Ph.D. Qualifying Exam, Real Analysis

Fall 2014, part I

Do all five problems. Write your solution for each problem in a separate blue book.

1 Prove or disprove: If $\mathcal{H}$ is an infinite dimensional separable Hilbert space, then $\mathcal{B}(\mathcal{H},\mathcal{H})$ is separable in the operator-norm topology.

2 Let (S,μ) be a σ-finite measure space. Let $f \in L^\infty(S,\mu)$ and let M_f be the bounded linear operator on $L^2(S,\mu)$, $M_f(g) = fg$.

a. Find a necessary and sufficient condition (in terms of f and μ) for M_f to have an eigenvector.

b. Find a necessary and sufficient condition (in terms of f and μ) for M_f to be compact.

3 Recall that a topological space S is first countable if at each point $p \in S$ there is a countable base for the topology, i.e. there exist O_n, $n \in \mathbb{N}$, open such that O open and $p \in O$ imply that for some n, $O_n \subset O$.

a. Suppose that X is a Banach space, $\{x_j\}_{j=1}^\infty$ is a sequence in X with the property that $\cup_{n=1}^\infty X_n = X$, where $X_n = \operatorname{span}\{x_1,\ldots,x_n\}$. Show that X is finite dimensional.

b. Show that if a Banach space is first countable in its weak topology then it is finite dimensional. (*Hint:* show that the dual space has the property in part (i).)

4 Recall that the Fourier transform of an $L^1(\mathbb{R}^n)$ function f is $(\mathcal{F}f)(\xi) = \int e^{-ix\cdot\xi} f(x)\,dx$, where $\cdot$ is the standard inner product on $\mathbb{R}^n$.

Let A be a real symmetric matrix, and define the function f by $f(x) = e^{-iAx\cdot x/2}$, which is a tempered distribution. Find (with proof) its Fourier transform if $\det(A) \neq 0$. Make sure to give an explicit formula in terms of A, without involving any limits. (*Hint:* Write A as the limit of complex symmetric matrices with negative definite imaginary part.)

5 For $k \in \mathbb{R}$, let $\mathcal{H}_+^k$ denote the set of distributions f on $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ whose Fourier series $\hat{f} = \mathcal{F}f$ are rapidly decreasing as $n \to -\infty$ (i.e. for all N, $|\mathcal{F}f(n)| \le C_N |n|^{-N}$ for $n < 0$), and which satisfy $|\hat{f}(n)| \le C_k (n+1)^k$ for $n \ge 0$, with the natural seminorms induced by these estimates. Show that there is K such that for $f, g \in \mathcal{H}_+^k$, fg is well-defined in $\mathcal{H}_+^K$ (extending the usual multiplication on $C^\infty(\mathbb{T})$) and $\mathcal{H}_+^k \times \mathcal{H}_+^k \ni (f,g) \to fg \in \mathcal{H}_+^K$ is continuous.

Ph.D. Qualifying Exam, Real Analysis

Fall 2014, part II

Do all five problems. Write your solution for each problem in a separate blue book.

1 Two quick problems.

a. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function with the property that for all $k \in \mathbb{N}$ and for all $x \in \mathbb{R}$ there exists a polynomial $P_{x,k}$ on $\mathbb{R}$ such that $|f(y) - P_{x,k}(y)| \leq C_{x,k}|x - y|^{k+1}$ if $|x - y| < 1$. Is f infinitely differentiable? Prove this or give a counterexample.

b. Suppose that Y is a normed complex vector space with norm $\|.\|$, and $\lambda : Y \to \mathbb{C}$ is linear but is not continuous. Show that $N = \lambda^{-1}(\{0\})$ is dense in Y.

2 Suppose A is compact operator on a Hilbert space $\mathcal{H}$. Prove that if $zI - A$ is injective, and $z \neq 0$, then the image of $zI - A$ is a closed subspace. Is this also true even if $zI - A$ is not injective?

3 For $E \subset \mathbb{R}$ let $E + E = \{x + y : \ x, y \in E\}$, and define $E - E$ similarly. Show that if E is a measurable subset of $\mathbb{R}$ of positive Lebesgue measure then $E - E$ and $E + E$ contain non-empty open sets.

4 Consider the collection of seminorms $\rho_{k,m}(f) = \sup |x^{-k}\partial^m f|$, $k, m \geq 0$ integer, on the subspace X of C^∞ functions on $[0, 1]$ vanishing with all derivatives at 0.

a. Show that the locally convex topology induced by these seminorms on X is the same as the restriction of the C^∞ topology to X.

b. Show that every $\{\rho_{k,m} : \ k, m \geq 0\}$-continuous linear functional ℓ on X has an extension to a continuous linear functional λ on $C^\infty([-1, 1])$, with X identified with the subspace of $C^\infty([-1, 1])$ consisting of functions vanishing on $[-1, 0]$.

5 Let $\ell^2(\mathbb{Z})$ denote the Hilbert space of square summable bi-infinite complex valued sequences, and let L be the operator acting on $\ell^2(\mathbb{Z})$ defined by

$$(Lf)(n) = f(n) - \frac{1}{2}(f(n + 1) + f(n - 1)).$$

(L is called the discrete Laplacian.)

a. Show that L is a bounded symmetric operator, with spectrum $[0, 2]$, and find the eigenvalues of L.

b. Let V denote multiplication by a real-valued function v: $(Vf)(n) = v(n)f(n)$, and suppose $\lim_{|n|\to\infty} v(n) = 0$. Show that the spectrum of $H = L + V$ outside $[0, 2]$ is a discrete set.