ASYMPTOTIC PROPERTIES OF POLYNOMIALS ORTHOGONAL OVER MULTIPLY CONNECTED DOMAINS

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ABSTRACT

We investigate the asymptotic behavior of polynomials orthogonal over certain multiply connected domains. Each domain that we consider has an analytic boundary and is, in a strong sense, conformally equivalent to a canonical type of multiply connected domain called a circular domain. The two most general results involve the construction of a series expansion and an integral representation for these polynomials. We show that the integral representation can be utilized to derive more specific results when the domain of orthogonality is circular. In this case, we shed light on the manner in which the holes in the domain of orthogonality influence the polynomials.

DEDICATION

To Linley Clark Henegan.

LIST OF ABBREVIATIONS AND SYMBOLS

$\hat{\mathbb{C}}$	$\mathbb{C} \cup \{\infty\}$		
Λ_s	$\{1, 2, 3, \ldots, s\}$	$s \in \mathbb{N}$	
$\mathbb{D}_r$	$\{z \in \mathbb{C} : \lvert z\rvert < r\}$	$r \in (0, \infty)$	
$\mathbb{T}_r$	$\{z \in \mathbb{C} : \lvert z\rvert = r\}$	$r \in (0, \infty)$	
Δ_r	$\{z \in \hat{\mathbb{C}} : \lvert z\rvert > r\}$	$r \in (0, \infty)$	
$\mathbb{A}_r$	$\{z : r < \lvert z\rvert < 1/r\}$	$r \in (0, 1)$	
$D_{c,r}$	$\{z : \lvert z - c\rvert \leq r\}$	$c \in \mathbb{C}$	$r \in (0, \infty)$

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TABLE OF CONTENTS

LIST OF FIGURES

1 INTRODUCTION AND MAIN RESULTS

1.1 Introduction

Let G be a bounded, simply connected domain in the complex plane and let $\mathcal{G}$ be a Lebesgue measurable subset of G. Let $\mathcal{A}^2(G, \mathcal{G})$ denote the set of all functions f which are analytic in G and square-integrable with respect to area measure over $\mathcal{G}$:

$$\int_{\mathcal{G}} |f(z)|^2 \, dA(z) < \infty.$$

The subsets $\mathcal{G}$ that we consider in this dissertation are such that the collection $\mathcal{A}^2(G, \mathcal{G})$ becomes a Hilbert space when endowed with the inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{\mathcal{G}} f(z) \, \overline{g(z)} \, dA(z). \tag{1.1.1}$$

Consequently, we may apply the Gram-Schmidt orthonormalization process to the linearly independent sequence of monomials $\{z^n\}_{n=0}^{\infty}$ to construct the unique sequence of *orthonormal* polynomials $\{p_n(z)\}_{n=1}^{\infty}$ characterized by the properties

$$p_n(z) = \kappa_n z^n + \ldots, \qquad \kappa_n > 0 \quad (n \geq 0)$$

and

$$\langle p_n, p_m \rangle = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases} .$$

If we divide p_n by its leading coefficient κ_n, then we obtain the nth *monic polynomial orthogonal with respect to* (or *over*) $\mathcal{G}$, which we will denote by the symbol $P_{n,\mathcal{G}}$. We call $\mathcal{G}$ the *domain of orthogonality* for the polynomials $\{P_{n,\mathcal{G}}\}_{n=1}^{\infty}$. If the domain of orthogonality is understood, then we will simply write P_n in place of $P_{n,\mathcal{G}}$.

We emphasize that, for the domains of orthogonality $\mathcal{G}$ which we will consider, the existence and uniqueness of the polynomials $\{P_{n,\mathcal{G}}\}_{n=1}^{\infty}$ are guaranteed. Moreover, the Gram-Schmidt process provides us with a constructive method for finding these polynomials. However, it is only in exceptional cases that either the Gram-Schmidt process or any other known method leads to an explicit representation for the polynomials. One such case is considered in the following paragraph.

Let G and $\mathcal{G}$ equal the unit disk $\mathbb{D}$. Then the monic polynomials orthogonal over $\mathbb{D}$ are given by

$$P_{n,\mathbb{D}}(z) = z^n, \quad n \geq 0. \tag{1.1.2}$$

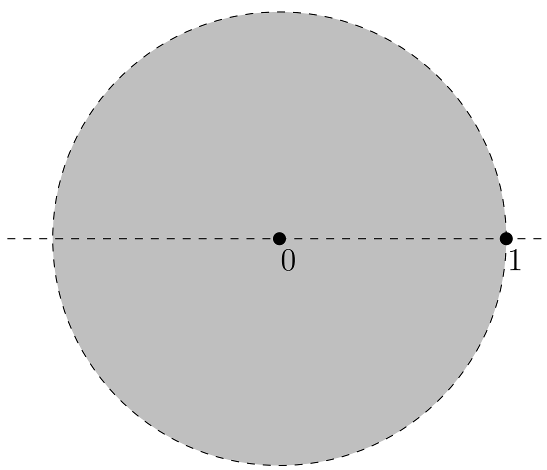


Figure 1.1: The unit disk $\mathbb{D}$

We highlight two aspects of this case which make it exceptional. First, the set $G = \mathbb{D}$ is the canonical example of a bounded, simply connected domain. Second, since $\mathcal{G} = G$, we have that the domain of orthogonality $\mathcal{G}$ is simply connected.

Now let G be a bounded, simply connected domain in the complex plane whose boundary is an analytic Jordan curve. For the choice $\mathcal{G} = G$, the polynomials $\{P_{n,\mathcal{G}}\}_{n=1}^{\infty}$ were first studied by T. Carleman in [2]. More recently, a thorough investigation of their asymptotic properties as $n \to \infty$ was accomplished in the series of papers [4, 5, 6, 11]. Broadly speaking, this dissertation extends the results of these papers to cases where $\mathcal{G}$ is a multiply connected subset of G. To be more precise, we must introduce the concept of a *circular multiply connected domain* (CMCD). In order to obtain an example of a CMCD, one begins with the unit disk and removes a finite number of mutually disjoint, closed subdisks.

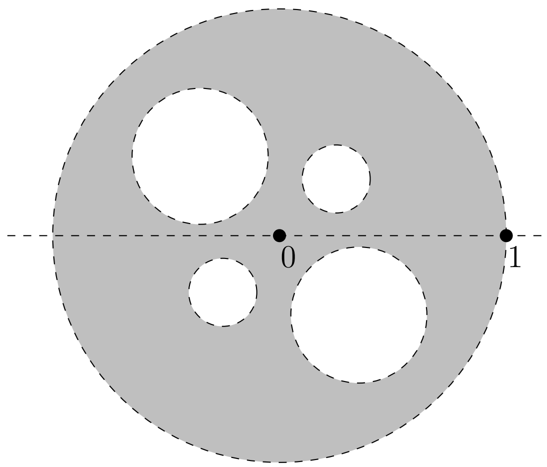


Figure 1.2: An example of a circular multiply connected domain

Let G be a bounded, simply connected domain in the complex plane whose boundary is an analytic Jordan curve and let φ be a conformal map of G onto the unit disk. Let $\mathcal{D}$ be a CMCD and put $\mathcal{G} := \varphi^{-1}(\mathcal{D})$. Finally, let $\{P_n\}_{n=1}^{\infty}$ denote the monic polynomials orthogonal over $\mathcal{G}$. The purpose of this dissertation is to initiate an investigation into understanding how the holes of the domain of orthogonality $\mathcal{G}$ influence the behavior of the resulting polynomials $\{P_n\}_{n=1}^{\infty}$. In the next section, we provide an outline of our main results.

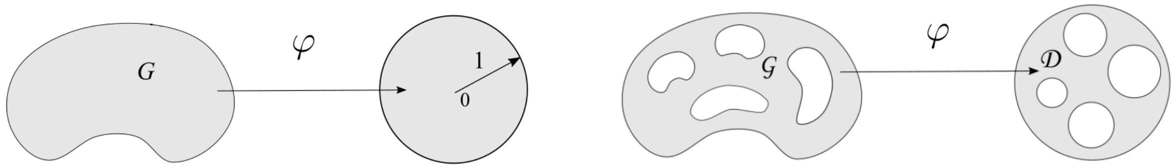


Figure 1.3: The construction of a domain of orthogonality $\mathcal{G}$

1.2 Outline of main results

In cases where $\mathcal{D}$ satisfies a certain condition (Assumption 1.3.1), we obtain a series representation for P_n for all n sufficiently large. This is stated in Theorem 1.6.1. From that theorem it follows that we can obtain an integral representation for P_n as n approaches infinity, which is precisely described in Theorem 1.6.2. These are the two most general theorems, from which more specific results may be derived.

The construction of the expansions in Theorem 1.6.1 depends on a so-called *reproducing kernel* associated with the space $\mathcal{A}^2(G, \mathcal{G})$. An essential result related to Theorem 1.6.1 involves providing an explicit representation for this kernel, which is the content of Theorem 1.4.1.

Perhaps the most attractive portion of our investigation emerges when orthogonality is considered over the canonical case of a CMCD. Analysis of the resulting polynomials is far from trivial, which stands in stark contrast to the the simplicity of the polynomials (1.1.2) orthogonal over the unit disk. The precise statements for this case are contained in Theorem 1.7.2. Here, we will be able to reveal the very interesting way in which the removed disks influence the behavior of the orthogonal polynomials.

Finally, we demonstrate the existence of a variety of CMCDS for which Assumption 1.3.1 holds. This is achieved in Theorem 1.9.1.

The remainder of this chapter is devoted to stating the main results which have been outlined above. We will begin by introducing preliminary material essential to this description.

1.3 Preliminaries

Let G_1 be a bounded, simply connected domain in the complex plane whose boundary L_1 is an analytic Jordan curve. Let φ be a conformal map of G_1 onto the unit disk. Let $\{D_{c_j,r_j}\}_{j=1}^s$ be a collection of $s \geq 1$ mutually disjoint, closed disks contained within the unit disk:

$$D_{c_j,r_j} = \{z \in \mathbb{C} : |z - c_j| \leq r_j\}, \quad 1 \leq j \leq s.$$

Let

$$\mathcal{D} := \mathbb{D} \setminus \bigcup_{j=1}^{s} D_{c_j,r_j}$$

be the circular multiply connected domain (CMCD) complementary to $\bigcup_{j=1}^s D_{c_j,r_j}$. Put

$$\mathcal{G} := \varphi^{-1}(\mathcal{D}).$$

For each $j \in \Lambda_s = \{1, 2, \ldots, s\}$, there exists a unique pair of numbers $a_j \in \mathbb{D}$ and $\sigma_j \in (0, 1)$ such that the Möbius transformation

$$\chi_j(z) := \frac{z + a_j}{1 + \overline{a_j} z}$$

maps the open disk centered at the origin of radius σ_j onto the interior of D_{c_j,r_j}:

$$\chi_j(\mathbb{D}_{\sigma_j}) = \mathring{D}_{c_j,r_j}.$$

The function χ_j is a conformal map of the unit disk onto itself (i.e., an *automorphism of the unit disk*) whose inverse is given by

$$\Phi_j(z) := \frac{z - a_j}{1 - \overline{a_j} z}.$$

For each $j \in \Lambda_s$, we use the constants a_j and σ_j to define the transformation

$$T_j(z) := \chi_j(\sigma_j^2 \Phi_j(z)), \quad z \in \hat{\mathbb{C}}.$$

We also set

$$T_0(z) := z, \quad z \in \hat{\mathbb{C}}.$$

We associate to $\mathcal{D}$ the family $\mathcal{T}^*$ of all finite compositions of the transformations T_j:

$$\mathcal{T}^* := \{T_{j_n} T_{j_{n-1}} \cdots T_{j_2} T_{j_1} : n \in \mathbb{N} \text{ and } j_k \in \Lambda_s \text{ for each } 1 \leq k \leq n\}.$$

Finally, we let

$$\mathcal{T} := \mathcal{T}^* \cup \{T_0\}.$$

Each $\tau \in \mathcal{T}$ is analytic in an open set containing the closed unit disk. We will use $\mathcal{T}$ to construct an asymptotic expansion for P_n in cases where the following assumption holds.

Assumption 1.3.1. *There exists some $\rho \in (0,1)$ such that the function series $\sum_{\tau \in \mathcal{T}} |\tau'|$ converges uniformly on each compact subset of $\mathbb{D}_{1/\rho}$.*

In this work, we will demonstrate that Assumption 1.3.1 is true for certain CMCDs. These are described with precision in Section 1.9. Here we remark that (i) the assumption always holds for CMCDs which have at most two removed disks and (ii) for any natural number s, there exists a CMCD with s removed disks for which the assumption is true.

If $\mathcal{D}$ is a CMCD for which Assumption 1.3.1 holds, then, by choosing a larger $\rho \in (0,1)$ if necessary, we can guarantee that the following conditions are satisfied:

- $\sum_{\tau \in \mathcal{T}} |\tau'|$ converges uniformly on every compact subset of $\mathbb{D}_{1/\rho}$,
- φ^{-1} has an analytic and univalent continuation to $\mathbb{D}_{1/\rho}$,
- $\mathbb{D}_{1/\rho} \subset \bigcap_{j \in \Lambda_s} \chi_j(\mathbb{D}_{1/\sigma_j})$, and
- $\bigcup_{j \in \Lambda_s} D_{c_j, r_j} \subset \mathbb{D}_\rho$.

The fact that condition two can be satisfied is a result of the discussion in Section 8.2.3. Meanwhile, the third condition can be satisfied since, for every $j \in \Lambda_s$, we have

$$\overline{\mathbb{D}} = \chi_j(\overline{\mathbb{D}}) \subsetneq \chi_j(\mathbb{D}_{1/\sigma_j}).$$

The purpose of the final condition above is to ensure that all of the holes of $\mathcal{D}$ belong to the interior of $\mathbb{T}_\rho$, the circle centered at the origin of radius ρ.

1.4 The reproducing kernel $\mathcal{K}_\mathcal{G}$ and the related kernel $\mathcal{M}_\mathcal{G}$

Since $\mathcal{A}^2(G_1, \mathcal{G})$ is a Hilbert Space under the inner product (1.1.1), the Riesz Representation Theorem asserts the existence of a unique function $\mathcal{K}_\mathcal{G}(z, \zeta)$ defined for z and ζ in G_1, analytic in z and anti-analytic in ζ, which has the so-called *reproducing property*

$$g(z) = \frac{1}{\pi} \int_\mathcal{G} g(\zeta)\, \mathcal{K}_\mathcal{G}(z, \zeta)\, dA(\zeta), \quad (z, g) \in G_1 \times \mathcal{A}^2(G_1, \mathcal{G}). \tag{1.4.1}$$

Our first theorem provides an explicit representation for this *reproducing kernel* $\mathcal{K}_\mathcal{G}$.

Theorem 1.4.1. *If $\mathcal{D}$ is a CMCD for which Assumption 1.3.1 holds, then we have the representation*

$$\mathcal{K}_\mathcal{G}(z, \zeta) = \sum_{\tau \in \mathcal{T}} \frac{\frac{d}{dz}[(\tau \circ \varphi)(z)]\, \overline{\varphi'(\zeta)}}{[1 - (\tau \circ \varphi)(z)\, \overline{\varphi(\zeta)}]^2}, \quad (z, \zeta) \in \varphi^{-1}(\mathbb{D}_{1/\rho}) \times \varphi^{-1}(\mathbb{D}_{1/\rho}).$$

From the kernel $\mathcal{K}_{\mathcal{G}}$ we construct the meromorphic kernel $\mathcal{M}_{\mathcal{G}}$, given by

$$\mathcal{M}_{\mathcal{G}}(z,\zeta) := \sum_{\tau\in\mathcal{T}} \left[\frac{\varphi'(\zeta)}{\varphi(\zeta)-\tau(0)} \cdot \frac{(\tau\circ\varphi)(z)-\tau(0)}{\varphi(\zeta)-(\tau\circ\varphi)(z)} \right], \quad (z,\zeta) \in \varphi^{-1}(\mathbb{D}_{1/\rho}) \times \varphi^{-1}(\mathbb{D}_{1/\rho}).$$

In a moment we will show how $\mathcal{M}_{\mathcal{G}}$ can lead to a series expansion of the P_n, but first we must first introduce a pair of maps.

1.5 The exterior maps ψ and ϕ

Let $\psi(w)$ be the unique conformal map of the exterior of the unit circle onto the exterior of L_1 which maps the point at infinity to itself and has a positive derivative at infinity:

$$\psi(\infty) = \infty \quad \text{and} \quad \psi'(\infty) := \lim_{w\to\infty} \frac{\psi(w)}{w} > 0.$$

Since L_1 is an analytic Jordan curve, there exists some $\varrho < 1$ such that ψ admits an analytic and univalent continuation to $\{w : |w| > \varrho\}$. We will continue to use the symbol ψ to denote this analytic continuation. For every $r \in [\varrho, \infty)$, we set

$$\Omega_r := \psi(\Delta_r), \qquad L_r := \partial\Omega_r, \qquad G_r := \mathbb{C} \setminus \overline{\Omega}_r,$$

so that, for every $r \in (\varrho, \infty)$, we have that L_r is an analytic Jordan curve. Finally, we let ϕ denote the inverse of ψ:

$$\phi : \Omega_\varrho \to \Delta_\varrho.$$

Now we have all of the tools necessary to construct a series expansion of the P_n. This is done in the following section.

1.6 Series expansion and integral representation of the P_n

For this section, we will suppose that $\mathcal{D}$ is a CMCD for which Assumption 1.3.1 holds. By choosing a larger $\rho \in (0,1)$ from Assumption 1.3.1 if necessary, we can guarantee that, in addition to the four conditions mentioned in Section 1.3, we also have $\rho \geq \varrho$, where ϱ is the number described in section 1.5.

Let us fix some $r \in (\rho, 1)$ such that $G_{1/r} \subset \varphi^{-1}(\mathbb{D}_{1/\rho})$. Next, we fix some $t \in (r,1)$. We record the relationships between ρ, r, and t below for reference:

$$0 < \rho < r < t < 1 < \frac{1}{t} < \frac{1}{r} < \frac{1}{\rho}.$$

Then, for each $n \in \mathbb{N}$, we recursively define a sequence $\{f_{n,k}\}_{k=0}^{\infty}$ in the following manner. First, we set

$$f_{n,0}(z) := 0, \quad z \in \hat{\mathbb{C}}.$$

Then, for $k \geq 0$, we put

$$f_{n,2k+1}(z) := -\frac{1}{2\pi i} \oint_{L_t} f_{n,2k}(\zeta)\, \mathcal{M}_{\mathcal{G}}(z,\zeta)\, [\phi(\zeta)]^{n+1}\, d\zeta, \quad z \in \varphi^{-1}(\mathbb{D}_{1/\rho}) \setminus L_t$$

and

$$f_{n,2k+2}(z) := \frac{1}{2\pi i} \oint_{L_{1/t}} f_{n,2k+1}(\zeta) \frac{\phi'(\zeta)\, [\phi(\zeta)]^{-n-1}}{\phi(\zeta) - \phi(z)}\, d\zeta, \quad z \in \Omega_r \setminus L_{1/t}.$$

We will show that, for n large enough, the two series

$$\sum_{k=0}^{\infty} f_{n,2k} \qquad \text{and} \qquad \sum_{k=0}^{\infty} f_{n,2k+1} \tag{1.6.1}$$

converge absolutely and normally in the sets $\Omega_r \setminus L_{1/t}$ and $G_{1/r} \setminus L_t$, respectively.

We let P_n denote the monic polynomials orthogonal over $\mathcal{G} = \varphi^{-1}(\mathcal{D})$ and we let κ_n denote the leading coefficient of the corresponding orthonormal polynomials. We may now state our results.

Theorem 1.6.1. *For n sufficiently large, we have the series expansion*

$$(n+1)[\phi'(\infty)]^{n+1}P_n(z) = \frac{d}{dz}\begin{cases} [\phi(z)]^{n+1}\displaystyle\sum_{k=0}^{\infty} f_{n,2k}(z) & z \in \Omega_{1/t}, \\ [\phi(z)]^{n+1}\displaystyle\sum_{k=0}^{\infty} f_{n,2k}(z) - \sum_{k=0}^{\infty} f_{n,2k+1}(z), & z \in \Omega_t \cap G_{1/t}, \\ -\displaystyle\sum_{k=0}^{\infty} f_{n,2k+1}(z), & z \in G_t \end{cases}$$

and

$$(n+1)[\phi'(\infty)]^{2n+2}\kappa_n^{-2} = 1 + \sum_{k=0}^{\infty} \frac{1}{2\pi i}\oint_{L_{1/t}} f_{n,2k+1}(\zeta)\phi'(\zeta)[\phi(\zeta)]^{-n-2}\, d\zeta.$$

By looking at the dominant terms of the the expansions in Theorem 1.6.1, we arrive at the following theorem.

Theorem 1.6.2. *For n sufficiently large, we have the integral representation*

$$P_n(z) = \frac{[\phi'(\infty)]^{-n-1}}{2\pi i}\int_{\mathbb{T}_1} w^n\,[1+K_n(w)]\,\sum_{\tau\in\mathcal{J}} \frac{\frac{d}{dz}[(\tau\circ\varphi)(z)]}{w-(\tau\circ\varphi)(z)}\,dw, \quad z \in G_1,$$

where $K_n(w)$ is analytic in $|w| < 1/t$ and $K_n(w) = O(t^{2n})$ locally uniformly as $n \to \infty$ in $|w| < 1/t$.

We remark that the expressions in the formula above depend largely on the geometry of the domains involved. Consequently, Theorem 1.6.2 can lead to more specific formulas for $P_n(z)$, provided that we know more about the domains. This is explored in the next section.

1.7 Polynomials orthogonal over circular multiply connected domains

We now consider what happens in Theorem 1.6.2 in the canonical case when $\mathcal{G} = \mathcal{D}$ and ϕ and φ are the identity maps. Then the equation in that theorem simplifies to

$$P_n(z) = \frac{1}{2\pi i}\int_{\mathbb{T}_1} w^n\,[1+K_n(w)]\,\sum_{\tau\in\mathcal{J}} \frac{\tau'(z)}{w-\tau(z)}\,dw, \quad z \in \mathbb{D}.$$

For each $(z, \tau) \in \mathbb{D} \times \mathcal{T}$, the expression $\frac{\tau'(z)}{w - \tau(z)}$, viewed as a function of w, is meromorphic in the extended complex plane, where its only singularity is a simple pole at the point $w = \tau(z)$. Then, by the Residue Theorem and by the fact that $\tau(\mathbb{D}) \subset \mathbb{D}$ for all $\tau \in \mathcal{T}$, we have the following result.

Corollary 1.7.1. *If $\mathcal{G} = \mathcal{D}$ and if ϕ and φ are the identity maps, then for n sufficiently large, we have*

$$P_n(z) = \sum_{\tau \in \mathcal{T}} [\tau(z)]^n \cdot \tau'(z) \cdot [1 + (K_n \circ \tau)(z)], \quad z \in \mathbb{D},$$

where $K_n(\zeta)$ is analytic in $|\zeta| < 1/t$ and $K_n(\zeta) = O(t^{2n})$ locally uniformly as $n \to \infty$ in $|\zeta| < 1/t$. Therefore, we have

$$P_n(z) = z^n \cdot [1 + K_n(z)] + \sum_{j=1}^{s} P_n(T_j(z)) \cdot T_j'(z), \quad z \in \mathbb{D}.$$

In order to make more precise statements about the asymptotic behavior of $P_n(z)$ for $z \in \mathbb{D}$, we will need several definitions. We let $\mathcal{H} := \{z : \mathrm{Re}(z) > 0\}$ denote the right half plane. For each $1 \leq j \leq s$, we make the following definitions.

- $\theta_j := \operatorname{Arg} a_j$
- $\mathcal{H}e^{i\theta_j} := \{ze^{i\theta_j} : z \in \mathcal{H}\}$
- $\beta_j := \frac{1}{\overline{a_j}} - a_j$
- $\mathcal{T}_j := \{T_j \tau : \tau \in \mathcal{T}\}$

We also define the constant

$$\alpha := \max\{|a_j| : j \in \Lambda_s\}.$$

After possible relabeling, we may assume that there exists some $\omega \in \Lambda_s$ such that

$$\alpha = |a_1| = |a_2| = \cdots = |a_\omega| > |a_j|, \quad \omega < j \leq s. \tag{1.7.1}$$

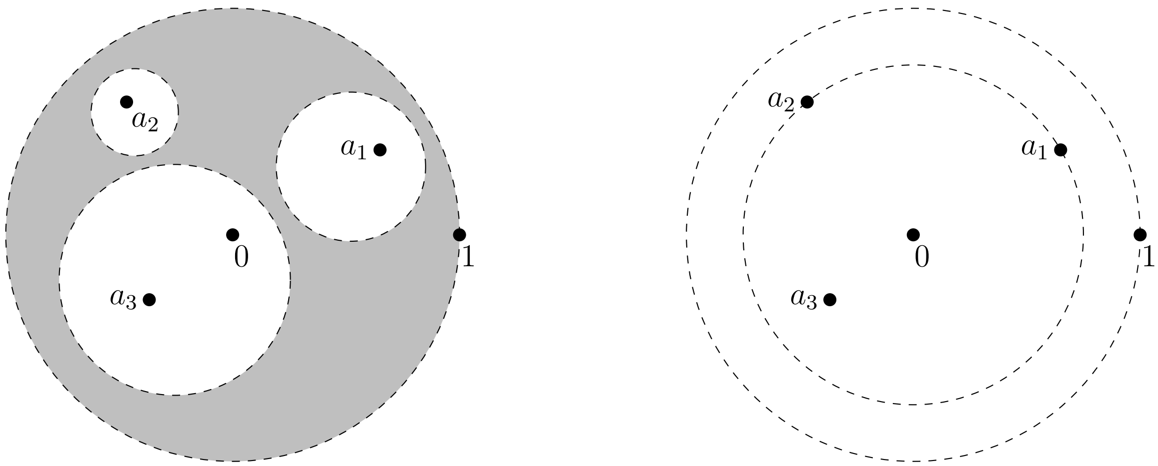


Figure 1.4: A CMCD with $\alpha = |a_1| = |a_2| > |a_3|$. The circle $\mathbb{T}_\alpha$.

We will see that the behavior of the polynomials changes dramatically across the circle $\mathbb{T}_\alpha$ that passes through those a_j with largest moduli. The behavior of the polynomials inside $\mathbb{T}_\alpha$ is described in terms of functions which we now introduce. For each $1 \le j \le \omega$, we define

$$\Theta_j(t) = t \sum_{v \in \mathbb{Z}} \sigma_j^{2v} \exp(-\overline{\beta_j} \sigma_j^{2v} t), \quad t \in \mathcal{H} e^{i\theta_j}$$

and

$$\mathcal{J}_{j,n}(z) := \sum_{\tau \in \mathcal{T} \setminus \mathcal{T}_j} (\alpha^2 - 1) \cdot e^{ni\theta_j} \cdot \frac{\Phi_j'(\tau(z))}{\Phi_j(\tau(z))} \cdot \Theta_j(-n\Phi_j(\tau(z))) \cdot \tau'(z), \quad z \in \mathbb{D}_\alpha.$$

We remark that, for each $1 \le j \le \omega$, the function $\mathcal{J}_{j,n}$ is bounded on compact subsets of $\mathbb{D}_\alpha$. We may now describe the behavior of P_n at each point in the complex plane.

Theorem 1.7.2. *The behavior of $P_n(z)$ for $z \in \mathbb{C}$ is as follows.*

(i) If $r > \alpha$, then there exists some $\nu \in (0, r)$ such that for every $z \in \mathbb{T}_r$, we have

$$P_n(z) = z^n + O(\nu^n).$$

(ii) Inside $\mathbb{T}_\alpha$, we have

$$P_n(z) = \frac{\alpha^n}{n} \cdot \sum_{j=1}^{\omega} \mathfrak{J}_{j,n}(z) + O\left(\frac{\alpha^n}{n^2}\right)$$

normally for $z \in \mathbb{D}_\alpha$ as $n \to \infty$.

(iii) If $z = a_j$ for some $1 \leq j \leq \omega$, then

$$P_n(a_j) = \frac{a_j^n}{1 - \sigma_j^2} + O\left(\frac{\alpha^n}{n}\right).$$

Otherwise, we have

$$P_n(z) = z^n + O\left(\frac{\alpha^n}{n}\right)$$

uniformly on compact subsets of $\mathbb{T}_\alpha \setminus \bigcup_{j=1}^{\omega}\{a_j\}$.

This theorem shows that, in terms of the asymptotic behavior of the polynomials orthogonal over a CMCD, the most relevant removed disks D_{c_j,r_j} are those for which the corresponding quantity a_j is of maximal magnitude (i.e., those for which we have $|a_j| = \alpha$). However, we may not disregard the other removed disks entirely, as they are used in the definitions of the functions $\mathfrak{J}_{j,n}$.

1.8 Polynomials orthogonal over circular doubly connected domains

In fact, we can say even more when the domain of orthogonality is a circular *doubly* connected domain. Then $\mathcal{D}$ can be represented as

$$\mathcal{D} = \mathbb{D} \setminus D,$$

where D is a closed disk contained within the unit disk. The exposition will be a little more clear if we assume that D is centered on the positive real axis.

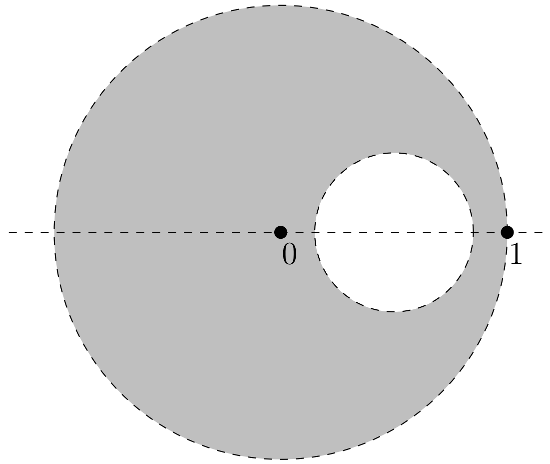


Figure 1.5: An example of a circular doubly connected domain

Let $a \in (0,1)$ be the unique number such that the Möbius transformation

$$\Phi(z) = \frac{z-a}{1-az}$$

maps the interior of D onto a circle centered at the origin and let σ denote the radius of $\Phi(D)$. If P_n are the monic polynomials orthogonal over $\mathcal{D}$, then we have

$$\lim_{n\to\infty} \frac{P_n(z)}{z^n} = 1, \quad |z| \geq a, \quad z \neq a,$$

$$\lim_{n\to\infty} \frac{P_n(a)}{a^n} = \frac{1}{1-\sigma^2}.$$

For the asymptotics on $|z| < a$, we need the function

$$F(w) = w \sum_{v\in\mathbb{Z}} \sigma^{2v} e^{(a-a^{-1})\sigma^{2v} w}, \quad \mathrm{Re}(w) > 0.$$

By Theorem 1.7.2, we have

$$P_n(z) = \frac{a^n}{n} \cdot (a^2-1) \cdot \frac{\Phi'(z)}{\Phi(z)} \cdot F(-n\Phi(z)) + O(a^n/n^2) \tag{1.8.1}$$

uniformly on compact subsets of $|z| < a$. Thus, in order to understand the limiting behavior of $P_n(z)$ for $|z| < a$, we must also understand the limiting behavior of $F(n\Phi(z))$.

Corollary 1.8.1. *Let* $\{n_k\}_{k=1}^{\infty}$ *be a subsequence of the natural numbers. The sequence* $\{F(n_k w)\}_{k=1}^{\infty}$ *converges normally on* $\mathrm{Re}(w) > 0$ *if and only if*

$$\lim_{k\to\infty} \langle \log_{\sigma^2} n_k \rangle = q$$

for some $q \in [0,1)$*, in which case*

$$\lim_{k\to\infty} \frac{n_k P_{n_k}(z)}{a^{n_k}} = f_q(z) := \frac{\Phi'(z)(a^2-1)}{\Phi(z)} F(-\sigma^{2q}\Phi(z)), \quad z \in \mathbb{D}_a.$$

Moreover, $f_q \neq f_p$ *for* $0 \leq q < p < 1$*, and since the sequence* $\{\langle \log_{\sigma^2} n_k \rangle\}$ *is dense in* $[0,1)$*, it follows that the sequence* $\{na^{-n}P_n(z)\}_{n\geq 0}$ *has for normal limit points on* $|z| < a$ *the following continuous one-parameter family of functions:*

$$\left\{ \frac{\Phi'(z)(a^2-1)}{\Phi(z)} F(-\sigma^{2q}\Phi(z)) : \; q \in [0,1) \right\}.$$

1.9 Cases where Assumption 1.3.1 holds true

Let

$$\mathcal{D} = \mathbb{D} \setminus \bigcup_{j\in\Lambda_s} D_{c_j,r_j}$$

be a CMCD and let a_j and σ_j be the constants introduced in section 1.3. We define the additional constants

- $a := \min_{j\in\Lambda_s} |a_j|$,
- $\sigma := \max_{j\in\Lambda_s} \sigma_j$,
- $m := \min_{j\in\Lambda_s} \left(\frac{1}{|a_j|} \cdot \frac{1-|a_j|^2\sigma_j^2}{1-\sigma_j^2} \right)$,

- $\lambda := \max_{j\in\Lambda_s}\{|c_j| + r_j\} = \max_{j\in\Lambda_s} \dfrac{|a_j| + \sigma_j}{1 + |a_j|\sigma_j}$, and

- $\mathcal{N}_{\mathcal{D}} := \left(\dfrac{\sigma}{1-\sigma^2} \cdot \dfrac{1-a^2}{a}\right)^2 \cdot \dfrac{1}{(m-\lambda)^2}$.

We comment that m represents the minimum of the magnitudes of the poles of the basic transformations T_j.

Theorem 1.9.1. *Assumption 1.3.1 holds for a CMCD $\mathcal{D}$ which satisfies any one of the following conditions:*

1) $c_j \neq 0$ *for each* $j \in \Lambda_s$ *and* $s < 1/\mathcal{N}_{\mathcal{D}}$,

2) $0 \in \mathcal{D}$ *and* $c_j \in (-1, 1)$ *for each* $j \in \Lambda_s$, *or*

3) $\mathcal{D} = \Psi(\tilde{\mathcal{D}})$, *where* $\tilde{\mathcal{D}}$ *is a CMCD that satisfies Assumption 1.3.1 and* Ψ *is an automorphism of the unit disk.*

For every $s \in \mathbb{N}$, there exists a CMCD with s removed disks which satisfies Case 1 above. Indeed, if $|a_j| = a$ and $\sigma_j = \sigma$ for every $j \in \Lambda_s$, then we have

$$\frac{1}{\mathcal{N}_{\mathcal{D}}} = \left(\frac{1 + a\sigma^3}{1 + a\sigma} \cdot \frac{1}{\sigma}\right)^2,$$

from which we infer that $1/\mathcal{N}_{\mathcal{D}} \to \infty$ as $\sigma \to 0$.

For the CMCD in Figure 1.6, we have $|a_j| = a \approx 0.456961$ and $\sigma_j = \sigma \approx 0.138475$ for each $j \in \Lambda_s$. In this case, we have $1/\mathcal{N}_{\mathcal{D}} \approx 46.2399$.

A CMCD which meets the second case in 1.9.1 may be referred to as a CMCD which *contains the origin and is symmetric about the real axis.* Once again, for every $s \in \mathbb{N}$, there exists a CMCD with s removed disks which satisfies the second case.

If $\mathcal{D}$ does not contain the origin but is symmetric about the real axis, then we can find some automorphism of the unit disk Ψ and some CMCD $\tilde{\mathcal{D}}$ which *does* contain the origin and is symmetric about the real axis such that $\Psi(\tilde{\mathcal{D}}) = \mathcal{D}$.

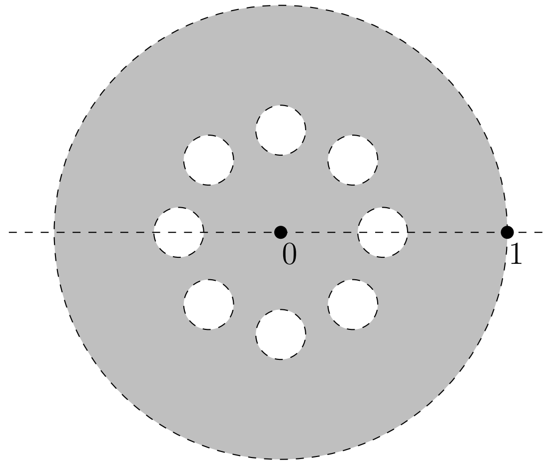


Figure 1.6: A CMCD where $|a_j| = a$ and $\sigma_j = \sigma$ for each $j \in \Lambda_s$.

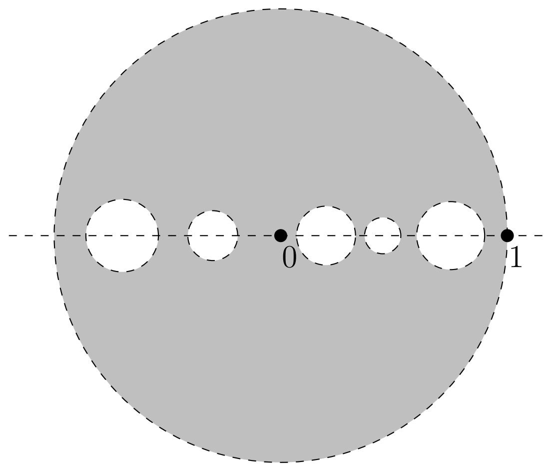


Figure 1.7: A CMCD which contains the origin and is symmetric about the real axis

Finally, we remark that if $\mathcal{D}$ is a CMCD with either one or two removed disks, then there exists some CMCD $\tilde{\mathcal{D}}$ which satisfies case two and there exists some automorphism of the unit disk Ψ such that $\Psi(\tilde{\mathcal{D}}) = \mathcal{D}$. In other words, Assumption 1.3.1 holds for every CMCD which has either one or two removed disks.

$\cdots$

We organize the forthcoming material as follows. In Chapter 2, we develop some preliminary material which is essential for a good understanding of the theory. Chapters 3 through 7 contain the proofs of the main theorems. Chapter 8, which serves as an Appendix, contains some supplementary material for the reader who is interested in some of the more technical aspects of the analysis.

2 PRELIMINARIES

In this chapter, we establish some facts about the family of compositions $\mathcal{T}$ associated with a given CMCD

$$\mathcal{D} = \mathbb{D} \setminus \bigcup_{j \in \Lambda_s} D_{c_j, r_j}.$$

We begin by investigating the transformations T_j which generate $\mathcal{T}$. By establishing these properties first, the proofs of the main theorems will become more transparent.

2.1 The transformations T_j

2.1.1 Basic properties of the transformations

We recall the construction of the operators T_j as described in Section 1.3. We associate to each $j \in \Lambda_s$ the unique pair of numbers $a_j \in \mathbb{D}$ and $\sigma_j \in (0, 1)$ such that the function

$$\chi_j(z) = \frac{z + a_j}{1 + \overline{a_j} z}$$

maps the open disk centered at the origin of radius σ_j onto the interior of D_{c_j, r_j}:

$$\chi_j(\mathbb{D}_{\sigma_j}) = \mathring{D}_{c_j, r_j}. \tag{2.1.1}$$

The function

$$\Phi_j(z) = \frac{z - a_j}{1 - \overline{a_j} z}$$

is the inverse of χ_j.

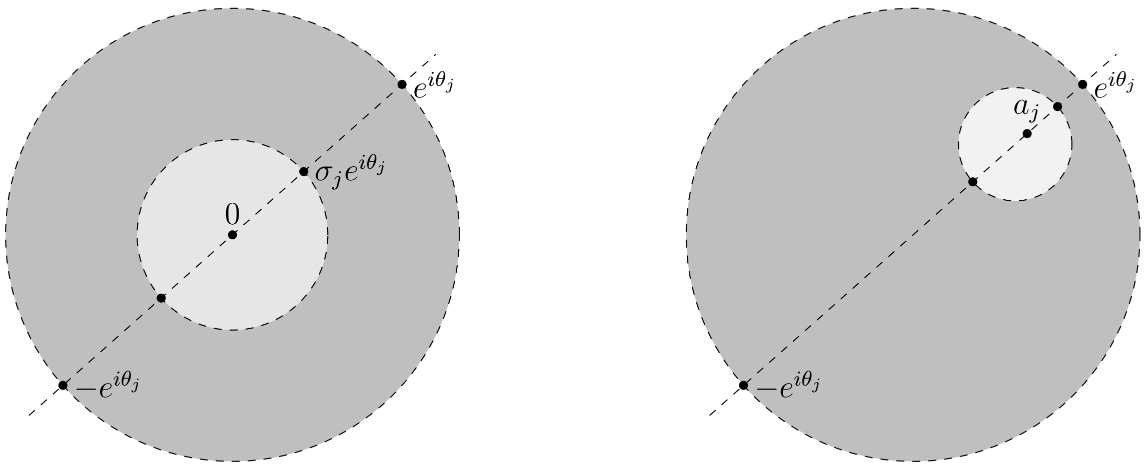


Figure 2.1: The effect of χ_j on the unit disk

In the figure, the function $\chi_j = \Phi_j^{-1}$ maps the figure on the left to the one on the right. Here, θ_j denotes the principal argument of a_j.

For each $j \in \Lambda_s$, we have $a_j = 0$ if and only if the disk D_{c_j,r_j} is centered at the origin:

$$a_j = 0 \quad \Leftrightarrow \quad c_j = 0.$$

In this case, the functions χ_j and Φ_j are the identity function:

$$\chi_j(z) = \Phi_j(z) = z.$$

If $a_j \neq 0$, then the functions χ_j and Φ_j are hyperbolic Möbius transformations whose fixed points are $e^{i\theta_j}$ and $-e^{i\theta_j}$, where θ_j denotes the principal argument of a_j. Any circle or line L which passes through both of these points is invariant under both χ_j and Φ_j, in the sense that $\chi_j(L) = \Phi_j(L) = L$. In particular, if L_{a_j} denotes the extended line (i.e., it includes the point at infinity) passing through the origin and a_j, then we have

$$\chi_j(L_{a_j}) = L_{a_j} \qquad \text{and} \qquad \Phi_j(L_{a_j}) = L_{a_j}.$$

We remark that if $a_j \neq 0$, then the pole of χ_j occurs at the point $-1/\overline{a_j}$ and the pole of Φ_j occurs at the point $1/\overline{a_j}$:

$$\chi_j(-1/\overline{a_j}) = \infty \qquad \text{and} \qquad \Phi_j(1/\overline{a_j}) = \infty. \tag{2.1.2}$$

In Section 1.3, we defined the transformation $T_j(z)$ as

$$T_j(z) = \chi_j(\sigma_j^2 \Phi_j(z)), \quad z \in \hat{\mathbb{C}}.$$

It will be helpful to have an alternate representation for this function. To this end, for each $j \in \Lambda_s$, we define

$$s_j(z) := \sigma_j^2 z.$$

Then we may write

$$T_j(z) = (\chi_j \circ s_j \circ \Phi_j)(z).$$

Since χ_j, s_j, and Φ_j are Möbius transformations, we see that T_j is a Möbius transformation as well. We will often use the following fact about Möbius transformations: *if T is a Möbius transformation, and if K is a circle or line in the extended complex plane, then $T(K)$ is a circle or line in the extended complex plane.*

We note that, since χ_j is the inverse of Φ_j, we have

$$T_j(\chi_j(\mathbb{D}_{1/\sigma_j})) = (\chi_j \circ s_j \circ \Phi_j \circ \chi_j)(\mathbb{D}_{1/\sigma_j}) = (\chi_j \circ s_j)(\mathbb{D}_{1/\sigma_j}) = \chi_j(\mathbb{D}_{\sigma_j}) = \mathring{D}_{c_j, r_j}, \tag{2.1.3}$$

by the definition of s_j and by relation (2.1.1).

2.1.2 Strings of transformations

Next we will consider compositions of transformations. We will write expressions such that $T_j T_k$ in place of $T_j \circ T_k$.

By a *string of transformations* (or simply a *string*), we mean an expression of the form

$$T_{j_n} T_{j_{n-1}} \cdots T_{j_2} T_{j_1},$$

where we have $j_k \in \Lambda_s$ for each $1 \leq k \leq n$. We say that the *length* of this string is n, we say that the *terminal transformation* of this string is T_{j_n}, and we say that the *initial transformation* of this string is T_{j_1}. Note that each string is a Möbius transformation. In particular, each string is analytic throughout $\hat{\mathbb{C}}$ except for a single point where it has a simple pole.

In the sequel, we will be interested in knowing how a given string τ acts on certain sets A in the complex plane. In other words, we will want to have knowledge of the image $\tau(A)$. We will also be interested in knowing where the pole of a given string resides. Our first lemma shows that, generally speaking, the terminal transformation of a string influences its images while the initial transformation influences the location of its pole.

Lemma 2.1.1. *If $\tau = T_{j_m} T_{j_{m-1}} \cdots T_{j_2} T_{j_1}$ is a string, then*

(i) $\tau(\chi_{j_1}(\mathbb{D}_{1/\sigma_{j_1}})) \subset \mathring{D}_{c_{j_m}, r_{j_m}}$ *and*

(ii) the pole of τ belongs to $\chi_{j_1}(\Delta_{1/\sigma_{j_1}})$.

Consequently, if $\rho \in (0,1)$ satisfies the condition $\mathbb{D}_{1/\rho} \subset \bigcap_{j \in \Lambda_s} \chi_j(\mathbb{D}_{1/\sigma_j})$, then

(iii) $\tau(\mathbb{D}_{1/\rho}) \subset \mathring{D}_{c_{j_m}, r_{j_m}}$ *and*

(iv) τ is analytic on $\mathbb{D}_{1/\rho}$.

