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OREGON STATE UNIVERSITY - CBEE
DEPARTMENT OF CHEMICAL ENGINEERING

CHE 331
Transport Phenomena I

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Application of Navier-Stokes Equations

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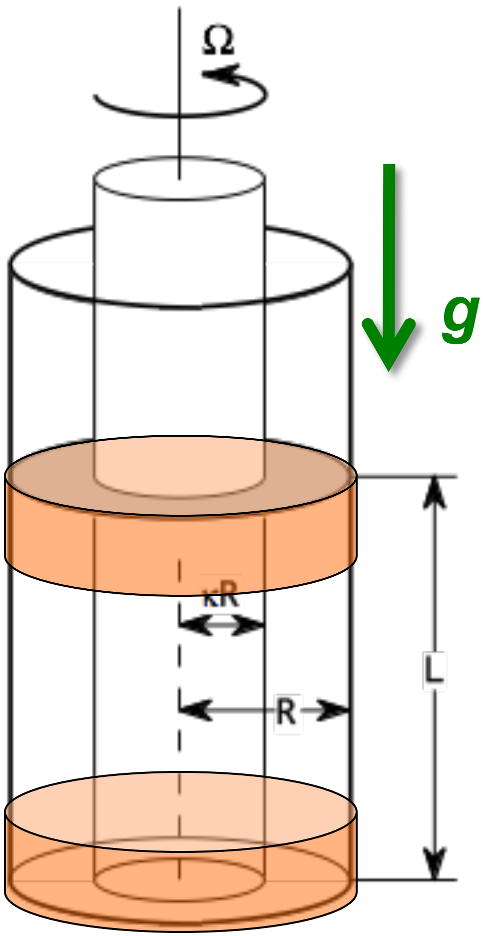
CHE 331
Fall 2022

Problem Statement

Find a solution for the steady state velocity vector as a function of position in flow between concentric cylinders (see illustration below). The inner cylinder rotates steadily about its axis at Ω [rad/s].

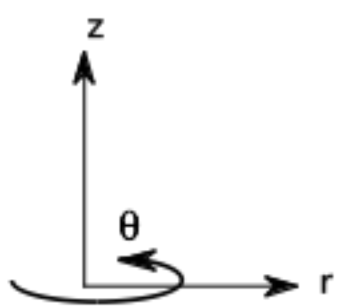


Assumptions:

The cylinders are long compared to the radii L/R >> 1

The fluid is Newtonian; the flow is isothermal, and at steady state.

The flow is laminar and unidirectional.



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Solution Strategy

Start your consideration from the Navier-Stokes equations.

Clearly state all assumptions and list all consequences that materialize from these assumptions.

Establish the mathematical model [differential equation(s) that precipitate from the analysis of the Navier-Stokes equations + appropriate boundary conditions] that represents the flow between concentric cylinders.

Solve the differential equations and obtain the velocity profile.

Find the expression for the torque required to maintain rotational speed of the inner cylinder.

Navier-Stokes Equations In Cylindrical Coordinates (r, θ, z) :

$$\rho\left[\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}\right]=$$

$$=-\frac{\partial P}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial^2 u_r}{\partial z^2}\right]+\rho g_r$$

