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Multiple Choice

Question 1. (A) is false by considering $a = b = -1$ and $c = -2$ which gives $LHS = -10, RHS = -6$. (B) is true. We will prove it by considering cases. If none of a, b, c are 0, by 4-variable AM-GM,

$$\frac{a^4 + a^4 + b^4 + c^4}{4} \geq \sqrt[4]{a^4a^4b^4c^4} = |abc(a)| \geq abc(a)$$

and summing cyclically gets the desired result. If at least one of them is 0, then RHS is 0 and LHS is non-negative as it is a sum of squares, still true.

Question 2.

Question 3. (B).
We use the substitution $u = \pi - x$ to get

$$\begin{aligned}
\int_0^{2\pi} \frac{x}{\phi - \cos^2 x}\,dx &= \int_{-\pi}^{\pi} \frac{\pi - x}{\phi - \cos^2 x}\,dx \\
&= \int_{-\pi}^{\pi} \frac{\pi}{\phi - \cos^2 x}\,dx && \left(\frac{x}{\phi - \cos^2 x} \text{ is odd}\right) \\
&= 2\pi \int_{-\pi}^{\pi} \frac{1}{\phi - \cos^2 x}\,dx && \left(\frac{1}{\phi - \cos^2 x} \text{ is even}\right) \\
&= 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\phi - \sin^2 x}\,dx && \left(\text{using } u = \frac{\pi}{2} - x\right) \\
&= 4\pi \int_0^{\frac{\pi}{2}} \frac{1}{\phi - \sin^2 x}\,dx && \left(\frac{1}{\phi - \sin^2 x} \text{ is even}\right) \\
&= 4\pi \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{\phi + (\phi - 1)\tan^2 x}\,dx \\
&= \frac{4\pi}{\sqrt{\phi^2 - \phi}} \tan^{-1}\left(\frac{\sqrt{\phi - 1}\tan x}{\sqrt{\phi}}\right)\Bigg|_0^{\frac{\pi}{2}} \\
&= \frac{4\pi}{\sqrt{\phi^2 - \phi}} \times \frac{\pi}{2} = 2\pi^2,
\end{aligned}$$

since $\phi^2 - \phi = 1$.

Question 4.

Question 5.

Question 6.

Question 7. Let $a_1, a_2, \ldots, a_9$ be the number of chairs between each consecutive pair of people

Question 8. (A).
Let A, B, C, D, E, F represent the complex numbers $z_1, z_2, z_3, 1/z_1, 1/z_2, 1/z_3$ respectively. Assume A, B, C lie on the line in that order. Thus,

$$\begin{aligned}
\angle ODE &= \arg\left(\frac{1}{z_1} - \frac{1}{z_2}\right) - \arg\left(\frac{1}{z_1}\right) \\
&= \arg\left(\frac{\frac{1}{z_1} - \frac{1}{z_2}}{\frac{1}{z_1} - 0}\right) \\
&= \arg\left(\frac{z_2 - z_1}{z_2}\right) \\
&= \arg(z_2 - z_1) - \arg(z_2) \\
&= \angle OBA
\end{aligned}$$

Similarly, $\angle OFE = \angle OBC$. Thus,

$$\angle ODE + \angle OFE = \angle OBA + \angle OBC = \pi$$

because A, B, C are collinear. Thus, O, D, E, F lie on a circle because if a quadrilateral's opposite angles are supplementary, its vertices lie on a circle.

Question 9. (B).
We solve for $x, y \in \mathbb{Z}$ in the equation

$$\begin{aligned} \frac{1}{x} + \frac{1}{y} &= \frac{1}{2019} \\ 2019x + 2019y &= xy \\ xy - 2019x + 2019y + 2019^2 &= 2019^2 \\ (x-2019)(y-2019) &= 2019^2 \end{aligned}$$

Notice that $2019 = 3 \times 673$ where $3, 673$ are primes so it has factors $1, 3, 673, 2019$. Thus, the pair $(x - 2019, y - 2019)$ could be

$$\begin{aligned} (x, y) = &(1, 2019), (3, 673), (673, 3), (2019, 1), \\ &(-1, -2019), (-3, -673), (-673, -3), (-2019, -1). \end{aligned}$$

which give 8 solutions.

Question 10. (C).
We have $z\bar{z} = |z|^2$. Notice that

$$|AB|^2 = |a-b|^2 = (a-b)(\overline{a-b}) = a\overline{a} + b\overline{b} + a\overline{b} + \overline{a}b = |a|^2 + |b|^2 + a\overline{b} + \overline{a}b = 2 + a\overline{b} + \overline{a}b$$

so $a\overline{b} + \overline{a}b = |AB|^2 - 2$. Similarly,

$$\begin{aligned} a\overline{b} + \overline{a}b &= |AB|^2 - 2 \\ b\overline{c} + \overline{b}c &= |BC|^2 - 2 \\ c\overline{a} + \overline{c}a &= |CA|^2 - 2 \end{aligned}$$

Since a, b, c lie on the unit circle, $|a| = |b| = |c| = 1$. Therefore,

$$\begin{aligned} |OH|^2 &= |a+b+c|^2 \\ &= (a+b+c)(\overline{a+b+c}) \\ &= a\overline{a} + b\overline{b} + c\overline{c} + a\overline{b} + b\overline{c} + c\overline{a} + \overline{a}b + \overline{b}c + \overline{c}a \\ &= |a|^2 + |b|^2 + |c|^2 + a\overline{b} + b\overline{c} + c\overline{a} + \overline{a}b + \overline{b}c + \overline{c}a \\ &= 3 + (2 - |AB|^2) + (2 - |BC|^2) + (2 - |BC|^2) \\ &= 9 - |AB|^2 - |BC|^2 - |CA|^2. \end{aligned}$$