Proof. To establish claim (i), we induct on the length of the string τ. If the length of τ is one, then we have $\tau = T_j$ for some $j \in \Lambda_s$. Then we may write

$$T_j(\chi_j(\mathbb{D}_{1/\sigma_j})) \subset \mathring{D}_{c_j, r_j}$$

by relationship (2.1.3). This establishes the base case of the induction argument for claim (i). Now suppose there exists some $k \in \mathbb{N}$ such that claim (i) holds for any string τ_k of length k. If τ is a string of length $k+1$, then we can write $\tau = T_j \tau_k$ for some $j \in \Lambda_s$ and some string $\tau_k = T_{j_k} T_{j_{k-1}} \cdots T_{j_2} T_{j_1}$ of length k . By the induction hypothesis, we have

$$T_j \tau_k(\chi_{j_1}(\mathbb{D}_{1/\sigma_{j_1}})) \subset T_j(\mathring{D}_{c_{j_k}, r_{j_k}}).$$

Recalling the definition of τ and noting that $\mathring{D}_{c_{j_k}, r_{j_k}} \subset \mathbb{D} \subset \chi_j(\mathbb{D}_{1/\sigma_j})$, this gives

$$\tau(\chi_{j_1}(\mathbb{D}_{1/\sigma_{j_1}})) \subset T_j(\chi_j(\mathbb{D}_{1/\sigma_j})) \subset \mathring{D}_{c_j, r_j}.$$

Therefore, claim (i) holds for all strings of length $k+1$. By the principle of mathematical induction, claim (i) holds for all strings.

Since $(\tau \circ \chi_{j_1})$ is a Möbius transformation, and hence a homeomorphism, the first claim implies that

$$\tau(\chi_{j_1}(\overline{\mathbb{D}}_{1/\sigma_1})) \subset D_{c_{j_m}, r_m}.$$

This shows that $|\tau(z)| < 1$ for all $z \in \chi_{j_1}(\overline{\mathbb{D}}_{1/\sigma_1})$, whence the pole of τ belongs to the complement of $\chi_{j_1}(\overline{\mathbb{D}}_{1/\sigma_1})$ in the extended complex plane. Since

$$\hat{\mathbb{C}} = \chi_{j_1}(\mathbb{D}_{1/\sigma_{j_1}}) \cup \chi_{j_1}(\overline{\Delta}_{1/\sigma_{j_1}})$$

with the union being disjoint, we see that the pole of τ belongs to the set $\chi_{j_1}(\Delta_{1/\sigma_1})$. This settles the second claim.

Next, if ρ satisfies the condition stated in the lemma, then we may write

$$\tau(\mathbb{D}_{1/\rho}) \subset \tau(\chi_{j_1}(\mathbb{D}_{1/\sigma_{j_1}})) \subset \mathring{D}_{c_{j_m}, r_{j_m}}$$

by claim one. This establishes claim (iii).

Finally, by claim (ii) we know that the pole of τ belongs to the set $\bigcup_{j\in\Lambda_s} \chi_j(\overline{\Delta}_{1/\sigma_j})$. By De Morgan's law, we have

$$\hat{\mathbb{C}} \setminus \bigcup_{j\in\Lambda_s} \chi_j(\overline{\Delta}_{1/\sigma_j}) = \bigcap_{j\in\Lambda_s} \chi_j(\mathbb{D}_{1/\sigma_j}).$$

Therefore, if ρ satisfies the stated condition, then the pole of τ does not belong to $\mathbb{D}_{1/\rho}$. This establishes claim (iv) and completes the proof of the lemma.

□

It will be convenient to establish some terminology for the next lemma. We say that two strings

$$\gamma_1 = T_{j_n} T_{j_{n-1}} \cdots T_{j_2} T_{j_1} \qquad \text{and} \qquad \gamma_2 = T_{k_m} T_{k_{m-1}} \cdots T_{m_2} T_{m_1}$$

are *equivalent* if we have $n = m$ and $j_\ell = k_\ell$ for each $1 \le \ell \le n$. Meanwhile, we say that γ_1 and γ_2 have a *common terminal string* if their terminal transformations are equivalent:

$$j_n = k_m.$$

Note that this occurs if and only if there exists some non-negative integer r such that

$$j_{n-\ell} = k_{m-\ell}, \quad 0 \le \ell \le r.$$

If this is the case, then the set

$$\{r \in \mathbb{N} \cup \{0\} : j_{n-\ell} = k_{m-\ell}, 0 \le \ell \le r\}$$

has a largest number t, and we call

$$\gamma = T_{j_n} T_{j_{n-1}} \cdots T_{j_{n-t}}$$

the *maximal common terminal string* of γ_1 and γ_2. Note that the length of γ is less than or equal to the minimum of the lengths of γ_1 and γ_2. Finally, we may define

$$\gamma_1^* := \gamma^{-1} \circ \gamma_1 \qquad \text{and} \qquad \gamma_2^* := \gamma^{-1} \circ \gamma_2. \tag{2.1.4}$$

For example, if $\gamma_1 = T_2T_7T_1T_8T_2T_8$ and $\gamma_2 = T_2T_7T_1T_8T_1T_8$ are strings, then the maximal common terminal string of γ_1 and γ_2 is $\gamma = T_2T_7T_1T_8$. If we apply the definitions (2.1.4) to this example, then we have $\gamma_1^* = T_2T_8$ and $\gamma_2^* = T_1T_8$.

Let γ_1 and γ_2 be strings. Suppose that the length of γ_1 is greater than or equal to the length of γ_2. Let γ be the maximal common terminal string of γ_1 and γ_2, and define γ_1^* and γ_2^* as in (2.1.4). Note that γ_1^* must be a string, since we are supposing that the length of γ_1 is greater than or equal to the length of γ_2. Meanwhile, if the length of γ is strictly less than the length of γ_2, then γ_2^* will also be a string. Moreover, the terminal transformations of γ_1^* and γ_2^* will not be equivalent. However, if the length of γ_2 equals the length of γ, then γ_2^* will be the identity function. In either case, we can write

$$\gamma_1 = \gamma \circ \gamma_1^* \qquad \text{and} \qquad \gamma_2 = \gamma \circ \gamma_2^*.$$

It may be helpful to keep these comments in mind as we prove the next lemma.

Lemma 2.1.2. *Suppose that*

- $\rho \in (0,1)$ *satisfies the condition* $\mathbb{D}_{1/\rho} \subset \bigcap_{j\in\Lambda_s} \chi_j(\mathbb{D}_{1/\sigma_j})$,
- $A \subset \mathbb{D}_{1/\rho} \setminus \mathbb{D}$,
- $\gamma_1 = T_{j_n} \cdots T_{j_1}$, *and*
- $\gamma_2 = T_{k_m} \cdots T_{k_1}$.

If the strings γ_1 and γ_2 are not equivalent, then $\gamma_1(A) \cap \gamma_2(A) = \emptyset$.

We comment that this lemma implies that strings which are not equivalent correspond to distinct functions.

Proof. By Lemma 2.1.1, we have $\gamma_1(\mathbb{D}_{1/\rho}) \subset \mathring{D}_{c_{j_n},r_{j_n}}$ and $\gamma_2(\mathbb{D}_{1/\rho}) \subset \mathring{D}_{c_{k_m},r_{k_m}}$. This proves the lemma in the case where the terminal transformations of γ_1 and γ_2 are not equivalent. Therefore, we assume that γ_1 and γ_2 have a common terminal string and we let γ denote the maximal common terminal string of γ_1 and γ_2. Write $\gamma_1 = \gamma \circ \gamma_1^*$ and $\gamma_2 = \gamma \circ \gamma_2^*$, where γ_1^* and γ_2^* are defined as in (2.1.4). Without loss of generality, we may assume that the length of γ_1 is greater than or equal to the length of γ_2. Then one of the following must be true:

(i) the length of γ is strictly less than the length of γ_2, so that γ_1^* and γ_2^* are strings, or

(ii) the length of γ equals the length of γ_2, so that γ_1^* is a string and γ_2^* is the identity function.

For case (i), let T_a and T_b denote the terminal transformations of γ_1^* and γ_2^*, respectively. Then by Lemma 2.1.1, we have $\gamma_1^*(\mathbb{D}_{1/\rho}) \subset \mathring{D}_{c_a,r_a}$ and $\gamma_2^*(\mathbb{D}_{1/\rho}) \subset \mathring{D}_{c_b,r_b}$. Since γ was the maximal common string of γ_1 and γ_2, we must have $a \neq b$, whence

$$\gamma_1^*(\mathbb{D}_{1/\rho}) \cap \gamma_2^*(\mathbb{D}_{1/\rho}) = \emptyset.$$

Then, since the Möbius transformation γ is injective, we have

$$\gamma_1(\mathbb{D}_{1/\rho}) \cap \gamma_2(\mathbb{D}_{1/\rho}) = (\gamma \circ \gamma_1^*)(\mathbb{D}_{1/\rho}) \cap (\gamma \circ \gamma_2^*)(\mathbb{D}_{1/\rho}) = \emptyset,$$

and case (i) is established.

In case (ii), we will have $\gamma_1^*(A) \cap A = \emptyset$, since $\gamma_1^*(A) \subset \mathbb{D}$ by Lemma 2.1.1 while $A \subset \mathbb{D}_{1/\rho} \setminus \mathbb{D}$. It follows that

$$\gamma_1(A) \cap \gamma_2(A) = (\gamma \circ \gamma_1^*)(A) \cap (\gamma \circ \gamma_2^*)(A) = \emptyset$$

since γ_2^* is the identity function and since γ is injective. This establishes the second case and completes the proof of the lemma.

$\square$

Next we will consider what happens when a transformation T_j is composed with itself. To this end, we define the function

$$T_j^v(z) := \chi_j(\sigma^{2v}\Phi_j(z)), \quad v \in \mathbb{Z}$$

for each $j \in \Lambda_s$. Note that, since χ_j is the inverse of Φ_j, we have

$$\underbrace{T_j T_j \cdots T_j T_j}_{v \text{ copies}} = T_j^v.$$

In particular, this means that each T_j^v belongs to the family of compositions $\mathcal{T}$.

In the following section, we find explicit expressions for T_j^v and its derivative $(T_j^v)'$.

2.1.3 T_j^v and its derivative

If $a_j = 0$, then χ_j and Φ_j are the identity functions, whence

$$T_j^v(z) = \sigma_j^{2v} z \qquad \text{and} \qquad \frac{d}{dz}\left[T_j^v(z)\right] = \sigma_j^{2v}.$$

For the remainder of this section, we will suppose that we have $a_j \neq 0$. To find expressions for T_j^v and $(T_j^v)'$, we will exploit a relationship that exists between Möbius transformations and matrices. We review this idea in the following section.

Möbius transformations and matrices

Let $\mathrm{Aut}(\hat{\mathbb{C}})$ denote the group of all Möbius transformations and let $\mathrm{SL}_2(\mathbb{C})$ denote the group of all 2×2 invertible matrices with determinant one whose entries are elements of $\mathbb{C}$. If we define the map $f : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Aut}(\hat{\mathbb{C}})$ by the relation

$$f\left(\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\right) = \frac{\alpha z+\beta}{\gamma z+\delta}, \tag{2.1.5}$$

then f is a group homomorphism of $\mathrm{SL}_2(\mathbb{C})$ onto $\mathrm{Aut}(\hat{\mathbb{C}})$. Consequently, if A_1 and A_2 are matrices in $\mathrm{SL}_2(\mathbb{C})$ such that $f(A_1) = T_1$ and $f(A_2) = T_2$, then we have

$$f(A_1A_2) = f(A_1)\circ f(A_2).$$

Suppose the matrix

$$A = \begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}$$

belongs to $\mathrm{SL}_2(\mathbb{C})$. If $\gamma \neq 0$ and if $f(A) = T$ for some $T \in \mathrm{Aut}(\hat{\mathbb{C}})$, then we may write

$$T'(z) = \frac{d}{dz}\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) = \frac{\alpha\delta-\beta\gamma}{(\gamma z+\delta)^2} = \frac{1}{(\gamma z+\delta)^2} = \frac{1}{\gamma^2(z+\delta/\gamma)^2}, \tag{2.1.6}$$

since the determinant $\alpha\delta - \beta\gamma$ of A is one.

We comment that the map f is surjective, but not injective. If I denotes the identity matrix, then the kernel of f is $\{I, -I\}$. Therefore, if $f(A) = T$ as above, then we have $f^{-1}(T) = \{A, -A\}$.

T_j^v and its derivative

If we define

$$s_j^v(z) := \sigma_j^{2v},$$

then we can use matrix multiplication to find expressions for $T_j^v(z)$ and its derivative, since

$$T_j^v(z) = (\chi_j \circ s_j^v \circ \Phi_j)(z).$$

To this end, we define the constant

$$\lambda_j := \frac{1}{\sqrt{1 - |a_j|^2}}$$

for each $j \in \Lambda_s$. With the map $f : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Aut}(\hat{\mathbb{C}})$ defined by (2.1.5), we see that

$$f\left(\lambda_j \begin{bmatrix} 1 & a_j \\ \overline{a_j} & 1 \end{bmatrix}\right) = \chi_j(z),$$

while

$$f\left(\begin{bmatrix} \sigma_j^v & 0 \\ 0 & \sigma_j^{-v} \end{bmatrix}\right) = s_j^v(z),$$

and

$$f\left(\lambda_j \begin{bmatrix} 1 & -a_j \\ -\overline{a_j} & 1 \end{bmatrix}\right) = \Phi_j(z).$$

Consequently, one may use matrix multiplication to determine that

$$T_j^v(z) = f\left(\frac{\lambda_j^2}{\sigma_j^v} \begin{bmatrix} \sigma_j^{2v} - |a_j|^2 & a_j(1 - \sigma_j^{2v}) \\ -\overline{a_j}(1 - \sigma_j^{2v}) & 1 - |a_j|^2 \sigma_j^{2v} \end{bmatrix}\right). \tag{2.1.7}$$

In other words, we have

$$T_j^v(z) = \frac{(\sigma_j^{2v} - |a_j|^2)z + a_j(1 - \sigma_j^{2v})}{\overline{a_j}(\sigma_j^{2v} - 1)z + 1 - |a_j|^2 \sigma_j^{2v}}.$$

Since, for the moment, we are assuming $a_j \neq 0$, then by (2.1.6) and (2.1.7), we can write

$$\frac{d}{dz}[T_j^v(z)] = \left(\frac{\sigma_j^v}{1 - \sigma_j^{2v}} \frac{1 - |a_j|^2}{\overline{a_j}}\right)^2 \cdot \left(z - \frac{1}{\overline{a_j}} \cdot \frac{1 - |a_j|^2 \sigma_j^{2v}}{1 - \sigma_j^{2v}}\right)^{-2}. \tag{2.1.8}$$

We refer to (2.1.6) again to note that the only singularity of T_j^v occurs at the same point as the only singularity of $(T_j^v)'$. This point is given by

$$p_j^v := \frac{1}{\overline{a_j}} \cdot \frac{1 - |a_j|^2 \sigma_j^{2v}}{1 - \sigma_j^{2v}}. \tag{2.1.9}$$

We comment that another representation for p_j^v is given by

$$p_j^v = \chi_j \left(-\frac{1}{\overline{a_j} \sigma_j^{2v}} \right).$$

One way to see this is by may be found by reasoning as follows:

$$\begin{aligned}
T_j^v(p_j^v) = \infty \quad & \Leftrightarrow & (\chi_j \circ s_j^v \circ \Phi_j)(p_j^v) = \infty \\
& \Leftrightarrow & (s_j^v \circ \Phi_j)(p_j^v) = \Phi_j(\infty) \\
& \Leftrightarrow & \sigma_j^{2v} \Phi_j(p_j^v) = -\frac{1}{\overline{a_j}} \\
& \Leftrightarrow & \Phi_j(p_j^v) = -\frac{1}{\overline{a_j} \sigma_j^{2v}} \\
& \Leftrightarrow & p_j^v = \chi_j \left(-\frac{1}{\overline{a_j} \sigma_j^{2v}} \right).
\end{aligned}$$

We examine the sequence of poles $\{p_j^v\}_{v=1}^{\infty}$ in the next section.

2.1.4 The poles of T_j^v

If $a_j = 0$, then the function $T_j^v(z) = \sigma_j^{2v} z$ is an entire function. For the rest of this section, we suppose that $a_j \neq 0$.

Since we have $0 < \sigma_j^{2(v+1)} < \sigma_j^{2v} < 1$ for every $v \in \mathbb{N}$, and since the function

$$f(x) := \frac{1 - \alpha x}{1 - x}, \quad x \in [0, 1), \quad \alpha \in (0, 1)$$

is increasing on $(0, 1)$ with $f(0) = 1$, we have

$$\frac{1}{|a_j|} < \frac{1}{|a_j|} \cdot \frac{1 - |a_j|^2 \sigma_j^{2(v+1)}}{1 - \sigma_j^{2(v+1)}} < \frac{1}{|a_j|} \cdot \frac{1 - |a_j|^2 \sigma_j^{2v}}{1 - \sigma_j^{2v}}, \quad v \in \mathbb{N}.$$

In other words, referring to definition 2.1.9, we have

$$\frac{1}{|a_j|} < |p_j^{v+1}| < |p_j^v|, \quad v \in \mathbb{N}.$$

Moreover, since $\sigma^{2v} \to 0$ as $v \to \infty$, we find that

$$\lim_{v \to \infty} |p_j^v| = \frac{1}{|a_j|}.$$

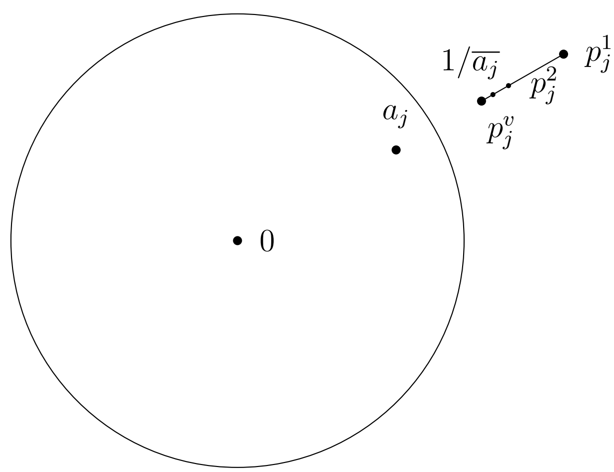


Figure 2.2: The sequence of poles $\{p_j^v\}_{v=1}^{\infty}$.

Therefore, the sequence $\{p_j^v\}_{v=1}^{\infty}$ of poles begins at the point

$$p_j^1 = \frac{1}{\overline{a_j}} \cdot \frac{1 - |a_j|^2 \sigma_j^2}{1 - \sigma_j^2}$$

and heads, in the direction of the origin, towards the accumulation point $1/\overline{a_j}$.

One consequence of this discussion is that each T_j^v is analytic on $\mathbb{D}_{1/|a_j|}$. Another consequence is that for each $\epsilon > 0$, there are infinitely many poles p_j^v which belong to $\mathbb{D}_{\epsilon+1/|a_j|}$.

2.1.5 The geometry of T_j

Recall that any Möbius transformation other than the identity function has at most two fixed points. If $a_j = 0$, then $T_j(z) = \sigma_j^2 z$ is a non-trivial dilation of the complex plane with respect to the origin and its fixed points are 0 and ∞. The family of invariant circles of this transformation are the extended lines which pass through the origin.

For the remainder of the section, we suppose that we have $a_j \neq 0$. The function T_j is a *hyperbolic* Möbius transformation since it is, by definition, conjugate to a non-trivial dilation with respect to the origin. We note that the points a_j and $1/\overline{a_j}$ are fixed by T_j:

$$T_j(a_j) = (\chi_j \circ s_j \circ \Phi_j)(a_j) = (\chi_j \circ s_j)(0) = \chi_j(0) = a_j$$

$$T_j(1/\overline{a_j}) = (\chi_j \circ s_j \circ \Phi_j)(1/\overline{a_j}) = (\chi_j \circ s_j)(\infty) = \chi_j(\infty) = 1/\overline{a_j}.$$

Therefore, the family of invariant circles of T_j consists of all circles or lines in $\hat{\mathbb{C}}$ passing through the points a_j and $1/\overline{a_j}$. In particular, if L_{a_j} denotes the line passing through the origin and a_j (including the point at infinity), then we have

$$T_j(L_{a_j}) = L_{a_j}.$$

Moreover, we can use equation (2.1.8) to compute

$$\left(\frac{d}{dz}[T_j(z)]\right)\bigg|_{z=a_j} = \sigma_j^2.$$

Since this quantity is less than one, we have that a_j is the *attracting* fixed point of T_j and that $1/\overline{a_j}$ is the *repelling* fixed point of T_j. This leads us to the beautiful formula

$$\frac{T_j^v(z) - a_j}{T_j^v(z) - 1/\overline{a_j}} = \sigma^{2v}\left(\frac{z - a_j}{z - 1/\overline{a_j}}\right), \quad v \in \mathbb{Z}, \quad z \neq 1/\overline{a_j}, \tag{2.1.10}$$

which can be used to give analytic arguments which are geometrically clear by the concepts of attraction and repulsion. Alternatively, one can use the fact that T_j^v is conjugate to a dilation with respect to the origin to make these arguments. This is done in the following section.

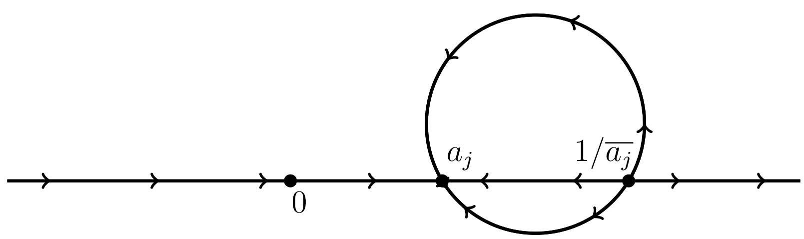


Figure 2.3: Attraction, repulsion, and the invariant circles for T_j

Before moving on, we comment that we can gain an intuitive understanding of the geometry of T_j by considering the sets

$$T_j^v(K), \quad v \in \mathbb{Z},$$

where K is a circle that is perpendicular to every T_j-invariant circle. If we let K denote the perpendicular bisector of the line segment connecting a_j and $1/\overline{a_j}$, then K meets this requirement. In fact, we have

$$K = \chi_j(\mathbb{T}_{1/|a_j|}),$$

where $\mathbb{T}_{1/|a_j|}$ denotes the circle centered at the origin of radius $1/|a_j|$.

In the following figure, the vertical line $K = \chi_j(\mathbb{T}_{1/|a_j|})$ is the perpendicular bisector of the line segment connecting a_j and $1/\overline{a_j}$.

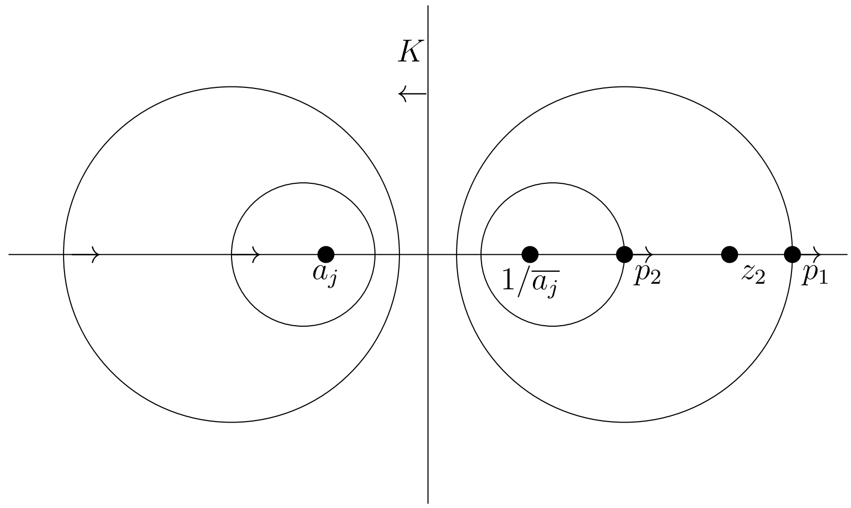


Figure 2.4: $T_j^v(K)$ for $-2 \leq v \leq 2$, where K is the perpendicular bisector if a_j and $1/\overline{a_j}$.

Beginning with the circle closest to $-1/\overline{a_j}$ and ending with the circle closest to a_j, the circles are the sets $T_j^{-2}(K)$, $T_j^{-1}(K)$, $T_j^1(K)$, and $T_j^2(K)$. The point p_1 is the pole of T_j^1, and the point p_2 is the pole of T_j^2. The point z_2 is the zero of T_j^2. The poles p_v and the zeros z_v of T_j^v approach $1/\overline{a_j}$ as $v \to \infty$, with $|z_{v+1}| < |p_v| < |z_v|$ for every $v \in \mathbb{N}$.

For every $z \in \hat{\mathbb{C}} \setminus \{1/\overline{a_j}\}$, we have $T_j^v(z) \to a_j$ as v approaches positive infinity. Meanwhile, for every $z \in \hat{\mathbb{C}} \setminus \{a_j\}$, we have $T_j^v(z) \to 1/\overline{a_j}$ as v approaches negative infinity.

Next to each circle, there is an arrow that points in the direction of $T_j^v(\chi_j(\mathbb{D}_{1/|a_j|}))$. In particular, we have that $T_j^v(\chi_j(\mathbb{D}_{1/|a_j|}))$ is the interior of $T_j^v(K) = T_j^v(\chi_j(\mathbb{T}_{1/|a_j|}))$ for $v = 1$ and $v = 2$. Meanwhile, we have that $T_j^v(\chi_j(\mathbb{D}_{1/|a_j|}))$ is the exterior of $T_j^v(\chi_j(\mathbb{T}_{1/|a_j|}))$ for $v = -1$ and $v = -2$. More generally, we have that $T_j(\chi_j(\mathbb{D}_t))$ is bounded if and only if $t < 1/|a_j|$.

2.1.6 The effect of T_j on open disks centered at the origin

Let $j \in \Lambda_s$ be such that $a_j \neq 0$. Suppose that r satisfies $|a_j| < r < 1/|a_j|$. Recall that $\Phi_j(a_j) = 0$. Since a_j belongs to the interior of $\overline{\mathbb{D}}_r$, and since Φ_j is a homeomorphism, we see that the origin belongs to the interior of $\Phi_j(\overline{\mathbb{D}}_r)$. Furthermore, since the pole of Φ_j (i.e., the point $-1/\overline{a_j}$) belongs to Δ_r, the set $\Phi_j(\overline{\mathbb{D}}_r)$ is a closed disk. Hence, we have that

$\Phi_j(\overline{\mathbb{D}}_r)$ is a convex set whose interior contains the origin. Therefore, if $s_j(z) = \sigma_j^2 z$, then we have

$$s_j(\Phi_j(\overline{\mathbb{D}}_r)) \subset \Phi_j(\mathbb{D}_r).$$

If we apply χ_j to both sides of the previous equation, then we obtain

$$(\chi_j \circ s_j \circ \Phi_j)(\overline{\mathbb{D}}_r) \subset (\chi_j \circ \Phi_j)(\mathbb{D}_r).$$

In effect, we have shown that we have

$$T_j(\overline{\mathbb{D}}_r) \subset \mathbb{D}_r, \quad |a_j| < r < 1/|a_j|. \tag{2.1.11}$$

Next, we investigate $T_j(\overline{\mathbb{D}}_{|a_j|})$. To this end, consider the set

$$A := \overline{\mathbb{D}}_{|a_j|} \setminus \{a_j\}.$$

Note that $\Phi_j(\overline{\mathbb{D}}_{|a_j|})$ is a convex set (a closed disk, to be precise) such that

$$0 \in \Phi_j(\overline{\mathbb{D}}_{|a_j|}) \setminus \Phi_j(A) \qquad \text{and} \qquad \Phi_j(A) \subset \Phi_j(\overline{\mathbb{D}}_{|a_j|}).$$

Therefore, we have

$$s_j(\Phi_j(A)) \subset \Phi_j(\mathbb{D}_{|a_j|}),$$

where $s_j(z) = \sigma_j^2 z$. If we apply χ_j to both sides of the previous equation, then we obtain

$$(\chi_j \circ s_j \circ \Phi_j)(A) \subset (\chi_j \circ \Phi_j)(\mathbb{D}_{|a_j|}).$$

In other words, we have

$$T_j(A) \subset \mathbb{D}_{|a_j|}. \tag{2.1.12}$$

Since $A = \overline{\mathbb{D}}_{|a_j|} \setminus \{a_j\}$ by definition, and since $T_j(a_j) = a_j$, we have established

$$T_j(\overline{\mathbb{D}}_{|a_j|}) \subset \mathbb{D}_{|a_j|} \cup \{a_j\}. \tag{2.1.13}$$

Finally, we investigate $T_j(\overline{\mathbb{D}}_{1/|a_j|})$. To this end, define

$$B := \overline{\mathbb{D}}_{1/|a_j|} \setminus \{1/|a_j|\}.$$

Then the set $\Phi_j(B)$ is a half plane (not including the point at infinity) whose interior contains the origin. Then $(s_j \circ \Phi_j)(B)$ belongs to the interior of $\Phi_j(B)$. Therefore $T_j(B)$ belongs to the interior of B. Hence

$$T_j(\overline{\mathbb{D}}_{1/|a_j|} \setminus \{1/|a_j|\}) \subset \mathbb{D}_{1/|a_j|}.$$

In effect, we have shown that

$$T_j(\overline{\mathbb{D}}_{1/|a_j|}) \subset \mathbb{D}_{1/|a_j|} \cup \{1/\overline{a_j}\}. \tag{2.1.14}$$

We will use some of these observations in the lemma below.

Lemma 2.1.3. *Let $j \in \Lambda_s$. If $a_j \neq 0$ and if r satisfies $|a_j| \leq r \leq 1/|a_j|$, then we have*

$$T_j(\mathbb{D}_r) \subset \mathbb{D}_r. \tag{2.1.15}$$

Meanwhile, if $a_j = 0$, then (2.1.15) holds for any $r \in (0, \infty)$.

Proof. First we suppose $a_j \neq 0$. We have already established the case where $|a_j| < r < 1/|a_j|$ with relation (2.1.11) in the comments preceding the lemma. We have also shown that

$$T_j(\overline{\mathbb{D}}_{|a_j|}) \subset \mathbb{D}_{|a_j|} \cup \{a_j\}. \tag{2.1.16}$$

Since T_j is a homeomorphism, it maps interiors of sets to interiors of sets. Then the equation above implies

$$T_j(\mathbb{D}_{|a_j|}) \subset \mathbb{D}_{|a_j|}.$$

Likewise, relation (2.1.14) implies

$$T_j(\mathbb{D}_{1/|a_j|}) \subset \mathbb{D}_{1/|a_j|}.$$

Finally, if $a_j = 0$, then we have $T_j(z) = \sigma_j^2 z$ with $\sigma_j \in (0, 1)$, in which case it is clear that (2.1.15) holds for any $r \in (0, \infty)$. The proof is complete.

□

2.2 The family of compositions $\mathcal{T}$

We recall the definition of $\mathcal{T}^*$ given in Section 1.3

$$\mathcal{T}^* = \{T_{j_n} T_{j_{n-1}} \cdots T_{j_2} T_{j_1} : n \in \mathbb{N} \text{ and } j_k \in \Lambda_s \text{ for each } 1 \leq k \leq n\}.$$

In that section, we also defined

$$\mathcal{T} = \mathcal{T}^* \cup \{T_0\},$$

where

$$T_0(z) = z$$

is the identity function.

It will be helpful to have an alternate representations for $\mathcal{T}^*$ and $\mathcal{T}$. To this end, for each natural number n, we let $\mathcal{E}_n$ denote the set of all strings of length n:

$$\mathcal{E}_n := \{T_{j_n} T_{j_{n-1}} \cdots T_{j_2} T_{j_1} : j_k \in \Lambda_s \text{ for each } 1 \leq k \leq n\}.$$

Note that, by Lemma 2.1.2, there are exactly s^n elements of $\mathcal{E}_n$. Also, note that

$$\mathcal{T}^* = \bigcup_{n=1}^{\infty} \mathcal{E}_n$$

and

$$\mathcal{E}_{n+1} = \bigcup_{j\in\Lambda_s} \{T_j\tau : \tau \in \mathcal{E}_n\} = \bigcup_{j\in\Lambda_s} \{\tau T_j : \tau \in \mathcal{E}_n\}, \quad n \in \mathbb{N}.$$

Next, by defining

$$\mathcal{E}_0 := \{T_0\},$$

we may also write

$$\mathcal{T} = \bigcup_{n=0}^{\infty} \mathcal{E}_n.$$

Finally, we note that we have the representations

$$\mathcal{T}^* = \bigcup_{n=0}^{\infty} \mathcal{E}_{n+1} = \bigcup_{n=0}^{\infty} \bigcup_{j\in\Lambda_s} \{T_j\tau : \tau \in \mathcal{E}_n\} = \bigcup_{\tau\in\mathcal{T}} \bigcup_{j\in\Lambda_s} \{T_j\tau\} \tag{2.2.1}$$

and

$$\mathcal{T}^* = \bigcup_{n=0}^{\infty} \mathcal{E}_{n+1} = \bigcup_{n=0}^{\infty} \bigcup_{j\in\Lambda_s} \{\tau T_j : \tau \in \mathcal{E}_n\} = \bigcup_{\tau\in\mathcal{T}} \bigcup_{j\in\Lambda_s} \{\tau T_j\} \tag{2.2.2}$$

We proceed by cataloging some facts about the elements of $\mathcal{T}^*$.

Lemma 2.2.1. *If $\tau \in \mathcal{T}^*$, then*

(i) there exists a unique natural number n and a unique n-tuple $\langle k_1, k_2, \dots k_n\rangle \in \Lambda_s^n$ such that $\tau = T_{k_n}T_{k_{n-1}}\cdots T_{k_2}T_{k_1}$.

Furthermore, if $\rho \in (0,1)$ satisfies $\mathbb{D}_{1/\rho} \subset \bigcap_{j\in\Lambda_s} \chi_j(\mathbb{D}_{1/\sigma_j})$ and $\bigcup_{j\in\Lambda_s} D_{c_j,r_j} \subset \mathbb{D}_\rho$, then

(ii) $|\tau(z)| \cdot |\zeta| < 1$ for every $(z,\zeta) \in \mathbb{D}_{1/\rho} \times \mathbb{D}_{1/\rho}$.

Proof. The existence part claim (i) follows from the definition of $\mathcal{T}^*$ while uniqueness follows from Lemma 2.1.2. To prove claim (ii), note that Lemma 2.1.1 gives

$$\tau(z) \in \mathring{D}_{k_n}.$$

Then the condition $\bigcup_{j\in\Lambda_s} D_{c_j,r_j} \subset \mathbb{D}_\rho$ gives $|\tau(z)| < \rho$, whence

$$|\tau(z)| \cdot |\zeta| < \rho \cdot 1/\rho = 1.$$

This establishes claim (ii) and completes the proof of the lemma. $\square$

2.2.1 Open disks centered at the origin and the transformations of $\mathcal{T}^*$

Analyticity

Let $\rho \in (0,1)$ satisfy $\mathbb{D}_{1/\rho} \subset \bigcap_{j\in\Lambda_s} \chi_j(\mathbb{D}_{1/\sigma_j})$. By Lemmas 2.1.1 and 2.2.1, we know that each $\tau \in \mathcal{T}^*$ is analytic on $\mathbb{D}_{1/\rho}$. Here we find the largest open disk G centered at the origin such that every $\tau \in \mathcal{T}^*$ is analytic on G. To this end, we define

$$\alpha := \max\{|a_j| : j \in \Lambda_s\}.$$

Note that, for every $j \in \Lambda_s$ such that $a_j \neq 0$, we have

$$|a_j| < 1/\alpha \leq 1/|a_j|.$$

If $\tau = T_{j_n}T_{j_{n-1}}\cdots T_{j_2}T_{j_1}$ is a string, then by Lemma 2.1.3, we have

$$\tau(\mathbb{D}_{1/\alpha}) = T_{j_n}T_{j_{n-1}}\cdots T_{j_2}T_{j_1}(\mathbb{D}_{1/\alpha}) \subset T_{j_n}T_{j_{n-1}}\cdots T_{j_2}(\mathbb{D}_{1/\alpha}) \subset \cdots \subset T_{j_n}(\mathbb{D}_{1/\alpha}) \subset \mathbb{D}_{1/\alpha}.$$

In particular, we have $\tau(z) < \infty$ for every $z \in \mathbb{D}_{1/\alpha}$. This implies that every $\tau \in \mathcal{T}^*$ is analytic on $\mathbb{D}_{1/\alpha}$.

Next, let ϵ be any positive number and let $j \in \Lambda_s$ be such that $|a_j| = \alpha$. Recall that, by the discussion in section 2.1.4, there exists some natural number N such that, whenever $v > N$, the operator T_j^v has a pole on $\mathbb{D}_{\epsilon+1/\alpha}$.

In conclusion, the set $\mathbb{D}_{1/\alpha}$ is the largest open disk G centered at the origin having the property that each $\tau \in \mathfrak{T}^*$ is analytic on G.

Contraction

Now we record a lemma that we will use later on.

Lemma 2.2.2. *If* $\alpha := \max_{j \in \Lambda_s} |a_j|$, *then for every* $\tau \in \mathfrak{T}$, *we have*

$$\tau(\mathbb{D}_\alpha) \subset \mathbb{D}_\alpha \qquad \textit{and} \qquad \tau(\overline{\mathbb{D}}_\alpha) \subset \overline{\mathbb{D}}_\alpha.$$

Proof. We induct on the length k of τ. If $k = 0$, then τ is the identity function, whence it it clear that the claim holds. Now suppose that the claim holds for each string τ_k in $\mathfrak{T}$ of length k. Let τ be a string of length $k+1$. Then we can write $\tau = T_j \tau_k$ for some $j \in \Lambda_s$. Then

$$\tau(\overline{\mathbb{D}}_\alpha) = (T_j \circ \tau_k)(\overline{\mathbb{D}}_\alpha) \subset T_j(\overline{\mathbb{D}}_\alpha)$$

by the induction hypothesis. Next, if j is such that $|a_j| = \alpha$, then we have $T_j(\overline{\mathbb{D}}_\alpha) \subset (\overline{\mathbb{D}}_\alpha)$ by relations (2.1.13). Otherwise, we have $T_j(\overline{\mathbb{D}}_\alpha) \subset (\overline{\mathbb{D}}_\alpha)$ by relation (2.1.11). In either case, we have $T_j(\mathbb{D}_\alpha) \subset \mathbb{D}_\alpha$ by Lemma 2.1.3. Then we have

$$\tau(\mathbb{D}_\alpha) \subset \mathbb{D}_\alpha \qquad \text{and} \qquad \tau(\overline{\mathbb{D}}_\alpha) \subset \overline{\mathbb{D}}_\alpha,$$

whence the proof is complete by the principle of mathematical induction. □

2.2.2 On maximal $|a_j|$ values

Recall that $\mathcal{T}_j$ is the set which consists of all operators whose terminal transformation is T_j. We begin by recording a lemma.

Lemma 2.2.3. *Let $j \in \Lambda_\omega$. If $r < \alpha$ with r sufficiently close to α, then we have*

$$(T_j \circ \tau)(\overline{\mathbb{D}}_r) \subset \overline{\mathbb{D}}_r, \quad \tau \in \mathcal{T}^* \setminus \mathcal{T}_j. \tag{2.2.3}$$

Proof. Fix some $j \in \Lambda_\omega$. Consider the set $\overline{\mathbb{D}}_\alpha \setminus \mathring{D}_{c_j,r_j}$. Since $\mathring{D}_{c_j,r_j}$ does not contain the origin, we have that the set $\overline{\mathbb{D}}_\alpha \setminus \mathring{D}_{c_j,r_j}$ is non-empty. In particular, we have $0 \in \overline{\mathbb{D}}_\alpha$ but $0 \notin \mathring{D}_{c_j,r_j}$.

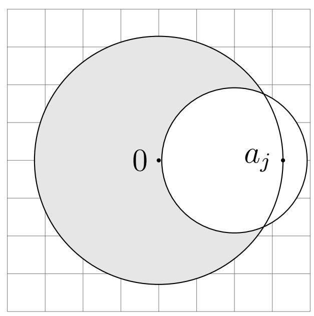


Figure 2.5: The set $\overline{\mathbb{D}}_{|a_j|} \setminus \mathring{D}_{c_j,r_j}$

It follows from relation (2.1.16) and from the fact that $a_j \in \mathring{D}_{c_j,r_j}$ that $T_j(\overline{\mathbb{D}}_\alpha \setminus \mathring{D}_{c_j,r_j})$ is a compact set contained in $\mathbb{D}_\alpha$. Therefore, we can find some $\varrho \in (0,\alpha)$ such that

$$T_j(\overline{\mathbb{D}}_\alpha \setminus \mathring{D}_{c_j,r_j}) \subset \overline{\mathbb{D}}_\varrho.$$

Note that this implies

$$T_j(\mathbb{D}_\alpha \setminus D_{c_j,r_j}) \subset \overline{\mathbb{D}}_\varrho. \tag{2.2.4}$$

Now, let $\varsigma \in (0,\alpha)$ be such that

$$|a_j| < \varsigma, \quad j \in \{\omega+1,\ldots,s\}.$$

Finally, set $r := \max\{\varrho, \varsigma\}$. We may now show that for every $\tau \in \mathcal{T}^* \setminus \mathcal{T}_j$, we have

$$(T_j \circ \tau)(\overline{\mathbb{D}}_r) \subset (\overline{\mathbb{D}}_r).$$

Indeed, $\tau \in \mathcal{T}^* \setminus \mathcal{T}_j$. It follows from Lemma 2.2.2 and from Lemma 2.1.1 that we will have

$$\tau(\mathbb{D}_\alpha) \subset \mathbb{D}_\alpha \setminus D_{c_j, r_j}$$

Then by relation (2.2.4) and the definition of r, we have

$$(T_j \circ \tau)(\overline{\mathbb{D}}_r) \subset (T_j \circ \tau)(\mathbb{D}_\alpha) \subset T_j(\mathbb{D}_\alpha \setminus D_{c_j, r_j}) \subset \overline{\mathbb{D}}_r.$$

This establishes the lemma. $\square$

Before we end this section, we record another lemma that will be used later on. Here, we use the notation

$$\mathcal{T}_j^* := \{T_j \tau : \tau \in \mathcal{T} \setminus \mathcal{T}_j\}.$$

Lemma 2.2.4. *Let $j \in \Lambda_\omega$. If K is a compact subset of $\mathbb{D}_\alpha$, then there exists a compact subset K^* of $\mathbb{D}_\alpha$ such that*

$$\tau(K) \subset K^*, \quad \tau \in \mathcal{T}_j^*.$$

Proof. First, we choose $r \in (0, \alpha)$ sufficiently close to α so that we have $K \subset \overline{\mathbb{D}}_r$ and so that relationship (2.2.3) holds. Note that, by Lemma 2.2.2, we have that $T_j(\overline{\mathbb{D}}_r)$ is a compact subset of $\mathbb{D}_\alpha$. Put $K^* = T_j(\overline{\mathbb{D}}_r) \cup \overline{\mathbb{D}}_r$. Since we have

$$\mathcal{T}_j^* = \{T_j\} \cup \{T_j \tau : \tau \in \mathcal{T}^* \setminus \mathcal{T}_j\},$$

the claim follows from the definition of K^* and from Lemma 2.2.3. $\square$

3 THE PROOF OF THEOREM 1.4.1

Throughout this chapter, we will suppose that $\mathcal{D} = \mathbb{D} \setminus \bigcup_{j=1}^{s} D_{c_j,r_j}$ is a CMCD for which assumption 1.3.1 holds. We will continue the notation introduced in Section 1.3. In particular, we recall that

- $\mathcal{T}$ is the family of compositions associated with $\mathcal{D}$,
- G_1 is a bounded, simply connected domain in the complex plane whose boundary L_1 is an analytic Jordan curve,
- φ is a conformal map of G_1 onto the unit disk, and
- $\mathcal{G} = \varphi^{-1}(\mathcal{D})$.

Furthermore, we will assume that $\rho \in (0, 1)$ has been selected to meet the four conditions mentioned in Section 1.3:

- $\sum_{\tau\in\mathcal{T}} |\tau'|$ converges uniformly on every compact subset of $\mathbb{D}_{1/\rho}$,
- φ^{-1} has an analytic and univalent continuation to $\mathbb{D}_{1/\rho}$,
- $\mathbb{D}_{1/\rho} \subset \bigcap_{j\in\Lambda_s} \chi_j(\mathbb{D}_{1/\sigma_j})$, and
- $\bigcup_{j\in\Lambda_s} D_{c_j,r_j} \subset \mathbb{D}_\rho$.

The purpose of this chapter is to prove Theorem 1.4.1. In other words, we will show that the function defined by the formula

$$\mathcal{K}_{\mathcal{G}}(z,\zeta) = \sum_{\tau\in\mathcal{T}} \frac{\frac{d}{dz}[(\tau\circ\varphi)(z)]\,\overline{\varphi'(\zeta)}}{[1-(\tau\circ\varphi)(z)\,\overline{\varphi(\zeta)}]^2}, \quad (z,\zeta) \in \varphi^{-1}(\mathbb{D}_{1/\rho}) \times \varphi^{-1}(\mathbb{D}_{1/\rho})$$

satisfies the relation

$$g(z) = \frac{1}{\pi} \int_{\mathcal{G}} g(\zeta)\, \mathcal{K}_{\mathcal{G}}(z,\zeta)\, dA(\zeta), \quad z \in G_1, \tag{3.0.1}$$

for every function g analytic on G_1 and square integrable over $\mathcal{G}$. In the introduction, we used the notation $\mathcal{A}^2(G_1, \mathcal{G})$ to represent the class of all functions g which fit this description. The space $\mathcal{A}^2(G_1, \mathcal{G})$ is an example of a *weighted Bergman space*. In order to prove Theorem 1.4.1, we will use some properties general Bergman spaces. We begin this section by reviewing some of these properties.

