Ph.D. Qualifying Exam, Real Analysis

Spring 2018, part I

Do all five problems. Write your solution for each problem in a separate blue book.

1 Two short problems.

a. Show that there is a closed subset E of $[0,1]$ of positive Lebesgue measure and with empty interior.

b. Show that if $f:[0,1]\to\mathbb{R}$ is absolutely continuous and A is Lebesgue measurable with measure 0 then $f(A)$ is measurable with measure 0.

2 Let S be a closed subspace of $C[0,1]$ (with the sup norm). Suppose that $f\in S$ implies that f is continuously differentiable. Prove that S is finite dimensional.

3 Let $K\in L^2_{\mathbb{R}}(\mathbb{R}^2)$, i.e. real valued element of the L^2-space on $\mathbb{R}^2$. Define $T:L^2_{\mathbb{R}}(\mathbb{R})\to L^2_{\mathbb{R}}(\mathbb{R})$ by

$$(Tf)(x)=\int_{\mathbb{R}}K(x,y)f(y)\,\mathrm{d}y.$$

a. Prove that $T:L^2_{\mathbb{R}}(\mathbb{R})\to L^2_{\mathbb{R}}(\mathbb{R})$ is bounded and moreover that T is compact.

b. For $\alpha\in\mathbb{R}$ and $g\in L^2_{\mathbb{R}}(\mathbb{R})$, consider the following equation where $f\in L^2_{\mathbb{R}}(\mathbb{R})$ is an unknown:

$$f(x)+\alpha\int_{\mathbb{R}}K(x,y)f(y)\,\mathrm{d}y=g(x). \tag{1}$$

Prove that there exists $\epsilon>0$ (depending only on K) such that if $|\alpha|<\epsilon$, then (1) admits a unique solution $f\in L^2_{\mathbb{R}}(\mathbb{R})$.

c. Suppose that $\int_{\mathbb{R}}\int_{\mathbb{R}}h(x)K(x,y)h(y)\,\mathrm{d}x\,\mathrm{d}y\geq 0$ for all $h\in L^2_{\mathbb{R}}(\mathbb{R})$. Prove that for all $\alpha\geq 0$, (1) admits a unique solution $f\in L^2_{\mathbb{R}}(\mathbb{R})$.

4 Let $L^2=L^2((0,\infty),x^{-1}\,dx)$, i.e. $\|f\|^2_{L^2}=\int_0^\infty|f(x)|^2\frac{dx}{x}$. For each $s\in\mathbb{R}$ consider the statement: there exists $C>0$ such that for all $u\in C_0^\infty((0,\infty))$, $\|x^{s-1}u\|_{L^2}\leq C\|x^s\partial_x u\|_{L^2}$. Find, with proof, the values of s for which this statement holds. (Hint: rewrite $x^s\partial_x u=(x^{s-1}(x\partial_x)x^{-(s-1)})(x^{s-1}u)$. Let $t=\log x$, and rewrite the estimate in terms of $L^2(\mathbb{R},dt)$. Then use the Fourier transform.)

5 Let X be a Banach space over $\mathbb{C}$ and M and N closed subspaces of X. Write $M+N=\{x\in X:\exists\, m\in M,\ n\in N,\ x=m+n\}$.

a. Show that $M+N$ is closed if and only if there exists $C>0$ such that for all $x\in M+N$ there exist $m\in M, n\in N$ such that $x=m+n$ and $\|m\|+\|n\|\leq C\|x\|$.

b. Suppose that $\ell_M:M\to\mathbb{C}$ and $\ell_N:N\to\mathbb{C}$ are continuous linear functionals and $\ell_M|_{M\cap N}=\ell_N|_{M\cap N}$. Show that if $M+N$ is closed, then there exists $\ell\in X^*$ such that $\ell|_M=\ell_M$ and $\ell|_N=\ell_N$.

c. Give an example of a Banach space X and closed subspaces M,N such that $M\cap N=\{0\}$ but $M+N$ is *not* closed.

