Factors Affecting the Damping Constant of a Spring System

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Abstract

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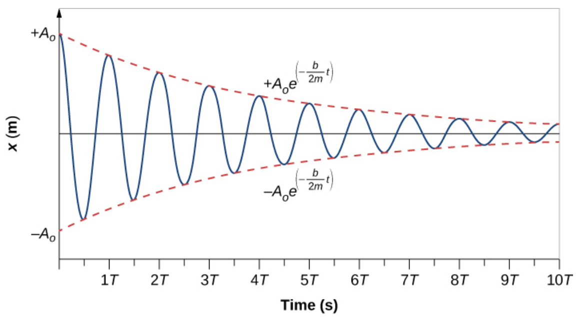


Figure 1: Displacement–time graph illustrating the characteristics of an underdamped harmonic oscillator.

1. Introduction

A vertical mass–spring system provides a simple model for studying oscillatory motion and energy dissipation in mechanical systems. An ideal mass–spring system undergoes simple harmonic motion with constant amplitude, but in real conditions mechanical energy is dissipated through a variety of means, causing oscillations to gradually decrease in amplitude. This phenomenon is known as damping, defined as "the loss of energy of an oscillating system by dissipation."

The degree of damping can be characterised by the decay constant γ, which specifies the rate at which oscillations diminish; larger values correspond to faster energy loss and more rapid amplitude reduction. The value of this constant depends on both the properties of the oscillating mass and the restoring system.

1.1. Aim and Hypothesis

The aim of this investigation is to determine how spring configuration and attached mass influence the rate of damping, quantified by the exponential decay constant γ, in a vertically oscillating mass-spring system.

It is hypothesised that if the attached mass is increased, there will be a decrease in the decay constant γ proportional to ${m_{\text{load}}}^{-1}$, and if the spring system is configured in series, there will also be a decrease in the decay constant.

The primary sources of damping are expected to be internal friction within the spring in conjunction with resistive forces such as air drag on the mass. For simplicity, it will be assumed that these factors can be modelled by a damping force which is linearly dependent on velocity, as with a standard viscous dampening model.

1.2. Variables

Independent Variables: The mass of the load attached to the end of the spring, and the configuration of the spring system.

Dependent Variable: The curve-fitting coefficients of the damped harmonic motion displacement-time graph.

Controlled Variables: The positioning of the retort stand and clamps, motion sensor alignment and sampling frequency, mass cross-sectional geometry, and ambient air conditions (temperature, pressure, humidity).

2. Theory

To model the motion mathematically, Newton's Second Law can be applied to the system for the applied, restoring, and damping forces. This results in a second-order differential equation that can be solved to find the displacement of the mass and the rate of amplitude decay, establishing the basis for determining k and b experimentally.

$$
\begin{aligned}
\sum \overrightarrow{F} = m\overrightarrow{a} &= \overrightarrow{F}_{\text{restoring}} + \overrightarrow{F}_{\text{damping}} \\
m\overrightarrow{g} &= -k\overrightarrow{x} - b\overrightarrow{v} \\
0 &= k\overrightarrow{x} + b\frac{\mathrm{d}\overrightarrow{x}}{\mathrm{d}t} + m\frac{\mathrm{d}^2\overrightarrow{x}}{\mathrm{d}t^2} \\
0 &= k\overrightarrow{x} + b\frac{\mathrm{d}\overrightarrow{x}}{\mathrm{d}t} + m\frac{\mathrm{d}^2\overrightarrow{x}}{\mathrm{d}t^2} \\
0 &= \frac{k}{m}\overrightarrow{x} + \frac{b}{m}\frac{\mathrm{d}\overrightarrow{x}}{\mathrm{d}t} + \frac{\mathrm{d}^2\overrightarrow{x}}{\mathrm{d}t^2} \\
0 &= \frac{k}{m}e^{\lambda t} + \frac{b}{m}\frac{\mathrm{d}(e^{\lambda t})}{\mathrm{d}t} + \frac{\mathrm{d}^2(e^{\lambda t})}{\mathrm{d}t^2} \\
0 &= k + bu + m\lambda^2 \\
\lambda &= \frac{-b \pm \sqrt{b^2 - 4km}}{2m} \\
\lambda &= -\frac{b}{2m} \pm \frac{\sqrt{b^2 - 4km}}{2m}
\end{aligned}
$$