3.1 Preliminaries for the proof of Theorem 1.4.1

3.1.1 Bergman spaces and reproducing kernels

Let Ω be a domain in the complex plane. The *Bergman space* $\mathcal{B}^2(\Omega)$ is defined to be the set of all functions f which satisfy the following properties:

- f is analytic on Ω,
- $\displaystyle\int_{\Omega} |f(z)|^2\, dA(z) < \infty.$

It can be shown that point-evaluation is a bounded linear functional in $\mathcal{B}^2(\Omega)$ (see, e.g., Duren [15]). One consequence of this fact is that $\mathcal{B}^2(\Omega)$ is a Hilbert space with inner product

$$\langle f, g\rangle_{\Omega} = \int_{\Omega} f(z)\, \overline{g(z)}\, dA(z).$$

For each $z \in \Omega$, the Riesz representation theorem for Hilbert spaces establishes the existence of a unique function k_z in $\mathcal{B}^2(\Omega)$ such that $f(z) = \langle f, k_z\rangle_{\Omega}$ for every $f \in \mathcal{B}^2(\Omega)$. The function $K_{\Omega} : \Omega \times \Omega \to \mathbb{C}$ defined by

$$K_{\Omega}(z,\zeta) := \overline{k_z(\zeta)}$$

is called the *reproducing kernel*, or the *Bergman kernel*, of the domain Ω. The Bergman kernel of Ω enjoys the reproducing property

$$f(z) = \int_{\Omega} f(\zeta)\; K_{\Omega}(z,\zeta)\; dA(\zeta), \quad (z,f) \in \Omega \times \mathcal{B}^2(\Omega).$$

We will use the well-established fact that the Bergman kernel of $\mathbb{D}$ is

$$K_{\mathbb{D}}(z,\zeta) = \frac{1}{\pi}\frac{1}{(1-z\bar{\zeta})^2}, \tag{3.1.1}$$

and we comment that the explicit formula illustrating the reproducing property for this kernel is

$$f(z) = \frac{1}{\pi}\int_{\mathbb{D}} \frac{f(\zeta)}{(1-z\bar{\zeta})^2}\; dA(\zeta), \quad (z,f) \in \mathbb{D} \times \mathcal{B}^2(\mathbb{D}). \tag{3.1.2}$$

We will also use the following lemma, whose proof is given in Duren's book [15].

Lemma 3.1.1. *Let Ω and D be domains in the complex plane and let K_D be the reproducing kernel of D. If $\Psi : \Omega \to D$ is a conformal map, then the reproducing kernel of Ω is*

$$K_{\Omega}(z,\zeta) = K_D[\Psi(z),\Psi(\zeta)]\; \Psi'(z)\; \overline{\Psi'(\zeta)}.$$

One consequence of this lemma is that, for each $j \in \Lambda_s$, we have

$$\frac{1}{(1-z\bar{\zeta})^2} = \frac{\Phi_j'(z)\overline{\Phi_j'(\zeta)}}{[1-\Phi_j(z)\overline{\Phi_j(\zeta)}]^2}, \quad (z,\zeta) \in \mathbb{D} \times \mathbb{D}. \tag{3.1.3}$$

To see this, set $\Omega = D = \mathbb{D}$ in Lemma 3.1.1. Then, since $\Phi_j : \mathbb{D} \to \mathbb{D}$ is a conformal map, relationship (3.1.3) follows from Lemma 3.1.1 and equation (3.1.1).

Next, recall that $\mathcal{A}^2(G_1, \mathcal{G})$ represents the space of complex valued functions g analytic in G_1 which satisfy

$$\frac{1}{\pi}\int_{\mathcal{G}} |g(z)|^2\; dA(z) < \infty.$$

The following lemma shows that $\mathcal{A}^2(G_1, \mathcal{G})$ is closely related to $\mathcal{B}^2(\mathbb{D})$.

Lemma 3.1.2. *If $g \in \mathcal{A}^2(G_1, \mathcal{G})$, then there exists a function $f \in \mathcal{B}^2(\mathbb{D})$ such that*

$$g(z) = (f \circ \varphi)(z)\, \varphi'(z), \quad z \in G_1.$$

Proof. Recall that φ is a conformal map of G_1 onto $\mathbb{D}$. Let

$$\lambda : \mathbb{D} \to G_1$$

denote the inverse of φ. Consider the function

$$f(w) := g(\lambda(w))\lambda'(w), \quad w \in \mathbb{D}.$$

We note that, since g is analytic in G_1, we have that f is analytic in $\mathbb{D}$. Also, we have

$$f(\varphi(z))\varphi'(z) = g(z)\lambda'(\varphi(z))\varphi'(z) = g(z), \quad z \in G_1.$$

It remains to be seen that

$$\int_{\mathbb{D}} |f(w)|^2 \, dA(w) < \infty.$$

Since $|\varphi'(z)|^2$ is the Jacobian of the conformal mapping φ, we have

$$\int_{\mathcal{D}} |f(w)|^2 \, dA(w) = \int_{\mathcal{G}} |f(\varphi(z))|^2 |\varphi'(z)|^2 dA(z) = \int_{\mathcal{G}} |g(z)|^2 \, dA(z) < \infty,$$

since $g \in \mathcal{A}^2(G_1, \mathcal{G})$. Then, since f is analytic in $\mathbb{D}$, the previous calculation implies

$$\int_{\mathbb{D} \setminus \mathcal{D}} |f(w)|^2 \, dA(w) < \infty$$

by the structure of $\mathcal{D}$ and the Maximum Modulus principle. Finally, since

$$\int_{\mathbb{D}} |f(w)|^2 \, dA(w) = \int_{\mathcal{D}} |f(w)|^2 \, dA(w) + \int_{\mathbb{D}\setminus\mathcal{D}} |f(w)|^2 \, dA(w),$$

the proof is complete.

$\square$

3.1.2 Auxiliary computations

Here we record some calculations that will be used in the proof of Theorem 1.4.1.

Lemma 3.1.3. *For each* $(j, z, f) \in \Lambda_s \times \mathring{D}_{c_j,r_j} \times \mathcal{B}^2(\mathbb{D})$*, we have*

$$f(z) = \frac{1}{\pi} \int_{\mathring{D}_{c_j,r_j}} f(\zeta) \, \frac{\sigma_j^2 \, \Phi_j'(z)\overline{\Phi_j'(\zeta)}}{[\sigma_j^2 - \Phi_j(z)\overline{\Phi_j(\zeta)}]^2} \, dA(\zeta).$$

Proof. We begin by defining

$$\Psi(z) := \frac{\Phi_j(z)}{\sigma_j}.$$

Once we recall that

$$\chi_j(\mathbb{D}_{\sigma_j}) = \mathring{D}_{c_j,r_j}$$

and note that Φ_j is the inverse of χ_j, then we see that

$$\Psi : \mathring{D}_{c_j,r_j} \to \mathbb{D}$$

is a conformal map. Then, by Lemma 3.1.1, we find that the Bergman kernel of $\mathring{D}_{c_j,r_j}$ is

$$K_{\mathring{D}_{c_j,r_j}}(z,\zeta) = K_{\mathbb{D}}[\Psi_j(z), \Psi_j(\zeta)] \; \Psi_j'(z)\overline{\Psi_j'(\zeta)} = \frac{1}{\pi} \frac{\sigma_j^2 \, \Phi_j'(z)\overline{\Phi_j'(\zeta)}}{[\sigma_j^2 - \Phi_j(z)\overline{\Phi_j(\zeta)}]^2}.$$

In other words, for every $g \in \mathcal{B}^2(\mathring{D}_{c_j,r_j})$, we have

$$g(z) = \frac{1}{\pi} \int_{\mathring{D}_{c_j,r_j}} g(\zeta) \, \frac{\sigma_j^2 \, \Phi_j'(z)\overline{\Phi_j'(\zeta)}}{[\sigma_j^2 - \Phi_j(z)\overline{\Phi_j(\zeta)}]^2} \, dA(\zeta).$$

Then the claim is established once we note that $f \in \mathcal{B}^2(\mathbb{D})$ implies $f \in \mathcal{B}^2(\mathring{D}_{c_j,r_j})$.

$\square$

Lemma 3.1.4. *For each $(j, z, \tau, f) \in \Lambda_s \times \mathbb{D} \times \mathcal{T} \times \mathcal{B}^2(\mathbb{D})$, we have*

$$\frac{1}{\pi} \int_{\mathring{D}_{c_j,r_j}} \frac{f(\zeta) \, \tau'(z)}{[1 - \tau(z)\overline{\zeta}]^2} \, dA(\zeta) = \frac{d}{dz}\{T_j[\tau(z)]\} \cdot f\{T_j[\tau(z)]\}.$$

Proof. First we use the definition of T_j (i.e., $T_j(z) = \chi_j[\sigma_j^2 \Phi_j(z)]$) and the chain rule to write

$$\frac{d}{dz}\{T_j[\tau(z)]\} = \chi_j'\{\sigma_j^2 \Phi_j[\tau(z)]\} \cdot \sigma_j^2 \cdot \frac{d}{dz}\{\Phi_j[\tau(z)]\} = \frac{\sigma_j^2 \cdot \frac{d}{dz}\{\Phi_j[\tau(z)]\}}{\Phi_j'[\chi_j\{\sigma_j^2 \Phi_j[\tau(z)]\}]}, \quad z \in \mathbb{D}.$$

In other words, we have

$$\frac{d}{dz}\{T_j[\tau(z)]\} = \frac{\sigma_j^2 \cdot \Phi_j'[\tau(z)] \cdot \tau'(z)}{\Phi_j'\{T_j[\tau(z)]\}}, \quad z \in \mathbb{D}.$$

Note that this calculation implies the relation

$$\sigma_j^2 \cdot \Phi_j'[\tau(z)] \cdot \tau'(z) = \frac{d}{dz}\{T_j[\tau(z)]\} \cdot \Phi_j'\{T_j[\tau(z)]\}, \quad z \in \mathbb{D}. \tag{3.1.4}$$

Now we fix some $z \in \mathbb{D}$ and begin to work on the integral

$$I := \frac{1}{\pi} \int_{\mathring{D}_{c_j,r_j}} \frac{f(\zeta) \, \tau'(z)}{[1 - \tau(z)\overline{\zeta}]^2} \, dA(\zeta).$$

First, we use relation (3.1.3) to write

$$\begin{aligned}
I &= \tau'(z) \cdot \frac{1}{\pi} \int_{\mathring{D}_{c_j,r_j}} f(\zeta)\, \frac{\Phi_j'[\tau(z)]\overline{\Phi_j'(\zeta)}}{[1-\Phi_j[\tau(z)]\overline{\Phi_j(\zeta)}]^2}\, dA(\zeta) \\
&= \tau'(z) \cdot \frac{1}{\pi} \int_{\mathring{D}_{c_j,r_j}} f(\zeta) \cdot \frac{\sigma_j^4}{\sigma_j^4} \cdot \frac{\Phi_j'[\tau(z)]\overline{\Phi_j'(\zeta)}}{[1-\Phi_j[\tau(z)]\overline{\Phi_j(\zeta)}]^2}\, dA(\zeta) \\
&= \tau'(z) \cdot \frac{1}{\pi} \int_{\mathring{D}_{c_j,r_j}} f(z) \cdot \frac{\sigma_j^2 \cdot \sigma_j^2 \Phi_j'[\tau(z)]\overline{\Phi_j'(\zeta)}}{[\sigma_j^2 - \sigma_j^2 \Phi_j[\tau(z)]\overline{\Phi_j(\zeta)}]^2}\, dA(\zeta).
\end{aligned}$$

Then, by relation (3.1.4), we have

$$I = \frac{d}{dz}\{T_j[\tau(z)]\} \cdot \frac{1}{\pi} \int_{\mathring{D}_{c_j,r_j}} f(z) \cdot \frac{\sigma_j^2 \cdot \Phi_j'\{T_j[\tau(z)]\}\overline{\Phi_j'(\zeta)}}{[\sigma_j^2 - \Phi_j\{T_j[\tau(z)]\}\overline{\Phi_j(\zeta)}]^2}\, dA(\zeta).$$

We note that $(T_j \circ \tau)(z) \in \mathring{D}_{c_j,r_j}$ by Lemma 2.2.1. Then by Lemma 3.1.3, we have

$$I = \frac{d}{dz}\{T_j[\tau(z)]\} \cdot f\{T_j[\tau(z)]\},$$

and the proof is complete.

□

3.2 Proof of Theorem 1.4.1

In this section, we prove Theorem 1.4.1. In other words, we show that the function defined by the formula

$$\mathcal{K}_{\mathcal{G}}(z,\zeta) = \sum_{\tau \in \mathcal{T}} \frac{\frac{d}{dz}[(\tau \circ \varphi)(z)]\ \overline{\varphi'(\zeta)}}{[1-(\tau\circ\varphi)(z)\ \overline{\varphi(\zeta)}]^2}, \quad (z,\zeta) \in \varphi^{-1}(\mathbb{D}_{1/\rho}) \times \varphi^{-1}(\mathbb{D}_{1/\rho})$$

satisfies the relation

$$g(z) = \frac{1}{\pi} \int_{\mathcal{G}} g(\zeta)\ \mathcal{K}_{\mathcal{G}}(z,\zeta)\ dA(\zeta) \tag{3.2.1}$$

for every $(z,g) \in G_1 \times \mathcal{A}^2(G_1,\mathcal{G})$.

To simplify the exposition, we introduce the auxiliary kernel

$$\mathcal{K}_{\mathcal{D}}(z,\zeta) := \sum_{\tau\in\mathcal{T}} \frac{\tau'(z)}{[1-\tau(z)\bar{\zeta}]^2}, \quad (z,\zeta)\in \mathbb{D}_{1/\rho}\times\mathbb{D}_{1/\rho}.$$

Note that we have

$$\mathcal{K}_{\mathcal{G}}(z,\zeta) = \mathcal{K}_{\mathcal{D}}[\varphi(z),\varphi(\zeta)]\varphi'(z)\overline{\varphi'(\zeta)}, \quad (z,\zeta)\in \varphi^{-1}(\mathbb{D}_{1/\rho})\times\varphi^{-1}(\mathbb{D}_{1/\rho}). \tag{3.2.2}$$

The crux of the proof lies in the claim that we have

$$f(w) = \frac{1}{\pi}\int_{\mathcal{D}} f(\zeta)\, \mathcal{K}_{\mathcal{D}}(w,\zeta)\, dA(\zeta) \tag{3.2.3}$$

for every $(w,f)\in\mathbb{D}\times\mathcal{B}^2(\mathbb{D})$. To see this, fix some $f\in\mathcal{B}^2(\mathbb{D})$. Then, for $w\in\mathbb{D}$, we write

$$\begin{aligned} I(w) :&= \frac{1}{\pi}\int_{\mathcal{D}} f(\zeta)\, \mathcal{K}_{\mathcal{D}}(w,\zeta)\, dA(\zeta) \\ &= \frac{1}{\pi}\int_{\mathbb{D}} f(\zeta)\, \mathcal{K}_{\mathcal{D}}(w,\zeta)\, dA(\zeta) - \sum_{j\in\Lambda_s}\frac{1}{\pi}\int_{D_j} f(\zeta)\, \mathcal{K}_{\mathcal{D}}(w,\zeta)\, dA(\zeta) \\ &= \frac{1}{\pi}\int_{\mathbb{D}} \sum_{\tau\in\mathcal{T}} \frac{f(\zeta)\,\tau'(w)}{[1-\tau(w)\bar{\zeta}]^2}\, dA(\zeta) - \sum_{j\in\Lambda_s}\frac{1}{\pi}\int_{\mathring{D}_j} \sum_{\tau\in\mathcal{T}} \frac{f(\zeta)\,\tau'(w)}{[1-\tau(w)\bar{\zeta}]^2}\, dA(\zeta). \end{aligned}$$

Then by relation (3.1.2) and Lemma (3.1.4), we may write

$$\begin{aligned} I(w) &= \sum_{\tau\in\mathcal{T}} \tau'(w)\cdot f[\tau(w)] - \sum_{j\in\Lambda_s}\sum_{\tau\in\mathcal{T}} \frac{d}{dz}\{T_j[\tau(w)]\}\cdot f\{T_j[\tau(w)]\} \\ &= \sum_{\tau\in\mathcal{T}} \tau'(w)\cdot f[\tau(w)] - \sum_{\tau\in\mathcal{T}^*} \tau'(w)\cdot f[\tau(w)], \end{aligned}$$

by relationship (2.2.1). Since the only function in $\mathcal{T}\setminus\mathcal{T}^*$ is the identity function, we have

$$I(w) = f(w),$$

and equation (3.2.3) is established.

Now we prove Theorem 1.4.1 by fixing some $g \in \mathcal{A}^2(G_1, \mathcal{G})$. By Lemma 3.1.2, we can find some $f \in \mathcal{B}^2(\mathbb{D})$ such that

$$g(z) = f(\varphi(z)) \cdot \varphi'(z), \quad z \in G_1.$$

By making the change of variables $w = \varphi(z)$ in equation (3.2.3), we obtain

$$f(\varphi(z)) = \frac{1}{\pi} \int_{\mathcal{G}} f(\varphi(\zeta))\ \mathcal{K}_{\mathcal{D}}[\varphi(z), \varphi(\zeta)]\ |\varphi'(\zeta)|^2\ dA(\zeta), \quad z \in G_1.$$

Multiplying both sides of this equation by $\varphi'(z)$ and using the relationship 3.2.2 gives

$$f(\varphi(z))\ \varphi'(z) = \frac{1}{\pi} \int_{\mathcal{G}} f(\varphi(\zeta))\ \varphi'(\zeta)\ K_{\mathcal{G}}(z, \zeta)\ dA(\zeta), \quad z \in G_1.$$

In other words, we have

$$g(z) = \frac{1}{\pi} \int_{\mathcal{G}} g(\zeta)\ K_{\mathcal{G}}(z, \zeta)\ dA(\zeta),$$

and the proof is complete.

3.3 The symmetry property

Before we move on, we a record a property enjoyed by $\mathcal{K}_{\mathcal{G}}$. First, we need a lemma.

Lemma 3.3.1. *Let $z \in \varphi^{-1}(\mathbb{D}_{1/\rho})$. The function*

$$h(\zeta) := \overline{\mathcal{K}_{\mathcal{G}}(z, \zeta)}$$

belongs to $\mathcal{A}^2(G_1, \mathcal{G})$.

Proof. First, note that we have the representation

$$h(\zeta) = \frac{\overline{\varphi'(z)}\ \varphi'(\zeta)}{[1 - \overline{\varphi(z)}\varphi(\zeta)]^2} + \sum_{\tau \in \mathcal{T}^*} \frac{\overline{\frac{d}{dz}[(\tau \circ \varphi)(z)]}\ \varphi'(\zeta)}{[1 - \overline{(\tau \circ \varphi)(z)}\ \varphi(\zeta)]^2}.$$

By Lemma 3.1.1, the Bergman kernel for G_1 is given by

$$K(\zeta, z) = \frac{\overline{\varphi'(z)}\ \varphi'(\zeta)}{[1 - \overline{\varphi(z)}\varphi(\zeta)]^2}.$$

In particular, we have $K \in \mathcal{A}^2(G_1, \mathcal{G})$. Therefore, it suffices to show that

$$h^*(\zeta) := \sum_{\tau \in \mathcal{T}^*} \frac{\overline{\frac{d}{dz}[(\tau \circ \varphi)(z)]}\ \varphi'(\zeta)}{[1 - \overline{(\tau \circ \varphi)(z)}\ \varphi(\zeta)]^2}$$

belongs to $\mathcal{A}^2(G_1, \mathcal{G})$ since $h = K + h^*$ and since $\mathcal{A}^2(G_1, \mathcal{G})$ is a Hilbert space.

If we define

$$M := \max\{|c_j| + r_j : j \in \Lambda_s\},$$

then, by Lemma 2.2.1, we have

$$|(\tau \circ \varphi)(z)| \cdot |\varphi(\zeta)| < M \cdot |\varphi(\zeta)| \leq M < 1, \quad (\tau, \zeta) \in \mathcal{T} \times \overline{G}_1.$$

This shows that each term in the function series defining $h^*(\zeta)$ is analytic for $\zeta \in G_1$. It also lets us estimate

$$\sum_{\tau \in \mathcal{T}^*} \frac{|\frac{d}{dz}[(\tau \circ \varphi)(z)]| \cdot |\varphi'(\zeta)|}{|1 - \overline{(\tau \circ \varphi)(z)}\ \varphi(\zeta)|^2} < \frac{1}{(1-M)^2} \sum_{\tau \in \mathcal{T}^*} \left| \frac{d}{dz}[(\tau \circ \varphi)(z)] \right|$$

for $\zeta \in \overline{G_1}$. Since the series on the right converges by Assumption 1.3.1, we have that the series on the left converges uniformly for $\zeta \in \overline{G_1}$ by the Weierstrass M-test. Therefore, $h(\zeta)$

is analytic for $\zeta \in G_1$. Furthermore, the estimate above lets us write

$$\int_{\mathcal{G}} |h^*(\zeta)|^2 \, dA(\zeta) \leq \int_{G_1} |h^*(\zeta)|^2 \, dA(\zeta) < \frac{\pi}{(1-M)^2} \sum_{\tau \in \mathcal{T}^*} \left| \frac{d}{dz}[(\tau \circ \varphi)(z)] \right|^2 < \infty$$

by Assumption 1.3.1. Therefore, we have $h^* \in \mathcal{A}^2(G_1, \mathcal{G})$, and the proof is complete.

$\square$

Now we can prove the symmetry property of the kernel $\mathcal{K}_{\mathcal{G}}$.

Corollary 3.3.2. *For every $(z, w) \in G_1 \times G_1$, we have*

$$\mathcal{K}_{\mathcal{G}}(z, w) = \overline{\mathcal{K}_{\mathcal{G}}(w, z)}.$$

Proof. Fix some $(z, w) \in G_1 \times G_1$. Then by Lemma 3.3.1, the functions

$$f(\zeta) := \overline{\mathcal{K}_{\mathcal{G}}(w, \zeta)}, \quad \zeta \in G_1$$

and

$$g(\zeta) := \overline{\mathcal{K}_{\mathcal{G}}(z, \zeta)}, \quad \zeta \in G_1$$

belong to $\mathcal{A}^2(G_1, \mathcal{G})$. By Theorem 1.4.1, we may write

$$\overline{\mathcal{K}_{\mathcal{G}}(w, z)} = f(z) = \frac{1}{\pi} \int_{\mathcal{G}} f(\zeta) \, \mathcal{K}_{\mathcal{G}}(z, \zeta) \, dA(\zeta) = \overline{\frac{1}{\pi} \int_{\mathcal{G}} \overline{f(\zeta) \, \mathcal{K}_{\mathcal{G}}(z, \zeta)} \, dA(\zeta)}.$$

In other words, we have

$$\overline{\mathcal{K}_{\mathcal{G}}(w, z)} = \overline{\frac{1}{\pi} \int_{\mathcal{G}} \mathcal{K}_{\mathcal{G}}(w, \zeta) \, g(\zeta) \, dA(\zeta)} = \overline{g(w)} = \mathcal{K}_{\mathcal{G}}(z, w),$$

and the proof is complete.

$\square$

4 THE PROOFS OF THEOREMS 1.6.1 AND 1.6.2

Throughout this chapter, we will suppose that $\mathcal{D} = \mathbb{D} \setminus \bigcup_{j=1}^{s} D_{c_j,r_j}$ is a CMCD for which assumption 1.3.1 holds. We will continue to use the notation introduced in Section 1.3. In particular, we recall that

- G_1 is a bounded, simply connected domain in the complex plane whose boundary L_1 is an analytic Jordan curve,
- $\mathcal{T}$ is the family of compositions associated with $\mathcal{D}$,
- φ is a conformal map of G_1 onto the open unit disk,
- ϕ is the conformal map (described in Section 1.5) of the exterior of L_1 onto the exterior of the unit disk, and
- $\mathcal{G} = \varphi^{-1}(\mathcal{D})$.

By selecting a larger $\rho \in (0, 1)$ from Assumption 1.3.1 if necessary, we can guarantee that the following conditions are satisfied:

- $\sum_{\tau \in \mathcal{T}} |\tau'|$ converges uniformly on every compact subset of $\mathbb{D}_{1/\rho}$,
- φ^{-1} has an analytic and univalent continuation to $\mathbb{D}_{1/\rho}$,
- ϕ^{-1} has an analytic and univalent continuation to $\{w : |w| > \rho\}$,
- $\mathbb{D}_{1/\rho} \subset \bigcap_{j \in \Lambda_s} \chi_j(\mathbb{D}_{1/\sigma_j})$, and
- $\bigcup_{j \in \Lambda_s} D_{c_j,r_j} \subset \mathbb{D}_\rho$.

We recall the definition of the kernel $\mathcal{M}_{\mathcal{G}}$:

$$\mathcal{M}_{\mathcal{G}}(z,\zeta) := \sum_{\tau\in\mathcal{T}} \left[\frac{\varphi'(\zeta)}{\varphi(\zeta)-\tau(0)} \cdot \frac{(\tau\circ\varphi)(z)-\tau(0)}{\varphi(\zeta)-(\tau\circ\varphi)(z)} \right], \quad (z,\zeta)\in\varphi^{-1}(\mathbb{D}_{1/\rho})\times\varphi^{-1}(\mathbb{D}_{1/\rho}).$$

We also recall the construction of the functions $\{f_{n,k}\}_{k=1}^{\infty}$. We fix some $r\in(\rho,1)$ such that $G_{1/r}\subset\varphi^{-1}(\mathbb{D}_{1/\rho})$. Next, we fix some $t\in(r,1)$. Then, for each $n\in\mathbb{N}$, we recursively define the sequence $\{f_{n,k}\}_{k=0}^{\infty}$ in the following manner. First, we set

$$f_{n,0}(z) := 0, \quad z\in\hat{\mathbb{C}}.$$

Then, for $k\geq 0$, we put

$$f_{n,2k+1}(z) := -\frac{1}{2\pi i}\oint_{L_t} f_{n,2k}(\zeta)\, \mathcal{M}_{\mathcal{G}}(z,\zeta)\, [\phi(\zeta)]^{n+1}\, d\zeta, \quad z\in\varphi^{-1}(\mathbb{D}_{1/\rho})\setminus L_t$$

and

$$f_{n,2k+2}(z) := \frac{1}{2\pi i}\oint_{L_{1/t}} f_{n,2k+1}(\zeta)\frac{\phi'(\zeta)\, [\phi(\zeta)]^{-n-1}}{\phi(\zeta)-\phi(z)}\, d\zeta, \quad z\in\Omega_r\setminus L_{1/t}.$$

In this section, we will prove that these functions are in fact well defined and that for n large enough, the two series

$$\sum_{k=0}^{\infty} f_{n,2k} \qquad \text{and} \qquad \sum_{k=0}^{\infty} f_{n,2k+1} \tag{4.0.1}$$

converge absolutely and locally uniformly in $\Omega_r\setminus L_{1/t}$ and $G_{1/r}\setminus L_t$, respectively.

The purpose of this chapter is to prove Theorem 1.6.1. In other words, if P_n denote the monic polynomials orthogonal over $\mathcal{G}=\varphi^{-1}(\mathcal{D})$, then we sill show that for n sufficiently large we have the series expansion

$$(n+1)[\phi'(\infty)]^{n+1}P_n(z) = \frac{d}{dz}\begin{cases} [\phi(z)]^{n+1}\displaystyle\sum_{k=0}^{\infty} f_{n,2k}(z) & z \in \Omega_{1/t}, \\ [\phi(z)]^{n+1}\displaystyle\sum_{k=0}^{\infty} f_{n,2k}(z) - \sum_{k=0}^{\infty} f_{n,2k+1}(z), & z \in \Omega_t \cap G_{1/t}, \\ -\displaystyle\sum_{k=0}^{\infty} f_{n,2k+1}(z), & z \in G_t \end{cases}.$$

Before we prove Theorem 1.6.1, we establish some preliminary facts. In Section 4.1.1, we develop an understanding of some kernels related to $\mathcal{K}_{\mathcal{G}}$. In Section 4.1.2, we establish some estimates. In Section 4.1.3, we show that the function

$$\mathcal{P}_n(z) := \begin{cases} [\phi(z)]^{n+1}\displaystyle\sum_{k=0}^{\infty} f_{n,2k}(z) & z \in \Omega_{1/t}, \\ [\phi(z)]^{n+1}\displaystyle\sum_{k=0}^{\infty} f_{n,2k}(z) - \sum_{k=0}^{\infty} f_{n,2k+1}(z), & z \in \Omega_t \cap G_{1/t}, \\ -\displaystyle\sum_{k=0}^{\infty} f_{n,2k+1}(z), & z \in G_t. \end{cases} \tag{4.0.2}$$

is a polynomial, provided that n is sufficiently large. Establishing these concepts first will put us in a good position to prove Theorem 1.6.1.

4.1 Preliminaries for the proof of Theorem 1.6.1

4.1.1 Kernels related to $\mathcal{K}_{\mathcal{G}}$

In this chapter, we use the notation

$$\mathbb{A}_\rho := \{\zeta \in \mathbb{C} : \rho < |\zeta| < 1/\rho\}.$$

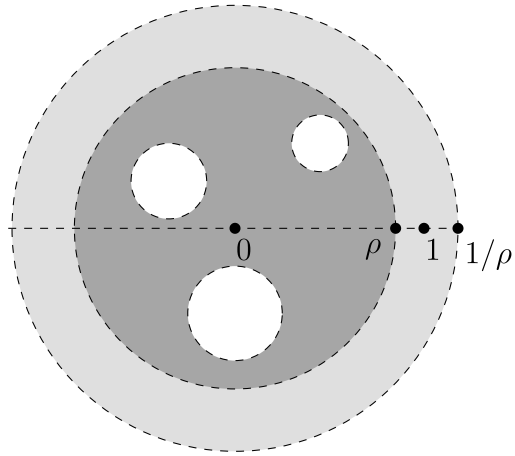


Figure 4.1: The sets $\mathbb{A}_\rho$ and $\mathbb{D}_\rho \setminus \bigcup_{j\in\Lambda_s} D_{c_j,r_j}$.

We recall that we have selected $\rho \in (0,1)$ close enough to 1 so that the condition

$$\bigcup_{j\in\Lambda_s} D_{c_j,r_j} \subset \mathbb{D}_\rho$$

is satisfied. Let m denote the distance from $\mathbb{A}_\rho$ to $\bigcup_{j\in\Lambda_s} D_{c_j,r_j}$:

$$m := \min\left\{ |z-\zeta| : (z,\zeta) \in \mathbb{A}_\rho \times \bigcup_{j\in\Lambda_s} D_{c_j,r_j} \right\}.$$

Note that, by Lemma 2.1.1 and the definition of m, we have

$$|\tau(z)-\zeta| > m, \quad (\tau,z,\zeta) \in \mathcal{T}^* \times \mathbb{D}_{1/\rho} \times \mathbb{A}_\rho. \tag{4.1.1}$$

In particular, this follows from the fact that Lemma 2.1.1 gives $|\tau(z)| < \rho$ whenever $\tau \in \mathcal{T}^*$ and $z \in \mathbb{D}_{1/\rho}$.

The Kernel $\mathcal{M}_{\mathcal{G}}$

To develop our understanding of the kernel $\mathcal{M}_{\mathcal{G}}$, we introduce an auxiliary kernel $\mathcal{M}^*_{\mathcal{D}}$. For $(z, \zeta) \in \mathbb{D}_{1/\rho} \times \mathbb{D}_{1/\rho}$, we define

$$\mathcal{M}^*_{\mathcal{D}}(z, \zeta) := \sum_{\tau \in \mathcal{T}^*} \left[\frac{1}{\zeta - \tau(0)} \cdot \frac{\tau(z) - \tau(0)}{\zeta - \tau(z)} \right].$$

We establish some observations about this function.

Lemma 4.1.1. *The kernel $\mathcal{M}^*_{\mathcal{D}}$ enjoys the following properties.*

- *For a fixed $\zeta \in \mathbb{A}_\rho$, the kernel $\mathcal{M}^*_{\mathcal{D}}$, viewed as a function of z, is analytic in $\mathbb{D}_{1/\rho}$.*
- *For a fixed $z \in \mathbb{D}_{1/\rho}$, the kernel $\mathcal{M}^*_{\mathcal{D}}$, viewed as a function of ζ, is analytic in $\mathbb{A}_\rho$.*

Proof. First, fix some $\zeta \in \mathbb{A}_\rho$. By relation (4.1.1), we see that each term in the function series defining $\mathcal{M}^*_{\mathcal{D}}$ is analytic for $z \in \mathbb{D}_{1/\rho}$. We will show that this series converges normally in that set. To this end, let K be a compact subset of $\mathbb{D}_{1/\rho}$. Then, for every $z \in K$, we can use relation (4.1.1) to write

$$\begin{aligned} \sum_{\tau \in \mathcal{T}^*} \left| \frac{1}{\zeta - \tau(0)} \cdot \frac{\tau(z) - \tau(0)}{\zeta - \tau(z)} \right| &< \frac{1}{m^2} \sum_{\tau \in \mathcal{T}^*} |\tau(z) - \tau(0)| \\ &= \frac{1}{m^2} \sum_{\tau \in \mathcal{T}^*} \left| \int_0^z \tau'(t)\, dt \right| \\ &< \frac{|z|}{m^2} \sum_{\tau \in \mathcal{T}^*} \max_{t \in [0,z]} |\tau'(t)|. \end{aligned}$$

Note that the final term is finite by Assumption 1.3.1. By the Weierstrass M-test, we have that the series defining $\mathcal{M}^*_{\mathcal{D}}$ converges uniformly on K. Therefore the series converges normally in $\mathbb{D}_{1/\rho}$, whence $\mathcal{M}^*_{\mathcal{D}}$ is analytic in that set. This proves the first claim.

To see the second claim, we fix some $z \in \mathbb{D}_{1/\rho}$. Once again, the estimate (4.1.1) informs us that each term in the function series defining $\mathcal{M}^*_{\mathcal{D}}$ is analytic for $\zeta \in \mathbb{A}_\rho$. We will demonstrate that the series converges normally there. Let K be a compact subset of $\mathbb{A}_\rho$. Then, for every $\zeta \in K$, we have

$$\sum_{\tau \in \mathcal{T}^*} \left| \frac{1}{\zeta - \tau(0)} \cdot \frac{\tau(z) - \tau(0)}{\zeta - \tau(z)} \right| < \frac{|z|}{m^2} \sum_{\tau \in \mathcal{T}^*} \max_{t \in [0,z]} |\tau'(t)| < \infty$$

by Assumption 1.3.1. The Weierstrass M-test tells us that the series defining $\mathcal{M}_{\mathcal{D}}^*$ converges uniformly on K. Therefore the series converges normally in $\mathbb{A}_\rho$, whence $\mathcal{M}_{\mathcal{D}}^*$ is analytic in that set. This settles the second claim and completes the proof of the lemma.

□

Next, for $(z, \zeta) \in \mathbb{D}_{1/\rho} \times \mathbb{D}_{1/\rho}$, we define the function

$$\mathcal{M}_{\mathcal{D}}(z, \zeta) := \sum_{\tau \in \mathcal{T}} \left[\frac{1}{\zeta - \tau(0)} \cdot \frac{\tau(z) - \tau(0)}{\zeta - \tau(z)} \right].$$

Since the identity function is the only element of $\mathcal{T} \setminus \mathcal{T}^*$, we may also write

$$\mathcal{M}_{\mathcal{D}}(z, \zeta) = \frac{1}{\zeta} \cdot \frac{z}{\zeta - z} + \mathcal{M}_{\mathcal{D}}^*(z, \zeta). \tag{4.1.2}$$

Now we recall the definition of $M_{\mathcal{G}}$. For $(z, \zeta) \in \varphi^{-1}(\mathbb{D}_{1/\rho}) \times \varphi^{-1}(\mathbb{D}_{1/\rho})$, we have

$$\mathcal{M}_{\mathcal{G}}(z, \zeta) = \sum_{\tau \in \mathcal{T}} \left[\frac{\varphi'(\zeta)}{\varphi(\zeta) - \tau(0)} \cdot \frac{\tau[\varphi(z)] - \tau(0)}{\varphi(\zeta) - \tau[\varphi(z)]} \right].$$

Note that, by the definition of $\mathcal{M}_{\mathcal{D}}$, we have

$$\mathcal{M}_{\mathcal{G}}(z, \zeta) = \mathcal{M}_{\mathcal{D}}[\varphi(z), \varphi(\zeta)]\, \varphi'(\zeta). \tag{4.1.3}$$

Therefore, we may appeal to (4.1.2) to write

$$\mathcal{M}_{\mathcal{G}}(z, \zeta) = \frac{\varphi'(\zeta)}{\varphi(\zeta)} \cdot \frac{\varphi(z)}{\varphi(\zeta) - \varphi(z)} + \varphi'(\zeta) \cdot \mathcal{M}_{\mathcal{D}}^*[\varphi(z), \varphi(\zeta)]. \tag{4.1.4}$$

The following lemma is an immediate consequence of the representation (4.1.4) and Lemma 4.1.1.

Lemma 4.1.2. *The function $\mathcal{M}_{\mathcal{G}}$ enjoys the following properties.*

- *For a fixed $\zeta \in \varphi^{-1}(\mathbb{A}_\rho)$, the kernel $\mathcal{M}_{\mathcal{G}}$, viewed as a function of z, is analytic in $\varphi^{-1}(\mathbb{D}_{1/\rho})$ except at the point $z = \zeta$, where it has a simple pole.*
- *For a fixed $z \in \varphi^{-1}(\mathbb{D}_{1/\rho}) \setminus \varphi^{-1}(\mathbb{A}_\rho)$, the kernel $\mathcal{M}_{\mathcal{G}}$, viewed as a function of ζ, is analytic in $\varphi^{-1}(\mathbb{A}_\rho)$.*
- *For a fixed $z \in \varphi^{-1}(\mathbb{D}_{1/\rho}) \cap \varphi^{-1}(\mathbb{A}_\rho)$, the kernel $\mathcal{M}_{\mathcal{G}}$, viewed as a function of ζ, is analytic in $\varphi^{-1}(\mathbb{A}_\rho)$ except at the point $\zeta = z$, where it has a simple pole.*

The Kernel $\mathcal{L}_{\mathcal{G}}$

In a moment, we will introduce a kernel that will help illustrate the connection between $\mathcal{M}_{\mathcal{G}}$ and $\mathcal{K}_{\mathcal{G}}$. The kernel we will introduce, $\mathcal{L}_{\mathcal{G}}$, is best understood by first considering a simpler auxiliary kernel $\mathcal{L}_{\mathcal{D}}$. For $(z,\zeta) \in \mathbb{D}_{1/\rho} \times \mathbb{D}_{1/\rho}$, we define the function

$$\mathcal{L}_{\mathcal{D}}(z,\zeta) := \sum_{\tau \in \mathcal{T}} \frac{\tau'(z)}{[\zeta - \tau(z)]^2}.$$

Note that we, for $(z,\zeta) \in \mathbb{D}_{1/\rho} \times \mathbb{D}_{1/\rho}$, we have the relation

$$\frac{\partial}{\partial z}\mathcal{M}_{\mathcal{D}}(z,\zeta) = \mathcal{L}_{\mathcal{D}}(z,\zeta). \tag{4.1.5}$$

To see this, it is sufficient to write

$$\begin{aligned}
\frac{\partial}{\partial z}\left[\frac{1}{\zeta - \tau(0)} \cdot \frac{\tau(z) - \tau(0)}{\zeta - \tau(z)}\right] &= \frac{1}{\zeta - \tau(0)} \cdot \frac{\partial}{\partial z}\left[\frac{\tau(z) - \tau(0)}{\zeta - \tau(z)}\right] \\
&= \frac{1}{\zeta - \tau(0)} \cdot \frac{[\zeta - \tau(z)][\tau'(z)] + [\tau(z) - \tau(0)][\tau'(z)]}{[\zeta - \tau(z)]^2} \\
&= \frac{\tau'(z)}{\zeta - \tau(0)} \cdot \frac{\zeta - \tau(z) + \tau(z) - \tau(0)}{[\zeta - \tau(z)]^2} = \frac{\tau'(z)}{[\zeta - \tau(z)]^2},
\end{aligned}$$

for this implies relation (4.1.5).

We may now introduce our final kernel. For $(z,\zeta) \in \varphi^{-1}(\mathbb{D}_{1/\rho}) \times \varphi^{-1}(\mathbb{D}_{1/\rho})$, we define the function

$$\mathcal{L}_{\mathcal{G}}(z,\zeta) := \mathcal{L}_{\mathcal{D}}[\varphi(z),\varphi(\zeta)]\ \varphi'(z)\varphi'(\zeta).$$

Note that we have the representation

$$\mathcal{L}_{\mathcal{G}}(z,\zeta) = \sum_{\tau\in\mathcal{T}} \frac{\tau'[\varphi(z)]\cdot\varphi'(z)\varphi'(\zeta)}{\{\varphi(\zeta)-\tau[\varphi(z)]\}^2} = \sum_{\tau\in\mathcal{T}} \frac{\frac{d}{dz}\{\tau[\varphi(z)]\}\varphi'(\zeta)}{\{\varphi(\zeta)-\tau[\varphi(z)]\}^2}.$$

We record some of the properties of this function.

Lemma 4.1.3. *The function $\mathcal{L}_{\mathcal{G}}$ enjoys the following properties.*

- *For a fixed $\zeta \in \varphi^{-1}(\mathbb{A}_\rho)$, the kernel $\mathcal{L}_{\mathcal{G}}$, viewed as a function of z, is analytic in $\varphi^{-1}(\mathbb{D}_{1/\rho})$ except at the point $z=\zeta$, where it has a double pole.*

- *For a fixed $z \in \varphi^{-1}(\mathbb{D}_{1/\rho})\backslash\varphi^{-1}(\mathbb{A}_\rho)$, the kernel $\mathcal{L}_{\mathcal{G}}$, viewed as a function of ζ, is analytic in $\varphi^{-1}(\mathbb{A}_\rho)$.*

- *For a fixed $z \in \varphi^{-1}(\mathbb{D}_\rho)\cap\varphi^{-1}(\mathbb{A}_\rho)$, the kernel $\mathcal{L}_{\mathcal{G}}$, viewed as a function of ζ, is analytic in $\varphi^{-1}(\mathbb{A}_\rho)$ except at the point $\zeta = z$, where it has a double pole.*

Proof. First, we note that we can write

$$\mathcal{L}_{\mathcal{D}}(z,\zeta) = \frac{1}{(\zeta-z)^2} + \frac{\partial}{\partial z}\mathcal{M}^*_{\mathcal{D}}(z,\zeta), \quad (z,\zeta)\in\varphi^{-1}(\mathbb{D}_{1/\rho})\times\varphi^{-1}(\mathbb{D}_{1/\rho}).$$

Then the lemma immediately follows from the definition of $\mathcal{L}_{\mathcal{G}}$ and Lemma 4.1.1.

$\square$

Relationships between the kernels

Lemma 4.1.4. *For* $(z, \zeta) \in \varphi^{-1}(\mathbb{D}_{1/\rho}) \times \varphi^{-1}(\mathbb{A}_\rho)$*, we have*

$$\mathcal{L}_{\mathcal{G}}(z,\zeta) = \frac{\varphi'(\zeta)\;\mathcal{K}_{\mathcal{G}}[z, \nu_\varphi(\zeta)]}{[\varphi(\zeta)]^2\overline{\varphi'[\nu_\varphi(\zeta)]}}, \quad \textit{where } \nu_\varphi(\zeta) := \varphi^{-1}\left[\frac{1}{\overline{\varphi(\zeta)}}\right].$$

Proof. Since

$$\mathcal{K}_{\mathcal{G}}[z, \nu_\varphi(\zeta)] = \sum_{\tau \in \mathcal{T}} \frac{\frac{d}{dz}\{\tau[\varphi(z)]\} \cdot \overline{\varphi'[\nu_\varphi(\zeta)]}}{\{1 - \tau[\varphi(z)]\;/\varphi(\zeta)\}^2},$$

we have

$$\frac{\varphi'(\zeta)\;\mathcal{K}_{\mathcal{G}}[z, \nu_\varphi(\zeta)]}{[\varphi(\zeta)]^2\overline{\varphi'[\nu_\varphi(\zeta)]}} = \sum_{\tau \in \mathcal{T}} \frac{\frac{d}{dz}\{\tau[\varphi(z)]\}\varphi'(\zeta)}{\{\varphi(\zeta) - \tau[\varphi(z)]\}^2},$$

and the claim is verified.

□

Lemma 4.1.5. *For* $(z, \zeta) \in \varphi^{-1}(\mathbb{D}_{1/\rho}) \times \varphi^{-1}(\mathbb{D}_{1/\rho})$*, we have*

$$\frac{\partial}{\partial z}\mathcal{M}_{\mathcal{G}}(z,\zeta) = \mathcal{L}_{\mathcal{G}}(z,\zeta).$$

Proof. We appeal to relation (4.1.3), relation (4.1.5), and to the chain rule in order to write

$$\frac{\partial}{\partial z}\mathcal{M}_{\mathcal{G}}(z,\zeta) = \frac{\partial}{\partial z}\{\mathcal{M}_{\mathcal{D}}[\varphi(z), \varphi(\zeta)]\;\varphi'(\zeta)\} = \mathcal{L}_{\mathcal{D}}[\varphi(z), \varphi(\zeta)]\;\varphi'(z)\varphi'(\zeta).$$

Then, by the definition of $\mathcal{L}_{\mathcal{G}}$, the claim is established.

□

4.1.2 Estimates

Recall that we have fixed some $r \in (\rho, 1)$ close enough to 1 so that the condition $G_{1/r} \subset \varphi^{-1}(\mathbb{D}_{1/\rho})$ is satisfied. We have also fixed some $t \in (r, 1)$.

