

# HST 190: Introduction to Biostatistics and Epidemiology

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## Contents

<b>Lecture 1: Probability (7/30/18)</b>	<b>3</b>
Definitions . . . . .	3
Properties . . . . .	3
Diagnostic testing . . . . .	4
Measure definitions . . . . .	4
Receiver operating characteristic (ROC) . . . . .	4
Random variables and distributions . . . . .	4
Distribution properties . . . . .	4
Binomial: $X \sim \text{Bin}(n, p)$ . . . . .	4
Normal: $X \sim \mathcal{N}(\mu, \sigma^2)$ . . . . .	4
<b>Lecture 2: CTL and One-Sample Inference (8/1/18)</b>	<b>5</b>
Central limit theorem . . . . .	5
Confidence intervals . . . . .	5
One-sample z-test . . . . .	6
<b>Lecture 3: Two-Sample Inference and Power (8/13/18)</b>	<b>6</b>
Maximum likelihood estimation (MLE) . . . . .	6
Power and error . . . . .	7
Two-sample $t$ -test . . . . .	9
Paired data . . . . .	9
Unpaired data . . . . .	9
<b>Lecture 4: Nonparametric Tests and Clinical Trials (8/13/2018)</b>	<b>10</b>
Nonparametric tests . . . . .	10
Wilcoxon signed-rank test . . . . .	10
Wilcoxon rank sum test (Mann-Whitney) . . . . .	10
Clinical trial study design . . . . .	11
Design process . . . . .	11
Design factors . . . . .	11
<b>Lecture 5: Linear Regression (8/15/2018)</b>	<b>12</b>
Correlation . . . . .	12
Simple linear regression . . . . .	12
Basic model . . . . .	12
Assumptions for residuals (LINE): . . . . .	12
Fitting the model . . . . .	13
Predictions and intervals . . . . .	13
Inference about the slope ( $\beta_1$ ) . . . . .	14
Multiple linear regression . . . . .	14
Basic model . . . . .	14
Categorical variables . . . . .	14

<b>Lecture 6: Multiple Linear Regression Cont. (8/17/18)</b>	<b>15</b>
Describing variation . . . . .	15
Sums of squares (SS) . . . . .	15
Coefficient of determination ( $R^2$ ) . . . . .	15
F-test . . . . .	15
ANOVA . . . . .	16
Multiple comparisons . . . . .	17
Bonferroni correction . . . . .	17
False discovery rate . . . . .	17
<b>Lecture 7: Inference for Categorical Data (8/19/18)</b>	<b>18</b>
One-proportion inference . . . . .	18
Two-proportion comparisons . . . . .	18
Chi-square test . . . . .	19
Contingency table . . . . .	19
Pearson's chi-square test . . . . .	19
Odds ratios and relative risk . . . . .	20
Case-control (retrospective) study: . . . . .	21
Odds ratio . . . . .	21
Mantel-Haenszel method . . . . .	21
Chi-square test for homogeneity: . . . . .	21
<b>Lecture 8: Logistic Regression and Survival Analysis (8/22/18)</b>	<b>23</b>
Logistic regression . . . . .	23
Survival analysis . . . . .	23
Kaplan-Meier estimation . . . . .	24
Hazard function . . . . .	25
Log-rank test . . . . .	25
Modeling survival with regression . . . . .	25
<b>Lecture 9: Optimizing Linear Models (8/24/2018)</b>	<b>26</b>
Mixed-effects model . . . . .	26
Model checking . . . . .	26
Variable selection . . . . .	27
Cross-validation . . . . .	27
<b>Flowcharts from Fundamentals of Biostatistics (Rosner 7th ed.)</b>	<b>28</b>

# Lecture 1: Probability (7/30/18)

## Definitions

1. Deductive reasoning: general to specific
2. Inductive reasoning: specific to general
3. Sample space: the set of all possible outcomes
4. Event: any subset of the sample space

## Properties

1. Probabilities are additive (with inclusion-exclusion principle)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

2. Law of total probability: if  $S$  can be divided into mutually exclusive events  $B_1, B_2, \dots, B_n$

$$\begin{aligned} P(A) &= P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3) \dots \\ &= \sum_{i=1}^n P(A \cap B_i) \\ P(A) &= P(B_1)P(A | B_1) + P(B_2)P(A | B_2) + P(B_3)P(A | B_3) \dots \\ &= \sum_{i=1}^n P(B_i)P(A | B_i) \end{aligned}$$

3. Conditional probability

$$P(A \cap B) = P(B)P(A | B) = P(A)P(B | A)$$

4. Independence

$$P(A \cap B) = P(A)P(B)$$

If  $B$  is not the null event, the following also applies:

$$P(A | B) = P(A)$$

5. Bayes's Theorem

$$P(B | A) = \frac{P(B)P(A | B)}{P(A)}$$

## Diagnostic testing

### Measure definitions

$$\begin{aligned}\text{Prevalence} &= P(D^+) \\ \text{Sensitivity} &= P(T^+ | D^+) \\ \text{Specificity} &= P(T^- | D^-)\end{aligned}$$

$$\begin{aligned}\text{Positive Predictive Value (PPV, PV}^+ &= P(D^+ | T^+) = \frac{P(D^+)P(T^+ | D^+)}{P(T^+)} \\ &= \frac{(\text{sensitivity})(\text{prevalence})}{(\text{sensitivity})(\text{prevalence}) + (1 - \text{specificity})(1 - \text{prevalence})}\end{aligned}$$

$$\begin{aligned}\text{Negative Predictive Value (NPV, PV}^- &= P(D^- | T^-) = \frac{P(D^-)P(T^- | D^-)}{P(T^-)} \\ &= \frac{(\text{specificity})(1 - \text{prevalence})}{(1 - \text{sensitivity})(\text{prevalence}) - (\text{specificity})(1 - \text{prevalence})}\end{aligned}$$

### Receiver operating characteristic (ROC)

- ROC curves plot TPR (sensitivity) vs. FPR (1-specificity) as the prediction threshold is changed.
- The area under the curve (AUC) is a useful summary of a classifier's predictive power.

