Class Problem– Lubricating Flow

***Problem Statement:** Consider two immiscible fluids 'Fluid-I' and 'Fluid-II' flowing coaxially in a horizontal pipe under isothermal and fully developed laminar flow conditions as illustration below. Develop algebraic expressions for the velocity profiles of both fluids $v_z^I(r)$ and $v_z^{II}(r)$ for a given pressure drop along the pipe ($\Delta P/\Delta L$).*

Develop:

a) velocity profiles inside the pipe for the both fluids $v_z^I(r)$ and $v_z^{II}(r)$

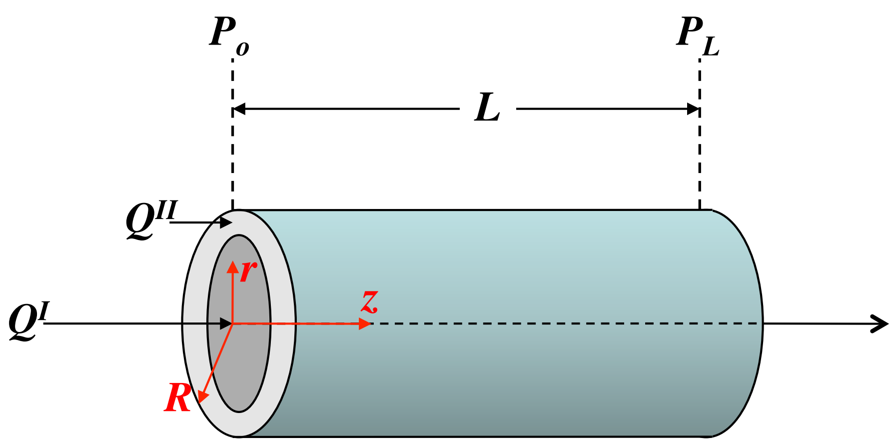
b) the expressions for the volumetric flow rate for each fluid Q^I and Q^{II} through the pipe if the following data is given:

Known Data:

$$\mu^I, \mu^{II}, \frac{\Delta P_z}{\Delta L}, \rho^I, \rho^{II}, \Delta L, R, h.$$

Assume:

$$\mu^I > \mu^{II}, \; \rho^I = \rho^{II}$$



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Assumptions

i) the tube is circular

ii) flow is isothermal

iii) density of both fluids are equal and $\rho = const.$ *(incompressible fluid)*

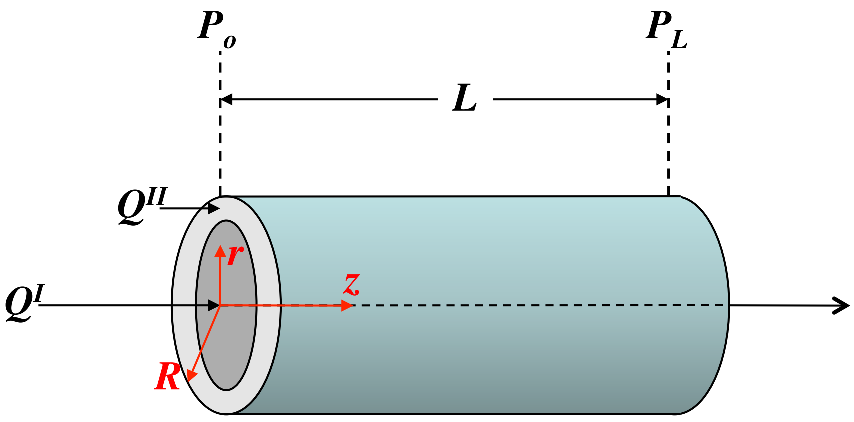
iv) steady state case

v) flow is laminar, unidirectional and fully developed:

$$\mathbb{V} = v(0,0,v_z); \quad \underbrace{v_z \neq f(z)}_{\frac{\partial v_z}{\partial z} = 0} \quad \underbrace{v_z = f(r)}_{\frac{\partial v_z}{\partial r} \neq 0}$$

vi) the axis of fluid flow is perpendicular to the gravity vector i.e. $g_Z = 0$.

vii) the fluid is Newtonian.



