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OREGON STATE UNIVERSITY-CBEE
DEPARTMENT OF CHEMICAL ENGINEERING

CHE 331
Transport Phenomena I

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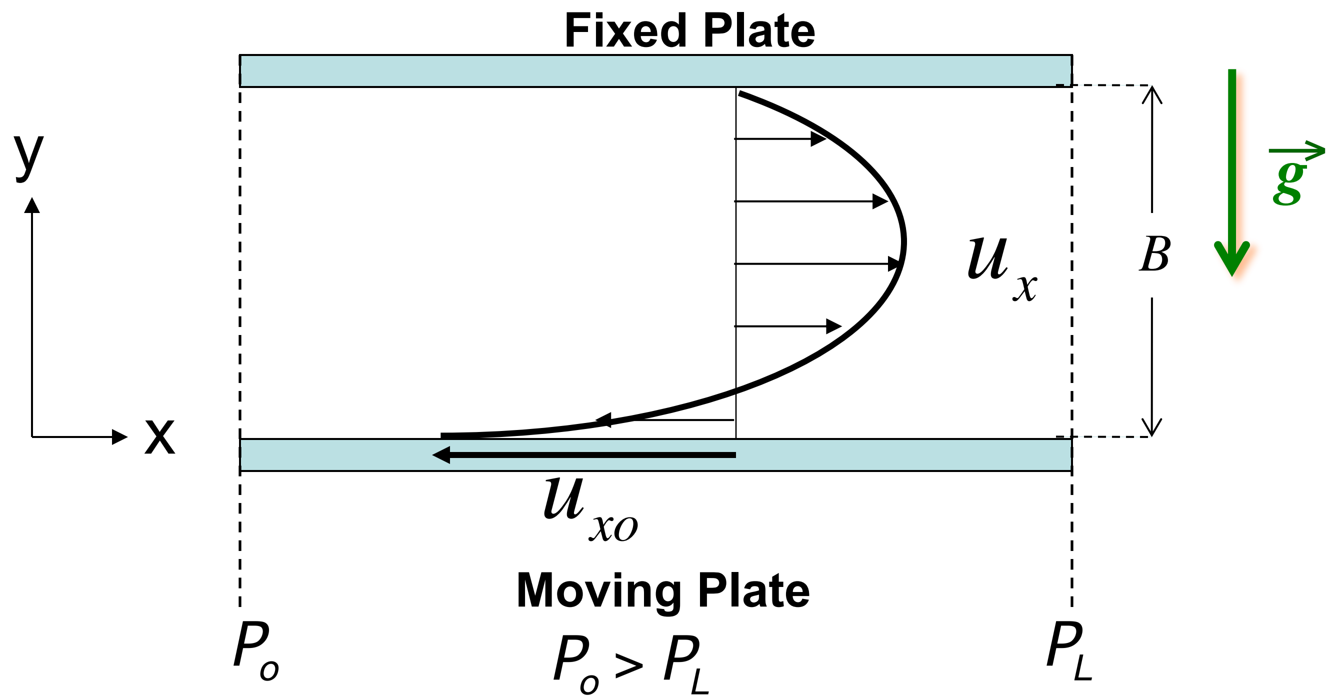
Application of Navier-Stokes Equations

Please turn-off cell phones

A viscous Newtonian fluid is confined between large parallel plates, as shown in Figure below. The motion of fluid is combination of drag and pressure flow



- **Derive the expression for the velocity profile of the fluid between the plates as a function of the fluid properties (ρ & μ), distance between the plates (*B*), plate velocity (u_{xo}), and pressure drop ($\Delta P/L$).**

- **Find the position (y) of the maximum velocity u_{max} if u_{xo}<U*.**

- **Find an expression for the value of the velocity u_{xo} = U* (the velocity of the bottom plate) that yields a zero shear stress at the fixed plate.**

- **Sketch the velocity profile of the fluid if the velocity of the bottom plate is u_{xo}=U*.**

Hint: Start from the Navier –Stokes equations. List all pertinent assumptions and boundary conditions. Find the steady state velocity profile. Assume isothermal, incompressible, laminar flow of Newtonian fluid

Find velocity profile between moving and fixed plate.

