Privacy, Data Privacy and Differential Privacy

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Tübingen

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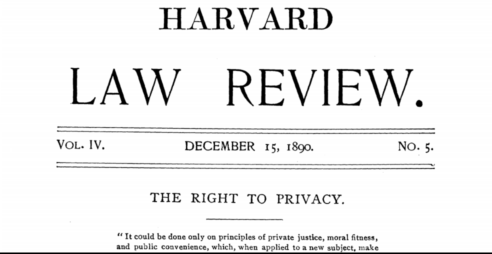
HARVARD

LAW REVIEW.

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THE RIGHT TO PRIVACY.

"It could be done only on principles of private justice, moral fitness, and public convenience, which, when applied to a new subject, make

The right to be ***let alone*****.**

Samuel D. Warren II

Louis Brandeis

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- Philosophy: "**Privacy . . . is a concept in disarray.** ... Currently privacy is a sweeping concept. . . . Philosophers . . . have frequently lamented the great difficulty in reaching a satisfying conception of privacy."
  Solove (2008) *Understanding Privacy.* Harvard University Press.

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- Releasing statistics while maintaining privacy

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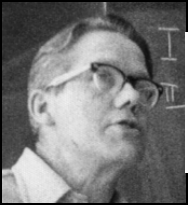
  - Dalenius (1977), Duncan & Lambert (1986):

    If the release of the statistics T makes it possible to determine [a record X_i] more accurately than is possible without access to T, a disclosure has taken place.



Towards a methodology for statistical disclosure control

by Tore Dalenius[1]

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- To produce useful statistics, we must allow for some (ideally small) amount of disclosure.
- Measure "amount of disclosure" by how much $\pi(X_i)$ and $\pi(X_i \mid T)$ differ.

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Thinking about T as a function of the dataset $\boldsymbol{x}$, its derivative is

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$\underline{d_{\mathrm{Pr}}}$: (ε, δ)-approximate DP (Dwork, Kenthapadi, et al., 2006) Rényi DP (Mironov, 2017) concentrated DP (Bun & Steinke, 2016) f-divergence privacy (Barber & Duchi, 2014; Barthe & Olmedo, 2013) f-DP (including Gaussian DP) (Dong et al., 2022)

$\underline{d_{\mathcal{X}}}$: $(\mathcal{R}, \varepsilon)$-generic DP (Kifer & Machanavajjhala, 2011a) edge vs node privacy (Hay et al., 2009; McSherry & Mahajan, 2010) d-metric DP (Chatzikokolakis et al., 2013) Blowfish privacy (He et al., 2014) element level DP (Asi et al., 2022) distributional privacy (Zhou et al., 2009) event-level vs user-level DP (Dwork et al., 2010)

$\underline{\mathscr{D}}$: privacy under invariants (Ashmead et al., 2019; Gong & Meng, 2020; Gao et al., 2022; Dharangutte et al., 2023) conditioned or empirical DP (J. M. Abowd et al., 2013; Charest & Hou, 2016) personalized DP (Ebadi et al., 2015; Jorgensen et al., 2015) individual DP (Soria-Comas et al., 2017; Feldman & Zrnic, 2022) bootstrap DP (O'Keefe & Charest, 2019) stratified DP (Bun et al., 2022) per-record DP (Seeman et al., 2023+) per-instance DP (Wang, 2018; Redberg & Wang, 2021)

$\underline{\mathcal{X}}$: DP for network data (Hay et al., 2009) for geospatial data (Andrés et al., 2013) Pufferfish DP (Kifer & Machanavajjhala, 2014) noiseless privacy (Bhaskar et al., 2011) privacy under partial knowledge (Seeman et al., 2022) privacy amplification (Beimel et al., 2010; Balle et al., 2020; Bun et al., 2022)

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The classic choice: pure ε-DP (Dwork, McSherry, et al., 2006)

- d_{Pr} is the *max. log-likelihood ratio* $d_{\mathrm{MULT}}(\mathrm{P}_{\boldsymbol{x}}, \mathrm{P}_{\boldsymbol{x}'}) = \sup_t \left| \log \frac{\mathrm{p}_{\mathbf{x}}(T=t)}{\mathrm{p}_{\mathbf{x}'}(T=t)} \right|$
- $d_{\mathcal{X}}$ is the *Hamming distance*

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Ex: $T_n = 0.45$, $p = 0.6$

$$\hat{p}_{\text{cheat}} = \frac{0.45 + 0.6 - 1}{2 \times 0.6 - 1} = 0.25$$

What is the loss of information or the gain in privacy?