## Random variables and distributions

### Distribution properties

- Probability distribution: assigns a probability to each outcome in the sample space.
- Expected value:

$$E(X) = \sum_i x_i P(X = x_i) = \int x f(x) dx$$

- Variance:

$$Var(X) = \sum_i (x_i - \mu_x)^2 P(X = x_i) = \int (x - \mu_x)^2 f(x) dx$$

**Binomial:**  $X \sim Bin(n, p)$

$$P(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

**Normal:**  $X \sim \mathcal{N}(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x - \mu_x)^2}{2\sigma^2}\right)$$

- Rule of thumb: ~68.25% within 1 sd, ~95.50% within 2 sd, ~99.75 within 3 sd.
- If  $X \sim \mathcal{N}(\mu, \sigma^2)$ , then  $Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$

## Lecture 2: CTL and One-Sample Inference (8/1/18)

### Central limit theorem

For large  $n$ :

$$\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim \mathcal{N}(0, 1)$$

- The distribution converges to normality more quickly when the population distribution is also normal.
- Standardization of the sample mean is the first step for computing  $p$ -values and confidence intervals.
- If sample size ( $n$ ) is unknown, we have to approximate  $\sigma$  with  $s = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{X})^2}$ . As a result, the sampling distribution actually converges to the  $t_{n-1}$  distribution.

$$E(s^2) = \sigma^2$$

$$Var(s^2) = \frac{2\sigma^4}{n-1}$$

### Confidence intervals

To compute a 95% confidence interval, we start with:

$$\begin{aligned} 0.95 &= P\left(-1.96 \leq \frac{\bar{x} - \mu_x}{\frac{\sigma}{\sqrt{n}}} \leq 1.96\right) \\ &= P\left(-1.96 \left(\frac{\sigma}{\sqrt{n}}\right) - \bar{x} \leq -\mu_x \leq 1.96 \left(\frac{\sigma}{\sqrt{n}}\right) - \bar{x}\right) \\ &= P\left(\bar{x} + 1.96 \left(\frac{\sigma}{\sqrt{n}}\right) \geq \mu_x \geq \bar{x} - 1.96 \left(\frac{\sigma}{\sqrt{n}}\right)\right) \end{aligned}$$

The confidence interval is:

$$\bar{x} \pm z_{1-\frac{\alpha}{2}} \left(\frac{\sigma}{\sqrt{n}}\right) \quad \text{OR} \quad \bar{x} \pm t_{n-1, 1-\frac{\alpha}{2}} \left(\frac{s}{\sqrt{n}}\right)$$

- If confidence = 90%,  $z = 1.65$
- If confidence = 95%,  $z = 1.96$
- If confidence = 99%,  $z = 2.58$

## One-sample z-test

### Assumptions

1. Random sampling
2. Independence
3. Approximately normal (or large sample)

### Procedure

1. Define hypotheses
  - $H_0 : \mu = \mu_0$
  - $H_1 : \mu \neq \mu_0$
2. Standardize the observed sample mean
  - $z = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$
3. Collect the  $p$ -value: probability of sample estimate as extreme or higher, if  $H_0$  is true. Reject  $H_0$  if  $p$ -value < significance level  $\alpha$ .

### Critical value

- The critical value is the test statistic for which  $p$ -value =  $\alpha$
- It determines the acceptance and rejection regions (exactly like a confidence interval).

## Lecture 3: Two-Sample Inference and Power (8/13/18)

### Maximum likelihood estimation (MLE)

Let  $X_n$  denote a vector of  $n$  independent observations and  $\theta_k$  denote  $k$  parameters to be estimated

$$\begin{aligned} L(X_n \mid \theta_k) &= P(x_1 \mid \theta_k)P(x_2 \mid \theta_k)\dots P(x_n \mid \theta_k) \\ &= \prod_{i=1}^n P(x_i \mid \theta_k) \end{aligned}$$

The maximum likelihood estimator is:

$$\begin{aligned} \hat{\theta}_{\text{MLE}} &= \arg \max_{\theta} [L(X_n \mid \theta_k)] \\ &= \arg \max_{\theta} [\log L(X_n \mid \theta_k)] \end{aligned}$$

### Rationale

- MLE chooses the parameter values that maximize the probability of observing the given data.
- MLE is consistent:  $\hat{\theta}_{\text{MLE}} \rightarrow \theta_0$  as  $n \rightarrow \infty$ .
- MLE is asymptotically normal:  $\hat{\theta}_{\text{MLE}} \rightarrow \mathcal{N}(\theta_0, \sigma^2)$  as  $n \rightarrow \infty$ .

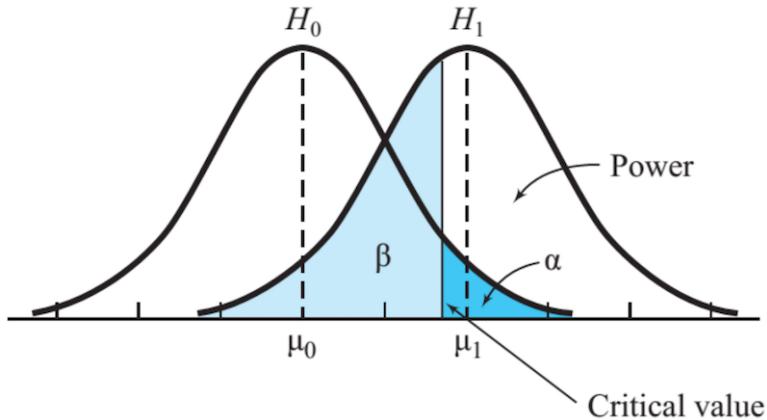


Figure 1: Statistical Power Graphic

## Power and error

### Definitions

Type I Error( $\alpha$ ) =  $P(H_0 \text{ true and falsely reject } H_0) = \text{false alarm}$

Type II Error( $\beta$ ) =  $P(H_0 \text{ false and fail to reject } H_0) = \text{alarm failure}$

Power ( $1 - \beta$ ) =  $P(\text{reject } H_0 \mid H_0 \text{ false})$

### Power of a z-test

$$1 - \beta = \Phi \left( -z + (\mu_1 - \mu_0) \frac{\sqrt{n}}{\sigma} \right)$$

- One-sided:  $z = z_{1-\alpha}$
- Two-sided:  $z = z_{1-\alpha/2}$

### Factors affecting power

1. Significance level ( $\alpha$ )
2. Effect size ( $\mu_1 - \mu_0$ )
3. Sample size ( $n$ )
4. Population standard deviation ( $\sigma$ )

Proof:

$$\begin{aligned}
\text{Power} &= P(\text{Reject } H_0 \mid H_1) \\
&= P(Z > z_{1-\alpha} \mid \mu = \mu_1) \\
&= P\left(\frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} > z_{1-\alpha} \mid \mu = \mu_1\right) \\
&= P\left(\bar{X} > z_{1-\alpha} \frac{\sigma}{\sqrt{n}} + \mu_0 \mid \mu = \mu_1\right) \\
&= P\left(\frac{\bar{X} - \mu_1}{\frac{\sigma}{\sqrt{n}}} > z_{1-\alpha} + (\mu_0 - \mu_1) \frac{\sqrt{n}}{\sigma}\right) \\
&= 1 - \Phi\left(z_{1-\alpha} + (\mu_0 - \mu_1) \frac{\sqrt{n}}{\sigma}\right) \\
\text{Power} &= \Phi\left(-z_{1-\alpha} + (\mu_1 - \mu_0) \frac{\sqrt{n}}{\sigma}\right) \quad \text{if } \mu_1 > \mu_0 \\
\text{Power} &= \Phi\left(z_{1-\alpha} - (\mu_1 - \mu_0) \frac{\sqrt{n}}{\sigma}\right) \quad \text{if } \mu_1 < \mu_0
\end{aligned}$$

