Sequences

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By now I hope all of you know what sequences are. There isn't really anything I need to teach about them, so I guess today's lecture will just be solving assorted sequences problems from algebra and number theory.

> **Rules for this lecture**
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> - Solve the questions and alert me when you have done so.
> - In the following questions, N is meant to be any given positive integer.

1. Is it possible to enumerate $\mathbb{Q}$ as $q_1, q_2, q_3, \ldots$ such that $\sum_n (q_n - q_{n+1})^2$ is finite?
2. Do there exist two nonzero real sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ such that, for every n, $b_n \le c_n$ and $x^2 + b_n x + c_n = (x - b_{n+1})(x - c_{n+1})$?
3. Do there exist integers $\{a_n\}_{n=1}^{N}$ such that $\frac{1}{a_1} + \frac{1}{2a_2} + \cdots + \frac{1}{Na_N} = 1$?
4. The real sequence $\{a_i\}_{i=0}^{N}$ satisfy $a_k = \frac{a_{k-1}}{\sqrt{1+N\cdot a_{k-1}^2}}$ for all $1 \le k \le N$. Show that $|Na_N| < 1$.
5. Find all integer sequences $\{a_n\}_{n=1}^{\infty}$ such that, for any $n \ge 1$, $f(n) \mid n^3$ and $\sum_{k=1}^{n} f(k)$ is a square.
6. Does there exist a non-constant integer sequence $\{a_n\}_{n=-\infty}^{\infty}$ such that, for **any** surjective integer sequence $\{b_n\}_{n=-\infty}^{\infty}$, the sequence $\{a_n + b_n\}_{n=-\infty}^{\infty}$ is also surjective?
7. Find all integer sequences $\{a_n\}_{n=1}^{\infty}$ that satisfy $\gcd(a_m + n, a_n + m) > \frac{m+n}{100}$ for any integers m, n.
8. The integer sequence $\{a_i\}_{i=1}^{n+1}$ satisfy $a_{k+1} = a_k^2 - a_k + 1$ for any $2 \le k \le n$. Prove $\gcd(a_{n+1}, 2n+1) = 1$.
9. Let $f(x) \in \mathbb{R}[x]$ such that $f(x) > 0$ for any $x > 0$. Given a positive real sequence $\{a_n\}_{n=1}^{\infty}$ satisfying $0 < a_{n+1} - a_n < 100$ for all n, let $\{s_n\}_{n=1}^{\infty}$ be the infinite string of base-10 digits formed when the digits of $\lfloor f(a_1) \rfloor, \lfloor f(a_2) \rfloor, \lfloor f(a_3) \rfloor, \ldots$ are written consecutively.

   Show that for any n, the set $S_n \stackrel{\text{def}}{=} \left\{ \overline{s_{n(k-1)+1} s_{n(k-1)+2} \ldots s_{nk}} \ : \ k \ge 1 \right\}$ contains all n-digit numbers.