Oregon State University

OREGON STATE UNIVERSITY-CBEE
DEPARTMENT OF CHEMICAL ENGINEERING

CHE 331
Transport Phenomena I

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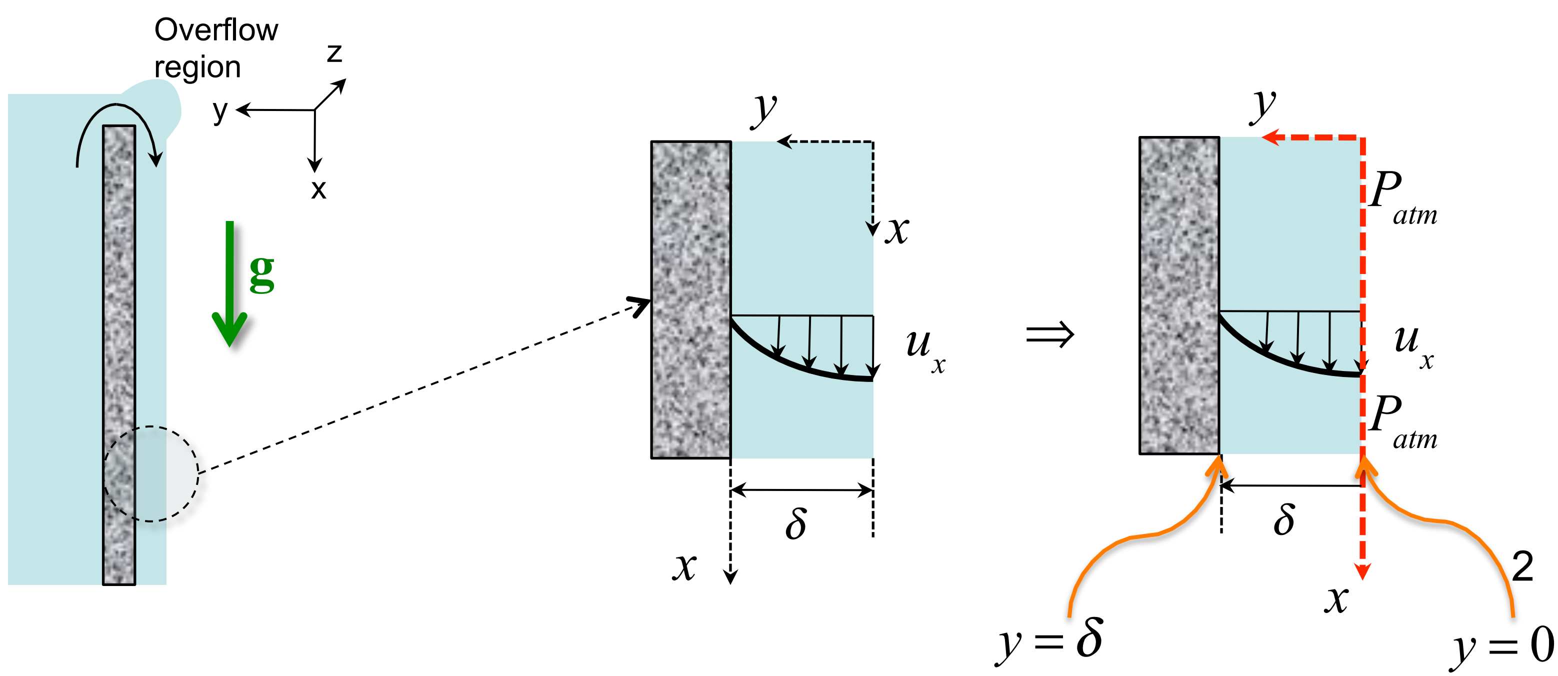
Application of Navier-Stokes Equations

Please turn-off cell phones

A viscous Newtonian fluid is flowing under gravity only, down a vertical plate as shown in the illustration below.

Liquid is bounded on one side by solid surface, and on the other side by a free surface. Flow is driven by a body force (gravity), not by imposition of a moving surface or pressure difference.

Derive expressions for: i) a steady state velocity profile $u_x(y)$, and ii) for volumetric flow rate Q $[m^3/s]$.



