# Colourings and Invariants/Monovariants

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## 1 Introduction

As the name implies, an *invariant* is something which doesn't change. They are a powerful, simple and rigorous method mainly used to show that something is impossible. (Note: to show that something is possible, we just need to provide one example, but to show that something is impossible, we must show that every single case will not produce a solution.

Invariants can come up in numerous shapes and forms, and so they often require a bit of exploring before they can be found. This set of notes will explore some common types of invariants using some examples to illustrate such ideas.

## 2 Numerical Invariants (Whiteboard Questions)

**Problem 2.1**

On a whiteboard are the numbers $1, 2, \ldots, 100$. In each second, Andy selects two numbers $a, b$ already on the board, removes them and writes the number $a + b$ on the board.

(a) How many moves does it take for exactly one number to remain on the board?

(b) Prove that no matter how Andy does his moves, the final number is always the same (and find this number).

*Proof.* Intuitively speaking, every time we do an operation, the number of numbers on the board decreases by exactly one since we delete two numbers and write one. This means by the time we get to only 1 number remaining, we must have done $100 - 1 = 99$ operations.

If we operate $a$ and $b$, then we get $a+b$. Then, if we operate the numbers $a+b$ and $c$, then we get $a+b+c$; in short, it is obvious that the **sum** of all numbers on the board never changes (is **invariant**) regardless of the operation. It follows that the final number will always be equal to $1 + 2 + \cdots + 100 = 5050$.

**Problem 2.2**

In a whiteboard, the numbers $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{2020}$ are written. At any stage, James may pick two numbers $a, b$ on the board, delete them and write the number $ab+a+b$. He repeats this until there is only one number remaining. What is this number?

*Solution.* This doesn't look as simple as the previous example, since the invariant isn't immediately obvious to us. However, notice that $ab + a + b$ looks very much like $(a+1)(b+1)$, except it's missing a $-1$. Hence the operation is $a, b \longrightarrow (a+1)(b+1) - 1$, or

$$(a+1) - 1, (b+1) - 1 \longrightarrow (a+1)(b+1) - 1.$$

It now becomes clear that operating any three numbers $a, b, c$ in any order then gives $(a+1)(b+1)(c+1) - 1$, and so on. Hence, the final number on the board will always be equal to

$$(1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\dots\left(1+\frac{1}{2020}\right) - 1 = \frac{2}{1} \times \frac{3}{2} \times \frac{4}{3} \times \cdots \times \frac{2021}{2020} - 1 = 2021 - 1 = 2020.$$

**Problem 2.3.** Same question as **Problem 2.1**, except that Andy writes the number $ab$ on the board instead of $a + b$ on every move.

**Problem 2.4.** Dmitry the Magician has a set of 2020 magical rods of length $1, 2, \dots, 2020$. Every night, he picks two rods of lengths $a, b$ and holds them at right angles to each other. He then exclaims "hypotenate!", which makes the two rods instantly disappear and are replaced by a third rod that would have completed the triangle in the air. After many nights, Dmitry has only one rod remaining. Find the length of this rod.

*For clarity, the operation is taking two rods length $a, b$ and replacing them with $\sqrt{a^2 + b^2}$.*

**Problem 2.5.** Thanom has the numbers $1, 2, \dots, 2020$ written on the whiteboard. In each move, she takes two numbers $a, b$, and replaces them by the number $ab - a - b + 2$. She repeats this until only one number remains. What is this final number?

**Problem 2.6.** Andy is back! On the whiteboard are the numbers $1, 2, 3, \dots, 10$. This time, he selects two numbers $a$ and $b$ and replaces them with $\dfrac{4a+3b}{5}$ and $\dfrac{4b-3a}{5}$. If at some point all the numbers are equal, Andy will buy everyone chocolates. Will everyone get chocolates?

**Problem 2.7.** On the whiteboard is one 3, two 7's, four 15's and so on until sixty-four 255's (so basically for each $k \leq 7$, the number $2^{k+1} - 1$ is written $2^{k-1}$ times). We are allowed to take any two numbers $a, b$ and replace them with $\frac{ab-1}{a+b+2}$. Find the last number remaining.

# 3 Parity/Modulo Invariance

**Problem 3.1**

On a sheet of paper are the numbers $1, 2, \dots, 100$. Andy picks two numbers $a, b$ and replaces them by $|a - b|$. He repeats this until there is only one number remaining. Prove that the final number must be even.

Unfortunately, this question isn't quite as simple as the $a + b$ variant since depending on how Andy chooses his moves, the final answer may not be the same (although despite that, it's always even). How do we address this?

