1 Introduction

2 Background

2.1 Notation

2.1.1 Hamming Distance

How to define the Hamming distance with the notation in this paper, which emphasizes units (rather than indices) of records in datasets.

Ways which are kind of cheating:

1. Don't define the Hamming distance precisely (basically what I've done at the moment).
2. Define the Hamming distance $d_{\mathrm{Ham}}(\mathfrak{d}, \mathfrak{d}')$ as the number of records that differ between $\mathfrak{d}$ and $\mathfrak{d}'$:
$$d_{\mathrm{Ham}}(\mathfrak{d}, \mathfrak{d}') = \sum_i \mathbb{1}\{\mathfrak{d}_i \neq \mathfrak{d}'_i\}.$$

Other ways (more rigorous but perhaps overkill):

1. Require that the unit sets are the same. That is, define
$$d_{\mathrm{Ham}}(\mathfrak{d}, \mathfrak{d}') = \begin{cases} \sum_{i \in \mathcal{U}(\mathfrak{d})} \mathbb{1}\{\mathfrak{d}_i \neq \mathfrak{d}'_i\} & \text{if } \mathcal{U}(\mathfrak{d}) = \mathcal{U}(\mathfrak{d}'), \\ \infty & \text{otherwise.} \end{cases}$$
I think this is the definition I take (implicitly?) in the proof of the privacy semantics results.
2. Use set notation:
$$d_{\mathrm{Ham}}(\mathfrak{d}, \mathfrak{d}') = \begin{cases} \frac{1}{2}|\mathfrak{d} \Delta \mathfrak{d}'| & \text{if } |\mathfrak{d}| = |\mathfrak{d}'|, \\ \infty & \text{otherwises.} \end{cases}$$
   - Question on this approach: Are the units (or unit identifiers) part of the record, or not? I.e. if two records agree on all attributes but belong to different units, does the set difference cancel them out? My answer: records are just a set of attributes; the unit is not part of the record. So if, for example, $\mathfrak{d}$ and $\mathfrak{d}'$ each contained a single record which had the same attributes but belonged to different units, then $d_{\mathrm{Ham}}(\mathfrak{d}, \mathfrak{d}') = 0$.
   - Thus I'm essentially saying to strip the dataset of its units before taking the Hamming distance.
   - How does this fit with the idea in DP that a unit's records can counterfactually change (not the unit changing, just the unit's records changing)? It seems like approach 1.

(requiring that the unit sets are the same) is more closely aligned to this idea.

3. Assume that there is some ordering of the records $i = 1, \dots, n$ in $\mathfrak{d}$ and $\mathfrak{d}'$ which is unrelated to the records' units. Write $\mathfrak{d}_i$ for the i-th record of $\mathfrak{d}$ in this ordering. Then define

$$d_{\text{Ham}}(\mathfrak{d}, \mathfrak{d}') = \begin{cases} \sum_{i=1}^n \mathbb{1}\{\mathfrak{d}_i \neq \mathfrak{d}'_i\} & \text{if } |\mathfrak{d}| = |\mathfrak{d}'|, \\ \infty & \text{otherwise.} \end{cases}$$

2.2 Differential Privacy

DP studies data-release mechanisms – functions T which take as input a dataset $\mathfrak{d}$ and a random seed ω, and output a stochastic summary $T(\mathfrak{d}, \omega)$ of $\mathfrak{d}$. **[JB: Do we need to talk about the random seed ω? Can we drop this for this paper?]**

Definition 2.1. A *data-release mechanism* is a function $T : \mathcal{D}_0 \times \Omega \to \mathcal{T}$ where **[JB: Get rid of the seed]**

- $\mathcal{D}_0$ is the data space, the set of all theoretically-possible datasets $\mathfrak{d}$; **[JB: Move DP specification first?]**
- Ω is the probability space of the seed ω with σ-algebra $\mathscr{F}_\Omega$ and probability P;
- $\mathcal{T}$ is equipped with a σ-algebra $\mathscr{F}_{\mathcal{T}}$ **[JB: Get rid of this – just reference Bailie et al.]**; and
- $T(\mathfrak{d}, \cdot)$ is measurable for all $\mathfrak{d} \in \mathcal{D}_0$.

