Oregon State University

OREGON STATE UNIVERSITY CBEE
DEPARTMENT OF CHEMICAL ENGINEERING

CHE 331
Transport Phenomena I

Dr. Goran Jovanovic

Laminar Flow in Pipes

Please turn-off cell phones

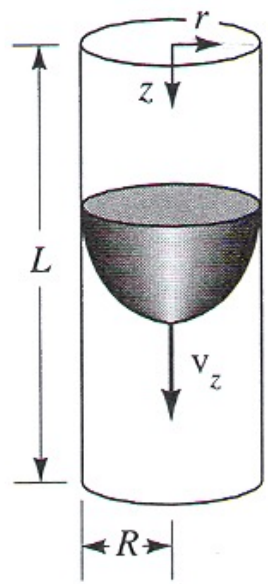
CHE 331
Fall 2022

Laminar Flow in Pipe – Problem Statement

For a steady, laminar, isothermal unidirectional flow . . . develop a relationship (mathematical model) between the pressure difference, $\Delta P_z[Pa]$, along a circular pipe and,

a) the velocity profile inside the pipe, $v_z(r)\left[\dfrac{m}{s}\right]$;

b) the volumetric flow rate of the fluid trough the pipe, $Q_z\left[\dfrac{m^3}{s}\right]$;



Before we proceed further let us consider some simple facts of physics.

Oregon State University

Laminar Flow in Pipe

$$momentum \quad \left\{ (m \cdot v) \; \langle = \rangle \left[\frac{kg \;\; m}{s} \right] \right. \quad \Leftrightarrow \quad \left\{ (F \cdot \Delta t) \; \langle = \rangle \left[\frac{kg \;\; m}{s} \right] \right. \quad impulse$$

$$rate \;\; of \;\; change \;\; of \;\; momentum \quad \left\{ \underbrace{\frac{\partial (m \cdot v)}{\partial t}}_{momentum \;\; rate} = m \frac{\partial v}{\partial t} = m \cdot a = Force \; \langle = \rangle \left[\frac{kg \;\; m}{s^2} \right] \right.$$

$$momentum \;\; flux \quad \left\{ \frac{\frac{\partial (m \cdot v)}{\partial t}}{A} = \frac{F}{A} = \tau \; \langle = \rangle \left[\frac{kg}{m \cdot s^2} \right] \right. \quad \Leftrightarrow$$

We could say that a stress is equal to the momentum flux.

Also:

$$shear \;\; rate \quad \left\{ \frac{\partial v_z}{\partial r} \; \langle = \rangle \left[\frac{1}{s} \right] \right.$$

Newton's Law

$$we \;\; also \;\; postulated \;\; that \quad \left\{ \tau_{zr} = -\mu \frac{\partial v_z}{\partial r} \; \langle = \rangle \left[\frac{kg}{m \cdot s^2} \right] \right.$$

Laminar Flow in Pipe

Back to our problem; we will make the following assumptions;

i) the tube is circular;

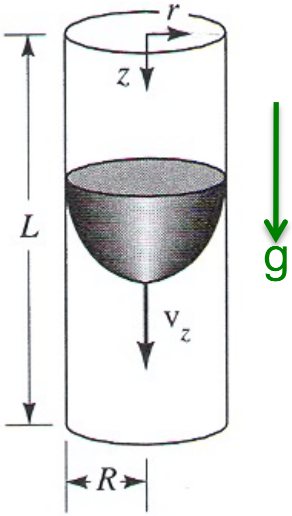
ii) flow is isothermal;

iii) density is constant ρ = const. (incompressible fluid);

iv) steady state case;

v) flow is laminar, unidirectional and fully developed:

$$\mathbb{V} = v(v_r, v_\theta, v_z) = v(0,0,v_z); \qquad \underbrace{v_z \neq f(z)}_{\displaystyle \frac{\partial v_z}{\partial z} = 0} \qquad \underbrace{v_z = f(r)}_{\displaystyle \frac{\partial v_z}{\partial r} \neq 0}$$



vi) the axis of fluid flow is collinear with the gravity vector;

vii) the fluid is Newtonian. $\tau_{zr} = -\mu \dfrac{\partial v_z}{\partial r}$

Laminar Flow in Pipe

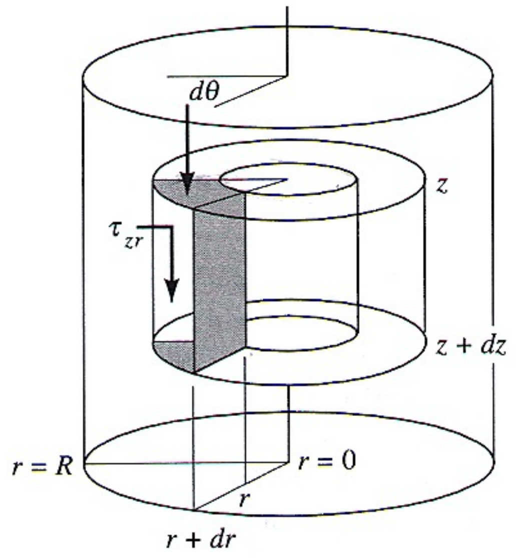
Momentum is subject to the first principle like mass and energy. In addition, momentum is, like force and velocity, a vector- therefore we will consider a conservation of momentum in a specific direction. We choose '*z*' direction.