*Proof.* We consider the number modulo 2 (which makes sense because we want to show the final number is even). It is trivial to check that $|a - b| \equiv a + b \pmod 2$, regardless of whether $a > b$ or $a < b$.

The operation of the problem then becomes the following. Andy has the numbers $1, 2, \dots, 100$ on the sheet of paper, and on each turn he takes $a, b$ and replaces them with a number with the same parity as $a + b$. It becomes obvious that the final number has the same parity as $1 + 2 + \cdots + 100 = \frac{100 \cdot 101}{2}$, which is an even number. It follows that the final number must always be even.

**Problem 3.2**

In the planet of Dmitros, there are $3n$ chameleons for some $n \geq 1$: $n$ reds, $n$ blues and $n$ yellows. In each day, it is possible for exactly one pair of chameleons of different colours to merge and turn into a chameleon of the third colour (so for example, a red and blue may combine to become a single yellow). This operation continues until no more such moves can be made. Is it possible that exactly one chameleon remains?

*Proof.* Let's consider the number of chameleons of each colour modulo 2. It is quite easy to show that in any of the three types of possible moves, the number of red, blue and yellow chameleons at any step must have the same parity. This cannot occur for the final configuration, since the numbers in (1,0,0) cannot all have the same parity.

**Problem 3.3.** The numbers $1, 2, \ldots, 100$ are written on the whiteboard. Andy may choose three numbers, say $a, b, c$, and replace them with the number $|a + b - 2c|$. He keeps doing this until exactly two numbers remain. Is it possible that the sum of the two final numbers is divisible by 3?

**Problem 3.4.** Many handshakes are exchanged on the first day of class. To attempt to make things interesting, the following rule is proposed. If a person has finished exchanging an odd number of handshakes, he or she must be wearing a party hat; otherwise, they must not be wearing a hat. Prove that at any moment there is an even number of party hats being worn in the room.

**Problem 3.5.** Consider an $8 \times 8$ chessboard with regular colouring. In a move we may take one row or column, and invert the colour of each square (each black square becomes white and each white square becomes black). Is it possible to reach a situation where there is exactly one black square?

**Problem 3.6.** Andy has a single pile of 100 chocolates in front of him. In any move, he may select any pile with more than 2 chocolates, eat one chocolate and then split the remaining pile into two smaller piles. Would it be possible for Andy to reach a situation where every pile has exactly three chocolates?

**Problem 3.7.** In the planet of Jamos are 10 red, 15 blue and 20 yellow chameleons. Whenever two chameleons of different colours meet, they both change to the third colour. Is it possible that at some point in time all the chameleons are of the same colour?

# 4 Chessboard/Colouring Invariance

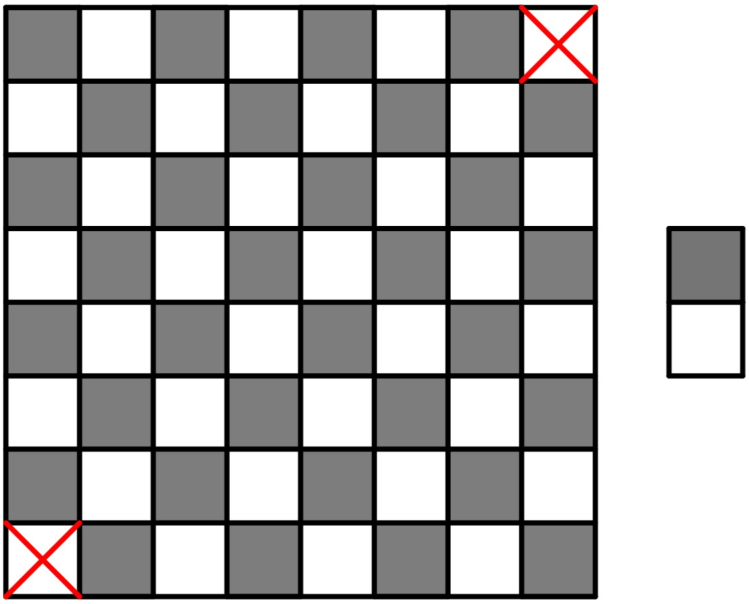
**Problem 4.1**

Consider an $8 \times 8$ chessboard with the top-right and bottom-left corners removed. Is it possible to tile the resultant chessboard with $1 \times 2$ dominoes?

*Solution.* No, it is not possible to do so.

Consider the following chessboard colouring (on the following page) with the two missing squares on white squares: it is easy to see that any domino must cover exactly one black square and one white square (as the blacks and whites are adjacent).

However, simply counting the number of black and white squares gives that there are 32 black squares and 30 white squares. Since they are the same, it follows that the remaining board cannot be tiled with dominoes with no spaces.



