F-BUNDLES AND BLOWUPS

NOTES ON A SERIES OF TALKS BY TONY YUE YU
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Abstract. From the abstract: "I will discuss the basic ideas and properties of F-bundles and non-commutative Hodge structures, as well as applications to birational geometry. Joint work with Katzarkov, Kontsevich and Pantev."

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Introduction

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1. Review of Gromov–Witten theory

For the time being, we work with smooth, projective varieties over $\mathbf{C}$.

Example 1.1. Let N_d be the number of rational curves in $\mathbf{P}^2$ of degree d, passing through $3d-1$ points in general position.

$$N_d = 1, 1, 12, 620, 87304, 26312976, 14616808192, \ldots$$

Some of the numbers are classical, but it turns out that in general, one may compute them using Kontsevich's recursive formula:

$$N_d = \sum_{d_1+d_2} N_{d_1} N_{d_2} d_1^2 d_2 \left(\binom{3d-4}{3d_1-2} - d_1 \binom{3d-4}{3d_1-1} \right).$$

To set up any curve-counting theory, there are two issues to take care of:

(1) Compactness
(2) Transversality

Different strategies for solving these issues lead to different curve-counting theories. We will use the oldest theory, Gromov–Witten theory, which also happens to be the most general (e.g., it works for varieties of arbitrary dimension).

Gromov–Witten theory. Consider maps

$$f : (C, p_1, \ldots, p_n) \to X,$$

where $(C, p_1, \ldots, p_n)$ is an n-pointed proper nodal curve, and the automorphism group $\mathrm{Aut}(f)$ is required to be finite. Such a map is called a *stable map*.

The moduli stack $\bar{\mathcal{M}} = \bar{\mathcal{M}}_{g,n}(X, \beta)$ of genus g stable maps to X, where $f_*[C] = \beta \in \mathrm{H}_2(X, \mathbf{Z})$. The expected dimension of $\bar{\mathcal{M}}$ is

$$\mathrm{vdim}(\bar{\mathcal{M}}) = (1-g)(\dim X - 3) + \int_\beta c_1(T_X) + n.$$

Then $\bar{\mathcal{M}}$ carries a virtual fundamental class $[\bar{\mathcal{M}}]^{\mathrm{vir}}$ in $\mathrm{CH}_{\mathrm{vdim}}(\bar{\mathcal{M}})$.

Rational Gromov–Witten invariants. Let $\phi_i \in \mathrm{H}^2(X, \mathbf{Q})$. Define

$$\langle \phi_1, \ldots, \phi_n \rangle_\beta = \int_{[\bar{\mathcal{M}}]^{\mathrm{vir}}} \prod_{i=1}^n \mathrm{ev}_i^* \phi_i \in \mathbf{Q},$$

where $\mathrm{ev}_i : \bar{\mathcal{M}} \to X$ is the evaluation at p_i. These numbers satisfy a number of interesting properties:

(1) Symmetry: $\langle \phi_{\sigma_1}, \ldots, \phi_{\sigma_n} \rangle_\beta = (-1)^\sigma \langle \phi_1, \ldots, \phi_n \rangle_\beta$.
(2) Support: $\langle \phi_1, \ldots, \phi_n \rangle_\beta \neq 0$ implies that $\beta = 0$ or β is an effective curve class.
(3) Constant maps: $\langle \phi_1, \ldots, \phi_n \rangle_0 \neq 0$ implies that $n = 3$, and

$$\langle \phi_1, \phi_2, \phi_3 \rangle_0 = \int_{[X]} \phi_1 \cup \phi_2 \cup \phi_2.$$

