Ph.D. Qualifying Exam, Real Analysis

Spring 2014, part I

Do all five problems. Write your solution for each problem in a separate blue book.

1 Recall that for any $0 < \alpha < 1$, the space $C^\alpha([0,1])$ is the set of continuous functions on $[0,1]$ with

$$\|f\|_{C^\alpha} = \sup|f| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty,$$

equipped with the norm $\|.\|_{C^\alpha}$.

a. Show that the unit ball of $C^\alpha([0,1])$ has compact closure in $C([0,1])$.

b. Show that $C^\alpha([0,1])$ is of first category in $C([0,1])$.

2 Two short problems.

a. Show that there exists a continuous function $f : [0,1] \to \mathbb{R}$ and a Lebesgue measureable set $A \subset [0,1]$ with measure $m(A) = 0$ such that $f(A)$ is measurable but has measure $m(f(A)) > 0$.

b. Show that if $f : [0,1] \to \mathbb{R}$ is absolutely continuous and A is Lebesgue measurable with measure 0 then $f(A)$ is measurable with measure 0.

3 Suppose X, Y are Hilbert spaces and $A \in \mathcal{L}(X, Y)$, the vector space of all bounded linear operators from X to Y.

a. Show that there is a unique operator $A^* : Y \to X$ such that $\langle A^* y, x\rangle_X = \langle y, Ax\rangle_Y$ for all $x \in X$ and $y \in Y$. Show further that $A^* \in \mathcal{L}(Y, X)$ with $\|A^*\|_{\mathcal{L}(Y,X)} = \|A\|_{\mathcal{L}(X,Y)}$.

b. Show that A has closed range if and only if $A^* \in \mathcal{L}(Y, X)$ has closed range.

4 Let $f \in \mathcal{S}(\mathbb{R})$ be a Schwartz function (i.e. $f \in C^\infty(\mathbb{R})$ and $x^j \partial^k f$ is bounded for all $j, k \in \mathbb{N}_0$). Suppose too that $\int_{\mathbb{R}} |f(x)|^2\, dx = 1$. Recall the Fourier transform of f, $\hat{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixy} f(x)\, dx$. Show that

$$\left(\int_{\mathbb{R}} x^2\, |f(x)|^2\, dx \right) \left(\int_{\mathbb{R}} y^2\, |\hat{f}(y)|^2\, dy \right) \geq \frac{1}{4}.$$

(*Hint*: Use Plancherel's theorem and the Cauchy-Schwarz inequality. The main point is to provide a positive lower bound which is independent of f; if you cannot obtain the constant $\frac{1}{4}$, try to find some lower bound with a positive constant c.)

5 Let X be a Banach space over $\mathbb{C}$ and M and N closed subspaces of X. Write $M + N = \{x \in X : \exists\, m \in M,\ n \in N,\ x = m + n\}$.

a. Show that $M + N$ is closed if and only if there exists $C > 0$ such that for all $x \in M + N$ there exist $m \in M, n \in N$ such that $x = m + n$ and $\|m\| + \|n\| \leq C\|x\|$.

b. Suppose that $\ell_M : M \to \mathbb{C}$ and $\ell_N : N \to \mathbb{C}$ are continuous linear functionals and $\ell_M|_{M \cap N} = \ell_N|_{M \cap N}$. Show that if $M + N$ is closed, then there exists $\ell \in X^*$ such that $\ell|_M = \ell_M$ and $\ell|_N = \ell_N$.

c. Give an example of a Banach space X and closed subspaces M, N such that $M \cap N = \{0\}$ but $M + N$ is *not* closed.

Ph.D. Qualifying Exam, Real Analysis

Spring 2014, part II

Do all five problems. Write your solution for each problem in a separate blue book.

1 Let $(X, \|\cdot\|)$ be a Banach space.

a. Show that every closed subspace of X is weakly closed.

b. Show that if X is infinite dimensional then the weak closure of the unit sphere, $S = \{x \in X : \|x\| = 1\}$, is the entire unit ball $B = \{x \in X : \|x\| \leq 1\}$.

