2020 Training Set T9 Notes

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August 7, 2020

Abstract

Overall, I doubt T9 would be a good representative for a paper in the IMO. Questions 1 and 3 are standard and can be solved with enough experience, and the difficulty of question 2 was on the lower end for a Q2.

I would like to thank Angelo and Sampson for giving me the opportunity to take part in the Mentor Program for the 2020 International Mathematical Olympiad, and to all 10 students who submitted solutions for this problem set.

1 Question 1

Find all triples (a, b, c) of positive integers such that $a^3 + b^3 + c^3 = (abc)^2$.

Overall, this problem was well done amongst all students. However, I was quite disheartened to see a lot of people getting a reasonable (yet somewhat large) bound for b, c and proceeding to bash out numerous cases, when this could have been easily avoided had the student bothered to spend a bit more effort getting a better bound. This resulted in inflated Question 1 page number counts for many students.

There were also some students who either did not solve the question or performed a lot of working out (mostly not to do with bounding). Even with those who solved the question, the proofs ranged from being fairly long to multidimensional case bashes repeating the same logic multiple times which were irritating to read. This question should by now be a very standard exercise, as these types of questions have come up numerous times in past competitions.

Answer: All permutations of $(1, 2, 3)$.

Motivation for solution: The total degree of the right side is 6 and the left side is 3. However, the degree of a on the right side is 2 and on the left side is 3. This implies that, if a is the largest of the three, then a must grow at a pace significantly faster than the other two. You can then use divisibility (or other means) to bound a from above, and combine with bounds attained before.

Solution 1 (James Bang (also Fredy, Jason and Zian up to $b \times c^4 \leq 18$)).
Since the equation is symmetric, we can assume $a \geq b \geq c$. Then,

$$(abc)^2 = a^3 + b^3 + c^3 \leq 3a^3 \implies a \geq \frac{b^2c^2}{3}.$$

However, we also have

$$a^2 \mid b^3 + c^3 \implies 2b^3 \geq b^3 + c^3 \geq a^2 \geq \left(\frac{b^2c^2}{3}\right)^2 \implies b \cdot c^4 \leq 18.$$

Since $b \geq c$ by assumption, we have $c^5 \leq 18$, giving $\boxed{c = 1}$.

Hence, the equation becomes $a^3 + b^3 + 1 = a^2b^2$. We do a similar bounding trick to the above: we have

$$a^2b^2 = a^3 + b^3 + 1 \leq 2a^3 + 1 \implies b^2 \leq 2a + \frac{1}{a^2}$$

which yields $b^2 \leq 2a$, as $1/a^2 < 1$ unless $a = b = c = 1$ which doesn't work. Also,

$$a^2 \mid b^3 + 1 \implies b^3 + 1 \geq a^2 \geq \left(\frac{b^2}{2}\right)^2 \implies b^4 \leq 4(b^3 + 1).$$

Standard analysis shows that this implies $b \leq 4$, as else if $b > 4$ then $b^3(b - 4) \leq 4$ which is impossible as $b^3 > 4^3 > 4$ and $b - 4 \geq 1$.

Finally, since $a^2 \mid b^3 + 1$ we just need to check for all $b \leq 4$ such that $b^3 + 1$ has a square factor greater than 1, which is only possible when $b = 2, a = 3$ or $b = 3, a = 2$. The latter is impossible since $b > a$, and thus the only solution (when $a \geq b \geq c$) is $(a, b, c) = (3, 2, 1)$.

Solution works, since $a^3 + b^3 + c^3 = 36 = (3 \times 2 \times 1)^2$.

Comment 1.1. It is possible to deduce from the earlier bound $b \cdot c^4 \leq 18 \implies b \leq 18$. From here, it is straightforward to check all b from 1 to 18, which some students (partially) did. But again, why not just get the better bound

Solution 2 (Andres, Kevin, Yasiru, Official Solution 4)
Assume $a \geq b \geq c$. Then, we have $0 < b^3 + c^3 = a^2b^2c^2 - a^3 = a^2(b^2c^2 - a)$, so $a < b^2c^2$.

