Ph.D. Qualifying Exam, Real Analysis

Spring 2023, part I

Do all five problems. Write your solution for each problem in a separate blue book.

1 Prove that if H is a nonseparable Hilbert space and A is a compact operator on H, then the nullspace $\ker(A)$ is itself a nonseparable Hilbert space.

2 Compute $\sup \int_0^1 x^3 g(x)\mathrm{d}x$, where the supremum is taken over $g \in L^2([0,1])$ with

$$\|g\|_{L^2([0,1])} = 1, \quad \int_0^1 g(x)\,\mathrm{d}x = \int_0^1 xg(x)\,\mathrm{d}x = \int_0^1 x^2 g(x)\,\mathrm{d}x = 0.$$

Justify your answer.

3 Let μ be a Borel measure on $\mathbb{R}^n$. Suppose that μ is singular with respect to the Lebesgue measure m_n and define

$$M\mu(x) = \sup_{r>0} \frac{\mu(B(x,r))}{m_n(B(x,r))}.$$

Prove that $\mu\{x : M\mu(x) < \infty\} = 0$.

4 Let $\mathbb{R}_+ = [0,\infty)$ and let $C_c^\infty(\mathbb{R}^+)$ be the set of functions which are restrictions of $C_c^\infty(\mathbb{R})$ functions to $\mathbb{R}_+$. For $f \in C_c^\infty(\mathbb{R}_+)$, define

$$Tf(z) = \int_0^\infty e^{-ixz} f(x)\,\mathrm{d}x, \quad z \in \mathbb{C}.$$

Define also $T_\eta f(\xi) = Tf(\xi + i\eta)$ for $\xi, \eta \in \mathbb{R}$.

a. For $\eta \le 0$, show that T_η, a priori defined on $C_c^\infty(\mathbb{R}_+)$, extends to a bounded map $\widehat{T}_\eta : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ and satisfies

$$\int_\infty^\infty |\widehat{T}_\eta f|^2(\xi)\,\mathrm{d}\xi \le 2\pi \int_0^\infty |f|^2(x)\,\mathrm{d}x, \quad \forall \eta \le 0.$$

b. Let $a > 0$ and define the space $L^2(\mathbb{R}_+, e^{2ax}\mathrm{d}x)$ to be the closure of $C_c^\infty(\mathbb{R}_+)$ under the norm $\int_0^\infty |f|^2 e^{2ax}\,\mathrm{d}x$. Prove that for any $f \in L^2(\mathbb{R}_+, e^{2ax}\mathrm{d}x)$, the function $Tf(z)$ extends to be holomorphic in the half-plane $\{z \in \mathbb{C} : Im(z) < a\}$.

5

a. Prove that the following inequality holds for any $f \in \mathcal{S}(\mathbb{R})$:

$$\|f\|_{L^2}^2 \le 2\|xf\|_{L^2}\|f'\|_{L^2}.$$

(Hint: apply the Cauchy–Schwarz inequality to the function xff'.)

b. Show that equality holds in the inequality in part (a) if and only if $f(x) = ae^{-bx^2/2}$ for some $b \in \mathbb{C}$ with positive real part and some $a \in \mathbb{C}$.

c. Given $f \in \mathcal{S}(\mathbb{R})$, denote its Fourier transform by $\hat{f}$. Prove that there exists $C > 0$ such that the following inequality holds for any $c, d \in \mathbb{R}$ and for any $f \in \mathcal{S}(\mathbb{R})$,

$$\left(\int_{-\infty}^\infty |f|^2(x)\,\mathrm{d}x\right)^2 \le C\left(\int_{-\infty}^\infty (x-c)^2|f(x)|^2\,\mathrm{d}x\right)\left(\int_{-\infty}^\infty (\xi-d)^2|\hat{f}(\xi)|^2\,\mathrm{d}\xi\right).$$

Ph.D. Qualifying Exam, Real Analysis

Spring 2023,   part II

Do all five problems. Write your solution for each problem in a separate blue book.

