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OREGON STATE UNIVERSITY - CBEE
DEPARTMENT OF CHEMICAL ENGINEERING

CHE 331
Transport Phenomena I

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CONSERVATION OF MOMENTUM
Navier-Stokes Equations

Please turn-off cell phones

Navier Stokes Equation

CHE 331
Fall 2022

Conservation of Momentum

Momentum is subject to the first principle like mass and energy. In addition momentum is, like force and velocity - a vector; therefore, we will consider a conservation of momentum in a specific direction. We choose 'x_1' direction.

There are two types of forces acting on the element of fluid:

a) ***Surface Forces*** *b)* ***Body Forces***

Among surface forces we recognize:

a_1) ***Shear Forces*** a_2) ***Normal Forces***

Momentum - Force - Stress relationship:

$$\text{Momentum}: \quad (m \cdot u) \langle = \rangle \left[\frac{kg \;\, m}{s} \right]$$

$$\text{Rate of change of momentum}: \quad \frac{\partial (m \cdot u)}{\partial t} = m \frac{\partial u}{\partial t} = m \cdot a = \text{Force} \langle = \rangle \left[\frac{kg \;\, m}{s^2} \right]$$

$$\text{Flux of the rate of change of momentum}: \frac{\left[\dfrac{\partial (m \cdot u)}{\partial t} \right]}{\text{Area}} = \frac{\text{Force}}{\text{Area}} = \tau \langle = \rangle \left[\frac{kg}{m \cdot s^2} \right]$$

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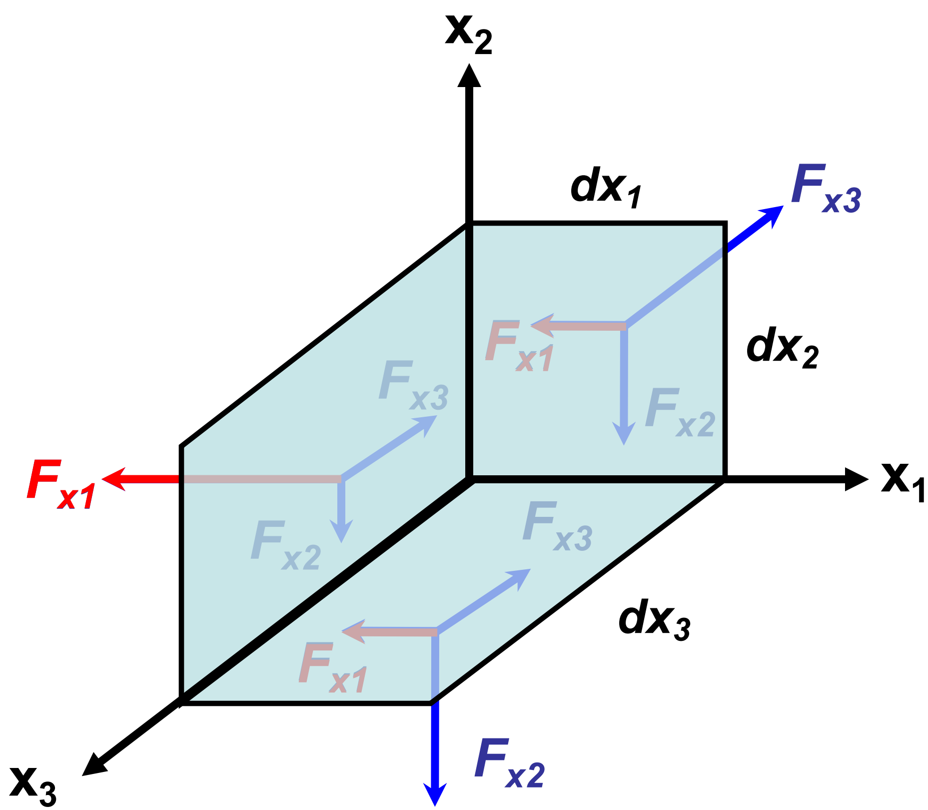
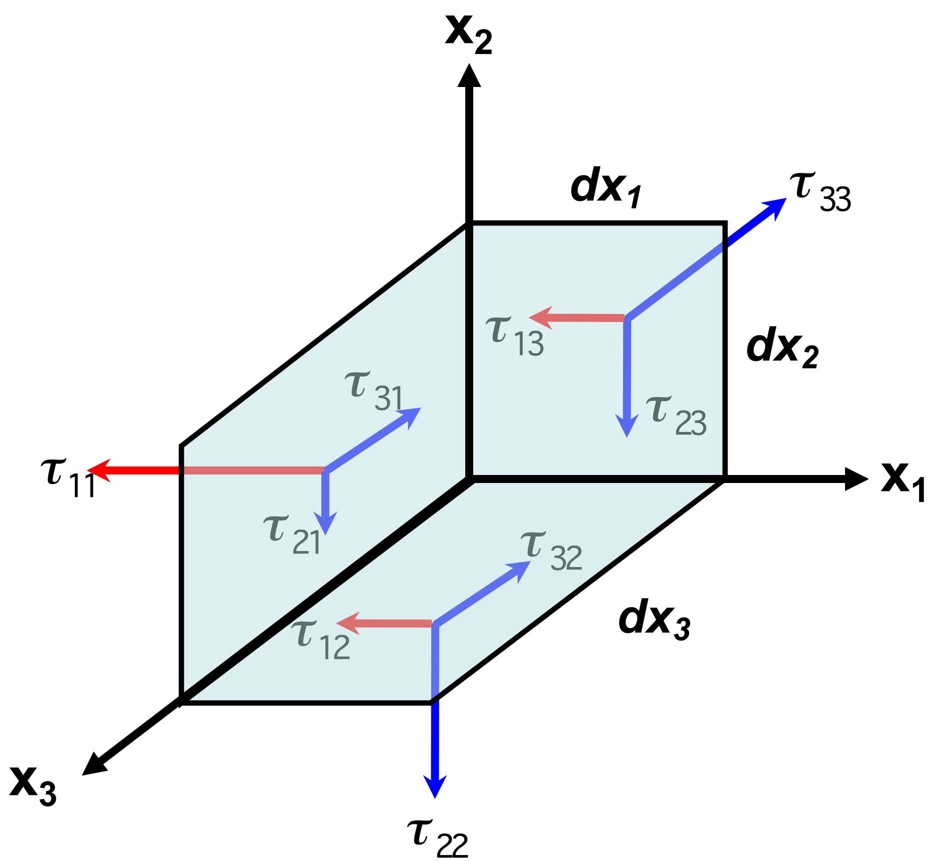
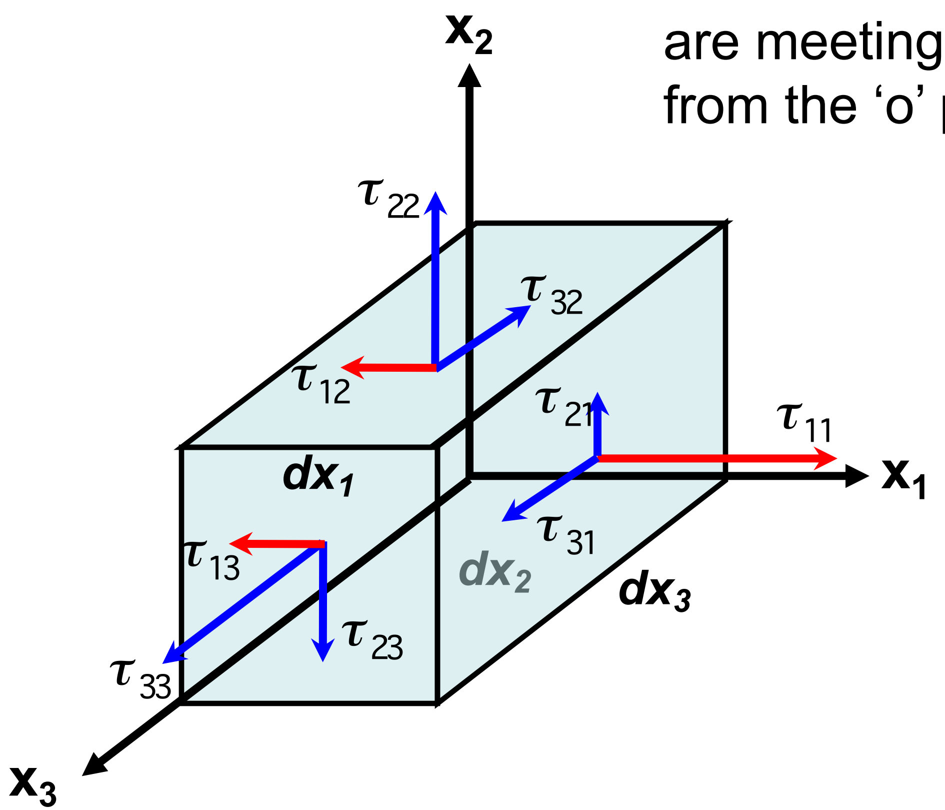
Conservation of Momentum - Stresses



