Linear Algebra Study Guide

Chapter 3: Complete Matrix Concepts with Explanations

Contents

1 Introduction to Matrices

A matrix is a rectangular array of numbers, symbols, or expressions arranged in rows and columns. Matrices are fundamental tools in linear algebra for organizing data and performing linear transformations.

A matrix A with m rows and n columns is written as:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where a_{ij} represents the element in the i-th row and j-th column.

Matrix Dimension: The dimension of matrix A is $m \times n$ (read as "m by n").

Example: An insurance company tracks policies sold by agents, where each row represents an agent and each column represents a type of policy (life, car, home).

2 Matrix Addition and Subtraction

2.1 Matrix Addition - Concept Explanation

Matrix addition combines two matrices by adding their corresponding elements. Both matrices must have the same dimension.

Mathematical Definition: For matrices A and B of dimension $m \times n$:

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{pmatrix}$$

Business Example: If matrix J represents January sales and matrix F represents February sales, then $J + F$ gives total sales for both months.

Properties of Matrix Addition:

1. Commutative: $A + B = B + A$
2. Associative: $(A + B) + C = A + (B + C)$
3. Identity Element: $A + 0 = A$ where 0 is the zero matrix

2.2 Matrix Subtraction - Concept Explanation

Matrix subtraction $A - B$ subtracts corresponding elements of matrix B from matrix A.

$$A - B = A + (-B)$$

Important Note: Matrix subtraction is **not commutative**: $A - B \neq B - A$ in general.

3 Scalar Multiplication

3.1 Scalar Multiplication - Concept Explanation

Scalar multiplication involves multiplying every element of a matrix by a real number (scalar).

For scalar c and matrix A:

$$cA = \begin{pmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{pmatrix}$$

Effects of Scalar Multiplication:

- If $c > 1$: All elements are scaled up
- If $0 < c < 1$: All elements are scaled down
- If $c < 0$: All elements change sign and are scaled
- If $c = 0$: Results in the zero matrix

Example: Doubling all sales figures: $2\begin{pmatrix} 10 & 5 \\ 8 & 12 \end{pmatrix} = \begin{pmatrix} 20 & 10 \\ 16 & 24 \end{pmatrix}$

4 Matrix Multiplication

4.1 Matrix Multiplication - Concept Explanation

Matrix multiplication is more complex than addition. For matrices A (dimension $m \times p$) and B (dimension $p \times n$), the product AB has dimension $m \times n$.

Rule: The number of columns in the first matrix must equal the number of rows in the second matrix.

Formula: The element in row i and column j of AB is:

$$(AB)_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj}$$

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1 \cdot 5 + 2 \cdot 7 & 1 \cdot 6 + 2 \cdot 8 \\ 3 \cdot 5 + 4 \cdot 7 & 3 \cdot 6 + 4 \cdot 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$$

Important: Matrix multiplication is **not commutative**: $AB \neq BA$ in general.

5 Special Types of Matrices

5.1 Identity Matrix

The identity matrix I_n is an $n \times n$ square matrix with 1s on the main diagonal and 0s elsewhere.

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Property: For any compatible matrix A: $AI = IA = A$

5.2 Zero Matrix

The zero matrix has all elements equal to zero: $0_{m\times n}$

Property: $A + 0 = A$ for any matrix A

5.3 Triangular Matrices

Upper Triangular: All elements below the main diagonal are zero. **Lower Triangular:** All elements above the main diagonal are zero. **Diagonal Matrix:** All non-diagonal elements are zero.

5.4 Singular vs. Non-singular Matrices

Non-singular Matrix: A square matrix that has an inverse. **Singular Matrix:** A square matrix that does not have an inverse.

6 Matrix Transpose

6.1 Matrix Transpose - Concept Explanation

The transpose of matrix A, denoted A^T, is formed by interchanging rows and columns.

If A is $m \times n$, then A^T is $n \times m$ with $(A^T)_{ij} = A_{ji}$

Example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Properties:

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(AB)^T = B^T A^T$
4. $(cA)^T = cA^T$

7 Matrix Inverse

7.1 Matrix Inverse - Concept Explanation

For a square matrix A, the inverse A^{-1} (if it exists) satisfies:

$$AA^{-1} = A^{-1}A = I$$

Existence: Not all matrices have inverses. A matrix has an inverse if and only if it is non-singular.

7.2 Finding Inverse of 2×2 Matrix

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Condition: The inverse exists if and only if $ad - bc \neq 0$.
Example:

$$A = \begin{pmatrix} 4 & -7 \\ 2 & -3 \end{pmatrix}$$

$$ad - bc = (4)(-3) - (-7)(2) = -12 + 14 = 2$$

$$A^{-1} = \frac{1}{2} \begin{pmatrix} -3 & 7 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} -3/2 & 7/2 \\ -1 & 2 \end{pmatrix}$$

8 Row Operations and Row Reduction

8.1 Elementary Row Operations

Three types of elementary row operations can be performed on matrices:

1. **Row Interchange:** $R_i \leftrightarrow R_j$ (swap rows i and j)
2. **Row Scaling:** $cR_i \to R_i$ (multiply row i by nonzero scalar c)
3. **Row Addition:** $cR_i + R_j \to R_j$ (add c times row i to row j)

8.2 Finding Inverse Using Row Reduction

To find the inverse of an $n \times n$ matrix A:

1. Form the augmented matrix $[A|I]$
2. Use row operations to transform the left side to the identity matrix
3. The right side becomes A^{-1}: $[I|A^{-1}]$

Process: $[A|I] \xrightarrow{\text{row operations}} [I|A^{-1}]$

9 Properties of Matrix Operations

9.1 Addition Properties

1. Commutative: $A + B = B + A$
2. Associative: $(A + B) + C = A + (B + C)$
3. Additive Identity: $A + 0 = A$
4. Additive Inverse: $A + (-A) = 0$

9.2 Multiplication Properties

1. Associative: $(AB)C = A(BC)$
2. Left Distributive: $A(B + C) = AB + AC$
3. Right Distributive: $(A + B)C = AC + BC$
4. Multiplicative Identity: $AI = IA = A$
5. **Not Commutative:** $AB \neq BA$ in general

9.3 Properties of Invertible Matrices

For invertible matrices A and B:

1. $(A^{-1})^{-1} = A$
2. $(AB)^{-1} = B^{-1}A^{-1}$
3. $(A^T)^{-1} = (A^{-1})^T$
4. If A is invertible, then A^T is also invertible

10 Summary and Connections

All matrix concepts work together to form a powerful mathematical framework:

- Matrix operations (addition, multiplication) follow specific rules that differ from scalar arithmetic
- Row operations provide a systematic method for solving linear systems and finding inverses
- The transpose operation connects matrices to their dual representations
- Matrix inverses enable solving matrix equations and represent "undoing" linear transformations
- Special matrices (identity, zero, triangular) have unique properties that simplify calculations

These matrix concepts are essential for understanding linear transformations, solving systems of equations, and many applications in engineering, computer science, economics, and physics.