CHAPTER 28: CONVECTIVE MASS TRANSFER

Introduction

Convection is the transfer of mass in the form of flux from a *SOURCE* maintained at a high concentration to a *SINK* maintained at a lower concentration across a boundary layer formed by a fluid (either gas or liquid) flowing over a boundary surface (liquid or solid). The rate of mass transfer is promoted by the rate of fluid flow over the boundary surface.

If the material containing a high concentration of transferring species ***A*** forming the boundary surface is the SOURCE, the fluid containing a lower concentration of ***A*** flowing over the surface is the SINK.

If a fluid containing a high concentration of transferring species ***A*** flowing over the boundary surface is the SOURCE, then the material containing a lower concentration of ***A*** that forms the boundary surface is the SINK.

In *forced convection*, the fluid flow is externally generated by a fan or pump.

In *natural convection*, fluid in contact with a solid boundary that is at a temperature different than the fluid creates a temperature-driven density difference in the fluid. This density difference generates a circulation pattern within the fluid phase.

Forced Convection:

Fluid flow over a flat plate (characteristic length = length L in direction of flow)

[draw picture here]

Fluid flow around a sphere (characteristic length = diameter D)

[draw picture here]

Fluid flow inside a tube (characteristic length = inner diameter D)

[draw picture here]

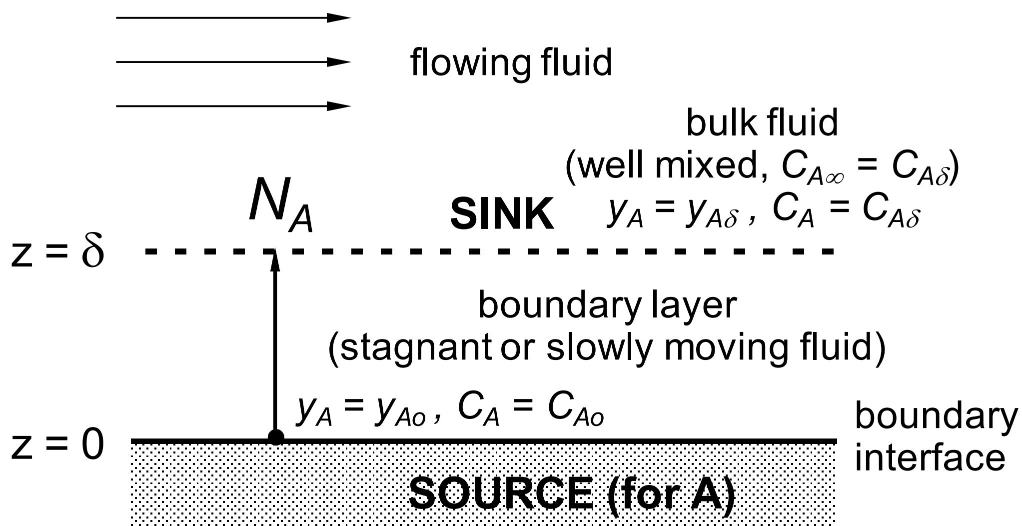
Fluid crossflow around the outside of a tube (characteristic length = outer diameter D)

[draw picture here]

Ultimately, our goal is to estimate the convective mass transfer coefficient "k_c" for all of these fluid flow situations using dimensionless groups for convection mass transfer.

28.1 Fundamental Considerations in Convective Mass Transfer

Film Theory presumes that the concentration gradient for mass transfer completely lies in a stagnant, quiescent fluid film formed at the boundary between a flowing fluid and a surface or interface



$N_A \propto$ *concentration difference* across film (boundary layer)

$$N_A = k_c\,(C_{Ao} - C_{A\delta}) = k_c\,C\,(y_{Ao} - y_{A\delta}) = k_y\,(y_{Ao} - y_{A\delta})$$

$$k = \text{convective mass transfer coefficient} \equiv \frac{\text{Flux}}{\text{concentration } \textbf{\textit{difference}} \text{ (driving force)}}$$

Film Theory Concept of Convective Mass Transfer (continued)

Consider

- Steady state process
- Binary mixture of species A (solute) and B (the carrier fluid)
- 1-D flux *across the film* along position *z*

$$N_A = -CD_{AB}\frac{dy_A}{dz} + y_A(N_A + N_B)$$

Case 1: UMD Process ($N_B = 0$)

$$N_A = -C\,D_{AB}\frac{dy_A}{dz} + y_A N_A$$

$$N_A = -\frac{C\,D_{AB}}{1-y_A}\frac{dy_A}{dz}$$ note $N_{A,z}$ is constant along z

δ = thickness of the “film” or hydrodynamic boundary layer

Integrate from y_{Ao} at $z = 0$ to $y_{A\delta}$ at $z = \delta$

$$N_A = \frac{C\,D_{AB}}{\delta}\ln\left[\frac{1-y_{A\delta}}{1-y_{Ao}}\right]$$

Define the "log mean" composition for stagnant species B in the film ($y_B = 1 - y_A$)

$$y_{B,lm} = (1-y_A)_{lm} = \frac{(1-y_{A\delta})-(1-y_{Ao})}{\ln\left[\dfrac{1-y_{A\delta}}{1-y_{Ao}}\right]}$$

$$N_A = \frac{C\,D_{AB}}{\delta\,(1-y_A)_{lm}}\left[y_{Ao} - y_{\delta}\right] = \frac{D_{AB}}{\delta\,(1-y_A)_{lm}}\left[C_{Ao} - C_{A\delta}\right] = k_c\left[C_{Ao} - C_{A\delta}\right]$$

Case 2. EMCD Process ($N_{A,z} = -N_{B,z}$)

$$N_A = - C\, D_{AB} \frac{dy_A}{dz}$$

Integrate from y_{Ao} at $z = 0$ to $y_{A\delta}$ at $z = \delta$

$$N_A = \frac{C\, D_{AB}}{\delta}\left[y_{Ao} - y_{A\delta} \right] = \frac{D_{AB}}{\delta}\left[C_{Ao} - C_{A\delta} \right] = k_c^o \left[C_{Ao} - C_{A\delta} \right]$$

Let

$$k_c^o = \frac{D_{AB}}{\delta} \quad and \quad k_c = \frac{D_{AB}}{\delta\,(1 - y_A)_{lm}} \quad as \quad \delta \downarrow, \quad k_c \uparrow$$

and

$$k_y^o = \frac{C\, D_{AB}}{\delta} \quad and \quad k_y = \frac{C\, D_{AB}}{\delta\,(1 - y_A)_{lm}} \quad as \quad y_A \to 0, \quad k_y^o \to k_y$$

Compare UMD to EMCD:

$$k_y^o = k_y (1 - y_A)_{lm}$$

Dilute system with respect to species A ($y_A < 0.05$ as a "rule of thumb")

$$(1 - y_A)_{lm} \approx 1 - y_A$$

Very dilute system with respect to species A ($y_A < 0.01$ as a "rule of thumb")

$$(1 - y_A)_{lm} \approx 1$$

For dilute solutions, mass transfer coefficients will be estimated by considering

- The hydrodynamic conditions of the fluid/boundary surface that determine "δ"
- The physical properties of the species in the mixture (e.g. fluid density, fluid viscosity, diffusion coefficients of solute A in fluid B)
- "Dimensionless Groups" for mass transfer

28.2 & 28.3 Significant Parameters / Dimensional Analysis of Convective Mass Transfer

Dimensionless Groups in Convective Mass Transfer

New friend: Sherwood Number, *Sh*
(also called Nusselt Number for Mass Transfer, Nu_{AB})

$$Sh = \frac{k_c L}{D_{AB}} = Nu_{AB} = \frac{\textit{convective mass transfer rate}}{\textit{diffusion mass transfer rate}}$$

k_c = convective mass transfer coefficient based on concentration (C_A) difference, m/sec

L = "characteristic length" defined by hydrodynamic conditions and dimensional analysis, m

D_{AB} = molecular diffusion coefficient of transferring solute A in carrier medium B, m^2/sec

Compare to an old friend: *Nusselt Number* for Heat Transfer

$$Nu = \frac{h L}{k} = \frac{\textit{convective heat transfer rate}}{\textit{conduction heat transfer rate}}$$

h = convective heat transfer coefficient, J/sec$\cdot$ m^2 $\cdot$K

k = thermal conductivity of the fluid medium (A and B), J/sec$\cdot$ m $\cdot$K

New friend: Schmidt Number, *Sc*

$$Sc = \frac{\nu}{D_{AB}} = \frac{\mu}{\rho D_{AB}} = \frac{\textit{momentum diffusivity}, \nu = \frac{\mu}{\rho}}{\textit{molecular diffusivity}}$$

ρ = mass density of fluid medium (A and B), kg/m^3

μ = viscosity of fluid medium (A and B), kg/m-sec

ν = "kinematic viscosity" of the fluid medium (A and B), m^2/sec

For dilute solutions, note

$$Sc = \;\cong \frac{\mu_B}{\rho_B\, D_{AB}}$$

Compare to an old friend: Prandtl Number for Heat Transfer (Pr)

$$Pr = \frac{\nu}{\alpha} = \frac{\mu C_p}{k} = \frac{\textit{momentum diffusivity}, \nu = \dfrac{\mu}{\rho}}{\textit{thermal diffusivity}, \alpha = \dfrac{k}{\rho C_p}}$$

α = thermal diffusivity of fluid medium (A and B), m^2/sec

C_p = mean heat capacity of fluid medium (A and B), J/kg-K

Lewis Number, Le

$$Le = \frac{\alpha}{D_{AB}} = \frac{\textit{thermal diffusivity}}{\textit{molecular diffusivity}}$$

