F-BUNDLES AND BLOWUPS

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Abstract. "I will discuss the basic ideas and properties of F-bundles and non-commutative Hodge structures, as well as applications to birational geometry. Joint work with Katzarkov, Kontsevich and Pantev."

Introduction

Here was the initial plan. In the end, the last topic was not covered.

(1) Review of Gromov–Witten theory
(2) nc-Hodge structures and mirror symmetry
(3) F-bundles
(4) Blowup decomposition
(5) Atoms
(6) Framing and uniqueness

1. Review of Gromov–Witten theory

For the time being, we work with smooth, projective varieties over $\mathbf{C}$.

Example 1.1. Let N_d be the number of rational curves in $\mathbf{P}^2$ of degree d, passing through $3d-1$ points in general position.

$$N_d = 1, 1, 12, 620, 87304, 26312976, 14616808192, \ldots$$

The first few numbers are classical, but it turns out that in general, one may compute them using Kontsevich's recursive formula:

$$N_d = \sum_{d_1+d_2=d} N_{d_1} N_{d_2} d_1^2 d_2 \left(\binom{3d-4}{3d_1-2} - d_1 \binom{3d-4}{3d_1-1} \right).$$

The formula doesn't look symmetric in d_1 and d_2, but it turns out to be.

To set up any curve-counting theory, there are two issues to take care of:

(1) Compactness
(2) Transversality

Different strategies for solving these issues lead to different curve-counting theories. We will use the oldest theory, Gromov–Witten theory, which also happens to be the most general (e.g., it works for varieties of arbitrary dimension).

Gromov–Witten theory. Consider maps

$$f : (C, p_1, \dots, p_n) \to X,$$

where $(C, p_1, \dots, p_n)$ is an n-pointed proper nodal curve, and the automorphism group $\mathrm{Aut}(f)$ is required to be finite. Such a map is called a *stable map.*

We consider the moduli stack $\bar{\mathcal{M}} = \bar{\mathcal{M}}_{g,n}(X, \beta)$ of genus g stable maps to X, where $f_*[C] = \beta \in \mathrm{H}_2(X, \mathbf{Z})$. It turns out that $\bar{\mathcal{M}}$ is a proper DM stack. The expected dimension of $\bar{\mathcal{M}}$ is

$$\mathrm{vdim}(\bar{\mathcal{M}}) = (1-g)(\dim X - 3) + \int_\beta c_1(T_X) + n.$$

Then $\bar{\mathcal{M}}$ carries a virtual fundamental class $[\bar{\mathcal{M}}]^{\mathrm{vir}}$ in $\mathrm{CH}_{\mathrm{vdim}}(\bar{\mathcal{M}}, \mathbf{Q})$.

Rational Gromov–Witten invariants. Let $\phi_i \in \mathrm{H}^*(X, \mathbf{Q})$. Define

$$\langle \phi_1, \dots, \phi_n \rangle_\beta = \int_{[\bar{\mathcal{M}}]^{\mathrm{vir}}} \prod_{i=1}^n \mathrm{ev}_i^* \phi_i \in \mathbf{Q},$$

where $\mathrm{ev}_i : \bar{\mathcal{M}} \to X$ is the evaluation at p_i. These numbers satisfy a number of interesting properties:

(1) Symmetry: $\langle \phi_{\sigma_1}, \dots, \phi_{\sigma_n} \rangle_\beta = \mathrm{sgn}(\sigma) \langle \phi_1, \dots, \phi_n \rangle_\beta$, for a permutation σ.
(2) Support: $\langle \phi_1, \dots, \phi_n \rangle_\beta \neq 0$ implies that $\beta = 0$ or β is an effective curve class.
(3) Constant maps: $\langle \phi_1, \dots, \phi_n \rangle_0 \neq 0$ implies that $n = 3$ and
$$\langle \phi_1, \phi_2, \phi_3 \rangle_0 = \int_{[X]} \phi_1 \cup \phi_2 \cup \phi_2.$$
(4) Dimension: $\langle \phi_1, \dots, \phi_n \rangle_\beta \neq 0$ implies $\sum \deg \phi_i = 2\mathrm{vdim}$.
(5) Unit: $\langle 1_X, \phi_2 \dots, \phi_n \rangle_\beta = 0$ if $\beta \neq 0$.
(6) Divisor: $\langle \phi_1, \dots, \phi_n, \phi_{n+1} \rangle_\beta = \langle \phi_1, \dots, \phi_n \rangle_\beta \cdot \int_\beta \phi_{n+1}$ if $\deg \phi_{n+1} = 2$ and $n \geq 3$.
(7) WDVV:
$$\sum_{\beta' + \beta'' = \beta} \quad \sum_{I' \cup I'' = \{5, \dots, n\}} \sum_i \langle \phi_1, \phi_2, \phi_{I'}, T_i \rangle_{\beta'} \cdot \langle \phi_5, \phi_4, \phi_{I''}, T^i \rangle_{\beta''}$$
is symmetric under swapping ϕ_2 and ϕ_3. Here, T_i and T^i are dual bases of $\mathrm{H}^*(X, \mathbf{Q})$.

