Linear Algebra Study Guide

Chapter 2: Complete Vector Concepts with Explanations

Contents

1 Introduction to Vectors

A **vector** is a mathematical object that has both magnitude (length) and direction. Vectors can represent physical quantities like velocity, force, displacement, or acceleration. In this study guide, we focus on vectors in two-dimensional space ($\mathbb{R}^2$).

A vector $\mathbf{v}$ in two-space is written as: $\boxed{\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}$

where v_1 is the horizontal component and v_2 is the vertical component.

Example: A velocity vector of 30 km/h east and 40 km/h north would be written as $\begin{bmatrix} 30 \\ 40 \end{bmatrix}$.

2 Vector Addition and Subtraction

2.1 Vector Addition - Concept Explanation

Vector addition combines two vectors by adding their corresponding components. Geometrically, this is equivalent to placing the tail of the second vector at the head of the first vector; the sum is the vector from the tail of the first to the head of the second.

Mathematical Definition: $\boxed{\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}}$

Physical Interpretation: Vector addition represents the cumulative effect of multiple vector quantities. For example, if you apply two forces to an object, the resulting force is the vector sum of the individual forces.

Navigation Example: If you walk 3 km east and 2 km north, then continue 1 km west and 4 km north, your total displacement is:

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

You end up 2 km east and 6 km north of your starting point.

Properties of Vector Addition:

1. **Commutative:** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. **Associative:** $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. **Identity Element:** $\mathbf{v} + \mathbf{0} = \mathbf{v}$ where $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

2.2 Vector Subtraction - Concept Explanation

Vector subtraction $\mathbf{u} - \mathbf{v}$ is equivalent to adding the negative of the second vector: $\mathbf{u} + (-\mathbf{v})$.

$\boxed{\mathbf{u} - \mathbf{v} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \end{bmatrix}}$

Geometric Interpretation: The vector $\mathbf{u} - \mathbf{v}$ points from the tip of $\mathbf{v}$ to the tip of $\mathbf{u}$ when both vectors are drawn from the same starting point.

Relative Position Example: If your friend is at position $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ *and you're at* $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$*, the vector from you to your friend is:*

$$\begin{bmatrix} 5 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

3 Scalar Multiplication and Magnitude Scaling

3.1 Scalar Multiplication - Concept Explanation

Scalar multiplication involves multiplying a vector by a real number (scalar). This operation changes the magnitude of the vector and potentially its direction.

$$\boxed{c\mathbf{v} = c \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}}$$

Effects of Scalar Multiplication:

- If $c > 1$: **Dilation** - the vector is stretched (magnitude increases)
- If $0 < c < 1$: **Contraction** - the vector is compressed (magnitude decreases)
- If $c < 0$: **Direction reversal** - the vector points in the opposite direction
- If $c = 0$: Results in the zero vector

Speed Scaling: If a car's velocity is $\begin{bmatrix} 60 \\ 0 \end{bmatrix}$ *km/h eastward, tripling the speed gives:*

$$3 \begin{bmatrix} 60 \\ 0 \end{bmatrix} = \begin{bmatrix} 180 \\ 0 \end{bmatrix} \; km/h$$

3.2 Vector Magnitude - Concept Explanation

The **magnitude** (or length) of a vector represents its size, regardless of direction. It's always a non-negative real number.

$$\boxed{\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}}$$

For three-dimensional vectors: $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

Geometric Interpretation: The magnitude is the distance from the origin to the point (v_1, v_2) in the coordinate plane, calculated using the Pythagorean theorem.

Distance Calculation: The magnitude of vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ *is:*

$$\left\| \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

3.3 Magnitude Under Scalar Multiplication

When a vector is multiplied by a scalar, its magnitude scales by the absolute value of that scalar:

$\boxed{\|c\mathbf{v}\| = |c| \cdot \|\mathbf{v}\|}$

Proof:

$$\|c\mathbf{v}\| = \left\| c \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix} \right\| \tag{1}$$

$$= \sqrt{(cv_1)^2 + (cv_2)^2} = \sqrt{c^2 v_1^2 + c^2 v_2^2} \tag{2}$$

$$= \sqrt{c^2(v_1^2 + v_2^2)} = |c|\sqrt{v_1^2 + v_2^2} = |c|\|\mathbf{v}\| \tag{3}$$

Force Scaling: If a force vector has magnitude 15 N and is doubled, the new magnitude is $|2| \times 15 = 30$ *N.*

4 Triangle Inequality

4.1 Triangle Inequality - Concept Explanation

The **triangle inequality** states that the magnitude of the sum of two vectors is less than or equal to the sum of their individual magnitudes.

