

Performative Learning Theory

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Abstract

Performative predictions influence the very outcomes they aim to forecast. We study performative predictions that affect a sample (e.g., only existing users of an app) and/or the whole population (e.g., all potential app users). This raises the question of how well models generalize under performativity. For example, how well can we draw insights about new app users based on existing users when both of them react to the app’s predictions? We address this question by embedding performative predictions into statistical learning theory. We prove generalization bounds under performative effects on the sample, on the population, and on both. A key intuition behind our proofs is that in the worst case, the population negates predictions, while the sample *deceptively* fulfills them. We cast such self-negating and self-fulfilling predictions as min-max and min-min risk functionals in Wasserstein space, respectively. Our analysis reveals a fundamental trade-off between performatively changing the world and learning from it: the more a model affects data, the less it can learn from it. Moreover, our analysis results in a surprising insight on how to improve generalization guarantees by retraining on performatively distorted samples. We illustrate our bounds in a case study on prediction-informed assignments of unemployed German residents to job trainings, drawing upon administrative labor market records from 1975 to 2017 in Germany.

1 Introduction

Machine learning (ML) and Statistics has evolved from merely analyzing the world to shaping it. Their predictions have real-world effects on what these systems aim to predict in the first place (Liu et al., 2025). Building on work by Grunberg & Modigliani (1954); Morgenstern (1928) in the social sciences, Perdomo et al. (2020) formalize such feedback loops as *performative* predictions (PP). Examples range from self-fulfilling prophecies in financial markets (MacKenzie, 2008; Neurath, 1911; Soros, 1994) and strategic behavior in selection processes (Khandani et al., 2010; Vo et al., 2024)

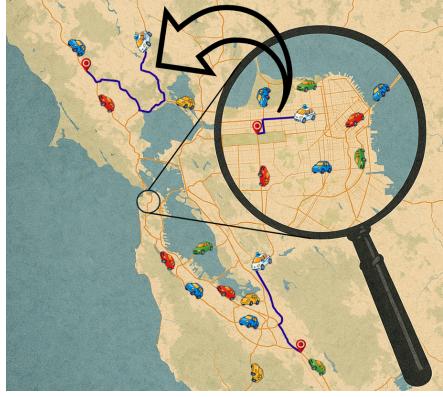
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[†]This work was partially conducted while James Bailie was a PhD student in the Department of Statistics at Harvard University.

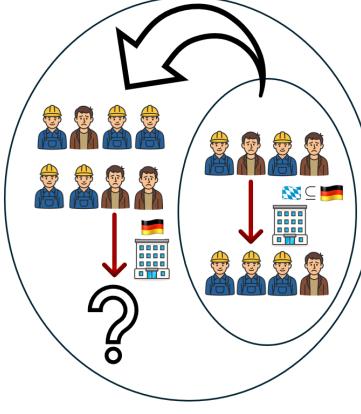
to self-negating traffic route predictions (Benenati & Grammatico, 2024), gaming recommender systems (Bian et al., 2023) and early warning systems in high schools (Perdomo et al., 2025). We refer to Hardt & Mendler-Dünner (2025) for a recent survey. The common thread in these examples are predictions of outcomes that have performative effects on the actual outcomes. Both the initial framework (Perdomo et al., 2020) and subsequent work (Brown et al., 2022; Miller et al., 2021; Mofakhami et al., 2023; Perdomo, 2023, 2024, 2025) define these actual outcomes as—statistically speaking—the whole population of units. Since their goal is to study stability and optimality of PP, these works do not address learnability of the population from a sample under performativity.

In the article at hand, we generalize this setting by studying performative effects on a sample drawn from the population, on the population itself, as well as on both. In many applications, only a finite sample of training data is available instead of complete access to the whole population. Performative effects might arise both in and out of this sample, see Sections 1.1 and 4 for real-world examples. In other words, while prior works focus on repeated *risk* minimization under *population* performativity, we turn our attention additionally to repeated *empirical risk* minimization (ERM, Vapnik 1998) under *empirical* and *population* performativity. This raises the question whether and how well models can generalize from train to test data if the train or test data (or both) are subject to performative effects. We prove bounds on the generalization error, the generalization gap and the excess risk in all these scenarios under minimal assumptions on the performative effects. In particular, we do not assume any specific knowledge about the performative effects—besides being continuous as in Brown et al. (2022); Perdomo et al. (2020). This is in contrast to strategic classification (assuming best-response under explicit costs), causal modeling (structural assumptions), or algorithmic game theory and mechanism design (assuming equilibrium-based response models with rational agents), see discussion of related work in Section 5.

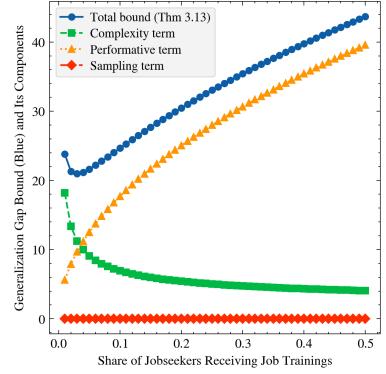
Summary of Contributions: The main objective of this work is to study generalization in the context of performativity. A necessary first step is to develop a conceptual grasp of what generalization actually means if the sample, population or both performatively react to predictions. Section 2 achieves this, triggering four main research questions **RQ1** through **RQ4**. We formally embed PP into learning theory, allowing us to answer **RQ1– RQ3**. Specifically, we prove generalization bounds on PP, requiring only a compact parameter space and a subset of the assumptions in the original PP framework (Brown et al., 2022; Perdomo et al., 2020). Moreover, we give tighter bounds under additional assumptions on the hypothesis class. We find that two factors complicate generalization under performativity: Not only can the population negate your predictions (self-defeating), the sample can also deceive you by doing the opposite and reinforce them (self-fulfilling), creating an *empirical echo chamber*. Technically, we cast these self-negating and self-fulfilling predictions as min-max and min-min learning problems in Wasserstein space, respectively. This allows us to leverage empirical process theory for dual characterization of locally-distributionally robust (Gao & Kleywegt, 2023) and favorable (Jiang & Xie, 2025) learning, respectively. Conceptually, our bounds highlight a fundamental trade-off between performatively changing data and drawing reliable conclusions from it. We further provide a corollary that allows for improving generalization bounds by retraining on performatively distorted samples. We illustrate our results in a case study on performative effects of unemployment risk predictions, harnessing administrative labor market records from 1975 to 2017 in Germany (raw data with over 60 million rows) provided by the German Federal Employment Agency.



1.1 Example (A)



1.2 Example (B)



1.3 Jobseeker Data (Section 4)
Shows Change vs. Learn
Trade-Off

Figure 1: 1.1 **Example (A):** Route predictions are known to have performative effects: Drivers avoid routes with predicted congestion, thereby rendering these predictions less accurate. Can routing apps still generalize from San Francisco (sample) to the whole Bay Area (population)?

1.2 **Example (B):** A job center in Bavaria assigns job training programs to those among the unemployed that have high risks of long-term unemployment according to ML predictions. As a result of the job training, their probability of finding a job increases, a textbook example of a performative effect, see Section 4. Can the job center’s ML model trained on performatively shifted data from Bavaria generalize to the whole German population, which will in turn react to the predictions? 1.3 **Theorem 3.13 applied to Jobseekers:** Growth of generalization gap bound (blue, logistic loss units) shows trade-off between performatively changing a population (by assigning more people to job trainings) and reliably learning its properties. Details on the bound’s components (red, green, yellow) can be found in Section 4.

1.1 Running Examples

We introduce two prototypical examples of generalization under performativity. In Example (A), performative effects are induced by the strategic behavior of agents (endogenous), while in Example (B), they result from treatments (exogenous). Our results address both scenarios. Both examples involve a model $\hat{\theta}_{t+1}$ trained on a sample \hat{d}_t , which was drawn from a population d_t . As is customary, we assume $\hat{d}_0 \stackrel{\text{iid}}{\sim} d_0$ for the initial sample \hat{d}_0 and the initial population d_0 . Both quantities can change over time, and hence are indexed by $t \in \{1, \dots, T\}$.

(A) Routing App: A company offers mapping services that help users find locations and get directions. Their route planning is known to have performative effects on the users. For instance, drivers avoid routes with predicted traffic jams, jeopardizing these prediction’s accuracy by altering the ground truth—a classic example of *self-negating* performative predictions (Bagabaldo et al., 2024; Benenati & Grammatico, 2024; Cabannes, 2019). Suppose the company now wants to use a new model for the route predictions. In order to test it, they first run the model $\hat{\theta}_1$ in beta stage on selected users (sample \hat{d}_0), say, only in San Francisco (see Figure 1). The performative effects of $\hat{\theta}_1$ alter this sample \hat{d}_0 and thus the subsequently updated models, giving $\hat{d}_1, \dots, \hat{d}_T$ and $\hat{\theta}_1, \dots, \hat{\theta}_T$. Before releasing the new model to all users d_0 in, say, the whole Bay Area (Figure 1), the company

would like to know the worst case error of their current model $\hat{\theta}_t \in \{\hat{\theta}_1, \dots, \hat{\theta}_T\}$ on all users d_0 (or performatively affected d_t). Our bounds allow for nuanced answers to this question. They tell the company the generalization error that their model will not exceed, depending on the sample size n of d_0 and on the (observable) number m of drivers in that sample that changed their route due to the predictions, giving rise to what we call the *performative response rate* $\frac{m}{n}$. The latter serves as an estimator of the performative response rate in the whole Bay Area.

(B) Profiling of Jobseekers (Section 4): Many public employment services (PES) attempt to predict the risk of newly registered job seekers becoming long-term unemployed (Allhutter et al., 2020; Junquera & Kern, 2025; Kern et al., 2021). These risk predictions—generated by caseworkers or ML models—determine access to scarce job training programs, since job centers assume high-risk individuals to benefit the most from those trainings (Ernst et al., 2024; Fischer-Abaigar et al., 2025). Thus, the predictions exhibit a performative effect: If the model predicts a high risk of prolonged unemployment for a person, they are more likely to be allocated to a training program, leading to a quicker reintegration in the labor market (if the training is effective). Suppose the German PES tests a new ML model on selected job centers (e.g., only in Bavaria, see Figure 1) and observes how this model’s predictions affect (via assignments to training programs) whether job seekers exit unemployment. Can we help the agency by giving guarantees on their model’s generalization error when rolled out to all job centers? Section 4 answers affirmatively by illustrating our generalization bounds on real data from the German PES.

2 Generalization Under Performativity

We assume our data live in a bounded subset $\mathcal{Z} = \mathcal{Y} \times \mathcal{X}$ of \mathbb{R}^ν with $\mathcal{Y} \subset \mathbb{R}^{\nu_y}$, $\mathcal{X} \subset \mathbb{R}^{\nu_x}$ and $\nu = \nu_y + \nu_x$. Random variables on \mathcal{Z} will be denoted by Z_1, \dots, Z_n , probability distributions on \mathcal{Z} by d , and the set of all such d by Δ . Let $\mathcal{F} := \{f_\theta : \mathcal{X} \rightarrow \mathcal{Y} \mid \theta \in \Theta\}$ be our hypothesis class, whose parameter space Θ we assume to be a compact subset of Euclidean space. We equip $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ and Θ with the Euclidean (L_2) norm $\|\cdot\|_2$, and define $F := \sup_{\theta \in \Theta, x \in \mathcal{X}} \|f_\theta(x)\|_2$. Observe that F is finite because \mathcal{Y} is bounded.

Definition 2.1 (Risk (Minimizer) and Loss). The *risk* of the parameter $\theta \in \Theta$ on data generated according to the distribution $d \in \Delta$ is defined as $\mathcal{R}(d, \theta) = \mathbb{E}_{Z \sim d} [\ell(Z, \theta)]$, where $\ell : \mathcal{Z} \times \Theta \rightarrow \mathbb{R}$ is a *loss function*. For any distribution $d \in \Delta$, we call

$$G(d) = \arg \min_{\theta \in \Theta} \mathcal{R}(d, \theta),$$

the *risk minimizer*, which we assume to be unique.

Given independent and identically distributed (i.i.d.) training data Z_1, \dots, Z_n drawn from a true but unknown distribution d_0 , we seek some $\theta \in \Theta$ whose risk $\mathcal{R}(d_0, \theta)$ is close to the true but unknown minimum risk $\inf_{\theta \in \Theta} \mathcal{R}(d_0, \theta)$, often referred to as *Bayes risk*. The risk $\mathcal{R}(d_0, \theta)$ of θ with respect to the true d_0 is known as *generalization error* of θ .

Definition 2.2 (Excess Risk). $\mathcal{R}(d, \theta) - \inf_{\theta' \in \Theta} \mathcal{R}(d, \theta')$ is called the *excess risk* of θ .

A popular learning strategy is ERM (Vapnik, 1998). Let $\widehat{d} := \frac{1}{n} \sum_{i=1}^n \delta_{Z_i}$ be the *empirical distribution* of Z_1, \dots, Z_n , where δ_{Z_i} is the Dirac measure at $Z_i \in \mathcal{Z}$. We denote the *empirical risk minimizer* $G(\widehat{d})$ by $\widehat{\theta}$. We now introduce the general setting of *performative predictions* (Perdomo et al., 2020), where the deployment of a predictive model influences the distribution of future data. For maximal generality, we adopt the *stateful* extension of performative prediction in which model predictions and the current data distribution jointly induce tomorrow's distribution (Brown et al., 2022). Formally, the evolution of the distribution is governed by an unknown transition map Tr as follows:

In each round $t \in \{1, \dots, T\}$, an institution deploys θ_t and subsequently observes the induced distribution

$$d_t = \text{Tr}(\theta_t, d_{t-1}).$$

Note that the *stateless* $d_t = \text{Tr}_s(\theta_t)$ by Perdomo et al. (2020) is a special case. The *stateful performative risk* of a θ is defined as $\mathbb{E}_{Z \sim d_t}[\ell(Z, \theta)] = \mathbb{E}_{Z \sim \text{Tr}(\theta_t, d_{t-1})}[\ell(Z, \theta)]$.

Definition 2.3 (Repeated (Empirical) Risk Minimization). A natural approach in the performative setting is to update the model through *repeated risk minimization (RRM)*. At each round t , the new model parameter $\theta_{t+1} = G(d_t)$ is chosen to minimize the (true) risk on the current distribution d_t . As hinted at above, we additionally study *repeated empirical risk minimization (RERM)* which is defined analogously for $Z_1, \dots, Z_n \sim \widehat{d}_t = \text{Tr}(\widehat{\theta}_t, \widehat{d}_{t-1})$ as $\widehat{\theta}_{t+1} = G(\widehat{d}_t)$.

In previous works, Brown et al. (2022); Perdomo et al. (2020) study stability and optimality of θ in these sequences $\{\theta\}_{t=1}^T$ and $\{\widehat{\theta}\}_{t=1}^T$. We explain how stable θ 's and optimal θ 's are subsumed by our analysis in Appendix A.

2.1 On the Meaning of Performative Generalization

To what extent can models generalize from a finite sample to the population in a performative world? We study this question under performativity in the sample, in the population or in both. Interestingly, performative generalization can differ across these scenarios, depending on the data on which the model is (re)trained: the sample, the population, or (sequentially or simultaneously) on both. Table 1 summarizes all of the resulting scenarios, including familiar paradigms like ERM, online learning and classical PP as well as four open research questions (RQs).

(RQ1) Can we bound the (classical) *excess risk* (Def. 2.2) $\mathcal{R}(d_0, \widehat{\theta}_t) - \inf_{\theta \in \Theta} \mathcal{R}(d_0, \theta)$ for models $\widehat{\theta}_t$ iteratively retrained on samples $\widehat{d}_{t-1} = \text{Tr}(\widehat{\theta}_{t-1}, \widehat{d}_{t-2})$? In this scenario, a model exhibits performative effects on the sample (or subpopulation) it is trained on, but not on the whole population. As illustration, consider Example (A) in which a navigation app's route predictions are being made visible only to existing registered users (sample), not to all users (population).

(RQ2) Can we bound the *performative excess risk*

$$\mathcal{R}(\text{Tr}(d_0, \widehat{\theta}_t), \widehat{\theta}_t) - \inf_{\theta \in \Theta} \mathcal{R}(\text{Tr}(d_0, \widehat{\theta}_t), \theta)$$

for any $t \in \{1, \dots, T\}$? This is the performative version of the classic excess risk (Def. 2.2). The first term is the *performative generalization error*. It describes the error a model $\widehat{\theta}_t$ (trained on a sample \widehat{d}_{t-1}) suffers when being deployed on the population d_0 under performativity. In this case,

Table 1: Overview of Research Questions RQ1–RQ4 on generalization under performativity. The setup with no retraining and no performativity is the classical learning theory setup, where ERM is Bayes optimal. Retraining on new samples without performativity is a special case of online learning. The performative prediction (PP) setup established by [Perdomo et al. \(2020\)](#), [Brown et al. \(2022\)](#) and [Perdomo \(2025\)](#) considers performativity on the whole population and retraining of models on the whole population. [Perdomo et al. \(2020, Sec. 3.4\)](#) also considers retraining on data sampled anew (*i.i.d.*) in each iteration from the performatively changed population. Dashes (“–”) mark nonsensical or inapplicable combinations. For instance, retraining on a sample without performative effects is nonsensical since the sample does not change and so retraining would correspond to doing the exact same thing multiple times. Retraining on both the sample and the population is relevant when we have access to a sample first, and then later to the whole population (sequentially) or for statistical analysis with a hypothetical process on the assumed population in parallel (simultaneously).

		Retraining				
Performative Effects	None	On Sample		Population	Both, ¹ Sequ.	Both, ¹ Simult.
		Drawn Once	Drawn Anew			
None	ERM	–	Online Learning	–	–	–
On Sample	–	RQ1	–	PP	–	–
On Population	RQ2 (a) [*]	RQ2 (b)	PP (Sec. 3.4)	PP	RQ3 (a)	–
On Both	RQ2 (c)	RQ2 (d)	PP (Sec. 3.4)	PP	RQ3 (b)	RQ4 ^{**}

¹ With the sample being drawn once.

^{*} Partly answered by [Kirev et al. \(2025\)](#).

^{**} Asymptotically answered by [Li et al. \(2025b\)](#).

the performative effect of $\hat{\theta}_t$ will shift d_0 to $d_1 = \text{Tr}(d_0, \hat{\theta}_t)$. The model is then evaluated on this performatively shifted distribution. The second term is the Bayes risk on this shifted d_1 within the “hypothesis class” Θ . **RQ2 (a):** In case of no retraining, $\hat{\theta}_1$ is found via standard ERM. After being deployed on the population, it causes a performative shift $d_1 = \text{Tr}(d_0, \hat{\theta}_1)$. This special case is partly answered by [Kirev et al. \(2025\)](#) for binary classification under linear performative shift of $Y \mid X$ and marginal X . Our results are more general in two ways. First, we account for all (Lipschitz-)continuous transition maps as in [Brown et al. \(2022\)](#); [Perdomo et al. \(2020\)](#). Second, we subsume performative shifts on any subsets of Y and X . **RQ2 (b):** Retraining on the same \hat{d}_1 (due to the absence of performative effects) gives $\hat{\theta}_1 = \hat{\theta}_2 = \dots = \hat{\theta}_T$. **RQ2 (c)** is subsumed by (a), as performative effects on the sample are irrelevant (do not change $\hat{\theta}_1$) due to lack of retraining. **RQ2 (d)** combines **RQ1** and **RQ2 (a):** In addition to (multi-shot) sample performativity $\hat{d}_t = \text{Tr}(\hat{d}_{t-1}, \hat{\theta}_t)$ as in **RQ1**, we have to account for (single-shot) population-level performative shifts $d_1 = \text{Tr}(d_0, \hat{\theta}_t)$ as in **RQ2 (a)**. Like the standard excess risk, the performative excess risk is subject to uncertainty about the unknown distribution d_0 . However, the performative excess risk is also subject to uncertainty about unknown $\text{Tr}(\cdot, \cdot)$. For scenario **RQ2 (d)**, consider again

Example (A): The routing model is (re)trained on registered users, but predictions are made public to all users in d_0 , causing a performative shift to d_1 . As we do not retrain on the population, we do not have to account for performative effects beyond the one shifting d_0 to d_1 . This changes in **RQ3**.

(RQ3) Can we bound the *cumulative performative excess risk* of a model that was first trained T times on the sample and then $\tilde{T} - T$ times on the population

$$\sum_{t=T}^{\tilde{T}} \mathcal{R}(d_t, \theta_t) - \inf_{\theta \in \Theta} \mathcal{R}(d_t, \theta)$$

with $d_t = \text{Tr}(d_{t-1}, \theta_t)$ for $t \geq T$, $\theta_T = \hat{\theta}_T$ and $d_{T-1} = d_0$? Note that we cannot disregard the retraining on shifted samples, because this affects the population via the initial performative effect of the deployed $\hat{\theta}_T$. While **RQ2** asks for the generalization error of a model on a population reacting to the model's predictions *once*, **RQ3** considers the accumulated generalization error on a population reacting to $(\tilde{T} - T$ times) repeatedly retrained models θ_t on correspondingly changing populations. Like **RQ2**, **RQ3** generalizes **RQ1** in the sense that **RQ3** is equivalent to **RQ1** when $\tilde{T} = 1$. We can assume $T = 1$ for **RQ3** (a), since there is no point in retraining on the sample because there are no performative effects on the sample, while T can be any positive integer in **RQ3** (b). Unlike **RQ2**, however, **RQ3** is only relevant if the whole population d_t is available to the institution.

