Asymptotic Properties of Polynomials Orthogonal over Multiply Connected Domains

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Dissertation Defense
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Introduction

We begin by establishing a framework, in the form of some language and notation, in order to introduce the main topic of this presentation.

Let G be a bounded, simply connected domain in the complex plane.

Let $\mathcal{G}$ be a Lebesgue measurable subset of G.

Let $\mathcal{A}^2(G, \mathcal{G})$ denote the set of all functions f such that

- f is analytic in G and
- f is square-integrable with respect to area measure over $\mathcal{G}$:

$$\int_{\mathcal{G}} |f(z)|^2 \, dA(z) < \infty.$$

Introduction

The subsets $\mathcal{G}$ that we will consider in this presentation are such that this collection $\mathcal{A}^2(G, \mathcal{G})$ becomes a Hilbert space when endowed with the inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{\mathcal{G}} f(z)\, \overline{g(z)}\, dA(z).$$

We can use this inner product, along with the Gram-Schmidt process, to construct the unique sequence of *orthonormal* polynomials $\{p_n(z)\}_{n=1}^{\infty}$ of a complex variable z characterized by the properties

$$p_n(z) = \kappa_n z^n + \ldots, \qquad \kappa_n > 0 \quad (n \geq 0)$$

and

$$\langle p_n, p_m \rangle = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases} .$$

Introduction

If we divide the orthonormal polynomial p_n by its leading coefficient κ_n, then we obtain the nth *monic* polynomial orthogonal with respect to $\mathcal{G}$, which we will denote by the symbol P_n.

We call $\mathcal{G}$ the *domain of orthogonality* for the polynomials P_n since the orthogonality condition depends on the inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{\mathcal{G}} f(z) \, \overline{g(z)} \, dA(z),$$

which, in turn, depends on $\mathcal{G}$.

Introduction

We emphasize that, for the domains of orthogonality $\mathcal{G}$ which we will consider, the existence and uniqueness of the polynomials P_n are guaranteed.

Moreover, the Gram-Schmidt process provides us with a constructive method for finding these polynomials.

However, it is only in exceptional cases that either the Gram-Schmidt process or any other known method leads to an explicit representation for the polynomials.

One such case is considered in the following slide.

Introduction

The monic polynomials orthogonal over the unit disk $\mathbb{D}$ are given by

$$P_n(z) = z^n, \quad n \geq 0.$$

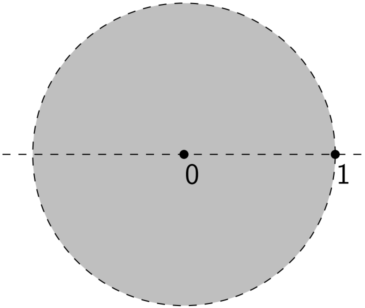
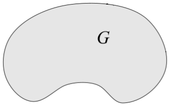


Figure: The unit disk $\mathbb{D}$

Introduction

Let G be a bounded, simply connected domain in the complex plane whose boundary is an analytic Jordan curve.



The monic polynomials P_n orthogonal over G were first studied by Carleman in 1922.

More recently, a thorough investigation of their asymptotic properties as $n \to \infty$ was accomplished by Miña-Díaz and Dragnev.

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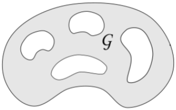

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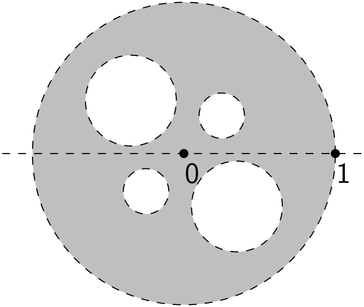
My dissertation considers cases where the domain of orthogonality $\mathcal{G}$ is a multiply connected subset of G. By increasing the scope of the investigation in this manner, we encounter new features which have no analogy in the simply connected case.

Introduction

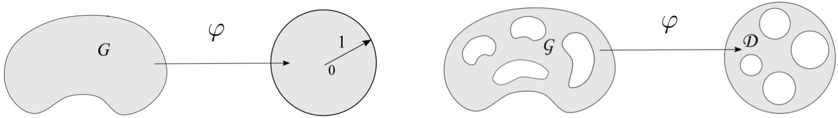
To be more precise, we must introduce the concept of a *circular multiply connected domain* (CMCD).

In order to obtain an example of a CMCD, one begins with the unit disk and removes a finite number of mutually disjoint, closed subdisks.



Introduction: Statement of Hypotheses

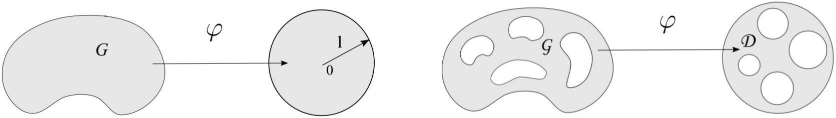
We now have the language to articulate the main problem of investigation.



Let

- G be a bounded, simply connected domain in the complex plane whose boundary is an analytic Jordan curve
- φ be a conformal map of G onto the unit disk
- $\mathcal{D}$ be a CMCD
- $\mathcal{G} := \varphi^{-1}(\mathcal{D})$
- $\{P_n\}_{n=1}^{\infty}$ denote the monic polynomials orthogonal over $\mathcal{G}$

Introduction: Statement of Hypotheses



The purpose of this dissertation is to begin to understand how the holes in the domain of orthogonality $\mathcal{G}$ influence the behavior of the resulting polynomials $\{P_n\}_{n=1}^{\infty}$.

To the best of our knowledge, this is an issue that has not been addressed in the literature.

Next, we give an outline of our main results.

Scope of Main Results

1. In cases where $\mathcal{D}$ satisfies a certain Assumption, we obtain a series representation for P_n for all n sufficiently large.

2. From the series representation, we can obtain an integral representation for P_n as n approaches infinity.

These are the two most general theorems, from which additional results may be derived. These additional results are more specific in nature but are still far from trivial.

