Oregon State University

OREGON STATE UNIVERSITY CBEE
Department of Chemical Engineering

CHE 331
Transport Phenomena I

Dr. Goran Jovanovic

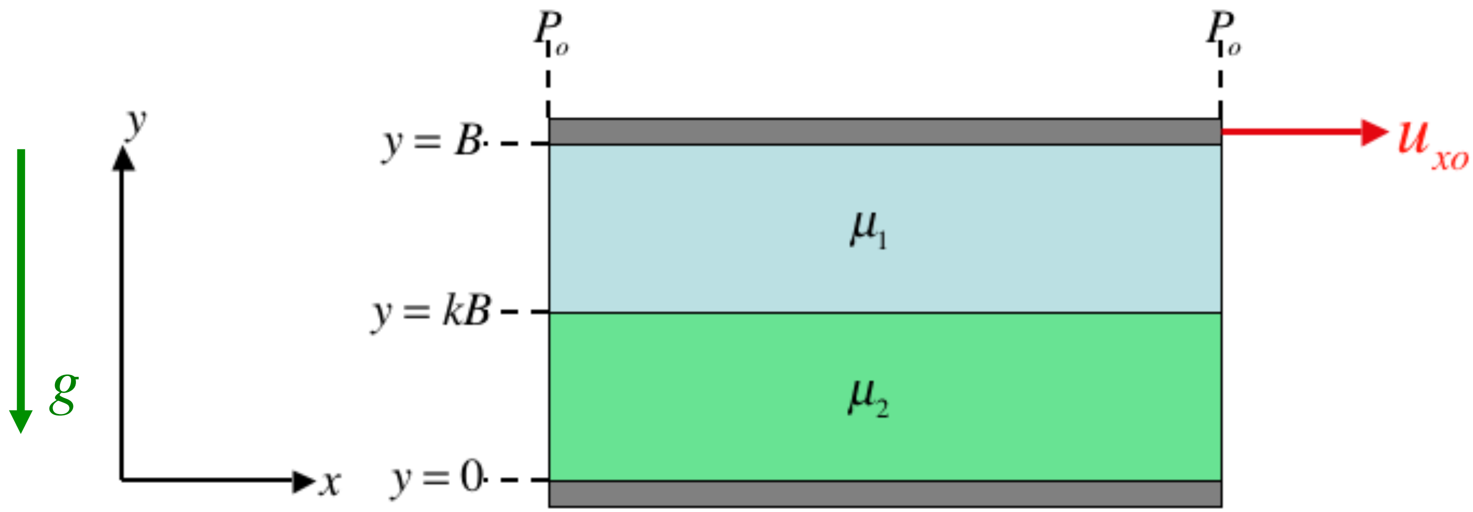
Applications of Navier-Stokes Equations

Please turn-off cell phones

CHE 331
Fall 2022

Oregon State University

A pair of viscous liquids is initially layered between two parallel plates, as illustrated below. The lower plate is stationary and the upper plate moves in its own plane at uniform velocity u_{xo}.