**Required sample size for desired power**

$$n = \frac{\sigma^2(z_{1-\beta} + z_{1-\alpha})^2}{(\mu_1 - \mu_0)^2}$$

Proof:

$$\begin{aligned}
1 - \beta &= \Phi\left(-z_{1-\alpha} + (\mu_1 - \mu_0) \frac{\sqrt{n}}{\sigma}\right) \\
z_{1-\beta} &= -z_{1-\alpha} + (\mu_1 - \mu_0) \frac{\sqrt{n}}{\sigma} \\
\frac{\sqrt{n}}{\sigma} &= \frac{z_{1-\beta} + z_{1-\alpha}}{\mu_1 - \mu_0} \\
n &= \frac{\sigma^2(z_{1-\beta} + z_{1-\alpha})^2}{(\mu_1 - \mu_0)^2}
\end{aligned}$$

## Two-sample $t$ -test

### Paired data

- Each data point on one sample is related to a unique data point in the other sample.
- This is actually one sample (of differences) in disguise.

$$t = \frac{\bar{d}}{s_d/\sqrt{n}} \sim t_{n-1}$$

### Unpaired data

Pooled variance estimator (if population variances are equal):

$s_p^2$  is the df-weighted average of  $s_1^2$  and  $s_2^2$

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

$s_d^2$  is the sum of the variances of the sample averages

$$\begin{aligned} t &= \frac{(\bar{x}_2 - \bar{x}_1) - (\mu_2 - \mu_1)}{\sqrt{\frac{s_p^2}{n_1} + \frac{s_p^2}{n_2}}} \sim t_{n_1+n_2-2} \\ &(\bar{x}_2 - \bar{x}_1) \pm t_{df, 1-\alpha/2} \sqrt{\frac{s_p^2}{n_1} + \frac{s_p^2}{n_2}} \\ df &= n_1 + n_2 - 2 \end{aligned}$$

Separate variance estimators (if population variances are unequal):

$$\begin{aligned} t &= \frac{(\bar{x}_2 - \bar{x}_1) - (\mu_2 - \mu_1)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \sim t_{df} \\ df &= \left\lfloor \frac{\left( \frac{s_1^2}{n_1} + \frac{s_2^2}{n_2} \right)^2}{\frac{(s_1^2/n_1)^2}{n_1-1} + \frac{(s_2^2/n_2)^2}{n_2-1}} \right\rfloor \\ &(\bar{x}_2 - \bar{x}_1) \pm t_{df, 1-\alpha/2} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \end{aligned}$$

# Lecture 4: Nonparametric Tests and Clinical Trials (8/13/2018)

## Nonparametric tests

- Agnostic to the underlying population distribution.
  - More robust to non-normality, small sample sizes, outliers, ordinary data, and measurement error.
  - Loss of power from reduced information.
- Ranked methods only apply to hypotheses about population medians, not means.

### Wilcoxon signed-rank test

#### Rationale

- Replaces observations with signed ranks (robust to outliers and retains relative magnitudes)
- Used in lieu of the 1-sample  $t$ -test
- Tests whether the population median is equal to a value (usually 0)

#### Procedure

1. Rank the differences
  - Arrange the differences  $d_i$  in order of absolute value.
  - Count the number of differences with the same absolute value.
  - Ignore the observations where  $d_i = 0$  and rank the remaining observations as 1-n for low-high.
  - If there is a group of several observations with the same absolute value, then assign the average rank for the whole group.
2. Compute the rank sum  $R_1$  of the positive differences and the corresponding  $t$ -statistic and  $p$ -value based on the underlying distribution (boxed area only necessary for  $g > 0$  number of ties):

$$R \sim \mathcal{N} \left( \mu = \frac{n(n+1)}{4}, \sigma = \sqrt{\frac{n(n+1)(2n+1)}{24 - \boxed{\sum_{i=1}^g (t_i^3 - t_i)/48}}} \right)$$

3. Apply the continuity correction (boxed) and compute  $T$ :

$$T = \frac{|R_1 - \mu_R| - \boxed{\frac{1}{2}}}{\sigma_R} \sim \mathcal{N}(0, 1)$$

### Wilcoxon rank sum test (Mann-Whitney)

- Replaces observations with ranks in lieu of the unpaired two-sample  $t$ -test
- Assumes that the two distributions have the same shape

#### Procedure

1. Rank the differences
  - Same procedure as before
2. Compute the rank sum  $R_1$  from the first sample and the corresponding  $t$ -statistic and  $p$ -value based on the underlying distribution (boxed area only necessary for  $g > 0$  number of ties):

$$R \sim \mathcal{N} \left( \mu = \frac{n_1(n_1 + n_2 + 1)}{2}, \sigma = \sqrt{\frac{n_1 n_2}{12} \left[ n_1 + n_2 + 1 - \boxed{\frac{1}{(n_1+n_2)(n_1+n_2-1)} \sum_{i=1}^g t_i^3 - t_i} \right]} \right)$$

3. Apply the continuity correction (boxed) and compute  $T$ :

$$T = \frac{|R_1 - \mu_R| - \boxed{\frac{1}{2}}}{\sigma_R} \sim \mathcal{N}(0, 1)$$

## Clinical trial study design

### Design process

1. Define questions → aims → endpoints
2. Define study populations (eligibility)
3. Design and plan study: concepts → details (protocol)
4. Implement and monitor study (randomization)
5. Analyze and interpret interim and final data

Phase	Objective	Sample Size
I	safety, dosage	~15-30
II	safety, efficacy	~100
III	safety, efficacy	~100-1000s
IV	post-marketing surveillance	Depends

### Design factors

1. Choosing a target population (easy to accrue, compliant, likely treatment benefit)
2. Hypotheses
  - Superiority and non-inferiority
  - Predefine as: hazard(A)/hazard(B) < H, or  $\mu_A - \mu_B > D$
3. Endpoints
  - Primary endpoint: type I error determines power/sample size
  - Secondary endpoints: may be accounted for in powering, but not always
4. Randomization
  - Simple random sampling: can be inefficient
  - Stratification : institution, gender, severity, past exposure
  - Blocking: treatment assignment
  - Adaptive: based on responses (play the winner)
5. Blinding:
  - Single-blind: subject does not know which group they're in
  - Double-blind: researchers also don't know which groups the subjects are in
  - Triple-bind: monitoring committee also doesn't know which groups the subjects are in
6. Interim monitoring: safety, efficacy
  - Data Safety Monitoring Board (DSMB) regularly reviews study conduct and data
  - Problem: loses info on secondary endpoints
7. Statistical details:
  - Sample size, type I error, power, desired effect size
  - Stopping rule, number of interim analyses, drop-out rate
8. Statistical Model
  - Translate data and hypotheses into a statistical model; fit the model and interpret results
  - Intention-to-treat: unbiased and conservative with non-compliance/drop-out
  - Per-protocol: biased and optimistic (higher power) with non-compliance/drop-out
9. Analyses
  - Missing data: last observation carried forward (LOCF) and multiple imputation.
  - Report all subgroup analyses; settles multiplicity problem.
  - Adjust for subgroup heterogeneity using an interaction test
  - Adjust for dependencies (longitudinal/repeated measurements, adjacent anatomical locations, adjacent genetic loci)