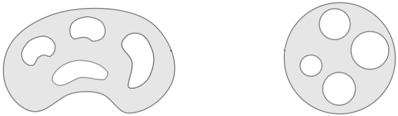
The construction of the series expansions depends on a so-called *reproducing kernel* associated with the space $\mathcal{A}^2(G, \mathcal{G})$.

3. We provide an explicit representation for the reproducing kernel in cases where $\mathcal{D}$ satisfies the Assumption.

Comment: The precise nature of the Assumption will be described in the next segment of the presentation.

Scope of Main Results

Perhaps the most attractive portion of our investigation emerges when orthogonality is considered over the canonical case of a CMCD.



Scope of Main Results

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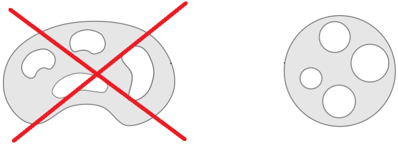


4. Here, we will be able to reveal the very interesting way in which the removed disks influence the behavior of the orthogonal polynomials.

Scope of Main Results

5. Finally, we demonstrate the existence of a variety of CMCDS for which the aforementioned Assumption holds.

The remainder of the presentation is devoted to stating the main results which have been outlined above.

We will begin by introducing preliminary material essential to this description.

Our first objective is to describe the Assumption mentioned above.

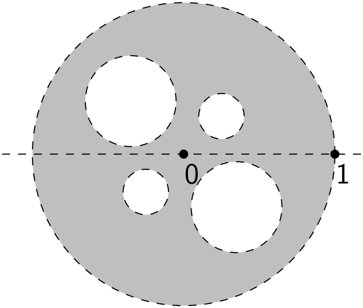
Preliminaries

Suppose our CMCD $\mathcal{D}$ has s removed disks.

Then $\mathcal{D}$ can be represented as

$$\mathcal{D} = \mathbb{D} \setminus \bigcup_{j=1}^{s} D_j,$$

where the s closed disks D_j are mutually disjoint and contained within the unit disk.

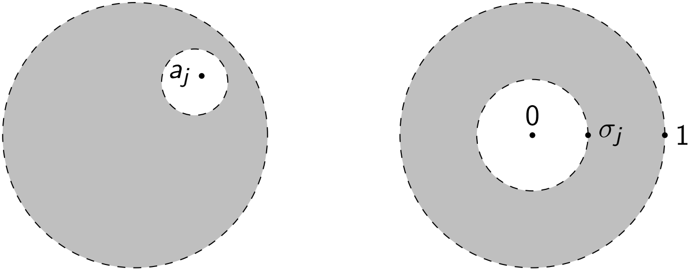


Preliminaries

For each $j \in \{1, 2, \dots, s\}$, there exists a unique $a_j \in \mathbb{D}$ such that the Möbius transformation

$$\Phi_j(z) = \frac{z - a_j}{1 - \overline{a_j} z}$$

maps the interior of D_j onto an open disk centered at the origin.



We let σ_j denote the radius of this open disk $\Phi_j(D_j)$ centered at the origin.

Preliminaries

For each $j \in \{1, 2, \ldots, s\}$, we use the constants a_j and σ_j to define the transformation

$$T_j(z) := \Phi_j^{-1}(\sigma_j^2 \Phi_j(z)), \quad z \in \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}.$$

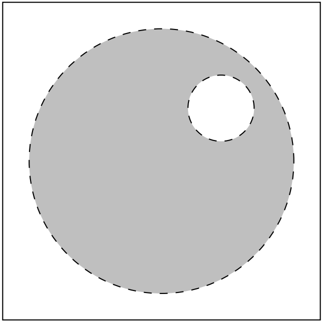


Figure: The set $K_j := \mathbb{D} \setminus D_j$

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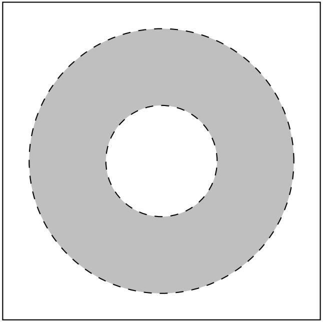

Figure: $\Phi_j(K_j)$

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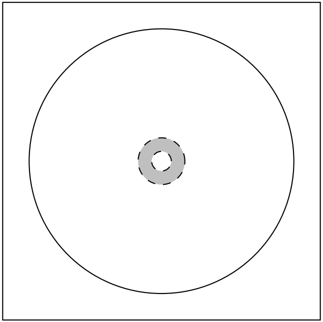

Figure: $\sigma_j^2 \Phi_j(K_j)$

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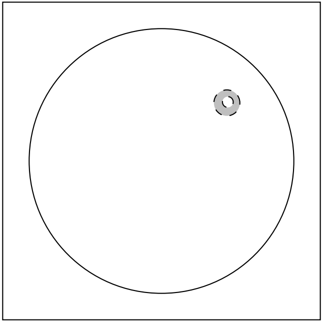

Figure: $\Phi_j^{-1}(\sigma_j^2 \Phi_j(K_j))$

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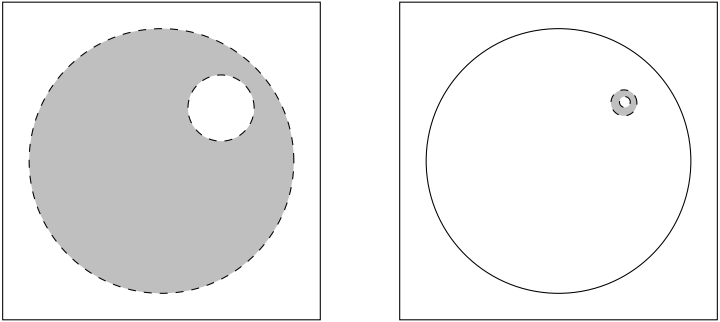


Figure: K_j on the left and $\Phi_j^{-1}(\sigma_j^2 \Phi_j(K_j))$ on the right

Preliminaries

We associate to $\mathcal{D}$ the family $\mathcal{T}^*$ of all finite compositions of the transformations T_j:

$$\mathcal{T}^* := \{\, T_{j_n} T_{j_{n-1}} \cdots T_{j_2} T_{j_1} : 1 \leq j_k \leq s \}.$$

We can view $\mathcal{T}^*$ as a *semigroup* under the binary operation of function composition.