(RQ4) The last research question is *statistical* in nature. It compares RERM $\{\hat{\theta}_t\}_{t=1}^T$ to a simultaneous (*hypothetical*) RRM $\{\theta_t\}_{t=1}^T$ on the population. Recall $\hat{\theta}_t \in G(\hat{d}_t)$ and $\theta_t \in G(d_t)$. Can we bound the *inferential gap* $\sum_{t=1}^T \mathcal{R}(d_t, \hat{\theta}_t) - \mathcal{R}(d_t, \theta_t)$ between the two? This question is partly answered by the central limit theorem (CLT) on statistical inference under *stateless* performativity by [Li et al. \(2025b\)](#), which states that, for any t , $\sqrt{n}(\hat{\theta}_t - \theta_t) \xrightarrow{D} N(0, \Sigma_t)$ as $n \rightarrow \infty$, where Σ_t is an identifiable covariance matrix. **RQ4** gives the standard CLT ([de Moivre, 1738](#)) for $t = 1$ as a special case. Obviously, this result implies that the inferential gap goes to zero as $n \rightarrow \infty$, under continuity of the risk. Finite sample analysis as well as the extension to the *stateful* case is yet to be conducted.

3 Performative Generalization Bounds

The previous section provides a conceptual understanding of what generalization means in a performative world. We now turn to a technical embedding of performativity into statistical learning theory ([Vapnik, 1991, 1998, 1999](#); [Vapnik & Chervonenkis, 1968](#)). We answer **RQ1–RQ3** under minimal assumptions on the nature of performative shifts. In fact, besides compact Θ , our bounds only require a subset of those in the original PP setup. Specifically, we do not assume any particular functional form of Tr like, e.g., [Kirev et al. \(2025\)](#); [Mendler-Dünner et al. \(2022\)](#). The price we pay for this generality is the fact that our bounds are looser than they would be otherwise (yet insightful, see Corollary 3.11). To address this, we will include more conditions to obtain tighter bounds later. To begin with, however, we only need the following subset of conditions in [Brown et al. \(2022\)](#); [Perdomo et al. \(2020\)](#):

Condition 3.1. The loss $\ell(z, \theta)$ is γ -strongly convex in θ .

Condition 3.2. The transition map Tr is (ε, p) -jointly sensitive for some $\varepsilon > 0$ and some $1 \leq p \leq 2$. That is, $W_p(\text{Tr}(d, \theta), \text{Tr}(d', \theta')) \leq \varepsilon W_p(d, d') + \varepsilon \|\theta - \theta'\|_2$, where W_p denotes the p -Wasserstein distance.

Condition 3.3. The loss function $\ell(z, \theta)$ and the hypothesis space \mathcal{F} are chosen such that $\ell(z, \theta)$ is κ -continuously differentiable with respect to parameters θ . That is, at every $z \in \mathcal{Z}$, the gradient $\nabla_\theta \ell(z, \theta)$ exists and is κ -Lipschitz continuous (with respect to the L_2 norm on domain and codomain) in each of θ and z .¹

Condition 3.1 implies that the loss is L_ℓ -Lipschitz on compact Θ by the mean value theorem. Together with Condition 3.3, this implies that the function G (Definition 2.1) is L_a -Lipschitz with $L_a > 0$ via the implicit function theorem, see Appendix C.1. Our strategy for proving generalization bounds is twofold. In order to generalize from \widehat{d}_T to d_0 (**RQ1**), we bound $W_p(\widehat{d}_0, d_0)$ (Lemma 3.4) and $W_p(\widehat{d}_0, \widehat{d}_T)$ (Lemma 3.5), before relating these divergence bounds to expectation difference bounds via the Kantorovich-Rubinstein Lemma (Kantorovich & Rubinstein, 1958).

Lemma 3.4 (Convergence in Wasserstein Space; Fournier & Guillin 2015). *Let $\mathcal{D}_{\mathcal{Z}} := \sup_{z, z'} \|z - z'\|_2 < \infty$. Then, for any $\beta_0 \in (0, \infty)$, we have $W_p(\widehat{d}_0, d_0) \leq \beta_0$, with probability at least $C_a \exp(-C_b n \beta_0^{\nu/p})$, where C_a and C_b are constants which depend on $p, \nu > 2p$, and $\mathcal{D}_{\mathcal{Z}}$ only.*

Lemma 3.5 (In-Sample Performative Shift Bound). *Assume that at most m units (in the sample of size n) change in response to predictions at each iteration t .² If the transition map is (ε, p) -jointly sensitive (Cond. 3.2), then*

$$W_p(\widehat{d}_0, \widehat{d}_T) \leq \frac{\varepsilon^T - 1}{\varepsilon - 1} \left(\frac{m}{n} \right)^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}} \tilde{L}_a,$$

pointwise in \widehat{d}_0 (i.e., for any fixed \widehat{d}_0) with $\tilde{L}_a = (1 + L_a)^{-1}$.

Lemma 3.4 is an application of Theorem 2, Case 1 in Fournier & Guillin (2015) with $\mathcal{E}_{\alpha, \gamma}(\mu) < \infty$ and $\alpha = \nu$. It implies \widehat{d}_0 is arbitrarily close to d_0 in Wasserstein space as $n \rightarrow \infty$ with high probability. Proofs of all our results, including Lemma 3.5, can be found in Appendix C.

RQ2–RQ4 involve performative shifts of the true law d_0 , which implies we might have to evaluate f_θ outside the support of d_0 . This is why we choose covering numbers (Kolmogorov & Tikhomirov, 1959; Talagrand, 2014) over popular Rademacher or Gaussian complexities (Bartlett & Mendelson, 2002) as a complexity measure to quantify the richness of the hypothesis class \mathcal{F} .

Definition 3.6 (Covering Number Entropy Integrals). Let $\|\cdot\|$ denote a norm on \mathcal{F} , such as the uniform norm $\|f\|_\infty = \sup_{x \in \mathcal{X}} \|f(x)\|_2$, or the L_2 norm $\|f\|_{L_2(d)} = \sqrt{\mathbb{E}_{X \sim d}[\|f(X)\|_2^2]}$ with respect to $d \in \Delta$. For $\epsilon > 0$, define the covering number $\mathcal{N}(\mathcal{F}, \|\cdot\|, \epsilon)$ as the minimal N such that there exists $\theta_1, \dots, \theta_N \in \Theta$ satisfying $\min_{1 \leq k \leq N} \|f_\theta - f_{\theta_k}\| \leq \epsilon$ for all $\theta \in \Theta$. The covering

¹This is also a non-vacuous condition on the hypothesis space \mathcal{F} in case of $\ell(z, \theta) = \ell(y, f_\theta(x))$, in particular, on the parameter mapping $\theta \rightarrow f_\theta$.

²Formally, denote by $m_t = n - |\text{supp}(\widehat{d}_t) \cap \text{supp}(\widehat{d}_{t-1})|$ the number of units changing in t with supp the (finite) support of some (empirical) \widehat{d}_t . Then, $m = \max\{m_1, \dots, m_T\}$.

number entropy integrals for the uniform and L_2 norm are

$$\begin{aligned}\mathfrak{C}_\infty(\mathcal{F}) &:= \int_0^\infty \sqrt{\log \mathcal{N}(\mathcal{F}, \|\cdot\|_\infty, \varepsilon)} d\varepsilon, \\ \mathfrak{C}_{L_2}(\mathcal{F}) &:= \sup_{d \in \Delta} \int_0^1 \sqrt{\log \mathcal{N}(\mathcal{F}, \|\cdot\|_{L_2(d)}, \varepsilon)} d\varepsilon.\end{aligned}$$

3.1 RQ1: Generalization Under Sample Performativity

We are now ready to answer **RQ1** with the following result.

Theorem 3.7 (Excess Risk Bound, **RQ1**). *The excess risk $\mathcal{R}(d_0, \hat{\theta}_T) - \inf_{\theta \in \Theta} \mathcal{R}(d_0, \theta)$ of a model $\hat{\theta}_T$ trained on performative samples $\hat{d}_0, \dots, \hat{d}_T$, in which at most $m \leq n$ units change in response to the predictions, is upper bounded by*

$$\begin{aligned}L_\ell \left(\frac{\log(C_a/\delta)}{C_b n} \right)^{p/\nu} + L_\ell L_a \frac{(\varepsilon\kappa/\gamma)^T - 1}{\varepsilon\kappa/\gamma - 1} \left(\frac{m}{n} \right)^{1/p} \mathcal{D}_{\mathcal{Z}} \\ + \frac{FL_\ell}{\sqrt{n}} \left(24\mathfrak{C}_{L_2}(\mathcal{F}) + 2\sqrt{2\ln(1/\delta)} \right),\end{aligned}$$

for any $T \in \mathbb{N}$, with probability (w.p.) over d_0 of at least $1 - \frac{\delta}{2}$, under Cond. 3.1–3.3.

Our setting's generality requires a generic proof strategy: We bound $\mathcal{R}(d, \hat{\theta}_T) - \mathcal{R}(\hat{d}_0, \hat{\theta}_T)$, $\mathcal{R}(\hat{d}_0, \hat{\theta}_T) - \mathcal{R}(\hat{d}_0, \hat{\theta}_0)$ and $\mathcal{R}(\hat{d}_0, \hat{\theta}_0) - \inf_{\theta \in \Theta} \mathcal{R}(d, \theta)$ with high probability over d_0 and then combine via the union bound (see Appendix C.2 for details and the complete proof). We can already make three interesting observations. First, the bound in Theorem 3.7 goes to zero as $n \rightarrow \infty$ if and only if either $m = o(n[\varepsilon\kappa/\gamma]^{-Tp})$ and $\varepsilon\kappa/\gamma \geq 1$; or $m = o(n)$ and $\varepsilon\kappa/\gamma < 1$. Second, this condition of $\varepsilon\kappa/\gamma < 1$ is weaker than what is required for convergence (“stability”) of PP (Brown et al., 2022, Theorem 4). Third, the bound generally grows in m , hinting at a fundamental trade-off between manipulating a sample and learning from it. For instance, if the job center in Example (B) wants to help more unemployed people (i.e., increase m) among their clients n , this comes at the cost of generalizing their model to new, unseen clients. Section 4 will dive further into this trade-off. Intuitively, this trade-off also holds beyond the setting of Theorem 3.7, because changing more units of an i.i.d. sample in an unknown way cannot guarantee improved generalization. Besides, our excess risk bound directly implies a data-dependent bound on the generalization error (see proof in Appendix C.3):

Corollary 3.8. *In the setting of Thm. 3.7, under Cond. 3.1–3.3,*

$$\mathcal{R}(d_0, \hat{\theta}_T) - \mathcal{R}(\hat{d}_{T-1}, \hat{\theta}_T) \leq L_\ell \left(\frac{\log(C_a/\delta)}{C_b n} \right)^{\frac{p}{\nu}} + \frac{(\varepsilon^{T-1} - 1)}{L_\ell^{-1}(\varepsilon - 1)} \left(\frac{m}{n} \right)^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}} \tilde{L}_a$$

w.p. over d_0 of at least $1 - \frac{\delta}{2}$.

3.2 RQ2: Generalization Under Full Performativity

In order to answer **RQ2**, we bound the excess risk of learning under full (i.e, sample and population) performativity. Truth is elusive now: The learning target is no longer static, but changes from d_0 to d_1, d_2, \dots in response to predictions. As a first step, we thus have to bound $W_p(d_0, d_T)$ in addition to $W_p(\hat{d}_0, d_0)$ (Lemma 3.4) and $W_p(\hat{d}_0, \hat{d}_T)$ (Lemma 3.5).

Lemma 3.9 (Performative Population Shift Bound). *Assume $s \in [0, 1]$ is the share of units in d_0 reacting to predictions (the “performative response rate”). Then $s < \frac{m}{n} + q(\delta) \sqrt{\frac{m}{n^2}(1 - \frac{m}{n})}$ w.p. $1 - \delta$, where $q(\delta)$ is the $(1 - \frac{1}{2}\delta)$ -quantile of the standard normal distribution.*

This lemma follows directly from Wald’s method (Brown et al., 2001) and treating the observed fraction $\frac{m}{n}$ as an estimator of an unobserved Bernoulli parameter s . We can now bound the excess risk under full performativity.

Theorem 3.10 (Performative Excess Risk Bound, **RQ2**). *Under Conditions 3.1–3.3, the performative excess risk $\mathcal{R}(\text{Tr}(d_0, \hat{\theta}_T), \hat{\theta}_T) - \inf_{\theta \in \Theta} \mathcal{R}(\text{Tr}(d_0, \hat{\theta}_T), \theta)$ of $\hat{\theta}_T$ is upper bounded by*

$$L_\ell \left[2A(m, n) + C(n) + \frac{2F}{\sqrt{n}} (12\mathfrak{C}_{L_2}(\mathcal{F}) + \sqrt{2 \ln(1/\delta)}) \right. \\ \left. + L_a \mathcal{D}_{\mathcal{Z}}(A(m, n) + K(T, m, n)) \right]$$

for any $T \in \mathbb{N}$ w.p. $1 - \frac{\delta}{4}$. $A(m, n)$, $C(n)$ and $K(T, m, n)$ depend only on constants as well as m , n and T .

The proof in Appendix C.5 provides explicit expressions for $A(m, n)$, $C(n)$ and $K(T, m, n)$, which show that the bounds in Theorems 3.7 and 3.10 grow at similar rates.

Crucially, these theorems point to two ways of improving generalization (guarantees) under performativity. The first one is straight-forward: I) Without knowledge of Tr , retraining on $\hat{d}_0, \dots, \hat{d}_T$ results in models that generalize worse than initial $\hat{\theta}_0$, because the excess risk bounds grow in T , both via $m = \max\{m_1, \dots, m_T\}$ (see footnote to Lemma 3.5) and via $\varepsilon\kappa/\gamma$ (if $\varepsilon\kappa/\gamma > 1$). The second one is less intuitive: II) Retraining on $\hat{d}_0, \dots, \hat{d}_T$ can still improve generalization bounds. While it worsens $\hat{\theta}_t$ ’s generalization capabilities, it allows for estimating Tr from observed $\hat{d}_1, \dots, \hat{d}_T$ more efficiently than Lemma 3.9. This tightens the bound in Theorem 3.10. Lemma 3.9 conservatively relies on m from that iteration, in which the most units react. But we have more information, as we observe the number m_t of units changing at each t (see footnote to Lemma 3.5).

Corollary 3.11 (Improving Bounds Under Performativity). **I**) $\hat{\theta}_0$ yields the tightest performative excess risk bound among $\{\hat{\theta}_t\}$ if $\varepsilon\kappa/\gamma \geq 1$. **II**) it holds w. p. $1 - \delta$ that $s < \frac{M_T}{Tn} + q(\delta) \sqrt{\frac{M_T}{(Tn)^2}(1 - \frac{M_T}{Tn})}$ with $M_T = \sum_{t=1}^T m_t$, giving an as-tight or tighter performative excess risk bound.

Corollary 3.11 constitutes a remarkably simple and readily applicable insight: If predictions exhibit performative effects on both the sample and the population, use I) the initial fit $\hat{\theta}_0$ on out-of-sample data and II) estimate the performative shift it will cause from the sample shifts you have observed, in order to obtain the tightest generalization guarantees.

These bounds, however, will still exhibit some slack due to their generality. This naturally begs the question whether we can prove tighter bounds under slightly stricter assumptions than Conditions 3.1–3.3. The following two results give affirmative answers. While Theorem 3.13 requires the strong Condition 3.12 of all functions in \mathcal{F} to be Lipschitz, Theorem 3.15 makes do with the substantially weaker Condition 3.14. Both Theorems bound the generalization gap, but a bound on the respective excess risk directly follows from the proofs of Theorems 3.7 and 3.10 (see Appendix C).

Condition 3.12 (Regularity of Models). All functions f_θ in \mathcal{F} are upper semi-continuous and L_f -Lipschitz.

Theorem 3.13 (Generalization Gap I, RQ2). *Under Conditions 3.1–3.3 and 3.12, the performativity generalization gap $\mathcal{R}(\text{Tr}(d_0, \hat{\theta}_T), \hat{\theta}_T) - \mathcal{R}(\text{Tr}(\hat{d}_0, \hat{\theta}_T), \hat{\theta}_T)$ is upper bounded w.p. $1 - \delta$ over d_0 by*

$$\frac{48}{\sqrt{n}} (\mathfrak{C}_\infty(\mathcal{F}) + L_\ell L_f R^{1-p} \mathcal{D}_{\mathcal{Z}}^p) + F \sqrt{\frac{2 \log(2/\delta)}{n}} + 2L_\ell R$$

with R the maximum of the bounds in Lemma 3.5 and 3.9.

Condition 3.14 (Weaker Regularity). There exist $f_0 \in \mathcal{F}$, $B \geq 0$, $\mathfrak{d} \in \mathbb{N}$ and $x_0 \in \mathcal{X}$, such that $\ell(f_0(x), y) \leq B \|x - x_0\|_2^\mathfrak{d}$ for all $x \in \mathcal{X}, y \in \mathcal{Y}$.

Theorem 3.15 (Generalization Gap II, RQ2). *Under Conditions 3.1–3.3 and 3.14, we obtain the same bound on $\mathcal{R}(\text{Tr}(d_0, \hat{\theta}_T), \hat{\theta}_T) - \mathcal{R}(\text{Tr}(\hat{d}_0, \hat{\theta}_T), \hat{\theta}_T)$ as in Theorem 3.13 but with $B 2^{p-1} (1 + \mathcal{D}_{\mathcal{Z}}/R)^p$ instead of $L_\ell L_f R^{1-p}$.*

The key technique in both proofs is to leverage empirical process theory for dual characterizations of locally inf-sup and inf-inf risk functionals in Wasserstein space (Gao & Kleywegt, 2023; Lee & Raginsky, 2018). Such functionals are commonly used in Distributionally Robust Optimization (DRO) (Shapiro et al., 2021) and distributionally favorable optimization (Jiang & Xie, 2025), respectively. In PP, they arise from the fact that we do not assume the transition map Tr to have a particular functional form, but we can nevertheless bound the shift induced by Tr . The worst-case generalization error of $\hat{\theta}_T$ then is $\mathcal{R}(\text{Tr}(d_0, \hat{\theta}_T), \hat{\theta}_T) = \sup_{d \in \mathcal{A}} \mathcal{R}(d, \hat{\theta}_T)$, where $\mathcal{A} = \{d : W_p(d_0, d) \leq b\}$ with b a bound on the shift induced by Tr as per Lemma 3.9. The conceptual difference to DRO is that the worst case generalization gap between $\mathcal{R}(\text{Tr}(d_0, \hat{\theta}_T), \hat{\theta}_T)$ and $\mathcal{R}(\text{Tr}(\hat{d}_T, \hat{\theta}_T), \hat{\theta}_T)$ under performativity does not only entail the worst case (supremum) risk on the population, but also the best case (infimum) risk on the sample (see Equation 78 in the proof of Theorem 3.13).

Structural insight: In this worst case, the sample “deceives” you by producing *self-fulfilling* performative effects, while the population acts in a *self-negating* way. In other words, RERM corresponds to fitting self-fulfilling models³ here, creating an empirical echo chamber, which results in models “far away” from good ones on the population, since the latter reacts in the opposite way. In Example (A) of routing apps, this means that drivers in San Francisco (the sample) fully trust the app and change their behavior to fulfill the predictions, while Bay Area drivers (the population) do the exact opposite and negate the predictions.

³That is, solving $\arg \inf_\theta \inf_{d \in \mathcal{A}'} \mathcal{R}(d, \theta)$ with $\mathcal{A}' = \{d : W_p(\hat{d}_0, d) \leq b' \wedge |\text{supp}(d)| < \infty\}$ and b' informed by Lemma 3.5.

3.3 Cumulative Performative Excess Risk Bound, RQ3

In addition to the error when generalizing from the sample to the performatively shifted population as in **RQ2** above, **RQ3** asks for the error when we retrain the model on this shifted population. In this scenario, the institution first trains a model T times on a sample before rolling it out to the population and retraining there $\tilde{T} - T$ times. Crucially, the performative effect of the (sample-trained) model $\hat{\theta}_T$ on the population happens *before* the the model is trained on the population for the first time, see Section 2. For details on (and the proof of) Theorem 3.16, we refer to Appendix C.9.

Theorem 3.16 (Cumulative Performative Excess Risk Bound). *Under Conditions 3.1–3.3, we have for any $\tilde{T} \geq T$*

$$\sum_{t=T}^{\tilde{T}} \left(\mathcal{R}(d_t, \theta_t) - \inf_{\theta \in \Theta} \mathcal{R}(d_t, \theta) \right) \leq \mathcal{B}(T, m, n) + \\ (\tilde{T} - T + 1) L_\ell L_a \left(\frac{m}{n} + q(\delta) \sqrt{\frac{\frac{m}{n}(1 - \frac{m}{n})}{n}} \right)^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}}.$$

w.p. $1 - \frac{\delta}{5}$, where $d_t = \text{Tr}(d_{t-1}, \theta_t)$ for $t \geq T$, $\theta_T = \hat{\theta}_T$ and $d_{T-1} = d_0$. $\mathcal{B}(T, m, n)$ is the performative excess risk bound from Theorem 3.10.

The idea of the proof (Appendix C.9) is to leverage the recursive argument that θ_t is the (population) risk minimizer under d_{t-1} , which determines d_t via unknown Tr.