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Model Development

Before we proceed further let us consider the mathematical model for the laminar flow of a single fluid in cylindrical pipe that we developed previously:

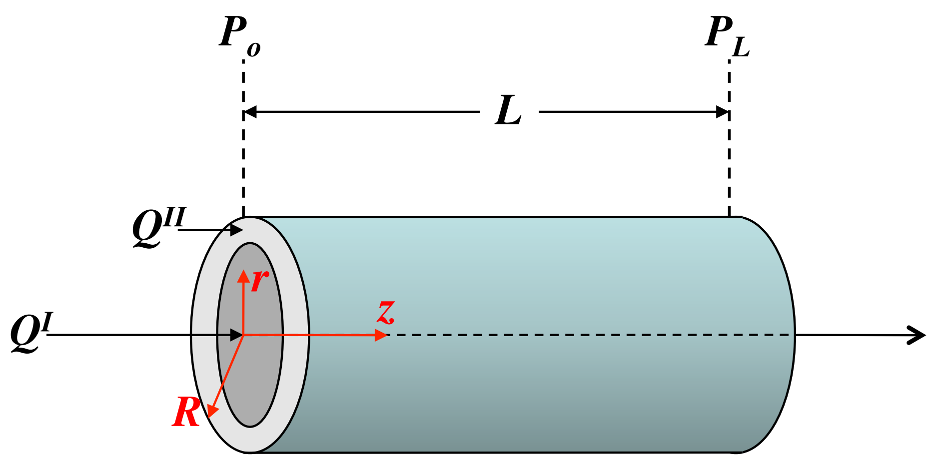
$$\boxed{-\frac{1}{r}\frac{\partial(r\cdot\tau_{zr})}{\partial r}=\frac{\partial P_z}{\partial z}-\rho g_z}$$

Eq: 1

$$BC1)\ \left.\frac{\partial v_z}{\partial r}\right|_{r=0}=0\ \ at\ \ r=0;$$

$$BC2)\ \left.v_z\right|_{r=R}=0\ \ at\ \ r=R;$$

$$BC3)\ \left.\begin{matrix}P_z=P_o\ \ at\ \ z=0\\ P_z=P_L\ \ at\ \ z=L\end{matrix}\right\}\frac{\Delta P}{\Delta L}$$



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The solution of the above model for the given boundary conditions was:

$$\boxed{v_z = \frac{\left[\left(P_o - P_L\right) + \rho g_z L\right] \cdot R^2}{4 \cdot \mu \cdot L} \cdot \left[1 - \frac{r^2}{R^2}\right]} \Rightarrow v_z = \frac{\overbrace{\left[-\Delta P_z + \rho g_z L\right]}^{\Delta\mathbb{P}} \cdot R^2}{4 \cdot \mu \cdot L} \cdot \left[1 - \frac{r^2}{R^2}\right]$$

and for horizontal pipe ($g_Z = 0$) we would have obtained:

$$v_z = \frac{\left[\left(P_o - P_L\right) + \rho \cancel{g_z} L\right] \cdot R^2}{4 \cdot \mu \cdot L} \cdot \left[1 - \frac{r^2}{R^2}\right] = \frac{\left[\left(P_o - P_L\right)\right] \cdot R^2}{4 \cdot \mu \cdot L} \cdot \left[1 - \frac{r^2}{R^2}\right]$$

$$\boxed{v_z = \frac{R^2}{4 \cdot \mu}\left(-\frac{\Delta P_z}{\Delta L}\right) \cdot \left[1 - \frac{r^2}{R^2}\right]}$$