Increased variance

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$$\frac{\Pr\left(T_i = 0 \mid X_i = 1\right)}{\Pr\left(T_i = 0 \mid X_i = 0\right)} = \frac{1-p}{p} = e^{-\varepsilon}$$

$$e^{-\varepsilon} \leq \frac{\Pr\left(T_i = t \mid X_i = 1\right)}{\Pr\left(T_i = t \mid X_i = 0\right)} \leq e^{\varepsilon}, \quad \text{for } t = 0, 1.$$

What is the loss of information or the gain in privacy?

Increased variance

$$\mathrm{Var}(\hat{p}_{\text{cheat}}) = \frac{1}{n}\frac{p_T(1-p_T)}{(2p-1)^2} \leq \frac{1}{16n}\frac{1}{(p-0.5)^2}$$

Control relative risk of disclosure via controlling the likelihood ratio

$$\frac{\pi(X_i=1|T_i)}{\pi(X_i=0|T_i)} = \frac{\Pr(T_i|X_i=1)}{\Pr(T_i|X_i=0)}\frac{\pi(X_i=1)}{\pi(X_i=0)} \Leftrightarrow \frac{\pi(X_i=1|T_i)}{\pi(X_i=1)}\Big/\frac{\pi(X_i=0|T_i)}{\pi(X_i=0)} = \frac{\Pr(T_i|X_i=1)}{\Pr(T_i|X_i=0)}$$

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$$d_{\text{MULT}}(\mathrm{P}_{\boldsymbol{x}}, \mathrm{P}_{\boldsymbol{x}'}) \leq \varepsilon d_{\text{Ham}}(\boldsymbol{x}, \boldsymbol{x}'), \quad \text{for } \boldsymbol{x}, \boldsymbol{x}' \in \{0, 1\}^n.$$

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Ex: randomised response (cont.)

Recall $T_i = 1_{\{X_i = U_i\}}$.

Suppose an adversary's prior for X_1 is $\pi(X_1 = 1) = \theta$. Given $t \in \{0, 1\}$,

$$\begin{aligned} C_\theta(t) &:= \frac{\pi(X_1 = 1 | T_1 = t)}{\pi(X_1 = 1)} = \frac{\Pr(T_1 = t | X_1 = 1)}{\Pr(T_1 = t)} \\ &= \frac{LR(t)}{LR(t)\theta + (1 - \theta)}, \quad \text{where } LR(t) = \frac{\Pr\left(T_1 = t \mid X_1 = 1\right)}{\Pr\left(T_1 = t \mid X_1 = 0\right)} \end{aligned}$$

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The prior-to-posterior semantic for differential privacy:

$$e^{-\varepsilon} \leq C_\theta(t) \leq e^{\varepsilon} \quad \text{for all } \theta \text{ if and only if} \quad e^{-\varepsilon} \leq LR(t) \leq e^{\varepsilon} \quad \text{for all } t$$

However, what if X_1 and X_2 are *a priori* dependent?

Suppose our prior for (X_1, X_2) is $\pi(X_1 = a, X_2 = b) = \theta_{ab}$. Let

$$C_\pi(t_1, t_2) := \frac{\Pr(X_1 = 1 | T_1 = t_1, T_2 = t_2)}{\Pr(X_1 = 1)} = \frac{\Pr(T_1 = t_1, T_2 = t_2 | X_1 = 1)}{\Pr(T_1 = t_1, T_2 = t_2)}$$

Transferring the bound on likelihood ratio to posterior-to-prior ratio

$$C_\theta(t_1, t_2) = \frac{LR(t_1, t_2)}{LR(t_1, t_2)\theta_{1\cdot} + (1 - \theta_{1\cdot})}, \quad \theta_{1\cdot} = \pi(X_1 = 1) = \theta_{11} + \theta_{10}$$

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Consider the case $t_1 = 1, t_2 = 1$, and recall $e^\varepsilon = p/(1 - p)$

$$LR(1, 1) = \frac{e^\varepsilon \frac{\theta_{11}}{\theta_{1\cdot}} + \frac{\theta_{10}}{\theta_{1\cdot}}}{\frac{\theta_{01}}{\theta_{0\cdot}} + e^{-\varepsilon} \frac{\theta_{00}}{\theta_{0\cdot}}}$$

The dependence is a big trouble maker

This means that when $\theta_{10} = \theta_{01} = 0$, $LR(1, 1) = e^{2\varepsilon} > e^{\varepsilon}$.

- But $\theta_{10} = \theta_{01} = 0$ means that $X_2 = X_1$, hence X_1 can be learned from the information for X_2. Consequently, the "individual information unit" for X_1 should be the pair $\{X_1, X_2\}$, not merely X_1.

