Problem Set 10 Solutions

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1 Problem Set 10

1.1 Question 1

Problem 1.1

Let a, b, c be the roots of the polynomial $P(x) = 2x^3 - 3x^2 + 4x - 5$. Evaluate the following numbers in closed form (i.e. express them as a single rational number).

(a) $a + b + c$;

(b) $a^2 + b^2 + c^2$;

(c) $(a + b)(b + c)(c + a)$.

Solution. We have $2x^3 - 3x^2 + 4x - 5 = 2(x - a)(x - b)(x - c)$, and so $\boxed{a + b + c = \frac{3}{2}}$, $ab + bc + ca = 2$ and $abc = \frac{5}{2}$. Hence, $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca) = (\frac{3}{2})^2 - 2(2) = \frac{7}{4}$.
We also have $(a + b)(b + c)(c + a) = (\frac{3}{2} - a)(\frac{3}{2} - b)(\frac{3}{2} - c) = \frac{1}{2}P(\frac{3}{2}) = \frac{1}{2}$.

1.2 Question 2

Problem 1.2

Let $x \neq y$ be integers. Show, by factorisation or otherwise, that $x - y \mid x^n - y^n$. Hence show that $x - y \mid P(x) - P(y)$ whenever $P(x)$ is an integer polynomial.

Solution. We have $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-1} + y^n)$. The second larger bracket is an integer since x, y are integers. It follows that $x - y \mid x^n - y^n$.

We also have

$$P(x) - P(y) = \sum_{k=0}^{n} a_k x^k - \sum_{k=0}^{n} a_k y^k = \sum_{k=0}^{n} a_k (x^k - y^k).$$

Since we have $x - y \mid x^k - y^k$ for each k, it follows that $x - y \mid P(x) - P(y)$.

1.3 Question 3

Problem 1.3

Let $P(x)$ be a **cubic** polynomial with roots r_1, r_2, r_3. It is given that

$$\frac{P(\frac{1}{2}) + P(-\frac{1}{2})}{P(0)} = 1012.$$

Find the value of $\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1}$.

Solution. For any polynomial P satisfying this, notice that the polynomial $c \cdot P$ for any nonzero constant c would also satisfy the equation. Hence, we can assume P is monic, i.e. $P(x) = x^3 + ax^2 + bx + c$ for some reals a, b, c. Then, we have

$$\frac{P(\frac{1}{2}) + P(-\frac{1}{2})}{P(0)} = \frac{((\frac{1}{2})^3 + a(\frac{1}{2})^2 + b(\frac{1}{2}) + c) + ((-\frac{1}{2})^3 + a(-\frac{1}{2})^2 + b(-\frac{1}{2}) + c)}{c} = \frac{a/2 + 2c}{c} = 1012$$

and so $a/2 + 2c = 1012c$ and so $a/c = 2020$. However, notice by Vieta that

$$a = -(r_1 + r_2 + r_3), \ c = -r_1 r_2 r_3$$

and so

$$\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1} = \frac{r_1 + r_2 + r_3}{r_1 r_2 r_3} = \frac{a}{c} = 2020.$$

1.4 Question 4

Problem 1.4

A nonzero real number a is given. Find all polynomials $P(x)$ with real coefficients for which $P(x+a) = P(x) + a$ for every real number x.

Hint. What can you say about the polynomial $Q(x) = P(x) - x$?

Solution. Let $Q(x) = P(x) - x$. Then, we have $P(x) = Q(x) + x$, and so plugging in we get

$$Q(x+a) + (x+a) = Q(x) + x + a \implies Q(x+a) = Q(x).$$

This means Q is a periodic polynomial, which implies Q is constant. It follows that $P(x) = x + c$ for some constant c.

1.5 Question 5

Problem 1.5

Find all polynomials $P(x)$ with real coefficients such that

$$(x-27)P(3x) = 27(x-1)P(x).$$

Solution. Plugging $x = 1$ gives $P(3) = 0$, and so $x - 3$ is a factor of P. Also plugging $x = 27$ gets $P(27) = 0$, and so $x - 27$ is also a factor of P. Hence, letting $P(x) = (x-3)(x-27)Q(x)$, we obtain

$$(x-27)(3x-3)(3x-27)Q(3x) = 27(x-1)(x-3)(x-27)Q(x)$$

$$(x-9)Q(3x) = 3(x-3)Q(x).$$

Plugging $x = 3$ gives $Q(9) = 0$, and so $x-9$ is a factor of Q. Hence, letting $Q(x) = (x-9)R(x)$, we obtain

$$(x-9)(3x-9)R(3x) = 3(x-3)(x-9)R(x)$$

$$R(3x) = R(x).$$

Suppose R is nonconstant: then if r was a root of R, then $3r$ is also a root, and so is $9r, 27r, 81r, \ldots$. This means R has infinitely many roots, which is a contradiction. It follows R is constant, and thus all polynomials $P(x)$ are of the form

$$P(x) = C(x-3)(x-9)(x-27)$$

for some constant C.