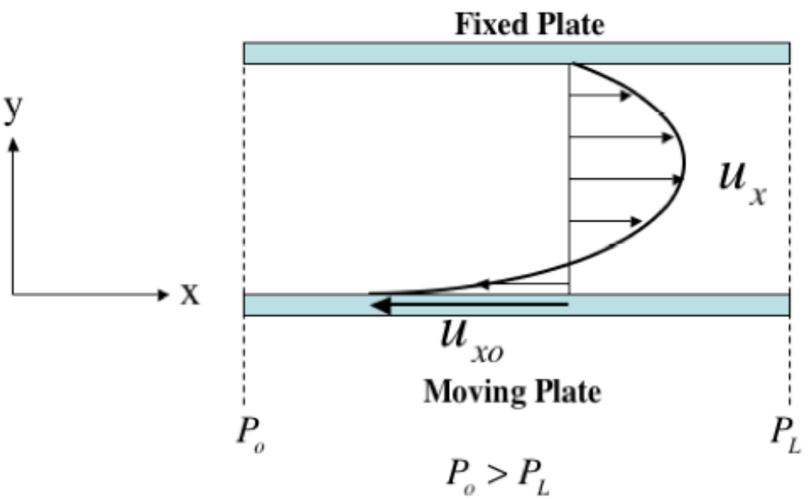
Start by simplifying Navier-Stokes Equations in rectangular coordinates using the following assumptions:

Steady state, fully developed, isothermal flow between parallel plates.

Laminar flow of a Newtonian incompressible fluid.

No-slip conditions at both plates. L>>B, end effects are negligible.

Flow is horizontal (gravity is negligible) and unidirectional in *X* direction



$$u_z = 0; \quad \frac{\partial u_z}{\partial x} = 0; \quad \frac{\partial u_z}{\partial y} = 0; \quad \frac{\partial u_z}{\partial z} = 0$$

$$u_y = 0; \quad \frac{\partial u_y}{\partial x} = 0; \quad \frac{\partial u_y}{\partial y} = 0; \quad \frac{\partial u_y}{\partial z} = 0$$

$$u_x \neq 0; \quad \frac{\partial u_x}{\partial x} = 0; \quad \frac{\partial u_x}{\partial z} = 0; \quad \frac{\partial u_x}{\partial y} \neq 0$$

$$\frac{\partial P}{\partial z} = 0; \quad \frac{\partial P}{\partial y} = -\rho g_y; \quad -\frac{\partial P}{\partial x} = \frac{\Delta P}{L} = \frac{P_o - P_L}{L}$$

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We Start from the Navier-Stokes equations for a fluid, and establish and apply appropriate boundary conditions.

In X direction:

$$\rho\left[\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}+u_z\frac{\partial u_x}{\partial z}\right]=-\frac{\partial P}{\partial x}+\mu\left[\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right]+\rho g_x$$

In Y direction:

$$\rho\left[\frac{\partial u_y}{\partial t}+u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}+u_z\frac{\partial u_y}{\partial z}\right]=-\frac{\partial P}{\partial y}+\mu\left[\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}+\frac{\partial^2 u_y}{\partial z^2}\right]+\rho g_y$$

In Z direction:

$$\rho\left[\frac{\partial u_z}{\partial t}+u_x\frac{\partial u_z}{\partial x}+u_y\frac{\partial u_z}{\partial y}+u_z\frac{\partial u_z}{\partial z}\right]=-\frac{\partial P}{\partial z}+\mu\left[\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_z}{\partial y^2}+\frac{\partial^2 u_z}{\partial z^2}\right]+\rho g_z$$

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For Newtonian fluid, incompressible, isothermal, laminar, 1-dimensional, fully developed, and steady state flow, we can obtain from the Navier-Stokes equation:

In ***X*** direction:

$$\rho\left[\cancel{\frac{\partial u_x}{\partial t}}+\cancel{u_x\frac{\partial u_x}{\partial x}}+\cancel{u_y}\frac{\partial u_x}{\partial y}+u_z\cancel{\frac{\partial u_x}{\partial z}}\right]=-\frac{\partial P}{\partial x}+\mu\left[\cancel{\frac{\partial^2 u_x}{\partial x^2}}+\frac{\partial^2 u_x}{\partial y^2}+\cancel{\frac{\partial^2 u_x}{\partial z^2}}\right]+\cancel{\rho g_x}$$

