Ph.D. Qualifying Exam, Real Analysis

Spring 2010, part I

Do all five problems. Write your solution for each problem in a separate blue book.

1 Suppose that X is a Banach space.

a. Suppose that the weak and the weak-* topologies on X^* are the same. Show that X is reflexive.

b. Show that every closed subspace of X (closed with respect to the norm topology) is weakly closed.

2 Suppose that K is a bounded measurable function on $\{(x,y) : 0 \leq y \leq x \leq 1\} \subset [0,1]^2$. Let T be the Volterra operator

$$(Tf)(x) = \int_0^x K(x,y)\, f(y)\, dy,$$

$f \in L^p([0,1])$, $1 < p < \infty$. Show that T is well-defined, and $T : L^p([0,1]) \to L^p([0,1])$ is bounded. Show moreover that the spectrum of T is $\{0\}$.

3 Let $S(\mathbb{R}^n)$, resp. $S'(\mathbb{R}^n)$, denote the set of Schwartz functions, resp. tempered distributions, on $\mathbb{R}^n$.

a. Suppose that $u \in S'(\mathbb{R}^n)$ and $x_j u = 0$ for $j = 1, \ldots, n$. Show that there exists $c \in \mathbb{C}$ such that $u = c\delta_0$.

b. Suppose that $f \in S'(\mathbb{R})$, $\psi_0 \in S(\mathbb{R})$ with $\int_{\mathbb{R}} \psi_0(x)\, dx \neq 0$, and $a \in \mathbb{R}$. Show that there is a unique $u \in S'(\mathbb{R})$ such that $u' = f$ and $u(\psi_0) = a$.

4 A number $\xi \in \mathbb{R}$ is *diophantine* (with exponent k, $k > 0$) if for some constant $C > 0$ there are no rational numbers p/q ($p, q \in \mathbb{Z}$, $q > 0$) such that $|\xi - p/q| < Cq^{-k}$.

ξ is a *Liouville number* if it is not diophantine, i.e. if for every $k > 0$ there exist integers p, q such that $q > k$ and $|\xi - p/q| \leq q^{-k}$.

a. Prove that the set of diophantine numbers is of the first category in $\mathbb{R}$; equivalently: the set of Liouville numbers is residual in $\mathbb{R}$.

Hint: Write $E_k = \cap_N \cup_{p,q\in\mathbb{Z},\, q>N} \{\xi : |\xi - p/q| < q^{-k}\}$, and show that $\mathcal{L} = \cap_k E_k$.

b. Prove that the set $\mathcal{L}$ of Liouville numbers has zero Hausdorff dimension. This means: for every $\epsilon > 0$ and finite interval I there are intervals I_n, $n \in \mathbb{N}$, such that $I \cap \mathcal{L} \subset \cup_{n\in\mathbb{N}} I_n$, and $\sum_n |I_n|^\epsilon < \epsilon$.

5 Suppose that $P(\xi) = \sum_{|\alpha|\leq m} a_\alpha \xi^\alpha$, $a_\alpha \in \mathbb{C}$, is a polynomial of degree m on $\mathbb{R}^n$; here for $\alpha \in \mathbb{N}^n$, $|\alpha| = \sum_{j=1}^n \alpha_j$, and $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}$. Let $P(D)$ be the corresponding differential operator, $P(D) = \sum_{|\alpha|\leq m} a_\alpha D^\alpha$, $D_j = -i\partial_j$, $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$. We say that P is elliptic if $\mathbb{R}^n \ni \xi \neq 0$ implies $\sum_{|\alpha|=m} a_\alpha \xi^\alpha \neq 0$.

a. Show that if P is elliptic, $u \in S'(\mathbb{R}^n)$ and $Pu \in S(\mathbb{R}^n)$ then $u \in C^\infty(\mathbb{R}^n)$.

b. Recall that for $m \geq 0$, $H^m(\mathbb{T}^n)$ is the subset of $L^2(\mathbb{T}^n)$ consisting of functions whose Fourier coefficients satisfy $\sum_{k\in\mathbb{Z}^n} (1 + |k|^2)^m |\hat{f}(k)|^2 < \infty$. Here $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ and $\hat{f}(k) = (2\pi)^{-n/2} \int e^{-ix\cdot k} f(x)\, dx$, $k \in \mathbb{Z}^n$.

