Inequalities

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1 What are inequalities?

In high school you would have seen inequalities like:

Find all real x such that $2x + 5 \geqslant 11$.

However, Olympiad inequalities are very different from this, and generally require you to prove that a certain statement holds where the variables involved can take on a huge range of different values, such as:

Show that $a^2 + b^2 + c^2 \geqslant ab + bc + ca$ for all real numbers a, b, c.

This means that we have to *show* that regardless of which three real numbers a, b, c we choose, $a^2+b^2+c^2 \geqslant ab+bc+ca$ holds. It is not enough to simply find one triple (a, b, c) that satisfies $a^2 + b^2 + c^2 \geqslant ab+bc+ca$ – we need to show that this is true for **all** triples of reals! So how do we do this?

2 Squares of real numbers are nonnegative

We know that $x^2 \geqslant 0$ for all real x. This is very useful as it allows us to prove lots of things e.ge $a^2+b^2 \geqslant 2ab$ for all reals a, b, as this is equivalent to $(a - b)^2 \geqslant 0$. We can sneakily apply the idea that the squares of reals are nonnegative to solve problems. Let's take the problem mentioned in the introduction for example. It looks pretty similar to $a^2 + b^2 \geqslant 2ab$, just that we have a few other terms floating around. It also looks like $b^2 + c^2 \geqslant 2bc$ and $c^2 + a^2 \geqslant 2ca$ should be useful, as these terms appear in the inequality as well, with the 'mixed' ab, bc, ca terms on the RHS and the 'squared' terms a^2, b^2, c^2 on the LHS. Now we want a way to combine all these inequalities together, which we can do by adding them together:

$$(a^2 + b^2) + (b^2 + c^2) + (c^2 + a^2) \geqslant 2ab + 2bc + 2ca.$$

We now collect like terms on the LHS and divide both sides by 2, and we're done!

Problem 2.1. Show that for any $x > 0$, the inequality $x + \frac{1}{x} \geqslant 2$ holds, and find all cases of equality.

Problem 2.2. Prove that if $x \geqslant 0$ and a is real, then $x(a - x) \leqslant \frac{a^2}{4}$.

Problem 2.3. For an arbitrary real number T, show that $T^4 - T + \frac{1}{2} \geqslant 0$.

Problem 2.4. For all $x > 0$, show that $\dfrac{x^2 + 2}{\sqrt{x^2 + 1}} \geqslant 2$.

Problem 2.5. Prove that if $a > 0$, then $\dfrac{1 + a^2}{1 + a^4} \leqslant \dfrac{1}{a}$.

Problem 2.6. For all reals $x_1, x_2, \ldots, x_n$ such that $x_k \geqslant k^2$ for each $1 \leqslant k \leqslant n$, show that

$$\frac{x_1 + x_2 + \cdots + x_n}{2} \geqslant \sqrt{x_1 - 1^2} + 2\sqrt{x_2 - 2^2} + \cdots + n\sqrt{x_n - n^2}.$$

Problem 2.7. Prove that for all real numbers x, y, z, we have $x^2 + y^2 + z^2 - xy - yz - zx \geqslant \frac{3}{4}(x - y)^2$.

3 Power Mean Inequalities

3.1 AM–GM Inequality

The AM-GM Inequality is one of the most important and useful inequalities. Here it is:

Let $a_1, a_2, \ldots, a_n$ be n nonnegative reals. Then

$$\boxed{\frac{a_1 + a_2 + \cdots + a_n}{n} \geqslant \sqrt[n]{a_1 a_2 \ldots a_n}}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Now this may seem totally random, but it is in fact very intuitive. When you think about it, the AM-GM Inequality is basically a theorem about maximising the product of nonnegative real numbers with a fixed sum. Now you all have probably seen the problem stating the minimum area of a rectangle with fixed perimeter occurs when it is a square – this is basically the $n = 2$ case of the AM-GM Inequality! Also note that the $n = 2$ and $n = 3$ cases follow from the factorisations:

$$a^2 + b^2 - 2ab = (a - b)^2 \geqslant 0,$$

$$a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a + b + c)((a - b)^2 + (b - c)^2 + (c - a)^2).$$

The idea you should be getting is that if you 'push' two numbers closer together, while keeping their sum constant, their product increases.

3.1.1 Proof of AM–GM

There are, of course, several different proofs of the AM–GM Inequality. However, the most instructive proof appears using Mathematical Induction.

Step 1: Base Case. For $n = 1$, the statement is an equality; for $n = 2, 3$, the proof follows from the factorisation as per above. Hence, the statement holds for multiple base cases.

Step 2: Inductive Step (1) We show that if the AM–GM Inequality holds for some $n = k$, it must also hold for $n = 2k$. To do this, we have

$$\begin{aligned}
\frac{a_1 + a_2 + \cdots + a_{2k}}{2k} &= \frac{1}{2}\left(\frac{a_1 + a_2 + \cdots + a_k}{k} + \frac{a_{k+1} + a_{k+2} + \cdots + a_{2k}}{k}\right) \\
&\geqslant \frac{1}{2}\left(\sqrt[k]{a_1 a_2 \cdots a_k} + \sqrt[k]{a_{k+1} a_{k+2} \cdots a_{2k}}\right) \quad \text{(by Inductive Hypothesis)} \\
&\geqslant \sqrt{\sqrt[k]{a_1 a_2 \cdots a_k} \times \sqrt[k]{a_{k+1} a_{k+2} \cdots a_{2k}}} \quad \text{(Two variable AM–GM)} \\
&= \sqrt[2k]{a_1 a_2 \cdots a_{2k}}.
\end{aligned}$$

Hence, the statement is also true for $n = 2k$, if it is true for some $n = k$.

Step 3: Inductive Step (2) We show that if the AM–GM Inequality holds for some $n = k$, it must also hold for $n = k - 1$. By the assumption, we have

$$\frac{a_1 + a_2 + \cdots + a_k}{k} \geqslant \sqrt[k]{a_1 a_2 \cdots a_k}.$$

Hence, this inequality still remains true when we plug in $a_k = \frac{a_1 + \cdots + a_{k-1}}{k-1}$ (since this inequality is true for **all** positive reals $a_1, \ldots, a_k$):

$$\frac{a_1 + \cdots + a_{k-1} + \frac{a_1 + \cdots + a_{k-1}}{k-1}}{k} \geqslant \sqrt[k]{a_1 a_2 \cdots a_{k-1}\left(\frac{a_1 + \cdots + a_{k-1}}{k-1}\right)}$$

$$\iff \frac{a_1+\cdots+a_{k-1}}{k-1} \geqslant \sqrt[k]{a_1a_2\cdots a_{k-1}\left(\frac{a_1+\cdots+a_{k-1}}{k-1}\right)}$$

and dividing both sides by $\sqrt[k]{\frac{a_1+\cdots+a_{k-1}}{k-1}}$ and raising both sides to the power of $k/(k-1)$ yields the AM–GM Inequality for $n = k-1$.