(See Bailie et al., 2025 for a slightly more general definition and for additional context.) $\diamond$

Intuitively speaking, $\mathfrak{d}$ is the data that is considered confidential and hence must not be disclosed by the summary $T(\mathfrak{d}, \omega)$. DP measures how the probabilistic noise induced by the seed ω masks this input dataset $\mathfrak{d}$.

It is convenient to think of a data-release mechanism as a function $\mathfrak{d} \mapsto \mathsf{P}_{\mathfrak{d}}(T \in \cdot)$. Here the probability distribution $\mathsf{P}_{\mathfrak{d}}(T \in \cdot)$ of the summary $T(\mathfrak{d}, \omega)$ is the push-forward measure induced by the distribution P of the random seed $\omega \in \Omega$, taking $\mathfrak{d}$ as fixed:

$$\mathsf{P}_{\mathfrak{d}}(T \in E) := \mathsf{P}(\{\omega \in \Omega : T(\mathfrak{d}, \omega) \in E\}),$$

where $E \in \mathscr{F}_{\mathcal{T}}$ is any measurable subset of the output space $\mathcal{T}$. DP is the condition that the data-release mechanism is Lipschitz continuous – i.e. that the distance $d_{\text{Pr}}(\mathsf{P}_{\mathfrak{d}}, \mathsf{P}_{\mathfrak{d}'})$ between outputs $\mathsf{P}_{\mathfrak{d}}$ and $\mathsf{P}_{\mathfrak{d}'}$ is at most a multiplicative factor of the distance $d_{\mathcal{D}_0}(\mathfrak{d}, \mathfrak{d}')$ between the corresponding inputs $\mathfrak{d}$ and $\mathfrak{d}'$.

By definition, a data-release mechanism T satisfies DP if it is Lipschitz continuous. There are different *flavors* (i.e. types or versions) of DP; each of these flavors correspond to different ways to specify continuity. For our purposes, there are four components to the specification of Lipschitz continuity. Most obviously, there are the premetrics $d_{\mathcal{D}_0}(\mathfrak{d}, \mathfrak{d}')$ and $d_{\mathrm{Pr}}(\mathsf{P}_{\mathfrak{d}}, \mathsf{P}_{\mathfrak{d}'})$. These premetrics measure the 'distance' between any two inputs $\mathfrak{d}$ and $\mathfrak{d}'$, or between any two output probabilities $\mathsf{P}_{\mathfrak{d}}$ and $\mathsf{P}_{\mathfrak{d}'}$. Secondly, there is the domain $\mathcal{D}_0$ of the data-release mechanism, which – as we shall see – serves as the parameter space of the attacker's inferential model.[1]

Definition 2.2 (Bailie et al., 2025)**.** A *differential privacy flavor* is a quadruple $(\mathcal{D}_0, \mathscr{D}, d_{\mathcal{D}_0}, d_{\mathrm{Pr}})$ where:

1. The *domain* $\mathcal{D}_0$ is the *data space* – the set of all (theoretically-possible) input datasets.
2. The *multiverse* $\mathscr{D} \subset 2^{\mathcal{D}_0}$ is a set of *universes*, which are denoted by $\mathcal{D}$ or $\mathcal{D}'$.
3. The *input premetric* $d_{\mathcal{D}_0}$ is a premetric on $\mathcal{D}_0$ – i.e. a function $\mathcal{D}_0 \times \mathcal{D}_0 \to \mathbb{R}^{\geq 0}$ such that $d_{\mathcal{D}_0}(\mathfrak{d}, \mathfrak{d}) = 0$ for all $\mathfrak{d} \in \mathcal{D}_0$.
4. The *output premetric* d_{Pr} is a premetric on the space of all probability distributions $\mathcal{P}$ – i.e. a function $\mathcal{P} \times \mathcal{P} \to \mathbb{R}^{\geq 0}$ of probabilities $\mathsf{P}, \mathsf{Q} \in \mathcal{P}$ such that
   - $d_{\mathrm{Pr}}(\mathsf{P}, \mathsf{P}) = 0$ for all $\mathsf{P} \in \mathcal{P}$; and
   - $d_{\mathrm{Pr}}(\mathsf{P}, \mathsf{Q}) = \infty$ for probabilities P, Q which live on different measurable spaces. ◇

