# Ph.D. Qualifying Exam, Real Analysis

## Spring 2024, part I

**Do all five problems. Write your name on the solutions. Use separate pages for separate problems.**

You may write on the both sides of a page. If you use more than one page for a problem, please staple them together with the stapler provided and make sure that you are stapling pages in the correct order.

**1** Suppose that $f$ is a Schwartz function on $\mathbb{R}$ with $\int_{-\infty}^{\infty} x^k f(x)\,\mathrm{d}x = 0$ for all $k \in \{0\} \cup \mathbb{N}$. Is $f$ the zero function? Prove this or given a counterexample.

**2** Let $1 < p < \infty$. Suppose $\{f_n\}_{n=1}^{\infty} \subset L^p([0,1])$ are functions such that for each $n \in \mathbb{N}$, $f_n(x) \geq 0$ for a.e. $x$. If $f_n$ converges weakly (in $L^p$) to a function $f \in L^p([0,1])$, prove that $f(x) \geq 0$ for a.e. $x$.

**3** For $k \in \mathbb{N}$, define $I_k = [-k, k] \subset \mathbb{R}$. Endow $C^\infty(\mathbb{R})$ with the locally convex space topology given by the semi-norms

$$\|f\|_k = \sum_{j=0}^{k} \sup_{x \in I_k} |f^{(j)}(x)|, \quad k \in \mathbb{N}.$$

Denote by $(C^\infty(\mathbb{R}))^*$ its topological dual. Prove that $\Lambda \in (C^\infty(\mathbb{R}))^*$ if and only if there exists a tempered distribution $\lambda \in (\mathcal{S}(\mathbb{R}))^*$ such that $\operatorname{supp}(\lambda) \subset \mathbb{R}$ is a bounded set and

$$\lambda(f) = \Lambda(f), \quad \forall f \in \mathcal{S}(\mathbb{R}).$$

(Recall that for a linear map $\lambda : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$, we say $x \notin \operatorname{supp}(\lambda)$ if $\exists$ open set $U_x$, $x \in U_x$ such that $\operatorname{supp}(f) \subset U_x \implies \lambda(f) = 0$.)

**4** Let $\mathcal{F}$ denote the Fourier transform on $\mathbb{R}^n$.

**a.** Prove that there exists $C > 0$ such that the following holds for all $f \in \mathcal{S}(\mathbb{R}^n)$:

$$\|\mathcal{F}f\|_{L^{p'}} \leq C\|f\|_{L^p}, \quad \forall 1 \leq p \leq 2, \quad \frac{1}{p'} + \frac{1}{p} = 1.$$

**b.** Prove that there exists $C > 0$ such that the following holds. For any Lebesgue measurable $E \subset \mathbb{R}^n$ with $\mathcal{L}^n(E) < \infty$ and for all $f \in \mathcal{S}(\mathbb{R}^n)$ with $\operatorname{supp}(\mathcal{F}f) \subset E$:

$$\|f\|_{L^q} \leq C(\mathcal{L}^n(E))^{\frac{1}{p} - \frac{1}{q}} \|f\|_{L^p} \quad \forall 1 \leq p \leq q \leq \infty,\ 1 \leq p \leq 2.$$

**5** Let $L^0([0,1])$ be the vector space of Lebesgue measurable functions. Let $d$ be the metric on $L^0([0,1])$ given by

$$d(f,g) = \int_0^1 \frac{|f-g|(x)}{1 + |f-g|(x)}\,\mathrm{d}x.$$

**a.** Prove that $f_n \to f$ in the metric $d$ if and only if $f_n \to f$ in measure.

**b.** Let $\mathcal{U} \subset L^0([0,1])$ be a non-empty open convex neighborhood of $0$. Prove that $\mathcal{U} = L^0([0,1])$.

**c.** Suppose $T : (L^0([0,1]), d) \to \mathbb{R}$ is a continuous linear function. Prove that $T$ is the zero map.

**Ph.D. Qualifying Exam, Real Analysis**

**Spring 2024, part II**

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**1** Let $H$ be a Hilbert space.

**a.** Suppose $\{x_n\}_{n=1}^{\infty} \subset H$, $x \in H$. Prove that $x_n \to x$ in norm if and only if $x_n \to x$ weakly and $\|x_n\| \to \|x\|$.

**b.** Let $\{T\}_{n=1}^{\infty} \in \mathcal{L}(H)$. Prove that $T_n \to T$ in the strong operator topology if and only if $T_n \to T$ in the weak operator topology and $T_n^* T_n \to T^* T$ in the weak operator topology.

**2** Let $f_n : [0,1] \to [0,1]$ be a sequence of Lebesgue measurable functions such that $\lim_{n\to\infty} f_n(x) = 0$ almost everywhere. Let

$$Mf_n(x) = \sup_{x \in I \subset [0,1]} \frac{1}{|I|} \int_I |f(y)| dy,$$

where the maximum is taken over closed intervals, be the Hardy–Littlewood maximal function. Show that $\lim_{n\to\infty} Mf_n = 0$ a.e.

**3** Suppose $X_1$, $X_2$, $Y$ are reflexive Banach spaces and $A_j : X_j \to Y$ are bounded linear maps. Suppose also that there is $C > 0$ such that

$$\|\lambda\|_{Y^*} \leq C(\|A_1^* \lambda\|_{X_1^*} + \|A_2^* \lambda\|_{X_2^*}), \quad \forall \lambda \in Y^*,$$

where $X_1^*$, $X_2^*$, $Y^*$ denote the dual spaces of $X_1$, $X_2$, $Y$, respectively, and $A_i^*$ is the adjoint of $A_i$.

Show that for all $y \in Y$ there exist $x_j \in X_j$ such that $A_1 x_1 + A_2 x_2 = y$.

**4** If $U \subset \mathbb{R}^n$ is a bounded open set and $\delta > 0$, prove that there is a countable collection of closed balls $\{\overline{B(x_i, \rho_i)}\}_{\mathrm{j}=1}^{\infty}$ such that $\rho_i \in (0, \delta)$ for all $i$, $\overline{B(x_i, \rho_i)} \cap \overline{B(x_j, \rho_j)} = \emptyset$ whenever $i \neq j$ and $\mathcal{L}^n(U \setminus \cup_{i=1}^{\infty} \overline{B(x_i, \rho_i)}) = 0$, where $\mathcal{L}^n$ denotes the Lebesgue measure.

**5** Let $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. Given $t \in \mathbb{R}$ and any Borel measure $\mu$ on $\mathbb{T}$, define its translate $\mu_t$ by $\mu_t(A) = \mu(A_t)$, where $A_t = \{x : x + t \in A\}$ for any Borel set $A$.

**a.** Is the map $t \mapsto \mu_t$ necessarily continuous in the topology given by the dual norm, where measures are viewed as the dual of $C(\mathbb{T})$? Prove or disprove.

**b.** Is the map $t \mapsto \mu_t$ necessarily continuous in the weak-* topology on measures as the dual of $C(\mathbb{T})$? Prove or disprove.

**c.** Is the map $t \mapsto \mu_t(A)$ necessarily continuous when $A$ is Borel? Prove or disprove.

**d.** Is the map $t \mapsto \mu_t(A)$ necessarily continuous if we in addition assume that $\mu$ is absolutely continuous with respect to the Lebesgue measure? Prove or disprove.