Ph.D. Qualifying Exam, Real Analysis

Fall 2024, part I

Do all five problems. Write your name on the solutions. Use separate pages for separate problems.

You may write on both sides of a page. If you use more than one page for a problem, please staple them together with the stapler provided and make sure that you are stapling pages in the correct order.

1 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an infinitely differentiable function. Suppose $f(x, y) = 0$ whenever $xy = 0$. Prove that there exists $C > 0$ such that

$$|f(x, y)| \leq C|x||y|, \quad \text{for all } (x, y) \in B(0, 1).$$

2 Prove that there exists $\Lambda \in (\ell_\infty)^*$ such that the following all hold:

a. $\|\Lambda\|_{(\ell_\infty)^*} = 1$.

b. If $a = (a_i)_{i=1}^\infty$ with $a_i = 1$ for all i, then $\Lambda(a) = 1$.

c. If $b = (b_i)_{i=1}^\infty \in \ell_\infty$ and $c = (c_i)_{i=1}^\infty$ is given by $c_i = b_{i+1} - b_i$ for all i, then $\Lambda(c) = 0$.

3 Let $\{f_n\}_{n=1}^\infty \subset L^2([0, 1])$. Suppose $\|f_n\|_{L^2} \leq 2024\|f_n\|_{L^1}$ for all $n \in \mathbb{Z}_{>0}$. If $\lim_{n\to\infty} \|f_n\|_{L^1} = \infty$, prove that there exists a sequence $y_n \to \infty$ such that

$$\inf_n |\{x : |f_n(x)| \geq y_n\}| > 0.$$

4

a. Define the linear map $S_1 : C_c^\infty(\mathbb{R}; \mathbb{R}) \to \mathbb{R}$ by

$$S_1(f) = \int_{-\infty}^{\infty} f(x)e^x \, \mathrm{d}x \quad \forall f \in C_c^\infty(\mathbb{R}; \mathbb{R}).$$

Prove that there exists <u>no</u> tempered distribution $T_1 \in \mathcal{S}'(\mathbb{R}; \mathbb{R})$ such that $T_1(f) = S_1(f)$ for all $f \in C_c^\infty(\mathbb{R}; \mathbb{R})$.

b. Define the linear map $S_2 : C_c^\infty(\mathbb{R}; \mathbb{R}) \to \mathbb{R}$ by

$$S_2(f) = \int_{-\infty}^{\infty} f(x)e^x \cos(e^x) \, \mathrm{d}x, \quad \forall f \in C_c^\infty(\mathbb{R}; \mathbb{R}).$$

Prove that there exists a tempered distribution $T_2 \in \mathcal{S}'(\mathbb{R}; \mathbb{R})$ such that $T_2(f) = S_2(f)$ for all $f \in C_c^\infty(\mathbb{R}; \mathbb{R})$.

5 Suppose $f : [0, 1] \to [0, \infty]$ has the property that if $f(y) \neq \infty$, then $\liminf_{x\to y} f(x) > f(y)$. Prove that $\{y \in [0, 1] : f(y) < \infty\}$ is (at most) countable.

Ph.D. Qualifying Exam, Real Analysis

Fall 2024, part II

Do all five problems. Write your name on the solutions. Use separate pages for separate problems.

You may write on both sides of a page. If you use more than one page for a problem, please staple them together with the stapler provided and make sure that you are stapling pages in the correct order.

1 Suppose $f \in L^1(\mathbb{R};\mathbb{R})$ and $\mathcal{F}f$ has compact support, where $\mathcal{F}$ denotes the Fourier transform. Find a function $g \in L^1(\mathbb{R};\mathbb{R}) \cap C(\mathbb{R};\mathbb{R})$ such that $f * g = 0$ (where $*$ denotes the convolution) but g is non-zero Lebesgue almost everywhere.

2 Suppose $\phi : \mathbb{Z} \to \mathbb{C}$ satisfies the property that for every $f \in L^1(\mathbb{R}/(2\pi\mathbb{Z}))$, there exists $g \in L^1(\mathbb{R}/(2\pi\mathbb{Z}))$ such that

$$\mathcal{F}g(n) = \phi(n)\mathcal{F}f(n), \quad \forall n \in \mathbb{Z}, \tag{1}$$

where $\mathcal{F}f(n) = \int e^{-inx} f(x)\,\mathrm{d}x$. Define $T : L^1(\mathbb{R}/(2\pi\mathbb{Z})) \to L^1(\mathbb{R}/(2\pi\mathbb{Z}))$ by $Tf = g$, where g is given by (1). Prove that T is a bounded linear operator on $L^1(\mathbb{R}/(2\pi\mathbb{Z}))$.

3 Let $f : [0,1] \to [0,\infty)$ be a measurable function. Show that the following statements are equivalent:

a. $\int_{[0,1]\times[0,1]} f(x^2y)dxdy < +\infty$.

b. $\int_{[0,1]\times[\frac{1}{2},1]} f(x^2y)dxdy < +\infty$.

4 Let $\{f_n(x,y,z)\}_{n=1}^{\infty}$ be a sequence of continuously differentiable functions on $\mathbb{R}^3$ which satisfy

$$\left(\int_{\mathbb{R}^3} (|\nabla f_n|^2 + |f_n|^2)(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \right)^{\frac{1}{2}} \leq 1.$$

a. Prove that $\{f_n(x,y,0)\}_{n=0}^{\infty}$ is a bounded sequence on the $\{z = 0\}$ plane with respect to the $L^2(\mathrm{d}x\mathrm{d}y)$ norm.

b. Suppose that

$$\left(\int_{\mathbb{R}^3} (|\nabla f_n|^2 + (1 + x^2 + y^2 + z^2)|f_n|^2)(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \right)^{\frac{1}{2}} \leq 10.$$

Prove that there is a subsequence $\{f_{n_k}(x,y,0)\}_{k=0}^{\infty}$ which is convergent on the $\{z = 0\}$ plane with respect to the $L^2(\mathrm{d}x\mathrm{d}y)$ norm.

5 Suppose $T : X \to Y$ is a bounded linear operator between Hilbert spaces and T^*T is Fredholm (i.e., it has closed range, as well as finite-dimensional kernel and co-kernel).

a. Show that T has closed range.

b. Is T necessarily Fredholm? Prove it or give a counterexample.