

Impact of Centre Vortices on the Gluon Propagator



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I would like to dedicate this thesis to

Declaration

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And I would like to acknowledge ...

Abstract

This is where you write your abstract ...

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Chapter 1

Introduction

Chapter 2

Lattice QCD

Since the first efforts to construct a non-perturbative approach to QCD in 1974[1], lattice QCD has developed over the past 40 years into a powerful tool used to probe the low-energy behaviour of the strong nuclear force. Rather than treat spacetime as a set of continuous axes, it is instead discretised into a finite set of points on a four-dimensional hypercube. This prescription allows for the explicit calculation of path integrals present in QCD, at the cost of introducing finite-spacing errors that must be systematically accounted for. In this chapter we will discuss the behaviour of QCD when spacetime is continuous (hereafter referred to as the continuum), and demonstrate how the transition can be made to a finite set of coordinates on a lattice. We will also describe the two choices of gauge used in this research, and how they are applied to the lattice.

2.1 QCD in the Continuum

2.1.1 Quarks and Gauge Invariance

QCD is the gauge field theory that describes the interactions of quarks and gluons. Like all gauge theories, it has an internal symmetry group which the Lagrangian is invariant under. In the case of QCD there are three quark colours, which leads to the symmetry group being $SU(3)$, the group of 3×3 unitary matrices of determinant 1¹. We can see this $SU(3)$ symmetry by inspecting the QCD quark Lagrangian

$$\mathcal{L} = \bar{\psi}(x) (i\not{D} - m) \psi(x). \quad (2.1)$$

¹This description is only true in the fundamental representation of the group, but this is the symmetry we observe in the Lagrangian and is a useful way to visualise the group symmetry.

If we apply an $SU(3)$ transformation Ω to the colour indices of the quark, $\psi(x)$, and anti-quark, $\bar{\psi}(x)$, fields, we see that

$$\begin{aligned}\mathcal{L} \rightarrow \mathcal{L}' &= \bar{\psi}(x) \Omega^\dagger (i\cancel{D} - m) \Omega \psi(x) \\ &= \bar{\psi}(x) (i\cancel{D} - m) \psi(x) \Omega^\dagger \Omega \\ &= \bar{\psi}(x) (i\cancel{D} - m) \psi(x) \\ &= \mathcal{L}.\end{aligned}$$

Here we have made use of the unitarity property $U U^\dagger = I$. If this symmetry were all we required then we would be done and our theory would be pleasantly simple. However, we find that we need our gauge symmetry to be *local*; that is, we demand that our gauge transformation itself be a function of x [2]. In this case, we find that the derivative in Eq. 2.1 results in a loss of $SU(3)$ symmetry. We can write an arbitrary $SU(3)$ gauge transformation as an exponential of the traceless, Hermitian group generators λ_a (known as the Gell-Mann matrices) such that $\Omega = \exp\left(i\omega^a(x) \frac{\lambda_a}{2}\right)$. Using this form for Ω , we find that under a gauge transformation the Lagrangian is now

$$\begin{aligned}\mathcal{L} \rightarrow \mathcal{L}' &= \bar{\psi}(x) \Omega^\dagger(x) (i\cancel{D} - m) \Omega(x) \psi(x) \\ &= \bar{\psi} \Omega^\dagger \left[-\frac{\lambda_a}{2} (\cancel{D} \omega^a(x)) + i \Omega (\cancel{D} \psi) - m \Omega \psi \right] \\ &= \mathcal{L} - \bar{\psi} \Omega^\dagger \frac{\lambda_a}{2} (\cancel{D} \omega^a(x)) \Omega \psi\end{aligned}\tag{2.2}$$

To amend this, we introduce the notion of the gauge-covariant derivative

$$D_\mu = \partial_\mu - igA_\mu(x),\tag{2.3}$$

where $A_\mu(x) = A_\mu^a(x) \frac{\lambda_a}{2}$, and $A_\mu^a(x)$ are eight new ‘gauge potentials’. Making the substitution $\partial_\mu \rightarrow D_\mu$, we introduce a new term into the Lagrangian that gives rise to an interaction between our quark and gauge fields.

$$\mathcal{L}_{\text{int}} = g \bar{\psi} A_\mu(x) \psi\tag{2.4}$$

To preserve the gauge invariance of the Lagrangian, we need the gauge transformation property of Eq. 2.4 to counteract the last term of Eq. 2.2. Hence we require that

$$g \bar{\psi} A_\mu(x) \psi \rightarrow g \bar{\psi} A_\mu(x) \psi + \bar{\psi} \Omega^\dagger \frac{\lambda_a}{2} (\partial_\mu \omega^a(x)) \Omega \psi. \quad (2.5)$$

Making use of the transformation properties of ψ and $\bar{\psi}$, this implies that

$$A_\mu(x) \rightarrow \Omega A_\mu(x) \Omega^\dagger - \frac{i}{g} (\partial_\mu \Omega) \Omega^\dagger. \quad (2.6)$$

This transformation property can also be expressed in terms of the covariant derivative. Doing so, we find that

$$\begin{aligned} D_\mu \psi &\rightarrow (\partial_\mu - ig \Omega A_\mu(x) \Omega^\dagger - (\partial_\mu \Omega) \Omega^\dagger) \Omega \psi \\ &= (\partial_\mu \Omega) \psi + \Omega (\partial_\mu \psi) - ig \Omega A_\mu(x) \psi - (\partial_\mu \Omega) \psi \\ &= \Omega D_\mu \psi. \end{aligned}$$

And therefore

$$D_\mu \rightarrow \Omega D_\mu \Omega^\dagger \quad (2.7)$$

This result tells us that the covariant derivative of a quark field transforms in the same way as the quark field itself. The covariant derivative can then be thought of as a connection between two points that may have a different underlying gauge. For example, if we consider an infinitesimal translation in the quark field

$$d\psi(x) = \psi(x + dx) - \psi(x),$$

we note that the gauge at the point x and at $x + dx$ in general will differ, so it doesn't make sense to compare the field values through the usual understanding of the derivative. Instead, the covariant derivative accounts for this underlying gauge structure, 'transporting' the field from one position to another.

2.1.2 Gluon Field

Using local gauge invariance as a guide, we can now seek other gauge invariant terms to insert into the Lagrangian. If we consider the commutator of the covariant derivative,

we have

$$\begin{aligned}
[D_\mu, D_\nu] &\rightarrow [\Omega D_\mu \Omega^\dagger, \Omega D_\nu \Omega^\dagger] \\
&= \Omega D_\mu D_\nu \Omega^\dagger - \Omega D_\nu D_\mu \Omega^\dagger \\
&= \Omega [D_\mu, D_\nu] \Omega^\dagger.
\end{aligned}$$

We therefore define the gluon field strength tensor to be

$$F_{\mu\nu} = \frac{i}{g} [D_\mu, D_\nu] \quad (2.8)$$

Expanding this definition, $F_{\mu\nu}$ may also be written

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \quad (2.9)$$

To obtain a gauge invariant quantity, we take the trace of the contracted field strength tensor. This allows us to make use of the cyclic property of the trace to obtain

$$\begin{aligned}
\text{Tr}(F_{\mu\nu} F^{\mu\nu}) &\rightarrow -\frac{1}{g^2} \text{Tr}(\Omega [D_\mu, D_\nu] \Omega^\dagger \Omega [D^\mu, D^\nu] \Omega^\dagger) \\
&= -\frac{1}{g^2} \text{Tr}(\Omega^\dagger \Omega [D_\mu, D_\nu] [D^\mu, D^\nu]) \\
&= \text{Tr}(F_{\mu\nu} F^{\mu\nu})
\end{aligned}$$

Thus we define the full gauge invariant QCD Lagrangian to be

$$\mathcal{L}_{\text{QCD}} = \bar{\psi}(x) (i\not{D} - m) \psi(x) - \frac{1}{2} \text{Tr}(F_{\mu\nu}(x) F^{\mu\nu}(x)) \quad (2.10)$$

This gluon term is not the only gauge invariant quantity we could construct; for example, $\bar{\psi}\psi\bar{\psi}\psi$ is clearly gauge invariant. However, it turns out that there is a further condition that must be satisfied by each term in the Lagrangian; each term must be *renormalisable*[2]. A complete discussion of renormalisation is unnecessary for this work, but renormalisability can be quickly summarised by looking at the dimensionality of each term in the Lagrangian. The Lagrangian must have units of (Energy)⁴, which in natural units is (mass)⁴ (hereafter referred to as just dimension 4). We therefore require that each term and its accompanying coupling constant give the same dimensionality. The fermion field has dimension $\frac{3}{2}$, the gauge potential has

dimension 1 and ∂_μ has dimension 1. Then we see that the terms present in Eq. 2.10 have dimension

$$\begin{aligned} D[\bar{\psi}(x) \gamma^\mu \partial_\mu \psi(x)] &= \frac{3}{2} + 1 + \frac{3}{2} = 4 \\ D[\bar{\psi}(x) \gamma^\mu A_\mu \psi(x)] &= \frac{3}{2} + 1 + \frac{3}{2} = 4 \\ D[\bar{\psi}(x) \psi(x)] &= 1 + \frac{3}{2} + \frac{3}{2} = 4 \\ D[F_{\mu\nu} F^{\mu\nu}] &= 2 + 2 = 4 \end{aligned}$$

as required. This also tells us that the coupling constant g is dimensionless. If a new gauge invariant term $h\bar{\psi}\psi\bar{\psi}\psi$ with coupling constant h is introduced, we then require that h have dimension -2 . It turns out that if the dimensionality of the coupling constant is less than 0 then the term is non-renormalisable. This means that integrals involving this new term will diverge in such a way that they cannot be systematically made finite through use of an ultraviolet cutoff, and hence they cannot form part of any physical theory. By applying the requirements of gauge invariance and renormalisability, it is apparent that Eq. 2.10 is the full QCD Lagrangian.