$$\boxed{\frac{\partial P}{\partial x}=\mu\frac{\partial^2 u_x}{\partial y^2}}$$

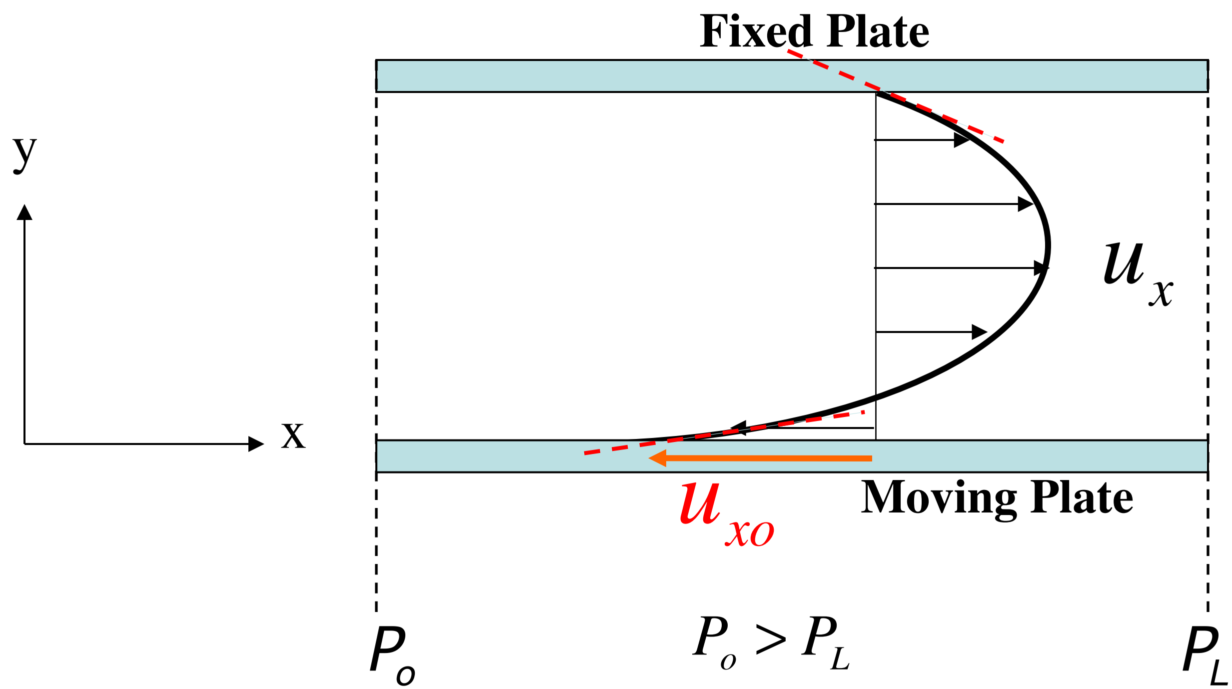
Similarly in ***Y*** direction:

$$\rho\left[\cancel{\frac{\partial u_y}{\partial t}}+u_x\cancel{\frac{\partial u_y}{\partial x}}+u_y\cancel{\frac{\partial u_y}{\partial y}}+u_z\cancel{\frac{\partial u_y}{\partial z}}\right]=-\frac{\partial P}{\partial y}+\mu\left[\cancel{\frac{\partial^2 u_y}{\partial x^2}}+\cancel{\frac{\partial^2 u_y}{\partial y^2}}+\cancel{\frac{\partial^2 u_y}{\partial z^2}}\right]+\rho g_y$$

$$0=-\frac{\partial P}{\partial y}+\rho g_y \quad\Rightarrow\quad \boxed{\frac{\partial P}{\partial y}=\rho g_y=-\rho g}$$

And finally in ***Z*** direction:

$$\cancel{\rho\left[\frac{\partial u_z}{\partial t}+u_x\frac{\partial u_z}{\partial x}+u_y\frac{\partial u_z}{\partial y}+u_z\frac{\partial u_z}{\partial z}\right]=-\frac{\partial P}{\partial z}+\mu\left[\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_z}{\partial y^2}+\frac{\partial^2 u_z}{\partial z^2}\right]}+\rho g_z$$



In summary:

$$\boxed{\frac{\partial P}{\partial x}=\mu\frac{\partial^2 u_x}{\partial y^2}}$$

$$\boxed{\begin{array}{llll} @ & y=0 & \Rightarrow & u_x=u_{xo} \\ @ & y=B & \Rightarrow & u_x=0 \\ @ & x=L & \Rightarrow & P=P_L \end{array}}$$

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$$\boxed{\frac{\partial P}{\partial x} = \mu \frac{\partial^2 u_x}{\partial y^2}} \Rightarrow f(x) = \varphi(y) \Rightarrow f(x) = C = \varphi(y)$$

$$\frac{\partial P}{\partial x} = C = \mu \frac{\partial^2 u_x}{\partial y^2} \Rightarrow \frac{dP}{dx} = C \Rightarrow dP = C dx \Rightarrow \int_{P_o}^{P_L} dP = C \int_0^L dx$$

$$P_L - P_0 = C \cdot L \Rightarrow \boxed{C = -\frac{P_o - P_L}{L}}$$

$$\mu \frac{d^2 u_x}{dy^2} = C \Rightarrow u_x = \frac{C}{2\mu} y^2 + C_1 \cdot y + C_2$$

