap (ADO) = \{A, P\}$. Let X, Y be the second intersections of Γ with PE, PF respectively.

    Prove that $XY \parallel BC$.

13. Triangle ΔABC has circumcentre O and incentre I. Points D, E, F are the incircle touchpoints with sides BC, CA, AB respectively, and A' is the reflection of A over O. Let M be the midpoint of EF. Suppose $(ABC) \cap (A'EF) = \{A, G\}$ and $(AMG) \cap (A'EF) = \{G, H\}$.

    Prove that if $T \stackrel{\text{def}}{=} GH \cap EF$, then $DT \perp EF$.

14. Triangle ΔABC has circumcentre O and incentre I. Point D is the incircle touchpoint with BC, and S is the point on (ABC) such that BC, OI, AS are concurrent. Point T is on (ABC) such that $\angle ATI = 90$, and H be the orthocentre of triangle BIC.

    Prove that $DTHS$ is cyclic.

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