Ph.D. Qualifying Exam, Real Analysis

Fall 2012, part I

Do all five problems. Write your solution for each problem in a separate blue book.

1 Two short problems.

a. Suppose u is a distribution on $\mathbb{R}$ and $x^k u = 0$ for some $k \in \mathbb{N}$, $k \geq 1$. Show that there exists $a_j \in \mathbb{C}$ such that $u(\phi) = \sum_{j=0}^{k-1} a_j \phi^{(j)}(0)$.

b. Suppose that (X, μ) is a measure space, $1 < p < \infty$, $u_n \in L^p(X, d\mu)$ for $n \in \mathbb{N}$, and for all $\phi \in L^q(X, d\mu)$, $q^{-1} + p^{-1} = 1$, $\lim_{n\to\infty} \int_X u_n \phi \, d\mu$ exists. Show that there exists $C \geq 0$ such that $\int_X |u_n|^p \, d\mu \leq C$ for all n.

2 Let D_N, $N \geq 1$ integer, be the Dirichlet kernel

$$D_N(\theta) = \frac{1}{2\pi} \frac{\sin(N + \frac{1}{2})\theta}{\sin \frac{1}{2}\theta}.$$

Let $L_N = \int_0^{2\pi} |D_N(\theta)| d\theta$. Prove that there exist $C_1, C_2 > 0$ such that for all $N \geq 2$,

$$C_1 \log N \leq L_N \leq C_2 \log N.$$

3 Two short problems.

a. Show that there is a closed subset E of $[0, 1]$ with positive Lebesgue measure and with empty interior.

b. Show that if $f : [0, 1] \to \mathbb{R}$ is absolutely continuous and $A \subset [0, 1]$ is Lebesgue measurable with measure 0 then $f(A)$ is measurable with measure 0.

4 Suppose that X, Y are Banach spaces, and let $\mathcal{T}_s$ denote the norm topology on X, $\mathcal{U}_s$ the norm topology on Y. Let $\mathcal{T}_w$ denote the weak topology on X, and $\mathcal{U}_w$ denote the weak topology on Y.

a. Show that $(X, \mathcal{T}_w)$ has the following property, sometimes called (T$3\frac{1}{2}$) or *completely regular*: if $x \in X$ then $\{x\}$ is $\mathcal{T}_w$-closed, and given any $x \in X$ and $C \subset X$ $\mathcal{T}_w$-closed with $x \notin C$, there is a continuous function $f : X \to [0, 1]$ with $f(x) = 1$ and f identically 0 on C.

b. Show that a linear map $T : X \to Y$ is continuous as a map from $(X, \mathcal{T}_s)$ to $(Y, \mathcal{U}_s)$ if and only if it is continuous as a map from $(X, \mathcal{T}_w)$ to $(Y, \mathcal{U}_w)$.

5 Suppose that X, Y are Hilbert spaces. An operator $A \in \mathcal{L}(X, Y)$ is *Fredholm* if A has closed range, and $\mathrm{Ker}A$ as well as $Y/\mathrm{Ran}A$ are finite dimensional.

a. Show that $A \in \mathcal{L}(X, Y)$ is Fredholm if and only if there are finite dimensional vector spaces V, W and a finite rank operator $P \in \mathcal{L}(W \oplus X, V \oplus Y)$ such that $\tilde{A} + P$ is invertible, where $\tilde{A} \in \mathcal{L}(W \oplus X, V \oplus Y)$ is defined by $\tilde{A}(w, x) = (0, Ax)$, $x \in X$, $w \in W$.

b. Suppose $A \in \mathcal{L}(X, Y)$ is Fredholm. Show that there exists $\delta > 0$ such that if $R \in \mathcal{L}(X, Y)$ with $\|R\|_{\mathcal{L}(X,Y)} < \delta$ then $A + R$ is Fredholm.