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- In fact as soon as $\text{Cov}(X_1, X_2) > 0$, $LR(1,1) > e^{\varepsilon}$. This is because

$$LR(1,1) > e^{\varepsilon} \quad \Longleftrightarrow \quad \pi(X_2 = 1|X_1 = 1) > \pi(X_2 = 1|X_1 = 0)$$

But

$$\begin{aligned} \text{Cov}(X_1, X_2) &= \pi(X_1 = 1, X_2 = 1) - \pi(X_1 = 1)\Pr(X_2 = 1) \\ &= [\pi(X_2 = 1|X_1 = 1) - \pi(X_2 = 1|X_1 = 0)]\,\pi(X_1 = 0)\pi(X_1 = 1). \end{aligned}$$

Data are *accidental* representation, not *essential* information

Manipulating data values without considering their interdependence is not a legitimate information operation in general

Does pure ε-DP control disclosure?

For a general prior π,

$$\frac{\pi(X_i = x_i \mid T = t)}{\pi(X_i = x_i)} = \frac{\pi(X_i = x_i) \int \mathrm{p}_{\boldsymbol{x}}(T = t) d\pi(\boldsymbol{X}_{-i} = \boldsymbol{x}_{-i} \mid X_i = x_i)}{\pi(X_i = x_i) \int \mathrm{p}_{\boldsymbol{x}'}(T = t) d\pi(\boldsymbol{X} = \boldsymbol{x}')}$$

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&\le e^{n\varepsilon},
\end{aligned}
$$

with equality as the records of $\boldsymbol{X}$ become totally dependent. (n is the number of records in $\boldsymbol{X}$.) (Dwork, McSherry, et al., 2006; Kifer & Machanavajjhala, 2011b)

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- Thus the guaranteed limit e^{ε} is only for the **unique individual information**: variations unexplained by anyone else in the database or by knowledge on (and beyond) the database population.

A Bayesian characterisation of pure ε-DP (Bailie, Gong & Meng, 2024+)

A random statistic $T \in \mathbb{R}^d$ is ε-DP if and only if for every prior π on $\boldsymbol{X}$, every sub-σ field $\mathcal{F}$ of the corresponding full σ-field σ_π, every $B \in \mathscr{B}\left(\mathbb{R}^d\right)$, every i, and every $A \in \mathscr{B}(\Theta_i)$, where Θ_i is the state space of x_i, we have

$$e^{-c_i\varepsilon}\pi(X_i \in A \mid \mathcal{F}) \leq \pi\left(X_i \in A \mid T \in B; \mathcal{F}\right) \leq e^{c_i\varepsilon}\pi(X_i \in A \mid \mathcal{F}), \tag{1}$$

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- *MIC* $= C_{-i} \cup \{X_i\}$: $C_{-i} \subset \boldsymbol{X}_{-i}$ is the *Markov boundary* for X_i, that is, the smallest subset of $\boldsymbol{X}_{-i}$ such that
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- MIC is the X_i's "information family" – knowing any one of them will provide information about X_i, in addition to public knowledge coded into $\mathcal{F}$.

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- MIC is the X_i's "information family" – knowing any one of them will provide information about X_i, in addition to public knowledge coded into $\mathcal{F}$.
- Protecting *relative* risk against "strongest attacker" is the easiest — the more the attacker's prior information, the less left for protection.

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Ex: privacy amplification by sampling

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Traditional statistical disclosure control attacker models

- The *nosy neighbor:* Knows that a record is in the sample.
- The *journalist:* Wants to learn about *any* record, so picks one in the sample.

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Ex: privacy amplification by sampling

- Suppose $T(\boldsymbol{x}, U)$ is ε-DP and $\mathcal{S}(\boldsymbol{x}, U')$ randomly samples f fraction of $\boldsymbol{x}$.
- $T'(\boldsymbol{x}, U, U') := T(\mathcal{S}(\boldsymbol{x}, U'), U)$ is ε'-DP with $\varepsilon' \approx f\varepsilon < \varepsilon$.
- So the (conditional) prior-to-posterior ratio of T' should be in the interval $[e^{-\varepsilon'}, e^{\varepsilon'}]$.

Traditional statistical disclosure control attacker models

- The *nosy neighbor:* Knows that a record is in the sample.
- The *journalist:* Wants to learn about *any* record, so picks one in the sample.

For these attackers, the (conditional) prior-to-posterior ratio of T' is in the interval $[e^{-\varepsilon}, e^{\varepsilon}]$, *not* the interval $[e^{-\varepsilon'}, e^{\varepsilon'}]$ (Bailie & Drechsler, 2024+).

