Coherent laser-ion interactions

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Contents

1.1 Interaction picture

The Hamilton operator H governs the dynamics described by the Schrödinger equation

$$i\hbar\partial_t\psi = H\psi.$$

If we split H into a time-independent part H_0 and second (potentially time-dependent) part H_1, so that $H = H_0 + H_1$, then the wave function

$$\psi_I = e^{\frac{i}{\hbar}H_0 t}\psi = {U_0}^\dagger\psi \tag{1}$$

is a solution of the Schrödinger equation

$$i\hbar\partial_t\psi_I = H_I\psi_I, \tag{2}$$

where

$$H_I = e^{\frac{i}{\hbar}H_0 t}H_1 e^{-\frac{i}{\hbar}H_0 t} = U_0^\dagger H_1 U_0 \tag{3}$$

To prove this,we calculate

$$\begin{aligned} i\hbar\partial_t\psi_I &= i\hbar(\frac{i}{\hbar}H_0\psi_I + e^{\frac{i}{\hbar}H_0 t}\partial_t\psi) \\ &= -H_0\psi_I + e^{\frac{i}{\hbar}H_0 t}(H_0 + H_1)\psi \\ &= (-H_0 + e^{\frac{i}{\hbar}H_0 t}(H_0 + H_1)e^{-\frac{i}{\hbar}H_0 t})\psi_I \\ &= e^{\frac{i}{\hbar}H_0 t}H_1 e^{-\frac{i}{\hbar}H_0 t}\psi_I \\ &= H_I\psi_I. \end{aligned}$$

We can interpret this in the following way: In the interaction picture defined with respect to H_0, we rewind the time evolution induced by H_0 in order to study only the dynamics induced by H_1. Having calculated the time evolution in the interaction picture, we can go back to the Schrödinger picture where we find

$$\psi(t) = U_0\psi_I(t) = U_0 U_I\psi_I(0)\,.$$

If H_1 is time-independent, we have $U_I = e^{-\frac{i}{\hbar}H_I t}$ so that

$$\psi(t) = e^{-\frac{i}{\hbar}H_0 t}e^{-\frac{i}{\hbar}H_I t}\psi_I(0) = e^{-\frac{i}{\hbar}H_0 t}e^{-\frac{i}{\hbar}H_I t}e^{+\frac{i}{\hbar}H_0 t}\psi(0)\,. \tag{4}$$

In many cases of interest, $\psi(0)$ is an eigenstate of H_0 in which case

$$\psi(t) \propto e^{-\frac{i}{\hbar}H_0 t}e^{-\frac{i}{\hbar}H_I t}\psi(0)$$

where the proportionality sign accounts for a global phase that is of no importance in many cases.

1.2 Two-level atom interacting with a laser beam

1.2.1 Hamiltonian

In this section, we will focus on the case of a two-level atom held at a fixed location in space and excited by a monochromatic laser beam. We label the atom's ground state $|\downarrow\rangle$ and its excited state $|\uparrow\rangle$ and assume that we can neglect spontaneous emission for the time scales we are interested in. In the absence of laser light, the atom is described by the Hamiltonian

$$H_a = \frac{\omega_0}{2}\sigma_z$$

where ω_0 is the atomic transition frequency and where we have set $\hbar = 1$ for the sake of simplicity. The total Hamiltonian including the atom-light interaction is $H = H_a + H_{al}$ where

$$H_{al} = \Omega(\sigma_+ + \sigma_-)\cos\omega_L t$$

where ω_L is the laser frequency and Ω is the Rabi frequency, the coupling constant accounting for the strength of the atom-light interaction. As we are only interested in resonant or near-resonant interactions, it is convenient to introduce the detuning $\Delta = \omega_L - \omega_0$ of the laser frequency from the atomic transition frequency. Now, there are two different options for going into an interaction picture.

Interaction picture 1: The first choice consists in splitting the Hamiltonian $H = H_0 + H_1$ into a part

$$H_0 = \frac{\omega_L}{2}\sigma_z$$

describing the bare atom, so that H_1 is given by

$$H_1 = -\frac{\Delta}{2}\sigma_z + \Omega\sigma_x\cos\omega_L t\,.$$

Using (3) and (27), we obtain for the interaction Hamiltonian

$$\begin{aligned} H_I &= e^{i\frac{\omega_L t}{2}\sigma_z}(-\frac{\Delta}{2}\sigma_z + \Omega\sigma_x\cos\omega_L t)e^{-i\frac{\omega_L t}{2}\sigma_z} \\ &= -\frac{\Delta}{2}\sigma_z + \Omega\cos\omega_L t\,(\cos\omega_L t\,\sigma_x - \sin\omega_L t\,\sigma_y), \end{aligned}$$

Because of $\cos^2 \omega_L t = \frac{1}{2}(1 + \cos 2\omega_L t)$ and $\cos \omega_L t \sin \omega_L t = \frac{1}{2} \sin 2\omega_L t$, the Hamiltonian H_I contains terms oscillating at twice the laser frequency. These rapidly oscillating terms hardly contribute[1] to the time evolution of $\psi_I(t)$. Therefore, the rotating wave approximation consists in neglecting these terms so that the interaction Hamiltonian becomes

$$H_I = -\frac{\Delta}{2}\sigma_z + \frac{\Omega}{2}\sigma_x. \tag{5}$$

This picture has the advantage of leading to a time-independent interaction Hamiltonian. However, it might be less intuitive to define the atomic evolution with respect to a phase reference oscillating at a frequency that can be different from the atomic transition frequency. For example, if we describe a Ramsey experiment in this picture, the atomic state keeps evolving in the time between the two Ramsey pulses under the action of the Hamiltonian $H_I = -\frac{\Delta}{2}\sigma_z$. In this case, it might be preferable to use an interaction picture where we define the phase reference with respect to the atomic transition frequency.

Interaction picture 2: The second option consists in splitting the Hamiltonian into the parts

$$H_0 = \frac{\omega_0}{2}\sigma_z$$

and

$$H_1 = \Omega\sigma_x \cos \omega_L t\,.$$

Carrying out the same calculations as in the first case and making the rotating wave approximation, we obtain the Hamiltonian

$$H_I = \frac{\Omega}{2}(\cos \Delta t\, \sigma_x + \sin \Delta t\, \sigma_y)\,. \tag{6}$$

Using the raising and lowering operators, the Hamiltonian can also be written as

$$H_I = \frac{\Omega}{2}(\sigma_+ e^{-i\Delta t} + \sigma_- e^{i\Delta t}).$$

[1] This statement assumes $\Omega \ll \omega_0$ which is true unless the atom is excited with ultrafast, intense laser pulses. In the regime of ultrastrong coupling, the counter-rotating terms are important.

