# Variational Data Assimilation (4D - Var)

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- Smoothing for large, deterministic systems.
- Commonly used in numerical weather prediction.

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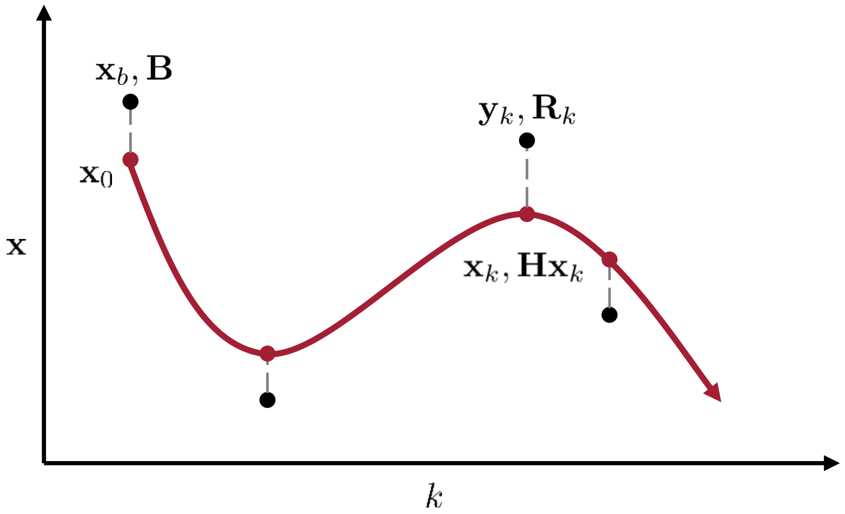
| | |
|---|---|
| $\mathbf{x}_{k+1} = M_k(\mathbf{x}_k)$ | Deterministic Dynamics |
| $\mathbf{x}_b, \mathbf{B}$ | Prior for Initial Condition (Background) |
| $\mathbf{y}_k, \mathbf{R}_k$ | Noisy Observations at times $k$ |

- Key idea: the best initial condition creates a trajectory that minimizes

$$J(\mathbf{x}_0) = \underbrace{\|\mathbf{x}_0 - \mathbf{x}_b\|}_{\text{Distance from Prior}}{}_{\underbrace{\mathbf{B}^{-1}}_{\text{Prior Precision}}} + \sum_k \underbrace{\|\mathbf{y}_k - \mathbf{H}\mathbf{x}_k\|}_{\text{Distance from Data}}{}_{\underbrace{\mathbf{R}_k^{-1}}_{\text{Data Precision}}}$$



# Computational Considerations

## How do we do this?

- $J$ depends on all of the $\mathbf{x}_k$, but the optimization requires $\mathrm{d}J/\mathrm{d}\mathbf{x}_0$
- Adjoint Model (Linearization):

$$\begin{cases} \mathbf{x}_{j+1} = M_j(\mathbf{x}_j) & \text{Nonlinear Dynamics} \\ \mathbf{M}_j = M_j'(\mathbf{x}_j) & \text{Jacobian Matrix} \end{cases}$$

Forward Pass: Dynamics

$$\mathbf{x}_k = [M_k \circ M_{k-1} \circ \ldots \circ M_2 \circ M_1](\mathbf{x}_0)$$

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{x}_0} = \mathbf{M}_1^T \cdot \mathbf{M}_2^T \cdot \ldots \cdot \mathbf{M}_{k-1}^T \cdot \mathbf{M}_k^T \cdot \frac{\mathrm{d}}{\mathrm{d}\mathbf{x}_k}$$

Backward/Adjoint Pass: Gradients

## Algorithm:

Initialize $\mathbf{x}_0$.
Loop until $||\mathrm{d}J/\mathrm{d}\mathbf{x}_0||$ is small:

- Forward pass: calculate $J(\mathbf{x}_0)$ and $\{\mathrm{d}J/\mathrm{d}\mathbf{x}_k\}$
- Adjoint pass: calculate $\mathrm{d}J/\mathrm{d}\mathbf{x}_0$ from $\{\mathrm{d}J/\mathrm{d}\mathbf{x}_k\}$
- Update $\mathbf{x}_0$ based on $\mathrm{d}J/\mathrm{d}\mathbf{x}_0$

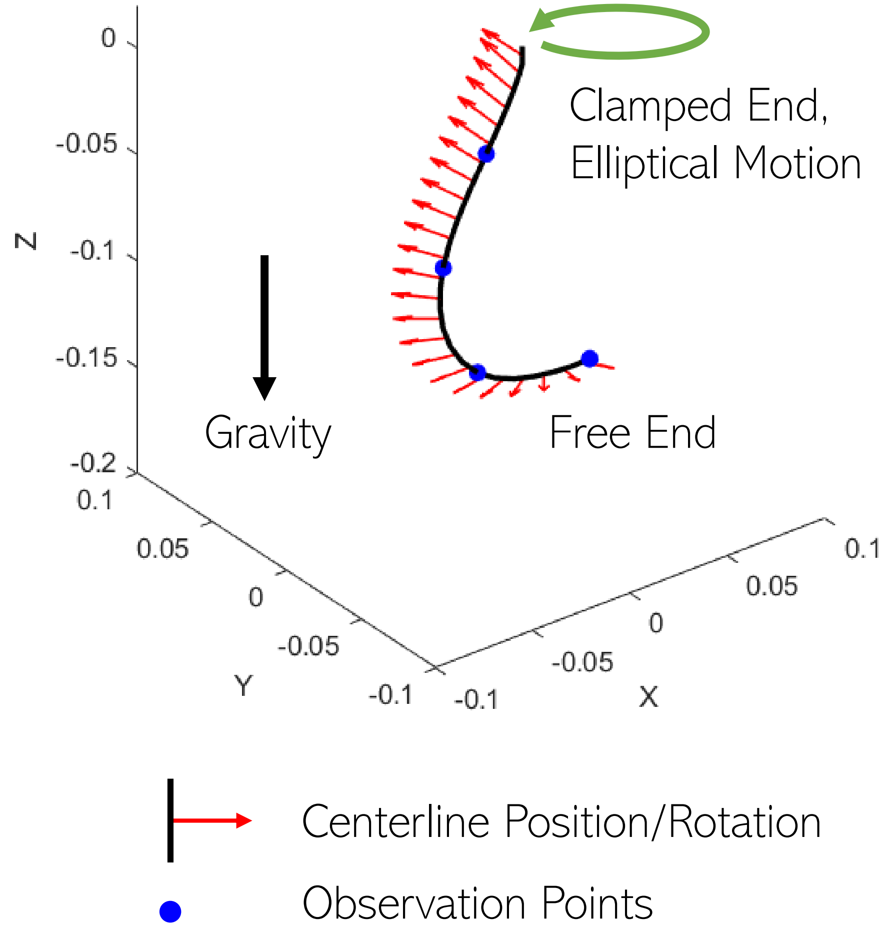
## Why do we do this?

- Large, nonlinear, deterministic systems.
- Minimization can be “sloppy” – approximate gradients, reduced models

# Example Application

## Infer the trajectory of an elastic rod

- System of constrained nonlinear ODEs
- Noisy observations of marked points
- See write-up for details on adjoints, covariance localization




**In one sentence:**
Variational Data Assimilation (4D-Var) smooths large deterministic systems using gradient-based minimization and adjoint models.