Ph.D. Qualifying Exam, Real Analysis

Fall 2017, part I

Do all five problems. Write your solution for each problem in a separate blue book.

1 Let X be a Banach space, and let S denote the unit sphere $S = \{x \in X : \|x\| = 1\}$.

a. Suppose $y_n \in S$ for all n, and $y_n \to y \in X$ weakly. Show that $\|y\| \leq 1$.

b. Suppose that X is a separable infinite dimensional Hilbert space, $y \in X$ and $\|y\| \leq 1$. Show that there exists a sequence $\{y_n\}_{n=1}^{\infty}$ with $y_n \in S$ for all n such that $y_n \to y$ weakly.

2 Let $1 < p < \infty$. Suppose that $f_n \in L^p([0,1])$ and $\|f_n\|_p \leq 1$ for all n. Assuming that $f_n(x) \to 0$ a.e., prove that $f_n \to 0$ weakly in L^p.

3 Let $e_\xi(x) = e^{ix\cdot\xi}$, $x, \xi \in \mathbb{R}^n$. The Fourier transform on tempered distributions $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ is defined by duality with Schwartz functions $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$: $(\mathcal{F}u)(\phi) = u(\mathcal{F}\phi)$ for $u \in \mathcal{S}'$, $\phi \in \mathcal{S}$, where $\mathcal{F}\phi(\xi) = \int e_{-\xi}(x)\,\phi(x)\,dx$, $\phi \in \mathcal{S}$.

For compactly supported distributions, i.e. distributions u for which $u = \chi u$ for some $\chi \in C_c^\infty(\mathbb{R}^n)$, $u(e_{-\xi}) = u(\chi e_{-\xi})$ is directly defined.

Show that for such distributions, $\mathcal{F}u$ is the tempered distribution given by the polynomially bounded function $\xi \mapsto u(\chi e_{-\xi})$ (include a proof of this polynomially bounded statement: there exist $C > 0$, $N \in \mathbb{N}$ such that $|u(\chi e_{-\xi})| \leq C(1 + |\xi|)^N$).

4 Suppose that X, Y, Z are Banach spaces $P : X \to Y$, $X \subset Z$ with the inclusion map compact. Suppose also that there is a constant $C > 0$ such that for all $x \in X$, $\|x\|_X \leq C(\|Px\|_Y + \|x\|_Z)$.

a. If P is injective, show that there is another constant $C' > 0$ such that for all $x \in X$, $\|x\|_X \leq C'\|Px\|_Y$.

b. Without assuming that P is injective show that there is a complementary subspace W of $\mathrm{Ker}P$, i.e. a closed subspace W of X with $W \oplus \mathrm{Ker}P = X$, such that there is another constant $C' > 0$ such that for all $x \in W$, $\|x\|_X \leq C'\|Px\|_Y$.

5 Let Q be the unit cube in $\mathbb{R}^n$, and let $H_0^2(Q)$, be the closure of C^∞ functions supported in Q with respect to the norm whose square is $\|f\|^2_{H_0^2(Q)} = \int_Q |f|^2 + |\nabla f|^2 + |\nabla^2 f|^2$.

a. Prove that for every $\epsilon > 0$, there exists a constant C_ϵ so that for all $f \in H_0^2(Q)$,

$$\int_Q |\nabla f|^2 \leq \epsilon \int_Q |\nabla^2 f|^2 + C_\epsilon \int_Q |f|^2.$$

b. Suppose that u is in the Hölder space $C^{1,\alpha}(I)$, $\alpha \in (0,1)$, $I = [a,b]$. Prove that for every $\epsilon > 0$ there exists C_ϵ such that

$$\sup_I |u'(x)| \leq \epsilon [u]_{I;1,\alpha} + C_\epsilon \sup_I |u|, \qquad \text{where } [u]_{I;1,\alpha} = \sup_{x \neq y, x,y \in I} \frac{|u'(x) - u'(y)|}{|x-y|^\alpha}.$$

c. Let $L = L_0 + \sum_{i=1}^n b_i(x)\frac{\partial}{\partial x_i} + c(x)$, $L_0 = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}$, with C^2 coefficients. Suppose that for every $u \in H_0^2(Q)$, $\|u\|^2_{H_0^2(Q)} \leq C(\|L_0 u\|^2_{L^2(Q)} + \|u\|^2_{L^2(Q)})$. Prove, using a) that a similar estimate holds if L_0 is replaced by L.

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Fall 2017, part II

Do all five problems. Write your solution for each problem in a separate blue book.

1 Two quick problems.

a. Show that a subspace of a Banach space is closed in the norm topology if and only if it is closed in the weak topology.

b. Suppose that H is a Hilbert space and $A \in \mathcal{L}(H)$ (bounded linear operator on H) is self-adjoint. Show that $\|A\| = \sup\{|\lambda| : \lambda \in \sigma(A)\}$ (with $\sigma(A)$ denoting the spectrum).

2 **a.** Is it possible to find uncountably many disjoint measurable subsets of $\mathbb{R}$ with strictly positive Lebesgue measure? Give an example or show that this is impossible.

b. Suppose $f \in L^1(\mathbb{R})$. Show that for any $\lambda > 0$, $\lim_{n\to\infty} n^{-\lambda} f(nx) = 0$ for almost all $x \in \mathbb{R}$, where $n \in \mathbb{N}^+$ is understood in the limit.

3 Let $K \subset \mathbb{R}^n$ be a compact set, and $f : K \to \mathbb{R}$ be continuous. Show that there exists a continuous function $\bar{f} : \mathbb{R}^n \to \mathbb{R}$ such that $\bar{f}(x) = f(x)$ for all $x \in K$.

4 Let $\hat{f}(\xi) = \int e^{-ix\xi} f(x)\,dx$ be the Fourier transform on Schwartz functions $f \in \mathcal{S}(\mathbb{R})$.

Suppose $f \in \mathcal{S}(\mathbb{R})$ satisfies $f(2\pi n) = 0$ and $\hat{f}(n) = 0$ for all integers n. Prove or disprove that f must be the zero function.

5 For $u \in \mathcal{S}'(\mathbb{R})$ let Du denote the distributional derivative of u, and let Mu be the distribution xu (i.e. $(Mu)(\phi) = u(x\phi)$, $\phi \in \mathcal{S}(\mathbb{R})$). Let $H_1 = \{u \in L^2(\mathbb{R}) : Du \in L^2,\ Mu \in L^2\}$, equipped with the norm $\|u\|_{H_1}^2 = \|u\|_{L^2}^2 + \|Du\|_{L^2}^2 + \|Mu\|_{L^2}^2$, and let $H_2 = \{u \in \mathcal{S}'(\mathbb{R}) : \exists u_0, u_1, u_2 \in L^2(\mathbb{R}) : u = u_0 + Du_1 + Mu_2\}$ with norm $\|u\|_{H_2} = \inf\{(\|u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2)^{1/2} : u = u_0 + Du_1 + Mu_2\}$.

a. Show that H_1 and H_2 are Banach spaces.

b. Show that the map $j : H_2 \to H_1^*$, $j(v)(u) = \langle u, v\rangle = \langle u, v_0\rangle + \langle Du, v_1\rangle + \langle Mu, v_2\rangle$, $v = v_0 + Dv_1 + Mv_2$, $v_j \in L^2(\mathbb{R})$, is well defined, and is an invertible (conjugate) linear map between Banach spaces. *Note: You may use without proof that $\mathcal{S}(\mathbb{R})$ is dense in H_1.*