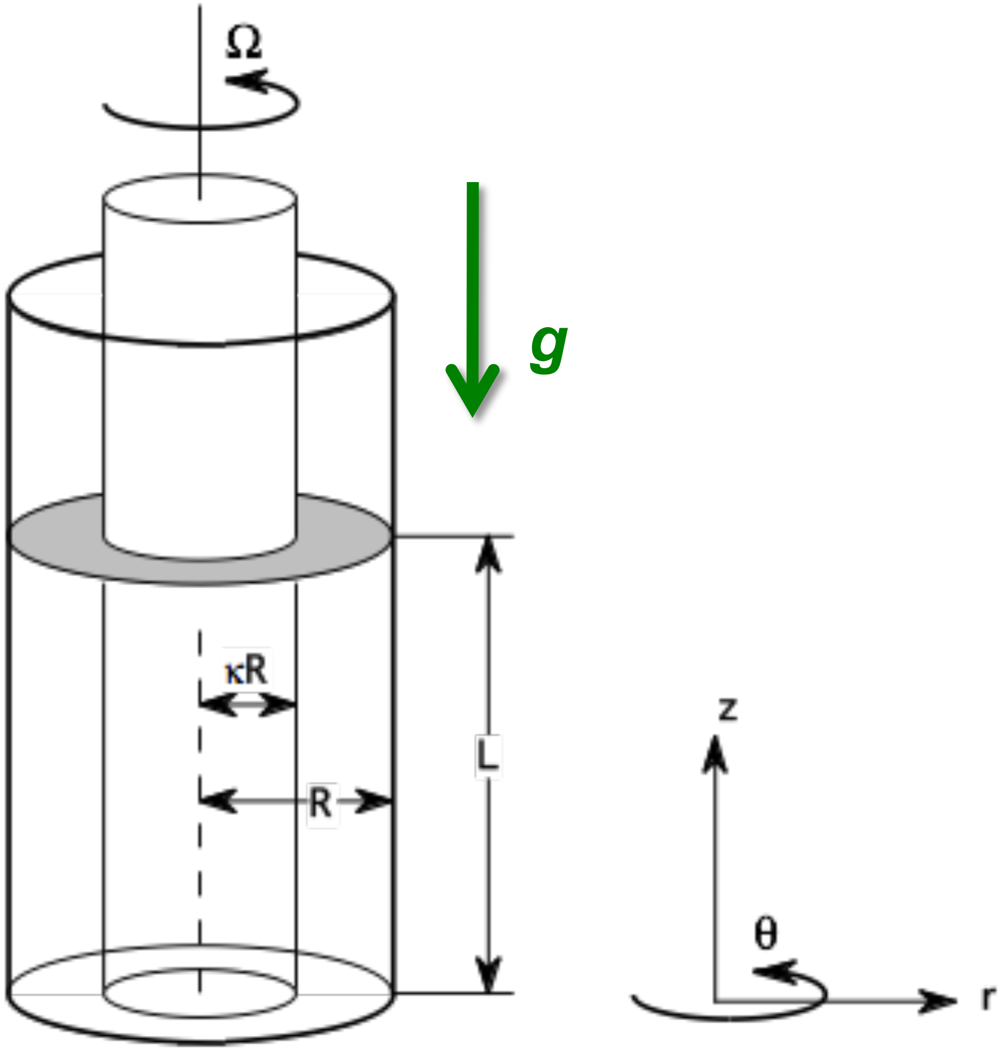
$$\rho\left[\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta}{r}+u_z\frac{\partial u_\theta}{\partial z}\right]=$$

$$=-\frac{1}{r}\frac{\partial P}{\partial \theta}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right]+\rho g_\theta$$

$$\rho\left[\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right]=$$

$$=-\frac{\partial P}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right]+\rho g_z$$

Navier-Stokes Equations In Cylindrical Coordinates (r, θ, z) :



$$All \quad \frac{\partial}{\partial t} = 0$$

$$All \quad \frac{\partial}{\partial \theta} = 0$$

$$u_r = 0$$

$$u_z = 0$$

$$u_\theta \neq f(z)$$

$$u_\theta \neq f(\theta)$$

$$u_\theta = f(r)$$

In Cartesian ***x, y, z***, coordinates with constant fluid density:

$$\nabla \cdot U = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

In cylindrical ***r***, θ, ***z***, coordinates with constant fluid density:

$$\nabla \cdot U = \frac{1}{r}\frac{\partial\left(r\,u_r\right)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$

In spherical ***r***, θ , ϕ , coordinates with constant fluid density:

$$\nabla \cdot U = \frac{1}{r^2}\frac{\partial\left(r^2 u_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(u_\theta \sin\theta\right)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}$$

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In cylindrical ***r***, θ, ***z***, coordinates with steady state and constant fluid density:

Continuity Equation

$$\nabla \cdot U = 0 \qquad \Rightarrow \qquad \frac{1}{r}\frac{\partial\left(r\ u_r\right)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

And, unidirectional flow: $u_r = 0 \quad and \quad u_z = 0$

$$\frac{1}{r}\frac{\partial\left(\cancel{r\ u_r}\right)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \cancel{\frac{\partial u_z}{\partial z}} = 0 \qquad \Rightarrow \qquad \frac{\partial u_\theta}{\partial \theta} = 0$$