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- In the 2010s, the US Census Bureau determined that swapping did not provide sufficient privacy protection.
- For the 2020 Census, disclosure avoidance was overhauled with the primary aim of satisfying *differential privacy*.
- They use two bespoke DP methods: the *TopDown Algorithm* (J. Abowd et al., 2022) and *SafeTabs* (Tumult Labs, 2022).

DP revisited: Five building blocks

A general DP specification (Bailie et al., 2024+)

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- The **standard of protection** (*how* to measure protection): the divergence d_{Pr} on probabilities;
- The **intensity of protection** (*how much* protection is afforded): privacy loss budget $\varepsilon_{\mathcal{D}} \in \mathbb{R}^{\geq 0}$, for each data universe $\mathcal{D}$.

Data swapping visualisation

State	Location	Number of adults	Number of children	Age1	Race1	$\cdots$
MA	Cambridge	2	2	45	White	$\cdots$
TX	Houston	1	0	28	Hispanic	$\cdots$
WA	Tacoma	5	0	67	Asian	$\cdots$
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$\boldsymbol{V}_{\text{Stratify}}$

$\boldsymbol{V}_{\text{Swap}}$

$\boldsymbol{V}_{\text{Rest}}$

Data swapping visualisation

Massachusetts: Location by Race (head of household) Contingency Table

	White	Hispanic	Asian	Black	...
Boston					
Cambridge					
Brookline					
Somerville					
Watertown					
⋮					

Data swapping visualisation

Massachusetts: Location by Race (head of household) Contingency Table

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Changes: Interior cells of $\mathbf{V}_{\text{Rest}} \times \mathbf{V}_{\text{Swap}}$.

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Invariants:

1. $\mathbf{V}_{\text{Stratify}} \times \mathbf{V}_{\text{Rest}}$
2. $\mathbf{V}_{\text{Stratify}} \times \mathbf{V}_{\text{Swap}}$

Swapping satisfies DP, subject to its invariants

Permutation swapping

Input: a dataset $\boldsymbol{x}$.
Define strata as groups of records which match on the swap key $\mathbf{V}_{\text{Stratify}}$.
Within each stratum:

1. Select each record independently with probability p (the swap rate).
2. Randomly permute swapping variable $\mathbf{V}_{\text{Swap}}$ of selected records.

Output: the *swapped* dataset $\boldsymbol{w}$.

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Permutation swapping is DP subject to its invariants, with input divergence $d_{\mathcal{X}} = d^{u}_{\text{Ham}}$, output divergence $d_{\text{Pr}} = d_{\text{MULT}}$ and budget

$$\varepsilon = \begin{cases} \ln(b+1) - \ln o & \text{if } 0 < p \leq 0.5, \\ \max\left\{\ln o, \ln(b+1) - \ln o\right\} & \text{if } 0.5 < p < 1, \end{cases}$$

where $o = p/(1-p)$ and b is the maximum stratum size.

Comparisons: US Decennial Censuses

	d_{Pr}	$d_{\mathcal{X}}$ (Unit)	Invariants	Privacy Loss Budget
TopDown*	D_{nor}	d^{p}_{Ham} (person)	Population (state) Total housing units (block) Occupied group quarters (block) Structural zeros	PL & DHC: $\rho^2 = 15.29$ $\varepsilon = 52.83$ ($\delta = 10^{-10}$)
SafeTab**	D_{nor}	d^{p}_{Ham} (person)	None	DDHC-A: $\rho^2 = 19.776$ DDHC-B & S-DHC: *TBD.*
Swapping	$d_{\textsc{Mult}}$	d^{h}_{Ham} (household)	Varies but greater than TDA	ε between 9.37-19.38

*(J. Abowd et al., 2022)

**(Tumult Labs, 2022)

- $\mathcal{X}$ is always the space of possible Census Edited Files, $\mathcal{X}_{\mathrm{CEF}}$.
- $D_{\mathsf{nor}}(P, Q) = \mathsf{sup}_{\alpha > 1} \frac{1}{\sqrt{\alpha}} \max \left[\sqrt{D_\alpha(P||Q)}, \sqrt{D_\alpha(Q||P)} \right]$ is the normalised Rényi metric [zero concentrated DP] (with D_α the Rényi divergence of order);
- $d_{\textsc{Mult}}(P, Q) = \mathsf{sup}_{S \in \mathcal{F}} \left| \ln \frac{P(S)}{Q(S)} \right|$ is the multiplicative distance (pure DP); and
- d^{u}_{Ham} is the Hamming distance (on units u).

