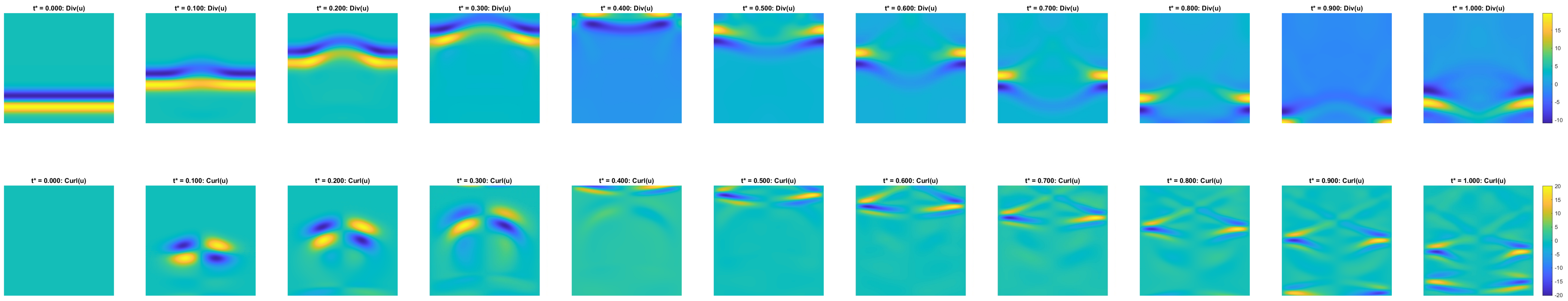
An HDG method for Linear Elastodynamics



MIT 16.930, Spring 2021 Final Project

James Gabbard, May 18th 2021

Linear Elasticity

Governing Equations

u_i Displacement

$\sigma_{ij} = C_{ijkl} q_{ij}$ Stress

q_{ij} Disp. Gradient

$t_i = \sigma_{ij} n_j$ Traction

$$q_{ij} - \frac{\partial u_i}{\partial x_j} = 0 \quad \text{in } \Omega$$

$$-\frac{\partial}{\partial x_j}\left(C_{ijkl} q_{kl}\right) = b_i \quad \text{in } \Omega$$

$$u_i = \bar{u}_i \quad \text{on } \partial\Omega_D$$

$$C_{ijkl} n_j q_{kl} = \bar{t}_i \quad \text{on } \partial\Omega_N$$

Local Problem

$$(\boldsymbol{q}, \boldsymbol{\zeta})_K + (\boldsymbol{u}, \nabla \cdot \boldsymbol{\zeta})_K - \langle \hat{\boldsymbol{u}}, \boldsymbol{\zeta} \boldsymbol{n} \rangle_{\partial K} = 0$$

$$(\boldsymbol{C}\boldsymbol{q}, \nabla \boldsymbol{w})_K - \langle \hat{\boldsymbol{\sigma}} \boldsymbol{n}, \boldsymbol{w} \rangle_{\partial K} = (\boldsymbol{b}, \boldsymbol{w})_K$$

$$\boldsymbol{\zeta} : K \to \mathbb{R}^{n_c \times d}, \boldsymbol{w} : K \to \mathbb{R}^{n_c}$$

Global Problem

$$\langle \hat{\boldsymbol{\sigma}} \boldsymbol{n}, \boldsymbol{\mu} \rangle_{\partial T_h \setminus \partial\Omega_D} + \langle \boldsymbol{u} - \hat{\boldsymbol{u}}, \boldsymbol{\mu} \rangle_{\partial\Omega_D} = \langle \bar{\boldsymbol{t}}, \boldsymbol{\mu} \rangle_{\partial\Omega_N}$$

$$\boldsymbol{\mu} : \partial T_h \to \mathbb{R}^{n_c}$$

Flux

$$\hat{\boldsymbol{\sigma}} = \boldsymbol{C}\boldsymbol{q} - \tau(\boldsymbol{u} - \hat{\boldsymbol{u}}) \otimes \boldsymbol{n}$$

Element Matrices

Kinematics (No coupling) Stabilization (No coupling) Elastic Moduli (coupling)

$$A = (\varphi_a, \varphi_b)_K$$
$$B_j = (\partial_j \varphi_a, \varphi_b)_K$$
$$C_j = \langle n_j \varphi_a, \varphi_b \rangle_{\partial K}$$

$$E = \langle \tau \varphi_a, \varphi_b \rangle_{\partial K}$$
$$F = \langle \tau \varphi_a, \phi_b \rangle_{\partial K}$$
$$I = \langle \tau \phi_a, \phi_b \rangle_{\partial K}$$

