# CHE331 – Transport Phenomena I

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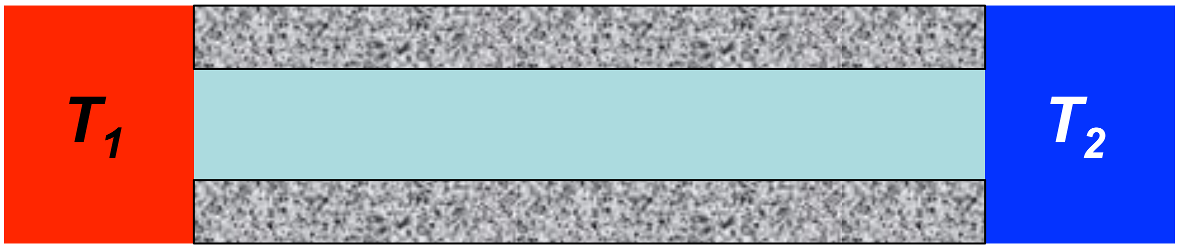
## Introduction to Mathematical Modeling II

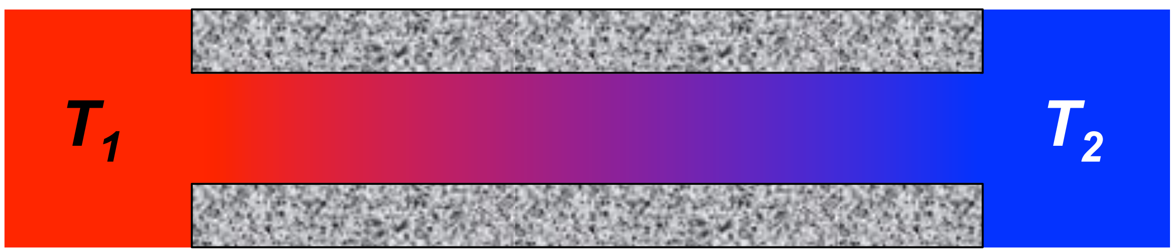
**(Please turn you cell-phones off)**

# Mathematical Modeling

Let us now consider a metal bar positioned between two (large) heat sinks as shown in illustration below.




After reasonably long time, i.e. after a steady state condition is reached, we may expect to find:



# Mathematical Modeling

***Question:***

What is the temperature distribution in the metal rod at steady state?

***Answer:***

We will set up the energy (heat) balance equation for the rod.

***List of Variables***:

$L$ (=) length of the rod [m]

$k$ (=) thermal conductivity of the metal rod [J/mK]

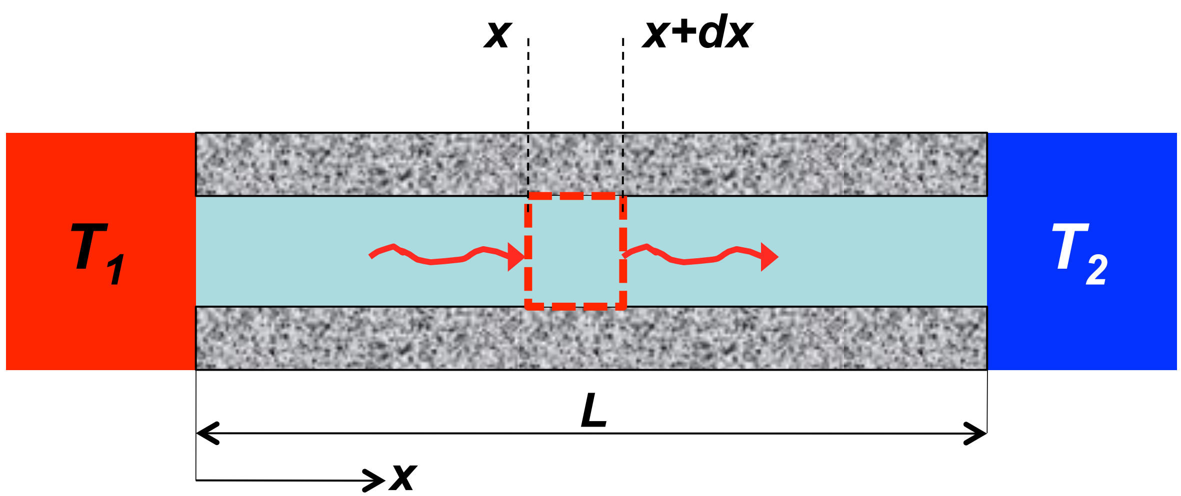
$d$ (=) diameter of the rod [m]

