Ph.D. Qualifying Exam, Real Analysis

Fall 2019, part I

Do all five problems. Write your solution for each problem in a separate blue book.

1 Let $\mathcal{H}$ be a Hilbert space. We say that a linear operator A on $\mathcal{H}$ is bounded below if there exists $c > 0$ such that for all $x \in \mathcal{H}$, $c\|x\| \leq \|Ax\|$.

Suppose $T : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator. Prove that T is invertible if and only if T and T^* are both bounded below.

2 Let $X \subseteq \mathbb{R}$ be a Borel set and μ be the Lebesgue measure. Suppose there exist $1 \leq p < q < +\infty$ such that $L^q(X,\mu) \subseteq L^p(X,\mu)$. Prove that $\mu(X) < +\infty$. *(Hint: First show that the inclusion is continuous.)*

3 Prove that a closed linear subspace of a reflexive Banach space is reflexive.

4 Suppose $f \in \mathcal{D}'(\mathbb{R})$, with $\mathcal{D}'(\mathbb{R})$ denoting the space of distributions on $\mathbb{R}$.

a. Show that there exists $u \in \mathcal{D}'(\mathbb{R})$ such that $u' = f$.

b. Show that if $v' = f$ as well, then $u - v$ is a distribution given by a constant function.

5 For $s \geq 0$ define $H^s(\mathbb{R}^n)$ to be the subspace of $L^2(\mathbb{R}^n)$ consisting of $f \in L^2(\mathbb{R}^n)$ with

$$\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi < \infty,$$

and let $\|f\|_{H^s(\mathbb{R}^n)} := \|(1 + |\xi|^2)^{\frac{s}{2}} \hat{f}(\xi)\|_{L^2(\mathbb{R}^n)}$, where $\hat{f}$ denotes the Fourier transform of f.

a. Prove that there is no continuous map

$$P : L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

so that $P(u,v) = uv$ for $u, v \in C_0^\infty(\mathbb{R}^n)$ (compactly supported smooth functions).

b. On the other hand, for $s > \frac{n}{2}$, show that there is a continuous map

$$P : H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$$

so that $P(u,v) = uv$ for $u, v \in C_0^\infty(\mathbb{R}^n)$. *(Hint: First prove that for $p > 0$, there is $C = C(p)$ so that*

$$(1 + |\xi|^2)^p \leq C(1 + |\xi - \xi'|^2)^p + C(1 + |\xi'|^2)^p,$$

for any $\xi, \xi' \in \mathbb{R}^n$.)

Ph.D. Qualifying Exam, Real Analysis

Fall 2019, part II

Do all five problems. Write your solution for each problem in a separate blue book.

1 Two short problems.

a. For a topological space M, let $C(M)$ denote the vector space of real valued continuous functions on M.

Suppose X, Y are compact Hausdorff topological spaces. Let D be the linear span of functions of the form $u(x, y) = \phi(x)\psi(y)$, $\phi \in C(X)$, $\psi \in C(Y)$. Show that D is dense in $C(X \times Y)$.

b. Let X be a separable Hilbert space. Show that if K is a compact operator on X, then K is the norm limit of finite rank operators.

2 Let X be a complex vector space. Suppose that $\{\rho_\alpha : \alpha \in A\}$ is a collection of seminorms on X such that for each $x \in X \setminus \{0\}$ there is $\alpha \in A$ such that $\rho_\alpha(x) \neq 0$, and $B : X \times X \to \mathbb{C}$ is a (jointly) continuous bilinear map in the locally convex topology generated by the ρ_α. Show that there exist $\alpha_1, \ldots, \alpha_n \in A$, $C > 0$, such that for all $x, y \in X$,

$$|B(x, y)| \leq C(\rho_{\alpha_1}(x) + \ldots + \rho_{\alpha_n}(x))(\rho_{\alpha_1}(y) + \ldots + \rho_{\alpha_n}(y)).$$

3 For (X, μ) measure space with $\mu(X) < \infty$, show that if $f_i \to f$ in measure and $\sup_i \|f_i\|_{L^p} < \infty$ for some $p > 1$ then $f_i \to f$ in L^1.

4 Suppose that $f : [0, \infty) \to [0, \infty)$ is continuous and for any x in $[0, \infty)$ the sequence $f(x), f(2x), f(3x), \ldots$ tends to zero. Show that $\lim_{x\to\infty} f(x) = 0$.

5

a. Show that for each $L > 0$ there exists C_L so that if $f \in C_c^\infty(\mathbb{R})$ and the support of f is contained inside the interval $[-L, L]$ then

$$\int_{-L}^{L} |f(x)|^2 dx \leq C_L \int_{-L}^{L} |f'(x)|^2 dx.$$

b. Assume that an inequality of the form $\|f\|_{L^2} \leq C\|f\|_{L^1}^a \|\nabla f\|_{L^2}^b$ holds for all f in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. Find the only possible values of a and b — note that they depend on the dimension n.

c. Use the Plancherel identity to show that if $\hat{f}(\xi) = 0$ for $|\xi| \leq R$ then $\|f\|_{L^2}^2 \leq \frac{C_1}{R^2}\|\nabla f\|_{L^2}^2$, and that if $\hat{f}(\xi) = 0$ for $|\xi| \geq R$, then $\|f\|_{L^2}^2 \leq C_1 R^n \|f\|_{L^1}^2$, with a constant C_1 that depends only on the dimension n. Combine these estimates to prove an inequality of the form $\|f\|_{L^2} \leq C\|f\|_{L^1}^a \|\nabla f\|_{L^2}^b$ with a and b you have found in part (b), and a constant C that depends only on the dimension n.