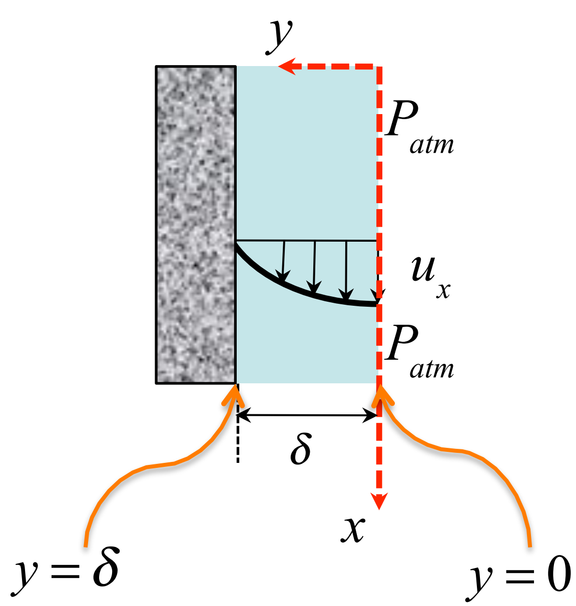
- **Steady state has been reached;**
- **The liquid is incompressible, Newtonian, and isothermal;**
- **The flow is laminar and unidirectional;**
- **The liquid film thickness δ is not function of x;**
- **There is no shear (no momentum exchange) between liquid and air**

$$\boxed{u_z = 0; \quad \frac{\partial u_z}{\partial x} = 0; \quad \frac{\partial u_z}{\partial y} = 0; \quad \frac{\partial u_z}{\partial z} = 0}$$

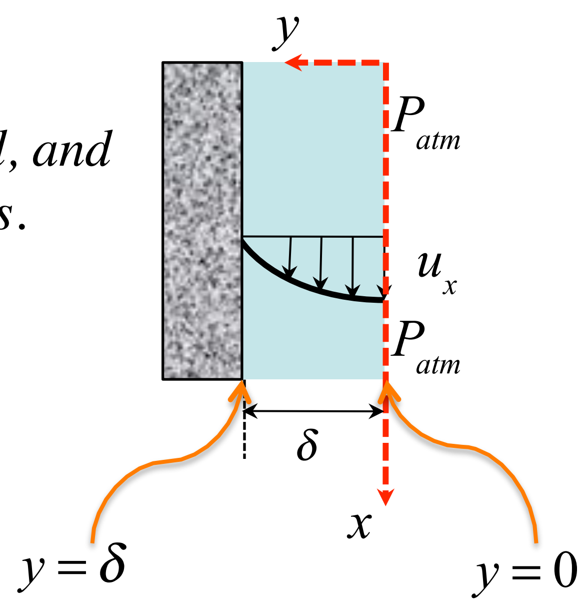
$$\boxed{u_y = 0; \quad \frac{\partial u_y}{\partial x} = 0; \quad \frac{\partial u_y}{\partial y} = 0; \quad \frac{\partial u_y}{\partial z} = 0}$$

$$\boxed{u_x \neq 0; \quad \frac{\partial u_x}{\partial x} = 0; \quad \frac{\partial u_x}{\partial z} = 0; \quad \frac{\partial u_x}{\partial y} \neq 0}$$

$$\boxed{\frac{\partial P}{\partial z} = 0; \quad \frac{\partial P}{\partial y} = 0; \quad \frac{\partial P}{\partial x} = 0}$$



We Start from the Navier-Stokes equations for a fluid, and establish and apply appropriate boundary conditions.



In X direction:

$$\rho\left[\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}+u_z\frac{\partial u_x}{\partial z}\right]=-\frac{\partial P}{\partial x}+\mu\left[\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right]+\rho g_x$$

In Y direction:

$$\rho\left[\frac{\partial u_y}{\partial t}+u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}+u_z\frac{\partial u_y}{\partial z}\right]=-\frac{\partial P}{\partial y}+\mu\left[\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}+\frac{\partial^2 u_y}{\partial z^2}\right]+\rho g_y$$

In Z direction:

$$\rho\left[\frac{\partial u_z}{\partial t}+u_x\frac{\partial u_z}{\partial x}+u_y\frac{\partial u_z}{\partial y}+u_z\frac{\partial u_z}{\partial z}\right]=-\frac{\partial P}{\partial z}+\mu\left[\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_z}{\partial y^2}+\frac{\partial^2 u_z}{\partial z^2}\right]+\rho g_z$$