Gromov–Witten potential. Choose a basis T_i of $\mathrm{H}^*(X, \mathbf{Q})$. The *Gromov–Witten potential* is given by

$$\Phi = \sum_{n \geq 0, \beta} \frac{q^\beta}{n!} \sum_{i_1, \dots, i_n} \langle T_{i_1}, \dots, T_{i_n} \rangle_\beta \cdot t_{i_1} \cdots t_{i_n} \in \mathbf{Q}[\![\mathrm{NE}(X, \mathbf{Z})]\!][\![\{t_i\}]\!].$$

We denote the coefficient ring by R. The potential leads to the quantum product

$$\mathrm{H} \otimes \mathrm{H} \to \mathrm{H} \otimes R, \quad T_i \star T_j := \sum_k \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial k} T^k,$$

where $\mathrm{H} = \mathrm{H}^*(X, \mathbf{Z})$. Then WDVV implies that $\star$ is associative. In fact, WDVV is equivalent to the associativity of the quantum product.

Conjecture 1.2. *The series Φ is convergent for small $|q|, |t_i|$.*

2. nc-Hodge structures and mirror symmetry

nc-Hodge structures. We package Gromov–Witten theory further into some differential-geometric data, with the goal of bringing in ideas from mirror symmetry.

Definition 2.1. A *rational pure nc-Hodge structure* consists of a tuple $(\mathrm{H}, \mathcal{E}_B, \mathrm{iso})$, where:

- H is a $\mathbf{Z}/2$-graded algebraic vector bundle on $\mathbf{A}^1$ with coordinate u,
- $\mathcal{E}_B$ is a local system of finite-dimensional $\mathbf{Q}$-vector spaces on $\mathbf{A}^1 - 0$,
- iso is an isomorphism
$$\mathcal{E}_B \simeq \mathcal{O}^{\mathrm{an}}_{\mathbf{A}^1-0} \simeq \mathrm{H}^{\mathrm{an}}|_{\mathbf{A}^1-0},$$
so that H^{an} inherits a flat holomorphic connection ∇ on $\mathbf{A}^1 - 0$.

These are required to satisfy the following conditions:

(1) nc-filtration axiom: ∇ is a meromorphic connection on H with a pole of order ≤ 2 at $u = 0$, and a regular singularity at ∞.
(2) $\mathbf{Q}$-structure axiom: $\mathcal{E}_B$ is compatible with "Stokes data."
(3) Opposedness axiom (omitted).

Example 2.2 (Classical Hodge structures). Let $(V, \mathrm{F}^\bullet V, V_{\mathbf{Q}})$ be a pure $\mathbf{Q}$-Hodge structure of weight w. Then the corresponding nc-Hodge structure is given as follows:

- $\mathrm{H} = \sum u^{-1}\mathrm{F}^i V[u]$, considered as a submodule of $\mathbf{C}[u, u^{-1}] \otimes V$.
- The connection is given by
$$\nabla = d - \frac{w}{2}\frac{du}{u}.$$
- The local system is the extension of $V_{\mathbf{Q}}$, which we think of as a vector subspace of the fiber of H at $1 \in \mathbf{A}^1$, to a $\mathbf{Q}$-local system in $\mathrm{H}^{\mathrm{an}}|_{\mathbf{A}^1-0}$ by parallel transport.

In this example, the pole at 0 has order 1.