There are two types of forces acting on any element of fluid:

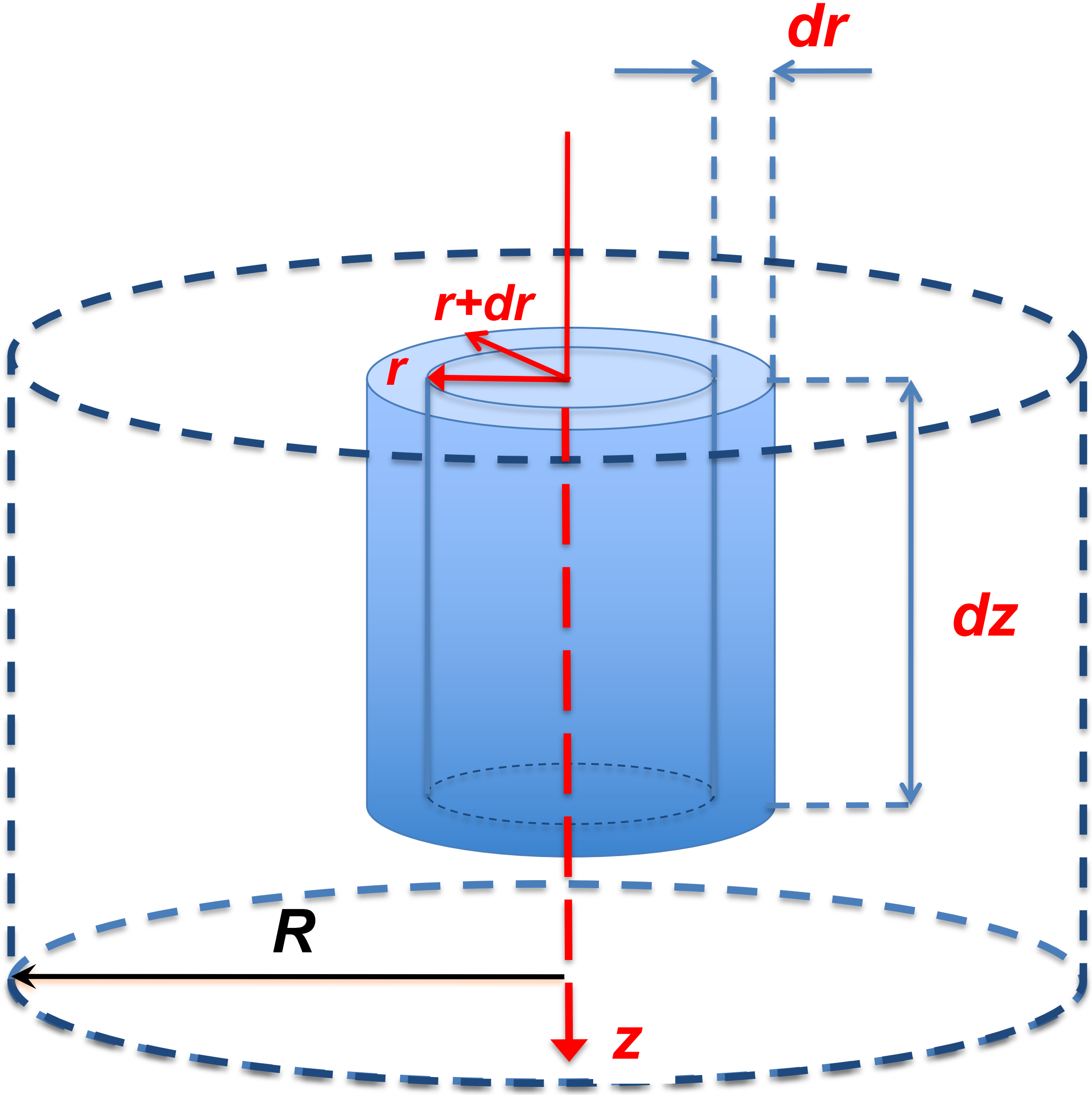
a) *Surface Forces,* and b) *Body Forces;*

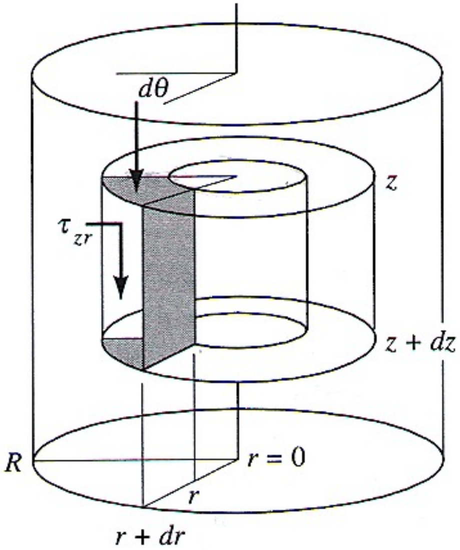
Among *Surfaces Forces* we recognize:

***i) Shear Forces,* and**
ii) Normal (pressure) Forces

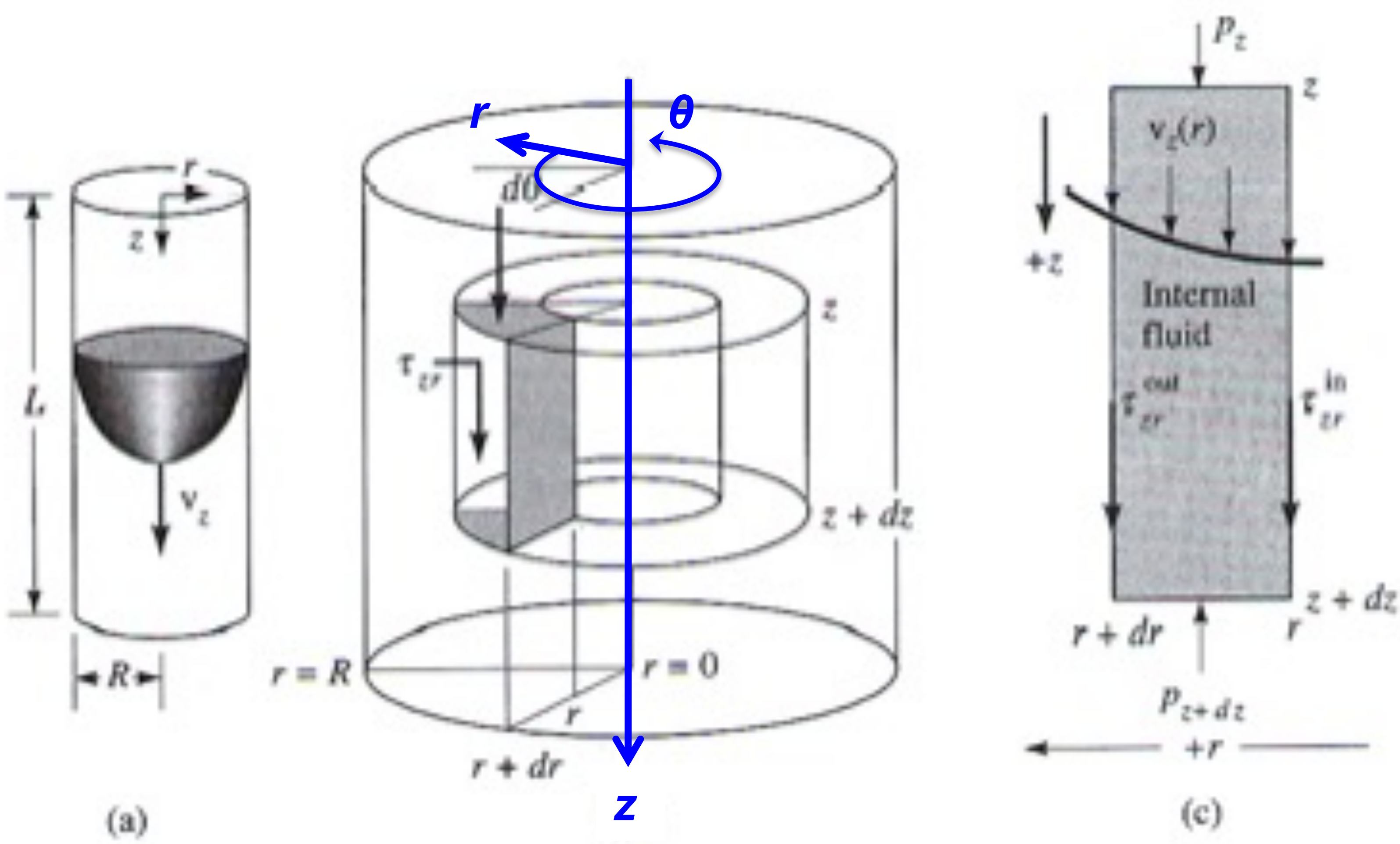


Laminar Flow in Pipe





Laminar Flow in Pipe



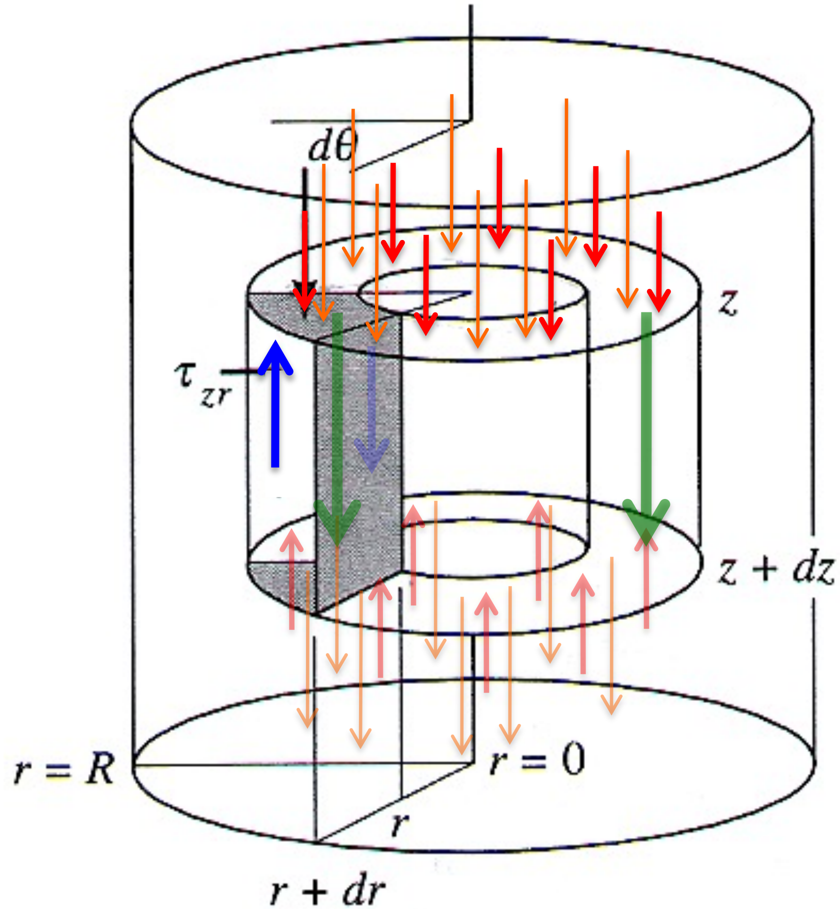
Laminar Flow in Pipe

$$\underbrace{\underbrace{dF_z\big|_r \Delta t + dF_z\big|_{r+dr} \Delta t}_{\text{momentum due to shear forces}} + \underbrace{dF_z\big|_z \Delta t + dF_z\big|_{z+dz} \Delta t}_{\text{momentum due to pressure (normal) forces}}}_{\text{momentum due to surface forces}} +$$

$$+ \underbrace{\underbrace{dF_g \Delta t}_{\text{momentum due to gravity force}}}_{\text{momentum due to body forces}} +$$

$$+ \underbrace{\underbrace{dF_v\big|_z \Delta t + dF_v\big|_{z+dz} \Delta t}_{\text{momentum due to flow of fluid through the boundary of the fluid element}}}_{\text{momentum due flow of fluid}} =$$

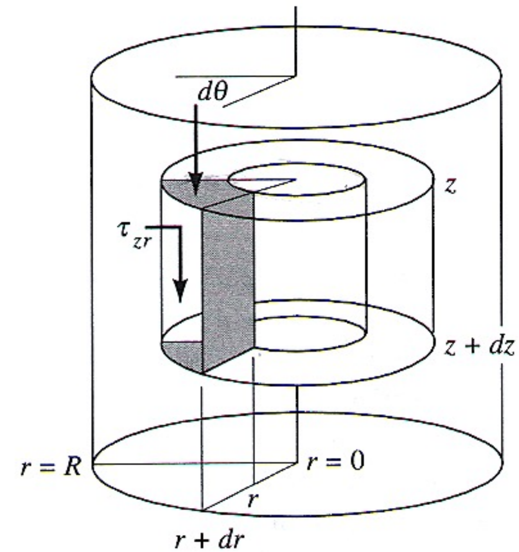
$$= \underbrace{mv_z\big|_{t+dt} - mv_z\big|_t}_{\text{accumulation of momentum}}$$



Laminar Flow in Pipe

$$dF_z\big|_r \cancel{\Delta}t + dF_z\big|_{r+dr} \cancel{\Delta}t + dF_z\big|_z \cancel{\Delta}t + dF_z\big|_{z+dz} \cancel{\Delta}t + dF_g \cancel{\Delta}t + dF_v\big|_z \cancel{\Delta}t + dF_v\big|_{z+dz} \cancel{\Delta}t =$$