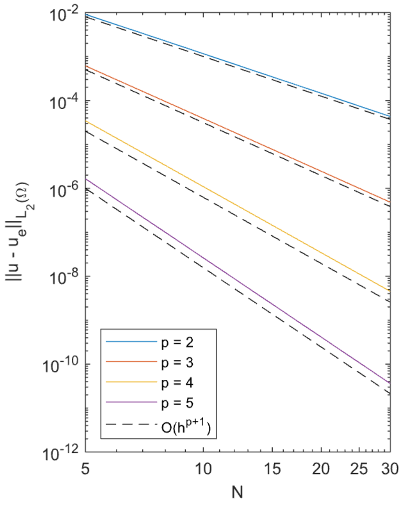
$$D_{i,kl} = (C_{ijkl} \partial_j \varphi_a, \varphi_b)_K - \langle C_{ijkl} n_j \varphi_a, \varphi_b \rangle_{\partial K} \qquad G_{i,kl} = \langle C_{ijkl} n_j \phi_a, \varphi_b \rangle_{\partial K}$$

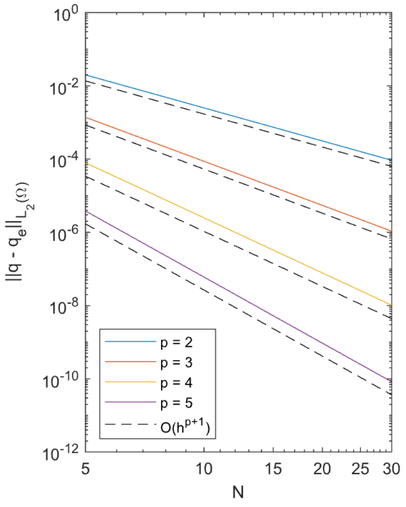
$$\left[\begin{array}{cccc|cc|cc}
A & & & & B_1 & & -C_1 & \\
 & A & & & & B_2 & & -C_2 \\
 & & A & & B_1 & & -C_1 & \\
 & & & A & & B_2 & & -C_2 \\
\hline
D_{1,11} & D_{1,12} & D_{1,21} & D_{1,22} & E & & -F & \\
D_{2,11} & D_{2,12} & D_{2,21} & D_{2,22} & & E & & -F \\
\hline
G_{1,11} & G_{1,12} & G_{1,21} & G_{1,22} & -F^T & & I & \\
G_{2,11} & G_{2,12} & G_{2,21} & G_{2,22} & & -F^T & & I
\end{array}\right]
\left[\begin{array}{c}
Q_{11} \\ Q_{12} \\ Q_{21} \\ Q_{22} \\ \hline U_1 \\ U_2 \\ \hline \hat{U}_1 \\ \hat{U}_2
\end{array}\right]
=
\left[\begin{array}{c}
0 \\ 0 \\ 0 \\ 0 \\ \hline (\varphi_a, b_1) \\ (\varphi_a, b_2) \\ \hline \langle \varphi_\alpha, \bar{t}_1 \rangle \\ \langle \varphi_\alpha, \bar{t}_2 \rangle
\end{array}\right]$$

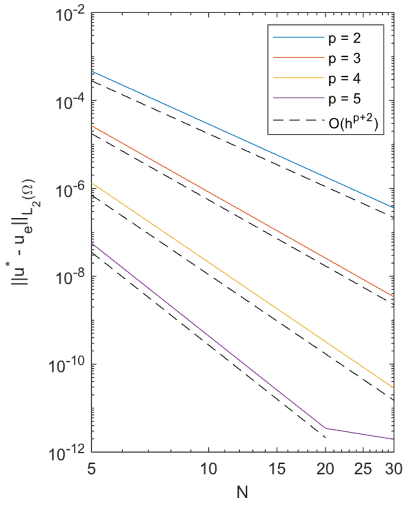
Results

Convergence test on the unit square.
(Manufactured solution)

$$\mathbf{u}_e = [u_1, u_2, u_3]\phi(x,y), \quad \phi = \sin(\pi x)\sin(\pi y)$$



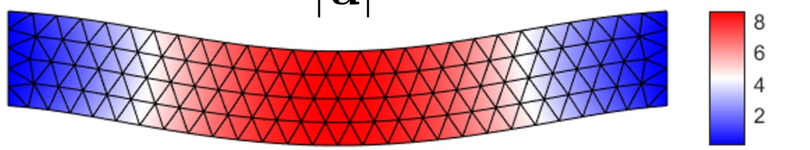




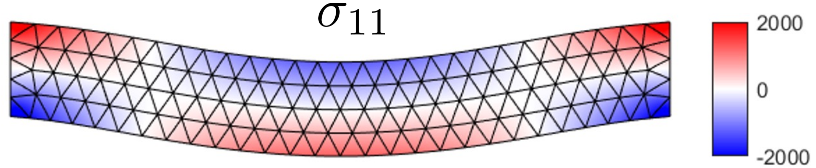
$$||\mathbf{u}-\mathbf{u_e}|| \sim \mathcal{O}(h^{p+1}), \quad ||\mathbf{q}-\mathbf{q_e}|| \sim \mathcal{O}(h^{p+1}), \quad ||\mathbf{u}^*-\mathbf{u_e}|| \sim \mathcal{O}(h^{p+2}),$$