1 Let $\{(X_j, d_j)\}_{j=1,2}$ be metric spaces and suppose there is a continuous surjection $f : (X_1, d_1) \to (X_2, d_2)$ satisfying $d_2(f(y), f(x)) \geq d_1(y, x)$, $\forall x, y \in X_1$. Prove or give a counterexample:

a. If (X_1, d_1) is complete, then so is (X_2, d_2).

b. If (X_2, d_2) is complete, then so is (X_1, d_1).

2 Let $f \in L^1([0,1])$ and let $1 < p < \infty$. Prove that $f \in L^p([0,1])$ if and only if

$$\sup_{\{I_j\}} \sum_j |I_j| \left(\frac{1}{|I_j|} \int_{I_j} |f| \right)^p < \infty,$$

where the supremum is taken over all finite partitions of $[0,1]$ into intervals $\{I_j\}$.

3 Suppose that $\mathcal{H}$ is a Hilbert space, $A \in \mathcal{L}(\mathcal{H})$ is self-adjoint, and for $t \in \mathbb{R}$ define $U(t) = f_t(A)$ via the functional calculus where $f_t(s) = e^{its}$.

a. Show that for $t \in \mathbb{R}$, $x_0 \in \mathcal{H}$, $x(t) = U(t)x_0$ satisfies $x \in C^1(\mathbb{R}; \mathcal{H})$, $x(0) = x_0$, $\frac{dx}{dt} = iAx$.

b. Show also that $U(t)$ is unitary for $t \in \mathbb{R}$, and $U(t) - I$ is compact for all $t \in \mathbb{R}$ if and only if A is compact.

4 For $\epsilon > 0$, define $u_{\pm,\epsilon} : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ by $u_{\pm,\epsilon}(\phi) = \int_{\mathbb{R}} (x \pm i\epsilon)^{-1} \phi(x)\, dx$, $\phi \in \mathcal{S}(\mathbb{R})$.

a. Show that for all $\epsilon > 0$, $u_{\pm,\epsilon} \in \mathcal{S}'(\mathbb{R})$, and that there exist $u_\pm \in \mathcal{S}'(\mathbb{R})$ such that for all $\phi \in \mathcal{S}(\mathbb{R})$, $u_{\pm,\epsilon}(\phi) \to u_\pm(\phi)$ as $\epsilon \to 0$.

b. Compute $u_+ - u_-$, and show that it can be represented by a locally finite Borel measure, i.e., there exists a locally finite Borel measure μ such that $u_+(\phi) - u_-(\phi) = \int_{\mathbb{R}} \phi \, \mathrm{d}\mu, \quad \forall \phi \in \mathcal{S}(\mathbb{R})$.

c. Prove that u_+ itself can<u>not</u> be represented by a locally finite Borel measure.

5 Let $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$ with coordinates (x_1, x_2). Define the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ to be the completions of $C^\infty(\mathbb{T}^2)$ under the respective inner products

$$\langle f, g \rangle_{\mathcal{H}_1} = \langle f, g \rangle_{L^2(\mathbb{T}^2)} + \langle \partial_{x_1} f, \partial_{x_1} g \rangle_{L^2(\mathbb{T}^2)}, \quad \langle f, g \rangle_{\mathcal{H}_2} = \langle f, g \rangle_{L^2(\mathbb{T}^2)} + \langle \partial_{x_2} f, \partial_{x_2} g \rangle_{L^2(\mathbb{T}^2)}.$$

a. Prove that if $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_2$, then $fg \in L^2$.

(Hint: first show that $\mathcal{H}_1$ embeds continuously into the $L^2_{x_2} L^\infty_{x_1}$ space, i.e., the normed space with norm $(\int_0^{2\pi} \mathrm{esssup}_{x_1} |f|^2(x_1, x_2)\, \mathrm{d}x_2)^{\frac{1}{2}}$.)

b. Suppose $\|f_n\|_{\mathcal{H}_1} \leq 1$, $\|g_n\|_{\mathcal{H}_2} \leq 1$, $f_n \to 0$ weakly in $\mathcal{H}_1$ and $g_n \to 0$ weakly in $\mathcal{H}_2$. Prove that $\langle f_n, g_n \rangle_{L^2(\mathbb{T}^2)} \to 0$.