2.1.3 Pure Gauge Action

For the purpose of this research, we are interested in the behaviour of gluons in the absence of any quarks, and as such we need to develop a description of pure gauge fields. In the continuum, a pure gauge field has the Lagrangian[3]

$$\mathcal{L}_{\text{gluon}} = \frac{1}{2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}), \quad (2.11)$$

which we observe to be the last term in Eq. 2.10. This Lagrangian has the corresponding action

$$\mathcal{S} = \int d^4x \mathcal{L}_{\text{gluon}}, \quad (2.12)$$

When considering the path integral formulation of a gauge field theory, integrals such as the generating functional,

$$\mathcal{Z} = \int \mathcal{D}A_\mu \exp(i\mathcal{S}[A_\mu(x)]), \quad (2.13)$$

and others of a similar form appear frequently. This integral closely resembles the partition function found in statistical mechanics, $Z_{\text{classical}} = \int d^3x d^3p \exp(-\beta H(x, p))$,

with the notable exception of the factor of i in the exponential. From the statistical mechanics perspective, the $\exp(\dots)$ term in the generating functional is a probability weighting for each ‘path’ through gauge space. The factor of i in Eq. 2.13 results in an oscillatory weighting, rendering numerical simulations untenable. To ensure that the weight factor is purely real, it is necessary to perform a Wick rotation into Euclidean space[1, 4]

$$x_0 \rightarrow -ix_0$$

The generating functional now becomes

$$\mathcal{Z}_{\text{Eucl}} = \int \mathcal{D}A_\mu \exp(-\mathcal{S}_{\text{Eucl}}[A_\mu(x)]). \quad (2.14)$$

The Wick rotation also has the consequence of reducing the metric $g_{\mu\nu}$ to the identity, meaning that there is no longer any differentiation between covariant and contravariant tensors.

2.2 Lattice Discretisation

Within this framework, we can now consider discretising spacetime into a finite lattice with N_s lattice sites in the spacial directions and N_t sites in the time direction. Each lattice site is separated by a spacing a , resulting in a total lattice volume $V = (N_s a)^3 \times N_t a$. This discretisation is shown in two dimensions in Fig. 2.1. We also must impose periodic boundary conditions, such that $x + (N + 1)a\hat{\mu} = x$.

When spacetime is discretised, it becomes necessary to consider derivatives as finite differences and integrals as finite sums.

$$\begin{aligned} \partial_\mu f(x) &\rightarrow \frac{f(x + a\hat{\mu}) - f(x)}{a} \\ \int d^4x f(x) &\rightarrow a^4 \sum_x f(x) \end{aligned}$$

For example, we can construct the lattice form of Eq. 2.9 as

$$F_{\text{Lat}}^{\mu\nu}(x) = \frac{A_\nu(x + a\hat{\mu}) - A_\nu(x)}{a} - \frac{A_\mu(x + a\hat{\nu}) - A_\mu(x)}{a} - ig[A_\mu(x), A_\nu(x)]. \quad (2.15)$$

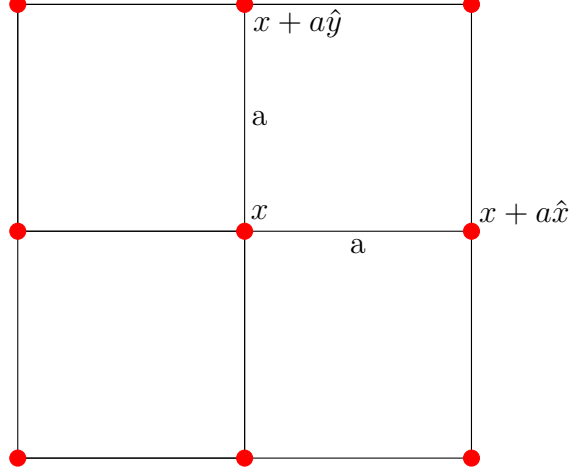


Fig. 2.1 An example of a 2D lattice with lattice spacing a . If we have a position x then we can define $x + a\hat{\mu}$ to represent the next lattice site in the $\hat{\mu}$ direction.

The notation $A_\nu(x + a\hat{\mu})$ denotes the field A_ν located at the site one lattice spacing in the $\hat{\mu}$ direction from x . We could continue to reformulate our lattice theory by imposing this method of discretisation, and indeed this is historically how the lattice framework was constructed[1]. However, it is useful to instead formulate our lattice theory in terms of the gauge *links*. Analogous to how we introduced the covariant derivative to compensate for the fact that the quark field at infinitesimally different points in space has a different underlying gauge, we now want to have a mechanism for comparing quark fields at some finite separation. This requires us to solve the parallel transport equation of our gauge field [5]

$$\frac{dx_\mu(t)}{dt} D_\mu U(x_\mu(t)) = 0, \quad t \in [0, 1], \quad (2.16)$$

where $U(x_\mu(t))$ is an $SU(3)$ element satisfying $U(x_\mu(0)) = I$. Using the explicit parametrisation $x_\mu = y_\mu + a t \hat{\sigma}$, where y_μ is a fixed position and σ is a fixed direction, we have

$$\begin{aligned} a \delta_\mu^\sigma (\partial_\mu - ig A_\mu) U(y_\mu + a t \hat{\sigma}) &= 0 \\ a \partial_\sigma U(y_\mu + a t \hat{\sigma}) &= iag A_\sigma U(y_\mu + a t \hat{\sigma}) \\ \frac{\partial}{\partial t} U(y_\mu + a t \hat{\sigma}) &= ig A_\sigma U(y_\mu + a t \hat{\sigma}). \end{aligned}$$

This is precisely the differential equation solved by the path-ordered exponential, so we find for each direction the gauge links

$$U_\mu(x) = \mathcal{P} \exp \left(ig \int_x^{x+a\hat{\mu}} dx' A_\mu(x') \right). \quad (2.17)$$

From this definition we also see that we can write the gauge link in the opposite direction, i.e. from $x + a\hat{\mu}$ to x , as

$$\begin{aligned} \mathcal{P} \exp \left(ig \int_{x+a\hat{\mu}}^x dx' A_\mu(x') \right) &= \mathcal{P} \exp \left(-ig \int_x^{x+a\hat{\mu}} dx' A_\mu(x') \right) \\ &= U_\mu^\dagger(x). \end{aligned}$$

These gauge links have the simple gauge transformation property [6]

$$U_\mu \rightarrow \Omega(x) U_\mu(x) \Omega^\dagger(x + a\hat{\mu}). \quad (2.18)$$

Making use of this gauge transformation property, we can construct gauge invariant Wilson loops by taking the product of the U_μ 's around a closed loop. The simplest such loop, the 1×1 square, is called the *plaquette*, and is defined as

$$P_{\mu\nu}(x) = U_\mu(x) U_\nu(x + a\hat{\mu}) U_\mu^\dagger(x + a\hat{\mu} + a\hat{\nu}) U_\nu^\dagger(x). \quad (2.19)$$

Taking the trace of this loop we see that by the cyclic property of the trace this is gauge invariant

$$\begin{aligned} \text{Tr}(P_{\mu\nu}(x)) &\rightarrow \text{Tr} \left(\Omega(x) U_\mu(x) \Omega^\dagger(x + a\hat{\mu}) \Omega(x + a\hat{\mu}) U_\nu(x + a\hat{\mu}) \Omega^\dagger(x + a\hat{\mu} + a\hat{\nu}) \right. \\ &\quad \left. \Omega(x + a\hat{\mu} + a\hat{\nu}) U_\mu^\dagger(x + a\hat{\mu}) \Omega^\dagger(x + a\hat{\nu}) \Omega(x + a\hat{\nu}) U_\nu^\dagger(x) \Omega^\dagger(x) \right) \\ &= \text{Tr}(P_{\mu\nu}(x)). \end{aligned}$$

We now return to the lattice formulation, making use of the gauge links to define our quantities of interest. Firstly, we approximate our gauge links using a midpoint approximation, such that

$$U_\mu^{\text{lat}}(x) = \exp \left(iag A_\mu \left(x + \frac{a}{2} \hat{\mu} \right) \right). \quad (2.20)$$

From this definition, we can also recover the the midpoint gauge potential [7, 8]

$$A_\mu \left(x + \frac{a}{2} \hat{\mu} \right) = \frac{1}{2ia g} \left(U_\mu(x) - U_\mu^\dagger(x) \right) - \frac{1}{6ia g} \text{Tr} \left(U_\mu(x) - U_\mu^\dagger(x) \right) I + \mathcal{O}(a^2). \quad (2.21)$$

We then note that we can write $F_{\mu\nu}$ in terms of the plaquette by Taylor expanding Eq. 2.19 to obtain [9]

$$P_{\mu\nu} = I + ia^2 g F_{\mu\nu} - \frac{a^4 g^2}{2} F_{\mu\nu} F^{\mu\nu} + \mathcal{O}(a^6), \quad (2.22)$$

and hence

$$a^4 \text{Tr} (F_{\mu\nu} F^{\mu\nu}) = \frac{2}{g^2} \text{Tr} \left(I - \frac{1}{2} (P_{\mu\nu} + P_{\mu\nu}^\dagger) \right). \quad (2.23)$$

We have now arrived at a definition of the contracted field strength tensor that can be used to define our lattice action. As we sum over the 6 unique plaquettes, we take into account a factor of 2 to arrive at the definition

$$\mathcal{S}_{\text{lat}} = \frac{6}{g^2} \sum_x \sum_{\mu < \nu} \frac{1}{3} \text{Tr} \left(I - \frac{1}{2} (P_{\mu\nu} + P_{\mu\nu}^\dagger) \right). \quad (2.24)$$

To remove higher order errors from the lattice action, it is possible to take into account terms containing larger Wilson loops, following procedure similar to the one outlined above [10–12]. For the purpose of this work, the gauge fields were generated using the $\mathcal{O}(a^2)$ -improved Lüscher and Weisz action [13], which consists of a linear combination of 1×1 plaquettes and 2×1 rectangles.