$$= \frac{mv_z\big|_{t+dt} - mv_z\big|_t}{\Delta t}$$



Divide whole equation with Δt :

$$\left(dF_z\big|_r + dF_z\big|_{r+dr}\right) + \left(dF_z\big|_z + dF_z\big|_{z+dz}\right) + dF_g + \left(dF_v\big|_z + dF_v\big|_{z+dz}\right) = \frac{mv_z\big|_{t+dt} - mv_z\big|_t}{\Delta t}$$

Shear Force — **Normal Force** — **Gravity Force** — **Flow Force** — **Accumulation of momentum**

Laminar Flow in Pipe

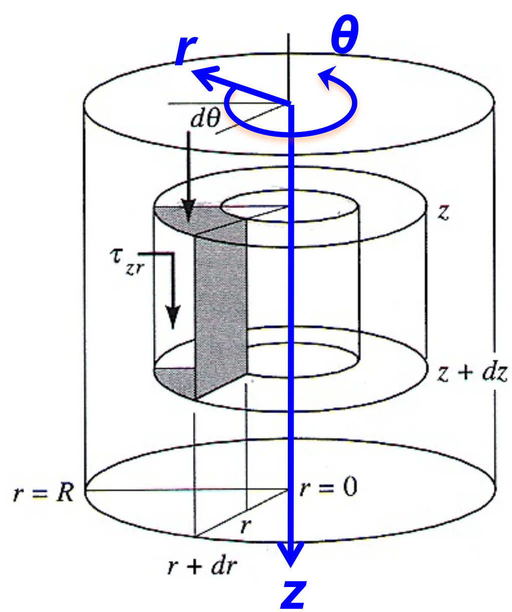
$$\left(dF_z\big|_r + dF_z\big|_{r+dr}\right) + \left(dF_z\big|_z + dF_z\big|_{z+dz}\right) + dF_g + \left(dF_v\big|_z + dF_v\big|_{z+dz}\right) = \frac{mv_z\big|_{t+dt} - mv_z\big|_t}{\Delta t}$$

$$\underbrace{\frac{kg}{m\cdot s^2}\times m^2 = \frac{kg\cdot m}{s^2}}_{\tau_{zr}\big|(dA_1)} = Force$$

$$\underbrace{\frac{kg}{m\cdot s^2}\times m^2 = \frac{kg\cdot m}{s^2}}_{P_z\big|(dA_2)} = Force$$

$$\rho g_z(dV)$$

$$\underbrace{\frac{kg}{m^3}\times\frac{m}{s}\times\frac{m^3}{s} = \frac{kg\cdot m}{s^2}}_{\rho v_z(dQ)\big|} = Force$$