Ph.D. Qualifying Exam, Real Analysis

Spring 2018, part II

Do all five problems. Write your solution for each problem in a separate blue book.

1 Suppose f is a non-negative Lebesgue measurable function on $[0,1]$ such that $f > 0$ almost everywhere. Show that for any $\epsilon > 0$ there is $\delta > 0$ such that if E is a Lebesgue measurable subset of $[0,1]$ with measure $m(E) \geq \epsilon$, then $\int_E f(x)\,dx \geq \delta$.

2 Let X be a non-zero Banach space and $T \in \mathcal{L}(X)$. Let $\rho(T)$ be the resolvent set, $\sigma(T)$ the spectrum of T.

a. Suppose $\{\lambda_n\}_{n=1}^{\infty} \subset \rho(T)$ and $\lambda \in \sigma(T)$ such that $\lambda_n \to \lambda$. Prove that

$$\sup_n \|(\lambda_n I - T)^{-1}\|_{\mathcal{L}(X)} = +\infty.$$

b. Using part (a), or otherwise, show that there exists $\lambda \in \sigma(T)$ such that $\lambda I - T$ is not bounded below, i.e. for every $c > 0$, there exists $x \in X \setminus \{0\}$ such that $\|(\lambda I - T)x\|_X \leq c\|x\|_X$.

3 Recall that the Fourier transform of an $L^1(\mathbb{R}^n)$ function f is $(\mathcal{F}f)(\xi) = \int e^{-ix\cdot\xi} f(x)\,dx$, where $\cdot$ is the standard inner product on $\mathbb{R}^n$.

Let A be a real symmetric matrix, and define the function f by $f(x) = e^{-iAx\cdot x/2}$. Show that f is a tempered distribution, and find (with proof) its Fourier transform if $\det(A) \neq 0$. Make sure to give an explicit formula in terms of A, without involving any limits. (*Hint:* Write A as the limit of complex symmetric matrices with negative definite imaginary part.)

4 **a.** Let X be a Banach space and $E \subset X^*$ be a subspace of X^* which is closed in the weak-* topology. Suppose moreover that $\cap_{\lambda\in E}\ker(\lambda) = \{0\}$. Prove that $E = X^*$.

b. Let X, Y be Banach spaces, $T \in \mathcal{L}(X,Y)$. Let $T' \in \mathcal{L}(Y^*, X^*)$ be the adjoint defined by $T'(\lambda)(x) = \lambda(T(x))$ for $x \in X$, $\lambda \in Y^*$. Prove that T is injective if and only if $\mathrm{im}(T') \subset X^*$ is dense in the weak-* topology.

5 Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Then A acts on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ by matrix multiplication, and thus on $C(\mathbb{T}^2)$ via pullback: $(\Phi_A f)(x) = f(Ax)$, $f \in C(\mathbb{T}^2)$, $x \in \mathbb{T}^2$.

a. Show that the action on $C(\mathbb{T}^2)$ extends to a weak-* continuous action on $\mathcal{D}'(\mathbb{T}^2)$, i.e. on distributions on the torus.

b. Express the Fourier coefficients $\widehat{\Phi_A u}(k)$, $k \in \mathbb{Z}^2$, of $\Phi_A u \in \mathcal{D}'(\mathbb{T}^2)$ in terms of the Fourier coefficients of u. (Recall that the Fourier series of f on $\mathbb{T}^2$ is of the form $\sum_{k\in\mathbb{Z}^2} \hat{f}(k) e^{2\pi i k\cdot x}$.)

c. Show that if $f \in L^1(\mathbb{T}^2)$ and $\Phi_A f = f$, then f is an a.e. constant function.

d. Show that the space of invariant distributions, i.e. $u \in \mathcal{D}'(\mathbb{T}^2)$ such that $\Phi_A u = u$, is however infinite dimensional.