If we let T_0 represent the identity function $T_0(z) = z$, then we can form the *monoid*

$$\mathcal{T} := \mathcal{T}^* \cup \{\, T_0 \}.$$

Preliminaries

We will use the monoid

$$\mathcal{T} = \{ T_0 \} \cup \{ T_{j_n} T_{j_{n-1}} \cdots T_{j_2} T_{j_1} : 1 \le j_k \le s \}$$

in a series of constructions in cases where our CMCD satisfies the following condition.

Assumption

There exists some $\rho \in (0,1)$ such that the function series

$$\sum_{\tau \in \mathcal{T}} |\tau'(z)|$$

converges uniformly on each compact subset of $|z| < 1/\rho$.

Preliminaries

At the end of the presentation, we describe CMCDs which satisfy the Assumption.

We need the Assumption for an original representation for the reproducing kernel associated with $\mathcal{A}^2(G, \mathcal{G})$.

The Reproducing Kernel

By the Riesz Representation Theorem, there exists a unique function $\mathcal{K}_{\mathcal{G}}(z, \zeta)$ defined for z and ζ in G, analytic in z and anti-analytic in ζ, which has the so-called *reproducing property*

$$g(z) = \frac{1}{\pi} \int_{\mathcal{G}} g(\zeta)\, \mathcal{K}_{\mathcal{G}}(z, \zeta)\, dA(\zeta), \quad (z, g) \in G \times \mathcal{A}^2(G, \mathcal{G}).$$

Theorem (Reproducing Kernel)

We have the representation

$$\mathcal{K}_{\mathcal{G}}(z, \zeta) = \sum_{\tau \in \mathcal{T}} \frac{\frac{d}{dz}[(\tau \circ \varphi)(z)]\, \overline{\varphi'(\zeta)}}{[1 - (\tau \circ \varphi)(z)\, \overline{\varphi(\zeta)}]^2}, \quad (z, \zeta) \in G^* \times G^*$$

Here, G^* is a domain which contains $\overline{G}$, the closure of G.

Preliminaries

We use the reproducing kernel $\mathcal{K}_{\mathcal{G}}$ to construct a meromorphic kernel

$$\mathcal{M}_{\mathcal{G}}(z,\zeta) := \sum_{\tau\in\mathcal{T}} \left[\frac{\varphi'(\zeta)}{\varphi(\zeta)-\tau(0)} \cdot \frac{(\tau\circ\varphi)(z)-\tau(0)}{\varphi(\zeta)-(\tau\circ\varphi)(z)} \right]$$

defined for z and ζ belonging to G^*.

We remind the audience that

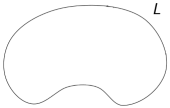
$$\mathcal{K}_{\mathcal{G}}(z,\zeta) = \sum_{\tau\in\mathcal{T}} \frac{\frac{d}{dz}[(\tau\circ\varphi)(z)]\,\overline{\varphi'(\zeta)}}{[1-(\tau\circ\varphi)(z)\,\overline{\varphi(\zeta)}]^2}, \quad (z,\zeta)\in G^*\times G^*$$

and

$$\mathcal{T} = \{T_0\} \cup \{T_{j_n}T_{j_{n-1}}\cdots T_{j_2}T_{j_1} : 1 \le j_k \le s\}$$

Preliminaries

We will use $\mathcal{M}_{\mathcal{G}}$ to construct our series expansion, but first we must introduce a map.



Let L denote the boundary of G.

The sets $\mathbb{C} \setminus L$ consists of exactly two components.

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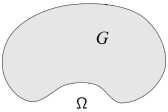


Let L denote the boundary of G.

The sets $\mathbb{C} \setminus L$ consists of exactly two components.

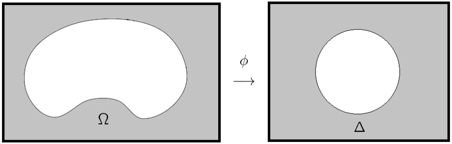
There is a bounded component, called the *interior* of L, which is G.

There is an unbounded component, called the *exterior* of L, which we will denote by Ω.

Preliminaries

By Riemann's mapping theorem, there exists a unique conformal map ϕ of Ω onto the exterior of the unit circle (which we denote by Δ) satisfying the conditions $\phi(\infty) = \infty$ and $\phi'(\infty) > 0$, where

$$\phi'(\infty) := \lim_{z \to \infty} \frac{\phi(z)}{z}.$$



In fact, it is possible to extend the domain of definition for ϕ to a domain Ω^* which includes the closure of Ω.

Preliminaries

For each $n \in \mathbb{N} \cup \{0\}$, we recursively define a sequence $\{f_{n,k}\}_{k=0}^{\infty}$ of functions in the following manner:

$$f_{n,0}(z) := 1, \quad z \in \overline{\mathbb{C}},$$

$$f_{n,2k+1}(z) := -\frac{1}{2\pi i} \oint_{L_t} f_{n,2k}(\zeta)\, \mathcal{M}_{\mathcal{G}}(z,\zeta)\, [\phi(\zeta)]^{n+1}\, d\zeta, \quad z \in G^* \setminus L_t,$$

$$f_{n,2k+2}(z) := \frac{1}{2\pi i} \oint_{L_{1/t}} \frac{f_{n,2k+1}(\zeta)\, \phi'(\zeta)\, [\phi(\zeta)]^{-n-1} d\zeta}{\phi(\zeta) - \phi(z)}, \quad z \in \Omega^* \setminus L_{1/t}.$$

Here, L_t and $L_{1/t}$ are level curves of ϕ^{-1} and belong respectively to the interior and exterior of L.