$\boxed{\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|}$

Geometric Interpretation: In any triangle, the length of one side is always less than or equal to the sum of the lengths of the other two sides. When vectors $\mathbf{u}$ and $\mathbf{v}$ form two sides of a triangle, their sum $\mathbf{u} + \mathbf{v}$ forms the third side.

When Equality Holds: The equality $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$ occurs when the vectors point in the same direction (one is a positive scalar multiple of the other).

Travel Distance: The direct distance between two cities is always less than or equal to the distance traveled via any intermediate city. If you travel from city A to city B via city C, the direct distance $\|\mathbf{AB}\|$ *satisfies:* $\|\mathbf{AB}\| \leq \|\mathbf{AC}\| + \|\mathbf{CB}\|$*.*

4.2 Numerical Example

Let $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$$\|\mathbf{u}\| = \sqrt{3^2 + 4^2} = 5 \tag{4}$$

$$\|\mathbf{v}\| = \sqrt{1^2 + 2^2} = \sqrt{5} \approx 2.24 \tag{5}$$

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 4 \\ 6 \end{bmatrix} \tag{6}$$

$$\|\mathbf{u} + \mathbf{v}\| = \sqrt{4^2 + 6^2} = \sqrt{52} \approx 7.21 \tag{7}$$

Checking: $7.21 \leq 5 + 2.24 = 7.24$

5 Geometric and Arithmetic Means

5.1 Relationship to Triangle Inequality

The triangle inequality is related to the relationship between geometric and arithmetic means of positive numbers.

For positive numbers a and b:

- **Geometric Mean:** $\sqrt{ab}$
- **Arithmetic Mean:** $\frac{a+b}{2}$

$$\boxed{\sqrt{ab} \leq \frac{a+b}{2}}$$

This inequality can be proven using the triangle inequality with carefully chosen vectors.

Investment Returns: If two investment portfolios have returns of 4% and 16%, then:

- *Geometric mean:* $\sqrt{4 \times 16} = \sqrt{64} = 8\%$
- *Arithmetic mean:* $\frac{4+16}{2} = 10\%$

The geometric mean (8%) is less than the arithmetic mean (10%), demonstrating the inequality.

6 Dot Product (Inner Product)

6.1 Dot Product - Concept Explanation

The **dot product** (also called inner product) is a scalar-valued operation that takes two vectors and returns a real number.

$$\boxed{\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2}$$

Alternative notation: $\mathbf{u}^T\mathbf{v}$ (transpose notation)

Geometric Interpretation: The dot product measures how much two vectors "align" with each other. It relates to the angle between the vectors and their magnitudes.

$$\boxed{\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta}$$

where θ is the angle between the vectors.

6.2 Properties of the Dot Product

1. **Commutative:** $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. **Distributive:** $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
3. **Scalar multiplication:** $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$
4. **Positive definite:** $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 \geq 0$, with equality only when $\mathbf{v} = \mathbf{0}$

Work Calculation: If force $\mathbf{F} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$ *N is applied over displacement* $\mathbf{d} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ *m, the work done is:*

$$W = \mathbf{F} \cdot \mathbf{d} = 10(3) + 5(2) = 30 + 10 = 40 \; J$$

7 Orthogonality

7.1 Orthogonal Vectors - Concept Explanation

Two vectors are **orthogonal** (perpendicular) if their dot product equals zero.

$\boxed{\mathbf{u} \perp \mathbf{v} \iff \mathbf{u} \cdot \mathbf{v} = 0}$

Geometric Significance: Orthogonal vectors form a 90-degree angle. This is fundamental in many applications, including coordinate systems, projections, and decompositions.