Ph.D. Qualifying Exam, Real Analysis

Fall 2012, part II

Do all five problems. Write your solution for each problem in a separate blue book.

1 Two short problems.

a. Show that the spectrum of a bounded linear operator A on a Banach space X is non-empty.

b. Show that if $1 < p < \infty$ then for $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, $p^{-1} + q^{-1} = 1$, $f * g$ defined by $(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy$ is a bounded continuous function with sup norm $\leq \|f\|_{L^p}\|g\|_{L^q}$.

2 Two short problems.

a. Let P denote the space of continuous piecewise affine functions on $[0,1]$. Show that any $f \in \mathcal{C}([0,1])$ is a uniform limit of elements of P.

b. Show that the inclusion map $\iota : C([0,1]) \to L^2([0,1])$ is *not* compact.

3 Let $\ell^2(\mathbb{Z})$ denote the space of square summable bi-infinite sequences. Let L denote the operator of multiplication by n, and let $\mathcal{H} = \{\{a_n\}_{n=-\infty}^{\infty} \in \ell^2(\mathbb{Z}) : \{na_n\}_{n=-\infty}^{\infty} \in \ell^2(\mathbb{Z})\} \subset \ell^2(\mathbb{Z})$ with $\|\{a_n\}_{n=1}^{\infty}\|_{\mathcal{H}}^2 = \sum_{n\in\mathbb{Z}}(1+n^2)|a_n|^2$. Let $R \in \mathcal{L}(\ell^2(\mathbb{Z}), \ell^2(\mathbb{Z}))$.

a. Show that there is a discrete set $D \subset \mathbb{C}$ such that $L + R - \lambda I : \mathcal{H} \to \ell^2$ is invertible for $\lambda \notin D$.

b. With $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, show that if $V \in C(\mathbb{T})$ then the set of $\lambda \in \mathbb{C}$ for which there exists $f \in C^1(\mathbb{T})$, not identically 0, for which $f' + Vf = \lambda f$, is discrete.

4 In this problem, let $\|.\|_p$ be the $L^p(\mathbb{R}^n)$ norm, $1 \leq p \leq \infty$, and let $\mathcal{C}_0^\infty(\mathbb{R})$ be the set of compactly supported $\mathcal{C}^\infty$ functions.

a. Show, including the explicit constant, that for $\phi \in \mathcal{C}_0^\infty(\mathbb{R})$, $\|\phi\|_\infty \leq \frac{1}{2}\int_{-\infty}^{\infty} |\phi'(t)|dt$.

b. Suppose that there is $C > 0$ such that for all functions $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ there is an inequality of the form $\|\phi\|_q \leq C\|\nabla\phi\|_p$. Show that one would necessarily then have the relationship $q^{-1} = p^{-1} - n^{-1}$. (Hint: consider the functions ϕ_t defined by $\phi_t(x) = \phi(tx)$.)

c. When $n = 2$, part b) suggests that one might have an inequality of the form $\|\phi\|_\infty \leq C\|\nabla\phi\|_2$. Show that there is no $C > 0$ such that this inequality holds for all $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$.

5 Let Ω be a compact polygonal domain in $\mathbb{R}^2$, i.e. Ω is an open set with compact closure such that each $x \in \partial\Omega$ has a neighborhood O_x such that $\Omega \cap O_x$ is given by the intersection of one or two half-planes with O_x.

Show that there is $C > 0$ such that the Fourier transform of the characteristic function χ_Ω of Ω satisfies $|(\mathcal{F}\chi_\Omega)(\xi)| \leq C(1 + |\xi|)^{-1}$, $\xi \in \mathbb{R}^2$. (Hint: reduce to the case of the Fourier transform of a product of a cutoff function with one or two characteristic functions of half-planes.)

Are there non-trivial open cones Σ in $\mathbb{R}^2 \setminus \{0\}$ such that a faster decay estimate holds as $\Sigma \ni \xi \to \infty$?