It is assumed that the spring will be underdamped, meaning the damping is weak enough that the system oscillates around equilibrium before settling. Physically, this requires that $b^2 - 4km < 0$, so the displacement equation has complex roots and produces harmonic motion.

$$\implies \lambda = -\frac{b}{2m} \pm \frac{i\sqrt{4km - b^2}}{2m}$$

Given $\gamma = \frac{b}{2m}$ and $\omega' = \frac{\sqrt{4km-b^2}}{2m}$

$$\implies \lambda = -\gamma \pm i\omega'$$

The general solution for $\overrightarrow{x} = e^{\lambda t}$ is a linear combination of the solutions corresponding to each root λ, as required for a second-order differential equation.

$$
\begin{aligned}
\overrightarrow{x} &= \alpha e^{(-\gamma + i\omega')t} + \beta e^{(-\gamma - i\omega')t} \\
&= e^{-\gamma t}\left[\alpha e^{i\omega' t} + \beta e^{-i\omega' t}\right] \\
&= e^{-\gamma t}\left[(\alpha + \beta)\cos(\omega' t) + i(\alpha - \beta)\sin(\omega' t)\right] \\
&= 2\sqrt{\alpha\beta}\, e^{-\gamma t}\cos(\omega' t + \phi)
\end{aligned}
$$

Letting $A = 2\sqrt{\alpha\beta}$:

$$\therefore \overrightarrow{x} = Ae^{-\gamma t}\cos(\omega' t + \phi)$$

The oscillatory behaviour of a mass-spring system arises from the continual transformation between kinetic and potential energy, represented by the cosine term in the displacement equation. When the spring is displaced from its equilibrium position, potential energy is stored in the spring, and the speed of the mass decreases toward a minimum for that cycle. As the mass passes through equilibrium, this potential energy is converted almost entirely into kinetic energy, producing the maximum speed for the cycle.

However, this process is not perfectly efficient as the damping force removes energy from the system, predominantly as heat, which causes the amplitude of oscillations to decrease gradually over time. This decay is captured by the exponential term in the displacement equation. This process drives the sinusoidal and decaying components of motion, which are quantified by the damped angular frequency ω' and decay constant γ

The damped angular frequency ω' is dependent on both the spring constant k and the damping coefficient b, as evident in its formula $\omega' = \frac{\sqrt{4km-b^2}}{2m}$.

$$
\begin{aligned}
\omega' &= \frac{\sqrt{4km - b^2}}{2m} \\
\omega'^2 &= \frac{k}{m} - \frac{b^2}{4m^2} \\
\omega'^2 &= \frac{k}{m} - \gamma^2 \qquad \text{with } \gamma = \frac{b}{2m} \\
\omega'^2 + \gamma^2 &= k\left(\frac{1}{m}\right)
\end{aligned}
$$

As shown above, plotting $\omega'^2 + \gamma^2$ against m^{-1} should give a straight line with gradient k. The spring constant k represents the stiffness of the spring, and quantifies the relationship between the spring's restoring force and resulting displacement according to Hooke's law ($\overrightarrow{F}_{\text{restoring}} = -k\overrightarrow{x}$).

$$
\begin{aligned}
\gamma &= \frac{b}{2m} \\
&= \frac{b}{2}\left(\frac{1}{m}\right)
\end{aligned}
$$

Similarly, plotting the decay constant γ against m^{-1} should produce a straight line with gradient $b/2$, allowing determination of the damping coefficient b. This coefficient characterises the velocity-relative magnitude of the resistive force opposing motion, responsible for gradual reduction in oscillation amplitude.