(4) Dimension: $\langle \phi_1, \ldots, \phi_n \rangle_\beta \neq 0$ implies $\sum \deg \phi_i = 2\mathrm{vdim}$.
(5) Unit: $\langle 1_X, \phi_2 \ldots, \phi_n \rangle_\beta = 0$ if $\beta \neq 0$.
(6) Divisor: $\langle \phi_1, \ldots, \phi_n, \phi_{n+1} \rangle_\beta = \langle \phi_1, \ldots, \phi_n \rangle_\beta \cdot \int_\beta \phi_{n+1}$ if $\deg \phi_{n+1}$ and $n \geq 3$.
(7) WDVV:

$$\sum_{\beta' + \beta'' = \beta} \sum_{I' \cup I'' = \{5, \ldots, n\}} \sum_i \langle \phi_1, \phi_2, \phi_{I'}, T_i \rangle_{\beta'} \cdot \langle \phi_5, \phi_4, \phi_{I''}, T^i \rangle_{\beta''}$$

is symmetry under swapping ϕ_2 and ϕ_3. Here, T_i and T^i are dual bases of $\mathrm{H}^*(X, \mathbf{Q})$.

Gromov–Witten potential. Choose a basis T_i of $\mathrm{H}^*(X, \mathbf{Q})$. The *Gromov–Witten potential* is given by

$$\Phi = \sum_{n \geq 0, \beta} \frac{q^\beta}{n!} \sum_{i_1, \ldots, i_n} \langle T_{i_1}, \ldots, T_{i_n} \rangle_\beta \in \mathbf{Q}[[\mathrm{NE}(X, \mathbf{Z})]][[\{t_i\}]].$$

We denote the coefficient ring by R. The potential leads to the quantum product

$$H \otimes H \to H \otimes R, \quad T_i \star T_j := \sum_k \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial k} T^k.$$

Then WDVV implies that $\star$ is associative. In fact, WDVV is equivalent to the associativity of the quantum product.

Conjecture 1.2. *The series Φ is convergent for small $|q|, |t_i|$.*

2. nc-Hodge structures and mirror symmetry

nc-Hodge structures. We package Gromov–Witten theory further, with the goal of bringing in ideas from mirror symmetry. A reference for some of the following material is [**?**].

Definition 2.1. A *rational pure nc-Hodge structure* consists of a tuple (H, E_B, iso), where:

- H is a $\mathbf{Z}/2$-graded algebraic vector bundle on $\mathbf{A}^1$ with coordinate u,
- E_B is a local system of finite-dimensional $\mathbf{Q}$-vector spaces on $\mathbf{A}^1 - 0$,
- iso is an isomorphism

$$E_B \simeq \mathcal{O}^{\mathrm{an}}_{\mathbf{A}^1 - 0} \simeq H^{\mathrm{an}}|_{\mathbf{A}^1 - 0},$$

  so that H^{an} inherits a flat holomorphic connection ∇ on $\mathbf{A}^1 - 0$.

such that the following conditions hold:

(1) nc-filtration axiom: ∇ is meromorphic on H with a pole of order ≤ 2 at $u = 0$, and a regular singularity at ∞.
(2) $\mathbf{Q}$-structure axiom: $\mathcal{E}_B$ is compatible with Stokes data.
(3) Opposedness axiom (omitted).

Example 2.2 (Classical Hodge structures)**.** Let $(V, \mathrm{F}^\bullet V, V_{\mathbf{Q}})$ be a pure $\mathbf{Q}$-Hodge structure of weight w. Then the corresponding nc-Hodge structure is given as follows:

- $H = \sum u^{-1} \mathrm{F}^i V[u]$, considered as a submodule of $\mathbf{C}[u, u^{-1}] \otimes V$.
- The connection is given by

$$\nabla = d - \frac{w}{2} \frac{du}{u}.$$

- The local system is the extension of $V_{\mathbf{Q}}$, which we think of as a vector subspace of the fiber of H at $1 \in \mathbf{A}^1$, to a local system in H by parallel transport.

In general, meromorphic connections with poles of order 2 are difficult, so we focus instead on a class where the pole is reasonable:

Definition 2.3. An nc-Hodge structure is of *exponential type* with exponents $c_1, \ldots, c_n$ if, after tensoring with $\mathbf{C}((u))$, it splits into a direct sum

$$\bigoplus (\mathcal{E}^{c_i/u} \otimes R_i),$$

where $\mathcal{E}^{c_i/u}$ is a rank one vector bundle with connection $d - d\left(\frac{c_i}{u}\right)$, and R_i has regular singularities.

