Ph.D. Qualifying Exam, Real Analysis

Spring 2015, part I

Do all five problems. Write your solution for each problem in a separate blue book.

1 Suppose $A, B \subset \mathbb{R}/\mathbb{Z}$ are measurable of positive Lebesgue measure: $m(A), m(B) > 0$.

a. Show that there exists $y \in \mathbb{R}/\mathbb{Z}$ such that $m((A + y) \cap B) > 0$.

b. Show that in fact there exists $y \in \mathbb{R}/\mathbb{Z}$ such that $m((A + y) \cap B) \geq m(A)m(B)$.

2 Let X, Y be Banach spaces.

a. Show that if $T_n \in \mathcal{L}(X, Y)$ are compact, and $T_n \to T \in \mathcal{L}(X, Y)$ in norm, then T is compact.

b. Show that if X, Y are separable Hilbert spaces then every compact operator $T \in \mathcal{L}(X, Y)$ is the norm limit of finite rank operators.

3 Let $\mathcal{S}(\mathbb{R})$ be the set of Schwartz functions on $\mathbb{R}$, i.e. the set of C^∞ functions ϕ on $\mathbb{R}$ with $x^\alpha \partial_x^\beta \phi$ bounded for all $\alpha, \beta \in \mathbb{N}$.

a. With the Fourier transform given by $(\mathcal{F}\phi)(\xi) = \int e^{-ix\xi}\, \phi(x)\, dx$, show the Poisson summation formula for $\phi \in \mathcal{S}(\mathbb{R})$:

$$2\pi \sum_{n\in\mathbb{Z}} \phi(x + 2n\pi) = \sum_{n\in\mathbb{Z}} (\mathcal{F}\phi)(n) e^{inx}, \qquad x \in \mathbb{R}.$$

b. Show that for any $t > 0$ we have

$$\sum_{n\in\mathbb{Z}} \exp(-t(2\pi n + \pi)^2/2) = \frac{1}{\sqrt{2\pi t}} \sum_{k\in\mathbb{Z}} (-1)^k e^{-k^2/(2t)}.$$

4 Let $\mathcal{S}(\mathbb{R}^n)$ denote set of Schwartz functions, and $\mathcal{S}'(\mathbb{R}^n)$ the dual space of tempered distributions. For $u \in \mathcal{S}'(\mathbb{R}^n)$ let $D_j u$ denote the distributional derivative of u in the jth coordinate. Let $\mathcal{H} = \{u \in L^2(\mathbb{R}^2) :\ D_2 u \in L^2(\mathbb{R}^2)\}$, equipped with the norm $\|u\|_{\mathcal{H}}^2 = \|u\|_{L^2}^2 + \|D_2 u\|_{L^2}^2$.

a. Show that $\mathcal{H}$ is a Hilbert space (with the norm being induced by the inner product), and $\mathcal{S}(\mathbb{R}^2)$ is dense in $\mathcal{H}$.

b. Show that the restriction map $R : \mathcal{S}(\mathbb{R}^2) \to \mathcal{S}(\mathbb{R})$, $(R\phi)(x_1) = \phi(x_1, 0)$, to $x_2 = 0$, has a unique continuous extension to a map $\mathcal{H} \to L^2(\mathbb{R})$.

5 Suppose that X is a Banach space, and let $B = \{x \in X :\ \|x\|_X \leq 1\}$ be the unit ball in X.

a. Suppose Z is a finite dimensional subspace of X. Show that there exists a closed subspace W of X such that $X = Z \oplus W$ (direct sum).

b. If X is infinite dimensional, show that B is not compact in the norm topology.

c. Suppose that X, Y are Banach spaces, $X \subset Y$ with the inclusion map $\iota : X \to Y$ continuous and compact. Let $T \in \mathcal{L}(X, Y)$, and suppose that for all $x \in X$, $\|x\|_X \leq C(\|Tx\|_Y + \|x\|_Y)$. Show that $\mathrm{Ker}T$ is finite dimensional, $\mathrm{Ran}T$ is closed, and the induced map $X/\mathrm{Ker}T \to \mathrm{Ran}T$ is invertible as a bounded linear map.

Ph.D. Qualifying Exam, Real Analysis

Spring 2015, part II

Do all five problems. Write your solution for each problem in a separate blue book.

1 Suppose that $f \in L^1([0,1])$. Prove that there are nondecreasing sequences of continuous functions, $\{\varphi_k\}_{k=1}^\infty$ and $\{\psi_k\}_{k=1}^\infty$, on $[0,1]$ such that for a.e. $x \in [0,1]$ (with respect to Lebesgue measure), both $\varphi_k(x)$ and $\psi_k(x)$ are bounded sequences, and moreover,

$$f(x) = \lim_{k\to\infty} \varphi_k(x) - \lim_{k\to\infty} \psi_k(x).$$

2 Let X be a vector space over $\mathbb{C}$, $\mathcal{F}$ a vector space of linear maps $X \to \mathbb{C}$, and equip X with the weakest topology in which all members of $\mathcal{F}$ are continuous. Show that the only continuous linear maps $X \to \mathbb{C}$ are those in $\mathcal{F}$.

3 Consider the partial sums $(S_N f)(x) = \sum_{|n|\leq N} c_n e^{inx}$ of the Fourier series of continuous functions f on $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, where c_n are the Fourier coefficients of f, and recall that $S_N f$ is given by the convolution of the Dirichlet kernel D_N with f.

a. Show that $\|D_N\|_{L^1} \to \infty$ as $N \to \infty$.

b. Show that there exists a continuous function f on $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ such that the Fourier series of f does not converge uniformly to f, i.e. $S_N f$ does not converge uniformly to f as $N \to \infty$.

4 Suppose $\mathcal{H}$ is a Hilbert space. Recall that $U \in \mathcal{L}(\mathcal{H})$ is unitary if $UU^* = I = U^*U$.

a. Show that if U is unitary then $\overline{\mathrm{Ran}(I-U)} \oplus \mathrm{Ker}(I-U) = \mathcal{H}$ (orthogonal direct sum).

b. Let P be orthogonal projection to $\mathrm{Ker}(I-U)$. Let $S_n = \frac{1}{n}\sum_{j=0}^{n-1} U^j$. Show that $S_n \to P$ in the strong operator topology (i.e. $S_n f \to Pf$ in $\mathcal{H}$ for all $f \in \mathcal{H}$). (This is the *von Neumann*, or *mean ergodic theorem.*)

c. Give an example of a unitary operator U on ℓ^2 such that S_n does not converge to P in norm.

5 Let $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ be the unit circle, and consider the integral $I(r) = \int_{\mathbb{T}} e^{ir\cos\theta}\varphi(\theta)\,d\theta$, $\varphi \in C^\infty(\mathbb{T})$, where $d\theta$ is the Lebesgue measure on $\mathbb{T}$. Show that there exists $C > 0$ such that $|I(r)| \leq Cr^{-1/2}$, $r \geq 1$. *Hint:* Show that if φ is supported away from $[0], [\pi] \in \mathbb{R}/(2\pi\mathbb{Z})$, then $I(r)$ is rapidly decreasing as $r \to \infty$; then assume φ is supported near $[0]$ or $[\pi]$, and change variables to obtain an integral of the form $\int e^{\pm irs^2}\tilde{\varphi}(s)\,ds$ (times a prefactor). (Note: $I(r)$ is essentially the Fourier transform, evaluated at $(r,0)$, of a delta distribution on the unit circle in $\mathbb{R}^2$.)