4 Illustration of Bounds on Job Seeker Data

To get a feel for how our generalization bounds look like in practice, we illustrate them on administrative data from the German Federal Employment Agency. The dataset contains a 2% sample of German labor market records spanning 1975 to 2017, containing over 60 million rows in its raw form. We emphasize that the goal here is a mere *illustration* rather than an application of our bounds. The latter would require historical records of predictions that are usually unavailable (Mendler-Dünner et al., 2022). We consider binary prediction tasks: will a job seeker remain continuously unemployed for some period of time? Such predictions typically have performative effects on whether people actually remain unemployed, since only those with high predicted risks of staying unemployed receive job trainings that help them to actually find a job (see Junquera & Kern (2025) and Section 1.1). The administrative data set contains 28 features on labor market histories, education, skill level and demographic information. To illustrate our bounds, we use a Lipschitz continuous (Cond. 3.12) logistic regression $f_\theta^{\log}(x) = \sigma(\theta^\top x)$, where $\sigma(t) := (1 + e^{-t})^{-1}$ with continuously differentiable (Cond. 3.3) logistic loss $-(y \log f_\theta(x) + (1 - y) \log(1 - f_\theta(x)))$, to assign job trainings, triggering jointly sensitive (Cond. 3.2) performative shifts. We employ standard L_2 -regularization, which renders the loss strongly convex (Cond. 3.1). All details on data pre-processing, model training and bound computation can be found in Appendix B.⁴

⁴Code to replicate our case study is available at <https://anonymous.4open.science/r/plt-jobseekers>

4.1 Historical Data (Sample Performativity, RQ1):

We exemplarily choose a sampled cohort of $n = 60147$ German residents $\hat{d}_0 \stackrel{\text{i.i.d.}}{\sim} d_0$, where $t = 0$ corresponds to when they lost their job in 2012. Their job center assigns some of them to trainings based on a fitted $f_{\hat{\theta}_1}^{\log}$, performatively shifting \hat{d}_0 to $\hat{d}_1 = \text{Tr}(\hat{d}_0, \hat{\theta}_1)$, where $t = 1$ corresponds to 14 days after losing their job, where we observe that $m = 1816$ residents change at least one feature. Refitting on this \hat{d}_1 gives $f_{\hat{\theta}_2}^{\log}$, whose generalization gap $\mathcal{R}(d_0, \hat{\theta}_2) - \mathcal{R}(\hat{d}_1, \hat{\theta}_2)$ we can upper bound by $0.012 + 0.29 = 0.302$ with probability 0.95 per Corollary 3.8 of Theorem 3.7; the first summand is the sampling term (small due to large n), the second is driven by the observed performative response rate $\frac{m}{n} = \frac{1816}{60147}$. The bound tells the job center that the generalization error of their model will not exceed the training error by more than 0.302 nats (logistic loss units) with 95% confidence.

4.2 Semi-Simulation (Full Performativity, RQ2):

On another sampled cohort $\hat{d}_0 \stackrel{\text{i.i.d.}}{\sim} d_0$ of $n = 41585$ German residents that just lost their jobs in 2012, we set $t = 0$ to 60 days after losing their job, and again $t = 1$ to 14 days later. We simulate assignments to job trainings based on predicted risks of prolonged unemployment by fitted logistic regression $f_{\hat{\theta}_1}^{\log}$. We assign the ξn units with the highest predicted risks among the initial sample to job trainings with policies $\xi \in \{0.01, 0.02, \dots, 0.5\}$. For ease of exposition, we assume these trainings to be fully effective⁵, i.e., flip targets from `jobseeking` to `not-jobseeking` for all ξn units, and no further changes, i.e., $m = \xi n$. Retraining on so shifted samples $\hat{d}_{1,\xi}$ gives $f_{\hat{\theta}_{2,\xi}}^{\log}$. We then simulate the employment agency to roll out the risk-based assignments to all of Germany d_0 , again with varying ξ , triggering a population shift $d_{1,\xi} = \text{Tr}_\xi(d_0, \hat{\theta}_2)$. Theorem 3.13 allows bounding the performative generalization gap $\mathcal{R}(d_{1,\xi}, \hat{\theta}_{2,\xi}) - \mathcal{R}(\text{Tr}(\hat{d}_1, \hat{\theta}_{2,\xi}), \hat{\theta}_{2,\xi})$ as a function of the agency's policy ξ . Figure 1.3 shows the total bound that holds with probability of $1 - \delta = 0.95$ for varying ξ , along with the bound's components $\frac{48}{\sqrt{n}} (\mathfrak{C}_\infty(\mathcal{F}) L_\ell L_f R^{1-p} \mathcal{D}_Z^p)$ (adaptive complexity term), $2L_\ell R$ (performative term) and $F \sqrt{(2 \log(2/\delta))/n}$ (sampling term). The more people receive job trainings, the less reliable the model's predictions become due to performative shifts both in and out of the sample—an instantiation of the principled trade-off between learning and changing we found in Section 3.

5 Related Work, Limitations, Discussion and Outlook

Table 1 facilitates positioning ours and related work as well as an outlook to future work—all at once. For example, Table 1 hints at Li et al. (2025b) answering **RQ4** for $n \rightarrow \infty$, as detailed in Section 2. Thus, answering **RQ4** for finite n is a promising avenue for future work.

⁵Otherwise, our bounds would still be valid, since ξn is an upper bound for the number of changed units either way.

5.1 Related work

Besides those works already discussed in Section 2, *calibration* under performativity (Boeken et al., 2025; Li et al., 2025a; Zalouk et al., 2025) is conceptually related to *generalization* under performativity. They investigate the discrepancy between predicted and true *conditional probabilities* under performativity, whereas our work studies the discrepancy between empirical and true *risk* under performativity.

Zhang & Conitzer (2021) study generalization under strategic reactions to predictions, i.e., performative shifts of X . Kirev et al. (2025) study PAC learnability of binary classification under (linear) performative shifts of $Y \mid X$ and marginal X , which is a special case of **RQ2** (a), see Table 1. Our framework is more general than both works in two ways. First, it accounts for all Lipschitz-continuous transition maps as in Perdomo et al. (2020). Second, it entails performative shifts on any subsets of Y and X .

Mendler-Dünner et al. (2022) study the causal effects of predictions on outcomes Y . If valid causal estimates can be obtained from (typically) observational data, they can improve generalization, see also König et al. (2025). Akin to the trade-off between performatively changing a sample and learning from it, Wilder & Welle (2025) identify a trade-off between treating those in need and learning about population level quantities. Generally, causal inference (Pearl, 2009; Peters et al., 2017) can model performative effects through structural causal models. It typically requires strong identifiability assumptions. Our approach treats Tr as a black box, connecting to work on distribution shift (Gama et al., 2014; Quionero-Candela et al., 2009; Shimodaira, 2000) and domain adaptation (Muandet et al., 2013; Zhang et al., 2013). However, unlike passive distribution shift, performative prediction involves active shift where the learner’s model causes the change. Recent advances in learning under distribution shift via kernel mean embeddings (Muandet et al., 2017, 2021) and domain generalization (Blanchard et al., 2021; Muandet et al., 2013) share our distributional perspective, but our focus is on finite-sample bounds under performative feedback rather than adapting across fixed domains.

On a technical level, our analysis is related to distributionally robust performative optimization (Jia et al., 2025) and prediction (Xue & Sun, 2024). These works use Wasserstein ambiguity sets to robustify optimization and prediction, respectively, against performative distribution shifts. Besides studying generalization instead of prediction or optimization, our work differs in the way ambiguity sets are specified: we estimate them from the sample’s performative reactions instead of specifying them *a priori*. Also technically related, Rodemann et al. (2024) and Rodemann & Bailie (2025) study learning from samples that are adaptively shifted by models (see also Hazan et al., 2025). While relying on Wasserstein ambiguity sets just like Theorem 3.13 and 3.15, their work is conceptually different: the models there have full control over samples, while performative effects here have to be estimated. Another technical tangent to our analysis is the line of work on distributionally robust performative optimization (Jia et al., 2025) and prediction (Xue & Sun, 2024), see also Section 5. These works use ambiguity sets to robustify optimization and prediction, respectively, against performative distribution shifts. Besides studying generalization instead of prediction or optimization, our work differs in the way ambiguity sets are specified: we estimate them from the sample’s performative reactions instead of specifying them *a priori*.

Cutler et al. (2024) analyze stochastic approximation with decision-dependent sampling distributions, but do not study finite-sample generalization from samples to populations under performativity (like the article at hand). Bracale et al. (2025) develop a framework for micro-foundation inference in strategic prediction, focusing on recovering agents’ payoff structures from observed strategic responses rather than on learning-theoretic guarantees. Complementary to both, we derive non-asymptotic generalization and excess-risk bounds for empirical risk minimization when sample and population distributions evolve in response to predictions.

Our work differs from strategic classification (Dong et al., 2018; Hardt et al., 2016; Miller et al., 2020), which assumes agents best-respond to predictions with explicit manipulation costs, and from mechanism design (Nisan et al., 2007; Roughgarden, 2016), which assumes rational agents and equilibrium concepts. We instead provide generalization guarantees under minimal structural assumptions—requiring only Lipschitz continuity of the transition map—making our results applicable when the exact nature of performative effects (strategic, behavioral, or intervention-based) is unknown.

The RERM framework roughly relates to online learning (Cesa-Bianchi & Lugosi, 2006; Hazan, 2016), where models are updated sequentially as new data arrives. However, standard online learning assumes the data distribution is either fixed or evolves independently of the learner’s actions, whereas performative prediction creates endogenous non-stationarity. This connects our work to active learning (Balcan et al., 2010; Hanneke, 2007, 2014; Settles, 2009), where the learner adaptively selects which data to observe, and to recent work on learning with feedback loops (Chaney et al., 2018; Steinke & Zakythinou, 2020). Our bounds reveal that retraining under performativity differs fundamentally from classical non-stationary online learning (Besbes et al., 2015; Russac et al., 2019): the distribution shift is bounded but adversarial (in the sense of creating worst-case generalization gaps), and the learner partially controls this shift through deployment of the model (e.g., through the share ξ of units receiving a treatment, see Section 5). Recent advances in adaptive experimentation (Hadad et al., 2021; Zhan et al., 2021; Zhang et al., 2021) and sequential decision-making under distribution shift (Chewi et al., 2025; Lu et al., 2021) study related phenomena, but typically assume the ability to randomize interventions or maintain a control group, luxuries often unavailable in performative settings where models are deployed system-wide.

Our identification of a fundamental trade-off between changing data and learning from it relates to work on the value of predictions in decision-making (Kleinberg et al., 2018, 2015) and optimal resource allocation under uncertainty (Bastani, 2021). While these works study how prediction quality affects downstream decisions, we show that the act of deploying predictions can degrade future prediction quality through performative effects. This connects to the exploration-exploitation trade-off in bandit problems (Lattimore & Szepesvári, 2020; Rodemann & Augustin, 2024; Slivkins et al., 2019), where learning requires balancing information gathering with reward maximization. Similarly, recent work on intervention-aware machine learning (Zhang & Bareinboim, 2020) and prediction-informed resource allocation (Kallus, 2018) recognizes that predictive models can serve dual purposes: informing decisions and causing outcomes. Our bounds formalize the cost of this duality, showing that more aggressive interventions (larger m/n) improve outcomes but worsen generalization.

5.2 Limitations and General Discussion

This article revealed an explicit trade-off between changing the world and learning from it: The more a model affects data, the less is to be learned from it. We further showed that so-changed samples can still be useful in estimating the performative shifts out of sample under natural assumptions, giving rise to a *practical take-away*: If you retrain under performativity, combine the initial model and the information from all performatively shifted samples to get the tightest generalization bound, i.e., the best guarantee for out-of-sample application of your model under performativity. On a more principled level, our bounds show that generalizing in a performative world is hard for two reasons: Not only can the population invalidate your predictions (*self-negation*), the sample might also deceive you by doing the exact opposite and confirm your predictions (*self-fulfilling*), effectively creating an empirical echo chamber. Our bounds require Cond. 3.1–3.3 from the original PP framework (Brown et al., 2022; Perdomo et al., 2020). Cond. 3.1 (strongly convex loss) is arguably restrictive. For tighter bounds, we even need that \mathcal{F} contains at least one f_0 that is Lipschitz. However, we emphasize that these assumption refer to something (loss and model class) we have complete control over. In other words, they are *verifiable*. Their strength allows to abstain from specific assumptions on unknown Tr , a population quantity we have no control over and whose properties we can only estimate.

5.3 Outlook

This trade-off (restricting the models the analysis applies to vs. making assumptions about the unknown performative shift) leaves room for future work on the other side of this trade-off. E.g., adopting the setup in Mofakhami et al. (2023) would allow dropping the convexity condition on ℓ at the price of stronger assumptions on Tr .

Impact Statement

Performative predictions (per definition) have societal consequences. The overall research direction of performativity aims to understand and mitigate potential harms that arise when machine learning systems shape the very outcomes they predict. By developing theoretical foundations for generalization under performativity, our line of work contributes to more responsible deployment of predictive systems in high-stakes domains such as employment services, education, criminal justice, and finance.

Our work reveals a trade-off between performatively changing the world and learning from it, which has important implications for practitioners: systems that heavily influence their environment may simultaneously undermine their own ability to make accurate predictions about that environment. This insight could inform more cautious deployment strategies and highlight the need for careful monitoring when predictive systems are rolled out.

The specific generalization bounds we derive could help institutions better understand the limitations of their models under performative effects, potentially preventing overconfident deployment decisions that could harm vulnerable populations. For instance, our analysis of job training allocation (Section 4) demonstrates how performative effects complicate generalization in unemployment prediction systems. The latter constitutes a domain where prediction errors can have significant consequences for individuals' livelihoods.

Beyond this illustrative use case in Section 4, however, the paper at hand makes fundamental theoretical contributions and is very unlikely to have *direct* ethical or social consequences. The work is primarily mathematical in nature and does not introduce new applications. Any societal impact would be indirect, i.e., mediated through how practitioners and researchers use these theoretical insights to design and deploy performative prediction systems.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding this research.

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References

- Allhutter, D., Cech, F., Fischer, F., Grill, G., & Mager, A. (2020). Algorithmic profiling of job seekers in Austria: How austerity politics are made effective. *Frontiers in Big Data*, 3, 5.
- Bach, R. L., Kern, C., Mautner, H., & Kreuter, F. (2023). The impact of modeling decisions in statistical profiling. *Data & Policy*, 5, e32.
- Bagabaldo, A. R., Gan, Q., Bayen, A. M., & González, M. C. (2024). Impact of navigation apps on congestion and spread dynamics on a transportation network. *Data science for Transportation*, 6(2), 12.
- Balcan, M.-F., Hanneke, S., & Vaughan, J. W. (2010). The true sample complexity of active learning. *Machine learning*, 80(2), 111–139.
- Bartlett, P. L. & Mendelson, S. (2002). Rademacher and Gaussian complexities: Risk bounds and structural results. *Journal of Machine Learning Research*, 3(Nov), 463–482.
- Bastani, H. (2021). Predicting with proxies: Transfer learning in high dimension. *Management Science*, 67(5), 2964–2984.
- Benenati, E. & Grammatico, S. (2024). Probabilistic game-theoretic traffic routing. *IEEE Transactions on Intelligent Transportation Systems*, 25(10), 13080–13090.
- Besbes, O., Gur, Y., & Zeevi, A. (2015). Non-stationary stochastic optimization. *Operations Research*, 63(5), 1227–1244.
- Bian, H., Li, E., Liu, L., & Wang, Z. (2023). The influencer copycats. *PBCSF-NIFR Research Paper*.
- Blanchard, G., Deshmukh, A. A., Dogan, Ü., Lee, G., & Scott, C. (2021). Domain generalization by marginal transfer learning. *Journal of Machine Learning Research*, 22(2), 1–55.
- Boeken, P., Zoeter, O., & Mooij, J. M. (2025). Conditional forecasts and proper scoring rules for reliable and accurate performative predictions. *NeurIPS*.
- Bousquet, O. & Elisseeff, A. (2002). Stability and generalization. *The Journal of Machine Learning Research*, 2, 499–526.
- Bracale, D., Maity, S., et al. (2025). Microfoundation inference for strategic prediction. In Y. Li, S. Mandt, S. Agrawal, & E. Khan (Eds.), *Proceedings of The 28th International Conference on Artificial Intelligence and Statistics*, volume 258 of *Proceedings of Machine Learning Research* (pp. 919–927).: PMLR.
- Brown, G., Hod, S., & Kalemaj, I. (2022). Performative prediction in a stateful world. In *International conference on artificial intelligence and statistics* (pp. 6045–6061).: PMLR.
- Brown, L. D., Cai, T. T., & DasGupta, A. (2001). Interval estimation for a binomial proportion. *Statistical science*, 16(2), 101–133.
- Cabannes, T. (2019). *Capturing the impact of navigational app usage on road traffic from a game theory point of view*. PhD thesis, UC Berkeley.

- Cesa-Bianchi, N. & Lugosi, G. (2006). *Prediction, learning, and games*. Cambridge University Press.
- Chaney, A. J. B., Stewart, B. M., & Engelhardt, B. E. (2018). How algorithmic confounding in recommendation systems increases homogeneity and decreases utility. In *Proceedings of the 12th ACM Conference on Recommender Systems*, RecSys '18 (pp. 224–232). New York, NY, USA: Association for Computing Machinery.
- Chewi, S., Niles-Weed, J., & Rigollet, P. (2025). *Statistical Optimal Transport: École d'Été de Probabilités de Saint-Flour XLIX – 2019*, volume 2364 of *Lecture Notes in Mathematics*. Springer.
- Cutler, J., Díaz, M., & Drusvyatskiy, D. (2024). Stochastic approximation with decision-dependent distributions: Asymptotic normality and optimality. *Journal of Machine Learning Research*, 25, 90:1–90:49.
- de Moivre, A. (1738). *The Doctrine of Chances: Or, a Method of Calculating the Probabilities of Events in Play*. London: H. Woodfall, 2 edition.
- Dong, J., Roth, A., Schutzman, Z., Waggoner, B., & Wu, Z. S. (2018). Strategic classification from revealed preferences. In *Proceedings of the 2018 ACM Conference on Economics and Computation* (pp. 55–70).
- Dudley, R. (1987). Universal Donsker classes and metric entropy. *The Annals of Probability*, 15(4), 1306–1326.
- Ernst, S., Mueller, A. I., & Spinnewijn, J. (2024). Risk scores for long-term unemployment and the assignment to job search counseling. *AEA Papers and Proceedings*, 114, 572–576.
- Fischer-Abaigar, U., Kern, C., & Perdomo, J. C. (2025). The value of prediction in identifying the worst-off. In *Forty-second International Conference on Machine Learning*.
- Fournier, N. & Guillin, A. (2015). On the rate of convergence in Wasserstein distance of the empirical measure. *Probability theory and related fields*, 162(3), 707–738.
- Gama, J., Žliobaitė, I., Bifet, A., Pechenizkiy, M., & Bouchachia, A. (2014). A survey on concept drift adaptation. *ACM Computing Surveys*, 46(4), 1–37.
- Gao, R. & Kleywegt, A. (2023). Distributionally robust stochastic optimization with Wasserstein distance. *Mathematics of Operations Research*, 48(2), 603–655.
- Grunberg, E. & Modigliani, F. (1954). The predictability of social events. *Journal of Political Economy*, 62(6), 465–478.
- Hadad, V., Hirshberg, D. A., Zhan, R., Wager, S., & Athey, S. (2021). Confidence intervals for policy evaluation in adaptive experiments. *Proceedings of the National Academy of Sciences*, 118(15).
- Hanneke, S. (2007). A bound on the label complexity of agnostic active learning. In *Proceedings of the 24th international conference on Machine learning* (pp. 353–360).
- Hanneke, S. (2014). Theory of disagreement-based active learning. *Foundations and Trends in Machine Learning*, 7(2-3), 131–309.

- Hardt, M., Megiddo, N., Papadimitriou, C., & Wootters, M. (2016). Strategic classification. In *Proceedings of the 2016 ACM Conference on Innovations in Theoretical Computer Science* (pp. 111–122).
- Hardt, M. & Mendler-Dünner, C. (2025). Performative prediction: Past and future. *Statistical Science*, 40(3), 417–436.
- Hazan, E. (2016). Introduction to online convex optimization. *Foundations and Trends in Optimization*, 2(3-4), 157–325.
- Hazan, E., Shwartz, S. S., & Srebro, N. (2025). Research program: Theory of learning in dynamical systems. *arXiv preprint arXiv:2512.19410*.
- Jia, Z., Wang, Y., Dong, R., & Hanasusanto, G. A. (2025). Distributionally robust performative optimization. In *The Thirty-ninth Annual Conference on Neural Information Processing Systems*.
- Jiang, N. & Xie, W. (2025). On tractability, complexity, and mixed-integer convex programming representability of distributionally favorable optimization. *Mathematical Programming*, (pp. 1–38).
- Junquera, Á. F. & Kern, C. (2025). From rules to forests: Rule-based versus statistical models for jobseeker profiling. *Journal for Labour Market Research*, 59(1), 26.
- Kallus, N. (2018). Balanced policy evaluation and learning. *Advances in neural information processing systems*, 31.
- Kantorovich, L. V. & Rubinstein, S. (1958). On a space of totally additive functions. *Vestnik of the St. Petersburg University: Mathematics*, 13(7), 52–59.
- Kern, C., Bach, R. L., Mautner, H., & Kreuter, F. (2021). Fairness in algorithmic profiling: A German case study. *arXiv preprint arXiv:2108.04134*.
- Khandani, A. E., Kim, A. J., & Lo, A. W. (2010). Consumer credit-risk models via machine-learning algorithms. *Journal of Banking & Finance*, 34(11), 2767–2787.
- Kirev, I., Baltadzhiev, L., & Konstantinov, N. (2025). PAC learnability in the presence of performance. *arXiv preprint arXiv:2510.08335*.
- Kleinberg, J., Lakkaraju, H., Leskovec, J., Ludwig, J., & Mullainathan, S. (2018). Human decisions and machine predictions. *The quarterly journal of economics*, 133(1), 237–293.
- Kleinberg, J., Ludwig, J., Mullainathan, S., & Obermeyer, Z. (2015). Prediction policy problems. *American Economic Review*, 105(5), 491–495.
- Kolmogorov, A. N. & Tikhomirov, V. M. (1959). ε -entropy and ε -capacity of sets in function spaces. *Uspekhi Matematicheskikh Nauk*, 14(2), 3–86.
- König, G., Fokkema, H., Freiesleben, T., Mendler-Dünner, C., & von Luxburg, U. (2025). Performative validity of recourse explanations. In *The Thirty-ninth Annual Conference on Neural Information Processing Systems*.
- Lattimore, T. & Szepesvári, C. (2020). *Bandit algorithms*. Cambridge University Press.

