# Functional Equations (Algebra)

James Bang

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## 1 Introduction

Functional equations are essentially questions in which you are given identities/properties of functions, and you need to either find all possible functions or to show a given property. The initial definitions may be quite confusing at first and functional equations are difficult in general, but they are usually fun to solve (plugging in tonnes of things!). Some of the hardest questions ever given at the IMO were functional equations.

N.B. I've never tried teaching functional equations to a student in this school, let alone a class, so pls try your best to focus in the first few lessons. If people find this too hard, we can instead learn something else.

**How to use this document:** First pls listen to me explaining this in class, because trust me it is **much** quicker for someone to explain the basics to you first than to read it yourself. Then have a skim read over the sections, start attempting the questions, and if you are stuck, read over again to look for new techniques to use. If you are stuck for a long time, then just find me somewhere and I'll give some more hints (anything but q5.1.40).

## 2 Basic Definitions

A functional equation question usually looks like this:

*Find all functions* $f : A \to B$ such that

$$f(\text{blah blah} + \text{blah blah}) + \text{blah} = \text{blah}$$

*for any two numbers* $x, y$ *in* $A$ (*or* $x, y \in A$).

- $f$ is our **function**: it could be thought of a "magic black box" such that for any input (in the domain), there is exactly one output. Examples of functions include $f(x) = x, x^2$ and 2018.
- $A$ is the **domain** of the function, and $B$ is the **co-domain** or **codomain**. Some people tend to use the words codomain and image instead, but I don't know why.
- The notation $f : A \to B$ means that the function $f(x)$ is defined whenever $x \in A$ ($x$ is an element of $A$), and whenever $x \in A$, $f(x) \in B$.
- The symbol $\mathbb{R}$ means the set of all real numbers. In a similar way, $\mathbb{N}$ (or $\mathbb{Z}^+$) is the natural numbers (**NOT INCLUDING 0**), $\mathbb{Z}$ is all integers, $\mathbb{Q}$ is the rational numbers, $\mathbb{C}$ is the complex numbers and $\mathbb{P}$ is the prime numbers. Positive real numbers can be represented by either $\mathbb{R}^+$ or $(0, \infty)$.

There are lots of other things we need to know to solve the hardest functional equations, but this will do for now. Let's do a question!

**Problem 2.0.1**

Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that for any reals $x, y$, we have

$$f(x+y) = y + f(x).$$

The question essentially says that we can plug *any* real numbers $x, y$ into the above equation and it must stay true. Always a good substitution is make something equal to 0. What about $y = 0$? We get

$$f(x+0) = 0 + f(x).$$

Damn it! We got a trivially true statement. This means $y = 0$ is a bad substitution, how about $x = 0$?

$$f(0+y) = y + f(0).$$

If we let $c = f(0)$ (which is a fixed constant), then the above turns into $f(y) = y + c$, for all reals $y$. This means $f$ is linear!

**DO NOT FORGET TO CHECK YOUR FUNCTION!!!!1!11** Olympiad markers almost surely will dock marks for not doing so. To check the function, all you have to do is plug $f(x) = x + c$ back into the original equation:

$$(x+y) + c = y + (x+c),$$

which fortunately comes out to be true. Hence $f(x) = x + c$ is the only solution, where $c$ can be any fixed real number. Easy, right?

# 3 Techniques to solve

Now that we (hopefully) have a basic understanding of what a functional equation actually is, we can start looking at some approaches to solving such equations.

## 3.1 Simple Substitution

This is essentially the simplest type of functional equation (why else is it called "simple"? Hmmm...), similar to that of Problem 2.1. However, I must stress the importance of substitution in a functional equation: **No subs, No solve !!!** Common substitutions for functional equations include:

- $x = y = 0$, $x, y = 0$, $x = y$, $x = -y$, $x = f(y)$, etc etc (assuming the substitutions are valid)
- making things cancel (or **DUHH WE WANT STUFF TO CANCEL**): if you have an equation like $f(x+a) = f(2x+b) + c$, then you can possibly substitute $x = a - b$ to "cancel" the two $f$'s.
- **Shifts**: if you have some equation true for all real $x$, then you can substitute $x$ for $x + t$ where $t$ is any real number (because, after all, $x + t$ is also a real number).

The following functional equations are solved by simple substitutions, so see if you can find them.

**Problem 3.1.1.** Does there exist a function $f : \mathbb{R} \to \mathbb{R}$, such that $f(x+y) = x + 5$ for all $x, y \in \mathbb{R}$?

**Problem 3.1.2.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x) + y) = x + y$, for all $x, y \in \mathbb{R}$.

**Problem 3.1.3.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = f(1-x) + x$, for all $x, y \in \mathbb{R}$.

**Problem 3.1.4.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x - f(y)) = 1 - x - y$, for all $x, y \in \mathbb{R}$.

**Problem 3.1.5.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(a + b^2) = 3 - f(a) + f(ab)$, for all $a, b \in \mathbb{R}$.