Laminar Flow in Pipe – Sign Convention

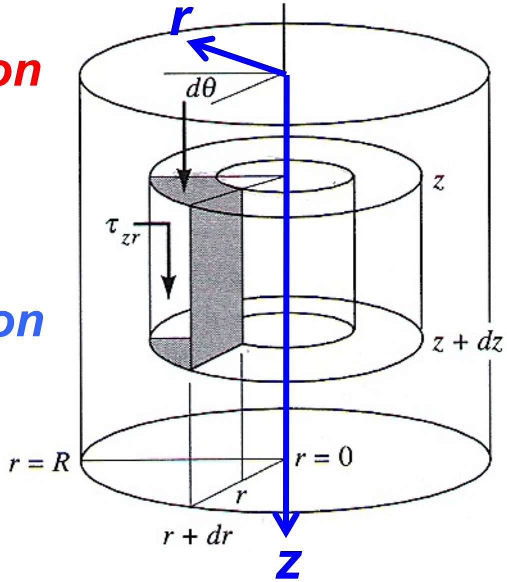
Signs in the above equation (except for the accumulation term) are not determined yet. They will reflect particular physical situation and orientation of the coordinate system.

SIGN CONVENTION

When passing across the boundary of the volume element, from external fluid to the internal fluid, if we move in the positive direction of the coordinate axis, we use a + sign on the force acting on that boundary.

Momentum that contributes to the acceleration of the observed element of fluid will have positive (+) sign.

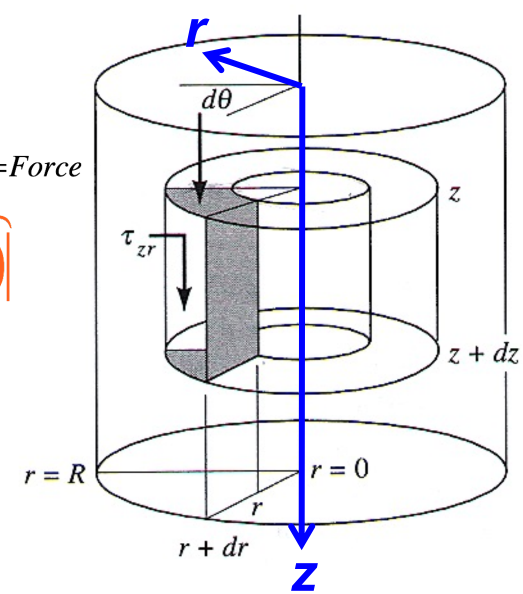
Momentum that contributes to the deceleration of the observed element of fluid will have negative (-) sign.



Laminar Flow in Pipe

$$\left(dF_z\big|_r + dF_z\big|_{r+dr}\right) + \left(dF_z\big|_z + dF_z\big|_{z+dz}\right) + dF_g + \left(dF_v\big|_z + dF_v\big|_{z+dz}\right) = \frac{mv_z\big|_{t+dt} - mv_z\big|_t}{\Delta t}$$

$$\overbrace{\tau_{zr}\big|(dA_1)}^{\frac{kg}{m\cdot s^2}\times m^2=\frac{kg\cdot m}{s^2}=Force} \qquad \overbrace{P_z\big|(dA_2)}^{\frac{kg}{m\cdot s^2}\times m^2=\frac{kg\cdot m}{s^2}=Force} \qquad \rho g_z(dV) \qquad \overbrace{\rho v_z(dQ)\big|}^{\frac{kg}{m^3}\times\frac{m}{s}\times\frac{m^3}{s}=\frac{kg\cdot m}{s^2}=Force}$$



$$\overbrace{\left(\tau_{zr}[2\pi r dz]\right)\big|_r}^{\frac{kg}{m\cdot s^2}\times m^2=\frac{kg\cdot m}{s^2}=Force} - \left(\tau_{zr}[2\pi r dz]\right)\big|_{r+dr} + \overbrace{P_z\big|_z(2\pi r dr)}^{\frac{kg}{m\cdot s^2}\times m^2=\frac{kg\cdot m}{s^2}=Force} - P_z\big|_{z+dz}(2\pi r dr) +$$

$$\rho g_z[2\pi r dr dz] + \underbrace{\rho v_z}_{\text{momentum per unit volume}} \underbrace{(v_z 2\pi r dr)}_{\text{volumetric flow rate}}\big|_z - \underbrace{\rho v_z(v_z 2\pi r dr)\big|_{z+dz}}_{\frac{kg}{m^3}\times\frac{m}{s}\times\frac{m}{s}\times m^2=\frac{kg\cdot m}{s^2}=Force} = \underbrace{\frac{mv_z\big|_{t+dt} - mv_z\big|_t}{\Delta t}}_{\text{accumulation of momentum; the rate of change of momentum; force}}$$

