Understanding the Fourier Transform: Purpose and Intuition

Purpose of the Fourier Transform

The Fourier Transform is a powerful mathematical tool used to break down a complex signal into simpler components, specifically **sine** and **cosine** waves. These waves serve as the basic building blocks of the signal, allowing us to:

- Identify which frequencies are present in the signal.
- Determine the strength (magnitude) of each frequency.
- Understand the timing or phase relationship of each frequency.

This is similar to analyzing a musical chord by identifying the individual notes that make it up, with each note corresponding to a specific frequency. To understand this concept in depth, we need to explore some foundational tools.

Convolutions and Correlation

What is a Convolution?

A convolution is a mathematical operation that determines how one function "overlaps" with another as it shifts across the domain. It provides a measure of how two signals are correlated over time or space. Correlation, in this context, refers to the degree to which the two functions align, meaning how similar their shapes are when one is shifted relative to the other.

To introduce this idea, let’s start with a simple example involving probabilities.

Discrete Convolution: Probability of Rolling a Sum

Scenario

Imagine rolling two weighted dice, each with an arbitrary number of sides. The first and second dice outcomes are represented by the values V_1 and V_2, respectively. Each die has a distinct probability distribution, which can be thought of as a discrete function $P(V_1)$ and $P(V_2)$, where:

- $P(V_1)$ is the probability of rolling a value V_1 on the first die.
- $P(V_2)$ is the probability of rolling a value V_2 on the second die.

We want to calculate the probability of rolling a specific sum $X = V_1 + V_2$. Since there are multiple possible combinations of V_1 and V_2 that can result in the same sum X, the total probability for this sum is the sum of the individual probabilities of all those combinations. For simplicity, we can write $V_2 = X - V_1$

Discrete Convolution Formula:

The convolution in this scenario is expressed as:

$$P(X) = \sum_{V_1} P(V_1) \cdot P(X - V_1)$$

Where:

- $P(V_1)$ is the probability of rolling V_1 on the first die.
- $P(X - V_1)$ is the probability of rolling $X - V_1$ or V_2 on the second die.

Explanation

- If we want to calculate the probability of getting a sum X, we check all possible values of V_1 (the outcome of the first die).
- For each V_1, we find the corresponding $V_2 = X - V_1$ (the outcome of the second die) and multiply their individual probabilities $P(V_1)$ and $P(V_2)$.
- The total probability $P(X)$ is the sum of these individual probabilities for all valid combinations of V_1 and V_2 that give the sum X.

Transition to Continuous Convolution

In a continuous setting, we generalize the discrete convolution to work with continuous functions instead of discrete probabilities. Here, we measure how much two functions align or "match" as one slides across the other. This process involves multiplication and integration, replacing the summation in the discrete case.

Setup

Considering the functions $f(t)$ and $g(t)$ that are defined along all real numbers, our definition of the discrete convolution along all integers is given by:

$$(f * g)[n] = \sum_{k=-\infty}^{\infty} f(k) \cdot g(k - n)$$

The continuous generalization of $f(t)$ and $g(t)$ naturally follows an integral instead of a discrete sum:

Convolution Definition:

$$(f * g)[t] = \int_{-\infty}^{\infty} f(\tau) \cdot g(t - \tau) d\tau$$

Building the Convolution in Stages

Given the continuous case, the equation can be seen as the natural calculus generalization of the discrete sum. While this might seem ambiguous due to the integral's complexity, we consider a more purpose: measuring how two functions correlate. To better understand, let’s generalize further with two functions, f and g, outside the constraints of probability. Let's continue by building up this integral.

Product: $f(t) \cdot g(t)$

If we multiply f and g for all t, you might notice that we can determine how much the functions match. This is a very trivial tool to find points along f and g measuring local correlations at a given t

Key considerations:

- Where both f and g have high magnitudes, their product will also be large.
- Where either f of g is near zero, the product will be close to zero.

Symmetry: $f(t) \cdot g(-t)$

Next, we introduce symmetry by reflecting the input space of $g(t)$ to $g(-t)$. This step ensures that the convolution is consistent with probabilistic interpretations and Fourier analysis, where reversing captures alignment across all phase relationships. Reflecting also simplifies the mathematical framework, ensuring symmetry in measuring overlap.

