Graph Theory (Combinatorics)

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1 | Introduction

Combinatorics refers to the vast area of mathematics concerned with counting. As an olympiad topic, it is (subjectively, in my opinion) the most difficult area because there's no universal method to attack such problems, unlike functional equations, Diophantine equations and even geometry to some extent.

Graph Theory is a method of converting many types of combinatorial problems into a simpler restatement involving dots (**vertices**) and **edges** connecting the vertices. There are several techniques which exist for solving such graph theory problems which you require to solve many difficult combinatorics problems. Many of them are detailed in this handout.

How to use this document: Read the handout, and try to make sense of as much as you can before I go over it in class. Also, to solve any combinatorics problem at this stage, you WILL need to use paper. Try drawing graphs for every problem (small cases) and even for the problems that I provide solutions to, as diagrams are a lot easier to understand than a bunch of words and symbols.

2 | Definitions: What is a "graph"??

A **graph** G, in simplest forms, is just a collection of points, known as **vertices**, some of which are joined together by line segments called **edges**. We denote the set of vertices as V, and the set of edges as E.

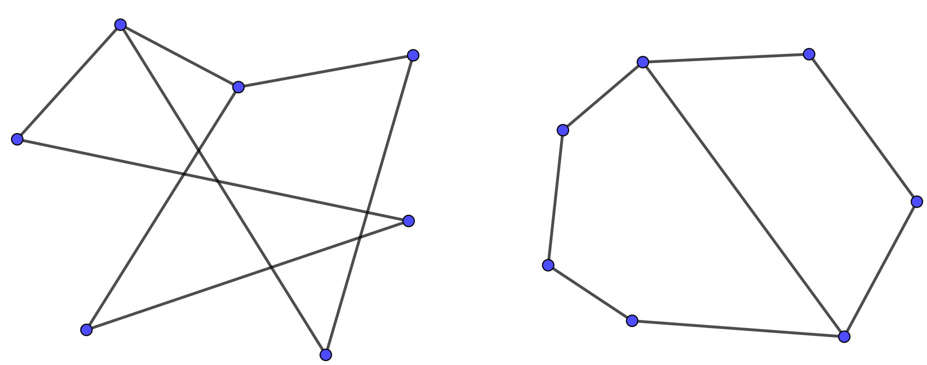

Figure 1: Two graphs I have randomly generated.

In drawing a graph, unless explicitly specified, we do not care about the lengths of the edges or the positions of the vertices (the geometry of the figure is more or less irrelevant).

Two graphs G and G' are identical (or **isomorphic**) if they are "the same": they have the same number n of vertices, and it is possible to label the vertices of both G and G' with each of the numbers $1, 2, \ldots, n$

such that vertices i, j are adjacent in G if and only if i, j are adjacent in G'. In other words, you can move around some vertices of G to get G'.

Problem 2.0.1

Show that in Figure 1, the two graphs are identical to each other.

All that suffices to solve this problem is to number the vertices of both graphs so that two vertices are adjacent in one graph if and only if they are adjacent in the other graph. This is not hard, so it is left an exercise to the reader.

In most cases, we usually refer to **simple graphs** as "graphs". A simple graph is essentially one where there is at most one edge between any two vertices, and furthermore edges to itself (**loops**) are not allowed. We call graphs where multiple edges and loops are allowed as **multigraphs**.