- Lee, J. & Raginsky, M. (2018). Minimax statistical learning with Wasserstein distances. *Advances in Neural Information Processing Systems*, 31.
- Li, V., Chen, B., Mao, Y., Lei, Q., & Deng, Z. (2025a). Performative risk control: Calibrating models for reliable deployment under performativity. *NeurIPS*.
- Li, X., Li, Y., Zhong, H., Lei, L., & Deng, Z. (2025b). Statistical inference under performativity. In *The Thirty-ninth Annual Conference on Neural Information Processing Systems*.
- Liu, L. T., Raji, I. D., et al. (2025). Bridging prediction and intervention problems in social systems. *arXiv preprint arXiv:2507.05216*.
- Lu, Y., Meisami, A., Tewari, A., & Yan, W. (2021). Regret analysis of bandit problems with causal background knowledge. In *Conference on Uncertainty in Artificial Intelligence* (pp. 1351–1361).: PMLR.
- MacKenzie, D. (2008). *An engine, not a camera: How financial models shape markets*. MIT Press.
- McDiarmid, C. et al. (1989). On the method of bounded differences. *Surveys in combinatorics*, 141(1), 148–188.
- Mendler-Dünner, C., Ding, F., & Wang, Y. (2022). Anticipating performativity by predicting from predictions. *Advances in neural information processing systems*, 35, 31171–31185.
- Miller, J., Milli, S., & Hardt, M. (2020). Strategic classification is causal modeling in disguise. In *International Conference on Machine Learning* (pp. 6917–6926).: PMLR.
- Miller, J. P., Perdomo, J. C., & Zrnic, T. (2021). Outside the echo chamber: Optimizing the performative risk. In *International Conference on Machine Learning* (pp. 7710–7720).: PMLR.
- Mofakhami, M., Mitliagkas, I., & Gidel, G. (2023). Performative prediction with neural networks. In *International Conference on Artificial Intelligence and Statistics* (pp. 11079–11093).: PMLR.
- Morgenstern, O. (1928). Wirtschaftsprägnose: Eine Untersuchung ihrer Voraussetzungen und Möglichkeiten, Wien 1928, cited after: G. betz (2004), Empirische und aprioristische Grenzen von Wirtschaftsprägnosen: Oskar Morgenstern nach 70 Jahren. *Wissenschaftstheorie in Ökonomie und Wirtschaftsinformatik*, Deutscher Universitäts-Verlag, Wiesbaden, (pp. 171–190).
- Muandet, K., Balduzzi, D., & Schölkopf, B. (2013). Domain generalization via invariant feature representation. In *International Conference on Machine Learning* (pp. 10–18).
- Muandet, K., Fukumizu, K., Sriperumbudur, B., & Schölkopf, B. (2017). Kernel mean embedding of distributions: A review and beyond. *Foundations and Trends in Machine Learning*, 10(1-2), 1–141.
- Muandet, K., Kanagawa, M., Saengkyongam, S., & Marukata, S. (2021). Counterfactual mean embeddings. In *International Conference on Machine Learning* (pp. 7107–7117).: PMLR.
- Nagler, T. (2024). *Statistical Learning Theory – Lecture Notes*. Technical report, LMU Munich.
- Neurath, O. (1911). *Nationalökonomie und Wertlehre: eine systematische Untersuchung*, volume 20.

- Nisan, N., Roughgarden, T., Tardos, E., & Vazirani, V. V. (2007). *Algorithmic game theory*. Cambridge University Press.
- Pearl, J. (2009). *Causality*. Cambridge University Press.
- Perdomo, J. (2023). *Performative Prediction: Theory and Practice*. PhD thesis, UC Berkeley.
- Perdomo, J., Zrnic, T., Mendler-Dünner, C., & Hardt, M. (2020). Performative prediction. In *International Conference on Machine Learning* (pp. 7599–7609).: PMLR.
- Perdomo, J. C. (2024). The relative value of prediction in algorithmic decision making. In *Proceedings of the 41st International Conference on Machine Learning* (pp. 40439–40460).
- Perdomo, J. C. (2025). Revisiting the predictability of performative, social events. *arXiv preprint arXiv:2503.11713*.
- Perdomo, J. C., Britton, T., Hardt, M., & Abebe, R. (2025). Difficult lessons on social prediction from Wisconsin public schools. In *Proceedings of the 2025 ACM Conference on Fairness, Accountability, and Transparency* (pp. 2682–2704).
- Peters, J., Janzing, D., & Schölkopf, B. (2017). *Elements of causal inference: foundations and learning algorithms*. MIT Press.
- Quionero-Candela, J., Sugiyama, M., Schwaighofer, A., & Lawrence, N. D. (2009). *Dataset shift in machine learning*. MIT Press.
- Rodemann, J. & Augustin, T. (2024). Imprecise bayesian optimization. *Knowledge-Based Systems*, 300, 112186.
- Rodemann, J. & Bailie, J. (2025). Generalization bounds and stopping rules for learning with self-selected data. *arXiv preprint arXiv:2505.07367*.
- Rodemann, J., Jansen, C., & Schollmeyer, G. (2024). Reciprocal learning. *Advances in Neural Information Processing Systems*, 37.
- Ross, J. & Nyström, D. W. (2018). Differentiability of the argmin function and a minimum principle for semiconcave subsolutions. *arXiv preprint arXiv:1808.04402*.
- Roughgarden, T. (2016). *Twenty lectures on algorithmic game theory*. Cambridge University Press.
- Russac, Y., Vernade, C., & Cappe, O. (2019). Weighted linear bandits for non-stationary environments. In *Advances in Neural Information Processing Systems*, volume 32.
- Schmucker, A. & vom Berge, P. (2023a). Factually anonymous version of the sample of integrated labour market biographies (SIAB-Regional File) – version 7521 v1. Research Data Centre of the Federal Employment Agency (BA) at the Institute for Employment Research (IAB). Research data.
- Schmucker, A. & vom Berge, P. (2023b). Sample of integrated labour market biographies regional file (siab-r) 1975–2021. FDZ-Datenreport. FDZ-Datenreport, 07/2023 (en), Nürnberg.
- Settles, B. (2009). Active learning literature survey. *University of Wisconsin-Madison Department of Computer Sciences*.

- Shalev-Shwartz, S. & Ben-David, S. (2014a). Part I: Foundations. *Understanding Machine Learning: From Theory to Algorithms*, (pp. 11–86).
- Shalev-Shwartz, S. & Ben-David, S. (2014b). *Understanding machine learning: From theory to algorithms*. Cambridge university press.
- Shapiro, A., Dentcheva, D., & Ruszczynski, A. (2021). *Lectures on stochastic programming: modeling and theory*. SIAM.
- Shimodaira, H. (2000). Improving predictive inference under covariate shift by weighting the log-likelihood function. *Journal of Statistical Planning and Inference*, 90(2), 227–244.
- Slivkins, A. et al. (2019). Introduction to multi-armed bandits. *Foundations and Trends® in Machine Learning*, 12(1-2), 1–286.
- Soros, G. (1994). *The Alchemy of Finance: Reading the Mind of the Market*. John Wiley & Sons.
- Steinke, T. & Zakynthinou, L. (2020). Reasoning about generalization via conditional mutual information. In *Conference on Learning Theory* (pp. 3437–3452).: PMLR.
- Talagrand, M. (1995). Concentration of measure and isoperimetric inequalities in product spaces. *Publications Mathématiques de l'Institut des Hautes Etudes Scientifiques*, 81, 73–205.
- Talagrand, M. (2014). *Upper and lower bounds for stochastic processes*, volume 60. Springer.
- Vapnik, V. (1991). Principles of risk minimization for learning theory. *Advances in neural information processing systems*, 4.
- Vapnik, V. (1998). Statistical learning theory. *John Wiley & Sons*, 2, 831–842.
- Vapnik, V. (1999). *The Nature of Statistical Learning Theory*. Springer Science & Business Media.
- Vapnik, V. N. & Chervonenkis, A. Y. (1968). The uniform convergence of frequencies of the appearance of events to their probabilities. *Doklady Akademii Nauk*, 181(4), 781–783.
- Vo, K. Q., Aadil, M., Chau, S. L., & Muandet, K. (2024). Causal strategic learning with competitive selection. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 38 (pp. 15411–15419).
- Von Luxburg, U. & Schölkopf, B. (2011). Statistical learning theory: Models, concepts, and results. In *Handbook of the History of Logic*, volume 10 (pp. 651–706). Elsevier.
- Wilder, B. & Welle, P. (2025). Learning treatment effects while treating those in need. In *Proceedings of the 26th ACM Conference on Economics and Computation* (pp. 448–473).
- Xue, S. & Sun, Y. (2024). Distributionally robust performative prediction. In *The Thirty-eighth Annual Conference on Neural Information Processing Systems*.
- Zalouk, S., Marx, C., Belakaria, S., De Sa, C., & Ermon, S. (2025). Multivariate calibration is performative: A perspective on pitfalls and progress. In *1st ICML Workshop on Foundation Models for Structured Data*.

- Zhan, R., Hadad, V., Hirshberg, D. A., & Athey, S. (2021). Off-policy evaluation via adaptive weighting with data from contextual bandits. In *Proceedings of the 27th ACM SIGKDD Conference on Knowledge Discovery & Data Mining* (pp. 2125–2135).
- Zhang, H. & Conitzer, V. (2021). Incentive-aware PAC learning. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 35 (pp. 5797–5804).
- Zhang, J. & Bareinboim, E. (2020). Designing optimal dynamic treatment regimes: A causal reinforcement learning approach. In *International Conference on Machine Learning* (pp. 11012–11022).: PMLR.
- Zhang, K., Janson, L., & Murphy, S. (2021). Statistical inference with m-estimators on adaptively collected data. *Advances in neural information processing systems*, 34, 7460–7471.
- Zhang, K., Schölkopf, B., Muandet, K., & Wang, Z. (2013). Domain adaptation under target and conditional shift. In *International Conference on Machine Learning* (pp. 819–827).

A Generalization Bounds on Stable and Optimal Models

Under appropriate regularity conditions (strong convexity and smoothness of the loss function, as well as Lipschitz continuity of the distribution mapping), the sequence of RRM converges to a *stable pair* (θ_S, d_S) where $d_S = \text{Tr}(\theta_S, d_S)$ is a fixed-point distribution for θ_S and $\theta_S \in G(d_S)$. Brown et al. (2022, Theorem 6) further show that θ_S is close to the optimal θ_{OPT} , which is defined as the parameter θ that achieves the minimal long-run loss $\mathbb{E}_{Z \sim d_\theta} \ell(Z, \theta)$ on its fixed-point distribution d_θ resulting from repeatedly applying Tr to θ (d_θ is unique—i.e., it does not depend on the starting distribution d_0 —due to the Banach fixed-point theorem). In other words, R(E)RM under performativity might retrieve stable models $\widehat{\theta}_S$ (stable on sample) and θ_S (stable on population) or even optimal ones $\widehat{\theta}_{\text{OPT}}$ and θ_{OPT} under conditions detailed in Brown et al. (2022); Perdomo et al. (2020).

These models are subsumed by our analysis. They constitute special cases of $\widehat{\theta}_t$ and θ_t , respectively. For instance, the performative excess risks of stable and optimal models $\widehat{\theta}_S$ and $\widehat{\theta}_{\text{OPT}}$ are

$$\mathcal{R}(\text{Tr}(d_0, \widehat{\theta}_S), \widehat{\theta}_S) - \inf_{\theta \in \Theta} \mathcal{R}(\text{Tr}(d_0, \widehat{\theta}_S), \theta) \quad (1)$$

and

$$\mathcal{R}(\text{Tr}(d_0, \widehat{\theta}_{\text{OPT}}), \widehat{\theta}_{\text{OPT}}) - \inf_{\theta \in \Theta} \mathcal{R}(\text{Tr}(d_0, \widehat{\theta}_{\text{OPT}}), \theta), \quad (2)$$

respectively.

B Details on Case Study

B.1 Data Source, Pre-Processing and General Setup

We make use of a real dataset on German jobseekers. The data contains information on jobseeker demographics, employment history, and benefits receipt—a rich panel spanning a 2% sample of all administrative labor market records in Germany. Data access was provided via a Scientific Use File from the Research Data Centre (FDZ) of the German Federal Employment Agency (BA) at the Institute for Employment Research (IAB) (Schmucker & vom Berge, 2023a).

Reproducibility note: The data are not publicly available due to their sensitive nature. Access can be requested through the Research Data Centre (FDZ) of the Institute for Employment Research (IAB).⁶ We provide all preprocessing and analysis code to enable replication for authorized users. For further details on the sampling procedure, feature construction, data sources, and weak anonymization procedures, see the official data report (Schmucker & vom Berge, 2023b).

We construct a prediction task of unemployment duration from this data source, following (Kern et al., 2021). We follow the spell aggregation procedures and feature construction described in Bach et al. (2023); Fischer-Abaigar et al. (2025). We construct covariates capturing demographics (e.g., gender and educational background), information on the last job prior to unemployment, and broader labor market and benefit histories. Some features are frozen at the start of the unemployment spell. Features that evolve mechanically over time (e.g., age and elapsed unemployment duration) are discretized into bins to obtain a more stable jobseeker cohort and to facilitate the analysis of our bounds. In total, we use 8 ordinal and 20 categorical features. Categorical variables are one-hot encoded, and ordinal variables are mapped to the unit interval $[0, 1]$.

We train a simple logistic regression model in scikit-learn with L2 regularization (regularization strength $C = 1$). Predictive performance is approximately 0.60–0.65 AUC on test sets. Low predictive accuracy is broadly consistent with results in the related literature. Importantly, our objective was not to build the best possible predictor and feature set, but to mimic the modeling choices and data constraints typically faced in employment offices.

B.1.1 Setup for Generalization Gap Bound on Historical Jobseeker Data (See B.2)

For our analysis of the generalization gap bound computed on historical data (Appendix B.2), we predict over a relatively short time span of 14 days to analyze a fairly stable jobseeker cohort, mimicking an employment office that would continuously retrain and re-predict unemployment outcomes. The interval between model retraining and re-evaluation is also set to 14 days. We further applied a Principal Component Analysis (PCA), reducing the feature set for 28 features to four main features. We focus on a cohort of jobseekers who entered unemployment during 2012—selected as a year after major labor market reforms in Germany and before the COVID pandemic. This yields a cohort of roughly 50,000–60,000 individual jobseekers, which we further split into training and test sets.

⁶<https://iab.de/en/unit/?id=17>

B.1.2 Setup for Bound on Semi-Simulated Data (See B.3)

For the semi-simulation study (Appendix B.3), we focus on individuals 60 days after they entered unemployment to avoid large instabilities at the beginning of spells, as many jobseekers find employment very quickly or spend only a short time unemployed. Here, we do not conduct a PCA and work with the full set of 28 features (see below). We set $t = 0$ at the start of the unemployment spell but pre-filter the cohort to individuals who remain unemployed for at least 7 days.

B.2 Details on the Generalization Gap Bound on Historical Jobseeker Data

As discussed in Section 4, we are using the popular logistic loss $\ell(y, x, \theta) = \log(1 + \exp(-y\langle\theta, x\rangle))$, which is κ -continuously differentiable as per Condition 3.3. Its gradient with respect to θ is given by $\nabla_\theta \ell(y, x, \theta) = -y \sigma(-y\langle\theta, x\rangle)x$, where $\sigma(u) = \frac{1}{1+e^{-u}}$. Using $\|\theta\| \leq \mathcal{D}_\Theta < \infty$ for all $\theta \in \Theta$ and $\|x\| \leq \mathcal{D}_Z < \infty$ for all $x \in Z$, so that $|y\langle\theta, x\rangle| \leq \mathcal{D}_Z \mathcal{D}_\Theta$, $\nabla_\theta \ell(y, x, \theta) = -y \sigma(-y\langle\theta, x\rangle)x$ is upper bounded by

$$L_\ell = \frac{\mathcal{D}_Z}{1 + e^{-\mathcal{D}_Z \mathcal{D}_\Theta}},$$

which proves the loss is L_ℓ -Lipschitz in θ .

We apply Principal Component Analysis (PCA) to the feature set, resulting in four features, scaled into $[0, 1]$, which (together with $\mathcal{Y} = \{0, 1\}$) gives $\mathcal{D}_Z = \sqrt{5}$. Moreover, we set $\Theta = [-1000, 1000]^5$, which has a diameter of $\mathcal{D}_\Theta = 2000\sqrt{5}$. Plugging this into $L_\ell = \frac{\mathcal{D}_Z}{1 + e^{-\mathcal{D}_Z \mathcal{D}_\Theta}}$ gives

$$L_\ell = \frac{\sqrt{5}}{1 + e^{-\sqrt{5} \cdot 2000\sqrt{5}}} = \frac{\sqrt{5}}{1 + e^{-10000}} \approx \sqrt{5} \approx 2.236.$$

We further have (per PCA reduction) $\nu = 4$. We use Wasserstein order $p = 2$ and take $C_a = C_b = 1$ (Fournier & Guillin, 2015). As explained in Section 4 and B.1, we employ L2-regularization with regularization strength $C = 1$, rendering the loss γ -strongly convex with $\gamma = \frac{1}{C}$, i.e.,

$$\mathcal{R}(d, \theta) = \mathbb{E}_d[\ell(\theta; z)] + \frac{\lambda}{2}\|\theta\|^2.$$

A standard sensitivity bound for strongly convex objectives gives

$$\|G(d) - G(d')\| \leq \frac{1}{\gamma} \sup_{\theta} \|\nabla_\theta \mathcal{R}(d, \theta) - \nabla_\theta \mathcal{R}(d', \theta)\|,$$

Moreover, for logistic regression, the per-example gradient is bounded by the feature norm:

$$\|\nabla_\theta \ell(\theta; (x, y))\| \leq \|x\| \leq D_X = \sqrt{4} = 2,$$

so a conservative Lipschitz constant of G is

$$L_a \lesssim \frac{D_X}{\gamma} = \frac{2}{1} = 2,$$

which implies

$$\tilde{L}_a = \frac{1}{1 + L_a} \approx \frac{1}{3}.$$

Now recall Corollary 3.8, which states that the setting of Thm. 3.7, under Cond. 3.1–3.3, we have

$$\mathcal{R}(d_0, \hat{\theta}_T) - \mathcal{R}(\hat{d}_{T-1}, \hat{\theta}_T) \leq L_\ell \left(\frac{\log(C_a/\delta)}{C_b n} \right)^{\frac{p}{\nu}} + \frac{(\varepsilon^{T-1} - 1)}{L_\ell^{-1}(\varepsilon - 1)} \left(\frac{m}{n} \right)^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}} \tilde{L}_a$$

w.p. over d_0 of at least $1 - \frac{\delta}{2}$.

The first summand is the sampling term. In our setup, using $n = 60147$, $m = 1816$, $L_\ell = 2.236$, $\mathcal{D}_{\mathcal{Z}} = \sqrt{5}$, and $\tilde{L}_a = \frac{1}{3}$, with $p = 2$, $\nu = 4$, $C_a = C_b = 1$, $\delta = 0.1$, and $T = 2$, the sampling term becomes

$$L_\ell \left(\frac{\log(1/\delta)}{n} \right)^{1/2} = 2.236 \sqrt{\frac{\log 10}{60147}} \approx 0.0123746.$$

Further using $\varepsilon = m/n$ and $L_\ell^{-1} = 1/2.236$ as well as $\mathcal{D}_{\mathcal{Z}} = \sqrt{5}$, the second summand simplifies for $T = 2$ to

$$\frac{(\varepsilon^{T-1} - 1)}{L_\ell^{-1}(\varepsilon - 1)} \left(\frac{m}{n} \right)^{1/p} \mathcal{D}_{\mathcal{Z}} \tilde{L}_a = L_\ell \left(\frac{m}{n} \right)^{1/2} \mathcal{D}_{\mathcal{Z}} \tilde{L}_a = \frac{5}{3} \sqrt{\frac{m}{n}}.$$

Plugging in $m = 1816$ and $n = 60147$,

$$\frac{5}{3} \sqrt{\frac{1816}{60147}} \approx 0.290207.$$

Therefore, with probability at least $1 - \delta/2 = 0.95$,

$$\mathcal{R}(d_0, \hat{\theta}_2) - \mathcal{R}(\hat{d}_1, \hat{\theta}_2) \leq 0.0123746 + 0.290207 = 0.302582.$$

B.3 Details on the Semi-Simulation Study on Jobseeker Data

As detailed in Section 4, we train a binary logistic regression model using, just like detailed in Appendix B.2, the classic logistic loss $\ell(y, x, \theta) = \log(1 + \exp(-y\langle\theta, x\rangle))$, which is κ -continuously differentiable as per Condition 3.3. Its gradient with respect to θ is given by $\nabla_{\theta}\ell(y, x, \theta) = -y\sigma(-y\langle\theta, x\rangle)x$, where $\sigma(u) = \frac{1}{1+e^{-u}}$. Using $\|\theta\| \leq \mathcal{D}_{\Theta} < \infty$ for all $\theta \in \Theta$ and $\|x\| \leq \mathcal{D}_{\mathcal{Z}} < \infty$ for all $x \in \mathcal{Z}$, so that $|y\langle\theta, x\rangle| \leq \mathcal{D}_{\mathcal{Z}}\mathcal{D}_{\Theta}$, $\nabla_{\theta}\ell(y, x, \theta) = -y\sigma(-y\langle\theta, x\rangle)x$ is upper bounded by $L_\ell = \frac{\mathcal{D}_{\mathcal{Z}}}{1+e^{-\mathcal{D}_{\mathcal{Z}}\mathcal{D}_{\Theta}}}$, which proves the loss is L_ℓ -Lipschitz in θ . Again, we use $p = 2$ -Wasserstein distance.