Consider the flow in *r* direction :

$$\frac{\partial}{\partial t}=0;\ \ \frac{\partial}{\partial \theta}=0;\ \ u_r=0;\ \ u_z=0;\ \ u_\theta \neq f(z);\ \ u_\theta \neq f(\theta);\ \ u_\theta = f(r)$$

$$\rho\left[\cancel{\frac{\partial u_r}{\partial t}}+u_r\cancel{\frac{\partial u_r}{\partial r}}+\cancel{\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}}-\frac{u_\theta^2}{r}+u_z\cancel{\frac{\partial u_r}{\partial z}}\right]=$$

$$=-\frac{\partial P}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\cancel{u_r})\right)+\frac{1}{r^2}\cancel{\frac{\partial^2 u_r}{\partial \theta^2}}-\frac{2}{r^2}\cancel{\frac{\partial u_\theta}{\partial \theta}}+\cancel{\frac{\partial^2 u_r}{\partial z^2}}\right]+\cancel{\rho g_r}$$

$$-\rho\frac{u_\theta^2}{r}=-\frac{\partial P}{\partial r} \quad \Rightarrow \quad \frac{\partial P}{\partial r}=\rho\frac{u_\theta^2}{r}$$

$$\boxed{dP=\rho\frac{u_\theta^2}{r}dr \quad \Rightarrow \quad \int_{P_{\kappa R}}^{P_r} dP=\rho\int_{\kappa R}^{r}\frac{u_\theta^2}{r}dr}$$

Consider the flow in θ direction :

$$\frac{\partial}{\partial t}=0;\ \ \frac{\partial}{\partial \theta}=0;\ \ u_r=0;\ \ u_z=0;\ \ u_\theta \neq f(z);\ \ u_\theta \neq f(\theta);\ \ u_\theta = f(r)$$

$$\rho\left[\cancel{\frac{\partial u_\theta}{\partial t}}+\cancel{u_r\frac{\partial u_\theta}{\partial r}}+\cancel{\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}}+\cancel{\frac{u_r u_\theta}{r}}+\cancel{u_z\frac{\partial u_\theta}{\partial z}}\right]=$$

$$=\cancel{-\frac{1}{r}\frac{\partial P}{\partial \theta}}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right)+\cancel{\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}}-\cancel{\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}}+\cancel{\frac{\partial^2 u_\theta}{\partial z^2}}\right]+\cancel{\rho g_\theta}$$

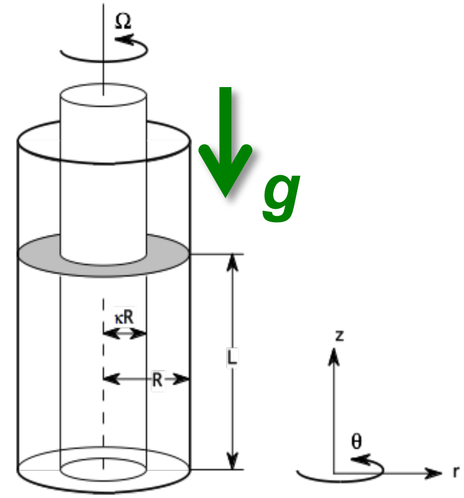
$$\boxed{0=\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right)\right]}$$

Consider the flow in *z* direction :

$$\frac{\partial}{\partial t}=0;\ \ \frac{\partial}{\partial \theta}=0;\ \ u_r=0;\ \ u_z=0;\ \ u_\theta \neq f(z);\ \ u_\theta \neq f(\theta);\ \ u_\theta = f(r)$$

$$\rho\left[\cancel{\frac{\partial u_z}{\partial t}}+\cancel{u_r\frac{\partial u_z}{\partial r}}+\cancel{\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}}+\cancel{u_z\frac{\partial u_z}{\partial z}}\right]=$$

$$=-\frac{\partial P}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\cancel{r\frac{\partial u_z}{\partial r}}\right)+\cancel{\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}}+\cancel{\frac{\partial^2 u_z}{\partial z^2}}\right]+\rho g_z$$



$$\boxed{g_z=-g}$$

$$0=-\frac{\partial P}{\partial z}+\rho g_z \Rightarrow dP=\rho g_z dz \Rightarrow dP=-\rho g dz$$