2.1 Other important graph theory definitions

- We call two vertices **adjacent** if there exists an edge that connects them. A vertex is adjacent to an edge if the edge is connected to that vertex (so the vertex is an endpoint of that edge).
- The **degree** of a vertex v is the number of vertices which are adjacent to v. (This set of adjacent vertices does NOT include vertex v.) We denote the degree of v by $\deg(v)$.
- A **path** in a graph G between two distinct vertices v, w is a sequence $v_1, v_2, \ldots, v_n$ of *distinct* vertices with $v_1 = v$ and $v_n = w$ such that v_i and v_{i+1} are adjacent for each $i = 1, \ldots, n-1$. Put simply, you can travel from v to w by following a bunch of edges. A **cycle** is a path where the first and last vertices are also adjacent. The **length** of a path/cycle is the number of edges in it.
- A graph G is **connected** if for every pair v, w of vertices, there exists a path between v and w. The graph is called k**-connected** if removing any $k-1$ edges still leaves a connected graph.
- A **tree** T is a connected graph with no cycles. It can be shown that any tree with n vertices has exactly $n-1$ edges.
- A **subgraph** H of a vertex G is essentially a subset of the vertices and edges of G, such that every edge in H is an endpoint of exactly two vertices in H. A subgraph H is called an **induced subgraph** if you can obtain H just by deleting some vertices (and all its adjacent edges) from G.
- The **complete graph** K_n is a graph on n vertices such that every possible pair of vertices are adjacent. Similarly, an **independent set** is a graph where no two vertices are adjacent to each other (so no edges).
- A graph G is **bipartite** if it is possible to partition $G = A \cup B$ such that any edge occurs between a vertex $a \in A$ and some vertex $b \in B$. A bipartite graph $K_{m,n}$ is called **complete** if it can be split into $A \cup B$ with $|A| = m, |B| = n$ and every possible edge between A and B is drawn (and no other edges exist). Similarly we can define the complete multipartite graph $K_{n_1,\ldots,n_k}$.
- A graph G is called **planar** if it is possible to draw G on the plane such that no two edges intersect each other. For instance, the graph in Figure 1 is planar because the second one (isomorphic to the first by Problem 2.0.1) has no crossing edges.
- A graph G is called k**-regular** if every vertex has degree exactly k. (The name may be misleading in that k-regular graphs may not necessarily have any geometric symmetry at all)

- The **complement** of a graph G, denoted by $\overline{G}$, has exactly all the edges that G is missing.

2.2 Definitions and Notations for Directed Graphs

A **Directed Graph** is essentially just a graph where each edge is replaced with an arrow. Such graphs are useful in representations of wins/losses in games or tournaments.

- The **in-degree** of a vertex is the number of edges pointing into the vertex. Similarly, the **out-degree** is the number of edges pointing out of the vertex.
- A **tournament** is the complete graph on n vertices, except that every edge is directed.
- A **directed path** is a sequence of vertices $v_1, \ldots, v_k$ for which v_i points towards v_{i+1} for each $i = 1, \ldots, k-1$. A **directed cycle** is a directed path except that v_k points to v_1 as well.

3 | Double Counting / Counting in 2 Ways

Double counting works as follows. Suppose you want to count something: if you use two different methods and obtain A and B as their values (and both are finite), then $A = B$.

For such an obvious result (trivial to prove since A and B count the same thing), this is actually very useful when used with the right methods.

Problem 3.0.1: Binomial Expansion

Let n be a positive integer, and $\binom{n}{k}$ is the number of ways to choose k objects from a possible n. Show that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

I mean, we all know the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, but substituting into the above gives no reasonable cancellation at all (big sad). Hence, we have to resort to combinatorial methods as opposed to algebra.

The right side, 2^n, denotes the number of ways to choose a subset out of $\{1, \ldots, n\}$ of arbitrary size (a subset size 0 or n is allowed). The left side counts the number of ways to choose a subset of size k from $\{1, \ldots, n\}$, summed from $k = 0$ to n. It becomes clear that both sides both count the number of ways to choose a subset of $\{1, \ldots, n\}$, which means both sides are equal to each other. $\square$

Next, we present a very useful lemma used all the time.

Problem 3.0.2: Handshake Lemma

In a party of n people, the k'th person shakes hands with exactly d_k other people for $k = 1, \ldots, n$. If E is the total number of handshakes, show that

$$d_1 + d_2 + \cdots + d_n = 2E.$$

By using the natural graph theoretic transformation, the problem is equivalent to the following: In a graph G of n vertices and E edges, we have

$$\sum_{v \in G} \deg(v) = 2|E|.$$

For each edge, count the number of times it appears in the left. Notice that it is counted exactly two times, one on each of the vertices that it is adjacent to. Hence, the left side counts twice the number of all edges in the graph, and the result follows. □

Directed graphs also have their analogue to the Handshake Lemma.

Problem 3.0.3: Handshake Lemma for Directed Graphs

In a directed graph G, each vertex v_i has in-degree w_i and out-degree ℓ_i for each $i = 1, \ldots, n$. Show that $\sum w_i = \sum \ell_i$.