**Problem 4.2.** Consider the following table:

| | | | |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 |
| 1 | $-1$ | 1 | 1 |

In one move, we may switch the signs of all the numbers in a row, column or a parallel to one of the diagonals (including the corner squares). Show that there will always be at least one $-1$.

**Problem 4.3.** Extend **Problem 4.1** by finding all $m, n$ for which it is possible to tile a $m \times n$ board with two missing opposite corner squares using only dominoes.

**Problem 4.4.** Arch-rivals Thanom and Thanos are playing a game on a board with an infinitely long line of squares. They take turns placing their symbol, either an $M$ or $S$ in the squares and the first person to have three of their symbols in a row wins. If Thanom goes first, does either player have a winning strategy?

**Problem 4.5.** Suppose we have an $m \times n$ chessboard coloured in the usual alternating way black and white. A move consists of switching the colour of every square in the same row, column or $2 \times 2$ square (so black goes to white and vice versa). For what $m, n$ is it possible to end up with exactly one black square?

**Problem 4.6.** We are tiling a $8 \times 8$ chessboard with $1 \times 3$ tiles, such that exactly one square is left untiled. Find all possible positions of this square.

**Problem 4.7.** On every square of a $4 \times 4$ grid is a coin. Precisely one coin is heads up. A move consists of flipping all coins in a row, column or line parallel to the diagonal (in particular, this means we can flip any corner square by itself). Find all positions of this heads coin if after a finite number of moves it is possible for all coins to be oriented the same way.

# 5 Monovariance

**Problem 5.1**

There are 1,000,000 people distributed amongst the rooms of a 123,456-room mansion. Each minute, as long as not all the people are in the same room, one person walks from one room to a different room with **at least as many people** as his initial room. Prove that eventually all the people are gathered in one room.

This result seems fairly plausible, since the most heavily populated rooms will get even more popular, while the rooms with fewer people will become less popular, so we should expect that all the people would gather into the largest room. To show that this case, we use a **monovariant.**

*Proof.* Let $S$ denote the **sum of squares** of the number of people in each of the rooms. We claim that as by each operation, $S$ must strictly increase. Let's suppose someone walks from a room of $n$ people (himself included) to somewhere with $m$ people. Then we have $m \geq n$, and $S$ changes by

$$(m+1)^2 + (n-1)^2 - [m^2 + n^2] = (2m+1) + (-2n+1) = 2(m-n) + 2 > 0,$$

since $m - n \geq 0$ and $2 > 0$.

Now, notice that $S$ cannot keep increasing forever since it's trivially bounded by the square of the number of people in the mansion $(1,000,000^2)$. It follows that eventually $S$ cannot keep increasing. However, as long as there are two nonempty rooms in the mansion, $S$ can always increase. It follows that when this procedure terminates, all the people will be in the same room. □

**Problem 5.2**

A $m \times n$ grid of real numbers is given. If there exists a row or column with a negative sum, we may switch the sign of all the numbers in that row/column (which obviously makes the sum positive). Suppose this operation is iterated. Prove that at some point, there exist no rows or columns with negative sums.

*Proof.* Intuitively, this makes sense since after we flip the numbers, the total sum of numbers increase, and so it seems plausible that the sum cannot keep increasing.

Let $S$ be the sum of all the numbers in the grid; it's clear that $S$ has at most $2^{m \times n}$ possible values since any combination of signs of numbers in the $m \times n$ grid will produce at most one legal value of $S$. It follows that $S$ has finitely many possible values, so the operation cannot be done infinitely (since any operation strictly increases the sum).

When this operation terminates, suppose by contradiction that there exists a row or column with a negative sum. Then we can strictly increase $S$ by flipping this row or column, which is a contradiction. It follows that when the operation terminates, none of the rows or columns can have a negative sum, and thus they are all nonnegative.

**Problem 5.3.** Three amoebae are situated at the points $(0,0), (0,1), (1,0)$ on the Cartesian plane. Each second, one of the amoebae reproduces by splitting into two amoebae, one to the right and one north. No two amoebae may occupy the same position. Is it possible to vacate the original three points in a finite amount of time?

**Problem 5.4.** On each integer coordinate $(x, y)$ on the Cartesian plane with $y \leq 0$, a checker is placed. You may alter this configuration by jumping a checker over an adjacent checker (provided this new square is empty before the move), and removing the checker that was jumped over. Is it ever possible for the checker to reach the point $(0, 5)$?