For Newtonian fluid, and incompressible, isothermal, laminar, one dimensional, fully developed, and steady state flow we can obtain from Navier-Stokes equation:

In ***X*** direction:

$$\rho\left[\cancel{\frac{\partial u_x}{\partial t}}+u_x\cancel{\frac{\partial u_x}{\partial x}}+\cancel{u_y}\frac{\partial u_x}{\partial y}+\cancel{u_z\frac{\partial u_x}{\partial z}}\right]=-\cancel{\frac{\partial P}{\partial x}}+\mu\left[\cancel{\frac{\partial^2 u_x}{\partial x^2}}+\frac{\partial^2 u_x}{\partial y^2}+\cancel{\frac{\partial^2 u_x}{\partial z^2}}\right]+\rho g_x$$

$$\boxed{0=\mu\frac{\partial^2 u_x}{\partial y^2}+\rho g_x}$$

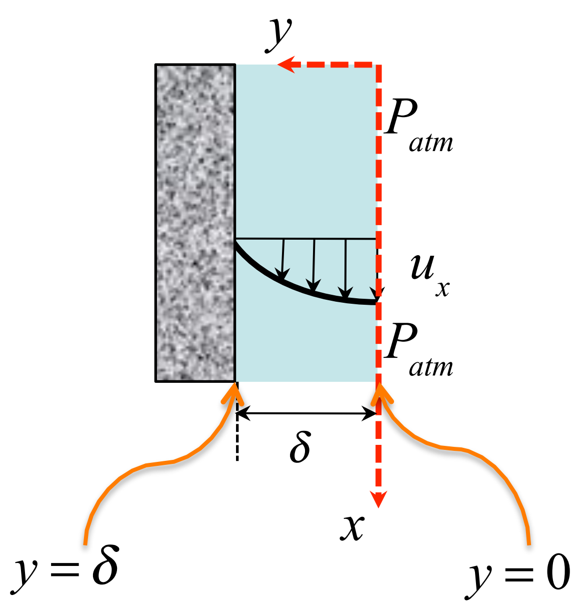
Similarly in ***Y*** direction:

$$\rho\left[\cancel{\frac{\partial u_y}{\partial t}}+u_x\cancel{\frac{\partial u_y}{\partial x}}+u_y\cancel{\frac{\partial u_y}{\partial y}}+u_z\cancel{\frac{\partial u_y}{\partial z}}\right]=-\cancel{\frac{\partial P}{\partial y}}+\mu\left[\cancel{\frac{\partial^2 u_y}{\partial x^2}}+\cancel{\frac{\partial^2 u_y}{\partial y^2}}+\cancel{\frac{\partial^2 u_y}{\partial z^2}}\right]+\cancel{\rho g_y}$$

$$\boxed{0=0}$$

And finally in ***Z*** direction:

$$\cancel{\rho\left[\frac{\partial u_z}{\partial t}+u_x\frac{\partial u_z}{\partial x}+u_y\frac{\partial u_z}{\partial y}+u_z\frac{\partial u_z}{\partial z}\right]=-\frac{\partial P}{\partial z}+\mu\left[\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_z}{\partial y^2}+\frac{\partial^2 u_z}{\partial z^2}\right]+\rho g_z}$$



$$\boxed{\begin{array}{l} @\ y=0 \quad \Rightarrow \quad \tau_{xy}(0)=-\mu\left(\frac{\partial u_x}{\partial y}\right)_{y=0}=0 \\ \text{if } \tau_{xy}(0)=0 \quad \Rightarrow \quad \left(\frac{\partial u_x}{\partial y}\right)_{y=0}=0 \end{array}}$$

In summary:

$$\boxed{0=\mu\frac{\partial^2 u_x}{\partial y^2}+\rho g_x}$$

$$\boxed{@\ y=\delta \quad \Rightarrow \quad u_x=0}$$

$$\boxed{@\ y=0 \quad \Rightarrow \quad \left(\frac{\partial u_x}{\partial y}\right)_{y=0}=0}$$