This lattice framework provides the tools necessary to explicitly calculate quantities of interest. Firstly, the gauge links are generated by monte-carlo methods, using $\exp(-\mathcal{S})$ as a probability weighting for a given configuration. Once these configurations are generated, gauge fixing can be performed (Sec. 2.3.1, 3.2.1), and quantities such as the gluon propagator (Chapter 4) can be obtained.

2.3 Gauge Fixing

The choice of gauge is crucial when performing calculations on the lattice, or more generally in any gauge field theory calculation. There are two choices of gauge relevant to this study: Landau gauge and maximal centre gauge. Maximal centre gauge is best

explored in the context of centre vortices, and will therefore be detailed in chapter 3.2.1, however the Landau gauge fixing condition provides a good introduction to the gauge fixing procedure, and as such will be described here.

2.3.1 Landau Gauge

In the continuum, Landau gauge corresponds to imposing the condition

$$\partial_\mu A^\mu = 0. \quad (2.25)$$

On the lattice, we can approximate this condition by imposing

$$\Delta(x) = \sum_\mu A_\mu \left(x + \frac{a}{2} \hat{\mu} \right) - A_\mu \left(x - \frac{a}{2} \hat{\mu} \right) = 0. \quad (2.26)$$

Here the fact that we have defined the lattice gauge potential to be at the midpoint of the link produces an improved continuum limit when we consider Eq. 2.26 in momentum space [8]. Performing a discrete Fourier transform, we see that

$$\begin{aligned} \Delta(p) &= \sum_x \Delta(x) e^{-i p \cdot x} \\ &= \sum_{x, \mu} e^{-i p \cdot x} \left(A_\mu \left(x + \frac{a}{2} \hat{\mu} \right) - A_\mu \left(x - \frac{a}{2} \hat{\mu} \right) \right) \\ &= \sum_{x, \mu} e^{i p \cdot \frac{a}{2} \hat{\mu}} e^{-i p \cdot (x + \frac{a}{2} \hat{\mu})} A_\mu \left(x + \frac{a}{2} \hat{\mu} \right) - e^{-i p \cdot \frac{a}{2} \hat{\mu}} e^{-i p \cdot (x - \frac{a}{2} \hat{\mu})} A_\mu \left(x - \frac{a}{2} \hat{\mu} \right) \\ &= \sum_\mu \left(e^{i p \cdot \frac{a}{2} \hat{\mu}} - e^{-i p \cdot \frac{a}{2} \hat{\mu}} \right) A_\mu(p) \\ &= \sum_\mu 2i \sin \left(\frac{a}{2} p_\mu \right) A_\mu(p) = 0. \end{aligned} \quad (2.27)$$

This is to be compared to the momentum space Landau gauge condition that can be obtained from the gauge potential defined on the lattice sites, which has the form [8]

$$\sum_\mu [(\cos(a p_\mu) - 1) - i \sin(a p_\mu)] A'_\mu(p) = 0. \quad (2.28)$$

In the limit as $a \rightarrow 0$, it can be seen that Eq. 2.27 exhibits $\mathcal{O}(a^2)$ improvement, whereas Eq. 2.28 has only $\mathcal{O}(a)$ improvement.

The Landau Gauge condition is imposed on the Lattice by finding extrema of the functional [14]

$$\mathcal{F} = \frac{4}{3}\mathcal{F}_1 - \frac{1}{12u_0}\mathcal{F}_2, \quad (2.29)$$

where

$$\begin{aligned} \mathcal{F}_1 &= \sum_{\mu,x} \frac{1}{2} \text{Tr} \left\{ U_\mu^G(x) + U_\mu^G(x)^\dagger \right\} \\ \mathcal{F}_2 &= \sum_{\mu,x} \frac{1}{2} \text{Tr} \left\{ U_\mu^G(x) U_\mu^G(x + a\hat{\mu}) + \text{h.c.} \right\} \\ u_0 &= \left(\frac{1}{3} \text{Re Tr} \langle U_{\text{pl}} \rangle \right)^{\frac{1}{4}}. \end{aligned}$$

The u_0 term contains the average value of the gauge links in the lattice, $\langle U_{\text{pl}} \rangle$, necessary to restore the continuum limit of \mathcal{F}_2 as $a \rightarrow 0$ [15]. We also explicitly write U_μ^G to indicate that we are considering gauge links under as-yet unknown gauge transformation

$$\Omega(x) = \exp \left(i\omega^a(x) \frac{\lambda_a}{2} \right) \quad (2.30)$$

It becomes apparent why this we seek the extrema of this particular functional when we take the functional derivative with respect to the free parameters of the gauge transformation, $\omega^a(x)$.

$$\frac{\delta \left\{ \frac{4}{3}\mathcal{F}_1 - \frac{1}{12u_0}\mathcal{F}_2 \right\}}{\delta \omega^a(x)} = g a^2 \sum_{\mu} \text{Tr} \left\{ \left[\partial_\mu A_\mu(x) - \frac{4}{360} a^4 \partial_\mu^5 A_\mu(x) + \mathcal{O}(a^6) \right] \frac{\lambda^a}{2} \right\} + \mathcal{O}(g^3 a^4) \quad (2.31)$$

If Eq. 2.31 is at an extrema, then

$$\sum_{\mu} \partial_\mu A_\mu(x) = \sum_{\mu} \frac{4}{360} a^4 \partial_\mu^5 A_\mu(x) + \mathcal{O}(a^6) + \mathcal{O}(g^3 a^4).$$

Hence at order $\mathcal{O}(a^4)$, finding the extrema of Eq. 2.31 is equivalent to satisfying the continuum Landau gauge condition, Eq. 2.25.

2.4 Lattice units

In the previous section, we have explicitly detailed how a variety of lattice quantities are constructed. For clarity, it is useful to remove extraneous constants by utilising so-called ‘lattice units’. Transforming to lattice units is done by setting $a = g = 1$,

which gives the following transformations

$$\begin{aligned}
 A_\mu \left(x + \frac{a}{2} \hat{\mu} \right) &\rightarrow A_\mu \left(x + \frac{\hat{\mu}}{2} \right) \\
 U_\mu(x) &\rightarrow \exp \left(i A_\mu \left(x + \frac{\hat{\mu}}{2} \right) \right) \\
 P_{\mu\nu}(x) &\rightarrow U_\mu(x) U_\nu(x + \hat{\mu}) U_\mu^\dagger(x + \hat{\nu}) U_\nu^\dagger(x)
 \end{aligned}$$

Chapter 3

Topology of the Lattice

3.1 Confinement

The confinement property of QCD, and the accompanying notion of asymptotic freedom, is one of the defining low-energy features of the theory of the strong interaction. With Gell-mann and Zweig's concurrent proposal of quarks as the elementary constituents of baryons and mesons [16, 17], it is natural to then attempt to observe these new particles in isolation. However, efforts to observe quarks proved impossible. Early experiments testing the behaviour of electron-proton collisions demonstrated that protons scatter elastically, behaving as though they are finite-sized particles recoiling electromagnetically from the incident electron [18]. These experiments indicated no further substructure to the proton, inconsistent with the quark model. As accelerator energies improved, later experiments [19, 20], using electron energies of 7 and 10 GeV found that inelastic scattering effects became dominant, with electrons behaving as though they were scattering off of loosely bound constituent particles. To explain this behaviour, Feynman proposed what is known as the 'parton' model [21], treating the proton as being comprised of non-interacting electrically charged particles in the limit that the incident electron energy tends towards infinity. This is precisely the notion of asymptotic freedom; at large distance scales the partons are tightly bound, whereas at short distances they behave as free particles. It did not take long for the separate theories of quarks and partons to be recognised as complementary, and by the early 70's the quark-parton model of hadrons accurately explained the experimental results observed in particle colliders.

These experimental and theoretical results led in part to the development of the non-Abelian gauge field theory of QCD, as introduced in Chapter 2. The proof that

non-Abelian gauge theories are asymptotically free was discovered in 1973 [22], and experimental evidence of the existence of 3 quark colours through study of the cross section of e^+e^- collisions supports the initial $SU(3)$ colour symmetry anticipated by Gell-Mann and Zweig. At high energies, QCD has consistently explained the behaviour of hadronic matter, and has become the accepted theory of the strong nuclear interaction. However, the question of whether QCD is indeed a confining theory still remains. As confinement is a low-momentum property of QCD, it is apparent that any analytic proof of confinement must take place far from the asymptotic limit. To date, no such analytic proof has been found.

Lattice calculations are currently the only method by which it is possible to investigate low-energy QCD phenomena from first-principles. Calculations of the static potential between two massive quarks, both recent and old [23–26], have shown that the potential rises linearly at sufficiently large separation distances. This behaviour is precisely what is expected of a colour confining theory. Other confinement mechanisms have also been proposed on the lattice, including mechanisms based on the behaviour of the gluon propagator at $q = 0$ [27] and the behaviour of the pion mass and Polyakov loop at light quark masses [28]. All lattice results so far have indicated that QCD is in fact a confining theory at low energy.

There is good evidence that confinement has its roots in the topological properties of the QCD vacuum. It is well understood that the QCD vacuum, unlike the QED vacuum, admits non-trivial instanton solutions: solutions of the vacuum field configurations that are all minima of the classical action, yet are distinguished from one another by a topological quantum number [29]. The presence of instanton solutions was significant in resolving the $U(1)$ anomaly [30], and provides an excellent method of calculating the ground state hadron spectrum [4]. The non-trivial topology of the QCD vacuum, and the success of topological features in resolving QCD anomalies, motivates the search for a topological explanation of confinement. A variety of features have been proposed, including: Abelian monopoles [31, 32], merons [33] and dual superconductors [34, 35]. One of the most promising models in recent years is known as the *Centre Vortex Model*, and it is this model that forms the backbone of this research.

