Ph.D. Qualifying Exam, Real Analysis

Fall 2013, part I

Do all five problems. Write your solution for each problem in a separate blue book.

1 Suppose that K is a bounded measurable function on $\{(x,y):\ 0 \le y \le x \le 1\} \subset [0,1]^2$. Let T be the Volterra operator

$$(Tf)(x) = \int_0^x K(x,y)\, f(y)\, dy,$$

$f \in L^p([0,1])$, $1 < p < \infty$. Show that for $f \in L^p([0,1])$, $1 < p < \infty$, this expression indeed defines a function $Tf \in L^p([0,1])$, and $T : L^p([0,1]) \to L^p([0,1])$ is bounded in this range of p. Show moreover that the spectrum of T is $\{0\}$.

2 Let $0 < \alpha < 1$. Show that there exists a continuous real-valued function f on $[0,1]$ such that for all $x \in [0,1]$ and for all $\delta > 0$ and $C > 0$ there is $y \in [0,1] \cap (x-\delta, x+\delta)$ such that the estimate $|f(y) - f(x)| \le C|y-x|^\alpha$ does *not* hold.

3 Suppose $f \in C^\infty(\mathbb{R}^n)$ is real valued and $K \subset \mathbb{R}^n$ is compact. Show that if the differential of f does not vanish on K then for all $u \in C^\infty(\mathbb{R}^n)$ with support in K and for all $N \ge 0$ there is $C > 0$ such that

$$\Big| \int e^{i\omega f(x)} u(x)\, dx \Big| \le C\omega^{-N},\ \omega > 1.$$

4 Show that the set of Schwartz functions, $\mathcal{S}(\mathbb{R})$, consisting of C^∞ functions such that the seminorms $\rho_{k,\ell}(\phi) = \sup_{x\in\mathbb{R}} |x^\ell \partial^k \phi|$, $\ell, k \in \mathbb{N}$, are finite, is not a normed space, i.e. there exists no norm on $\mathcal{S}(\mathbb{R})$ giving rise to the locally convex topology induced by these seminorms. (Here $\mathbb{N}$ is the set of non-negative integers.)

5 Let $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ denote the circle. Thus, $C^k(\mathbb{T})$ can be identified with the space of 2π-periodic C^k functions on $\mathbb{R}$. Let $M : C^k(\mathbb{T}) \to C^k(\mathbb{T})$ denote the multiplication operator by e^{ix}: $(Mf)(x) = e^{ix} f(x)$, and for $k \ge 1$ (including $k = \infty$) let $\frac{d}{dx} : C^k(\mathbb{T}) \to C^{k-1}(\mathbb{T})$ be the usual derivative.

Suppose that $T : C^\infty(\mathbb{T}) \to C^\infty(\mathbb{T})$ is a linear map (no continuity of any kind is assumed!) and T satisfies $T\frac{d}{dx} = \frac{d}{dx}T$ and $TM = MT$ (on $C^\infty(\mathbb{T})$). Show that there exists $c \in \mathbb{C}$ such that $T = c\,\mathrm{Id}$, i.e. $Tf = cf$ for all $f \in C^\infty(\mathbb{T})$.

(Hint: First show that if $y \in \mathbb{T}$ and $f(y) = 0$ then $(Tf)(y) = 0$; this shows that T is multiplication by a function.)

Ph.D. Qualifying Exam, Real Analysis

Fall 2013, part II

Do all five problems. Write your solution for each problem in a separate blue book.

1 Two short problems:

a. Show that $C^\infty([0,1])$ is dense in $L^p([0,1])$ for $1 \le p < \infty$.

b. Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{x_n\}_{n=1}^\infty$. Let $\{y_n\}$ be a sequence in $\mathcal{H}$ and prove that the following two statements are equivalent.

1. $\lim_{n\to\infty}(x, y_n) = 0$ for all $x \in \mathcal{H}$.
2. $\{\|y_n\|\}_{n=1}^\infty$ is bounded in $\mathbb{R}$, and $\lim_{n\to\infty}(x_m, y_n) = 0$ for each $m \ge 1$.

2 Suppose X, Y are Hilbert spaces.

a. Show that if $A \in \mathcal{L}(X, Y)$ has closed range and finite dimensional kernel then there is a constant C and a finite rank orthogonal projection $F \in \mathcal{L}(X)$ such that $\|x\|_X \le C(\|Ax\|_Y + \|Fx\|_X)$.

b. Show that if in addition $Y/\mathrm{Ran}A$ is finite dimensional, then there is a constant $\tilde{C}$ and a finite rank orthogonal projection $\tilde{F} \in \mathcal{L}(Y)$ such that $\|y\|_Y \le \tilde{C}(\|A^*y\|_X + \|\tilde{F}y\|_Y)$.

3 Let $\mathcal{S}(\mathbb{R})$ denote set of Schwartz functions, and $\mathcal{S}'(\mathbb{R})$ the dual space of tempered distributions. For $u \in \mathcal{S}'(\mathbb{R})$ let Du denote the distributional derivative of u, and let Mu be the distribution xu (i.e. $(Mu)(\phi) = u(x\phi)$, $\phi \in \mathcal{S}(\mathbb{R})$). Let $\mathcal{H} = \{u \in L^2(\mathbb{R}) : \ Du \in L^2, \ Mu \in L^2\}$, equipped with the norm $\|u\|_{\mathcal{H}}^2 = \|u\|_{L^2}^2 + \|Du\|_{L^2}^2 + \|Mu\|_{L^2}^2$. Show that $\mathcal{H}$ is a Hilbert space (with the norm being induced by the inner product), $\mathcal{S}(\mathbb{R})$ is dense in $\mathcal{H}$, and the inclusion map $\iota : \mathcal{H} \to L^2(\mathbb{R})$ is compact.

4 Recall that with $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ the circle, the dual space of $C(\mathbb{T})$ is the Banach space $\mathcal{M}$ of finite Borel measures. Show that $L^1(\mathbb{T})$ is closed in $\mathcal{M}$ in the Banach topology induced by this duality, but is not closed in $\mathcal{M}$ if the latter is equipped with the weak-* topology. Is $L^1(\mathbb{T})$ closed in $\mathcal{M}$ if $\mathcal{M}$ is equipped with the weak topology?

5 Suppose that $\{a_n : \ n = 0, 1, 2, \ldots\}$ is *any* sequence of real numbers. Show that there exists a real valued function $f \in C^\infty(\mathbb{R})$ such that $f^{(n)}(0) = a_n$. (Hint: let $\chi \in C_c^\infty(\mathbb{R})$ identically 1 near 0, and choose $\epsilon_n > 0$ appropriately so that $\lim_{n\to\infty} \epsilon_n = 0$ and

$$\sum_{n=0}^\infty \frac{a_n x^n}{n!}\, \chi(x/\epsilon_n)$$

converges in C^∞.)