A dear friend: Reynolds Number, Re (forced convection)

$$Re = \frac{\rho\, v_\infty L}{\mu} = \frac{v_\infty L}{\nu} = \frac{\textit{inertial flow}}{\textit{viscous flow}}$$

v_∞ = bulk fluid velocity, m/sec

Sh = f (Re, Sc) for forced convection

Nu = f (Re, Pr) for forced convection

Peclet Number for Forced Convection Mass Transfer, Pe_{AB}

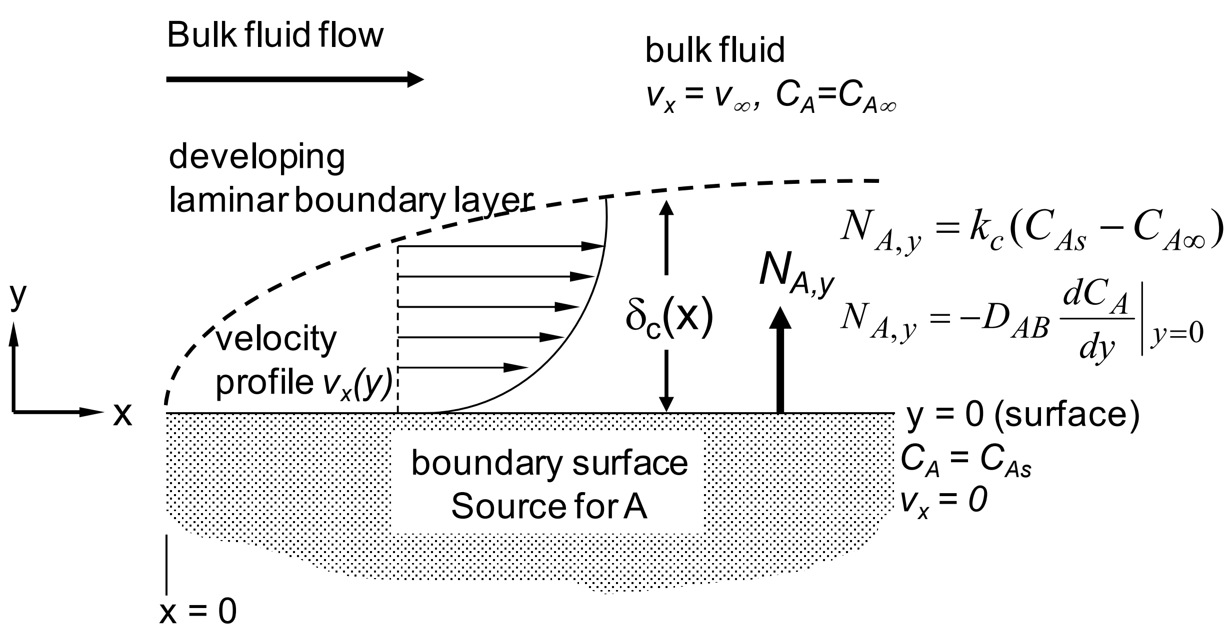
$$Pe_{AB} = Re\, Sc = \frac{\textit{transfer of matter with flow}}{\textit{diffusion transport}} = \frac{v_\infty L}{D_{AB}}$$

NOTE: For dilute systems (solute "A" transferring through flowing fluid medium "B"), the physical properties μ , ρ , C_p , k can be approximated as those of the fluid medium "B", although theoretically they represent the properties of a mixture of "A" and "B".

Summary of Dimensionless Groups used in Mass Transfer

Name	Symbol	Physical Meaning	Dimensionless Group
Reynolds number	***Re***	$\dfrac{inertial\ flow}{viscous\ flow}$	$Re = \dfrac{\rho\, v_\infty L}{\mu} = \dfrac{v_\infty L}{\nu}$
Sherwood number	***Sh***	$\dfrac{convective\ mass\ transfer}{diffusion\ mass\ transfer}$	$Sh = \dfrac{k_c L}{D_{AB}}$
Schmidt number	***Sc***	$\dfrac{momentum\ diffusivity}{molecular\ diffusivity}$	$Sc = \dfrac{\nu}{D_{AB}} = \dfrac{\mu}{\rho\, D_{AB}}$
Lewis number	***Le***	$\dfrac{thermal\ diffusivity}{molecular\ diffusivity}$	$Le = \dfrac{\alpha}{D_{AB}}$
Prandtl number	***Pr***	$\dfrac{momentum\ diffusivity}{thermal\ diffusivity}$	$Pr = \dfrac{\nu}{\alpha} = \dfrac{\mu C_p}{k}$
Peclet number	***Pe***	$\dfrac{transfer\ of\ matter\ with\ flow}{diffusion\ transport}$	$Pe_{AB} = Re\, Sc = \dfrac{v_\infty L}{D_{AB}}$
Stanton number	***St***	$\dfrac{mass\ transport\ by\ convection}{mass\ transport\ by\ flow}$	$St = \dfrac{k_c}{v_\infty}$
Grashof number	***Gr***	$\dfrac{bouyancy\ force}{viscous\ force}$	$Gr = \dfrac{L^3 \rho_L\, g\,(\rho_L - \rho_G)}{\mu_L^2}$
Mass transfer j-factor	j_D	*heat-mass transfer analogy* $j_H = j_C$	$j_D = \dfrac{k_c}{v_\infty} Sc^{2/3}$
Heat transfer j-factor	j_H	*heat-mass transfer analogy* $j_H = j_C$	$j_H = \dfrac{h}{\rho\, C_p\, v_\infty} Pr^{2/3}$

28.4 Exact Analysis of the Laminar Concentration Boundary Layer



In boundary layer analysis, the "stagnant film" abovet he boundary surface is generalized to the developing boundary layer

The boundary layer thickness (δ) increases with position x

Two boundary layers

δ = hydrodynamic boundary layer thickness

δ_c = concentration boundary layer thickness

28.4 Exact Analysis of the Laminar Concentration Boundary Layer (cont.)

The big picture: relate "hydrodynamic boundary layer" to "concentration boundary layer", for laminar flow over a flat plate, then develop fundamental relationship for Sh with Re and Sc.

Hydrodynamic boundary layer (δ) | *Concentration boundary layer(δ_c)*

Boundary Conditions across the Boundary Layer

at *y = 0* (fluid-surface boundary, "no slip" at the wall)

$v_x = v_{x,s} = 0$ | $C_A = C_{As}$ (for all x)

$$\frac{v_x}{v_\infty} = \frac{v_x - v_{x,s}}{v_\infty - v_{x,s}} = 0 \qquad \frac{C_A - C_{As}}{C_{A\infty} - C_{As}} = 0$$

at *y = ∞* (bulk fluid)

$v_x = v_\infty$ | $C_A = C_{A\infty}$ (for all x)

$$\frac{v_x}{v_\infty} = \frac{v_x - v_{x,s}}{v_\infty - v_{x,s}} = 1 \qquad \frac{C_A - C_{As}}{C_{A\infty} - C_{As}} = 1$$

Conservation Equations

Navier Stokes (Momentum Transfer) | General Differential Equation for Mass Transfer ($R_A = 0$, steady state)

v(x,y) | C_A(x,y)

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_y}{\partial y} = \upsilon \frac{\partial^2 v_y}{\partial y^2} \quad \textbf{(1)}$$

$$v_x \frac{\partial C_A}{\partial x} + v_y \frac{\partial C_A}{\partial y} = D_{AB} \frac{\partial^2 C_A}{\partial y^2} \quad \textbf{(2)}$$

Note symmetry between fluid flow and mass transfer

Analytical Solution of Conservation Equations for Momentum & Mass Transfer

Recall Blasius Solution for Eq. (1) (W^3-R p. 156-160)

Two parameters, f ʿand η

$$f' = 2\frac{v_x}{v_\infty} \qquad \eta = \frac{y}{2}\sqrt{\frac{v_\infty}{\upsilon\, x}} = \frac{y}{2x}\sqrt{\frac{xv_\infty}{\upsilon}} = \frac{y}{2x}\sqrt{\frac{xv_\infty}{\upsilon}} = \frac{y}{2x}\sqrt{\mathrm{Re}_x}$$

with $\mathrm{Re} = \frac{x\,v_\infty}{\nu}$ the "local" Reynolds number at position "x"

Blasius showed that at the surface ($y = 0$)

$$\frac{df'}{d\eta} = f''(0) = 1.328 = \frac{\frac{d}{dy}\left(\frac{2v_x}{v_\infty}\right)\Big|y=0}{\frac{d}{dy}\left(\frac{y}{2x}\sqrt{\mathrm{Re}_x}\right)\Big|y=0}$$

Consider the similar situation for mass transfer at the surface ($y = 0$)

at $\boldsymbol{y = 0}$ (surface)

$$f' = 2\frac{v_x}{v_\infty} = 2\frac{v_x - v_{x,s}}{v_\infty - v_{x,s}} = 2\frac{C_A - C_{As}}{C_{A\infty} - C_{As}}$$

$$\frac{df'}{d\eta} = f''(0) = 1.328 = \frac{\frac{d}{dy}\left(2\frac{C_A - C_{As}}{C_{A\infty} - C_{As}}\right)\Big|y=0}{\frac{d}{dy}\left(\frac{y}{2x}\sqrt{\mathrm{Re}_x}\right)\Big|y=0}$$