Variation of nc-Hodge structures. There is a notion of a variation of nc-Hodge structure. We omit the precise definition. Suffice it to say that the definition consists of the following ingredients:

- A $\mathbf{Z}/2$-graded vector bundle on $\mathbf{A}^1 \times B$, which is algebraic in the $\mathbf{A}^1$ direction.
- A local system E_B of $\mathbf{Z}/2$-graded vector spaces on $\mathbf{G}_m \times B$.
- An isomorphism
$$\mathrm{iso} : E_B \otimes \mathcal{O}_{\mathbf{G}_m \times S} \simeq H_{\mathbf{G}_m \times S},$$

satisfying several conditions:

(1) nc-filtration axiom
(2) Griffiths transversality
(3) $\mathbf{Q}$-structure axiom
(4) Opposedness

A-model nc-Hodge structure. Let $B = \mathrm{Spf}(\mathbf{Q}[[\mathrm{NE}(X, \mathbf{Z})]][[\{t_i\}]])$, which serves as the base of the variation. Let H be the trivial bundle over $B \times \mathbf{A}^1_u$ with fiber $\mathrm{H} = \mathrm{H}^*(X, \mathbf{Q})$. We take ∇ to be the *quantum connection*:

$$\nabla = \begin{cases} \nabla_{\partial_u} = \partial_u + \frac{1}{u^2} K + \frac{1}{u} G \\ \nabla_{\partial_{t_i}} = \partial_{t_i} + \frac{1}{u} A_i \\ \nabla_{q_j \partial_{q_j}} = q_j \partial_{q_j} + \frac{1}{u} A_j \end{cases}$$

Here, when we write q_j, we mean the variable q^β associated that an effective curve class in a chosen basis. The quantum operators are:

$$\begin{aligned} K &= \left(K_X + \sum_{i : \deg T_i \neq 2} \frac{\deg T_i - 2}{2} t_i T_i \right) \star (-) \\ G &= \sum_{i=0}^{\dim X} \frac{i - \dim X}{2} \mathrm{id}_{\mathrm{H}^i(X, \mathbf{Q})} \\ A_i &= T_i \star (-). \end{aligned}$$

These operators preserve the $\mathbf{Z}/2$-grading.

Conjecture 2.4. *(H, ∇) has exponential type.*

Let us assume convergence (Conjecture 1.2), and explain some of the other aspects of the (conjectural) variation. The $\mathbf{Q}$-structure is induced by the image

$$\mathrm{H}^k(X, \mathbf{Q}) \xrightarrow{(2\pi i)^{k/2}} \mathrm{H}^k(X, \mathbf{C}) \xrightarrow{\hat{\Gamma}(X) \wedge -} \mathrm{H}^k(X, \mathbf{C}),$$

where

$$\hat{\Gamma}(X) = \prod \Gamma(1+\lambda_i)$$
$$= \exp\left(\gamma \cdot \mathrm{ch}_i(T_X)) + \sum_{n\geq 2} \frac{\zeta(n)}{n} \mathrm{ch}_n(T_X)\right).$$

Here, λ_i are the Chern roots of T_X.

What is γ?

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Conjecture 2.5 (Gamma conjecture)**.** *For this specific* $\mathbf{Q}$*-structure, the* $\mathbf{Q}$*-structure axioms and opposedness axioms for a variation of nc-Hodge structure are satisfied.*

B-model nc-Hodge structures. We now explain the B-model nc-Hodge structure associated to a Landau–Ginzburg model: Y is smooth and quasi-projective over $\mathbf{C}$, and $w : Y \to \mathbf{A}^1$ is a function such that $\mathrm{Crit}(f)$ is proper.

The twisted de Rham cohomology is

$$\mathcal{H} = \mathrm{R}\Gamma(Y, \Omega_Y^\bullet[u], ud - (dw \wedge -))$$

It is a difficult theorem of Sabbah et. al. that this is a $\mathbf{Z}/2$-graded vector bundle over $\mathbf{A}^1$. The connection is the Gauss–Manin connection

$$\nabla = \partial_u - \frac{w}{u^2}.$$

Finally, the $\mathbf{Q}$-structure is given by *rapid decay cohomology*:

$$E_\theta = \mathrm{H}^*(Y, w^{-1}(Y_\theta)).$$

Here, θ is an angle, and $w^{-1}(Y_\theta)$ is a fiber over a point of angle θ, sufficiently far from the origin.