3. Equipment and Method

3.1. Equipment

- 1 × LabQuest 2 and cables
- 1 × Vernier Motion Detector (±1 mm)
- 2 × Stiff springs
- 1 × Light spring
- 1 × Vernier Hanging Mass Set (250 g/50 g)
- 1 × Retort stand
- 1 × Bosshead clamp, spring hanger, and rod
- 1 × Scientific scale (±0.01 g)
- 1 × Computer with Logger Pro 3

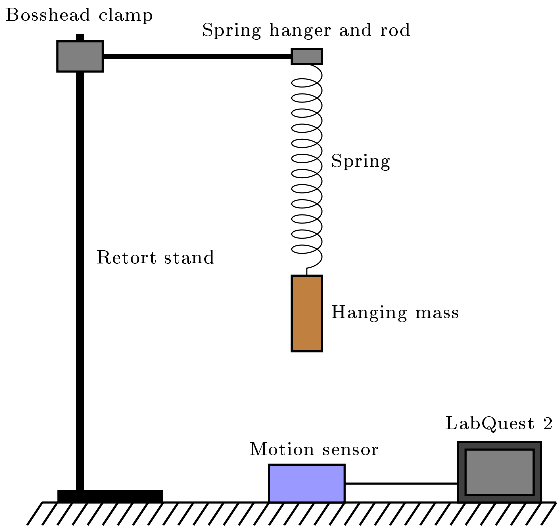


Figure 2: Experimental setup diagram for a vertical mass–spring system utilising a motion sensor.

3.2. Method

1. Set up equipment as depicted in Fig. 2, with the light spring hooked onto the spring hanger and base of the retort stand towards the mass.
2. Initialise the LabQuest 2 and Motion Detector and configure the mode to be displacement–time, sampling period to be 60 s, and sample rate to be 60 Hz.
3. Connect the LabQuest 2 wirelessly to a computer with Logger Pro running.
4. Measure and record the mass of the hanger from the Hanging Mass Set on a scientific scale.
5. Attach the mass to the end of the spring and release it in a controlled manner, ensuring there is no horizontal displacement of the system before starting the data recording from the computer.
6. Allow the 60 s data collection period to finish without disturbing the system before removing the mass from the spring system.
7. Apply a 'Damped Harmonic' curve fit in Logger Pro and record coefficients A, B, and C, and the R^2 value.
8. Place a 50 g mass from the Hanging Mass set onto the hanger and measure and record the new total mass.
9. Repeat steps 5 to 8 until at least 5 different masses have been trialled.
10. Replace the light spring with a stiff spring and repeat steps 5 to 9.
11. Replace the stiff spring with two stiff springs in series and repeat steps 5 to 9.

4. Results

Table 1: Regression parameters for the light spring.

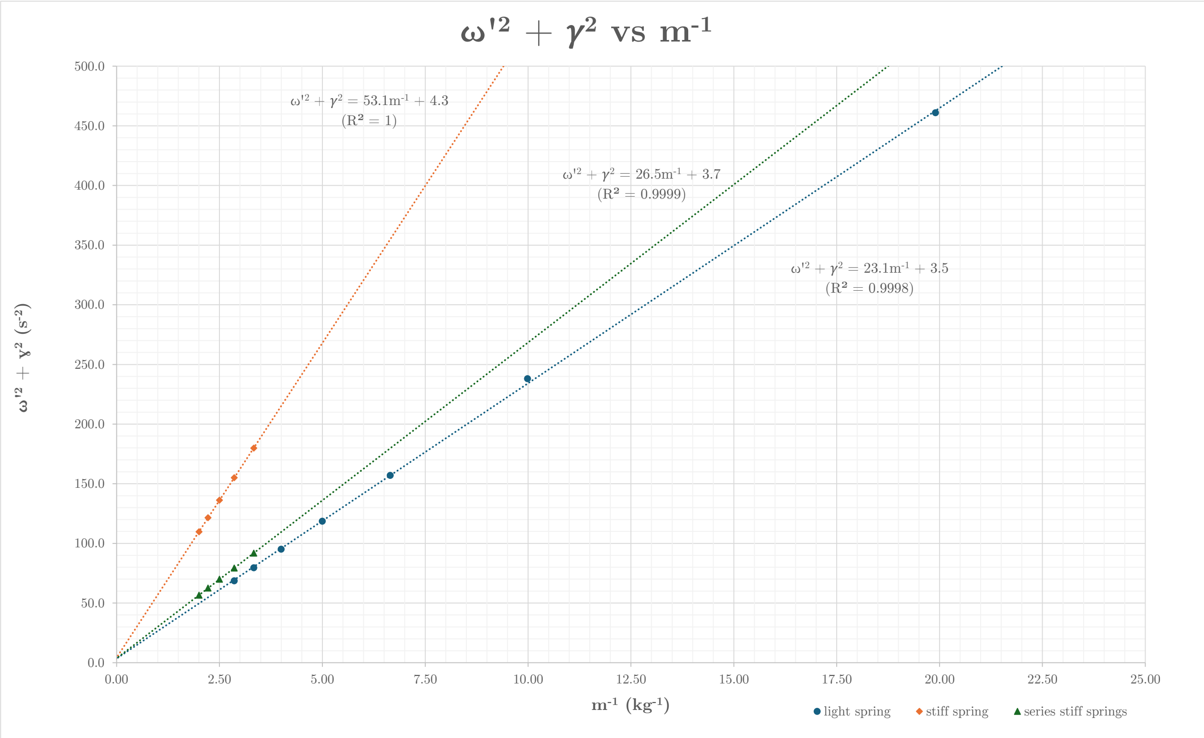
m_{load} (kg)	γ (s^{-1})	ω' (rad s^{-1})	R^2	${m_{\text{load}}}^{-1}$ (kg^{-1})	$\omega'^2+\gamma^2$ (s^{-2})
0.050 25	0.024 240	21.47	0.9990	19.90	461.0
0.100 14	0.019 520	15.43	0.8385	9.9860	238.1
0.150 44	0.015 780	12.53	0.9947	6.6472	157.0
0.200 11	0.012 860	10.89	0.9996	4.9973	118.6
0.250 19	0.012 236	9.75	0.9894	3.9970	95.1
0.300 12	0.011 830	8.92	0.9999	3.3320	79.5
0.349 91	0.011 380	8.28	0.9999	2.8579	68.6