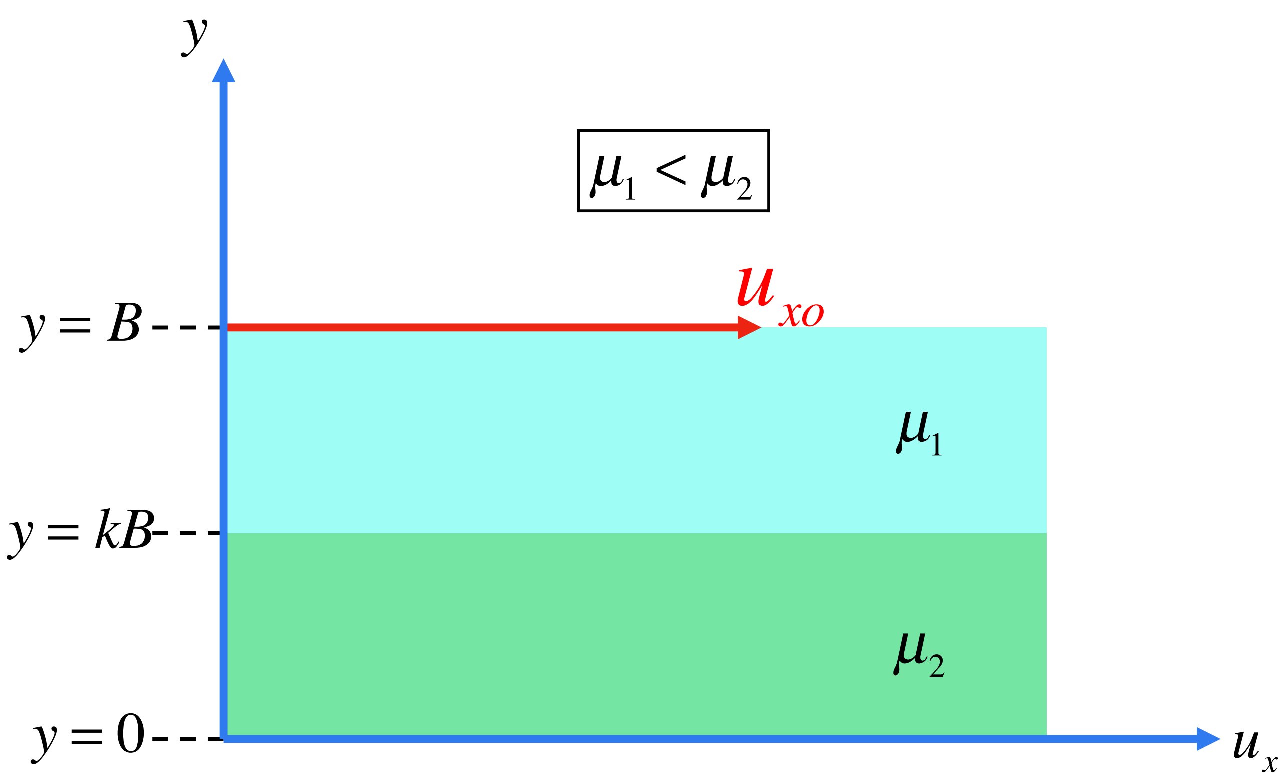
The viscosities of the two fluids are μ_1 and μ_2 respectively. No pressure difference is imposed externally on the fluid, i.e. this is strictly a **drag induced flow**. The obvious consequence is that pressure P is not function of 'x', i.e.:

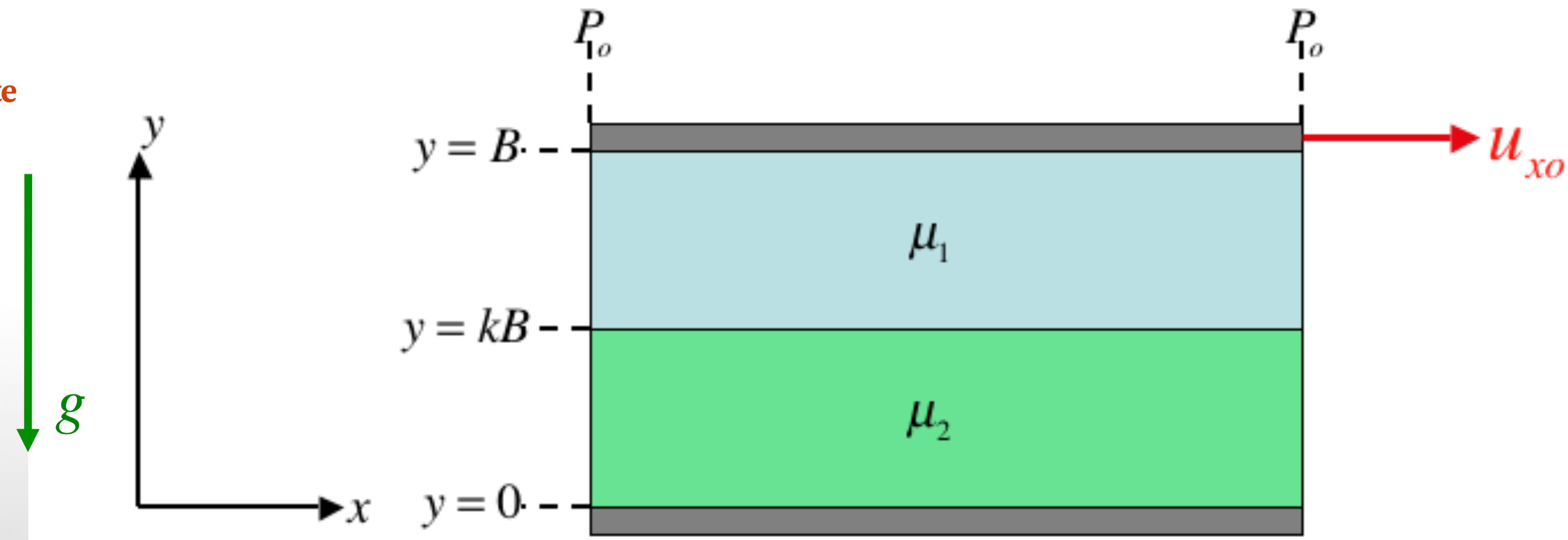
$$\boxed{\frac{\partial P}{\partial x} = 0}$$

