2a) Let $g \sim \Gamma(3, \gamma)$

$$\frac{f(x;\gamma)}{g(x;\beta)} = \frac{(1/k(\gamma))x^2\exp\{-\frac{x^2}{2}-\gamma x\}}{(\beta^3/\Gamma(3))x^2\exp\{-\beta x\}}$$

$$= \frac{\Gamma(3)}{\beta^3 k(\gamma)} \exp\{-\frac{x^2}{2}-(\gamma-\beta)x\}$$

$$= h(x;\beta,\gamma)$$

Find the supremum of $h(x)$

$$\frac{\partial h}{\partial x} = \underbrace{\frac{\Gamma(3)}{\beta^3 k(\gamma)}}_{>0}\left(-x-(\gamma-\beta)\right)\underbrace{\exp\{-\frac{x^2}{2}-(\gamma-\beta)x\}}_{>0}$$

$$\frac{\partial h}{\partial x} = 0 \Rightarrow -x+\beta-\gamma = 0$$

$$\Rightarrow x = \beta-\gamma$$

Show this is a supremum

$$\frac{\partial^2 h}{\partial x^2} = \underbrace{\frac{\Gamma(3)}{\beta^3 k(\gamma)}}_{>0}\underbrace{\exp\{-\frac{x^2}{2}-(\gamma-\beta)x\}}_{>0}\left((\beta-\gamma-x)^2-1\right)$$

$(\beta-\gamma-x)^2-1 < 0$ when $x=\beta-\gamma$

$\Rightarrow$ maximum when $x=\beta-\gamma$

Now calculate M

$$M = h(\beta-\gamma;\beta,\gamma)$$

$$= \frac{\Gamma(3)}{\beta^3 k(\gamma)}\exp\{-\frac{(\beta-\gamma)^2}{2}-(\gamma-\beta)(\beta-\gamma)\}$$

$$= \frac{2}{\beta^3 k(\gamma)}\exp\{\frac{(\beta-\gamma)^2}{2}\}$$

Now ~~[illegible]~~ minimise M w.r.t. β

$$\frac{\partial M}{\partial \beta} = \frac{-6}{\beta^4 k(\gamma)} \exp\left\{\frac{(\beta-\gamma)^2}{2}\right\} + \frac{2(\beta-\gamma)}{\beta^3 k(\gamma)} \exp\left\{\frac{(\beta-\gamma)^2}{2}\right\}$$

$$= \underbrace{\frac{2}{\beta^3 k(\gamma)}}_{>0} \underbrace{\exp\left\{\frac{(\beta-\gamma)^2}{2}\right\}}_{>0} \left(\frac{-3}{\beta} + \beta - \gamma\right)$$

$$\frac{\partial M}{\partial \beta} = 0 \implies \frac{-3}{\beta} + \beta - \gamma = 0$$

$$\implies \beta^2 - \gamma\beta - 3 = 0$$

$$\implies \beta = \frac{\gamma \pm \sqrt{\gamma^2 + 12}}{2}$$

Since $\beta > 0$, the positive solution is required.

$$\hat{\beta} = \frac{\gamma + \sqrt{\gamma^2 + 12}}{2}$$

Sub back in to M:

$$M(\gamma) = \frac{2}{\hat{\beta}^3 k(\gamma)} \exp\left\{\frac{(\hat{\beta}-\gamma)^2}{2}\right\}$$

$M(\gamma)$ is the expected number of attempts required to produce an accepted from f.