Beam with clamped ends,
uniform distributed load

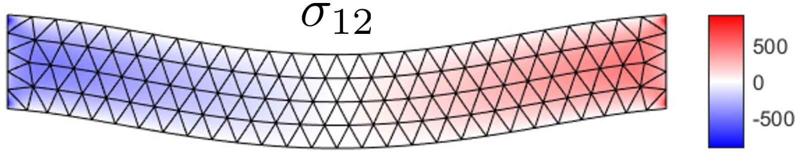
$|\mathbf{u}|$



σ_{11}



σ_{12}



Linear Elastodynamics

Governing Equations

$$v_i = \frac{\partial u_i}{\partial t}$$ Velocity

$$\frac{\partial q_{ij}}{\partial t} - \frac{\partial v_i}{\partial x_j} = 0 \quad \text{in } \Omega$$

$$\rho \frac{\partial v_i}{\partial t} - \frac{\partial}{\partial x_j}\left(C_{ijkl} q_{kl}\right) = b_i \quad \text{in } \Omega$$

$$v_i = \frac{\partial \bar{u}_i}{\partial t} \quad \text{on } \partial\Omega_D$$

$$C_{ijkl} n_j q_{kl} = \bar{t}_i \quad \text{on } \partial\Omega_N$$

Physical expectations:

In an isotropic linear homogeneous medium, the divergence of the displacement obeys

$$\partial_t^2 \theta + c_p^2 \nabla^2 \theta = 0, \quad c_p = \sqrt{(\lambda + 2\mu)/\rho)}$$

These are P-waves (pressure). The curl of displacement obeys

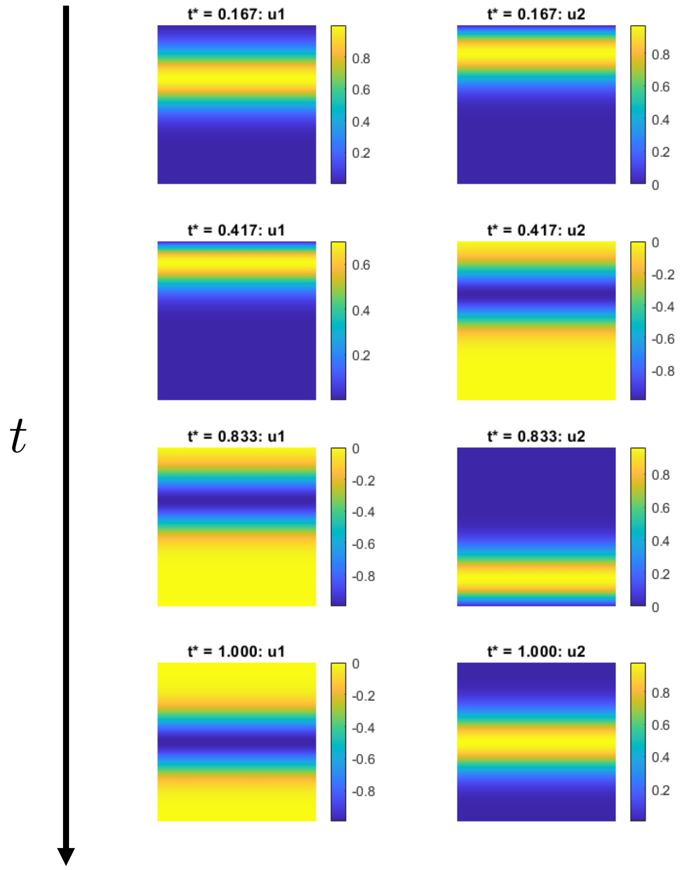
$$\partial_t^2 \omega + c_s^2 \nabla^2 \omega = 0, \quad c_s = \sqrt{\mu/\rho}$$

These are S-waves (shear) that propagate independently of P-waves at a lower speed.

