Ph.D. Qualifying Exam, Real Analysis

Spring 2016, part I

Do all five problems. Write your solution for each problem in a separate blue book.

1 Two short problems.

a. Suppose $(X, \mathcal{B}, \mu)$ is a measure space. If $\mu(X) < \infty$, are there any inclusions among the spaces $L^1(X, \mu)$, $L^2(X, \mu)$, $L^\infty(X, \mu)$? (List any inclusions you can, and provide a proof for these.) If $\mu(X) = \infty$, but μ is σ-finite, can the *reverse* of these inclusions hold? (Give an example or provide a proof to the contrary.)

b. Suppose E_k, $k = 1, 2, \ldots$, are measurable subsets of $\mathbb{R}^n$ with $\sum_k \mu(E_k) < \infty$, where μ is the Lebesgue measure. Show that almost all x lie in E_k for finitely many k, i.e. that

$$\mu\{x : \ x \in E_k \text{ for infinitely many } k\} = 0.$$

(Make sure that you show the measurability of any set whose measure you use.)

2 Prove that a weakly convergent sequence $x_n \in \ell^1$ also converges strongly.

3 Recall that for $s \geq 0$ the Sobolev space $H^s(\mathbb{R}^n)$ consists of $u \in L^2(\mathbb{R}^n)$ with

$$\int (1 + |\xi|^2)^s |(\mathcal{F}u)(\xi)|^2 \, d\xi < \infty,$$

where $\mathcal{F}$ is the Fourier transform. Let $R : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^{n-1})$ denote the restriction map $(Ru)(x') = u(x', 0)$, $x' \in \mathbb{R}^{n-1}$. Show that for $s > 1/2$, R has a unique continuous extension to a map $R : H^s(\mathbb{R}^n) \to H^{s-1/2}(\mathbb{R}^{n-1})$.

4 Let $\mathbb{T}^n = (\mathbb{R}/(2\pi\mathbb{Z}))^n$ be the n-torus, and let $\mathcal{D}'(\mathbb{T}^n)$ denote the set of distributions on the torus.

a. For $\phi \in C^\infty(\mathbb{T}^{n+m})$ let $\phi_x(y) = \phi(y, x)$, $(y, x) \in \mathbb{T}^n \times \mathbb{T}^m$, so $\phi_x \in C^\infty(\mathbb{T}^n)$. Show that if $u \in \mathcal{D}'(\mathbb{T}^n)$ and $\phi \in C^\infty(\mathbb{T}^{n+m})$ then the function $f : \mathbb{T}^m \to \mathbb{C}$ defined by $f(x) = u(\phi_x)$ is C^∞.

b. For $\phi \in C(\mathbb{T}^m; C^\infty(\mathbb{T}^n))$ (i.e. $x \mapsto \phi_x$ is continuous as a map $\mathbb{T}^m \to C^\infty(\mathbb{T}^n)$) show that $\int_{\mathbb{T}^m} u(\phi_x) \, dx = u(\int_{\mathbb{T}^m} \phi_x \, dx)$.

5 Suppose that w is a measurable function on $\mathbb{R}^n$ which is finite and strictly positive almost everywhere. Suppose that K is a measurable function on $\mathbb{R}^{2n}$ such that

$$\int_{\mathbb{R}^n} |K(x, y)| w(y) \, dy \leq A w(x), \qquad \int_{\mathbb{R}^n} |K(x, y)| w(x) \, dx \leq A w(y)$$

for almost every x, and for almost every y, respectively. Prove that the integral operator

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) \, dy$$

is bounded on $L^2(\mathbb{R}^n)$ with $\|T\| \leq A$.

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1

a. Suppose that Y is a normed complex vector space with norm $\|.\|$, and $\lambda : Y \to \mathbb{C}$ is linear but is not continuous. Show that $N = \lambda^{-1}(\{0\})$ is dense in Y.

b. Show that ℓ^2 has an orthonormal basis $\{x_n\}$, where each x_n actually lies in ℓ^1, and has the property that if we list the components of each x_n as $x_n^{(k)}$, then $\sum_{k=1}^{\infty} x_n^{(k)} = 0$.

2 Suppose that $1 < p < \infty$, $f, f_n \in L^p([0,1])$, $n \in \mathbb{N}$, $\|f_n\|_{L^p} \leq 1$ for all n, and $f_n \to f$ almost everywhere. Show that $f_n \to f$ weakly and $\|f\|_{L^p} \leq 1$.

3

a. Give an example of Hilbert spaces X, Y and an operator $A \in \mathcal{L}(X,Y)$ such that $\mathrm{Ran} A$ is not closed.

b. Show that if X, Y are Hilbert spaces, $A \in \mathcal{L}(X,Y)$, and $\mathrm{Ran} A$ is closed then $\mathrm{Ran} A^*$ is closed (where $A^* \in \mathcal{L}(Y,X)$ is the Hilbert space adjoint). (Hint: reduce to the case when A is invertible.)

c. Show that if there exists $C > 0$ such that for all $y \in Y$, $\|y\|_Y \leq C\|A^* y\|_X$ then A is surjective. Is the conclusion still true if only A^* injective is assumed? (Give a proof or provide a counterexample.)

4 Let $\mathcal{S}(\mathbb{R}^n)$, resp. $\mathcal{S}'(\mathbb{R}^n)$, denote the set of Schwartz functions, resp. tempered distributions, on $\mathbb{R}^n$.

a. Suppose that $f \in \mathcal{S}'(\mathbb{R})$, $\psi_0 \in \mathcal{S}(\mathbb{R})$ with $\int_{\mathbb{R}} \psi_0(x)\,dx \neq 0$, and $a \in \mathbb{R}$. Show that there is a unique $u \in \mathcal{S}'(\mathbb{R})$ such that $u' = f$ and $u(\psi_0) = a$.

b. For $\epsilon > 0$, $k \in \mathbb{N}^+$, define $u_{\pm,\epsilon} : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ by $u_{\pm,\epsilon}(\phi) = \int_{\mathbb{R}} (x \pm i\epsilon)^{-k} \phi(x)\,dx$, $\phi \in \mathcal{S}(\mathbb{R})$. Show that for all $\epsilon > 0$, $u_{\pm,\epsilon} \in \mathcal{S}'(\mathbb{R})$, and that there exist $u_\pm \in \mathcal{S}'(\mathbb{R})$ such that for all $\phi \in \mathcal{S}(\mathbb{R})$, $u_{\pm,\epsilon}(\phi) \to u_\pm(\phi)$ as $\epsilon \to 0$, and compute $u_+ - u_-$.

5 Let $\phi : \mathbb{R} \to \mathbb{C}$ be 2π-periodic and Hölder continuous with exponent α. Let $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(x)\, e^{-inx}\,dx$ denote the n-th Fourier coefficient of ϕ.

a. Show that there exists $C > 0$ such that

$$\sum_{n=-\infty}^{\infty} (1 - \cos(nh))\, |c_n|^2 = \frac{1}{4\pi} \int_{-\pi}^{\pi} |\phi(x+h) - \phi(x)|^2 \leq C|h|^{2\alpha}$$

for each $h \in \mathbb{R}$.

b. Suppose now that $\alpha > \frac{1}{2}$. Show that $\sum_{n=-\infty}^{\infty} |c_n| < \infty$.