In general, meromorphic connections with poles of order 2 are difficult, so we focus instead on a class where the pole is reasonable:

Definition 2.3. An nc-Hodge structure is of *exponential type* with exponents $c_1, \ldots, c_n$ if, after tensoring with $\mathbf{C}((u))$, it splits into a direct sum
$$\bigoplus (\mathcal{E}^{c_i/u} \otimes R_i),$$
where $\mathcal{E}^{c_i/u}$ is a rank one vector bundle with connection $d - d\left(\frac{c_i}{u}\right)$, and R_i has regular singularities.

Variation of nc-Hodge structures. There is a notion of a variation of nc-Hodge structure. We omit the precise definition. Suffice it to say that the definition consists of the following ingredients:

- A $\mathbf{Z}/2$-graded vector bundle H on $\mathbf{A}^1 \times B$, which is algebraic in the $\mathbf{A}^1$ direction.
- A local system $\mathcal{E}_B$ of $\mathbf{Z}/2$-graded vector spaces on $\mathbf{G}_m \times B$.
- An isomorphism
$$\mathrm{iso} : \mathcal{E}_B \otimes \mathcal{O}_{(\mathbf{A}^1-0)\times S} \simeq \mathrm{H}|_{(\mathbf{A}^1-0)\times S},$$

satisfying several conditions:

(1) nc-filtration axiom
(2) Griffiths transversality
(3) $\mathbf{Q}$-structure axiom
(4) Opposedness

A-model nc-Hodge structure. We explain how Gromov–Witten invariants give rise to a variation of nc-Hodge structure.

Let $B = \mathrm{Spf}(\mathbf{Q}[\![\mathrm{NE}(X,\mathbf{Z})]\!][\![\{t_i\}]\!])$, which serves as the base of the variation. Let H be the trivial bundle over $B \times \mathbf{A}^1_u$ with fiber $\mathrm{H} = \mathrm{H}^*(X, \mathbf{Q})$. We take ∇ to be the *quantum connection*:

$$\nabla = \begin{cases} \nabla_{\partial_u} = \partial_u + \frac{1}{u^2}K + \frac{1}{u}G \\ \nabla_{\partial_{t_i}} = \partial_{t_i} + \frac{1}{u}A_i \\ \nabla_{q_j\partial_{q_j}} = q_j\partial_{q_j} + \frac{1}{u}A_j \end{cases}$$

Here, when we write q_j, we mean the variable q^β associated that an effective curve class in a chosen basis. The quantum operators are:

$$\begin{aligned} K &= \left(K_X + \sum_{i:\deg T_i \neq 2} \frac{\deg T_i - 2}{2} t_i T_i \right) \star (-) \\ G &= \sum_{i=0}^{\dim X} \frac{i - \dim X}{2} \mathrm{id}_{\mathrm{H}^i(X,\mathbf{Q})} \\ A_i &= T_i \star (-). \end{aligned}$$

These operators preserve the $\mathbf{Z}/2$-grading.

Conjecture 2.4. (H, ∇) *has exponential type.*

Let us assume convergence (Theorem 1.2), and explain some of the other aspects of the (conjectural) variation. The $\mathbf{Q}$-structure is induced by the image of the composition

$$\mathrm{H}^k(X,\mathbf{Q}) \xrightarrow{(2\pi i)^{k/2}} \mathrm{H}^k(X,\mathbf{C}) \xrightarrow{\hat{\Gamma}(X)\wedge -} \mathrm{H}^k(X,\mathbf{C}),$$

where

$$\begin{aligned} \hat{\Gamma}(X) &= \prod \Gamma(1+\lambda_i) \\ &= \exp\left(\gamma \cdot \mathrm{ch}_1(T_X)) + \sum_{n \geq 2} \frac{\zeta(n)}{n} \mathrm{ch}_n(T_X) \right). \end{aligned}$$

Here, λ_i are the Chern roots of T_X, i.e., the total Chern class $c(T_X)$ is equal to $\prod(1+\lambda_i)$. Note that γ is the Euler–Mascheroni constant.

Conjecture 2.5 (Gamma conjecture)**.** *For this specific* $\mathbf{Q}$*-structure, the* $\mathbf{Q}$*-structure axioms and opposedness axioms for a variation of nc-Hodge structure are satisfied.*

B-model nc-Hodge structures. We now explain the B-model nc-Hodge structure associated to a Landau–Ginzburg model: Y is smooth and quasi-projective over $\mathbf{C}$, and $w : Y \to \mathbf{A}^1$ is a function such that $\mathrm{Crit}(w)$ is proper.