**Problem 3.1.6.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x^2 - 1) = f(x) + x$, for all $x \in \mathbb{R}$.

## 3.2 Swapping stuff around

Let's say we have the following functional equation:

**Problem 3.2.1**

Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that $f(f(x) + f(y)) = f(x) + y$, for all $x, y \in \mathbb{R}$.

You can try plugging $x = 0$ or $x = -y$ or whatever, but this one appears to resist the so-called "simple substitution" method. However, this problem can be instantly solved by realising that the entire left side function bracket is symmetric in $x, y$, so you can switch $x, y$ in the equation to also get

$$f(f(x) + f(y)) = f(y) + x.$$

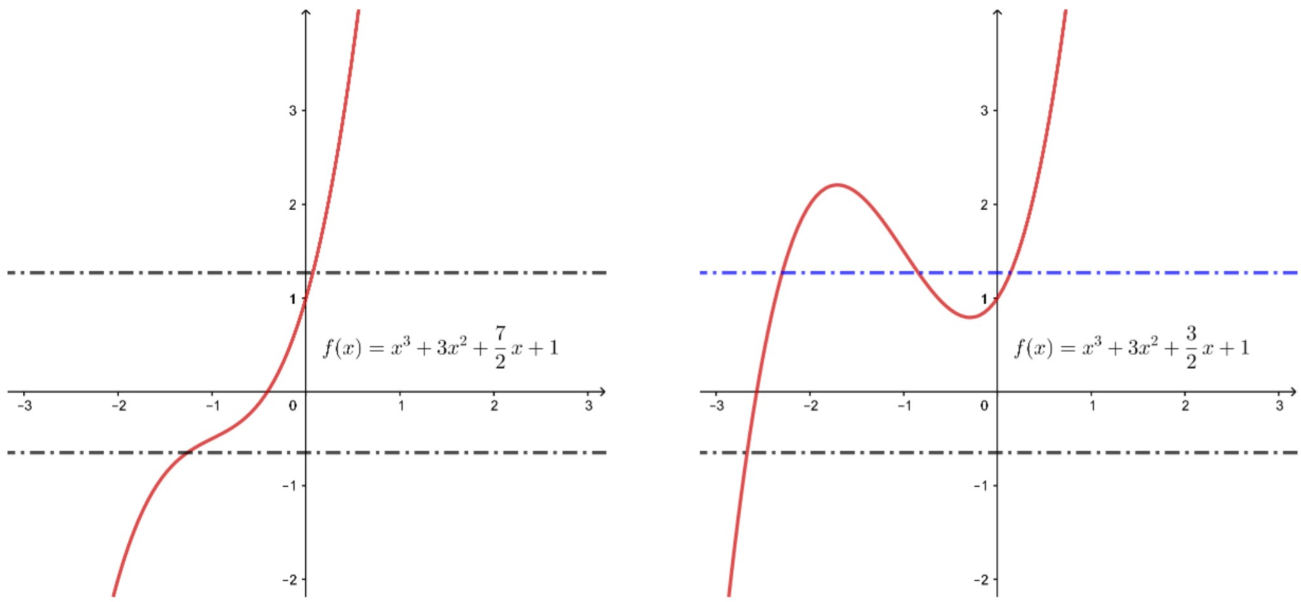
**Exercise:** Finish off the problem.

(This was actually a problem from my first AMOC math camp in year 9. I got 7/7 for this problem but used a significantly longer way oops)

No extra problems for this section; however, many of the problems in section 5 require this technique, so try those for practice.

## 3.3 Injectivity

A function is said to be **injective** if, for any two values $a, b$ such that $f(a) = f(b)$, we also have $a = b$. This means if you draw the graph and any random horizontal line, then it intersects the graph **at most once**. Examples of injective functions include $f(x) = x,\ x^3,\ \log(x)$ (in $(0, \infty)$), $e^x$ and $1/x$ (in $\mathbb{R}\backslash\{0\}$). Examples of **non-injective** functions (yes they exist!) include $f(x) = x^2$, $0$ and $|x|$. Nearly all functions $f : \mathbb{R} \to \mathbb{R}$ are **not** injective, so if you want to use injectivity in a problem, you must prove it first.




(a) Injective function

(b) Non-injective function

Many beginners to functional equations unfortunately assume injectivity for all functions and proceed to "solve" problems (we usually call this a "fakesolve"), so do not fall into this trap! In this section, I will show you how to show a function is injective.

**Example:** Show that the function $f(x) = x^3$ is injective.

This example is just for the sake of familiarity with the concept. To do this, we often say "Suppose there exist real numbers $a, b$ such that $f(a) = f(b)$" (i.e. there is a horizontal line intersecting the graph at two points $x = a$ and $x = b$), and try to show $a = b$. We know $f(a) = f(b)$, so

$$a^3 = b^3 \iff (a-b)(a^2+ab+b^2) = 0.$$

If $a \neq b$, then we must have $a^2 + ab + b^2 = 0$; however this cannot be the case since

$$a^2 + ab + b^2 = \frac{a^2 + b^2 + (a+b)^2}{2} > 0,$$

which is bad. Hence $f(x) = x^3$ is injective. Yay! Now let's look at an actual functional equation.