Preliminaries

We show that, for n large enough, the two series

$$\sum_{k=0}^{\infty} f_{n,2k} \quad \text{and} \quad \sum_{k=0}^{\infty} f_{n,2k+1}$$

converge absolutely and locally uniformly in their respective domains of definition.

Theorem (Series Expansion)

Let $\{P_n\}_{n=0}^{\infty}$ denote the monic orthogonal polynomials over $\mathcal{G}$ and let κ_n denote the leading coefficient of the corresponding orthonormal polynomials. Then, for n large enough, we have

$$\frac{(n+1)P_n(z)}{[\phi'(\infty)]^{-(n+1)}} = \frac{d}{dz}\begin{cases} [\phi(z)]^{n+1}\sum_{k=0}^{\infty} f_{n,2k}(z), & z \in \operatorname{ext}(L_{1/t}), \\ [\phi(z)]^{n+1}\sum_{k=0}^{\infty} f_{n,2k}(z) - \sum_{k=0}^{\infty} f_{n,2k+1}(z), & z \in \operatorname{int}(L_{1/t}) \cap \operatorname{ext}(L_t), \\ -\sum_{k=0}^{\infty} f_{n,2k+1}(z), & z \in \operatorname{int}(L_t) \end{cases}$$

and

$$(n+1)[\phi'(\infty)]^{2n+2}\kappa_n^{-2} = 1 + \sum_{k=0}^{\infty}\frac{1}{2\pi i}\oint_{L_{1/t}} f_{n,2k+1}(\zeta)\phi'(\zeta)[\phi(\zeta)]^{-n-2}\, d\zeta.$$

Here is a sketch of the proof:

- Verify that all functions are well-defined and that the two series $\sum_{k=0}^{\infty} f_{n,2k}$ and $\sum_{k=0}^{\infty} f_{n,2k+1}$ converge for n sufficiently large. Convergence is established by using the Weierstrass M-test.

- Denote by $\mathcal{P}_n$ the function on the right of the differential operator on the previous slide. Show that $\mathcal{P}_n$ is a polynomial of degree $n+1$. This is done by proving that it has an analytic continuation to $\mathbb{C}$ and then by applying Liouville's theorem.

- Demonstrate that $\mathcal{P}_n'$ satisfies the orthogonality relation. This is achieved by using contour deformation and Fubini's theorem. Ultimately, the critical calculation relies on the reproducing property enjoyed by the function $\mathcal{K}_{\mathcal{G}}(z, \zeta)$.

We can use the series expansions to obtain the following integral representation.

Theorem (Integral Representation)

For n sufficiently large, we have

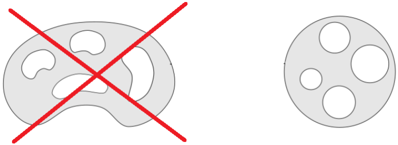
$$P_n(z) = \frac{[\phi'(\infty)]^{-n-1}}{2\pi i} \int_{|w|=1} w^n \, [1 + K_n(w)] \sum_{\tau \in \mathcal{T}} \frac{\frac{d}{dz}[(\tau \circ \varphi)(z)]}{(\varphi \circ \phi^{-1})(w) - (\tau \circ \varphi)(z)} \, dw$$

for $z \in G$, where $K_n(w)$ is analytic in $|w| < 1/t$ and $K_n(w) = O(t^{2n})$ locally uniformly as $n \to \infty$ in $|w| < 1/t$.

Here, t is a number belonging to the interval $(\rho, 1)$. This formula indicates the manner in which P_n is somehow encoded inside of the geometry of the domains involved. In particular, we can observe how P_n emerges from the interplay between the inner and outer conformal maps (φ and ϕ) and the transformations τ of the monoid $\mathcal{T}$.

OPs over CMCDs

As we mentioned before, the series expansions and the integral representation are our most general results. They can lead to more specific formulas for $P_n(z)$ in more specialized circumstances. In particular, suppose that the domain of orthogonality is a CMCD.



In this case, we may take φ and ϕ to be the identity map:

$$\varphi(z) = \phi(z) = z.$$

This simplifies the expression in the Integral Representation and leads to the following result.

Corollary (OPs over CMCDs)

For n sufficiently large, we have

$$P_n(z) = \sum_{\tau \in \mathcal{T}} [\tau(z)]^n \cdot \tau'(z) \cdot [1 + (K_n \circ \tau)(z)], \quad z \in \mathbb{D},$$

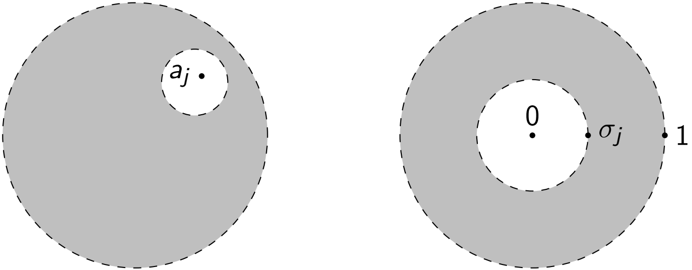
where $K_n(\zeta)$ is analytic in $|\zeta| < 1/t$ and $K_n(\zeta) = O(t^{2n})$ locally uniformly as $n \to \infty$ in $|\zeta| < 1/t$. This leads to the functional equation

$$P_n(z) = z^n \cdot [1 + K_n(z)] + \sum_{j=1}^{s} P_n(T_j(z)) \cdot T_j'(z), \quad z \in \mathbb{D}.$$

We can utilize this Corollary to make more precise statements about the asymptotic behavior of $P_n(z)$ for $z \in \mathbb{C}$. But first, we will need some more definitions.