$\rho$ (=) density of the metal [$kg/m^3$]

$C_p$ (=) specific heat of the metal rod [J/kgK]

# Mathematical Modeling

*Since we are interested in the temperature distribution along the rod we will set up a differential energy balance across one differential element at any non-specific position in the rod.*

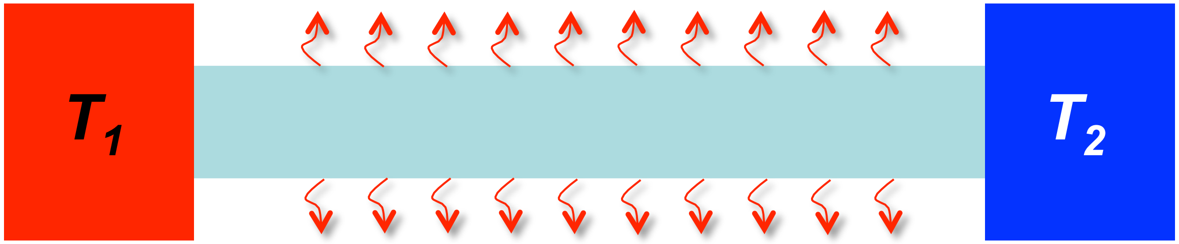


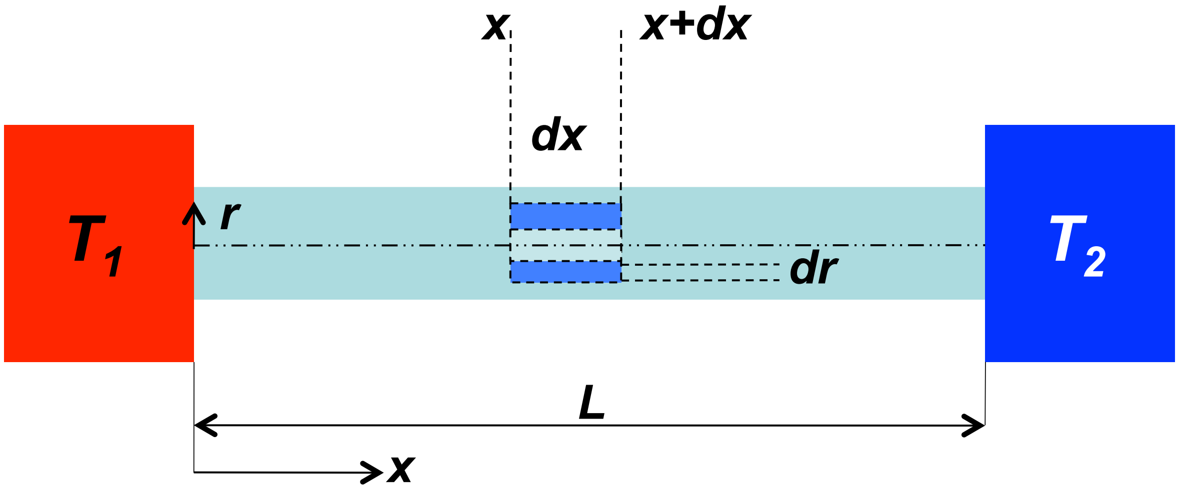

Apply the energy conservation law:

**Input - Output = Accumulation [J]**

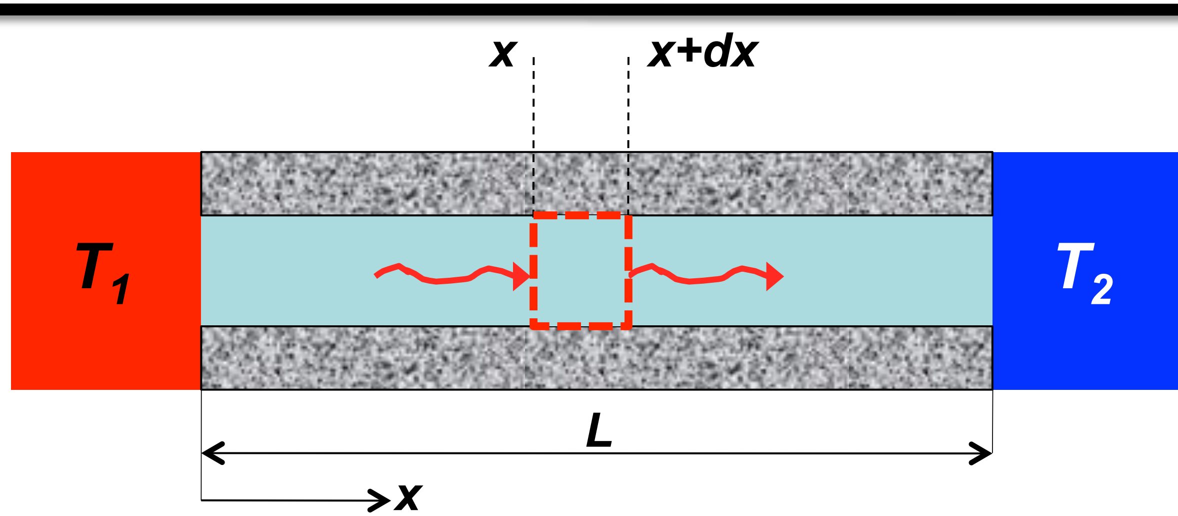
# Mathematical Modeling

*Since we are interested in the temperature distribution along the rod we will set up a differential energy balance across one differential element at any non-specific position in the rod.*

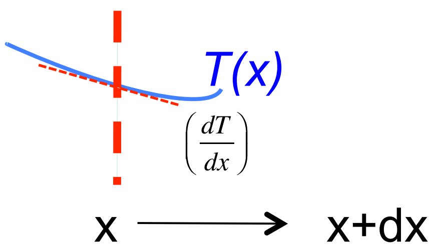


# Mathematical Modeling




According to the Fourier "law" the flux of heat energy passing through a surface is:

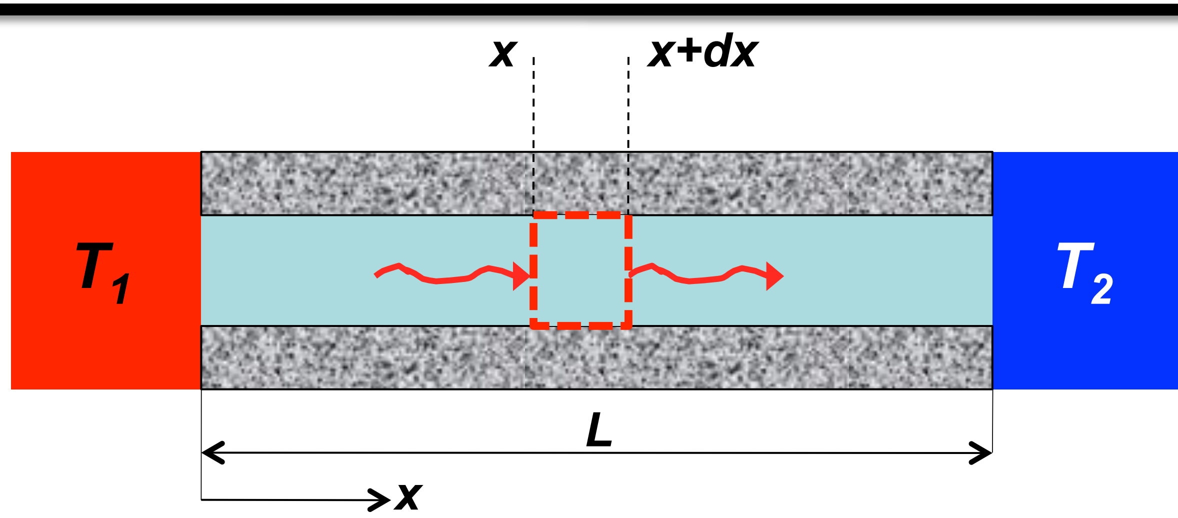



$$\Phi_{heat}\Big|_x = k\left(-\frac{dT}{dx}\right)\Bigg|_x \quad \left[\frac{J}{m^2 \cdot s}\right] \quad \textit{flux of energy transported at } x$$

$$q\Big|_x = k\left(\frac{\pi d^2}{4}\right)\left(-\frac{dT}{dx}\right)\Bigg|_x \quad \left[\frac{J}{s}\right] \quad \textit{rate of energy transported at } x$$

$$Q\Big|_x = k\left(\frac{\pi d^2}{4}\right)\left(-\frac{dT}{dx}\right)\Bigg|_x \Delta t \quad [J] \quad \textit{energy}$$

# Mathematical Modeling




**Input - Output = Accumulation [J]**

$$k\left(\frac{\pi d^2}{4}\right)\left(-\frac{dT}{dx}\right)\Bigg|_x \Delta t - k\left(\frac{\pi d^2}{4}\right)\left(-\frac{dT}{dx}\right)\Bigg|_{x+\Delta x} \Delta t = \rho C_p T\left(\frac{\pi d^2}{4}\right)\Delta x\Bigg|_{t+\Delta t} - \rho C_p T\left(\frac{\pi d^2}{4}\right)\Delta x\Bigg|_t \quad [J]$$

$$\left(\frac{J}{m\cdot K\cdot s}\right)\left(m^2\right)\left(\frac{K}{m}\right)\left(s\right) \qquad \left(\frac{kg}{m^3}\right)\left(\frac{J}{kg\cdot K}\right)\left(K\right)\left(m^2\right)\left(m\right)$$

# Mathematical Modeling

$$k\left(\cancel{\frac{\pi d^2}{4}}\right)\left(-\frac{dT}{dx}\right)\bigg|_{x}\Delta t - k\left(\cancel{\frac{\pi d^2}{4}}\right)\left(-\frac{dT}{dx}\right)\bigg|_{x+\Delta x}\Delta t = \rho C_p T\left(\cancel{\frac{\pi d^2}{4}}\right)\Delta x\bigg|_{t+\Delta t} - \rho C_p T\left(\cancel{\frac{\pi d^2}{4}}\right)\Delta x\bigg|_{t}$$

$$k\left(\frac{dT}{dx}\right)\bigg|_{x+\Delta x}\Delta t - k\left(\frac{dT}{dx}\right)\bigg|_{x}\Delta t = \rho C_p T\Delta x\Big|_{t+\Delta t} - \rho C_p T\Delta x\Big|_{t} \qquad \textit{Devide both side with } \rho C_p \Delta t \Delta x$$

