Domain Coloring

and the Visualization of Complex-Valued Functions

Graduate Student Seminar

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Motivation

The ability to visualize real-valued functions of a single real variable often facilitates one's understanding of the subject we normally refer to as **calculus**. For example, the content of the **Intermediate Value Theorem** appears trivial when one considers its geometric interpretation. Although it is no substitute for analysis and rigor, visualization has proven to be an invaluable tool for the student of calculus. It provides insight into certain problems, it makes formulas easier to memorize, and it connects the abstract with the concrete.

Statement of the Problem - Part 1

The student of **complex analysis** may seek methods of function visualization which are analogous to the methods utilized in the study of calculus. However, it soon becomes clear that the “old methods” must be updated. Let’s see why this is.

Recall that if $f : X \longrightarrow Y$ is a function, then the **graph** of f is defined to be $\{(x, f(x)) \mid x \in X\}$. Note that the graph of f will be a subset of $X \times Y$.

In general, we hope that our **visualization** of f contains all the information that is encoded in the graph of f.

Statement of the Problem - Part 2

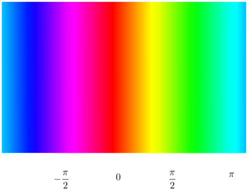
If, as is often the case in calculus, f maps a subset of $\mathbb{R}$ into $\mathbb{R}$, then the graph of f will be a subset of $\mathbb{R} \times \mathbb{R}$. This makes it relatively easy to visualize f. However, the typical function g from complex analysis will map a subset of $\mathbb{C}$ into $\mathbb{C}$. This means that the graph of g will be a subset of $\mathbb{C} \times \mathbb{C}$. If we identify each complex number z with an ordered pair of real numbers, say $\left(\operatorname{Re} z, \operatorname{Im} z\right)$, then it looks as though the graph of g will be a subset of $\mathbb{R}^2 \times \mathbb{R}^2$. This is what makes it relatively difficult to visualize g.

Approach to a Solution - Part 1

It is possible to visualize complex valued functions of a complex variable through a technique known as **domain coloring**.

At its core, domain coloring makes use of the fact that any complex number w may be described by its modulus and principal argument. In particular, if $\rho = |w|$ and $\theta = \operatorname{Arg} w$, then we have $w = \rho e^{i\theta}$. Note that $\rho \in [0, \infty)$ while $\theta \in (-\pi, \pi]$. In terms of visualization, this is useful because we may now consider the graph of a complex-valued function as a subset of $\mathbb{C} \times [0, \infty) \times (-\pi, \pi]$.

The term "domain coloring" comes from the following idea: it is possible to map interval $(\pi, \pi]$ into the set of spectral colors (Newton's color wheel) in a continuous and one-to-one fashion.



Approach to a Solution - Part 2

Let f be a complex valued function defined on some subset U of the complex plane. One specific way to use domain coloring is given below.

Here we imagine that our domain U resides at the bottom of some three dimensional space. Then for each $z \in U$, we represent $f(z)$ by plotting a point directly above z such that

(i) the height of this point is determined by $|f(z)|$; and
(ii) the color of this point is determined by $\operatorname{Arg} f(z)$.

We will refer to the method described above as **lifted domain coloring.** Today we will use lifted domain coloring to visualize a variety of complex-valued functions.

Classes of Functions we will Consider and their Relevant Concepts

1. **Entire Functions**: A Proposition Regarding Polynomials, the Properties of the Exponential Function, and Liouville's Theorem
2. **Functions with Poles**: A Characterization of Poles and the Validity of the Laurent Series Representation
3. **Functions with Essential Singularities**: Picard's Great Theorem

Furthermore, the **Maximum Modulus Principle** will be relevant to every function we see.

We will recall the details of these theorems and concepts as we go along. But first, let's familiarize ourselves with "lifted domain coloring" by considering the **identity map**:

$$z \mapsto z \text{ for every } z \in \mathbb{C}.$$

A Property of Polynomials

If $P : \mathbb{C} \to \mathbb{C}$ is a polynomial, then $|P(z)| \to \infty$ as $|z| \to \infty$.

Proof. Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ with $a_n \neq 0$.

Then for any $z \neq 0$, we may estimate

$$\begin{aligned} |P(z)| &= |a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0| \\ &= |z|^n \cdot \left| a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n} \right| \\ &\geq |z|^n \left(|a_n| - \frac{|a_{n-1}|}{|z|} - \cdots - \frac{|a_1|}{|z|^{n-1}} - \frac{|a_0|}{|z|^n} \right). \end{aligned}$$

From this inequality we infer the desired result.

The Exponential Function

Among the properties of the exponential function which are "easy to visualize", we have

- $\left|e^z\right| = e^{\operatorname{Re} z}$;
- $\arg\left(e^z\right) = \operatorname{Im} z$; and
- $e^z = e^w \Leftrightarrow w = 2k\pi i$ for some integer k.

Here's one way to think about the second property:

Color[e^z]:$\approx \operatorname{Arg} e^z = \operatorname{Im} z + 2\pi ik$ for some $k \in \mathbb{Z} \Rightarrow$ Color[e^z] $\equiv \operatorname{Im} z \pmod{2\pi ik}$;

. . . .

"This is the most important function in mathematics."

-Walter Rudin

Real and Complex Analysis (Prologue)

Liouville's Theorem

Statement

The only bounded entire functions are the constant functions on $\mathbb{C}$.

Corollary

If $f : \mathbb{C} \to \mathbb{C}$ is entire and non-constant, then the set $\{|f(z)| : z \in \mathbb{C}\}$ is not bounded from above by any real number.

The Maximum Modulus Principle

Theorem

Let a function f be analytic in a domain D. Suppose that there exists a point z_0 of D with the property that $\left|f(z)\right| \leq \left|f(z_0)\right|$ for every z in D. Then f is constant in D.

Corollary

Let D be a bounded domain in the complex plane, and let $f : \overline{D} \longrightarrow \mathbb{C}$ be a continuous function that is analytic in D. Then $\left|f(z)\right|$ reaches its maximum at some point on the boundary of D.

Characterization of Poles

Theorem

Let a function f have an isolated singularity at a point z_0. The singularity is a pole if and only if

$$\lim_{z \longrightarrow z_0} \left| f\left(z\right) \right| = \infty.$$

Validity of the Laurent Series Representation

Proposition

Suppose that ρ_0 and ρ_I are the outer and inner radii of convergence of a Laurent series

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

centered at z_0. The series diverges for any z satisfying $|z - z_0| > \rho_0$ or $|z - z_0| < \rho_I$.

A Modification to our Technique

On occasion, we will be interested in studying the zeroes of a complex-valued function. When this is the case, it may be helpful to modify our original procedure of "lifted domain coloring" in the following manner.

Let f be a complex valued function defined on some subset U of the complex plane. We imagine that our domain U resides *in the middle* of some three dimensional space. Then for each $z \in U$, we represent $f(z)$ by plotting a point directly above (or below) z such that

(i) the height of this point is determined by $\operatorname{Log}|f(z)|$; and
(ii) the color of this point is determined by $\operatorname{Arg} f(z)$.

Now if $|f(z)| \longrightarrow 0$, then the "height" of the point which represents $f(z)$ will approach $-\infty$. This will make it easy to identify the zeroes of the function we are visualizing.

Functions with Essential Singularities

Picard's Great Theorem

If a function f is analytic in a punctured disk $\Delta^* = \Delta^*(z_0, r)$ and has an essential singularity at its center, then the set $\mathbb{C} \sim f(\Delta^*)$ contains at most one point.