If $z \in G_{1/r} \setminus L_t$ is fixed, then, by Lemma 4.1.2, the kernel $\mathcal{M}_{\mathcal{G}}(z, \zeta)$, viewed as a function of ζ, is continuous on the compact set L_t. Therefore, we have

$$\max_{\zeta \in L_t} |\mathcal{M}_{\mathcal{G}}(z, \zeta)| < \infty, \quad z \in G_{1/r} \setminus L_t.$$

Furthermore, since $L_{1/t} \subset G_{1/r} \setminus L_t$ and since $\mathcal{M}_{\mathcal{G}}(z, \zeta)$ is continuous on the compact set $L_{1/t} \times L_t$, we have

$$\max_{(z,\zeta) \in L_{1/t} \times L_t} |\mathcal{M}_{\mathcal{G}}(z, \zeta)| < \infty.$$

If we define the constant

$$M_t := \max_{(z,\zeta) \in L_{1/t} \times L_t} |\mathcal{M}_{\mathcal{G}}(z, \zeta)|,$$

then, by the remarks above, we have $M_t < \infty$.

Next, we recall the definitions of the integral transforms introduced in Section 1.6. First, we set

$$f_{n,k}(z) = 0$$

for every $z \in \hat{\mathbb{C}}$. Next, for $k \geq 0$, we put

$$f_{n,2k+1}(z) := -\frac{1}{2\pi i} \oint_{L_t} f_{n,2k}(\zeta)\, \mathcal{M}_{\mathcal{G}}(z, \zeta)\, [\phi(\zeta)]^{n+1}\, d\zeta, \quad z \in G_{1/r} \setminus L_t$$

and

$$f_{n,2k+2}(z) := \frac{1}{2\pi i} \oint_{L_{1/t}} f_{n,2k+1}(\zeta) \frac{\phi'(\zeta)\, [\phi(\zeta)]^{-n-1}}{\phi(\zeta) - \phi(z)}\, d\zeta, \quad z \in \Omega_r \setminus L_{1/t}.$$

Fix some $n \in \mathbb{N} \cup \{0\}$. Note that, for each $k \geq 0$, the function $f_{n,2k+1}$ is analytic in $G_{1/r} \setminus L_t$ and the function $f_{n,2k+2}$ is analytic in $\Omega_r \setminus L_{1/t}$.

Next we record some estimates.

Lemma 4.1.6. *Let $\Upsilon_t := \ell(L_t)/2\pi t$, where $\ell(L_t)$ is the length of the curve L_t. Then, for every integer $k \geq 0$, we have both*

$$|f_{n,2k+1}(z)| \leq \Upsilon_t t^{n+2} \left[\frac{\Upsilon_t M_t t^{2n+2}}{1/t - t}\right]^k \max_{\zeta \in L_t} |\mathcal{M}_{\mathcal{G}}(z,\zeta)|, \quad z \in G_{1/r} \setminus L_t \tag{4.1.6}$$

and

$$|f_{n,2k+2}(z)| \leq \frac{\Upsilon_t M_t t^{2n+2}}{|1/t - |\phi(z)||} \left[\frac{\Upsilon_t M_t t^{2n+2}}{1/t - t}\right]^k, \quad z \in \Omega_r \setminus L_{1/t}. \tag{4.1.7}$$

Proof. First, we record the calculation

$$\int_{L_t} |\phi(\zeta)|^{n+1} \, |d\zeta| = t^{n+1} \int_{L_t} |d\zeta| = t^{n+1} \, \ell(L_t) = 2\pi \Upsilon_t t^{n+2}. \tag{4.1.8}$$

This helps us obtain an estimate for the modulus of the function

$$f_{n,1}(z) = -\frac{1}{2\pi i} \int_{L_t} \mathcal{M}_{\mathcal{G}}(z,\zeta) \, [\phi(\zeta)]^{n+1} \, d\zeta.$$

In particular, we may write

$$|f_{n,1}(z)| \leq \Upsilon_t \, t^{n+2} \max_{\zeta \in L_t} |M_{\mathcal{G}}(z,\zeta)|, \quad z \in G_{1/r} \setminus L_t. \tag{4.1.9}$$

In other words, (4.1.6) holds for $k = 0$.

Combining (4.1.9) with the definition of M_t, we get

$$\max_{\zeta \in L_{1/t}} |f_{n,1}(\zeta)| \leq \max_{\zeta \in L_{1/t}} \left\{ \Upsilon_t \, t^{n+2} \max_{\xi \in L_t} |\mathcal{M}_{\mathcal{G}}(\zeta,\xi)| \right\} = \Upsilon_t t^{n+2} M_t. \tag{4.1.10}$$

Meanwhile, we also have

$$\int_{L_{1/t}} \frac{|\phi'(\zeta)|}{|\phi(\zeta) - \phi(z)|} \; |\phi(\zeta)|^{-n-1} \; |d\zeta| \le \frac{2\pi t^n}{|1/t - |\phi(z)||}. \tag{4.1.11}$$

By the definition of $f_{n,2}$ and by observations (4.1.11) and (4.1.10), we discover that

$$|f_{n,2}(z)| \le \frac{1}{2\pi} \int_{L_{1/t}} \left| \frac{f_{n,1}(\zeta)\phi'(\zeta)}{\phi(\zeta) - \phi(z)} \right| \; |\phi(\zeta)|^{-n-1} \; |d\zeta| \le \frac{\Upsilon_t M_t t^{2n+2}}{|1/t - |\phi(z)||} \tag{4.1.12}$$

for $z \in \Omega_r \setminus L_{1/t}$. This shows that (4.1.7) holds for $k = 0$ as well.

Let us then assume that both (4.1.6) and (4.1.7) hold for some $k \ge 0$, and we will demonstrate that they also hold when k is replaced by $k + 1$.

According to the definition of $f_{n,2k+3}$, we have

$$f_{n,2k+3}(z) = -\frac{1}{2\pi i} \int_{L_t} f_{n,2k+2}(\zeta) \; \mathcal{M}_{\mathcal{G}}(z, \zeta) \; [\phi(\zeta)]^{n+1} \; d\zeta, \qquad z \in G_{1/r} \setminus L_t. \tag{4.1.13}$$

By (4.1.7), the condition $\zeta \in L_t$ implies

$$|f_{n,2k+2}(\zeta)| \le \frac{\Upsilon_t M_t t^{2n+2}}{|1/t - |\phi(\zeta)||} \left[\frac{\Upsilon_t M_t t^{2n+2}}{1/t - t} \right]^k = \left[\frac{\Upsilon_t M_t t^{2n+2}}{1/t - t} \right]^{k+1}. \tag{4.1.14}$$

Combining (4.1.13), (4.1.8), and (4.1.14) results in the inequality (4.1.6) holding when k is replaced by $k + 1$.

Similarly, the definition of $f_{n,2k+4}$ gives

$$f_{n,2k+4}(z) = \frac{1}{2\pi i} \int_{L_{1/t}} \frac{f_{n,2k+3}(\zeta) \; \phi'(\zeta) \; [\phi(\zeta)]^{-n-1}}{\phi(\zeta) - \phi(z)} \; d\zeta, \qquad z \in \Omega_r \setminus L_{1/t}, \tag{4.1.15}$$

and the inequality (4.1.6) tells us that for $\zeta \in L_{1/t}$, we have

$$|f_{n,2k+3}(\zeta)| \le \Upsilon_t t^{n+2} \left[\frac{\Upsilon_t M_t t^{2n+2}}{1/t - t} \right]^{k+1} \max_{\zeta \in L_t} |\mathcal{M}_{\mathcal{G}}(z, \zeta)|. \tag{4.1.16}$$

Then, combining (4.1.15), (4.1.16), and (4.1.11) yields (4.1.7) for k replaced by $k+1$. The proof is complete by the principle of mathematical induction.

$\square$

Next, we define the number

$$N_t := \min\left\{ n \in \mathbb{N} : \frac{\Upsilon_t M_t t^{2(n+1)}}{1/t - t} < 1 \right\}. \tag{4.1.17}$$

Lemma 4.1.7. *For all $n > N_t$, we have that*

- $\sum_{k=0}^{\infty} f_{n,2k}$ *converges absolutely and normally on* $\Omega_r \setminus L_{1/t}$
- $\sum_{k=0}^{\infty} f_{n,2k+1}$ *converges absolutely and normally on* $G_{1/r} \setminus L_t$

Consequently, for all $n > N_t$, we have

- $\sum_{k=0}^{\infty} f_{n,2k}(z)$ *is analytic in* $\Omega_r \setminus \L_{1/t}$
- $\sum_{k=0}^{\infty} f_{n,2k+1}(z)$ *is analytic in* $G_{1/r} \setminus \L_t$.

Proof. In the first two claims, absolute convergence is a direct consequence of Lemma 4.1.6. We will prove that the series $\sum_{k=0}^{\infty} f_{n,2k}$ converges normally in $\Omega_r \setminus L_{1/t}$ by showing that it converges uniformly on each closed disk that is contained in that set. To this end, let K be a closed disk contained in $\Omega_r \setminus L_{1/t}$. Let d denote the distance from $\phi(K)$ to the point $1/t$, which cannot belong to $\phi(K)$. Then we have

$$|1/t - |\phi(z)|| \geq d > 0, \quad z \in K.$$

Then by Lemma 4.1.6, we have

$$|f_{n,2k+2}(z)| \leq \frac{\Upsilon_t M_t t^{2n+2}}{d} \left(\frac{\Upsilon_t M_t t^{2n+2}}{1/t - t} \right)^k, \quad z \in K.$$

By the Weierstrass M-test, $\sum_{k=0}^{\infty} f_{n,2k}$ converges absolutely and uniformly on K. This shows that $\sum_{k=0}^{\infty} f_{n,2k}$ converges normally in $\Omega_r \setminus L_{1/t}$.

Next, we will show that the series $\sum_{k=0}^{\infty} f_{n,2k+1}$ converges normally in $G_{1/r} \setminus L_t$. Let K be a closed disk contained in $G_{1/r} \setminus L_t$. By Lemma 4.1.2, we have

$$M_K := \max_{(z,\zeta)\in K\times L_t} |\mathcal{M}_{\mathcal{G}}(z,\zeta)| < \infty.$$

Then by Lemma 4.1.6, we have

$$|f_{n,2k+1}(z)| \le \Upsilon_t t^{n+2} \left(\frac{\Upsilon_t M_t t^{2n+2}}{1/t - t}\right)^k M_K, \quad z \in K.$$

By the Weierstrass M-test, $\sum_{k=0}^{\infty} f_{n,2k+1}$ converges absolutely and uniformly on K. This shows that $\sum_{k=0}^{\infty} f_{n,2k+1}$ converges normally in $G_{1/r} \setminus L_t$. The final two claims are a direct result of the first two.

$\square$

4.1.3 The function $\mathcal{P}_n$

For each $n > N_t$, we define the function

$$\mathcal{P}_n(z) = \begin{cases} [\phi(z)]^{n+1} \displaystyle\sum_{k=0}^{\infty} f_{n,2k}(z) & z \in \Omega_{1/t}, \\ [\phi(z)]^{n+1} \displaystyle\sum_{k=0}^{\infty} f_{n,2k}(z) - \sum_{k=0}^{\infty} f_{n,2k+1}(z), & z \in \Omega_t \cap G_{1/t}, \\ -\displaystyle\sum_{k=0}^{\infty} f_{n,2k+1}(z), & z \in G_t. \end{cases} \tag{4.1.18}$$

It will be helpful to have alternate representations for $\mathcal{P}_n$. For example, if $z \in \Omega_{1/t}$, then we have

$$\mathcal{P}_n(z) = [\phi(z)]^{n+1} \left[1 + \sum_{k=1}^{\infty} f_{n,2k}(z)\right]$$

$$
\begin{aligned}
&= [\phi(z)]^{n+1} \left[1 + \sum_{k=0}^{\infty} f_{n,2k+2}(z) \right] \\
&= [\phi(z)]^{n+1} \left[1 + \sum_{k=0}^{\infty} \frac{1}{2\pi i} \oint_{L_{1/t}} \frac{\phi'(\zeta)\ f_{n,2k+1}(\zeta)}{[\phi(\zeta) - \phi(z)]\ [\phi(\zeta)]^{n+1}}\, d\zeta \right] \\
&= [\phi(z)]^{n+1} \left[1 + \frac{1}{2\pi i} \oint_{L_{1/t}} \sum_{k=0}^{\infty} \frac{\phi'(\zeta)\ f_{n,2k+1}(\zeta)}{[\phi(\zeta) - \phi(z)]\ [\phi(\zeta)]^{n+1}}\, d\zeta \right].
\end{aligned}
$$

Also, note that

$$
\lim_{z\to\infty} \left[\frac{\phi(z)}{z} \right]^{n+1} = [\phi'(\infty)]^{n+1}
$$

and

$$
\lim_{z\to\infty} \sum_{k=0}^{\infty} f_{n,2k}(z) = 1.
$$

Therefore, we have

$$
\lim_{z\to\infty} \frac{\mathcal{P}_n(z)}{z^{n+1}} = [\phi'(\infty)]^{n+1} > 0. \tag{4.1.19}
$$

Meanwhile, if $z \in G_t$, then we may write

$$
\begin{aligned}
\mathcal{P}_n(z) = -\sum_{k=0}^{\infty} f_{n,2k+1}(z) &= \sum_{k=0}^{\infty} \frac{1}{2\pi i} \oint_{L_t} f_{n,2k}(\zeta)\ \mathcal{M}_{\mathcal{G}}(z,\zeta)\ [\phi(\zeta)]^{n+1}\, d\zeta \\
&= \frac{1}{2\pi i} \oint_{L_t} \sum_{k=0}^{\infty} f_{n,2k}(\zeta)\ \mathcal{M}_{\mathcal{G}}(z,\zeta)\ [\phi(\zeta)]^{n+1}\, d\zeta.
\end{aligned}
$$

We will utilize these alternate representations in the sequel.

Next, for each $n > N_t$, we define the function

$$
\mathcal{H}_n(z) := [\phi(z)]^{n+1} \left[1 + \frac{1}{2\pi i} \oint_{L_1} \frac{\phi'(\zeta)\ \sum_{k=0}^{\infty} f_{n,2k+1}(\zeta)}{[\phi(\zeta) - \phi(z)]\ [\phi(\zeta)]^{n+1}}\, d\zeta \right], \quad z \in \Omega_1.
$$

Lemma 4.1.8. *The function $\mathcal{H}_n(z)$ has the following properties:*

- $\mathcal{H}_n$ *is analytic in* Ω_1
- $\mathcal{H}_n(z) = \mathcal{P}_n(z)$ *for each* $z \in \Omega_{1/t}$

- $\mathcal{H}_n(z) = \mathcal{P}_n(z)$ *for each* $z \in G_{1/t} \cap \Omega_1$

Thus $\mathcal{H}_n(z)$ *is the analytic continuation of* $\mathcal{P}_n(z)$ *from* $\Omega_{1/t} \cup (G_{1/t} \cap \Omega_1)$ *to* Ω_1.

Proof. The fact that $\mathcal{H}_n(z)$ is analytic on Ω_1 follows from Lemma 4.1.7. To see the second claim, fix some $z \in \Omega_{1/t}$. Then we may write

$$\begin{aligned}
\mathcal{P}_n(z) &= [\phi(z)]^{n+1}\left[1 + \frac{1}{2\pi i}\oint_{L_{1/t}} \frac{\phi'(\zeta)\ \sum_{k=0}^{\infty} f_{n,2k+1}(\zeta)}{[\phi(\zeta)-\phi(z)]\ [\phi(\zeta)]^{n+1}}\ d\zeta\right] \\
&= [\phi(z)]^{n+1}\left[1 + \frac{1}{2\pi i}\oint_{L_{1}} \frac{\phi'(\zeta)\ \sum_{k=0}^{\infty} f_{n,2k+1}(\zeta)}{[\phi(\zeta)-\phi(z)]\ [\phi(\zeta)]^{n+1}}\ d\zeta\right] \\
&= \mathcal{H}_n(z),
\end{aligned}$$

which validates the second claim.

To see the third claim, fix some $z \in \Omega_1 \cap G_{1/t}$. To assist with notation, we write

$$F_n(\zeta) = \frac{\phi'(\zeta)\ \sum_{k=0}^{\infty} f_{n,2k+1}(\zeta)}{[\phi(\zeta)]^{n+1}}, \quad \zeta \in G_{1/r} \setminus L_t.$$

We see that $[\phi(\zeta) - \phi(z)]^{-1}F_n(\zeta)$, viewed as a function of ζ, is meromorphic in the intersection of Ω_ρ and $G_{1/r} \setminus L_t$, where its only singularity is a simple pole at the point $\zeta = z$. Then we compute

$$\begin{aligned}
\mathcal{H}_n(z) &= [\phi(z)]^{n+1}\left[1 + \frac{1}{2\pi i}\oint_{L_1} \frac{F_n(\zeta)}{\phi(\zeta)-\phi(z)}\ d\zeta\right] \\
&= [\phi(z)]^{n+1}\left[1 + \frac{1}{2\pi i}\oint_{L_{1/t}} \frac{F_n(\zeta)}{\phi(\zeta)-\phi(z)}\ d\zeta - \mathrm{Res}_{\zeta=z}\frac{F_n(\zeta)}{\phi(\zeta)-\phi(z)}\right] \\
&= [\phi(z)^{n+1}]\left[1 + \frac{1}{2\pi i}\oint_{L_{1/t}} \frac{\phi'(\zeta)\ \sum_{k=0}^{\infty} f_{n,2k+1}(\zeta)}{[\phi(\zeta)-\phi(z)]\ [\phi(\zeta)]^{n+1}}\ d\zeta - \frac{F_n(z)}{\phi'(z)}\right] \\
&= [\phi(z)]^{n+1}\sum_{k=0}^{\infty} f_{n,2k}(z) - \sum_{k=0}^{\infty} f_{n,2k+1}(z) \\
&= \mathcal{P}_n(z).
\end{aligned}$$

This proves the third claim.

Note that the first two claims show that $\mathcal{H}_n$ is the analytic continuation of $\mathcal{P}_n(z)$ from $\Omega_{1/t}$ to Ω_1, while claims one and three show that $\mathcal{H}_n$ is the analytic continuation of $\mathcal{P}_n(z)$ from $G_{1/t} \cap \Omega_1$ to Ω_1. This proves the lemma. □

Now, for each $n > N_t$, we define the function

$$\mathcal{J}_n(z) := \frac{1}{2\pi i} \oint_{L_1} \sum_{k=0}^{\infty} f_{n,2k}(\zeta)\, \mathcal{M}_{\mathcal{G}}(z,\zeta)\, [\phi(\zeta)]^{n+1}\, d\zeta, \quad z \in G_1.$$

Lemma 4.1.9. *The function $\mathcal{J}_n$ has the following properties:*

- *$\mathcal{J}_n(z)$ is analytic on G_1*
- *$\mathcal{J}_n(z) = \mathcal{P}_n(z)$ for each $z \in G_t$*
- *$\mathcal{J}_n(z) = \mathcal{P}_n(z)$ for each $z \in G_1 \cap \Omega_t$*

Thus $\mathcal{J}_n(z)$ is the analytic continuation of $\mathcal{P}_n(z)$ from $G_t \cup (G_1 \cap \Omega_t)$ to G_1.

Proof. The fact that $\mathcal{J}_n(z)$ is analytic on G_1 follows from Lemma 4.1.7 and Lemma 4.1.2.

To prove the second claim, fix some $z \in G_t$. Then we use Lemma 4.1.7 and Lemma 4.1.2 again to write

$$\begin{aligned}\mathcal{P}_n(z) &= -\sum_{k=0}^{\infty} f_{n,2k+1}(z) = \frac{1}{2\pi i} \oint_{L_t} \sum_{k=0}^{\infty} f_{n,2k}(\zeta)\, \mathcal{M}_{\mathcal{G}}(z,\zeta)\, [\phi(\zeta)]^{n+1}\, d\zeta \\ &= \frac{1}{2\pi i} \oint_{L_1} \sum_{k=0}^{\infty} f_{n,2k}(\zeta)\, \mathcal{M}_{\mathcal{G}}(z,\zeta)\, [\phi(\zeta)]^{n+1}\, d\zeta = \mathcal{J}_n(z).\end{aligned}$$

This establishes the second claim.

For the third claim, we fix some $z \in G_1 \cap \Omega_t$. To assist with notation, we write

$$F_n(\zeta) := [\phi(\zeta)]^{n+1} \sum_{k=0}^{\infty} f_{n,2k}(\zeta), \quad \zeta \in \Omega_r \setminus L_{1/t}.$$

By Lemmas 4.1.7 and 4.1.2, we see that $F_n(\zeta)\, \mathcal{M}_{\mathcal{G}}(z,\zeta)$, viewed as a function of ζ, is meromorphic in the intersection of $\Omega_r \setminus L_{1/t}$ and $G_{1/r}$, where its only singularity is a simple pole at the point $\zeta = z$. In fact, by relation (4.1.4), we may write

$$F_n(\zeta) \cdot \mathcal{M}_{\mathcal{G}}(z,\zeta) = F_n(\zeta) \left\{ \frac{\varphi'(\zeta)}{\varphi(\zeta)} \cdot \frac{\varphi(z)}{\varphi(\zeta) - \varphi(z)} + \varphi'(\zeta) \cdot \mathcal{M}^*_{\mathcal{D}}[\varphi(z), \varphi(\zeta)] \right\},$$

where $F_n(\zeta)$ and $\mathcal{M}^*_{\mathcal{D}}[\varphi(z), \varphi(\zeta)]$ are analytic in the intersection of $\Omega_r \setminus L_{1/t}$ and $G_{1/r}$. Therefore, we have

$$\mathrm{Res}_{\zeta=z}[F_n(\zeta)\, \mathcal{M}_{\mathcal{G}}(z,\zeta)] = \lim_{\zeta \to z}[(\zeta - z) \cdot F_n(\zeta) \cdot M_{\mathcal{G}}(z,\zeta)] = F_n(z).$$

Then we can compute

$$\begin{aligned} \mathcal{J}_n(z) &= \frac{1}{2\pi i} \oint_{L_1} F_n(\zeta)\, \mathcal{M}_{\mathcal{G}}(z,\zeta)\, d\zeta \\ &= \frac{1}{2\pi i} \oint_{L_t} F_n(\zeta)\, \mathcal{M}_{\mathcal{G}}(z,\zeta)\, d\zeta + \mathrm{Res}_{\zeta=z}[F_n(\zeta)\, \mathcal{M}_{\mathcal{G}}(z,\zeta)] \\ &= \frac{1}{2\pi i} \oint_{L_t} F_n(\zeta)\, \mathcal{M}_{\mathcal{G}}(z,\zeta)\, d\zeta + F_n(z) \\ &= -\sum_{k=0}^{\infty} f_{n,2k+1}(z) + [\phi(z)]^{n+1} \sum_{k=0}^{\infty} f_{n,2k}(z) = \mathcal{P}_n(z). \end{aligned}$$

This proves the third claim.

Note that the first two claims show that $\mathcal{J}_n(z)$ is the analytic continuation of $\mathcal{P}_n(z)$ from G_t to G_1, while claims one and three show that $\mathcal{J}_n(z)$ is the analytic continuation of $\mathcal{P}_n(z)$ from $G_1 \cap \Omega_t$ to G_1. This proves the lemma.

□

Lemma 4.1.10. *For $n > N_t$, the function $\mathcal{P}_n$ is a polynomial of degree $n+1$ with leading coefficient $[\phi'(\infty)]^{n+1}$.*

Proof. Lemmas 4.1.8 and 4.1.9 show that $\mathcal{P}_n$ admits an analytic continuation to the entire complex plane. By (4.1.19), there exists some $R > 0$ such that

$$\frac{|\mathcal{P}_n(z)|}{|z|^{n+1}} < 2[\phi'(\infty)]^{n+1}, \quad |z| > R.$$

If we define $M_k := \max\{|\mathcal{P}_n(z)| : z \in \overline{\mathbb{D}}_R\}$, then we will have

$$|\mathcal{P}_n(z)| \leq M_k + 2|z \cdot \phi'(\infty)|^{n+1}, \quad z \in \mathbb{C}.$$

By the Extended Liouville Theorem, we have that $\mathcal{P}_n(z)$ is a polynomial of degree at most $n+1$. We appeal to (4.1.19) once again to verify that the true degree of $\mathcal{P}_n$ is $n+1$ and that the leading coefficient is $[\phi'(\infty)]^{n+1}$.

□

4.2 Proof of Theorem 1.6.1.

The following uses a pair of ideas from the classical theory of orthogonal polynomials. If p_n denotes the orthonormal polynomial corresponding to P_n and if κ_n denotes the leading coefficient of p_n, then we have

$$\frac{1}{\kappa_n^2} = \frac{1}{\pi} \int_{\mathcal{G}} P_n(z)\, \overline{z}^n \, dA(z). \tag{4.2.1}$$

Also, in order to show that the polynomials $\mathcal{P}_n'(z)$ are orthogonal over $\mathcal{G}$ (for n large), it is sufficient to show that we have

$$\frac{1}{\pi} \int_{\mathcal{G}} \mathcal{P}_n'(z)\, \overline{z}^m \, dA(z), \quad 0 \leq m < n.$$

Proof. Let $n > N_t$ and let m be an integer satisfying $0 \leq m \leq n$. We define

$$J_m := \int_{\mathcal{G}} \mathcal{P}_n'(z)\, \overline{z}^m \, dA(z).$$

It follows from the proof of Lemma 4.1.9 that for any $\lambda \in (1, 1/t)$, we have

$$\mathcal{P}_n(z) = \frac{1}{2\pi i} \oint_{L_\lambda} \sum_{k=0}^{\infty} f_{n,2k}(\zeta)\, \mathcal{M}_{\mathcal{G}}(z,\zeta)\, [\phi(\zeta)]^{n+1}\, d\zeta, \quad z \in G_\lambda.$$

Fix a particular $\lambda \in (1, 1/t)$. Then by Lemma 4.1.5, we have

$$\mathcal{P}'_n(z) = \frac{1}{2\pi i} \oint_{L_\lambda} \sum_{k=0}^{\infty} f_{n,2k}(\zeta)\ \mathcal{L}_{\mathcal{G}}(z,\zeta)\ [\phi(\zeta)]^{n+1}\ d\zeta, \quad z \in G_\lambda.$$

Then with the help of Fubini's theorem, we have

$$\begin{aligned} J_m &= \frac{1}{2\pi i} \int_{\mathcal{G}} \left[\int_{L_\lambda} \sum_{k=0}^{\infty} f_{n,2k}(\zeta)\ \mathcal{L}_{\mathcal{G}}(z,\zeta)\ [\phi(\zeta)]^{n+1}\ d\zeta \right] \overline{z}^m\ dA(z) \\ &= \frac{1}{2\pi i} \int_{L_\lambda} \int_{\mathcal{G}} \sum_{k=0}^{\infty} f_{n,2k}(\zeta)\ \mathcal{L}_{\mathcal{G}}(z,\zeta)\ [\phi(\zeta)]^{n+1}\ \overline{z}^m\ dA(z)\ d\zeta \\ &= \frac{1}{2\pi i} \int_{L_\lambda} \sum_{k=0}^{\infty} f_{n,2k}(\zeta) [\phi(\zeta)]^{n+1} \left\{ \int_{\mathcal{G}} \mathcal{L}_{\mathcal{G}}(z,\zeta)\ \overline{z}^m\ dA(z) \right\} d\zeta. \end{aligned}$$

Defining

$$\nu_\varphi(\zeta) := \varphi^{-1}\left[\frac{1}{\overline{\varphi(\zeta)}} \right],$$

we call upon Lemmas 4.1.4 and 3.3.2 so that we may work on the integral

$$\begin{aligned} \int_{\mathcal{G}} \mathcal{L}_{\mathcal{G}}(z,\zeta)\ \overline{z}^m\ dA(z) &= \frac{\varphi'(\zeta)}{[\varphi(\zeta)]^2 \overline{\varphi'[\nu_\varphi(\zeta)]}} \int_{\mathcal{G}} \mathcal{K}_{\mathcal{G}}[z, \nu_\varphi(\zeta)]\ \overline{z}^m\ dA(z) \\ &= \frac{\varphi'(\zeta)}{[\varphi(\zeta)]^2 \overline{\varphi'[\nu_\varphi(\zeta)]}} \int_{\mathcal{G}} \overline{\mathcal{K}_{\mathcal{G}}[\nu_\varphi(\zeta), z]}\ \overline{z}^m\ dA(z) \\ &= \frac{\varphi'(\zeta)}{[\varphi(\zeta)]^2 \overline{\varphi'[\nu_\varphi(\zeta)]}} \overline{\int_{\mathcal{G}} \mathcal{K}_{\mathcal{G}}[\nu_\varphi(\zeta), z]\ z^m\ dA(z)} \\ &= \frac{\pi\ \varphi'(\zeta)\ \overline{[\nu_\varphi(\zeta)]}^m}{[\varphi(\zeta)]^2\ \overline{\varphi'[\nu_\varphi(\zeta)]}}, \end{aligned}$$

by Theorem 1.4.1. This computation lets us write

$$\begin{aligned} J_m &= \frac{1}{2i} \int_{L_\lambda} \sum_{k=0}^{\infty} f_{n,2k}(\zeta) [\phi(\zeta)]^{n+1} \frac{\varphi'(\zeta)\ \overline{[\tau_\varphi(\zeta)]}^m}{[\varphi(\zeta)]^2 \overline{\varphi'[\tau_\varphi(\zeta)]}}\ d\zeta \\ &= \frac{1}{2i} \int_{L_1} \sum_{k=0}^{\infty} f_{n,2k}(\zeta) [\phi(\zeta)]^{n+1} \frac{\varphi'(\zeta)\ \overline{\zeta}^m}{[\varphi(\zeta)]^2 \overline{\varphi'(\zeta)}}\ d\zeta, \end{aligned}$$

since, for $\zeta \in L_1$, we have $\tau_\varphi(\zeta) = \zeta$. Now we use the differential relationship from Section 8.4.1 to deduce that

$$\begin{aligned} J_m = & -\frac{1}{2i}\int_{L_1}\sum_{k=0}^{\infty} f_{n,2k}(\zeta)[\phi(\zeta)]^{n+1}\overline{\zeta}^m\overline{d\zeta} \\ &= -\frac{1}{2i}\overline{\int_{L_1}\sum_{k=0}^{\infty}\overline{f_{n,2k}(\zeta)}\,\overline{\phi(\zeta)}^{n+1}\zeta^m d\zeta}. \end{aligned}$$

From the definition of $f_{n,2k}$, we see that

$$f_{n,2k}(\psi(w)) = \frac{1}{2\pi i}\oint_{\mathbb{T}_{1/t}} \frac{f_{n,2k-1}(\psi(\xi))\,\xi^{-n-1}}{\xi - w}\,d\xi, \quad k \geq 1,$$

is analytic on $\hat{\mathbb{C}} \setminus \mathbb{T}_{1/t}$. Then, for each $k \geq 0$, we have

$$\begin{aligned} \oint_{L_1}\overline{f_{n,2k}(\zeta)}\,\overline{\phi(\zeta)}^{n+1}\zeta^m d\zeta &= \oint_{\mathbb{T}_1}\overline{f_{n,2k}[\psi(w)]}\,\overline{w}^{n+1}\,[\psi(w)]^m\,\psi'(w)\,dw \\ &= -\overline{\oint_{\mathbb{T}_1} f_{n,2k}[\psi(w)]\,w^{n-1}\,\overline{[\psi(w)]^m}\,\overline{\psi'(w)}\,dw} \\ &= \begin{cases} 0, & 0 \leq m < n \\ \overline{-2\pi i f_{n,2k}[\psi(0)]\cdot[\psi'(\infty)]^{n+1}}, & m = n. \end{cases} \\ &= \begin{cases} 0, & 0 \leq m < n \\ 2\pi i[\psi'(\infty)]^{n+1}\overline{f_{n,2k}[\psi(0)]} & m = n \end{cases} \end{aligned}$$

by Lemmas 8.2.3 and 8.4.2. Thus $J_m = 0$ for $0 \leq m < n$, whence

$$P_n(z) = \frac{\mathcal{P}_n'(z)}{(n+1)[\phi'(\infty)]^{n+1}} \tag{4.2.2}$$

by Lemma 4.1.10. When $m = n$, we have

$$J_n = -\frac{1}{2i}\overline{\int_{L_1}\sum_{k=0}^{\infty}\overline{f_{n,2k}(\zeta)}\,\overline{\phi(\zeta)}^{n+1}\zeta^n d\zeta}$$

$$
\begin{aligned}
&= -\frac{1}{2i}\sum_{k=0}^{\infty}\overline{2\pi i[\psi'(\infty)]^{n+1}\overline{f_{n,2k}[\psi(0)]}} \\
&= \pi[\psi'(\infty)]^{n+1}\sum_{k=0} f_{n,2k}[\psi(0)] \\
&= \pi[\psi'(\infty)]^{n+1}\cdot\left[1+\sum_{k=0}\frac{1}{2\pi i}\oint_{\mathbb{T}_{1/t}}\frac{f_{n,2k+1}(\psi(\xi))\ \xi^{-n-1}}{\xi}\right] \\
&= \pi\frac{1}{[\phi'(\infty)]^{n+1}}\cdot\left[1+\sum_{k=0}\frac{1}{2\pi i}\oint_{L_{1/t}} f_{n,2k+1}(\zeta)\ \phi'(\zeta)\ [\phi(\zeta)]^{-n-2}\ d\zeta\right].
\end{aligned}
$$

Then by relation(4.2.1), relation (4.2.2), the definition of J_n, and the work above, we have

$$
\begin{aligned}
\frac{1}{\kappa_n^2} &= \frac{1}{\pi}\int_{\mathcal{G}} P_n(z)\ \overline{z}^n\ dA(z) \\
&= \frac{1}{(n+1)\cdot[\phi'(\infty)]^{n+1}}\cdot\frac{1}{\pi}\int_{\mathcal{G}}\mathcal{P}_n'(z)\ \overline{z}^n\ dA(z) \\
&= \frac{1}{(n+1)\cdot[\phi'(\infty)]^{n+1}}\cdot\frac{J_n}{\pi} \\
&= \frac{1}{(n+1)\cdot[\phi'(\infty)]^{2n+2}}\cdot\left[1+\sum_{k=0}\frac{1}{2\pi i}\oint_{L_{1/t}} f_{n,2k+1}(\zeta)\ \phi'(\zeta)\ [\phi(\zeta)]^{-n-2}\ d\zeta\right],
\end{aligned}
$$

whence

$$
(n+1)[\phi'(\infty)]^{2n+2}\kappa_n^{-2} = 1+\sum_{k=0}^{\infty}\frac{1}{2\pi i}\oint_{L_{1/t}} f_{n,2k+1}(\zeta)\phi'(\zeta)[\phi(\zeta)]^{-n-2}\ d\zeta.
$$

This completes the proof.

□

4.3 Proof of Theorem 1.6.2

Proof. By definition,

$$
f_{n,2k+2}(\zeta) = \frac{1}{2\pi i}\oint_{L_{1/t}}\frac{f_{n,2k+1}(s)\ \phi'(s)\ [\phi(s)]^{-n-1}ds}{\phi(s)-\phi(\zeta)}, \quad \zeta\in\Omega_r\setminus L_{1/t}.
$$

Making the change of variables $\zeta = \psi(w)$ and $s = \psi(\xi)$ in this integral yields

$$(f_{n,2k+2} \circ \psi)(w) = \frac{1}{2\pi i} \int_{\mathbb{T}_{1/t}} \frac{f_{n,2k+1}[\psi(\xi)]\, t^{-n-1}}{\xi - w}\, d\xi, \qquad w \in \Delta_r \setminus \mathbb{T}_{1/t}.$$

This shows that the function $f_{n,2k+2} \circ \psi$ has an analytic continuation $f^*_{n,2k+2}(w)$ to $\hat{\mathbb{C}} \setminus \mathbb{T}_{1/t}$ given by the very same formula:

$$f^*_{n,2k+2}(w) := \frac{1}{2\pi i} \int_{\mathbb{T}_{1/t}} \frac{f_{n,2k+1}[\psi(\xi)]\, t^{-n-1}}{\xi - w}\, d\xi, \qquad |w| \neq 1/t. \tag{4.3.1}$$

If we define

$$F_n(w) := \sum_{k=0}^{\infty} f^*_{n,2k+2}(w), \qquad |w| \neq 1/t, \qquad n \geq 0,$$

then by (4.3.1) and Lemma (4.1.6), we have

$$F_n(w) = O(t^{2n}) \qquad \text{and} \qquad F'_n(w) = O(t^{2n})$$

locally uniformly as $n \to \infty$ on $|w| \neq 1/t$.

By Theorem 1.6.1, for $z \in G_t$ and n large, we have

$$\begin{aligned}(n+1)[\phi'(\infty)]^{n+1} P_n(z) &= -\sum_{k=0}^{\infty} f'_{n,2k+1}(z) \\ &= \frac{1}{2\pi i} \int_{L_t} \sum_{k=0}^{\infty} f_{n,2k}(\zeta)\, L_{\mathcal{G}}(z,\zeta)\, [\phi(\zeta)]^{n+1}\, d\zeta.\end{aligned}$$

For $z \in G_t$, we can change the contour of integration from L_t to L_1 without changing the value of the latter integral. This will leave a function that is analytic in G_1, and by uniqueness of the analytic continuation, we must have

$$(n+1)[\phi'(\infty)]^{n+1} P_n(z) = \frac{1}{2\pi i} \int_{L_1} \sum_{k=0}^{\infty} f_{n,2k}(\zeta)\, L_{\mathcal{G}}(z,\zeta)\, [\phi(\zeta)]^{n+1}\, d\zeta, \quad z \in G_1. \tag{4.3.2}$$

If we define

$$\tilde{M}_{\mathcal{G}}(z,\zeta) := -\sum_{\tau\in\mathcal{T}} \frac{\frac{d}{dz}\{\tau[\varphi(z)]\}}{\varphi(\zeta) - (\tau\circ\varphi)(z)},$$

then we have the relation

$$\frac{\partial}{\partial\zeta}\tilde{M}_{\mathcal{G}}(z,\zeta) = L_{\mathcal{G}}(z,\zeta),$$

and so, by the chain rule, we have

$$\frac{\partial}{\partial w}\tilde{M}_{\mathcal{G}}[z,\psi(w)] = L_{\mathcal{G}}[z,\psi(w)]\ \psi'(w).$$

Hence, making the change of variables $\zeta = \psi(w)$ and integrating by parts in (4.3.2), we obtain

$$(n+1)[\phi'(\infty)]^{n+1}P_n(z) = -\frac{1}{2\pi i}\int_{\mathbb{T}_1} \tilde{M}_{\mathcal{G}}[z,\psi(w)][(1+F_n(w))w^{n+1}]'\, dw, \quad z\in G_1.$$

We can now write

$$[(1+F_n(w))w^{n+1}]' = (n+1)w^n\left[1+K_n(w)\right],$$

with

$$K_n(w) = F_n(w) + \frac{wF_n'(w)}{n+1},$$

so that

$$P_n(z) = -\frac{[\phi'(\infty)]^{-n-1}}{2\pi i}\int_{\mathbb{T}_1} w^n\left[1+K_n(w)\right]\tilde{M}_{\mathcal{G}}[z,\psi(w)]\, dw, \quad z\in G_1,$$

proving the theorem. □

5 THE PROOF OF THEOREM 1.7.2

Let $\mathcal{D}$ be a CMCD for which Assumption 1.3.1 holds. By choosing a larger $\rho \in (0,1)$ if necessary, we can guarantee that the following conditions are satisfied:

- $\sum_{\tau\in\mathcal{T}} |\tau'|$ converges uniformly on every compact subset of $\mathbb{D}_{1/\rho}$,
- $\mathbb{D}_{1/\rho} \subset \bigcap_{j\in\Lambda_s} \chi_j(\mathbb{D}_{1/\sigma_j})$, and
- $\bigcup_{j\in\Lambda_s} D_{c_j,r_j} \subset \mathbb{D}_\rho$.

Let $\{P_n\}_{n=1}^{\infty}$ denote the monic polynomials orthogonal over $\mathcal{D}$. Let $t \in (\rho, 1)$. By Corollary 1.7.1, for n sufficiently large, we have

$$P_n(z) = \sum_{\tau\in\mathcal{T}} [\tau(z)]^n \cdot \tau'(z) \cdot [1 + (K_n \circ \tau)(z)], \quad z \in \mathbb{D}, \tag{5.0.1}$$

where $K_n(\zeta)$ is analytic in $|\zeta| < 1/t$ and $K_n(\zeta) = O(t^{2n})$ locally uniformly as $n \to \infty$ in $|\zeta| < 1/t$. Here we show how equation (5.0.1) leads to the formula

$$P_n(z) = z^n \cdot [1 + K_n(z)] + \sum_{j=1}^{s} P_n(T_j(z)) \cdot T_j'(z), \quad z \in \mathbb{D}. \tag{5.0.2}$$

Since we have $T_j(\mathbb{D}) \subset \mathbb{D}$ for each $j \in \Lambda_s$, equation (5.0.1) implies that, for each $j \in \Lambda_s$, we have

$$P_n(T_j(z)) = \sum_{\tau\in\mathcal{T}} [(\tau \circ T_j)(z)]^n \cdot \tau'(T_j(z)) \cdot [1 + (K_n \circ \tau \circ T_j)(z)], \quad z \in \mathbb{D}. \tag{5.0.3}$$

Meanwhile, since the only transformation in $\mathcal{T} \setminus \mathcal{T}^*$ is the identity function, equation (5.0.1) also implies that we may write

$$P_n(z) = z^n \cdot [1 + K_n(z)] + \sum_{\tau \in \mathcal{T}^*} [\tau(z)]^n \cdot \tau'(z) \cdot [1 + (K_n \circ \tau)(z)], \quad z \in \mathbb{D}. \tag{5.0.4}$$

Finally, by (2.2.2), we have $\mathcal{T}^* = \bigcup_{j=1}^{s} \{\tau T_j : \tau \in \mathcal{T}\}$. Therefore, we may also write

$$P_n(z) = z^n \cdot [1 + K_n(z)] + \sum_{j=1}^{s} \sum_{\tau \in \mathcal{T}} [(\tau \circ T_j)(z)]^n \cdot (\tau \circ T_j)'(z) \cdot [1 + (K_n \circ \tau \circ T_j)(z)].$$

Combining this equation with equation (5.0.3) and the chain rule yields (5.0.2).

In order to facilitate some of the arguments which will appear in this chapter, we fix a positive constant M for which the following estimate holds:

$$|K_n(z)| \leq M t^{2n}, \quad z \in \overline{\mathbb{D}}, \quad n \in \mathbb{N}. \tag{5.0.5}$$

Recall the definition of α:

$$\alpha := \max\{|a_j| : j \in \Lambda_s.\}$$

The main purpose of this chapter is to analyze the behavior of $P_n(z)$ for $z \in \mathbb{C}$. We will begin this study by analyzing the behavior of $P_n(z)$ for z satisfying $\alpha < |z| < 1$. This is done in Section 5.1. The main result of that section is Proposition 5.1.2. Then we use Proposition 5.1.2 to describe the behavior of $P_n(z)$ for any z satisfying $|z| \geq 1$. The result is recorded in Proposition 5.2.1. In Section 5.3, we analyze the behavior of $P_n(z)$ for $z \in \mathbb{D}_\alpha$. The main result of that Section is Proposition 5.3.1. Finally, in section 5.4, we address the behavior of $P_n(z)$ for $z \in \mathbb{T}_\alpha$, where the main result is Proposition 5.4.1. Then Theorem 1.7.2 is the consolidation of Propositions 5.1.2, 5.2.1, 5.3.1, and 5.4.1.

5.1 On the behavior of $P_n(z)$ for z satisfying $\alpha < |z| < 1$

For every $r \in (0, 1/\alpha)$ and for every $j \in \Lambda_s$, we define the constant

$$\rho_j(r) := \max_{z \in \overline{\mathbb{D}}_r} |T_j(z)| \qquad \text{and} \qquad M_r := \max_{j \in \Lambda_s} \rho_j(r).$$

We comment that, if θ_j denotes the principal argument of a_j, then we have

$$\rho_j(r) = |T_j(re^{i\theta_j})|, \quad j \in \Lambda_s.$$

Lemma 5.1.1. *For every r satisfying $\alpha < r < 1/\alpha$ and for every $\tau \in \mathcal{T}^*$, we have*

$$|\tau(z)| \leq M_r < r, \quad z \in \overline{\mathbb{D}}_r.$$

Proof. Let r satisfy $\alpha < r < 1/\alpha$. First we show that we have

$$M_r < r. \tag{5.1.1}$$

Note that relationship (2.1.11) gives

$$T_j(\overline{\mathbb{D}}_r) \subset \mathbb{D}_r, \quad j \in \Lambda_s.$$

In other words, we have

$$|T_j(z)| < r, \quad (j, z) \in \Lambda_s \times \overline{\mathbb{D}}_r. \tag{5.1.2}$$

Then, by the definition of $\rho_j(r)$, we have

$$\rho_j(r) < r, \quad j \in \Lambda_s.$$

Then the definition of M_r yields (5.1.1).