Table 2: Regression parameters for the stiff spring.

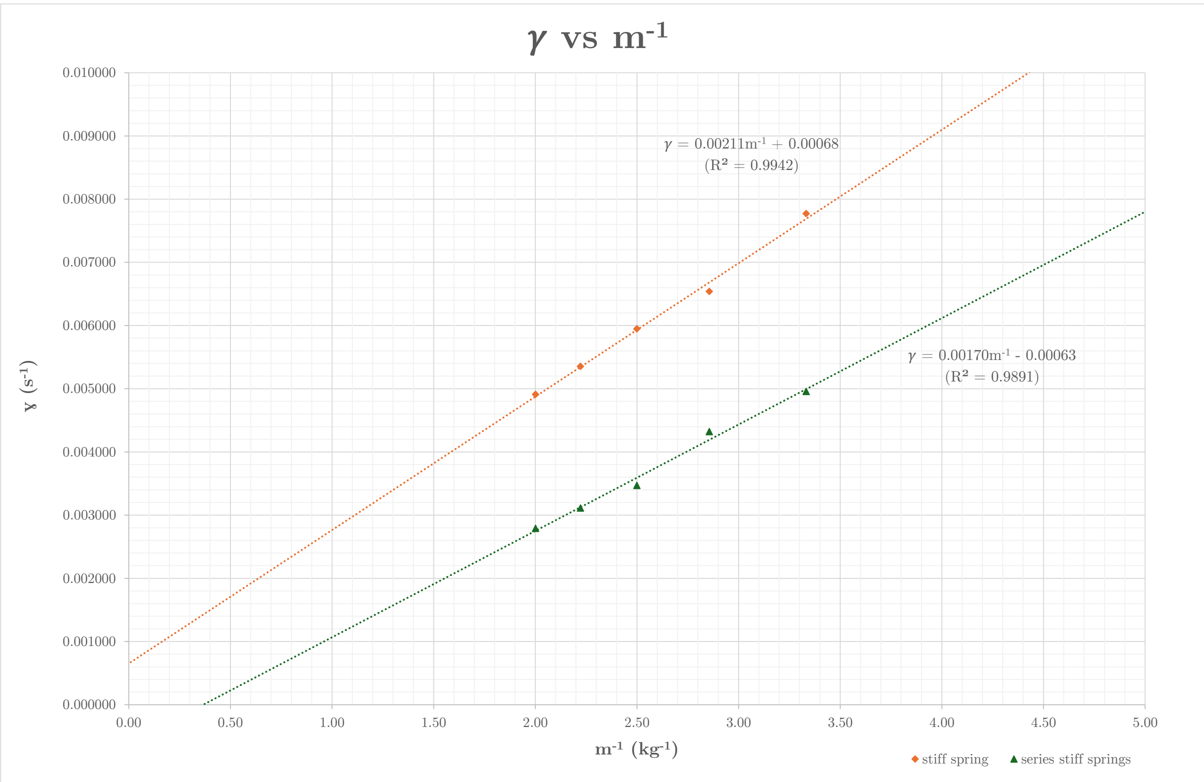
m_{load} (kg)	γ (s^{-1})	ω' (rad s^{-1})	R^2	${m_{\text{load}}}^{-1}$ (kg^{-1})	$\omega'^2+\gamma^2$ (s^{-2})
0.300 10	0.007 771	13.41	0.9971	3.3322	179.8
0.350 14	0.006 539	12.45	0.9955	2.8560	155.0
0.400 14	0.005 946	11.67	0.9993	2.4991	136.2
0.450 04	0.005 352	11.02	0.9993	2.2220	121.4
0.499 84	0.004 909	10.47	0.9995	2.0006	109.6