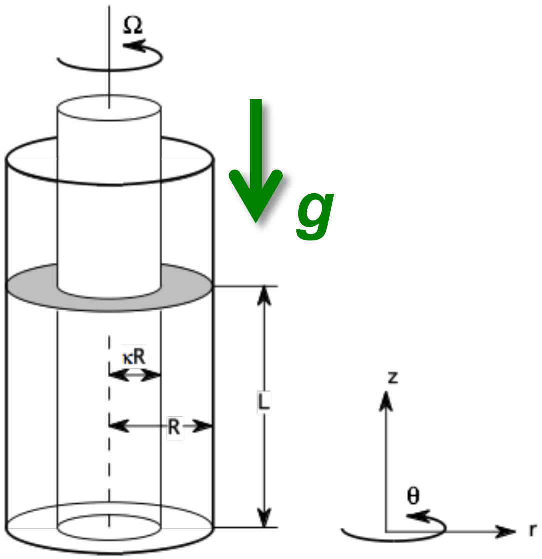
$$\int_{P_L}^{P_z} dP=-\rho g\int_{z=L}^{z} dz \Rightarrow P_z-P_L=\rho g(L-z) \Rightarrow \boxed{P_z=P_{atm}+\rho g(L-z)}$$

Complete Mathematical Model

$$dP_z = -\rho g dz \quad \Rightarrow \quad \boxed{P_z = P_{atm} + \rho g(L-z)}$$

$$\boxed{dP_r = \rho \frac{u_\theta^2}{r} dr \quad \Rightarrow \quad \int_{P_{\kappa R}}^{P_r} dP_r = \rho \int_{\kappa R}^{r} \frac{u_\theta^2}{r} dr}$$

$$\boxed{0 = \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r u_\theta) \right) \right]}$$



The appropriate boundary conditions are:

Only inner cylinder is rotating:

$@ \quad r = R \quad \Rightarrow \quad u_\theta = 0$

$@ \quad r = \kappa R \Rightarrow \quad u_\theta = \kappa R \Omega$

Both cylinders are rotating at **Ω**:

$@ \quad r = R \quad \Rightarrow \quad u_\theta = R\Omega$

$@ \quad r = \kappa R \Rightarrow \quad u_\theta = \kappa R \Omega$

Solution

The only equation left to integrate is:

$$0=\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right)\right] \Rightarrow \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right)=0$$

The appropriate boundary conditions are:

$$@ \quad r=R \Rightarrow u_\theta=0$$
$$@ \quad r=\kappa R \Rightarrow u_\theta=\kappa R\Omega$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right)=0 \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)=a \Rightarrow ru_\theta=\frac{a\cdot r^2}{2}+b \Rightarrow \boxed{u_\theta=\frac{a\cdot r}{2}+\frac{b}{r}}$$

Apply boundary conditions to obtain: $\boxed{u_\theta=\frac{\kappa R\Omega}{\kappa-\frac{1}{\kappa}}\left(\frac{r}{R}-\frac{R}{r}\right)}$

Solution

$$\boxed{u_\theta = \frac{a\cdot r}{2} + \frac{b}{r}}$$

$$@\quad r = R \quad\Rightarrow\quad u_\theta = 0$$

$$@\quad r = \kappa R \Rightarrow\quad u_\theta = \kappa R\Omega$$

$$u_\theta = 0 = \frac{a\cdot R}{2} + \frac{b}{R} \qquad\Rightarrow\qquad b = -\frac{a\cdot R^2}{2}$$