Different to Appendix B.2, we do not use PCA to reduce feature set dimension. Instead, we work with the full set of 28 features as described in Section 4 and Appendix B.1. After normalizing (see Appendix B.1), this yields:

$$\begin{aligned}\mathcal{D}_{\mathcal{Z}} &= \sqrt{(1-0)^2 + (1-0)^2 + \cdots + (1-0)^2} \\ &= \sqrt{\underbrace{1+1+\cdots+1}_{28 \text{ times}}} \\ &= \sqrt{28} \\ &= 2\sqrt{7}\end{aligned}$$

So the diameter is $\mathcal{D}_{\mathcal{Z}} = \sqrt{28} = 2\sqrt{7} \approx 5.29$.

and store predicted probabilities on the train and test sets in CSV files: `train_predictions.csv` and `test_predictions.csv`. Each file contains two columns: the true label $y_i \in \{0, 1\}$ and the predicted probability $\hat{p}_i \in (0, 1)$.

We use confidence level $\delta = 0.05$ and yielding a normal quantile of

$$q(\delta) = \Phi^{-1}(1 - \delta/2) \approx 1.96. \quad (3)$$

Furthermore, we use a normalized, compact parameter set $\Theta = \{\theta : \|\theta\|_2 \leq 1\}$. For logistic regression predictions $f_{\theta}(x) = \sigma(\theta^\top x)$, since $\sigma'(t) \leq 1/4$,

$$\|\nabla_x f_{\theta}(x)\|_2 \leq \frac{1}{4} \|\theta\|_2 \leq \frac{1}{4}, \quad (4)$$

we get

$$L_f = \frac{1}{4}. \quad (5)$$

As in the case of historical data (see Appendix B.2), the loss Lipschitz constant is $\approx \mathcal{D}_{\mathcal{Z}}$. That is,

$$L_{\ell} \approx \mathcal{D}_{\mathcal{Z}} = \sqrt{28}, \quad (6)$$

and the uniform bound for the prediction range to be

$$F = 1, \quad (7)$$

since $f_{\theta}(x) \in [0, 1]$.

We further use the integral complexity term (as defined in Section 3) denoted by $C_{\infty}(\mathcal{F})$. For our logistic regression class with compact Θ and the chosen normalization, we use the numerical value

$$C_{\infty}(\mathcal{F}) \approx 7.3855, \quad (8)$$

which results from upper bounding $\log N(\mathcal{F}, \|\cdot\|_{\infty}, \varepsilon)$ using a Lipschitz-in-parameter covering argument and integrate $C_{\infty}(\mathcal{F}) = \int_0^1 \sqrt{\log N(\mathcal{F}, \|\cdot\|_{\infty}, \varepsilon)} d\varepsilon$.

Theorem 3.13 and 3.15 define the auxiliary radius

$$R(\xi) = \max \left\{ \left(q(\delta) \sqrt{\frac{\xi(1-\xi)}{n}} \right)^{1/p} D_Z, \xi^{1/p} D_Z \right\}, \quad p = 2. \quad (9)$$

with $\xi \in \{0.01, \dots, 0.5\}$ from Section 4.

We compute two bounds, from Theorem 3.13 as well as from 3.15

Theorem 3.13 bounds the excess risk under assumption 3.12 (which is fulfilled by logistic regression, see Section 4) by

$$\frac{48}{\sqrt{n}} \left(C_\infty(\mathcal{F}) + L_\ell L_f R(r)^{1-p} D_Z^p \right) + F \sqrt{\frac{2 \log(2/\delta)}{n}} + 2L_\ell R(\xi), \quad (10)$$

We decompose the total bound its three summands:

$$\textbf{Complexity term: } \text{Comp}(r) = \frac{48}{\sqrt{n}} \left(C_\infty(\mathcal{F}) + L_\ell L_f / R(\xi) \right), \quad (11)$$

$$\textbf{Sampling term: } \text{Samp}(\xi) = F \sqrt{\frac{2 \log(2/\delta)}{n}}, \quad (12)$$

$$\textbf{Performative term: } \text{Perf}(\xi) = 2L_\ell R(r), \quad (13)$$

$$\textbf{Total bound: } \text{Comp}(r) + \text{Samp}(\xi) + \text{Perf}(\xi). \quad (14)$$

Code to compute those terms for datasets with varying ξ and plot them can be found in <https://anonymous.4open.science/r/plt-jobseekers/README.md>

Theorem 3.15 is variant that replaces the $L_\ell L_f R^{1-p} D_Z^p$ term by a Condition-3.13 constant B :

$$\frac{48}{\sqrt{n}} \left(C_\infty(\mathcal{F}) + B 2^{p-1} \left(1 + \frac{D_Z}{R(r)} \right)^p \right) + F \sqrt{\frac{2 \log(2/\delta)}{n}} + 2L_\ell R(\xi) \quad (15)$$

we compute this bound exemplarily for varying ξ and $B = 10^{-3}$

We decompose this bound analogously into

$$\text{Comp}_B(r) + \text{Samp}(r) + \text{Perf}(\xi), \quad (16)$$

changing only

$$\text{Comp}_B(\xi) = \frac{48}{\sqrt{n}} \left(C_\infty(\mathcal{F}) + B \cdot 2 \cdot (1 + 1/R(\xi))^2 \right), \quad (17)$$

For each ξ , we plot the three components and the total bound. The x-axis is the sweep parameter $\xi = m/n$ (share of jobseekers receiving job trainings). The y-axis is the generalization-gap bound (and its components), all in logistic-loss units. Code to compute those terms for datasets with varying ξ and plot them can be found in <https://anonymous.4open.science/r/plt-jobseekers/README.md>

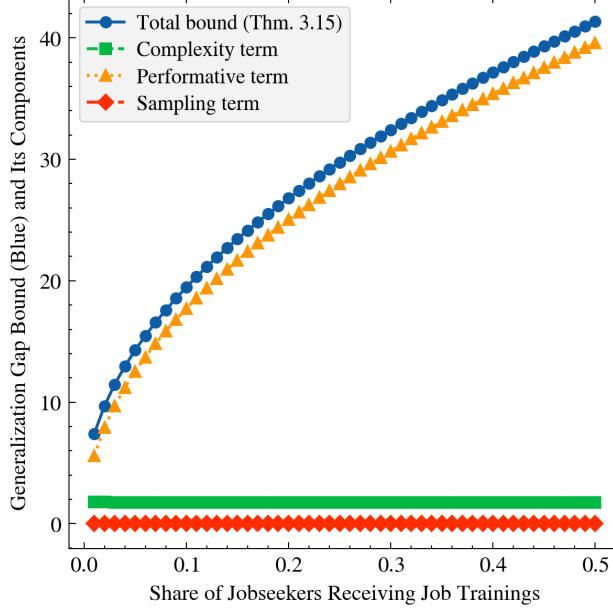


Figure 2: Generalization gap bound from Theorem 3.15 and its decomposition as a function of ξ (fraction of jobseekers receiving training). The total bound (blue) is decomposed into complexity term $\text{Comp}_B(\xi)$, sampling term $\text{Samp}(\xi)$, and performative term $\text{Perf}(\xi)$ (see details in Appendix B). All values are in logistic loss units. Parameters: n training samples, $B = 10^{-3}$, $\delta = 0.05$, $p = 2$.

B.4 Further Results from the Semi-Simulation Study on Jobseeker Data

While Figure 1 showing the bound from Theorem 3.13 for different $\xi = \frac{m}{n}$ is in the main paper, we include visualization of the how the generalization gap bound from Theorem 3.15 behaves for varying ξ in what follows.

Namely, Figure 2 visualizes the bound from Theorem 3.15:

$$\frac{48}{\sqrt{n}} \left(C_\infty(\mathcal{F}) + B 2^{p-1} \left(1 + \frac{D_Z}{R(\xi)} \right)^p \right) + F \sqrt{\frac{2 \log(2/\delta)}{n}} + 2L_\ell R(\xi) \quad (18)$$

we compute this bound exemplarily for varying ξ and $B = 10^{-3}$ as detailed above.

We decompose this bound analogously into

$$\text{Comp}_B(r) + \text{Samp}(r) + \text{Perf}(\xi), \quad (19)$$

changing only

$$\text{Comp}_B(\xi) = \frac{48}{\sqrt{n}} \left(C_\infty(\mathcal{F}) + B \cdot 2 \cdot (1 + 1/R(\xi))^2 \right), \quad (20)$$

For each ξ (share of Jobseekers receiving trainings), we plot the three components and the total bound. The x-axis is the sweep parameter $\xi = m/n$ (share of jobseekers receiving job trainings). The y-axis is the generalization-gap bound (and its components), all in logistic-loss units. We

can see that the performative term is dominated by $\frac{48}{\sqrt{n}} C_\infty(\mathcal{F})$ due to our choice of B . Code to compute those terms for datasets with varying ξ and plot them can be found in <https://anonymous.4open.science/r/plt-jobseekers/README.md>

C Proofs

C.1 Proof of Lemma 3.5

We restate the result for ease of exposition.

Lemma (In-Sample Performative Shift Bound) Assume that at most m units (in the sample of size n) change in response to predictions at each iteration t . If the transition map is (ε, p) -jointly sensitive (Cond. 3.2), then

$$W_p(\hat{d}_0, \hat{d}_T) \leq \frac{\varepsilon^T - 1}{\varepsilon - 1} \left(\frac{m}{n} \right)^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}} \tilde{L}_a,$$

pointwise in \hat{d}_0 , i.e., for any fixed \hat{d}_0 , with $\tilde{L}_a = (1 + L_a)^{-1}$.

Proof. By triangle inequality,

$$W_p(\hat{d}_0, \hat{d}_T) \leq \sum_{t=0}^{T-1} W_p(\hat{d}_t, \hat{d}_{t+1}). \quad (21)$$

for any \hat{d}_0 . By joint (ε, p) -sensitivity of Tr (Condition 3.2), we further have

$$W_p(\hat{d}_{t+1}, \hat{d}_t) \leq \varepsilon \cdot W_p(\hat{d}_t, \hat{d}_{t-1}) + \varepsilon \cdot \|\hat{\theta}_{t+1} - \hat{\theta}_t\| \quad (22)$$

and thus

$$\begin{aligned} W_p(\hat{d}_{t+1}, \hat{d}_t) &\leq \varepsilon \cdot \left(W_p(\hat{d}_t, \hat{d}_{t-1}) + \|\hat{\theta}_{t+1} - \hat{\theta}_t\| \right) \\ &\leq \varepsilon^2 \cdot \left(W_p(\hat{d}_{t-1}, \hat{d}_{t-2}) + \|\hat{\theta}_t - \hat{\theta}_{t-1}\| \right) \\ &\leq \dots \\ &\leq \varepsilon^{T-1} \cdot \left(W_p(\hat{d}_1, \hat{d}_0) + \|\hat{\theta}_2 - \hat{\theta}_1\| \right). \end{aligned} \quad (23)$$

Summing both sides over $0, \dots, T-1$ yields

$$\sum_{t=0}^{T-1} W_p(\hat{d}_{t+1}, \hat{d}_t) \leq \sum_{t=0}^{T-1} \varepsilon^{T-1} \cdot \left(W_p(\hat{d}_1, \hat{d}_0) + \|\hat{\theta}_2 - \hat{\theta}_1\| \right) = \frac{\varepsilon^T - 1}{\varepsilon - 1} \left(W_p(\hat{d}_1, \hat{d}_0) + \|\hat{\theta}_2 - \hat{\theta}_1\| \right). \quad (24)$$

Since the risk is strongly convex and κ -continuously differentiable as an integral over strongly convex (Cond. 3.1) and κ -continuously differentiable (Cond. 3.3) loss ℓ , the function $G : \Delta \rightarrow \Theta; d \mapsto \arg \min_{\theta \in \Theta} \mathcal{R}(d, \theta)$ (Definition 2.1) is κ -continuously differentiable. This follows from the Implicit Function Theorem, see e.g., [Ross & Nyström \(2018\)](#). Since \mathcal{Z} is compact, the domain of the function G is compact. By the standard Mean Value Theorem, κ -continuously differentiable functions with compact domain are Lipschitz. Thus, $G : \Delta \rightarrow \Theta; d \mapsto \arg \min_{\theta \in \Theta} \mathcal{R}(d, \theta)$ is Lipschitz in d . We denote its Lipschitz constant by L_a and use $\|\hat{\theta}_2 - \hat{\theta}_1\| \leq L_a \cdot W_p(\hat{d}_1, \hat{d}_0)$ in what follows.

Further observe that $W_p(\widehat{d}_1, \widehat{d}_0) = W_p(\text{Tr}(\widehat{\theta}_0, \widehat{d}_0), \widehat{d}_0)$. Since Tr changes $m < n$ units in the sample, we can further bound

$$W_p(\text{Tr}(\widehat{\theta}_0, \widehat{d}_0), \widehat{d}_0) \leq \left(\frac{m}{n}\right)^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}} \quad (25)$$

by definition of Wasserstein- p distance and $\mathcal{D}_{\mathcal{Z}} = \sup_{z,z'} d_{\mathcal{Z}}(z, z') < \infty$. Together, this gives

$$W_p(\widehat{d}_1, \widehat{d}_0) + \|\widehat{\theta}_2 - \widehat{\theta}_1\| \leq (1 + L_a) \left(\frac{m}{n}\right)^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}} \quad (26)$$

By combining (21), (24) and (26), the result follows immediately. \square

C.2 Proof of Theorem 3.7

We restate the result for ease of exposition.

Theorem: The excess risk $\mathcal{R}(d_0, \widehat{\theta}_T) - \inf_{\theta \in \Theta} \mathcal{R}(d_0, \theta)$ of a model $\widehat{\theta}_T$ trained on performatve samples $\widehat{d}_0, \dots, \widehat{d}_T$, in which at most $m \leq n$ units change in response to the predictions, is upper bounded by

$$\begin{aligned} & L_\ell \left(\frac{\log(C_a/\delta)}{C_b n} \right)^{p/\nu} + L_\ell L_a \frac{(\varepsilon \kappa / \gamma)^T - 1}{\varepsilon \kappa / \gamma - 1} \left(\frac{m}{n} \right)^{1/p} \mathcal{D}_{\mathcal{Z}} \\ & + \frac{FL_\ell}{\sqrt{n}} \left(24 \mathfrak{C}_{L_2}(\mathcal{F}) + 2\sqrt{2 \ln(1/\delta)} \right) \end{aligned}$$

for any $T \in \mathbb{N}$, with probability over d_0 of at least $1 - \frac{\delta}{2}$ and L_a the Lipschitz constant of G , under Conditions 3.1–3.3.

Proof. The idea of the proof is as follows: Bound 1. $\mathcal{R}(d, \widehat{\theta}_T) - \mathcal{R}(\widehat{d}_0, \widehat{\theta}_T)$, 2. $\mathcal{R}(\widehat{d}_0, \widehat{\theta}_T) - \mathcal{R}(\widehat{d}_0, \widehat{\theta}_0)$, as well as 3. $\mathcal{R}(\widehat{d}_0, \widehat{\theta}_0) - \inf_{\theta \in \Theta} \mathcal{R}(d, \theta)$ (all with high probability over d_0) and then combine via the union bound.

1. We need the Kantorovich-Rubinstein Lemma (proved in [Kantorovich & Rubinstein 1958](#)):

$$W_1(\mu, \nu) = \frac{1}{K} \sup_{\|f\|_L \leq K} \mathbb{E}_{x \sim \mu}[f(x)] - \mathbb{E}_{y \sim \nu}[f(y)]$$

with measures μ, ν , Lipschitz-continuous $f(\cdot)$ and $\|\cdot\|_L$ the Lipschitz-norm, rendering K a Lipschitz-constant of f

By taking $f(\cdot)$ to be the loss function $\ell(\cdot, \theta)$ which is L_ℓ -Lipschitz by Condition 3.1 on compact Θ , the Kantorovich-Rubinstein Lemma implies

$$|\mathcal{R}(d, \widehat{\theta}_T) - \mathcal{R}(\widehat{d}_0, \widehat{\theta}_T)| \leq L_\ell W_1(d, \widehat{d}_0), \quad (27)$$

where L_ℓ is the Lipschitz constant of ℓ .

We have $W_1(d, \widehat{d}_0) \leq W_p(d, \widehat{d}_0)$ for $1 \leq p \leq 2$. In other words, we can bound the risk difference $\mathcal{R}(d, \widehat{\theta}_T) - \mathcal{R}(\widehat{d}_0, \widehat{\theta}_T)$ via the p -Wasserstein distance between its respective distributions $W_p(d, \widehat{d}_0)$. We thus have to bound $W_p(d, \widehat{d}_0)$.

Further recall Lemma 3.4:

$$W_p(d_0, \widehat{d}_0) \leq \left(\frac{\log(C_a/\delta)}{C_b n} \right)^{p/\nu} \quad (28)$$

with probability over \widehat{d}_0 of at least $1 - \delta$.

Combining Equations 27 and 28 gives

$$\mathcal{R}(d_0, \widehat{\theta}_T) \leq \mathcal{R}(\widehat{d}_0, \widehat{\theta}_T) + L_\ell \left(\frac{\log(C_a/\delta)}{C_b n} \right)^{p/\nu} \quad (29)$$

with probability over \widehat{d}_0 of at least $1 - \delta$.

2. We will now bound $\|\widehat{\theta}_0 - \widehat{\theta}_T\|$, which will subsequently allow us to bound $\mathcal{R}(\widehat{d}_0, \widehat{\theta}_T) - \mathcal{R}(\widehat{d}_0, \widehat{\theta}_0)$ via Lipschitz-continuity of the loss in θ .

By triangle inequality, we get

$$\|\widehat{\theta}_0 - \widehat{\theta}_T\| \leq \sum_{t=0}^{T-1} \|\widehat{\theta}_t - \widehat{\theta}_{t+1}\|. \quad (30)$$

It holds per Lemma 2 in [Brown et al. \(2022\)](#) that the ERM-part of stateful performative prediction is $\varepsilon \frac{\kappa}{\gamma}$ -Lipschitz under condition 3.1. (See also equation (60) in [Rodemann et al. \(2024\)](#).) We thus have for all $t \in \{1, \dots, T\}$

$$\|\widehat{\theta}_{t+1} - \widehat{\theta}_t\| \leq \varepsilon \frac{\kappa}{\gamma} \cdot \|\widehat{\theta}_t - \widehat{\theta}_{t-1}\| \quad (31)$$

and thus

$$\|\widehat{\theta}_{t+1} - \widehat{\theta}_t\| \leq \varepsilon \frac{\kappa}{\gamma} \cdot \|\widehat{\theta}_t - \widehat{\theta}_{t-1}\| \leq \left(\varepsilon \frac{\kappa}{\gamma} \right)^2 \cdot \|\widehat{\theta}_{t-1} - \widehat{\theta}_{t-2}\| \leq \dots \leq \left(\varepsilon \frac{\kappa}{\gamma} \right)^t \cdot \|\widehat{\theta}_1 - \widehat{\theta}_0\|. \quad (32)$$

Summing both sides over $1, \dots, T$ yields

$$\sum_{t=0}^{T-1} \|\widehat{\theta}_{t+1} - \widehat{\theta}_t\| \leq \sum_{t=0}^{T-1} \left(\varepsilon \frac{\kappa}{\gamma} \right)^t \cdot \|\widehat{\theta}_1 - \widehat{\theta}_0\| = \frac{(\varepsilon \frac{\kappa}{\gamma})^T - 1}{(\varepsilon \frac{\kappa}{\gamma}) - 1} \|\widehat{\theta}_1 - \widehat{\theta}_0\|. \quad (33)$$

Observe that $\widehat{\theta}_0 = \arg \min_{\theta \in \Theta} \mathcal{R}(\widehat{d}_0, \theta)$ and $\widehat{\theta}_1 = \arg \min_{\theta \in \Theta} \mathcal{R}(\widehat{d}_1, \theta)$. Since the risk is strongly convex and continuously differentiable as an integral over strongly convex (Cond. 3.1) and continuously differentiable (Cond. 3.3) loss ℓ (in θ , x , and y), the function $G : \Delta \rightarrow \Theta; d \mapsto \arg \min_{\theta \in \Theta} \mathcal{R}(d, \theta)$ (Definition 2.1) is continuously differentiable. This

follows from the implicit function theorem, see e.g., [Ross & Nyström \(2018\)](#). Since \mathcal{Z} is compact, the domain of the function G is compact. By the Mean Value Theorem, continuously differentiable functions with compact domain are Lipschitz. Thus, $G : \Delta \rightarrow \Theta; d \mapsto \arg \min_{\theta \in \Theta} \mathcal{R}(d, \theta)$ is Lipschitz in d . We denote its Lipschitz constant by L_a .