Aside:

$$\frac{df'}{d\eta} = f''(0) = 1.328 = \frac{\frac{d}{dy}\left(2\frac{C_A - C_{As}}{C_{A\infty} - C_{As}}\right)\Big|_{y=0}}{\frac{d}{dy}\left(\frac{\mathrm{y}}{2\mathrm{x}}\sqrt{\mathrm{Re}_x}\right)\Big|_{y=0}} = \frac{\left(\frac{2}{C_{A\infty} - C_{As}}\right)\left(\frac{dC_A}{dy}\right)\Big|_{y=0}}{\left(\frac{1}{2\mathrm{x}}\sqrt{\mathrm{Re}_x}\right)\Big|_{y=0}}$$

$$\therefore \frac{dC_A}{dy}\Big|_{y=0} = \frac{1.328}{2 \cdot 2} = \frac{(C_{A\infty} - C_{As})}{x}\mathrm{Re}_x^{1/2}$$

$$\frac{dC_A}{dy}\Big|_{y=0} = (C_{A\infty} - C_{As})\left[\frac{0.332}{x}\mathrm{Re}_x^{1/2}\right]$$

For a dilute system (w.r.t. A) at steady state, convective flux through the boundary layer is equal to the diffusive flux at the surface (*y = 0*)

$$N_{A,y} = k_c(C_{As} - C_{A\infty}) = -D_{AB}\frac{dC_A}{dy}\Big|_{y=0}$$

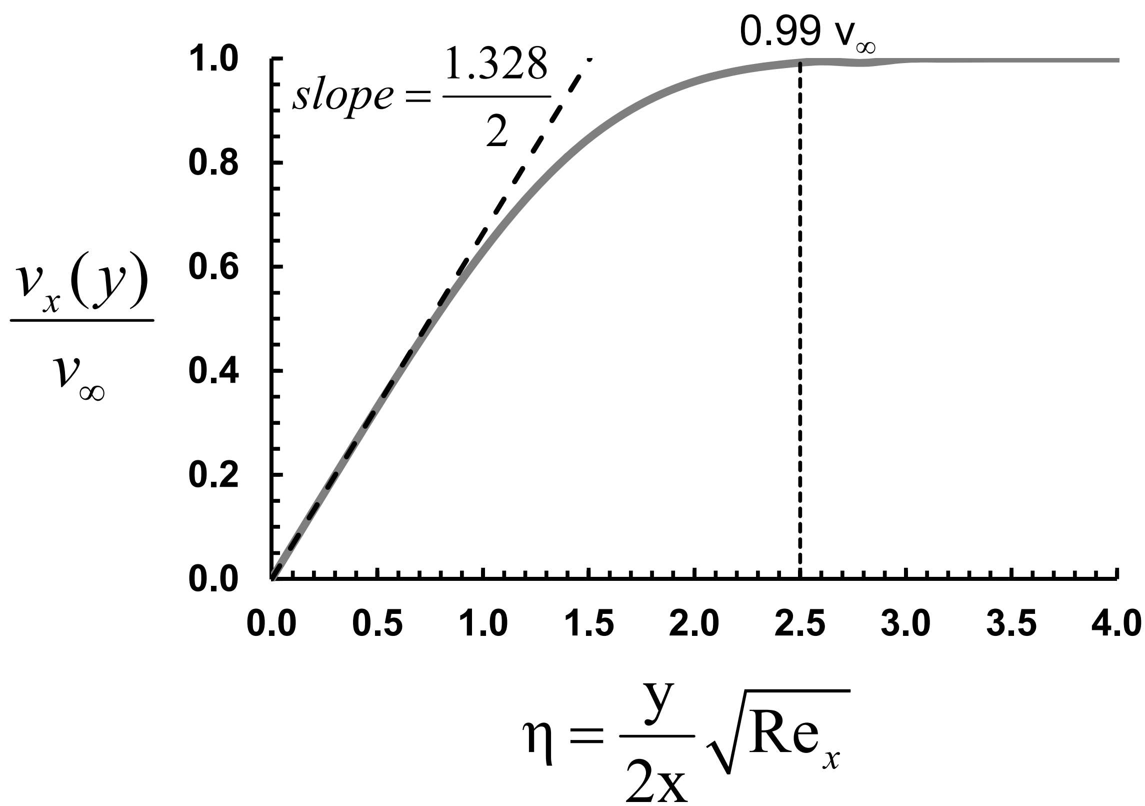
$$k_c(C_{As} - C_{A\infty}) = -D_{AB}\frac{dC_A}{dy}\Big|_{y=0} = -D_{AB}(C_{A\infty} - C_{As})\left[\frac{0.332}{x}\mathrm{Re}_x^{1/2}\right]$$

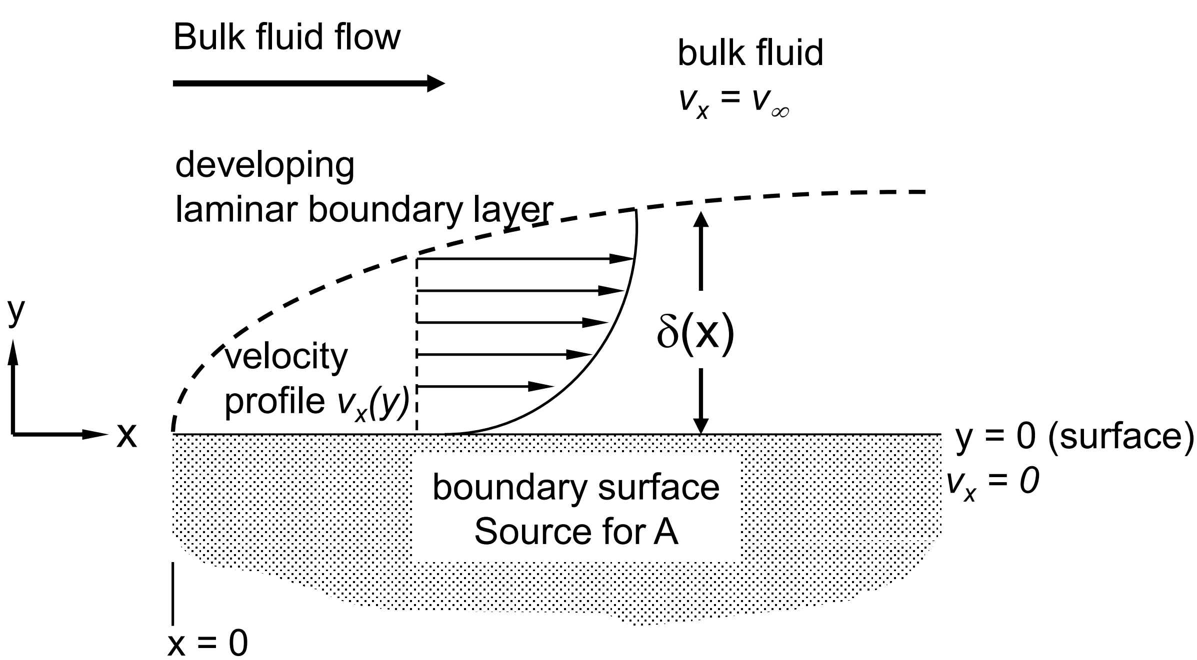
$$k_c = D_{AB}\left[\frac{0.332}{x}\mathrm{Re}_x^{1/2}\right]$$

Define Sh_x the “local” Sherwood Number with “local” mass transfer coefficient k_c

$$Sh_x = \frac{k_c\, x}{D_{AB}} = 0.332\,\mathrm{Re}_x^{1/2}$$

Aside: Blasius Solution for laminar flow over a flat plate





For laminar flow over a flat plate, the hydrodynamic boundary layer thickness increases as position ***x*** increases:

$$\text{Note at } v_x(y) = 0.99\, v_\infty,\ \mathrm{y} \approx \delta \text{ and } 2.5 = \eta = \frac{\mathrm{y}}{2\mathrm{x}}\sqrt{\mathrm{Re}_x}$$

$$\therefore \frac{\delta}{x} = \frac{5}{\sqrt{\mathrm{Re}_x}} \quad \text{OR} \quad \delta = \frac{5\,x^{1/2}}{\sqrt{\frac{v_\infty}{\upsilon}}} \quad \text{i.e. } \delta \propto x^{1/2}$$

The above analysis assumes

$$\frac{\delta}{\delta_c} = \frac{\text{hydrodynamic laminar boundary layer thickness}}{\text{concentration laminar boundary layer thickness}} = 1$$

But Recall from Heat Transfer, for laminar flow over a flat plate

$$\frac{\delta}{\delta_h} = \frac{\text{hydrodynamic laminar boundary layer thickness}}{\text{thermal laminar boundary layer thickness}} = \mathrm{Pr}^{1/3}$$

$\therefore$ by analogy for Mass Transfer $\quad \dfrac{\delta}{\delta_c} = \mathrm{Sc}^{1/3}$

Finally, for laminar flow over a flat plate, the "local" transfer coefficients at position "x" along the length of the plate are