Notice that $\sum w_i$ is the total in-degree of G. Hence, we want to show that the total in-degree is equal to the total out-degree. Let's remove the edges of G, one at a time. Each edge has exactly one in-degree and one out-degree, so the difference of the sums are still the same. When there are no edges left, the total in-degree and out-degree are both 0. Hence, they must be equal from the start. □

Finally, we present a harder, more brutal and algebraic problem.

Problem 3.0.4: Mantel's Theorem, $n = 3$ case of Turán's Theorem

A graph G has no triangles (cycles of length 3). Show that it has at most $n^2/4$ vertices.

You can try the usual "suppose $> n^2/4$ edges" and the usual proof by contradiction, but it really doesn't seem to go anywhere. Hmmm.

To solve this question, we first need to find a better way to characterise a triangle. A triangle is essentially when two adjacent vertices are both adjacent to a third. So for two vertices v, w which are adjacent, they are together adjacent to $\deg(v) + \deg(w) - 2$ vertices (other than each other). Since there are $n - 2$ vertices other than v, w in G, it follows that if $\deg(v) + \deg(w) > n$, they are both adjacent to some common vertex, forming a triangle which is bad. Hence, we have $\deg(v) + \deg(w) \leq n$ whenever vw is an edge of G. Summing this over all edges gives

$$\sum_{vw \in E} (\deg(v) + \deg(w)) \leq n|E|.$$

Now we have to count the left side.

Lemma:

$$\sum_{vw \in E} (\deg(v) + \deg(w)) = \sum_{v \in G} (\deg(v))^2.$$

Proof: For each vertex v, count the number of times $\deg(v)$ shows up on the left hand side. It shows up every time v is part of some edge vw, and so it turns up exactly the number of edges v is adjacent to, which is $\deg(v)$ times. Hence, the left side is equal to the sum of $\deg(v) \cdot \deg(v) = (\deg(v))^2$ over v, which is equal to the right side.

It remains to show that $\sum_{v \in G} (\deg(v))^2 \leq n|E|$ implies $|E| \leq n^2/4$. Recall the quadratic-arithmetic mean inequality, which states if a_i are nonnegative real numbers then

$$\sqrt{\frac{a_1^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + \cdots + a_n}{n} \iff a_1^2 + \cdots + a_n^2 \geq \frac{(a_1 + \cdots + a_n)^2}{n}.$$

Hence, using this gives

$$\sum_{v\in G}(\deg(v))^2 \geq \frac{(\sum \deg(v))^2}{n} = \frac{4|E|^2}{n},$$

where the second equality occurs from Problem 3.0.2. Hence, rearranging this yields

$$\frac{4|E|^2}{n} \leq n|E| \implies |E| \leq \frac{n^2}{4},$$

which is as desired. $\square$

Problem 3.0.5. Prove the generalisation of Problem 3.0.1, the general case of Newton's binomial formula which states $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$, using a similar double counting method.

Problem 3.0.6. Use a similar method to Problem 3.0.3 to obtain a lower bound for the number of triangles: specifically, if G is a graph with n vertices, E edges and T triangles, then $T \geq \frac{E(4E-n^2)}{3n}$.

Problem 3.0.7. Given a graph G, denote $\chi(G)$ as the **chromatic number** of G, i.e. the minimal number of colours required to colour the vertices of G such that no 2 adjacent vertices are the same colour. Show that if G has E edges, then $\chi(G) \leq \frac{1}{2} + \sqrt{2E + \frac{1}{4}}$.

Problem 3.0.8. Let G be a graph with n vertices and E edges which does not contain a 4-cycle (a cycle of length 4). Show that $E \leq \frac{n}{4}(1+\sqrt{4n-3})$.

Problem 3.0.9. There are n points in the plane, no three of which are collinear. Show that there are at most $\frac{2}{3} \cdot n(n-1)$ triangles of area 1.

Problem 3.0.10. Let n, k be positive integers and S is a set of positive integers in the plane, such that:

1. No three points of S are collinear
2. For every point $P \in S$, there are at least k points in S equidistant from P.

Show that $k < \frac{1}{2} + \sqrt{2n}$.

4 | The Pigeonhole Principle

The **Pigeonhole Principle** (PHP) is the statement that if you have $n+1$ objects ("pigeons") in n or less sets ("pigeonholes"), then there will exist two sets which contain at least two objects. This is fairly easy to show: suppose you have n pigeonholes (if less than n, then insert random empty pigeonholes). If each set contains at most one object, then there is at most n objects altogether. However we have $n+1$ objects, which is a contradiction.