Show that if P is elliptic of order m then P considered as an operator $P : H^m(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$ has finite dimensional nullspace.

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Spring 2010, part II

Do all five problems. Write your solution for each problem in a separate blue book.

1 Two short problems:

a. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is increasing, and is differentiable almost everywhere with respect to the Lebesgue measure. Show that $\int_0^1 f'(x)\, dx \leq f(1) - f(0)$.

b. Let $S(\mathbb{R})$, resp. $S'(\mathbb{R})$, denote the set of Schwartz functions, resp. tempered distributions, on $\mathbb{R}$.

For $\phi \in S(\mathbb{R})$, $\epsilon > 0$, $k \in \mathbb{Z}$, define $u_\epsilon : S(\mathbb{R}) \to \mathbb{C}$ by $u_\epsilon(\phi) = \int_{\mathbb{R}} (x + i\epsilon)^{-k} \phi(x)\, dx$. Show that for all $\epsilon > 0$, $u_\epsilon \in S'(\mathbb{R})$, and that there exists $u \in S'(\mathbb{R})$ such that for all $\phi \in S(\mathbb{R})$, $u_\epsilon(\phi) \to u(\phi)$ as $\epsilon \to 0$.

2 Suppose $(X, \mathcal{T})$ is a compact Hausdorff topological space, and f_j, $j \in \mathbb{N}$, are real valued continuous functions on X that separate points, i.e. if $x, y \in X$, $x \neq y$ then there exists $j \in \mathbb{N}$ such that $f_j(x) \neq f_j(y)$. Show that $(X, \mathcal{T})$ is metrizable.

3

a. Show that on $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, $f \in L^p(\mathbb{T})$, $1 \leq p < \infty$ implies that $\int_{\mathbb{T}} |f(x+h) - f(x)|^p\, dx \to 0$ as $h \to 0$.

b. Suppose that $f \in L^p(\mathbb{T})$, $1 < p < \infty$, and $\sup_{h \neq 0,\, |h|<1} \int_{\mathbb{T}} \left| \frac{f(x+h)-f(x)}{h} \right|^p dx < \infty$. Show that the distributional derivative of f, f', satisfies $f' \in L^p(\mathbb{T})$.

4 Let $C_c(\mathbb{R}^N)$ denote the set of continuous real-valued functions of compact support on $\mathbb{R}^N$. Let μ_n, $n \in \mathbb{N}$, and μ be locally finite Borel measures on $\mathbb{R}^N$ such that $\int f\, d\mu_n \to \int f\, d\mu$ for all $f \in C_c(\mathbb{R}^N)$.

a. Show that if $U \subset \mathbb{R}^N$ is open then $\mu(U) \leq \liminf_{n\to\infty} \mu_n(U)$.

b. Show that if S is a bounded Borel set and if $\mu(\partial S) = 0$ then $\mu(S) = \lim_{n\to\infty} \mu_n(S)$.

5 A Banach space X is *uniformly convex* if for every $\epsilon \in (0, 1)$ there exists $\eta < 1$ such that if $x, y \in X$, $\|x\| = \|y\| = 1$ and $\|x - y\| > 2\epsilon$ then $\|\frac{1}{2}(x + y)\| < \eta$.

a. Show that every Hilbert space is uniformly convex.

b. Let X be a uniformly convex Banach space. Suppose that C is a closed convex subset of X, and $z \in X$. Show that there exists a unique $x \in C$ such that $\|x - z\| = \inf\{\|y - z\| :\ y \in C\}$.

c. Give an example showing that the property in (b) can fail if X is not uniformly convex.