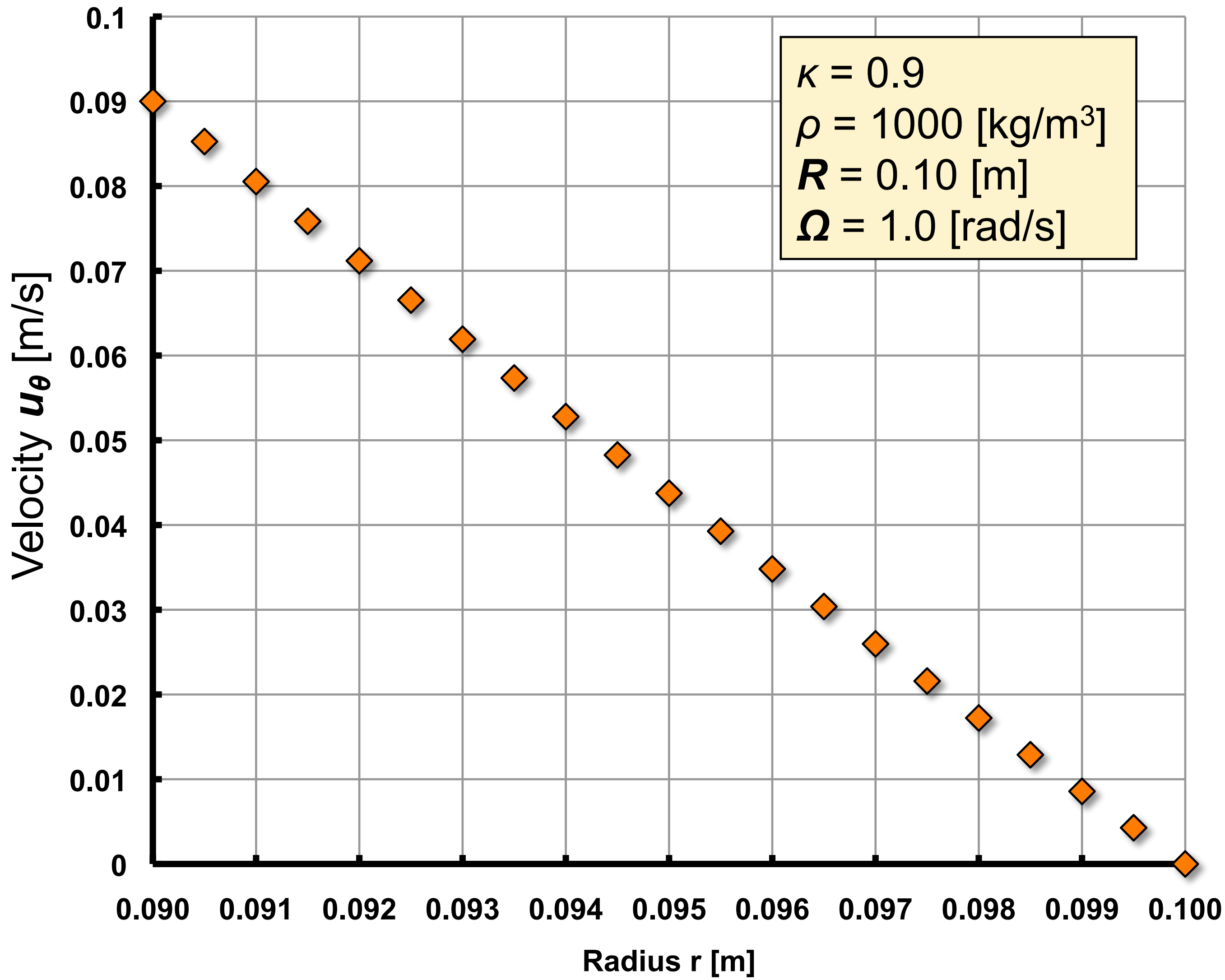
$$u_\theta = \kappa R\Omega = \frac{a\cdot\kappa R}{2} + \frac{b}{\kappa R} \qquad\Rightarrow\qquad \kappa R\Omega = \frac{a\cdot\kappa R}{2} - \frac{aR^2}{2\kappa R}$$

$$\boxed{a = \frac{2\kappa^2\Omega}{\kappa^2 - 1}} \qquad\Rightarrow\qquad \boxed{b = -\frac{\kappa^2\Omega\cdot R^2}{\kappa^2 - 1}}$$

Finally, we obtain:

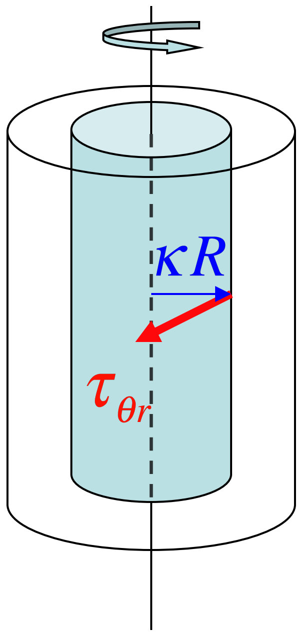
$$\boxed{u_\theta = \frac{\kappa^2 R\Omega}{(1-\kappa^2)}\left(\frac{R}{r} - \frac{r}{R}\right)}$$

Velocity Profile $u_\theta(r)$



Solution

Find the expression for the torque required to maintain rotational speed of the inner cylinder.