The TopDown Algorithm (TDA) (J. Abowd et al., 2022)

Two-step procedure:

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where $\boldsymbol{W} \sim \mathcal{N}_{\mathbb{Z}}(0, \boldsymbol{\Sigma})$, so that $\boldsymbol{T}$ satisfies $\mathrm{DP}(\mathcal{X}_{\mathrm{CEF}}, \{\mathcal{X}_{\mathrm{CEF}}\}, d^{p}_{\mathrm{Ham}}, D_{\mathrm{nor}})$ with budget ρ_{TDA} (Canonne et al., 2022).

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2. “Post-process”: find dataset $\boldsymbol{z}$ with $\boldsymbol{q}(\boldsymbol{z})$ close to $\boldsymbol{T}(\boldsymbol{x})$ such that $\boldsymbol{c}_{\text{TDA}}(\boldsymbol{z}) = \boldsymbol{c}_{\text{TDA}}(\boldsymbol{x})$.

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Theorem: TDA satisfies DP, subject to its invariants

Let $\boldsymbol{c}_{\text{TDA}} : \mathcal{X}_{\text{CEF}} \to \mathbb{R}^l$ be the invariants of TDA and let $\mathscr{D}_{\boldsymbol{c}_{\text{TDA}}}$ be the induced data multiverse:

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- Let $\boldsymbol{c}'$ be any proper subset of TDA's invariants. TDA does not satisfy $\mathrm{DP}(\mathcal{X}_{\mathrm{CEF}}, \mathscr{D}_{\boldsymbol{c}'}, d_{\mathcal{X}}, D_{\mathrm{nor}})$ with any finite budget ρ.

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Ex: $\bar{Y}_n = 0.45$, $p = 0.6$

$$\hat{p}_{\text{cheat}} = \frac{0.45 + 0.6 - 1}{2 \times 0.6 - 1} = 0.25$$

What is the loss of information or the gain in privacy?

Increased Variance

$$\mathrm{Var}(\hat{p}_{\text{cheat}}) = \frac{1}{n}\frac{p_Y(1-p_Y)}{(2p-1)^2} \leq \frac{1}{16n}\frac{1}{(p-0.5)^2}$$

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Control Relative Risk via Controlling Likelihood Ratio

$$\frac{\Pr(X_i = 1 | Y_i)}{\Pr(X_i = 0 | Y_i)} = \frac{\Pr(Y_i | X_i = 1)}{\Pr(Y_i | X_i = 0)} \frac{\Pr(X_i = 1)}{\Pr(X_i = 0)}$$

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The "first" example of *differential privacy*

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What is the loss of information or the gain in privacy?

Increased Variance

$$\mathrm{Var}(\hat{p}_{\mathrm{cheat}}) = \frac{1}{n}\frac{p_Y(1-p_Y)}{(2p-1)^2} \leq \frac{1}{16n}\frac{1}{(p-0.5)^2}$$

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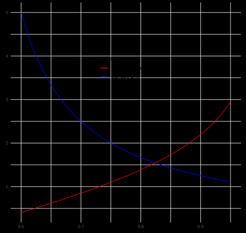

Define *Pure* DP: Dwork et al. (2006) vs Dwork et al. (2016)

Let the database $\boldsymbol{X} = \{x_1, \ldots, x_n\}$ be a vector of n entries from some domain D, typically of the form $\{0, 1\}^d$ or $\mathbb{R}^d$. Let $T_{\mathcal{A}}$ be a random mechanism (map) from D^n to a state space $\mathcal{T}$, corresponding to a query from an adversary $\mathcal{A}$.

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Definition 1 of Dwork, McSherry, et al. (2006)

A mechanism is ε-indistinguishable if for all pairs $\boldsymbol{X}, \boldsymbol{X}' \in D^n$ which differ in only one entry, for all adversaries $\mathcal{A}$, and for all transcripts t:

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Definition 2.1 of Dwork et al. (2016)

A noninteractive mechanism $\mathcal{M}$ is ε-differentially private (with respect to a given distance measure) if for all neighboring datasets $\boldsymbol{X}, \boldsymbol{X}' \in \mathbb{N}^{|D|}$, and for all events (measurable sets) S in the space of outputs of $\mathcal{M}$:

$$\Pr\left(M(\boldsymbol{X}) \in S\right) \leq e^{\varepsilon} \Pr\left(M(\boldsymbol{X}') \in S\right).$$

The probabilities are over the coin flips of M.

Differential Privacy for the 2020 U.S. Census: Can We Make Data Both Private and Useful?

