Ph.D. Qualifying Exam, Real Analysis

Spring 2012, part I

Do all five problems. Write your solution for each problem in a separate blue book.

1 Let μ denote the Lebesgue measure on $[0,1]$. Suppose that f_k, $k \in \mathbb{N}$, are Lebesgue measurable (real-valued) functions on $[0,1]$, and let $m_{kn} = \mu(\{x \in [0,1] :\ |f_k| \in (2^{n-1}, 2^n]\})$ for $n \in \mathbb{N}$.

a. Suppose that $\sum_{n=1}^{\infty} n2^n m_{kn} \leq 1$ for all k and suppose that $f_k \to 0$ a.e. Show that $\int f_k \, d\mu \to 0$.

b. Give an example of f_k as above such that $\sum_{n=1}^{\infty} 2^n m_{kn} \leq 1$ for all k, $f_k \to 0$ a.e, but $\int f_k \, d\mu$ does not tend to 0.

2 Let $\mathcal{S}'(\mathbb{R}^n)$ denote the space of tempered distributions on $\mathbb{R}^n$. A locally L^1 function f on $\mathbb{R}^n$ (i.e. the restriction to compact sets is L^1) is said to lie in $\mathcal{S}'(\mathbb{R}^n)$ if the map $C_c^\infty(\mathbb{R}^n) \ni \phi \mapsto \int_{\mathbb{R}^n} f\phi \in \mathbb{C}$ has a (necessarily unique) continuous extension to an element of $\mathcal{S}'(\mathbb{R}^n)$. Here $C_c^\infty(\mathbb{R}^n)$ is the set of compactly supported C^∞ functions on $\mathbb{R}^n$.

a. Show that the function $|x|^{-(n-\beta)}$ lies in the space $\mathcal{S}'(\mathbb{R}^n)$ if $0 < \beta < n$. For what values of β is it a sum of an L^1 and a L^2 function?

b. Show that the function $e^x \cos(e^x)$ on $\mathbb{R}$ lies in $\mathcal{S}'(\mathbb{R})$.

c. Prove that there is a locally $L^1(\mathbb{R}^n)$ function that does not lie in $\mathcal{S}'(\mathbb{R}^n)$.

3 Suppose that $(X, \|.\|_X)$ is a normed vector space, M and N are (not necessarily closed) subspaces equipped with norms $\|.\|_M$, resp. $\|.\|_N$ such that the identity maps $(M, \|.\|_M) \to (M, \|.\|_X)$, resp. $(N, \|.\|_N) \to (N, \|.\|_X)$ are continuous. Let $M+N$ be the algebraic sum: $M + N = \{m + n \in X : m \in M,\ n \in N\}$. For $x \in M+N$, let

$$\|x\|_{M+N} = \inf\{\|m\|_M + \|n\|_N :\ m \in M,\ n \in N,\ x = m+n\}.$$

a. Show that $\|.\|_{M+N}$ is a norm on $M+N$.

b. Show that if $(M, \|.\|_M)$ and $(N, \|.\|_N)$ are complete then $(M+N, \|.\|_{M+N})$ is complete.

4 Suppose that f, g are holomorphic functions on a non-empty open connected set $\Omega \subset \mathbb{C}$, and suppose that $|f|^2 + |g|^2$ is constant on Ω. Show that f and g are constant on Ω.

5 Suppose that X, Y are reflexive separable Banach spaces, X^*, Y^* the duals, $P \in \mathcal{L}(X,Y)$, and suppose that its adjoint $P' \in \mathcal{L}(Y^*, X^*)$ satisfies the following property: There is a Banach space Z and a compact map $\iota : Y^* \to Z$ and $C > 0$ such that for all $\phi \in Y^*$,

$$\|\phi\|_{Y^*} \leq C(\|P'\phi\|_{X^*} + \|\iota\phi\|_Z).$$

a. Show that $\mathrm{Ker} P'$ is finite dimensional.

b. Let V be a closed subspace of Y^* with $V \oplus \mathrm{Ker} P' = Y^*$. Show that there is $C' > 0$ such that for $\phi \in V$, $\|\phi\|_{Y^*} \leq C' \|P'\phi\|_{X^*}$.

c. Show that for $f \in Y$ such that $\ell(f) = 0$ for all $\ell \in \mathrm{Ker} P'$, there is $u \in X$ such that $Pu = f$.