Conservation of Momentum - Stresses

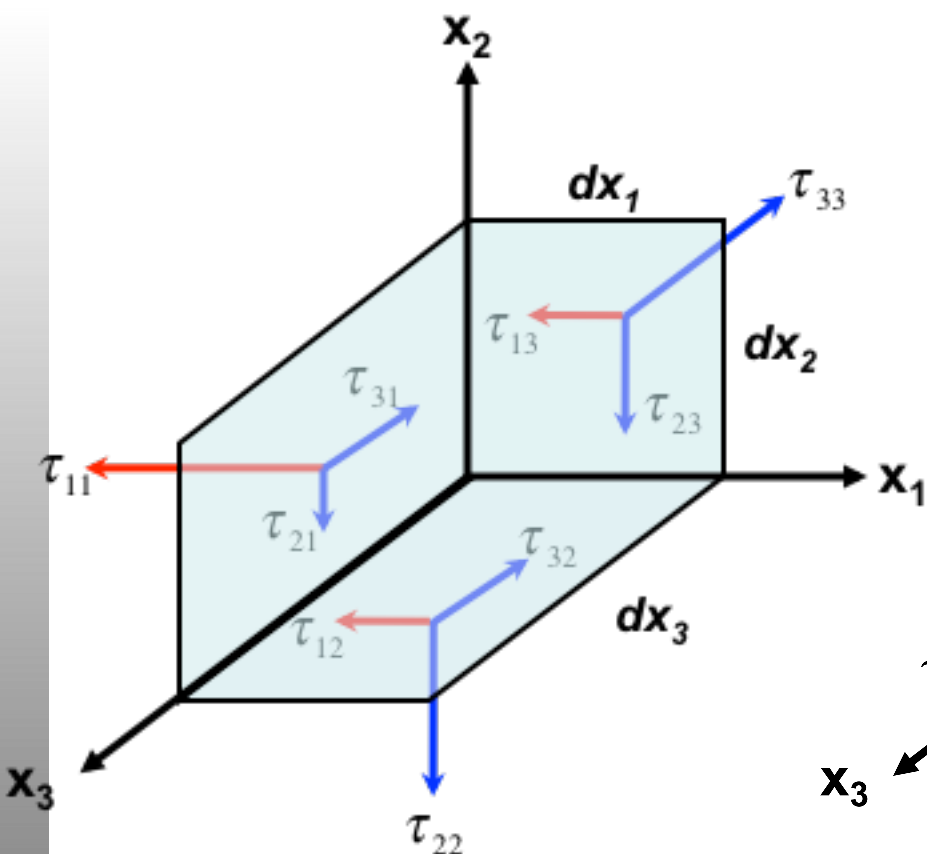


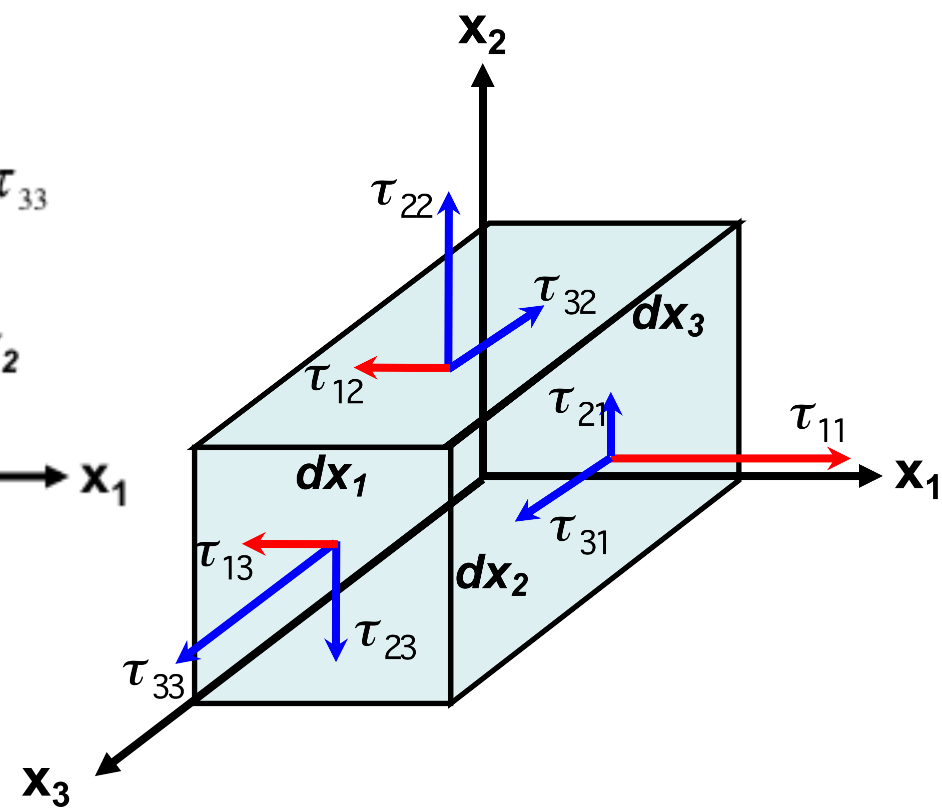
Conservation of Momentum - Stresses



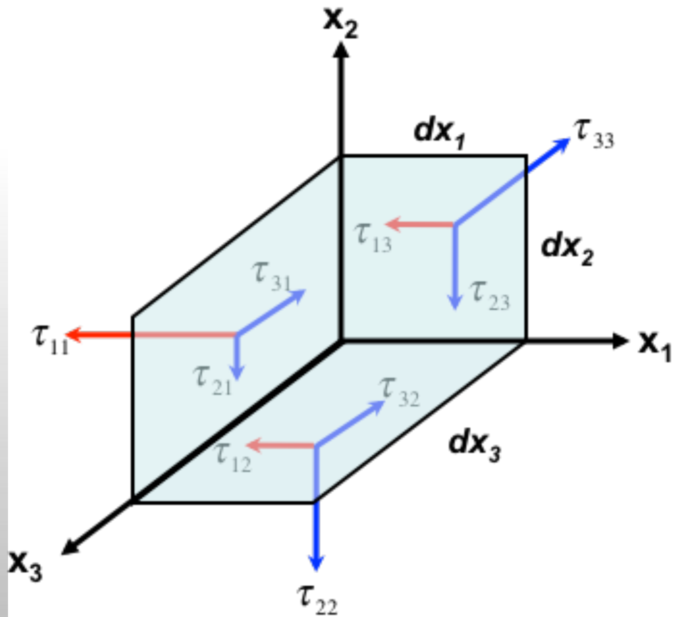
These three surfaces are meeting ***dx*** away from the 'o' point.

Conservation of Momentum - Stresses





Conservation of Momentum in x_1 Direction-Stresses:



$$S_{1,o} = \left(\tau_{11}dx_2dx_3 + \tau_{12}dx_1dx_3 + \tau_{13}dx_1dx_2\right)_o$$

$$S_{1,dx} = \left(\tau_{11}dx_2dx_3 + \tau_{12}dx_1dx_3 + \tau_{13}dx_1dx_2\right)_{dx}$$

$$S_{1,net} = \left(S_{1,o} - S_{1,dx}\right)$$

Observe this difference

$$S_{1,net} = \left(S_{1,o} - S_{1,dx}\right)$$

$$S_{1,net} = \left(\tau_{11}\big|_o - \tau_{11}\big|_{dx_1}\right)dx_2dx_3 + \left(\tau_{12}\big|_o - \tau_{12}\big|_{dx_2}\right)dx_1dx_3 + \left(\tau_{13}\big|_o - \tau_{13}\big|_{dx_3}\right)dx_1dx_2$$

If ***dX***(dx_1, dx_2, dx_3) vector is small we can use Taylor expansion to estimate any function in the vicinity of a point **O**$(0,0,0)$.

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Navier Stokes Equation

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Conservation of Momentum in x_1 Direction-Stresses:

$$\tau_{11}\big|_{dx_1} = \tau_{11}\big|_o + \frac{\partial \tau_{11}}{\partial x_1} dx_1 \qquad \left(\tau_{11}\big|_o - \tau_{11}\big|_{dx_1}\right) = -\frac{\partial \tau_{11}}{\partial x_1} dx_1$$

$$\tau_{12}\big|_{dx_2} = \tau_{12}\big|_o + \frac{\partial \tau_{12}}{\partial x_2} dx_2 \quad \Leftrightarrow \quad \left(\tau_{12}\big|_o - \tau_{12}\big|_{dx_2}\right) = -\frac{\partial \tau_{12}}{\partial x_2} dx_2$$