1.2.2 Definition of the Rabi frequency

In the literature, one encounters two different definitions of the Rabi frequency. If Ω is defined as in (5) or (6), the Rabi frequency is the frequency at which the population oscillates between the ground and the excited state. This can be seen by looking at the time evolution

$$U_I(\tau) = e^{-iH_I\tau} = e^{-i\frac{\Omega\tau}{2}\sigma_x} = \cos(\frac{\Omega\tau}{2})\mathcal{I} - i\sin(\frac{\Omega\tau}{2})\sigma_x,$$

which shows that in order to carry out a π-pulse, the laser needs to be applied for a duration τ satisfying $\Omega\tau = \pi$. Most Innsbruck publications use this convention. An alternative definition of Ω that is for example used in (all?) publications of the Boulder ion trapping group, is to write

$$H_I = \Omega(\cos\Delta t\sigma_x + \sin\Delta t\sigma_y)$$

instead of (6). If this definition is used, Ω describes the frequency at which the wave function returns to its original state (here, one needs to keep in mind that after a 2π rotation, the wave function is multiplied by -1, so that a 4π rotation is required to come back to the original state.).

1.2.3 Dynamical Stark effect

Non-resonant excitation of a two-level atom leads to an ac-Stark shift of its energy levels. This shift can be calculated by looking at the eigenvalues of the Hamiltonian (5). Setting $|H_I - \lambda\mathbf{I}| = 0$, we find

$$0 = \begin{vmatrix} \lambda + \frac{\Delta}{2} & -\frac{\Omega}{2} \\ -\frac{\Omega}{2} & \lambda - \frac{\Delta}{2} \end{vmatrix} = \lambda^2 - \frac{\Delta^2}{4} - \frac{\Omega^2}{4}$$

and $\lambda_\pm = \pm\frac{1}{2}\sqrt{\Delta^2 + \Omega^2}$. The ac-Stark shift of the levels is then given by

$$\delta = \lambda_\pm - \lambda_\pm(\Omega = 0) = \pm\frac{\Omega^2}{4\Delta}.$$

For a blue-detuned laser ($\Delta > 0$), the levels shift in such a way that the detuning becomes even bigger, i.e the upper level is shifted downwards and the lower level upwards. The ac-Stark shift of the transition frequency is twice as big as the individual shifts of the atomic levels.

$$\delta_{transition} = -\frac{\Omega^2}{2\Delta}.$$

1.3 Trapped ion interacting with a laser beam

1.3.1 Hamiltonian

Now, we want to consider a slightly more involved problem where the two-level atom interacting with the laser beam is no longer held at a fixed position but is trapped in a harmonic potential with oscillation frequency ν. In this case, the total Hamiltonian consists of three parts,

$$H = H^{(a)} + H^{(m)} + H^{(al)}$$

where

$$\begin{aligned} H^{(a)} &= \frac{\omega_0}{2}\sigma_z & (7)\\ H^{(m)} &= \nu a^\dagger a & (8)\\ H^{(al)} &= \Omega(\sigma_+ + \sigma_-)\cos(k\hat{x} - \omega_L t - \phi) & (9) \end{aligned}$$

describe the internal states of the atom, its motion in the trap and the atom-light interaction respectively. As before, ω_L denotes the laser frequency and ϕ the phase of the laser field[2]. In the definition of (9), we could have also chosen another phase convention[3] by using the argument $k\hat{x} - \omega_L t + \phi$. We now pass into an interaction picture defined with respect to the Hamiltonian $H_0 = H^{(a)} + H^{(m)}$ and calculate the interaction Hamiltonian

$$\begin{aligned} H_I &= e^{i(H^{(a)}+H^{(m)})t}\left\{\frac{\Omega}{2}(\sigma_+ + \sigma_-)(e^{i(k\hat{x}-\omega_L t-\phi)} + e^{-i(k\hat{x}-\omega_L t-\phi)})\right\}e^{-i(H^{(a)}+H^{(m)})t}\\ &= \frac{\Omega}{2}\left\{e^{iH^{(a)}t}(\sigma_+ + \sigma_-)e^{-iH^{(a)}t}\right\}\left\{e^{iH^{(m)}t}(e^{i(k\hat{x}-\omega_L t-\phi)} + e^{-i(k\hat{x}-\omega_L t-\phi)})e^{-iH^{(m)}t}\right\} \end{aligned}$$

Using (28), we simplify the first bracketed term to

$$e^{iH^{(a)}t}(\sigma_+ + \sigma_-)e^{-iH^{(a)}t} = e^{i\omega_0 t}\sigma_+ + e^{-i\omega_0 t}\sigma_- \,.$$

[2]For the sake of simplicity, the phase was set to zero in the previous section. However, it becomes important once we want to carry out composite pulse sequences where the phase of the laser might be switched from one pulse to the next by changing the optical path length using an acousto-optical or electro-optical modulator.

[3]The chosen definition has the following consequence: if the laser frequency is slightly detuned from an atomic transition frequency ω_0 by an amount $\Delta = \omega_L - \omega_0$, then we might want to write the cosine term as $\cos(k\hat{x} - \omega_0 t - \phi)$ where the time-dependent phase that is accumulated is given by $\phi(t) = \Delta t + \phi_0$, i. e. for a blue-detuned laser the phase picked up is positive.