## Lecture 5: Linear Regression (8/15/2018)

### Correlation

Given paired observations  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ , the Pearson product-moment correlation is:

$$\begin{aligned}\rho &= E\left[\left(\frac{X - \mu_X}{\sigma_X}\right)\left(\frac{Y - \mu_Y}{\sigma_Y}\right)\right] \\ &= \frac{E[(X - \mu_X)(Y - \mu_Y)]}{\sigma_X \sigma_Y} \\ &= \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}\end{aligned}$$

The sample correlation coefficient is:

$$r = \frac{1}{n-1} \sum_{i=1}^n \left( \frac{x_i - \bar{X}}{s_X} \right) \left( \frac{y_i - \bar{Y}}{s_Y} \right)$$

- $r$  is sensitive to outliers and highly non-normal distributions
- An alternative is Spearman's rank correlation, which replaces the data values with their relative ranks.
- To test the null hypothesis  $H_0 : \rho = 0$  vs.  $H_1 : \rho \neq 0$ , we compute the test statistic:

$$t = r \sqrt{\frac{n-2}{1-r^2}} \sim t_{n-2}$$

### Simple linear regression

#### Basic model

$$Y = \beta_0 + \beta_1 X + e, \quad e \sim \mathcal{N}(0, \sigma_e^2)$$

$$\begin{aligned}Y | X &\sim \mathcal{N}(\beta_0 + \beta_1 X, \sigma_e^2) \\ E[Y | X] &= \beta_0 + \beta_1 X \\ Var[Y | X] &= \sigma_e^2\end{aligned}$$

**Assumptions for residuals (LINE):**

1. Linearity
2. Independence
3. Normal errors
4. Equal variance

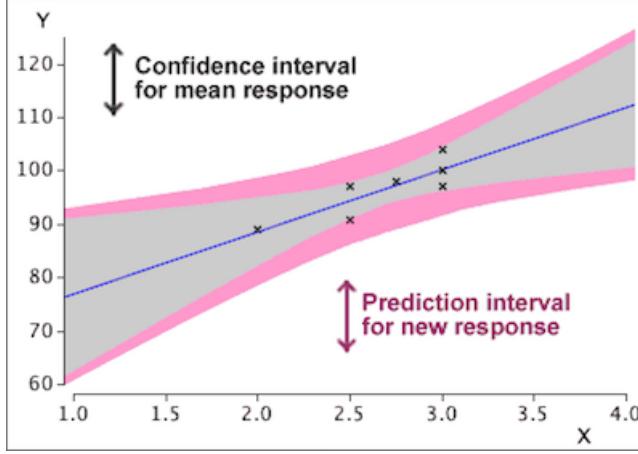


Figure 2: Confidence and Prediction Intervals

### Fitting the model

The optimal parameters  $(\hat{\beta}_0, \hat{\beta}_1)$  will minimize the squared error. This can be solved using differential calculus (taking partial derivatives), linear algebra (solving the normal equations), or MLE (algebraically or by gradient descent).

$$\text{Squared error} = \sum_{i=1}^n (y_i - \hat{y}_i)^2 = \sum_{i=1}^n (y_i - (\beta_0 + \beta_1 x_i))^2$$

$$(\hat{\beta}_0, \hat{\beta}_1) = \arg \min_{\beta_0, \beta_1} \sum_{i=1}^n (y_i - (\beta_0 + \beta_1 x_i))^2$$

$$\beta_1 = r \left( \frac{s_Y}{s_X} \right)$$

$$\beta_0 = \bar{Y} - \beta_1 \bar{X}$$

### Predictions and intervals

1. Confidence interval: estimates the mean response:

$$\hat{y} \pm t_{n-k, 1-\alpha/2} \times \hat{\sigma} \sqrt{\frac{1}{n} + \frac{(x_{new} - \bar{x})^2 / s_x^2}{(n-k)}}$$

where:

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^n (y_i - \hat{y}_i)^2}{n-k}$$

2. Prediction interval: estimates a new response:

$$\hat{y} \pm t_{n-k, 1-\alpha/2} \times \hat{\sigma} \sqrt{\boxed{1} \frac{1}{n} + \frac{(x_{new} - \bar{x})^2 / s_x^2}{(n-k)}}$$

This accounts for the additional uncertainty of the error term itself.  
Caution: do not extrapolate to x-values beyond where you have data!

## Inference about the slope ( $\beta_1$ )

$$\begin{aligned}\text{S.E. } (\beta_1) &= \frac{\hat{\sigma}}{\sqrt{s_x^2(n-1)}} \\ b \pm t_{n-k, 1-\alpha/2} \times \text{S.E. } (\beta_1) \\ t &= \frac{\beta_1}{\text{S.E. } (\beta_1)} \sim t_{n-k}\end{aligned}$$

## Multiple linear regression

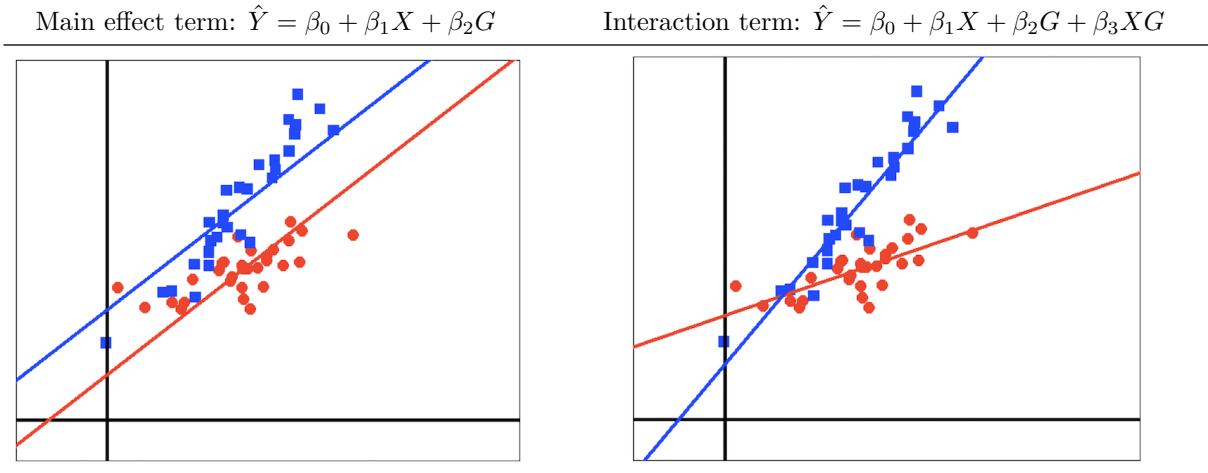
### Basic model

$$Y = \beta_0 + \beta_1 X + \beta_2 x_2 + \cdots + \beta_k x_k + e, \quad e \sim \mathcal{N}(0, \sigma_e^2)$$