Step 4: Conclusion. By **Step 2,** we know that the AM–GM Inequality must hold for arbitrarily large n (since $n = 2$ works, and so must $n = 4, 8, 16, \ldots, 2^N, \ldots$). By **Step 3,** we know that if the inequality is true for some n, it must also be true for $n-1$. Since we have the inequality true for arbitrarily large n, it must also be true for every integer smaller than this arbitrarily large n. It follows that the AM-GM Inequality is true for all positive integers n. $\square$

3.2 GM–HM Inequality

Let $a_1, a_2, \ldots, a_n$ be positive reals. Then

$$\boxed{\sqrt[n]{a_1a_2\ldots a_n} \geqslant \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}.}$$

The RHS is the **Harmonic Mean** of $a_1, a_2, \ldots, a_n$.

3.2.1 Proof of GM–HM

The GM–HM Inequality happens to be equivalent to the AM–GM Inequality: rewrite the inequality in the form

$$\frac{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}{n} \geqslant \sqrt[n]{\frac{1}{a_1}\cdot\frac{1}{a_2}\cdot\cdots\cdot\frac{1}{a_n}},$$

and at this point the inequality becomes the AM–GM Inequality of the numbers $1/a_1, 1/a_2, \ldots, 1/a_n$. $\square$

3.3 QM–AM Inequality

Let $a_1, a_2, \ldots, a_n$ be positive reals. Then

$$\boxed{\sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}} \geqslant \frac{a_1+a_2+\cdots+a_n}{n},}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

The LHS is the **Quadratic Mean** of $a_1, a_2, \ldots, a_n$.

QM-AM can be proved by expanding and applying AM-GM and Sum of Squares. If you're having trouble seeing how this is done, expand for $n = 2, 3, 4$ and carefully observe what happens. Alternatively, it is a simple consequence of the Cauchy-Schwarz Inequality (discussed below).

3.4 Putting it all together...

The inequalities together give the QM–AM–GM–HM Inequality: Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Then the following chain of inequalities hold:

$$\sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}} \geqslant \frac{a_1+a_2+\cdots+a_n}{n} \geqslant \sqrt[n]{a_1a_2\ldots a_n} \geqslant \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}.$$

This can be generalised even further. For given positive real numbers $a_1, \ldots, a_n$, we define the **Power Mean Function** $M : \mathbb{R} \to \mathbb{R}$ as:

$$M(r) = \begin{cases} \left(\dfrac{a_1^r + a_2^r + \cdots + a_n^r}{n} \right)^{1/r} & r \neq 0 \\ \sqrt[n]{a_1 a_2 \cdots a_n} & r = 0. \end{cases}$$

Then, the **Power Mean Inequality** states that $M(r)$ is an increasing function, and furthermore it is always strictly increasing unless $a_1 = a_2 = \cdots = a_n$. In particular, the cases for $r = 2, 1, 0, -1$ yields the QM–AM–GM–HM Inequality as per above (check this!)

Problem 3.1. Prove that if a, b, c are positive real numbers, then $(a+b)(b+c)(c+a) \geqslant 8abc$.

Problem 3.2. For positive real numbers $x_1, x_2, \ldots, x_n$, prove that $\dfrac{x_1}{x_2} + \dfrac{x_2}{x_3} + \cdots + \dfrac{x_n}{x_1} \geqslant n$.

Problem 3.3. Let $a, b, c > 0$. Prove that $\dfrac{a+b}{c} + \dfrac{b+c}{a} + \dfrac{c+a}{b} \geqslant 6$.

Problem 3.4. Prove that if a, b, c are real, then

$$a^4(1+b^4) + b^4(1+c^4) + c^4(1+a^4) \geqslant 6a^2b^2c^2.$$

Problem 3.5. Let $a, b, c > 0$. Show that $(ab)^2 + (bc)^2 + (ca)^2 \geqslant abc(a+b+c)$.

Problem 3.6. Let $a, b, c > 0$. Show that $\dfrac{a+b+c}{abc} \leqslant \dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2}$.

Problem 3.7. Let $x, y, z > 0$. Prove that

$$\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} \geqslant \max\left\{ \frac{x}{y} + \frac{y}{z} + \frac{z}{x}, \frac{y}{x} + \frac{z}{y} + \frac{x}{z} \right\}.$$

Problem 3.8. Consider positive reals a, b, c such that $a^3b^2c = 1$. Find the minimum possible value of $2a + 7b + c$.

Problem 3.9. Prove that for all reals a, b, c, we have $a^4 + b^4 + c^4 \geqslant a^2b^2 + b^2c^2 + c^2a^2 \geqslant abc(a+b+c)$.

Problem 3.10. Prove that for positive reals a, b, c, $a^5 + b^5 + c^5 \geqslant abc(a^2 + b^2 + c^2)$.

4 Cauchy-Schwarz

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be reals. Then

$$\boxed{(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geqslant (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2.}$$

While this may look random, it's not too bad. This should be true as all the stronger terms are on the LHS, and the mixed (weaker) terms are on the RHS. Expand this for small n to see how it decomposes into a sum of squares!

If we let $a_i = x_i/\sqrt{y_i}$ and $b_i = \sqrt{y_i}$ for positive reals x_i, y_i, we obtain a version of Cauchy-Schwarz that is very useful when dealing with fractions:

$$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + \cdots + \frac{x_n^2}{y_n} \geqslant \frac{(x_1 + x_2 + \cdots + x_n)^2}{y_1 + y_2 + \cdots + y_n}.$$

When we have lots of fractions and want to use this theorem, it helps if the numerators are squares, as otherwise we will have lots of square roots floating around. So it is often a good idea to multiply the numerator and denominator by something so that this happens.