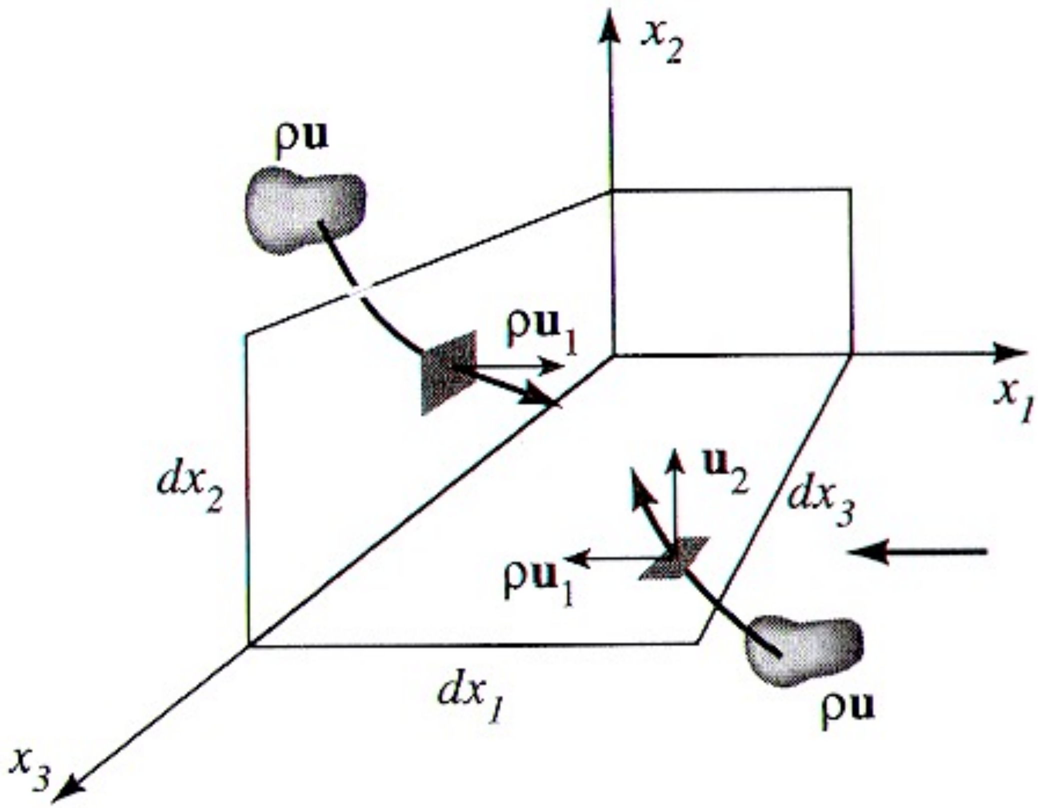
$$\tau_{13}\big|_{dx_3} = \tau_{13}\big|_o + \frac{\partial \tau_{13}}{\partial x_3} dx_3 \qquad \left(\tau_{13}\big|_o - \tau_{13}\big|_{dx_3}\right) = -\frac{\partial \tau_{13}}{\partial x_3} dx_3$$

$$S_{1,net} = -\frac{\partial \tau_{11}}{\partial x_1} dx_1 dx_2 dx_3 - \frac{\partial \tau_{12}}{\partial x_2} dx_2 dx_1 dx_3 - \frac{\partial \tau_{13}}{\partial x_3} dx_3 dx_1 dx_2$$

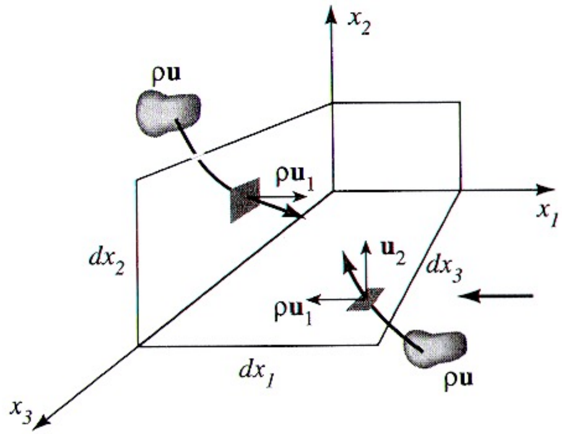
$$\boxed{S_{1,net} = -\frac{\partial \tau_{11}}{\partial x_1} dV - \frac{\partial \tau_{12}}{\partial x_2} dV - \frac{\partial \tau_{13}}{\partial x_3} dV}$$

Net force due to shear and normal stresses acting in ***dx*₁** direction.

Conservation of Flow Momentum in x_1 Direction



Conservation of Flow Momentum in *x1* Direction



$$mu\,(=)\,\frac{kg\cdot m}{s} \leftarrow \text{Momentum}$$

$$\rho u\,(=)\,\frac{kg\cdot m}{m^3 s} \leftarrow \text{Momentum density}$$

$$\rho u\cdot Q\,(=)\,\frac{kg\cdot m}{m^3 s}\cdot\frac{m^3}{s}=\frac{kg\cdot m}{s^2} \leftarrow \text{Force}$$

Volumetric Flow Rate Through Surface dx_2dx_3 in u_1 direction

Volumetric Flow Rate Through Surface dx_1dx_3 in u_1 direction

Volumetric Flow Rate Through Surface dx_1dx_2 in u_1 direction

$$C_{1,0}=\left(\rho u_1\right)_{dx_2dx_3}u_1dx_2dx_3+\left(\rho u_1\right)_{dx_1dx_3}u_2dx_1dx_3+\left(\rho u_1\right)_{dx_1dx_2}u_3dx_1dx_2$$

$$C_{1,dx}=\left(\rho u_1\right)_{dx_2dx_3}u_1dx_2dx_3+\left(\rho u_1\right)_{dx_1dx_3}u_2dx_1dx_3+\left(\rho u_1\right)_{dx_1dx_2}u_3dx_1dx_2$$

The net Flow Force in dx_1 direction: $\boxed{C_{1,net}=C_{1,o}-C_{1,dx}}$

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Conservation of Flow Momentum in *x1* Direction

$$C_{1,net} = C_{1,0} - C_{1,dx} = \left[\left(\rho u_1 u_1\right)_o - \left(\rho u_1 u_1\right)_{dx_1} \right] dx_2 dx_3 +$$

$$+ \left[\left(\rho u_1 u_2\right)_o - \left(\rho u_1 u_2\right)_{dx_2} \right] dx_1 dx_3 +$$

$$+ \left[\left(\rho u_1 u_3\right)_o - \left(\rho u_1 u_3\right)_{dx_3} \right] dx_1 dx_2$$

If **dX**(dx_1, dx_2, dx_3) vector is small we can use Taylor expansion
To estimate any function in the vicinity of a point **O**($0,0,0$).

$$\left.\begin{aligned} \left(\rho u_1 u_1\right)_{dx_1} &= \left(\rho u_1 u_1\right)_O + \frac{\partial\left(\rho u_1 u_1\right)}{dx_1} dx_1 \\ \left(\rho u_1 u_2\right)_{dx_2} &= \left(\rho u_1 u_2\right)_O + \frac{\partial\left(\rho u_1 u_2\right)}{dx_2} dx_2 \\ \left(\rho u_1 u_3\right)_{dx_3} &= \left(\rho u_1 u_3\right)_O + \frac{\partial\left(\rho u_1 u_3\right)}{dx_3} dx_3 \end{aligned}\right\}$$

$$\left(\rho u_1 u_1\right)_O - \left(\rho u_1 u_1\right)_{dx_1} = -\frac{\partial\left(\rho u_1 u_1\right)}{dx_1} dx_1$$

$$\left(\rho u_1 u_2\right)_O - \left(\rho u_1 u_2\right)_{dx_2} = -\frac{\partial\left(\rho u_1 u_2\right)}{dx_2} dx_2$$

$$\left(\rho u_1 u_3\right)_O - \left(\rho u_1 u_3\right)_{dx_3} = -\frac{\partial\left(\rho u_1 u_3\right)}{dx_3} dx_3$$