### Categorical variables

- Indicator/dummy variables are created to capture categories. You need  $k - 1$  indicators for  $k$  variables.
- The associated regression parameters represent constant differences between each category and the baseline category.
- Interaction terms involve the product of predictor variables.
- This leads to different slopes when conditioning on a given  $x_k$ ; the divergence in slopes is given by  $\beta_k$ .
- When fitting a linear model computationally,  $t$ -statistics and  $p$ -values for  $\beta_j$  are computed with respect to  $H_0 : \beta_j = 0$  and all other  $\beta_{\neq j}$  are fixed as their point estimates.

$$\frac{\beta_j}{\text{S.E. } (\beta_j)} \sim t_{n-k-1}$$



## Lecture 6: Multiple Linear Regression Cont. (8/17/18)

### Describing variation

#### Sums of squares (SS)

$$SS_{total} = SS_{regression} + SS_{residual}$$

$$\sum_{i=1}^n (y_i - \bar{y})^2 = \sum_{i=1}^n (\hat{y}_i - \bar{y})^2 + \sum_{i=1}^n (y_i - \hat{y})^2$$

Total variation = From model + From noise/other

#### Coefficient of determination ( $R^2$ )

Definition:

$$R^2 = 1 - \frac{SS_{resid}}{SS_{total}}$$

= proportion of variation explained by the model  
= correlation coefficient squared ( $r^2$ )

Adjusted  $R^2$ :

$$R_{adj}^2 = 1 - \left( \frac{n-1}{n-k-1} \right) (1 - R^2)$$

$$\lim_{n \rightarrow \infty} R_{adj}^2 = R^2$$

- $n$  = number of data points and  $k$  = number of predictor variables
- $R^2$  always increases when a new variable is added; does not account for model complexity

### F-test

For the model:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k + e$$

Test  $H_0$  (intercept only) vs.  $H_1$  (full model), where:

$$H_0 : \beta_1 = 0 \text{ and } \beta_2 = 0 \text{ and } \dots \text{ and } \beta_k = 0$$

$$H_1 : \beta_1 \neq 0 \text{ or } \beta_2 \neq 0 \text{ or } \dots \text{ or } \beta_j \neq 0$$

Under  $H_0$ , the F-statistic follows the F-distribution:

$$\begin{aligned}
 MS_{reg} &= \frac{RSS_{reg}}{k} = \frac{\sum_{i=1}^k \sum_{j=1}^{n_i} (\bar{Y}_i - \bar{Y})^2}{k} \sim \chi_k^2 \\
 MS_{resid} &= \frac{RSS_{resid}}{n-k-1} = \frac{\sum_{i=1}^k \sum_{j=1}^{n_i} (\bar{Y}_{ij} - \bar{Y}_i)^2}{n-k-1} \sim \chi_{n-k-1}^2 \\
 F &= \frac{MS_{reg}}{MS_{resid}} \\
 &= \frac{\text{explained variance}}{\text{unexplained variance}} = \frac{\text{between-group variability}}{\text{within-group variability}} \\
 F &\sim F_{k, n-k-1}
 \end{aligned}$$

Reject  $H_0$  if  $F > F_{n-k-1, 1-\alpha}$

## ANOVA

Suppose we had  $k$  different populations, each roughly normal with common variance  $\sigma^2$ , and we wanted to test for equality:

$$\begin{aligned}
 H_0 : \mu_1 &= \mu_2 = \cdots = \mu_k \\
 H_1 : \text{at least one } \mu &\text{ is different from the others}
 \end{aligned}$$

### Assumptions

- Homoscedasticity (equal variance  $\sigma^2$ )
- Units in  $k$  samples are independent (within and between samples)
- Populations are approximately normal

### Basic model

$$\begin{aligned}
 y &= \beta_0 + \beta_1 I_1 + \beta_2 I_2 + \cdots + \beta_k I_k + e \\
 \beta_0 &= \bar{y}_0 \\
 \beta_{i|i \neq 0} &= \bar{y}_{i|i \neq 0} - \bar{y}_0
 \end{aligned}$$

Non-parametric model: apply ANOVA to the sample ranks (Kruskal-Wallis)

$$\begin{aligned}
 H_0 : \text{median}(y_1) &= \text{median}(y_2) = \cdots = \text{median}(y_k) \\
 H_1 : \text{at least one median} &\text{ is different from the others} \\
 \text{The final test statistic is } KW &\sim \chi_{k-1}^2
 \end{aligned}$$

## Multiple comparisons

When comparing 2 of  $k$  groups, use the pooled variance estimator if all group variances are considered equal:

$$s_p^2 = \frac{SS_{resid}}{n - k} = \frac{1}{n - k} \sum_{i=0}^k (n_i - 1) s_i^2$$

This gives the  $t$ -statistic:

$$t = \frac{\bar{x}_2 - \bar{x}_1}{s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{n-k}$$

### Bonferroni correction

$$\alpha^* = P(\text{at least 1 type I error}) \leq \frac{\alpha}{\text{No. of tests}}$$

### False discovery rate

Controls the expected proportion of incorrectly rejected null hypotheses:  $P(\text{reject each } H_0 \mid \text{all } H_0 \text{ true})$ .

#### Procedure

1. Rank tests by  $p$ -value ( $p_1 \leq p_2 \leq \dots \leq p_k$ )
2. Define  $q_i = \frac{k}{i} p_i$
3. Define  $FDR_i = \min(q_i, \dots, q_k)$ . These will be ranked in increasing order.
4. Reject all hypotheses with  $FDR_i < FDR^*$ .

#### Example (for $n = 50$ )

$i$ (rank)	Test	$p$ -value	$q_i$	$FDR_i$
1	#31	0.0001	0.0001(50/1) = 0.0050	0.0050
2	#21	0.0015	0.0015(50/2) = 0.0375	0.0317
3	#49	0.0019	0.0019(50/3) = 0.0317	0.0317
4	#50	0.0170	0.0170(50/4) = 0.2125	0.1800
5	#4	0.0180	0.0180(50/5) = 0.1800	0.1800

## Lecture 7: Inference for Categorical Data (8/19/18)

### One-proportion inference

Suppose that we count  $X$  successes and  $N - X$  failures from a sample size of  $N$ .

$$\begin{aligned} H_0 : p &= p_0 & H_0 : p &> p_0 \\ H_1 : p &\neq p_0 & \text{or} & H_1 : p \leq p_0 \\ && \hat{p} = X/N \\ && X \sim \text{Bin}(p_0, N) \\ && P(X = k) = \binom{N}{k} p_0^k (1 - p_0)^{N-k} \end{aligned}$$