The only force resisting the rotation of the inner cylinder is tangential force exerted by the fluid. ***This Shear*** stress acts in θ direction on the surface normal to the ***r*** direction.

$$\tau_{\theta r} = \left[P + \frac{2}{3}\mu(\nabla \cdot U) \right] \cancelto{0}{\delta_{\theta r}} - \mu\Delta_{\theta r}$$

$$\tau_{\theta r}\Big|_{r=\kappa R} = -\mu\Delta_{\theta r}\Big|_{r=\kappa R} = -\mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \cancelto{0}{\frac{1}{r}\frac{\partial u_r}{\partial \theta}} \right]_{r=\kappa R}$$

$$\tau_{\theta r}\Big|_{r=\kappa R} = -\mu\left[r\frac{d}{dr}\left(\frac{u_\theta}{r}\right) \right]_{r=\kappa R}$$

Solution

$$\tau_{\theta r}\Big|_{r=\kappa R} = -\mu\left[r\frac{d}{dr}\left(\frac{u_\theta}{r}\right)\right]_{r=\kappa R}$$

$$\tau_{\theta r}\Big|_{r=\kappa R} = -\mu\left[r\frac{d}{dr}\left(\frac{\frac{\kappa R\Omega}{\kappa - 1/\kappa}\left(\frac{r}{R}-\frac{R}{r}\right)}{r}\right)\right]_{r=\kappa R} \Rightarrow \boxed{\tau_{\theta r}\Big|_{r=\kappa R} = \frac{2\mu\Omega}{1-\kappa^2}}$$

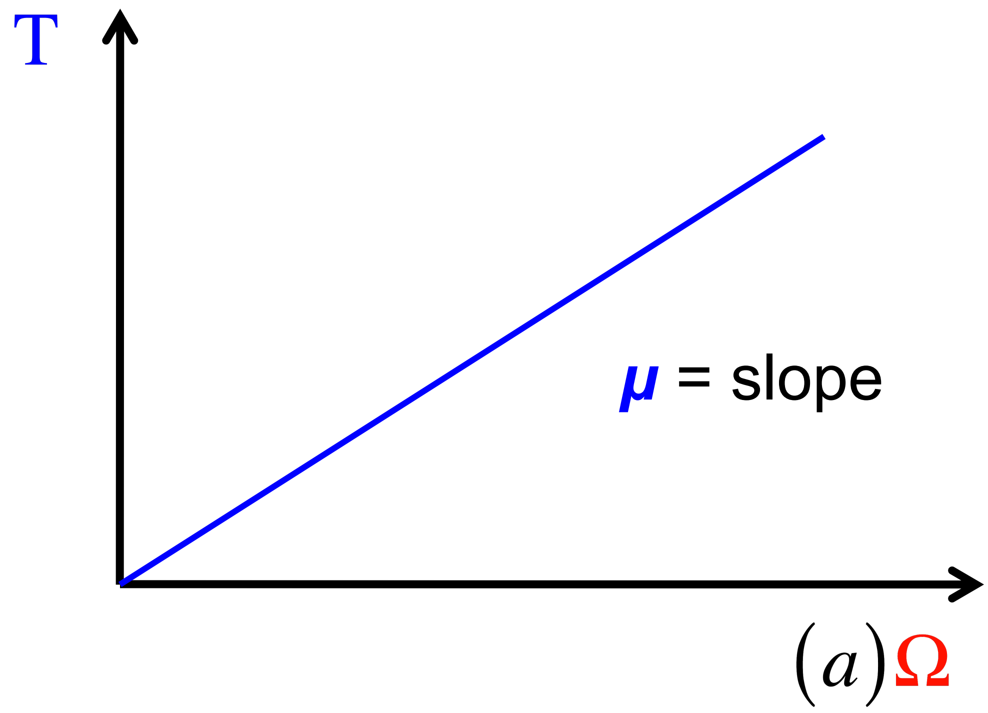
The torque, T, is defined as:

$$\mathrm{T} = \tau_{\theta r}\Big|_{r=\kappa R}\cdot A\cdot\kappa R = \frac{2\mu\Omega}{1-\kappa^2}\cdot 2\pi\kappa RL\cdot\kappa R$$

$$\boxed{\mathrm{T} = \mu\left(\frac{4\pi L\kappa^2 R^2\Omega}{1-\kappa^2}\right)}$$