The twisted de Rham cohomology is

$$\mathcal{H} = \mathrm{R}\Gamma(Y, \Omega_Y^\bullet[u], ud - (dw \wedge -))$$

It is a difficult theorem of Sabbah and others that this is a $\mathbf{Z}/2$-graded vector bundle over $\mathbf{A}^1$. The connection is the Gauss–Manin connection

$$\nabla = \partial_u - \frac{w}{u^2}.$$

Finally, the $\mathbf{Q}$-structure is given by *rapid decay cohomology*:

$$E_\theta = \mathrm{H}^*(Y, w^{-1}(p_\theta)).$$

Here, θ is an angle, and p_θ is a point of angle θ, sufficiently far from the origin.

Conjecture 2.6. *With the data above, one has a nc-Hodge structure.*

The main difficulty seems to be opposedness.

Hodge-theoretic mirror symmetry. Mirror symmetry is usually thought of as a duality between Calabi–Yau varieties of the same dimension:

$$X \iff \check{X}$$

Nowadays, we can take X to be either Calabi–Yau or Fano, and the mirror to be a Landau–Ginzburg model $(Y, w : Y \to \mathbf{C})$. Using nc-Hodge structures, one obtains a precise version of mirror symmetry:

Conjecture 2.7 (Hodge-theoretic mirror symmetry)**.** *The A-model nc-Hodge structure for X is isomorphic to the B-model nc-Hodge structure for* (Y, w).

The fact that either side is an nc-Hodge structure in the first place is conjectural, as indicated above.

Remark 2.8. Under the conjecture, the critical values $\mathrm{Crit}(w)$ correspond to the eigenvalues of the quantum operator K.

A different version of mirror symmetry is the homological mirror symmetry conjecture of Kontsevich, which predicts that there is an equivalence of categories

$$\mathrm{D^b}(X) \simeq \mathrm{FS}(Y, w).$$

The right-hand side has a *natural* semiorthogonal decomposition up to mutation, which suggests that the left-hand side should also have a *natural* semiorthogonal decomposition up to mutation.

The main takeaway is the following series of implications:

$$\begin{array}{c}\text{Decomposition of B-model nc-Hodge structure}\\ \rightsquigarrow \\ \text{Decomposition of A-model nc-Hodge structure}\\ \rightsquigarrow \\ \text{Decomposition of classical Hodge structures of } X\end{array}$$

You might say this is on very shaky ground, since it is built on a web of difficult conjectures. To return to firmer (and more scientific) ground, we discard the categorical point of view and work directly with differential equations.

2.1. **Analytic decomposition via the Fourier transform.** We have the following diagram:

$$\begin{tikzcd} (1) \arrow[r, "\text{Fourier}"] \arrow[d, "\text{irr RH}"] & (2) \arrow[d, "RH"] \\ (3) \arrow[r, "\text{top. RH}"] & (4). \end{tikzcd}$$

where:

(1) de Rham data: An algebraic vector bundle $\mathcal{H}$ with connection ∇ on $\mathbf{G}_m$, singularities of exponential type at 0, regular singularities at ∞.
(2) regular holonomic D-module on $\mathbf{A}^1$ with regular sing, and vanishing de Rham cohomology.
(3) $\mathbf{Q}$-Stokes structure (I, I_S) of exponential type.
(4) Constructible sheaf F of $\mathbf{Q}$-vector spaces on $\mathbf{A}^1$ such that $\mathrm{R}\Gamma(F) = 0$.

Comments:

- Having exponential type at 0, reg. sing. at ∞ is equivalent to having regular singularities after Fourier transform.
- The condition that (2) has vanishing de Rham cohomology implies that the corresponding perverse sheaf is concentrated in a single degree.