Convergence Results

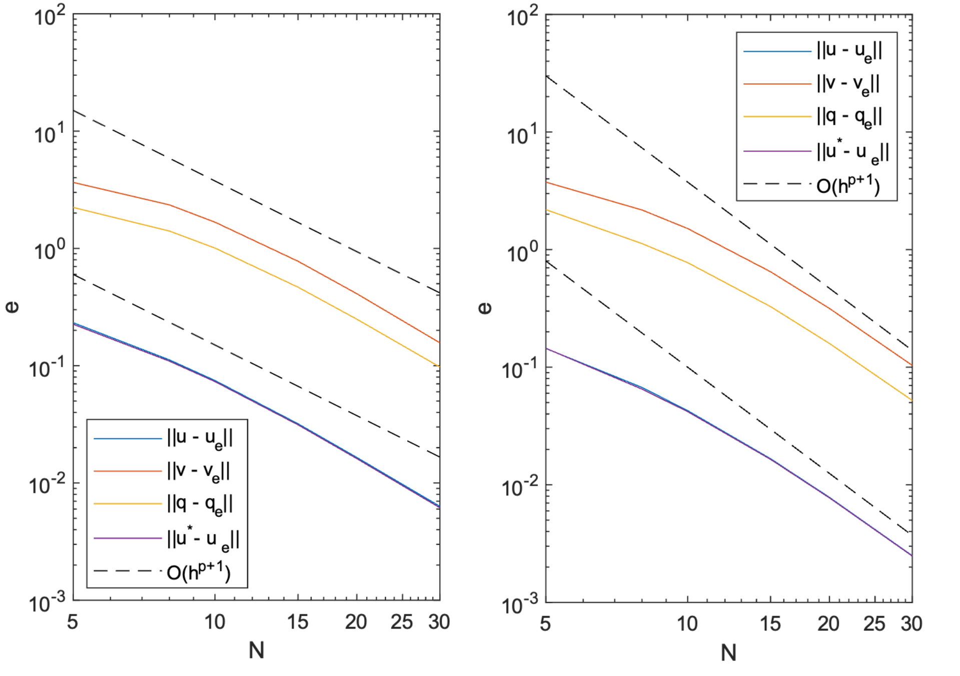
S-wave,
1 reflection

P-wave,
2 reflections



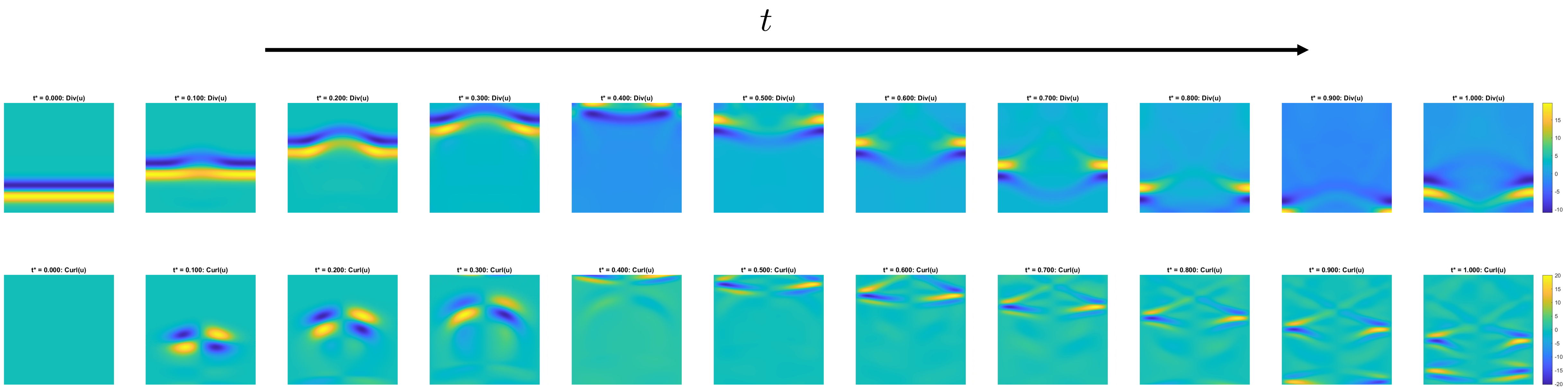
$p = 2$
$c_p \Delta t / h = 0.5$

$p = 1$
$c_p \Delta t / h = 0.067$



- Integrated with DIRK(3,3)
- For constant CFL, max convergence order is $\mathcal{O}(h^3)$
- Expected but did not observe $\mathcal{O}(h^3)$ convergence after post-processing for $p = 1$.

Physics Results: Reflection and Refraction



Top row: $\nabla \cdot \mathbf{u}$ Bottom row: $\nabla \times \mathbf{u}$

- In isotropic homogeneous media, P-waves and S-waves propagate independently
- Spatial variation in the elastic moduli can convert one into the other, as can oblique reflections
- Above: linear moduli, with Young's modulus doubled in a region centered on $(0.5, 0.5)$.
- Periodic side boundaries, clamped top/bottom boundaries, $p = 4$, $h = 0.05$.