Convective Heat Transfer

$$Nu_x = \frac{h\,x}{k} = 0.332\,\mathrm{Re}_x^{1/2}\,\mathrm{Pr}^{1/3}$$

Convective Mass Transfer

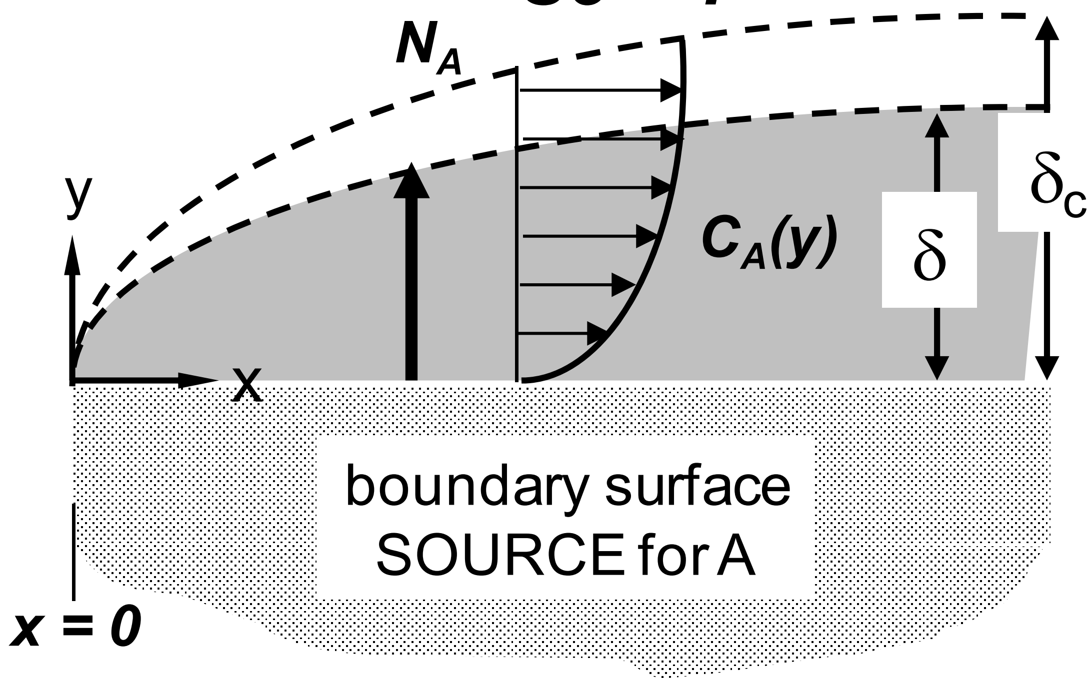
$$\mathrm{Sh_x} = \frac{\mathrm{k_c\,x}}{\mathrm{D_{AB}}} = 0.332\mathrm{Re_x}^{1/2}\mathrm{Sc}^{1/3}$$

Show in your notes that

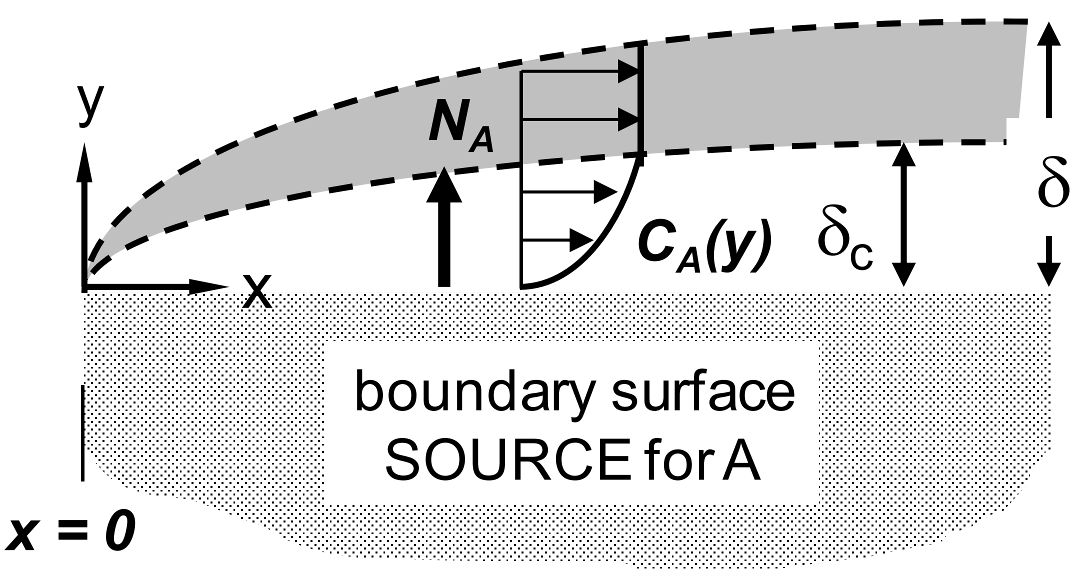
$$k_{c,x} \propto x^{-1/2}$$

Boundary layer thickness

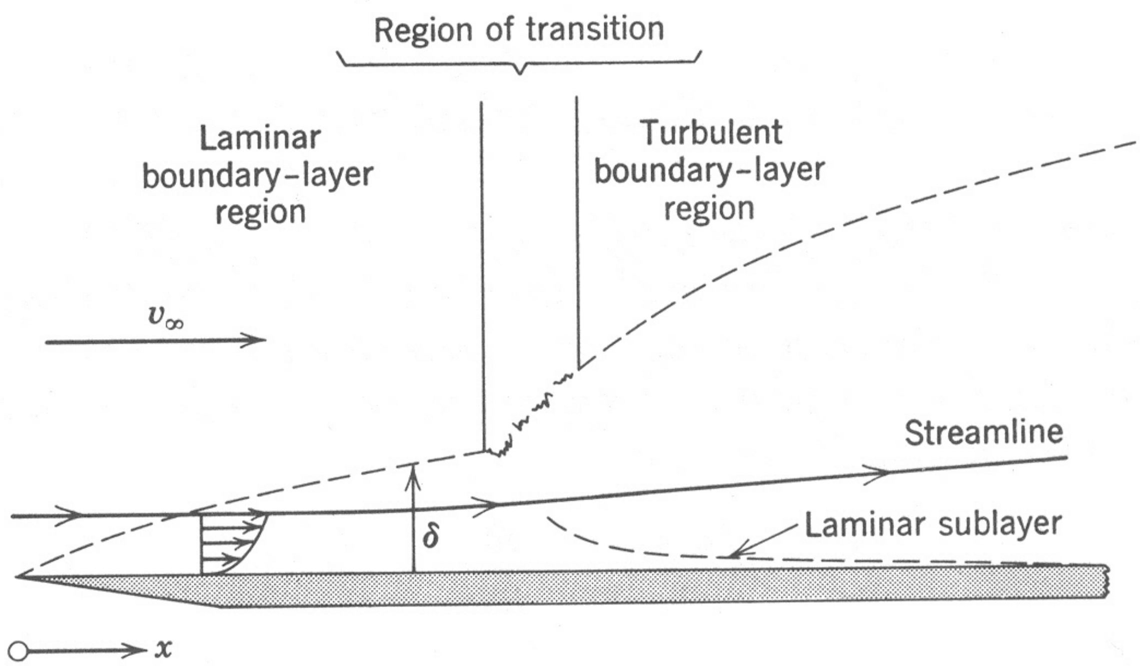
Sc < 1



Sc > 1



28.5 Approximate Analysis of the Concentration Boundary Layer



Approximate analysis of boundary layer for flow over a flat plate

	Local (at x)	***Overall (from x to L)***
laminar: (approximate) $Re \le 2 \times 10^5$	$Sh_x = 0.323\, Re_x^{1/2}\, Sc^{1/3}$	$Sh = 0.646\, Re_L^{1/2}\, Sc^{1/3}$
laminar: (exact)	$Sh_x = 0.332\, Re_x^{1/2}\, Sc^{1/3}$	$Sh = 0.664\, Re_L^{1/2}\, Sc^{1/3}$
	$Sh_x = \dfrac{k_c\, x}{D_{AB}}\ ,\ \ Re_x = \dfrac{v_\infty x}{\nu}$	$Sh = \dfrac{k_c\, L}{D_{AB}}\ ,\ \ Re_L = \dfrac{v_\infty L}{\nu}$
fully turbulent: $Re \ge 3 \times 10^6$	$Sh_x = 0.0292\, Re_x^{4/5}\, Sc^{1/3}$	$Sh = 0.0365\, Re_L^{4/5}\, Sc^{1/3}$

Transition regime: $2 \times 10^5 < Re_x < 3 \times 10^6$

Local vs. Average Mass Transfer Coefficients

For external fluid flow over a flat plate, three regimes

Flow	***Re*** Regime	Mass Transfer Coefficient
Laminar	$0 < \mathbf{Re_x} \le 2.0 \times 10^5$ Average k_c $k_c = \dfrac{\int_0^L k_{c,\mathrm{lam}}(x)\,dx}{L}$	*Local* $Sh_x = 0.332\,\mathrm{Re}_x^{1/2}\,Sc^{1/3}$ $k_{c,x} = 0.332\,\dfrac{D_{AB}}{x}\left(\dfrac{v_\infty \cdot x}{\upsilon_B}\right)^{1/2} Sc^{1/3}$ *Average* $Sh_L = 0.664\,\mathrm{Re}_L^{1/2}\,Sc^{1/3}$ $\mathrm{Re}_L = \dfrac{v_\infty \cdot L}{\upsilon_B}$ and $Sh_L = \dfrac{k_c \cdot L}{D_{AB}}$
Transition	$2.0 \times 10^5 < \mathbf{Re_x}$ $3.0 \times 10^6 < \mathbf{Re_x}$ Transition length $L_t = 2.0 \cdot 10^5\,\dfrac{\upsilon_B}{v_\infty}$	*Local* – No good description *Average* $k_c = \dfrac{1}{L}\left[\int_0^{L_t} k_{c,lam}(x)dx + \int_{L_t}^{L} k_{c,turb}(x)dx\right]$
Turbulent	$\mathbf{Re_x} \ge 3.0 \times 10^6$	*Local* $Sh_x = 0.0292\,\mathrm{Re}_x^{4/5}\,Sc^{1/3}$ $k_{c,x} = 0.0292\,\dfrac{D_{AB}}{x}\left(\dfrac{v_\infty \cdot x}{\upsilon_B}\right)^{4/5} Sc^{1/3}$ *Average* $Sh_L = 0.0365\,\mathrm{Re}_L^{4/5}\,Sc^{1/3}$ (neglect laminar contribution)