Binomial exact method:

$$\begin{aligned} \text{If } \hat{p} \leq p_0 : p\text{-value} &= 2 \sum_{k=0}^X P(X = k) \\ \text{If } \hat{p} > p_0 : p\text{-value} &= 2 \sum_{k=X}^N P(X = k) \end{aligned}$$

Normal approximation method (valid when  $\text{Var}(X) = np(1 - p) \geq 5$ ):

$$\begin{aligned} \hat{p} &\sim \mathcal{N}\left(p, \frac{p(1-p)}{N}\right) \\ Z &= \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{N}}} \sim \mathcal{N}(0, 1) \\ n &= \frac{p_0(1-p_0) \left(z_{1-\alpha/2} + z_{1-\beta} \sqrt{\frac{p_1(1-p_1)}{p_0(1-p_0)}}\right)^2}{(p_1 - p_0)^2} \end{aligned}$$

### Two-proportion comparisons

$$\begin{aligned} H_0 : p_1 &= p_2 \\ H_1 : p_1 &\neq p_2 \\ \text{If } p_1 = p_2 : \hat{p}_{\text{pooled}} &= \frac{X_1 + X_2}{N_1 + N_2} \\ p_1 - p_2 &\sim \mathcal{N}\left(0, \sqrt{\frac{\hat{p}_1(1 - \hat{p}_1)}{n_1} + \frac{\hat{p}_2(1 - \hat{p}_2)}{n_2}}\right) \end{aligned}$$

If variances are assumed equal, replace  $\hat{p}_1$  and  $\hat{p}_2$  with the pooled proportion estimate  $\hat{p}$

## Chi-square test

### Contingency table

Data is cross-classified according to discrete/categorical variables:

	Positive	Negative	Total
Sharing needles	12	28	40
Not sharing needles	11	49	60
Total	23	77	100

### Pearson's chi-square test

Tests association between 2 categorical variables:

- $H_0$ : variables are not associated (joint = product of marginals)
- $H_1$ : variables are associated

$$\begin{aligned}
 E_{ij} &= \frac{O_i O_j}{N} \\
 X^2 &= \sum_{i=1}^R \sum_{j=1}^C \frac{(\text{observed}_{ij} - \text{expected}_{ij})^2}{\text{expected}_{ij}} \\
 &= \frac{N \left( |ad - bc| - \frac{N}{2} \right)^2}{(a+b)(c+d)(a+c)(b+d)} \\
 X^2 &\sim \chi^2_{df, 1-\alpha} \\
 df &= (R-1)(C-1)
 \end{aligned}$$

### Details

- 2x2 is valid only if all  $E_{ij} \geq 5$
- RxC is valid only if all  $E_{ij} \geq 1$  and at least 80% of  $E_{ij} \geq 5$
- Subtract 0.5 for the Yates continuity correction
- Right-tail integral gives the  $p$ -value for a 2-sided alternative

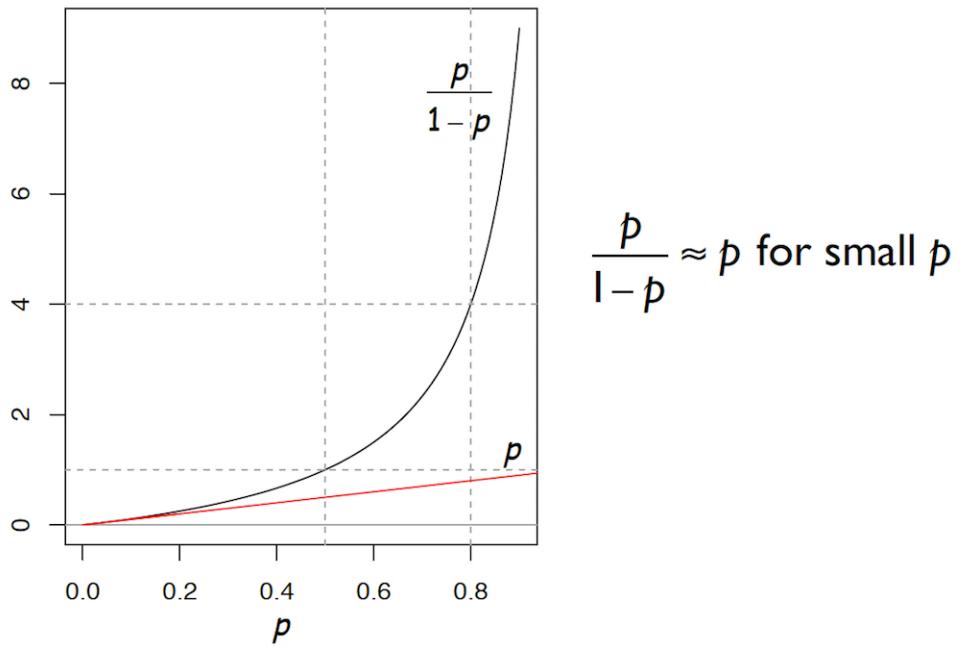


Figure 3: Odds Ratio

### Odds ratios and relative risk

	Disease <sup>+</sup>	Disease <sup>-</sup>	Total
Exposure <sup>+</sup>	A	B	A+B
Exposure <sup>-</sup>	C	D	C+D
Total	A+C	B+D	N

Odds ratios and relative risk measure the *magnitude* of association between 2 categorical variables.

$$p_1 = P(\text{disease} \mid \text{exposed})$$

$$p_2 = P(\text{disease} \mid \text{NOT exposed})$$

$$\text{Risk Difference} = p_1 - p_2$$

$$\text{Risk Ratio} = \frac{p_1}{p_2}$$

$$\begin{aligned} \text{Odds ratio} &= \frac{p_1}{1-p_1} / \frac{p_2}{1-p_2} \\ &= ad/bc \end{aligned}$$

### Case-control (retrospective) study:

- To compute  $\hat{p}_1$  and  $\hat{p}_2$ , we need to sample patients on exposure and classify on disease
- Instead, a case-control study samples patients on disease status and classifies on exposure.
- The case-control odds-ratio is the same as sampling by exposure and taking the ratio of the odds for  $\hat{p}_1$  and  $\hat{p}_2$ .
- If  $p \ll 1$  (low prevalence),  $\frac{p}{1-p} \approx p$  and odds-ratio  $\approx$  risk ratio

### Odds ratio

$OR > 1$ : exposure  $\rightarrow$  higher disease risk

$OR < 1$ : exposure  $\rightarrow$  lower disease risk

$OR = 1$ : no association between exposure and disease risk

$$\ln(\widehat{OR}) \sim \mathcal{N}\left(\ln(OR), \sqrt{\text{Var}(\ln OR)}\right)$$

$$\text{Var}(\ln \widehat{OR}) \approx \frac{1}{n_1 \hat{p}_1 (1 - \hat{p}_1)} + \frac{1}{n_2 \hat{p}_2 (1 - \hat{p}_2)} \approx \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$$

$$\ln(OR) \text{ CI: } \ln(\widehat{OR}) \pm z_{1-\alpha/2} \sqrt{\text{Var}(\ln \widehat{OR})} = [c_{\text{lower}}, c_{\text{upper}}]$$

$$OR \text{ CI: } [e^{c_{\text{lower}}}, e^{c_{\text{upper}}}]$$

Note: the OR confidence interval is NOT symmetric about the point estimate.