For such a simple statement, the pigeonhole principle has far-reaching consequences, and in fact many difficult olympiad problems can be solved by choosing the right pigeons and pigeonholes. Of course, the difficult part lies within choosing such objects.

Problem 4.0.1

In a school of $n \geq 2$ people, any pair are either friends or not frieds (friendship is mutual). Show that there exists two people with the same number of friends.

We use the natural graph theoretical interpretation: we are given a simple graph of n vertices where $n \geq 2$, and want to show there exists two vertices with the same degree.

A vertex can have degree $0, 1, 2, \ldots, n-1$: it can't have at least n because there are only $n-1$ vertices other than that vertex itself (a person can't be friend with themselves). However, since there are only n vertices in the graph we currently cannot use the pigeonhole principle (there are n objects (vertices) n pigeonholes (the possible degree).

Even though PHP does not directly apply, we can still use the general idea. If there are n objects and n sets, and no 2 objects are in the same set, then each set must have exactly one object. Hence, supposing there is no 2 people with the same number of friends, for every number between 0 and $n-1$ there exists a vertex with that degree. In particular, there exists two vertices with degree 0 and $n-1$ respectively. However, the vertex with degree $n-1$ must be adjacent to every vertex, including the vertex with degree 0, which is a contradiction. Hence, there are two people with the same number of friends. $\square$

Next, we present a slightly more obscure application of PHP.

Problem 4.0.2

There are 6 people in a room, every two of whom are either friends or strangers. Show that there exists three people who are either all friends with each other or all strangers with each other.

The natural graph theoretic translation is that if we colour all the edges of the complete graph K_6 of 6 vertices in either red or blue (friends or strangers), then there exists a monochromatic triangle[1].

Consider a vertex v from the graph. It has 5 neighbours, and each of the 5 adjacent edges are either red or blue. Hence, by PHP one of the two colours occur at least $\lceil \frac{5}{2} \rceil = 3$ times. WLOG assume this colour is red.

Let A, B, C be the three vertices adjacent to v such that Av, Bv, Cv are all red. Then if AB is red, then vAB is a completely red triangle, and so AB must be blue. Similarly BC, CA are also blue. However, then ABC is a completely blue triangle. Hence, in all possible cases there must exist some monochromatic triangle. $\square$

Problem 4.0.3. There are 1001 people at a party. Suppose that for every three people, some two are friends (friendship is mutual). Show that someone is friends with at least 500 other people. Can you show there are at least 501 such people who are friends with at least 500 others?

Problem 4.0.4. In a graph G of n vertices, denote the **independence number** $\alpha(G)$ as the maximal size of an independent subgraph of G. Show that

$$\alpha(G) \cdot \chi(G) \geq n,$$

where $\chi(G)$ is the chromatic number.

Problem 4.0.5. In a school of $n^2 + 1$ people ($n \geq 5$), every person is friends with exactly n other people. Show that there exists two people who are not friends and share no common friends.
(**HARD**: Show that n can be lowered to 4.)

5 | More on Planar Graphs

Recall that a graph is planar if it can be drawn in such a way that none of its edges intersect. Planar graphs hence look a bit like a polyhedron, except they could be disconnected and perhaps have vertices of degree $0, 1$ which make them not look like polyhedrons. Of the theorems in planar graph theory, perhaps one of the most important results is the following:

[1]A **monochromatic triangle** is a triangle of which all its edges are coloured the same.

Problem 5.0.1: Euler's Characteristic

Let G be a connected planar graph, with E edges and V vertices. Also let F be the number of "faces", i.e. the number of regions (including the outermost, unbounded region) bounded by edges of the graph. Then the following equation holds:

$$V + F = E + 2.$$

To prove Euler's formula we use some induction. When there are no cycles, G becomes a tree and so $E = n - 1$, $V = n$. Also there's exactly one face (the infinite region) so $F = 1$. Hence $V + F - E = n + 1 - (n - 1) = 2$, and so the equation holds in this case.

Otherwise, some cycle exists. If we pick an edge e out of some cycle and delete it, clearly V stays the same, E decreases by one (we deleted 1 edge), and F decreases by one (since the two faces on either side of the graph merge, so net loss is 1). Hence $V + F - E$ remains constant, and thus by induction it is equal to 2. $\square$

Using Euler's Characteristic, we can find some basic, necessary results for a graph to be planar.