Once we have specified the four components for Lipschitz continuity via a DP flavor, we also need to specify the multiplicative constant (known as the Lipschitz constant) which controls the rate between input and output variations. Together, choices for these five components are called a DP specification:

Definition 2.3. A *differential privacy specification* is a quintuple $(\mathcal{D}_0, \mathscr{D}, d_{\mathcal{D}_0}, d_{\mathrm{Pr}}, \varepsilon_{\mathcal{D}})$ consisting of a DP flavor $(\mathcal{D}_0, \mathscr{D}, d_{\mathcal{D}_0}, d_{\mathrm{Pr}})$ and a privacy-loss budget $\varepsilon_{\mathcal{D}} : \mathscr{D} \to \mathbb{R}^{\geq 0}$. We denote a DP specification by $\varepsilon_{\mathcal{D}}$-DP$(\mathcal{D}_0, \mathscr{D}, d_{\mathcal{D}_0}, d_{\mathrm{Pr}})$.

A data-release mechanism $T : \mathcal{D}_0 \times \Omega \to \mathcal{T}$ *satisfies* the DP specification $\varepsilon_{\mathcal{D}}$-DP$(\mathcal{D}_0, \mathscr{D}, d_{\mathcal{D}_0}, d_{\mathrm{Pr}})$ if, for all data universes $\mathcal{D} \in \mathscr{D}$, and all $\mathfrak{d}, \mathfrak{d}' \in \mathcal{D}$,

$$d_{\mathrm{Pr}}\left[\mathsf{P}_{\mathfrak{d}}(T \in \cdot), \mathsf{P}_{\mathfrak{d}'}(T \in \cdot)\right] \leq \varepsilon_{\mathcal{D}} d_{\mathcal{D}_0}(\mathfrak{d}, \mathfrak{d}'). \tag{2.1}$$

Let $\mathcal{M}(\mathcal{D}_0, \mathscr{D}, d_{\mathcal{D}_0}, d_{\mathrm{Pr}}, \varepsilon_{\mathcal{D}})$ denote the set of data-release mechanisms which satisfy the DP specification $\varepsilon_{\mathcal{D}}$-DP$(\mathcal{D}_0, \mathscr{D}, d_{\mathcal{D}_0}, d_{\mathrm{Pr}})$. ◇

[1] This discussion is still missing at this point, but will be included in the final version of the paper.

For the purposes of understanding DP in the context of survey sampling, the relevant components of a DP flavor $(\mathcal{D}_0, \mathscr{D}, d_{\mathcal{D}_0}, d_{\mathrm{Pr}})$ are its domain $\mathcal{D}_0$ and its multiverse $\mathscr{D}$.

Example 2.4. **[JBN: Reword this: Pure ϵ-DP corresponds to the choice... (similar to this wording: " pure ϵ-DP, approximate (ϵ, δ)-DP, and ρ-zero concentrated DP (ρ-zCDP) specify choices for the output premetric d_{Pr} only. This is evidenced by the fact that there are both bounded and unbounded versions of these three notions"]** For pure ε-DP, as defined in Dwork et al. (2006b), the multiplicative factor is ε; the distance between inputs $\mathfrak{d}$ and $\mathfrak{d}'$ is the Hamming distance; and the distance between outputs $\mathsf{P}_{\mathfrak{d}}$ and $\mathsf{P}_{\mathfrak{d}'}$ is the multiplicative distance:

$$d_{\mathrm{MULT}}(\mathsf{P}_{\mathfrak{d}}, \mathsf{P}_{\mathfrak{d}'}) = \sup_{E \in \mathscr{F}_{\mathcal{T}}} \left| \ln \frac{\mathsf{P}_{\mathfrak{d}}(T \in E)}{\mathsf{P}_{\mathfrak{d}'}(T \in E)} \right|,$$

(For readers that are familiar with the definition of pure ε-DP in terms of neighboring datasets $\mathfrak{d}$ and $\mathfrak{d}'$, the Lipschitz condition (2.1) for non-neighbors is implied by group privacy. Hence, the neighbor definition of pure ε-DP is the equivalent to the above definition.) **[JD: Why do we suddenly have a different notation ($\mathsf{P}_{\mathfrak{d}}(T\ldots)$ instead of $P(T(\mathfrak{d}))$)? Is this just old notation? And what is $\mathscr{F}_{\mathcal{T}}$?] [JB: Sorry for the confusion – this is indeed the old notation. I was in the middle of redrafting this section.]**

