Pell's Equation stuff

james bang

November 17, 2019

1 Introduction

For an integer N **not** a perfect square, the corresponding **Pell's Equation** is the Diophantine equation

$$x^2 - Ny^2 = 1.$$

Basically, the Greeks somehow found a contradiction in the equation $x^2 - 2y^2 = 0$ (since rearranging gives $\sqrt{2} = x/y$ which contradicts the irrationality of $\sqrt{2}$), and were like hmmm what about let's say $x^2 - 2y^2 = 1$? (and whilst we're at it, we might as well study the general case where 2 is replaced by N.) Then, the quotient x/y gives increasingly better rational approximations to $\sqrt{2}$ (or $\sqrt{N}$), since as $y \to \infty$,

$$\frac{x}{y} = \sqrt{2 + \frac{1}{y^2}} \to \sqrt{2}.$$

It turns out that for each non-square N, there exists infinitely many solutions, which are generated quite nicely: there exists a **fundamental solution** (x_0, y_0) to the equation with $x_0, y_0 > 0$ (which is basically the "smallest" such solution of $x^2 - Ny^2 = 1$, having smallest positive x and y values), and for every other solution (x, y) with $x, y > 0$, there exists a positive integer k for which

$$x + y\sqrt{N} = (x_0 + y_0\sqrt{N})^k.$$

The other solutions for $x < 0$ or $y < 0$ can be found simply by flipping signs of the above solutions.

For equations of the form $x^2 - Ny^2 = a$ for general integers a, known as **Pell-type Equations**, the solution set is often not as concise as the case when $a = 1$, and may have multiple families of solutions (or none at all!) depending on the values of a and N.

2 Why can't N be a square?

Well, basically it can, but it's a lot less interesting than the case when N is not a square.

Problem 2.1

Suppose $N = n^2$ for some positive integer n. Then the only solutions (x, y) to $x^2 - Ny^2 = 1$ is $(\pm 1, 0)$.

Not very hard: you literally factor the thing,

$$1 = x^2 - n^2y^2 = (x + ny)(x - ny).$$

This means $x + ny$ and $x - ny$ are either both $+1$ or -1; either way, they are the same, so subtracting gives $y = 0$. Plug into the equation to get $x = \pm 1$. $\square$

Since the case for N being a square is thus not very interesting, for the rest of this handout we will assume N is not a square.

3 Norm stuff

In basic high school level mathematics, you learn these things called "complex numbers", which are numbers of the form $a+bi$: the a is the real part, while b is the imaginary part. The a and b are independent of each other since i is not real, and so two complex numbers a_1+b_1i and a_2+b_2i are the same if and only if $a_1=a_2$ and $b_1=b_2$.

Similarly, this same notion can be used to define the **Ring of Quadratic Integers** $N \mapsto \mathbb{Z}[\sqrt{N}]$, which is basically the set of all integers of the form $a+b\sqrt{N}$ where a, b are integers. Likewise for complex numbers, for each $\alpha = a+b\sqrt{N} \in \mathbb{Z}[\sqrt{N}]$, there exists a **conjugate** $\overline{\alpha} = a-b\sqrt{N}$ and a **norm** $N(\alpha) = \alpha \cdot \overline{\alpha} = a^2 - Nb^2$. The following basic facts follow by definition:

- $N(\alpha) = N(\overline{\alpha})$ since $N(\alpha) = a^2 - Nb^2 = a^2 - N(-b)^2 = N(\overline{\alpha})$
- $\overline{\alpha \cdot \beta} = \overline{\alpha} \cdot \overline{\beta}$, since
$$\overline{\alpha \cdot \beta} = \overline{(a+b\sqrt{N})(c+d\sqrt{N})} = (ac+Nbd) - \sqrt{N}(ad+bc) = (a-b\sqrt{N})(c-d\sqrt{N}) = \overline{\alpha} \cdot \overline{\beta}$$
- $N(\alpha) \cdot N(\beta) = N(\alpha \cdot \beta)$, since
$$N(\alpha \cdot \beta) = \alpha\beta \cdot \overline{\alpha\beta} = \alpha\overline{\alpha} \cdot \beta\overline{\beta} = N(\alpha) \cdot N(\beta).$$

In particular, consider the equation $x^2 - Ny^2 = 1$: the left side is simply the norm of $\alpha = x + y\sqrt{N}$; hence, the Pell's Equation is equivalent to the statement $N(\alpha) = 1$ for $\alpha = x + y\sqrt{N} \in \mathbb{Z}[\sqrt{N}]$. This will considerably help in cleaning up notation throughout the handout.