a) Find the steady state velocity profiles in the two fluids.

Hint: Start from the Navier-Stokes equations for each fluid. Do not forget to define boundary conditions for both fluids. State clearly all assumptions!

b) As you develop the solution (velocity profile in both fluids) for the above problem make a graphical sketch of the velocity profiles:





$$\boxed{u_z \equiv 0 \quad \Rightarrow \quad \frac{\partial u_z}{\partial x} = \frac{\partial u_z}{\partial y} = \frac{\partial u_z}{\partial z} \equiv 0}$$

$$\boxed{\frac{\partial P}{\partial x} = 0}$$

$$\boxed{u_y \equiv 0 \quad \Rightarrow \quad \frac{\partial u_y}{\partial x} = \frac{\partial u_y}{\partial y} = \frac{\partial u_y}{\partial z} \equiv 0}$$

$$\boxed{u_x \neq 0 \qquad \frac{\partial u_x}{\partial x} = 0 \qquad \frac{\partial u_x}{\partial z} = 0 \qquad \frac{\partial u_x}{\partial y} \neq 0}$$

Oregon State University

We start from the Navier-Stokes equations for each fluid, and establish and apply appropriate boundary conditions.

In X direction:

$$\rho\left[\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}+u_z\frac{\partial u_x}{\partial z}\right]=-\frac{\partial P}{\partial x}+\mu\left[\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right]+\rho g_x$$

In Y direction:

$$\rho\left[\frac{\partial u_y}{\partial t}+u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}+u_z\frac{\partial u_y}{\partial z}\right]=-\frac{\partial P}{\partial y}+\mu\left[\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}+\frac{\partial^2 u_y}{\partial z^2}\right]+\rho g_y$$

In Z direction:

$$\rho\left[\frac{\partial u_z}{\partial t}+u_x\frac{\partial u_z}{\partial x}+u_y\frac{\partial u_z}{\partial y}+u_z\frac{\partial u_z}{\partial z}\right]=-\frac{\partial P}{\partial z}+\mu\left[\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_z}{\partial y^2}+\frac{\partial^2 u_z}{\partial z^2}\right]+\rho g_z$$

For Newtonian fluid, and incompressible, isothermal, laminar, one dimensional, fully developed, and steady state flow we can obtain from the Navier-Stokes equation:

$$\rho\left[\cancel{\frac{\partial u_x}{\partial t}} + u_x \cancel{\frac{\partial u_x}{\partial x}} + \cancel{u_y} \frac{\partial u_x}{\partial y} + \cancel{u_z} \cancel{\frac{\partial u_x}{\partial z}}\right] = -\cancel{\frac{\partial P}{\partial x}} + \mu\left[\cancel{\frac{\partial^2 u_x}{\partial x^2}} + \frac{\partial^2 u_x}{\partial y^2} + \cancel{\frac{\partial^2 u_x}{\partial z^2}}\right] + \cancel{\rho g_x}$$

$$\boxed{\frac{\partial^2 u_x}{\partial y^2} = 0}$$

The above equation is true for both fluids, thus we can write:

$$Fluid\ \ I \quad \boxed{\frac{\partial^2 u_x^I}{\partial y^2} = 0} \qquad\qquad Fluid\ \ II \quad \boxed{\frac{\partial^2 u_x^{II}}{\partial y^2} = 0}$$