The notion of ρ-zero concentrated DP (zCDP) (Bun & Steinke, 2016) is the condition that the output premetric is the *normalized Rényi metric* D_{nor} for d_{Pr}:

$$D_{\mathrm{nor}}(\mathsf{P}, \mathsf{Q}) = \sup_{\alpha > 1} \frac{1}{\sqrt{\alpha}} \max\left\{ \sqrt{D_\alpha(\mathsf{P}||\mathsf{Q})}, \sqrt{D_\alpha(\mathsf{Q}||\mathsf{P})} \right\},$$

where D_α is the Rényi divergence of order α:

$$D_\alpha(\mathsf{P}||\mathsf{Q}) = \begin{cases} \frac{1}{\alpha-1} \ln \int \left[\frac{d\mathsf{P}}{d\mathsf{Q}}\right]^\alpha d\mathsf{Q}, & \text{if } \mathsf{P} \text{ is absolutely continuous wrt. } \mathsf{Q}, \\ \infty & \text{otherwise.} \end{cases}$$

For approximate (ε, δ)-DP (Dwork et al., 2006a), the multiplicative factor is again ε; the distance between inputs is given by

$$d_{\mathcal{D}_0}^{\mathrm{neighbors}}(\mathfrak{d}, \mathfrak{d}') = \begin{cases} 0 & \text{if } \mathfrak{d} = \mathfrak{d}', \\ 1 & \text{if } \mathfrak{d} \text{ and } \mathfrak{d}' \text{ are neighbors}, \\ \infty & \text{otherwise;} \end{cases}$$

and the distance between outputs is given by

$$d^{\delta}_{\text{MULT}}(\mathsf{P}_{\mathfrak{d}}, \mathsf{P}_{\mathfrak{d}'}) = \sup_{E \in \mathscr{F}_{\mathcal{T}}} \left\{ \ln \frac{[\mathsf{P}_{\mathfrak{d}}(T \in E) - \delta]^+}{\mathsf{P}_{\mathfrak{d}'}(T \in E)}, \ln \frac{[\mathsf{P}_{\mathfrak{d}'}(T \in E) - \delta]^+}{\mathsf{P}_{\mathfrak{d}}(T \in E)}, 0 \right\},$$

(where $[x]^+ = \max\{x, 0\}$). Note that $d^{\text{neighbors}}_{\mathcal{D}_0}$ and d^{δ}_{MULT} are not distances in the mathematical sense of a metric; we will instead refer to them as *premetrics* from herein. Since d^{δ}_{MULT} does not satisfy the triangle inequality, approximate (ε, δ)-DP's group privacy budget does not increase linearly with the group size; hence we cannot replace $d^{\text{neighbors}}_{\mathcal{D}_0}$ with the Hamming distance, as we did for pure ε-DP. ◇

2.2.1 Technical definitions:

Definition 2.5. **[JB: Do we need this definition?]** Given a DP flavor $(\mathcal{D}_0, \mathscr{D}, d_{\mathcal{D}_0}, d_{\text{Pr}})$, two datasets $\mathfrak{d}, \mathfrak{d}' \in \mathcal{D}_0$ are *comparable* when 1) $\mathfrak{d} \neq \mathfrak{d}'$; 2) $d_{\mathcal{D}_0}(\mathfrak{d}, \mathfrak{d}') < \infty$ or $d_{\mathcal{D}_0}(\mathfrak{d}', \mathfrak{d}) < \infty$; and 3) there exists a data universe $\mathcal{D} \in \mathscr{D}$ such that $\mathfrak{d}, \mathfrak{d}' \in \mathcal{D}$. ◇

[JB: To write:] Note that comparable datasets are those that have direct conditions placed on them by DP. But there may be indirect conditions imposed on datasets which are not comparable. There is an implicit restriction between $T(\mathfrak{d})$ and $T(\mathfrak{d}')$ if and only if $\mathfrak{d}$ and $\mathfrak{d}'$ are connected. **[JB: Do I include all of this?]**