Eq. 2

We came to the above solution by successfully arguing that the partial differential equation Eq. 1 represents two ordinary differential equations:

$$\boxed{-\frac{1}{r}\frac{d\left(r \cdot \tau_{zr}\right)}{dr} = C = \frac{dP_z}{dz} - \rho \cancel{g_z} \Rightarrow -\frac{1}{r}\frac{d\left(r \cdot \tau_{zr}\right)}{dr} = C; \quad and \quad \frac{dP_z}{dz} - \rho \cancel{g_z} = C}$$

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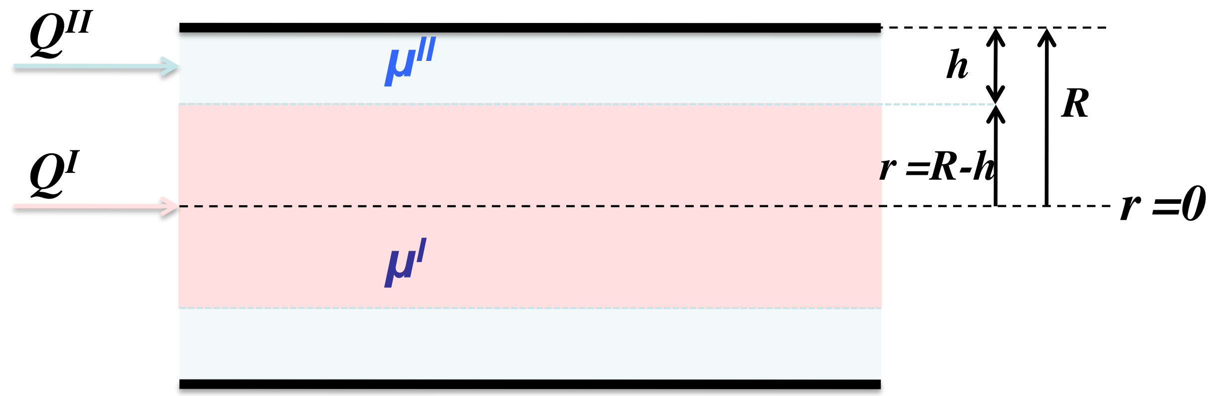
(Look in the Appendix A of your handout for the above solving procedure/argument)

There are no reasons why we would not be able to apply Eq. 1 and Eq. 2 for each of the two fluids as described in the problem statement. One just has to set up the differential equation for each fluid separately and define the boundary conditions that realistically reflect given physical situation.



Mathematical Model Fluid I:

Eq. 3

$$\boxed{-\frac{1}{r}\frac{d\left(r\cdot\tau_{zr}^{I}\right)}{dr}=C^{I}=\frac{dP_{z}^{I}}{dz}\quad\Rightarrow\quad-\frac{1}{r}\frac{d\left(r\cdot\tau_{zr}^{I}\right)}{dr}=C^{I};\quad and\quad\frac{dP_{z}^{I}}{dz}=C^{I}}$$

Boundary Conditions for Fluid I:

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Boundary Conditions for Fluid I:

$$\boxed{(BC1)\quad \tau_{zr}^{I} \neq \infty \quad at\ \ r=0}$$

$$\boxed{(BC2)\ v_z^{I} = v_z^{II} \quad at\ \ r=R-h}$$

$$\boxed{(BC3)\quad \left.\begin{array}{l} P_z = P_o \ \ at\ \ z=0 \\ P_z = P_L \ \ at\ \ z=L \end{array}\right\} \frac{\Delta P}{\Delta L}}$$

Eq. 3a

$$(BC4)\ \tau_{zr}^{I} = \tau_{zr}^{II} \ \Rightarrow \underbrace{-\mu^{I}\left(\frac{dv_z^{I}}{dr}\right) = -\mu^{II}\left(\frac{dv_z^{II}}{dr}\right)}_{} \quad at\ \ r=R-h$$

$$\frac{\mu^{I}}{\mu^{II}} = \frac{\left(\frac{dv_z^{II}}{dr}\right)}{\left(\frac{dv_z^{I}}{dr}\right)}$$

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One could now integrate the above equations if they were not coupled with the similar equations for the Fluid II through the boundary conditions (BC2) and (BC4).

