Sequences

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Last time when I took this lecture, someone complained that my problem set was too hard and I didn't give enough time to solve the questions. I've made the sheet slightly easier than last time (maybe).

Rules for this lecture

- Solve the questions and alert me when you have done so.
- In the following questions, N is meant to be any given positive integer.

1. Let $(a_n)_{n\geq 1}$ and $(p_n)_{n\geq 1}$ be two sequences of positive integers with $a_1 \geq 2$, such that p_n is the smallest prime divisor of a_n, and $a_{n+1} = a_n + \frac{a_n}{p_n}$ for all $n \geq 1$. Show that there exists $M \in \mathbb{N}$ such that $a_{n+3} = 3 \times a_n$ for all $n \geq M$.

2. Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two infinite arithmetic positive integer sequences. Suppose that the set

$$S \stackrel{\text{def}}{=} \{(i,j) \in \mathbb{N} : \ 0 \leq j - i \leq 2021 \text{ and } a_i \mid b_j\}$$

has infinite size. Show that $\forall i \geq 1, \ \exists j \geq 1$ such that $(i,j) \in S$.

3. Suppose $\mathbb{N} = \bigcup_{j=1}^{N} S_j$ where S_j is an infinite arithmetic progression with common difference d_j. Show that there is a unique $i \leq N$ such that $\prod_{j\neq i} d_j \in S_i$.

4. Let $(a_n)_{n\geq 0}$ be a sequence of nonzero integers, and denote $P_n(x) \stackrel{\text{def}}{=} a_0 + a_1 x + \cdots + a_n x^n$. Show that there exists $n \geq 2021$ such that all real roots of $P_n(x)$ have modulus less than $2 + \frac{1}{2021}$.

5. Suppose $S \subset \mathbb{N}$ with $3 \leq |S| < \infty$. Show that it is possible to label $S = \{a_1, \ldots, a_N\}$ with $N = |S|$, such that a_i does **not** divide $a_{i-1} + a_{i+1}$ for each $2 \leq i \leq N$.

6. Let $n > 1$ and $(a_i)_{0\leq i<n}$ be positive integers, such that $a_i \equiv i \pmod{n}$ for all $i < n$. Show the existence of an infinite sequence $(b_i)_{i\in\mathbb{N}}$, such that $0 \leq b_i < n$ and $\sum_{k\in\mathbb{N}} a_{b_k} n^{-k} \in \mathbb{N}$.

7. Let $(a_n)_{n\geq 1}$ be a sequence of positive integers such that:

   - $\gcd(a_i, a_j) \leq \gcd(i,j)^{2021}$ for all $i, j \geq 1$
   - $0 \leq a_n - n \leq 2021$ for all $n \geq 1$.

   Show that $\exists M \in \mathbb{N}$ such that $f(n) = n, \ \forall n \geq M$.