Problem 5.0.2

Suppose G is a planar graph with E edges and V vertices, and there exists some $\ell \geq 3$ such that G has at least ℓ edges and every cycle of G has length $\geq \ell$. Then

$$|E| \leq \frac{\ell}{\ell - 2}(V - 2).$$

To show this, notice that every edge can be adjacent to at most two edges (one on each side of the edge). Furthermore, every face is adjacent to at least ℓ edges. Consider the number T of pairs (face, edge) where the edge is part of the face. Then $T \leq 2E$ since every edge is adjacent to at most two faces. On the other hand, $T \geq \ell \cdot F$ since every face is adjacent to at least ℓ edges. Thus, we have $\ell \cdot F \leq 2E$.

Substitute into the Euler Characteristic to obtain

$$V - 2 = E - F \geq E\left(1 - \frac{2}{\ell}\right) \implies |E| \leq \frac{\ell}{\ell - 2}(V - 2). \quad \square$$

A very useful corollary of Problem 5.0.2 is setting $\ell = 3$, a planar graph G has at most $3V - 6$ edges.

Problem 5.0.3

Prove that the complete graph K_5 and the complete bipartite graph $K_{3,3}$ are not planar.

The latter we've seen countless times as the "Three Utilities Problem". Suppose there are three cottages and each needs to be connected to some gas, water and electricity companies. Is there a way to make all nine connections without any of the lines crossing each other?

To show K_5 is not planar, notice that it has 5 vertices and 10 edges. This violates $E \leq 3V - 6$ since $10 > 3(5) - 6 = 9$.

To show $K_{3,3}$ is not planar, notice that $K_{3,3}$ does not have any 3-cycles (triangles); this is easy to see by inspection. Hence, we can set $\ell = 4$ so $E \leq \frac{4}{2}(V - 2) = 2(V - 2)$, which is violated since $K_{3,3}$ has 9 edges and $9 > 2(6 - 2) = 8$. $\square$

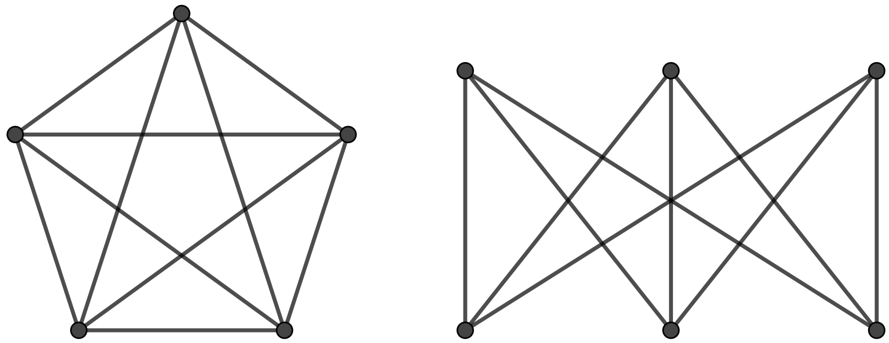

Figure 2: The graphs K_5 and $K_{3,3}$.

In fact, **Kuratowski's Theorem** states that a graph G is planar **if and only if** it does not contain K_5 or $K_{3,3}$ as a subgraph. This theorem is very difficult to prove.

6 More on Trees

Remember that a tree is a connected graph on n vertices with exactly $n-1$ vertices. It turns out that trees can be characterised in many different ways, including some below.

Problem 6.0.1

A connected graph G is a tree, if and only if:

1. G has exactly $n-1$ edges (by definition)
2. G does not contain any cycles
3. The removal of any edge disconnects the graph
4. There exists exactly one path joining any two vertices.

By definition, (1) is true. To show (1) $\Longrightarrow$ (2), if G has a cycle you can remove an edge of that cycle and still have a connected graph with $\geq n-1$ edges, so it can't be a tree. To show (2) $\Longrightarrow$ (1), start with the graph on n vertices and 0 edges. This has exactly n "connected components". At every second, let's add in some edge to reconstruct G. Then since there are no cycles, the number of connected components decreases by 1 every time. Hence, the first time G is connected, there's exactly one connected component so we have added in $n-1$ edges.