Mathematical Model Fluid II:

$$\boxed{-\frac{1}{r}\frac{d\left(r\cdot\tau_{zr}^{II}\right)}{dr} = C^{II} = \frac{dP_z^{II}}{dz} \quad \Rightarrow \quad -\frac{1}{r}\frac{d\left(r\cdot\tau_{zr}^{II}\right)}{dr} = C^{II}; \quad and \quad \frac{dP_z^{II}}{dz} = C^{II}}$$

Eq. 4

Boundary Conditions for Fluid II:

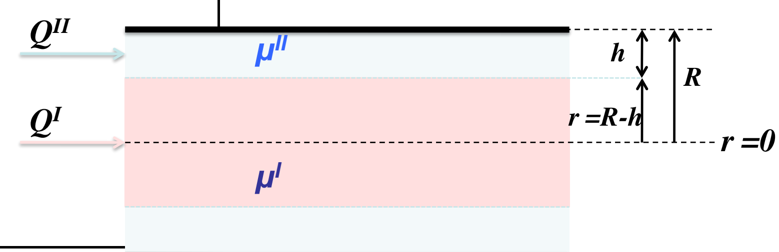
$$\boxed{(BC1) \quad v_z^{II} = 0 \quad at \;\; r = R}$$

$$\boxed{(BC2) \quad v_z^{I} = v_z^{II} \quad at \;\; r = R-h}$$

$$\boxed{(BC3) \quad \left.\begin{aligned} P_z &= P_o \;\; at \;\; z = 0 \\ P_z &= P_L \;\; at \;\; z = L \end{aligned}\right\} \frac{\Delta P}{\Delta L}}$$

$$(BC4) \quad \tau_{zr}^{I} = \tau_{zr}^{II} \quad \Rightarrow \underbrace{-\mu^{I}\left(\frac{dv_z^{I}}{dr}\right) = -\mu^{II}\left(\frac{dv_z^{II}}{dr}\right)}_{\frac{\mu^{I}}{\mu^{II}} = \frac{\left(\frac{dv_z^{II}}{dr}\right)}{\left(\frac{dv_z^{I}}{dr}\right)}} \quad at \;\; r = R-h$$

Eq. 4a



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One could now integrate the above equations if they were not coupled with the If one counts all boundary conditions that are defined it is easily seen that only five boundary conditions are independent; boundary conditions 2, 3, and 4 are shared between two sets of equations and only boundary conditions 1 (for each fluid) are independent.

We can now start integrating the Eq. 3 and Eq. 4.

<u>For Fluid I (inner fluid) apply BC1:</u>

$$-\frac{1}{r}\frac{d\left(r\cdot\tau_{zr}^{I}\right)}{dr}=C^{I}\ \Rightarrow\ -\tau_{zr}^{I}=\frac{C^{I}\cdot r}{2}+\frac{E^{I}}{r}\quad apply\quad \boxed{(BC1)\ \ \tau_{zr}^{I}\neq\infty\ \ at\ \ r=0}$$

Newton's Law

$$\Rightarrow\ E^{I}\equiv 0;\quad \Rightarrow\ -\tau_{zr}^{I}=\frac{C^{I}\cdot r}{2}\quad also\quad -\tau_{zr}^{I}=\mu^{I}\left(\frac{dv_{z}^{I}}{dr}\right)\Rightarrow\ \mu^{I}\left(\frac{dv_{z}^{I}}{dr}\right)=\frac{C^{I}\cdot r}{2}$$

$$\Rightarrow\ after\ integrating\ the\ last\ expression:\ \boxed{v_{z}^{I}=\frac{C^{I}\cdot r^{2}}{4\mu^{I}}+F^{I}}\quad for\ \ all\ \ r\in\left[0\leq r\leq R-h\right]$$