**Problem 5.5.** Consider 2009 cards, each having one gold side and one black side, lying in parallel on a long table. Initially all cards show their gold sides. Two players, standing by the same long side of the table, play a game with alternating moves. Each move consists of choosing a block of 50 consecutive cards, the leftmost of which is showing gold, and turning them all over, so those which showed gold now show black and vice versa. The last player who can make a legal move wins.

(a) Does the game necessarily end?

(b) Does there exist a winning strategy for the starting player?

# 6 Geometric Invariance

**Problem 6.1**

A biologist has a $10 \times 10$ array of cells, 9 of which are infected. If one healthy cell has 2 infected neighbours, then it too will become infected. If this process repeats itself, can the infection spread to every cell?

*Solution.* For any configuration, we consider the **perimeter** of all the infected squares, as the number of unit edges for which an infected square touches a non-infected square. Let $P$ be the perimeter: we show that $P$ never increases.

For a square to become infected, it must be adjacent to at least two other infected squares. However, when this square becomes infected, the total perimeter must either stay the same or decrease, since at most two other edges of the perimeter form, while exactly two edges are destroyed with the new infected cell.

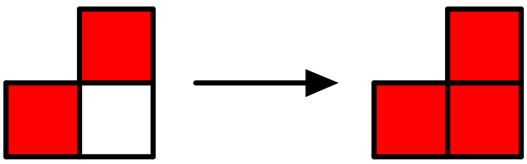


Figure 1: The perimeter stays the same in this case.

Initially, the perimeter of the shape is at most $4 \times 9 = 36$, since each of the four edges of the square contribute at most one edge to the perimeter. If it were possible for the entire grid to be infected, the perimeter must be exactly 40. However, this is impossible since the perimeter must never increase. It follows that the infection cannot spread to every cell.

**Problem 6.2.** Given a polygon $\mathcal{P}$, a new polygon $\mathcal{Q}$ can be created using the following procedure. The polygon $\mathcal{P}$ is divided into two polygons by a straight cut, then one of the resulting polygons is flipped over and joined back to the other polygon along the edge created by the cut. This procedure is only allowed if the two polygons do not overlap after being joined. Is it possible to transform $\mathcal{P} = \square$ into $\mathcal{Q} = \triangle$ by applying the procedure a finite amount of times?

**Problem 6.3.** An $n \times n$ square is tiled with $1 \times 1$ tiles, some of which are coloured. Thanom is allowed to colour in any uncoloured tile that shares edges with at least three coloured tiles. She discovers that by repeating this process all tiles will eventually be coloured. Show that initially there must have been more than $\frac{n^2+2n}{3}$ coloured tiles.

# 7 More Problems

**Problem 7.1.** Suppose that all the camp attendees are sitting around in a circle with a pile of 2015 chocolates in the middle. Each person takes some number of chocolates from the pile, and by coincidence each person either has 6 less or twice as many chocolates as the person to their right. Prove that not all the chocolates were taken.

**Problem 7.2.** In a game a tetris, tetrominos (a shape made up of 4 unit squares joined together) continuously fall one by one, in a screen of dimensions $12 \times 20$, such that if each unit square in a particular row of the screen is occupied by the squares of the fallen tetrominos, then all these squares in this row are removed, and all squares in rows above it move down one unit. Is it possible to have a situation where there are exactly 30 squares on the screen?

**Problem 7.3.** On a cube, there are seven vertices marked 0 and one marked 1. It is permitted to add 1 to any two neighbouring vertices (that is, two vertices connected by an edge). Is it possible that all the numbers are divisible by 3 after a finite number of steps?

**Problem 7.4.** Andy writes the numbers $1, 2, 3, \cdots, 10$ on a blackboard. In a particular move, he replaces two numbers $a$ and $b$ with the number $\dfrac{ab}{a+b}$. This is repeated until there is only one number. Andy will buy you a chocolate if the final fraction is greater than $\dfrac{1}{2}$. Will Victor buy you a chocolate?

**Problem 7.5.** Players $A$ and $B$ play a game where there is a pile of 1000 chocolates and at every move a player subtracts $n$ chocolates where $n$ is some power of 2 (including $2^0 = 1$) under the condition that a player cannot subtract more chocolates than are present at any given stage. The player to take the last chocolate wins. If player $A$ starts the game and if both players play optimally, who would win the game?

**Problem 7.6 [Fifteen puzzle]** The fifteen puzzle is a game in which 15 of the 16 squares of a $4 \times 4$ frame are filled with numbered sliding pieces, leaving one space in which to slide one piece at a time. Is it possible to begin with the configuration below and finish with the pieces numbered 14 and 15 swapped?

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 5 | 6 | 7 | 8 |
| 9 | 10 | 11 | 12 |
| 13 | 14 | 15 | |