Now let $\tau \in \mathcal{T}^*$ and let $z \in \overline{\mathbb{D}}_r$. We will show that we have

$$|\tau(z)| \leq M_r.$$

We induct on the length k of τ. If $k = 1$, then $\tau = T_j$ for some $j \in \Lambda_s$. Therefore, we have

$$|\tau(z)| = |T_j(z)| \leq \rho_j(r) \leq M_r$$

by the definitions of $\rho_j(r)$ and M_r. Now we suppose that we have $|\tau_k(z)| \leq M_r$ for each string τ_k of length k. Let τ be a string of length $k+1$. Then we may write $\tau = T_j \tau_k$ for some $j \in \Lambda_s$. By the induction hypothesis, we have $|\tau_k(z)| \leq M_r$. In other words, we have $\tau_k(z) = w$ for some $w \in \overline{\mathbb{D}}_{M_r}$. Then relationship (5.1.1) tells us that we also have $w \in \mathbb{D}_r$. Therefore

$$|\tau(z)| = |T_j(\tau_k(z))| = |T_j(w)| \leq \rho_j(r) \leq M_r$$

by the definitions of $\rho_j(r)$ and M_r. The proof is complete by the principle of mathematical induction. $\square$

Proposition 5.1.2. *Let $z \in \mathbb{T}_r$ for some r satisfying $\alpha < r < 1$. Define*

$$\eta := \max\{rt^2, M_r\}.$$

Then we have $0 < \eta < r$ and

$$P_n(z) = z^n + O(\eta^n), \quad n \to \infty.$$

Proof. It follows from (5.0.4) that we may write

$$P_n(z) - z^n = z^n \cdot K_n(z) + \sum_{\tau \in \mathcal{T}^*} [\tau(z)]^n \cdot \tau'(z) \cdot [1 + (K_n \circ \tau)(z)]. \tag{5.1.3}$$

First, we note that

$$z^n \cdot K_n(z) = O((rt^2)^n). \tag{5.1.4}$$

Indeed, it follows from the estimate (5.0.5) and from the fact that $z \in \mathbb{T}_r$ that we may write

$$|z^n \cdot K_n(z)| = r^n \cdot |K_n(z)| \leq r^n \cdot Mt^{2n} = M(rt^2)^n, \quad n \to \infty.$$

Next, it follows from Lemma 2.1.1 that for every $\tau \in \mathfrak{T}^*$, we have $\tau(\mathbb{D}) \subset \mathbb{D}$. Combining this with the estimate (5.0.5) gives

$$|(K_n \circ \tau)(z)| \leq M, \quad z \in \mathbb{D}, \quad \tau \in \mathfrak{T}^*, \quad n \in \mathbb{N}.$$

Furthermore, Lemma 5.1.1 gives us the estimate

$$|\tau(z)| < M_r, \quad z \in \mathbb{T}_r, \quad \tau \in \mathfrak{T}n$$

Therefore, we may write

$$\begin{aligned}\left|\sum_{\tau \in \mathfrak{T}^*} [\tau(z)]^n \cdot \tau'(z) \cdot [1 + (K_n \circ \tau)(z)]\right| &\leq (1+M) \cdot \sum_{\tau \in \mathfrak{T}^*} |\tau(z)|^n \cdot |\tau'(z)| \\ &\leq (1+M) \cdot (M_r)^n \cdot \sum_{\tau \in \mathfrak{T}^*} |\tau'(z)| \\ &\leq (1+M) \cdot (M_r)^n \cdot K\end{aligned}$$

for some $K \in (0, \infty)$ by Assumption 1.3.1. In other words, we have

$$\sum_{\tau \in \mathfrak{T}^*} [\tau(z)]^n \cdot \tau'(z) \cdot [1 + (K_n \circ \tau)(z)] = O((M_r)^n), \quad n \to \infty. \tag{5.1.5}$$

Since $t \in (\rho, 1)$, we have $0 < rt^2 < r$. Meanwhile, Lemma 5.1.1 gives $0 \leq M_r < r$. Then by the definition of η, we have $0 < \eta < r$. Then the proposition follows from (5.1.3), (5.1.4), and (5.1.5). $\square$

5.2 On the behavior of $P_n(z)$ for $|z| \geq 1$

Proposition 5.2.1. *Let $z \in \mathbb{T}_r$ for some r satisfying $r \geq 1$. There exists some $\nu \in (0, r)$ such that*

$$P_n(z) = z^n + O(\nu^n), \quad n \to \infty.$$

Proof. Let $\varsigma \in (\alpha, 1)$. By Proposition 5.1.2, there exists some $\eta \in (0, \varsigma)$ such that

$$P_n(z) = z^n + O(\eta^n)$$

for $z \in \mathbb{T}_\varsigma$. Then there exists some $M_\varsigma \in (0, \infty)$ such that

$$\left| \frac{P_n(z)}{z^n} - 1 \right| \leq M_\varsigma \left(\frac{\eta}{\varsigma} \right)^n \tag{5.2.1}$$

for every $z \in \mathbb{T}_\varsigma$ and for every $n \in \mathbb{N}$. Note that, for every $n \in \mathbb{N}$, the function

$$F_n(z) = \frac{P_n(z)}{z^n} - 1$$

is analytic in $\hat{\mathbb{C}} \setminus \{0\}$. Then by the Maximum Modulus Principle, we have

$$\max_{z \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}_\varsigma} |F_n(z)| \leq \max_{z \in \hat{\mathbb{C}} \setminus \overline{\mathbb{T}}_\varsigma} |F_n(z)|$$

for every $n \in \mathbb{N}$. Therefore, the estimate (5.2.1) actually holds for every $z \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}_\varsigma$ and for every $n \in \mathbb{N}$. Hence, for every $z \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}_\varsigma$ and for every $n \in \mathbb{N}$, we have

$$|P_n(z) - z^n| \leq M_\varsigma \left(\frac{\eta \cdot |z|}{\varsigma} \right)^n.$$

Therefore, if $z \in \mathbb{T}_r$ with $r \geq 1$, then we may write

$$|P_n(z) - z^n| \leq M_\varsigma \left(\frac{\eta \cdot r}{\varsigma}\right)^n = M_\varsigma \nu^n, \quad \nu := \frac{\eta r}{\varsigma}$$

for every $n \in \mathbb{N}$. Since we have $0 < \eta < \varsigma$, this gives

$$P_n(z) = z^n + O(\nu^n)$$

with $0 < \nu < r$, as claimed.

$\square$

5.3 On the behavior of $P_n(z)$ for $z \in \mathbb{D}_\alpha$

Now we define

$$\mathfrak{T}_j := \{T_j \tau : \tau \in \mathfrak{T}\}.$$

These are all of the strings in $\mathfrak{T}$ whose terminal operator is T_j. Note that we have

$$\mathfrak{T}^* = \bigcup_{j \in \Lambda_s} \mathfrak{T}_j,$$

with the union being disjoint. Then we may write

$$P_n(z) = z^n \cdot [1 + K_n(z)] + \sum_{j=1}^{s} \sum_{\tau \in \mathfrak{T}_j} [\tau(z)]^n \cdot \tau'(z) \cdot [1 + (K_n \circ \tau)(z)], \quad z \in \mathbb{D}.$$

After possible relabeling, we may assume that there exists some $\omega \in \Lambda_s$ such that

$$\alpha = |a_1| = |a_2| = \cdots = |a_{\omega-1}| = |a_\omega| > |a_j|, \quad \omega < j \leq s.$$

For $z \in \mathbb{D}$ and $n \in \mathbb{N}$, we define

$$\mathcal{X}_n(z) := z^n \cdot [1 + K_n(z)],$$

$$\mathcal{Y}_n(z) := \sum_{j=1}^{\omega} \sum_{\tau \in \mathcal{T}_j} [\tau(z)]^n \cdot \tau'(z) \cdot [1 + (K_n \circ \tau)(z)],$$

and

$$\mathcal{Z}_n(z) := \sum_{j=\omega+1}^{s} \sum_{\tau \in \mathcal{T}_j} [\tau(z)]^n \cdot \tau'(z) \cdot [1 + (K_n \circ \tau)(z)].$$

Then we may write

$$P_n(z) = \mathcal{X}_n(z) + \mathcal{Y}_n(z) + \mathcal{Z}_n(z). \tag{5.3.1}$$

First we describe the behavior of $P_n(z)$ for $z \in \mathbb{D}_\alpha$. If $z \in \mathbb{D}_\alpha$, then there exists some $r \in [0, \alpha)$ such that $z \in \mathbb{T}_r$. Then we have

$$\mathcal{X}_n(z) = O(r^n) \tag{5.3.2}$$

since, by the estimate (5.0.5), we have

$$|\mathcal{X}_n(z)| = |z|^n \cdot |[1 + K_n(z)]| \leq r^n \cdot (1 + M), \quad n \in \mathbb{N}.$$

Meanwhile, in Section 5.3.1, we show that there exists some $\nu \in (0, \alpha)$ such that

$$\mathcal{Z}_n(z) = O(\nu^n) \tag{5.3.3}$$

for $z \in \overline{\mathbb{D}}_\alpha$ as $n \to \infty$.

To describe the behavior of $\mathcal{Y}_n(z)$ for $z \in \mathbb{D}_\alpha$, we use the notation

$$\Lambda_\omega = \{1, 2, \dots \omega\}.$$

Then for each $j \in \Lambda_\omega$, we define

$$\mathcal{T}_j := \{T_j\tau : \tau \in \mathcal{T}\}.$$

Next, for each $j \in \Lambda_\omega$, let θ_j denote the principal argument of a_j. We define the sets

$$\mathcal{H} := \{z : \mathrm{Re}(z) > 0\} \qquad \text{and} \qquad \mathcal{H}e^{i\theta_j} := \{ze^{i\theta_j} : z \in \mathcal{H}\}$$

for each $j \in \Lambda_\omega$. Next, we define the constants

$$\beta_j := \frac{1}{\overline{a_j}} - a_j, \quad j \in \Lambda_\omega,$$

and the functions

$$\Theta_j(t) = t \sum_{v \in \mathbb{Z}} \sigma_j^{2v} \exp(-\overline{\beta_j}\sigma_j^{2v} t), \quad t \in \mathcal{H}e^{i\theta_j}, \quad j \in \Lambda_\omega.$$

Finally, we define

$$\mathcal{J}_{j,n}(z) := \sum_{\tau \in \mathcal{T}\setminus\mathcal{T}_j} (\alpha^2 - 1) \cdot e^{ni\theta_j} \cdot \frac{\Phi_j'(\tau(z))}{\Phi_j(\tau(z))} \cdot \Theta_j(-n\Phi_j(\tau(z))) \cdot \tau'(z), \quad j \in \Lambda_\omega,$$

In Section 5.3.2, we show that

$$\mathcal{Y}_n(z) = \frac{\alpha^n}{n} \cdot \sum_{j=1}^{\omega} \mathcal{J}_{j,n}(z) + O\left(\frac{\alpha^n}{n^2}\right) \tag{5.3.4}$$

normally for $z \in \mathbb{D}_\alpha$ as $n \to \infty$. Then the following proposition is an immediate consequence of relationships (5.3.1), (5.3.2), (5.3.3), and (5.1.5).

Proposition 5.3.1. *We have*

$$P_n(z) = \frac{\alpha^n}{n} \cdot \sum_{j=1}^{\omega} \mathcal{J}_{j,n}(z) + O\left(\frac{\alpha^n}{n^2}\right)$$

normally for $z \in \mathbb{D}_\alpha$ as $n \to \infty$.

To prove Proposition 5.3.1, we must establish relationships (5.3.3) and (5.1.5). As mentioned above, we prove (5.3.3) in Section 5.3.1 and we prove (5.1.5) in Section 5.3.2.

5.3.1 On the function $\mathcal{Z}_n(z)$ for $z \in \overline{\mathbb{D}}_\alpha$

Define

$$\nu := \max_{\omega < j \leq s} \rho_j(\alpha).$$

We will see in the proof of the following lemma that we have

$$0 < \nu < \alpha.$$

Lemma 5.3.2. *There exists some $K \in (0, \infty)$ such that, for every $n \in \mathbb{N}$, we have*

$$\left| \sum_{j=\omega+1}^{s} \sum_{\tau \in \mathcal{T}_j} [\tau(z)]^n \cdot \tau'(z) \cdot [1 + (K_n \circ \tau)(z)] \right| \leq K\nu^n.$$

Proof. We begin by writing

$$\sum_{j=\omega+1}^{s} \sum_{\tau \in \mathcal{T}_j} [\tau(z)]^n \cdot \tau'(z) \cdot [1 + (K_n \circ \tau)(z)]$$
$$= \sum_{j=\omega+1}^{s} \sum_{\tau \in \mathcal{T}} [(T_j \circ \tau)(z)]^n \cdot (T_j \circ \tau)'(z) \cdot [1 + (K_n \circ T_j \circ \tau)(z)].$$

It follows from the definition of $\rho_j(\alpha)$ given at the beginning of section 5.1 that, for each $j \in \Lambda_s$, we have

$$|T_j(z)| \leq \rho_j(\alpha), \quad z \in \overline{\mathbb{D}}_\alpha. \tag{5.3.5}$$

Now, since we have $|a_j| < \alpha$ for each j satisfying $\omega < j \leq s$, we note that relationship (2.1.11) gives

$$T_j(\overline{\mathbb{D}}_\alpha) \subset \mathbb{D}_\alpha, \quad \omega < j \leq s.$$

In other words, we have

$$|T_j(z)| < \alpha, \quad z \in \overline{\mathbb{D}}_\alpha \tag{5.3.6}$$

for each $\omega < j \leq s$. Then we have

$$\rho_j(\alpha) < \alpha, \quad \omega < j \leq s. \tag{5.3.7}$$

Note that this gives

$$0 < \nu < \alpha.$$

Meanwhile, Lemma 2.2.2 tells us that we have

$$\tau(z) \in \overline{\mathbb{D}}_\alpha, \quad \tau \in \mathfrak{T}, \quad z \in \overline{\mathbb{D}}_\alpha.$$

Combining this with (5.3.5) and (5.3.7) gives

$$(T_j \circ \tau)(z) \leq \rho_j(\alpha) < \alpha, \quad \omega < j \leq s,$$

for every $z \in \overline{\mathbb{D}}_\alpha$. Thus, for every $\omega < j \leq s$ and for every $z \in \overline{\mathbb{D}}_\alpha$, we have

$$\begin{aligned} &\left|\sum_{\tau \in \mathfrak{T}} [(T_j \circ \tau)(z)]^n \cdot (T_j \circ \tau)'(z) \cdot [1 + (K_n \circ T_j \circ \tau)(z)]\right| \\ &\qquad\qquad \leq (1 + M) \cdot \sum_{\tau \in \mathfrak{T}} |[(T_j \circ \tau)(z)]^n| \cdot |(T_j \circ \tau)'(z)| \end{aligned}$$

$$
\begin{aligned}
&\leq (1+M)\cdot[\rho_j(\alpha)]^n \sum_{\tau\in\mathcal{T}} |(T_j\circ\tau)'(z)| \\
&= (1+M)\cdot[\rho_j(\alpha)]^n \sum_{\tau\in\mathcal{T}_j} |\tau'(z)|
\end{aligned}
$$

Therefore, for every $z \in \overline{\mathbb{D}}_\alpha$, we have

$$
\begin{aligned}
&\left|\sum_{j=\omega+1}^{s}\sum_{\tau\in\mathcal{T}}[(T_j\circ\tau)(z)]^n\cdot(T_j\circ\tau)'(z)\cdot[1+(K_n\circ T_j\circ\tau)(z)]\right| \\
&\qquad = (1+M)\cdot\sum_{j=\omega+1}^{s}[\rho_j(\alpha)]^n\sum_{\tau\in\mathcal{T}_j}|\tau'(z)| \\
&\qquad \leq (1+M)\cdot\nu^n\cdot\sum_{j=\omega+1}^{s}\sum_{\tau\in\mathcal{T}_j}|\tau'(z)| \\
&\qquad \leq (1+M)\cdot\nu^n\cdot K_0
\end{aligned}
$$

for some constant $K_0 \in (0,\infty)$, by Assumption 1.3.1. By setting

$$
K = (1+M)\cdot K_0,
$$

the proof is complete. $\square$

5.3.2 On the function $\mathcal{Y}_n(z)$ for $z \in \mathbb{D}_\alpha$

In this section, we examine the function

$$
\mathcal{Y}_n(z) = \sum_{j=1}^{\omega}\sum_{\tau\in\mathcal{T}_j}[\tau(z)]^n\cdot\tau'(z)\cdot[1+(K_n\circ\tau)(z)], \quad z\in\mathbb{D}.
$$

To be more precise, we will show that we have

$$
\mathcal{Y}_n(z) = \frac{\alpha^n}{n}\cdot\sum_{j=1}^{\omega}\mathcal{J}_{j,n}(z) + O\left(\frac{\alpha^n}{n^2}\right)
$$

normally for $z \in \mathbb{D}_\alpha$ as $n \to \infty$. First, we write

$$\sum_{\tau \in \mathcal{T}_j} [\tau(z)]^n \cdot \tau'(z) \cdot [1 + (K_n \circ \tau)(z)] = \sum_{\tau \in \mathcal{T}_j} [\tau(z)]^n \cdot \tau'(z) + \sum_{\tau \in \mathcal{T}_j} [\tau(z)]^n \cdot \tau'(z) \cdot (K_n \circ \tau)(z) \quad (5.3.8)$$

We claim that, for $z \in \overline{\mathbb{D}}_\alpha$, we have

$$\sum_{\tau \in \mathcal{T}_j} [\tau(z)]^n \cdot \tau'(z) \cdot (K_n \circ \tau)(z) = O((\alpha t^2)^n), \quad n \to \infty. \quad (5.3.9)$$

Indeed, by Lemma 2.2.2, we have

$$|\tau(z)| \leq \alpha, \quad z \in \overline{\mathbb{D}}_\alpha, \quad \tau \in \mathcal{T}.$$

Therefore, by the estimate (5.0.5), we have for all $z \in \overline{\mathbb{D}}_\alpha$ and $n \in \mathbb{N}$, the estimate

$$\begin{aligned}
\left| \sum_{\tau \in \mathcal{T}_j} [\tau(z)]^n \cdot \tau'(z) \cdot (K_n \circ \tau)(z) \right| &\leq \sum_{\tau \in \mathcal{T}_j} |\tau(z)|^n \cdot |\tau'(z)| \cdot |(K_n \circ \tau)(z)| \\
&\leq \alpha^n \cdot M t^{2n} \cdot \sum_{\tau \in \mathcal{T}_j} |\tau'(z)| \\
&\leq \alpha^n \cdot M t^{2n} \cdot K
\end{aligned}$$

for some constant $K \in (0, \infty)$ by Assumption 1.3.1. This establishes relation (5.3.9).

Recall that $\mathcal{T}_j$ denotes the set of all transformations whose terminal operator is T_j. Then the set $\mathcal{T} \setminus \mathcal{T}_j$ is the collection of all the transformations with a terminal operator different from T_j, together with the identity transformation $T_0(z) = z$. Note that we have

$$\mathcal{T}_j = \bigcup_{v=1}^{\infty} \{T_j^v \tau : \tau \in \mathcal{T} \setminus \mathcal{T}_j\}.$$

Therefore, for each $j \in \Lambda_\omega$, we can write

$$\begin{aligned}
\sum_{\tau \in \mathcal{T}_j} [\tau(z)]^n \cdot \tau'(z) &= \sum_{\tau \in \mathcal{T} \setminus \mathcal{T}_j} \sum_{v=1}^{\infty} [(T_j^v \circ \tau)(z)]^n \cdot (T_j^v \circ \tau)'(z) \\
&= \sum_{\tau \in \mathcal{T} \setminus \mathcal{T}_j} \sum_{v=0}^{\infty} [(T_j^v \circ T_j \circ \tau)(z)]^n \cdot (T_j^v \circ T_j \circ \tau)'(z)) \\
&= \sum_{\tau \in \mathcal{T} \setminus \mathcal{T}_j} \sum_{v=0}^{\infty} [(T_j^v \circ T_j \circ \tau)(z)]^n \cdot (T_j^v)'[(T_j \circ \tau)(z)] \cdot (T_j \circ \tau)'(z).
\end{aligned}$$

Defining the function

$$\mathcal{R}_{j,n}(z) := \sum_{v=0}^{\infty} [T_j^v(z)]^n \cdot (T_j^v)'(z), \quad z \in \mathbb{D}, \quad j \in \Lambda_\omega,$$

we have

$$\sum_{\tau \in \mathcal{T}_j} [\tau(z)]^n \cdot \tau'(z) = \sum_{\tau \in \mathcal{T} \setminus \mathcal{T}_j} \mathcal{R}_{j,n}[(T_j \circ \tau)(z)] \cdot (T_j \circ \tau)'(z) \tag{5.3.10}$$

In section 8.4.3, we prove the following proposition, which is a generalization of a Lemma from the paper [6].

Proposition 5.3.3. *For each $j \in \Lambda_\omega$, we have*

$$\mathcal{R}_{j,n}(z) = \frac{\Phi_j'(z)}{\Phi_j(z)} \cdot (\alpha^2 - 1) \cdot \frac{a_j^n}{n} \cdot \Theta_j(-n\Phi_j(z)) + O\left(\frac{\alpha^n}{n^2}\right)$$

normally for $z \in \mathbb{D}_\alpha$ as $n \to \infty$.

Next, we define the set of transformations

$$\mathcal{T}_j^* : \{T_j \tau : \tau \in \mathcal{T} \setminus \mathcal{T}_j\}, \quad j \in \Lambda_\omega.$$

Resuming our analysis of equation (5.3.10), it follows from Proposition 8.4.4 and Lemma 2.2.3 that we have

$$\begin{aligned}
\sum_{\tau\in\mathcal{T}_j}[\tau(z)]^n\cdot\tau'(z) &= \sum_{\tau\in\mathcal{T}_j^*}(\mathcal{R}_{j,n}\circ\tau)(z)\cdot\tau'(z)\\
&= \sum_{\tau\in\mathcal{T}_j^*}\left[-\frac{\Phi_j'(\tau(z))}{\Phi_j(\tau(z))}\cdot(1-\alpha^2)\cdot\frac{a_j^n}{n}\cdot\Theta_j(-n\Phi_j(\tau(z)))+O\left(\frac{\alpha^n}{n^2}\right)\right]\cdot\tau'(z)\\
&= \sum_{\tau\in\mathcal{T}_j^*}-\frac{\Phi_j'(\tau(z))}{\Phi_j(\tau(z))}\cdot(1-\alpha^2)\cdot\frac{a_j^n}{n}\cdot\Theta_j(-n\Phi_j(\tau(z)))\cdot\tau'(z)+O\left(\frac{\alpha^n}{n^2}\right)\\
&= \frac{\alpha^n}{n}\cdot\sum_{\tau\in\mathcal{T}_j^*}(\alpha^2-1)\cdot\frac{\Phi_j'(\tau(z))}{\Phi_j(\tau(z))}\cdot e^{in\theta_j}\cdot\Theta_j(-n\Phi_j(\tau(z)))\cdot\tau'(z)+O\left(\frac{\alpha^n}{n^2}\right)
\end{aligned}$$

normally for $z\in\mathbb{D}_\alpha$ as $n\to\infty$. Next, using the fact that $(\Phi_j\circ T_j)(z)=\sigma_j^2\Phi_j(z)$ along with the fact that, for every $j\in\Lambda_\omega$ and for every $t\in\mathcal{H}e^{i\theta_j}$, we have

$$\begin{aligned}
\Theta_j(\sigma_j^2t) &= \sigma_j^2t\sum_{v\in\mathbb{Z}}\sigma_j^{2v}\exp(-\overline{\beta_j}\sigma_j^{2v}\cdot\sigma_j^2t)\\
&= t\sum_{v\in\mathbb{Z}}\sigma_j^{2(v+1)}\exp(-\overline{\beta_j}\sigma_j^{2(v+1)}t)\\
&= \Theta_j(t),
\end{aligned}$$

we note that

$$\begin{aligned}
&\sum_{\tau\in\mathcal{T}_j^*}\frac{\Phi_j'(\tau(z))}{\Phi_j(\tau(z))}\cdot\tau'(z)\cdot\Theta_j(-n\Phi_j(\tau(z)))\\
&\quad= \sum_{\tau\in\mathcal{T}\setminus\mathcal{T}_j}\frac{\Phi_j'((T_j\circ\tau)(z))}{(\Phi_j\circ T_j\circ\tau)(z)}\cdot(T_j\circ\tau)'(z)\cdot\Theta_j(-n\cdot(\Phi_j\circ T_j\circ\tau)(z))\\
&\quad= \sum_{\tau\in\mathcal{T}\setminus\mathcal{T}_j}\frac{(\Phi_j\circ T_j\circ\tau)'(z)}{(\Phi_j\circ T_j\circ\tau)(z)}\cdot\Theta_j(-n\sigma_j^2\cdot(\Phi_j\circ\tau)(z))\\
&\quad= \sum_{\tau\in\mathcal{T}\setminus\mathcal{T}_j}\frac{(\Phi_j\circ\tau)'(z)}{(\Phi_j\circ\tau)(z)}\cdot\Theta_j(-n\cdot(\Phi_j\circ\tau)(z))\\
&\quad= \sum_{\tau\in\mathcal{T}\setminus\mathcal{T}_j}\frac{\Phi_j'(\tau(z))}{\Phi_j(\tau(z))}\cdot\tau'(z)\cdot\Theta_j(-n\Phi_j(\tau(z))).
\end{aligned}$$

Hence, we have

$$\sum_{\tau\in\mathcal{T}_j}[\tau(z)]^n\cdot\tau'(z)=\frac{\alpha^n}{n}\cdot\sum_{\tau\in\mathcal{T}\setminus\mathcal{T}_j}(\alpha^2-1)\cdot\frac{\Phi_j'(\tau(z))}{\Phi_j(\tau(z))}\cdot e^{in\theta_j}\cdot\Theta_j(-n\Phi_j(\tau(z)))\cdot\tau'(z)+O\left(\frac{\alpha^n}{n^2}\right)$$

normally for $z\in\mathbb{D}_\alpha$ as $n\to\infty$. Therefore, using the definition

$$\mathcal{J}_{j,n}(z):=\sum_{\tau\in\mathcal{T}\setminus\mathcal{T}_j}(\alpha^2-1)\cdot e^{ni\theta_j}\cdot\frac{\Phi_j'(\tau(z))}{\Phi_j(\tau(z))}\cdot\Theta_j(-n\Phi_j(\tau(z)))\cdot\tau'(z),\quad j\in\Lambda_\omega,$$

for each $j\in\Lambda_\omega$, we have

$$\sum_{\tau\in\mathcal{T}_j}[\tau(z)]^n\cdot\tau'(z)=\frac{\alpha^n}{n}\cdot\mathcal{J}_{j,n}(z)+O\left(\frac{\alpha^n}{n^2}\right)$$

normally for $z\in\mathbb{D}_\alpha$ as $n\to\infty$. Then by (5.3.8) and (5.3.9), we have

$$\sum_{\tau\in\mathcal{T}_j}[\tau(z)]^n\cdot\tau'(z)\cdot[1+(K_n\circ\tau)(z)]=\frac{\alpha^n}{n}\cdot\mathcal{J}_{j,n}(z)+O\left(\frac{\alpha^n}{n^2}\right)$$

Finally, we have

$$\begin{aligned}\mathcal{Y}_n(z)&=\sum_{j=1}^{\omega}\sum_{\tau\in\mathcal{T}_j}[\tau(z)]^n\cdot\tau'(z)\cdot[1+(K_n\circ\tau)(z)]\\&=\sum_{j=1}^{\omega}\left[\frac{\alpha^n}{n}\cdot\mathcal{J}_{j,n}(z)+O\left(\frac{\alpha^n}{n^2}\right)\right]\\&=\frac{\alpha^n}{n}\cdot\sum_{j=1}^{\omega}\mathcal{J}_{j,n}(z)+O\left(\frac{\alpha^n}{n^2}\right)\end{aligned}$$

normally for $z\in\mathbb{D}_\alpha$ as $n\to\infty$.

5.4 On the behavior of $P_n(z)$ for $z \in \mathbb{T}_\alpha$

In this section it will be convenient to define the set

$$\mathcal{C} := \mathbb{T}_\alpha \setminus \bigcup_{j=1}^{\omega} \{a_j\}$$

We note that if $z \in \mathbb{T}_\alpha$, then we have

$$z^n \cdot K_n(z) = O((\alpha t^2)^n). \tag{5.4.1}$$

If z belongs to a compact subset K of $\mathcal{C}$, then there exists a compact subset K^* of $\mathbb{D}_\alpha$ such that, for each $j \in \Lambda_s$, the point $T_j(z)$ belongs to K^*. Then, by Proposition 5.3.1, we have

$$P_n(T_j(z)) = O\left(\frac{\alpha^n}{n}\right), \quad j \in \Lambda_s$$

uniformly on compact subsets of $\mathcal{C}$. Combining this observation with (5.0.4) and (5.4.1) gives

$$P_n(z) = z^n + O\left(\frac{\alpha^n}{n}\right)$$

uniformly on compact subsets of $\mathcal{C}$.

Now let $z = a_j$ for some $j \in \Lambda_\omega$. After possible relabeling, we may assume that $z = a_1$. Then, by equation (5.0.4), we have

$$P_n(a_1) = a_1^n[1 + K_n(z)] + P_n(T_1(a_1)) \cdot T_1'(a_1) + \sum_{j=2}^{s} P_n(T_j(a_1)) \cdot T_j'(a_1).$$

For j satisfying $2 \le j \le s$ the point $T_j(a_1)$ belongs to $\mathbb{D}_\alpha$. Then, by Proposition 5.3.1, we have

$$P_n(T_j(a_1)) = O\left(\frac{\alpha^n}{n}\right), \quad 2 \le j \le s.$$

Combining this with (5.4.1) and with the relations $T_1(a_1) = a_1$ and $T_1'(a_1) = \sigma_1^2$, we have

$$P_n(a_1) = a_1^n + P_n(a_1) \cdot \sigma_1^2 + O\left(\frac{\alpha^n}{n}\right),$$

which, in turn, yields

$$P_n(a_1) = \frac{a_1^n}{1 - \sigma_1^2} + O\left(\frac{\alpha^n}{n}\right).$$

In effect, we have proved the following proposition.

Proposition 5.4.1. *Let $z \in \mathbb{T}_\alpha$. If $z = a_j$ for some $j \in \Lambda_\omega$, then*

$$P_n(a_j) = \frac{a_j^n}{1 - \sigma_j^2} + O\left(\frac{\alpha^n}{n}\right).$$

Otherwise, we have

$$P_n(z) = z^n + O\left(\frac{\alpha^n}{n}\right)$$

uniformly on compact subsets of $\mathbb{C}$.

6 THE PROOF OF COROLLARY 1.8.1

Let $\mathcal{D}$ be the circular doubly connected domain

$$\mathcal{D} = \mathbb{D} \setminus D,$$

where D is a closed disk contained within the unit disk which is centered on the positive real axis. Let $a \in (0,1)$ be the unique number such that the Möbius transformation

$$\Phi(z) = \frac{z-a}{1-az}$$

maps the interior of D onto a circle centered at the origin and let σ denote the radius of $\Phi(D)$. Let P_n denote the monic polynomials orthogonal over $\mathcal{D}$. Define

$$F(w) = w\sum_{v\in\mathbb{Z}} \sigma^{2v} e^{(a-a^{-1})\sigma^{2v}w}, \quad \mathrm{Re}(w) > 0.$$

Let $\{n_k\}_{k=1}^{\infty}$ be a subsequence of the natural numbers. Here, we show that the sequence of functions $\{F(n_k w)\}_{k=1}^{\infty}$ converges normally on $\mathrm{Re}(w) > 0$ if and only if

$$\lim_{k\to\infty} e^{2\pi i(\log_{\sigma^2} n_k - q)} = 1 \tag{6.0.1}$$

for some $q \in [0,1)$.

The "if" part of the claim follows directly from the representation

$$F(nw) = \beta \cdot \frac{\sigma^2}{1-\sigma^2} \cdot w^2 \int_0^\infty x\sigma^{-2\langle \log_{\sigma^2}(x/n)\rangle} \, e^{(a-a^{-1})wx} \, dx, \quad \mathrm{Re}(w) > 0, \quad n \in \mathbb{N}.$$

In particular, we see that the functions $\{F(nw)\}_{n=1}^{\infty}$ are uniformly bounded on compact subsets of $\mathrm{Re}\, w > 0$.

For the "only if" part, suppose that the functions $F(n_k w)$ converge as $k \to \infty$, but that (6.0.1) holds true for no $q \in [0,1)$. Then we can find two subsequences of $\{n_k\}_{k=1}^{\infty}$, say $\mathcal{N}_1$ and $\mathcal{N}_2$, such that

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{N}_1}} e^{2\pi i \log_{\sigma^2}(n)} = e^{2\pi i q}, \quad \lim_{\substack{n\to\infty \\ n\in\mathcal{N}_2}} e^{2\pi i \log_{\sigma^2}(n)} = e^{2\pi i p}$$

for some $0 \le q < p < 1$. But this and the convergence of $F(n_k t)$ leads to a contradiction, for we shall now prove that

$$F(\sigma^{2q} w) \not\equiv F(\sigma^{2p} w)$$

(or equivalently, that $f_q \not\equiv f_p$) for $0 \le q < p < 1$.

$$F(w) = w \sum_{v\in\mathbb{Z}} \sigma^{2v} e^{(a-a^{-1})\sigma^{2v} w}, \quad \mathrm{Re}(w) > 0.$$

$$w = \sigma^{2x+1}$$

$$\varrho = \sigma^2$$

$$g_q(x) = F(\varrho^{q+x}) = F(\sigma^{2(q+x)}) = F(\sigma^{2q} \cdot \sigma^{2x}), \quad x \in \mathbb{R}$$

$$g_p(x) = F(\varrho^{p+x}) = F(\sigma^{2(p+x)}) = F(\sigma^{2p} \cdot \sigma^{2x}), \quad x \in \mathbb{R}$$

It suffices to show that $g_q(x) - g_p(x) \not\equiv 0$ for $0 \le q < p < 1$.

The function $g_q(x) - g_p(x)$ is analytic and 1-periodic on $\mathbb{R}$.

We can expand $h(x)$ in an exponential Fourier series

$$h(x) = \sum_{v \in \mathbb{Z}} c_v e^{2\pi i v x}$$

where the v-th Fourier coefficient of h is

$$c_v = \int_0^1 h(x) e^{-2\pi i v x} \, dx$$

We have $c_v = g(v)$, where $g(\alpha)$ is the Fourier transform of $h(x)$

$$g(\xi) = \int_{-\infty}^{\infty} h(x) e^{-2\pi i x \xi} \, dx$$

$$\int_{-\infty}^{\infty} h(x) e^{-2\pi i x \xi} \, dx = \int_{-\infty}^{\infty} (g_q(x) - g_p(x)) e^{-2\pi i x \xi} \, dx$$

$$\begin{aligned} c_v &= \int_{-\infty}^{\infty} (g_q(x) - g_p(x)) e^{-2\pi i x v} \, dx \\ &= \int_{-\infty}^{\infty} (F(\sigma^{2q} \cdot \sigma^{2x}) - F(\sigma^{2p} \cdot \sigma^{2x})) e^{-2\pi i x v} \, dx \end{aligned}$$

$$F(w) = w \sum_{v \in \mathbb{Z}} \sigma^{2v} e^{(a - a^{-1}) \sigma^{2v} w}, \quad \mathrm{Re}(w) > 0.$$

$$F(\sigma^{2(q+x)}) = 2(q+x) \sum_{v \in \mathbb{Z}} \sigma^{2v} e^{2(a - a^{-1}) \sigma^{2v} (q+x)}$$

$$
\begin{aligned}
F(\sigma^{2(p+x)}) &= 2(p+x)\sum_{v\in\mathbb{Z}}\sigma^{2v}e^{2(a-a^{-1})\sigma^{2v}(p+x)} \\
&= 2p\sum_{v\in\mathbb{Z}}\sigma^{2v}e^{2(a-a^{-1})\sigma^{2v}(p+x)} + 2x\sum_{v\in\mathbb{Z}}\sigma^{2v}e^{2(a-a^{-1})\sigma^{2v}(p+x)}
\end{aligned}
$$

7 THE PROOF OF THEOREM 1.9.1

Let

$$\mathcal{D} = \mathbb{D} \setminus \bigcup_{j\in\Lambda_s} D_{c_j,r_j}$$

be a CMCD and let a_j and σ_j be the constants introduced in section 1.3. We define the additional constants

- $a := \min\limits_{j\in\Lambda_s} |a_j|$,
- $\sigma := \max\limits_{j\in\Lambda_s} \sigma_j$,
- $m := \min\limits_{j\in\Lambda_s} \left(\frac{1}{|a_j|} \cdot \frac{1-|a_j|^2\sigma_j^2}{1-\sigma_j^2} \right)$,
- $\lambda := \max_{j\in\Lambda_s}\{|c_j| + r_j\} = \max\limits_{j\in\Lambda_s} \frac{|a_j|+\sigma_j}{1+|a_j|\sigma_j}$, and
- $\mathcal{N}_{\mathcal{D}} := \left(\frac{\sigma}{1-\sigma^2} \cdot \frac{1-a^2}{a} \right)^2 \cdot \frac{1}{(m-\lambda)^2}$.

Recall that Assumption 1.3.1 states that there exists some $\rho \in (0,1)$ such that the series $\sum_{\tau\in\mathcal{T}} |\tau'|$ converges uniformly on each compact subset of $\mathbb{D}_{1/\rho}$. The purpose of this chapter is to prove Theorem 1.9.1. In other words, we will show that Assumption 1.3.1 holds whenever $\mathcal{D}$ which satisfies any of the following cases:

1) $c_j \neq 0$ for each $j \in \Lambda_s$ and $s < 1/\mathcal{N}_{\mathcal{D}}$,

2) $0 \in \mathcal{D}$ and $c_j \in (-1,1)$ for each $j \in \Lambda_s$, or

3) there exists some CMCD $\mathcal{D}^*$ that satisfies Assumption 1.3.1 and there exists some automorphism of the unit disk Ψ such that $\Psi(\mathcal{D}^*) = \mathcal{D}$.

7.1 Case 1 of Theorem 1.9.1

We suppose that we have $c_j \neq 0$ for each $j \in \Lambda_s$. This means that we also have $a_j \neq 0$ for each $j \in \Lambda_s$. Note that, for each $j \in \Lambda_s$, we may use (2.1.8) to write

$$T_j'(z) = \left(\frac{\sigma_j}{1-\sigma_j^2} \cdot \frac{1-|a_j|^2}{\overline{a_j}} \right)^2 \cdot \left(z - \frac{1}{\overline{a_j}} \cdot \frac{1-|a_j|^2\sigma_j^2}{1-\sigma_j^2} \right)^{-2}. \tag{7.1.1}$$

7.1.1 Preliminaries

Since we plan on estimating functions related to T_j', we will use the constants

$$a = \min_{k \in \Lambda_s} |a_k|, \quad \text{and} \quad \sigma = \max_{k \in \Lambda_s} \sigma_k.$$

First we make a quick observation. We claim that, for each $k \in \Lambda_s$, we have

$$0 < \frac{1-|a_k|^2}{|a_k|} \leq \frac{1-a^2}{a} \qquad \text{and} \qquad 0 < \frac{\sigma_k}{1-\sigma_k^2} \leq \frac{\sigma}{1-\sigma^2}. \tag{7.1.2}$$

Indeed, the function

$$f(t) = \frac{1-t^2}{t}$$

is decreasing on the positive real axis. This follows from the fact that

$$\frac{d}{dt}\left(\frac{1-t^2}{t}\right) = -\frac{1}{t^2} - 1 < 0, \quad 0 < t < \infty.$$

Furthermore, we have $f(t) > 0$ for $t \in (0,1)$. Therefore, if $0 < x_1 \leq x_2 < 1$, then we have

$$0 < \frac{1-x_2^2}{x_2} \leq \frac{1-x_1^2}{x_1}$$

which, in turn, gives

$$0 < \frac{x_1}{1-x_1^2} \leq \frac{x_2}{1-x_2^2}.$$

Since we have $0 < a \leq |a_k| < 1$ and $0 < \sigma_k \leq \sigma < 1$, we see that claim (7.1.2) is established.

Now we prepare to prove our first lemma. We begin by defining

$$\rho := \max_{j \in \Lambda_s} |a_j|.$$

By the discussion in Section 2.2.1, we know that each T_j is analytic on $\mathbb{D}_{1/\rho}$. Meanwhile, by the discussion in Section 2.1.4, the quantity

$$\frac{1}{|a_j|} \cdot \frac{1 - |a_j|^2 \sigma_j^2}{1 - \sigma_j^2}$$

represents the distance from the origin to the pole of the transformation T_j. Therefore, if we define

$$m := \min_{j \in \Lambda_s} \left(\frac{1}{|a_j|} \cdot \frac{1 - |a_j|^2 \sigma_j^2}{1 - \sigma_j^2} \right),$$

then we have

$$1 < 1/\rho < m.$$

In particular, note that if $z \in \mathbb{D}_{1/\rho}$, then, for each $j \in \Lambda_s$, we have

$$\begin{aligned} \left| z - \frac{1}{\overline{a_j}} \cdot \frac{1 - |a_j|^2 \sigma_j^2}{1 - \sigma_j^2} \right| &\geq \left| |z| - \frac{1}{|a_j|} \cdot \frac{1 - |a_j|^2 \sigma_j^2}{1 - \sigma_j^2} \right| \\ &= \frac{1 - |a_j|^2 \sigma_j^2}{1 - \sigma_j^2} - |z| \\ &> m - 1/\rho. \end{aligned}$$

We are now in a position where we can prove our first lemma.

Lemma 7.1.1. *If $j \in \Lambda_s$, then, for every $z \in \mathbb{D}_{1/\rho}$, we have*

$$|T_j'(z)| < \left(\frac{\sigma}{1 - \sigma^2} \cdot \frac{1 - a^2}{a} \right)^2 \cdot \frac{1}{(m - 1/\rho)^2}.$$

Proof. By (7.1.1), we may write

$$|T_j'(z)| = \left(\frac{\sigma_j}{1-\sigma_j^2} \cdot \frac{1-|a_j|^2}{|a_j|} \right)^2 \cdot \left| z - \frac{1}{\overline{a_j}} \cdot \frac{1-|a_j|^2\sigma_j^2}{1-\sigma_j^2} \right|^{-2}.$$

By relationship 7.1.2, we have

$$\frac{\sigma_j}{1-\sigma_j^2} \cdot \frac{1-|a_j|^2}{|a_j|} \leq \frac{\sigma}{1-\sigma^2} \cdot \frac{1-a^2}{a}.$$

Meanwhile, by the comments preceding the lemma, we also have

$$\left| z - \frac{1}{\overline{a_j}} \cdot \frac{1-|a_j|^2\sigma_j^2}{1-\sigma_j^2} \right| > m - 1/\rho,$$

and the lemma is established. □

Next, we recall the constant

$$\lambda := \max_{j \in \Lambda_s} \{|c_j| + r_j\}.$$

We also recall a term that was defined earlier. If $\tau = T_{k_n} T_{k_{n-1}} \cdots T_{k_2} T_{k_1} \in \mathcal{T}^*$, then we say that τ is of *length* n.

Lemma 7.1.2. *If $\tau \in \mathcal{T}^*$ is of length n, then, for every $z \in \mathbb{D}_{1/\rho}$, we have*

$$|\tau'(z)| < K \cdot \mathcal{N}_{\mathcal{D}}^{n-1},$$

where

$$K := \left(\frac{\sigma}{1-\sigma^2} \cdot \frac{1-a^2}{a} \right)^2 \cdot \frac{1}{(m-1/\rho)^2}.$$

Proof. We induct on n. Note that Lemma 7.1.1 proves the base case (when $n = 1$).

Now suppose the claim holds for all transformations τ_0 of length $n-1$ (where $n \geq 2$) and let $\tau \in \mathcal{T}$ be of length n. Then we can find some $j \in \Lambda_s$ such that $\tau = T_j\tau_0$, where τ_0 is of length $n-1$.

By the chain rule, we have

$$|\tau'(z)| = |T_j'[\tau_0(z)]| \cdot |\tau_0'(z)|.$$

Now, by relationships (7.1.1) and 7.1.2, we may write

$$|T_j'(\tau_0(z))| < \left(\frac{\sigma}{1-\sigma^2} \cdot \frac{1-a^2}{a}\right)^2 \cdot \left|\tau_0(z) - \frac{1}{\overline{a_j}} \cdot \frac{1-|a_j|^2\sigma_j^2}{1-\sigma_j^2}\right|^{-2}.$$

Meanwhile, by Lemma 2.2.1 and the definitions of λ and m, we also have

$$|\tau_0(z)| < \lambda < 1 < m \leq \frac{1}{|a_j|} \cdot \frac{1-|a_j|^2\sigma_j^2}{1-\sigma_j^2}.$$

Therefore, we have

$$|T_j'(\tau_0(z))| < \left(\frac{\sigma}{1-\sigma^2} \cdot \frac{1-a^2}{a}\right)^2 \cdot \frac{1}{(m-\lambda)^2} = \mathcal{N}_{\mathcal{D}}.$$

Therefore, by the induction hypothesis and the comments above, we have

$$\begin{aligned}
|\tau'(z)| &= |T_j'[\tau_0(z)]| \cdot |\tau_0'(z)| \\
&< \mathcal{N}_{\mathcal{D}} \cdot \left(\frac{\sigma}{1-\sigma^2} \cdot \frac{1-a^2}{a}\right)^2 \cdot \frac{\mathcal{N}_{\mathcal{D}}^{n-2}}{(m-1/\rho)^2} \\
&= \left(\frac{\sigma}{1-\sigma^2} \cdot \frac{1-a^2}{a}\right)^2 \cdot \frac{\mathcal{N}_{\mathcal{D}}^{n-1}}{(m-1/\rho)^2},
\end{aligned}$$

and the proof is complete by the principle of mathematical induction.

□

7.1.2 Proof for Case 1

In what follows, we suppose that we have $s < 1/\mathcal{N}_{\mathcal{D}}$.

