Generalized P-Series Test

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1 Abstract

Throughout MATH 152 (Calculus II at Texas A&M University) students are regularly tasked with proving the convergence of infinite series in the following form:

$$\sum_{n=1}^{\infty} \frac{an^{\mu} + ...}{bn^{\nu} + ...}$$

where μ, ν, a, and b are all positive real numbers, and the "..." represents other polynomial or trigonometric terms of *lesser degree*. With this form, students are typically expected to use some sort of comparison test to show convergence of the series because the traditional P-Series Test fails in these cases. However, this paper gives a more general form of the P-Series Test that can prove convergence of series in the form listed above.

2 A Necessary Preliminary Proof

While this proof may seem rather arbitrary for the time being, it will hold to be completely necessary for extending the P-Series Test in the next section.

Given a function with arbitrary positive, real number constants μ, ν, a, and b and "..." representing further polynomial or trigonometric terms of *lesser degree*

$$\frac{ax^{\mu} + ...}{bx^{\nu} + ...}$$

prove that there exists a function positive real number constant C

$$C\,\frac{x^{\mu}}{x^{\nu}}$$

such that for all $\mathbb{R}$ greater than some finite number N the following holds:

$$C\,\frac{x^{\mu}}{x^{\nu}} > \frac{ax^{\mu} + ...}{bx^{\nu} + ...}$$

Begin by using properties of exponents to rearrange the exponent on the right side of the inequality:

$$\frac{C}{x^{\nu-\mu}} > \frac{ax^{\mu} + ...}{bx^{\nu} + ...}$$

Now, multiply both sides of the equation by $x^{\nu-\mu}$

$$C \; > \frac{x^{\nu-\mu}[ax^{\mu} + ...]}{bx^{\nu} + ...}$$

Simplify the right side a bit by distributing

$$C \; > \frac{ax^{\nu} + ...}{bx^{\nu} + ...}$$

Notice now that the only thing of interest is proving that this inequality is true for all $\mathbb{R}$ greater than some finite constant N. Thus, if it can be proven that

$$\lim_{x\to\infty} C \; > \lim_{x\to\infty} \frac{ax^{\nu} + ...}{bx^{\nu} + ...}$$

then it must be true that the original inequality is true for all $\mathbb{R}$ greater than some arbitrary, yet finite, number N.
Now, evaluate the limit to get the following

$$C \ > \ \frac{a}{b}$$

which will be true as long as the value of C is chosen to be greater than $\frac{a}{b}$. Thus, so long as the value of C is chosen to be sufficiently large, it is in general true that for all $\mathbb{R}$ greater than some arbitrary number N the following relationship must exist

$$C\ \frac{x^{\mu}}{x^{\nu}} > \frac{ax^{\mu} + ...}{bx^{\nu} + ...}$$

With this proof complete, the P-Series Test is now ready to be extended.

3 Expanding P-Series Convergence

To expand the convergence of the P-Series Test, the following must be completed: prove that below series converges for all $\nu - \mu > 1$ given that all constants are positive real numbers and that "..." represents further polynomial or trigonometric terms of *lesser degree*.

$$\text{Figure 1} = \sum_{n=1}^{\infty} \frac{an^{\mu} + ...}{bn^{\nu} + ...}$$

Taking what was concluded in the first proof, it can be stated that the following inequality holds for all $\mathbb{R}$ greater than some arbitrary constant N

$$C\,\frac{x^{\mu}}{x^{\nu}} > \frac{ax^{\mu} + ...}{bx^{\nu} + ...}$$

This inequality can easily be re-written to better conform to P-Series Test standards. First, rearrange the left side of the relation to look like this

$$\frac{C}{x^{\nu-\mu}} > \frac{ax^{\mu} + ...}{bx^{\nu} + ...}$$

Next, define a variable P such that $P = \nu - \mu$. Substitute P for $\nu - \mu$

$$\frac{C}{x^{P}} > \frac{ax^{\mu} + ...}{bx^{\nu} + ...}$$

Now, if convergence can be proven for

$$\text{Figure 2} = \sum_{n=1}^{\infty} \frac{C}{n^{p}}$$

the comparison test could be used to show that Figure 1 also converges because the inequality holds for all $\mathbb{R} > N$.