Next, we express the position operator in terms of creation and annihilation operators and introduce the Lamb-Dicke parameter η, defined as

$$k\hat{x} = \eta(a + a^\dagger), \tag{10}$$

or equivalently

$$\eta = kx_0 = k\sqrt{\frac{\hbar}{2m\nu}}$$

where x_0 denotes the width of the ground state. Now we can also simplify the second term with the help of (35) by noting that

$$\begin{aligned} e^{iH^{(m)}t}e^{+i(\eta(a+a^\dagger)-\omega_L t-\phi)}e^{-iH^{(m)}t} &= e^{+i(\eta(ae^{-i\nu t}+a^\dagger e^{i\nu t})-\omega_L t-\phi)} \\ e^{iH^{(m)}t}e^{-i(\eta(a+a^\dagger)-\omega_L t-\phi)}e^{-iH^{(m)}t} &= e^{-i(\eta(ae^{-i\nu t}+a^\dagger e^{i\nu t})-\omega_L t-\phi)} \end{aligned}$$

This leads to the interaction Hamiltonian

$$H_I = \frac{\Omega}{2}(e^{i\omega_0 t}\sigma_+ + e^{-i\omega_0 t}\sigma_-)(e^{+i(\eta(ae^{-i\nu t}+a^\dagger e^{i\nu t})-\omega_L t-\phi)} + e^{-i(\eta(ae^{-i\nu t}+a^\dagger e^{i\nu t})-\omega_L t-\phi)}).$$

As in the previous section, we will make a rotating-wave approximation that gets rid of terms oscillating at optical frequencies:

$$H_I = \frac{\Omega}{2}(\sigma_+ e^{-i(\Delta t+\phi)}e^{+i\eta(ae^{-i\nu t}+a^\dagger e^{i\nu t})} + \sigma_- e^{i(\Delta t+\phi)}e^{-i\eta(ae^{-i\nu t}+a^\dagger e^{i\nu t})}) \tag{11}$$

In the following, we will make the Lamb-Dicke approximation, i.e. assume that the extent of the motional wave packet is small compared to the wavelength of the laser exciting the transition. This last assumption allows us to Taylor-expand the exponential containing creation and annihilation operators to first order.

$$e^{i\eta(ae^{-i\nu t}+a^\dagger e^{i\nu t})} = \mathbb{1} + i\eta(ae^{-i\nu t} + a^\dagger e^{i\nu t}) + \mathcal{O}(\eta^2) \tag{12}$$

In addition, we will assume that we operate in the resolved sideband limit. This means that the Rabi frequency is small compared to the oscillation frequency of the ions in the trap. In this case, when the laser frequency is set such that either $\Delta \approx 0$ or $\Delta \approx \pm\nu$, we can make a second rotating wave approximation by neglecting all terms oscillating with frequencies that are of order ν. This leads to the following interesting cases:

Carrier transition: we excite the ion with a detuning $\Delta = 0$ and the Hamiltonian takes on the form

$$H_{CAR} = \frac{\Omega}{2}(\sigma_+ e^{-i\phi} + \sigma_- e^{i\phi}). \tag{13}$$

In this case, the laser manipulates the qubit without affecting the motional state. The laser couples the quantum states pairwise on the transition $|\downarrow, n\rangle \leftrightarrow |\uparrow, n\rangle$, $n = 0, 1, 2, \ldots$.

Red sideband transition: we excite the ion with a detuning $\Delta = -\nu$ and obtain the Hamiltonian

$$H_{RSB} = \frac{i\eta\Omega}{2}(\sigma_+ a e^{-i\phi} - \sigma_- a^\dagger e^{i\phi}). \tag{14}$$

In this case, when we excite the ion to the upper state, we also decrease the vibrational quantum number by one quantum of motion. The laser couples the quantum states pairwise on the transition $|\downarrow, n+1\rangle \leftrightarrow |\uparrow, n\rangle$ with the exception of the state $|\downarrow, 0\rangle$ that remains uncoupled. In resolved sideband cooling, we take advantage of this fact by exciting the ion on the red sideband transition from the lower to the upper state (thus reducing the vibrational quantum number) and bringing back the ion from the upper to the lower state by a dissipative process that is unlikely to change the vibrational quantum number.

Blue sideband transition: we excite the ion with a detuning $\Delta = +\nu$ and obtain the Hamiltonian

$$H_{BSB} = \frac{i\eta\Omega}{2}(\sigma_+ a^\dagger e^{-i\phi} - \sigma_- a e^{i\phi}). \tag{15}$$

Whenever we excite the ion to the upper state, we also increase the vibrational quantum number by one. The laser couples the quantum states pairwise on the transition $|\downarrow, n\rangle \leftrightarrow |\uparrow, n+1\rangle$ with the exception of the state $|\uparrow, 0\rangle$ that remains uncoupled. The coupling strength $\Omega_{n,n+1} \propto \sqrt{n+1}$ depends on n. This makes it possible to measure the vibrational state populations $p_0, p_1, \ldots$... by exciting the ion on the blue sideband and measuring the evolution of the upper state population as a function of time.

Starting from a ground state cooled ion, application of a $\pi/2$ pulse on the blue or red sideband can maximally entangle the motional and electronic degree of freedom. For example, a $\pi/2$ pulse on the blue sideband maps the state $|\downarrow\rangle|0\rangle$ to $\frac{1}{2}(|\downarrow\rangle|0\rangle + |\uparrow\rangle|1\rangle)$. If instead we apply π-pulse to a ground state cooled ion, the electronic qubit state $(\alpha|\downarrow\rangle| + \beta|\uparrow\rangle)|0\rangle$ will be mapped to a qubit stored in the two lowest vibrational states: $|\uparrow\rangle(\beta|0\rangle + \alpha|1\rangle)$. This operation is one essential step of the Cirac-Zoller CNOT gate operation.

1.3.2 Corrections to the Lamb-Dicke approximation

In some cases, we have also to consider terms in eq. (11) that are of higher order in η. If we expand eq. (12) further, the second order term is given by

$$-\frac{\eta^2}{2}(ae^{-i\nu t} + a^\dagger e^{i\nu t})^2 = -\frac{\eta^2}{2}(a^2 e^{-i2\nu t} + a^{\dagger 2} e^{i2\nu t} + aa^\dagger + a^\dagger a)$$

The first two terms can be made resonant by choosing a detuning $\Delta = \pm 2\nu$. This correspond to a situation where the atom is excited the on a second-order sideband where each photon absorption or emission is accompanied by a change of the vibrational state by two vibrational quanta; however, the coupling strength of order $\eta^2\Omega$ will be fairly weak. The last two terms are resonant when the ion is excited on the carrier transition and they will reduce the coupling strength. Writing $aa^\dagger + a^\dagger a = 2a^\dagger a + 1$, we see that the Rabi frequency Ω in (13) needs to be replaced by a coupling strength

$$\Omega_{n,n} = \Omega(1 - \eta^2(n + \frac{1}{2}))$$

when exciting the $|\downarrow, n\rangle \leftrightarrow |\uparrow, n\rangle$ transition. This means that Rabi oscillations on the carrier will damp out if the initial state is a thermal state with a wide distribution of vibrational quanta. The damping of Rabi oscillations on the carrier can therefore be used as a means of assessing the quality of Doppler cooling.