### Mantel-Haenszel method

- Confounding: stratifying results by a confounding variable may affect disease-exposure association
- Simpson's paradox:
  - A factor associated with both treatment assignment and outcome may reverse the direction of association
  - Example: compared to open procedures, percutaneous procedures are associated with higher success rate overall ( $OR > 1$ ), but lower success rate ( $OR < 1$ ) when outcomes are stratified by small stones and large stones

### Chi-square test for homogeneity:

1. Stratify your data into  $k$  strata (RxC tables)
2. Compute the statistic  $X_{homo}^2$

$$H_0 : OR_1 = OR_2 = \dots = OR_k \text{ (homogeneity)}$$

$$H_1 : \text{at least one } OR \text{ is different (heterogeneity)}$$

$$X_{homo}^2 = \sum_{i=1}^k w_i \left( \ln \widehat{OR}_i - \ln \overline{OR} \right)^2 \sim \chi_{k-1}^2$$

$$w_i = \left( \frac{1}{a_i} + \frac{1}{b_i} + \frac{1}{c_i} + \frac{1}{d_i} \right)^{-1}$$

$$\ln \widehat{OR}_i = \ln \left( \frac{a_i d_i}{b_i c_i} \right)$$

$$\ln \overline{OR} = \sum_{i=1}^k w_i \ln \widehat{OR}_i \Big/ \sum_{i=1}^k w_i$$

3. If you conclude homogeneity, compute the Mantel-Haenszel estimator of the common odds ratio:

$$\widehat{OR}_{MH} = \sum_{i=1}^k \frac{a_i d_i}{n_i} \Big/ \sum_{i=1}^k \frac{b_i c_i}{n_i}$$

4. Compute the confidence interval

- Check the following assumptions:

$$\sum_{i=1}^k \frac{(a_i + c_i)(a_i + b_i)}{n_i} \geq 5, \quad \sum_{i=1}^k \frac{(a_i + c_i)(c_i + d_i)}{n_i} \geq 5$$

$$\sum_{i=1}^k \frac{(b_i + d_i)(a_i + b_i)}{n_i} \geq 5, \quad \sum_{i=1}^k \frac{(b_i + d_i)(c_i + d_i)}{n_i} \geq 5$$

- Compute the CI as:

$$\ln OR_{MH} \text{ CI: } \ln \widehat{OR}_{MH} \pm z_{1-\alpha/2} \left( \frac{1}{\sqrt{\sum_{i=1}^k w_i}} \right) = [c_{\text{lower}}, c_{\text{upper}}]$$

$$OR_{MH} \text{ CI: } [e^{c_{\text{lower}}}, e^{c_{\text{upper}}}]$$

5. Perform a hypothesis test on:

$$H_0 : OR = 1$$

$$H_1 : OR \neq 1$$

Compute:

$$X_{MH}^2 = \frac{(|O - E| - 0.5)^2}{V}$$

$$O = \sum_{i=1}^k O_i = \sum_{i=1}^k a_i$$

$$E = \sum_{i=1}^k E_i = \sum_{i=1}^k \frac{(a_i + b_i)(a_i + c_i)}{n_i}$$

$$V = \sum_{i=1}^k V_i = \sum_{i=1}^k \frac{(a_i + b_i)(c_i + d_i)(a_i + c_i)(b_i + d_i)}{n_i^2(n_i - 1)}$$

If  $H_0$  is true and  $V \geq 5$ :  $X_{MH}^2 \sim \chi^2_{(R-1)(C-1)}$

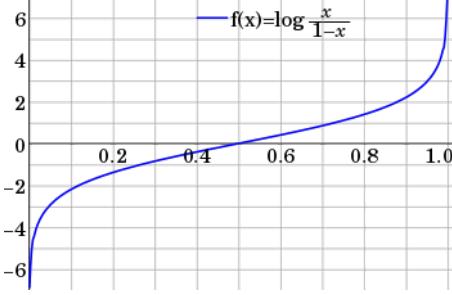


Figure 4: The logit function

## Lecture 8: Logistic Regression and Survival Analysis (8/22/18)

### Logistic regression

$$\begin{aligned}\text{logit}(p) &= \ln\left(\frac{p}{1-p}\right) \\ &= \beta_0 + \beta_1 X_1 + \cdots + \beta_k X_k \\ p &= \frac{e^{\beta_0 + \beta_1 X_1 + \cdots + \beta_k X_k}}{1 + e^{\beta_0 + \beta_1 X_1 + \cdots + \beta_k X_k}}\end{aligned}$$

- The  $\beta_j$  coefficients are log-odds (an increase of 1 unit means that the odds increase by a factor of  $e^1$ ).
- In other words,  $\ln \widehat{OR} \sim \mathcal{N}(\hat{\beta}_j, \text{S.E.}(\hat{\beta}_j))$ .

Recall:

$$\text{S.E.}(\beta_k) = \frac{\hat{\sigma}}{\sqrt{s_x^2(n-1)}}, \quad \hat{\sigma} = \frac{\sum_{i=1}^n (y_i - \hat{y}_i)^2}{n-k}$$

Additional assumption for logistic regression:

$$Y \sim \text{Bern}\left(p = \frac{\exp(\beta_0 + \beta_1 x)}{1 + \exp(\beta_0 + \beta_1 x)}\right)$$

### Survival analysis

- Survival analysis covers “time-to-event.”
- Often described by a survival curve  $S(t)$ , which plots 1–CDF(failure).
- Censored data: when you stop receiving data on a subject before failure.
  - You only know they survived at least as long as they did.
  - Assume that censoring is noninformative and unbiased, i.e., that being lost is unrelated to prognosis.

## Kaplan-Meier estimation

The Kaplan-Meier product-limit estimator:  $\hat{S}(t)$

$$\begin{aligned} S(t_i) &= P(\text{alive at } t_i \mid \text{alive at } t_{i-1}) \\ &\quad \times P(\text{alive at } t_1 \mid \text{alive at } t_{i-2}) \\ &\quad \dots \\ &\quad \times P(\text{alive at } t_2 \mid \text{alive at } t_1) \\ &\quad \times P(\text{alive at } t_1) \end{aligned}$$

Estimate each "step survival" probability as:

$$\begin{aligned} P(S_{i-1, i}(t)) &= \frac{N_{\text{alive and not censored at } t_i}}{N_{\text{alive and not censored at } t_{i-1}}} \\ &= 1 - \frac{N_{\text{died at } t_i}}{N_{\text{alive and not censored at } t_{i-1}}} \end{aligned}$$

That gives us  $\hat{S}(t)$ :

$$\hat{S}_{KM}(t_i) = \left(1 - \frac{d_1}{S_0}\right) \times \left(1 - \frac{d_2}{S_1}\right) \times \dots \times \left(1 - \frac{d_i}{S_{i-1}}\right) = \prod_{j=1}^i 1 - \frac{d_j}{S_{j-1}}$$

This estimator jumps at event times only; goes to zero if there are no more events.