Question 11

(a) First, we evaluate the integral $\int_0^1 x^k \ln x \, dx$. We use the substitution $u = \ln x$ and so,

$$\begin{aligned}
\int_0^1 x^k \ln x \, dx &= \int_{-\infty}^0 u e^{(k+1)u} \, du \\
&= \left. \frac{u e^{(k+1)u}}{k+1} \right|_{-\infty}^0 - \int_{-\infty}^0 \frac{e^{(k+1)u}}{k+1} \, du \\
&= \left. \frac{e^{(k+1)u}}{(k+1)^2} \right|_{-\infty}^0 \\
&= -\frac{1}{(k+1)^2},
\end{aligned}$$

using integration by parts and $\lim_{x \to -\infty} x e^x = 0$. Now, we apply the substitution $u = \tan x$ to the original integral to get

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \frac{\ln(\cot\theta)}{(\cos^2\theta)^n} \, d\theta &= \int_0^{\frac{\pi}{4}} -\ln(\tan\theta)(1+\tan^2\theta)^{n-1} \sec^2\theta \, d\theta \\
&= \int_0^1 -\ln(u)(1+u^2)^{n-1} \, du \\
&= -\int_0^1 \ln(u) \left(\sum_{k=0}^{n-1} \binom{n-1}{k} u^{2k} \right) du \\
&= \sum_{k=0}^{n-1} \frac{\binom{n-1}{k}}{(2k+1)^2},
\end{aligned}$$

using the result we derived earlier.

(b) (i) First we note that a rearrangement of $r^2 - 2r\pi + 1 = 0$ gives $\frac{1}{2r} + \frac{r}{2} = \pi$. Then, we note that

$$\frac{r}{2}\left(1+\frac{z}{r}\right)\left(1+\frac{1}{zr}\right) = \frac{r}{2}\left(1+\frac{z}{r}+\frac{1}{rz}+\frac{1}{r^2}\right) = \frac{r}{2}+\frac{1}{2r}+\cos x = \pi + \cos x.$$

Thus

$$I = \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)(\pi+\cos x)} = \frac{2}{r}\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)\left(1+\frac{z}{r}\right)\left(1+\frac{1}{zr}\right)}.$$

(ii) We have

$$\begin{aligned}
\frac{2r}{r^2-1}\int_{-\infty}^{\infty} \frac{1}{1+x^2} \sum_{N=-\infty}^{\infty} \frac{(-z)^N}{r^{|N|}} \, dx &= \frac{2r}{r^2-1}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\left(\sum_{N=0}^{\infty} \frac{(-z)^N}{r^N} + \sum_{N=0}^{\infty} \frac{(-z)^{-N}}{r^N} - 1 \right) dx \\
&= \frac{2r}{r^2-1}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\left(\frac{1}{1+\frac{z}{r}} + \frac{1}{1+\frac{1}{zr}} - 1 \right) dx \\
&= \frac{2r}{r^2-1}\int_{-\infty}^{\infty} \frac{1-\frac{1}{r^2}}{(1+x^2)\left(1+\frac{z}{r}\right)\left(1+\frac{1}{zr}\right)} \, dx \\
&= \frac{2}{r}\int_{-\infty}^{\infty} \frac{1}{(1+x^2)\left(1+\frac{z}{r}\right)\left(1+\frac{1}{zr}\right)} \, dx
\end{aligned}$$

(iii) First, we note that

$$
\begin{aligned}
I &= \frac{2r}{r^2-1}\int_{-\infty}^{\infty}\frac{1}{1+x^2}\sum_{N=-\infty}^{\infty}\frac{(-z)^N}{r^{|N|}}\,dx\\
&= \frac{2r\pi}{r^2-1}\sum_{N=\infty}^{-\infty}\left(e^{-|N|}\frac{(-1)^N}{r^{|N}}\right)\\
&= \frac{2r\pi}{r^2-1}\left(1+2\left(-\frac{1}{er}+\frac{1}{e^2r^2}-\frac{1}{e^3r^3}+\cdots\right)\right)\\
&= \frac{2\pi r}{r^2-1}\left(\frac{-\frac{2}{er}}{1+\frac{1}{er}}+1\right)\\
&= \frac{2\pi r}{r^2-1}\times\frac{er-1}{er+1}\\
&= \pi\left(\frac{2}{r-\frac{1}{r}}\right)\times\frac{er-1}{er+1}\\
&= \pi\left(\frac{2}{\pi+\sqrt{\pi^2-1}-(\pi-\sqrt{\pi^2-1})}\right)\times\frac{e(\pi+\sqrt{\pi^2-1})-1}{e(\pi+\sqrt{\pi^2-1})+1}\\
&= \pi\left(\frac{2}{2\sqrt{\pi^2-1}}\right)\times\frac{e(\pi+\sqrt{\pi^2-1})-1}{e(\pi+\sqrt{\pi^2-1})+1}\\
&= \frac{\pi}{\sqrt{\pi^2-1}}\times\left(1-\frac{2}{\pi e+\sqrt{\pi^2-1}e+1}\right)\\
&= \frac{\pi}{\sqrt{\pi^2-1}}\times\left(1-\frac{4}{2\pi e+2\sqrt{\pi^2-1}e+2}\right)\\
&= \frac{\pi}{\sqrt{\pi^2-1}}\times\left(1-\frac{2(\sqrt{\pi+1}-\sqrt{\pi-1})(\sqrt{\pi+1}+\sqrt{\pi-1})}{(e+1)(\pi+1)+(e-1)(\pi-1)+2e\sqrt{\pi^2-1}}\right)\\
&= \frac{\pi}{\sqrt{\pi^2-1}}\times\left(1-\frac{2(\sqrt{\pi+1}-\sqrt{\pi-1})(\sqrt{\pi+1}+\sqrt{\pi-1})}{((e+1)\sqrt{\pi+1}+(e-1)\sqrt{\pi-1})(\sqrt{\pi+1}-\sqrt{\pi-1})}\right)\\
&= \frac{\pi}{\sqrt{\pi^2-1}}\times\left(1-\frac{2\sqrt{\pi+1}-2\sqrt{\pi-1}}{(e+1)\sqrt{\pi+1}+(e-1)\sqrt{\pi-1}}\right)\\
&= \frac{\pi}{\sqrt{\pi^2-1}}\times\left(\frac{(e-1)\sqrt{\pi+1}+(e+1)\sqrt{\pi-1}}{(e+1)\sqrt{\pi+1}+(e-1)\sqrt{\pi-1}}\right)\\
&= \frac{\pi}{\sqrt{\pi^2-1}}\times\frac{\sqrt{\pi-1}+\frac{e+1}{e-1}\sqrt{\pi+1}}{\sqrt{\pi+1}+\frac{e+1}{e-1}\sqrt{\pi-1}}\\
&= \frac{\pi}{\sqrt{\pi^2-1}}\times\frac{\sqrt{\pi-1}+\sqrt{\pi+1}\tanh\left(\frac{1}{2}\right)}{\sqrt{\pi+1}+\sqrt{\pi-1}\tanh\left(\frac{1}{2}\right)}
\end{aligned}
$$

as required.