Local Mass Transfer Coefficients – Flow over a Flat Plate

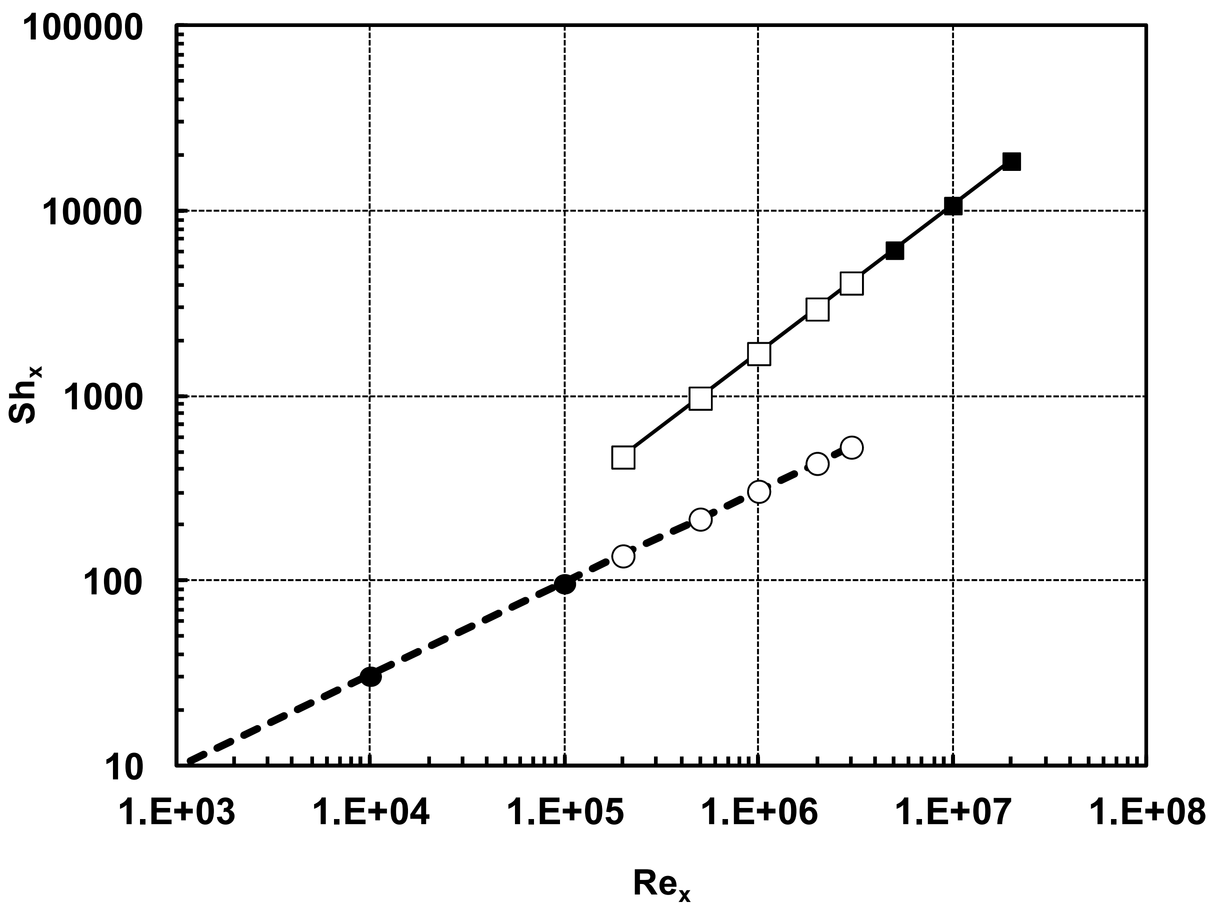
D_{AB} =	0.10	cm^2/sec
ν_B =	0.08	cm^2/sec
Sc =	0.80	
v_∞ =	50.0	cm/sec

$$Re = \frac{x\, v_\infty}{\nu}$$

Transition regime: $2 \times 10^5 < Re_x < 3 \times 10^6$

$$Re_t = 2 \times 10^5$$

At fixed $\boldsymbol{v_\infty}$ **,** looking towards increasing $\boldsymbol{x}$ shows the following dependence of local ***Sh*** on position in laminar and turbulent regimes



Local Mass Transfer Coefficients – Flow over a Flat Plate (cont.)

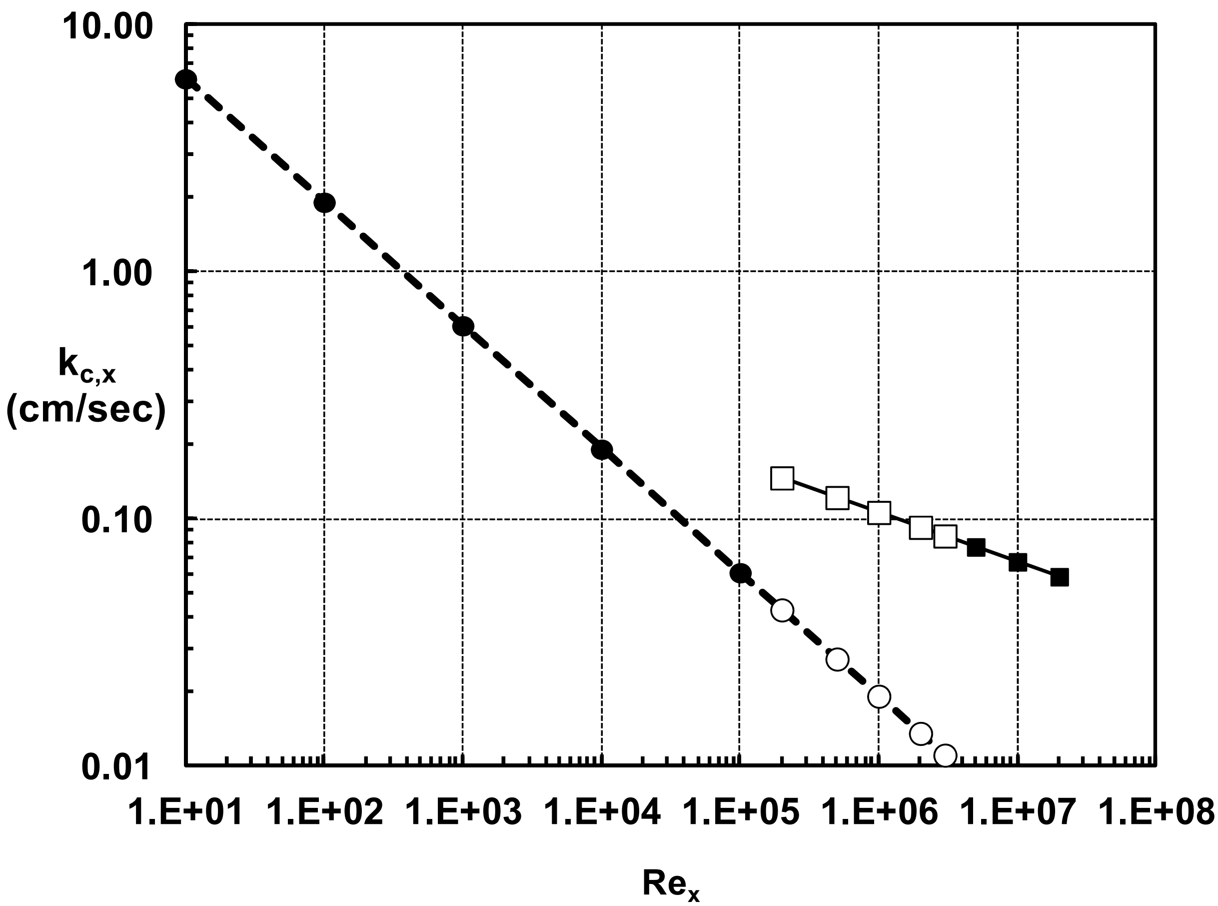
D_{AB} =	0.10	cm^2/sec
ν_B =	0.08	cm^2/sec
Sc =	0.80	
v_∞ =	50.0	cm/sec

$$Re = \frac{x\, v_\infty}{\nu}$$

Transition regime: $2 \times 10^5 < Re_x < 3 \times 10^6$

$$Re_t = 2 \times 10^5$$

At fixed $\boldsymbol{v_\infty}$ = 50 cm/sec, moving towards increasing $\boldsymbol{x}$ shows the following dependence of local $\boldsymbol{k_{c,x}}$ on position in laminar and turbulent regimes