1.3.3 Light shifts on sideband transitions

When exciting an ion on the blue or red sidebands, light shifts induced by off-resonant carrier excitation are an issue. For example, if the coupling strength on the sideband is $\eta\Omega = (2\pi)\,20$ kHz, obtained by choosing $\Omega = (2\pi)\,200$ kHz for $\eta = 0.1$ and the trap frequency is $\nu = (2\pi)\,1$ MHz, then the sideband transitions are shifted by $\delta = \frac{\Omega^2}{2\nu} = (2\pi)\,40$ kHz due to off-resonant excitation of the carrier transition.

1.3.4 Multiple vibrational modes

The description given so far is valid for a laser beam impinging on the ion along a direction that coincides with its direction of oscillation. If this condition is not fulfilled, the term $\cos(k\hat{x} - \omega_L t - \phi)$ in the atom-light Hamiltonian of eq. (9) needs to be replaced by $\cos(\mathbf{k}\hat{\mathbf{r}} - \omega_L t - \phi)$. This means that in the definition of the Lamb-Dicke factor, k becomes the projection of $\mathbf{k}$ on the direction of oscillation or equivalently

$$\eta = |\mathbf{k}| \cos\alpha \sqrt{\frac{\hbar}{2m\nu}}$$

where α is the angle between the laser beam and the direction of ion motion. For $\alpha = 90°$, the coupling to the sideband vanishes as no momentum can be transferred from the light field to the ion motion.

Moreover, for a three-dimensional laser-ion geometry, there exist three vibrational modes along the directions x,y,z, each of which is described by their own creation and annihilation operators $a_i^\dagger$, a_i and a Lamb Dicke factor η_i accounting for the coupling strength of the laser to the respective sidebands. We can take all these modes into account by replacing the exponential factor in (11) describing the coupling to the sidebands by a product of such factors each describing the coupling to one particular mode.

If the ion the laser couples to is embedded in a string of ions, then each harmonic oscillator that turns up in the laser-ion coupling describes one of the joint vibrational modes of the ion string. For a more detailed treatment of this case, see the paper by Daniel James (Appl. Phys. B **66**, 181 (1998)).

1.3.5 Spontaneous emission processes

What happens to the vibrational state in spontaneous emission processes? To answer this question, we can use the standard treatment of spontaneous emission where the atom is coupled to all the modes of the electromagnetic field using a Jaynes-Cummings Hamiltonian. Similar to what happens when we excite the atom with a laser beam, the atom can decay from the excited state to the ground state by emitting a photon either on the carrier transition (no change in vibrational quantum number) or on a sideband. For an atom in the state $|\uparrow, n\rangle$ and one particular electromagnetic field mode, the relative probabilities of decaying to the states $|\uparrow, n-1\rangle$, $|\uparrow, n\rangle$, $|\uparrow, n+1\rangle$ are $\eta^2 n$, 1, and $\eta^2(n+1)$, respectively (to calculate the probabilities, we have to take

the squares of the coupling strengths). To calculate the overall probability of vibrational state changing processes, we have to average over all electromagnetic field modes. If we just consider one field mode along the axis of oscillations and two modes that are perpendicular to it, the relative probabilities would become $\frac{1}{3}\eta^2 n$, 1, and $\frac{1}{3}\eta^2(n+1)$. For a proper treatment, one would have to take into account the spatial emission pattern of the particular electronic transition one is interested in. In any case, however, the simple argument shows that if the ion is in the Lamb-Dicke regime with respect to the wavelength of spontaneous emission, the emission process is not very likely to change the vibrational state.

1.4 Bichromatic excitation and entangling quantum gates

1.4.1 Hamiltonian

An interesting situation occurs if we consider exciting an ion simultaneously on the red and the blue sideband with a bichromatic laser beam of equal strength. In this case, we have to sum up the two Hamiltonians (14) and (15) to obtain

$$H_{BIC} = \frac{i\eta\Omega}{2}(\sigma_+ a e^{-i\phi_R} - \sigma_- a^\dagger e^{i\phi_R}) + \frac{i\eta\Omega}{2}(\sigma_+ a^\dagger e^{-i\phi_B} - \sigma_- a e^{i\phi_B}).$$

This equation can be written in a more useful form as

$$H_{BIC} = \frac{\eta\Omega}{2}\left(\cos\phi_+\,\sigma_x + \sin\phi_+\,\sigma_y\right)\left(\cos\phi_-\,(a^\dagger + a) + \sin\phi_-\,(i(a^\dagger - a))\right) \tag{16}$$

where

$$\phi_+ = \frac{\phi_r + \phi_b}{2} - \frac{\pi}{2} \quad \text{and} \quad \phi_- = \frac{\phi_r - \phi_b}{2},$$

or $\phi_b = \phi_+ - \phi_- + \frac{\pi}{2}$ and $\phi_r = \phi_+ + \phi_- + \frac{\pi}{2}$. To see that both definitions of H_{BIC} are equal, note that

$$\begin{aligned}\cos\phi_+\,\sigma_x + \sin\phi_+\,\sigma_y &= \cos\phi_+\,(\sigma_+ + \sigma_-) - i\sin\phi_+\,(\sigma_+ - \sigma_-)\\ &= \sigma_+ e^{-i\phi_+} + \sigma_- e^{i\phi_+}\end{aligned}$$

and

$$\cos\phi_-\,(a^\dagger + a) + \sin\phi_-\,(i(a^\dagger - a)) \;=\; a^\dagger e^{i\phi_-} + a e^{-i\phi_-}$$

The bichromatic Hamiltonian is quite different from the Hamiltonian describing excitation of a red or blue sideband. It does not give rise to a pairwise coupling but instead couples all states

$$|\downarrow,0\rangle \leftrightarrow |\uparrow,1\rangle \leftrightarrow |\downarrow,2\rangle \leftrightarrow |\uparrow,3\rangle \leftrightarrow \ldots$$

with each other and similarly the states

$$|\uparrow,0\rangle \leftrightarrow |\downarrow,1\rangle \leftrightarrow |\uparrow,2\rangle \leftrightarrow |\downarrow,3\rangle \leftrightarrow \ldots.$$

