Ph.D. Qualifying Exam, Real Analysis

Fall 2018, part I

Do all five problems. Write your solution for each problem in a separate blue book.

1 Two short problems.

a. Define the function ν from subsets of $\mathbb{R}$ to $[0,\infty]$ as follows. If $A \subset \mathbb{R}$, let $\nu(A) = \infty$ if 0 is in the closure of A, and let $\nu(A) = 0$ otherwise. Show that ν is finitely additive but it is not countably additive.

b. Let m be the Lebesgue measure on $\mathbb{R}$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is measurable and let $S = \{x \in \mathbb{R} : m(f^{-1}(\{x\})) > 0\}$. Show that S is countable, and give an example to show that S can be dense in $\mathbb{R}$.

2 Given a Schwartz function $f \in \mathcal{S}(\mathbb{R})$, define the $H^s(\mathbb{R})$ norm by $\|f\|_{H^s(\mathbb{R})} := \|(1+|\xi|^2)^{\frac{s}{2}} \hat{f}(\xi)\|_{L^2(\mathbb{R})}$, where $\hat{f}$ denotes the Fourier transform of f and $s \in \mathbb{R}$.

a. Prove that for every $s > \frac{1}{2}$, there exists a constant $C > 0$ (depending only on s) such that for every $f \in \mathcal{S}$,

$$\|f\|_{L^\infty(\mathbb{R})} \leq C\|f\|_{H^s(\mathbb{R})}.$$

b. Let $K = \{f \in \mathcal{S}(\mathbb{R}) : \exists d > 0 \text{ such that } \hat{f}(\xi) \neq 0 \Rightarrow d \leq |\xi| \leq 2d\}$. Prove that there exists a constant $C > 0$ (independent of d) such that for every $f \in K$,

$$\|f\|_{L^\infty(\mathbb{R})} \leq C\|f\|_{H^{\frac{1}{2}}(\mathbb{R})}.$$

3 Suppose X is a vector space over $\mathbb{C}$ and $\mathcal{F}$ is a collection of linear maps $X \to \mathbb{C}$. Equip X with the $\mathcal{F}$-weak topology, i.e. the weakest topology in which all elements of $\mathcal{F}$ are continuous.

a. Show that the vector space operations $+ : X \times X \to X$ and $\cdot : \mathbb{C} \times X \to X$ are continuous in this topology (where $X \times X$ and $\mathbb{C} \times X$ are equipped with the product topology).

b. Suppose that $\rho : X \to [0,\infty)$ is continuous and is a seminorm. Show that there exist $k \in \mathbb{N}$, $\ell_1, \ldots, \ell_k \in \mathcal{F}$ and $C > 0$ such that $\rho(x) \leq C \sum_{j=1}^k |\ell_j(x)|$ for all $x \in X$.

4 Suppose that X, Y are Hilbert spaces, $P \in \mathcal{L}(X,Y)$, the set of bounded linear operators $X \to Y$. Suppose also that there is $C \geq 0$ such that $\|x\|_X \leq C\|Px\|_Y$ for all $x \in X$.

a. Show that P is surjective if and only if P^* is injective.

b. Show that if P is surjective then $\|v\|_{Y^*} \leq C\|P^*v\|_{X^*}$ for all $v \in Y^*$, where C is as above.

5 Let $\chi_{[\alpha,\beta]}$ denote the characteristic (indicator) function of the interval $[\alpha,\beta]$, as well as the corresponding multiplication operator. Below let $\mathcal{F}$ be the Fourier transform on $L^2(\mathbb{R})$.

a. Show that for $a, b > 0$,

$$\mathcal{F}^{-1}\chi_{[-b,b]}\mathcal{F}\chi_{[-a,a]} \in \mathcal{L}(L^2(\mathbb{R}))$$

is compact.

b. Show that for $a, b > 0$,

$$\mathcal{F}^{-1}(1 - \chi_{[-b,b]})\mathcal{F}\chi_{[-a,a]} \in \mathcal{L}(L^2(\mathbb{R}))$$

is *not* compact.

Ph.D. Qualifying Exam, Real Analysis

Fall 2018, part II

Do all five problems. Write your solution for each problem in a separate blue book.

1 Two short problems.

a. Show that a Hilbert space is separable as a metric space (i.e. has a countable dense subset) if and only if it has a countable complete orthonormal set (basis).

b. Show that in a separable Hilbert space the closed unit ball is weakly sequentially compact, i.e. any sequence $\{x_n\}_{n=1}^{\infty}$ in it has a weakly convergent subsequence.

2 Suppose X, Y and Banach spaces, $A_j, A \in \mathcal{L}(X, Y)$, $j \in \mathbb{N}$, and A is invertible. (Here $\mathcal{L}(X, Y)$ is the set of bounded linear operators $X \to Y$.)

a. Suppose that $A_j \to A$ in norm in $\mathcal{L}(X, Y)$. Show that there exists N such that for $j \geq N$, A_j is also invertible.

b. Suppose now that $A_j \to A$ in the strong operator topology on $\mathcal{L}(X, Y)$. Does it follow that A_j is invertible for sufficiently large j? Prove it, or give a counterexample.

3 Suppose that $K \in L^p([0,1] \times [0,1])$, $1 < p < \infty$. Let q be the dual exponent, $p^{-1} + q^{-1} = 1$.

a. For $f \in L^q([0,1])$, let $(Af)(x) = \int K(x,y) f(y) dy$. Show that $(Af)(x)$ indeed exists for almost every x and $A \in \mathcal{L}(L^q([0,1]), L^p([0,1]))$.

b. Suppose that for every $f \in L^q([0,1])$, $(Af)(x) = 0$ for almost every x. Show that $K = 0$ a.e.

4 Suppose $T : X \to Y$ is a Fredholm map between two complex Banach spaces, i.e. has closed range, and $\mathrm{Ker} T$, $Y / \mathrm{Ran} T$ are finite dimensional.

a. Show that there exist $m, n \geq 0$ integers such that $X \oplus \mathbb{C}^m$ is isomorphic to $Y \oplus \mathbb{C}^n$ as Banach spaces (there is an invertible continuous linear map between them).

b. Prove that X is separable if and only if Y is separable.

5 Let $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ be the unit circle, and consider the integral $I(r) = \int_{\mathbb{T}} e^{ir\cos\theta} \varphi(\theta)\, d\theta$, $\varphi \in C^\infty(\mathbb{T})$, where $d\theta$ is the Lebesgue measure on $\mathbb{T}$. Show that there exists $C > 0$ such that $|I(r)| \leq C r^{-1/2}$, $r \geq 1$. *Hint:* Show that if φ is supported away from $[0], [\pi] \in \mathbb{R}/(2\pi\mathbb{Z})$, then $I(r)$ is rapidly decreasing as $r \to \infty$; then assume φ is supported near $[0]$ or $[\pi]$, and change variables to obtain an integral of the form $\int e^{\pm irs^2} \tilde{\varphi}(s)\, ds$ (times a prefactor). (Note: $I(r)$ is essentially the Fourier transform, evaluated at $(r, 0)$, of a delta distribution on the unit circle in $\mathbb{R}^2$.)