3.2 Centre Vortices

Originally proposed by 't Hooft in 1978 [36, 37], centre vortices are closed two-dimensional surfaces present in four-dimensional Euclidean space. The key property of a centre vortex is that in three dimensions any Wilson loop (see Sec. 2.2) that encloses a vortex line will acquire a centre phase, such that

$$W(C) \rightarrow z W(C), \quad (3.1)$$

where z is an element of $Z(3)$, the centre of $SU(3)$. The centre of a group is the subgroup that contains all the elements of the group that commute with all other elements. In the case of $SU(3)$ this corresponds to

$$Z(3) = \left\{ \exp\left(\frac{m\pi i}{3}\right) I \mid m = 0, \pm 1 \right\}. \quad (3.2)$$

3.2.1 Maximal Centre Gauge

3.2.2 Centre Projection

3.3 Instantons

The notion of a non-empty QCD vacuum is intrinsically tied to the existence of instanton solutions to the gluonic equations of motion. An instanton solution is simply a non-trivial solution to the equations of motion that has zero total energy

3.3.1 Topological Charge

Chapter 4

Landau Gauge Gluon Propagator

In a gauge field theory, the position-space propagator, $D_{\mu\nu}(x - y)$, of the gauge boson is the two-point correlation function, which in the case of $SU(3)$ can be interpreted as the probability of a gluon being created at position x , propagating to y , and then being annihilated. The propagator therefore serves as a useful measure of the behaviour of gluons as a function of distance; or, correspondingly, as a function of momentum in the momentum-space representation. In this chapter we detail how the Landau gauge gluon propagator is calculated on the lattice.

4.1 Lattice Definition of the Gluon Propagator

We begin with the definition of the coordinate space propagator [27, 38, 39].

$$D_{\mu\nu}^{ab}(x) = \langle A_\mu^a(x) A_\nu^b(0) \rangle. \quad (4.1)$$

The propagator in momentum space is simply related by the discrete Fourier transform,

$$D_{\mu\nu}^{ab}(p) = \sum_x e^{-ip \cdot x} \langle A_\mu^a(x) A_\nu^b(0) \rangle. \quad (4.2)$$

Noting that the coordinate space propagator $D_{\mu\nu}^{ab}(x - y)$ only depends on the difference $x - y$, such that

$$\langle A_\mu^a(x) A_\nu^b(0) \rangle = \langle A_\mu^a(x + y) A_\nu^b(y) \rangle, \quad (4.3)$$

we can make use of translational invariance to average over the four-dimensional volume to obtain the form for the momentum space propagator.

$$\begin{aligned}
D_{\mu\nu}^{ab}(p) &= \frac{1}{V} \sum_{x,y} e^{-ip \cdot x} \langle A_\mu^a(x+y) A_\nu^b(y) \rangle \\
&= \frac{1}{V} \sum_{x,y} \langle e^{-ip \cdot (x+y)} A_\mu^a(x+y) e^{+ip \cdot y} A_\nu^b(y) \rangle \\
&= \frac{1}{V} \langle A_\mu^a(p) A_\nu^b(-p) \rangle.
\end{aligned} \tag{4.4}$$

Hence we find that the momentum space gluon propagator on a finite lattice with four-dimensional volume V is given by

$$D_{\mu\nu}^{ab}(p) \equiv \frac{1}{V} \langle A_\mu^a(p) A_\nu^b(-p) \rangle. \tag{4.5}$$

In the continuum, the Landau-gauge momentum-space gluon propagator has the following form [7, 40]

$$D_{\mu\nu}^{ab}(q) = \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \delta^{ab} D(q^2), \tag{4.6}$$

where $D(q^2)$ is the scalar gluon propagator. Contracting Gell-Mann index b with a and Lorentz index ν with μ one has

$$D_{\mu\mu}^{aa}(q) = (4-1)(n_c^2-1) D(q^2), \tag{4.7}$$

such that the scalar function can be obtained from the gluon propagator via

$$D(q^2) = \frac{1}{3(n_c^2-1)} D_{\mu\mu}^{aa}(q), \tag{4.8}$$

where $n_c = 3$ is the number of colours.

As the lattice gauge links $U_\mu(x)$ naturally reside in the 3×3 fundamental representation of $SU(3)$, we now wish to work in the matrix representation of $A_\mu(x)$, as introduced in Eq. 2.3. Using the orthogonality relation $\text{Tr}(\lambda_a \lambda_b) = \delta_{ab}$ for the Gell-Mann matrices, it is straightforward to see that

$$2 \text{Tr}(A_\mu A_\mu) = A_\mu^a A_\mu^a, \tag{4.9}$$

which can be substituted into equation 4.8 to obtain the final expression for the lattice scalar gluon propagator,

$$D(p^2) = \frac{2}{3(n_c^2 - 1)V} \left\langle \text{Tr } A_\mu(p) A_\mu(-p) \right\rangle. \quad (4.10)$$

As defined in Eq. 2.21, we make use of the midpoint definition of the gauge potential in terms of the lattice link variables. Once the link variables are fixed to Landau gauge following the procedure described in Sec. 2.3.1, we obtain the momentum-space gauge potential

$$A_\mu(p) = \sum_x e^{-ip \cdot (x + \hat{\mu}/2)} A_\mu(x + \hat{\mu}/2). \quad (4.11)$$

4.2 Momentum Variables

4.3 Lattice Parameters and Data Cuts

Chapter 5

Smoothing

Topological objects are inherently geometric in nature, and as a result they can suffer substantially from ambiguities brought about by the discretisation of spacetime. Hence, when considering the behaviour of topological objects on the lattice, it has been proven to be necessary to remove high frequency fluctuations in the gauge fields [41]. Furthermore, when investigating the long distance behaviour of the lattice it is also beneficial to filter off the short distance fluctuations to better reveal the physics in the region of interest [42]. The process of removing these fluctuations is known as smoothing, which in turn falls into two sub-categories: cooling and smearing.

The purpose of both these methods is similar, however the algorithms used to implement them differ greatly. Cooling assesses each link, replacing the existing link with one that locally minimises some choice of action (see e.g. the Wilson action in Eq. 2.24). Smearing does not depend on the choice of action, and instead replaces each link with a weighted average of its nearest neighbours. Once every link in the lattice has been updated, the configuration is said to have had one sweep of smoothing applied. The process can then be repeated to an arbitrary number of sweeps. Due to the differences in the routines, it is important to compare the results from both to observe how they each perform and quantitatively observe how they alter the measured quantities.

5.1 Smoothing Methods

5.1.1 Cooling

5.1.2 Over-Improved Smearing

5.2 Results from the Gluon Propagator

It has previously been shown that smoothing is necessary to obtain agreement between the untouched and vortex only string tension, mass function and instanton content [43–45]. Motivated by these results, we now investigate the effect of both $\mathcal{O}(a^4)$ -improved cooling [46] and over-improved stoutlink smearing [42]. Following the results of Ref. [42], the over-improved smearing parameters are $\rho = 0.06$ and $\epsilon = 0.25$ to best preserve the size of instantons on the lattice. To accomplish a similar preservation of topological objects under cooling, we used the three-loop improved algorithm as described in Ref. [46].

We first plot the untouched propagator after 0, 1, 2, 4 and 8 sweeps of cooling in Fig. 5.1. In gauge fixing, each sweep has been preconditioned by the Landau gauge transformation of the prior sweep in descending order (i.e. the transformation for sweep 10 preconditions sweep 9). This preconditioning is done to ensure that the Landau gauge functional is near the same local minima for each cooling sweep. We observe the expected removal of short distance fluctuations that is typical of smoothing, resulting in a suppressed propagator at large q . This is complemented by an amplification in the infra-red region which can be attributed to the increase in low momentum modes arising from the smoothing of the gauge fields.

To compare the effects of cooling and over-improved smearing, the untouched gluon propagator is plotted in Fig. 5.2 after either over-improved smearing or cooling. By comparing the smeared and cooled propagator we can see that cooling has a more rapid effect, related to the well-known fast removal of action from the lattice. The qualitative shape of the propagator remains the same however, and it can be seen that, for example, 4 smearing sweeps produces a propagator remarkably similar to 1 cooling sweep. More generally, we observe that in regards to the shape of the propagator, $n_{\text{sm}} \approx 4 n_{\text{cool}}$. Following the observation made in Ref. [47] that the number of over-improved stoutlink