Definition 2.6. Given a DP flavor $(\mathcal{D}_0, \mathscr{D}, d_{\mathcal{D}_0}, d_{\text{Pr}})$, denote the protection objects *connected* to $\mathfrak{d} \in \mathcal{D}_0$ by

$$[\mathfrak{d}] = \{\mathfrak{d}' \in \mathcal{D}_0 : d_{\mathcal{D}_0}(\mathfrak{d}, \mathfrak{d}') < \infty\}.$$

[JB: Do we need also $d_{\mathcal{D}_0}(\mathfrak{d}', \mathfrak{d})$?] **[JB: Maybe not, maybe we actually want something small. I think we don't actually want this]** **[JB: Define connected components.]** Then the *completion* $\overline{\mathscr{D}}$ of the data multiverse $\mathscr{D}$ is defined as

$$\overline{\mathscr{D}} = \{\mathcal{D} \cap [\mathfrak{d}] : \mathcal{D} \in \mathscr{D}, \mathfrak{d} \in \mathcal{D}\}.$$

[JB: This is the right definition of completion (we want the universes to be as small as possible). But I think it is the wrong definition of connectedness. Connectedness should be symmetric.]

[JB: Better way to define this: Comparability is an equivalence relation. Then take the completion of $\mathscr{D}$ as the partition of $\mathcal{D}_0$ under this equivalence relation.] ◇

Lemma 2.7. **[JB: Update notation and proof. Can this partially be proved using the above Proposition 2.4?]** *Let $(\mathcal{D}_0, \mathscr{D}, d_{\mathcal{D}_0}, d_{\text{Pr}})$ be a DP flavor where $d_{\mathcal{D}_0}$ is a metric* **[JB: Do we just need symmetry? Maybe we don't need anything, if we get the definition of complete correct.]**. *Then, the completion $\overline{\mathscr{D}}$ of*

$\mathscr{D}$ is complete and, for all budgets $\varepsilon_{\mathcal{D}} : \mathscr{D} \to \mathbb{R}^{\geq 0}$,

$$\mathcal{M}(\mathcal{D}_0, \overline{\mathscr{D}}, d_{\mathcal{D}_0}, d_{\mathrm{Pr}}, \varepsilon_{\mathcal{D}'}) = \mathcal{M}(\mathcal{D}_0, \mathscr{D}, d_{\mathcal{D}_0}, d_{\mathrm{Pr}}, \varepsilon_{\mathcal{D}}),$$

where

$$\varepsilon_{\mathcal{D}'} = \inf\{\varepsilon_{\mathcal{D}} : \mathcal{D} \in \mathscr{D} \text{ s.t. } \mathcal{D}' \subset \mathcal{D}\}.$$

All the above definitions are properties just of $(\mathcal{D}_0, \mathscr{D}, d_{\mathcal{D}_0})$ and not d_{Pr} – i.e. they are properties not of a DP flavor $(\mathcal{D}_0, \mathscr{D}, d_{\mathcal{D}_0}, d_{\mathrm{Pr}})$ but of the first three components of a DP flavor.

Definition 2.8. A multiverse $\mathscr{D}$ is *singular* if $|\mathcal{D} \cap \mathcal{D}'| \leq 1$ for all distinct $\mathcal{D} \neq \mathcal{D}' \in \mathscr{D}$. $\diamond$

WLOG, we assume throughout that $\mathscr{D}$ is singular. (Why can we do this? Because we can replace any DP flavor with an equivalent, singular flavor, right?)

$\mathscr{D}$ being singular is nice to have, but not necessary, right? It's nice to have because any pair of datasets $(\mathfrak{d}, \mathfrak{d}')$ belong to at most one universe, so you only have to deal with one budget at a time. You don't have to consider all $\mathcal{D} \in \mathscr{D}$ with $\mathfrak{d}, \mathfrak{d}' \in \mathcal{D}$ and then take the infimum of $\varepsilon_{\mathcal{D}}$ over all such $\mathcal{D}$.

Result: If $\mathscr{D}$ is singular, then its completion $\overline{\mathscr{D}}$ is also singular. (But it is definitely not the case that any completion $\overline{\mathscr{D}}$ is singular.)