(3) $\Longleftrightarrow$ (4) is obvious since if there's exactly one path between any two vertices, then deleting some edge will disconnect the two vertices on either side of that edge. (2) $\Longleftrightarrow$ (4) is also obvious since if there is a cycle, then there exists two distinct paths between two vertices within that cycle. $\square$

6.1 Spanning Trees

For a connected graph G, a **spanning tree** is a tree which contains every vertex of G. It can be seen as the "backbone" of the graph G. Some graphs can have multiple spanning trees, however every connected graph must have at least one spanning tree.

Problem 6.1.1

Every connected graph G has a spanning tree.

To prove this, if G is already a tree then we are good. Otherwise it is not a tree, so by (3) in Problem 6.0.1, the graph must have some cycle. Remove some edge in this cycle: then the graph still remains connected, and it loses one edge. Keep doing this until there are no cycles left, and at this point you are left with a tree. This tree is a spanning tree of G. $\square$

Let's start solving problems based on trees!

Problem 6.1.2

Show that in every connected graph G, there exists a vertex v such that its removal (alongside all its edges) $G \setminus v$ remains connected.

It looks quite hard at first, but if you consider the spanning tree H of G, then you really only need to show the result for trees (if for some vertex v you have $H \setminus v$ is connected, then $G \setminus v$ is also connected since $H \setminus v \subset G \setminus v$).

To prove this result for trees H, all you really need to do is to find a vertex v of H that has degree 1, as removal will clearly leave H connected. Suppose every vertex has degree ≥ 2: then the total degree of all vertices of H is $\geq 2n$ (since n vertices). However, the Handshake Lemma shows that the total degree of all vertices is also $2E = 2(n-1) < 2n$, which is a contradiction. $\square$

Next, we present a very useful lemma.

Problem 6.1.3: Balancing Trees

A connected graph G with V vertices has every degree at most Δ. Show that G can be partitioned into two connected subgraphs, each having at least $\frac{V-1}{\Delta}$ vertices.

So we may WLOG assume G to be a tree by otherwise taking the spanning tree of G: this loses no information, as every vertex still has degree at most Δ.

For each vertex v, denote $f(v)$ as the largest size of a connected component created upon the deletion of v (and all of its edges). Take the vertex v for which $f(v)$ is minimal. Then $f(v) \geq \frac{V-1}{\Delta}$ by PHP (since there are $\leq \Delta$ connected components formed by the deletion of v), but also $f(v) \leq V/2$ since otherwise take the first vertex w adjacent to v in the largest connected component, then clearly w has a smaller $f(w)$ as the largest connected component has $f(v) - 1$ vertices, contradicting minimality. Hence, split the tree at vw so you will get two connected subgraphs both with size $\geq \frac{V-1}{\Delta}$. $\square$

Problem 6.1.4. Let G be a graph where every vertex has degree $\geq \delta$ for some integer $\delta \geq 1$. Show that any tree T of $\delta + 1$ vertices is a subgraph of G.

Problem 6.1.5. In the infinite grid $\mathbb{Z}^2$, call a figure consisting of a bunch of squares "connected" if for every two cells, you can travel from one to another by a path of squares, every two sharing a common side. A *dinosaur* is a connected figure with ≥ 2019 cells. We call a dinosaur *primitive* if it cannot be partitioned into two or more dinosaurs. Find the maximal number of cells in a primitive dinosaur.

Problem 6.1.6. Consider a (not necessarily convex, but not self-intersecting) polygon with n vertices. Show that it is possible to draw a diagonal completely within this polygon, which splits its vertices into two sets both with at least $\frac{n}{3} - 1$ vertices.

7 Miscellaneous Graph Theory Problems

In lots of occasions, graph theory turns up in some rather unexpected places. Here are some examples of problems that do not appear very combinatorial, but in fact have a solution based on graph theory.

Problem 7.0.1

Given a function f, denote $f^k(n)$ to be f applied n times (so $\underbrace{f(f(\dots(f}_{k}(n))\dots))$. Does there exist a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$, such that

$$f^n(n) = n + 1$$

for each integer n?