Conjecture 2.6. *With the data above, one has a nc-Hodge structure.*

The main difficulty seems to be opposedness.

Hodge-theoretic mirror symmetry. Mirror symmetry is usually thought of as a duality between Calabi–Yau varieties of the same dimension:

$$X \iff \check{X}$$

Nowadays, we can take X to be either Calabi–Yau or Fano, and the mirror to be a Landau–Ginzburg model $(Y, w : Y \to \mathbf{C})$. Using nc-Hodge structures, one obtains a precise version of mirror symmetry:

Conjecture 2.7 (Hodge-theoretic mirror symmetry)**.** *The A-model nc-Hodge structure for* X *is isomorphic to the B-model nc-Hodge structure for* (Y, w)*.*

The fact that either side is an nc-Hodge structure in the first place is conjectural, as indicated above.

Remark 2.8. Under the conjecture, the critical values $\mathrm{Crit}(f)$ correspond to the eigenvalues of the quantum operator K.

A different version of mirror symmetry is the homological mirror symmetry conjecture of Kontsevich, which predicts that there is an equivalence of categories

$$\mathrm{D^b}(X) \simeq \mathrm{FS}(Y, w).$$

The right-hand side has a *natural* semiorthogonal decomposition up to mutation, which suggests that the left-hand side should also have a *natural* semiorthogonal decomposition up to mutation.

The main takeaway is the following series of implications:

$$\begin{array}{c} \text{Decomposition of B-model nc-Hodge structure} \\ \rightsquigarrow \\ \text{Decomposition of A-model nc-Hodge structure} \\ \rightsquigarrow \\ \text{Decomposition of classical Hodge structures of } X \end{array}$$

You might say this is on very shaky ground, since it is built on a web of difficult conjectures. To return to firmer (and more scientific) ground, we discard the categorical point of view and working directly with differential equations.

2.1. **Analytic decomposition via the Fourier transform.** We have the following diagram:

$$\begin{tikzcd} (1) \arrow[r, "\text{Fourier}"] \arrow[d, "\text{irr RH}"] & (2) \arrow[d, "RH"] \\ (3) \arrow[r, "\text{top. RH}"] & (4). \end{tikzcd}$$

where:

(1) de Rham data: An algebraic vector bundle $\mathcal{H}$ with connection ∇ on $\mathbf{G}_m$, singularities of exponential type at 0, regular singularities at ∞.
(2) regular holonomic D-module on $\mathbf{A}^1$ with regular sing, and vanishing de Rham cohomology.
(3) $\mathbf{Q}$-Stokes structure (I, I_S) of exponential type.
(4) Constructible sheaf F of $\mathbf{Q}$-vector spaces on $\mathbf{A}^1$ such that $\mathrm{R}\Gamma(F) = 0$.

Comments:

- Having exponential type at 0, reg. sing. at ∞ is equivalent to having regular singularities after Fourier transform.
- The condition that (2) has vanishing de Rham cohomology implies that the corresponding perverse sheaf is concentrated in a single degree.

Definition 2.9. A *vanishing cycle decomposition* of F is a collection of finite-dimensional $\mathbf{Q}$-vector spaces $U_1, \dots, U_n$, and a collection of maps $T_{ij} : U_i \to U_j$. Here, $U_i = F_{c_0}/F_{c_i}$, where we have chosen some points $c_0, c_1, \dots, c_n$.

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Incomplete. This is supposed to be the generalization of the classical picture of a perverse sheaf on a disc.

