Ph.D. Qualifying Exam, Real Analysis

Fall 2009, part I

Do all five problems. Write your solution for each problem in a separate blue book.

1 Two short problems:

a. For a topological space M, let $C(M)$ denote the vector space of real valued continuous functions on M.

Suppose X, Y are compact Hausdorff topological spaces. Let D be the linear span of functions of the form $u(x, y) = \phi(x)\psi(y)$, $\phi \in C(X)$, $\psi \in C(Y)$. Show that D is dense in $C(X \times Y)$.

b. Suppose that X is a Banach space, $\{x_j\}_{j=1}^{\infty}$ is a sequence in X with the property that $\cup_{n=1}^{\infty} X_n = X$, where $X_n = \mathrm{Span}\{x_1, \dots, x_n\}$. Show that X is finite dimensional.

2 Let X be a Banach space, and let S denote the unit sphere $S = \{x \in X : \|x\| = 1\}$.

a. Suppose $y_n \in S$ for all n, and $y_n \to y \in X$ weakly. Show that $\|y\| \le 1$.

b. Suppose that X is a separable infinite dimensional Hilbert space, $y \in X$ and $\|y\| \le 1$. Show that there exists a sequence $\{y_n\}_{n=1}^{\infty}$ with $y_n \in S$ for all n such that $y_n \to y$ weakly.

3 Let $f, g \in L^1(\mathbb{T})$, $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, and assume that for any $\psi \in C^{\infty}(\mathbb{T})$,

$$\int f(t)\psi'(t)\,dt = -\int g(t)\psi(t)\,dt.$$

(If this holds, it is common to say that "g is a weak derivative of f".) Prove that f is absolutely continuous and $f' = g$ a.e.

4 Suppose that X, Y are Banach spaces, $T \in \mathcal{L}(X, Y)$ is compact.

a. If $S \in \mathcal{L}(X, X)$, $R \in \mathcal{L}(Y, Y)$, show that $TS, RT \in \mathcal{L}(X, Y)$ are compact.

b. Show that $T^* \in \mathcal{L}(Y^*, X^*)$ is compact.

5 A Banach space B is *uniformly convex* if for every $\epsilon \in (0, 1)$ there exists $\eta < 1$ such that if $x, y \in B$, $\|x\| = \|y\| = 1$ and $\|x - y\| > 2\epsilon$ then $\|\frac{1}{2}(x + y)\| < \eta$.

Let B be a uniformly convex Banach space.

a. Assume that $x_n \in B$ for $n = 1, 2, \dots$ and $x_n \to x_0$ in the weak topology, and $\|x_n\| \to \|x_0\|$. Prove that $x_n \to x_0$ in norm.

Hint: Assume that $\{x_n\}$ is not a Cauchy sequence, and for suitable pairs n_j, m_j, $n_j < m_j$, $n_j \to \infty$, consider $y_j = (x_{n_j} + x_{m_j})/2$.

b. Give an example that the statement in (a) is false for general Banach spaces.

Ph.D. Qualifying Exam, Real Analysis

Fall 2009, part II

Do all five problems. Write your solution for each problem in a separate blue book.

1 Two short problems:

a. Suppose that X is a finite dimensional real vector space. Show that all norms on X are equivalent.

b. Let X be a complex Hilbert space, Y a subspace of X (with the induced norm), $f : Y \to \mathbb{C}$ continuous linear. Show that f has a *unique* continuous linear extension $F : X \to \mathbb{C}$ with the same norm as f.

2 Let μ denote the Lebesgue measure on $[0,1]$. For $1 < p < \infty$ construct

a. a subspace of $L^p([0,1],\mu)$ which is not dense in L^p but is dense in L^r for all $r < p$.

b. a subspace of $L^\infty([0,1],\mu)$ which is dense in L^p but not dense in L^s for any $s > p$.

3 For $f \in C([0,1])$, $x \in [0,1]$, let $(Tf)(x)$ be given by

$$(Tf)(x) = \int_0^x f(y)\,dy.$$

Show that $T : C([0,1]) \to C([0,1])$ is bounded, find (with proof) $\|T\|$ and the spectral radius of T.

4 Suppose that X is a locally convex vector space with topology $\mathcal{T}$ generated by a family $\{\rho_a :\ a \in A\}$ of seminorms.

a. If $\|.\|$ is a continuous seminorm on X, show that there exist $a_1, \ldots, a_n \in A$ and $C > 0$, such that $\|x\| \leq C(\rho_{a_1}(x) + \ldots + \rho_{a_n}(x))$ for all $x \in X$.

b. Let $X = C^\infty(\mathbb{T})$, $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. Show that X does not have a norm generating its standard topology (given by $\{\rho_k = \|.\|_{C^k} :\ k \geq 0,\ k \in \mathbb{Z}\}$).

5 Let $f : \mathbb{R} \to \mathbb{C}$ be a Schwartz function (i.e. f is C^∞ and $\sup |x^N \frac{d^k f}{dx^k}| < \infty$ for all N and k) satisfying $\int_{\mathbb{R}} |f(x)|^2\,dx = 1$. Let $\hat{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy}\, f(x)\,dx$ denote the Fourier transform of f.

a. Show that

$$\left(\int_{\mathbb{R}} (x - x_0)^2\, |f(x)|^2\,dx \right) \left(\int_{\mathbb{R}} (y - y_0)^2\, |\hat{f}(y)|^2\,dy \right) \geq \frac{1}{4}$$

for all $x_0, y_0 \in \mathbb{R}$.

Hint: You may find it easier to reduce to, and then work with, the case $x_0 = y_0 = 0$.

b. Suppose that the equality sign holds in (a). Show that there exists a real number $\lambda > 0$ and a complex number c such that $|c| = 1$ and

$$f(x) = c\,(\pi\lambda)^{-\frac{1}{4}}\, \exp\left(-\frac{(x - x_0)^2}{2\lambda} + ixy_0 \right)$$

for all $x \in \mathbb{R}$.