Ph.D. Qualifying Exam, Real Analysis

Fall 2011, part I

Do all five problems. Write your solution for each problem in a separate blue book.

1 Two short problems.

a. Suppose that $(X, \mathcal{A}, \mu)$ is a finite measure space and $f : X \to \mathbb{R}$ is an $\mathcal{A}$-measurable function. Prove that $\|f\|_{L^\infty} = \lim_{p\to\infty} \|f\|_{L^p}$.

b. Let $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, and identify it with the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ in the complex plane. For $\delta \in (0, 1)$, let $\Omega_\delta = \{z \in \mathbb{C} : 1 - \delta < |z| < 1\}$. Suppose that $\phi \in C(\mathbb{T})$ and there exist $\delta \in (0, 1)$ and $f \in C(\overline{\Omega_\delta})$, f holomorphic on Ω_δ such that $f|_{\mathbb{T}} = \phi$. Let $(\mathcal{F}\phi)_n = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} e^{-in\theta} \phi(\theta)\, d\theta$ be the nth Fourier coefficient of ϕ. Show that for all $k > 0$ there exists $C > 0$ such that $|(\mathcal{F}\phi)_n| \le C|n|^{-k}$ for $n \le -1$.

2 Suppose that $f : \mathbb{N} \to \mathbb{C}$ is a bounded function (where $\mathbb{N}$ is the set of the positive integers), and define $M_f \in \mathcal{L}(\ell^p)$, $1 \le p < \infty$, by $M_f(\{a_n\}_{n=1}^\infty) = \{f(n)a_n\}_{n=1}^\infty$.

a. Find (with proof) the spectrum of M_f.

b. State and prove a necessary and sufficient condition in terms of f for M_f being a compact operator.

c. Can there be any points in the spectrum of M_f that are not eigenvalues? Give an example or prove the contrary.

3 In this problem X, Y are Hilbert spaces, $\mathcal{L}(X, Y)$ the set of bounded linear operators from X to Y. Prove or disprove the following statements: (a) There exists $T \in \mathcal{L}(X, Y)$ such that T is a bijection, but the set theoretic inverse T^{-1} is not in $\mathcal{L}(Y, X)$. (b) There exists $T \in \mathcal{L}(X, Y)$ such that T is injective, but there is no left inverse $S \in \mathcal{L}(Y, X)$ for T (i.e. there is no $S \in \mathcal{L}(Y, X)$ such that ST is the identity on X). (c) There exists $T \in \mathcal{L}(X, Y)$ such that T is surjective, but there is no right inverse $S \in \mathcal{L}(Y, X)$ for T (i.e. there is no $S \in \mathcal{L}(Y, X)$ such that TS is the identity on Y).

4 Suppose that X is a separable reflexive Banach space.

a. Show that $B = \{x \in X : \|x\| \le 1\}$ is compact in the weak topology on X.

b. Give (and prove) a necessary and sufficient condition for $\{x \in X : \|x\| = 1\}$ to be compact in the weak topology on X.

c. Prove that $B = \{x \in X : \|x\| \le 1\}$ is sequentially compact in the weak topology of X, i.e. every sequence in B has a weakly convergent subsequence.

5 Recall that $\mathrm{SL}_2(\mathbb{R}) \subset \mathrm{GL}_2(\mathbb{R})$, consisting of matrices of determinant 1, acts on $\mathbb{R}^2$ by $\mathrm{SL}_2(\mathbb{R}) \times \mathbb{R}^2 \ni (A, x) \mapsto Ax \in \mathbb{R}^2$. Find all Baire measures μ on $\mathbb{R}^2$ (i.e. Borel measures finite on compact sets) which are invariant under the $\mathrm{SL}_2(\mathbb{R})$-action, i.e. such that $\mu(S) = \mu(AS)$ (with $AS = \{Ax : x \in S\}$) for all $A \in \mathrm{SL}_2(\mathbb{R})$ and S Borel.

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Fall 2011, part II

Do all five problems. Write your solution for each problem in a separate blue book.

1 Suppose $A \subset \mathbb{R}$ is Borel, T is a dense subset of $\mathbb{R}$ and $\tau_t(A) \setminus A$ has Lebesgue measure zero for each $t \in T$, where $\tau_t : \mathbb{R} \to \mathbb{R}$ is the translation $x \mapsto x + t$. Prove that either A or $\mathbb{R} \setminus A$ has Lebesgue measure zero.

