# Derivation of Softmax Random Walk Algorithm

## Supplementary Method

### Objective:

To compute the probability of moving from a current position $(x, y)$ to a neighboring position $(i, j)$ by blending terrain slope and movement inertia, considering valid data and avoiding immediate return to the last position.

### Preliminaries and Definitions:

#### Notations:

- $(x, y)$: Current Position
- $(i, j)$: Neighboring Position

#### Parameters:

- $\phi$: Weighting factor for slope.
- $\psi$: Weighting factor for inertia.
- $\omega$: Blending factor between inertia and slope.

#### Cosine Similarity:

Given two vectors $A$ and $B$, the cosine similarity, $\text{cosine\_similarity}(A, B)$, is:

$$\text{cosine\_similarity}(A, B) = \frac{A \cdot B}{\|A\|_2 \|B\|_2}$$

#### Softmax Transformation:

For a vector $z$ with components $z_1, z_2, \ldots z_k$, the softmax function, denoted as $\text{softmax}(z)$, is defined as:

$$\text{softmax}(z)_j = \frac{e^{z_j}}{\sum_{l=1}^{k} e^{z_l}}$$

## Elements Calculation:

### Slope, $S_{i,j}$:

Given:

- $e_c$ and $e_n$ as elevations at current position and neighbor.
- $dx$ and $dy$ as absolute differences in x and y coordinates.
  The slope $S_{i,j}$ between positions is calculated as follows, considering the handling of invalid data and the distance for diagonal vs. orthogonal moves:

$$S_{i,j} = \begin{cases} \text{np.NINF} & \text{if } e_c \text{ or } e_n \text{ is invalid or } e_c = e_n = \text{no\_data\_value} \\ \frac{e_c - e_n}{\sqrt{2}} & \text{if } dx = dy \text{ (diagonal move)} \\ \frac{e_c - e_n}{1} & \text{if } dx \neq dy \text{ (orthogonal move)} \end{cases}$$

### Inertia, $I_{i,j}$:

Using the last movement vector $V_{last} = [dx_{last}, dy_{last}]$ and the direction vector $V_{direction} = [i - x, j - y]$.

$$I_{i,j} = \left( \frac{1 + \text{cosine\_similarity}(V_{last}, V_{direction})}{2} \right)$$

## Combining Slope and Inertia:

### Raw Combined Weight:

Blend slope and inertia using:

$$W_{i,j}^{raw} = (1 - \omega)\psi I_{i,j} + \omega\phi S_{i,j}$$

## Probabilistic Decision Model:

### Softmax Transformation:

Compute the transition probabilities by applying the softmax function to the combined weights directly:

$$P((x,y) \to (i,j)) = \text{softmax}(W^{raw})_{i,j} \to \frac{e^{W_{i,j}^{raw}}}{\sum_{(k,l)\in\text{neighbors}} e^{W_{k,l}^{raw}}}$$

Here, $(k,l)$ ranges over all positions neighboring $(x,y)$ that are potential candidates for movement, excluding the current position itself and any positions that would constitute an immediate return to the last position.

## Computation:

1. For $(x,y)$, calculate $S_{i,j}$ and $I_{i,j}$ for all valid neighbors, excluding the immediate last position to prevent back-and-forth movement.
2. Compute the raw combined weight, $W_{i,j}^{raw}$, for each neighbor by blending inertia and slope.
3. Apply the softmax function to the combined weights to get probabilities for each neighbor.
4. Select a neighbor $(i,j)$ based on these probabilities.
5. Update $(x,y)$ to $(i,j)$ and continue, ensuring to handle masked or invalid elevation data appropriately.