Laminar Flow in Pipe

$$\left(\tau_{zr}[2\pi r dz]\right)\Big|_{r} - \left(\tau_{zr}[2\pi r dz]\right)\Big|_{r+dr} + P_z\big|_z(2\pi r dr) - P_z\big|_{z+dz}(2\pi r dr) +$$

$$+\rho g_z[2\pi r dr dz] + \rho v_z(v_z 2\pi r dr)\big|_z - \rho v_z(v_z 2\pi r dr)\big|_{z+dz} = \frac{mv_z\big|_{t+dt} - mv_z\big|_t}{\Delta t}$$

$$\left[\tau_{zr} r\big|_r - \tau_{zr} r\big|_{r+dr}\right](2\pi dz) + \left[P_z\big|_z - P_z\big|_{z+dz}\right](2\pi r dr) + \rho g_z[2\pi r dr dz] +$$

$$+\left[\rho v_z v_z\big|_z - \rho v_z v_z\big|_{z+dz}\right](2\pi r dr) = \frac{mv_z\big|_{t+dt} - mv_z\big|_t}{\Delta t}$$

$$/\ (2\pi dr dz)$$

$$\frac{\left[\tau_{zr} r\big|_r - \tau_{zr} r\big|_{r+dr}\right](\cancel{2\pi}\cancel{dz})}{(\cancel{2\pi}\cdot dr\cdot \cancel{dz})} + \frac{\left[P_z\big|_z - P_z\big|_{z+dz}\right](\cancel{2\pi}r\cancel{dr})}{(\cancel{2\pi}\cdot \cancel{dr}\cdot dz)} + \frac{\rho g_z(\cancel{2\pi}r\cdot\cancel{dr}\cdot\cancel{dz})}{(\cancel{2\pi}\cdot\cancel{dr}\cdot\cancel{dz})} +$$

$$+\frac{\left[\rho v_z v_z\big|_z - \rho v_z v_z\big|_{z+dz}\right](\cancel{2\pi}r\cancel{dr})}{(\cancel{2\pi}\cdot\cancel{dr}\cdot dz)} = \underbrace{\frac{mv_z\big|_{t+dt} - mv_z\big|_t}{\Delta t(2\pi\cdot dr\cdot dz)}}_{0}$$

0

Steady State

Oregon State University

Laminar Flow in Pipe

$$\frac{\left[\tau_{zr}r\big|_{r}-\tau_{zr}r\big|_{r+dr}\right]}{dr}+\frac{\left[P_z\big|_{z}-P_z\big|_{z+dz}\right](r)}{dz}+\rho g_z(r)+\frac{\left[\rho v_z v_z\big|_{z}-\rho v_z v_z\big|_{z+dz}\right](r)}{dz}=0$$

$$\frac{-\left[\tau_{zr}r\big|_{r+dr}-\tau_{zr}r\big|_{r}\right]}{dr}-\frac{\left[P_z\big|_{z+dz}-P_z\big|_{z}\right](r)}{dz}+\rho g_z(r)-\frac{\left[\rho v_z v_z\big|_{z+dz}-\rho v_z v_z\big|_{z}\right](r)}{dz}=0$$

$$\lim_{\substack{dz\to 0\\ dr\to 0}}\left\{\frac{-\left[\tau_{zr}r\big|_{r+dr}-\tau_{zr}r\big|_{r}\right]}{dr}-\frac{\left[P_z\big|_{z+dz}-P_z\big|_{z}\right](r)}{dz}+\rho g_z(r)-\frac{\left[\rho v_z v_z\big|_{z+dz}-\rho v_z v_z\big|_{z}\right](r)}{dz}\right\}=0$$

$$-\frac{\partial(\tau_{zr}r)}{\partial r}-\frac{r\partial(P_z)}{\partial z}+\rho g_z(r)-\frac{r\partial(\rho v_z v_z)}{\partial z}=0$$

But remember that:

$$\frac{\partial v_z}{\partial z}=0$$

$$-\frac{\partial(\tau_{zr}r)}{\partial r}-\frac{r\partial(P_z)}{\partial z}+\rho g_z(r)-\cancel{\frac{r\rho\partial(v_z v_z)}{\partial z}}=0$$

0

v_z is not function of z

Laminar Flow in Pipe

$$-\frac{\partial(\tau_{zr} r)}{\partial r} - \frac{\cancel{r}\,\partial(P_z)}{\partial z} + \rho g_z (\cancel{r}) = 0 \quad \Rightarrow \quad -\frac{1}{r}\frac{\partial(\tau_{zr} r)}{\partial r} - \frac{\partial(P_z)}{\partial z} + \rho g_z = 0$$

$$\boxed{\frac{1}{r}\frac{\partial(\tau_{zr} r)}{\partial r} = -\frac{\partial(P_z)}{\partial z} + \rho g_z}$$

We also know that shear stress in *z* direction is defined by Newton's Law:

$$\tau_{zr} = -\mu \frac{\partial(v_z)}{\partial r}$$

$$-\mu\left[\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r}\right]$$

$$\frac{1}{r}\frac{\partial\left(-\mu \frac{\partial v_z}{\partial r} r\right)}{\partial r} = -\frac{\partial(P_z)}{\partial z} + \rho g_z$$

Laminar Flow in Pipe

$$\frac{1}{r}\frac{\partial\left(-\mu\frac{\partial v_z}{\partial r}r\right)}{\partial r}=-\frac{\partial\left(P_z\right)}{\partial z}+\rho g_z \quad\Rightarrow\quad \frac{\mu}{r}\frac{\partial\left(\frac{\partial v_z}{\partial r}r\right)}{\partial r}=\frac{\partial\left(P_z\right)}{\partial z}-\rho g_z$$

We originally assumed that the flow is *laminar and unidirectional*, which means that there are ***no radial*** (in the direction of '*r*') and ***no circular*** (in the direction of '*θ*') components of the velocity vector ***V*** $\left(v_r \equiv 0 \quad and \quad v_\theta \equiv 0\right)$; neither is there any net force or acceleration acting in the radial and circular direction. Thus, there can be no pressure variation in the radial and circular direction.