Writing the Hamiltonian in the form of (16) shows that it describes a spin-dependent force: it couples a spin-projection operator acting on the qubit to quadrature component of the harmonic oscillator. If we label the eigenstates of the spin projection operators $|+\rangle$ and $|-\rangle$, we see that the Hamiltonian couples the states

$$|+,0\rangle \leftrightarrow |+,1\rangle \leftrightarrow |+,2\rangle \leftrightarrow |+,3\rangle \leftrightarrow \ldots$$

and

$$|-,0\rangle \leftrightarrow |-,1\rangle \leftrightarrow |-,2\rangle \leftrightarrow |-,3\rangle \leftrightarrow \ldots$$

If the qubit is in an eigenstate of the operator acting on the qubit, the Hamiltonian displaces the state in phase space. If in addition, the motional state is initially in the ground state, the Hamiltonian creates a coherent state of motion. If, on the contrary, the qubit is in an equal superposition of eigenstates, the Hamiltonian will create a Schrödinger cat state as the two eigenstates will be displaced into opposite directions in phase space.

Example 1:

We set $\phi_+ = \phi_- = 0$. Then, the Hamiltonian $H_{BIC} = \frac{\eta\Omega}{2}\sigma_x(a^\dagger + a)$ acting on the initial state $|\psi\rangle = |+\rangle_x|0\rangle$ for a duration t will displace motional state without altering the qubit state. The bichromatic laser pulse creates the state

$$|\psi(t)\rangle = e^{-iH_{BIC}t}|+\rangle_x|0\rangle = |+\rangle_x\hat{D}(-\frac{i\eta\Omega t}{2})|0\rangle \equiv |+\rangle_x|\alpha\rangle$$

where $\hat{D}$ is the displacement operator creating the coherent state $|\alpha\rangle$ with $\alpha = -\frac{i\eta\Omega t}{2}$. For the definition of displacement operators and coherent states, see the appendix 1.8.3. Application of the Hamiltonian creates a state with a mean vibrational quantum number of $\bar{n} = |\alpha|^2 = (\eta\Omega t)^2/4$.

Example 2:

Once again, we set $\phi_+ = \phi_- = 0$. This time, the initial state is $|\psi\rangle = |\downarrow\rangle|0\rangle$ which we rewrite as $|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle_x - |-\rangle_x)|0\rangle$. Starting from this state, the laser-ion interaction creates the state

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}(|+\rangle_x|\alpha\rangle - |-\rangle_x|-\alpha\rangle)$$

which is a superposition of coherent states with $\alpha = -\frac{i\eta\Omega t}{2}$ that are maximally entangled with the qubit states $|\pm\rangle_x$. During the interaction, the electronic upper state population increases from zero to 0.5 for $\alpha \to \infty$. It is a simple exercise to calculate that

$$\langle\psi(t)|\sigma_z|\psi(t)\rangle = e^{-2|\alpha|^2}$$

which shows that for $\bar{n} = |\alpha|^2 = 0.5$, the upper state population reaches already more than 0.3.

1.4.2 Bichromatic two-ion quantum gate

An interesting situation occurs if we apply the bichromatic Hamiltonian to two trapped ions simultaneously. We assume that the coupling strength is the same for both ions and will focus on coupling to the sidebands of the centre-of-mass mode of the ions. This configuration, first investigated by A. Sørensen and K. Mølmer, allows to entangle the internal states of the ions by an interaction that is mediated by the coupling to a joint vibrational mode of the ions. Our goal is to engineer an interaction that leaves the initial motional state unaltered at the end of the gate operation. It turns out that one way to achieve this goal is to detune the two frequencies of the bichromatic laser beam slightly from the sideband frequencies.

More precisely, we will choose a detuning of $\Delta = \nu + \delta$ for the laser beam coupling to the upper motional sideband and set the detuning of the light coupling to the lower motional sideband to $\Delta = -(\nu + \delta)$, i.e. the sum of the two laser frequencies is equal to twice the qubit transition frequency. This means that the laser phases ϕ_r, ϕ_b of the Hamiltonians (14) and (15) describing the coupling to the red and blue sideband respectively become time-dependent with $\phi_b = \phi_{b,0} + \delta t$ and $\phi_r = \phi_{r,0} - \delta t$. Therefore, the phases of the bichromatic Hamiltonian (16) become

$$\phi_+ = (\phi_r + \phi_b - \pi)/2 = (\phi_{r,0} + \phi_{b,0} - \pi)/2$$

and

$$\phi_- = (\phi_r - \phi_b)/2 = (\phi_{r,0} - \phi_{b,0})/2 - \delta t.$$

To keep the discussion simple, we assume that $\phi_{r,0} = \phi_{b,0} = \pi/2$. Then, the Mølmer-Sørensen Hamiltonian becomes

$$\begin{aligned} H_{MS} &= \frac{\eta\Omega}{2}\left(\sigma_x^{(1)} + \sigma_x^{(2)}\right)\left(\cos\delta t\,(a^\dagger + a) - i\sin\delta t\,(a^\dagger - a)\right) \\ &= \frac{\eta\Omega}{2}\left(\sigma_x^{(1)} + \sigma_x^{(2)}\right)(a^\dagger e^{-i\delta t} - a e^{i\delta t}). \end{aligned} \tag{17}$$

If both qubits are in an eigenstate of σ_x, the Hamiltonian only displaces the motional state without modifying the qubit states. However, contrary to the case of a resonant bichromatic coupling to the sideband, here the displacement does not occur on a straight line in phase space. Rather, we are dealing with an off-resonantly excited harmonic oscillator.

