Ph.D. Qualifying Exam, Real Analysis

Spring 2017, part I

Do all five problems. Write your solution for each problem in a separate blue book.

1 Two short problems.

a. Does the fact that $\lim_{a\to 0}\int_a^1 f\,dx$ exists guarantee that f is in $L^1[0,1]$? Prove this or give a counterexample.

b. Suppose that $f:\mathbb{R}\to\mathbb{R}$ and for each $x\in\mathbb{R}$ there exists a quadratic polynomial P_x, $P_x(y)=a_x(y-x)^2+b_x(y-x)+c_x$, such that $\lim_{y\to x}|y-x|^{-2}|f(y)-P_x(y)|=0$. Does this imply that f is twice differentiable? Prove this or give a counterexample.

2 Let $T_t: L^2(\mathbb{R})\to L^2(\mathbb{R})$, $t\in\mathbb{R}$, be the operator given by $(T_tf)(x)=f(x-t)$.

a. Suppose that $\|T_t-T_s\|<2$, where the norm is that of $\mathcal{L}(L^2(\mathbb{R}))$, the space of bounded operators on $L^2(\mathbb{R})$. Show that $t=s$.

b. Give (with proof) a locally convex topology on $\mathcal{L}(L^2(\mathbb{R}))$ in which the map $\mathbb{R}\ni t\mapsto T_t\in\mathcal{L}(L^2(\mathbb{R}))$ is continuous.

3 Let ℓ^∞, resp. ℓ^2, denote the vector spaces of bounded, resp. square summable, complex valued sequences with the standard norms.

a. Let $\{a_n\}_{n\in\mathbb{N}}\in\ell^\infty$. Define $T:\ell^2\to\ell^2$ by $(Tx)_n=a_nx_n$. Find (with proof) $\sigma(T)$ (the spectrum of T).

b. Let K be a closed bounded non-empty subset of $\mathbb{C}$. Show that there exist H a Hilbert space and $T:H\to H$ a bounded linear map such that $\sigma(T)=K$.

4 Let $H^1([0,\infty))$ (Sobolev space on the half line) denote the completion of $C_c^1([0,\infty))$ (C^1 functions on $[0,\infty)$ which vanish outside a compact set, but not necessarily at 0) in the norm $\|f\|_{H^1([0,\infty))}=\|f\|_{L^2([0,\infty))}+\|f'\|_{L^2([0,\infty))}$, and $H^1(\mathbb{R})$ be the standard Sobolev space (consisting of functions in $L^2(\mathbb{R})$ whose Fourier transform satisfies $\int(1+|\xi|^2)|\mathcal{F}f(\xi)|^2\,d\xi<\infty$).

a. Show that the restriction map $R: C_c^1(\mathbb{R})\to C_c^1([0,\infty))$ extends to a continuous linear map (still denoted by R) $R: H^1(\mathbb{R})\to H^1([0,\infty))$.

b. Show that there is a continuous linear map $E: H^1([0,\infty))\to H^1(\mathbb{R})$ such that $RE=I$ (identity map). (Hint: consider a map of the form $Ef(x)=f(x)$ when $x\geq 0$, $Ef(x)=\sum_{j=1}^k a_jf(-jx)$ when $x<0$, where k and a_j are appropriately chosen.)

5 Suppose that A is a bounded operator on a Hilbert space H and K be a compact operator on H.

a. If A has closed range, finite dimensional kernel and infinite dimensional cokernel, then show that $A+K$ also has all the same properties.

b. If A has closed range, infinite dimensional kernel, finite dimensional cokernel, then show that $A+K$ has same properties.

c. Is it true that if A has closed range but kernel and cokernel both infinite dimensional, then $A+K$ still has closed range? Prove or give a counterexample.

Ph.D. Qualifying Exam, Real Analysis

Spring 2017, part II

Do all five problems. Write your solution for each problem in a separate blue book.

1 Two short problems.

a. Suppose that X is a Banach space. Show that every closed subspace of X (closed with respect to the norm topology) is weakly closed.

b. Let X, Y, Z be Banach spaces, $T : X \to Y$, $S : Y \to Z$ be linear maps. Suppose $S \circ T$ is bounded and S is both bounded and injective. Show that T is bounded.

2 Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions on a measurable space $(E, \mathcal{E})$. Show that the following functions are also measurable: $\inf_n f_n$, $\sup_n f_n$, $\liminf_n f_n$, $\limsup_n f_n$. Show also that $\{x \in E : f_n(x)$ converges as $n \to \infty\} \in \mathcal{E}$.

3 Let $C^\infty(\mathbb{T})$ denote the vector space of infinitely differentiable complex-valued functions on $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, and let $\|\phi\|_{C^k(\mathbb{T})} = \sum_{j\le k} \sup |\partial^j \phi|$, $\phi \in C^\infty(\mathbb{T})$, be the C^k norm of ϕ. Let $\mathcal{T}$ be the weakest topology on $C^\infty(\mathbb{T})$ in which the functions

$$f_{k,\psi}(\phi) = \|\phi - \psi\|_{C^k(\mathbb{T})} : C^\infty(\mathbb{T}) \to [0, \infty),$$

$k \ge 0$, $\psi \in C^\infty(\mathbb{T})$, are continuous.

a. Show that $\mathcal{T}$ is metrizable, and write down an explicit metric giving rise to the topology $\mathcal{T}$.

b. Show that there exists no norm $\|.\|$ on $C^\infty(\mathbb{T})$ such that $\mathcal{T}$ is the topology given by the norm $\|.\|$.

4 Suppose $f \in C^\infty(\mathbb{R}^n)$ complex-valued with $\operatorname{Im} f \ge 0$ and $K \subset \mathbb{R}^n$ is compact. Suppose that for all points $x \in K$ with $\operatorname{Im} f(x) = 0$, the differential of f does not vanish at x. Show that for all $u \in C^\infty(\mathbb{R}^n)$ with support in K and for all $N \ge 0$ there is $C > 0$ such that

$$\Big| \int e^{i\omega f(x)} u(x)\, dx \Big| \le C\omega^{-N},\ \omega > 1.$$

5 Let X_j, $j = 1, 2, \ldots$, be real-valued, independent and identically distributed random variables, such that the probability density $p(x)$ of each X_j is a Schwartz class function, and set $Z_n = \frac{X_1+\cdots+X_n}{n}$. Show that the probability distribution function of Z_n is $p_Z(x) = n[p \star p \star \ldots p](nx)$, and there exists a constant s such that for any Schwartz function $g(x)$ we have $E(g(Z_n)) \to g(s)$.

A word on notation: a *random variable* X is a measurable function on a measure space Ω equipped with a (non-negative) measure μ such that $\int_\Omega d\mu = 1$. A Borel measurable function $p(x)$ is the *probability density* of X if for any Borel set $B \subseteq \mathbb{R}$ we have $\mu(\omega \in \Omega :\ X(\omega) \in B) = \int_B p(x)dx$. The *expected value* of a random variable Z is $E(Z) = \int_{\mathbb{R}} xp(x)dx$. Finally, random variables $X_1, X_2, \ldots, X_n$ are *independent* if for any collection of the Borel sets $B_1, \ldots, B_n$ we have

$$\mu(\omega \in \Omega :\ X_1(\omega) \in B_1, \ldots, X_n(\omega) \in B_n) = \mu(\omega \in \Omega :\ X_1(\omega) \in B_1) \cdot \cdots \cdot \mu(\omega \in \Omega :\ X_n(\omega) \in B_n).$$