Theorem 2.10 (Analytic decomposition theorem, KKP). *An nc-Hodge structure of exponential type is equivalent to the following data:*

- *a finite set* $\{c_1, \dots, c_n\} \subset \mathbf{C}$
- *a collection of nc-Hodge structures* (H_i, E_B, iso)*,* $i = 1, \dots, n$
- *Gluing data and maps* $T_{ij} : (E_{B,j})_{c_o} \to (E_{B,i})_{c_0}$*.*

3. F-bundles

From here on out, there will be no more conjectures—only theorems. The idea here is to ignore both $\mathbf{Q}$-structure and issues of convergence. It turns out that the trick for this is to work over a non-archimedean field.

We begin with a smooth rigid k-analytic variety, where k is an algebraically closed non-archimedean field. Let $\mathbf{D}$ be the germ of $0 \in \mathbf{A}^1_u$.

Definition 3.1. An *F-bundle* $(H, \nabla)/B$ consists of a vector bundle H over $B \times \mathbf{D}$ with a meromorphic flat connection ∇, such that:

(1) ∇_{∂_u} has a pole of order ≤ 2 along $u = 0$,
(2) For any tangent vector field ξ in B, ∇_ξ has a pole of order ≤ 1 along $u = 0$.

Example 3.2. Here is an example:

$$\nabla_{\partial_{t_i}} = \frac{1}{u}\begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, \quad \nabla_{\partial_u} = \partial_u - \frac{1}{u^2}\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}$$

For any $b \in B$, we obtain a map

$$\mu_b : T_b B \to \mathrm{End}(H_{b,0}), \quad v \mapsto \nabla_{uv}|_{H_{b,0}}$$

Flat implies that the image consists of commuting operators.

Definition 3.3. An F-bundle (H, D) is *overmaximal* (resp., *maximal*) at $b \in B$ if there exists $h \in H_{b,0}$ such that

$$T_b B \to H_{b,0}, \quad v \mapsto \mu_b(v)h$$

is an epimorphisms (resp., isomorphism).

If the F-bundle is maximal, then there is a commutative product on the tangent bundle TB given by

$$\mu_b(v_1 \star v_2) = \mu_b(v_2) \circ \mu_b(v_1)(h) \quad \text{quantum product in A-model}$$

We also obtain an Euler vector field Eu on B, such that

$$\mu_b(\mathrm{Eu}) = K_b = \nabla_{u^2\partial_u}|_{u=0}.$$

Here, K_b is the same K as in quantum cohomology.

Example 3.4.

$$\partial_{t_i} \star \partial_{t_j} = \delta_{ij}\partial_{t_i}$$
$$\mathrm{Eu} = \sum_j t_i \partial_{t_i}.$$

3.1. **Nonarchimedean decomposition theorem.** Let $(H, \nabla)/B$ be a maximal F-bundle, and let $b \in B$ be a point. At b, there is a generalized eigenspace decomposition

$$H_{b,0} = \bigoplus E_i. \tag{3.1}$$

Theorem 3.5 (Nonarchimedean decomposition theorem)**.** *Then $(H, \nabla)/B$ locally splits into a product of maximal F-bundles $(H_i, \nabla)/B_i$, extending (3.1).*

Remark 3.6 (Comparison)**.** It is possible to make a comparison with the vanishing cycle decomposition for nc-Hodge structures. One can show that there is a choice of paths from c_i to c_0, such that the associated vanishing cycle decomposition

4. Blowup decomposition

Let X be smooth, projective over $\mathbf{C}$, $Z \subset X$ a smooth subvariety of pure codimension $r \geq 2$, and $\tilde{X} \to X$ the blowup of X along Z. Recall that we have a decomposition of cohomology:

$$\mathrm{H}^*(\tilde{X}) \simeq \mathrm{H}^*(X) \oplus \bigoplus_{i=1}^{r} \mathrm{H}^*(Z)[-2i]$$

One goal might be to extend this to a decomposition of nc-Hodge structures. But this is too conjectural, so we instead try to extend it to a decomposition of F-bundles.