Special Issue 2

FROM THE EDITORS

Harnessing the Known Unknowns: Differential Privacy and the 2020 Census

by Ruobin Gong, Erica L. Groshen, and Salil Vadhan

Published: Jun 24, 2022

Special Issue 2: Differential Privacy for the 2020 U.S. Census

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Disclosure Protection in the Context of Statistical Agency Operations: Data Quality and Related Constraints

by John L. Eltinge

Published: Jun 24, 2022

Implementing Differential

Does DP control the posterior-to-prior ratio ?

Revisit the Random Response Mechanism: $Y_i = 1_{\{X_i = R_i\}}$.

Suppose an adversary's prior for X_1 is $\Pr(X_1 = 1) = \pi$.

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C_\pi(y) &\equiv \frac{\Pr(X_1 = 1 | Y_1 = y)}{\Pr(X_1 = 1)} = \frac{\Pr(Y_1 = y | X_1 = 1)}{\Pr(Y_1 = y)} \\
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The prior-to-posterior semantic for differential privacy:

$$e^{-\varepsilon} \leq C_\pi(y) \leq e^{\varepsilon} \quad \text{for all } \pi \text{ if and only if } \quad e^{-\varepsilon} \leq LR(y) \leq e^{\varepsilon}$$

However, what if X_1 and X_2 are *a priori* dependent?

Suppose our prior for (X_1, X_2) is $\Pr(X_1 = a, X_2 = b) = \pi_{ab}$. Let

$$C_\pi(y_1, y_2) \equiv \frac{\Pr(X_1 = 1 | Y_1 = y_1, Y_2 = y_2)}{\Pr(X_1 = 1)} = \frac{\Pr(Y_1 = y_1, Y_2 = y_2 | X_1 = 1)}{\Pr(Y_1 = y_1, Y_2 = y_2)}$$

Transferring the bound on likelihood ratio to posterior-to-prior ratio

$$C_\pi(y_1, y_2) = \frac{LR(y_1, y_2)}{LR(y_1, y_2)\pi_{1\cdot} + (1 - \pi_{1\cdot})}, \quad \pi_{1\cdot} = \Pr(X_1 = 1) = \pi_{11} + \pi_{10}$$

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Consider the case $y_1 = 1, y_2 = 1$, and recall $e^\varepsilon = p/(1 - p)$

$$LR(1, 1) = \frac{e^\varepsilon \frac{\pi_{11}}{\pi_{1\cdot}} + \frac{\pi_{10}}{\pi_{1\cdot}}}{\frac{\pi_{01}}{\pi_{0\cdot}} + e^{-\varepsilon} \frac{\pi_{00}}{\pi_{0\cdot}}}$$

The dependence is a big trouble maker

This means that when $\pi_{10} = \pi_{01} = 0$, $LR(1,1) = e^{2\varepsilon} > e^{\varepsilon}$.

- But $\pi_{10} = \pi_{01} = 0$ means that $X_2 = X_1$, hence X_1 can be learned from the information for X_2. Consequently, the "individual information unit" for X_1 should be the pair $\{X_1, X_2\}$, not merely X_1.

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- In fact as soon as $\text{Cov}(X_1, X_2) > 0$, $LR(1,1) > e^{\varepsilon}$. This is because

$$LR(1,1) > e^{\varepsilon} \quad \Longleftrightarrow \quad \Pr(X_2 = 1 | X_1 = 1) > \Pr(X_2 = 1 | X_1 = 0)$$

But

$$\begin{aligned}\text{Cov}(X_1, X_2) &= \Pr(X_1 = 1, X_2 = 1) - \Pr(X_1 = 1)\Pr(X_2 = 1) \\ &= [\Pr(X_2 = 1 | X_1 = 1) - \Pr(X_2 = 1 | X_1 = 0)]\Pr(X_1 = 0)\Pr(X_1 = 1).\end{aligned}$$

Data are *accidental* representation, not *essential* information itself

Manipulating data values without considering their interdependence is not a legitimate information operation in general

In general, what does DP actual guarantee?

An attacker A is interested in learning about $\boldsymbol{X}_A = \{x_i, i \in I_A\}$ in a database $\boldsymbol{X} = \{X_i, i \in I\}$, where I_A could contain a single individual or everyone in I. Suppose the attacker has prior knowledge about the entire $\boldsymbol{X}$ in the form of $\pi(\boldsymbol{X})$.

