Ph.D. Qualifying Exam, Real Analysis

Spring 2011, part I

Do all five problems. Write your solution for each problem in a separate blue book.

1 Consider ℓ^2 with norm $\|\{a_n\}_{n=1}^\infty\|^2 = \sum_{n=1}^\infty |a_n|^2$. Let T be the operator on ℓ^2 given by $T(\{a_1, a_2, \ldots\}) = \{a_2, \frac{1}{2}a_3, \frac{1}{3}a_4, \ldots\}$, i.e. $T(\{a_n\}_{n=1}^\infty) = \{n^{-1}a_{n+1}\}_{n=1}^\infty$. Show that T is compact, and find the spectrum of T. Does T have any eigenvalues?

2

a. Show that the set of points A at which a function $f : [0, 1] \to \mathbb{R}$ is continuous is a G_δ set (countable intersection of open sets).

b. Give an example of a function $f : [0, 1] \to \mathbb{R}$ that is continuous at all the irrationals and is discontinuous at all the rationals.

c. Show that there is no function $f : [0, 1] \to \mathbb{R}$ which is continuous at all the rationals and is discontinuous at all the irrationals.

3 Let $\mathbb{T}^n = (\mathbb{R}/(2\pi\mathbb{Z}))^n$ be the n-torus, e_j the image of the jth unit vector in $\mathbb{R}^n$ in the torus, and let $\mathcal{D}'(\mathbb{T}^n)$ denote the set of distributions on the torus, equipped with the weak-* topology.

a. Show that for all $y \in \mathbb{T}^n$ the map $L_y : C(\mathbb{T}^n) \to C(\mathbb{T}^n)$ defined by $(L_y f)(x) = f(x + y)$ has a unique continuous extension to a map $\mathcal{D}'(\mathbb{T}^n) \to \mathcal{D}'(\mathbb{T}^n)$.

b. If $u \in \mathcal{D}'(\mathbb{T}^n)$ (i.e. u is a distribution on the torus) then for all j, $u_h = h^{-1}(L_{he_j}u - u)$ converges to the distributional derivative $\partial_j u$ as $h \to 0$.

c. For $\phi \in C^\infty(\mathbb{T}^{n+m})$ let $\phi_x(y) = \phi(y, x)$, $(y, x) \in \mathbb{T}^n \times \mathbb{T}^m$, so $\phi_x \in C^\infty(\mathbb{T}^n)$. Show that if $u \in \mathcal{D}'(\mathbb{T}^n)$ and $\phi \in C^\infty(\mathbb{T}^{n+m})$ then the function $f : \mathbb{T}^m \to \mathbb{C}$ defined by $f(x) = u(\phi_x)$ is C^∞.

4 For $E \subset \mathbb{R}$ let $E + E = \{x + y : \ x, y \in E\}$, and define $E - E$ similarly. Show that if E is a measurable subset of $\mathbb{R}$ of positive Lebesgue measure then $E - E$ and $E + E$ contain non-empty open sets.

5 Let $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, and for $f \in L^2(\mathbb{T}^n)$ let $\hat{f}(k) = (2\pi)^{-n/2} \int_{\mathbb{T}^n} e^{-ix\cdot k} f(x)\, dx$, $k \in \mathbb{Z}^n$, denote the Fourier coefficients of f. For $m \geq 0$, $H^m(\mathbb{T}^n)$ is the subset of $L^2(\mathbb{T}^n)$ consisting $f \in L^2(\mathbb{T}^n)$ with Fourier coefficients satisfying $\sum_{k\in\mathbb{Z}^n}(1 + |k|^2)^m |\hat{f}(k)|^2 < \infty$, and with norm $\|f\|^2_{H^m(\mathbb{T}^n)} = \sum_{k\in\mathbb{Z}^n}(1 + |k|^2)^m |\hat{f}(k)|^2$.

Suppose that $s > r \geq 0$, $m \geq 0$, and $P : H^s(\mathbb{T}^n) \to H^m(\mathbb{T}^n)$ is a continuous linear map satisfying

$$\|u\|_{H^s(\mathbb{T}^n)} \leq C(\|Pu\|_{H^m(\mathbb{T}^n)} + \|u\|_{H^r(\mathbb{T}^n)}),\ u \in H^s(\mathbb{T}^n).$$

Show that the nullspace of P is a finite dimensional subspace of $H^s(\mathbb{T}^n)$, and its range is closed in $H^m(\mathbb{T}^n)$.

Ph.D. Qualifying Exam, Real Analysis

Spring 2011, part II

Do all five problems. Write your solution for each problem in a separate blue book.

1 A projection $P \in \mathcal{L}(X)$ on a Hilbert space X is an operator with $P^2 = P$. Show that such a projection P is an orthogonal projection (i.e. $x - P(x) \in (P(X))^\perp$ for all $x \in X$) if and only if P is self-adjoint.

2 Let X be a vector space over $\mathbb{C}$, $\mathcal{F}$ a vector space of linear maps $X \to \mathbb{C}$, and equip X with the weakest topology in which all members of $\mathcal{F}$ are continuous. Show that the only continuous linear maps $X \to \mathbb{C}$ are those in $\mathcal{F}$.

3 Suppose X, Y are [reflexive] Banach spaces, $A_n \in \mathcal{L}(X, Y)$ for $n \in \mathbb{N}$. Suppose that for all $x \in X$ and for all $\ell \in Y^*$, $\lim_{n\to\infty} \ell(A_n x)$ exists. Show that there exists $A \in \mathcal{L}(X, Y)$ such that $A_n \to A$ in the weak operator topology. Give an example of A_n satisfying these assumptions such that A_n does not converge to A in the strong operator topology.

4 Let $C^\alpha([0,1])$, $0 < \alpha < 1$, denote the space of Hölder continuous functions on $[0,1]$, i.e. continuous functions such that $\|f\|_\alpha = \sup|f| + \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty$, equipped with the norm $\|.\|_\alpha$.

a. Suppose $0 < \alpha < 1$, X is a subspace of $C^\alpha([0,1])$, and X is closed as a subspace of $C([0,1])$. Show that X is finite dimensional.

b. Suppose that X is a subspace of $C([0,1])$ and X is closed as a subspace of $L^2([0,1])$ (with the standard norm). Show that X is finite dimensional.

5 We say that a sequence $\{a_n\}_{n=1}^\infty$ in $[0,1]$ is equidistributed (in $[0,1]$) if for all intervals $[c,d] \subset [0,1]$, the proportion of elements $a_n \in [c,d]$ for $n \leq N$ converges as $N \to \infty$ to $d - c$.

a. Show that $\{a_n\} \subset [0,1]$ is equidistributed if and only if the measures $\mu_N = \frac{1}{N}\sum_{1\leq n\leq N} \delta_{a_n}$ (with δ_{a_n} the unit mass at a_n) converge in the weak-* topology to the Lebesgue measure.

b. Prove that the sequence $a_n = n\alpha \mod 1$, $n \geq 1$, is equidistributed if and only if α is irrational.