Ph.D. Qualifying Exam, Real Analysis

Fall 2023, part I

Do all five problems. Write your name on the solutions. Use separate pages for separate problems.

You may write on the both sides of a page. If you use more than one page for a problem, please staple them together with the stapler provided and make sure that you are stapling pages in the correct order.

1 Let μ be a finite Borel measure on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ such that $\lim_{n\to\infty} \int_{\mathbb{T}} z^n \mathrm{d}\mu(z) = 0$. Prove that for any $f : \mathbb{T} \to \mathbb{C}$, $f \in L^1(\mathbb{T}, \mu)$, we have

$$\lim_{n\to\infty} \int_{\mathbb{T}} z^n f(z)\, \mathrm{d}\mu(z) = 0.$$

2 Fix $f \in L^1(\mathbb{T})$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. For each $\tau \in \mathbb{T}$, define f_τ, a translate of f by τ, using $f_\tau(x) = f(x - \tau)$. Consider the sets $\mathcal{B}_f$ and $\mathcal{M}_f$ defined by

$$\mathcal{B}_f = \Big\{ \sum_{i=1}^{N} a_i f_{\tau_i}(x) : N \in \mathbb{N},\, a_i \in \mathbb{R},\, \tau_i \in \mathbb{T} \Big\}, \quad \mathcal{M}_f = \{ f \star g : g \in L^1(\mathbb{T}) \}.$$

Show that $\overline{\mathcal{M}_f} = \overline{\mathcal{B}_f}$, where the overlines denote the $L^1(\mathbb{T})$-closures.

3 Let X be a Banach space and $\mathcal{I} \subset \mathcal{L}(X, X)$ be the set of invertible bounded linear operators.

a. Prove that $\mathcal{I} \subset \mathcal{L}(X, X)$ is open with respect to the operator norm topology.

b. Is $\mathcal{I} \subset \mathcal{L}(X, X)$ necessarily open with respect to the strong operator topology? Prove this or give a counterexample.

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a. Prove that there is a constant $C_1 > 0$ such that $\int_0^1 w^2(t)\, \mathrm{d}t \le C_1 \int_0^1 (w')^2(t)\, \mathrm{d}t$ for all $w \in C^\infty([0, 1])$ satisfying $w(0) = 0 = w(1)$.

b. Prove that there is a constant $C_2 > 0$ such that $\int_{-\infty}^{\infty} u^6(t)\, \mathrm{d}t \le C_2 \int_{-\infty}^{\infty} (u')^2(t)\, \mathrm{d}t$ for all $u \in C_c^\infty(\mathbb{R})$ satisfying $\int_{-\infty}^{\infty} u^2(t)\, \mathrm{d}t = 1$. (Hint: Justify the change of variables $s = \int_{-\infty}^{t} u^2(y)\, \mathrm{d}y$.)

5 For $E \subset \mathbb{R}^n$ and $f : E \to \mathbb{R}^n$, let

$$F = \{x \in E : \text{there is } \{x_k\}_{k=1}^{\infty} \subset E \setminus \{x\} \text{ with } x_k \to x \text{ and } f(x_k) \to f(x)\}.$$

Prove that $E \setminus F$ is at most countable.

Ph.D. Qualifying Exam, Real Analysis

Fall 2023, part II

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You may write on the both sides of a page. If you use more than one page for a problem, please staple them together with the stapler provided and make sure that you are stapling pages in the correct order.

1 Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ denote the spaces of Schwartz functions and tempered distributions, respectively.

a. For every $t > 0$, define $P_t : \mathcal{S}(\mathbb{R}^n) \to S(\mathbb{R}^n)$ by $(P_t f)(x) = f(tx)$. Show that P_t extends (weak-*) continuously to a map $\bar{P}_t : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$.

b. We say that $u \in \mathcal{S}'(\mathbb{R}^n)$ is homogeneous of degree d if $\bar{P}_t u = t^d u$ for all $t > 0$. Show that if $u \in \mathcal{S}'(\mathbb{R}^n)$ is homogeneous of degree d, then its Fourier transform is homogeneous of degree $-n-d$.

c. Show that if $u \in \mathcal{S}'(\mathbb{R}^n)$ is a compactly supported distribution which is homogeneous of some degree d, then u is a differentiated delta distribution, and d is an integer $\leq -n$.

2 For $f \in C^2(\mathbb{R})$, let $M_k = \sup_x |f^{(k)}(x)|$.

a. Prove that $M_1 \leq 2\sqrt{M_0 M_2}$.

b. Show that if equality holds in (a), then f is a constant function.

3 Let $\mathcal{L}f$ denote the Laplace transform

$$\mathcal{L}f(s) = \int_0^\infty e^{-xs} f(x)\,\mathrm{d}x.$$

Prove that $\mathcal{L}$ is a bounded operator on $L^p([0, +\infty))$ if and only if $p = 2$.

4 Suppose H is a closed subspace of $L^2([0,1])$ such that $H \subset C([0,1])$. Prove that H is finite-dimensional.

5 Let H be a separable Hilbert space and $\{e_j\}_{j=1}^\infty$ be an orthonormal basis. A bounded operator A on H is called Hilbert–Schmidt if

$$\sum_{j=1}^\infty ||Ae_j||^2 < \infty. \tag{1}$$

a. Show that (1) implies that $\sum_{j=1}^\infty ||Ae_j'||^2 < \infty$ for *any* orthonormal basis $\{e_j'\}_{j=1}^\infty$ of H.

b. Prove that the set of Hilbert–Schmidt operators on H is itself a Hilbert space, with inner product $\langle A, B\rangle = \sum_{j=1}^\infty \langle Ae_j, Be_j\rangle$.

c. If $H = L^2([0,1]; \mathrm{d}x)$, prove that the Volterra operator

$$Vu(x) = \int_0^x u(y)\,\mathrm{d}y, \quad x \in [0,1],$$

is Hilbert–Schmidt.