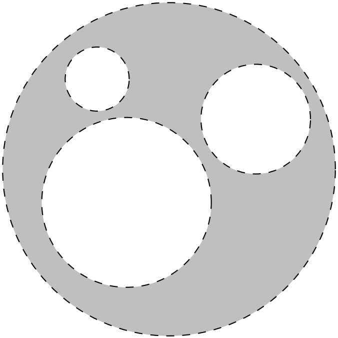
OPs over CMCDs

Recall that we have associated a pair of constants (a_j, σ_j) to each removed disk D_j.



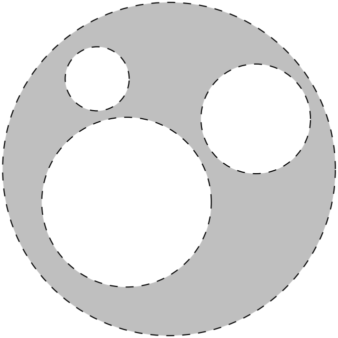
$$\Phi_j(z) = \frac{z - a_j}{1 - \overline{a_j} z}$$

OPs over CMCDs



Which of the removed disks has the greatest influence over P_n?

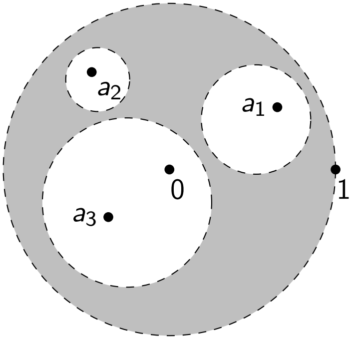
OPs over CMCDs

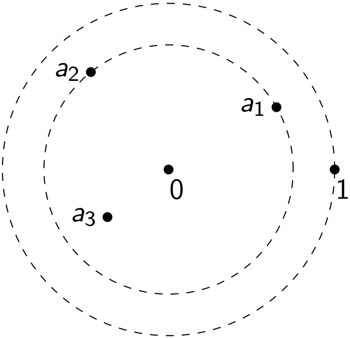


Surprisingly, the most important disks are the ones that have an a_j value of maximal modulus. We define the constant

$$\alpha := \max\{|a_j| : 1 \leq j \leq s\}.$$

OPs over CMCDs





After possible relabeling, we may assume that there exists some $\omega \in \{1, 2, \dots s\}$ such that

$$\alpha = |a_1| = |a_2| = \cdots = |a_\omega| > |a_j|, \quad \omega < j \leq s.$$

OPs over CMCDs

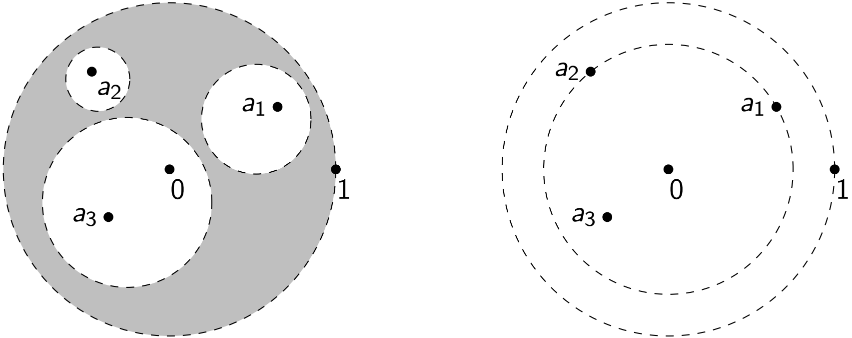


Figure: A CMCD with $\alpha = |a_1| = |a_2| > |a_3|$. The circle $\mathbb{T}_\alpha := \{z : |z| = \alpha\}$.

We will see that the behavior of the polynomials changes dramatically across the circle $\mathbb{T}_\alpha$ that passes through those a_j with largest moduli.

OPs over CMCDs

In order to describe behavior of the polynomials inside $\mathbb{T}_\alpha$, we need some definitions.

We let $\mathcal{H} := \{z : \mathrm{Re}(z) > 0\}$ denote the right half plane.

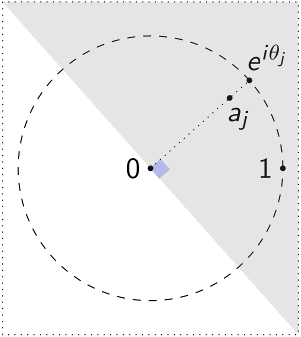
For each $1 \leq j \leq \omega$, we define

- $\theta_j := \mathrm{Arg}\, a_j$
- $\mathcal{H}e^{i\theta_j} := \{ze^{i\theta_j} : z \in \mathcal{H}\}$
- $\beta_j := \frac{1}{\overline{a_j}} - a_j$
- $\mathcal{T}_j := \{T_j\tau : \tau \in \mathcal{T}\}$

OPs over CMCDs

For each $1 \leq j \leq \omega$, we define

$$\Theta_j(w) = w \sum_{v \in \mathbb{Z}} \sigma_j^{2v} \exp(-\overline{\beta_j} \sigma_j^{2v} w), \quad w \in \mathcal{H} e^{i\theta_j}$$



OPs over CMCDs

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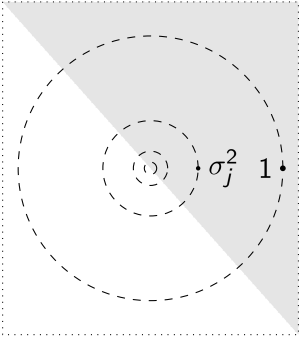


For $w \in \mathcal{H} e^{i\theta_j}$, we have

$$\Theta_j(\sigma_j^2 w) = \Theta_j(w)$$

OPs over CMCDs

For each $1 \leq j \leq \omega$, we define

$$\mathcal{J}_{j,n}(z) := -e^{in\theta_j} \cdot \sum_{\tau \in \mathcal{T} \setminus \mathcal{T}_j} \frac{\Phi_j'(\tau(z))}{\Phi_j(\tau(z))} \cdot \Theta_j(-n\Phi_j(\tau(z))) \cdot \tau'(z), \quad z \in \mathbb{D}_\alpha.$$

We remark that, for each $1 \leq j \leq \omega$, the function $\mathcal{J}_{j,n}$ is bounded on compact subsets of $\mathbb{D}_\alpha := \{z : |z| < \alpha\}$.