Solution

$$\mathrm{T} = \mu\left(\frac{4\pi L\kappa^2 R^2\Omega}{1-\kappa^2}\right) = \mu\left(\frac{4\pi L\kappa^2 R^2}{1-\kappa^2}\right)\Omega = \mu(a)\Omega$$



Solution - Centrifugal Pressure P_r

$$\int_{P_{\kappa R}}^{P_r} dP = P_r - P_{\kappa R} = \rho \int_{\kappa R}^{r} \frac{u_\theta^2}{r} dr = \rho \int_{\kappa R}^{r} \frac{1}{r} \left[\frac{\kappa^2 R\Omega}{(\kappa^2 - 1)} \left(\frac{r}{R} - \frac{R}{r} \right) \right]^2 dr$$

$$P_r = P_{\kappa R} + \rho \int_{\kappa R}^{r} \frac{u_\theta^2}{r} dr = P_{\kappa R} + \rho \int_{\kappa R}^{r} \frac{1}{r} \left[\frac{\kappa^2 R\Omega}{(\kappa^2 - 1)} \left(\frac{r}{R} - \frac{R}{r} \right) \right]^2 dr$$

$$P_r = P_{\kappa R} + \rho \frac{\kappa^2 R\Omega}{(\kappa^2 - 1)} \int_{\kappa R}^{r} \left[\frac{r}{R^2} - \frac{2}{r} + \frac{R^2}{r^3} \right] dr =$$

$$P_r = P_{\kappa R} + \rho \frac{\kappa^2 R\Omega}{(\kappa^2 - 1)} \left[\frac{r^2}{2R^2} - 2\ln(r) - \frac{R^2}{2r^2} \right]_{\kappa R}^{r} =$$

$$P_r = P_{\kappa R} + \rho \frac{\kappa^2 R\Omega}{(\kappa^2 - 1)} \left[\frac{r^2}{2R^2} - 2\ln(r) - \frac{R^2}{2r^2} - \frac{\kappa^2}{2} + 2\ln(\kappa R) + \frac{1}{2\kappa^2} \right]$$

Solution - Centrifugal Pressure P_r

$$P_r = P_{\kappa R} + \rho \frac{\kappa^2 R\Omega}{\left(\kappa^2 - 1\right)}\left[\frac{r^2}{2R^2} - 2\ln(r) - \frac{R^2}{2r^2} - \frac{\kappa^2}{2} + 2\ln(\kappa R) + \frac{1}{2\kappa^2}\right]$$

$$P_{r=\kappa R} = P_{\kappa R} + \rho \frac{\kappa^2 R\Omega}{\left(\kappa^2 - 1\right)}\left[\cancel{\frac{\kappa^2}{2}} \cancel{- 2\ln(\kappa R)} \cancel{- \frac{1}{2\kappa^2}} \cancel{- \frac{\kappa^2}{2}} \cancel{+ 2\ln(\kappa R)} \cancel{+ \frac{1}{2\kappa^2}}\right]$$

$$\boxed{P_{r=\kappa R} = P_{\kappa R}}$$

$$P_{r=R} = P_{\kappa R} + \rho \frac{\kappa^2 R\Omega}{\left(\kappa^2 - 1\right)}\left[\frac{1}{2} - 2\ln(R) - \frac{1}{2} - \frac{\kappa^2}{2} + 2\ln(\kappa R) + \frac{1}{2\kappa^2}\right]$$

$$P_{r=R} = P_{\kappa R} + \rho \frac{\kappa^2 R\Omega}{\left(\kappa^2 - 1\right)} \frac{1}{2\kappa^2}\left[-\kappa^4 + 4\kappa^2 \ln(\kappa) + 1\right]$$

$$\boxed{P_{r=R} = P_{\kappa R} + \rho \frac{R\Omega}{2\left(\kappa^2 - 1\right)}\left[1 + 4\kappa^2 \ln(\kappa) - \kappa^4\right]}$$

Solution - Centrifugal Pressure P_r