Eq. 5

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Similarly for Fluid II (outer fluid) we can write:

$$-\frac{1}{r}\frac{d\left(r\cdot\tau_{zr}^{II}\right)}{dr}=C^{II} \Rightarrow -\tau_{zr}^{II}=\frac{C^{II}\cdot r}{2}+\frac{E^{II}}{r} \quad \text{for all } r\in\left[R-h\le r\le R\right]$$

note that we cannot now claim that E^{II} *is zero. Actually* $E^{II}\neq 0$.

$$Also,\ -\tau_{zr}^{II}=\mu^{II}\left(\frac{dv_z^{II}}{dr}\right) \Rightarrow \mu^{II}\left(\frac{dv_z^{II}}{dr}\right)=\frac{C^{II}\cdot r}{2}+\frac{E^{II}}{r}$$

after integrating the last expression:

$$\Rightarrow \boxed{v_z^{II}=\frac{C^{II}\cdot r^2}{4\mu^{II}}+\frac{E^{II}}{\mu^{II}}\cdot\ln(r)+F^{II}} \quad \forall\ r\in\left[R-h\le r\le R\right] \qquad \textbf{Eq. 6}$$

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We can now find the values of C^I and C^{II} by applying the BC3 for each of the fluids. From Eq. 3 and Eq. 4 we have:

$$\frac{dP_z^I}{dz} = C^I \quad and \quad \frac{dP_z^{II}}{dz} = C^{II}; \quad apply \quad (BC3) \quad \left.\begin{array}{l} P_z = P_o \;\; at \;\; z = 0 \\ P_z = P_L \;\; at \;\; z = L \end{array}\right\} \frac{\Delta P}{\Delta L}$$

$$\underbrace{\int_{P_o}^{P_L} dP_z^I = \int_0^L C^I\, dz \quad and \quad \int_{P_o}^{P_L} dP_z^{II} = \int_0^L C^{II}\, dz}$$

$$\Downarrow$$

$$\left.\begin{array}{l} -C^I = \dfrac{P_0 - P_L}{L} \quad and \quad -C^{II} = \dfrac{P_0 - P_L}{L} \end{array}\right\} \Rightarrow \boxed{C^I = C^{II} = -\frac{P_0 - P_L}{L} = \frac{\Delta P}{\Delta L}}$$

Eq. 7

Thus the two constants <u>C^I and C^{II}</u> are now defined.

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Now, we can apply BC4 for both Fluid I and Fluid II:

$$(BC4)\ \tau^{I}_{zr} = \tau^{II}_{zr} \Rightarrow -\mu^{I}\left(\frac{dv^{I}_{z}}{dr}\right) = -\mu^{II}\left(\frac{dv^{II}_{z}}{dr}\right)\quad at\ \ r = R-h$$

$$\underbrace{\left.\left(\frac{\mu^{I} d\left(\frac{C^{I}\cdot r^{2}}{4\mu^{I}} + F^{I}\right)}{dr}\right)\right|_{r=R-h} = \left.\left(\frac{\mu^{II} d\left(\frac{C^{II}\cdot r^{2}}{4\mu^{II}} + \frac{E^{II}}{\mu^{II}}\cdot \ln(r) + F^{II}\right)}{dr}\right)\right|_{r=R-h}}\quad at\ \ r = R-h$$

$$\Downarrow$$

$$\cancel{\mu}^{I}\frac{C\cdot(R-h)}{2\cancel{\mu}^{I}} = \cancel{\mu}^{II}\frac{C\cdot(R-h)}{2\cancel{\mu}^{II}} + \cancel{\mu}^{II}\frac{E^{II}}{\cancel{\mu}^{II}(R-h)} \Rightarrow \frac{C\cdot(R-h)}{2} - \frac{C\cdot(R-h)}{2} = \frac{E^{II}}{(R-h)}$$

