The Reproducing Kernel of a Weighted Bergman Space

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Overview

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Bergman Spaces

Let Ω be a domain (i.e., an open, connected set) in the complex plane.

For every positive number p, we let $A^p(\Omega)$ denote the collection of all functions $f : \Omega \to \mathbb{C}$ such that the following properties hold:

- f is analytic in Ω
- $\displaystyle\int_\Omega |f(z)|^p \, dA(z) < \infty$

We call $A^p(\Omega)$ the *Bergman Space* of p-power over Ω.

Bergman Spaces as Banach Spaces ($p \geq 1$)

It turns out that for $p \geq 1$, $A^p(\Omega)$ is a Banach space (i.e., a complete normed vector space).

The underlying field of scalars is $\mathbb{C}$ and the norm is

$$\|f\|_p = \left\{ \int_\Omega |f(z)|^p \, dA(z) \right\}^{1/p}.$$

Bergman Spaces as Hilbert Spaces ($p = 2$)

Furthermore, one finds that $A^2(\Omega)$ is a Hilbert space.

This means that $A^2(\Omega)$ has an *inner product* that induces the $\|\cdot\|_2$ norm.

Our inner product is

$$\langle f, g \rangle = \int_\Omega f(z)\,\overline{g(z)}\,dA(z).$$

The fact that $A^2(\Omega)$ is a Hilbert space is very important for what follows.

To see why, let's review some concepts from functional analysis.

Linear Functionals

A linear functional is a linear transformation from a vector space to its field of scalars.

Example. For each $z \in \Omega$, we define $\ell_z : A^2(\Omega) \to \mathbb{C}$ by the relation

$$\ell_z(f) = f(z).$$

Then we have

$$\begin{aligned}
\ell_z(\alpha f + \beta g) &= (\alpha f + \beta g)(z) \\
&= (\alpha f)(z) + (\beta g)(z) \\
&= \alpha \cdot f(z) + \beta \cdot g(z) \\
&= \alpha \ell_z(f) + \beta \ell_z(g).
\end{aligned}$$

for any $\alpha, \beta \in \mathbb{C}$ and $f, g \in A^2(\Omega)$.

Bounded Linear Functionals

Definition

Suppose that $\mathcal{H}$ is a Hilbert space over $\mathbb{C}$ with norm $\|\cdot\|$. We say that a linear functional $\ell : \mathcal{H} \to \mathbb{C}$ is *bounded* if there exists an $M > 0$ such that

$$|\ell(f)| \leq M\|f\|, \quad f \in \mathcal{H}.$$

As it turns out, for each $z \in \Omega$, the mapping $\ell_z : A^2(\Omega) \to \mathbb{C}$ is a bounded linear functional since we have the estimate

$$|\ell_z(f)| = |f(z)| \leq \frac{\|f\|_2}{\sqrt{\pi \cdot \delta(z)}}, \quad f \in A^2(\Omega),$$

where $\delta(z)$ is the distance from z to the boundary of Ω.

[A proof of this claim can be found in Duren's book.]

The Riesz Representation Theorem

Theorem

If ℓ is a bounded linear functional on a Hilbert space $\mathcal{H}$, then there exists a unique $g \in \mathcal{H}$ such that

$$\ell(f) = \langle f, g \rangle, \quad f \in \mathcal{H}.$$

Therefore, for each $z \in \Omega$, there exists a unique function g_z in $A^2(\Omega)$ such that $f(z) = \langle f, g_z \rangle$ for every $f \in A^2(\Omega)$.

The Bergman Kernel Function and its Reproducing Property

Now consider the function K (of two variables) defined by the relation

$$K(z,\zeta) = \overline{g_z(\zeta)}, \quad (z,\zeta) \in \Omega \times \Omega.$$

This is called the *Bergman kernel function* of the domain Ω. It has several nice properties.

For example, for any $(f,z) \in A^2(\Omega) \times \Omega$, we have

$$\int_\Omega f(\zeta)\, K(z,\zeta)\, dA(\zeta) = \int_\Omega f(\zeta)\, \overline{g_z(\zeta)}\, dA(\zeta) = \langle f, g_z \rangle = f(z).$$

This is called the *reproducing property* of the Bergman kernel function.

Here is another nice property of the Bergman kernel function.

Theorem

If $\{\varphi_n\}_{n=1}^{\infty}$ is an orthonormal basis of $A^2(\Omega)$, then the kernel function has the representation

$$K(z,\zeta) = \sum_{n=1}^{\infty} \varphi_n(z)\,\overline{\varphi_n(\zeta)}.$$

In particular, if one happens to know what the orthonormal polynomials over Ω look like, then one can find a representation for the kernel function.