First, we need to care of curve classes, since these are part of the coefficients of F-bundles. To simplify notation, let

Iritani's theorem.

$$\mathbf{Q}[Q] = \mathbf{Q}[[\mathrm{NE}(X, Z)]] = \mathbf{Q}[[Q^d, d \in \mathrm{NE}(X, Z)]]$$
$$\mathbf{Q}[\mathcal{Q}] = \mathbf{Q}[[Q^d, xy^{-1}, Q^{\phi_* \tilde{d}} y^{-[E]\tilde{d}} : d \in \mathrm{NE}(X, \mathbf{Z}), \tilde{d} \in \mathrm{NE}(\tilde{X}, \mathbf{Z})]]$$
$$\mathbf{Q}[[\tilde{Q}]] = \mathbf{Q}[[\mathrm{NE}(\tilde{X}, \mathbf{Z})]]$$
$$\mathbf{Q}[[Q_Z]] = \mathbf{Q}[[\mathrm{NE}(Z, \mathbf{Z})]]$$

Here $\mathrm{NE}(X, Z)$ are the effective curve classes supported on Z, etc.

From embedding everything into $\mathrm{Bl}_{Z \times 0}(X \times \mathbf{P}^1)$, we get embeddings:

$$\mathbf{Q}[[Q]] \to \mathbf{Q}((q^{-1/5}))[[\mathcal{Q}]] \quad s = \begin{cases} r-1 & r \equiv 0 \mod 2 \\ 2(r-1) & r \equiv 1 \mod 2 \end{cases}$$
$$\mathbf{Q}[[\tilde{Q}]] \to \mathbf{Q}((q^{-1/5}))[\mathcal{Q}] \qquad \tilde{Q}^{\tilde{d}} \mapsto Q^{\phi_* d} q^{-[E]\tilde{d}}$$
$$\mathbf{Q}[[Q_Z]] \to \mathbf{Q}((q^{-1/5}))[\mathcal{Q}], \quad Q_Z^d \mapsto Q_Z^d \mapsto Q^{\varphi_* d} q^{-c_1(N_{Z/X})d/(r-1)}$$

Theorem 4.1 (Iritani)**.** *After pullback to $\mathbf{C}((q^{-1/5}))[[\mathcal{Q}]]$, there exists a formal invertible change of variavles*

$$\mathrm{H}^*(\tilde{X}) \mapsto \mathrm{H}^*(X) \oplus \mathrm{H}^*(Z)^{\oplus r-1}, \quad \tau \mapsto (\tau(\tilde{\tau})), \{\zeta_j \cdot (\tilde{\tau})\}_{0 \leq j \leq r-2}$$

defined over $\mathbf{C}((q^{-1/5}))[[\mathcal{Q}]]$, *and an isomorphism of formal F-bundles*

$$(H_{\tilde{X}}, \nabla_{\tilde{X}}) \simeq \tau^*(H_X, \nabla_X) \oplus_{j=0}^{r-2} \zeta_j^*(H_Z, \nabla_Z)$$

This is a really interesting theorem, and is a "packaged" way of answering the question: How do the Gromov–Witten invariants of X change under a blowup?

Non-archimedean F-bundles. Consider $(\tilde{H}, \tilde{\nabla})/\tilde{B}^{\max}$ for $\tilde{X}$, $(H', \nabla')_{B',\max}$. Here,

$$X' = X \sqcup_{r-1} \bigsqcup Z.$$

Theorem 4.2. *There exists unique isomorphism of maximal F-bundles between* $(\tilde{X}, \tilde{\nabla})$ *and* (H', ∇') *over an analytic domain* $(\tilde{U})$ *in* $\tilde{B}^{\max}$, *and the analytic domain* U' *in* $B^{',\max}$. *The union of different choices of* $\tilde{U}$ *is connected and nonempty; same for* U'.

Remark 4.3 ($B^{\max}$)**.** When X is smooth, projective, we had a formal base B_{formal} for the variation of nc-Hodge structure. For $B^{\max}$, we first consider let k be a non-archimedean base field. Then we consider $\mathrm{H}^2(X, k^*)$, which has a valuation map to $\mathrm{H}^2(X, \mathbf{R})$ (taking the valuation of the coefficient). In $\mathrm{H}^2(X, \mathbf{R})$, one has the ample cone. The preimage in $\mathrm{H}^2(X, k^*)$ is denoted B^2. Then

$$B^{\max} = B^2 \times (\text{open unit disc in all } t_i : \deg t_i \neq 2).$$

5. Atoms

Here, the goal is to relate F-bundles with motives. We will work with three different fields. First, K is any field; we will work with varieties over K. Then we consider k, which has characteristic 0; it will be the coefficient field of the cohomology theory. Finally, we will consider an algebraically closed nonarchimedean field, $\mathbf{k}$. Previously, this was denoted k. Sorry!