(c) (i) By Pythagoras' Theorem,

$$
\begin{aligned}
|P(z)|^2 &= \mathrm{Re}(P(z))^2+\mathrm{Im}(P(z))^2\\
&= (\cos 2\theta-\cos\theta+\alpha)^2+(\sin 2\theta-\sin\theta)^2\\
\implies \frac{\mathrm{d}}{\mathrm{d}x}|P(z)|^2 &= 8\alpha x-2-2\alpha \qquad (x=\cos\theta)\\
\frac{\mathrm{d}^2}{\mathrm{d}x^2}|P(z)|^2 &= 8\alpha>0
\end{aligned}
$$

so $|P(z)|^2$ has a minimum turning point at $\cos\theta = \frac{1+\alpha}{4\alpha}$. Thus,

$$\begin{aligned}
|P(z)|^2 &= 4\alpha x^2 - 2(\alpha+1)x + \alpha^2 - 2\alpha + 2 \\
&\geq \frac{(\alpha+1)^2 - 2(\alpha+1)^2 + 4\alpha^3 - 8\alpha^2 + 8\alpha}{4\alpha} \\
&= \frac{4\alpha^3 - 9\alpha^2 + 6\alpha - 1}{4\alpha} \\
&= \frac{4\alpha(\alpha^2 - 2\alpha + 1) - (\alpha^2 - 2\alpha + 1)}{4\alpha} \\
&= \frac{(1-\alpha)^2(4\alpha - 1)}{4\alpha}
\end{aligned}$$

for all $0 < \alpha < 1$. It remains now to prove $|P(z)|^2 \geq \alpha^2$ for $0 < \alpha \leq \frac{1}{3}$. When $0 < \alpha \leq \frac{1}{3}$, $\cos\theta = \frac{1+\alpha}{4\alpha}$ is not attainable when $\alpha \leq \frac{1}{3}$. Thus, it suffices to test that $|P(z)| \geq \alpha^2$ at $\cos\theta$'s extremal points, that is, $\cos\theta = -1$ or $\cos\theta = 1$. We have

$$|P(z)|^2 \geq min\{4\alpha - 2\alpha - 2 + \alpha^2 - 2\alpha + 2, 4\alpha + 2\alpha + 2 + \alpha^2 - 2\alpha + 2\}$$

so $|P(z)|^2 \geq min\{\alpha^2, \alpha^2 + 4\alpha + 4\} \geq \alpha^2$ as desired.

(ii) Let $|\omega| = r \geq 1$. We compute $|P(\omega)|^2$. Let $\omega = r(\cos\beta + i\sin\beta)$. Thus,

$$\begin{aligned}
|P(\omega)|^2 &= (\omega^2 - \omega + \alpha)(\overline{\omega^2 - \omega + \alpha}) \\
&= (\omega^2 - \omega + \alpha)(\overline{\omega}^2 - \overline{\omega} + \alpha) \\
&= r^4 - \omega r^2 - \overline{\omega} r^2 + r^2 + \alpha(\omega^2 + \overline{\omega}^2 - \omega - \overline{\omega}) + \alpha^2 \\
&= r^4 - \omega r^2 - \overline{\omega} r^2 + r^2 + \alpha(\omega^2 + 2\omega\overline{\omega} + \overline{\omega}^2 - \omega - \overline{\omega}) - 2r^2\alpha + \alpha^2 \\
&= \alpha^2 + r^4 - 2\text{Re}(\omega)r^2 + r^2 - 2r^2\alpha + \alpha(2\text{Re}(\omega))(2\text{Re}(\omega) - 1) \\
&= \alpha^2 + r^4 - 2\cos\beta r^3 + r^2 - 2r^2\alpha + \alpha(2r\cos\beta)(2r\cos\beta - 1).
\end{aligned}$$

Letting $x = \cos\beta$ gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}|P(\omega)|^2 &= 8\alpha r^2 x - 2r\alpha - 2r^3 \\
\frac{\mathrm{d}^2}{\mathrm{d}x^2}|P(\omega)|^2 &= 8\alpha r^2 > 0
\end{aligned}$$

so $|P(\omega)|^2$ has a minimum turning point at $\cos\beta = \frac{\alpha + r^2}{4\alpha r}$. Substituting gives

$$\begin{aligned}
|P(\omega)|^2 &\geq \alpha^2 + r^4 - 2r^3\left(\frac{\alpha + r^2}{4\alpha r}\right) + r^2 - 2r^2\alpha + \alpha 4r^2\left(\frac{\alpha + r^2}{4\alpha r}\right)^2 - 2r\left(\frac{\alpha + r^2}{4\alpha r}\right)\alpha \\
\implies \frac{\mathrm{d}}{\mathrm{d}(r^2)}|P(\omega)|^2 &= 2r^2 - \frac{1}{2} - \frac{r^2}{\alpha} + 1 - 2\alpha + \frac{1}{2} + \frac{r^2}{\alpha} - \frac{1}{2} \\
&= 2r^2 + \frac{1}{2} - 2\alpha \\
\frac{\mathrm{d}^2}{\mathrm{d}(r^2)^2}|P(\omega)|^2 &= 2 > 0
\end{aligned}$$

so $|P(\omega)|^2$ has a global minimum at $r^2 = \alpha - \frac{1}{4} < 1$ so $|P(\omega)|^2$ is increasing with respect to r^2 when $r^2 \geq 1$. Thus, choosing $z = \cos\theta + i\sin\theta$ such that $\cos\theta = \frac{1+\alpha}{4\alpha}$ gives $|P(z)|^2 \leq |P(\omega)|^2$ because $r^2 = 1$ on LHS but $r^2 \geq 1$ on RHS, so $|P(z)| \leq |P(\omega)|$ as desired.