Recall the following definition. For each natural number n, we let $\mathcal{E}_n$ denote the set of all strings of length n:

$$\mathcal{E}_n := \{T_{j_n} T_{j_{n-1}} \cdots T_{j_2} T_{j_1} : j_k \in \Lambda_s \text{ for each } 1 \leq k \leq n\}.$$

Note that for each $n \in \mathbb{N}$, there are exactly s^n elements of $\mathcal{E}_n$.

By Lemma 7.1.2, for every $n \in \mathbb{N}$ and for every $z \in \mathbb{D}_{1/\rho}$, we have

$$\sum_{\tau \in \mathcal{E}_n} |\tau'(z)| < s^n \cdot K \cdot \mathcal{N}_{\mathcal{D}}^{n-1},$$

where

$$K = \left(\frac{\sigma}{1 - \sigma^2} \cdot \frac{1 - a^2}{a} \right)^2 \cdot \frac{1}{(m - 1/\rho)^2}.$$

Therefore, we may write

$$\begin{aligned} \sum_{\tau \in \mathcal{T}^*} |\tau'(z)| &= \sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{E}_n} |\tau'(z)| \\ &< sK \sum_{n=1}^{\infty} (s\mathcal{N}_{\mathcal{D}})^{n-1}. \end{aligned}$$

Since we are assuming that $s\mathcal{N}_{\mathcal{D}} < 1$, the Weierstrass M-test informs us that $\sum_{\tau \in \mathcal{T}} |\tau'|$ converges uniformly only $\mathbb{D}_{1/\rho}$.

7.2 Case 2 of Theorem 1.9.1

In what follows, we will use the fact that the numbers c_j, r_j, a_j, and σ_j satisfy

$$c_j = \frac{a_j(1-\sigma_j^2)}{1-|a_j|^2\sigma_j^2} \qquad \text{and} \qquad r_j = \frac{\sigma_j(1-|a_j|^2)}{1-|a_j|^2\sigma_j^2}. \tag{7.2.1}$$

These relationships are established in the Appendix. In particular, we remark that we have

$$|c_j| - r_j = \frac{|a_j| - \sigma_j}{1-|a_j|\sigma_j} \tag{7.2.2}$$

for each $j \in \Lambda_s$.

7.2.1 Preliminaries

Consequences of having $c_j \in (-1,1)$ for each $j \in \Lambda_s$

Let $j \in \Lambda_s$ and suppose that $c_j \in (-1,1)$. By the relationship between c_j and a_j given in equation (7.2.1), we see that we also have $a_j \in (-1,1)$. Then, by the discussion in Section 2.1.5, we see that the extended real axis is invariant under the transformation T_j:

$$T_j(\mathbb{R} \cup \{\infty\}) = \mathbb{R} \cup \{\infty\}.$$

Consequently, if we have $c_j \in (-1,1)$ for each $j \in \Lambda_s$, then we have

$$\tau(\mathbb{R} \cup \{\infty\}) = \mathbb{R} \cup \{\infty\} \tag{7.2.3}$$

for every $\tau \in \mathcal{T}$.

Consequences of having $0 \in \mathcal{D}$

In this section, we will suppose that we have $0 \in \mathcal{D}$. Note that this is equivalent to the condition

$$0 \notin D_{c_j,r_j}, \quad j \in \Lambda_s.$$

For the moment, fix some $j \in \Lambda_s$. Since we have $0 \notin D_{c_j,r_j}$, we may write

$$0 < |c_j| - r_j.$$

Then by relation 7.2.2, we have

$$0 < \frac{|a_j| - \sigma_j}{1 - |a_j|\sigma_j}.$$

Note that this implies $0 < |a_j| - \sigma_j$. In other words, we have

$$0 < \sigma_j < |a_j| < 1,$$

which, in turn, gives

$$1 < \frac{1}{|a_j|} < \frac{1}{\sigma_j} < \infty. \tag{7.2.4}$$

Now, recall that the pole of χ_j occurs at the point $-1/\overline{a_j}$:

$$\chi_j(-1/\overline{a_j}) = \infty.$$

By (7.2.4), we see that $\chi_j(\mathbb{T}_{1/\sigma_j})$ is a circle and not a line. Furthermore, since the circle $\mathbb{T}_{1/\sigma_j}$ is contained in the Δ_1 (i.e., the exterior of the unit circle) and since $\chi_j(\Delta_1) = \Delta_1$, we have

$$\chi_j(\mathbb{T}_{1/\sigma_j}) \subset \Delta_1. \tag{7.2.5}$$

Now, in the extended complex plane, the complement of the circle $\chi_j(\mathbb{T}_{1/\sigma_j})$ is the set

$$\chi_j(\mathbb{D}_{1/\sigma}) \cup \chi_j(\Delta_{1/\sigma}),$$

which consists of two components. By definition, the interior of $\chi_j(\mathbb{T}_{1/\sigma_j})$ is the bounded component while the exterior is the unbounded one. Since (7.2.4) gives

$$-\frac{1}{\overline{a_j}} \in \mathbb{D}_{1/\sigma_j}$$

and since we have $\chi_j(-1/\overline{a_j}) = \infty$, we find that

$$\infty \in \chi_j(\mathbb{D}_{1/\sigma_j}). \tag{7.2.6}$$

In other words, the set $\chi_j(\mathbb{D}_{1/\sigma_j})$, being unbounded, is the exterior of $\chi_j(\mathbb{T}_{1/\sigma_j})$. Therefore, the set $\chi_j(\Delta_{1/\sigma_j})$ is the interior of $\chi_j(\mathbb{T}_{1/\sigma_j})$ and is bounded.

Consequently, if we define

$$M := \max\left\{ |z| : z \in \bigcup_{j\in\Lambda_s} \chi_j(\mathbb{T}_{1/\sigma_j}) \right\}, \tag{7.2.7}$$

then we will have $M < \infty$ and

$$\bigcup_{j\in\Lambda_s} \chi_j(\Delta_{1/\sigma_j}) \subset \mathbb{D}_M. \tag{7.2.8}$$

Next, we record a consequence of the discussion above.

Lemma 7.2.1. *Suppose we have* $0 \in \mathcal{D}$. *If* $\tau = T_{j_n}T_{j_{n-1}}\cdots T_{j_2}T_{j_1}$ *is a string, then*

$$\tau(\infty) \in \mathring{D}_{c_{j_n}, r_{j_n}}$$

Proof. We induct on the length n of τ. If $n = 1$, then we have $\tau = T_j$ for some $j \in \Lambda_s$. Then relationships (7.2.6) and (2.1.3) imply

$$T_j(\infty) \in T_j(\chi_j(\mathbb{D}_{1/\sigma_j})) = \mathring{D}_{c_j,r_j}.$$

This establishes the base case for claim (iii). Next, suppose that claim (iii) holds for all strings of length k. Let τ be a string of length $k+1$. Then we can write $\tau = T_j\tau_k$ for some $j \in \Lambda_s$ and some string τ_k of length k. By the induction hypothesis, we have $\tau_k(\infty) \in \mathbb{D}$. Therefore,

$$\tau(\infty) = T_j\tau_k(\infty) \in T_j(\mathbb{D}) \subset T_j(\chi_j(\mathbb{D}_{1/\sigma_j})) = \mathring{D}_{c_j,r_j}$$

by relationship (2.1.3). The claim is established by the principle of mathematical induction and the proof of the lemma is complete.

□

Next, recall our discussion from Section 2.1.3 regarding Möbius transformations and matrices. In particular, if we let $\text{Aut}(\hat{\mathbb{C}})$ denote the group of all Möbius transformations and if we let $\text{SL}_2(\mathbb{C})$ denote the group of all 2×2 invertible matrices with determinant one whose entries are elements of $\mathbb{C}$, then the map $f : \text{SL}_2(\mathbb{C}) \to \text{Aut}(\hat{\mathbb{C}})$ defined by the relation

$$f\left(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\right) = \frac{\alpha z + \beta}{\gamma z + \delta},$$

a group homomorphism of $\text{SL}_2(\mathbb{C})$ onto $\text{Aut}(\hat{\mathbb{C}})$. Furthermore, if $\gamma \neq 0$ and if $f(A) = T$ for some $T \in \text{Aut}(\hat{\mathbb{C}})$, then we may write

$$T'(z) = \frac{d}{dz}\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right) = \frac{1}{\gamma^2(z + \delta/\gamma)^2},$$

as previously demonstrated in the series of calculations (2.1.6).

For the remainder of this section, we fix some $\rho \in (0, 1)$ such that

$$\mathbb{D}_{1/\rho} \subset \bigcap_{j \in \Lambda_s} \chi_j(\mathbb{D}_{1/\sigma_j}). \tag{7.2.9}$$

Lemma 7.2.2. *Suppose the we have $0 \in \mathcal{D}$. If $\tau \in \mathcal{T}^*$, then there exist complex numbers γ and δ such that*

$$\tau'(z) = \frac{1}{\gamma^2(z + \delta/\gamma)^2}, \quad z \in \mathbb{C} \setminus \{-\delta/\gamma\}. \tag{7.2.10}$$

Furthermore, if ρ be defined as in (7.2.9) and if M is defined as in (7.2.7), then we have

$$1/\rho < |\delta/\gamma| < M$$

whenever γ and δ satisfy relation (7.2.10).

Proof. Since τ is a Möbius transformation and since f is onto, there exists some matrix

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

belonging to $\mathrm{SL}_2(\mathbb{C})$ such that $f(A) = \tau$. In other words, we have

$$\tau(z) = \frac{\alpha z + \beta}{\gamma z + \delta}.$$

By Lemma 7.2.1, we have $\tau(\infty) \in \mathbb{D}$. In particular, we see that the point at infinity is not fixed by τ. Therefore, we have $\gamma \neq 0$. Hence, we may write

$$\tau'(z) = \frac{1}{\gamma^2(z + \delta/\gamma)^2}.$$

By Lemma 2.1.1, the pole of τ belongs to

$$\bigcup_{j \in \Lambda_s} \chi_j(\Delta_{1/\sigma_j}).$$

Since $|\delta/\gamma|$ is the magnitude of the pole of τ, relation (7.2.8) gives

$$|\delta/\gamma| < M.$$

Furthermore, we know that τ is analytic on $\mathbb{D}_{1/\rho}$ by Lemmas 7.2.1 and 2.1.1. Therefore, we have

$$1/\rho < |\delta/\gamma|.$$

This completes the proof of the lemma. □

7.2.2 Proof for Case 2

Let $\mathcal{D}$ be a CMCD such that $0 \in \mathcal{D}$ and $c_j \in (-1, 1)$ for each $j \in \Lambda_s$. Let M be defined as in (7.2.7) and let ρ be defined as in (7.2.9). Let K be a compact subset of $\mathbb{D}_{1/\rho}$. Let $r \in (1, 1/\rho)$ satisfy $K \subset \overline{\mathbb{D}}_r$. In this section, we will show that the series $\sum_{\tau \in \mathcal{T}^*} |\tau'|$ converges uniformly on $\overline{\mathbb{D}}_r$. Our proof will be an easy consequence of three lemmas.

The first lemma

First, fix some $m \in (1, r)$. We record the relationships between m, r, ρ, and M below for reference:

$$1 < m < r < 1/\rho < M.$$

Lemma 7.2.3. *Let $\tau \in \mathcal{T}^*$ and let γ and δ be complex numbers which satisfy*

$$\tau'(z) = \frac{1}{\gamma^2(z + \delta/\gamma)^2} \quad z \in \mathbb{D}_{1/\rho}.$$

For every $\zeta \in \overline{\mathbb{D}}_r$ and for every t belonging to the interval $[1, m]$, we have

$$0 < \frac{1}{|\zeta + \delta/\gamma|} < \frac{m + M}{1/\rho - r} \cdot \frac{1}{|t + \delta/\gamma|}. \tag{7.2.11}$$

Proof. First, we note that, by Lemma 7.2.2, we have

$$|\zeta + \delta/\gamma| \quad \geq \quad |\delta/\gamma| - |\zeta| \quad \geq \quad 1/\rho - r \quad > \quad 0.$$

Therefore, we may write

$$1 \leq \frac{|\zeta + \delta/\gamma|}{1/\rho - r}.$$

Next, we note that $|t + \delta/\gamma| \neq 0$, since $t \in [1, m]$ and

$$|t + \delta/\gamma| \quad \geq \quad |\delta/\gamma| - t \quad \geq \quad 1/\rho - m \quad > \quad 0.$$

Then we may write

$$0 \quad < \quad |t + \delta/\gamma| \quad \leq \quad |t| + |\delta/\gamma| \quad < \quad m + M$$

by Lemma 7.2.2.

Next, by multiplying the two inequalities

$$0 < 1 \leq \frac{|\zeta + \delta/\gamma|}{1/\rho - r} \qquad \text{and} \qquad 0 < |t + \delta/\gamma| < m + M,$$

we obtain

$$0 < |t + \delta/\gamma| < \frac{m + M}{1/\rho - r} \cdot |\zeta + \delta/\gamma|,$$

which, in turn, implies (7.2.11) and completes the proof of the lemma.

□

The second lemma

Next, let I denote the interval $[1, m]$:

$$I := [1, m].$$

Lemma 7.2.4. *If $\tau_0 \in \mathcal{T}^*$, then $\tau_0(I)$ is a closed interval contained in $(-1,1)$. Moreover, if $l[\tau(I)]$ denotes the length of $\tau(I)$, then we have*

$$\sum_{\tau \in \mathcal{T}^*} l[\tau(I)] \leq 2.$$

Proof. First, note that we have $I \subset \mathbb{D}_{1/\rho} \setminus \mathbb{D}$ and $I \subset \mathbb{R}$. Next, recall relation (7.2.3):

$$\tau(\mathbb{R} \cup \{\infty\}) = \mathbb{R} \cup \{\infty\}, \quad \tau \in \mathcal{T}^*.$$

Meanwhile, by Lemma 2.2.1, we also have

$$\tau(\mathbb{D}_{1/\rho}) \subset \mathbb{D} \quad \tau \in \mathcal{T}^*.$$

When we combine these observations with the fact that τ_0 is continuous, we see that $\tau_0(I)$ is some closed interval contained in $(-1,1)$. Therefore, we may write

$$\bigcup_{\tau \in \mathcal{T}^*} \tau(I) \subset (-1,1),$$

which, in turn, gives

$$\mu\left(\bigcup_{\tau \in \mathcal{T}^*} \tau(I)\right) \leq l[(-1,1)] = 2,$$

where μ denotes one-dimensional Lebesgue measure.

Next, by Lemma 2.1.2, we have

$$\tau_1(I) \cap \tau_2(I) = \emptyset$$

whenever τ_1 and τ_2 are distinct elements of $\mathcal{T}^*$. Therefore, we actually have

$$\sum_{\tau \in \mathcal{T}^*} l[\tau(I)] = \mu\left[\bigcup_{\tau \in \mathcal{T}^*} \tau(I)\right] \leq 2,$$

which establishes the lemma. $\square$

The third lemma

For the final lemma of this section, we define the constant

$$C := \left(\frac{m+M}{1/\rho - r}\right)^2 \cdot \frac{1}{m-1},$$

Lemma 7.2.5. *If $\zeta \in \overline{\mathbb{D}}_r$ and if $\tau \in \mathcal{T}^*$, then we have*

$$|\tau'(\zeta)| < C \cdot l[\tau(I)].$$

Proof. First, by Lemma 7.2.2, we can find some complex numbers γ and δ such that

$$\tau'(\zeta) = \frac{1}{\gamma^2(\zeta + \delta/\gamma)^2}, \quad \zeta \in \mathbb{D}_{1/\rho}.$$

Next, recall that $I = [1, m]$. By the mean value theorem for integrals, there exists some $t_0 \in (1, m)$ such that

$$\frac{1}{m-1} \int_I \frac{1}{|t + \delta/\gamma|^2}\, dt = \frac{1}{|t_0 + \delta/\gamma|^2}. \tag{7.2.12}$$

Meanwhile, with the help of Lemma 7.2.3, we may write

$$|\tau'(\zeta)| = \frac{1}{|\gamma|^2} \cdot \frac{1}{|\zeta + \delta/\gamma|^2} < \frac{1}{|\gamma|^2} \cdot \left(\frac{m+M}{1/\rho - r}\right)^2 \cdot \frac{1}{|t_0 + \delta/\gamma|^2}.$$

Then we can use (7.2.12) to obtain

$$\begin{aligned} |\tau'(\zeta)| &< \left(\frac{m+M}{1/\rho - r}\right)^2 \cdot \frac{1}{|\gamma|^2} \cdot \frac{1}{m-1} \int_I \frac{1}{|t + \delta/\gamma|^2}\, dt \\ &= C \cdot \int_I |\tau'(t)|\, dt \end{aligned}$$

$$= C \cdot l\tau(I)],$$

and the proof is complete.

$\square$

Uniform convergence for $\sum_{\tau\in\mathfrak{T}^*}|\tau'|$ on $\overline{\mathbb{D}}_r$

By Lemma 7.2.5, we have the estimate

$$|\tau'(\zeta)| < C \cdot l[\tau(I)]$$

for every $\tau \in \mathfrak{T}^*$ and for every $\zeta \in \overline{\mathbb{D}}_r$. Meanwhile, by Lemma 7.2.4, we have

$$\sum_{\tau\in\mathfrak{T}^*} l[\tau(I)] \leq 2.$$

Therefore, the series $\sum_{\tau\in\mathfrak{T}^*}|\tau'(\zeta)|$ converges uniformly on $\overline{\mathbb{D}}_r$ by the Weierstrass M-test. This completes the proof for Case 2 of Theorem 1.9.1.

7.3 Case 3 of Theorem 1.9.1

Let $\tilde{\mathcal{D}}$ be the CMCD complementary to $\bigcup_{j=1}^{s} D_{c_j,r_j}$. Let $\tilde{\mathfrak{T}}$ denote the family of transformations associated with $\tilde{\mathcal{D}}$. Suppose there exists some $\tilde{\rho} \in (0,1)$ such that the series of functions $\sum_{\tilde{\tau}\in\tilde{\mathfrak{T}}}|\tilde{\tau}'|$ converges uniformly on each compact subset of $\mathbb{D}_{1/\tilde{\rho}}$. Let Ψ be a conformal map of the unit disk to itself. Let

$$\mathcal{D} := \Psi(\tilde{\mathcal{D}}) = \mathbb{D} \setminus \left[\bigcup_{j=1}^{s} \Psi(D_{c_j,r_j})\right]$$

be the CMCD complementary to $\bigcup_{j=1}^{s}\Psi(D_{c_j,r_j})$. Let $\mathfrak{T}$ denote the family of transformations associated with $\Psi(\mathcal{D})$. We will show that there exists somes $\rho \in (0,1)$ such that $\sum_{\tau\in\mathfrak{T}}|\tau'|$ converges uniformly on each compact subset of $\mathbb{D}_{1/\rho}$.

We begin by noting that the function $f : \tilde{\mathfrak{T}} \to \mathfrak{T}$ defined by

$$f(\tilde{\tau}) = \Psi \circ \tilde{\tau} \circ \Psi^{-1}$$

is an isomorphism between the semigroups $\tilde{\mathfrak{T}}$ and $\mathfrak{T}$. This follows from the fact that if $\tilde{\mathfrak{T}}$ is generated by the transformations $\tilde{T}_j$, then $\mathfrak{T}$ is generated by the transformations

$$T_j = \Psi \circ \tilde{T}_j \circ \Psi^{-1}.$$

Therefore, we may write

$$\begin{aligned}\sum_{\tau \in \mathfrak{T}} |\tau'(z)| &= \sum_{\tilde{\tau} \in \tilde{\mathfrak{T}}} \left| \frac{d}{dz} (\Psi \circ \tilde{\tau} \circ \Psi^{-1})(z) \right| \\ &= \sum_{\tilde{\tau} \in \tilde{\mathfrak{T}}} |\Psi'((\tilde{\tau} \circ \Psi^{-1})(z))| \cdot |\tilde{\tau}'(\Psi^{-1}(z))| \cdot |(\Psi^{-1})'(z)|.\end{aligned}$$

We claim that both $|(\Psi^{-1})'(z)|$ and $|\Psi'((\tilde{\tau} \circ \Psi^{-1})(z))|$ are bounded on $\overline{\mathbb{D}}$. It is clear that $|(\Psi^{-1})'(z)|$ is bounded on $\overline{\mathbb{D}}$, since Ψ^{-1} is a conformal map of the unit disk to itself. To see that $|\Psi'((\tilde{\tau} \circ \Psi^{-1})(z))|$ is bounded on $\overline{\mathbb{D}}$, note that, for every $z \in \overline{\mathbb{D}}$, we have

$$(\tilde{\tau} \circ \Psi^{-1})(z) \subset \overline{\mathbb{D}}, \quad \tilde{\tau} \in \tilde{\mathfrak{T}}^*.$$

Therefore $|\Psi'((\tilde{\tau} \circ \Psi^{-1})(z))|$ is bounded on $\overline{\mathbb{D}}$ since $|\Psi'(z)|$ is bounded on is bounded on $\overline{\mathbb{D}}$.

Then by continuity, the product

$$|(\Psi^{-1})'(z)| \cdot |\Psi'[(\tilde{\tau} \circ \Psi)(z)]|$$

is bounded by a positive constant M on some closed disk K_1 centered at the origin with

$$\overline{\mathbb{D}} \subsetneq K_1 \subset \mathbb{D}_{1/\tilde{\rho}}$$

so that we have the estimate

$$z \in K_1 \quad \Rightarrow \quad |(\Psi^{-1})'(z)| \cdot |\Psi'[(\tilde{\tau} \circ \Psi^{-1})(z)]| \leq M.$$

Meanwhile, we can also find a closed disk K_2 centered at the origin with

$$\overline{\mathbb{D}} \subsetneq K_2 \subset \Psi(\mathbb{D}_{1/\tilde{\rho}})$$

Then, by hypothesis, we have that

$$\sum_{\tilde{\tau} \in \tilde{\mathfrak{T}}} |\tilde{\tau}'(\Psi^{-1}(z))|$$

converges uniformly on K_2.

Put $K = K_1 \cap K_2$. Then the series $\sum_{\tau \in \mathfrak{T}} |\tau'|$ converges uniformly for $z \in K$ since

$$\begin{aligned}\sum_{\tau \in \mathfrak{T}} |\tau'(z)| &= \sum_{\tilde{\tau} \in \tilde{\mathfrak{T}}} |\Psi'((\tilde{\tau} \circ \Psi^{-1})(z))| \cdot |\tilde{\tau}'(\Psi^{-1}(z))| \cdot |(\Psi^{-1})'(z)| \\ &\leq M \cdot \sum_{\tilde{\tau} \in \tilde{\mathfrak{T}}} |\tilde{\tau}'(\Psi^{-1}(z))|\end{aligned}$$

for every $z \in K$. Since $\overline{\mathbb{D}} \subsetneq K$, we readily see that there exists some $\rho \in (0, 1)$ such that

$$\mathbb{D}_{1/\rho} \subset K,$$

and that $\sum_{\tau \in \mathfrak{T}} |\tau'|$ will converge uniformly on every compact subset of $\mathbb{D}_{1/\rho}$. This completes the proof.

8 APPENDIX

The purpose of the Appendix is to catalog some technical information which supports the theory of the preceding chapters. It is included here mainly for reference, although much of the material can be found in the literature.

The main purpose of Section 8.1 is to establish the relationships between c_j, r_j, a_j, and σ_j.

8.1 Automorphisms of the unit disk

A conformal map of the unit disk onto itself is called an *automorphism of the unit disk*. A well known theorem states that if Ψ is an automorphism of the unit disk, then

$$\exists a \in \mathbb{D} \quad \& \quad \exists \theta \in \mathbb{R} \qquad \text{such that} \qquad \Psi(z) = e^{i\theta}\,\frac{z+a}{1+\overline{a}z}.$$

The entire collection $\mathrm{Aut}(\mathbb{D})$ of automorphisms of the unit disk forms a group whose operation is the composition of functions.

The functions ψ_a

For the moment, we will restrict our attention to elements of $\mathrm{Aut}(\mathbb{D})$ which take the form

$$\psi_a(z) := \frac{z+a}{1+\overline{a}z}, \quad a \in \mathbb{D}.$$

Put $R := \{\psi_a : a \in \mathbb{D}\}$. The set R contains the identity of the group $\mathrm{Aut}(\mathbb{D})$, which is the function $\psi_0(z) = z$. Furthermore, the inverse of ψ_a is ψ_{-a}, since

$$\psi_a[\psi_{-a}(z)] = z \quad \text{and} \quad \psi_{-a}[\psi_a(z)] = z.$$

In other words, the set R is closed under inverses.

However, R fails to be a subgroup of $\mathrm{Aut}(\mathbb{D})$ as it is not closed under the operation of function composition. Indeed, if a and b are elements of $\mathbb{D}$, then we have

$$\psi_a[\psi_b(z)] = e^{i\theta}\frac{z+c}{1+\bar{c}z}, \quad \text{where} \quad c = \psi_b(a) \quad \text{and} \quad \theta = 2\,\mathrm{Arg}(1+a\bar{b}). \tag{8.1.1}$$

This means that $\psi_a\psi_b \in R$ if and only if $a\bar{b} \in (-1,1)$. Provided that both a and b are nonzero, this occurs if and only if the line passing through the origin and a is precisely the same line as the one passing through the origin and b.

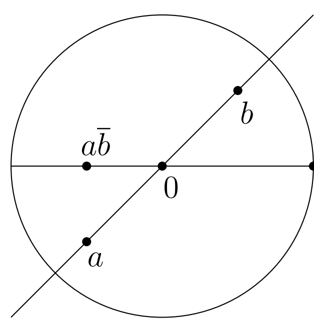


Figure 8.1: The points a, b, and $a\bar{b}$.

Consequently, if we fix some $w \in \mathbb{D} \setminus \{0\}$ and let L_w denote the line passing through the origin and w, then the collection of automorphisms

$$R_w := \{\psi_a : a \in \mathbb{D} \cap L_w\}$$

is a subgroup of $\mathrm{Aut}(\mathbb{D})$.

The functions ψ_a will play an essential role in the upcoming material. Here we record a few observations about them.

Lemma 8.1.1. *If $\alpha \in [0,1)$, then we have $\psi_{-\alpha}(x) \leq x$ for every $x \in (0,1)$, with equality holding if and only if $\alpha = 0$.*

Proof. Consider the function

$$f(x) := x - \psi_{-\alpha}(x) = \frac{x - \alpha x^2}{1 - \alpha x} - \frac{x - \alpha}{1 - \alpha x} = \frac{\alpha(1 - x^2)}{1 - \alpha x}.$$

If $\alpha \in (0,1)$, then we have $f(x) > 0$ for $x \in (0,1)$. If $\alpha = 0$, then we have $f(x) = 0$. This establishes the lemma.

□

Lemma 8.1.2. *If $a \in \mathbb{D} \setminus \{0\}$, then*

- *$\psi_{|a|}(x)$ is strictly increasing on $\mathbb{R} \setminus \{-1/|a|\}$*
- *$\psi_{|a|}(x) > 0$ for $x \in (0,\infty)$*

Proof. We have

$$\psi_{|a|}(x) = \frac{x + |a|}{1 + |a|x} \quad \Rightarrow \quad \psi'_{|a|}(x) = \frac{1 - |a|^2}{(1 + |a|x)^2}, \quad x \in \mathbb{R} \setminus \{-1/|a|\}.$$

Thus $\psi'_{|a|}(x) > 0$ for $x \in \mathbb{R} \setminus \{-1/|a|\}$, and claim one is established. Claim two follows from the definition of $\psi_{|a|}$, and the proof is complete.

□

The functions ψ_a as Möbius transformations

We remark that any element of $\mathrm{Aut}(\mathbb{D})$ also belongs to $\mathrm{Aut}(\hat{\mathbb{C}})$, the the collection of Möbius transformations. We will frequently make use of the following concept: *a Möbius Tranformation maps circles or lines in $\hat{\mathbb{C}}$ to circles or lines in $\hat{\mathbb{C}}$.*

If τ is a Möbius transformation, then we can find some $(a, b, c, d) \in \mathbb{C}^4$ such that $ad - bc = 1$ and $\tau(z) = (az + b)/(cz + d)$. In this case, we say that (a, b, c, d) *represents* τ *in normalized form.*

Lemma 8.1.3. *If* $a \in \mathbb{D} \setminus \{0\}$*, then the function*

$$\psi_a(z) := \frac{z + a}{1 + \overline{a}z}$$

is a hyperbolic Möbius transformation. Its fixed points are $a/|a|$ *and* $-a/|a|$*. Moreover, we have*

- $\psi_a(\infty) = 1/\overline{a}$
- $\psi_a(-1/\overline{a}) = \infty$

Proof. In order to express ψ_a in normalized form, we write

$$\psi_a(z) = \frac{\lambda z + \lambda a}{\lambda \overline{a} z + \lambda}, \quad \text{where} \quad \lambda := \frac{1}{\sqrt{1 - |a|^2}}.$$

Since $2\lambda = \frac{2}{\sqrt{1-|a|^2}} > 2$, we recognize that ψ_a is hyperbolic. The computations

$$\psi_a\left(\frac{a}{|a|}\right) = \frac{\frac{a}{|a|} + a}{1 + \overline{a}\frac{a}{|a|}} = \frac{a}{|a|} \cdot \frac{1 + |a|}{1 + |a|} = \frac{a}{|a|}$$

and

$$\psi_a\left(-\frac{a}{|a|}\right) = \frac{-\frac{a}{|a|} + a}{1 - \overline{a}\frac{a}{|a|}} = -\frac{a}{|a|} \cdot \frac{1 - |a|}{1 - |a|} = -\frac{a}{|a|}$$

show that $a/|a|$ and $-a/|a|$ are fixed by ψ_a. Since a Möbius transformation that is not the identity can have at most two fixed points, these are the only fixed points of the transformation.

For $z \neq 0$, we may write

$$\psi_a(z) = \frac{z + a}{1 + \overline{a}z} = \frac{1 + \frac{a}{z}}{\frac{1}{z} + \overline{a}},$$

which implies $\psi_a(\infty) = 1/\overline{a}$. Finally, we note that $1 + \overline{a_j}z = 0$ if and only if $\overline{a_j}z = -1$, which happens if and only if $z = -1/\overline{a_j}$. This establishes the lemma.

□

Circles and lines in $\hat{\mathbb{C}}$

In the literature, the phrase *circle in the extended complex plane* may refer to either a true circle (a subset of $\mathbb{C}$) or a line (a subset of $\mathbb{C} \cup \{\infty\}$). Here, when we speak of a circle K, we always mean a true circle. We will often write $K \subset \mathbb{C}$ to emphasize this idea.

Definition If $a \in \mathbb{C} \setminus \{0\}$, then let L_a^* denote the line in the complex plane passing through the origin and a. We define $L_a := L_a^* \cup \{\infty\}$.

The family of invariant circles of ψ_a

If $a \in \mathbb{D} \setminus \{0\}$, then Lemma 8.1.3 indicates that the so-called *family of invariant circles of ψ_a* consists of L_a as well as any circle $K \subset \mathbb{C}$ such that $\{a/|a|, -a/|a|\} \subset K$. The reader is referred to page 410 of [14] for more details. In particular, we have

$$\psi_a(L_a) = L_a, \quad a \in \mathbb{C} \setminus \{0\}.$$

We say that a circle $K \subset \mathbb{C}$ is symmetric about the line L if L contains the center of K. In this case, the line L intersects K at exactly two points. If we let z and w denote these two points, then the radius of K is given by $|z - w|/2$ and the center by $(z + w)/2$.

The proof of the following lemma utilizes the symmetry principle for Möbius transformations. The reader may consult page eleven of [16] for more details.

Lemma 8.1.4. *Let $a \in \mathbb{D} \setminus \{0\}$. If $K_{c,r} \subset \mathbb{C}$ is a circle symmetric about L_a such that $-1/\overline{a} \notin K_{c,r}$, then $\psi_a(K_{c,r})$ is also a circle symmetric about L_a.*

Proof. Since ψ_a is a Möbius Transformation and $K_{c,r}$ a circle, $\psi_a(K_{c,r})$ is either a circle or a line. Since $-1/\overline{a} \notin K_{c,r}$, we have $\psi_a(K_{c,r}) \subset \mathbb{C}$ by Lemma 8.1.3. Therefore $\psi_a(K_{c,r})$ is indeed a circle. Let $K_{\kappa,\rho}$ denote the circle $\psi_a(K_{c,r})$.

It remains to be seen that $K_{\kappa,\rho}$ is actually symmetric about L_a. To establish this, we must show that $\kappa \in L_a$. Since $K_{c,r}$ is symmetric about L_a, we have $c \in L_a$. Choose some $\zeta \in L_a \setminus \{c\}$ and let ζ^* denote the inverse of ζ with respect to $K_{c,r}$.

Note that we have $\zeta^* \in L_a$. To see this, observe that the line which passes through ζ and ζ^* must also pass through the c. (This follows from the fact that ζ^* is the inverse of ζ with respect to $K_{c,r}$.) But this line is L_a, since L_a contains both c and ζ. Hence $\zeta^* \in L_a$, as claimed.

By the symmetry principle for Möbius transformations, we have that $\psi_a(\zeta^*)$ is the inverse of $\psi_a(\zeta)$ with respect to $\psi_a(K_{c,r}) = K_{\kappa,\rho}$. Thus, the line L^* that passes through $\psi_a(\zeta^*)$ and $\psi_a(\zeta)$ must also pass through κ. Meanwhile, since we have both $\{\zeta, \zeta^*\} \subset L_a$ and $\psi_a(L_a) = L_a$, we see that $\{\psi_a(\zeta), \psi_a(\zeta^*)\} \subset L_a$. Therefore $L^* = L_a$, whence L_a passes through κ. This validates the assertion that $K_{\kappa,\rho} = \psi_a(K_{c,r})$ is symmetric about L_a and completes the proof.

□

Perturbations of circles centered at the origin

Theorem 8.1.5. *For every $(a, \sigma) \in \mathbb{D} \times (0, \infty) \setminus \{1/|a|\}$, there exists a unique circle $K_{c,r} \subset \mathbb{C}$ such that $\psi_a(\mathbb{T}_\sigma) = K_{c,r}$. The values for c and r are given by the relations*

$$c = \frac{a(1-\sigma^2)}{1-|a|^2\sigma^2} \quad \textit{and} \quad r = \frac{\sigma(1-|a|^2)}{|1-|a|^2\sigma^2|}.$$

Proof. The uniqueness claim is clear: if $\psi_a(\mathbb{T}_\sigma) = K_{c_1,r_1}$ and $\psi_a(\mathbb{T}_\sigma) = K_{c_2,r_2}$, then $K_{c_1,r_1} = K_{c_2,r_2}$. Here, we establish existence and demonstrate that the formulas given above are correct.

We note that when $a = 0$, ψ_a is the identity function, and so the lemma holds. Therefore, we suppose $a \neq 0$ and we let θ denote the principal argument of a.

Since $\mathbb{T}_\sigma$ is a circle symmetric about L_a and since $-1/\overline{a} \notin \mathbb{T}_\sigma$, Lemma 8.1.4 tells us that $\psi_a(\mathbb{T}_\sigma) \subset \mathbb{C}$ is a circle symmetric about L_a. We let $K_{c,r}$ denote this circle.

Since $K_{c,r}$ is symmetric about L_a, we can write $L_a \cap K_{c,r} = \{w_1, w_2\}$, where

$$c = \frac{w_1 + w_2}{2} \quad \text{and} \quad r = \frac{|w_1 - w_2|}{2}.$$

Now we note that

$$\begin{aligned}
\{\psi_a(e^{i\theta}\sigma), \psi_a(-e^{i\theta}\sigma)\} &= \psi_a(\{e^{i\theta}\sigma, -e^{i\theta}\sigma\}) \\
&= \psi_a(L_a \cap \mathbb{T}_\sigma) \\
&\subset \psi_a(L_a) \cap \psi_a(\mathbb{T}_\sigma) = L_a \cap K_{c,r} = \{w_1, w_2\}.
\end{aligned}$$

Therefore, we may write

$$c = \frac{\psi_a(e^{i\theta}\sigma) + \psi_a(-e^{i\theta}\sigma)}{2} = \frac{1}{2}\left[\frac{e^{i\theta}\sigma + a}{1 + \overline{a}e^{i\theta}\sigma} + \frac{-e^{i\theta}\sigma + a}{1 - \overline{a}e^{i\theta}\sigma}\right].$$

Since $a = |a|e^{i\theta}$, we have $\overline{a}e^{i\theta} = |a|$. Therefore, we find

$$\begin{aligned}
c &= \frac{1}{2}\left[\frac{e^{i\theta}\sigma + a}{1 + |a|\sigma} + \frac{-e^{i\theta}\sigma + a}{1 - |a|\sigma}\right] \\
&= \frac{1}{2}\left[\frac{(e^{i\theta}\sigma + a)(1 - |a|\sigma) + (-e^{i\theta}\sigma + a)(1 + |a|\sigma)}{1 - |a|^2\sigma^2}\right] \\
&= \frac{1}{2}\left[\frac{e^{i\theta}\sigma - e^{i\theta}|a|\sigma^2 + a - a|a|\sigma - e^{i\theta}\sigma - e^{i\theta}|a|\sigma^2 + a + a|a|\sigma}{1 - |a|^2\sigma^2}\right] \\
&= \frac{1}{2}\left[\frac{-e^{i\theta}|a|\sigma^2 + a - e^{i\theta}|a|\sigma^2 + a}{1 - |a|^2\sigma^2}\right] \\
&= \frac{1}{2}\left[\frac{2(a - e^{i\theta}|a|\sigma^2)}{1 - |a|^2\sigma^2}\right] = \frac{a - a\sigma^2}{1 - |a|^2\sigma^2} = \frac{a(1 - \sigma^2)}{1 - |a|^2\sigma^2}.
\end{aligned}$$

Similarly, the computation

$$\begin{aligned}\frac{\psi_a(e^{i\theta}\sigma)-\psi_a(-e^{i\theta}\sigma)}{2} &= \frac{1}{2}\left[\frac{e^{i\theta}\sigma+a}{1+|a|\sigma}-\frac{-e^{i\theta}\sigma+a}{1-|a|\sigma}\right]\\ &= \frac{1}{2}\left[\frac{(e^{i\theta}\sigma+a)(1-|a|\sigma)-(-e^{i\theta}\sigma+a)(1+|a|\sigma)}{1-|a|^2\sigma^2}\right]\\ &= \frac{1}{2}\left[\frac{2e^{i\theta}\sigma-2a|a|\sigma}{1-|a|^2\sigma^2}\right]\\ &= \frac{1}{2}\left[\frac{2e^{i\theta}\sigma(1-ae^{-i\theta}|a|)}{1-|a|^2\sigma^2}\right]=\frac{e^{i\theta}\sigma(1-|a|^2)}{1-|a|^2\sigma^2}\end{aligned}$$

shows that

$$r=\frac{|\psi_a(e^{i\theta}\sigma)-\psi_a(-e^{i\theta}\sigma)|}{2}=\frac{\sigma(1-|a|^2)}{|1-|a|^2\sigma^2|}.$$

This completes the proof of the theorem.

□

Corollary 8.1.6. *For every $(a,\sigma)\in\mathbb{D}\times(0,1)$, there exists a unique circle $K_{c,r}\subset\mathbb{D}$ such that $\psi_a(\mathbb{T}_\sigma)=K_{c,r}$. The values of c and r are given by the formulas*

$$c=\frac{a(1-\sigma^2)}{1-|a|^2\sigma^2}=\chi_{-a\sigma^2}(a)\quad \textit{and}\quad r=\frac{\sigma(1-|a|^2)}{1-|a|^2\sigma^2}=\chi_{-\sigma|a|^2}(\sigma).$$

Furthermore, we have

- $|c|\le|a|$, *with equality holding if and only if* $a=0$
- $r\le\sigma$, *with equality holding if and only if* $a=0$
- $|c|+r=\dfrac{|a|+\sigma}{1+|a|\sigma}$
- $|c|-r=\dfrac{|a|-\sigma}{1-|a|\sigma}$

Moreover, if θ denotes the principal argument for $a\neq 0$, then we have

- $\psi_a(\sigma e^{i\theta})=c+re^{i\theta}$

- $\psi_a(-\sigma e^{i\theta}) = c - re^{i\theta}$

Proof. Since $\psi_a \in \mathrm{Aut}(\mathbb{D})$ and since $\mathbb{T}_\sigma \subset \mathbb{D}$, we have $K_{c,r} = \psi_a(\mathbb{T}_\sigma) \subset \mathbb{D}$. Also, since $|a^2\sigma^2| < 1$, we have $|1 - |a|^2\sigma^2| = 1 - |a|^2\sigma^2$. This justifies expression for r given in the corollary.

Note that we may write

$$|c| = \frac{|a| - |a|\sigma^2}{1 - |a|^2\sigma^2} = \chi_{-|a|\sigma^2}(|a|).$$

By Lemma 8.1.1, we have $\psi_{-|a|\sigma^2}(|a|) \leq |a|$, with equality holding if and only if $a = 0$. This proves the first statement. Similarly, we have $r = \psi_{-\sigma|a|^2}(\sigma) \leq \sigma$, with equality holding if and only if $a = 0$. This proves the second statement.

Note that we may write

$$|c| - r = \frac{|a|(1-\sigma^2)}{1-|a|^2\sigma^2} - \frac{\sigma(1-|a|^2)}{1-|a|^2\sigma^2} = \frac{(1+|a|\sigma)(|a|-\sigma)}{(1+|a|\sigma)(1-|a|\sigma)} = \frac{|a|-\sigma}{1-|a|\sigma}$$

and

$$|c| + r = \frac{|a|(1-\sigma^2)}{1-|a|^2\sigma^2} + \frac{\sigma(1-|a|^2)}{1-|a|^2\sigma^2} = \frac{(1-|a|\sigma)(|a|+\sigma)}{(1+|a|\sigma)(1-|a|\sigma)} = \frac{|a|+\sigma}{1+|a|\sigma}.$$

This establishes claims three and four.

If $a \neq 0$, then let θ denote the principal argument of a. Note that the formulas from Theorem 8.1.5 imply that θ is also the principal argument of c. In this case, we may write

$$c + re^{i\theta} = e^{i\theta}(|c| + r) = \frac{a + \sigma e^{i\theta}}{1 + |a|\sigma} = \frac{a + \sigma e^{i\theta}}{1 + \overline{a}e^{i\theta}\sigma} = \psi_a(\sigma e^{i\theta})$$

and

$$c - re^{i\theta} = e^{i\theta}(|c| - r) = \frac{a - \sigma e^{i\theta}}{1 - |a|\sigma} = \frac{a - \sigma e^{i\theta}}{1 - \overline{a}e^{i\theta}\sigma} = \psi_a(-\sigma e^{i\theta}),$$

and the proof is complete. $\square$

Corollary 8.1.7. *Let $a \in \mathbb{D} \setminus \{0\}$ and let $\varsigma \in (1/|a|, \infty)$. There exists a unique circle $K_{c,r} \subset \Delta_1$ such that $\psi_a(\mathbb{T}_\sigma) = K_{c,r}$. The values of c and r are given by the formulas*

$$c = \frac{a(\varsigma^2 - 1)}{|a|^2\varsigma^2 - 1} \quad \textit{and} \quad r = \frac{\varsigma(1 - |a|^2)}{|a|^2\varsigma^2 - 1}.$$

Furthermore, we have

- $|c| - r = \dfrac{|a| + \varsigma}{1 + |a|\varsigma}$

Moreover, if θ denotes the principal argument for a, then we have

- $\psi_a(\varsigma e^{i\theta}) = c - re^{i\theta}$

Proof. Since $\psi_a \in \mathrm{Aut}(\mathbb{D})$ and since $\mathbb{T}_\varsigma \subset \Delta_1$, we have $K_{c,r} = \psi_a(\mathbb{T}_\varsigma) \subset \mathbb{D}$. Also, since we have

$$1 < 1/|a| < \varsigma,$$

we have $|1 - |a|^2\varsigma^2| = |a|^2\varsigma^2 - 1$. This justifies expression for r given in the corollary. Note that we may write

$$|c| = \frac{|a| \cdot |1 - \varsigma^2|}{|1 - |a|^2\varsigma^2|} = \frac{|a|(\varsigma^2 - 1)}{|a|^2\varsigma^2 - 1}.$$

Then we have

$$|c| - r = \frac{|a|(\varsigma^2 - 1)}{|a|^2\varsigma^2 - 1} - \frac{\varsigma(1 - |a|^2)}{|a|^2\varsigma^2 - 1} = \frac{(|a|\varsigma - 1)(|a| + \varsigma)}{(|a|\varsigma + 1)(|a|\varsigma - 1)} = \frac{|a| + \varsigma}{1 + |a|\varsigma}$$

If $a \neq 0$, then let θ denote the principal argument of a. Note that θ is also the principal argument of c. Therefore, we may write

$$c - re^{i\theta} = e^{i\theta}(|c| - r) = \frac{a + \varsigma e^{i\theta}}{1 + |a|\varsigma} = \frac{a + \varsigma e^{i\theta}}{1 + \bar{a}e^{i\theta}\varsigma} = \psi_a(\varsigma e^{i\theta}),$$

and the proof is complete. $\square$

Disks contained in $\mathbb{D}$

Lemma 8.1.8. *Let $(a,\sigma) \in \mathbb{D}\times(0,1)$. If $K_{c,r} \subset \mathbb{D}$ is the circle which satisfies $\psi_a(\mathbb{T}_\sigma) = K_{c,r}$, then we have*

- $\psi_a(\mathbb{D}_\sigma) = \mathring{D}_{c,r}$
- $\psi_a(\{z : \sigma < |z| < 1\}) = \mathbb{D} \setminus D_{c,r}$
- $\psi_a(\overline{\mathbb{D}}_\sigma) = D_{c,r}$
- $a \in \mathring{D}_{c,r}$

Proof. Note that the set $\mathbb{D} \setminus K_{c,r}$ consists of two components. One of these, $\mathring{D}_{c,r}$, is simply connected, while the other, $\mathbb{D} \setminus D_{c,r}$, is doubly connected. If we apply the function ψ_a to both sides of the equation

$$\mathbb{D} \setminus \mathbb{T}_\sigma = \mathbb{D}_\sigma \cup \{z : \sigma < |z| < 1\},$$

then we obtain $\mathbb{D} \setminus K_{c,r} = \psi_a(\mathbb{D}_\sigma) \cup \psi_a(\{z : \sigma < |z| < 1\})$. Since this represents $\mathbb{D} \setminus K_{c,r}$ as the disjoint union of two open sets, we see that these must be the components of $\mathbb{D} \setminus K_{c,r}$. Since $\psi_a(\mathbb{D}_\sigma)$ is conformally equivalent to the simply connected domain $\mathbb{D}_\sigma$ while $\psi_a(\{z : \sigma < |z| < 1\})$ is conformally equivalent to the doubly connected domain $\{z : \sigma < |z| < 1\}$, we conclude that $\psi_a(\mathbb{D}_\sigma) = \mathring{D}$ and

$$\psi_a(\{z : \sigma < |z| < 1\}) = \mathbb{D} \setminus D_{c,r}.$$

The third claim is true by the the fact that ψ_a is a homeomorphism.