**Problem 3.3.1**

Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that $f(x) + xy = yf(x) + f(f(x))$, for all $x, y \in \mathbb{R}$.

So, seeing the section, you would think we need to prove injectivity for this problem, and you are right. How do we do this? Well, once again, "suppose there exist real numbers $a, b$ such that $f(a) = f(b)$", and we will try to show $a = b$. In this case, if we let $x = a$, and $x = b$, we get

$$f(a) + ay = yf(a) + f(f(a)), \quad f(b) + by = yf(b) + f(f(b)).$$

Now the magical thing is since we assumed $f(a) = f(b)$, we know $f(a) = f(b)$, $yf(a) = yf(b)$ and $f(f(a)) = f(f(b))$. Hence subtracting the two equations gives us

$$ay = by \implies a = b.$$

Hence it's injective. Yay! Oh wait... How do we use this injectivity then?

Well, in this case, it becomes fairly obvious once we plug in $y = 0$, because we are left with $f(x) = f(f(x))$. Injectivity! We can thus "remove" an '$f$' from both sides, leaving us with $x = f(x)$. Yay!

As you can see, problem 3.3.1 looks a lot more intimidating than our Example, but they are both essentially using the same idea of showing that $f(a) = f(b)$ implies $a = b$, and solving problems using this.

**Note:** Sometimes you cannot show a function is injective, because perhaps it isn't! (for instance $f(x) = x^2$ from $\mathbb{R} \to \mathbb{R}$) However, a lot of the time you can show a weaker version of injectivity, for instance if $f(a) = f(b)$ then $a^2 = b^2$, which doesn't quite show $a = b$ but is good enough to solve the question. Problem 3.3.5 illustrates this idea.

**Problem 3.3.2.** Suppose we are given two functions $f, g : \mathbb{R} \to \mathbb{R}$, such that both $f$ and $g$ are injective. Show that the function $f(g(x))$ is also injective.

**Problem 3.3.3.** Show that a **strictly increasing** function $f : \mathbb{R} \to \mathbb{R}$ (that is, any two values $x, y$ with $x > y$ satisfies $f(x) > f(y)$) is injective.

**Problem 3.3.4.** Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that $f(x + f(y)) + f(y) = f(f(x)) + 2y$.

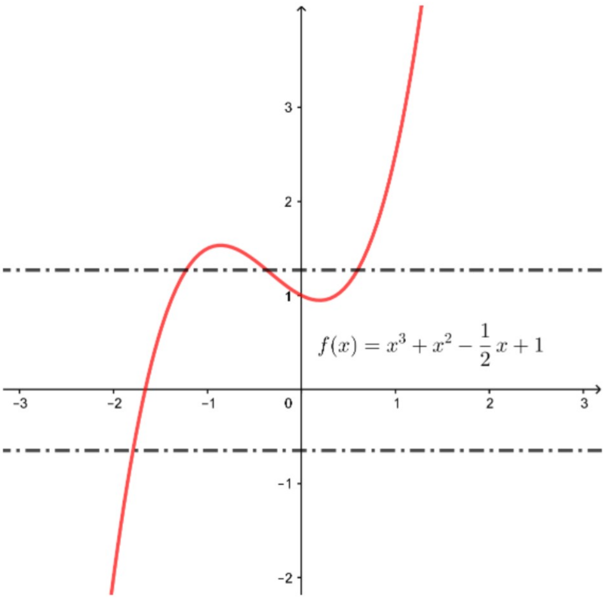
**Problem 3.3.5.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)^2 + yf(x)) + y^2 = f(f(x)f(y) + y^2) + x^2$.

Try problem 5.1.39 for a big challenge!

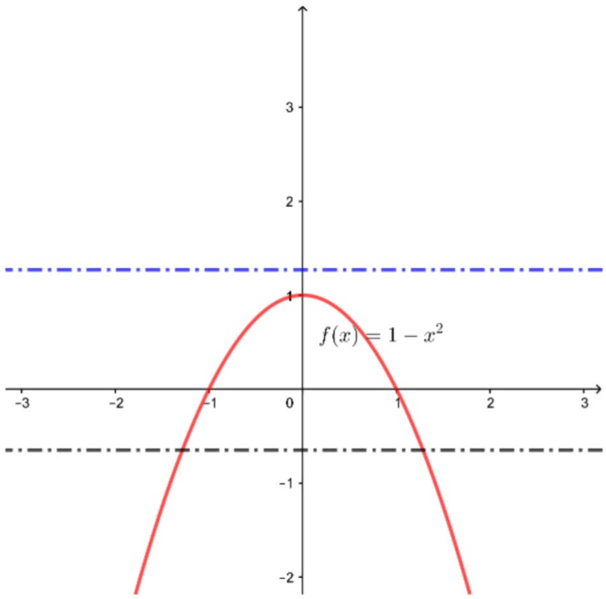
## 3.4 Surjectivity

A function $f : A \to B$ is said to be **surjective** if for any $b \in B$, there exists $a \in A$ such that $f(a) = b$. In a graphical way, if a function is surjective, then any horizontal line will intersect the graph in **at least one point**.