Definition 2.9. A *vanishing cycle decomposition* of a perverse sheaf F on $\mathbf{A}^1$ is a collection of finite-dimensional $\mathbf{Q}$-vector spaces $U_1, \ldots, U_n$, and a collection of maps $T_{ij} : U_i \to U_j$. Here, $U_i = F_{c_0}/F_{c_i}$, where we have some points $c_0, c_1, \ldots, c_n$ in $\mathbf{A}^1$.[1]

Theorem 2.10 (Analytic decomposition theorem, KKP)**.** *An nc-Hodge structure of exponential type is equivalent to the following data:*

- *a finite set* $\{c_1, \ldots, c_n\} \subset \mathbf{C}$
- *a collection of nc-Hodge structures* $(\mathrm{H}_i, \mathcal{E}_B, \mathrm{iso})$*,* $i = 1, \ldots, n$
- *Gluing data: paths* $c_0 \rightsquigarrow c_i$ *and maps* $T_{ij} : (\mathcal{E}_{B,j})_{c_o} \to (\mathcal{E}_{B,i})_{c_0}$*.*

3. F-bundles

From here on out, there will be no more conjectures—only theorems. The idea here is to ignore both $\mathbf{Q}$-structure and issues of convergence. It turns out that the trick for this is to work over a non-archimedean field.

We begin with a smooth rigid $\mathbf{k}$-analytic variety B, where $\mathbf{k}$ is an algebraically closed non-archimedean field. Let $\mathbf{D}$ be the germ of $0 \in \mathbf{A}^1_u$.

Definition 3.1. An *F-bundle* $(\mathrm{H}, \nabla)/B$ consists of a vector bundle H over $B \times \mathbf{D}$ with a meromorphic flat connection ∇, such that:

[1]This is a generalization of the standard description of a perverse disc on $\mathbf{A}^1$, perverse with respect to the stratification $\{0\} \subset \mathbf{A}^1$, in terms of a pair of vector spaces with maps between them satisfying certain conditions.

(1) ∇_{∂_u} has a pole of order ≤ 2 along $u = 0$,
(2) For any tangent vector field ξ in B, ∇_ξ has a pole of order ≤ 1 along $u = 0$.

Example 3.2. Here is an example:

$$\nabla_{\partial_{t_i}} = \frac{1}{u}\begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, \quad \nabla_{\partial_u} = \partial_u - \frac{1}{u^2}\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}$$

For any $b \in B$, we obtain a map

$$\mu_b : T_bB \to \mathrm{End}(\mathrm{H}_{b,0}), \quad v \mapsto \nabla_{uv}|_{\mathrm{H}_{b,0}}$$

Flatness of the connection implies that the image consists of commuting operators.

Definition 3.3. An F-bundle $(\mathrm{H}, \nabla)/B$ is *overmaximal* (resp., *maximal*) at $b \in B$ if there exists $h \in \mathrm{H}_{b,0}$ such that

$$T_bB \to \mathrm{H}_{b,0}, \quad v \mapsto \mu_b(v)h$$

is an epimorphism (resp., isomorphism).

If an F-bundle $(\mathrm{H}, \nabla)/B$ is maximal, then there is a commutative product on the tangent bundle TB given by

$$\mu_b(v_1 \star v_2) = \mu_b(v_2) \circ \mu_b(v_1)(h).$$

For example, the quantum product in the A-model arises in such a way.

We also obtain an Euler vector field Eu on B, such that

$$\mu_b(\mathrm{Eu}) = K_b = \nabla_{u^2\partial_u}|_{u=0}.$$

Here, K_b is the same K as in quantum cohomology.

Example 3.4.

$$\partial_{t_i} \star \partial_{t_j} = \delta_{ij}\partial_{t_i}$$
$$\mathrm{Eu} = \sum_j t_i\partial_{t_i}.$$

3.1. **Nonarchimedean decomposition theorem.**

Theorem 3.5 (Nonarchimedean decomposition theorem)**.** *Let* $(\mathrm{H}, \nabla)/B$ *be a maximal F-bundle, and let* $b \in B$ *be a point. At* b*, consider the generalized eigenspace decomposition*

$$\mathrm{H}_{b,0} = \bigoplus E_i. \tag{3.1}$$

Then $(\mathrm{H}, \nabla)/B$ *locally splits into a product of maximal F-bundles* $(\mathrm{H}_i, \nabla)/B_i$*, extending* (3.1)*.*

Remark 3.6 (Comparison)**.** It is possible to make a comparison with the vanishing cycle decomposition for nc-Hodge structures. The vanishing cycle decomposition relies on a choice of paths, and apparently one should take so-called *Gabrielov paths.*

4. Blowup decomposition

Let X be smooth, projective over $\mathbf{C}$, $Z \subset X$ a smooth subvariety of pure codimension $r \geq 2$, and $\tilde{X} \to X$ the blowup of X along Z. Recall that we have a decomposition of cohomology:

$$\mathrm{H}^*(\tilde{X}) \simeq \mathrm{H}^*(X) \oplus \bigoplus_{i=1}^{r} \mathrm{H}^*(Z)[-2i]$$

One goal might be to extend this to a decomposition of nc-Hodge structures. But this is too conjectural, so we instead try to extend it to a decomposition of F-bundles.