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Spring 2012, part II

Do all five problems. Write your solution for each problem in a separate blue book.

1 Suppose X is a compact metric space, and ℓ is a positive linear functional on $C(X)$, the Banach space of real valued continuous functions, i.e. $f \geq 0$ implies $\ell(f) \geq 0$. Suppose $K \subset X$ compact. Let O_j, $j \in \mathbb{N}$, be open sets satisfying $O_j \supset \overline{O_{j+1}}$ and $\cap_{j\in\mathbb{N}} O_j = K$, and let $f_j \in C(X;[0,1])$ be such that $f_j = 1$ on K, $f_j = 0$ on the complement of O_j. Show directly from first principles (without quoting a major theorem) that $\lim_{j\to\infty} \ell(f_j)$ exists, and is independent of the choice of O_j and f_j satisfying these conditions.

2 Suppose $p \in (0,1)$, $0 < a < p$, and suppose $N \in \mathbb{N}$, $A_1, \ldots, A_N$ are Lebesgue measurable subsets of $[0,1]$ with average measure $\frac{1}{N}\sum_{i=1}^N \mu(A_i) \geq p$. Let $E = \{x \in [0,1] :\ x \in A_i \text{ for at least } aN \text{ values of } i\}$.

a. Show that $\mu(E) \geq \frac{p-a}{1-a}$.

b. Show that if $c > \frac{p-a}{1-a}$ then there exist $N \in \mathbb{N}$ and sets $A_1, \ldots, A_N$ with $\mu(A_i) \geq p$ for all i such that $\mu(E) < c$.

3 Let $\mathcal{S}$ denote the set of Schwartz functions on $\mathbb{R}^n$. The Fourier transform on $\mathcal{S}$, and more generally on L^1, is given by $(\mathcal{F}\phi)(\xi) = \int e^{-ix\cdot\xi}\phi(x)\,dx$, while on tempered distributions $v \in \mathcal{S}'$ it is given by $(\mathcal{F}v)(\phi) = v(\mathcal{F}\phi)$, $\phi \in \mathcal{S}$.

a. Show that there exists a compactly supported C^∞ function ϕ on $\mathbb{R}$ such that $\phi \geq 0$, $\phi(0) > 0$, and $\mathcal{F}\phi$ is non-negative, $(\mathcal{F}\phi)(0) > 0$. (Hint: when is the Fourier transform of a function real?)

b. Show that if u is a compactly supported distribution then the Fourier transform (as defined above) is C^∞, and there exists $N \in \mathbb{R}$ such that for all $\alpha \in \mathbb{N}^n$ there exist $C_\alpha > 0$ with $|(D^\alpha \mathcal{F}u)(\xi)| \leq C_\alpha (1+|\xi|)^N$.

4 Suppose X is a vector space over $\mathbb{C}$ and $\mathcal{F}$ is a collection of linear maps $X \to \mathbb{C}$. Equip X with the $\mathcal{F}$-weak topology, i.e. the weakest topology in which all elements of $\mathcal{F}$ are continuous.

a. Show that the vector space operations $+ : X \times X \to X$ and $\cdot : \mathbb{C} \times X \to X$ are continuous in this topology (where $X \times X$ and $\mathbb{C} \times X$ are equipped with the product topology).

b. Suppose that $\rho : X \to [0,\infty)$ is continuous and is a seminorm. Show that there exist $k \in \mathbb{N}$, $\ell_1, \ldots, \ell_k \in \mathcal{F}$ and $C > 0$ such that $\rho(x) \leq C\sum_{j=1}^k |\ell_j(x)|$ for all $x \in X$.

5 For $s \geq 0$, let $H^s(\mathbb{T})$ be the space of L^2 functions f on the circle $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ whose Fourier coefficients $\hat{f}_n = \int e^{-inx} f(x)\,dx$ satisfy $\sum (1+n^2)^s |\hat{f}_n|^2 < \infty$, with norm $\|f\|_s^2 = (2\pi)^{-1}\sum(1+n^2)^s|\hat{f}_n|^2$.

a. Show that for $r > s \geq 0$, the inclusion map $\iota : H^r(\mathbb{T}) \to H^s(\mathbb{T})$ is compact.

b. Show that if $s > 1/2$, then $H^s(\mathbb{T})$ includes continuously into $C(\mathbb{T})$, and indeed the inclusion map is compact.