2 Two short problems.

a. Suppose f_n is a sequence in $L^2([0,1])$ with $\|f_n\|_{L^2} \leq 1$ for all n, and $f \in L^2([0,1])$. Consider the following two statements: (i) $f_n \to f$ in L^2, (ii) $f_n(x) \to f(x)$ for almost every $x \in [0,1]$. For each of the implications (i)⇒(ii) and (ii)⇒(i) either give a proof or give a counterexample.

b. Show that there are Banach spaces X, Y and operators $A, A_n \in \mathcal{L}(X, Y)$, $n \in \mathbb{N}$, such that for every n, $\operatorname{Ran} A_n = D$, and $A_n \to A$ in the strong operator topology, but D is a proper subspace of $\operatorname{Ran} A$. Can such an example exist if D is closed in Y?

3

a. Suppose that $f \in L^1(\mathbb{R})$, $f \geq 0$, and that $f \neq 0$ as an element of L^1. Let $\hat{f}(\xi) = \int e^{-ix\cdot\xi} f(x)\,dx$ be the Fourier transform of f. Show that $\sup |\hat{f}|$ is attained exactly at 0.

b. Suppose $f \in \mathcal{S}(\mathbb{R})$, $f \geq 0$, $\int f(x)\,dx = 1$, $\int x f(x)\,dx = 0$, and let $f_1 = f$, $f_k = f_{k-1} * f$, $k \geq 2$. Show that with $g_k(x) = k f_k(kx)$, $g_k \to \delta_0$ in distributions as $k \to \infty$. (You may use part (a) even if you have not proved it.)

4 Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and let μ be the Lebesgue measure on it. For $A \subset \mathbb{T}$ and $y \in \mathbb{T}$ write $A + y = \{x + y \in \mathbb{T} : x \in A\}$. Suppose $A \subset \mathbb{T}$ is measurable, and for each $n \in \mathbb{N}^+$, $A + 2^{-n} = A$ (i.e. A is invariant under dyadic translations). Show that either $\mu(A) = 0$ or $\mu(A) = 1$.

5 Consider $\mathcal{S}(\mathbb{R})$, the space of Schwartz functions, and its dual $\mathcal{S}'(\mathbb{R})$, the space of tempered distributions.

a. Suppose $u \in \mathcal{S}'(\mathbb{R})$ and $x^k u = 0$ for some k (i.e. $u(x^k\phi) = 0$ for all $\phi \in \mathcal{S}(\mathbb{R})$). Show that u is a finite sum of derivatives of the delta distribution at 0, i.e., there exists $k \geq 1$ and $a_j \in \mathbb{C}$ such that $u(\phi) = \sum_{j=0}^{k-1} a_j \phi^{(j)}(0)$ for all $\phi \in \mathcal{S}(\mathbb{R})$.

b. One says that $x_0 \notin \operatorname{supp} u$ if x_0 has a neighborhood U such that for all $\phi \in \mathcal{S}(\mathbb{R})$ with $\operatorname{supp}\phi \subset U$, $u(\phi) = 0$. Suppose now that $u(\phi) = 0$ if $0 \notin \operatorname{supp}\phi$, so $\operatorname{supp} u \subset \{0\}$. Show that u is again a finite sum of derivatives of the delta distribution. (*Hint*: prove that if $\phi \in \mathcal{S}(\mathbb{R})$ and $\phi^{(j)}(0) = 0$ for $j \leq k$ then there are $\phi_n \in \mathcal{S}(\mathbb{R})$, $n \geq 1$, such that $0 \notin \operatorname{supp}\phi_n$, $\phi_n(x) = \phi(x)$ for $|x| \geq 1$ and $\sum_{j\leq k} \sup_{x\in\mathbb{R}} |\phi_n^{(j)}(x) - \phi^{(j)}(x)| \to 0$ as $n \to \infty$.)