Ph.D. Qualifying Exam, Real Analysis

Spring 2022, part I

Do all five problems. Write your solution for each problem in a separate blue book.

1 Suppose that $(X, \mathcal{A}, \mu)$ is a measure space, with $\mu(X) = 1$, f a measurable function on X such that $f > 0$ μ-almost everywhere. Show that for all $\epsilon > 0$ there is $\delta > 0$ such that $A \in \mathcal{A}$, $\mu(A) \geq \epsilon$ implies $\int_A f\, d\mu \geq \delta$.

2 Consider $c_0 \subset \ell_\infty$, where c_0 is the space of real valued sequences converging to 0, equipped with the supremum norm (and ℓ_∞ is the space of bounded sequences, with the $\sup$ norm, as usual).

a. Is $c_0 \subset \ell_\infty$ dense in the weak topology? Prove or disprove.

b. Viewing $\ell_\infty = (\ell_1)^*$, we can endow ℓ_∞ with the weak-* topology. Is $c_0 \subset \ell_\infty$ dense in the weak-* topology? Again, prove or disprove.

3

a. Let $G(\varphi) = \lim_{\epsilon \to 0^+} \int_{|x| \geq \epsilon} \frac{\varphi(x)}{x}\, dx$ for $\varphi \in \mathcal{S}(\mathbb{R})$. Show that G is a well-defined tempered distribution.

b. Define what one means by "the Fourier transform of G" and compute it.

4 Recall that for $s \geq 0$ the Sobolev space $H^s(\mathbb{R}^n)$ consists of $u \in L^2(\mathbb{R}^n)$ with

$$\int (1 + |\xi|^2)^s |(\mathcal{F}u)(\xi)|^2\, d\xi < \infty,$$

where $\mathcal{F}$ is the Fourier transform. Let $\mathbb{R}^n = \mathbb{R}^{n-k} \oplus \mathbb{R}^k$, $0 < k \leq n$, be the standard decomposition (corresponding to the standard basis). Let $R : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^{n-k})$ denote the restriction map $(Ru)(x') = u(x', 0)$, $x' \in \mathbb{R}^{n-k}$, where $(x', 0) = \mathbb{R}^{n-k} \oplus \mathbb{R}^k = \mathbb{R}^n$. Show that for $s > k/2$, R has a unique continuous extension to a map $R : H^s(\mathbb{R}^n) \to H^{s-k/2}(\mathbb{R}^{n-k})$.

5 Consider a metric space (X, d). For $S \subset (X, d)$ write $\mathrm{Lim}(S)$ for the set of limit points of S, $\mathrm{Lim}^2(S) = \mathrm{Lim}(\mathrm{Lim}(S))$, and $\mathrm{Lim}^N(S)$ for the limit point operation iterated N times. Assume that $S \subset (X, d)$ is compact and has $\mathrm{Lim}^N(S) = \emptyset$ for some N. Prove that for any $\epsilon > 0$ there is a finite set $x_1, ..., x_m$ in S and $r_j < \epsilon$ so that $S \subset \cup_{j=1}^m B_{r_j}(x_j)$ and the open balls $B_{r_j}(x_j)$ are pairwise disjoint.

Ph.D. Qualifying Exam, Real Analysis

Spring 2022, part II

Do all five problems. Write your solution for each problem in a separate blue book.

1 Prove that there is a signed Radon measure ν on $[0,1]$ such that $\int_{[0,1]} p\,d\nu = p'(0)$ for every polynomial of degree at most 100. Is there a signed Radon measure μ such that $\int_{[0,1]} p\,d\mu = p'(0)$ for every polynomial p? Prove or disprove.

2 Suppose $\mathcal{H}$ is an infinite dimensional separable Hilbert space.

a. Give an example of a bounded operator, $A \in \mathcal{L}(\mathcal{H})$, such that A is non-compact and $A^2 = 0$.

b. Show that if $K \in \mathcal{L}(\mathcal{H})$ is compact, $K = K^*$ and $K^n = 0$ for some $n \geq 2$ then $K = 0$.

c. Suppose $A \in \mathcal{L}(\mathcal{H})$ is bounded, $AA^* = A^*A$ and $A^n = 0$ for some $n \geq 2$. Does it follow that $A = 0$? Prove or disprove.

3 Suppose that $K \in L^p([0,1] \times [0,1])$, $1 < p < \infty$. Let q be the dual exponent, $p^{-1} + q^{-1} = 1$.

a. For $f \in L^q([0,1])$, let $(Af)(x) = \int K(x,y)f(y)dy$. Show that $(Af)(x)$ indeed exists for almost every x and $A \in \mathcal{L}(L^q([0,1]), L^p([0,1]))$.

b. Suppose that for every $f \in L^q([0,1])$, $(Af)(x) = 0$ for almost every x. Show that $K = 0$ a.e.

4 Suppose that $\{a_n : n = 0, 1, 2, \ldots\}$ is *any* sequence of real numbers. Show that there exists a real valued function $f \in C^\infty(\mathbb{R})$ such that $f^{(n)}(0) = a_n$.

5 Consider the operator $P = -\frac{d}{dx} p \frac{d}{dx} + q$ on $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, $p, q \in C^0(\mathbb{T})$ are real valued (acting as multiplication operators), and where $p > 0$. Let $\langle ., . \rangle$ be the $L^2(\mathbb{T})$ inner product.

a. Show that $P : H^1(\mathbb{T}) \to H^{-1}(\mathbb{T})$ is continuous, and $\langle Pu, v \rangle = \langle u, Pv \rangle$ holds for $u, v \in H^1(\mathbb{T})$.

b. Show that there exist $C > 0$ and $C' \geq 0$ such that for $u \in H^1(\mathbb{T})$, $\langle Pu, u \rangle \geq C\|u\|_{H^1}^2 - C'\|u\|_{L^2}^2$, and if $q > 0$, one can take $C' = 0$.

c. Show that $\|u\|_{H^1} \leq C_1(\|Pu\|_{H^{-1}} + \|u\|_{L^2})$, and if $C' = 0$ then in fact $\|u\|_{H^1} \leq C_1\|Pu\|_{H^{-1}}$.

d. Show that if $C' = 0$ then $P : H^1 \to H^{-1}$ is invertible.

e. Show that even if $C' \neq 0$, $\mathrm{Ker}P \subset H^1$ is finite dimensional, $\mathrm{Ran}P \subset H^{-1}$ is closed, and has finite codimension.