4.1 Hölder's Inequality

The Cauchy–Schwarz Inequality be generalised for three or more sequences of real numbers, for example

$$(a_1^3 + a_2^3 + \cdots + a_n^3)(b_1^3 + b_2^3 + \cdots + b_n^3)(c_1^3 + c_2^3 + \cdots + c_n^3) \geqslant (a_1b_1c_1 + a_2b_2c_2 + \cdots + a_nb_nc_n)^3$$

and similar for higher number of variables. Similar to Cauchy, the stronger (unmixed) terms a_i^3 are on the LHS and the weaker (mixed) terms are on the RHS, and so we expect this inequality to be true as well (although it's more difficult to prove by straight expansion than Cauchy simply due to the increasing number of terms). For example, Hölder's Inequality allows us to prove the inequality

$$\left(\frac{1}{a(a+b)} + \frac{1}{b(b+c)} + \frac{1}{c(c+a)}\right)(a+b+c)((a+b)+(b+c)+(c+a)) \geqslant (1+1+1)^3 = 27$$

$$\implies \frac{1}{a(a+b)} + \frac{1}{b(b+c)} + \frac{1}{c(c+a)} \geqslant \frac{27}{2(a+b+c)^2}$$

which may be difficult to prove using the other methods above.

5 Smoothing / Jensen's Inequality

5.1 Convex Functions

A function $f : [a, b] \to \mathbb{R}$ is said to be **convex** if the second derivative $f''(x)$ exists is and is nonnegative. Geometrically (as shown below), for any two points $A = (\alpha, f(\alpha))$ and $B = (\beta, f(\beta))$ within its range, any point on the segment joining A and B lies above corresponding point on the graph (which means, the segment AB lies completely above the function). Hence, this gives us the inequality:

$$\lambda f(\alpha) + (1-\lambda)f(\beta) \geqslant f(\lambda\alpha + (1-\lambda)f(\beta))$$

whenever $0 \leqslant \alpha \leqslant 1$ (this can be seen geometrically by looking at the graph). For instance, the function $f(x) = x^2$ (as shown in the figure) is convex because

$$\begin{aligned}\lambda f(\alpha) + (1-\lambda)f(\beta) - f(\lambda\alpha + (1-\lambda)\beta) &= \lambda\alpha^2 + (1-\lambda)\beta^2 - (\lambda\alpha + (1-\lambda)\beta)^2 \\ &= \alpha^2\lambda(1-\lambda) + \beta^2(1-\lambda)\lambda - 2\alpha\beta \cdot \lambda(1-\lambda) \\ &= \lambda(1-\lambda) \cdot (\alpha - \beta)^2 \geqslant 0.\end{aligned}$$

(Alternatively, it's convex because $f''(x) = 2 > 0$.)

5.2 Jensen's Inequality

For convex functions, the inequality readily generalises:

(Weighted) Jensen's Inequality. Let $f : [a, b] \to \mathbb{R}$ be a convex function, and $a_1, a_2, \ldots, a_n \in [a, b]$ and some weights $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\sum_{k=1}^n \lambda_k = 1$. Then the following inequality holds:

$$\boxed{f(\lambda_1a_1 + \lambda_2a_2 + \cdots + \lambda_na_n) \leqslant \lambda_1f(a_1) + \lambda_2f(a_2) + \cdots + \lambda_nf(a_n).}$$

If the function is **concave** instead (the segment AB is **below** the function for any two points A, B within the domain), then this above inequality is flipped. Also notice that for $n = 2$, the inequality becomes the one discussed above: the generalised and $n = 2$ cases turn out to be equivalent to each other.

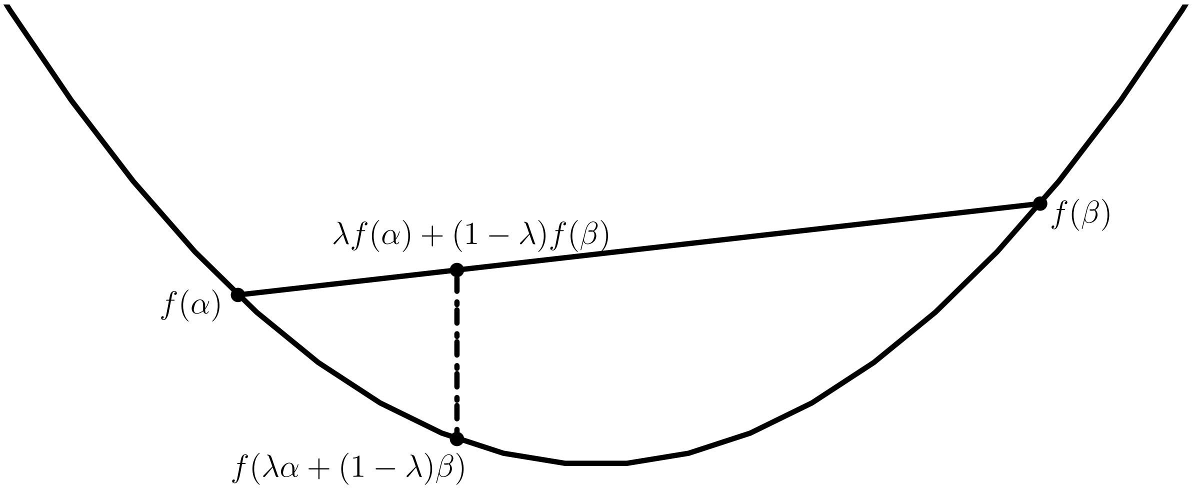

Figure 1: The function $y = x^2$ is convex.

In the case when we have $\lambda_k = 1/n$ for each k, we get **Jensen's Inequality** (unweighted):

$$\boxed{f\left(\frac{a_1+a_2+\cdots+a_n}{n}\right) \leqslant \frac{f(a_1)+f(a_2)+\cdots+f(a_n)}{n}.}$$

This form is also very useful: for instance, it can be used to prove the AM–GM inequality.

Problem 5.2.1. Use $f(x) = \log(x)$ to prove AM–GM Inequality using Jensen.