Conservation of Momentum in x_1 Direction

$$C_{1,net} = -\frac{\partial(\rho u_1 u_1)}{\partial x_1} dx_1 dx_2 dx_3 - \frac{\partial(\rho u_1 u_2)}{\partial x_2} dx_1 dx_2 dx_3 - \frac{\partial(\rho u_1 u_3)}{\partial x_3} dx_1 dx_2 dx_3$$

$$\boxed{C_{1,net} = \left[-\frac{\partial(\rho u_1 u_1)}{\partial x_1} - \frac{\partial(\rho u_1 u_2)}{\partial x_2} - \frac{\partial(\rho u_1 u_3)}{\partial x_3} \right] dV}$$

In the x_1 direction we may have a body force from an external field (gravity for example).

$$\boxed{B_{1,net} = (\rho f_1) dx_1 dx_2 dx_3}$$

Where f_1 is acceleration in x_1 direction

Finally, we can sum-up all forces in x_1 direction and obtain:

$$\boxed{\sum_{net} F = \text{Accumulation}}$$

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Navier Stokes Equation

Net force due to ***Shear and Normal Stresses*** acting in the x_1 direction on the surfaces of the differential element of the fluid:

$$\boxed{S_{1,net} = -\frac{\partial \tau_{11}}{\partial x_1} dV - \frac{\partial \tau_{12}}{\partial x_2} dV - \frac{\partial \tau_{13}}{\partial x_3} dV}$$

Net force in the x_1 direction arising from ***Convective Flow of Momentum*** across the surfaces of the differential element of the fluid:

$$\boxed{C_{1,net} = -\frac{\partial(\rho u_1 u_1)}{\partial x_1} dV - \frac{\partial(\rho u_1 u_2)}{\partial x_2} dV - \frac{\partial(\rho u_1 u_3)}{\partial x_3} dV}$$

Body Force acting on the fluid within the differential volume of the fluid:

$$\boxed{B_1 = (\rho f_1) dV}$$

Where f_1 is the acceleration in x_1 direction, created by the field (for gravitational field $f_1 = g_1$).

Conservation of Momentum in x_1 Direction

It is also possible for the flow field to be unsteady in time in the observed element of the fluid (i.e. *net force ≠ 0*) When this occurs, there is a time change of momentum of fluid within the differential volume element.

$$\boxed{A_1 = \frac{\partial(\rho u_1)}{\partial t} dV}$$

According to Newton's second law one can write:

$$\begin{bmatrix} \text{Rate of change} \\ \text{of momentum} \end{bmatrix} = \begin{bmatrix} \text{Sum of Surface} \\ \text{and Body Forces} \end{bmatrix} + \begin{bmatrix} \text{Sum of forces due} \\ \text{convective flow} \end{bmatrix}$$

$$[A_1] = [S_{1,net} + B_1] + [C_{1,net}]$$

Conservation of Momentum in x_1 Direction

$$\left[C_{1,net} + S_{1,net} + B_1\right] = \left[A_1\right]$$

$$-\frac{\partial(\rho u_1 u_1)}{\partial x_1}dV - \frac{\partial(\rho u_1 u_2)}{\partial x_2}dV - \frac{\partial(\rho u_1 u_3)}{\partial x_3}dV +$$

$$-\frac{\partial \tau_{11}}{\partial x_1}dV - \frac{\partial \tau_{12}}{\partial x_2}dV - \frac{\partial \tau_{13}}{\partial x_3}dV + (\rho f_1)dV = \frac{\partial(\rho u_1)}{\partial t}dV$$

The force balance above is calculated in x_1 direction. We can develop the force balance equations in x_2 and x_3 directions. The above equation can be rearranged to obtain:

Conservation of Momentum in *All* Directions

$$\frac{\partial(\rho u_i)}{\partial t}+\overbrace{\left[\frac{\partial(\rho u_i u_1)}{\partial x_1}+\frac{\partial(\rho u_i u_2)}{\partial x_2}+\frac{\partial(\rho u_i u_3)}{\partial x_3}\right]}^{\text{Convective momentum term}}=$$

$$=(\rho f_i)-\left[\frac{\partial\tau_{i1}}{\partial x_1}+\frac{\partial\tau_{i2}}{\partial x_2}+\frac{\partial\tau_{i3}}{\partial x_3}\right]\quad where\ i=1,2,3$$

From the above expression one can generate force balance equations in all three directions x_1, x_2, and x_3.

But first, let's see if we can simplify the convective momentum term.

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Conservation of Momentum in x_1 Direction

One can perform **differentiation** of the convective momentum term with ρ = const., and obtain:

$$\frac{\partial(\rho u_1 u_1)}{\partial x_1}+\frac{\partial(\rho u_1 u_2)}{\partial x_2}+\frac{\partial(\rho u_1 u_3)}{\partial x_3}=$$

$$=\rho\left[u_1\frac{\partial u_1}{\partial x_1}+u_1\frac{\partial u_1}{\partial x_1}+u_1\frac{\partial u_2}{\partial x_2}+u_2\frac{\partial u_1}{\partial x_2}+u_1\frac{\partial u_3}{\partial x_3}+u_3\frac{\partial u_1}{\partial x_3}\right]$$

$$=\rho\left[u_1\cancel{\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)}+u_1\frac{\partial u_1}{\partial x_1}+u_2\frac{\partial u_1}{\partial x_2}+u_3\frac{\partial u_1}{\partial x_3}\right]$$

$\nabla\cdot\mathbf{U}=0$ Continuty

$$=\rho\left[u_1\frac{\partial u_1}{\partial x_1}+u_2\frac{\partial u_1}{\partial x_2}+u_3\frac{\partial u_1}{\partial x_3}\right]$$

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Conservation of Momentum in *All* Directions