Hence we conclude that $\boldsymbol{P_z}$ does not depend on '*r*' or '*θ* '.

Laminar Flow in Pipe

$$\frac{\mu}{r}\frac{\partial\left(\frac{\partial v_z}{\partial r}r\right)}{\partial r}=\frac{\partial\left(P_z\right)}{\partial z}-\rho g_z$$

Therefore, the right-hand-side is function of 'z' only, and the left-hand side is function of 'r' only.

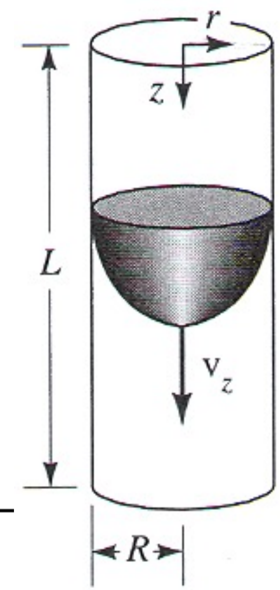
What is the consequence of the above observation? How could it be that left-hand-side of the equation is function of 'r' only and the right-hand-side of the equation is function of 'z' only, and still the two sides of the equation are always equal to each other? Remember that both 'r' and 'z' are independent variables, thus they can take any value independently of a current value of the other variable.

The only possible solution supporting this situation is that both sides of the equation are equal to a constant C.

Laminar Flow in Pipe

Therefore we can write:
$$\boxed{\frac{\mu}{r}\frac{\partial\left(\frac{\partial v_z}{\partial r}r\right)}{\partial r} = C = \frac{\partial(P_z)}{\partial z} - \rho g_z}$$

$$\frac{\mu}{r}\frac{d\left(\frac{dv_z}{dr}r\right)}{dr} = C \quad \text{and} \quad \frac{d(P_z)}{dz} - \rho g_z = C$$



$$\frac{d\left(\frac{dv_z}{dr}r\right)}{dr} = \frac{Cr}{\mu}$$

$$BC1)\quad \left.\frac{dv_z}{dr}\right|_{r=0} = 0 \quad @ \quad r = 0$$

$$BC2)\quad v_z\big|_{r=R} = 0 \quad @ \quad r = R$$

$$\frac{d(P_z)}{dz} - \rho g_z = C$$

$$BC3)\quad P = P_o \quad @ \quad z = 0$$

Laminar Flow in Pipe

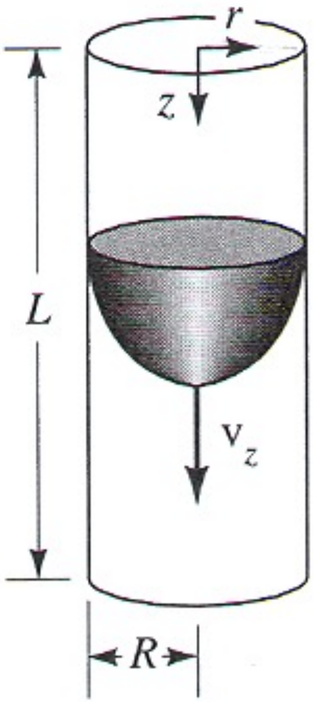
Or, alternatively we could show the mathematical model in somewhat "condensed" form:

$$\frac{\mu}{r}\frac{\partial\left(\frac{\partial v_z}{\partial r} r\right)}{\partial r} = \frac{\partial\left(P_z\right)}{\partial z} - \rho g_z$$

$$BC1)\quad \left.\frac{dv_z}{dr}\right|_{r=0} = 0 \quad @ \quad r = 0$$

$$BC2)\quad \left. v_z\right|_{r=R} = 0 \quad @ \quad r = R$$

$$BC3)\quad P = P_o \quad @ \quad z = 0$$



Laminar Flow in Pipe

Now we can integrate the two equations; first the equation which is function of '*r*' only:

$$\boxed{\frac{d\left(\frac{dv_z}{dr}r\right)}{dr}=\frac{Cr}{\mu}} \Rightarrow \frac{dv_z}{dr}r=\frac{Cr^2}{2\mu}+E \Rightarrow \frac{dv_z}{dr}=\frac{Cr}{2\mu}+\frac{E}{r}$$

Now apply BC1: $\boxed{\left.\frac{dv_z}{dr}\right|_{r=0}=0 \quad @ \quad r=0} \Rightarrow \boxed{E\equiv 0}$

Therefore: $\frac{dv_z}{dr}=\frac{Cr}{2\mu} \Rightarrow v_z=\frac{Cr^2}{4\mu}+F$

Oregon State University

Laminar Flow in Pipe

Now apply BC2: $v_z\big|_{r=R} = 0 \quad @ \quad r = R$ $\qquad v_z = \dfrac{Cr^2}{4\mu} + F$

$$0 = \frac{CR^2}{4\mu} + F \qquad \Rightarrow \qquad \boxed{F = -\frac{CR^2}{4\mu}}$$

Finally one can obtain for *v_z*:

$$v_z = \frac{Cr^2}{4\mu} - \frac{CR^2}{4\mu} \qquad \Rightarrow \qquad \boxed{v_z = \frac{C}{4\mu}\left(r^2 - R^2\right)}$$

We still have to determine the separation constant *C* ?