Example: The Bergman Kernel Function of the Unit Disk

The functions

$$\varphi_n(z) = \sqrt{\frac{n+1}{\pi}} z^n, \quad n = 0, 1, 2, \ldots,$$

form an orthonormal basis in $A^2(\mathbb{D})$. This implies that

$$\sum_{n=0}^{\infty} \varphi_n(z)\overline{\varphi_n(\zeta)} = \frac{1}{\pi}\sum_{n=0}^{\infty}(n+1)(z\bar{\zeta})^n = \frac{1}{\pi}\frac{1}{(1-z\bar{\zeta})^2}$$

is the Bergman kernel function of the unit disk. By the reproducing property, we have

$$f(z) = \frac{1}{\pi}\int_{\mathbb{D}} \frac{f(\zeta)}{(1-\bar{\zeta}z)^2}\, dA(\zeta), \quad (f, z) \in A^2(\mathbb{D}) \times \mathbb{D}.$$

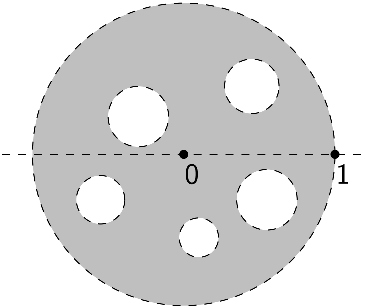
Review

Given a domain Ω, we can construct the Hilbert Space $A^2(\Omega)$.

We know that the Bergman Kernel Function for $A^2(\Omega)$ exists, but don't always have an explicit representation for it.

Circular Multiply Connected Domains

Let $\{D_{c_j,r_j}\}_{j=1}^{s}$ be a collection of mutually disjoint, closed disks contained within the unit disk.



We call the set $\mathcal{D} = \mathbb{D} \setminus \bigcup_{j=1}^{s} D_{c_j,r_j}$ the *circular multiply connected domain* (CMCD) complementary to $\bigcup_{j=1}^{s} D_{c_j,r_j}$.

The Weighted Bergman Space $A^2_{\mathcal{D}}(\mathbb{D})$

Consider the set $A^2_{\mathcal{D}}(\mathbb{D})$ of all functions $f : \mathbb{D} \to \mathbb{C}$ such that

- f is analytic in $\mathbb{D}$
- $\displaystyle\int_{\mathcal{D}} |f(z)|^2 \, dA(z) < \infty.$

One finds that $A^2_{\mathcal{D}}(\mathbb{D})$ is a Hilbert space with inner product

$$\langle f, g \rangle_w = \int_{\mathcal{D}} f(z)\overline{g(z)} \, dA(z).$$

We call $A^2_{\mathcal{D}}(\mathbb{D})$ a *weighted Bergman space*.

The Orthonormal Polynomials over $\mathcal{D}$

We can use this inner product to construct the *orthonormal polynomials (OPs) over* $\mathcal{D}$.

To be more precise: with the inner product in hand, we may apply the Gram-Schmidt orthonormalization process to the linearly independent sequence of monomials $\{z^n\}_{n=0}^{\infty}$ to construct the unique sequence of polynomials $\{p_n(z)\}_{n=1}^{\infty}$characterized by the properties

$$p_n(z) = \kappa_n z^n + \ldots, \quad \kappa_n > 0 \quad \text{for } n \geq 0$$

$$\text{and} \quad \langle p_n, p_m \rangle_w = \begin{cases} 0, & n \neq m \\ 1, & n = m. \end{cases}$$

Using the Kernel to Express the OPs

Although the Gram-Schmidt process guarantees the existence of the OPs p_n for every $n \in \mathbb{N}$, it is impractical to use this method to find an explicit representation for these polynomials in the general case. This is especially true for large values of n.

An alternate approach would be to find a representation the reproducing kernel for $A^2_{\mathcal{D}}(\mathbb{D})$.

Since the kernel is closely related to the OPs, we will then be able to find an asymptotic ($n \to \infty$) representation for the p_n.

Objective

We are looking for the function $K_{\mathcal{D}}$ with the reproducing property

$$f(z) = \int_{\mathcal{D}} f(\zeta)\, K_{\mathcal{D}}(z,\zeta)\, dA(\zeta), \quad (f,z) \in A^2_{\mathcal{D}}(\mathbb{D}) \times \mathbb{D}.$$

Recall that we have been given a collection $\{D_{c_j,r_j}\}_{j=1}^{s}$ of s mutually disjoint, closed disks contained within the unit disk.

To establish some notation, we write $\Lambda_s = \{1, 2, \ldots, s\}$.

Also, for each $a \in \mathbb{D}$, put

$$\psi_a(z) = \frac{z+a}{1+\bar{a}z}, \quad z \in \mathbb{C}.$$

For a given a, then map ψ_a is a conformal map of the unit disk onto itself (i.e., an automorphism of the unit disk).