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Let $\pi_A(X_i)$ be the marginal prior, and $\pi_A(X_i|\boldsymbol{X}_{-i})$ be the conditional prior, conditioning on $\boldsymbol{X}_{-i} = \{X_j, j \neq i\}$. Upon learning $M = m$,

- Does ε-DP guarantees the marginal posterior-to-prior ratio

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Yes

- Thus the guaranteed limit e^{ε} is only for the **unique individual information**: variations unexplained by anyone else in the database or by knowledge on (and beyond) the database population.

Theorem (Bailie, Gong & Meng, 2023)

A random map M delivers ε-DP under Hamming distance if and only if for every prior π on $\mathcal{D}$, every sub-σ field $\mathcal{F}$ of the corresponding full σ-field $\sigma_\pi(\mathcal{X})$, every $B \in \mathscr{B}\left(\mathbb{R}^d\right)$, every i, and every $A \in \mathscr{B}(\Theta_i)$, where Θ_i is the state space of x_i, we have

$$e^{-c_i\varepsilon}\pi(X_i \in A \mid \mathcal{F}) \leq \Pr\left(X_i \in A \mid M \in B; \mathcal{F}\right) \leq e^{c_i\varepsilon}\pi(x_i \in A \mid \mathcal{F}), \tag{2}$$

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- *MIC* $= C_{-i} \cup \{X_i\}$: $C_{-i} \subset \boldsymbol{X}_{-i}$ is the *Markov boundary* for X_i, that is, the smallest subset of $\boldsymbol{X}_{-i}$ such that
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- MIC is the X_i's "information family" – knowing any one of them will provide information about X_i, in addition to public knowledge coded into $\mathcal{F}$.

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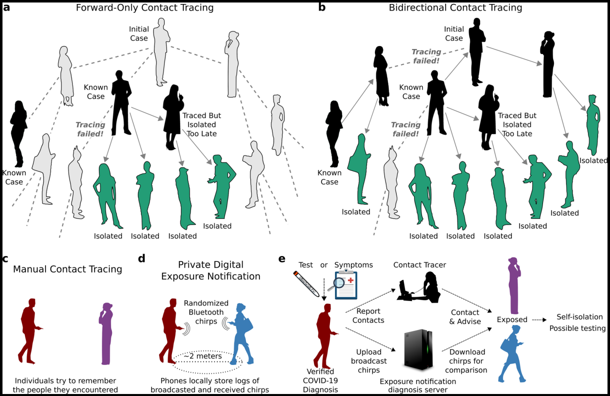
- MIC is the X_i's "information family" – knowing any one of them will provide information about X_i, in addition to public knowledge coded into $\mathcal{F}$.
- Protecting *relative* risk against "strong adversary" is the easiest — the more the adversary's prior information, the less left for protection.

Information spreads like a virus — we need to quarantine not only the infected individual but also everyone they've come into contact with.



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Let the probability space for $M(\boldsymbol{X})$ be $\{\mathcal{M}, \mathcal{F}, P_{\boldsymbol{X}}\}$ (with $P_{\boldsymbol{X}}(S) = \Pr(M(\boldsymbol{X}) \in S|\boldsymbol{X})$)

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"Differential" comes from "derivative", essential for studying *changes*

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A general DP Specification (Bailie et al., 2023)

A data-release mechanism $M : \mathcal{X} \to \mathcal{M}$ satisfies a *DP specification* $(\mathcal{X}, \mathscr{D}, d_{\mathcal{X}}, d_{\Pr}, \varepsilon_{\mathcal{D}})$ if

$$d_{\Pr}\left[P_{\boldsymbol{X}}, P_{\boldsymbol{X}'}\right] \le \varepsilon_{\mathcal{D}} d_{\mathcal{X}}(\boldsymbol{X}, \boldsymbol{X}'), \tag{3}$$

for all $\boldsymbol{X}, \boldsymbol{X}'$ in every data universe $\mathcal{D}$ in the data multiverse $\mathscr{D}$.

Five Building Blocks

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- The **standard of protection** (*how* to measure protection): the divergence d_{Pr} on probabilities;
- The **intensity of protection** (*how much* protection is afforded): privacy loss budget $\varepsilon_{\mathcal{D}} \in \mathbb{R}^{\geq 0}$, for each data universe $\mathcal{D}$.

Examples in the Literature

4. d_{Pr}: (ε, δ)-approximate DP (Dwork, Kenthapadi, et al., 2006) Rényi DP (Mironov, 2017) concentrated DP (Bun & Steinke, 2016) f-divergence privacy (Barber & Duchi, 2014; Barthe & Olmedo, 2013) f-DP (including Gaussian DP) (Dong et al., 2022).