(d) Since $\frac{\pi}{3} \leq \arg(x) - \arg(y) \leq \frac{5\pi}{3}$, the isosceles triangle formed by points $x, y, 0$ has angle θ at least $\frac{\pi}{3}$ at the point 0. By cosine rule,

$$|x-y|^2 = |x|^2 + |y|^2 - 2|x||y|\cos\theta \geq |x|^2 + |y|^2 - |x||y| \geq 2 - 2 \cdot \frac{1}{2} = 1$$

because $\cos x$ is decreasing over $[0, \pi]$, so $|y - x| \geq 1$. Thus, by triangle inequality,

$$|z| + |z-x| + |y-z| \geq |z| + |z-x+y-z| \geq |z| + 1 = |z| + \left|-\frac{x}{y}\right| \geq \left|z - \frac{x}{y}\right|$$

as required.

(e) (i) By the AM-GM Inequality,

$$\frac{a^4 + a^4 + a^4 + b^4}{4} \geq \sqrt[4]{a^4 a^4 a^4 b^4} = a^3 b$$

Similarly,

$$\frac{3}{4}a^4 + \frac{1}{4}b^4 \geq a^3 b$$
$$\frac{3}{4}b^4 + \frac{1}{4}c^4 \geq b^3 c$$
$$\frac{3}{4}c^4 + \frac{1}{4}a^4 \geq c^3 a$$

Summing gives

$$a^4 + b^4 + c^4 \geq a^3 b + b^3 c + c^3 a$$

(ii) Let $t^{x-1/4} = a, t^{y-1/4} = b, t^{z-1/4} = c$. From (i),

$$a^4 + b^4 + c^4 \geq a^3 b + b^3 c + c^3 a$$
$$t^{4x-1} + t^{4y-1} + t^{4z-1} \geq t^{3x+y-1} + t^{3y+z-1} + t^{3z+x-1}$$
$$\int_0^1 t^{4x-1}dt + \int_0^1 t^{4y-1}dt + \int_0^1 t^{4z-1}dt \geq \int_0^1 t^{3x+y-1}dt + \int_0^1 t^{3y+z-1}dt + \int_0^1 t^{3z+x-1}dt$$

Using the fact that $\int t^k dt = \frac{1}{k+1}t^{k+1}$, we have $\int_0^1 t^k dt = \frac{1}{k+1}$. Thus,

$$\frac{1}{4x-1+1} + \frac{1}{4y-1+1} + \frac{1}{4z-1+1} \geq \frac{1}{3x+y-1+1} + \frac{1}{3y+z-1+1} + \frac{1}{3z+x-1+1}$$

so

$$\frac{1}{4x} + \frac{1}{4y} + \frac{1}{4z} \geq \frac{1}{3x+y} + \frac{1}{3y+z} + \frac{1}{3z+x}$$

as desired.

Question 13

(a) (i) From the given formula,

$$\begin{aligned} rs = \text{Area}(ABC) &= \frac{1}{2}|AH||BC| \\ |DI| \times \frac{a+b+c}{2} &= \frac{1}{2}|AH|a \\ \frac{|DD_1|}{2} \times \frac{a+b+c}{2} &= \frac{1}{2}|AH|a \\ \frac{|DD_1|}{|AH|} &= \frac{2a}{a+b+c} \end{aligned}$$

as required.

(ii) Let the incircle touch AB and AC at X and Y. Since tangent segments are equal in length, let $|AX| = |AY| = x, |BX| = |BD| = y, |CY| = |CD| = z$. We have

$$\begin{aligned} y + z &= a \\ z + x &= b \\ x + y &= c \\ y - x &= a - b \\ 2y &= a - b + c \\ |CE| = |BD| = y &= \frac{a-b+c}{2} \end{aligned}$$

Thus,

$$|DE| = |BC| - |BD| - |CE| = a - 2\left(\frac{a-b+c}{2}\right) = b - c$$

as required. We also have

$$\begin{aligned} |EH| &= |BC| - |CE| - |BH| \\ &= a - \frac{a-b+c}{2} - c\cos B \\ &= \frac{a+b-c}{2} - \frac{a^2+c^2-b^2}{2a} && \text{(Cosine rule)} \\ &= \frac{a^2+ab-ac-a^2-c^2+b^2}{2a} \\ &= \frac{ab+b^2+bc-ac-bc-c^2}{2a} \\ &= \frac{(b-c)(a+b+c)}{2a} \end{aligned}$$

as desired. We now deduce

$$\frac{|DE|}{|EH|} = \frac{b-c}{\frac{(b-c)(a+b+c)}{2a}} = \frac{2a}{a+b+c} = \frac{|DD_1|}{|EH|}$$

and $\angle AHE = \angle D_1DE = 90°$ by definition and tangent line. Thus, by SAS, $\triangle AHE \sim \triangle D_1DE$. Since corresponding angles in similar triangles are equal,

$$\angle AEH = \angle D_1ED = \angle D_1EH$$

so A, D_1, H are collinear.

(b) (i) Let $2\alpha, 2\beta, 2\gamma$ be the angles $\angle A, \angle B, \angle C$ respectively. Since chords subtend equal angles,

$$\begin{aligned}\angle IBS &= \angle IBC + \angle CBS \\ &= \frac{1}{2}\angle ABC + \angle CAS \\ &= \beta + \frac{1}{2}\angle CAB \\ &= \beta + \alpha \\ &= \frac{1}{2}(2\alpha + 2\beta + 2\gamma) - \gamma \\ &= 90^\circ - \gamma\end{aligned}$$

due to angle sum of triangle ABC. However,

$$\angle ISB = \angle ASB = \angle ACB = 2\gamma$$

so by angle sum of $\triangle IBS$,

$$\angle BIS = 180^\circ - \angle IBS - \angle ISB = 180^\circ - (90^\circ - \gamma) - 2\gamma = 90^\circ - \gamma = \angle IBS$$

Thus, $\triangle IBS$ is isosceles. Similarly, $\triangle ICS$ is isosceles. Thus, $|BS| = |IS| = |CS|$ as desired.