Oregon State University

Similarly in "y" direction:

$$\rho\left[\cancel{\frac{\partial u_y}{\partial t}}+u_x\cancel{\frac{\partial u_y}{\partial x}}+u_y\cancel{\frac{\partial u_y}{\partial y}}+u_z\cancel{\frac{\partial u_y}{\partial z}}\right]=-\frac{\partial P}{\partial y}+\mu\left[\cancel{\frac{\partial^2 u_y}{\partial x^2}}+\cancel{\frac{\partial^2 u_y}{\partial y^2}}+\cancel{\frac{\partial^2 u_y}{\partial z^2}}\right]+\rho g_y$$

$$0=-\frac{\partial P}{\partial y}+\rho g_y \quad\Rightarrow\quad \boxed{\frac{\partial P^I}{\partial y}=\rho g_y \quad\Rightarrow\quad \frac{\partial P^{II}}{\partial y}=\rho g_y}$$

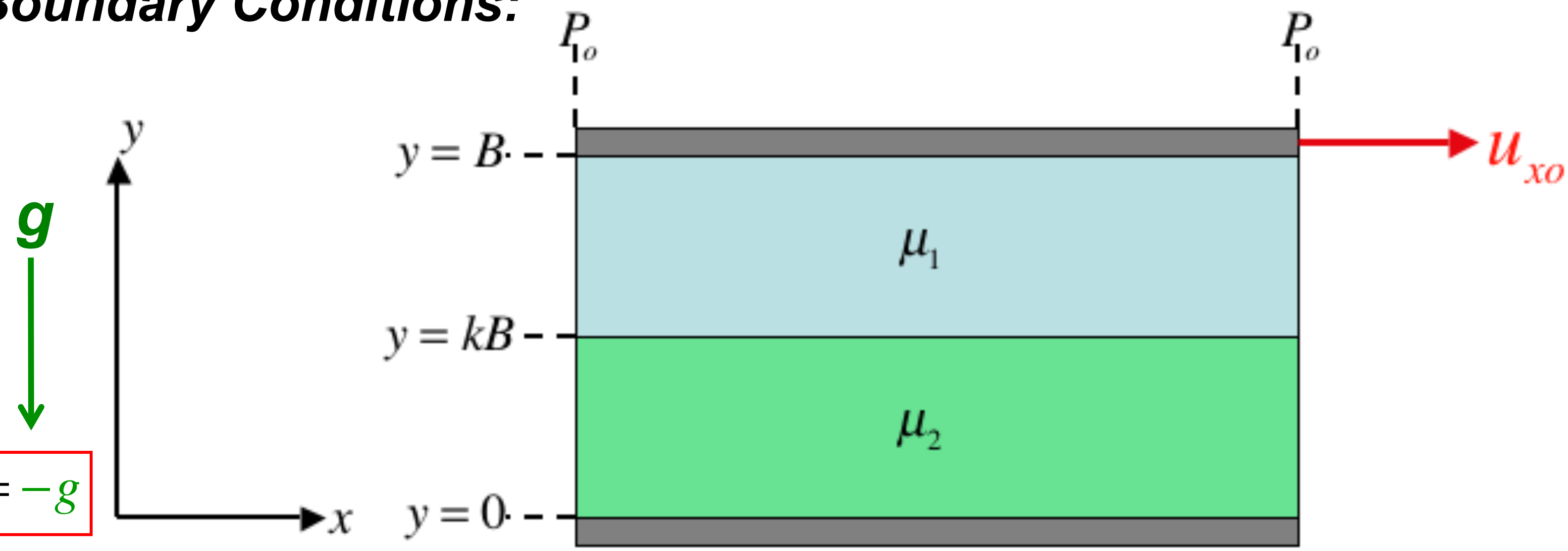
And finally in "z" direction:

$$\cancel{\rho\left[\frac{\partial u_z}{\partial t}+u_x\frac{\partial u_z}{\partial x}+u_y\frac{\partial u_z}{\partial y}+u_z\frac{\partial u_z}{\partial z}\right]=-\frac{\partial P}{\partial z}+\mu\left[\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_z}{\partial y^2}+\frac{\partial^2 u_z}{\partial z^2}\right]}+\rho g_z$$