Average Mass Transfer Coefficients – Flow over a Flat Plate

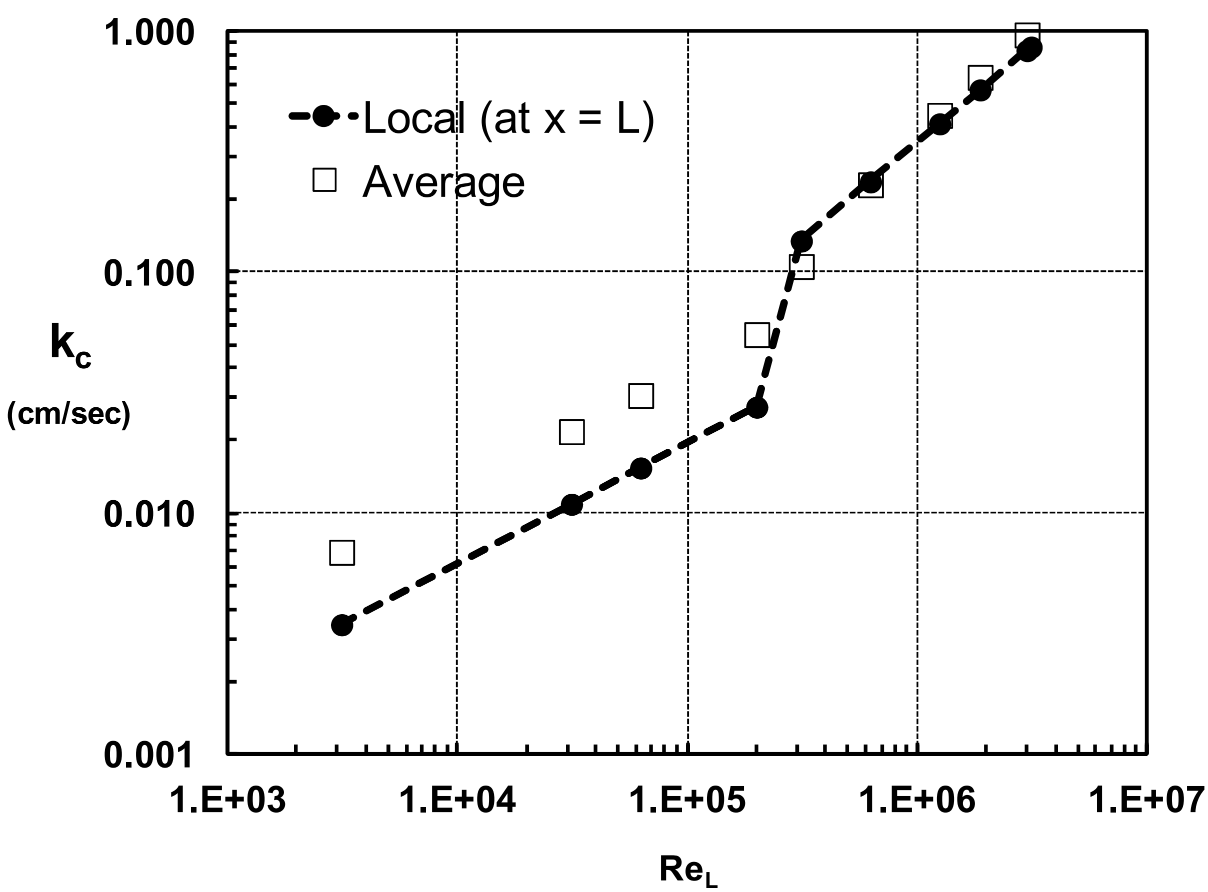
D_{AB} =	0.10	cm^2/sec
ν_B =	0.08	cm^2/sec
Sc =	0.80	
L =	500	cm

$$Re_L = \frac{v_\infty L}{\nu}$$

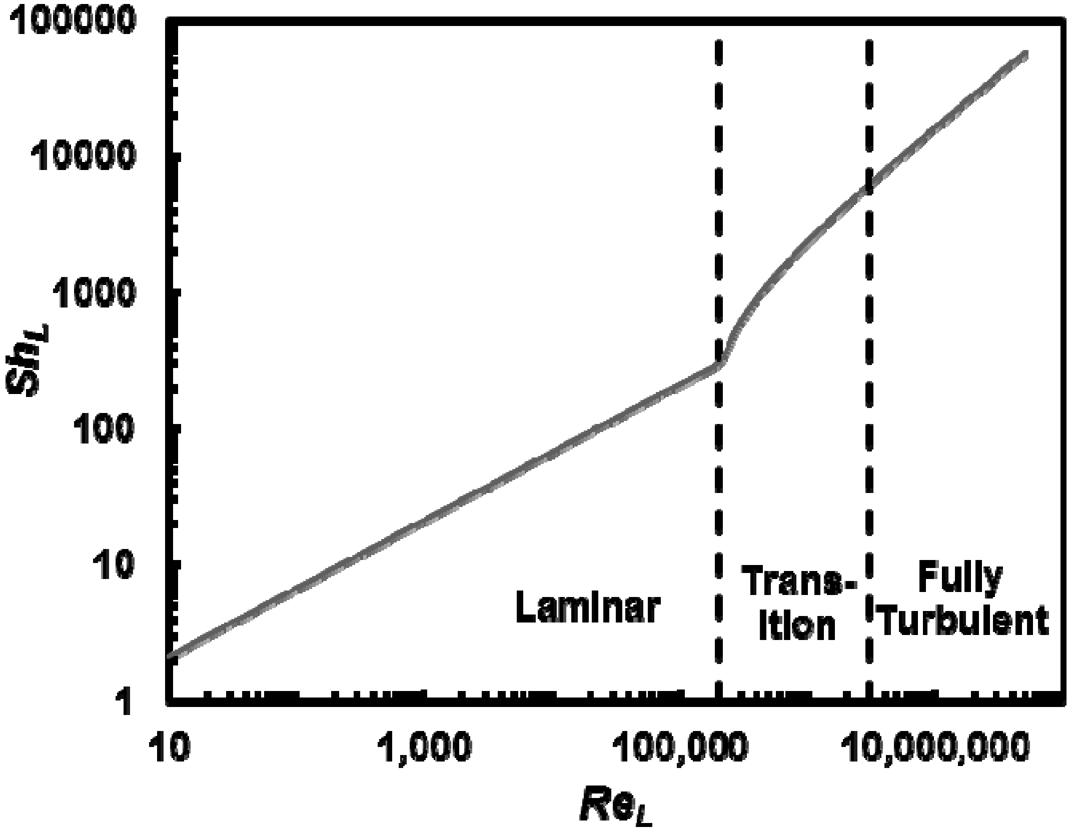
Transition regime: $2 \times 10^5 < Re_x < 3 \times 10^6$

$$Re_t = 2 \times 10^5$$

At fixed $\boldsymbol{x = L}$ = 500 cm, increasing $\boldsymbol{v_\infty}$ shows the following dependence of local $\boldsymbol{k_{c,x}}$ and overall (average) $\boldsymbol{k_c}$ on flow in the laminar and turbulent regimes



Average Mass Transfer Coefficients – Flow over a Flat Plate

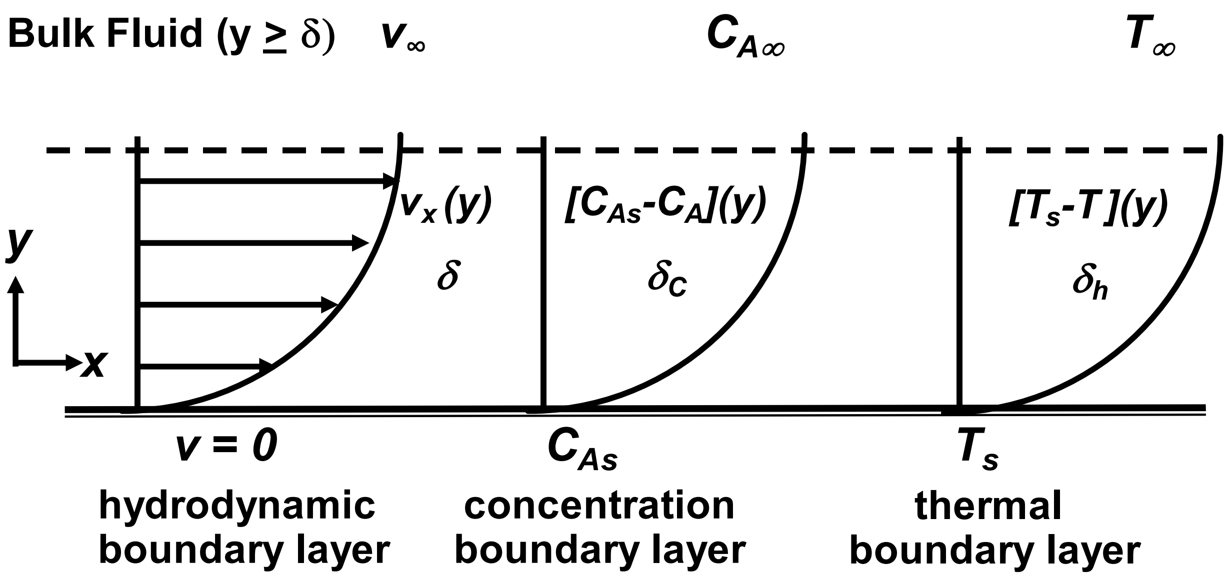


28.6 Mass, Energy, and Momentum Transfer Analogies

In convective mass transfer, heat transfer coefficients (h) for convection can be used to estimate mass transfer coefficients (k_c) for convection (and vice versa)

All "Transport Analogies" require:

- Constant physical properties of the mixture (can be evaluated at "film" average temperature and solute concentration)
- No homogeneous reaction within the boundary layer
- Velocity profile is not distorted by high mass transfer flux in concentrated solution (i.e. valid only for dilute A in carrier medium B)