(ii) Since chords subtend equal angles and $SB = SC$,

$$\angle BKS = \angle BAS = \angle SAC = \angle SBC = \angle SBL$$

so by converse of alternate segment theorem, (KBL) is tangent to BS. By power of a point,

$$|SL||SK| = |SB|^2 = |IS|^2$$

since $|BS| = |IS|$. By power of a point, (KIL) is tangent to SI so it is tangent to line AS.

(c) (i) Let D_1 be the second intersection of line DI with the incircle, and let E be the point on side BC such that $|BD| = |CE|$. From (a), A, D_1, E are collinear. Also, P is midpoint of AD by definition, I is midpoint of DD_1 because radii are equal, and M is midpoint of DE since

$$|DM| = |BM| - |BD| = |CM| - |CE| = |EM|$$

Thus,

$$\frac{|DP|}{|DI|} = \frac{|DA|}{|DD_1|}$$

and $\angle PDI = \angle ADD_1$ is a common angle. Thus, $\triangle ADD_1 \sim \triangle PDI$. Similarly, $\triangle D_1DE \sim \triangle IDM$. Since corresponding angles of similar triangles are equal,

$$180^\circ = \angle AD_1E = \angle AD_1D + \angle DD_1E = \angle PID + \angle IMD = \angle PIM$$

so P, I, M are collinear.

(ii) Setting $K = M$ and $L = Q$, (QIM) is tangent to line AS. By alternate segment theorem, $\angle AIQ = \angle IMQ = \angle MID$ since alternate angles on parallel lines $DI \parallel QM$ (since both are perpendicular to BC) are equal. Since QS perpendicularly bisects BC, the centre of (ABC) lies on QS so QS is a diameter. Thus, $\angle IAQ = \angle SAQ = 90^\circ$ and $\angle IDM = 90^\circ$ since MD is tangent to the incircle. Thus, $\triangle IDM \sim \triangle IAQ$ as they are equiangular.

(iii) Since AA' and QM share midpoint N, $AQA'M$ is a parallelogram. Since opposite sides of a parallelogram are equal, $MA' = AQ$. Since $\triangle IDM \sim \triangle IAQ$ and matching sides of similar triangles have equal ratio,

$$\frac{|DM|}{|MA'|} = \frac{|DM|}{|AQ|} = \frac{|DI|}{|IA|}$$

as required. Since $AQ \parallel MA'$ and alternate angles are equal,

$$\begin{aligned}\angle A'MD &= \angle A'MQ + \angle QMD \\ &= \angle AQM + 90^\circ \\ &= 90^\circ - \angle ASQ + 90^\circ \\ &= \angle 180^\circ - \angle ISQ \\ &= 180^\circ - \angle SID \\ &= \angle AID\end{aligned}$$

so $\triangle DMA' \sim \triangle DIA$ by SAS. Thus, $\angle A'DM = \angle ADI$ since matching angles in similar triangles are equal. Thus, rearranging gives

$$\frac{|DM|}{|DI|} = \frac{|MA'|}{|IA|}$$

so by SAS, $\triangle AA'D \sim \triangle IMD$.

(iv) Since P, N are the midpoints of AD, AA',

$$\frac{|AP|}{|AD|} = \frac{1}{2} = \frac{|AN|}{|AA'|}$$

and $\angle PAN = \angle DAA'$ is common. Thus, $\triangle APN \sim \triangle ADA'$ so $\angle APN = \angle ADA' = 90^\circ$ since matching angles in similar triangles are equal. Thus, NP is perpendicular to AD and bisects it, so N lies on the perpendicular bisector of AD as required.

(d) Let A' be the reflection of A over BC, which lies on (ABC) since BC is a diameter. Let D' be the reflection of D over BC. By power of a point, $|PA|^2 = |PD||PE|$ and $\angle DPA$ is common so $\triangle PDA \sim \triangle PAE$ by SAS. Similarly, since $PA = PA'$ and tangent segments are equal, PA' is tangent so $\triangle PDA' \sim \triangle PA'E$. Since similar triangles' matching sides have equal ratio,

$$\frac{|EA|}{|AD|} = \frac{|PE|}{|PA|} = \frac{|PE|}{|PA'|} = \frac{|EA'|}{|A'D|} \implies |EA||A'D| = |AD||EA'| \qquad (*)$$

Let ED intersect AA' at K. Since chords subtend equal angles, $\angle KEA = \angle DEA = \angle DA'A = \angle KA'D$ and similarly $\angle A'DK = \angle EAK$. Thus, $\triangle EAK \sim \triangle A'DK$ as they are equiangular and similarly, $\triangle EA'K =\sim \triangle AKD$. Therefore,

$$\begin{aligned}\frac{|AK|}{|KD|} &= \frac{|EA|}{|DA'|} \quad \text{and} \quad \frac{|DK|}{|KA'|} = \frac{|DA|}{|A'E|} \\ \implies \frac{|AK|}{|KD|}\cdot\frac{|DK|}{|KA'|} &= \frac{|EA|}{|DA'|}\cdot\frac{|DA|}{|A'E|} \\ \frac{|AK|}{|KA'|} &= \frac{|EA|^2|EA'||AD|}{|A'D||EA||EA'|^2} \\ \frac{|AK|}{|KA'|} &= \frac{|EA|^2}{|EA'|^2} \qquad \text{(from } (*)\text{)}\end{aligned}$$

Let M be the point on segment AA' such that $\angle AEK = \angle MEA'$. By sine rule on $\triangle EKA, \triangle EKA'$, we have

$$\frac{|AK|}{|EA|} = \frac{\sin\angle AEK}{\sin\angle EKA} \quad \text{and} \quad \frac{|A'K|}{|EA'|} = \frac{\sin\angle A'EK}{\sin\angle EKA'} = \frac{\sin\angle A'EK}{\sin\angle EKA}$$

since $\angle EKA + \angle EKA' = 180°$. Similarly, by sine rule on $\triangle EMA, \triangle EMA'$,

$$\frac{|AM|}{|EA|} = \frac{\sin \angle AEM}{\sin \angle EMA} = \frac{\sin \angle A'EK}{\sin \angle EMA} \quad \text{and} \quad \frac{|A'M|}{|EA'|} = \frac{\sin \angle A'EM}{\sin \angle EMA'} = \frac{\sin \angle AEK}{\sin \angle EMA}$$

because $\angle AEK = \angle MEA'$ and thus $\angle AEM = \angle KEA'$. Therefore,

$$\frac{|AK|}{|A'K|} \cdot \frac{|AM|}{|A'M|} = \frac{|EA|^2}{|EA'|^2} = \frac{|AK|}{|A'K|}$$

so $AM = A'M$, so M is the midpoint of AA' and is therefore on BC. We now proceed with the main proof.