To prove the convergence of Figure 2, the integral test can be used. The series satisfies the conditions of the integral test because it is positive (recall that C is a positive real number constant) and decreasing because the denominator is growing while the numerator remains constant. Now that the conditions for the integral test have been met, apply the test

$$\int_{1}^{\infty} \frac{C}{x^{P}} dx$$

Assuming that $P \neq 1$ (there is no interest in allowing $P = 1$ because that would mean the series diverges anyways) the reverse power rule can be used to easily evaluate the integral.

$$(\frac{C}{1-P}) \lim_{a\to\infty} (x^{1-P})\Big|_1^a$$

Substitute the limits of integration into the expression.

$$(\frac{C}{1-P}) \lim_{a\to\infty} [a^{1-P} - 1]$$

The coefficient of

$$\frac{C}{1-P}$$

can easily be discarded seeing that it plays no role in the infinite limit. Likewise, the term of (-1) can also be eliminated because it also has no affect on the ultimate convergence of the limit. Thus, the remaining expression to be evaluated is

$$\lim_{a\to\infty} a^{1-P}$$

Now the following question must be considered: For what values of P does the above expression converge to a finite value?

It is easy to logically see that the limit will only converge when the exponent is less than or equal to 0. Thus, values for P that are greater than or equal to 1 will cause the expression to converge. However, $P = 1$ is not considered for convergence of the limit because the original P-Series Test already dictates the $P = 1$ will cause Figure 2 to diverge. Thus, the accepted values of P that cause cause the original integral to converge are members of the following set:

$$\{P : P > 1, P \in \mathbb{R}\}$$

Now that it has been shown that values of $P > 1$ will cause the integral to converge, it has likewise been shown that values of $P > 1$ will also cause

$$\sum_{n=1}^{\infty} \frac{C}{x^n}$$

to converge for all $P > 1$. Now because

$$C\,\frac{x^\mu}{x^\nu} > \frac{ax^\mu + ...}{bx^\nu + ...}$$

holds for all $\mathbb{R}$ greater than some arbitrary, yet finite, number N, it has been shown, by comparison, that

$$\sum_{n=1}^{\infty} \frac{an^{\mu} + ...}{bn^{\nu} + ...}$$

also converges for all $P > 1$ (Recall that $P = \nu - \mu$). Thus, through a circuitous process involving a preliminary proof, the integral test, and the comparison test, the P-Series Test, for $P > 1$, has been expanded to prove convergence for all series that take on the general form of

$$\sum_{n=1}^{\infty} \frac{an^{\mu} + ...}{bn^{\nu} + ...}$$

regardless of the number of following terms included in the "...", so long as the terms in "..." are all trigonometric or polynomial terms of lesser degree.

4 A Quick Explanation of the Constant N

Throughout this proof, it has been stated numerous times that some inequality holds for all $\mathbb{R}$ greater than some arbitrary, unknown, yet finite constant N. This repeated statement is entirely motivated by the comparison test for showing convergence. Essentially, it does not matter if one series is temporarily larger than another, so long as it is smaller than it for infinitely many values of x. In taking the limit of two sides of an inequality one can show that one side is smaller than another for infinitely many values of x, even if it may be larger for certain small values of x. All in all, mentioning that one function is less than another for all $\mathbb{R}$ greater than some arbitrary constant N allows for the usage of the comparison test when those functions are used in an infinite series.

5 Conclusion

This paper has proven, through a sequence of a preliminary proof, integral test, and comparison test, that series of the following form

$$\sum_{n=1}^{\infty} \frac{an^{\mu} + ...}{bn^{\nu} + ...}$$

converge so long as all constants are positive, real numbers, P (or $\nu - \mu$) is greater than one, and all terms in the "..." are either trigonometric or polynomial terms of lesser degree. Ideally, the logic of this proof could quite easily be expanded to include divergence of P-Series rational series and convergence and divergence of improper P-Test integrals, but those proofs are not contained in this paper.

6 Personal Commentary

To me, this proof has always been somewhat self-apparent, even though I had never set out to rigorously prove it until now. It was always true in my head by the same logic that only the terms of highest degree in a limit of a rational function have any real say in the limit: as x (or n for series) approaches infinity, only terms of the highest degree have any significance. Thus, even though a complicated rational function may have several terms in both the numerator and denominator, the only terms that really matter as x approaches infinity are the terms of highest degree, regardless of whether you are considering limits, infinite series, or improper integrals (or any other mathematical operation that allows some variable to approach infinity). Problems similar to question #11 from Fall 2016 Exam 3A always frustrated me because there really is no reason to use a comparison test at all when a generalized P-Series Test would be much easier to use. On a more productive note, this logic of a generalized P-Series Test series can easily (but I got lazy and decided not to prove these) be extended to include divergence of P-Series rational expression and convergence and divergence of improper P-Test integrals.