Consider two analogies

1. Reynolds Analogy
2. Chilton-Colburn Analogy

Reynolds Analogy

Recall from Exact Boundary Layer Analysis

at *y = 0* (fluid-surface boundary, "no slip" at the wall)

Fluid	Mass	Heat
$v_x = v_{x,s} = 0$	$C_A = C_{As}$	$T = T_s$
$\dfrac{v_x}{v_\infty} = \dfrac{v_x - v_{x,s}}{v_\infty - v_{x,s}} = 0$	$\dfrac{C_A - C_{As}}{C_{A\infty} - C_{As}} = 0$	$\dfrac{T - T_s}{T_\infty - T_s} = 0$

Reynolds postulated that the mechanisms for momentum and mass transfer were identical

$$\frac{\partial}{\partial y}\left(\frac{v_x - v_{x,s}}{v_\infty - v_{x,s}}\right) = \frac{\partial}{\partial y}\left(\frac{v_x}{v_\infty}\right) = \frac{1}{v_\infty}\frac{\partial v_x}{\partial y}\Big|_{y=0} = \frac{\partial}{\partial y}\left(\frac{C_A - C_{As}}{C_{A\infty} - C_{As}}\right) = \frac{1}{C_{A\infty} - C_{As}}\frac{\partial C_A}{\partial y}\Big|_{y=0}$$

$$\frac{\partial}{\partial y}\left(\frac{v_x - v_{x,s}}{v_\infty - v_{x,s}}\right) = \frac{\partial}{\partial y}\left(\frac{v_x}{v_\infty}\right) = \frac{1}{v_\infty}\frac{\partial v_x}{\partial y}\Big|_{y=0} \frac{\partial}{\partial y}\left(\frac{T - T_s}{T_\infty - T_s}\right) = \frac{1}{T_\infty - T_s}\frac{\partial T}{\partial y}\Big|_{y=0}$$

Relate Fluids to Mass Transfer at Boundary Surface (***y = 0***)

$$\frac{1}{v_\infty}\frac{\partial v_x}{\partial y}\Big|_{y=0} = \frac{1}{C_{A\infty} - C_{As}}\frac{\partial C_A}{\partial y}\Big|_{y=0} \quad \text{or} \quad \frac{C_{A\infty} - C_{As}}{v_\infty}\frac{\partial v_x}{\partial y}\Big|_{y=0} = \frac{\partial C_A}{\partial y}\Big|_{y=0}$$

Mass transfer: $N_{A,y} = k_c(C_{As} - C_{A\infty}) = -D_{AB}\dfrac{dC_A}{dy}\Big|_{y=0}$

(Fick's Law)

Fluid flow: $\tau_{xy}\big|_{y=0} = -\mu\dfrac{\partial v_x}{\partial y}\Big|_{y=0}$ and $\tau_o = \dfrac{C_f}{2}\rho \cdot v_\infty^2$ (by convention $\tau_{xy} = -\tau_o$)

(Newton's First Law)

C_f = coefficient of friction (y = 0)

$$\frac{\partial v_x}{\partial y}\Big|_{y=0} = \frac{C_f}{2}\frac{\rho \cdot v_\infty^2}{\mu}$$

τ_{xy} = fluid shear stress (kg/m·sec^2)

Therefore $\frac{dC_A}{dy}\Big|_{y=0} = +\frac{k_c(C_{A\infty} - C_{As})}{D_{AB}} = \frac{C_{A\infty} - C_{As}}{\mathrm{v}_\infty}\frac{\partial \mathrm{v}_x}{\partial y}\Big|_{y=0}$

and so

$k_c = \frac{D_{AB}}{\mathrm{v}_\infty}\frac{\partial \mathrm{v}_x}{\partial y}\Big|_{y=0}$ or $k_c = \frac{D_{AB}}{\mathrm{v}_\infty}\frac{C_f}{2}\frac{\rho \cdot \mathrm{v}_\infty^2}{\mu}$ or $k_c = \mathrm{v}_\infty \frac{C_f}{2}\frac{\rho\, D_{AB}}{\mu}$

SET $Sc = \frac{\mu}{\rho\, D_{AB}} = 1$ so that $\delta = \delta_c$ then $k_c = \mathrm{v}_\infty \frac{C_f}{2}$

Relate Fluids to Heat Transfer at Boundary Surface (***y = 0***)

$\frac{1}{\mathrm{v}_\infty}\frac{\partial \mathrm{v}_x}{\partial y}\Big|_{y=0} = \frac{1}{T_\infty - T_s}\frac{\partial T}{\partial y}\Big|_{y=0}$ or $\frac{T_\infty - T_s}{\mathrm{v}_\infty}\frac{\partial \mathrm{v}_x}{\partial y}\Big|_{y=0} = \frac{\partial T}{\partial y}\Big|_{y=0}$

Recall

Heat transfer: $q_y = h(T_s - T_\infty) = -\kappa \frac{dT}{dy}\Big|_{y=0}$

(Fourier's Law) κ = thermal conductivity of fluid (e.g. units J/m-sec-K)

h = convective heat trans. coeff. (e.g. units J/m^2-sec-K)

Fluid flow: $\tau_o = -\mu \frac{\partial \mathrm{v}_x}{\partial y}\Big|_{y=0} = \frac{C_f}{2}\rho \cdot \mathrm{v}_\infty^2$ or $\frac{\partial \mathrm{v}_x}{\partial y}\Big|_{y=0} = \frac{C_f}{2}\frac{\rho \cdot \mathrm{v}_\infty^2}{\mu}$

(Newton's Law) C_f = coefficient of friction (y = 0)

Therefore $\frac{dT}{dy}\Big|_{y=0} = +\frac{h(T_\infty - T_s)}{\kappa} = \frac{(T_\infty - T_s)}{\mathrm{v}_\infty}\frac{\partial \mathrm{v}_x}{\partial y}\Big|_{y=0}$

and so

$h = \frac{\kappa}{\mathrm{v}_\infty}\frac{\partial \mathrm{v}_x}{\partial y}\Big|_{y=0}$ or $h = \frac{\kappa}{\mathrm{v}_\infty}\frac{C_f}{2}\frac{\rho \cdot \mathrm{v}_\infty^2}{\mu}$ or $h = \mathrm{v}_\infty \frac{C_f}{2}\frac{\rho \cdot \kappa}{\mu}$

SET $\Pr = \dfrac{\mu C_p}{\kappa} = 1$ so that $\delta = \delta_h$ then $h = \dfrac{C_f}{2} \mathrm{v}_\infty \rho C_p$

C_p = heat capacity of the fluid (e.g. units J/kg-K)

Finally the Reynolds Analogy for Fluids-Heat-Mass is (Sc = 1, Pr = 1)

$$\frac{C_f}{2} = \frac{h}{\mathrm{v}_\infty \rho C_p} = \frac{k_c}{\mathrm{v}_\infty}$$

Chilton-Colburn Analogy

What if Sc ≠ 1 and Pr ≠ 1?

From Blasius Solution for fluid flow over a flat plate
(Eq. (12-29) W^3-R)

$$\left.\frac{\partial \mathrm{v}_x}{\partial y}\right|_{y=0} = \mathrm{v}_\infty \frac{0.332}{x} \mathrm{Re}_x^{1/2} \quad (1)$$

Remember $\left.\dfrac{\partial \mathrm{v}_x}{\partial y}\right|_{y=0} = \dfrac{C_f}{2} \dfrac{\rho \cdot \mathrm{v}_\infty^2}{\mu}$ and $\mathrm{Re}_x = \dfrac{\rho \cdot \mathrm{v}_\infty x}{\mu}$

and so $\left.\dfrac{\partial \mathrm{v}_x}{\partial y}\right|_{y=0} = \dfrac{C_f}{2x} \mathrm{v}_\infty \mathrm{Re}_x \quad (2)$

Combining (1) and (2) $\dfrac{C_f}{2} = \dfrac{0.332}{\mathrm{Re}_x^{1/2}}$

Now consider $\mathrm{Sh}_x = 0.332 \mathrm{Re}_x^{1/2} \mathrm{Sc}^{1/3}$

Therefore $\dfrac{C_f}{2} = \dfrac{0.332}{\mathrm{Re}_x^{1/2}} = \dfrac{\mathrm{Sh}_x}{\mathrm{Re}_x \cdot \mathrm{Sc}^{1/3}}$

Expand terms
$$\frac{C_f}{2} = \frac{\mathrm{Sh_x}}{\mathrm{Re_x} \cdot \mathrm{Sc}^{1/3}} = \frac{\left(\dfrac{k_c\ x}{D_{AB}}\right) Sc^{2/3}}{\left(\dfrac{\rho\, \mathrm{v}_\infty x}{\mu}\right)\left(\dfrac{\mu}{\rho D_{AB}}\right)}$$

Finally
$$\frac{C_f}{2} = \frac{k_c\ Sc^{2/3}}{\mathrm{v}_\infty} = j_D$$
Colburn "j factor" for mass transfer

Note $\dfrac{\mathrm{k_c}}{\mathrm{v}_\infty} = \mathrm{St} = \text{Stanton number}$

Similarly, for convective heat transfer now consider $\mathrm{Nu_x} = 0.332 \mathrm{Re_x}^{1/2} \mathrm{Pr}^{1/3}$

Therefore
$$\frac{C_f}{2} = \frac{0.332}{\mathrm{Re}_x^{1/2}} = \frac{\mathrm{Nu_x}}{\mathrm{Re_x} \cdot \mathrm{Pr}^{1/3}}$$