Finally, note that $\psi_a(0) = a$. Since $0 \in \mathbb{D}_\sigma$, we have $a = \psi_a(0) \subset \psi_a(\mathbb{D}_\sigma) = \mathring{D}$, and the proof is complete.

□

Centering circles contained in $\mathbb{D}$ to the origin

Next, we set out to show that for every $K_{c,r} \subset \mathbb{D}$, there exists a unique $(a, \sigma) \in \mathbb{D} \times (0,1)$ such that $\psi_a(\mathbb{T}_\sigma) = K_{c,r}$. This is accomplished via a series of lemmas.

Lemma 8.1.9. *Let $K_{c,r} \subset \mathbb{D}$ with $c \in (0,1)$. If we define $F : [-1,1] \to [-1,1]$ by*

$$F(t) := \frac{\psi_{-t}(c+r) + \psi_{-t}(c-r)}{2} = \frac{1}{2}\left[\frac{c+r-t}{1-(c+r)t} + \frac{c-r-t}{1-(c-r)t}\right],$$

then there exists a unique $a \in [-1,1]$ such that $F(a) = 0$. More precisely, we have $a \in (0,1)$.

Proof. We will begin by showing that

- F is continuous on $[-1,1]$
- $F(0) > 0$
- $F(1) < 0$

Then existence will follow by Bolzano's Theorem. To establish uniqueness, we will show that F is strictly decreasing on $(-1,1)$.

Existence To see that F is continuous on $[-1,1]$, we write

$$\frac{c+r-t}{1-(c+r)t} = -\frac{t-(c+r)}{1-(c+r)t} = \psi_{-(c+r)}(t) \quad \text{and} \quad \frac{c-r-t}{1-(c-r)t} = \psi_{-(c-r)}(t).$$

Since $\psi_{-(c+r)}$ and $\psi_{-(c-r)}$ are elements of $\mathrm{Aut}(\mathbb{D})$, we see that

$$F(t) = \frac{\psi_{-(c+r)}(t) + \psi_{-(c-r)}(t)}{2} \tag{8.1.2}$$

is indeed continuous on its domain.

Meanwhile, since ψ_0 is the identity function, we have $F(0) = c > 0$.

Also, if we write $k_1 := c + r$ and $k_2 := c - r$, then we can compute

$$\begin{aligned}
F(1) &= \frac{1}{2}\left(\frac{c+r-1}{1-(c+r)} + \frac{c-r-1}{1-(c-r)}\right) \\
&= \frac{1}{2}\left(\frac{k_1-1}{1-k_1} + \frac{k_2-1}{1-k_2}\right) \\
&= \frac{1}{2}\left[\frac{(k_1-1)(1-k_2)+(k_2-1)(1-k_1)}{(1-k_1)(1-k_2)}\right] \\
&= \frac{1}{2}\left[\frac{-(1-k_1)(1-k_2)-(1-k_2)(1-k_1)}{(1-k_1)(1-k_2)}\right] \\
&= \frac{1}{2}\left[\frac{-2(1-k_1)(1-k_2)}{(1-k_1)(1-k_2)}\right] = -1,
\end{aligned}$$

and the existence portion of the proof is complete.

Uniqueness Direct computation shows that for $t \in (-1, 1)$, we have

$$\psi'_{-(c+r)}(t) = -\frac{1-(c+r)^2}{[t(c+r)-1]^2} \quad \text{and} \quad \psi'_{-(c-r)}(t) = -\frac{1-(c-r)^2}{[t(c-r)-1]^2}.$$

Now, the conditions $K_{c,r} \subset \mathbb{D}$ and $c \in (0, 1)$ imply both $(c+r)^2 < 1$ and $(c-r)^2 < 1$. Then we have both $\psi'_{-(c+r)}(t) < 0$ and $\psi'_{-(c-r)}(t) < 0$ on $(-1, 1)$. If we apply this information to the representation of F given in (8.1.2), then we see that F is strictly decreasing on $(-1, 1)$. This establishes uniqueness and completes the proof of the lemma.

□

Lemma 8.1.10. *If $K_{c,r} \subset \mathbb{D}$ with $c \in (0, 1)$, then there exists a unique $(a, \sigma) \in (-1, 1) \times (0, 1)$ such that $\psi_a(\mathbb{T}_\sigma) = K_{c,r}$. More specifically, we have $a \in (0, 1)$.*

Proof. Put $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$. For each $t \in (-1, 1)$, we have

- $\psi_{-t}(\overline{\mathbb{R}}) = \overline{\mathbb{R}}$
- $\psi_{-t}(K_{c,r})$ is a circle contained in $\mathbb{D}$ symmetric about the real axis.

The second observation implies that $\psi_{-t}(K_{c,r})$ intersects the real axis at two points, say x_1 and x_2, and that the center of $\psi_{-t}(K_{c,r})$ is given by $(x_1 + x_2)/2$. Now note that

$$\begin{aligned}\{\psi_{-t}(c-r), \psi_{-t}(c+r)\} &= \psi_{-t}(\{c-r, c+r\}) \\ &= \psi_{-t}(K_{c,r} \cap \overline{\mathbb{R}}) \\ &\subset \psi_{-t}(K_{c,r}) \cap \psi_{-t}(\overline{\mathbb{R}}) \\ &= \psi_{-t}(K_{c,r}) \cap \overline{\mathbb{R}} \\ &= \{x_1, x_2\}.\end{aligned}$$

This implies that the center of $\psi_{-t}(K_{c,r})$ is given by the expression

$$\frac{\psi_{-t}(c-r) + \psi_{-t}(c+r)}{2} = F(t),$$

where F is the function from lemma 8.1.9. By that lemma, there exists a unique $a \in (-1, 1)$ such that $F(a) = 0$. In fact, we know that $a \in (0, 1)$. This means that $\psi_{-a}(K_{c,r})$ is a circle centered at the origin. Therefore, there exists a unique $\sigma \in (0, 1)$ such that $\psi_{-a}(K_{c,r}) = \mathbb{T}_\sigma$. Since ψ_a is the inverse of ψ_{-a}, the proof is complete.

□

Theorem 8.1.11. *For every $K_{c,r} \subset \mathbb{D}$, there exists a unique $(a, \sigma) \in \mathbb{D} \times (0, 1)$ such that $\psi_a(\mathbb{T}_\sigma) = K_{c,r}$. If $c = 0$, then $a = 0$ and $\sigma = r$. If $c \neq 0$, then*

$$a = \frac{1 + |c|^2 - r^2 - \sqrt{(1 + |c|^2 - r^2)^2 - 4|c|^2}}{2\overline{c}} \quad \textit{and} \quad \sigma = |\psi_{-a}(c + re^{i\theta})|,$$

where θ denotes the principal argument of c.

Proof. Existence If $c = 0$, then we can take $a = c$ and $\sigma = r$, since ψ_0 is the identity function. Therefore, we assume $c \neq 0$. Let θ denote the principal argument of c and define $\rho(z) := e^{-i\theta} z$. Then $\rho(K_{c,r})$ is a circle contained in the unit disk centered on $(0, 1)$. By

Lemma 8.1.10, there exists a unique $(\alpha,\sigma)\in(-1,1)\times(0,1)$ such that $\psi_\alpha(\mathbb{T}_\sigma)=\rho(K_{c,r})$. Moreover, we know that $\alpha>0$. Since $\mathbb{T}_\sigma=\rho(\mathbb{T}_\sigma)$, we may write the previous equation as $\psi_\alpha[\rho(\mathbb{T}_\sigma)]=\rho(K_{c,r})$. Applying ρ^{-1} to both sides of this equation gives $(\rho^{-1}\circ\psi_\alpha\circ\rho)(\mathbb{T}_\sigma)=K_{c,r}$. We compute

$$(\rho^{-1}\circ\psi_\alpha\circ\rho)(z)=(\rho^{-1}\circ\psi_\alpha)(e^{-i\theta}z)=\rho^{-1}\left(\frac{e^{-i\theta}z+\alpha}{1+\overline{\alpha}e^{-i\theta}z}\right).$$

Since $\alpha\in(0,1)$, we have $\overline{\alpha}=\alpha$. Therefore, we can write

$$\begin{aligned}(\rho^{-1}\circ\psi_\alpha\circ\rho)(z)=\rho^{-1}\left(\frac{e^{-i\theta}z+\alpha}{1+\alpha e^{-i\theta}z}\right)&=e^{i\theta}\frac{e^{-i\theta}z+\alpha}{1+\alpha e^{-i\theta}z}\\&=\frac{z+\alpha e^{i\theta}}{1+\alpha e^{-i\theta}z}=\frac{z+a}{1+\overline{a}z}=\psi_a,\end{aligned}$$

where $a:=\alpha e^{i\theta}$. We note that $\mathrm{Arg}(a)=\mathrm{Arg}(c)$.

Then the previously established equation $(\rho^{-1}\circ\psi_\alpha\circ\rho)(\mathbb{T}_\sigma)=K_{c,r}$ becomes $\psi_a(\mathbb{T}_\sigma)=K_{c,r}$, and existence is established.

Uniqueness Suppose that the elements (a,σ) and (b,τ) in $\mathbb{D}\times 0,1$ satisfy $\psi_a(\mathbb{T}_\sigma)=K_{c,r}$ and $\psi_b(\mathbb{T}_\tau)=K_{c,r}$. Then we can use the formula for r from Theorem **??** to write

$$r=\frac{\sigma(1-|a|^2)}{1-|a|^2\sigma^2}\quad\text{and}\quad r=\frac{\tau(1-|b|^2)}{1-|b|^2\tau^2}. \tag{8.1.3}$$

Now, the equations $\psi_a(\mathbb{T}_\sigma)=K_{c,r}$ and $\psi_b(\mathbb{T}_\tau)=K_{c,r}$ give $\psi_b^{-1}\psi_a(\mathbb{T}_\sigma)=\mathbb{T}_\tau$. In other words, $\psi_b^{-1}\psi_a$ is a conformal map of $\{z:\sigma<|z|<1\}$ onto $\{z:\tau<|z|<1\}$. This is possible if and only if $\sigma=\tau$. [Nehari, pg. 334] If we apply this information to (8.1.3), then we have

$$\frac{1-|a|^2}{1-|a|^2\sigma^2}=\frac{1-|b|^2}{1-|b|^2\sigma^2}. \tag{8.1.4}$$

We claim that this implies $|a| = |b|$. To see this, define the function $f : [0, 1] \to [0, 1]$ by the relation

$$f(x) := \frac{1 - x^2}{1 - x^2\sigma^2}.$$

Direct computation shows that

$$f'(x) = \frac{-2x(1 - \sigma^2)}{(1 - x^2\sigma^2)^2}, \quad x \in (0, 1),$$

which implies that f is strictly decreasing on $(0, 1)$. This implies that (8.1.4) holds if and only if $|a| = |b|$.

Next, we can use the formula for c from Theorem **??** to write

$$c = \frac{a(1 - \sigma^2)}{1 - |a|^2\sigma^2} \quad \text{and} \quad c = \frac{b(1 - \tau^2)}{1 - |b|^2\tau^2}.$$

But since we know that $\sigma = \tau$ and $|a| = |b|$, we must also have $a = b$. In other words, we have $(a, \sigma) = (b, \tau)$, and uniqueness is verified.

Computation We have shown that there exists a unique $(a, \sigma) \in \mathbb{D} \times (0, 1)$ such that $\psi_a(\mathbb{T}_\sigma) = K_{c,r}$. If $c = 0$, then we can take $a = c$ and $\sigma = r$. If $c \neq 0$, then we know that the principal argument of a agrees with that of c. Here we find an expression for a.

Let θ denote the principal argument of c. By Corollary 8.1.6, we have

$$\psi_a(\sigma e^{i\theta}) = c + re^{i\theta} \quad \text{and} \quad \psi_a(-\sigma e^{i\theta}) = c - re^{i\theta}.$$

Let k_1 denote $c + re^{i\theta}$ and let k_2 denote $c - re^{i\theta}$. Then the relationships above imply $\psi_{-a}(k_1) + \psi_{-a}(k_2) = 0$. In other words, we have

$$\begin{aligned} & \frac{k_1 - a}{1 - \overline{a}k_1} + \frac{k_2 - a}{1 - \overline{a}k_2} = 0 \\ \Rightarrow \quad & (k_1 - a)(1 - \overline{a}k_2) + (k_2 - a)(1 - \overline{a}k_1) = 0 \end{aligned}$$

$$\Rightarrow \qquad k_1 - \overline{a}k_1k_2 - a + |a|^2k_2 + k_2 - \overline{a}k_1k_2 - a + |a|^2k_1 = 0$$

$$\Rightarrow \qquad (|a|^2 + 1)(k_1 + k_2) - 2(\overline{a}k_1k_2 + a) = 0$$

Note that $k_1 + k_2 = 2c$. Also, note that

$$k_1k_2 = (c + re^{i\theta})(c - re^{i\theta}) = c^2 - r^2e^{2i\theta} = e^{2i\theta}(|c|^2 - r^2)$$

$$\Rightarrow \qquad \overline{a}k_1k_2 = |a|e^{-i\theta} \cdot e^{2i\theta}(|c|^2 - r^2) = a(|c|^2 - r^2)$$

$$\Rightarrow \qquad \overline{a}k_1k_2 + a = a(|c|^2 - r^2) + a = a(|c|^2 - r^2 + 1).$$

Returning to our main line of computation, we see that

$$\frac{k_1 - a}{1 - \overline{a}k_1} + \frac{k_2 - a}{1 - \overline{a}k_2} = 0$$

$$\Rightarrow \qquad 2c(|a|^2 + 1) - 2a(|c|^2 - r^2 + 1) = 0$$

$$\Rightarrow \qquad c(|a|^2 + 1) - a(|c|^2 - r^2 + 1) = 0$$

$$\Rightarrow \qquad c\,|a|^2 + (r^2 - |c|^2 - 1)a + c = 0$$

Now we write $c|a|^2 = |c|e^{i\theta}|a|^2 = |c|e^{-i\theta}e^{2i\theta}|a|^2 = \overline{c}a^2$. Then our equation becomes

$$\overline{c}a^2 + (r^2 - |c|^2 - 1)a + c = 0$$

$$\Rightarrow \qquad a = \frac{1 + |c|^2 - r^2 \pm \sqrt{(r^2 - |c|^2 - 1)^2 - 4|c|^2}}{2\overline{c}}$$

$$\Rightarrow \qquad a = e^{i\theta}\frac{1 + |c|^2 - r^2 \pm \sqrt{(r^2 - |c|^2 - 1)^2 - 4|c|^2}}{2|c|}$$

It is shown in Lemmas 8.1.12, 8.1.13, and 8.1.14 that taking the "+" option would lead to $|a| > 1$ while taking the "−" option would lead to $|a| < 1$. Since we need the latter outcome, we have

$$a = e^{i\theta}\,\frac{1+|c|^2-r^2-\sqrt{(1+|c|^2-r^2)^2-4|c|^2}}{2|c|}.$$

Now that we have a, we can find σ by using the relation $\sigma e^{i\theta} = \psi_{-a}(c+re^{i\theta})$. The proof is complete.

□

8.1.1 Auxiliary computations

Lemma 8.1.12. *If $K_{c,r} \subset \mathbb{D}$, then $1+|c|^2-r^2 > 2|c|$.*

Proof. Note that the claim holds if and only if $|c|^2-2|c|-r^2+1 > 0$. Put $p(t) := t^2-2t-r^2+1$. We will show that $p(t) > 0$ for $t \in [0, 1-r)$. This will establish the claim, since $K_{c,r} \subset \mathbb{D}$ if and only if $0 \le |c| < 1-r$.

The quadratic formula tells us that $p(t)$ has its zeros at $t = 1+r$ and $t = 1-r$. Since $p(0) = 1-r^2 > 0$, we conclude that $p(t) > 0$ for $t \in [0, 1-r)$, and the proof is complete.

□

Lemma 8.1.13. *If $K_{c,r} \subset \mathbb{D}$, then*

$$-1 < \frac{1+|c|^2-r^2-\sqrt{(1+|c|^2-r^2)^2-4|c|^2}}{2|c|} < 1.$$

Proof. By Lemma 8.1.12, we have $1+|c|^2-r^2 > 2|c|$. Therefore, we can write

$$\begin{aligned}
(1+|c|^2-r^2)-(2|c|) &= \sqrt{[(1+|c|^2-r^2)-(2|c|)]^2} \\
&< \sqrt{[(1+|c|^2-r^2)-(2|c|)]\cdot[(1+|c|^2-r^2)+(2|c|)]} \\
&= \sqrt{(1+|c|^2-r^2)^2-4|c|^2} \\
&< \sqrt{[(1+|c|^2-r^2)+2|c|]\cdot[(1+|c|^2-r^2)+2|c|]} \\
&= \sqrt{[(1+|c|^2-r^2)+2|c|]^2]}
\end{aligned}$$

$$= (1 + |c|^2 - r^2) + 2|c|.$$

Now,

$$(1 + |c|^2 - r^2) - 2|c| < \sqrt{(1 + |c|^2 - r^2)^2 - 4|c|^2} < (1 + |c|^2 - r^2) + 2|c|$$

implies

$$-(1 + |c|^2 - r^2) + 2|c| > -\sqrt{(1 + |c|^2 - r^2)^2 - 4|c|^2} > (1 + |c|^2 - r^2) - 2|c|,$$

which, in turn, yields

$$2|c| > (1 + |c|^2 - r^2) - \sqrt{(1 + |c|^2 - r^2)^2 - 4|c|^2} > -2|c|.$$

This implies

$$-1 < \frac{1 + |c|^2 - r^2 - \sqrt{(1 + |c|^2 - r^2)^2 - 4|c|^2}}{2|c|} < 1,$$

and the proof is complete. □

Lemma 8.1.14. *If $K_{c,r} \subset \mathbb{D}$, then*

$$\frac{1 + |c|^2 - r^2 + \sqrt{(1 + |c|^2 - r^2)^2 - 4|c|^2}}{2|c|} > 1.$$

The two solutions to

$$c\,|a|^2 + (r^2 - |c|^2 - 1)a + c = 0$$

are

$$a_1 = e^{i\theta}\,\frac{1 + |c|^2 - r^2 - \sqrt{(1 + |c|^2 - r^2)^2 - 4|c|^2}}{2|c|}$$

and

$$a_2 = e^{i\theta}\,\frac{1 + |c|^2 - r^2 + \sqrt{(1 + |c|^2 - r^2)^2 - 4|c|^2}}{2|c|}$$

We see that

$$\frac{a_1 + a_2}{2} = e^{i\theta} \frac{1 + |c|^2 - r^2}{2|c|}.$$

By Lemma 8.1.12, we have

$$\left| \frac{a_1 + a_2}{2} \right| = \left| e^{i\theta} \frac{1 + |c|^2 - r^2}{2|c|} \right| = \frac{1 + |c|^2 - r^2}{2|c|} > 1.$$

Note that this gives $2 < |a_1 + a_2|$. Then by Lemma 8.1.13, we have

$$1 + |a_1| < 1 + 1 = 2 < |a_1 + a_2| \leq |a_1| + |a_2|,$$

which gives $|a_2| > 1$ and completes the proof.

8.2 Material related to the curve L_1 and its associated maps

8.2.1 Analytic Jordan curves, reflection, and the Schwarz function

An *analytic Jordan curve* L is a subset of the complex plane which is the image of a path $f(e^{i\theta})$, $0 \leq \theta \leq 2\pi$, where f is analytic and univalent (i.e., conformal) on some annulus $\mathbb{A}_r := \{z : r < |z| < 1/r\}$, with $0 < r < 1$. We call the function f a *parametrization* of the curve L. By the Jordan curve theorem, $\hat{\mathbb{C}} \setminus L$ consists of exactly two connected components; one is bounded, the other is not. They are called the *interior* and *exterior* of L, respectively, and we will denote these sets by the symbols $\mathrm{int}(L)$ and $\mathrm{ext}(L)$.

If $U_r := f(\mathbb{A}_r)$, then the transformation $z \to z^*$ given by

$$z^* := f\left(\frac{1}{\overline{f^{-1}(z)}} \right), \quad z \in U_r,$$

is called the *reflection* of z about L. It generalizes the notion of reflection about a line segment or a circle.

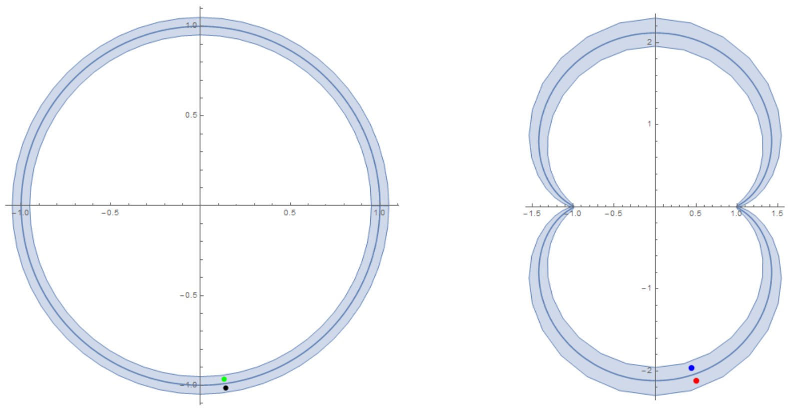


Figure 8.2: Reflection about an analytic Jordan curve.

In the figure above, we see the annulus $\mathbb{A}_{20/21}$ and its image under the map $f(z) = \sin(10\pi z/21)$. The unit circle $\mathbb{T}_1$ is shown in the plot on the left and the analytic jordan curve $f(\mathbb{T}_1)$ is shown on the right. Let $w_0 := .14 - 1.015i$ and $z_0 := f(w_0)$. Then

- z_0 is the red dot on the right
- $f^{-1}(z_0)$ is the black dot on the left
- $1/\overline{f^{-1}(z_0)}$ is the green dot on the left
- z_0^* is the blue dot on the right

The reflection map and the domain U_r have the following properties $(*)$:

- the reflection map $z \to z^*$ is univalent (i.e., injective) for all $z \in U_r$
- for each $z \in L$, we have $z^* = z$
- if $z \in U_r \cap \mathrm{ext}(L)$, then $z^* \in U_r \cap \mathrm{int}(L)$
- if $z \in U_r \cap \mathrm{int}(L)$, then $z^* \in U_r \cap \mathrm{ext}(L)$
- $(z^*)^* = z$ for all $z \in U_r$

Any domain containing an analytic Jordan curve having the properties $(*)$ mentioned above will be called a *reflection domain* for the reflection map generated by the curve.

The reflection map is closely related to the important function

$$S(z) := \overline{z^*} = \overline{f\left(\frac{1}{\overline{f^{-1}(z)}}\right)}, \quad z \in U_r.$$

This is called the *Schwarz function* of the curve L. By the first property in $(*)$, it is univalent in U_r. Furthermore, it is analytic in U_r and it has the property that $S(z) = \overline{z}$ for $z \in L$. By the Principle of Analytic Continuation (page 307 of [14]), there can be only one function analytic in U_r with this property. Hence, despite the apparent dependence on the parametrization f, both the the Schwarz function S and the reflection map $z \to z^*$ are univocally determined by L.

Now suppose that $U \subset \hat{\mathbb{C}}$ is a domain containing U_r such that S has a meromorphic, but not necessarily univalent, extension to U. Then one can extend the reflection map $z \to z^*$ to the whole of U by simply defining

$$z^* := \overline{S(z)}, \quad z \in U.$$

Generally speaking, this extended notion of reflection on U acts as an "actual reflection" only for z sufficiently close to L. More precisely, the properties $(*)$ enjoyed by the reflection map for $z \in U_r$ do not necessarily generalize to this extended notion of reflection on U. *Nevertheless, given an analytic Jordan curve L, we are guaranteed the existence of a neighborhood G of L such that the Schwarz function generated by L is analytic and univalent throughout G.*

When reflection about more than one curve is being discussed, we will write S_L for the Schwarz function of the curve L.

Our primary interest in the Schwarz function and the extended concept of reflection associated with an analytic Jordan Curve L comes from the fact that they can be used to extend the domain of definition for certain functions associated with L. This topic is addressed in the following section.

8.2.2 Meromorphic continuation across analytic curves

Let L be an analytic Jordan curve and let be $U \subset \hat{\mathbb{C}}$ be a domain containing L such that the Schwarz function S_L generated by L extends meromorphically to U. Denote by U_e and U_i, respectively, the intersection of U with the exterior and the interior of L. Now let τ be another analytic Jordan curve and $V \subset \hat{\mathbb{C}}$ a domain containing τ such that the Schwarz function S_τ generated by τ extends meromorphically to V.

Suppose that U has the property that, for all $z \in U_i$, we have $\overline{S_L(z)} \in U_e$. If $h_e : U_e \cup L \to V$ is a continuous function that is meromorphic in U_e such that $h_e(L) \subset \tau$, then h_e can be meromorphically continued to U by setting

$$h_e(z) := \overline{(S_\tau \circ h_e)\left[\overline{S_L(z)}\right]}, \quad z \in U_i.$$

Notice that if S_L, h_e, and S_τ are univalent in U_i, $U_e \cup L$, and V, respectively, then the meromorphic continuation of h_e to U is univalent as well. *In particular, if we know that h_e is univalent in $U_e \cup L$, then we are guaranteed some neighborhood O of L such that the meromorphic continuation of h_e to $U_e \cup O$ is univalent.* This follows from the fact that S_L and S_τ are guaranteed to be univalent in some neighborhoods of L and τ, respectively.

Similarly, if for every $z \in U_e$, we have $\overline{S_L(z)} \in U_i$, and if $h_i : U_i \cup L \to V$ is a continuous function that is meromorphic in U_i such that $h(L) \subset \tau$, then we can meromorphically extend h_i to the whole of U by setting

$$h_i(z) := \overline{(S_\tau \circ h_i)\left[\overline{S_L(z)}\right]}, \quad z \in U_e.$$

Once again, if we know that h_i is univalent in $U_i \cup L$, then we are guaranteed some neighborhood O of L such that the meromorphic continuation of h_i to $U_i \cup O$ is univalent.

8.2.3 Conformal maps associated with L_1

Let L_1 be an analytic Jordan curve. Let Ω_1 denote the exterior of L_1. By the Riemann Mapping Theorem, there exists a unique conformal map

$$\psi : \Delta_1 \to \Omega_1$$

of the exterior of the unit circle onto the exterior of L_1 which has a positive derivative at infinity and which maps the point ∞ to itself:

$$\psi'(\infty) := \lim_{w\to\infty} \frac{\psi(w)}{w} > 0, \quad \psi(\infty) = \infty.$$

Since L_1 is a Jordan curve, Carathéodory's theorem (Ch. IX, Theorem 4.9 in [14]) on the boundary correspondence of conformal maps implies that ψ admits a continuous and univalent extension to $\overline{\Delta}_1$ such that $\psi(\mathbb{T}_1) = L_1$. But the curve L_1 is also analytic, whence there exists some $\mu \in [0, 1)$ such that ψ has an analytic and univalent continuation to Δ_μ. Thus $\psi(e^{i\theta})$, $0 \le \theta \le 2\pi$, provides a parametrization of L_1 and the set

$$U_\mu := \{z = \psi(w) : \mu < |w| < 1/\mu\}$$

is a reflection domain for the reflection map generated by L_1.

For every $r \in [\mu, \infty)$, set

$$\Omega_r := \psi(\Delta_r), \qquad L_r := \partial\Omega_r, \qquad G_r := \mathbb{C} \setminus \overline{\Omega}_r,$$

so that for $r > \mu$, we have that L_r is an analytic Jordan curve. Observe that

$$U_\mu = \Omega_\mu \cap G_{1/\mu}.$$

We can now define

$$\phi : \Omega_\mu \to \Delta_\mu$$

to be the inverse of ψ, so that reflection about L_1 on U_μ is given by the map $z \to z^*$, where

$$z^* := \psi\left(1/\overline{\phi(z)}\right), \quad z \in U_\mu.$$

Having established the exterior conformal map ϕ, we now discuss the interior one. Let

$$\varphi : G_1 \to \mathbb{D}$$

be a conformal map of the interior of L_1 onto the unit disk, which we know extends continuously and univalently to $\overline{G}_1$. Moreover, reflection about L_1 allows us to extend φ meromorphically and univalently to $G_{1/\mu}$ by setting

$$\varphi(z) := \frac{1}{\overline{\varphi(z^*)}}, \quad z \in G_{1/\mu} \setminus \overline{G}_1.$$

Let us consider the behavior of φ on $G_{1/\mu}$. To this end, let z_0 denote the unique point in G_1 such that $\varphi(z_0) = 0$. Then either $z_0 \in G_1 \setminus \Omega_\mu$ or $z_0 \in G_1 \cap \Omega_\mu$. If $z_0 \in G_1 \setminus \Omega_\mu$, then φ is analytic in $G_{1/\mu}$. But if $z_0 \in G_1 \cap \Omega_\mu$, then φ is analytic in $G_{1/\mu} \setminus \{z_0^*\}$ with a simple pole at z_0^*. To see why, first observe that

$$
\begin{aligned}
z \in G_{1/\mu} \setminus \overline{G}_1 &\Leftrightarrow 1 < |\phi(z)| < \frac{1}{\mu} \\
&\Leftrightarrow \mu < \frac{1}{|\phi(z)|} < 1 \\
&\Leftrightarrow \psi\left(1/\overline{\phi(z)}\right) \in G_1 \cap \Omega_\mu \\
&\Leftrightarrow z^* \in G_1 \cap \Omega_\mu.
\end{aligned}
$$

On the one hand, if $z_0 \in G_1 \setminus \Omega_\mu$, then by the chain of equivalences above, we have $\varphi(z^*) \neq 0$ for all $z \in G_{1/\mu} \setminus \overline{G}_1$. Hence $\varphi(z) = 1/\overline{\varphi(z^*)}$ is analytic throughout $G_{1/\mu} \setminus \overline{G}_1$.

On the other hand, assume that $z_0 \in G_1 \cap \Omega_\mu$. Since $(z_0^*)^* = z_0$, the chain of equivalences above implies that $z_0^* \in G_{1/\mu} \setminus \overline{G}_1$ and that φ has a simple pole at z_0^*.

To summarize the results of this section, we view ϕ as a conformal map of Ω_μ onto Δ_μ for some $\mu \in [0, 1)$. Meanwhile, we view φ as a univalent, meromorphic function defined on $G_{1/\mu}$ taking G_1 to $\mathbb{D}$ in a conformal fashion. Furthermore, φ is known to be analytic on some domain containing $\overline{G_1}$.

8.2.4 The Map ψ

Lemma 8.2.1. *There exists a unique conformal map ψ of Δ_1 onto $\hat{\mathbb{C}} \setminus \overline{G}_1$ with $\psi(\infty) = \infty$ and*

$$b := \lim_{z \to \infty} \frac{\psi(z)}{z} > 0.$$

Furthermore, the Laurent expansion of ψ is of the form

$$\psi(z) = bz + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n}, \quad z \to \infty.$$

Proof. To establish existence, let $w_0 \in G_1$ and put

$$r(z) := \frac{1}{z - w_0}, \quad z \in \hat{\mathbb{C}}.$$

Then r is a conformal map of $\hat{\mathbb{C}} \setminus \overline{G}_1$ onto $\hat{\mathbb{C}} \setminus r(\overline{G}_1)$ and

$$r^{-1}(z) = \frac{1}{z} + w_0$$

is a conformal map of $\hat{\mathbb{C}} \setminus r(\overline{G}_1)$ onto $\hat{\mathbb{C}} \setminus \overline{G}_1$. Note that $r^{-1}(0) = \infty$.

We claim that the set $\hat{\mathbb{C}} \setminus r(\overline{G}_1)$ is a simply connected domain contained in the complex plane containing the origin. Since $\overline{G}_1$ is a connected set and since r is a continuous function, we see that $r(\overline{G}_1)$ is a connected set in the topology of the extended complex plane (page 356 of [14]). Since $r(\overline{G}_1)$ is connected, we know that $\hat{\mathbb{C}} \setminus r(\overline{G}_1)$ is simply connected (Theorem 3.6 on page 422 of [14].) Meanwhile, since we have $w_0 \in \overline{G}_1$ and $r(w_0) = \infty$, we find that $\infty \notin \hat{\mathbb{C}} \setminus r(\overline{G}_1)$. In other words, $\hat{\mathbb{C}} \setminus r(\overline{G}_1) \subset \mathbb{C}$. Furthermore, since $r(\infty) = 0$ and $\infty \notin \overline{G}_1$, we see that $\hat{\mathbb{C}} \setminus r(\overline{G}_1)$ contains the origin. This validates the claim that we have $0 \in \hat{\mathbb{C}} \setminus r(\overline{G}_1) \subset \mathbb{C}$.

Define $s(z) := 1/z$. Then s is a conformal map of Δ_1 onto $\mathbb{D}$ with $s(\infty) = 0$. Now, by the Riemann Mapping Theorem, there exists a unique conformal map f of $\mathbb{D}$ onto $\hat{\mathbb{C}} \setminus r(\overline{G}_1)$ with $f(0) = 0$ and $f'(0) > 0$. Define $\psi := r^{-1} \circ f \circ s$. Then ψ is a conformal map of Δ_1 onto $\hat{\mathbb{C}} \setminus \overline{G}_1$. Furthermore, we have

$$\psi(\infty) = (r^{-1} \circ f \circ s)(\infty) = \infty = (r^{-1} \circ f)(0) = r^{-1}(0) = \infty.$$

Now, in order to obtain a better understanding of the map ψ, we write

$$\psi(z) = (r^{-1} \circ f \circ s)(z) = (r^{-1} \circ f)(1/z) = r^{-1}[f(1/z)] = w_0 + \frac{1}{f(1/z)}.$$

Then we have

$$\lim_{z\to\infty} \frac{\psi(z)}{z} = \lim_{z\to\infty} \left[\frac{w_0}{z} + \frac{1}{z} \cdot \frac{1}{f(1/z)}\right] = \lim_{w\to 0} \frac{w}{f(w)} = \frac{1}{f'(0)} > 0.$$

This completes the existence portion of the proof.

To demonstrate uniqueness, let ψ^* be another conformal map of Δ_1 onto $\hat{\mathbb{C}} \setminus \overline{G}_1$ with $\psi^*(\infty) = \infty$ and $\lim_{z\to\infty} \psi^*(z)/z > 0$. Then $g := r \circ \psi^* \circ s^{-1}$ is a conformal map of $\mathbb{D}$ onto $\hat{\mathbb{C}} \setminus r(\overline{G}_1)$. Note that we have

$$g(0) = (r \circ \psi^* \circ s^{-1})(0) = (r \circ \psi^*)(\infty) = r(\infty) = 0.$$

Meanwhile, since

$$g(z) = (r \circ \psi^* \circ s^{-1})(z) = (r \circ \psi^*)(1/z) = \frac{1}{\psi^*(1/z) - w_0},$$

we also have

$$\begin{aligned} g'(0) &= \lim_{z\to 0} \frac{g(z) - g(0)}{z - 0} = \lim_{z\to 0} \frac{g(z)}{z} = \lim_{z\to 0} \frac{1}{z} \cdot \frac{1}{\psi^*(1/z) - w_0} = \lim_{w\to\infty} \frac{w}{\psi^*(w) - w_0} \\ &= \lim_{w\to\infty} \left(\frac{\psi^*(w) - w_0}{w} \right)^{-1} = \left(\lim_{w\to\infty} \frac{\psi^*(w) - w_0}{w} \right)^{-1} = \left[\lim_{w\to\infty} \left(\frac{\psi^*(w)}{w} - \frac{w_0}{w} \right) \right]^{-1} \\ &= \left[\lim_{w\to\infty} \frac{\psi^*(w)}{w} \right]^{-1} > 0. \end{aligned}$$

By the uniqueness of f, we must have $f = g$. This, in turn, yields $r \circ \psi \circ s^{-1} = r \circ \psi^* \circ s^{-1}$, which gives $\psi = \psi^*$. This proves the uniqueness claim.

Next, since f is analytic in $\mathbb{D}$ with $f(0) = 0$, we may write

$$f(z) = a_1 z + O(z^2) = z[a_1 + O(z)], \quad z \in \mathbb{D},$$

where $f'(0) = a_1 > 0$. Next, for $z \to 0$, we may write

$$\frac{1}{f(z)} = \frac{1}{z} \cdot \frac{1}{a_1 + O(z)} = \frac{1}{z} \cdot \frac{1}{a_1} \cdot \frac{1}{1 + O(z)} = \frac{1}{z} \cdot \frac{1}{a_1} \cdot [1 + O(z)].$$

Then for $z \to \infty$, we have

$$\frac{1}{f(1/z)} = \frac{z}{a_1} \cdot [1 + O(1/z)] = \frac{z}{a_1} + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots, \quad z \to \infty,$$

which gives

$$\psi(z) = w_0 + \frac{1}{f(1/z)} = \frac{z}{a_1} + w_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots, \quad z \to \infty.$$

Finally, we observe that

$$\frac{1}{a_1} = \frac{1}{f'(0)} = \lim_{z\to\infty} \frac{\psi(z)}{z},$$

and the proof is complete.

□

Corollary 8.2.2. *We have*

- $\lim_{w\to 0}[w \cdot \overline{\psi(1/\overline{w})}] = b$
- $\lim_{w\to 0} \overline{\psi'(1/\overline{w})} = b$

Proof. The Laurent expansion for ψ given in Lemma 8.2.1 gives

$$\overline{\psi(1/\overline{w})} = \frac{\bar{b}}{w} + \overline{b_0} + \sum_{n=1}^{\infty} \overline{b_n} w^n, \quad w \to 0.$$

Since $b > 0$, this implies the first claim. Meanwhile, the same expansion implies

$$\lim_{z\to\infty} \overline{\psi'(z)} = \bar{b} = b \quad \Rightarrow \quad \lim_{w\to 0} \overline{\psi'(1/\overline{w})} = b.$$

This establishes the second claim and completes the proof of the lemma.

□

Lemma 8.2.3. *Let $n \in \mathbb{N}$ and let m be an integer satisfying $0 \leq m \leq n$.*

- *If F is a function analytic in some neighborhood of $\overline{\mathbb{D}}$ with $F(0) = 0$, then*

$$\frac{1}{2\pi i}\oint_{\mathbb{T}_1} F(w)\, w^{n-1}\, \overline{[\psi(w)]}^m\, \overline{\psi'(w)}\, dw = 0, \quad 0 \le m \le n.$$

- *If F is a function analytic in some neighborhood of $\overline{\mathbb{D}}$ with $F(0) \neq 0$, then*

$$\frac{1}{2\pi i}\oint_{\mathbb{T}_1} F(w)\, w^{n-1}\, \overline{[\psi(w)]}^m\, \overline{\psi'(w)}\, dw = \begin{cases} 0, & 0 \le m < n \\ F(0)\cdot b^{n+1}, & m = n. \end{cases}$$

Proof. We begin by writing

$$\begin{aligned} I &:= \frac{1}{2\pi i}\oint_{\mathbb{T}_1} F(w)\, w^{n-1}\, \overline{[\psi(w)]}^m\, \overline{\psi'(w)}\, dw \\ &= \frac{1}{2\pi i}\oint_{\mathbb{T}_1} F(w)\, w^{n-1}\, \overline{[\psi(1/\overline{w})]}^m\, \overline{\psi'(1/\overline{w})}\, dw. \end{aligned}$$

Since L_1 is an analytic Jordan curve, there exists a number μ, with $0 \le \mu < 1$, such that ψ has an analytic and univalent continuation to Δ_μ. This follows from the discussion Subsection 8.2.2. Therefore $\overline{[\psi(1/\overline{w})]}^m\, \overline{\psi'(1/\overline{w})}$ is meromorphic in some neighborhood of $\overline{\mathbb{D}}$, where its only singularity occurs at the origin. More specifically, the Laurent expansion for ψ given in Lemma 8.2.1 implies that this singularity is a pole order m.

Suppose $F(0) = 0$, so that we can write $F(w) = w \cdot G(w)$, where G is analytic in some neighborhood of $\overline{\mathbb{D}}$. If $0 \le m \le n$, then the integrand in the final integral above represents a function analytic in some neighborhood of $\overline{\mathbb{D}}$. Then we have $I = 0$ by Cauchy's Theorem.

Now suppose that $F(0) \neq 0$. If $0 \le m < n$, then the integrand still represents a function analytic in some neighborhood of $\overline{\mathbb{D}}$, and we still have $I = 0$ by Cauchy's Theorem.

Meanwhile, if $F(0) \neq 0$ and if $m = n$, then the integrand represents a function meromorphic in some neihghborhood of $\overline{D}$, where its only singularity is a simple pole at the point $w = 0$. We compute

$$\begin{aligned}
\mathrm{Res}_{w=0}[F(w)w^{n-1}\,\overline{[\psi(1/\overline{w})]^n}\,\overline{[\psi'(1/\overline{w})]}] &= \lim_{w\to 0}[w\cdot F(w)w^{n-1}\,\overline{[\psi(1/\overline{w})]^n}\,\overline{[\psi'(1/\overline{w})]}] \\
&= \lim_{w\to 0}[F(w)\cdot w^n\cdot \overline{\psi(1/\overline{w})}^n\cdot \overline{\psi'(1/\overline{w})}] \\
&= \lim_{w\to 0}\{F(w)\cdot [w\cdot \overline{\psi(1/\overline{w})}]^n\cdot \overline{\psi'(1/\overline{w})}\} \\
&= F(0)\cdot b^n\cdot b \\
&= F(0)\cdot b^{n+1}.
\end{aligned}$$

Then the the lemma follows by the Residue Theorem. □

8.3 CMCDs and weighted Bergman spaces

Let $\mathcal{D} = \mathbb{D}\setminus\{D_{c_j,r_j}\}_{j=1}^s$ be a circular multiply connected domain. Consider the space X of all functions $f : \mathbb{D}\to\mathbb{C}$ which satisfy the following two conditions:

- f is analytic in $\mathbb{D}$
- $\displaystyle\int_{\mathcal{D}} |f(z)|^2\, dA(z) < \infty$

Note that $\mathcal{A}^2(\mathbb{D}) \subset X$ since

$$\int_{\mathcal{D}} |g(z)|^2\, dA(z) \le \int_{\mathbb{D}} |g(z)|^2\, dA(z) < \infty, \quad g\in\mathcal{A}^2(\mathbb{D}).$$

Meanwhile, we also have $X\subset\mathcal{A}^2(\mathbb{D})$. Indeed, if f is analytic in $\mathbb{D}$ with $\int_{\mathcal{D}} |f(z)|^2\, dA(z) < \infty$, then we may write

$$\begin{aligned}
\int_{\mathbb{D}} |f(z)|^2\, dA(z) &= \int_{\mathcal{D}} |f(z)|^2\, dA(z) + \sum_{j=1}^s \int_{D_{c_j,r_j}} |f(z)|^2\, dA(z) \\
&\le \int_{\mathcal{D}} |f(z)|^2\, dA(z) + \sum_{j=1}^s \pi r_j^2 \max_{z\in K_{c_j,r_j}} |f(z)|^2
\end{aligned}$$

by the Maximum Modulus Principle. Since this quantity is finite, we have $f \in \mathcal{A}^2(\mathbb{D})$. Therefore, the sets X and $\mathcal{A}^2(\mathbb{D})$ are identical.

Now, we know that $\mathcal{A}^2(\mathbb{D})$ is a Hilbert space with inner product

$$\langle f, g\rangle_{\mathbb{D}} = \int_{\mathbb{D}} f(z)\overline{g(z)}\, dA(z).$$

Here we demonstrate that X is a Hilbert space with inner product

$$\langle f, g\rangle = \int_{\mathcal{D}} f(z)\overline{g(z)}\, dA(z).$$

The crux of the matter consists of showing that X is complete in the metric $d(f,g) = \|f-g\|$, where $\|f\| = \langle f, f\rangle^{1/2}$. In other words, we must show that every Cauchy sequence $\{f_n\} \subset X$ converges in norm to some function in X. Since $X \subset L^2(\mathbb{D})$ and since $L^2(\mathbb{D})$ is complete, we only have to show that X is a closed subspace of $L^2(\mathbb{D})$. In other words, we have to show that if $\{f_n\} \subset X$ converges to some f in $L^2(\mathbb{D})$, then f also belongs to X. Essentially, this is accomplished in Lemma 8.3.5. We prepare for that proof by recording a few auxiliary lemmas.