1.4.3 Driven harmonic oscillator

Before addressing the problem of calculating the gate dynamics of the Mølmer-Sørensen gate, we will investigate the dynamics of a driven quantum harmonic oscillator. As in the classical case, we expect the quantum harmonic oscillator to return to its initial state if the drive frequency does not coincide with its oscillation frequency after a time that is the inverse of the difference between drive and oscillation frequency. The Hamiltonian we want to study is given by

$$H = i(\gamma(t)a^\dagger - \gamma^*(t)a). \tag{18}$$

Each application of the Hamiltonian in the small time interval $[t, t + dt]$ displaces the quantum state by an amount $d\alpha = \gamma(t)dt$. Because of the multiplication rules for displacement operations (38), the propagator describing the dynamics in the interval $[0, t]$ will be a displacement followed by a multiplication with a phase factor. For the propagator, we have

$$U(t) = \lim_{n\to\infty}\prod_{k=1}^{n}\exp(-\frac{i}{\hbar}H(t_k)\Delta t) = \hat{D}(\alpha(t))\exp(i\Phi(t)) \tag{19}$$

where $\Delta t = t/n$, $t_k = k\Delta t$ and

$$\begin{aligned} \alpha(t) &= \int_0^t dt'\gamma(t'), \\ \Phi(t) &= \mathrm{Im}\int_0^t dt'\gamma(t')\int_0^{t'} dt''\gamma^*(t''). \end{aligned}$$

In case of a driving force with constant amplitude, $\gamma(t) = \Omega e^{i\delta t}$, one obtains $\alpha(t) = i\left(\frac{\Omega}{\delta}\right)(1 - e^{i\delta t})$ and $\Phi = \left(\frac{\Omega}{\delta}\right)^2(\delta t - \sin\delta t)$. After a time $\tau_N = 2\pi N/|\delta|$, $N = 1, 2, \ldots$, the coherent state returns to its initial state in phase space with its phase changed by an amount $\Phi(\tau_N) = 2\pi N\left(\frac{\Omega}{\delta}\right)^2 \mathrm{sign}(\delta)$.

1.4.4 Mølmer-Sørensen gate

By making this phase change dependent on the internal states of a pair of ions, an entangling gate operation can be achieved. For

$$H = i(\gamma(t)a^\dagger - \gamma^*(t)a)\mathcal{O} \tag{20}$$

where $\mathcal{O}$ is an operator acting on the qubit states, the propagator (19) is replaced by

$$U_\gamma(t) = \hat{D}(\alpha(t)\mathcal{O})\exp(i\Phi(t)\mathcal{O}^2). \tag{21}$$

Choosing the interaction time τ such that $\alpha(\tau) = 0$ thus realizes a propagator that depends nonlinearly on $\mathcal{O}$ and does not alter the vibrational state. For the Mølmer-Sørensen gate, the operator $\mathcal{O}$ is set to

$$\mathcal{O} = \sigma_x^{(1)} + \sigma_x^{(2)}$$

or any other spin projection in the equatorial plane of the Bloch sphere. As a result, an entangling gate operation

$$U_{MS} = e^{i\frac{\pi}{4}\sigma_x^{(1)}\otimes\sigma_x^{(2)}} \tag{22}$$

is achieved after a time $\tau = 2\pi/\delta$ if the Rabi frequency is adjusted so that $\Phi = \pi/8$ in (21). For the times $\tau, 2\tau, 3\tau \ldots$, the propagator looks as if it was induced by an effective spin-spin interaction describes by a Hamiltonian

$$H_{eff} \propto \sigma_x^{(1)} \otimes \sigma_x^{(2)}$$

that flips spin of ion 1 only if also the spin of ion 2 is flipped. Another thing to note is that (22) contains only qubit operators. Therefore, the gate is capable of entangling the ions independent of the vibrational state they are in. No ground state cooling is required even though is is advisable because of effects that do not show up in this simple treatment.

A more detailed treatment and information about the practical implementation of these gate operations can be found here:

C. Roos, *Ion trap quantum gates with amplitude-modulated laser beams*, New J. Phys. **10**, 013002 (2008).

G. Kirchmair *et al.*, *Deterministic entanglement of ions in thermal states of motion*, New J. Phys. **11**, 023002 (2009).

1.5 Coupling strengths on dipole and quadrupole transitions

1.5.1 Gaussian laser beams

The electric field of a plane-wave Gaussian TEM_{00} beam propagating in vacuum in the z-direction is given by

$$E = E_0 e^{-\frac{r^2}{w(z)^2}} \cos(kz - \omega t)$$

with $w(z) = w_0\sqrt{1 + \left(\frac{z}{z_0}\right)^2}$ where w_0 denotes the laser waist and $z_0 = \frac{\pi w_0^2}{\lambda}$ the Rayleigh length. The intensity of the beam in the transverse direction falls off as $e^{-\frac{2r^2}{w^2}}$. To calculate the maximum electric field strength as a function of the laser beam parameters (P, w_0), we need the Poynting vector

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} = \epsilon_0 c^2 \mathbf{E} \times \mathbf{B}$$

which for a plane wave is given by $\mathbf{S} = \epsilon_0 c^2 \mathbf{E_0} \times \mathbf{B_0} \cos^2(kz - \omega t)$. As $B_0 = \frac{1}{c}E_0$, the intensity which is given by the time-averaged magnitude of the Poynting vector is

$$I = \langle S \rangle_T = \frac{c\epsilon_0}{2} E_0(r)^2.$$

To calculate laser power P, we have to integrate the intensity over the beam profile, giving

$$P = \int dA I = \frac{c\epsilon_0}{2} E_0^2 \int_0^{2\pi} d\phi \int_0^{\infty} r dr\, e^{-\frac{2r^2}{w_0^2}} = \frac{\pi w_0^2}{2} c\epsilon_0 E_0^2.$$

Therefore, the maximum electric field is related to the laser power and its waist by

$$E_0 = \left(\frac{2P}{\pi w_0^2 c\epsilon_0}\right)^2. \tag{23}$$

1.6 Rabi frequency on dipole transitions

In D. James' paper published in Appl. Phys. B,**66**, 181 (1998), the Rabi frequency on a dipole transition is given as

$$\Omega_0 = \frac{e|E|}{\hbar\sqrt{c\alpha}}\sqrt{\frac{A}{k^3}}\sigma$$

with

$$\sigma^{(E_1)} = \sqrt{\frac{3(2j'+1)}{4}}\left|\sum_{q=-1}^{1}\begin{pmatrix} j & 1 & j' \\ -m_j & q & m'_j \end{pmatrix} c_i^{(q)}\epsilon_i\right|$$

where $c_i^{(q)}$ are the normalized spherical vectors $c^{(1)} = -\frac{1}{\sqrt{2}}(1,-i,0)$, $c^{(0)} = (0,0,1)$, and $c^{(-1)} = \frac{1}{\sqrt{2}}(1,i,0)$ and ϵ_i the polarization vector; the summation includes a Wigner 3-j symbol accounting the magnetic quantum numbers m, m' of the lower and upper state. Similarly, the angular momenta of the lower and the upper state are denoted by j and j', respectively. The 3-j symbol is related to the Clebsch-Gordan coefficient via