4 Generating solutions

In a Pell equation, having one nontrivial solution is quite powerful, since it allows us to generate infinitely many different solutions to the equation.

Lemma 4.0.1: Brahmagupta's Identity

For given N, the set of all numbers of the form $x^2 - Ny^2$ is closed under multiplication: more explicitly, we have
$$(a^2 - Nb^2)(c^2 - Nd^2) = (ac + Nbd)^2 - N(ad + bc)^2.$$

To prove this, you literally expand and everything cancels out (oh how cool is that). It turns out this statement is equivalent to $N(\alpha \cdot \beta) = N(\alpha) \cdot N(\beta)$ (hmmm I wonder why there's so much resemblance).

Hence, taking any nontrivial solution (x, y) to the Pell equation, we can generate more solutions simply by expanding out α^k for any positive integer k, since $\alpha^k = X_k + Y_k\sqrt{N}$ for some integers X_k, Y_k, and also
$$N(X_k + Y_k\sqrt{N}) = N(\alpha^k) = N(\alpha)^k = 1^k = 1$$
and thus $X_k + Y_k\sqrt{N}$ must also be a solution to the Pell equation. This allows us to generate infinitely many solutions given a single nontrivial solution (a similar technique was used by the Greeks to find better approximations to $\sqrt{2}$). In fact, it turns out we can prove even more:

Lemma 4.0.2: Every solution is generated by a single solution

Suppose that a nontrivial solution to the Pell equation exists: choose the solution (x_1, y_1) for which $x_1, y_1 > 0$ and $\alpha = x_1 + y_1\sqrt{N} > 1$ is minimal (which must exist since ordering of $\mathbb{N}$). Then, **every** other solution (x_k, y_k) where $x_k, y_k > 0$ must be of the form

$$x_k + y_k\sqrt{N} = (x_1 + y_1\sqrt{N})^k \ (= \alpha^k).$$

Proof: Take a solution (x, y) such that $x, y > 0$ and $x^2 - Ny^2 = 1$. Then by minimality, we must have $x + y\sqrt{N} \geq x_1 + y_1\sqrt{N} = \alpha$: thus, we can find a positive integer k, for which $\alpha^k \leq x < \alpha^{k+1}$. Then, consider the number $\beta = (x + y\sqrt{N})/\alpha^k$: clearly, we have $1 \leq \beta < \alpha$, and also

$$N(\beta) = N\left(\frac{x + y\sqrt{N}}{\alpha^k}\right) = \frac{N(x + y\sqrt{N})}{N(\alpha)^k} = \frac{1}{1^k} = 1,$$

and so $N(\beta) = 1$, meaning it's a solution to the Pell equation smaller than α. Furthermore, $\beta \in \mathbb{Z}[\sqrt{N}]$, since

$$\beta = \frac{x + y\sqrt{N}}{\alpha^k} = \frac{(x + y\sqrt{N})(\overline{\alpha})^k}{(\alpha \cdot \overline{\alpha})^k} = (x + y\sqrt{N})(\overline{\alpha})^k \in \mathbb{Z}[\sqrt{N}].$$

By minimality, it must be that $\beta = 1$, which means $x + y\sqrt{N} = \alpha^k$, as desired. □

5 Why does it have a nontrivial solution?

For most small values of N, it is possible to just bash through some algebra and find the minimal solution, from which it's easy to find the rest of the solutions by the method described above. However, sometimes these solutions are pretty damn huge, even for relatively small N:

N	x	y
13	649	180
29	9801	1820
53	66249	9100
61	1766319049	226153980
109	158070671986249	15140424455100

At other times, in an Olympiad math problem, it suffices to just show a solution exists without explicitly constructing one. What guarantee do we have that such a fundamental solution actually exists?

Lemma 5.0.1: Dirichlet's Approximation

Let α be an **irrational** real number, and $n \geq 1$. Then, there exists integers p and $1 \leq q \leq n$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q(n+1)}.$$

Proof. Notice the statement is equivalent to $|q \cdot \alpha - p| < \frac{1}{n+1}$. As always, denote $\{x\}$ as the *fractional part* of x, the smallest nonnegative number formed by adding/subtracting an integer from x.