Problem 5.2.2. Show that if a, b are positive real numbers with $a + b = 1$, then

$$\left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2 \geqslant \frac{25}{2}.$$

Problem 5.2.3. Let $-1 < x_1, x_2, \ldots, x_n$ be real numbers. Show that

$$\frac{(1+x_1)(1+x_2)\ldots(1+x_n)}{1+x_1x_2\cdots x_n} \leqslant 2^{n-1}.$$

Problem 5.2.4. Prove that for positive real numbers a, b, c such that $a + b + c = 1$,

$$a\sqrt{2b+1} + b\sqrt{2c+1} + c\sqrt{2a+1} \leq \sqrt{2-(a^2+b^2+c^2)}.$$

Problem 5.2.5. Prove that for all real numbers a, b, c, the following inequality holds:

$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geqslant 1.$$

5.3 Smoothing

Suppose for some n-variable function f, you want to prove an inequality of the form

$$f(x_1, x_2, \ldots, x_n) \geqslant C$$

with an imposed constraint $x_1 + \cdots + x_n = 1$. Suppose further you know an equality case is when they are all equal, then you could try to make $f(x_1, \ldots, x_n)$ smaller by "moving the x_i closer together". Of course, you will need to prove a rigorous inequality to which you decrease the sum (without breaking any of the imposed conditions!), which could include (but not limited to):

$$f(x_1, x_2, \ldots, x_n) \geqslant f\left(\frac{1}{n}, x_1 + x_2 - \frac{1}{n}, x_3, \ldots, x_n\right),$$
$$f(x_1, x_2, \ldots, x_n) \geqslant f\left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}, x_3, \ldots, x_n\right).$$

However, when using inequalities of the second such form, you need to be careful to ensure the "halving" process stops after a certain number of moves, or give a reason as to why repeating the process infinitely many times will approach the value C.

Problem 5.3.1. Prove the AM–GM Inequality using smoothing.

Problem 5.3.2. Let $x_1, x_2, \ldots, x_n$ be real. Find the minimum value of $|x - x_1| + |x - x_2| + \cdots + |x - x_n|$.

Problem 5.3.3. For any three **distinct** real numbers x, y, z, let

$$E(x, y, z) = \frac{(|x| + |y| + |z|)^3}{|(x - y)(y - z)(z - x)|}.$$

Determine the minimum possible value of $E(x, y, z)$.

Problem 5.3.4. Let $n \geqslant 1$ be a given integer, and let $a_1, a_2, \ldots, a_n$ be real numbers such that $|a_i - a_j| \geqslant 1$ whenever $i \neq j$. Find the minimum value of $|a_1|^3 + |a_2|^3 + \cdots + |a_n|^3$.

Problem 5.3.5. For $x, y, z > 0$ and $x + y + z = 1$, prove that $0 \leqslant xy + yz + zx - 2xyz \leqslant \frac{7}{27}$.

Problem 5.3.6. Let $a, b, c \geqslant 0$ with $a + b + c = 1$. Show that $a^5 b + b^5 c + c^5 a \leqslant 5^5/6^6$.

Problem 5.3.7. Let $x, y, z \geqslant 0$. Show that $x^3 + y^3 + z^3 - 3xyz \geqslant \frac{3}{4}|x - y| \cdot |y - z| \cdot |z - x|$.

Problem 5.3.5. For $n \geqslant 2$, let $a_1, a_2, \ldots, a_n$ be positive real numbers such that

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \leqslant \left(n + \frac{1}{2}\right)^2.$$

Prove that $\max\{a_1, a_2, \ldots, a_n\} \leqslant 4 \min\{a_1, a_2, \ldots, a_n\}$.

6 Majorisation Inequalities

Denote two sequences of positive real numbers as $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_n)$. Then, we denote $\alpha \succ \beta$ if and only if:

1. The sums are equal, i.e. $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$
2. They are in decreasing order, i.e. $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$, and $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n$
3. For each $1 \leqslant k \leqslant n$, we have $a_1 + a_2 + \cdots + a_k \geqslant b_1 + b_2 + \cdots + b_k$.

For example:

- $(\frac{7}{3}, \frac{1}{3}, \frac{1}{3}) \succ (1, 1, 1)$ (the numbers may not necessarily be integers)
- $(a + b + c, a + b + c, 0) \succ (a + b, b + c, c + a)$ (the numbers may be variables)
- $(2019) \succ (2019)$ (trivial majorisation)
- $(3, 0, 0) \not\succ (2, 0, 0)$, since the sums are not the same
- $(4, 3, 1, 0) \not\succ (3, 3, 2)$, since the two sequences are not the same length
- $(2, 1, 0) \not\succ (3, 0, 0)$, since $2 < 3$.

6.1 Muirhead's Inequality

The statement of Muirhead's Inequality has far too many symbols and looks much more complicated than it really is. For sake of completeness, here it is anyway:

Muirhead's Inequality. Suppose that $(a_1, a_2, \ldots, a_n) \succ (b_1, b_2, \ldots, b_n)$, and $x_1, \ldots, x_n$ are any n **positive** real numbers. Then the following inequality holds:

$$\boxed{\sum_{\text{sym}} x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} \geqslant \sum_{\text{sym}} x_1^{b_1} x_2^{b_2} \ldots x_n^{b_n},}$$

where the symmetric sum $\sum_{\text{sym}}$ is taken over **all** $n!$ permutations of $x_1, x_2, \ldots, x_n$.

Now, what does this actually mean? Let's say we want to show the inequality

$$a^3b + a^3c + b^3a + b^3c + c^3a + c^3b \geqslant 2(a^2bc + b^2ca + c^2ab).$$

Of course, it is possible to apply AM–GM in a nontrivial way to solve this, but it is just a trivial consequence of Muirhead's Inequality: the left side represents the sequence $(3, 1, 0)$ (simply take a term, say a^3b, and count the degrees of a, b, c in descending order), while the right side represents $(2, 1, 1)$. However, trivially $(3, 1, 0) \succ (2, 1, 1)$, and so the inequality is done by Muirhead. $\square$

Muirhead is for SYMMETRIC inequalities only, NOT CYCLIC!! In the inequality

$$a^4b + b^4c + c^4d + d^4a \geqslant abcd(a + b + c + d),$$

it may be tempting to "use" Muirhead to say this is equivalent to $(4, 1, 0, 0) \succ (2, 1, 1, 1)$. However, this is not true as $(4, 1, 0, 0)$ represents the **symmetric** sum of a^4b, while the LHS is only cyclic (not symmetric).[1]