First, we need to care of curve classes, since these are part of the coefficients of F-bundles. To simplify notation, let:[2]

$$\begin{aligned}\mathbf{Q}[Q] &= \mathbf{Q}[\![\mathrm{NE}(X, \mathbf{Z})]\!] = \mathbf{Q}[\![Q^d, d \in \mathrm{NE}(X, \mathbf{Z})]\!] \\ \mathbf{Q}[\mathcal{Q}] &= \mathbf{Q}[\![Q^d, xy^{-1}, Q^{\phi_* \tilde{d}} y^{-[E]\tilde{d}} : d \in \mathrm{NE}(X, \mathbf{Z}), \tilde{d} \in \mathrm{NE}(\tilde{X}, \mathbf{Z})]\!] \\ \mathbf{Q}[\![\tilde{Q}]\!] &= \mathbf{Q}[\![\mathrm{NE}(\tilde{X}, \mathbf{Z})]\!] \\ \mathbf{Q}[\![Q_Z]\!] &= \mathbf{Q}[\![\mathrm{NE}(Z, \mathbf{Z})]\!]\end{aligned}$$

Here $\mathrm{NE}(X, Z)$ are the effective curve classes supported on Z, etc.

Iritani's theorem. By embedding everything into $\mathrm{Bl}_{Z \times 0}(X \times \mathbf{P}^1)$, we can get embeddings:

$$\begin{aligned}\mathbf{Q}[\![Q]\!] &\to \mathbf{Q}((q^{-1/s}))[\![\mathcal{Q}]\!] \quad \text{in the obvious way} \\ \mathbf{Q}[\![\tilde{Q}]\!] &\to \mathbf{Q}((q^{-1/s}))[\![\mathcal{Q}]\!], \quad \tilde{Q}^{\tilde{d}} \mapsto Q^{\phi_* d} q^{-[E]\tilde{d}} \\ \mathbf{Q}[\![Q_Z]\!] &\to \mathbf{Q}((q^{-1/s}))[\![\mathcal{Q}]\!], \quad Q_Z^d \mapsto Q_Z^d \mapsto Q^{\varphi_* d} q^{-c_1(N_{Z/X})d/(r-1)} \\ s &= \begin{cases} r-1 & r \equiv 0 \mod 2 \\ 2(r-1) & r \equiv 1 \mod 2 \end{cases}\end{aligned}$$

Theorem 4.1 (Iritani). *After pullback to $\mathbf{C}((q^{-1/5}))[\![\mathcal{Q}]\!]$, there exists a formal invertible change of variavles*

$$\mathrm{H}^*(\tilde{X}) \mapsto \mathrm{H}^*(X) \oplus \mathrm{H}^*(Z)^{\oplus r-1}, \quad \tilde{\tau} \mapsto (\tau(\tilde{\tau})), \{\zeta_j(\tilde{\tau})\}_{0 \leq j \leq r-2}$$

defined over $\mathbf{C}((q^{-1/s}))[\![\mathcal{Q}]\!]$, and an isomorphism of formal F-bundles

$$(H_{\tilde{X}}, \nabla_{\tilde{X}}) \simeq \tau^*(H_X, \nabla_X) \oplus_{j=0}^{r-2} \zeta_j^*(H_Z, \nabla_Z)$$

This is a really interesting theorem, and is a "packaged" way of answering the question: How do the Gromov–Witten invariants of X change under a blowup?

Non-archimedean F-bundles. Back to our nonarchimedean setup. Consider the nonarchimedean A-model F-bundle $(\tilde{\mathrm{H}}, \tilde{\nabla})/\tilde{B}^{\max}$ for the blowup $\tilde{X}$, and the similarly the F-bundle $(\mathrm{H}', \nabla')_{B',\max}$ for the disjoint union

$$X' = X \amalg \coprod_{i=1}^{r-1} Z.$$

[2] In the first line, $\mathrm{NE}(X, Z)$ was written on the board, but after checking Iritani's paper, I think that $\mathrm{NE}(X, \mathbf{Z})$ was intended.