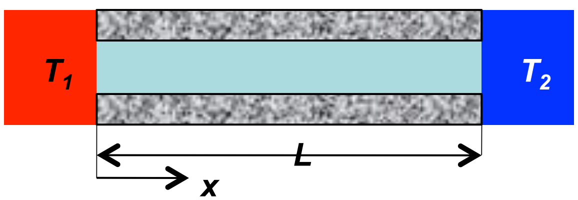
$$\left(\frac{k}{\rho C_p}\right)\frac{\left(\frac{dT}{dx}\right)\Big|_{x+\Delta x} - \left(\frac{dT}{dx}\right)\Big|_{x}}{\Delta x} = \frac{T\big|_{t+\Delta t} - T\big|_{t}}{\Delta t}$$

$$\left(\frac{k}{\rho C_p}\right)\lim_{\Delta x\to 0}\left(\frac{\left(\frac{dT}{dx}\right)\Big|_{x+\Delta x} - \left(\frac{dT}{dx}\right)\Big|_{x}}{\Delta x}\right) = \lim_{\Delta t\to 0}\left(\frac{T\big|_{t+\Delta t} - T\big|_{t}}{\Delta t}\right) \quad \Rightarrow \quad \boxed{\left(\frac{k}{\rho C_p}\right)\frac{d^2T}{dx^2} = \frac{dT}{dt}}$$

# Mathematical Modeling

At steady state: $\left(\frac{k}{\rho C_p}\right)\frac{d^2T}{dx^2} = \cancel{\frac{dT}{dt}} \Rightarrow \left(\frac{k}{\rho C_p}\right)\frac{d^2T}{dx^2} = 0 \Rightarrow \boxed{\frac{d^2T}{dx^2} = 0}$

Thus, mathematical model in differential form:




$$\boxed{\begin{array}{c} \frac{d^2T}{dx^2} = 0 \\ BC1: @ \quad x = 0 \quad T = T_1 \\ BC2: @ \quad x = L \quad T = T_2 \end{array}}$$

Now we can integrate (twice) the governing differential equation:

$$\frac{d^2T}{dx^2} = 0 \quad \Rightarrow \quad \frac{dT}{dx} = a \quad \Rightarrow \quad T = ax + b$$

# Mathematical Modeling

Integration constants *a* and *b*, are determined from boundary conditions *BC1* and *BC2*;

$$T = ax + b \qquad \Rightarrow \qquad b = T_1 \qquad \Rightarrow \qquad a = \frac{T_2 - T_1}{L}$$

$$\boxed{\begin{array}{lll} BC1: @ & x = 0 & T = T_1 \\ BC2: @ & x = L & T = T_2 \end{array}}$$

Thus, the mathematical model in integral form:

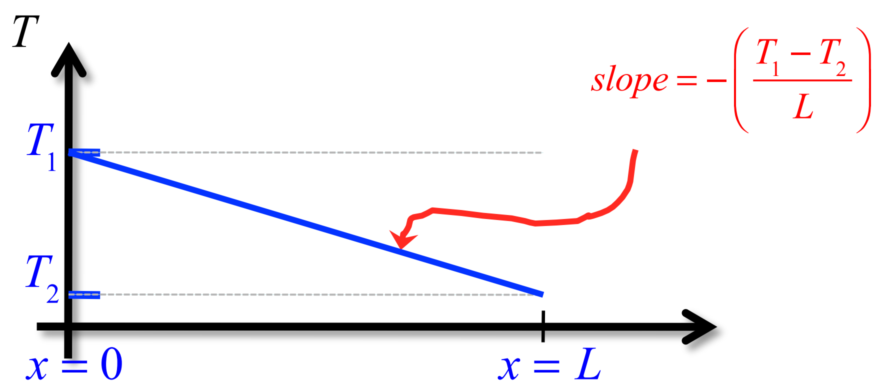
$$\boxed{T = -\left(\frac{T_1 - T_2}{L}\right)x + T_1}$$

# Mathematical Modeling

Notice, that the form of the solution (for steady state condition) does not depend on physical properties ($k, \rho, C_p$) of the metal rod;

$$\boxed{T = -\left(\frac{T_1 - T_2}{L}\right)x + T_1}$$

And, the solution in graphical form;



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*People. Ideas. Innovation.*

*Thank you for your attention!*