Consider the $n+2$ distinct real numbers $0, \{\alpha\}, \{2\alpha\}, \ldots, \{n\alpha\}, 1$, which are all between 0 and 1 (inclusive). By the Pigeonhole Principle, there exists two of these, which differ by at most $\frac{1}{n+1}$:

- If the two numbers are $\{k\alpha\}, \{\ell\alpha\}$ for some $k \neq \ell$, then set $q = |k - \ell|$: since $|\{k\alpha\} - \{\ell\alpha\}| < \frac{1}{n+1}$, we either have $\{|k-\ell| \cdot \alpha\} < \frac{1}{n+1}$ or $> 1 - \frac{1}{n+1}$. In either case, simply choosing p as the closest integer to $q\alpha$ gives $|q\alpha - p| < \frac{1}{n+1}$.
- If the two numbers are 0 and $\{k\alpha\}$, then set $q = k$ and p as the closest integer to $q\alpha$.
- If the two numbers are $\{k\alpha\}$ and 1, then also just choose $q = k$ and p as the closest integer to $q\alpha$.

In any case, there exists some $q \leq n$ and a corresponding p such that this statement holds. □

An obvious corollary of this result is that there exists infinitely many pairs (p, q) of positive integers, such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2},$$

since we have $n + 1 > q$, and furthermore each (p, q) satisfies the Dirichlet inequality for finitely many n (and thus there must be infinitely many such pairs (p, q)).

Anyway, in the corollary we set $\alpha = \sqrt{N}$ for our positive integer N not a square number; then there must be infinitely many pairs (x, y) for which

$$\left|\sqrt{N} - \frac{x}{y}\right| < \frac{1}{y^2}, \text{ or } |y\sqrt{N} - x| < \frac{1}{y}.$$

Notice that this bound looks very familiar: the bounded term $y\sqrt{N} - x$ is simply a factor of $x^2 - Ny^2 = (x - y\sqrt{N})(x + y\sqrt{N})$, so it seems like a good idea to bound the other term:

$$|x + y\sqrt{N}| \leq |x - y\sqrt{N}| + |2y\sqrt{N}| \leq \frac{1}{y} + 2y\sqrt{N} < 1 + 2y\sqrt{N},$$

and so

$$|x^2 - Ny^2| = |x - y\sqrt{N}| \cdot |x + y\sqrt{N}| < \frac{1}{y}(1 + 2y\sqrt{N}) = \frac{1}{y} + 2\sqrt{N} < 1 + 2\sqrt{N}.$$

Hence, the term $|x^2 - Ny^2|$ is bounded by a fixed constant $1 + 2\sqrt{N}$: by the infinite Pigeonhole Principle, this means there exists infinitely many pairs of positive integers (x, y) such that $x^2 - Ny^2 = c$, where c is a fixed integer with $|c| < 1 + 2\sqrt{N}$.

Since there are infinitely many pairs (x, y) which satisfies these equations, again by the infinite Pigeonhole Principle, there exists two solutions (x_1, y_1) and (x_2, y_2), for which $x_1 \equiv x_2 \pmod{c}$ and $y_1 \equiv y_2 \pmod{c}$: basically, first find infinitely many solution pairs with the x-term having a constant modulo c term, and from these infinitly many pairs, another infinitely many of them must also have a y-term which is constant modulo c, and just choose two such solutions $(x_1, y_1), (x_2, y_2)$ [1]. Denote $z_1 = x_1 - y_1\sqrt{N}$ and $z_2 = x_2 - y_2\sqrt{N}$: we can WLOG assume $z_1 > z_2$ (as they are assumed to be distinct). Then:

$$X + Y\sqrt{N} = \frac{x_1 - y_1\sqrt{N}}{x_2 - y_2\sqrt{N}} = \frac{(x_1 - y_1\sqrt{N})(x_2 + y_2\sqrt{N})}{x_2^2 - Ny_2^2} = \frac{x_1x_2 - Ny_1y_2}{c} + \sqrt{N} \cdot \frac{x_1y_2 - x_2y_1}{c}.$$

