FYS4150, project 1

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1 Project 1a)

The aim of this report is to solve the one-dimensional Poisson equation:

$$-u''(x) = f(x), \qquad x \in (0,1), \qquad u(0) = u(1) = 0.$$

where the discrete approximation to the analytic function $u(x)$ is named v_i. Euler's method will be used to numerically solve this differential equation. Hence, v_i has points $x_i = ih$ in the interval of $x_0 = 0$ to $x_{n+1} = 1$, and where h is the step length, $h = 1/(n+1)$. The second derivative of $u(x)$ takes the numerical form:

$$-\frac{v_{i+1} + v_{i-1} - 2v_i}{h^2} = f_i \qquad \text{for } i = 1, \ldots, n, \tag{1}$$

where $f_i = f(x_i)$.

This can be written as a linear set of equations:

$$\mathbf{A}\mathbf{v} = \tilde{\mathbf{b}},$$

where $\tilde{b}_i = h^2 f_i$ and A is:

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & \ldots & \ldots & 0 \\ -1 & 2 & -1 & 0 & \ldots & \ldots \\ 0 & -1 & 2 & -1 & 0 & \ldots \\ & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & & -1 & 2 & -1 \\ 0 & \ldots & & 0 & -1 & 2 \end{bmatrix}$$

This can be shown in the following equation, where the result is the same as Eq. 1:

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & \ldots & \ldots & 0 \\ -1 & 2 & -1 & 0 & \ldots & \ldots \\ 0 & -1 & 2 & -1 & 0 & \ldots \\ & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & & -1 & 2 & -1 \\ 0 & \ldots & & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \ldots \\ \ldots \\ \ldots \\ v_n \end{bmatrix} = h^2 \begin{bmatrix} -(v_2 + v_0 - 2v_1) \\ -(v_3 + v_1 - 2v_2) \\ \ldots \\ \ldots \\ \ldots \\ -(v_{n+1} + v_{n-1} - 2v_n) \end{bmatrix}.$$

In this report $f(x)$ will be assumed to be $f(x) = 100e^{-10x}$, which has the solution $u(x) = 1 - (1 - e^{-10})x - e^{-10x}$. This can be shown by inserting the solution $u(x)$ into the Poisson equation. The first derivative of $u(x)$ with respect to x is therefore:

$$\frac{du(x)}{dx} = -(1 - e^{-10}) + 10e^{-10x}$$

and the second derivative is:

$$\frac{d^2u(x)}{dx^2} = -100e^{-10x}$$

and hence $-u'' = 100e^{-10x}$.