2 Two short problems.

a. Let $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ be the circle, and let $C^1(\mathbb{T})$ denote the Banach space of complex valued continuously differentiable functions on $\mathbb{T}$ with the norm $\|f\|_{C^1} = \sup|f| + \sup|f'|$. Suppose that $u : C^1(\mathbb{T}) \to \mathbb{C}$ is continuous and has the property that $\phi \in C^1(\mathbb{T})$, $\phi \geq 0$ on $\mathbb{T}$, imply $u(\phi) \geq 0$. Show that there is a finite Borel measure μ on $\mathbb{T}$ such that $u(\phi) = \int \phi \, d\mu$ for all $\phi \in C^1(\mathbb{T})$.

b. Suppose that $\mathcal{H}$ is a Hilbert space, $\mathcal{L}(\mathcal{H})$ the set of bounded operators on $\mathcal{H}$, $A \in \mathcal{L}(\mathcal{H})$ is self-adjoint, and for $t \in \mathbb{R}$ define $U(t) = f_t(A)$ via the functional calculus where $f_t(s) = e^{its}$. Show that $U(t) - I$ is compact for all $t \in \mathbb{R}$ if and only if A is compact.

3 Let $\mathcal{L}(L^2(\mathbb{R}))$ denote the set of bounded linear operators on $L^2(\mathbb{R})$. Consider the following operators Λ_s, $s \in \mathbb{R}$, on $L^2(\mathbb{R})$: $(\Lambda_s f)(\xi) = (1+|\xi|^2)^{is/2} f(\xi)$. Prove or disprove each of the following statements for the map $\Lambda : \mathbb{R} \to \mathcal{L}(L^2(\mathbb{R}))$ given by $\Lambda(s) = \Lambda_s$: (a) Λ is continuous when $\mathcal{L}(L^2(\mathbb{R}))$ is equipped with the norm topology. (b) Λ is continuous when $\mathcal{L}(L^2(\mathbb{R}))$ is equipped with the strong operator topology. (c) Λ is continuous when $\mathcal{L}(L^2(\mathbb{R}))$ is equipped with the weak operator topology.

4 Let $\mathcal{D}'(\mathbb{T})$ denote the set of distributions on $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, i.e. the dual of $C^\infty(\mathbb{T})$, equipped with the 'weak-* topology', i.e. the weakest topology in which the maps $E_\phi : \mathcal{D}'(\mathbb{T}) \to \mathbb{C}$, $E_\phi(u) = u(\phi)$ are continuous for all $\phi \in C^\infty(\mathbb{T})$.

a. Show that the multiplication map $M : C(\mathbb{T}) \times C(\mathbb{T}) \to C(\mathbb{T})$ given by $(M(f,g))(x) = f(x)g(x)$ has no continuous extension to a map $\tilde{M} : \mathcal{D}'(\mathbb{T}) \times \mathcal{D}'(\mathbb{T}) \to \mathcal{D}'(\mathbb{T})$.

b. Show that, on the other hand, the convolution map $* : C(\mathbb{T}) \times C(\mathbb{T}) \to C(\mathbb{T})$ given by $*(f,g)(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x-y)g(y)\,dy$ (usually written as $(f * g)(x)$) has a unique [separately] continuous extension to a map $\tilde{*} : \mathcal{D}'(\mathbb{T}) \times \mathcal{D}'(\mathbb{T}) \to \mathcal{D}'(\mathbb{T})$.

5 For $s \geq 0$, let $H^s(\mathbb{T})$ be the space of L^2 functions f on the circle $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ whose Fourier coefficients $\hat{f}_n = \int e^{-inx} f(x)\,dx$ satisfy $\sum(1+n^2)^s|\hat{f}_n|^2 < \infty$, and define the norm on $H^s(\mathbb{T})$ by $\|f\|_s^2 = (2\pi)^{-1}\sum(1+n^2)^s|\hat{f}_n|^2$.

Consider $A = -\frac{d^2}{dx^2}$, $A \in \mathcal{L}(H^2(\mathbb{T}), L^2(\mathbb{T}))$, suppose $V \in \mathcal{L}(L^2(\mathbb{T}))$, and let $L = A + V \in \mathcal{L}(H^2(\mathbb{T}), L^2(\mathbb{T}))$. With $R(\lambda) = (\lambda I - L)^{-1} : L^2(\mathbb{T}) \to H^2(\mathbb{T})$ when $\lambda I - L : H^2(\mathbb{T}) \to L^2(\mathbb{T})$ is invertible, show that $\mathbb{C} \ni \lambda \mapsto R(\lambda)$ is a meromorphic operator-valued function, and that there exists a half-plane $\Omega = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < -C\}$ such that $R(\lambda)$ is holomorphic on Ω.