Since $z_1 > z_2$ by assumption, we have $X + Y\sqrt{N} > 1$, and thus this does not correspond to the trivial solution $(\pm 1, 0)$. Furthermore, X, Y are actually integers, since

$$x_1x_2 - Ny_1y_2 \equiv x_1^2 - Ny_1^2 \equiv c \equiv 0 \pmod{c},$$

[1] Alternatively, there are only c^2 possible pairs (x, y) with both x, y taken modulo c, so by the Pigeonhole Principle one of them must occur infinitely many times.

since we had $x_1 \equiv x_2 \pmod{c}$ and $y_1 \equiv y_2 \pmod{c}$; similarly, it can be shown Y is also an integer, since

$$x_1y_2 - x_2y_1 \equiv x_1(y_1) - (x_1)y_1 \equiv 0 \pmod{c}.$$

However, from earlier, we had $x_1^2 - Ny_1^2 = x_2^2 - Ny_2^2 = c$, and since $X = \frac{x_1x_2 - Ny_1y_2}{c}$, and $Y = \frac{x_1y_2 - x_2y_1}{c}$:

$$\begin{aligned} X^2 - NY^2 &= \left(\frac{x_1x_2 - Ny_1y_2}{c}\right)^2 - N\left(\frac{x_1y_2 - x_2y_1}{c}\right)^2 \\ &= \frac{(x_1x_2)^2 - 2N(x_1x_2)(y_1y_2) + N^2(y_1y_2)^2 - N[(x_1y_2)^2 - 2x_1x_2y_1y_2 + (x_2y_1)^2]}{c^2} \\ &= \frac{x_1^2x_2^2 - Nx_1^2y_1^2 - Nx_2^2y_1^2 + N^2y_1^2y_2^2}{c^2} \\ &= \frac{(x_1^2 - Ny_1^2)(x_2^2 - Ny_2^2)}{c^2} \\ &= \frac{c^2}{c^2} = 1. \end{aligned}$$

It follows that (X, Y) is a nontrivial solution to the Pell's Equation $x^2 - Ny^2 = 1$. Hence, it follows that a nontrivial solution to Pell's equation exists for each N not a square. $\square$

6 Continued Fraction stuff

Note: I won't prove much stuff in this section because I don't know how to prove them yay what a noob

A **Continued Fraction** of an irrational number α is a representation as an infinite number of nested fractions, i.e.

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}} = [a_0; a_1, a_2, a_3, \ldots].$$

It turns out that in the case of $\alpha = \sqrt{N}$ for positive integer N, the continued fraction must be **periodic** and symmetric along each period: that is, it takes the form

$$\sqrt{N} = [a_0; \overline{a_1, a_2, \ldots, a_{k-1}, a_k}],$$

where $a_k = 2a_0$, and $a_j = a_{k-j}$ for each $1 \le j \le k-1$. It turns out that the fundamental solution is related to the continued fraction in the following way: if (x, y) is the fundamental solution to $x^2 - Ny^2 = 1$, then we have

$$\frac{x}{y} = \begin{cases} [a_0; a_1, a_2, \ldots, a_{k-1}] & \text{if } 2 \mid k \\ [a_0; a_1, a_2, \ldots, a_{2k-1}] & \text{if } 2 \nmid k. \end{cases}$$

For example, the continued fraction for $\sqrt{7}$ is $[2; \overline{1, 1, 1, 4}]$, meaning $k = 4$ (which is even), and so

$$\frac{x}{y} = \mathbf{2} + \cfrac{1}{\mathbf{1} + \cfrac{1}{\mathbf{1} + \cfrac{1}{\mathbf{1}}}} = \frac{8}{3},$$

meaning $(8, 3)$ is the fundamental solution to $x^2 - 7y^2 = 1$ (which it is). This gives a much more efficient means of computing the fundamental solution than to manually check each $x \ge 2$.

7 What about other Pell-type equations?

As mentioned before, **Pell-type Equations** are equations of the form $x^2 - Ny^2 = a$, where $a \in \mathbb{Z}/\{0\}$. Similar to Pell equations, they can be written in the form $N(\alpha) = a$, where $\alpha = x + y\sqrt{N}$; distinctively, they may not have any integer solutions at all depending on N, a (for example $x^2 - 7y^2 = 2019$ has no solutions, since $2019 \equiv 3$ is not a quadratic residue mod 7) or may have multiple families of solutions.