Therefore, the force balance equation for ρ=const. takes the following form:

$$in\ x_1\ direction$$

$$\rho\left[\frac{\partial u_1}{\partial t}+u_1\frac{\partial u_1}{\partial x_1}+u_2\frac{\partial u_1}{\partial x_2}+u_3\frac{\partial u_1}{\partial x_3}\right]=\left(\rho f_1\right)-\left[\frac{\partial \tau_{11}}{\partial x_1}+\frac{\partial \tau_{12}}{\partial x_2}+\frac{\partial \tau_{13}}{\partial x_3}\right]$$

$$in\ x_2\ direction$$

$$\rho\left[\frac{\partial u_2}{\partial t}+u_1\frac{\partial u_2}{\partial x_1}+u_2\frac{\partial u_2}{\partial x_2}+u_3\frac{\partial u_2}{\partial x_3}\right]=\left(\rho f_2\right)-\left[\frac{\partial \tau_{21}}{\partial x_1}+\frac{\partial \tau_{22}}{\partial x_2}+\frac{\partial \tau_{23}}{\partial x_3}\right]$$

$$in\ x_3\ direction$$

$$\rho\left[\frac{\partial u_3}{\partial t}+u_1\frac{\partial u_3}{\partial x_1}+u_2\frac{\partial u_3}{\partial x_2}+u_3\frac{\partial u_3}{\partial x_3}\right]=\left(\rho f_3\right)-\left[\frac{\partial \tau_{31}}{\partial x_1}+\frac{\partial \tau_{32}}{\partial x_2}+\frac{\partial \tau_{33}}{\partial x_3}\right]$$

$$\boxed{\rho\left[\frac{\partial \mathbf{U}}{\partial t}+\mathbf{U}\cdot\nabla\mathbf{U}\right]=\rho f-\left[\nabla\cdot\mathbf{T}\right]}$$

Conservation of Momentum in *All* Directions

$$\rho\left[\frac{\partial u_1}{\partial t}+u_1\frac{\partial u_1}{\partial x_1}+u_2\frac{\partial u_1}{\partial x_2}+u_3\frac{\partial u_1}{\partial x_3}\right]=\left(\rho f_1\right)-\left[\frac{\partial \tau_{11}}{\partial x_1}+\frac{\partial \tau_{12}}{\partial x_2}+\frac{\partial \tau_{13}}{\partial x_3}\right]$$

$$\rho\left[\frac{\partial u_2}{\partial t}+u_1\frac{\partial u_2}{\partial x_1}+u_2\frac{\partial u_2}{\partial x_2}+u_3\frac{\partial u_2}{\partial x_3}\right]=\left(\rho f_2\right)-\left[\frac{\partial \tau_{21}}{\partial x_1}+\frac{\partial \tau_{22}}{\partial x_2}+\frac{\partial \tau_{23}}{\partial x_3}\right]$$

$$\rho\left[\frac{\partial u_3}{\partial t}+u_1\frac{\partial u_3}{\partial x_1}+u_2\frac{\partial u_3}{\partial x_2}+u_3\frac{\partial u_3}{\partial x_3}\right]=\left(\rho f_3\right)-\left[\frac{\partial \tau_{31}}{\partial x_1}+\frac{\partial \tau_{32}}{\partial x_2}+\frac{\partial \tau_{33}}{\partial x_3}\right]$$

$$\nabla\cdot\mathbf{U}=0 \quad\Rightarrow\quad \left[\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right]=0$$

We have 4 equations and 12 unknown dependent variables; (3 velocities and 9 stress elements).

We obviously need more equations or constitutive relationships.

There is a large class of fluids for which the stress components τ_{ij} of the stress tensor are linearly related to shear rates, i.e., the components of the rate deformation tensor Δ_{ij}

Newtonian Fluids

For Newtonian fluids by definition:

$$\boxed{\begin{aligned}\tau_{ij} &= \delta_{ij} P - \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \delta_{ij}\frac{2}{3}\mu \nabla \cdot \mathbf{U} \\ \tau_{ij} &= \delta_{ij} P - \mu\ \Delta_{ij} - \delta_{ij}\frac{2}{3}\mu \nabla \cdot \mathbf{U}\end{aligned}}$$

Where:

$$\boxed{\begin{aligned}\delta_{ij} &= 0 \quad if \quad i \neq j \\ \delta_{ij} &= 1 \quad if \quad i = j\end{aligned}}$$

From the above expression it is obvious that $\tau_{ij} = \tau_{ji}$ and, for example:

$$\tau_{11} = P - \mu\ \Delta_{11} - \frac{2}{3}\mu \nabla \cdot \mathbf{U} = P - \mu\left(2\frac{\partial u_i}{\partial x_i}\right) - \frac{2}{3}\mu \nabla \cdot \mathbf{U}$$

$$\tau_{12} = -\mu\ \Delta_{12} = -\mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

Conservation of Momentum in x_1 Direction

If we substitute the definition for Newtonian fluids into stress tensor elements, we can obtain:

$$\tau_{11} = P - \mu\left(2\frac{\partial u_1}{\partial x_1}\right) - \frac{2}{3}\mu\nabla\cdot\mathbf{U} \quad\Rightarrow\quad \frac{\partial \tau_{11}}{\partial x_1} = \frac{\partial P}{\partial x_1} - 2\mu\frac{\partial^2 u_1}{\partial x_1^2} - \frac{\partial}{\partial x_1}\left(\frac{2}{3}\mu\nabla\cdot\mathbf{U}\right)$$

$$\tau_{12} = -\mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \quad\Rightarrow\quad \frac{\partial \tau_{12}}{\partial x_2} = -\mu\left(\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_1\,\partial x_2}\right)$$

$$\tau_{13} = -\mu\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \quad\Rightarrow\quad \frac{\partial \tau_{13}}{\partial x_3} = -\mu\left(\frac{\partial^2 u_1}{\partial x_3^2} + \frac{\partial^2 u_3}{\partial x_1\,\partial x_3}\right)$$

$$-\left[\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3}\right] = -\frac{\partial P}{\partial x_1} + \mu\left[2\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_1\,\partial x_2} + \frac{\partial^2 u_1}{\partial x_3^2} + \frac{\partial^2 u_3}{\partial x_1\,\partial x_3}\right] - \frac{\partial}{\partial x_1}\left(\frac{2}{3}\mu\nabla\cdot\mathbf{U}\right)$$