Problem 6.1.1. Let $a_1, \ldots, a_n$ be positive real numbers. Show, using Muirhead's Inequality, that

$$(a_1 + a_2 + \cdots + a_n)^2 \leqslant n(a_1^2 + a_2^2 + \cdots + a_n^2).$$

Problem 6.1.2. Prove that for arbitrary positive numbers a, b, c, the following inequality holds:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leqslant \frac{a^8 + b^8 + c^8}{a^3b^3c^3}.$$

Problem 6.1.3. Let n, k be natural numbers and let $a_1, a_2, \ldots, a_n$ be positive real numbers with sum 1. Prove that

$$a_1^{-k} + a_2^{-k} + \cdots + a_n^{-k} \geqslant n^{k+1}.$$

6.2 Karamata's Inequality

Let $f : [a, b] \to \mathbb{R}$ be a convex function, and $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_n)$ in the usual fashion. Suppose $\alpha \succ \beta$. Then, the following inequality holds:

$$\boxed{f(a_1) + f(a_2) + \cdots + f(a_n) \geqslant f(b_1) + f(b_2) + \cdots + f(b_n).}$$

The inequality is flipped when f is a concave function.

This is very useful when the same function occurs on both the left side and the right side of an inequality. For example, we can use Karamata to prove

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geqslant 2\left(\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x}\right) \geqslant \frac{9}{x+y+z},$$

[1] Funnily enough, this inequality is true nonetheless. Try to prove it!

since the problem becomes equivalent to

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geqslant \frac{1}{(x+y)/2}+\frac{1}{(y+z)/2}+\frac{1}{(z+x)/2} \geqslant 3 \cdot \frac{3}{(x+y+z)/3}.$$

We can now assume $x \geqslant y \geqslant z$ and use Karamata on $f(x)=1/x$, since the conclusion follows from

$$(x, y, z) \succ \left(\frac{x+y}{2}, \frac{y+z}{2}, \frac{z+x}{2}\right) \succ \left(\frac{x+y+z}{3}, \frac{x+y+z}{3}, \frac{x+y+z}{3}\right).$$

Be very careful that α and β have their components sorted in increasing order. For instance, one could claim to show

$$(a^2+2bc)^N+(b^2+2ca)^N+(c^2+2ab)^N \leqslant (a^2+b^2+c^2)^N+2(ab+bc+ca)^N$$

by "using" Karamata on the function $f(x)=x^N$ (obviously convex since 2nd derivative is either 0 or $N(N-1)x^{N-2}>0$) and claiming $(a^2+b^2+c^2, ab+bc+ca, ab+bc+ca) \succ (a^2+2bc, b^2+2ca, c^2+2ab)$ whenever $a \geqslant b \geqslant c$. However, this solution does not work because $(a^2+2bc, b^2+2ca, c^2+2ab)$ may not be correctly sorted, for instance when $a=2, b=1, c=0$ it is $(4,1,4)$ which is incorrectly sorted, and thus the application of Karamata is invalid.

Problem 6.2.1. Let a, b, c be side lengths of a triangle. Prove that

$$\sqrt{a+b-c}+\sqrt{b+c-a}+\sqrt{c+a-b} \leqslant \sqrt{a}+\sqrt{b}+\sqrt{c}.$$

7 Chunking (Isolated Fudging)

This is a very powerful technique that is probably not talked about enough. Basically, if you have an inequality of the form

$$f(a, b, c)+f(b, c, a)+f(c, a, b) \geqslant \text{stuff},$$

then you're done if you can find a function g such that $f(x, y, z) \geqslant g(x, y, z)$ and $g(a, b, c)+g(b, c, a)+g(c, a, b)=\text{stuff}$. Of course you can apply a similar idea when dealing with functions of one variable, two variables etc.

There are two really common types of chunks. Firstly, in the three variable case where stuff $=1$, something like $g(a, b, c)=\frac{a}{a+b+c}$ is good to try, and at times it can help to try something like $g(a, b, c)=\frac{a^k}{a^k+b^k+c^k}$. Different chunks work for different problems: as an example, consider the inequality

$$\frac{a_1^3}{a_1^2+a_2a_3}+\frac{a_2^3}{a_2^2+a_3a_4}+\cdots+\frac{a_n^3}{a_n^2+a_1a_2} \geqslant \frac{1}{2}(a_1+a_2+\cdots+a_n).$$

It turns out that the chunking

$$\frac{a_1^3}{a_1^2+a_2a_3} \geqslant a_1-\frac{a_2+a_3}{4}$$

works since expanding gives this is equivalent to $(a_1^2+a_2a_3)(a_2+a_3) \geqslant 4a_1a_2a_3$, which is true simply by AM–GM. Summing across all n such inequalities gets the desired result.

Problem 7.0.1. Let $a, b, c>0$ satisfy $abc=1$. Show that

$$\frac{1}{a^5+b^5+ab}+\frac{1}{b^5+c^5+bc}+\frac{1}{c^5+a^5+ca} \leqslant 1.$$

7.1 Linear Approximations / Tangent Line Trick

The other chunk that comes in handy a lot is approximating by linear functions, assuming 'stuff' is a linear function. In this case, our problem is usually in one or two variables. Of course, depending on the form of 'stuff', we might be able to approximate by quadratic functions, etc.

8 Homogenising/Dehomogenising

An inequality is **homogeneous** if everything is of the same degree. Another way to think of this is that if the inequality is the same if we scale each variable by a nonzero factor of t, and then cancel out any factors of t from both sides. Sometimes we would like to homogenise an inequality as a lot of inequality theorems work nicely when things are homogeneous. We can do this when we are given a condition such as $abc = 1$. However, there are also situations when we want to dehomogenise an inequality, and we can do this by assuming without loss of generality that $a + b + c = 1$, for example. This may enable us to chunk nicely, or use Jensen.

As an example of homogenisation, suppose $a, b, c > 0$ such that $a+b+c \geqslant \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$, and we want to show

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geqslant \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}.$$

The condition looks fairly horrible, and it doesn't look like something we want to work with. Hence, to homogenise the inequality, it suffices to show

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geqslant \left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right) \cdot \frac{a+b+c}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \geqslant \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}.$$

However we can rearrange the inequality into

$$\left(\frac{b(abc)}{c}+\frac{c(abc)}{a}+\frac{a(abc)}{b}\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geqslant (a+b+c)^2,$$

which is true since it's a straightforward application of the Cauchy–Schwarz Inequality.