Theorem 4.2 (Blowup formula)**.** *There exists a unique isomorphism of maximal F-bundles between* $(\tilde{\mathrm{H}}, \tilde{\nabla})$ *and* (H', ∇') *over an analytic domain* $\tilde{U}$ *in* $\tilde{B}^{\max}$*, and the analytic domain* U' *in* $B'^{,\max}$*. The union of different choices of* $\tilde{U}$ *is connected and nonempty; same for* U'*.*

Remark 4.3 (Defining $B^{\max}$)**.** When X is smooth, projective, we had a formal base B_{formal} for the variation of nc-Hodge structure. For $B^{\max}$, we first consider let $\mathbf{k}$ be a non-archimedean base field. Then we consider $\mathrm{H}^2(X, \mathbf{k}^*)$, which has a valuation map to $\mathrm{H}^2(X, \mathbf{R})$ (taking the valuation of the coefficient). In $\mathrm{H}^2(X, \mathbf{R})$, one has the ample cone. The preimage in $\mathrm{H}^2(X, \mathbf{k}^*)$ is denoted B^2. Then

$$B^{\max} = B^2 \times (\text{open unit disc in all } t_i : \deg t_i \neq 2).$$

5. Atoms

Here, the goal is to relate F-bundles with motives. We will work with three different fields. First, K is any field; we will work with varieties over K. Then we consider k, which has characteristic 0; it will be the coefficient field of the cohomology theory (e.g., $\mathbf{Q}$ is fine). Finally, we will consider an algebraically closed nonarchimedean field, $\mathbf{k}$.

Let $\mathcal{C}$ be a semisimple neutral Tannakian category over k, so that $\mathcal{C} = \mathrm{Rep}(G)$, for G a pro-reductive group. Assume that G has a central element $\epsilon \in G$ of order 2. Let H^* be a Weil cohomology theory of projective K-varieties, taking values in $\mathcal{C}$, satisfying a *Mumford–Tate normalization* condition:

(1) If $\mathrm{H}^2(\mathbf{P}^1)$ is a trivial rank 1 G-module.
(2) For any smooth, projective K-variety X and any $i \in \mathbf{Z}$, ϵ acts on $\mathrm{H}^i(X)$ by $(-1)^i$.

Example 5.1. Let $k = \mathbf{Q}$, $\mathcal{C}_0$ be the category of pure, polarizable $\mathbf{Q}$-Hodge structures. Then $\mathcal{C}_0 = \mathrm{Rep}(G_0)$, where G_0 is the Mumford–Tate group. There is a natural homomorphism from G_0 to $\mathbf{G}_m$ corresponding to the action of G_0 on $\mathrm{H}^2(\mathbf{P}^1)$; the kernel G satisfies the conditions above. Then ϵ comes from the Deligne torus.

Example 5.2. One can take $C = \mathrm{Rep}(G)$ to be Andre's category of motivated cycles (so $\mathrm{H}^*(X)^G$ is the subgroup spanned by motivated cycles).

5.1. **The general cubic fourfold is irrational.**

Definition 5.3. Let B be the germ of a smooth $\mathbf{k}$-analytic space at a rigid point (a point corresponding to a maximal ideal). A G-equivariant maximal F-bundle is called a G-atom if the action by the Euler field Eu has a single eigenvalue. (Eu is the residue of the second order pole of the connection.)

Two G-atoms are *equivalent* if they come from a G-equivariant F-bundle over a connected base (a connected smooth $\mathbf{k}$-analytic space).

For any G-equivariant maximal F-bundle $(H, \nabla)/B$, then we consider the locus $B_0 \subset B^G$ where the number of distinct eigenvalues of Eu is maximal. Finally, we apply Theorem 3.5: Consider the finite étale covering $\mathbf{B} \to B_0$. Then each connected component of $\mathbf{B}$ gives an equivalence class of G-atom.

We will ultimately look at a coarser invariant. Each G-atom gives an isomorphism class of finite-dimensional G-representation.