Consider the graph of $f(x) = x^2(b^2c^2 - x)$ in the range $x \in [b, b^2c^2 - 1]$. It is easy to see that the function only has at most 1 maximum turning point in its range and no minimum turning points, and thus $f(x)$ attains its minimum on the boundary, i.e.

$$f(x) \leq \min\left(f(b), f(b^2c^2 - 1)\right).$$

Case 1 Minimum is $f(b)$. Then, $2 \cdot b^3 \geq b^3 + c^3 = f(a) \geq f(b) = b^2(b^2c^2 - b)$, which gets $3b^3 \geq b^4c^2$ or $b \cdot c^2 \leq 3$. This means $c = 1$, and $b \in \{1, 2\}$. Both cases have been tested in Solution 1, to find the solution $(a, b, c) = (3, 2, 1)$.
Case 2 Minimum is $f(b^2c^2 - 1)$. Then, $2 \cdot b^3 \geq b^3 + c^3 = f(a) \geq f(b^2c^2 - 1) = (b^2c^2 - 1)^2$. If $c > 1$, then

$$2b^3 \geq (b^2c^2 - 1)^2 = (4b^2 - 1)^2 > 16b^4 - 8b^2 \implies 16b^2 - 2b - 8 < 0$$

which has no positive integer solutions b. Thus $c = 1$, and similarly

$$2b^3 \geq (b^2 - 1)^2 > b^4 - 2b^2 \implies b^2 - 2b - 2 < 0$$

and therefore $b \leq 2 \implies b \in \{1, 2\}$. All of these cases have once again been tested in Solution 1. □
Comment 1.2. While this solution is more complex than the straightforward bounding from Solution 1, it does have the advantage of resulting in lower bounds for b, which I guess is more convenient.
Comment 1.3. It is also possible (as some of the students did) to instead use the bound $a < b^2c^2/3$ from before, and instead use the inequality

$$f(x) \leq \min\left(f\left(\frac{b^2c^2}{3}\right), f(b^2c^2 - 1)\right).$$

While it does work, the bounding is slightly uglier than just using b.

Marking Criteria:
7- Criteria

- Any full solution will receive **7 marks.**
- Each significant error (i.e. not checking solutions, minor incorrect bounds, etc) will **lose 1 mark**, down to a minimum of 5 marks.

0+ Criteria

- Guessing the full correct solution, i.e. $(a, b, c) = (1, 2, 3)$ and its permutations, and checking the solution will receive **1 mark**.
- From Solution 1: (cumulative)
  - Finds a suitable inequality involving a, b, c: **1 Mark**
  - Correctly proves $c = 1$ and finds an upper bound for b - **4 Marks**
  - Resolving the final cases - **1 Mark**
- From Solution 2: (cumulative)
  - Explicitly considers $f(x) = x^2(b^2c^2 - x)$ and notes where the minimum occurs - **2 Marks**
  - Correctly addresses both cases arising from the boundaries - **2 Marks each**
- In the case where a student has ideas from both Solution 1 and 2, the maximum mark from the approaches will be rewarded (the marks are not cumulative)

2 Question 2

Let $n \geq 2$ be a given integer. Leo the leopard lives in some cell of a $3n \times 3n$ grid. Per jump, he lands on another cell of the grid which is either one cell up, or one cell to the right, or one cell diagonally down-left from his current cell. He makes several jumps, never visiting a cell twice, and with his final move returns to his starting cell.

Determine the largest number of jumps Leo could possibly have made.

This question was done extremely well amongst all students (that is, those who submitted something). It was amusing to see many people being unable to write "$9n^2 - 3$" on their paper – many students wrote $9n^3 - 3$ or even $9n^2 - 3n$ whilst meaning the correct thing.

Answer: The maximum number of squares traversed is $9n^2 - 3$.

Motivation for solution: Small cases – they allow you to see that the answer is $9n^2 - 3$ (which is probably the hardest part of this problem, to guess the answer correctly). Proving that anything higher isn't possible should just naturally flow after that.

Solution (Almost everyone who got it right) To show that $9n^2 - 3$ is possible, simply draw up the following diagram and say that it generalises.