Put in another way, having surjectivity is useful in expressions such as $f(f(x)) - f(x)^3 = 0$ where you can just replace $f(x)$ with $x$ to get $f(x) - x^3 = 0$. Examples of surjective functions include $f(x) = x, x^3, \log(x)$ (in $(0, \infty)$) and $1/x$ (in $\mathbb{R}\backslash\{0\}$). Examples of **non-surjective** functions (yes they exist) include $f(x) = x^2, 0, |x|$ and $e^x$. Most functions from reals to reals are not surjective, so you need to prove it before using it. In this section I will show you how to show a function is surjective by using some examples.




(a) Surjective function




(b) Non-surjective function

**Problem 3.4.1**

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x) + f(y) + x) = y$ for all reals $x, y$.

Once again, the section tells it all, we probably have to show this function is surjective. In this case, however, the "intermediate" step of rearranging to show surjectivity has already been done, because it's in the form $f(\text{bleh}) = y$. We can then say $f$ is surjective, because for each real number $y$, there exists a number $z = f(x) + f(y) + x$ such that $f(z) = y$, thus satisfying the definition of surjectivity as per above. Yay... wait... now what?

One of the powerful things about surjectivity is how it tells us that there exists a constant $c$ such that $f(c) = 0$ (i.e. the line $y = 0$ intersects the function somewhere). Plugging $x = y = c$ gives

$$f(f(c) + f(c) + c) = c \implies f(0 + 0 + c) = c \implies c = 0,$$

so $f(0) = 0$. Then plugging $x = 0$ gives $\boxed{f(f(y)) = y}$, and finally "$f$"ing both sides (yes this is allowed!) gives

$$f(f(f(x) + f(y) + x)) = f(y) \implies f(x) + f(y) + x = f(y)$$

and thus finally $f(x) = -x$ for all reals $x$. Yay! (Don't forget to check the function though)

Although this problem is in the "surjectivity" section, it is in fact possible to solve this alternatively by proving injectivity, plugging $y = f(z)$ for some $z$, and canceling $f$'s from both sides. Try this!

**Problem 3.4.2.** Show that any quadratic function (i.e. $f(x) = Ax^2 + Bx + C$ for $A \neq 0$) is **not** surjective, while any cubic function ($f(x) = Ax^3 + Bx^2 + Cx + D$ for $A \neq 0$) is surjective.[1]

**Problem 3.4.3.** A function $f$ satisfies $f(x + f(y)) = f(x) + y^3$. Show that $f$ is surjective.

**Problem 3.4.4.** Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that $f(f(x)^2) + yf(x) + y = f(xf(x) + y)$.

## 3.5 Cauchy's Functional Equation

Cauchy's functional equation is the equation

$$f(x + y) = f(x) + f(y).$$

Functions that satisfy this condition are called *additive*. The functions $f(x) = kx$ and 0 are solutions to Cauchy's functional equation, and in particular they are the **only** solutions in many cases such as functions from $\mathbb{Q}$ to itself or $\mathbb{R} \to \mathbb{R}^+$.

Let's solve the following question:

**Problem 3.5.1**

Suppose a function $f : \mathbb{R} \to \mathbb{R}$ is a solution to $f(x + y) = f(x) + f(y)$. Then show that $f(n) = Cn$ for any integer $n$, where $C$ is a fixed number.

Well, where could we possibly start? Plugging $x = y = 0$ in the equation gives $f(0) = 2f(0)$, so $f(0) = 0$. What about $x = y$? That gives us

$$f(2x) = f(x) + f(x) = 2f(x).$$

In a similar way, $y = 2x$ gives us

$$f(3x) = f(x) + f(2x) = f(x) + 2f(x) = 3f(x),$$

and it can be seen that $f(nx) = f(x) + f((n-1)x) = f(x) + f(x) + f((n-2)x) = \cdots = nf(x)$ whenever $n$ is an integer, and in particular, if we set $x = 1$ we get $f(n) = nf(1) = Cn$ where $C$ is fixed. Yay!

**Problem 3.5.2.** Solve the above question except prove instead that $f(q) = Cq$ for any rational $q$.

Given the above results, we would be inclined to believe that $f(x) = Cx$ for all reals $x$. However, this is not true: there are very nasty non-linear additive functions $f$ which are so crazy that a graph of such a function would be completely black! (These functions can only be defined by something called the Axiom of Choice.)

However, in an olympiad environment additive functions are more likely to be linear than not, and any of the following conditions are enough to imply that $f$ is linear:

- $f$ is bounded below or above on an interval (there exists real numbers $a < b$ and $N$ such that $f(x) < N$ for all $a < x < b$, or that $f(x) > -N$ for all $a < x < b$)

[1] In fact, any polynomial $f(x) = \sum_{k=0}^{n} a_k x^k$, when $a_n \neq 0$, is surjective **if and only if** $n$ is odd. Prove this if you can!