In summary:

$$\boxed{\frac{\partial^2 u_x^I}{\partial y^2}=0} \quad \boxed{\frac{\partial^2 u_x^{II}}{\partial y^2}=0} \quad \boxed{\frac{\partial P^I}{\partial y}=\rho g_y} \quad \boxed{\frac{\partial P^{II}}{\partial y}=\rho g_y}$$

Boundary Conditions:



$$\boxed{g_y = -g}$$

$$\frac{\partial^2 u_x^I}{\partial y^2} = 0$$

$$\text{at } y = B \quad u_x^I = u_{xo}$$

$$@ \; y = kB; \quad u_x^I = u_x^{II}$$

$$\frac{\partial^2 u_x^{II}}{\partial y^2} = 0$$

$$\text{at } y = 0 \quad u_x^{II} = 0$$

$$@ \; y = kB; \quad \mu^I \frac{\partial u_x^I}{\partial y} = \mu^{II} \frac{\partial u_x^{II}}{\partial y}$$

$$\frac{\partial P^I}{\partial y} = \rho g_y$$

$$\int_{P_{y=kB}}^{P_{y=B}} dP^I = \rho g_y \int_{y=kB}^{y=B} dy \quad \Rightarrow \quad \Delta P^I = -\rho g B(1-k)$$

$$\frac{\partial P^{II}}{\partial y} = \rho g_y$$

$$\int_{P_{y=0}}^{P_{y=kB}} dP^{II} = \rho g_y \int_{y=0}^{y=kB} dy \quad \Rightarrow \quad \Delta P^{II} = -\rho g (kB)$$

Oregon State University

The solutions of the Navier-Stokes Equations are:

Fluid I:	*Fluid II:*
$u_x^I = a^I y + b^I$	$u_x^{II} = a^{II} y + b^{II}$
$at \quad y = B \quad u_x^I = u_{xo}$	$at \quad y = 0 \quad u_x^{II} = 0$
$\boxed{u_{xo} = a^I B + b^I}$	$0 = a^{II} 0 + b^{II} \quad \Rightarrow \quad \boxed{b^{II} = 0}$

$$at \quad y = kB \quad u_x^I = u_x^{II} \quad \Rightarrow \quad \boxed{a^I (kB) + b^I = a^{II} (kB) + \cancel{b}^{II}}$$

$$at \quad y = kB \quad \mu^I \frac{\partial u_x^I}{\partial y} = \mu^{II} \frac{\partial u_x^{II}}{\partial y} \quad \Rightarrow \quad \boxed{\mu^I a^I = \mu^{II} a^{II}}$$

From the above four equations one can calculate a^I, a^{II}, b^I, and b^{II}.