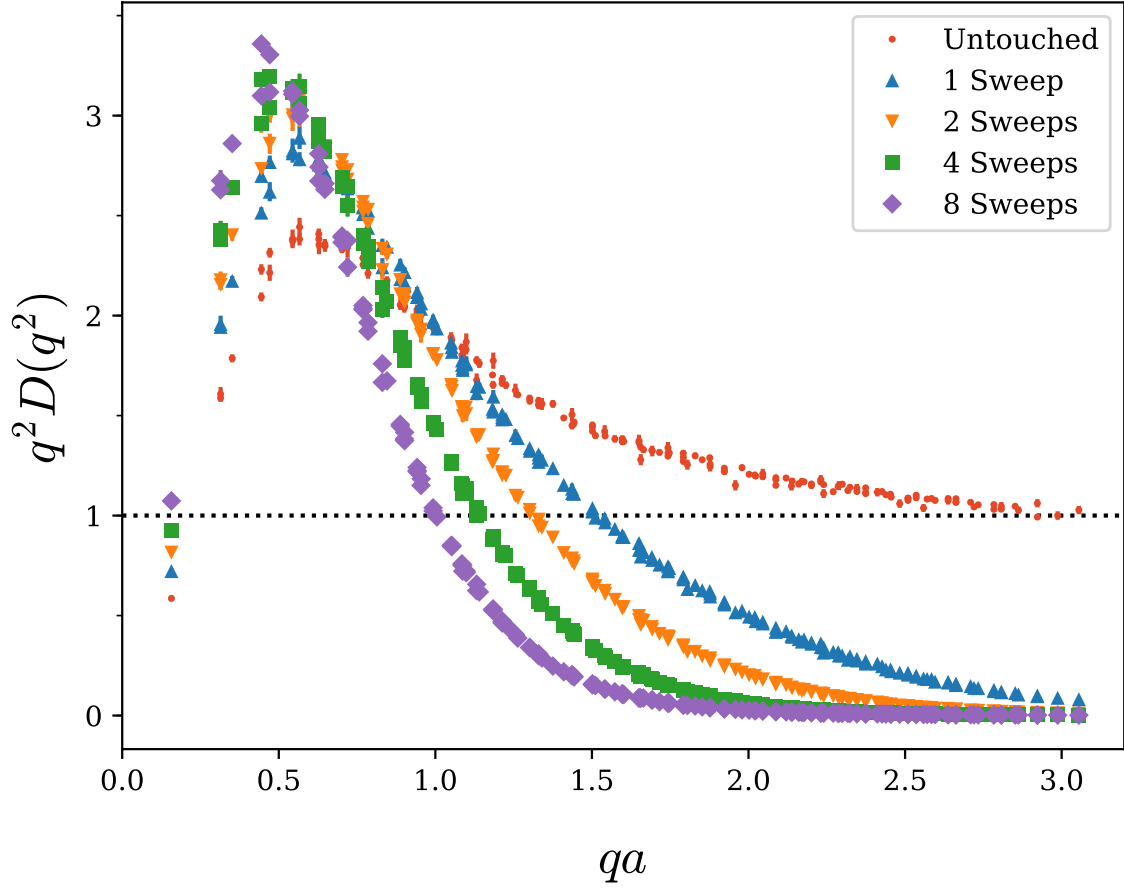


Fig. 5.1 Comparison of the gluon propagator on the untouched configurations after cooling. For clarity we have selected a sample of sweeps between 1 and 8.

smearing sweeps is related to the gradient flow time by

$$t \approx \rho n_{\text{sm}} , \quad (5.1)$$

we deduce that the relationship between gradient flow time and cooling is

$$t \approx 0.24 n_{\text{cool}} . \quad (5.2)$$

It is well understood that smoothing alters the vortex background, and based on previous work [43, 48, 49] we anticipate that the vortices identified on smoothed configurations would differ to those identified on the unsmoothed configurations.

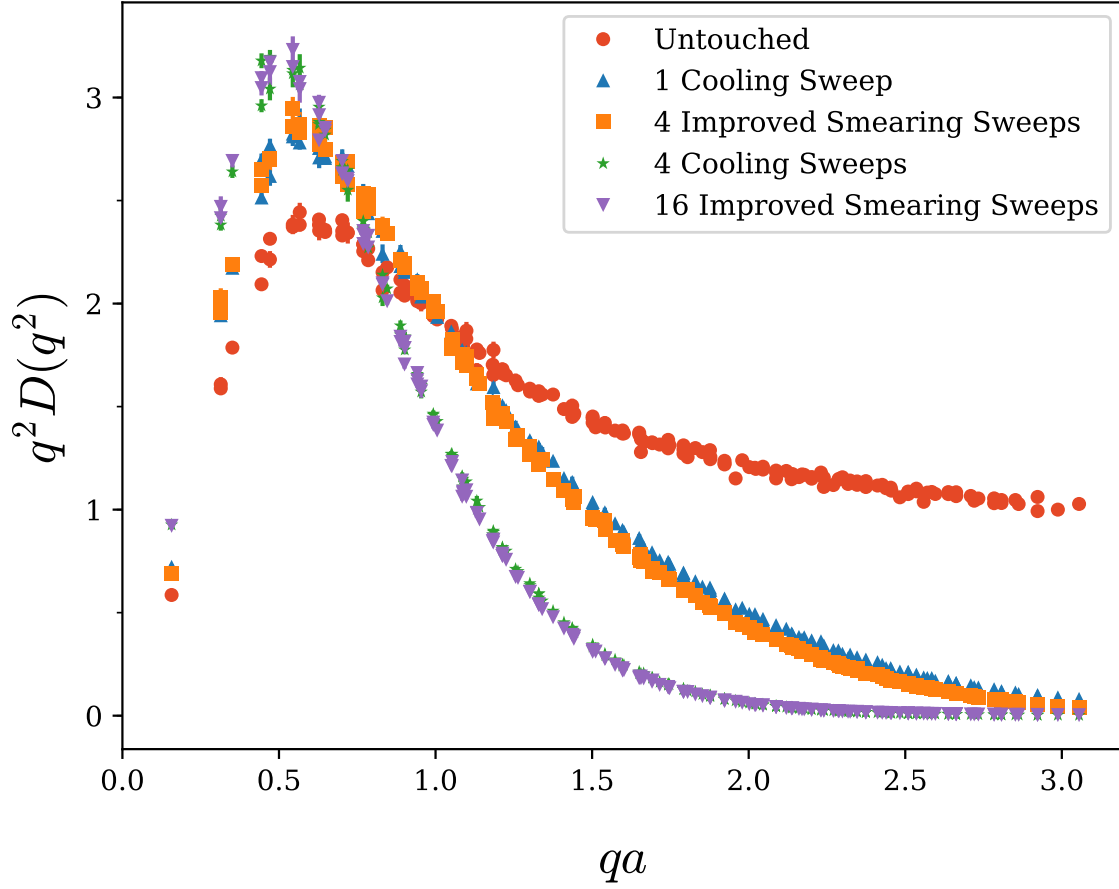


Fig. 5.2 The gluon propagator after cooling or improved smearing. We see that the shape of the plot changes minimally between the smoothing routines. However cooling requires fewer sweeps to produce the same effect when compared to smearing.

We therefore perform vortex identification only on the original configurations, with smoothing then being performed independently on the untouched, vortex-only and vortex-removed configurations. We choose to use cooling as the smoothing algorithm for the results presented in this paper, however it is worth noting that similar results can be obtained with the use of over-improved smearing.

Chapter 6

Gluon Propagator on Vortex-Modified Backgrounds

Here we present the results from the gluon propagator, calculated according to method outlined in Chapter 4, on our three vortex-modified configurations:

1. Original ‘untouched’ fields, $U_\mu(x)$,
2. Projected vortex-only fields, $Z_\mu(x)$,
3. Vortex-removed fields, $R_\mu(x) = Z_\mu^\dagger(x) U_\mu(x)$.

6.1 Preliminary Results

Calculating the scalar propagator on untouched, vortex-removed and vortex only configurations gives the results illustrated in Fig. 6.1. To make contact with the tree-level propagator at large q^2 , we renormalise such that $q^2 D(q^2) = 1$ for $qa = 3.0$ on the original configurations, and apply this same renormalisation factor to the vortex removed and vortex only propagators. The vortex removed configurations display the expected behaviour, with vortex removal corresponding to significant infrared suppression of the propagator when compared to the untouched propagator, in agreement with the results of Ref. [50]. The increased roughness of the gauge fields after vortex removal is evidenced by the enhancement of the propagator at large q . This reflects the increase in short-distance fluctuations that have been introduced to the gauge fields by the vortex removal procedure.

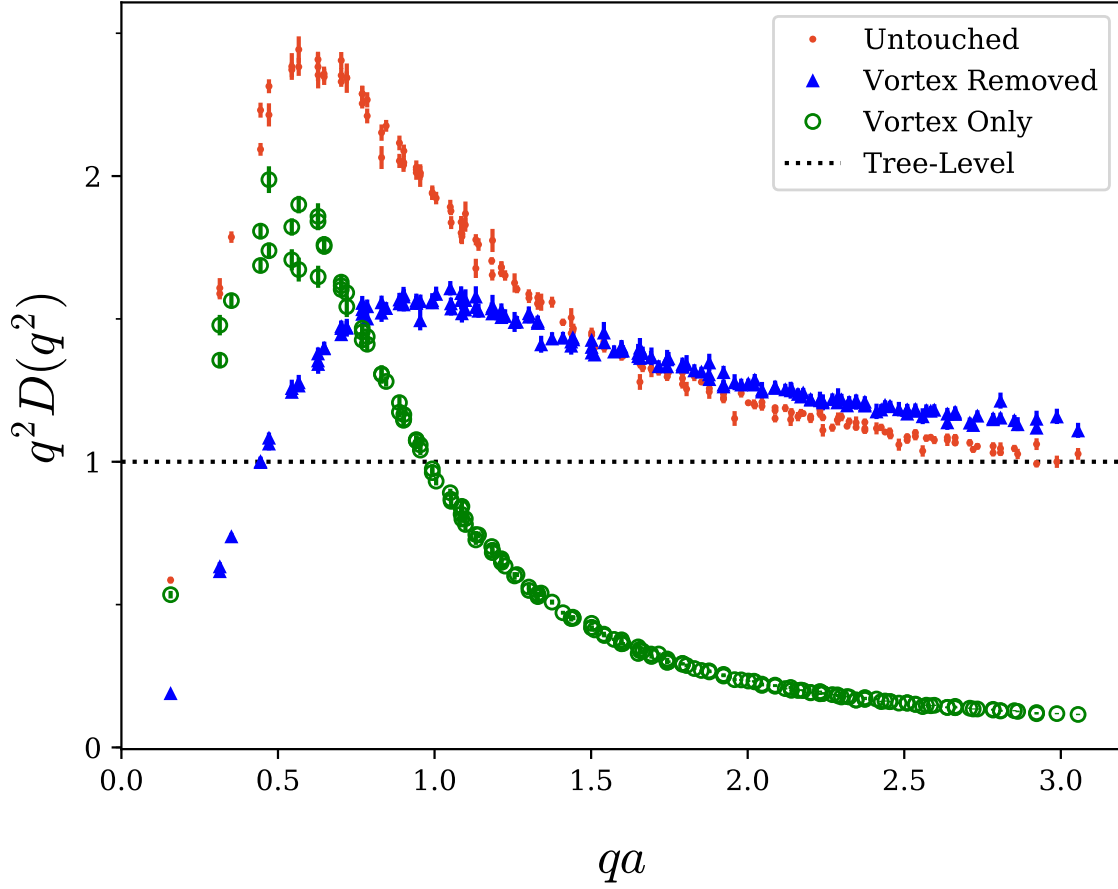


Fig. 6.1 The gluon propagator calculated from the original untouched (red dots), shown with the vortex removed (blue triangles) and vortex only (green open circles) results. Here, the renormalisation factor for the vortex removed and vortex only propagators is chosen to be the same as for the untouched propagator.