Expand terms
$$\frac{C_f}{2} = \frac{\mathrm{Nu_x}}{\mathrm{Re_x} \cdot \mathrm{Pr}^{1/3}} = \frac{\left(\dfrac{h\ x}{\kappa}\right) \mathrm{Pr}^{2/3}}{\left(\dfrac{\rho\, \mathrm{v}_\infty x}{\mu}\right)\left(\dfrac{\mu C_p}{\kappa}\right)}$$

Finally
$$\frac{C_f}{2} = \frac{h}{\rho\, C_p} \frac{\mathrm{Pr}^{2/3}}{\mathrm{v}_\infty} = j_H$$
Colburn "j factor" for heat transfer

Combine, knowing $\boldsymbol{j_D = j_H}$

$$\mathrm{k_c} = \frac{\mathrm{h}}{\rho\, \mathrm{C_p}} \left(\frac{\mathrm{Pr}}{\mathrm{Sc}}\right)^{2/3}$$
note the intimacy of Pr and Sc

Summary of Heat-Mass Transfer Analogies for Convection

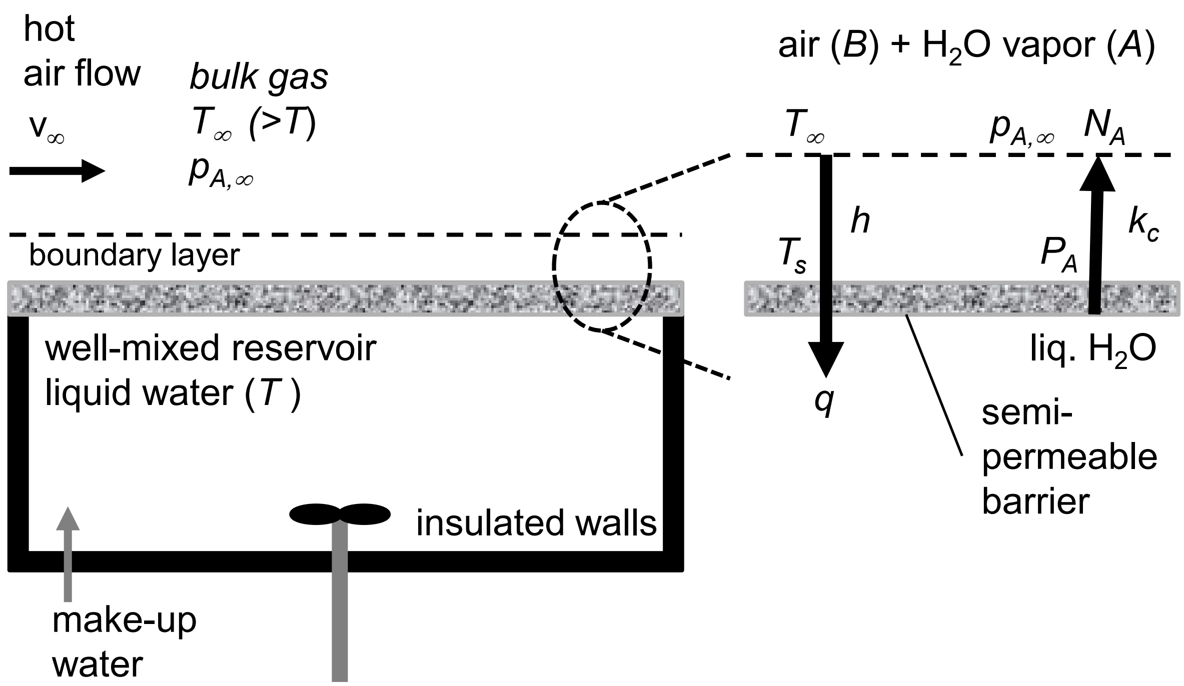
Analogy	*Heat Transfer*	*Mass Transfer*
Reynolds (Pr = Sc = 1) $k_c = \dfrac{h}{\rho\, C_p}$	$h = \dfrac{C_f}{2} v_\infty \rho\, C_p$	$k_c = v_\infty \dfrac{C_f}{2}$
Chilton-Colburn (Pr ≠ Sc) $k_c = \dfrac{h}{\rho\, C_p}\left(\dfrac{Pr}{Sc}\right)^{2/3}$	$\dfrac{C_f}{2} = \dfrac{h}{\rho\, C_p}\dfrac{Pr^{2/3}}{v_\infty} = j_H$	$\dfrac{C_f}{2} = \dfrac{k_c\, Sc^{2/3}}{v_\infty} = j_D$

Combined Heat and Mass Transfer

An important process illustrating the coupling of heat and mass transfer is *evaporative cooling*

- A thin, semi-permeable barrier allows water vapor, but liquid water, to readily pass through it

- A small amount of liquid vaporizes, using heat transferred from the hot flowing gas stream



Energy Balance

$$\Delta H = Q$$

$$\begin{pmatrix}\text{flow of water} \\ \text{vapor}\end{pmatrix}\begin{pmatrix}\text{enthalpy of} \\ \text{phase change}\end{pmatrix} = \begin{pmatrix}\text{convective heat transfer rate} \\ \text{across boundary layer}\end{pmatrix}$$

$$\left(N_A S\right)\left(\Delta H_{v,A}\right) = S\,h\,(T_\infty - T_s)$$

$$k_c\left(C_{As} - C_{A,\infty}\right)\Delta H_{v,A} = h(T_\infty - T_s)$$

Remember $C_{A\infty} = \dfrac{P_A}{RT}, \quad C_{As} = \dfrac{P_A^*(T_s)}{RT}$

S is the surface area of the semi-permeable barrier surface

$\Delta H_{v,A}$ is the latent heat of vaporization of the liquid at temperature T_s

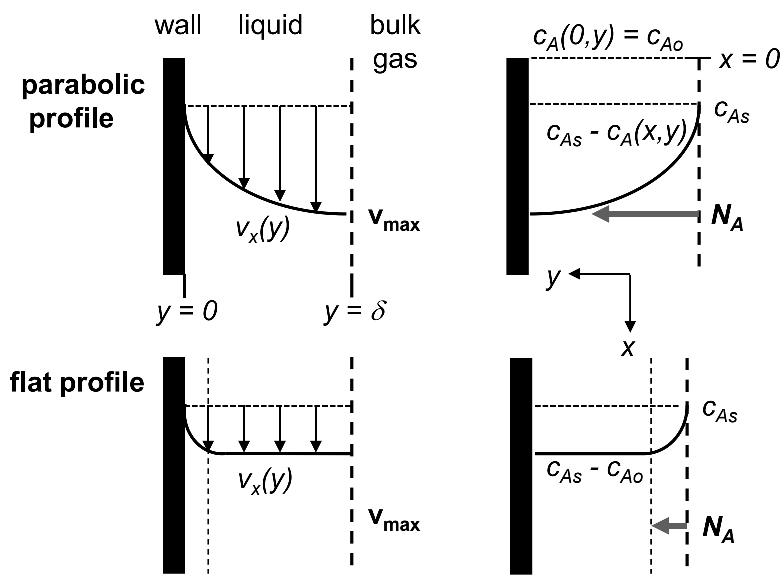
Chilton-Colburn analogy

$$h = k_c\, C_{p,B}\, \rho_B \left(\frac{Sc}{\mathrm{Pr}} \right)^{2/3}$$

Combine with Energy Balance – what happens?

28.7 Models for Convective Mass Transfer Coefficients (dilute systems)

Laminar Falling Liquid Film – Simultaneous Momentum and Mass Transfer



From Topic 26.4, at the gas-liquid interface ($y = \delta$), the local flux of solute A from the gas phase into the liquid phase at position x down the length of the falling liquid film is

$$N_A = \left(c_{As} - c_{Ao}\right)\sqrt{\frac{D_{AB}\mathrm{v}_{max}}{\pi x}}$$

Recall definition of convective mass transfer

$$N_A = k_{c,x}\left(c_{As} - c_{Ao}\right)$$

Therefore, for a falling liquid film

$$k_{c,x} = \sqrt{\frac{D_{AB}\mathrm{v}_{max}}{\pi x}}$$

Maximum velocity $\mathrm{v}_{max} = \dfrac{\rho g \delta^2}{2\mu}$ (all parameters refer to liquid)

The average f $k_{c,x}$ down the length L of the falling liquid film

$$k_c = \frac{1}{L}\int_0^L k_{c,x}\,dx = \sqrt{\frac{4D_{AB}\mathrm{v}_{max}}{\pi L}}$$

28.7 Models for Convective Mass Transfer Coefficients (dilute systems)

Model	Basic Form	f (D_{AB})	Notes
Film Theory	$k_c = \frac{D_{AB}}{\delta}$	$k_c \propto D_{AB}$	film layer thickness δ not known up front
Penetration Theory	$k_c = \sqrt{\frac{4D_{AB}\mathrm{v}_{\max}}{\pi L}}$	$k_c \propto D_{AB}^{1/2}$	good model if there is homogeneous chemical reaction within the boundary layer (small penetration depth)
Boundary Layer Theory (e.g. Laminar Flow over a Flat Plate)	$k_c = 0.664\frac{D_{AB}}{L}\left(\frac{\mathrm{v}_\infty L}{\nu}\right)^{1/2}\left(\frac{\nu}{D_{AB}}\right)^{1/3}$	$k_c \propto D_{AB}^{2/3}$ (if $Sh \propto Sc^{1/3}$)	best way to "scale" mass transfer coefficients from one solute to another within a mixture exposed to the same hydrodynamic situation