Further recall that $W_p(\hat{d}_1, \hat{d}_0) = W_p(\text{Tr}(\cdot, \hat{d}_0), \hat{d}_0)$. Since Tr changes m units in the sample and \mathcal{Z} is bounded, we can further bound (with reasoning analogous to Lemma 3.5)

$$\|\hat{\theta}_0 - \hat{\theta}_1\| \leq L_a \cdot W_p(\hat{d}_1, \hat{d}_0) = L_a \cdot W_p(\text{Tr}(\cdot, \hat{d}_0), \hat{d}_0) \leq L_a \cdot \left(\frac{m}{n}\right)^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}}. \quad (34)$$

Combining equation 30, equation 33, and equation 34 gives

$$\|\hat{\theta}_0 - \hat{\theta}_T\| \leq L_a \frac{(\varepsilon \frac{\kappa}{\gamma})^T - 1}{(\varepsilon \frac{\kappa}{\gamma}) - 1} \left(\frac{m}{n}\right)^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}}. \quad (35)$$

Since the loss is L_ℓ -Lipschitz, see above, we have

$$\mathcal{R}(\hat{d}_0, \hat{\theta}_T) - \mathcal{R}(\hat{d}_0, \hat{\theta}_0) \leq L_\ell \|\hat{\theta}_0 - \hat{\theta}_T\| \leq L_\ell L_a \frac{(\varepsilon \frac{\kappa}{\gamma})^T - 1}{(\varepsilon \frac{\kappa}{\gamma}) - 1} \left(\frac{m}{n}\right)^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}} \quad (36)$$

3. Define $\mathcal{L} := \ell \circ \mathcal{F}$ as the loss class and recall $\sup_{\theta \in \Theta, x \in \mathcal{X}} \|f_\theta(x)\|_2 \leq F < \infty$ with $f_\theta \in \mathcal{F}$ (see Section 3). As ℓ is L_ℓ -Lipschitz (see 1.), we have that \mathcal{L} is uniformly bounded by $2FL_\ell$. Define

$$\mathfrak{R}_n(\mathcal{F}) := \mathbb{E} \left[\sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^n \mathcal{E}_i f(X_i) \right] \quad (37)$$

as the expected Rademacher average⁷ of any function class \mathcal{F} on \mathcal{X} with Rademacher random variables $\mathcal{E}_1, \dots, \mathcal{E}_n$ independent of X_1, \dots, X_n . It follows with the standard symmetrization argument (see, e.g., Chapter 26 in [Shalev-Shwartz & Ben-David \(2014b\)](#) or [Nagler \(2024\)](#)) for any function class \mathcal{F} uniformly bounded by B that

$$\frac{1}{2} \mathfrak{R}_n(\mathcal{F}) - \frac{B}{\sqrt{n}} \leq \mathbb{E} \left[\sup_{f \in \mathcal{F}} [\mathbb{E}_{X \sim d}(f(X)) - \mathbb{E}_{X \sim \hat{d}_0}(f(X))] \right] \leq 2\mathfrak{R}_n(\mathcal{F}) \quad (38)$$

Together with McDiarmid's inequality ([McDiarmid et al., 1989](#)) we thus have for the standard empirical risk minimizer $\hat{\theta}_0$ on $\hat{d}_0 \stackrel{\text{i.i.d.}}{\sim} d$ and function the class $\mathcal{F} = \mathcal{L}$ with $B = FL_\ell$

$$\mathcal{R}(\hat{d}_0, \hat{\theta}_0) - \inf_{\theta \in \Theta} \mathcal{R}(d, \theta) \leq 2\mathfrak{R}_n(\mathcal{L}) + 2FL_\ell \sqrt{\frac{2 \ln(1/\delta)}{n}} \quad (39)$$

⁷See, e.g., [Bousquet & Elisseeff \(2002\)](#) and [Von Luxburg & Schölkopf \(2011\)](#).

with probability over \hat{d}_0 of at least $1 - \delta$. Talagrand's contraction lemma states that for L_ℓ -Lipschitz loss (given by 1.), we have $\mathfrak{R}_n(\mathcal{L}) \leq L_\ell \mathfrak{R}_n(\mathcal{F})$, see (Talagrand, 1995). We further have that $\mathfrak{R}_n(\mathcal{F}) = F \mathfrak{R}_n(\mathcal{F}/F)$ by definition of \mathfrak{R}_n . Hence, $2\mathfrak{R}_n(\mathcal{L}) \leq 2FL_\ell \mathfrak{R}_n(\mathcal{F}/F)$ in equation 39. Since all $f_\theta \in \mathcal{F}/F$ are upper-bounded by 1, we can use Dudley's theorem (Dudley, 1987) to obtain

$$\mathfrak{R}_n(\mathcal{F}/F) \leq \frac{12}{\sqrt{n}} \sup_P \int_0^1 \sqrt{\log \mathcal{N}(\mathcal{F}, \|\cdot\|_{L_2(P)}^2, \varepsilon)} d\varepsilon \quad (40)$$

with \mathcal{N} the covering number from definition 3.6, where $\|\cdot\|_{L_2(P)}^2$ was defined as $\|f\|_{L_2(P)}^2 = \mathbb{E}_{X \sim P}[f(X)^2]$ for some measure P .

Recall $\mathfrak{C}_{L_2}(\mathcal{F}) := \sup_P \int_0^1 \sqrt{\log \mathcal{N}(\mathcal{F}, \|\cdot\|_{L_2(P)}^2, \varepsilon)} d\varepsilon$. Thus,

$$\mathcal{R}(\hat{d}_0, \hat{\theta}_0) - \inf_{\theta \in \Theta} \mathcal{R}(d, \theta) \leq 2FL_\ell \frac{12}{\sqrt{n}} \mathfrak{C}_{L_2}(\mathcal{F}) + 2FL_\ell \sqrt{\frac{2 \ln(1/\delta)}{n}} \quad (41)$$

with probability over \hat{d}_0 of at least $1 - \delta$.

The claim then follows by 1., 2., 3., together with the union bound. \square

C.3 Proof of Corollary 3.8

Once again, we restate the result for ease of exposition.

Corollary: In the setting of Theorem 3.7, $\mathcal{R}(d_0, \hat{\theta}_T) \leq \mathcal{R}(\hat{d}_{T-1}, \hat{\theta}_T) + L_\ell \left(\frac{\log(C_a/\delta)}{C_b n} \right)^{p/\nu} + \frac{L_\ell(\varepsilon^{T-1})}{\varepsilon-1} \left(\frac{m}{n} \right)^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}} \tilde{L}_a$.

Proof. Recall from Condition 3.1 that the loss $\ell(z, \theta)$ is L_ℓ -Lipschitz in θ . By the Kantorovich-Rubinstein Lemma (Kantorovich & Rubinstein, 1958), we have

$$|\mathcal{R}(d_0, \hat{\theta}_T) - \mathcal{R}(\hat{d}_0, \hat{\theta}_T)| \leq L_\ell W_1(d_0, \hat{d}_0), \quad (42)$$

where L_ℓ is the Lipschitz constant of ℓ .

Since $W_1(d_0, \hat{d}_0) \leq W_p(d_0, \hat{d}_0)$ for $1 \leq p \leq 2$, we can bound the risk difference via the Wasserstein- p distance between the distributions.

By the triangle inequality, we have

$$W_p(d_0, \hat{d}_0) \leq W_p(d_0, \hat{d}_0) + W_p(\hat{d}_0, \hat{d}_{T-1}). \quad (43)$$

However, since we want to relate $\mathcal{R}(d_0, \hat{\theta}_T)$ to $\mathcal{R}(\hat{d}_{T-1}, \hat{\theta}_T)$, we observe that by triangle inequality:

$$W_p(d_0, \hat{d}_{T-1}) \leq W_p(d_0, \hat{d}_0) + W_p(\hat{d}_0, \hat{d}_{T-1}). \quad (44)$$

From Lemma 3.4, we have

$$W_p(d_0, \hat{d}_0) \leq \left(\frac{\log(C_a/\delta)}{C_b n} \right)^{p/\nu} \quad (45)$$

with probability at least $1 - \delta$ over \hat{d}_0 .

From Lemma 3.5, we have

$$W_p(\hat{d}_0, \hat{d}_{T-1}) \leq \frac{\varepsilon^{T-1} - 1}{\varepsilon - 1} \left(\frac{m}{n} \right)^{1/p} \mathcal{D}_{\mathcal{Z}} \tilde{L}_a. \quad (46)$$

Note that for the bound involving \hat{d}_{T-1} and $\hat{\theta}_T$, we use the fact that $T - 1$ iterations have occurred to reach \hat{d}_{T-1} , thus the exponent is $T - 1$ rather than T in the geometric series.

Combining these via the Kantorovich-Rubinstein Lemma yields

$$\begin{aligned} \mathcal{R}(d_0, \hat{\theta}_T) - \mathcal{R}(\hat{d}_{T-1}, \hat{\theta}_T) &\leq L_\ell W_p(d_0, \hat{d}_{T-1}) \\ &\leq L_\ell \left(\frac{\log(C_a/\delta)}{C_b n} \right)^{p/\nu} + L_\ell \frac{\varepsilon^{T-1} - 1}{\varepsilon - 1} \left(\frac{m}{n} \right)^{1/p} \mathcal{D}_{\mathcal{Z}} \tilde{L}_a. \end{aligned} \quad (47)$$

with probability at least $1 - \delta$, which was to be shown. \square

C.4 Proof of Lemma 3.9

We restate the result for ease of readability.

Lemma (Performative Population Shift Bound) Assume $s \in [0, 1]$ is the share of units in d_0 reacting to predictions (the “performative response rate”). Then

$$s < \frac{m}{n} + q(\delta) \sqrt{\frac{m}{n^2} \left(1 - \frac{m}{n}\right)}$$

with probability $1 - \delta$, where $q(\delta)$ is the $(1 - \frac{1}{2}\delta)$ -quantile of the standard normal distribution.

Proof. We have per Lemma 3.5 for any initial $\hat{\theta}_0$

$$W_p(\hat{d}_0, \hat{d}_T) \leq \frac{\varepsilon^T - 1}{\varepsilon - 1} \left(\frac{m}{n} \right)^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}}. \quad (48)$$

It is easy to see that the same holds for populations d_0, d_T

$$W_p(d_0, d_T) \leq \frac{\varepsilon^T - 1}{\varepsilon - 1} s^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}}, \quad (49)$$

where $s \in [0, 1]$ is the share of units in the population reacting to the predictions, i.e., the population analog of m/n . By treating m/n as an estimator of the parameter s of a Bernoulli distribution (modeling whether statistical units react or not), we have per Wald method (see e.g., Brown et al. 2001) that

$$s < \frac{m}{n} + q(\delta) \sqrt{\frac{\frac{m}{n}(1 - \frac{m}{n})}{n}} \quad (50)$$

with probability $1 - \delta$, where $q(\delta) = q_{1 - \frac{1}{2}\delta}$ is the $(1 - \frac{1}{2}\delta)$ -quantile of a standard normal distribution. \square

C.5 Proof of Theorem 3.10

We restate the theorem for ease of exposition.

Theorem: The *performative excess risk* $\mathcal{R}(\text{Tr}(d_0, \hat{\theta}_T), \hat{\theta}_T) - \inf_{\theta \in \Theta} \mathcal{R}(\text{Tr}(d_0, \hat{\theta}_T), \theta)$ of a model $\hat{\theta}_T$ trained on performative samples $\hat{d}_T, \dots, \hat{d}_T$, in which $m \leq n$ units react to the predictions, is upper bounded by

$$\begin{aligned} L_\ell \left(\frac{\log(C_a/\delta)}{C_b n} \right)^{\frac{p}{\nu}} + L_\ell L_a \frac{\left(\frac{\varepsilon\kappa}{\gamma} \right)^T - 1}{\frac{\varepsilon\kappa}{\gamma} - 1} \left(\frac{m}{n} \right)^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}} \\ + \frac{FL_\ell}{\sqrt{n}} \left(24 \mathfrak{C}_{L_2}(\mathcal{F}) + 2\sqrt{2 \ln(1/\delta)} \right) + 2L_\ell \mathcal{D}_{\mathcal{Z}} \left(\frac{m}{n} + \frac{q(\delta)\sqrt{m(n-m)}}{n^{3/2}} \right)^{1/p}. \end{aligned} \quad (51)$$

for any $tT \in \mathbb{N}$, with probability of at least $1 - \frac{\delta}{4}$ under Conditions 3.1–3.3. Or equivalently, it is upper bounded by

$$\begin{aligned} \mathcal{R}(d_1, \hat{\theta}_T) - \inf_{\theta \in \Theta} \mathcal{R}(d_1, \theta) \leq L_\ell \left[2A(m, n) + C(n) \right. \\ + \frac{2F}{\sqrt{n}} \left(12 \mathfrak{C}_{L_2}(\mathcal{F}) + \sqrt{2 \ln(1/\delta)} \right) \\ \left. + L_a \mathcal{D}_{\mathcal{Z}} (A(m, n) + K(T, m, n)) \right], \end{aligned} \quad (52)$$

for any $T \in \mathbb{N}$ with probability $1 - \frac{\delta}{4}$, where $A(m, n)$, $C(n)$ and $K(T, m, n)$ depend on constants and m, n, T only.

Proof. Fix $d_1 = \text{Tr}(d_0, \hat{\theta}_T)$. Denote the Bayes-optimal model (from the hypothesis class \mathcal{F} induced by Θ) on d_0 by $\theta_{0,B} := \inf_{\theta \in \Theta} \mathcal{R}(d_0, \theta)$ and the one on d_1 by $\theta_{1,B} := \inf_{\theta \in \Theta} \mathcal{R}(d_1, \theta)$, respectively. Invoking the Kantorovich-Rubinstein Lemma (see proof of Theorem 3.7) again, we have

$$\mathcal{R}(d_0, \theta_{0,B}) - \mathcal{R}(d_1, \theta_{0,B}) \leq L_\ell W_1(d_0, d_1), \quad (53)$$

where L_ℓ is the Lipschitz constant of ℓ .

We have per Lemma 3.5 for any initial $\hat{\theta}_0$

$$W_p(\hat{d}_0, \hat{d}_T) \leq \frac{\varepsilon^T - 1}{\varepsilon - 1} \left(\frac{m}{n} \right)^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}} \tilde{L}_a. \quad (54)$$

It is easy to see that the same holds for populations d_0, d_T

$$W_p(d_0, d_T) \leq \frac{\varepsilon^T - 1}{\varepsilon - 1} s^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}} \tilde{L}_a, \quad (55)$$

where $s \in [0, 1]$ is the share of units in the population reacting to the predictions (i.e., the population analog of $\frac{m}{n}$). By treating $\frac{m}{n}$ as an estimator of the parameter s of a Bernoulli distribution (modeling whether statistical units react or not), we have per Wald method (see e.g., Brown et al. 2001) that

$$s < \frac{m}{n} + q(\delta) \sqrt{\frac{\frac{m}{n}(1 - \frac{m}{n})}{n}} \quad (56)$$

with probability $1 - \delta$, where $q(\delta) = q_{1 - \frac{1}{2}\delta}$ is the $(1 - \frac{1}{2}\delta)$ -quantile of a standard normal distribution, see Lemma 3.9. Hence,

$$W_p(d_0, d_T) \leq \frac{\varepsilon^T - 1}{\varepsilon - 1} \left(\frac{m}{n} + \frac{q(\delta)\sqrt{m(n-m)}}{n^{3/2}} \right)^{1/p} \mathcal{D}_{\mathcal{Z}} \tilde{L}_a \quad (57)$$

with probability $1 - \delta$. Combined with Equation 53, this gives for $T = 1$

$$\mathcal{R}(d_0, \theta_{0,B}) - \mathcal{R}(d_1, \theta_{0,B}) \leq L_\ell \left(\frac{m}{n} + \frac{q(\delta)\sqrt{m(n-m)}}{n^{3/2}} \right)^{1/p} \mathcal{D}_{\mathcal{Z}} \tilde{L}_a \quad (58)$$

with probability $1 - \delta$.

We will now bound $\|\theta_{B,0} - \theta_{B,1}\|$, which will allow us to bound $\mathcal{R}(d_1, \theta_{B,0}) - \mathcal{R}(d_1, \theta_{B,1})$ via Lipschitz-continuity of the loss in θ . In the next step, this bound on $\mathcal{R}(d_1, \theta_{B,0}) - \mathcal{R}(d_1, \theta_{B,1})$ together with the probabilistic bound on $\mathcal{R}(d_0, \theta_{0,B}) - \mathcal{R}(d_1, \theta_{0,B})$ from Equation 58 will yield a high probability bound on $\mathcal{R}(d_0, \theta_{B,0}) - \mathcal{R}(d_1, \theta_{B,1}) := \inf_{\theta \in \Theta} \mathcal{R}(d_0, \theta) - \inf_{\theta \in \Theta} \mathcal{R}(\text{Tr}(d_0, \hat{\theta}_T), \theta)$. (Recall we fixed $d_1 = \text{Tr}(d_0, \hat{\theta}_T)$.)

Since Tr changes a share $s \in [0, 1]$ of the population and \mathcal{Z} is bounded, we can bound (with reasoning analogous to Lemma 3.5)

$$\|\theta_{B,0} - \theta_{B,1}\| \leq L_a \cdot s^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}} \tilde{L}_a. \quad (59)$$

where L_a is the Lipschitz constant of $\Delta \rightarrow \Theta : d \mapsto \arg \min_{\theta \in \Theta} \mathcal{R}(d, \theta)$ as above. With reasoning analogous to Equation 57, we can upper bound s by our tail distribution of our estimator $\frac{m}{n}$ with probability $1 - \delta$, yielding

$$s < \frac{m}{n} + q(\delta) \sqrt{\frac{\frac{m}{n}(1 - \frac{m}{n})}{n}} \quad (60)$$

with probability $1 - \delta$, where again $q(\delta) = q_{1 - \frac{1}{2}\delta}$ is the $(1 - \frac{1}{2}\delta)$ -quantile of a standard normal distribution. Hence,

$$\|\theta_{B,0} - \theta_{B,1}\| \leq L_a \cdot s^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}} \leq L_a \left(\frac{m}{n} + \frac{q(\delta) \sqrt{m(n-m)}}{n^{3/2}} \right)^{1/p} \mathcal{D}_{\mathcal{Z}} \tilde{L}_a \quad (61)$$

with probability $1 - \delta$. It follows from the linearity of expectation and the loss being Lipschitz in θ that

$$\mathcal{R}(d_1, \theta_{B,0}) - \mathcal{R}(d_1, \theta_{B,1}) \leq L_\ell \|\theta_{B,0} - \theta_{B,1}\|. \quad (62)$$

Thus,

$$\mathcal{R}(d_1, \theta_{B,0}) - \mathcal{R}(d_1, \theta_{B,1}) \leq L_\ell L_a \cdot \left(\frac{m}{n} + \frac{q(\delta) \sqrt{m(n-m)}}{n^{3/2}} \right)^{1/p} \mathcal{D}_{\mathcal{Z}} \tilde{L}_a. \quad (63)$$

with probability $1 - \delta$.

Together with Equation 58, this gives an upper bound w.h.p. on $\mathcal{R}(d_0, \theta_{B,0}) - \mathcal{R}(d_1, \theta_{B,1}) := \inf_{\theta \in \Theta} \mathcal{R}(d_0, \theta) - \inf_{\theta \in \Theta} \mathcal{R}(\text{Tr}(d_0, \hat{\theta}_T), \theta)$. Namely,

$$\begin{aligned} \inf_{\theta \in \Theta} \mathcal{R}(d_0, \theta) - \inf_{\theta \in \Theta} \mathcal{R}(d_1, \theta) &\leq L_\ell L_a \left(\frac{m}{n} + \frac{q(\delta) \sqrt{m(n-m)}}{n^{3/2}} \right)^{1/p} \mathcal{D}_{\mathcal{Z}} \tilde{L}_a \\ &\quad + L_\ell \left(\frac{m}{n} + \frac{q(\delta) \sqrt{m(n-m)}}{n^{3/2}} \right)^{1/p} \mathcal{D}_{\mathcal{Z}} \tilde{L}_a. \end{aligned} \quad (64)$$

with probability $1 - \delta$. Or equivalently,

$$\inf_{\theta \in \Theta} \mathcal{R}(d_0, \theta) - \inf_{\theta \in \Theta} \mathcal{R}(d_1, \theta) \leq L_\ell \mathcal{D}_{\mathcal{Z}} \tilde{L}_a \left(\frac{m}{n} + \frac{q(\delta) \sqrt{m(n-m)}}{n^{3/2}} \right)^{1/p} (1 + L_a) \quad (65)$$

with probability $1 - \delta$, which simplifies to

$$\inf_{\theta \in \Theta} \mathcal{R}(d_0, \theta) - \inf_{\theta \in \Theta} \mathcal{R}(d_1, \theta) \leq L_\ell \mathcal{D}_{\mathcal{Z}} \left(\frac{m}{n} + \frac{q(\delta) \sqrt{m(n-m)}}{n^{3/2}} \right)^{1/p} \quad (66)$$

with probability $1 - \delta$ due to the definition of $\tilde{L}_a = 1/(1 + L_a)$. In complete analogy to the proof of Theorem 3.7, we can now bound 1. $\mathcal{R}(d_0, \hat{\theta}_T) - \mathcal{R}(\hat{d}_0, \hat{\theta}_T)$, 2. $\mathcal{R}(\hat{d}_0, \hat{\theta}_T) - \mathcal{R}(\hat{d}_0, \hat{\theta}_0)$, as well as 3. $\mathcal{R}(\hat{d}_0, \hat{\theta}_0) - \inf_{\theta \in \Theta} \mathcal{R}(d_0, \theta)$ (all with high probability over \hat{d}_0) and then combine via the union bound to get

$$L_\ell \left(\frac{\log(C_a/\delta)}{C_b n} \right)^{p/\nu} + L_\ell L_a \frac{(\varepsilon \kappa / \gamma)^T - 1}{\varepsilon \kappa / \gamma - 1} \left(\frac{m}{n} \right)^{1/p} \mathcal{D}_{\mathcal{Z}} + \frac{F L_\ell}{\sqrt{n}} \left(24 \mathfrak{C}_{L_2}(\mathcal{F}) + 2 \sqrt{2 \ln(1/\delta)} \right) \quad (67)$$

as an upper bound on $\mathcal{R}(d_0, \hat{\theta}_T) - \inf_{\theta \in \Theta} \mathcal{R}(d_0, \theta)$ for any $T \in \mathbb{N}$ with probability of at least $1 - \frac{\delta}{2}$ under conditions 3.1 and 3.2.