$$\frac{E^{II}}{(R-h)} = 0 \Rightarrow \boxed{E^{II} \equiv 0}$$

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Therefore we can write:

$$\boxed{v_z^I = \frac{C \cdot r^2}{4\mu^I} + F^I} \quad for\ all\ r \in [0 \le r \le R - h]$$

$$\boxed{v_z^{II} = \frac{C \cdot r^2}{4\mu^{II}} + F^{II}} \quad for\ all\ r \in [R - h \le r \le R]$$

Now we can apply BC1 for fluid II:

$$(BC1) \quad v_z^{II} = 0 \quad at\ r = R$$

$$v_z^{II} = \frac{C \cdot r^2}{4\mu^{II}} + F^{II} \quad \Rightarrow \quad 0 = \frac{C \cdot R^2}{4\mu^{II}} + F^{II} \quad \Rightarrow \quad \boxed{F^{II} = -\frac{C \cdot R^2}{4\mu^{II}}}$$

Therefore:

$$v_z^{II} = \frac{C \cdot r^2}{4\mu^{II}} - \frac{C \cdot R^2}{4\mu^{II}} \quad \Rightarrow \quad \boxed{v_z^{II} = \frac{C \cdot R^2}{4\mu^{II}}\left(\frac{r^2}{R^2} - 1\right)}$$

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Therefore we can write:

$$\boxed{v_z^I = \frac{C\cdot r^2}{4\mu^I} + F^I} \qquad for\ all\ r \in [0 \le r \le R-h]$$

$$\boxed{v_z^{II} = \frac{C\cdot R^2}{4\mu^{II}}\left(\frac{r^2}{R^2} - 1\right)} \qquad for\ all\ r \in [R-h \le r \le R]$$

Now we can apply BC2 for both fluids:

$$(BC2) \quad v_z^I = v_z^{II} \quad at\ r = R-h \quad \Rightarrow \quad \frac{C\cdot r^2}{4\mu^I} + F^I = \frac{C\cdot R^2}{4\mu^{II}}\left(\frac{r^2}{R^2} - 1\right)\Bigg|_{r=R-h}$$

$$\frac{C\cdot (R-h)^2}{4\mu^I} + F^I = -\frac{C}{4\mu^{II}}R^2\left[1 - \frac{(R-h)^2}{R^2}\right] \Rightarrow$$

$$\Rightarrow \quad \boxed{F^I = -\frac{C\cdot (R-h)^2}{4\mu^I} - \frac{C}{4\mu^{II}}R^2\left[1 - \frac{(R-h)^2}{R^2}\right]}$$

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Finally after some algebra, and with the following substitution: $M = \frac{\mu^I}{\mu^{II}};\ \eta = \frac{h}{R};\ s = \frac{r}{R}$

We can now obtain:

$$\boxed{\begin{aligned} v_z^I &= \frac{(P_o - P_L)\cdot R^2}{4\mu^I L}\left[(1-\eta)^2 + M\cdot\eta(2-\eta) - s^2\right] \quad \textit{for all } r \in [R-h \le r \le R] \\ v_z^{II} &= \frac{(P_o - P_L)\cdot R^2}{4\mu^I L}\cdot M\left(1-s^2\right) \quad \textit{for all } r \in [0 \le r \le R-h] \end{aligned}}$$

Sketch the velocity profile of both fluids in a graph v_z v.s. r. Suggested values:

$\mu^I = 0.1[kg/ms]$	$\mu^{II} = 0.025[kg/ms]$	$M = 4$
$\Delta P = 192[Pa]$	$L = 1.0[m]$	$R = 0.01[m]$
$h = 0.002[m]$		

