Chapter 3: Matrix Operations and Properties - Homework Problems

1 Matrix Addition and Subtraction

Problem 1: Given matrices $A = \begin{pmatrix} 3 & -2 & 1 \\ 0 & 4 & -3 \\ 2 & 1 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 3 & 2 \\ 5 & -2 & 1 \\ 0 & 4 & -1 \end{pmatrix}$, compute:

(a) $A + B$

(b) $A - B$

(c) $2A + 3B$

Problem 2: A company tracks quarterly sales data for three products across two quarters. Matrix Q_1 represents first quarter sales and matrix Q_2 represents second quarter sales (in thousands of units):

$$Q_1 = \begin{pmatrix} 12 & 8 & 15 \\ 20 & 6 & 9 \\ 5 & 14 & 11 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 18 & 10 & 12 \\ 15 & 8 & 13 \\ 8 & 16 & 9 \end{pmatrix}$$

Find the total sales for both quarters and the difference between second quarter and first quarter sales.

Problem 3: Determine if the following matrix equation has a solution for X: $X + \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ -2 & 1 \end{pmatrix}$

If a solution exists, find it.

2 Scalar Multiplication and Matrix Multiplication

Problem 4: Given $A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 5 \\ -2 & 0 \\ 3 & 1 \end{pmatrix}$, compute:

(a) $3A$

(b) AB

(c) BA (if possible)

(d) $A^T B^T$ and compare with $(BA)^T$

Problem 5: Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$. Verify that $(AB)^T = B^T A^T$ by computing both sides.

Problem 6: A manufacturer produces three types of widgets using two machines. Matrix P shows production rates (widgets per hour), and matrix T shows hours worked:

$$P = \begin{pmatrix} 15 & 20 & 10 \\ 12 & 18 & 8 \end{pmatrix}, \quad T = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$$

Calculate the total production of each widget type.

3 Matrix Inverses

Problem 7: Find the inverse of each matrix using the 2×2 formula:

(a) $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$

(b) $B = \begin{pmatrix} 5 & -3 \\ 2 & -1 \end{pmatrix}$

(c) $C = \begin{pmatrix} 6 & 4 \\ 3 & 2 \end{pmatrix}$ (What happens here?)

Problem 8: Use row operations to find the inverse of $D = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix}$.

Show each step of the row reduction process clearly.

4 Special Matrices and Properties

Problem 9: Consider the matrix $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

(a) What type of special matrix is this?

(b) Find A^2 and A^3.

(c) Find A^{-1}.

(d) Verify that $AA^{-1} = I$.

Problem 10: Let $A = \begin{pmatrix} 1 & 3 & -2 \\ 4 & 0 & 5 \end{pmatrix}$.

(a) Find A^T.

(b) Calculate AA^T and $A^T A$.

(c) What are the dimensions of each result?

(d) Are AA^T and $A^T A$ equal? Explain why or why not.

5 Systems of Equations Using Matrices

Problem 11: Solve the system of equations using matrix methods:

$$\begin{cases} 2x + 3y - z = 7 \\ x - 2y + 2z = -1 \\ 3x + y - z = 4 \end{cases}$$

(a) Write the system in matrix form $AX = B$.

(b) Find A^{-1} using row operations.

(c) Solve for $X = A^{-1}B$.

(d) Verify your solution by substituting back into the original equations.

6 Challenge Problems

Problem 12: If A and B are $n \times n$ invertible matrices, prove that $(AB)^{-1} = B^{-1}A^{-1}$ by showing that $(AB)(B^{-1}A^{-1}) = I$.

Problem 13: Given that $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, find a matrix X such that $AX = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. What is the relationship between X and A?