Problem 8.1. Let $a, b, c > 0$ with $a + b + c = 3$. Show that $(3-2a)(3-2b)(3-2c) \leqslant abc$.

Problem 8.2. Let $a, b, c > 0$ with $a + b + c = 3$. Show that $(3-2a)(3-2b)(3-2c) \leqslant (abc)^2$.

Problem 8.3. Let $a, b, c > 0$ with $abc = 1$. Show that $\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a} \geqslant a+b+c$.

Problem 8.4. Let $a, b, c > 0$ with $a + b + c = 2$. Prove that

$$\frac{a^2}{2-a}+\frac{b^2}{2-b}+\frac{c^2}{2-c} \geqslant 1.$$

Problem 8.5. Let $a, b, c > 0$ with $a + b + c = 1$. Prove that

$$\sqrt{\frac{ab}{ab+c}}+\sqrt{\frac{bc}{bc+a}}+\sqrt{\frac{ca}{ca+b}} \leqslant \frac{3}{2}.$$

Problem 8.6. Let $x, y, z > 0$ with $x + y + z = 1$. Prove that

$$\frac{\sqrt{xyz}}{x^2+y^2+z^2-x^3-y^3-z^3} \leqslant \sqrt{\frac{xy}{(1-z)^2}+\frac{yz}{(1-x)^2}+\frac{zx}{(1-y)^2}}.$$

Problem 8.7 Find the maximum value of the real number k such that for any $a, b, c > 0$ with $a+b+c = ab+bc+ca$, the following inequality holds:

$$(a+b+c)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}-k\right) \geqslant k.$$

Problem 8.8. Let $a, b, c > 0$ with $a + b + c = 3$. Show that $\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2} \geqslant a^2+b^2+c^2$.

Problem 8.9. Let $a, b, c > 0$ with $a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$. Show that

$$\frac{1}{(2a+b+c)^2}+\frac{1}{(a+2b+c)^2}+\frac{1}{(a+b+2c)^2} \leqslant \frac{3}{16}.$$

9 Substitutions

Sometimes we can transform an inequality into something simpler by making a substitution. There are a variety of substitutions that can considerably simplify problems, including but not limited to:

- $abc = 1$
  - $\boxed{a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}}$, or $\boxed{a = \frac{x^2}{yz}, b = \frac{y^2}{zx}, c = \frac{z^2}{xy}}$ (no conditions on x, y, z)
  - $\boxed{a = \frac{1}{x}, b = \frac{1}{y}, c = \frac{1}{z}}$ (where $xyz = 1$)
- ΔABC is a triangle: $\boxed{a = y+z, b = z+x, c = x+y}$ where $x, y, z \geqslant 0$ (since incircle)
- Trigonometric Substitutions for Exotic conditions (A, B, C are angles of an acute triangle)
  - $a+b+c = abc$: $\boxed{a = \tan A, b = \tan B, c = \tan C}$
  - $ab+bc+ca = 1$: $\boxed{a = \cot A, b = \cot B, c = \cot C}$
  - $a^2+b^2+c^2+2abc = 1$: $\boxed{a = \cos A, b = \cos B, c = \cos C}$

For example, let's suppose we want to show the inequality

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leqslant 1,$$

where a, b, c are positive real numbers with $abc = 1$. Given the inequality is deeply asymmetric, we may want to try making it more symmetric via a substitution. Hence, let $a = x/y$ and similar for b, c. Then:

$$\begin{aligned}\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) &= \left(\frac{x}{y} - 1 + \frac{x}{z}\right)\left(\frac{y}{z} - 1 + \frac{y}{x}\right)\left(\frac{z}{x} - 1 + \frac{z}{y}\right) \\ &= \frac{(-x+y+z)(x-y+z)(x+y-z)}{xyz} \leqslant 1.\end{aligned}$$

However, this inequality follows from the fact that $2x = (x-y+z)+(x+y-z) \geqslant 2\sqrt{(x-y+z)(x+y-z)}$, and multiplying the similar inequalities yield the desired result. $\square$

Problem 9.1. Let a, b, c be the side lengths of a triangle. Prove that

$$a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) \leqslant 3abc.$$

Problem 9.2. Let a, b, c be the side lengths of a triangle. Prove that

$$\frac{1}{-a+b+c} + \frac{1}{a-b+c} + \frac{1}{a+b-c} \geqslant \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Problem 9.3. Let a, b, c be the side lengths of a triangle. Prove that $(a+b+c)^2 \leqslant 4(ab+bc+ca)$.

Problem 9.4. Let a, b, c be the side lengths of a triangle. Prove that

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geqslant 0.$$

Problem 9.5. Let a, b, c be positive real numbers satisfying $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geqslant \frac{3}{2}.$$

Problem 9.6. Let $a, b, c \geqslant 0$ satisfy $\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} = 2$. Prove that $ab+bc+ca \leqslant \frac{3}{2}$.

Problem 9.7. Let x, y, zbe positive reals such that $x+y+z = xyz$. Prove that

$$x^7(yz-1) + y^7(zx-1) + z^7(xy-1) \geqslant 162\sqrt{3}.$$

10 Other (Miscellaneous) Inequalities that may be useful

- **Weighted AM–GM Inequality.** Let $a_1, a_2, \ldots, a_n > 0$, and consider n "weights" $\omega_1, \omega_2, \ldots, \omega_n > 0$ with sum 1. Then the following inequality holds:

$$\boxed{a_1\omega_1 + a_2\omega_2 + \cdots + a_n\omega_n \geqslant a_1^{\omega_1} a_2^{\omega_2} \ldots a_n^{\omega_n}.}$$

In the case where $\omega_1 = \omega_2 = \cdots = \omega_n = \dfrac{1}{n}$, we get the classical AM–GM Inequality.

- **Bernoulli's Inequality.** Let $x \geqslant -1$, and $r \geqslant 1$ be real numbers. Then, $\boxed{(1+x)^r \geqslant 1 + rx.}$

This inequality still holds if we have $x \geqslant -1$ and $r < 0$ instead.