Conservation of Momentum in x_1 Direction

$$-\left[\frac{\partial \tau_{11}}{\partial x_1}+\frac{\partial \tau_{12}}{\partial x_2}+\frac{\partial \tau_{13}}{\partial x_3}\right]=-\frac{\partial P}{\partial x_1}+\mu\left[\frac{\partial^2 u_1}{\partial x_1^2}+\frac{\partial^2 u_1}{\partial x_1^2}+\frac{\partial^2 u_1}{\partial x_2^2}+\frac{\partial^2 u_2}{\partial x_1\,\partial x_2}+\frac{\partial^2 u_1}{\partial x_3^2}+\frac{\partial^2 u_3}{\partial x_1\,\partial x_3}\right]-\frac{\partial}{\partial x_1}\left(\frac{2}{3}\mu\nabla\cdot\mathbf{U}\right)$$

$$-\left[\frac{\partial \tau_{11}}{\partial x_1}+\frac{\partial \tau_{12}}{\partial x_2}+\frac{\partial \tau_{13}}{\partial x_3}\right]=-\frac{\partial P}{\partial x_1}+\mu\left\{\left[\frac{\partial^2 u_1}{\partial x_1^2}+\frac{\partial^2 u_1}{\partial x_2^2}+\frac{\partial^2 u_1}{\partial x_3^2}\right]+\left[\frac{\partial}{\partial x_1}\underbrace{\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)}_{\nabla\cdot\mathbf{U}=0}\right]\right\}-\cancelto{0}{\frac{\partial}{\partial x_1}\left(\frac{2}{3}\mu\nabla\cdot\mathbf{U}\right)}$$

$$\boxed{-\left[\frac{\partial \tau_{11}}{\partial x_1}+\frac{\partial \tau_{12}}{\partial x_2}+\frac{\partial \tau_{13}}{\partial x_3}\right]=-\frac{\partial P}{\partial x_1}+\mu\left[\frac{\partial^2 u_1}{\partial x_1^2}+\frac{\partial^2 u_1}{\partial x_2^2}+\frac{\partial^2 u_1}{\partial x_3^2}\right]}$$

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Conservation of Momentum in *All* Directions

$$\rho\left[\frac{\partial u_1}{\partial t}+u_1\frac{\partial u_1}{\partial x_1}+u_2\frac{\partial u_1}{\partial x_2}+u_3\frac{\partial u_1}{\partial x_3}\right]=\left(\rho f_1\right)-\left[\frac{\partial \tau_{11}}{\partial x_1}+\frac{\partial \tau_{12}}{\partial x_2}+\frac{\partial \tau_{13}}{\partial x_3}\right]$$

$$\Downarrow$$

$$\rho\left[\frac{\partial u_1}{\partial t}+u_1\frac{\partial u_1}{\partial x_1}+u_2\frac{\partial u_1}{\partial x_2}+u_3\frac{\partial u_1}{\partial x_3}\right]=\rho g_1-\frac{\partial P}{\partial x_1}+\mu\left[\frac{\partial^2 u_1}{\partial x_1^2}+\frac{\partial^2 u_1}{\partial x_2^2}+\frac{\partial^2 u_1}{\partial x_3^2}\right]$$

Similarly, in x_2 and x_3 direction:

$$\rho\left[\frac{\partial u_2}{\partial t}+u_1\frac{\partial u_2}{\partial x_1}+u_2\frac{\partial u_2}{\partial x_2}+u_3\frac{\partial u_2}{\partial x_3}\right]=\rho g_2-\frac{\partial P}{\partial x_2}+\mu\left[\frac{\partial^2 u_2}{\partial x_1^2}+\frac{\partial^2 u_2}{\partial x_2^2}+\frac{\partial^2 u_2}{\partial x_3^2}\right]$$

$$\rho\left[\frac{\partial u_3}{\partial t}+u_1\frac{\partial u_3}{\partial x_1}+u_2\frac{\partial u_3}{\partial x_2}+u_3\frac{\partial u_3}{\partial x_3}\right]=\rho g_3-\frac{\partial P}{\partial x_3}+\mu\left[\frac{\partial^2 u_3}{\partial x_1^2}+\frac{\partial^2 u_3}{\partial x_2^2}+\frac{\partial^2 u_3}{\partial x_3^2}\right]$$

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Navier Stokes Equations

If we rename variables $\boldsymbol{x_1=x}$, $\boldsymbol{x_2=y}$, and $\boldsymbol{x_3=z}$, as well as $u_1= u_x$, $u_2= u_y$, $u_3= u_z$, we can obtain:

$$\rho\left[\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}+u_z\frac{\partial u_x}{\partial z}\right]=-\frac{\partial P}{\partial x}+\mu\left[\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right]+\rho g_x$$

$$\rho\left[\frac{\partial u_y}{\partial t}+u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}+u_z\frac{\partial u_y}{\partial z}\right]=-\frac{\partial P}{\partial y}+\mu\left[\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}+\frac{\partial^2 u_y}{\partial z^2}\right]+\rho g_y$$

$$\rho\left[\frac{\partial u_z}{\partial t}+u_x\frac{\partial u_z}{\partial x}+u_y\frac{\partial u_z}{\partial y}+u_z\frac{\partial u_z}{\partial z}\right]=-\frac{\partial P}{\partial z}+\mu\left[\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_z}{\partial y^2}+\frac{\partial^2 u_z}{\partial z^2}\right]+\rho g_z$$

The above equations are **Navier-Stokes Equations** in rectangular coordinate system.

Navier Stokes Equations - In Cylindrical Coordinates (r, θ, z)

$$\rho\left[\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}\right]=$$

$$=-\frac{\partial P}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_r)\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial^2 u_r}{\partial z^2}\right]+\rho g_r$$

$$\rho\left[\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta}{r}+u_z\frac{\partial u_\theta}{\partial z}\right]=$$