It is interesting to note that the vortex only propagator retains approximately two thirds of the untouched propagator's peak strength. This is comparable to previous work showing partial recovery of the string tension on vortex only configurations [43, 45, 51?]. Despite only recovering a portion of the original strength, the infrared peak is still considerably greater than the peak observed in the vortex removed propagator. The loss of strength is most likely in part because of the known imperfections in the vortex identification algorithm that results in some vortex matter remaining in the vortex removed configurations. The vortex only configurations also exhibit a loss of short range strength, due to the absence of the high frequency modes that are instead contained within the vortex removed field.

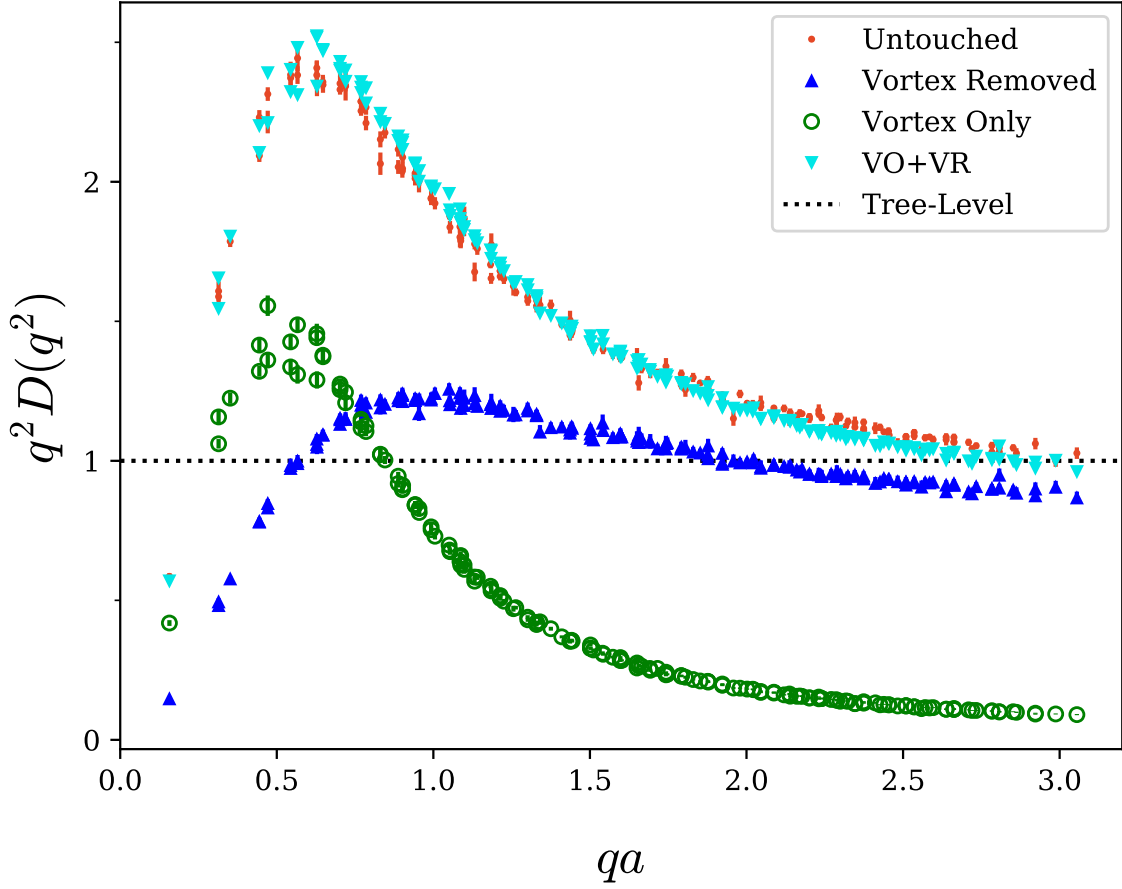


Fig. 6.2 The gluon propagator from the original untouched ensemble as in Fig. 6.1, now shown with the independently renormalised sum (cyan triangles) of the vortex removed and vortex only propagators. The two vortex modified propagators are also shown, but here their renormalisation factor is chosen to be the same as for the summed propagator.

If we sum the vortex only and vortex removed propagators and independently renormalise such that $q^2 D(q^2) = 1$ at $qa = 3.0$, we obtain the result shown in Fig. 6.2. Here we observe agreement between the untouched and summed propagators. This indicates that vortex modification effectively partitions the lattice configuration into short range physics on the vortex removed configurations and long range physics on the vortex only configurations, up to errors in the vortex identification procedure.

6.1.1 Partitioning

This partitioning is expected if the vortex removed and vortex only configurations are orthogonal. To see how this behaviour emerges, suppose that we can decompose the

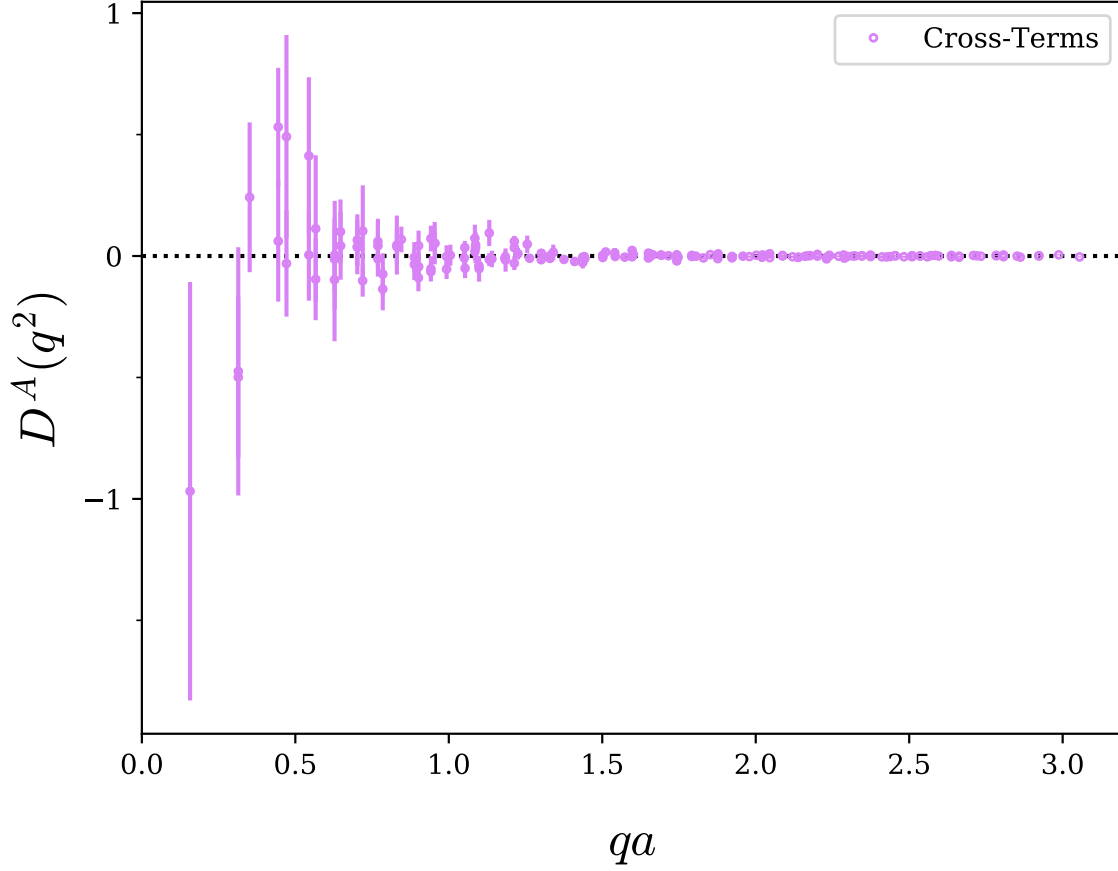


Fig. 6.3 Calculation of the cross-terms arising from Eq. 6.2

gluon field A_μ into two independent fields as follows

$$A_\mu(p) = B_\mu(p) + C_\mu(p). \quad (6.1)$$

In the context of this work, we associate B_μ with the background field of short-range gluon fluctuations and C_μ with the centre vortex field. Note also that if B and C are in Landau gauge then so is A . Using this partitioning it follows that the gluon propagator

for A can be written as the sum of the respective gluon propagators for B and C ,

$$\begin{aligned}
 D_{\mu\nu}^A(p) &= \frac{1}{V} \langle A_\mu(p) A_\nu(-p) \rangle \\
 &= \frac{1}{V} \left(\langle B_\mu(p) B_\nu(-p) \rangle + \langle C_\mu(p) C_\nu(-p) \rangle \right. \\
 &\quad \left. + \langle B_\mu(p) C_\nu(-p) + C_\mu(p) B_\nu(-p) \rangle \right) \\
 &= D_{\mu\nu}^B(p) + D_{\mu\nu}^C(p),
 \end{aligned} \tag{6.2}$$

where we have made use of the fact that B and C represent orthogonal degrees of freedom in the gauge field and hence in the ensemble average the cross-correlations should vanish. These cross-correlations are explicitly calculated by evaluating

$$D_{\text{cross-terms}}(p^2) = \frac{2}{3(n_c^2 - 1)V} \langle \text{Tr} (B_\mu(p) C_\mu(-p) + C_\mu(p) B_\mu(-p)) \rangle, \tag{6.3}$$

analogous to the scalar gluon propagator derived in Chapter 4. The results of this calculation are shown in Fig. 6.3. As can be clearly seen, in the ensemble average Eq. 6.3 vanishes, indicating that the vortex only and vortex removed configurations truly do represent a orthogonal degrees of freedom.