If, on the other hand, $0 < r < 1$, then the inequality is flipped: $\boxed{(1+x)^r \leqslant 1 + rx.}$

- **Rearrangement Inequality.** Say we have two sequences of real numbers, $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$ and $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n$. Take a random permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$. Then

$$\boxed{\sum_{k=1}^{n} a_k b_{n-k} \leqslant \sum_{k=1}^{n} a_k b_{\sigma_k} \leqslant \sum_{k=1}^{n} a_k b_k.}$$

How to think of this inequality: There are a large supply of coins of value $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$, but you are only allowed to pick exactly b_i coins of a single, different value for each $i \leqslant n$. To maximise the total value, you would pick most of the highest-value coins and least of the smallest value coins. To minimise the total value, you do the opposite: pick most of the lowest value coins. Every other value must lie somewhere in between.

- **Minkowski's Inequality.** Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be $2n$ **positive** real numbers and $T > 1$. Then the following inequality holds:

$$\boxed{\left(\sum_{k=1}^{n} x_k^T\right)^{1/T} + \left(\sum_{k=1}^{n} y_k^T\right)^{1/T} \geqslant \left(\sum_{k=1}^{n} (x_k + y_k)^T\right)^{1/T}.}$$

Think of this as a generalised Triangle Inequality: for $n = 2$ and $T = 2$, this is equivalent to

$$\sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2} \geqslant \sqrt{(x_1 + y_1)^2 + (y_1 + y_2)^2},$$

which is the Triangle Inequality on a parallelogram with one of the vertices at the origin (check this!).

- **Schur's Inequality.** Let a, b, c be nonnegative real numbers, and $r > 0$. Then the following inequality holds:

$$\boxed{a^r(a-b)(a-c) + b^r(b-a)(b-c) + c^r(c-a)(c-b) \geqslant 0.}$$

Schur's Inequality is rather strong because it has two distinct equality cases, namely $a = b = c$, or that $a = 0$ and $b = c$ (and cyclic permutations). The expanded form for $r = 1$ is also very useful: we have

$$\boxed{a^3 + b^3 + c^3 + 3abc \geqslant a^2b + a^2c + b^2a + b^2c + c^2a + c^2b,}$$

and this allows us to combine a weaker, more "mixed" term abc with stronger values such as a^3, and get something reasonably strong on the right side.

- **Chebyshev's Inequality.** Let $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant 0$ and $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_n \geqslant 0$ be real numbers. Then, the following inequalities hold:

$$\boxed{\frac{x_1y_1 + x_2y_2 + \cdots + x_ny_n}{n} \geqslant \left(\frac{x_1 + \cdots + x_n}{n}\right)\left(\frac{y_1 + \cdots + y_n}{n}\right) \geqslant \frac{x_1y_n + x_2y_{n-1} + \cdots + x_ny_1}{n}.}$$

11 Tips for solving Inequalities

- Do lots of problems, as you will then develop a strong sense of intuition as to what will work (and doesn't work) when you are presented with a problem.
- **Do NOT blindly memorise inequalities** and just try them in the most obvious way possible in an Olympiad inequalities question. Every worthy problem has its own somewhat-unique step, making it impossible for a set of generic inequalities to completely solve every inequality the same way.
- Don't forget to factorise! Sometimes an inequality can be greatly simplified by a factorisation.
- In terms of trying to get good at inequalities, you should start by getting a strong handle on AM-GM as this is probably the most widely used technique when solving these problems. The other means inequalities mentioned in this handout are useful as well, but they really do follow from AM-GM, so if you get really good at AM-GM you will then be good at the others. Then you can aim at mastering other tricks such as Cauchy-Schwarz and chunking.
- There are, unfortunately, times when you have to just expand the inequality. Don't hesitate to do so, unless the expansion is so messy it can't be done within an hour or two.

12 Problems

1. Prove that $\sqrt{\frac{1}{n+1}+\frac{2}{n+1}+\cdots+\frac{n}{n+1}} \geqslant 1$ for $n \geqslant 2$.
2. Prove that for all real numbers x, we have $x^4+6x^2+1 \geqslant 4x(x^2+1)$. When does inequality hold?
3. Prove that for all positive reals a, b, c, we have $\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a} \geqslant 3$.
4. Prove that for all positive reals x, y, we have $x^3+y^3 \geqslant x^2y+xy^2$.
5. What is the maximum value of $a^5(1-a)$ for $0<a<1$?
6. Prove that for all positive reals a, b, c, we have $\dfrac{a^2}{b}+\dfrac{b^2}{c}+\dfrac{c^2}{a} \geqslant a+b+c$.
7. For any positive real numbers $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$, prove that
$$\frac{1}{x_1y_1}+\frac{1}{x_2y_2}+\cdots+\frac{1}{x_ny_n} \geqslant \frac{4n^2}{(x_1+y_1)^2+(x_2+y_2)^2+\cdots+(x_n+y_n)^2}.$$
8. For $a, b, c, d \geqslant 0$, show that $\sqrt{(a+c)(b+d)} \geqslant \sqrt{ab}+\sqrt{cd}$.
9. Let a, b, c be real. Prove that
$$a^4(1+b^4)+b^4(1+c^4)+c^4(1+a^4) \geqslant 6a^2b^2c^2.$$
10. Let x, y be nonnegative reals with sum 2. Prove that $x^2y^2(x^2+y^2) \leqslant 2$.
11. Let $a, b, c, d, e > 0$. Show that $\left(\dfrac{a}{b}\right)^4+\left(\dfrac{b}{c}\right)^4+\left(\dfrac{c}{d}\right)^4+\left(\dfrac{d}{e}\right)^4+\left(\dfrac{e}{a}\right)^4 \geqslant \dfrac{b}{a}+\dfrac{c}{b}+\dfrac{d}{c}+\dfrac{e}{d}+\dfrac{a}{e}$.
12. Let $x_1, x_2, \ldots, x_n$ be positive reals with product 1. Show that $(1+x_1)(1+x_2)\ldots(1+x_n) \geqslant 2^n$.

13. Show that for all reals a, b, c, $a^4 + b^4 + c^2 \geqslant 2\sqrt{2}abc$.

14. Suppose a, b, c are positive reals. Show that $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{4}{c} + \dfrac{16}{d} \geqslant \dfrac{64}{a+b+c+d}$.