# Supplementary Note
# data dictionary
# blue color denotes icesat-2 measurement
For Supplementary Tables 1 and 2
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Avulsion Name: name of the avulsion.
geometry: geometry of the avulsion (WKT point).
source: source of the avulsion data.
num_icesat_measurements: number of ICESat measurements used.
num_fabdem_measurements: number of FABDEM measurements used.
method_used: method used to calculate the channel depth based on the relationship between A and B. 1 = A<B, 2 = 1<A<1.5B, 3 = A>1.5B.
dist_mtn_front_to_avulsion_m: distance from the mountain front to the avulsion point (m).
total_transport_distance_m: total transport distance from mountain-valley exit to sink (m).
normalized_distance: X_N from the manuscript, calculated as total_transport_distance_m / dist_mtn_front_to_avulsion_m (-)
sar1: alluvial ridge slope measurement 1 (m/m).
sar2: alluvial ridge slope measurement 2 (m/m).
sar3: alluvial ridge slope measurement 3 (m/m).
sar_mean: mean alluvial ridge slope (m/m).
sm1: main channel slope measurement 1 (m/m).
sm2: main channel slope measurement 2 (m/m).
sm3: main channel slope measurement 3 (m/m).
are1_m: alluvial ridge elevation measurement 1 (m).
fpe1_m: floodplain elevation measurement 1 (m).
wse1_m: water surface elevation measurement 1 (m).
are2_m: alluvial ridge elevation measurement 2 (m).
fpe2_m: floodplain elevation measurement 2 (m).
wse2_m: water surface elevation measurement 2 (m).
are3_m: alluvial ridge elevation measurement 3 (m).
fpe3_m: floodplain elevation measurement 3 (m).
wse3_m: water surface elevation measurement 3 (m).
width_m: channel width (m).
discharge_min_cms: minimum yearly discharge from RiverATLAS (m3/s).
discharge_mean_cms: mean yearly discharge from RiverATLAS (m3/s).
discharge_max_cms: maximum yearly discharge from RiverATLAS (m3/s).
se_1: superelevation calculation from _1 measurements (m).
se_2: superelevation calculation from _2 measurements (m).
se_3: superelevation calculation from _3 measurements (m).
se_std: standard deviation of superelevation measurements (m).
sm_mean: mean main channel slope (m/m).
gamma: ratio of alluvial ridge slope to main channel slope (-).
spr: slope position ratio X_N/S_M (-).
gamma_error_upper: upper bound of gamma error (-).
gamma_error_lower: lower bound of gamma error (-).
corrected_discharge: RiverATLAS corrected discharge (m3/s).
xgb_depth: BASED depth estimate of channel depth (m).

har1_m: alluvial ridge height calculation from _1 measurements (m).
har2_m: alluvial ridge height calculation from _2 measurements (m).
har3_m: alluvial ridge height calculation from _3 measurements (m).
a: mean of (alluvial ridge elevation - water surface elevation) for all three measurements (m).
b: same as xgb_depth, repeated for clarity (m).
a1: (alluvial ridge elevation - water surface elevation) from _1 measurements (m).
b1: same as xgb_depth, repeated for clarity (m).
a2: (alluvial ridge elevation - water surface elevation) from _2 measurements (m).
b2: same as xgb_depth, repeated for clarity (m).
a3: (alluvial ridge elevation - water surface elevation) from _3 measurements (m).
b3: same as xgb_depth, repeated for clarity (m).
a_over_b: a / b (-).
beta: superelevation, calculated according to method_used column, see manuscript for details.
beta_sem: standard error of beta.
lambda: product of beta and gamma (-).
ab_flag: flag for whether or not a > b.
beta_uncertainty: uncertainty in beta from monte carlo simulation, see manuscript for details.

**For Supplementary Table 1
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all the same as above apart from the USED_FLAG column which denotes whether or not the measurement was used in the beta and gamma analysis. 1 = used, 0 = not used.

For Supplementary Table 3
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avulsion_name: name of the river avulsion
IS2_har_m: Height of alluvial ridge in meters measured with ICESat-2
FABDEM_har_m: Height of alluvial ridge in meters measured with FABDEM
IS2_sar: Slope of alluvial ridge (m/m) measured with ICESat-2
FABDEM_sar: Slope of alluvial ridge (m/m) measured with FABDEM