To elucidate the connection to the unitary formulation of the lattice gauge links, we suppose that we can transform A to an “ideal centre gauge” such that in lattice units the field C consists purely of centre phases,

$$C_\mu(x) = k \frac{2\pi}{3} I, \quad k \in \{-1, 0, +1\}. \tag{6.4}$$

On a continuous manifold we can write the Wilson line corresponding to a lattice link as a path-ordered exponential,

$$U_\mu(x) = \mathcal{P} e^{i \int_0^1 d\lambda A_\mu(x + \lambda \hat{\mu})}. \tag{6.5}$$

The lattice midpoint approximation replaces the integral as follows,

$$U_\mu(x) = e^{i A_\mu(x + \hat{\mu}/2)}. \tag{6.6}$$

As $A = B + C$ it immediately follows that we can write

$$U_\mu(x) = e^{i B_\mu(x + \hat{\mu}/2)} e^{i C_\mu(x + \hat{\mu}/2)}, \tag{6.7}$$

noting that in our ideal centre gauge $[B, C] = 0$ so the Baker-Campbell-Hausdorff relation is trivial. Identifying

$$Z_\mu(x) = e^{iC_\mu(x+\hat{\mu}/2)} \quad (6.8)$$

as the vortex-projected field, and

$$R_\mu(x) = e^{iB_\mu(x+\hat{\mu}/2)} \quad (6.9)$$

as the background remainder field we thus recover the decomposition of the links used herein,

$$U_\mu(x) = Z_\mu(x) \cdot R_\mu(x). \quad (6.10)$$

In practise, on the lattice the maximal centre gauge fixing that is implemented will differ from the ideal centre gauge postulated here due to apparent numerical difficulties in simultaneously identifying all vortex matter within an $SU(3)$ gauge field. What this means is that the projected field Z may not capture all of the vortex matter such that there is some non-trivial topological structures that remain in the background field R . The infrared enhancement in the vortex removed results in Fig. 6.1 suggests this is the case.

6.1.2 Renormalisation

In Fig. 6.2 it proved necessary to independently renormalise the untouched and summed propagators such that they agree at $qa = 3.0$. The necessity of this renormalisation is worth discussing, as it is important to motivate why comparison between the original and reconstructed propagators is valid.

Renormalisation steps:

1. Calculation contains divergent integrals that must be isolated.
2. Dimensional regularisation isolates the infinities in the integral.
3. Redefine the mass/coupling to absorb these infinities. The new constant is *finite*, whereas the original constant is taken to be *infinite*.
4. Alternatively, the counter term method can be used.
5. The counter term method states that the original constant is *finite*, that are modified by infinite constants.

6. Upon calculating physical quantities, the infinite constants cancel off the infinities arising from the loop integrals.

Questions:

1. At what scale does $D_{\text{QED}}(q^2) = \frac{1}{q^2}$?
2. At what scale can we suppose that $D_{\text{QED}}(q^2) = D_{\text{QCD}}(q^2)$?
3. Is it reasonable to say that the condition that must be enforced on the combined propagator is $D(q^2) = \frac{1}{q^2}$ at some large momentum, and as such this is the normalisation that should be applied to the VR and VO propagators?
4. Does $D_{\text{QCD}}(q^2)$ equal the bare continuum propagator at a fixed cutoff?
5. What does the renormalised propagator correspond to in the continuum?

6.2 Impact of Cooling

After performing 10 sweeps of cooling on the untouched, vortex-removed and vortex-only ensembles, we obtain the results shown in Fig. 6.4. As is typical of cooling, the removal of short range structures means that all three ensembles tend to zero as $q \rightarrow \infty$. There is now a noticeable improvement in the agreement between the untouched and vortex only configurations; however there is still a difference present, especially in the $qa \approx 0.5$ and $qa \approx 1.5$ regions.

We perform the same analysis of the vortex only propagator under cooling as performed in Sec. 5.2 on the untouched propagator. Once again in gauge fixing, each sweep is preconditioned by the Landau gauge transformation of the previous sweep in descending order. The result of this analysis is shown in Fig. 6.5. This figure shows a similar change in the vortex only propagator when compared to the untouched propagator in Fig. 5.1, with an enhancement in the infrared and suppression in the UV modes. The UV suppression is less noticeable in this case due to the prior removal of short range effects brought about by the vortex identification.

We observe that the vortex only and untouched propagators in Fig. 6.4 resemble the gluon propagator under a differing number of sweeps of cooling, as shown in Fig. 5.1 and Fig. 6.5. The vortex only propagator has a peak that sits below the untouched

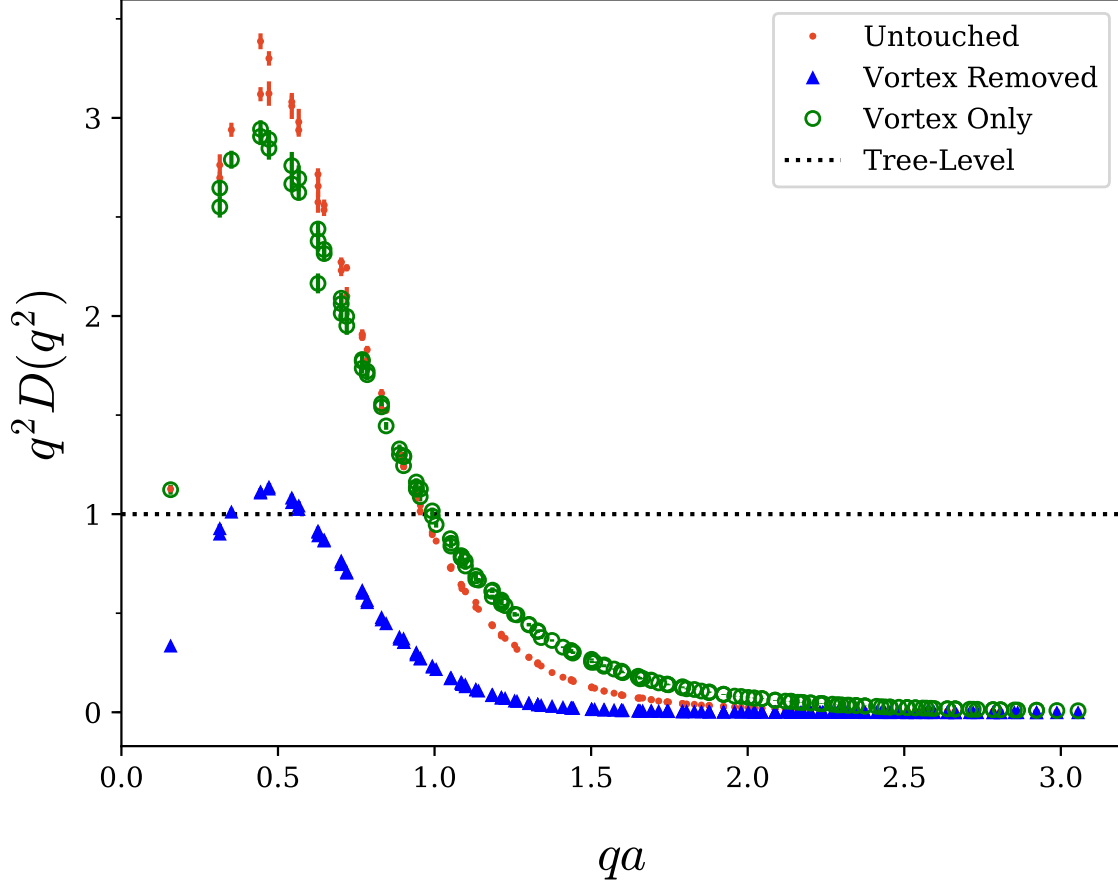


Fig. 6.4 The gluon propagator calculated on the three ensembles after 10 sweeps of cooling. We now observe an improved agreement between the untouched and vortex only propagators.

propagator, and the untouched propagator is further suppressed in the $qa \approx 1.5$ region. Following the trend in Fig. 5.1 and Fig. 6.5, this indicates that further cooling on the vortex only propagator would align it with the untouched propagator. This follows from an understanding that the vortex-only configurations are initially much rougher than their untouched counterparts [44], and should therefore require additional cooling to obtain agreement with the untouched configurations.

We take the average $\mathcal{O}(a^4)$ three-loop improved action of the lattice divided by the single instanton action $S_0 = \frac{8\pi^2}{g^2}$, denoted \bar{S}/S_0 , to be a measure of roughness. We observe that for $n < 20$ cooling sweeps the vortex-only configurations have a significantly higher action than their untouched counterparts after the same number of sweeps of cooling, as illustrated in Fig. 6.6. We therefore seek to find the number