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1-j_2-m_3}}{\sqrt{2j_3+1}}\langle j_1 m_1 j_2 m_2 | j_3 - m_3\rangle.$$

1.7 Appendix 1: Pauli spin matrix algebra

The Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

have eigenvectors $|\pm\rangle_n$ corresponding to the eigenvalues ± 1. It can be easily checked that

$$|+\rangle_z = \begin{pmatrix}1\\0\end{pmatrix},\ |-\rangle_z = \begin{pmatrix}0\\1\end{pmatrix}$$

as well as

$$|+\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix},\ |-\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

and

$$|+\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix},\ |-\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}.$$

In this document, often the alternative notation $|\uparrow\rangle = |+\rangle_z$ and $|\downarrow\rangle = |-\rangle_z$ are also used to denote the eigenstates of σ_z. With the help of eq. (26), it is easy to show that application of $\exp(-i\frac{\pi}{4}\sigma_y)$ to $|+\rangle_z$ maps the vector onto $|+\rangle_x$ and that similarly the operations $\exp(-i\frac{\pi}{4}\sigma_z)$ map $|+\rangle_x$ to $|+\rangle_y$ and $\exp(-i\frac{\pi}{4}\sigma_x)\ |+\rangle_y$ to $|+\rangle_z$. There are phase factors that occur in the mapping which one can change by choosing different definitions of the eigenstates (for example by choosing $|-\rangle_x = (-1,1)/\sqrt{2}$ instead of $(1,-1)/\sqrt{2}$). However, whatever the definitions are one chooses, it is not possible to make these phase factors disappear in all the mappings simultaneously[4].

The Pauli matrices satisfy the (anti-) commutation relations

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \tag{24}$$

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} \tag{25}$$

where ϵ_{ijk} is the Levi-Civita symbol defined as $\epsilon_{ijk} = 1$ for even permutations of the indices, $\epsilon_{ijk} = -1$ for odd permutations and zero otherwise and $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ otherwise is the Kronecker symbol. Summing up the two relations, we have

$$\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k\,.$$

Special cases of this formula are $\sigma_x\sigma_y = i\sigma_z$, $\sigma_y\sigma_z = i\sigma_x$, $\sigma_z\sigma_x = i\sigma_y$ and so on. Using the commutation relations given above, it is easy to show that

$$e^{i\theta\sigma_j} = \cos\theta + i\sin\theta\,\sigma_j \tag{26}$$

and for $k \neq j$

$$\begin{aligned} e^{i\frac{\theta}{2}\sigma_i}\sigma_j e^{-i\frac{\theta}{2}\sigma_i} &= \cos\theta\,\sigma_j + \frac{i}{2}\sin\theta\,[\sigma_i, \sigma_j] \\ &= \cos\theta\,\sigma_j - \sin\theta\,\epsilon_{ijk}\sigma_k\,. \end{aligned} \tag{27}$$

where we used $\cos^2\alpha - \sin^2\alpha = \cos(2\alpha)$ and $2\sin\alpha\cos\alpha = \sin(2\alpha)$. For the raising and lowering operators

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

[4]The reason is that the composite operation $\exp(-i\frac{\pi}{4}\sigma_y)\exp(-i\frac{\pi}{4}\sigma_z)\exp(-i\frac{\pi}{4}\sigma_x)$ which maps the $+1$ eigenstate from z to x, then to y and finally back to z is by itself a rotation and thus multiplies $|+\rangle_z$ with a phase factor $e^{i\phi}$ and $|-\rangle_z$ with $e^{-i\phi}$.

we have $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$ or

$$\begin{aligned} \sigma_x &= \sigma_+ + \sigma_- \\ \sigma_y &= -i(\sigma_+ - \sigma_-) \\ \sigma_z\sigma_\pm &= \pm\sigma_\pm \\ \sigma_\pm\sigma_z &= \mp\sigma_\pm. \end{aligned}$$

and

$$e^{i\frac{\theta}{2}\sigma_z}\sigma_\pm e^{-i\frac{\theta}{2}\sigma_z} = e^{\pm i\theta}\sigma_\pm\,. \tag{28}$$

1.8 Appendix 2: Harmonic oscillator algebra

1.8.1 Quantum harmonic oscillator

The harmonic oscillator

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\nu^2\hat{x}^2$$

of a particle of mass m and oscillation frequency ν is conveniently described in terms of the creation and annihilation operators defined as

$$a = \sqrt{\frac{m\nu}{2\hbar}}(\hat{x} + \frac{i}{m\nu}\hat{p}) \tag{29}$$

$$a^\dagger = \sqrt{\frac{m\nu}{2\hbar}}(\hat{x} - \frac{i}{m\nu}\hat{p}). \tag{30}$$

Conversely, position and momentum operators can be defined in terms of a and $a^\dagger$ as

$$\hat{x} = \sqrt{\frac{\hbar}{2m\nu}}(a^\dagger + a) \tag{31}$$

$$\hat{p} = \sqrt{\frac{\hbar m\nu}{2}}i(a^\dagger - a). \tag{32}$$

In terms of a, $a^\dagger$, the Hamiltonian is written as

$$H = \hbar\nu(a^\dagger a + \frac{1}{2}).$$

From the commutation relation $[\hat{x}, \hat{p}] = i\hbar$ of the position and momentum operator it follows that the annihilation and creation operators satisfy the commutation relation

$$[a, a^\dagger] = 1.$$

The rms size of the ground state wave function of the harmonic oscillator defines a characteristic length scale x_0 given by

$$x_0{}^2 \equiv \langle 0|\hat{x}^2|0\rangle = \frac{\hbar}{2m\nu}\langle 0|(a^\dagger + a)^2|0\rangle = \frac{\hbar}{2m\nu}.$$

1.8.2 Useful multiplication rules

For going from one interaction picture to another one, the following relations are important:

$$e^{i\theta a^\dagger a} a e^{-i\theta a^\dagger a} = e^{-i\theta} a \tag{33}$$

$$e^{i\theta a^\dagger a} a^\dagger e^{-i\theta a^\dagger a} = e^{i\theta} a^\dagger \tag{34}$$

