**Ph.D. Qualifying Exam, Real Analysis**

**Fall 2021, part I**

Do all five problems. Write your solution for each problem in a separate blue book.

**1** Two short problems.

**a.** Suppose $(X, \mathcal{B}, \mu)$ is a measure space and $f$ is a non-negative measureable function. Show that

$$\lim_{n\to\infty} \int_X n \log(1 + f/n)\, d\mu = \int_X f\, d\mu.$$

**b.** Suppose that $f \in L^1(\mathbb{R})$. Show that the Fourier transform of $f$, given by $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx$, $\xi \in \mathbb{R}$, is continuous and $\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0$.

**2**

**a.** Suppose $X$ is a Banach space, $A_n \in \mathcal{L}(X)$ are compact operators, $n \in \mathbb{N}$. Suppose $\{A_n\}_{n\in\mathbb{N}}$ converges to $A \in \mathcal{L}(X)$ in the operator norm topology. Show that $A$ is compact.

**b.** Give (with proof) an example of a sequence $\{A_n\}_{n\in\mathbb{N}}$ of compact operators on a Banach space $X$ converging to a *non-compact* operator $A \in \mathcal{L}(X)$ in the strong operator topology.

**3** Recall that $W^{1,p} = W^{1,p}(\mathbb{R})$ is the collection of $L^p(\mathbb{R})$ functions $f$ such that the distributional derivative $f'$ of $f$ is also in $L^p$. If $u \in W^{1,p}$ show that $|u| \in W^{1,p}$ and compute its distributional derivative.

**4** Let $\mathcal{D}'(\mathbb{T})$ denote the set of distributions on $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, i.e. the dual of $C^\infty(\mathbb{T})$, equipped with the 'weak-* topology', i.e. the weakest topology in which the maps $E_\phi : \mathcal{D}'(\mathbb{T}) \to \mathbb{C}$, $E_\phi(u) = u(\phi)$ are continuous for all $\phi \in C^\infty(\mathbb{T})$.

**a.** For $\phi \in C^\infty(\mathbb{T}^2)$ let $\phi_x(y) = \phi(y, x)$, $(y, x) \in \mathbb{T} \times \mathbb{T}$, so $\phi_x \in C^\infty(\mathbb{T})$. Show that if $u \in \mathcal{D}'(\mathbb{T})$ and $\phi \in C^\infty(\mathbb{T}^2)$ then the function $f : \mathbb{T} \to \mathbb{C}$ defined by $f(x) = u(\phi_x)$ is $C^\infty$.

**b.** Show that the convolution map $* : C(\mathbb{T}) \times C(\mathbb{T}) \to C(\mathbb{T})$ given by

$$*(f, g)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x - y) g(y)\, dy$$

(usually written as $(f * g)(x)$) has a unique separately continuous extension to a map $\tilde{*} : \mathcal{D}'(\mathbb{T}) \times \mathcal{D}'(\mathbb{T}) \to \mathcal{D}'(\mathbb{T})$.

**5** Suppose that $H_1$ and $H_2$ are separable Hilbert spaces and $A : H_1 \to H_2$ is a bounded linear operator. Suppose that there exist $B \in \mathcal{L}(H_2, H_1)$ and a compact operator $E$ on $H_1$, such that $BA = I - E$, where $I$ is the identity operator on $H_1$.

**a.** Show that the nullspace of $A$ is finite dimensional and the range of $A$ is closed in $H_2$.

**b.** Give an example of such operators $A, B, E$ (and spaces $H_1, H_2$) such that the orthocomplement of the range of $A$ is infinite dimensional.

**c.** Show that if in addition $A^*$ is injective then in fact $A$ is surjective.

**Ph.D. Qualifying Exam, Real Analysis**

**Fall 2021, part II**

Do all five problems. Write your solution for each problem in a separate blue book.

**1** Suppose that $X$ is a Banach space.

**a.** Show that the natural inclusion $\iota : X \to X^{**}$, given by $(\iota(x))(\lambda) = \lambda(x)$, $x \in X$, $\lambda \in X^*$, is an isometry.

**b.** Show that if $X$ is a Banach space and $X^* = X^{***}$ (under the natural inclusion) then $X = X^{**}$.

**2** Let $\mu$ be the Lebesgue measure on $\mathbb{R}^n$, $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$. For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, define the maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{\mu(B_r)} \int_{B_r(x)} |f|.$$

**a.** Show that there is $C > 0$ (depending only on $n$) so that if $f \in L^1(\mathbb{R}^n)$ then

$$\mu(\{x : Mf > t\}) \le Ct^{-1}\|f\|_{L^1}.$$

**b.** For any non-zero $f \in L^1(\mathbb{R}^n)$, show that $Mf$ is not in $L^1(\mathbb{R}^n)$.

**3** Suppose $X$ is a Banach space, $A \in \mathcal{L}(X)$ and $\|A\| < 1$. Show that there exists $B \in \mathcal{L}(X)$ such that $B^2 = I + A$.

**4** Let $C^\alpha([0,1])$, $0 < \alpha < 1$, denote the space of Hölder continuous functions on $[0,1]$, i.e. continuous functions such that $\|f\|_\alpha = \sup|f| + \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty$, equipped with the norm $\|.\|_\alpha$.

Suppose that $0 < \beta < \alpha < 1$, $\|f_n\|_{C^\alpha([0,1])} \le 1$, $n \in \mathbb{N}$. Show that there is a subsequence $\{f_{n_i}\}_{i=1}^\infty$ that converges in $C^\beta([0,1])$.

**5** Suppose that $P(\xi) = \sum_{|\alpha|\le m} a_\alpha \xi^\alpha$, $a_\alpha \in \mathbb{C}$, is a polynomial of degree $m$ on $\mathbb{R}^n$; here for $\alpha \in \mathbb{N}^n$, $|\alpha| = \sum_{j=1}^n \alpha_j$, and $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$. Let $P(D)$ be the corresponding differential operator, $P(D) = \sum_{|\alpha|\le m} a_\alpha D^\alpha$, $D_j = -i\partial_j$, $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$. We say that $P$ is elliptic if $\mathbb{R}^n \ni \xi \neq 0$ implies $\sum_{|\alpha|=m} a_\alpha \xi^\alpha \neq 0$.

**a.** Show that if $P$ is elliptic, $u \in S'(\mathbb{R}^n)$ and $Pu \in S(\mathbb{R}^n)$ then $u \in C^\infty(\mathbb{R}^n)$.

**b.** Recall that for $m \ge 0$, $H^m(\mathbb{T}^n)$ is the subset of $L^2(\mathbb{T}^n)$ consisting of functions whose Fourier coefficients satisfy $\sum_{k\in\mathbb{Z}^n}(1+|k|^2)^m|\hat{f}(k)|^2 < \infty$. Here $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ and $\hat{f}(k) = (2\pi)^{-n/2}\int e^{-ix\cdot k} f(x)\,dx$, $k \in \mathbb{Z}^n$.

Show that if $P$ is elliptic of order $m$ then $P$ considered as an operator $P : H^m(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$ has finite dimensional nullspace.