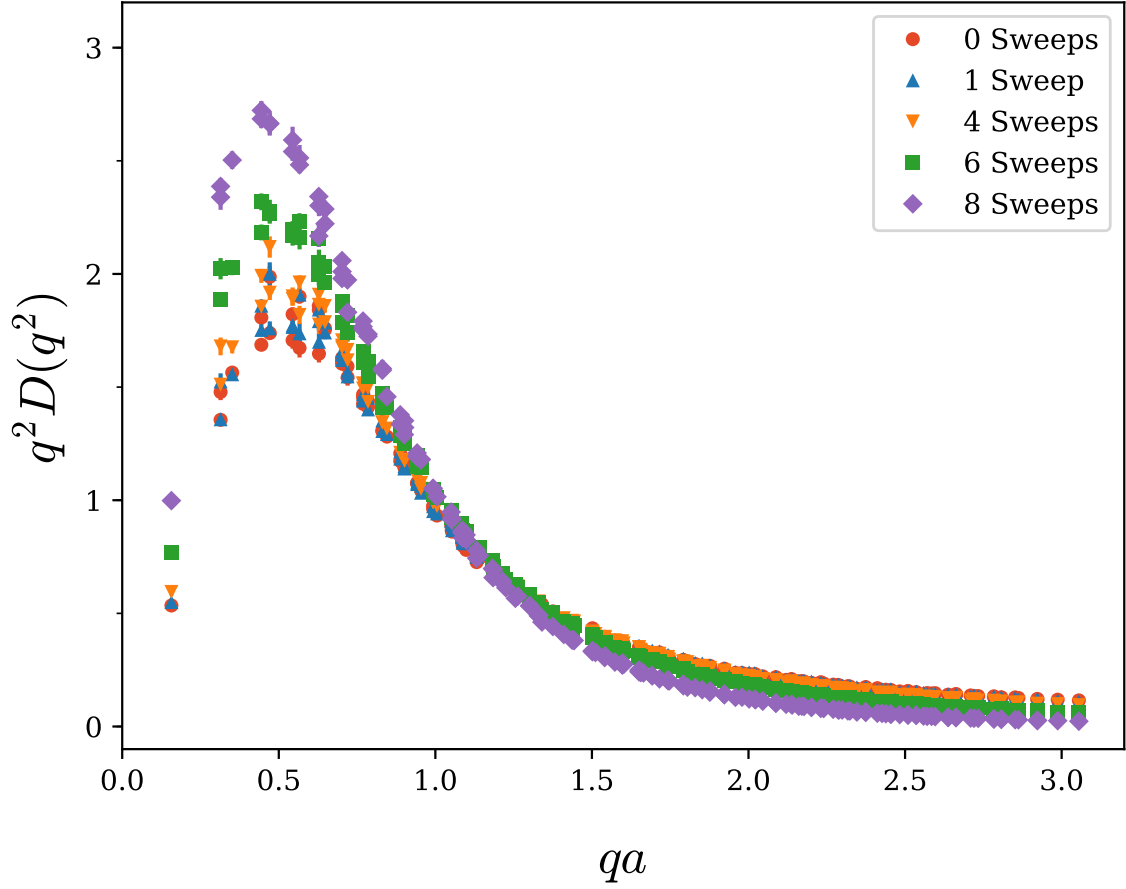


Fig. 6.5 The vortex only propagator after different sweeps of cooling. A trend similar to Fig. ?? is observed, with enhancement in the infrared and suppression in the UV region.

of sweeps required to best match the action between the vortex-only and untouched configurations. The results of this procedure are shown in Table 6.1. If we now plot these matched configurations, we obtain the results shown in Fig. 6.7. Here we have truncated the plot at large qa to better show the agreement in the mid- qa region. By matching the actions as closely as possible with an integer number of cooling sweeps, we see that there is a better agreement between the untouched and vortex-only gluon propagators.

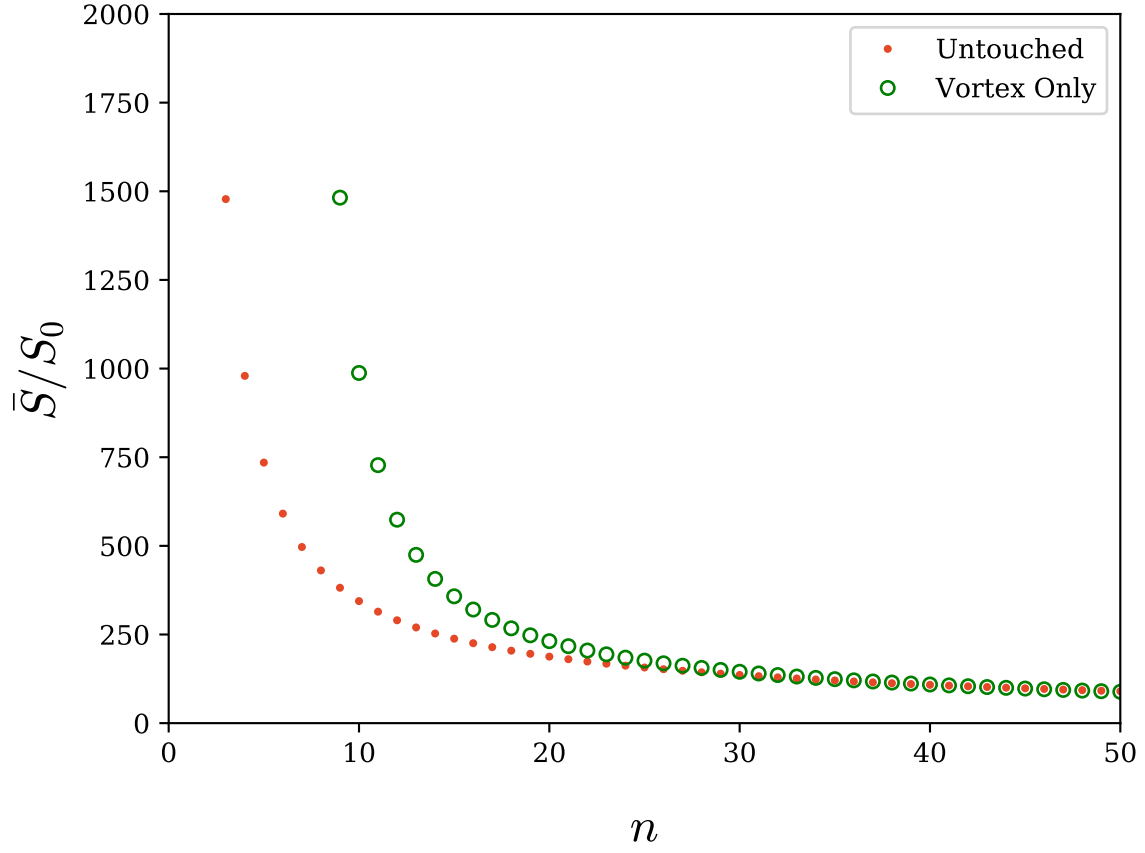


Fig. 6.6 The average action calculated on the untouched and vortex-only configurations as a function of cooling sweeps, n . The vortex only configurations are initially rougher than the untouched, as evidenced by the higher average action.

Table 6.1 Comparison of the number of cooling sweeps on the untouched (n_U) and vortex only (n_{VO}) configurations required to match the average action.

n_U	\bar{S}/S_0	n_{VO}	\bar{S}/S_0
5	734.83	11	727.67
10	344.22	15	357.68
15	238.21	20	231.19
20	187.55	24	184.68
25	156.92	28	155.72
30	135.91	32	135.61
35	120.29	36	120.66
40	107.08	40	109.02

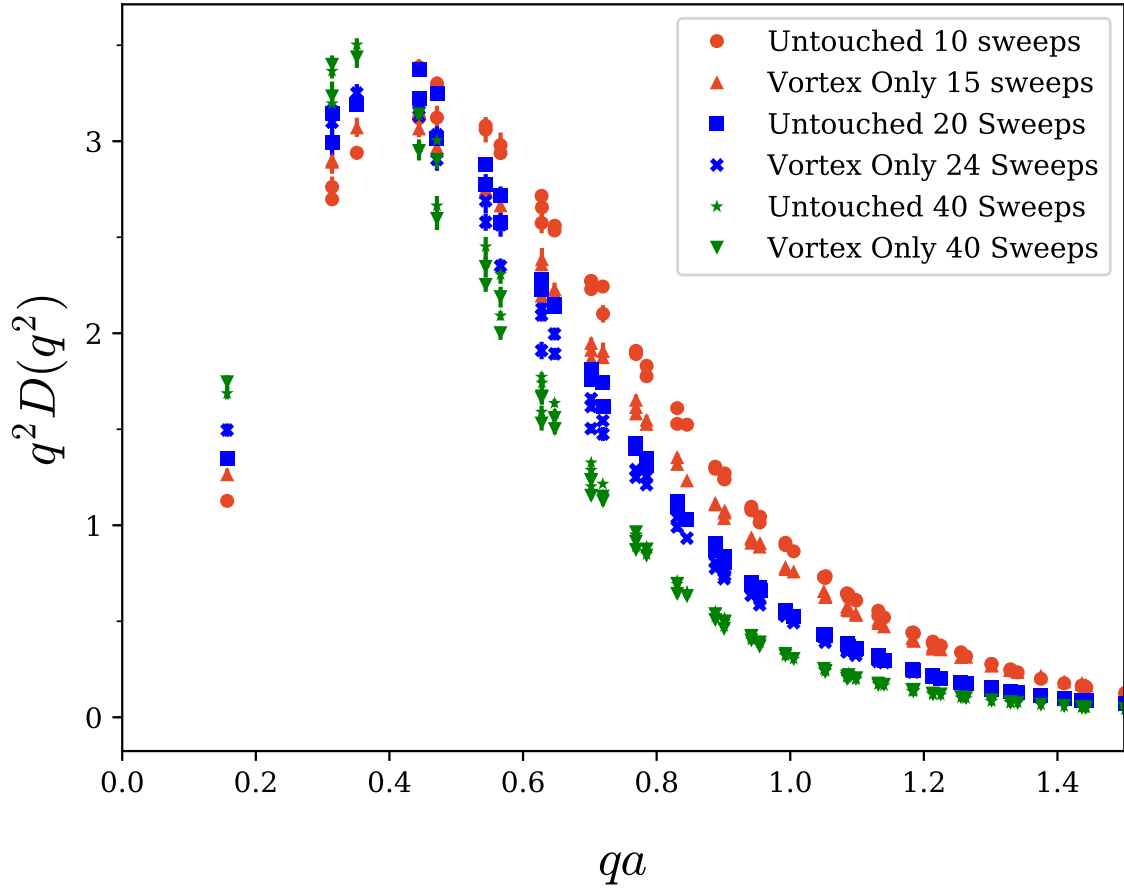


Fig. 6.7 Comparison of the gluon propagator on the untouched and vortex only configurations after tuning the number of cooling sweeps to best match the average plaquette action. This procedure gives a much better agreement in the shape of the gluon propagator from the two configurations.

Chapter 7

Centre Vortex Visualisations

7.1 3D Models

7.1.1 Time Slices

7.1.2 Time-Oriented Links

7.1.3 Topological Charge

7.2 Centre Vortices and Topological Charge

Chapter 8

Conclusion

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