We may now describe the behavior of P_n at each point in the complex plane.

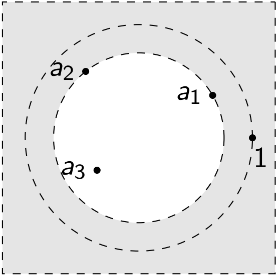
Theorem

The behavior of $P_n(z)$ for $z \in \mathbb{C}$ is as follows.

(i) *For every $r > \alpha$, there exists some $\nu \in (0, r)$ such that*

$$P_n(z) = z^n + O(\nu^n)$$

uniformly for $z \in \mathbb{T}_r$.



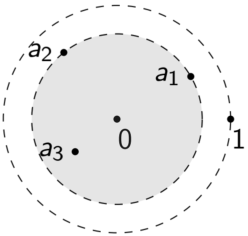
Theorem

The behavior of $P_n(z)$ for $z \in \mathbb{C}$ is as follows.

(ii) *Inside $\mathbb{T}_\alpha$, we have*

$$P_n(z) = (1 - \alpha^2) \cdot \frac{\alpha^n}{n} \cdot \sum_{j=1}^{\omega} \mathcal{J}_{j,n}(z) + O\left(\frac{\alpha^n}{n^2}\right)$$

normally for $z \in \mathbb{D}_\alpha$ as $n \to \infty$.



Theorem

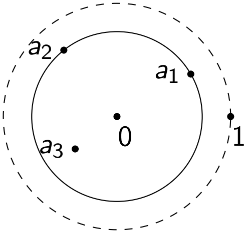
(iii) *On the circle $\mathbb{T}_\alpha$, if $z = a_j$ for some $1 \le j \le \omega$, then*

$$P_n(z) = \frac{a_j^n}{1-\sigma_j^2} + O\left(\frac{\alpha^n}{n}\right).$$

Otherwise, we have

$$P_n(z) = z^n + O\left(\frac{\alpha^n}{n}\right)$$

uniformly on compact subsets of $\mathbb{T}_\alpha \setminus \{a_1, a_2, \ldots, a_\omega\}$.



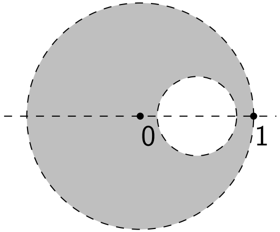
The Case of a Circular Doubly Connected Domain

In fact, we can say even more when the domain of orthogonality is a circular *doubly* connected domain.

Then $\mathcal{D}$ can be represented as

$$\mathcal{D} = \mathbb{D} \setminus D,$$

where D is a closed disk contained within the unit disk.



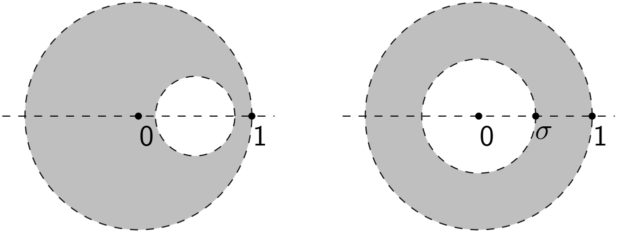
The exposition will be more straightforward if we assume that D is centered on the positive real axis.

The Case of a Circular Doubly Connected Domain

Let $a \in (0,1)$ be the unique number such that the Möbius transformation

$$\Phi(z) = \frac{z-a}{1-az}$$

maps the interior of D onto a circle centered at the origin and let σ denote the radius of $\Phi(D)$.



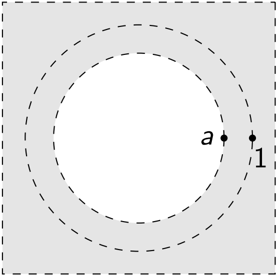
The Case of a Circular Doubly Connected Domain

The previously mentioned (more general) results imply

$$\lim_{n\to\infty} \frac{P_n(z)}{z^n} = 1, \quad |z| \geq a, \quad z \neq a,$$

and

$$\lim_{n\to\infty} \frac{P_n(a)}{a^n} = \frac{1}{1-\sigma^2}.$$

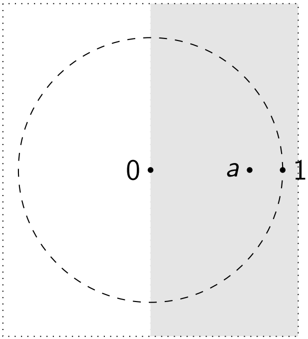


The Case of a Circular Doubly Connected Domain

To understand P_n on the rest of the plane, all we need is the function

$$\Theta(w) = w \sum_{v \in \mathbb{Z}} \sigma^{2v} \exp(-\beta \sigma^{2v} w), \quad w \in \mathcal{H},$$

where we have defined $\beta := 1/a - a$.

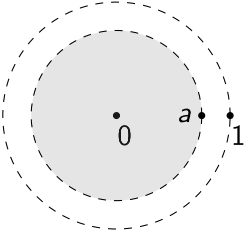


The Case of a Circular Doubly Connected Domain

We have

$$P_n(z) = -(1-a^2) \cdot \frac{\Phi'(z)}{\Phi(z)} \cdot \frac{a^n}{n} \cdot \Theta(-n\Phi(z)) + O\left(\frac{a^n}{n^2}\right)$$

uniformly on compact subsets of $|z| < a$.



Thus, in order to understand the limiting behavior of $P_n(z)$ for $|z| < a$, we must also understand the limiting behavior of Θ.