Perpendicular Forces: Vectors $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ *are orthogonal because:*

$$\mathbf{u} \cdot \mathbf{v} = 3(4) + 4(-3) = 12 - 12 = 0$$

7.2 Orthogonal Projections

The orthogonal projection of vector $\mathbf{u}$ onto vector $\mathbf{v}$ is:

$\boxed{\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}}$

This gives the component of $\mathbf{u}$ in the direction of $\mathbf{v}$.

8 Angles Between Vectors

8.1 Angle Formula - Concept Explanation

The angle θ between two non-zero vectors can be found using the dot product formula:

$\boxed{\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}}$

Therefore: $\boxed{\theta = \arccos\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)}$

Range: The angle θ is always between 0 and π radians (0° and 180°).

8.2 Special Cases

- If $\mathbf{u} \cdot \mathbf{v} > 0$: The angle is acute ($0 < \theta < \frac{\pi}{2}$)
- If $\mathbf{u} \cdot \mathbf{v} = 0$: The angle is right ($\theta = \frac{\pi}{2}$) - vectors are orthogonal

- If $\mathbf{u} \cdot \mathbf{v} < 0$: The angle is obtuse ($\frac{\pi}{2} < \theta < \pi$)

$$\mathbf{u} \cdot \mathbf{v} = 2(-1) + 6(5) = -2 + 30 = 28 \tag{8}$$
$$\|\mathbf{u}\| = \sqrt{2^2 + 6^2} = \sqrt{40} = 2\sqrt{10} \tag{9}$$
$$\|\mathbf{v}\| = \sqrt{(-1)^2 + 5^2} = \sqrt{26} \tag{10}$$
$$\cos\theta = \frac{28}{2\sqrt{10} \cdot \sqrt{26}} = \frac{28}{2\sqrt{260}} = \frac{14}{\sqrt{260}} \approx 0.868 \tag{11}$$
$$\theta \approx \arccos(0.868) \approx 0.518 \text{ radians} \approx 29.7 \tag{12}$$

9 Cauchy-Schwarz Inequality

9.1 Cauchy-Schwarz Inequality - Concept Explanation

The **Cauchy-Schwarz inequality** is a fundamental inequality in linear algebra that relates the dot product of two vectors to their magnitudes.

$\boxed{|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|}$

Equivalent Form: $(\mathbf{u} \cdot \mathbf{v})^2 \leq \|\mathbf{u}\|^2\|\mathbf{v}\|^2$

When Equality Holds: The equality $|\mathbf{u} \cdot \mathbf{v}| = \|\mathbf{u}\|\|\mathbf{v}\|$ occurs if and only if the vectors are linearly dependent (one is a scalar multiple of the other).

9.2 Relationship to Triangle Inequality

The Cauchy-Schwarz inequality is actually the key to proving the triangle inequality:

Taking square roots: $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$

9.3 Applications of Cauchy-Schwarz

Statistical Correlation: In statistics, the Cauchy-Schwarz inequality ensures that correlation coefficients are always between -1 and 1. If $\mathbf{x}$ and $\mathbf{y}$ are data vectors, then:

$$\frac{|\mathbf{x} \cdot \mathbf{y}|}{\|\mathbf{x}\|\|\mathbf{y}\|} \leq 1$$

$$\mathbf{u} \cdot \mathbf{v} = 1(3) + 2(1) = 5 \tag{13}$$
$$\|\mathbf{u}\| = \sqrt{1^2 + 2^2} = \sqrt{5} \tag{14}$$
$$\|\mathbf{v}\| = \sqrt{3^2 + 1^2} = \sqrt{10} \tag{15}$$
$$\|\mathbf{u}\|\|\mathbf{v}\| = \sqrt{5} \cdot \sqrt{10} = \sqrt{50} \tag{16}$$

Checking Cauchy-Schwarz: $|5| = 5 \leq \sqrt{50} \approx 7.07$

10 Summary and Connections

All these vector concepts are interconnected:

- **Vector addition** gives us the parallelogram law and triangle inequality
- **Scalar multiplication** controls magnitude and direction
- **Dot product** connects geometry (angles) with algebra (coordinates)
- **Cauchy-Schwarz inequality** provides the foundation for the triangle inequality
- **Orthogonality** gives us perpendicular relationships essential for coordinate systems

These concepts form the foundation for understanding vector spaces, linear transformations, and many applications in physics, engineering, computer graphics, and data science.