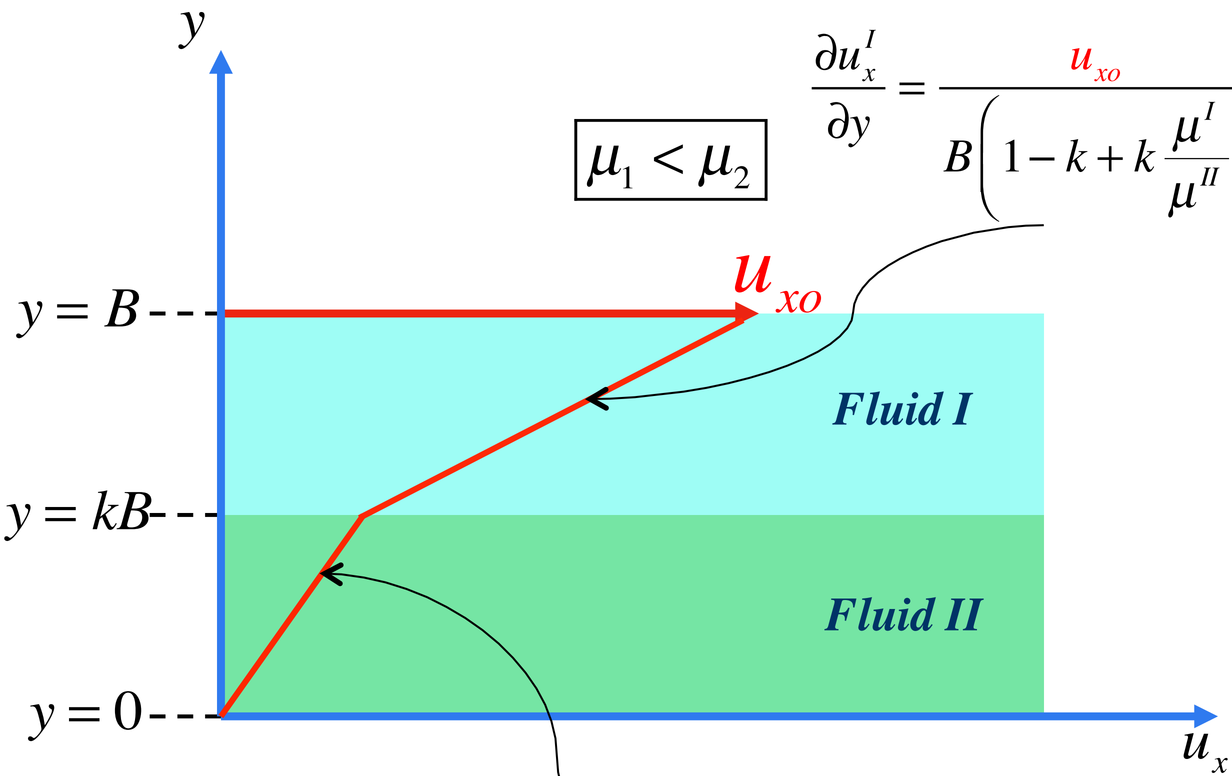
$$a^I = \frac{\left(\frac{u_{xo}}{B}\right)}{1-k+k\frac{\mu^I}{\mu^{II}}} \qquad a^{II} = \left(\frac{\mu^I}{\mu^{II}}\right)\frac{\left(\frac{u_{xo}}{B}\right)}{1-k+k\left(\frac{\mu^I}{\mu^{II}}\right)}$$

$$b^I = u_{xo} - \frac{u_{xo}}{1-k+k\left(\frac{\mu^I}{\mu^{II}}\right)} \qquad b^{II} = 0$$

Finally:

$$\boxed{\frac{u_x^I}{u_{xo}} = \frac{1}{1-k+k\left(\frac{\mu^I}{\mu^{II}}\right)}\left[\frac{y}{B}\right] + \left(1 - \frac{1}{1-k+k\left(\frac{\mu^I}{\mu^{II}}\right)}\right)}$$

$$\boxed{\frac{u_x^{II}}{u_{xo}} = \frac{\left(\frac{\mu^I}{\mu^{II}}\right)}{1-k+k\left(\frac{\mu^I}{\mu^{II}}\right)}\left[\frac{y}{B}\right]}$$



$$\frac{\partial u_x^I}{\partial y} = \frac{u_{xo}}{B\left(1-k+k\dfrac{\mu^I}{\mu^{II}}\right)}$$

$$\frac{\partial u_x^{II}}{\partial y} = \frac{u_{xo}\dfrac{\mu^I}{\mu^{II}}}{B\left(1-k+k\dfrac{\mu^I}{\mu^{II}}\right)}$$

$$\mu_1 < \mu_2 \quad \Rightarrow \quad \frac{\partial u_x^I}{\partial y} > \frac{\partial u_x^{II}}{\partial y}$$

If fluid 1 and fluid 2 are the same fluid, then:

$$\boxed{\mu^{I} = \mu^{II}}$$

$$\frac{u_x^I}{u_{xo}} = \frac{1}{1-k+k\left(\cancel{\frac{\mu^I}{\mu^{II}}}\right)}\left[\frac{y}{B}\right] + \left(1 - \frac{1}{1-k+k\left(\cancel{\frac{\mu^I}{\mu^{II}}}\right)}\right) \Rightarrow \frac{u_x^I}{u_{xo}} = \left[\frac{y}{B}\right]$$

1 1

$$u_x^I = u_{xo}\left[\frac{y}{B}\right]$$