Recall we want to bound the *performative excess risk*

$$\mathcal{R}(d_1, \hat{\theta}_T) - \inf_{\theta \in \Theta} \mathcal{R}(d_1, \theta) = \mathcal{R}(\text{Tr}(d_0, \hat{\theta}_T), \hat{\theta}_T) - \inf_{\theta \in \Theta} \mathcal{R}(\text{Tr}(d_0, \hat{\theta}_T), \theta) \quad (68)$$

of a model $\hat{\theta}_T$ trained on performative samples $\hat{d}_0, \dots, \hat{d}_T$, in which $m \leq n$ units react to the predictions. Obviously,

$$\begin{aligned} \mathcal{R}(d_1, \hat{\theta}_T) - \inf_{\theta \in \Theta} \mathcal{R}(d_1, \theta) &= \mathcal{R}(d_1, \hat{\theta}_T) - \mathcal{R}(d_0, \hat{\theta}_T) \\ &\quad + \mathcal{R}(d_0, \hat{\theta}_T) - \inf_{\theta \in \Theta} \mathcal{R}(d_0, \theta) \\ &\quad + \inf_{\theta \in \Theta} \mathcal{R}(d_0, \theta) - \inf_{\theta \in \Theta} \mathcal{R}(d_1, \theta). \end{aligned} \quad (69)$$

We have per reasoning from Equations 58 and 53 as well as per symmetry of the Wasserstein-distance that

$$\mathcal{R}(d_1, \theta) - \mathcal{R}(d_0, \theta) \leq L_\ell \left(\frac{m}{n} + \frac{q(\delta) \sqrt{m(n-m)}}{n^{3/2}} \right)^{1/p} \mathcal{D}_{\mathcal{Z}} \quad (70)$$

for any $\theta \in \Theta$ with probability $1 - \delta$. Applying this to θ_T and using Equation 66 as well as Equation 67, we can upper bound our target $\mathcal{R}(d_1, \hat{\theta}_T) - \inf_{\theta \in \Theta} \mathcal{R}(d_1, \theta)$ (Equation 69) by

$$\begin{aligned} &\underbrace{\left(\frac{m}{n} + \frac{q(\delta) \sqrt{m(n-m)}}{n^{3/2}} \right)^{1/p}}_{\geq \mathcal{R}(d_1, \theta) - \mathcal{R}(d_0, \theta)} \\ &\quad + \underbrace{L_\ell \left(\frac{\log(C_a/\delta)}{C_b n} \right)^{\frac{p}{\nu}} + L_\ell L_a \frac{\left(\frac{\varepsilon \kappa}{\gamma} \right)^T - 1}{\frac{\varepsilon \kappa}{\gamma} - 1} \left(\frac{m}{n} \right)^{1/p} \mathcal{D}_{\mathcal{Z}}}_{\geq \mathcal{R}(d_0, \hat{\theta}_T) - \inf_{\theta \in \Theta} \mathcal{R}(d_0, \theta)} \\ &\quad + \underbrace{\frac{F L_\ell}{\sqrt{n}} \left(24 \mathfrak{C}_{L_2}(\mathcal{F}) + 2 \sqrt{2 \ln(1/\delta)} \right)}_{\geq \mathcal{R}(d_0, \hat{\theta}_T) - \inf_{\theta \in \Theta} \mathcal{R}(d_0, \theta)} \\ &\quad + \underbrace{L_\ell \mathcal{D}_{\mathcal{Z}} \left(\frac{m}{n} + \frac{q(\delta) \sqrt{m(n-m)}}{n^{3/2}} \right)^{1/p}}_{\geq \inf_{\theta \in \Theta} \mathcal{R}(d_0, \theta) - \inf_{\theta \in \Theta} \mathcal{R}(d_1, \theta)}. \end{aligned} \quad (71)$$

for any $T \in \mathbb{N}$ with probability $1 - \frac{\delta}{4}$.

Equivalently, $\mathcal{R}(d_1, \hat{\theta}_T) - \inf_{\theta \in \Theta} \mathcal{R}(d_1, \theta)$ is upper bounded by

$$\begin{aligned} & L_\ell \left(\frac{\log(C_a/\delta)}{C_b n} \right)^{\frac{p}{\nu}} + L_\ell L_a \frac{\left(\frac{\varepsilon\kappa}{\gamma} \right)^T - 1}{\frac{\varepsilon\kappa}{\gamma} - 1} \left(\frac{m}{n} \right)^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}} \\ & + \frac{FL_\ell}{\sqrt{n}} \left(24 \mathfrak{C}_{L_2}(\mathcal{F}) + 2\sqrt{2 \ln(1/\delta)} \right) + 2L_\ell \mathcal{D}_{\mathcal{Z}} \left(\frac{m}{n} + \frac{q(\delta)\sqrt{m(n-m)}}{n^{3/2}} \right)^{1/p}. \end{aligned} \quad (72)$$

Simplifying further, we obtain

$$\begin{aligned} \mathcal{R}(d_1, \hat{\theta}_T) - \inf_{\theta \in \Theta} \mathcal{R}(d_1, \theta) & \leq L_\ell \left[2A(m, n) + C(n) \right. \\ & + \frac{2F}{\sqrt{n}} \left(12 \mathfrak{C}_{L_2}(\mathcal{F}) + \sqrt{2 \ln(1/\delta)} \right) \\ & \left. + L_a (A(m, n) + K(T, m, n)) \right]. \end{aligned} \quad (73)$$

for any $T \in \mathbb{N}$ with probability $1 - \frac{\delta}{4}$, where

$$\begin{aligned} A(m, n) &:= 2 \mathcal{D}_{\mathcal{Z}} \left(\frac{m}{n} + \frac{q(\delta)\sqrt{m(n-m)}}{n^{3/2}} \right)^{1/p}, \\ C(n) &:= \left(\frac{\log(C_a/\delta)}{C_b n} \right)^{\frac{p}{\nu}}, \\ K(T, m, n) &:= \frac{\left(\frac{\varepsilon\kappa}{\gamma} \right)^T - 1}{\frac{\varepsilon\kappa}{\gamma} - 1} \left(\frac{m}{n} \right)^{1/p}. \end{aligned}$$

□

C.6 Proof of Corollary 3.11

We restate the corollary first.

Corollary (Improving Bounds Under Performativity)

I) $\hat{\theta}_0$ yields the tightest performative excess risk bound among all $\{\hat{\theta}_t\}$. **II**) it holds with probability $1 - \delta$ that $s < q(\delta) \sqrt{\frac{M_T}{Tn} (1 - \frac{M_T}{Tn}) / (Tn)}$ with $M_T = \sum_{t=1}^T m_t$, giving an as tight or tighter performatve excess risk bound.

Proof. We now prove both parts separately.

I) From Theorem 3.10, the performative excess risk bound has the form

$$R(\text{Tr}(d_0, \hat{\theta}_T), \hat{\theta}_T) - \inf_{\theta \in \Theta} R(\text{Tr}(d_0, \hat{\theta}_T), \theta) \leq B_1 + B_2(T) + B_3,$$

where $B_2(T)$ contains the term

$$L_\ell L_a \frac{(\varepsilon\kappa/\gamma)^T - 1}{\varepsilon\kappa/\gamma - 1} \left(\frac{m}{n}\right)^{1/p} D_Z.$$

Since $\varepsilon\kappa/\gamma < 1$ (by assumption), the function $T \mapsto \frac{(\varepsilon\kappa/\gamma)^T - 1}{\varepsilon\kappa/\gamma - 1}$ is strictly increasing in T . Therefore, $B_2(T)$ is minimized at $T = 0$, meaning $\hat{\theta}_0$ yields the tightest bound among all $\{\hat{\theta}_t\}_{t=0}^T$.

II) In Lemma 3.9, we bounded s (the population performative response rate) using a single observation m/n :

$$s < \frac{m}{n} + q(\delta) \sqrt{\frac{m/n(1-m/n)}{n}}.$$

However, when retraining T times, we observe m_1, \dots, m_T at each iteration. Define $M_T = \sum_{t=1}^T m_t$. By treating each m_t/n as an independent Bernoulli estimate of s and pooling across iterations, we obtain $M_T/(Tn)$ as an improved estimator of s based on effective sample size Tn .

Applying Wald's method to this estimator:

$$s < \frac{M_t}{Tn} + q(\delta) \sqrt{\frac{M_T/(Tn) \cdot (1 - M_T/(Tn))}{Tn}} = \frac{M_t}{Tn} + q(\delta) \sqrt{\frac{M_T(Tn - M_T)}{T^2 n^2}}.$$

Since $M_T \leq T \cdot \max\{m_1, \dots, m_T\} = Tm$ and the variance term decreases with larger sample size, this bound is at least as tight as (and typically tighter than) the single-iteration bound from Lemma 3.9.

Substituting this improved bound into Theorem 3.10 yields a tighter performative excess risk bound. \square

C.7 Proof of Theorem 3.13

We restate the theorem for ease of exposition.

Theorem: The performative generalization gap $\mathcal{R}(\text{Tr}(d_0, \hat{\theta}_T), \hat{\theta}_T) - \mathcal{R}(\text{Tr}(\hat{d}_0, \hat{\theta}_T), \hat{\theta}_T)$ is upper bounded by

$$\frac{48}{\sqrt{n}} (\mathfrak{C}_\infty(\mathcal{F}) + L_\ell L_f R^{1-p} \mathcal{D}_{\mathcal{Z}}^p) + F \sqrt{\frac{2 \log(2/\delta)}{n}} + 2L_\ell R$$

with probability $1 - \delta$ and $R := \max \left\{ \left(q(\delta) \sqrt{\frac{\frac{m}{n}(1-\frac{m}{n})}{n}} \right)^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}}, \left(\frac{m}{n} \right)^{1/p} \mathcal{D}_{\mathcal{Z}} \tilde{L}_a \right\}$.

Proof. We want to bound the generalization gap under performativity

$$\mathcal{R}(\text{Tr}(d_0, \hat{\theta}_T), \hat{\theta}_T) - \mathcal{R}(\text{Tr}(\hat{d}_0, \hat{\theta}_T), \hat{\theta}_T).$$

Recall from Lemma 3.5

$$W_p(\hat{d}_0, \hat{d}_T) \leq \frac{\varepsilon^T - 1}{\varepsilon - 1} \left(\frac{m}{n} \right)^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}} \tilde{L}_a,$$

and analogously on the population

$$W_p(d_0, d_T) \leq \frac{\varepsilon^T - 1}{\varepsilon - 1} (s)^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}}.$$

Slightly overloading notation, denote $d_1 = \text{Tr}(d_0, \hat{\theta}_T)$ and $\hat{d}_1 = \text{Tr}(\hat{d}_0, \hat{\theta}_T)$, respectively. We have $W_p(d_0, d_1) \leq (s)^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}}$ and $W_p(\hat{d}_0, \hat{d}_1) \leq \left(\frac{m}{n} \right)^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}}$. This implies

$$\mathcal{R}(\text{Tr}(d_0, \hat{\theta}_T), \hat{\theta}_T) - \mathcal{R}(\text{Tr}(\hat{d}_0, \hat{\theta}_T), \hat{\theta}_T) \leq \sup_{d: W_p(d_0, d) \leq s^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}}} \mathcal{R}(d, \hat{\theta}_T) - \inf_{d: W_p(\hat{d}_0, d) \leq \left(\frac{m}{n} \right)^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}}} \mathcal{R}(d, \hat{\theta}_T). \quad (74)$$

Note that by reasoning from the Proof of Theorem 3.10 we have that

$$s < \frac{m}{n} + q(\delta) \sqrt{\frac{\frac{m}{n}(1 - \frac{m}{n})}{n}} \quad (75)$$

with probability $1 - \delta$, where $q(\delta) = q_{1 - \frac{1}{2}\delta}$ is the $(1 - \frac{1}{2}\delta)$ -quantile of a standard normal distribution.

Informed by this result, choose a common radius

$$R := \max \left\{ \left(\frac{m}{n} + q(\delta) \sqrt{\frac{\frac{m}{n}(1 - \frac{m}{n})}{n}} \right)^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}}, \left(\frac{m}{n} \right)^{1/p} \mathcal{D}_{\mathcal{Z}} \tilde{L}_a \right\}.$$

Then trivially

$$\{d : W_p(\hat{d}_0, d) \leq r_2\} \subseteq \{d : W_p(\hat{d}_0, d) \leq R\}. \quad (76)$$

It also holds with probability $1 - \delta$ that

$$\{d : W_p(d_0, d) \leq r_1\} \subseteq \{d : W_p(d_0, d) \leq R\}. \quad (77)$$

Therefore,

$$\begin{aligned} & \mathcal{R}(\text{Tr}(d_0, \hat{\theta}_T), \hat{\theta}_T) - \mathcal{R}(\text{Tr}(\hat{d}_0, \hat{\theta}_T), \hat{\theta}_T) \\ & \leq \sup_{W_p(d_0, d) \leq R} \mathcal{R}(d, \hat{\theta}_T) - \inf_{W_p(\hat{d}_0, d) \leq R} \mathcal{R}(d, \hat{\theta}_T). \end{aligned} \quad (78)$$

with probability $1 - \delta$. The right hand side is equivalent to

$$\underbrace{\sup_{W_p(d_0,d) \leq R} \mathcal{R}(d, \hat{\theta}_T) - \sup_{W_p(\hat{d}_0,d) \leq R} \mathcal{R}(d, \hat{\theta}_T)}_{(a)} + \underbrace{\sup_{W_p(\hat{d}_0,d) \leq R} \mathcal{R}(d, \hat{\theta}_T) - \inf_{W_p(\hat{d}_0,d) \leq R} \mathcal{R}(d, \hat{\theta}_T)}_{(b)} \quad (79)$$

As you might have already guessed, we will first bound (a), then (b). In (a), we are looking at two supremum risk functionals with respect to two Wasserstein balls with same radius R , but with different centers \hat{d}_0 and d_0 . In (b), we are looking at two infimum/supremum risk functionals with respect to the same Wasserstein ball (same radius R and same center \hat{d}_0).

- (a) Observe that $\sup_{W_p(d_0,d) \leq R} \mathcal{R}(d, \hat{\theta}_T)$ and $\sup_{W_p(\hat{d}_0,d) \leq R} \mathcal{R}(d, \hat{\theta}_T)$ are distributionally robust risk functionals. We can thus use the following dual characterization of a distributionally robust risk from [Gao & Kleywegt \(2023\)](#) for supremum risks centered around some d_c :

$$\sup_{W_p(d,d_c) \leq R} \mathcal{R}(d, \theta) = \min_{\lambda \geq 0} \{ \lambda R^p + \mathbb{E}_{d_c} [\varphi_{\lambda, \theta}(Z)] \}, \quad (80)$$

where

$$\varphi_{\lambda, \theta}(z) := \sup_{z' \in \mathcal{Z}} \{ \ell(f_\theta(x'), y') - \lambda \cdot d_{\mathcal{Z}}^p(z, z') \} \quad (81)$$

for any $\theta \in \Theta$ and any $\lambda \geq 0$ and $d_{\mathcal{Z}}$ a metric on \mathcal{Z} . [Gao & Kleywegt \(2023\)](#) prove this characterization holds for any upper semi-continuous function $\ell \circ f_\theta : \mathcal{Z} \rightarrow \mathbb{R}$ and for any d with p -finite moments, which holds in our setup with $1 \leq p \leq 2$. Since f_θ is continuous per condition 3.12 and ℓ is continuous in the data per condition 3.3, we can apply this result to (a) with respect to both true law d_0 and sample \hat{d}_0 and write

$$\begin{aligned} \sup_{W_p(d_0,d) \leq R} \mathcal{R}(d, \hat{\theta}_T) - \sup_{W_p(\hat{d}_0,d) \leq R} \mathcal{R}(d, \hat{\theta}_T) &= \min_{\lambda \geq 0} \left\{ \lambda R^p + \mathbb{E}_{d_0} [\varphi_{\lambda, \hat{\theta}_T}(Z)] \right\} \\ &\quad - \min_{\lambda \geq 0} \left\{ \lambda R^p + \mathbb{E}_{\hat{d}_0} [\varphi_{\lambda, \hat{\theta}_T}(Z)] \right\}. \end{aligned} \quad (82)$$

Defining $\hat{\lambda} := \arg \min_{\lambda} \left\{ \lambda R^p + \mathbb{E}_{\hat{d}_0} [\varphi_{\lambda, \hat{\theta}_T}(Z)] \right\}$, we get

$$\begin{aligned} &\min_{\lambda \geq 0} \left\{ \lambda R^p + \int_z \varphi_{\lambda, \hat{\theta}_T}(z) d_0(dz) \right\} - \min_{\lambda \geq 0} \left\{ \lambda R^p + \int_z \varphi_{\lambda, \hat{\theta}_T}(z) \hat{d}_0(dz) \right\} \\ &= \min_{\lambda \geq 0} \left\{ \lambda R^p + \int_z \varphi_{\lambda, \hat{\theta}_T}(z) d_0(dz) \right\} - \left(\hat{\lambda} R^p + \int_z \varphi_{\hat{\lambda}, \hat{\theta}_T}(z) \hat{d}_0(dz) \right) \\ &\leq \int_z \varphi_{\hat{\lambda}, \hat{\theta}_T}(z) (d_0 - \hat{d}_0)(dz) \end{aligned} \quad (83)$$

We will now show—closely following Lemma 1 by (Lee & Raginsky, 2018)—that $\widehat{\lambda} \leq L_\ell L_f R^{1-p}$, where $L_\ell L_f$ is the Lipschitz constant of $\ell \circ f_\theta$ as above.

It trivially holds

$$\widehat{\lambda} \cdot R^p \leq \widehat{\lambda} \cdot R^p + \mathbb{E}_{\widehat{d}_0} \left[\sup_{z' \in \mathcal{Z}} \left\{ f_{\widehat{\theta}_T} \circ \ell(z') - f_{\widehat{\theta}_T} \circ \ell(Z) - \widehat{\lambda} \cdot d_z^p(z, z') \right\} \right]. \quad (84)$$

Since $\widehat{\lambda}$ is optimal with respect to $\widehat{\theta}_T$ (see Definition above) and due to $\ell \circ f_\theta$ being $L_\ell L_f$ -Lipschitz we have $\forall \lambda \geq 0$:

$$\begin{aligned} & \widehat{\lambda} \cdot R^p + \mathbb{E}_{\widehat{d}_0} \left[\sup_{z' \in \mathcal{Z}} \left\{ f_{\widehat{\theta}_T} \circ \ell(z') - f_{\widehat{\theta}_T} \circ \ell(Z) - \widehat{\lambda} \cdot d_z^p(z, z') \right\} \right] \\ & \leq \lambda \cdot R^p + \mathbb{E}_{\widehat{d}_0} \left[\sup_{z' \in \mathcal{Z}} \left\{ f_{\widehat{\theta}_T} \circ \ell(z') - f_{\widehat{\theta}_T} \circ \ell(Z) - \lambda \cdot d_z^p(z, z') \right\} \right] \\ & \leq \lambda \cdot R^p + \mathbb{E}_{\widehat{d}_0} \left[\sup_{z' \in \mathcal{Z}} \left\{ L_\ell L_f \cdot d_z(z, z') - \lambda \cdot d_z^p(z, z') \right\} \right] \\ & \leq \lambda \cdot R^p + \sup_{t \geq 0} \{ L_\ell L_f \cdot t - \lambda \cdot t^p \}, \end{aligned}$$

where we parametrize $t = d_z(z, z')$ as in (Lee & Raginsky, 2018). If $p = 1$, we have

$$\widehat{\lambda} \cdot R \leq L_\ell L_f \cdot R + \sup_{t \geq 0} \{ L_\ell L_f \cdot t - L_\ell L_f \cdot t \} = L_\ell L_f \cdot R,$$

by setting $L_\ell L_f = \lambda$, which gives $\widehat{\lambda} \leq L_\ell L_f$. In case of $p > 1$, we have

$$\widehat{\lambda} \cdot R \leq \lambda \cdot R^p + L_\ell L_f^{\frac{p}{p-1}} p^{-\frac{p}{p-1}} (p-1) \lambda^{-\frac{1}{p-1}}$$

via $\arg \sup_{t \geq 0} \{ L_\ell L_f \cdot t - L_\ell L_f \cdot t \} = (L_\ell L_f / p \lambda)^{1/(p-1)}$. Minimizing the right-hand side over $\lambda \geq 0$ with the choice of $\lambda = L_\ell L_f / p R^{p-1}$, we get the above stated bound

$$\widehat{\lambda} \leq L_\ell L_f R \cdot R^{-p}.$$

In analogy to Theorem 2 by (Lee & Raginsky, 2018), we can now define the function class

$$\Phi := \{ \varphi_{\lambda, \theta} : \lambda \leq L_\ell L_f R^{1-p}, f_\theta \in \mathcal{F} \},$$

such that

$$\begin{aligned} \sup_{W_p(d_0, d) \leq R} \mathcal{R}(d, \widehat{\theta}_T) - \sup_{W_p(\widehat{d}_0, d) \leq R} \mathcal{R}(d, \widehat{\theta}_T) & \leq \int_{\mathcal{Z}} \varphi_{\widehat{\lambda}, \widehat{\theta}_T}(z) (d_0 - \widehat{d}_0)(dz) \\ & \leq \sup_{\varphi \in \Phi} \int_{\mathcal{Z}} \varphi(z) (d_0 - \widehat{d}_0)(dz). \end{aligned} \quad (85)$$

where we used equations 82 and 83.