7.1 $a = -1$ (Negative Pell Equation)

It is clear that for $x^2 - Ny^2 = -1$, the number N must not be divisible by any prime $p \equiv 3 \pmod 4$ as otherwise we can take modulo p which gets $x^2 + 1 \equiv 0 \pmod p$, which is impossible. Hence, N must be divisible only by primes $p \equiv 1 \pmod 4$ (or equivalently, $N = s^2 + t^2$ for some coprime integers s, t). However, this condition is not sufficient (oh no!) – for example, $x^2 - 2y^2 = 5$ has no solution modulo 25.

Problem 7.1

Suppose the equation $x^2 - Ny^2 = 1$ has the fundamental solution $\alpha = x + y\sqrt{N}$. Then, the corresponding negative Pell equation $x^2 - Ny^2 = -1$ has solutions if and only if there exists $\beta \in \mathbb{Z}[\sqrt{N}]$ for which $\beta^2 = \alpha$.

Proof. The forward direction is trivial: if $\alpha = \beta^2$ for some $\beta \in \mathbb{Z}[\sqrt{d}]$, we have $1 = N(\alpha) = N(\beta^2) = N(\beta)^2$: however, $N(\beta) \neq 1$ due to minimality of α, and hence $N(\beta) = -1$.

For the "only-if" direction, choose the smallest solution $\beta = x + y\sqrt{N}$ for which $x, y > 0$ and $x^2 - Ny^2 = -1$. Then, since $N(\beta^2) = N(\beta)^2 = 1$, we must have $\beta^2 = \alpha^k$ for some positive integer k. Then $k = 1$ since otherwise if $k \geq 3$ odd then β is not minimal, and if k even then we must have $N(\beta) = 1$, which is a contradiction. □

This allows us to solve the Negative Pell Equation in certain cases when the fundamental solution factorises. For example, the equation $x^2 - 2y^2 = 1$ has the fundamental solution $\alpha = 3 + 2\sqrt{2} = (1 + \sqrt{2})^2$, and this leads to the general solution of the Negative Pell equation $x^2 - 2y^2 = -1$:

$$(x, y) = \left(\frac{(1+\sqrt{2})^{2n+1} - (1+\sqrt{2})^{-(2n+1)}}{2}, \frac{(1+\sqrt{2})^{2n+1} + (1+\sqrt{2})^{-(2n+1)}}{2\sqrt{2}} \right).$$

In general, the equation $x^2 - Ny^2 = -1$ has a solution if and only if $N = s^2 + t^2$ for some coprime positive integers s, t, for which there exists a primitive Pythagorean triple (A, B, C) where $A^2 + B^2 = C^2$ and $|As - Bt| = 1$. This is sufficient, as setting $x = At + Bs$ and $y = C$ gives

$$Ny^2 = (s^2 + t^2)(A^2 + B^2) = (As - Bt)^2 + (At + Bs)^2 = x^2 + 1.$$

The necessity, on the other hand...

7.2 General a

In general, if there exists a single solution (x, y) to $x^2 - Ny^2 = a$ (or $N(\beta) = a$ for $\beta \in \mathbb{Z}[\sqrt{-N}]$), then it is possible to generate infinitely many solutions by taking the fundamental solution $N(\alpha) = 1$, and simply multiply this to get $N(\beta \cdot \alpha^k) = N(\beta) \cdot N(\alpha)^k = a \cdot 1^k = a$ where $k \in \mathbb{Z}$ (allowed to be negative). This generates solutions with arbitrarily large x and y values.

While a closed-form solution for Pell equations in general seem impossible (especially considering just the values $a = \pm 1$), it is possible to find the general solution to a Pell-type equation when specific values of a, N are given, with a bit of algebra and trial-and-error (given the values of a, N and the fundamental solution coefficients aren't enormous):

Lemma 7.2.1: Bounds for Solutions to Pell-type equations

For a given Pell-type equation $x^2 - Ny^2 = a$, suppose the fundamental solution of the corresponding Pell equation $x^2 - Ny^2 = 1$ is given by (x_1, y_1) (with $x_1, y_1 > 0$) and let $\alpha = x_1 + y_1\sqrt{N} > 1$. Then, if there exists a solution (x, y) to $x^2 - Ny^2 = a$, then we must have

$$|x| \leq \frac{\sqrt{|a|}}{2}\left(\sqrt{\alpha} + \frac{1}{\sqrt{\alpha}}\right),$$

and the similar inequality $y \leq \sqrt{\frac{x^2 - a}{N}}$.