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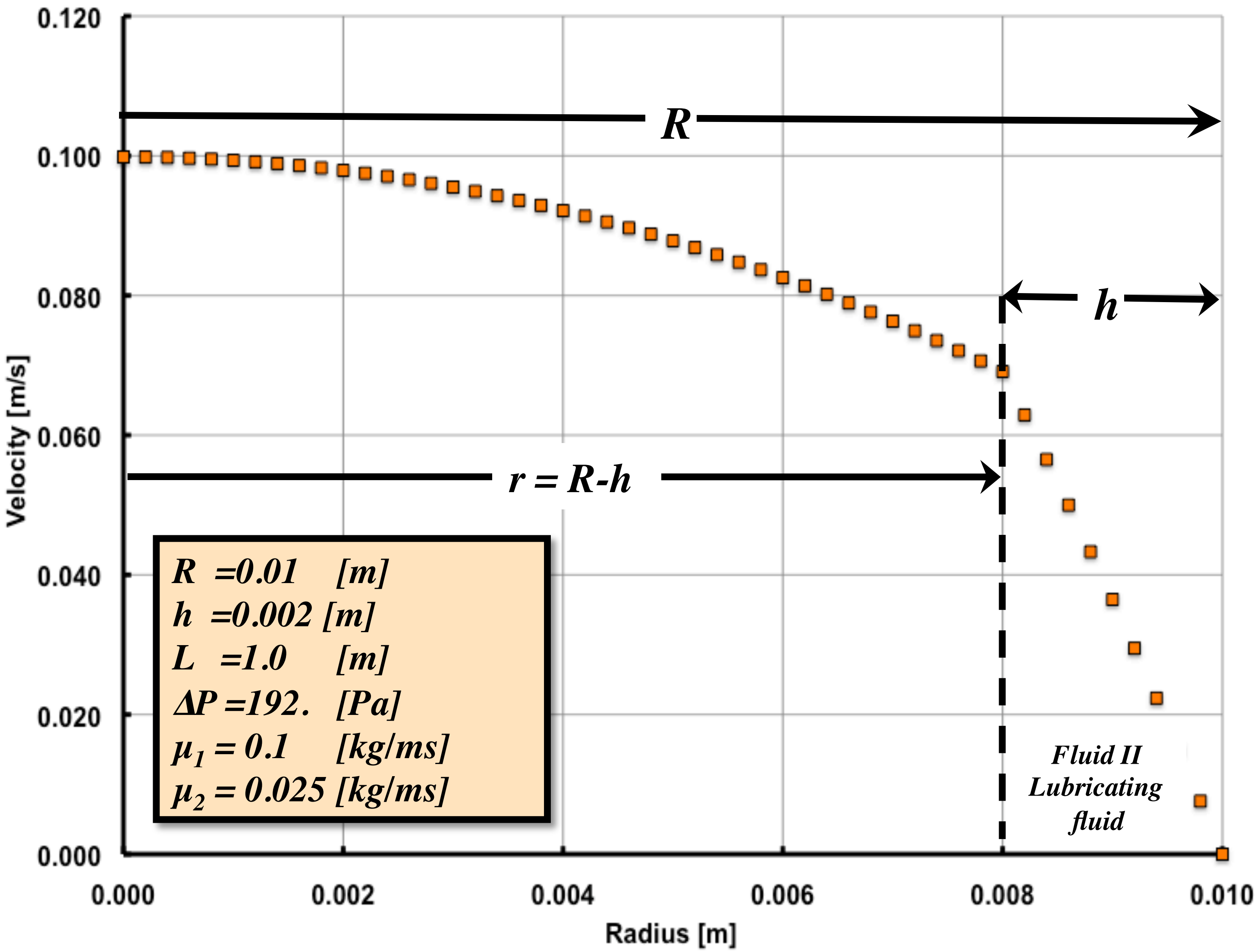

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b) Determine the volumetric flow rate of the fluid trough the pipe Q^{I} *and* $Q^{II}\left[\frac{m^3}{s}\right]$.

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Thank you for your attention!