Accumulation: $\int_{-\infty}^{\infty} f(\tau) \cdot g(-\tau) d\tau$

We integrate the product over all real numbers denoted by the integrating variable τ. This step aggregates the pointwise correlations into a single scalar value that represents the overall alignment across all inputs:

Key considerations:

- If f and g are uncorrelated, the integral will approach zero because uncorrelated functions lack a consistent pattern of matching amplitude or sign. Over an infinite range, the lack of coordination between their values results in relatively equal positive and negative contributions tending to cancel each other out, leading to a net value around zero.
- If f and g are highly correlated, the integral will yield a high-magnitude value because correlated functions constructively interfere, with their amplitudes and signs consistently reinforcing each other to produce high-magnitude values.

Offset Definition: $\int_{-\infty}^{\infty} f(\tau) \cdot g(t - \tau)d\tau$

In convolution analysis, the scalar result obtained from integrating the product of f and g when they are centered at zero represents the alignment of these functions under a specific phase relationship. This scalar quantifies the degree of correlation at that alignment. However, this value is not invariant; it changes with offsets or phase shifts between f and g. By introducing a variable phase shift, t, into the relationship, the convolution can be expressed as a function of correlation dependent on the offset between f and g. Traditionally the function g is modified to $g(t - \tau)$ which is computationally the same as $f(t - \tau)$, allowing the convolution to capture the correlation as f and g shift relative to one another.

Signal Processing and Kernels

In signal processing, f represents the input signal, g and serves as the kernel. While a kernel can be any function, it is often specifically designed to filter, smooth, or extract features from the signal.

Defining Kernels

A kernel in signal processing is a predefined function that interacts with the input signal to produce a desired transformation. Unlike arbitrary functions, kernels can be defined over a finite domain or specific region to focus their effect on localized features of the input signal. This restricted domain ensures computational efficiency and aligns with practical applications, where only a portion of the signal is typically analyzed or transformed. Different kernels are tailored to achieve specific outcomes, such as noise reduction, feature extraction, or signal enhancement.

Laplace Transform and Fourier Transform

The Laplace Transform

The Laplace Transform is a generalization of the Fourier Transform that is particularly useful for solving differential equations. Laplace Transform is similar to a convolution given by:

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt \quad \text{where: } s = \sigma + i\omega$$

note integrating bounds are $[0, \infty]$.

This transform converts linear differential equations into algebraic equations, simplifying their analysis and solution. From a convolution perspective, the Laplace Transform is a versatile tool for analyzing a system containing periodic components, particularly in systems with significant exponential growth or decay.

The Fourier Transform as a Special Case

The Fourier Transform is a special case of the Laplace Transform where $s = i\omega$ and is evaluated along the bounds $[-\infty, \infty]$. Specialized for periodic signals with consistent amplitude and invariant time features. The Fourier transform of f denoted by $\hat{f}$ is defined as:

$$\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

The kernel $g(t) = e^{-i\omega t}$ can be separated into $g(t) = cos(\omega t) - i\, sin(\omega t)$ separate integrals, allowing an alternative interpretation of the Fourier transform as a sum of two integrals, allowing a simpler intuition.

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)cos(\omega t)dt - i\int_{-\infty}^{\infty} f(t)sin(\omega t)dt$$

The real part of the Fourier Transform, $RE(\hat{f})$, is a measure of the correlation of $f(t)$ and a cosine wave at the frequency ω with a zero-phase offset corresponding to the cosine wave basis function. The imaginary component of the Fourier Transform, $IM(\hat{f})$ measures the correlation between $f(t)$ and a wave of the frequency ω with a $\frac{\pi}{2}$-phase offset, which is the sine wave basis function

$\hat{f}(\omega)$ can be interpreted as $\hat{f}(\omega) = (r(\omega), \varphi(\omega))$ where $r(\omega)$ is wave amplitude and $\varphi(\omega)$ is wave phase given by

$$r(\omega) = \left| \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \right|, \ \varphi(\omega) = arctan\left(\frac{-i \int_{-\infty}^{\infty} f(t) sin(\omega t) dt}{\int_{-\infty}^{\infty} f(t) cos(\omega t) dt}\right)$$