Proof. The method of proof is essentially the same with what we've seen before. Suppose there exists a solution $\beta = X + Y\sqrt{N}$ to $N(\beta) = a$ (so $X^2 - NY^2 = a$). Then notice that $N(\beta \cdot \alpha^k) = N(\beta) \cdot N(\alpha)^k = a$ and so $\beta \cdot \alpha^k$ is also a solution. Hence, there must exist some integer k for which

$$\sqrt{\frac{|a|}{\alpha}} \leq \beta \cdot \alpha^k < \sqrt{|a| \cdot \alpha}.$$

Let $x_1 + y_1\sqrt{N} = \beta \cdot \alpha^k = z$. Then, we also have

$$x_1 - y_1\sqrt{N} = \frac{x_1^2 - Ny_1^2}{x_1 + y_1\sqrt{N}} = \frac{a}{\beta \cdot \alpha^k} = \frac{a}{z} \implies 2x_1 = z + \frac{a}{z}.$$

However, it's easy to see using some basic calculus or otherwise that the function $f(z) = z + \frac{a}{z}$ across the domain $\sqrt{\frac{|a|}{\alpha}} \leq z < \sqrt{|a| \cdot \alpha}$ reaches its maximum on the endpoints (since it only has one stationary point at $z = \sqrt{\alpha}$, which is a minimum turning point) which give the same value, $\sqrt{|a|}\left(\sqrt{\alpha} + \frac{1}{\sqrt{\alpha}}\right)$. Hence, the bound for $|x_1|$ follows. The bound for y_1 follows by subbing the x_1-bound into the given equation. $\square$

It is worth noting that a similar technique to the one employed above gives a nicer, albeit weaker bound

$$|x_1| < \sqrt{|a| \cdot \alpha}, \quad |y_1| < \sqrt{\frac{|a| \cdot \alpha}{N}}.$$

8 Example Problems

Problem 8.1

Solve the Pell equation $x^2 - 2019y^2 = 1$.

Pretty straightforward: you literally use the continued fraction or otherwise, to get the Fundamental solution as $674 + 15\sqrt{2019}$. Then every positive solution is of the form $x_k + y_k\sqrt{2019} = (674 + 15\sqrt{2019})^k$, where $k \geq 1$. Flip signs to get all the negative solutions (and $k = 0$ gets the trivial solution).

Problem 8.2: Albania MO

Find all solutions (m, n) of **positive** integers to the equation $1000m^2 + m = 999n^2 + n$.

Solution. Let $G = \gcd(m, n)$, with $m = G \cdot m_1$ and $n = G \cdot n_1$. The equation becomes

$$m_1(1000G \cdot m_1 + 1) = n_1(999G \cdot n_1 + 1).$$

Hence, there must exist some integer K (obviously coprime with G) such that $1000G \cdot m_1 + 1 = Kn_1$ and $999G \cdot n_1 + 1 = Km_1$. Subtracting gives $G(1000m_1 - 999n_1) = K(n_1 - m_1)$, and since $\gcd(G, K) = 1$, we have $G \mid m_1 - n_1$, and so there must exist some ℓ with $n_1 = m_1 + \ell \cdot G$ (with $\ell > 0$ clearly because $n_1 > m_1$ from the equations). Hence:

$$G(1000m_1 - 999(m_1 + \ell \cdot G)) = K((m_1 + \ell \cdot G) - m_1) \implies m_1 = 999\ell \cdot G + K\ell.$$

Since $\ell \mid m_1$, we must also have $\ell \mid n_1$ (since $n_1 = m_1 + \ell \cdot G$); however, since they are coprime, we must have $\boxed{\ell = 1}$. This gives us $n_1 = m_1 + G$, and so

$$m_1 = 999G + K, \quad n_1 = 1000G + K.$$

Plugging into $999G \cdot n_1 + 1 = Km_1$ gives

$$999G \cdot (1000G + K) + 1 = K(999G + K) \implies K^2 - 999000G^2 = 1,$$

which is a Pell equation with minimal solution $(K, G) = (1999, 2)$. Hence,

$$K + G\sqrt{999000} = \left(1999 + 2\sqrt{999000}\right)^k$$

for some positive k. It easily follows all solutions (m, n) to the above equation are of the form

$$(m, n) = (G(999G + K), G(1000G + K))$$

where K, G are defined as per above. □

Problem 8.3

Show that there are infinitely many positive integer solutions to the equation $a^2 + b^3 = c^4$.