Let $\mathcal{C}$ be a semisimple neutral Tannakian category over k, so that $\mathcal{C} = \mathrm{Rep}(G)$, for G a pro-reductive group. Assume that G has a central element $\epsilon \in G$ of order 2. Let H^* be a Weil cohomology theory of projective K-varieties, taking values in $\mathcal{C}$, satisfying a *Mumford–Tate normalization* condition:

(1) If $\mathrm{H}^2(\mathbf{P}^1)$ is a trivial rank 1 G-module.
(2) For any smooth, projective K-variety X and any $i \in \mathbf{Z}$, ϵ acts on $\mathrm{H}^i(X)$ by $(-1)^i$.

Example 5.1. Let $k = \mathbf{Q}$, $\mathcal{C}_0$ be the category of pure, polarizable $\mathbf{Q}$-Hodge structures. Then $\mathcal{C}_0 = \mathrm{Rep}(G_0)$, where G_0 is the Mumford–Tate group. There is a natural homomorphism from G_0 to $\mathbf{G}_m$ corresponding to the action of G_0 on $\mathrm{H}^2(\mathbf{P}^1)$; the kernel G satisfies the conditions above. Then ϵ comes from the Deligne torus.

Example 5.2. One can take $C = \mathrm{Rep}(G)$ to be Andre's category of motivated cycles (so $\mathrm{H}^*(X)^G$ is the subgroup spanned by motivated cycles).

The proof.

Definition 5.3. Let B be the germ of a smooth $\mathbf{k}$-analytic space at a rigid point (a point corresponding to a maximal ideal). A G-equivariant maximal F-bundle is called a G-atom if the action by the Euler field Eu has a single eigenvalue. (Eu is the residue of the second order pole of the connection.)

Two G-atoms are *equivalent* if they come from a G-equivariant F-bundle over a connected base (a connected smooth $\mathbf{k}$-analytic space).

For any G-equivariant maximal F-bundle $(H, \nabla)/B$, then we consider the locus $B_0 \subset B^G$ where the number of distinct eigenvalues of Eu is maximal. Finally, we apply Theorem 3.5: Consider the finite étale covering $\mathbf{B} \to B_0$. Then each connected component of $\mathbf{B}$ gives an equivalence class of G-atom.

We can look at a coarser invariant. Each G-atom gives an isomorphism class of finite-dimensional G-representation.

Let G be in examples 1 or 2. Then G acts on non-archimedean A-model F-bundle $(H, \nabla)/B$ associated to a smooth, projective variety X, $\mathrm{Atom}(X)$. The blowup decomposition theorem says that

$$\mathrm{Atom}(\tilde{X}) = \mathrm{Atom}(X) + \sum_{r-1} \mathrm{Atom}(Z).$$

This is a sum of multisets.

Theorem 5.4. *If K_X is nef, then* $\mathrm{Atom}(X)$ *is a singleton.*

Consider a general cubic 4-fold. The eigenvalues of Eu at a specific point are given by:

$$\begin{array}{ccccc} 1 & & & & \\ 1 & & & & \\ 1 & + & 1 & 22 & 1 \\ 1 & & & & \\ 1 & & & & \end{array}$$

$\dim V = 24$, $\dim V^G = 2$. This is computed at a specific point.

When we go to a general point, V may split further into $V' \oplus \cdots$. But

$$\dim V'^{p-q=2} \geq 1, \quad \dim V'^G \leq 2.$$

But the claim is that such a V' does not appear in the F-bundle of any variety S with $\dim S \leq 2$. This is because for surfaces with $h^{2,0} \neq 0$, there is a birational model with K_S nef. Then $\mathrm{Atom}(S)$ is a singleton, and $\dim V'^G \geq 3$.