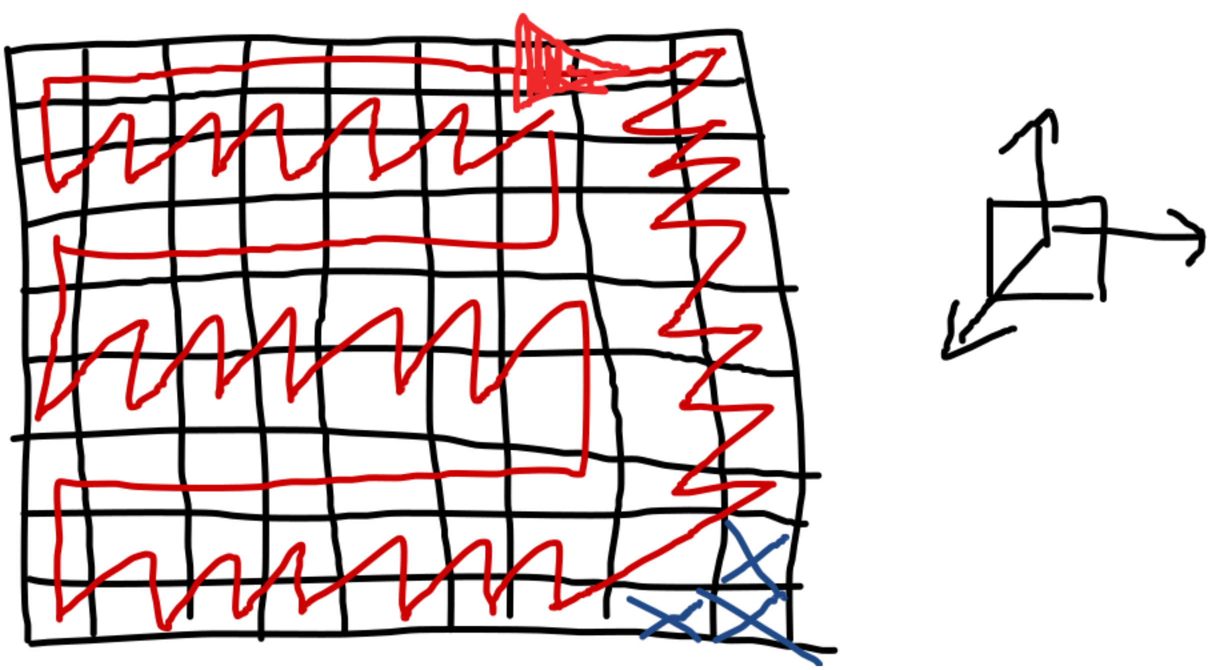
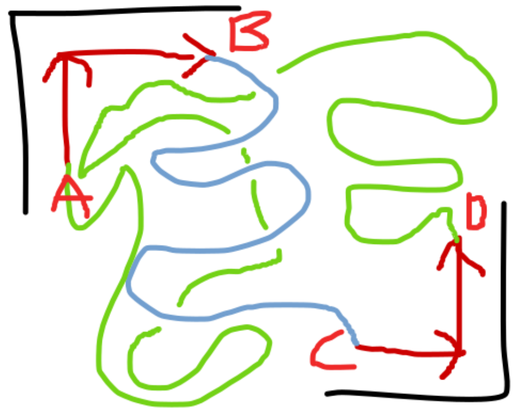


Figure 1: This was drawn in the GNU Drawing app, not MS Paint.

To prove that you cannot do any better, first up notice that the three vectors specified in the problem are pairwise linearly independent, yet their sum is $\vec{0}$, and since the entire loop forms a cycle it follows that there are the same number of each vector (and thus the number of squares traversed is divisible by 3). Hence, to beat the construction above, you need to have $9n^2$ squares, i.e. ALL squares on the board need to be covered.

Hence, it can be assumed that the top right and bottom left squares are covered. However, there is only one way we can enter and exit these squares, and it is shown on the diagram. To form a complete cycle, we need the point labelled A to connect with D (green path), and a path from B to D (blue path). This is obviously impossible, because this (by the diagram) requires the paths to intersect, but no two arrows can intersect anywhere apart from their endpoints on the centre of a square, and the paths cannot intersect at the endpoints because then that square would have two arrows pointing to/away from it. □



Marking Criteria:
7− Criteria

- Any full solution will receive **7 marks.**
- Each significant error will result in a **deduction of 1 mark**, down to a minimum of 5 marks.

0+ Criteria

- Finding the correct construction for $9n^2 - 3$ for all n: **2 Marks**
- Showing that the total number of moves is a multiple of 3 - **1 Mark**
- Finishing the proof - **4 Marks**

3 Question 3

Let I be the incentre of acute angled triangle ABC. Let the incircle meet BC, CA, AB at D, E, F respectively. Let line EF intersect the circumcircle of the triangle at P and Q, such that F lies between E and P.

Prove that $\angle DPA + \angle AQD = \angle QIP$.

This problem was done reasonably well amongst students who submitted solutions for this question, and I guess the majority of students found it quite difficult. However, honestly I personally thought that for such a simple diagram, all solutions that were shown to me (including the official solutions) introduced too many points and/or did too many complicated things than the problem deserves.