Laminar Flow in Pipe

We will determine the constant '*C*' from the last boundary condition and the second equation, which is a function of '*z*' only:

$$\frac{dP_z}{dz} - \rho g_z = C \quad \text{at } z = 0 \quad P = P_o \quad \text{and } z = L \quad P = P_L \quad \Rightarrow$$

$$dP_z = \left(C + \rho g_z\right) dz \quad \Rightarrow \quad \int_{P_o}^{P_L} dP_z = \int_0^L \left(C + \rho g_z\right) dz \quad \Rightarrow$$

$$P_L - P_o = C \cdot L + \rho g_z \cdot L \quad \Rightarrow \quad \boxed{C = -\left[\frac{\left(P_o - P_L\right) + \rho g_z L}{L}\right]}$$

We have evaluated all unknown constants and we can finally assemble the expression for the velocity profile:

Laminar Flow in Pipe

$$v_z = \frac{C}{4 \cdot \mu} \cdot \left[r^2 - R^2 \right] \quad \Rightarrow \quad v_z = \frac{-\left[\frac{(P_o - P_L) + \rho g_z L}{L} \right]}{4 \cdot \mu} \cdot \left[r^2 - R^2 \right]$$

$$v_z = \frac{(P_o - P_l) + \rho g_z L}{4 \cdot \mu \cdot L} \cdot \left[R^2 - r^2 \right] \quad \Rightarrow$$

$$v_z = \frac{\left[(P_o - P_L) + \rho g_z L \right] \cdot R^2}{4 \cdot \mu \cdot L} \cdot \left[1 - \frac{r^2}{R^2} \right] \quad \Rightarrow \quad \boxed{v_z = \frac{\Delta \mathbb{P} \cdot R^2}{4 \cdot \mu \cdot L} \cdot \left[1 - \frac{r^2}{R^2} \right]}$$

$$\boxed{v_z = \frac{R^2}{4 \cdot \mu} \left(\frac{\Delta \mathbb{P}}{L} \right) \cdot \left[1 - \frac{r^2}{R^2} \right]} \qquad @ \quad r = 0 \quad v_z = \boxed{v_{z\,\max} = \frac{R^2}{4 \cdot \mu} \left(\frac{\Delta \mathbb{P}}{L} \right)}$$

Laminar Flow in Pipe

We can now determine the volumetric flow rate of the fluid trough the pipe Q [m³/s]:

$$Q = \int_0^R v_z \cdot 2\pi r \cdot dr = \int_0^R \frac{\left[\left(P_o - P_L\right) + \rho g_z L\right]}{4 \cdot \mu \cdot L} \cdot R^2 \cdot \left[1 - \frac{r^2}{R^2}\right] \cdot 2\pi r \cdot dr$$

$$Q = \frac{\cancel{2}\pi\left[\left(P_o - P_L\right) + \rho g_z L\right]}{2\cancel{4} \cdot \mu \cdot L} \cdot R^2 \int_0^R r \cdot \left[1 - \frac{r^2}{R^2}\right] \cdot dr$$

$$Q = \frac{\pi\left[\left(P_o - P_L\right) + \rho g_z L\right]}{2 \cdot \mu \cdot L} \cdot R^2 \cdot \left[\frac{R^2}{2} - \frac{R^4}{4R^2}\right]$$

$$\boxed{Q = \frac{\pi\left[\left(P_o - P_L\right) + \rho g_z L\right]}{8 \cdot \mu \cdot L} \cdot R^4} \quad \text{or} \quad \boxed{Q = \frac{\pi \cdot R^4}{8 \cdot \mu} \cdot \left(\frac{\Delta \mathbb{P}}{L}\right)}$$

where $\Delta \mathbb{P} = \left(P_o - P_L\right) + \rho g_z L$

Laminar Flow in Pipe

Using the result obtained above we can determine the average velocity in the pipe:

$$\bar{v}_z = \frac{Q}{A} = \frac{\dfrac{\cancel{\pi}\left[\left(P_o - P_L\right) + \rho g_z L\right]}{8 \cdot \mu \cdot L} \cdot R^{\cancel{4}2}}{\cancel{\pi}\, \cancel{R^2}} \Rightarrow \boxed{\boxed{\bar{v}_z = \frac{\left[\left(P_o - P_L\right) + \rho g_z L\right] \cdot R^2}{8 \cdot \mu \cdot L}}}$$

$$\boxed{\bar{v}_z = \frac{R^2}{8 \cdot \mu}\left(\frac{\Delta \mathbb{P}}{L}\right)}$$

$$\boxed{v_{z\max} = \frac{R^2}{4 \cdot \mu}\left(\frac{\Delta \mathbb{P}}{L}\right)}$$

$$\boxed{v_{z\max} = 2\bar{v}_z}$$

Oregon State University OSU College of Engineering

People. Ideas. Innovation.

Thank you for your attention!