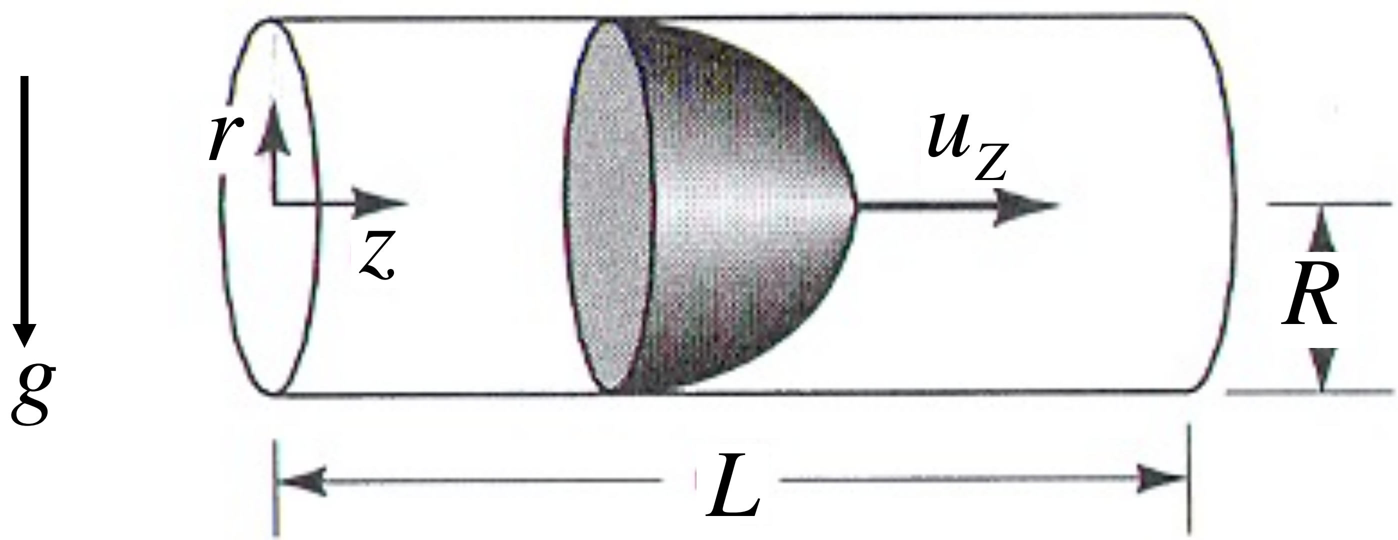
$$=-\frac{1}{r}\frac{\partial P}{\partial \theta}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right]+\rho g_\theta$$

$$\rho\left[\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right]=$$

$$=-\frac{\partial P}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right]+\rho g_z$$

Application of Navier Stokes Equations

Consider a case of laminar flow in a horizontal pipe. Apply Navier-Stokes equations to generate an appropriate mathematical model.



ASSUMPTIONS

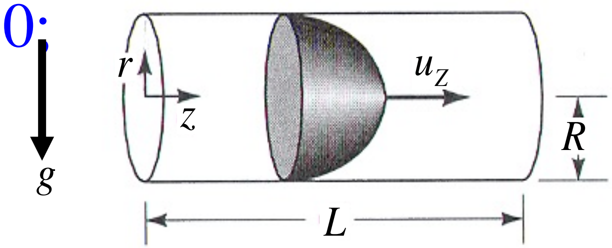
1. The tube is circular
2. Flow is isothermal
3. Density ρ=const
4. Steady state flow
5. Fluidi is Newtonian
6. Flow is LAMINAR unidirectional and fully developed.

Application of Navier Stokes Equations

Laminar, unidirectional, developed, steady state flow:

$$\mathbf{U} = \mathrm{U}\left(u_r, u_\theta, u_z\right); \quad u_r = 0; \quad u_\theta = 0; \quad u_z \neq 0 \text{ and } u_z = u_z(r)$$

$$\frac{\partial u_z}{\partial t} = 0; \quad \frac{\partial u_z}{\partial r} \neq 0; \quad \frac{\partial u_z}{\partial z} = 0; \quad \frac{\partial u_z}{\partial \theta} = 0;$$



$$\rho\left[\cancel{\frac{\partial u_r}{\partial t}} + \cancel{u_r\frac{\partial u_r}{\partial r}} + \cancel{\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}} - \cancel{\frac{u_\theta^2}{r}} + \cancel{u_z\frac{\partial u_r}{\partial z}}\right] =$$

$$= -\frac{\partial P}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(\cancel{r u_r}\right)\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}} - \cancel{\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}} + \cancel{\frac{\partial^2 u_r}{\partial z^2}}\right] + \rho g_r$$

$$\boxed{0 = -\frac{\partial P}{\partial r} + \rho g_r \quad \Rightarrow \quad \frac{\partial P}{\partial r} = \rho g_r}$$

Application of Navier Stokes Equations

Consider flow θ direction:

$$\mathbf{U} = \mathrm{U}\left(u_r, u_\theta, u_z\right); \quad u_r = 0; \quad u_\theta = 0; \quad u_z \neq 0 \text{ and } u_z = u_z(r)$$

$$\frac{\partial u_z}{\partial t} = 0; \quad \frac{\partial u_z}{\partial r} \neq 0; \quad \frac{\partial u_z}{\partial z} = 0; \quad \frac{\partial u_z}{\partial \theta} = 0;$$

$$\cancel{\rho\left[\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} + u_z \frac{\partial u_\theta}{\partial z}\right] =}$$

$$\cancel{= -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r u_\theta\right)\right) + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} + \frac{\partial^2 u_\theta}{\partial z^2}\right] + \rho g_\theta}$$

Application of Navier Stokes Equations

Consider flow in "z" direction:

$$\mathbf{U} = \mathrm{U}\left(u_r, u_\theta, u_z\right); \quad u_r = 0; \quad u_\theta = 0; \quad u_z \neq 0 \text{ and } u_z = u_z(r)$$

$$\frac{\partial u_z}{\partial t} = 0; \quad \frac{\partial u_z}{\partial r} \neq 0; \quad \frac{\partial u_z}{\partial z} = 0; \quad \frac{\partial u_z}{\partial \theta} = 0;$$

$$\rho\left[\cancel{\frac{\partial u_z}{\partial t}} + \cancel{u_r \frac{\partial u_z}{\partial r}} + \cancel{\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}} + u_z \cancel{\frac{\partial u_z}{\partial z}}\right] =$$

$$= -\frac{\partial P}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 u_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 u_z}{\partial z^2}}\right] + \cancel{\rho g_z}$$

$$0 = -\frac{\partial P}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)\right] \quad \Rightarrow \quad \boxed{\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = \frac{\partial P}{\partial z}}$$

$$\boxed{-\frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{z,r}\right) = \frac{\partial P}{\partial z}}$$

Oregon State University OSU College of Engineering

People. Ideas. Innovation.

Thank you for your attention!