Example:

Year ( $t$ )	Failed ( $d_i$ )	Censored ( $I_i$ )	Survived ( $S_i$ )	Total ( $S_{i-1}$ )	$\hat{S}(t)$
2	7	2	$100 - 7 - 2 = 91$	100	$\hat{S}(2) = 1 - \frac{7}{100}$
4	16	5	$91 - 16 - 5 = 70$	91	$\hat{S}(4) = \left(1 - \frac{16}{91}\right) \hat{S}(2)$
6	19	8	$70 - 19 - 8 = 43$	70	$\hat{S}(6) = \left(1 - \frac{19}{70}\right) \hat{S}(4)$

Confidence intervals:

$$\begin{aligned} \ln \hat{S}(t_i) &\sim \mathcal{N} \left( \ln S(t_i), \sum_{j=1}^i \frac{d_j}{S_{j-1}(S_{j-1} - d_j)} \right) \\ \text{CI for } \ln \hat{S}(t_i) : \quad &\ln \hat{S}(t_i) \pm z_{1-\alpha/2} \sqrt{\sum_{j=1}^i \frac{d_j}{S_{j-1}(S_{j-1} - d_j)}} = [c_{\text{lower}}, c_{\text{upper}}] \\ \text{CI for } \hat{S}(t_i) : \quad &[e^{c_{\text{lower}}}, e^{c_{\text{upper}}}] \end{aligned}$$

## Hazard function

- The hazard function can be considered to be:
  - An instantaneous conditional death rate
  - The probability of an event at time  $t$  given no event up to time  $t$
- $h(t)$  often represents survival distributions; constant for the exponential distribution

$$h(t) = \frac{\lim \Delta \rightarrow 0 \left( \frac{S(t) - S(t + \Delta t)}{\Delta t} \right)}{S(t)} = \frac{\text{instantaneous death rate}}{\text{fraction of individuals still alive}}$$

## Log-rank test

The log-rank test compares survival functions with the following hypotheses:

- $H_0: h_1(t) = h_2(t)$  for all  $t$  in the study (or  $h_1(t)/h_2(t) = 1$ )
- $H_1: h_1(t) \neq h_2(t)$  for all  $t$  in the study

The log-rank test is a direct application of the Mantel-Haenszel test:

- Divide the study period into  $k$  intervals
- Create a 2x2 table for each interval (group 1/2 vs. death/survival)

	Death	Survived or Censored	Total
Group 1	$a_i$	$b_i$	$n_{i,1}$
Group 2	$c_i$	$d_i$	$n_{i,2}$
Total	$a_i + c_i$	$b_i + d_i$	$n_i$

- Compute the  $\chi^2_{LR} \sim \chi^2_1$  test statistic (see chi-square test for homogeneity on page 22)

## Modeling survival with regression

Cox model (proportional-hazards model) is semiparametric and fits diverse survival distributions.

- Models  $h(t) = h_0(t) \exp(\beta_1 x_1 + \beta_2 x_2 + \dots + \beta_k x_k)$
- In other words,  $\ln h(t)/h_0(t) = \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_k x_k$

Interpreting the Cox Model

- $h_0(t)$  is the “baseline hazard rate” =  $h(t)$  when all  $x_i = 0$ .
- A unit increase of any covariate will scale  $h(t)$  by  $\exp(\Delta\beta_i)$
- Use the hypothesis  $H_0: \beta_i = 0$  to test whether a covariate affects survival time
- Make sure the final KM curves are proportional (not converging or diverging)

# Lecture 9: Optimizing Linear Models (8/24/2018)

## Mixed-effects model

Basic model:

$$y_{ij} = \alpha_i + \beta_0 + \beta_1 x_j + e_{ij}$$

$y_{ij}$  = response at time  $j$  for person  $i$

$x_j$  = time point

$\alpha_i$  = random effect: intercept adjustment  
 $\sim \mathcal{N}(0, \sigma_A^2)$

$\beta_1$  = fixed effect: slope

$e_{ij} \sim \mathcal{N}(0, \sigma^2)$

## Examples

- Longitudinal studies (group by same subject, same time point, etc.)
- Multi-stage sampling (counties → households → individuals)

## Rationale

- Accounts for correlations/dependencies without adding  $i$  parameters for  $i$  subjects (as fixed effects does)
- Properly indicates smaller coefficient S.E. values
- Gives an estimate for  $\sigma_A$ ; how much subject intercepts differ

## Alternative

- Sandwich estimator: estimates variance of  $\hat{\beta}$  as a function of the covariates  $(x_1, \dots, x_k)$  and  $\text{Var}(y_i)$ .

## Model checking

1. Linearity
  - Diagnose using pairwise scatterplots. Take vertical slices and look for (a) means in straight line and (b) SDs approximately equal
  - Consider transformations (log, inverse, square, exp, or interaction terms)
  - Use splines, polynomial regression, or generalized additive models
  - Coefficients are biased
2. Independence of Errors
  - Diagnose on plot of residuals vs. each X
  - Problems occur with interacting units, spatial/temporal proximity, common data source, or clustering effects
  - Consider modeling dependencies with random-effects or time-series models
  - Coefficients are unbiased, but standard errors are affected.
  - If residuals are positively correlated, we have less information and our confidence intervals will be optimistic
3. Normality of errors
  - Diagnose using QQ-plot
  - Consider conducting regression with t-distributed errors
4. Equal variance of residuals
  - Diagnose on plot of residuals vs. each X
  - Consider “squashing” transformations or weighted regression (observations weighted by 1/variance)
  - Heteroscedasticity does not bias coefficients, but standard errors are affected

## Variable selection

- Methods: forward, backward, stepwise, all subsets
- Criteria:
  - General form:  $f(\hat{\sigma}^2) + g(p)$
  - Adjusted-R<sup>2</sup> (minimize residual variance):  $1 - \left( \frac{n-1}{n-k-1} \right) (1 - R^2)$
  - AIC:  $n \ln(SS_{\text{resid}}/n) + 2p$
  - BIC:  $n \ln(SS_{\text{resid}}/n) + p \ln n$
- Also include:
  - Predictors that are significant or with the expected sign
  - Interaction terms for predictors with large effect sizes

## Cross-validation

Divide the data into three parts:

1. Training set: to fit the model
2. Validation set: to estimate hyperparameters and refine the model
3. Test set: to provide an unbiased estimate of predictive capacity

K-fold cross-validation

- Split the data into  $k$  components
- Conduct cross-validation  $k$  times, with a different component as the validation set each time.
- This gives  $k$  error measures, which can be combined into mean and variance estimates.

## Flowcharts from Fundamentals of Biostatistics (Rosner 7th ed.)