Solution (James Bang).

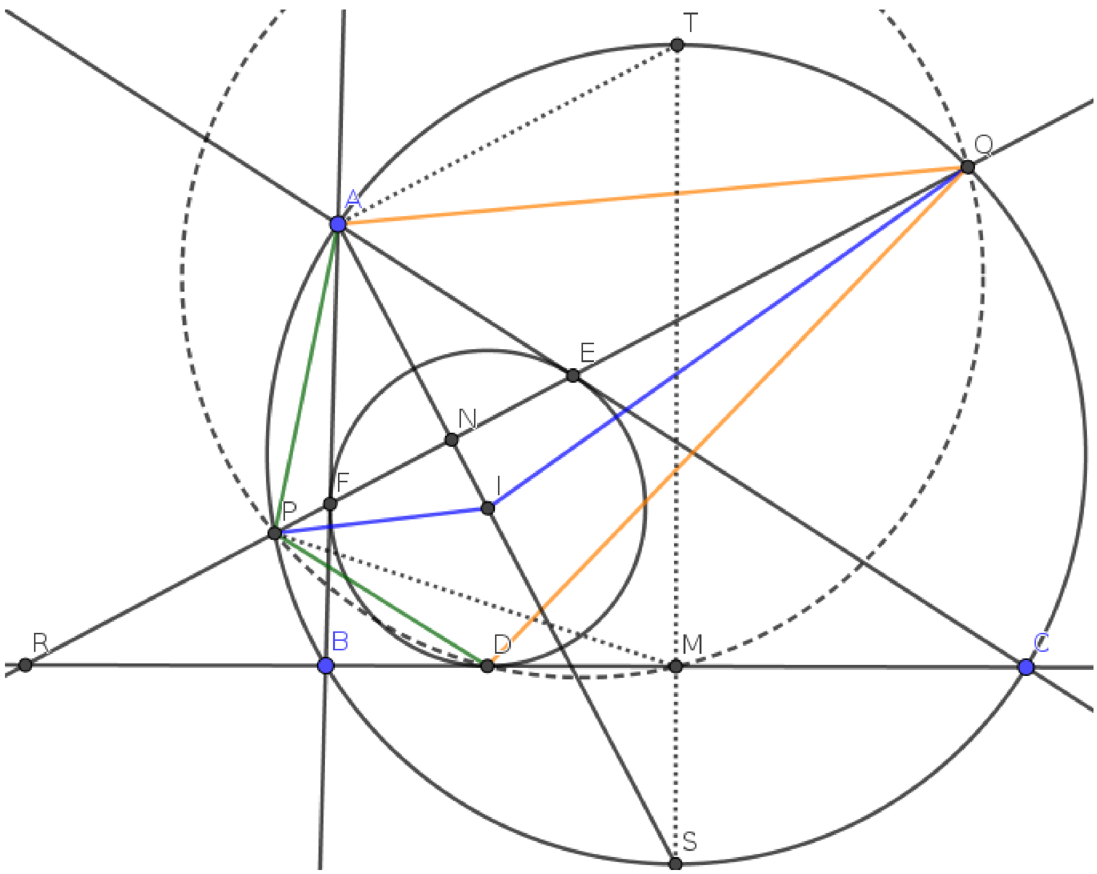


Figure 2: Yes, that's all we need!

First, it should be common knowledge that $PQMD$ is a cyclic quadrilateral – proving this is standard projective geometry (or length bash if you really want). Thus, we have

$$\angle APD + \angle AQD = \angle API + \angle IPD + \angle AQI + \angle IQD = (\angle API + \angle AQI) + (\angle PIQ - \angle PDQ).$$

Since we want $\angle APD + \angle AQD = \angle PIQ$, and have $\angle PDQ = \angle PMQ$, we need $\angle APD + \angle AQD = \angle PMQ$. To prove this, we conjecture that $\angle API = \angle PMT$. We have $\angle PAI = \angle PAS = \angle PTS = \angle PTM$, so it would suffice to show $\Delta API \sim \Delta TMP$, or equivalently

$$\frac{AP}{AI} = \frac{TM}{TP} \Leftrightarrow AP \times TP = AI \times TM.$$

This is true, since $\Delta ANP \sim \Delta TPS$ (giving us $AP \times TP = AN \times TS$), and also $AFIN \sim TBSM$ (giving us $AN \times TS = AI \times TM$). $\square$