The Case of a Circular Doubly Connected Domain

To this end, let $\langle x \rangle$ denote the fractional part of the real number x and let $\{n_k\}_{k=1}^{\infty}$ be a subsequence of the natural numbers.

The sequence $\{\Theta(n_k t)\}_{k=1}^{\infty}$ converges normally on $\mathcal{H}$ if and only if

$$\lim_{k\to\infty} \langle \log_{\sigma^2} n_k \rangle = q$$

for some $q \in [0, 1)$. In this case,

$$\lim_{k\to\infty} \Theta(n_k t) = \Theta(\sigma^{2q} t),$$

and thus, by the previously mentioned asymptotic formula, we have

$$\lim_{k\to\infty} \frac{n_k P_{n_k}(z)}{a^{n_k}} = -(1 - a^2) \cdot \frac{\Phi'(z)}{\Phi(z)} \cdot \Theta(-\sigma^{2q}\Phi(z)).$$

The Case of a Circular Doubly Connected Domain

Therefore, the sequence

$$\left\{ \frac{nP_n(z)}{a^n} \right\}_{n=1}^{\infty}$$

has for normal limit points on $|z| < a$ the following continuous one-parameter family of functions:

$$\left\{ -(1-a^2) \cdot \frac{\Phi'(z)}{\Phi(z)} \cdot \Theta(-\sigma^{2q}\Phi(z)) : \; q \in [0,1) \right\}.$$

We close the presentation by stating the cases where we have shown that the Assumption holds.

Cases where the Assumption Holds

Let's first recall what the Assumption says.

Assumption

There exists some $\rho \in (0,1)$ *such that the function series* $\sum_{\tau \in \mathfrak{T}} |\tau'(z)|$ *converges uniformly on each compact subset of* $|z| < 1/\rho$*.*

Cases where the Assumption Holds: Case 1

The precise statement of Case 1 is fairly technical, but its essential implication is relatively easy to communicate.

Let s be any natural number.

Let $\{D_j\}_{j=1}^{s}$ be a collection of s mutually disjoint, closed disks contained within the unit disk and let

$$\mathcal{D} = \mathbb{D} \setminus \bigcup_{j=1}^{s} D_j.$$

Case 1 implies that $\mathcal{D}$ satisfies the Assumption provided that the radii of the removed disks D_j are sufficiently small. A more precise statement of Case 1 is given on the next slide.

Cases where the Assumption Holds: Case 1

Define

- $a := \min |a_j|, \quad \sigma := \max \sigma_j,$
- $m := \min \left(\frac{1}{|a_j|} \cdot \frac{1 - |a_j|^2 \sigma_j^2}{1 - \sigma_j^2} \right), \quad \lambda := \max \frac{|a_j| + \sigma_j}{1 + |a_j| \sigma_j},$

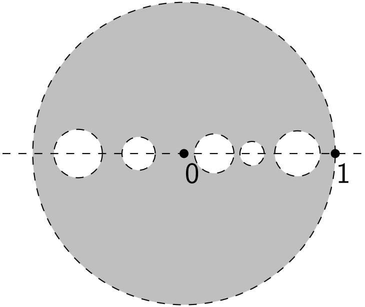
Finally, define

$$\mathcal{N}_{\mathcal{D}} := \left(\frac{\sigma}{1 - \sigma^2} \cdot \frac{1 - a^2}{a} \right)^2 \cdot \frac{1}{(m - \lambda)^2}.$$

Case 1. If $\mathcal{N}_{\mathcal{D}} < 1/s$, then the Assumption holds.

Cases where the Assumption Holds: Case 2

Case 2. If $\mathcal{D}$ is symmetric about the real axis, then the Assumption holds.

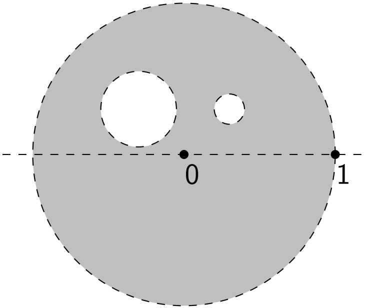


Cases where the Assumption Holds: Case 3

We will use the term *automorphism of the unit disk* to refer to a conformal map of the unit disk onto itself.

Case 3. If $\tilde{\mathcal{D}}$ is a CMCD that satisfies the Assumption and if Ψ is an automorphism of the unit disk, then $\mathcal{D} = \Psi(\tilde{\mathcal{D}})$ also satisfies the Assumption.

Consequently, any CMCD with at most two removed disks satisfies the Assumption.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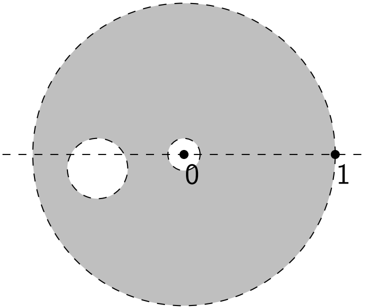
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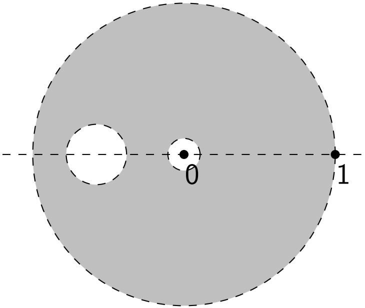
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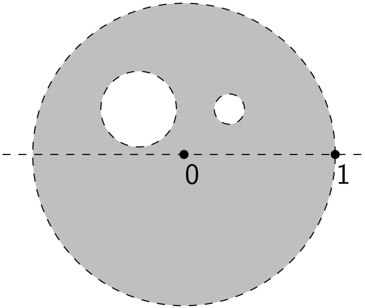
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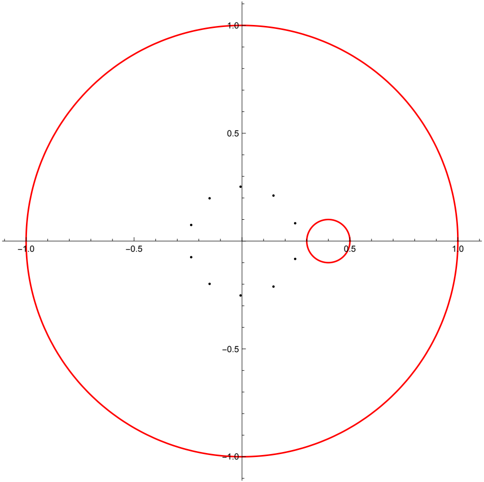
Thank you for your attention!