We can eventually obtain the following Rademacher bound by standard symmetrization, following the proof of Theorem 3 in [Lee & Raginsky \(2018\)](#)

$$\sup_{\varphi \in \Phi} \int_{\mathcal{Z}} \varphi(Z) (d_0 - \hat{d}_0) \leq 2\mathfrak{R}_n(\Phi) + F \sqrt{\frac{2 \log(2/\delta)}{n}} \quad (86)$$

with probability of at least $1 - \delta/2$. We used that F is the upper bound of all $f_\theta \in \mathcal{F}$, see Section 3, and define

$$\mathfrak{R}_n(\Phi) := \mathbb{E} \left[\sup_{\varphi \in \Phi} \frac{1}{n} \sum_{i=1}^n \varepsilon_i \varphi(Z_i) \right] \quad (87)$$

as the standard empirical Rademacher average (see, e.g., chapter 26 in [Shalev-Shwartz & Ben-David, 2014a](#)) or ([Bousquet & Elisseeff, 2002](#); [Von Luxburg & Schölkopf, 2011](#)) of the above defined function class Φ with Rademacher random variables $\varepsilon_1, \dots, \varepsilon_n$ independent of the random variables describing the data.

All in all, we obtain for part (a):

$$\sup_{W_p(d_0, d) \leq R} \mathcal{R}(d, \hat{\theta}_T) - \sup_{W_p(\hat{d}_0, d) \leq R} \mathcal{R}(d, \hat{\theta}_T) \leq 2\mathfrak{R}_n(\Phi) + F \sqrt{\frac{2 \log(2/\delta)}{n}} \quad (88)$$

(b) Recall the classic Kantorovich-Rubinstein Lemma (see proof of Theorem 3.7)

$$\mathcal{R}(d', \theta) - \mathcal{R}(d'', \theta) \leq L_\ell W_1(d', d''), \quad (89)$$

for any θ and any d', d'' , where L_ℓ is the Lipschitz constant of ℓ . Also, it trivially holds that for any d', d'' such that $W_p(\hat{d}_0, d') \leq R$ and $W_p(\hat{d}_0, d'') \leq R$ (i.e., for any d', d'' in the Wasserstein ball around \hat{d}_0 of Radius R) that $W_p(d', d'') \leq 2R$. This directly implies that

$$\sup_{W_p(\hat{d}_0, d) \leq R} \mathcal{R}(d, \hat{\theta}_T) - \inf_{W_p(\hat{d}_0, d) \leq R} \mathcal{R}(d, \hat{\theta}_T) \leq 2L_\ell R \quad (90)$$

Combining (a) and (b), we retrieve that $\mathcal{R}(\text{Tr}(d_0, \hat{\theta}_T), \hat{\theta}_T) - \mathcal{R}(\text{Tr}(\hat{d}_0, \hat{\theta}_T), \hat{\theta}_T) \leq$

$$\underbrace{\sup_{W_p(d_0, d) \leq R} \mathcal{R}(d, \hat{\theta}_T) - \sup_{W_p(\hat{d}_0, d) \leq R} \mathcal{R}(d, \hat{\theta}_T)}_{\leq 2\mathfrak{R}_n(\Phi) + F \sqrt{\frac{2 \log(2/\delta)}{n}}} + \underbrace{\sup_{W_p(\hat{d}_0, d) \leq R} \mathcal{R}(d, \hat{\theta}_T) - \inf_{W_p(\hat{d}_0, d) \leq R} \mathcal{R}(d, \hat{\theta}_T)}_{\leq 2L_\ell R} \quad (91)$$

with probability $1 - \delta$. All that is left to do now is to bound the Rademacher average $\mathfrak{R}_n(\Phi)$ of the above defined function class $\Phi := \{\varphi_{\lambda,\theta} : \lambda \leq L_\ell L_f R^{1-p}, f_\theta \in \mathcal{F}\}$. To this end, define the Φ -indexed process $X = (X_\varphi)_{\varphi \in \Phi}$ via $X_\varphi := \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i \varphi(Z_i)$, with $\mathbb{E}[X_\varphi] = 0$ for all $\varphi \in \Phi$. Following Lemma 5 in Lee & Raginsky (2018), we first show that X is a subgaussian process with respect to a suitable pseudometric.

For $\varphi = \varphi_{\lambda,f}$ and $\varphi' = \varphi_{\lambda',f'}$, define $d_\Phi(\varphi, \varphi') := \|f - f'\|_\infty + \mathcal{D}_{\mathcal{Z}}^p |\lambda - \lambda'|$, for which it is easy to see that $\|\varphi - \varphi'\|_\infty \leq d_\Phi(\varphi, \varphi')$. Using Hoeffding's lemma and the fact that (ε_i, Z_i) are i.i.d., we obtain for any $c \in \mathbb{R}$

$$\begin{aligned} \mathbb{E}[\exp(c(X_\varphi - X_{\varphi'}))] &= \mathbb{E}\left[\exp\left(\frac{c}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i (\varphi(Z_i) - \varphi'(Z_i))\right)\right] \\ &= \left\{\mathbb{E}\left[\exp\left(\frac{c}{\sqrt{n}} \varepsilon_1 (\varphi(Z_1) - \varphi'(Z_1))\right)\right]\right\}^n \\ &\leq \exp\left(\frac{c^2 d_\Phi(\varphi, \varphi')^2}{2}\right). \end{aligned} \quad (92)$$

Hence, X is subgaussian with respect to $d_\Phi(\varphi, \varphi')$, and therefore the Rademacher average $R_n(\Phi)$ can be upper-bounded by the Dudley entropy integral (see e.g., Talagrand (2014)):

$$R_n(\Phi) \leq \frac{12}{\sqrt{n}} \int_0^\infty \sqrt{\log \mathcal{N}(\Phi, d_\Phi, u)} du,$$

where $\mathcal{N}(\Phi, d_\Phi, \cdot)$ denotes the covering numbers of (Φ, d_Φ) .

From the definition of $d_\Phi(\varphi, \varphi')$ above we have

$$\mathcal{N}(\Phi, d_\Phi, u) \leq \mathcal{N}\left(\mathcal{F}, \|\cdot\|_\infty, \frac{u}{2}\right) \cdot \mathcal{N}\left([0, L_\ell L_f R^{1-p}], |\cdot|, \frac{u}{2\mathcal{D}_{\mathcal{Z}}^p}\right),$$

and therefore

$$R_n(\Phi) \leq \frac{12}{\sqrt{n}} \left\{ \int_0^\infty \sqrt{\log \mathcal{N}\left(\mathcal{F}, \|\cdot\|_\infty, \frac{u}{2}\right)} du + \int_0^\infty \sqrt{\log \mathcal{N}\left([0, L_\ell L_f R^{1-p}], |\cdot|, \frac{u}{2\mathcal{D}_{\mathcal{Z}}^p}\right)} du \right\}.$$

Since $[0, L_\ell L_f R^{1-p}]$ is a compact interval, it is straightforward to upper-bound the second integral:

$$\begin{aligned} \int_0^\infty \sqrt{\log \mathcal{N}\left([0, L_\ell L_f R^{1-p}], |\cdot|, \frac{u}{2\mathcal{D}_{\mathcal{Z}}^p}\right)} du &\leq 2 L_\ell L_f R^{1-p} \mathcal{D}_{\mathcal{Z}}^p \int_0^{1/2} \sqrt{\log(1/u)} du \\ &= 2c L_\ell L_f R^{1-p} \mathcal{D}_{\mathcal{Z}}^p, \end{aligned} \quad (93)$$

where $L_\ell L_f R^{1-p}$ is the length of the interval $[0, L_\ell L_f R^{1-p}]$, and

$$c = \frac{1}{2} \sqrt{\log 2} + \sqrt{\pi} \operatorname{erfc}(\sqrt{\log 2}) < 1.$$

Thus,

$$R_n(\Phi) \leq \frac{12}{\sqrt{n}} \left\{ \int_0^\infty \sqrt{\log \mathcal{N}(\mathcal{F}, \|\cdot\|_\infty, \frac{u}{2})} du + 2 L_\ell L_f R^{1-p} \mathcal{D}_{\mathcal{Z}}^p \right\}.$$

Using definition 3.6 of $\mathfrak{C}_\infty(\mathcal{F}) := \int_0^\infty \sqrt{\log \mathcal{N}(\mathcal{F}, \|\cdot\|_\infty, \varepsilon)} d\varepsilon$ and integration by substitution (reverse chain rule), we obtain

$$R_n(\Phi) \leq \frac{24}{\sqrt{n}} (\mathfrak{C}_\infty(\mathcal{F}) + L_\ell L_f R^{1-p} \mathcal{D}_{\mathcal{Z}}^p).$$

Plugging this into Equation 91 yields

$$\begin{aligned} \mathscr{R}(\text{Tr}(d_0, \hat{\theta}_T), \hat{\theta}_T) - \mathscr{R}(\text{Tr}(\hat{d}_0, \hat{\theta}_T), \hat{\theta}_T) &\leq \frac{48}{\sqrt{n}} (\mathfrak{C}_\infty(\mathcal{F}) + L_\ell L_f R^{1-p} \mathcal{D}_{\mathcal{Z}}^p) \\ &\quad + F \sqrt{\frac{2 \log(2/\delta)}{n}} + 2L_\ell R. \end{aligned} \tag{94}$$

with probability $1 - \delta$, which was to be shown. \square

C.8 Proof of Theorem 3.15

We restate the Theorem for better readability.

Theorem (Generalization Gap II, RQ2) Under Conditions 3.1–3.3 and 3.14, we obtain the same bound on $\mathscr{R}(\text{Tr}(d_0, \hat{\theta}_T), \hat{\theta}_T) - \mathscr{R}(\text{Tr}(\hat{d}_0, \hat{\theta}_T), \hat{\theta}_T)$ as in Theorem 3.13 but with $B2^{p-1}(1 + \mathcal{D}_{\mathcal{Z}}/R)^p$ instead of $L_\ell L_f R^{1-p}$.

Proof. The reasoning mirrors the one in the proof of Theorem 3.13 except that we upper bound $\hat{\lambda}$ by $B2^{p-1}(1 + \mathcal{D}_{\mathcal{Z}}/R)^p$, where B is from Condition 3.14 and

$$R := \max \left\{ \left(\frac{m}{n} + q(\delta) \sqrt{\frac{\frac{m}{n}(1 - \frac{m}{n})}{n}} \right)^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}}, \left(\frac{m}{n} \right)^{1/p} \mathcal{D}_{\mathcal{Z}} \tilde{L}_a \right\}$$

as in the proof of Theorem 3.13.

Recall $\hat{\lambda} := \arg \min_{\lambda} \left\{ \lambda R^p + \mathbb{E}_{\hat{d}_0} [\varphi_{\lambda, \hat{\theta}_T}(Z)] \right\}$. Since $\varphi_{\lambda, \hat{\theta}_T} \geq 0$ for all, we clearly have $\hat{\lambda} \leq \sup_{W_p(d_0, d) \leq R} \mathscr{R}(d, \theta)/R^p$

Further recall from equation 80 the dual characterization of a locally supremum risk by Gao & Kleywegt (2023)

$$\sup_{W_p(d_0, d) \leq R} \mathscr{R}(d, \theta) = \min_{\lambda \geq 0} \{ \lambda R^p + \mathbb{E}_{d_c} [\varphi_{\lambda, \theta}(Z)] \}, \tag{95}$$

where

$$\varphi_{\lambda, \theta}(z) = \sup_{z' \in \mathcal{Z}} \{ \ell(f_\theta(x'), y') - \lambda \cdot d_{\mathcal{Z}}^p(z, z') \} \tag{96}$$

Thus,

$$\inf_{\theta} \sup_{W_p(d_0, d) \leq R} \mathcal{R}(d, \theta) = \inf_{\theta} \min_{\lambda \geq 0} \{ \lambda R^p + \mathbb{E}_{d_c} [\varphi_{\lambda, \theta}(Z)] \} \quad (97)$$

Hence, we can apply Lemma 2 by Lee & Raginsky (2018) to yield $\widehat{\lambda} \leq B2^{p-1}(1 + \mathcal{D}_Z/R)^p$. We can define the function class

$$\tilde{\Phi} := \{ \varphi_{\lambda, \theta} : \lambda \leq B2^{p-1}(1 + \mathcal{D}_Z/R)^p, f_\theta \in \mathcal{F} \} \quad (98)$$

instead of

$$\Phi := \{ \varphi_{\lambda, \theta} : \lambda \leq L_\ell L_f R^{1-p}, f_\theta \in \mathcal{F} \} \quad (99)$$

as in the proof of Theorem 3.13. The remainder directly follows from applying the reasoning in the proof of Theorem 3.13 to $\tilde{\Phi}$ instead of applying it to Φ .

In particular, in place of Equation 93, we get

$$\begin{aligned} & \int_0^\infty \sqrt{\log \mathcal{N}\left([0, B2^{p-1}(1 + \mathcal{D}_Z/R)^p], |\cdot|, \frac{u}{2\mathcal{D}_Z^p}\right)} du \\ & \leq 2B2^{p-1}(1 + \mathcal{D}_Z/R)^p \mathcal{D}_Z^p \int_0^{1/2} \sqrt{\log(1/u)} du \\ & = 2c B2^{p-1}(1 + \mathcal{D}_Z/R)^p \mathcal{D}_Z^p. \end{aligned} \quad (100)$$

where $B2^{p-1}(1 + \mathcal{D}_Z/R)^p$ is the length of the interval $[0, B2^{p-1}(1 + \mathcal{D}_Z/R)^p]$, and

$$c = \frac{1}{2} \sqrt{\log 2} + \sqrt{\pi} \operatorname{erfc}(\sqrt{\log 2}) < 1.$$

Thus,

$$R_n(\tilde{\Phi}) \leq \frac{12}{\sqrt{n}} \left\{ \int_0^\infty \sqrt{\log \mathcal{N}(\mathcal{F}, \|\cdot\|_\infty, \frac{u}{2})} du + 2B2^{p-1}(1 + \mathcal{D}_Z/R)^p \mathcal{D}_Z^p \right\}.$$

Using definition 3.6 of $\mathfrak{C}_\infty(\mathcal{F}) := \int_0^\infty \sqrt{\log \mathcal{N}(\mathcal{F}, \|\cdot\|_\infty, \varepsilon)} d\varepsilon$ and integration by substitution, we obtain

$$R_n(\tilde{\Phi}) \leq \frac{24}{\sqrt{n}} (\mathfrak{C}_\infty(\mathcal{F}) + B2^{p-1}(1 + \mathcal{D}_Z/R)^p \mathcal{D}_Z^p).$$

Applying this to the remainder of the proof of Theorem 3.13, we get

$$\begin{aligned} \mathcal{R}(\operatorname{Tr}(d_0, \widehat{\theta}_T), \widehat{\theta}_T) - \mathcal{R}(\operatorname{Tr}(\widehat{d}_0, \widehat{\theta}_T), \widehat{\theta}_T) & \leq \frac{48}{\sqrt{n}} \left(\mathfrak{C}_\infty(\mathcal{F}) + B2^{p-1}(1 + \mathcal{D}_Z/R)^p \right) \mathcal{D}_Z^p \\ & \quad + F \sqrt{\frac{2 \log(2/\delta)}{n}} + 2L_\ell R. \end{aligned} \quad (101)$$

with probability $1 - \delta$, which was to be shown. □

C.9 Proof of Theorem 3.16

For ease of exposition, we start by restating the claim.

Theorem: Under Conditions 3.1–3.3, we have for any $\tilde{T} \geq T$

$$\begin{aligned} \sum_{t=T}^{\tilde{T}} \left(\mathcal{R}(d_t, \theta_t) - \inf_{\theta \in \Theta} \mathcal{R}(d_t, \theta) \right) &\leq \mathcal{B}(T, m, n) + \\ (\tilde{T} - T + 1) L_\ell L_a &\left(\frac{m}{n} + q(\delta) \sqrt{\frac{\frac{m}{n}(1 - \frac{m}{n})}{n}} \right)^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}}. \end{aligned}$$

with probability $1 - \frac{\delta}{5}$, where $d_t = \text{Tr}(d_{t-1}, \theta_t)$ for $t \geq T$, $\theta_T = \hat{\theta}_T$ and $d_{T-1} = d_0$. $\mathcal{B}(T, m, n)$ is the performative excess risk bound from Theorem 3.10.

Proof. The idea of the proof is to leverage the recursive argument that θ_t is the (population) risk minimizer under d_{t-1} , which determines d_t via unknown Tr .

We want to bound the *cumulative performative excess risk*

$$\sum_{t=T}^{\tilde{T}} \mathcal{R}(d_t, \theta_t) - \inf_{\theta \in \Theta} \mathcal{R}(d_t, \theta)$$

with $d_t = \text{Tr}(d_{t-1}, \theta_t)$, $\theta_T = \hat{\theta}_T$ and $d_{T-1} = d_0$.

This term is equivalent to

$$\left(\mathcal{R}(\text{Tr}(d_0, \hat{\theta}_T), \hat{\theta}_T) - \inf_{\theta \in \Theta} \mathcal{R}(\text{Tr}(d_0, \hat{\theta}_T), \theta) \right) + \sum_{t=T+1}^{\tilde{T}} \mathcal{R}(d_t, \theta_t) - \inf_{\theta \in \Theta} \mathcal{R}(d_t, \theta). \quad (102)$$

We have an upper bound with high probability on $\mathcal{R}(\text{Tr}(d_0, \hat{\theta}_T), \hat{\theta}_T) - \inf_{\theta \in \Theta} \mathcal{R}(\text{Tr}(d_0, \hat{\theta}_T), \theta)$ per Theorem 3.10. In order to bound $\sum_{t=T+1}^{\tilde{T}} \mathcal{R}(d_t, \theta_t) - \inf_{\theta \in \Theta} \mathcal{R}(d_t, \theta)$, we first observe from definition 2.3 that θ_t is the (population) risk minimizer on d_{t-1} . In other words,

$$\theta_t \in \arg \inf_{\theta \in \Theta} \mathcal{R}(d_{t-1}, \theta), \quad (103)$$

and analogously

$$\theta_{t+1} \in \arg \inf_{\theta \in \Theta} \mathcal{R}(d_t, \theta). \quad (104)$$

We can thus write $\sum_{t=T+1}^{\tilde{T}} \mathcal{R}(d_t, \theta_t) - \inf_{\theta \in \Theta} \mathcal{R}(d_t, \theta) = \sum_{t=T+1}^{\tilde{T}} \mathcal{R}(d_t, \arg \inf_{\theta \in \Theta} \mathcal{R}(d_{t-1}, \theta)) - \inf_{\theta \in \Theta} \mathcal{R}(d_t, \theta)$, or more helpfully,

$$\sum_{t=T+1}^{\tilde{T}} \mathcal{R}(d_t, \theta_t) - \inf_{\theta \in \Theta} \mathcal{R}(d_t, \theta) = \sum_{t=T+1}^{\tilde{T}} \mathcal{R}(d_t, \theta_t) - \mathcal{R}(d_t, \theta_{t+1}). \quad (105)$$

Since Tr modifies a share $s \in [0, 1]$ of the population and \mathcal{Z} is bounded, an argument analogous to the in-sample case gives

$$\|\theta_t - \theta_{t+1}\| \leq L_a s^{1/p} \mathcal{D}_{\mathcal{Z}}, \quad (106)$$

where L_a is the Lipschitz constant of the map $P \mapsto \arg \min_{\theta \in \Theta} \mathcal{R}(P, \theta)$. We need to estimate the performativity propensity in the population, i.e., the share s of the population reacting to predictions. Lemma 3.9 gives

$$s < \frac{m}{n} + q(\delta) \sqrt{\frac{m}{n^2} (1 - \frac{m}{n})} \quad (107)$$

with probability $1 - \delta$, where $q(\delta) = q_{1 - \frac{1}{2}\delta}$ is the $(1 - \frac{1}{2}\delta)$ -quantile of a standard normal distribution and $\frac{m}{n}$ as above. Using this high-probability upper bound on s as above, we obtain

$$\|\theta_t - \theta_{t+1}\| \leq L_a \left(\frac{m}{n} + q(\delta) \sqrt{\frac{m(1 - \frac{m}{n})}{n}} \right)^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}} \quad (108)$$

with probability at least $1 - \delta$. Since ℓ is Lipschitz in θ (and thus \mathcal{R} as an integral over ℓ), we have

$$\mathcal{R}(d_t, \theta_t) - \mathcal{R}(d_t, \theta_{t+1}) \leq L_\ell \|\theta_t - \theta_{t+1}\|, \quad (109)$$

and hence

$$\mathcal{R}(d_t, \theta_t) - \mathcal{R}(d_t, \theta_{t+1}) \leq L_\ell L_a \left(\frac{m}{n} + q(\delta) \sqrt{\frac{m(1 - \frac{m}{n})}{n}} \right)^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}}. \quad (110)$$

with probability at least $1 - \delta$. This implies

$$\sum_{t=T-1}^{\tilde{T}} \mathcal{R}(d_t, \theta_t) - \mathcal{R}(d_t, \theta_{t+1}) \leq (\tilde{T} - T + 1) L_\ell L_a \left(\frac{m}{n} + q(\delta) \sqrt{\frac{m(1 - \frac{m}{n})}{n}} \right)^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}}. \quad (111)$$

with probability at least $1 - \delta$.

Together with the bound on $\mathcal{R}(\text{Tr}(d_0, \hat{\theta}_T), \hat{\theta}_T) - \inf_{\theta \in \Theta} \mathcal{R}(\text{Tr}(d_0, \hat{\theta}_T), \theta)$ per Theorem 3.10, we get

$$\sum_{t=T}^{\tilde{T}} \mathcal{R}(d_t, \theta_t) - \inf_{\theta \in \Theta} \mathcal{R}(d_t, \theta) \leq \mathcal{B}(T, m, n) + (\tilde{T} - T + 1) L_\ell L_a \left(\frac{m}{n} + q(\delta) \sqrt{\frac{m(1 - \frac{m}{n})}{n}} \right)^{\frac{1}{p}} \mathcal{D}_{\mathcal{Z}}. \quad (112)$$

with probability greater or equal than $1 - \frac{\delta}{5}$, where $\mathcal{B}(T, m, n)$ is the performativity excess risk bound from Theorem 3.10. \square