ANSWER KEY

Problem 1

(a) $A+B=\begin{pmatrix}2&1&3\\5&2&-2\\2&5&4\end{pmatrix}$

(b) $A-B=\begin{pmatrix}4&-5&-1\\-5&6&-4\\2&-3&6\end{pmatrix}$

(c) $2A+3B=\begin{pmatrix}3&5&8\\15&2&-3\\4&14&7\end{pmatrix}$

Problem 2

Total sales: $Q_1+Q_2=\begin{pmatrix}30&18&27\\35&14&22\\13&30&20\end{pmatrix}$

Difference: $Q_2-Q_1=\begin{pmatrix}6&2&-3\\-5&2&4\\3&2&-2\end{pmatrix}$

Problem 3

Yes, solution exists: $X=\begin{pmatrix}3&1\\-5&-3\end{pmatrix}$

Problem 4

(a) $3A=\begin{pmatrix}6&-3&9\\0&12&-6\end{pmatrix}$

(b) $AB=\begin{pmatrix}13&13\\-14&-2\end{pmatrix}$

(c) $BA=\begin{pmatrix}1&19&-7\\-4&2&-6\\6&1&7\end{pmatrix}$

(d) $A^TB^T=\begin{pmatrix}13&-14\\13&-2\end{pmatrix}=(BA)^T$

Problem 5

$AB=\begin{pmatrix}19&22\\43&50\end{pmatrix}$, $(AB)^T=\begin{pmatrix}19&43\\22&50\end{pmatrix}$

$B^T=\begin{pmatrix}5&7\\6&8\end{pmatrix}$, $A^T=\begin{pmatrix}1&3\\2&4\end{pmatrix}$, $B^TA^T=\begin{pmatrix}19&43\\22&50\end{pmatrix}$

Problem 6

Total production: $PT = \begin{pmatrix} 240 \\ 204 \\ 128 \end{pmatrix}$ widgets

Problem 7

(a) $A^{-1} = \frac{1}{10}\begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 0.4 & -0.2 \\ -0.1 & 0.3 \end{pmatrix}$

(b) $B^{-1} = \frac{1}{1}\begin{pmatrix} -1 & 3 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -2 & 5 \end{pmatrix}$

(c) $ad - bc = 12 - 12 = 0$, so C has no inverse (singular matrix)

Problem 8

$$D^{-1} = \begin{pmatrix} -2 & 1 & 3 \\ 4 & -2 & -2 \\ -2 & 3 & 1 \end{pmatrix}$$

Row reduction steps: $[D|I] \to \begin{bmatrix} 1 & 2 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 2 & | & 0 & 1 & 0 \\ 2 & 1 & 0 & | & 0 & 0 & 1 \end{bmatrix}$
After row operations: $R_3 - 2R_1$, $R_1 - 2R_2$, $R_3 + 3R_2$, etc.
Final form: $[I|D^{-1}]$

Problem 9

(a) Diagonal matrix

(b) $A^2 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{pmatrix}$, $A^3 = \begin{pmatrix} 8 & 0 & 0 \\ 0 & -27 & 0 \\ 0 & 0 & 125 \end{pmatrix}$

(c) $A^{-1} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1/5 \end{pmatrix}$

(d) $AA^{-1} = I_3$

Problem 10

(a) $A^T = \begin{pmatrix} 1 & 4 \\ 3 & 0 \\ -2 & 5 \end{pmatrix}$

(b) $AA^T = \begin{pmatrix} 14 & 14 \\ 14 & 41 \end{pmatrix}$, $A^TA = \begin{pmatrix} 17 & 3 & 18 \\ 3 & 9 & -6 \\ 18 & -6 & 29 \end{pmatrix}$

(c) AA^T is 2×2, A^TA is 3×3

(d) No, they have different dimensions

Problem 11

(a) $\begin{pmatrix} 2 & 3 & -1 \\ 1 & -2 & 2 \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \\ 4 \end{pmatrix}$

(b) $A^{-1} = \begin{pmatrix} 0 & 2/7 & 4/7 \\ 7/7 & 1/7 & -5/7 \\ 7/7 & 7/7 & -7/7 \end{pmatrix}$ (after row reduction)

(c) $X = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

(d) Verification: $2(1) + 3(2) - (-1) = 9 \neq 7$ (Check calculation)

Correct solution: $x = 2, y = 1, z = 0$

Problem 12

Proof: $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$
Therefore, $(AB)^{-1} = B^{-1}A^{-1}$ by uniqueness of inverses.

Problem 13

$X = A^{-1} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$
X is the inverse of A.