Wow, this totally does not look like graph theory at all! However, it turns out to have a natural graph theoretic interpretation. Let G be a directed graph where each vertex is labelled $1, 2, \dots$ (so each positive integer appears exactly once). We assign directed edges to some pairs of vertices to G, such that each vertex has exactly one out-degree, and for each $n \geq 1$, applying f exactly n times to n gives $n+1$. Does there exist any such graph?

Lemma: f is surjective for each $n \geq f(1)$.

Proof: Induct on n. For $n = f(1)$, well obviously it lies in the range of f. For $n = k \geq f(1)$, suppose there exists u with $f(u) = k$. Then $f^k(f(u)) = f^k(k) = k + 1$, so there exists v with $f(v) = k + 1$, and so the inductive step is done. Hence f is surjective for $n \geq f(1)$.

In particular, this means that we cannot get stuck in a cycle since we must be able to attain all sufficiently large positive integers just by following the arrows.

Anyway, plugging $n = 1$ you get $f(1) = 2$, so in graph G we have an arrow from 1 to 2. Then since $n = 2$ gives $f(f(2)) = 3$, in graph G there's some arrow from 2 to some k, and from k to 3. If $k = 3$ then $f(3) = 3$ so we are stuck in a loop around 3 forever, which is a contradiction. Otherwise $k \geq 4$, but then following the arrows you have $2 \to k \to 3 \to \cdots \to 4 \to \cdots \to k$, and we get stuck in another cycle. Hence, no such graph G exists, and hence no such f exists as well. $\square$

Problem 7.0.2. Let n be a positive integer. Find all polynomials P, Q with real coefficients such that $P(Q(x)) = P(x)^n$ holds for every real number x.

8 Other Useful Theorems

- **Four Colour Theorem:** The vertices of a planar graph can be coloured in 4 colours such that any two adjacent vertices have different colours. ($\chi(G) \leq 4$)
- **Hall's Marriage Theorem:** There are n boys and n girls in a school. For each subset S of the boys (possibly size 1), denote $\Gamma(S)$ as the set of girls that at least one of the boys in S likes. Then, a boy can be paired to a girl he likes (so no two boys paired to same girl) if and only if for each nonempty set S of boys, the following holds:

$$|\Gamma(S)| \geq |S|.$$

9 | Problem Section

With any Olympiad-level mathematics problem, the following questions will require some level of trying and creativity as well as some subset of the above techniques to solve. Problems should become harder as you go down the list, and stars indicate relative difficulty. Good luck!

Problem 9.1. Find the number of graphs that can be drawn from n distinguishable vertices.

Problem 9.2. Prove that a graph G is bipartite if and only if it has no odd cycles.

Problem 9.3. You are visiting a palace. The travel guide tells you that there are 2019 rooms, each room having 10 doors. One of these doors is the royal exit. Show that the travel guide is a liar.

Problem 9.4. In a chess tournament, every pair of players play exactly one game. It turns out that there were no draws, and no player won all of their games. Show that there exists three players A, B, C for which A beat B, B beat C and C beat A.

Problem 9.5. In a certain country there are n cities, where each pair of cities is connected by a one-way road. Show that there exists some city C which you can get to from any other city, using at most one intermediate city.

Problem 9.6. Suppose there are n cities, where the distance between any two cities is distinct. Suppose further that any city is connected only to its closest city (other than itself) by a one-way road. Show that no two roads intersect, and there cannot exist a cycle of roads.

Problem 9.7. Given is a connected graph G with every vertex having even degree. Show that it is possible to start at a vertex, walk across every edge exactly once, and return to the original vertex.[2]

Problem 9.8. In a country of n cities, each pair is connected by either train or bus. Show that it is possible to traverse from any city to any other using exactly one method of transport.

Problem 9.9. In a tournament of n people $P_1, \ldots, P_n$, suppose player P_i has w_i wins and ℓ_i losses. Show that

$$\sum_{k=1}^{n} w_k^2 = \sum_{k=1}^{n} \ell_k^2.$$

Problem 9.10. Suppose every vertex of a graph G has degree at most Δ. Show that it is possible to colour the vertices using at most $\Delta + 1$ colours, such that no edge has its vertices coloured the same. ($\chi(G) \leq \Delta + 1$)

Problem 9.11. There are 20 football teams taking part in a tournament. On day 1, all teams play one match. On day 2, all teams play another match with a different team. Show that it is possible to select 10 teams where no two have played one another.