Lemma 8.3.1. *We have $\mathcal{A}^2(\mathbb{D}) \subset \mathcal{A}^2(\mathcal{D})$.*

Proof. Let $f \in \mathcal{A}^2(\mathbb{D})$. Then $\int_{\mathcal{D}} |f(z)|^2\, dA(z) \leq \int_{\mathbb{D}} |f(z)|^2\, dA(z) < \infty$, since $f \in \mathcal{A}^2(\mathbb{D})$. Next, we note that, since f is analytic in $\mathbb{D}$, we have that f is analytic in $\mathcal{D}$. Thus $f \in \mathcal{A}^2(\mathcal{D})$, and the proof is complete. □

Lemma 8.3.2. *For each $z \in \mathbb{D}$, the operator $F_z : \mathcal{A}^2(\mathbb{D}) \to \mathbb{C}$ defined by*

$$F_z(f) := f(z)$$

is a linear functional.

Proof. Let $f, g \in \mathcal{A}^2(\mathbb{D})$. Then

$$F_z(f+g) = (f+g)(z) = f(z) + g(z) = F_z(f) + F_z(g).$$

Also, if $\alpha \in \mathbb{C}$, then we have

$$F_z(\alpha f) = (\alpha f)(z) = \alpha f(z) = \alpha F_z(f).$$

This shows that F_z is a linear functional. $\square$

Lemma 8.3.3. *For each $z \in \mathbb{D}$, there exists a constant $M_z > 0$ such that*

$$|F_z(f)| \leq M_z \|f\|, \quad f \in \mathcal{A}^2(\mathbb{D}).$$

Proof. *Part One* If $z \in \mathcal{D}$, then we have

$$|F_z(f)| \leq \frac{1}{\pi} \frac{\|f\|}{\delta(z)}, \quad f \in \mathcal{A}^2(\mathcal{D}),$$

where $\delta(z)$ denotes the distance from z to the boundary of $\mathcal{D}$. This is Theorem 1 on page 7 of [15]. By Lemma 8.3.1, we know that $f \in \mathcal{A}^2(\mathbb{D})$ gives $f \in \mathcal{A}^2(\mathcal{D})$. Thus, we may write

$$|F_z(f)| \leq \frac{\|f\|}{\pi^2 \delta(z)}, \quad f \in \mathcal{A}^2(\mathbb{D}).$$

This proves the lemma in the case where $z \in \mathcal{D}$.

Part Two Now suppose $z \in \mathbb{D} \setminus \mathcal{D}$. Let $r < 1$ be large enough so that the following two conditions are satisfied:

- $D_{c_k, r_k} \subset \mathbb{D}_r, \quad k \in \{1, 2, \ldots, s\}$
- $\delta(w) = 1 - r, \quad w \in \mathbb{T}_r$

The first condition guarantees that $\mathbb{T}_r \subset \mathcal{D}$ and that $\mathbb{D} \setminus \mathcal{D}$ belongs to the interior of $\mathbb{T}_r$. Meanwhile, the second condition ensures that $\mathbb{T}_r$ is closer to the unit circle than it is to any of the removed disks D_{c_k,r_k}.

For the moment, fix some $f \in \mathcal{A}^2(\mathbb{D})$. By the Maximum Modulus Principle, there exists some $z_0 \in \mathbb{T}_r \subset \mathcal{D}$ such that $|f(z)| \leq |f(z_0)|$. Then, with the help of the first part of this lemma, we may write

$$|F_z(f)| = |f(z)| \leq |f(z_0)| \leq \frac{\|f\|}{\pi^2 \delta(z_0)} = \frac{\|f\|}{\pi^2(1-r)}.$$

Since f was arbitrary, we have

$$|F_z(f)| \leq \frac{\|f\|}{\pi^2(1-r)}, \quad f \in \mathcal{A}^2(\mathbb{D}).$$

Thus the lemma holds in the case where $z \in \mathbb{D} \setminus \mathcal{D}$, and the proof is complete.

□

Lemma 8.3.4. *If f_n and f are in $\mathcal{A}^2(\mathbb{D})$ and $\|f_n - f\| \to 0$ as $n \to \infty$, then $f_n(z) \to f(z)$ uniformly on each compact subset of $\mathbb{D}$.*

Proof. Let K be a compact subset of $\mathbb{D}$. Let $M = \max_{z \in K} M_z$, where M_z is the constant from Lemma 8.3.3. Let $\epsilon > 0$ be given. Note that we have

$$|f_n(z) - f(z)| = |(f_n - f)(z)| = |F_z(f_n - f)| \leq M\|f_n - f\|, \quad (n, z) \in \mathbb{N} \times K.$$

Since $\|f_n - f\| \to 0$ as $n \to \infty$, there exists some $N \in \mathbb{N}$ such that

$$\|f_n - f\| < \frac{\epsilon}{M}, \quad n \geq N.$$

Then, when $n \geq N$, we have

$$|f_n(z) - f(z)| < \epsilon, \quad z \in K,$$

proving the lemma. $\square$

Lemma 8.3.5. *If* $\{f_n\} \subset \mathcal{A}^2(\mathbb{D})$ *and if* $f \in L^2(\mathbb{D})$ *with* $\|f_n - f\| \to 0$*, then there exists some* $g \in \mathcal{A}^2(\mathbb{D})$ *with* $f(z) = g(z)$ *for almost every* $z \in \mathbb{D}$*.*

Proof. On the one hand, the fact that $\{f_n\}$ converges to f in norm means that we can extract some subsequence $\{f_{n_k}\}$ with $f_{n_k}(z)$ converging to $f(z)$ for almost every $z \in \mathbb{D}$.

On the other hand, the fact that $\{f_n\}$ converges in the norm $\|\cdot\|$ implies that $\{f_n\}$ is a Cauchy sequence in norm. In particular, $\{f_n\}$ is a locally uniform Cauchy sequence in $\mathbb{D}$. Then $\{f_n\}$ converges locally uniformly in $\mathbb{D}$. Then there is a function $g : \mathbb{D} \to \mathbb{C}$ such that $f_n \to g$ pointwise in $\mathbb{D}$. Since $\{f_n\}$ converges locally uniformly in $\mathbb{D}$, we have that g is analytic in $\mathbb{D}$.

Now, the fact that $f_n \to g$ pointwise in $\mathbb{D}$ implies that $f_{n_k}(z) \to g(z)$ for every $z \in \mathbb{D}$. Meanwhile, we had already said that $f_{n_k}(z) \to f(z)$ for almost every $z \in \mathbb{D}$. We conclude that $g(z) = f(z)$ for almost every $z \in \mathbb{D}$, and so the proof is complete. $\square$

8.4 Auxiliary relationships

8.4.1 A differential relationship

Lemma 8.4.1. *For every continuous function* f *on* L_1*, we have*

$$\overline{\int_{L_1} f(z)dz} = \int_{L_1} \overline{f(z)}\,\overline{dz},$$

with

$$\overline{dz} = -\frac{\varphi'(z)dz}{\overline{\varphi'(z)}[\varphi(z)]^2}. \tag{8.4.1}$$

Proof. Writing $w = e^{i\theta}$, we have

$$dw = ie^{i\theta}d\theta, \quad |dw| = |ie^{i\theta}|d\theta = d\theta,$$

and so

$$dw = iw|dw|.$$

Hence, if φ^{-1} is the inverse of φ, then

$$\begin{aligned}
\overline{\int_{L_1} f(z)\, dz} &= \overline{\int_{\mathbb{T}_1} f[\varphi^{-1}(w)]\, \varphi^{-1\prime}(w)\, dw} \\
&= \overline{\int_{\mathbb{T}_1} f[\varphi^{-1}(w)]\, \varphi^{-1\prime}(w)\, iw\, |dw|} \\
&= -\int_{\mathbb{T}_1} \overline{f[\varphi^{-1}(w)]}\, \overline{\varphi^{-1\prime}(w)} i\overline{w}|dw| \\
&= -\int_{\mathbb{T}_1} \overline{f[\varphi^{-1}(w)]}\, \overline{\varphi^{-1\prime}(w)} \frac{1}{w^2} dw \\
&= -\int_{L_1} \overline{f(z)}\, \frac{\varphi'(z)dz}{\overline{\varphi'(z)}[\varphi(z)]^2}.
\end{aligned}$$

This completes the proof. $\square$

Corollary 8.4.2. *For every continuous function f on $\mathbb{T}_1$, we have*

$$\overline{\int_{\mathbb{T}_1} f(w)dw} = \int_{\mathbb{T}_1} \overline{f(w)}\,\overline{dw},$$

with

$$\overline{dw} = -\frac{dw}{w^2}. \tag{8.4.2}$$

8.4.2 Residue theorem corollary

Lemma 8.4.3. *Suppose that*

- *a function f is analytic in an open set U, modulo an isolated singularity at the point z*

- *L_I is a Jordan contour in U such that z belongs to the exterior of L_I*

- *L_O is a Jordan contour in U such that z belongs to the interior of L_O*

Then we have

$$\int_{L_I} f(z)\,dz = \int_{L_O} f(z)\,dz - 2\pi i \operatorname{Res}(z, f)$$

Proof. Let L_I^*, L_0^*, and L_S be Jordan contours such that

- L_I^* and L_I are homologous in $U \setminus \{z\}$

- L_O^* and L_O are homologous in $U \setminus \{z\}$

- $L_S := L_O^* - L_I^*$ is a Jordan contour such that z belongs to the interior of L_S

Then properties one and two give

$$\int_{L_I} f(z)\,dz = \int_{L_I^*} f(z)\,dz \quad \text{and} \quad \int_{L_O} f(z)\,dz = \int_{L_O^*} f(z)\,dz$$

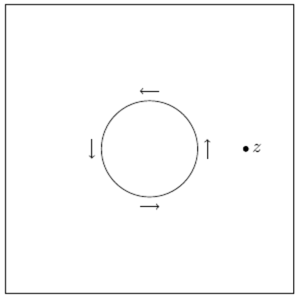
while the third property gives

$$\int_{L_O^*} f(z)\,dz - \int_{L_I^*} f(z)\,dz = \int_{L_S} f(z)\,dz.$$

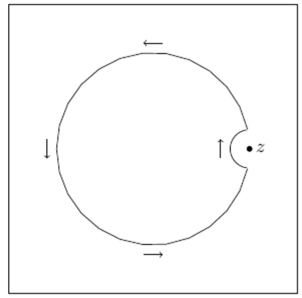
Therefore, we have

$$\int_{L_I} f(z)\,dz = \int_{L_O} f(z) - \int_{L_S} f(z)\,dz = \int_{L_O} f(z) - 2\pi i \operatorname{Res}(z, f),$$

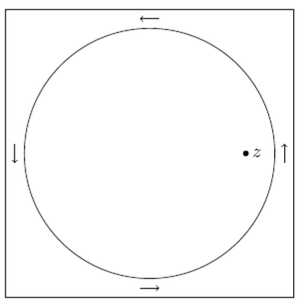
by the Residue Theorem. □



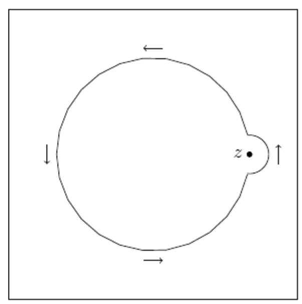
The oriented curve L_I



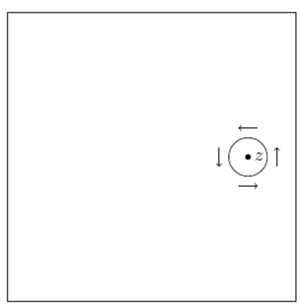
The oriented curve L_I^*



The oriented curve L_O



The oriented curve L_O^*



The oriented curve L_S

Figure 8.3: The Residue Theorem Corollary

8.4.3 On the Function $\mathcal{R}_{j,n}$

In this section, we prove the following proposition.

Proposition 8.4.4. *For each $j \in \Lambda_\omega$, we have*

$$\mathcal{R}_{j,n}(z) = \frac{\Phi_j'(z)}{\Phi_j(z)} \cdot (\alpha^2 - 1) \cdot \frac{a_j^n}{n} \cdot \Theta_j(-n\Phi_j(z)) + O\left(\frac{\alpha^n}{n^2}\right)$$

normally for $z \in \mathbb{D}_\alpha$ as $n \to \infty$.

We comment that much of the work in this section is a generalization of a series of Lemmas which can be found in the paper [6].

The functions λ_j, $\mathcal{G}_{j,n}$, and $\mathcal{R}_{j,n}$ and some of their properties

For each $j \in \Lambda_s$, we define

$$\lambda_j(z) := -\frac{z - a_j}{1 - \overline{a_j}z}$$

We record some of the relationships between χ_j, Φ_j, and λ_j. We will use the fact that

$$\Phi_j^{-1} = \chi_j. \tag{8.4.3}$$

Lemma 8.4.5. *We have*

(i) $\lambda_j(z) = \chi_j(-z)$

(ii) $\lambda_j(z) = -\Phi_j(z)$

(iii) $\Phi_j(\lambda_j(z)) = -z$

(iv) $\lambda_j'(t) = -\chi_j'(-t)$

(v) $\Phi_j'(\lambda_j(t)) = -\dfrac{1}{\lambda_j'(t)}$

Proof. For (i) and (ii), we write

$$\chi_j(-z) = \frac{-z + a_j}{1 - \overline{a_j} z} = \frac{z - a_j}{-1 + \overline{a_j} z} = \lambda_j(z) = -\frac{z - a_j}{1 - \overline{a_j} z} = -\Phi_j(z).$$

For (iii), we use (i) and (8.4.3) to write

$$\Phi_j(\lambda_j(z)) = \Phi_j(\chi_j(-z)) = -z.$$

Property (iv) follows from property (i) and the chain rule. For (v), we use (8.4.3) followed by (iii) and (iv) to write

$$\Phi_a'(\lambda_a(t)) = \frac{1}{\chi_a'(\Phi_a(\lambda_a(t)))} = \frac{1}{\chi_a'(-t)} = \frac{-1}{\lambda_a'(t)}.$$

This completes the proof.

$\square$

Lemma 8.4.6. *We have*

$$\lambda_j(\mathbb{D}_{|a_j|}) = \left\{ z : \left| z - \frac{a_j}{1 + |a_j|^2} \right| < \frac{|a_j|}{1 + |a_j|^2}. \right\}$$

Proof. Let L_{a_j} denote the extended line passing through the origin and the point a_j. Note that we have $\lambda_{a_j}(L_{a_j}) = L_{a_j}$. Meanwhile, the circle $\mathbb{T}_{|a_j|}$ is symmetric about L_{a_j}. Furthermore, the line segments connecting the points a_j and $-a_j$ serves as a diameter for the circle $\mathbb{T}_{|a_j|}$. It follows that the line segment connecting $\lambda_j(a_j)$ and $\lambda_j(-a_j)$ is a diameter for the circle $\lambda_j(\mathbb{T}_{|a_j|})$. We have

$$\lambda_j(a_j) = 0 \qquad \text{and} \qquad \lambda_{a_j}(-a_j) = \frac{2a_j}{1 + |a_j|^2}.$$

Therefore, the center of $\lambda_{a_j}(\mathbb{T}_{a_j})$ is given by

$$\frac{\lambda_j(a_j)+\lambda_j(-a_j)}{2}=\frac{a_j}{1+|a_j|^2}$$

and the radius of $\lambda_j(\mathbb{T}_{a_j})$ is given by

$$\frac{|\lambda_j(a_j)-\lambda_j(-a_j)|}{2}=\frac{|a_j|}{1+|a_j|^2}.$$

This completes the proof. $\square$

We define the function

$$\mathcal{G}_{j,n}(t) := \left[\frac{\lambda_j(t)}{a_j}\right]^n \cdot \lambda_j'(t).$$

Lemma 8.4.7. *We have*

(i) $T_j^v(\lambda_j(t)) = \lambda_j(\sigma_j^{2v}t)$

(ii) $(T_j^v)'(\lambda_j(t)) = \dfrac{\sigma_j^{2v}\lambda_j'(\sigma_j^{2v}t)}{\lambda_j'(t)}$

Proof. To prove (i), we use parts (iii) and (i) and of Lemma 8.4.5 to write

$$T_j^v(\lambda_j(t)) = \chi_j(\sigma_j^{2v}\Phi_j(\lambda_j(t)) = \chi_j(-\sigma_j^{2v}t) = \lambda_j(\sigma_j^{2v}t).$$

To prove (ii), we first note that

$$(T_j^v)'(z) = \chi_j'(\sigma_j^{2v}\Phi_j(z)) \cdot \sigma_j^{2v} \cdot \Phi_j'(z).$$

Using this relationship with Lemma 8.4.5 gives

$$(T_j^v)'(\lambda_j(t)) = \chi_j'(\sigma_j^{2v}\Phi_j(\lambda_j(t))) \cdot \sigma_j^{2v} \cdot \Phi_j'(\lambda_j(t))$$

$$= \chi_j'(-\sigma_j^{2v}t) \cdot \sigma_j^{2v} \cdot \frac{-1}{\lambda_j'(t)}$$
$$= \frac{\sigma_j^{2v}\lambda_j'(\sigma_j^{2v}t)}{\lambda_j'(t)}.$$

This completes the proof.

□

Lemma 8.4.8. *We have*

$$\mathcal{R}_{j,n}(\lambda_j(t)) = \frac{a_j^n}{\lambda_j'(t)} \sum_{v=0}^{\infty} \sigma_j^{2v} \mathcal{G}_{j,n}(\sigma_j^{2v}t).$$

Proof. By the definition of $\mathcal{R}_{j,n}$ and Lemma 8.4.7, we have

$$\begin{aligned}
\mathcal{R}_{j,n}(\lambda_j(t)) &= \sum_{v=0}^{\infty} [T_j^v(\lambda_j(t))]^n \cdot (T_j^v)'(\lambda_j(t)) \\
&= \sum_{v=0}^{\infty} [\lambda_j(\sigma_j^{2v}t)]^n \cdot \frac{\sigma_j^{2v}\lambda_j'(\sigma_j^{2v}t)}{\lambda_j'(t)}.
\end{aligned}$$

In other words, we have

$$\begin{aligned}
\mathcal{R}_{j,n}(\lambda_j(t)) &= \frac{1}{\lambda_j'(t)} \sum_{v=0}^{\infty} \sigma_j^{2v} [\lambda_j(\sigma_j^{2v}t)]^n \cdot \lambda_j'(\sigma_j^{2v}t) \\
&= \frac{a_j^n}{\lambda_j'(t)} \sum_{v=0}^{\infty} \sigma_j^{2v} a_j^{-n} [\lambda_j(\sigma_j^{2v}t)]^n \cdot \lambda_j'(\sigma_j^{2v}t) \\
&= \frac{a_j^n}{\lambda_j'(t)} \sum_{v=0}^{\infty} \sigma_j^{2v} \mathcal{G}_{j,n}(\sigma_j^{2v}t),
\end{aligned}$$

and the proof is complete. □

The step function S_j and some of its properties

We let $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ denote the floor function. In other words, $\lfloor x \rfloor$ is the greatest integer less than or equal to the real number x. We define

$$S_j(x) := \sigma_j^{2\lfloor \log_{\sigma_j^2}(x) \rfloor}, \quad x \in (0, \infty).$$

This function is such that for every $(x, v) \in (0, \infty) \times \mathbb{Z}$, we have

$$x \in (\sigma_j^{2(v+1)}, \sigma_j^{2v}] \quad \Leftrightarrow \quad S_j(x) = \sigma_j^{2v}.$$

This follows from the fact that $\log_{\sigma_j^2}$ is decreasing on $(0, \infty)$. In particular, for every $(x, v) \in (0, \infty) \times \mathbb{Z}$, we have

$$\sigma_j^{2(v+1)} < x \leq \sigma_j^{2v} \quad \Leftrightarrow \quad v + 1 > \log_{\sigma_j^2}(x) \geq v \quad \Leftrightarrow \quad v = \lfloor \log_{\sigma_j^2}(x) \rfloor.$$

Therefore S_j is a step function on $(0, \infty)$.

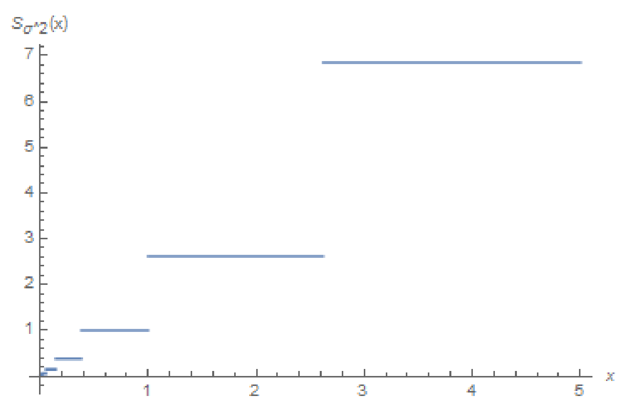


Figure 8.4: Plot of $S_j(x)$.

Next we record some properties that the function S_j enjoys. Since we have $|S_j(x)| \leq 1$ for $x \in (0,1)$ and since $(0,1] = \bigcup_{v=0}^{\infty} (\sigma_j^{2(v+1)}, \sigma_j^{2v}]$, we have

$$\int_0^1 S_j(x)\, f(x)\, dx = \sum_{v=0}^{\infty} \int_{\sigma_j^{2v+2}}^{\sigma_j^{2v}} S_j(x)\, f(x)\, dx = \sum_{v=0}^{\infty} \int_{\sigma_j^{2v+2}}^{\sigma_j^{2v}} \sigma_j^{2v}\, f(x)\, dx$$

whenever $\int_0^1 f(x)\, dx$ exists. Similarly, if $\int_0^\infty S_j(x)\, g(x)\, dx$ exists, then

$$\int_0^\infty S_j(x)\, g(x)\, dx = \sum_{v\in\mathbb{Z}} \int_{\sigma_j^{2v+2}}^{\sigma_j^{2v}} S_j(x)\, g(x)\, dx = \sum_{v\in\mathbb{Z}} \int_{\sigma_j^{2v+2}}^{\sigma_j^{2v}} \sigma_j^{2v}\, g(x)\, dx.$$

Defining $\langle \cdot \rangle : \mathbb{R} \to [0,1)$ via the relation $\langle X \rangle := X - \lfloor X \rfloor$, we may write $\lfloor X \rfloor = X - \langle X \rangle$. Therefore we have the following representation.

Lemma 8.4.9. *For every $x \in (0,\infty)$, we have*

$$S_j(x) = x \sigma_j^{-2\langle \log_{\sigma_j^2}(x) \rangle}, \qquad x \in (0,\infty).$$

Proof. We write

$$\begin{aligned} S_j(x) &= \sigma_j^{2\lfloor \log_{\sigma_j^2}(x) \rfloor} \\ &= \sigma_j^{2(\log_{\sigma_j^2}(x) - \langle \log_{\sigma_j^2}(x) \rangle)} \\ &= x \sigma_j^{-2\langle \log_{\sigma_j^2}(x) \rangle}, \end{aligned}$$

and the proof is complete. □

Auxiliary Lemmas

Lemma 8.4.10. *For any $(m,n) \in \mathbb{N} \times \mathbb{N}$ and for any $t \in \lambda_j(\mathbb{D}_a)$, we have*

$$\sum_{v=0}^{\infty} \sigma_j^{2v} \mathcal{G}_{j,m}(\sigma_j^{2v} t) = \frac{\mathcal{G}_{j,m}(t)}{1-\sigma_j^2} - \frac{\sigma_j^2}{1-\sigma_j^2} \cdot \frac{t}{n} \int_0^n S_j(x/n)\, \mathcal{G}'_{j,m}(xt/n)\, dx.$$

Proof. Let $K \in \mathbb{N}$. An application of the summation by parts formula gives

$$\sum_{v=0}^{K} \sigma_j^{2v} \mathcal{G}_{j,m}(\sigma_j^{2v} t) = \frac{\mathcal{G}_{j,m}(t)}{1-\sigma_j^2} - \frac{\sigma_j^{2(K+1)} \mathcal{G}_{j,m}(\sigma_j^{2(K+1)} t)}{1-\sigma_j^2}$$
$$+ \sum_{v=0}^{K} \frac{\sigma_j^{2(v+1)}}{1-\sigma^2} \{\mathcal{G}_{j,m}[\sigma_j^{2(v+1)} t] - \mathcal{G}_{j,m}(\sigma_j^{2v} t)\}.$$

Letting $K \to \infty$, we obtain

$$\sum_{v=0}^{\infty} \sigma_j^{2v} \mathcal{G}_{j,m}(\sigma_j^{2v} t) = \frac{\mathcal{G}_{j,m}(t)}{1-\sigma_j^2} + \sum_{v=0}^{\infty} \frac{\sigma_j^{2(v+1)}}{1-\sigma_j^2} \{\mathcal{G}_{j,m}[\sigma_j^{2(v+1)} t] - \mathcal{G}_{j,m}(\sigma_j^{2v} t)\}.$$

Meanwhile, by the First Fundamental Theorem of Calculus, we may write

$$\mathcal{G}_{j,m}[\sigma_j^{2(v+1)} t] - \mathcal{G}_{j,m}(\sigma_j^{2v} t) = \int_{\sigma_j^{2v}}^{\sigma_j^{2(v+1)}} \frac{\partial \mathcal{G}_{j,m}(st)}{\partial s} \, ds.$$

Then we have

$$\begin{aligned}\sum_{v=0}^{\infty} \sigma_j^{2v} \mathcal{G}_{j,m}(\sigma_j^{2v} t) &= \frac{\mathcal{G}_{j,m}(t)}{1-\sigma_j^2} + \sum_{v=0}^{\infty} \frac{\sigma_j^{2(v+1)}}{1-\sigma_j^2} \int_{\sigma_j^{2v}}^{\sigma_j^{2(v+1)}} \frac{\partial \mathcal{G}_{j,m}(st)}{\partial s} ds \\ &= \frac{\mathcal{G}_{j,m}(t)}{1-\sigma_j^2} + \sum_{v=0}^{\infty} \frac{\sigma_j^{2(v+1)}}{1-\sigma_j^2} \int_{\sigma_j^{2v}}^{\sigma_j^{2(v+1)}} t \mathcal{G}'_{j,m}(st) \, ds,\end{aligned}$$

which is to say

$$\sum_{v=0}^{\infty} \sigma_j^{2v} \mathcal{G}_{j,m}(\sigma_j^{2v} t) = \frac{\mathcal{G}_{j,m}(t)}{1-\sigma_j^2} - \frac{\sigma_j^2 t}{1-\sigma_j^2} \sum_{v=0}^{\infty} \int_{\sigma_j^{2(v+1)}}^{\sigma_j^{2v}} \sigma_j^{2v} \mathcal{G}'_{j,m}(st) \, ds. \tag{8.4.4}$$

Now we note that the integral

$$\int_0^1 \mathcal{G}'_{j,m}(st) \, ds$$

exists. Thus the properties of the step function S_j mentioned in Section 8.4.3 and a change of variables via the relationship $x = ns$ allow us to write

$$\begin{aligned}\sum_{v=0}^{\infty} \int_{\sigma_j^{2(v+1)}}^{\sigma_j^{2v}} \sigma_j^{2v} \mathcal{G}'_{j,m}(st)\, ds &= \int_0^1 S_j(s)\, \mathcal{G}'_{j,m}(st)\, ds \\ &= \frac{1}{n} \int_0^n S_j(x/n)\, \mathcal{G}'_{j,m}(xt/n)\, dx.\end{aligned}$$

Combining this result with (8.4.4) completes the proof of the lemma. $\square$

$$\Theta_j(t) = \overline{\beta} \cdot \frac{\sigma^2}{1-\sigma^2} \cdot t^2 \int_0^\infty S_j(x)\ \exp(-\overline{\beta} t x)\ dx. \tag{8.4.5}$$

$$\Theta_j(\sigma^{2p} t) = \overline{\beta} \cdot \frac{\sigma^2}{1-\sigma^2} \cdot \sigma^{4p} t^2 \int_0^\infty S_j(x)\ \exp(-\overline{\beta} \sigma^{2p} t x)\ dx. \tag{8.4.6}$$

$$\Theta_j(\sigma^{2q} t) = \overline{\beta} \cdot \frac{\sigma^2}{1-\sigma^2} \cdot \sigma^{4q} t^2 \int_0^\infty S_j(x)\ \exp(-\overline{\beta} \sigma^{2q} t x)\ dx. \tag{8.4.7}$$

$$\Theta_j(\sigma^{2p} t) - \Theta_j(\sigma^{2q} t) = \overline{\beta} \cdot \frac{\sigma^2}{1-\sigma^2} \cdot t^2 \int_0^\infty S_j(x) [\sigma^{4p} \exp(-\overline{\beta} \sigma^{2p} t x) - \sigma^{4q} \exp(-\overline{\beta} \sigma^{2q} t x)]$$

What about when $t = 1/\beta$?

$$\Theta_j(\sigma^{2p}(1/\beta)) - \Theta_j(\sigma^{2q}(1/\beta)) = 1/\beta \cdot \frac{\sigma^2}{1-\sigma^2} \cdot \int_0^\infty S_j(x) [\sigma^{4p} \exp(-\sigma^{2p} x) - \sigma^{4q} \exp(-\sigma^{2q} x)]\ dx$$

Lemma 8.4.11. *If $n \in \mathbb{N}$ and if $t \in \mathcal{H}e^{i\theta_j}$, then we have*

$$\Theta_j(nt) = \overline{\beta_j} \cdot \frac{\sigma_j^2}{1-\sigma_j^2} \cdot nt^2 \int_0^\infty S_j(x/n)\ \exp(-\overline{\beta_j}tx)\ dx.$$

Proof. First, we will show that

$$\Theta_j(t) = \overline{\beta_j} \cdot \frac{\sigma_j^2}{1-\sigma_j^2} \cdot t^2 \int_0^\infty S_j(x)\ \exp(-\overline{\beta_j}tx)\ dx. \tag{8.4.8}$$

This will essentially prove the lemma since it implies

$$\Theta_j(nt) = \overline{\beta_j} \cdot \frac{\sigma_j^2}{1-\sigma_j^2} \cdot n^2t^2 \int_0^\infty S_j(s)\ \exp(-\overline{\beta_j}nts)\ ds,$$

which, by making the change of variables $s = x/n$, yields

$$\Theta_j(nt) = \overline{\beta_j} \cdot \frac{\sigma_j^2}{1-\sigma_j^2} \cdot nt^2 \int_0^\infty S_j(x/n)\ \exp(-\overline{\beta_j}tx)\ dx.$$

Now, in order to establish (8.4.8), we use the properties of S_j mentioned in Section 8.4.3 to write

$$-\overline{\beta_j}t \int_0^\infty S_j(x)\ \exp(-\overline{\beta_j}tx)\ dx = -\overline{\beta_j}t \sum_{v\in\mathbb{Z}} \sigma_j^{2v} \int_{\sigma_j^{2v+2}}^{\sigma_j^{2v}} \exp(-\overline{\beta_j}tx)\ dx. \tag{8.4.9}$$

Next, since we have

$$\frac{d}{dx}\left[-\frac{\exp(-\overline{\beta_j}tx)}{\overline{\beta_j}t}\right] = \exp(-\overline{\beta_j}tx),$$

we may write

$$\begin{aligned} &-\overline{\beta_j}t \sum_{v\in\mathbb{Z}} \sigma_j^{2v} \int_{\sigma_j^{2v+2}}^{\sigma_j^{2v}} \exp(-\overline{\beta_j}tx)\ dx \\ &\qquad = \sum_{v\in\mathbb{Z}} \sigma_j^{2v} \exp(-\overline{\beta_j}t\sigma_j^{2v}) - \sum_{v\in\mathbb{Z}} \sigma_j^{2v} \exp(-\overline{\beta_j}t\sigma_j^{2v+2}) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{v\in\mathbb{Z}} \sigma_j^{2v} \exp(-\overline{\beta_j} t \sigma_j^{2v}) - \frac{1}{\sigma_j^2} \sum_{v\in\mathbb{Z}} \sigma_j^{2v+2} \exp(-\overline{\beta_j} t \sigma_j^{2v+2}) \\
&= \frac{\Theta_j(t)}{t} - \frac{\Theta_j(t)}{t\sigma_j^2} \\
&= \frac{\Theta_j(t)}{t} \left(\frac{\sigma_j^2 - 1}{\sigma_j^2} \right).
\end{aligned}
$$

Combining this with (8.4.9) gives

$$
-\overline{\beta_j} t \int_0^\infty S_j(x)\ \exp(-\overline{\beta_j} t x)\, dx = \frac{\Theta_j(t)}{t} \left(\frac{\sigma_j^2 - 1}{\sigma_j^2} \right).
$$

This implies (8.4.8) and completes the proof. □

Establishing an estimate

The lemma below is a generalization of a lemma used in the paper [6].

Lemma 8.4.12. *For every compact subset K of $\lambda_j(\mathbb{D}_{|a_j|})$, there exist positive constants m and M such that for every integer $n \geq 1$, we have*

$$
\left| e^{-\overline{\beta_j} w s} - a_j^{-n} \lambda_j^n (ws/n) \right| \leq \frac{M s^2 e^{-ms}}{n}, \quad w \in K, \quad 0 \leq s \leq n.
$$

To establish the lemma above, we create a dictionary cataloging relationships between expressions used in the paper and expressions used here.

First of all, the function λ in the paper corresponds to the function

$$
F_j(z) := -\frac{z - |a_j|}{1 - |a_j| z}
$$

here. Next, the quantity μ in the paper corresponds to the quantity $|a_j|$ here.

Note that, by Lemma 8.4.6, we have

$$
F_j(\mathbb{D}_{|a_j|}) = \left\{ z : \left| z - \frac{|a_j|}{1 + |a_j|^2} \right| < \frac{|a_j|}{1 + |a_j|^2}. \right\}
$$

Writing $a_j = |a_j|e^{i\theta_j}$, we have

$$\lambda_j(\mathbb{D}_{|a_j|}) = e^{i\theta_j} F_j(\mathbb{D}_{|a_j|})$$

We mention that, in the paper, the quantity μ is related to a quantity $R > 2$ by the formulas

$$\mu = \frac{R - \sqrt{R^2 - 4}}{2} \qquad \text{and} \qquad R = \frac{1 + \mu^2}{\mu}.$$

Note that

$$\begin{aligned} \{t : |1 - Rt| < 1\} &= \left\{t : \left|\frac{1}{R} - t\right| < \frac{1}{R}\right\} \\ &= \left\{t : \left|\frac{\mu}{1+\mu^2} - t\right| < \frac{\mu}{1+\mu^2}\right\} \end{aligned}$$

Therefore, the set

$$\{t : |1 - Rt| < 1\}$$

in the paper corresponds to the set $F_j(\mathbb{D}_{|a_j|})$ here.

What follows is a translation of the lemma from the paper into the language we are using.

Lemma 8.4.13. *For every compact set $E \subset F_j(\mathbb{D}_{|a_j|})$, there exist positive constants m and M such that for every integer $n \geq 1$, we have*

$$\left| e^{-(|a_j|^{-1} - |a_j|)ts} - |a_j|^{-n} F_j^n(ts/n) \right| \leq \frac{M s^2 e^{-ms}}{n}, \quad t \in E, \quad 0 \leq s \leq n.$$

Now we can prove Lemma 8.4.12.

Proof. Let K be a compact subset of $\lambda_j(\mathbb{D}_{|a_j|})$. Then there exists some compact subset E of $F_j(\mathbb{D}_{|a_j|})$ such that

$$K = Ee^{i\theta_j}.$$

In other words, we have $w \in K$ if and only if $w = te^{i\theta_j}$ for some $t \in E$.

Note that we have

$$\begin{aligned}\lambda_j(te^{i\theta_j}) &= -\frac{te^{i\theta_j} - a_j}{1 - \overline{a_j}te^{i\theta_j}} \\ &= e^{i\theta_j}\left(-\frac{t - |a_j|}{1 - |a_j|t}\right) \\ &= e^{i\theta_j}F_j(t).\end{aligned}$$

More generally, if we replace t with ts/n in the calculation above, then we have

$$\frac{\lambda_j(te^{i\theta_j}s/n)}{a_j} = \frac{e^{i\theta_j}F_j(ts/n)}{a_j} = \frac{F_j(ts/n)}{|a_j|}. \tag{8.4.10}$$

Meanwhile, it follows

$$\overline{\beta}e^{i\theta_j} = \left(\frac{1}{a_j} - \overline{a_j}\right)e^{i\theta_j} = \frac{e^{i\theta}}{a_j} - e^{i\theta}\overline{a_j} = \frac{1}{|a_j|} - |a_j|.$$

that we also have

$$-\overline{\beta_j}ws = -\overline{\beta_j}te^{i\theta_j}s = -(|a_j|^{-1} - |a_j|)ts$$

Therefore, for every $w \in K$, we have, with $w = te^{i\theta_j}$,

$$\begin{aligned}\left|e^{-\overline{\beta_j}ws} - a_j^{-n}\lambda_j^n(ws/n)\right| &= \left|e^{-\overline{\beta_j}te^{i\theta_j}s} - a_j^{-n}\lambda_j^n(te^{i\theta_j}s/n)\right| \\ &= \left|e^{-(|a_j|^{-1}-|a_j|)ts} - |a_j|^{-n}F_j^n(ts/n)\right| \\ &\le \frac{Ms^2e^{-ms}}{n}\end{aligned}$$

for every $0 \le s \le n$. This concludes the proof of Lemma 8.4.12. $\square$

Using the estimate

Lemma 8.4.14. *We have*

$$\int_0^n nS_j(x/n)\ \mathcal{G}'_{j,n+1}(xt/n)\ dx = \frac{(1-|a_j|^2)^2}{|a_j|} \cdot \frac{1-\sigma_j^2}{\sigma_j^2} \cdot \frac{(n+1)[\Theta_j(nt)+O(1/n)]}{\overline{\beta_j}t^2}$$

normally on $\lambda_j(\mathbb{D}_{|a_j|})$ as $n \to \infty$.

Proof. We define the function

$$\mathcal{L}_{j,n}(t) := [\lambda_j'(t)]^2 + \frac{\lambda_j(t) \cdot \lambda_j''(t)}{n}$$

Straightforward calculations show that we have

$$\mathcal{L}_{j,n}(t) = \frac{(1-|a_j|^2)^2}{(1-\overline{a_j}t)^4} + \frac{2\overline{a_j}(t-a_j)(1-|a_j|^2)}{n(1-\overline{a_j}t)^4} \tag{8.4.11}$$

and

$$\mathcal{G}'_{j,n+1}(t) = \frac{n+1}{a_j^{n+1}} \cdot [\lambda_j(t)]^n \cdot \mathcal{L}_{j,n+1}(t).$$

Let E be a compact subset of $\lambda_j(\mathbb{D}_{|a_j|})$. By equation (8.4.11), we have

$$\begin{aligned}\mathcal{L}_{j,n+1}(xt/n) &= \frac{(1-|a_j|^2)^2}{(1-|a_j|xt/n)^4} + \frac{2|a_j|(xt/n-|a_j|)(1-|a_j|^2)}{n(1-|a_j|xt/n)^4} \\ &= (1-|a_j|^2)^2[1+O(s/n)+O(1/n)]\end{aligned}$$

uniformly for $(t,x) \in E \times [0,n]$ as $n \to \infty$. Therefore, we have

$$\begin{aligned}&\int_0^n nS_j(x/n)\ \exp(-\overline{\beta_j}tx)\ \mathcal{L}_{j,n+1}(xt/n)\ dx \\ &\qquad = (1-|a_j|^2)^2 \int_0^n nS_j(x/n)\ \exp(-\overline{\beta_j}tx)\ dx + O(1/n)\end{aligned}$$

uniformly for $(t,s) \in E \times [0,n]$ as $n \to \infty$. Hence, defining

$$\mathcal{J}_n(x,t) := nS_j(x/n)\ \exp(-\overline{\beta_j}tx),$$

we see that there exist positive constants m and M' such that

$$\begin{aligned}
&\left|\int_0^n nS_j(x/n)\ \mathcal{G}'_{j,n+1}(xt/n)\ dx - \frac{(n+1)(1-|a_j|^2)^2}{a_j}\int_0^n \mathcal{J}_n(x,t)\ dx\right| \\
&\leq \frac{n+1}{|a_j|}\int_0^n nS_j(x/n)\left|\frac{[\lambda_j(tx/n)]^n}{|a_j|^n} - \exp(-\overline{\beta_j}tx)\right|\cdot|\mathcal{L}_{j,n+1}(xt/n)|\ dx + O(1) \\
&\leq M'\int_0^\infty x^3 e^{-ms}\ dx + O(1)
\end{aligned}$$

uniformly for $t \in E$ as $n \to \infty$.

Now, we note that

$$\Theta_j(nt) = \overline{\beta_j}\cdot\frac{\sigma_j^2}{1-\sigma_j^2}\cdot t^2\int_0^\infty \mathcal{J}_n(x,t)\ dx.$$

Therefore, we have

$$\int_0^n nS_j(x/n)\ \mathcal{G}'_{j,n+1}(xt/n)\ dx = \frac{(1-|a_j|^2)^2}{a_j}\cdot\frac{1-\sigma_j^2}{\sigma_j^2}\cdot\frac{(n+1)[\Theta_j(nt)+O(1/n)]}{\overline{\beta_j}t^2}$$

normally on $\lambda_j(\mathbb{D}_{|a_j|})$ as $n \to \infty$. Next, we use the fact that

$$\overline{\beta_j} = \frac{1-|a_j|^2}{a_j}$$

to see that we have

$$\int_0^n nS_j(x/n)\ \mathcal{G}'_{j,n+1}(xt/n)\ dx = (1-|a_j|^2)\cdot\frac{1-\sigma_j^2}{\sigma_j^2}\cdot\frac{(n+1)[\Theta_j(nt)+O(1/n)]}{t^2}$$

$\square$

Proof of Proposition 8.4.4

Here we show that

$$\mathcal{R}_{j,n}(z) = -\frac{\Phi_j'(z)}{\Phi_j(z)} \cdot (1-|\alpha|^2) \cdot \frac{a_j^n}{n}\Theta_j(-n\Phi_j(z)) + O\left(\frac{\alpha^n}{n^2}\right),$$

normally for $z \in \mathbb{D}_\alpha$.

Proof. We have

$$\begin{aligned}\sum_{v=0}^{\infty} \sigma_j^{2v}\mathcal{G}_{j,n+1}(\sigma_j^{2v}t) &= \frac{\mathcal{G}_{j,n+1}(t)}{1-\sigma_j^2} - \frac{\sigma_j^2}{1-\sigma_j^2} \cdot \frac{t}{n^2}\int_0^n nS_j(x/n)\ \mathcal{G}_{j,n+1}'(xt/n)\,dx \\ &= -\frac{(n+1)(1-|a_j|^2)[\Theta_j(nt)+O(1/n)]}{n^2 t}\end{aligned}$$

normally for t on $\lambda_j(\mathbb{D}_{|a_j|})$ as $n \to \infty$. This gives

$$\begin{aligned}\mathcal{R}_{j,n}(\lambda_j(t)) &= \frac{a_j^n}{\lambda_j'(t)}\sum_{v=0}^{\infty}\sigma_j^{2v}\mathcal{G}_{j,n}(\sigma_j^{2v}t) \\ &= -\frac{a_j^n}{\lambda_j'(t)}\frac{1-|a_j|^2}{t}\frac{n}{(n-1)^2}[\Theta_j((n-1)t)+O(1/n)]\end{aligned}$$

normally for t on $\lambda_j(\mathbb{D}_{|a_j|})$ as $n \to \infty$. Next, the correspondence $z = \lambda_j(t)$ gives

$$\mathcal{R}_{j,n}(z) = -\frac{\Phi_j'(z)}{\Phi_j(z)} \cdot (1-|a_j|^2) \cdot \frac{a_j^n \cdot n}{(n-1)^2}[\Theta_j(-(n-1)\Phi_j(z)) + O(1/n)]$$

locally uniformly for $z \in \mathbb{D}_{|a_j|}$. In other words, we have

$$\mathcal{R}_{j,n+1}(z) = -\frac{\Phi_j'(z)}{\Phi_j(z)} \cdot (1-|a_j|^2) \cdot \frac{a_j^{n+1}(n+1)}{n^2}[\Theta_j(-n\Phi_j(z)) + O(1/n)]$$

locally uniformly for $z \in \mathbb{D}_{|a_j|}$. Next, we have

$$\mathcal{R}_{j,n+1}(z) = -\frac{\Phi_j'(z)}{\Phi_j(z)} \cdot (1-|\alpha|^2) \cdot \frac{(n+1)a_j^{(n+1)}}{n^2}\Theta_j(-n\Phi_j(z))$$

$$
\begin{aligned}
&+ -\frac{\Phi_j'(z)}{\Phi_j(z)} \cdot (1-|\alpha|^2) \cdot \frac{(n+1)a_j^{(n+1)}}{n^2} \Theta_j(-n\Phi_j(z)) \cdot O(1/n) \\
&= -\frac{\Phi_j'(z)}{\Phi_j(z)} \cdot (1-|\alpha|^2) \cdot \frac{(n+1)a_j^{(n+1)}}{n^2} \Theta_j(-n\Phi_j(z)) + O\left(\frac{\alpha^n}{n^2}\right) \\
&= -\frac{\Phi_j'(z)}{\Phi_j(z)} \cdot (1-|\alpha|^2) \cdot \frac{a_j^{(n+1)}}{n} \Theta_j(-n\Phi_j(z)) + O\left(\frac{\alpha^n}{n^2}\right)
\end{aligned}
$$

normally for $z \in \mathbb{D}_\alpha$ as $n \to \infty$. Finally, we use the fact that

$$\Theta_j(-n\Phi_j(z)) - \Theta_j(-(n+1)\Phi_j(z)) = O(1/n)$$

normally for $z \in \mathbb{D}_\alpha$ to write

$$\mathcal{R}_{j,n}(z) = -\frac{\Phi_j'(z)}{\Phi_j(z)} \cdot (1-|\alpha|^2) \cdot \frac{a_j^n}{n} \Theta_j(-n\Phi_j(z)) + O\left(\frac{\alpha^n}{n^2}\right),$$

which holds normally for $z \in \mathbb{D}_\alpha$. This completes the proof of the proposition. $\square$

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