- $f$ is continuous at a single point
- $f$ is increasing (for any reals $x < y$, you have $f(x) \leq f(y)$) or decreasing.

**Problem 3.5.2.** The functional equation $f : \mathbb{R} \to \mathbb{R}$, satisfying for all real $x, y$

$$\frac{f(x) + f(y)}{2} = f\left(\frac{x+y}{2}\right),$$

is called **Jensen's Functional Equation**. Show that any function that satisfies Jensen's equation and $f(0) = 0$ is also additive.

**Problem 3.5.3.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$.

**Problem 3.5.4.** Find all functions $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x+y)f(x-y) = [f(x)f(y)]^2$.

## 3.6 Periodic Functions

A function is **periodic** if there is a real number $t$ such that $f(x+t) = f(x)$ for any real $x$. It is *called* periodic since after $|t|$, the function starts repeating again. If you can show a function is periodic, then

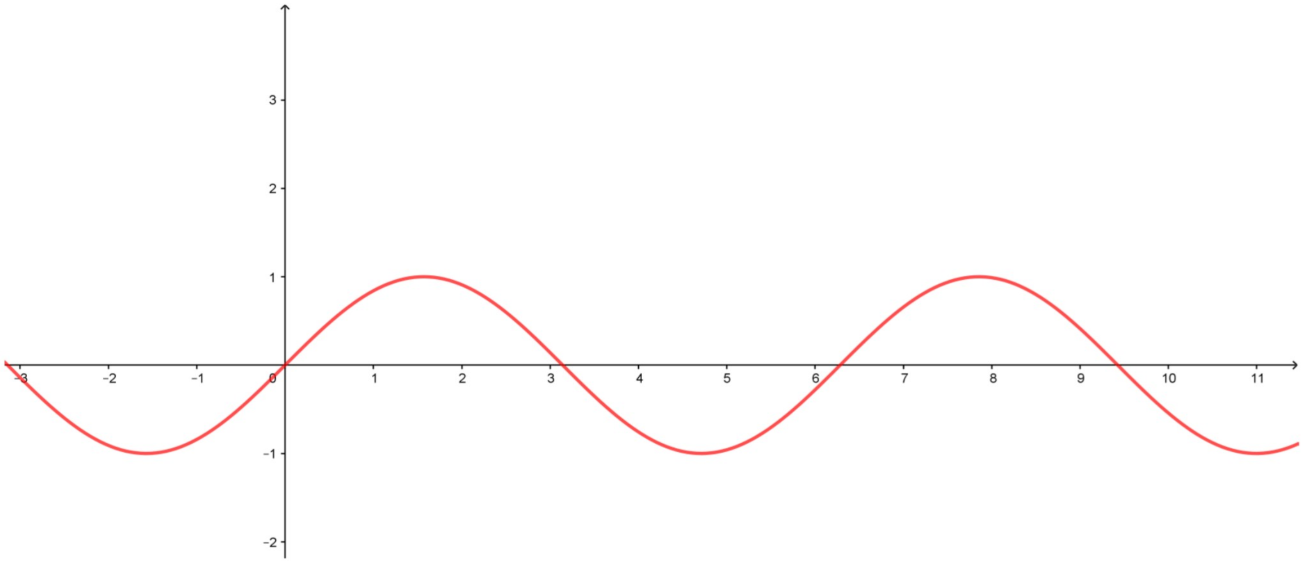



Figure 3: A periodic function ($f(x) = \sin x$).

this is an extremely strong property since you only need to determine the values inside one period. However there are many, very unusual properties that periodic functions could have, so be careful!

- A function $f : \mathbb{R} \to \mathbb{R}$ **may not necessarily have a "minimal period"**. Notice that in our $f(x) = \sin x$ graph, the minimal period is $2\pi$; there are functions out there which have successively smaller periods tending to 0. A classic example is the Dirichlet's function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$$

  which essentially says "$f(x)$ is 1 for rationals and 0 for irrationals". This function is periodic with any rational number as a period (!).

Apparently many people fell into this trap when solving the following functional equation, so let's look at it together.

**Problem 3.6.1**

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y^2 - f(y)) = f(x)$ for all reals $x, y$ .

The problem essentially says that if there exists $y$ with $f(y) - y^2 \neq 0$, then $f$ is periodic, and it has $y^2 - f(y)$ as a period. How strange.

So if I let $t \neq 0$ be a period (so $f(x+t) = f(x)$, where $t$ could possibly be $y^2 - f(y)$), then what happens? I could shift $x \to x + t$, but this gives a trivial statement. How about $y \to y + t$? We get

$$f(x + y^2 - f(y)) = f(x) = f(x + (y + t)^2 - f(y + t)) = f(x + (y + t)^2 - f(y)),$$

and we can then shift $x \to x - y^2 + f(y)$ to get

$$f(x) = f(x + (y + t)^2 - y^2) = f(x + (2yt + t^2)).$$

We assumed $t \neq 0$. But that means $2yt + t^2$ ranges across all real numbers, so $f(x) = f(x + y)$ for any real $y$. This implies $f$ is constant! (Unless, of course, no such period $t$ exists in which case you must have $y^2 - f(y) = 0$ for any $y$, giving $f(x) = x^2$ as the other solution.)