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1. $\mathscr{X}$: DP for network data (Hay et al., 2009) for geospatial data (Andrés et al., 2013) Pufferfish DP (Kifer & Machanavajjhala, 2014) noiseless privacy (Bhaskar et al., 2011) privacy under partial knowledge (Seeman et al., 2022) privacy amplification (Beimel et al., 2010; Balle et al., 2020; Bun et al., 2022).

Examples from the US Decennial Censuses

	d_{Pr}	$d_{\mathcal{X}}$ (Unit)	Invariants	Privacy Loss Budget
TopDown*	D_{nor}	d^{p}_{Ham} (person)	Population (state) Total housing units (block) Occupied group quarters (block) Structural zeros	PL & DHC: $\rho^2 = 15.29$ $\varepsilon = 52.83$ ($\delta = 10^{-10}$)
SafeTab**	D_{nor}	d^{p}_{Ham} (person)	None	DDHC-A: $\rho^2 = 19.776$ DDHC-B & S-DHC: *TBD.*
Swapping	$d_{\textsc{Mult}}$	d^{h}_{Ham} (household)	Varies but greater than TDA	ε between 9.37-19.38

*(J. Abowd et al., 2022)

**(Tumult Labs, 2022)

- $\mathcal{X}$ is always the space of possible Census Edited Files, $\mathcal{X}_{\text{CEF}}$.
- $D_{\text{nor}}(P, Q) = \sup_{\alpha > 1} \frac{1}{\sqrt{\alpha}} \max \left[\sqrt{D_\alpha(P||Q)}, \sqrt{D_\alpha(Q||P)} \right]$ is the normalised Rényi metric [zero concentrated DP] (with D_α the Rényi divergence of order);
- $d_{\textsc{Mult}}(P, Q) = \sup_{S \in \mathcal{F}} \left| \ln \frac{P(S)}{Q(S)} \right|$ is the multiplicative distance (pure DP); and
- d^{u}_{Ham} is the Hamming distance (on units u).

Swapping Satisfies DP, Subject to its Invariants

Permutation Swapping

Input: a dataset $\boldsymbol{x}$.
Define strata as groups of records which match on the swap key $\mathbf{V}_{\text{Stratify}}$.
Within each stratum:

1. Select each record independently with probability p (the swap rate).
2. Randomly derange swapping variable $\mathbf{V}_{\text{Swap}}$ of selected records.

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Permutation Swapping is DP subject to its invariants, with input divergence $d_{\mathcal{X}} = d^{u}_{\text{Ham}}$, output divergence $d_{\mathsf{Pr}} = d_{\text{MULT}}$ and budget

$$\varepsilon = \begin{cases} \ln(b+1) - \ln o & \text{if } 0 < p \leq 0.5, \\ \max\left\{\ln o, \ln(b+1) - \ln o\right\} & \text{if } 0.5 < p < 1, \end{cases}$$

where $o = p/(1-p)$ and b is the maximum stratum size.

The TopDown Algorithm (TDA) (J. Abowd et al., 2022)

Two-step procedure:

0. Start with a Census edited file $\boldsymbol{x} \in \mathcal{X}_{\mathrm{CEF}}$.

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$$\boldsymbol{T}(\boldsymbol{x}) = \boldsymbol{q}(\boldsymbol{x}) + \boldsymbol{W},$$

where $\boldsymbol{W} \sim \mathcal{N}_{\mathbb{Z}}(0, \boldsymbol{\Sigma})$, so that $\boldsymbol{T}$ satisfies $\mathrm{DP}(\mathcal{X}_{\mathrm{CEF}}, \{\mathcal{X}_{\mathrm{CEF}}\}, d^{p}_{\mathrm{Ham}}, D_{\mathrm{nor}})$ with budget ρ_{TDA} (Canonne et al., 2022).

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TDA satisfies $\text{DP}(\mathcal{X}_{\text{CEF}}, \mathscr{D}_{\boldsymbol{c}_{\text{TDA}}}, d^p_{\text{Ham}}, D_{\text{nor}})$ with budget ρ_{TDA}.

Theorem: TDA Satisfies DP, Subject to its Invariants

Let $\boldsymbol{c}_{\mathrm{TDA}} : \mathcal{X}_{\mathrm{CEF}} \to \mathbb{R}^l$ be the invariants of TDA and let $\mathscr{D}_{\boldsymbol{c}_{\mathrm{TDA}}}$ be the induced data multiverse:

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- Let $\boldsymbol{c}'$ be any proper subset of TDA's invariants. TDA does not satisfy $\text{DP}(\mathcal{X}_{\text{CEF}}, \mathscr{D}_{\boldsymbol{c}'}, d_{\mathcal{X}}, D_{\text{nor}})$ with any finite budget ρ.

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