Let Q' be the centre of circle (EPM). The goal is to show that Q' is in fact Q, and so it suffices to show Q' has all the desired properties:

- $Q'E$ is tangent to (ABC);
- Q' lies on AC;
- $\angle PQ'C = 90°$.

Since $\angle AEM = \angle KEA'$ as proven and $\angle EAM = \angle EAA' = \angle EDA'$ since chords subtend equal angles, $\triangle EAM \sim \triangle EDA'$ as they are equiangular. We have, by matching angles in similar triangles,

$$\begin{aligned}
\angle EA'D &= \angle AME \\
&= \angle PME - 90° \\
&= 180° - \frac{1}{2}\angle PQ'E - 90° && \text{(angle at centre is twice angle at circumference)} \\
&= 90° - \frac{1}{2}\angle PQ'E \\
&= \angle Q'EP
\end{aligned}$$

so $Q'E$ is tangent to (ABC) by the converse of alternate segment theorem, proving the first fact. Now,

$$\begin{aligned}
\angle PQ'M &= 2\angle PEM && \text{(angle at centre is twice angle at circumference)} \\
&= 2\angle DED' \\
&= 2\angle ACB && (AD = DB = BD' = D'A') \\
&= \angle PAM \\
&= \angle PAB + \angle BAM \\
&= 2\angle ACB && \text{(alternate segment theorem)}
\end{aligned}$$

so $(PQ'AM)$ is cyclic. Now, since opposite angles in cyclic quadrilaterals are supplementary,

$$\begin{aligned}
\angle Q'AM &= 180° - \angle Q'PM \\
&= 90° + 90° - \angle Q'PM \\
&= 90° + \frac{1}{2}\angle PQ'M \\
&= 90° + \angle PEM && \text{(angle at centre is twice angle at circumference)} \\
&= 90° + \angle DED' \\
&= 90° + \angle ACB && (AD = DB = BD' = D'A') \\
&= 90° + 90° - \angle MAC && \text{(angle sum } \triangle AMC) \\
&= 180° - \angle MAC
\end{aligned}$$

so $\angle Q'AM + \angle CAM = 180°$ so Q' lies on AC. This proves the second fact. Since $(PQ'AM)$ cyclic, $\angle PQ'C = \angle PQ'A = 180° - \angle PMA = 90°$ as required, proving the third fact.

Question 14

(a) (i) By integrating both components of $(\ddot{x}(t), \ddot{y}(t)) = (0, -g)$ twice we obtain the general projectile motion solution

$$x = x_0 + \dot{x}_0 t; \qquad y = y_0 + \dot{y}_0 t - \frac{1}{2}gt^2. \tag{1}$$

The release points for particles P and Q are $(x_0, y_0) = \frac{1}{\sqrt{2}}(1, -1)$ and $(x_0, y_0) = \frac{1}{\sqrt{2}}(-1, 1)$ respectively. If the release speeds are V_P and V_Q respectively, then since the direction vectors of their release velocity is $\frac{1}{\sqrt{2}}(1, 1)$, we obtain

$$x_P = \frac{r}{\sqrt{2}} + \frac{1}{\sqrt{2}}V_P t; \qquad y_P = -\frac{r}{\sqrt{2}} + \frac{1}{\sqrt{2}}V_P t - \frac{1}{2}gt^2$$

and

$$x_Q = \frac{r}{\sqrt{2}} + \frac{1}{\sqrt{2}}V_Q t; \qquad y_Q = -\frac{r}{\sqrt{2}} + \frac{1}{\sqrt{2}}V_Q t - \frac{1}{2}gt^2$$

To determine their initial speeds, we use a conservation of energy argument which is slightly out of syllabus, although one can also simply integrate the equations of motion. As the only force doing work on these particles is their weight (the tension force is orthogonal to velocity), we have conservation of total energy.

From the condition that the particles have speed (not velocity, which is vector valued) U at $(\pm r, 0)$, we have that this constant energy is given by

$$E = \frac{1}{2}mU^2$$

and so when these particles are released at height h, the release speed V satisfies

$$\frac{1}{2}mV^2 = \frac{1}{2}mU^2 + mgh \Rightarrow V = \sqrt{U^2 + 2gh}.$$

Substituting in the release heights for P, Q we obtain

$$V_P = \sqrt{U^2 + \sqrt{2}gr}; \qquad V_Q = \sqrt{U^2 - \sqrt{2}gr}.$$

(ii) First we compute the directrix equation for the general parabola $y = ax^2 + bx + c$. By completion of the square, we get

$$(x + \frac{b}{2a})^2 = \frac{1}{a}\left(y + \frac{b^2 - 4ac}{4a}\right).$$

So up to translation in x, the parabola is

$$x^2 = 4 \cdot \left(\frac{1}{4a}\right)\left(y + \frac{\Delta}{4a}\right)$$

which is now in standard form and we can read off that the directrix when $a < 0$ (as will be the case in this question) is given by

$$y = \frac{1}{4|a|}(1 + \Delta).$$

Next we compute the discriminant for the general projectile motion trajectory. From (1) we eliminate t to get the locus

$$y = \frac{2y_o\dot{x}_0^2 - 2x_0\dot{x}_0\dot{y}_0 - gx_0^2}{2\dot{x}_0^2} + \frac{\dot{x}_0\dot{y}_0 + gx_0}{\dot{x}_0^2}x - \frac{g}{2\dot{x}_0^2}x^2. \tag{2}$$