**Problem 3.6.2.** A function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x+2) = f(x-1) \cdot f(x+5)$ for all real $x$. Show that $f$ is periodic.

**Problem 3.6.3.** A function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x+2) = f(x-1) + f(x+5)$ for all real $x$. Show that $f$ is periodic.

**Problem 3.6.4.** A function $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x)| \leq 1$ for any real $x$, and

$$f\left(x + \frac{13}{42}\right) + f(x) = f\left(x + \frac{1}{6}\right) + f\left(x + \frac{1}{7}\right).$$

Prove that $f$ is a periodic function.

**Problem 3.6.5.** Show that the functional equation in Problem 5.1.16 satisfies the weaker form of injectivity ($f(a) = f(b)$ implies $a^2 = b^2$), and thus solve the problem. (Hint: show that if $f$ does not satisfy weak injectivity as above, then $f$ is periodic.)

## 3.7 Transformations (or so-called "Angelo's guess and hope Method")

This technique is transforming a function into another in order to remove clutter and/or to simplify the question. For instance, suppose you think the function $f(x) = x^2$ is the only solution; then you can try considering functions such as $g(x) = f(x) - x^2$, $f(x)/x^2$, or even $\sqrt{f(x)}$ (if $f(x) \geq 0$ for all $x$). As the name suggests, none of the functions may work, so some guessing (and hoping) may be required.

**Problem 3.7.1**

Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(x + y) = f(x) + f(y) + 2xy$, $x, y \in \mathbb{Q}$.

Well well. It looks a bit like Cauchy, right? Oh wait there's that pesky $2xy$ term on the right, which kinda stuffs it up a bit.

Playing around with the function a bit, it may occur that the function $f(x) = x^2$ is a valid function; it may also occur that $f(x) = x^2 + kx$ is also valid whenever $k$ is a fixed real number. So we may want to perhaps try use the function $g(x) = f(x) - x^2$? Plugging in, we get

$$g(x+y) + (x+y)^2 = g(x) + x^2 + g(y) + y^2 + 2xy$$

$$g(x+y) = g(x) + g(y).$$

Tada! It's Cauchy's functional equation, and we can continue as we did above!

**Problem 3.7.2.** Find all functions $f : \mathbb{Q} \to \mathbb{Q}$, such that $f((x+y)^3) = f(x^3) + f(y^3) + x^2y + xy^2$.

**Problem 3.7.3.** Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$, such that $f(x + f(y)) = f(x+y) + f(y)$ whenever $x, y > 0$.

Challenge: Solve Problem 5.1.27!

# 4 Common Traps, Pitfalls and things

There are many traps and pitfalls when solving a functional equation. **Most of these arise because one assumes something "intuitive" which may not always hold true.** When solving a functional equation, you must not give the function the "benefit of the doubt" and must rigorously show each manipulation to be awarded full marks. (Reals to reals is a big place after all)

## 4.1 "Pointwise Trap"

Suppose you get the identity $f(x)^2 = x^2$ for all real $x$. You may be tempted to jump to the "logical" conclusion that $f(x) = x$ or $f(x) = -x$ are the only solutions. However, this is not true, because you could have $f(x) = x$ for some $x$, and $f(x) = -x$ for some others. This is called the "pointwise trap", and many beginners make this mistake.

**Problem 4.1.1**

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y)f(x-y) = (f(x) + f(y))^2 - 4x^2 f(y).$$

Ok then. So first, $x = y = 0$ (always a good substitution) gives $f(0)^2 = 4f(0)^2 - 0$ so $f(0) = 0$. Then $x = y$ (another good sub) gives $f(2x)f(0) = 4f(x)^2 - 4x^2 f(x)$, or as $f(0) = 0$, we have

$$f(x)^2 = x^2 f(x).$$

At this point a beginner may jump to the conclusion that "Hey we must have either $f(x) = 0$ for all $x$, or that $f(x) = x^2$ for all $x$ !!!". Oops, as you can have $f(x) = x^2$ for some $x$ and 0 for others.