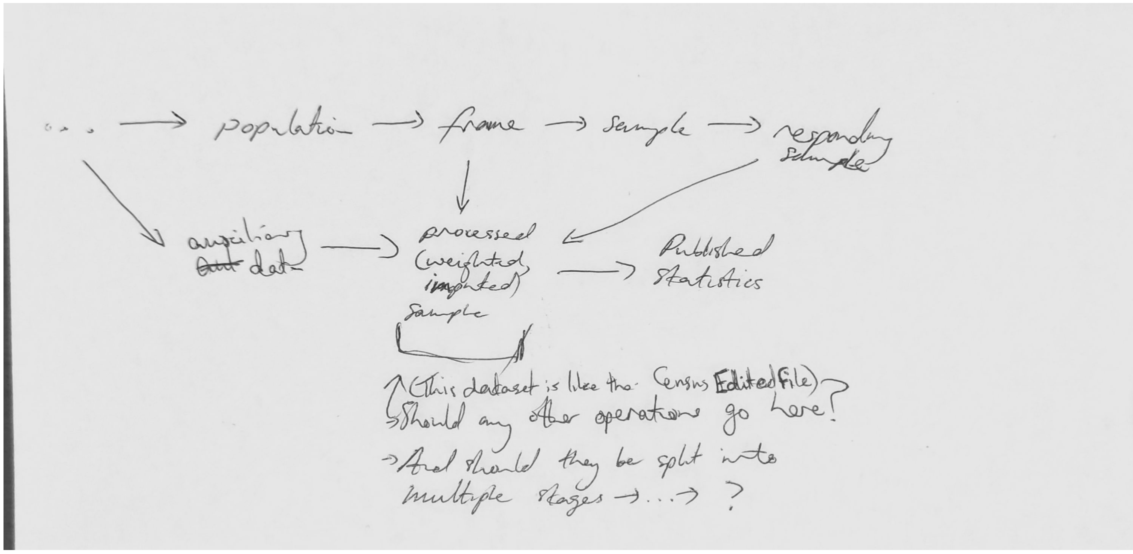

Figure 3.1: Caption. **[JBN: Should the auxiliary data arrow come from out of the population? Maybe there should be arrows into auxiliary data from both ... and the population]**

2.3 Survey Sampling

2.3.1 The Augmented Frame

2.4 Survey Weights

3 DP Flavors for Survey Statistics

3.1 Moved from Section 2

3.2 Recap: Survey Statistics Data Pipeline/Lifecycle

4 Utility Considerations

4.1 Privacy Amplification via Sampling

4.2 Privacy Amplification for Different DP Flavors

4.3 Weighting

4.4 Sensitivity Reduction from the Sampling Design

4.5 Utility Implications for the Horvitz-Thompson Estimator

4.5.1 The Laplace Mechanism for the Horvitz-Thompson Estimator

4.5.2 Sensitivity of the Horvitz-Thompson Estimator

5 Privacy Considerations

5.1 Privacy Semantics

5.1.1 Posterior-to-Posterior Comparisons

5.1.2 Implications for the Different Settings

5.1.3 No Privacy Amplification if the Attacker Knows that Unit i Is in the Sample

5.1.4 The Journalist and Sampling Amplification

5.2 Amplification by Sampling and Composition

5.3 DP and the Sample Design

5.4 Use of auxiliary data

6 Discussion

Acknowledgements

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A Proofs

Proof of Lemma 2.7. **[JB: This proof needs to be completely rewritten since we now have different definitions and lemma statements. E.g. there is nothing to do with connectedness anymore.]** Let $\mathcal{D}' \in \overline{\mathscr{D}}$. Then there exists some $\mathcal{D} \in \mathscr{D}$ and $\boldsymbol{x} \in \mathcal{D}$ such that $\mathcal{D}' = \mathcal{D} \cap [\boldsymbol{x}]$. Since every $\boldsymbol{x}', \boldsymbol{x}'' \in [\boldsymbol{x}]$ are connected, it follows that every $\boldsymbol{x}', \boldsymbol{x}'' \in \mathcal{D}'$ are also connected. This proves that $\overline{\mathscr{D}}$ is complete.