The discriminant is then

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= \frac{(\dot{x}_0\dot{y}_0 + gx_0)^2}{x_0^4} + \frac{2g}{\dot{x}_0^2}\left(\frac{2y_o\dot{x}_0^2 - 2x_0\dot{x}_0\dot{y}_0 - gx_0^2}{2\dot{x}_0^2}\right)\\ &= \dot{x}_0^{-4}(\dot{x}_0^2\dot{y}_0^2 + 2gx_0\dot{x}_0\dot{y}_0 + g^2x_0^2 + 2gy_0\dot{x}_0^2 - 2gx_0\dot{x}_0\dot{y}_0 - g^2x_0^2)\\ &= \frac{\dot{y}_0^2 + 2gy_0}{\dot{x}_0^2}.\end{aligned}$$

Hence the directrix equation is given by

$$y = \frac{1}{4|a|}(1+\Delta) = \frac{\dot{x}_0^2}{2g}\left(1 + \frac{\dot{y}_0^2 + 2gy_0}{\dot{x}_0^2}\right) = \frac{1}{mg}E$$

for both particles.

(Indeed this shows that any release point other than those where the parabola degenerates would give the same directrix.)

(iii) From the quadratic formula, we have the for the general projectile motion we have

$$\begin{aligned}R &= \frac{-\frac{\dot{x}_0\dot{y}_0+gx_0}{\dot{x}_0^2} - \sqrt{\frac{\dot{y}_0^2+2gy_0}{\dot{x}_0^2}}}{-\frac{g}{\dot{x}_0^2}}\\ &= \frac{\dot{x}_0\dot{y}_0 + gx_0 + \sqrt{\dot{x}_0^2\dot{y}_0^2 + 2gy_0\dot{x}_0^2}}{g}.\end{aligned}$$

Substituting in the initial conditions for P, Q we get

$$\begin{aligned}R_P &= \frac{\frac{1}{2}(U^2 + \sqrt{2}gr) + \frac{gr}{\sqrt{2}} + \sqrt{\frac{(U^2+\sqrt{2}gr)^2}{4} - \sqrt{2}gr\frac{(U^2+\sqrt{2}gr)}{2}}}{g}\\ &= \frac{U^2 + 2\sqrt{2}gr + \sqrt{U^4 - 2g^2r^2}}{2g}\end{aligned}$$

and

$$\begin{aligned}R_Q &= \frac{\frac{1}{2}(U^2 - \sqrt{2}gr) - \frac{gr}{\sqrt{2}} + \sqrt{\frac{(U^2-\sqrt{2}gr)^2}{4} + \sqrt{2}gr\frac{(U^2-\sqrt{2}gr)}{2}}}{g}\\ &= \frac{U^2 - 2\sqrt{2}gr + \sqrt{U^4 - 2g^2r^2}}{2g}.\end{aligned}$$

(iv) If we let the trajectories be given by $y = f_P(x), f_Q(x)$ respectively, we have $f_Q(R_Q) = 0 < f_P(R_Q)$ as $R_Q < R_P$ but on the other hand $f_Q(-r/\sqrt{2}) = r/\sqrt{2} > 0 > f_P(-r/\sqrt{2})$. Hence $f_P - f_Q$ changes sign and must have a root S_x in the interval $[-r/\sqrt{2}, R_Q]$. Moreover as $y_Q > 0$ on this interval, we have $S_y = y_Q(S_x) > 0$.

(v) Substituting the initial conditions for P, Q into the locus equation (2) we get

$$y_P = -\sqrt{2}r - \frac{gr^2}{2V_P^2} + (1 + \frac{\sqrt{2}gr}{V_P^2})x_P - \frac{g}{V_P^2}x_P^2$$

and

$$y_Q = \sqrt{2}r - \frac{gr^2}{2V_Q^2} + (1 - \frac{\sqrt{2}gr}{V_Q^2})x_Q - \frac{g}{V_Q^2}x_Q^2$$

Subtracting the coefficients in the second equation from the first we get a quadratic with roots the x-coordinates of the points of intersection of the trajectories of P, Q.

$$\begin{aligned} g\frac{V_P^2 - V_Q^2}{V_P^2 V_Q^2}x^2 + \sqrt{2}gr(\frac{V_P^2 + V_Q^2}{V_P^2 V_Q^2})x - 2\sqrt{2}r + \frac{gr^2}{2}\frac{V_P^2 - V_Q^2}{V_P^2 V_Q^2} &= 0 \\ \Rightarrow 2\sqrt{2}g^2rx^2 + 2\sqrt{2}grU^2x - 2\sqrt{2}r(U^4 - 2g^2r^2) + \sqrt{2}g^2r^3 &= 0 \\ \Rightarrow 2g^2x^2 + 2gU^2x + 5g^2r^2 - 2U^4 &= 0 \end{aligned}$$

and so

$$\begin{aligned} x &= \frac{-2gU^2 \pm \sqrt{4g^2U^4 - 8g^2(5g^2r^2 - 2U^4)}}{4g^2} \\ &= \frac{-2gU^2 \pm \sqrt{20g^2U^4 - 40g^4r^2}}{4g^2} \\ &= \frac{-U^2 \pm \sqrt{5(U^4 - 2g^2r^2)}}{2g}. \end{aligned}$$

As the smaller root is negative, we conclude that

$$S_x = \frac{-U^2 + \sqrt{5(U^4 - 2g^2r^2)}}{2g}.$$

Substituting into our locus equation for P we get

$$\begin{aligned} S_y &= y_P(S_x) \\ &= blah \\ &= \frac{-2U^2 + \sqrt{5(U^4 - 2g^2r^2)}}{g}. \end{aligned}$$

(vi)

(vii)

(b) (i) By Vieta's formulae we have

$$-\frac{1}{a_{2020}} = \sum_{i=1}^{2020} \prod_{j \neq i} r_j = \sum_{i=1}^{2020} \frac{1}{r_i} \prod_{j=1}^{2020} r_j = \frac{1}{a_{2020}} \sum_{i=1}^{2020} \frac{1}{r_i}$$

and

$$\frac{1}{a_{2020}} = \sum_{i<j} \prod_{k \notin \{i,j\}} r_k = \sum_{i<j} \frac{1}{r_i r_j} \prod_{k=1}^{2020} r_k = \frac{1}{a_{2020}} \sum_{i<j} \frac{1}{r_i r_j}.$$

Hence

$$\sum_{i=1}^{2020} \frac{1}{r_i} = -1$$

and

$$\sum_{i<j} \frac{1}{r_i r_j} = 1.$$

(ii) We have

$$\sum_{i=1}^{2020} \frac{1}{r_i^2} = \left(\sum_{i=1}^{2020} \frac{1}{r_i} \right)^2 - 2\sum_{i<j} \frac{1}{r_i r_j} = 1 - 2 = -1$$

from the previous part. But the LHS is non-negative if all r_i are real, so at least one r_i is non-real. Furthermore, by the conjugate root theorem this root has a conjugate which is also non-real. Hence at most 2018 of the 2020 complex roots of f can be real.