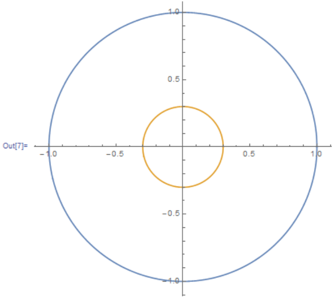
Constructing the Kernel

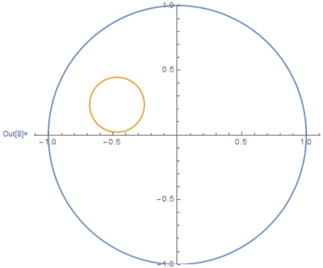
Step 1. *Find the constants a_j and σ_j.*

For each $j \in \Lambda_s$, there exists a unique $(a_j, \sigma_j) \in \mathbb{D} \times (0, 1)$ such that ψ_{a_j} maps the disk centered at the origin of radius σ_j onto the interior of D_{c_j, r_j}.

On the right, we see the unit circle (blue) and $\partial D_{c_j, r_j}$.

On the left, we see the unit circle (blue) and $\mathbb{T}_{\sigma_j}$ (yellow), the circle centered at the origin of radius $\sigma_j = 0.3$.





The function ψ_{a_j} (with $a_j = -\frac{1}{2} + \frac{i}{4}$) is a conformal map of the unit disk onto itself with $\psi_{a_j}(\mathbb{T}_{\sigma_j}) = \partial D_{c_j, r_j}$, the yellow circle on the right.

The Automorphism of the Unit Disk ψ_{a_j}

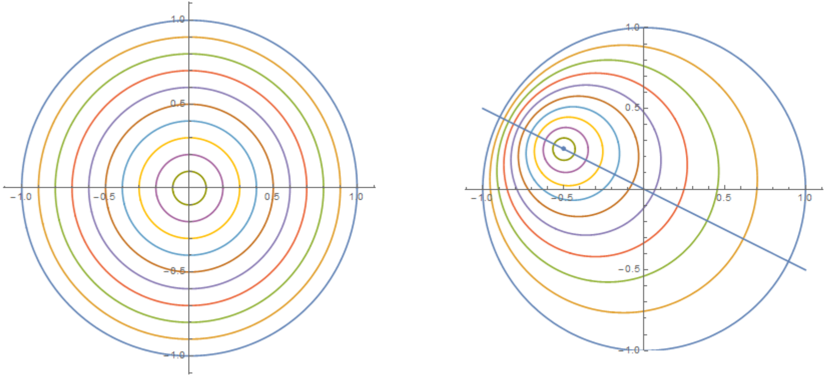


Figure: A more detailed look at how ψ_{a_j} (from the previous slide) affects the unit disk.

Constructing the Kernel

Step 2. *Define the operators T_j.*

We use the constants a_j and σ_j to define the operators

$$T_j(z) := \psi_{a_j}[\sigma_j^2 \psi_{a_j}^{-1}(z)], \quad (j, z) \in \Lambda_s \times \mathbb{C} \quad \text{and} \quad T_0(z) := z, \quad z \in \mathbb{C}.$$

Step 3. *Define the Family of Compositions $\mathcal{F}$.*

Next, we define

$$\mathcal{F}^* = \{ T_{j_n} T_{j_{n-1}} \cdots T_{j_2} T_{j_1} : n \in \mathbb{N} \text{ and } j_k \in \Lambda_s \text{ for each } 1 \leq k \leq n \}$$

and $\mathcal{F} = \mathcal{F}^* \cup \{ T_0 \}$.

An Assumption

We are almost ready to present the reproducing kernel.

But first, we must make an assumption.

Let us assume that there exists some $\rho > 1$ such that the series

$$\sum_{\gamma \in \mathcal{F}} \gamma'$$

converges absolutely and uniformly on $\overline{\mathbb{D}}_\rho = \{z : |z| \leq \rho\}$.

We have shown that this condition is true for certain CMCDs; we will return to this idea in a moment.

The Reproducing Kernel for $A^2_{\mathcal{D}}(\mathbb{D})$

Definition

For $(z, \zeta) \in \mathbb{D} \times \mathbb{D}$, we define the function

$$K_{\mathcal{D}}(z, \zeta) := \frac{1}{\pi} \sum_{\gamma \in \mathcal{F}} \frac{\gamma'(z)}{[1 - \gamma(z)\overline{\zeta}]^2}.$$

Theorem

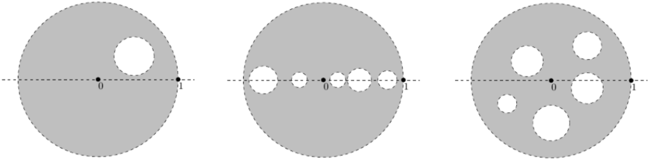
For every $(z, f) \in \mathbb{D} \times A^2_{\mathcal{D}}(\mathbb{D})$, we have

$$f(z) = \int_{\mathcal{D}} f(\zeta)\, K_{\mathcal{D}}(z, \zeta)\, dA(\zeta)$$

Comments

This kernel leads to an asymptotic representation of the OPs over $\mathcal{D}$, but we will not go into detail about that for this presentation.

We have shown that the previously mentioned assumption does holds for certain CMCDs.



In the future, it would be nice to show that the assumption holds for all CMCDs.

Thank You

Thank you for your time and attention.

Now go grade some exams!