Let G be as in Theorem 5.1 or Theorem 5.2. Then G acts on nonarchimedean A-model F-bundle $(H, \nabla)/B$ associated to a smooth, projective variety X.

Definition 5.4. Let $\mathrm{Atoms}(X)$ be the multiset of equivalence classes of G-atoms from the nonarchimedean A-model F-bundle (H, ∇).

The blowup decomposition theorem says that

$$\mathrm{Atoms}(\tilde{X}) = \mathrm{Atoms}(X) + \sum_{r-1} \mathrm{Atoms}(Z). \tag{5.1}$$

To emphasize, this is a sum of multisets. It just means that elements of the multiset $\mathrm{Atoms}(Z)$ are added, with multiplicity, to the multiset $\mathrm{Atoms}(X)$.

Theorem 5.5. *If K_X is nef, then* $\mathrm{Atoms}(X)$ *is a singleton.*

Consider a general cubic 4-fold. Here, general means that $\mathrm{H}^4_{\mathrm{prim}}(X)$ does not contain any Hodge classes; we will show that all of these are irrational. The Hodge diamond of X has the following form:

$$\underbrace{\begin{matrix} & 1 & \\ & 1 & \\ 1 & 21 & 1 \\ & 1 & \\ & 1 & \end{matrix}}_{\mathrm{H}^*(X)} = \underbrace{\begin{matrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{matrix}}_{\mathrm{H}^*(\mathbf{P}^5)|_X} + \underbrace{\begin{matrix} 1 & 20 & 1 \end{matrix}}_{\mathrm{H}^4_{\mathrm{prim}}(X)}$$

Here, we have written $\mathrm{H}^*(X) = \mathrm{H}^*(\mathbf{P}^4) + \mathrm{H}^4_{\mathrm{prim}}(X)$.

A classical computation of Givental (for some special parameters) shows that the spectrum of Eu has the following shape:

$$\underbrace{\begin{matrix} & 1 & \\ & & \\ & 24 & \\ 1 & & 1 \end{matrix}}_{\mathrm{H}} = \underbrace{\begin{matrix} & 1 & \\ & & \\ & 2 & \\ 1 & & 1 \end{matrix}}_{\mathrm{H}^*(\mathbf{P}^5)|_X} + \underbrace{\begin{matrix} & 0 & \\ & & \\ & 22 & \\ 0 & & 0 \end{matrix}}_{\mathrm{H}^4(X)_{\mathrm{prim}}}$$

Here, the displayed numbers refer to the dimensions of eigenspaces, and the position of each number is supposed to indicate the corresponding eigenvalue (with 0 at the center). For instance, on the left, 24 refers to the dimension of the eigenspace for the eigenvalue 0.

Let V be the eigenspace for the eigenvalue 0, so $\dim V = 24$ and $\dim V^G = 2$. The eigenvalue computation above could have occured at a point $p \in B^G$ which is not contained in the locus B_0 where the number of eigenvalues is maximal. In other words, it could happen that as we vary parameters to move p into B_0, V splits further into some G-representations

$$V = V' \oplus \cdots .$$

But certainly for some V' in the decomposition, we have $\dim V'^{0,2} > 0$ and $\dim V'^G \leq 2$.

The key observation is that V' does not appear in the F-bundle of any variety S with $\dim S \leq 2$. Indeed, such a variety must have $h^{2,0} > 0$, hence it is a surface. But then it is known that there is a minimal model S with K_S nef. By Theorem 5.5, $\mathrm{Atoms}(S)$ is a singleton, so V' must be the whole G-representation $\mathrm{H}^{2*}(S)$. But V' does not contain enough Hodge classes to be the even cohomology of a surface: $\dim V'^G \leq 2$.

The irrationality of the cubic fourfold then follows immediately from the blowup formula for atoms, Equation (5.1). For instance, assuming for the sake of contradiction that X is rational, one can take a roof

$$
\begin{tikzcd}
 & Y \arrow[dl, "g"'] \arrow[dr, "f"] & \\
\mathbf{P}^5 & & X,
\end{tikzcd}
$$

where both f and g are compositions of blowups at smooth centers. Applying the blowup decomposition (5.1) for f, we see that $\mathrm{Atoms}(Y)$ contains the atom corresponding to V'. Applying the blowup decomposition for g, we see that V' must come from the F-bundle of a variety of dimension ≤ 2, which we have ruled out above.