To derive these relations, we use $a(a^\dagger a)^k = (aa^\dagger)^k a$ and $(a^\dagger a)^k a^\dagger = a^\dagger (aa^\dagger)^k$ and the Taylor expansion $e^{-i\theta a^\dagger a} = \sum_{n=0}^{\infty} \frac{(-i\theta)^n}{n!}(a^\dagger a)^n$ in order to write

$$a e^{-i\theta a^\dagger a} = a + a\sum_{n=1}^{\infty} \frac{(-i\theta)^n}{n!}(a^\dagger a)^n = a + \sum_{n=1}^{\infty} \frac{(-i\theta)^n}{n!}(aa^\dagger)^n a = e^{-i\theta aa^\dagger} a\,.$$

Then, by multiplying $a e^{-i\theta a^\dagger a} = e^{-i\theta(a^\dagger a+1)} a$ with $e^{i\theta a^\dagger a}$ from the left, we obtain eq. (33) and eq. (34) by a similar calculation. Another useful equation is

$$e^{i\theta a^\dagger a} e^{i\eta(a+a^\dagger)} e^{-i\theta a^\dagger a} = e^{i\eta(ae^{-i\theta} + a^\dagger e^{i\theta})} \tag{35}$$

which is derived by using the Taylor expansion of $e^{i\eta(a+a^\dagger)}$ to write the left hand side as

$$e^{i\theta a^\dagger a}\sum_{k=0}^{\infty} \frac{(i\eta)^k}{k!}(a + a^\dagger)^k e^{-i\theta a^\dagger a} = \mathbb{1} + \sum_{k=1}^{\infty} \frac{(i\eta)^k}{k!}\left[e^{i\theta a^\dagger a}\,(a + a^\dagger)\,e^{-i\theta a^\dagger a}\right]^k$$

and making use of equations (33) and (34) to obtain the expression on the right hand side.

1.8.3 Displacement operations in phase space and coherent states

In phase space, the displacement operator is defined as

$$\hat{D}(\alpha) = e^{\alpha a^\dagger - \alpha^* a}. \tag{36}$$

For the displacement operator, the following equation is interesting:

$$a\hat{D}(\alpha) = \hat{D}(\alpha)(a + \alpha). \tag{37}$$

Note that because of the definition $\hat{D}^\dagger(\alpha) = \hat{D}(-\alpha)$. If we apply this equation to the ground state $|0\rangle$, we see that $\hat{D}(\alpha)|0\rangle$ is an eigenstate of the annihilation operator because of

$$a\,\hat{D}(\alpha)|0\rangle = \hat{D}(\alpha)(a + \alpha)|0\rangle = \alpha\,\hat{D}(\alpha)|0\rangle.$$

Therefore, the displacement operator creates the coherent state $|\alpha\rangle$ when acting on the ground state:

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle$$

To prove (37), one uses $a(\alpha a^\dagger + \alpha^* a) = \alpha + (\alpha a^\dagger + \alpha^* a)a$ to prove by induction that $a(\alpha a^\dagger + \alpha^* a)^n = n\alpha(\alpha a^\dagger + \alpha^* a)^{n-1} + (\alpha a^\dagger + \alpha^* a)^n a$ for $n \geq 1$ which in turn allows to convert the left hand side of the equation into the right hand side by using the Taylor series of the exponential function. The form one usually encounters the equality (37) is

$$\begin{aligned} \hat{D}^\dagger(\alpha) a \hat{D}(\alpha) &= (a + \alpha) \\ \hat{D}^\dagger(\alpha) a^\dagger \hat{D}(\alpha) &= (a^\dagger + \alpha^*). \end{aligned}$$

Another important equation concerns the multiplication of displacement operators:

$$\hat{D}(\alpha)\hat{D}(\beta) = \hat{D}(\alpha + \beta)e^{\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)} = \hat{D}(\alpha + \beta)e^{i\mathrm{Im}(\alpha\beta^*)}. \tag{38}$$

This relation follows from the application of the Baker-Campbell-Hausdorff equation

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]} \tag{39}$$

valid for operators A and B, that commute with the commutator $[A, B]$, that is [A,[A,B]]=[B,[A,B]]=0. Another application of the Baker-Campbell-Hausdorff formula leads to the representation of the displacement operator as

$$\hat{D}(\alpha) = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} e^{-\alpha^* a}. \tag{40}$$

Formula (40) allows us to write the coherent states in the Fock state basis as

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger}|0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{k=0}^{\infty} \frac{\alpha^k}{\sqrt{k!}}|k\rangle \tag{41}$$

where we used $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. This shows that the probability of occupying the n*th* level is given by a Poissonian distribution with average $|\alpha|^2$. In particular, the overlap of a coherent state $|\alpha\rangle$ with the ground state is given by

$$\langle 0|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}.$$

Another way to calculate the average phonon number of the state $|\alpha\rangle$ is by applying eq. (37) to calculate

$$\begin{aligned} \langle\alpha|\hat{n}|\alpha\rangle &= \langle\alpha|a^\dagger a\hat{D}(\alpha)|0\rangle = \langle\alpha|\hat{D}(\alpha)(a^\dagger+\alpha^*)(a+\alpha)|0\rangle \\ &= \langle 0|(a^\dagger+\alpha^*)(a+\alpha)|0\rangle = |\alpha|^2. \end{aligned}$$

More information on coherent states can be found here:

R. Glauber, *Coherent and incoherent states of the radiation field*, Phys. Rev. **131**, 2766 (1963).

S. Haroche, J. M. Raimond, *Exploring the quantum*, Oxford University Press (2006).

Proof of the Glauber identity For the case where the operators A and B commute with $[A, B]$, the Baker-Campbell-Haussdorff formula

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$$

is also called Glauber identity in the context of quantum optics. To prove this identity, is to take the function

$$f(x) = e^{Ax}e^{Bx}$$

and to calculate its derivative

$$\begin{aligned} \frac{df}{dx} &= f(x)(e^{-Bx}Ae^{Bx} + B) \\ &= f(x)(x[A,B] + A + B) \end{aligned} \tag{42}$$

where the last step made use of an operator identity valid for the case where $[A, B]$ is a complex number. This equation can also be read as a differential equation for the function f. A solution to this differential equation is given by

$$g(x) = e^{(A+B)x+\frac{x^2}{2}[A,B]}$$

since

$$\frac{dg}{dx} = g(x)(A + B + x[A, B]).$$

As $f(0) = g(0)$, we can conclude that $f(x) = g(x)$ for all values of x and in particular for $x = 1$ from which follows the validity of (39).