(iii) **No.** Suppose for the sake of contradiction that we could. Then the roots of f are $r_1, \dots, r_{2018}, \alpha, \bar{\alpha}$ with all r_i real.

The identities in (i) give

$$\frac{1}{\alpha} + \frac{1}{\bar{\alpha}} = -\sum_{i=1}^{2018} \frac{1}{r_i} - 1$$

and

$$\sum_{1 \le i < j \le 2018} \frac{1}{r_i r_j} + \left(\frac{1}{\alpha} + \frac{1}{\bar{\alpha}}\right) \sum_{i=1}^{2018} \frac{1}{r_i} = 1.$$

Substituting the first equation into the second and putting $t = \sum_{i=1}^{2018} r_i^{-1}$ we get

$$\frac{1}{2}(t^2 - \sum_{i=1}^{2018} \frac{1}{r_i^2}) - t(t+1) = 1$$

and so

$$-\frac{1}{2}t^2 - t - 1 = \frac{1}{2} \sum_{i=1}^{2018} \frac{1}{r_i^2} > 0$$

but this is absurd for real t as the quadratic in t is negative-definite (as its leading coefficient and discriminant are negative).

Question 15

(a) (i)

(b) (i) By the binomial expansion,

$$\begin{aligned}(a+p\ell)^n-(a^n+n\cdot p\ell a^{n-1}) &= \sum_{i=0}^{n}\binom{n}{i}p^i\ell^i a^{n-i}-(a^n+n\cdot p\ell a^{n-1})\\ &= a^n+\binom{n}{1}a^{n-1}p\ell+\left(\sum_{i=2}^{n}\binom{n}{i}p^i\ell^i a^{n-i}\right)-(a^n+n\cdot p\ell a^{n-1})\\ &= p^2\sum_{i=2}^{n}\binom{n}{i}p^{i-2}\ell^i a^{n-i}\\ &= p^2M\end{aligned}$$

where $M=\sum_{i=2}^{n}\binom{n}{i}p^{i-2}\ell^i a^{n-i}$ is an integer since $\mathbb{Z}$ is closed under addition and multiplication. Thus, $p^2 \mid (a+p\ell)^n-(a^n+n\cdot p\ell a^{n-1})$.

(ii) Notice that

$$a^{p-1}+\ldots+(a+p\ell)^{p-1}=\frac{(a+p\ell)^p-a^p}{(a+p\ell)-a}=\frac{(a+p\ell)^p-a^p}{p\ell}$$

From (i), $(a+p\ell)^n$ has remainder $a^n+n\cdot p\ell a^{n-1}$ when divided by p^2. Thus, $a^{p-i-1}(a+p\ell)^i$ has remainder

$$a^{p-i-1}(a^i+n\cdot p\ell a^{i-1})=a^{p-1}+n\cdot p\ell a^{p-2}$$

when divided by p^2. Therefore, $\sum_{i=0}^{p-1}a^{p-i-1}(a+p\ell)^i$ has remainder

$$\sum_{i=0}^{p-1}a^{p-1}+n\cdot p\ell a^{p-2}=p(a^{p-1}+n\cdot p\ell a^{p-2})=p\cdot a^{p-1}+p^2\ell a^{p-2}$$

which is the same as $p\cdot a^{p-1}$ since p^2 divides $p^2\ell a^{p-2}$, as desired.

(iii) For $n=0$, $p^{0+1}\mid a-b$ and $p^{0+2}\nmid a-b$ as given. Assume it is true for $n=k\in\mathbb{Z}$ with $k\geq 0$. Thus,

$$p^{k+1}\mid a^{p^k}-b^{p^k},\ \text{but} p^{k+2}\nmid a^{p^k}-b^{p^k}$$

Consider the factorisation

$$a^{p^{k+1}}-b^{p^{k+1}}=\left(a^{p^k}\right)^p-\left(b^{p^k}\right)^p=\left(a^{p^k}-b^{p^k}\right)\left(a^{p(p^k-1)}+\ldots+b^{p(p^k-1)}\right)$$

From (ii), we have that $a^{p(p^k-1)}+\ldots+b^{p(p^k-1)}$ has a remainder of $a^{p(p^k-1)}\cdot p$ when divided by p^2 and a is not divisible by p. Thus, it is divisible by p but not p^2. Therefore,

$$a^{p^{k+1}}-b^{p^{k+1}}=\left(a^{p^k}\right)^p-\left(b^{p^k}\right)^p=\left(a^{p^k}-b^{p^k}\right)\left(a^{p(p^k-1)}+\ldots+b^{p(p^k-1)}\right)$$

is divisible by $p^{k+1}\cdot p=p^{k+2}$ but not $p^{k+1}\cdot p^2=p^{k+3}$ since $p^{k+2}\nmid a^{p^k}-b^{p^k}$, as desired.

Therefore, by the principle of mathematical induction, it is true for all integers $n\geq 0$.

(iv)

(v) If $p=2$, we can check that $20^2+19^2=761$ which a prime, so $k=1$ as needed. If $p\geq 3$, then p is odd so $20^p+19^p=20^p-(-19)^p$. Since $3\mid 39=20-(-19)$, and p odd, we can factorise

$$20^p+19^p=(20+19)(20^{p-1}-20^{p-2}19+\ldots+19^{p-1})$$

Question 15

(a) (i)