Ph.D. Qualifying Exam, Real Analysis

Fall 2022, part I

Do all five problems. Write your solution for each problem in a separate blue book.

1 Let $Y \subset \mathbb{R}$ be a non-empty compact subset. Show that there is a Hilbert space H and a self-adjoint bounded linear operator $T : H \to H$ such that the spectrum of T is exactly Y.

2 Consider $C^\infty(\mathbb{T})$ (where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$) with the usual Fréchet topology given by the seminorms $\rho_k(f) = \sup_{x\in\mathbb{T}} |f^{(k)}(x)|$. Prove that there is no norm on $C^\infty(\mathbb{T})$ which induces the same topology.

3 Consider $f \in L^1_{loc}(\mathbb{R}^n)$ and assume that the (uncentered) maximal function Mf (given by $Mf(x) = \sup\limits_{B\ni x} \dfrac{1}{|B|}\displaystyle\int_B |f|$, where the supremum is taken over all balls containing x) satisfies $Mf > \lambda > 0$ on $B_1(0)$. Show that $Mf > c\lambda$ on $B_2(0)$ for some $c > 0$ depending only on the dimension n. (Here, $B_R(0) = \{x \in \mathbb{R}^n : |x| < R\}$.)

4 Suppose X, Y are Hilbert spaces, $P : X \to Y$ continuous linear. Suppose Z is a dense subspace of Y and for each $z \in Z$ there exists $x \in X$ such that $Px = z$. Consider in addition the following two estimates: i) (for all $x \in X$) $\|x\|_X \le C\|Px\|_Y$, ii) (for all $v \in Y$) $\|v\|_Y \le C\|P^*v\|_X$. Does either one of these imply that $P : X \to Y$ is invertible? For both inequalities either give a proof of invertibility or provide a counterexample.

5 For every $f \in \mathcal{S}(\mathbb{R}^2)$, define $Tf(x, y, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(x\xi_1+y\xi_2+z|\xi|)} f(\xi_1, \xi_2)\, d\xi_1\, d\xi_2$, where $|\xi| = \sqrt{|\xi_1|^2 + |\xi_2|^2}$.

a. Prove that there exists $C > 0$ such that the following holds for all $f \in \mathcal{S}(\mathbb{R}^2)$ and for all $z \in \mathbb{R}$:

$$\Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Tf(x, y, z)|^2\, dx\, dy\Big)^{1/2} =: \|Tf(\cdot, z)\|_{L^2(\mathbb{R}^2)} \le C\|f\|_{L^2(\mathbb{R}^2)}.$$

b. Prove that there exists $C > 0$ such that the following holds for all $f \in \mathcal{S}(\mathbb{R}^2)$ satisfying $\mathrm{supp}(f) \subset \{(\xi_1, \xi_2) : |\xi| \le 2|\xi_1|\}$ and for all $x \in \mathbb{R}$:

$$\Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Tf(x, y, z)|^2\, dy\, dz\Big)^{1/2} \le C\|f\|_{L^2(\mathbb{R}^2)}.$$

c. Prove that for any $(x, y) \in \mathbb{R}^2$ and every $f \in \mathcal{S}(\mathbb{R}^2)$, there is a $C > 0$ (depending on (x, y) and f) such that the following holds for all $z \in \mathbb{R}$:

$$|Tf(x, y, z)| \le C(1 + |z|)^{-1}.$$

Ph.D. Qualifying Exam, Real Analysis

Fall 2022, part II

Do all five problems. Write your solution for each problem in a separate blue book.

1 If $A \subset [0, 2\pi]$ is measurable, prove that $\lim_{n\to\infty} \int_A \cos nx \, \mathrm{d}x = 0$.

2 Does there exist $f \in C([0,1])$ with $\int_0^1 x f(x)\, \mathrm{d}x = 1$ and $\int_0^1 x^n f(x)\, \mathrm{d}x = 0$ for $n = 0, 2, 3, \cdots$? Give an example or prove that no such f exists.

3 Let V be the vector space of all real-valued Borel measurable functions on $[0,1]$. Show that there is no seminorm $\|\cdot\|$ on V such that a sequence $\{f_n\}$ converges in (the Lebesgue) measure to f on $[0,1]$ if and only if $\lim_{n\to\infty} \|f - f_n\| = 0$. (Hint: assume that such a seminorm exists and show that for each $\epsilon > 0$ there is a collection of functions $g_1, \cdots g_n$ in V with $\|g_j\| \le \epsilon$ and $n^{-1}\sum_j g_j = 1$.)

4 Let $\ell \subset \ell^\infty$ be the space of convergent real sequences $a = \{a_n\}_{n=1}^\infty$. Define $T_0 : \ell \to \mathbb{R}$ by $T_0(a) = \lim_{n\to\infty} a_n$.

a. Prove that there exists a linear functional $T : \ell^\infty \to \mathbb{R}$ such that $T = T_0$ on ℓ and for $a = \{a_n\}_{n=1}^\infty$,

$$\liminf_{n\to\infty} a_n \le T(a) \le \limsup_{n\to\infty} a_n.$$

(Hint: first ignore the inequality involving $\liminf_{n\to\infty} a_n$.)

b. Fix a T as in part (a). For each $E \subset \mathbb{N}$, define $a_E = \{a_{E,n}\}_{n=1}^\infty$ by setting $a_{E,n} = 1$ when $n \in E$ and $a_{E,n} = 0$ otherwise. Define the function $\kappa : 2^{\mathbb{N}} \to \mathbb{R}$ by $\kappa(E) = T(a_E)$. Show that κ is finitely additive but not countably additive.

5 Recall that for $s \ge 0$ the Sobolev space $H^s(\mathbb{R}^n)$ consists of $u \in L^2(\mathbb{R}^n)$ with $\int (1+|\xi|^2)^s |(\mathcal{F}u)(\xi)|^2 \, \mathrm{d}\xi < \infty$, where $\mathcal{F}$ is the Fouier transform.

a. Prove that for every $s > \frac{n}{2}$, there exists a constant $C_1 > 0$ (depending only on s and n) such that for every $f \in H^s(\mathbb{R}^n)$,

$$\|f\|_{L^\infty(\mathbb{R}^n)} \le C_1 \|f\|_{H^s(\mathbb{R}^n)}.$$

b. Prove that for every $s > \frac{n}{2}$, there exists a constant $C_2 > 0$ (depending only on s and n) such that for every $f,\, g \in H^s(\mathbb{R}^n)$,

$$\|fg\|_{H^s(\mathbb{R}^n)} \le C_2 \|f\|_{H^s(\mathbb{R}^n)} \|g\|_{H^s(\mathbb{R}^n)}.$$

(Hint: first prove that for every $p > 0$, there is $C = C(p)$ so that

$$(1+|\xi|^2)^p \le C(1+|\xi-\xi'|^2)^p + C(1+|\xi'|^2)^p, \quad \forall \xi,\, \xi' \in \mathbb{R}^n.)$$

c. Let $s > \frac{n}{2}$ and $f \in H^s(\mathbb{R}^n)$. Prove that there exists $\epsilon > 0$ such that if $\|f\|_{H^s(\mathbb{R}^n)} \le \epsilon$, then $h(x) = \frac{1}{1-f(x)} - 1$ is well-defined (i.e. $1 - f(x) \ne 0$ for a.e. x) and $h \in H^s(\mathbb{R}^n)$.