Inverse of a 3×3 Matrix - The Cofactor Method

1 Problem Statement

Find the inverse of the following 3×3 matrix:

$$A = \begin{pmatrix} 7 & -6 & 3 \\ 4 & -5 & -4 \\ 2 & 1 & 8 \end{pmatrix}$$

2 Step 1: Find the Determinant of Matrix A

To find the determinant of a 3×3 matrix, we use the expansion along the first row with alternating signs (+, -, +):

$$\det(A) = 7 \begin{vmatrix} -5 & -4 \\ 1 & 8 \end{vmatrix} - (-6) \begin{vmatrix} 4 & -4 \\ 2 & 8 \end{vmatrix} + 3 \begin{vmatrix} 4 & -5 \\ 2 & 1 \end{vmatrix}$$

Calculate each 2×2 determinant:

$$\begin{vmatrix} -5 & -4 \\ 1 & 8 \end{vmatrix} = (-5)(8) - (-4)(1) = -40 + 4 = -36 \tag{1}$$

$$\begin{vmatrix} 4 & -4 \\ 2 & 8 \end{vmatrix} = (4)(8) - (-4)(2) = 32 + 8 = 40 \tag{2}$$

$$\begin{vmatrix} 4 & -5 \\ 2 & 1 \end{vmatrix} = (4)(1) - (-5)(2) = 4 + 10 = 14 \tag{3}$$

Substituting back:

$$\det(A) = 7(-36) + 6(40) + 3(14) = -252 + 240 + 42 = 30$$

3 Step 2: Find the Cofactors

The cofactor C_{ij} is given by $C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the minor (determinant of the 2×2 matrix obtained by removing row i and column j).

The sign pattern for a 3×3 matrix is:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

3.1 Visual Representation of 2×2 Matrices for Each Position

To visualize the process, here are all the 2×2 matrices that correspond to each position in the original 3×3 matrix:

$$\begin{pmatrix} \begin{vmatrix} -5 & -4 \\ 1 & 8 \end{vmatrix} & \begin{vmatrix} 4 & -4 \\ 2 & 8 \end{vmatrix} & \begin{vmatrix} 4 & -5 \\ 2 & 1 \end{vmatrix} \\ \\ \begin{vmatrix} -6 & 3 \\ 1 & 8 \end{vmatrix} & \begin{vmatrix} 7 & 3 \\ 2 & 8 \end{vmatrix} & \begin{vmatrix} 7 & -6 \\ 2 & 1 \end{vmatrix} \\ \\ \begin{vmatrix} -6 & 3 \\ -5 & -4 \end{vmatrix} & \begin{vmatrix} 7 & 3 \\ 4 & -4 \end{vmatrix} & \begin{vmatrix} 7 & -6 \\ 4 & -5 \end{vmatrix} \end{pmatrix}$$

Each 2×2 matrix is formed by eliminating the row and column of the corresponding position from the original 3×3 matrix.

3.2 Calculate all cofactors:

$$C_{11} = (+1)\begin{vmatrix} -5 & -4 \\ 1 & 8 \end{vmatrix} = +(-36) = -36 \tag{4}$$

$$C_{12} = (-1)\begin{vmatrix} 4 & -4 \\ 2 & 8 \end{vmatrix} = -(40) = -40 \tag{5}$$

$$C_{13} = (+1)\begin{vmatrix} 4 & -5 \\ 2 & 1 \end{vmatrix} = +(14) = 14 \tag{6}$$

$$C_{21} = (-1)\begin{vmatrix} -6 & 3 \\ 1 & 8 \end{vmatrix} = -((-6)(8) - (3)(1)) = -(-48 - 3) = 51 \tag{7}$$

$$C_{22} = (+1)\begin{vmatrix} 7 & 3 \\ 2 & 8 \end{vmatrix} = +((7)(8) - (3)(2)) = +(56 - 6) = 50 \tag{8}$$

$$C_{23} = (-1)\begin{vmatrix} 7 & -6 \\ 2 & 1 \end{vmatrix} = -((7)(1) - (-6)(2)) = -(7 + 12) = -19 \tag{9}$$

$$C_{31} = (+1)\begin{vmatrix} -6 & 3 \\ -5 & -4 \end{vmatrix} = +((-6)(-4) - (3)(-5)) = +(24 + 15) = 39 \tag{10}$$

$$C_{32} = (-1)\begin{vmatrix} 7 & 3 \\ 4 & -4 \end{vmatrix} = -((7)(-4) - (3)(4)) = -(-28 - 12) = 40 \tag{11}$$

$$C_{33} = (+1)\begin{vmatrix} 7 & -6 \\ 4 & -5 \end{vmatrix} = +((7)(-5) - (-6)(4)) = +(-35 + 24) = -11 \tag{12}$$

The cofactor matrix is:

$$C = \begin{pmatrix} -36 & -40 & 14 \\ 51 & 50 & -19 \\ 39 & 40 & -11 \end{pmatrix}$$

4 Step 3: Find the Adjugate Matrix

The adjugate matrix is the transpose of the cofactor matrix:

$$\text{adj}(A) = C^T = \begin{pmatrix} -36 & 51 & 39 \\ -40 & 50 & 40 \\ 14 & -19 & -11 \end{pmatrix}$$

5 Step 4: Calculate the Inverse

The inverse of matrix A is given by:

$$A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A)$$

Substituting our values:

$$A^{-1} = \frac{1}{30} \begin{pmatrix} -36 & 51 & 39 \\ -40 & 50 & 40 \\ 14 & -19 & -11 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{-36}{30} & \frac{51}{30} & \frac{39}{30} \\ \frac{-40}{30} & \frac{50}{30} & \frac{40}{30} \\ \frac{14}{30} & \frac{-19}{30} & \frac{-11}{30} \end{pmatrix}$$

6 Step 5: Simplify the Fractions

Reducing to lowest terms:

$$A^{-1} = \begin{pmatrix} -\frac{6}{5} & \frac{17}{10} & \frac{13}{10} \\ -\frac{4}{3} & \frac{5}{3} & \frac{4}{3} \\ \frac{7}{15} & -\frac{19}{30} & -\frac{11}{30} \end{pmatrix}$$

7 Summary

The complete process for finding the inverse of a 3×3 matrix using the cofactor method involves:

1. Calculate the determinant of the original matrix
2. Find all nine cofactors using the appropriate sign pattern

3. Transpose the cofactor matrix to get the adjugate matrix
4. Multiply the adjugate matrix by $\frac{1}{\det(A)}$
5. Simplify the resulting fractions

Final Answer:

$$A^{-1} = \begin{pmatrix} -\frac{6}{5} & \frac{17}{10} & \frac{13}{10} \\ -\frac{4}{3} & \frac{5}{3} & \frac{4}{3} \\ \frac{7}{15} & -\frac{19}{30} & -\frac{11}{30} \end{pmatrix}$$