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$$\boxed{\mu \frac{\partial^2 u_x}{\partial y^2} = -\rho g_x} \Rightarrow \frac{\partial^2 u_x}{\partial y^2} = -\frac{\rho g_x}{\mu} \Rightarrow \boxed{\frac{\partial u_x}{\partial y} = -\frac{\rho g_x}{\mu} y + C_1}$$

$$\Downarrow$$

$$\boxed{@\ y = 0 \quad \Rightarrow \left(\frac{\partial u_x}{\partial y}\right)_{y=0} = 0}$$

$$\Downarrow$$

$$\boxed{C_1 = 0}$$

$$\boxed{\frac{\partial u_x}{\partial y} = -\frac{\rho g_x}{\mu} y + \cancel{C_1}}$$

$$\Downarrow$$

$$\boxed{u_x = -\frac{\rho g_x}{2\mu} y^2 + C_2}$$

$$\Downarrow$$

$$\boxed{@\ y = \delta \quad \Rightarrow \quad u_x = 0}$$

$$\Downarrow$$

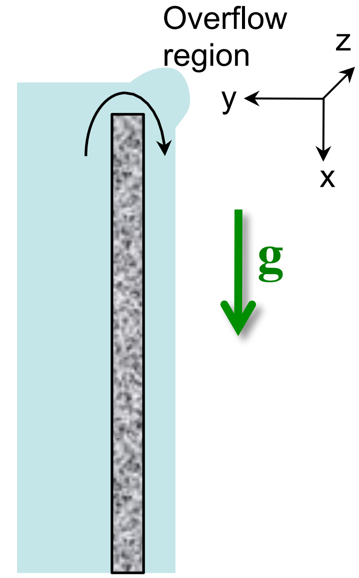
$$\boxed{C_2 = \frac{\rho g_x}{2\mu}\delta^2} \Rightarrow \boxed{u_x = -\frac{\rho g_x}{2\mu} y^2 + \frac{\rho g_x}{2\mu}\delta^2}$$

$$\Uparrow$$

$$\boxed{\boxed{u_x = \frac{\rho g_x}{2\mu}\left(\delta^2 - y^2\right)}}$$

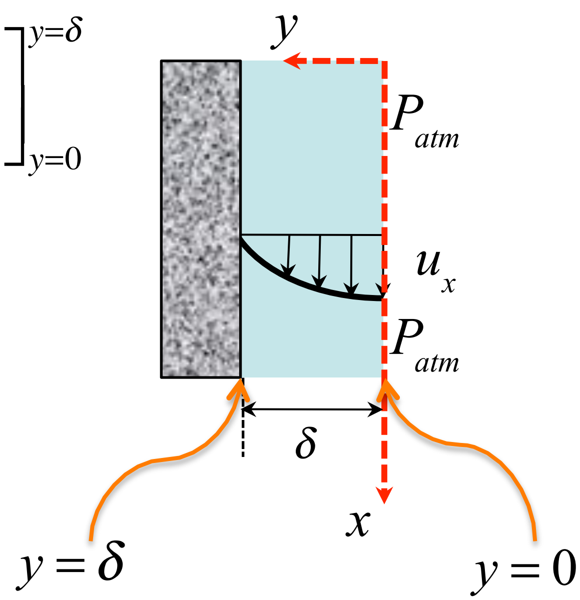
Finally we can obtain expression for the volumetric flow rate Q

$$Q = \int u_x \, dA \quad \Rightarrow \quad Q = W \int_{y=0}^{y=\delta} u_x \, dy$$



$$Q = W \int_{y=0}^{y=\delta} \frac{\rho g_x}{2\mu}\left(\delta^2 - y^2\right) dy = \frac{W \rho g_x}{2\mu} \int_{y=0}^{y=\delta} \left(\delta^2 - y^2\right) dy$$

$$Q = \frac{W \rho g_x}{2\mu} \int_{y=0}^{y=\delta} \left(\delta^2 - y^2\right) dy = \frac{W \rho g_x}{2\mu} \left[y\delta^2 - \frac{1}{3} y^3 \right]_{y=0}^{y=\delta}$$



Finally:

$$\boxed{Q = \frac{\rho g_x W}{3\mu} \delta^3}$$

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Thank you for your attention!