15. Let a, b, c be positive reals. Prove that $\dfrac{a}{b+c} + \dfrac{b}{c+a} + \dfrac{c}{a+b} \geqslant \dfrac{3}{2}$.

16. Let $n > 1$ be an integer. Show that $\dfrac{1}{n} + \dfrac{1}{n+1} + \cdots + \dfrac{1}{n^2} > 1$.

17. Let a, b, c be positive reals. Show that $(a^3 + b^3 + c^3)(a + b + c) \geqslant (a^2 + b^2 + c^2)^2$.

18. Prove **Schur's Inequality** for $n = 3$.

19. Show that if the real numbers $a_1, a_2, \ldots, a_n$ satisfy the inequality

$$a_1 + a_2 + ... + a_n \geqslant \sqrt{(n-1)\left(a_1^2 + a_2^2 + ... + a_n^2\right)},$$

then all of these numbers $a_1, a_2, \ldots, a_n$ are non-negative.

20. Let $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 1$. Show that $\dfrac{x}{y+1} + \dfrac{y}{x+1} \leqslant 1$.

21. (AMO 1997) Let $m, n > 1$ be positive integers. Show that $\dfrac{1}{\sqrt[m]{n+1}} + \dfrac{1}{\sqrt[n]{m+1}} > 1$.

22. Let $a, b, c > 0$ be the side lengths of a triangle. Show that $\dfrac{a}{b+c-a} + \dfrac{b}{a+c-b} + \dfrac{c}{a+b-c} \geqslant 3$.

23. Let $a, b, c > 0$ be the side lengths of a triangle. Show that

$$(-a+b+c)(a-b+c) + (a-b+c)(a+b-c) + (a+b-c)(-a+b+c) \leqslant \sqrt{abc}(\sqrt{a} + \sqrt{b} + \sqrt{c}).$$

24. For positive $a_1, a_2, \ldots, a_n$ and $n \geqslant 2$, show that

$$(a_1^3 + 1)(a_2^3 + 1)\ldots(a_n^3 + 1) \geqslant (a_1^2 a_2 + 1)(a_2^2 a_3 + 1)\ldots(a_n^2 a_1 + 1).$$

25. Let $n \geqslant 2$ be an integer. Show that $\dfrac{1}{n+1}\left(1 + \dfrac{1}{3} + \cdots + \dfrac{1}{2n-1}\right) \geqslant \dfrac{1}{n}\left(\dfrac{1}{2} + \dfrac{1}{4} + \cdots + \dfrac{1}{2n}\right)$.

26. Show that whenever $n \geqslant 1$, we have $n^n \leqslant (n!)^2 \leqslant \left(\dfrac{(n+1)(n+2)}{6}\right)^n$.

27. Let $a, b, c \geqslant 0$ with $a + b + c \geqslant abc$. Show that $a^2 + b^2 + c^2 \geqslant abc$.

28. Let $x, y \geqslant 0$ with $x + y = 2$. Show that $x^2y^2(x^2 + y^2) \leqslant 2$.

29. Let $x, y, z \geqslant 0$. Show that $\sqrt{x^2 - xy + y^2} + \sqrt{y^2 - yz + z^2} \geqslant \sqrt{z^2 + zx + x^2}$.

30. (2013 T-sets) For positive reals a, b, c, prove that $\dfrac{a^3 + 3b^3}{5a + b} + \dfrac{b^3 + 3c^3}{5b + c} + \dfrac{c^3 + 3a^3}{5c + a} \geqslant \dfrac{2}{3}(a^2 + b^2 + c^2)$.

31. (April 2012 S4Q3) Let a, b, c be nonnegative real numbers such that $a + b + c = 1$. Find the maximum value of

$$\frac{1}{a^2 - 4a + 9} + \frac{1}{b^2 - 4b + 9} + \frac{1}{c^2 - 4c + 9}.$$

32. Let a, b, c be positive reals such that $abc = 1$. Prove that $\dfrac{1}{a+b+1} + \dfrac{1}{b+c+1} + \dfrac{1}{c+a+1} \leqslant 1$.

33. Let $n \geqslant 2$ be a positive integer and let $x_1, x_2, \ldots, x_n$ be positive real numbers. Define $x_{n+1} = x_1$. Prove that
$$\sum_{k=1}^{n} \frac{x_k^3 - x_{k+1}^3}{x_k + x_{k+1}} \leqslant \sum_{k=1}^{n} (x_k - x_{k+1})^2.$$

34. Let a, b, c be positive real numbers. Show that $\dfrac{1}{a^3+b^3+abc} + \dfrac{1}{b^3+c^3+abc} + \dfrac{1}{c^3+a^3+abc} \leqslant \dfrac{1}{abc}$.

35. (Japan 1997) Let a, b, c be positive real numbers. Show that
$$\frac{(b+c-a)^2}{(b+c)^2+a^2} + \frac{(c+a-b)^2}{(c+a)^2+b^2} + \frac{(a+b-c)^2}{(a+b)^2+c^2} \geqslant \frac{3}{5}.$$

36. Let a, b, c be positive real numbers such that $abc = 1$. Prove that
$$\frac{1}{(a+1)^2+b^2+1} + \frac{1}{(b+1)^2+c^2+1} + \frac{1}{(c+1)^2+a^2+1} \leqslant \frac{1}{2}.$$

37. (April 2005 S2Q3) Real numbers x, y, z satisfy $xyz = -1$. Show that
$$x^4 + y^4 + z^4 + 3(x+y+z) \geqslant \frac{x^2}{y} + \frac{x^2}{z} + \frac{y^2}{x} + \frac{y^2}{z} + \frac{z^2}{x} + \frac{z^2}{y}.$$

38. Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Prove the inequality
$$\binom{n}{2} \sum_{i<j} \frac{1}{a_i a_j} \geqslant 4 \left(\sum_{i<j} \frac{1}{a_i + a_j} \right)^2.$$

39. Let x, y, z be positive real numbers such that $5 \cdot \min\{x, y, z\} \geqslant x + y + z$. Prove that
$$\frac{1}{(x^2+xy+y^2)^2} + \frac{1}{(y^2+yz+z^2)^2} + \frac{1}{(z^2+zx+x^2)^2} \geqslant \frac{3}{(xy+yz+zx)^2}.$$

40. For two given positive integers $m, n > 1$, let a_{ij} $(i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, m)$ be nonnegative real numbers, not all zero. Find the maximum and minimum values of f, where
$$f_{a_{ij}} = \frac{n \sum_{i=1}^{n} (\sum_{j=1}^{m} a_{ij})^2 + m \sum_{j=1}^{m} (\sum_{i=1}^{n} a_{ij})^2}{(\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij})^2 + mn \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}^2}.$$