Table 3: Regression parameters for two stiff springs in series.

m_{load} (kg)	γ (s^{-1})	ω' (rad s^{-1})	R^2	${m_{\text{load}}}^{-1}$ (kg^{-1})	$\omega'^2+\gamma^2$ (s^{-2})
0.300 10	0.004 957	9.581	0.9996	3.3322	91.80
0.350 14	0.004 321	8.907	0.9995	2.8560	79.33
0.400 14	0.003 469	8.363	0.9991	2.4991	69.94
0.450 04	0.003 112	7.907	0.9995	2.2220	62.52
0.499 84	0.002 791	7.517	0.9996	2.0006	56.51



Graph 1: Relationship between $\omega'^2 + \gamma^2$ and $m_{\text{load}}{}^{-1}$ for different spring configurations, where the gradient represents the spring constant (k).



Graph 2: Relationship between γ and $m_{\text{load}}{}^{-1}$ for different spring configurations, where the gradient represents half the damping coefficient ($\frac{b}{2}$).

For **Graph 1**: k = gradient.

$$\begin{aligned} k_{\text{light}} &= \frac{\Delta(\omega'^2+\gamma^2)}{\Delta m_{\text{load}}^{-1}} \\ &= \frac{315.0-165.0}{13.50-7.00} \\ &= 23.1 \text{ kg s}^{-2} \text{ (3 s.f.)} \end{aligned}$$

$$\begin{aligned} k_{\text{stiff}} &= \frac{\Delta(\omega'^2+\gamma^2)}{\Delta m_{\text{load}}^{-1}} \\ &= \frac{375.0-30.0}{7.00-0.50} \\ &= 53.1 \text{ kg s}^{-2} \text{ (3 s.f.)} \end{aligned}$$

$$\begin{aligned} k_{\text{series}} &= \frac{\Delta(\omega'^2+\gamma^2)}{\Delta m_{\text{load}}^{-1}} \\ &= \frac{480.0-255.0}{18.00-9.50} \\ &= 26.5 \text{ kg s}^{-2} \text{ (3 s.f.)} \end{aligned}$$

For **Graph 2**: $b = 2 \times$ gradient.

$$\begin{aligned} b_{\text{stiff}} &= 2 \times \frac{\Delta\gamma}{\Delta m_{\text{load}}^{-1}} \\ &= 2 \times \frac{(9.375-4.688)\times 10^{-3}}{4.12-1.90} \\ &= 4.22 \times 10^{-3} \text{ kg s}^{-1} \text{ (3 s.f.)} \end{aligned}$$

$$\begin{aligned} b_{\text{series}} &= 2 \times \frac{\Delta(\omega'^2+\gamma^2)}{\Delta m_{\text{load}}^{-1}} \\ &= 2 \times \frac{(6.875-1.250)\times 10^{-3}}{4.40-1.10} \\ &= 3.40 \times 10^{-3} \text{ kg s}^{-1} \text{ (3 s.f.)} \end{aligned}$$

Figure 3: Calculations of k and b values from graphs.