To get around this trap, we need to **assume there exists real numbers** $a, b \neq 0$ **such that** $f(a) = a^2$ **and** $f(b) = 0$. Plugging $y = b$ into the original gives

$$f(x+b)f(x-b) = f(x)^2.$$

Now supposing you have $f(x) \neq 0$ (for instance $x = a$), you must have $f(x+b)f(x-b) \neq 0$. Since $f(x)$ is either 0 or $x^2$, this means $f(x+b) = (x+b)^2$ and $f(x-b) = (x-b)^2$, which gives

$$(x+b)^2(x-b)^2 = f(x)^2 = x^4 \;\Rightarrow\; (x^2-b^2)^2 = x^4 \;\Rightarrow\; x = \pm b/\sqrt{2}.$$

In particular, this means that if $t \neq \pm b/\sqrt{2}$, then $f(t) = 0$, so plugging $y = t$ gives

$$f(x+t)f(x-t) = f(x)^2$$

for all $t \neq \pm b/\sqrt{2}$. Then you can let $x = b/\sqrt{2}$ and $b + t$ larger than $b/\sqrt{2}$ which gives $f(x+t) = 0$ and so $f(x) = 0$. Similar for $x = -b/\sqrt{2}$. Hence if such $b$ exists, then $f(x)$ must be 0. If $b$ does not exist, then $f(x) = x^2$. Hence $f(x) = x^2$ and $f(x) = 0$ are the only solutions.

**Problem 4.1.2.** Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any real numbers $a, b, c, d > 0$ with $abcd = 1$, $(f(a) + f(b))(f(c) + f(d)) = (a + b)(c + d)$.

**Problem 4.1.3.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(xf(x) + f(y)) = f(x)^2 + y$.

## 4.2 Other pitfalls / Common errors

- The two sets $\mathbb{N} = \mathbb{Z}^+$ and $\mathbb{R}^+ = (0, \infty)$ **DOES NOT CONTAIN 0**
- Make sure your substitution is valid; for instance, you cannot sub in $\sqrt{2}$ if it is $f : \mathbb{Q} \to \mathbb{R}$.
- **BEWARE OF THE POINTWISE TRAP!**
- A periodic function (one in which there exists real $t$ such that $f(x + t) = f(x)$ for all $x$, for instance $\sin x$) DOES NOT necessarily have a "minimal period". In particular it is possible for a non-constant function $f$ to have both $\pi$ and $e$ as periods.
- Some functional equations, even from real olympiad competitions and sources, do not have a nice set of solutions. The most extreme example of such a problem is the 2004 problem $f : \mathbb{R} \to \mathbb{R}$ with $f(x^2 + y^2 + 2f(xy)) = (f(x + y))^2$, which has the solutions $f(x) = 0$, $f(x) = x$ as well as

$$f(x) = \begin{cases} -1 & \text{if } x \in R \\ 1 & \text{if } x \notin R \end{cases}$$

  whenever $R$ is any subset of $(-\infty, -2/3)$. (There are no more solutions.) Wow!
- **CHECK YOUR FUNCTION!!**

# 5 Problems

Problems should get harder as you go down the list. Some of these are rather hard, and the final ones may require a lot of thinking before any progress is made: for instance, I could not solve 5.1.40 within a week.

Stars indicate relative difficulty; they may not be completely accurate. Have fun!

**Problem 5.1.1.** Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that $f(x + y)^2 = f(x)^2 + f(y)^2$.

**Problem 5.1.2.** Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that $f(x^2 + y) = f(x^{27} + 2y) + f(x^4)$.
(Hint: **"DUHH WE WANT STUFF TO CANCEL"**)

**Problem 5.1.3.** Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that

$$f(x - y)f(x + y) - (x + y)f(x - y) = 4xy(x^2 - y^2).$$

(Hint: we have $x + y$ and $x - y$ terms everywhere, what substitution can we make to get rid of them?)

**Problem 5.1.4.** Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that $f(x + y^2) = f(x^2) + f(y)$.

**Problem 5.1.5.** Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that $x^2 f(x) + f(1 - x) = 2x - x^4$.

**Problem 5.1.6.** Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that $xf(y) + yf(x) = (x + y)f(x)f(y)$.

**Problem 5.1.7.** Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that $f(x^2 + y) = xf(x) + f(y)$.

**Problem 5.1.8.** Find all functions $f : \mathbb{R}\backslash\{0, 1\} \to \mathbb{R}$ (i.e. you cannot plug $x = 0$ or 1) such that

$$f(x) + \frac{1}{2x} f\left(\frac{1}{1 - x}\right) = 1.$$

**Problem 5.1.9.** Find all functions $f : \mathbb{R}\backslash\{\frac{2}{3}\} \to \mathbb{R}$ such that

$$504x - f(x) = \frac{1}{2} f\left(\frac{2x}{3x - 2}\right).$$

**Problem 5.1.10.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(2018x - f(0)) = 2018x^2$.

***Problem 5.1.11.** Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) + f(f(n)) = 2n$.

***Problem 5.1.12.** Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that $f((x - y)^2) = x^2 - 2yf(x) + (f(y))^2$.

***Problem 5.1.13.** Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that for all $x, h \in \mathbb{R}$, $|f(x + h) - f(x)| \le h^2$.

***Problem 5.1.14.** A function $f : \mathbb{N} \to \mathbb{N}$ satisfies $f(n) < f(n + 1)$ and $f(f(n)) = 3n$ for each $n \in \mathbb{N}$. Find $f(2016)$.