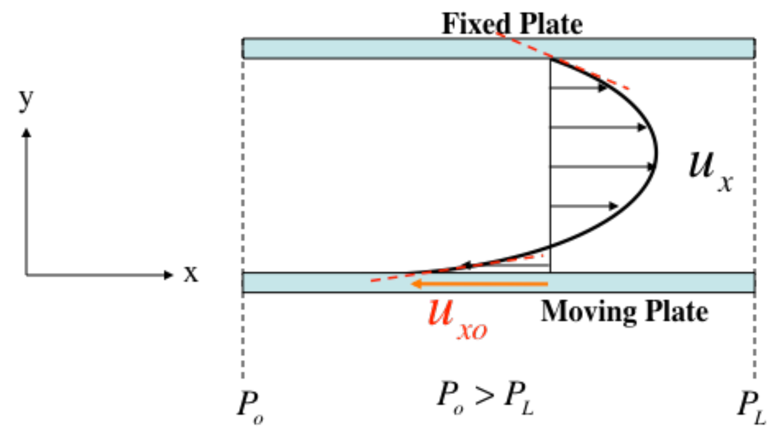
$$\boxed{C_1 = \frac{(P_o - P_L)}{2L\mu} B - \frac{u_{xo}}{B}}$$

$$@ \quad y = 0 \Rightarrow u_x = u_{xo} \Rightarrow \boxed{C_2 = u_{xo}}$$

$$@ \quad y = B \Rightarrow u_x = 0 \Rightarrow 0 = \frac{-(P_o - P_L)}{2L\mu} B^2 + C_1 B + u_{xo}$$

Finally we can obtain expression for the velocity profile"

$$\boxed{u_x = \frac{-(P_o - P_L)}{2L\mu} y^2 + \left[\frac{(P_o - P_L)}{2L\mu} B - \frac{u_{xo}}{B} \right] \cdot y + u_{xo}}$$



Find the position (y) of the maximum velocity u_{max} (if u_{xo}<U*)

$$\frac{\partial u_x}{\partial y} = 0 = \frac{-(P_o - P_L)}{L\mu} y_{\max} + \frac{(P_o - P_L)}{2L\mu} B - \frac{u_{xo}}{B}$$

$$\Downarrow$$

$$\boxed{y_{\max} = \frac{B}{2} - \frac{u_{xo} L \cdot \mu}{B(P_o - P_L)}}$$

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Find an expression for the value of the velocity u_{xo} = U* (the velocity of the bottom plate) that yields a zero-shear stress at the fixed plate.

$$\left.\frac{\partial u_x}{\partial y}\right|_{y=B} = 0$$

$$\boxed{u_x = \frac{-(P_o - P_L)}{2L\mu} y^2 + \left[\frac{(P_o - P_L)}{2L\mu} B - \frac{u_{xo}}{B}\right] \cdot y + u_{xo}}$$

$$\frac{\partial u_x}{\partial y} = \frac{-(P_o - P_L)}{L\mu} y + \frac{(P_o - P_L)}{2L\mu} B - \frac{u_{xo}}{B}$$

$$\Downarrow$$

$$\left.\frac{\partial u_x}{\partial y}\right|_{y=B} = 0 = \frac{-(P_o - P_L)}{L\mu} B + \frac{(P_o - P_L)}{2L\mu} B - \frac{U^*}{B}$$

$$\Downarrow$$

$$\boxed{U^* = \frac{-(P_o - P_L)}{2L\mu} B^2}$$

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What is the shape of the velocity profile when u_{xo} = U* (the velocity of the bottom plate) that yields a zero shear stress at the fixed plate.

$$u_x = \frac{-(P_o - P_L)}{2L\mu} y^2 + \left[\frac{(P_o - P_L)}{2L\mu} B - \frac{u_{xo}}{B} \right] \cdot y + u_{xo}$$

$$U^* = \frac{-(P_o - P_L)}{2L\mu} B^2$$

$$u_x = \frac{-(P_o - P_L)}{2L\mu} y^2 + \left[\frac{(P_o - P_L)}{2L\mu} B - \frac{\frac{-(P_o - P_L)}{2L\mu} B^2}{B} \right] \cdot y + \frac{-(P_o - P_L)}{2L\mu} B^2$$

$$\Downarrow$$

$$u_x = \left[\frac{-(P_o - P_L)}{2L\mu} \right] y^2 + \left[\frac{(P_o - P_L)}{L\mu} B \right] \cdot y - \frac{(P_o - P_L)}{2L\mu} B^2$$

Sketch the velocity profile of the fluid if the velocity of the bottom plate is $u_{xo}=U^*$