5. Analysis of Results

Analysis of the data in presented by **Graph 1** finds reasonably clear patterns emerging. By plotting the sum of the damped angular frequency squared and decay constant squared ($\omega'^2+\gamma^2$) against m_{load}^{-1}, a highly linear relationship between the variables can be observed. Forcing a linear line of best fit, the gradient is expected to yield the spring constant k. From the recorded data, the calculated spring constants are:

$$\begin{aligned} k_{\text{light}} &= 23.1 \text{kg s}^{-2} \\ k_{\text{single stiff}} &= 53.1 \text{kg s}^{-2} \\ k_{\text{series stiff}} &= 26.5 \text{kg s}^{-2} \end{aligned}$$

Given spring constant reflects the force required to stretch the spring a certain distance, it follows that the stiff spring has a higher k value than the light spring. Furthermore, the precision of the data can be testified by application of the general spring constant equation that states that the $k_{\text{equivalent}}$ of two identical springs in series is equal to $\frac{1}{2}k_{\text{single stiff}}$. From the measured $k_{\text{single stiff}}$, the theoretical equivalent series spring constant is $\frac{1}{2}(53.1) = 26.6$ (3 s.f.). This possesses a highly respectable percentage error of -0.376% between the calculated and empirical values, solidifying the relative precision between calculations.

Graph 2 depicts a slightly more ambiguous pattern. Unlike the clear proportionality observed in **Graph 1**, the relationship between γ and m_{load}^{-1} shows greater scatter, and the physical interpretation of the linear fit is less direct. Both the stiff spring and series stiff spring datasets still yield strong linear correlations ($R^2 = 0.9942$ and 0.9891 respectively), but the fitted lines have small but non-negligible intercepts, one positive and one negative. These offsets suggest random influences outside the idealised damping model, such as frictional or measurement effects that contribute to a baseline damping constant independent of mass.

Let $m_{\text{eq}} = m_{\text{load}} + \frac{1}{3}m_{\text{spring}}$ and $\lambda = m_{\text{load}}^{-1}$

$$\begin{aligned} \gamma &= \frac{b}{2m_{\text{eq}}} \\ &= \frac{b}{2\left(m_{\text{load}}+\frac{1}{3}m_{\text{spring}}\right)} \\ &= \frac{b}{2\left(\frac{1}{\lambda}+\frac{1}{3}m_{\text{spring}}\right)} \\ &= \frac{b}{2}\frac{\lambda}{1+\frac{1}{3}m_{\text{spring}}\,\lambda} \\ \therefore \frac{d\gamma}{d\lambda} &= \frac{b}{2}\frac{1}{\left(1+\frac{1}{3}m_{\text{spring}}\,\lambda\right)^2} \end{aligned}$$

$$\bar{b}_{\text{single}} = 4.22 \times 10^{-3}\text{kg}\,\text{s}^{-1}, s.e.\left(b_{\text{single}}\right) = 0.00018$$

$$\begin{aligned}
C_{95\%}\left(b_{\text{single}}\right) &= \bar{b}_{\text{single}} \pm \left(1.96 \times s.e.\left(b_{\text{single}}\right)\right) \\
&= 4.22 \times 10^{-3} \pm (1.96 \times 0.00018) \\
&= (4.22 \pm 0.353) \times 10^{-3} \\
&= (3.87,\ 4.57) \times 10^{-3}\ \text{kg}\,\text{s}^{-1}
\end{aligned}$$

$$\bar{b}_{\text{series}} = 3.40 \times 10^{-3}\text{kg}\,\text{s}^{-1}, s.e.\left(b_{\text{series}}\right) = 0.00020$$

$$\begin{aligned}
C_{95\%}\left(b_{\text{series}}\right) &= \bar{b}_{\text{series}} \pm \left(1.96 \times s.e.\left(b_{\text{series}}\right)\right) \\
&= 3.40 \times 10^{-3} \pm (1.96 \times 0.00020) \\
&= (3.40 \pm 0.392) \times 10^{-3} \\
&= (3.01,\ 3.79) \times 10^{-3}\ \text{kg}\,\text{s}^{-1}
\end{aligned}$$

Figure 4: Calculations of 95% confidence intervals for the b values of the single and series configurations.

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6. Conclusion

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