Ph.D. Qualifying Exam, Real Analysis

Spring 2019, part I

Do all five problems. Write your solution for each problem in a separate blue book.

1 Prove that if $f \in L^1$ and its distributional derivative f' is also in L^1, then f is absolutely continuous.

2 Let (X, μ) be a measure space such that $\mu(X) = 1$. Assume that the functions $f_n(x)$, $x \in X$ are measurable and for each $y \in \mathbb{R}$ and $n \in \mathbb{N}$ we have

$$\mu\{x : f_n(x) < y\} = \int_{-\infty}^{y} e^{-t^2/2} \frac{dt}{\sqrt{2\pi}}.$$

Show that for almost every $x \in X$ (with respect to the measure μ) there exist only finitely many $n \in \mathbb{N}$ such that $|f_n(x)| > 2019\sqrt{\log n}$. (Hint: use the Borel-Cantelli lemma.)

3 A form of the Hahn-Banach theorem states that if X is a normed vector space over either $\mathbb{F} = \mathbb{R}$ (real version) or $\mathbb{F} = \mathbb{C}$ (complex version), Y a subspace, $\ell : Y \to \mathbb{F}$ a $\mathbb{F}$-linear map with $|\ell(y)| \leq \|y\|$ for all $y \in Y$ then there is a $\mathbb{F}$-linear map $L : X \to \mathbb{F}$ such that $L|_Y = \ell$ and $|L(x)| \leq \|x\|$ for all $x \in X$. Given the real version of the Hahn-Banach theorem, prove the complex version.

4 Recall that A is a Hilbert-Schmidt operator (on a separable Hilbert space H) if $\sum_{n\in\mathbb{N}} \|Ae_n\|^2 < \infty$ for an orthonormal basis $\{e_n : \ n \in \mathbb{N}\}$.

a. Prove that every Hilbert-Schmidt operator is compact, but there exist compact operators which are not Hilbert-Schmidt.

b. Let $H = L^2((0,1))$ and suppose that $(Af)(x) = \int_0^x f(y)dy$. Prove that A is Hilbert-Schmidt.

5 For $s \geq 0$ define $H^s(\mathbb{R})$ to be the subspace of $L^2(\mathbb{R})$ consisting of $f \in L^2(\mathbb{R})$ with $\int_{\mathbb{R}}(1+|\xi|^2)^s|\hat{f}(\xi)|^2\, d\xi < \infty$, and let $\|f\|_{H^s(\mathbb{R})} := \|(1 + |\xi|^2)^{\frac{s}{2}} \hat{f}(\xi)\|_{L^2(\mathbb{R})}$, where $\hat{f}$ denotes the Fourier transform of f.

a. Prove that for every $s > \frac{1}{2}$, there exists a constant $C > 0$ (depending only on s) such that for every $f \in H^s(\mathbb{R})$,

$$\|f\|_{L^\infty(\mathbb{R})} \leq C\|f\|_{H^s(\mathbb{R})}.$$

b. Show if $u, v \in H^1(\mathbb{R})$ then their product $uv \in H^1(\mathbb{R})$ and there is $C' > 0$ such that

$$\|uv\|_{H^1(\mathbb{R})} \leq C'\|u\|_{H^1(\mathbb{R})}\|v\|_{H^1(\mathbb{R})}.$$

(Hint: show and use that $\|u\|_{H^1(\mathbb{R})}$ is a constant multiple of $(\|u\|_{L^2}^2 + \|u'\|_{L^2}^2)^{1/2}$.)

Ph.D. Qualifying Exam, Real Analysis

Spring 2019, part II

Do all five problems. Write your solution for each problem in a separate blue book.

1

a. Prove that there is an uncountable number of disjoint balls of radius 1 in ℓ^∞, and hence ℓ^∞ is not separable.

b. Show that on the other hand ℓ^1 is separable.

2 Show that for each $k \in \mathbb{R} \setminus \{0\}$ the limit

$$I(k) := \lim_{R \to +\infty} \int_{-R}^{R} \frac{e^{ikx}}{1+|x|} dx$$

exists, and find all $m \in \mathbb{R}$ such that $\lim_{k\to 0} |k|^m I(k) = 0$.

3 Suppose X, Y are reflexive Banach spaces, $A_n \in \mathcal{L}(X,Y)$ for $n \in \mathbb{N}$. Suppose that for all $x \in X$ and for all $\ell \in Y^*$, $\lim_{n\to\infty} \ell(A_n x)$ exists. Show that there exists $A \in \mathcal{L}(X,Y)$ such that $A_n \to A$ in the weak operator topology. Give an example of A_n satisfying these assumptions such that A_n does not converge to A in the strong operator topology.

4 Let $H^1(\mathbb{R}^2)$ be the subspace of $L^2(\mathbb{R}^2)$ consisting of $f \in L^2(\mathbb{R}^2)$ with the distributional derivatives of f satisfying $\partial_j f \in L^2(\mathbb{R}^2)$, $j = 1, 2$. Define X_j as the completion of $\mathcal{S}(\mathbb{R}^2)$ (Schwartz functions) with respect to the norm

$$\|u\|_j = (\|u\|_{L^2}^2 + \|\partial_j u\|_{L^2}^2)^{1/2},$$

where ∂_j is the jth partial, $j = 1, 2$. Show that X_j is (naturally identified with) a subspace of $L^2(\mathbb{R}^2)$ (i.e. the inclusion from $\mathcal{S}(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2)$ extends to an injective continuous linear map from X_j to $L^2(\mathbb{R}^2)$), and $X_1 \cap X_2 = H^1(\mathbb{R}^2)$.

5 Suppose that X, Y are reflexive separable Banach spaces, X^*, Y^* the duals, $P \in \mathcal{L}(X,Y)$, and suppose that its adjoint $P^* \in \mathcal{L}(Y^*, X^*)$ satisfies the following property: There is a Banach space Z and a compact map $\iota : Y^* \to Z$ and $C > 0$ such that for all $\phi \in Y^*$,

$$\|\phi\|_{Y^*} \le C(\|P^*\phi\|_{X^*} + \|\iota\phi\|_Z).$$

a. Show that $\mathrm{Ker} P^*$ is finite dimensional.

b. Let V be a closed subspace of Y^* with $V \oplus \mathrm{Ker} P^* = Y^*$. Show that there is $C' > 0$ such that for $\phi \in V$, $\|\phi\|_{Y^*} \le C' \|P^*\phi\|_{X^*}$.

c. Show that for $f \in Y$ such that $\ell(f) = 0$ for all $\ell \in \mathrm{Ker} P^*$, there is $u \in X$ such that $Pu = f$.