Ok, I'll admit there are numerous methods of solving this problem, many of which don't even require the Pell's equation at all[2]. However, there is indeed a pretty cool solution using Pell's:

Solution. Recall the well-known formula

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 \implies \left(\frac{n(n-1)}{2}\right)^2 + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

and thus it suffices to show that $n(n+1)/2 = k^2$ is a perfect square for infinitely many n (which would yield the required a, b, c). It turns out this is a simple Pell's equation:

$$\frac{n(n+1)}{2} = k^2 \iff (2n+1)^2 - 2(2k)^2 = 1$$

which can be solved using standard methods. □

[2] For example, $a = s(r^4 - s^2)^4, b = (r^4 - s^2)^3, c = r(r^4 - s^2)^2$ works for suitable $r, s \in \mathbb{Z}$

9 Problems

Problem 9.0. Call a number n *fat* if for any $p \mid n$, we also have $p^2 \mid n$. Does there exist a pair of consecutive fat numbers, both exceeding 2019^{2019}? **Problem 9.1.** Find an explicit formula for the general solution of $x^2 - 4xy + y^2 = 1$.

Problem 9.2. Find an explicit formula for the general solution (m, n) of $m + (m+1) + \cdots + n = mn$ and $m < n$.

Problem 9.3. Find an explicit formula for the general solution (m, n) of $(2m^2 - 1)^2 - 8n^2 = 1$.

Problem 9.4. Find an explicit formula for the general solution (x, y) of $3x^2 = y^2(y^2 - 1)$.

Problem 9.5. Define the number $x = \displaystyle\sum_{k=0}^{n} 2^k \binom{2n+1}{2k+1}$. Show that $x^2 + 1 = 2a(a+1)$ for some integer a.

Problem 9.6. Prove that the Negative Pell equation $x^2 - py^2 = -1$ always has a solution whenever $p \equiv 1 \pmod 4$ is a prime number.

Problem 9.7. The sequence $\{a_n\}_{n \geq 1}$ is defined by

$$a_1 = 1, a_{n+1} = 2a_n + \sqrt{3a_n^2 + 1}.$$

Show that a_n is an integer for every n.

Problem 9.8. Suppose N is not a perfect square. Show that for any $k \geq 1$, there exists infinitely many pairs (x, y) such that $x^2 - Ny^2 = 1$ and $k \mid y$.

Problem 9.9. Show that any solution (x, y) to the equation $x(x+y) = y^2 + 1$ is in the form (F_{2n-1}, F_{2n}) where F denotes the Fibonacci number.

Problem 9.10. Prove that the only positive integer solution (a, b) to $5^a - 3^b = 2$ is $a = b = 1$.

Problem 9.11. (EGMO 2016/6) Let S be the set of all positive integers n such that n^4 has a divisor in the range $n^2+1, n^2+2, ..., n^2+2n$. Prove that there are infinitely many elements of S of each of the forms $7m, 7m+1, 7m+2, 7m+5, 7m+6$ and no elements of S of the form $7m+3$ and $7m+4$, where m is an integer.

Problem 9.12. (China TST 2013) For a positive integer n, define $f(n) = \min\limits_{m \in \mathbb{Z}} \left|\sqrt{2} - \frac{m}{n}\right|$. Let $\{n_i\}$ be a strictly increasing sequence of positive integers. C is a constant such that $f(n_i) < \dfrac{C}{n_i^2}$ for all $i \in \{1, 2, \ldots\}$. Show that there exists a real number $q > 1$ such that $n_i \geqslant q^{i-1}$ for all $i \in \{1, 2, \ldots\}$.

Problem 9.13. (Iran) Prime $p = n^2 + 1$ is given. Find the sets of solutions to the below equation:

$$x^2 - (n^2+1)y^2 = n^2.$$