Suppose that $T \in \mathcal{M}(\mathcal{X}, \overline{\mathscr{D}}, d_{\mathcal{D}_0}, d_{\mathrm{Pr}}, \varepsilon_{\mathcal{D}'})$. Take some $\mathcal{D} \in \mathscr{D}$ and some $\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{D}$. We wish to show that

$$d_{\mathrm{Pr}}(\mathsf{P}_{\mathfrak{d}}, \mathsf{P}_{\mathfrak{d}'}) \leq \varepsilon_{\mathcal{D}} d_{\mathcal{D}_0}(\boldsymbol{x}, \boldsymbol{x}'). \tag{A.1}$$

We may assume without loss of generality that $d_{\mathcal{D}_0}(\boldsymbol{x}, \boldsymbol{x}') < \infty$. Define $\mathcal{D}' = \mathcal{D} \cap [\boldsymbol{x}]$. Since $\mathcal{D}' \in \overline{\mathscr{D}}$ and $\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{D}'$, we know that

$$d_{\mathrm{Pr}}(\mathsf{P}_{\mathfrak{d}}, \mathsf{P}_{\mathfrak{d}'}) \leq \varepsilon_{\mathcal{D}'} d_{\mathcal{D}_0}(\boldsymbol{x}, \boldsymbol{x}').$$

(A.1) then follows by observing that $\varepsilon_{\mathcal{D}'} \leq \varepsilon_{\mathcal{D}}$.

Suppose that $T \in \mathcal{M}(\mathcal{D}_0, \mathscr{D}, d_{\mathcal{D}_0}, d_{\mathrm{Pr}}, \varepsilon_{\mathcal{D}})$. Take some $\mathcal{D}' \in \overline{\mathscr{D}}$ and some $\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{D}'$. Then

$$\begin{aligned} d_{\mathrm{Pr}}(\mathsf{P}_{\mathfrak{d}}, \mathsf{P}_{\mathfrak{d}'}) &\leq \inf\{\varepsilon_{\mathcal{D}} d_{\mathcal{D}_0}(\boldsymbol{x}, \boldsymbol{x}') : \mathcal{D} \in \mathscr{D} \text{ s.t. } \boldsymbol{x}, \boldsymbol{x}' \in \mathcal{D}\} \\ &\leq \inf\{\varepsilon_{\mathcal{D}} d_{\mathcal{D}_0}(\boldsymbol{x}, \boldsymbol{x}') : \mathcal{D} \in \mathscr{D} \text{ s.t. } \mathcal{D}' \subset \mathcal{D}\} \\ &= \varepsilon_{\mathcal{D}'} d_{\mathcal{D}_0}(\boldsymbol{x}, \boldsymbol{x}'). \end{aligned}$$

□

Proof of Theorem **??**. Let $\mathcal{D} \in \mathscr{D}$ and $\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{D}$. The density of $\mathsf{P}_{\mathfrak{d}}(T \in \cdot)$ is

$$f_{\boldsymbol{x}}(t) = (2\Delta_q([\boldsymbol{x}]_{\mathscr{D}}))^{-k} \exp\left(-\frac{\|t - q(\boldsymbol{x})\|_1}{\Delta_q([\boldsymbol{x}]_{\mathscr{D}})}\right).$$

Thus,

$$\begin{aligned} d_{\textsc{Mult}}(\mathsf{P}_{\mathfrak{d}}, \mathsf{P}_{\mathfrak{d}'}) &= \sup_{t \in \mathbb{R}} \left| \ln \frac{f_{\boldsymbol{x}}(t)}{f_{\boldsymbol{x}'}(t)} \right| \\ &= \sup_{t \in \mathbb{R}} \left| \frac{\|t - q(\boldsymbol{x}')\|_1 - \|t - q(\boldsymbol{x})\|_1}{\Delta_q([\boldsymbol{x}]_{\mathscr{D}})} \right| \\ &\leq \varepsilon d_{\mathcal{D}_0}(\boldsymbol{x}, \boldsymbol{x}'), \end{aligned}$$

where the first line follows by Proposition 38 of Bailie and Gong, 2024, the second because $[\boldsymbol{x}]_{\mathscr{D}} = [\boldsymbol{x}']_{\mathscr{D}}$ and the third by the reverse triangle inequality. □