***Problem 5.1.15.** Find all functions $f : \mathbb{N} \to \mathbb{N}$, such that $n + f(m) \mid f(n) + nf(m)$ for all $m, n \in \mathbb{N}$.

****Problem 5.1.16.** Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that $f(x^2 + y) - f(f(x) - y) = 4yf(x)$. (Hint: Similar idea to 5.1.2)

****Problem 5.1.17.** Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$, such that $f(xf(y)) = f(xy) + x$.

****Problem 5.1.18.** Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that $f(x + y) + y \le f(f(f(x)))$.

****Problem 5.1.19.** Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$, such that for all $x, y \in \mathbb{R}$

$$f(x)f(y) = f(y)f(xf(y)) + \frac{1}{xy}.$$

****Problem 5.1.20.** Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$, such that $x^2(f(x) + f(y)) = (x + y)f(yf(x))$.

****Problem 5.1.21.** Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that $f(x + y) = f(x) + f(y)$ and $f(x^{2018}) = f(x)^{2018}$.

****Problem 5.1.22.** Find all **injective** functions $f : \mathbb{R} \to \mathbb{R}$, such that for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\left| \sum_{i=1}^{n} i\left(f(x + i + 1) - f(f(x + i))\right) \right| < 2018.$$

****Problem 5.1.23.** Does there exist a function $f : \mathbb{N} \to \mathbb{N}$, such that $f(f(n)) = n + 2019$ for any $n \in \mathbb{N}$?

*****Problem 5.1.24.** Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$, such that for all $w, x, y, z$ with $wx = yz$,

$$\frac{f(w)^2 + f(x)^2}{f(y^2) + (z^2)} = \frac{w^2 + x^2}{y^2 + z^2}.$$

*** **Problem 5.1.25.** Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that for any $x, y \in \mathbb{R}$,

$$f(\lfloor x \rfloor \cdot y) = f(x) \cdot \lfloor f(y) \rfloor.$$

(Note: $\lfloor x \rfloor$ is the "floor" function: it is the largest integer not exceeding $x$, for instance $\lfloor 3 \rfloor = 3$ and $\lfloor \pi \rfloor = 3$.)

*** **Problem 5.1.26.** Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that $f(x^2 + yf(x)) = xf(x + y)$.

*** **Problem 5.1.27.** Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that

$$f(x + f(y)) - f(x) = (x + f(y))^4 - x^4.$$

*** **Problem 5.1.28.** Determine all functions $f : \mathbb{N} \to \mathbb{N}$, such that for all positive integers $a, b$ there exists a non-degenerate triangle with side lengths $a, f(b)$ and $f(b + f(a) - 1)$.

*** **Problem 5.1.29.** Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that $x(f(x + y) - f(x - y)) = 4yf(x)$.

*** **Problem 5.1.29.** Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that $f(x^2 + f(y)) = y + [f(x)]^2$.

*** **Problem 5.1.30.** Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$, such that

$$xf(x^2)f(f(y)) + f(yf(x)) = f(xy)\left(f(f(x^2)) + f(f(y^2))\right).$$

**** **Problem 5.1.31.** Do there exist two functions $f, g : \mathbb{R} \to \mathbb{R}$, such that for all real $x$:

1. $f(g(x)) = x^2$ and $g(f(x)) = x^3$ ?
2. $f(g(x)) = x^3$ and $g(f(x)) = x^4$ ?

**** **Problem 5.1.32.** Let $G$ be a set of nonconstant, linear functions such that for all $f, g \in G$, we have $g(f(x)), f^{-1}(x) \in G$ and $f$ has a fixed point. Show that there exists a real $t$, for which $f(t) = t$ for all $f \in G$. (A *fixed point* $t$ of a function is just a point where $f(t) = t$.)

**** **Problem 5.1.33.** Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that $f(yf(x + y) + f(x)) = 4x + 2f(x + y)$.

**** **Problem 5.1.34.** Find all functions $f : \mathbb{Q}^+ \to \mathbb{R}^+$, such that $f(xy) = f(x + y)(f(x) + f(y))$.

**** **Problem 5.1.35.** Find all functions $f : \mathbb{Q} \to \mathbb{R}$, such that $f(x^{2019} + (f(y))^{2019}) = (f(x))^{2019} + y^{2019}$.

******** **Problem 5.1.36.** Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that

$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x).$$

************ **Problem 5.1.37.** Let $f$ be any function that maps the set of real numbers to the set of real numbers (i.e. $f : \mathbb{R} \to \mathbb{R}$). Prove that there exists real numbers $x, y$ such that

$$f(x - f(y)) > yf(x) + x.$$

******************** **Problem 5.1.38.** Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1.$$

****************************** **Problem 5.1.39.** Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that

$$f(f(x)f(y)) + f(x + y) = f(xy).$$

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**Problem 5.1.40.** A function $f : \mathbb{R} \to \mathbb{R}$ has the following property:

For every $x, y \in \mathbb{R}$ such that $(f(x) + y)(f(y) + x) > 0$, we have $f(x) + y = f(y) + x$.

Prove that $f(x) + y \leq f(y) + x$ whenever $x > y$.