Numerical Experiments

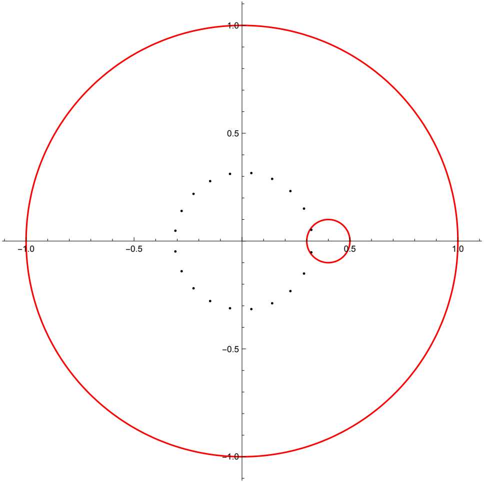
$\mathcal{D} := \mathbb{D} \setminus D$, where $D = \{z : |z - 0.4| \leq 0.1\}$.



Zeros of P_n for $n = 10$

Numerical Experiments

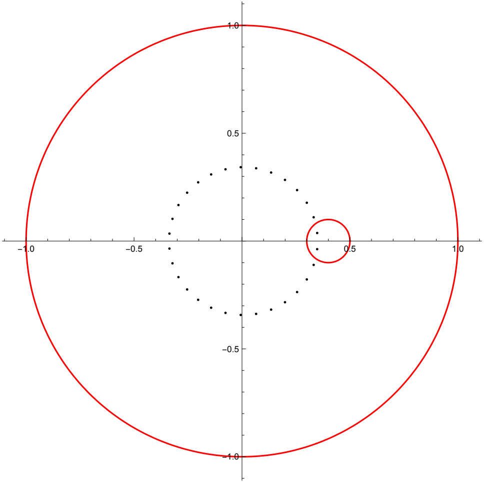
$\mathcal{D} := \mathbb{D} \setminus D$, where $D = \{z : |z - 0.4| \leq 0.1\}$.



Zeros of P_n for $n = 20$

Numerical Experiments

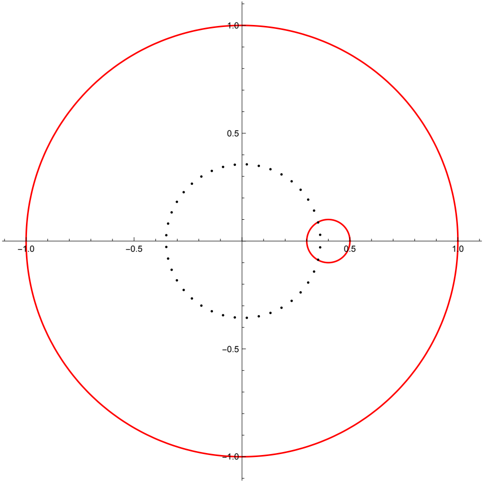
$\mathcal{D} := \mathbb{D} \setminus D$, where $D = \{z : |z - 0.4| \leq 0.1\}$.



Zeros of P_n for $n = 30$

Numerical Experiments

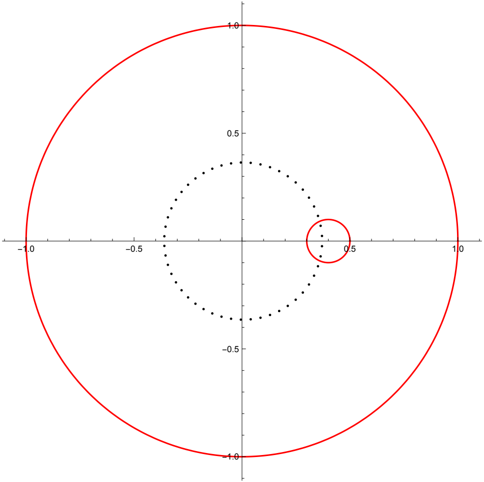
$\mathcal{D} := \mathbb{D} \setminus D$, where $D = \{z : |z - 0.4| \leq 0.1\}$.



Zeros of P_n for $n = 40$

Numerical Experiments

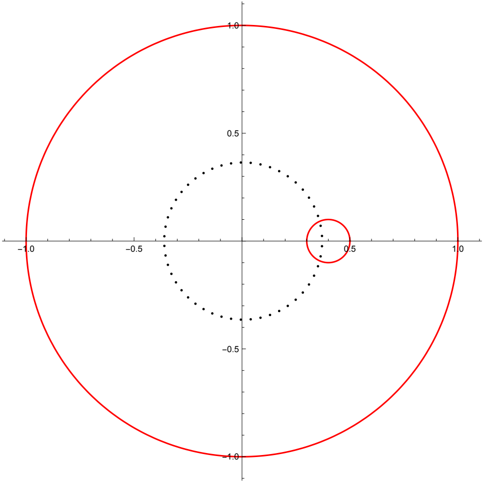
$\mathcal{D} := \mathbb{D} \setminus D$, where $D = \{z : |z - 0.4| \leq 0.1\}$.



Zeros of P_n for $n = 50$

Numerical Experiments

$\mathcal{D} := \mathbb{D} \setminus D$, where $D = \{z : |z - 0.4| \leq 0.1\}$.



Zeros of P_n for $n = 50$

$a \approx 0.404831$