Problem 9.12. Show that every tournament on n vertices contains a Hamiltonian path, i.e. a sequence of vertices $v_1, \ldots, v_n$ for which $v_i \to v_{i+1}$ for each $i = 1, \ldots, n-1$.

***Problem 9.13.** There are $2n + 1$ people in a school. It is known that for each set S of at most n people in the school, there's some person not in S who is friends with everyone in S. Show that someone is friends with everyone else.

***Problem 9.14.** Let G_1, G_2, G_3 be three (possibly overlapping) graphs on the same vertex set. Suppose $\chi(G_1) = 2, \chi(G_2) = 3$ and $\chi(G_3) = 4$. Let $G = G_1 \cup G_2 \cup G_3$ be the graph formed by the union of all edges of G_1, G_2, G_3. Show that $\chi(G) \leq 24$.

[2] This is known as an **Eulerian Circuit**.

***Problem 9.15.** Each edge of a planar graph G is oriented with an arrow such that every vertex has at least one in-degree and at least one out-degree. Show that some face of G has its edges oriented in a cycle.

***Problem 9.16.** Given a graph G of V vertices and E edges, show that there exists an induced subgraph H of G such that each vertex has degree at least E/V.

***Problem 9.17.** Let G be a connected, planar graph. Show that it is possible to orient the edges of G (give it a direction) such that each vertex has at most 3 arrows pointing away from it.

***Problem 9.18.** A house has an even number of lamps distributed among its rooms in such a way that there are at least three lamps in every room. Each lamp shares a switch with exactly one other lamp, not necessarily from the same room. Each change in the switch shared by two lamps changes their states simultaneously. Prove that for every initial state of the lamps there exists a sequence of changes in some of the switches at the end of which each room contains lamps which are on as well as lamps which are off.

***Problem 9.19.** (**Ramsey's Theorem.**) Given positive integers r, s, denote $R(r, s)$ as the smallest n for which any colouring of the edges of K_n using red/there contains either a red K_r or a blue K_s as a subgraph. Show that $R(r, s) \leq R(r-1, s) + R(r, s-1)$. Deduce that $R(r, s)$ is always finite.

(The problem of determining $R(r, s)$ for most values of r, s remain an open question. Problem 4.0.2 shows $R(3, 3) \leq 6$, and in fact equality holds by a simple colouring of K_5. We currently do not know the value of $R(5, 5)$, although it's known that $43 \leq R(5, 5) \leq 48$.)

****Problem 9.20.** (**Turàn's Theorem**.) Let G be a graph on n vertices, E edges which does not contain K_k for some integer k. Then show that

$$E \leq \left\lfloor \left(\frac{n}{k}\right)^2 \cdot \binom{k}{2} \right\rfloor.$$

Furthermore show that equality is achieved for the complete k-partite graph $G = K_{m_1, m_2, \ldots, m_k}$ where the m_i pairwise differ by at most 1 and $\sum m_i = n$.

****Problem 9.21.** Find the maximum number E such that the following holds: There is an edge-colored graph with 60 vertices and E edges, with each edge colored either red or blue, such that there is no monochromatic cycles of length 3 or 5.

****Problem 9.22.** For a pair $A = (x_1, y_1), B = (x_2, y_2)$ of points in the coordinate plane, let $d(A, B) = |x_1 - x_2| + |y_1 - y_2|$. The pair (A, B) is called harmonic if $1 < d(A, B) \leq 2$. Determine the maximum number of harmonic points amongst 100 points in the plane.

****Problem 9.23.** (**Dirac's Theorem**.) Show that a graph G on n vertices, where every vertex has degree at least $n/2$, contains a Hamiltonian cycle.

****Problem 9.24.** In a country, there are N airlines that offer two-way flights between pairs of cities. Each airline offers exactly one flight from each city in such a way that it is possible to travel between any two cities in the country through a sequence of flights, possibly from more than one airline. If $N - 1$ flights are cancelled, all from different airlines, show that it is still possible to travel between any two cities.

*****Problem 9.25.** A group of n people are at a party. No one is friends with everyone else. Every pair of strangers have exactly one common friend. No three people are mutually friends. Prove that everybody has the same number of friends.

*****Problem 9.26.** In every cell of a square table is a number. The sum of the largest two numbers in each row is a and the sum of the largest two numbers in each column is b. Prove that $a = b$.