2c) We wish to approximate f by a $N(\mu, \sigma^2)$ distribution. Using the Laplace Approximation, we have

$$F_X(x;\gamma) = \int_0^x f(y;\gamma)\,dy$$
$$= \int_0^x \exp\{-q(y)\}\,dy$$

$$q(y) = -\log(f(y;\gamma))$$
$$= -\log\left(\frac{1}{k(\gamma)} y^2 \exp\left\{\frac{-y^2}{2} - \gamma y\right\}\right)$$
$$= -\log\left(\frac{1}{k(\gamma)}\right) - \log(y^2) - \log\left(\exp\left\{-\frac{y^2}{2} - \gamma y\right\}\right)$$
$$= \log(k(\gamma)) - 2\log(y) + \frac{y^2}{2} + \gamma y$$

Minimise $q(y)$ w.r.t. y.

$$\frac{\partial q}{\partial y} = \frac{-2}{y} + y + \gamma$$

Find the stationary points

$$\frac{\partial q}{\partial y} = 0 \implies \frac{-2}{y} + y + \gamma = 0$$
$$\implies y^2 + \gamma y - 2 = 0$$
$$\implies y = \frac{-\gamma \pm \sqrt{\gamma^2 + 8}}{2}.$$

$y > 0$ so take the positive root

$$\hat{y} = \frac{-\gamma + \sqrt{\gamma^2 + 8}}{2}$$

~~Check~~ Check this is the minimum

$$\frac{\partial^2 q}{\partial y^2} = \frac{2}{y^2} + 1 > 0, \quad \forall y \implies \hat{y} \text{ is a minimum.}$$

$$q(\hat{y}) = \log(k(\gamma)) - 2\log(\hat{y}) + \frac{\hat{y}^2}{2} + \gamma\hat{y}$$

Since this is a 1D problem

$$H = \frac{\partial^2 q}{\partial y^2}\bigg|_{y=\hat{y}} = 2\hat{y}^{-2} + 1 = \frac{2+\hat{y}^2}{\hat{y}^2}$$

Also note that

$$y^2 + \gamma y - 2 = 0$$
$$\Rightarrow \quad y^2 = 2 - \gamma y$$
$$\Rightarrow \quad \hat{y}^2 = 2 - \gamma \hat{y}$$

So

$$H = \frac{2+\hat{y}^2}{\hat{y}^2} = \frac{-\gamma\hat{y}+4}{-\gamma\hat{y}+2} = \frac{4-\gamma\hat{y}}{2-\gamma\hat{y}}$$

The normal distribution for the Laplace approximation has the following parameters.

$$\mu = \hat{y} = \frac{-\gamma + \sqrt{\gamma^2+8}}{2}$$

$$\sigma^2 = \frac{1}{H} = \frac{\hat{y}^2}{\hat{y}^2+2} = \frac{2-\gamma\hat{y}}{4-\gamma\hat{y}}$$

However, the distribution $N(\mu=\hat{y}, \sigma^2 = \frac{1}{H})$ is not suitable for the rejection algorithm. To see this consider, $g(x) \sim N(\hat{y}, 1/H)$, to find where $\frac{f(x)}{g(x)}$ is maximised.

$$h(x) = \frac{f_1(x)}{g_1(x)} = \frac{x^2 \exp\{-\frac{x^2}{2} - \gamma x\}}{\exp\{\frac{-1}{2}\left(\frac{x-\hat{y}}{H^{-1/2}}\right)^2\}}$$

$$= \frac{x^2 \exp\{-\frac{x^2}{2} - \gamma x\}}{\exp\{H(-\frac{1}{2}x^2 + \hat{y}x - \frac{\hat{y}^2}{2})\}}$$

$$= x^2 \exp\left\{-\frac{x^2}{2} - \gamma x - H\left(-\frac{1}{2}x^2 + \hat{y}x - \frac{\hat{y}^2}{2}\right)\right\}$$

$$= x^2 \exp\left\{ x^2\left(\frac{H-1}{2}\right) - x(\gamma + H\hat{y}) + \frac{H\hat{y}^2}{2} \right\}$$
$$= x^2 p(x)$$

Find the maximum.

$$\frac{\partial h}{\partial x} = 2xp(x) + x^2 p(x)\left(x(H-1) - \gamma - H\hat{y} \right)$$
$$= xp(x)\left(2 + x^2(H-1) - \gamma x - xH\hat{y} \right)$$

$p(x) > 0$, and $x > 0$ is required by f.
Hence

$$\frac{\partial h}{\partial x} = 0 \implies x^2(H-1) - x(\gamma + H\hat{y}) + 2 = 0$$
$$\implies x = \frac{\gamma + H\hat{y} \pm \sqrt{(\gamma + H\hat{y})^2 - 8(H-1)}}{2(H-1)}$$

The solution only exists if
$$(\gamma + H\hat{y})^2 - 8(H-1) \geq 0$$

Suppose we could choose H so that
$$-8(H-1) \geq 0$$
$$\implies H - 1 \leq 0$$
$$\implies H \leq 1 \quad (*)$$
$$\implies \frac{2 + \hat{y}^2}{\hat{y}^2} \leq 1 \implies 2 \leq 0 \implies \Rightarrow\Leftarrow$$

~~No such~~ Since H is determined by γ, this isn't possible. Instead, introduce a new parameter λ and draw from $g(x) \sim N(\hat{y}, \lambda/H)$ where we can choose λ. The parameter continues throughout all calculations as $H \to H/\lambda$. We return to the line indicated by (*)

$$(*) \implies \frac{H}{\lambda} \leq 1$$
$$\implies H \leq \lambda$$
$$\implies \lambda \geq \frac{\gamma^2 - \gamma\sqrt{\gamma^2+8} + 8}{\gamma^2 - \gamma\sqrt{\gamma^2+8} + 4}$$

We can choose any λ that satisfies the inequality. $\circledast$
Now need to find the supremum, we require the positive root

$$\hat{x} = \frac{\gamma + \frac{H}{\lambda}\hat{y} - \sqrt{\left(\gamma + \frac{H}{\lambda}\hat{y}\right)^2 - 8\left(\frac{H}{\lambda} - 1\right)}}{2\left(\frac{H}{\lambda} - 1\right)}$$

Take negative option because $\frac{H}{\lambda} - 1 \leq 0$

$M_1(\gamma) = h(\hat{x})$ is the bound on $\frac{f_1(x)}{g_1(x)}$.

This is required for the rejection algorithm.